NOETHER-LEFSCHETZ THEORY AND NÉRON-SEVERI GROUP

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Abstract. Let \( Z \) be a closed subscheme of a smooth complex projective complete intersection variety \( Y \subseteq \mathbb{P}^N \), with \( \dim Y = 2r + 1 \geq 3 \). We describe the Néron-Severi group \( NS_r(X) \) of a general smooth hypersurface \( X \subset Y \) of sufficiently large degree containing \( Z \).

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1. Introduction

After Noether and Lefschetz \([1]\) one knows that the intermediate Néron-Severi group (i.e. the image of the cycle map \( A_r(X) \to H_m(X; \mathbb{Z}) \), \([6]\), §19.1) of a general hypersurface \( X \subset \mathbb{P}^{m+1} = \mathbb{P}^{m+1}(\mathbb{C}) \), \( m = 2r \geq 2 \), is generated by the \( r \)-th power of the hyperplane class, as soon as \( \deg X \geq 2 + 2/r \). On the other hand a result of Lopez \([10]\), inspired by a previous work of Griffiths and Harris \([7]\), gives a recipe for the computation of the Néron-Severi group of a general complex surface of sufficiently large degree in \( \mathbb{P}^3 \) containing a given smooth curve. One of the main purposes of our paper is to generalize Lopez’s result to higher Néron-Severi groups.

More generally let \( Y \subseteq \mathbb{P}^N \) be a smooth complex projective complete intersection of dimension \( m + 1 = 2r + 1 \geq 3 \), \( Z \) be a closed subscheme of \( Y \), and \( \delta \) be a positive integer such that \( I_{Z,Y}(\delta) \) is generated by global sections. Assume that for \( d \gg 0 \) the general divisor \( X \in |I^0(Y,I_{Z,Y}(d))| \) is smooth. This implies that \( 2 \dim Z \leq m \) and that, for any \( d \geq \delta \), there exists a smooth hypersurface of degree \( d \) containing \( Z \) \([11]\). Improving \([11], 0.4. \) Theorem), in \([2]\) we proved that if \( d \geq \delta + 1 \) then the monodromy representation on \( H^m(X; \mathbb{Q})_{\perp Z} \) for the family of smooth divisors \( X_t \in |I^0(Y,O_Y(d))| \) containing \( Z \) as above is irreducible (here we denote by \( H^m(X; \mathbb{Q})_{\perp Z} \) the orthogonal complement in \( H^m(X; \mathbb{Q}) \) of the subspace \( H^m(Y; \mathbb{Q}) + H^m(X; \mathbb{Q})_Z \), where \( H^m(X; \mathbb{Q})_Z \) denotes the subspace of \( H^m(X; \mathbb{Q}) \) generated by the cycle classes of the maximal dimensional irreducible components of \( Z \) if \( m = 2 \dim Z \), and \( H^m(X; \mathbb{Q})_Z = 0 \) otherwise). Such a result does not apply directly to Néron-Severi groups since they involve integral homology, so this problem is more subtle.

Unfortunately, the techniques generally used around questions of this kind (semistable degenerations, mixed Hodge structures, intersection (co)homology, etc.) may have a worse behaviour in passing from \( \mathbb{Q} \) to \( \mathbb{Z} \). Our approach consists in looking at \( X \) as the general hyperplane section of a very special variety with isolated singularities, and to reduce the problem to a question about the homology groups of such a variety, which can be expressly computed.

Our first result is the following:
Theorem 1.1. Let $Y$ and $Z$ be as above, and let $X \in |H^0(Y, \mathcal{O}_Y(d))|$ be a general divisor containing $Z$, with $d \geq \delta + 1$. Let $Z_1, \ldots, Z_\rho$ be the irreducible components of $Z$ of dimension $r$ (if there are). Assume that $h^{r-m-i}(X) \neq 0$ for some $i \neq r$. Then the Néron-Severi group $NS_r(X)$ of $X$ is of rank $\rho + 1$ freely generated by $Z_1, \ldots, Z_\rho$ and by the class $H^i_X \in H^m(X; \mathbb{Z})$ of the linear section of $X$.

Theorem above follows from the more general Theorem 3.2 which improves the main result of [2], and also provides us the appropriate context in order to extend Lopez Theorem to higher Néron-Severi groups:

Theorem 1.2. Let $Y$ be as above. Let $X, G_1, \ldots, G_r$ be a regular sequence of smooth divisors in $Y$, with $X \in |H^0(Y, \mathcal{O}_Y(d))|$, $G_l \in |H^0(Y, \mathcal{O}_Y(k_l))|$ for $1 \leq l \leq r$, and $d > k_1 > \cdots > k_r \geq 1$. Let $Z$ be a closed subscheme of the complete intersection $X \cap G_1 \cap \cdots \cap G_r$. Let $X_t \in |H^0(Y, \mathcal{O}_Y(d))|$ be a general divisor containing $Z$, and assume that $h^{r-m-i}(X) \neq 0$ for some $i \neq r$. Denote by $Z_1, \ldots, Z_\rho, R_1, \ldots, R_\sigma$ the irreducible components of $X_t \cap G_1 \cap \cdots \cap G_r$, where $Z_1, \ldots, Z_\rho$ denote the irreducible components of $Z$ of maximal dimension $r$. Then $NS_r(X_t)$ is of rank $\rho + \sigma$ freely generated by $H^i_X, Z_1, \ldots, Z_\rho, R_1, \ldots, R_\sigma$.

