HARDY’S OPERATOR AND NORMABILITY OF
GENERALIZED LORENTZ - MARCINKIEWICZ SPACES,
with sharp or weakly sharp constant estimation.

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Abstract.
We introduce a Banach rearrangement invariant (tail) quasi - norm by means of Hardy’s (Cesaro) average on the (measurable) functions defined on some measurable space which is a slight generalization of classical Lorentz - Marcinkiewicz norm and find for it an equivalent norm expression.

Key words and phrases: Tail function, rearrangement invariant norm and spaces, weight, fundamental function, random variable, Hardy (Cesaro) operator, slowly and regular varying functions, examples, integral operator, factorization, upper and lower estimates, right and left inverse function, ordinary and Grand Lebesgue spaces, Lorentz, Marcinkiewicz norm and spaces, exactness and weak exactness.

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1 Notations. Statement of problem.

Let \((X = \{x\}, \mathcal{A}, \mu)\) be measurable space with non-trivial sigma-finite measure \(\mu\). We will suppose without loss of generality in the case \(\mu(X) < \infty\) that \(\mu(X) = 1\) (the probabilistic case) and denote \(x = \omega, \ P = \mu\).

Define as usually for arbitrary measurable function \(f : X \to R\) its distribution function (more exactly, tail function)

\(T_f(t) = \mu\{x : |f(x)| \geq t\}, \ t \geq 0,\)

\[\|f\|_p = \left[\int_X |f(x)|^p \mu(dx)\right]^{1/p}, \ p \geq 1; \ L_p = L_{p,\mu} = \{f, \|f\|_p < \infty\},\]
\[ \|f\|_\infty = \text{vraisup}_x |f(x)| \mod \mu, \]

and denote by \( f^*(t) = T_f^{-1}(t) \) the left inverse to the tail function \( T_f(t) \):

\[ f^{**}(t) \overset{\text{def}}{=} t^{-1} \int_0^t f^*(s) \, ds, \quad t > 0. \]

It will be presumed in the case \( X \subset \mathbb{R} \) that the measure \( \mu \) is classical Lebesgue measure: \( \mu(dx) = dx \).

We will denote the set of all tail functions as \( \{T\} \); obviously, the set \( \{T\} \) contains all the functions which are right continuous, monotonically non-increasing with values in the set \([0, \mu(X)]\).

Let also \((V, \| \cdot \|_V)\) be arbitrary Banach complete function space over the functions defined on the open semi-axis \( \mathbb{R}_+ = (0, \infty) \), not necessary to be rearrangement invariant. This imply in particular that if \( f_1, f_2 \in V \) and \( |f_1(t)| \leq |f_2(t)| \), then \( \|f_1\|_V \leq \|f_2\|_V \).

**Definition 1.1.**

We introduce the following quasy - norms \( \|\|f\||_Y^* = \|\|f\||_{Y,V}^* \) and \( \|\|f\||_Y = \|\|f\||_{Y,V} \)
for the measurable functions \( f : X \to \mathbb{R} \) as follows:

\[
\|\|f\||_Y^* = \|\|f\||_{Y,V}^* \overset{\text{def}}{=} \|f^*\|_V, \quad (1.1)
\]

\[
\|\|f\||_Y = \|\|f\||_{Y,V} \overset{\text{def}}{=} \|f^{**}\|_V, \quad (1.2)
\]

and correspondingly the following spaces

\[
(Y, \| \cdot \|_Y) = \{ f, \|\|f\||_Y < \infty \}, \quad (Y_*, \| \cdot \|_{Y^*}) = \{ f, \|\|f\||_{Y^*} < \infty \}. \quad (1.3)
\]

These spaces (or similar) are named by D.E.Edmund and B.Opic in [19] as "Lorentz-Karamata spaces", by B.Opic and L.Pick in [43] as "Lorentz-Zygmund spaces", by E.Pustylnik in [53] as "Ultrasymmetric spaces".

We will prove in this article that under some simple conditions the space \((Y, \| \cdot \|_Y)\) is (complete) rearrangement invariant Banach space and that the quasy - norm \( \|\|f\||_Y^* \) and the norm \( \|\|f\||_Y \) are linear equivalent:

\[
K_1(V) \|\|f\||_Y^* \leq \|\|f\||_Y \leq K_2(V) \|\|f\||_Y^*, \quad (1.4)
\]

where \( K_1(V), K_2(V) \) are finite positive constants (more exactly, function on \( V \)) depending only on the space \((V, \| \cdot \|_V)\) but not on the function \( f \), and will find sharp (exact) or as a minimum weak sharp (i.e. up to multiplicative constant) values of these functions.

In the articles [48], [50] the inequalities of a view (1.4) were applied in the theory of Probability and further - in Statistics and in the Monte-Carlo method in order to characterize the tail behavior of random variables and sums of random variables.

This problem in less general statement see in [14], [15], [17], [18], [28], [29], [30], [31], [37], [38], [39], [48], [50], [61]; see also reference therein.
The another but near statement of problem based on the Cesaro average and Cesaro spaces see in the recent article of S.V.Astashkin and L.Maligranda [4].

**Example 1.1.** (See [48]).

Let \( w = w(s), s \geq 0 \) be any continuous strictly increasing numerical function (weight) defined on the set \( s \in (0, \infty) \) such that

\[
    w(s) = 0 \leftrightarrow s = 0; \quad \lim_{s \to \infty} w(s) = \infty. \tag{1.5}
\]

We impose here on the set of all such a functions \( W = \{ w \} \) the following restriction:

\[
    \forall w \in W \exists T \in \{ T \} \Rightarrow w(T(s)) = 1/s.
\]

Let us introduce the following important functional

\[
    \gamma(w) = \sup_{t > 0} \left[ \frac{w(t)}{t} \int_0^t \frac{du}{w(u)} \right] \tag{1.6}
\]

and the following quasi-norms:

\[
    ||f||^*_w = \sup_{t > 0} [w(t) f^*(t)], \tag{1.7}
\]

\[
    ||f||_w = \sup_{t > 0} [w(t) f^{**}(t)], \tag{1.8}
\]

The necessary and sufficient condition for finiteness of the functional \( \gamma(w) \) see, e.g. in the article [3].

**Remark 1.1.** Note that

\[
    ||f||^*_w = \sup_{t > 0} [tw(T_f(t))],
\]

so that if \( ||f||^*_w \in (0, \infty) \), then

\[
    T_f(t) \leq w^{-1}(||f||^*_w/t).
\]

Therefore the functional \( f \to ||f||^* \) may called ”the tail quasinorm”.

**Remark 1.2.** As long as

\[
    f^{**}(t) = t^{-1} \sup_{\mu(E) \leq t} \int_E |f(x)| \mu(dx), \tag{1.9}
\]

we can rewrite the expression for \( ||f||_w \) as follows:

\[
    ||f||_w = \sup_{t > 0} \left[ \frac{w(t)}{t} \cdot \sup_{E: \mu(E) \leq t} \int_E |f(x)| \mu(dx) \right]. \tag{1.10}
\]

If the measure \( \mu \) has not atoms, then the expression (1.10) may be rewritten as follows:

\[
    ||f||_w = \sup_{E: \mu(E) \leq \infty} \left[ \frac{w(\mu(E))}{\mu(E)} \cdot \int_E |f(x)| \mu(dx) \right]. \tag{1.11}
\]
It follows from equality (1.10) that $||f||_w$ is natural rearrangement invariant norm and the space $L_w = \{f : ||f||_w < \infty\}$ is complete Banach functional rearrangement invariant space with Fatou property. The proof is similar to one in the case $w(t) = t^{1/p}$, $p \geq 1$; see [8], chapters 1,2; [62], chapter 8.

