AN EXPLICIT CONSTRUCTION OF RULED SURFACES

ALBERTO ALZATI AND FABIO TONOLI

Abstract. The main goal of this paper is to give a general algorithm to compute, via computer algebra systems, an explicit set of generators of the ideals of the projective embeddings of ruled surfaces, i.e. projectivizations of rank two vector bundles over curves, such that the fibres are embedded as smooth rational curves.

There are two different applications of our algorithm. Firstly, given a very ample linear system on an abstract ruled surface, our algorithm allows to compute the ideal of the embedded surface, all the syzygies, and all the algebraic invariants which are computable from its ideal as, for instance, the $k$-regularity. Secondly, it is possible to prove the existence of new embeddings of ruled surfaces.

The method can be implemented over any computer algebra system able to deal with commutative algebra and Gröbner-basis computations. An implementation of our algorithms for the computer algebra system Macaulay2 (cf. [G-S]) and explicit examples are enclosed.

Introduction and Notation

Let $E$ be a rank 2 vector bundle over a smooth, genus $q$, curve $C$. It is known that any such vector bundle $E$, regarded as a sheaf, is an extension of invertible sheaves. If $E$ is a normalized vector bundle, i.e. $H^0(C, E) \neq 0$ but $H^0(C, E \otimes G) = 0$ for any line bundle $G$ of negative degree, then $E$ fits into a short exact sequence

\begin{equation}
0 \to O_C \to E \to L \to 0,
\end{equation}

and $L = \text{det}(E)$.

We consider the geometrically ruled surface $X := \mathbb{P}(E)$, endowed with the natural projection $p : \mathbb{P}(E) \to C$. In this case $\text{Pic}(X) \cong \mathbb{Z} \oplus p^* \text{Pic}(C)$, where $\mathbb{Z}$ is generated by the tautological divisor of $X$, i.e. a divisor $C_0$, image of a section $\sigma_0 : C \to X$ with minimal self-intersection. According to this notation, every divisor on $X$ is linearly (resp. numerically) equivalent to $aC_0 + p^*B$ where $B$ is a degree $b$ divisor of $C$ (resp. $aC_0 + bf$, where $f$ is the numerical class of a fiber of $p$).

We choose a very ample divisor $A$ on $X$ and we consider the polarized ruled surface $(X, A)$, i.e., $X$ embedded in $\mathbb{P}^n(X, A) := H^0(X, A)$, the symmetric algebra of $V = H^0(X, A)$. The algorithm requires the knowledge of the following data: a set of generators of the ideal $I_X$ in the ring $S(V) := \oplus_{i \geq 0} S^i(V)$, the symmetric algebra of $V$; the divisor $B$ (as Weil divisor on $C$); and the extension class giving $E$ (in turn, we will need to give a specific morphism between two modules corresponding to the extension class given by the equation (0.1)).

Ampleness conditions for the divisor $A$ are classical and well known (cf. e.g. [H]). In particular, by the Nakai’s criterion, denoting with $e := -\deg E$ the invariant of

1991 Mathematics Subject Classification. 14J26, 14Q10.

This work was written within the framework of the national research project “Geometry of Algebraic Varieties” Cofin 2006 of MIUR and the DFG-Forschungsschwerpunkt “Globale Methoden in der komplexen Geometrie”.

1
X, an ample divisor A is numerically equivalent to aC₀ + bf with a ≥ 1 and b > ae if e ≥ 0 or b > ae/2 if e < 0. On the contrary the very ampleness condition for A has to be checked case by case with some criteria, e.g. Reider’s criterion (cf. [R]) or by looking at the image of X by |A|.

Let \( \mathbb{K} \) be the base field. Our main result is the following theorem, the algorithm being included in its proof:

**Main Theorem.** (char \( \mathbb{K} = 0 \) or \( q \leq 1 \) for \( k > 1 \)). Let \( C \subset \mathbb{P}^m \) be a smooth curve of genus \( q \), \( B \) a divisor on \( C \) and \( L \) a line bundle over \( C \). Consider a normalized rank \( 2 \) vector bundle \( E \in \text{Ext}^1(L, \mathcal{O}_C) \) over \( C \) given by an extension \( 0 \to \mathcal{O}_C \to E \to L \to 0 \) and suppose that, for some positive integer \( k \), the divisor \( A = kC₀ + p^*B \) on the surface \( X = \mathbb{P}(E) \) is very ample.

Then there is an algorithm yielding a set of generators of the ideal \( I_X \) of the embedded \( X \) in \( \mathbb{P}^n(X, A) \) by \( |A| \).

A few words about the algorithm. We already listed at the beginning the data required by the algorithm. To be more precise, the algebraic data needed by the algorithm are:

1. the ideal \( I_C \) of the embedded curve \( C \subset \mathbb{P}^m \);
2. three divisors on \( C \): \( B \), an auxiliary effective divisors \( D \), satisfying some technical conditions (see (1.3) and (3.1)), such that \( D - B \) is effective and \( |L \otimes \mathcal{O}_C(D)| \neq \emptyset \), and another auxiliary divisor \( D' \in |L \otimes \mathcal{O}_C(D)| \);
3. an element in \( \text{Ext}^1_S(I_D', I_D) \).

Let us see how these algebraic data are related to the data at the beginning. Let \( S \) be the coordinate ring of \( C \). The first technical condition for \( D \), i.e., (1.3), will guarantee that the graded \( S \)-module \( H^0_C(L, \mathcal{O}_C(D)) = \bigoplus_{i \geq 0} H^0(C, L \otimes \mathcal{O}_C(i)) \) is an extension in \( \text{Ext}^1_S(H^0_C(L, \mathcal{O}_C(i)), H^0_C(\mathcal{O}_C(D))) \). We will moreover see that \( H^0_C(L, \mathcal{O}_C(D)) = I_D' \) and \( H^0_C(\mathcal{O}_C(D)) = I_D \). Therefore, there is a \( 1:1 \) correspondence between the elements in \( \text{Ext}^1_S(I_D', I_D) \) as in (3) and the vector bundles \( E \in \text{Ext}^1(L, \mathcal{O}_C) \).

The technical conditions for \( D \) will be explicitly given and we will see that it is always possible to determine a minimal degree for \( D \) such that they will hold. Notice also that the polarized surface \( (X, A) \) is independent of the choices of the projective model for \( C \), of \( D \), and of \( D' \in |L - D| \).

We also remark that two assumptions in the Main Theorem can be replaced by computational checks: the assumption char \( \mathbb{K} = 0 \) (if \( q > 1 \)) and the very ampleness assumption for \( A = kC₀ + p^*B \).

The restriction on the characteristic in the statement of the main theorem for \( k \geq 2 \) is required only to guarantee the \( k \)-normality of an isomorphic image of \( X \) involved in the algorithm and depending on the choice of \( D \). In finite characteristic the algorithm is still valid modulo performing a computational check of the \( k \)-normality and eventually changing the choice of \( D \) (see remark (3.6)).

Even without the very ampleness assumption for \( A \), the algorithm computes the ideal of the image of \( X \) by the rational map \( \phi_{|A|} \) associated to \( |A| \). A computational check that \( \phi_{|A|}(X) \) is a smooth surface of degree \( A^2 \) and that \( |A| \) has empty base locus ensures then that \( \phi_{|A|} \) is an embedding (see remarks (3.7) and (3.8)).

The algorithm is straightforward in the case of scrolls, when \( k = 1 \). Instead, in the case \( k \geq 2 \) it firstly computes a suitable scroll and then its image by a rational map associated to a suitable linear system. Hence it is useful to divide the main theorem into two steps. In section 1 we will give and prove theorem A, concerning the case of scrolls. In section 3 we will give and prove theorem B, concerning the cases \( k \geq 2 \). The main theorem will follow from theorems A and B.
In sections 2 and 4 we will give applications of the algorithm to study some properties of surfaces whose existence is known. In section 2 we describe a family of scrolls of degree 8 with sectional genus 2 (cf. [C]) and a family of scrolls of degree 6 with sectional genus 1, while in section 4 we describe conic bundles of degree 8 with sectional genus 3, cubic bundles of degree 9 with sectional genus 4 and a ruled surface having cubic fibres.

In section 5 we describe some varieties related to ruled surfaces with conic fibres.

In section 6 we use our algorithm to construct some embedded surfaces whose existence was not known. To do this, we will consider a large set of ruled surfaces $X$ treated in [Li, §3]. There the author proved the very ampleness of some divisors $\mathcal{L}_0 \equiv a\mathcal{C}_0 + b\mathcal{F}$ by applying Reider’s criterion in a convenient way, but she was unable to decide whether also the subsystems given by $\mathcal{L} = \mathcal{L}_0 - \sum p_i$ are very ample for low values of $t$ and generic choices of the points $p_i$’s. This amounts to ask whether some projections of $(X, \mathcal{L}_0)$ are smooth. By our algorithm we will be able to get a set of generators for the ideal of $(X, \mathcal{L}_0)$, to project $(X, \mathcal{L}_0)$ from some random points on it, and then to test directly whether such projections are smooth.

In this way we prove that there exist geometrically ruled surfaces $X$, over genus $q$ curves, of invariant $e$, such that the linear systems $|\mathcal{L}|$ in the following table are very ample. $X$ will be embedded by $|\mathcal{L}|$ as a surface of degree $d$ and sectional genus $g = 7$.

| $g$ | $d$ | $h^0(\mathcal{L})$ | $q$ | $|\mathcal{L}|$ |
|-----|-----|-------------------|-----|----------------|
| 9   | 7   | $\sum p_i$       | 1   | $3C_0 + 3f - \sum p_i$ |
| 10  | 7   | $\sum p_i$       | 1   | $4C_0 - \sum p_i$ |
| 11  | 7   | $\sum p_i$       | 1   | $5C_0 - \sum p_i$ |
| 13  | 7   | $\sum p_i$       | 2   | $2C_0 + 2f - \sum p_i$ |
| 13' | 7   | $\sum p_i$       | 2   | $2C_0 + 3f - \sum p_i$ |

There are other methods to get explicit equations of embedded ruled and (non ruled) surfaces, but they work only for low codimension and they do not allow to have some control over the “geometry” of the polarized surface (the curve $C$, the extension giving $E$, and the divisor $B$).

Indeed, if $X$ has codimension 2, i.e. $X \subset \mathbb{P}^4$, then there exist two sheaves $\mathcal{F}$ and $\mathcal{G}$ with $\text{rk}\mathcal{G} = \text{rk}\mathcal{F} + 1$ and a map $\Phi : \mathcal{F} \to \mathcal{G}$ such that the Eagon-Northcott complex defined by the maximal minors of $\Phi$ identifies $\text{coker}\Phi$ with the ideal sheaf of $X$. The sheaves $\mathcal{F}$ and $\mathcal{G}$ are then constructed starting from the cohomology table of $\mathcal{I}_X$. This constructing method was introduced in [DES] and it is largely used to construct surfaces in $\mathbb{P}^4$ (c.f. also [DS] for a further description and a nearly up-to-date list of references). If $X$ has codimension 3, this type of construction can be still performed using the Pfaffian complex instead of the Eagon-Northcott complex: if $X$ is a codimension 3 subcanonical scheme in $\mathbb{P}^5$ a locally free resolution of its ideal sheaf is still known (c.f. [W]).

A final remark. The algorithm requires the knowledge of the embedded curve $C$ which is the base of the ruling of the surface. If instead, in an example, we do not have a projective model of $C$ there is the further mathematical problem to construct such a model, at least for a random curve $C$ (random in its moduli space). There is no general method to determine embedded curves at random, but it is a classical topic how to parametrize curves of low genus ($g \leq 10$) by using nodal plane models and an explicit method to parametrize smooth space curves up to genus 14 is known and illustrated in [S-T].

We use the computer–algebra program Macaulay2 ([G-S]) to implement the algorithms and the examples.
Acknowledgments

An algorithm for a particular case of scrolls (ruled surfaces over a curve of genus 2 embedded as scrolls of degree 8 in $\mathbb{P}^5$) was partially developed in [C].

We thank F.-O. Schreyer for a useful conversation about symmetric algebras of graded modules over the coordinate ring of a variety and their presentation as modules.

Notation Table

| Notation | Description |
|----------|-------------|
| $K$      | base field, usually $\mathbb{C}$ (but also finite fields are considered) |
| $\mathbb{P}^n$ | projective n-dimensional space over $K$ |
| $\mathbb{P}(E)$ | projectivization of the rank 2 vector bundle $E$ over a smooth curve $C$, $C_0$ is its tautological divisor, $p : \mathbb{P}(E) \to C$ the natural projection, and $f$ the numerical class of a generic fibre of $p$ |
| $c_i(E)$ | $i$-th Chern class of $H^0(W,F \otimes O_W(t))$ for any sheaf $F$ on $W \subset \mathbb{P}^n$ |
| $\mathbf{F}_{e,q}$ | ruled surface of invariant $e := -\deg(c_1(E)) \geq -q$ over a smooth, genus $q$, curve $C$ |
| $\equiv$ | numerical equivalence |
| $|D|$ | means duality |
| $I_W$ ($\mathcal{I}_W$) | ideal (ideal sheaf) of a projective variety $W \subset \mathbb{P}^n$ |
| $K_W$ | canonical divisor of a smooth variety $W$ |
| $g(W)$ | sectional genus of a smooth variety $W \subset \mathbb{P}^n$ |
| $H^0(W,F)$ | sheaf of $\mathcal{O}_W$-modules associated to any $S$-module $M$, where $S$ is the coordinate ring of a smooth variety $W$ |
| $S(V)$ | symmetric algebra of the vector space $V$ |
| $S(E)$ | symmetric $\mathcal{O}_W$-algebra of the vector bundle $E$ over a variety $W$ |
| $\mu(E)$ | $\deg E / \text{rk } E$, slope of the vector bundle $E$ |
| $\mu^-(E)$ | $\min(\mu(Q)|E \to Q \to 0)$ |

1. Construction of scrolls

In this section we develop an algorithm to compute a set of generators of the ideal of embedded scroll surfaces:

**Theorem A.** Let $C \subset \mathbb{P}^n$ be a smooth curve $C$ of genus $q$, $B$ be a divisor on $C$ and $L$ a line bundle over $C$. Consider a normalized rank 2 vector bundle $E \in \text{Ext}^1(L, \mathcal{O}_C)$ over $C$ given by an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ and suppose that the divisor $A = C_0 + p^*B$ on the surface $X = \mathbb{P}(E)$ is very ample, where $p : \mathbb{P}(E) \to C$ is the natural projection.

