\section{Introduction}

Matroids are combinatorial abstractions of hyperplane arrangements that have served as a nexus between algebraic geometry and combinatorics. One such interaction concerns the Tutte polynomial of a matroid, an important invariant first defined for graphs by Tutte \cite{Tut67} and then for matroids by Crapo \cite{Cra69}.

\begin{definition}
Let $M$ be a matroid of rank $r$ on a finite set $[n] = \{1, 2, \ldots, n\}$ with rank function $\rk_M : 2^n \to \mathbb{Z}_{\geq 0}$. Its \textbf{Tutte polynomial} $T_M(x, y)$ is a bivariate polynomial in $x, y$ defined by
\begin{equation*}
T_M(x, y) := \sum_{S \subseteq [n]} (x - 1)^{r - \rk_M(S)} (y - 1)^{|S| - \rk_M(S)}.
\end{equation*}
\end{definition}

An algebro-geometric interpretation of the Tutte polynomial was given in \cite{FS12} via the $K$-theory of the Grassmannian. Let $Gr(r; n)$ be the Grassmannian of $r$-dimensional linear subspaces in $\mathbb{C}^n$, and more generally let $Fl(r; n)$ be the flag variety of flags of linear spaces of dimensions $r = (r_1, \ldots, r_k)$. A point $L \in Gr(r; n)$ on the Grassmannian corresponds to a realization of a matroid, and its torus-orbit closure defines a $K$-class $[\mathcal{O}_L] \in K^0(Gr(r; n))$ that depends only on the matroid. In general, a matroid $M$ of rank $r$ on $[1, \ldots, n]$ defines a $K$-class $y(M) \in K^0(Gr(r; n))$. Fink and Speyer related $y(M)$ to the Tutte polynomial $T_M(x, y)$ via the diagram

\begin{equation}
\begin{array}{ccc}
Gr(r; n) & \xrightarrow{\pi_r} & Fl(1, r, n - 1; n) \\
\pi_{(n-1)1} & & \\
& & \xrightarrow{} \\
& & Gr(n - 1; n) \times Gr(1; n) \longleftrightarrow (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1},
\end{array}
\end{equation}

where $\pi_r$ and $\pi_{(n-1)1}$ are maps that forget appropriate subspaces in the flag.

\begin{theorem}[FS12, Theorem 5.1]
Let $O(1)$ be the line bundle on $Gr(r; n)$ of the Plücker embedding $Gr(r; n) \hookrightarrow \mathbb{P}^{n(r)}$. With notations as above, we have
\begin{equation*}
T_M(\alpha, \beta) = (\pi_{(n-1)1})_* \pi_r^*(y(M) \cdot [O(1)]) \in K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}) \simeq \mathbb{Q}[\alpha, \beta]/(\alpha^n, \beta^n),
\end{equation*}
where $\alpha, \beta$ are the $K$-classes of the structures sheaves of the hyperplanes of $(\mathbb{P}^{n-1})^\vee, \mathbb{P}^{n-1}$.
\end{theorem}

We extend this relation between the $K$-theory of Grassmannians and matroids to a relation between $K$-theory of flag varieties and flag matroids. As a result, we show that there are (at least) \textit{two} different generalizations of the Tutte polynomial to flag matroids, each with its own merits.
A flag matroid is a sequence of matroids $M = (M_1, \ldots, M_k)$ on a common ground set such that every circuit of $M_i$ is a union of circuits of $M_{i-1}$ for all $2 \leq i \leq k$. The rank of $M$ is the sequence $(\text{rk}(M_1), \ldots, \text{rk}(M_k))$. For the most of the paper, we will concern the case of $k = 2$. In this case, the two-step flag matroids $(M_1, M_2)$ are often called matroid morphisms or matroid quotients. They are combinatorial abstractions of graph homomorphisms, linear surjections, and embeddings of graphs on surfaces. See §2.3 for details on flag matroids and matroid quotients.

Many features of matroids naturally generalize to flag matroids. For instance, just as a point on a Grassmannian corresponds to a realization of a matroid, a point $L$ on the flag variety $Fl(r; n)$ corresponds to a realization of a flag matroid. The torus-orbit closure of $L$ defines a $K$-class $[O_{TL}] \in K^0(Fl(r; n))$ that depends only on the flag matroid. In general, a flag matroid $M$ of rank $r$ on a ground set $\{1, \ldots, n\}$ defines a $K$-class $y(M)$ of the flag variety $Fl(r; n)$. See §2.3 or [CDMS20, §8.5] for details.

At this point, however, extending the constructions on matroids to flag matroids splits into several strands, for there are (at least) two distinguished ways to generalize the diagram (1).

- The "flag-geometric" diagram:

$$
\begin{array}{ccc}
Fl(1, r, n - 1; n) & \pi_r & \pi_{(n-1)1} \\
\downarrow & & \downarrow \\
Fl(r; n) & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}
\end{array}
$$

where $\pi_r$ and $\pi_{(n-1)1}$ are maps that forget appropriate subspaces in the flag.

- The "Las Vergnas" diagram:

$$
\begin{array}{ccc}
\tilde{Fl}(1, r, n - 1; n) & \tilde{\pi}_r & \tilde{\pi}_{(n-1)1} \\
\downarrow & & \downarrow \\
Fl(r; n) & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}
\end{array}
$$

where $\tilde{Fl}(1, r, n - 1; n)$ is the variety defined as

$$
\tilde{Fl}(1, r, n - 1; n) := \left\{ \text{linear subspaces } (\ell_1, \ell_2, \ldots, \ell_k, H) \mid \dim \ell = 1, \dim H = n - 1, (L_1, \ldots, L_k) \in Fl(r; n), \text{ and } \ell \subseteq L_k \text{ and } L_1 \subseteq H \right\},
$$

and $\tilde{\pi}_r$ and $\tilde{\pi}_{(n-1)1}$ are maps that forget appropriate subspaces in the flag.

Let us first consider the construction (\tilde{Fl}). While the construction (\tilde{Fl}) may seem geometrically unnatural, since $\tilde{Fl}(1, r, n - 1; n)$ is not a flag variety, it leads to the previously established notion of Las Vergnas’ Tutte polynomials of morphisms of matroids, defined as follows.

**Definition 1.3.** Let $M = (M_1, M_2)$ be a two-step flag matroid on a ground set $[n] = \{1, \ldots, n\}$. For $i = 1, 2$, let us write $r_i$ for the rank of $M_i$, and $r_i(S)$ for the rank of $S \subseteq [n]$ in $M_i$. The Las Vergnas Tutte polynomial of $(M_1, M_2)$ is a polynomial in three variables $x, y, z$ defined by

$$
LV_T M(x, y, z) := \sum_{S \subseteq [n]} (x - 1)^{r_1(S)} (y - 1)^{r_2(S)} z^{r_2(S) - r_1(S) - r_1(S)}.
$$

Las Vergnas introduced this generalization of the Tutte polynomial in [LV75], and studied its properties in a series of subsequent works [LV80, LV84, LV99, ELV04, LV07, LV13]. Our first main theorem is a $K$-theoretic interpretation of the Las Vergnas Tutte polynomial.
Theorem 5.2. Let \( M = (M_1, M_2) \) be a flag matroid with \( r_1 = \text{rk}(M_1), r_2 = \text{rk}(M_2) \). Let \( \mathcal{O}(0,1) \) be the line bundle on \( Fl(r_1, r_2; n) \) of the map \( Fl(r_1, r_2; n) \to \text{Gr}(r_2; n) \hookrightarrow \mathbb{P}^{(r_2)}_{[n]}-1 \), and let \( S_2/S_1 \) be the vector bundle on \( Fl(r_1, r_2; n) \) whose fiber over a point \((L_1, L_2) \in Fl(r_1, r_2; n)\) is \( L_2/L_1 \). Then,

\[
LV_T(M, \beta, w) = \sum_{m=0}^{r_2-r_1} (\pi_{(n-1)n}) * \pi_{(r_2)}^*(y(M)[\mathcal{O}(0,1)][\wedge(S_2/S_1)]) w^m
\]

as elements in \( K^0((\mathbb{P}^{n-1})^n \times \mathbb{P}^n) [w] \simeq \mathbb{Q}[a, \beta, w] / (a^n, \beta^n) \).

Let us now consider the construction \((\tilde{F})\). It leads to the following different generalization of the Tutte polynomial, which was first defined in the review \cite{CDMS20}.

Definition 6.1. Let \( M \) be a flag matroid of rank \( r = (r_1, \ldots, r_k) \) on \( \{1, \ldots, n\} \), and let \( \mathcal{O}(1) \) be the line bundle of the embedding \( Fl(r; n) \hookrightarrow \text{Gr}(r_1; n) \times \cdots \times \text{Gr}(r_k; n) \hookrightarrow \mathbb{P}^{(r)}_{[n]}-1 \times \cdots \times \mathbb{P}^{(r)}_{[n]}-1 \). The **flag-geometric Tutte polynomial** of \( M \), denoted \( K_T(M, x, y) \), is the unique bivariate polynomial in \( x, y \) of bi-degree at most \((n-1, n-1)\) such that

\[
K_T(M, \alpha, \beta) = (\pi_{(n-1)n}) * \pi_{(r)}^*(y(M) \cdot [\mathcal{O}(1)]) \in K((\mathbb{P}^{n-1})^n \times \mathbb{P}^n) \simeq \mathbb{Q}[a, \beta] / (a^n, \beta^n).
\]

While the construction \((\tilde{F})\) may be more geometrically natural than the construction \((\tilde{F})\), many combinatorial properties of \( K_T(M) \) remain unclear. Here, we make progress on two fronts.

The first concerns a search for a "corank-nullity formula" for \( K_T(M) \). Both the usual Tutte polynomial and the Las Vergnas Tutte polynomial can be expressed as a summation over all subsets of the ground set, with terms involving coranks and nullities of subsets. As a result, for a matroid \( M \) or a flag matroid \((M_1, M_2)\) on \( \{1, \ldots, n\} \), one has \( T_T(M_2, 2) = 2^n \) and \( LV_T(M_1, M_2)(2, 2, 1) = 2^r \).

We show that the value of \( K_T(M_2, 2) \) is more intricate.

Theorem 6.7. Let \( M \) be a two-step flag matroid \( M = (M_1, M_2) \) on a ground set \( [n] = \{1, \ldots, n\} \). Let \( pB(M) \) be the set of subsets \( S \subseteq [n] \) such that \( S \) is spanning in \( M_1 \) and independent in \( M_2 \). Then with \( q \) as a formal variable, we have

\[
K_T(M, 1 + q^{-1}, 1 + q) = q^{-r_2} \cdot 2^n \cdot \left( \sum_{S \in pB(M)} q^{|S|} \right),
\]

and in particular, \( K_T(M_2, 2) = 2^n \cdot |pB(M)| \).

The second concerns a search for analogues of the deletion-contraction recursion that the Tutte polynomial and the Las Vergnas Tutte polynomial both satisfy. Unlike the two, the flag-geometric Tutte polynomial \( K_T(M) \) does not satisfy the usual deletion-contraction recursion. We instead show the following deletion-contraction-like relation.

Theorem 6.8. Let \( M \) be a matroid on a ground set \( \{0, 1, \ldots, n\} \) such that the element 0 is neither a loop nor a coloop in \( M \). Then we have

\[
K_T(M, M, x, y) = K_T(M/M\langle 0 \rangle, M\langle 0 \rangle, x, y) + K_T(M/M\langle 0 \rangle, M\langle 0 \rangle, x, y) + K_T(M/M\langle 0 \rangle, M\langle 0 \rangle, x, y).
\]
1.2. **Computation.** At https://github.com/chrisweur/kTutte, the reader can find a Macaulay2 code for computations with torus-equivariant $K$-classes and flag matroids. In particular, it computes the polynomials $LV_{T(M_1,M_2)}$ and $KT_M$ and their torus-equivariant versions.

1.3. **Notation.** Throughout we set $[n] := \{1, \ldots, n\}$. For $i = 1, \ldots, n$, we set $e_i$ to be the standard coordinate vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and write $e_S := \sum_{i \in S} e_i$ for a subset $S \subseteq [n]$. Let $\langle x, y \rangle$ be the standard inner product on $\mathbb{R}^n$. Cardinality of a set $S$ is denoted by $|S|$, and disjoint unions by $\sqcup$. A variety is a reduced and irreducible proper scheme over $\mathbb{C}$.

### 2. Preliminaries: flag matroids and their $K$-classes on flag varieties

Here we review flag matroids and their (torus-equivariant) $K$-classes on flag varieties. Most of the material in this section is described in more detail in the review [CDMS20].

#### 2.1. Matroid quotients and flag matroids

We assume familiarity with the fundamentals of matroid theory, and point to [Oxl11, Whi86, Wel76] as references. We write $U_{r,n}$ for the uniform matroid of rank $r$ on $[n]$. For a linear subspace $L \subseteq \mathbb{C}^n$, let $M(L)$ denote the linear matroid whose ground set is the image of $\{e_1, \ldots, e_n\}$ under the dual map $\mathbb{C}^n \to L^\vee$. For a matroid $M$ on a ground set $[n]$ we set:

- $\text{rk}_M : 2^{|n|} \to \mathbb{Z}$ to be the rank function of $M$, with $r(M) := \text{rk}_M([n])$,
- $M|\mathcal{S}$, $M \setminus \mathcal{S}$ and $M/\mathcal{S}$ to be the restriction to, the deletion of, and the contraction by a subset $\mathcal{S} \subseteq [n]$ (respectively),
- $B(M)$ to be the set of bases of $M$, and
- $Q(M) \subseteq \mathbb{R}^n$ to be the base polytope of $M$, which is the convex hull of $\{e_B \mid B \in B(M)\}$.

