PROOFS OF TWO FORMULAS OF VLADETA JOVOVIC

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Abstract. In this paper, we first provide an analytic and a bijective proof of a formula stated by Vladeta Jovovic in the OEIS sequence A117989. We also provide a bijective proof of another interesting result stated by him on the same page concerning integer partitions with fixed differences between the largest and smallest parts.

Keywords. Partition, part, largest part, smallest part, fixed difference

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1. Introduction

A partition \( \pi \) of a positive integer \( n \) is a non-increasing sequence of natural numbers \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \) such that \( \sum_{i=1}^{r} \lambda_i = n \). \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are called the parts of the partition \( \pi \). We call \( \lambda_1 \) and \( \lambda_r \) to be the largest and smallest parts of the partition \( \pi \) and denote \( p(n) \) to be the number of (unrestricted) partitions of \( n \). For example, \( p(5) = 7 \) where the seven
partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1. Their corresponding largest parts are 5, 4, 3, 3, 2, 2, and 1 respectively and smallest parts are 5, 1, 2, 1, 1, 1, and 1 respectively. Throughout the paper, we consider $|q| < 1$ and adopt the usual notation for the conventional $q$-Pochammer symbols:

$$ (a)_n = (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), $$

$$ (a)_\infty = (a; q)_\infty := \lim_{n \to \infty} (a; q)_n. $$

We define $(a)_n$ for all real numbers $n$ by

$$ (a)_n := \frac{(a)_\infty}{(aq^n)_\infty}. $$

By convention, we take $(a)_0 = 1$ and $(0)_\infty = 1$.

2. Main Formulas

Our starting point is the sequence A117989 in the Online Encyclopedia of Integer Sequences [3]. The sequence in question, $a(n)$, counts the number of partitions of $n$ where the smallest part occurs at least twice. For example, $a(6) = 7$ where the relevant partitions are $4 + 1 + 1$, $3 + 3$, $3 + 1 + 1 + 1$, $2 + 2 + 2$, $2 + 2 + 1 + 1$, $2 + 1 + 1 + 1 + 1$, and $1 + 1 + 1 + 1 + 1 + 1$.

Also, on the page of A117989, we find the following formula of $a(n)$ by Vladeta Jovovic:

**Formula 1**: (Vladeta Jovovic, July 21 2006)

$$ a(n) = 2p(n) - p(n + 1) \forall n \geq 1 \tag{2.1} $$

where $p(n)$ denotes the partition function.

We denote $b(n) = 2p(n) - p(n + 1)$. By (2.1), we have $a(6) = 7 = 22 - 15 = 2p(6) - p(7) = b(6)$.

We also find another interesting result on the same page posted by Vladeta Jovovic which states:

**Formula 2**: (Vladeta Jovovic, May 09 2008)

$$ a(n) = p(2n, n) \forall n \geq 1 \tag{2.2} $$

where $p(2n, n)$ denotes the number of partitions of $2n$ with fixed difference equal to $n$ between the largest and smallest parts.
In sections 3 and 4, we give a q-theoretic and a bijective proof of (2.1) respectively and finally we provide a bijective proof of (2.2) in section 5.

3. Analytic Proof of Formula 1

In this section, we prove (2.1) using generating functions and elementary infinite series-product identities from the theory of q-hypergeometric series.

To begin with, we define the generating functions for $a(n)$ and $b(n)$ to be

$$A(q) := \sum_{n=1}^{\infty} a(n)q^n \quad \text{and} \quad B(q) := \sum_{n=1}^{\infty} b(n)q^n \quad (3.1)$$

respectively. We then have

$$A(q) = \sum_{k=1}^{\infty} q^{k+k} \cdot (1 + q^k + q^{2k} + \ldots) \cdot (1 + q^{k+1} + q^{2(k+1)} + \ldots) \cdot \ldots$$

$$= \sum_{k=1}^{\infty} q^{2k} = \frac{1}{(q)_{\infty}} \sum_{k=1}^{\infty} \frac{(q)_{\infty} q^{2k}}{(q^2)_{\infty}} = \frac{1}{(q)_{\infty}} \sum_{k=1}^{\infty} (q)_k q^{2k}$$

