Homology cycles in manifolds with locally standard torus actions

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Abstract. Let $X$ be a $2n$-manifold with a locally standard action of a compact torus $T^n$. If the free part of action is trivial and proper faces of the orbit space $Q$ are acyclic, then there are three types of homology classes in $X$: (1) classes of face submanifolds; (2) $k$-dimensional classes of $Q$ swept by actions of subtori of dimensions $k$; (3) relative $k$-classes of $Q$ modulo $\partial Q$ swept by actions of subtori of dimensions $\geq k$. The submodule of $H_\ast(X)$ spanned by face classes is an ideal in $H_\ast(X)$ with respect to the intersection product. It is isomorphic to $(\mathbb{Z}[S_\mathcal{Q}]/\Theta)/W$, where $\mathbb{Z}[S_\mathcal{Q}]$ is the face ring of the Buchsbaum simplicial poset $S_\mathcal{Q}$ dual to $Q$; $\Theta$ is the linear system of parameters determined by the characteristic function; and $W$ is a certain submodule, lying in the socle of $\mathbb{Z}[S_\mathcal{Q}]/\Theta$. Intersections of homology classes different from face submanifolds are described in terms of intersections on $Q$ and $T^n$.

1. Introduction

An action of a compact torus $T^n$ on a smooth compact manifold $M$ of dimension $2n$ is called locally standard if it is locally isomorphic to the standard action of $T^n$ on $\mathbb{C}^n$. The orbit space $Q = M/T^n$ has a natural structure of a manifold with corners in which open $k$-dimensional faces of $Q$ correspond to $k$-dimensional orbits of an action. Every manifold with locally standard torus action is equivariantly homeomorphic to the quotient model $X = Y/\sim$, where $Y$ is a principal $T^n$-bundle over $Q$ and $\sim$ is an equivalence relation determined by the characteristic function on $Q$ [17].

This paper is the third in a series of works, where we study topology of $X$ under the assumption that proper faces of the orbit space are acyclic and $Y$ is a trivial

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bundle. Previous works \cite{1,2} were devoted to the homological spectral sequence associated with the filtration of $X$ by orbit types. In this paper we give a geometrical description of homology cycles on $X$.

In the case when all faces of $Q$ including $Q$ itself are acyclic, the topology of the corresponding manifold $X$ is known (see \cite{9}). In this case the equivariant cohomology ring is isomorphic to the face ring of the simplicial poset $S_Q$ dual to $Q$: $H^*_T(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q]$. As a ring, it is generated by equivariant cycles, dual to face submanifolds of $X$ (these generators correspond to the standard generators of the face ring). The spectral sequence of the Borel fibration $E_T^n \times_T X \to BT^n$ collapses at a second page. A fiber-inclusion map $\nu: X \hookrightarrow E_T^n \times_T X$ induces a surjective ring homomorphism $\nu^*: H^*_T(X; \mathbb{Z}) \to H^*(X; \mathbb{Z})$, whose kernel is the image of $H^>0(BT^n; \mathbb{Z})$ under $\pi^*$. Thus, $H^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q]/(\theta_1, \ldots, \theta_n)$, where $\theta_i$ are the images of generators $v_i$ of the ring $H^*(BT^n; \mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_n]$. The sequence $(\theta_1, \ldots, \theta_n)$ is a linear system of parameters in $\mathbb{Z}[S_Q]$. Since $S_Q$ is Cohen–Macaulay, this is a regular sequence and $\dim H^{2k}(X) = h_k(S_Q)$.

In the case when only proper faces of $Q$ are acyclic, this approach is inapplicable. The spectral sequence of the Borel fibration does not collapse at a second page. We still have the ring homomorphism $H^*_T(X)/(\pi^*H^>0(BT^n)) \to H^*(X)$, but it is neither injective nor surjective.

Nevertheless, there is an apparent connection between topology of spaces with torus actions and the theory of face rings. In \cite{3} we proved that there exists an isomorphism of rings (and $H^*(BT^n)$-modules):

\begin{equation}
H^*_T(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q] \oplus H^*(Q; \mathbb{Z})
\end{equation}

(the units of the rings in the direct sum are identified).

When proper faces of $Q$ are acyclic, the dual simplicial poset $S_Q$ is Buchsbaum \cite[Cor.6.3]{11}. There is a standard tool in combinatorics and commutative algebra devised to study Buchsbaum simplicial complexes, namely, the $h'$-vector. By definition, the $h'$-numbers of a Buchsbaum simplicial poset $S$ are the dimensions of homogeneous components of the quotient algebra $k[S]/(\theta_1, \ldots, \theta_n)$, where $\theta_1, \ldots, \theta_n$ is any linear system of parameters. These numbers do not depend on a linear system of parameters, and can be expressed in terms of the ordinary $h$-numbers and Betti numbers of $S$ (see \cite{14,11} or Definition \cite{2,8} below). In \cite[Th.3]{2} we proved that $\dim(E_X)_{q,q} = h'_{n-q}(S_Q)$, where $(E_X)^*_{s,s}$ is the homological spectral sequence associated with the orbit type filtration of $X$.

In this paper we describe the geometrical structure of homology cycles on $X$.

**Theorem 1.** Homology classes of $X$ have three different types:

1. the classes of face submanifolds (we call them face classes);
2. the classes, represented by $k$-cycles of $Q$, swept by an action of a subtorus of dimension $< k$ (these classes will be called spine classes);
(3) the classes, represented by relative \( k \)-cycles of \( Q \) modulo \( \partial Q \) with \( k < n \), swept by an action of a subtorus of dimension \( \geq k \) (these classes will be called diaphragm classes).

Linear relations on face classes are of two types: the relations appearing in the ring \( k[S_Q]/(\theta_1, \ldots, \theta_n) \), and additional relations lying in a socle of \( k[S_Q]/(\theta_1, \ldots, \theta_n) \).

Intersections of face classes are encoded by the multiplication in the face ring of \( S_Q \). Proper face classes span the ideal of \( H_\partial(X) \) with respect to the intersection product. Intersections of other classes are described by means of the intersection products on \( Q \) and \( T^n \).

Precise statements are given in Propositions 2.9, 4.1, 5.6, 6.1, 6.2, 6.3 and Theorem 2.

Face classes and the elements of \( H_k(Q, \partial Q) \) swept by the action of the whole group \( T^n \) are equivariant. This gives an independent geometrical evidence for the formula (1.1).

The paper may be briefly outlined as follows. Section 2 contains basic definitions and outlines the previous results. In Section 3 we make technical preparations for Section 4 which is devoted to linear relations on face classes. In Section 5 we realize non-face classes of \( X \) as embedded pseudomanifolds. These geometrical constructions imply a partial description of intersection theory on \( X \), which is done in Section 6.

Two examples of computations are discussed in Section 7. A very particular 4-dimensional example is worked out, and the reader is encouraged to refer to it while reading other parts of the paper. The second example is more general: we apply our technique to the class of orientable toric origami manifolds with acyclic proper faces of the orbit space, and rediscover some results of [3]. A supplementary space \( \hat{X} \) is introduced in the last section. This space can be considered as a \( T^n \)-invariant tubular neighborhood of the union of characteristic submanifolds in \( X \). By using intersection theory on \( \hat{X} \), we prove that certain elements of \( k[S_Q]/(\theta_1, \ldots, \theta_n) \) lie in the socle of this module. This gives a geometrical interpretation of the result obtained by Novik–Swartz [11].

2. Preliminaries and previous results

2.1. Manifolds with locally standard torus actions. An action of \( T^n \) on a (compact connected smooth) manifold \( M^{2n} \) is called locally standard, if \( M \) has an atlas of \( T^n \)-invariant charts, each equivalent to an open \( T^n \)-invariant subset of the standard action of \( T^n \) on \( \mathbb{C}^n \). The reader is referred to [6] or [17] for the precise definition. The orbit space of a locally standard action is a compact connected \( n \)-dimensional manifold with corners with the property that every codimension \( k \) face of \( Q \) lies in exactly \( k \) facets of \( Q \) (such manifolds with corners were called nice in [9], or manifolds with faces elsewhere).
**Definition 2.1.** A finite partially ordered set (poset) $S$ is called simplicial if 
(1) there is a minimal element $\hat{0} \in S$; (2) for each element $J \in S$ the lower order ideal $\{ I \in S \mid I \leq J \}$ is isomorphic to the poset of faces of a $k$-simplex for some number $k$, called the dimension of $I$.

The elements of $S$ are called simplices. Simplices of dimension 0 are called vertices. The number $|I| = \dim I + 1$ is equal to the number of vertices of $I$ and is called the rank of $I$. The set of vertices of a simplicial poset or a simplex is denoted by $\text{Vert}(\cdot)$.

Every manifold with corners $Q$ determines a dual poset $S_Q$ whose elements are the faces of $Q$ ordered by the reversed inclusions. When $Q$ is a nice connected manifold with corners, $S_Q$ is a simplicial poset. We denote abstract elements of $S_Q$ by $I$, $J$, etc. and the corresponding faces of $Q$ by $F_I$, $F_J$, etc. There holds $\dim F_I = n - |I|$. The minimal element of $S_Q$ corresponds to the maximal face of $Q$, i.e. $Q$ itself. Vertices of $S_Q$ correspond to facets of $Q$. The set of facets of $Q$ is denoted by $\text{Fac}(Q)$.

