Sharp blow-up result for the intercritical Inhomogeneous NLS equation

Yuan Li

Abstract

In this paper, we consider the intercritical inhomogeneous nonlinear Schrödinger equation. For the radial symmetry initial data, we construct the ring blow-up solutions and obtain blow-up speed. This result implies that the upper bound on the blow-up speed given by Cardoso and Farah [J. Funct. Anal.,281(8) No.109134, (2021)] is sharp.

Keywords: Inhomogeneous NLS equation; Intercritical regime; Ring blow-up solution; Sharp upper bound

1 Introduction and Main Result

We consider the following inhomogeneous nonlinear Schrödinger equation

\[ \begin{cases} 
i u_t + \Delta u + |x|^{-\sigma} |u|^{p-1} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(t_0, x) = u_0(x), & \end{cases} \tag{INLS} \]

where \( u(t, x) \) is a complex-valued function, \( 0 < \sigma < \min\{ \frac{N}{2}, 2 \} \) and \( t_0 \in \mathbb{R} \). This kind of problem arises naturally in nonlinear optics for the propagation of laser beams. We refer the reader to Gill [13], Liu and Tripathi [18] for more details.

Let us review some well-known results about (INLS). From Genoud and Stuart [9], also see Guzman [15], given \( u_0 \in H^1(\mathbb{R}^N) \), there exists a unique solution \( u \in C([t_0, T), H^1(\mathbb{R}^N)) \) to (INLS) and there holds the blowup alternative:

\[ T < +\infty, \quad \text{implies} \quad \lim_{t \to \infty} \| u(t) \|_{H^1} = +\infty. \]

Moreover, the \( H^1 \) flow admits the conservation laws:

Mass:

\[ M(u(t)) = \int |u(t, x)|^2 \, dx = M(u_0); \]

Energy:

\[ E_\sigma(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 \, dx - \frac{1}{p + 1} \int |x|^{-\sigma} |u(t, x)|^{p+1} \, dx = E_\sigma(u_0). \]
In addition, equation (INLS) has the following symmetries
\[ u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) e^{i\gamma}, \quad \lambda > 0, \ \gamma \in \mathbb{R}. \]
It leaves invariant the norm in the homogeneous Sobolev space \( \dot{H}^{s_c} \), where \( s_c = \frac{N}{2} - \frac{2-\sigma}{p-1} \).

If \( s_c = 0 \), the problem is mass-critical, if \( s_c = 1 \) it is energy-critical and if \( 0 < s_c < 1 \), it is mass super-critical and energy subcritical or just intercritical. The existence of solution with finite maximal time of existence is already known in \( H^1 \) for the equation (INLS). As for the classical NLS equation this is a consequence of the following Virial identity
\[ \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 = 8((p-1)s_c + 2)E(u_0) - 4(p-1)s_c ||\nabla u(t)||_{L^2}^2, \]
satisfied by solutions to (INLS) with initial data
\[ u_0 \in \Sigma = H^1 \cap \{ xu \in L^2 \}. \]

This was obtained by Farah [8] and Dinh [7]. From above identity, the solution blows up in finite time if \( E_\sigma(u_0) < 0 \). In particular, Cardoso and Farah [3, 4] obtained that if the initial data \( u_0 \in H^1 \) and the maximal time of existence \( T^* > 0 \) for the corresponding solution \( u \in ([0, T), H^1) \) of (INLS) is finite, then the following space-time upper bound holds
\[ \int_t^{T^*} (T^* - \tau) ||\nabla u(\tau)||_{L^2}^2 d\tau \leq C(u_0)(T^* - t) \frac{2(3-\sigma)}{(p-1)(N-2+2\sigma+(5-\sigma))}, \]
for \( t \) close enough to \( T^* \) (\( N \geq 3 \) and \( 0 < \sigma < 1 \)). For more details about the existence of blow-up solution and blow-up rate, one can see [1, 4, 20, 23] and the references therein.

If \( \sigma = 0 \), then (INLS) is the classical NLS equation
\[ \begin{cases} 
i u_t + \Delta u + |u|^{p-1} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
u(t_0, x) = u_0(x). \end{cases} \]

From Ginibre and Velo [14], given \( u_0 \in H^1 \), there exists a unique solution \( u \in C([t_0, T), H^1) \) and there holds the blow-up alternative. Moreover, the \( H^1 \) flow satisfies the conservation laws of mass, energy, momentum. Further more, a group of \( H^1 \) symmetries leaves the flow invariant: if \( u(t, x) \) solves (1.2) then for any \( (\lambda_0, \tau_0, x_0, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \), so does
\[ v(t, x) \mapsto \lambda_0^{\frac{2}{p-1}} u(\lambda_0^2 t + \tau_0, \lambda_0 x + x_0) e^{i\gamma_0}, \]
and the Galilean drift
\[ v(t, x) \mapsto u(t, x - \beta_0 t) e^{i\frac{\beta_0}{2} \left( x - \frac{\beta_0 t}{2} \right)}, \quad \beta_0 \in \mathbb{R}^N. \]
Notice that by the scaling invariant homogeneous Sobolev space \( \dot{H}^{s_c^{\sigma}} \), we see that \( s_c^{\sigma} = \frac{N}{2} - \frac{2}{p-1} > 0 \). If \( 0 < s_c^{\sigma} < 1 \), this means that the problem (1.2) is mass super-critical and energy-subcritical. If the initial data \( u_0 \in \Sigma = H^1 \cap \{ xu \in L^2 \} \), then from [5], the following Virial identity holds
\[ \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 4N(p-1)E_0(u_0) - \frac{16s_c^{\sigma}}{N-2s_c^{\sigma}} \int |\nabla u|^2 \leq 4N(p-1)E_0(u_0). \]
This implies that if the initial energy $E_0(u_0) < 0$, then the solution of problem (1.2) blows up in finite time. Also, (1.3) implies that there exists a universal upper bound on blow-up rate (see [5]):

$$\int_0^T (T - t) \| \nabla u(t) \|_{L^2}^2 dt < +\infty.$$ 

Furthermore, Merle, Raphaël and Szeftel [19] obtained the upper bound on the blow-up speed for the radial data ($N \geq 2, \ p \leq 5$),

$$\int_t^T (T - \tau) \| \nabla u(\tau) \|_{L^2}^2 d\tau \leq C(T - t)^{\frac{2(5-p)}{(p-1)(N-1)+5-p}}. \quad (1.4)$$

They also constructed the collapsing ring blow-up solution ($N \geq 2, 1 + \frac{2}{N} < p < \min \{ \frac{N+2}{N-2}, 5 \}$),

$$u(t, x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} [Q e^{-i\beta_\infty y}] \left( \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma},$$

where

$$\lambda(t) \sim (T - t)^{1+\frac{5-p}{(p-1)(N-1)}}, \quad \alpha(t) \sim (T - t)^{\frac{(5-p)}{(p-1)(N-1)+5-p}},$$

and $Q$ (see [12, 16]) is the one-dimensional unique positive ground state of

$$-\Delta Q + Q + Q^p = 0, \quad (1.5)$$

and proved that the upper bound (1.4) is sharp.

Inspired by [19], in this paper, we aim to construct the ring blowup solution to the inhomogeneous NLS equation

$$iu_t + \Delta u + |x|^{-\sigma} |u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad (1.6)$$

and prove the upper bound (1.1) is sharp.

Throughout this paper, we define

$$a_\infty = \frac{1}{2(\sigma + 1)}, \quad \beta_\infty = \sqrt{\frac{1}{3(1+\sigma)}}, \quad (1.7)$$

Now we state our main result.

**Theorem 1.1.** Let $0 < \sigma < 1$ and $Q$ be the one-dimensional mass subcritical ground state solution to (1.5) ($p = 3$). Then there exist a time $t_0 < 0$ and a solution $u \in C([t_0, 0), H^1)$ of (1.6) with radial symmetry which blows up at time $T = 0$. More precisely, it holds that

$$u(t, x) - (\alpha(t))^{\frac{2}{p}} \frac{1}{\lambda(t)} [Q e^{-i\beta_\infty y}] \left( \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)} \rightarrow 0 \quad \text{in} \ L^2(\mathbb{R}^3) \quad \text{as} \ t \rightarrow 0^-,$$
where \( y = \frac{r-\alpha(t)}{\lambda(t)} \) and
\[
\lambda(t) \sim |t|^\frac{1}{1-\sigma \alpha \infty}, \quad \alpha(t) \sim |t|^\frac{1}{1-\sigma \alpha \infty}, \quad \gamma(t) \sim |t|^\frac{1}{1-\sigma \alpha \infty}, \quad \text{as} \quad t \to 0^-.
\]

Moreover, the blowup speed is given by
\[
\|\nabla u(t)\|_{L^2} \sim \frac{1}{|T^* - t|^{\frac{p}{2} + \frac{1}{2}}} \quad \text{as} \quad t \to T^*.
\] (1.8)

**Comments:**
1. Extension. Similar result can be addressed for the problem (INLS) with \( N \geq 2, \sigma \in (0, 1) \) and \( 1 + \frac{4-2\sigma}{N} < p < \min \left\{ \frac{N+2-2\sigma}{N-2}, 5 \right\} \). We claim that blowup solution have the following form
\[
u(t, x) - (\alpha(t))^{\frac{2}{p-1}} \frac{1}{\lambda^{p-1}(t)} \left[ Q e^{-i\beta_\infty^1 y} \right] \left( \frac{r-\alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to 0 \text{ in } L^2(\mathbb{R}^N) \quad \text{as} \quad t \to 0^-,
\]
where \( y = \frac{r-\alpha(t)}{\lambda(t)} \) and
\[
\lambda(t) \sim |t|^\frac{1}{1-\sigma \alpha \infty}, \quad \alpha(t) \sim |t|^\frac{1}{1-\sigma \alpha \infty}, \quad \gamma(t) \sim |t|^\frac{1}{1-\sigma \alpha \infty}, \quad \text{as} \quad t \to 0^-.
\]

Here
\[
a_\infty^1 = \frac{5 - p}{(p-1)(N-1) + 4\sigma}, \quad \beta_\infty^1 = \sqrt{\frac{2(p-1)(5-p)}{(p+3)((p-1)(N-1) + 4\sigma)}}.
\]

The corresponding blowup speed speed is
\[
\|\nabla u(t)\|_{L^2} \sim \frac{1}{|T^* - t|^{\frac{p}{2} + \frac{1}{2}}} \quad \text{as} \quad t \to T^*.
\]

The proof carries over verbatim except for some technicalities when the nonlinearity \(|x|^{-\sigma}|u|^{p-1}u\) fails to be smooth.

2. Sharp upper bound on the blowup speed. The blowup rate (1.8) of ring solutions saturates the upper bound (1.1), which is therefore optimal in the radial setting.

3. Although in [2, 21], they considered the minimal mass blowup solution of inhomogeneous mass critical NLS equation, where the inhomogeneous factor is smooth, bounded and integrable. However, in our case, the inhomogeneous term \( \frac{1}{|x|^\sigma} \) is singular, so we need to use some new techniques to handle it. Fortunately, our problem (INLS) has the scaling symmetry compared to the equations in [2, 21]. This is very useful for us to construct the approximate profile.

4. Due to the inhomogeneous factor, the parameters are different from the classical NLS equation, see Lemma 2.1 and 3.1, so we need to be careful to obtain the sharp result.
Remark 1.2. Notice that in Theorem 1.1, $Q$ is the ground state of (1.5) instead of the ground state of

$$-\Delta u + u + \frac{1}{|x|^\sigma} u^p = 0. \quad (1.9)$$

One reason is that the inhomogeneous factor $\frac{1}{|x|^\sigma}$ is a singular term, the ground state of equation (1.9) is undefined at origin. Another important reason is that it is very difficult to construct approximate solutions by using the ground state of (1.9). In our argument, we treat $\frac{1}{|x|^\sigma}$ as a weight function, so we can use the ground state $Q$ to construct the approximate solution of the equation (1.6). Actually, it is a naturally choice to construct the approximate solution by using the ground state solution $Q$ of equation (1.5) (see Section 2).

Notations

For the positive $a$ and $b$, the notation $a \lesssim b$ means that $a \leq cb$ holds for a universal constant $c > 0$, $a \gtrsim b$ means that $a \geq cb$ holds for a universal constant $c > 0$, $a \sim b$ means that $a \lesssim b$ and $a \gtrsim b$. Let $(\cdot, \cdot)$ denote the scalar product on $L^2$,

$$(f, g) = \int f(x)\overline{g(x)} \, dx,$$

for $f, g$ two complex valued functions in $L^2$.

Denote the operator

$$\Lambda f = f + x \cdot \nabla f.$$ 

Let $L = (L_+, L_-)$ be the linearized operator around $Q$,

$$L_+ = -\partial_y^2 + 1 - 3Q^2, \quad L_- = -\partial_y^2 + 1 - Q^2,$$

where $Q$ be the one-dimensional mass sub-critical ground state solution to (1.5).

The rest of the paper is as follows. In section 2, we construct an approximate solution $Q_P$ of the renormalized solution. In section 3, we decompose the solution and estimate the modulation parameters. In section 4, we establish a refined energy/virial type estimate. In section 5, we apply the energy estimate to establish a bootstrap argument that will be needed in the construction of solutions that ring blow up. In section 6, we give the proof of the Theorem 1.1. The finally section is the appendix.

2 The approximate solution

In this section, we aim to construct the approximate solution at any order. Let us consider the general modulated

$$u(t, x) = \alpha \tilde{f}(t) \frac{1}{\lambda(t)} v \left( s, \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)} + \frac{ds}{dt} = \frac{1}{\lambda^2}.$$ 

Injecting this into the problem (1.6) yields:

\[
i \partial_s v + v_{yy} + \frac{\lambda}{\alpha} \frac{2}{1 + \frac{1}{\alpha} y} \frac{1}{2} v_y - v + \frac{1}{\lambda + 1} |v|^2 v = i \frac{\lambda_s}{\lambda} \Lambda v + i \frac{\alpha_s}{\lambda} \left( v_y - \frac{\sigma \lambda}{2 \alpha} v \right) + \tilde{\gamma}_s v, \tag{2.1}
\]

where \( \tilde{\gamma}_s = \gamma_s - 1 \).