The norm $||f||_w$ is named Marcinkiewicz’s norm, see [33], chapter 2, section 2.

It is proved in [48] that if $w \in W$, $\gamma(w) < \infty$, then

$$1 \cdot ||f||_w^* \leq ||f||_w \leq \gamma(w) \cdot ||f||_w^*, \quad (1.13)$$

and both the coefficients ”1” and ”$\gamma(w)$” in (1.13) are the best possible.

On the other word, the space $Y = Y_w$, $w \in W$ in this example consists on all the right continuous functions defined on the set $[0, \mu(X)]$ equipped with the norm

$$||g||_{Y_w} = \sup_t ||g(t)||_{w(t)}$$

and moreover the exact values of constants $K_1(Y_w)$, $K_2(V_w)$ are correspondingly: $K_1(Y_w) = 1$, $K_2(Y_w) = \gamma(w)$.

We use the symbols $C(X,Y)$, $C(p,q;\psi)$, etc., to denote positive finite constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X,Y)$ and $C_2(X,Y)$.

**Layout of the paper.** The paper is organized as follows. In the next section we study the estimations of Hardy operators in weighted Lebesgue-Riesz spaces. Third section is devoted to the multidimensional generalization of Hardy operators estimations.

The fourth section contains the main result of offered article: sufficient conditions for normability of generalized Lorentz spaces. In the fifth section we investigate the boundedness of Hardy operator in the so-called anisotropic Grand Lebesgue spaces.

In addition, the last section contains a few review about properties of offered here spaces, in particular, calculation its fundamental function.

\[\square\]

### 2 Auxiliary facts: estimations of Hardy operators.

Let again $(V,|| \cdot ||_V)$ be the Banach functional space defined on the set $R^+ = (0, \infty)$. Recall that the classical Hardy’s operator $H = H[f] = H[f](t)$ (on the other term, Cesaro average) is defined as follows:

$$H[f](t) = \frac{1}{t} \int_0^t f(s)ds. \quad (2.1)$$
It will be presumed that the Hardy’s operator is defined on the space $V$ and is bounded therein:

$$\|H\|_{V \to V} \overset{\text{def}}{=} \sup_{f \in V, \|f\|_V = 1} \|H[f]\|_V < \infty. \quad (2.2)$$

For instance, if $V = L_p(R^+)$, $1 < p \leq \infty$, then $\|H\|_{L_p(R^+) \to L_p(R^+)} \leq p' := p/(p-1)$, and the last estimation is not improvable [23].

There are many estimations for the norm of Hardy’s operators in different spaces, for instance, in weight Lebesgue spaces

$$L_1, L_\infty;$$

in generalized weight Lorentz spaces [15], [65], [66] etc.; in Grand Lebesgue spaces [22], [49]; in generalized weight Lorentz spaces [15], [65], [66] etc.

Note that in the article of S.Bloom and R.Kerman [12] is considered more general operator of a view, e.g.

$$I_{\alpha,\beta}[f](x) = x^{-\beta} \int_0^x (x - y)^\alpha f(y)dy.$$ 

P.R.Beessack in [5] established the following result. Let $s = s(t) \geq 0, t > 0$ be non-negative decreasing function such that

$$S(x) := \int_0^x s(t)dt > 0, \ x > 0;$$

$f = f(t)$ be any function from the space $L_p(1), \ p \in (1, \infty)$. Then

$$\left[ \left( \frac{1}{S(x)} \int_0^x s(x-t)f(t)dt \right)^p \right]^{1/p} \leq \frac{p^2}{(p-1)} \left[ \int_0^\infty |f(t)|^pdt \right]^{1/p}.$$ 

We refer here also the famous result of J.S. Bradley [13]. The inequality of a view with weights $u^p(x)$, $v^p(x)$

$$\left( \int_0^\infty \left( \int_0^x f(s) \ ds \right)^p u^p(x) \ dx \right)^{1/p} \leq C_p(u, v) \left( \int_0^\infty |f(x)|^p \ v^p(x) \ dx \right)^{1/p} \quad (2.4)$$

is true for arbitrary (measurable) function $f : R^+ \to R$, where constant $C_p(u, v)$ does not depend on $f, \ f \in L_p(v^p)$ iff

$$B_p(u, v) := \sup_{r > 0} \left[ \left( \int_0^r u^p(x)dx \right)^{1/p} \cdot \left( \int_0^r v^{-p'}(x) \ dx \right)^{1/p'} \right] < \infty \quad (2.5)$$

and moreover

$$B_p(u, v) \leq C_p(u, v) \leq p^{1/p} (p')^{1/p'} B_p(u, v).$$

Note that $1 \leq p^{1/p} (p')^{1/p'} \leq 2$, therefore $B_p(u, v) \leq C_p(u, v) \leq 2B_p(u, v)$.

For the different powers, i.e. for the inequality of a view
\[
\left( \int_0^\infty \left| \int_0^x f(s) \, ds \right|^q u^q(x) \, dx \right)^{1/q} \leq C_{p,q}(u,v) \left( \int_0^\infty |f(x)|^p v^p(x) \, dx \right)^{1/p} \]  

(2.6)

J.S. Bradley proved in [13] that if \( 1 < p \leq q < \infty \) and

\[
B_{p \leq q}(u,v) \overset{\text{def}}{=} \sup_{r > 0} \left[ \left( \int_r^\infty u^q(x) \, dx \right)^{1/q} \cdot \left( \int_0^r v^{-p'}(s) \, ds \right)^{1/p'} \right] < \infty, \quad (2.6a)
\]

then \( C_{p,q}(u,v) := C_{p \leq q}(u,v) < \infty \); moreover

\[
B_{p \leq q}(u,v) \leq C_{p \leq q}(u,v) \leq p^{1/q} \left( p' \right)^{1/p'} B_{p \leq q}(u,v).
\]

The case \( 1 \leq q < p < \infty \) in the inequality (2.6) was investigated by V.G.Maz’ja in [29], chapter 11. Indeed, the assertion (2.6) holds true under condition \( 1 \leq q < p < \infty \) for arbitrary admissible function \( f : (0, \infty) \to R \) iff

\[
B_{p > q}(u,v) := \left\{ \int_0^\infty \left[ \left( \int_0^x |v(y)|^{-p'} \, dy \right)^{q-1} \right]^{p/(p-q)} \frac{dx}{|v(x)|^{p'}} \right\}^{(p-q)/(pq)} < \infty
\]

herewith

\[
\left[ \frac{p-q}{p-1} \right]^{(q-1)/q} q^{1/q} B_{p > q}(u,v) \leq C_{p > q}(u,v) \leq \left[ \frac{p}{p-1} \right]^{(q-1)/q} q^{1/q} B_{p > q}(u,v). \quad (2.7a)
\]

