Two-dimensional Stationary Wake Fields in Vortexfree Cold Plasma. I
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1 Introduction

In one dimensional model of wake field generation by a relativistic electron bunch in a cold plasma (the beam-generator is modelled by an infinite layer moving in the direction perpendicular to its surface) a number of interesting results are obtained. In particular, possibility of acceleration of electrons (positrons) by wake fields is shown with the acceleration rate proportional to $\gamma^{1/2} n_0^{-2}$ at $n \approx n_0$; (see also the references cited in [2]) possibility of generation of the fields exceeding the wave breaking limit by a special combination of bunches (generator-invertor-dumper) is shown in [3], etc.

However the obtained data require some refinement on the subject of finite transverse sizes of bunches, since the bunches of existing and designing accelerators and storage rings mainly have longitudinal geometry, or some times comparable longitudinal and transverse sizes.

In present paper master system of equations for two-dimensional finite charged relativistic bunch (of electrons or positrons) moving in cold, collisionless, vortex free, stationary plasma with immobile ions are formulated (see also [4]-[8]). The system of nonlinear equations in partial derivatives is then reduced to the form convinient for further numerical analysis.
To check the validity of obtained system of equations two limiting cases are considered - for infinite wide (transversal) bunch and for infinite long (longitudinal) bunch. Both cases admit exact analytical solutions, obtained formerly in [1], [2] and [4]. Considered limiting cases of the general system as one should expect, coincide with the main equations obtained in [2], [4]. The analysis of solutions of mentioned limiting cases from viewpoint of development of numerical algorithm for solution of the main general system of equations is also presented. Ranges of parameters values are determined, where non-stable or nonphysical ($n_e < 0$) solutions take place, when doing numerical calculations, in particular, at joining the solutions in space regions occupied by a limited bunch and that free from it, removal of abovementioned limitations on plasma properties, probably will promote to the clearing up the nature of these nonstable or nonphysical states.

The third obvious check of the validity of the numerical calculations is the passage to the linear limit, which is considered in a number of papers (see e.g. [5], [6], [8] and references therein).

The work is presented in the three parts. Present paper (part I) is devoted to formulation of the system of the master equations of the problem and analytical considerations of the limiting cases, which have a exact solutions [1]-[3] (transversal and longitudinal flat bunches, linear approximation). Presented consideration gives some hints to the construction of proper algorithm for numerical calculations.

The second part (II) will be devoted to the construction of the algorithm for numerical calculations of the system of basic equations of the problem for the general case of the bunch of finite transversal and longitudinal dimensions.

The third part (III) will present the results of numerical calculations, their analysis and comparison with the existing analytical results.
2 The Master Equations

The equation of motion of relativistic plasma electrons in electric $\vec{E}$ and magnetic $\vec{H}$ fields is written as:

$$\frac{\partial \vec{p}}{\partial t} + (\vec{v} \frac{\partial}{\partial \vec{r}})\vec{p} = -e\vec{E} - \frac{e}{c}[\vec{v} \times \vec{H}], \quad (1)$$

where $\vec{p}$ is momentum, $\vec{v}$ is the velocity of plasma electrons in lab system, the electron charge is set $-e$. Introducing scalar $\varphi$ and vector $\vec{A}$ potentials for electromagnetic fields through formulas

$$\vec{E} = -\text{grad} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{H} = \text{rot} \vec{A} \quad (2)$$

one can represent the equation (1) in the form:

$$\frac{\partial}{\partial t}(\vec{p} - \frac{e}{c}\vec{A}) = e\text{grad} \varphi - mc^2 \text{grad} \gamma - [\text{rot}(\vec{p} - \frac{e}{c}\vec{A}) \times \vec{v}], \quad (3)$$

where $\gamma$ is the Lorentz-factor of plasma electrons.

One can note, that the eq. (3) has a partial solution

$$\vec{p} = \frac{e}{c}\vec{A} + \text{grad} \chi, \quad (4)$$

$$mc^2 \gamma = e\varphi - \frac{\partial \chi}{\partial t}, \quad (5)$$

where $\chi$ is an arbitrary calibrating function. Since from (4) follows $\text{rot}(\vec{p} - \frac{e}{c}\vec{A}) = 0$, this solution corresponds to the vortex free motion of the plasma electrons considered in [7], [8], [4]. The eqs. (3)(4) along with Maxwell’s equations for the field potentials, with charges and currents corresponding to the motion in plasma bunch charges (which is supposed to be given—rigid bunch approximation) form a complete system of equations of the cold vortex free hydrodynamic plasma model, with immobile ions.

