ON THE SMALL BALL INEQUALITY IN THREE DIMENSIONS

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Abstract. Let $h_R$ denote an $L^\infty$ normalized Haar function adapted to a dyadic rectangle $R \subset [0, 1]^3$. We show that there is a positive $\eta < \frac{1}{2}$ so that for all integers $n$, and coefficients $a(R)$ we have

$$2^{-n} \sum_{|R|=2^{-n}} |a(R)| \lesssim n^{1-\eta} \left\| \sum_{|R|=2^{-n}} a(R) h_R \right\|_\infty.$$ 

This is an improvement over the ‘trivial’ estimate by an amount of $n^{-\eta}$, while the Small Ball Conjecture says that the inequality should hold with $\eta = \frac{1}{2}$. There is a corresponding lower bound on the $L^\infty$ norm of the Discrepancy function of an arbitrary distribution of a finite number of points in the unit cube in three dimensions. The prior result, in dimension 3, is that of József Beck [1], in which the improvement over the trivial estimate was logarithmic in $n$. We find several simplifications and extensions of Beck’s argument to prove the result above.

1. The Principal Conjecture and the Main Results

In one dimension, the class of dyadic intervals in the unit interval is $\mathcal{D} := \{(j2^{-k}, (j + 1)2^{-k}) : j, k \in \mathbb{N}, 0 \leq j \leq 2^k - 1\}$. Each dyadic interval has a left and right half, which are also dyadic. Define the Haar functions

$$h_I := -1_{I_{\text{left}}} + 1_{I_{\text{right}}}.$$ 

Note that we use an $L^\infty$ normalization of these functions, which will make some formulas seem odd to a reader accustomed to the $L^2$ normalization.

In dimension $d$, a dyadic rectangle in the unit cube $[0, 1]^d$ is a product of dyadic intervals, thus an element of $\mathcal{D}^d$. A Haar function associated to $R$ is defined as a product of the Haar functions associated with each side of $R$, namely

$$h_{R_1 \times \cdots \times R_d}(x_1, \ldots, x_d) := \prod_{j=1}^d h_{R_j}(x_j).$$ 

This is the usual ‘tensor’ definition.

We will concentrate on rectangles with fixed volume. This is the ‘hyperbolic’ assumption, that pervades the subject. Our concern is the following Theorem and Conjecture concerning a lower bound on the $L^\infty$ norm of sums of hyperbolic Haar functions:
1.1. **Theorem** (Talagrand [13], Temlyakov [16]). In dimension $d = 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim \left\| \sum_{|R|\geq 2^{-n}} \alpha(R)h_{R} \right\|_{\infty}. \tag{1.2}$$

Here, the sum on the right is taken over all rectangles with area at least $2^{-n}$.

1.3. **Small Ball Conjecture.** For dimension $d \geq 3$ we have the inequality

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim n^{\frac{1}{2}(d-2)} \left\| \sum_{|R|\geq 2^{-n}} \alpha(R)h_{R} \right\|_{\infty}. \tag{1.4}$$

This conjecture is, by one square root of $n$, better than the trivial estimate available from the Cauchy-Schwartz inequality, see §2. As well, see that section for an explanation as to why the conjecture is sharp. The case of $d = 2$ (with a sum over $|R| = 2^{-n}$ on the right-hand side) was resolved by Talagrand [13]. Temlyakov has given an easier proof of the inequality in its present form [14], [16], which resonates with the ideas of Roth [9], Schmidt [10], and Halász [6].

Perhaps, it is worthwhile to explain the nomenclature ‘Small Ball’ at this point. The name comes from the probability theory. Assume that $X_t : T \rightarrow \mathbb{R}$ is a canonical Gaussian process indexed by a set $T$. The **Small Ball Problem** is concerned with estimates of $\mathbb{P}(\sup_{t \in T} |X_t| < \varepsilon)$ as $\varepsilon$ goes to zero, i.e. the probability that the random process takes values in an $L^\infty$ ball of small radius. The reader is advised to consult a paper by Kuelbs and Li [7] for a survey of this type of questions. A particular question of interest to us deals with the Brownian Sheet, that is, a centered Gaussian process indexed by the points in the unit cube $[0, 1]^d$ and characterized by the covariance relation $\mathbb{E}X_s \cdot X_t = \prod_{j=1}^{d} \min(s_j, t_j)$. The conjectured form of the aforementioned probability in this case is the following:

1.5. **The Small Ball Conjecture for the Brownian Sheet.** In dimensions $d \geq 2$, for the Brownian Sheet $B$ we have

$$-\log \mathbb{P}(\|B\|_{C([0, 1]^d)} < \varepsilon) \approx \varepsilon^{-(\log 1/\varepsilon)^2 + d-2}, \quad \varepsilon \downarrow 0.$$ 

In dimension $d = 2$, this conjecture has been resolved by Talagrand in the already cited paper [13], in which he used a version of (1.2) for continuous wavelets in place of Haars to prove the lower bound in the inequality above. In higher dimensions, the upper bounds are established and the known lower bounds miss the conjecture by a single power of the logarithm.

Kuelbs and Li [7] have discovered a tight connection between the Small Ball probabilities and the properties of the reproducing kernel Hilbert space corresponding to the process, which in the case of the Brownian Sheet is $WM^2_d$, the Sobolev space of the functions on $[0, 1]^d$ with mixed derivative in $L^2$. In Approximation Theory, the covering number $N(\varepsilon)$ is defined as the smallest number of $L^\infty$ balls of radius $\varepsilon$ needed to cover the unit ball of $WM^2_d$, i.e. the cardinality of the smallest $\varepsilon$-net, a quantification of compactness of the unit ball in the uniform metric. The result of Kuelbs and Li states that
1.6. **Theorem.** In dimension $d \geq 2$, as $\varepsilon \downarrow 0$ we have

$$- \log P(\|B\|_{C([0,1]^d)} < \varepsilon) \approx \varepsilon^{-2}(\log \frac{1}{\varepsilon})^\beta \iff \log N(\varepsilon) \approx \varepsilon^{-1}(\log \frac{1}{\varepsilon})^{\beta/2}.$$  

This theorem together with Talagrand’s work shows that the Small Ball Conjecture 1.3 for continuous wavelets implies the lower bound in the conjectured asymptotics of the covering numbers $N(\varepsilon)$ (the upper bounds are known). It is also not very hard to show this implication directly. The Small Ball Conjecture for the Haar functions implies a lower bound for the covering numbers of the space $WM^1_d$. A detailed discussion of the connections of the Small Ball Conjecture to the Approximation Theory and other related areas can be found in [15], [17].

Even though all of the mentioned questions had been completely resolved in dimension $d = 2$, there has been very little progress in higher dimensions. The main result of the present paper is a partial resolution of the three dimensional case of the Small Ball Conjecture. We extend and simplify an approach of J. Beck [1], establishing the following theorem:

1.7. **Theorem.** In dimension $d = 3$, there is a positive $\eta > 0$ for which we have the estimate

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \leq n^{1-\eta} \left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_\infty.$$  

Beck [1] established this inequality with $n^{-\eta}$ replaced by a term logarithmic in $n$, although Beck himself did not state the result this way, as the principal concern of that paper is on the question of Irregularities of Distribution, another area relevant to the Small Ball Conjecture.

In this subject one takes $A_N$ to be $N$ points in the $d$-dimensional unit cube, and considers the Discrepancy Function

$$D_N(x) = \#A_N \cap [\vec{0}, \vec{x}) - N|[\vec{0}, \vec{x}].$$  

Here $[\vec{0}, \vec{x}) = \prod_{j=1}^d [0, x_j)$ is a rectangle with antipodal corners being $\vec{0}$ and $\vec{x}$. We will typically suppress the dependence upon the selection of points $A_N$. A set of points will be well distributed if this function is small in some appropriate function space. Thus, the principal concern are various lower bounds for the $L^p$ norm of $D_N$. Many variants of this question are interesting; readers are encouraged to consult one of the excellent references in this area, e.g. [2]. The connection\(^1\) to the Small Ball Conjecture lies in the ‘hyperbolic orthogonal function’ method initiated by Roth [9] when he proved that for all dimensions $d \geq 2$,

$$\|D_N\|_2 \gtrsim (\log N)^{\frac{d}{d+1}}.$$  

Later, Schmidt [10] has shown that in dimension 2, the $L^\infty$ norm of the discrepancy function is much bigger than what the $L^2$ estimate gives us:

$$\|D_N\|_\infty \gtrsim \log N.$$  

\(^1\)One expects extremal point distributions $A_N$ to have about one point in each cube of volume about $N^{-1}$. Thus the Haar functions adapted to dyadic rectangles of about this volume are important.

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Notice that, just like in the Small Ball Conjecture 1.3, this beats the $L^2$ bound by one square root.

Using our method of proof, and well known facts in the literature on Irregularities of Distribution ([1,2]), we obtain following theorem:

1.10. **Theorem.** There is a choice of $0 < \eta < \frac{1}{2}$ for which the following estimate holds for all collections $\mathcal{A}_N \subset [0,1]^3$:

$$\|D_N\|_{\infty} \gtrsim (\log N)^{1+\eta}.$$  

Beck’s result is as above, with $(\log N)\eta$ replaced by a doubly logarithmic term in $N$. There is no further result known to the authors about the Small Ball Problem, nor the $L^\infty$ norm of the Discrepancy Function in higher dimensions.

Concerning the value of $\eta$ for which our Theorems hold, it is computable, but we do not carry out this step, as the particular $\eta$ we would obtain is certainly not optimal. Instead, the point of this proof is that the methods pioneered by József Beck are more powerful than originally suspected. We expect more efficient organization of the proof, and less ad hoc constructions, will yield quantifiable and substantive improvements to the results of this paper.\(^2\)

The organization of the proof, at the highest level, and outlined in §7, is that of József Beck [1]. At the same time, both the exact construction and subsequent details are in many respects easier than in Beck’s paper. In particular, the construction in that section is a Riesz product construction, following the lines of §3. But, the product, with our current understanding, must be taken to be ‘short,’ a dictation to us from the third dimension: the ‘product rule’ 3.1 does not hold in dimension three. This unfortunate, and critical fact, forces the definition of ‘strongly distinct’ on us. See Definition 6.4. Still, our Riesz product is defined in a way to facilitate the use of Littlewood Paley inequalities and conditional expectation arguments, which is the source of our simplification and strengthening of Beck’s argument.

The principal argument begins in §6. The earlier sections of the paper include a brief discussion of prerequisites for the proof.