We refer also the following important for us result belonging to V.D.Stepanov [63] relative the non-increasing non-negative function \( f \). Consider the inequality of a view

\[
||H[f]||_{L_q(w)} \leq C(w,v) \cdot ||f||_{L_p(v)}, \quad 0 < C(w,v) < \infty,
\]

and define \( p' = p/(p-1) \), \( p > 1 \); \( V(x) = \int_0^x v(s)ds \), \( 1/r = 1/q - 1/p \),

\[
A_0 = \sup_{t>0} \left[ \left( \int_0^t w(x) \, dx \right)^{1/q} \cdot \left( \int_0^t v(x) \, dx \right)^{-1/p} \right],
\]

\[
A_1 = \sup_{t>0} \left[ \left( \int_t^\infty x^{-q} w(x) \, dx \right)^{1/q} \cdot \left( \int_0^t x^{p'} V^{-p'}(x) v(x) \, dx \right)^{1/p'} \right],
\]

\[
B_0 = \left\{ \int_0^\infty \left[ \left( \int_0^t w(x)dx \right)^{1/p} \cdot \left( \int_0^t v(x)dx \right)^{-1/p} \right]^r w(t) dt \right\}^{1/r}, \quad (2.8)
\]

\[
B_1 = \left\{ \int_0^\infty \left[ \left( \int_0^\infty x^{-q} w(x)dx \right)^{1/q} \cdot \left( \int_0^t x^{p'} V^{-p'}(x) v(x)dx \right)^{1/q} \right]^r t^{p'} V^{-p'}(t) v(t) \, dt \right\}^{1/p}. \quad (2.9)
\]

If \( 1 < p \leq q < \infty \), then
\( \alpha_1(p,q)(A_0 + A_1) \leq C(w,v) \leq \alpha_2(p,q)(A_0 + A_1), \) 0 < \( \alpha_1(p,q) \leq \alpha_2(p,q) < \infty \); \hspace{1cm} (2.10)

if 1 < q < p < \infty, then

\( \beta_1(p,q)(B_0 + B_1) \leq C(w,v) \leq \beta_2(p,q)(B_0 + B_1), \) 0 < \( \beta_1(p,q) \leq \beta_2(p,q) < \infty \). \hspace{1cm} (2.11)

The "equal" case when \( p = q \) and \( u = v \) was considered, e.g. in [1], [9], [41]. Namely, the inequality

\[ \|H[f]\|_{L_p(w)} \leq C_p(w) \cdot \|f\|_{L_p(w)} \] \hspace{1cm} (2.12)

holds true iff

\[ \exists D = D(p,w) \in (0, \infty), \int_0^\infty s^{-p}w(s)ds \leq D(p,w) t^{-p} \int_0^t w(s)ds. \] \hspace{1cm} (2.13)

See also [1], [9], [41] where are obtained alike result without constants computation.

In the theses of L.Arendarenko [2] and O.Popova [52] there is a comprehensive review about this problem and are offered some new results.

**Example 2.1.**

Let us consider an inequality of a view:

\[ \|x^\alpha H[f]\|_q \leq K_{\alpha,\beta}(p) \|x^\beta f\|_p, \alpha, \beta = \text{const}, \] \hspace{1cm} (2.14)

or equally

\[ \|H_{\alpha,\beta}[g]\|_q \leq K_{\alpha,\beta}(p) \|g\|_p, \alpha, \beta = \text{const}, \] \hspace{1cm} (2.14a)

where

\[ H_{\alpha,\beta}[g](x) \overset{\text{def}}{=} x^\alpha H[x^{-\beta} g](x). \]

In the capacity of the value \( K_{\alpha,\beta}(p) \) we understood its minimal (and implied to be finite) value:

\[ K_{\alpha,\beta}(p) = \sup_{0 < \|x^\beta f\|_p < \infty} \frac{\|x^\alpha H[f]\|_q}{\|x^\beta f\|_p} = \sup_{0 < \|g\|_p < \infty} \frac{\|H_{\alpha,\beta}[g]\|_q}{\|g\|_p}. \] \hspace{1cm} (2.15)

**A.** We investigate first of all the case \( \alpha \geq \beta \) or equally \( q \geq p \) (case "Bradley").

Denote for simplicity

\[ p_0 = 1/(1 - \beta), \ p_+ = 1/(\alpha - \beta), \ q_0 = 1/(1 - \alpha), \ q_+ = +\infty, \ \delta = \alpha - \beta. \]

It is clear that 0 < \( \delta \leq 1 \) and \( p_0 < p \leq p_+ \Leftrightarrow q_0 < q \leq \infty. \)

It follows from Bradley’s inequality then (2.14) holds iff

\[ 0 \leq \alpha, \beta < 1, \ \delta = \alpha - \beta = \frac{1}{p} - \frac{1}{q}, \] \hspace{1cm} (2.16a)
\[ p > \frac{1}{1-\beta} = p_0, \quad q > \frac{1}{1-\alpha} = q_0. \quad (2.16b) \]

Since \( q < \infty \), we conclude in the considered case \( p \leq 1/(\alpha - \beta) = p_+ \). So,

\[ p_0 < p \leq p_+, \quad q_0 < q \leq q_. \]

**Remark 2.1.** Note that the values \( \alpha, \beta, p_0, p_+, q_0, q_+ \) presumed to be constants, but the values \( p, q \) are variable.

We deduce from the Bradley’s inequality:

\[ K_{\alpha,\beta}(p) \leq C(\alpha, \beta) \left[ \frac{p}{p-p_0} \right]^{1-\alpha+\beta}, \quad p_0 < p \leq p_. \quad (2.16c) \]

As a particular case: as \( \alpha = \beta \)

\[ K_{\alpha,\alpha}(p) \leq \frac{C_2(\alpha) p}{p-p_0}, \quad p_0 < p \leq p_. \quad (2.16d) \]

We will prove further the inverse inequality. Thus:

\[ K_{\alpha,\alpha}(p) \asymp \frac{p}{p-p_0}, \quad p_0 < p \leq p_. \]

Note that in the case \( \alpha = \beta = 0 \) we obtain the weak version of the classical Hardy’s inequality with coefficient

\[ K_{0,0}(p) \asymp \frac{p}{p-1}, \quad p \in (1, \infty], \]

which is exact up to multiplicative constant.

**B. Case "Maz'ja".** Let now \( \alpha < \beta \) or equally \( q < p \).

Note that in the considered here restriction and when \( u(x) := u_\alpha(x) = x^\alpha, \ v(x) := v_\beta(x) = x^\beta \)

\[ B_{p>q}(u_\alpha(\cdot), v_\beta(\cdot)) = +\infty. \]

Therefore, the inequality (2.6) with \( u(x) = u_\alpha(x) = x^\alpha, \ v(x) = v_\beta(x) = x^\beta \) or equally \( C_{q>p}(u_\alpha(\cdot), v_\beta(\cdot)) < \infty \) may be true iff \( \alpha > \beta \) or equally \( q > p \).

