Stochastic modeling of nonlinear random vibrations using heavy-tailed mixture distribution

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Abstract

In this study, we propose a simple probability density function (PDF) model of Gaussian–Laplacian mixture type, which is capable of parameterizing a heavy-tailed data easily. We construct our model of a convex combination of Gaussian and Laplacian PDFs and derive a minimal parameterization of it. Next, we conduct least-squares fitting of our model to a heavy-tailed data generated by a random Duffing oscillator and obtain over 94% of residual sum of squares (RSS) fitness. The resulting model is applied to predicting transient moment responses and yields over 97% of RSS fitness to Monte–Carlo simulation results of the original system.

1 Introduction

Random vibrations in the real world sometimes exhibit heavy-tailed fluctuations whose probability density functions (PDFs) have heavier tails than the Gaussian PDF. Huge amount of physical examples can be found in the fields of earth dynamics [1], economic dynamics [2], human dynamics [3–5], and so on.

From an engineering point of view, how to design and control such heavy-tailed fluctuations will become important issues. However, calculating heavy-tailed PDFs is not necessarily easy task. One reason is that in general, their explicit mathematical expressions are hardly obtainable as well as their variances can be mathematically infinite [6]. Of course, numerical methods [7] are effective for such problems. However, if closed expressions of heavy-tailed PDFs are available even approximately, further engineering applications will follow.

In this study, we propose a simple PDF model of Gaussian–Laplacian mixture (GLM) type, which is capable of parameterizing a heavy-tailed data easily. The resulting PDF model is applied to prediction of transient moment responses of the original system.

The rest of the paper is organized as follows. In Sec. 2, we describe an example of heavy-tailed PDF data observed in a random Duffing oscillator. In Sec. 3, we propose the GLM model of the PDF data. In Sec. 4, we fit our GLM model to the PDF data. In Sec. 5, we apply the fitted GLM model to improving prediction of transient moment responses of the original system. In Sec. 6, we present our conclusions.

2 Heavy-tailed nonlinear vibration

To provide a simple example, we consider a random Duffing oscillator in the following form:

\[ \ddot{x} + c \dot{x} + x + b x^3 = s_a w_1(t) + s_m w_2(t) x, \]

where \( \dot{x} := dx/dt \), \( w_1(t) \) and \( w_2(t) \) are independent Gaussian white noises with zero mean and unit variance, and \( s_a \) and \( s_m \) are additive and multiplicative intensities, respectively. By using a state vector \( \mathbf{x} = (x_1, x_2)^T := (x, \dot{x})^T \) (T denotes transpose), (1) can be rewritten in a first-order form:

\[ \dot{x} = f(x) + G(x) w(t), \quad f(x) = \begin{bmatrix} x_2 \\ -x_1 - b x_1^3 - c x_2 \end{bmatrix}, \]

\[ G(x) = \begin{bmatrix} 0 & 0 \\ s_a & s_m x_1 \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}. \]

In Fig. 1, the small circles indicate a PDF data generated by (2) for \( b = 1 \), \( c = 0.7 \), \( s_a = 0.01 \), and \( s_m = 0.99 \).
their transient responses decayed. The sample paths togram of 10 data was obtained by normalizing two-dimensional his-

x (i.e., the multiplicative term is dominant), plotted as the marginal PDFs with respect to $x_1$ and $x_2$. The PDF data was obtained by normalizing two-dimensional histogram of 10$^3$ sample paths of (2) at $t = 50$ s, after their transient responses decayed. The sample paths are generated along separate sample paths of $w_1(t)$ and $w_2(t)$. The histogram was built on $51 \times 51$ uniform grid points inscribed in rectangle domain $(x_1, x_2) \in [-0.5, 0.5] \times [-0.5, 0.5]$.

Furthermore, the broken and dashed lines in Fig. 1 indicate the corresponding one-dimensional Gaussian and Laplacian PDFs whose mean and variance values are set to those obtained from the PDF data.

