Transonic Potential Flows in A
Convergent–Divergent Approximate Nozzle

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Abstract

In this paper we prove existence, uniqueness and regularity of certain perturbed (subsonic–supersonic) transonic potential flows in a two-dimensional Riemannian manifold with “convergent–divergent” metric, which is an approximate model of the de Laval nozzle in aerodynamics. The result indicates that transonic flows obtained by quasi-one-dimensional flow model in fluid dynamics are stable with respect to the perturbation of the velocity potential function at the entry (i.e., tangential velocity along the entry) of the nozzle. The proof is based upon linear theory of elliptic–hyperbolic mixed type equations in physical space and a nonlinear iteration method.

Key words: potential flow equation, transonic flow, Riemannian manifold, hyperbolic–elliptic mixed equation, de Laval nozzle

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1 Introduction

Understanding flow patterns in a convergent–divergent nozzle (the so called de Laval nozzle in engineering) is a prominent issue in aerodynamics and partial differential equations due to their numerous applications in practice, and closely connection with many difficult mathematical problems, such as mixed type equations and free boundary problems \[6,13,14,17\]. Since these flow patterns are genuinely nonlinear, various physically significant special solutions of the corresponding mathematical problems play an important role in the theoretical analysis. For example, for the nearly spherical symmetric transonic shocks and transonic shocks in a slowly varying nozzle, there are works of Chen et.al. \[1,5,15,22,24\] on potential flow equation and complete Euler system based upon in essence two classes of special solutions \[25\]. There are also many progresses in the analysis of subsonic nozzle flows, see, for instance, \[16,21\] and reference therein. However, since no simple and physical special transonic–flow solution is available, presently the study of subsonic–supersonic transonic flow mainly utilized the methods of compensated compactness (see \[3,4,17,21\] and references therein).

In \[23\], motivated by a significant work of Sibners \[20\], Yuan constructed various interesting special solutions in a two–dimensional Riemannian manifold with “convergent–divergent” metric, which may be regarded as an approximation of the de Laval nozzle. This paper also studied several boundary value problems of subsonic flows in such a manifold. In the present paper we will further investigate the subsonic–supersonic flow via the potential flow equation. We show that the special subsonic–supersonic transonic flows are stable with respect to the small perturbation of the velocity potential function (i.e., the tangential velocity) at the entry (see Theorem 5).

The potential flow equation is a second order equation of elliptic–hyperbolic mixed type for transonic flow. For such equations, presently one of the main tool is the theory of positive symmetric systems and techniques of energy estimates, see, for instance, \[8,13,14\] and references therein. In this paper we employ the theory developed in \[14\] to show the solvability of linear problem, and then a nonlinear iteration argument to solve the nonlinear problem.

We remark that in recent years there are many breakthroughs on partial differential equations of mixed type and degenerate elliptic type arising in differential geometry and physics, see, for example, \[2,8,9,10,11,12,14,17\]. For earlier developments in this field, one may also consult \[17,18,19\] and references therein.

The rest of the paper is organized as follows. In Section 2 we formulate the problem, and study the properties of the coefficients of the potential flow
equations in the manifold. In Section 3 we solve the linear problem, and finally in Section 4 we state the main result, Theorem 5 and prove it.

2 Formulation of the problem

Let \( S^1 \) be the standard unit circle in \( \mathbb{R}^2 \), and \( M \) be the Riemannian manifold \( \{(x^1, x^2) \in [-1, 1] \times S^1\} \) with a metric \( G = g_{ij}dx^i \otimes dx^j = dx^1 \otimes dx^1 + n(x^1)^2 dx^2 \otimes dx^2 \). Here \( n(t) \) is a positive smooth function on \( [-1, 1] \) satisfies:

1. \( n''(t) > 0 \); 2. \( n'(t) < 0 \) for \( t \in (-1, 0) \), \( n'(t) > 0 \) for \( t \in (0, 1) \).

Such a manifold \( M \) may be regarded as an approximation of a two–dimensional convergent–divergent nozzle, with \( M^{\pm} = M \cap \{x^1 \gtrless 0\} \) respectively the divergent and convergent part. We also call \( \Sigma_k = \{k\} \times S^1 \), \( k = -1, 0, 1 \) respectively the entrance, throat and exit of \( M \). Obviously \( \partial M = \Sigma_{-1} \cup \Sigma_1 \).

Let \( p, \rho \) be functions in \( M \) represent respectively the pressure and density of gas flow in \( M \), and \( v \) be a vector field in \( M \) represent the velocity of the flow. We consider polytropic and isentropic gas flows, then \( p = \kappa \rho^\gamma \) with \( \kappa > 0 \), \( \gamma > 1 \) two constants, and the speed of sound is \( c = \sqrt{\kappa \gamma \rho^{\gamma-1}} \). Let \( \tilde{v} \) be the 1-form corresponding to \( v \) under the metric \( G \). The flow is irrotational if \( \tilde{v} \) is exact; That is, there exists a function \( \varphi \) in \( M \) such that \( \tilde{v} = d\varphi \). Substituting this in the equation of conservation of mass \( \text{div}(\rho v) = -d^*(\rho \tilde{v}) = 0 \), where \( \text{div} \) and \( d^* \) are respectively the divergence operator and codifferential operator in \( M \), then by the formula \( d^*(\rho \tilde{v}) = \rho d^* \tilde{v} - \langle d\rho, \tilde{v} \rangle \), with \( \langle \cdot, \cdot \rangle \) the inner product of forms in \( M \), we have

\[
\rho \Delta \varphi = \langle d\rho, d\varphi \rangle, \tag{1}
\]

where \( \Delta = dd^* + d^*d \) is the Hodge Laplacian of forms. (Note that \( d^* \varphi = 0 \).) By the Bernoullli’s law which represents conservation of energy:

\[
\frac{1}{2} \langle d\varphi, d\varphi \rangle + \frac{\kappa \gamma}{\gamma - 1} \rho^{\gamma - 1} = c_0, \tag{2}
\]

where \( c_0 \) is a positive constant, \( \rho \) in (1) can be expressed in terms of \( d\varphi \). So we may write (1) as a second order equation of \( \varphi \).

Indeed, in the \((x^1, x^2)\) coordinates, we have

\[
\Delta \varphi = -\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j \varphi) = -\frac{1}{n(x^1)} \left( \partial_1 (n(x^1) \partial_1 \varphi) + \frac{1}{n(x^1)} \partial_2 \varphi \right).
\]
Here $\sqrt{g} = \sqrt{\det(g_{ij})}$, $(g'^{ij})$ is the inverse of $(g_{ij})$, and $\partial_i = \partial_{x^i}$, $\partial_{ij} = \partial_i \partial_j$. By differentiating (2) we have

$$
\frac{c^2}{\rho} d\rho = -\left( \frac{1}{2} \partial_i \varphi \partial_j \varphi \partial_k g^{ij} + \partial_{ik} \varphi \partial_j \varphi g^{ij} \right) dx^k.
$$

Then by a straightforward calculation we obtain

$$
n(x^1)^2(c^2 - (\partial_1 \varphi)^2) \partial_{11} \varphi - 2 \partial_1 \varphi \partial_2 \varphi \partial_1 \varphi + \left( c^2 - \frac{1}{n(x^1)^2} (\partial_2 \varphi)^2 \right) \partial_{22} \varphi + n(x^1)n'(x^1) \left( c^2 + \frac{1}{n(x^1)^2} (\partial_2 \varphi)^2 \right) \partial_1 \varphi = 0.
$$

(3)

Direct computation yields that this equation is of elliptic type if the flow is subsonic ($c^2 > (\partial_1 \varphi)^2 + (\partial_2 \varphi)^2/n(x^1)^2$), and is of hyperbolic type if the flow is supersonic ($c^2 < (\partial_1 \varphi)^2 + (\partial_2 \varphi)^2/n(x^1)^2$).