Now we define

\[
a = \frac{2 \beta \lambda}{\alpha \lambda}. \tag{2.2}
\]

Then, (2.1) is equivalent to

\[
i \partial_s v + v_{yy} + \frac{2 a_\infty a}{1 + \frac{2 a_\infty a}{2 \beta} y} \frac{1}{2 \beta} v_y - v + \frac{1}{1 + \frac{2 a_\infty a}{2 \beta} y} |v|^2 v = i \frac{\lambda_s}{\lambda} \Lambda v + i \frac{\alpha_s}{\lambda} \left( v_y - \frac{\sigma a_\infty a}{4 \beta} v \right) + \tilde{\gamma}_s v,
\]

Let

\[
w(s, y) = v(s, y) e^{i \beta y},
\]

which satisfies

\[
i \partial_s w + w_{yy} - w + \frac{1}{1 + \frac{2 a_\infty a}{2 \beta} y} |w|^2 w + \frac{a_\infty a}{\beta} \frac{1}{1 + \frac{2 a_\infty a}{2 \beta} y} (w_y - i \beta w) + a(i \Lambda w + \beta y w) - \frac{i \sigma a_\infty a w}{2} = - \beta_s w + \left( \frac{\lambda_s}{\lambda} + a \right) (i \Lambda w + \beta y w) + \left( \frac{\alpha_s}{\lambda} + 2 \beta \right) \left( iw_y + \beta w - \frac{i \sigma a_\infty a}{4 \beta} w \right) + (\tilde{\gamma}_s - \beta^2) w. \tag{2.3}
\]

Let

\[
\beta = \beta_\infty + \tilde{\beta}.
\]

We look for an approximate solution to (2.1) of the form

\[
v(s, y) = Q_P(y), \quad \text{where} \quad P = (a, \tilde{\beta}).
\]

Following the slow modulated ansatz strategy developed in [17, 19, 21], we freeze the modulation

\[
\frac{\lambda_s}{\lambda} = -a + A_1(P), \quad \frac{\alpha_s}{\lambda} = -2 \beta, \quad \tilde{\gamma}_s = \beta^2, \quad \beta_s = A_2(P),
\]

where \( A_1 \) and \( A_2 \) are polynomials in \( P \), which will be chosen later to ensure suitable solvability conditions. From the definition of \( a \) (see (2.2)), we have the following relation

\[
a_s + (1 - a_\infty) a^2 - \frac{a}{\beta} A_2 - a A_1 = \frac{a}{\beta} (\tilde{\beta}_s - A_2) + a \left( \frac{\lambda_s}{\lambda} + a - A_1 \right) - \frac{a_\infty a^2}{2 \beta} \left( \frac{\alpha_s}{\lambda} + 2 \beta \right).
\]
From the above, we have
\[
\begin{align*}
i \left( -(1 - a_{\infty})a^2 + \frac{a}{\beta}A_2 + aA_1 \right) \partial_a Q_P + iA_2 \partial_\beta Q_P \\
- (1 + \beta^2)Q_P + i(a - A_1)(1 + y\partial_y)Q_P + 2i\beta \partial_y Q_P - i\sigma a_{\infty}^2 a Q_P \\
+ \partial_y^2 Q_P + \frac{a_{\infty}a}{\beta} \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} \partial_y Q_P + \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} |Q_P|^2 Q_P = -\Psi_P. \tag{2.4}
\end{align*}
\]

Let
\[
Q_P(y) = P_P e^{-i\beta y - ia_{\infty}^2 y^2},
\]
so that (2.4)
\[
\begin{align*}
&i \left( -(1 - a_{\infty})a^2 + \frac{a}{\beta}A_2 + aA_1 \right) \partial_a P_P + iA_2 \partial_\beta P_P \\
&+ \partial_y^2 P_P - P_P + \frac{a_{\infty}a}{\beta} \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} \partial_y P_P + \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} |P_P|^2 P_P \\
&- iA_1(1 + y\partial_y)P_P - A_1 \left( \beta y + \frac{ay^2}{2} \right) P_P + A_2 y P_P \\
&+ \left[ a\beta y + \left( a_{\infty}a^2 + \frac{a}{\beta}A_2 + aA_1 \right) y^2 - i \left( \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} \frac{a_{\infty}a^2}{2\beta} (1 - a_{\infty})y \right) - \frac{i\sigma a_{\infty}^2}{2} \right] P_P \\
&= -\Psi_P e^{i\beta y - ia_{\infty}^2 y^2}. \tag{2.5}
\end{align*}
\]

Next Lemma we will construct the approximate solution to (2.5).

**Lemma 2.1.** (Approximate Profile). Let an integer \( l > 5 \), then there exist polynomials \( A_1 \) and \( A_2 \) of the following form
\[
\begin{align*}
A_1(P) &= \sigma a_{\infty}a + \sum_{2 \leq j + k \leq l - 1} c_{1,j,k} a^j \tilde{\beta}^k, \\
A_2(P) &= \sigma a_{\infty}\beta_{\infty}a + \sum_{2 \leq j + k \leq l - 1} c_{2,j,k} a^j \tilde{\beta}^k, \tag{2.6}
\end{align*}
\]
where \( c_{1,j,k} \) and \( c_{2,j,k} \) are the constants and smooth well-localized solutions \( (T_{j,k}, S_{j,k}) \), such that
\[
P_P = Q + \sum_{1 \leq j + k \leq l - 1} a^j \tilde{\beta}^k (T_{j,k} + iS_{j,k}), \tag{2.7}
\]
where \( Q \) is the one dimensional ground state solution to (1.5)(\( p = 3 \)), is a approximate solution to (2.5) with \( \Psi_P \) smooth and well-localized in \( y \) satisfying the following decay estimate
\[
\Psi_P = O(a_{\infty}^l |y|^q e^{-|y|}).
\]
Moreover, the approximate profile holds the decay estimate
\[
|P_P| \lesssim (1 + |y|^2) e^{-|y|}. \tag{2.8}
\]
Proof. From [6, 22], the kernel of \( L = (L_+, L_-) \) is explicit

\[
\ker L_+ = \text{span}\{Q'\}, \quad \ker L_- = \text{span}\{Q\}.
\] (2.9)

It follows from the kernel properties (2.9) of \( L_+ \) and \( L_- \) and well-known properties of the Helmholtz kernel that

\[
\forall g \in H^1(\mathbb{R}), \quad (g, Q') = 0, \quad \exists f_+ \in H^1(\mathbb{R}), \quad \text{s. t.} \quad L_+ f_+ = g,
\]

\[
\forall g \in H^1(\mathbb{R}), \quad (g, Q) = 0, \quad \exists f_- \in H^1(\mathbb{R}), \quad \text{s. t.} \quad L_+ f_- = g.
\]

We also give the following Pohozaev identities

\[
-2 \int (Q')^2 + 2 \int Q^2 = \int Q^4,
\]

\[
\int (Q')^2 + \int Q^2 = \int Q^4.
\] (2.10)

From the above identities (2.10), we have

\[
3 \int (Q')^2 = \int Q^2 = \frac{3}{4} \int Q^4.
\] (2.11)

We recall that \( L = (L_+, L_-) \) has a generalized nullspace characterized by the following algebraic identities generated by the symmetry group:

\[
L_- Q = 0, \quad L_- yQ = -2 \nabla Q,
\]

\[
L_+ Q' = 0, \quad L_+ \Lambda Q = -2 Q.
\]

The proof proceeds by injecting the expansion (2.7) into (2.5), identifying the terms with the same homogeneity, and inverting the corresponding operator. Notice that we have the following Taylor expansion

\[
(1 + x)^b = 1 + bx + b(b - 1) \frac{x^2}{2!} + b(b - 1)(b - 2) \frac{x^3}{3!} + \ldots.
\]

Now we divide the proof into the following steps.

**Step 1:** General case. Let \( j + k \geq 1 \). Assume that \( T_{p,q}, S_{p,q}, c_1_{p,q} \) and \( c_2_{p,q} \) for \( q + p \leq j + k - 1 \) have been constructed. Then, identifying the terms homogeneous of order \((j, k)\) in (2.5) yields a linear system of the following type

\[
\begin{cases}
L_+ T_{j,k} = h_{1,j,k} - c_{1,j,k} \beta yQ + c_{2,j,k} yQ,
L_- S_{j,k} = h_{2,j,k} - c_{1,j,k} \Lambda Q,
\end{cases}
\] (2.12)

where \( h_{1,j,k} \) and \( h_{2,j,k} \) can be computed explicitly and only depend on \( T_{p,q}, S_{p,q}, c_1_{p,q} \) and \( c_2_{p,q} \) for \( q + p \leq j + k - 1 \). According to (2.9), the invertibility of (2.12) must be satisfy the orthogonality condition

\[
\begin{cases}
(h_{1,j,k} - c_{1,j,k} \beta yQ + c_{2,j,k} yQ, Q') = 0,
(h_{2,j,k} - c_{1,j,k} \Lambda Q, Q) = 0.
\end{cases}
\] (2.13)
Now we claim: For all $1 \leq j + k \leq l - 1$, let

$$c_{1,j,k} = \frac{1}{(Q, \Lambda Q)} (h_{2,j,k}, Q), \quad (2.14)$$

$$c_{2,j,k} = \frac{2}{\|Q\|_{L^2}^2} (h_{1,j,k}, Q') + \frac{\beta_\infty}{(Q, \Lambda Q)} (h_{2,j,k}, Q). \quad (2.15)$$

Then, there exist $(T_{j,k}, S_{j,k})$ solution of $(2.5)$ for all $1 \leq j + k \leq l - 1$. Furthermore, $T_{j,k}$ and $S_{j,k}$ are smooth and decay as

$$T_{j,k}, S_{j,k} = \mathcal{O}(|y|^{2(j+k)} e^{-|y|}).$$

In fact, to be able to solve for $(T_{j,k}, S_{j,k})$. If we choosing $c_{1,j,k}$ and $c_{2,j,k}$ as in $(2.14)$ and $(2.15)$, respectively, we may solve for $(T_{j,k}, S_{j,k})$ solution of $(2.12)$.

Next, we investigate the smoothness and decay properties of $(T_{j,k}, S_{j,k})$. Identifying the terms homogeneous of order $j + k$ in $(2.5)$, we have for $h_{1,j,k}$ and $h_{2,j,k}$ defined in $(2.12)$

$$h_{1,j,k} = \sum_{p+q \leq j+k-1} \left( a_{1,p,q} y^{j+k-p-q} T_{p,q} + a_{2,p,q} y^{j+k-p-q} S_{p,q} + a_{3,p,q} y^{j+k-p-q} T'_{p,q} \right) + NL_j^{(1)},$$

$$h_{2,j,k} = \sum_{p+q \leq j+k-1} \left( a_{4,p,q} y^{j+k-p-q} T_{p,q} + a_{5,p,q} y^{j+k-p-q} S_{p,q} + a_{6,p,q} y^{j+k-p-q} T'_{p,q} \right) + NL_j^{(2)},$$

where $T_{0,0} = Q$, $a_{n,p,q}$ are real numbers which may be explicitly computed, and where $NL_j^{(1)}$ and $NL_j^{(2)}$ are the contributions coming from the Taylor expansion of the term $\frac{1}{(1+\frac{1}{2\beta_\infty} |y|^2)^{p} P_p}$. Then, by the similar argument as $(19)$, we can obtain the smoothness and decay properties of $T_{j,k}$ and $S_{j,k}$. Here we omit it.

**Step 2**: Computation of $c_{1,1,0}$ and $c_{2,1,0}$. From the terms homogeneous of order $(1,0)$ in $(2.5)$, we get

$$\begin{cases}
L_+ T_{1,0} = a_0 y + \beta_\infty y Q - c_{1,1,0} \beta_\infty y Q + c_{2,1,0} y Q + \frac{\sigma a_\infty}{\beta_\infty} Q^3, \\
L_- S_{1,0} = -c_{1,1,0} \Lambda Q - \frac{\sigma a_\infty}{2} Q.
\end{cases}$$

From $(2.9)$ and $(2.13)$, the solvability conditions for $T_{1,0}$ and $S_{1,0}$ are equivalent to

$$\begin{cases}
(\frac{a_0}{\beta_\infty} y + \beta_\infty y Q - c_{1,1,0} \beta_\infty y Q + c_{2,1,0} y Q + \frac{\sigma a_\infty}{\beta_\infty} Q^3, Q') = 0, \\
(-c_{1,1,0} \Lambda Q - \frac{\sigma a_\infty}{2} Q, Q) = 0.
\end{cases}$$

By the second equation, we can obtain

$$c_{1,1,0} = \frac{\sigma a_\infty}{2(\Lambda Q, Q)} \|Q\|_{L^2}^2 = \sigma a_\infty.$$

Notice that

$$\left( \frac{a_\infty}{\beta_\infty} Q' + \beta_\infty y Q, Q' \right) = \frac{a_\infty}{\beta_\infty} \int (Q')^2 - \frac{\beta_\infty}{2} \int Q^2 = \left( \frac{a_\infty}{\beta_\infty} - \frac{3\beta_\infty}{2} \right) \int (Q')^2 = 0,$$
where we used the equality (2.11) and the definition of \( a_\infty \) and \( \beta_\infty \) (see (1.7)). Hence, we can obtain that

\[
c_{2,1,0} = \beta_\infty c_{1,1,0},
\]

since \( Q \) is even function and \( Q' \) is odd.

**Step 3:** Computation of \( c_{1,0,1} \) and \( c_{2,0,1} \). From the terms homogeneous of order \((0, 1)\) in (2.5) and get

\[
\begin{align*}
L_+ T_{0,1} &= -c_{1,0,1} \beta_\infty y Q + c_{2,0,1} y Q,
L_- S_{0,1} &= -c_{1,0,1} \Lambda Q.
\end{align*}
\]

By (2.13), we can obtain

\[
c_{1,0,1} = 0 = c_{2,0,1}.
\]

**Step 4:** Computation of \( c_{1,2,0} \) and \( c_{2,2,0} \). From the terms homogeneous of order \((2, 0)\) in (2.5) and get

\[
\begin{align*}
L_+ T_{2,0} &= \left( -(1 - a_\infty) + c_{1,1,0} + \frac{c_{2,1,0}}{\beta_\infty} \right) S_{1,0} + \frac{a_{\infty}}{\beta_\infty} T_{1,0} - \frac{a_\infty}{2\beta_\infty} y Q' + 3Q T_{1,0}^2 + Q S_{1,0}^2 + \frac{a_\infty}{2\beta_\infty} Q^2 T_{1,0} \\
&\quad + \frac{a_\sigma}{2\beta_\infty} Q^3 + \beta_\infty y T_{1,0} + \left( a_\infty + \frac{c_{2,1,0}}{\beta_\infty} + c_{1,1,0} \right) \frac{1}{2} y^2 Q + \frac{a_{\infty}}{2} S_{1,0} \\
&\quad - \frac{a_\infty}{2\beta_\infty} y Q + c_{2,2,0} y Q + \frac{a_\infty}{2} S_{1,0},
L_- S_{1,0} &= \left( -(1 - a_\infty) + c_{1,1,0} + \frac{c_{2,1,0}}{\beta_\infty} \right) T_{1,0} + \frac{a_{\infty}}{\beta_\infty} S_{1,0} + 2Q T_{1,0} S_{1,0} + \frac{a_{\infty}}{2\beta_\infty} Q^2 S_{1,0} + \beta_\infty y S_{1,0} \\
&\quad - \frac{a_\infty}{2\beta_\infty} (1 - \frac{2a_\infty}{\beta_\infty} T_{1,0} - a_\infty) y Q - c_{1,2,0} \Lambda Q.
\end{align*}
\]

Notice that \( T_{1,0} \) is an odd function, while \( S_{1,0} \) is an even function. From the solvability conditions for \( T_{2,0} \) and \( S_{2,0} \), we can obtain

\[
c_{1,2,0} \neq 0, \quad c_{2,2,0} \neq 0.
\]

**Step 5:** Conclusion. The error term \( \Psi_P \) consists of polynomial in \((T_{j,k}, S_{j,k})_{j+k\leq l-1} \) with lower-order \( l \), the error between the Taylor expansion of the potential terms \( \frac{a_{\infty}}{2l+1} \frac{2}{2l+1} y \) and \( \frac{2}{1+\frac{2a_{\infty}}{\beta_\infty} y} \) in (2.5), and between the Taylor expansion of the nonlinear term \( \frac{1}{|y|} |P_p|^2 P_p \) and \( \frac{1}{|y|^l} \). Using the exponential decay of \( P_p \), we can easily estimate these terms.

