A unified (classical-quantum-statistical) formalism for continuous spectrum systems.

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A unified (classical-quantum-statistical) formalism for a system with continuous spectrum is introduced. For this kind of systems ergodicity behavior and the existence of microcanonical and canonical (KMS) equilibrium is proved. It is argued that the continuous spectrum condition is essential for the thermodynamical behavior.

I. INTRODUCTION.

Systems with continuous spectrum have been studied at the classical, quantum, and statistical level. In papers [1], [2], and [3] we have contributed to this research. Precisely, based on the ideas of van Hove [4] we have introduced a method to deal with these systems, that we have successfully used to treat decaying phenomena, statistical mechanics problems and to find the classical limit of quantum systems [5]. In this paper, using the same technique, we would like to present a panoramic version of the different "limits" between physical levels. We will see that we can use our method (complemented by other ideas) as a unified formalism that allows to go from one level to the other. Even if some features of the problem are well known from a long time we believe that a unified treatment of these matters, like ours, was missing. Furthermore, most probably, it will be quite useful. Essentially we believed it is useful to see how the interplay of classical and quantum concepts can be used to solve problems.

So let us begin looking at the whole panorama. Physics studies two kind of systems:

A.- Individual systems, like particles or single physical systems. This is the case in classical mechanics, electromagnetism, special, and general relativity.

B.- Statistical systems, like ensembles or sets with many copies of a single physical system. This is the case in quantum mechanics, statistical (classical or quantum) mechanics, thermodynamics, and quantum field theory.

In category A the limits between different levels are simple and well known:

A1.- We can go from general relativity to special relativity if we consider just flat space-times.

A2.- In the "classical limit" ($\beta = \frac{1}{T} \to 0$) special relativity becomes classical mechanics.

The limit between the two categories, e.g.: of statistical mechanics towards classical mechanics is produced by the phenomenon of localization that we have studied in the just quoted paper [5], where the necessary conditions under which this phenomenon take place (localizing potential, adequate initial conditions, etc.) are given for some examples [6]. So we will just make a short comment of this problem in the paper.

In category B the limits are much more involved and will be considered in this paper. So it is organized as follows:

B1.- The thermalization of a quantum system is considered in section II. We will study it in two steps:

B1.1.- A first step is decoherence (section II.A). This section, as section III, is a brief resume of ref. [5] included in the just quoted paper.

B1.2.- We will see how we can obtain a KMS equilibrium after decoherence (section II.B).

B2.- The quantum to classical statistical mechanic limit is studied in section III.

B3.- The statistical classical mechanics relation with the theory of dynamical system and with thermodynamics will be considered in two steps in section IV:

B3.1.- In the first step the classical statistical system is considered as dynamical one. Then the micro-canonical ensembles appear as a consequence of dynamical systems theorems (section IV.A).

B3.2.- Finally, in the second step, the canonical ensembles and thermodynamic are introduced (section IV.B).

Other limit can be obtained combining those above. Knowing all these matters we can foresee the final fate of a quantum system (to remain a quantum one, or a classical one, or to end in thermodynamical equilibrium, etc.).

1 Usually the relations between physical levels are called "limits", even though only in some cases they are just simple mathematical limits.

2 The study of the localization phenomena is a deep and complicated subject, since in its classical version is related with the problem of the "limit circle" (see [4], cap II). This is one of the problems that Hilbert listed for the XX century mathematicians and it remains still unsolved.
Section V is devoted to the localization problem (namely the $B \rightarrow A$ limit).
We will state our conclusion in section VI.

**II. QUANTUM TO STATISTICAL LIMIT.**

We will demonstrate that for a wide set of quantum systems non-diagonal density operators can be considered as the transient phase while diagonal density operators is the final regime and a permanent state\(^3\). We will find a basis where exact density operators decoherence appears for these final states. Then we will see how the quantum systems thermalize.

**A. Decoherence.**

1. Decoherence in the energy.

Let us consider a closed and isolated quantum system with $N + 1$ dynamical variables and a Hamiltonian endowed with a continuous spectrum and just one bounded ground state. So the discrete part of the spectrum of $H$ has only one value $\omega_0 < 0$\(^4\) and the continuous spectrum is, let say, $0 \leq \omega < \infty$. Let us assume that it is possible to diagonalize the Hamiltonian $H$, together with $N$ observables $O_i$ ($i = 1, ..., N$). The operators $(H, O_1, ..., O_N)$ form a complete set of commuting observables (CSCO). For simplicity we also assume a discrete spectrum for the $N$ observables $O_i$.

Therefore we write

$$H = \omega_0 \sum_m |\omega_0, m\rangle\langle \omega_0, m| + \int_0^\infty \omega \sum_m |\omega, m\rangle\langle \omega, m|d\omega$$

(1)

where $\omega_0 < 0$ is the energy of the ground state, and $m \doteq \{m_1, ..., m_N\}$ labels a set of discrete indexes which are the eigenvalues of the observables $O_1, ..., O_N$. $\{|\omega_0, m\rangle, |\omega, m\rangle\}$ is a basis of simultaneous generalized eigenvectors of the CSCO:

$$H|\omega_0, m\rangle = \omega_0 |\omega_0, m\rangle, \quad H|\omega, m\rangle = \omega |\omega, m\rangle,$$

$$O_i|\omega_0, m\rangle = m_i |\omega_0, m\rangle, \quad O_i|\omega, m\rangle = m_i |\omega, m\rangle.$$

As at the statistical quantum level we only measure mean values of observables\(^5\) let us define the observables we will use. The most general observable that we are going to consider in our model reads:

$$O = \sum_{mm'} O(\omega_0)_{mm'} |\omega_0, m\rangle\langle \omega_0, m'| + \sum_{mm'} \int_0^\infty d\omega O(\omega)_{mm'} |\omega, m\rangle\langle \omega, m'| +$$

$$+ \sum_{mm'} \int_0^\infty d\omega' O(\omega, \omega')_{mm'} |\omega_0, m\rangle\langle \omega', m'| +$$

$$+ \sum_{mm'} \int_0^\infty \int_0^\infty d\omega d\omega' O(\omega, \omega')_{mm'} |\omega, m\rangle\langle \omega', m'|,$$

(2)

where $O^\dagger = O$ and $O(\omega)_{mm'}$, $O(\omega, \omega)_{mm'}$, $O(\omega_0, \omega)_{mm'}$ and $O(\omega, \omega')_{mm'}$ are ordinary functions of the real variables $\omega$ and $\omega'$ (these functions must have some mathematical properties in order to develop the theory; these properties

