Equilibrating dynamics in quenched Bose gases: characterizing multiple time regimes

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We address the physics of equilibration in ultracold atomic gases following a quench of the interaction parameter. Our work is based on a bath model which generates damping of the bosonic excitations. We illustrate this dissipative behavior through the momentum distribution of the excitations, $n_k$, observing that larger $k$ modes have shorter relaxation times $\tau(k)$; they will equilibrate faster, as has been claimed in recent experimental work. We identify three time regimes. At short times $n_k$ exhibits oscillations; these are damped out at intermediate times where the system appears to be in a false or slowly converging equilibrium. Finally, at longer times, full equilibration occurs. This false-equilibrium is, importantly, associated with the $k$ dependence in $\tau(k)$ and has implications for experiment.

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Introduction - Recent interaction quench experiments in cold bosonic gases are providing unique perspectives into the behavior of out-of-equilibrium dynamics of quantum systems [1–5]. These perspectives were hitherto not available in the quantum fluids of condensed matter. The extent to which equilibrium is accessible and the time constants for equilibration are all open questions. Equally of interest is the nature of metastable states, which are so produced. Considerable theoretical attention has gone into this subject, albeit characterizing the post-quench physics entirely in terms of oscillatory behavior [4, 6, 7, 9–11].

How long do these oscillations persist and how does equilibration proceed for different momentum states is a complicated problem that is the focus of the present paper. Here we discuss the different time scales associated with dissipation and equilibration in the context of the evolution of the momentum distribution $n_k$ for a three-dimensional Bose gas. While we use a specific bath model to derive detailed results for $n_k(t)$, our central results can almost be anticipated by making use of empirical observations in previous quench experiments [4, 5]. As emphasized in both experiments, the equilibration dynamics is rather strongly dependent on the momentum of the state under consideration. An unpublished analysis [12] of the experiments in Ref. 4, led to the conclusion that damping at large momentum had to be included. Also notable is the claim that “it is perhaps not unexpected that higher momenta dynamics saturate faster” [5]. This demonstration that large momentum $k$, high energy, states equilibrate more rapidly than those at small $k$ is the aim of this paper. It leads to a multi-step equilibration process, assuming, as is reasonable, that the condensate also evolves in time as the system re-equilibrates.

The important point at issue is that the relaxation times $\tau(k)$ disperse with $k$. At some intermediate time after the quench, there will always be higher energy $k$ states which will be able to follow quasi-adiabatically the (necessarily) slower relaxation of the condensate. But lower energy states, as well as the condensate, will not yet have equilibrated. This suggests, as has been claimed in the literature [3] that, after the initial time period in which $n_k$ oscillates, there will be a two stage equilibration process, associated with the false equilibrium of the high $k$ states, and the ultimate true equilibration of the full system.

The fact that large $k$ is observed [5] to equilibrate first suggests that theoretical calculations of the short distance behavior should not be characterized entirely by an oscillatory time dependence as might be associated with short time evolution. Arriving at an understanding of the short distance behavior should also include dissipation mechanisms. More generally, a description of the post quench behavior entirely in terms of non-dissipative oscillatory contributions (although they may not be that apparent after integration over momentum) is argued here to be inadequate. In this regard, we differ from the literature [4, 9, 10].

In this paper we focus on including this dissipation and will demonstrate that the $k$ dependence claimed in Ref. 5 is consistent with our calculations. We will use a simple bath model [1], but before doing so, we begin at a more heuristic level, using another point of view, that of the dissipative versions of the Gross-Pitaevski equation (DGPE) [14–17]. It should be stressed that the subject matter of this paper does not concern the DGPE as such, except as a back-of-the-envelope method for arriving at the results of the bath model. Nevertheless, this approach will give us the prototypical time-scales, showing the generality of our arguments. In these approaches, the equation of motion of the (mostly condensate) field is given by $i\partial_t \phi(x, t) =$

$$[1 - i\gamma(x)] \left\{ - \frac{\nabla^2}{2m} - \mu + V(x) + g|\phi(x, t)|^2 \right\} \phi(x, t),$$

where $V(x)$ is the trap potential and $g$ the two-body interaction strength. Here $\gamma$ describes the dissipation processes and its specific form depends on the model used to derive the DGPE. Here and in the following, we work in units such that $\hbar = k_B = 1$. 