Later on we will work with dimensionless potentials

$$\vec{a} = \vec{A}/(mc^2/e), \quad (6)$$

$$f = \varphi/(mc^2/e) \quad (7)$$
and we will normalize the density of plasma electrons and bunch charges on plasma ions density $n_0$. Introduce also plasma frequency $\omega_p$ and wavelength $\lambda_p$

$$\omega_p = (4\pi e^2 n_0 / m)^{1/2}, \quad \lambda_p = c / \omega_p$$

we will use dimensionless coordinates (in units of $\lambda_p$) and dimensionless time (in units of $\omega_p^{-1}$); 4-momenta of plasma electrons represent in the form $(\epsilon, \vec{p} c) = m c^2 (\gamma, \vec{\beta} \gamma)$, Lorentz factor and velocity of bunch charges denote as $\gamma_0, \vec{\beta}_0 c$.

Let’s choose a calibrating function $\chi = 0$. As a result we obtain a full system of equations for the considered plasma-electron bunch model:

$$\gamma = f,$$  \hspace{1cm} (9)

$$\vec{\beta} \gamma = \vec{a},$$  \hspace{1cm} (10)

$$\Box f + \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial t} + \text{div} \vec{a} \right) = -1 + n_e + n_b,$$  \hspace{1cm} (11)

$$\Box \vec{a} - \text{grad} \left( \frac{\partial f}{\partial t} + \text{div} \vec{a} \right) = n_e \vec{\beta} + n_b \vec{\beta}_0,$$  \hspace{1cm} (12)

where $n_e$ is the plasma electrons density, $n_b$ is the bunch charges density (positive for electrons and negative for bunches consisting of positively charged particles).

As one can see from relations (9) and (10) components of 4-potential are connected on mass surface of plasma electrons:

$$f^2 = 1 + a_y^2 + a_z^2$$  \hspace{1cm} (13)

This follows from the selection of calibration function $\chi = 0$; i.e. (13) is a calibrating condition on potentials. Calibration of potentials (13) we will call ”energetic”.

From eqs. (11) and (12) also follows continuity equation for plasma electrons:

$$\text{div}(n_e \vec{\beta}) + \frac{\partial n_e}{\partial t} = 0$$  \hspace{1cm} (14)

We are interested in steady state wake field solutions of eqs. (8-12), i.e. only on those in which longitudinal dependence of all variables is determined by the longitudinal position of point of observation relative to the bunch. Assuming that the bunch propagates along
the axis z, suppose all the physical variables depending on combination $z - \beta_0 t$, which we also will denote as $z$. Let us consider the case when all the physical variables depend on transverse coordinate $y$ only (flat bunch with horizontal dimensions much more greater than vertical one). Then we obtain three scalar eqs. for three values $f, a_y, a_z$ and unknown plasma electrons density function $n_e$, obeying the continuity eq. (14):

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \beta_0 \frac{\partial a_y}{\partial z} = -1 + n_e + n_b,$$  \hspace{1cm} (15)

$$\beta_0 \frac{\partial^2 f}{\partial z \partial y} + (1 - \beta_0^2) \frac{\partial a_y}{\partial z} - \beta_0 \frac{\partial a_z}{\partial y} = n_e \frac{a_y}{f},$$  \hspace{1cm} (16)

$$\beta_0 \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_z}{\partial y \partial z} - \beta_0^2 \frac{\partial^2 a_z}{\partial z^2} = n_e \frac{a_z}{f} + n_b \beta_0,$$  \hspace{1cm} (17)

It is convenient to introduce the new unknown functions

$$v = f - \beta_0 a_z,$$  \hspace{1cm} (18)

$$u = \beta_0 f - a_z,$$

Then eqs (15-17) convert to the form:

$$\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} + (1 - \beta_0^2) \frac{\partial a_y}{\partial z} \right) = -\beta_0 + n_e (1 - \beta_0^2) \frac{u}{v - \beta u},$$  \hspace{1cm} (19)

$$\frac{\partial^2 v}{\partial y^2} + (1 - \beta_0^2) \frac{\partial^2 v}{\partial z^2} = -1 + n_b (1 - \beta_0^2) + n_e (1 - \beta_0^2) \frac{v}{v - \beta u},$$  \hspace{1cm} (20)

$$\frac{\partial}{\partial z} \left( \frac{\partial u}{\partial y} + (1 - \beta_0^2) \frac{\partial a_y}{\partial z} \right) = n_e (1 - \beta_0^2) \frac{a_y}{v} \frac{1}{v - \beta u},$$  \hspace{1cm} (21)