\(^3\)The arguments of this section are fully developed in paper \(^4\), where the philosophy of the method is explained at large.
\(^4\)The case of many bound states is considered in paper \(^5\). Here we consider the case with only one bound state because in this case the system usually decoheres.
\(^5\)Other quantum measurements imply a limit from category B to category A (cf. \(^3\)) which is outside the scope of this paper.
are listed in paper [2]). We will say that these observables belong to a space $O$ (which is contained in the algebra studied in [4]). This space has the basis \{|$\omega_0$, $mm'$\}, |$\omega$, $mm'$\}, |$\omega_0$, $mm'$\}, |$\omega_0$, $mm'$\}, |$\omega_0$, $mm'$\}, |$\omega_0$, $mm'$\}, |$\omega_0$, $mm'$\}:

$$
|\omega_0, mm'\rangle \doteq |\omega_0, m\rangle \langle \omega_0, m'|, \quad |\omega, mm'\rangle \doteq |\omega, m\rangle \langle \omega, m'|
$$

\begin{align}
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|,
|\omega_0, mm'\rangle & \doteq |\omega_0, m\rangle \langle \omega_0, m'|
\end{align}

The quantum states $\rho$ are measured by the observables just defined, computing the mean values of these observable in the quantum states, i.e. in the usual notation: $\langle O \rangle_{\rho} = Tr(\rho O)$. We can consider that mean values are the more primitive objects of quantum theory [7]. These mean values, generalized as in paper [3], can be considered as linear functionals $I$ defined as functionals by the equations:

$$
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle,
|\omega, mm'\rangle \doteq \delta_{m'n'}|\omega, mm'\rangle,
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle,
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle,
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle,
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle,
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle,
|\omega_0, mm'\rangle \doteq \delta_{m'n'}|\omega_0, mm'\rangle.
$$

and all other $\langle .| . \rangle$ are zero. Then, a generic quantum state reads:

$$
\rho = \sum_{mm'} \rho(\omega_0)_{mm'}|\omega_0, mm'\rangle \langle \omega_0, mm'| + \sum_{mm'} \int_0^\infty d\omega \rho(\omega)_{mm'}|\omega, mm'\rangle \langle \omega, mm'|
$$

\begin{align}
+ \sum_{mm'} \int_0^\infty d\omega \rho(\omega, \omega_0)_{mm'}|\omega_0, mm'\rangle \langle \omega_0, mm'| + \\
+ \sum_{mm'} \int_0^\infty d\omega \rho(\omega, \omega'_0)_{mm'}|\omega_0, mm'\rangle \langle \omega_0, mm'| + \\
+ \sum_{mm'} \int_0^\infty d\omega \int_0^\infty d\omega' \rho(\omega, \omega')_{mm'}|\omega_0, mm'\rangle \langle \omega_0, mm'|
\end{align}

$$
\text{where}

$$
\rho(\omega, \omega_0)_{mm'} = \rho(\omega, \omega)_{mm'}, \quad \rho(\omega, \omega'_0)_{mm'} = \rho(\omega', \omega)_{mm'},
$$

and $\rho(\omega_0)_{mm}$ and $\rho(\omega)_{mm}$ are real and non negative satisfying the total probability condition

$$
(\rho|I) = \sum_m \rho(\omega_0)_{mm} + \sum_m \int_0^\infty d\omega \rho(\omega)_{mm} = 1,
$$

where $I = \sum_m |\omega_0, m\rangle \langle \omega_0, m| + \int_0^\infty d\omega \sum_m |\omega, m\rangle \langle \omega, m|$ is the identity operator in $O$. Eq. (5) is the extension to state functionals of the usual condition $Tr \rho I = 1$, used when $\rho$ is a density operator. Thus, from now on, $Tr \rho \doteq (\rho|I)$.

The time evolution of the quantum state $\rho$ reads:
Thus, any quantum state weakly goes to a linear combination of the energy diagonal states (the energy “off-diagonal” states (ωω, mm) are not present in ρ∗. This is the case if we observe and measure the system evolution with any possible observable of space Ω (albeit the discussion in section IV.A.3). Then, from the observational point of view, we have decoherence of the energy levels, even that, from the strong limit point of view the off-diagonal terms never vanish, they just oscillate, since we cannot directly use the Riemann-Lebesgue theorem in the operator equation (II).

As we have already said at the statistical quantum level we can only measure mean values of observables in quantum states, i.e.:

\[ \langle O \rangle_{\rho(t)} = \langle \rho(t) | O | \rangle = \]

\[ = \sum_{mm'} \overline{\rho(\omega_0)}_{mm'} O(\omega_0)_{mm'} + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{mm'} O(\omega)_{mm'} + \]

\[ + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega, \omega_0)}_{mm'} e^{i(\omega - \omega_0)t} O(\omega, \omega_0)_{mm'} + \]

\[ + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega, \omega')}_{mm'} e^{i(\omega - \omega')t} O(\omega, \omega')_{mm'} + \]

\[ + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega, \omega')}_{mm'} e^{i(\omega - \omega')t} O(\omega, \omega')_{mm'}, \]

(7)

Let us now consider the fate of these mean values when \( t \to \infty \), using the Riemann-Lebesgue theorem we obtain the limit, for all \( O \in \Omega \)

\[ \lim_{t \to \infty} \langle O \rangle_{\rho(t)} = \langle O \rangle_{\rho_*} \]

(9)

where we have introduced the diagonal asymptotic or equilibrium state functional \[ \rho_* = \sum_{mm'} \overline{\rho(\omega_0)}_{mm'} (\omega_0, mm') + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{mm'} (\omega, mm') \]

(10)

Therefore, in a weak sense we have:

\[ W \lim_{t \to \infty} \rho(t) = \rho_* \]

(11)

Thus, any quantum state weakly goes to a linear combination of the energy diagonal states (ω0, mm’) and (ω, mm’) (the energy “off-diagonal” states (ωω0, mm’), (ωω’, mm’) and (ωω’, mm’) are not present in \( \rho_* \). This is the case if we observe and measure the system evolution with any possible observable of space \( \Omega \) (albeit the discussion in section IV.A.3). Then, from the observational point of view, we have decoherence of the energy levels, even that, from the strong limit point of view the off-diagonal terms never vanish, they just oscillate, since we cannot directly use the Riemann-Lebesgue theorem in the operator equation (II).

2. Decoherence in the other "momentum" dynamical variables.

Having established the decoherence in the energy levels we must consider the decoherence in the other dynamical variables \( O_i \), of the CSCO where we are working. We will call these variables "momentum variables". As the

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7 The three of diagonal terms of eqs. (7) or (8) disappear. This happens because we have just one bound state and would not be the case for more than one bound state.
expression of \( \rho_* \) given in eq. (10) involve only the time independent components of \( \rho(t) \), it is impossible that a different decoherence process take place to eliminate the off-diagonal terms in the remaining \( N \) dynamical variables. Therefore, the only thing to do is to find if there is a basis where the off-diagonal components of \( \rho(\omega)_m \) and \( \rho(\omega, \omega') \) vanish at any time.