If the flow depends only on $x^1$, then (3) indicates that $\varphi_b = \varphi_b(x^1)$ satisfies the equation

$$
n(x^1)(c_b^2 - (\partial_1 \varphi_b)^2) \partial_{11} \varphi_b + n'(x^1)c_b^2 \partial_1 \varphi_b = 0,
$$

(4)

where $c_b$ is the sonic speed corresponding to $\varphi_b$. It can be shown that there are special flows $\varphi_b \in C^5(\mathcal{M})$ which are subsonic in $\mathcal{M}^-$ and supersonic in $\mathcal{M}^+$, and $\partial_1 \varphi_b > 0$, $\partial_{11} \varphi_b > 0$ in $\mathcal{M}$ (see [23]). We call such flows background solutions. The aim of this paper is to study stability of certain background solutions under perturbations of $\varphi$ on the entrance $\Sigma^{-1}$ of $\mathcal{M}$.

Let $\hat{\varphi} = \varphi - \varphi_b$. By subtracting (4) from (3), we have

$$
k(D\varphi)\partial_{11} \hat{\varphi} + b(D\varphi)\partial_{12} \hat{\varphi} + \partial_{22} \hat{\varphi} - \alpha(x^1)\partial_1 \hat{\varphi} = f(D\varphi),
$$

(5)

where

$$
k(D\varphi) := \frac{n(x^1)^2(c^2 - (\partial_1 \varphi)^2)}{c^2 - \frac{1}{n(x^1)^2} (\partial_2 \varphi)^2},
$$

(6)

$$
b(D\varphi) := -\frac{2 \partial_1 \varphi \partial_2 \varphi}{c^2 - \frac{1}{n(x^1)^2} (\partial_2 \varphi)^2},
$$

(7)

$$
\alpha(x^1) := \frac{n(x^1)^2 \partial_{11} \varphi_b}{\partial_1 \varphi_b} \cdot \frac{c_b^4 + c_b^2 (\partial_1 \varphi_b)^2 + (\gamma - 1)(\partial_1 \varphi_b)^4}{c_b^4},
$$

(8)
\[ f(D\varphi) := \left\{ \frac{\gamma - 1}{2} \left( \partial_{11} \varphi_b + \frac{n'(x^1)}{n(x^1)} \partial_{1} \varphi_b \right) \left( (\partial_1 \hat{\varphi})^2 + \frac{1}{n(x^1)^2} (\partial_2 \hat{\varphi})^2 \right) \right. \]
\[ + \partial_{11} \varphi_b (\partial_1 \hat{\varphi})^2 - \frac{n'(x^1)}{n(x^1)} (c^2 - c_b^2) \partial_1 \hat{\varphi} - \frac{n'(x^1)}{n(x^1)^2} \partial_1 \varphi_b (\partial_2 \hat{\varphi})^2 \left. \right\} \cdot \frac{n(x^1)^2}{c^2 - \frac{1}{n(x^1)^2}(\partial_2 \varphi_b)^2} \]
\[ + \left[ \frac{n(x^1)^2 \partial_{11} \varphi_b}{c_b^2 \partial_1 \varphi_b} \cdot \frac{c_b^4 + c_b^2 (\partial_1 \varphi_b)^2 + (\gamma - 1) (\partial_1 \varphi_b)^4}{c^2 - \frac{1}{n(x^1)^2} (\partial_2 \varphi_b)^2} - \alpha(x^1) \right] \partial_1 \hat{\varphi}. \]

We will investigate the following problem:

\[ \text{Eq. (5) in } \mathcal{M} \text{ with } \varphi = \hat{\varphi} + \varphi_b, \tag{10} \]
\[ \hat{\varphi} = g(x^2) \text{ on } \Sigma^{-1} \text{ with } \|g\|_{H^5(S^1)} \text{ small.} \tag{11} \]

This is a Dirichlet problem of an elliptic–hyperbolic mixed equation.

For background solution \( \varphi_b \), let \( \tau = (\partial_1 \varphi_b/c_b)^2 \) be the square of Mach number. By Bernoulli’s law we may compute

\[ \partial_1 (k(D\varphi_b)) = -\frac{nn'}{\tau - 1} [(\gamma + 1) \tau^2 - 2\tau + 2] < 0 \tag{12} \]

in \( \mathcal{M} \). Here we also note the fact that \( \partial_{11} \varphi_b = (\partial_1 \varphi_b)n'/(n \cdot (\tau - 1)) \), and \( n'/(\tau - 1) > 0 \) in \( \mathcal{M} \), especially at the throat \( \Sigma^0 \) we have \( n'/(\tau - 1) = \sqrt{nn''/(\gamma + 1)} \) by L’Hospital’s rule in calculus. We also may write

\[ \alpha(x^1) = \frac{nn'[1 + \tau + (\gamma - 1) \tau^2]}{\tau - 1} > 0, \]

so there holds

\[ 2\alpha - l\partial_1 (k(D\varphi_b)) > \delta_1 > 0 \tag{13} \]

in \( \mathcal{M} \) for a fixed number \( \delta_1 \) and all positive number \( l \). In addition,

\[ 2\alpha + \partial_1 (k(D\varphi_b)) = \frac{nn'\tau}{\tau - 1} [(\gamma - 3) \tau + 4] > \delta_2 > 0 \tag{14} \]

in \( \mathcal{M} \). We remark that the constants \( \delta_1, \delta_2 \) depend only on the specific background solution \( \varphi_b \).
3 Solvability of linear problem

Let \( k, b, a, \alpha, f \) be functions in \( M = [-1, 1] \times S^1 \) with flat metric. In this section we investigate the solvability of the following linear problem:

\[
Lu := k \partial_{11} u + b \partial_{12} u + a \partial_{22} u - \alpha(x^1) \partial_1 u = f \quad \text{in} \ M, \quad \text{on} \ \Sigma^{-1}. \tag{15}
\]

\[
u = 0 \quad \text{on} \ \Sigma^{-1}. \tag{16}
\]

We use \( H^j \) to denote the Sobolev space \( W^{j,2}(M) \), and the corresponding norm is written as \( \| \cdot \|_j \). Note that \( C^j(M) \) denotes the usual space of \( j \) times continuous differential functions in \( M \).

**Theorem 1** Let \( k > 0 \) on \( \Sigma^{-1} \) and \( k < 0 \) on \( \Sigma^1 \), and \( k, b, a, \alpha \in C^4(M) \). Suppose \( f \in H^s(M), 0 \leq s \leq 3 \) and there is a positive constant \( \delta \) such that

\[
a \geq \delta > 0, \quad \partial_1 a \leq -\delta < 0 \quad \text{in} \ M, \tag{17}
\]

\[
2\alpha - (2p - 1) \partial_1 k \geq \delta > 0 \quad \text{in} \ M \quad \text{for} \ p = 0, 1, \ldots, s. \tag{18}
\]

Then there is a \( \nu > 0 \) such that if

\[
\| b \|_3 \leq \nu, \quad \| \partial_2 a \|_{C^3} \leq \nu \quad \text{in} \ M, \tag{19}
\]

then there exists uniquely one solution \( u \in H^1 \) to problem (15) and (16) and there holds the estimate

\[
\| u \|_{s+1} \leq C \| f \|_s. \tag{20}
\]

To prove this, we follow the celebrated ideas presented in [13,14]. That is, one first solves a boundary value problem of a mixed type equation which is elliptic at both the entry and exit. Then the above theorem can be demonstrated by an extension technique.

