SUPERCOHERENT STATES OF CALOGERO-SUTHERLAND OSCILLATOR

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Abstract

Supersymmetric quantum mechanical model of Calogero-Sutherland singular oscillator is constructed. Supercoherent states are defined with the help of supergroup displacement operator. They are proper states of a fermionic annihilation operator. Their coordinate and superholomorphic representations are considered. The supermeasure on superunit disc which realizes the resolution of the unity is calculated. The cases of exact and spontaneously broken supersymmetry are treated separately. As an example the supersymmetric partners of the input Hamiltonian expressed in terms of elementary functions are given.

1 Introduction

The singular oscillator is a quantum mechanical system described by the Hamiltonian

$$h_0 = -\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{b}{x^2}.$$  \hfill (1)

Several quantum mechanical problems and number of quantum field theory problems are reduced to the Schrödinger equation with the Hamiltonian (1). We can first cite the Coulomb problem and the problem of describing of quantum system consisting of three (in general $N$) kinematically similar particles in one dimension interacting pairwise via quadratic and centrifugal potentials (Calogero problem [1]). The Hamiltonian (1) attracts now a considerable interest in connection with its usage for describing of spin chains [2, 3], quantum Hall effect [4], fractional statistics and anyons [5, 6].

The Hamiltonian (1) has only discrete spectrum on the half-line $[0, \infty)$. The systems of coherent states for this Hamiltonian are well known and studied (see for example [7]). These states are defined with the help of the displacement operator for the group $SU(1.1)$ which is a dynamical symmetry group for the system under consideration and represent a continuous basis in the Hilbert space of functions square integrable on the half-line. Coherent states can also be defined using ladder operator as the eigenstates of the destruction operator. The difference of these two definitions was recently analyzed in [8].

It is known that using the Darboux transformation one can construct the superhamiltonian (see, for example, [9]) of the Witten supersymmetric quantum mechanics [10]. One can find the review of the results obtained in supersymmetric quantum mechanics in papers [11, 12]. This method was recently generalized to a time-dependent Schrödinger equation [13, 14, 15] and applied to constructing of coherent states of anharmonic oscillator Hamiltonians with equidistant and quasiequidistant spectra [14, 15] and coherent states of the Hamiltonian of the one soliton potential [16].

The notion of coherent states is generalized in supersymmetric quantum mechanics up to supercoherent ones. Supercoherent states for supersymmetric harmonic oscillator have been introduced in [17, 18] and then generalized to arbitrary Lie supergroups [19] in the frame of Perelomov’s coherent states [7].
These states defined with the help of the supergroup displacement operator was recently applied for the description of coherent states of a spinning particle in varying electromagnetic field \[^21, 22\]. We will mention as well the papers \[^21, 22\] in which the detailed analysis of supercoherent states and their underlying geometric structures for the supergroups \(OSP(1/2)\) and \(OSP(2/2)\) have been made.

In this paper we use the Darboux transformation operator to construct the supersymmetric partner of the Hamiltonian \(h_0\). Using this transformation we obtain two copies of the Hilbert space with a positive definite quadratic form (scalar product) and a representation of the algebra \(su(1.1)\) defined on them. These two Hilbert spaces are essential elements in constructing of the coordinate representation of a linear superspace which becomes the Hilbert superspace after definition on it a superscalar product. Then one defines superalgebra generators which form a coordinate representation of the dynamical superalgebra of the system with given superhamiltonian. The supercoherent states are defined with the help of the supergroup displacement operator. Their coordinate representation is obtained. The representation of these states in the space of the functions superholomorphic in a superunit disc is considered. Supermeasure which realizes resolution of the identity operator in the given superspace is calculated. The possibility of the spontaneous supersymmetry breaking down is taken into account. In conclusion concrete examples of transformations which give exact and spontaneously broken supersymmetry are given.

### 2 Coherent States of Singular Oscillator

We shall give now a brief survey of the well known \[^7\] properties of coherent states of the singular oscillator. The dynamical symmetry algebra for the system with Hamiltonian (1) is \(su(1.1)\). Its Cartan-Weil basis in coordinate representation is expressed via the harmonic oscillator annihilation \(a = d/dx + x/2\) and creation \(a^+ = -d/dx + x/2\) operators as follows

\[
k_0 = (1/2)h_0, \quad k_+ = (1/2) [(a^*)^2 - b/x^2], \quad k_- = (1/2) [a^2 - b/x^2],
\]

\[
[k_0, k_{\pm}] = \pm k_{\pm}, \quad [k_+, k_-] = 2k_0.
\]

Casimir operator in this representation takes a constant value \(C = \frac{1}{2} |k_+, k_-, k_- k_+| - k_0^2 = 3/16 - b/4\).