Let us consider the following change of basis

\[
|\omega_0, r⟩ = \sum_m U(\omega_0)_{mr} |\omega_0, m⟩,
|\omega, r⟩ = \sum_m U(\omega)_{mr} |\omega, m⟩,
\]

(12)

where \( r \) and \( m \) are short notations for \( r = \{r_1, ..., r_N\} \) and \( m = \{m_1, ..., m_N\} \), and \([U(\Omega)]^{-1}_{mr} = U(\Omega)_{rm} \) (\( \Omega \) denotes either \( \omega_0 < 0 \) or \( \omega \in \mathbb{R}^+ \)).

The new basis \( \{|\omega_0, r⟩, |\omega, r⟩\} \) verifies the generalized orthogonality conditions

\[
⟨\omega_0, r|\omega_0, r'⟩ = \delta_{rr'}, \quad ⟨\omega, r|\omega', r'⟩ = \delta(\omega - \omega')\delta_{rr'},
⟨\omega_0, r|\omega, r'⟩ = ⟨\omega, r|\omega_0, r'⟩ = 0.
\]

As \( \rho(\omega)_m \) and \( \rho(\omega, \omega') \) vanish, it is possible to choose \( U(\omega_0) \) and \( U(\omega) \) in such a way that the off-diagonal parts of \( \rho(\omega)_m \) and \( \rho(\omega, \omega') \) vanish, i.e.

\[
\rho(\omega)_m = \rho(\omega) \delta_{rr'}, \quad \rho(\omega, \omega') = \rho(\omega) \delta_{rr'}.
\]

(13)

Therefore, there is a final pointer basis for the observables given by \( \{|\omega_0, r⟩, |\omega, r⟩\} \) and defined as in eq. (3). The corresponding final pointer basis for the states \( \{|\omega_0, r⟩, |\omega, r⟩, |\omega_0, rr⟩, |\omega, rr⟩, |\omega_0', rr⟩, |\omega_0', rr⟩, \} \) diagonalizes the time independent part of \( \rho(t) \) and therefore it diagonalizes the final state \( \rho_* \)

\[
\rho_* = W \lim_{t \to \infty} \rho(t) = \sum_r \rho_r(\omega_0)|\omega_0, rr⟩ + \sum_r \int_0^\infty d\omega \rho_r(\omega)|\omega, rr⟩.
\]

(14)

Now we can define the final exact pointer observables

\[
P_i = \sum_r P_i^r(\omega)|\omega_0, r⟩⟨\omega_0, r| + \int_0^\infty d\omega \sum_r P_i^r(\omega)|\omega, r⟩⟨\omega, r|.
\]

(16)

As \( H \) and \( P_i \) are diagonal in the basis \( \{|\omega_0, r⟩, |\omega, r⟩\} \), the set \( \{H, P_1, ..., P_N\} \) is precisely the complete set of commuting observables (CSCO) related to this basis, where \( \rho_* \) is diagonal in the corresponding co-basis for the states. For simplicity we define the operators \( P_i \) such that \( P_i^r(\omega) = P_i^r(\omega) = r_i \), thus

\[
P_i|\omega_0, r⟩ = r_i|\omega_0, r⟩, \quad P_i|\omega, r⟩ = r_i|\omega, r⟩.
\]

(17)

Therefore \( \{|\omega_0, r⟩, |\omega, r⟩\} \) is the observers' final pointer basis were there is a perfect decoherence in the corresponding state co-basis. Moreover the generalized states \( |\omega_0, rr⟩ \) and \( |\omega, rr⟩ \) are constants of the motion, and therefore these exact pointer observables have a constant statistical entropy and will be "at the top of the list" of Zurek’s "predictability sieve".

Therefore:

i.- Decoherence in the energy is produced by the time evolution.

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8If we would like to just have integrals and treat all spectra as continuous we would write

\[
\sum_r \int_0^\infty d\omega = \int_0^\infty ... \int_0^\infty d\omega dR_1 ... dR_N \delta(r_1 - R_1)... \delta(r_N - R_N) = \int d\omega \int d\mu(R)
\]

(15)

9In usual quantum mechanical system this set is numerable.
Decoherence in the other dynamical variables can be seen if we choose an adequate basis, namely the final pointer basis.

Our main result is eq. (14): When $t \to \infty$ then $\rho(t) \to \rho_*$ and in this state the dynamical variables $H, P_1, ..., P_N$ are well defined. Therefore the eventual conjugated variables to these momentum variables (namely: configuration variables, if they exist) are completely undefined.

In fact, calling by $L_i$ the generator of the displacements along the eventual configuration variable conjugated to $P_i$, we have $(L_i \rho_* O) = (\rho_* [L_i O]) = 0$ for all $O \in \mathcal{O}$ as it can be proved by direct computation [5]. Then $\rho_*$ is homogeneous in these configuration variables.

From the preceding section we may have the feeling that the process of decoherence must be found in all the physical systems, and therefore, all of them eventually would become classical when $\hbar \to 0$. It is not so, this evolution only happens under certain conditions, as explained in [5]. I. e., if there is more than one bound state or the decoherence time is infinite the system does not decohere.

B. Quantum thermalization.

1. The minimally biased state.

At this point we can find the final quantum minimally biased state according to information theory [13]. For simplicity let us just use one observable in our CSCO, the hamiltonian $H$ with no bound state $\omega^{010}$. Then eq. (14) reads

$$\langle \rho_\omega | = \int_0^\infty \rho(\omega) |\omega\rangle d\omega$$

(18)

$(\rho_\omega |$ has trace one, so from eq. (18) we know that

$$Tr \rho_* = (\rho_* | I) = \int_0^\infty \rho(\omega) d\omega = 1$$

(19)

The mean value of the energy reads

$$\langle H \rangle_{\rho_*} = (\rho_* | H) = \int_0^\infty \omega \rho(\omega) d\omega = E$$

(20)

If this is the only data available to obtain the minimal biased state we must maximizes the (Shannon) missing information

$$I = \mathcal{H}[\rho_*] = -K \int_0^\infty \rho(\omega) \log \rho(\omega) d\omega$$

(21)

constrained by eqs. (19) and (20). The solution of this variational problem is $\rho(\omega) \sim e^{-\beta \omega}$ so the unbiased $(\rho_* |$ is

$$\langle \rho_\omega | = Z^{-1} \int_0^\infty e^{-\beta \omega} |\omega\rangle d\omega$$

(22)

where the constants $Z$ and $\beta$ can be computed from eqs. (19) and (20).

The next problem is to represent this minimal biased $(\rho_\omega |$ as $e^{-\beta H}$. But an equation like $(\rho_\omega | = e^{-\beta H}$ is impossible since $H$ is a "vector" (ket) and $(\rho_* |$ is a "functional" (bra). To solve this problem we must go back to eq. (19) and see that the trace of the state can be written as

$$Tr \rho_* = (\rho | I) = \int_0^\infty |\omega\rangle d\omega \quad |I| = \int_0^\infty |\omega\rangle d\omega$$

(23)

10 In fact, in our model $H$ is the only member of the CSCO with continuous spectrum, so all the problem is contained in $H$.