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### 2. The Trivial Bounds

**Notation.** The language and notation of probability and expectation is used throughout. Thus,

$$E f = \int_{[0,1]^d} f(x) \, dx$$

\(^2\)Additional steps that one could take to optimize the proof are known to the authors; others are the subject of speculation.
and $\mathbb{P}(A) = \mathbb{E}1_A$. This serves to keep formulas simpler. As well, certain conditional expectation arguments are essential to us. We use the notation

$$\mathbb{P}(B \mid A) = \mathbb{P}(A)^{-1}\mathbb{P}(A \cap B), \quad \mathbb{E}(B \mid A) = \mathbb{P}(A)^{-1}\mathbb{E}(A \cap B).$$

For a sigma field $\mathcal{F}$, $\mathbb{E}(f \mid \mathcal{F})$ is the conditional expectation of $f$ given $\mathcal{F}$. In all instances, $\mathcal{F}$ will be generated by a finite collection of atoms $\mathcal{F}_{\text{atoms}}$, in which case

$$\mathbb{E}(f \mid \mathcal{F}) = \sum_{A \in \mathcal{F}_{\text{atoms}}} \mathbb{P}(A)^{-1}\mathbb{E}(f 1_A) \cdot 1_A.$$

We suppress many constants which do not affect the arguments in essential ways. $A \lesssim B$ means that there is an absolute constant so that $A \leq KB$. Thus $A \lesssim 1$ means that $A$ is bounded by an absolute constant. And $A \simeq B$ means $A \lesssim B \lesssim A$.

The inequality (1.2) with an extra square root of $n$ is easy to prove.

2.1. Lemma. It is the case that

$$\sum_{|R|=2^{-n}} |\alpha(R)| \cdot |R| \lesssim n^{1/2(d-1)} \sum_{|R| \geq 2^{-n}} |\alpha(R)| h_R \|h_R\|_\infty.$$

Proof. Each point $x \in [0, 1]^d$, is in at most $n^{d-1}$ possible rectangles. This is the essential point dictated by the hyperbolic nature of the problem. Using this, and the Cauchy–Schwartz inequality, we have

$$\sum_{|R|=2^{-n}} |\alpha(R)| \cdot |R| = \left| \sum_{|R|=2^{-n}} |\alpha(R)| 1_R \right|_1 \leq n^{1/2(d-1)} \left| \sum_{|R|=2^{-n}} |\alpha(R)|^2 1_R \right|_1^{1/2} \lesssim n^{1/2(d-1)} \left| \sum_{|R| \geq 2^{-n}} \alpha(R) h_R \right|_2 \leq n^{1/2(d-1)} \left| \sum_{|R| \geq 2^{-n}} \alpha(R) h_R \right|_\infty,$$

$\square$

Let us also see that the Small Ball Conjecture is sharp. Indeed, we take the $\alpha(R)$ to be random choices of signs. It is immediate that

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \simeq n^{d-1}.$$

On the other hand, for fixed $x \in [0, 1]^d$, by the properties of Rademacher functions we have

$$\mathbb{E} \left| \sum_{|R|=2^{-n}} \alpha(R) h_R(x) \right| \simeq n^{1/2(d-1)}.$$
It is also well known that sums of Rademacher random variables obey a sub–Gaussian distributional estimate. The supremum of such sums admits easily estimated upper bounds. In particular, it is enough to test the $L^\infty$ norm of the sum at a grid of $2^{nd}$ points in the unit cube, hence we have

$$ \mathbb{E} \left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_\infty \lesssim \sqrt{\log 2} \cdot \sup_x \mathbb{E} \left| \sum_{|R|=2^{-n}} \alpha(R) h_R(x) \right| \lesssim n^{d/2}. $$

Comparing these two estimates shows that the Small Ball Conjecture is sharp. In the trigonometric case, a similar remark has appeared in [18].

### 3. Proof of Talagrand’s Theorem

In this section we sketch the proof of V. Temlyakov [16] to the stronger inequality (1.2) in the case of $d = 2$, as this will help understand our construction for $d = 3$. The line of reasoning is similar to that of Schmidt [10].

The decisive point in two dimensions is that one has a ‘product rule’:

**3.1. Product Rule in Dimension 2.** Let $R, R'$ be two dyadic rectangles of the same area. Then, $h_R \cdot h_{R'} \in \{0, 1_R, \pm d_R \cap d_{R'}\}$. More generally, let $R_1, R_2, \ldots, R_k$ be dyadic rectangles of equal area and distinct lengths in e.g. their first coordinates. Then $\prod_{j=1}^k h_R \in \{0, \pm d_{R_1 \cap \cdots \cap R_k}\}$.

The fact that this ‘product rule’ fails in higher dimensions is the most essential complication to the resolution of the Small Ball Conjecture.

The proof of (1.2) is by duality. Fix

$$ H = \sum_{|R|=2^{-n}} \alpha(R) h_R. $$

We will construct a function $\Psi$ with $L^1$ norm at most 1, for which the inner product

$$ (H, \Psi) = 2^{-n-1} \sum_{|R|=2^{-n}} |\alpha(R)|. $$

(3.2) This clearly implies Theorem 1.1. Moreover, the function $\Psi$ is defined as a Riesz product:

$$ \Psi := \prod_{s=1}^n (1 + \frac{1}{2} \psi_s), $$

$$ \psi_s = \sum_{R : |R_1|=2^{-s}, |R_2|=2^{-s+s}} \text{sgn}(\alpha(R)) h_R. $$

Of course $\Psi$ is non–negative. Moreover, it has $L^1$ norm one: expanding the product, the leading term is 1. All products of $\psi_s$ are, by Proposition 3.1, a sum of Haar functions, hence have mean zero. A similar argument implies (3.2). The proof is complete.
4. Littlewood-Paley Theory

In this section we review some basic facts from the Littlewood-Paley Theory, which will be used repeatedly in subsequent sections. We state the main inequalities here to make the exposition self-contained. We also remind the reader that the Haar functions are normalized to have $L^\infty$ norm one, so that our formulas are different from most of our references.

It is important to our applications that we consider the Haar basis as one for vector valued functions. The vector space should be a Hilbert space $H$, and by $L^p_H$ we mean the class of measurable functions $f : [0, 1] \to H$ such that $\|f\|_{L^p_H} < \infty$.

The Haar Square Function is

$$S(f) := \left[ \|E f\|_H^2 + \sum_{I \in D} \frac{\langle f, h_I \rangle_{H}^2}{|I|^2} 1_I \right]^{1/2}.$$  

Here, $\langle f, h_I \rangle = \int h_I(x)f(x)dx$ and $Ef$ should be understood as Bochner integrals, and we are taking the Hilbert space norm of those terms that involve $f$. We shall be applying the Square Function in the cases when $f$ is a finite linear combination of Haars, i.e. $f = \sum_{I \in I} a_I h_I$, where $I$ is a finite subset of $D$ and $(a_I)_{I \in I} \subset H$. In this case, $f$ has mean zero and the Square Function takes the form

$$S(f) = \left[ \sum_{I \in I} |a_I|_{H}^2 1_I \right]^{1/2}.$$  

Of course we have $\|f\|_2 = \|S(f)\|_2$ just due to the fact that $\{1_{[0,1]}\} \cup \{h_I : I \in D\}$ is an orthogonal basis.

The Littlewood-Paley Inequalities are a extension of this equality, to an approximate version that holds on all $L^p$, $1 < p < \infty$.

4.1. Littlewood Paley Inequalities. For $1 < p < \infty$ there are absolute constants $0 < A_p < B_p < \infty$ so that

$$\|f\|_p \leq B_p \|S(f)\|_p, \quad 1 < p < \infty$$

$$B_p \leq 1 + \sqrt{p} \quad \text{for} \quad p \geq 2.$$  

In the reverse direction, we have

$$A_p \|S(f)\|_p \leq \|f\|_p, \quad 1 < p < \infty,$$

$$A_p \approx 1 + \frac{1}{\sqrt{p-1}}.$$  

We stress that these results are delicate.\(^3\) Burkholder [3] has shown that the best constants in the inequality above for general martingales are $A_p^{-1} = B_p = \max\{p, q\} - 1$. However, a

\(^3\)To prove our Theorems, we only need these inequalities with constant $B_p \leq p'$ for some fixed power of $t$. But, the power of $t = \frac{1}{2}$ is the sharp result, so we use it here.
Har series is not a general martingale; it is dyadic, which forces conditional symmetry. See [4,5,19].

The constants above are sharp. To see that $B_p \approx \sqrt{p}$ is sharp for $p$ large, just use the Central Limit Theorem for Rademacher random variables.

5. Exponential Moments

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a symmetric convex function with $\psi(x) = 0$ iff $x = 0$. Define the Orlicz norm

$$\|f\|_\psi := \inf \{ C > 0 : \mathbb{E}\psi(f/C) \leq 1 \}.$$  

We take the infimum of the empty set to be $+\infty$, and denote by $L_\psi$ to be the collection of functions for which $\|f\|_\psi < \infty$. If $\psi(x) = x^p$, then $\|\cdot\|_p$ is the usual $L_p$ norm.

We are especially interested in the class of $\psi$ given by $\psi_\alpha(x) = e^{\|x\|_\alpha}$, $\|x\|_\alpha \gg 1$. We will write $L_{\psi_\alpha} = \exp(L_\alpha)$. These are the exponential Orlicz classes. The following equivalence is well known and is based on Taylor series and Stirling’s formula:

5.2. Proposition. We have the equivalence of norms

$$\|f\|_{\exp(L_\alpha)} \approx \sup_{p \geq 1} p^{-1/\alpha} \|f\|_p \approx \sup_{\lambda > 0} \lambda^\alpha [\log \mathbb{P}(|f| > \lambda)].$$

The following distributional estimate holds for hyperbolic sums of Haar functions:

5.3. Theorem. In dimension $d \geq 2$ we have the estimate

$$\left\| \sum_{\|R\| = 2^{-n}} \alpha(R) h_R \right\|_{L_{\psi_{2d/(d-1)}}} \lesssim \left\| \left( \sum_{\|R\| = 2^{-n}} \alpha(R)^2 \mathbf{1}_R \right)^{1/2} \right\|_{L_\infty}.$$  

Of principal relevance to us is the three dimensional case, where the estimate above asserts that the hyperbolic sums are exponentially integrable.

Proof. The tool is the vector valued Littlewood Paley inequality, with sharp rate of growth in the constants as $p \to \infty$, stated in the previous section. As such the proof is a standard one, see [5,8]. We will make use of similar arguments more than once in this paper.