When $Y = \mathbb{P}^3$, and $Z \subseteq Y$ is an irreducible smooth curve, then previous Theorem 1.2 reduces to the main result in [10], of which we now give a different proof.

We note that the proof of the asserted independence of the generators will follow from the injectivity (proved in Theorem 2.3) of the push-forward $H_m(W; \mathbb{Z}) \rightarrow H_m(X; \mathbb{Z})$, where $Y = X \cap G$ is a complete intersection with $G$ smooth hypersurface such that $deg G < deg X$ (condition which implies that $W$ has at worst isolated singularities [5]). We believe this fact of independent interest since it can be false when $G$ is not smooth, even for nodal $W$’s. Indeed a general hypersurface $X \subset \mathbb{P}^5$ contains non-factorial nodal threefolds of the form $G \cap X$, with $deg G < deg X$.

2. Preliminaries

Notations 2.1. (i) Let $Y \subseteq \mathbb{P}^N$ be a smooth complex projective variety, complete intersection in $\mathbb{P}^N$, of odd dimension $m + 1 = 2r + 1 \geq 3$. Fix integers $1 \leq k < d$, and smooth divisors $G \in |H^0(Y, \mathcal{O}_Y(k))|$ and $X \in |H^0(Y, \mathcal{O}_Y(d))|$. $X$ is a smooth complete intersection, hence $H^m(X; \mathbb{Z})$ is torsion free and $H^m(X; \mathbb{Z}) \subset H^m(X; \mathbb{Q})$. Put $W := G \cap X$. By [3], p. 133, Proposition 4.2.6. and proof) we know that $W$ has only isolated singularities.

(ii) For any smooth divisor $X_t \in |H^0(Y, \mathcal{O}_Y(d))|$ we will denote by PD the Poincaré dualities $H_m(X_t; \mathbb{Z}) \cong H^m(X_t; \mathbb{Z})$ and $H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q})$.

(iii) Let $P = Bl_W(Y)$ be the blowing-up of $Y$ along $W$. We refer to [2], 4.1, for a geometric description of $P$. Here we recall some facts. $P$ has at worst isolated singularities, which are isolated hypersurface singularities because $P$ is a divisor in the smooth variety $\mathbb{P}(\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d))$. For the strict transform $\tilde{G}$ of $G$ in $P$ we have $\tilde{G} \cong G$, and $G \subseteq P \setminus Sing(P)$. The same holds for any smooth divisor $X_t \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $W$.

(iv) Let $\{X_t\}_{t \in L}$ ($L = \mathbb{P}^1$) be a general pencil of divisors $X_t \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $W$, and let $B_L$ be its base locus (apart from $W$), which we may consider as contained in $P \setminus Sing(P)$. Notice that $B_L \cong X_t \cap M_L$ for a suitable general $M_L \in |H^0(Y, \mathcal{O}_Y(d - k))|$. Denote by $P_L$ the blowing-up of $P$ along $B_L$, and consider the natural map $f : P_L \rightarrow L$. Let $A \subseteq L$ be the set of the critical values.
of \( f \). This is a finite set, and there is a point \( a_{\infty} \in A \) corresponding to the unique reducible fibre of \( f \). For any \( t \in L \setminus \{ a_{\infty} \} \) \( f^{-1}(t) \cong X_t \in |H^0(Y, \mathcal{O}_Y(d))| \). When \( a \in A \setminus \{ a_{\infty} \} \) then \( X_a \) is an irreducible divisor with a unique singular point \( q_a \in P_L \), and \( \text{Sing}(P_L) \subseteq \{ q_a : a \in A \setminus \{ a_{\infty} \} \} \).

(v) Fix two regular values \( t, t_1 \in L \setminus A \), with \( t \neq t_1 \), and for any critical value \( a \in A \) of \( L \) fix a closed disk \( \Delta_a \subset L \setminus \{ t \} \cong \mathbb{C} \) with center \( a \) and radius \( 0 < \rho \ll 1 \). As in \( [8] \), (5.3.1) and (5.3.2), one proves that

\[
H_*(P_L \setminus X_t, X_{t_1}; \mathbb{Z}) \cong \oplus_{a \in A} H_*(f^{-1}(\Delta_a), X_{a, \rho}; \mathbb{Z}).
\]