Then there is an algorithm yielding a set of generators of the ideal $I_X$ of the embedded $X$ in $\mathbb{P}^{n+1}(X, A) = \mathbb{P}(H^0(X, A)^*)$ by $|A|$.

A description of the explicit form of the data required by the algorithm is given in the introduction, immediately after the statement of the Main Theorem.

At first, let us point out the basic idea of the method. Let $I_C$ be the ideal of the curve $C$ in $\mathbb{P}^n = \text{Proj}(R)$, where $R = K[x_0,\ldots,x_m]$, and let $S := R/I_C$ be the coordinate ring of $C \subset \mathbb{P}^n$. Our strategy will be to get a presentation of the $S$-module $M$ defined as

$$M := H^0(C,E \otimes \mathcal{O}_C(B)) = \oplus_{i\geq 0} H^0(C, E \otimes \mathcal{O}_C(B + iH)),$$

where $H$ is a hyperplane divisor of $C \subset \mathbb{P}^n$. Then we will apply a straightforward computation to get a set of generators of $I_X$. Details will be given later in the proof of the theorem. The short exact sequence (0.1) implies the exactness of

$$0 \to \mathcal{O}_C(B) \to E \otimes \mathcal{O}_C(B) \to L \otimes \mathcal{O}_C(B) \to 0,$$

from which we will derive the desired presentation of $M$.

Now, we give here some Lemmas needed for the proof of the theorem.
Lemma 1.1. Let $I_C$ be the ideal of a smooth curve $C$ in a projective space $\mathbb{P}^m$ and let $S$ be the coordinate ring of $C$. Let $D$ be an effective divisor on $C$. Then the $S$-modules $H^i_0(C, \mathcal{O}_C(D))$ and $(I_D)^* := \text{Hom}_S(I_D, S)$ are naturally isomorphic as (graded) $S$-modules, where $I_D \subset S$ is the ideal of the divisor $D$.

Proof. Let us recall the following well-known result on local cohomology (cf. [E, Thm. 3.1]). Let $S$ be a graded noetherian ring with degree 0 part a field, $m$ the maximal ideal generated by the degree 1 part of $S$ and $M$ a finitely generated graded $S$-module. Then there is a natural exact sequence

$$0 \rightarrow H^0_m(M) \rightarrow M \rightarrow \oplus_{i \geq 0} H^0(\text{Proj} \ S, \widetilde{M}(i)) \rightarrow H^1_m(M) \rightarrow 0,$$

where $H^i_m(M)$ denotes the $i$-th local cohomology group of $M$ with respect to $m$. By using this sequence, whenever $W \subset \mathbb{P}^m$ is a variety, $S$ is its coordinate ring, and $M$ is any graded $S$-module, one obtains that $M \cong H^0_0(W, \mathcal{M})$ if $H^0_m(M) = H^1_m(M) = 0$. Moreover a sufficient condition for obtaining the two vanishings $H^0_m(M) = H^1_m(M) = 0$ is that $\text{depth}(m, M) \geq 2$ by [H, Ch. III, Ex. 3.4 and 3.3]. For more details and related results, see [G] (in particular Prop. 2.2 and Thm. 3.8).

In our case $W = C$, $S := \mathbb{K}[x_0, \ldots, x_m]/I_C$ is the coordinate ring of the curve $C$ in $\mathbb{P}^m$, $m := (x_0, \ldots, x_m)$ is the image of the irrelevant ideal of $\mathbb{K}[x_0, \ldots, x_m]$ in $S$, and $M = (I_D)^*$. Recall that $S$ is the coordinate ring of an affine cone over a curve, and therefore $I_D$, as well as $(I_D)^*$, is not necessarily a projective $S$-module, since it may be not locally free if you localize it at the vertex of the cone. Therefore we proceed as follows.

Firstly, notice that $\text{depth}(m, S) = 2$ and that a regular sequence for $S$ is also a regular sequence for $I_D$, since $I_D$ is a submodule of $S$, thus $\text{depth}(m, I_D) \geq \text{depth}(m, S) = 2$ and $I_D = H^0_0(C, \mathcal{O}_C(-D))$.

Secondly, we claim that, if $t_1, \ldots, t_d \in m$ is a regular sequence for $S$, then it is also a regular sequence for $(I_D)^* = \text{Hom}(I_D, S)$. We argue by contradiction. Suppose that $t_i$ is a 0-divisor for $(I_D)^*$ mod $(t_1, \ldots, t_{i-1})$. Then there exists a non-zero morphism $\varphi \in (I_D)^*$ mod $(t_1, \ldots, t_{i-1})$ s.t. $t_i \varphi = 0$ mod $(t_1, \ldots, t_{i-1})$. Take an $x \in I_D$ s.t. $\varphi(x) \neq 0$ in $S/(t_1, \ldots, t_{i-1})$: from $t_i \varphi(x) = 0$ in $S/(t_1, \ldots, t_{i-1})$ we get that $t_i$ is a 0-divisor in $S/(t_1, \ldots, t_{i-1})$, a contradiction. Moreover $t_1, \ldots, t_d \in m$, hence we have $(t_1, \ldots, t_d)(I_D)^* \neq (I_D)^*$. Indeed, if this is not the case, then $m(I_D)^* = (I_D)^*$ and therefore there exists an element $r \in m$ such that $(1-r)(I_D)^* = 0$, cf. [E, Cor. 4.7]. In particular, considering the inclusion $i : I_D \rightarrow S$, we have $(1-r)i = 0$ and therefore $1 - r$ is a 0-divisor in $S$. Since $S$ is an integral domain, it follows that $r = 1$, which is absurd since $r \in m$. We conclude that $\text{depth}(m, (I_D)^*) \geq \text{depth}(m, S) = 2$ and therefore we get $(I_D)^* = H^0_0(C, [(I_D)^*]) = H^0_0(C, \mathcal{O}_C(D))$. ∎

Lemma 1.2. Let $S$ be a commutative ring, $F$ and $G$ be two $S$-modules with free resolutions:

$$F^* : \quad \cdots \rightarrow F_3 \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\psi} F \rightarrow 0,$$

$$G^* : \quad \cdots \rightarrow G_3 \xrightarrow{\phi_3} G_2 \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0 \xrightarrow{\psi} G \rightarrow 0.$$

Then any morphism $\varphi \in \text{Hom}_S(F_1, G_0)$ satisfying $\psi \circ \varphi \circ \phi_2 = 0$, i.e. inducing a morphism in $\text{Hom}_S(\ker \phi, G) = \text{Hom}_S(\text{im} \phi_1, \text{coker} \psi_1)$, determines an extension $M \in \text{Ext}_S^2(F, G)$ and, conversely, any extension is determined by such a morphism.

Moreover, the module $M \in \text{Ext}_S^2(F, G)$ corresponding to $\varphi$ has presentation

$$\begin{pmatrix} \phi_1 & 0 \\ \phi & \psi_1 \end{pmatrix} : \ F_1 \oplus G_1 \rightarrow F_0 \oplus G_0.$$

Proof. The proof is standard. For completeness, since this proof is important to implement our algorithm, we shortly repeat it here (for some references cf. [G-H,
Let \( \text{rk } \mathrm{M} \) be the module \( \text{Hom}_S(F_0, G) \rightarrow \text{Hom}_S(K, G) \rightarrow \text{Ext}_S^1(F, G) \rightarrow \text{Ext}_S^1(F_0, G) = 0 \). Therefore \( \text{Ext}_S^1(F, G) \cong \text{Hom}_S(K, G)/\text{Hom}_S(F_0, G) \).

In the same way the short exact sequence \( 0 \rightarrow \text{ker } \phi_1 \rightarrow F_1 \rightarrow \text{ker } \phi_2 \rightarrow 0 \) gives \( \text{Hom}_S(K, G) \rightarrow \text{Hom}_S(F_1, G) \rightarrow \text{Hom}_S(\text{ker } \phi_1, G) \). Hence \( \text{Hom}_S(K, G) \) is the kernel of the second map and we can identify \( \text{Hom}_S(K, G) \) with the set of morphisms \( \eta \in \text{Hom}_S(F_1, G) \) whose restriction to \( \text{ker } \phi_1 \) is zero or equivalently, since \( \text{ker } \phi_1 = \text{im } \phi_2 \), whose composition \( \eta \circ \phi_2 \) is zero. Since \( F_1 \) is projective, the surjectivity of \( G_0 \rightarrow G \) gives the surjectivity of \( \text{Hom}_S(F_1, G_0) \rightarrow \text{Hom}_S(F_1, G) \).

Conversely, a morphism \( \varphi \in \text{Hom}_S(F_1, G_0) \) satisfying the hypothesis determines by composition a morphism \( \eta \in \text{Hom}_S(F_1, G) \) satisfying \( \eta \circ \phi_2 = 0 \) which therefore is an element of \( \text{Hom}_S(K, G) \): its equivalence class in \( \text{Hom}_S(K, G)/\text{Hom}_S(F_0, G) \) determines an extension \( M \in \text{Ext}_S^1(F, G) \), as desired.

To compute a presentation of such an extension \( M \), let us denote with \( i \) the inclusion \( K \rightarrow F_0 \), and with \( \varphi' \in \text{Hom}_S(K, G) \) the morphism induced by \( \varphi \). Then the module \( M \) is the quotient \( (F_0 \oplus G)/\text{im}(i \oplus \varphi') \), which is the cokernel of the morphism \( \begin{pmatrix} \phi_1 \\ \varphi \\ 0 \\ \psi_1 \end{pmatrix} : F_1 \oplus G_1 \rightarrow F_0 \oplus G_0 \).

**Lemma 1.3.** Let \( W \subset \mathbb{P}^m \) be a smooth algebraic variety and let \( \mathcal{E} \) be a locally free sheaf on \( W \). Denote by \( S := \mathbb{K}[x_0, \ldots, x_m]/I_W \) the coordinate ring of \( W \) in \( \mathbb{P}^m \). Suppose further that the tautological bundle \( \tau_{\mathbb{P}^m}(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^m}(1) \) of \( \mathbb{P}^m \) is very ample. Then, given a presentation of the \( S \)-module \( M := H^0(W, \mathcal{E}) \), there is an algorithm yielding a set of generators of the ideal \( I_{\mathbb{P}^m}(\mathcal{E}) \) of the embedded variety \( \mathbb{P}(\mathcal{E}) \) by the complete linear system \( \tau_{\mathbb{P}^m}(\mathcal{E}) \).

**Proof.** Notice that \( H^0(\mathbb{P}(\mathcal{E}), \tau_{\mathbb{P}^m}(\mathcal{E})) = H^0(W, \mathcal{E}) \) and let \( h^0(W, \mathcal{E}) = n + 1 \). The embedding \( i \) associated to \( \tau_{\mathbb{P}^m}(\mathcal{E}) \) comes with a map of sheaves of rings \( i^* : \mathcal{O}_{\mathbb{P}^m} \rightarrow \mathcal{O}_{\mathbb{P}^m}(1) \), induced by mapping \( n + 1 \) new variables \( y_0, \ldots, y_n \) to a basis of \( H^0(W, \mathcal{E}) = H^0(\mathbb{P}(\mathcal{E}), \tau_{\mathbb{P}^m}(\mathcal{E})) \). The ideal sheaf \( I_{\mathbb{P}^m}(\mathcal{E}) \) is the kernel of this map.

Let \( M' \subset M \) be the \( S \)-submodule generated by a basis of \( H^0(W, \mathcal{E}) \). If \( \phi \) is the given free presentation of \( M \), we can compute a free presentation \( \phi' \) of \( M' \):

\[
M_1 \xrightarrow{\phi'} M_0 \rightarrow M' \rightarrow 0,
\]
where \( \text{rk } M_0 = n + 1 \) and the generators of \( M_0 \) map to the chosen basis of \( H^0(W, \mathcal{E}) \).

Let us consider in \( \mathbb{S}[y_0, \ldots, y_n] \) the ideal \( I \) given by

\[
I := (y_0 \ldots y_n) \cdot \phi',
\]
where, by abuse of language, we use again \( \phi' \) to denote the matrix associated to the map \( \phi' \). The ideal \( I_{\mathbb{P}^m}(\mathcal{E}) \) is given by the polynomial relations among the \( y_0, \ldots, y_n \) in the saturation of \( I \) with respect to the ideal \( (x_0, \ldots, x_m) \subset \mathbb{S}[y_0, \ldots, y_n] \). Therefore, \( I_{\mathbb{P}^m}(\mathcal{E}) \) can be obtained by saturating \( I \) with respect to the ideal \( (x_0, \ldots, x_m) \) and intersecting this new ideal with the subring \( \mathbb{K}[y_0, \ldots, y_n] \).

**Remark 1.4.** Equation (1.2) yields a presentation, as a \( \mathbb{S}[y_0, \ldots, y_n] \)-module, of the \( S \)-algebra generated by \( H^0(W, \mathcal{E}) \) in \( S(H^0_0(W, \mathcal{E})) \). If \( M \) is generated by \( H^0(W, \mathcal{E}) \), then \( M \) admits a presentation \( \bigoplus_{j=0}^S S(-l_j) \xrightarrow{\phi} \bigoplus_{j=0}^S M \rightarrow 0 \) and the \( S \)-algebra \( S(H^0_0(W, \mathcal{E})) \) has a presentation

\[
\bigoplus_{j=0}^S \mathbb{S}[y_0, \ldots, y_n][(-l_j) \xrightarrow{\sum_{i=0}^n w_i \phi_{ij} \ldots} \mathbb{S}[y_0, \ldots, y_n] \rightarrow S(H^0_0(W, \mathcal{E})) \rightarrow 0.
\]
If this is not the case, i.e. $M$ is not generated by the minimal degree part, let \( \bigoplus_{j=0}^s S(-l_j) \xrightarrow{\phi} \bigoplus_{i=0}^n S(-h_i) \to M \to 0 \) be a presentation of $M$. Then the $S$-algebra $S(H^0_i(W, E))$ has still a presentation as above, but now the the ring $S[y_0, \ldots, y_n]$ is weighted, $y_i$ having weight $h_i$.