In this paper, by **morphisms of matroids** we will mean matroid quotients, as defined below\(^1\). They generalize the graph homomorphisms, linear maps, and graphs embedded on surfaces; see [EH20] for illustrations of these examples.

**Definition 2.1.** Let $M_1, M_2$ be two matroids on a common ground set $[n]$. We say that $M_1$ is a **matroid quotient** of $M_2$, written $M_1 \twoheadleftarrow M_2$, if any of the following equivalent conditions are met [Bry86, Proposition 7.4.7]:

1. every circuit of $M_2$ is a union of circuits of $M_1$,
2. $\text{rk}_{M_1}(B) - \text{rk}_{M_2}(A) \geq \text{rk}_{M_1}(B) - \text{rk}_{M_2}(A)$ for any $A \subseteq B \subseteq [n]$,
3. there exists a matroid $N$ on a ground set $[n] \sqcup \mathcal{S}$ with $|\mathcal{S}| = r(M_2) - r(M_1)$ such that $M_1 = N/\mathcal{S}$ and $M_2 = N \setminus \mathcal{S}$.

**Example 2.2.** Matroid quotients are combinatorial abstractions of linear maps of maximal rank. An inclusion of linear subspaces $L_1 \subseteq L_2 \subseteq \mathbb{C}^n$, or equivalently a quotient $\mathbb{C}^n \twoheadrightarrow L_2^\vee \twoheadrightarrow L_1^\vee$, defines matroids $M(L_1)$ and $M(L_2)$, which form a matroid quotient $M(L_1) \twoheadleftarrow M(L_2)$.

**Example 2.3** (Canonical matroid quotients). Just as any linear space $L$ has two canonical linear maps, the identity $L \to L$ and the zero map $L \to 0$, any matroid $M$ has two canonical matroid quotients, the identity $M \twoheadrightarrow M$ and the trivial quotient $M \twoheadleftarrow U_{0,n}$.

A matroid quotient $M_1 \twoheadleftarrow M_2$ is an **elementary quotient** if $r(M_2) - r(M_1) = 1$. Every matroid quotient $M_1 \twoheadleftarrow M_2$ can be realized as a composition of a series of elementary quotients. A canonical one is given by the **Higgs factorization** $M_1 = M^{(r_2-r_1)} \twoheadleftarrow \cdots \twoheadleftarrow M^{(1)} \twoheadleftarrow M^{(0)} = M_2$, defined by $B(M^{(i)}) = \{S \subseteq [n] \mid |S| = r(M_2) - i, S$ spans $M_1$ and is independent in $M_2\}$. The subsets $S \subseteq [n]$ that span $M_1$ and are independent in $M_2$ are called **pseudo-bases** of $(M_1, M_2)$.

For a more on matroid quotients, we refer the reader to [Bry86, §7.4] or [Oxl11, §7.3].

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\(^1\)The behavior of a morphism of matroids, in a more general sense of [EH20] or [HP18], is largely governed by an associated matroid quotient [EH20, Lemma 2.4].
Definition 2.4. A flag matroid is a sequence of matroids \( M = (M_1, \ldots, M_k)^2 \) on a ground set \([n]\) such that \( M_i \leftarrow M_{i+1} \) for all \( i = 1, \ldots, k-1 \). The matroids \( M_i \) are constituents of \( M \), and the rank of \( M \) is the sequence of ranks of its constituents \( (r(M_1), \ldots, r(M_k)) \). The set of bases of \( M \), denoted \( B(M) \), is the set of all \( k \)-flags of subsets \( (B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k) \) such that \( B_i \in B(M_i) \).

Example 2.5 (Linear flag matroids). A sequence of matroids \((M(L_1), \ldots, M(L_k))\) defined by a flag \( L \) of linear subspaces \( L_1 \subseteq \cdots \subseteq L_k \subseteq C^n \) is a flag matroid. We denote this flag matroid by \( M(L) \). Flag matroids arising in this way are called (linear or realizable) flag matroids.

For \( S = (S_1, \ldots, S_k) \) a flag of subsets of \([n]\), we write \( e_S = e_{S_1} + \cdots + e_{S_k} \). The base polytope \( Q(M) \) of a flag matroid \( M \) is the convex hull of \( \{e_B \mid B \in B(M)\} \), whose vertices are in bijection with the bases of \( M \). The polytope \( Q(M) \) is also the Minkowski sum of the base polytopes \( Q(M_i) \) of the constituents of \( M \). The classical theorem of Gelfand, Goresky, MacPherson, and Serganova [GGMS87] characterizes base polytopes of matroids. The analogue for flag matroids holds:

Theorem 2.6. [BGW03, Theorem 1.11.1] A lattice polytope \( P \subseteq \mathbb{R}^n \) is the base polytope of a rank \((r_1, \ldots, r_k)\) flag matroid on \([n]\) if and only if the following two conditions hold:

1. every vertex of \( P \) is a \( \mathfrak{S}_n \)-permutation of \( e_{1,2,\ldots,r_1} + \cdots + e_{1,2,\ldots,r_k} \), and
2. every edge of \( P \) is parallel to \( e_i - e_j \) for some \( i, j \in [n] \).

In particular, the normal fan of the base polytope \( Q(M) \) of a flag matroid is a coarsening of the braid arrangement, which is the normal fan of the zonotope \( \sum_{1 \leq i < j \leq n} \text{Conv}(e_i, e_j) \).

Consequently, every face of a base polytope of a flag matroid is again a base polytope of a flag matroid. The faces can be described explicitly. For \( u \in \mathbb{R}^n \) and a polytope \( Q \subseteq \mathbb{R}^n \), let \( Q^u := \{x \in Q \mid \langle x, u \rangle = \max_{y \in Q} \langle y, u \rangle\} \) be the face maximizing in the direction of \( u \).

Proposition 2.7. Let \( M = (M_1, \ldots, M_k) \) be a flag matroid on \([n]\) or rank \( r = (r_1, \ldots, r_k) \), and let \( S = S_1 \subseteq \cdots \subseteq S_m \) be a flag of subsets of \([n]\). Then \( Q(M)^S \) is the base polytope of a flag matroid whose \( i \)-th constituent (for \( i = 1, \ldots, k \)) is

\[
M_i|S_1 \oplus M_i|S_2 \oplus \cdots \oplus M_i|S_{m}/S_{m-1} \oplus M_i/S_m.
\]

In other words, the bases of the flag matroid of \( Q(M)^S \) are bases \( B = (B_1, \ldots, B_k) \) of \( M \) such that \( \text{rk}_{M_i}(S_j) = |B_i \cap S_j| \) for all \( 1 \leq i \leq k \) and \( 1 \leq j \leq m \).

Proof. Note that if \( Q = \sum_{i=1}^k Q_i \) is a Minkowski sum of polytopes, then for any \( u \in \mathbb{R}^n \), the face \( Q^u \) is the Minkowski sum \( \sum_{i=1}^k Q_i^u \) of faces. The proof of the proposition is thus reduced to the case of \( M \) being a matroid. In this case, the statement is an immediate consequence of the greedy algorithm structure for matroids.

2.2. Torus-equivariant K-theory of flag varieties. We will study combinatorial properties of flag matroids through the geometry of (partial) flag varieties and their (torus-equivariant) K-theory. We refer to [CDMS20, §8] or [FS10, §2] (and references therein) for a detailed exposition of equivariant K-theory of flag varieties.

We begin by describing torus-equivariant K-theory and the method of localization. Let \( T = (\mathbb{C}^*)^n \), and write \( \mathbb{Z}[t^\pm] := \mathbb{Z}[t_1^\pm, \ldots, t_n^\pm] = \mathbb{Z}[\mathbb{Z}^{[n]}] \) for the character ring of \( T \). Let \( X \) be a smooth variety with a \( T \)-action, and let \( E \) be a \( T \)-equivariant vector bundle on \( X \). We write:

- \( K^0(X) \) for the Grothendieck ring of vector bundles on \( X \), which is isomorphic to the Grothendieck group of coherent sheaves \( K_0(X) \) since \( X \) is smooth,
- \( K^0_T(X) \) for the \( T \)-equivariant Grothendieck ring,
- \([E] \in K^0(X) \) for the K-class of \( E \) and \([E]^T \in K^0_T(X) \) for its \( T \)-equivariant K-class,
• $f_*$ for the (derived) pushforward map and $f^*$ for the pullback map of $K$-classes along a proper map $f : X \to X'$ of smooth varieties,
• $\chi$ for the pushforward along the structure map $X \to \text{Spec} \mathbb{C}$, and
• $\chi^T$ for the $T$-equivariant pushforward to $K^0_T(pt) = \mathbb{Z}[t^\pm]$, the Lefschetz trace [Nie74, §4].

We now restrict to the case when $X$ is equivariantly formal and contracting, the precise definition of which we will not need. Examples of such $X$ include flag varieties and smooth toric varieties. By definition, the set $X^T$ of $T$-fixed points of $X$ is finite, and for each $x \in X^T$ there is a $T$-invariant affine neighborhood $U_x \simeq \mathbb{A}^\text{dim}_X$ whose characters $\{\lambda_1(x), \ldots, \lambda_\text{dim}_X(x)\} \subset \mathbb{Z}^n$ generate a pointed semigroup. Fundamental results from the method of equivariant localization are collected in the following theorem.

**Theorem 2.8.** Let $X$ be a equivariantly formal and contracting smooth $T$-variety. Then:

1. [VV03, Corollary 5.12], [KR03, Corollary A.5] (cf. [Nie74, Theorem 3.2], [Tho87, Theorem 2.7]) The restriction map
   
   $$K^0_T(X) \hookrightarrow K^0_T(X^T) \simeq (\mathbb{Z}[t^\pm])^{X^T}, \quad \epsilon \mapsto \epsilon(\cdot)$$

   is injective. Moreover, an element $\epsilon(\cdot) \in (\mathbb{Z}[t^\pm])^{X^T}$ is in the image if and only if for every one-dimensional $T$-orbit in $X$ with boundary points $x, y \in X^T$ in the closure, the function $\epsilon(\cdot) : X^T \to \mathbb{Z}[t^\pm]$ satisfies
   
   $$\epsilon(x) \equiv \epsilon(y) \mod 1 - \lambda$$

   where $\lambda$ is the character of the action of $T$ on the one-dimensional orbit.

2. [FS10, Theorem 2.6], [MS05, Theorem 8.34] Let $\mathcal{E}$ be a $T$-equivariant coherent sheaf on $X$, and let $x \in X^T$. The image $[\mathcal{E}]^T(x)$ of $[\mathcal{E}]^T$ under the restriction $K^0_T(X) \to K^0_T(x) \simeq \mathbb{Z}[t^\pm]$ is $K(\mathcal{E}(U_x); t)$ where
   
   $$\text{Hilb}(\mathcal{E}(U_x)) := \frac{K(\mathcal{E}(U_x); t)}{\prod_{i=1}^{\text{dim}_X}(1 - t^{-\lambda_i(x)})}$$

   is the multigraded Hilbert series of the $\mathcal{O}_X(U_x)$-module $\mathcal{E}(U_x)$ [MS05, Theorem 8.20].

3. [CG10, Theorem 5.11.7] (cf. [Nie74, §4]) Let $f : X \to Y$ be a proper $T$-equivariant map of equivariantly formal, contracting, and smooth $T$-varieties, and let $\alpha \in K^0_T(X), \beta \in K^0_T(Y)$. Then we have

   $$(f^* \beta)(x) = \beta(f(x))$$

   for every $x \in X^T$, and

   $$(f_* \alpha)(y) = \left(\prod_{i=1}^{\text{dim}_Y}(1 - t^{-\lambda_i(y)})\right) \left(\sum_{x \in X^T \cap f^{-1}(y)} \frac{\alpha(x)}{\prod_{i=1}^{\text{dim}_X}(1 - t^{-\lambda_i(x)})}\right)$$

   for every $y \in Y^T$.

We now specialize our discussion of $K$-theory to flag varieties. For a sequence of non-negative integers $\mathbf{r} = (r_1, \ldots, r_k)$ such that $0 < r_1 \leq \cdots \leq r_k < n$, denote by $Fl(\mathbf{r}; n)$ the flag variety

$$Fl(\mathbf{r}; n) := \{L = (L_1 \subseteq \cdots \subseteq L_k \subseteq \mathbb{C}^n) \text{ linear subspaces with } \dim L_i = r_i \forall 1 \leq i \leq k\}.$$ 

For each $i = 1, \ldots, k$, we have the tautological sequence of vector bundles on $Fl(\mathbf{r}; n)$

$$0 \to S_i \to \mathbb{C}^n \to Q_i \to 0$$

where $S_i$ is the $(i$-th) universal subbundle. It is a vector bundle whose fiber at a point $L \in Fl(\mathbf{r}; n)$ is the subspace $L_i$. For $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we denote by $\mathcal{O}(\mathbf{a})$ the line bundle $\mathcal{O}(a_1, \ldots, a_k)$, and by $\mathcal{O}(1)$ the line bundle $\mathcal{O}(1, 1, \ldots, 1)$ on $Fl(\mathbf{r}; n)$. The torus $T := (\mathbb{C}^*)^n$ acts on $Fl(\mathbf{r}; n)$ by its action on $\mathbb{C}^n$ where $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1^{-1}x_1, \ldots, t_n^{-1}x_n)$. With this $T$-action, a flag variety is an equivariantly formal and contracting space with the following structure:

• Each $T$-fixed point $x_\mathbf{r}$ of $Fl(\mathbf{r}; n)$ is in bijection with a flag $\mathbf{S}$ of subsets $S_1 \subseteq \cdots \subseteq S_k \subseteq [n]$ with $|S_i| = r_i$ for all $i = 1, \ldots, k$. 