**Theorem 2** Under the assumptions of Theorem 1, but supposing \( k > 0 \) on both \( \Sigma^{-1} \) and \( \Sigma^1 \), then the following problem

\[
Lu = f \quad \text{in} \ M, \tag{21}
\]

\[
u = 0 \quad \text{on} \ \Sigma^{-1}, \tag{22}
\]

\[
\partial_1 u = 0 \quad \text{on} \ \Sigma^1 \tag{23}
\]

has uniquely one solution \( u \in H^1 \) and it also satisfies (20).
Proof of Theorem 1. Step 1. Uniqueness. Let \( f \equiv 0 \). Multiplying to \((15)\) by \( \partial_1 u \) and integrating the expression in \([-1, 1] \times S^1 \), Note that \( \partial_2 u = 0 \) on \( \Sigma^{-1} \), we have

\[
\int_M \left[ (\alpha + \frac{1}{2} \partial_1 k + \frac{1}{2} \partial_2 b)(\partial_1 u)^2 - \frac{1}{2} \partial_1 a(\partial_2 u)^2 + \partial_2 a \partial_1 u \partial_2 u \right] \, dx \, dx^2 \\
= \int_{\Sigma} \left[ \frac{k}{2} (\partial_1 u)^2 - \frac{a}{2} (\partial_2 u)^2 \right] \, dx^2 - \int_{\Sigma^{-1}} \left[ \frac{k}{2} (\partial_1 u)^2 - \frac{a}{2} (\partial_2 u)^2 \right] \, dx^2 \\
\leq 0
\]

via the integration by parts and divergence theorem. Since \( \partial_1 a \leq -\delta < 0 \) in \( M \), we infer that, by choosing \( \nu = \delta/4 \),

\[
(2\alpha + \partial_1 k + \partial_2 b)(\partial_1 u)^2 - \partial_1 a(\partial_2 u)^2 + 2\partial_2 a \partial_1 u \partial_2 u \\
\geq (\delta - 2\nu)(\partial_1 u)^2 + (\delta - \nu)(\partial_2 u)^2 \geq \frac{\delta}{2} ((\partial_1 u)^2 + (\partial_2 u)^2).
\]

Hence we have

\[
(\delta/2) \int_M [(\partial_1 u)^2 + (\partial_2 u)^2] \, dx \, dx^2 \leq 0
\]

by \((21)\) and therefore \( u \equiv 0 \) in \( M \) due to \((16)\).

Step 2. Existence. Let \( M^* = [-1, 2] \times S^1 \). We may extend \( k, b, a, \alpha, f \) to \( M^* \) such that they still satisfy \((18) \sim (19)\) and other assumptions in Theorem 2, especially \( k > 0 \) on the new exit \( \Sigma^2 = \{2\} \times S^1 \), and \( \|f\|_{H^s(M^*)} \leq C \|f\|_{H^s(M)} \). Denoting the obtained extended operator in \( M^* \) as \( L \) again, we consider the problem \((21) (22)\) together with boundary condition \( \partial_1 u = 0 \) on \( \Sigma^2 \). By Theorem 2 there is a unique solution \( u^* \) and \( \|u^*\|_{s+1} \leq C \|f\|_{H^s(M^*)} \). Obviously \( u := u^*|_M \) is also a solution to problem \((15)\) and \((16)\). This finishes the proof.

The proof of Theorem 2 also follows in a similar way of \([14]\) (Theorem 1.1, pp.9–18), but needs some modifications to deal with the mixed derivative term \( b \partial_{12} u \), the non-divergence term \( a \partial_{22} u \), and no lateral boundary in our case. For completeness and convenience of the readers, we sketch out the proofs. Some of the details in the proof are important in the analysis of the nonlinear problems.

Proof of Theorem 2. Step 1. Uniqueness. This may be proved by a similar method as in deriving \((25)\).

Step 2. Approximate problem. To show existence of a \( H^1 \) weak solution to problem \((21) \sim (23)\), as in \([14]\), we consider the following singular perturbation problem:
\[ L_\epsilon u := Lu + \epsilon \partial_{111} u = f_\epsilon \quad 1 > \epsilon > 0 \text{ in } M, \]  
\[ u(-1, x^2) = \partial_1 u(-1, x^2) = 0, \]  
\[ \partial_1 u(1, x^2) = 0, \]

where \( f_\epsilon \in C^1 \) and \( f_\epsilon \to f \) in \( L^2 \) as \( \epsilon \to 0 \).

**Step 2.1. Galerkin’s method.** To show existence of a solution \( u^\epsilon \) to problem (26)-(28), we employ the Galerkin’s method of finite dimensional approximation. Let \{\( Y_i(x^2) \)\} \( (i = 1, 2, \cdots) \) be a complete system in \( H^2(S^1) \) and orthogonal in \( L^2(S^1) \). We may also assume that each \( Y_i(x^2) \) is smooth. Set

\[
u^{N,\epsilon}(x^1, x^2) = \sum_{i=1}^{N} X_i^{N,\epsilon}(x^1) Y_i(x^2), \quad N = 1, 2, \cdots.
\]

The functions \( X_i^{N,\epsilon} \) \( (i = 1, \cdots, N) \) are to be determined by a boundary value problem of a system of third order ODEs:

\[
\int_{S^1} [L_\epsilon u^{N,\epsilon} - f_\epsilon] Y_j(x^2) \, dx^2 = 0, \quad j = 1, 2, \cdots, N,
\]

\[
X_i^{N,\epsilon}(-1) = (X_i^{N,\epsilon})'(-1) = 0, \quad (X_i^{N,\epsilon})'(1) = 0.
\]

Equation (29) can be written explicitly as

\[
\epsilon (\int_{S^1} dx^2) (X_j^{N,\epsilon})''' + \sum_{i=1}^{N} \left[ \left( \int_{S^1} k Y_i Y_j \, dx^2 \right) (X_i^{N,\epsilon})'' \right. \\
+ \left. \left( \int_{S^1} (b Y_i' Y_j - a Y_i Y_j') \, dx^2 \right) (X_i^{N,\epsilon})' \right. \\
+ \left. \left( \int_{S^1} a Y_i''' Y_j \, dx^2 \right) (X_i^{N,\epsilon}) \right] = \int_{S^1} f_\epsilon Y_j(x^2) \, dx^2.
\]

**Step 2.1.1. Uniqueness.** Now we show the solution to problem (29)-(31) is unique. Indeed, multiplying to (29) by \( (X_j^{N,\epsilon})' \), summing up for \( j \) from 1 to \( N \) and integrating with respect to \( x^1 \) on \([-1, 1]\), we have

\[- \int_{M} [L_\epsilon u^{N,\epsilon} \partial_1 u^{N,\epsilon}] \, dx^1 dx^2 = - \int_{M} f_\epsilon \partial_1 u^{N,\epsilon} \, dx^1 dx^2.
\]

Writing \( u^{N,\epsilon} \) simply as \( w \), then integrating by parts and using \( \partial_1 w(-1, x^2) = \partial_1 w(1, x^2) = 0 \), we obtain that
\[-2 \int_M \left[ L_w \partial_1 w \right] dx^1 dx^2 = \int_M \left[ (2\alpha + \partial_1 k + \partial_2 b)(\partial_1 w)^2 - (\partial_1 a)(\partial_2 w)^2 \right. \\
+ 2\partial_2 a \partial_1 u \partial_2 u \right] dx^1 dx^2 + 2\epsilon \int_M (\partial_1 w)^2 dx^1 dx^2 + \int_{\Sigma} \left[ a(\partial_2 u)^2 \right] dx^2.\]