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One can deduce some simple physical results from this dissipative GP equation. Throughout we ignore trap effects as has been argued to be appropriate for times smaller than the inverse trap frequency \[4\]. The DGPE for a perturbation \(\delta \phi \) from the equilibrium solution \(\phi_0\) is schematically of the form

\[i \partial_t \delta \phi_k = (1 - i \gamma) \left[\left(\epsilon_k + gn_0\right) \delta \phi_k + gn_0 \delta \phi_k^\ast\right], \tag{2}\]

where \(\epsilon_k = \frac{k^2}{2m}\), the condensate \(n_0 = |\phi_0|^2\), and \(\delta \phi_k\) represents an excited state having momentum \(k\). Thus it will qualitatively behave as

\[\delta \phi_k(t) \propto e^{-i \sqrt{E_k^2 - (\gamma gn_0)^2} t - \gamma (\epsilon_k + gn_0) t}, \tag{3}\]

with Bogoliubov energy \(E_k = \sqrt{\epsilon_k (\epsilon_k + 2gn_0)}\). This simple analysis shows that there are two distinct time dependences: an oscillatory contribution which is proportional to the energy \(E_k\) (for sufficiently large momentum) and a damping contribution which scales with the energy \(\omega_k = \epsilon_k + gn_0\), and is multiplied by a dissipative factor \(\gamma\) as well.

To make these heuristic arguments we presume (for the moment) that the condensate \(n_0\) has little or no time dependence. Under this assumption, we may read off from Eq. (3) the relaxation time associated with the damping of oscillations

\[\tau_{\text{interm}}(k) \propto \frac{1}{\gamma \omega_k}.\]

Importantly, from this equation we note that higher energy or larger \(k\) modes will equilibrate faster. By contrast, we associate the short time (i.e. undamped) dynamics with the characteristic time

\[\tau_{\text{short}}(k) \propto \frac{1}{E_k},\]

provided \(k\) is sufficiently large \[18\]. This typical time-scale is associated with the oscillation period of observables as predicted by Bogoliubov theory \[1, 4, 9-11\].

We see that as long as \(t \ll \tau_{\text{interm}}(k)\), we can ignore the damping, and the system behaves as if it were described by Bogoliubov theory and its variants. Here there is always a range in time where the short time undamped dynamics is correct, but this range gets smaller and smaller as \(k\) increases. However, this undamped dynamics is meaningful only if there is a clear separation of scales \(\tau_{\text{interm}}(k) \sim \frac{\gamma}{\omega_k} \ll 1\). This also implies that if one works at fixed time as is done in the experiments (call that time \(t_{\text{exp}}\), the short time dynamics will not be able to describe the physics for momenta such that \(\tau_{\text{interm}}(k) \ll t_{\text{exp}}\). This has implications for extracting the Tan contact parameter \[19\].

Figure 1 shows the typical behavior for the short \(\tau_{\text{short}}(k)\) (dashed line) and intermediate \(\tau_{\text{interm}}(k)\) (solid) relaxation times. Both are peaked at small \(k\), and equilibration is very fast at short distances. The inset is from Ref. 5. This inset sets up the underlying experimental challenge that “the momentum dependence of the time scales remains to be understood”. This will be discussed in more detail below.

In the above heuristic argument we have not considered the possibility that the relaxational dynamics of the condensate may contribute an additional time-scale. Indeed, because of the wide spread, associated with the \(k\)-dependence of the excitation relaxation times, the full equilibration process is more complex.

At a simple level we can characterize the time dependence of the condensate in terms of a single relaxation time \(\gamma_0^{-1}\)

\[n_0(t) = n_{0,f} + h(\gamma_0 t)(n_{0,i} - n_{0,f}), \tag{4}\]

where we introduce a damping function \(h(\gamma_0 t)\) which, for example, can be taken as a simple exponential, \(e^{-\gamma_0 t}\). Here \(n_{0,i}\) (\(n_{0,f}\)) is the initial (final) value of the condensate, associated with a quench.

The details of this phenomenon are in no way essential to the arguments in this paper. We present it here for illustration purposes. Indeed, one could contemplate very non-monotonic functional forms. What is essential here is rather the behavior at the end-stage of condensate evolution, where presumably the condensate approaches equilibrium in a monotonic fashion. This condensate evolution represents another time-scale in the equilibration process

\[\tau_{\text{long}} \propto \gamma_0^{-1}.\]

For sufficiently large momentum, we have \(\tau_{\text{interm}}(k) \ll \tau_{\text{long}}\), implying that the high energy modes will equilibrate faster than the condensate. In the spirit of simplicity, we adopt Eq (4), with an exponential damping function, for definiteness. Here
we presume that after an interaction quench, particularly near unitarity as in Ref. 5, the condensate density \( n_0(t) \) evolves and most probably decreases as the system reaches a new equilibrium state.

