PATHWISE SOLUTIONS FOR FULLY NONLINEAR FIRST- AND
SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH
MULTIPLICATIVE ROUGH TIME DEPENDENCE

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Abstract. The notes are an overview of part of the theory of pathwise weak solutions to two classes
of scalar fully nonlinear first- and second-order degenerate parabolic partial differential equations
with multiplicative rough time dependence, a special case being Brownian. These are Hamilton-
Jacobi-Isaacs-Bellman and quasilinear divergence form equations including multi-dimensional scalar
conservation laws. If the time dependence is “regular”, the weak solutions are respectively the
viscosity and entropy/kinetic solutions. The results presented here are about the wellposedness of
the solutions. Some concrete applications are also discussed. The material for the first class of
problems are part of the ongoing development of the theory in collaboration with P.-L. Lions. The
results about quasilinear divergence form equations are based on joint work with P.-L. Lions, B.
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0. Introduction

This is an overview of part of the theory of pathwise weak solutions to two classes of scalar fully
nonlinear first- and second-order degenerate parabolic stochastic partial differential equations (spde
for short) with multiplicative rough time dependence, a special case being Brownian. These are
Hamilton-Jacobi and quasilinear divergence form pde including multidimensional scalar conservation
laws. If the time dependence is “regular”, the weak solutions are respectively the viscosity and entropy/kinetic solutions. The results presented here are about the wellposedness of the solutions. Some concrete applications are also discussed both to motivate as well as to show the scope of
the theory. The material about the first class of problems are part of the ongoing development
of the theory in collaboration with P.-L. Lions [57, 58, 59, 60, 61, 62, 54, 55]. The results about
quasilinear divergence form equations are based on joint work with P.-L. Lions, B. Perthame and
B. Gess [49, 50, 51, 29, 28, 30, 27].

Problems of the type discussed here arise in several applied contexts and models for a wide variety
of phenomena and applications including mean field games, turbulence, phase transitions and front
propagation with random velocity, nucleations in physics, macroscopic limits of particle systems,
pathwise stochastic control theory, stochastic optimization with partial observations, stochastic
selection, etc..

The theory is about the general Hamilton-Jacobi-Bellman-Isaacs evolution equations

(0.1) \[ du = F(D^2 u, Du, u, x, t)dt + \sum_{i=1}^m H_i(Du, u, x, t) \cdot dB_i \text{ in } Q_T := \mathbb{R}^N \times (0, T], \]

and the scalar divergence form quasilinear problem

(0.2) \[ du + \sum_{i=1}^N \partial_x A_i(u, x) \cdot dB_i - \text{div}(A(u, x)Du)dt = 0 \text{ in } Q_T, \]
with initial condition

\[(0.3)\quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N.\]

Here \(F, H_1, \ldots, H_m, A_1, \ldots, A_N\) and \(A\) are (at least) continuous functions of their arguments (exact assumptions will be shown later), \(F\) and \(A\) are respectively degenerate elliptic and monotone, \(B := (B_1, \ldots, B_m)\) is a continuous rough path in time, “.” simply denotes the way \(B\) acts on the \(H_i\) and \(A_i\). When \(B\) it is a Brownian path, “.” becomes the usual Stratonovich differential “o”, something justified by the fact that the pathwise solutions to the equations may be obtained as the limit of solutions to the equations with \(B\) replaced by smooth approximations. Finally, \(B\) can be taken to be a finite mode approximation of “colored white noise.” Given the notation used here, its smooth spatial dependence is taken to be part of the \(H_i\)’s and the \(A_i\)’s.

When \(B\) is either smooth or has bounded variation, then “\(d\)” is the regular time derivative and \((0.1)\) and \((0.2)\) are classical “deterministic equations” which have been studied using respectively the classical viscosity and entropy/kinetic theories.

The theory presented in these notes is a pathwise one and simply treats \(B\) as the time derivative of a continuous function. When the \(H_i\)’s and \(A_i\)’s are independent of \(x\), the general qualitative theory does not need any other assumption but continuity. When there is spatial dependence then it is necessary to use more and the theory of rough paths comes in handy as it will become clear later.

There is a vast literature for linear and quasilinear versions of \((0.1)\) as well as work for some versions of \((0.2)\). Listing all the references is not possible in this introduction but connections will be made in the next sections.

**A warning.** These notes are by no means comprehensive. They represent a summary of the kind of results that have been obtained in the past as well as a reference to the problems and open questions. A reader interested in the area should definitely consult all the references.

**Organization of the notes.** Concrete examples where \((0.1)\) and \((0.2)\) arise are presented in Section 1. Section 2 discusses the main difficulties and explains why the Stratonovich formulation is more appropriate. Section 3 is devoted to some special cases, a classical result about stochastic Hamilton-Jacobi equations in the smooth regime as well as nonlinear equations with linear rough path dependence. Section 4 is about fully nonlinear equations with semilinear rough path dependence. Section 5 discusses formulae or the lack thereof for Hamilton-Jacobi equations with time dependence. Section 6 is about the simplest possible nonlinear pde with rough time signals as the limit of regular approximations. Results about the wellposedness of the pathwise solutions to nonlinear first order pde with nonsmooth Hamiltonians and rough signals is presented in Section 7. Section 8 is devoted to the second order problems with smooth Hamiltonians. Section 8 summarizes some upcoming results and ongoing research. Section 9 is about stochastic conservation laws. Finally, the Appendix summarizes few basic things from the classical theory of viscosity solutions that are used in the notes.

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1. Motivation and some examples

A discussion follows of a number of results that have been proved or may be solved using the theory presented in here. In several places in this section, to keep the discussion simple, the presentation is informal.

Motion of interfaces. An important question in pde and geometry as well as applications like phase transitions is the understanding of the long time behavior of solutions to reaction-diffusion equations and the properties of the interface developing (for large times) separating the regions where the solutions approach the different equilibria of the equation.

A classical and well studied problem in this context is the asymptotic behavior of the solutions \( u^\varepsilon \) to the so-solutions to called Allen-Cahn equation

\[
\frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon + \frac{1}{\varepsilon} W'(u^\varepsilon) = 0 \quad \text{in} \; Q_T,
\]

where \( W : \mathbb{R} \to \mathbb{R} \) is a double-well potential with wells of equal depth located, for example, at \( \pm 1 \). It is well known that as, \( \varepsilon \to 0 \), \( u^\varepsilon \to \pm 1 \) inside and outside an interface moving with normal velocity \( V = -\kappa \), where \( \kappa \) is the mean curvature. The interface is the zero-level-set of the solution to the level-set pde

\[
v_t - (I - \frac{Dv}{|Dv|} \otimes \frac{Dv}{|Dv|}) : D^2 v = 0 \quad \text{in} \; Q_T,
\]

where for \( A, B \in S^N \), \( A : B := \text{tr} AB \) and \( I \) is the identity matrix in \( \mathbb{R}^N \).

For the applications, however, it is interesting to consider potentials with wells at locations which change with the scale \( \varepsilon \) and depend on \( x \) and \( t \). An example of such a problem is

\[
\frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon + \frac{1}{\varepsilon} (W'(u^\varepsilon) + c(t) \varepsilon^{1/2}) = 0 \quad \text{in} \; Q_T,
\]

for some smooth \( c \), which leads, as \( \varepsilon \to 0 \), to an interface moving with normal velocity \( V = -\kappa + \alpha c(t) \) for some \( \alpha \in \mathbb{R} \) which depends only on the potential \( W \).

A natural question is what happens if \( c \) is irregular and, in particular, if \( c = dB \), where \( B \) is a Brownian path. It turns that in this case the oscillations of the wells of \( W + \varepsilon^{1/2} dB \) are too extreme for the system to stabilize. However, if \( B \) is replaced by a “mild” approximation \( B^\varepsilon \), then the asymptotic interface moves with normal velocity

\[
V = -\kappa + \alpha dB,
\]

and is characterized as a level-set of the solution to the “stochastic” level-set pde

\[
dv - (I - \frac{Dv}{|Dv|} \otimes \frac{Dv}{|Dv|}) : D^2 v dt + \alpha |Dv| \circ dB = 0 \quad \text{in} \; Q_T.
\]

The asymptotic behavior of the solutions to the perturbed Allen-Cahn equation with \( c = dB \) and \( B \) a space time Brownian motion was conjectured by Otha, Jasnow and Kawasaki \[66\], while the unperturbed equation was proposed by Allen and Cahn \[1\] as a model to study phase transitions. The rigorous justification of the conjectured behavior for the latter as well as its perturbed version with \( c \) smooth was obtained by Evans, Soner and Souganidis \[21\] and Barles and Souganidis \[7\]; see Souganidis \[8, 81\] for a comprehensive overview of the theory. For the former problem with \( c = dB^\varepsilon \) and \( B^\varepsilon \) a mild approximation of a time Brownian Yip \[84\] and Funaki \[25\] obtained results for short time and convex initial interfaces. Lions and Souganidis \[54\] proved the full global in time
result in this case and they also showed that the behavior conjectured in [66] cannot be correct if $c = dB$ and $B$ a Brownian motion in time. The explanation for the latter is that the oscillations due to the presence of $dB$ are so strong that they interfere with the stability properties of the equilibria of the potential.

A stochastic selection principle. A classical question in the theory of level-set interfacial motions is whether there is “fattening”, that is, if there are configurations (initial data) such that the zero level-set of the solution $v$ to (1.1) develops interior. For the motion by mean curvature it is known that, if the initial datum is two touching balls, then, for positive times the evolving front is a “surface” that looks like the boundary of either two separated shrinking balls or some connected open set which moves in time, and there are well defined minimal and maximal moving boundaries. The reason behind such behavior is that the curvature is not well defined at the touching point and the initial surface can be thought as the boundary of either two balls which are infinitesimally separated or an open set which is “almost” pinched.

As it is often the case the introduction of stochasticity resolves this ambiguity and provides a definitive selection principle. Indeed it was proved by Souganidis and Yip [82] that the solutions $v^{\pm\varepsilon}$ of the stochastically perturbed level-set pde

$$dv^{\pm\varepsilon} - (I - \frac{Dv^{\pm\varepsilon}}{|Dv^{\pm\varepsilon}|} \otimes \frac{D^2v^{\pm\varepsilon}}{|Dv^{\pm\varepsilon}|}) : D^2v^{\pm\varepsilon} dt \pm \varepsilon |Dv^{\pm\varepsilon}| \circ dB = 0 \text{ in } Q_T,$$

with initial data two touching balls, never develop interior and, as $\varepsilon \to 0$, their zero level-set converges in the Hausdorff distance to the maximal interface of the unperturbed problem.

Pathwise stochastic control theory. A typical stochastic control theory problem consists of a controlled stochastic (sde for short) differential equation

$$dX_t = b(X_t, \alpha_t) dt + \sqrt{2}\sigma(X_t) dB \text{ for } t > 0 \text{ and } X_0 = x,$$

where $(\alpha_t)_{t \geq 0} \in A$, the set of control processes satisfying appropriate measurability properties and taking values in some compact set $A$, and a pay-off functional, which, to simplify the presentation, here is taken to be

$$J(x, t) = g(X_t),$$

the goal being to minimize the pay-off over $A$.

If the minimization takes place in a pathwise sense, it was conjectured in Lions and Souganidis [58] and shown in [54] (see also Buckdahn and Ma [9] for a special case) that the associated value function

$$u(x, t) := \inf_A g(X_t)$$

is the pathwise solution to the nonlinear stochastic pde

$$du + \sup_{\alpha \in A} (-b(x, \alpha) \cdot Du) dt - \sqrt{2}\sigma Du \circ dB = 0 \text{ in } Q_T \quad u(\cdot, 0) = g \text{ on } \mathbb{R}^N,$$

which is a special case of (0.1) with $F$ nonlinear and $H$ linear.

The classical stochastic control theory is about the minimization of the mean of the payoff and the value function

$$w(x, t) = \inf_A Eg(X_t)$$
is the viscosity solution to the "deterministic" Bellman pde

\[ w_t + \sup_{\alpha \in A} (-b(x, \alpha) \cdot Dw) - \sigma \sigma^T : D^2w = 0 \text{ in } Q_T \quad u(\cdot, 0) = g \text{ on } \mathbb{R}^N. \]

Notice that formally it is possible to obtain the pde above by taking expectations to the stochastic pde satisfied by \( v \). This argument requires commuting expectations with nonlinearities which is, however, far from rigorous.

**Mean field games.** A typical example of the Lasry-Lions mean field theory \[41, 39, 40\] is the study of the asymptotic behavior, as \( L \to \infty \), of the law \( \mathcal{L}(X^1_t, \ldots, X^L_t) \) of the solution to the sde

\[ dX^i = \sigma(X^i, \frac{1}{L-1} \sum_{j \neq i} \delta_{X^j}) \cdot dB \quad (i = 1, \ldots, L). \]

Here \( \delta_y \) is the Dirac mass at \( y \) and \( \sigma \in C^{0,1}(\mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N); \mathcal{S}^N) \), \( \mathcal{S}^N \) and \( \mathcal{P}(X) \) being, respectively, the sets of symmetric \( N \times N \) matrices and probability measures on \( X \).

The result (see Lions \[43\]) is that, as \( L \to \infty \), in the sense of measures and all \( t > 0 \),

\[ \mathcal{L}(X^1_t, \ldots, X^L_t) \to \pi_t \in \mathcal{P}(\mathcal{P}(\mathbb{R}^N)), \]

and the density \( (m_t)_{t \geq 0} \) of the evolution in time of \( (\pi_t)_{t \geq 0} \) defined, for all \( U \in C(\mathcal{P}(\mathbb{R}^N)) \), by

\[ \int U(m) d\pi_t(m) = E[U(m_t)], \]

solves the stochastic conservation law

\[ dm + \text{div}_x(\sigma^T(m, x) \cdot dB) = 0, \]

which is a special case of \[0.2\]; here \( \sigma^T \) is the transpose of the matrix \( \sigma \).

2. **The Main Difficulties and Stratonovich vs Itô’s formulation.**

**Main difficulties.** Given that, in general and even without rough signals, \[0.1\] and \[0.2\] do not have global smooth solutions, it is natural to expect that this is the case in the presence of rough time dependence.

It is also not possible to use directly the standard viscosity and entropy solutions of the “deterministic” theory, since they depend on inequalities satisfied either at some special points or after integration. For example, the weak entropy inequality \( dS(u) + Q(u)_x \cdot dB \leq 0 \) for all pairs \((S, Q)\) of convex entropy \( S \) and entropy flux \( Q \), where for simplicity it is assumed that \( N = 1 \) and \( A = 0 \) in \[0.2\], does not appear to make sense.

Moreover, the lack of regularity does not allow to express the solutions in any form involving time integration as is the case for sde. For example, if \( u \) is the viscosity solution to \( u_t = H(Du) \) in \( Q_T \), the expression \( u(x, t) = u(x, s) + \int_s^t H(Du(x, \tau)) d\tau \) does not make sense due to the lack of regularity.

Another option, at least for \( m = 1 \), is to take advantage of the multiplicative noise to change time and obtain an equation without rough parts. For example, formally, if \( du + H(Du) \cdot dB = 0 \), the change of time \( u(x, t) = U(x, B(t)) \) yields that \( U \) must be a global smooth solution to the forward-backward time homogeneous Hamilton-Jacobi equation \( U_t + H(DU) = 0 \) in \( \mathbb{R}^N \times (-\infty, \infty) \). It is, of course, well known that such solutions do not exist in general. Behind this difficulty is the basic fact that the nonlinear problems develop shocks which are not reversible, while the changing sign of the rough signals, in some sense, forces the solutions to move forward and backward in time. Of
course the time change works in intervals where $dB$ does not change sign. More details about this are given later in the notes.

A natural question is whether it is possible to solve the equations in law. Recall that solving the sde $dX = \sqrt{2}\sigma(X_t)dB$ in law is equivalent to understanding, for all smooth $\phi$ and $T > 0$, the solutions $u$ to the initial value problem

$$u_t = \sigma \sigma^T : D^2u \text{ in } Q_T \quad u(\cdot,0) = \phi \text{ on } \mathbb{R}^N.$$ 

For the equations here the state variable would be functions in a suitable function space and the corresponding sde is set in infinite dimensions. For example, the infinite dimensional pde describing the law of $du = \sqrt{2} H(Du) \circ dB$ is, formally,

$$U_t = D^2U(H(Df),H(Df)).$$

The problem is that the Hessian $D^2U$ is an unbounded operator independently of the choice of the base space. Such pdes are far away from the theory of viscosity solutions in infinite dimensions developed by Crandall and Lions [14, 15].

Solving linear stochastic pde in law is related to the martingale approach which has been used successfully in linear and some quasilinear settings. A partial list of references is Chueshov and Vuillermot [11, 12], Da Prato, Ianneli and Tubaro [16], Huang and Kushner [33], Krylov [36], Krylov and Röckner [37], Rozovskiĭ [75, 76], Pardoux [69, 67, 68, 70], Watanabe [83], etc.. The methodology requires some tightness (compactness) which typically follows from estimates on the derivatives of the solutions. In general, the latter are not available for nonlinear problems.

**Stratonovich vs Itô.** In the study of sde it is important to decide if the equations are considered in the Stratonovich or Itô sense, each of which having advantages and disadvantages; more regularity and chain rule for the former and less regularity but no chain rule for the latter.

At first glance, the choice of calculus does not seem to be relevant for the nonlinear problems discussed here due to the lack of regularity the solutions. This is, however, not the case. The actual formulation plays an important role in the interpretation, wellposedness, stability and the construction of the solutions, which, typically, are obtained as limits of solutions with regular time dependence. The discussion below addresses this issue.

The advantage of the Stratonovich formulation can be seen in the following rather simple example. Consider, for $\lambda \geq 0$, the Itô form spde

$$du = \sqrt{2}\lambda u_{xx}dt + u_x dB \text{ in } Q_T.$$ 

The change of variables $u(x,t) = v(x + B(t),t)$ yields that $v$ satisfies the (deterministic) pde

$$v_t = (\lambda - 1)v_{xx},$$

which is well-posed if and only if $\lambda > 1$.

Of course this is not an issue if the spde was in Stratonovich form to begin with. In that case the change of variables yields the equation

$$v_t = \lambda v_{xx},$$

which is well posed if and only if $\lambda > 0$. And this more natural, since it is the case when $B$ is a smooth path.
Consider, for example, a family \((B^\varepsilon)_{\varepsilon>0}\) of smooth approximations of the Brownian motion \(B\) and the solution \(u^\varepsilon\) to
\[
u^\varepsilon_t = \sqrt{2\lambda} u^\varepsilon_{xx} + u^\varepsilon_x \dot{B}^\varepsilon.
\]
It is obvious that \(u^\varepsilon(x,t) = v(x + B^\varepsilon(t),t)\) with \(v\) solving \(v_t = \lambda v_{xx}\). Then, letting \(\varepsilon \to 0\) yields that \(u^\varepsilon \to u\), which solves
\[
\frac{du}{dt} = \sqrt{2\lambda} u_{xx} + u_x \circ dB.
\]
Another example, where the use of Stratonovich appears to be necessary, is the application to front propagation via level-set pde. One of the critical elements of the theory is that the moving interfaces depend only on the initial one and not the particular choice of the initial datum of the pde. This is equivalent to the requirement that the equations are invariant under increasing changes of the unknown.

Consider, for example, the pde
\[
\frac{D}{Dt} + |Du| = 0.
\]
Arguing as if the solution \(u\) were smooth (the argument can be made rigorous using viscosity solutions), it is straightforward to check that, for nondecreasing \(\phi\), \(\phi(u)\) is also a solution; note that the monotonicity of \(\phi\) is important when dealing with viscosity solutions.

Assume that the correct formulation of the level-set pde of the interfacial motion \(V = dB\) with \(B\) a Brownian motion is
\[
\frac{D}{Dt} = |Du|dB.
\]
If \(u\) is a smooth solution and \(\phi : \mathbb{R} \to \mathbb{R}\) is smooth and nondecreasing, Itô’s formula yields that
\[
\frac{d\phi(u)}{dt} = |D\phi(u)|dB + \frac{1}{2} \phi''(u)|Du|^2,
\]
which clearly is not the same equation as the one satisfied by \(u\). This is of course not the case if the level-set pde was written in the Stratonovich form, which, however, requires a priori additional regularity which is not available here. Indeed, if \(du = H(Du) \circ dB\), then, in Itô’s form
\[
\frac{d\phi(u)}{dt} = H(Du)dB + \frac{1}{2} <D^2uDH(Du), DH(Du)> dt,
\]
where for \(x, y \in \mathbb{R}^N, <x, y>\) is the usual inner product.

In the context of second- and first-order (deterministic) pde the difficulties due to the lack of regularity are overcome using viscosity solutions. Their definition is based on inequalities which, as mentioned earlier, cannot be expected to make sense in the presence of rough signals.

There is, however, a definition for viscosity solutions, which, at first glance, appears to be more conducive to stochastic calculus. Unfortunately its Stratonovich formulation fails since it requires regularity which is not available, while, given in Itô’s form, it leads to the wrong conclusions. Indeed, for \(B\) smooth, consider again the equation
\[
\frac{D}{Dt} = H(Du, x)\dot{B}
\]
and recall that the classical definition of viscosity sub-solutions is equivalent to the requirement that, for any smooth \(\phi : QT \to \mathbb{R}\), the map \(t \to \max(u - \phi)\) satisfies, in the viscosity sense, the differential inequality
\[
\frac{d}{dt} \max(u(\cdot, t) - \phi) \leq \sup_{\hat{x}(t)}(H(D\phi(\hat{x}(t)), \hat{x}(t))\dot{B}),
\]
where $\bar{x}(t)$ denotes a point where $\max(u(\cdot, t) - \phi)$ is achieved — there may, of course, exist several such points, etc.

If $B$ is a Brownian motion, then, assuming that there exists a unique maximum point $\bar{x}(t)$ of $u(\cdot, t) - \phi$, the Stratonovich formulation should be

$$\frac{d}{dt} \max(u(\cdot, t) - \phi) \leq H(D\phi(\bar{x}(t)), \bar{x}(t)) \circ dB,$$

a fact which completely breaks down due to the lack of regularity in $t$ of the map $t \mapsto \bar{x}(t)$.

When $\dot{B} \in L^1((0, T))$, the above inequality is meaningful and has been used by Lions and Perthame [47] and Ishii [34] to study viscosity solutions to Hamilton-Jacobi equations with $L^1$-time dependence.

Requiring that the inequality holds in Itô’s sense also contradicts the classical fact that the maximum of two subsolutions is a subsolution.

Recall that, if $u$ and $v$ are actually differentiable with respect to $t$, then

$$\frac{d}{dt} (\max(u,v)) = 1_{\{u(\cdot, t) > v(\cdot, t)\}} u_t + 1_{\{u(\cdot, t) \leq v(\cdot, t)\}} v_t,$$

where $1_A$ denotes the characteristic function of the set $A$, and, if

$$u_t = H(Du), \quad v_t = Hv, \quad H(0) = 0,$$

it follows that

$$\frac{d}{dt} \max(u, v) \leq H(D\max(u, v)),$$

and, hence, $\max(u, v)$ is a sub-solution.

Checking the same claim in the Itô’s formulation yields

$$d_I \max(u, v) \geq 1_{\{u(\cdot, t) > v(\cdot, t)\}} du + 1_{\{u(\cdot, t) \leq v(\cdot, t)\}} dv,$$

where $d_I$ denotes the Itô differential. The above inequality suggests, always formally, that $\max(u, v)$ is not necessarily a subsolution.

The final justification for considering the Stratonovich formulation when studying, for example, the equation

$$du = H(Du, x) \cdot dB$$

(2.1)

comes from the family of approximate problems

$$u^\varepsilon_t = H(Du^\varepsilon, x) \dot{B}^\varepsilon,$$

where $B^\varepsilon$ are smooth approximations of the Brownian motion $(B_t)_{t \geq 0}$. If $u^\varepsilon$ and $u$ are smooth and, as $\varepsilon \to 0$, $u^\varepsilon \to u$ in $C^2(\mathbb{R}^N \times (0, \infty))$, it is not difficult to see that $u$ must solve (2.1) in the Stratonovich sense.

Note that, under suitable assumptions on the initial datum of the regularized equation and the Hamiltonian, it is possible to show, using arguments from the theory of viscosity solutions, the solutions $u^\varepsilon$ are, uniformly in $\varepsilon$, bounded and Lipschitz continuous in $x$, and, hence, converge uniformly along subsolutions for each $t$. This observation is the starting point of the theory.
3. Single versus multiple signals, the method of characteristics and nonlinear pde with linear rough dependence on time.

**Single versus multiple signals.** The next simple example illustrates that there is a difference between one single and many signals and indicates the role that rough paths play in the theory. Consider two smooth paths $B_1$ and $B_2$ and the linear pde

$$u_t = u_x \dot{B}_1 + f(x) \dot{B}_2 \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}. \tag{3.1}$$

It is immediate that $v(x,t) = u(x - B_1(t), t)$ solves

$$v_t = f(x - B_1(t)) \dot{B}_2 \text{ in } Q_T \quad v(\cdot, 0) = u_0 \text{ on } \mathbb{R},$$

and, hence,

$$u(x, t) = v(x + B_1(t), t) = u_0(x + B_1(t)) + \int_0^t f(x + B_1(t) - B_1(s)) \dot{B}_2(s) \, ds.$$

To extend this expression to non smooth paths, it is necessary to deal with integrals of the form

$$\int_a^b g(B_1(s)) \, dB_2(s),$$

which is one of the ingredients of Lyons's theory of rough paths; see, for example, Qian and Lyons [64], Lyons [65, 63], Lejay and Lyons [42], etc..

**Nonlinear pde with linear rough dependence on time.** The calculation above suggests, however, a possible way to study general linear/nonlinear equations with linear rough dependence, that is equations of the form

$$du = F(D^2 u, Du, x) \, dt + <a(x), Du> \cdot dB_1 + c(x) u \cdot dB_2 \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N. \tag{3.2}$$

Consider the system

$$\begin{cases}
\frac{dX}{dt} = -a(X) \cdot dB_1 \\
\frac{dP}{dt} = <Da(X), P> \cdot dB_1 + <Dc(X), P> U \cdot dB_2, \\
\frac{dU}{dt} = c(X) U \cdot dB_2,
\end{cases} \tag{3.3}$$

which, in view of the theory of rough paths, has a solution for any initial datum $(x, p, u)$. Then (3.3) with initial condition $X(0) = x, P(0) = Du_0(x), U(0) = u_0(x)$ can be easily recognized as the characteristic equations for the linear Hamilton-Jacobi equation

$$du = <a(x), Du> \cdot dB_1 + c(x) u \cdot dB_2 \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N. \tag{3.4}$$

Note that, due to linearity of the problem, it is immediate that the map $x \mapsto X(x, t)$ is invertible for all $t$.

The next step is to make the ansatz that the solution $u$ to (3.2) has the form

$$u(x, t) = v(X^{-1}(x, t), t), \tag{3.4}$$

and to find the equation satisfied by $v$.

Substituting in (3.2), arguing formally (the calculation can be made rigorous using viscosity solutions when $B_1$ and $B_2$ are smooth), and rewriting (3.1) as

$$u(\cdot, t) = S(t)v(\cdot, t),$$
where, for any \( v_0 \), \( S(t)v_0 \) is the solution to the linear Hamilton-Jacobi equation with initial datum \( v_0 \), yields
\[
du = d(S(t)v(\cdot, t)) = dS(t)v(\cdot, t) + S(t)dv(\cdot, t)
\]
\[
= <a(x), DS(t)v(\cdot, t) > \cdot dB_1 + c(x)S(t)v(\cdot, t) \cdot dB_2 + S(t)(v(\cdot, t))
\]
\[
= <a(x), DS(t)v(\cdot, t) > \cdot dB_1 + c(x)S(t)v(\cdot, t) \cdot dB_2
\]
\[
+ F(D^2S(t)v(\cdot, t), DS(t)v(\cdot, t), S(t)v(\cdot, t), x),
\]
and, hence,
\[
S(t)dv(\cdot, t) = F(D^2S(t)v(\cdot, t), DS(t)v(\cdot, t), S(t)v(\cdot, t), x),
\]
and
\[
(3.5) \quad dv = S^{-1}(t)F(D^2S(t)v(\cdot, t), DS(t)v(\cdot, t), S(t)v(\cdot, t), x).
\]
Since the last equation does not contain any singular time dependence, it is convenient to replace \( dv \) by \( v_t \) and to rewrite (3.6) as
\[
(3.6) \quad v_t = S^{-1}(t)F(D^2S(t)v(\cdot, t), DS(t)v(\cdot, t), S(t)v(\cdot, t), x).
\]
Although (3.6) appears to be more complicated than (3.2), but this is only due to the notation. The point is that (3.6) actually is simpler since the transformation eliminates the troublesome term
\[
<a(x), Du > \cdot dB_1 + c(x)u \cdot dB_2.
\]
The new equation
\[
v_t = \tilde{F}(D^2v, Dv, v, x, t) \text{ in } Q_T \quad v(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N
\]
can be studied using the viscosity theory as long as \( \tilde{F} \) satisfies the appropriate conditions for wellposedness.