Therefore by Hölder inequality and Young inequality, due to (19), we have

\[\epsilon \left\| \partial_{11} w \right\|_0^2 + \left\| Dw \right\|_0^2 \leq C \left\| f \right\|_0^2 \leq C' \left\| f \right\|_0^2.\]  
(33)

Since \( w(-1, x^2) = 0 \), it follows that

\[ w(x^1, x^2) = w(x^1, x^2) - w(-1, x^2) = \int_{-1}^{x^1} \partial_1 w(t, x^2) dt. \]

By Hölder inequality,

\[ |w(x^1, x^2)|^2 \leq \int_{-1}^{x^1} 1^2 dt \int_{-1}^{x^3} |\partial_1 w(t, x^2)|^2 dt \leq 2 \int_{-1}^{1} |\partial_1 w(x^1, x^2)|^2 dx^1. \]

Hence, we have

\[ \left\| w \right\|_0^2 = \int_M |w(x^1, x^2)|^2 dx^1 dx^2 \leq 4 \int_M |\partial_1 w(x^1, x^2)|^2 dx^1 dx^2 = 4\left\| \partial_1 w \right\|_0^2. \]  
(34)

Now it easily follows from (33) and (34) that

\[ \epsilon \left\| \partial_{11} w \right\|_0^2 + \left\| w \right\|_1^2 \leq C \left\| f \right\|_0^2 \]  
(35)

for a constant \( C \) independent of \( N \) and \( \epsilon \). This proves uniqueness of solution to problem (29)–(31). Note the above estimate also holds for the solution \( u^\epsilon \) of problem (26)–(28).

**Step 2.1.2. Existence and regularity.** Now by Fredholm alternative of boundary value problems of ODEs, we can infer that problem (29)–(31) has a solution \( u^{N,\epsilon} \) which satisfies (35). Indeed, we may write (29)–(31) equivalently as a boundary value problem of a first order system of ODEs with \( 3n \) unknowns

\[ \frac{d\mathbf{\chi}}{dx^1} = A\mathbf{\chi} + \mathbf{F}, \]
\[ B\mathbf{\chi} = 0. \]  
(36)  
(37)

Here \( \mathbf{\chi}, \mathbf{F} \in \mathbb{R}^{3n} \) are column vectors, and \( A, B \) are \( 3n \times 3n \) matrices. Let \( \mathbf{\chi}_1, \cdots, \mathbf{\chi}_{3n} \) be a set of \( 3n \) linearly independent solutions of the homogeneous
equation $\frac{dX}{dx} = AX$, and $X_0$ be a special solution of the nonhomogeneous equation $\frac{dX}{dx} = AX + F$, then to show existence, we need to find $3n$ numbers $c_1, \cdots, c_{3n}$ such that

$$X = \sum_{j=1}^{3n} c_j X_j + X_0$$

satisfies $BX = 0$, or equivalently,

$$(BX_1, \cdots, BX_{3n})(c_1, \cdots, c_{3n})^T = -BX_0.$$  \hfill (38)

This is a linear algebraic system, and it is well known that it is uniquely solvable if and only if for $X_0 = 0$, then $c_1 = \cdots = c_{3n} = 0$. However, this is guaranteed by uniqueness we proved in step 2.1.1.

Note that all the coefficients in the left side of (32) belong to $C^3$ and the right side of (32) belongs to $C^1$, so $u^{N,\epsilon} \in C^4$.

**Step 2.2. Solution of problem (26)-(28).** By the uniform estimate (35), there exists a subsequence $\{u^{N,j,\epsilon}\}$ converges weakly in $H^1$ to a $u^\epsilon \in H^1$ as $j \to \infty$, and $\partial_{11}u^{N,j,\epsilon}$ converges weakly in $L^2$ to $\partial_{11}u^\epsilon \in L^2$. We show that $u^\epsilon$ is a weak solution of problem (26)-(28).

For $\chi_j(x^1) \in C_0^\infty([0, 1])$, by multiplying it to (29), summing for $j$ from 1 to $N$, and integrating with respect to $x^1$ on $[-1, 1]$, one has

$$\int_M \{[\partial_1(k\chi^N) + \partial_2(b\chi^N) + \alpha\chi^N - \epsilon\partial_{11}\chi^N]\partial_1 u^{N,\epsilon} + \partial_2(a\chi^N)\partial_2 u^{N,\epsilon} + f\chi^N\} \, dx^1 dx^2 = 0$$

after integration by parts, where $\chi^N = \sum_{j=1}^N \chi_j Y_j$. Supposing $\chi^N \to \chi$ strongly in $H^2$, let $N \to \infty$ in the above equality; we have

$$\int_M \{[\partial_1(k\chi) + \partial_2(b\chi) + \alpha\chi - \epsilon\partial_{11}\chi]\partial_1 u^\epsilon + \partial_2(a\chi)\partial_2 u^\epsilon + f\chi\} \, dx^1 dx^2 = 0$$ \hfill (39)

for all $\chi \in H^2 \cap H_0^1$. Therefore $u^\epsilon$ is a weak solution to (26)-(28).

Next we show $u^\epsilon$ satisfies (27) and (28). Indeed, since $H^1(M) \hookrightarrow L^2(\partial M)$, $u^{N,\epsilon}(-1, x^2) = 0$ and $u^{N,\epsilon} \rightharpoonup u^\epsilon$ weakly in $H^1$ indicate that $u^\epsilon(-1, x^2) = 0$. Since $\epsilon$ is fixed presently, and $W^{1,2}([-1, 1]; L^2(S^1)) \hookrightarrow C([-1, 1]; L^2(S^1))$ (see Theorem 2, p.286 of [7]), (35) implies that $\partial_1 u^{N,\epsilon} \in C([-1, 1]; L^2(S^1))$. Therefore by (30) and (31), we obtain that (27) and (28) hold.

**Step 3. Existence of weak solution of problem (15) and (16).**
Step 3.1. Weak solution. Now for $\epsilon \in (0, 1]$ we have constructed weak solutions of problem (26)–(28) which also satisfy (35). Then there is a subsequence $\{u^\epsilon_j\}$ converges weakly in $H^1$ to a $u \in H^1$ as $\epsilon_j \to 0$. Obviously we have

$$\|u\|_1 \leq C \|f\|_0. \tag{40}$$

We claim $u$ is a weak solution of problem (15) and (16). To verify this, let us take $\epsilon = \epsilon_j \to 0$ in (39) for any $\chi \in H^2 \cap H^1_0$. Note that

$$\left| \int_M \epsilon \partial_{11} \chi \partial_1 u^\epsilon \, dx^1 \, dx^2 \right| \leq C \epsilon \|f\|_0 \to 0,$$

we get

$$\int_M \{[\partial_1(k \chi) + \partial_2(b \chi) + \alpha \chi] \partial_1 u + \partial_2(a \chi) \partial_2 u + f \chi \} \, dx^1 \, dx^2 = 0. \tag{41}$$

By approximation this holds for all $\chi \in H^1_0$.

Step 3.2. Boundary conditions. The fact that $u$ satisfies the boundary condition $u = 0$ on $\Sigma^{-1}$ can be deduced by the same argument as in step 2.2.