Denote the combination involved in both eqs. (19) and (21) by

$$\mu \equiv \frac{\partial u}{\partial y} + (1 - \beta_0^2) \frac{\partial a_y}{\partial z},$$  \hspace{1cm} (22)

one can symmetrize derivatives in (13-21) introducing "tilded" argument and variables:

$$\tilde{z} = z \gamma_0, \quad \tilde{v} = v \gamma_0, \quad \tilde{u} = u \gamma_0, \quad \tilde{\mu} = \mu \gamma_0$$  \hspace{1cm} (23)

Instead of $n_e$ introduce a modified electron density $N$ by the formula:

$$N = \frac{n_e}{\gamma_0 (v - \beta_0 \tilde{u})}$$  \hspace{1cm} (24)
The system of equations for variables $a_y, \tilde{u}, \tilde{v}, \tilde{\mu}, N$ describing considered plasma electron bunch model, is written as

\[
\frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial^2 \tilde{v}}{\partial z^2} = \gamma_0(-1 + n_b(1 - \beta_0^2)) + N\tilde{v}
\]  

(25)

\[
\frac{\partial \tilde{u}}{\partial y} + \frac{\partial a_y}{\partial z} = \tilde{\mu}
\]  

(26)

\[
\frac{\partial \tilde{\mu}}{\partial y} = -\beta_0 \gamma_0 + N\tilde{u}
\]  

(27)

\[
\frac{\partial \tilde{\mu}}{\partial z} = N a_y
\]  

(28)

\[
1 + a_y^2 + \tilde{u}^2 = \tilde{v}^2
\]  

(29)

The last equation is the condition of “energetic” calibration of potentials (13).

One can easily see that in this representation the continuity equation (14) is resulted from (27), (28) and has the form

\[
\frac{\partial}{\partial y}(N a_y) - \frac{\partial}{\partial z}(N\tilde{u}) = 0
\]  

(30)

At numeric modelling of the system of equations (25-29) one of eqs (27), (28) can be replaced by the eq. (30).

One can reduce eqs (25-29) to eqs. for $a_y, \tilde{v}, \tilde{\mu}$. To do so, one can express $N$ e.g. from (27) and define $\tilde{u}$ from (29). Eventually we obtain the system of equations

\[
a_y \frac{\partial a_y}{\partial y} - \sqrt{\tilde{v}^2 - 1 - a_y^2} \cdot \frac{\partial a_y}{\partial z} = \tilde{v} \frac{\partial \tilde{v}}{\partial y} - \sqrt{\tilde{v}^2 - 1 - a_y^2} \cdot \tilde{\mu}
\]  

(31)

\[
a_y \frac{\partial \tilde{\mu}}{\partial y} - \sqrt{\tilde{v}^2 - 1 - a_y^2} \frac{\partial \tilde{\mu}}{\partial z} = -\beta_0 \gamma_0 a_y
\]  

(32)

\[
\frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial^2 \tilde{v}}{\partial z^2} = n_b \gamma_0^{-1} - \gamma_0 + \frac{\tilde{v}}{\sqrt{\tilde{v}^2 - 1 - a_y^2}} \left(\beta_0 \gamma_0 + \frac{\partial \tilde{\mu}}{\partial y}\right)
\]  

(33)

Below we will use bunch models symmetrical with respect to $y$ axis. In this case one can easily note, that functions $a_y$ and $\tilde{\mu}$ are odd functions on $y$, $\tilde{v}$ is an even one.

Also note, that nondisturbed plasma (no charged particles bunches and wake fields) corresponds to the following values of variables:

\[
\tilde{v} = \gamma_0, \quad \tilde{u} = \beta_0 \gamma_0, \quad N = 1, \quad a_y = \tilde{\mu} = 0
\]  

(34)
It is also interesting to write the equations in the form when the calibrating condition (29) is an integral of motion of the differential equations. This can be done by involving the plasma electrons density among the unknown functions. Monitoring of this parameter is useful also from the physical viewpoint. Let’s write these eqs. for deviations of functions \( \tilde{u}, \tilde{v} \) and \( N \) from the vacuum values:

\[
\tilde{u} = \beta \gamma + \chi, \quad (35)
\]

\[
\tilde{v} = \gamma + \lambda, \quad (36)
\]

\[
N = 1 + \nu. \quad (37)
\]