The discussion above gives an alternative way to find pathwise solutions to all the equations studied using the martingale method as well as scalar quasilinear equations of divergence form, always with linear rough time dependence.

**Stochastic characteristics.** The analysis in the previous subsection suggests that to handle equations with nonlinear rough dependence, it may be useful to look at the associated system of characteristics, assuming that \( H \) is smooth. When the time signals are smooth this is classical. In the particular case that the rough dependence is Brownian, the stochastic characteristics were used in the work of Kunita [38] on stochastic flows. In what follows statements are made without any assumptions and the details are left to the reader.

The characteristics of the Hamilton-Jacobi equation
\[
(3.7) \quad du = \sum_{i=1}^m H_i(Du, u, x, t) \cdot dB_i \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N,
\]
are the solutions to the following system of differential equations:

\[
\begin{align*}
   dX &= -\sum_{i=1}^{m} D_p H_i(P, U, X, t) \cdot dB_i, \\
   dP &= \sum_{i=1}^{m} (D_x H_i(P, U, X, t) + D_u H_i(P, U, X, t) P) \cdot dB_i, \\
   dU &= \sum_{i=1}^{m} (H_i(Du, x, t) - \frac{1}{2} D_p H_i(Du, x, t), P > \cdot dB_i, \\
   X(x, 0) &= x, \quad P(x, 0) = Du_0(x), \quad U(x, 0) = u_0(x).
\end{align*}
\]

(3.8)

The connection between (3.7) and (3.8) is made through the relationship

\[ U(x, t) = u(X(x, t), t) \quad \text{and} \quad P(x, t) = Du(X(x, t), t). \]

The method of characteristics works as long as it is possible to invert the map \( t \mapsto X(x, t) \). This can always be done in some interval \(( -T^*, T^* )\) for small \( T^* > 0 \), which depends on bounds on \( H, Du_0 \) and the signal, and, in general, is difficult to estimate in a sharp way.

It then follows that \( u(x, t) = U(X^{-1}(x, t), t) \) is a smooth solution to (3.7) in \( \mathbb{R}^N \times ( -T^*, T^* ) \). The latter means, for all \( s, t \in ( -T^*, T^* ) \) with \( s < t \) and \( x \in \mathbb{R}^N \),

\[ u(x, t) = u(x, s) + \int_s^t \sum_{i=1}^{m} H_i(Du(x, r), u(x, r), x, r) \cdot dB_i(r). \]

If \( m = 1 \), it is possible to express the solutions to (3.8) using in the characteristics of the “deterministic” equation

\[ u_t = H(Du, u, x, t) \] in \( QT \quad u(\cdot, 0) = u_0 \) on \( \mathbb{R}^N \).

Indeed if \( (X_d, P_d, U_d) \) is the solution to

\[
\begin{align*}
   \dot{X}_d &= -D_p H(P_d, U_d, X_d, t) \\
   \dot{P}_d &= (D_x H_i(P_d, U_d, X_d, t) + D_u H_i(P_d, U_d, X_d, t) P_d), \\
   \dot{U}_d &= H_i(P_d, U_d, X_d, t) - \frac{1}{2} D_p H(P_d, U_d, X_d, t), P_d >, \\
   X_d(x, 0) &= x, \quad P_d(x, 0) = Du_0(x), \quad U_d(x, 0) = u_0(x),
\end{align*}
\]

(3.9)

then \( X(x, t) = X_d(x, B(t)), P(x, t) = P_d(x, B(t)) \), and \( U(x, t) = U_d(x, B(t)) \), and the inversion is possible as long as \( |B(t)| < T_d^* \), the maximal time for which \( X_d \) is invertible. This simple expression for the solution to (3.8) is not valid for \( m \geq 2 \) unless the Hamiltonian \( H \) satisfies the involution relationship

\[ \{ H_i, H_j \} := D_x H_i D_p H_j - D_x H_j D_p H_i = 0 \quad \text{for all} \quad i, j = 1, \ldots, m. \]

The latter yields that the solutions to the system of the characteristics commute, that is

\[ X(x, t) = X^1_d(\cdot, B_1(t)) \circ X^2_d(\cdot, B_2(t)) \circ \cdots \circ X^m_d(\cdot, B_m(t))(x) \]
where, for \( i = 1, \ldots, m \), \((X^i_d, P^i_d, U^i_d)\) is the solution to (3.8) with \( H \equiv H_i \) and \( B_i(t) = 1 \) and \( \circ \) stands for the composition of maps.

For example, if, for all \( i = 1, \ldots, m \), the \( H_i \)'s are independent of \( x, u \) and \( t \) in which case the involution relationship is satisfied, (3.8) reduces to

\[
dX = -\sum_{i=1}^{m} DH_i(P) \cdot dB_i, \quad dP = 0, \quad dU = \sum_{i=1}^{m} \left[ H_i(P) - \langle D_p H_i(P), P \rangle \right] \cdot dB_i.
\]

and the \( X \)-characteristic is given by

\[
X(x,t) = x - \sum_{i=1}^{m} D_x H_i(Du_0(x)) B_i(t).
\]

Finally, either for \( m = 1 \) or for space homogeneous Hamiltonians when \( m \geq 2 \), it is possible to find \( X, P \) and \( U \) for any continuous \( B \).

4. Fully nonlinear equations with semilinear stochastic dependence

Consider the initial value problem

\[
(4.1) \quad du = F(D^2 u, Du, u)dt + \sum_{i=1}^{m} H_i(u) \cdot dB_i \quad \text{in} \ Q_T; \quad u(\cdot, 0) = u_0 \text{ on} \ \mathbb{R}^N,
\]

with \( u_0 \in BUC(\mathbb{R}^N) \), \( B = (B_1, \ldots, B_m) \) continuous, \( F \in C(S^N \times \mathbb{R}^N) \) degenerate elliptic, that is, for all \((p, u) \in \mathbb{R}^N \times \mathbb{R} \) and \( X, Y \in S^N \),

\[
\text{if } X \leq Y, \text{ then } F(X, p, u) \geq F(Y, p, u),
\]

and

\[
H = (H_1, \ldots, H_m) \in (C^{3,1}(\mathbb{R}))^m.
\]

The results presented here also apply to the more general equations

\[
(4.4) \quad du = F(D^2 u, Du, u, x, t)dt + \sum_{i=1}^{m} H_i(u, x, t) \cdot dB_i \quad \text{in} \ Q_T;
\]

the exact assumptions will become clear below.

For \( v \in \mathbb{R} \) consider the differential equation

\[
(4.5) \quad d\Phi = \sum_{i=1}^{m} H_i(\Phi) \cdot dB_i \quad \text{in} \ (0, \infty); \quad \Phi(v, 0) = v.
\]

It is assumed that

\[
(4.6) \quad \begin{cases}
\text{there exists a unique solution } \Phi : \mathbb{R} \times (0, \infty) \to \mathbb{R} \text{ to (4.5)} \quad \text{such that, for all } T > 0, \\
\Phi \in C([0, T]; C^3(\mathbb{R})) \quad \text{and} \quad M(T) := \sup_{0 \leq t \leq T} \left[ |\Phi(0, t)| + \sum_{i=1}^{3} \| D^i \Phi(\cdot, t) \|_{\infty} \right] < \infty.
\end{cases}
\]

When \( m > 1 \), the \( B_i \)'s are taken to be geometric rough paths and (4.3) is interpreted in the rough path sense. If \( m = 1 \), then, for all \( t > 0 \),

\[
\Phi(v, t) = \tilde{\Phi}(v, B(t)),
\]
where \( \hat{\Phi} \) solves the ode

\[
\dot{\hat{\Phi}} = H(\hat{\Phi}) \quad \text{in} \quad \mathbb{R}, \quad \hat{\Phi}(v,0) = v.
\]

It is then straightforward to obtain (4.6) from the analogous properties of \( \hat{\Phi} \). Moreover, it is also clear that (4.6) and (4.7) also hold, if \( B \) is any continuous and not only Brownian path.

Define \( \tilde{F} : S^N \times \mathbb{R} \times [0, \infty) \times \Omega \rightarrow \mathbb{R} \) by

\[
\tilde{F}(X,p,v,t) := \frac{1}{\Phi'(v,t)} F(\Phi'(v,t)X + \Phi''(v,t)(p \otimes p), \Phi'(v,t)p, \Phi(v,t)),
\]

where, to simplify the presentation, “\( ' \)” denotes the partial derivatives of \( \Phi \) with respect to \( v \).

The following definitions are motivated by the strategy described in Section 3 which amounts to inverting the characteristics. For (4.1) the latter are the solutions to (4.5), which, in view of the semilinear form of the coefficient of the rough path, are globally invertible.

For each \( \phi \in C^2(Q_T) \), set

\[
\Psi(x,t) := \Phi(\phi(x,t),t),
\]

and note that \( \Psi \) is smooth in \( x \).

The definition of weak solution to (4.1) is:

**Definition 4.1.** Fix \( T > 0 \). Then \( u \in BUC(Q_T) \) is a pathwise sub-(resp. super-) solution to (4.1), if, for all \( \phi \in C^2(Q_T) \) and all local maximum (resp. minimum) points \( (x_0,t_0) \in Q_T \) of \( (x,t) \rightarrow u(x,t) - \Phi(\phi(x,t),t) \),

\[
\phi_t(x_0,t_0) \leq \tilde{F}(D^2\phi(x_0,t_0),D\phi(x_0,t_0),u(x_0,t_0),t_0),
\]

(resp.

\[
\phi_t(x_0,t_0) \geq \tilde{F}(D^2\phi(x_0,t_0),D\phi(x_0,t_0),u(x_0,t_0),t_0)).
\]

A function \( u \in BUC(Q_T) \) is a pathwise (viscosity) solution to (4.1), if it is both sub- and super-solution to (4.1).

Since the characteristics are globally invertible, it is possible to introduce a global change of the unknown without going through test functions.

The second definition is:

**Definition 4.2.** For \( T > 0 \) fix. Then \( u \in BUC(Q_T) \) is a pathwise sub-(resp. super-) solution to (4.1), if the function \( v : \mathbb{R}^N \times [0,T] \times \Omega \rightarrow \mathbb{R} \) defined by

\[
u_t = \tilde{F}(D^2v,Dv,v,t) \quad \text{in} \quad Q_T \quad v(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^N.
\]

A function \( u \in BUC(Q_T) \) is a pathwise solution to (4.1) if it is both sub- and super-solution.

The two definitions are equivalent, and, moreover, for smooth \( B \)'s, the solutions introduced in Definitions 4.1 and 4.2 coincide with the classical viscosity solution.

In view of the above, the wellposedness of solutions to (4.1) reduces to the study of the analogous questions for (4.13).
After the work described above was announced, Buckdahn and Ma \[8, 9\] used the map (4.12), which is known as the Sussman-Doss transformation, to study equations similar to (4.1) but for a more restrictive class of $F$'s and proved wellposedness under the assumption that the transformed initial value problem admits a comparison principle.

If $H$ is linear in $u$, the problem can be treated using the arguments of the previous section since $F$ is independent of $v$. Hence the rest of the section is about nonlinear Hamiltonians. To simplify the presentation, it is moreover assumed that $F$ is independent of $u, x$ and $t$ and $\Phi$ dependents only of $u$.

To deal with the nontrivial dependence of $\tilde{F}$ it is necessary to assume that
\[
F \in C^{0,1}(S^{N} \times \mathbb{R}^{N}),
\]
and
\[
\begin{cases}
\text{there exists a constant } C > 0 \text{ such that, for almost every } (X, p), \\
\text{either } < D_{X}F(X, p), X > + < D_{p}F(X, P), P > - F \leq C \\
\text{or } < D_{X}F(X, p), X > + < D_{p}F(X, P), P > - F \geq -C.
\end{cases}
\]

It is easy to see that any linear $F$ satisfies (4.15). Moreover, (4.14) implies that $F$ can be written as the minmax of linear functions, that is
\[
F(X, p) = \sup_{a \in A} \inf_{b \in B} (a_{\alpha, \beta}, X + < b_{\alpha, \beta}, p > + h_{\alpha, \beta}),
\]
for $A \subset S^{N}$ and $B \subset \mathbb{R}^{N}$ bounded and $a_{\alpha, \beta} \in S^{N}$ and $b_{\alpha, \beta} \in \mathbb{R}^{N}$ such that
\[
\sup_{a \in A} \inf_{b \in B} [||a_{\alpha, \beta}|| + |b_{\alpha, \beta}|] < \infty.
\]

Since $< D_{X}F(X, p), X > + < D_{p}F(X, P), P > - F$ is formally the derivative, at $\lambda = 1$, of the map $\lambda \to F(\lambda X, \lambda p) - \lambda F(X, P)$, it follows that (4.15) is related to, a uniform in $\alpha, \beta$, one sided bound of $\lambda^{-1} h_{\alpha, \beta} - h_{\alpha, \beta}$ in a neighborhood of $\lambda = 1$.

The need for an assumption like (4.15) is discussed next. Two explanations are presented. The first is based on considerations from the method of characteristics. The second relies on viscosity solutions arguments.

Consider the following first-order versions of (4.1) and (4.13), namely
\[
\begin{align*}
(i) \; du &= F(Du)dt + H(u) \cdot dB \\
(ii) \; v_{t} &= \tilde{F}(Dv, v, t),
\end{align*}
\]
with
\[
\tilde{F}(p, v, t) = \frac{1}{\Phi'(v, t)} F(\Phi'(v, t)p)
\]
and assume that $F, H, B$ and, hence, $\tilde{F}$ are smooth.

The characteristics of the equations in (4.16)(i) and (4.16)(ii) are respectively
\[
\begin{cases}
\dot{X} = -D_{p}F(P), \\
\dot{P} = H'(U)PB, \\
\dot{U} = F(P) - < DF(P), P > + H(U)\dot{B},
\end{cases}
\]
with
\[
\tilde{F}(p, v, t) = \frac{1}{\Phi'(v, t)} F(\Phi'(v, t)p).
\]
\[\dot{Y} = -D_Q \tilde{F}(Q, V) = -D_P F(\Phi^{-1}(V), Q),\]
\[\dot{Q} = \tilde{F}_V Q = \Phi''(V)(\Phi^{-1}(V))^{-2}Q[D_P F(\Phi'(V)Q), \Phi'(V)Q] - F(\Phi'(V)Q),\]
\[\dot{V} = \tilde{F} - <D_Q \tilde{F}(Q, V), Q >= (\Phi'(V))^{-1}[F(\Phi'(V)Q) - <D_P F(\Phi'(V)Q), \Phi'(V)Q>].\]

Of course, (4.18) and (4.19) are equivalent after a change of variables. Moreover, it is also clear that some additional hypotheses are needed in order for (4.18), and, hence, (4.19) to have unique solutions. For example, the right hand side of the \(P\)-equation in (4.18) may not be Lipschitz continuous. On the other hand, the right hand side of the equations for \(Q\) and \(V\) in (4.19) contain the quantity \(<D_p F, P > - F\) appearing in (4.15) and an, at least one-sided, Lipschitz condition necessary to yield existence and uniqueness.

For the second explanation, the comparison principle for the pathwise viscosity solution to (4.1) will result from the comparison in \(BUC(Q_T)\) of viscosity solutions to (4.13). The latter does not follow directly from the existing theory unless something more is assumed; see, for example, Barles \[5\] and Crandall, Ishii and Lions \[13\]. Indeed one of the conditions needed to prove comparison results for viscosity solutions to equations like (4.13) is that, for each \(R > 0\), there exists \(C_R > 0\) such that, for all \(X \in S^N\), \(p \in \mathbb{R}^N\), \(v \in [-R, R]\) and \(t \in [0, T]\),

\[\frac{\partial \tilde{F}}{\partial v}(X, p, v, t) \leq C_R.\]

A straightforward calculation, using (4.15), yields that, for all \(X, p, v\) and \(t\),

\[\frac{\partial \tilde{F}}{\partial v} = \frac{\Phi''}{(\Phi')^2}[D_X F \cdot (\Phi' X + \Phi'' p \otimes p) + D_P F, \Phi'(p)] - F + \Phi'\left(\frac{\Phi''}{\Phi'}\right)'D_X F \cdot p \otimes p;\]

note that to keep the formula simpler, the explicit dependence of \(F\) and its derivatives on \(\Phi' X + \Phi'' p\) is omitted.

It is immediate that \(\frac{\partial \tilde{F}}{\partial v}\) does not satisfy (4.20) without an extra assumption on \(F\) and control on the size of \(p\). If a bound on \(p\) is not available, it is necessary to know that \(\Phi'(\Phi''(\Phi')^{-1})' \geq 0\). The last point that needs explanation is that (4.21) is nonlocal, in the sense that it depends on \(v\) through \(\Phi\), while (4.15) is a local one, that is \(\Phi\) plays no role whatsoever. This can be handled in the proof by working in uniformly small time intervals using the local time behavior of \(\Phi\) and iterating in time.

The result is:

\textbf{Theorem 4.1.} Assume (4.2), (4.3), (4.6), (4.14) and (4.15). For each \(T > 0\) and any geometric rough path \(B\), there exists a constant \(C = C(F, H, B, T) > 0\) such that, if \(v \in BUC(Q_T)\) and \(\nu \in BUC(Q_T)\) are respectively sub- and super-solution to (4.1), then, for all \(t \in [0, T]\),

\[\sup_{\mathbb{R}^N}(\nu(\cdot, t) - \nu(\cdot, 0))_+ \leq C \sup_{\mathbb{R}^N}(\nu(\cdot, 0) - \nu(\cdot, 0))_.\]

\textbf{Proof.} To simplify the presentation, it is assumed that \(F\) is a smooth. The actual proof follows by writing finite differences instead of taking derivatives and using regularizations. Since \(\Phi(v, 0) = v\), (4.6) yields that, for each \(\delta > 0\), there exists \(h > 0\) so small that

\[\sup_{0 \leq t \leq h} [\|\Phi(v, t) - v\| + |\Phi'(v, t) - 1| + |\Phi''(v, t)| + |\Phi''(v, t)|] \leq \delta.\]
Next consider the new change of variables
\[ v = \phi(z) = z + \delta \psi(z) \quad \text{with} \quad \phi' > 0. \]

If \( v \) is a sub- (resp. super-) solution to \((4.13)\), then \( z \) is a sub- (resp. super-) solution to
\[
(4.23) \quad z_t = \tilde{F}(D^2 z, Dz, z),
\]
with
\[
(4.24) \quad \begin{cases}
\tilde{F}(X, p, z) = \Phi'(\phi(z), t)\phi'(z)^{-1} F(\Phi'(\phi(z), t)[\phi'(z)z + \phi''(z)(p \otimes p) \\
+ \Phi''(\phi(z), t)(\phi'(z))^2(p \otimes p), \Phi'(\phi(z), t)\phi'(z)p). 
\end{cases}
\]

The comparison result follows from the classical theory of viscosity solutions, if there exists \( C = C_R > 0 \), for \( R = \max(||\bar{v}||, ||\underline{v}||) \), such that, for all \( X, p \) and \( z \),
\[
(4.25) \quad \frac{\partial}{\partial z} \tilde{F}(X, p, z) \leq C.
\]

But
\[
\frac{\partial}{\partial z} \tilde{F}(X, p, z) = -\frac{(\Phi'\phi')'}{(\Phi'\phi')^2} F + \frac{1}{(\Phi'\phi')^2} \left[ <D_X F, [(\Phi'\phi')'X + [(\Phi'\phi'')' + (\Phi''\phi')^2](p \otimes p) > + <D_p F, (\Phi'\phi')'p) > \right]
\]
\[
= \frac{(\Phi'\phi')'}{(\Phi'\phi')^2} \left[ - F + <D_X F, (\Phi'\phi'X + (\Phi'\phi'' + \Phi''\phi')(p \otimes p)) > + <D_p F, (\Phi'\phi')p) > \right]
\]
\[
+ <D_X F, \frac{[\Phi''\phi' + (\Phi'\phi')^2]}{\Phi''\phi'} - \frac{(\Phi'\phi'')' - (\Phi'\phi'')(\Phi'\phi')''}{(\Phi'\phi')^2} (p \otimes p) >,
\]
where again, to simplify the notation, the arguments of \( F, D_p F, D_X F, \Phi', \Phi'', \phi' \) and \( \phi'' \) are omitted.

In view of \((4.2)\) and \((4.14)\), to obtain \((4.25)\) it suffices to choose \( \phi \) so that
\[
(4.26) \quad \frac{(\Phi'\phi'' + (\Phi'\phi')^2)' - (\Phi'\phi'')' - (\Phi'\phi'')(\Phi'\phi')''}{(\Phi'\phi')^2} \leq 0
\]
and, if \((4.15)\)(ii) holds,
\[
(4.27) \quad \frac{(\Phi'\phi')'}{(\Phi'\phi')^2} \leq 0
\]
or, if \((4.15)\)(i) holds,
\[
(4.28) \quad \frac{(\Phi'\phi')'}{(\Phi'\phi')^2} \geq 0.
\]

Assumption \((4.22)\) and the special choice of \( \phi \) yield that \((4.26)\) is satisfied if \( \psi'' \leq -1 \), and that \((4.27)\) (resp. \((4.28)\)) holds, if \( \psi'' \leq -1 \) (resp. \( \psi'' \geq 1 \)). It is a simple exercise to find \( \psi \) so that \((4.26)\) and either \((4.27)\) or \((4.28)\) hold in its domain of definition.

The classical comparison result for viscosity solutions then yields that, if \( \overline{v}(\cdot, 0) \leq \underline{v}(\cdot, 0) \) on \( \mathbb{R}^N \), then \( \overline{v} \leq \underline{v} \) on \( \mathbb{R}^N \times [0, h] \). The same argument then yields the comparison in \([h, 2h]\), etc..
The existence of the pathwise solutions to (4.1) is based on approximating (4.1) by
\[ u^\varepsilon_t = F(D^2 u^\varepsilon, D u^\varepsilon) + \sum_{i=1}^{M} H_i(u^\varepsilon) \dot{B}^\varepsilon_i \text{ in } Q_T \quad u^\varepsilon(\cdot, 0) = u^\varepsilon_0 \text{ on } \mathbb{R}^N, \]
where \( u^\varepsilon_0 \in BUC(\mathbb{R}^N) \), and
\[ B^\varepsilon = (B^\varepsilon_1, \ldots, B^\varepsilon_m) \in C^1([0, \infty); \mathbb{R}^m), \]
\[ \text{and, for all } T > 0, \text{ as } \varepsilon \to 0, \quad B^\varepsilon \to B \quad \text{uniformly in } [0, T]. \]

The result is:

**Theorem 4.2.** Assume (4.2), (4.3), (4.6), (4.14) and (4.15) and fix \( T > 0 \). Let \((\zeta^\varepsilon)_{\varepsilon > 0}\) and \((\xi^\eta)_{\eta > 0}\) satisfy (4.30) and consider the solutions \( u^\varepsilon, v^\eta \in BUC(Q_T) \) of (4.29) with initial datum \( u^\varepsilon_0 \) and \( v^\eta_0 \) respectively. If, as \( \varepsilon, \eta \to 0 \), \( u^\varepsilon_0 - v^\eta_0 \to 0 \) uniformly on \( \mathbb{R}^N \), then, as \( \varepsilon, \eta \to 0 \), \( u^\varepsilon - v^\eta \to 0 \) uniformly on \( Q_T \). In particular, each family \( (u^\varepsilon)_{\varepsilon > 0} \) is Cauchy in \( Q_T \). Hence, it converges uniformly to \( u \in BUC(Q_T) \), which is a pathwise viscosity solution to (4.1). Moreover, all approximate families converge to the same limit.

The proof of Theorem 4.2 follows from the comparison between sub- and super-solutions to (4.13) for different approximations \((\zeta^\varepsilon)_{\varepsilon > 0}\) and \((\xi^\eta)_{\eta > 0}\). Since a similar theorem will be proved later when dealing with nonlinear gradient dependent \( H \), the proof is omitted.

The last result is about the Lipschitz continuity of the solutions. Its proof is based on the comparison estimate obtained in Theorem 4.1 and, hence, it is omitted.

**Proposition 4.1.** Fix \( T > 0 \) and assume (4.2), (4.3), (4.6), (4.14) and (4.15) and let \( u \in BUC(Q_T) \) be the unique pathwise solution to (4.1) for \( u_0 \in C^{0,1}(\mathbb{R}^N) \). Then \( u(\cdot, t) \in C^{0,1}(\mathbb{R}^N) \) for all \( t \in [0, T] \), and there exists \( C = C(F, H, B, T) > 0 \) such that, for all \( t \in [0, T] \), \( ||Du(\cdot, t)|| \leq C \).

### 5. The simplest nonlinear pde with rough signals as a limit of regular approximations

In this section it is assumed again that \( m = 1 \). However, in view of the discussion in Section 3 about when the characteristics commute, the results easily extend to the multi-path setting.

Fix \( T > 0 \) and \( B \in C([0, \infty)) \) with \( B(0) = 0 \) and consider a family \((B^\varepsilon)_{\varepsilon > 0}\) of smooth paths with \( B^\varepsilon(0) = 0 \) approximating \( B \) in \( C([0, \infty)) \), that is, as \( \varepsilon \to 0 \), \( B^\varepsilon \to B \) locally uniformly in \([0, \infty)\), and the family of Hamilton-Jacobi equations
\[ u^\varepsilon_t = H(Du^\varepsilon) \dot{B}^\varepsilon \text{ in } Q_T \quad u^\varepsilon(\cdot, 0) = u^\varepsilon_0 \text{ on } \mathbb{R}^N, \]
which, for each \( u^\varepsilon_0 \in BUC(\mathbb{R}^N) \) and merely continuous \( H \), have a unique solution \( u^\varepsilon \in BUC(Q_T) \).

It is shown here that, if the \( u^\varepsilon_0 \)'s converge uniformly to \( u_0 \) and the Hamiltonian has some additional regularity, then the \( u^\varepsilon \)'s converge, as \( \varepsilon \to 0 \), uniformly in \( Q_T \) to \( u \in BUC(Q_T) \), which is independent of the particular approximation of the path and \( T > 0 \). This limit will turn out to be the pathwise solution to
\[ du = H(Du) \cdot dB \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N. \]
The interpretation of the result is that the solution operator for smooth paths has a unique extension to the set of continuous paths.

The result is:

**Theorem 5.1.** Fix $B \in C([0, \infty))$ with $B(0) = 0$ and assume that $H \in C^1_{loc}(\mathbb{R}^N)$ and $u_0 \in BUC(\mathbb{R}^N)$. Let $(B^\varepsilon)_{\varepsilon > 0}$ in $C^1((0, \infty)) \cap C([0, \infty))$ with $B^\varepsilon(0) = 0$ and $(u^\varepsilon_0)_{\varepsilon > 0}$ in $BUC(\mathbb{R}^N)$ be such that, as $\varepsilon \to 0$, $B^\varepsilon \to B$ in $C([0, \infty))$ and $u^\varepsilon_0 \to u_0$ in $BUC(\mathbb{R}^N)$. There exists $u \in C(\mathbb{R}^N \times [0, \infty))$ such that, for each $\varepsilon, \eta$ such that, as $\varepsilon \to 0$, $u^\varepsilon \to u$ in $BUC(\overline{Q}_T)$ and, if $u^\varepsilon \in BUC(\overline{Q}_T)$ is the solution to (5.1), then, as $\varepsilon \to 0$, $u^\varepsilon \to u$ in $BUC(\overline{Q}_T)$.

Proof. Let $(B^\varepsilon)_{\varepsilon > 0}$ and $B^\eta_{\eta > 0}$ be two $C^1$-approximations of $B$ in $C([0, \infty))$ such that $B^\varepsilon(0) = B^\eta(0) = 0$, and consider two approximations $u^\varepsilon_0$ and $u^\eta_0$ of $u_0$ in $BUC(\mathbb{R}^N)$. Fix $T > 0$ and let $u^\varepsilon, \tilde{u}^\eta \in BUC(\overline{Q}_T)$ be the viscosity solutions to (5.1) with paths $B^\varepsilon, B^\eta$ and initial datum $u^\varepsilon_0, \tilde{u}^\eta_0$. The claim follows if it is shown that, as $\varepsilon, \eta \to 0$, $u^\varepsilon - \tilde{u}^\eta \to 0$ in $\mathbb{R}^N \times [0, T]$.