(vi) Fix \( a \in A \setminus \{ a_{\infty} \} \). By \( [9] \), p. 28, we know that near to the isolated singular point \( q_a \in P_L \) the pencil \( f : P_L \to L \) defines a Milnor fibration with Milnor fiber \( X_{a, \rho} \cap D_a \), where \( D_a \) denotes a closed ball of \( \mathbb{P}(\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d)) \) with center \( q_a \) and small radius \( \epsilon \) with \( \rho << \epsilon << 1 \). The Milnor fiber \( X_{a, \rho} \cap D_a \) has the homotopy type of a bouquet of \( m \)-spheres. In particular

\[
H_m(X_{a, \rho} \cap D_a; \mathbb{Z}) \text{ is torsion free, and } H_{m+1}(X_{a, \rho} \cap D_a; \mathbb{Z}) = 0.
\]

Moreover, since \( f^{-1}(\Delta_a) \setminus D_a^0 \to \Delta_a \) is a trivial fibre bundle (\( D_a^0 := \text{interior of } D_a \)), Excision Axiom and Leray-Hirsch Theorem \( [13] \) imply that the inclusion \( (X_{a, \rho}, X_{a, \rho} \cap D_a) \subseteq (f^{-1}(\Delta_a), f^{-1}(\Delta_a) \cap D_a) \) induces a natural isomorphism \( H_*(X_{a, \rho}, X_{a, \rho} \cap D_a; \mathbb{Z}) \cong H_*(f^{-1}(\Delta_a); f^{-1}(\Delta_a) \cap D_a; \mathbb{Z}) \). Therefore, from the conic structure of \( f^{-1}(\Delta_a) \cap D_a \) (\( [9] \), Lemma (2.10)), we deduce a natural isomorphism for any \( l > 0 \)

\[
H_l(X_{a, \rho}, X_{a, \rho} \cap D_a; \mathbb{Z}) \cong H_l(f^{-1}(\Delta_a); \mathbb{Z}).
\]

(vii) The fibre \( f^{-1}(a_{\infty}) \) is the union \( \bar{G} \cup \bar{M}_L \) of the strict transforms of \( G \) and \( M_L \). Since \( \bar{G} \cong G \) and \( \bar{M}_L \) is isomorphic to the blowing-up of \( M_L \) along the smooth complete intersection \( M_L \cap W \), and \( G \cap M_L \cong G \cap M_L \), then the map \( f^{-1}(\Delta_a) \to \Delta_a \cap D_a \) is a semi-stable degeneration. Hence \( f^{-1}(\Delta_{a_{\infty}}) \) retracts onto \( f^{-1}(a_{\infty}) \) (\( [12] \), p. 185), and we have

\[
H_*(f^{-1}(\Delta_{a_{\infty}}); \mathbb{Z}) \cong H_*(f^{-1}(a_{\infty}); \mathbb{Z}).
\]

(viii) For any regular value \( t \in L \setminus A \) the natural inclusion map \( i_t : X_t \to P_L \) induces Gysin maps \( i_t^* : H_{m+2}(P_L; \mathbb{Z}) \to H_m(X_t; \mathbb{Z}) \) and \( i^*_{t, Q} : H_{m+2}(P_L; \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \). Now let \( I_W \) be the subspace of the invariant cocycles in \( H^m(X_t; \mathbb{Q}) \) with respect to the monodromy representation on \( H^m(X_t; \mathbb{Q}) \) for the family of smooth divisors \( X_t \in |H^0(Y, \mathcal{O}_Y(d))| \) containing \( W \). Denote by \( i_W : W \to X_t \) the inclusion map. It induces push-forward maps \( i_{W*} : H_m(W; \mathbb{Z}) \to H_m(X_t; \mathbb{Z}) \) and \( i_{W, Q*} : H_m(W; \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \). Put \( H^m(X_t; \mathbb{Z})_W := \text{Im}(PD \circ i_{W*}) \), and \( H^m(X_t; \mathbb{Q})_W := \text{Im}(PD \circ i_{W, Q*}) \). Observe that \( H^m(Y; \mathbb{Z}) + H^m(X_t; \mathbb{Z})_W \) is contained in \( \text{Im}(PD \circ i_t^*) \). By \( [2] \), (11), we also know that

\[
I_W = H^m(Y; \mathbb{Q}) + H^m(X_t; \mathbb{Q})_W.
\]

**Lemma 2.2.** The following properties hold: (a) \( \text{Im}(PD \circ i_t^*) = H^m(Y; \mathbb{Z}) + H^m(X_t; \mathbb{Z})_W \); (b) \( \text{Coker}(PD \circ i_t^*) \) is torsion free; (c) \( \dim \ker(PD \circ i_t^*) \leq 2 \).