**Remark 1.5.** (char $\mathbb{K} = 0$ or $q \leq 1$). Let us assume that $W = C \subset \mathbb{P}^m$ is a smooth projective curve of genus $g$ and $E$ is a vector bundle over $C$. If $\mu(E) \geq 2q$ and $\deg(OC(1)) \geq 2q$ and moreover one of these two inequalities is strict, then $H^0(C, E)$ generates $H^2_s(C, E)$ as $S$-module, where $S$ denotes the coordinate ring of $C$ in $\mathbb{P}^m$.

**Proof.** This is a direct application of Theorem 2.1 of [Bu]: under these assumptions the map $H^0(C, E) \otimes H^0(C, OC(t)) \to H^0(C, E \otimes OC(t))$ is surjective $\forall t \geq 0$.

**Proof of Thm. A.** Let $S$ be the coordinate ring of $C$ in $\mathbb{P}^m$ and let $H$ be the divisor induced on $C$ by a hyperplane section of $\mathbb{P}^m$. Recall the assigned exact sequence in the statement: $0 \to OC \to E \to L \to 0$ and let $D$ be any effective divisor on $C$ such that $D - B$ is effective, $|L \otimes OC(D)| \neq \emptyset$, and

\[(1.3) \quad H^1(C, OC(D + jH)) = 0 \quad \forall j \geq 0.
\]

Note that by choosing the degree of $D$ big enough, it is always possible to find such a divisor $D$.

By condition (1.3) there is a short exact sequence of $S$-modules

\[0 \to H^1_s(C, OC(D)) \to H^1_s(C, E \otimes OC(D)) \to H^1_s(C, L \otimes OC(D)) \to 0,
\]

implying that $H^1_s(C, E \otimes OC(D))$ can be obtained as an extension in $Ext^1_s(H^1_s(C, L \otimes OC(D)); H^1_s(C, OC(D)))$.

Since $|L \otimes OC(D)| \neq \emptyset$, there exists an effective divisor $D' \in |L \otimes OC(D)|$. Applying Lemma 1.1 to the divisors $D$ and $D'$, we easily get explicit free resolutions of the $S$-modules $H^1_s(C, OC(D)) = I^s_D$ and $H^1_s(C, L \otimes OC(D)) = I^s_D$.

Given the two resolutions of $I^s_D$ and of $I^s_{D'}$, since $H^1_s(C, E \otimes OC(D))$ is an extension in $Ext^1_s(I^s_{D'}, I^s_D)$, we can apply Lemma 1.2 to get a presentation of $H^1_s(C, E \otimes OC(D))$. The tensorization of $H^1_s(C, E \otimes OC(D))$ with $I_{D-B} = H^0_s(C, OC(-D+B))$ yields an $S$-module $M'$ whose associated coherent sheaf is $E \equiv B$.

With a computer algebra system we compute a presentation of the module of the global sections of the coherent sheaf associated to $M'$: this precisely the module $M := H^0_s(C, E \otimes OC(B)) = H^0_s(C, M')$.

Finally, from a presentation of $M$, Lemma 1.3 explains how to get a set of generators of $I_X$, where $X = \mathbb{P}(E)$ is embedded by the very ample divisor $A = C_n + p B$ (the tautological divisor of $\mathbb{P}(E \otimes OC(B))$).

The algorithm based on the above considerations is therefore the following:

1. choose on $C \subset \mathbb{P}^m$ an effective divisor $D$ satisfying (1.3) and such that $D - B$ is effective, $|L \otimes OC(D)| \neq \emptyset$;
2. choose a divisor $D' \in |L \otimes OC(D)|$ and, from the ideal of $C$ and from $L$, compute $S$ and a set of generators of the ideals $I_D$ and $I_{D'}$ (as $S$-modules);
3. identify the extension in $Ext^1_s(I_{D'}, I_D)$ corresponding to the given extension of $E$ as an element in $Ext^1(L, OC) \cong Ext^1(L \otimes OC(D), OC \otimes OC(D))$ and compute a presentation of the module $N = H^0_s(C, E \otimes OC(D))$ as explained in Lemma 1.2;
4. compute a presentation of $M' = N \otimes I_{D-B} = H^0_s(C, E \otimes OC(D)) \otimes I_{D-B}$;
5. compute a presentation of $M = H^0_s(C, M')$, which is the module $H^0_s(C, E \otimes OC(B))$ and proceed as explained in Lemma 1.3 to compute a set of generators of $I_X$.

\[\square\]
Remark 1.6. Suppose that $|B|$ and $|L \otimes B|$ contain effective divisors $D \in |B|$ and $D' \in |L \otimes B|$, and that $\deg B > 2q - 2$ and $h^1(C, O_C(B + jH)) = 0$ for all $j \geq 0$. Then in the proof of the theorem we can choose $D = B$, i.e. the module $M = H^0(C, E \otimes O_C(B))$ can be directly obtained as an extension in $Ext^2_B((I_D)^*, (I_D)^*)$.

Remark 1.7. The theorem can be used to obtain examples of scroll surfaces by considering random effective divisors $D, D'$ with fixed degrees such that $\deg D > 2q - 2$ and a random extension class in $Ext^2_B((I_D)^*, (I_D)^*)$. Defining $L$ as the sheaf $O_C(D' - D)$, the previous extension class determines an element in $Ext^2_B(L, O_C) \cong H^1(C, L^*)$ and the condition $\deg D > 2q - 2 = 0$ ensures that every extension in $Ext^2_B(L, O_C)$ can be obtained starting from an extension in $Ext^2_B((I_D)^*, (I_D)^*)$.

2. Some examples of “interesting” ruled surfaces

In this section we will construct some examples of “interesting” ruled surfaces by applying Theorem A, where by “interesting” we mean that these surfaces have some particular properties. We will use the computer algebra system Macaulay2 to execute all the computations described in the algorithm of Theorem A. In order to get the whole module $M = H^0(C, E \otimes O_C(B))$, rather than the submodule $M'$, we will use the corresponding implemented command in the computer algebra system Macaulay2. If this command is not available in the computer algebra system under use, the practical computation may have some difficulties, but, in any case, there is no problem when the assumptions in the Remark (1.6) are valid because the assumptions allow us to choose $D = B$.

2.1. First example. Let $C$ be a smooth curve of genus 2. Let $E$ be a normalized rank 2 vector bundle of degree 2, so that we have the following exact sequence:

$$0 \to O_C \to E \to L \to 0$$

where $L = \det(E) = c_1(E), \deg L = 2$. Let $B$ be any degree 3 divisor of $C$. On the surface $X = \mathbb{P}(E)$ we can consider the divisor $A = C_0 + p^*B \equiv C_0 + 3f$. $A$ is a very ample divisor, whatever $B$ is chosen, $h^0(X, A) = h^0(C, E \otimes B) = 6$ and it embeds $X$ in $\mathbb{P}^5$ as a smooth scroll of degree 8 (see [I-2]); $g(X) = 2$. It is easy to see that $X$ is 2-normal if and only if it is not contained in a quadric.

About this surface we have the following proposition (see [A-B-B-1]):

Proposition 2.1. Let $X$ be the surface above. $X$ is contained in a rank 4 quadric cone whose vertex is a 4-secant line for $X$ and therefore $X$ is not 2-normal.

Note that in [A-B-B-1] the proposition is proved by using geometric arguments and it is not considered the $k$-normality of $X$ for $k \geq 3$ or the problem to determine a free resolution of the ideal $I_X$. Some more information about $X$ can be found in [C], the article which suggested to us to approach the problem.

We now explicitly describe how to construct such an example by applying the algorithm in Theorem A. The first step is to get a projective model of a smooth curve $C$ of genus 2. Following the method described in [S-T] for obtaining projective space models of general curves of genus 11, we choose a random space curve $C$ of genus 2 and degree 5 by the function (see the proof of Lemma 6.3):

```plaintext
randomGenus2Curve = (R) -> {
  correctCodimAndDegree:=false;
  while not correctCodimAndDegree do {
    I:=ideal syz transpose random(R^{-2,2:-3},R^{-2:-4});
    correctCodimAndDegree=(codim I==2 and degree I==5);)
  }
}
```

We check the smoothness of $C$ by means of the jacobian criterion:
isSmoothSpaceCurve = (I) -> (  
  singI:=I+minors(2,jacobian I);  
codim singI==4);

Proceeding in the construction of the starting data of the algorithm, we also need to choose \( t \) random points on \( C \). This step can be performed by separating the points of a good hyperplane section (on a non algebraically closed field it can happen that these points are not separated):

\[
\text{randomPoint} = (C) \rightarrow (  
  R:=\text{ring } C;  
isSinglePoint:=false;  
  \text{while not isSinglePoint do (}  
  \text{hypsection}:=C+\text{ideal random}(R^{1},R^{-1});  
  pt:=(\text{decompose hypsection})#0;  
isSinglePoint=(\text{degree pt}==1);};  
pt);  
\text{randomPoints} = (C,t) \rightarrow (  
  pt:=\text{randomPoint } C; i:=t-1;  
  \text{while } i!=0 \text{ do } (pti=\text{randomPoint } C; pt=\text{intersect}(pt,pti); i=i-1););  
pt);  
\]

We are now ready to explicitly compute such an example by following the algorithm of Theorem A: we choose a random smooth genus 2 curve \( C \in \mathbb{P}^3 \) as explained above, we apply random choices for \( D, D', N \in \text{Ext}^1(H^0(D'),H^0(D)) \), \( B \) and, finally, we compute a set of generators of the ideal of \( X \) embedded via the linear system \( |A| = |C_0 + p^* B| \) corresponding to our random choices.

The following input lines computes the ideal \( C \) of such a random curve \( C \):

\[
K=\mathbb{Z}/101;  
R=K[x_0..x_3]  
C=\text{randomGenus2Curve } R  
isSmoothSpaceCurve(C)  
\text{betti res } C  
\]

We choose \( L \) and \( B \) as effective divisors of degree 2 and 3 respectively. By Remark 1.6, we can therefore choose \( D = B \). The points of an effective divisor are chosen via the function \( \text{randomPoints}() \), which returns their ideals \( L_{\text{dual}} \) and \( D_{\text{dual}} \) in \( \mathbb{P}^3 \). Their intersection is called \( D'_{\text{dual}} \).

\[
L_{\text{dual}}=\text{randomPoints}(C,2)  
D_{\text{dual}}=\text{randomPoints}(C,3)  
D'_{\text{dual}}=\text{intersect}(L_{\text{dual}},D_{\text{dual}})  
\]

We then compute the \( S \)-modules \( H^0_2(D) \) and \( H^0_0(D') \), called resp. \( DS \) and \( D'S \).

\[
S=R/C  
DS=\text{substitute}(D_{\text{dual}},S);DS=\text{Hom}(DS_{\text{dual}},S);  
D'S=\text{substitute}(D'_{\text{dual}},S);D'S=\text{Hom}(D'S_{\text{dual}},S);  
\]

We compute a presentation \( \phi \) of a random module \( N \) in \( \text{Ext}^1(H^0_0(D'),H^0_0(D)) \) as explained in Lemma 1.2. For this purpose, we define the function \( \text{randomExt}() \)

\[
\text{randomExt} = (A,B) \rightarrow (  
  phia:=\text{presentation } A;  
  phib:=\text{presentation } B;  
  \text{Homom}:=\text{Hom}(image \phi,\text{coker } phib);  
  \phiab:=\text{homomorphism random}(\text{Homom},S^{-1});\phiab=\text{matrix } \phiab;  
  \phiNull:=0*\text{random}(\text{target } \phi,\text{source } \phiib);  
  \phi:=(\phiab|\phiab)(\phiNull|\phiib);  
  \text{coker } \phi)  
\]

and we apply this function to \( D'S \) and \( DS \).

\[
N=\text{randomExt}(D'S,DS)  
\]
As explained in the proof of Theorem A, the choice of $N$ as extension class in $\text{Ext}^1(H^0_*(D'), H^0_*(D))$ corresponds to the choice of a bundle $E$ as extension class in $\text{Ext}^1(L, O_C)$ such that $N = H^0_*(C, E \otimes O_C(D))$.

We should now compute $M' = N \otimes I_{D-B}$ and $M = H^0_*(\tilde{M}')$. Here these two steps are not needed because of our choice $D = B$, and so $M = N = H^0_*(C, E \otimes O_C(D))$.

However, given a module $N$, the command to compute $H^0_*(\tilde{N})$ is:

$$M=\text{HH}^0(\text{sheaf } N)(>=0);$$

We are now ready to compute an explicit set of generators of the ideal $I_X$ of $X \subset \mathbb{P}^5$. We proceed as explained in the proof of Lemma 1.3 and we define for this purpose the function `scrollIdeal()`:

```plaintext
scrollIdeal = (M) -> (  
  phi=presentation prune image basis(0,M);  
  T=K[y_1..y_(numgens target phi)];  
  R:=ring phi;TR:=T**R;  
  Phi:=substitute(phi,TR);  
  IS:=ideal(substitute(vars T,TR)*Phi);  
  J:=saturate(IS, ideal substitute(vars R,TR));  
  ideal mingens substitute(J,T))
```

Note that the first line computes a presentation of the submodule generated by the elements of degree 0 of $M$, i.e. by $H^0_*(C, E \otimes O_C(B))$ (in this example this step is not needed by Remark 1.5 since $\deg C = 5$, hence $M$ is generated in degree 0).

The ideal $I_X$ is called $J$ in the script.