A simple density argument implies that it is enough to consider $u_0, u^\varepsilon_0, u^\eta_0 \in C^{0,1}(\mathbb{R}^N)$ with $\max(\|Du^\varepsilon_0\|, \|Du^\eta_0\|) \leq C$ for some $C > 0$. Since $H$ is independent of $x$, it follows that

$$\max(\|Du^\varepsilon(\cdot, t)\|, \|D\tilde{u}^\eta(\cdot, t)\|) \leq \max(\|Du^\varepsilon_0\|, \|D\tilde{u}^\eta_0\|),$$

and, hence, without any loss of generality, it may be assumed that $H \in C^{1,1}(\mathbb{R}^N)$.

For each $\varepsilon$ and $\eta$, $u^\varepsilon$ and $\tilde{u}^\eta$ are actually also Lipschitz continuous with respect to $t$ but with Lipschitz constants depending on $|B^\varepsilon|$ and $|B^\eta|$, and, hence, not bounded uniformly in $\varepsilon, \eta$. This is, of course, one of the main reasons behind the difficulties here.

Finally, to keep the arguments simple, it is assumed that $u_0, u^\varepsilon_0$ and $u^\eta_0$, and, hence, $u^\varepsilon$ and $\tilde{u}^\eta$ are periodic in the unit cube $\mathbb{T}^N$. This simplification allows not to be concerned about infinity, and, more precisely, the possibility that suprema below are not achieved. The periodicity can be eliminated as an assumption by introducing appropriate penalizations at infinity that force the suprema to be actually maxima.

The general strategy in the theory of viscosity solutions to show that, as $\varepsilon, \eta \to 0$, $u^\varepsilon - \tilde{u}^\eta \to 0$ uniformly in $\overline{Q}_T$, is to double the variables and to consider the function

$$z(x, y, t) := u^\varepsilon(x, t) - \tilde{u}^\eta(y, t),$$

which satisfies the so-called “doubled” equation

$$z_t = H(D_x z) B^\varepsilon - H(-D_y z) \tilde{B}^\eta \text{ in } \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty) \quad z(\cdot, \cdot, 0) = z_0 \text{ in } \mathbb{R}^N \times \mathbb{R}^N, \tag{5.3}$$

with

$$z_0(x, y) = u^\varepsilon_0(x) - \tilde{u}^\eta_0(y).$$

The assumptions on $u^\varepsilon_0$ and $\tilde{u}^\eta_0$ yield that there exists $\theta(\lambda) \to 0$ as $\lambda \to \infty$, such that

$$z_0(x, y) \leq \lambda |x - y|^2 + \theta(\lambda) + \sup(u^\varepsilon_0 - \tilde{u}^\eta_0).$$

To conclude, it suffices to show that there exists some $U^{\varepsilon, \eta, \lambda} : \mathbb{R}^N \times \mathbb{R}^N \times [0, T] \to \mathbb{R}$ such that, as $\varepsilon, \eta \to 0$ and $\lambda \to \infty$,

$$U^{\varepsilon, \eta, \lambda}(x, x, t) \to 0 \text{ uniformly in } \overline{Q}_T \text{ and } z \leq U^{\varepsilon, \eta, \lambda} \text{ in } \mathbb{R}^N \times \mathbb{R}^N \times [0, T].$$

It then follows that

$$\min_{\varepsilon, \eta} \max_{\mathbb{R}^N \times [0, T]} z(x, x, t) \to 0,$$
which is one part of the claim. The other direction is proved similarly.

Again as in the general theory, it is natural to try to show that there exists some $C > 0$ and $a(\lambda) > 0$ such that $a(\lambda) \to 0$ as $\lambda \to \infty$ and

$$U^{\varepsilon, \eta, \lambda}(x, y, t) = \lambda C |x - y|^2 + a(\lambda).$$

This is, however, the main difficulty, since both the $C$ and $a(\lambda)$ will depend on $|\dot{B}^\varepsilon|$ and $|\dot{B}^\eta|$.

The new idea to circumvent this difficulty is to find sharper upper bounds by considering the solution $\phi = \phi^{\lambda, \varepsilon, \eta}$ of (5.3) with $\phi_0(x, y) = \lambda |x - y|^2$.

Note that, since $H$ depends only on the gradient and $\phi_0$ depends only on $x - y$,

$$\phi^{\lambda, \varepsilon, \eta}(x, y, t) = \Phi^{\lambda, \varepsilon, \eta}(x - y, t),$$

where $\Phi = \Phi^{\lambda, \varepsilon, \eta}$ solves, for $\Phi_0(z) = \lambda |z|^2$,

$$\Phi_t = H(D\Phi)(\dot{\Phi}^\varepsilon - \dot{\Phi}^\eta) \text{ in } Q_T, \quad \Phi(\cdot, 0) = \Phi_0 \quad \text{on } \mathbb{R}^N.$$

This initial value problem has, for each $\varepsilon$ and $\eta$, a unique solution. Moreover, the standard comparison principle for viscosity solutions yields, for all $x, y \in \mathbb{R}^N$, $t \geq 0$ and $\lambda > 0$

$$z(x, y, t) \leq \phi^{\lambda, \varepsilon, \eta}(x, y, t) + \max_{x, y, t \in \mathbb{R}^N} (z(x, y, 0) - \lambda |x - y|^2),$$

and, hence, in view of (5.3), for all $x \in \mathbb{R}^N$ and $t \geq 0$,

$$u^\varepsilon(x, t) - u^\eta(x, t) \leq \phi^{\lambda, \varepsilon, \eta}(x, x, t) + \Theta(\lambda) + \sup(u_0^\varepsilon - u_0^\eta).$$

To conclude, it is necessary to show that there exists $\Theta(\lambda) > 0$ such that $\lim_{\lambda \to \infty} \Theta(\lambda) = 0$ and

$$\lim_{\varepsilon, \eta \to 0} \sup_{x \in \mathbb{R}^N} \phi^{\lambda, \varepsilon, \eta}(x, x, t) \leq \Theta(\lambda),$$

a fact that apriori may present a problem since the “usual” viscosity theory yields the existence of $\phi^{\lambda, \varepsilon, \eta}$ but not the desired estimate.

Here comes the second new idea, namely, to use the characteristics to construct a smooth solution $\phi$, at least for a small time, which, of course, depends on $\varepsilon$ and $\eta$. The aim then will be to show that, as $\varepsilon, \eta \to 0$, the interval of existence becomes of order one.

The characteristics of the doubled equation (5.3) with initial datum $\lambda |x - y|^2$ solve the odes

$$\begin{cases}
\dot{X} = -DH(P)\dot{B}^\varepsilon, & \dot{Y} = -DH(Q)\dot{B}^\eta \\
\dot{P} = 0, & \dot{Q} = 0, \\
\dot{U} = (H(P) - <D_pH(P), P>)\dot{B}^\varepsilon - (H(Q) - <D_H(Q), Q>)\dot{B}^\eta, \\
X(0) = x, & Y(0) = y, \quad P(0) = Q(0) = 2\lambda(x - y), \quad U(0) = \lambda|x - y|^2;
\end{cases}$$

(5.6)

note that to keep the equations simpler the system is written for $Q(x, t) = -Du^\eta(Y(t), t)$.

Recall that the method of characteristics provides a classical solution to the associated pde for some short time $T^{\varepsilon, \eta, \lambda}$, as long the map $(x, y) \mapsto (X(t), Y(t))$ is invertible. One way to guarantee this is to show that the Jacobian of the map, which, at $t = 0$, is 1, does not vanish in $[0, T^{\varepsilon, \eta, \lambda})$.

Finding a sharp estimate for the first time the Jacobian vanishes is, in general, a difficult problem. In the case at hand, however, there is no need for sharpness since the “crudest” known estimate
does the job. It is, however, one of the major technical difficulties when trying to study equations which depend on $x$ and have multiple paths.

The special structure of (5.6), which follows from the fact that $H$ depends only on the gradient, yields that, for all $t \geq 0$,

$$P(t) = Q(t) = 2\lambda(x - y),$$

and

$$(X - Y)(t) = (x - y) - DH(2\lambda(x - y))(B^\varepsilon_t - \tilde{B}^\eta_t).$$

To simplify the notation, let $z = x - y$ and $Z(t) = X(t) - Y(t)$, in which case the last equation is rewritten as

$$Z(t) = z - DH(2\lambda z)(B^\varepsilon_t - \tilde{B}^\eta_t).$$

Note that $z \mapsto Z(z, t)$ is the position characteristic associated with the simplified initial value problem (5.5), and, in the problem at hand, is the only map that needs to be inverted. A straightforward computation yields that its gradient is

$$I + 2\lambda D^2 H(2\lambda z)(B^\varepsilon_t - \tilde{B}^\eta_t),$$

and clearly its Jacobian does not vanish as long as

$$\sup_{t \in [0, T]} |B^\varepsilon_t - \tilde{B}^\eta_t| \|D^2 H\|_{\infty} < (2\lambda)^{-1}.$$

This is, of course, possible for any $T$ and $\lambda$ as long as $\varepsilon$ and $\eta$ are small, since, as $\varepsilon, \eta \to 0$, $B^\varepsilon \to \tilde{B}^\eta$ in $C([0, \infty))$.

The above estimates depend on having $H \in C^2$. Since the interval of existence depends only on the $C^{1,1}$ norm of $H$, it can be assumed that $H$ has the assume regularity and then conclude introducing yet another level of approximations. If $H$ is less regular than $C^{1,1}$, then there is a nontrivial interaction between the regularities of $H$ and $W$. This is described and analyzed in detail in the next section.

Next observe that the characteristics yield

$$\phi^{\lambda, \varepsilon, \eta}(X(t), Y(t), t) = \lambda|x - y - D_p H(2\lambda(x - y))(B^\varepsilon_t - \tilde{B}^\eta_t)|^2$$

$$+ [H(2\lambda(x - y)) - <D_p H(2\lambda(x - y)), 2\lambda(x - y) >](B^\varepsilon_t - \tilde{B}^\eta_t).$$

Since $|B^\varepsilon_t - \tilde{B}^\eta_t| \|D^2 H\|_{\infty} < (2\lambda)^{-1}$, it is now immediate that, for an appropriate $C > 0$ which depends on $\|H\|_{C^2}$,

$$|\phi^{\lambda, \varepsilon, \eta}(X(t), Y(t), t) - \lambda|x - y|^2| \leq \lambda C \sup_{0 \leq i \leq T} |B^\varepsilon_i - \tilde{B}^\eta_i|.$$

Returning to $x, y$ variables, the above estimate gives that, for each fixed $\lambda > 0$ and $T > 0$ and as $\varepsilon, \eta \to 0$,

$$\sup_{x, y \in \mathbb{R}^N} \phi^{\lambda, \varepsilon, \eta}(x, y, t) - \lambda|x - y|^2 \to 0.$$
Since the strategy and the arguments of the proof above are used several times in the theory, it is helpful to present a brief summary of the main points.

The conclusion of the theorem is that it is possible to construct, using the classical theory of viscosity solutions, a (unique) \( u \in BUC(Q_T) \) which is the candidate for the solution of (5.2) for any \( B \) continuous as long as \( H \in C^{1,1}_{loc} \).

The key technical step in the proof was the fact that, if, as \( \varepsilon, \eta \to 0 \), \( B_\varepsilon - B_\eta \to 0 \) in \( C([0, \infty)) \), then, for each \( \lambda > 0 \) and \( T > 0 \), as \( \varepsilon, \eta \to 0 \)

\[
\sup_{z \in \mathbb{R}^N \atop t \in [0,T]} \left| v_{\varepsilon, \eta}^\lambda(z, t) - \lambda|z| \right| \to 0,
\]

where \( v = v_{\varepsilon, \eta}^\lambda \) is the solution to

\[
v_t = H(Dv)(\dot{B}_\varepsilon - \dot{B}_\eta) \quad \text{in} \quad Q_T \quad v(z,0) = \lambda|z|.
\]

The proof presented earlier used \( \lambda|z|^2 \) as initial datum. It is not hard to see, however, that the same argument will work for initial datum \( \lambda|z| \). Indeed it is enough to consider regularizations like \((\delta + |z|^2)^{1/2}\) and to observe that the estimate on \( u^\varepsilon(\cdot, t) - \tilde{u}^\eta(\cdot, t) \) is uniform on \( \delta \) in view of the assumption that \( H \in C^2 \). The conclusion for \( \lambda|z| \) follows from the stability properties of viscosity solutions.

The estimate can be recast in the following way. Fix \( T > 0 \), let \( B_n \in C^1([0, \infty)) \) be such that, \( B_n(0) = 0 \) and, as \( n \to \infty \), \( B_n \to 0 \) uniformly in \([0, T]\), and consider the solution \( v_n^\lambda \) to

\[
v_{n,t}^\lambda = H(Dv_n^\lambda)\dot{B}_n \quad \text{in} \quad Q_T \quad v_n^\lambda(z,0) = \lambda|z|.
\]

Then, as \( n \to \infty \) and \( \lambda \to \infty \) and for all \( T > 0 \),

\[
\sup_{(z,t) \in \mathbb{R}^N \times [0,T]} (v_n^\lambda(z, t) - \lambda|z|) \to 0 .
\]

Assuming that \( C^{1,1}_{loc} \) is, however, rather restrictive. For example, the typical Hamiltonian \( H(p) = |p| \) arising in front propagation does not have this regularity. On the other hand, the only assumption made on \( B \) is continuity. It turns out that it is actually possible to relax the regularity of \( H \), if more is assumed about \( B \).

6. Formulae for solutions to time dependent first-order initial value problems and pathwise solutions for equations with non-smooth Hamiltonians

**Formulae for solutions.** It is a very natural question to investigate whether there are any simple formulae available for the pathwise viscosity solutions. In the deterministic theory the simplest available expressions, when the Hamiltonian depends only on the gradient, are the so called Lax-Oleinik and Hopf formulae which are discussed below. If the Hamiltonian depends even smoothly in time, such simple expressions are not available; see Appendix for a discussion of this. Hence, in the setting here it is not possible to approximate the signals and then pass to the limit. Instead it is necessary to work on intervals of monotonicity of the continuous signal and then to try to iterate the outcomes.

The Lax-Oleinik formula applies to problems with either convex or concave Hamiltonians; see Lions [41] for a proof. The Hopf formula requires that the initial datum is either convex or concave and imposes no condition on \( H \); see Appendix as well as Lions [41] 53 and Bardi and Evans [3].
Consider the initial value problem

(6.1) \[ u_t = H(Du) \text{ in } Q_T, \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N. \]

If \( H \) is concave (resp. concave), the Lax-Oleinik formula is

(6.2) \[ u(x, t) = \inf_{y \in \mathbb{R}^N} \left[ u_0(y) + tH^*(\frac{y-x}{t}) \right] \quad (\text{resp. } u(x, t) = \sup_{y \in \mathbb{R}^N} \left[ u_0(y) - tH^*(\frac{y-x}{t}) \right]). \]

When \( H \) is convex (resp. concave), \( H^* \) is its convex (resp. concave) dual, that is \( H^*(p) = \sup \langle p, q \rangle - H(q) \) (resp. \( H^*(p) = \inf \langle p, q \rangle + H(q) \)).

The Hopf formula is the “dual” of the Lax-Oleinik one. If \( u_0 \) is convex (resp. concave), then

(6.3) \[ v(x, t) = \sup_{p \in \mathbb{R}^N} \left[ \langle p, x \rangle + tH(p) - u_0^*(p) \right] \quad (\text{resp. } v(x, t) = \inf_{p \in \mathbb{R}^N} \left[ \langle p, x \rangle + tH(p) + u_0^*(p) \right]); \]

as before when \( u_0 \) is convex (resp. concave), then \( u_0^* \) is its convex (resp. and concave) dual.

If the Hamiltonian depends on time, it is not possible to have either kind of formulae for the solution to

(6.4) \[ du = H(Du) \cdot dB \text{ in } Q_T, \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N, \]

unless \( B \) is either increasing or decreasing, in which case it is possible to change the time.

Indeed, if \( H \in C(\mathbb{R}^N), B \in C^1 \) and \( u_0 \) convex, the natural extension of (A.17) should be

\[ \sup_{p \in \mathbb{R}^N} \left[ \langle p, x \rangle + B(t)H(p) - u_0^*(p) \right]. \]

But, in general, this is not a solution to (6.4), unless \( dB \) has a fixed sign, but rather a subsolution, since, for each fixed \( p \), \( \langle p, x \rangle + B(t)H(p) - u_0^*(p) \) is a solution to (6.4). The explanation for this is that the shocks that develop for positive times are not reversible.

For example, if \( H(p) = |p| \) and \( u_0(x) = |x| \), then

\[ \sup_{p \in \mathbb{R}^N} \left( \langle p, x \rangle + B(t)|p| - |\cdot|^*(p) \right) = (|x| + B(t))_. \]

On the other hand, the following is true.

**Proposition 6.1.** The unique viscosity solution to (6.4) with \( B \in C^1 \), \( B(0) = 0 \), \( H(p) = |p| \) and \( u_0(x) = |x| \) is

(6.5) \[ u(x, t) = \max \left[ |x| + B(t)_+, \max_{0 \leq s \leq t} B(s) \right]. \]

Although the regularity of \( B \) is used in the proof of (6.5), the actual formula extends by density to arbitrary continuous \( B \)'s.

It is possible to give two different proofs for (6.3). One is based on dividing \([0, T]\) into intervals where \( dB \) is positive or negative and iterating the Hopf formula. The second is a direct justification that (6.3) is the viscosity solution to the problem. Since both arguments are lengthy, the proof is omitted.
Pathwise solutions for nonlinear first order pde with nonsmooth Hamiltonians and rough signals. When $H$ is less regular than in Theorem 5.1, it is also possible to prove the unique extension property for the solution operator for smooth paths, but the argument is different and does not rely on the characteristics. It is, however, possible to use the general strategy summarized after the proof of Theorem 5.1.

Indeed, it suffices to consider, for each fixed $T > 0$, the initial value problems
\[ v_{n,t} = H(Dv_n)B_n \text{ in } Q_T \quad v_n(z,0) = \lambda |z|, \]
with $B_n$ smooth and $B_n(0) = 0$ and to show that, if $B_n \to 0$ in $C([0,\infty))$, then
\[ \lim_{n \to \infty} \sup_{(z,t) \in \mathbb{R}^N \times [0,T]} |v_n(z,t) - \lambda |z|| \to 0. \]

It turns out that to prove (6.6), it is necessary to investigate the nontrivial interplay between the regularities of $H$ and $B$, which has very much the flavor of interpolation. Indeed, it is shown that a necessary condition to have a unique extension for all continuous $B$ is that $H$ is the difference of two convex functions. Formal interpolation suggests that, if $H \in C^0,\alpha$ for some $\alpha \in (0,1)$, then one needs $H \in C^{2(1-\alpha),2(1-\alpha)-2(1-\alpha)}$, where $[x]$ denotes the greatest integer part of $x \in \mathbb{R}$. So far, the only known result in this direction is that there is an extension if $H \in C^\beta - \beta$ for $\beta > 2(1-\alpha)$. The extensions for $B \in C^{0,1}$ and $H \in C^{0,1}$, or $B$ a Brownian motion and $H \in C^1$ or $H \in C^{0,1}$ are still an open problem, while it is possible to have a theory for $B \in C^{0,1}$ and $H \in C^1$.

Identifying the class of Hamiltonians which can be as written is the difference of two convex functions is an interesting question. When $N = 1$, this property is equivalent to $H' \in BV$, while no such necessary and sufficient property is known in higher dimensions. Of course, if $H = H_1 - H_2$ with $H_1, H_2$ convex, then, as for $N = 1$, $DH \in BV$. Functions with gradients in $BV$ do not have directional derivatives at every point, while differences of convex functions do. On the other hand, if $H \in C^{1,1}$, then $H$ is clearly the difference of convex functions. Indeed since, for some $c > 0$, $D^2H \geq -2c I_N$, $I_N$ being the $N \times N$ identity matrix, then $H = H_1 - H_2$ with $H_1(p) = H(p) + c|p|^2$ and $H_2(p) = c|p|^2$.

The first result discussed here is that
\[ H = H_1 - H_2 \text{ with } H_1, H_2 \text{ convex}, \]
is a necessary condition to have an extension.

**Proposition 6.2.** Assume that (6.6) holds for any sequence $B_n \in C([0,\infty))$ such that $B_n(0) = 0$ and, as $n \to 0$, $B_n \to 0$ in $C([0,\infty))$. Then $H$ must be the difference of two convex functions.

**Proof.** The proof is based on constructing a sequence of piecewise linear functions $B_n$ as in the claim with the additional property that $\dot{B}_n = \pm \mu$ in $2n$ successive intervals of length $1/2n$; for definiteness it is assumed that $\dot{B}_n = \mu$ in the first interval. Then
\[ \sup_{t \in [0,1]} |B_n(t)| \leq \frac{\mu}{2n} \quad \text{and} \quad B_n \to 0 \text{ in } C([0,\infty)) \text{ if } \mu/2n \to 0. \]

It is shown that, if for some $\delta > 0$, $\mu = 2n\delta$, then the solutions $v_n$ actually blow up as $n \to \infty$ if $H$ is not the difference of two convex functions in a ball of radius $\lambda$.
Recall that, in each time interval of length $1/2n$, the equations are

\[ v_{n,t} = 2n\delta H(Dv_n) \]  \quad \text{or}  \quad \text{or, after rescaling,}

\[ v_{n,t} = -2n\delta H(Dv_n) \]

The solutions are then constructed by a repeated iteration of Hopf’s formula. This procedure yields sequences $(V_{2k+1})_{k=0}^{\infty}$ and $(V_{2k})_{k=0}^{\infty}$ which, as $k \to \infty$, either blow up or converge (uniformly in $B_\lambda$) to $\bar{V}_1$ and $\bar{V}_2$ respectively. In the latter case it follows that

\[ \bar{V}_2^* = (\bar{V}_1^* - \delta H)^{**} \quad \text{and} \quad \bar{V}_1^* = (\bar{V}_2^* + \delta H)^{**}, \]

and, therefore,

\[ \delta H = \bar{V}_1^* - \bar{V}_2^*, \]

which yields that $H$ is the difference of two convex functions.

If the sequences $(V_{2k+1})_{k=0}^{\infty}$ and $(V_{2k})_{k=0}^{\infty}$ blow up, then a diagonal argument, in the limit $\delta \to 0$, shows that (6.6) cannot hold.

Indeed, since, for each $\delta > 0$ and as $k \to \infty$,

\[ V_{2k+1}^* \to -\infty \quad \text{and} \quad V_{2(k+1)}^* \to -\infty \quad \text{in} \quad B_\lambda, \]

choosing $\delta = 1/m$ along a sequence $k_m \to \infty$ yields $V_{2(k_m+1)}^* \leq -1$.

Going back to the original scaled problem, it follows that $v_{k_m} \leq -1$, while $B_{k_m} \to 0$ in $C([0,\infty))$ and $v_{k_m}(0,0) = 0$. \qed

The other direction is:

**Proposition 6.3.** Assume (6.7). Then (6.6) holds for any sequence $B_n \in C([0,\infty))$ such that $B_n(0) = 0$ and, as $n \to 0$, $B_n \to 0$ in $C([0,\infty))$.

The proof of Proposition 6.3 is more complicated, since it is not possible to use the characteristics as it was done in the proof of Theorem 5.1. Instead, it is necessary to find a way to control the oscillations of the solutions in time to deal with the difficulty that $\dot{B}$ may not be defined.

This is very much related to the fact that, due to the formation of shocks, the equations are not reversible. In other words, solving the problem backwards does not give the same function. On the other hand, “some memory” remains, resulting in cancellations taking place as it can be seen in the next result, which is about initial value problems of the form

\[ \begin{cases} (i) & u_t = \sum_{i=1}^{m} H_i(Du)B^i \quad \text{in} \quad Q_T \quad u(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^N, \\ (ii) & v^i_t = H_i(Dv^i)B^i \quad \text{in} \quad Q_T \quad v^i(\cdot,0) = v_0^i \quad \text{on} \quad \mathbb{R}^N, \end{cases} \]

where, for each $i = 1, \ldots, m$,

\[ H_i \in C(\mathbb{R}^N), \quad B^i \in C^1([0,\infty)) \quad \text{and} \quad u_0, v_0^i \in \text{BUC}(\mathbb{R}^N). \]

It is known that the both initial value problems in (6.8) have unique viscosity solutions. In the statement below, $S_{H_i}$ is the solution operator to (6.8) (ii) with $B_i(t) \equiv t$. 

Proposition 6.4. Assume, in addition to \([6.9]\), that, for each \(i = 1, \ldots, m\), \(H_i\) is convex and \(DH_i(p_i)\) exists for some \(p_i \in \mathbb{R}^N\), and let \(u \in \text{BUC}(\overline{Q}_T)\) be the viscosity solution of \([6.8]\). Then, for all \((x, t) \in \overline{Q}_T\),
\[
\prod_{i=1}^m S_{H_i}(-\min_{0 \leq s \leq t} B^i(s))u_0 \left( x + \sum_{i=1}^m DH_i(p_i)(\min_{0 \leq s \leq t} (B^i(s) - B^i(t))) \right) \\
+ \sum_{i=1}^m H_i(p_i)(\min_{0 \leq s \leq t} (B^i(s) - B^i(t))) \leq u(x, t) \leq \\
\prod_{i=1}^m S_{H_i}(\max_{0 \leq s \leq t} B^i(s))u_0 \left( x + \sum_{i=1}^m DH_i(p_i)(\min_{0 \leq s \leq t} (B^i(s) - B^i(t))) \right) \\
- \sum_{i=1}^m H_i(p_i)(\max_{0 \leq s \leq t} (B^i(s) - B^i(t))).
\]
(6.10)

The proof of Proposition 6.4, which is based on repeated use of the Lax-Oleinik and Hopf formulae, is complicated. The details are not presented here.

It is, however, interesting to observe that \(6.10\) is sharp. Indeed, recall that in the particular case
\[
H(p) = |p|, \quad u_0(x) = |x| \quad \text{and} \quad m = 1,
\]
the solution to \([6.8]\) is given by
\[
u(x, t) = \max([|x| + B(t)]_+, \max_{0 \leq s \leq t} B(s)).
\]
Evaluating the formula at \(x = 0\) yields that the upper bound in Proposition 6.4 is sharp, since, in this case,
\[
S_{H}(\max_{0 \leq s \leq t} B(s))u_0(0) = \max_{0 \leq s \leq t} B(s) \quad \text{and} \quad \max(B_+(t), \max_{0 \leq s \leq t} B(s)) = \max_{0 \leq s \leq t} B(s).
\]
Using Proposition 6.4, it is now possible to prove Proposition 6.3. 

The proof of Proposition 6.3. Fix \(T > 0\). As discussed earlier, it is enough to show that, if
\[
\phi^{\lambda, \varepsilon, \eta}_t = \sum_{i=1}^m H_i(D\phi^{\lambda, \varepsilon, \eta})(\xi^{i, \varepsilon} - \xi^{i, \eta}) \text{ in } Q_T \quad \phi^{\lambda, \varepsilon, \eta}(z, 0) = \phi_0(z) = \lambda|z|,
\]
where \((\xi^{i, \varepsilon})_{\varepsilon > 0}\) and \((\xi^{i, \eta})_{\eta > 0}\), with \(\xi^{i, \varepsilon}(0) = \xi^{i, \eta}(0) = 0\), are smooth approximations in \(C([0, \infty))\) of the given \(B \in C([0, \infty))\), then, for each \(\lambda > 0\),
\[
\lim_{\varepsilon, \eta \to 0} \max_{z \in \mathbb{R}^N, t \in [0, T]} |\phi^{\lambda, \varepsilon, \eta}(z, t) - \lambda|z||.
\]
To simplify the presentation it is assumed that \(H^1_1(0) = H^2_1(0) = \min H^1_t = \min H_i = 0 \text{ and } DH^1_1(0) = DH^2_1(0)\), in which case \([6.10]\) takes the form
\[
\prod_{i=1}^m S_{H_i}(-\min_{0 \leq s \leq t} B^i(s))u_0(x) \leq u(x, t) \leq \prod_{i=1}^m S_{H_i}(\max_{0 \leq s \leq t} B^i(s))u_0(x).
\]
(6.11)
Rewriting the equation satisfied by $\phi^{\lambda,\varepsilon,\eta}$ as

$$
\phi^{\lambda,\varepsilon,\eta}_t = \sum_{i=1}^{m} H_i^1(D\phi^{\lambda,\varepsilon,\eta})(\xi^{i,\varepsilon} - \xi^{i,\eta}) + \sum_{i=1}^{m} H_i^2(D\phi^{\lambda,\varepsilon,\eta})(\dot{\xi}^{i,\varepsilon} - \dot{\xi}^{i,\eta}),
$$

and using (6.11) yields, for all $x \in \mathbb{R}^N$,

$$
\prod_{i=1}^{m} S_{H_i^1}(\min_{0 \leq s \leq t} (\xi^{i,\varepsilon} - \xi^{i,\eta})(s)) \prod_{i=1}^{m} S_{H_i^2}(\max_{0 \leq s \leq t} (\xi^{i,\varepsilon} - \xi^{i,\eta})(s)) \phi_0(x)
$$

and the claim now follows since, $\lim_{\varepsilon,\eta \to 0} \max_{s \in [0,T]} |\xi^{\varepsilon} - \xi^{\eta}| = 0$. \qed

Proposition 6.2 and Proposition 6.3 are then combined in the following theorem.