Using eqs. (25-28), as well as the continuity eq. (29) for the values \( a_y, \chi, \lambda, \nu \) one can obtain the following eqs.:

\[
(1 + \nu)(\Delta a_y - a_y) = (\beta_0 \gamma_0 + \chi) \frac{\partial^2 \nu}{\partial y \partial \tilde{z}} + \nu a_y (1 + \nu) - a_y \frac{\partial^2 \nu}{\partial y^2} - 2 \frac{\partial \nu \partial a_y}{\partial y \partial \tilde{z}} + \frac{\partial \nu \partial \chi}{\partial \tilde{z} \partial \tilde{y}} + \frac{\partial \nu \partial \chi}{\partial \tilde{z} \partial \tilde{y}}, \quad (38)
\]

\[
(1 + \nu)(\Delta \chi - \chi) = \nu (1 + \nu) (\beta_0 \gamma_0 + \chi) \frac{\partial^2 \nu}{\partial \tilde{z}^2} + a_y \frac{\partial^2 \nu}{\partial \tilde{z}^2} - 2 \frac{\partial \nu \partial \chi}{\partial \tilde{z} \partial \tilde{y}} + \frac{\partial \nu \partial a_y}{\partial \tilde{z} \partial \tilde{y}} + \frac{\partial \nu \partial a_y}{\partial \tilde{z} \partial \tilde{y}}, \quad (39)
\]

\[
\Delta \lambda - \lambda = \frac{\nu b}{\gamma_0} + \nu (\gamma_0 + \lambda) \quad (40)
\]

\[
a_y^2 \frac{\partial^2 \nu}{\partial y^2} - 2 a_y (\beta_0 \gamma_0 + \chi) \frac{\partial^2 \nu}{\partial y \partial \tilde{z}} + (\beta_0 \gamma_0 + \chi)^2 \frac{\partial^2 \nu}{\partial \tilde{z}^2} = -(1 + \nu)(\gamma_0 + \lambda) \frac{\nu b}{\gamma_0} + (1 + \nu)(-\nu + \gamma_0 (\lambda - \beta_0 \chi)) + (\beta_0 \gamma_0 + \chi)(\frac{\partial \nu \partial a_y}{\partial \tilde{z} \partial \tilde{y}} + \frac{\partial \nu \partial \chi}{\partial \tilde{z} \partial \tilde{y}} + \frac{\partial \nu \partial \chi}{\partial \tilde{z} \partial \tilde{y}} - 2 \frac{\partial \nu \partial a_y}{\partial \tilde{z} \partial \tilde{y}}) - (1 + \nu) \left[ \left( \frac{\partial \lambda}{\partial \tilde{y}} \right)^2 + \left( \frac{\partial \lambda}{\partial \tilde{z}} \right)^2 - \left( \frac{\partial \chi}{\partial \tilde{y}} \right)^2 - \left( \frac{\partial \chi}{\partial \tilde{z}} \right)^2 - \left( \frac{\partial a_y}{\partial \tilde{y}} \right)^2 - \left( \frac{\partial a_y}{\partial \tilde{z}} \right)^2 \right] \quad (41)
\]

The calibrating condition is written as:

\[
2\gamma_0 (\lambda - \beta_0 \chi) + \lambda^2 - a_y^2 - \chi^2 = 0 \quad (42)
\]
In spite of rather unwieldy form of eqs. (38-41) they have a number of advantages for numerical calculations. Firstly, the relationship (42) follows from eqs. (38-41) and it can used for verification of calculations. Secondly, from eqs. (38-41) one can easily come to the linear case corresponding to the bunches with low density. In so doing the linearized eqs. are written as:

\[ \Delta a_y - a_y = \beta_0 \gamma_0 \frac{\partial^2 \nu}{\partial y \partial \tilde{z}} \]  
(43)

\[ \Delta \chi - \chi = \beta_0 \gamma_0 (\nu - \frac{\partial^2 \nu}{\partial \tilde{z}}) \]  
(44)

\[ \Delta \lambda - \lambda = \frac{n_b}{\gamma_0} + \gamma_0 \nu \]  
(45)

\[ \beta_0^2 \gamma_0^2 \frac{\partial^2 \nu}{\partial \tilde{z}^2} = -n_b - \nu \]  
(46)

The calibration relationship

\[ \lambda - \beta_0 \chi = 0 \]  
(47)

can be easily obtained from eqs. (44-46).