It is clear in Fig. 1 that the PDF data is not fitted well by both PDFs and exhibits an intermediate feature between them. In particular, the PDF data yields significantly lower peaks near the origin than the Laplacian PDFs and higher peaks than the Gaussian PDFs. These are caused by coexistence of additive and multiplicative terms ($s_a, s_m \neq 0$) and such distributions are easily found in real-world random behaviors [3, 5].

3 GLM representation of heavy-tailed density functions

As shown in Fig. 1, the vibration system (2) can produce an intermediate distribution between Gaussian and Laplacian. To describe this, we propose a GLM representation of PDFs.

3.1 General formulation

We propose a GLM model of PDFs in the following form:

$$p_{\text{mix}}(x; \mu, \Sigma_N, \Sigma_L, \kappa) := \kappa N(x; \mu, \Sigma_N) + (1 - \kappa) L(x; \mu, \Sigma_L),$$

$$0 \leq \kappa \leq 1,$$  (3)

where $N$ and $L$ are $n$-dimensional Gaussian [8] and Laplacian [9] PDFs defined below, $\Sigma_N$ and $\Sigma_L$ denote respective covariance matrices, $\mu$ a common mean vector, and $\kappa$ a mixture ratio. The convex combination used in (3) guarantees

$$\int_{-\infty}^{\infty} p_{\text{mix}}(x; \mu, \Sigma_N, \Sigma_L, \kappa) dx = 1.$$  (4)

Here, the $n$-dimensional Gaussian distribution [8] is well known to be given by

$$N(x; \mu, \Sigma) := \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}.$$  (5)

On the other hand, one can find different definitions of $n$-dimensional Laplace distributions [9, 10]. In this study, we employ a finite-valued type of them, in the following form [9]:

$$L(x; \mu, \Sigma) := \frac{(n + 1) \Gamma \left(\frac{r}{2}\right)}{2 \pi^{\frac{r}{2}} \Gamma(n)} |\Sigma|^{-\frac{r}{2}} \times \exp \left\{ -\sqrt{(n + 1)(x - \mu)^T \Sigma^{-1} (x - \mu)} \right\}$$  (6)

where $\Gamma$ denotes the gamma function. The Laplace distribution (6) provides a rare example of heavy-tailed distribution that is able to be parameterized by mean $\mu$ and variance $\Sigma$ only.

The characteristic functions of (5) and (6) are given by, respectively,

$$\psi_N(t; \mu, \Sigma) := \exp \left\{ it^T \mu \right\} \exp \left\{ -\frac{1}{2} t^T \Sigma t \right\},$$  (7)

$$\psi_L(t; \mu, \Sigma) := \exp \left\{ it^T \mu \right\} \left(1 + t^T \Sigma t \right)^{-\frac{n+1}{2}},$$  (8)

where $i := \sqrt{-1}$. Respective moment-generating functions are given by

$$M_N(t; \mu, \Sigma) := \psi_N(-it; \mu, \Sigma),$$  (9)

$$M_L(t; \mu, \Sigma) := \psi_L(-it; \mu, \Sigma).$$  (10)

Note that unfortunately, the marginal PDF of multi-
dimensional Laplace PDFs in (6) does not match $L(x; \mu, \Sigma)_{n=1}$. This consistency problem seems still open in the field of heavy-tailed distributions. Therefore, in our numerical examples below, we obtain one-dimensional marginal plots of multi-dimensional PDFs by numerically integrating them.

3.2 Two-dimensional case

Hereafter, $\langle \cdot \cdot \cdot \rangle$ denotes an ensemble average, $\mu_i := \langle x_i \rangle$ a mean, and $\sigma_{ij} := \langle (x_i - \mu_i)(x_j - \mu_j) \rangle$ a variance, for $1 \leq i \leq j \leq 2$.