11 I. e., the continuous version of eq. (2.3) or [13] or eq. (5.6) of [14], postulating that when no information or probabilities are available we should consider the cells equally probable.
Symmetrically we can define the trace of an operator $A$ as
\[ \text{Tr} A = (I|A) = \int_0^\infty (\omega|d\omega|A); \quad (I) = \int_0^\infty (\omega|d\omega) \tag{24} \]

We will see that this definition is completely reasonable at the end of section IV.B. Using it we can write the functional that corresponds to $(\rho_\star|A)$ as
\[ w_\beta[A] = (\rho_\star|A) = \frac{(I|e^{-\beta H}A)}{(I|e^{-\beta H})} \tag{25} \]

We can immediately see that the trace of this functional is one since $w_\beta[I] = (\rho_\star|I) = 1$. Moreover, the last definition coincides with the one of eq. (22) since $e^{-\beta H} = \int e^{-\beta \omega} |\omega\rangle d\omega$ and
\[ Z = (I|e^{-\beta H}) = \int \int (\omega|e^{-\beta \omega'} |\omega') d\omega d\omega' = \int e^{-\beta \omega} \delta(\omega - \omega') d\omega \tag{26} \]

Also
\[ (I|e^{-\beta H} A) = \int e^{-\beta \omega} (|\omega\rangle A) d\omega \tag{27} \]

From the two last equations it is evident that (22) coincides with (25) and we can write the minimally biased canonical final state as
\[ (\rho_\star) = Z^{-1}(I|e^{-\beta H}) \tag{28} \]

It is easy to generalize the above reasoning from $H$ to the CSCO $\{H, P_1, ..., P_N\}$. Then we would obtain
\[ (\rho_\star) = Z^{-1}(I|e^{-\beta H - \gamma_1 P_1 - ... - \gamma_N P_N}) \tag{29} \]

that corresponds to a generalized grand-canonical ensemble.

2. KMS equilibrium.

As a demonstration that $w_\beta[A]$ is the good equilibrium limit at temperature $\beta^{-1}$ we will prove that it satisfies the KMS condition [16]. The time evolution of operator $A$ is given by
\[ \alpha_t(A) = e^{iHt} A e^{-iHt} \tag{30} \]

( eq. (7), for the state evolution, is deduced from this equation). Then we must check the analyticity properties of
\[ w_\beta[\alpha_t(A)]B = \frac{(I|e^{-\beta H} \alpha_t(A) B)}{(I|e^{-\beta H})} = \frac{(I|e^{-\beta H} e^{iHt} A e^{-iHt} B)}{(I|e^{-\beta H})} \tag{31} \]

We must first prove that definition (24) has the cyclic trace property. Since the algebra $O$ is associative [1] we only need to demonstrate the commutativity. In fact [7]
\[ \text{Tr}(AB) = (I|AB) = \int (\omega|d\omega|AB) = \int A_\omega B_\omega d\omega = (I|BA) = \text{Tr}(BA) \tag{32} \]

then
\[ w_\beta[\alpha_t(A)]B = \frac{(I|B e^{iHt + i\beta} A e^{-iHt})}{(I|e^{-\beta H})} = \frac{(I|B e^{-\beta H} e^{iHt} A e^{-iHt} e^{\beta H})}{(I|e^{-\beta H})} = w_\beta(B \alpha_{t+i\beta}(A)) \tag{33} \]

All these probabilities naturally appear if we postulate that the characteristic algebra is a nuclear one as we will do in the rigorous treatment of the subject elsewhere.
Then calling (16) eq. (V.1.7)

\[
F_{A,B}^{(\beta)}(z) = w_{\beta}[B\alpha_z(A)]
\]

\[
G_{A,B}^{(\beta)}(z) = w_{\beta}[\alpha_z(A)B]
\]  

(34)

where \(z = t + i\gamma\), we obtain

\[
F_{A,B}^{(\beta)}(z) = \frac{(I|e^{-\beta H}Be^{iHz}Ae^{-iHz})}{(I|e^{-\beta H})} = \frac{(I|Be^{iHt}e^{-\gamma H}Ae^{-iHt}e^{-(\beta-\gamma)H})}{(I|e^{-\beta H})}
\]  

(35)

But from eq. (24)

\[
Tr e^{-\beta H} = (I|e^{-\beta H}) = \int e^{-\beta\omega}d\omega
\]  

(36)

which is only convergent and for \(\beta > 0\). Analogously, the r.h.s. of eq. (35) reads

\[
\int \int B(\omega, \omega')e^{i\omega't}e^{-\gamma\omega'}A(\omega', \omega)e^{-i\omega't}e^{-(\beta-\gamma)\omega}d\omega d\omega'/\int e^{-\beta\omega}d\omega
\]

Thus in order that \(F_{A,B}^{(\beta)}(z)\) be analytic in \(z\) these integrals must be convergent and therefore the three real parts of the exponents: \(\beta, \gamma\) and, \(\beta - \gamma\) must be positive. Then the operator in (35) is analytic for

\[0 < \gamma < \beta\]  

(37)

Also from (33)

\[G_{A,B}^{(\beta)}(t) = F_{A,B}^{(\beta)}(t + i\beta)\]  

(38)

and both functions are analytic for the \(z = t + i\gamma\) satisfying condition (37). Therefore KMS condition is satisfied.

We have proved that the thermal equilibrium state exists in the system but we have not proved if the system spontaneously goes to this equilibrium (it "thermalizes") and we have not said under what conditions this phenomenon takes place. The reason is that the two powerful devises we have used: the diagonal form of hamiltonian \(H\) and information theory allow us to reach the above conclusion but they yield to a static and already thermalized diagonal of the density matrix. To study thermalization we must consider at least two subsystems and their interaction before diagonalization and show how they reach the same temperature. In this case the method can be also used to show that, if the system is endowed with an adequate interaction, it naturally evolves to state (28), i.e. it "thermalizes". Thus the kind of interaction defines if the system thermalize or not. In fact, we have proved in paper [3] that the thermodynamic limit can be formalized with our method and we have developed a simple model with linear interaction (Friedrichs model) where an oscillator is thermalized by a bath for small interaction. The case of big interactions will be treated elsewhere.

Let us finally remark that a KMS state has a trivial explicit definition in a box. But in an unbounded space it was substituted by a state that satisfies KMS conditions with no explicit expression [16]. Our method has allow us to find this explicit expression, namely (28).

