Quantum Corrections to Synchrotron Radiation from Wave-Packet

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We calculate the radiated energy to $O(\hbar)$ from a charged wave-packet in the uniform magnetic field. In the high-speed and weak-field limit, while the non-commutativity of the system reduces the classical radiation, the additional corrections originated from the velocity uncertainty of the wave-packet leads to an enhancement of the radiation.

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I. INTRODUCTION

Radiation spectra such as the synchrotron radiation by accelerated electrons had been derived for many years by applying the semi-classical formalism. In the conventional treatments using the relativistic quantum mechanics, while the electromagnetic field is quantized as a field, the electrons or point-charges are quantized as particles. Such a formalism is particularly suitable for studying the semi-classical theories in accelerated frames, because the classical concepts such as the trajectory can emerge in the form of the expectation values and help to define an accelerated frame clearly.

To compare directly with those discussions in accelerated frame, the expectation value of the position of the electron has to evolve. This means that we have to choose a four-dimensional wave-packet as the quantum state of a moving electron, though the quantum states usually used in calculating the synchrotron radiation are “energy” eigenstates in the proper frame. Since a single electron in the accelerator behaves like a moving particle rather than an eigenstate, it is interesting to check the discrepancy between our choice of quantum states and those in conventional calculations.

In the present paper, we will work out the $O(\hbar)$ corrections to the radiated energies emitted from a wave-packet moving in a uniform magnetic field. The paper is organized as follows. The semi-classical formalism for calculating the radiated energy of charges will be given in Section II. Following this formalism, the synchrotron radiation from charges in a uniform magnetic field will be calculated in Section III. Then we will discuss the cases in the static limit and the high-speed, weak-field limit. Finally, our conclusion will be given in Section IV.

II. RADIATED ENERGY FROM ACCELERATED WAVE-PACKETS

The relativistic Lorentz electron with the charge $e$ and the mass $m$ is described by the action,

$$S = \int d\tau \frac{m}{2} v_\mu v^\mu + \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu(x) A^\mu(x) \right], \quad (1)$$

where $v^\mu \equiv dy^\mu(\tau)/d\tau$ is the proper velocity, $F_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu$ is the electromagnetic field tensor, and the current of the point-like electron is defined by

$$j_\mu(x) \equiv e \int d\tau v_\mu(\tau) \delta^4(x - y(\tau)) \left[-g\right]^{-1/2}, \quad (2)$$

with the conservation law $j^\mu_{\mu} = 0$. Below we use the Cartesian coordinate $d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2$ and natural units $c = \hbar = 1$ for convenience but keep $\hbar$ in our expressions.

In quantum electrodynamics, the energy radiated from a charged particle via photons is the photon energy $\hbar \omega$ multiplied by the transition probability $I$ as

$$\mathcal{E} = \int \frac{d^3k}{(2\pi)^3} \hbar \omega \frac{d^2I}{d\omega d\Omega}. \quad (3)$$

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From the interacting action $S_{\text{int}} \sim \int j_{\mu} A^\mu$ in (I), the transition probability $\mathcal{I}$ up to the first order of perturbation is given by

$$
\mathcal{I} = \frac{1}{\hbar^2} \text{Re} \left\{ \left| \int d^4x \sqrt{-g} j_{\mu}(x) A^\mu(x) \right|^2 \right\}.
$$

(4)

The quantum states for $j_{\mu}$ could be represented by Klein-Gordon wave functions $\phi$, or some linear combinations of them, in the semi-classical treatment. The trajectory of the charge in our semi-classical theory is thus related to the expectation values of the operators for the motion of the source.