In two-dimensional case ($n = 2$), formulas (5), (6), (9), and (10) can be written down in the following forms, respectively,

$$N(x_1, x_2; \mu, \Sigma) = \frac{1}{2\pi \sqrt{C}} \exp \left( -\frac{Q}{2C} \right),$$  (11)

$$L(x_1, x_2; \mu, \Sigma) = \frac{3}{2\pi \sqrt{C}} \exp \left( -\sqrt{3Q} \right),$$  (12)

where $C := \sigma_{11} \sigma_{22} - \sigma_{12}^2, Q := \sigma_{22}(x_1 - \mu_1)^2 - 2\sigma_{12}(x_1 - \mu_1)(x_2 - \mu_2) + \sigma_{11}(x_2 - \mu_2)^2$, and

$$M_N(t_1, t_2; \mu, \Sigma) = \exp \{ \mu_1 t_1 + \mu_2 t_2 \} \times \exp \left\{ \frac{\sigma_{11} t_1^2 + 2\sigma_{12} t_1 t_2 + \sigma_{22} t_2^2}{2} \right\},$$  (13)

$$M_L(t_1, t_2; \mu, \Sigma) = \exp \{ \mu_1 t_1 + \mu_2 t_2 \} \times \left(1 - \frac{\sigma_{11} t_1^2 + 2\sigma_{12} t_1 t_2 + \sigma_{22} t_2^2}{3} \right)^{-3/2}.$$  (14)
In this case, GLM model (3) becomes

\[ p_{\text{mix}}(x_1, x_2; \mu, \Sigma_\nu, \Sigma_L, \kappa) := \kappa N(x_1, x_2; \mu, \Sigma_\nu) + (1 - \kappa) L(x_1, x_2; \mu, \Sigma_L), \quad 0 \leq \kappa \leq 1. \quad (15) \]

4 GLM fitting to PDF data

Let \( \hat{p}(x_1, x_2) \) denote the PDF data shown in Fig. 1 whose mean and variance are written as

\[
\mu = \begin{bmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} \\ \bar{\sigma}_{12} & \bar{\sigma}_{22} \end{bmatrix}. \quad (16)
\]

In this section, we consider how to obtain the best-fitting \( p_{\text{mix}}(x_1, x_2; \mu, \Sigma_\nu, \Sigma_L, \kappa) \) to \( \hat{p}(x_1, x_2) \) in terms of least squares.

4.1 Parameterization of \( p_{\text{mix}} \)

We introduce a specification for fitting as follows.

- The fitted \( p_{\text{mix}} \) has the same values of mean and variance as those of \( \hat{p}(x_1, x_2) \).

As for mean values, this specification can easily be fulfilled by substituting \( \mu = \bar{\mu} \) into \( p_{\text{mix}} \) of (15).

On the other hand, as for variances, some other steps are required. Let \( \Sigma_{\text{mix}} \) denote a covariance matrix of \( p_{\text{mix}} \). This is calculated as

\[
\Sigma_{\text{mix}} = \int_{-\infty}^{\infty} xx^T p_{\text{mix}}(x_1, x_2; \mu, \Sigma_\nu, \Sigma_L, \kappa) dx_1 dx_2
\]

\[
= \kappa \int_{-\infty}^{\infty} xx^T N(x_1, x_2; \mu, \Sigma_\nu) dx_1 dx_2
\]

\[
+ (1 - \kappa) \int_{-\infty}^{\infty} xx^T L(x_1, x_2; \mu, \Sigma_L) dx_1 dx_2
\]

\[
= \kappa \Sigma_\nu + (1 - \kappa) \Sigma_L. \quad (17)
\]

Hence we have the specification for variances as

\[
\Sigma = \Sigma_{\text{mix}} = \kappa \Sigma_\nu + (1 - \kappa) \Sigma_L. \quad (18)
\]

