Noisy Intermediate-Scale Quantum (NISQ) technology will be available in the near future. Quantum computers with 50-100 qubits may be able to perform tasks which surpass the capabilities of today’s classical digital computers, but noise in quantum gates will limit the size of quantum circuits that can be executed reliably. NISQ devices will be useful tools for exploring many-body quantum physics, and may have other useful applications, but the 100-qubit quantum computer will not change the world right away — we should regard it as a significant step toward the more powerful quantum technologies of the future. Quantum technologists should continue to strive for more accurate quantum gates and, eventually, fully fault-tolerant quantum computing.
Review of Quantum Mechanics Part 1

Note: Everyone is assumed to be familiar with grad level QM

Review of 2-level systems, Tensor Products of States, Operators, and Hilbert Spaces. Density Matrix formalism

State vectors ("Rays" in Preskill)
Unique quantum state $\leftrightarrow$ unique state vector

$|\psi\rangle \in \mathcal{E}$ $\rightarrow$ State Space

Linear Operators

$\forall \psi \in \mathcal{E}: A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$

Projectors

$P_\psi = |\psi\rangle \langle \psi|$ $\leftarrow$ Projector on $|\psi\rangle$

$P_{\mathcal{E}_\psi} = \sum_{i=1}^{q} |\psi_i\rangle \langle \psi_i|$ $\leftarrow$ projector on subspace $\mathcal{E}_\psi$

Basis in $q$ dimensional $\mathcal{E}_\psi$

Hermitian Operators

$A^+ = A$

Adjoint $|\psi'\rangle = A|\psi\rangle$ $\leftrightarrow$ $\langle \psi'| = \langle \psi|A^+$

Physical (measurable) quantities!
Review of Quantum Mechanics Part 1

### Linear Operators
\[ \forall \psi \in \mathcal{E} : A\psi = \lambda \psi \]

### Projectors
\[ P_\psi = \psi \langle \psi | \leftarrow \text{Projector on } \psi \]
\[ P_{\mathcal{E}_q} = \sum_{i=1}^{q} |\psi_i \rangle \langle \psi_i | \leftarrow \text{projector on subspace } \mathcal{E}_q \]

**Basis in q dimensional } \mathcal{E}_q**

### Hermitian Operators
\[ A^+ = A \]

**Adjoint**
\[ \langle \psi' | = \langle A \psi | \leftarrow <\psi'| = <\psi|A^+ \]

**Physical (measurable) quantities!**

### Eigenvalue Equation
\[ A\psi = \lambda \psi \]

* **A** Hermitian
* **Eigenvalues of** \( A \) **are real-valued**
* **Eigenvectors** \( A\psi = \lambda \psi \) **are orthogonal**
  \[ A\phi = \mu \phi \text{ if } \lambda \neq \mu \]
* **Eigenvectors of** \( A \) **form orthonormal basis in** \( \mathcal{E} \)

### Commuting Observables
\[ [A,B] = AB - BA = 0 \]

\[ \exists \text{ orthonormal basis in } \mathcal{E} \text{ of common eigenvectors of } A, B \]
Review of Quantum Mechanics Part 1

Eigenvalue Equation
\[ A |\psi \rangle = \lambda |\psi \rangle \]

- A is Hermitian
- Eigenvectors of A are real-valued
- Eigenvectors are orthogonal if \( \lambda \neq \mu \)
- Eigenvectors of A form orthonormal basis in \( \mathcal{E} \)

C.S.C.O (Complete set of commuting observables)
Set \( A, B, C, \ldots \) such that basis \( \exists \) in \( \mathcal{E} \) of eigenvectors \( |a_m, b_m, c_n \ldots \rangle \) uniquely labeled by the set of eigenvalues \( a_m, b_m, c_m \)
Example \( H, L_z, L_\perp \) for the Hydrogen atom

Commuting Observables
\[ [A, B] = AB - BA = 0 \]
\( \exists \) orthonormal basis in \( \mathcal{E} \) of common eigenvectors of \( A, B \)

Unitary Operators
\( U \) is unitary
\[ U^{-1} = U^* \quad U^* U = U U^* = 1 \]
Scalar product invariant:
\[ \langle \psi | \phi \rangle = \langle U | U^* \psi | \phi \rangle \]
\( U \) is a change of basis in \( \mathcal{E} \)
\[ U |\psi \rangle = \lambda |\psi \rangle \]
\( \lambda = e^{i\theta} \)
eigenvecs for \( \lambda \neq \lambda' \) are orthogonal
Review of Quantum Mechanics Part 1