The linearized eqs. are also a convenient testing area for numerical methods in going to the nonlinear eqs.

### 3 Limiting Cases of the Master Equations

Let us consider two limiting cases of equations (25-27): previously considered case of bunch with infinite transverse sizes and case of bunch with finite transverse and infinite longitudinal sizes. One can note, that in both cases from the formulation of the problem follows, that there is no plasma electrons flow along the axis \( y \), i.e \( \beta_y = 0 \) and

\[ a_y = 0. \]  
(48)

Taking into account (48) one can reduce (31-33) to

\[ \tilde{v} \frac{\partial \tilde{v}}{\partial y} - \sqrt{\tilde{v}^2 - 1} \tilde{\mu} = 0, \]  
(49)

\[ \sqrt{\tilde{v}^2 - 1} \frac{\partial \tilde{\mu}}{\partial \tilde{z}} = 0, \]  
(50)
\[
\frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial^2 \tilde{v}}{\partial z^2} = n_b \gamma_0^{-1} - \gamma_0 + \frac{\tilde{v}}{\sqrt{\tilde{v}^2 - 1}} \left( \beta_0 \gamma_0 + \frac{\partial \tilde{\mu}}{\partial y} \right). \tag{51}
\]

In the case of infinite transverse size there are no dependence of all variables on \(y\). This case is described by a single equation for \(\tilde{v}\):

\[
\frac{\partial^2 \tilde{v}}{\partial z^2} = n_b \gamma_0^{-1} - \gamma_0 + \beta_0 \gamma_0 \frac{\tilde{v}}{\sqrt{\tilde{v}^2 - 1}} \tag{52}
\]

Analogous equation was obtained previously in [2] and was a subject on analysis in a series of papers e.g. [7],[8],[3]. Note, that condition (8) of absence of vortex

\[
\text{rot}(\vec{p} - \frac{e}{c} \vec{A}) = 0
\]

is fulfilled automatically.

If substitute \(\tilde{v}\) and \(\tilde{z}\) by \(v\) and \(z\) from (23), then eq. (52) coincides with eq. (8) in [3], having an integral

\[
\epsilon = \frac{1}{2} v'^2 + \gamma^2 [v - \beta (v^2 - \gamma^{-2})^{1/2}] - n_b v, \tag{53}
\]

which allows to consider equation of motion of plasma electrons and Maxwell’s eqs (Coulomb’s Law) as an equation of motion of a point with a unit mass, coordinate \(v\), velocity \(v'\), moving in potential

\[
U = \gamma_0^2 [v - \beta_0 (v^2 - \gamma^{-2})^{1/2}] - n_b v. \tag{54}
\]
On Fig. 1 $U$ is represented as function of $v$ for different values $n_b$ and $\gamma_0 = 10$. Negative values of $n_b$ correspond to bunches of positively charged particles. Boundary values $v = 1, U' = 0, \epsilon = 1 - n_b$ are reached at front of bunch $z = d$. One can see from Fig. 1, that for $n_b \leq 0$ the motion is always periodical, it is also periodical for $0 \leq n_b \leq 1/2$ and is nonperiodic for $n_b > 1/2$. 

On Fig. 2a. Phase-plane portrait for $n_b = -10, \gamma = 10$ for different energies $G_45$. $G_45 = 11$ is the separatrix.

On Fig. 2b. Phase-plane portrait for $n_b = 0.4, \gamma = 10$. Separatrix: $G_45 = 9.96$. 

Fig. 1. Curves of potential $U$ for different $n_b$. 

Fig. 2a. Phase-plane portrait for $n_b = 10, \gamma = 10$ for different energies $G_45$. $G_45 = 11$ is the separatrix. 

Fig. 2b. Phase-plane portrait for $n_b = 0, \gamma = 10$. Separatrix: $G_45 = 9.96$. 

\[ v = \gamma^{-1} \]

\[ n_b = -10 \]

\[ n_b = -6 \]

\[ n_b = -2 \]

\[ n_b = 0 \]

\[ n_b = 0.2 \]

\[ n_b = 0.4 \]

\[ n_b = 2 \]

\[ n_b = 6 \]

\[ n_b = 10 \]
On Fig. 2 phase space portraits of system cold plasma—onedimensional flat bunch of length $d$ with infinite transverse sizes, $n_b = -10; 0.4; 10, \gamma = 10$; boundary values are not fixed (different $\epsilon$) are presented. One can use these trajectories, in particular, to form solutions for combined bunches moving in plasma and generating no wake fields [3].