### III. THE CLASSICAL STATISTICAL LIMIT.

In this section we will use the Wigner integrals that introduce an isomorphism between quantum observables \(O\) and states \(\rho\) and their classical analogues \(O^W(q, p)\) and \(\rho^W(q, p)\) [17]:

13Or after diagonalization but in an indirect way as in [14] or [15]
\[ O^W(q, p) = \int d\lambda \left( q - \frac{\lambda}{2} (Oq + \frac{\lambda}{2}) \exp(\lambda p) \right) \]
\[ \rho^W(q, p) = \frac{1}{\pi^{N+1}} \int d\lambda \left( \rho(q + \lambda) \langle q - \lambda \rangle \right) \exp(2i\lambda p). \]  

(39)

It is possible to prove that \( \int dq dp \rho^W(q, p) = \langle \rho | I \rangle = 1 \), but in general \( \rho^W \) is not always non negative. It is also possible to deduce that

\[ (\rho^W | O^W) = \int dq dp \rho^W(q, p)O^W(q, p) = \langle \rho | O \rangle, \]  

and therefore: to the mean value in the classical space corresponds the mean value in the quantum space. Moreover, calling \( L \) the classical Liouville operator, and \( L \) the quantum Liouville-Von Neumann operator, we have

\[ L \rho^W(q, p) = [L\rho]^W(q, p) + O(h/s), \]  

(41)

where \( L \rho^W(q, p) = i \{ H^W(q, p), \rho^W(q, p) \} \) \( \rho_B \), \( s \) is an action with the characteristic dimension of the system, and

\[ (L\rho | O) = \langle \rho | [H, O] \rangle. \]  

(42)

Finally, if \( O = O_1O_2 \), where \( O_1 \) and \( O_2 \) are two quantum observables, we have

\[ O^W(q, p) = O_1^W(q, p)O_2^W(q, p) + O(h/s). \]  

(43)

We will prove that the distribution function \( \rho^W_q (q, p) \), that corresponds to the state functional \( \rho \), via the Wigner integral is a non negative function of the classical constants of the motion, in our case \( \hbar \), obtained from the corresponding quantum operators \( H, P_1, ..., P_N \).

From eq. (44) we have:

\[ \rho_\omega = W \lim_{t \to \infty} \rho(t) = \sum_r \rho_r(\omega_0, rr) + \sum_r \int_0^\infty d\omega \rho_r(\omega, rr), \]  

(44)

so we must compute:

\[ \rho_{\omega r}^W(q, p) \equiv \pi^{-N-1} \int (\omega, rr)\langle q + \lambda \rangle \langle q - \lambda \rangle e^{2ip\lambda} d\lambda \]  

(45)

We know from \( III \) section II. C, (or we can prove directly from eqs.(14-7)) that

\[ (\omega_0, rr|H^n) = \omega_0^n, \quad (\omega, rr|H^n) = \omega^n, \]
\[ (\omega_0, rr|P_i^n) = r_i^n, \quad (\omega, rr|P_i^n) = r_i^n, \quad i = 1, ..., N \]  

(46)

for \( n = 0, 1, 2, ... \) Using the relation (43) between quantum and classical products of observables and relation (40) between quantum and classical mean values, in the limit \( \hbar \to 0 \) (precisely when \( s \to \infty \), i.e. when the dimension of the system are very large compared with \( \hbar \)), we deduce that the characteristic property of the distribution \( \rho_{\omega r}^W(q, p) \), that corresponds to the state functional \( \langle \omega, rr \rangle \), is \( r_i^n \).

\[ \int \rho_{\omega r}^W(q, p)[H^W(q, p)]^n dq dp = \omega^n, \quad \int \rho_{\omega r}^W(q, p)[P_i^W(q, p)]^n dq dp = r_i^n, \]  

(47)

---

14 We can prove that \( H^W(q, p), P_1^W(q, p), ..., P_N^W(q, p) \) are constants of the motion using Heisenberg version of Liouville equation and eq. (41).

15 We will consider that we always take this limit when we refer to classical equations below.

16 To simplify the demonstration we will consider that all the spectra are continuous as explained in footnote 7.
for any natural number \( n \). Thus \( \rho^W_{\omega r}(q,p) \) must be the functional\(^7\)

\[
\rho^W_{\omega r}(q,p) = \delta(H^W(q,p) - \omega)\delta(P_1^W(q,p) - r_1)\cdots\delta(P_N^W(q,p) - r_N).
\]

(48)

For the distribution \( \rho^W_{\omega r}(q,p) \) corresponding to the state functional \( (\omega_0,r_r) \), we obtain

\[
\rho^W_{\omega_0 r}(q,p) = \delta(H^W(q,p) - \omega_0)\delta(P_1^W(q,p) - r_1)\cdots\delta(P_N^W(q,p) - r_N).
\]

(49)

Therefore, going back to eq. (44) and since the Wigner relation is linear, we have\(^5\)

\[
\rho^*_r(q,p) = \sum_r \rho_r(\omega_0)\rho^W_{\omega r}(q,p) + \sum_r \int_0^\infty d\omega \rho_r(\omega)\rho^W_{\omega r}(q,p) = \rho_0(H^W(q,p), P_1^W(q,p), \ldots, P_N^W(q,p)) + \rho(H^W(q,p), P_1^W(q,p), \ldots, P_N^W(q,p))
\]

(50)

Also we obtain \( \rho^W_r(q,p) \geq 0 \), because \( \rho_r(\omega_0) \) and \( \rho_r(\omega) \) are non-negative.

Therefore, the classical state \( \rho^W_r(q,p) \) is a linear combination of the generalized classical states \( \rho^W_{\omega_0 r}(q,p) \) (where \( \Omega \) is either \( \omega_0 \) or \( \omega \)), having well defined values \( \Omega, r_1, \ldots, r_N \) of the classical observables \( H^W(q,p), P_1^W(q,p), \ldots, P_N^W(q,p) \).

The corresponding classical canonically conjugated variables are completely undefined since neither \( \rho^W_{\omega_0 r}(q,p) \) nor \( \rho^W_r(q,p) \) is a function of these variables. So we reach, in the classical case, to the same conclusion than in the quantum case (see end of subsection II.A.2). But now all the classical canonically conjugated variables \( a_0, a_1, \ldots, a_N \) do exist since they can be found solving the corresponding Poisson brackets differential equations. We can also expand the densities given in eqs. (48, 50) in terms of classical motions as shown in [5] in great detail. In fact, as the momenta

\(^7\)Let us precisely define the \( \delta \) of the next equation. In eq. (17) we can make the canonical transformation \( q,p \to Q^W, P^W \) then it reads

\[
\int \rho^W_{\omega r}(Q^W_i, P^W_i)\prod_i P^W_i dQ^W_i dP^W_i = r^*_i, \quad i = 0,1,\ldots,N
\]

Then:

\[
\rho^W_{\omega r}(Q^W, P^W) = (V_Q)^{-1} \prod_i \delta(P^W_i - r_i)
\]

where

\[
V_Q = \int dQ^W
\]

is the volume of configuration space that we will consider as bounded for simplicity. Nevertheless the case \( V_Q \to \infty \) can be studied with the techniques of paper [3].