Next, we introduce a variance-ratio \( \lambda_{ij} \) and put

\[
(\Sigma_\nu)_{ij} = \lambda_{ij} (\Sigma_L)_{ij}, \quad 1 \leq i \leq j \leq 2, \quad (19)
\]

where \((\cdot)_{ij}\) denotes \((i, j)\)-element of a matrix. As \((\Sigma)_{ij} = \kappa \lambda_{ij} (\Sigma_L)_{ij} + (1 - \kappa) (\Sigma_\nu)_{ij}\), we have

\[
(\Sigma_\nu)_{ij} = \frac{\lambda_{ij} (\Sigma_L)_{ij}}{\kappa \lambda_{ij} + (1 - \kappa)}, \quad (\Sigma_L)_{ij} = \frac{(\Sigma)_{ij}}{\kappa \lambda_{ij} + (1 - \kappa)}. \quad (20)
\]

These obviously satisfy (18).

As the result, we can parameterize our \( p_{\text{mix}} \) by

\[
p_{\text{mix}}(x_1, x_2; \varphi), \quad \varphi := (\lambda_{11}, \lambda_{12}, \lambda_{22}, \kappa), \quad (21)
\]

based on the measured \( \bar{\mu} \) and \( \bar{\Sigma} \) via formula (20). We call \( \varphi \) a GLM parameter.

4.2 Fitting method and condition

Based on the above, we fit \( p_{\text{mix}}(x_1, x_2; \varphi) \) to \( \hat{p}(x_1, x_2) \) by solving the following optimization problem:

\[
q^* = \arg \min_\varphi E(\varphi), \quad (22)
\]

for a residual sum of squares (RSS) of them:

\[
E(\varphi) := \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} \left( p_{\text{mix}}(x_1^i, x_2^j; \varphi) - \hat{p}(x_1^i, x_2^j) \right)^2
\]

\[
\times \left( \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} \hat{p}(x_1^i, x_2^j)^2 \right)^{-1}, \quad (23)
\]

where

\[
x_k^i := x_k^i + (i - 1) \frac{x_k^2 - x_k^1}{N_k - 1}, \quad i = 1, \ldots, N_k, \quad k = 1, 2.
\]

Here, \( N_k \) denotes the number of grids along \( k \)-th direction and \([x_k^1, x_k^2] \times [x_k^2, x_k^1] \) is the domain of \( \hat{p}(x_1, x_2) \).

Since this problem has a strong nonlinearity with respect to \( \varphi \), we solve it by means of particle swarm optimization (PSO) \([11, 12]\).

4.3 Fitting results

Table 1 shows the GLM fitting result for \( q^* \). The resulting \( p_{\text{mix}}(x_1, x_2; q^*) \) is shown in Fig. 2, plotted as marginal PDFs. It is clear that drastically improved fitting is obtained as compared with those in Fig. 1. The resulting \( E(q^*) \approx 0.588 \) indicates that over 94.1% of RSS fitness:

\[
\{1 - E(q^*)\} \times 100\% \quad (25)
\]

was obtained by our GLM approach.

| Table 1: GLM fitting result. |
|--------------------------------|
| (a) Fitted GLM parameter \( q^* \) |
| \( \lambda_{11} \) | 2.22637926e-01 |
| \( \lambda_{12} \) | 2.00000000e+00 |
| \( \lambda_{22} \) | 2.87570863e-01 |
| \( \kappa \) | 5.10748491e-01 |
| \( E(q) \) | 5.88203091e-02 |
| (b) External parameter values from \( \hat{p} \) |
| \( \mu_1 \) | -1.47555100e-05 |
| \( \mu_2 \) | -1.56345300e-06 |
| \( \sigma_{11} \) | 1.69044700e-02 |
| \( \sigma_{12} \) | 2.03478000e-06 |
| \( \sigma_{22} \) | 1.90911200e-02 |

“\( \times 10^y \)” denotes \( \times 10^y \).
Fig. 2: The fitted GLM model $p_{\text{mix}}(x_1, x_2; q^*)$. The RSS fitness marks over 94.1%.

5 Application to moment response prediction

In this final section, we demonstrate an application of the fitted $p_{\text{mix}}(x_1, x_2; q^*)$. Here, we use it to improve stochastic equivalent linearization (SEL) of moment differential equation (MDE).