C.S.C.O (Complete set of commuting observables)
Set \( \{A_i, B, C, \ldots\} \) such that basis \( \exists \) in \( \mathcal{E} \) of eigenvectors \( \{a_m, b_m, c_m, \ldots\} \) uniquely labeled the set of eigenvalues \( a_m, b_m, c_m \)
Example \( H, L^2, L^2_z \) for the Hydrogen atom

Representation and bases
The set \( \{|\mu_i\rangle\} \) forms a basis in \( \mathcal{E} \) if the expansion
\[ |\psi\rangle = \sum |\mu_i\rangle \langle \mu_i| |\psi\rangle \quad \forall |\psi\rangle \in \mathcal{E} \]

Unitary Operators
\( U \) is unitary \( \implies U^{-1} = U^\dagger \iff U^\dagger U = UU^\dagger = 1 \)
Scalar product invariant: \( \langle \psi | \phi \rangle = \langle \psi | U^\dagger U | \phi \rangle \)
\( U \) is a change of basis in \( \mathcal{E} \)
\( U|\psi\rangle = \lambda|\psi\rangle \quad \lambda = e^{i\Theta} \)
eigenvectors for \( \lambda \neq \lambda' \) are orthogonal

States
\[ |\psi\rangle \leftarrow \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{bmatrix} \]

Operators
\[ A \leftarrow \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \]
Review of Quantum Mechanics Part 1

Postulates of Quantum Mechanics

1. At a fixed time \( t \) the state of a physical system is defined by specifying a ket \( |\psi(t)\rangle \) belonging to the state space \( \mathcal{E} \).

2. Every measurable physical quantity \( \mathcal{A} \) is described by an operator \( \hat{A} \) acting in \( \mathcal{E} \); this operator is an observable.

3. The only possible result of a measurement of a physical quantity \( \mathcal{A} \) is one of the eigenvalues of the corresponding observable \( \hat{A} \).

4. (Discrete non-degenerate spectrum) When the physical quantity \( \mathcal{A} \) is measured on a system in the normalized state \( |\psi\rangle \), the probability \( P(a_n) \) of obtaining the non-degenerate eigenvalue \( a_n \) of the observable \( A \) is:
\[
P(a_n) = |\langle a_n |\psi\rangle|^2 = \langle \psi | P_n |\psi\rangle
\]
where \( |a_n\rangle \) is the normalized eigenvector of \( A \) associated with the eigenvalue \( a_n \), and \( P = |a_n\rangle \langle a_n| \) is the projector onto \( |a_n\rangle \).

5. If the measurement of the physical quantity \( \mathcal{A} \) on the system in state \( |\psi\rangle \) gives the result \( a_n \), then the state immediately after the measurement is the normalized projection of \( |\psi\rangle \) onto \( |a_n\rangle \):
\[
|\psi_{\text{after}}\rangle = \frac{P_n |\psi\rangle}{\langle \psi | P_n |\psi\rangle}
\]
Degenerate case: use projector onto the Subspace associated with \( a_n \).

6. The time evolution of the state vector \( |\psi(t)\rangle \) is governed by the Schrödinger equation:
\[
i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle
\]
where \( H(t) \) is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for “classical” probability distributions

9-17-2024
Postulates of Quantum Mechanics

(5) If the measurement of the physical quantity \( A \) on the system in state \( |\psi\rangle \) gives the result \( a_n \), then the state immediately after the measurement is the normalized projection of \( |\psi\rangle \) onto \( |a_n\rangle \):

\[
|\psi\text{after}\rangle = \frac{\rho_n |a_n\rangle}{\langle a_n |a_n\rangle}
\]

Degenerate case: use projector onto the Subspace associated with \( a_n \).

(6) The time evolution of the state vector \( |\psi(t)\rangle \) is governed by the Schrödinger equation:

\[
i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle
\]

where \( \hat{H}(t) \) is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for “classical” probability distributions.

Quantum Mechanics of systems that consist of multiple parts

\[
\mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{E}
\]

System 1 \hspace{1cm} System 2 \hspace{1cm} Joint system

Def: Let \( \mathcal{E}_1, \mathcal{E}_2 \) be vector spaces of dimension \( N_1, N_2 \).