Discrete representation of \(su(1.1)\) algebra corresponding to the value of \(k = 1/2 + (1/4)\sqrt{1 + \frac{4b}{9}}\) defined by the conditions \(k_- | 0 \rangle = 0, \quad k_0 | 0 \rangle = k | 0 \rangle, \quad k_+ | n \rangle = \sqrt{(n + 1)(n + 2k)} | n + 1 \rangle\). Let \(H_0\) be the Hilbert space of the solutions of the time-dependent Schrödinger equation with the Hamiltonian (1) and \(| n, t \rangle\) be its discrete basis vectors. In coordinate representation every function \(\psi_0(x, t) = (x | n, t)\) satisfies the zero boundary condition on half-line \([0, \infty)\). We shall consider the functions \(| n \rangle = | n, 0 \rangle\) defined by the action of the raising operator \(k_+\):

\[
| n \rangle = [(n!)^{-1}\Gamma^{-1}(2k + n)\Gamma(2k)]^{1/2} (k_+)^n | 0 \rangle, \quad k_0 | n \rangle = (k + n) | n \rangle.
\]

Their coordinate representation is as follows:

\[
\psi_n(x) = [n!2^{-2k}\Gamma^{-1}(n + 2k)]^{1/2} x^{2k-1/2} \exp(-x^2/4)L_n^{2k-1}(x^2/2), \quad (2)
\]

where \(L_n^k(z)\) is the Laguerre polynomial.

Coherent states \(| z \rangle\) are defined by the action of the displacement operator \(D_z\) for the group \(SU(1,1)\)

\[
D_z = e^{z \kappa^+} \exp[\ln(1 - z\bar{z})k_0] e^{-\kappa^-}, \quad z \in \mathbb{C}, \quad |z| < 1
\]

on the vacuum state

\[
| z \rangle = D_z | 0 \rangle = (1 - z\bar{z})^k \sum_{n=0}^{\infty} [(n!)^{-1}\Gamma^{-1}(2k)\Gamma(2k + n)]^{1/2} z^n | n \rangle.
\]

In the coordinate representation we have

\[
\psi_z(x) = (x | z) = 2^{1/2-k}\Gamma^{-1/2}(2k)(1 - z)^{-2k}(1 - z\bar{z})^k x^{2k-1/2} \exp\left[\frac{(1+z)\bar{z}^2}{4(1-z)}\right]. \quad (3)
\]

As \(\bar{z}\) we denote the value complex conjugated to \(z\).
The functions \( \psi_n(x) \) realize in the space \( H_0 \) an overcomplete basis set far which the resolution of the unity reads as follows

\[
\int_{|z|<1} |z\rangle\langle z| d\mu(z) = 1, \quad d\mu(z) = (1/\pi)(2k - 1)(1 - z^2)^{-2}dzd\bar{z}.
\]

The decomposition coefficients \( C_n \) of any \( |\psi\rangle \in H_0 \) in terms of the discrete basis \( |n\rangle \), \( |\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \) define its holomorphic representation \( \psi(z) \) in the space of functions holomorphic in the unit disk \( |z|<1 \). Scalar product in this space is defined as follows

\[
\langle \psi_1(z) | \psi_2(z) \rangle = \int_{|z|<1} e^{-f} \psi_1(\bar{z})\psi_2(z)d\mu(z),
\]

where \( f = \ln |\langle 0 | z \rangle|^{-2} = \ln(1 - z^2)^{-2k} \).

### 3 Supersymmetry of Singular Oscillator

Following the papers \([9, 23, 24]\) we construct a new exactly solvable potential \( h_1 = h_0 + A(x) \) with the help of the Darboux transformation which is defined by a nodeless in half-line \((0, \infty)\) real solution \( u(x) \) of the initial Schrödinger equation \( h_0u = \alpha u \) called transformation function, where \( \alpha(< 2k) \) is an arbitrary constant. The transformation function \( u(x) \) completely defines the potential difference

\[
A(x) = -2 \ln[u(x)]''', 
\]

where the prime denotes the derivative with respect to \( x \) and transformation operator is defined as follows:

