Operator ultra-amenability

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Abstract

Extending M. Daws’ definition of ultra-amenable Banach algebras, we introduce the notion of operator ultra-amenability for completely contractive Banach algebras. For a locally compact group $G$, we show that the operator ultra-amenability of $A(G)$ imposes severe restrictions on $G$. In particular, it forces $G$ to be a discrete, amenable group with no infinite abelian subgroups. For various classes of such groups, this means that $G$ is finite.

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Introduction

In ([11]), B. E. Johnson characterized the amenable locally compact groups $G$ in terms of a cohomological triviality condition of their group algebras $L^1(G)$. This cohomological triviality condition can be extended to arbitrary Banach algebras and defines the class of amenable Banach algebras.

In ([3]), M. Daws defined—the notion of ultra-amenability of a Banach algebra. Given a Banach algebra $\mathfrak{A}$ and an ultrafilter $\mathcal{U}$ over an arbitrary index set, the ultrapower $(\mathfrak{A})_\mathcal{U}$ is again a Banach algebra ([9, Proposition 3.1(i)]). Consequently, Daws defined $\mathfrak{A}$ to be ultra-amenable if $(\mathfrak{A})_\mathcal{U}$ is amenable in the sense of ([11]) for every ultrafilter $\mathcal{U}$ over any any index set. Daws proved that a $C^*$-algebra is ultra-amenable if and only if it is subhomogeneous and that, for a locally compact group $G$ satisfying certain properties, the group algebra $L^1(G)$ is ultra-amenable only if $G$ is finite ([3, Theorem 5.17]), e.g., if $G$ is abelian, compact, or discrete ([3, Theorems 5.9 and 5.11]). He strongly suspected that $L^1(G)$ is ultra-amenable if and only if $G$ is finite for every locally compact group $G$.

In this note, we introduce a notion of ultra-amenability in the operator space context and focus, in particular, on the Fourier algebra of a locally compact group.

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1 Operator ultra-amenability for completely contractive Banach algebras

In [15], Z.-J. Ruan initiated the theory of abstract operator spaces: these spaces can be completely isometrically be represented as subspace of \( B(\mathfrak{H}) \) for some Hilbert space \( \mathfrak{H} \). It is straightforward that the ultraproduct construction in the Banach space category carries over to the category of operator spaces ([5, Section 10.3]). In [16], Ruan introduced the notion of a completely contractive Banach algebra: this is an algebra equipped with an operator space structure such that multiplication is a completely contractive bilinear map. Consequently, Ruan defined the notion of an operator amenable, completely contractive Banach algebra by requiring the bounded derivations in Johnson’s original definition of an amenable Banach algebra to be completely bounded.

If \( \mathfrak{A} \) is a completely contractive Banach algebra, and \( \mathcal{U} \) is an ultrafilter over an arbitrary index set, then \( (\mathfrak{A})_{\mathcal{U}} \) is also a completely contractive Banach algebra (as is easy to see).

We thus define:

Definition 1.1. Let \( \mathfrak{A} \) be a completely contractive Banach algebra. We say that \( \mathfrak{A} \) is operator ultra-amenable if the ultrapower \( (\mathfrak{A})_{\mathcal{U}} \) is operator amenable for any ultrafilter \( \mathcal{U} \) over an arbitrary index set.

There are various canonical functors from the category of Banach spaces into the category of operator spaces. One of them is the max functor that assigns to each Banach space the largest operator structure there is on it. By [5, Proposition 1.5], a Banach algebra \( \mathfrak{A} \) is amenable if and only if the completely contractible Banach algebra \( \text{max} \mathfrak{A} \) is operator amenable.

We shall show that this is compatible with the ultraproduct construction:

Proposition 1.2. Let \( E \) be a Banach space and let \( \mathcal{U} \) be an ultrafilter over any index set. Then the identity map on \( (E)_{\mathcal{U}} \) induces a completely isomorphic isomorphism from \( (\text{max} E)_{\mathcal{U}} \) to \( \text{max}(E)_{\mathcal{U}} \).

Proof. Let \( S \) be the closed unit ball of \( E \). It is obvious that \( E \) is a quotient of \( \ell^1(S) \). Both the ultrapower construction and the max functor are compatible with taking quotients. It is thus sufficient to show that the canonical map from \( \text{max}(\ell^1(S))_{\mathcal{U}} \) to \( (\text{max} \ell^1(S))_{\mathcal{U}} \) is a completely isomorphic isomorphism.

This, however, is easily accomplished by adapting the proof of ([5, Proposition 10.3.8]).

\[ \square \]
Corollary 1.3. Let $\mathfrak{A}$ be a Banach algebra. Then $\mathfrak{A}$ is ultra-amenable if and only if $\max \mathfrak{A}$ is operator ultra-amenable.