$$= \frac{q^2}{(q)_{\infty}} \sum_{k=0}^{\infty} (q)_k q^{2k} = \frac{q^2}{(q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q)_k (q)_k q^{2k}}{(q)_k}$$

$$= \frac{q^2 (q^2)_{\infty}}{(q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^2)_k q^{2k}}{(q^2)_k} (3.2)$$

$$= \sum_{k=0}^{\infty} \frac{q^{k+2}}{(q)_k (1 - q^{k+2})} = \sum_{k=0}^{\infty} \frac{q^{k+2} (1 - q^{k+1})}{(q)_k}$$

$$= \sum_{k=0}^{\infty} \frac{q^{k+2} (1 - q^{k+3})}{(q)_k} - \sum_{k=0}^{\infty} \frac{q^{2k+3} (q)_k}{(q)_k} = \sum_{k=0}^{\infty} \frac{q^{k+2}}{(q)_k} - \frac{1}{q} \sum_{k=0}^{\infty} \frac{(q^2)_{k+2}}{(q)_k}$$

$$= \left( \frac{1}{(q)_{\infty}} - \frac{1}{1 - q} \right) - \frac{1}{q} \left( \frac{1}{(q^2)_{\infty}} - \frac{1 - q + q^2}{1 - q} \right) (3.3)$$

$$= \frac{1}{(q)_{\infty}} - \frac{1}{1 - q} - \frac{1 - q}{q(q)_{\infty}} + \frac{1 - q + q^2}{q(1 - q)} = \frac{2}{(q)_{\infty}} - \frac{1}{q(q)_{\infty}} + \frac{1}{q} - 1$$

which is $B(q)$, the generating function for $b(n) = 2p(n) - p(n+1) \forall n \geq 1$.

Note that (3.2) follows by replacing $a = q$, $b = q$, $c = 0$ and $t = q^2$ in Heine’s transformation \[1\], p.19, Corollary 2.3 and (3.3) follows by replacing $a = 0$ and $t = q$ in Cauchy’s identity \[1\], p.17, Theorem 2.1. \[\Box\]

4. Bijective Proof of Formula 1

In this section, we provide a bijective proof of (2.1).
From (2.1), we have

\[ a(n) = 2p(n) - p(n + 1) \]

\[ = p(n) - (p(n + 1) - p(n)) \]

which implies

\[ p(n) - a(n) = p(n + 1) - p(n). \quad (4.1) \]

Let us now define \( c(n) = p(n) - a(n) \) and \( d(n) = p(n + 1) - p(n) \). Thus, from (4.1), it suffices to prove that \( c(n) = d(n) \forall n \geq 1 \).

We note that \( c(n) \) denotes the number of partitions of \( n \) where the smallest part occurs exactly once. This follows straightforward from the definition of \( a(n) \). We also note that \( d(n) \) denotes the number of partitions of \( n + 1 \) which do not contain 1 as a part because every partition of \( n + 1 \) which contains 1 as a part can be obtained by adjoining 1 as a part to every partition of \( n \).

Let \( C_n \) be the set of all partitions of \( n \) where the smallest part occurs exactly once and \( D_n \) be the set of all partitions of \( n + 1 \) not containing 1 as a part. So, \( \#C_n = c(n) \) and \( \#D_n = d(n) \). Thus, it is clear that we will now produce a bijection between \( C_n \) and \( D_n \) to get the desired result.

Firstly, we consider a partition \( \pi \in C_n \) and consider two cases pertaining to \( \pi \): If 1 is a part of \( \pi \), since it is the smallest part, it occurs exactly once. Now, add 1 to the 1 already in \( \pi \) to get a new partition \( \pi' \in D_n \) whose smallest part is now 2. Hence, \( \pi' \) does not contain 1 as a part. Now, if 1 is not a part of \( \pi \), then the smallest part of \( \pi \) is greater than or equal to 2 and hence it does not contain 1 as a part. On adding 1 to the smallest part of \( \pi \), we get a new partition \( \pi' \) of \( n + 1 \) which does not contain 1 as a part. Hence, \( \pi' \in D_n \).