\( \tilde{L} = -u'(x)/u(x) + d/dx \). Operator \( \tilde{L} \) transforms the solutions \( \psi(x) \) of the input Schrödinger equation

\[
h_0\psi(x) = E\psi(x)
\]

into the solutions of the transformed one

\[
h_1\varphi(x) = E\varphi(x), \quad \varphi(x) = \tilde{L}\psi(x)
\]

corresponding to the same eigenvalue \( E \). The operator \( \tilde{L}^+ \) adjoint to \( \tilde{L} \) with respect to scalar product for which \( (d/dx)^+ = -d/dx \) realizes the transformation in the inverse direction, i.e. from the solutions \( \varphi(x) \) of the equation (8) to the solutions \( \psi(x) \) of the equation (7). The products of the operators \( \tilde{L} \) and \( \tilde{L}^+ \) are symmetry operators for the equations (7) and (8) and are expressed through the Hamiltonians \( h_0 \) and \( h_1 \): \( \tilde{L}^+\tilde{L} = h_0 - \alpha, \tilde{L}\tilde{L}^+ = h_1 - \alpha \). Moreover the operators \( \tilde{L}^+ \) and \( \tilde{L} \) intertwine the Hamiltonians \( h_0 \) and \( h_1 \)

\[
\tilde{L}h_0 = h_1\tilde{L}, \quad h_0\tilde{L}^+ = \tilde{L}^+h_1.
\]

If the functions \( \psi(x) = \psi_n(x) \) with eigenvalues \( E_n \) are normalized to unity we easily obtain the norm of the functions \( \varphi_n(x) = \tilde{L}\psi_n(x) \): \( \langle \varphi_n | \varphi_n \rangle = E_n - \alpha \).

If the function \( u \) becomes infinity on the bounds of the interval \([0, \infty)\) then the equation (8) has the following function \( \varphi_{-1}(x) = u^{-1}(x) \in H_1 \) as its solution. It is square integrable on this interval and satisfies the boundary condition for the discrete spectrum eigenfunctions of the equation (8). We denote as \( H_1 \) a Hilbert space of the solutions of the time-dependent Schrödinger equation with the Hamiltonian \( h_1 \), so \( \varphi_n(x) = \varphi_n(x,0), \varphi_n(x,t) \in H_1 \). There exist in \( H_0 \) such functions \( \psi_n(x) \) that \( \tilde{L}\psi_n(x) \in H_1 \).

The function \( \varphi_{-1}(x) \) will be the ground state function of the Hamiltonian \( h_1 \) and we shall obtain the exact supersymmetry \([10, 11]\). We shall as well consider the case when \( u(0) = 0 \) and \( u(\infty) = \infty \) so that \( u^{-1}(x) \notin H_1 \). In this case we obtain the spontaneously broken supersymmetry. It should be noted that the functions \( u \) and \( \varphi_{-1}(x) \) form the kernels of the operators \( \tilde{L} \) and \( \tilde{L}^+ \): \( \tilde{L}u = 0, \tilde{L}^+\varphi_{-1}(x) = 0 \). The case \( u = \psi_0(x) \) reduces to the considered one after the replacement \( h_0 \leftrightarrow h_1 \).

We will use the means of the supermathematics \([23, 24, 27]\) together with the transformation operators

\[
L = \tilde{L}(h_0 - \alpha)^{-1/2} = (h_1 - \alpha)^{-1/2} \tilde{L}
\]
and its conjugate
\[ L^+ = \bar{L}^+ (h_1 - \alpha)^{-1/2} = (h_0 - \alpha)^{-1/2} \bar{L}^+. \]

These operators are such that \( LL^+ = 1 \) and \( L^+ L = 1 \) in the corresponding spaces. If the supersymmetry is broken \( L\psi \neq 0, \forall \psi \in H_0, \) \( L^+ \varphi \neq 0, \forall \varphi \in H_1 \) and also \( h_0 - \alpha \neq 0 \) in \( H_0 \) and \( H_1 \) respectively. Operators \( L \) and \( L^+ \) are in this case unitary operators and consequently they transform one orthonormal basis set into another one.

If supersymmetry is exact \( \langle \psi_{-1} = u^{-1} \in H_1 \rangle \) we can decompose the space \( H_1 \) into a direct sum \( H_1 = H_1^0 \oplus H_1^1 \) where \( H_1^0 = \text{span} \{ \varphi_{-1} \} \) and \( H_1^1 = \text{span} \{ \varphi_n = L\psi_n, \forall \psi_n \in H_0 \} \). The symbol \( \text{span} \) stands for the linear hull over the complex number field \( \mathbb{C} \). The relations \( LL^+ = 1 \) and \( h_1 - \alpha \neq 0 \) are valid for any \( \varphi \in H_1^1 \).