**Proof.** (a) We only have to prove that

\[
\text{Im}(PD \circ i_t^*) \subseteq H^m(Y; \mathbb{Z}) + H^m(X_t; \mathbb{Z})_W.
\]
Consider the following natural commutative diagram:

\[
\begin{array}{ccc}
H_{m+2}(P_L; \mathbb{Z}) & \overset{\rho_1}{\longrightarrow} & H_{m+2}(\widetilde{W} \cup \widetilde{B_L}; \mathbb{Z}) \\
\alpha_1 \downarrow & & \downarrow \alpha_1 \\
H_{m+2}(Y; \mathbb{Z}) & \overset{\rho_2}{\longrightarrow} & H_{m+2}(Y \setminus (W \cup (X_t \cap M_L)); \mathbb{Z}) \\
\beta_1 \downarrow & & \downarrow \beta_1 \\
H_m(X_t; \mathbb{Z}) & \overset{\rho_3}{\longrightarrow} & H_m(X_t \setminus (W \cup (X_t \cap M_L)); \mathbb{Z})
\end{array}
\]

where \( \alpha \) and \( \alpha_1 \) are push-forward maps, \( \beta \) and \( \beta_1 \) are Gysin maps, and \( \rho_1, \rho_2 \) and \( \rho_3 \) are restriction maps to open subset in Borel-Moore homology ([6], p. 371). Fix a general, then \( W \cap M_L \) and \( X_t \cap M_L \) are smooth complete intersections in \( \mathbb{P}^N \), of dimension \( m-2 \) and \( m-1 \). In particular \( H_{m-1}(W \cap M_L; \mathbb{Z}) = 0 \). From the Mayer-Vietoris sequence of the pair \((W, X_t \cap M_L)\) we deduce that the natural map \( H_m(W; \mathbb{Z}) \oplus H_m(X_t \cap M_L; \mathbb{Z}) \to H_m(W \cup (X_t \cap M_L); \mathbb{Z}) \) is surjective. Therefore to prove (b) it suffices to prove that \( \text{Im}(H_m(X_t \cap M_L; \mathbb{Z}) \to H_m(X_t; \mathbb{Z}) \cong H^m(X_t; \mathbb{Z})) \) is contained in \( H^m(Y; \mathbb{Z}) \). This is obvious because \( H_m(X_t \cap M_L; \mathbb{Z}) \) is generated by the linear section class.

(b) Since \( X_t \subseteq P_L \setminus \text{Sing}(P_L) \), using Excision Axiom and Duality Theorem ([13], p. 296) we get a natural isomorphism \( H_{m+2}(P_L, P_L \setminus X_t; \mathbb{Z}) \cong H^m(X_t; \mathbb{Z}) \). Therefore \( PD \circ i_t^* \) identifies with the second map of the natural exact sequence

\[
(7) \quad H_l(P_L \setminus X_t; \mathbb{Z}) \to H_l(P_L; \mathbb{Z}) \to H_l(P_L, P_L \setminus X_t; \mathbb{Z}) \to H_{l-1}(P_L \setminus X_t; \mathbb{Z})
\]

when \( l = m + 2 \), and the proof of (b) amounts to show that \( H_{m+1}(P_L \setminus X_t; \mathbb{Z}) \) is torsion free. Since \( H_{m+1}(X_t; \mathbb{Z}) = 0 \), from the natural exact sequence

\[
(8) \quad H_l(X_t; \mathbb{Z}) \to H_l(P_L \setminus X_t; \mathbb{Z}) \to H_l(P_L \setminus X_t, X_t; \mathbb{Z})
\]

we see that \( H_{m+1}(P_L \setminus X_t; \mathbb{Z}) \) is contained in \( H_{m+1}(P_L \setminus X_t, X_t; \mathbb{Z}) \). Hence, by (1), to prove (b) it is enough to show that \( H_{m+1}(f^{-1}(\Delta_a), X_{a+\rho}; \mathbb{Z}) \) is torsion free for any \( a \in A \). To this aim, consider the exact sequence of the pair \((f^{-1}(\Delta_a), X_{a+\rho})\)

\[
(9) \quad H_l(X_{a+\rho}; \mathbb{Z}) \to H_l(f^{-1}(\Delta_a); \mathbb{Z}) \to H_l(f^{-1}(\Delta_a), X_{a+\rho}; \mathbb{Z}) \to H_{l-1}(X_{a+\rho}; \mathbb{Z})
\]

Since \( H_{m+1}(X_{a+\rho}; \mathbb{Z}) = 0 \) and \( H_m(X_{a+\rho}; \mathbb{Z}) \) is torsion free then it suffices to prove that

\[
(10) \quad H_{m+1}(f^{-1}(\Delta_a); \mathbb{Z}) \text{ is torsion free for any } a \in A.
\]

When \( a \neq a_{\infty} \) this follows by (2) and (3), taking into account the natural exact sequence of the pair \((X_{a+\rho}, X_{a+\rho} \cap D_a)\)

\[
(11) \quad H_l(X_{a+\rho}; \mathbb{Z}) \to H_l(X_{a+\rho}, X_{a+\rho} \cap D_a; \mathbb{Z}) \to H_{l-1}(X_{a+\rho} \cap D_a; \mathbb{Z}),
\]

and that \( H_{m+1}(X_{a+\rho}; \mathbb{Z}) = 0 \).