\[
\square
\]

### 3 Modulation Estimate

In this section, we aim to give the parameters estimate that will be useful to prove the existence of the blowup solution. Now we introduce a smooth cut-off function

\[
\xi(y) = \begin{cases} 
0 & \text{for } y \leq -2, \\
1 & \text{for } y \geq -1.
\end{cases}
\]
Let \( \xi_a(y) = \xi(\sqrt{ay}) \) and define
\[
\mathcal{Q}_P(y) = \xi_a(y)P_\mathcal{P}(y)e^{-i\beta y - ia^2 2}. \tag{3.1}
\]

Given \( C^1 \) modulation parameters \( (\lambda(t), \alpha(t), \gamma(t), \tilde{\beta}(t)) \) such that \( 0 < a = \frac{2\beta}{a_\infty} \frac{\lambda(t)}{\alpha(t)} \ll 1 \), let
\[
\tilde{\mathcal{Q}}(t, x) = \alpha^2(t) \frac{1}{\lambda(t)} \mathcal{Q}_P \left( s, \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)}. \tag{3.2}
\]

Then, by Lemma 2.1, we can obtain that \( \tilde{\mathcal{Q}} \) is a smooth radially symmetric function which satisfies
\[
i \partial_t \tilde{\mathcal{Q}} + \Delta \tilde{\mathcal{Q}} + |x|^{-\sigma}|\tilde{\mathcal{Q}}|^2 \tilde{\mathcal{Q}} = \psi = \alpha^2(t) \frac{1}{\lambda^3(t)} \Psi \left( t, \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)} \tag{3.3}
\]
with
\[
\Psi = - (\gamma_s - 1 - \beta^2) \mathcal{Q}_P - i \left( \frac{\lambda_s}{\lambda} - a - A_1 \right) (\Lambda \mathcal{Q}_P - a \partial_a \mathcal{Q}_P)
- i \left( \frac{\alpha_s}{\lambda} + 2\beta \right) \left( \partial_y \mathcal{Q}_P + \frac{a_\infty}{2\beta} a^2 \partial_a \mathcal{Q}_P - \frac{\sigma a_\infty a}{4\beta} \mathcal{Q}_P \right)
+ \left( a_s + (1 - a_\infty) a^2 - \frac{a}{\beta} A_1 - a A_1 \right) \partial_a \mathcal{Q}_P
+ i(\beta_s - A_2) \left( \partial_\beta \mathcal{Q}_P + \frac{a}{\beta} \partial_a \mathcal{Q}_P \right)
+ \mathcal{O} \left( \frac{e^{-|y|}}{a^{\sigma_1} 1_{y \sim \frac{1}{\sigma}} + |P|^4 \xi_a |y|^c e^{-|y|}} \right). \tag{3.4}
\]

3.1 The approximation of the parameters

In this subsection, we give the exact modulation equations formally predicted by the \( \mathcal{Q}_P \) construction. Now we have the following lemma.

**Lemma 3.1.** There exist \( t_0 < 0 \) small enough and a solution \( (\lambda, a, \tilde{\beta}, \alpha, \gamma) \) to the dynamical system
\[
\begin{align*}
\frac{\lambda_s}{\lambda} &= -a + A_1(\mathcal{P}), & \mathcal{P} &= (a, \tilde{\beta}), \\
\frac{\alpha_s}{\lambda} &= -2\beta, & \beta &= \beta_\infty + \tilde{\beta}, \\
\gamma_s &= 1 + \beta^2, \\
\frac{\partial_a}{\partial t} &= \frac{1}{\lambda_s}, \\
a &= \frac{2\beta}{a_\infty} \frac{\lambda_s}{\alpha_s}.
\end{align*} \tag{3.5}
\]
which is defined on $[t_0, 0)$. Moreover, this solution satisfies the following bounds,

$$a(t) = \frac{1}{(1 - (2\sigma + 1)a_\infty)} \frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty} B_2 \frac{1 - \sigma a_\infty}{1 - (\sigma - 1)a_\infty} \frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty} \left(1 + O\left(\log|t|/|t|\right)\right),$$

(3.6)

$$|\tilde{\beta}| = O\left(|t| \frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty}\right)$$

(3.7)

$$\lambda(t) = B_2 \frac{1 - \frac{1}{1 - (\sigma - 1)a_\infty}}{1 - \frac{1}{1 - (\sigma - 1)a_\infty}} \frac{1 - \frac{1}{1 - (\sigma + 1)a_\infty}}{1 - \frac{1}{1 - (\sigma + 1)a_\infty}} |t| \frac{1 - \frac{1}{1 - (\sigma - 1)a_\infty}}{1 - \frac{1}{1 - (\sigma - 1)a_\infty}} \left(1 + O\left(\log|t|/|t|\right)\right),$$

(3.8)

$$\alpha(t) = b\lambda \frac{a_\infty}{1 - (\sigma + 1)a_\infty} = b_\infty \frac{1 - \frac{1}{1 - (\sigma - 1)a_\infty}}{1 - \frac{1}{1 - (\sigma - 1)a_\infty}} B_2 \frac{1 - \frac{1}{1 - (\sigma + 1)a_\infty}}{1 - \frac{1}{1 - (\sigma + 1)a_\infty}} |t| \frac{1 - \frac{1}{1 - (\sigma + 1)a_\infty}}{1 - \frac{1}{1 - (\sigma + 1)a_\infty}} \left(1 + O\left(\log|t|/|t|\right)\right),$$

(3.9)

$$\gamma(t) = (1 + \beta_\infty^2) B_1 \frac{1 - \frac{1}{1 - (\sigma - 1)a_\infty}}{1 - \frac{1}{1 - (\sigma - 1)a_\infty}} B_2 \frac{1 - \frac{1}{1 - (\sigma + 1)a_\infty}}{1 - \frac{1}{1 - (\sigma + 1)a_\infty}} |t| \frac{1 - \frac{1}{1 - (\sigma + 1)a_\infty}}{1 - \frac{1}{1 - (\sigma + 1)a_\infty}} + O(\log|t|),$$

(3.10)

where

$$B_1 = \frac{1 - (\sigma - 1)a_\infty}{1 - (\sigma + 1)a_\infty}, \quad B_2 = \frac{a_\infty}{2(1 - (2\sigma + 1)a_\infty)b_\infty}$$

and a universal constant

$$|b_\infty - 1| \ll 1.$$

To prove Lemma 3.1, we need to prove the following lemma.

**Lemma 3.2.** There exists a universal constant $s_0 \gg 1$ such that the following holds. Let

$$\frac{1}{2} < b_0 < 1, \quad \gamma_0 \in \mathbb{R}, \quad a_0 = \frac{1}{(1 - (2\sigma + 1)a_\infty)s_0}.$$  \hspace{1cm} (3.11)

Then the solution $(\lambda, a, \tilde{\beta}, \alpha, \gamma)$ to the dynamical system

$$\begin{cases}
\frac{\lambda}{\lambda} = -a + A_1(P), \quad P = (a, \tilde{\beta}), \\
\frac{\alpha}{\lambda} = -2\beta, \quad \beta = \beta_\infty + \beta, \\
\beta_\lambda = A_2, \\
\gamma_\lambda = 1 + \beta^2, \\
\frac{ds}{dt} = \frac{1}{\lambda}_r, \\
a = \frac{2\beta_\lambda}{a_\infty}/a,
\end{cases}
$$

with

$$\begin{cases}
\frac{a(s_0)}{a_\infty} = b_0, \\
\lambda_\infty^{-\sigma a_\infty}(s_0) = a(s_0) = a_0, \\
\beta(s_0) = \frac{1}{s_0}, \\
\gamma(s_0) = \gamma_0.
\end{cases}$$

(3.12)

is defined on $[s_0, +\infty)$. Moreover, there exists $b_\infty > 0$ with

$$b_\infty = b_0 + o(s_\infty),$$

(3.13)

such that the following asymptotics hold on $[s_0, +\infty)$:

$$a(s) = \frac{1}{(1 - (2\sigma + 1)a_\infty)s} + O\left(\frac{1}{s^2}\right), \quad |\tilde{\beta}(s)| \lesssim \frac{1}{s},$$

(3.14)
\[
\lambda(s) = \left( \frac{a_\infty}{2(1 - (2\sigma + 1)a_\infty)b_\infty} \right)^{\frac{1}{1 - \sigma a_\infty}} \left( 1 + O \left( \frac{\log s}{s} \right) \right), \tag{3.15}
\]

\[
\alpha(t) = b\lambda^{1-s_a}\infty = b_\infty \left( \frac{a_\infty}{2(1 - (2\sigma + 1)a_\infty)b_\infty} \right)^{\frac{1}{1 - \sigma a_\infty}} \left( 1 + O \left( \frac{\log s}{s} \right) \right), \tag{3.16}
\]

\[
\gamma(s) = (1 + \beta^2_\infty) s + O(||s\log||). \tag{3.17}
\]

**Proof. Step 1:** Bootstrap bounds. From the Cauchy-Lipschitz theorem, we can obtain the local existence. To control the solution on large positive times, let us introduce the auxiliary function

\[
b = \frac{\alpha}{\lambda^{1-s_a}\infty},
\]

which from (3.12) satisfies

\[
\frac{db}{ds} = \frac{a_s \lambda a}{\lambda^{1-s_a}\infty} - \frac{a_\infty \lambda a}{(1 - \sigma a_\infty)\lambda^{1-s_a}\infty} + 1 = \frac{\alpha_s \alpha a}{\lambda^{1-s_a}\infty} - \frac{a_\infty \lambda a}{(1 - \sigma a_\infty)\lambda^{1-s_a}\infty} + 1
\]

\[
= - \frac{a_\infty \lambda a}{1 - \sigma a_\infty} \left( \frac{\lambda}{\lambda} + (1 - \sigma a_\infty)a \right) = - \frac{a_\infty}{1 + \sigma a_\infty} b(\lambda a - \sigma a_\infty a).
\]

Hence, system (3.12) is equivalent to

\[
\begin{align*}
\frac{db}{ds} &= - \frac{a_\infty}{1 - \sigma a_\infty} b(\lambda a - \sigma a_\infty a), \\
\frac{d}{ds} \left( a_s + (1 - a_\infty) a^2 \right) &= \frac{2}{\beta} A_2 + a A_1, \\
\frac{d}{ds} \beta_s &= A_2, \\
a &= \frac{2\lambda}{a_\infty \alpha}, \quad \beta = \lambda + \tilde{\beta},
\end{align*}
\]

We assume the following priori bounds: for \( s_0 \leq s \leq s_1 \),

\[
|b(s)| \leq 1 + 2b_0, \quad |\beta(s)| \leq \frac{1}{s}, \quad \left| a(s) - \frac{1}{(1 - (2\sigma + 1)a_\infty)s} \right| \leq \frac{\log s^2}{s^2}. \tag{3.19}
\]

**Step 2.** Closing the bootstrap. We claim that the bounds (3.19) can be improved on \([s_0, s_1]\) provided \( s_0 \) has been chosen large enough. Indeed, let us close the \( a \) bound. From (3.18) and (3.19), we have

\[
\left| \frac{1}{a(s)} - (1 - a_\infty - 2\sigma a_\infty)s \right| \lesssim \left| \frac{1}{a(s_0)} - (1 - a_\infty - 2\sigma a_\infty)s_0 \right| + \int_{s_0}^{s} \frac{1}{s} \lesssim \log s,
\]

where in the Penultimate step we use the definition of \( A_1 \) and \( A_2 \) (see (2.6)) and thus using the boundary condition on \( a \) at \( s_0 \) and the initialization (3.11), we have

\[
\left| \frac{1}{a(s)} - (1 - a_\infty - 2\sigma a_\infty)s \right| \lesssim \left| \frac{1}{a(s_0)} - (1 - a_\infty - 2\sigma a_\infty)s_0 \right| + \int_{s_0}^{s} \frac{1}{s} \lesssim \log s,
\]
where we assume \( s \geq s_0 \gg 1 \). Hence,

\[
\left| a(s) - \frac{1}{(1 - a_{\infty} - 2\sigma a_{\infty})s} \right| \lesssim \frac{\log s}{s^2}. \tag{3.20}
\]

Next, we consider \( \tilde{\beta} \). Since

\[
A_2 = \sigma a_{\infty} \beta_0 a + O \left( a^2 + |\tilde{\beta}|^2 \right),
\]

we obtain,

\[
\beta_s = \sigma a_{\infty} \beta_0 a + O \left( \frac{1}{s^2} \right).
\]

From (3.19) and (3.20), we have

\[
\left| d \frac{d}{ds} \left( s^{1-a_{\infty}-2\sigma a_{\infty}} \tilde{\beta} \right) \right| \lesssim s^{1-a_{\infty}-2\sigma a_{\infty}} \left( \frac{\log s}{s^2} |\tilde{\beta}| + \frac{1}{s^2} \right) \lesssim s^{1-a_{\infty}-2\sigma a_{\infty}}. \]

Using the boundary condition (3.18) and \( \frac{2}{1-a_{\infty}-2\sigma a_{\infty}} - 2 > 0 \), we get

\[
\left| s^{1-a_{\infty}-2\sigma a_{\infty}} \tilde{\beta}(s) \right| \lesssim \left| s^{1-a_{\infty}-2\sigma a_{\infty}} \tilde{\beta}(s_0) \right| + s^{1-a_{\infty}-2\sigma a_{\infty}} - 1 - s^{1-a_{\infty}-2\sigma a_{\infty}} \lesssim s^{1-a_{\infty}-2\sigma a_{\infty}} - 1,
\]

and thus

\[
|\tilde{\beta}(s)| \lesssim \frac{1}{s}. \tag{3.21}
\]

Finally, we estimate \( b \). From (3.18), (3.20) and (3.21), we have

\[
\left| \frac{d}{ds} b \right| \lesssim \frac{1 + 2b_0}{s^2}.
\]

Thus, by (3.18), we deduce

\[
|b(s)| \leq b_0 + C \frac{1 + 2b_0}{s_0} \leq \frac{1}{2} + \frac{3}{2} b_0, \tag{3.22}
\]

for \( s_0 \) large enough. The bounds (3.20), (3.21) and (3.22) improve (3.19), and thus from a standard continuity argument, the bounds (3.20), (3.21) and (3.22) hold on \([s_0, \infty)\) and the solution is global.