According to our random choices of $C$, $D = B$, $D'$, and $N$, the resulting surface $X \subset \mathbb{P}^5$ is computable by the following lines and has the following properties (a line beginning with `oNN` is the output line number NN of the program)

```plaintext
J=scrollIdeal(M)  
dim J, degree J  
o26 = (3, 8)  
codim (J+minors(3,jacobian J))  
o27 = 6  
betti res J  
o28 = total: 1 8 15 13 6 1  
0: 1 . . . .  
1: . 1 . . . .  
2: . 6 7 . . .  
3: . 1 8 13 6 1
```

In particular, $X$ is a smooth surface of degree 6 and $I_X$ has free resolution

$$
\begin{align*}
&O(-2) \\
0 \leftarrow I_X \leftarrow 6O(-3) \leftarrow \oplus 13O(-6) \leftarrow 6O(-7) \leftarrow O(-8) \leftarrow 0. \\
&\oplus 8O(-5) \\
&O(-4)
\end{align*}
$$

Moreover, $X$ is contained in only one quadric hypersurface $Q$, which is a rank 4 quadric cone having a 4-secant line $L$ as vertex, according to Proposition 2.1:

```plaintext
Q=(gens J)_{0}  
rank jacobian transpose jacobian Q  
o30 = 4  
singQ=ideal Q+ideal jacobian Q  
L=saturate(singQ)  
codim(L+J),degree(L+J)  
o33 = (5, 4)
```

The $k$-normality of $X$ can be investigated by computing the difference between the dimension of the degree $k$ part of the coordinate ring of $X \subset \mathbb{P}^5$ and
Given the ideal of a non-degenerate surface $X \subset \mathbb{P}^r$, degree $d$, it follows from the Castelnuovo bound that $X$ is $k$-normal for $k \geq k_0 = d - 2 + r$, cf. [La]. The $k$-normality for $k < k_0$ can then be checked by computing the Hilbert function of $X$ up to degree $k_0 - 1$.

### 2.2. Second example

Let $C$ be a smooth curve of genus 1. Let $E$ be a normalized rank 2 vector bundle of degree 0. Then we have one of the following cases:

1. $E = \mathcal{O}_C \oplus \mathcal{O}_C$ and $\mathbb{P}(E) = C \times \mathbb{P}^1$
2. $E = \mathcal{O}_C \oplus L$, where $L \neq \mathcal{O}_C$ but $\deg L = 0$
3. $E$ is given by the unique not trivial extension $0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow \mathcal{O}_C \rightarrow 0$.

Let us call $X_i$ for $i = 0, 1, 2$ the three surfaces $\mathbb{P}(E)$. It is known that, if we consider any degree 3 divisor $B$ over $C$, $X_i$ is embedded in $\mathbb{P}^5$ by $A = C_0 + p \ast B \equiv C_0 + 3f$ as a smooth scroll surface of degree 6 (see [I-1]) and $g(X_i) = 1$. Moreover, $C_0 \simeq C$ is embedded as a smooth plane curve of degree 3 and $h^0(X_i, C_0) = 2$, $h^0(X_i, C_0) = 1$ for $i = 1, 2$.

About this surface we have the following proposition (see [A-B-B-2]):

**Proposition 2.3.** Every $X_i$ is projectively normal and it is contained exactly in only one net of quadrics $\Lambda_i \simeq \mathbb{P}^2$. Moreover: i) $\Lambda_0$ contains only rank 4 quadrics whose line vertex is generically disjoint from $X_0$; in $\Lambda_0$ there is a smooth plane curve $\simeq C$ whose points correspond to the quadrics of $\Lambda_0$ whose vertex contained in $X_0$.

ii) The generic quadric of $\Lambda_0$ is smooth; the only singular quadrics in $\Lambda_0$ have rank 4 and they are parametrized by a smooth plane curve $C \simeq C$; the discriminant divisor in $\Lambda_0 \simeq \mathbb{P}^2$ is a reducible plane sextic $D = 2C$.

iii) The generic quadric of $\Lambda_2$ has rank 5; the only rank 4 quadrics in $\Lambda_2$ are parametrized by $C_0$; in fact their vertices are lines, tangent to $C_0$ with multiplicity 2.

Since here $q = 1$, we can take a smooth plane cubic as $C$. Moreover, in order to satisfy the assumptions of Theorem A and Remark 1.6, we can choose $D = B$ and $D'$ as effective divisors of degree 3, with $L = \mathcal{O}_C(D' - D)$. We then compute the modules $H^0_*(D)$ and $H^0_*(L \otimes \mathcal{O}_C(D))$, called resp. $\text{DS}$ and $D' \ast S$:

```plaintext
k=ZZ/101;
R=R[x_0..x_2];
C=ideal random(R^1,R^[0,3])
codim (C*ideal jacobian C)
Ddual=randomPoints(C,3)
D' dual=randomPoints(C,3)
S=R/C
DSdual=substitute(Ddual,S);DS=Hom(DSdual,S);
D'Sdual=substitute(D' dual,S);D'S=Hom(D'Sdual,S)
```

#### 2.2.1. Case 0. $E = \mathcal{O}_C \oplus \mathcal{O}_C$. Here $M = DS \oplus DS$:

```plaintext
M=DS++DS
```
and we proceed as in the previous example. \( \mathcal{I}_{X_0} \) has free resolution:

\[
0 \to \mathcal{I}_{X_0} \to 3\mathcal{O}(-2) \to 4\mathcal{O}(-3) \to 2\mathcal{O}(-3) \oplus 9\mathcal{O}(-4) \to 6\mathcal{O}(-5) \to \mathcal{O}(-6) \to 0.
\]

The resolution of \( \mathcal{I}_{X_0} \) shows that \( X_0 \) is contained in a net of quadrics, which we denote \( \Lambda_0 \), and it suggests that these quadrics have 2 independent linear relations among them. Indeed, it is straightforward to check that \( \Lambda_0 \) is the space generated by the \( 2 \times 2 \) minors of the \( 3 \times 2 \) matrix of linear forms given by the morphism \( 3\mathcal{O}(-2) \to 2\mathcal{O}(-3) \) appearing in the above resolution. Moreover, the ideal generated by \( \Lambda_0 \) defines a smooth scroll of dimension 3 and degree 4 in \( \mathbb{P}^5 \).

We verify that \( \Lambda_0 \) contains only rank 4 quadrics by checking that, in a space of parameters \( \mathbb{P}^2 \) for \( \Lambda_0 \), the locus where the \( 6 \times 6 \) matrix representing the generic quadric has rank \( \leq 4 \) (resp. \( \leq 3 \)) is the whole \( \mathbb{P}^2 \) (resp. empty).

Then it is possible to compute the vertex locus in \( \mathbb{P}^2 \times \mathbb{P}^5 \) of the net of quadrics \( \Lambda_0 \) and the locus \( G \subset \mathbb{P}^2 \) of the quadrics \( \Gamma_0 \) whose vertex line is contained in \( X_0 \), checking that indeed this is a smooth plane cubic. The fact that the curve \( G \) is isomorphic to \( C \) is a geometric consequence of the construction, since each vertex line is a line of the scroll \( X_0 \), which projects in a point of \( C \).

2.2.2. Case 1. Here \( M \) is again a direct sum, namely \( M = DS \oplus D'S \):

\[
M=DS+D'S
\]

and we proceed as in the previous example. \( \mathcal{I}_{X_1} \) has free resolution:

\[
0 \to \mathcal{I}_{X_1} \to 3\mathcal{O}(-2) \oplus 2\mathcal{O}(-3) \to 9\mathcal{O}(-4) \to 6\mathcal{O}(-5) \to \mathcal{O}(-6) \to 0.
\]

As in the previous subcase, we can compute the matrix representing a generic quadric in the net \( \Lambda_1 \) and the discriminant divisor. Then we can check that it is indeed the square of a cubic form \( G \) and that the singular quadrics, parametrized by \( G \), have all rank 4.

We want to point out that, for all the constructed examples of surfaces \( X_1 \), we got the following nice geometric configuration, not shown by Proposition 2.3 and completely unexpected. The locus \( Y_1 \) of the lines in \( \mathbb{P}^5 \), which are vertices of the singular quadrics in the net \( \Lambda_1 \) of quadrics containing \( X_1 \), is again a geometrically ruled surface of degree 6. According to the classification in Proposition 2.3, \( Y_1 \) is of the same type as \( X_1 \). Moreover, the intersection \( X_1 \cap Y_1 \) consists of two plane cubic curves lying in disjoint planes, one of them being \( C_0 \), the tautological divisor of \( \mathbb{P}(E) \).

In order to compute \( C_0 \), we start by defining a function to compute the ideal of the fiber in \( \mathbb{P}^5 \) of an effective divisor over \( C \).

\[
\text{pullbackIdeal} = (I) \to (\text{R:=ring I;TR:=ring IS; J:=substitute(I,TR)+IS; J=saturate(J,ideal substitute (vars R, TR))); ideal mingens substitute(J,T))
\]

With this function we compute the ideal \( H \) corresponding to the divisor \( p^*(D) \) (3 lines). The ideal \( \mathcal{C}_0 \) of \( C_0 \) may be computed as the quotient of the ideal of a hyperplane section of \( X_1 \) containing \( p^*(D) \) by \( H \), the ideal of \( p^*(D) \):

\[
\mathcal{C}_0 = (\text{ideal H,0+J}) : H
\]

2.2.3. Case 2. Here \( M \) is an extension in \( \text{Ext}^1(DS,DS) \), where \( DS \) is the module constructed as in Case 0 corresponding to an effective divisor \( D \) of degree 3. We apply the function \texttt{randomExt()} defined in section 2.1:

\[
\mathcal{M=\text{randomExt(DS,DS)}}
\]

Then we proceed as in the previous example. \( \mathcal{I}_{X_2} \) has free resolution:

\[
0 \to \mathcal{I}_{X_2} \to 3\mathcal{O}(-2) \to 2\mathcal{O}(-3) \to 9\mathcal{O}(-4) \to 6\mathcal{O}(-5) \to \mathcal{O}(-6) \to 0.
\]
As in the previous case, we can compute the matrix representing a generic quadric in the net \( \Lambda_2 \), we can check that all the quadrics have rank \( \leq 5 \), and we can compute the divisor of the rank 4 quadrics in \( \Lambda_2 \), a smooth cubic \( G \).

According to Proposition 2.3, it is possible to check that the vertex of any rank 4 quadric in \( \Lambda_2 \) is a line tangent to \( X_2 \) at a point of \( C_0 \).

Again, there is a nice geometric configuration, not shown by Proposition 2.3 and completely unexpected. The locus \( Y_2 \) of the lines in \( \mathbb{P}^3 \), which are vertices of the singular quadrics in the net \( \Lambda_2 \) of quadrics containing \( X_2 \), is a geometrically ruled surface of degree 6. According to the classification in Proposition 2.3, \( Y_2 \) is of the same type as \( X_2 \). Moreover, the intersection \( X_2 \cap Y_2 \) consists of the cubic \( C_0 \) counted twice. Therefore a vertex of a quadric in \( \Lambda_2 \) is a line \( L \) tangent to \( X_2 \) at the point of \( C_0 \) given by the intersection of \( L \) with \( C_0 \).

3. Embeddings with fibres of higher degree

In this section we develop a method to construct explicitly the ideals of surfaces which are \( \mathbb{P}^1 \)-bundles over a smooth curve \( C \), embedded in such a way that every fibre is a rational curve of degree \( k \geq 2 \). In other words, we consider polarized surfaces \((X, A)\), where \( X := \mathbb{P}(E) \) as in §1 and \( A = kC_0 + p^*B \) is very ample. We aim to prove the following:

**Theorem B.** (char \( K = 0 \) or \( q \leq 1 \)). Let \( C \subset \mathbb{P}^m \) be a smooth curve \( C \) of genus \( q \), \( B \) a divisor on \( C \) and \( L \) a line bundle over \( C \). Consider a normalized rank 2 vector bundle \( E \in \text{Ext}^1(L, \mathcal{O}_C) \) over \( C \) given by an extension \( 0 \to \mathcal{O}_C \to E \to L \to 0 \) and suppose that, for some positive integer \( k \), the divisor \( A = kC_0 + p^*B \) on the surface \( X = \mathbb{P}(E) \) is very ample.

Then there is an algorithm yielding a set of generators of the ideal \( I_X \) of the embedded \( X \) in \( \mathbb{P}^N(X,A)^{-1} = \mathbb{P}(H^0(X,A)^*) \) by \( |A| \).

A description of the explicit form of the data required by the algorithm is given in the introduction, immediately after the statement of the Main Theorem.

Let us say a few words about the algorithm which we are going to present. We adopt here the choice to proceed in a geometric way as clear and straightforward as it is possible rather than to optimize the computational aspect. This choice allows us an easy implementation, but it may be computationally not a good one: for example, the computational complexity could be a problem for large values of \( k \) (see Remark 3.5). We also point out that the restriction on char \( K = 0 \) (if \( q > 1 \)) can be bypassed with a computational check (see Remark 3.6).

Firstly, let us develop a technical criterion needed in the sequel. We recall the following theorem of Butler:

**Theorem 3.1.** [Bu, Thm. 5.1A] (char \( K = 0 \) or \( q \leq 1 \)). Let \( E \) be a vector bundle over a smooth projective curve \( C \) of genus \( q \), \( p : E \to C \) the projection, and let \( X = \mathbb{P}(E) \). If \( Z \) is a \((-1)\) \( p \)-regular line bundle over \( X \) with \( \mu^{-}(p, Z) > 2q \), then \( Z \) is normally generated.

We will use this result in the form of the following corollary:

**Corollary 3.2.** (char \( K = 0 \) or \( q \leq 1 \)). Let \( E \) be a vector bundle over a smooth projective curve \( C \) of genus \( q \), \( p : E \to C \) the projection, and let \( X = \mathbb{P}(E) \). If \( \mu^{-}(E) > 2q \), the tautological divisor \( \tau \) of \( X \) is very ample and \( X \subset \mathbb{P}(H^0(X, \tau)^*) \) is projectively normal.

*Proof.* It is well known that the condition \( \mu^{-}(E) > 2q \) implies that the tautological divisor \( \tau \) of \( X \) is very ample, cf. Lemma 1.12 of [Bu]. Since \( \tau \) is very ample, it is enough to prove that \( X \) is normally generated in \( \mathbb{P}(H^0(X, \tau)^*) \).
For this we apply Theorem 3.1. We recall that a divisor $Z$ is called $(-1)$ $p$-regular if, for every fibre $F$ of $p$ over $C$, $H^i(F, Z_f(-1 - i)) = 0$, for all $i > 0$. In our case these groups are $H^i(\mathbb{P}^d, \mathcal{O}_p(-i)) = 0$, where $h = \text{rk} E - 1$, hence $\tau$ is automatically $(-1)$ $p$-regular. For the second condition of Theorem 3.1, we remark that $\mu^* (p, \tau) = \mu^* (E)$. □

Now we are ready to give the required criterium.