We shall construct the linear superspace \( H_s \) with the help of the spaces \( H_0 \) and \( H_1 \) and the generators \( \theta \) and \( \bar{\theta} \) of the four-dimensional Grassman algebra: \( \bar{\theta} = \theta, \theta^2 = \bar{\theta}^2 = \theta \bar{\theta} = 0 \) using so-called homogeneous realization: \( H_s = H_{1\overline{1}} \oplus H_T \). The even subspace \( H_{1\overline{1}} \) is defined as follows \( H_{1\overline{1}} = \text{span} \{ \Psi^1_n(x, \theta, \overline{\theta}) = \psi_n(x), n = 0, 1, 2, \ldots \} \). When one defines the odd component \( H_T \) it is necessary to distinguish the cases of broken and exact supersymmetry. For spontaneously broken supersymmetry the space \( H_T \) is defined as follows \( H_T = \text{span} \{ \Psi^2_n(x, \theta, \overline{\theta}) = \theta \varphi_n(x), n = 0, 1, 2, \ldots \} \). For the supersymmetry is exact the same linear hull defines the space \( H_T^+ \). The space \( H_T^+ \) is in this case a direct sum \( H_T^+ = H_0^0 \oplus H_T^1 \) where \( H_0^0 = \text{span} \{ \theta \varphi_{-1}(x) \} \). The whole space \( H_s \) is in the latter case as well a direct sum \( H_s = H_1^0 \oplus H_T^1 \), where \( H_1^0 = H_{1\overline{1}} \oplus H_T^+ \). The elements of the spaces \( H_{1\overline{1}} \) and \( H_T^+ \) are called homogeneous elements. For every homogeneous element \( v \in H_s \) one defines its parity \( \varepsilon(v) \) as follows: \( \varepsilon(v) = 0 \) if \( v \in H_{1\overline{1}} \) and \( \varepsilon(v) = 1 \) if \( v \in H_T^+ \).

For the purpose of construction of coherent states we need the notion of the Grassmann envelop of the second kind. To define this notion we shall consider the four-dimensional Grassmann algebra \( \Lambda_2 \) with the generators \( \alpha \) and \( \overline{\alpha} \) \( \overline{\alpha} = \alpha, \alpha^2 = \overline{\alpha} \overline{\alpha} = 0 \). We define the Grassmann envelop of the second kind \( H_s \) of the superspace \( H_s \) as a left \( \Lambda_2 \)-module [24]: \( H_s = (\Lambda_2 \otimes H_{1\overline{1}}) \oplus (\Lambda_2^0 \otimes H_T^1) \).

Here \( \Lambda_2^0 \) is the commutative subalgebra of the even elements of the algebra \( \Lambda_2 \) (c-numbers), \( \Lambda_2^0 = \text{span} \{ 1, \epsilon, \underline{\epsilon}, \overline{\epsilon}, \overline{\underline{\epsilon}} \} \). The symbol \( \Lambda_2^0 \) and \( \Lambda_2^1 \) are in this case a module \[ \Lambda_2 = \text{span} \{ 1, \epsilon, \underline{\epsilon}, \overline{\epsilon}, \overline{\underline{\epsilon}} \} \] and \( \Lambda_2 = \Lambda_2^0 \oplus \Lambda_2^1 \) (supernumbers) [27]. We notice the following property: \( \beta v = (-1)^{\varepsilon(\beta)\varepsilon(v)} v \beta \) which holds for homogeneous elements \( v \in H_s \) and \( \beta \in \Lambda_2 \).

To define the space \( H_s \) dual to \( H_s \) we need the definition of the complex conjugation. We define the complex conjugation by the following property
\[ \overline{\beta v} = \overline{\beta} \overline{v}, \quad \forall v \in H_s, \quad \forall \beta \in \Lambda_2. \]

This definition differs from the widely used condition \[ \overline{\beta_1 \beta_2} = \overline{\beta_2} \overline{\beta_1} \] in the latter case the quantity \( \beta \overline{\beta} \) is real independently of the parity of an element \( \beta \in \Lambda_2 \). We follow the definition (10) since in the other case one faces a number of inconsistencies. In particular, the super Kähler 2-form is neither real nor imaginary [22] and it is difficult to establish in this case the correspondence between the real observables and self superadjoint operators.