Solution for a bunch with given $d, n_b, \beta_0$ must be joined with wave solution for a free plasma beyond the bunch (wake waves with $n_b = 0$, phase velocity of waves coincides with velocity of the bunch $v_{ph} = v_0$ [1], [2]). In doing so no additional limitations arise if $n_b < 0$, or $0 \leq n_b \leq \frac{1}{1+\beta_0}$. For nonperiodical solutions inside bunch, when $n_b > \frac{1}{1+\beta_0}$, $n_e$ may become negative; it remains positive for values of dimensionless pulses $\rho(0) = \frac{p e\gamma_0}{m c}$ of plasma electrons on the rear edge of the bunch $-\beta_0\gamma_0 \leq \rho(0) \leq 0$, i.e. for certain bunch lengths $0 \leq d \leq d_{cr}, \rho(0)$ is limited from above. Negative $n_e$ usually interpreted as a consequence of violation of steady state condition (see e.g. [2], [3]); in particular, it is possible that the wake field will break behind the bunch if the wave amplitude is large enough imidiately behind the bunch.

In the other (longitudinal) limiting case there is no dependence of all variables on $\tilde{z}$. Physically such conditions can be realized for long enough bunches far from the bunch’s head. The linearized problem for semi-infinite bunch in cylindrical geometry was considered in [12]. A tensor of dielectric permeability was introduced to connect the...
plasme electrons current density with electric field. The essential contribution was defined by concomitant fields. The transition fields arising at injection decrease exponentially. In the case when the bunch transverse size is much smaller than the plasma wave length, a return current, concentrated in the bunch transverse section, arises. In the other limiting case, the transverse section of the return current is much greater than the bunch transverse section.

From (49-51) we obtain

\[
\frac{\partial^2 \tilde{v}}{\partial y^2} = n_b \gamma_0^{-1} - \gamma_0 + \frac{\tilde{v}}{\sqrt{\gamma_0^2 - 1}} \left( \beta_0 \gamma_0 + \frac{\partial^2}{\partial y^2} \sqrt{\gamma_0^2 - 1} \right) \] (55)

It is interesting to note, that eq. (55) (the longitudinal limiting case) differs from (52) (the transverse limiting case) by the term \( \frac{\partial^2}{\partial y^2} \sqrt{\gamma_0^2 - 1} \) in brackets in the right side of eq. (55).

By substitution

\[
\sqrt{\gamma_0^2 - 1} = shR, \quad \tilde{v} = chR \] (56)

we obtain the following equation for \( R \)

\[
R'' = -(n_b \gamma_0^{-1} - \gamma_0) shR - \beta_0 \gamma_0 chR \] (57)

Prime in (52) means derivative with respect to \( y \). Note that by substitution

\[
R = ln\gamma_0 (1 + \beta_0) - \chi \] (58)

one can obtain following equation for \( \chi \):

\[
\chi'' = n_b \beta_0 ch\chi + (1 - n_b) sh\chi \] (59)

This equation is obtained in work [4], dedicated to the transportation of longitudinally infinite bunches of charged particles with current, exceeding the Alfvén limit through a cold vortexless plasma.

The first integral of (59) when \( n_b = \text{const} \) is

\[
E = \frac{1}{2} (R')^2 + (n_b \gamma_0^{-1} - \gamma_0) chR + \beta_0 \gamma_0 shR = \text{const} \] (60)
Thus in this case also the problem is reduced to the study of massive (unit mass) point motion in potential field $U(R)$:

$$U(R) = \gamma [\beta_0 shR - (n_b \gamma_0^{-2} - 1)chR]$$  \hspace{1cm} (61)

Form of function $U(R)$ essentially depends on the regions of parameter values $n_b$ and $\beta_0$.

When

$$n_b > \frac{1}{1 - \beta_0},$$  \hspace{1cm} (62)

the function $U(R)$ has a single minimum and forms the potential well.

If

$$n_b < \frac{1}{1 + \beta_0},$$  \hspace{1cm} (63)

the function $U$ has one maximum and looks like a "hill".

In region of values

$$\frac{1}{1 + \beta_0} < n_b < \frac{1}{1 - \beta_0},$$  \hspace{1cm} (64)

the function has no extrema and has a characteristic bend, and varies from $-\infty$ to $+\infty$.