The vector space \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \) is called the Tensor Product of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) iff

\[
\forall \text{ pairs } |\psi_1\rangle \in \mathcal{E}_1, |\psi_2\rangle \in \mathcal{E}_2, \exists \text{ vector } |\xi\rangle \in \mathcal{E}
\]

such that

1. The association is linear with respect to multiplication with complex numbers

\[
\lambda |\psi_1\rangle \otimes |\psi_2\rangle = \lambda |\xi\rangle
\]
Review of Quantum Mechanics Part 1

Quantum Mechanics of systems that consist of multiple parts

\[ \mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{E} \]

System 1  System 2  Joint system

**Def:** Let \( \mathcal{E}_1, \mathcal{E}_2 \) be vector spaces of dimension \( N_1, N_2 \).

The vector space \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \) is called the **Tensor Product** of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) iff

\[ \forall \text{ pairs } |\psi(1)\rangle \in \mathcal{E}_1, |\chi(2)\rangle \in \mathcal{E}_2, \exists \text{ vector } |\psi\rangle \in \mathcal{E} \]

such that

1. The association is linear with respect to multiplication with complex numbers

\[ \lambda |\psi(1)\rangle \otimes |\chi(2)\rangle = \lambda \mu |\psi(1)\rangle \otimes |\chi(2)\rangle \]

2. Distributive

\[ |\psi(1)\rangle \otimes \left[ a |x_1(2)\rangle + b |x_2(2)\rangle \right] = a |\psi(1)\rangle \otimes |x_1(2)\rangle + b |\psi(1)\rangle \otimes |x_2(2)\rangle \]

3. Bases \( \{ |\psi(1)\rangle \} \) in \( \mathcal{E}_1 \), \( \{ |\psi(2)\rangle \} \) in \( \mathcal{E}_2 \)

\[ \{ |\psi(1)\rangle \otimes |\psi(2)\rangle \} \] is a basis in \( \mathcal{E} \)

Iff \( N_1, N_2 \) are finite, then \( \text{Dim} (\mathcal{E}) = N_1 \times N_2 \)

**These properties**

The usual linear algebra works in \( \mathcal{E} \)

**Analogy:** Tensor product of 1D & 2D geometrical space

\[ \mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D \text{ geom. space} \]

\( \delta \Phi \) of vectors in \( \mathcal{E}_1 \) w/vectors in \( \mathcal{E}_2 \) not defined
**Review of Quantum Mechanics Part 1**

**Quantum Mechanics of systems that consist of multiple parts**

\[ \mathcal{E}_1 \times \mathcal{E}_2 = \mathcal{E} \]

**System 1**  
**System 2**  
**Joint system**

**Def:** Let \( \mathcal{E}_1, \mathcal{E}_2 \) be vector spaces of dimension \( N_1, N_2 \)

The vector space \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \) is called the **Tensor Product** of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) iff

\[ \forall \text{ pairs } |\psi(1)\rangle \in \mathcal{E}_1, |\chi(2)\rangle \in \mathcal{E}_2, \exists \text{ vector } |\psi\rangle \otimes |\chi\rangle \text{ in } \mathcal{E} \]

such that

1. The association is linear with respect to multiplication with complex numbers

\[ \lambda |\psi(1)\rangle \otimes |\chi(2)\rangle = \lambda \mu |\psi(1)\rangle \otimes |\chi(2)\rangle \]

2. Distributive

\[ |\psi(1)\rangle \otimes [a |X_1(2)\rangle + b |X_2(2)\rangle] = a |\psi(1)\rangle \otimes |X_1(2)\rangle + b |\psi(1)\rangle \otimes |X_2(2)\rangle \]

3. Bases \( \{ |\mu_1(1)\rangle \} \text{ in } \mathcal{E}_1, \{ |\nu_2(2)\rangle \} \text{ in } \mathcal{E}_2 \)

\[ \{ |\mu_1(1)\rangle \otimes |\nu_2(2)\rangle \} \text{ is a basis in } \mathcal{E} \]

If \( N_1, N_2 \) are finite, then \( \text{Dim}(\mathcal{E}) = N_1 \times N_2 \)

**These properties**  
**The usual linear algebra works in \( \mathcal{E} \)**

**Analogy:** Tensor product of 1D & 2D geometrical space

Note: \( \mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D \) geom. space  
\( \delta P \) of vectors in \( \mathcal{E}_1 \) w/vectors in \( \mathcal{E}_2 \) not defined
Review of Quantum Mechanics Part 1