Applying the one dimensional Littlewood Paley inequality in the coordinate $x_1$ we see that

$$\left\| \sum_{\|R\| = 2^{-n}} \alpha(R) h_R \right\|_p \lesssim \sqrt{p} \left\| \left( \sum_{r_1=1}^n \left( \sum_{\|R\| = 2^{-n}} \sum_{\|R_1\| = 2^{-r_1}} \alpha(R) h_R \right)^2 \right)^{1/2} \right\|_p.$$  

If we are in dimension 2, note that due to the hyperbolic assumption, all the rectangles satisfying the conditions of the summation are disjoint, and thus we have:

$$\left( \sum_{\|R\| = 2^{-n}} \alpha(R) h_R \right)^2 = \sum_{\|R\| = 2^{-n}} |\alpha(R)|^2 \mathbf{1}_R,$$
so our proof is complete in this case.

In the higher dimensional case, the key point is to observe that the last term can be viewed as an \( \ell^2 \) space valued function, that is if we fix all the coordinates except \( x_2 \) and define an \( \ell^2 \)-valued function

\[
F(x_2) = \sum_{R_2} \left\{ \sum_{|R| = 2^{-n}} \alpha(R) \prod_{j \neq 2} h_{R_j}(x_j) \right\}^{n} h_{R_2}(x_2),
\]

then the expression inside the \( L^p \) norm on the right hand side of (5.5) is exactly \( |F|_{\ell^2} \). Thus, the Hilbert space valued Littlewood Paley inequality applies to the second coordinate, to give us

\[
\left\| \sum_{|R| = 2^{-a}} \alpha(R) h_{R_j} \right\|_{p} \lesssim \left( \sum_{n=1}^{n} \sum_{r_1=1}^{n} \sum_{|R| = 2^{-j}} \alpha(R) h_{R_j} \right)^{1/2} \left\| \sum_{|R| = 2^{-j}} \alpha(R) \right\|_{p}. \]

Observe that we have a full power of \( p \), due to the two applications of the Littlewood Paley inequalities. And if \( d = 3 \), then analog of (5.6) holds, completing the proof in this case.

In the case of dimension \( d \geq 4 \) note that we can continue applying the Littlewood Paley inequalities inductively. They need only be used \( d - 1 \) times due to the hyperbolic assumption. Thus, we have the inequality

\[
\left\| \sum_{|R| = 2^{-a}} \alpha(R) h_{R_j} \right\|_{p} \lesssim \left( \sum_{n=1}^{n} \sum_{r_1=1}^{n} \sum_{|R| = 2^{-j}} \alpha(R) h_{R_j} \right)^{1/2} \left\| \sum_{|R| = 2^{-j}} \alpha(R) \right\|_{p}, \quad 2 \leq p < \infty.
\]

The implied constant depends upon dimension; the main point we are interested in is the rate of growth of the \( L^p \) norms. Assuming that the Square Function of the sum is bounded in \( L^\infty \), the \( L^p \) norms can only grow at the rate of \( p^{(d-1)/2} \), which completes the proof. \( \square \)

This theorem illustrates a thesis of A. Zygmund, which says that the estimates on product domains are controlled by the effective number of parameters, which in our hyperbolic setting is \( d - 1 \). The method of iteration of the one parameter inequalities, in the vector valued setting, is a common technique in the subject, see for instance [11, 12]. We shall repeatedly make use of this technique in the present paper.

6. Definitions and Initial Lemmas for Dimension Three

As it has been already pointed out, the principal difficulty in three and higher dimensions is that the product of Haar functions is not necessarily a Haar function. On this point, we have the following higher dimensional analogue of the ‘product rule’ (3.1):

6.1. Proposition. Suppose that \( R_1, \ldots, R_k \) are rectangles such that there is no choice of \( 1 \leq j < j' \leq k \) and no choice of coordinate \( 1 \leq t \leq d \) for which we have \( R_{j,t} = R_{j',t} \). Then, for a choice of
\[ \text{sign } \varepsilon \in \{\pm 1\} \text{ we have} \]

\begin{equation}
(6.2) \quad \prod_{j=1}^{k} h_R = \varepsilon h_{S_j}, \quad S = \bigcap_{j=1}^{k} R_k.
\end{equation}

**Proof.** Expand the product as

\[ \prod_{m=1}^{\ell} h_{R_m}(x_1, \ldots, x_d) = \prod_{m=1}^{\ell} \prod_{t=1}^{d} h_{R_m}(x_t). \]

Our assumption is that for each \( t \), there is exactly one choice of \( 1 \leq m_0 \leq \ell \) such that \( R_{m_0,t} = S_t \). And moreover, since the minimum value of \(|R_{m,t}|\) is obtained exactly once, for \( m \neq m_0 \), we have that \( h_{R_{m,t}} \) is constant on \( S_t \). Thus, in the \( t \) coordinate, the product is

\[ h_{S_t}(x_t) \prod_{1 \leq m \neq m_0 \leq \ell} h_{R_{m,t}}(S_t) = \varepsilon_t h_{S_t}(x_t), \quad \text{where } \varepsilon_t \in \{\pm 1\}. \]

This proves our Lemma. \( \square \)

**Remark.** It is also a useful observation, that the products of Haar functions have mean zero, if the minimum value of \(|R_{m,t}|\) is unique for at least one coordinate \( t \).

Let \( \vec{r} \in \mathbb{N}^3 \) be a partition of \( n \), thus \( \vec{r} = (r_1, r_2, r_3) \), where the \( r_j \) are non negative integers and \( |\vec{r}| := \sum r_t = n \). Denote all such vectors as \( \mathbb{H}_n \). (‘\( \mathbb{H} \)’ for ‘hyperbolic.’) These vectors will specify the geometry of the rectangles, i.e. we set \( \mathcal{R}_\vec{r} = \{R \in \mathcal{D}^n : |R_j| = 2^{-r_j}, \ j = 1, 2, 3\} \).

We call a function \( f \) an \( \vec{r} \) function with parameter \( \vec{r} \) if

\begin{equation}
(6.3) \quad f = \sum_{R \in \mathcal{R}_\vec{r}} \varepsilon_R h_R , \quad \varepsilon_R \in \{\pm 1\}.
\end{equation}

We will use \( f_\vec{r} \) to denote a generic \( \vec{r} \) function. A fact used without further comment is that \( f_\vec{r}^2 \equiv 1 \).

**6.4. Definition.** For vectors \( \vec{r}'_j \in \mathbb{N}^3 \), say that \( \vec{r}'_1, \ldots, \vec{r}'_j \) are strongly distinct iff for coordinates \( 1 \leq t \leq 3 \) the integers \( \{r'_{j,t} : 1 \leq j \leq J\} \) are distinct. The product of strongly distinct \( \vec{r} \) functions is also an \( \vec{r} \) function, which follows from ‘the product rule’ (6.1).

The \( \vec{r} \) functions we are interested in are

\begin{equation}
(6.5) \quad f_\vec{r} := \sum_{R \in \mathcal{R}_\vec{r}} \text{sgn}(\alpha(R)) h_R ,
\end{equation}

where \( H_n = \sum_{|R| \geq 2^{-n}} \alpha(R) h_R \).
7. József Beck’s Short Riesz Product

Let us define relevant parameters by

\begin{align*}
q &= an^\varepsilon, \\
b &= \frac{1}{6}, \\
\bar{\rho} &= aq^b n^{-1}, \\
\rho &= \sqrt{qn}^{-1}.
\end{align*}

Here, \( a \) are small positive constants, we use the notation of \( b = \frac{1}{6} \) throughout, so as not to obscure those aspects of the argument that that dictate this choice of \( b \). \( \bar{\rho} \) is a ‘false’ \( L^2 \) normalization for the sums we consider, while the larger term \( \rho \) is the ‘true’ \( L^2 \) normalization. Our ‘gain over the trivial estimate’ in the Small Ball Conjecture is \( q^b = n^{\varepsilon/6} \). \( 0 < \varepsilon < 1 \) is a small constant; the exact determination of what we could take \( \varepsilon \) equal to in this proof doesn’t seem to be worth calculating as it surely will not be optimal.

In Beck’s paper, the value of \( q = q_{\text{Beck}} = \log n / \log \log n \) was much smaller than our value of \( q \). The point of this choice is that \( q_{\text{Beck}} \approx n \), with the term \( q \) controlling many of the combinatorial issues concerning the expansion of the Riesz product.\(^4\) With our substantially larger value of \( q \), we need to introduce additional tools to control the combinatorics. These tools are

- A Riesz product that will permit us to implement various conditional expectation arguments.
- Attention to \( L^p \) estimates of various sums, and their growth rates in \( p \).
- Systematic use of the Littlewood-Paley inequalities, with the sharp constants in \( p \).

Divide the integers \( \{1, 2, \ldots, n\} \) into \( q \) disjoint increasing intervals \( I_1, \ldots, I_q \), and let \( A_t := \{ \vec{r} \in \mathbb{H}_n : r_1 \in I_t \} \). Let

\[ F_t = \sum_{\vec{r} \in A_t} f_{\vec{r}}. \]

The Riesz product is now a ‘short product.’

\[ \Psi := \prod_{t=1}^{q} (1 + \bar{\rho} F_t). \]

The ‘false’ \( L^2 \) normalization implies that the product is, with high probability, positive, and thus \( \|\Psi\|_1 \approx \mathbb{E}\Psi \), with expectations being typically easier to estimate. This heuristic is made precise below.

Proposition 6.1 suggests that we should decompose the product \( \Psi \) into

\[ \Psi = 1 + \Psi^{sd} + \Psi^-, \]

\(^4\)Specifically, \( q^{Cq} \) is a naive bound for the number of admissible graphs, as defined in § 10.
where the two pieces are the ‘strongly distinct’ and ‘not strongly distinct’ pieces. To be specific, for integers \(1 \leq u \leq q\), let

\[
\Psi^{sd}_u := \tilde{\rho}^u \sum_{1 \leq v_1 < v_2 \leq q} \sum_{\vec{r}_t \in A_{v_t}} \prod_{t=1}^{u} f_{\vec{r}_t},
\]

where \(\sum^{sd}\) is taken to be over all \(\vec{r}_t \in A_{v_t} 1 \leq t \leq u\) such that:

(7.5) the vectors \(\{\vec{r}_t : 1 \leq t \leq u\}\) are strongly distinct.