Next, let $M'_\sigma = M \cap \{(x^1, x^2) : 1 - \sigma < x^1 < 1\}$. Since $k > 0$ on $\Sigma^1$ by assumption, for rather small $\sigma$, the operator $L$ is elliptic in $M'_{3\sigma}$. We claim that

$$\|\partial_1 u^\epsilon\|_{H^3(M'_{3\sigma})} \leq C \|f\|_0. \tag{42}$$

If this is true, then clearly $\partial_1 u = 0$ on $\Sigma^1$.

Now we prove (42). Let $\eta \in C^\infty([-1, 1])$ such that

$$\eta(x^1) = \begin{cases} 0, & -1 < x^1 \leq 1 - 3\sigma, \\ e^{\mu x^1}, & 1 - 2\sigma \leq x^1 \leq 1, \end{cases} \tag{43}$$

where $\mu$ is a large positive constant such that $\partial_1(a \eta) \geq 0$ on $\Sigma^1$.

Denote $u^{N,\epsilon}$ by $w$. Multiplying (29) by $\eta \partial_{11}(X_j^{N,\epsilon})$, summing up for $j$ from 1 to $N$, and integrating the equality on $[-1, 1]$, we have
\[
\int_{M} f \eta \partial_{11} w \, dx^{1} \, dx^{2} = \int_{M} \left\{ \partial_{1} \left[ \frac{1}{2} \epsilon \eta (\partial_{11} w)^{2} + \frac{1}{2} \partial_{1} (\eta \alpha) (\partial_{2} w)^{2} - \frac{1}{2} \alpha \eta (\partial_{1} w)^{2} - a \eta \partial_{2} \partial_{11} w \right] \\
+ \partial_{2} (a \eta \partial_{2} \partial_{11} w) + (k \eta - \frac{1}{2} \partial_{1} \eta) (\partial_{11} w)^{2} + b \eta \partial_{11} w \partial_{12} w + \frac{1}{2} \partial_{1} (\alpha \eta) (\partial_{1} w)^{2} \\
- \partial_{2} (a \eta) \partial_{2} \partial_{11} w - \frac{1}{2} \partial_{11} (a \eta) (\partial_{2} w)^{2} + a \eta (\partial_{12} w)^{2} \right\} \, dx^{1} \, dx^{2} \\
\geq \int_{M} \eta [(k - \nu - \varrho) (\partial_{11} w)^{2} + (a - \nu) (\partial_{12} w)^{2}] \, dx^{1} \, dx^{2} - C_{r, \varrho} \|w\|^{2} \, dx^{1} \, dx^{2}. \tag{44}
\]

Here we used the fact that \( \partial_{1} w = \partial_{12} w = 0 \) on \( \Sigma^{1} \), and (35) to control the term \(- (1/2) \epsilon \partial_{1} \eta (\partial_{11} w)^{2} \). Hence by applying Young inequality to the left side of the above inequality and choosing \( \nu, \varrho \) small enough, we have

\[
\int_{M} \eta [(\partial_{11} w)^{2} + (\partial_{12} w)^{2}] \, dx^{1} \, dx^{2} \leq C \|f\|^{2}. \tag{45}
\]

This implies (42) by taking \( N \to \infty \).

**Step 4.** \( H^{2} \) Regularity.

**Step 4.1. Regularity in elliptic region.** Let \( M_{\sigma} = M \cap \{(x^{1}, x^{2}) : -1 < x^{1} < -1 + \sigma\} \). Since \( k > 0 \) on \( \Sigma^{-1} \), we may choose \( \sigma > 0 \) small such that the operator \( L \) is elliptic in \( M_{2\sigma} \). By standard theory of elliptic equations (i.e., the interior estimate and the estimate near boundary of elliptic equations) we have

\[
\|u\|_{H^{2}(M_{2\sigma})} \leq C (\|u\|_{0} + \|f\|_{0}) \leq C \|f\|_{0}, \tag{46}
\]

\[
\|u\|_{H^{2}(M_{2\sigma})} \leq C \|f\|_{0}. \tag{47}
\]

Next we derive an estimate of \( w = u^{N, \epsilon} \) similar to (35) near \( \Sigma^{-1} \). Let \( \xi = \xi(x^{1}) \in C_{0}^{\infty}([-1, 1]) \) satisfy

\[
\xi(x^{1}) = \begin{cases}
0, & x^{1} \in [-1, -1 + \sigma/4] \cup [-1 + 2\sigma, 1], \\
1, & x^{1} \in [-1 + \sigma/2, -1 + 3\sigma/2].
\end{cases} \tag{48}
\]

By multiplying \( \xi \partial_{11} (X_{j}^{N, \epsilon}) \) to (29), summing up for \( j \) from 1 to \( N \) and then integrating on \([-1, 1]\) with respect to \( x^{1} \), by the first equality in (44) we have similarly
\[
\int_M f\xi \partial_{11}w \, dx^1 \, dx^2 \geq (\delta/16) \int_M \xi[(\partial_{11}w)^2 + (\partial_{12}w)^2] \, dx^1 \, dx^2 - C \|w\|_1^2,
\]
hence
\[
\int_M \xi[(\partial_{11}w)^2 + (\partial_{12}w)^2] \, dx^1 \, dx^2 \leq C \|f\|_0^2. \tag{49}
\]

**Step 4.2. Regularity in mixed-type region.** Let \( \vartheta \in C_0^\infty([-1,1]) \) such that
\[
0 \leq \vartheta \leq 1 \quad \text{and} \quad \vartheta(x^1) = \begin{cases} 
0, & -1 \leq x^1 \leq -1 + \sigma/2, \\
1, & -1 + 3\sigma/2 \leq x^1 \leq 1 - 3\sigma/2, \\
0, & 1 - \sigma/2 \leq x^1 \leq 1.
\end{cases}
\]

Multiplying (29) by \( \vartheta \partial_{111}(X_j^{N,e}) \), summing up for \( j \) from 1 to \( N \), and integrating the equality on \([-1,1]\), we get
\[
-\int_M \partial_1(f\vartheta)\partial_{11}w \, dx^1 \, dx^2 = \int_M \{(\alpha - \frac{1}{2} \partial_1 k)\vartheta(\partial_{11}w)^2 - \frac{1}{2} k\vartheta'(\partial_{11}w)^2 \\
- \frac{3}{2} \partial_1 a\vartheta(\partial_{12}w)^2 - \frac{3}{2} a\vartheta'(\partial_{12}w)^2 - \partial_{11}(a\vartheta)\partial_2w\partial_{12}w - \frac{1}{2} \partial_{11}(a\vartheta)(\partial_1w)^2 \\
+ \partial_{12}(a\vartheta)\partial_2w\partial_{11}w - \partial_1(b\vartheta)\partial_{12}w\partial_{11}w + \epsilon\vartheta(\partial_{11}w)^2 \\
+ \frac{1}{2} \partial_2(b\vartheta)(\partial_{11}w)^2 + \partial_2(a\vartheta)\partial_{12}w\partial_{11}w\} \, dx^1 \, dx^2. \tag{50}
\]

Here we may estimate those terms involving \( \vartheta', \vartheta'' \) by the estimates (45) and (49), since \((\text{supp } \vartheta') \subset [-1 + \sigma/2, -1 + 3\sigma/2] \cup [1 - 3\sigma/2, 1 - \sigma/2]\). Therefore
\[
-\int_M \left[ \frac{1}{2} k\vartheta'(\partial_{11}w)^2 + \frac{3}{2} a\vartheta'(\partial_{12}w)^2 \right] \, dx^1 \, dx^2 \leq C \|f\|_0. \tag{51}
\]
Hence by Hölder inequality and Young inequality, the right side of (50) is bounded below by
\[
(\delta/4) \int_M \vartheta|D(\partial_1w)|^2 \, dx^1 \, dx^2 - C(\|w\|_1^2 + \|f\|_1^2). \tag{52}
\]
Therefore, it is easy to get
\[
\int_M \vartheta|D(\partial_1w)|^2 \, dx^1 \, dx^2 \leq C \|f\|_1^2.
\]
with $C$ independent of $N, \epsilon$. Letting $N \to \infty$ and $\epsilon \to 0$, we obtain that

$$\int_M \vartheta |D(\partial_1 u)|^2 \, dx_1 \, dx_2 \leq C \|f\|_1^2.$$  

By (40), (46), (47) and the above inequality, we obtain that

$$\|\partial_1 u\|_1 \leq C \|f\|_1.$$  

Since $a \geq \delta > 0$ in $M$, by (21) we may estimate $\partial_{22} u$. Hence we conclude that

$$\|u\|_2 \leq C \|f\|_1.$$  

**Remark 3** We observe that in deriving the $H^2$ estimate, we used the assumption that $D^2 a, D^2 \alpha \in L^\infty$, but just required that $Dk, Db \in L^\infty$.