**Proposition 3.3.** (char $\mathbb{K} = 0$ or $q \leq 1$). Let $E$ be a vector bundle over a (smooth) curve $C$ of genus $q$ and let $D$ be an effective divisor of degree $d$ on $C$. If the condition
\begin{equation}
\mu^* (E) + d > 2q
\end{equation}
is satisfied, the divisor $C_0 + p^*(D)$ is very ample on $X = \mathbb{P}(E)$ and the image $X'$ of $X$, given by the linear system $|C_0 + p^*(D)|$, is projectively normal.

**Proof.** Just apply Corollary 3.2 to $E' := E \otimes \mathcal{O}_C(D)$ and recall that $\mu^* (E') = \mu^* (E) + d$. □

**Remark 3.4.** Condition (3.1) is not an evident numerical condition, since it is not clear how to compute $\mu^* (E)$ for a given vector bundle $E$.

However, if the genus $q$ of the curve $C$ satisfies $q \geq 2$, the set of points in $\text{Ext}^1(L, \mathcal{O}_C)$ parametrizing a semi-stable vector bundle $E$ is a Zariski open set, see the classical [N-S, Thm. 2]. Hence for a general choice of such an extension the corresponding $E$ is semistable and $\mu^* (E) = \mu (E) = \frac{-e}{2}$.

For the case $q = 1$, it is known that if $E$ is indecomposable then $E$ is semi-stable. If instead $E$ is decomposable, say $E = L \oplus L'$, then $\mu^* (E) = \min(\text{deg}(L), \text{deg}(L'))$. By [A-B-B-3, Lemma 2.8]. In the case $q = 0$, $E$ is necessarily of the type $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ and $\mu^* (E) = \min(a, b)$.

**Proof of Thm. B.** Chose an effective divisor $D$ on $C$ of degree $d$ such that $D - B$ is effective, $[L \otimes \mathcal{O}_C(D)] \neq \emptyset$, and satisfying conditions (1.3) and (3.1). Note that this is always possible by choosing the degree of $D$ big enough. By Proposition (3.3), the divisor $C_0 + p^*(D)$ on the surface $X = \mathbb{P}(E)$ is very ample. Let $X' \subset \mathbb{P}^r$ be the image of the embedding $i: X = \mathbb{P}(E) \hookrightarrow \mathbb{P}^r$ given by $[C_0 + p^*(D)]$, where $r = h^0(C_0 + p^*(D)) - 1$. By applying Theorem A (with the same choice of $D$), we can obtain a set of generators of the ideal $I_{X'}$ of $X' \subset \mathbb{P}^r$.

Let $R$ be a section of the sheaf $i_* \mathcal{O}_C(kD - B)$ and let $H$ be the hyperplane divisor of $\mathbb{P}^r$. Since $R$ is an effective divisor of $X'$ we have
\[\mathcal{I}_{X'} \rightarrow \mathcal{I}_R \rightarrow \mathcal{I}_{R, X'} \rightarrow 0,\]
where $\mathcal{I}_{R, X'}$ is the relative ideal sheaf of $R$ in $X'$.

Notice that the surface $X'$ is projectively normal by Proposition 3.3. Therefore
\[H^1(\mathbb{P}^r, I_{X'}(kH)) = 0\]
and the above sequence, tensorized with $\mathcal{O}_{\mathbb{P}^r}(kH)$, gives
\[0 \rightarrow H^0(\mathbb{P}^r, I_{X'}(kH)) \rightarrow H^0(\mathbb{P}^r, I_R(kH)) \rightarrow H^0(\mathbb{P}^r, I_{R, X'}(kH)) \rightarrow 0.\]

Since $I_{R, X'}(kH) = \mathcal{O}_{X'}(kH - R) \cong \mathcal{O}_X(kC_0 + p^*(B)) = \mathcal{O}_X(A)$, we have
\[H^0(X, A) \cong \frac{H^0(\mathbb{P}^r, I_R(kH))}{H^0(\mathbb{P}^r, I_{X'}(kH))}.\]

Let $f_0, \ldots, f_n$ be a set of polynomials in $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k))$ whose equivalence classes form a basis of this quotient space ($n = h^0(X, A) - 1$). The image of $X$ under the linear system $|A|$ is then given in the following way: if $y_0, \ldots, y_n$ are indeterminates and $V$ is the coordinate ring of $X' \subset \mathbb{P}^r$, the ideal $I_X$ is the kernel of the map $K[y_0, \ldots, y_n] \rightarrow V$ obtained by sending $y_i$ to $[f_i]$, the class of $f_i$ in $V$.

The algorithm based on the above considerations is therefore the following:
(1) choose on \( C \subset \mathbb{P}^m \) an effective divisor \( D \) satisfying conditions (1.3) and (3.1) and such that \( D - B \) is effective, \( |L \otimes \mathcal{O}_C(D)| \neq \emptyset; \\
(2) choose a divisor \( D' \in |L \otimes \mathcal{O}_C(D)| \) and, from the ideal of \( C \) and from \( L \), compute \( S \) and a set of generators of the ideals \( I_D \) and \( I_{D'} \) (as \( S \)-modules); \\
(3) identify the extension in \( \text{Ext}^1_L(I_D, I_D) \) corresponding to the given extension of \( E \) as an element in \( \text{Ext}^1(L, \mathcal{O}_C) \cong \text{Ext}^1(L \otimes \mathcal{O}_C(D), \mathcal{O}_C \otimes \mathcal{O}_C(D)) \) and compute a presentation of the module \( N = H^0(C, E \otimes \mathcal{O}_C(D)) \) as explained in Lemma 1.2; \\
(4) proceed as explained in Lemma 1.3 to compute a set of generators of \( I'_X \), where \( X' \) is the embedding of \( \mathbb{P}(E) \) by \( |C_0 + p^*(D)|; \\
(5) for each point \( P' \) (with multiplicity) of a fixed element of \( |kD - B| \), compute the ideal of the fiber of \( X' \) over such point. This ideal can be computed in the same way used to compute \( I'_X \), as explained in Lemma 1.3, but adding to the relations, besides the ones coming from the presentation of \( N \), also the generators of the ideal of the point \( P' \) (with multiplicity); \\
(6) the intersection of all the ideals computed in the previous step gives the ideal \( I_R \) of the divisor \( R \in \{ i_*p^* \mathcal{O}_C(kD - B) \}; \\
(7) compute a set of homogeneous polynomials of degree \( k \) \( f_0, \ldots, f_n \) whose equivalence classes form a basis of \( H^0(\mathbb{P}^r, \mathcal{I}_R(kH))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(kH)); \\
(8) compute the ideal of \( X \) as the kernel of the map \( \mathbb{K}[y_0, \ldots, y_n] \rightarrow S \) which sends \( y_i \) to the equivalence class of \( f_i \) in the coordinate ring \( S \) of \( X' \). \( \square \)

Remark 3.5. In the previous algorithm, a difficult point is to compute the system of hypersurfaces of degree \( k \) in \( \mathbb{P}^r \) through \( \text{deg}(kD - B) \) lines of the scroll \( X' \). We do not know how hard is this task computationally when \( k \) or the degree of \( D \) increase.

Remark 3.6. The assumption \( \text{char} \, \mathbb{K} = 0 \) (if \( q > 1 \)) of Theorem B can be replaced by a computational check.

Proof. This assumption ensures that the scroll \( X' \), corresponding to a choice of \( D \) as in the proof of Theorem B, is \( k \)-normal. Therefore the assumption is not required if \( X' \) is \( k \)-normal.

Once the auxiliary divisor \( D \) is chosen and the resulting surface \( X' \) is determined, \( k \)-normality of \( X' \) can be computationally checked. If \( X' \) is not \( k \)-normal, change \( D \) and repeat the check. \( \square \)

Remark 3.7. Even without the very ampleness assumption on \( A = kC_0 + p^*B \), the algorithm of Theorem B still holds for computing the ideal of the image of \( X \) by the rational map \( \phi_{|A|} \) associated to \( |A| \) (even for \( k = 1 \)).

Proof. Perform steps (1)-(7) of the algorithm described in Theorem B and compute a set of homogeneous polynomials \( f_0, \ldots, f_n \) of degree \( k \) whose equivalence classes form a basis of \( H^0(X, A) \cong H^0(\mathbb{P}^r, \mathcal{I}_R(kH))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(kH)); \) on \( X' \) the map associated to \( |A| \) corresponds to the rational map \( (f_0 : \ldots : f_n) \) and step (8) computes the ideal of the image of \( X \) by \( \phi_{|A|}. \) \( \square \)

Remark 3.8. Given a divisor \( A = kC_0 + p^*B \), it is possible to decide computationally whether this divisor is very ample.

Proof. As explained in the above remark, apply the algorithm of Theorem B and compute the image of \( X \), say \( Y \), by the rational map \( \phi_{|A|} \) associated to \( |A| \). The point is to determine computationally whether \( \phi_{|A|} \) is an embedding.

This can be done in the following way: \( \phi_{|A|} \) is an embedding if and only if \( Y \) is a smooth surface of degree \( A^2 \) and \( |A| \) has empty base locus. Indeed, assume that \( Y \) is smooth of degree \( A^2 \) and that \( \phi_{|A|} \) is a morphism. Then \( \phi_{|A|} \) is a birational
morphism and, since $X$ is a minimal surface being geometrically ruled, a posteriori $\phi|_A$ is an isomorphism (the case $X = \mathbb{F}_{1,0}$ has to be considered separately).

These conditions can easily be checked computationally: the base locus of $A$ is determined by the vanishing set of $(f_0, \ldots, f_n)$ on $X'$, while the dimension, smoothness, and degree of $Y$ are directly computable from its ideal. \hfill $\Box$

4. Examples of ruled surfaces with higher degree fibers

4.1. An example of conic bundles. Let $C$ be a smooth curve of genus 1. Let $E$ be a normalized rank 2 vector bundle of degree 1 which is given by the only non trivial extension:

$$ 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(P) \to 0 $$

where $\det(E) = c_1(E) = \mathcal{O}_C(P)$ and $P$ is a fixed point of $C$. Let $Q$ be any other point of $C$, eventually $Q = P$. It is known that, on the surface $X = \mathbb{P}(E)$, the divisor $A = 2C_0 + p^*Q \equiv 2C_0 + f$ is very ample, whatever $Q$ is chosen, $h^0(X, A) = h^0(C, S^2(E) \otimes \mathcal{O}_C(Q)) = 6$ and $A$ embeds $X$ in $\mathbb{P}^3$ as a smooth ruled surface of degree 8, whose fibres are embedded as smooth plane conics (see [I-1]); $g(X) = 3$. In this case, $C_0$ is embedded as a smooth plane cubic and $h^0(C, C_0) = 1$.

We now apply the algorithm of Theorem B to compute a set of generators of the ideal $I_X$. We set $B = Q$ and we can choose $D = Q + Q'$, where $Q'$ is a further point, so that $D - B$ is effective and $2D - B = Q + 2Q'$.

We fix a choice of a smooth plane cubic curve $C \subset \mathbb{P}^2$, named $C$, and of three points $p$, $q$, $q'$, which correspond respectively to the three points $P, Q, Q'$ on $C$.

Let $S$ be coordinate ring of $C$. The necessary steps to compute a presentation of the $S$-module $M = H^0_2(C, E \otimes \mathcal{O}_C(D))$ are as usual:

```
K=QQ;
R=K[x_0..x_2]
C=ideal (x_0*(x_2)^2-x_1*(x_1+x_0)*(x_1+2*x_0))
p=ideal (x_1,x_2);q=ideal (x_1,x_0);q'=ideal (x_1+x_0,x_2);
Ddual=intersect(p,Ddual);
D'sdual=intersect(p,Ddual);
S=R/C
D'Sdual=Hom(D'Sdual,S);
D'S=Hom(D'Sdual,S);
M=randomExt(D'Sdual,S);
```

We then compute the ideal $J$ of $X' \subset \mathbb{P}^4$, the embedding of $\mathbb{P}(E)$ by the linear system $|C_0 + p^*(D)|$ as explained in section 2:

```
J=scrollIdeal(M) --ideal of X'
```

$X'$ is a smooth surface (of degree 5), and $\mathcal{I}_{X'}$ has free resolution:

$$ 0 \to \mathcal{I}_{X'} \to 5\mathcal{O}(-3) \to 5\mathcal{O}(-4) \to \mathcal{O}(-5) \to 0. $$

The next step is to compute $I_R$, where $R$ is the pullback on $X'$ of the divisor $2D - B = Q + 2Q' \in \text{Div}(C)$.

```
quFiber=intersect(q'squareFiber,qFiber);
```

We now perform the step (7) of the algorithm in the proof of Theorem B, i.e., we compute a set of homogeneous polynomials of degree 2 $f_0, \ldots, f_n$ whose equivalence classes form a basis of the quotient space $H^0(\mathbb{P}^r, \mathcal{I}_R(2H))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(2H))$. The resolution of $I_X'$ shows that $X'$ is not contained in any quadric. Therefore, in this
case, the vector space $H^0(\mathbb{P}^r, \mathcal{I}_R(2H))/ H^0(\mathbb{P}^r, \mathcal{I}_X(2H))$ is just $H^0(\mathbb{P}^4, \mathcal{I}_R(2H))$, the space of quadrics of $\mathbb{P}^4$ passing through the fibers over $Q + 2Q'$:

\[ Q = \text{super basis}(2, A) -- \text{the linear system } |2H-2D+B| \]

Finally, we compute $X$ as the image of $X'$ via the embedding given by the linear system of quadrics $Q$ obtained above:

\[ \mathbb{Z}[z_0 \ldots z_5]; S' = \mathbb{X}/J; f = \text{map}(S', \mathbb{Z}, \text{substitute}(Q, S')) \]

The ideal $I$ will be the ideal of the surface $X$. It is straightforward to check that $X \subset \mathbb{P}^5$ has degree 8 and $\mathcal{I}_X$ has resolution:

\[ 0 \to \mathcal{I}_X \to \mathcal{O}(-2) \oplus 8\mathcal{O}(-3) \to 15\mathcal{O}(-4) \to 8\mathcal{O}(-5) \to \mathcal{O}(-6) \to 0. \]

The Hilbert function of $X$ is given by:

\[ h^0(X, \mathcal{O}_X(t)) = 1, 6, 20, 42, 72, 110, 156, 210, 272, 342 \ldots \text{ for } t = 0, 1, 2 \ldots \]

4.2. **A ruled surfaces with cubic fibres.** Let $C$ be a smooth curve of genus 1. Let $E$ be a normalized rank 2 vector bundle of degree 1 which is given by the only non trivial extension:

\[ 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(P) \to 0 \]

where $\det(E) = c_1(E) = \mathcal{O}_C(P)$ and $P$ is a fixed point of $C$. On the surface $X = \mathbb{P}(E)$, the divisor $A = 3C_0$ is very ample: the numerical criterion of Reider is satisfied (cf. [RI]). $h^0(X, A) = h^0(C, S^3(E)) = 6$ and $A$ embeds $X$ in $\mathbb{P}^5$ as a smooth ruled surface of degree 9, whose fibers are embedded as twisted cubics (see [I-1]);

\[ g(X) = 4. \text{ In this case, } C_0 \text{ is embedded as a smooth plane cubic as } h^0(X, 2C_0) = 3. \]

We have $B = 0$ and we can choose $D = Q + Q'$, where $Q, Q'$ are any couple of points. The divisor $3D - B$ is then $3D = 3Q + 3Q'$.