**Step 3.** Conclusion. From (3.20) and (3.21), the bound (3.14) is proved. Moreover, from (3.18), we have

\[
\int_{s_0}^{\infty} \left| \frac{d}{ds} b \right| \leq \int_{s_0}^{\infty} \frac{1}{s^2} = o(1) \text{ as } s_0 \to \infty,
\]

and hence there exists \( b_\infty \) satisfying (3.13) such that for any \( s \geq s_0 \),

\[
|b(s) - b_\infty| \lesssim \frac{1}{s}. \tag{3.23}
\]
From (3.18), (3.19) and (3.23), we get

\[ \lambda(s) = \frac{a_{\infty}a_{\alpha}}{2^{\beta}} = \frac{a_{\infty}}{2(1 - (2\sigma + 1)a_{\infty})\beta_{\infty}s} b_{\infty} \lambda^{1 - a_{\infty}} \left( b_{\infty} \lambda^{1 - a_{\infty}} \right) \left( 1 + O \left( \frac{\log s}{s} \right) \right). \]

This means that

\[ \lambda(s) = \left( \frac{a_{\infty}}{2(1 - (2\sigma + 1)a_{\infty})\beta_{\infty}s} b_{\infty} \right)^{1 - a_{\infty}} \left( 1 + O \left( \frac{\log s}{s} \right) \right). \]

By (3.23), we obtain

\[ \alpha(t) = b\lambda^{1 - a_{\infty}} = b_{\infty} \left( \frac{a_{\infty}}{2(1 - (2\sigma + 1)a_{\infty})\beta_{\infty}s} b_{\infty} \right)^{1 - a_{\infty}} \left( 1 + O \left( \frac{\log s}{s} \right) \right). \]

Finally, it remains to estimate \( \gamma \). In view of (3.5) and (3.21), we have

\[ \frac{d\gamma}{ds} = 1 + \beta_{\infty}^2 + O \left( \frac{1}{s} \right), \]

Integration between \( s_0 \) and \( s \), we have

\[ \gamma(s) = (1 + \beta_{\infty}^2)s + O(|\log s|). \]

This complete the proof of Lemma 3.2.

Proof of Lemma 3.1. Notice that from (3.15), we have

\[ \int_{s_0}^{\infty} \lambda^2 < +\infty. \]

Thus, since \( \frac{d}{dt} = \frac{1}{\lambda(t)} \), the time of existence of the dynamical system in time \( t \) is finite, and we may choose the origin of time \( t \) such that the final time is 0. Then, for all \( t_0 \leq t < 0 \), we have

\[ -t = \int_{s_0}^{\infty} \lambda^2 , \]

which together with (3.15) yields

\[ \frac{1}{s} = \left( \frac{1 - (\sigma - 1)a_{\infty}}{1 - (\sigma + 1)a_{\infty}} \right)^{1 - a_{\infty}} \left( \frac{a_{\infty}}{b_{\infty}} \right)^{2^{1 - a_{\infty}} \left( 1 + O \left( \log |t||t|^{1 - (\sigma - 1)a_{\infty}} \right) \right). \]

Injecting (3.24) into (3.14), (3.15), (3.16) and (3.17), we get the estimates (3.6), (3.7), (3.8), (3.9) and (3.10). This concludes the proof of Lemma 3.1.
3.2 Geometrical decomposition

In this subsection, from the standard argument, we show that there exists a unique decomposition which relies on the implicit function theorem, the mass subcritical nondegeneracy \((\Lambda Q, Q) \neq 0\) and the modulation parameters is sufficiently small.

**Lemma 3.3.** There exists a universal constant \(\delta > 0\) such that the following holds. Let \(u\) be a radially symmetric function of the form

\[
u(t, x) = \alpha_0^2(t) \frac{1}{\lambda_0(t)} Q_{(\alpha_0, \beta_0)} \left( t, \frac{r - \alpha_0}{\lambda_0} \right) e^{i\gamma_0} + \hat{u}_0(t, x)
\]

with

\[
\lambda_0, \alpha_0 > 0, \quad \beta_0 = \beta_\infty + \tilde{\beta}_0, \quad a_0 = \frac{2\beta_0 \lambda_0}{a_\infty \alpha_0},
\]

the a priori bound

\[
\frac{\alpha_0}{\lambda_0^{1/n_\infty}} \geq 1, \quad (3.25)
\]

and

\[
0 < |a_0| + |\tilde{\beta}_0| + \|\hat{u}_0\|_{L^2} < \delta. \quad (3.26)
\]

Then there exists a unique decomposition

\[
u(t, x) = \alpha_1^2(t) \frac{1}{\lambda_1(t)} Q_{(\alpha_1, \beta_1)} \left( t, \frac{r - \alpha_1}{\lambda_1} \right) e^{i\gamma_1} + \hat{u}_1(t, x)
\]

with

\[
\beta_1 = \beta_\infty + \tilde{\beta}_1, \quad a_1 = \frac{2\beta_1 \lambda_1}{a_\infty \alpha_1},
\]

such that

\[
\hat{u}_1(x) = \alpha_1^2(t) \frac{1}{\lambda_1(t)} \tilde{e}_{(a_1, \beta_1)} \left( \frac{r - \alpha_1}{\lambda_1} \right) e^{i\gamma_1}
\]

satisfies the orthogonality conditions

\[
(\Re \tilde{e}_1, \xi_{a_1} y Q) = (\Re \tilde{e}_1, \xi_{a_1} Q) = (\Im \tilde{e}_1, \xi_{a_1} \Lambda Q) = (\Im \tilde{e}_1, \xi_{a_1} \partial_y Q) = 0.
\]

Moreover, there holds the smallness

\[
\left| \frac{\lambda_1}{\lambda_0} - 1 \right| + \frac{\alpha_0 - \alpha_1}{\lambda_0} + |\beta_0 - \tilde{\beta}_0| + |\gamma_0 - \gamma_1| + \|\hat{u}_1\|_{L^2} \lesssim \delta.
\]

This is a standard consequence of the Implicit function theorem, For the convenience of readers’, we will provide proof in Appendix A.
3.3 Modulation equations and estimates

Let \( u(t, x) \in H^1 \) be the radially solution to (1.6) on a time interval \([t_0, t_1]\), \( t_1 < 0 \). From Lemma 3.3, we assume that \( u(t) \) admits on \([t_0, t_1]\) a unique decomposition

\[
u(t, x) = \alpha \frac{x}{\lambda(t)} \frac{1}{\lambda(t)} v \left( t, \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)},
\]

where

\[
a(t) = \frac{2 \beta \lambda}{a_\infty}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)},
\]

and holds the unique decomposition

\[
w(s, y) = v(s, y)e^{i\beta y} = Q_p e^{i\beta y} + \tilde{\epsilon}(t, y), \quad \tilde{\epsilon} = \tilde{\epsilon}_1 + i\tilde{\epsilon}_2, \quad \tilde{\epsilon} = \epsilon e^{i\beta}
\]

with the orthogonality conditions

\[
(\tilde{\epsilon}_1, \xi_\alpha Q) = (\tilde{\epsilon}_1, \xi_\alpha) = (\tilde{\epsilon}_2, \xi_\alpha \Lambda Q) = (\tilde{\epsilon}_2, \xi_\alpha Q') = 0.
\]

From (3.27), we have

\[
a_s + (1 - a_\infty) a^2 - \frac{a}{\beta} A_2 - b A_1 = \frac{a}{\beta} (\beta_s - A_2) + a \left( \frac{\lambda_s}{\lambda} + a - A_1 \right) - \frac{a_\infty}{2\beta} a^2 \left( \frac{\alpha_s}{\lambda} + 2\beta \right).
\]

The modulation equations are a consequence of the orthogonality conditions (3.29) and require the derivation of the equation for \( \tilde{\epsilon} \). Recall the equation (2.3) satisfied by \( w \):

\[
i \partial_s w + w_{yy} - w + \frac{1}{1 + \frac{a_\infty}{\beta} y} |w|^2 w + \frac{a_\infty a}{\beta} + \frac{1}{1 + \frac{a_\infty}{\beta} y} (w_y - i\beta w) + a(i\Lambda w + \beta yw) - \frac{i\sigma a_\infty a}{2} w
\]

\[
= - \beta_s w + \left( \frac{\lambda_s}{\lambda} + a \right) (i\Lambda w + \beta yw) + \left( \frac{\alpha_s}{\lambda} + 2\beta \right) \left( i w_y + \beta w - \frac{i\sigma a_\infty a}{4\beta} w \right) + (\tilde{\epsilon}_s - \beta^2) w.
\]

We inject the decomposition (3.28), which we rewrite using (3.1):

\[
w = \xi_\alpha Q_p e^{-ia \frac{x^2}{4}} + \tilde{\epsilon}.
\]

into (2.3), using the formula (3.30), (3.4) and the fact that \( P_\beta = Q + \mathcal{O}(ae^{-c|\eta|}) \), then we obtain the following system

\[
\partial_s \tilde{\epsilon}_1 - M_\beta \tilde{\epsilon}_2 = - \frac{a_\infty}{\beta} + \frac{2}{1 + \frac{a_\infty}{\beta} y} (\partial_y \tilde{\epsilon}_2 - \beta \tilde{\epsilon}_1) - \beta_s y \tilde{\epsilon}_2 + \left( \frac{\lambda_s}{\lambda} + a - A_1 \right) \Lambda Q
\]

\[
+ \frac{\lambda_s}{\lambda} (\Lambda \tilde{\epsilon}_1 + \beta y \tilde{\epsilon}_2) + \left( \frac{\alpha_s}{\lambda} + 2\beta \right) \left( \partial_y Q + \partial_y \tilde{\epsilon}_1 - \frac{\sigma a_\infty a}{4\beta} (Q + \tilde{\epsilon}_1) \right)
\]

\[
+ \Gamma \tilde{\epsilon}_2 - \Im R(\tilde{\epsilon}) + \mathcal{O}(a|\tilde{\epsilon}| + a^l + a \text{Mod}) e^{-c|\eta|},
\]

(3.31)
\[ \partial_s \tilde{e}_2 + M_+ \tilde{e}_1 = -\frac{a_{\infty}a}{2\beta} \left( -\partial_y \tilde{e}_1 - \beta \tilde{e}_2 \right) + (\beta_s - A_2) y Q + \tilde{\beta}_s y \tilde{e}_1 \]
\[ - \beta \left( \frac{\lambda_s}{\lambda} a - A_1 \right) y Q + \frac{\lambda_s}{\lambda} (\Lambda \tilde{e}_2 - \beta y \tilde{e}_1) + \left( \frac{\alpha_s}{\lambda} + 2\tilde{\beta} \right) \left( \partial_y \tilde{e}_2 - \frac{\sigma a_{\infty}a}{4\beta} \tilde{e}_2 \right) \]
\[ - \Gamma (Q + \tilde{e}_1) + \Re R(\tilde{e}) + \mathcal{O}(a|\tilde{e}| + a^l + a \Mod) e^{-c|y|}, \] (3.32)

where \((M_+, M_-)\) are small deformation of the linearized operator \((L_+, L_-)\) close to \(Q\):

\[ M_+(\tilde{e}) = -\partial_y^2 \tilde{e}_1 + \tilde{e}_1 - 3 \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} \sigma^2 \tilde{e}_1, \]
\[ M_-(\tilde{e}) = -\partial_y^2 \tilde{e}_2 + \tilde{e}_2 - 3 \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} \sigma^2 \tilde{e}_2, \]

and

\[ \Gamma = (\tilde{\gamma}_s - \beta^2) + \beta \left( \frac{\alpha_s}{\lambda} + 2\tilde{\beta} \right) \] (3.33)

and the nonlinear term is given by

\[ R(\tilde{e}) = \frac{1}{1 + \frac{a_{\infty}a}{2\beta} y} \left[ (Q + \tilde{e})(\tilde{e}_1^2 + \tilde{e}_2^2) + 2Q \tilde{e}_1 \tilde{e}_2 \right]. \] (3.34)

We now give the following modulation estimate.

**Lemma 3.4.** There holds the bounds

\[ |\Mod(t)| \lesssim a \|\tilde{e}\|_{H^l} + a^l, \] (3.35)
\[ a_s + (1 - a_{\infty}) a^2 - \frac{a}{\beta} A_2 - a A_1 \lesssim a^2 \|\tilde{e}\|_{H^l} + a^l + 1. \] (3.36)
**Proof.** Except for dealing with the new terms that come from the inhomogeneous factor, the proof of this lemma is similar to [17, 19, 21]. For the reader convenience, we give the proof of this lemma.

We divide the proof into the following steps.

**Step 1:** Estimate $|\frac{\alpha_s}{\lambda} + 2\beta|$.

We multiply the equation of $\tilde{\epsilon}_1$ (3.31) by $\xi_\alpha yQ$ and integrate by parts. Using the orthogonality conditions (3.29), identity $L_-(yQ) = -2Q'$ and the following relation

$$(\partial_y Q, \xi_\alpha yQ) = -\frac{1}{2}\|Q\|_{L^2}^2 + O(e^{-\frac{\sqrt{\alpha}}{2}}),$$

(3.37)

we obtain

$$\left|\frac{\alpha_s}{\lambda} + 2\beta\right| \lesssim a\|\tilde{\epsilon}\|_{L^2_\mu} + \text{Mod}(a + \|\tilde{\epsilon}\|_{L^2_\mu}) + a' + \int |y|^\alpha |R(\tilde{\epsilon})|\eta_0 e^{-|y|}$$

$$\lesssim a\|\tilde{\epsilon}\|_{L^2_\mu} + \text{Mod}(a + \|\tilde{\epsilon}\|_{L^2_\mu}) + a' + \|\tilde{\epsilon}\|_{L^2_\mu}^2,$$

(3.38)

where we used (3.34), the decay estimate (2.8), $\tilde{\epsilon}$ is small, the following bound

$$\|\tilde{\epsilon}\|_{L^\infty(y \geq -\frac{\delta}{\lambda})} \lesssim \|\tilde{\epsilon}\|_{L^2(y \geq -\frac{\delta}{\lambda})} + \|\tilde{\epsilon}\|_{L^2_\mu} \lesssim \|\tilde{\epsilon}\|_{H^\mu_\lambda}$$

(3.39)

and

$$\int |y|^\alpha |R(\tilde{\epsilon})|\xi_\alpha e^{-|y|} \lesssim \int |y|^\alpha \xi_\alpha^2 e^{-2|y|} \tilde{\epsilon}^2 + \int |\tilde{\epsilon}|^3 \xi_0$$

$$\lesssim \|\tilde{\epsilon}\|_{L^2_\mu}^2 + \|\tilde{\epsilon}\|_{L^\infty} \|\tilde{\epsilon}\|_{L^2_\mu} \lesssim \|\tilde{\epsilon}\|_{L^2_\mu}^2.$$

Here we also used the decay estimates (2.8) and (3.39).