Let $Q$ be the orbit space of locally standard action, and let $x \in F^0$ be a point in the interior of a facet $F \in \text{Fac}(Q)$. Then the stabilizer of $x$, denoted by $\lambda(F)$, is a 1-dimensional toric subgroup in $T^n$. If $F_I$ is a codimension $k$ face of $Q$, contained in facets $F_1, \ldots, F_k \in \text{Fac}(Q)$, then the stabilizer of $x \in F_I$ is the $k$-dimensional torus $T_I = \lambda(F_1) \times \ldots \times \lambda(F_k) \subset T^n$, where the product is free inside $T^n$. This puts a specific restriction on subgroups $\lambda(F)$. In general, a map

$$\lambda: \text{Fac}(Q) \to \{ \text{1-dimensional toric subgroups of } T^n \}$$

is called a characteristic function, if, whenever facets $F_1, \ldots, F_k$ have nonempty intersection, the map

$$\lambda(F_1) \times \ldots \times \lambda(F_k) \to T^n,$$

induced by inclusions $\lambda(F_i) \hookrightarrow T^n$, is injective and splits. This condition is called (\*)-condition. Let $i \in \text{Vert}(S_Q)$ be the vertex of $S_Q$, and $T_i = \lambda(F_i)$ be the value of characteristic function. Let $\omega_i \in H_1(T^n; k) \cong k^n$ be the fundamental class of $T_i$. This class is defined uniquely up to sign.

Let $\mu: M \to Q$ be the projection to the orbit space. The free part of the action has the form $\mu|_{Q^c}: \mu^{-1}(Q^c) \to Q^c$, where $Q^c = Q \setminus \partial Q$ is the interior of the manifold with corners. It is a principal torus bundle over $Q^c$ which can be uniquely extended over $Q$; it determines a principal $T^n$-bundle $\rho: Y \to Q$. Therefore any manifold with locally standard action defines three objects: a nice manifold with corners $Q$, a principal torus bundle $\rho: Y \to Q$, and a characteristic function $\lambda$. One can recover the manifold $M$, up to equivariant homeomorphism, from these data by the following standard construction.

**Construction 2.2 (Quotient construction).** Let $\rho: Y \to Q$ be a principal $T^n$-bundle over a nice manifold with corners, and $\lambda$ be a characteristic function on $\text{Fac}(Q)$. Consider the space $X \overset{\text{def}}{=} Y/\sim$, where $y_1 \sim y_2$ if and only if $\rho(y_1) = \rho(y_2) \in$
$F_i^q$ for some face $F_i$ of $Q$, and $y_1, y_2$ lie in the same $T_I$-orbit of the $T^n$-action on $Y$. Let $f: Y \to X$ be the quotient map.

Every manifold $M$ with locally standard torus action is equivariantly homeomorphic to its model $X$ ([17 Cor.2]). In the rest of the paper we use the model $X$ instead of $M$.

Remark 2.3. In the paper we work with a smooth manifold with corners $Q$ and smooth manifolds $X \cong M$, but this is done basically to simplify the exposition. The quotient model $X = (Q \times T^n)/\sim$ can obviously be defined for a larger class of spaces. If $Q$ is a homology manifold with a simple stratification of the boundary, in which faces are homology manifolds with boundaries, then $X$ is a closed homology manifold. All results of the paper are valid in this setting as can be seen from the proofs.

2.2. Filtrations. There are natural topological filtrations on $Q$, $Y$ and $X$. Namely, $Q_k \subseteq Q$ is the union of $k$-dimensional faces of $Q$, $Y_k = \rho^{-1}(Q_k) \subseteq Y$, and $X_k = f(Y_k) \subseteq X$ is the union of toric orbits of dimension at most $k$. The maps $\mu: X \to Q$, $\rho: Y \to Q$ and $f: Y \to X$ respect these filtrations. The homological spectral sequences produced by these filtrations are denoted $(E_Q)^{*,*,*}_*\otimes_i$, $(E_Y)^{*,*,*}_*\otimes_i$, and $(E_X)^{*,*,*}_*\otimes_i$. The map $f$ induces the morphism of spectral sequences $f^*_{\ast\ast}: (E_Y)^{\ast\ast}_{\ast\ast} \to (E_X)^{\ast\ast}_{\ast\ast}$.

The subsets $\rho^{-1}(F_I) \subseteq Y$ and $\mu^{-1}(F_I) \subseteq X$ which cover the face $F_I \subseteq Q$ are denoted $Y_I$ and $X_I$ respectively. Note that the subset $X_I$ is a closed submanifold of $X$ of codimension $2|I|$. It is called a face submanifold. Face submanifolds of codimension 2 are called characteristic submanifolds. They correspond to facets of $Q$.

The first page of $(E_Q)^{*,*,*}_*$ has the form

$$(E_Q)^{1} = H_{p+q}(Q, Q_{p-1}) \cong \bigoplus_{I \in \mathcal{S}_Q, \dim F_I = p} H_{p+q}(F_I, \partial F_I).$$

and the first differential $(d_Q)^1$ is the sum of the maps

$$(2.2) \quad m^q_{I,J}: H_{q+\dim F_I}(F_I, \partial F_I) \to H_{q+\dim F_{I-1}}(\partial F_I) \to H_{q+\dim F_{I-1}}(\partial F_I, \partial F_I \setminus F_J^q) \cong H_{q+\dim F_J}(F_J, \partial F_J),$$

defined for every face $F_I$ and $F_J \in \text{Fac}(F_I)$. Here the first map is the connecting homomorphism in the homology exact sequence of $(F_I, \partial F_I)$, and the last isomorphism is due to excision.

2.3. Almost acyclic case. Let $k$ be a ground ring. When coefficients in the notation of (co)homology are omitted, they are supposed to be in $k$. From now on we impose two restrictions on $X$ mentioned in the introduction. First, $Q$ is an orientable manifold and all its proper faces are acyclic (over $k$). Second, the principal torus bundle $Y \to Q$ is trivial. Thus $X = (Q \times T^n)/\sim$. The following propositions were proved in [1, 2].
Proposition 2.4. The poset $S_Q$ is a Buchsbaum simplicial poset (over $\mathbb{K}$).

Proposition 2.5. There exists a homological spectral sequence $(\hat{E}_Q)_r^{*,*} = H_{p+q}(Q)$, $(\hat{d}_Q)^r: (\hat{E}_Q)_r^{*,*} \rightarrow (\hat{E}_Q)_{r-r+q,r}^{*,*}$ with the properties:

1. $(\hat{E}_Q)^1 = H((E_Q)^1, d_Q^1)$, where the differential $d_Q^1: (E_Q)^1_p \rightarrow (E_Q)^1_{p-1}$ coincides with $(d_Q)^1$ for $p < n$, and vanishes otherwise.
2. The module $(\hat{E}_Q)_r^{*,*}$ coincides with $(E_Q)_r^{*,*}$ for $r > 2$.
3. $(\hat{E}_Q)_r^{*,*} = \begin{cases} H_p(\partial Q), & \text{if } q = 0, p < n; \\
H_{q+n}(Q, \partial Q), & \text{if } p = n, q \leq 0; \\
0, & \text{otherwise.}
\end{cases}$
4. Nontrivial differentials for $r \geq 1$ have pairwise different domains and targets. They have the form $(\hat{d}_Q)^r: (\hat{E}_Q)_r^{n+1-r} \rightarrow (\hat{E}_Q)_{n-r,0}^r$ and coincide with the connecting homomorphisms $\delta_{n+1-r}: H_{n+1-r}(Q, \partial Q) \rightarrow H_{n-r}(\partial Q)$.

Let $\Lambda_*$ denote the homology module of a torus: $\Lambda_* = \bigoplus \Lambda_s$, $\Lambda_s = H_s(T^n)$.

Proposition 2.6. There exists a homological spectral sequence $(\hat{E}_Y)_r^{*,*} = H_{p+q}(Y)$ such that:

1. $(\hat{E}_Y)^1 = H((E_Y)^1, d_Y^1)$, where the differential $d_Y^1: (E_Y)^1_p \rightarrow (E_Y)^1_{p-1}$ coincides with $(d_Q)^1$ for $p < n$, and vanishes otherwise.
2. $(\hat{E}_Y)^r = (E_Y)^r$ for $r > 2$.
3. $(\hat{E}_Y)_r^{*,*} = \bigoplus \Lambda_q \Lambda_\delta$ and $(\hat{d}_Y)^r = (d_Q)^r \otimes id_{\Lambda}$ for $r > 1$.

Proposition 2.7. There exists a homological spectral sequence $(\hat{E}_X)_r^{*,*} = H_{p+q}(X)$ and the morphism of spectral sequences $f_\#^*: (\hat{E}_Y)_r^{*,*} \rightarrow (\hat{E}_X)_r^{*,*}$ such that:

1. $(\hat{E}_X)^1 = H((E_X)^1, d_X^1)$ where the differential $d_X^1: (E_X)^1_p \rightarrow (E_X)^1_{p-1}$ coincides with $(d_Y)^1$ for $p < n$, and vanishes otherwise. The map $f_\#^1: (E_X)^1 \rightarrow (E_X)^1$ is induced by $f_\#^1: (E_Y)^1 \rightarrow (E_X)^1$.
2. $(\hat{E}_X)^r = (E_X)^r$ for $r > 2$.
3. $(E_X)_r^{*,*} = (E_X)_r^{*,*} = 0$ for $p < q$.
4. $f_\#^1: (\hat{E}_X)^1_{r,q} \rightarrow (\hat{E}_X)^1_{r,q}$ is an isomorphism for $p > q$ and injective for $p = q$.
5. As a consequence of previous items, for $r \geq 1$, the differentials $(\hat{d}_X)^r$ are either isomorphic to $(d_Y)^r$ (when they hit the cells with $p > q$), or isomorphic to the composition of $(\hat{d}_Y)^r$ with $f_\#^1$ (when they hit the cells with $p = q$), or zero (otherwise).
6. The ranks of diagonal terms at a second page are the $h'$-numbers of the poset $S_Q$ dual to the orbit space: $\dim(\hat{E}_X)^2_{q,q} = \dim(E_X)^2_{q,q} = h_{n-q}(S_Q)$.