We stress the contrast here with fermionic superfluids, where on the basis of Bogoliubov-de Gennes theory, the order parameter dynamics is obtained from the excitation dynamics. Indeed, it has been argued that the same should apply to quenched Bose gases, through use of the number equation \([9, 11]\). In this way the condensate dynamics are derived from, and therefore somewhat secondary to that of the excitations. Our emphasis in this paper is on the inclusion of dissipation which is presumably rather independent of the condensate (and more directly associated with higher energy states not included in Bogoliubov theory). The spirit here is closer to that of conventional bosonic Bogoliubov theory where the condensate has an intrinsic dynamics, distinct from that of the excited states.

**Overview of Bath Approach.** We now characterize the time evolution of the equilibration dynamics concretely through the study of the momentum distribution \( n_k(t) \) using a bath model. We implement quantum dissipation following the work of Caldeira and Leggett \([20]\). In this seminal paper \([20]\), dissipation was induced by coupling the system to a bath composed of an infinite set of harmonic oscillators. A connection between the DGPE approach and that of Ref. 20 was proposed by Stoof \([15]\). If the system is either a free particle or a particle confined in an harmonic oscillator, the Hamiltonian is quadratic and one can solve the equations of motion exactly.

The introduction of the bath, as well as its parameters, has to be seen as mainly phenomenological. Nevertheless, the bath is often viewed as reflecting the incoherent (high energy) modes that are integrated out in other approaches (such as the higher-harmonics modes of the trap in the stochastic GPE approaches). These allow energy to dissipate. The bath can be thought of as incorporating the interactions between the different modes of the full many-body interacting system that would allow equilibration if treated beyond mean-field (Bogoliubov theory).

The coupling between these extra degrees of freedom and the Bogoliubov modes is characterized by the so-called spectral function of the bath \( \Sigma_2(\omega) \). For an Ohmic bath \([3]\), \( \Sigma_2(\omega) \propto \Gamma \omega \), where \( \Gamma \) describes the strength of the coupling that plays a similar role to that of \( \gamma \) in Eq. (1) (for \( \Gamma = 0 \), one recovers Bogoliubov theory and the results of Ref. \([4]\)). In our previous study \([1]\) of the quench dynamics of a two-dimensional Bose gas, we have shown that a value \( \Gamma \approx 0.1 \) was consistent with the experiment of Ref. 4. This is also the value that we will use here in our numerical results to illustrate the equilibration dynamics \([22]\).

The technical details of our calculation, as well as the explicit expression for the momentum distribution are given in the Supplemental Materials. The initial conditions corresponds to that of an ideal Bose gas at zero temperature that is quenched to a finite interaction strength.

**Dynamics of the momentum distribution.** To discuss the behavior of the momentum density after an interaction quench, it is convenient to introduce a characteristic time-scale \( t_0 = (\gamma n_{0,f})^{-1} \) corresponding to the characteristic energy scale of a Bose condensate \( g n_{0,f} \). We first discuss the case with constant condensate density \( n_0(t) = n_{0,i} = n_{0,f} \), which is appropriate for small quenches \([1, 4]\). Fig. 2 shows the momentum density as a function of \( \epsilon_k \) at different times. One observes that at smaller momentum, \( k \)-dependent oscillations appear after sufficient wait-times, but at larger \( k \), there is no perceptible time dependence; the system has equilibrated. (We cannot extract the Tan contact \([19]\) from the large \( k \) tails, since the bath model treats high energy states as a dissipation mechanism.)

The inset shows the time evolution of the momentum distribution for a typical \( \Gamma = 0.1 \) as well as the results obtained from Bogoliubov theory \([4]\). One sees that without dissipation, the momentum distribution, including the Tan contact, has unphysical undamped oscillations, as reported by other groups (who also introduced a time dependent condensate) \([9, 10]\); it was argued that, because these oscillations disappear upon integrating over momenta, they are less problematic; here we maintain that these integrated quantities are not representative of a metastable state.