Outcome:

**Proposition 2.1.** The constant \( K_{\alpha,\beta}(p) = K_{\alpha,\beta}(p,q) \) from inequality (2.15) is finite iff

\[ 0 \leq \alpha, \beta < 1, \quad \alpha > \beta, \quad (2.17a) \]

\[ \alpha - \beta = \frac{1}{p} - \frac{1}{q}, \quad (2.17b) \]

\[ p_0 = \frac{1}{1-\beta} < p \leq \frac{1}{\alpha - \beta} = p_+, \quad q_0 = \frac{1}{1-\alpha} < q \leq \infty = q_. \quad (2.17c) \]
More general weight inequality with exact value of the constant belongs to G.Hardy ([8], p. 124-125):

\[
\left\{ \int_0^\infty \left( t^\nu H[f](t) \right)^q \frac{dt}{t} \right\}^{1/q} \leq \frac{1}{1 - \nu} \left\{ \int_0^\infty \left( t^\nu f(t) \right)^q dt/t \right\}^{1/q}, \quad \nu = \text{const} < 1.
\]

The accuracy calculation used Bradley’s estimation tell us that \( K_{\alpha,\beta}(p) \geq K^0_{\alpha,\beta}(p) \), where

\[
K^0_{\alpha,\beta}(p) = (1 - \beta)^\delta - 1 \cdot (p - 1)^{1-1/p} \cdot \delta^{1/p-\delta} \cdot \frac{[p_+ - p]^{1/p-\delta}}{[p - p_0]^{1-\delta}} =
\]

\[
(1 - \beta)^{\alpha - 1} \cdot (p - 1)^{1-1/p} \cdot [p - 1/(1 - \beta)]^{\alpha - 1} \cdot [1 + p(\beta - \alpha)]^{\alpha - 1/p},
\]

and

\[
K_{\alpha,\beta}(p) \leq p^{1/q} \left( p' \right)^{1/p'} \cdot K^0_{\alpha,\beta}(p), \quad q = q(p) = p/(1 - p\delta).
\]

As long as

\[
p^{1/q} \left( p' \right)^{1/p'} = \left( p' \right)^{1/p'} \cdot p^{1/p + \beta - \alpha} \leq \left( p' \right)^{1/p'} \cdot p^{1/p} \cdot p^{\beta - \alpha} \leq 2,
\]

we conclude

\[
K_{\alpha,\beta}(p) \leq K^+_{\alpha,\beta}(p) \overset{\text{def}}{=} 2 \cdot K^0_{\alpha,\beta}(p).
\]

**Our hypothesis:** under our restrictions (2.17a), (2.17b), (2.17c)

\[
K_{\alpha,\beta}(p) = C(\alpha, \beta) \cdot K^0_{\alpha,\beta}(p), \quad 1 \leq C(\alpha, \beta) \leq 2.
\]

See also [42], pp. 211-221.

**Example 2.2.**

More generally, consider the inequality of a view

\[
||x^\alpha L(x) H[f]||_q \leq K_{L,M;\alpha,\beta}(p) \cdot ||x^\beta M(x) f||_p, \quad \alpha, \beta = \text{const}, \quad (2.18)
\]

where \( L(x), M(x) \) are slowly varying simultaneously as \( x \to 0^+ \) and as \( x \to \infty \) continuous in the semi-axis \((0, \infty)\) positive functions. As ordinary, in the capacity of the value \( K_{L,M;\alpha,\beta}(p) \) we understood its minimal value, presumed to be finite.

We conclude using at the same method as in last example that the estimate (2.18) holds true iff \((\alpha, \beta, p, q)\) satisfy the conditions (2.17a), (2.17b), (2.17c) and

\[
0 < \inf_{x > 0} \left[ \frac{L(x)}{M(x)} \right] \leq \sup_{x > 0} \left[ \frac{L(x)}{M(x)} \right] < \infty. \quad (2.18a)
\]

Moreover, under these conditions

\[
K_{L,M;\alpha,\beta}(p) \overset{\text{def}}{=} C(L, M; \alpha, \beta) \cdot \left\{ \frac{p}{[p - 1/(1 - \beta)]} \right\}^{1-\alpha+\beta}. \quad (2.18b)
\]
We use the multidimensional generalization of the so-called dilation, or scaling method, see [64], [62], chapter 3. Indeed, let us introduce the following dilation operator $T_\lambda[f](x) := f(\lambda x)$, $\lambda \in (0, \infty)$. Suppose the inequality (2.18) is true for arbitrary function $f$ from the Schwartz space $S(0, \infty)$, and substitute in (2.18) the function $T_\lambda[f] \in S(0, \infty)$ instead $f$, $f \neq 0$:

$$||x^\alpha L(x) H[T_\lambda[f]]||_q \leq K_{L,M,\alpha,\beta}(p) ||x^\beta M(x) T_\lambda[f]||_p.$$  \hfill (2.18c)

We deduce consequently:

$$||x^\beta M(x) T_\lambda[f]||_p = \int_0^\infty x^{\beta p} M^p(x)|f(\lambda x)|^p dx =$$

$$\lambda^{-1-\beta p} \int_0^\infty y^{\beta p} M^p(y/\lambda)|f(y)|^p dy \asymp \lambda^{-1-\beta p} M^p(1/\lambda) \int_0^\infty y^{\beta p} |f(y)|^p dy;$$

$$||x^\beta M(x) T_\lambda[f]||_p \asymp \lambda^{-1/p-\beta} M(1/\lambda)|| x^\beta f ||_p;$$

$$H[T_\lambda[f]] = T_\lambda[H[f]];$$

$$||x^\alpha L(x) H[T_\lambda[f]]||_q \asymp \lambda^{-1/q-\alpha} L(1/\lambda)|| x^\alpha H[f] ||_q;$$

$$\lambda^{-1/q-\alpha} L(1/\lambda)|| x^\alpha H[f] ||_q \asymp \lambda^{-1/p-\beta} M(1/\lambda)|| x^\beta f ||_p;$$

therefore

$$1/q + \alpha = 1/p + \beta, \quad L(1/\lambda) \asymp M(1/\lambda),$$

which is equivalent to our assertion.

The passing to the limit as $\lambda \to 0+$ or $\lambda \to \infty$ in the considered case is grounded in [35], [45].

The relation (2.18b) follows immediately from Bradley’s estimation by means if properties of slowly and regular varying functions, see [11], chapter 3; [56], chapter 2.

**Example 2.3.**

The lower bound in the inequality (2.16c) may be obtained even without restriction $\beta < \alpha$ by means of consideration of an example

$$f_0(x) = x^{-1} (\log x)^\Delta I_{(1,\infty)}(x).$$  \hfill (2.19)

Here and further $I_A(x) = 1$, $\lambda \in A$, $I_A(x) = 0$, $x \notin A$.