2. Distributive
\[ |\psi\rangle \otimes \left[ a |x_1\rangle + b |x_2\rangle \right] = a |\psi\rangle \otimes |x_1\rangle + b |\psi\rangle \otimes |x_2\rangle \]

3. Bases
\[ \{ |u_i \rangle \}_{i=1}^N \text{ in } \mathcal{E}, \quad \{ |v_j \rangle \}_{j=1}^N \text{ in } \mathcal{E}_2 \]

\[ \{ |u_i \rangle \otimes |v_j \rangle \}_{i,j=1}^{N_1 \times N_2} \text{ is a basis in } \mathcal{E} \]

If $N_1, N_2$ are finite, then $\text{Dim}(\mathcal{E}) = N_1 \times N_2$

These properties

The usual linear algebra works in $\mathcal{E}$

Hugely important:

There are vectors in $\mathcal{E}$ that are not tensor products of vectors from $\mathcal{E}_1, \mathcal{E}_2$

General vector $\psi \in \mathcal{E}$ can be written as
\[ |\psi\rangle = \sum_{i,e} c_{i,e} |u_i \rangle \otimes |v_e \rangle \]

How to see? There are $N_1 \times N_2$ prob. ampl’s $c_{i,e}$

These cannot all be written as $a_i \times b_e$ where the sets $\{a_i\}$, $\{b_e\}$ are valid probability amplitudes.

Vectors in $\mathcal{E}$
Let
\[ |\psi\rangle = \sum a_i |u_i \rangle \]
\[ |x_1\rangle = \sum b_e |v_e \rangle \]

Then
\[ |\psi\rangle \otimes |x_1\rangle = \sum_{i,e} a_i b_e |u_i \rangle \otimes |v_e \rangle \]
**Review of Quantum Mechanics Part 1**

**Vectors in $\mathcal{E}$**

Let

$$|\psi(1)\rangle = \sum_i a_i |\mu_i(1)\rangle$$

$$|\chi(2)\rangle = \sum_i b_i |\nu_i(2)\rangle$$

Then

$$|\psi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,\ell} a_i b_\ell |\mu_i(1)\rangle \otimes |\nu_\ell(2)\rangle$$

**Hugely important:**

There are vectors in $\mathcal{E}$ that are not tensor products of vectors from $\mathcal{E}_1, \mathcal{E}_2$.

General vector $\psi \in \mathcal{E}$ can be written as

$$|\psi\rangle = \sum_{i,\ell} c_{i\ell} |\mu_i(1)\rangle \otimes |\nu_\ell(2)\rangle$$

How to see? There are $N_1 \times N_2$ prob. ampl’s $c_{i\ell}$.

These cannot all be written as $a_i \times b_\ell$ where the sets $\{a_i\}$, $\{b_\ell\}$ are valid probability amplitudes.

**Example:** $\mathcal{E}_1$, $\mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$$|\psi(1)\rangle = a_1 |\mu_1(1)\rangle + a_2 |\mu_2(1)\rangle$$

$$|\chi(2)\rangle = b_1 |\nu_1(2)\rangle + b_2 |\nu_2(2)\rangle$$

2 real-valued variables each

In basis $\{|\mu_i(1)\rangle \otimes |\nu_\ell(2)\rangle\}$

**Product state**

$$\begin{bmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{bmatrix}$$

4 real-valued variables

**General state**

$$\begin{bmatrix}
  c_{11} \\
  c_{12} \\
  c_{21} \\
  c_{22}
\end{bmatrix}$$

6 real-valued variables

$N$ qubits $\implies$

- product state $\xrightarrow[]{} 2N$ real variables
- general state $\xrightarrow[]{} 2^{N+1} - 2$ real variables
Example: $\psi_1, \psi_2$ are qubits, $N_1 = N_2 = 2$