Then define

(7.6) \[\Psi^{sd} := \sum_{u=1}^{q} \Psi^{sd}_u.\]

With this definition, it is clear that we have

\[
\langle H_n, \Psi^{sd} \rangle = \langle H_n, \Psi^{sd}_1 \rangle \geq q^b \cdot n^{-1} \cdot 2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)|,
\]

(7.7) \[H_n = \sum_{|R|\geq 2^{-n}} \alpha(R)h_R.\]

\(q^b\) is our ‘gain over the trivial estimate’, once we prove that \(\|\Psi^{sd}\|_1 \leq 1\) (estimate (7.14) below). Proving this inequality is the main goal of the technical estimates of the following Lemma:

7.8. Lemma. We have these estimates:

(7.9) \[P(\Psi < 0) \leq \exp(-Aq^{1/2-b});\]

(7.10) \[\|\Psi\|_2 \leq \exp(a'q^{2b});\]

(7.11) \[E\Psi = 1;\]

(7.12) \[\|\Psi\|_1 \leq 1;\]

(7.13) \[\|\Psi^{-}\|_1 \leq 1;\]

(7.14) \[\|\Psi^{sd}\|_1 \leq 1.\]

Here, \(0 < a' < 1\), in (7.10), is a small constant, decreasing to zero as \(a\) in (7.1) goes to zero; and \(A > 1\), in (7.9) is a large constant, tending to infinity as \(a\) in (7.1) goes to zero.

Proof. We give the proof of the Lemma, assuming our main inequalities proved in the subsequent sections.

Proof of (7.9). We first note that Theorem 5.3 implies that \(\rho F_i\) is in \(\exp(L)\). Then using the distributional estimate of Proposition 5.2, we estimate
\[ \mathbb{P}(\Psi < 0) \leq \sum_{i=1}^{q} \mathbb{P}(\tilde{\rho} F_i < -1) \]
\[ = \sum_{i=1}^{q} \mathbb{P}(\rho F_i < -a^{-1}q^{1/2-b}) \]
\[ \leq \exp(-ca^{-1}q^{1/2-b}). \]

**Proof of (7.10).** The proof of this is detailed enough and uses the results of subsequent sections, so we postpone it to Lemma 9.1 below.

It is important for our purposes in the proof of the current Lemma to note that Lemma 9.1 proves a uniform estimate, namely

\[ \sup_{V \subset \{1, \ldots, q\}} \mathbb{E} \prod_{v \in V} (1 + \tilde{\rho}F_v) \leq \exp(a'q^{2b}). \]  

(7.15)

**Proof of (7.11).** Expand the product in the definition of \( \Psi \). The leading term is one. Every other term is a product \( \prod_{k \in V} \tilde{\rho} F_k \), where \( V \) is a non-empty subset of \( \{1, \ldots, q\} \). This product is in turn a linear combination of products of \( r \) functions. Among each such product, the maximum in the first coordinate is unique. This fact tells us that the expectation of these products of \( r \) functions is zero. So the expectation of the product above is zero. The proof is complete.

**Proof of (7.12).** We use the first two estimates of our Lemma. Observe that

\[ \|\Psi\|_1 = \mathbb{E}\Psi - 2\mathbb{E}\Psi 1_{\Psi < 0} \]
\[ \leq 1 + 2\mathbb{P}(\Psi < 0)^{1/2}\|\Psi\|_2 \]
\[ \leq 1 + \exp(-Aq^{1/2-b}/2 + a'q^{2b}). \]

We have taken \( b = 1/6 \) so that \( 1/2 - b = 2b \). For sufficiently small \( a \) in (7.1), we will have \( A \geq a' \). We see that (7.12) holds.

In light of the estimate (7.15), we see that the argument above proves

\[ \sup_{V \subset \{1, \ldots, q\}} \|\prod_{v \in V} (1 + \tilde{\rho}F_v)\|_1 \leq 1. \]  

(7.16)

**Proof of (7.13).** The primary facts are (7.16) and Theorem 10.1; we use the notation devised for that Theorem.
Note that the Inclusion-Exclusion principle gives us the identity
\[ \Psi^- = \sum_{V \subset \{1, \ldots, q\}, |V| \geq 2} (-1)^{|V|+1} \text{Prod}(\text{NSD}(V)) \cdot \prod_{t \in \{1, \ldots, q\} - V} (1 + \tilde{\rho} F_t). \]

We use the triangle inequality, the estimates of Lemma 9.1, Hölder’s inequality, with indices \(1 + 1/q^{2b}\) and \(q^{2b}\), and the estimate of (10.2) in the calculation below. Notice that we have
\[
\sup_{V \subset \{1, \ldots, q\}} \left\| \prod_{t \in V} (1 + \tilde{\rho} F_t) \right\|_{1+q^{-2b}} \leq \sup_{V \subset \{1, \ldots, q\}} \left\| \prod_{t \in V} (1 + \tilde{\rho} F_t) \right\|_{1+q^{-2b}}^{(1-q^{-2b})/(1+q^{-2b})} 
\times \left\| \prod_{t \in V} (1 + \tilde{\rho} F_t) \right\|_2^{q^{-2b}/(1+q^{-2b})} 
\leq \exp(a'/(1 + q^{-2b})) \leq 1. 
\]

We now estimate
\[
\|\Psi^-\|_1 \leq \sum_{V \subset \{1, \ldots, q\}, |V| \geq 2} \left\| \text{Prod}(\text{NSD}(V)) \cdot \prod_{t \in \{1, \ldots, q\} - V} (1 + \tilde{\rho} F_t) \right\|_1 
\leq \sum_{V \subset \{1, \ldots, q\}, |V| \geq 2} \left\| \text{Prod}(\text{NSD}(V)) \right\|_{q^{2b}} \cdot \left\| \prod_{t \in \{1, \ldots, q\} - V} (1 + \tilde{\rho} F_t) \right\|_{1+q^{-2b}} 
\leq \sum_{v=2}^q \left[ q^C n^{-x} \right]^v \leq n^{-c'} \leq 1. 
\]

Proof of (7.14). This follows from (7.13) and (7.12) and the identity \( \Psi = 1 + \Psi^{sd} + \Psi^- \) and the triangle inequality.

\[\square\]

8. The Beck Gain in the Simplest Instance

Beck considered sums of products of \(r\) functions that are not strongly distinct, and observed that the \(L^2\) norm of the same are smaller than one would naively expect. This is what we call the Beck Gain. A product of \(r\) functions will not be strongly distinct if the product involves two or more vectors which agree in one or more coordinates. In this section, we study the sums of products of two \(r\) functions which are not strongly distinct. A later section, §10, will study the general case. The results of this Section are critical to the next section, in which we bound the \(L^2\) norm of our Riesz product.

In this section, and again in §10, we will use this notation. For a subset \(C \subset \mathbb{H}^k_n\), let
\[
(8.1) \quad \text{Prod}(C) := \sum_{(\tilde{r}_1, \ldots, \tilde{r}_k) \in C} \prod_{j=1}^k f_{\tilde{r}_j}. 
\]
In this section, we are exclusively interested in $k = 2$.

Let $C(2) \subset H_n^2$ consist of all pairs of distinct vectors $\{\vec{r}_1, \vec{r}_2\}$ for which $r_{1,2} = r_{2,2}$. J. Beck calls such terms ‘coincidences’ and we will continue to use that term. We need norm estimates on the sums of products of such vectors.

8.2. Lemma. [The Simplest Instance of the Beck Gain.] We have these estimates for arbitrary subsets $C \subset C(2)$

$$\|\text{Prod}(C)\|_p \lesssim p^{5/4} n^{7/4}.$$  
(8.3)

Moreover, if we have $C = C(2) \cap A_s \times A_t$ for some $0 \leq s, t \leq q$ we have

$$\|\text{Prod}(C)\|_p \lesssim p^{3/2} n^{3/2}.$$  
(8.4)

The second estimate of the Lemma appears to be sharp, in that the collection $C(2)$ has three free parameters, and the estimates is in terms of $n^{3/2}$. Note that for $p \approx n$ we have

$$\|\text{Prod}(C_2)\|_{n} \approx \|\text{Prod}(C_2)\|_{\infty}.$$  

And the latter term can be as big as $n^3$, which matches the bound above. Thus we only need to deal with the case $p \leq n$.

The proof of the Lemma requires we pass through an intermediary collection of four tuples of vectors. Let $B(4) \subset H_n^4$ be four tuples of distinct vectors $(\vec{r}, \vec{s}, \vec{t}, \vec{u})$ for which (i) $r_2 = s_2$ and $t_2 = u_2$; and (ii) in the first and third coordinate the maximum is achieved twice.

Proof. The method of proof is probably best explained by considering first the case of $p = 2$. Observe that

$$\|\text{Prod}(C)\|_2^2 = \mathbb{E} \text{Prod}(B) + \mathbb{E} \text{Prod}(\tilde{B}),$$

where $B = C \times C \cap B(4)$ and $\tilde{B}$ is a collection of four-tuples in $C \times C$ in which some of the vectors completely coincide. Indeed, the main point is that

$$\mathbb{E} f_{\vec{r}_1} \cdot f_{\vec{r}_2} \cdot f_{\vec{s}_3} \cdot f_{\vec{s}_4} \neq 0$$

iff the maximum is not unique in each coordinate. But, if the vectors are distinct, this is the definition of $B(4)$. Thus the case $p = 2$ follows almost immediately from Lemma 8.6 below, since $\mathbb{E} \text{Prod}(\tilde{B})$ is easy to estimate.

Now, let us consider $p \geq 4$. Each pair $(\vec{r}, \vec{s}) \in C$ must be distinct in the first and third coordinates. Therefore, we can apply the Littlewood Paley inequalities in those coordinates, very much in the same fashion as it was done in the proof of Theorem 5.3, to estimate

$$N(p) := \|\text{Prod}(C)\|_p \lesssim p \left\| \left( \sum_{a,b} \left| \sum_{(\vec{r},\vec{s}) \in C \atop \max[r_{1,1}] = a \atop \max[r_3,s_3] = b} f_\vec{r} \cdot f_\vec{s} \right|^2 \right|^{1/2} \right\|_p.$$
Here, we have a full power of $p$, as we apply the Littlewood Paley inequalities twice. Observe that

$$
\sum_{a,b} \left| \sum_{(\vec{r},\vec{s}) \in C} f_\vec{r} \cdot f_\vec{s} \right|^2 = \#C + \sum_{i \neq j \in \{1,2,3,4\}} \text{Prod}(C_{i,j}) + \text{Prod}(B).
$$

The term $\#C$ arises from the diagonal of the square. The terms $C_{i,j}$ are

$$
C_{i,j} := \{ (\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) \in C \times C : \vec{r}_i = \vec{r}_j, \text{and the other two vectors are distinct} \}.
$$

Note that by definition, $C_{1,2} = C_{3,4} = \emptyset$, in other cases, the $C_{i,j}$ are of the same class of objects as $C$. The term $B$ we have already defined.