Namely, it is easy to compute that under formulated before conditions and restrictions and as $\Delta = \text{const} >> 1$ there holds

$$\frac{||x^\alpha H[f_0]||_q}{||x^\beta f_0||_p} \asymp \left[ \frac{p}{p - 1/(1 - \beta)} \right]^{1-\alpha+\beta}, \quad p > 1/(1 - \beta).$$  \hfill (2.19)

In detail,

$$|| x^\beta f_0 ||_p = \int_1^\infty x^{\beta p - p} \log^\Delta x \ dx = \frac{\Gamma(\Delta p + 1)}{|p(1 - \beta) - 1|^{\Delta+1/p}};$$  \hfill (2.20)
\[ x^\alpha H[f_0](x) = x^{\alpha-1} \int_1^x s^{-1} \log s \, ds \, I_{(1,\infty)}(x) = \]
\[ (1 + \Delta)^{-1} I_{(1,\infty)}(x) \, x^{\alpha-1} \log^{\Delta+1} x; \tag{2.21} \]
\[ ||x^\alpha H[f_0]||_q = \frac{\Gamma^{1/q}((\Delta + 1)q + 1)}{\Gamma^{1/q}((\Delta + 1)q + 1) - 1} \cdot \left( \int_1^x s^{-1} \log s \, ds \right) I_{(1,\infty)}(1, \infty) \cdot (x); \tag{2.22} \]
\[ \lim_{\Delta \to \infty} \frac{||x^\alpha H[f_0]||_q}{||x^\beta f_0||_p} = \left[ \frac{1}{p - 1/(1 - \beta)} \right]^{1-\alpha+\beta} \cdot (1 - \beta)^{\alpha-\beta-1} \cdot \frac{p}{[1 - p(\alpha - \beta)]^{\alpha-\beta-1}}. \tag{2.23} \]

We used the Stirling’s formula for the Gamma function \( \Gamma(z), \ z \to \infty. \)

**Corollary 2.1.**
\[ K_{\alpha,\beta}(p) \geq \left[ \frac{1}{p - 1/(1 - \beta)} \right]^{1-\alpha+\beta} \cdot (1 - \beta)^{\alpha-\beta-1} \cdot \frac{p}{[1 - p(\alpha - \beta)]^{\alpha-\beta-1}}. \tag{2.24} \]

**Corollary 2.2.**
As long as \( K_{\alpha,\beta}(p) \) is less than the right-hand side of inequality (2.24), we conclude that in general case the function \( K_{\alpha,\beta}(p) \) is not exact lower bound in the Bradley bilateral inequality.

\[ \square \]

## 3 Multidimensional case.

We consider further in this section the so-called \( d \)-dimensional Hardy’s operator \( H_d[f] \) defined on the functions defined on the ”octant” \( R^d_+ = (R^1_+)^d \) by a formula
\[ H_d[f](x_1, x_2, \ldots, x_d) = \frac{1}{x_1 x_2, \ldots, x_d} \cdot \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_d} f(y_1, y_2, \ldots, y_d) \, dy_1 dy_2 \cdots dy_d. \tag{3.1} \]

The \( L_p(b) \) estimations for the norm of \( H_d[\cdot] \) see in [54], [55], [57].

Another approach (”spherical definition”) see in [16], [20].

In [20] are described in addition an applications of these estimations into the quantum mechanic.

We intent to investigate in this section the inequality of a view
\[ ||w(x) H_d[f](x)||_q \leq C(w, v) \cdot ||v(x) f(x)||_p, \ 0 < C(w, v) < \infty, \tag{3.2} \]

where the weight function \( w(x) = w_\alpha(x) \) is homogeneous of degree \( \alpha \) continuous on the unit sphere positive function, the weight function \( v(x) = v_\beta(x) \) is homogeneous of degree \( \beta \) continuous on the unit sphere positive function.
Proposition 3.1. Suppose the inequality (2.15) holds true for each non-zero function from the Schwartz class $S(R^d_+): f \in S(R^d_+)$. Then

$$\alpha - \beta = d \left( \frac{1}{p} - \frac{1}{q} \right). \tag{3.3}$$

Proof. We will use again the so-called dilation, or equally scaling method, see [62], chapter 10; [64]. Namely, let us define the family of dilation operators $T_\lambda = T_\lambda[f]$ as follows:

$$T_\lambda[f](x) = f(\lambda x), \lambda \in (0, \infty). \tag{3.4}$$

Evidently, $f_\lambda := T_\lambda[f] \in S(R^d_+)$. We have:

$$|| w_\alpha(x) H_d[f_\lambda](x) ||_q \leq || v_\beta(x) f(\lambda x), ||_p. \tag{3.5}$$

Note that

$$|| v_\beta(x) f(\lambda x) ||_p^p = \int_0^\infty v_\beta^p(x) |f(\lambda x)|^p \, dx = \lambda^{-\beta p - d} \int_0^\infty v_\beta^p(y) |f(y)|^p \, dy = \lambda^{-\beta p - d} ||v_\beta(x) f(x)||_p^p, \tag{3.6}$$

therefore

$$|| v_\beta(x) f(\lambda x) ||_p = \lambda^{-\beta - d/p} ||v_\beta(x) f(x)||_p. \tag{3.7}$$

Further, $H_d[T_\lambda f] = T_\lambda H_d[f]$,

$$||w_\alpha(x) H_d[T_\lambda[f]]||_q^q = ||w_\alpha(x) T_\lambda[H_d[f]]||_q^q = \lambda^{-\alpha q - d} ||w_\alpha(x) H_d[f_\lambda](x)||_q^q,$n

$$||w_\alpha(x) H_d[T_\lambda[f]]||_q = \lambda^{-\alpha - d/q} ||w_\alpha(x) H_d[f](x)||_q. \tag{3.8}$$

We get substituting into inequality (3.4):

$$\lambda^{-\alpha - d/q} ||w_\alpha(x) H_d[f](x)||_q \leq C(w_\alpha, v_\beta) \cdot \lambda^{-\beta - d/p} ||v_\beta(x) f(x)||_p. \tag{3.9}$$

Since the value $\lambda$ is arbitrary positive, we conclude from (3.9)

$$\alpha + d/q = \beta + d/p,$$

which is equivalent to (3.3).

Now we investigate the inequality (2.14) without assumption of homogeneity of a functions $w(x), v(x)$. Namely, we suppose the existence of finite constants $\alpha(0), \alpha(\infty), \beta(0), \beta(\infty)$ for which the following functions

$$v_0(x) := \sup_{\lambda \in (0,1)} \frac{v(\lambda x)}{\lambda^\beta(0)}, v_\infty(x) := \inf_{\lambda \in (1,\infty)} \frac{v(\lambda x)}{\lambda^\beta(\infty)},$$

$$w_0(x) := \inf_{\lambda \in (0,1)} \frac{w(\lambda x)}{\lambda^\alpha(0)}, w_\infty(x) := \sup_{\lambda \in (1,\infty)} \frac{w(\lambda x)}{\lambda^\alpha(\infty)},$$

are non-trivial: non-zero and integrable.
Proposition 3.2. Suppose in addition the inequality (3.2) holds true for each non-zero function from the Schwartz class $S(R^d_+)$: $f \in S(R^d_+)$. Then

$$\alpha(0) - \beta(0) \geq d \left( \frac{1}{p} - \frac{1}{q} \right).$$

$$\alpha(\infty) - \beta(\infty) \leq d \left( \frac{1}{p} - \frac{1}{q} \right).$$

Proof is at the same as in proposition (2.1) and may be omitted.