We fix a choice of a smooth plane cubic curve $C \subset \mathbb{P}^2$, named $C$, and of three points $p, q, q'$, which correspond respectively to the three points $P, Q, Q'$ on $C$. Let $S$ be coordinate ring of $C$. The necessary steps to compute a presentation of the $S$-module $M = H^0(C, E \otimes \mathcal{O}_C(D))$ are as usual. For shortness, we perform the same choices as in (4.1) and we start with the ideal $J$.

We now compute the ideal $I_R$, where $R$ is the pullback on $X'$ of the divisor $3D - B = 3Q + 3Q' \in \text{Div}(C)$:

```
qcubefiber=pullbackIdeal(q^3)
q=cubefiber=pullbackIdeal(q''^3)
A=intersect(qcubefiber,q''cubefiber); betti A
o42 = generators: total: 1 11
   0: 1
   1: .
   2: . 11
```

Since the ideal $J$ of $X'$ contains a five dimensional space of cubics, we need to find a set of representatives cubics $Q$ for a basis of the quotient space $H^0(\mathbb{P}^r, \mathcal{I}_R(3H))/ H^0(\mathbb{P}^r, \mathcal{I}_X(3H))$:

\[ a3=\text{super basis}(3, A)
 j3=\text{super basis}(3, J)
 Q=\text{super basis}(3, \text{ideal a3/ideal j3})
 Q=\text{matrix}(T, \text{entries } Q) \]

Finally, we compute $X$ as the image of $X'$ via the embedding given by the linear system of cubics $Q$ obtained above:

\[ \mathbb{Z}[z_0 \ldots z_5]; S' = \mathbb{X}/J; f = \text{map}(S', \mathbb{Z}, \text{substitute}(Q, S')) \]

I=ker f --ideal of X
The projective normality of the previous surface is proved in the rather
H
S
X
| D
X
z
by the above assumptions, we have

\[ h^0(X, \mathcal{O}_X(t)) = 1, 6, 21, 45, 78, 120, 171, 231, 300, 378, 465 \ldots \] for \( t = 0, 1, 2 \ldots \)

Remark 4.1. The projective normality of the previous surface is proved in the rather long Proposition 4.6 of [B-DR]. As a preparatory lemma, in [B-DR] the authors proved that the projective normality of \( X \) is equivalent to the 2-normality of \( X \), which is equivalent to the fact that \( X \) does not lie on any quadric (cf. Lemma 4.3). This property can be immediately verified for any constructed example, by looking at the resolution of \( I_X \).

5. Some Varieties Related to Conic Bundles

5.1. Let \( D \) be a degree \( d \) effective divisor of \( C \) satisfying (3.1) and \( E' = E \otimes \mathcal{O}_C(D) \).

We want to describe shortly some varieties related to \( \mathbb{P}(E') \) and their geometric correlations.

Let us define \( E_1 := S^2(E') \) and let us consider the image \( X_1 \) of the 3-fold \( \mathbb{P}(E_1) \) in \( \mathbb{P}^r := \mathbb{P}^N \) via the linear system given by the tautological divisor \( T_1 \) of \( E_1 \): as \( D \) satisfies (3.1), it is easy to see that \( S^2(E') \) is very ample by Lemma 1.12 of [Bu], because \( \mu^-(S^2(E')) = 2 \mu^-(E') > 4q \).

On the other side, let \( X' \subset \mathbb{P}^r \) be the image of \( \mathbb{P}(E') \) embedded via its tautological bundle, i.e. via \( |C_0 + p^*(D)| \), which is very ample because \( D \) satisfies (3.1). Let \( X'' \) be the image of \( X' \) under the 2-Veronese embedding \( \nu: \mathbb{P}^r \hookrightarrow \mathbb{P}^N \), where \( N = \binom{r + 1}{2} - 1 \), i.e. the image of \( X' \) under the composition \( \nu \circ \iota \) where \( \iota \) is the embedding of \( X \) in \( \mathbb{P}^r \). Then \( X'' \) is the image of \( X \) via the map associated to the linear system \( |2C_0 + p^*(2D)| \). Algebrically, the map is given as follows. Let \( y_1, \ldots, y_r \) be a basis of \( \mathcal{H}^0(C, E') \); then \( y_0^2, y_0 y_1, \ldots, y_r^2 \) is a basis of \( S^2(\mathcal{H}^0(C, E')) \) and, defining \( z_{i,j} \) for \( i \leq j \) as a set of coordinates for \( \mathbb{P}^N \), the composition \( \nu \circ \iota \) is given by mapping \( z_{i,j} \) to the product \( y_i y_j \), considered as an element in \( \mathcal{O}_{\mathbb{P}(E')}(2) \).

We want to describe a method to compute a set of generators of the ideal of the image \( X_1 \) of \( \mathbb{P}(E_1) \) and also of the ideal of the image \( X'' \) of \( \mathbb{P}(E) \). Notice that \( X'' \) is a surface contained in \( X_1 \). Indeed, the projective normality of \( X'' \) (see the proof of Thm. B) implies the exactness of the sequence

\[
\begin{align*}
0 & \to \mathcal{H}^0(\mathbb{P}^r, \mathcal{I}_{X'}(2)) \to \mathcal{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \to \mathcal{H}^0(X', \mathcal{O}_{X'}(2)) \to 0,
\end{align*}
\]

where \( \mathcal{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \cong S^2(\mathcal{H}^0(C, E')) \) and \( \mathcal{H}^0(X', \mathcal{O}_{X'}(2)) \cong \mathcal{H}^0(C, S^2(E')) \); hence \( S^2(\mathcal{H}^0(C, E')) \) surjects to \( \mathcal{H}^0(C, S^2(E')) \).

For sake of simplicity, let us now assume that \( C \subset \mathbb{P}^m \) satisfies

\[
de\deg(C) \geq 2g + 1,
\]

so that \( C \) is projectively normal too and its coordinate ring is \( S = \oplus_{t \geq 0} \mathcal{H}^0(C, \mathcal{O}_C(t)) \).

Then \( \mathcal{H}^0(C, E') \) is generated by \( \mathcal{H}^0(C, E') \), as graded module over \( S \). In fact, by the above assumptions, we have \( \deg(\mathcal{O}_C(t)) \geq 2q \) for \( t \geq 1 \) and \( \mu^-(E') > 2q \), and the surjectivity of the natural map \( \mathcal{H}^0(C, E') \otimes \mathcal{H}^0(C, \mathcal{O}_C(t)) \to \mathcal{H}^0(C, E'(t)) \) is an application of Theorem 2.1 of [Bu]. In the same way \( \mathcal{H}^0(C, S^2(E')) \) generates the module \( \mathcal{H}^0_n(C, S^2(E')) \).

We now describe how to determine the equations of \( X_1 \). As first step, notice that \( S^2(\mathcal{H}^0(C, E')) \) surjects to \( \mathcal{H}^0(C, S^2(E')) \). Indeed, since \( S^2(\mathcal{H}^0(C, E')) \) surjects to \( \mathcal{H}^0(C, S^2(E')) \) and \( \mathcal{H}^0(C, S^2(E')) \) generates \( \mathcal{H}^0_n(C, S^2(E')) \) by (5.2), the images of \( y_0^2, y_0 y_1, \ldots, y_r^2 \) in \( \mathcal{H}^0_n(C, S^2(E')) \) is a set of generators of this module.
Let the following one be a free presentation of the $S$-module $H^0_0(C, E')$:
\begin{equation}
M_2 \rightarrow M_1 \rightarrow H^0_0(C, E') \rightarrow 0.
\end{equation}
Then $S^2(H^0_0(C, E'))$ has a free presentation
\begin{equation}
(M_2 \otimes M_1) \xrightarrow{\phi} S^2M_1 \rightarrow S^2(H^0_0(C, E')) \rightarrow 0,
\end{equation}
where, according to the notation introduced at the beginning of this subsection, $S^2M_1$ is the free $S$-module with generators $z_{i,j}$ for $i \leq j$ and the map $S^2M_1 \rightarrow S^2(H^0_0(C, E'))$ is determined by sending $z_{i,j}$ to the element corresponding to $y_iy_j$ in $S^2(H^0_0(C, E'))$.

We now compute the polynomials in $\mathbb{K}[z_{i,j}]$ which are zero in $S(H^0_0(C, S^2E'))$. Let $N$ be the set of elements $\sum \alpha_{i,j}z_{i,j} \in S^2M_1$ ($\alpha_{i,j} \in \mathbb{K}$) such that $\sum \alpha_{i,j}y_iy_j \in H^0(\mathbb{P}^r, I_{X'}(2))$. By (5.1), the symmetric algebra $S(H^0_0(C, S^2E'))$ is isomorphic to the symmetric algebra $S(V)$, where $V$ is the vector space $S^2(H^0_0(C, E'))/N$. Hence a polynomial $g(z_{i,j}) \in S(S^2M_1)$ whose image is nonzero in $S(S^2(H^0_0(C, E')))$ is zero in $S(H^0_0(C, S^2E'))$ if and only if $g$ is in the ideal generated by $N$.

A basis of $N$ over $\mathbb{K}$ can be computed in the following way. By following our algorithm we compute $I_{X'}$ and a basis of $H^0(\mathbb{P}^r, I_{X'}(2))$, and for each element $\sum \alpha_{i,j}y_iy_j$ (written with $i \leq j$) in this basis we consider the corresponding element $\sum \alpha_{i,j}z_{i,j} \in S^2M_1$: the set of all these elements forms a basis of $N$.

We are therefore able to get a set of generators of the ideal of $X_1 \subset \mathbb{P}^N$ in the following way. In the ring $\mathbb{K}[z_{i,j}, x_0, \ldots, x_m]$ we multiply the matrix associated to the map $\phi$ in (5.4) with a row vector given by the variables $z_{i,j}$ (by considering their exact order), and we consider the ideal generated by these elements and by the linear forms in $N$. Then we saturate this ideal with respect to the irrelevant ideal $(x_0, \ldots, x_m)$ and we consider the intersection of this ideal with the subring $\mathbb{K}[z_{i,j}]$. Note that $X_1$ is degenerate if $H^0(\mathbb{P}^r, I_{X'}(2)) \neq 0$: $X_1$ lies in the $\mathbb{P}^s \subset \mathbb{P}^N$ given by the linear equations coming from $N$, obtained from $H^0(\mathbb{P}^r, I_{X'}(2))$.

The ideal of the surface $X''$ can be obtained by adding to the ideal of $X_1$ the forms of degree 2 (in the $z_{i,j}$) lying in the kernel of the map
\[ S^2(S^2(H^0_0(C, E')))) \rightarrow S^4(H^0_0(C, E')). \]
Indeed, these forms generate the ideal of the Veronese image of $\mathbb{P}^r$ in $\mathbb{P}^N$.

In terms of sheaves on $C$, we have the following exact sequence:
\[ 0 \rightarrow \mathcal{L} \rightarrow S^2(S^2(E')) \rightarrow S^4(E') \rightarrow 0, \]
where $\mathcal{L}$ is a line bundle, being $E'$ of rank 2. The quadratic forms obtained above are global sections of $\mathcal{L}$. We call these quadrics the relative Veronese quadrics, since they give fiberwise the ideal of the Veronese embedding $\mathbb{P}^1 \rightarrow \mathbb{P}^2$.

Remark 5.1. If $\deg(C) \leq 2q$ and $C \subset \mathbb{P}^n$ is not projectively normal, then we can still find explicit equations of $X_1$, by arguing as already done in the proof of Lemma 1.3. Indeed, instead of the presentation (5.3), we can take a presentation of the submodule $M' \subset H^0_0(C, E')$ generated by $H^0_0(C, E')$ and the corresponding presentation $\phi$ of the submodule generated by $S^2(H^0_0(C, E'))$ in $S^2(H^0_0(C, E'))$. Since $X'$ is projectively normal, $S^2(H^0_0(C, E'))$ surjects to $H^0_0(C, S^2E')$ and we can proceed as above.

5.2. An alternative algorithm for conic bundles. From the short exact sequence $0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow \mathcal{L} \rightarrow 0$ one can derive two other short exact sequences where $E_1$ fits, namely:
\begin{equation}
0 \rightarrow E \otimes \mathcal{O}_C(2D) \rightarrow E_1 \rightarrow L^2 \otimes \mathcal{O}_C(2D) \rightarrow 0
\end{equation}
and
\begin{equation}
0 \to \mathcal{O}_C(2D) \to E_1 \to E \otimes L \otimes \mathcal{O}_C(2D) \to 0.
\end{equation}

In fact, we have \(0 \to E \otimes \mathcal{O}_C \to S^2(E) \to S^2(L) \to 0\), which can be rewritten as \(0 \to E \to S^2(E) \to L^2 \to 0\) and from this sequence we can proceed in two ways: either we tensorize with \(\mathcal{O}_C(2D)\) and we get (5.5), or we dualize it and we get \(0 \to L^{-2} \to S^2(E^*) \to E^* \to 0\), where \(L^{-i}\) denotes the \(i\)-th tensor power of \(L^*\). Since \(E^* \cong E \otimes L^*\), we obtain (5.6) by tensorizing with \(L^2 \otimes \mathcal{O}_C(2D)\).