$|\psi(1)\rangle = a_1 |\psi_1(1)\rangle + a_2 |\psi_2(1)\rangle$

$|\psi(2)\rangle = b_1 |\psi_1(2)\rangle + b_2 |\psi_2(2)\rangle$

2 real-valued variables each

In basis $\{ |\psi_1(1)\rangle \otimes |\psi_2(2)\rangle \}$

Product state

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

General state

$$\begin{bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{bmatrix}$$

4 real-valued variables

6 real-valued variables

Note: States in $\mathcal{E}$ that are not product states are known as

Entangled States or Correlated States

Back to the Linear Algebra engine of QM

Scalar product:

$$\langle \psi'(1) | \otimes | \psi'(2) \rangle \langle | \psi(1) \rangle \otimes | \psi(2) \rangle$$

$$= \langle \psi'(1) | \psi(1) \rangle \langle \psi'(2) | \psi(2) \rangle$$

Operators: Let $A(\ell)$ act in $\mathcal{E}(\ell)$

The Extension $\tilde{A}(\ell)$ acting in $\mathcal{E}$ is defined by

$$\tilde{A}(\ell) [ | \psi(\ell) \rangle \otimes | \psi(2) \rangle ] = [ A(\ell) | \psi(\ell) \rangle ] \otimes | \psi(2) \rangle$$

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar
Review of Quantum Mechanics Part 1

**Note:** States $\psi \in \mathcal{E}$ that are not product states are known as

**Entangled States** or **Correlated States**

---

Back to the Linear Algebra engine of QM

**Scalar product:**

\[
\langle \phi'(1) | \phi(1) \rangle \langle \chi'(2) | \chi(2) \rangle = \langle \phi'(1) | \phi(1) \rangle \langle \chi'(2) | \chi(2) \rangle
\]

**Operators:** Let $A(1)$ act in $\mathcal{E}(1)$

The Extension $\tilde{A}(1)$ acting in $\mathcal{E}$ is defined by

\[
\tilde{A}(1) | \phi(1) \rangle \otimes | \chi(2) \rangle = (A(1) | \phi(1) \rangle) \otimes | \chi(2) \rangle
\]

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar

---

**Tensor Product of Operators**

\[
[A(1) \otimes B(2)] [ | \phi(1) \rangle \otimes | \chi(2) \rangle ] = [ A(1) | \phi(1) \rangle] \otimes [ B(2) | \chi(2) \rangle ]
\]

\[
A(1) \otimes B(2) = \tilde{A}(1) \tilde{B}(2)
\]

special case:

\[
\tilde{A}(1) = A(1) \otimes 1(2) \quad \tilde{B}(2) = 1(1) \otimes B(2)
\]

**Commutator**

\[
[\tilde{A}(1), \tilde{B}(2)] = 0 \quad \text{because} \quad [A(1), 1(1)] = [B(2), 1(2)] = 0
\]

**Notation:** Obvious from context

\[
| \phi(1) \rangle \otimes | \chi(2) \rangle \leftrightarrow | \phi(1) \rangle | \chi(2) \rangle \leftrightarrow | \phi(1) \chi(2) \rangle
\]

\[
A(1) \otimes B(2) \leftrightarrow A(1) B(2)
\]

\[
\tilde{A}(1) \leftrightarrow A(1)
\]
Review of Quantum Mechanics Part 1

Tensor Product of Operators

\[
[\hat{A}(1) \otimes \hat{B}(2)] [\ket{\Psi(1)} \otimes \ket{X(2)}] = [\hat{A}(1) \ket{\Psi(1)}] \otimes [\hat{B}(2) \ket{X(2)}]
\]

\[\hat{A}(1) \otimes \hat{B}(2) = \hat{A}(1) \hat{B}(2)\]

special case:

\[\hat{A}(1) = \hat{A}(1) \otimes 1(2)\]

\[\hat{B}(2) = 1(1) \otimes \hat{B}(2)\]

Eigenvalue problem in \(E\)

Let

\[\hat{A}(1) \ket{\Phi_n^i(t)} = \lambda_n \ket{\Phi_n^i(t)}\quad i = 1, \ldots, g_n\]

\[\hat{A}(1) \ket{\Phi_n^i(t)} \ket{X(2)} = \lambda_n \ket{\Phi_n^i(t)} \ket{X(2)} \quad \forall \ket{X(2)} \in E_2\]

Can choose \(\ket{X(2)}\) orthonormal basis in \(E_2\)

\[g; \times N_2\ - \text{fold degeneracy of } \lambda_n \text{ in } E\]