Then, we can estimate by the triangle inequality, and the sub-additivity of $x \mapsto \sqrt{x}$,

$$
p^{-1}||\text{Prod}(C)||_p \leq (\#C)^{1/2} + \sum_{i < j \in \{1,2,3,4\}} ||\text{Prod}(C_{i,j})||_p^{1/2} + ||\text{Prod}(B)||_p^{1/2}.
$$

This inequality is useful for induction.

Let us consider the case of (8.4). We have already seen that $N(2) \leq n^{3/2}$. Hence (8.5) implies that for $p = 2^{n+1}$

$$
N(2^{n+1}) \leq 2^{n+1} \left\{ n^{3/2} + 4N(2^n)^{1/2} \right\}.
$$

Clearly, this can be recursively applied, to yield a proof of (8.4) in the case $p \leq n$. But the case of $p \geq n$ is trivial, as the $L^\infty$ norm of the terms we are estimating are at most $n^3$.

\[ \square \]

8.6. **Lemma.** For any subset $B \subset B(4)$

$$
||\text{Prod}(B)||_p \leq \sqrt{p} n^{7/2}.
$$

If we do not consider arbitrary subsets, the estimate improves. We have the following

$$
||\text{Prod}(B(4) \cap (A_2 \times A_4))||_p \leq p n^3,
$$

This Lemma, with exponents on $n$ being $n^{7/2}$ appears in Beck’s paper [1], in the case of $p = 2$. The $L^p$ variants, following from consequences of Littlewood-Paley inequalities, are important for us.

The first estimate is recorded, as it is interesting that it applies to arbitrary subsets of $B(4)$. We will rely upon the second estimate. Pointed out to us by Mihalis Kolountzakis, this estimate is better for all ranges of $p \leq n$.

**Proof.** We discuss (8.7). The proof is a case analysis, depending upon the number of $\{r, s, t, u\}$ at which the maximums occur in the first and third coordinates. We proceed immediately to the cases.

Let $B_2 \subset B$ consist of those four–tuples $\{r, s, t, u\}$ for which

$$
r_1 = t_1 = \max\{r_1, s_1, t_1, u_1\}, \quad r_3 = t_3 = \max\{r_3, s_3, t_3, u_3\}.
$$


This collection is empty, for necessarily we must have \( r_2 = s_2 = t_2 = u_2 \), but then \( \vec{r} = \vec{s} \), as the parameters of all vectors is \( n \). This violates the definition of \( \mathcal{B} \).

Let \( \mathcal{B}_3 \subset \mathcal{B} \) consist of those four-tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \) for which
\[
    r_1 = t_1 = \max\{r_1, s_1, t_1, u_1\}, \quad r_3 = u_3 = \max\{r_3, s_3, t_3, u_3\}.
\]
That is, the maximal values involve three distinct vectors. These four vectors can be depicted as
\[
    \vec{r} = \begin{pmatrix} r_1(\Box) \\ r_2 \\ r_3 \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} * \\ r_2 \\ \Box \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} r_1 \\ t_2 \\ \Box \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} \Box \\ t_2 \\ r_3 \end{pmatrix}.
\]

A \( \Box \) denotes a parameter which is determined by other choices. It is essential to note that choices of \( r_2 \) and \( r_3 \) determine the value of \( r_1 \) (hence the \( \Box \) in the first coordinate for \( \vec{r} \)), and so the vector \( \vec{r} \). The only free parameters are (say) \( s_1 \), denoted by an \( * \) above.

But, note that we must then have \( |s| = s_1 + s_2 + s_3 < n \). Therefore this case is empty.

Let \( \mathcal{B}_4 \) be those four-tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathcal{B} \) such that \( s_1 = u_1 \) and \( r_3 = t_3 \). That is there are four vectors involved in the maximum of the second and third coordinates. These four vectors can be represented as
\[
\begin{align*}
    (8.9) \quad \vec{r} &= \begin{pmatrix} \Box \\ r_2 \\ r_3 \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} s_1 \\ r_2 \\ \Box \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} \Box \\ t_2 \\ r_3 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} s_1 \\ t_2 \\ \Box \end{pmatrix}.
\end{align*}
\]

The next argument proves (8.7). Let \( \mathcal{B}_4(a, a', b) \) be those four tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathcal{B} \) such that
\[
    r_2 = s_2 = a, \quad t_2 = u_2 = a', \quad s_1 = u_1 = b.
\]
The point to observe is that
\[
    ||\text{Prod}(\mathcal{B}_4(a, a', b))||_p \leq C \sqrt{p} \sqrt{n}.
\]
As there at most \( \leq n^3 \) choices for \( a, a', b \) this will prove the Lemma.

Indeed, we have not specified \( r_3 = t_3 \). Since all vectors are distinct, we can assume without loss of generality that \( a < a' \) (and thus \( r_1 > t_1 \)) and in considering the norm above, we ignore \( \vec{s} \) and \( \vec{u} \), as they are completely specified by the datum \( (a, a', b) \). We apply the Littlewood-Paley inequality in the first coordinate to the product \( f_{\vec{r}} \cdot f_{\vec{t}} \)
\[
    ||f_{\vec{r}} \cdot f_{\vec{t}}||_p \leq \sqrt{p} \left| \left| \sum_{\vec{r}, \vec{t} : t_1 < r_1 = c} \sum_{\epsilon} f_{\vec{r}} \cdot f_{\vec{t}} \right|^{1/2} \right|_p = \sqrt{p} \sqrt{n},
\]
since \( r \) and \( t \) are completely specified once \( r_1 \) is fixed. The proof of (8.7) is finished.
We turn to the proof of (8.8), arguing similarly. We have already seen that the only non-empty case is $\mathbb{B}_4$. Let $\mathbb{B}_4(a, a')$ be those four tuples $[\vec{r}, \vec{s}, \vec{t}, \vec{u}] \in \mathbb{B}_4$ such that

$$r_2 = s_2 = a, \quad t_2 = u_2 = a'.$$

The point to observe is that

$$\|\text{Prod}(\mathbb{B}_4(a, a'))\|_p \leq Cpn.$$

As there at most $\lesssim n^2$ choices for $a, a'$ this proves the Lemma.

The point is that $\text{Prod}(\mathbb{B}_4(a, a'))$ almost splits into a product. Namely, if we define

$$\text{Prod}(\mathbb{B}_{4,1}(a, a')) := \{[\vec{r}, \vec{t}] : r_2 = a, t_2 = a', r_3 = t_3\},$$

$$\text{Prod}(\mathbb{B}_{4,2}(a, a')) := \{[\vec{s}, \vec{u}] : s_2 = a, u_2 = a', s_1 = u_1\},$$

we will have

(8.10) $\text{Prod}(\mathbb{B}_4(a, a')) = \text{Prod}(\mathbb{B}_{4,1}(a, a')) \cdot \text{Prod}(\mathbb{B}_{4,2}(a, a')) - \text{Prod}(\mathcal{M}),$

where $\mathcal{M} \subset (\mathbb{B}_{4,1}(a, a')) \times (\mathbb{B}_{4,2}(a, a'))$ consists of quadruples in which the coincidence either in the first or the third coordinate is not a maximum in that coordinate.

We first prove the estimate

(8.11) $\|\text{Prod}(\mathbb{B}_{4,k}(a, a'))\|_{2p} \lesssim \sqrt{p} \cdot n^{1/2}, \quad k = 1, 2.$

We may assume without loss of generality that $k = 1$, and $a > a'$. The pairs in $\text{Prod}(\mathbb{B}_{4,1}(a, a'))$ consist of the two vectors $\vec{r}$ and $\vec{t}$ in (8.9). These two vectors are parameterized by $t_1$, say. Since $a = r_2 < a' = t_2$, and $r_3 = t_3$, the hyperbolic assumption implies $t_1$ is the maximal coordinate. Therefore, the Littlewood-Paley inequality in this coordinate applies.

Now we deal with the term $\text{Prod}(\mathcal{M})$. For this, assume that in the first coordinate the maximum is achieved at $r_1$. This situation is depicted below:

$$\vec{r} = \begin{pmatrix} \max \left( a \atop r_3 \right) \\ a \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} s_1 \\ a \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} * \\ a' \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} s_1 \\ a' \end{pmatrix}.$$

Notice that in this situation the maximum in the third coordinate cannot be $r_3 = t_3$, for we would then have $s_1 + s_2 + s_3 < r_1 + r_2 + r_3 = n$. So, the maximum in this coordinate is $s_3$ or $u_3$. Also notice, that with $a$ and $a'$ fixed, choosing the values of $r_1$ and $s_3$ (or $u_3$) completely determines the quadruple of vectors. Thus we can apply the Littlewood-Paley inequality twice in the first and the third coordinates, which would yield

(8.13) $\|\text{Prod}(\mathcal{M})\|_p \lesssim (\sqrt{p} \sqrt{n})^2 = pn.$

Combining (8.10), (8.11) and (8.13), we see that we have proved

$$\|\text{Prod}(\mathbb{B}_4(a, a'))\|_p \lesssim pn.$$

The proof is complete.
There is another corollary to the proof above required at a later stage of the proof. For an integer \( a \), let \( \mathbb{B}_a(4) \subset \mathbb{H}_n^4 \) be four tuples of distinct vectors \((\vec{r}, \vec{s}, \vec{t}, \vec{u})\) for which (i) \( r_2 = s_2 \) and \( t_2 = u_2 \); and (ii) in the first coordinate we have \( s_1 = u_1 = a \); and (iii) two of the four vectors agree in the third coordinate.

8.14. **Lemma.** For any integer \( a \), and subset \( \mathcal{B} \subset \mathbb{B}_a(4) \) we have

\[
\| \text{Prod}(\mathcal{B}) \|_p \lesssim pn^{5/2}.
\]

The point of this estimate is that we reduce the number of parameters of \( \mathbb{B}(4) \) by one, and gain a full power of \( n \) in the size of the \( L^p \) norm, as compared to the estimate in (8.7).

**Proof.** In the proof of Lemma 8.6, in the analysis of the terms \( \mathcal{B}_4 \) we used the triangle inequality over the term \( b = s_1 = u_1 \). Treating this coordinate as fixed, we gain a term \( n^{-1} \) in the previous proof, hence proving the Lemma above.

\[\square\]

A further sub-case of the inequality (8.3) demands attention. Using the notation of Lemma 8.2, let

\[
(8.16) \quad \mathcal{C}_{2,b} := \{ (\vec{r}_1, \vec{r}_2) \in \mathcal{C}_2 : r_{1,1} = b \}, \quad 1 \leq a \leq n.
\]

Thus, this collection consists of pairs of distinct vectors, with a coincidence in the second coordinate, and the first coordinate of \( \vec{r}_1 \) is fixed. Note that these collections of variables have two free parameters. At \( L^2 \) we find a 1/4 gain over the ‘naive’ estimate.