Obviously, when $\alpha(0) - \beta(0) = \alpha(\infty) - \beta(\infty)$, then

$$\alpha(0) - \beta(0) = \alpha(\infty) - \beta(\infty) = d \left( \frac{1}{p} - \frac{1}{q} \right).$$

We recall here the definition of the so-called anisotropic Lebesgue (Lebesgue-Riesz) spaces, or equally the spaces with mixed norms. More detail information about this spaces see in the books of Besov O.V., Ilin V.P., Nikolskii S.M. [10], chapter 16,17; Leoni G. [27], chapter 16,17; using for us theory of operators interpolation in this spaces see in [10], chapter 17,18.

Let $(X_1, A_1, \mu_1), j = 1, 2, \ldots, d$ be measurable spaces with sigma-finite non-trivial measures $\mu_j$. Let also $p = (p_1, p_2, \ldots, p_d)$ be $d-$dimensional vector such that $1 \leq p_j \leq \infty$.

Recall that the anisotropic Lebesgue space $L_p$ consists on all the total measurable real valued function $f = f(x_1, x_2, \ldots, x_d) = f(\vec{x})$,

$$f : \otimes_{j=1}^d X_j \rightarrow R$$

with finite norm $|f|_{p} \overset{df}{=} \left( \left( \int_{X_{d-1}} \mu_d(dx_d) \left( \int_{X_{d-2}} \mu_{d-1}(dx_{d-1}) \cdots \left( \int_{X_1} |f(\vec{x})|^{p_1} \mu_1(dx_1) \right)^{p_2/p_1} \right)^{p_3/p_2} \cdots \right)^{1/p_d} \right)^{1/p_d}. \ (3.10)$

Note that in general case $|f|_{p_1, p_2} \neq |f|_{p_2, p_1}$, but $|f|_{p, p} = |f|_{p}$.

Observe also that if $f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2)$ (condition of factorization), then $|f|_{p_1, p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2}$, (formula of factorization).

We use here the case $X_j = R_+$, $\mu_j(dx_j) = dx_j$.

We consider in this section the weight multidimensional (vector): $d \geq 2$ generalization of weight Hardy’s $L_p(b_1) \rightarrow L_q(b_2)$ estimations.

In this section $x = \vec{x} \in R^d$ be $d-$dimensional vector, $d = 2, 3, \ldots$ which consists on the $d$ coordinates $x_j, j = 1, 2, \ldots, d$:

$$x = (x_1, x_2, \ldots, x_d),$$

$$\alpha = \vec{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}, \beta = \vec{\beta} = \{\beta_1, \beta_2, \ldots, \beta_d\}.$$

We denote as ordinary

$$x^\alpha = \vec{x}^{\vec{\alpha}} = \prod_{j=1}^d x_j^{\alpha_j}, \quad y^\beta = \vec{y}^{\vec{\beta}} = \prod_{j=1}^d y_j^{\beta_j}.$$
Let $f, f : \mathbb{R}^d \to \mathbb{R}$ be (total) measurable function. Let also
\[ \alpha_j, \beta_j = \text{const} \in [0, 1), \quad \alpha_j > \beta_j, \quad j = 1, 2, \ldots, d; \quad p_j \in ((1/(1 - \beta_j), 1/(\alpha_j - \beta_j)). \quad (3.11) \]

We define the function $q_j = q_j(p_j)$ as follows:
\[ \frac{1}{p_j} - \frac{1}{q_j} = \alpha_j - \beta_j. \quad (3.11a) \]

The equality (3.11a) defines the dependance between $\vec{p}$ and $\vec{q}$; we will denote this functions as follows:
\[ \vec{p} = \vec{p}(\vec{q}), \quad \vec{q} = \vec{q}(\vec{p}). \]

Obviously, two functions $\vec{p} = \vec{p}(\vec{q})$ and $\vec{q} = \vec{q}(\vec{p})$ are reciprocal inverse.

**Theorem 3.2.**

The conditions (2.17a), (2.17b), (2.17c) for the variables $(\alpha_j, \beta_j, p_l, q_j)$ are necessary and sufficient for the existence of non-trivial coefficient $K(d; \vec{\alpha}, \vec{\beta}, \vec{p})$ for the following estimate:
\[ ||\vec{x}^\vec{\alpha} H_d[f](x)||_\vec{q} \leq K(d; \vec{\alpha}, \vec{\beta}; \vec{p}) \cdot ||\vec{x}^\vec{\beta} f(x)||_\vec{p}, \quad 0 < K(d, \vec{\alpha}, \vec{\beta}) < \infty, \quad (3.12) \]
and under this conditions for the minimal value of coefficient $K(d; \vec{\alpha}, \vec{\beta}; \vec{p})$ there hold the following equality:
\[ K(d; \vec{\alpha}, \vec{\beta}; \vec{p}) = \prod_{j=1}^{d} K_{\alpha_j, \beta_j}(p_j), \quad (3.13a) \]
and $K(d; \vec{\alpha}, \vec{\beta}; \vec{p}) = \infty$ in other case.

The proof is at the same as one in [51] for weight Riesz and Fourier integral transform. Namely, let us introduce the following one-dimensional operators
\[ H^{(j)}[f](x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) = \frac{1}{x_j} \int_{0}^{x_j} f(x_1, x_2, \ldots, x_{j-1}, s_j, x_{j+1}, \ldots, x_d) \, ds_j, \quad (3.14) \]
then
\[ H_d = \otimes_{j=1}^{d} H^{(j)}. \quad (3.15) \]
It is sufficient to consider only the two-dimensional case $d = 2$. Denote
\[ z(x_1, x_2) = H_2[f](x_1, x_2) = \frac{1}{x_1} \int_{0}^{x_1} g(s_1; x_2) \, ds_1 = H^{(1)}[g](x_1; x_2), \quad (3.16) \]
where
\[ g(s_1; x_2) = H^{(2)}[f](s_1, x_2) = \frac{1}{x_2} \int_{0}^{x_2} f(s_1, x_2) \, ds_1. \quad (3.17) \]
We have using the one-dimensional estimate (2.16c) for $u(\cdot, \cdot)$:
\[ \| x_1^{\alpha_1} z(\cdot, x_2) \|_{q_1, x_1} \leq K_{\alpha_1, \beta_1}(p_1) \cdot \| x_1^{\beta_1} g(\cdot, x_2) \|_{p_1, x_1}. \]  

(3.18)

Now we apply the triangle inequality for the \( L(q_2) \) norm and the one-dimensional estimate (2.16c):

\[ \| x_1^{\alpha_1} x_2^{\alpha_2} z(\cdot, \cdot) \|_{q_1, q_2} = \| x_1^{\alpha_1} x_2^{\alpha_2} u(\cdot, x_2) \|_{q_1, x_1} \leq \]

\[ K_{\alpha_1, \beta_1}(p_1) \cdot K_{\alpha_2, \beta_2}(p_2) \cdot \| x_1^{\beta_1} x_2^{\beta_2} f(\cdot, x_2) \|_{p_1, x_1} \|_{p_2, x_2} = \]

\[ K_{\alpha_1, \beta_1}(p_1) \cdot K_{\alpha_2, \beta_2}(p_2) \cdot \| x_1^{\beta_1} x_2^{\beta_2} f \|_{p_1, p_2}. \]  

(3.19)