The definition of the space \( H_{1\overline{1}} \) dual to \( H_s \) is equivalent to the definition of the bilinear form on \( H_s \) with respect to which the space \( H_{1\overline{1}} \) will be the space of the linear functionals defined on \( H_s \). We define first the dual space in such a way that bilinear form on \( H_s \) be expressed in terms of bilinear forms (scalar products) defined in \( H_0 \) and \( H_1 \) in accordance with the following property [22]
\[ (v_1 | v_2) = \langle \psi_1 | \psi_2 \rangle + i \langle \varphi_1 | \varphi_2 \rangle, \]

where \( v_1 = \psi_1 + \theta \varphi_1, v_2 = \psi_2 + \theta \varphi_2 \) are elements of the space \( H_s \) in their homogeneous realization and \( \langle v_1 | v_2 \rangle, \langle \varphi_1 | \varphi_2 \rangle \) are the usual scalar products defined in the spaces \( H_0 \) and \( H_1 \). For this purpose consider the homogeneous decomposition of the dual space \( \overline{H_s} = \overline{H_{1\overline{1}}} \oplus \overline{H_T^+} \). We suppose that if we pass from \( H_s \) to \( \overline{H_s} \) the parity of the elements does not change. This condition can be realized if with every element \( \Psi^1_n(x, \theta, \overline{\theta}) = \psi_n \in \overline{H_s} \) one associates an element \( \overline{\Psi}^1_n(x, \theta, \overline{\theta}) = \theta \varphi_n \in \overline{H_T^+} \) and with every element \( \Psi^2_n(x, \theta, \overline{\theta}) = \theta \varphi_n \in \overline{H_T^+} \) one associates an element \( \overline{\Psi}^2_n(x, \theta, \overline{\theta}) = \theta \varphi_n \in \overline{H_T^+} \). For spontaneously broken supersymmetry the spaces \( \overline{H_{1\overline{1}}} \) and \( \overline{H_T^+} \) are defined as the linear hulls over the complex number field \( \mathbb{C} \) of the vectors \( \overline{\Psi}^0_n \) and \( \overline{\Psi}^1_n, n = 0, 1, 2, \ldots \) respectively. For exact supersymmetry we shall have
the direct sum $H_{\Pi} = \overline{H}_{\Pi} \oplus \overline{H}_{\Pi}$, where $\overline{H}_{\Pi} = \text{span}\{\overline{\Psi}_{-1} = \overline{\sigma}_{-1}(x)\}$ and $\overline{H}_{\Pi} = \text{span}\{\overline{\Psi}_{n},\ n = 0, 1, 2, \ldots\}$. When the space $H_{s}$ is considered as left $\Lambda_{2}$ - module one has to take into account the property (10).

Since we have the one-to-one correspondence between the elements of the spaces $H_{s}$ and $\overline{H}_{s}$ we can define the bilinear form $\langle \cdot | \cdot \rangle$: $H_{s} \otimes H_{s} \rightarrow \mathbb{C}$ as follows

$$\langle \Psi_{1} | \Psi_{2} \rangle = \int \overline{\Psi}_{1}(x, \theta, \overline{\beta})\Psi_{2}(x, \theta, \overline{\beta})dxd\theta d\overline{\beta} \in \mathbb{C}. \quad (11)$$

The integration over the Grassmann variables is understood in the sense of Berezin [25] when the only non-zero integral is $\int \theta d\theta d\overline{\beta} = 1$. When $H_{s}$ is considered as left $\Lambda_{2}$ - module this definition realizes the following correspondence: $H_{s} \otimes \overline{H}_{s} \rightarrow \Lambda_{2}^{0}$. We notice the property $\langle v | w \rangle = (-1)^{\varepsilon(v)\varepsilon(w)}\langle w | v \rangle$ which holds for homogeneous elements $v, w \in H_{s}$. One should work with supernumbers according to the following rule: $\langle \beta_{1}v | \beta_{2}w \rangle = (-1)^{\varepsilon(v)\varepsilon(\beta_{2})}\overline{\beta}_{1}\beta_{2}\langle v | w \rangle$, where $\beta_{2}$ and $v$ are homogeneous elements of $\Lambda_{2}$ and $H_{s}$ respectively.

The comparison of the bilinear form (11) with the super Hermitian form defined in an abstract super vector space [24] shows that we have a concrete (coordinate) realization of the abstract super Hilbert space with (11) as the super Hermitian form. It is worth noticing that there exist other definitions of super Hilbert spaces, see for example [29].

We pass now to definition of the coordinate representation of generators of the superalgebra acting in the space $H_{s}$. Let us put

$$K_{0} = k_{0}(\partial/\partial\theta)\theta + Lk_{0}L^{+}\theta(\partial/\partial\theta), \quad K_{\pm} = k_{\pm}(\partial/\partial\theta)\theta + Lk_{\pm}L^{+}\theta(\partial/\partial\theta). \quad (12)$$

Here $\theta$ is an operator of the left multiplication on the element $\theta$ and $\partial/\partial\theta$ is an operator of the left differentiation. (We define the left action of the operators on the vectors $\Psi(x, \theta, \overline{\beta}) \in H_{s}$).