For a flag $S$, denote by $Ex(S)$ the set of $(i,j) \in [n] \times [n]$ such that $i \in S_\ell$ and $j \notin S_\ell$ for some $1 \leq \ell \leq k$. Then the set of characters of the $T$-neighborhood $U_S$ of $x_S$ is

$$\{e_i - e_j \mid (i,j) \in Ex(S)\}.$$  

The sign-convention we have adopted for the action of $T$ ensures that $T$ acts on the sections of $S_i^\vee$ by positive characters. For instance, we have $[S_i^\vee]^T(x_S) = \sum_{j \in S_i} t_j$ and $[Q_i^\vee]^T(x_S) = \sum_{j \in [n]\setminus S_i} t_j$, and moreover $[\bigwedge S_i^\vee]^T(x_S) = \sum_{A \subseteq S_i, |A| = p} t^{e_A}$ and $[\bigwedge S_i]^T(x_S) = \sum_{A \subseteq S_i, |A| = p} t^{-e_A}$ (likewise for $\bigwedge Q_i^\vee$, $\bigwedge Q_i$).

### 2.3. $K$-class of a flag matroid

Flag matroids enter into the $K$-theory of flag varieties as $T$-equivariant $K$-classes as follows. Let $M$ be a flag matroid of rank $r$ on a ground set $[n]$. For $B$ a basis of $M$, define a polyhedral cone $\text{Cone}_B(M) := \text{Cone}(Q(M) - e_B) \subseteq \mathbb{R}^n$, also known as the tangent cone of $Q(M)$ at the vertex $e_B$, and let $\text{Hilb}_B(M)$ be the multigraded Hilbert series of $C[(\lambda) \mid \lambda \in \text{Cone}_B(M) \cap \mathbb{Z}^n]$ (see [MS05, Theorem 8.20]).

**Definition 2.9.** [CDMS20, Definition 8.19] Let $M$ be a flag matroid of rank $r$ on a ground set $[n]$. Then define $y(M)^T(\cdot) \in K^0_T(Fl(r; n)^T)$ by

$$y(M)^T(x_S) := \begin{cases} \text{Hilb}_S(M) \cdot \prod_{(i,j) \in Ex(S)} (1 - t_i^{-1} t_j) & \text{if } S \text{ a basis of } M \\ 0 & \text{otherwise.} \end{cases}$$

By combining Theorem 2.8.(1) and Theorem 2.6, one observes that $y(M)^T$ can be considered as a class in $K^0_T(Fl(r; n))$ [CDMS20, Proposition 8.20]. We will write $y(M)$ for the underlying non-equivariant $K$-class. The geometric motivation for this $K$-class constitutes the remark below.

**Remark 2.10.** Recall from Example 2.5 that a point $L \in Fl(r; n)$ defines a flag matroid $M := M(L)$ of rank $r$. One observes that the torus-orbit closure $T \cdot L$ is isomorphic to the toric variety of the base polytope $Q(M)$, and then by applying Theorem 2.8.(2) one shows that the class $[\text{O}_{T \cdot L}]^T \in K^0_T(Fl(r; n))$ satisfies $[\text{O}_{T \cdot L}]^T(\cdot) = y(M)^T(\cdot)$. See [CDMS20, §8.5] for details.

We also remark that the assignment $M \mapsto y(M)$ is a valuative invariant of flag matroids under flag matroid polytope subdivisions.

**Remark 2.11.** Let $P(\text{FMat}_{r,n})$ be a group generated by the indicator functions $1(Q) : \mathbb{R}^n \to \mathbb{R}$ of base polytopes $Q$ of rank $r$ flag matroids on $[n]$. A function $\varphi$ from the set of flag matroids of rank $r$ on $[n]$ to an abelian group $A$ is (strongly) valuative if it factors through $P(\text{FMat}_{r,n})$. As taking tangent cones and taking Hilbert series are valuative, it follows easily from the definition that the assignment $M \mapsto y(M)$ is valuative.

When $r = (r)$ (that is, we are concerned with the Grassmannian $Gr(r; n)$ and hence matroids of rank $r$ on $[n]$), invariants of a matroid $M$ built from $y(M)$ were explored in [Spe09] and [FS12] as follows. To avoid confusion we write $\mathbb{P}^{n-1}$ for $Gr(1; n)$ and $(\mathbb{P}^{n-1})^\vee$ for $Gr(n-1; n)$. Recall the diagram:

$$(2) \quad \begin{array}{ccc} Fl(1, r, n - 1; n) & \rightarrow & \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \\ \pi_r & \downarrow & \pi_{(n-1)} \\ Gr(r; n) & \rightarrow & (\mathbb{P}^{n-1})^\vee. \end{array}$$
Let \( \alpha \) be the \( K \)-class of the structure sheaf of a hyperplane in \( (\mathbb{P}^{n-1})^\vee \) and \( \beta \) the likewise \( K \)-class from \( \mathbb{P}^{n-1} \). We remark that our notation of \( \alpha, \beta \) is flipped from the notation in [FS12]\(^3\). Recall that \( K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}) \cong \mathbb{Q}[x, y]/(x^n, y^n) \).

**Theorem 2.12.** [FS12, Theorem 5.1] Let \( M \) be a matroid of rank \( r \) on \( [n] \), and let \( T_M(x, y) \) be its Tutte polynomial. Then we have

\[
T_M(\alpha, \beta) = (\pi_{(n-1)1})_* \pi^* \left( y(M) \cdot [O(1)] \right).
\]

We will generalize this \( K \)-theoretic formulation of Tutte polynomials of matroids to flag matroids in two different ways in subsequent sections. In both cases, similarly to Theorem 2.12, the Tutte polynomials of flag matroids are formulated via diagrams like (2), which we introduce in the next section.

### 3. Two diagrams and a fundamental computation

The main goal of this section is to prove Proposition 3.1, which relates a pushforward of a pullback of \( K \)-classes to Euler characteristics of certain associated sheaves. As this section is closely adapted from [FS12, \S 4], we only give sketches of proofs, save for the modified parts.

Let \( r = (r_1, \ldots, r_k) \) be a sequence of non-negative integers. For each \( i = 1, \ldots, k \), recall that we have tautological bundles \( S_i \) and \( Q_i \) on \( Fl(r; n) \) fitting into the short exact sequences

\[
0 \to S_i \to C^n \to Q_i \to 0.
\]

For two vector bundles \( E, F \) on \( X = Fl(r; n) \), we write \( \pi : \text{BiProj}(E, F) \to X \) for the bi-projectivization of the direct sum \( E \oplus F \). That is, \( \text{BiProj}(E, F) := \text{Proj}(\text{Sym}^\cdot E) \times_X \text{Proj}(\text{Sym}^\cdot F) \), so that for each point \( x \in X \), the fiber \( \pi^{-1}(x) \) is \( \mathbb{P}(E_x^\vee) \times \mathbb{P}(F_x^\vee) \). We consider the following two distinguished cases; note that the two cases are identical when \( k = 1 \) (i.e. when \( Fl(r; n) \) is a Grassmannian \( Gr(r; n) \)).

- **BiProj\((S_1^r, Q_k) \cong Fl(1, r, n - 1; n) \).** In this case, we have maps:

\[
Fl(1, r, n - 1; n) \xrightarrow{\pi_{r}} Fl(1, r, n; n) \xrightarrow{\pi_{(n-1)1}} (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}
\]

where \( \pi_r \) and \( \pi_{(n-1)1} \) are given by forgetting the linear spaces of appropriate dimensions.

- **BiProj\((S_1^r, Q_1) \cong Fl(1, r, n - 1; n) \) where \( \bar{Fl}(1, r, n - 1; n) \) is a variety

\[
\bar{Fl}(1, r, n - 1; n) := \left\{ \text{linear subspaces } \left( \ell, L_1, \ldots, L_k, H \right) \mid \begin{array}{ll}
\text{dim } \ell = 1, \text{dim } H = n - 1, \text{ } & L \subseteq Fl(r; n), \\
\text{and } \ell \subseteq L_k \text{ and } L_1 \subseteq H
\end{array} \right\}.
\]

In this case, we also have maps:

\[
Fl(1, r, n - 1; n) \xrightarrow{\bar{\pi}_r} \bar{Fl}(1, r, n - 1; n) \xrightarrow{\bar{\pi}_{(n-1)1}} (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}
\]

where \( \bar{\pi}_r \) and \( \bar{\pi}_{(n-1)1} \) are given by forgetting the linear spaces of appropriate dimensions.

---

\(^3\)In [FS12], the authors consider \( \pi_{(n-1)} : Fl(1, r, n - 1; n) \to \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \), and set \( \alpha \) and \( \beta \) as the \( K \)-classes of the structure sheaves of hyperplanes from \( \mathbb{P}^{n-1} \) and \( (\mathbb{P}^{n-1})^\vee \) (respectively). Our flipped naming of \( \alpha, \beta \) is to remedy a minor error in the proof of [FS12, Lemma 4.1] (bottom three lines on pg. 2709), which accidentally flips the correspondence of \( \alpha, \beta \) to appropriate \( K \)-classes.
As before, let \( \alpha = [\mathcal{O}_{H_1}] \) be the \( K \)-class of the structure sheaf of a hyperplane in \( (\mathbb{P}^{n-1})^\vee \) and \( \beta = [\mathcal{O}_{H_2}] \) the likewise \( K \)-class from \( \mathbb{P}^{n-1} \). The main statement of this section is as follows.

**Proposition 3.1.** Let \( \epsilon \in K^0(\text{Fl}(r; n)) \). With \( u \) and \( v \) as formal variables, define polynomials

\[
R_\epsilon(u, v) := \sum_{p, q} \chi \left( \epsilon \cdot [\bigwedge^p S_k][\bigwedge^q Q_1^\vee] \right) u^pv^q \quad \text{and} \quad \bar{R}_\epsilon(u, v) := \sum_{p, q} \chi \left( \epsilon \cdot [\bigwedge^p S_k][\bigwedge^q Q_k^\vee] \right) u^pv^q.
\]

Then we have the following identities in \( K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}) \).

\[
R_\epsilon(\alpha - 1, \beta - 1) = (\tau_{(n-1)_1})_*, \tau_\epsilon^*(\epsilon) \quad \text{and} \quad \bar{R}_\epsilon(\alpha - 1, \beta - 1) = (\bar{\tau}_{(n-1)_1})_*, \bar{\tau}_\epsilon^*(\epsilon).
\]

When \( k = 1 \) (i.e. \( \text{Fl}(r; n) \) is a Grassmannian), Proposition 3.1 reduces to [FS12, Lemma 4.1]. We remark that, just as in [FS12], Proposition 3.1 is an identity in the non-equivariant \( K \)-theory.

The proof of Proposition 3.1 is a minor modification of the proof of [FS12, Lemma 4.1]. Here, as a lemma, we separate out (and also fix a minor error in) the part of the proof in [FS12] that needs modification.

**Lemma 3.2.** Denote \( \eta_1 := (1 - \alpha)^{-1} = [\mathcal{O}(1, 0)] \) and \( \eta_2 := (1 - \beta)^{-1} = [\mathcal{O}(0, 1)] \), and let \( t \) be a formal variable. Then the following identities hold in \( K^0(\text{Fl}(r; n))[[t]] \).

\[
\sum_p \left[ \bigwedge^p S_k \right] t^p = (1 + t)^n(\tau_\epsilon)_*, \tau_\epsilon^*(\epsilon) \left( \frac{1}{1 + t\eta_1} \right) \quad \text{and} \quad \sum_q \left[ \bigwedge^q Q_1^\vee \right] t^q = (1 + t)^n(\tau_\epsilon)_*, \tau_\epsilon^*(\epsilon) \left( \frac{1}{1 + t\eta_2} \right).
\]

And likewise,

\[
\sum_p \left[ \bigwedge^p S_k \right] t^p = (1 + t)^n(\bar{\tau}_\epsilon)_*, \bar{\tau}_\epsilon^*(\epsilon) \left( \frac{1}{1 + t\eta_1} \right) \quad \text{and} \quad \sum_q \left[ \bigwedge^q Q_k^\vee \right] t^q = (1 + t)^n(\bar{\tau}_\epsilon)_*, \bar{\tau}_\epsilon^*(\epsilon) \left( \frac{1}{1 + t\eta_2} \right).
\]

**Proof.** For each \( i = 1, \ldots, k \) note that

\[
\left( \sum_{\ell} \left[ \bigwedge^\ell S_i \right] t^\ell \right) \left( \sum_m \left[ \bigwedge^m Q_i \right] t^m \right) = (1 + t)^n,
\]

which follows from the short exact sequence (3) and [Eis95, A2.2.(c)]. We also have an identity

\[
\left( \sum_{\ell} \left[ \bigwedge^\ell S_i \right] t^\ell \right) \left( \sum_m (-1)^m \left[ \text{Sym}^m S_i \right] t^m \right) = 1
\]

and likewise identities for \( Q_i \) and the duals \( S_i^\vee, Q_i^\vee \), which follow from the exactness of the Koszul complex [Eis95, A2.6.1]. Now, we note by [Har77, Exercise III.8.4] that

\[
(\tau_\epsilon)_*, \tau_\epsilon^*(\eta_2^2\eta_1^m) = [\text{Sym}^\ell S_i^\vee \otimes \text{Sym}^m Q_k] \quad \text{and} \quad (\bar{\tau}_\epsilon)_*, \bar{\tau}_\epsilon^*(\eta_2^2\eta_1^m) = [\text{Sym}^\ell S_i^\vee \otimes \text{Sym}^m Q_1].
\]

Combining (4), (5), and (6) then yields the desired identities. \( \square \)

**Sketch of proof of Proposition 3.1.** One combines Lemma 3.2 with the projection formula for \( K \)-theory [Ful98, §15.1]. Then by expanding the power series in \( u \) and \( v \), which is in fact a finite sum, comparing coefficients yields the desired identity. See the proof in [FS12] for details. \( \square \)
4. Summations of lattice point generating functions

The method of equivariant localization §2.2, aided by Proposition 3.1, will reduce our K-theoretic computations to summations of lattice point generating functions. Here we collect some useful results concerning summations of lattice point generating functions arising from polyhedra, along with variants that are suitable for our purposes. Our main novel contribution is Theorem 4.7, which is a useful variant of the method of flipping cones.