8.17. **Lemma.** For any \( b \) and any subset \( \mathcal{C} \subset \mathcal{C}_{2,b} \) we have the estimates

\[
(8.18) \quad \| \text{Prod}(\mathcal{C}) \|_p \lesssim p \cdot n^{5/4}, \quad 2 \leq p < \infty.
\]

**Proof.** As in the proof of Lemma 8.2, we begin with the case \( p = 2 \). Observer that

\[
\| \text{Prod}(\mathcal{C}) \|^2 = \mathbb{E} \text{Prod}(\mathcal{B}),
\]

where \( \mathcal{B} = \mathcal{C}_{2,b} \times \mathcal{C}_{2,b} \cap \mathcal{B}_b(4) \), with the last collection defined in Lemma 8.14. Therefore, the Lemma in this case follows from that Lemma.

More generally, no pair of vectors in \( \mathcal{C}_{2,b}(2) \) can have a coincidence in the third coordinate, so we can use the Littlewood Paley inequalities in that coordinate to estimate

\[
\| \text{Prod}(\mathcal{C}) \|_p \leq \sqrt{p} \left[ \sum_c \sum_{(\vec{r}_1, \vec{r}_2) \in \mathcal{C}} \sum_{\max(r_{1,3}, r_{2,3}) = c} f_{\vec{r}_1} \cdot f_{\vec{r}_2} \right]^{1/2} \left\| \frac{1}{p} \right\|.
\]

Observe that

\[
(8.19) \quad \sum_c \sum_{(\vec{r}_1, \vec{r}_2) \in \mathcal{C}} \sum_{\max(r_{1,3}, r_{2,3}) = c} f_{\vec{r}_1} \cdot f_{\vec{r}_2} = \# \mathcal{C} + \sum_{i < j \in \{1, 2, 3, 4\}} \text{Prod}(\mathcal{C}_{ij}) + \text{Prod}(\mathcal{B}).
\]
Similar to before, we define the collections \( C_{i,j} \) as follows.

\[
C_{i,j} := \{(r_1, r_2, r_3, r_4) \in \mathbb{C} \times \mathbb{C} : r_i = r_j, \text{ and the other two vectors are distinct}\}.
\]

In this case, observe that five of these collections are empty, namely

\[
C_{1,2} = C_{2,3} = C_{1,4} = C_{2,3} = C_{2,4} = \emptyset.
\]

The only non-empty collection is \( C_{1,3} \). Yet, in \( C_{1,3} \), the vectors \( r_2 \) and \( r_4 \) have a coincidence in the first coordinate. Thus, Lemma 8.2 applies to \( C_{1,3} \), so that we have the estimate

\[
\|\text{Prod}(C_{1,3})\|_p \lesssim p^{5/4} n^{7/4}.
\]

Let us prove (8.18). Combining these observations with (8.19) and Lemma 8.14 we see that

\[
p^{-1/2}\|\text{Prod}(C)\|_p \lesssim n + \|\text{Prod}(C_{1,3})\|^{1/2} + \|\text{Prod}(B)\|^{1/2}
\]

\[
\lesssim n + p^{5/8} n^{7/8} + p^{1/2} n^{5/4}.
\]

Concerning the right hand side, note that for \( 2 < p < n^3 \), we have \( p^{5/8} n^{7/8} < p^{1/2} n^{5/4} \). Hence we have proved

\[
\|\text{Prod}(C)\|_p \lesssim pn^{5/4}, \quad 1 < p < n^3.
\]

Yet, for \( p \geq n \) the \( L^p \) norm above is comparable to the \( L^\infty \) norm, so we have finished the proof of (8.18).

\[\square\]

9. The \( L^2 \) Norm of the Riesz Product

We now prove a central estimate of the proof.

9.1. Lemma. The estimate (7.10) holds. Moreover, we have

\[
\sup_{V \subset \{1, \ldots, q\}} E \prod_{t \in V} (1 + \tilde{\rho}F_t)^2 \lesssim \exp(a' q^{2k}).
\]

Here, \( \tilde{\rho} \) is as in (7.2), and \( a' \) is a fixed constant times \( 0 < a < 1 \), the small constant that enters into the definition of \( \tilde{\rho} \).

Remark. A conditional expectation argument is essential to this proof. This Lemma is also proved in Beck’s paper, using a much more involved argument: his more complicated Riesz product precludes our simpler line of reasoning.

Proof. The supremum over \( V \) will be an immediate consequence of the proof below, and so we don’t address it specifically.

Let us give the initial, essential observation. We expand

\[
E \prod_{t=1}^q (1 + \tilde{\rho}F_t)^2 = E \prod_{t=1}^q (1 + 2\tilde{\rho}F_t + (\tilde{\rho}F_t)^2).
\]
Hold the $x_2$ and $x_3$ coordinates fixed, and let $\mathcal{F}$ be the sigma field generated by $F_1, \ldots, F_{q-1}$. We have

$$
\mathbb{E}(1 + 2\tilde{\rho}F + (\tilde{\rho}F)^2 \mid \mathcal{F}) = 1 + \mathbb{E}((\tilde{\rho}F)^2 \mid \mathcal{F})
$$

$$
\quad = 1 + a^2q^{2b-1} + \tilde{\rho}^2\Gamma_q,
$$

(9.3)

where $\Gamma_i := \sum_{\tilde{r} \in \mathcal{A}_i} f_{\tilde{r}} \cdot f_{\tilde{r}}$.

Then, we see that

$$
\mathbb{E} \prod_{t=1}^{q}(1 + 2\tilde{\rho}F_t + (\tilde{\rho}F_t)^2) = \mathbb{E}\left(\prod_{t=1}^{q-1}(1 + 2\tilde{\rho}F_t + (\tilde{\rho}F_t)^2) \times \mathbb{E}(1 + 2\tilde{\rho}F_q + (\tilde{\rho}F_q)^2 \mid \mathcal{F})\right)
$$

(9.4)

$$
\quad \leq (1 + a^2q^{2b-1})\mathbb{E} \prod_{t=1}^{q-1}(1 + 2\tilde{\rho}F_t + (\tilde{\rho}F_t)^2)
$$

$$
\quad + \left|\mathbb{E} \prod_{t=1}^{q-1}(1 + 2\tilde{\rho}F_t + (\tilde{\rho}F_t)^2) \cdot \tilde{\rho}^2\Gamma_q\right|.
$$

(9.5)

This is the main observation: one should induct on (9.4), while treating the term in (9.5) as an error, as the ‘Beck Gain’ estimate (8.4) applies to it.

Let us set up notation to implement this line of approach. Set

$$
N(V; r) := \left\|\prod_{t=1}^{V}(1 + \tilde{\rho}F_t)\right\|_r, \quad V = 1, \ldots, q.
$$

We will use the trivial inequality available from the exponential moments

$$
N(V; 4) \leq \prod_{t=1}^{V}\|1 + \tilde{\rho}F_t\|_4
$$

$$
\quad \leq (1 + Cq^{b-1/2}V)^V
$$

$$
\quad \leq (Cq)^V.
$$

This of course is a terrible estimate, but we now use interpolation, noting that

$$
N(V; 2(1 - 1/q)^{-1}) \leq N(V; 2)^{1-1/q} \cdot N(V; 4)^{1/q}.
$$

(9.6)

We see that (9.4), (9.5) and (9.6) give us the inequality

$$
N(V + 1; 2) \leq (1 + a^2q^{2b-1})^{1/2}N(V; 2) + C \cdot N(V; 2(1 - 1/q)^{-1}) \cdot \|\tilde{\rho}^2\Gamma_q\|_q
$$

(9.7)

$$
\quad \leq (1 + a^2q^{2b-1})^{1/2}N(V; 2) + CN(V; 2)^{1-1/q} \cdot N(V; 4)^{1/q}\|\tilde{\rho}^2\Gamma_q\|_q
$$

$$
\quad \leq (1 + a^2q^{2b-1})^{1/2}N(V; 2) + Cq^{Cn^{-1/2}}N(V; 2)^{1-1/q}.
$$

In the last line we have used the inequality (8.4).
Of course we only apply this as long as \( N(V; 2) \geq 1 \). Assuming this is true for all \( V \geq 1 \), we see that
\[
N(q; 2) \leq (1 + a^2 q^{2b-1} + Cq^{-n^{-1/2}})^q \\
\leq e^{q^2}.
\]
Here of course we need \( Cq^{-n^{-1/2}} \leq aq^{2b-1} \), which we certainly have for large \( n \).

\[ \square \]

10. The Beck Gain

Let us state the main result of this section. Given \( V \subset \{1, \ldots, q\} \) let
\[
\text{NSD}(V) := \left\{ \vec{r}_j : j \in V \right\} \times_{j \in V} A_j \text{ for each } j \in V, \text{ there is a choice of } j' \in V - \{j\} \text{ and } \ell = 2, 3 \text{ so that } r_{j, \ell} = r_{j', \ell} \right\}.
\]
That is, we take tuples of \( r \) vectors, indexed by \( V \), requiring that each \( \vec{r}_j \) be in a coincidence. Such sums admit a favorable estimate on their \( L^2 \) norms.

10.1. Theorem. [The Beck Gain.] There are positive constants \( C_0, C_1, C_2, C_3, \kappa \) for which we have the estimate
\[
(10.2) \quad \left\| \prod(\text{NSD}(V)) \right\|_p \lesssim [C_0|V|^{C_1} p^{C_2} q^{C_3} n^{-\kappa}]^{|V|}, \quad V \subset \{1, \ldots, q\}.
\]

Remark. The novelty in this estimate is that we find that (a) the gain can be given in a manner proportional to \( |V| \) and (b) the gain also holds in \( L^p \) norms. In application, \( p \lesssim q^{2b} = q^{1/3} \approx n^{\varepsilon'} \), so the polynomial growth in \( p \) and in \( q \) is acceptable to us. \footnote{Beck \cite{1} found a gain in \( L^2 \) norm of order \( n^{-1/4} \), for all \( V \). Such a small gain of course forces a much shorter Riesz product.}

The proof of this Theorem requires a careful analysis of the variety of ways that a product can fail to be strongly distinct. That is, we need to understand the variety of ways that coincidences can arise, and how coincidences can contribute to a smaller norm.