In [16], Ruan showed that a $C^*$-algebra is operator amenable if and only if it amenable. We show that the same holds true for operator ultra-amenability:

Proposition 1.4. For a $C^*$-algebra $\mathfrak{A}$, the following are equivalent:

(i) $\mathfrak{A}$ is operator ultra-amenable;
(ii) $\mathfrak{A}$ is ultra-amenable;
(iii) $\mathfrak{A}$ is subhomogeneous.

Proof. (ii) $\iff$ (iii) is from [3] (see the corrigendum).

(ii) $\implies$ (i) is trivial in view of Corollary 1.3.

For (i) $\implies$ (ii), let $\mathfrak{A}$ be an operator ultra-amenable $C^*$-algebra, and let $U$ be an ultrafilter over an arbitrary index set. Then $(\mathfrak{A})_U$ is operator amenable. However, $(\mathfrak{A})_U$ is also $C^*$-algebra by [9, Proposition 3.1(ii)] and is therefore amenable by [10, Theorem 5.1]. As $U$ was arbitrary, $\mathfrak{A}$ is ultra-amenable. $\square$

2 The case of the Fourier algebra

It is fair to say that the notion of operator amenability was introduced with the Fourier algebra in mind. In [6], the first and the second named author showed that, for a locally compact group $G$, its Fourier algebras $A(G)$ is amenable only $G$ is the finite extension of an abelian group. On the other hand, Ruan showed in [16] that $A(G)$ is operator amenable if and only if $G$ is amenable—a much less restrictive property.

We now turn to the question for which locally compact groups $G$, the Fourier algebra $A(G)$ is operator ultra-amenable. In view of [3], it is very plausible to conjecture that, for any locally compact group $G$, the operator ultra-amenability of $A(G)$ would force $G$ to be finite. We shall not be able to prove this conjecture in full generality, but still establish it for important special cases.

We start with a simple hereditary property:

Lemma 2.1. Let $G$ be a locally compact group such that $A(G)$ is operator ultra-amenable. Then $A(H)$ is operator ultra-amenable for every closed subgroup $H$ of $G$.

Proof. Let $H$ be a closed subgroup of $G$. Then the restriction map

$$A(G) \rightarrow A(H), \quad f \mapsto f|_H$$

is a complete quotient map (and an algebra homomorphism).
Let $\mathcal{U}$ be any ultrafilter over an arbitrary index set, so that $(A(G))_{\mathcal{U}}$ is operator amenable. Since the ultrafilter construction preserves (complete) quotient maps, the completely contractive Banach algebra $(A(H))_{\mathcal{U}}$ is also operator amenable. As $\mathcal{U}$ was arbitrary, this means that $A(H)$ is operator ultra-amenable.

**Corollary 2.2.** Let $G$ be a locally compact group such that $A(G)$ is operator ultra-amenable. Then every abelian subgroup of $G$ is finite.

**Proof.** Let $H$ be an abelian subgroup of $G$, and suppose without loss of generality that it is closed in $G$. As the Fourier transform between $A(H)$ and $L^1(\hat{H})$ is a completely isometric isomorphism, it follows that $\hat{H}$—being abelian—has to be finite by ([3, Theorem 5.9]) as has $H$.

We can now collect (quite) a few examples for of locally compact groups for which $A(G)$ is not operator ultra-amenable:

**Proposition 2.3.** Let $G$ be a locally compact group and let $H$ be an infinite closed subgroup of $G$ such that either of the following holds:

(a) $H$ is compact;

(b) $H$ is connected.

Then $A(G)$ is not operator ultra-amenable.

**Proof.** Both of these cases can be shown to follow from Corollary 2.2.

Case (a): This follows from Corollary 2.2 and from a result by E. I. Zelmanov, which asserts that every infinite compact group contains an infinite abelian subgroup (see [19]).

Case (b): Assume that $A(G)$ is operator ultra-amenable. Then this is also true for $A(H)$. As $H$ is a connected group, there are, by [13, Theorem 4.13], abelian subgroups $H_1, \ldots, H_n$ of $H$, as well as a compact subgroup $K$ of $H$, such that

$$H_1 \times \cdots \times H_n \times K \to H, \quad (x_1, \ldots, x_n, y) \mapsto x_1 \cdots x_n y$$

is a homeomorphism. Corollary 2.2 and Proposition 2.3(a) thus yield that $H_1, \ldots, H_n$, and $K$ must be all be finite subgroups of $H$, so that ultimately $H$ is finite, which is a contradiction.

**Theorem 2.4.** Let $G$ be a locally compact group such that $A(G)$ operator ultra-amenable. Then $G$ is discrete and amenable, and contains no infinite abelian subgroup.