Here we want to use (5.6). If \(\mu^-(E) + 2d > 2q - e\), then by Lemma 1.12 of [Bu] \(E \otimes L \otimes \mathcal{O}_C(2D)\) is very ample and its tautological bundle \(\pi\) defines an embedding \(\Sigma\) of \(\mathbb{P}(E)\). The sequence (5.6) implies both \(T_{1|\Sigma} = \tau\) and \(\Sigma \in [T_1 - p_1^*(2D)]\), where \(p_1\) is the projection of \(X_1\) to \(C\) (\(X_1\) is the image of \(\mathbb{P}(E)\)) by the tautological divisor \(T_1\) of \(E_1\), see the beginning of subsection 5.1. If furthermore \(d > q - 1\), then \(h^1(C, 2D) = 0\) and \(\Sigma\) is linearly normal in \(X_1\).

We can also proceed by using \(\Sigma\) to compute a set of generators of the ideal \(I_X\), where, as usual, \(X\) denotes the image of \(\mathbb{P}(E)\) via the map associated to the linear system \(|A|\) in Theorem B for the case \(k = 2\). In this case, further conditions on \(D\) are needed. Let us suppose that \(D\) satisfy, besides the (already required) conditions \(\mu^-(E) + d > 2q - e\) and \(d > q - 1\), the further condition
\begin{equation}
\mathcal{H}^0(C, L^2 \otimes \mathcal{O}_C(4D - B)) \neq 0.
\end{equation}

Consider a divisor \(D' \in |L \otimes \mathcal{O}_C(2D)|\), so that \(\tau = C_0 + p^*(D')\). Then the sheaf \(\mathcal{O}_{X_1}(2T_1 - p_1^*(2D' - B))\) on \(X_1\) restricts to \(\Sigma\) to the sheaf
\(\mathcal{O}_{\Sigma}(2\tau - p^*(2D' - B)) \cong \mathcal{O}_X(2(C_0 + p^*(D') - p^*(2D' - B)) \cong \mathcal{O}_X(2C_0 + p^*B) = \mathcal{O}_X(A)\).

Since \(\mu^-(E_1) = 2\mu^-(E) + 2d > 4q\), \(X_1\) is projectively normal by Corollary 3.2. Hence, if
\begin{equation}
|2T_1 - p_1^*(2D' - B)| \to |2\tau - p^*(2D' - B)|
\end{equation}
is surjective, we can proceed as in the proof of Theorem B: we take an effective divisor \(R_1 \in |p_1^*(2D' - B)|\) and we have
\(\mathcal{H}^0(X, \mathcal{O}_X(A)) \cong \frac{\mathcal{H}^0(\mathbb{P}^n, \mathcal{I}_{R_1}(2H_1))}{\mathcal{H}^0(\Sigma, \mathcal{I}_{\Sigma}(2H_1))}\),
where \(H_1\) denotes a hyperplane of \(\mathbb{P}^n \subset \mathbb{P}^N\), and again it is straightforward to compute a set of generators of \(I_X\).

Assumption (5.8) can be translated into numerical conditions:

**Proposition 5.2.** The restriction map \(|2T_1 - p_1^*(2D' - B)| \to |2\tau - p^*(2D' - B)|\) is surjective if one of the following conditions are satisfied:
\begin{equation}
\begin{cases}
\begin{aligned}
h^1(\mathcal{O}_C(B)) &= h^1(\mathcal{O}_C(D + B) \otimes L^{-2}) = h^1(\mathcal{O}_C(D + B) \otimes L^{-1}) = 0, \\
2\mu^-(E) + \deg(B) > 2q - 2 + 2e
\end{aligned}
\end{cases}
\end{equation}
where \(L^{-1}\) denotes the \(i\)-th tensor power of \(L^*\).

**Proof.** Recall that \(\Sigma \in |T_1 - p_1^*(\mathcal{O}_C(2D))|\). Therefore \(H^1(X_1, \mathcal{I}_E(2T_1 - p_1^*(2D' - B)) = H^1(X_1, \mathcal{O}_E(T_1 - p_1^*(\mathcal{O}_C(2D - B) \otimes L^2))) \cong H^1(C, E_1 \otimes \mathcal{O}_C(B - 2D) \otimes L^{-2})\) and the surjectivity follows from the vanishing of \(H^1(C, E_1 \otimes \mathcal{O}_C(B - 2D) \otimes L^{-2})\).

By tensorizing the exact sequences (5.5) and (5.6) with \(\mathcal{O}_C(B - 2D) \otimes L^{-2}\) and by using suitable tensorizations of the sequence \(0 \to \mathcal{O}_C \to E \to L \to 0\) we get the first two conditions.

Alternatively, one can again work with \(\mu^-(E)\). By Lemma 1.12 of [Bu], it is enough that \(\mu^-(E_1 \otimes \mathcal{O}_C(B - 2D) \otimes L^{-2}) > 2q - 2\), i.e., the third condition. □
6. Some new embeddings for ruled surfaces

In previous sections we considered some very ample line bundles $A$ over some geometrically ruled surfaces $X = \mathbb{P}(E)$ and we used our algorithm to compute a set of generators of the ideals of the embeddings of $X$ by $|A|$. In this last section we consider, on some ruled surfaces $X$, some linear systems $|L|$ whose very ampleness is unknown and we investigate their very ampleness by using our algorithm. In this way, when $|L|$ is very ample, we get some new embeddings of $X$.

Let us consider the list of ruled surfaces presented in §3 of [Li]. There the author uses Reider’s method to investigate the very ampleness of some linear systems on ruled surfaces $X$ which could give rise to embedded surfaces with low sectional genus: namely, $g(X) \leq 7$. Obviously Reider’s method does not work in every case and there are many candidate pairs $(X, L)$ such that the very ampleness of $|L|$ is not proved by the author. These are the pairs in column $D$ of the list in §3 of [Li].

When $g(X) = 7$ some of the pairs were excluded in [B-L] (in particular, the case number (12)), but, up to now, there is still a lot of open cases. Here we consider only some of them: cases (9), (10), (11) and the two ones in (13), to show that our algorithm can be used to prove very ampleness of linear systems. The part of the list in §3 of [Li] describing these cases is the following:

|   |   |   |   |   |
|---|---|---|---|---|
| $g$ | $d$ | $h^0(|L|)$ | $q$ | $|L|$ | $e$ | $t$ |
| (9) | 7 | 18, ..., 10 | 12, ..., 6 | 1 | $3C_0 + 3f - \sum^t_{i=1} p_i$ | 0 | $1 \leq t \leq 8$ |
| (10) | 7 | 16, ..., 10 | 10, ..., 6 | 1 | $4C_0 - \sum^t_{i=1} p_i$ | $-1$ | $2 \leq t \leq 6$ |
| (11) | 7 | 15, ..., 10 | 9, ..., 6 | 1 | $5C_0 - \sum^t_{i=1} p_i$ | $-1$ | $1 \leq t \leq 5$ |
| (13) | 7 | 16, ..., 10 | 9, ..., 6 | 2 | $2C_0 + 2f - \sum^t_{i=1} p_i$ | $-2$ | $2 \leq t \leq 6$ |
| (13') | 7 | 16, ..., 10 | 9, ..., 6 | 2 | $2C_0 + 3f - \sum^t_{i=1} p_i$ | $-1$ | $1 \leq t \leq 6$ |

Let $E$ be any rank 2 vector bundle over a smooth curve $C$ of genus $q$ with $\deg(E) = \deg(L) = -e$. Let $X$ be $\mathbb{P}(E)$ as usual, and let us consider any divisor $L_0 \equiv ac_0 + bf$ obtained from the previous table by omitting $\sum^t_{i=1} p_i$. In [Li] it is proved that $L_0$ is always very ample and embeds $X$ as a surface of degree equal to the maximal value of $d$ in the table, with the exception of case (13) for which condition (0.14) of [Li] is requested.

On the other hand, the author is not able to decide if the linear subsystems $|L| = |L_0 - \sum^t_{i=1} p_i|$ are very ample too. Note that this is equivalent to prove that the projections of $X$, embedded by $|L_0|$, from $t$ suitable points $p_1, \ldots, p_t$ of $X$ are smooth for the values of $t$ in the above table. Thanks to our algorithm, we are able to check this smoothness at least for random choices of $X$ and of the points $p_1, \ldots, p_t$. In this way we give many examples of pairs $(X, L)$ as in the previous table where $|L|$ is very ample, proving the existence of new embedded surfaces with sectional genus $g = 7$.

Our strategy will be the following: for any considered case, firstly we give a set of generators of the ideal of a random surface $X$ embedded by $|L_0|$, then we choose $t$ random points on $X$, we determine the ideal of the projected surface $\tilde{X}$ from these points and we check whether $\tilde{X}$ is smooth or not.

**Proposition 6.1.** According to our notation, there exist surfaces $X = \mathbb{P}(E)$ such that, for generic points $p_1, \ldots, p_t$ of $X$, the linear systems $|L|$ in the following table
are very ample:

|   |  g  |  d  | $h^0(\mathcal{L})$ | $|\mathcal{L}|$ | $e$ | $t$ |
|---|---|---|---|---|---|---|
| (9) | 7 | 18 - $t$ | 12 - $t$ | 1 | $3C_0 + 3f - \sum t_i p_i$ | 0 | $1 \leq t \leq 5$ |
| (10) | 7 | 16 - $t$ | 10 - $t$ | 1 | $4C_0 - \sum t_i p_i$ | -1 | $2 \leq t \leq 3$ |
| (11) | 7 | 15 - $t$ | 9 - $t$ | 1 | $5C_0 - \sum t_i p_i$ | -1 | $1 \leq t \leq 2$ |
| (13) | 7 | 16 - $t$ | 9 - $t$ | 2 | $2C_0 + 3f - \sum t_i p_i$ | -2 | $t = 2$ |
| (13') | 7 | 16 - $t$ | 9 - $t$ | 2 | $2C_0 + 3f - \sum t_i p_i$ | -1 | $1 \leq t \leq 2$ |

**Remark 6.2.** At least experimentally, no new smooth surface is expected for values of $t$ higher than the ones in table (6.1). Indeed, the projections of a random pair $(X, L_0)$ in table (6.1) from $t$ random points is singular for these higher values of $t$.

**Proof.** Firstly we need a technique to pick random points on a ruled surface $X$ embedded by $|L_0| = |L + \sum t_i p_i|$. We adopt the following procedure for choosing a random point on $X$. The intersection of a fiber $f$ over a random point of $C$ with a random hyperplane consists of $k$ points. We then compute the primary decomposition of their ideal: if in the primary decomposition there is an ideal corresponding to a single point, i.e., separated from the other $k - 1$ ones, we choose this ideal, otherwise we change the choice of the random hyperplane.

Let $C$ denote the base curve $C$ and let $f$ be the map from the scroll $X'$ to $X$, where $X'$ is the embedding of $|C_0 + p^*(D)|$, (see the proof of Theorem B for the definitions of $D$ and $f$). Then the following function chooses a random point on $X$:

```plaintext
randomPointOnS = (C,f) -> (
  p=randomPoint(C);
  p'=pullbackIdeal(p);
  p''=pullback(p',f);
  isSinglePoint=false;
  Z:=source f;
  while not isSinglePoint do
    hypsection:=p''+ideal random(Z^1,Z^{-1});
    pt:=(decompose hypsection)#0;
    isSinglePoint=(degree pt==1);)
  pt);
```

This procedure works without problems only over finite fields, otherwise we have the problem of separating the points of the intersection of a hyperplane and a fiber.

Therefore, we prove Proposition 6.1 in two steps. The first step is to construct an example of $(X, L_0)$ over a finite field, by using our algorithm. The second one is to ensure the existence of a lift $(X, L)$ of $(X, L_0)$ in characteristic 0; this will be done in lemma 6.3. From now on in this proof, we will omit the overline to denote objects constructed over a finite field, using again the notation introduced so far.

By [Li], we know that $L_0$ is very ample on $X$. We use our algorithm to compute a model of $(X, L_0)$. Examples of case (9) of the table can be obtained as follows:

- fix a generic smooth plane elliptic curve $C$;
- define a rank 2 vector bundle $E$ over $C$ as an extension
  
  \[ 0 \to \mathcal{O}_C \to E \to L \to 0 \]

  with $\deg(E) = \deg(L) = 0$ (note that $\mu^-(E) = 0$);
- to define $L$ choose an effective divisor $B$ of degree 3, put $D = B$ (so that all conditions to use the constructive proof of Thm. B are satisfied) and choose another effective divisor $D'$ of degree 3 so that $L = D' - D$;
– according to our algorithm, construct $H^0(C, E \otimes \mathcal{O}_C(D))$ as an extension in $\text{Ext}^1(I^*_D, I^*_D)$ and compute a set of generators of $I_X$;
– compute (a set of generators of) the ideal of $kD - B = 3D - B = 2D$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^1$.

Then we find $t$ random points on $(X, \mathcal{L}_0)$ by using the procedure explained above and we compute the projection $(X, \mathcal{L})$ of $(X, \mathcal{L}_0)$ from the linear space spanned by the $t$ points.

Finally we check that $(X, \mathcal{L})$ is smooth, by a partial application of the jacobian criterion: if $I$ is the ideal of the surface and $J$ the jacobian matrix of the generators of $I$, we compute just a few random minors of $J$ and verify that the obtained ideal, together with $I$, defines an empty projective variety.

The other cases are similar:

(10) – $C$ is a smooth plane elliptic curve;
– $\deg(E) = \deg(L) = 1$, $L = \mathcal{O}_C(P)$;
– $\deg(B) = 0$, say $B = \mathcal{O}_C$;
– $\mu^-(E) = 1/2$. We choose $D$ effective of degree 2 and set $D' = L + D = P + D$;
– compute the ideal of $4D - B = 4D$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^9$.

(11) – $C$ is a smooth plane elliptic curve;
– $\deg(E) = \deg(L) = 1$, $L = \mathcal{O}_C(P)$
– $\deg(B) = -1$, say $B = -R_1$;
– $\mu^-(E) = 1/2$. We choose $D$ effective of degree 2 and set $D' = L + D$ of degree 3;
– compute the ideal of $5D - B = 5D + R_1$ (11 fibers) and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^8$.

(13) – $C$ is a smooth space curve of genus 2 and degree 5;
– $\deg(E) = \deg(L) = 2$, $L = \mathcal{O}_C(P_1 + P_2)$;
– $\deg(B) = 2$, say $B = R_1 + R_2$;
– $\mu^-(E) = 1$. We choose $D$ effective of degree 4 such that $D - B$ is effective, say $D = R_3 + R_2 + R_4$, and set $D' = L + D$ of degree 6;
– compute the ideal of $2D - B = R_1 + R_2 + 2R_3 + 2R_4$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^8$.