Operators (12) realize evidently the coordinate representation of the algebra $su(1, 1)$ in the space $H_{s}$. The identity operator in this space has the form $I = \frac{\partial}{\partial\theta} \theta + \theta \frac{\partial}{\partial\theta}$. We note as well another even operator: $B_{0} = \frac{\partial}{\partial\theta} \theta - \theta \frac{\partial}{\partial\theta}$. It has the property $B_{0}v = (-1)^{\varepsilon(v)}v$ for all homogeneous $v \in H_{s}$.

All the operators introduced do not change the parity of the elements and consequently they are even operators. We define odd operators by the relations

$$Q_{-} = L^{+}(\partial/\partial\theta), \quad Q_{+} = L\theta. \quad (13)$$

They commute with all even operators and $\{Q_{-}, Q_{+}\} = I$. It follows from this that the linear hull $sal = \text{span}\{K_{0}, K_{+}, K_{-}, I, Q_{+}, Q_{-}\}$ represents a superalgebra. For this superalgebra we can write the following direct sums $sal = sal_{\overline{\Pi}} \oplus sal_{\overline{\Pi}}$, $sal_{\overline{\Pi}} = su(1, 1)_{\overline{\Pi}} \oplus e_{\overline{\Pi}}$, where $su(1, 1)_{\overline{\Pi}} = \text{span}\{K_{0}, K_{+}, K_{-}\}$, $e_{\overline{\Pi}} = \text{span}\{I\}$ and $sal_{\overline{\Pi}} = \text{span}\{Q_{+}, Q_{-}\}$.

Having in hand the super Hermitian form (11) on $H_{s}$ we define operator $A^{+}$ super adjoint to $A$ [22],

$$\langle A^{+}v | w \rangle = (-1)^{\varepsilon(v)\varepsilon(A)}\langle v | Aw \rangle, \quad \forall v, w \in H_{s}, \quad A \in sal$$

$v$ and $A$ being homogeneous elements of $H_{s}$ and $sal$ respectively. The conjugation operation so defined has the properties $(A^{+})^{+} = A$, $(AB)^{+} = (-1)^{\varepsilon(A)\varepsilon(B)}B^{+}A^{+}$ and $[A, B]^{+} = -[A^{+}, B^{+}]$, where $[,]$ is superalgebra bracket [25] with the property $[\beta_{1}A, \beta_{2}B] = (-1)^{\varepsilon(\beta_{1})\varepsilon(\beta_{2})}\beta_{1}\beta_{2}[A, B]$. Self super adjoint operator is defined as follows $A^{+} = A$. It is easily seeing that $K_{0}^{+} = K_{0}$, $K_{\pm}^{+} = K_{\mp}$, $I^{+} = I$, $B_{0}^{+} = B_{0}$, $Q_{\pm}^{+} = iQ_{\mp}$. Operator $\mathcal{H} = 2K_{0}$ can be treated as superhamiltonian since $\mathcal{H}^{\dagger} = \mathcal{H}$ and the spectral problem for it considered in the spaces $H_{\Pi}$ and $\overline{H}_{\Pi}$ is reduced to the solution of the input and output Schrödinger equations. Superalgebra $sal$ is evidently its dynamical supersymmetry algebra. It is worth noticing that the operators (12), (13) and $I$ in the case of spontaneously broken supersymmetry realize irreducible representation of this superalgebra in the space $H_{s}$. If the supersymmetry is exact this representation is reducible one and the direct decomposition $H_{s} = H_{s}^{1} \oplus H_{s}^{0}$ represents the decomposition of this representation in terms of irreducible ones. The representation of the superalgebra $sal$ in the space $H_{s}^{0}$ is trivial: $gu = 0, \forall g \in sal, \forall u \in H_{s}^{0}$. The whole spectrum of the superhamiltonian $\mathcal{H}$ is two-fold degenerate in the case of the spontaneously broken supersymmetry. In the case of the exact supersymmetry the ground state level
\[ E = \alpha \] of the superhamiltonian \( \mathcal{H} \) is nondegenerate and the vacuum state is described by the function \( \Psi_{-1}(x, \theta, \bar{\theta}) = \theta \varphi_{-1}(x) \). If we define superunitary operator by the condition \( U^{-1} = U^+ \) we can construct superunitary symmetry operators which form local supergroup with superalgebra sal.