**Proof.** That $G$ has no infinite abelian subgroup is, of course, the statement of Corollary 2.2. That $G$ is amenable follows from fact that $A(G)$ is also operator amenable (see [16]).
To see that $G$ is discrete, we first observe that Proposition 2.3(b) forces $G$ to be totally disconnected. It follows from the Gleason–Yamabe Theorem (see [18]), that every totally disconnected locally compact group has a neighborhood base of the identity consisting of open, compact subgroups. Since any such subgroup must be finite by Proposition 2.3(a), this forces $G$ to be discrete.

In view of this theorem, we shall suppose for the remainder of the paper that all groups considered are discrete.

Historically, O. Yu. Šmidt conjectured that every infinite group contained an infinite abelian subgroup. An affirmative answer to Šmidt’s conjecture would thus have allowed us to show that that operator ultra-amenability of $A(G)$ would indeed force the group $G$ to be finite. However, S. I. Adian and P. S. Novikov (see [1]) showed that Šmidt’s conjecture is false, i.e., there are infinite groups without infinite abelian subgroups. Nevertheless, there are a number of important classes of groups $G$ for which we shall be able to show that the operator ultra-amenability of $A(G)$ does indeed imply that $G$ is finite.

Recall that a group called

- periodic if each of its element has finite order and
- locally finite if each of its finite subsets generates a finite subgroup.

Clearly, every locally finite group is periodic whereas the converse is obviously false. One of the fundamental properties of locally finite groups is that if $G$ is infinite and locally finite, then $G$ contains an infinite abelian subgroup. This was established independently by P. Hall and C. R. Kulatilaka ([8]) as well as M. I. Kargarpolov ([12]). As the operator amenability of $A(G)$ for every group $G$, forces $G$ to be periodic by Corollary 2.2, this means:

**Proposition 2.5.** Let $G$ be a locally finite group such that $A(G)$ is operator ultra-amenable. Then $G$ is finite.

Amongst all discrete amenable groups, perhaps the simplest and best understood are the **elementary amenable groups**. Recall that the elementary amenable groups form the smallest class $\mathcal{E}$ of groups $G$ such that:

(a) $\mathcal{E}$ contains all finite and all abelian groups;

(b) if $G \in \mathcal{E}$ and $H$ is a group isomorphic to $G$, then $H \in \mathcal{E}$;

(c) $\mathcal{E}$ is closed under forming subgroups, quotients, and extensions;

(d) $\mathcal{E}$ is closed under directed unions.
Recall that a group is called \textit{linear} if it is isomorphic to a subgroup of \(\text{GL}(n, \mathbb{F})\) for some field \(\mathbb{F}\). Suppose that \(G\) is linear and amenable, and let \(H\) be a finitely generated subgroup of \(G\). As a consequence, of the main result of \cite{17}, \(H\) either contains a copy of the free group on two generators or has a solvable subgroup of finite index. Since \(H\) is amenable, only the latter is possible. Now, solvable groups are elementary amenable, as are finite extensions of solvable groups. Since \(G\) is the directed union of all of its finitely generated subgroups, it follows that \(G\) is elementary amenable. Moreover, C. Chou (\cite{2, Theorem 2.3}) showed that a periodic elementary amenable group is always locally finite.

We thus obtain the following corollary to Proposition \ref{prop:elementary amenability}:

\begin{corollary}
Let \(G\) be a linear group or an elementary amenable group such that \(A(G)\) is operator ultra-amenable. Then \(G\) is finite.
\end{corollary}

Finally, we say that a group \(G\) has \textit{polynomial growth} if, for each finite set \(F \subseteq G\), there is \(p \in \mathbb{N}\) such that \(|F^n| = O(n^p)\) for all \(n \in \mathbb{N}\). (Here, \(F^n\) denotes the set \(\{x_1 \cdots x_n : x_1, \ldots, x_n \in H\}\).)

We have:

\begin{proposition}
Let \(G\) be a group with polynomial growth such that \(A(G)\) is operator ultra-amenable. Then \(G\) is finite.
\end{proposition}

\begin{proof}
Let \(H\) be a finitely generated subgroup of \(G\). Then \(H\) also has polynomial growth. It follows from a result of M. L. Gromov \cite{7} that \(H\) has a nilpotent subgroup \(N\) of finite index in \(H\). If \(N\) were infinite, it would contain an infinite abelian subgroup, which is impossible by Lemma \ref{lem:finite index}. Consequently \(N\) must be finite, as must be \(H\). This shows that \(G\) is locally finite. We conclude from Proposition \ref{prop:elementary amenability} that \(G\) is finite.
\end{proof}

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