**Step 5. Higher regularity.** The regularity in (20) for $s = 3, 4$ in the elliptic region is obvious (also can be obtained by multiplier $\xi \partial_1^k w, \eta \partial_1^k w$ for integer $k = 4, 6$), and in the mixed-type region can be obtained in the same fashion as those to (45) by multiplier $\vartheta \partial_1^k w$ for integer $k = 5, 7$.

For example, for $H^3$ estimate, by the above energy estimate technique, we have

$$- \int_M \partial_{11}(f \vartheta) \partial_{111} w \, dx_1 \, dx_2 = \int_M c \vartheta (\partial_{111} w)^2 \, dx_1 \, dx_2$$

$$+ \int_M \{(\alpha \vartheta - \frac{3}{2} \partial_1(k \vartheta) + \frac{1}{2} \partial_2(b \vartheta))(\partial_{111} w)^2 - \frac{5}{2} \partial_1(a \vartheta)(\partial_{112} w)^2\} \, dx_1 \, dx_2$$

$$+ \int_M \{\partial_2(a \vartheta) \partial_{112} w \partial_{111} w - 2 \partial_1(b \vartheta) \partial_{112} w \partial_{111} w\} \, dx_1 \, dx_2$$

$$+ \int_M \{- \partial_{11}(a \vartheta) \partial_{22} w \partial_{111} w + 2 \partial_{12}(a \vartheta) \partial_{12} w \partial_{111} w - 2 \partial_{11}(a \vartheta) \partial_{12} w \partial_{112} w$$

$$+ \partial_{11}(\alpha \vartheta) \partial_1 w \partial_{111} w - \partial_{11}(k \vartheta) \partial_1 w \partial_{111} w - \partial_{11}(b \vartheta) \partial_1 w \partial_{112} w\} \, dx_1 \, dx_2$$

$$- \int_M \partial_{11}(\alpha \vartheta)(\partial_{11} w)^2 \, dx_1 \, dx_2$$

$$\geq \int_M \vartheta \{((\vartheta - \nu - \sigma)(\partial_{111} w)^2 + (\partial_{112} w)^2\} \, dx_1 \, dx_2 - C \|f\|_2^2 \cdot (53)$$

An important fact is that the terms with underlines involve only $Db$, so we may use (19) (and $H^3 \hookrightarrow C^1$) to get the inequality. Therefore by choosing $\sigma$ small we get

$$\int_M \vartheta ((\partial_{111} w)^2 + (\partial_{112} w)^2) \, dx_1 \, dx_2 \leq C \|f\|_2^2 \cdot (54)$$

Combing an estimate obtained similarly in the elliptic domain, we have
\[ \| \partial_{11} w \|_1 \leq C \| f \|_2. \]  
(55)

Letting \( N \to \infty \) and \( \epsilon \to 0 \), the weak limit \( u \) of the sequence \( \{ w = u^{N, \epsilon} \} \), which is a solution of (21), also satisfies
\[ \| \partial_{11} u \|_1 \leq C \| f \|_2. \]  
(56)

By (21), we have
\[ \| \partial_{22} u \|_1 \leq C \| f \|_1 + C \| u \|_2 + C \| \partial_{11} u \|_1 + \left\| \frac{b}{a} \partial_{12} u \right\|_1 \leq C \| f \|_2 + \left\| \frac{b}{a} \partial_{11} u \right\|_0 + \left\| \frac{b}{a} \partial_{12} u \right\|_0 \]  
(57)

for a constant \( C \) depending on \( \delta \). By (19), we have \( \left\| \frac{b}{a} \partial_{12} u \right\|_0 \leq (\nu/\delta) \| \partial_{22} u \|_1 < \| \partial_{22} u \|_1 \). Also, there holds \( \left\| \frac{b}{a} \partial_{11} u \right\|_0 \leq C \| \partial_{11} u \|_1 \). So we get
\[ \| \partial_{22} u \|_1 \leq C \| f \|_2 \]  
(58)

and the \( H^3 \) regularity
\[ \| u \|_3 \leq \| u \|_2 + \| \partial_{11} u \|_1 + \| \partial_{22} u \|_1 \leq C \| f \|_2. \]  
(59)

Similar argument works for \( H^4 \) estimate. This finishes the proof. \( \square \)

**Remark 4** We see here that we can not apply directly Theorem 1 to (5) since \( \partial_1 a \leq -\delta < 0 \) in (17) does not hold in \( M \). However, we need this assumption to control the mixed derivative term \( b \partial_{12} u \).

An important observation to Theorem 1 is that its conditions are not invariant under multiplication of a positive function to the mixed type equation (15)! Therefore it is expected to find an appropriate multiplier to (5) such that Theorem 1 works.

Now we choose \( h(x^1) = e^{-\mu x^1} \) with \( \mu > 0 \) a small constant (depending only on \( \delta_2 \) and \( \| k \|_{L^\infty} \)), which is a bounded smooth positive function in \( \mathcal{M} \), then it is easy to check that there is a positive constant \( \delta^* \) such that
\[ h'(x^1) \leq -\delta^* < 0 \quad \text{in } \mathcal{M}, \]  
(60)
\[ 2h(x^1)\alpha(x^1) + \partial_1 (h(x^1)k(D\varphi_b)) \geq \delta^* > 0 \quad \text{in } \mathcal{M}, \]  
(61)
\[ 2\alpha h - l \partial_1 (hk(D\varphi_b)) \geq \delta^* > 0 \quad \text{in } \mathcal{M}, \]  
(62)
where \( l > 0 \) is less than a fixed number (say, \( l \leq 6 \)).

4 Solvability of nonlinear problem and main result

Now we prove the following stability result of transonic potential flows in \( M \).

**Theorem 5** Let \( \varphi_b \in C^5 \) be a background transonic flow in \( M \). Then there exist positive constant \( C \) and \( \varepsilon_0 \) depending only on \( \varphi_b \) such that if

\[
\varphi = \varphi_b + g(x^2) \quad \text{on} \quad \Sigma^{-1} \tag{63}
\]

for any \( g \in H^5(S^4) \) and \( \| g \|_5 \leq \varepsilon \leq \varepsilon_0 \), then problem \( (1), (2) \) and \( (63) \) has uniquely one solution \( \varphi \) with

\[
\| \varphi - \varphi_b \|_4 \leq C \varepsilon. \tag{64}
\]

This result shows that the subsonic–supersonic flow is stable under perturbations of the velocity potential function at the entry. More physically, since \( \partial_2 \varphi/n(-1) \) is the velocity of the flow along \( x^2 \) direction, so \( g(x^2) \) measures the perturbation of the tangential velocity along the entry. So we may claim that the special transonic flow is stable under small variation of the tangential velocity at the entrance.