**Step 2:** Estimate $|\frac{\alpha_s}{\lambda} + a - A_1|$.

We multiply the equation of $\tilde{\epsilon}_1$ (3.31) by $\xi_\alpha Q$ and integrate by parts. Using the orthogonality condition 3.29, $L_-Q = 0$ and the relation

$$\langle \xi_\alpha \Lambda Q, Q \rangle = \frac{1}{2} \int Q^2 + O(e^{-\frac{\sqrt{\alpha}}{2}}),$$

(3.40)

we have

$$\left|\frac{\alpha_s}{\lambda} + a - A_1\right| \lesssim a\|\tilde{\epsilon}\|_{L^2_\mu} + \text{Mod}(a + \|\tilde{\epsilon}\|_{L^2_\mu}) + a' + \|\tilde{\epsilon}\|_{L^2_\mu}^2.$$

(3.41)

**Step 3:** Estimate $\Gamma$, where $\Gamma$ is defined by (3.33).

Multiplying the equation of $\tilde{\epsilon}_2$ (3.32) by $\xi_\alpha \Lambda Q$ and using the orthogonality condition (3.29), $L_+\Lambda Q = -2Q$ and (3.40), we deduce

$$|\Gamma| \lesssim a\|\tilde{\epsilon}\|_{L^2_\mu} + \text{Mod}(a + \|\tilde{\epsilon}\|_{L^2_\mu}) + a' + \|\tilde{\epsilon}\|_{L^2_\mu}^2.$$

(3.42)

**Step 4:** Estimate $|\tilde{\beta}_s - A_2|$.

We multiply the equation of $\tilde{\epsilon}_2$ (3.32) by $\xi_\alpha \partial_y Q$ and use the orthogonality condition (3.29), $L_+(\partial_y Q) = 0$ and (3.37) to obtain

$$|\tilde{\beta}_s - A_2| \lesssim a\|\tilde{\epsilon}\|_{L^2_\mu} + \text{Mod}(a + \|\tilde{\epsilon}\|_{L^2_\mu}) + a' + \|\tilde{\epsilon}\|_{L^2_\mu}^2.$$

(3.43)

Combining the above estimates (3.38), (3.41), (3.42) and (3.43) yields (3.35). Estimate (3.36) follows from (3.35) and (3.30). We now complete the proof of this lemma.
4 Refine energy estimate

In this section, our aim is to derive a mixed energy/Morawetz type estimate which is very crucial for the blowup solution.

Let \( u(t, x) \) be a solution to (1.6) on \([t_0, 0)\) and have the following decomposition

\[
u(t, x) = \tilde{Q}(t, x) + \tilde{u}(t, x),
\]

where \( \tilde{Q}(t, x) \) defined by (3.2) and

\[
\tilde{u}(t, x) = \alpha \sigma(t) \frac{1}{\lambda(t)} \epsilon \left( s, \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)}, \quad \epsilon(s, y) = \epsilon(s, y) e^{i\beta y}
\]

which in view of (3.3), yields the equation for \( \tilde{u} \),

\[
i\partial_t \tilde{u} + \Delta \tilde{u} + \frac{1}{|x|^\sigma} \left( |u|^2 u - |	ilde{Q}|^2 \tilde{Q} \right) = -\psi = -\alpha \frac{\tilde{Q}}{\lambda(t)} \tilde{Q} \left( t, \frac{r - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)},
\]

with \( \Psi \) defined by (3.4). From the Lemma 3.3, the decomposition holds as long as

\[
\frac{\alpha(t)}{\lambda^{1 - \frac{\sigma a}{a_\infty}} \lambda(t)} \geq 1
\]

and

\[
|a(t)| + |\tilde{\beta}(t)| + \|\epsilon(t)\|_{L^2_{\mu}} < \delta.
\]

then we assume a priori bounds

\[
\|\epsilon\|_{H^1_{\mu}} < \min(a, \lambda) \delta, \quad 0 < a < \delta, \quad |\tilde{\beta}| \leq a,
\]

and

\[
\frac{b_0}{2} \leq \frac{\alpha(t)}{\lambda(t)^{1 - \frac{\sigma a}{a_\infty}}} \leq 2b_0,
\]

where \( b_0 \) is defined in Lemma 3.1.

Let \( \phi : [-1, +\infty) \to \mathbb{R} \) be a time-independent smooth compactly supported cutoff function which satisfies

\[
\phi(z) = \begin{cases} 
0 & \text{for } -1 \leq z \leq -\frac{1}{2} \text{ and for } z \geq \frac{1}{2}, \\
1 & \text{for } z \in \left( -\frac{1}{2}, \frac{1}{2} \right),
\end{cases}
\]

and

\[
\sup_{z \geq -1} |\phi(z)| < 2.
\]

Let

\[
F(u) = \frac{1}{4} |u|^4, \quad f(u) = |u|^2 u, \quad F'(u) \cdot h = \Re f(u) \tilde{h}.
\]

Now we give the following energy/Virial estimate.
Lemma 4.1. Let

$$J(\tilde{u}) = \frac{1}{2} \int |\nabla \tilde{u}|^2 + \frac{1 + \beta^2}{2} \int \frac{|	ilde{u}|^2}{\lambda^2} - \int \frac{1}{|x|^\sigma} [F(\tilde{Q} + \tilde{u}) - F(\tilde{Q}) - F'(\tilde{Q}) \cdot \tilde{u}]$$

$$+ \frac{\beta}{\lambda} \Im \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \partial_t \tilde{u} \bar{u},$$

and

$$K(\tilde{u}) = -\frac{1 + \beta^2}{2} \Im \left( \frac{1}{|x|^\sigma} (f(u) - f(\tilde{Q}), \bar{u}) - \frac{2\beta}{\lambda} \Re \left( \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\tilde{Q} + \tilde{u}) - f(\tilde{Q})) \partial_t \tilde{u} \right) \right)$$

$$- \Re \left( \partial_t \tilde{Q}, \frac{1}{|x|^\sigma} (f(\tilde{Q} + \tilde{u}) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{u}) \right).$$

Then the following holds:

$$\frac{d}{dt} J(\tilde{u}) = K(\tilde{u}) + O \left( \frac{\alpha}{\lambda^2} \|\epsilon\|^2_{H^1_0} + \frac{\alpha_1^d}{\lambda^4} \|\epsilon\|_{H^1_0} \right).$$

Proof. We divide the proof into two steps.

Step 1: Estimate the energy part. From (4.1), we have

$$\frac{d}{dt} \left\{ \frac{1}{2} \int |\nabla \tilde{u}|^2 + \frac{1 + \beta^2}{2} \int \frac{|	ilde{u}|^2}{\lambda^2} - \int \frac{1}{|x|^\sigma} [F(\tilde{Q} + \tilde{u}) - F(\tilde{Q}) - F'(\tilde{Q}) \cdot \tilde{u}] \right\}$$

$$= - \Re \left( \partial_t \tilde{u}, \Delta \tilde{u} - \frac{1 + \beta^2}{\lambda^2} \tilde{u} + \frac{1}{|x|^\sigma} (f(u) - f(\tilde{Q})) \right) - \frac{(1 + \beta^2)\lambda}{\lambda^3} \int |\tilde{u}|^2$$

$$+ \frac{\beta \lambda}{\lambda^2} \int |\tilde{u}|^2 - \Re \left( \partial_t \tilde{Q}, \frac{1}{|x|^\sigma} (f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{u}) \right)$$

$$= \Im \left( \psi, \Delta \tilde{u} - \frac{1 + \beta^2}{\lambda^2} \tilde{u} + \frac{1}{|x|^\sigma} (f(u) - f(\tilde{Q})) \right) - \frac{(1 + \beta^2)\lambda}{\lambda^3} \Im \left( \frac{1}{|x|^\sigma} (f(u) - f(\tilde{Q})), \tilde{u} \right)$$

$$- \frac{(1 + \beta^2)\lambda}{\lambda^3} \int |\tilde{u}|^2 + \frac{\beta \lambda}{\lambda^2} \int |\tilde{u}|^2$$

$$- \Re \left( \partial_t \tilde{Q}, \frac{1}{|x|^\sigma} (f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{u}) \right).$$

From (3.35) and the identity $\frac{\lambda}{\lambda^4} = \frac{\lambda}{\lambda^4} - \frac{\lambda}{\lambda^4} (\frac{\lambda}{\lambda^4} + a - A_1)$, we have

$$- \frac{\lambda}{\lambda^3} \int |\tilde{u}|^2 = \frac{\alpha}{\lambda^4} \int |\tilde{u}|^2 - \frac{A_1}{\lambda^4} \int |\tilde{u}|^2 - \frac{1}{\lambda^4} \left( \frac{\lambda}{\lambda^4} + a - A_1 \right) \int |\tilde{u}|^2$$

$$= \frac{1}{\lambda^4} O(a \|\epsilon\|^2_{L^2_0}),$$

where we used the priori estimate (4.2), (4.3) and Lemma 3.1. Also from (3.35), we have

$$\frac{\beta \lambda}{\lambda^2} \int |\tilde{u}|^2 = \frac{\beta A_2}{\lambda^4} \int |\tilde{u}|^2 + \frac{\beta \lambda}{\lambda^4} \int |\tilde{u}|^2 = \frac{1}{\lambda^4} O(a \|\epsilon\|^2_{L^2_0}).$$
From (3.4), (3.35) and (3.36), we have

\[ |\psi| \lesssim \xi \alpha (a^l + \text{Mod})(1 + |y|^{a^i}) e^{-|y|} + \frac{e^{-|y|}}{\alpha^i y^{-\frac{1}{\alpha^i}}} + \frac{1}{\alpha^i y^{-\frac{1}{\alpha^i}}}. \]

(4.11)

Now we estimate the first term in the right-hand side of (4.8), using the fact that \( f'(w) \cdot u = 2|w|^2 u + w^2 \bar{u} \) and the priori estimate (4.2) and (4.3) and (4.11), we have

\[
\begin{align*}
\| \psi, \Delta \bar{u} - \frac{1 + \beta^2}{\lambda^2} \bar{u} + \frac{1}{|x|^\sigma} (f(u) - f(\bar{Q})) \|_\mathbb{H}^2 \\
\leq & \| \psi, \Delta \bar{u} - \frac{1 + \beta^2}{\lambda^2} \bar{u} + \frac{1}{|x|^\sigma} (f(u) - f(\bar{Q}) - f'\bar{Q} \cdot \bar{u}) \|_\mathbb{H}^2 \\
\leq & \int \left( \Delta \bar{u} - \frac{1 + \beta^2}{\lambda^2} \psi + 2 \frac{1}{|x|^\sigma} |\bar{Q}|^2 \psi + \frac{1}{|x|^\sigma} \bar{Q}^2 \psi \right) \bar{u} + \int \| \psi, \text{Mod} \|_\mathbb{H}^2 (1 + |y|^{a^i}) e^{-|y|} + \frac{e^{-|y|}}{\alpha^i y^{-\frac{1}{\alpha^i}}} \|_\mathbb{H}^2 |\bar{Q}||\bar{u}|^2 + ||\bar{u}||^3
\end{align*}
\]

(4.12)

where in the penultimate and last step, we used the Sobolev embedding and Lemma 3.1. Injecting (4.9), (4.10) and (4.12) into (4.8) yields

\[
\begin{align*}
\frac{d}{dt} \left\{ \frac{1}{2} \int |\nabla \bar{u}|^2 + \frac{1 + \beta^2}{2} \int |\bar{u}|^2 \frac{1}{\lambda^2} \int \frac{1}{|x|^\sigma} \left[ F(\bar{Q} + \bar{u}) - F(\bar{Q}) - F'(\bar{Q}) \cdot \bar{u} \right] \right\} \\
= - \frac{1 + \beta^2}{\lambda^2} \int \left( \frac{1}{|x|^\sigma} (f(u) - f(\bar{Q}), \bar{u}) \right) - \Re \left( \partial_t \bar{Q}, \frac{1}{|x|^\sigma} (f(u) - f(\bar{Q}) - f'(\bar{Q}) \cdot \bar{u}) \right) \\
+ \frac{1}{\lambda^4} \mathcal{O}(a^l \| \epsilon \|_{\mathbb{H}^2} + a \| \epsilon \|_{\mathbb{H}^2}^2).
\end{align*}
\]

(4.13)

**Step 2:** Estimate the Virial part. We now estimate the other part in (4.5). Using (3.27), we have the following relation

\[
\frac{d}{dt} \left\{ \frac{1}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right\} = - \frac{\alpha r}{\alpha^2(t)} = - \frac{\alpha s}{\lambda} \frac{r}{\lambda^2} = \frac{a_\infty a(t)}{2 \beta \lambda^2 \alpha(t)} \frac{r}{\alpha(t)} \left( \frac{\alpha s}{\lambda} + 2 \beta \right).
\]

From the above relation, by the direct computation, we can deduce the following

\[
\begin{align*}
\frac{d}{dt} \left\{ \frac{\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \partial_t \bar{u} \right\} \\
= \frac{a_\infty \beta a}{\lambda^3} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \partial_t \bar{u}
\end{align*}
\]

22
where we used (3.35), Lemma 3.1 and the priori estimates (4.2) and (4.3), and the fact that
\[
\frac{1}{r} \sim \frac{1}{\alpha(t)} \quad \text{on the support of } \phi \left( \frac{r}{\alpha(t)} - 1 \right).
\]

Now from (4.1), (4.15) and the integration by parts, the first term in the right-hand side of (4.14) can be written as follows:

\[
\frac{\beta}{\lambda} \int \imath \partial_r \bar{u} \left( \frac{1}{\alpha(t)} \phi' \left( \frac{r}{\alpha(t)} - 1 \right) \bar{u} + 2\phi \left( \frac{r}{\alpha(t)} - 1 \right) \partial_r \bar{u} \right)
\]
\[
= \frac{a_\infty a}{\lambda^2} \int \phi' \left( \frac{r}{\alpha(t)} - 1 \right) |\partial_r \bar{u}|^2 - \frac{a_\infty a^2}{8\beta \lambda^3} \int \Delta \left( \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \right) \bar{u}^2
\]
\[
- \frac{2\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q}) \bar{\nabla} \bar{u})
\]
\[
- \frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q}) \bar{\nabla} \bar{u})
\]
\[
- \frac{2\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \psi \partial_r \bar{u} - \frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \psi \bar{u}
\]
\[
= - \frac{2\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q}) \bar{\nabla} \bar{u})
\]
\[
- \frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q}) \bar{\nabla} \bar{u})
\]
\[
- \frac{2\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \psi \partial_r \bar{u} - \frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \psi \bar{u} + O \left( \frac{a}{\lambda^4} \|e\|^2_{H^2} \right).
\]

Now we estimate the second term in the right hand side of (4.16)

\[
\left| - \frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q}) \bar{u}) \right|
\]
By the Gagliardo-Nirebergy inequality, we have
\[ \frac{1}{\lambda} \left\| \frac{\partial}{\partial r} \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \right\|_{L^\infty} \leq C \left( \frac{1}{\lambda} \left\| \frac{\partial}{\partial r} \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) f \right\|_{L^\infty} \right) \]
where we used the (4.15). Notice that $\tilde{Q}$ is localized in the region $r \geq \frac{\alpha(t)}{2}$ due to the cut-off function $\xi_a$ in its definition. Now, the region $r \geq \frac{\alpha(t)}{2}$ corresponds to $y \geq -\frac{\alpha(t)}{2\lambda(t)}$ and thus
\[ \mu \gtrsim 1 \quad \text{for} \quad r \geq \frac{\alpha(t)}{2}. \]

By the Sobolev embedding and priori bound (4.2) and Lemma 3.1, we have
\[ \int \frac{1}{|x|^\sigma} \left| \tilde{Q} \right|^3 \leq \frac{1}{\lambda^3} \int \left| \alpha \right|^{2\sigma+2} \left| \lambda y + \alpha^{-\sigma} |\epsilon|^3 |Q| \mu \right| \]
\[ \lesssim \frac{1}{\lambda^3} \int \left| \alpha \right|^{\sigma+2} \left| \frac{\lambda}{\alpha} y + 1 \right|^{-\sigma} |\epsilon|^3 \mu \]
\[ \lesssim \frac{1}{\lambda^2} \int \left| \alpha \right| y + 1 \left|^{-\sigma} |\epsilon|^3 \mu \right| \]
\[ \lesssim \frac{1}{\lambda^2} \| \epsilon \|_{L^\infty} \| \epsilon \|_{H^1_{\frac{1}{2}}} \lesssim \frac{1}{\lambda^2} \| \epsilon \|_{H^1_{\frac{1}{2}}} \| \epsilon \|_{L^2_{\alpha}} \]
\[ \lesssim \frac{\delta \| \epsilon \|_{H^1_{\frac{1}{2}}}^2}{\lambda^2}. \]
Similarly, we can obtain
\[ \int \frac{1}{|x|^\sigma} \left| \tilde{Q} \right|^2 \lesssim \frac{\| \epsilon \|_{H^1_{\frac{1}{2}}}^2}{\lambda^2}. \]