Recall the definition of $h'$-numbers.
DEFINITION 2.8. Let $S$ be a pure simplicial poset, $\dim S = n - 1$. Let $f_k$ be the number of $k$-dimensional simplices in $S$. The array $(f_{-1} = 0, f_0, \ldots, f_{n-1})$ is called the $f$-vector of $S$. Define $h$-numbers by the relation:

$$h_0 s^n + h_1 s^{n-1} + \ldots + h_n = f_{-1}(s - 1)^n + f_0(s - 1)^{n-1} + \ldots + f_{n-1}.$$ 

Let $\tilde{\beta}_k(S) = \dim \tilde{H}_k(S)$. Define $h'$-numbers by the relation

$$h'_k = h_k + \left( \frac{n}{k} \right) \left( \sum_{j=1}^{k-1} (-1)^{k-j-1} \tilde{\beta}_{j-1}(S) \right) \text{ for } 0 \leq k \leq n.$$ 

Propositions 2.5-2.7 yield the description of $H_\ast(X)$. Let $H_{k,l}(Y)$ denote the $\mathbb{k}$-module $H_k(Q) \otimes \Lambda_l$. By Künneth’s formula, $H_j(Y) \cong \bigoplus_{k+l=j} H_{k,l}(Y)$.

PROPOSITION 2.9. Over a field, there exists a decomposition $H_j(X) \cong \bigoplus_{k+l=j} H_{k,l}(X)$ and the $\mathbb{k}$-module homomorphisms $f_\ast : H_{k,l}(Y) \rightarrow H_{k,l}(X)$ with the following properties:

1. If $k > l$, then $f_\ast : H_{k,l}(Y) \rightarrow H_{k,l}(X)$ is an isomorphism. In particular, $H_{k,l}(X) \cong H_k(Q) \otimes \Lambda_l$.
2. If $k < l$, there exists an isomorphism $H_{k,l}(X) \cong H_k(Q, \partial Q) \otimes \Lambda_l$.
3. If $k < n$, the module $H_{k,k}(X)$ fits in the exact sequence $0 \rightarrow (\dot{E}_X)_{k,k}^\ast \rightarrow H_{k,k}(X) \rightarrow H_k(Q, \partial Q) \otimes \Lambda_k \rightarrow 0$.
4. $H_{n,n}(X) \cong \mathbb{k}$.

There holds bigraded Poincare duality: $H_{k,l}(X) \cong H_{n-k,n-l}(X)$.

3. Preliminary computations

3.1. Orientations. We use the notation $I \sim J$ whenever the simplices $I, J \in S$ satisfy $I \subset J$ and $|J| - |I| = k$. For each pair $I \sim J$, there are exactly two simplices $J' \neq J''$ between them: $I \sim J', J'' \sim J$. For every simplicial poset, there exists a “sign convention” which means that we can associate an incidence number $[J : I] = \pm 1$ to any pair $I \subset J \in S$ in such way that the relation $[J : J'] \cdot [J' : I] + [J : J''] \cdot [J'' : I] = 0$ holds for any $I \sim J$.

The choice of a sign convention is equivalent to the choice of orientations of all nonempty simplices. By the orientation of a simplex $I$ in an abstract simplicial poset we mean the rule which tells whether a given total ordering of the vertices of $I$ is positive or negative, so that even permutations of the order preserve the sign and odd permutations change it. If $I \subset J$, then there is exactly one vertex $i$ of $J$ which is not in $I$. Given the orientations of simplices $I$ and $J$, and given some positive ordering $i_1 < \ldots < i_s$ of the vertices of $I$, we set $[J : I]$ to be $+1$ if $i < i_1 < \ldots < i_s$ is a positive ordering on Vert($J$), and $-1$ if it is negative. The construction works
in the opposite direction in an obvious way: incidence signs determine orientations of all simplices by induction.

Fix arbitrary orientations of the orbit space Q and the torus $T^n$. Together they define an orientation of $Y = Q \times T^n$ and $X = Y/\sim$. Also choose an omniorientation, which means the orientations of all characteristic submanifolds $X_{\lambda}$ in $Q$. A choice of an omniorientation determines the characteristic values $\omega_i \in H_1(T^n; \mathbb{Z})$ without ambiguity of sign. To perform explicit calculations with the spectral sequences $(\hat{E}_X)^* \text{ and } (\hat{E}_Y)^*$ we also need to orient all faces of $Q$.

**Construction 3.1.** The orientation of a simplex $I \in S_Q$ determines the orientation of a face $F_I \subset Q$ by the following convention.

Let $i_1, \ldots, i_{n-q}$ be the vertices of $I$, listed in a positive order. The face $F_I$ lies in the intersection of facets $F_{i_1}, \ldots, F_{i_{n-q}}$. The normal bundles $\nu_i$ to facets $F_i$ have natural orientations, in which inward normal directions are positive. Orient $F_I$ in such way that $T_2 F_I \oplus \nu_{i_1} \oplus \ldots \oplus \nu_{i_{n-q}} \cong T_2 Q$ is positive in the orientation of $Q$.

Thus there are distinguished elements $[F_I] \in H_{\dim F_I}(F_I, \partial F_I)$. By checking the signs one can prove that the maps

$$m_{F_I}^0 : H_{\dim F_I}(F_I, \partial F_I) \to H_{\dim F_J}(F_J, \partial F_J)$$

(see (2.2)) send $[F_I]$ to $[J : I] \cdot [F_J]$.

The choice of omniorientation and orientations of simplices determines the orientation of each orbit $T^n/T_I$ by the following convention.

**Construction 3.2.** Let $i_1, \ldots, i_{n-q}$ be the vertices of $I$, listed in a positive order. The module $H_{1}(T^n/T_I)$ is naturally identified with $\Lambda_I/L_I$, where $L_I$ is a submodule generated by $\omega_{i_1}, \ldots, \omega_{i_{n-q}} \in \Lambda_I = H_1(T^n)$. The basis $[\gamma_1], \ldots, [\gamma_q] \in H_{1}(T^n/T_I), [\gamma_n] = \gamma_I + L_I$ is said to be positive if the basis $(\omega_{i_1}, \ldots, \omega_{i_{n-q}}, \gamma_1, \ldots, \gamma_q)$ is positive in $\Lambda_I$. This orientation of $T^n/T_I$ determines a distinguished fundamental cycle $\Omega_I \in H_q(T^n/T_I)$.

The omniorientation and the orientation of $S$ together determine the class of each face submanifold: $[X_I] = [F_I] \otimes \Omega_I$. Note that both orientations $[F_I]$ and $[\Omega_I]$ depend on the orientation of $I$ by construction. Thus $[X_I]$ does not actually depend on the sign convention on $S_Q$ and depends only on the omniorientation.

**3.2. Arithmetics of torus quotients.** Let us fix some coordinate representation of the torus $T^n = T^{(1)} \times \ldots \times T^{(n)}$, where each $T^{(i)}$ is a 1-dimensional torus with a chosen orientation. For a subset $A = \{j_1 < \ldots < j_q\} \subseteq [n]$ we denote the coordinate subtorus $T^{(j_1)} \times \ldots \times T^{(j_q)} \subseteq T^n$ by $T^{(A)}$.

The coordinate splitting gives a positive basis $e_1, \ldots, e_n$ of the module $\Lambda_1 = H_1(T^n)$, where $e_j$ is the fundamental class of $T^{(j)}$. For a vertex $i \in \text{Vert}(S)$ let $(\lambda_i,1, \ldots, \lambda_i,n)$ denote the coordinates of $\omega_i \in \Lambda_1$ in this basis.
Lemma 3.3. Let $I \in S_Q$, $I \neq \emptyset$ be a simplex with the vertices $\{i_1, \ldots, i_{n-q}\}$ listed in a positive order. Let $A = \{j_1 < \ldots < j_q\} \subset [n]$ be a subset of indices, and let $e_A = e_{j_1} \wedge \ldots \wedge e_{j_q} \in H_q(T^n; \mathbb{Z})$ be the fundamental class of $T^A$. Consider the map $\varrho : T^m \to T^m/T_I$. Then $\varrho_*(e_A) = C_{I,A} \Omega_I \in H_q(T^n/T_I; \mathbb{Z})$. The constant $C_{I,A}$ is equal to

$$\text{sgn}_A \det (\lambda_{i,j})_{j \in \{i_1, \ldots, i_{n-q}\}}$$

where $\text{sgn}_A = \pm 1$ depends only on $A \subset [n]$. When $q = 0$, the constant $C_{I,A}$ is equal to $\pm 1$ depending on the positivity of the basis $\omega_{i_1}, \ldots, \omega_{i_n}$ and coincides with the sign of the fixed point of the action.

Proof. When $q = 0$ the statement is simple. Let $q \neq 0$. Choose vectors $\gamma_1, \ldots, \gamma_q$ so that $(b_i) = (\omega_{i_1}, \ldots, \omega_{i_{n-q}}, \gamma_1, \ldots, \gamma_q)$ is a positive basis of the lattice $H_1(T^n, \mathbb{Z})$. Thus $b_i = U e_i$ with the matrix $U$ of the form

$$U = \begin{pmatrix}
\ell_{i,1} & \cdots & \ell_{i,n-q,1} & \cdots & * \\
\ell_{i,2} & \cdots & \ell_{i,n-q,2} & \cdots & * \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\ell_{i,n} & \cdots & \ell_{i,n-q,n} & \cdots & *
\end{pmatrix}
$$

We have $\det U = 1$ since both bases are positive. Consider the inverse matrix $V = U^{-1}$. Thus

$$e_A = e_{j_1} \wedge \ldots \wedge e_{j_q} = \sum_{M = \{a_1 < \ldots < a_q\} \subset [n]} \det (V_{j,\alpha})_{j \in A} b_{a_1} \wedge \ldots \wedge b_{a_q}.$$  

After taking the quotient $\Lambda/\langle \omega_{i_1}, \ldots, \omega_{i_{n-q}} \rangle$ all summands with $M \neq \{n-q+1, \ldots, n\}$ vanish. When $M = \{n-q+1, \ldots, n\}$, the element $b_{n-q-1} \wedge \ldots \wedge b_{n} = \gamma_1 \wedge \ldots \wedge \gamma_q$ goes to $\Omega_I$. Thus

$$C_{I,A} = \det (V_{j,\alpha})_{j \in A \atop \alpha \in \{n-q+1, \ldots, n\}}.$$  

Now apply Jacobi’s identity (see e.g. [4 Sect.4]):

$$\det (V_{j,\alpha})_{j \in A \atop \alpha \in \{n-q+1, \ldots, n\}} = \frac{(-1)^{\text{sgn}}} {\det U} \det (U_{r,s})_{r \in \{1, \ldots, n-q\} \atop s \in [n]\setminus A}$$

where $\text{sgn} = \sum_{r=1}^{n-q} r + \sum_{s \in [n]\setminus A} s$. Since first $n - q$ columns of $U$ are exactly the vectors $\lambda_{i,j}$, this observation completes the proof.