Following Beck, we will use the language of Graph Theory to describe these general patterns of coincidences, although there is no graph theoretical fact that we need. Rather, the use of this language is just a convenient way to do some bookkeeping.

The class of graphs that we are interested in satisfies particular properties. A graph \( G \) is the triple of \( (V(G), E_2, E_3) \), of the vertex set \( V(G) \subset \{1, \ldots, q\} \), and edge sets \( E_2 \) and \( E_3 \), of color 2 and 3 respectively. Edge sets are are subsets of
\[
E_j \subset V(G) \times V(G) - \{(k,k) | k \in V(G)\}.
\]
Edges are symmetric, thus if \( (v, v') \in E_j \) then necessarily \( (v', v) \in E_j \).

A clique of color \( j \) is a maximal subset \( Q \subset V(G) \) such that for all \( v \neq v' \in Q \) we have \( (v, v') \in E_j \). By maximality, we mean that no strictly larger set of vertices \( Q' \supset Q \) satisfies this condition.
Call a graph $G$ admissible iff
1. The edges sets, in both colors, decompose into a union of cliques.
2. Any two cliques $Q_2$ in color 2 and clique $Q_3$ in color 3 can contain at most one common vertex.
3. Every vertex is in at least one clique.

A graph $G$ is connected iff for any two vertices in the graph, there is a path that connects them. A path in the graph $G$ is a sequence of vertices $v_1, \ldots, v_k$ with an edge of either color, spanning adjacent vertices, that is $(v_j, v_{j+1}) \in E_2 \cup E_3$.

Reduction to Admissible Graphs. It is clear that admissible graphs as defined above are naturally associated to sums of products of $r$ functions. Given admissible graph $G$ on vertices $V$, we set $X(G)$ to be those tuples of $r$ vectors

\[
[\vec{r}_v : v \in V] \in \prod_{v \in V} A_v,
\]

so that if $(v, v')$ is an edge of color $j$ in $G$, then $r_{v,j} = r_{v',j}$.

We will prove the Lemma below in the following two subsections.

10.3. Lemma. For an admissible graph $G$ on vertices $V$ we have the estimate below for positive, finite constants $C_0, C_1, C_2, C_3, \kappa$:

\[
(10.4) \quad \rho^{|V|} \|\text{Prod}(X(G))\|_1 \leq [C_0 |V|^{C_1} p^{C_2} q^{C_3} n^{-\kappa}]^{|V|}, \quad 2 < p < \infty.
\]

Let us give the proof of Theorem 10.1 assuming this Lemma. Our tool is the Inclusion-Exclusion Principle, but to apply it we need additional concepts.

Given two admissible graphs $G_1, G_2$ on the same vertex set $V$, let $G_1 \land G_2$ be the smallest admissible graph which contains all the edges in $G_1$ and in $G_2$. By smallest, we mean the graph with the fewest number of edges; and such a graph may not be defined, in which case we take $G_1 \land G_2$ to be undefined. We recursively define $G_1 \land \cdots \land G_k := (G_1 \land \cdots \land G_{k-1}) \land G_k$. This wedge product is associative.

Let $G_0$ be the set admissible graphs on $V$ which are not of the form $G_1 \land G_2$ for admissible $G_1 \neq G_2$. These are the ‘prime’ graphs. (If $V$ is of cardinality 2 or 3, every graph is prime.) Now define $G_k$ to be those graphs which are equal to a wedge product $G_1 \land \cdots \land G_k$, with $G_j \in G_0$, and moreover, $k$ is the smallest integer for which this is true. Clearly, we only need to consider $k \leq q$.

Then, by the inclusion-exclusion principle,

\[
(10.5) \quad \text{Prod}(\text{NSD}(V)) = \sum_{k=0}^{q} (-1)^k \sum_{G \in G_k} \text{Prod}(X(G)).
\]

The number of admissible graphs on a set of vertices $V$ is at most $2^{|V||V|!} < 2^{|V||V|!}$. So that using (10.4) clearly implies Theorem 10.1.
Norm Estimates for Admissible Graphs. We begin this section with a further reduction to connected admissible graphs. Let us write \( G \in BG(C_0, C_1, C_2, C_3, \kappa) \) if the estimates (10.4) holds. (‘BG’ for ‘Beck Gain.’) We need to see that all admissible graphs are in \( BG(C_0, C_1, C_2, C_3, \kappa) \) for non-negative, finite choices of the relevant constants.

10.6. Lemma. Let \( C_0, C_1, C_2, C_3, \kappa \) be non-negative constants. Suppose that \( G \) is an admissible graph, and that it can be written as a union of subgraphs \( G_1, \ldots, G_k \) on disjoint vertex sets, where all \( G_j \in BG(C_0, C_1, C_2, C_3, \kappa) \). Then,

\[
G \in BG(C_0, C_1, C_2, C_2 + C_3, \kappa).
\]

With this Lemma, we will identify a small class of graphs for which we can verify the property (10.4) directly, and then appeal to this Lemma to deduce Theorem 10.1. Accordingly, we modify our notation. If \( \mathcal{G} \) is a class of graphs, we write \( \mathcal{G} \subset BG(\kappa) \) if there are constants \( C_0, C_1, C_2, C_3 \) such that \( \mathcal{G} \subset BG(C_0, C_1, C_2, C_3, \kappa) \).

Proof. We then have by Proposition 10.7

\[
\text{Prod}(X(G)) = \prod_{j=1}^{k} \text{Prod}(X(G_j)).
\]

Using Hölder’s inequality, we can estimate

\[
\rho^{\mid V \mid} \text{Prod}(X(G)) \leq \prod_{j=1}^{k} \rho^{\mid V_j \mid} \text{Prod}(X(G_j)) \leq \prod_{j=1}^{k} [C_0(kp)^{C_1}q^{C_2}n^{-\kappa}]^{\mid V_j \mid} \leq [C_0p^{C_1}q^{C_2}n^{C_1}n^{-\kappa}]^{\mid V \mid}.
\]

Here, we use the fact that since the graphs are non-empty, we necessarily have \( k \leq q \). \( \square \)

10.7. Proposition. Let \( G_1, \ldots, G_p \) be admissible graphs on pairwise disjoint vertex sets \( V_1, \ldots, V_p \). Extend these graphs in the natural way to a graph \( G \) on the vertex set \( V = \bigcup V_i \). Then, we have

\[
\text{Prod}(X(G)) = \prod_{i=1}^{p} \text{Prod}(X(G_i)).
\]

Connected Graphs Have the Beck Gain. We single out for special consideration the connected admissible graphs \( G \). Let \( \mathcal{G}_{\text{connected}} \) be the collection of all admissible connected graphs on \( V \subset \{1, \ldots, q\} \).

10.8. Lemma. We have \( \mathcal{G}_{\text{connected}} \subset BG(\frac{1}{15}) \).
We will have to pay special attention to the case of 2 and 3 vertices. It is important to observe that the first coordinates are necessarily distinct, and have the partial order inherited from the vertex set $V$. Namely, the vertex set $V \subset \{1, \ldots, q\}$, and $V$ inherits the order from the integers. By the construction of our Riesz product, the first coordinates inherit this same order.

**General Remarks on Littlewood-Paley Inequality.** These remarks are essential to our analysis of this lemma, and the Theorem we are proving. The vertex set $V$ is a subset of $\{1, \ldots, q\}$ and it inherits an order from that set. Moreover, the tuples of $r$ vectors do as well. Namely, writing $V = \{v_1 < \cdots < v_\ell\}$, for $\{\vec{r}_1, \ldots, \vec{r}_\ell\} \in X(G)$, we have, by construction, $r_{1,1} < \cdots < r_{\ell,1}$. This since $r_{m,1} \in I_{v_m}$, where $I_{v_m}$ is the increasing sequence of intervals of length equal to $n/q$ that partition $\{1, \ldots, n\}$.

There is a natural way to apply the Littlewood-Paley inequalities. For integer $b_\ell \in I_\ell$, let $X(G; b_\ell)$ be the tuple of $r$ vectors $\{\vec{r}_1, \ldots, \vec{r}_\ell\}$ such that $r_{\ell,1} = b_\ell$. We have

$$\|\text{Prod}(X(G))\|_p \leq \sqrt{p} \cdot \sup_{b_\ell \in I_\ell} \left[ \sum_{b_\ell \in I_\ell} |\text{Prod}(X(G; b_\ell))|^2 \right]^{1/2}.$$  

(10.9)

It is tempting to continue this procedure, by applying the Littlewood-Paley inequality again to the vertex $v_{\ell-1}$. Yet—and this in an important point—due to the nature of $r$ functions, this option is blocked to us. The vertex $v_\ell$ is in at least one clique $Q$ of, say, color 2. We could choose a value $c_Q$ for that clique, thereby specifying all coordinates of the vector $\vec{r}_\ell$. Set $X(G; b_\ell; c_Q)$ be the tuple of $r$ vectors $\{\vec{r}_1, \ldots, \vec{r}_\ell\}$ such that

$$\{\vec{r}_1, \ldots, \vec{r}_{\ell-1}, (b_\ell, c_Q, n - b_\ell - c_Q)\} \in X(G; b_\ell).$$

Here, $X(G; b_\ell; c_Q)$ consists of tuples of length $\ell - 1$, since the vector $\vec{r}_\ell$ is completely specified. Thus, we see that

$$\|\text{Prod}(X(G))\|_p \leq \sqrt{p} \cdot n \sup_{c_Q} \left[ \sum_{b_\ell} |\text{Prod}(X(G; b_\ell; c_Q))|^2 \right]^{1/2}.$$  

(10.10)

At this point, the (Hilbert space) Littlewood-Paley inequalities will again apply.

We will refer to the notation above. Keep in mind that $\vec{b}$ is for the coordinates specified by a Littlewood-Paley inequality; $\vec{c}$ are for the coordinates in a coincidence that we use the triangle inequality on. We shall return to these themes momentarily.

**Proof of Lemma 10.8.** We begin the proof with a discussion of the case of two and three vertices, which will not be susceptible to the general methods related to the Littlewood-Paley inequality outlined above.
The Case of Two Vertices. Notice that if \( G \) consists of only two vertices, the relevant estimate is (8.4). Namely, we have
\[
\| \text{Prod}(X(G)) \|_p \leq C P^{3/2} n^{3/2}.
\]
Equivalently, \( G \in \mathcal{B}G(C_{0}, 3/4, 0, 1/4) \).

The Case of Three Vertices. The case of \( G \in \mathcal{G}_{\text{connected}} \) having three vertices depends critically on the same phenomena behind the Beck Gain for graphs on two vertices. We will deduce this case as a corollary to the case of two vertices.