**Theorem 6.1.** Assume that $H \in C(\mathbb{R}^N)$ satisfies (6.7). Then the solution operator for 6.4 on the class of smooth paths has a unique extension to the space of continuous paths.

Another consequence of the “cancellation” estimates of Proposition 6.3 is an explicit error estimate for $u^\varepsilon - v^0$, where $u^\varepsilon$ and $v^0$ are the solutions to

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u^\varepsilon}{\partial t} = \sum_{i=1}^{m} H_i(Du^\varepsilon)\dot{\xi}^{i,\varepsilon} & \text{in } Q_T, \\
& u^\varepsilon(\cdot,0) = u_0 \text{ on } \mathbb{R}^N,
\end{array} \right.
\end{align*}
$$

and

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial v^0}{\partial t} = \sum_{i=1}^{m} H_i(Dv^0)\dot{\xi}^{i,\eta} & \text{in } Q_T, \\
& v^0(\cdot,0) = u_0 \text{ on } \mathbb{R}^N.
\end{array} \right.
\end{align*}
$$

**Theorem 6.2.** Assume that, for each $1, \ldots, m$, $H_i \in C(\mathbb{R}^N)$ is the difference of two convex non-negative functions $H_i^1$, $H_i^2 \in C^2(\mathbb{R}^N)$, $(\xi^{i,\varepsilon})_{\varepsilon>0}$ and $(\xi^{i,\eta})_{\eta>0}$ are two approximations in $C([0,+\infty))$ of $B_i \in C([0,\infty))$ with $\xi^{i,\varepsilon}(0) = \xi^{i,\eta}(0) = 0$, and $u_0 \in C^{0,1}(\mathbb{R}^N)$. Let $u^\varepsilon, v^0 \in BUC(Q_T)$ be the solutions to (6.12). There exists $C > 0$ depending on $\|u_0\|$ and $\|Du_0\|$ and the growth of $H_i$’s such that, for all $t > 0$,

$$
\sup_{x \in \mathbb{R}^N} |u^\varepsilon(x,t) - v^0(x,t)| \leq C \max_{i=1,\ldots,m} \max_{0 \leq s \leq t} |\xi^{i,\varepsilon}(s) - \xi^{i,\eta}(s)|.
$$

Moreover, if, for $u_0 \in C^{0,1}(\mathbb{R}^N)$ and $B_1^{i,1}, B_1^{i,2} \in C([0,+\infty))$, $u_1, u_2 \in BUC(Q_T)$ are the extensions obtained by Theorem 6.1 then,

$$
\sup_{x \in \mathbb{R}^N} |u_1(x,t) - u_2(x,t)| \leq C \max_{i=1,\ldots,m} \max_{0 \leq s \leq t} |B_1^1(s) - B_1^2(s)|.
$$

**Proof.** Only the estimate for $u^\varepsilon - v^0$ is shown here. The one for $u_1 - u_2$ follows by a density argument.

Let $L$ be the Lipschitz constant of $u_0$. Since the Hamiltonians are $x$-independent, it is immediate from the contraction property that, for all $t \geq 0$, $u^\varepsilon(\cdot,t), v^0(\cdot,t) \in C^{0,1}(\mathbb{R}^N)$ and

$$
\max(\|Du^\varepsilon(\cdot,t)\|, \|Du^0(\cdot,t)\|) \leq L.
$$

The standard comparison estimate for viscosity solutions implies that, for all $(x,t) \in \overline{Q}_T$,

$$
U^\varepsilon(x,t) - v^0(x,t) - \phi^{L,\varepsilon,\eta}(x,x,t) \leq \sup_{x,y \in \mathbb{R}^N} |u_0(x) - u_0(y) - L|x-y| \leq 0.
$$
In view of the above assumption on the $H_i$’s, basic estimates from the theory of viscosity solutions yield, for any $\tau > 0$ and $w \in C^{0,1}(\mathbb{R}^N)$ with $\|Du\| \leq L$,

$$\|S_{H_i}(\tau)w - w\| \leq (\max \max_{i} |H_i(p)|) \tau.$$ 

This implies that

$$\phi^{L,\varepsilon,0}(x, x, t) \leq L|x - x| + m \max_{1 \leq i \leq m} \max_{|p| \leq L} |H_i(p)| \max_{0 \leq s \leq t} |\xi^i_{\varepsilon}(s) - \zeta^i_{\eta}(s)|$$

$$= m \max_{1 \leq i \leq m} \max_{|p| \leq L} |H_i(p)| \max_{0 \leq s \leq t} |\xi^i_{\varepsilon}(s) - \zeta^i_{\eta}(s)|.$$ 

Combining the upper bounds for $u^\varepsilon(x, t) - \tilde{u}^\eta(x, t)$ and $\phi^{L,\varepsilon,0}(x, x, t)$ gives the claim. \qed

Another very natural question concerning the class of Hamilton-Jacobi equations under consideration here is whether there is a finite domain of dependence, a property otherwise known as finite speed of propagation.

In the context of the (deterministic) viscosity solutions this property says that, if $H$ is Lipschitz continuous with constant $L$, and $u^1, u^2 \in BUC(\overline{Q}^T)$ solve the initial value problems

$$u^1_t = H(Du^1) \text{ in } Q_T \quad u^1(x, 0) = u^1_0(x) \quad \text{and} \quad u^2_t = H(Du^2) \text{ in } Q_T \quad u^2(x, 0) = u^2_0(x),$$

then

if $u^1_0 = u^2_0$ in $B(0, R)$, then $u^1(\cdot, t) = u^2(\cdot, t)$ in $B(0, R - Lt)$.

The only positive but partial result in this direction, which yields something weaker than finite speed of propagation, is about the initial value problem

$$(6.13) \quad u_t = (H_1(Du) - H_2(Du))B \text{ in } Q_T,$$

with

$$(6.14) \quad H_1, H_2 \text{ convex and bounded from below.}$$

**Proposition 6.5.** Assume $u \in BUC(\overline{Q}^T)$ be a solution to $(6.13)$ such that, for some $A \in \mathbb{R}$ and $R_0 > 0$, $u(\cdot, 0) \equiv A$ in $B(0, R_0)$. Let $L > 0$ be the Lipschitz constant of $H_1$ and $H_2$ in $B(0, 2R_0)$. Then $u(\cdot, t) \equiv A$ in $B(0, R - L(\max_{0 \leq s \leq t} B(s) - \min_{0 \leq s \leq t} B(s)))$.

**Proof.** Without loss of generality, the problem may be reduced to Hamiltonians with the additional condition

$$(6.15) \quad H_1, H_2 \text{ nonnegative and } H_1(0) = H_2(0) = 0.$$ 

As long as $R > L(\max_{0 \leq s \leq t} B(s) - \min_{0 \leq s \leq t} B(s))$, and, since $H_1(0) = H_2(0) = 0$, the finite speed of propagation of the initial value problem with $B(t) = t$ yields

$$S_{H_1}(\max_{0 \leq s \leq t} B^+(s))u_0 = S_{H_2}(\max_{0 \leq s \leq t} B^+(s))u_0 = A,$$

and the claim then follows using the estimate in Proposition $6.3$. \qed

After this course, Gassiat [26] came with a nonconvex Hamiltonian for which the initial value problem does not have finite speed of propagation property. It remains, however, an open question, if this property holds for convex Hamiltonian, since in that case, as shown above, there are cancellations.
Another important open question is whether it is possible to control the cancellations for spatially dependent Hamiltonians. An important step towards such a conclusion is to show that, if $H$ is convex and nonnegative, then

$$S_H(a)S_{-H}(a)u \leq u \leq S_{-H}(a)S_H(a)u,$$

where $S_{\pm H}$ are the solution operators of the “deterministic” problems with Hamiltonians $\pm H$.

It can be shown for convex Hamiltonians, using the control interpretation of the problems, that the claim is true if $H^*(x,0) \equiv 0$. However, it is not clear what to do if this last condition does not hold. It is also not obvious how to reduce to this because of the $x$-dependence.

A summary follows of what is known so far about the interplay between the regularity of $H$ and the paths in order to have a unique extension.

The theory of viscosity solutions applies when $H \in C$ and $B \in C^1$. As a matter of fact it is possible to consider $B \in C^{1,1}$ or even discontinuous $B$ as long as $\tilde{B} \in L^1$. When $H \in C^{1,1}$ or, more generally, if $H$ is the difference of two convex (or half-convex) functions, there exists a unique extension for any $B \in C([0,\infty))$.

Arguments similar to the ones presented next yield a unique extension for $B \in C^{0,\alpha}([0,\infty))$ with $\alpha \in (0,1)$ and $H \in C^{2(1-\alpha)+\varepsilon}(\mathbb{R}^N)$ for $\varepsilon > 0$. It is not clear if the additional $\varepsilon$-regularity is necessary; recall that, for any $\beta \in [0,\infty)$, $C^{\beta}(\mathbb{R}^N)$ is the space $C^{[\beta],[\beta]}(\mathbb{R}^N)$.

The conclusion resembles nonlinear interpolation. Indeed, consider the solution mapping $T(B,H) = u$, which is a bounded map from $C^1 \times C^0$ into $C$ and $C^0 \times C^2$ into $C$. Typically, if $T$ is bilinear, abstract interpolation results would imply that $T$ must be a bounded map from $C^{0,\alpha} \times C^{[2(1-\alpha)],2(1-\alpha)-[2(1-\alpha)]} \rightarrow C$. But $T$ is far from being bilinear.

Next it is shown that, in the particular case $\alpha = 1/2$, it is possible to have a unique extension if $H \in C^{1,\delta}$ for $\delta > 0$. Of course, the goal is to show that is enough to have $H \in C^1$ or even $H \in C^{0,1}$. This is another open problem.

A sequence $(B_n)_{n \in \mathbb{N}}$ in $C^1([0,\infty))$ is said to approximate $B \in C([0,\infty))$ in $C^{0,1/2}$ if, as $n \rightarrow \infty$,

$$B_n \rightarrow B \text{ in } C([0,\infty)) \quad \text{and} \quad \sup_n \| \tilde{B}_n \| \| B_n - B \| < \infty.$$  

Given $B \in C^{0,1/2}([0,\infty))$, it is possible to find at least two classes of such approximations. The first uses convolution with a suitable smooth kernel, while the second relies on finite differences.

Let $\rho_n(t) = n\rho(nt)$ with $\rho$ a smooth nonnegative kernel with compact support in $[-1,1]$ such that

$$\int z\rho(z)dz = 0 \quad \text{and} \quad \int \rho(z)dz = 1,$$

and consider the smooth function $B_n = B * \rho_n$. If $C = (\|\rho'\| + \|\rho\| + 1)[B]_{0,1/2}$, then

$$\| \tilde{B}_n \| \leq C\sqrt{n} \quad \text{and} \quad \| B_n - B \| \leq C/\sqrt{n}.$$  

For the second approximation, subdivide $[0,T]$ into intervals of length $\Delta = T/n$ and construct $B_n$ by a linear interpolation of $(B_k\Delta)_{k=1,...,n}$. Then

$$| \tilde{B}_n | = \frac{| B_{(k+1)\Delta} - B_{k\Delta} |}{\Delta} \leq \frac{[B]_{0,1/2}}{\sqrt{\Delta}} = C\sqrt{n} \quad \text{and} \quad \| B - B_n \| \leq [B]_{0,1/2}\sqrt{n} = \frac{C}{\sqrt{n}}.$$
The next result says that $C^{0,rac{1}{2}}$-approximations of $C^{0,\frac{1}{2}}$ paths yield a unique extension for $H \in C^{1,\delta}(\mathbb{R}^N)$ with $\delta > 0$. As a matter of fact the result not only gives an extension but also an estimate.

**Theorem 6.3.** Assume that $B \in C^{0,1/2}([0, \infty))$ and, for some $\delta > 0$, $H \in C^{1,\delta}(\mathbb{R}^N)$ and fix $T > 0$ and $u_0 \in \text{BUC}(\mathbb{R}^N)$. For any $(\zeta_n)_{n \in \mathbb{N}}, (\zeta^m)_{m \in \mathbb{N}} \in C([0, \infty))$ and $u_{0,n}, v_{0,m} \in \text{BUC}(\mathbb{R}^N)$, which are respectively $C^{0,1/2}$-approximations of $B$ and $u_0$ in $\text{BUC}(\mathbb{R}^N)$, let $u_n, v_m \in \text{BUC}(\overline{Q}_T)$ be the solutions to the corresponding initial value problems. Then there exists $u \in \text{BUC}(\overline{Q}_T)$ such that, as $n, m \to \infty$, $u_n, v_m \to u$ in $\text{BUC}(\overline{Q}_T)$. Moreover, if $\|u_{0,n} - u_0\| \leq Cn^{-\beta}$ for some $C, \beta > 0$, then there exists $\gamma > 0$ such that $|u_n - u| \lesssim n^{-\gamma}$ in $\overline{Q}_T$.

The basic step of the proof, which is omitted because it is rather long and technical, is to consider the behavior, as $n \to \infty$, $m \to \infty$ and $\lambda \to \infty$, of

$$\sup_{z \in \mathbb{R}^N, t \in [0,T]} [\phi_{n,m}^n(z,t) - \lambda |z|],$$

where $\phi_{n,m}$ is the solution to

$$\phi_t^{n,m} = H(D\phi^{\lambda,n,m})(\tilde{B}_n - \tilde{B}^m) \text{ in } Q_T \quad \phi_{n,m}^n(z,0) = \lambda |z|.$$ 

A discussion follows about the need to have conditions on $H$. The key step in the proof of Theorems 6.3 can be reformulated as follows. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of $C^1$-functions such that, as $n \to \infty$,

$$(6.16) \quad B_n \to 0 \quad \text{and} \quad \sup_n \|B_n\| \|\tilde{B}_n\| < \infty,$$

and consider the solution $v_n$ to

$$v_{n,t} = H(Dv_n)\tilde{B}_n \text{ in } Q_T \quad v_n(x,0) = \lambda |x|.$$ 

It suffices to show that, for each fixed $T > 0$ and for all $(x,t) \in \overline{Q}_T$,

$$\lim_{n \in \mathbb{N}} \sup_{(x,t) \in \overline{Q}_T} [v_n(x,t) - \lambda |x|] \to 0.$$ 

Next let $\hat{B}$ be piecewise constant such that, for $t_i = \frac{T}{k} i$,

$$\hat{B} = \Delta_1 \text{ in } [t_{2k}, t_{2k+1}] \quad \text{and} \quad \hat{B} = -\Delta_2 \text{ in } [t_{2k+1}, t_{2k}],$$

and, for simplicity, take $\lambda = 1$. Arguments similar to the ones earlier in this section and the fact that $v_k$ is convex, since $v_k(\cdot, 0)$ is, yield a decreasing sequence $w_n = v_k^*$ such that

$$w_0 = \begin{cases} 
0 & \text{if } |p| \leq 1, \\
+\infty & \text{if } |p| \geq 1,
\end{cases}$$

and

$$w_{2k+1} = (w_{2k} + \Delta_{2k} H)^* \quad \text{and} \quad w_{2k} = (w_{2k-1} - \Delta_{2k-1} H)^*,$$

where

$$\Delta_i = k[B((i+1)T/k) - B((iT/k))].$$

Then convergence will follow if there is a lower bound for the $w_k$'s.

Consider next the particular case

$$H(p) = |p|^\theta$$
and assume that
\[ \Delta_i = \sqrt{k} \text{ for all } i. \]
If \( \tilde{w}_k \) is constructed similarly to \( w_k \) but with \( \Delta_i = 1 \), it is immediate that
\[ w_k = k^{-1/2}\tilde{w}_k, \]
and, since \( \tilde{w}_{k+1} = ((\tilde{w}_k \pm |p|^\theta)^{**} \mp |p|^\theta)^{**} \), it follows that \( \tilde{w}_{k+1} \leq \tilde{w}_k \) and \( \tilde{w}_k = +\infty \) if \( |p| > 1 \).

Let \( m_k = -\inf_{|p|<1} w_k(p) \). Since \( H \) is not the difference of two convex functions if \( \theta \in (0, 1/2) \), it must be that \( \lim_{k \to \infty} m_k = \infty \).

It turns out, and this is tedious computation, that there exists \( c > 0 \) such that
\[ \tilde{w}_k \leq -ck^{1-\theta}. \]
It follows that, if \( \theta < 1/2 \),
\[ w_k = k^{-1/2}\tilde{w}_k \leq -ck^{1/2-\theta} \to -\infty \text{ as } k \to \infty. \]
The above calculations show that, if \( H \in C^{0,\alpha}(\mathbb{R}^N) \) with \( \alpha \in (0, \frac{1}{2}) \) and \( \sup_{n} \|B_{n}\|_{C^{0,\frac{1}{2}}} < \infty \), then there is blow up, and, hence, not a good solution. On the other hand if \( H \in C^{0,\frac{1}{2}}(\mathbb{R}^N) \) then there is no blow up.

It is also possible to construct an example of a Hamiltonian \( H \), which is not the difference of two convex functions and has better regularity than mere continuity, for which there is no extension unless more regularity is assumed.

7. Pathwise solutions for fully nonlinear, second-order PDE with rough signals and smooth, spatially homogeneous Hamiltonians

Consider the initial value problem
\[
(7.1) \quad du = F(D^2u, Du, u, x, t) \, dt + \sum_{i=1}^{m} H_i(Du_i) \cdot dB_i \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ in } \mathbb{R}^N,
\]
with
\[
(7.2) \quad H_i \in C^2(\mathbb{R}^N) \quad \text{and} \quad B_i \in C([0, \infty)) \quad \text{for } i = 1, \ldots, m,
\]
and \( F \) degenerate elliptic.

The case of “irregular” Hamiltonians requires different arguments; see [54] for the details. Spatially dependent regular Hamiltonians are discussed later.

An important question is if the Hamiltonian’s can depend on \( u \) and \( Du \) at the same time. The theory for Hamiltonians depending only on \( u \) was developed in Section 4. Whether the theory can be used when \( H \) depends both on \( u \) and \( Du \) is an open problem with the exception of a few special cases, like, for example, linear dependence on \( u \) and \( p \), which are basically an exercise.

The theory of viscosity solutions for equations like (7.1) with \( H \equiv 0 \) is based on using smooth test functions to test the equation at appropriate points. As already discussed earlier this can not be applied directly to (7.1).

Finally, recall that, when \( H \) is sufficiently regular, it is possible to construct, using the characteristics, local in time smooth solutions to (5.2). These solutions, for special initial data, play the role of the smooth test functions for (7.1).
Definition 7.1. Fix $B \in C([0,\infty);\mathbb{R}^m)$ and $T > 0$. A function $u \in BUC(\overline{Q}_T)$ is a pathwise sub (resp. super)-solution to (7.1) if, for any maximum (resp. minimum) $(x_0,t_0) \in Q_T$ of $u - \Phi - \psi$, where $\psi \in C^1((0,\infty))$ and $\Phi$ is a smooth solution to $d\Phi = \sum_{i=1}^m H_i(D\Phi) \cdot dB_i$ in $\mathbb{R}^N \times (t_0-h,t_0+h)$ for some $h > 0$, then
\[
\psi'(t_0) \leq F(D^2\Phi(x_0,t_0), D\Phi(x_0,t_0), u(x_0,t_0), x_0, t_0)
\]
(7.3)
\[
(\text{resp. } \psi'(t_0) \geq F(D^2\Phi(x_0,t_0), D\Phi(x_0,t_0), u(x_0,t_0), x_0, t_0).)
\]
Finally, $u \in BUC(\overline{Q}_T)$ is a solution to (7.1) if it is both sub- and super-solution.

Although somewhat natural, the definition introduces several difficulties at the technical level. One of the advantages of the theory of viscosity solutions is the flexibility associated with the choice of the test functions. This is not, however, the case here. As a result it is necessary to work very hard to obtain facts which were almost trivial in the deterministic setting. For example, in the definition, it is often useful to assume that the max/min is strict. Even this fact, which is trivial for classical viscosity solutions, in the current setting requires a bit more work.

It is also useful to point out the relationship between the approach used for equations with linear dependence on $Du$ and the above definition. Heuristically in Definition 7.1 one inverts locally the characteristics in an attempt to “eliminate” the bad term involving $\dot{B}$. Since the problem is nonlinear and $u$ is not regular, it is, of course, not possible to do this globally. In a way consistent with the spirit of the theory of viscosity solutions, this difficulty is overcome by working at the level of the test functions, where, of course, it is possible to invert locally the characteristics. The price to pay for this construction is that the test functions used here are very robust and not as flexible as the ones used in the classical deterministic theory. This leads to several technical difficulties, since all the theory has to be revisited.

The fact that Definition 7.1 is good in the sense that it agrees with the classical (deterministic) one, if $B \in C^1$, is left as an exercise. There are also several other preliminary facts about short time behavior, etc., which are also omitted. Details can be found in [54].

The emphasis here is on establishing a comparison principle and some stability properties. The existence follows either by a density argument or by Perron’s method. The latter was established lately in a very general setting by Seeger [8] for $m \geq 1$.

The next result is about the stability properties of the pathwise viscosity solutions. Although it can be stated in a much more general form using “half relaxed limits” and lower- and upper-semicontinuous envelopes, here it is presented in a simplified form.

Proposition 7.1. Let $F_n, F$ be degenerate elliptic, $H_n, H \in C^2(\mathbb{R}^N;\mathbb{R}^m)$, $B_n, B \in C([0,\infty);\mathbb{R}^m)$ be such that $\sup_{i,n} \|D^2H_{i,n}\| < \infty$ and, as $n \to \infty$ and locally uniformly, $F_n \to F$, $H_n \to H$ in $C^2(\mathbb{R}^N;\mathbb{R}^m)$, and $B_n \to B$ in $C([0,\infty);\mathbb{R}^m)$. If $u_n$ is a pathwise viscosity to (7.1) with nonlinearity $F_n$, Hamiltonian $H_n$ and path $B_n$ and $u_n \to u$ in $C(\overline{Q}_T)$, then $u$ is a pathwise viscosity solution to (7.1).

The assumptions that $H_n \to H$ in $C^2(\mathbb{R}^N)$ instead of just in $C(\mathbb{R}^N)$ and $\sup_n \|D^2H_n\| < \infty$ are not needed for the “deterministic” theory. Here they are dictated by the nature of the test functions.

Proof of Proposition 7.1. Let $(x_0,t_0) \in \mathbb{R}^N \times (0,T)$ be a strict maximum of $u - \Phi - \psi$ where $\psi \in C^1((0,\infty))$ and, for some $h > 0$, $\Phi$ is a smooth solution to (5.2) in $(t_0-h,t_0+h)$.
Let \( \Phi_n \) be the smooth solution to

\[
\Phi_{nt} = H_n(D\Phi_n) \dot{B}_n \text{ in } \mathbb{R}^N \times (t_0 - h_n, t_0 + h_n), \quad \Phi_n(\cdot, t_0) = \Phi(\cdot, t_0) \text{ in } \mathbb{R}^N.
\]

The assumptions on the \( H_n \) and \( B_n \) imply that, as \( n \to \infty \), \( \Phi_n \to \Phi \), \( D\Phi_n \to D\Phi \) and \( D^2\Phi_n \to D^2\Phi \) in \( C(\mathbb{R}^N \times (t_0 - h', t_0 + h')) \), for some, uniform in \( n, h' \in (0, h) \); note that this is the place where \( H_n \to H \) in \( C^2(\mathbb{R}^N) \) and \( \sup_n \|D^2H_n\| < \infty \) are used.

Let \((x_n, t_n)\) be a maximum point of \( u_n - \Phi_n - \psi \) in \( \mathbb{R}^N \times [t_0 - h', t_0 + h'] \). Since \((x_0, t_0)\) is a strict maximum of \( u - \Phi - \psi \), there exists a subsequence such that \((x_n, t_n) \to (x_0, t_0)\). The definition of viscosity solution then gives

\[
\psi'(t_n) \leq F(D^2\Phi_n(x_n, t_n), D\Phi_n(x_n, t_n), u_n(x_n, t_n), x_n, t_n).
\]

Letting \( n \to \infty \) yields the claim. \( \square \)

The next result is about the comparison principle for pathwise viscosity solutions to the first-order initial value problem \((5.2)\) with \( x \)-independent smooth Hamiltonians.

**Theorem 7.1.** Assume that, for \( i = 1, \ldots, m \), \( H_i \in C^2(\mathbb{R}^N) \), \( B \in C([0, \infty); \mathbb{R}^m) \) with \( B(0) = 0 \) and \( u_0 \in BUC(\mathbb{R}^N) \). Then, for each \( T > 0 \), \((5.2)\) has a unique pathwise solution \( u \in BUC(\overline{Q}_T) \).

**Proof.** If \( u, v \in BUC(\mathbb{R}^N \times [0, T]) \) are two solutions (as before it is assumed that \( u \) and \( v \) are periodic in space) of \((5.2)\), then

\[
z(x, y, t) = u(x, t) - v(y, t)
\]

is a solution to the “doubled” stochastic equation

\[
dz = \sum_{i=1}^m H_i(Dxz) \cdot dB_i - H_i(-Dyz) \cdot dB_i \text{ in } \mathbb{R}^N \times \mathbb{R}^N \times (0, T).
\]

This is a simple consequence of the discussion in the previous sections about local time smooth solutions and the commutation of the characteristics.

The conclusion now follows arguing as in the proof of the unique extension, since it is immediate that, for \( \lambda > 0 \),

\[
z(x, y, t) = \lambda |x - y|^2
\]

is a smooth solution to \((7.3)\) with initial data \( \lambda |x - y|^2 \). \( \square \)

The next result is about the extension operator for \((7.1)\). As before, it is shown that the solutions to initial value problems \((7.1)\) with smooth time signal approximating the given rough one form a Cauchy family in \( BUC(\overline{Q}_T) \) and, hence, all converge to the same function which is a pathwise viscosity solution to \((7.1)\).

**Theorem 7.2.** Assume that \( F \) is degenerate elliptic, \( H \in C^2(\mathbb{R}^N; \mathbb{R}^m) \), \((\xi_{\varepsilon})_{\varepsilon > 0}, (\zeta_{\eta})_{\eta > 0} \in C^1([0, \infty))\) two families of approximations in \( C([0, \infty); \mathbb{R}^m) \) of \( B \in C([0, \infty); \mathbb{R}^m) \) and \((u_{0, \varepsilon})_{\varepsilon > 0}, (v_{0, \eta})_{\eta > 0} \in BUC(\mathbb{R}^N) \) are such that, as \( \varepsilon, \eta \to 0 \) and uniformly in \( \mathbb{R}^N \), \( u_{0, \varepsilon} - v_{0, \eta} \to 0 \). Let \((u_{\varepsilon})_{\varepsilon > 0}, (v_{\eta})_{\eta > 0} \in BUC(\overline{Q}_T) \) be the unique viscosity solutions to \((7.1)\) with signal and initial datum \( \xi_{\varepsilon}, u_{0, \varepsilon} \) and \( \zeta_{\eta}, v_{0, \eta} \) respectively. Then, for all \( T > 0 \), as \( \varepsilon, \eta \to 0 \), \( u_{\varepsilon} - \tilde{u}_{\eta} \to 0 \) uniformly in \( \overline{Q}_T \). In particular, the family \((u_{\varepsilon})_{\varepsilon > 0} \) is Cauchy in \( BUC(\overline{Q}_T) \) and all approximations converge to the same limit.
Proof. Fix $T > 0$ and consider the “doubled” initial value problem

$$
\begin{cases}
\frac{dZ^{\lambda,\varepsilon,\eta}}{dt} = \sum_{i=1}^{m} H(D_{x}Z^{\lambda,\varepsilon,\eta})\dot{\xi}_{i,\varepsilon} - \sum_{i=1}^{m} H(-D_{y}Z^{\lambda,\varepsilon,\eta})\dot{\xi}_{i,\eta} \quad \text{in} \quad \mathbb{R}^{N} \times \mathbb{R}^{N} \times (0, T), \\
Z^{\lambda,\varepsilon,\eta}(x, y, 0) = \lambda|x - y|^2.
\end{cases}
$$

(7.6)

It is immediate that

$$
Z^{\lambda,\varepsilon,\eta}(x, y, t) = \Phi^{\lambda,\varepsilon,\eta}(x - y, t),
$$

(7.7)

where

$$
\Phi^{\lambda,\varepsilon,\eta}_{t} = \sum_{i=1}^{m} H(D_{x}\Phi^{\lambda,\varepsilon,\eta})(\dot{\xi}_{i,\varepsilon} - \dot{\xi}_{i,\eta}) \quad \text{in} \quad QT \quad \Phi^{\lambda,\varepsilon,\eta}(z, 0) = \lambda|z|^2.
$$

(7.8)

As discussed earlier, there exists $T^{\lambda,\varepsilon,\eta} > 0$ such that $\Phi^{\lambda,\varepsilon,\eta}$ is given by the method of characteristics in $\mathbb{R}^{N} \times [0, T^{\lambda,\varepsilon,\eta}]$ and

$$
\lim_{\varepsilon,\eta \to 0} T^{\lambda,\varepsilon,\eta} = \infty \quad \text{and} \quad \lim_{\varepsilon,\eta \to 0} \sup_{(z, t) \in \mathbb{R}^{N} \times [0, T]} |\Phi^{\lambda,\varepsilon,\eta}(z) - \lambda|z|^2| = 0.
$$

(7.9)

The conclusion will follow as soon as it established that

$$
\lim_{\lambda \to \infty} \lim_{\varepsilon,\eta \to 0} \sup_{(x, y, t) \in \mathbb{R}^{2N} \times [0, T]} (u^{\varepsilon}(x, t) - \tilde{u}^{\eta}(y, t) - \lambda|x - y|^2) = 0.
$$

(7.10)

Consider next the function

$$
\Psi^{\lambda,\varepsilon,\eta}(x, y, t) = u^{\varepsilon}(x, t) - u^{\eta}(y, t) - \Phi^{\lambda,\varepsilon,\eta}(x - y, t).
$$

The classical theory of viscosity solutions yields that the map

$$
t \mapsto M^{\lambda,\varepsilon,\eta}(t) = \sup_{x, y \in \mathbb{R}^{N}} [u^{\varepsilon}(x, t) - \tilde{u}^{\eta}(y, t) - \Phi^{\lambda,\varepsilon,\eta}(x - y, t)]
$$

is nonincreasing in $[0, T^{\lambda,\varepsilon,\eta})$.