**PROOF.** The proof is based on a nonlinear iteration scheme.

**Step 1.** We define the iteration set as

\[
E_\kappa = \{ \varphi \in H^4(M) : \| \varphi - \varphi_b \|_4 \leq \kappa \leq \kappa_0 \},
\]

where \( \kappa_0 \) is a small positive constant to be specified later. It is straightforward to check that there hold

\[
\| f(D \varphi) \|_3 \leq C_0 \kappa^2, \tag{65}
\]

\[
\| f(D \varphi^{(1)}) - f(D \varphi^{(2)}) \|_2 \leq C_0 \kappa \| \varphi^{(1)} - \varphi^{(2)} \|_3 \tag{66}
\]

for any \( \varphi, \varphi^{(1)}, \varphi^{(2)} \in E_\kappa \).

**Step 2.** Let
\[(h \cdot M_\varphi)(\hat{\varphi}) := h(x^1)k(D\varphi)\partial_{11}\hat{\varphi} + h(x^1)b(D\varphi)\partial_{12}\hat{\varphi} + h(x^1)\partial_{22}\hat{\varphi} - h(x^1)\alpha(x^1)\partial_{1}\hat{\varphi} = h(x^1)f(D\varphi), \tag{67}\]

where \(\hat{\varphi} = \varphi - \varphi_b\), and \(f(D\varphi)\) is defined by \((10)\). By considering \(\varphi = \hat{\varphi} - g(x^2)\) as the unknown, problem \((1) \tag{2} \) and \((63)\) is equivalent to the following problem:

\[(hM_\varphi)(\varphi) = h(x^1)f(D\varphi) + (hM_\varphi)(g(x^2)) \text{ in } M, \tag{68}\]
\[\varphi = 0 \quad \text{on } \Sigma^{-1}. \tag{69}\]

Here \(\varphi = \phi + g(x^2) + \varphi_b\), \(\hat{\varphi} = \varphi - \varphi_b\) and note that

\[
\|M_\varphi(g)\|_3 \leq C_0\varepsilon, \tag{70}
\]
\[
\|M_{\varphi(1)}(g) - M_{\varphi(2)}(g)\|_2 \leq C_0\varepsilon\|\varphi^{(1)} - \varphi^{(2)}\|_3 \tag{71}
\]

for \(\varphi, \varphi^{(1)}, \varphi^{(2)} \in E_\kappa\).

**Step 3.** Now by \((60) \tag{62}\), we may choose \(\kappa_0\) so small such that for any \(\varphi \in E_\kappa\), there hold in \(M\) the following inequalities:

\[
h(x^1) \geq \delta^*/2 > 0, \tag{72}
\]
\[
2h(x^1)\alpha(x^1) + \partial_l(h(x^1)k(D\varphi)) \geq \delta^*/2 > 0, \tag{73}
\]
\[
\partial_lh(x^1) \leq -\delta^*/2 < 0, \tag{74}
\]
\[
2\alpha h_l - lh_l(hk(D\varphi)) \geq \delta^*/2 > 0 \quad l = 0, 1, \ldots, 5, \tag{75}
\]
\[
\|b(D\varphi)\|_3 \leq \nu^* = \delta^*/4. \tag{76}
\]

**Step 4.** Then for any \(\varphi \in E_\kappa\), we solve the following linear problem of \(\tilde{\varphi}\):

\[(hM_\varphi)(\tilde{\varphi}) = h(x^1)f(D\varphi) + (hM_\varphi)(g(x^2)) \text{ in } M, \tag{77}\]
\[\tilde{\varphi} = 0 \quad \text{on } \Sigma^{-1}. \tag{78}\]

By \((72) \tag{76}\) and Theorem \(\|\cdot\|_3\) and the analysis in step 6 below, there exists uniquely one solution \(\tilde{\varphi} \in H^1\) and it satisfies

\[
\|\tilde{\varphi}\|_4 \leq C_0(\kappa^2 + \varepsilon). \tag{79}
\]

Now choosing \(\varepsilon_0 \leq 1/(8C_0^2)\) and \(\kappa = 4C_0\varepsilon \leq \kappa_0\) (that is, \(C = 4C_0\)), we get a \(\bar{\varphi} = \tilde{\varphi} + g + \varphi_b\) with \(\|\bar{\varphi}\|_4 \leq \kappa\). Therefore we established a mapping \(T : \varphi \mapsto \bar{\varphi}\) on \(E_\kappa\).
Step 5. Next we will show that $T$ is contractive on $E_\kappa$ in $H^3$ norm.

Let $\varphi^{(i)} \in E_\kappa$, $T(\varphi^{(i)}) = \tilde{\varphi}^{(i)}$, and $\tilde{\varphi}^{(i)} = \varphi^{(i)} - g - \varphi_b$, $i = 1, 2$. Then $\tilde{\varphi}^{(1)} - \tilde{\varphi}^{(2)}$ satisfies the following problem

$$
(hM_{\varphi^{(1)}})(\tilde{\varphi}^{(1)} - \tilde{\varphi}^{(2)}) = -\left[ hM_{\varphi^{(1)}}(\varphi^{(2)}) - hM_{\varphi^{(2)}}(\varphi^{(1)}) \right] + h(x^1) \left[ f(D\varphi^{(1)}) - f(D\varphi^{(2)}) \right] + (hM_{\varphi^{(1)}})(g) - (hM_{\varphi^{(2)}})(g)
$$

in $M$,

$$
\tilde{\varphi}^{(1)} - \tilde{\varphi}^{(2)} = 0
$$
on $\Sigma^{-1}$.

Note that

$$
\left\| M_{\varphi^{(1)}}(\varphi^{(2)}) - M_{\varphi^{(2)}}(\varphi^{(2)}) \right\|_2 \leq C_0 C \| \varphi^{(1)} - \varphi^{(2)} \|_3,
$$

then Theorem 1 and the results in step 6 below yield

$$
\left\| \tilde{\varphi}^{(1)} - \tilde{\varphi}^{(2)} \right\|_3 \leq C_1 C \| \varphi^{(1)} - \varphi^{(2)} \|_3.
$$

By choosing $\varepsilon_0$ further small, we obtain contraction. Then by a simple generalized Banach fixed point theorem, we proved Theorem 5.

Step 6. Solving $H^4$ solution of (77) and (78) with lower regular coefficients.

For simplicity, we may write

$$
M_{\varphi}(\tilde{\varphi}) := [\tilde{k} + O_1(D\tilde{\varphi})] \tilde{\partial}_{11} \tilde{\varphi} + O_2(D\tilde{\varphi}) \tilde{\partial}_{12} \tilde{\varphi} + \tilde{\partial}_{22} \tilde{\varphi} - \alpha \tilde{\partial}_{1} \tilde{\varphi} = F(D\varphi),
$$

where $\tilde{k} = k(D\varphi_b)$, $O_1(D\tilde{\varphi}) = k(D\varphi) - k(D\varphi_b)$, $O_2(D\tilde{\varphi}) = b(D\varphi)$, $F = f(D\varphi) + M_{\varphi}(g)$, and there holds

$$
\|O_i\|_3 \leq C\kappa \leq C\kappa_0
$$

for $i = 1, 2$ and $\varphi \in E_\kappa$. Since $h$ is smooth and bounded away from zero, the solvability of (81) (77) is equivalent to (77) (78).