By the Gagliardo-Nirebergy inequality, we have
\[ \int \frac{1}{|x|^\sigma} |\tilde{u}|^4 \lesssim \| \nabla \tilde{u} \|_{L^2_{\alpha}} \| \tilde{u} \|_{L^2_{\alpha}}^{1-\sigma} = \frac{1}{\lambda^2} \| \nabla \epsilon \|_{L^2_{\alpha}}^{3+\sigma} \| \epsilon \|_{L^2_{\alpha}}^{1-\sigma} \lesssim \frac{\delta \| \epsilon \|_{H^1_{\frac{1}{2}}}^2}{\lambda^2}. \]

From above estimates, we can obtain
\[ \left| -\frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \frac{1}{|x|^\sigma} \left( f(\tilde{Q} + \tilde{u}) - f(\tilde{Q}) \right) \tilde{u} \right| \lesssim \mathcal{O} \left( \frac{\alpha}{\lambda^4} \| \epsilon \|_{H^1_{\frac{1}{2}}}^2 \right). \quad (4.17) \]

We estimate the terms that contain $\psi$ in (4.16). Using (4.11), we obtain
\[ \left| -\frac{2\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \psi \partial_r \tilde{u} - \frac{\beta}{\lambda} \int \partial_r \left( \phi \left( \frac{r}{\alpha(t)} - 1 \right) \right) \psi \tilde{u} \right| \]
\[ a^l + a \|\epsilon\|_{H^1} \|\epsilon\|_{H^1} \geq \frac{\lambda^4}{\lambda} \|\epsilon\|_{H^1}. \]  \quad (4.18)

Injecting (4.16), (4.17) and (4.18) into (4.14), we can obtain
\[
\frac{d}{dt} \left\{ \frac{\beta}{\lambda} \int \phi \left( \frac{r}{\alpha(t)} \right) \partial_r \tilde{u} \tilde{u} \right\} = -\frac{2\tilde{\beta}}{\lambda} \Re \int \phi \left( \frac{r}{\alpha(t)} \right) \left( f(\tilde{Q} + \tilde{u}) - f(\tilde{Q}) \right) \partial_r \tilde{u} + O \left( \frac{a^l}{\lambda^4} \|\epsilon\|_{H^1}^2 + \frac{a^l}{\lambda^4} \|\epsilon\|_{H^1} \right). \quad (4.19)
\]

Now combining the (4.13) and (4.19), we can obtain (4.7). This conclude the proof of Lemma 4.1. \qed

Next Lemma we will give the following upper bound and lower bound of (4.5).

**Lemma 4.2.** Let \( J \) defined by (4.5). Then
\[
c_1 \left( \|\nabla \tilde{u}\|_{L^2}^2 + \frac{1}{\lambda^2} \|\tilde{u}\|_{L^2}^2 \right) \geq J(\tilde{u}) \geq c_2 \left( \|\nabla \tilde{u}\|_{L^2}^2 + \frac{1}{\lambda^2} \|\tilde{u}\|_{L^2} \right) \quad (4.20)
\]
for some constants \( c_1, c_2 > 0 \).

**Proof.** To prove the lower bound. Notice that \( J(\tilde{u}) \) can be write as
\[
J(\tilde{u}) = \frac{1}{2\lambda^2} \left( \int |\partial_y \epsilon|^2 \mu + 2\beta \Im \phi(z) \partial_y \epsilon \epsilon \mu + (1 + \tilde{\beta}^2) \int |\epsilon|^2 \mu 
- 2 \int \frac{\lambda}{\alpha} y + 1 \int \mu \left( F(Q_P + \epsilon) - F(Q_P) - F'(Q_P) \cdot \epsilon \right) \right), \quad (4.21)
\]
where
\[ z = \frac{r}{\alpha} - 1 = \frac{a\infty}{2\beta} \frac{y}{y}, \quad \mu = (1 + z)^2 \]
and we used Lemma 3.1.

We now estimate the nonlinear term in (4.21).
\[
\bigr| \frac{\lambda}{\alpha} y + 1 \int \mu \left( F(Q_P + \epsilon) - F(Q_P) - F'(Q_P) \cdot \epsilon \right) \biggr| \\
= \int \frac{\lambda}{\alpha} y + 1 \int \mu \left( \frac{1}{2} |Q_P|^2 |\epsilon|^2 + \frac{1}{4} |\epsilon|^4 + |\epsilon|^2 \Re(Q_P \epsilon) \right) \mu \\
\geq \frac{1}{2} \int \frac{\lambda}{\alpha} y + 1 \int \mu \left( \xi Q^2 (3Q_1^2 + Q_2^2) + O(\delta C \|\epsilon\|_{H^1}^2) \right), \quad (4.22)
\]
where we used the fact that
\[ Q_P = \xi Q e^{-i\beta y} + O(ae^{-|y|}), \quad \tilde{\epsilon} = \epsilon e^{i\beta y}. \]
Combining the above estimate (4.22), priori estimate (4.2) and the definition of $\tilde{\beta}$, we can get

$$J(\bar{u}) = \frac{1}{2\lambda^2} \left\{ \int |\partial_y \epsilon|^2 \mu + 2\beta_\infty \Re \int \phi(z) \partial_y \epsilon \bar{\epsilon} \mu + (1 + \beta_\infty^2) \int |\epsilon|^2 \mu 
- \int \left| \frac{\lambda}{\alpha} y + 1 \right|^{-\sigma} \xi_a Q^2(3\epsilon_1^2 + \epsilon_2^2) + O(\delta^C_k \|\epsilon\|^2_{H^1_\mu}) \right\}$$

$$= \frac{1}{2\lambda^2} \left\{ I_1 + I_2 - \int \left| \frac{\lambda}{\alpha} y + 1 \right|^{-\sigma} \xi_a Q^2(3\epsilon_1^2 + \epsilon_2^2) + O(\delta^C_k \|\epsilon\|^2_{H^1_\mu}) \right\}.$$ (4.23)

where

$$I_1 = \int_{|y| \geq \frac{1}{\sqrt{a}}} |\partial_y \epsilon|^2 \mu + 2\beta_\infty \Re \int_{|y| \geq \frac{1}{\sqrt{a}}} \phi(z) \partial_y \epsilon \bar{\epsilon} \mu + (1 + \beta_\infty^2) \int_{|y| \geq \frac{1}{\sqrt{a}}} |\epsilon|^2 \mu,$$

$$I_2 = \int_{|y| \leq \frac{1}{\sqrt{a}}} |\partial_y \epsilon|^2 \mu + 2\beta_\infty \Re \int_{|y| \leq \frac{1}{\sqrt{a}}} \phi(z) \partial_y \epsilon \bar{\epsilon} \mu + (1 + \beta_\infty^2) \int_{|y| \leq \frac{1}{\sqrt{a}}} |\epsilon|^2 \mu.$$

By the definition of $\phi$ (see (4.4)) and the basic property of the quadratic form, we can obtain

$$I_1 \gtrsim \int_{|y| \geq \frac{1}{\sqrt{a}}} (|\partial_y \epsilon|^2 + |\epsilon|^2) \mu, \quad (4.24)$$

since $\beta_\infty^2 \psi^2(z) - (1 + \beta_\infty^2)^2 < 0$.

For the case $|y| \leq \frac{1}{\sqrt{a}}$, from (4.4), then $|\phi(z) - 1| \lesssim |z| \lesssim \sqrt{a}$, hence

$$I_2 = \int_{|y| \leq \frac{1}{\sqrt{a}}} (|\partial_y \epsilon|^2 + |\epsilon|^2) \mu + O(\sqrt{a} \|\epsilon\|^2_{H^1_\mu}). \quad (4.25)$$

Combining the above estimates (4.24), (4.25) and (4.23), we have

$$2\lambda^2 J(\bar{u}) = \int_{|y| \leq \frac{1}{\sqrt{a}}} |\partial_y \epsilon|^2 + |\epsilon|^2 - \int \left| \frac{\lambda}{\alpha} y + 1 \right|^{-\sigma} \xi_a Q^2(3\epsilon_1^2 + \epsilon_2^2)
+ \int_{|y| \geq \frac{1}{\sqrt{a}}} (|\partial_y \epsilon|^2 + |\epsilon|^2) \mu + O(\delta^C_k \|\epsilon\|^2_{H^1_\mu}).$$ (4.26)

On the other hand, by using the following property of the linearized operator $L = (L_+, L_-)$ (see [6])

$$(L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) \geq C_0 \|\epsilon\|^2_{H^1} - \frac{1}{C_0} \left( (\epsilon_1, Q)^2 + (\epsilon, yQ)^2 + (\epsilon_2, \Lambda Q)^2 \right),$$

and the orthogonality conditions (3.29), we can obtain

$$\int_{|y| \leq \frac{1}{\sqrt{a}}} |\partial_y \epsilon|^2 \mu + |\epsilon|^2 \mu - \int \left| \frac{\lambda}{\alpha} y + 1 \right|^{-\sigma} \xi_a Q^2(3\epsilon_1^2 + \epsilon_2^2) \mu$$
We have the bound
\[ \| \mathcal{Q} \|_{H^1} \leq \frac{\alpha}{\lambda^2} \| \epsilon \|_{H^2}^2. \]  
(4.28)

Proof. By using (3.2), we get
\[
\begin{align*}
\partial_t \tilde{Q} = & \gamma_\ell \tilde{Q} - \frac{\lambda t}{\lambda} \tilde{Q} - \frac{r - \alpha(t)}{\lambda} \frac{1}{\alpha} \frac{1}{\lambda} Q' \left( \frac{r - \alpha(t)}{\lambda} \right) e^{i\gamma} - \frac{\alpha_t}{\lambda} \frac{1}{\lambda} Q' \left( \frac{r - \alpha(t)}{\lambda} \right) e^{i\gamma} \\
& + a_t \frac{1}{\alpha} \frac{1}{\lambda} \partial_\sigma Q' \left( \frac{r - \alpha(t)}{\lambda} \right) e^{i\gamma} + \tilde{\beta_t} \frac{1}{\alpha} \frac{1}{\lambda} \partial_\beta Q' \left( \frac{r - \alpha(t)}{\lambda} \right) e^{i\gamma} - \frac{\sigma}{2} \tilde{Q} \\
= & \left( \frac{i(1 + \beta^2)}{\lambda^2} + \frac{a}{\lambda^2} \right) \tilde{Q} + \frac{a r - \alpha(t)}{\lambda} \partial_r \tilde{Q} + \frac{2 \beta}{\lambda} \partial_r \tilde{Q} - \frac{\sigma}{2} \frac{a}{\lambda^2} \tilde{Q} \\
& + \frac{1}{\lambda^3} \mathcal{O} \left( \left| a^2 + \text{Mod} + \left| a_s + \frac{1}{2} a^2 - \frac{a}{\beta} A_2 - a A_1 \right| \right| \right| \left| \xi_a |y|^{\gamma} e^{-|y|} \right| \\
= & \frac{i(1 + \beta^2)}{\lambda^2} \tilde{Q} + \frac{2 \beta}{\lambda} \partial_r \tilde{Q} + \frac{1}{\lambda^3} \mathcal{O} (a \xi_a |y|^{\gamma} e^{-|y|}),
\end{align*}
\]

where we used the modulation estimate (3.35) and the decay estimate (2.8). Then we have
\[
- \Re \left( \partial_\sigma \frac{1}{|x|^{\sigma}} \left( f(\tilde{Q} + \tilde{u}) - f(\tilde{Q} - f'(\tilde{Q}) \cdot \tilde{u}) \right) \right) \\
= - \Re \left( \partial_\sigma \frac{1}{|x|^{\sigma}} \left( \tilde{Q} \tilde{u}^2 + 2 \tilde{Q} |\tilde{u}|^2 + |\tilde{u}|^2 \tilde{u} \right) \right) \\
= - \frac{(1 + \beta^2)}{\lambda^2} \Im \left( \frac{1}{|x|^{\sigma}} \left( \tilde{Q} \tilde{u}^2 + 2 \tilde{Q} |\tilde{u}|^2 + |\tilde{u}|^2 \tilde{u} \right) \tilde{Q} \right) \\
& - \frac{2 \beta}{\lambda} \Re \left( \frac{1}{|x|^{\sigma}} \left( \tilde{Q} \tilde{u}^2 + 2 \tilde{Q} |\tilde{u}|^2 + |\tilde{u}|^2 \tilde{u} \right) \partial_\sigma \tilde{Q} \right) \\
& + \frac{1}{\lambda^3} \mathcal{O} \left( \left( \frac{1}{|x|^{\sigma}} a \xi_a |y|^{\gamma} e^{-|y|} \right) \left( |\tilde{Q} \tilde{u}^2 + 2 \tilde{Q} |\tilde{u}|^2 + |\tilde{u}|^2 \tilde{u} | \right) \right)
\]
Next, we need to estimate the remain terms in (4.29). Injecting this into (4.6), we get

\[ K(\bar{u}) = -\frac{1 + \beta^2}{\lambda^2} \int \frac{1}{|x|^\sigma} \left( \bar{u}^2 \tilde{Q}^2 + |\bar{u}|^2 (2\tilde{Q} \bar{u} + \tilde{Q} \bar{u}) \right) \]

\[-\frac{2\beta}{\lambda} \Re \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\tilde{Q} + \bar{u}) - f(\tilde{Q})) \bar{\partial}_r \bar{u} \]

\[-\frac{(1 + \beta^2)}{\lambda^2} \Re \int \frac{1}{|x|^\sigma} \left( \bar{Q} \bar{u}^2 + 2\tilde{Q} |\bar{u}|^2 + |\bar{u}|^2 \bar{u} \right) \bar{Q} \]

\[-\frac{2\beta}{\lambda} \Re \int \frac{1}{|x|^\sigma} \left( \bar{Q} \bar{u}^2 + 2\tilde{Q} |\bar{u}|^2 + |\bar{u}|^2 \bar{u} \right) \partial_r \bar{Q} + O \left( \frac{\alpha}{\lambda^2} \|\epsilon\|_{H^1_\lambda} \right) \]

\[ = I_1 + O \left( \frac{\alpha}{\lambda^2} \|\epsilon\|_{H^1_\lambda} \right). \tag{4.29} \]