Furthermore

\[\begin{align*}
\hat{A}(1) \ket{\Phi_n^i(t)} & = \lambda_n \ket{\Phi_n^i(t)} \\
\hat{B}(2) \ket{X(2)} & = \beta \ket{X(2)}
\end{align*}\]

\[\begin{align*}
(\hat{A}(1) + \hat{B}(2)) \ket{\Phi_n^i(t)} \ket{X(2)} & = (\lambda_n + \beta) \ket{\Phi_n^i(t)} \ket{X(2)} \\
\hat{A}(1) \hat{B}(2) \ket{\Phi_n^i(t)} \ket{X(2)} & = \lambda_n \beta \ket{\Phi_n^i(t)} \ket{X(2)} \\
\hat{f}(\hat{A}(1), \hat{B}(2)) \ket{\Phi_n^i(t)} \ket{X(2)} & = \hat{f}(\lambda_n, \beta) \ket{\Phi_n^i(t)} \ket{X(2)}
\end{align*}\]

Notation: Obvious from context

\[\ket{\Phi(1)} \otimes \ket{X(2)} \leftrightarrow \ket{\Phi(1)} X(2) \leftrightarrow \ket{\Phi(1)} X(2)\]

\[\hat{A}(1) \otimes B(2) \leftrightarrow A(1) B(2)\]

\[\hat{A}(1) \leftrightarrow A(1)\]

Commutator

\[\comm{\hat{A}(1)}{\hat{B}(2)} = 0 \quad \text{because} \quad [\hat{A}(1), 1(1)] = [\hat{B}(2), 1(2)] = 0\]

Postulates of QM apply in \(E_1, E_2\) and \(E\)

\[\Rightarrow \text{We are Done!}\]
Postulates of Quantum Mechanics

(1) At a fixed time $t$ the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{E}$.

(2) Every measurable physical quantity $\mathcal{A}$ is described by an operator $\hat{A}$ acting in $\mathcal{E}$; this operator is an observable.

(3) The only possible result of a measurement of a physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable $\hat{A}$.

(4) (Discrete non-degenerate spectrum) When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the non-degenerate eigenvalue $a_n$ of the observable $\hat{A}$ is:

$$P(a_n) = |\langle a_n | \psi \rangle|^2 = \langle \psi | P_n | \psi \rangle$$

where $|a_n\rangle$ is the normalized eigenvector of $\hat{A}$ associated with the eigenvalue $a_n$, and $P = |a_n \rangle \langle a_n|$ is the projector onto $|a_n\rangle$.

(5) If the measurement of the physical quantity $\mathcal{A}$ on the system in state $|\psi\rangle$ gives the result $a_n$, then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{\text{after}}\rangle = \frac{P_n |\psi\rangle}{\langle \psi | P_n | \psi \rangle}$$

Degenerate case: use projector onto the Subspace associated with $a_n$.

(6) The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation:

$$i \hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

where $\hat{H}(t)$ is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for “classical” probability distributions.
Bayes rule and the updating of probabilities

The Bayesian Update Rule

Consider two stochastic variables $A$ and $B$. The joint, conditional, and univariate probabilities are related as follows:

$$P(A,B) = P(A|B)P(B)$$

$$P(A,B) = P(B|A)P(A)$$

Therefore, we have:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Thus, with knowledge of $P(A)$ and $P(B)$ we can update our prior knowledge $P(B|A)$ when new information, $P(A|B)$, becomes available.

The Bayesian Update Rule generalizes like this:

$$P(\alpha|B) \; d\alpha = \frac{P(B|\alpha)p(\alpha) \; d\alpha}{P(B)}$$

where $P(B) = \int_{-\infty}^{\infty} P(B|\alpha)p(\alpha) \; d\alpha$ is a number

and therefore

$$P(\alpha|B) \propto P(B|\alpha)p(\alpha)$$

See https://math.mit.edu/~dav/05.dir/clss13-slidesall.pdf Page 17
Bayes rule and the updating of probabilities

The Bayesian Update Rule generalizes like this:

\[ p(\alpha | B) \, d\alpha = \frac{p(B | \alpha) \, p(\alpha) \, d\alpha}{\int_{-\infty}^{\infty} p(B | \alpha) \, p(\alpha) \, d\alpha} \]

where \[ p(B) = \int_{-\infty}^{\infty} p(B | \alpha) \, p(\alpha) \, d\alpha \] is a number

and therefore

\[ p(\alpha | B) \propto p(B | \alpha) \, p(\alpha) \]

See https://math.mit.edu/~dav/05.dir/clss13-slidesall.pdf Page 17

Bayesian Update of Classical Information

Consider a classical particle located somewhere on the \( \alpha \) – axis. The Bayesian interpretation holds that a probability distribution quantifies prior knowledge, in this example about the position of the particle.