We need the completeness condition of the basis set \( \{ \Psi^0_n, \Psi^1_n, n = 0, 1, \ldots \} \) in the space \( H_s \) (in \( H^1_s \) for the exact supersymmetry). To formulate this condition we define the projection operator \( P_v \): \( P_v w = \langle v | w \rangle \), where the superscalar product is defined by the formula (11). Let us introduce the notations: \( P_n^0 = P_{\Psi^0_n} P_{\Psi^0_n}^* = P_{\Psi^0_n} \). It is evident that \( P_n^0 P_n^1 = 0, \forall n, n' \) and operator \( P_n = P_n^0 - iP_n^1 \) projects on the two-dimensional space of the solutions of the super Schrödinger equation \( \mathcal{H}\Psi = E\Psi \) with given value of \( E \). The completeness condition of the basis \( \{ \Psi^0_n, \Psi^1_n \} \) reads as follows: \( \sum_{n=0}^{\infty} P_n = I \).

4 Supercoherent States

We define the supercoherent states using supergroup displacement operator \([19, 20, 21]\).

The state \( \Psi^0_0 \) is of maximal symmetry state since it is the proper state of the operators \( K_0 = \mathcal{H}, K_-, Q_- \), and \( I \) forming the algebra \( \mathcal{B} \) of the isotropy subgroup of the element \( \Psi^0_0 \). In this case the complexification of sal, sal\( ^c \), is decomposed in the direct sum \( sal^c = \mathcal{B} \oplus \mathcal{B}^* \), where \( \mathcal{B} = \text{span}\{K_0, I, K_+, Q_+\} \). This is the reason to take \( \Psi^0_0 \) as a cyclic vector (fiducial state) and apply to it the nonunitary supergroup translation operator.

\[ \Psi_{z\alpha}(x, \theta, \bar{\theta}) = N \exp (zK_+ - \alpha Q_+) \Psi^0_0, \quad z \in \mathbb{C}, \quad \alpha \in \Lambda_2. \]

The normalizing coefficient \( N \) should be calculated separately. Taking into account the commutativity of \( Q_+ \) and \( K_+ \) one finds

\[ \Psi_{z\alpha}(x, \theta, \bar{\theta}) = N (\psi_z(x) - \alpha \theta \varphi_z(x)), \quad \varphi_z(x) = L \psi_z(x). \] (14)

The direct calculation shows that this function is a proper function of (fermionic) annihilation operator: \( Q_- \Psi_{z\alpha} = \alpha \Psi_{z\alpha} \).

Using the fact that \( \psi_z(x) \) is normalized to unity and with the help of the formula (11) we find the normalization coefficient \( N = 1 + \frac{1}{2} i \alpha \bar{\alpha} = N \).

It is worth noticing that in the paper \([28]\) the analogous expression for coherent states of the harmonic oscillator potential has been obtained.

Using the completeness of the basis \( \{ \Psi^0_n, \Psi^1_n \} \) we can find the supermeasure \( d\mu(z, \zeta, \alpha, \bar{\alpha}) = I \) which gives the resolution of the identity in the space \( H_s \)

\[ \int_D P_{\Psi \zeta} d\mu(z, \zeta, \alpha, \bar{\alpha}) = I \]

where \( P_{\Psi \zeta} \) is the projection operator on the state \( \Psi_{\zeta \alpha}(x, \theta, \bar{\theta}) \) and \( D \) is a domain in the superspace with coordinates \((z, \alpha), |z| < 1\), called super unit disc.

A simple computation leads to

\[ d\mu(z, \zeta, \alpha, \bar{\alpha}) = i d\alpha d\bar{\alpha} d\mu(z, \zeta), \]

where \( d\mu(z, \zeta) \) is determined by the formula (4).

The resolution of the identity permits one to construct a superholomorphic representation of the vectors from \( H_s \). For this purpose we write an element \( \Psi(x, \theta, \bar{\theta}) \in H_s \) as follows

\[ \Psi(x, \theta, \bar{\theta}) = \sum_{n=0}^{\infty} \left[ C^0_n \psi_n(x) + C^1_n \theta \varphi_n(x) \right] = \psi(x) + \theta \varphi(x). \]