Hence, for $x, y \in \mathbb{R}^{N}$ and $t \in [0, T^{\lambda,\varepsilon,\eta})$,

$$
u^{\varepsilon}(x, t) - u^{\eta}(y, t) - \Phi^{\lambda,\varepsilon,\eta}(x - y, t) \leq \sup_{x, y \in \mathbb{R}^{N}} (\nu^{\varepsilon}_{0}(x) - u^{\eta}_{0}(y) - \lambda|x - y|^2).
$$

The claim now follows from the assumptions on $\nu^{\varepsilon}_{0}$ and $u^{\eta}_{0}$. □

The uniqueness of the pathwise viscosity solutions to (7.1) is considerably more complicated than the one for (5.2). This is consistent with the deterministic theory, where the uniqueness theory of viscosity solutions for second-order degenerate, elliptic equations is by far more complex than the one for Hamilton-Jacobi equations. For the same reasons as for the existence, the argument is presented omitting the dependence on $u, x$ and $t$. The proof follows the general strategy outlined in the “User’s Guide”. The actual arguments are, however, different and more complicated. Recall that in the background of the “deterministic” proof are the so called sup- and inf-convolutions. These are particular regularizations that yield approximations which have parabolic expansions almost everywhere and are also sub- and super-solutions to the nonlinear pde.

This is exactly where the pathwise case becomes different. The “classical” sup- and inf-convolutions of pathwise viscosity solutions do not have parabolic expansions. In accordance with the general strategy about pathwise viscosity solutions, these sup- and inf-convolutions are modified by replacing the quadratic weights by short time smooth solutions to the first-order part of (7.1). The new regularizations have now parabolic expansions—the reader should think that the new weights
remove the “singularities” due to the roughness of \( B \). The rest of the argument follows then the proof of the deterministic uniqueness, although the details are more intricate.

**Theorem 7.3.** Let \( u, v \in BUC(\overline{Q_T}) \) be respectively viscosity sub- and super-solutions to (7.11) and assume that \( H = (H_1, \ldots, H_m) \in C^3(\mathbb{R}^N; \mathbb{R}^m) \) and \( F \) degenerate elliptic. Then, for all \( t \in [0, T] \),

\[
(7.11) \sup_{x \in \mathbb{R}^N} (u - v)(x, t) \leq \sup_{x \in \mathbb{R}^N} (u - v)(x, 0).
\]

**Proof.** To simplify the presentation below it is assumed that \( m = 1 \). Recall that, for any \( \phi \cdot C^3(\mathbb{R}^N \times \mathbb{R}^N) \cap C^{0,1}(\mathbb{R}^N \times \mathbb{R}^N) \), there exists some \( a > 0 \) such that the doubled initial value problem

\[
dU = [H(D_x U) - H(-D_y U)] \cdot dB \quad \text{in} \quad \mathbb{R}^N \times (t_0 - a, t_0 + a) \quad U(x, y, t_0) = \phi(x, y),
\]

has a smooth solution which, for future use, is denoted by \( \tilde{S}_H(t - t_0, t_0)\phi \).

If \( \phi \) is of separated form, that is, \( \phi(x, y) = \phi_1(x) + \phi_2(y) \), it is immediate by making, necessary, the interval of existence smaller, that

\[
\tilde{S}(t - t_0, t_0)\phi(x, y) = S_H(t - t_0, t_0)\phi_1(x) + S_H(t - t_0, t_0)\phi_2(y),
\]

where, as before, \( S_{\pm}H \) denote the smooth short time solution operators to \( du = \pm H(Du) \cdot dB \).

Moreover, for any \( \lambda > 0 \) and \( t, t_0 \in \mathbb{R} \), it is obvious that

\[
\tilde{S}(t - t_0, t_0)(\lambda | \cdot - \cdot |^2)(x, y) = \lambda |x - y|^2.
\]

Finally, again for smooth solutions,

\[
S_H(t - t_0, t_0)\phi_2(y) = -S_H^+(t - t_0, t_0)(-\phi_2)(y).
\]

Fix \( \mu > 0 \). The claim is that, as \( \lambda \to \infty \),

\[
\Phi(x, y, t) = u(x, t) - u(y, t) - \lambda |x - y|^2 - \mu t
\]

cannot have a maximum in \( \mathbb{R}^N \times \mathbb{R}^N \times (0, T] \). This leads to the desired conclusion as in the classical proof of the maximum principle.

Arguing by contradiction, it is assumed that there exists \( (x_\lambda, y_\lambda, t_\lambda) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T] \) such that, for all \( (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T] \),

\[
(7.12) \quad \Phi(x, y, t) = u(x, t) - v(y, t) - \tilde{S}(t - t_\lambda, t_\lambda)(\lambda | \cdot - \cdot |^2)(x, y) - \mu t \leq \Phi(x_\lambda, y_\lambda, t_\lambda).
\]

To handle the behavior at infinity and assert the existence of a max, it is necessary to consider \( \tilde{S}(t - t_\lambda, t_\lambda)[| \lambda | \cdot - \cdot |^2 + \beta \nu(\cdot)](x, y) \) instead of \( \tilde{S}(t - t_\lambda)(| \lambda | \cdot - \cdot |^2) \) in (7.12), for \( t - t_\lambda \) small, \( \beta \to 0 \), and a smooth approximation \( \nu(x) \) of \( |x| \). Since this adds some tedious details which may obscure the main ideas of the proof, below it is assumed that a max exists.

Elementary computations and a straightforward application of the Cauchy-Schwarz inequality yield, for all \( \varepsilon > 0 \) and \( \xi, \eta \in \mathbb{R}^N \),

\[
|x - y|^2 - |x_\lambda - y_\lambda|^2 \leq 2 < x_\lambda - y_\lambda, x - x_\lambda - \xi > + 2 < x_\lambda - y_\lambda, y - y_\lambda - \eta >
\]
\[
+ 2 < x_\lambda - y_\lambda, \xi - \eta > + (2 + \varepsilon^{-1})(|x - x_\lambda - \xi|^2 + |y - y_\lambda - \eta|^2) + (1 + 2\varepsilon)|\xi - \eta|^2.
\]

Let

\[
p_\lambda = \lambda(x_\lambda - y_\lambda), \quad \lambda_\varepsilon = \lambda(2 + \varepsilon^{-1}) \quad \text{and} \quad \beta_\varepsilon = \lambda(1 + 2\varepsilon).
\]
The comparison of local in time smooth pathwise solutions to Hamilton-Jacobi equations, which are easily obtained by the method of characteristics, and the facts introduced before the beginning of the proof yield that the function

\[
\Psi(x, y, \xi, \eta, t) = u(x, t) - v(y, t) - S^+_H(t - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda_\epsilon | \cdot - x_\lambda - \xi|^2)(x)
- S^-_H(t - t_\lambda, t_\lambda)(-2 < p_\lambda, \cdot - y_\lambda - \eta > + \lambda_\epsilon | \cdot - y_\lambda - \eta|^2)(y)
- 2 < p_\lambda, \xi - \eta > - \beta_\epsilon |\xi - \eta|^2 - \mu t
\]

achieves, for \( h \leq h_0 = h_0(\lambda, \epsilon^{-1}) \), its maximum in \( \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times (t_\lambda - h, t_\lambda + h) \) at \((x_\lambda, y_\lambda, 0, 0, t_\lambda)\).

Note that here it is necessary to take \( t - t_\lambda \) sufficiently small to have local in time smooth solutions for the doubled as well as the \( H \) and \(-H\) equations given by the characteristics.

For \( t \in (t_\lambda - h, t_\lambda + h) \) define the modified sup- and inf-convolutions

\[
\tilde{u}(\xi, t) = \sup_{x \in \mathbb{R}^N} [u(x, t) - S^+_H(t - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda_\epsilon | \cdot - x_\lambda - \xi|^2)(x)]
\]

and

\[
v(\eta, t) = \inf_{y \in \mathbb{R}^N} [v(y, t) + S^-_H(t - t_\lambda, t_\lambda)(-2 < p_\lambda, \cdot - y_\lambda - \eta > + \lambda_\epsilon | \cdot - y_\lambda - \eta|^2)(y)].
\]

It follows that, for \( \delta > 0 \),

\[
G(\xi, \eta, t) = \tilde{u}(\xi, t) - v(\eta, t) - (\beta_\epsilon + \delta)|\xi - \eta|^2 - 2 < p_\lambda, \xi - \eta > - \mu t
\]

attains its maximum in \( \mathbb{R}^N \times \mathbb{R}^N \times (t_\lambda - h, t_\lambda + h) \) at \((0, 0, t_\lambda)\).

Observe next that there exists a constant \( K_{\epsilon, \lambda} > 0 \) such that, in \( \mathbb{R}^N \times (t_\lambda - h, t_\lambda + h) \),

(7.14)

\[
D^2_\xi \tilde{u} \geq -K_{\epsilon, \lambda}, \quad D^2_\eta v \leq K_{\epsilon, \lambda}, \quad \tilde{u}_t \leq K_{\epsilon, \lambda} \quad \text{and} \quad v_t \geq -K_{\epsilon, \lambda}
\]

with the inequalities understood both in the viscosity and distributional sense.

The one sided bounds of \( D^2_\xi \tilde{u} \) and \( D^2_\eta v \) are an immediate consequence of the definition of \( \tilde{u} \) and \( v \) and the regularity of the kernels, which imply that, for some \( K_{\epsilon, \lambda} > 0 \) and in \( \mathbb{R}^N \times (t_\lambda - h, t_\lambda + h) \),

\[
|D^2_\xi S^+_H(\cdot - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda_\epsilon | \cdot - x_\lambda - \xi|^2)|
+ |D^2_\eta S^-_H(\cdot - t_\lambda, t_\lambda)(-2 < p_\lambda, \cdot - y_\lambda - \eta > + \lambda_\epsilon | \cdot - y_\lambda - \eta|^2)| \leq K_{\epsilon, \lambda}.
\]

The bound for \( \tilde{u}_t \) is shown next; the argument for \( v_t \) is similar. Note that such a bounds cannot be expected to hold for \( u_t \) where, in view of the behavior of \( dB \), it is not possible to have anything of that form. Indeed take \( F \equiv 0 \) and \( H \equiv 1 \), in which case \( u(x, t) = B(t) \).

Assume that, for some smooth function \( g \) and \( \xi \) fixed, the map \( \xi \mapsto \tilde{u}(\xi, t) - g(t) \) has a max at \( \hat{\xi} \).

It follows that

\[
(x, t) \mapsto u(x, t) - S^+_H(t - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda_\epsilon | \cdot - x_\lambda - \xi|^2)(x) - g(t)
\]

has a max at \((\hat{x}, \hat{t})\), where \( \hat{x} \) is a point where that sup in the definition of \( \tilde{u}(\xi, t) \) is achieved, that is

\[
\tilde{u}(\xi, \hat{t}) = u(\hat{x}, \hat{t}) - S^+_H(\hat{t} - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda_\epsilon | \cdot - x_\lambda - \xi|^2)(\hat{x}).
\]

In view of the definition of the pathwise viscosity sub-solution, it follows that there exists some \( \bar{K}_{\epsilon, \lambda} \), depending on \( K_{\epsilon, \lambda} \) and \( H \), such that \( g(\hat{t}) \leq \bar{K}_{\epsilon, \lambda} \), and, hence, the claim.
The one-sided bounds \((\ref{7.14})\) yield the existence of \(p_n, q_n, \xi_n, \eta_n \in \mathbb{R}^N\) and \(t_n > 0\) such that, as \(n \to \infty\),

(i) \((\xi_n, \eta_n, t_n) \to (0, 0, t_\lambda)\), \(p_n, q_n \to 0\),

(ii) the map \((\xi, \eta, t) \mapsto \bar{u}(\xi, t) - \bar{v}(\eta, t) - \beta \varepsilon |\xi - \eta|^2 - <p_n, \xi> - <q_n, \eta> - 2 <p_\lambda, \xi - \eta> \mu t\) has a maximum at \((\xi_n, \eta_n, t_n)\),

(iii) \(\bar{u}\) and \(\bar{v}\) have parabolic second-order expansions from above and below at \((\xi_n, t_n)\) and \((\eta_n, t_n)\) respectively, that is, there exist \(a_n, b_n \in \mathbb{R}\) such that

\[
\bar{u}(\xi, t) \leq \bar{u}(\xi_n, t_n) + a_n(t - t_n) + \frac{1}{2} D_\xi^2 \bar{u}(\xi_n, t_n)(\xi - \xi_n) + o(|\xi - \xi_n|^2 + |t - t_n|),
\]

and

\[
\bar{v}(\eta, t) \geq \bar{v}(\eta_n, t_n) + b_n(t - t_n) + \frac{1}{2} D_\eta^2 \bar{v}(\eta_n, t_n)(\eta - \eta_n) + o(|\eta - \eta_n|^2 + |t - t_n|),
\]

and, finally,

(iv) \(a_n = b_n + \mu, \quad D_\xi \bar{u}(\xi_n, t_n) = p_n + 2p_\lambda + 2\beta \varepsilon (\xi_n - \mu_n), \quad D_\eta \bar{v}(\eta_n, t_n) = -q_n + 2p_\lambda 2\beta \varepsilon (\xi_n - \eta_n)\) and \(D_\xi^2 \bar{u}(\xi_n, t_n) \leq D_\eta^2 \bar{v}(\eta_n, t_n)\).

It follows that, for some \(\theta > 0\) fixed, \(t < t_n (\xi, t)\) near \((\xi_n, t_n)\) and \((\eta, t)\) near \((\eta_n, t_n)\), the maps

\[
(x, \xi, t) \mapsto u(x, t) - S_H(t - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda \varepsilon \cdot -x_\lambda - \xi|\xi|^2)(x) - \Phi(x, t),
\]

and

\[
(y, \eta, t) \mapsto v(y, t) + S_H(t - t_\lambda, t_\lambda)(-2 < p_\lambda, \cdot - y_\lambda - \eta > + \lambda \varepsilon \cdot -\eta|\eta|^2)(y) - \Psi(x, t),
\]

attain respectively a maximum at \((x_n, \xi_n, t_n)\) and a minimum at \((y_n, \eta_n, t_n)\), where

\[
\Phi(\xi, t) = \bar{u}(\xi_n, t_n) + (\bar{u}_t(\xi_n, t_n) - \theta)(t - t_n) + \frac{1}{2} (D_\xi^2 \bar{u}(\xi_n, t_n) + \theta I)(\xi - \xi_n),
\]

and

\[
\Psi(\eta, t) = \bar{v}(\eta_n, t_n) + (\bar{v}_t(\eta_n, t_n) + \theta)(t - t_n) + \frac{1}{2} (D_\eta^2 \bar{v}(\eta_n, t_n) + \theta I)(\eta - \eta_n).
\]

Next, for sufficiently small \(r > 0\), let

\[
\Phi(x, t) := \inf_{(\xi, t) \in B(\xi_n, r) \times (t_n - r, t_n)} \left\{ \Phi(\xi, t) + S_H^+(t - t_\lambda, t_\lambda)(2 < p_\lambda, \cdot - x_\lambda - \xi > + \lambda \varepsilon \cdot -x_\lambda - \xi|\xi|^2)(x) \right\},
\]

and

\[
\Psi(y, t) := \sup_{(\xi, t) \in B(\xi_n, r) \times (t_n - r, t_n)} \left\{ \Psi(\eta, t) - S_H(t - t_\lambda, t_\lambda)(-2 < p_\lambda, \cdot - y_\lambda - \eta > + \lambda \varepsilon \cdot -y_\lambda - \eta|\eta|^2)(y) \right\}.
\]

It follows that \(u - \Phi\) and \(v - \Psi\) attain a local max at \((x_n, t_n)\) and a local min at \((y_n, t_n)\). Moreover, \(\Phi\) and \(\Psi\) are smooth solutions of \(du = H(Du) \cdot dB\) for \((x, t)\) near \((x_n, t_n)\) and \(dv = -H(-D_yv) \cdot dB\) for \((y, t)\) near \((y_n, t_n)\). This last assertion for \(\Phi\) and \(\Psi\) follows, using the inverse function theorem,
from the fact that, at \((x_n, t_n)\) and \((y_n, t_n)\), there exists a unique minimum in the definition of \(\Phi\) and \(\Psi\). This in turn comes from the observation that, for \(\lambda > \lambda_0\), at \((\xi_n, x_n, t_n)\) and \((\eta_n, y_n, t_n)\),
\[
D^2\Phi(\xi_n, t_n) + \lambda \varepsilon I > 0 \quad \text{and} \quad D^2\Psi(\eta_n, t_n) + \lambda \varepsilon I < 0.
\]
Finally, elementary calculations also yield that
\[
D^2\Phi(\xi_n, t_n) \geq D^2\Phi(x_n, t_n) \quad \text{and} \quad D^2\Psi(\eta_n, t_n) \leq D^2\Psi(y_n, t_n).
\]
Applying now the definitions of the pathwise viscosity sub- and super-solution to \(u\) and \(v\) respectively yields
\[
\bar{u}_t(\xi_n, t_n) - \theta \leq F(D^2\Phi(x_n, t_n), D_x\Phi(x_n, t_n))
\]
\[
\leq F(D^2\Phi(\xi_n, t_n), D_\xi\Phi(\xi_n, t_n)) = F(D^2\bar{u}(\xi_n, t_n) + \theta I, D_\xi\bar{u}(\xi_n, t_n))
\]
and
\[
\bar{v}_t(\eta_n, t_n) + \theta \geq F(-D^2\bar{v}(\xi_n, t_n) - \theta I, D_{\eta}\bar{v}(\eta_n, t_n)).
\]
Hence
\[
\mu - 2\theta \leq a_n - b_n - 2\theta
\]
\[
\leq \sup |F(A + \theta I, p + q_n) - F(A - \theta I, p + q_n)| : |p_n|, |q_n| \leq n^{-1}, |A| \leq K_{\varepsilon, \lambda}.
\]
The conclusion now follows choosing \(\varepsilon = (2\lambda)^{-1}\) and letting \(\lambda \to \infty\) and \(\delta \to 0\). \(\square\)

It is worth remarking that, in the course of the previous proof, it was shown that, for \(0 < h \leq \hat{h}_0\), with \(\hat{h}_0 = \hat{h}_0(\lambda, \varepsilon) \leq h_0\), \(\bar{u}\) (resp. \(\bar{v}\)) is a viscosity sub-solution (resp. super-solution) of
\[
\bar{u}_t \leq F(D_\xi^2\bar{u}, D_\xi\bar{u}) \quad \text{and} \quad \bar{v}_t \geq F(D_\eta^2\bar{v}, D_{\eta}\bar{v}) \quad \text{in} \quad \mathbb{R}^N \times (t_\lambda - h, t_\lambda + h).
\]

8. Pathwise solutions to fully nonlinear first and second order pde with spatially dependent smooth Hamiltonians

The general problem, strategy and difficulties. This section is about pathwise solutions to initial value problems of the form
\[
(8.1) \quad du = F(D^2u, Du, u, x) + H(Du, x) \cdot dB \quad \text{in} \quad Q_T, \quad u(\cdot, 0) = u_0 \quad \text{on} \quad \mathbb{R}^N,
\]
with only one rough path and, as always, \(F\) is degenerate elliptic.

It was already discussed that it is not yet understood how to define solutions when the Hamiltonian depends nonlinearly both in \(Du\) and \(u\). When \(H\) depends linearly on the gradient, then a change of unknowns similar to the one described in Section 3 leads to problems with semilinear rough path dependence, which can be studied using the methodology of Section 4. When \(H\) depends linearly on \(u\), then again a change of variables to remove the \(u\)-dependence from the Hamiltonian leads to an equation that can be studied following the results of this section.

Extending the theory to equations with multiple rough time dependence had been an open problem for some time. Recently, however, Lions and Souganidis \([55]\) came up with a way to resolve the problem.

Finally, to study equations for nonsmooth Hamiltonians, it is necessary to modify the definition of the solution using now as test functions solutions to the double equations constructed for nonsmooth Hamiltonians as in Section 6; see \([54]\).
The strategy is similar to the one followed for spatially homogeneous Hamiltonians. The pathwise solutions are defined using as test functions smooth solutions to

\[ du = H(Du, x) \cdot dB \quad \text{in} \quad \mathbb{R}^N \times (t_0 - h, t_0 + h), \quad u(\cdot, t_0) = \phi \quad \text{on} \quad \mathbb{R}^N, \]

which under the appropriate assumptions on \( H \) exist for each \( t_0 > 0 \) and smooth \( \phi \) in \( (t_0 - h, t_0 + h) \) for some small \( h \).

The aim in this section is to prove that pathwise solutions are well posed. To avoid many technicalities the discussion is restricted to Hamilton-Jacobi initial value problems like

\[ du = H(Du, x) \cdot dB \quad \text{in} \quad Q_T \quad u(\cdot, 0) = u_0 \quad \text{on} \quad \mathbb{R}^N. \]

The general problem (8.1) is studied using the arguments of this and the previous sections; details can be found in [54] and Seeger [78, 77].

Similarly to the spatially homogeneous case, the main technical issue is to control the length of the interval of existence of smooth solutions to the doubled equation with quadratic initial datum and smooth approximations to \( B \), that is

\[
\begin{aligned}
dz = H(D_x z, x) \cdot dB^\epsilon - H(-D_y z, y) \cdot d\tilde{B}^\eta \\
&\text{in} \quad \mathbb{R}^N \times \mathbb{R}^N \times (t_0 - h, t_0 + h), \\
z(x, y, 0) = \lambda |x - y|^2,
\end{aligned}
\]

where \( B^\epsilon \) and \( \tilde{B}^\eta \) are regular approximations of \( B \).

As already seen in the earlier discussions, the most basic estimate is that \( h = O(\lambda^{-1}) \), which, as it is explained below, is too small to carry out the comparison proof. The challenge, therefore, is to take advantage of the cancellations, due to the special form of the initial datum as well as of the doubled Hamiltonian, to obtain smooth solutions in a longer time interval.

Since the smooth solutions to (8.4) are constructed by the method of characteristics, the technical issue is to control the length of the interval of invertibility of the characteristics that correspond to \( x \) and \( y \). This can be done by estimating the interval of time in which the Jacobian does not vanish. It is here that using a single path helps, because, after a change of time, the problem reduces to studying the analogous question for homogeneous in time odes.

To further simplify the presentation, the problem discussed in the sequel is not (8.4) but rather the doubled equation with the actual rough path, that is

\[ dz = H(D_x z, x) \cdot dB - H(-D_y z, y) \cdot dB \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^N \times (t_0 - h, t_0 + h), \quad z(x, y, 0) = \lambda |x - y|^2; \]

the reader who has come so far should be able to figure out how to deal with this approximate problem. Note that, to avoid cumbersome expressions, \( \frac{1}{2} |x - y|^2 \) is written as \( \lambda |x - y|^2 \) and it is left up to the reader to recall that \( \lambda = 2\lambda' \).

The smooth solutions to (8.5) are given by \( z(x, y, t) = Z(x, y, B(t) - B(t_0)) \), where \( Z \) is the short time smooth solution to the “deterministic” doubled initial value problem

\[ dZ = H(D_x Z, x) - H(-D_y Z, y) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^N \times (-T^*, T^*) \quad Z(x, y, 0) = \lambda |x - y|^2, \]

and \( T^* > 0 \) and \( h \) are such that \( \sup_{s \in (t_0 - h, t_0 + h)} |B(s) - B(t_0)| \leq T^* \); note that this is the place where assuming that there is only rough path enters. The argument for multiple paths is rather different.
The smooth solutions to (8.6) are constructed by inverting the map \((x, y) \mapsto (X(x, y, t), Y(x, y, t))\) of the corresponding system of characteristics, that is

\[
\begin{align*}
\dot{X} &= -D_y H(P, X) \quad \dot{Y} = -D_y H(Q, Y), \\
\dot{P} &= D_x H(P, X) \quad \dot{Q} = D_y H(Q, Y), \\
\dot{U} &= H(P, X) - D_y H(P, X), P > -H(Q, Y) + D_y H(Q, Y), Q >,
\end{align*}
\]

with initial data

\[
\begin{align*}
X(x, y, 0) &= x, \quad Y(x, y, 0) = y, \\
\text{and } U(x, y, 0) &= \lambda|x - y|^2.
\end{align*}
\]

A rough estimate, which does not take into account the special form of the system and the initial data, gives that the map \((x, y) \mapsto (X(x, y, t), Y(x, y, t))\) is invertible at least in a time interval of length \(O(\lambda^{-1})\) for some \(O\) depending on \(\|H\|_{C^2}\).

This implies that the characteristics of (8.5) are invertible as long as

\[
\sup_{s \in (t_0 - h, t_0 + h)} |B(s) - B(t_0)| \leq O(\lambda^{-1}).
\]

The discussion next aims to explain the need of intervals of invertibility that are longer than \(O(\lambda^{-1})\), and serves as a blueprint for the strategy of the actual proof.

Let \(u\) and \(v\) be respectively a subsolution and a supersolution of (8.2). As in the \(x\)-independent case, it is assumed that, for some \(\alpha > 0\) and \(\lambda > 0\), \((x_0, y_0, t_0) \in \mathbb{R}^N \times (0, \infty)\) is a maximum point of

\[
u(x, t) - v(y, t) - \lambda' |x - y|^2 - \alpha t.
\]

Then, for \(h \in (0, t_0)\) and all \(x, y \in \mathbb{R}^N\),

\[
u(x, t_0 + h) - v(y, t_0 + h) \leq \lambda' |x - y|^2 - \alpha h + u(x_0, t_0) - v(y_0, t_0) - \lambda' |x_0 - y_0|^2.
\]

Since \(w(x, y, t) = u(x, t) - v(y, t)\) solves the doubled equation (8.2), to obtain the comparison it is enough to compare \(w\) with the small time smooth solution \(z\) to (8.5) starting at \(t_0 - h\).

It follows that

\[
u(x_0, t_0) - v(y_0, t_0) \leq z(x_0, y_0, t_0) + u(x_0, t_0) - v(y_0, t_0) - z(x_0, y_0, t_0 - h) - \alpha h,
\]

and, hence,

\[
\alpha \leq \frac{z(x_0, y_0, t_0) - z(x_0, y_0, t_0 - h)}{h}.
\]

Recall that \(h\) depends on \(\lambda\) and to conclude this dependence must be such that

\[
\limsup_{\lambda \to \infty} \frac{z(x_0, y_0, t_0) - z(x_0, y_0, t_0 - h)}{h} \leq 0.
\]

On the other hand, it will be shown that, if \(z\) is a smooth solution to (8.5), then

\[
z(x_0, y_0, t_0) - z(x_0, y_0, t_0 - h) \lesssim \sup_{s \in (t_0 - h, t_0 + h)} |B(s) - B(t_0)|\lambda^{-\frac{1}{2}}.
\]

Combining the last two statements implies that, to get a contradiction, \(h = h(\lambda)\) must be such that

\[
\limsup_{\lambda \to \infty} \sup_{s \in (t_0 - h, t_0 + h)} |B(s) - B(t_0)|h^{-\frac{1}{2}} = 0.
\]
The next argument indicates that there is indeed a problem if the smooth solutions to the “deterministic” doubled problem exist only for times of order $O(\lambda^{-1})$.