To the first order approximation, since $A_\mu$ and $j_\mu$ are independent dynamical variables in the free field theory, we choose the quantum state of the system to be the direct product of the vacuum state for $A_\mu$ and the quantum states for $j_\mu = ev_\mu$. Therefore,

$$
\mathcal{I} = \frac{e^2}{\hbar^2} \text{Re} \int d\tau d\tau' \langle y | v_\mu(\tau) (0|A^\mu(y(\tau'))A^\nu(y(\tau'))|0) v_\nu(\tau') | y \rangle.
$$

(5)

Suppose the laboratory observer is in Minkowski frame. What the laboratory observer can record are thus Minkowski photons. Let us take the initial state for photons being the Minkowski vacuum $|0_M\rangle$, which is defined by the creation and annihilation operators in Minkowski space. Accordingly, the vector field could be decomposed into

$$
A^\mu(x) = \sum_{\lambda=1,2} \int \frac{d^4k}{(2\pi)^3} \left[ a_\lambda(k) e^{i\kappa x} \chi(\omega) e^{ikx} + a_\lambda^\dagger(k) e^{-i\kappa x} \chi^*(\omega) e^{-ikx} \right],
$$

(6)

where $\chi(\omega) = \sqrt{\hbar/2\omega} e^{-i\omega t}$ with $\omega = |k|$, such that

$$
\mathcal{I} = \frac{e^2}{16\pi^3\hbar} \text{Re} \int d\omega d\Omega \int d\tau d\tau' \langle y | v_\mu(\tau) e^{ik\alpha\nu(\tau)} e^{-ik\alpha\nu(\tau')} v^\mu(\tau') | y \rangle,
$$

(7)

hence the spectral and angular distribution of the radiated energy reads

$$
\frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{e^2\omega^2}{16\pi^3} \text{Re} \int d\tau d\tau' \langle y | v_\mu(\tau) e^{ik\alpha\nu(\tau)} e^{-ik\alpha\nu(\tau')} v^\mu(\tau') | y \rangle.
$$

(8)

Note that the above $\langle A^\mu A^\nu \rangle$ are functions of $y_\mu(\tau)$, which do not commute with $v_\mu = (-i\hbar\partial_\mu - eA_\mu)/m$. This non-commutativity is understood as an origin of the quantum correction. If we ignore the non-commutativity of the operators, the classical formula will be explicitly obtained by substituting classical current $v^\mu$ into Eq.

Let us concentrate on the expectation value in the differential radiated energy $d\mathcal{E}$, namely,

$$
\mathcal{A} \equiv \langle v_\mu(\tau) e^{ik\alpha\nu(\tau)} e^{-ik\alpha\nu(\tau')} v^\mu(\tau') \rangle.
$$

(9)

Denoting the fluctuations $\delta y^\mu(\tau) = y^\mu(\tau) - \langle y^\mu(\tau) \rangle$ and $\delta v^\mu(\tau) = v^\mu(\tau) - \langle v^\mu(\tau) \rangle$ and expanding $\mathcal{A}$ in terms of $\delta y^\mu$ and $\delta v^\mu$, one has

$$
\mathcal{A} = \langle v_\mu(\tau) \rangle \langle v^\mu(\tau') \rangle + \sum_{i=1}^3 f_i \left[ e^{ik\alpha\nu(\tau)} - \langle y^\nu(\tau) \rangle \right] + O(\hbar^2),
$$

(10)

in which the $O(\hbar)$ corrections $f_i$ are

$$
f_1 \equiv \langle \delta v_\mu(\tau) \delta v^\mu(\tau') \rangle,
$$

(11)

$$
f_2 \equiv -\frac{k_\mu k^\nu}{2} \left[ \langle \delta y^\mu(\tau) \delta y^\nu(\tau') \rangle + \langle \delta y^\mu(\tau') \delta y^\nu(\tau) \rangle - 2 \langle \delta y^\mu(\tau) \delta y^\nu(\tau') \rangle \langle v_\mu(\tau) \rangle \langle v^\nu(\tau') \rangle \right],
$$

(12)

$$
f_3 \equiv ik^\nu \left\{ \langle v^\mu(\tau') \rangle \langle \delta v_\mu(\tau) \delta y_\nu(\tau) - \delta y_\mu(\tau') \rangle \right\} + \langle v_\mu(\tau) \rangle \langle \delta y_\mu(\tau) - \delta y_\nu(\tau') \rangle \delta v^\nu(\tau').
$$

(13)

Because of the dependence on $k^\mu$, one expects that $f_2$ and $f_3$ will become more important than $f_1$ in the short wave-length regime.