Then to prove eqs. (48, 49) we can first write eq. (44) as

\[
\rho^*_r(q,p) = \int d\mu(R)\rho_R(\omega_0)\rho^W_{\omega_0 r}(q,p) + \\
\int d\mu(R) \int_0^\infty d\omega \rho_R(\omega)\rho^W_{\omega r}(q,p)
\]

where we have used eq. (15).

\(^8\)We will call:

\[
\rho_0(\omega, r_1, \ldots, r_N) = \rho_0(\omega_0) = \rho_{r_1,\ldots,r_N}(\omega_0)
\]

\[
\rho(\omega, r_1, \ldots, r_N) = \rho_r(\omega) = \rho_{r_1,\ldots,r_N}(\omega)
\]
$H^W, P^W_1, ..., P^W_N$, or any function of these momenta, that we will call generically $\Pi$, are also constant of the motion, then we have \( \frac{d}{dt}\Pi = -\frac{\partial H}{\partial \alpha} = 0 \), where $\alpha$ is the classically conjugated variable to $\Pi$. So $H$ is just a function of the $\Pi$ and:

\[
\frac{d}{dt}\alpha = \frac{\partial H(\Pi)}{\partial \Pi} = \varpi(\Pi) = \text{const.}
\]

so:

\[
\alpha_j(t) = \varpi_j(\Pi)t + \alpha_j(0), \quad j = 0, 1, ..., N.
\]

Thus (going back to the old coordinates $q, p$) in the set of classical motions contained in the densities \([48]\) and \([49]\) the momenta $H^W(q, p)$, $P^W_1(q, p)$, ..., $P^W_N(q, p)$, are completely defined and the origin of the corresponding motions, that we will respectively call $a_0(0)$, $a_1(0)$, ..., and $a_N(0)$, are completely undefined (since $\rho^W(q, p)$ does not contain these variables), in such a way that the motions represented in the last equation homogeneously fill the surface where $H^W$, $P^W_1$, ..., $P^W_N$, have constant values. If the system is bounded and integrable (see below), these surface turns out to be the usual torus of phase space. This is the interpretation that we give to the density \((50)\) which is just a function of the variables $H^W$, $P^W_1$, ..., $P^W_N$. The classical motions described by eqs. \((51)\) and \((52)\) will be used to characterize the dynamical system in the next section. Precisely, using these motions the evolution of the $\rho^W$, close to equilibrium $\rho^W_\text{eq}$ can be interpreted as Frobenius-Perron operators \(\text{FP}\).

In conclusion.

i.- We have shown that the quantum state functional $\rho(t)$ evolves to a diagonal state $\rho_\text{eq}$.
ii.- This quantum state $\rho_\text{eq}$ has its corresponding classical density $\rho^W_\text{eq}(q, p)$.
iii.- This classical density can be decomposed in classical densities that correspond to sets of classical motions where $H^W$, $P^W_1$, ..., $P^W_N$ remain constant. These motions have $H^W(q, p)$, $P^W_1(q, p)$, ..., $P^W_N(q, p) = \text{const}$ and initial conditions $a_0(0)$, $a_1(0)$, ..., $a_N(0)$ distributed in a homogeneous way.
iv.- From eqs. \((45)\) and \((46)\) we will obtain that\([50]\)

\[
\rho^W_\text{eq}(q, p) = \rho(H^W(q, p), P^W_1(q, p), ..., P^W_N(q, p)) \geq 0
\]

IV. RELATION BETWEEN CLASSICAL STATISTIC, DYNAMICAL SYSTEMS AND THERMODYNAMICS.

In this section we will characterize the classical motions we have obtained in the ergodic hierarchy and we will make contact with classical thermodynamics.

A. The microcanonical ensemble.

1. The notion of isolating and non-isolating constant of the motion.

The classical constants of the motion $H^W(q, p)$, $P^W_1(q, p)$, ..., $P^W_N(q, p)$ can be rigorously classified as \([21], [22], [23]\):

i.- **Global or isolating constant of the motion**, that we will call "$H$", precisely:

\[
H^W(q, p) = H^W(q, p), \quad H^W_1(q, p) = P^W_1(q, p), ..., H^W_A(q, p) = P^W_A(q, p)
\]

when the conditions

\[
H^W_i = r_i, \quad i = 0, ..., A
\]

\(^{19}\) This is an important step towards classicality since the system has lost its quantum characteristic (e. g. totality, contextuality and non-locality according to \([13]\)) and we can resolve the classical density matrix evolution in a set of particle motions.

\(^{20}\) From now on we will forget the bound eigenvalue $\omega_0$.  

11
(where \( r_0 = \Omega \)) define global sub manifolds (tori in the bounded case) \( \mathcal{M}(r_0, ..., r_A) \), of phase space where the trajectories necessarily move, for each set of constants \((r_0, ..., r_A)\). The dimension of \( \mathcal{M} \) is \( 2(N + 1) - (A + 1) = 2N - A + 1 \).

ii.- Local or non isolating constant of the motion, that we will call:

\[
J^W_j(q, p) = \mathcal{P}^W(q, p), \ldots, J^W_N(q, p) = \mathcal{P}^W_N(q, p)
\]  

(56)

when the conditions

\[
J^W_j = r_{j+A} \quad j = 1, \ldots, N - A
\]

(57)
do not define any global sub-manifold, since they are just local \( \mathcal{P}^W \). The \( J^W_j \) can only be considered as local coordinates on the manifold \( \mathcal{M}(r_0, ..., r_A) \).

When \( A = N \) we say that the system is integrable, when \( A < M \) we say that the system is not integrable. Let us consider both cases.

2. Integrable systems.

In this case all the \( \mathcal{P} \) are isolating constants of the motion \((H)\) and there are no \( J \). Then the situation is like the one described at the end of section III: condition (55) foliates phase space with submanifolds \( \mathcal{M}(r_0, ..., r_N) \), labelled by the constants \((r_0, ..., r_N)\). This submanifold would be tori if the system is bounded. As we have already said we will only consider the bounded case \( \mathcal{P}^W \). In the tori the motion of the configuration variables is given by eq. (52). In the generic case the \( \mathcal{P}_{j}(\Pi) \) are not rationally dependant (or non-commensurable). Then the trajectories of the configuration variable fill each torus in a dense way. Therefore the motion is ergodic in each torus. Moreover, we can independently see from eq. (53) that there is a unique equilibrium state in each torus

\[
\rho^W_*(q, p) = \rho(r_0, r_1, ..., r_N)
\]

(58)

which is constant in the submanifold. Therefore we have a microcanonical equilibrium in each torus.