3.3. Face ring and linear system of parameters. Recall the definition of a face ring of a simplicial poset $S$ (see [15 or 5]). For $I_1, I_2 \in S$ let $I_1 \vee I_2$ denote the set of least upper bounds, and $I_1 \cap I_2 \in S$ be the intersection of simplices (the intersection is well-defined and unique in the case when $I_1 \vee I_2 \neq \emptyset$).
**Definition 3.4.** The face ring $\mathbb{k}[S]$ is the quotient ring of $\mathbb{k}[v_I \mid I \in S]$, deg $v_I = 2|I|$ by the relations

$$v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \cup I_2} v_J, \quad v_{\emptyset} = 1.$$ 

The sum over an empty set is assumed to be 0.

Let $[m] = \{1, \ldots, m\}$ be the set of vertices of $S$ and let $\mathbb{k}[m] = \mathbb{k}[v_1, \ldots, v_m]$ be the graded polynomial ring with deg $v_i = 2$. The ring homomorphism $\mathbb{k}[m] \to \mathbb{k}[S]$ sending $v_i$ to $v_i$ defines a structure of $\mathbb{k}[m]$-module on $\mathbb{k}[S]$.

A characteristic function on $Q$ determines the set of linear forms $\{\theta_1, \ldots, \theta_n\} \subset \mathbb{k}[S_Q]$, where $\theta_j = \sum_{i \in \text{Vert}(S_Q)} \lambda_{i,j} v_i$. If $J \in S$ is a maximal simplex, $|J| = n$, then

$$\lambda_{i,j} = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{otherwise} \end{cases}$$

by the $(*)$-condition. This condition is equivalent to the statement that the sequence $\{\theta_1, \ldots, \theta_n\}$ is a linear system of parameters in $\mathbb{k}[S]$ (see e.g. [6 Lm.3.5.8]). It generates an ideal $(\theta_1, \ldots, \theta_n) \subset \mathbb{k}[S]$ denoted by $\Theta$.

A face ring $\mathbb{k}[S]$ is an algebra with straightening law (see, e.g. [6 §3.5]). As a $\mathbb{k}$-module, it has an additive basis

$$\{P_\sigma = v_{I_1} \cdot v_{I_2} \cdot \ldots \cdot v_{I_t} \mid \sigma = (I_1 \leq I_2 \leq \ldots \leq I_t \in S)\}.$$ 

**Lemma 3.5.** The elements $[v_I] = v_I + \Theta$ span the $\mathbb{k}$-module $\mathbb{k}[S]/\Theta$.

**Proof.** Take any element $P_\sigma$ with $|\sigma| \geq 2$. Using relations in the face ring, we can express $P_\sigma = v_{I_1} \cdot \ldots \cdot v_{I_t}$ as $v_i \cdot v_{I_1 \setminus i} \cdot \ldots \cdot v_{I_t}$ for a vertex $i \in I_1$. Indeed, for every $J \in i \cup (I_1 \setminus i)$ except $I_1$, the product $v_J \cdot v_{I_t}$ vanishes.

The element $v_i$ can be expressed as $\sum_{\alpha'} a_{\alpha'} v_{\alpha'}$ modulo $\Theta$ according to (3.1) (exclude all $v_i$ corresponding to the vertices of some maximal simplex $J \supseteq I_t$). Thus $v_i v_{I_t}$ is expressed as a combination of $v_{I_t^1}$ with $I_t^1 > I_t$. Therefore, up to ideal $\Theta$, the element $P_\sigma$ is expressed as a linear combination of elements $P_{\sigma'}$ which have either smaller length $t$ (in case $|I_1| = 1$) or smaller $I_1$ (in case $|I_1| > 1$). By iterating this descending process, we express the element $P_\sigma + \Theta \in \mathbb{k}[S]/\Theta$ as a linear combination of $[v_I]$. \( \square \)

### 4. Linear relations on face classes

Let $H^*_T(X)$ be a $T^n$-equivariant cohomology ring of $X$. Any proper face of $Q$ is acyclic, therefore any face has a vertex. Hence, there is an injective homomorphism

$$\mathbb{k}[S_Q] \hookrightarrow H^*_T(X),$$

which sends $v_I$ to the cohomology class, equivariant Poincare dual to $[X_I]$ (see [9 Lm.6.4]). The inclusion of a fiber in the Borel construction, $X \to X \times_T ET^n$, induces
the ring homomorphism $H^*_T(X) \to H^*(X)$. The subspace $H_*(X)$, Poincare dual to the image of the homomorphism

$$g: \mathbb{k}[S_Q] \hookrightarrow H^*_T(X) \to H^*(X)$$

is generated by the elements $[X_I]$, thus coincides with the submodule $\bigoplus_q (\dot{E}_X)^\infty_{q,q} \subset H_*(X)$. We call the classes $[X_I]$ the face classes (or face cycles) of $X$.

Note that the elements $[X_I] = [F_I] \otimes \Omega_I$ can also be considered as the free generators of the $\mathbb{k}$-module

$$\bigoplus_q (E_X)^1_{q,q} = \bigoplus_q \bigoplus_{|I| = n-q} H_q(F_I, \partial F_I) \otimes H_q(T^n/T_I).$$

In the following let $\langle [X_I] \rangle$ denote the free $\mathbb{k}$-module generated by the elements $[X_I], I \in S_Q$.

**Proposition 4.1.** Let $C_{I,A}$ be the constants defined in Lemma 3.3. The submodule of $H_*(X)$ generated by the face classes $[X_I]$ has relations of the following two types:

1. For each $J \in S$, $|J| = n - q - 1$, and $A \subset [n]$, $|A| = q$ there is a relation $R_{I,A} = 0$ where

$$R_{I,A} = \sum_{I \supset J, I \supset I} [I : J] C_{I,A}[X_I].$$

2. Let $q \leq n - 2$ and let $\beta \in H_q(\partial Q)$ be a homology class lying in the image of the connecting homomorphism $\delta_{q+1}: H_{q+1}(Q, \partial Q) \to H_q(\partial Q)$. Let $\sum_{I, |I| = n-q} B_I[F_I]$ be a cellular chain representing $\beta$ (such representation exists since every face of $\partial Q$ is acyclic, thus may be considered as a homological cell). Then, for each $A \subset [n], |A| = q$ we have a relation $R'_{I,A} = 0$, where

$$R'_{I,A} = \sum_{I, |I| = n-q} B_I C_{I,A}[X_I].$$

**Proof.** The proof follows from the structure of homological spectral sequences of $X$ and $Y$. The module $\bigoplus_q (E_X)^1_{q,q}$ is freely generated by $[X_I]$. Relations on $[X_I]$ in $H_*(X)$ appear as the images of the differentials hitting $\bigoplus_q (E_X)^1_{q,q}$. The relation of first type $R_{I,A}$ is the image of the generator

$$[F_I] \otimes [T^{(A)}] \in H_{q+1}(F_I, \partial F_I) \otimes H_q(T^n/T_I) \subset (E_X)^1_{q+1,q}$$

under the differential $(d_X)^1: (E_X)^1_{q+1,q} \to (E_X)^1_{q,q}$. Thus relations of the first type span the image of the first differentials hitting $(E_X)^1_{q,q}$.

Let us prove that images of higher differentials are generated by $R'_{I,A}$. Higher differentials $(d_q)^{\geq 2}$ coincide with $\delta_{q+1}: H_{q+1}(Q, \partial Q) \to H_*(\partial Q)$ by Proposition 2.5. The differentials $(d_Y)^{\geq 2}$ coincide with $\delta_{q+1} \otimes \text{id}_A$ by Proposition 2.6. Thus the image
of \((d_Y)^{>2}\) in \((E_Y)_{q,q}^2\) is generated by the elements \(\beta \otimes [T^{(A)}]\), which are, in turn, the homology classes of the elements

\[
\left( \sum_{I, |I|=n-q} B_I[F_I] \right) \otimes [T^{(A)}] \in (E_Y)_{q,q}^1.
\]

By Proposition 2.7, the differential \((d_X)^{*}\) which hits \((E_X)_{q,q}^*\) coincides with the composition of \((d_Y)^{*}\) and inclusion \(f_2^*\). The map \(f_2^*: (E_Y)_{q,q}^1 \to (E_X)_{q,q}^1\) is the sum of the maps

\[
id \otimes q: H_q(F_I, \partial F_I) \otimes H_q(T^n) \to H_q(F_I, \partial F_I) \otimes H_q(T^n/T_I)
\]

over all simplices \(I\) of rank \(n-q\). Thus

\[
f_2^*(\beta \otimes [T^{(A)}]) = \left[ f_2^* \left( \left( \sum_{I, |I|=n-q} B_I[F_I] \right) \otimes [T^{(A)}] \right) \right] = R_{\beta,A}^\prime
\]

by Lemma 3.3. \(\square\)

**Remark 4.2.** It follows from the spectral sequence that the element \(R_{\beta,A}^\prime \in \oplus_q (E_X)_{q,q}^2\) does not depend on a cellular chain, representing \(\beta\). Proposition 2.7 also implies that relations \(\{R_{\beta,A}^\prime\}\) are linearly independent in \((E_X)_{q,q}^2\) when \(\beta\) runs over some basis of \(\text{Im} \delta_q + 1\) and \(A\) runs over all subsets of \([n]\) of cardinality \(q\).

Next we want to check that relations of the first type are exactly the relations in the quotient ring \(k[S_Q]/\Theta\).