When \( a = a_{\infty} \) by (4) we see that to prove (10) it is enough to prove that \( H_{m+1}(f^{-1}(a_{\infty}); \mathbb{Z}) \) is torsion free. This follows from the Mayer-Vietoris sequence

\[
(12) \quad H_l(G; \mathbb{Z}) \oplus H_l(\tilde{M}_{L}; \mathbb{Z}) \to H_l(f^{-1}(a_{\infty}); \mathbb{Z}) \to H_{l-1}(G \cap M_L; \mathbb{Z})
\]
because $H_m(G \cap M_L; \mathbb{Z})$ is torsion free, $H_{m+1}(G; \mathbb{Z}) = 0$, and $H_{m+1}(\tilde{M}_L; \mathbb{Z}) = 0$ in view of the decomposition (11), p. 170, Théorème 7.31)

\[ H_1(\tilde{M}_L; \mathbb{Z}) \cong H_1(M_L; \mathbb{Z}) \oplus H_{1-2}(M_L \cap W; \mathbb{Z}). \]

(c) By the sequence (7) (tensored with $\mathbb{Q}$) we see that to prove (c) it is enough to show that $h_{m+2}(P_L \setminus X_1; \mathbb{Q}) \leq 2$. Hence, since $H_{m+2}(X_1; \mathbb{Z}) \cong \mathbb{Z}$, by (5) it suffices to prove that $h_{m+2}(P_L \setminus X_1, X_1; \mathbb{Q}) \leq 1$. Combining (2) and (3) with the exact sequences (9) and (11) with $l = m+2$, and with the fact that $H_{m+1}(X_{a+\rho}; \mathbb{Z}) = 0$, we get $H_{m+2}(f^{-1}(\Delta_a), X_{a+\rho}; \mathbb{Q}) = 0$ for any $a \in A \setminus \{a_{\infty}\}$. So, in view of the decomposition (11), the proof of (c) amounts to prove that $h_{m+2}(f^{-1}(\Delta_a), X_{a+\rho}; \mathbb{Q}) \leq 1$. To this aim observe that since $H_{m+2}(X_{a+\rho}; \mathbb{Q})$ is generated by the linear section class then the first map in (9) with $l = m+2$ is injective. It follows that $h_{m+2}(f^{-1}(\Delta_a), X_{a+\rho}; \mathbb{Q}) = h_{m+2}(f^{-1}(\Delta_a); \mathbb{Q}) - 1$ for $H_{m+1}(X_{a+\rho}; \mathbb{Z}) = 0$. Therefore, by (4), we see that it is enough to prove that $h_{m+2}(f^{-1}(\Delta_a); \mathbb{Q}) = 2$. This follows by the sequence (12) with $l = m+2$, taking into account that the kernel of its first map is $H_{m+2}(G \cap M_L; \mathbb{Z}) \cong \mathbb{Z}$, that $H_{m+1}(G \cap M_L; \mathbb{Z}) = 0$, and that $h_{m+2}(G; \mathbb{Q}) + h_{m+2}(\tilde{M}_L; \mathbb{Q}) = 3$ in view of (13).

**Theorem 2.3.** Let $Y \subseteq \mathbb{P}^N$ be a smooth complete intersection of odd dimension $m+1 = 2r+1 \geq 3$. Fix integers $1 \leq k < d$, and smooth divisors $G \in |H^0(Y, \mathcal{O}_Y(k))|$ and $X \in |H^0(Y, \mathcal{O}_Y(d))|$. Put $W := X \cap G$. Then the following properties hold:

(a) $I_W \cap H^m(X; \mathbb{Z}) = H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_W$ (see Notations 2.7 (viii));

(b) the push-forward $i_W^* : H_m(W; \mathbb{Z}) \rightarrow H_m(X; \mathbb{Z})$ is injective.

**Proof.** (a) By (5) we only have to prove that $I_W \cap H^m(X; \mathbb{Z}) \subseteq H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_W$. To this aim, consider $\xi \in I_W \cap H^m(X; \mathbb{Z})$. By (5) $\xi$ is a torsion element in $H^m(X; \mathbb{Z})$ modulo $H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_W$. So, by Lemma 2.2 (a) and (b), we deduce that $\xi \in H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_W$.

(b) Since $H_m(W; \mathbb{Z})$ is torsion free (3, (4.4) Corollary (i), p. 162)) then to prove (b) is equivalent to prove that $i_W^* : H^m(W; \mathbb{Z}) \rightarrow H^m(X; \mathbb{Z})$ is injective. By (5) we have $I_W = H^m(X; \mathbb{Z})_W$ because $H^m(Y; \mathbb{Z})$ is generated by the linear section class which belongs also to $H^m(X; \mathbb{Z})_W$. Therefore to show (b) it suffices to show that $h_m(W; \mathbb{Q}) \leq \dim_Q I_W$. On the other hand, by Lemma 2.2 (a) and (5) we have $I_W = \text{Im}(P_D \circ i^*_Q)$. Hence $\dim Q I_W = h_{m+2}(P_L; \mathbb{Q}) - \dim Q \text{Ker}(P_D \circ i^*_Q)$, and, in view of Lemma 2.2 (c), in order to prove (b) it is enough to show that

\[ h_{m+2}(P_L; \mathbb{Q}) \geq h_m(W; \mathbb{Q}) + 2. \]