At this point we know that there is a unique stationary equilibrium state \( \rho^W_*(q, p) = \rho(r_0, ..., r_N) \) but we do not even know if the evolution converges to \( \rho^W_*(q, p) \). So now we will prove, using the quantum equations (14) or (44), that when \( t \to \infty \), any \( \rho^W \to \rho^W_* \) in a weak way.

In the quantum case we have

\[
\lim_{t \to \infty} \langle \rho(t)|O \rangle = \langle \rho_*|O \rangle
\]

(59)

Now, quantum products can be changed into classical products according to eq. (10) so:

\[
\lim_{t \to \infty} \langle \rho^W(t)|O^W \rangle = \langle \rho^W_*|O^W \rangle
\]

(60)

and we have proved that:

\[
W \lim_{t \to \infty} \rho^W(t) = \rho^W_*
\]

(61)

Therefore any \( \rho^W \to \rho^W_* \) weakly when \( t \to \infty \).

---

21 Isolating constant of the motion would be the "simple" constant of the motion in [14] p. 60.

22 Let us list the introduced dimension:

i.- The total dimension is \( 2(N + 1) \).

ii.- The number of the isolating constants is \( A + 1 \).

iii.- The number of the non isolating constant is \( N - A \).

iv.- The number of configuration coordinates is \( N + 1 \).

v.- The dimension of \( \mathcal{M} \) is \( 2N - A + 1 \).

23 The unbounded case requires a new singular structure in the states (see [2]). This generalization will be studied elsewhere.
3. Non integrable case.

In this case not all the \( P \) are isolating constants of the motion so there are \( H \) and \( J \). Then the trajectories must be dense in a domain \( \mathcal{D}(r_0, ..., r_A) \subseteq \mathcal{M}(r_0, ..., r_A) \) of dimensions \( 2N - A + 1 \). If not the dimensions of \( \mathcal{D}(r_0, ..., r_A) \) would be \( < 2N - A + 1 \). In this case a new global constant must exist and there would be \( A + 2 \) global constants. But this is impossible since \( A + 1 \) is the total number of these constants. It is quite clear that, as the equilibrium classical density must be globally defined in \( \mathcal{D}(r_0, ..., r_A) \) the \( J^W_1(q, p), ..., J^W_{N-A}(q, p) \) cannot be explicit variables of \( \rho^W_*(q, p) \) so we must just have:

\[
\rho^W_*(q, p) = \rho(H^W_0(q, p), H^W_1(q, p), ..., H^W_A(q, p))
\]

But on \( \mathcal{D}(r_0, ..., r_A) \) we have that:

\[
H^W_0(q, p) = r_0 = \text{const.}, \ H^W_1(q, p) = r_1 = \text{const.}, ..., H^W_A(q, p) = r_A = \text{const.}
\]

so on \( \mathcal{D}(r_0, ..., r_A) \) it is:

\[
\rho^W_*(q, p) = \rho(r_0, r_1, ..., r_A).
\]

and we find the unique equilibrium for each \( \mathcal{D}(r_0, ..., r_A) \), namely for each set of constants \( (r_0, ..., r_A) \). Now we can follow the reasoning of the previous section. Phase space is now foliated by submanifolds \( \mathcal{M}(r_0, ..., r_A) \) of dimension \( 2N - A + 1 \). The changes are that now not all the coordinates of these submanifolds are configuration variables\(^{26}\), there are also momentum variables, therefore we cannot use the reasoning about the \( \pi_j(\Pi) \) not rationally related in this case. Nevertheless in each \( \mathcal{D}(r_0, ..., r_A) \subset \mathcal{M}(r_0, ..., r_A) \) there is a unique equilibrium state (64) so using theorem 4.3 of ref. \(^{18}\) we conclude that the motion is ergodic in \( \mathcal{D}(r_0, ..., r_A) \). Now we can repeat the reasoning of eqs. (59) to (63) to show that the equilibrium (64) is weakly reached. Since this equilibrium is a constant we again find a microcanonical ensemble.

But, at this point we may ask ourselves why function \( \rho(H^W_0(q, p), H^W_1(q, p), ..., H^W_A(q, p)) \) looses its \( J \) variables. To explain this fact we may say that really space \( \mathcal{O} \) must contain physical measurable observables. But only the CSCO can be measured in an independent way since their observables commute. Moreover as the classical momenta \( J^W_1(q, p), ..., J^W_{N-A}(q, p) \) have an ergodic motion so it is reasonable to consider that the quantum analogues \( J_1, ..., J_{N-A} \), cannot really be measured, even at the quantum level\(^{27}\). Then the set \{ \( H_0, ..., H_A \) \} is the relevant measurable CSCO, and it is only possible to measure the \( H, H_1, ..., H_A \). Then the isolating constant \{ \( H_0, ..., H_A \) \} remains as the only characters in the quantum or classical play.

Phrased in another words the \( J \) variables can only be diagonalized locally, while the diagonalization procedure of section II.A.2 was thought as a global one. So it is better to consider that the unitary operators \( U \) of eq. (13) only diagonalize the indices of the \( H \) (that we will call \( r \) and do not diagonalize the indices of the \( J \) (that we will call \( m \)). Then we obtain a basis where the coordinates of the stases read (\( \text{see (13)} \))

\[
\rho(\omega)_{rm} = \rho_{rm}(\omega)\delta_{rr'}
\]

Then eqs. (14) and (16) become}

\(^{24}\)We may say that \( A + 1 \) tori are not broken and that \( A \) tori are broken.

\(^{25}\)From its definition, in eq. (13) or in eq. (61), \( \rho \) is just an ordinary function of global variables. Therefore it cannot be considered as a function defined using local coordinates.

\(^{26}\)Regarding the configuration variables it is clear that, in the non-integrable case, none of them is a global constant of the motion, that further reduces the dimensions of \( \mathcal{M} \) (or \( \mathcal{D} \)). In fact:

i.- The preserve torii satisfy an irrationality condition (\( \text{eq. (3.4.12)} \)) so the corresponding ratios of the frequencies are irrational and the trajectories are dense in these torii.

ii.- In the broken torii the trajectories are chaotic.

Nevertheless, if in a particular case there is a configuration variable \( X \) that turns out to be a global constant of the motion it can be considered among the "\( H \)". Then we will work essentially in the manifold \( X = \text{const.} \) and nothing will change.

\(^{27}\)We can only measure dynamical variables when they can be considered as constants in time, at least in the period of measurement. If a constant is not global it is only constant in time in a local coordinate system, i.e. it is not really physically constant.
\[ \rho_* = W \lim_{t \to \infty} \rho(t) = \sum_{rmm'} \int d\omega \rho_{rmm'}(\omega, rmrm') \]  

and

\[ P_i = \sum_{rmm'} \int d\omega P_i^{rmm'}(\omega rm)|\omega rm'm\rangle, \quad i = 1, 2, \ldots, A \]

and so on for the rest of the equations.