If we choose the quantum state $|y\rangle$ to be a four-dimensional wave-packet centered at some trajectory $\langle y_\mu(\tau) \rangle$ with the four-velocity $\langle v_\mu(\tau) \rangle = d\langle y_\mu(\tau) \rangle/d\tau$, then the first term in the bracket of Eq.

(10) gives the classical radiated
energy by the charge moving in the trajectory \(\langle y_{\mu}(\tau)\rangle\), while the \(f_1\) term might be interpreted as the Unruh effect on the charged current \(\overline{\hbar}\).

To calculate the \(f_i\)'s, one needs the Hamiltonian for the charge motion. From the sector of the charge motion in the action \(\mathcal{I}_0\), the conjugate momentum for \(y^\mu\) is \(p^\mu = m v^\mu + e A^\mu_{\text{in}}\). Hence the “Hamiltonian” with respect to the proper time \(\tau\) reads

\[
\mathcal{H} = \frac{1}{2m} (p^\mu - e A^\mu_{\text{in}})(p^\nu - e A^\nu_{\text{in}}),
\]

after a Legendre transformation. The quantum mechanics for the Lorentz electron is given by the equal-proper-time commutation relation

\[
[y_{\mu}(\tau), p_{\nu}(\tau)] = i\hbar g_{\mu\nu}.
\]

It follows that

\[
[v^\mu(\tau), v^\nu(\tau')] = i\hbar \frac{e}{m^2} F_{\mu\nu}^{\text{in}},
\]

so \(v^\mu(\tau)\) and \(v^\nu(\tau')\) do not commute in the presence of the background electromagnetic field. This implies the uncertainty relation,

\[
\sqrt{\langle \delta v^\mu_1(\tau) \rangle \langle \delta v^\nu_2(\tau) \rangle} \geq \frac{\hbar}{2m^2} |e F_{\mu\nu}^{\text{in}}|.
\]

### III. SYNCHROTRON RADIATION IN UNIFORM MAGNETIC FIELD

For a charge moving in a uniform magnetic field \(B^3 = F_{\text{in}}^{12} = H\), the Heisenberg equation of motion gives the evolution of the operators as follows,

\[
\begin{align*}
v_1(\tau) &= \hat{v}_1 \cos \omega_0 \tau + \hat{v}_2 \sin \omega_0 \tau, \quad (18) \\
v_2(\tau) &= \hat{v}_2 \cos \omega_0 \tau - \hat{v}_1 \sin \omega_0 \tau, \quad (19) \\
v_0(\tau) &= \hat{v}_0, \quad v_3(\tau) = \hat{v}_3, \quad (20) \\
y_1(\tau) &= \hat{y}_1 + \frac{\hat{v}_1}{\omega_0} \sin \omega_0 \tau - \frac{\hat{v}_2}{\omega_0} (\cos \omega_0 \tau - 1), \quad (21) \\
y_2(\tau) &= \hat{y}_2 + \frac{\hat{v}_2}{\omega_0} \sin \omega_0 \tau + \frac{\hat{v}_1}{\omega_0} (\cos \omega_0 \tau - 1), \quad (22) \\
y_0(\tau) &= \hat{y}_0 + \hat{v}_0 \tau, \quad y_3(\tau) = \hat{y}_3 + \hat{v}_3 \tau, \quad (23)
\end{align*}
\]