**Proposition 4.3.** Let \(\varphi: \langle [X_I] \rangle \to k[S_Q]\) be the degree reversing linear map, which sends the generator \([X_I]\) to \(v_I\). Then \(\varphi\) descends to the isomorphism of \(k\)-modules

\[
\tilde{\varphi}: \langle [X_I] \rangle/\langle R_{J,A} \rangle \to k[S_Q]/\Theta.
\]

**Proof.** (1) First we prove that \(\tilde{\varphi}\) is well defined by showing that the element

\[
\varphi(R_{J,A}) = \sum_{I,J} I:J \left[ I,J \right] C_{I,A} v_I \in k[S_Q]
\]

lies in \(\Theta\). Let \(s = |J|\) and, consequently, \(|I| = s + 1\), \(|A| = n - s - 1\). Let \([n]\backslash A = \{\alpha_1 < \ldots < \alpha_{s+1}\}\) and let \(\{j_1, \ldots, j_s\}\) be the vertices of \(J\) listed in a positive order. Consider the \(s \times (s + 1)\) matrix:

\[
D = \begin{pmatrix}
\lambda_{j_1,\alpha_1} & \cdots & \lambda_{j_1,\alpha_{s+1}} \\
\vdots & \ddots & \vdots \\
\lambda_{j_s,\alpha_1} & \cdots & \lambda_{j_s,\alpha_{s+1}}
\end{pmatrix}
\]

Denote by \(D_l\) the square submatrix obtained from \(D\) by deleting \(l\)-th column and let \(a_l = (-1)^{l+1} \det D_l\). We claim that

\[
\varphi(R_{J,A}) = \pm v_J \cdot (a_1 \theta_{\alpha_1} + \ldots + a_{s+1} \theta_{\alpha_{s+1}}).
\]
Indeed, after expanding each $\theta_i$ as $\sum_{i \in \text{Vert}(S)} \lambda_i \iota_i v_i$, all elements of the form $v_J v_i$ with $i < J$ cancel (the coefficients at these terms are determinants of matrices with two coinciding rows). Other terms give

$$\sum_{I, I \supset J, i \in I \setminus J} (a_1 \lambda_{i, \alpha_1} + \ldots + a_{s+1} \lambda_{i, \alpha_{s+1}}) v_I.$$ 

The coefficient at $v_I$ is equal to the determinant of the matrix

$$\begin{pmatrix} \lambda_{i, \alpha_1} & \ldots & \lambda_{i, \alpha_{s+1}} \\ \lambda_{j_1, \alpha_1} & \ldots & \lambda_{j_1, \alpha_{s+1}} \\ \vdots & \ddots & \vdots \\ \lambda_{j_s, \alpha_1} & \ldots & \lambda_{j_s, \alpha_{s+1}} \end{pmatrix},$$

by the cofactor expansion along the first row. This determinant is equal to $\text{sgn}_A [I : J] C_{I, A}$ by the definition of $C_{I, A}$. Indeed, the number $C_{I, A}$ was defined as the determinant for some positive ordering of vertices of $I$. The ordering $i < j_1 < \ldots < j_s$ (used to order the rows of matrix (4.2)) is either positive or negative depending on the incidence sign $[I : J]$.

(2) $\tilde{\phi}$ is surjective by Lemma 3.5.

(3) Ranks of both spaces are equal. Indeed, $\dim \langle [X_I] \mid |I| = n - q \rangle / \langle R_{J, A} \rangle = \dim (E_X)_{q, q} = h'_{n-q} (S_Q)$ by Proposition 2.7. By Proposition 2.4 the poset $S_Q$ is Buchsbaum. Thus $\dim (k[S_Q] / \Theta)_{n-q} = h'_{n-q} (S_Q)$ by Schenzel’s theorem (see [14], [16] Ch.II.§8.2, or [11] Prop.6.3 for simplicial posets).

(4) If $k$ is a field, then we are done. Since the statement holds over any field, the case $k = \mathbb{Z}$ automatically follows. \hfill $\Box$

The Poincare duality in $X$ yields

**Corollary 4.4.** The map $g: k[S_Q] \to H^* (X)$ factors through $k[S_Q] / \Theta$ and the kernel of the homomorphism $\tilde{g}: k[S_Q] / \Theta \to H^* (X)$ is additively generated by the elements

$$L'_{\beta, A} = \sum_{I, |I| = n-q} B_I C_{I, A} v_I,$$

where $q \leq n - 2$, $\beta \in \text{Im} (\delta_{q+1}: H_{q+1} (Q, \partial Q) \to H_q (\partial Q))$, $\sum_{I, |I| = n-q} B_I [F_I]$ is a cellular chain in $\partial Q$ representing $\beta$, and $A \subseteq [n]$, $|A| = q$.

**Remark 4.5.** The ideal $\Theta \subset k[S_Q]$ coincides with the image of the homomorphism $H^{>0} (BT^n) \to H^* (X)$. So the fact that $\Theta$ vanishes in $H^* (X)$ is not surprising. An interesting thing is that $\Theta$ vanishes already in a second term of the spectral sequence, while other relations in $H^* (X)$ demonstrate the effects of higher differentials.
Note, that the elements $R_{p,A} = \sum_{l,|l|=q} B_q^{l,A}[X_l] \in (E_X)^{2,q}_A$ and $L_{p,A} = \sum_{l,|l|=q} B_q^{l,A}v_l \in k[S_\Theta]/\Theta$ can be defined for any homology class $\beta \in H_q(\partial Q)$ (not only for the image of the connecting homomorphism $\delta_{q+1}$).

Recall that the socle of a module $M$ over the polynomial ring $k[m]$ is the submodule

$$\text{Soc } M \overset{\text{def}}{=} \{y \in M \mid k[m]^+ \cdot y = 0\},$$

where $k[m]^+$ is the maximal graded ideal of $k[m]$.

**Theorem 2.** For every $\beta \in H_q(\partial Q)$, $q \leq n-1$ and $A \subset [n]$, $|A| = q$ the element $L_{p,A} \in k[S_\Theta]/\Theta$ lies in a socle of $k[S_\Theta]/\Theta$.

We postpone the proof to Section $\S$.

5. **Non-face cycles of $X$**

5.1. **Spine and diaphragm cycles.** In this section we give a geometrical description of homology classes of $Q$ different from face classes.

**Construction 5.1.** Let $\eta \in H_k(Q)$ be a cycle of $Q$ and let $a \in H_l(T^n)$, $l < k$ be a cycle of $T^n$ represented by a subtorus $T^{(a)} \subset T^n$. They determine the homology class $\eta \otimes [T^{(a)}] \in H_{k,l}(Y) \cong H_k(Q) \otimes H_l(T^n)$. Thus they determine the class $\text{Sp}_{\eta,a} \in H_{k+l}(X)$ via the isomorphism $f_*: H_{k,l}(Y) \to H_{k,l}(X)$ asserted by pt.(1) of Proposition $\S2.9$. The cycles of this form (which are the elements of $H_{k,l}(X)$ for $k > l$) will be called spine cycles (or spine classes).

Suppose that $\eta$ is represented by an embedded pseudomanifold $N \subset Q$. We may assume that $N$ lies in the interior of $Q$. Then the spine cycle $\text{Sp}_{\eta,a}$ is represented by an embedded pseudomanifold $N \times T^{(a)} \subset Q^{(a)} \times T^n \subset X$.

**Construction 5.2.** Suppose $k < n$ and let $\zeta \in H_k(Q,\partial Q)$ be a relative homology class. Assume for simplicity that $\zeta$ is represented by a submanifold (or, generally, embedded pseudomanifold) $L \subset Q$ of dimension $k$ with boundary $\partial L \subset \partial Q$ (which may be empty). Every proper face of $Q$ is acyclic, thus can be considered as a homological cell of $\partial Q$. Therefore without lost of generality we may assume $\partial L \subset Q_{k-1}$.

We also assume that $L \setminus \partial L \subset Q \setminus \partial Q$.

For each class $a \in H_l(T^n)$, represented by an $l$-dimensional subtorus $T^{(a)}$, consider the subset $Z_{L,a} = (L \times T^{(a)})/\sim \subset X$.

**Proposition 5.3.**

- If $l \geq k$, the subset $Z_{L,a}$ is a pseudomanifold. Thus it represents a well-defined element $Df_{L,a} \in H_{k+l}(X)$ which will be called a diaphragm class (or diaphragm cycle).
- If $l > k$, the class $Df_{L,a} \in H_{k+l}(X)$ depends only on the class $\zeta = [L] \in H_k(Q,\partial Q)$ but not on the particular representative $L$. 
PROOF. The set \(((L\setminus\partial L) \times T^{(a)})/\sim = (L\setminus\partial L) \times T^{(a)}\) is a manifold of dimension \(k+l\). The exceptional locus \((\partial L \times T^{(a)})/\sim\) has dimension at most \(k+l-2\). Indeed, we have \(\partial L \subset Q_{k-1}\), thus, under the projection map \((\partial L \times T^{(a)})/\sim \to \partial L\), every point \(x \in \partial L\) has a preimage of the form \(T^{(a)}/T_1\) with \(|I| \geq n-k+1\). This set has dimension at most \(l-1\) since \(l + (n-k+1) > n\). Thus the total dimension of exceptional locus is at most \(\dim \partial L + (l-1) = k+l-2\).

The second statement can be proved similarly. Let \((L_1, \partial L_1)\) and \((L_2, \partial L_2)\) be two manifolds representing the same element \(\zeta \in H_k(Q, \partial Q)\). There exists a pseudomanifold bordism between them, i.e. a pseudomanifold \(\Xi\) of dimension \(k+1\) with boundary and a map \(\phi: \Xi \to Q\) such that \(L_1, L_2\) are disjoint submanifolds of \(\partial \Xi\), the restriction of \(\phi\) to \(L_\xi\) is the inclusion of \(L_\xi \hookrightarrow Q\), and \(\phi(\partial \Xi \setminus (L_1 \cup L_2)) \subset \partial Q\) (this follows from the geometrical definition of homology, see [13, App. A.2]). Again, we may assume that \(\phi(\partial \Xi \setminus (L_1 \cup L_2)) \subset Q_k\). Similar to the first statement, we can consider the space \((\Xi \times T^{(a)})/\sim\) of dimension \(k+l+1\). This space is a pseudomanifold with boundary, and the boundary is exactly the difference \(Z_{L_1,a} - Z_{L_2,a}\). Thus \(\text{Df}_{L_1,a} = \text{Df}_{L_2,a}\) in \(H_k(X)\).