In order to follow this analysis we must "trace away" the \( m \) since the corresponding observables cannot we measured, i.e. they cannot be considered classical in a global way. Essentially we must consider the \( J \) operators and the \( m \) indices as inexistent in space \( O \). As \( O \) is the space of all measurable observables this fact must be considered as the following change in the observables of eq. (66)

\[ O(\omega)_{rmrm'} \rightarrow O(\omega)_{rr}\delta_{mm'} \]  

(we consider only the diagonal term since we are only concerned in the classical part and we neglect the \( \omega_0 \) term since we are systematically forgetting the ground state). In this way the \( m \) index has a "spherical symmetry" and they measure nothing. Then the relevant part of eq. (67) reads

\[ \langle O \rangle_{\rho} = \sum_{rr'} \int d\omega \rho(\omega)_{rr'}O(\omega)_{rr'} = \sum_{rr'} \int d\omega \rho(\omega)_{rr'}O(\omega)_{rr}\delta_{mm'} = \]

\[ \sum_{rr'} \int d\omega \left( \sum_m \rho(\omega)_{rr'm} \right) O(\omega)_{rr'} \]

Calling \( \sum_m \rho(\omega)_{rr'm} = \overline{\rho(\omega)_{rr'}} \) we obtain the "traced" equation:

\[ \langle O \rangle_{\rho} = \sum_{rr'} \int d\omega \overline{\rho(\omega)_{rr'}} O(\omega)_{rr'} \]

and the \( m \) indices have disappeared. From now on we can work with only the \( r \) indices \((r_0, \ldots, r_A)\) and in this way the \( \rho \) of eq. (68) becomes the \( \rho \) of eq. (70) solving the problem.

Two comments are in order:

i.- From what we have said eq. (14) can be considered as the quantum version of the microcanonical equilibrium. Therefore the conditions to obtain decoherence are equal to those to obtain microcanonical equilibrium states. But this state will be different according to the number of \( J \).

ii.- \( H \) has a continuous spectrum, so this is also the case of \( L \). (see (42)). Thus, albeit the \( O(h/s) \) of eq. (11), we can conclude that the classical \( L \) has a continuous spectrum. Moreover, normally quantum corrections make discrete the classical continuous spectra, and not viceversa. Even more, the spectrum of classical Liouville operators are usually continuous. Then \( L \), most likely, has a continuous spectrum. This is one of the characteristic properties of mixing (and therefore ergodic) systems, the spectrum of their evolution operators are continuous. This is another way to see that the flow is mixing in \( D(r_0, \ldots, r_A) \) [23], so the existence of a weak limit is natural.

B. The canonical and grand canonical ensemble.

We have proved that our system reach a microcanonical equilibrium. Then, the frequent presence of canonical equilibrium can be explained by at least three reasonings:

i.- The thermodynamic limit. In this limit microcanonical and canonical densities coincide (see e.g. [24]).

ii.- Canonical subsystem of a microcanonical system. A small subsystem of a big microcanonical system (that can be considered as a thermic bath) is canonical [24]. So, if the only isolating constant of the motion is the energy \( H \), we have reach to the notion of canonical equilibrium:

\[ \rho_*^W \sim e^{-\beta H^W} \]  

where \( \beta = T^{-1} \). In the more general case of section III we would arrive to the conclusion that (cf. [25])

\[ \rho_*^{W^{\prime}} \sim e^{-\beta H^{W^{\prime}}} \]
\[ \rho^W_\omega(q,p) = \delta(H^W(q,p) - \omega) \]  

so the classical object corresponding to \( I \) is

\[ I^W(q,p) = \int \delta(H^W(q,p) - \omega)d\omega = 1 \]

Therefore the classical trace of an operator corresponding to \( I|O \) is

\[ (I^W|O^W) = \int O^W(q,p)dqdp \]

namely the integral usually associated with the trace. Finally the classical equation corresponding to \( I|O \) reads

\[ w^W_\beta(A) = \frac{\int e^{-\beta H^W(q,p)} A^W(q,p)dqdp}{\int e^{-\beta H^W(q,p)} dqdp} \]

i.e. eq. (71), if we consider \( \rho^W_\omega \) as a functional of \( A \) and for the simplified CSCO \( \{H\} \). In the general case the functional version of (72) would be

\[ w^W_\beta(A) = \frac{\int e^{-\beta H^W(q,p)} - \gamma_1 H^W_1 - \ldots - \gamma_A H^W_A A^W(q,p)dqdp}{\int e^{-\beta H^W(q,p)} - \gamma_1 H^W_1 - \ldots - \gamma_A H^W_A dqdp} \]

i.e., the classical version of (28) restricted to the isolating members of the CSCO.

V. LOCALIZATION.

The fate of a classical statistical system is to remain unlocalized in the phase space or to localize around a classical trajectory, in which case the system becomes classical. This fate depends in the potential acting in the system (which can be localizing potentials or not) and the initial conditions. Let us consider a classical distribution \( \rho(q,p) \) with support of volume \( \Delta V^{(N+1)} \). It would localize if \( \Delta V^{(N+1)} \to 0 \) when \( t \to \infty \). But this is impossible since we are dealing with an hamiltonian system where, according to Poincaré theorem, the volume of phase space remains constant. Nevertheless we see localized classical objects. This is only possible of the system corresponds to those studied in section IV. A. 3 where there is a volume \( \Delta V^{(A+1)}_O \), that corresponds to the projection of the support of \( \rho(q,p) \) over the subspace of the observed dynamical variables \( H \), that vanishes when \( t \to \infty \), while the volume \( \Delta V^{(N-A)}_U \) corresponding to the unobservable \( J \) dynamical variables diverges when \( t \to \infty \) being

\[ \Delta V^{(N-A)}_U \Delta V^{(A+1)}_O \sim \Delta V^{(N+1)} = const. \]

Therefore the subsystem of the observable variables is not hamiltonian and the Poincaré theorem is by-passed. We conclude that only non integrable system become classical and that the ”non-observability” invoked in section IV.A.3 is essential to obtain a final classical mechanical limit.

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28 We will give examples of the localization phenomenon elsewhere.
VI. CONCLUSION.

In usual text books of thermodynamic or statistical mechanics the ergodicity of the system or the micro-canonical state are postulated. With our method we have proved these postulates for systems with evolution operator endowed with continuous spectrum only. Is this a limitation of our result? Quite on the contrary, it characterizes the systems where thermodynamic and statistical mechanics can be used: ergodic or mixing systems, being the latter one endowed with continuous evolution spectra. Of course if we work on a box the evolution operator may have a discrete spectrum and we cannot reach to this conclusion. But we know that in this case sooner or later we must expand the box to infinity. Then, either we use hand waving argument: e. g. that the distance between the values of the spectrum is very small etc., or we use the rigorous result that in the limit the spectrum is continuous, and from this fact deduce we the physical properties of the system, as we have done in this paper.

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