where \(\omega_0 = eH/m\), \(\hat{v}_\mu \equiv v_\mu(0)\) and \(\hat{y}_\mu \equiv y_\mu(0)\). Substituting above operators into \(\text{(11)-(13)}\), the \(f_i\)'s can be worked out straightforwardly, for instance,

\[
\begin{align*}
f_1 &= \langle v_\mu(\tau)v^\mu(\tau')\rangle - \langle v_\mu(\tau)\rangle \langle v^\mu(\tau')\rangle \\
&= -\langle \delta \hat{v}_0^2 \rangle + \langle \delta \hat{v}_3^2 \rangle + \left( \langle \delta \hat{v}_1^2 \rangle + \langle \delta \hat{v}_2^2 \rangle \right) \cos \omega_0 (\tau - \tau') - \\
&[\hat{v}_1, \hat{v}_2] \sin \omega_0 (\tau - \tau').
\end{align*}
\]

Note that the last term in the above equation is an odd function of \((\tau - \tau')\), and the techniques of integrations would be different from those for even functions of \((\tau - \tau')\). Below we call the radiated energies contributed by the odd and even functions of \((\tau - \tau')\) in \(f_i\)'s as the “odd” and “even” part of the radiated energy respectively.

Let us consider the initial condition \(\langle \delta \hat{v}^\mu \rangle = (\gamma, \gamma v_0, 0, 0)\) with \(\gamma^{-1} = \sqrt{1 - v^2}\) while \(\delta \hat{v}_3\) is zero. Then the classical radiated energy, contributed by \(\langle v_\mu(\tau)\rangle \langle v^\mu(\tau')\rangle\) in \(\text{(10)}\), follows immediately as

\[
P_0 = \frac{e^2 \omega_0^2}{4\pi} \frac{2}{3} \chi^2 v^2.
\]

Employing the same techniques given in Ref. \(\overline{\overline{\text{b}}}\), one obtains the radiated power per unit coordinate time up to \(O(\hbar)\), \(P \equiv \mathcal{E}/\int \gamma d(\tau + \tau')/2\), coming from the “even” part of \(f_i\)'s:

\[
P_{f_1}^{\text{even}} = \frac{e^2 \omega_0^2}{4\pi} \left\{ 1 - \gamma \frac{2}{3} \langle \delta \hat{v}_0^2 \rangle + \frac{2 + \gamma^2}{3} \left( \langle \delta \hat{v}_1^2 \rangle + \langle \delta \hat{v}_2^2 \rangle \right) \right\}.
\]
\[ \mathcal{P}^\text{even}_{f_2} = \frac{e^2 \omega_0^2}{4\pi} \left\{ (3 - 7\gamma^2 + 4\gamma^4) \langle \delta \hat{v}_0^2 \rangle + \frac{20}{3} - 11\gamma^2 + 4\gamma^4 \right\} + \left( \frac{4}{3} + \gamma^2 \right) \langle \delta \hat{v}_2^2 \rangle, \]
\[ \mathcal{P}^\text{even}_{f_3} = \frac{e^2 \omega_0^2}{4\pi} \left\{ \left( \frac{2 + 14\gamma^2}{3} - 4\gamma^4 \right) \langle \delta \hat{v}_0^2 \rangle + \left( \frac{16 + 4\gamma^2}{3} + 4\gamma^4 \right) \langle \delta \hat{v}_2^2 \rangle \right\}, \]
all of which are proportional to the velocity uncertainty of the wave-packet at \( \tau = 0 \). The odd part of the radiated energies are
\[ \mathcal{E}^\text{odd}_{f_1} = \frac{e^2}{16\pi^3} \text{Re} \int_0^{\pi/2} d\varphi \int_0^{\pi\omega_0} d\theta \sin \theta \int_0^{\infty} d\omega \int d\tau d\tau' f_1^\text{odd} e^{-i\omega \gamma(\tau - \tau')} \times \]
\[ \exp \frac{i}{\omega_0} \lambda v \sin \theta \cos \varphi \sin \omega_0(\tau - \tau') \sin \varphi \cos \omega_0(\tau - \tau') \sin \varphi \cos \omega_0(\tau - \tau'), \]
where
\[ f_1^\text{odd} = -i \frac{\omega_0}{m} \sin \omega_0(\tau - \tau'), \]
\[ f_2^\text{odd} = -i \frac{\omega^2}{2m\omega_0} \sin^2 \theta \left[ 1 - v^2 \cos \omega_0(\tau - \tau') \right] \left[ \omega_0(\tau - \tau') - \sin \omega_0(\tau - \tau') \right], \]
\[ f_3^\text{odd} = \frac{\hbar}{m} \omega \sin \theta \sin \omega_0(\tau - \tau') \left[ \cos \varphi \sin \omega_0(\tau - \tau') \sin \varphi \cos \omega_0(\tau - \tau') \right]. \]
They are coming from the non-commutativity [16] of the system and independent of the choice of the quantum state. Unfortunately, we did not succeed to evaluate \( \mathcal{E}^\text{odd}_{f_1} \) for general \( v \). In the following we will discuss the cases in the static limit and the high-speed, weak-field limit.