Now, we consider a partition \( \pi' \in D_n \). Thus, \( \pi' \) does not contain 1 as a part which implies that the smallest part of \( \pi' \) is greater than or equal to 2. Again, we consider two cases concerning \( \pi' \): If the smallest part of \( \pi' \) occurs exactly once, we subtract 1 from it to get a new partition \( \pi \in C_n \) and we are done. On the other hand, if the smallest part of \( \pi' \) occurs at least twice, subtract 1 from any one of the smallest parts to get a new partition \( \pi \in C_n \).

Thus, the process is reversible and hence \( C_n \) is bijection with \( D_n \forall n \geq 1 \). So, we have our desired result. \( \square \)
5. Bijective Proof of Formula 2

In this section, we provide a bijective proof of \((2.2)\).

Let \(A_n\) be the set of all partitions of \(n\) where the smallest part occurs at least twice and \(F_n\) be the set of all partitions of \(2n\) where the difference between the largest and smallest parts is \(n\). So, \(#A_n = a(n)\) and \(#F_n = p(2n, n)\). Now, we will provide a bijection between \(A_n\) and \(F_n\) to show that \(a(n) = p(2n, n)\).

Firstly, we consider a partition \(\pi \in A_n\). Then, we add \(n\) to any one of the smallest parts of \(\pi\) (since the smallest part of \(\pi\) occurs at least twice) to get a new partition \(\pi' \in F_n\) because the largest part of \(\pi'\) now is equal to \(n + \) the smallest part of \(\pi\) and the smallest part of \(\pi'\) is equal to the smallest part of \(\pi\) thus making the difference equal to \(n\).

For the other way, we now consider a partition \(\pi' \in F_n\). Note that the largest part of \(\pi'\) occurs exactly once. We then subtract \(n\) from the largest part of \(\pi'\) to get a new partition \(\pi\) whose smallest part is equal to the smallest part of \(\pi'\) and consider two cases pertaining to \(\pi'\): If the smallest part of \(\pi'\) occurs at least twice, we are done, i.e., \(\pi \in A_n\) since subtracting \(n\) from the largest part of \(\pi'\) does not affect the frequency of the smallest part of \(\pi\) (= the smallest part of \(\pi'\)) which still remains at least \(2\). Lastly, if the smallest part of \(\pi'\) occurs exactly once, subtracting \(n\) from the largest part of \(\pi'\) makes it equal to the the smallest part of \(\pi'\) and thus, the new partition \(\pi\) that we obtain has smallest part occurring at least twice since smallest parts of \(\pi'\) and \(\pi\) are equal. Hence, \(\pi \in A_n\).

Thus, the process is reversible and hence, we have a bijection between \(A_n\) and \(F_n\) giving our desired result. \(\square\)

6. Conclusion

Thus, we have \(p(2n, n) = a(n) = 2p(n) - p(n + 1) \forall n \geq 1\). We speculate that there is an interesting proof of \((2.2)\) using the generating function approach. Although in [2], Andrews, Beck, and Robbins gave the generating function for \(p(n, t)\) (which is the number of partitions of \(n\) with fixed difference equal to \(t\) between the largest and smallest parts), the generating function of \(p(2n, n)\) does not follow straightforward.

In the same spirit, we define \(G_m(q) := \sum_{n=1}^{\infty} a_m(n)q^n\) where \(a_m(n)\) denotes the number of partitions of \(n\) where the smallest part occurs at least \(m\).
times. On the page of A117989, we see that \( G_m(q) = \sum_{k=1}^{\infty} \frac{q^{mk}}{(q^k)_\infty} \). It will be very interesting to see if there is a closed formula analogous to (2.1) for \( a_m(n) \forall m \geq 3 \) and if there exists such a formula, then it would be nice to provide a combinatorial proof of it.

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References

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