The lower estimate in (3.13) may be obtained after consideration an example \( f_0(\vec{x}) \) of factorized function of a view

\[ f_0(\vec{x}) = \prod_{j=1}^{d} h_j(x_j). \]  

(3.20)

It may be considered analogously a more general case of inequality of a view

\[ \| u(\vec{x}) H_d[f] \| \leq C^{(d)}(u; \vec{p}, \vec{q}) \| v(\vec{x}) f \| \]  

where both the weight \( u(\vec{x}) \) and \( v(\vec{x}) \) are factorable:

\[ u(\vec{x}) = \prod_{j=1}^{d} u_j(x_j), \quad v(\vec{x}) = \prod_{j=1}^{d} v_j(x_j). \]  

(3.22)

Theorem 3.3. Let the weights functions \( u(\vec{x}), v(\vec{x}) \) be factorable in the sense (3.22). The inequality (3.21) holds true with the coefficient

\[ C^{(d)}(u; \vec{p}, \vec{q}) = \prod_{j=1}^{d} C_{p_j, q_j}(u_j, v_j), \]  

(3.23)

where the function \( C_{p,q}(u,v) \) is defined in (2.6).

Example 3.1. Suppose

\[ u_j(x_j) = x_j^{\alpha_j} L_j(x_j), \quad v_j(x_j) = x_j^{\beta_j} M_j(x_j), \quad j = 1, 2, \ldots, d, \]  

(3.24)

where \( L_j(y), M_j(y), y \in (0, \infty) \) are positive continuous slowly varying simultaneously as \( y \to 0^+ \) and as \( y \to \infty \) functions. The inequality (3.21) for these weights is valid if the parameters \( (\alpha_j, \beta_j; p_j, q_j) \) satisfy the conditions (3.11), (3.11a) and moreover

\[ 0 < \min_{j} \inf_{y>0} \left[ \frac{u_j(y)}{v_j(y)} \right] \leq \max_{j} \sup_{y>0} \left[ \frac{u_j(y)}{v_j(y)} \right] < \infty. \]  

(3.25)
4 Main result.

The following assertion is obvious.

**Proposition 4.1.** The space \((Y, \| \cdot \|_Y)\) is the Banach (complete) rearrangement invariant functional space defined on the set \((X = \{x\}, \mathcal{A}, \mu)\).

For instance, the triangle inequality and homogeneity of the norm \(\| \cdot \|_V\) follows immediately from the equality \((1.9)\).

Denote as \(D_+ = D_+(V)\) the set of all positive decreasing (measurable) functions \(f : V \to \mathbb{R}_+\). We define

\[
K(V) = K(V, H) \overset{\text{def}}{=} \sup_{0 \neq g \in D_+(V)} \frac{\|Hg\|_V}{\|g\|_V}.
\]

(4.1)

Obviously, \(K(V, H) \leq \|H\|_{(V \to V)}\).

**Theorem 4.1.**

\[
1 \cdot \|f\|_Y^* \leq \|f\|_Y \leq K(V, H) \cdot \|f\|_Y^*,
\]

where both the constants "1" and "\(K(V, H)\)" are the best possible.

**Proof. Inequalities.** The left-hand side of proposition \((3.2)\) follows immediately from the inequality \(f^*(t) \leq f^{**}(t)\). The main idea for proving of the right-hand side \((3.2)\) is following:

\[
f^{**}(t) = t^{-1} \int_0^t f^*(s) \, ds = H[f^*](t).
\]

(4.3)

Suppose \(K(V, H) < \infty\); other case it is nothing to prove.

Since the function \(t \to f^*(t)\) is positive and decreasing, we can use the definition of the constant \(K(V, H)\):

\[
\|f\|_Y^* = \|f^{**}(\cdot)\|_V \leq K(V, H) \cdot \|f^*(\cdot)\|_V = K(V, H) \cdot \|f\|_Y^*.
\]

(4.4)

**Proof. Exactness.** The exactness of the left constant "1" is true, e.g. for the spaces \(L_p, \ p \in (1, \infty)\), see [48]. But we can assert the exactness for each Banach functional space \((V, \| \cdot \|_V)\), as in the article [50]. Indeed, let us denote

\[
\overline{K}(V) = \inf_{f \neq 0} \left[ \frac{\|f\|_V}{\|(f)^*\|_V} \right].
\]

We introduce also the family of a functions of a view

\[
f^*_h(t) = h_\kappa(t) = 1 - t^\kappa, \ t \in (0, 1);
\]

then

\[
f^{**}_h(t) = 1 - t^\kappa/(\kappa + 1).
\]

Obviously, as \(\kappa \to 0+ \Rightarrow f^{**}_h(t)/f^{**}_h(t) \to 1\) a.e. Therefore,
\[
K(V) \leq \lim_{\kappa \to 0^+} \left[ \frac{|||f^*|||_{Y}}{|||f^*|||_{V}} \right] = \\
\lim_{\kappa \to 0^+} \left[ \frac{||1 - t^\kappa||_{V}}{||1 - t^\kappa/(\kappa + 1)||_{Y}} \right] = 1.
\]

As long as \(K(V) \leq 1\), we conclude \(K(V) = 1\).

It remains to prove the exactness of right constant \(K(V, H)\).

We can suppose without loss of generality the existence of a positive right continuous decreasing function \(g_0 = g_0(t)\) from the space \(V\) for which

\[
||g_0||_{V} = 1, \quad ||H[g_0]||_{V} = K(V, H)||g_0||_{V} = K(V, H).
\]

There exists a function \(f_0 = f_0(x), \ x \in [0, 1]\) such that \(f_0^*(t) = g_0(t)\) and following

\[
||f_0||_{Y} = ||H[g_0]||_{V} = K(V, H) \cdot ||g_0||_{V} = K(V, H) \cdot ||f_0||_{Y}^*.
\]

This completes the proof of theorem 4.1.

Therefore, we can use for constant \(K(V, H)\) estimate the results of second section.

Example 4.1.

Let \(V = L_p(u^p)\), where \(u = u(x)\) is weight function, then \(K(V, H)\) allows the following estimate:

\[
K(V, H) \leq 2C_p(u, u),
\]

where \(C_p(u, u)\) is defined in (2.4)-(2.5); and the estimate (4.7) is weakly exact.

As a consequence from theorem 4.1 in the case \(V = L_p(u^p)\):

\[
|||f|||_{Y}^* \leq |||f|||_{Y} \leq 2C_p(u, u) \cdot |||f|||_{Y}^*,
\]

Subexample 4.2.