4.1. Brion’s formula. Here we review the results in [Bri88, Ish90]. For a subset $S \subset \mathbb{R}^n$, denote by $\mathbb{1}(S) : \mathbb{Z}^n \to \mathbb{Q}$ its indicator function sending $x \mapsto 1$ if $x \in S$ and 0 otherwise. Let $\mathcal{P}_n$ be a vector space of $\mathbb{Q}$-valued functions on $\mathbb{Z}^n$ generated by $\{\mathbb{1}(P) \mid P \subset \mathbb{R}^n$ lattice polyhedra}. It follows from the Brianchon-Gram formula [Bri37, Gra74, She67] that $\mathcal{P}_n$ is generated by indicator functions of cones, and by triangulating one concludes that $\mathcal{P}_n$ is generated by indicator functions of smooth cones.

We will often consider elements of $\mathcal{P}_n$ as elements in the power series ring $\mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$ by identifying $\mathbb{1}(P)$ with $\sum_{\lambda \in P \cap \mathbb{Z}^n} t^\lambda$. The following fundamental theorem concerns convergence of these power series to a rational function.

**Theorem 4.1.** [Ish90, Theorem 1.2]$^4$ Consider $\mathcal{P}_n$ as a $\mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$-submodule of $\mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$, and let $\mathbb{Q}(t_1, \ldots, t_n)$ be the fraction field. There exists a unique $\mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$-linear map

$$\text{Hilb} : \mathcal{P}_n \to \mathbb{Q}(t_1, \ldots, t_n)$$

such that if $C = \text{Cone}(v_1, \ldots, v_k) \subset \mathbb{R}^n$ is a smooth cone with primitive ray generators $v_1, \ldots, v_k \in \mathbb{Z}^n$ then $\text{Hilb}(\mathbb{1}(C)) = \prod_{i=1}^k \frac{1}{1-t_i}$.

Two remarks about the above linear map Hilb follow:

(1) The notation Hilb agrees with our previous notion of Hilbert series: when $C$ is a pointed rational polyhedral cone, not necessarily smooth, $\text{Hilb}(\mathbb{1}(C))$ equals the multigraded Hilbert series of $\mathbb{C}[t^\lambda \mid \lambda \in C \cap \mathbb{Z}^n]$ in the sense of [MS05, Theorem 8.20].

(2) If $P$ is a lattice polyhedron with a non-trivial lineality space, then $\text{Hilb}(\mathbb{1}(P)) = 0$.

For $P$ a lattice polyhedron, we will often by abuse of notation write $\text{Hilb}(P)$ for $\text{Hilb}(\mathbb{1}(P))$. An important result on rational generating functions for cones is the formula of Brion [Bri88], which was extended to a slightly more general version in [Ish90]. Here we will only need the following special case of [Ish90, Theorem 2.3].

**Theorem 4.2.** Let $P \subset \mathbb{R}^n$ be a lattice polyhedron with a nonempty set of vertices (so $P$ has no lineality space), and let $C(P)$ be its recession cone. For every vertex $v$ of $P$, write $C_v$ for $\text{Cone}(P - v)$. Then we have

$$\text{Hilb}(P) = \sum_{v \in \text{Vert}(P)} \text{Hilb}(C_v + v) \quad \text{and} \quad \text{Hilb}(C(P)) = \sum_{v \in \text{Vert}(P)} \text{Hilb}(C_v).$$

4.2. Lawrence-Varchenko formula (flipping cones) and variants. Here we review the method of flipping cones [FS10, §6], [BHS09, (11)]. Our contribution is a generalization Theorem 4.7, which will serve as a key technical tool in subsequent sections.

Let $\zeta \in \mathbb{R}^n$. For every $a \in \mathbb{R}$, we will denote the hyperplane $\{x \in \mathbb{R}^n \mid \langle \zeta, x \rangle = a\}$ by $H_{\zeta=a}$ and the half-space $\{x \in \mathbb{R}^n \mid \langle \zeta, x \rangle \geq a\}$ by $H_{\zeta \geq a}$. For an element $f \in \mathcal{P}_n$, by considering $f$ as an element of $\mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$ we write $f|_{H_{\zeta=a}}$ for the sum of terms $c w$ in $f$ such that $\langle w, \zeta \rangle = a$.

**Definition 4.3.** A polyhedron $P \subset \mathbb{R}^n$ is **$\zeta$-pointed** if $P \subset H_{\zeta \geq a}$ for some $a \in \mathbb{R}$. Let $\mathcal{P}_n^\zeta$ be the vector space of $\zeta$-pointed elements in $\mathcal{P}_n$.

---

$^4$Fink and Speyer in [FS12] and Postnikov in [Pos09] cite [KP92], whereas Ishida in [Ish90] writes that the theorem is originally due to Brion.
We note the following useful observation: Let \( P \subseteq \mathbb{R}^n \) be a polyhedron with vertices \( \text{Vert}(P) \), and as before let \( C_v := \text{Cone}(P - v) \) for \( v \in \text{Vert}(P) \). For \( \zeta \in \mathbb{R}^n \), the cone \( C_v \) is \( \zeta \)-pointed if and only if \( v \) is a vertex of the face \( P - \zeta \) of \( P \) minimizing in the \( \zeta \) direction. If \( f \in \mathcal{P}_n^\zeta \), then one can compute \( \text{Hilb}(f) \) "slice-by-slice" in the following sense.

**Lemma 4.4.** Let \( f, g \in \mathcal{P}_n^\zeta \) and suppose that \( \text{Hilb}(f) = \text{Hilb}(g) \). Then for every \( a \in \mathbb{R} \), it holds that \( \text{Hilb}(f|_{H_{t-a}}) = \text{Hilb}(g|_{H_{t-a}}) \).

**Proof.** Write \( b = f - g \), and suppose by contradiction that there is an \( a \in \mathbb{R} \) with \( \text{Hilb}(b|_{H_{t-a}}) \neq 0 \).

Since \( b \in \mathcal{P}_n^\zeta \), there is a minimal such \( a \), which we will denote by \( a_0 \). By the Claim below, we can find a nonzero \( q = \sum_{c \in \mathbb{Z}^n} \lambda_c t^e \in \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}] \) such that \( q \cdot b \) has finite support, i.e. is a Laurent polynomial. So \( \text{Hilb}(q \cdot b) = q \text{Hilb}(b) = 0 \). Since \( q \cdot b \) has finite support, this implies that \( q \cdot b = 0 \). Let \( c = \min \{ \langle \zeta, e \rangle | \lambda_c \neq 0 \} \), and let \( q_0 = \sum_{c: \langle \zeta, c \rangle = c} \lambda_c t^e \). Then \( 0 = \text{Hilb}( (q \cdot b)|_{H_{t-a_0+c}}) = \text{Hilb}(b|_{H_{t-a_0}}) \neq 0 \), a contradiction.

**Claim:** For every \( f \in \mathcal{P}_n \), there is a nonzero Laurent polynomial \( q \in \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}] \) such that \( q \cdot f \), which is a priori an element of the power series ring \( \mathbb{Q}[[t_1^{\pm}, \ldots, t_n^{\pm}]] \), is a Laurent polynomial in \( \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}] \).

**Proof of claim:** If \( f = 1(C) \) for some smooth cone \( C \), one can take \( q(t) = \prod (1 - t^e) \), where the product is over the primitive ray generators of \( C \). Since \( \mathcal{P}_n \) is generated by smooth cones, the result follows.

Suppose that \( \zeta \) is chosen such that the \( \zeta_i \) are \( \mathbb{Q} \)-linearly independent (we say "\( \zeta \) is irrational" in this case). Then for every \( a \in \mathbb{R} \), the intersection \( H_{t=a} \cap \mathbb{Z}^n \) consists of at most one point. In this case Lemma 4.4 reduces to saying that \( \text{Hilb} : \mathcal{P}_n^\zeta \to \mathbb{Q}(t_1, \ldots, t_n) \) is injective, and we recover [FS10, Lemma 6.3].

We next recall the notion of cone flips. We begin with a lemma for their existence.

**Lemma 4.5.** [Haa05, Lemma 6], [FS12, Lemma 2.1] Assume \( \zeta \) is irrational. For every \( f \in \mathcal{P}_n \), there is a unique \( f^\zeta \in \mathcal{P}_n^\zeta \) such that \( \text{Hilb}(f) = \text{Hilb}(f^\zeta) \). The map \( f \mapsto f^\zeta \) is linear.

The map \( (\cdot)^\zeta \) in the lemma above can be described explicitly as follows. Let \( C \subseteq \mathbb{R}^n \) be a rational simplicial cone, i.e.

\[
C = \{ w + \sum_{i=0}^{n-1} a_i v_i \mid a_i \geq 0 \text{ for all } i \in [n] \}.
\]

Then the image \( C^\zeta \in \mathcal{P}_n^\zeta \) under the map of Lemma 4.5 is given by

\[
C^\zeta = (-1)^\ell \mathbb{1} \left( \left\{ w + \sum_{i=0}^{n-1} a_i v_i \mid a_i \geq 0 \text{ for all } i \text{ with } \langle \zeta, v_i \rangle > 0, \text{ and } a_i < 0 \text{ for all } i \text{ with } \langle \zeta, v_i \rangle < 0 \right\} \right),
\]

where \( \ell \) is the number of rays \( v_i \) for which \( \langle \zeta, v_i \rangle < 0 \). We will refer to \( C^\zeta \) as the cone flip of \( C \) in direction \( \zeta \). For a general pointed rational cone \( C \), one defines the flipped cone \( C^\zeta \in \mathcal{P}_n^\zeta \) by triangulating the cone.

**Remark 4.6.** The assumption that \( \zeta \) is irrational is essential for Lemma 4.5: if \( \zeta \) is not irrational then \( \mathcal{P}_n^\zeta \) contains some lattice polyhedron \( P \) with a non-trivial lineality space, and \( \text{Hilb}(P) = 0 = \text{Hilb}(0) \), contradicting uniqueness.

\[\text{We remark that calling } C^\zeta \text{ the "flipped cone" of } C \text{ is a slight abuse of terminology when } C \text{ is not simplicial, since } C^\zeta \text{ is not necessarily the support function of a polyhedron. It can be a genuine linear combination of some of those; see [FS10, Remark 6.7].}\]
Now, suppose we are given an expression
\begin{equation}
\varphi = \sum_{\lambda \in \Lambda} a_\lambda \text{Hilb}(C_\lambda) \in \mathbb{Q}(t_1, \ldots, t_n),
\end{equation}
which is a finite summation where the $C_\lambda$ are pointed cones with vertices not necessarily at the origin and $a_\lambda \in \mathbb{Q}$ are scalars. Suppose we know that $\varphi \in \mathbb{Q}(t_1, \ldots, t_n)$ is in fact a Laurent polynomial (for example, because $\varphi$ arose from a computation in $T$-equivariant K-theory). Then we can use cone-flipping to get partial information about the coefficients of $\varphi$. The following proposition is our "cone-flipping in slices" technique which will be used repeatedly in later sections.

**Theorem 4.7.** Suppose $\varphi = \sum_{\lambda} a_\lambda \text{Hilb}(C_\lambda)$ is a Laurent polynomial, i.e. $\varphi \in \mathbb{Q}[t_1^\pm, \ldots, t_n^\pm]$, and let $P$ be the convex hull of the vertices of the $C_\lambda$. For $\zeta \in \mathbb{R}^n$, not necessarily irrational, suppose that every cone $C_\lambda$ whose vertex $w_\lambda$ satisfies $\langle \zeta, w_\lambda \rangle < b$ is $\zeta$-pointed. Then
\[ \varphi|_{H_{\zeta-b}} = \sum_{C_\lambda \in \mathcal{P}_n^b} a_\lambda \text{Hilb}(C_\lambda \cap H_{\zeta-b}). \]

In particular, if $P \cap H_{\zeta-b}$ is the face $P^{-\zeta}$ of $P$ minimizing in the $\zeta$ direction, then
\[ \varphi|_{H_{\zeta-b}} = \sum_{\zeta\text{-pointed } C_\lambda \text{ whose vertex } w_\lambda \text{ is on } P^{-\zeta}} a_\lambda \text{Hilb}(C_\lambda \cap H_{\zeta-b}). \]

**Remark 4.8.** In the special case of Theorem 4.7 where $H_{\zeta-b} \cap P = \{w\}$ is a vertex of $P$, the coefficient of $t^w$ in $\varphi$ is equal to $\sum a_\lambda$, where the sum is over all $\lambda$ for which $C_\lambda \in \mathcal{P}_n^b$ and the vertex of $C_\lambda$ is at $w$. Moreover, if $v$ is a vertex of the Newton polytope $\text{Newt}(\varphi)$ of $\varphi$, then for any irrational $\zeta \in \mathbb{R}^n$ there must exist a cone $C_\lambda$ such that its vertex $w_\lambda$ satisfies $\langle \zeta, w_\lambda \rangle < b$ and $\langle \zeta, v \rangle \not\in \langle \zeta, w_\lambda \rangle$. In other words, Theorem 4.7 is a generalization of [FS10, Corollary 6.9], which states that the Newton polytope of $\varphi$ is contained in $P$.

We prepare for the proof by noting a useful feature of the cone-flipping operation, starting with the following notion.

**Definition 4.9.** Let $C$ be a pointed cone, and $\zeta \in \mathbb{R}^n$. We say that an irrational $\zeta' \in \mathbb{R}^n$ is an **irrational approximation** of $\zeta$ with respect to $C$, if for every ray generator $v \in \mathbb{R}^n$ of $C$ it holds that $\langle \zeta, v \rangle > 0 \implies \langle \zeta', v \rangle > 0$ and $\langle \zeta, v \rangle < 0 \implies \langle \zeta', v \rangle < 0$.