Indeed, if the interval of existence is of length $O(\lambda^{-1})$ and $B \in C^{0,\beta}([0, \infty))$, it follows that $h^\beta \approx \lambda^{-1}$ and, then,

$$\sup_{s \in (t_0-h,t_0+h)} |B(s) - B(t_0)| h^{-1} \lambda^{-\frac{1}{2}} \lesssim h^{\frac{3\beta}{2}} \lambda^{-1}.$$ 

Hence, for (8.9) to hold, it is necessary to have $\beta > 3/2$, which, of course, excludes Brownian motions.

**Improving the interval of existence of smooth solutions.** The problem is to find longer than $O(\lambda^{-1})$ intervals of existence of smooth solution to the doubled deterministic Hamilton-Jacobi equation

$$U_t = H(D_x U, x) - H(-D_y U, y) \quad \text{in} \quad \mathbb{R}^N \times (-T, T) \quad U(x, y, 0) = \lambda'|x - y|^2.$$ 

Two general sets of conditions will be modeled by two particular classes of Hamiltonians, namely separated and linear $H$'s.

To give the reader a flavor of the type of arguments that will be involved, it is convenient to begin with “separated” Hamiltonians of the form

$$H(p, x) = H(p) + F(x),$$ 

in which case the doubled equation and its characteristics are

$$U_t = H(D_x U) - H(-D_y U) + F(x) - F(y) \quad \text{in} \quad \mathbb{R}^N \times (-T, T) \quad U(x, y, 0) = \lambda'|x - y|^2,$$

and

$$\begin{align*}
\dot{X} &= -D_p H(P) - D_y H(Q), \\
\dot{Y} &= -D_P H(Q), \\
\dot{P} &= D_x F(X), \\
\dot{Q} &= D_y F(Y), \\
\dot{U} &= H(P) - H(Q), \\
X(0) &= x, \quad Y(0) = y, \quad P(0) = Q(0) = \lambda(x - y), \quad U(0) = \lambda'|x - y|^2.
\end{align*}$$

Let $J(t)$ denote the Jacobian of the map $(x, y) \mapsto (X(x, y, t), Y(x, y, t))$. In what follows, to avoid the rather cumbersome notation involving determinants etc., all the calculations are presented for $N = 1$, that is $x, y \in \mathbb{R}$; the reader, of course, can easily see how to extend everything to higher dimensions.

It follows that

$$J = \frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x} \quad \text{and} \quad J(0) = 1.$$ 

The most direct way to find an estimate for the time of existence of smooth solutions is, for example, to obtain a bound for the first time $t_\lambda$ such that $J(t_\lambda) = \frac{1}{2}$, and for this it is convenient to calculate and estimate the derivatives of $J$ with respect to time at $t = 0$.

Hence, it is necessary to derive the odes satisfied by $\frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}$ and $\frac{\partial Y}{\partial x}, \frac{\partial Y}{\partial y}$. Writing $\frac{\partial X}{\partial \alpha}, \frac{\partial Y}{\partial \alpha}, \frac{\partial P}{\partial \alpha}, \frac{\partial Q}{\partial \alpha}$ with $\alpha = x$ or $y$, differentiating (8.12) and omitting the subscripts for the derivatives of $H$ and $F$...
yields the systems

\[
\begin{align*}
\frac{\dot{X}}{\partial \alpha} &= -D^2H(P)\frac{\partial P}{\partial \alpha}, \\
\frac{\dot{P}}{\partial \alpha} &= D^2F(X)\frac{\partial X}{\partial \alpha}, \\
\frac{\partial X}{\partial x}(x,y,0) &= 1, \quad \frac{\partial P}{\partial x}(x,y,0) = \lambda, \\
\frac{\partial X}{\partial y}(x,y,0) &= 0, \quad \frac{\partial P}{\partial y}(x,y,0) = -\lambda, \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{\dot{Y}}{\partial \alpha} &= -D^2H(Q)\frac{\partial Q}{\partial \alpha}, \\
\frac{\dot{Q}}{\partial \alpha} &= D^2F(Y)\frac{\partial Y}{\partial \alpha}, \\
\frac{\partial Y}{\partial x}(x,y,0) &= 0, \quad \frac{\partial Q}{\partial x}(x,y,0) = \lambda, \\
\frac{\partial Y}{\partial y}(x,y,0) &= 1, \quad \frac{\partial Q}{\partial y}(x,y,0) = -\lambda.
\end{align*}
\]

**Proposition 8.1.** Assume that $DH, DF, D^2F, D^2H, |D^3H|(1 + |p|)$ and $D^4H$ are bounded. If $t_\lambda$ is the first time that $J(t_\lambda) = 1/2$, then, for some uniform constant $c > 0$ which depends on the bounds on the $H, F$ and their derivatives, and for all $x, y \in \mathbb{R}^N$,

\[
t_\lambda \geq c \min(1, \lambda^{-1/3}).
\]

**Proof.** Straightforward calculations that take advantage of the separated form of the Hamiltonian yield

\[
\dot{j} = -D^2H(P)(\frac{\partial P}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Y}{\partial x}) - D^2H(Q)(\frac{\partial X}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Q}{\partial x})
\]

and

\[
\dot{J} = -(D^2H(P)D^2F(X) + D^2H(Q)D^2F(Y))J + 2D^2H(P)D^2H(Q)(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x})
\]

\[
- D^3H(P)D^2F(X)(\frac{\partial P}{\partial y}\frac{\partial Y}{\partial x} - \frac{\partial P}{\partial x}\frac{\partial Y}{\partial y}) - D^3H(Q)D^2F(Y)(\frac{\partial X}{\partial y}\frac{\partial Q}{\partial x} - \frac{\partial X}{\partial x}\frac{\partial Q}{\partial y})
\]

To simplify the expressions for $\dot{j}$ and $\dot{J}$, it is convenient to express $\frac{\partial X}{\partial \alpha}, \frac{\partial Y}{\partial \alpha}, \frac{\partial P}{\partial \alpha}, \frac{\partial Q}{\partial \alpha}$ in terms of the solutions $(\eta_1, \phi_1, \eta_2, \psi_2)$ to the linearized system

\[
\begin{align*}
\dot{\xi}_1 &= -D^2H(P)\phi_1, \quad \xi_1(0) = 1, \\
\dot{\phi}_1 &= D^2F(X)\xi_1, \quad \phi_1(0) = 0, \\
\dot{\xi}_2 &= -D^2H(P)\phi_2, \quad \xi_2(0) = 0, \\
\dot{\phi}_2 &= D^2F(X)\xi_2, \quad \phi_2(0) = 1,
\end{align*}
\]

which are bounded in $[0,1]$ and satisfy

\[
\begin{align*}
\xi_1(t) = 1 + O(1)t, \quad \xi_2(t) = O(1)t, \\
\phi_1(t) = O(1)t, \quad \phi_2(t) = 1 + O(1)t,
\end{align*}
\]

and

\[
\begin{align*}
\dot{\eta}_1 &= -D^2H(Q)\psi_1, \quad \eta_1(0) = 1, \\
\dot{\psi}_1 &= D^2F(Y)\eta_1, \quad \psi_1(0) = 0, \\
\dot{\eta}_2 &= -D^2H(Q)\psi_2, \quad \eta_2(0) = 0, \\
\dot{\psi}_2 &= D^2F(Y)\eta_2, \quad \psi_2(0) = 1,
\end{align*}
\]

where $O(1)$ denotes different quantities for each functions which are uniformly bounded in $[0,1]$; note that the assumption that $D^2H$ and $D^2F$ are bounded is used here.

A direct substitution yields

\[
\begin{align*}
\frac{\partial X}{\partial x} = \xi_1 + \lambda \xi_2, \quad \frac{\partial X}{\partial y} = -\lambda \xi_2, \\
\frac{\partial P}{\partial x} = \phi_1 + \lambda \phi_2, \quad \frac{\partial P}{\partial y} = -\lambda \phi_2, \\
\frac{\partial Q}{\partial x} = \lambda \eta_2, \quad \frac{\partial Q}{\partial y} = \eta_1 - \lambda \eta_2,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial Y}{\partial x} = \lambda \psi_2, \quad \frac{\partial Y}{\partial y} = \psi_1 - \lambda \psi_2.
\end{align*}
\]
and
\[
\frac{\partial P}{\partial x} \frac{\partial P}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x} = (\phi_1 + \lambda \phi_2)(\psi_1 - \lambda \psi_2) - (-\lambda \psi_2)\lambda \psi_2
\]
\[
= \phi_1 \psi_1 + 2\lambda(\phi_2 \psi_1 - \phi_1 \psi_2) = O(1)(1 + 2\lambda t),
\]
since
\[
\phi_1 \psi_1 = O(1) \quad \text{and} \quad \phi_2 \psi_1 - \phi_1 \psi_2 = (1 + O(1)t)O(1)t - O(1)t(1 + O(1)t) = O(1)t.
\]
Similarly, since
\[
\phi_2 \eta_1 - \phi_1 \eta_2 = (1 + O(1)t)(1 + O(1)t) - O(1)tO(1)t = 1 + O(1)t \quad \text{and}
\]
\[
\xi_2 \psi_1 - \xi_1 \psi_2 = O(1)tO(1)t - (1 + O(1)t)(1 + O(1)t) = O(1)t - 1,
\]
it follows that
\[
\frac{\partial P}{\partial x} \frac{\partial Y}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Y}{\partial x} = (\phi_1 + \lambda \phi_2)(\eta_1 - \lambda \eta_2) - (-\lambda \psi_2)\lambda \eta_2
\]
\[
= \phi_1 \xi_1 + \lambda(\phi_2 \eta_1 - \phi_1 \eta_2) = O(1)(1 + \lambda t) + \lambda,
\]
and
\[
\frac{\partial X}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial X}{\partial y} \frac{\partial Q}{\partial x} = (\xi_1 + \lambda \xi_2)(\psi_1 - \psi_2) - (-\lambda \xi_2)\lambda \psi_2
\]
\[
= \xi_1 \psi_1 + \lambda(\xi_2 \psi_1 - \xi_1 \psi_2) = O(1)(1 + \lambda t) - \lambda.
\]
Inserting all the above in the expression for \( \bar{J} \) yields
\[
\bar{J} = O(1)J + O(1)(1 + \lambda t) + \lambda(D^3H(Q)DF(Y) - D^3H(P)DF(X)).
\]
Set
\[
A := (D^3H(Q) - D^3H(P))DF(Y) \quad \text{and} \quad D := D^3H(P)(DF(Y) - DF(X)).
\]
It is immediate that
\[
\lambda|A| \leq \lambda O(||DF||_\infty|Q - P|) = \lambda O(1)t,
\]
with the last estimate following from the observation that
\[
(P - Q)(t) = \lambda(x - y) + \int_0^t DF(X(s))ds - (\lambda(x - y)t)\int_0^t DF(Y(s))ds
\]
\[
= \int_0^t (DF(X(s)) - DF(Y(s)))ds = O(1)t.
\]
As far as \( D \) is concerned, observe that
\[
|D| \leq ||D^3H||_\infty|X - Y| \leq \frac{|X - Y|}{1 + |P|},
\]
and recall that
\[
|X - Y| \leq |x - y| + O(1)t \quad \text{and} \quad |P| = |\lambda(x - y) + O(1)t|.
\]
Hence,
\[
\lambda|D| \leq \left[ \frac{\lambda|x - y|}{1 + |\lambda(x - y) + O(1)t|} + \frac{\lambda O(1)t}{1 + |P|} \right] \leq \left[ \frac{|P(0)|}{1 + |P(0) + O(1)t|} + \frac{\lambda O(1)t}{1 + |P|} \right];
\]
the second term in the bound above comes from \( \lambda O(1)t \), while an additional argument is needed for the first.
Choose $t \leq t_1$ so that the $O(1)t$ term in $P$ is such that $|O(1)t| \leq \frac{1}{2}$. If $|P(0)| \leq 1$, then
\[
\frac{|P(0)|}{1 + |P(0) + O(1)t|} \leq 1
\]
while, if $|P(0)| > 1$,
\[
1 + |P(0) + O(1)t| \geq 1 + |P(0)| - |O(1)t| \geq |P(0)| + \frac{1}{2}
\]
and
\[
\frac{|P(0)|}{1 + |P(t)|} \leq \frac{|P(0)|}{\frac{1}{2} + |P(0)|} \leq 1.
\]
Combining the estimates on $\lambda A$ and $\lambda D$ gives
\[
\ddot{J} = O(1)J + O(1)\lambda t + O(1).
\]
On the other hand, it is immediate that $J(0) = 1$ and $\dot{J}(0) = 1$; this is another place where the separated form of the Hamiltonian and the symmetric form of the test function play role.

It follows that there exists $s_{\lambda} \in (0, t_{\lambda})$, such that
\[
\frac{1}{2} = 1 + \frac{1}{2}t_{\lambda}^2\dot{J}(s_{\lambda})
\]
and, hence,
\[
|t_{\lambda}^2\dot{J}(s_{\lambda})| = 1,
\]
which implies
\[
1 \lesssim t_{\lambda}^2(1 + \lambda t_{\lambda}).
\]
and, finally,
\[
1 \lesssim \lambda t_{\lambda}^3 + t_{\lambda}^2,
\]
and the claim is proved. $\square$

Having established a longer than $O(\lambda^{-1})$ interval of existence for the solution $U_{\lambda}$ of (8.11), it is now possible to obtain the following comparison result for pathwise solutions to Hamilton-Jacobi equations with separated Hamiltonians.

**Theorem 8.1.** Let $u \in BUC(Q_T)$ and $v \in BUC(Q_T)$ be respectively sub- and super-solutions to (8.2) in $Q_T$ with $H$ of separated form satisfying the assumptions of Proposition 8.1. Moreover, assume that $B \in C^{0,\beta}([0, \infty])$ with $\beta > 2/5$. Then, for all $t \in [0, T]$,
\[
\sup_{x \in \mathbb{R}^N} (u(x, t) - v(x, t)) \leq \sup_{x \in \mathbb{R}^N} (u(\cdot, 0) - v(\cdot, 0)).
\]

The following lemma, which is stated without a proof since it is rather classical, will be used in the proof of Theorem 8.1.

**Lemma 8.1.** Assume that $H \in C(\mathbb{R}^N)$ and $F \in C^{0,1}(\mathbb{R}^N)$ and let $U_{\lambda}$ be the viscosity solution to the doubled equation $w_t = H(D_x w) + F(x) - H(-D_y w, y) - F(y)$ in $Q_T$ with initial datum $\lambda|x-y|^2$. Then, for all $x, y \in \mathbb{R}^N$ and $t \in [0, T]$,
\[
|U_{\lambda}(x, y, t) - \frac{\lambda}{2}|x-y|^2| \leq t\|DF\||x-y|.
\]
The proof of Theorem 8.1. Assume that \((x_0, y_0, t_0)\) with \(t_0 > 0\) is a maximum point of \(u(x, t) - v(y, t) - \lambda|x - y|^2 - \alpha t\). Repeating the arguments at the end of the previous subsection and using Lemma 8.1 yields

\[
(8.13) \quad \alpha h \leq \|DF\|_{\infty} |x_0 - y_0| |B(t_0) - B(t_0 - h)|.
\]

Recall that, in view of Proposition 8.1, the above inequality holds as long as

\[
|B(t_0) - B(t_0 - h)| \approx \lambda^{-1/3}.
\]

Since \(B \in C^{0,\beta}(\mathbb{R}^N)\), the above yield

\[
\lambda^{-1} \approx h^{-3\beta}.
\]

Moreover, \((x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N\) being a maximum of \(u(x, t_0) - v(y, t_0) - \lambda|x - y|^2 \) yields \(\lambda|x_0 - y_0|^2 \leq \max(\|u\|, \|v\|)\) and, if \(\omega\) is the modulus of continuity of \(\omega\), \(\lambda|x_0 - y_0|^2 \leq \omega(\lambda^{-1/2} \max(\|u\|, \|v\|)^{1/2}) = O(1)\), and, hence, \(|x_0 - y_0| \lesssim \lambda^{-1/2} \).

Inserting all the observations above in (8.13) gives \(\alpha h \lesssim h^{\frac{5\rho}{2}}\), and, thus, \(\alpha \lesssim h^{\frac{5\rho}{2} - 1}\), which is a contradiction as \(\lambda \to \infty\). \(\square\)

Note that it is possible to assume less on \(B\) in Theorem 8.1 if more information is available for the modulus of continuity of either \(u\) or \(v\).

For general, that is not of separated form, Hamiltonians, the situation is more complicated. Indeed the “canonical” assumption on \(H\) for the deterministic theory is that, for some modulus \(\omega_H\) and all \(x, y, p \in \mathbb{R}^N\),

\[
(8.14) \quad |H(p, x) - H(p, y)| \leq \omega_H(|x - y|(1 + |p|)).
\]

On the other hand, it was shown in the course of the proof of the comparison

\[
\lambda|x_0 - y_0|^2 \leq 2 \max(\|u\|, \|v\|) \quad \text{and} \quad \lambda|x_0 - y_0|^2 \leq \max(\omega_\alpha(|x_0 - y_0|), \omega_\beta(|x_0 - y_0|)),
\]

and, hence,

\[
|x_0 - y_0|^2 \leq \lambda^{-1} \max(\omega_\alpha, \omega_\beta)(2(\lambda^{-1} \max(\|u\|, \|v\|)^{1/2})).
\]

Finally, if either \(u\) or \(v\) is Lipschitz continuous, then the above estimate can be improved to

\[
|x_0 - y_0| \leq \min(\|Du\|, \|Dv\|) \lambda^{-1}.
\]

The next technical result replaces Lemma 8.1. Its proof is again classical and it is omitted.

**Lemma 8.2.** Assume that \(H\) satisfies (8.14) with \(\omega_H(r) = Lr\). Let \(U_\lambda\) be the viscosity solution to the doubled equation \(w_t = H(D_x w) + F(x) - H(-D_y w) - F(y)\) in \(Q_T\) with initial datum \(\lambda|x - y|^2\).

Then there exists \(C > 0\) depending on \(L\) such that, for all \(x, y \in \mathbb{R}^N\) and \(t \in [0, T]\),

\[
(8.15) \quad |U_\lambda(x, y, t) - \lambda e^{Ct}|x - y|^2| \leq (e^{Ct} - 1)|x - y|.
\]

If, in addition \(|x - y| \lesssim \lambda^{-1}\), then

\[
|U_\lambda(x, y, t) - \lambda e^{Ct}|x - y|^2| \lesssim t \lambda^{-1}.
\]

The discussion follows about how to “improve” the length of the interval of existence of solutions given by the method of characteristics for \(H\) which are not separated. To keep the notation simple, it is again convenient to argue for \(N = 1\) and leave it to the reader to check the details in the general case.
The characteristic odes for the deterministic doubled pde \((8.11)\) are
\[
\begin{align*}
\dot{X} &= -D_p H(P, X), \\
\dot{Y} &= -D_q H(Q, Y), \\
\dot{P} &= D_p H(P, X), \\
\dot{Q} &= D_q H(Q, Y), \\
\dot{U} &= H(P, X) - <D_p H(P, X, P) > - H(Q, Y) + <D_q H(Q, Y) Q >, \\
X(0) &= x, \\ Y(0) &= y, \\ P(0) &= Q(0) = \lambda(x - y), \\ U(0) &= \lambda' |x - y|^2.
\end{align*}
\]
Recall that the Jacobian is given by
\[
J = \frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x},
\]
and, for \(\alpha = x\) or \(y\),
\[
\begin{align*}
\frac{\partial X}{\partial \alpha} &= -D_p^2 H(P, X) \frac{\partial P}{\partial \alpha} - D_{px}^2 H(P, X) \frac{\partial X}{\partial \alpha}, \\
\frac{\partial P}{\partial \alpha} &= D_{px}^2 H(P, X) \frac{\partial P}{\partial \alpha} + D_x^2 H(P, X) \frac{\partial X}{\partial \alpha}, \\
\frac{\partial X}{\partial \alpha}(0) &= \begin{cases} 1 & \text{if } \alpha = x, \\
0 & \text{if } \alpha \neq x.
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial Y}{\partial \alpha} &= -D_q^2 H(Q, Y) \frac{\partial Q}{\partial \alpha} - D_{qy}^2 H(Q, Y) \frac{\partial Y}{\partial \alpha}, \\
\frac{\partial Q}{\partial \alpha} &= D_{qy}^2 H(Q, Y) \frac{\partial Q}{\partial \alpha} + D_y^2 H(Q, Y) \frac{\partial Y}{\partial \alpha}, \\
\frac{\partial Y}{\partial \alpha}(0) &= \begin{cases} 0 & \text{if } \alpha = x, \\
1 & \text{if } \alpha = y.
\end{cases}
\end{align*}
\]
It is also necessary to consider, for \(i = 1, 2\) and \(z = x - y\), the auxiliary linearized systems equations
\[
\begin{align*}
\dot{\xi}_i &= -D_{pp}^2 H(P, X)(1 + \lambda|z|)\phi_i - D_{xp}^2 H(P, X)\xi_i, \\
\dot{\phi}_i &= D_{xp}^2 H(P, X)\phi_i + \frac{D_{xp}^2 H(P, X)\xi_i}{1 + \lambda|z|}, \\
\xi_1(0) &= 1, \quad \xi_2(0) = 0, \quad \phi_1(0) = 0, \quad \phi_1(0) = \frac{1}{1 + \lambda|z|},
\end{align*}
\]
and
\[
\begin{align*}
\dot{\eta}_i &= -D_{qy}^2 H(Q, Y)(1 + \lambda|z|)\psi_i - D_{qy}^2 H(Q, Y)\eta_i, \\
\dot{\psi}_i &= D_{qy}^2 H(Q, Y)\psi_i + \frac{D_{qy}^2 H(Q, Y)\eta_i}{1 + \lambda|z|}, \\
\eta_1(0) &= 1, \quad \eta_2(0) = 0, \quad \psi_1(0) = 0, \quad \psi_2(0) = \frac{1}{1 + \lambda|z|}.
\end{align*}
\]
It is immediate that
\[
\begin{align*}
\frac{\partial X}{\partial x} &= \xi_1 + \lambda \xi_2, & \frac{\partial X}{\partial y} &= -\lambda \xi_2, & \frac{\partial Y}{\partial x} &= -\lambda \eta_2, & \frac{\partial Y}{\partial y} &= -\lambda \eta_2, \\
\frac{\partial P}{\partial x} &= (\phi_1 + \lambda \phi_2)(1 + |z|), & \frac{\partial Q}{\partial x} &= -\lambda \psi_2(1 + |z|), \\
\frac{\partial P}{\partial y} &= -\lambda \phi_2(1 + |z|), & \frac{\partial Q}{\partial y} &= (\psi_1 + \lambda \psi_2)(1 + |z|).
\end{align*}
\]

Assume next that, for all \( p, x \in \mathbb{R}^N \),
\[
|D_{xp}^2 H(p, x)| \lesssim 1, \quad |D_{pp}^2 H(p, x)| \lesssim 1, \quad (1 + |p|)|D_{xp}^2 H(p, x)| \lesssim 1, \quad |D_{xpp}^3 H(p, x)| \lesssim 1,
\]
(8.18)
\[
(1 + |p|)|D_{xy}^2 H(p, x)| \lesssim 1 \quad \text{and} \quad (1 + |p|)^2|D_{pp}^3 H(p, x)| \lesssim 1.
\]
It follows that there exists \( C = C(T) > 0 \) such that, for all \( t \in [-T, T] \),
\[
|\xi_1(t)| \leq C, \quad |\eta_1(t)| \leq C, \quad |\xi_2(t)| \leq \frac{Ct}{1 + \lambda |z|} \quad \text{and} \quad |\eta_2(t)| \leq \frac{Ct}{1 + \lambda |z|}.
\]

Consider next the matrices
\[
A^x = \begin{pmatrix}
-D_{xp}^2 H(P, X) & -D_{pp}^2 H(P, X)(1 + \lambda |z|) \\
D_{xp}^2 H(P, X) & D_{xpp}^3 H(P, X)
\end{pmatrix}
\]
and
\[
A^y = \begin{pmatrix}
-D_{yy}^2 H(Q, Y) & -D_{qq}^2 H(Q, Y)(1 + \lambda |z|) \\
D_{yy}^2 H(Q, Y) & D_{yyq}^3 H(Q, Y)
\end{pmatrix}
\]

The next lemma, which is stated without proof, is important for the development of the rest of the theory here as well as for the theory of pathwise conservation laws.

**Lemma 8.3.** Assume that, in addition to \((\text{8.18})\), for all \( p, x \in \mathbb{R}^N \), \(|D_p H(p, x)|\) and \((1 + |p|)^{-1}|D_x H(p, x)|\) are bounded. Then there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for all \( t \in (0, \varepsilon_0) \),
\[
||A^x - A^y|| \leq C|z|.
\]

Lemma 8.3 gives that, for all \( t \in (0, \varepsilon_0) \),
\[
|\xi_1 - \eta_1| \leq C|z|t \quad \text{and} \quad |\xi_2 - \eta_2| \leq \frac{C|z|t}{1 + \lambda |z|},
\]
(8.20)
and, since
\[
\lambda(\xi_2 \eta_1 - \xi_1 \eta_2) = \lambda(\xi_2 - \eta_2)\eta_1 + \lambda \eta_2(\eta_1 - \xi_1),
\]
it follows using \((\text{8.19})\) and \((\text{8.20})\) that
\[
|\lambda(\xi_2 \eta_1 - \xi_1 \eta_2)| \leq Ct.
\]
(8.21)
Similar arguments allow to obtain an interval of invertibility of the characteristics that is uniform in \( \lambda \), and, hence, a \( O(1) \)-interval of existence of smooth solutions to the doubled equation in either one
Let \( \phi \) be such that
\[
\begin{align*}
&|D_x^2H| \lesssim 1, \quad |D_{xxx}^3H| \lesssim 1, \quad |D_x^2H| \lesssim 1, \quad |D_{xxp}^3H|(1 + |p|) \lesssim 1, \\
&|D_p^2H| \lesssim 1, \quad |D_{xp}^3H|(1 + |p|) \lesssim 1, \quad |D_{pp}^3H|(1 + |p|) \lesssim 1.
\end{align*}
\]
(8.22)

\[
\begin{align*}
&|D_x^2H| \lesssim 1, \quad |D_x^3H| \lesssim (1 + |p|), \quad |D_{xp}^2H| \lesssim 1, \quad |D_{xpp}^3H| \lesssim 1, \\
&|D_p^2H|(1 + |p|) \lesssim 1, \quad |D_{pp}^3H|(1 + |p|) \lesssim 1, \quad |D_p^3H|(1 + |p|^2) \lesssim 1.
\end{align*}
\]
(8.23)

\[
\begin{align*}
&|D_x^2H| \lesssim 1, \quad |D_p^2H| \lesssim 1, \quad |D_x^2H|(1 + |p|) \lesssim 1, \\
&|D_p^2H|(1 + |p|) \lesssim 1, \quad |D_{xpp}^3H|(1 + |p|^2) \lesssim 1, \quad |D_{xxp}^3H| \lesssim 1, \quad |D_x^3H|(1 + |p|) \lesssim 1.
\end{align*}
\]
(8.24)

Note that (8.22) contains the split variable case, and linear Hamiltonians are a special case of (8.23).

Calculations similar to the ones used in the split variable case
\[
|\xi_2\eta_1 - \eta_2\xi_1| \lesssim t^2
\]
and, as it was already seen, \( t_\lambda \equiv \lambda^{-1/3} \). Note that, if \( |DH|, |D^2H| \) and \( |D^3H| \) are all bounded, then \( t_\lambda \equiv \lambda^{-1/2} \).

**The necessity of the assumptions.** An important question is whether conditions like the ones stated above are actually necessary to have well posed problems for Hamiltonians that depend on \( p, x \). That some conditions are needed is natural since the argument is based on inverting characteristics and, hence, staying away from shocks. In view of this, assumptions that control the behavior of \( H \) and its derivatives for large \( |p| \) are to be expected.

On the other hand, some of the restrictions imposed are due to the specific choice of the initial datum of doubled equation, which, in principle, does not “interact well” with the cancellation properties of the given \( H \).