Let in addition \(u(x) = v(x) = x^\beta, \ \beta = \text{const} \in [0, 1]\) and \(p > 1/(1 - \beta)\); then

\[
|||f|||_{Y}^* \leq |||f|||_{Y} \leq C(\beta) \left[ \frac{p}{p - 1/(1 - \beta)} \right] \cdot |||f|||_{Y}^*,
\]

\[
\square
\]

5 Generalization on the anisotropic Grand Lebesgue spaces.

1. (Ordinary) Grand Lebesgue spaces.
Recently, see [22], [24], [25], [32], [35], [44], [45] etc. appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}, A \geq 1, A < B \leq \infty$, spaces consisting on all the measurable functions $f : X \to \mathbb{R}$ with finite norms

$$||f||_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \in (A,B)} [|f|_p / \psi(p)].$$

(5.1)

Here $\psi(\cdot)$ is some continuous positive on the open interval $(A, B)$ function such that

$$\inf_{p \in (A,B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (A, B) = \{p : \psi(p) < \infty, \}$$

The set of all $\psi$ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [8], and are used, for example, in the theory of probability [32], [44], [45]; theory of Partial Differential Equations [21], [25]; functional analysis [22], [24], [35], [45]; theory of Fourier series [44], theory of martingales [45], mathematical statistics [58], [59], [60]; theory of approximation [47] etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, \infty)$ and a function $p \to p \cdot \log \psi(p)$ is convex, then the space $G\psi$ coincides with some exponential Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

**Remark 5.1** If we introduce the discontinuous function

$$\psi_r(p) = 1, \quad p = r; \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in \mathbb{R}^1$, then the norm in the space $G(\psi_r)$ coincides with the $L_r$ norm:

$$||f||_{G(\psi_r)} = |f|_r.$$

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces $L_r$.

**Remark 5.2** The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \to \mathbb{R}$ such that $\exists (A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p.$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T, T$ is arbitrary set, be some family $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \xi(t, \cdot) : X \to \mathbb{R}$ such that

$$\exists (A, B) : 1 \leq A < B \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

(5.2)

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p.$$
The function $\psi_F(p)$ may be called as a natural function for the family $F$. This method was used in the probability theory, more exactly, in the theory of random fields, see [44], chapters 3,4.

2. Anisotropic Grand Lebesgue-Riesz spaces.

Let $Q$ be convex (bounded or not) subset of the set $\otimes_{j=1}^d[1, \infty]$. Let $\psi = \psi(\vec{p})$ be continuous in an interior $Q^\circ$ of the set $Q$ strictly positive function such that

$$\inf_{\vec{p} \in Q^\circ} \psi(\vec{p}) > 0; \inf_{\vec{p} \notin Q^\circ} \psi(\vec{p}) = \infty.$$  

We denote the set all of such a functions as $\Psi(Q)$.

The Anisotropic Grand Lebesgue Spaces $AGLS = AGLS(\psi)$ space consists on all the measurable functions

$$f : \otimes_{j=1}^d X_j \to R$$

with finite (mixed) norms

$$||f||_{AG\psi} = \sup_{\vec{p} \in Q^\circ} \left[ \frac{|f|_{\vec{p}}}{\psi(\vec{p})} \right].$$  \hspace{1cm} (5.3)

An application into the theory of multiple Fourier transform of these spaces see in articles [6] and [46], where are considered some problems of boundedness of singular operators in (weight) Grand Lebesgue Spaces and Anisotropic Grand Lebesgue Spaces. We intend to extend some results obtained in [6], [46].

3. Hardy’s operator in the anisotropic Grand Lebesgue spaces.

Let $Q$ be appropriate for considered problem Hardy’s operator estimates domain: convex (bounded or not) subset of the set $\otimes_{j=1}^d[1, \infty]$. Let $\psi = \psi(\vec{p})$ be continuous in an interior $Q^\circ$ of the set $Q$ strictly positive function such that

$$\inf_{\vec{p} \in Q^\circ} \psi(\vec{p}) > 0; \inf_{\vec{p} \notin Q^\circ} \psi(\vec{p}) = \infty.$$  

Let $f(\vec{x}) = f(x)$ be some function such that the product $\vec{x}^{\vec{\beta}} f(\cdot)$ lies in the space $AG\psi$. We denote the function $\vec{q} = q(\vec{p})$ as described before and denote the inverse function by $\vec{p} = p(\vec{q})$.

Define a new function

$$\nu_{R}(\vec{q}) = \psi(p(\vec{q})) \cdot K(d; \vec{\alpha}, \vec{\beta}; \vec{p}(\vec{q})), \hspace{1cm} (5.4)$$

where the parameters $(\vec{\alpha}, \vec{\beta}, \vec{p}, \vec{q})$ satisfy the conditions of theorem 3.2.

Theorem 5.1.

$$||\vec{x}^{\vec{\beta}} H_d[f]||_{AG\nu_{R}} \leq 1 \cdot ||\vec{x}^{\vec{\beta}} f||_{AG\psi},$$  \hspace{1cm} (5.5)

where the constant "1" is the best possible.

Proof. Let $\vec{x}^{\vec{\beta}} f(\cdot) \in AG\psi$; we can suppose without loss of generality $||f||_{AG\psi} = 1$. This imply that
We have denoting \( u = \vec{x}^\vec{\alpha} H_d[f] \):
\[
|| \vec{x}^\vec{\alpha} [f] ||_p \leq \psi(\vec{p}).
\]

As long as the variable \( \vec{p} \) is uniquely defined monotonic function on \( \vec{q} \), the inequality (5.5) is equivalent to the assertion of theorem 5.1.

The exactness of this estimation is proved in one-dimensional \( d = 1 \) in the article [46]; the multidimensional case \( d \geq 2 \) provided analogously.

\[\Box\]

6 Concluding remarks.

A. The complete investigation of some subclasses of considered here spaces: description of conjugate (= associate or dual) spaces, conditions of reflexivity and separability, description of compact subsets, conditions for absolutely continuity norm, density of simple functions, boundedness of integral (regular and singular) operators with some applications see, e.g. in [14], [15], [18], [19], [31], [39], [43], [53], [61] etc.

B. The fundamental functions \( \Phi_{Y^*}(\delta) \), \( \Phi_Y(\delta) \), \( \delta > 0 \) for the spaces \( (Y, ||| \cdot |||_Y) \) and \( (Y^*, ||| \cdot |||_{Y^*}) \) correspondingly may be calculated as follows. Denote by \( \phi_V(t) \) the fundamental function of the space \( (V, || \cdot ||_V) \).

**Proposition 6.1.**
\[
\Phi_{Y^*}(\delta) = \phi_V(\delta), \quad (6.1)
\]
\[
\Phi_Y(\delta) = \phi_V(\delta) + || I_{(\delta, \mu(X))}(t) \cdot (1/t) ||_V. \quad (6.2)
\]

**Proof.** Let \( A \) be measurable subset of \( X \) such that \( \mu(A) = \delta \). If we denote \( g(x) = I_A(x) \), then
\[
g^*(t) = I_{[0, \delta]}(t),
\]
hence
\[
\Phi_{Y^*}(\delta) = ||g^*(\cdot)||_V = || I_{[0, \delta]}(\cdot) ||_V = \phi_V(\delta).
\]
Further,
\[
g^{**}(t) = I_{[0, \delta]}(t) + I_{(\delta, \mu(X))}(t) \cdot (1/t),
\]
therefore
\[
\Phi_Y(\delta) = || I_{[0, \delta]}(\cdot) ||_V + || I_{(\delta, \mu(X))}(\cdot) \cdot (1/t) ||_V = \phi_V(\delta) + || I_{(\delta, \mu(X))}(\cdot) \cdot (1/t) ||_V.
\]
Recall that the fundamental function play a very important role in the investigation of integral operators and in the theory of Fourier series and transform, see [8], chapter 10.

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