### A. Static Limit

In the static limit \( v = 0, \gamma \to 1 \), the classical radiated power \( \mathcal{P}_0 \) in Eq. (24) vanishes. The sum of the even radiated power [20]-[28] now reads
\[ \lim_{v \to 0} \mathcal{P}_{f_1}^\text{even} + \mathcal{P}_{f_2}^\text{even} + \mathcal{P}_{f_3}^\text{even} = \frac{e^2 \omega_0^2}{4\pi} \left( \frac{2}{3} \right) \left( \langle \delta \hat{v}_1^2 \rangle + \langle \delta \hat{v}_2^2 \rangle \right), \]
in which \( \mathcal{P}_{f_i}^\text{even} \big|_{v \to 0} \) is actually zero. If the wave-packet is identical to the “ground state” wave function, \( \psi \) in Eq. (A2) with \( v = 0 \), the above even part of the radiated power with the velocity uncertainty [A7] will be exactly cancelled by the sum of the odd radiated power,
\[ \lim_{v \to 0} \mathcal{P}_{f_1}^\text{odd} + \mathcal{P}_{f_2}^\text{odd} + \mathcal{P}_{f_3}^\text{odd} = \frac{e^2 \omega_0^2}{4\pi} \frac{\hbar |\omega_0|}{m} \left( -1 + \frac{1}{3} + 0 \right). \]
Hence, as we expected, a static charge in the ground state does not radiate any photon.

Excited eigenstates are also static. Sandwiched by some excited eigenstate, the expectation values of the velocity \( \langle v_1(\tau) \rangle \) and \( \langle v_2(\tau) \rangle \) will be zero, while the velocity uncertainties \( \langle \delta \hat{v}_1^2 \rangle \) and \( \langle \delta \hat{v}_2^2 \rangle \) will be greater than \( \hbar |\omega_0|/2m \) in general. This makes the sum of Eq. (33) and (34) positive, which implies that a static charge in some excited eigenstate will radiate photon with the same power as the classical radiation by a charge moving in the same background with a speed \( \tilde{v} \) defined by
\[ \frac{m}{2(1 - \tilde{v}^2)} = \langle \mathcal{H} \rangle_{n,L_z} - \langle \mathcal{H} \rangle_{n,0,0}, \]
according to [A4]. Here \( n \) is the principal quantum number and \( L_z \) is the angular momentum parallel to the uniform magnetic field. This non-zero radiated energy actually corresponds to the spontaneous emission of the system [11][12]. Such an energy dissipation of the wave-packet continues until the system finally reaches the ground state.