By Sobolev embedding theorem we have $H^3 \hookrightarrow C^1$, so due to Remark 3 in Theorem 1, (81), (78) has uniquely one solution $\tilde{\varphi}$ and

$$
\| \tilde{\varphi} \|_2 \leq C \| F \|_1.
$$

However, we can not infer from Theorem 1 directly the existence of $H^4$ solutions and estimate like $\| \tilde{\varphi} \|_4 \leq C \| F \|_3$. So we need the following apriori estimates and approximation argument.
Step 6.1. An apriori $H^4$ estimate. Now suppose there is a solution $\tilde{\phi} \in H^4$ to problem (81)-(78), we show that if $\kappa_0$ is sufficiently small, then there holds the following estimate

$$
\| \tilde{\phi} \|_p \leq C \| F \|_{p-1}, \quad \text{for } p = 3, 4. \tag{84}
$$

The proof of the case $p = 3$ is similar to the following $p = 4$ case and even more simple (which should use the fact (83)). Now we suppose (84) holds for $p = 3$ and to prove $p = 4$ case.

First, for small $\sigma$ and $\kappa_0$, the operator $M_\phi$ is elliptic in $M_{3\sigma}$. By the regularity theory of elliptic equation, we have

$$
\| \tilde{\phi} \|_{H^4(M_{3\sigma})} \leq C \| F \|_3. \tag{85}
$$

Now let $\zeta \in C^\infty([-1, 1])$ satisfy

$$
0 \leq \zeta \leq 1 \quad \text{and} \quad \zeta(x^1) = \begin{cases} 
0, & -1 \leq x^1 \leq -1 + \sigma, \\
1, & -1 + 2\sigma \leq x^1 \leq 1.
\end{cases}
$$

Denote $\bar{u} = \zeta \tilde{\phi}$. Then it satisfies the following equation

$$
M_\phi(\bar{u}) = \bar{F} := \zeta F + \zeta'' k \tilde{\phi} + \zeta'(2k\partial_1 \tilde{\phi} + b\partial_2 \tilde{\phi} - \alpha \tilde{\phi}). \tag{86}
$$

Since $\text{ supp } \zeta' \subset (-1 + \sigma, -1 + 2\sigma)$, by (85), we have

$$
\| \bar{F} \|_3 \leq C \| F \|_3. \tag{87}
$$

Then by differentiating (86) with respect to $x^1$ twice, we get $w = \partial_{11} \bar{u}$ satisfies

$$
M_\phi(w) + 2\partial_1 k \partial_1 w = \partial_{11} \bar{F} + [\partial_{11} k \partial_{11} \bar{u} + \partial_{11} \alpha \partial_1 \bar{u} + 2\partial_1 \alpha \partial_{11} \bar{u}] \\
- [\partial_{11} O_1 \partial_{11} \bar{u} + \partial_{11} O_2 \partial_{12} \bar{u} + 2\partial_1 O_1 \partial_{111} \bar{u} + 2\partial_1 O_2 \partial_{112} \bar{u}] \\
:= \tilde{F} \tag{88}
$$

as well as $w = 0$ on $\Sigma^{-1}$ by the cut-off function $\zeta$.

We note that the operator $M_\phi + 2\partial_1 k \partial_1$ in the left hand side of the above equation also satisfies the assumptions of Theorem[II] (by multiplying a suitable positive function). So we have
\[ \|w\|_2 \leq C\|F\|_1. \]  

(89)

We now estimate \( \tilde{F} \) term by term.

(1) Obviously by (87) we have

\[ \|\partial_{11} \tilde{F}\|_1 \leq \|\tilde{F}\|_3 \leq C\|F\|_3. \]  

(90)

(2) Since \( \bar{k}, \alpha \in C^4 \), we get

\[ \|\partial_{11} \bar{k} \partial_{11} \bar{u} + \partial_{11} \alpha \partial_{11} \bar{u} + 2\partial_1 \alpha \partial_{11} \bar{u}\|_1 \leq C \|\bar{u}\|_3 \leq C\|F\|_2. \]  

(91)

Here we use (83) and (84) \((p = 3)\) for the second inequality.

(3) We first recall the inequality

\[ \|uv\|_1 \leq C\|u\|_2\|v\|_1 \]  

(92)

provided \( u \in H^2 \), \( v \in H^1 \) (see [14], p.73). Then by (82)

\[
\begin{align*}
\|\partial_{11} O_1 \partial_{11} \bar{u}\|_1 & \leq C\|O_1\|_3 \|\bar{u}\|_4 \leq C\kappa \|\bar{u}\|_4, \\
\|\partial_{11} O_2 \partial_{12} \bar{u}\|_1 & \leq C\|O_2\|_3 \|\bar{u}\|_4 \leq C\kappa \|\bar{u}\|_4, \\
\|\partial_1 O_1 \partial_{111} \bar{u}\|_1 & \leq C\|O_1\|_3 \|\bar{u}\|_4 \leq C\kappa \|\bar{u}\|_4, \\
\|\partial_1 O_2 \partial_{112} \bar{u}\|_1 & \leq C\|O_2\|_3 \|\bar{u}\|_4 \leq C\kappa \|\bar{u}\|_4.
\end{align*}
\]

In all, we get \( \|\tilde{F}\|_1 \leq C(\|F\|_3 + \kappa \|\bar{u}\|_4) \), hence

\[ \|\partial_{11} \bar{u}\|_2 \leq C(\|F\|_3 + \kappa \|\bar{u}\|_4). \]  

(93)

By (86), we may then estimate

\[ \|\partial_{22} \bar{u}\|_2 \leq C(\|F\|_3 + \kappa \|\partial_{12} \bar{u}\|_2 + \kappa \|\bar{u}\|_4) \leq C(\|F\|_3 + \kappa \|\bar{u}\|_4). \]

So there holds

\[ \|\bar{u}\|_4 \leq C(\|\partial_{11} \bar{u}\|_2 + \|\partial_{22} \bar{u}\|) + \|\bar{u}\|_3 \leq C(\|F\|_3 + \kappa \|\bar{u}\|_4). \]

By choosing \( \kappa_0 \) small, we can deduce that \( \|\bar{u}\|_4 \leq C\|F\|_3 \). Combing this with (85), we can get (84) for the case \( p = 4 \).
Step 6.2. Existence of $H^4$ solution by approximation.

For $\varphi \in E_k$, $O_i \in H^3, i = 1, 2$, we approximate $O_i$ by $\{O_i^{(l)}\}_{l=1}^{\infty} \subset C^4$ such that $O_i^{(l)} \to O_i(D\bar{\varphi})$ strongly in $H^3$ (so (82) holds). By Theorem 15 the problem

$$M_{\varphi}^{(l)}(\bar{\varphi}) := [k + O_1^{(l)}] \partial_{11} \bar{\varphi} + O_2^{(l)} \partial_{12} \bar{\varphi} + \partial_{22} \bar{\varphi} - \alpha \partial_1 \bar{\varphi} = F(D\varphi),$$

$$\bar{\varphi} = 0 \text{ on } \Sigma^{-1}$$

has uniquely one solution $\bar{\varphi}^{(l)} \in H^4$. By the apriori estimate (84) we have $\|\bar{\varphi}^{(l)}\|_4 \leq C \|F\|_3$ for $C$ independent of $l$. So there is a $H^4$ weak limit $\bar{\varphi} \in H^4$ of this sequence of approximate solutions. Then clearly $\bar{\varphi}$ is a $H^4$ solution of (81) and (78), and by Theorem 1 this solution is unique.

The proof of Theorem 5 is then finished.

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