To this purpose let $R \rightarrow P$ be a desingularization of $P$, and $R_L \rightarrow P_L$ be the induced map on the blowing-up. By (4), Proposition 5.4.4 p. 157, and Corollary 5.4.11 p. 161) and (13) (p. 170, Théorème 7.31) we see that $H^{m+2}(P; \mathbb{Q})$ and $H^{m+3}(P; \mathbb{Q})$ are naturally embedded in $H^{m+2}(R_L; \mathbb{Q})$ via pull-back. Therefore the pull-back $H^{m+2}(P; \mathbb{Q}) \rightarrow H^{m+2}(P_L; \mathbb{Q})$ is injective, hence the push-forward $H_{m+2}(P_L; \mathbb{Q}) \rightarrow H_{m+2}(P; \mathbb{Q})$ is surjective. This map cannot be injective because the class $P^1 \times H_{B_L}^{-1} \in H_{m+2}(P_L; \mathbb{Q})\setminus \{0\}$ ($H_{B_L}$ = hyperplane section of $B_L$) vanishes in $H_{m+2}(P; \mathbb{Q})$. We deduce

\[ h_{m+2}(P_L; \mathbb{Q}) \geq h_{m+2}(P; \mathbb{Q}) + 1. \]
Now let $\tilde{W}$ be the exceptional divisor in $P$, and consider the following exact sequences in Borel-Moore homology ([6], p. 371, (6)):

\[ 0 = H_{m+3}(Y; \mathbb{Q}) \rightarrow H_{m+3}(Y \setminus W; \mathbb{Q}) \rightarrow H_{m+2}(W; \mathbb{Q}) \rightarrow H_{m+2}(Y; \mathbb{Q}), \]

\[ H_{m+3}(P \setminus \tilde{W}; \mathbb{Q}) \rightarrow H_{m+2}(\tilde{W}; \mathbb{Q}) \rightarrow H_{m+2}(P; \mathbb{Q}). \]

By ([3], p. 161, (4.3) Theorem (i)) we know that $H_{m+2}(W; \mathbb{Q})$ is generated by the linear section class, so the push-forward $\nu$ is injective, and then the first sequence proves that $H_{m+3}(Y \setminus W; \mathbb{Q}) = 0$. Then also $H_{m+3}(P \setminus \tilde{W}; \mathbb{Q}) = 0$ because $P \setminus \tilde{W} \cong Y \setminus W$, and from the second sequence we obtain

\[ h_{m+2}(P; \mathbb{Q}) \geq h_{m+2}(\tilde{W}; \mathbb{Q}). \]

Finally, using Leray-Hirsch Theorem [19], we have

\[ h_{m+2}(\tilde{W}; \mathbb{Q}) \geq h_{m}(W; \mathbb{Q}) + 1. \]

Putting together ([15], [10] and [17]), we get ([14]). □

3. PROOF OF THE ANNOUNCED RESULTS

We keep the same notation we introduced before, and need further preliminaries.

Notations 3.1. (i) Let $X, G_1, \ldots, G_r$ be a regular sequence of divisors in $Y$, with $X \in |H^0(Y, \mathcal{O}_Y(d))|$ and $G_i \in |H^0(Y, \mathcal{O}_Y(k_i))|$ for $1 \leq i \leq r$. Let $\Delta := X \cap G_1 \cap \cdots \cap G_r$ be their complete intersection, hence $\dim \operatorname{Sing} X = r$. Denote by $C_1, \ldots, C_\omega$ the irreducible components of $\Delta$. Assume that $d > k_i$ for any $1 \leq l \leq r$, that $X$ and $G_1$ are smooth, and that for any $2 \leq l \leq r$ one has

\[ \dim \operatorname{Sing}(G_1 \cap \cdots \cap G_l) \leq l - 2 \quad \text{and} \quad \dim \operatorname{Sing}(X \cap G_1 \cap \cdots \cap G_l) \leq l - 1. \]

Observe that $\Delta$ verifies condition (0.1) in [11]. Put $W := X \cap G_1$.

(ii) Let $I_\Delta$ be the subspace of the invariant cocycles in $H^m(X; \mathbb{Q})$ with respect to the monodromy representation on $H^m(X; \mathbb{Q})$ for the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $\Delta$. Let $V_\Delta$ be its orthogonal complement in $H^m(X; \mathbb{Q})$. Denote by $H^m(X; \mathbb{Z})$ the image of $H_m(\Delta; \mathbb{Z})$ in $H^m(X; \mathbb{Z})$ via the natural map $H_m(W; \mathbb{Z}) \rightarrow H_m(X; \mathbb{Z}) \cong H^m(X; \mathbb{Z})$. In Notations 2.1 (viii), we already defined $I_W$ and $H^m(X; \mathbb{Z})_W$. Observe that $I_\Delta \subseteq I_W$ for the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $W$ is a subgroup of the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $\Delta$.