There are three distinct sub-cases. The more delicate of the two cases is as follows. The graph is depicted as
\[
(10.11)
\]
\[
\begin{array}{ccc}
  v_1 & v_2 & v_3 \\
  \square & \square & \square \\
  \bullet & = & \bullet
\end{array}
\]
where \( v_1 < v_2 < v_3 \). (The case of \( v_2 < v_1 < v_3 \) is entirely the same, and we don’t discuss it directly.)

By our general remarks on the Littlewood-Paley inequality, this inequality applies in the first coordinate, to the vertex \( v_3 \). Using the notation in (10.9), we have
\[
\| \text{Prod}(X(G)) \|_p \lesssim \sqrt{p} \left\| \sum_{b_3 \in I_{v_3}} |\text{Prod}(X(G; b_3))|^2 \right\|^{1/2}.
\]
The vectors \( v_2 \) and \( v_3 \) have a coincidence in the third coordinate. Therefore, we specify the value of the coincidence to be \( c_3 \) and estimate
\[
(10.12) \quad \| \text{Prod}(X(G)) \|_p \lesssim \sqrt{p} \cdot n \cdot \sup_{c_3} \left\| \sum_{b_3} \text{Prod}(X(G; b_3; c_3))^2 \right\|^{1/2}.
\]
Recall that \( X(G; b_3; c_3) \) consists only of pairs of vectors. This graph can be depicted as
\[
\begin{array}{ccc}
  v_1 & v_2 \\
  \square & \square \\
  \bullet & = \\
  c_3
\end{array}
\]
But this is the case considered in (8.18). From that inequality, we see that we have the estimate
\[
\| \text{Prod}(X(G; b_3; c_3)) \|_p \lesssim \sqrt{p} n^{5/4}.
\]
Therefore,
\[
\left\| \left( \sum_{b_3} \text{Prod}(X(G; b_3; c_3))^2 \right)^{1/2} \right\|_p \lesssim \sqrt{n} \sup_{b_3} \| \text{Prod}(X(G; b_3; c_3)) \|_p \lesssim \sqrt{p} \cdot n^{7/4}.
\]
Here we have crudely estimated the $\ell^2$ sum in (10.12). Combining the last estimate with (10.12), we see that

\begin{equation}
\|\text{Prod}(X(G))\|_p \lesssim p^{3/2} n^{11/4}.
\end{equation}

Recall that the point of comparison is to $\rho^{-3} = n^3 q^{-3/2}$, and the estimate above is smaller by $n^{-1/4}$. Thus the class of graphs given by (10.11) are contained in $BG(1/2^3)$.

The other case is when the graph can be depicted by

\begin{align*}
v_1 \quad v_3 \quad v_2 \\
\square \quad \square \quad \square \\
\bullet = \bullet \quad \bullet = \bullet
\end{align*}

where $v_3$, the maximal index is in both cliques. This case is much easier, as one application of the Littlewood Paley inequality, and the triangle inequality will determine the value of both cliques. It is very easy to see that this class of graphs is in $BG(1/6)$, and the details are omitted. The third case is even easier – it involves the graphs which have a clique of size three in one of the coordinates. Hence the discussion of graphs on three vertices is complete.

A General Estimate. We now present a general recursive estimate for the $L^p$ norm of $\text{Prod}(X(G))$, assuming that $G$ is a connected graph on at least four vertices. Write $V$ as

$$V = \{v_1 < \cdots < v_{\ell}\}.$$ 

The estimate is obtained recursively. Along the way we will construct two disjoint subsets $V_{3/2}, V_{1/2} \subset V$. $V_{3/2}$ will be the vertices to which we apply both the Littlewood Paley and triangle inequalities, thus these vertices contribute $n^{3/2} q^{-1/2}$ to our estimate. $V_{1/2}$ will be the vertices to which we apply only the Littlewood Paley inequality, thus these vertices contribute $(n/q)^{1/2}$ to our estimate. Those vertices not in $V_{3/2} \cup V_{1/2}$ will be those which are determined by earlier steps in the procedure. They contribute nothing to our estimate. In estimating an $L^p$ norm, the power of $p$ is one-half of the number of applications of the Littlewood-Paley inequality, namely $\frac{1}{2} \#(V_{3/2} \cup V_{1/2})$.

The purpose of these considerations is to prove the estimate

\begin{equation}
\|\text{Prod}(X(G))\|_p \leq (C \sqrt{p})^{\#(V_{3/2} \cup V_{1/2})} (n/q)^{\#(V_{3/2} \cup V_{1/2})/2} n^{|V_{3/2}|}.
\end{equation}

Initialize

$$V_{3/2} \leftarrow \emptyset, \quad V_{1/2} \leftarrow \emptyset, \quad Q_{\text{fixed}} \leftarrow \emptyset.$$ 

The last collection consists of those cliques which are specified by earlier stages of the argument.
At each stage, we will have an estimate for the form
\[
\| \text{Prod}(X(G)) \|_p \leq (C \sqrt{p})^{V_{3/2} + |V_{1/2}|} n^{V_{3/2}}
\]
\[
\times \sup_{\vec{c} \in \{1, \ldots, n\}^{Q_{\text{fixed}}}} \left\| \left[ \sum_{\vec{b} \in \{1, \ldots, n\}^{V_{3/2} \cup V_{1/2}}} \text{Prod}(X(G; \vec{b}; \vec{c})) \right]^{1/2} \right\|.
\]

**Base Case of the Recursion.** We update \(V_{3/2} \leftarrow \{v_\ell\}\), since it is the maximal element. We update \(Q_{\text{fixed}}\) to those cliques which contain \(v_\ell\). Then (10.15) is a consequence of (10.10).

**Recursive Case.** At this point, we have the datum \(V_{3/2}, V_{1/2}\), and \(Q_{\text{fixed}}\). We also have datum \(\vec{b} \in \{1, \ldots, n\}^{V_{3/2} \cup V_{1/2}}\), and \(\vec{c} \in \{1, \ldots, n\}^{Q_{\text{fixed}}}\). Notice that this datum can completely specify some \(r\) vectors associated to vertices not in \(V_{3/2} \cup V_{1/2}\)—think of a vertex that is in two cliques in \(Q_{\text{fixed}}\).

The recursion stops if every vertex \(v_k\) is determined by this datum. Otherwise, let \(k\) be the largest integer such that \(\vec{r}_{v_k}\) is *not* determined by this datum. If *no clique in* \(Q_{\text{fixed}}\) *contains* \(v_k\) update
\[
V_{3/2} \leftarrow V_{3/2} \cup \{v_k\},
\]
and update \(Q_{\text{fixed}}\) to include those cliques which contain \(v_k\). By application of the Littlewood-Paley inequality and the triangle inequality, the estimate (10.15) continues to hold for these updated values.

If *some clique in* \(Q_{\text{fixed}}\) *contains* \(v_k\), then there can be exactly one clique \(Q_{v_k}\) which does, for otherwise \(\vec{r}_{v_k}\) would have been completely specified by these two cliques. Update
\[
V_{1/2} \leftarrow V_{1/2} \cup \{v_k\},
\]
and update \(Q_{\text{fixed}}\) to include all cliques which contain \(v_k\). By application of the Littlewood-Paley inequality, the estimate (10.15) continues to hold for these updated values.

Once the recursion stops the inequality (10.15) holds. But note that we necessarily have
\[
\text{Prod}(X(G; \vec{b}; \vec{c}))^2 \equiv 1,
\]
as all \(r\) vectors are completely determined by \(\vec{b}\) and \(\vec{c}\). Therefore, we have proven (10.14).

**The Conclusion of the Proof.** Since \(V_{3/2}\) and \(V_{1/2}\) are disjoint subsets of \(V\), we have proven the inequality
\[
\rho^{|V|} \| \text{Prod}(X(G)) \|_p \leq (C \sqrt{p})^{V_{3/2} + |V_{1/2}|} n^{V_{3/2}} + \frac{1}{2} |V_{1/2}| - |V|.
\]

And the remaining analysis concerns the exponent on \(n\) above, namely we should see that
\[
|V|^{-1} \left[ \frac{3}{2} |V_{3/2}| + \frac{1}{2} |V_{1/2}| - |V| \right] \leq -\frac{1}{10},
\]
for a fixed positive choice of \(\kappa\), and all connected graphs \(G\) on at least four vertices. We would conclude that this collection of graphs is in \(\text{BG}(\frac{1}{10})\).
In order to make the left hand side of (10.17) as large as possible, we should maximize $V_{3/2}$. To continue, we note another formula. Let $E(G)$ be the total number of edges in the graph $G$, and let $E(v)$ be the number of edges in $G$ with one endpoint of the edge being $v$.

For $v \in V_{3/2} \cup V_{1/2}$, let $F(v)$ be the number of edges which are specified upon the selection of that vertex in our recursive procedure. It is clear that we have $E(v) = F(v)$ if $v \in V_{3/2}$. But also,

$$\sum_{v \in V_{3/2} \cup V_{1/2}} F(v) = E(G).$$

It follows that to maximize the cardinality of $V_{3/2}$, those vertices must be in small cliques. There are two different classes of graphs which are extremal with respect to these criteria.

The first extremal class consists of graphs $G$ with all cliques being of size 2, and the number of cliques is $|V| - 1$. For such graphs, $|V_{3/2}| \leq \lceil \frac{1}{2}|V| \rceil$, and if the value is maximal then $V_{1/2}$ is either 0 if $|V|$ is odd, and 1 if $|V|$ is even. It is straightforward to see that the maximum of (10.17) occurs at $|V| = 5$, and is $-\frac{1}{10}$. Here, it is vital that we have already discussed the case of two and three vertices!

The second class are graphs on an even number of vertices, with half the vertices in a clique $Q$, and each vertex $v \in Q$ is in one clique of size 2. One can depict such a graph on six vertices as

$v_1 \; v_2 \; v_3 \; v_4 \; v_5 \; v_6$

$* \; = \; * \; = \; *$

$a \; b \; c \; a \; b \; c$

The vertices are written in increasing order: $v_1 < v_2 < v_3 < v_4 < v_5 < v_6$. Note that $v_1, v_2, v_3$ form a single clique of color 2. There are three additional cliques of size 2, all of color 3. They are $\{v_j, v_{j+3}\}$ for $j = 1, 2, 3$. For such a graph, it is clear that $|V_{3/2}| = \frac{1}{2}|V|$, and $|V_{1/2}| = 1$. The term (10.17) behaves exactly like the first class of extremal graphs on an even number of vertices. Our proof is complete.

\[\Box\]

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