Note that an irrational approximation of $\zeta$ can always be obtained as a small perturbation of $\zeta$. The following is a minor generalization of [FS12, Lemma 2.3], with almost identical proof, which we have included for completeness.

**Lemma 4.10.** Let $\zeta \in \mathbb{R}^n$, let $C$ be a pointed cone with vertex at $w$, and let $\zeta' \in \mathbb{R}^n$ be an irrational approximation of $\zeta$. Then $C_{\zeta'}$ is supported in the half space $\{x | \langle \zeta, x \rangle \geq \langle \zeta, w \rangle \}$. Furthermore, if $C$ is not contained in $\{x | \langle \zeta, x \rangle \geq \langle \zeta, w \rangle \}$, then $C_{\zeta'}$ is supported in the open half space $\{x | \langle \zeta, x \rangle > \langle \zeta, w \rangle \}$; in particular $w \not\in C_{\zeta'}$.

**Proof.** If $C$ is simplicial, the result follows immediately from the construction of cone flips (7) and Definition 4.9. For general $C$, we can obtain the first statement by considering any triangulation of $C$. For the second one, choose a ray $v$ of $C$ such that $\langle \zeta, v \rangle < 0$ and a triangulation of $C$ such that every interior cone contains $v$. Such a triangulation can for instance be constructed by triangulating the faces of $C$ that do not contain $v$, and then coning that triangulation from $v$. Now $C = \sum_{F} (-1)^{\dim C - \dim F} \mathbb{1}(F)$ and $C_{\zeta'} = \sum_{F} (-1)^{\dim C - \dim F} \mathbb{1}(F)_{\zeta'}$, where the sum is over all interior cones of the triangulation. The result now follows from the simplicial case. \qed
Proof of Theorem 4.7. Since the summation defining $\varphi$ is over a finite collection of cones $\{C_{\lambda}\}_{\lambda \in \Lambda}$, there exists a $\zeta' \in \mathbb{R}$ which is an irrational approximation of $\xi$ with respect to every cone $C_{\lambda}$. By assumption $\varphi = \text{Hilb}(f)$, where $f \in \mathcal{P}_{\mathbb{R}}$ has finite support, in particular $f \in \mathcal{P}_{\mathbb{R}}^k$. Hence, by Lemma 4.4, $\varphi|_{\mathcal{H}_{\xi}} = \text{Hilb}(\sum a_{\lambda} 1(C_{\lambda} \cap H_{\xi_{\lambda}}))$. If $C_{\lambda} \notin \mathcal{P}_{\mathbb{R}}^k$, then by assumption the vertex $w_{\lambda}$ of $C_{\lambda}$ satisfies $\langle \zeta, w_{\lambda} \rangle \geq b$, and by Lemma 4.10 $C_{\lambda}^\prime$ is supported on the open half-space $\{ x | \langle \zeta, x \rangle > b \}$, in particular $C_{\lambda}^\prime \cap H_{\xi_{\lambda}} = \emptyset$. If $C_{\lambda} \in \mathcal{P}_{\mathbb{R}}^k$, then since $C_{\lambda}$ and $C_{\lambda}^\prime$ are both in $\mathcal{P}_{\mathbb{R}}^k$, it follows from Lemma 4.4 that $\text{Hilb}(C_{\lambda}^\prime \cap H_{\xi_{\lambda}}) = \text{Hilb}(C_{\lambda} \cap H_{\xi_{\lambda}})$. □

4.3. Flipping cones for base polytopes. Let us now specialize our discussion of summing lattice point generating functions to ones arising from flag varieties. For the rest of this section, let $M$ be a flag matroid of rank $r = (r_1, \ldots, r_k)$ on a ground set $[n]$, whose constituent matroids have rank functions $r_1, \ldots, r_k$. As before, for a basis $B$ of $M$ let us write $Cone_B(M) := Cone(Q(M) - e_B)$ and $\text{Hilb}_B(M) := \text{Hilb}(Cone_B(M))$.

Consider the expression below, which is a finite summation

$$\varphi = \sum_{\lambda \in \Lambda} a_{\lambda} t^{w_{\lambda}} \text{Hilb}_{B_{\lambda}}(M),$$

where $a_{\lambda} \in \mathbb{Q}$, $w_{\lambda} \in \mathbb{Z}^n$, and $B_{\lambda}$ a basis of $M$. We allow the same basis to occur several times in the sum. Note that $t^{w_{\lambda}} \text{Hilb}_{B_{\lambda}}(M) = \text{Hilb}(C_{\lambda})$, where $C_{\lambda}$ is a cone with vertex at $w_{\lambda}$, so (9) is a special case of (8). As before, we assume that $\varphi \in \mathbb{Q}[t_1^\pm, \ldots, t_n^\pm]$, i.e. $\varphi$ is a Laurent polynomial, and we write $P := \text{Conv}(w_{\lambda} | \lambda \in \Lambda)$ for the convex hull of the $w_{\lambda}$. We will assume that all $w_{\lambda}$ lie in $\mathbb{Z}_{\geq 0}^n$, and that there exists a $c \in \mathbb{Z}_{\geq 0}$ such that the sum of the entries of any $w_{\lambda}$ is equal to $c$. Let $\bar{P} := \text{Conv}(\sigma \cdot w_{\lambda} | \sigma \in S_n, \lambda \in \Lambda)$ be the convex hull of all points in $\mathbb{Z}_{\geq 0}^n$ that are equal to one of the $w_{\lambda}$ up to permuting entries. The following theorem will be repeatedly applied in the next sections.

**Theorem 4.11.** Let $\varphi$ and $\bar{P}$ be as above, and let $v$ be a vertex of $\bar{P}$. Write $v = e_{s_1} + \cdots + e_{s_m}$, with $S_1 \subseteq \ldots \subseteq S_m \subseteq [n]$. Fix a basis $B = (B_1, \ldots, B_k)$ of $M$ such that $e_B$ is a vertex of the face $Q(M)^v$ of $Q(M)$ maximizing the direction $v$, that is, a basis $B$ satisfying $|S_i \cap B_j| = r_{M_j}(S_i)$ for all $1 \leq i \leq m$ and $1 \leq j \leq k$ (Proposition 2.7). Then the coefficient of $t^v$ in $\varphi \in \mathbb{Q}[t_1^\pm, \ldots, t_n^\pm]$ is equal to the sum of all $a_{\lambda}$ for which $w_{\lambda} = v$ and $B_{\lambda} = B$.

**Proof.** If $v \notin P$, the coefficient is 0, and the result follows from Remark 4.8. So, we now consider the case $v \in P$. Let us write $v = (v_1, \ldots, v_n)$ and $e_B = (b_1, \ldots, b_n)$. By permuting the coordinates of $\mathbb{N}^n$, we may assume that $v_i \geq v_{i+1}$ for all $i \in [n]$, and that $b_i \geq b_{i+1}$ whenever $v_i = v_{i+1}$. Let

$$\zeta' := ne_1 + (n-1)e_2 + \cdots + 2e_{n-1} + e_n.$$ 

We claim that this $\zeta'$ has the following properties.

1. The vertex $\{v\}$ is the face of $P$ maximizing in the $\zeta'$ direction, and hence is the vertex of $P$ maximizing in the $\zeta'$ direction.

2. The vertex face of $Q(M)$ maximizing in the $\zeta'$ direction is $\{e_B\}$.

The first property is immediate from the construction of $\zeta'$, as we have assumed that $v_i \geq v_{i+1}$ for all $i = 1, \ldots, n$. For the second property, note that $\zeta'$ is an interior point in the cone

$$\text{Cone}(e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}) + \text{Re}_{[n]},$$

of which the cone

$$\text{Cone}(e_{s_1}, e_{s_2}, \ldots, e_{s_m}) + \text{Re}_{[n]}$$

is a face. This face contains $v$ in its relative interior. These two cones are cones in the braid arrangement, of which the normal fan of $Q(M)$ is a coarsening (Theorem 2.6). Thus, the vertex
face of $Q(M)$ maximizing in the $\zeta'$ direction is among the vertices of $Q(M)^\Gamma$, and our assumption $b_i \geq b_{i+1}$ for all $i = 1, \ldots, n$ such that $v_i = v_{i+1}$ ensures that $e_B$ is indeed the one. Now, applying Theorem 4.7 (in the form of Remark 4.8) with $\zeta = -\zeta'$ gives the desired statement. □

5. The Las Vergnas Tutte polynomial of a matroid quotient

In [LV75], Las Vergnas introduced a Tutte polynomial of a matroid quotient as follows, and studied its properties in a series of subsequent works [LV80, LV84, LV99, ELV04, LV07, LV13]. The reader may find the survey [LV80] particularly useful.

**Definition 5.1.** Let $M_1 \twoheadrightarrow M_2$ be a matroid quotient on a ground set $[n]$. For $i = 1, 2$ write $r_i$ for the rank of $M_i$ and $r_i(S)$ for the rank of $S \subseteq [n]$ in $M_i$. The Las Vergnas Tutte polynomial of $(M_1, M_2)$ is

\[
LVT_{M_1, M_2}(x, y, z) := \sum_{S \subseteq [n]} (x - 1)^{r_1 - r_1(S)}(y - 1)^{|S| - r_2(S)}z^{r_2(S) - r_1(S)}
\]

(10)

For the remainder of this section, we let $M$ be a two-step flag matroid, i.e. a matroid quotient $M_1 \twoheadrightarrow M_2$ on a ground set $[n]$. We show in this section that $LVT_M$ arises $K$-theoretically from $y(M)$. We start by recalling the construction ($\tilde{Fl}$) of $\tilde{Fl}(1, r_1, r_2, n - 1; n)$ in §3 with the maps

\[
\begin{align*}
\tilde{Fl}(1, r_1, r_2, n - 1; n) & \rightarrow S_2/F_1, \\
Fl(r_1, r_2; n) & \rightarrow (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}.
\end{align*}
\]

We have an inclusion of tautological vector bundles $0 \rightarrow S_1 \rightarrow S_2$ on the flag variety $Fl(r_1, r_2; n)$. Let $S_2/S_1$ be the quotient bundle.

**Theorem 5.2.** With the notations as above, we have

\[
LVT_M(\alpha - 1, \beta - 1, w) = \sum_{m = 0}^{r_2 - r_1} (\tilde{\pi}_{(n-1)1}^* \tilde{\pi}_r^* y(M)|\mathcal{O}(0,1)][\bigwedge (S_2/S_1)]^m w^m
\]

(11)

as elements in $K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1})[w].$

We will prove the stronger statement that the $T$-equivariant version of Theorem 5.2 holds. By Proposition 3.1, the statement of Theorem 5.2 is equivalent to stating

\[
LVT_M(u + 1, v + 1, w) = \sum_{p,q,m} \chi(y(M)|\mathcal{O}(0,1)][\bigwedge S_1][\bigwedge Q_2^p][\bigwedge (S_2/S_1)]^m u^p v^q w^m.
\]

(12)

We thus define the $T$-equivariant Las Vergnas Tutte polynomial of $M$ by

\[
LVT^T_M(u + 1, v + 1, w) := \sum_{p,q,m} \chi^T(y(M)^T|\mathcal{O}(0,1)^T)[\bigwedge S_1^T][\bigwedge Q_2^p][\bigwedge (S_2/S_1)]^T u^p v^q w^m.
\]

**Theorem 5.3.** With the notations as above, we have

\[
LVT^T_M(u + 1, v + 1, w) = \sum_{S \subseteq [n]} t^s \mu^{r_1(S)} s^{-r_1(S)} r_2(S)] w^{r_2(S) - r_1(S)} + r_1(S).
\]
\textbf{Proof.} First, it follows from Theorem 2.8.(3) that

\begin{equation}
\sum_{p,q,m} \chi^T(y(M)^T \mathcal{O}(0,1)]^T \bigwedge^p S_1^T \bigwedge^q Q_2^T \bigwedge^m (S_2/S_1)^T \bigwedge^p q^p \omega^m
\end{equation}

\begin{align*}
&= \sum_{B=(B_1,B_2), \ B \in \mathcal{B}(M)} \text{Hilb}({\text{Con}}_{B}(M)) \cdot t^{e_{B_1}} \left( \sum_{p' \subseteq B_1} t^{-e_{p'}} u^{[p']} \right) \left( \sum_{q \subseteq [n] \setminus B_2} t^{e_{q}} \omega^{[q]} \right) \left( \sum_{m' \subseteq B_2 \setminus B_1} t^{-e_{m'}} \omega^{[m']} \right) \\
&= \sum_{B \in \mathcal{B}(M)} \text{Hilb}({\text{Con}}_{B}(M)) \left( \sum_{p \subseteq B_1} t^{-e_{p}} u^{[p]} \right) \left( \sum_{q \subseteq [n] \setminus B_2} t^{e_{q}} \omega^{[q]} \right) \left( \sum_{m \subseteq B_2 \setminus B_1} t^{-e_{m}} \omega^{[m]} \right) \\
&= \sum_{B \in \mathcal{B}(M)} \text{Hilb}({\text{Con}}_{B}(M)) \sum_{p \subseteq B_1, \ m \subseteq B_2 \setminus B_1, \ q \subseteq [n] \setminus B_2, \ m \neq m} t^{e_{p} + e_{m} + e_{q}}.
\end{align*}

We can now apply Theorem 4.11 to compute the sum

\begin{equation}
\sum_{B \in \mathcal{B}(M)} \text{Hilb}({\text{Con}}_{B}(M)) \sum_{p \subseteq B_1, \ m \subseteq B_2 \setminus B_1, \ q \subseteq [n] \setminus B_2, \ m \neq m} t^{e_{p} + e_{m} + e_{q}}.
\end{equation}

for fixed $p, m, q$.