(iii) For any $1 \leq l \leq r - 1$ fix general divisor $H_l \in |H^0(Y, \mathcal{O}_Y(\mu_l))|$, with $0 \ll \mu_1 \ll \cdots \ll \mu_{r-1}$, and for any $0 \leq l \leq r - 1$ define $(Y_l, X_l, W_l, \Delta_l)$ as follows. For $l = 0$ put $(Y_0, X_0, W_0, \Delta_0) := (Y, X, W, \Delta)$. For $1 \leq l \leq r - 1$ put $Y_l := G_1 \cap \cdots \cap G_l \cap H_1 \cap \cdots \cap H_l$, $X_l := X \cap Y_l$, $W_l := X \cap Y_l \cap G_{l+1}$, and $\Delta_l := \Delta \cap Y_l$. Notice that $\dim Y_{r-1} = 3$ and that $\Delta_{r-1} = W_{r-1}$.

Theorem 3.2. Let $X \in |H^0(Y, \mathcal{O}_Y(d))|$ be a general divisor containing $\Delta$. Then $H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta$ is freely generated by $C_1, \ldots, C_{\omega-1}$ and the linear section $H^m_X$. Moreover $I_\Delta \cap H^m(X; \mathbb{Z}) = H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta = H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_W$, and the monodromy representation on $V_\Delta$ for the family of smooth divisors in $|H^0(Y, \mathcal{O}_Y(d))|$ containing $\Delta$ is irreducible.
Proof. Since $H_m(\Delta; \mathbb{Z})$ is freely generated by $C_1, \ldots, C_\omega$ then to prove the first assertion is equivalent to prove that the push-forward $H_m(\Delta; \mathbb{Z}) \to H_m(X; \mathbb{Z})$ is injective. When $r = 1$ this follows by Theorem 2.3 (b), because in this case $W = \Delta$. Now argue by induction on $r \geq 2$. Since $\Delta_1 = \Delta \cap H_1$ then $\Delta$ and $\Delta_1$ have the same number of components. Therefore the Gysin map $H_m(\Delta; \mathbb{Z}) \to H_{m-2}(\Delta_1; \mathbb{Z})$ is bijective, and its composition $\varphi$ with the push-forward $H_{m-2}(\Delta_1; \mathbb{Z}) \to H_{m-2}(X_1; \mathbb{Z})$ is injective by induction. On the other hand $\varphi$ is nothing but the composition of the push-forward $H_m(\Delta; \mathbb{Z}) \to H_m(W; \mathbb{Z})$ with the Gysin map $H_m(W; \mathbb{Z}) \to H_{m-2}(X_1; \mathbb{Z})$ (observe that $X_1 = W \cap H_1$). Hence the map $H_m(\Delta; \mathbb{Z}) \to H_m(W; \mathbb{Z})$ is injective, and so is the map $H_m(\Delta; \mathbb{Z}) \to H_m(X; \mathbb{Z})$ by Theorem 2.3 (b), again.

As for the remaining claims, note that since $I_\mathcal{W} \supseteq I_\Delta$ and $I_\Delta \cap H^m(X; \mathbb{Z}) \supseteq H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta$, by Theorem 2.3 (a), it suffices to prove that $H^m(X; \mathbb{Z})_\mathcal{W} \subseteq H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta$, and that $V_\Delta$ is irreducible. So it is enough to show that for any $0 \leq l \leq r - 1$ one has

$$H^m_l(X_l; \mathbb{Z})_\mathcal{W} \subseteq H^m_l(Y_l; \mathbb{Z}) + H^m_l(X_l; \mathbb{Z})_\Delta.$$  

($m_l := m - 2l$), and that the monodromy representation on $V_{\Delta_l}$ for the family of smooth divisors $X_l \in \mathcal{H}^0(Y_l, \mathcal{O}(d))$ containing $\Delta_l$ is irreducible. To this purpose we argue by decreasing induction on $l$. When $l = r - 1$ we have $\Delta_{r-1} = W_{r-1}$. In this case (19) is obvious, and by (2), Theorem 1.1 we know that $V_{\Delta_{r-1}}$ is irreducible. Now assume $0 \leq l < r - 1$. By induction we have

$$I_{\Delta_{l+1}} \cap H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) = H^{m_{l+1}}(Y_{l+1}; \mathbb{Z}) + H^{m_{l+1}}(X_{l+1}; \mathbb{Z})_{\Delta_{l+1}}.$$  