Next, we need to estimate the remain terms in (4.29). To estimate this, we need to introduce the cutoff function $\rho$. Let $\rho$ be a smooth compactly supported cutoff function which is 1 in the neighborhood of the support of $\phi$, and 0 in the neighborhood of $z = -1$. Now we compute

\[ I_2 = -\frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\tilde{Q} + \bar{u}) - f(\tilde{Q})) \bar{\partial}_r \bar{u} \]

\[-\frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} \left( \bar{Q} \bar{u}^2 + 2\tilde{Q} |\bar{u}|^2 + |\bar{u}|^2 \bar{u} \right) \partial_r \bar{Q} \]

\[ = -\frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} f(\tilde{Q} + \bar{u}) (\partial_r \bar{Q} + \partial_t \bar{u}) + \frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) a(r) f(\tilde{Q}) \partial_r \bar{Q} \]

\[ + \frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} \left( f(\tilde{Q}) \partial_r \bar{u} + f'(\tilde{Q}) \cdot \tilde{u} \bar{\partial}_r \bar{Q} \right) \]

\[ = -\frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} \partial_r \left( F(u) - F(\tilde{Q}) - f(\tilde{Q}) \bar{u} \right). \]

Integrating by parts in $r$ and using the properties of $\rho$, we have

\[ I_2 = \frac{2\beta}{\lambda} \Re \int \frac{1}{\alpha(t)} \rho' \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} \left( F(u) - F(\tilde{Q}) - f(\tilde{Q}) \bar{u} \right) \]

28
where we used the properties of \( \rho \),

\[
\frac{1}{r} \sim \frac{1}{\alpha(t)} \text{ on the support of } \rho \left( \frac{1}{\alpha(t)} - 1 \right).
\]

On the other hand, since \( \rho = 1 \) on the support of \( \phi \), we have

\[
\frac{2\beta}{\lambda} \Re \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q})) \partial_r \bar{u}
\]

Thus

\[
I_3 = - \frac{2\beta}{\lambda} \Re \int \phi \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q})) \partial_r \bar{u}
\]

\[
+ \frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q})) \partial_r \bar{u}
\]

\[
= - \frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \left[ \phi \left( \frac{r}{\alpha(t)} - 1 \right) - 1 \right] \frac{1}{|x|^\sigma} (f(\bar{Q} + \bar{u}) - f(\bar{Q})) \partial_r \bar{u}
\]

\[
+ \frac{2\beta}{\lambda} \Re \int \rho \left( \frac{r}{\alpha(t)} - 1 \right) \left[ \phi \left( \frac{r}{\alpha(t)} - 1 \right) - 1 \right] \frac{1}{|x|^\sigma} \left( 2|\bar{u}|^2 \bar{Q} + \bar{u}^2 \bar{Q} + |\bar{u}|^2 \bar{u} \right) \partial_r \bar{Q}
\]

\[
= \frac{2\beta}{\lambda} \Re \int \left( \frac{a\alpha\phi}{2\beta\lambda} \partial_z (\phi(z) - 1) \right) \frac{1}{|x|^\sigma} + \rho(z)(\phi(z) - 1) \partial_r \frac{1}{|x|^\sigma} \left[ F(\bar{Q} + \bar{u}) - F(\bar{Q}) - f(\bar{Q}) \right]
\]

where \( z = \frac{r}{\alpha(t)} - 1 = \frac{a\alpha}{2\beta} y \). Since \( \phi(0) = 1 \), from (2.2), we have

\[
|\phi(z) - 1| \lesssim |z| \lesssim a|y|, \quad |\partial_z (\phi(z) - 1)| \lesssim 1.
\]

Then, we deduce

\[
I_3 \lesssim \frac{a}{\lambda^4} \int (|\epsilon|^4 + a|y|^a \xi_0 e^{-2|y|}|\epsilon|^2) \mu + \frac{a}{\lambda^4} \int |y| \left( |\epsilon|^2 |y|^a \xi_0 e^{-2|y|} + \xi_0 |\epsilon|^3 e^{-a|y|} \right) \mu
\]
\[ I_2 - I_3 = \mathcal{O} \left( \frac{a}{\lambda^4} \| \epsilon \|_{H^\mu_1}^2 \right). \]  

(4.32)

Notice that the function \( 1 - \rho \) is supported by construction in \( y \leq -\frac{1}{a} \) where \( \tilde{Q} \) vanishes, hence from (4.32), we get

\[ I_1 = \mathcal{O} \left( \frac{a}{\lambda^4} \| \epsilon \|_{H^\mu_1}^2 \right). \]  

(4.33)

Injecting (4.33) into (4.29), we get

\[ K(\tilde{u}) = \mathcal{O} \left( \frac{a}{\lambda^4} \| \epsilon \|_{H^\mu_1}^2 \right). \]

This means that (4.28) holds and we complete the proof of this lemma.

\[ \square \]

5 Backwards propagation of smallness

In this section, we now apply the energy estimate of the previous section in order to establish a bootstrap argument that will be needed in the construction of the ring blowup solution.

From now on, we choose the integer \( l \) appearing in Lemma 2.1 such that \( l > 5 \). Given \( t_0 < t_1 < 0 \), let \( u(t) \) be the solution to (1.6) with initial data at \( t = t_1 \) given explicitly by

\[ u(t_1, r) = \alpha_1^a(t_1) \frac{1}{\lambda_1(t_1)} Q_{(a_1(t_1), \tilde{\beta}_1(t_1))} \left( r - \alpha_1(t_1) \right) \lambda_1(t_1)^{-1} e^{i\gamma_1(t_1)}. \]  

(5.1)

Our aim is to derive bounds on \( u \) backward on a time interval independent of \( t_1 \) as \( t_1 \to 0 \). From (5.1), we have the well-prepared data initialization

\[ \epsilon(t_1) = 0, \quad (\lambda, a, \tilde{\beta}, \alpha, \gamma)(t_1) = (\lambda_1, a_1, \tilde{\beta}_1, \alpha_1, \gamma_1)(t_1), \]

and we may thus consider a backward time \( t < t_1 \) such that the following bootstrap assumption

\[ \| \epsilon \|_{H^\mu_1} < \min \{ a, \lambda \} \delta, \quad 0 < a < \delta, \quad |\tilde{\beta}| \leq a, \]

\[ \frac{b_\infty}{2} \leq \frac{\alpha(t)}{\lambda^1_{\infty}} \leq 2b_\infty. \]

Now we claim the following estimates hold.
Lemma 5.1. Let $t_0$ be defined in Lemma 3.1. For any $t \in [t_0, t_1)$, $t_1 < 0$, such that the priori estimates (4.2) and (4.3) are satisfied on the interval $[t_0, t_1)$, there holds

$$\|\epsilon\|_{H^m} \lesssim \min \{ |t|^{\frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}}, \lambda \} |t|^{\frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}},$$

$$a(t) = \frac{1}{(1 - (2\sigma + 1)\lambda_0)} B_1^{1 - \frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} B_2^{1 - \frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} |t|^{1 - \frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} \left( 1 + O \left( \log |t| \|t\|^{1 - \frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} \right) \right),$$

$$|\tilde{\beta}| = O \left( |t|^{\frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} \right),$$

$$\lambda(t) = B_2^{-\frac{1}{1 - \sigma\lambda_0}} B_1^{1 - \frac{1}{1 - \sigma\lambda_0}} |t|^{1 - \frac{1}{1 - \sigma\lambda_0}} \left( 1 + O \left( \log |t| \|t\|^{1 - \frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} \right) \right),$$

$$a(t) = b \lambda_0^{\frac{a}{1 - \sigma\lambda_0}} = b \lambda_0^{\frac{\lambda_0}{1 - \sigma\lambda_0}} B_2^{1 - \frac{1}{1 - \sigma\lambda_0}} B_1^{1 - \frac{1}{1 - \sigma\lambda_0}} |t|^{1 - \frac{1}{1 - \sigma\lambda_0}} \left( 1 + O \left( \log |t| \|t\|^{1 - \frac{1-(\sigma+1)\lambda_0}{1-(\sigma+1)\lambda_0}} \right) \right),$$

where $B_1$, $B_2$ and $b_\infty$ are defined in Lemma 3.1.

Proof. We divide the proof into the two steps.

**Step 1:** Control of $\epsilon$. By (4.7) and (4.6), we have

$$\frac{d}{dt} J(\tilde{u}) \leq \frac{1}{\lambda^m} J(\tilde{u}) + m \frac{a}{\lambda^{2+m}} J(\tilde{u}) - m \frac{A_1}{\lambda^{2+m}} J(\tilde{u}) - \left( \frac{\lambda_0}{\lambda} + a - A_1 \right) \frac{J(\tilde{u})}{\lambda^{2+m}} + O \left( \frac{a}{\lambda^{4+m}} \|\epsilon\|^2_{H^m} + \frac{a^l}{\lambda^{4+m}} \|\epsilon\|^2_{H^m} \right).$$

From (3.35), (4.2) and (4.20), we get

$$\left| -m \frac{A_1}{\lambda^{2+m}} J(\tilde{u}) \right| - \left| \left( \frac{\lambda_0}{\lambda} - a - A_1 \right) \frac{J(\tilde{u})}{\lambda^{2+m}} \right| \lesssim \frac{a}{\lambda^{4+m}} \left( a^2 + a \|\epsilon\|_{H^m} + a^l \right) \|\epsilon\|^2_{H^m} \lesssim \frac{a^2 C}{\lambda^{4+m}} \|\epsilon\|^2_{H^m}.$$

Again, using (4.20), there exists $C > 0$ such that

$$\frac{d}{dt} J(\tilde{u}) \geq (c_0 m - C) \frac{a}{\lambda^{4+m}} \|\epsilon\|^2_{H^m} - C \frac{a^2 - 1}{\lambda^{4+m}}.$$

Then, if $c_0 m - C > 0$,

$$\frac{d}{dt} J(\tilde{u}) \geq -a^{2k-1} \lambda^{4+m} \lesssim -a \lambda^{2(4-1)(\frac{a}{\lambda^{4+m}})^{-4-m}}, \quad (5.2)$$

where in the last step we used $a \sim \lambda^{1 - \frac{a}{\lambda^{4+m}}}$, this can be obtained by the definition of (2.2), priori estimates (4.2) and (4.3), and Lemma 3.1. Integrating from $t_0$ to $t_1$, and using $J(\tilde{u}(t_1)) = 0$, we obtain

$$J(\tilde{u}) \lesssim \lambda^m \int_{t_0}^{t_1} a(t) \lambda(t)^{2l-1} (\frac{a}{\lambda^{4+m}})^{-4-m} d\tau.$$
From (3.35), (4.2) and (4.3), we deduce
\[
\left| \frac{\lambda}{\lambda} + a \right| \lesssim |A_1| + a\|H_1^a\| + a^t \lesssim \delta C^i a.
\]
This means that \(0 < a \lesssim -\lambda \lambda t\). Therefore, if choose \(l\) is sufficient large, we have
\[
J(\tilde{u}) \lesssim \lambda^2(t).
\]
By (4.20), we have
\[
\|\nabla \tilde{u}\|_L^2 + \frac{\|\tilde{u}\|_L^2}{\lambda^2} \lesssim \lambda^2(t).
\]  \hspace{1cm} (5.3)
This is equivalent to
\[
\|\varepsilon\|_{H_1^a}^2 \lesssim \lambda^4(t).
\]

**Step 2:** Control of the modulation parameters. From (5.2) and (5.3), we can obtain
\[
\|\varepsilon\|_{H_1^a}^2 \lesssim \lambda^{2+(l-1)(1 - \frac{\sigma}{1 - \sigma a_\infty}) - \frac{2}{2(l-1)}}.
\]
Together with (3.35) and \(a \sim \lambda^{1 - \frac{\sigma}{1 - \sigma a_\infty}}\), we have
\[
\text{Mod}(t) \lesssim a\|\varepsilon\|_{H_1^a} + a^t \lesssim a'.
\]
Let \(l > \frac{2}{1 - \sigma a_\infty} + 1\) and \(t_1\) be as defined in Lemma 3.1 and \(t_0 \leq t < t_1 < 0\). Let \((\lambda_0, a_0, \tilde{\beta}_0, a_0, \gamma_0)\) be the solution to the system (3.5). Let \((\lambda, a, \tilde{\beta}, \alpha, \gamma)(t_1)\) be the initial data as
\[
(\lambda, a, \tilde{\beta}, \alpha, \gamma)(t_1) = (\lambda_0, a_0, \tilde{\beta}_0, a_0, \gamma_0)(t_1)
\]
and be the solution of the following perturbed system of modulation equations on \([t_0, t_1)\):
\[
\begin{cases}
\frac{\lambda}{\lambda} + a - A_1 = \mathcal{O}(a'), \\
\frac{\alpha}{\lambda} + 2\beta = \mathcal{O}(a'), \\
\tilde{\beta} - A_2 = \mathcal{O}(a'), \\
a = \frac{2\beta \sqrt{\lambda}}{a_\infty}; \quad \beta = \beta_\infty + \tilde{\beta}, \\
\gamma_s = 1 + \beta^2 + \mathcal{O}(a').
\end{cases}
\]
Then we claim the following bounds hold on the time interval \([t_0, t_1)\):
\[
a(t) = \frac{1}{(1 - (2\sigma + 1)a_\infty)B_1^{1 - (\sigma + 1)a_\infty}} B_2^{-2 \frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty}} |t|^{\frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty}} \left(1 + \mathcal{O} \left(\log |t||t|^{\frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty}}\right)\right),
\]
\[
|\tilde{\beta}| = \mathcal{O} \left(|t|^{\frac{1 - (\sigma + 1)a_\infty}{1 - (\sigma - 1)a_\infty}}\right)
\]
\[ \lambda(t) = B_2^{-1} \left( \frac{1}{1 - \frac{\alpha}{2a_\infty}} \right) B_1^{1 - \frac{\alpha}{2a_\infty}} |t|^{-\frac{1}{1 - \frac{\alpha}{2a_\infty}}} \left( 1 + \mathcal{O} \left( \log |t| |t|^{-\frac{1}{1 - \frac{\alpha}{2a_\infty}}} \right) \right), \]

\[ \alpha(t) = b \lambda^{-\frac{\alpha}{2a_\infty}} = b_{\infty} B_2^{-1} \left( \frac{1}{1 - \frac{\alpha}{2a_\infty}} \right) B_1^{1 - \frac{\alpha}{2a_\infty}} |t|^{-\frac{1}{1 - \frac{\alpha}{2a_\infty}}} \left( 1 + \mathcal{O} \left( \log |t| |t|^{-\frac{1}{1 - \frac{\alpha}{2a_\infty}}} \right) \right), \]

\[ \gamma(t) = (1 + \beta_{\infty}^2) B_1^{1 - \frac{\alpha}{2a_\infty}} B_2^{1 + \frac{\alpha}{2a_\infty}} |t|^{-\frac{1}{1 - \frac{\alpha}{2a_\infty}}} + \mathcal{O}(\log |t|), \]

where

\[ B_1 = \frac{1 - (\sigma - 1)a_\infty}{1 - (\sigma + 1)a_\infty}, \quad B_2 = \frac{a_\infty}{2(1 - 3a_\infty)\beta_{\infty}}b_{\infty} \]

where \( b_{\infty} \) is defined in Lemma 3.1. By the similar argument as [19], we can obtain the above estimates. This complete the proof of this lemma. \( \square \)