To compute the coefficient of $t^{e_{s}}$, we pick a basis $(B_1, B_2)$ such that $|S \cap B_1| = \text{rk}_1(S)$ and $|S \cap B_2| = \text{rk}_2(S)$. Then we need to compute the number of terms in the sum above for which $B = (B_1, B_2)$ and $e_{p} + e_{m} + e_{q} = e_{s}$. But such a term needs to satisfy $p = S \cap B_1$, $p \cup m = S \cap B_2$, and $p \cup m \cup q = S$. In particular $p = \text{rk}_1(S)$, $p + m = \text{rk}_2(S)$, and $p + m + q = |S|$. If these three equalities are satisfied, there is indeed exactly one such term.

\textbf{Remark 5.4.} We remark that the Las Vergnas polynomial, and our $K$-theoretic interpretation of it, generalize the Tutte polynomial of a matroid in the following ways. Recall that any matroid $M$ has two canonical matroid quotients, $M \to M$ and $M \to U_{0,n}$.

- When $M = (M)$ (i.e. one constituent), the equation (11) reduces to the one in Theorem 2.12 [FS12, Theorem 5.1].
- When $M = (M, M)$, one can observe from (10) or (11) that $LVT_{M} (x, y, z) = T_{M} (x, y)$.
- When $M = (U_{0,n}, M)$, one can observe from (10) or (12) that $LVT_{M} (x, y, z) = T_{M} (z+1, y)$.

\textbf{Remark 5.5.} The Las Vegas Tutte polynomial satisfies a deletion-contraction relation similar to that of the Tutte polynomial [LV80, Proposition 5.1]. We remark that our “cone-flipping with slices” (Theorem 4.7) can be used to show deletion-contraction relation for $LVT_{M}$ and $T_{M}$. For example, if $i \in [n]$ is neither a loop nor a coloop of $M_2$,

\begin{equation}
LVT_{M_1,M_2} (x, y, z) = LVT_{M_1 \setminus i, M_2 \setminus i} (x, y, z) + LVT_{M_1 / i, M_2 / i} (x, y, z).
\end{equation}

This identity is obtained by applying Theorem 4.7 to (13) as follows. By considering $\xi = e_i$, we find that the terms in (13) that are not divisible by $t_i$ sum to $LVT_{M_1 \setminus i, M_2 \setminus i} (x, y, z)$, where $T' = (C^*)^{n-1}$. By considering $\xi = -e_i$, we find that the terms that are divisible by $t_i$ sum to $t_i LVT_{M_1 / i, M_2 / i} (x, y, z)$. We leave the details to the reader.

\textbf{Remark 5.6.} Unlike the Tutte polynomials of matroids, the constant term of $LVT_{M}$ is no longer necessarily zero. This reflects the fact that for most $L \in \mathcal{F}l(r_1, r_2; n)$, the map $\pi_{(n-1)}(\pi_{-1} (T \mathcal{T} \mathcal{L})) \to (\mathbb{P}^{n-1})^e \times \mathbb{P}^{n-1}$ is surjective. If further $r_2 - r_1 = 1$, then this map is a finite morphism, and the degree of the map is exactly given by the Crapo’s beta invariant $\beta(N)$ where $N$ is a matroid such that $M_1 = N/e$ and $M_2 = N \setminus e$. 
**Remark 5.7.** It follows from Remark 2.11 and Theorem 5.2 that the assignment $M \mapsto LV_T M$ is valuative.

6. flag-geometric Tutte polynomial of a flag matroid

In this section, we explore the behavior of another notion of Tutte polynomials of flag matroids that differs from that of Las Vergnas in the previous section. Here, instead of the construction ($\bar{F}I$), we consider the more geometrically natural construction (FI) in §3 with the maps

$$FI(1, r, n - 1; n) \xrightarrow{\pi_r} FI(r; n) \xrightarrow{\pi_{n-1}^*} (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}.$$

**Definition 6.1.** [CDMS20, Definition 8.23] Let $M$ be a flag matroid of rank $r = (r_1, \ldots, r_k)$ on $[n]$. Then the flag-geometric Tutte polynomial of $M$, denoted $K_T M(x, y) \in \mathbb{Z}[x, y]$, is the (unique) polynomial of bi-degree at most $(n - 1, n - 1)$ such that

$$K_T M(\alpha, \beta) = (\pi_{n-1}^*)^* \left(y(M) \cdot [O(1)]\right).$$

While the construction (FI) leading to $K_T^* M$ may be more geometrically natural than ($\bar{F}I$), the combinatorial properties of $K_T^* M$ seem more mysterious than those of $LV_T^* M$. For example, in contrast to $LV_T^* M$, the polynomial $K_T^* M$ does not readily reduce to the Tutte polynomial of $M$ when $M$ is one of the two canonical matroid quotients of a matroid $M$ (i.e. $M = \eta M$ and $U_{0,n} \rightarrow M$).

We illuminate some combinatorial structures of $K_T^* M$ as follows.

- There is no known (corank-nullity) combinatorial formula for $K_T^* M$ that is similar to (10) for $LV_T^* M$. Our result in §6.2, which in particular computes $K_T^* M(2, 2)$, can be considered as a first step in this direction.
- No deletion-contraction relation is known to hold for $K_T^* M$; one may construe this to be a consequence of the fact that the base polytope of a flag matroid generally has lattice points that are not vertices. In §6.3 we formulate and prove a deletion-contraction-like relation for elementary matroid quotients.

6.1. **First properties of $K_T^* M$.** Again, by Proposition 3.1, we have that

$$K_T^* M(u + 1, v + 1) = \sum_{p,q} \chi^T \left(y(M)^T \cdot [O(1)]^T \left[\bigwedge S_k^T \bigcup \bigwedge Q_1^T\right]\right) u^p v^q,$$

which leads us to the following $T$-equivariant version of $K_T^* M$.

**Definition 6.2.** The $T$-equivariant flag-geometric Tutte polynomial of a flag matroid $M$ is

$$K_T^* M(u + 1, v + 1) := \sum_{p,q} \chi^T \left(y(M)^T \cdot [O(1)]^T \left[\bigwedge S_k^T \bigcup \bigwedge Q_1^T\right]\right) u^p v^q.$$

Theorem 2.8.(3) again yields $K_T^* M$ as a sum of rational functions as follows via a similar computation as one in the proof of Theorem 5.2.

**Lemma 6.3.** For a flag matroid $M = (M_1, \ldots, M_k)$ on a ground set $[n]$, we have

$$K_T^* M(u + 1, v + 1) = \sum_{B \in M} \text{Hilb}(\text{Cone}_B(M)) \sum_{p \leq B_k \subseteq [n]} \sum_{q \leq B_1 \subseteq [B_1]} t^{e_{B_1} + \cdots + e_{B_{k-1}} + e_p} u^{r_k - r_{k-1} - |p|} v^{q}. \quad (15)$$
Many of our results on $KT_M$ will be obtained by manipulation with the equation (15). We start with the following example.

**Example 6.4.** For any matroid $M$ on $[n]$, we have $KT_{U_{0,n},M}(x,y) = y^n T_M(x,1)^6$. To verify this, we compute

$$KT_{U_{0,n},M}^T(u + 1, v + 1) = \sum_{B \in \mathcal{B}(M)} \sum_{p \leq B, q \leq [n]} t^{e_p + e_q} u^{r - |p|} v^{q |q|}$$

(16)

$$= \left( \prod_{i=1}^n (1 + t_i v) \right) \cdot \sum_{B \in \mathcal{B}(M)} \sum_{p \leq B} \text{Hilb}(\text{Cone}_B(M)) \cdot t^{e_p} u^{r - |p|}$$

$$= \left( \prod_{i=1}^n (1 + t_i v) \right) \cdot KT_M^T(u + 1, 1).$$

Setting $t_i = 1$, $u = x - 1$, and $v = y - 1$ yields the desired claim. This example shows that we cannot recover $T_M$ from $KT_{U_{0,n}}$, although $U_{0,n} \leftarrow M$ is a canonical matroid quotient of $M$.

**Proposition 6.5.** Let $M = (M_1, \ldots, M_k)$ be a flag matroid on $[n]$. The following properties hold for the flag-geometric Tutte polynomial $KT_M^T$:

1. (Direct sum) If $M$ is a direct sum $M' \oplus M''$ of two flag matroids on ground sets $A, B$ with $A \cup B = [n]$, then $KT_M^T(x,y) = KT_{M'}^T(x,y) \cdot KT_{M''}^T(x,y)$ (where $T' = (C^*)^A$, $T'' = (C^*)^B$).
2. (Loops & coloops) Let $\ell$ be the number of loops in $M_1$, and $c$ the number of coloops in $M_k$. Then $x^\ell y^c$ divides $KT_M^T(x,y)$.
3. (Duality) If $M^\vee$ is the dual flag matroid of $M$, whose constituents are matroid duals of the original, then $KT_M^T(y,x) = KT_{M^\vee}^T(x,y)$.
4. (Base polytope) $KT_M^T(1, 1) = \text{Hilb}(Q(M))$.
5. (Valuativeness) The map $M \mapsto KT_M^T$ is valuative.

**Proof:** The first two statements follow from manipulating with the identity (15) in a similar way as the computation (16) in Example 6.4. For the third statement, we claim that the $T$-equivariant version of the statement is $t^{e|y|} KT_M^{-1}(y,x) = KT_{M^\vee}(x,y)$ (where the $T^{-1}$ superscript means that we have replaced $t_i$ by $t_i^{-1}$). Verifying this identity is then another easy manipulation with (15). The fourth statement follows from Brion’s formula (Theorem 4.2). The last statement follows from Remark 2.11.

We can use Theorem 4.11 to compute some of the terms in (15):

**Theorem 6.6.** Let $M = (M_1, M_2)$ be a 2-step flag matroid and let $t^k u^{r^2 - i |\mathcal{B}|}$ be a monomial occurring in (15). Then $\sum_{i=1}^n k_i = r_1 + i + j$. Let $c$ denote the number of entries in $k$ that are equal to 1. If $c \leq (r_1 + j - i)$, the coefficient of $t^k u^{r^2 - i |\mathcal{B}|}$ is equal to

1. $1$, if $S_2$ is spanning for $M_1$, $S_1$ is independent in $M_2$, and $c = |r_1 + j - i|$,
2. $0$, otherwise,

where $S_1$ and $S_2$ are defined by $S_1 \subseteq S_2$ and $k = e_{S_1} + e_{S_2}$.

**Proof:** The equality $\sum_{i=1}^n k_i = r_1 + i + j$ follows immediately from (15). The coefficient of $t^k u^{r^2 - i |\mathcal{B}|}$ is equal to

$$\sum_{B \in \mathcal{B}(M)} \sum_{p \subseteq B, q \subseteq [n], |p| = i} t^{e_p} \cdot \text{Hilb}(\text{Cone}_B(M)) \cdot \sum_{i=1}^n t^{e_i} u^{r^2 - i |\mathcal{B}|}$$

$$= \sum_{B \in \mathcal{B}(M)} \sum_{p \subseteq B, q \subseteq [n], |p| = i} t^{e_p} \cdot \text{Hilb}(\text{Cone}_B(M)) \cdot \sum_{|q| = j} t^{e_q} u^{r^2 - j |\mathcal{B}|}$$

$$= \sum_{B \in \mathcal{B}(M)} \text{Hilb}(\text{Cone}_B(M)) \cdot \sum_{|q| = j} t^{e_q} u^{r^2 - j |\mathcal{B}|}.$$
where we have denoted \( J_1 := [n] \setminus B_1 \). It is not hard to see that the vertices of \( \bar{P} \) have \( |r_1 + j - i| \) entries equal to 1. This proves that the coefficient is 0 if \( c < |r_1 + j - i| \). So from now on we assume \( c = |r_1 + j - i| \).

Next, we apply Theorem 4.11. Writing \( k = e_{S_1} + e_{S_2} \) (note that \( |S_1| = \min(i, r_1 + j) \) and \( |S_2| = \max(i, r_1 + j) \)) we find a basis \( B = (B_1, B_2) \) of \( M \) for which \( r_j(S_i) = |S_i \cap B_j| \). We now need to compute the number of ways \( k \) can be written as a sum \( e_{B_1} + e_p + e_q \). If \( S_2 \) is not spanning for \( M_1 \), or if \( S_1 \) is not independent in \( M_2 \), there are no ways to do this, and the coefficient is 0. Otherwise, if \( i \leq r_1 + j \), we need to put \( p = S_1 \) and \( q = S_2 \setminus S_1 \). If \( i \geq r_1 + j \), we need to put \( q = S_1 \cap J_1 \) and \( p = S_1 \cup J_1 \). In both cases, there is just one way, so the coefficient is 1. □

6.2. Towards a corank-nullity formula. For a matroid \( M \) on \( [n] \), the corank-nullity formula for the Tutte polynomial \( T_M(x, y) = \sum_{S \subseteq [n]} (x - 1)^{r(S)} (y - 1)^{|S| - r(S)} \) expresses \( T_M \) as a sum over all subsets of \( [n] \). In particular, we have \( T_M(2, 2) = 2_n \); in fact, \( \mathcal{K}_{M_1}(2, 2) = \prod_{i=1}^n (1 + t_i) \). As a first step towards a similar formula for \( K_{M_1} \), we show the following for a two-step flag matroid.