In order to illustrate the effects of a time-varying condensate, we choose, \( n_{0,i} = 2 n_{0,f} \) to correspond to roughly the depletion which can be extrapolated from Ref. 5. The time-scales of the condensate \( \gamma_0^{-1} \) and of the excitation \( \tau_k = 1/(\epsilon_k + \gamma_0 n_0) \) are chosen such that (for definiteness) \( \Gamma = \gamma_0/g f n_0 = 0.1 \). The left panel of Fig. 3 compares the momentum distributions (solid curves) for two different \( k \). The arrows indicate the characteristic time \( \tau_{\text{interm}}(k) \).

That the solid curves in the left panel of Fig. 3 are still
Figure 3. Left: $n_k(t)$ versus $t/t_0$ for a time dependent condensate (Eq. (4)) with $n_{o_1} = 2n_{o_f}$ for $\epsilon_k = g_fn_{o_f}$ (top solid curve) and $\epsilon_k = g_fn_{o_f}$ (bottom solid curve). The arrows show the typical damping time of the oscillations $\tau_{interm}(k)$ for both energies. Right: Zoom on time-evolution of the $n_k(t)$ for $\epsilon_k = 3g_fn_{o_f}$, showing in more detail the long-time equilibration of $n_k(t)$ with time-dependent condensate. The dotted line is the equilibrium $n^\text{eq}_k[n_{o_f}(t)]$ and the dashed line corresponds to the quasi-adiabatic momentum distribution $n^\text{eq}_k[n_0(t)]$ (see Eqs. (5) and (4)). In both figures $\Gamma = \gamma_0t_0 = 0.1$.

far from their long time asymptotes is illustrated through a blow up of the lower curve in the right panel of Fig. 3. One sees that the momentum density has a very slow, non-oscillatory, dynamics, reflecting the instantaneous value of the condensate $n_0(t)$; that is, $n_k(t) \simeq n^\text{eq}_k[n_0(t)]$, where

$$n^\text{eq}_k[n_0] = \frac{1}{2} \left( \frac{\epsilon_k + g_fn_0}{\sqrt{\epsilon_k(\epsilon_k + 2g_fn_0)}} - 1 \right) \tag{5}$$

is the equilibrium value of the momentum distribution for a condensate density $n_0$ (and interaction strength $g_f$). This is a quasi-adiabatic process in which the large $k$ states are able to follow the condensate in time. Nevertheless until the condensate has reached its final value, the system is not fully equilibrated.

We comment now on the relation to the experimental data from Ref. 5 which was presented in Figure 1. Our results support the observation of these authors that “the higher momentum population saturates earlier”.

Given that we have argued there are multiple time-scales, it is important to infer which of these is represented by their data in the inset. With the caveat that our Bogoliubov-based theory may not be relevant to quenches to unitarity, we can nevertheless infer from their Figure 4, that experimentally, on the time-scales studied, the measured $n_k$ appears to be time dependent at small $k$. This suggests that the relevant measurement times correspond to $\tau_{interm} \ll t \ll \tau_{long}$.

We stress that the $k$ units used in our figures and that of Ref. 5 are of the same order of magnitude when the energy $\epsilon_k$ is normalized using the density as only length scale. To see this note that $\tilde{g}_f n_{o_f} = \epsilon_k = k_h^2/2m$ corresponds to the typical kinetic energy of the condensate after the quench ($k_h$ is the inverse healing length). At large $s$-wave scattering length $a$, the interaction strength $g_f = 4\pi a/m$ is not well defined. As in Ref. 9, one replaces in this case $g_f$ by an effective $T$-matrix $\tilde{g}_f = \frac{4\pi a/m}{1 + \alpha n(t)}$ with $n$ the density and $\alpha$ a numerical constant. At unitarity, one obtains $\tilde{g}_f \propto n^{-1/3}$, and therefore $\tilde{g}_f n \propto n^{2/3}/m$ is of the order of $\epsilon_k = \frac{\mp^{2/3}}{16\pi^2 n^{2/3}}$ (assuming $n_0 \simeq n$). Thus the energy (momentum) range of Fig. 1 and that of the figure of Ref. 5 shown in the inset are of the same order of magnitude.

**Conclusion**—In summary, in this paper, we have addressed the various time scales and the nature of the equilibration process in three dimensional Bose gases. We stress that these calculations, based on a bath approach, lead to characteristic time-scales which are quite general. They can be extracted for instance from the dissipative Gross-Pitaevskii scheme at a heuristic level.