Since $X_{l+1} = W_l \cap H_{l+1}$ then the inclusion map $i_{X_{l+1}} : X_{l+1} \to W_l$ defines a Gysin map $i_{X_{l+1}}^* : H^m_l(W_l; \mathbb{Z}) \to H^m_{l+1}(X_{l+1}; \mathbb{Z})$. Using the same argument as in the proof of (2), Lemma 2.3 we see that $PD \circ i_{X_{l+1}}^*$ is injective (recall that $H_m(W_l; \mathbb{Z})$ is torsion free by (3), (4.4) Corollary (i), p. 162), and that its image is globally invariant. Since $\Delta_l \subseteq W_l$, $\Delta_{l+1} = \Delta_l \cap H_{l+1}$ (hence the Gysin map $H_m(\Delta_l; \mathbb{Z}) \to H_{m+1}(\Delta_{l+1}; \mathbb{Z})$ is bijective because both groups are freely generated by the irreducible components), and by Lefschetz Hyperplane Theorem we have $H^{m_{l+1}}(Y_{l+1}; \mathbb{Z}) \cong H^{m_{l+1}}(Y_{l+1}; \mathbb{Z})$, then $Im(PD \circ i_{X_{l+1}}^*) \supseteq H^{m_{l+1}}(Y_{l+1}; \mathbb{Z}) + H^{m_{l+1}}(X_{l+1}; \mathbb{Z})_{\Delta_{l+1}}$. It follows that these groups are equal, i.e.

$$H^m_l(W_l; \mathbb{Z}) \cong Im(PD \circ i_{X_{l+1}}^*) = H^{m_{l+1}}(Y_{l+1}; \mathbb{Z}) + H^{m_{l+1}}(X_{l+1}; \mathbb{Z})_{\Delta_{l+1}}.$$  

In fact, otherwise, by (20) one would have $V_{\Delta_{l+1}} \cap Im(PD \circ i_{X_{l+1}}^*; \mathbb{Q}) \neq \{0\}$, and since $V_{\Delta_{l+1}}$ is irreducible it would follow that $H_m(W_l; \mathbb{Q}) \cong H^m_{l+1}(X_{l+1}; \mathbb{Q})$. This is impossible because, for $0 \ll \mu_1 \ll \cdots \ll \mu_{l+1}$, $h_{m_{l+1}}(X_{l+1}; \mathbb{Q})$ is arbitrarily large with respect to $h_m(W_l; \mathbb{Q})$. From (21) we get (19) for the natural map $H^m_l(W_l; \mathbb{Z}) \to H^m_l(X_l; \mathbb{Z}) \cong H^m_l(X_l; \mathbb{Z})$ sends $H^{m_{l+1}}(Y_{l+1}; \mathbb{Z})$ in $H^{m_{l+1}}(Y_{l+1}; \mathbb{Z})$ and $H^{m_{l+1}}(X_{l+1}; \mathbb{Z})_{\Delta_{l+1}}$ in $H^{m_{l+1}}(X_{l+1}; \mathbb{Z})_{\Delta_{l+1}}$. Finally note that by Theorem 2.3 and (19) it follows $I_{\Delta_l} = I_{W_l}$. So (2), Theorem 1.1, and (11) implies $V_{\Delta_l}$ is irreducible. □

We are in position to prove the results announced in the Introduction.

Proof of Theorem 1.7. Let $G_1, \ldots, G_r$ be general divisors in $|H^0(Y, \mathcal{O}(d))|$ containing $Z$. By (11), 1.2. Theorem we know that $X, G_1, \ldots, G_r$ is a regular sequence, verifying conditions (18). Put $\Delta := X \cap G_1 \cap \cdots \cap G_r$. Hence $\Delta$ is a complete intersection of dimension $r$ containing $Z$, and again by (11), 1.2. Theorem we know that $\Delta \setminus Z$ is smooth and connected. Observe that $\Delta \neq Z_1 \cup \cdots \cup Z_r$, etc.
otherwise $Z = \Delta$ and this is in contrast with the assumption that $I_{Z,Y}(\delta)$ is generated by global sections. Therefore, apart from $Z_1, \ldots, Z_{\rho}$, $\Delta$ has a unique residual irreducible component, and so $H^m(Y; Z) + H^m(X; Z)_{\Delta}$ is freely generated by $Z_1, \ldots, Z_{\rho}, H^r_X$ by Theorem 3.2. Again by Theorem 3.2 we deduce that $I_{\Delta}$ is equal to the subspace $I_Z$ of the invariant cocycles in $H^m(X; \mathbb{Q})$ with respect to the monodromy representation on $H^m(X; \mathbb{Q})$ for the family of smooth divisors in $|H^0(Y, O_Y(d))|$ containing $Z$. Since $I_Z \perp (= V_{\Delta})$ is irreducible, a standard argument shows that $NS(X) \subseteq I_{\Delta}$, i.e. $NS(X) = H^m(Y; Z) + H^m(X; Z)_{\Delta}$.

\begin{proof}[Proof of Theorem 1.2] Put $\Delta := X_t \cap G_1 \cap \cdots \cap G_r$. Since $X_t, G_1, \ldots, G_r$ are smooth and $d > k_1 > \cdots > k_r$ then the conditions (18) are verified \cite[p. 133, Proposition 4.2.6. and proof]{5}. Therefore Theorem 3.2 applies to $\Delta$. We deduce that $NS(X_t) = H^m(Y; Z) + H^m(X_t; Z)_{\Delta}$, hence $NS(X_t)$ is freely generated by $H^r_{X_t}, Z_1, \ldots, Z_{\rho}, R_1, \ldots, R_{\sigma-1}$.
\end{proof}

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