In the fig. 3 region of parameters $(n_b, \beta_0)$ and some potential curves for $\beta_0 = 0.85$ and a few values of $n_b$ are represented.
Phase trajectories in the plane \((R, R')\), corresponding to parameters \((n_b, \beta_0)\) from regions I \((n_b > 1/(1 - \beta_0))\), II \((1/(1 - \beta_0) > n_b > 1/(1 + \beta_0))\), III \((1/(1 + \beta_0) > n_b)\) are presented in figs. 4-6.

For the region I the phase trajectories are closed and there is a single stable immobile
point:

\[ R'_0 = 0, \quad R_0 = \text{arcth} \frac{\beta_0}{1 - n_b \gamma_0^2} \]  \hspace{1cm} (65)

In the region II all phase trajectories are not closed, there are no singular points, in the region III only hyperbolic motion around the point \(65\) can be realized; the point \(65\) in this case becomes unstable one.

Location of \(R_0\) for different values of parameters \(n_b\) are shown in fig. 7.

![Fig. 7. Location of immobile point of phase trajectories on the R axis depending on \(n_b\). Region of allowable values of \(R_0\) is defined by the strip \([-1, +1]\).](image)

Since finally we will be interested in restricted in transverse direction bunches \((n_b = 0\) at \(|y| > b, 2b\) is the width of flat bunch), the case \(n_b=0\) requires an additional consideration. Passing from the region occupied by bunch (\(|y| \leq b\)) to the region without it needs the physical solutions, corresponding to the null value of all physical quantities (electromagnetic fields, pulses of plasma electrons) at \(y \to \pm \infty\). These are the solutions which must be joined with the solutions inside the bunch, where \(n_b \neq 0\). For hyperbolic
phase trajectories (corresponding to the case \( n_b = 0 \)) the separatrices passing through the unstable (hyperbolic) immobile point are the only suitable trajectories. Passing along two branches of separatrices requires infinite long "time-development" on parameter \( y \).

Eqs. for the branches of separatrices are:

\[
R' = \pm \left[ \left( \frac{1 - \beta_0}{1 + \beta_0} \right)^{1/4} e^{R/2} - \left( \frac{1 + \beta_0}{1 - \beta_0} \right)^{1/4} e^{-R/2} \right]
\]  

(66)

Construction of solutions, corresponding to the bunches finite with respect to \( y \), can be done in the following way. Branches of separatrix \( n_b = 0 \), realizing the motion in \( y \) from the side surfaces of the bunch to the infinity are to be superimposed on phase trajectories corresponding to the position of \((\beta_0, n_b)\) in one of the regions I, II, III. Separatrix branches intersect many phase trajectories-different choices of intersection regions correspond to the different widths of bunches with the same values of parameters \((n_b, \beta_0)\). Figs. 8 and 9 represent the process for construction of solutions in regions II and III.

![Fig. 8. Formation of physical solutions for limited along y bunches; \( n_b=0.4, \beta_0=0.85 \). Branches of separatrix for \( n_b=0 \) are shown, which realize transition to \( \pm \infty \) along \( y \) axis. One of combined trajectories is shown, the interval of phase trajectory between branches of separatrix correspond to the solution inside the bunch.](image1)

![Fig. 9. Formation of physical solutions for limited bunches (\( n_b=0.7, \beta_0=0.85 \), II region) analogous to Fig. 7. Arrows show the limiting combined phase trajectory corresponding to the plasma electrons density \( n_e=0 \) on the \( R \) axis.](image2)

Note an important circumstance. There are no transverse motion of plasma electrons in adopted model and adjacent layers of plasma slide one with respect to other (because we neglect the plasma viscosity). In such a formulation the continuity equation is satisfied automatically and from the mathematical viewpoint there are solutions with any values
of \(n_e\), including negative ones, found, in particular in [2], [4].

However, clearly such solutions are not physical and are to be discarded. In [4] an illegitimate, (by our opinion), attempt is done to interpret the regions with negative \(n_e\) as channels from which all plasma electrons are displaced.

Using the formula (24) and relation (60) one can obtain the following expression for plasma electrons density:

\[
n_e = \gamma_0 c h R (1 - \beta_0 t h R) (-E + \frac{3}{2} (R')^2).
\]

(67)

From (67) follows that \(n_e\) can alter its sign, and it can take place only if \(E > 0\). Indeed, as it is seen from construction of physical solutions by means of joining of phase trajectories for \(n_b \neq 0\) with branches of separatrix at \(n_b = 0\), such a combined phase trajectory always intersects the \(R\) axis \((R' = 0)\). Thus, for positive \(E\) inside the bunch \(n_e < 0\). It is clear, that there are no physical solutions when \(n_b > 1/(1 - \beta_0)\) (region I of values \((n_b, \beta_0)\)), since the minimum values of corresponding potential pits are positive. In contrary, in the region III \((n_b < 1/(1 + \beta_0))\) the condition \(E < 0\) does not impose any additional limitations, since the regions for joining of solutions correspond to \(E < E_0\), where \(E_0 < 0\) is the maximum value of potential \(U(R)\).