Also intuitive should be our major conclusion which follows from the experimental observations [4, 5] that large $k$ excited states equilibrate most rapidly. This leads to an interesting phenomenon in which the system may appear to be equilibrated (at large $k$), even though it is not. This quasi-equilibrated phase corresponds to a situation in which the high energy (large momentum) excitations, which are damped out more rapidly than the condensate, are able to adiabatically follow the time evolution of the condensate. It is only when the condensate has reached its final time-independent state, at $t \gg \gamma_0^{-1}$, that full equilibration is reached.

Recent experiments have provided the first glimpse of unitary Bose gases [5] formed through a quench. Figure 4 from their paper suggests that the small momentum states may not have equilibrated. While the present approach is restricted to a Bogoliubov-based scheme, our work suggests that the time scales of this important experiment may correspond to $\tau_{interm}(k)$.

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SUPPLEMENTAL MATERIALS

We give here the technical details of our bath approach to the dynamics of a quench Bose gas that was developed in Ref. 1. Following Ref. 1, we compute the momentum distribution $n_k(t)$. Based on Bogoliubov theory, we consider the full Hamiltonian in the absence of a trap to be given by $\hat{H}(g; n_0) = \hat{H}_{\text{Bog}}(g; n_0) + \hat{H}_{\text{bath}} + \hat{H}_{\text{coup}}$ where

$$\hat{H}_{\text{Bog}}(g; n_0) = \sum_k \left[ \hat{\psi}_k^+ (\epsilon_k + g n_0) \hat{\psi}_k + \frac{g n_0}{2} \hat{\psi}_k \hat{\psi}_{-k} \right] + \frac{g n_0}{2} \hat{\psi}_k \hat{\psi}_{-k}$$

$$\hat{H}_{\text{bath}} = \sum_{i,k} \left[ \omega_{i,k} \hat{\psi}_{i,k}^+ \hat{\psi}_{i,k} + \nu_{i,k} \hat{\psi}_{i,k}^+ \hat{\psi}_{i,k} \right]$$

$$\hat{H}_{\text{coup}} = \sum_{i,k} \left[ \eta_{i,k} \hat{\psi}_{i,k}^+ \hat{\psi}_{i,k} + \zeta_{i,k} \hat{\psi}_{i,k}^+ \hat{\psi}_{i,k} + h.c. \right]$$

where $\hat{\psi}_k(t)$ annihilates (creates) an atom with momentum $k \neq 0$. Here $n_0$ is the condensate density and $g$ is the interaction strength. The bath is characterized by two kinds of bosonic modes, $\hat{\psi}_{i,k}(t)$ and $\hat{\psi}_{i,k}^+(t)$, which allow for a well-behaved spectral function [1, 2].

The dynamics of the system after an interaction quench from $g_i$ to $g_f$ is described by $i \partial_t \hat{\psi}_k(t) = \left[ \hat{\psi}_k(t), \hat{H}(g_f, n_0(t)) \right]$, etc., where we have allowed a time dependent condensate. One can solve the equation for the bath operators which in turn gives

$$i \partial_t \hat{\psi}_k(t) = \omega_k(t) \hat{\psi}_k(t) + g_f n_0(t) \hat{\psi}_{-k}(t) + \hat{D}_k(t)$$

$$\hat{D}_k(t) = -i \int_t^0 ds \gamma_k(t-s) \hat{\psi}_k(s)$$

$$i \partial_t \hat{\psi}_{-k}^+(t) = -\omega_k(t) \hat{\psi}_{-k}(t) - g_f n_0(t) \hat{\psi}_k(t) - \hat{D}_{-k}(t)$$

$$\hat{D}_{-k}(t) = -i \int_t^0 ds \gamma_k(t-s) \hat{\psi}_{-k}(s).$$

Here, $\omega_k(t) = \epsilon_k + g_f n_0(t)$, $\hat{D}_k(t) = \sum_j \eta_{j,k} e^{-i\omega_{j,k} t} \hat{W}_{j,k}(0)$, $\hat{D}_{-k}(t) = \sum_j \zeta_{j,k} e^{i\omega_{j,k} t} \hat{V}_{j,k}(0)$, and $\gamma_k(t) = \int_\omega \Sigma_\omega \hat{\psi}_{-k}(t) - \hat{\psi}_k(t) - \hat{D}_{-k}(t)$

In the following, we will use an Ohmic bath where $
\gamma_k(t) = 2\Gamma_k \omega f(\omega/\Omega)$

Note that in this framework, the bath parameter $\Gamma_k$ is independent of the temperature of the bath (it is only related to the microscopic coupling between the bath and the system).