5.1 Nonlinear MDE and equivalent gains

By the help of Itô calculus [13], the MDE of (2) is derived in the following form:

$$\frac{dm}{dt} = \begin{bmatrix} m_2 \\ -km_1 - cm_4 - b\langle x_1^3 \rangle \\ -km_3 - cm_4 + m_5 + bm_1\langle x_1^3 \rangle - b\langle x_1^4 \rangle \\ -2km_4 - 2cm_5 + 2bm_3\langle x_1^3 \rangle \\ -2b\langle x_2^3 \rangle x_2 + (m_1 + m_4)s_m^2 + s_n^2 \end{bmatrix}$$

(26)

where $m := (\mu_1, \mu_2, \sigma_{11}, \sigma_{12}, \sigma_{22})^T$.

Unfortunately, this MDE cannot be solved for $m$ as it contains nonlinear terms $\langle x_1^3 \rangle$, $\langle x_1^4 \rangle$, and $\langle x_1^3 x_2 \rangle$. Extending $m$ to contain such higher moments does not work because this causes more higher moments. Therefore, there is no choice but to solve MDE (26) approximately. SEL [14–16] provides one of the fundamental bases of such approximations [13, 17–19].

In a standard SEL approach, a Gaussian distribution is assumed to approximate nonlinear terms by

$$\langle h(x_1, x_2) \rangle \approx \varphi[N, h](m)$$

$$:= \int_{-\infty}^{\infty} h(x_1, x_2)N(x_1, x_2; \mu, \Sigma_N) \, dx_1 dx_2$$

(27)

where $h(x_1, x_2)$ is a scalar-valued function of random variables. As $N$ is a function of $m$, the result becomes a scalar-valued function $\varphi[N, h](m)$ of $m$, which is called a SEL gain. This approximation works well only if $x$ is almost Gaussian. One can replace $N$ with $L$ as Laplacian distribution $L$ is also parametrized by $m$. This also results in a scalar function of $m$ and works well only if $x$ is almost Laplacian.

Table 2 lists the resulting SEL gains derived based on Gaussian and Laplacian distributions via the moment-generating functions in (13) and (14), i.e.,

$$\varphi[p, x_1^h x_2^h](m) = \left. \frac{\partial^{k+l} M_p(t_1, t_2)}{\partial t_1^k \partial t_2^l} \right|_{t_1 = 0, t_2 = 0}, \quad p = N, L.$$

(28)

Table 2: SEL gains based on $N$ and $L$.

| $h(x_1, x_2)$ | $\varphi[N, h]$ | $\varphi[L, h]$ |
|----------------|----------------|----------------|
| $x_1^h$        | $3\mu_1 \sigma_{11}$ | $3\mu_2 \sigma_{11}$ |
| $x_2^h$        | $\sigma_{11}(6\mu_1^2 + 3\sigma_{11})$ | $\sigma_{11}(6\mu_2^2 + 5\sigma_{11})$ |
| $x_1^3 x_2$    | $3\mu_1 \sigma_{12} \sigma_{21}$ | $3\mu_2 \sigma_{12} \sigma_{21}$ |
| $x_2^3$        | $+ 3\mu_1^2 \sigma_{12} + 3\sigma_{11} \sigma_{12} + 3\mu_2^2 \sigma_{12} + 5\sigma_{11} \sigma_{12}$ |

5.2 GLM equivalent gains

The SEL gains in Table 2 cannot be expected to provide a good approximation as the PDF data has the distinct distribution from them. Instead, we use the well-fitted $p_{\text{mix}}(x_1, x_2; q^*)$ in Fig. 2 to propose a GLM equivalent gain in the following form:

$$\varphi[p_{\text{mix}}, h](m)$$

$$:= \int_{-\infty}^{\infty} h(x_1, x_2) p_{\text{mix}}(x_1, x_2; q^*) \, dx_1 dx_2$$

$$= \kappa \varphi[N, h](m_N) + (1 - \kappa) \varphi[L, h](m_L).$$

(29)