It follows from this relation that

\[ \langle \Psi_{\zeta \alpha} | \Psi \rangle = (1 - z \zeta) (1 - \zeta \alpha) \langle \psi(z, \alpha), \Psi(z, \alpha) \rangle = \psi(z) - i \alpha \varphi(z), \]

where \( \psi(z) \) and \( \varphi(z) \) are holomorphic representations of functions \( \psi(x) \) and \( \varphi(x) \). Superfunction \( \Psi(z, \alpha) \) is the holomorphic representation of the vector \( \Psi \).
In the space of superholomorphic functions we can define the scalar product

$$\langle \Psi_1(z, \alpha) | \Psi_2(z, \alpha) \rangle = \int D\overline{\Psi}_1(z, \alpha)\Psi_2(z, \alpha)e^{-f}d\mu,$$

where superfunction $f$ is defined by the condition \[23\] $f = f(z, \overline{z}, \alpha, \overline{\alpha}) = \ln |\langle \Psi_0^0 | \overline{\Psi}_{20} \rangle|^{-2}$ so that $\exp(-f) = (1 - i\alpha \overline{\alpha})(1 - z\overline{z})^{2k}$. The formula (15) leads to the same definition for the scalar product: $\langle \Psi_1 | \Psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle + i\langle \varphi_1 | \varphi_2 \rangle$, where $\langle \psi_1 | \psi_2 \rangle$ and $\langle \varphi_1 | \varphi_2 \rangle$ are scalar products defined for the holomorphic representations of the vectors $\psi$ and $\varphi$.

5 Example

As an example consider a solution of the elementary form of the Schrödinger equation with the Hamiltonian (1) as the transformation functions. The Hamiltonian $h_1$ will be expressed in this case in terms of elementary functions.

Singular at the bounds of the interval $[0, \infty)$ solutions of the Schrödinger equation with Hamiltonian (1) can be chosen as follows

$$u_p(x) = x^{3/2-2k}e^{x^2/4}L_p^{1-2k}(y), \quad y = -x^2/2, h_0u_p = 2(k - p - 1)u_p, \quad p = 0, 1, 2, \ldots.$$ (16)

With the help of the formula (6) we find out the potential difference

$$A_p(x) = -1 + \frac{3 - 4k}{x^2} + 2 \frac{xL_{p-1}^{2-2k}(y)}{L_p^{1-2k}(y)} - \frac{2x^2L^{3-2k}_p(y) + L^{2-2k}_{p-1}(y)}{L_p^{1-2k}(y)}.$$ (17)

The transformation operator which transforms the solutions (2) of the input equation (7) into the solutions of the equation (8) reads as follows

$$L = \left[ \frac{4k - 3}{2x} - \frac{x}{2} - \frac{xL_{p-1}^{2-2k}(y)}{L_p^{1-2k}(y)} - \frac{d}{dx} \right] (h_0 + 2 + 2p - 2k)^{-1/2}.$$ (18)

The solution of the equation (7) which can not be obtained by the action of the operator (18) to the functions from $H_0$ is $u_p^{-1}(x)$. This function square integrable on half-line $[0, \infty)$ only at even $p$. Its normalization constant can be calculated by the method described in [50]. One then obtains

$$\int_0^\infty u_p^{-2}(x)dx = (-1)^p2^{2k-2-p}!2k!.$$ (19)

At odd $p$ the potential differences has poles on half-line $[0, \infty)$. At even $p$ transformation function (16) generates the exact supersymmetry with vacuum state constructed using the normalized at unity ground state of the new Hamiltonian

$$\varphi^{-1}(x) = 2^{k-1}\sqrt{p!2k!}u_p^{-1}(x).$$

To construct the spontaneously broken supersymmetry we use the transformation function which vanishes at $x = 0$ and equals infinity at $x = \infty$

$$v_p(x) = x^{2k-1/2}e^{x^2/4}L_p^{2k-1}(y), \quad h_0v_p = -2(k + p)v_p, \quad p = 0, 1, 2, \ldots.$$ (20)

This function is not square integrable on half-line $[0, \infty)$ (the normalization integral diverges at $x = 0$), but the potential difference calculated with it

$$A_p(x) = -1 + \frac{4k - 1}{x^2} + 2x^2 \frac{L_{p-1}^{2k}(y)}{L_p^{2k-1}(y)} - \frac{2x^2L^{2k-1}_p(y) + L^{2k}_{p-1}(y)}{L_p^{2k-1}(y)}.$$
is a regular function in the interval \((0, \infty)\). One obtains the solutions of the new Schrödinger equation with the help of the transformation operator

\[
L = \left( \frac{1 - 4k}{2x} - \frac{x}{2} - \frac{xL_{2k-1}^k(y)}{L_{2k-1}^k(y)} - \frac{d}{dx} \right) (h_0 + 2k + 2p)^{-1/2}.
\]  

(19)

Transformation operators (18) and (19) define the superalgebra (12) and (15) and supercoherent states (14).

In conclusion we note that the method developed for the singular oscillator with dynamical algebra \(su(1.1)\) can be applied to other systems. We intend to consider them in subsequent publications.

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