Thus for \(k < l\) there is a well defined homology class \(\text{Df}_{\zeta,a} \overset{\text{def}}{=} \text{Df}_{L,a} \in H_{k,l}(X)\) depending on \(\zeta \in H_k(Q, \partial Q)\) and \(a \in H_l(T^n)\). These classes span the homology modules \(H_{k,l}(X)\) for \(k < l\) and correspond to pt.(2) of Proposition 2.9.

When \(k = l \leq n\) we call the classes \(\text{Df}_{L,a}\) extremal diaphragm classes. In this case the situation is a bit different: the classes \(\text{Df}_{L,a}\) depend not only on the homology class of \(L\) but on the representative \(L\) itself. Nevertheless, if \(L_1\) and \(L_2\) represent the same class in \(H_k(Q, \partial Q)\), then the classes \(\text{Df}_{L_1,a}, \text{Df}_{L_2,a} \in H_{k,k}(X)\) coincide modulo face classes, as proved below. Our goal is to derive exact formulas, thus we restrict to the case, when \(a \in H_l(T^n)\) is represented by a coordinate subtorus \(T^{(A)}\) for \(A \subset [n]\), \(|A| = l\).

**Construction 5.4.** Let \(\phi_\epsilon: (L_\epsilon, \partial L_\epsilon) \to (Q, \partial Q), \epsilon = 1, 2\), be two pseudomanifolds representing the same element \(\zeta \in H_k(Q, \partial Q), k < n\). As in the proof of Proposition 5.3 consider a pseudomanifold bordism \((\Xi, \partial \Xi)\) between \(L_1\) and \(L_2\), and the map \(\phi: \Xi \to Q\), which sends the boundary \(\partial L\) into the union of \(L_1, L_2\) and \(Q_k\). The skeletal stratification of \(Q\) induces a stratification on \(\Xi\). The restriction of the map \(\phi\) sends \(\Xi_{k-1}\) to \(Q_{k-1}\). Let \(\delta\) be the connecting homomorphism

\[\delta: H_{k+1}(\Xi, \partial \Xi) \to H_k(\partial \Xi, \Xi_{k-1})\]

in the long exact sequence of the triple \((\Xi, \partial \Xi, \Xi_{k-1})\). Consider the sequence of homomorphisms
\[ H_{k+1}(\Xi, \partial \Xi) \overset{\delta}{\longrightarrow} H_{n-1}(\partial \Xi, \Xi_{k-1}) \cong \]
\[ H_k(L_1, \partial L_1) \oplus H_k(L_2, \partial L_2) \oplus H_k(\partial \Xi \setminus (L_1^0 \cup L_2^0), \partial \Xi_{k-1}) \overset{\text{id} \oplus \text{id} \oplus \phi \circ}{\longrightarrow} \]
\[ H_k(L_1, \partial L_1) \oplus H_k(L_2, \partial L_2) \oplus H_k(Q_k, Q_{k-1}). \]

It sends the fundamental cycle \([\Xi] \in H_{k+1}(\Xi, \partial \Xi)\) to the element

\[
(5.1) \left( [L_1], -[L_2], \sum_{I, \dim F_I = k} D_I[F_I] \right)
\]

of the group \(H_k(L_1, \partial L_1) \oplus H_k(L_2, \partial L_2) \oplus H_k(Q_k, Q_{k-1})\), for some numbers \(D_I \in \mathbb{k}\).

**Proposition 5.5.** Let \(L_1, L_2\) be two manifolds representing the same class \(\zeta \in H_k(Q, \partial Q)\), \(k < n\). Consider any subset \(A \subset [n]\), \(|A| = k\) and let \(a \in H_k(T^n)\) be the fundamental class of the coordinate subtorus \(T_A^n\). Then there is a relation in \(H_{2k}(X)\):

\[
(5.2) \quad Df_{L_1,a} - Df_{L_2,a} + \sum_{I, \dim F_I = k} D_I C_{I,A}[X_I] = 0.
\]

The numbers \(D_I\) are given by \((5.1)\), and the numbers \(C_{I,A}\) were defined in Lemma 3.3.

**Proof.** Consider the space \((\Xi \times T(A))/\sim'\) and the map \(\phi \times \iota : (\Xi \times T(A))/\sim' \to X = (Q \times T^n)/\sim\), where the relation \(\sim'\) is induced from \(\sim\) by the map \(\phi\), and \(\iota : T(A) \to T^n\) is the inclusion map. The space \((\Xi \times T(A))/\sim'\) is a pseudomanifold with boundary, and its boundary represents the element \(Df_{L_1,a} - Df_{L_2,a} + \sum_{I, \dim F_I = k} D_I C_{I,A}[X_I]\) in \(H_{2k}(X)\) by Lemma 3.3. Thus this element vanishes in homology. \(\square\)

Therefore, up to face classes, the middle homology group \(H_{k,k}(X)\) coincides with \(H_k(Q, \partial Q) \otimes H_k(T^n)\) for \(k < n\). This was stated in equivalent form in pt.(3) of Proposition 2.9.

### 5.2. Integral coefficients

Proposition 2.9 was stated only over a field. On the other hand, the geometrical constructions of the previous subsection determine the additive homomorphisms

\[
\bigoplus_{k > l} H_k(Q) \otimes H_l(T^n) \to H_*(X), \quad \bigoplus_{k \leq l} H_k(Q, \partial Q) \otimes H_l(T^n) \to H_*(X)
\]

over \(\mathbb{Z}\) as well. Combining this with the inclusion of face cycles

\[
\langle [X_I] \rangle / (R_{I,A}, R'_{\beta, A}) \hookrightarrow H_*(X)
\]

we get a map

\[
(5.3) \quad \left( \bigoplus_{k > l} H_k(Q) \otimes H_l(T^l) \right) \oplus \left( \bigoplus_{k \leq l} H_k(Q, \partial Q) \right) \oplus \left( \langle [X_I] \rangle / (R_{I,A}, R'_{\beta, A}) \right) \to H_*(X)
\]
which is well defined for any coefficients and is an isomorphism over a field. Now it follows by induction from the universal coefficient theorem, that the map \((5.3)\) is an isomorphism over \(\mathbb{Z}\). This proves

**Proposition 5.6.** Proposition \(2.9\) holds over \(\mathbb{Z}\). Homology groups of \(X\) are generated by face classes, spine classes, and diaphragm classes. The groups \(H_{k,l}(X)\) for \(k > l\) are generated by spine classes; the groups \(H_{k,l}(X)\) for \(k < l\) are generated by non-extremal diaphragm classes; the short exact sequence

\[
0 \to (\hat{E}_X)_{k,k} \to H_{k,k}(X) \to H_k(Q, \partial Q) \otimes \Lambda_k \to 0.
\]

identifies the quotient of \(H_{k,k}(X)\) by the face classes with the group of extremal diaphragm classes. This short exact sequence splits, but the splitting is not canonical.

### 5.3. Auxiliary cycles and relations.

In construction \(5.1\) we defined the cycles \(\text{Sp}_{\eta,a}\) for each \(\eta \in H_k(Q)\) and \(a \in H_l(T^n)\) under assumption that \(k > l\). But the same construction can be applied for any \(k\) and \(l\). If \(\eta\) is represented by a pseudomanifold \(N\) in the interior of \(Q\) and \(a\) is represented by a subtorus, then the product \(N \times T^{(a)}\) is a submanifold in \(Q^* \times T^n \subset X\), thus represents an element \([N \times T^{(a)}] \in H_{k+l}(X)\).

Although for \(k \leq l\) the element \(N \times T^{(a)}\) is not a spine cycle, we keep denoting it \(\text{Sp}_{\eta,a}\). For \(k < l\) we have \(\text{Sp}_{\eta,a} = \text{Df}_{\eta',a}\), where \(\eta'\) is the image of \(\eta\) in \(H_k(Q, \partial Q)\).

### 6. Intersections in \(H_*(X)\)

Let \(\cap : H_k(M) \otimes H_l(M) \to H_{k+l-\dim M}(M)\) denote the intersection product on a closed manifold \(M\), i.e. an operation Poincare dual to the cup-product in cohomology. From the geometrical structure of \(X\) (and also from Proposition \(6.3\)) follows

**Proposition 6.1.** If \(I_1, I_2 \in S_Q\), then \([X_{I_1}] \cap [X_{I_2}] = [X_{I_1 \cap I_2}] \cap \sum_{I \in I_1 \cup I_2} [X_I]\).

Intersections of spine and diaphragm classes can also be described geometrically. There exists an intersection product, \(\cap : H_{k_1}(Q) \otimes H_{k_2}(Q, \partial Q) \to H_{k_1+k_2-n}(Q)\) dual to the cohomology product \(H^{n-k_1}(Q, \partial Q) \otimes H^{n-k_2}(Q) \to H^{2n-k_1-k_2}(Q, \partial Q)\). If the classes \(\eta \in H_{k_1}(Q)\) and \(\zeta \in H_{k_2}(Q, \partial Q)\) are represented by a smooth submanifold \(N\) and a smooth submanifold with boundary \((L, \partial L)\) respectively, and if \(N\) intersects \(L\) transversely, then \(\eta \cap \zeta\) is represented by a submanifold \(N \cap L\). There is also an intersection product in homology of torus \(\cap : H_{l_1}(T^n) \otimes H_{l_2}(T^n) \to H_{l_1+l_2-n}(T^n)\). From the geometrical construction of cycles in \(X\) we conclude that the intersection product on \(H_*(X)\) has the following structure.