Note that $\dot{D}_k(t)$ plays the role of a random force operator and $\gamma_k(t)$ reflects the damping. The relaxation to equilibrium will be insured by the satisfaction of the fluctuation-dissipation relation

$$\left[ \hat{D}_k(t), \hat{D}_{-k}^+(s) \right] = \gamma_k(t-s).$$

The equations of motion (6) can be formally solved by introducing a matrix Green’s function

$$M_k(t,s) = \left( \begin{array}{cc} M_{1,k}(t,s) & M_{2,k}(t,s) \\ M_{3,k}(t,s) & M_{4,k}(t,s) \end{array} \right)$$

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where

\[ i\hbar \dot{M}_k(t, s) = \begin{pmatrix} \omega_k(t) - i\gamma_k & g f n_0(t) \\ -g f n_0(t) & \omega_k(t) - i\gamma_k \end{pmatrix} M_k(t, s), \tag{9} \]

and \( \gamma_k * f(t, s) = \int_0^t du \gamma_k(t-u) f(u, s) \) for any function \( f(t, s) \). The initial condition is given by \( M_k(t, t) = -i\mathbb{T} \). One readily shows that \( M^*_{s,k}(t, s) = M_{1,k}(t, s) M^*_{s,k}(t, s) = M_{2,k}(t, s) \). The formal solution of Eq. (6)

can be written as

\[
\begin{pmatrix} \hat{\psi}_k(t) \\ \hat{\psi}^{\dagger}_{-k}(t) \end{pmatrix} = M_k(t, 0) \begin{pmatrix} i\hat{\psi}_{k,0} \\ i\hat{\psi}^{\dagger}_{-k,0} \end{pmatrix} + \int_0^t ds M_k(t, s) \begin{pmatrix} \hat{D}_k(s) \\ -\hat{D}^{\dagger}_{-k}(s) \end{pmatrix}. \tag{10} \]

Eq. (10) is a generalization of the time-dependent Bogoliubov-de Gennes equation, that includes both dissipation and equilibration. For a time-dependent condensate, \( M_k(t, s) \) depends on two times separately, whereas it is a function of \( t-s \) if the condensate is constant, as was studied in Ref. 1. Here one has to solve the time

evolution matrix numerically when the time dependence of the condensate is specified (see main text).

To compute an observable such as the momentum distribution \( n_k = \langle \hat{\psi}^{\dagger}_{k}(t)\hat{\psi}_k(t) \rangle \), one has to specify the initial state of the system, through the initial correlation functions \( \langle i\psi_{\pm,k,0}^{\dagger}(t)\psi_{\pm,k,0}^{\dagger}(0) \rangle \), \( \langle \psi_{\pm,k,0}^{\dagger}(t)\hat{W}_{\pm,k}(0) \rangle \), etc. In order to simplify both the discussion and the numerical calculations, we will assume that at \( t = 0 \), the system is an ideal Bose gas \( \langle n_0(t = 0^-) = n_{0,i} \rangle \) and \( g_i = 0 \) that does not interact with the bath, leading to the simplification that all cross-correlation functions such as \( \langle \hat{\psi}_{k,0}\hat{W}_{\pm,k}(0) \rangle \) vanish.

We will furthermore assume that \( \Gamma_k = \Gamma \) is momentum independent. For simplicity, the bath is assumed to be at zero temperature. Then, the momentum distribution is given by

\[
n_k(t) = -M_{k,3}(t, 0)M_{k,2}(t, 0) - \int_0^t ds \int_0^t du \mathcal{D}_k(s-u) \left[ M_{k,1}(t, u)M_{k,4}(t, s) + M_{k,2}(t, u)M_{k,3}(t, s) \right], \tag{11} \]

where \( \mathcal{D}_k(s-u) = \langle \hat{D}_k^{\dagger}(s)\hat{D}_k(u) \rangle = \langle \hat{D}_k(s)\hat{D}_k^{\dagger}(u) \rangle \) is given by \( \mathcal{D}_k(s-u) = \int_{\omega<0} \Sigma_2(k, \omega) e^{i\omega(s-u)} \). In the limit of vanishing system-bath coupling, we obtain the standard Bogoliubov results [4].

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