Here, different moment vectors $m_N$ and $m_L$ are introduced. These are to maintain our distributive formulation by variance-ratio $\lambda_{ij}$ with (20). Specifically, the solution $m = (\mu_1, \mu_2, \sigma_{11}, \sigma_{12}, \sigma_{22})^T$ of (26) are distributed to $m_N = (\mu_i^N, \mu_j^N, \sigma_{11}^N, \sigma_{12}^N, \sigma_{22}^N)^T$ and $m_L = (\mu_i^L, \mu_j^L, \sigma_{11}^L, \sigma_{12}^L, \sigma_{22}^L)^T$ in the following manner:

$$\mu_i^N = \mu_i,$$

$$\sigma_{ij}^N = \frac{\lambda_{ij} \sigma_{ij}}{\kappa \lambda_{ij} + (1 - \kappa)}, \quad \sigma_{ij}^L = \frac{\sigma_{ij}}{\kappa \lambda_{ij} + (1 - \kappa)},$$

$$1 \leq i < j \leq 2.$$  (30)

Therefore, we have the GLM equivalent gains that approximate the nonlinear terms as follows:

$$\langle x_1^3 \rangle \approx \varphi[p_{\text{mix}}, x_1^3](m), \quad \langle x_1^4 \rangle \approx \varphi[p_{\text{mix}}, x_1^4](m),$$

$$\langle x_1^3 x_2 \rangle \approx \varphi[p_{\text{mix}}, x_1^3 x_2](m).$$  (31)
6 Conclusion

In this study, we have proposed Gaussian–Laplacian mixture (GLM) probability density function (PDF) to describe a heavy-tailed PDF data generated by a nonlinear vibration system subjected to both additive and multiplicative random excitations.

First, we proposed a GLM-type PDF model $p_{\text{mix}}$ as a convex combination of the Gaussian and Laplacian PDFs with mixture-ratio $\kappa$. Then, we developed a method to fit the proposed $p_{\text{mix}}$ to the PDF data by parameterizing $p_{\text{mix}}$ with four-dimensional GLM parameter $q$ and by formulating the optimization problem to minimize the residual sum of squares (RSS) between our $p_{\text{mix}}$ and the PDF data. The results showed the following.

- Our proposed $p_{\text{mix}}$ drastically improved RSS fitness to the PDF data as compared with separate Gaussian and Laplacian PDFs.
- The resulting RSS fitness of our $p_{\text{mix}}$ was over 94%.

Next, we applied the resulting $p_{\text{mix}}$ to improve stochastic equivalent linearization (SEL) of moment differential equation (MDE) and obtained the following results.

- Our proposed GLM equivalent gains resulted in the best-fitting moment responses $m(t)$ as compared with those by conventional SEL methods.
- The resulting RSS fitness of moment responses was over 97% for all components of $m(t)$.

Given the above, we believe that our GLM approach provides a simple but effective way to obtain closed representation of heavy-tailed PDF data.

In future, we plan to investigate the applicability of our GLM to real-world heavy-tailed fluctuations.

5.3 Moment response prediction

In Fig. 3, the solid lines indicate the numerical solutions of MDE (26) with the GLM equivalent gains in (31) and the small circles indicate the Monte-Carlo simulation results on the original Duffing oscillator (2). The dotted and broken lines indicate the MDE solutions by conventional Gaussian SEL [15, 20] and Laplacian SEL, respectively. Furthermore, the values shown in each graph indicate RSS fitness of the MDE solutions to the Monte-Carlo simulation results.

The results clearly show that the best-fitting moment responses were obtained by our GLM approach that marks over 97.3% of RSS fitness for all components of $m(t)$. Our advantage was conspicuous especially for variances $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{22}$. The second performance is obtained by Laplacian SEL and the worst by conventional Gaussian SEL.

![Fig. 3: Transient moment responses predicted by GLM equivalent gains.](image)

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