### 6 Existence of the ring blowup solution

In this section, we aim to prove our main result Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \{t_n\}_{n \geq 0} \) be an increasing sequence of times \( t_n < 0 \) such that \( t_n \to 0^- \). Let \( u_n \) be the solution to (1.6) with the initial data at \( t = t_n \),

\[ u_n(t_n, x) = (\alpha_0(t_n))^{\frac{\sigma}{2}} \frac{1}{\lambda_0(t_n)} \mathcal{P}_0(t_n) \left( \frac{r - \alpha_0(t_n)}{\lambda_0(t_n)} \right) e^{i\beta_0(t_n)}, \]

with \( \mathcal{P}_0(t_n) = (\alpha_0(t_n), \bar{\beta}_0(t_n)) \). Let \( t_1 < 0 \) (see Lemma 5.1) which is independent of \( n \). The \( L^2 \) compactness of \( u_n(t_0) \) is a consequence of a standard localization procedure. Indeed, Lemma 5.1 ensures the uniform bound \( \|u_n\|_{H^1} \lesssim 1 \). We note that the uniform bound

\[ \left| \frac{d}{dt} \int \chi_R |u_n|^2 \right| = 2 \left| \Re \int (\nabla \chi_R \cdot \nabla u_n) \bar{u}_n \right| \lesssim \frac{1}{R}, \]

with a smooth cut-off function \( \chi_R(x) = \chi \left( \frac{x}{R} \right) \) where \( \chi(x) \equiv 0 \) for \( |x| \leq 1 \) and \( \chi(x) = 1 \) for \( |x| \geq 2 \). By integrating this bound from \( t_0 \) to \( t_1 \), we have

\[ \lim_{R \to \infty} \int_{|x| \geq R} |u_n(t_0)|^2 = 0, \]

which together with the \( L^2(|x| < R) \) compactness of \( u_n(t_0) \), up to a subsequence, we can obtain

\[ u_n(t_0) \to u(t_0) \quad \text{in} \quad L^2 \quad \text{as} \quad n \to +\infty. \]

Let \( u \in C([t_0, 0), H^1) \) be the solution to (1.6) with the initial data \( u(t_0) \), then using the uniform estimate in \( H^1 \) for \( u_n \) and the converge in \( L^2 \) of \( u_n(t_0) \), we have, for \( t \in [t_0, 0) \),

\[ u_n(t) \to u(t) \quad \text{as} \quad L^2. \]

33
Let \( u_n(t) \) admits a geometrical decomposition
\[
u_n(t, x) = (\alpha_n(t))^{\frac{\sigma}{2}} \frac{1}{\lambda_n(t)} [Q\mathcal{P}_n(t) + \epsilon_n] \left( \frac{r - \alpha_n(t)}{\lambda_n(t)} \right) e^{iy_n(t)},
\]
then \( u \) satisfied the following form
\[
u(t, x) = \alpha(t)^{\frac{\sigma}{2}} \frac{1}{\lambda(t)} [Q\mathcal{P}(t) + \epsilon] \left( \frac{r - \alpha(t)}{\lambda(t)} \right) e^{iy(t)},
\]
with \( \mathcal{P}_n \to \mathcal{P}, \gamma_n \to \gamma \) and \( \epsilon_n \to \epsilon \) in \( L^2 \) as \( n \to \infty \). In addition, by Lemma 5.1, the parameters and \( \epsilon \) have the following bounds,
\[
a(t) = \frac{1}{(1 - (2\sigma + 1)a_{\infty})} B_1 \frac{1}{1 - (\sigma - 1)a_{\infty}} B_2 - 2 \frac{1}{1 - (\sigma - 1)a_{\infty}} |t|^{1 - (\sigma - 1)a_{\infty}} \left( 1 + O \left( \log |t| |t|^{1 - (\sigma - 1)a_{\infty}} \right) \right),
\]
\[
|\tilde{\beta}| = O \left( |t|^{1 - (\sigma + 1)a_{\infty}} \right),
\]
\[
\lambda(t) = B_2 \frac{1}{1 - (\sigma - 1)a_{\infty}} |t|^{1 - (\sigma - 1)a_{\infty}} \left( 1 + O \left( \log |t| |t|^{1 - (\sigma - 1)a_{\infty}} \right) \right),
\]
\[
\alpha(t) = b\lambda_{1 - \sigma a_{\infty}} = b_{\infty} B_2 \frac{2}{1 - (\sigma - 1)a_{\infty}} B_1 \frac{1}{1 - (\sigma - 1)a_{\infty}} |t|^{1 - (\sigma - 1)a_{\infty}} \left( 1 + O \left( \log |t| |t|^{1 - (\sigma - 1)a_{\infty}} \right) \right),
\]
\[
\gamma(t) = (1 + \beta_{\infty}^2) B_1 \frac{2}{1 - (\sigma - 1)a_{\infty}} B_2 \frac{1}{1 - (\sigma - 1)a_{\infty}} |t|^{1 - (\sigma - 1)a_{\infty}} + O(\log |t|),
\]
and
\[
\|\epsilon\|_{H^1_b} \lesssim |t|^{\frac{1}{1 - (\sigma + 1)a_{\infty}}},
\]
where
\[
B_1 = \frac{1 - (\sigma - 1)a_{\infty}}{1 - (\sigma + 1)a_{\infty}}, \quad B_2 = \frac{a_{\infty}}{2(1 - 3a_{\infty})}\beta_{\infty} - b_{\infty},
\]
and \( b_{\infty} \) given by Lemma 3.1. Now by the standard argument as \([10, 11, 19, 21]\), we can obtain \( u \in C([0, T], H^1) \) and \( u \) blows up at time \( T = 0 \). The parameters estimates are the consequence of the above estimates for \((a, \lambda, \alpha, \gamma, \epsilon)\).

This complete the proof of Theorem 1.1. \( \square \)

### A Appendix

In this section, we aim to prove Lemma 3.3. This is a standard modulation lemma relies on the implicit function theorem and the mass subcritical nondegeneracy \((Q, \Lambda Q) = \frac{1}{2}||Q||_{L^2}^2 \neq 0\).

**Proof of Lemma 3.3.** By the assumption, we have
\[
u(x) = \alpha_0^{\frac{\sigma}{2}}(t) \frac{1}{\lambda_0(t)} Q_{(a_0, \tilde{\beta}_0)} \left( \frac{r - \alpha_0}{\lambda_0} \right) e^{iy_0} + \tilde{u}_0(x),
\]
and we wish to introduce a modified decomposition

\[ u(x) = \alpha_1^\beta(t) \frac{1}{\lambda_1(t)} Q_{(a_1, \beta_1)} \left( \frac{r - \alpha_1}{\lambda_1} \right) e^{i\gamma_1} + \tilde{u}_1(x). \]

Comparing the decompositions, we obtain the formula

\[ \tilde{u}_1(x) = \alpha_0^\beta(t) \frac{1}{\lambda_0(t)} Q_{(a_0, \beta_0)} \left( \frac{r - \alpha_0}{\lambda_0} \right) e^{i\gamma_0} - \alpha_1^\beta(t) \frac{1}{\lambda_1(t)} Q_{(a_1, \beta_1)} \left( \frac{r - \alpha_1}{\lambda_1} \right) e^{i\gamma_1} + \tilde{u}_0(x). \]

We now define the functional

\[ F_{z, \mu, \gamma, \tilde{\beta}, \nu}(y) = \nu^\gamma \mu Q_{(a_0, \beta_0)}(\mu y + z) e^{i\gamma_1 + i(\beta_0 + \tilde{\beta})y} - Q_{(a_1, \beta_1)}(y) e^{i\beta_1 y}, \]

with

\[ z = \frac{\alpha_1 - \alpha_0}{\lambda_0}, \quad \mu = \frac{\lambda_1}{\lambda_0}, \quad \gamma = \gamma_1 - \gamma_0, \quad \tilde{\beta} = \tilde{\beta}_1 - \tilde{\beta}_0, \quad \nu = \frac{\alpha_0}{\alpha_1}. \]

So that

\[ \tilde{c}_1(y) = F_{z, \mu, \gamma, \tilde{\beta}, \nu}(y) + \alpha_1^\beta \lambda_1 \tilde{u}_0(\lambda_1 y + \alpha_1) e^{i\gamma_1 + i\beta_1 y}. \]

We then define the scalar products, for \( j = 1, 2 \)

\[ \rho^{(j)} = \int_{-\infty}^{+\infty} \Re \tilde{c}_1(\xi_1) T^{(j)}(y) dy \]

\[ = \int_{-\infty}^{+\infty} \Re F_{z, \mu, \gamma, \tilde{\beta}, \nu}(y) \xi_1 T^{(j)}(y) dy + \Re \int \tilde{u}_0(r) \alpha_1^\beta(\xi_1 T^{(j)}) \left( \frac{r - \alpha_1}{\lambda_1} \right) e^{i\gamma_1 + \frac{r - \alpha_1}{\lambda_1}} dr, \]

and for \( j = 3, 4 \)

\[ \rho^{(j)} = \int_{-\infty}^{+\infty} \Im \tilde{c}_1(\xi_1) T^{(j)}(y) dy \]

\[ = \int_{-\infty}^{+\infty} \Im F_{z, \mu, \gamma, \tilde{\beta}, \nu}(y) \xi_1 T^{(j)}(y) dy + \Re \int \tilde{u}_0(r) \alpha_1^\beta(\xi_1 T^{(j)}) \left( \frac{r - \alpha_1}{\lambda_1} \right) e^{i\gamma_1 + \frac{r - \alpha_1}{\lambda_1}} dr, \]

where

\[ T^{(1)} = yQ, \quad T^{(2)} = Q, \quad T^{(3)} = \partial_\gamma Q, \quad T^{(4)} = \Lambda Q. \]

We now view \( \rho = (\rho^{(j)})_{1 \leq j \leq 4} \) as smooth functions of \((\tilde{u}_0, z, \mu, \tilde{\beta}, \gamma)\). Observe that the bound \( (3.25) \) ensures that

\[ |\rho(\tilde{u}_0, 0, 1, 0, 0)| \lesssim \delta. \]

Notice that

\[ a_1 = \frac{2\beta_1 \lambda_1}{a_\infty \alpha_1} = 2(\beta_0 + \tilde{\beta}) \frac{\lambda_0}{a_\infty a_0} \mu \frac{\alpha_0}{\alpha_1} \left( 1 + \frac{\tilde{\beta}}{\beta_0} \right) a_0 \mu \left( 1 + \frac{a_\infty a_0}{2\beta_0} \right)^{-1}. \]
Using

\[ Q_{\rho|\rho=0} = Q e^{-i\tilde{\beta}_\infty y} \]

we obtain

\[
\begin{aligned}
\partial_z F|_{(z=0,\mu=1,\tilde{\beta}=1,\gamma=0)} &= Q' - i\beta_\infty Q + O(|a_0| + |\tilde{\beta}_0|)e^{-c|y|}, \\
\partial_\mu F|_{(z=0,\mu=1,\tilde{\beta}=1,\gamma=0)} &= \Lambda Q - i\beta_\infty Q + O(|a_0| + |\tilde{\beta}_0|)e^{-c|y|}, \\
\partial_{\tilde{\beta}} F|_{(z=0,\mu=1,\tilde{\beta}=1,\gamma=0)} &= iyQ + O(|a_0| + |\tilde{\beta}_0|)e^{-c|y|}, \\
\partial_\gamma F|_{(z=0,\mu=1,\tilde{\beta}=1,\gamma=0)} &= -iQ + O(|a_0| + |\tilde{\beta}_0|)e^{-c|y|}.
\end{aligned}
\]

Then the Jacobian matrix of \( \rho \) at \((z = 0, \mu = 1, \tilde{\beta} = 1, \gamma = 0)\) is not zero from the smallness assumption (3.26). The existence of the desired decomposition now follows from the implicit function theorem, and the smallness of the parameters.

Acknowledgments

Y. Li was supported by China Postdoctoral Science Foundation (No. 2021M701365) and the funding of innovating activities in Science and Technology of Hubei Province.

References

[1] A. H. Ardila, V. D. Dinh, and L. Forcella. Sharp conditions for scattering and blow-up for a system of NLS arising in optical materials with \( \chi^3 \) nonlinear response. *Comm. Partial Differential Equations*, 46(11):2134–2170, 2021.

[2] V. Banica, R. Carles, and T. Duyckaerts. Minimal blow-up solutions to the mass-critical inhomogeneous NLS equation. *Comm. Partial Differential Equations*, 36(3):487–531, 2011.

[3] M. Cardoso and L. G. Farah. Blow-up of radial solutions for the intercritical inhomogeneous NLS equation. *J. Funct. Anal.*, 281(8):Paper No. 109134, 38, 2021.

[4] M. Cardoso and L. G. Farah. Blow-up solutions of the intercritical inhomogeneous NLS equation: the non-radial case. *Math. Z.*, 303(3):Paper No. 63, 18, 2023.

[5] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[6] S. M. Chang, S. Gustafson, K. Nakanishi, and T. P. Tsai. Spectra of linearized operators for NLS solitary waves. *SIAM J. Math. Anal.*, 39(4):1070–1111, 2007/08.

[7] V. D. Dinh. Blowup of \( H^1 \) solutions for a class of the focusing inhomogeneous nonlinear Schrödinger equation. *Nonlinear Anal.*, 174:169–188, 2018.
[8] Luiz G. Farah. Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation. *J. Evol. Equ.*, 16(1):193–208, 2016.

[9] F. Genoud and C. A. Stuart. Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves. *Discrete Contin. Dyn. Syst.*, 21(1):137–186, 2008.

[10] V. Georgiev and Y. Li. Blowup dynamics for mass critical half-wave equation in 3D. *J. Funct. Anal.*, 281(7):Paper No. 109132, 34, 2021.

[11] V. Georgiev and Y. Li. Nondispersive solutions to the mass critical half-wave equation in two dimensions. *Comm. Partial Differential Equations*, 47(1):39–88, 2022.

[12] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[13] T. S. Gill. Optical guiding of laser beam in nonuniform plasma. *Pramana*, 55(5):835–842, 2000.

[14] J. Ginibre and G. Velo. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Functional Analysis*, 32(1):1–32, 1979.

[15] C. M. Guzmán. On well posedness for the inhomogeneous nonlinear Schrödinger equation. *Nonlinear Anal. Real World Appl.*, 37:249–286, 2017.

[16] M. K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.

[17] Y. Li. Blowup dynamics for inhomogeneous mass critical half-wave equation. *arXiv preprint arXiv:2206.04938*, 2022.

[18] C. S. Liu and V. K. Tripathi. Laser guiding in an axially nonuniform plasma channel. *Physics of plasmas*, 1(9):3100–3103, 1994.

[19] F. Merle, P. Raphaël, and J. Szeftel. On collapsing ring blow-up solutions to the mass supercritical nonlinear Schrödinger equation. *Duke Math. J.*, 163(2):369–431, 2014.

[20] C. Peng and D. Zhao. Blow-up dynamics of $L^2$-critical inhomogeneous nonlinear Schrödinger equation. *Math. Methods Appl. Sci.*, 41(18):9408–9421, 2018.

[21] P. Raphaël and J. Szeftel. Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS. *J. Amer. Math. Soc.*, 24(2):471–546, 2011.

[22] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

[23] S. Zhu. Blow-up solutions for the inhomogeneous Schrödinger equation with $L^2$ supercritical nonlinearity. *J. Math. Anal. Appl.*, 409(2):760–776, 2014.
Yuan Li,
School of Mathematics and Statistics, Central China Normal University, Wuhan, PR China
E-mail: yli2021@ccnu.edu.cn