(13′) – $C$ is a smooth space curve of genus 2 and degree 5;
– $\deg(E) = \deg(L) = 1$, $L = \mathcal{O}_C(P)$;
– $\deg(B) = 3$, say $B = R_1 + R_2 + R_3$;
– $\mu^-(E) = 1/2$. We choose $D$ effective of degree 4 such that $D - B$ is effective, say $D = R_1 + R_2 + R_3 + R_4$, and set $D' = L + D$ of degree 5;
– compute the ideal of $2D - B = R_1 + R_2 + R_3 + 2R_4$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^8$.

Lemma 6.3. Consider pairs of surfaces described according to table (6.1). Given an example of smooth pair $(\overline{X}, \overline{\mathcal{L}})$ defined over a finite field as in table (6.1), there exists a smooth pair $(X, \mathcal{L})$ defined in characteristic 0 of the same type as $(\overline{X}, \overline{\mathcal{L}})$.

Proof. Consider the embedded curve $\overline{C} \subset \mathbb{P}^m$ ($m = 2, 3$) used to construct the pair $(\overline{X}, \overline{\mathcal{L}})$.

Claim: the generic curve $\overline{C}$ of genus 1 in $\mathbb{P}^2$ and of genus 2 in $\mathbb{P}^3$ can be obtained as the degeneracy locus of a generic map $\varphi$ of vector bundles defined over $\text{Spec} \, \mathbb{Z}$.

The claim is obvious for $g(\overline{C}) = 1$, where the map $\varphi \in \text{Hom}(\mathcal{O}_{p^2}(-3), \mathcal{O}_{p^3})$ is given by the cubic equation of $\overline{C}$.

For $g(\overline{C}) = 2$, the map $\varphi$ can be chosen in $\text{Hom}(2\mathcal{O}_{p^3}(-4), \mathcal{O}_{p^3}(-2) \oplus 2\mathcal{O}_{p^3}(-3))$.

To show this, we illustrate a script for choosing a genus 2 curve at random as a
space curve of degree 5. We construct \( \overline{C} \) starting from its Hartshorne-Rao module \( H^1(I_{\overline{C}}) \). This is a standard method, (cf. for example [S-T, §1.2]). Consider the exact sequence:

\[ 0 \to I_{\overline{C}} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\overline{C}} \to 0. \]

Suppose that the restriction map \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\overline{C}, \mathcal{O}_{\overline{C}}(m)) \) has maximal rank for each \( m \in \mathbb{Z} \). Then the Hilbert function of the Hartshorne-Rao module \( H^1(I_{\overline{C}}) \) is fixed. Moreover, the vector bundle \( \mathcal{G} \) associated to the first syzygy module of \( I_{\overline{C}} \), determined by the exact sequence

\[ 0 \to I_{\overline{C}} \to \oplus \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{G} \to 0, \]

has only intermediate cohomology \( H^2(\mathcal{G}) = H^1(I_{\overline{C}}) \). In our case \( g(\overline{C}) = 2 \), then \( H^1(I_{\overline{C}}) = 0 \) and thus \( \mathcal{G} \) is a direct sum of line bundles.

Under the further assumption that \( I_{\overline{C}} \) has minimal possible syzygies, i.e. \( I_{\overline{C}} \) has a minimal resolution

\[ 0 \to I_{\overline{C}} \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-3) \to 2\mathcal{O}_{\mathbb{P}^3}(-4) \to 0 \]

so that the Betti numbers are the minimal possible ones, then \( \mathcal{G} = 2\mathcal{O}_{\mathbb{P}^3}(-4) \), \( \overline{C} \) is the degeneracy locus of a map \( \overline{\varphi} \in \text{Hom}(2\mathcal{O}_{\mathbb{P}^3}(-4), \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-3)) \) and a set of generators of \( I_{\overline{C}} \) can be obtained by computing the kernel of \( \overline{\varphi}^t \in \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(3), 2\mathcal{O}_{\mathbb{P}^3}(4)) \).

Therefore, if we consider curves of genus 2 and degree 5 in \( \mathbb{P}^3 \) with “maximal rank” and “minimal syzygies”, their Hilbert scheme has dimension equal to \( \dim \text{Grass}(2, 2h^0(1)) + \dim \text{Grass}(1, 2h^0(2) - 2h^0(1)) - \dim SL(2) = 12 + 11 - 3 = 20 \), where \( h^0(i) \) stands for \( h^0(\mathcal{O}_{\mathbb{P}^3}(i)) \). But this dimension agrees with the one expected by Brill-Noether theory and thus a generic curve can be obtained as described above. This proves the claim.

Now, if in the example \( (X, \mathcal{L}) \) we choose \( \overline{C} \) as the degeneracy locus of a map \( \overline{\varphi} \) of vector bundles defined over Spec \( \mathbb{Z} \), then \( \overline{C} \) lifts to characteristic 0, as wanted.

The next step to get a model of a surface \( (X, \mathcal{L}) \) is the choice of the effective divisors \( \overline{D} \) and \( \overline{D}' \) on \( \overline{C} \) with \( \overline{L} = \mathcal{O}_{\overline{C}}(\overline{D}' - \overline{D}) \) and the choice of a rank 2 vector bundle \( \overline{E} \) in \( \text{Ext}^1(\mathcal{O}_{\overline{C}}(\overline{D}'), \mathcal{O}_{\overline{C}}(\overline{D}) \cong H^1(\mathcal{O}_{\overline{C}}(\overline{D}' - \overline{D}))) \). The divisors \( \overline{D} \) and \( \overline{D}' \) lift on \( C \), and since \( h^1(\mathcal{O}_{\overline{C}}(\overline{D} - \overline{D}')) \) is determined by Riemann-Roch because of degree reasons, the dimension of \( \text{Ext}^1(\mathcal{O}_{\overline{C}}(\overline{D}'), \mathcal{O}_{\overline{C}}(\overline{D})) \) is constant over Spec \( \mathbb{Z} \) and also the bundle \( \overline{E} \) lifts to a bundle \( E \) in characteristic 0. Hence we have proven that the scroll \( \overline{X}' \) lifts to characteristic zero.

Let \( \overline{L}_0 \) be the linear system obtained from \( \overline{L} \) by omitting \( \sum p_i \). To get the embedding \( (\overline{X}, \overline{L}_0) \) we have to project \( \overline{X}' \) (see the proof of Thm. B), and \( (\overline{X}, \overline{L}) \) is a projection of \( (\overline{X}, \overline{L}_0) \), so that both \( (\overline{X}, \overline{L}_0) \) and \( (\overline{X}, \overline{L}) \) lift to characteristic zero.

For the smoothness, simply note that if \( (\overline{X}, \overline{L}) \) is smooth, then a fortiori \( (X, \mathcal{L}) \) is smooth as well. \( \square \)

We show in details the script for case (9) (the other cases are similar). We start by computing the ideal \( J \) of the scroll \( X' \):

\[
\begin{align*}
K=\mathbb{Z}/101 \\
R=\mathbb{K}[x_0..x_2] \\
C=\text{ideal random}(R^{-1}, R^{-3}) \\
\text{codim}(C=\text{ideal jacobian} C) \\
Ddual=\text{randomPoints}(C,3) \\
D'dual=\text{randomPoints}(C,3) \\
S=\text{R/C} \\
DSdual=\text{substitute}(Ddual,S); DS=\text{Hom}(DSdual,S); \\
D'Ddual=\text{substitute}(D'dual,S); D'S=\text{Hom}(D'Ddual,S); \\
M=\text{randomExt}(DS,DS)
\end{align*}
\]
Recall the divisor $R = i_∗ p^∗ O_C(kD − B) = i_∗ p^∗ O_C(2D)$ in the proof of thm. B: we have to compute the cubic polynomials contained in the ideal of $R$ modulo the cubic polynomials contained in the ideal of $X'$:

\[ A = \text{pullbackIdeal}(\text{Ddual}^2); \text{betti}\ A \]

\[ a3 = \text{super basis}(3, A) \quad \text{-- the linear system } 3H-2D \]

\[ j3 = \text{super basis}(3, J); Q3 = \text{super basis}(3, \text{ideal } a3/\text{ideal } j3) \]

\[ Q3 = \text{matrix}(T, \text{entries } Q3); \text{betti } Q3 \quad \text{-- } h0 = 12 \]

The cubics in $Q$ define the embedding of $X$ by $|L_0|$ in $\mathbb{P}^{11}$:

\[ Z = K[z_0 \ldots z_11]; S' = T/J; f = \text{map}(S', Z, \text{substitute}(Q, S')) \]

\[ I = \text{saturate ker } f; \text{dim } I, \text{ degree } I \]

\[ o50 = (3, 18) \]

\[ \text{betti } I \]

\[ o51 = \text{generators: total: 1 38} \]

\[ 0: 1 . \]

\[ 1: . 36 \]

\[ 2: . 2 \]

This surface is the surface $X$ of degree $18$ in $\mathbb{P}^{11}$ corresponding to case (9) embedded by $\mathcal{L}_0$. Now, we project $X$ from 5 random points of $X$ and we control that the projection is smooth:

\[ p1 = \text{randomPointOnS}(C, f); p2 = \text{randomPointOnS}(C, f); p3 = \text{randomPointOnS}(C, f); \]

\[ p4 = \text{randomPointOnS}(C, f); p5 = \text{randomPointOnS}(C, f) \]

\[ E = \text{intersect}(p1, p2, p3, p4, p5); I = \text{super basis}(1, E) \]

\[ W = K[x_0 \ldots x_6]; Z' = Z/I; g = \text{map}(Z', W, \text{substitute}(L, Z')) \]

\[ I' = \text{saturate ker } g; \text{dim } I', \text{ degree } I' \]

\[ o83 = (3, 13) \]

\[ \text{betti } I' \]

\[ o116 = \text{generators: total: 1 18} \]

\[ 0: 1 . \]

\[ 1: . 1 \]

\[ 2: . 17 \]

\[ \text{codim } (SI' = \text{FewSmooth}(I', 1, 10)) = 7 \]

\[ o117 = \text{true} \]

Here, the function $\text{FewSmooth}(I, a, n)$ computes $n$ minors from the first $a$ rows of the corresponding jacobian matrix and $n$ minors from the remaining rows.

**References**

[A-B-B-1] A. Alzati, M. Bertolini, and G. M. Besana. Projective normality of varieties of small degree. *Comm. in Algebra*, 25(12): 3761–3771, 1997.

[A-B-B-2] A. Alzati, M. Bertolini, and G. M. Besana. Two dimensional scrolls contained in quadric cones in $\mathbb{P}^5$. *Abh. Math. Sem. Univ. Hamburg*, 74: 57–75, 2004.

[A-B-B-3] A. Alzati, M. Bertolini, and G. M. Besana. Numerical criteria for very ampleness of divisors on projective bundles over an elliptic curve. *Canad. J. Math.*, 48(6): 1121–1137, 1996.

[B-DR] G. M. Besana and S. Di Rocco. On the projective normality of smooth surfaces of degree nine. *Geom. Dedicata*, 74: 1–21, 1999.

[B-L] A. Biancofiore, E. L. Livorni. Algebraic ruled surfaces with low sectional genus. *Ricerche Mat.*, 36(1): 17–32, 1987.

[Bu] D. C. Butler. Normal generation of vector bundles over a curve. *J. Differential Geom.*, 39(1): 1–34, 1994.

[C] L. Costa. An explicit description of some surfaces of degree 8 in $\mathbb{P}^5$. *Le Matematiche*, 53: 99–112, 1988.

[DES] W. Decker, L. Ein, and F.-O. Schreyer. Construction of surfaces in $\mathbb{P}^4$. *J. Algebraic Geom.*, 2(2): 185–237, 1993.

[DS] W. Decker and F.-O. Schreyer. Non-general type surfaces in $\mathbb{P}^3$: some remarks on bounds and constructions. *J. Symbolic Comput.*, 29(4-5): 545–582, 2000.
[E] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *G.T.M.* Springer Verlag, 1995.

[G-S] Daniel R. Grayson and Michael E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/, 1993.

[G-H] P. Griffiths and J. Harris. *Principles of Algebraic geometry*. Wiley Interscience, New York, 1978.

[G] A. Grothendieck. *Local Cohomology*, volume 41 of *Lect. Notes Math.* Springer Verlag, 1967.

[H] R. Hartshorne. *Algebraic Geometry*, volume 52 of *G.T.M.* Springer Verlag, Berlin, 1977.

[I-1] P. Ionescu. Embedded projective varieties of small invariants. In *Proceedings of the Week of Algebraic Geometry, Bucharest 1982*, volume 1056 of *L.N.M.*, pages 142–186. Springer, 1984.

[I-2] P. Ionescu. Embedded projective varieties of small invariants III. In *Algebraic Geometry L’Aquila 1988*, volume 1417 of *L.N.M.*, pages 138–154. Springer, 1990.

[La] R. Lazarsfeld. A sharp Castelnuovo bound for smooth surfaces. *Duke Math. J.*, 55: 423–429, 1987.

[Li] E. L. Livorni. On the existence of some surfaces. In *Algebraic Geometry L’Aquila 1988*, volume 1417 of *L.N.M.*, pages 155–179. Springer, 1990.

[N-S] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math.*, 82(2): 540–567, 1965.

[R] I. Reider. Vector bundles of rank 2 and linear systems on algebraic surfaces. *Ann. of Math.*, 127(2): 309–319, 1988.

[S-T] F.-O. Schreyer and F. Tonoli. *Computations in algebraic geometry with Macaulay 2*, chapter Needles in a haystack: special varieties via small fields, pages 251–279. Number 8 in *Algorithms Comput. Math.* Springer, 2002.

[W] C. Walter. Pfaffian subschemes. *J. Algebraic Geom.*, 5(4): 671–704, 2000.

 Dipartimento di Matematica Univ. di Milano, via C. Saldini 50 20133-Milano (Italy)
 E-mail address: alzati@mat.unimi.it

 Dipartimento di Matematica Univ. di Trento, Via Sommarive, 14 38050-Trento (Italy)
 E-mail address: tonoli@science.unitn.it