Consider, for example, the Hamiltonian
\[
H(p, x) = F(a(x)p),
\]
with
\[
a, F \in C^2(\mathbb{R}) \cap C^{0,1}(\mathbb{R}) \quad \text{and} \quad a > 0.
\]

The characteristics are
\[
\dot{X} = -F'(a(X)P)a(X) \quad \text{and} \quad \dot{P} = F'(a(X)P)a'(X)P.
\]
Let \( \phi \in C^2(\mathbb{R}) \) be such that
\[
\phi' = \frac{1}{a},
\]
and consider the new characteristics
\[
\dot{X} = \phi(X) \quad \text{and} \quad \dot{P} = a(X)P.
\]
Then
\[
\dot{X} = \phi'(X)\dot{X} = -a^{-1}(X)F'(a(X)P)a(X) = -F'(\dot{P})
\]
and
\[ \dot{P} = a'(X)XP + a(X)\dot{P} = -a'(X)F'(\dot{P})a(X)P + a(X)F'(a(X)P)a'(X)P = 0. \]

At the level of the pde
\[ du = F(a(x)u_x) \cdot dB, \]
the above transformation yields that, if
\[ u(x,t) = U(\phi(x),t), \]
then
\[ dU = F(U_x) \cdot dB, \]
a problem which is, of course, homogeneous in space, and, hence, as already seen, there is a $O(1)$-interval of existence for the doubled pde.

This leads to the question if it is possible to find, instead of $\lambda|x-y|^2$, an initial datum for the doubled pde, which is still coercive, and, in the mean time, better adjusted to the structure of the doubled equation. This question was answered affirmatively for equations with quadratic Hamiltonians corresponding to Riemannian metrics in Friz, Gassiat, Lions and Souganidis [24], positively homogeneous and convex in $p$ Hamiltonians in Seeger [77] and, for general, convex in $p$ Hamiltonians in Lions and Souganidis [56].

9. Pathwise entropy/kinetic solutions for scalar conservation laws with multiplicative rough time signals.

**Introduction.** Ideas similar to the ones described up to the previous sections were used by Lions, Perthame and Souganidis [49, 50], Gess and Souganidis [29, 28, 30] and Gess, Perthame and Souganidis [27] to study pathwise entropy/kinetic solutions for scalar conservation laws with multiplicative rough time signals as well as their long time behavior, the existence of invariant measures, possible regularization by noise and the convergence of general relaxation schemes with error estimates.

To keep the ideas simple the presentation here is about the simplest possible case, that is the spatially homogeneous initial value problem
\begin{equation}
(9.1) \quad du + \sum_{i=1}^N A_i(u)x_i \cdot dB_i = 0 \quad \text{in } Q_T, \quad u_0(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^N,
\end{equation}
with
\begin{equation}
(9.2) \quad A = (A_1, ..., A_N) \in C^2(\mathbb{R}; \mathbb{R}^N)
\end{equation}
and merely continuous paths
\begin{equation}
(9.3) \quad B = (B_1, ..., B_N) \in C([0, \infty); \mathbb{R}^N).
\end{equation}
If $B \in C^1([0, \infty); \mathbb{R}^N)$, then (9.1) is a “classical” problem with a well known theory; see, for example, the books by Dafermos [17] and Serre [79]. The solution can develop singularities in the form of shocks (discontinuities). Hence it is necessary to consider entropy solutions which, although not regular, satisfy the $L^1$-contraction property established by Kruzkov [35].

Solutions to deterministic non-degenerate conservation laws have remarkable regularizing effects in Sobolev spaces of low order. It is an interesting question to see if they are still true in the present
case. This is certainly possible with different exponents as shown in Lions, Perthame and Souganidis
[50] and Gess and Souganidis [28].

Contrary to the Hamilton-Jacobi equation, the approach put forward for (9.1) does not work for
conservation laws with semilinear rough path dependence like

\[ du + \sum_{i=1}^{N} (A_i(u))_{x_i} dt = \Phi(u) \cdot d\tilde{B} \quad \text{in} \quad Q_T \quad u(.,0) = u_0 \quad \text{on} \quad \mathbb{R}^N, \]

for \( \Phi = (\Phi_1, ..., \Phi_m) \in C^2(\mathbb{R}; \mathbb{R}^m) \) and an \( m \)-dimensional path \( \tilde{B} = (\tilde{B}_1, ..., \tilde{B}_m) \).

Semilinear stochastic conservation laws in Itô’s form like

\[ du + \sum_{i=1}^{N} (A_i(u))_{x_i} dt = \Phi(u) d\tilde{B} \quad \text{in} \quad Q_T \]

have been studied by Debussche and Vovelle [18, 19, 20], Feng and Nualart [22], Chen, Ding and
Karlsen [10], and Hofmanova [31, 32]).

It turns out that pathwise solutions are natural in problems with nonlinear dependence. Indeed,
let \( u, v \) be solutions of the simple one dimensional problems

\[ du + A(u) \cdot dB = 0 \quad \text{and} \quad dv + A(v) \cdot dB = 0. \]

Then

\[ d(u - v) + (A(u) - A(v)) \cdot dB = 0. \]

Multiplying by the \( \text{sign}(u - v) \) and integrating over \( \mathbb{R} \) formally leads to

\[ d \int_{\mathbb{R}} |u - v| dx + \int_{\mathbb{R}} (\text{sign}(u - v)(A(u) - A(v)))_{x} \cdot dB = 0 \]

and, hence,

\[ d \int_{\mathbb{R}} |u - v| dx = 0. \]

On the other hand, if \( du = \Phi(u) \cdot dB \) and \( dv = \Phi(v) \cdot dB \), then the previous argument cannot be
used since the term \( \int_{\mathbb{R}} \text{sign}(u - v)(\Phi(u) - \Phi(v)) \cdot dB \) is neither 0 nor has a sign. More about this
is presented in the last subsection.

**A brief review of the kinetic theory when \( B \) is smooth.** To make the connection with the
“deterministic” theory, assume that \( B \in C^1((0, \infty); \mathbb{R}^N) \), in which case \( du \) stands for the usual
derivative and \( \cdot \) is the usual multiplication and, hence, should be ignored.

The entropy inequality, which guarantees the uniqueness of the weak solutions, is

\[ dS(u) + \sum_{i=1}^{N} (A_i^S(u))_{x_i} \cdot dB_i \leq 0 \quad \text{in} \quad Q_T \quad S(u) = S(u_0) \quad \text{on} \quad \mathbb{R}^N \times \{0\}, \]

for all \( C^2 \)-convex functions \( S \) and entropy fluxes \( Q \) defined by

\[
\left(A^S(u)\right)' = a(u)S'(u) \quad \text{with} \quad a = A'.
\]

It is by now well established that the simplest way to handle conservation laws is through their
kinetic formulation developed in a series of papers – see Perthame and Tadmor [73], Lions, Perthame...
and Tadmor [52], Perthame [71, 72], and Lions, Perthame and Souganidis [48]. The basic idea is to write a linear equation satisfied by the nonlinear function

\begin{equation}
\chi(x, \xi, t) = \chi(u(x, t), \xi) = \begin{cases} 
+1 & \text{if } 0 \leq \xi \leq u(x, t), \\
-1 & \text{if } u(x, t) \leq \xi \leq 0, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

The kinetic formulation states that the entropy inequalities (9.6) for all convex entropies \( S \) is equivalent to \( \chi \) solving, in the sense of distributions, \( \int S'(\xi) \chi(u(x, t), \xi) d\xi \),

\begin{equation}
d\chi + \sum_{i=1}^{N} a_i(\xi) \partial_{x_i} \chi \cdot dB_i = \partial_{\xi} m dt \quad \text{in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty),
\end{equation}

where

\begin{equation}
m \text{ is a nonnegative bounded measure in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty).
\end{equation}

One direction of this equivalence can be seen, at least formally, easily. Indeed since, for all \((x, t) \in \mathbb{R}^N \times (0, \infty), \)

\begin{equation}
S(u(x, t)) - S(0) = \int S'(\xi) \chi(u(x, t), \xi) d\xi,
\end{equation}

for smooth \( \psi \) and, for all smooth test functions \( \psi \),

\begin{equation}
\int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}} m(x, \xi, t) dx d\xi dt \leq \left\| u_0 \right\|_{L^2(\mathbb{R}^N)}^2,
\end{equation}

\begin{equation}
\int_0^\infty \int_{\mathbb{R}^N} m(x, \xi, t) dx dt \leq \left\| u_0 \right\|_{L^1(\mathbb{R}^N)}
\end{equation}

and, for all smooth test functions \( \psi \),

\begin{equation}
\frac{d}{d\xi} \int_{\mathbb{R}^N} \psi(x, t) m(x, \xi, t) dx dt \leq \left[ \left\| D_x \psi \right\|_{L^\infty(\mathbb{R}^N+1)} + \left\| \psi(\cdot, 0) \right\|_{L^\infty(\mathbb{R}^N)} \right] \left\| u_0^0 \right\|_{L^1(\mathbb{R}^N)}.
\end{equation}

The following observation is the backbone of the theory pathwise entropy/kinetic solutions. The reader will recognize ideas described already in the earlier parts of these notes. Since the flux in (9.11) is independent of \( x \), it is possible to use the characteristics associated with (9.8) to derive an identity which is equivalent to solving (9.8) in the sense of distributions. Indeed, choose

\begin{equation}
\rho_0 \in C^\infty(\mathbb{R}^N) \text{ such that } \rho_0 \geq 0 \text{ and } \int_{\mathbb{R}^N} \rho_0(x) dx = 1,
\end{equation}
and observe that
\begin{equation}
\rho(y, x, \xi, t) = \rho_0(y - x + a(\xi) B(t)),
\end{equation}
where
\begin{equation}
a(\xi) B(t) := (a_1(\xi) B_1(t), a_2(\xi) B_2(t), \ldots, a_N(\xi) B_N(t)),
\end{equation}
solves the linear transport equation (recall that in this subsection it is assumed that \( B \) is smooth)
\begin{equation}
d\rho + \sum_{i=1}^{N} a_i(\xi) \partial_{x_i} \rho \cdot dB_i = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R} \times (0, \infty),
\end{equation}
and, hence,
\begin{equation}
d(\rho(y, x, \xi, t)\chi(x, \xi, t)) + \sum_{i=1}^{N} a_i(\xi) \partial_{x_i}(\rho(y, x, \xi, t)\chi(x, \xi, t)) \cdot dB_i = \rho(y, x, \xi, t)\partial_{\xi} m(x, \xi, t) dt.
\end{equation}
Integrating \((9.19)\) with respect to \( x \) (recall that \( \rho_0 \) has compact support) yields that, in the sense of distributions in \( \mathbb{R} \times (0, \infty) \),
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^N} \chi(x, \xi, t) \rho(y, x, \xi, t) dx = \int_{\mathbb{R}^N} \rho(y, x, \xi, t) \partial_{\xi} m(x, \xi, t) dx.
\end{equation}
Observe that, although the regularity of the path was used to derive \((9.20)\), the actual conclusion does not need it. In particular, \((9.20)\) holds for paths which are only continuous. Moreover, \((9.20)\) is basically equivalent to the kinetic formulation, if the measure \( m \) satisfies \((9.9)\).
Finally, note that \((9.20)\) makes sense only after integrating with respect to \( \xi \) against a test function. This requires that \( a' \in C^1(\mathbb{R}; \mathbb{R}^N) \) as long as we only use that \( m \) is a measure. Indeed, integrating against a test function \( \Psi \), yields
\begin{equation}
\int_{\mathbb{R}^{N+1}} \Psi(\xi) \rho(y, x, \xi, t) \partial_{\xi} m(x, \xi, t) \ dx d\xi =
\end{equation}
\begin{equation}
= -\int_{\mathbb{R}^{N+1}} \Psi'(\xi) \rho(y, x, \xi, t) m(x, \xi, t) \ dx d\xi
\end{equation}
\begin{equation}
+ \int_{\mathbb{R}^{N+1}} \Psi(\xi)(\sum_{i=1}^{N} \partial_{x_i} \rho(y, x, \xi, t)a_i(\xi) B_i(t)) \ m(x, \xi, t) \ dx d\xi
\end{equation}
and all the terms make sense as continuous functions tested against a measure.

Some (new) estimates and identities, needed for the proof of the main results of this section and derived from \((9.20)\), are stated next. Here \( \delta \) denotes the Dirac mass at the origin.

**Proposition 9.2.** Assume \((9.2)\) and \( u_0 \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^N) \). Then, for all \( t > 0 \),
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^{N+1}} |\chi(x, \xi, t)| d\xi \ dx = -2 \int_{\mathbb{R}^N} m(x, 0, t) dx,
\end{equation}
and
\begin{equation}
\int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}^{2N}} \delta(\xi - u(z, t)) \rho(y, z, \xi, t) \rho(y, x, \xi, t) \ m(t, x, \xi) dxdydzd\xi
\end{equation}
\begin{equation}
= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^{N+1}} [\int_{\mathbb{R}^N} \chi(x, \xi, t) \rho(y, x, \xi, t) dx] ^2 - |\chi(y, \xi, t)| |dyd\xi|.
\end{equation}

**Proof.** The first identity is obtained (see \([71, 72]\)) from multiplying \((9.1)\) by \( \text{sign}(\xi) \) and using that the fact that \( \text{sign}(\xi)\chi(x, \xi, t) = |\chi(x, \xi, t)| \). Notice that taking the value \( \xi = 0 \) in \( m \) is allowed by the Lipschitz regularity in Proposition \((9.1)\).
The proof of (9.22) uses the regularization kernel along the characteristics (9.17). Indeed, (9.20) and the fact that \( \chi \left( z, \xi, t \right) = \delta (\xi) - \delta (\xi - u (z, t)) \) yield

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^{N+1}} \left( \int_{\mathbb{R}^N} \chi (x, \xi, t) \rho (y, x, \xi, t) dx \right)^2 dyd\xi
= \int_{\mathbb{R}^{N+1}} \left[ \int_{\mathbb{R}^N} \chi (z, \xi, t) \rho (y, z, \xi, t) dz \int_{\mathbb{R}^N} \rho (y, x, \xi, t) \partial_\xi m (x, \xi, t) dx \right] dyd\xi
\]

(9.23)

\[
= - \int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}^N} \left[ \delta (\xi) - \delta (\xi - u (z, t)) \right] \rho (y, z, \xi, t) \rho (y, x, \xi, t) m (x, \xi, t) dzdx dyd\xi
\]

\[
= - \int_{\mathbb{R}^N} m (x, 0, t) dx
+ \int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}^N} \delta (\xi - u (z, t)) \rho (y, z, \xi, t) \rho (y, x, \xi, t) m (x, \xi, t) dzdx dyd\xi.
\]

An important step in the calculation above is that, for all \( \xi \in \mathbb{R} \),

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi (z, \xi, t) [D_y \rho (y, z, \xi, t) \rho (y, x, \xi, t) + \rho (y, z, \xi, t) D_y \rho (y, x, \xi, t)] m (t, x, \xi) dzdx = 0,
\]

which follows from the observation that the integrand is an exact derivative with respect to \( y \).

Using (9.21) in (9.23) gives (9.22).

Dissipative solutions. The notion of dissipative solutions, which was introduced by Perthame and Souganidis [73], is equivalent to that of entropy solutions. The interest in them is twofold. Firstly, the definition resembles and enjoys the same flexibility as the one for viscosity solutions in, of course, the appropriate function space. Secondly, in defining them, it is not necessary to talk at all about entropies, shocks, etc.. Thirdly, and this was the motivation of [73], it provides a very convenient tool to prove asymptotic results.

It is said that \( u \in L^\infty ((0, T), (L^1 \cap L^\infty) (\mathbb{R}^N)) \) is a dissipative solution to (9.1), if, for all \( \Psi \in C ([0, \infty); C_c^\infty (\mathbb{R}^N)) \) and all \( \psi \in C_c^\infty (\mathbb{R}; [0, \infty)) \), where the subscript \( c \) means compactly supported, in the sense of distributions,

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi (k) (u - k - \Psi) \cdot dxdk \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi (k) \text{sign}_+ (u - k - \Psi) (-\Psi_t - \sum_{i=1}^N \partial_{x_i} (A_i (\Psi)) \cdot dB_i) dxdk.
\]

To provide an equivalent definition which will allow to go around the difficulties with inequalities mentioned earlier, it is necessary to take a small detour to recall the classical fact that, under the assumed regularity conditions on the flux and paths, for any \( \phi \in C_c^\infty (\mathbb{R}^N) \) and any \( t_0 > 0 \), there exists \( h > 0 \), which depends on \( \phi \), such that the problem

\[
(9.24) \quad d\Psi + \sum_{i=1}^N \partial_{x_i} (A_i (\Psi)) \cdot dB_i = 0 \quad \text{in} \quad \mathbb{R}^N \times (t_0 - h, t_0 + h) \quad \Psi = \phi \quad \text{on} \quad \mathbb{R}^N \times \{t_0\},
\]

has a smooth solution given by the method of characteristics.

It is left up to the reader to check that the definition of the dissipative solution is equivalent to saying that, for \( \phi \in C_c^\infty (\mathbb{R}^N) \), \( \psi \in C_c^\infty (\mathbb{R}; [0, \infty)) \) and any \( t_0 > 0 \), there exists \( h > 0 \), which depends on \( \phi \), such that, if \( \Psi \) and \( h > 0 \) are as in (9.24), then in the sense of distributions

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi (k) (u - k - \Psi) \cdot dxdk \leq 0 \quad \text{in} \quad (t_0 - h, t_0 + h).
\]
9.1. **Pathwise kinetic/entropy solutions.** Neither the notions of entropy and dissipative solutions nor the kinetic formulation can be used to study (9.1), since all involve either inequalities or quantities with sign which do not make sense.

The following definition is motivated by the theory of pathwise viscosity solutions.

**Definition 9.1.** Assume (9.2) and (9.3). Then \( u \in (L^1 \cap L^\infty)(Q_T) \) is a pathwise kinetic/entropy solution to (9.1), if there exists a nonnegative bounded measure \( m \) on \( \mathbb{R}^N \times \mathbb{R} \times (0, \infty) \) such that, for all test functions \( \rho \) given by (9.17) with \( \rho_0 \) satisfying (9.16), in the sense of distributions in \( \mathbb{R} \times (0, \infty) )

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \chi(x, \xi, t) \rho(y, x, \xi, t) dx = \int_{\mathbb{R}^N} \rho(y, x, \xi, t) \partial_\xi m(x, \xi, t) dx.
\]

The main result is:

**Theorem 9.1.** Assume (9.2), (9.3) and \( u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N) \). For all \( T > 0 \) there exists a unique pathwise entropy/kinetic solution \( u \in C([0, \infty); L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \) to (9.1) and (9.10), (9.11), (9.13), (9.14) and (9.15) hold. In addition, any pathwise entropy solutions \( u_1, u_2 \in C([0, \infty); L^1(\mathbb{R}^N)) \) to (9.1) satisfy, for all \( t > 0 \), the contraction property

\[
\|u_2(\cdot, t) - u_1(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_2(\cdot, 0) - u_1(\cdot, 0)\|_{L^1(\mathbb{R}^N)}.
\]

Moreover, there exists a uniform constant \( C > 0 \) such that, if, for \( i = 1, 2 \), \( u_i \) is the pathwise entropy/kinetic solution to (9.1) with path \( B_i \) and \( u_{i,0} \in BV(\mathbb{R}^N) \), then \( u_1 \) and \( u_2 \) satisfy, for all \( t > 0 \), the contraction’ property

\[
\|u_2(\cdot, t) - u_1(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_{2,0} - u_{1,0}\|_{L^1(\mathbb{R}^N)} + C[|a|^1_{BV(\mathbb{R}^N)} + |u_{1,0}|_{BV(\mathbb{R}^N)})](B_1 - B_2)(t) + (\sup_{s \in [0, t]} |(B_1 - B_2)(s)| |a|^1_{L^2(\mathbb{R}^N)})^{1/2}.
\]

Looking carefully into the proof of the (9.27) for smooth paths, it is possible to establish, after some approximations, an estimate similar to (9.27), for non BV-data, with a rate that depends on the modulus of continuity in \( L^1 \) of the initial data. It is also possible to obtain an error estimate for different fluxes. The details for both are left to the interested reader.

**Estimates for regular paths.** Following ideas from the earlier parts of the notes, the solution operator of (9.1) may be thought of as the unique extension of the solution operators of (9.1) with regular paths. It is therefore necessary to study first (9.1) with smooth paths and to obtain estimates that allow to prove that the solutions corresponding to any regularization of the same path converge to the same limit, which is a pathwise entropy/kinetic solution. The intrinsic uniqueness for the latter is proved later.

The key step is a new estimate, which depends only on the sup-norm of \( B \) and yields compactness with respect to time.

**Theorem 9.2.** Assume (9.2) and, for \( i = 1, 2 \), \( u_{i,0} \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^N) \). Consider two smooth paths \( B_1 \) and \( B_2 \) and the corresponding solutions \( u_1 \) and \( u_2 \) to (9.1). There exists a uniform constant \( C > 0 \) such that, for all \( t > 0 \), (9.27) holds.
The proof of Theorem 9.2, which is long and technical, can be found in [49]. It combines the uniqueness proof for scalar conservation laws based on the kinetic formulation of [71, 72] and the regularization method along the characteristics introduced for Hamilton-Jacobi equations in [57, 58, 59, 60, 54].

The proof of Theorem 9.1. The existence of a pathwise kinetic/entropy solution follows easily. Indeed, the estimate of Theorem 9.2 implies that, for every \( u_0 \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^N) \) and for every \( T > 0 \), the mapping \( B \in C([0,T]; \mathbb{R}^N) \mapsto u \in C([0,T]; L^1(\mathbb{R}^N)) \) is well defined and uniformly continuous with the respect to the norm of \( C([0,T]; \mathbb{R}^N) \). Therefore, by density, it has a unique extension to \( C([0,T]) \). Passing to the limit gives the contraction properties (9.26) and (9.27) as well as (9.1). Once (9.26) is available for initial data in \( BV(\mathbb{R}^N) \), the extension to general data is immediate by density.

The next step is to show that pathwise kinetic/entropy satisfying (9.1) are intrinsically unique in a stronger sense. The contraction property only proves uniqueness of the solution built by the above regularization process. It is, however, possible to prove that (9.25) implies uniqueness. Indeed, for \( BV \)-data, the estimates in the proof of Theorem 9.2 only use the equality of Definition 9.1. From there the only nonlinear manipulation needed is to check that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^{N+1}} \left( \int_{\mathbb{R}^N} \chi(x,\xi,t) \rho(y,x,\xi,t) \, dx \right)^2 \, dt = \int_{\mathbb{R}^{N+1}} \left( \int_{\mathbb{R}^N} \chi(x,\xi,t) \rho(y,x,\xi,t) \, dx \right) \frac{d}{dt} \int_{\mathbb{R}^{N+1}} \left( \int_{\mathbb{R}^N} \chi(x,\xi,t) \rho(y,x,\xi,t) \, dx \right).
\]

This is justified after time regularization by convolution because it has been assumed that solutions belong to \( C([0,T]; L^1(\mathbb{R}^N)) \) for all \( T > 0 \). This fact also allows to justify that the right hand side

\[
\int_{\mathbb{R}^{N+1}} \left( \int_{\mathbb{R}^N} \chi(x,\xi,t) \rho(y,x,\xi,t) \, dx \right) \int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}^N} \chi(z,\xi,t) \rho(y,z,\xi,t) \, dz \int_{\mathbb{R}^N} \rho(y,x,\xi,t) \partial_x \chi(x,\xi,t) \, dx
\]

can be analyzed by a usual integration by parts, because it is possible to add a convolution in \( \xi \) before forming the square. All these technicalities are standard and have been detailed in [71, 72]. The uniqueness for general data requires one more layer of approximation.

The semilinear problem. Based on the results of Section 4, it is natural to expect that the approach developed earlier will also be applicable to the semilinear problem (9.4) to yield a pathwise theory of stochastic entropy solutions. It turns out, however, that, as explained next, this is not the case.

To keep things simple, here it is assumed that \( N = 1 \), \( B = t \) and \( \tilde{B} \in C([0,\infty); \mathbb{R}) \) is a single rough path. Consider, for \( \Phi \in C^2(\mathbb{R}; \mathbb{R}) \), the problem

\[
(9.28) \quad du + \text{div} A(u) \, dt = \Phi(u) \cdot dB \quad \text{in} \quad Q_T \quad u = u_0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}.
\]

Following the earlier considerations as well as the analogous problem for Hamilton-Jacobi equations described in Section 4, it is assumed that, for each \( v \in \mathbb{R} \) and \( T > 0 \), the initial value problem

\[
(9.29) \quad d\Psi = \Phi(\Psi) \cdot d\tilde{B} \quad \text{in} \quad (0,\infty), \quad \Psi(0) = v,
\]

has a unique solution

\[
(9.30) \quad \Psi(v; \cdot) \in C([0,T]; \mathbb{R}) \quad \text{such that, for all} \quad t \in [0,T], \quad \Psi(\cdot, t) \in C^1(\mathbb{R}; \mathbb{R}).
\]
Following Section 4, to study (9.28) it is natural to consider a change of unknown given by the
Doss-Sussman-type transformation
(9.31) \[ u(x, t) := \Psi(v(x, t), t). \]
Assuming for a moment that \( \tilde{B} \) and, hence, \( \Psi \) are smooth with respect to \( t \) and (9.28) and (9.29)
have classical solutions, it follows, after a straightforward calculation, that
(9.32) \[ v_t + \text{div} \tilde{A}(v, t) = 0 \quad \text{in} \quad Q_T, \quad v = u_0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}, \]
where \( \tilde{A} \in C^{1,0}(\mathbb{R} \times [0, T]) \) is given by \( \tilde{A}'(v, t) = A'((v, t)). \)
Under the above assumptions on the flux and the forcing term, the theory of entropy solutions
toscalar conservation laws applies to (9.32) and yields the existence of a unique entropy solution.
Hence, exactly as in Section 4, it is tempting to define
(9.33) \[ u \in (L^1 \cap L^\infty)(\mathbb{R}^N \times (0, T)), \]
and, hence, considering the approximate equation
\[ u_t + \text{div} \tilde{A}(u) = \Phi(u) \cdot dB + \nu \Delta u, \]
and, after the transformation (9.31), the problem
\[ v_t + \langle a(\psi(x, t)), Dv \rangle > \nu \frac{\Delta \psi(x, t)}{\Psi_v(\psi(x, t), t)} = \nu \Delta v + \nu((\psi_v^{-1})(v(x, t), t)|Dv|^2). \]
If the approach based on (9.31) were correct, one would expect to get, after letting \( \nu \to 0 \), (rigorously) (9.32). This, however, does not seem to be the case due to the lack of the necessary a priori bounds to pass in the limit.
The problem is, however, not just a technicality but something deeper. Indeed the transformation
(9.31) does not, in general, preserve the shocks unless, as an easy calculation shows, the forcing is linear.
Assume that \( N = 1, B(t) = t, \) let \( H \) be the Heaviside step function and consider the semilinear
Burger’s equation
(9.33) \[ u_t + \frac{1}{2}(u^2)_x = \Phi(u) \quad \text{in} \quad Q_T \quad u_0(\cdot, 0) = H \quad \text{pn} \quad \mathbb{R}, \]
with \( \Phi \) such that
(9.34) \[ \Phi(0) = 0, \quad \Phi(1) = 0, \quad \text{and} \quad \Phi(u) > 0 \quad \text{for} \quad u \in (0, 1). \]
It is easily seen that the entropy solution to (9.33) is
\[ u(x, t) = \begin{cases} 1 \quad \text{for} \quad x < t/2, \\ 0 \quad \text{for} \quad x > t/2. \end{cases} \]
Next consider the transformation \( u = \Psi(v, t) \) with \( \Psi(v; t) = \Phi(\Psi(v; t)), \quad \Psi(v; 0) = v. \)
Since, in view of (9.31), \( \Psi(0; t) = \Psi(1; t) \equiv 1, \) and \( \Psi(v; t) > v \) for \( v \in (0, 1), \) it follows that the
flux for the equation for \( v \) is
\[ \tilde{A}(v, t) = \int_0^v \Psi(w; t) dw, \]
and the entropy solution with initial data \( u_0 \) is \( u(x,t) = H(x - \bar{x}(t)) \) with the Rankine-Hugoniot condition

\[
\dot{\bar{x}}(t) = \int_{0}^{1} \Psi(w; t) dw > \int_{0}^{1} w dw = \frac{1}{2},
\]

which shows that the shock waves are not preserved.