**Proposition 6.2.**

1. The cycles \(\text{Sp}_{\eta,a} \in H_{k_1,l_1}(X), k_1 > l_1\) and \(\text{Df}_{L,b} \in H_{k_2,l_2}(X), k_2 \leq l_2\) satisfy \(\text{Sp}_{\eta,a} \cap \text{Df}_{L,b} = \text{Sp}_{\eta \cap [L], a \cap b}\).
Since \( \dim(\eta \cap [L]) = k_1 + k_2 - n \) and \( \dim(a \cap b) = l_1 + l_2 - n \), the element \( \text{Sp}_{\eta \cap [L], a \cap b} \) is either a spine class (if \( k_1 + k_2 > l_1 + l_2 \)), or a diaphragm class determined in subsection \( \textbf{5.3} \) (if \( k_1 + k_2 = l_1 + l_2 \)).

(2) The cycles \( \text{Sp}_{\eta', a} \in H_{k_1,l_1}(X), \ k_1 > l_1 \) and \( \text{Sp}_{\eta', b} \in H_{k_2,l_2}(X), \ k_2 > l_2 \) satisfy

\[
\text{Sp}_{\eta', a} \cap \text{Sp}_{\eta', b} = \text{Sp}_{\eta' \cap a \cap b}.
\]

The result is a spine cycle.

(3) Spine cycles do not meet face cycles: \( \text{Sp}_{\eta,a} \cap [X_I] = 0 \).

The proof follows directly from the constructions.

**Proposition 6.3.** The linear span of proper face classes \([X_I]\) is an ideal of \( H_*(X) \) with respect to the intersection product.

**Proof.** Suppose \( I \neq \emptyset \) and let \( \iota: X_I \hookrightarrow X \) be the inclusion of a face submanifold. Let \( \kappa \) be a cohomology class Poincare dual to some diaphragm class \( Df \) in \( X \). Then \( [X_I] \cup Df = [X_I] \cap \kappa = \iota_*(\kappa^*) \cap [X_I] \). The class \( \iota^*(\kappa) \cap [X_I] \in H_*(X_I) \) is a linear combination of face classes since there are no other classes in \( H_*(X_I) \). Thus \( [X_I] \cap Df \) is a linear combination of face classes in \( H_*(X) \).

**Remark 6.4.** The description of intersections of diaphragm cycles with themselves and with face cycles is difficult in general. Nevertheless, in practice one can use the following trick (cf. the discussion of a similar problem in \( \textbf{8.3} \) Sect.8]). Suppose the task is to compute \( Df_{L,a} \cap [X_I] \). If \( L \cap F_I = \emptyset \), then the intersection product is 0. If not, find another submanifold with boundary \( L' \) such that \( [L'] = [L] \in H_*(Q, \partial Q) \) and \( L' \cap F_I = \emptyset \). Then, by Proposition \( \textbf{5.3} \) we have \( Df_{L,a} = Df_{L',a} + \Sigma \), where \( \Sigma \) is a linear combination of face classes. Thus we have

\[
Df_{L,a} \cap [X_I] = Df_{L',a} \cap [X_I] + \Sigma \cap [X_I] = \Sigma \cap [X_I].
\]

The last intersection can be computed by Proposition \( \textbf{6.1} \).

**7. Examples**

**7.1. A very concrete example.** This example is similar to the one studied in \( \textbf{12} \) Th.3.1]. For \( Q \) take a square with triangular hole. Orientations of facets and values of characteristic function are assigned to \( Q \) as shown on Fig.11 left.

Homology groups of the corresponding 4-dimensional manifold \( X = (Q \times T^2)/\sim \) are as follows.

(1) **Face classes.** These are the following: the fundamental class \([X] \in H_4(X)\); the classes of characteristic submanifolds

\[
[X_1], [X_2], \ldots, [X_7] \in H_{1,1}(X) \subset H_2(X),
\]

which correspond to the sides of \( Q \); and the classes of fixed points

\[
[X_{12}], [X_{23}], [X_{34}], [X_{45}], [X_{56}], [X_{67}], [X_{57}] \in H_{0,0}(X) = H_0(X),
\]
Figure 1. Structure of $Q$ and values of characteristic function

which correspond to vertices of $Q$. Relations on these classes are given by Proposition\[4\]. The first-type relations in $H_2(X)$ are:

$[X_1] + [X_3] + 3[X_4] + 2[X_5] + [X_6] - 3[X_7] = 0$

$[X_2] + [X_4] + 3[X_5] + 2[X_6] - 5[X_7] = 0$

(the coefficients are respectively first and second coordinates of the values of characteristic function). The first type relations on the classes $[X_{ij}]$ are encoded by the sides of of the orbit space. These relations are the following:

$[X_{12}] = -[X_{23}] = [X_{34}] = -[X_{14}]$

$[X_{56}] = [X_{67}] = [X_{57}]$

(the signs are due to the signs of fixed points). To find relations of the second type, we need to pick a homology class in the image of $\delta_1: H_1(Q, \partial Q) \to H_0(\partial Q)$. Take for example the class, represented by the chain $[F_{14}] - [F_{57}]$. It gives a relation of the second type

$[X_{14}] - [X_{57}] = 0$

Thus all fixed points up to sign represent the same generator $[pt] \in H_0(X)$. Of course, this follows easily from the connectivity of $X$, but here we wanted to emphasize the different nature of two types of relations.

(2) Spine cycles. Consider a submanifold $N \subset Q$ representing the generator $\eta \in H_1(Q)$ (Fig.1 right). Together with the class of a point $[T(\emptyset)] \in H_0(T^2)$ it determines a spine class $Sp_{\eta, \emptyset} \in H_{1,\emptyset}(X) = H_1(X)$. Geometrically, $Sp_{\eta, \emptyset}$ is represented by a submanifold $N \subset Q$ lifted by a zero-section map $Q \hookrightarrow X$.

(3) Diaphragm cycles. Consider the submanifold $L$ representing the generator of $H_1(Q, \partial Q)$ with the boundary lying in the 0-skeleton of $Q$ (see Fig.1 right). For each subset $A = \{1\}, \{2\}, \{1, 2\}$ we have a homology cycle in $H_{1,|A|}(X)$ represented
by a pseudomanifold $(L \times T^{(4)})/\sim$. Thus we have the generators
\[ Df_{L,1} = [(L \times T^{(1)})]/\sim, \quad Df_{L,2} = [(L \times T^{(2)})]/\sim \]
of $H_{1,1}(X) \subset H_{2}(X)$ and the generator
\[ Df_{L,(12)} = [(L \times T^{2})]/\sim \]
of $H_{1,2}(X) = H_{3}(X)$. Let $L'$ be another submanifold representing the same homology class in $H_{1}(Q, \partial Q)$ (see Figure 1). Consider a bordism $\Xi$ between $L$ and $L'$ shown on the figure. We have
\[ Df_{L,(12)} = Df_{L',(12)} \]
in $H_{3}(X)$ since $(\Xi \times T^{2})/\sim$ is a pseudomanifold bordism between $(L \times T^{2})/\sim$ and $(L' \times T^{2})/\sim$.

We have a relation $\delta \Xi = -[L_{1}] + [L_{2}] + [F_{4}] + [F_{6}] + [F_{7}]$. It generates the relations
\[
\begin{align*}
-Df_{L,1} + Df_{L',1} + 1[X_{4}] + 2[X_{6}] - 5[X_{7}] &= 0 \\
-Df_{L,2} + Df_{L',2} + 3[X_{4}] + 1[X_{6}] - 3[X_{7}] &= 0
\end{align*}
\]
in $H_{2}(X)$. These relations are the boundaries of $(\Xi \times T^{(1)})/\sim$ and $(\Xi \times T^{(2)})/\sim$ respectively. The coefficients are the complimentary coordinates of characteristic function: for the cycle encoded by the first coordinate subtorus, we take the second coordinates of characteristic function, and vice versa. In this computation we used the formula for the coefficients $C_{I,A}$ asserted by Lemma 3.3.

(4) **Intersections of cycles** can be seen from the picture. In particular, the cycles $Sp_{X,\emptyset}$ and $Df_{L,(1,2)}$ are transversal, and their intersection induces a nondegenerate pairing between $H_{1}(X)$ and $H_{3}(X)$.

![Figure 2. Different representatives of diaphragm classes](image)

For a nontrivial example, let us compute the intersection of $Df_{L,1}$ with $Df_{L,2}$ to demonstrate the idea sketched in Remark 6.4. Consider the auxiliary intervals $L'$ and $L''$ shown on Fig 2. Similar to the calculations above we have
\[
\begin{align*}
Df_{L,1} &= Df_{L',1} + 1[X_{4}] - 5[X_{7}] \\
Df_{L,2} &= Df_{L'',2} - 1[X_{1}] - 2[X_{5}]
\end{align*}
\]
Thus \( Df_{L,1} \cap Df_{L,2} = (Df_{L,1} + [X_4] - 5[X_7]) \cap (Df_{L,2} - [X_1] - 2[X_5]) = -[X_4] \cap [X_1] + 10[X_7] \cap [X_5] = -[X_4] + 10[X_5] = 9(pt) \in H_0(X). \)

### 7.2. Toric origami manifolds

In this subsection we apply the general method to a class of toric origami manifolds and derive some results proved in [3] in a different way.

Toric origami manifolds (see [7],[10]) appeared in differential geometry as generalizations of symplectic toric manifolds. The precise geometrical definition is in most part irrelevant to our study. Essential are the following properties: orientable toric origami manifold \( X \) is a manifold with locally standard torus action; its orbit space \( Q = X/T^n \) is homotopy equivalent to a graph \( \Gamma \), and inclusion of any face in \( Q \) is homotopy equivalent to the inclusion of a subgraph in \( \Gamma \).

As usual, there is a principal torus bundle \( Y \to Q \) such that \( X = Y/\sim \). Since \( Q \) is homotopy equivalent to a graph, \( H^2(Q, \mathbb{Z}^n) = 0 \), so the Euler class of \( Y \) vanishes. Thus in origami case we have \( Y = Q \times T^n \).