**Theorem 6.7.** Let \( M \) be a two-step flat matroid \( M = (M_1, M_2) \) or rank \( (r_1, r_2) \), and let \( pB(M) \) be the set of pseudo-bases of \( M \), i.e., subsets \( S \subseteq [n] \) such that \( S \) is spanning in \( M_1 \) and independent in \( M_2 \). With \( q \) as a formal variable, we have

\[
K_T^T(1 + q^{-1}, 1 + q) = q^{-r_2} \left( \prod_{i=1}^n (1 + t_i q) \right) \left( \sum_{S \in B(M)} t^{e_S} q^{|S|} \right),
\]

and in particular, we have

\[
K_T^T(1 + q^{-1}, 1 + q) = q^{-r_2} \cdot 2^n \cdot \left( \sum_{S \in B(M)} q^{|S|} \right),
\]

\[
K_T^T(2, 2) = \left( \prod_{i=1}^n (1 + t_i) \right) \left( \sum_{S \in B(M)} t^{e_S} \right), \text{ and}
\]

\[
K_T^T(2, 2) = 2^n |pB(M)|.
\]

**Proof.** Setting \( u = q^{-1} \) and \( v = q \) in (15) of Lemma 6.3 gives us

\[
kT^T(1 + q^{-1}, 1 + q) = \sum_{B = (B_1, B_2), \ B \in B(M)} \text{Hilb}(\text{Cone}_B(M)) \sum_{p \subseteq B_2, q \subseteq [n] \setminus B_1} t^{e_{B_1} + e_p + e_q} q^{|p| - |q| - r_2}
\]

\[
= \sum_{B = (B_1, B_2), \ B \in B(M)} \text{Hilb}(\text{Cone}_B(M)) \sum_{R \subseteq E} \sum_{S \subseteq B_2 \setminus B_1} t^{e_{B_1} + e_R + e_S} q^{|R| - |S| - r_2}
\]

\[
= q^{-r_2} \prod_{i \in E} (1 + t_i q) \sum_{B = (B_1, B_2), \ B \in B(M)} \text{Hilb}(\text{Cone}_F(M)) \sum_{S \subseteq B_2 \setminus B_1} t^{e_{B_1} + e_S} q^{|S|}.
\]

We now use Theorem 4.11 to compute the sum

\[
\varphi_r := \sum_{B = (B_1, B_2), \ B \in B(M)} \text{Hilb}(\text{Cone}_B(M)) \sum_{B_1 \subseteq p \subseteq B_2, \ |p| = r} t^{e_p}
\]

for a fixed \( r_1 \leq r \leq r_2 \). First, we note that the polytope \( \bar{P} = \text{Conv}(e_S \mid S \subseteq E, \ |S| = r) \), obtained as the convex hull of the \( S_n \)-orbit of \( \{ e_p \mid B_1 \subseteq p \subseteq B_2, \ |p| = r \} \), has no interior lattice points.

For \( S \subseteq E \) with \( |S| = r \), if \( S \) is not a pseudo-basis of \( M_1 \leftarrow M_2 \), then there is no basis \( B \) of \( M \) such that \( B_1 \subseteq S \subseteq B_2 \), and hence the coefficient of \( t^{e_S} \) is 0 in this case. Now, suppose \( S \) is a pseudo-basis of \( M_1 \leftarrow M_2 \), which by definition implies that there exists basis \( B = (B_1, B_2) \) of \( M \) with \( B_1 \subseteq S \subseteq B_2 \). This basis \( B \) is a vertex of the face \( Q(M)^{e_S} \) by Proposition 2.7, and thus by Theorem 4.11 the coefficient of \( t^{e_S} \) is equal to 1 in \( \varphi_r \). □
We do not know of analogues of Theorem 6.7 for flag matroids with more than two constituents.

6.3. A deletion-contraction-like relation. In this section, we consider $K_T M$ of an elementary quotient $M = (M_1, M_2)$. By definition we have $r(M_2) - r(M_1) = 1$, and in this case there is a unique matroid $M$ on a ground set $\overline{[n]} := \{0\} \cup \{1\}$ such that $M_1 = M/0$ and $M_2 = M \setminus 0$. Our main theorem of this subsection is the following deletion-contraction-like relation.

**Theorem 6.8.** Let $M$ be a matroid of rank $r$ on $\overline{[n]} := \{0\} \cup \{1\}$ such that the element 0 is neither a loop nor a coloop in $M$. Let $T = C^* \times T = (C^*)^{n+1}$ be the torus with character ring $Z[t_0^\pm, \ldots, t_n^\pm]$. Then we have

$$K_T^T M(x,y) = t_0^r K_T^T M/0, M/0(x,y) + t_0 K_T^T M/0, M/0(x,y) + K_T^T M/0, M/0(x,y).$$

In particular, we have $K_T^T M(x,y) = K_T^T M/0, M/0(x,y) + K_T^T M/0, M/0(x,y) + K_T^T M/0, M/0(x,y).$

We use $\{e_0, \ldots, e_n\}$ for the standard basis of $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$. For a polyhedron $P \subset \mathbb{R}^n$, we will often abuse the notation and write $P$ also for $\{0\} \times P \subset \mathbb{R} \oplus \mathbb{R}^n$. We prepare for the proof of Theorem 6.8 by an observation that motivated the theorem.

As the base polytope $Q(M)$ is a $(0,1)$-polytope (i.e. a lattice polytope contained in the Boolean cube $[0,1]^{n+1} \subset \mathbb{R}^{n+1}$), every lattice point is a vertex. Moreover, observe that the vertices of $Q(M)$ partition into two parts, the bases of $M/0$ and the bases of $M \setminus 0$.

As a result, the lattice points of $Q(M) = Q(M) + Q(M)$ partition into the following three parts, with $Q_1 = \frac{1}{2}(Q_0 + Q_2)$:

- $Q_2 := Q(M, M) \cap H_{e_0 = 2} = \{2e_0\} \times Q(M/0, M/0)$,
- $Q_1 := Q(M, M) \cap H_{e_0 = 1} = \{e_0\} \times Q(M/0, M \setminus 0)$, and
- $Q_0 := Q(M, M) \cap H_{e_0 = 0} = \{0\} \times Q(M \setminus 0, M \setminus 0)$.

The case of setting $x = y = 1$ (cf. Proposition 6.5(4)) in (18) of Theorem 6.8 witnesses this partition of the lattice points of $Q(M, M)$. The following lemma in preparation for the proof of Theorem 6.8 is a consequence of $Q_1 = \frac{1}{2}(Q_0 + Q_2)$.

**Lemma 6.9.** Let the notations be as above. Then for $B \in B(M)$ with $0 \not\in B$, we have

$$\text{Hilb}(\text{Cone}_B(Q(M, M)) \cap H_{e_0 = 1}) = \sum_{I \in B(M/0), I \subseteq B} t_0^{-1} \text{Hilb}(\text{Cone}_{(I,B)}(Q(M/0, M \setminus 0))).$$

$$\text{Hilb}(\text{Cone}_B(Q(M, M)) \cap H_{e_0 = 0}) = \sum_{I \in B(M/0), I \subseteq B} \text{Hilb}(\text{Cone}_{(I,B)}(Q(M/0, M \setminus 0))).$$

**Proof.** We have an equality of polyhedra

$$\text{Cone}_B(Q(M, M)) \cap H_{e_0 = 1} = \text{Cone}_B(Q(M \setminus 0)) + Q_1 - 2e_B.$$

We claim that $\text{Cone}_B(Q(M \setminus 0)) + Q_1$ has vertices $\{e_I + e_B\}$ for $I \in B(M/0)$ such that $I \subseteq B$. The two statements in the lemma then follow from Brion’s formula Theorem 4.2.

For the claim, we start by noting that if $I \in B(M/0)$ then there exists $B' \in B(M \setminus 0)$ such that $I \subseteq B'$ (since $M/0 \leftarrow M \setminus 0$). Consequently, if $e_B$ is the vertex of $Q(M \setminus 0)$ that minimizes $\langle v, e_B \rangle$ for some $v \in \mathbb{R}^n$, then a vertex of $Q(M/0)$ that minimizes $\langle v, \cdot \rangle$ must be $e_I$ satisfying $I \subseteq B$. Our claim now follows from $Q_1 = \frac{1}{2}(Q_0 + Q_2)$. \(\square\)

**Proof of Theorem 6.8.** Let us begin by noting that the equation (15) for $K_T^T M$ reads

$$K_T^T M(u+1, v+1) = \sum_{B \in B(M)} \text{Hilb}(\text{Cone}_B(Q(M, M))) \sum_{p \subseteq B} \sum_{q \subseteq [n] \setminus B} t^{e_B + e_q + e_u v r - |p| |v| q}.$$
We apply Theorem 4.7 with $\zeta = e_0$ and $L$ defined by $t_0 = 0$. Note that $\text{Cone}_B(Q(M, M)) \in \mathcal{P}^E_{19}$ if and only if $0 \not\in B$. Hence all cones occurring in (19) with vertex on $L$ are in $\mathcal{P}^E_{19}$, and we find that the terms in (19) not divisible by $t_0$ sum to

$$\sum_{B \in \mathcal{B}(M), 0 \not\in B} \text{Hilb}(\text{Cone}_B(Q(M, M)) \cap H_{e_0=0}) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} \in B \subset [n]\setminus B$$

$$= \sum_{B \in \mathcal{B}(M \setminus 0)} \text{Hilb}(\text{Cone}_B(M \setminus 0)) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} \in B \subset [n]\setminus B$$

$$= K_{T_{M \setminus 0, M \setminus 0}}^T(u + 1, v + 1).$$

A similar argument, with $\zeta = -e_0$, shows that the coefficient of $t_0^2$ in (19) is $K_{T_{M \setminus 0, M \setminus 0}}^T$.

Finally, we apply Theorem 4.7 once more, this time with $\zeta = e_0$ and $L = H_{e_0=1}$. We find that the terms in (19) divisible by $t_0$ but not by $t_0^2$ sum to

$$\left( \sum_{B \in \mathcal{B}(M), 0 \not\in B} \text{Hilb}(\text{Cone}_B(Q(M, M))) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} \right) \bigg|_{H_{e_0=1}}$$

$$= \left( \sum_{B \in \mathcal{B}(M), 0 \not\in B} \text{Hilb}(\text{Cone}_B(Q(M, M))) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} (1 + t_0)v u^{r - |p|, q} \right) \bigg|_{H_{e_0=1}}$$

$$= \sum_{B \in \mathcal{B}(M), 0 \not\in B} \text{Hilb}(\text{Cone}_B(Q(M, M)) \cap H_{e_0=1}) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} \in B \subset [n]\setminus B$$

$$+ t_0 \sum_{B \in \mathcal{B}(M), 0 \not\in B} \text{Hilb}(\text{Cone}_B(Q(M, M)) \cap H_{e_0=0}) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} + 1,$$

which by Lemma 6.9 is equal to

$$\sum_{B \in \mathcal{B}(M), I \in \mathcal{B}(M \setminus 0), I \subset B} t_0 t_{B \setminus I} \text{Hilb}(\text{Cone}_{(I,B)}(Q(M \setminus 0, M \setminus 0))) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} \in B \subset [n]\setminus B$$

$$+ t_0 \sum_{B \in \mathcal{B}(M), I \in \mathcal{B}(M \setminus 0), I \subset B} \text{Hilb}(\text{Cone}_{(I,B)}(Q(M \setminus 0, M \setminus 0))) \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} u^{r - |p|, q} + 1$$

$$= t_0 \sum_{(I,B) \in \mathcal{B}(M \setminus 0, M \setminus 0)} \text{Hilb}(\text{Cone}_{(I,B)}(Q(M \setminus 0, M \setminus 0))) \left( \sum_{p \leq B, q \leq [n] \setminus B} t_{e_p + e_q} (1 + t_{B \setminus I} v) u^{r - |p|, q} \right)$$

$$= t_0 \sum_{(I,B) \in \mathcal{B}(M \setminus 0, M \setminus 0)} \text{Hilb}(\text{Cone}_{(I,B)}(Q(M \setminus 0, M \setminus 0))) \left( \sum_{p \leq B, q \leq [n] \setminus I} t_{e_p + e_q} u^{r - |p|, q} \right)$$

$$= t_0 K_{T_{M \setminus 0, M \setminus 0}}^T(u + 1, v + 1),$$

as desired. \hfill \square

**Remark 6.10.** We remark that for a general flag matroid $M$, the slices $\{Q(M) \cap H_{e_k} \}_{k \in \mathbb{Z}}$ need not be flag matroid base polytopes. Moreover, even when they are, we do not observe an identity like the one in Theorem 6.8 that expresses $K_T^T_M$ in terms of the slices.

For example, consider $M = (U_{1,3}, U_{2,3})$. We have $K_{T_{M}}^T(x, y) = x^2 y^2 + x^3 y + x y^3 + x^2 + 2 x y + y^2$. In any coordinate direction, its three slices are $(U_{0,2}, U_{1,2})$, $(U_{1,2}, U_{1,2})$, and $(U_{1,2}, U_{2,2})$, whose $K_T$ are (respectively), $x y^2 + y^2$, $x y + x + y$, and $x^2 + y^2$. 
Remark 6.11. One can generalize Theorem 6.8 as follows. Denote by $M^\ell := (M, \ldots, M)$, the flag matroid whose constituents are $M$ repeated $\ell$ times. Then we have

$$K\mathcal{T}_{M^\ell}^T = t_0^\ell K\mathcal{T}_{(M/0)^\ell}^T + t_0^{\ell-1} K\mathcal{T}_{(M/0)^{\ell-1},M\setminus 0}^T + \cdots + K\mathcal{T}_{(M/0)^0}^T.$$  

The proof is essentially identical to one given for Theorem 6.8.

7. Future directions

7.1. $g$ and $h$ polynomials for flag matroids. For a matroid $M$, Speyer introduced in [Spe09] a polynomial invariant $g_M(t) \in \mathbb{Q}[t]$ and a close cousin $h_M(t) \in \mathbb{Q}[t]$, which is related to $g_M(t)$ by $h_M(t) = (-1)^c g_M(-t)$ where $c$ is the number of connected components of $M$. A K-theoretic interpretation of the polynomial $h_M$ was given in [FS12].