The final point is that, when \( B \) is a Brownian path, it is more natural to consider contractions in \( L^1(\mathbb{R}^N \times \Omega) \) instead of \( L^1(\mathbb{R}^N) \) a.s. in \( \omega \) for (9.4). To fix the ideas take \( A = 0 \) and \( B \) a Brownian motion and consider the stochastic initial value problem

\[
(9.35) \quad du = \Phi(u) \circ dB \quad \text{in} \quad (0, \infty), \quad u(\cdot,0) = u_0.
\]

If \( u_1, u_2 \) are solutions to (9.35) with initial data \( u_{1,0}, u_{2,0} \) respectively, then, subtracting the two equations, multiplying by \( \text{sign}(u_1 - u_2) \), taking expectations and using Itô’s calculus, gives, for some \( C > 0 \) depending on bounds on \( \Phi \) and its derivatives,

\[
E \int |u_1(\cdot, t) - u_2(\cdot, t)| \leq \exp(Ct)E \int |u_0^1 - u_0^2|,
\]

while it is not possible, in general, to get an almost sure inequality on \( \int |u_1(\cdot, t; \omega) - u_2(\cdot, t, \omega)| \).

**Appendix A. Some facts from the theory of viscosity solutions in the deterministic setting**

This is a summary of several facts about the theory of viscosity solutions of Hamilton-Jacobi equations that are used in these notes. At several places, an attempt is made to motivate the definitions and the arguments. This review is very limited in scope. Good references are the books by Bardi and Capuzzo-Dolceta [2], Barles [3], Fleming and Soner [23], the CIME notes [3] and the “User’s Guide” by Crandall, Ishii and Lions [13].

Consider the initial value problem

\[
(A.1) \quad u_t = H(Du, x) \quad \text{in} \quad Q_T \quad u(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^N.
\]

The classical method of characteristics yields, for smooth \( H \) and \( u_0 \), short time smooth solutions to (A.1). Indeed, assume that \( H, u_0 \in C^2 \). The characteristics associated with (A.1) are the solutions to the system of odes

\[
(A.2) \quad \dot{X} = -D_pH(P, X), \quad \dot{P} = D_xH(P, X), \quad \dot{U} = H(P, X) - < D_pH(P, X), P >
\]

with initial conditions

\[
(A.3) \quad X(x,0) = x, \quad P(x,0) = Du_0(x) \quad \text{and} \quad U(x,0) = u_0(x).
\]

The connection between (A.1) and (A.2) and (A.3) is made through the relationship

\[
U(t) = u(X(x,t), t) \quad \text{and} \quad P(t) = -Du(X(x,t), t).
\]

The issue is then the invertibility, with respect to \( x \), of the map \( x \mapsto X(x,t) \). A simple calculation involving the Jacobian of \( X \) shows that \( x \mapsto X(t,x) \) is a diffeomorphism in \((-T^*, T^*)\) with

\[
T^* = \left( \| D^2H \| \| D^2u_0 \| \right)^{-1}.
\]

Passing next to the issues of the definition and well-posedness of weak solutions, to keep the ideas simple, it is convenient to consider the two simple problems

\[
(A.4) \quad u_t = H(Du) \dot{B} \quad \text{in} \quad Q_T \quad u(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^N,
\]

\[
56 \quad \text{PANAGIOTIS E. SOUGANIDIS}
\]
and
\[(A.5) \quad u + H(Du) = f \text{ in } \mathbb{R}^N, \]
and to assume that \(H \in C(\mathbb{R}^N), \ B \in C^1(\mathbb{R})\) and \(f \in BUC(\mathbb{R}^N).\)

Nonlinear first-order equations do not have in general smooth solutions. This can be easily seen with explicit examples. On the other hand, it is natural to expect, in view of the many applications, like control theory, front propagation, etc., that global, not necessarily smooth solutions, must exist for all time and must satisfy a comparison principle. For \((A.4)\) this will mean that if \(u_0 \leq v_0,\) then \(u(\cdot, t) \leq v(\cdot, t)\) for all \(t > 0\) and, for \((A.5),\) if \(f \leq g,\) then \(u \leq v.\)

To motivate the definition of the viscosity solutions it is useful to proceed, in a formal way, to prove this comparison principle.

Beginning with \((A.5),\) it is assumed that, for \(i = 1, 2,\ u_i\) solves \((A.5)\) with right hand side \(f_i.\) To avoid further technicalities, it is further assumed that the \(u_i\)'s and \(f_i\)'s are periodic in the unit cube.

The "classical" proof consists of looking at \(\max(u_1 - u_2)\) which, in view of the assumed periodicity, is attained at some \(x_0 \in \mathbb{R}^N,\) that is,

\[
(u_1 - u_2)(x_0) = \max(u_1 - u_2).
\]

If both \(u_1\) and \(u_2\) are differentiable at \(x_0,\) then \(Du_1(x_0) = Du_2(x_0),\) and then it follows from the equations that

\[
(u_1 - u_2)(x_0) \leq (f_1 - f_2)(x_0).
\]

Observe that, to prove that \(u_1 \leq u_2,\) it is enough to have that

\[
u_1 + H(Du_1) \leq f_1 \quad \text{and} \quad u_2 + H(Du_2) \geq f_2,
\]

that is, it suffices for \(u_1\) and \(u_2\) to be respectively a subsolution and a supersolution.

Turning now to \((A.4),\) it is again assumed that the data is periodic in space. If, for \(i = 1, 2,\ u_i\) solves \((A.4)\) and \(u_1(\cdot, 0) \leq u_2(\cdot, 0),\) the aim is to show that, for all \(t > 0,\) \(u_1(\cdot, t) \leq u_2(\cdot, t).\)

Fix \(\delta > 0\) and let \((x_0, t_0)\) be such that

\[
(u_1 - u_2)(x_0, t_0) - \delta t_0 = \max_{(x,t) \in \mathbb{R}^N \times [0,T]} (u_1(x, t) - u_2(x, t) - \delta t).
\]

If \(t_0 \in (0, T)\) and \(u_1, u_2\) are differentiable at \((x_0, t_0),\) then

\[
Du_1(x_0, t_0) = Du_2(x_0, t_0) \quad \text{and} \quad u_{1,t}(x_0, t_0) \geq u_{2,t}(x_0, t_0) + \delta.
\]

Since evaluating the equations at \((x_0, t_0)\) yields a contradiction, it must be that \(t_0 = 0,\) and, hence,

\[
\max_{(x,t) \in \mathbb{R}^N \times [0,T]} ((u_1 - u_2)(x, t) - \delta t) \leq \max_{\mathbb{R}^N} (u_1(\cdot, 0) - u_2(\cdot, 0)) \leq 0.
\]

Letting \(\delta \to 0\) leads to the desired conclusion.

The previous arguments, of course, use strongly the fact that \(u_1\) and \(u_2\) are both differentiable at the maximum of \(u_1 - u_2,\) which is not the case in general. This is a major difficulty that is overcome using the notion of viscosity solution, which relaxes the need to have differentiable solutions.

The definition of the viscosity solutions for the general problems

\[(A.6) \quad u_t = F(D^2u, Du, u, x, t) \quad \text{in} \quad U \times (0, T),\]
Indeed, if, for a test function $\phi$ (resp. $u \in C(U \times (0, T))$) (resp. $u \in C(U)$) is a viscosity subsolution of (A.6) (resp. (A.7)), if, for all smooth test functions of $u - \phi$ and all maximum points $(x_0, t_0) \in U \times (0, T)$ (resp. $x_0 \in U$) of $u - \phi$

$$\phi_t + F(D^2 \phi, D\phi, u, x_0, t_0) \leq 0, \quad (\text{resp. } F(D^2 \phi, D\phi, u, x_0) \leq 0.)$$

(ii) $u \in C(U \times (0, T))$ (resp. $u \in C(U)$) is a viscosity supersolution of (A.6) (resp. (A.7)) if, for all smooth test functions $\phi$ and all minimum points $(x_0, t_0) \in U \times (0, T)$ (resp. $x_0 \in U$) of $u - \phi$,

$$\phi_t + F(D^2 \phi, D\phi, u, x_0, t_0) \geq 0, \quad (\text{resp. } F(D^2 \phi, D\phi, u, x_0) \geq 0.)$$

(iii) $u \in C(U \times (0, T))$ (resp. $u \in C(U)$) is a viscosity solution of (A.6) (resp. (A.7)) if it is both a sub- and super-solution of (A.6) (resp. (A.7)).

In the above definition, maxima (resp. minima) can be either global or local. Moreover, $\phi$ may have any regularity, $C^1$ being the minimum for first-order and $C^{2,1}$ for second-order equations.

Using the definition of viscosity solution, it is possible to make the previous heuristic proof rigorous and to show the well-posedness of the solutions.

A general comparison result for (A.4) and (A.5) is stated and proved next.

**Theorem A.1.** (i) Assume $H \in C(\mathbb{R}^N), f, g \in \text{BUC}(\mathbb{R}^N)$ and let $u, v \in \text{BUC}(\mathbb{R}^N)$ be respectively viscosity sub- and super-solutions of (A.5) with right hand side $f$ and $g$ respectively. Then $\sup_{\mathbb{R}^N}(u - v)_+ \leq \sup_{\mathbb{R}^N}(f - g)_+$.

(ii) Assume $H \in \mathbb{R}^N, B \in C^1(\mathbb{R})$, $u_0, v_0 \in \text{BUC}(\mathbb{R}^N)$ and let $u, v \in \text{BUC}(\overline{Q}_T)$ be respectively viscosity sub- and super-solutions to (A.4) with initial data $u_0$ and $v_0$ respectively. Then, for all $t \in [0, T], \sup_{\mathbb{R}^N}(u(\cdot, t) - v(\cdot, t))_+ \leq \sup_{\mathbb{R}^N}(u_0 - v_0)_+$.

**Proof.** To simplify the argument it is assumed throughout the proof that $f, g, u_0, v_0, u$ and $v$ are periodic in the unit cube. This assumption guarantees that all suprema in the statement are actually achieved and are therefore maxima. The general result is proved by introducing appropriate penalization at infinity, i.e., considering, in the case of (A.5), for example, $\sup(u(x) - v(x) - \alpha|x|^2)$ and then letting $\alpha \to 0$; see [3] and [4] for all the arguments and variations.

Consider first (A.5). The key technical step is to double the variables by introducing the new function $z(x, y) = u(x) - v(y)$ which solves the doubled equation

(A.8) $z + H(D_x z) - H(-D_y z) \leq f(x) - g(y)$.

Indeed if, for a test function $\phi, z - \phi$ attains a maximum at $(x_0, y_0)$, then $u(x) - \phi(x, y_0)$ and $v(y) + \phi(x_0, y_0)$ attain respectively a maximum at $x_0$, and a minimum at $y_0$. Therefore

$$u(x_0) + H(D_x \phi(x_0, y_0)) \leq f(x_0) \quad \text{and} \quad v(y_0) + H(-D_y \phi(x_0, y_0)) \geq f(y_0),$$

and the claim follows by subtracting these two inequalities.

To prove the comparison result, $z$ is compared with a smooth function, which is “almost” a solution, that is, in the case at hand, a function of $x - y$. 

(A.7) $F(D^2 u, Du, u, x) = 0$ in $U$, where $U$ is an open subset of $\mathbb{R}^N$, is introduced next.
It turns out that the most convenient choice is, for an appropriate \( a_\varepsilon \),
\[
\phi_\varepsilon(x, y) = \frac{1}{2\varepsilon} |x - y|^2 + a_\varepsilon.
\]
Indeed
\[
\phi_\varepsilon + H(D_x \phi_\varepsilon)) - H(-D_y \phi_\varepsilon) - (f(x) - g(y)) = \frac{1}{2\varepsilon} |x - y|^2 + a_\varepsilon - (f(x) - g(y)) \geq 0,
\]
if
\[
a_\varepsilon = \max(f - g) + \nu_\varepsilon \quad \text{and} \quad \nu_\varepsilon = \max(g(x) - g(y) - \frac{1}{2\varepsilon} |x - y|^2);
\]
note that, since \( g \) is uniformly continuous, \( \lim_{\varepsilon \to 0}\nu_\varepsilon = 0 \).

Let \((x_\varepsilon, y_\varepsilon)\) be such that
\[
z(x_\varepsilon, y_\varepsilon) - \phi(x_\varepsilon, y_\varepsilon) = \max_{\mathbb{R}^N \times \mathbb{R}^N} (z - \phi).
\]
Then
\[
z(x_\varepsilon, y_\varepsilon) + H(D_x \phi(x_\varepsilon, y_\varepsilon)) - H(-D_y \phi(x_\varepsilon, y_\varepsilon)) \leq f(x_\varepsilon) - g(y_\varepsilon).
\]
On the other hand, it is known that
\[
\phi(x_\varepsilon, y_\varepsilon) + H(D_x \phi(x_\varepsilon, y_\varepsilon)) - H(-D_y \phi(x_\varepsilon, y_\varepsilon)) \geq f(x_\varepsilon) - g(y_\varepsilon).
\]
It follows that \( z(x_\varepsilon, y_\varepsilon) \leq \phi(x_\varepsilon, y_\varepsilon) \) and, hence,
\[
z \leq \phi \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^N.
\]
Letting \( x = y \) in the above inequality yields
\[
u(x) - v(x) \leq \phi(x, x) = a_\varepsilon = \max(f - g) + \nu_\varepsilon
\]
and, after sending \( \varepsilon \to 0 \),
\[
\max(u - v) \leq \max(f - g).
\]
The comparison for (A.4) is proved similarly. In the course of the proof, however, it is not necessary to double the \( t \)-variable, since the equation is linear in the time derivative. This fact plays an important role in the analysis of the pathwise pde when \( B \) is merely continuous.

To this end, define the function
\[
z(x, y, t, s) = u(x, t) - v(y, s),
\]
and observe, as before, that
\[
z_t - s_t \leq H(D_x z)\dot{B}(t) - H(-D_y z)\dot{B}(s) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty) \times (0, \infty).
\]
On the other hand, it is possible to show that \( z(x, y, t) = u(x, t) - v(y, t) \) actually satisfies
\[
z_t \leq (H(D_x z) - H(-D_y z))\dot{B} \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty).
\]
Indeed, fix a smooth \( \phi \) and let \((x_0, y_0, t_0)\) be a (strict) local maximum of \((x, y, t) \mapsto z(x, y, t) - \phi(x, y, t)\). Since all functions are assumed to be periodic with respect to the spatial variable, the penalized function
\[
u(x, t) - v(y, s) - \phi(x, y, t) - \frac{1}{2\theta}(t - s)^2
\]
achieves a local maximum at \((x_0, y_0, t_0)\). It follows that, as \( \theta \to 0 \), \((x_\theta, y_\theta, t_\theta) \to (x_0, y_0, t_0)\).

Applying the definition to the function \( u(x, t) - v(y, s) \) gives at \((x_\theta, y_\theta, t_\theta, s_\theta)\)
\[
\phi_t + \frac{1}{\theta}(t_\theta - s_\theta) - \frac{1}{\theta}(t_\theta - s_\theta) \leq H(D_x \phi)\dot{B}(t_\theta) - H(-D_y \phi)\dot{B}(s_\theta)
\]
and, after letting $\theta \to 0$ and using the assumption that $B \in C^1$, at $(x_0, y_0, t_0)$,

$$\phi_t \leq (H(D_x\phi) - H(-D_y\phi))\dot{B}.$$  

Since

$$\phi_\varepsilon(x, y) = \frac{1}{2\varepsilon}|x - y|^2$$

is a smooth super-solution to (A.9), it follows immediately, after repeating an earlier argument, that

$$u(x, t) - v(y, t) \leq \frac{1}{2\varepsilon}|x - y|^2 + \max_{x,y \in \mathbb{R}^N} (u(x, 0) - v(y, 0) - \frac{1}{2\varepsilon}|x - y|^2)$$

and, after letting $\varepsilon \to 0$,

$$\max_{\mathbb{R}^N}(u(x, t) - v(x, t)) \leq \max_{\mathbb{R}^N}(u(x, 0) - v(x, 0)).$$

\[ \square \]

The next item in this review is the control interpretation of Hamilton-Jacobi equation. For simplicity here $\dot{B} \equiv 1$.

Consider the controlled system of ode

$$\dot{x}(t) = b(x(t), \alpha_t) \text{ in } (0, \infty) \quad x(0) = x \in \mathbb{R}^N,$$

where $b : \mathbb{R}^N \times A \to \mathbb{R}^N$ is bounded and Lipschitz continuous with respect to $x$ uniformly in $\alpha$, $A$ is a compact (for simplicity) subset of $\mathbb{R}^M$ for some $M$, $t \mapsto \alpha_t \in A$ is the control, which is taken to be measurable, and $(x_t)_{t \geq 0}$ is the state variable.

The associated cost function is given by

$$J(x, t, \alpha) = \int_0^t f(x(s), \alpha_s)ds + u_0(x(t)),$$

where $u_0 \in BUC(\mathbb{R}^N)$ is the terminal cost and $f : \mathbb{R}^N \times A \to \mathbb{R}$ is the running cost, which is also assumed to be bounded and Lipschitz continuous with respect to $x$ uniformly in $\alpha$.

The goal is to minimize — one can, of course, consider maximization — the cost function $J$ over all possible controls. The value function is

\[ (A.10) \]

$$u(x, t) = \inf_{\alpha} J(x, t, \alpha).$$

The key tool to study $u$ is the dynamic programming principle, which is nothing more than the semigroup property. It states that, for any $\tau \in (0, t),

\[ (A.11) \]

$$u(x, t) = \inf_{\alpha} \left[ \int_0^\tau f(x(s), \alpha_s)ds + u(x(\tau), t - \tau) \right].$$

Its proof, which is straightforward, is based on the elementary observation that when pieced together, optimal controls and paths in $[a, b]$ and $[b, c]$ form an optimal path for $[a, c]$.

The following formal argument, which can be made rigorous using viscosity solutions and test functions shows the connection between the dynamic programming and the Hamilton-Jacobi equation. Using the dynamic programming identity, with $\tau = h$ small, yields

$$u(x, t) \approx \inf_{\alpha} (hf(x, \alpha) + D_x u(x, t) \cdot b(x, \alpha)h) + u(x, t) - u_t(x, t)h,$$
and, hence,
\[ u_t + \sup_{\alpha}[-D_x u \cdot b(x,\alpha) - f(x,\alpha)] = 0 , \]
that is
\[ u_t + H(Du, x) = 0 \text{ in } \mathbb{R}^N \times (0, \infty) , \]
where the (convex) Hamiltonian \( H \) is given by the formula
\[ H(p, x) = \sup_{\alpha}[-p \cdot b(x,\alpha) - f(x,\alpha)] . \]
When \( H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) is convex, the previous discussion provides a formula for the viscosity solution to the Hamilton-Jacobi equation
\[ u_t + H(Du, x) = 0 \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N. \] (A.12)
Indeed recall that
\[ H(p, x) = \sup_q \langle p, q \rangle - H^*(q, x) \]
and consider the controlled system
\[ \dot{x}(t) = q(t) \quad x(0) = x, \]
and the pay-off
\[ J(x, t, q, \cdot) = u_0(x(t)) + \int_0^t H^*(q(s), x(s))ds. \]
The theory of viscosity solutions (see [5], [44]) yields that
\[ u(x, t) = \inf_q [u_0(x(t)) + \int_0^t H^*(q(s), x(s))ds]. \] (A.13)
When \( H \) does not depend on \( x \), then (A.13) can be simplified considerably. Indeed, applying Jensen’s inequality to the representation formula (A.13) of the viscosity solution \( u \) of
\[ u_t + H(Du) = 0 \text{ in } Q_T \quad u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^N, \] (A.14)
where \( H(p) = \sup_q \langle p, q \rangle - H^*(q) \) yields the Lax-Oleinik formula
\[ u(x, t) = \inf_{y \in \mathbb{R}^N} \left[ u_0(y) + tH^*\left(\frac{x - y}{t}\right) \right] . \] (A.15)
A similar argument, when \( H \) is concave, yields
\[ u(x, t) = \sup_{y \in \mathbb{R}^N} \left[ u_0(y) - tH^*\left(\frac{x - y}{t}\right) \right] . \] (A.16)
The existence of viscosity solution follows either directly using Perron’s method, which yields the solution as the maximal (resp. minimal) subsolution (resp. supersolution) or indirectly by considering regularizations of the equation, the most commonly used consisting of “adding” \(-\varepsilon \Delta u^\varepsilon \) to the equation and passing to the limit \( \varepsilon \to 0 \).
The results discussed earlier yield that there exists a unique solution \( u \in BUC(Q_T) \). In particular, \( u = S_H(t)u_0 \), with the solution operator \( S_H(t) : BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N) \) a strongly continuous semigroup, that is, for \( s, t > 0 \),
\[ S_H(t + s) = S_H(t)S_H(s) . \]
The time homogeneity of the equation also yields, for \( t > 0 \), the identity

\[ S_H(t) = S_H(1). \]

Moreover, \( S_H \) commutes with translations, additions of constants and is order-preserving, and, hence, a contraction in the sup-norm, that is,

\[ \| (S_H(t)u - S_H(t)v) \|_\infty \leq \| u - v \|_\infty. \]

If \( u_0 \in C^{0,1}(\mathbb{R}^N) \), the space homogeneity of \( H \) and the contraction property yield, that, for all \( t > 0 \), \( S_H(t)u \in C^{0,1}(\mathbb{R}^N) \) and, moreover,

\[ \| D S_H(t)u_0 \| \leq \| Du_0 \|. \]

It also follows from the order preserving property that, for all \( u, v \in BUC(\mathbb{R}^N) \) and \( t > 0 \),

\[ S_H(t) \max(u, v) \geq \max(S_H(t)u, S_H(t)v) \quad \text{and} \quad S_H(t) \min(u, v) \leq \min(S_H(t)u, S_H(t)v). \]

Finally, it can be easily seen from the definition of viscosity solutions that, if, for \( i \in I \), \( u_i \) is a sub- (resp. super-solution), then \( \sup_i u_i \) is a sub- (resp. inf \( i u_i \) a super-solution).

A natural question is whether there are any other explicit formulae for the solutions to \((A.14)\); recall that for \( H \) convex/concave, the solutions satisfy the Lax-Oleinik formula.

It turns out there exists another formula, known as the Hopf formula, which does not require \( H \) to have any concavity/convexity property as long as the initial datum is convex/concave.

For definiteness, here it is assumed that \( u_0 \) is convex, and denote by \( u_0^* \) its Legendre transform.

It is immediate that, for any \( p \in \mathbb{R}^N \), the function \( u_p(x,t) = (p,x) + tH(p) \) is a viscosity solution to \((A.14)\) and, hence, in view of the previous discussion,

\[
(A.17) \quad v(x,t) = \sup_{p \in \mathbb{R}^N} \left[ (p,x) + tH(p) - u_0^*(p) \right],
\]

is a subsolution to \((A.14)\).

The claim is that, if \( u_0 \) is convex, then \( v \) is actually a solution. Since this fact plays an important role in the analysis, it is stated as a separate proposition. For completeness, its proof is also discussed here; the result was first shown in \([53]\).

**Proposition A.1.** Let \( H \in C(\mathbb{R}^N) \) and assume that \( u_0 \in BUC(\mathbb{R}^N) \) is convex. The unique viscosity solution \( u \in BUC(Q_T) \) of \((A.14)\) is given by \((A.17)\).

**Proof.** If \( H \) is either convex or concave, the claim follows using the Lax-Oleinik formula. Assume for example that \( H \) is convex. Then

\[
\sup_{p \in \mathbb{R}^N} \left[ \langle p, x \rangle + tH(p) - u_0^*(p) \right] = \sup_{p \in \mathbb{R}^N} \left[ \langle p, x \rangle + t \sup_{q \in \mathbb{R}^N} ((p,q) - H^*(q)) - u_0^*(p) \right] \\
= \sup_{p \in \mathbb{R}^N} \sup_{q \in \mathbb{R}^N} \left[ \langle p, x \rangle + t(p,q) - tH^*(q) - u_0^*(p) \right] \\
= \sup_{q \in \mathbb{R}^N} \sup_{p \in \mathbb{R}^N} \left[ \langle p, x + tq \rangle - u_0^*(p) - tH^*(q) \right] \\
= \sup_{q \in \mathbb{R}^N} \left[ u_0(x + tq) - tH^*(q) \right] = \sup_{y \in \mathbb{R}^N} \left[ u_0(y) - tH^*(\frac{y-x}{t}) \right].
\]
If $H$ is concave, the argument is similar, provided the min-max theorem is used to interchange the sup and inf that appear in the formula.

For the general case the first step is that the map $F(t): BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ defined by

$$F(t)u_0(x) = \sup_{p \in \mathbb{R}^N} [<p, x > tH(p) - u_0^*(p)]$$

has the semi-group property, that is, $F(t+s) = F(t)F(s)$.

If $u_0$ is convex, then $F(t)u_0$ is also convex, since it is the sup of linear functions, and, moreover,

$$F(t)u_0 = (u_0^* - tH)^*.$$

In view of this observation and the fact that, for $w$ convex, $w = w^{**}$, the semigroup identity follows if it shown that

$$(u_0^* - (t + s)H)^* = ((u_0^* - sH)^* - tH)^*.$$

On the other hand, the definition of the Legendre transform, the min-max theorems and the fact that

$$\sup_{x \in \mathbb{R}^N} < z, x > = \begin{cases} +\infty & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

yield, for $\tau > 0$,

$$(u_0^* - \tau H)^*(y) = \sup_{x \in \mathbb{R}^N} [<y, x > - (u_0^* - \tau H)^*(x)] = \sup_{x \in \mathbb{R}^N} [<y, x > - \sup_{p \in \mathbb{R}^N} [<x, p > + \tau H(p) - u_0^*(p)]]$$

$$= \sup_{x \in \mathbb{R}^N} \inf_{p \in \mathbb{R}^N} [<y - p, x > - \tau H(p) + u_0^*(p)]$$

$$= \inf_{p \in \mathbb{R}^N} \sup_{x \in \mathbb{R}^N} [<y - p, x > - \tau H(p) + u_0^*(p)] = u_0^*(y) - \tau H(p).$$

It follows that

$$(u_0^* - (t + s)H)^* = u_0^* - (t + s)H = u_0^* - sH - tH = (u_0^* - sH)^* - tH = ((u_0^* - sH)^* - tH)^*.$$

Next it is shown that actually (A.17) yields is a viscosity solution. In view of the previous discussion, it is only needed to check the super-solution property.

Assume that, for some smooth $\phi$, $v - \phi$ attains a minimum at $(x_0, t_0)$ with $t_0 > 0$. Let $p = D\phi(x_0, t_0)$ and $\lambda = \phi_t(x_0, t_0)$. The convexity of $v$ yields that, for all $(x, t)$ and $h \in (0, t_0)$,

$$v(x, t_0 - h) \geq v(x_0, t_0) + <p, x - x_0 > - \lambda h + o(h).$$

Since

$$v(x_0, t_0) = F(h)v(\cdot, t_0 - h)(x_0),$$

it follows that

$$v(x_0, t_0) = F(h)(v(x_0, t_0) + <p, \cdot - x_0 >)(x_0) - \lambda h + o(h),$$

and, finally,

$$\lambda h \geq hH(p) + o(h).$$

Dividing by $h$ and letting $h \to 0$ gives $\lambda \geq H(p)$. \qed
The above proof is a typical argument in the theory of viscosity solutions which has been used by Lions [45] to give a characterization of viscosity solutions and Souganidis [80] and Barles and Souganidis [6] to prove convergence of approximations to viscosity solutions. Similar arguments were also used by Lions [46] in image processing and Barles and Souganidis [7] to study front propagation.

It is a natural question to investigate whether the Hopf formula can be used for more general Hamilton-Jacobi equations with possible dependence on $(u, x)$.

A first requirement for such formula to hold is that the equation must preserve convexity, that is, if $u_0$ is convex, then $u(\cdot, t)$ must be convex for all $t > 0$.

It turns out that the general form of Hamiltonian’s satisfying this latter property is

$$H(p, u, x) = \sum_{j=1}^{N} x_j H_j(Du) + u H_0(Du) + G(Du).$$

To establish a Hopf-type formula, it is necessary to look at solutions starting with linear initial data, that is, for some $p \in \mathbb{R}^N$ and $a \in \mathbb{R}$,

$$u_0(x) = \langle p, x \rangle + a.$$

If there is a Hopf-type formula, the solution $u$ starting with $u_0$ as above must be of the form

$$u(x, t) = P(t)x + A(t) \quad \text{with} \quad A(0) = a \quad \text{and} \quad P(0) = p.$$

A straightforward computation yields that $P$ and $A$ must satisfy, for $H = (H_1, \ldots, H_N)$, the ode

$$\dot{P} = H(P) + H_0(P)P \quad \text{and} \quad \dot{A} = H_0(P)A + G(P).$$

Whether the function

$$\sup_{p \in \mathbb{R}^N} \left[ \langle P(t), x \rangle + A(t) \right]$$

with $A(0) = -u_0^*(p)$ is a solution to the Hamilton-Jacobi equation is an open question in general. Some special cases can be analyzed under additional assumptions on the $H_i$’s, $H_0$, etc..

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