Now we restrict to the case when all proper faces of \( Q \) are acyclic. Since they are homotopy equivalent to graphs, it follows automatically that they are contractible. Let \( b_1 = \dim H_1(Q) = \dim H_1(\Gamma) \). Poincare–Lefchetz duality implies:

\[
H_q(Q, \partial Q) \cong H^{n-q}(Q) \cong \begin{cases} 
\mathbb{Z}, & \text{if } q = n; \\
\mathbb{Z}^{b_1}, & \text{if } q = n - 1; \\
0, & \text{otherwise.}
\end{cases}
\]

Let us describe the connecting homomorphisms \( \delta_i : H_i(Q, \partial Q) \to H_{i-1}(\partial Q) \). For simplicity we discuss the case \( n \geq 4 \); dimensions 2 and 3 can be done similarly. When \( n \geq 4 \), lacunas in the exact sequence of the pair \( (Q, \partial Q) \) imply that \( \delta_i : H_i(Q, \partial Q) \to H_{i-1}(\partial Q) \) is an isomorphism for \( i = n - 1, n \), and trivial otherwise. We have

\[
H_i(\partial Q) \cong \begin{cases} 
\mathbb{k}, & \text{if } i = 0, n - 1; \\
\mathbb{k}^{b_1}, & \text{if } i = 1, n - 2; \\
0, & \text{otherwise.}
\end{cases}
\]

Proposition [2,7] implies that \((E_X)_{p,q}^2\) has the form shown schematically on Figure 3. The differential \((d_X)^2\) hitting the marked position produces relations \( R_{\beta,A}^2 \) of the second type on the cycles \([X_i] \in H^{2n-4}(X) \). These relations are explicitly described by Proposition [4,1] and the number of independent relations is \( (\binom{n}{2})b_1 \). Dually, this consideration shows that the map \( \mathbb{k}[S]/\Theta \to H^*(X) \) has nontrivial kernel of dimension \( (\binom{n}{2})b_1 \) in degree 4.

In addition, we have the following non-face cycles:

1. There are \( b_1 \) one-dimensional spine classes, which appear as the liftings of cycles in \( \Gamma \).
(2) There are $b_1$ diaphragm classes of codimension 1 given by the generators of $H_{n-1}(Q, \partial Q) \cong H^1(\Gamma)$ swept by the action of a whole torus. These cycles are equivariant. They are dual to cycles (1).

(3) There are $n b_1$ extremal diaphragm classes of codimension 2. These are given by the generators of $H_{n-1}(Q, \partial Q)$ swept around by actions of $(n-1)$-dimensional subtori. A choice of these classes is not canonical.

8. Collar model

In this section we prove Theorem 2 using an auxiliary space $\hat{X}$.

Construction 8.1. Consider the space $\hat{Q} = \partial Q \times [0, 1]$. It is an $(n-1)$-dimensional manifold with the boundary $\partial \hat{Q}$ of the form $\partial_0 \hat{Q} \sqcup \partial_1 \hat{Q}$, where $\partial_\epsilon \hat{Q} = \partial Q \times \{\epsilon\}$, $\epsilon = 0, 1$. We may identify $\partial_0 \hat{Q}$ with $\partial Q$ and consider $\hat{Q}$ as a filtered topological space:

$$Q_0 \subset Q_1 \subset \ldots \subset Q_{n-1} = \partial Q = \partial_0 \hat{Q} \subset \hat{Q}.$$
One may think about $\hat{Q}$ as a collar of $\partial Q$ inside $Q$.

Consider the space $\hat{Y} = \hat{Q} \times T^n$ and the identification space $\hat{X} = \hat{Y}/\sim$. The relation $\sim$ identifies points over $\partial_0 \hat{Q}$ in the same way as it does for $\partial Q \subset Q$, while there are no identifications over $\partial_1 \hat{Q}$. The space $\hat{X}$ is a manifold with boundary. Its boundary consists of points over $\partial_1 Q$, hence is homeomorphic to $\partial Q \times T^n$. The space $\hat{X}$ can be considered as a $T^n$-invariant tubular neighborhood of the union of all characteristic submanifolds in $X$. There are natural topological filtrations on $\hat{Y}$ and $\hat{X}$ induced by the filtration on $\hat{Q}$.

In terminology of [2] the space $\hat{X}$ is a Buchsbaum pseudo-cell complex, thus Propositions 2.5, 2.6, and items (1)–(5) of Proposition 2.7 hold for $\hat{Q}$, $\hat{Y}$, and $\hat{X}$. The $n$-th column of all spectral sequences vanishes, since $H_* (\hat{Q}, \partial_0 \hat{Q}) = 0$. Thus all spectral sequences $(E^r_\hat{Q})^r$, $(\hat{E}^r_{\hat{X}})^r$ collapse at a first page and, consequently, the spectral sequences $(E^r_X)^r$, $\hat{(E}^r_{\hat{X}})^r$ collapse at a second page.

For each $I \in S_Q$, with dim $F_I = q < n$ there is a distinguished element $[X_I] \in H_{2q}(\hat{X}_q, \hat{X}_{q-1}) = (E^1_{\hat{X}})_{q,q}^I$. It survives in a spectral sequence, and gives the fundamental class of the face submanifold $X_I \subset \hat{X}$. Linear relations on classes $[X_I]$ in $H_* (\hat{X})$ are described similarly to Section 4. When $q = n - 1$ there are no relations on $[X_I]$, since there are no differentials landing at the cell $(E^1_{\hat{X}})^{n-1,n-1}_q$. For $q < n - 1$ the relations on $[X_I]$ are the images of $(d^1_{\hat{X}}): (E^1_{\hat{X}})^{q+1,q}_q \to (E^1_{\hat{X}})^{q,q}_q$. These differentials coincide with $(d^1_{\hat{X}})$ thus the relations on $[X_I]$ for $|I| > 1$ are exactly $R_{I,A}$ defined in Proposition 4.3. Thus Proposition 4.3 implies

**Lemma 8.2.** Let $V_*$ be a submodule of $H_*(\hat{X})$ generated by face classes $[X_I]$, $I \neq \emptyset$. Then there is a degree reversing linear map

$$\hat{\varphi}: V_{2q} \to (k[\Theta])_{2(n-q)},$$

sending $[X_I]$ to $v_I$. It is an isomorphism for $q < n - 1$ and surjective for $q = n - 1$. This map is a ring homomorphism with respect to the intersection product on $\hat{X}$.

Now we can give a geometric proof of Theorem 2.

**Proof.** We need to prove that the element $L'_{\beta,A}$ lies in a socle of $k[\Theta]$. Thus we need to show that $L'_{\beta,A} v_i = 0$ for every vertex $i \in Vert(S_Q)$, where $\beta \in H_q (\partial Q)$, $|A| = q$, and $q \leq n - 2$. According to Lemma 8.2 it is sufficient to show that $R'_{\beta,A} \cap [F_i] = 0$ in $H_* (\hat{X})$.

Consider a geometrical cycle $B \subset \partial_0 \hat{Q}$ representing $\beta$. Now we allow $B$ to be any representative, and do not require that it lies in the stratum $Q_q$. Consider the cycle $B \times T^n$ in $\partial_0 \hat{Q} \times T^n$, and the corresponding cycle $(B \times T^n)/\sim$ in $\hat{X}$. The latter cycle represents the class $R'_{\beta,A} \in H_{2q}(\hat{X})$ by definition. The space $\hat{Q}$ is homologous to $\partial_0 \hat{Q}$.
so we can move the cycle $B \subset \hat{Q}$ away from the boundary $\partial_B \hat{Q}$. Thus $B \cap [F_i] = \emptyset$ and therefore $R'_{\beta,A} \cap [X_i] = 0$ in $H_*(\hat{X})$.

**Remark 8.3.** The same argument proves that $L'_{\beta,A} v_I = 0$ for any simplex $I \in S \setminus \hat{0}$. This fact does not directly follow from Theorem [2] since the map $k[m] \to \mathbb{k}[S]/\Theta$ may be not surjective in general.

**Remark 8.4.** The only reason why we considered the collar model $\hat{X}$ instead of $X$ is that there are no additional relations in $H_*(\hat{X})$ compared with $k[S]/\Theta$. The space $\hat{X}$ captures the properties of $k[S]/\Theta$ more precisely than $X$. On the other hand, $\hat{X}$ is a manifold with boundary so there is a geometrical intersection theory on it. This makes it an object worth studying.

**Remark 8.5.** The classes $R'_{\beta,A} \in (E_X)_{q,q}^2$ are the images of the classes $\beta \times [T^A] \in H_q(\partial Q) \times H_q(T^n)$ under the homomorphism $f^1_*: (E_{\hat{X}})_{q,q}^1 \to (E_{\hat{X}})_{q,q}^1$. This homomorphism is injective by Proposition [2.7], thus the construction gives an inclusion

$$H_q(\partial Q) \otimes H_q(T^n) \hookrightarrow \text{Soc}(k[S]/\Theta)_{2(n-q)}$$

for each $q \leq n-2$. When $q = n-1$, the map $H_{n-1}(\partial Q) \otimes H_{n-1}(T^n) \to \text{Soc}(k[S]/\Theta)_{2(n-q)}$ has the kernel of the form $\langle [\partial Q] \rangle \otimes H_{n-1}(T^n)$, where $[\partial Q]$ denotes the fundamental class of $\partial Q$. Note that $\partial Q$ may be disconnected, thus there could exist classes in $H_{n-1}(\partial Q)$ different from $[\partial Q]$. So far, there exists an injective map

$$(H_{n-1}(\partial Q)/\langle [\partial Q] \rangle) \otimes H_{n-1}(T^n) \hookrightarrow \text{Soc}(k[S]/\Theta)_{2(n-q)}$$

These statements reprove the result of Novik–Swartz [11, Th.3.5] in case of homology manifolds.

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