Let \( p(\alpha) \) be the probability density for finding the particle at position \( \alpha \). We assume this pdf is a Gaussian centered at \( \alpha = 0 \).

\[ p(\alpha) = \frac{1}{\sqrt{2\pi}\sigma^2} \, e^{-\frac{\alpha^2}{2\sigma^2}} \]

Next, we measure the position of the particle without disturbing it. The measurement has finite resolution, i.e., there is a change of observing the particle at \( B \) even if the actual position is \( \alpha \). This resolution is quantified by the likelihood Function \( p(B | \alpha) \)
Bayes rule and the updating of probabilities

Bayesian Update of Classical Information

Consider a classical particle located somewhere on the $\alpha$ – axis. The Bayesian interpretation holds that a probability distribution quantifies prior knowledge, in this example about the position of the particle.

Let $p(\alpha)$ be the probability density for finding the particle at position $\alpha$. We assume this pdf is a Gaussian centered at $\alpha = 0$.

\[ p(\alpha) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{\alpha^2}{2\sigma^2}} \]

Next, we measure the position of the particle without disturbing it. The measurement has finite resolution, i.e., there is a change of observing the particle at $B$ even if the actual position is $\alpha$. This resolution is quantified by the likelihood Function $p(B|\alpha)$.

Bayesian Update of Classical Information, cont.

Let $p(B|\alpha)$ be a Gaussian,

\[ p(B|\alpha) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{B^2}{2\sigma^2}} \]

Post-measurement, we can use Bayes Rule to update our knowledge of the position of the particle given that we observed $B$:

\[ p(\alpha|B) \propto p(B|\alpha)p(\alpha) \]

The product of two Gaussians is a Gaussian, and therefore $G_{pm}$ is also a Gaussian.

Furthermore, there are exact expressions for the means and $\sigma$’s of the products, see, e.g. http://www.lucamartino.altervista.org/2003-003.pdf
Bayes rule and the updating of probabilities

Bayesian Update of Classical Information, cont.

Let $\mathcal{N}(B|\alpha)$ be a Gaussian,

$$P(B|\alpha) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-B^2/2\sigma^2}$$

Post-measurement, we can use Bayes Rule to update our knowledge of the position of the particle given that we observed $B$:

$$P(\alpha|B) \propto P(B|\alpha) P(\alpha)$$

The product of two Gaussians is a Gaussian, and therefore $G_{pm}$ is also a Gaussian.

Furthermore, there are exact expressions for the means and $\sigma$’s of the products, see, e. g. http://www.lucamartino.altervista.org/2003-003.pdf

Physical Interpretation, Sharp Measurement

Now let $\sigma_{B|\alpha} \ll \sigma_\alpha$. In that case the pdf’s will look like this:

$$P(\alpha|B) \approx \frac{1}{\sqrt{2\pi} \sigma^2} e^{-(\alpha-B)^2/2\sigma^2_{B|\alpha}}$$

Here we learn a lot from the measurement, and this leads to a large update of our Prior. In this example there will be a large change in the mean and uncertainty that we assign post-measurement. The resulting pdf looks much more like the resolution function than the pdf for the original Gaussian $P(\alpha)$. 


Bayes rule and the updating of probabilities

Physical Interpretation, Sharp Measurement

Now let $\sigma_{B|\alpha} \ll \sigma_{\alpha}$. In that case the pdf’s will look like this:

$$p(\alpha|B) \approx \frac{1}{\sqrt{2\pi}\sigma_{B|\alpha}} e^{-\frac{(\alpha - B)^2}{2\sigma_{B|\alpha}^2}}$$

Here we learn a lot from the measurement, and this leads to a large update of our Prior. In this example there will be a large change in the mean and uncertainty that we assign post-measurement. The resulting pdf looks much more like the resolution function than the pdf for the original Gaussian $p(\alpha)$.

Physical Interpretation, Unsharp Measurement

Now let $\sigma_{B|\alpha} \approx \sigma_{\alpha}$. In that case the pdf’s will look like this:

$$p(\alpha|B) \approx \frac{1}{\sqrt{2\pi}\sigma_{\alpha}} e^{-\frac{\alpha^2}{2\sigma_{\alpha}^2}}$$

Here we learn little from the measurement and this leads to at most a minor update of our Prior. In this example there will be at most a modest change in the mean and uncertainty that we assign post-measurement. The result looks like a slightly shifted and broadened version of the original $p(\alpha)$.