Theorem 7.1. [FS12, Theorem 6.1 & Theorem 6.5] Let $M$ be a matroid of rank $r$ on $[n]$ without loops or coloops. Let $\pi_r$, $\pi_{(n-1)r}$, $\alpha$, $\beta$ be as in §2.3. Then the polynomial $h_M$ is the (unique) univariate polynomial of degree at most $n - 1$ such that

$$(\pi_{(n-1)r})_\pi^\ast(y(M)) = h_M(\alpha + \beta - \alpha \beta).$$

For a flag matroid $M$ on $[n]$, this motivates us to consider $(\pi_{(n-1)r})_\pi^\ast(y(M))$, where the maps are as in the flag-geometric construction (FL). By Proposition 3.1, this is equal to

$$\sum_{p,q} \chi\left(y(M) \bigwedge^p S_k \bigwedge^q Q_{i}^\ell \right) (a - 1)^p (\beta - 1)^q.$$

Let us consider its torus-equivariant version

$$\sum_{p,q} \chi^T\left(y(M)^T \bigwedge^p S_k^T \bigwedge^q Q_{i}^\ell^T \right) u^p v^q$$

where $u$ and $v$ are formal variables. We show that this is a polynomial in $uv$, which thereby establishes that $(\pi_{(n-1)r})_\pi^\ast(y(M))$ is a polynomial in $\alpha + \beta - \alpha \beta$ (since the substitution $u = \alpha - 1, v = \beta - 1$ yields $1 - uv = \alpha + \beta - \alpha \beta$).

Lemma 7.2. (cf. [FS12, Lemma 6.2]) Let $M = (M_1, \ldots, M_k)$ be a flag matroid on $[n]$, and suppose every constituent of $M$ is both loopless and coloopless. Then

$$\sum_{p,q} \chi^T\left(y(M)^T \bigwedge^p S_k^T \bigwedge^q Q_{i}^\ell^T \right) u^p v^q \in \mathbb{Q}[u,v]$$

is a polynomial in $\mathbb{Q}[uv]$.

We remark that the condition about a flag matroid $M = (M_1, \ldots, M_k)$ being loopless or coloopless depends only on $M_1$ or $M_k$ (respectively). First, note that by the condition (2) in Definition 2.1, if $\ell \in [n]$ is a loop in $M_i$ then it is a loop in $M_{i-1}$ also. By duality, if $\ell \in [n]$ is a coloop in $M_i$ then it is a coloop in $M_{i+1}$ also. Hence, the flag matroid $M$ is loopless (coloopless) if and only if $M_1$ has no loops ($M_k$ has no coloops).

Proof. Once more by Theorem 2.8.(3), we get

$$\sum_{p,q} \chi^T\left(y(M)^T \bigwedge^p S_k^T \bigwedge^q Q_{i}^\ell^T \right) u^p v^q = \sum_{B \in M} \text{Hilb}(\text{Cone}_B(M)) \sum_{p \leq q \leq \sum_{i} k \setminus \sum_{j \leq |B_i|} \sum_{t} t^{e_p + e_q} u^{|p|} v^{|q|}.}$$
Fix \( |p| = i, |q| = j \), and consider the sum

\[
\varphi_{ij} = \sum_{B \in \mathcal{M}} \text{Hilb}(\text{Cone}_B(M)) \sum_{p \in B_i, q \in [n] \setminus B_i, |p| = i, |q| = j} t^{-e_p + e_q}.
\]

We need show that \( \varphi_{ij} \) is zero if \( i \neq j \). Let \( P \) be the convex hull of \( \{-e_p + e_q\} \) appearing in the summation (20). Note that \( P \) is contained in the intersection of \( H_{e_{[i]-j-i}} \) and the cube \( \{x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1 \forall i \in [n]\} \). By Theorem 4.7 (in the form of Remark 4.8), it thus suffices to show that \( \varphi_{ij}|_{H_{e_{[i]-j-i}}} = 0 \) and \( \varphi_{ij}|_{H_{e_{[i]-j-i}}} = 0 \) for all \( i \in [n] \).

Let us now fix any \( i \in [n] \). As none of the constituents have coloops (and in particular \( \ell \) is not a loop in \( M_k \)), the intersection \( Q(M) \cap H_{e_{[i]-i}} \) is a non-empty face of \( Q(M) \) minimizing in the \( e_{\ell} \) direction, consisting of bases \( B = (B_1, \ldots, B_k) \) such that \( \ell \notin B_k \). Thus, we have that \( \text{Cone}_B(M) \in \mathcal{P}_n^{e\ell} \) if and only if \( \ell \notin B_k \), and by Theorem 4.7 with \( \zeta = e_{\ell} \) we have

\[
\varphi_{ij}|_{H_{e_{[i]-j-i}}} = \sum_{\ell \notin B_k} \sum_{p \in B_i, q \in [n] \setminus B_i, |p| = i, |q| = j} \text{Hilb}((-e_p + e_q + \text{Cone}_B(M))|_{H_{e_{[i]-j-i}}}).
\]

But since \( \ell \notin B_k \) implies \( \ell \notin p \), every cone \(-e_p + e_q + \text{Cone}_B(M)\) occurring in the sum above will have vertex \( v \) with \( v_\ell > -1 \). Moreover, we have \( \text{Cone}_B(M) \in \mathcal{P}_n^{e\ell} \) for such cones, and hence we get \( \varphi_{ij}|_{H_{e_{[i]-j-i}}} = 0 \). A similar argument with \( \zeta = -e_{\ell} \), noting that \( \ell \) is not a loop in \( M_1 \), shows that \( \varphi_{ij}|_{H_{e_{[i]-j-i}}} = 0 \).

We thus make the following definition that generalizes the polynomial \( h_M \) of a matroid \( M \) to the setting of flag matroids. It is well-defined by Lemma 7.2.

**Definition 7.3.** Let \( M = (M_1, \ldots, M_k) \) be a flag matroid \([n]\) such that every constituent of \( M \) is both loopless and coloopless. Let \( \pi_{(n-1)} \), \( \pi_{\alpha}, \alpha, \beta \) be as in §3. Then the polynomial \( h_M \) is defined as the (unique) univariate polynomial of degree at most \( n - 1 \) such that

\[
(\pi_{(n-1)} \circ \pi_\alpha^\beta(y(M))) = h_M(\alpha + \beta - \alpha \beta).
\]

**Remark 7.4.** We have constructed the polynomial \( h_M \) via the flag-geometric diagram (F1). Although one may also consider a similar construction via the "Las Vergnas" diagram (F1), the analogue of Lemma 7.2 fails in this case.

In the case of matroids realizable over \( C \), the behavior of the polynomial \( g_M \) of a matroid \( M \), in particular the non-negativity of its coefficients, was used to establish a bound on the number of interior faces in a matroidal subdivision of a base polytope of a matroid [Spe09]. Extending these results to arbitrary matroids is so far open, but an announcement of a relevant forthcoming work has been made in [LdMRS20].

In another forthcoming work [BEZ20], the authors study flag-matroidal subdivisions of base polytopes of flag matroids, and extend the tropical geometry of matroids used in [Spe09] to the setting of flag matroids. We are thus led to ask the following.

**Question 7.5.** Does a suitable modification of our polynomial \( h_M \) give an analogue of the polynomial \( g_M \) for flag matroids, and does its behavior lead to a bound on the number of interior faces in a flag-matroidal subdivision of a base polytope of a flag matroid?

### 7.2. Characteristic polynomials of matroid morphisms.

A recent breakthrough in matroid theory is the log-concavity of the coefficients of the characteristic polynomial of a matroid [AHK18]. We consider here several candidates for characteristic polynomials of morphisms of matroids. We begin with the one coming from the flag-geometric Tutte polynomial.
Definition 7.6. For a flag matroid $M$, define the flag-geometric characteristic polynomial $K_{\chi M}(q)$ of $M$ by

$$K_{\chi M}(q) := (-1)^{r_1}KT_{M}(1-q,0).$$

Like the usual characteristic polynomial, the polynomial $KT_{M}$ satisfies $K_{\chi M}(q) = 0$ whenever the first constituent $M_1$ of $M$ has a loop (\cite[Proposition 6.5]{7}). The following conjecture is supported by computer computations. It suggests that the flag-geometric characteristic polynomial of a two-step flag matroid may contain little information about the flag matroid itself.

Conjecture 7.7. Let $M$ be a matroid of rank $r$ with no loops, so that $U_{1,r} \rightarrow M$ is a valid matroid quotient. Then $K\chi(U_{1,r},M)(q) = (q-1)^r$.

Let us now turn to the Las Vergnas Tutte polynomial. The last two bullet points of Remark 5.4 suggest two different ways of generalizing the characteristic polynomial of a matroid. The case of $M = (M,M_1)$ gives rise to the polynomial $p_M(q,s) := (-1)^{r_1}LVT_{M}(1-q,0,-s)$, which was studied by Las Vergnas as the Poincaré polynomial of a matroid quotient \cite[§4]{LV1}. Here we introduce another generalization following the case of $M = (U_{0,n}, M)$.

Definition 7.8. For a flag matroid $M = (M_1, M_2)$ on $[n]$, define its beta polynomial $\beta_M(q)$ by

$$\beta_M(q) := (-1)^{r_2-r_1}LVT_{M}(0,0,-q).$$

When $M = (U_{0,n}, M)$, it follows from $LVT_{M}(x,y,z) = T_{M}(z+1,y)$ that $\beta_M(q) = \chi_M(q)$, the characteristic polynomial of $M$. The terminology for $\beta_M(q)$ is motivated by the following observation:

Proposition 7.9. For a matroid $M$, let $\beta(M)$ be the Crapo’s beta invariant of $M$, and let $M_1 = M^{[r_2-r_1]} \leftarrow \cdots \leftarrow M^{(1)} \leftarrow M^{(0)} = M$ be the Higgs factorization of a matroid quotient $M_1 \leftarrow M_2$. The beta polynomial $\beta_{M_1,M_2}(q)$ is divisible by $(q-1)$, and the reduced beta polynomial of $M_1 \leftarrow M_2$, defined as $\tilde{\beta}_{M_1,M_2}(q) := \beta_{M_1,M_2}(q)/(q-1)$, satisfies

$$\tilde{\beta}_{M_1,M_2}(q) = \sum_{i=0}^{r_2-r_1-1} (-1)^{2r_1-1-i}(\beta(M^{(i)}) + \beta(M^{(i+1)}))q^i.$$

If $\tilde{M}^{(i)}$ is the (unique) matroid on $[n] \cup \{0\}$ such that $\tilde{M}^{(i)}/0 = M^{(i+1)}$ and $\tilde{M}^{(i)} \setminus 0 = M^{(i)}$, then $\beta(M^{(i)}) + \beta(M^{(i+1)}) = \beta(M^{(i)})$ (this is among the defining properties of the invariant $\beta$).

Proof. Let $M = (M_1, M_2)$ and $d = r_2 - r_1$. \cite[Theorem 3.1]{LV2} states that

$$LVT_{M}(x,y,z) = \sum_{i=0}^{d} t_i(M; x,y)z^i,$$

where

$$t_i(M; x,y) = \frac{1}{xy - x - y} \left( (y-1)T_{M^{(i-1)}}(x,y) + (-xy + x + y - 2)T_{M^{(i)}}(x,y) + (x-1)T_{M^{(i+1)}}(x,y) \right)$$

for $i = 1, \ldots, d-1$, and

$$t_0(M; x,y) = \frac{1}{xy - x - y} \left( -T_{M^{(0)}}(x,y) + (x-1)T_{M^{(1)}}(x,y) \right),$$

$$t_d(M; x,y) = \frac{1}{xy - x - y} \left( (y-1)T_{M^{(d-1)}}(x,y) + T_{M^{(d)}}(x,y) \right).$$
We now recall that the beta invariant $\beta(M)$ of a matroid $M$ of rank $r$ is defined as

$$
\beta(M) := (-1)^{r-1} \left( \frac{d}{dq} \chi_M(q) \right)_{q=1}
\leq \lim_{q \to 1} \frac{(-1)^r T_M(1-q, 0) - (-1)^r T_M(0, 0)}{q - 1}
\leq - \lim_{q \to 1} \frac{T_M(1-q, 0)}{q - 1}.
$$

As $LV T_M(x, y, z)$ is a polynomial, each $t_i(M; x, y)$ is also a polynomial. Hence, we have $t_i(M; 0, 0) = \lim_{q \to 1} t_i(M; 1 - q, 0)$, and thus the above expressions for $t_i(M; x, y)$ give

$$
t_i(M; 0, 0) = \beta(M^{(k-1)}) + 2\beta(M^{(k)}) + \beta(M^{(k+1)})
\quad \text{for } i = 1, \ldots, d - 1, \text{ and}
$$

$$
t_0(M; 0, 0) = \beta(M^{(0)}) + \beta(M^{(1)}),
$$

$$
t_d(M; 0, 0) = \beta(M^{(d-1)}) + \beta(M^{(d)}).
$$

As a result, we have

$$
(-1)^d LV T_M(0, 0, -q) = (q - 1) \left( \sum_{i=0}^{d-1} (-1)^{d-1-i} (\beta(M^{(i)}) + \beta(M^{(i+1)})) q^i \right),
$$

yielding the desired result for the reduced beta polynomial $\overline{\beta}_M(q)$. \hfill \Box

Log-concavity of the coefficients of the reduced characteristic polynomial $\overline{\chi}_M(q) = \frac{\chi_M(q)}{q-1}$ was established in [AHK18]. This motivates the following conjecture.

**Conjecture 7.10.** The coefficients of $\overline{\beta}_{M_1, M_2}(q)$ form a log-concave sequence. Consequently, the coefficients of $LV T_M(0, 0, q)$ form a log-concave sequence.

The coefficients of $LV T_M(1, 1, q)$ were shown to be (ultra) log-concave in [EH20].

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