Lastly, in the II region \((1/(1 - \beta_0) > n_b > 1/(1 + \beta_0))\) intervals of phase trajectories necessary for joining with branches of the immobile point of phase trajectories correspond to the energy values \(E > 0\), when \(n_b > 1\). Therefore at \(1/(1 - \beta_0) > n_b > 1\) inside bunch always \(n_e < 0\). Such solutions as it was mentioned above are not physical by our opinion. In the region \(1 > n_b > 1/(1 + \beta_0)\) the condition \(n_e > 0\) is not satisfied for a limited set of values of energy only \((n_b - 1 < E < 0)\). The maximum allowed bunch width corresponds to the ”time” which is necessary for passing the section of phase trajectory at \(E = 0\) between the branches of separatrix, where \(n_b = 0\).

In fig. 9 arrows indicate the limiting combined phase trajectory corresponding to the plasma electrons density \(n_e = 0\) in the middle of bunch. Phase trajectories right to the limiting one correspond to the nonphysical \((n_e < 0)\) solutions.

Complete sweeping of plasma electrons by the bunch, i.e. formation of a channel in
plasma take place at $(R')^2 = \frac{2}{3}E \geq 0$. In particular, this condition, as it was mentioned, is satisfied in the middle of bunch $(R' = 0)$ at $E = 0$, i.e. at $R = arcth(\beta_0/(1 - n_b\gamma_0^2))$ in an immobile point of the region I, (which is unstable for region III).

Thus, summarising the contents of the presented consideration one can note, that for a transversely infinite bunch at $n_b < 1/(1 + \beta_0)$ physical solutions exist for bunches of any width. In the case $1/(1 + \beta_0) < n_b < 1$ physical solutions exist for bunches of limited width only. When $n_b > 1$ in the middle of bunch $n_e < 0$ and we discard such a solutions as nonphysical ones for considered formulation of the problem (collisionless, stationary, vortexless, cold plasma).

Very likely that consideration of the problems which takes into account violation of stationary condition, thermal motion of plasma electrons and ions, plasma viscosity possibly will allow to clear out the physical nature of these nonstable or ”nonphysical” solutions.

It is necessary to remember that states with $n_e < 0$ are obtained in very long or very wide bunches. The finite dimensions of real bunches can change the domain of existence of these states or even eliminate them completely. Linear approximation to the limiting cases of formulated problem, discussed particularly in [3], [4], is valid, when bunch density is $n_b \ll 1$, and plasma electrons are nonrelativistic $|\vec{\beta}| \ll 1$. The last assumption provide that the condition $rot(\vec{p} - ze\vec{A}) = 0$ for the absence of the vortexes in plasma flow is fulfilled automaticaly, due to nonrelativistic equation of motion of plasma electrons and Faraday’s law [3]. Condition $n_b \ll 1$ allows to seek the plasma electron density as $n_e = 1 + n'_e, |n'_e| \ll 1$ and linearize the continuity eq. (14) and then, using nonrelativistic equation of plasma electron motion and Coulomb law, find a solution for $n'_e$. According to the condition $\beta \ll 1, f$ in (9) in linear approximation is $f = 1 + f', |f'| \ll 1, \vec{a} \approx \vec{\beta}$ and eqs. (11), (12) with the given right side have exact analytical solution, which can served as a test function for subsequent limiting case in computer simulation of the general problem. It is necessary to take into account that condition $\beta \ll 1$, can be fulfilled only for short enough bunches.

Obtained results of presented analytical consideration of general system of equations
for two-dimensional limited bunch of charged particles moving in a cold, collisionless, vortexless, stationary cold plasma, as well as the limiting cases corresponding to the infinitely wide (transverse) and infinitely long (longitudinal) bunches and linear approximation, valid for $n_b \ll 1, \beta \ll 1$, must serve as an analytical bases for formulation of an algorithm for numerical solution of the problem-obtaining of wake fields and focusing forces for bunches of an arbitrary longitudinal and transverse sizes.
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