### B. High-Speed, Weak-Field Limit

In the high-speed limit \( v \to 1 \) or \( \gamma \gg 1 \) with the weak background field \( |eH|/m^2 \ll 1 \), the radiated photons are mainly in the short wave-length regime. The radiation is concentrated in a narrow cone in this limit, such that only
a small part of the trajectory is effective in producing the radiation observed in a given direction. Explicitly, only the time interval \((\tau - \tau')\) with \(|\omega_0(\tau - \tau')|^2 \sim \gamma^{-2} = 1 - v^2 \approx 2(1 - v)\) is effective. According to this observation, one keeps the lowest order of \((\tau - \tau')\), for example,

\[
\begin{align*}
\tau - \tau' - \frac{2v}{\omega_0} \sin \frac{\omega_0}{2} (\tau - \tau') & \approx (1 - v) (\tau - \tau') + \frac{\omega_0^2}{24} (\tau - \tau')^3, \\
\tau - \tau' + \frac{2v}{\omega_0} \sin \frac{\omega_0}{2} (\tau - \tau') & \approx 2(\tau - \tau'),
\end{align*}
\]

in \(f_i\)'s to simplify the integration. By noting that

\[
\int d\omega \int_0^\infty dz \sin \omega z = 0,
\]

the odd part of the \(O(\hbar)\) radiated power becomes

\[
\lim_{v \to 1} P_{f_1}^{\text{odd}} + P_{f_2}^{\text{odd}} + P_{f_3}^{\text{odd}} = e^2 \frac{\omega_0^2}{4\pi} \frac{\hbar|\omega_0|}{m} \left( 0 + \frac{5\sqrt{3}}{24} \gamma^3 - \frac{5\sqrt{3}}{2} \gamma^3 + O(\gamma) \right) \approx e^2 \frac{\omega_0^2}{4\pi} \frac{55\sqrt{3}}{24} \frac{\hbar|\omega_0|}{m} \gamma^3.
\]

This is the well-known result in the semi-classical electrodynamics\[1, 2\]. The negative sign contributed by \(P_{fs}^{\text{odd}}\) indicates that this quantum effect tends to reduce the classical radiation. It can be interpreted as the absorption of the system from the background magnetic field\[3\].

The even part of the \(O(\hbar)\) radiated power\[26, 28\] in the same limit is, however, positive:

\[
\lim_{v \to 1} P_{f_1}^{\text{even}} + P_{f_2}^{\text{even}} + P_{f_3}^{\text{even}} = e^2 \frac{\omega_0^2}{4\pi} \left( 8\langle \delta \hat{v}_1^2 \rangle \gamma^4 + O(\gamma^2) \right).
\]

If one further chooses the initial wave-packet as the Gaussian wave-packet\[A2\] with the velocity uncertainty\[A7\], the negative result in \[39\] would be overpowered by \[40\] as \(\gamma \gg 1\), and the radiated power from the Gaussian wave-packet\[A2\] at \(\tau = 0\) would be enhanced by the quantum effect. This enhancement indicates that the energy dissipation via radiation would cause a dispersion of the wave-packet in addition to the decrease of the radius of its circular motion. By noting that \(\langle \delta \hat{v}_1^2 \rangle = m^{-2} \langle (\delta p_1 - eH \delta y_2/2)^2 \rangle |_{\tau=0}\), and \(y_2\)-axis is parallel to the radial direction of the circular motion at \(\tau = 0\), one sees that the quantum correction\[40\] tends to squeeze the Gaussian wave-packet\[A2\] along the radial axis. This is due to the tidal force induced by the different dissipation rates for different portions of the wave-packet with different radius in the circular motion. In our system, the particle in an outer orbit has a greater acceleration, hence a larger radiated power, and falls instantaneously faster than the particle in an inner orbit.

A typical synchrotron radiation\[X\]-ray source has \(H \sim 1\) Tesla produced by its bending magnets and \(\gamma \sim 10^4\) for the electrons in its storage ring\[14\]. This corresponds to a quantum correction \(P_{fs}^{\text{odd}}\) in \[39\] with the magnitude about \(10^{-6}\) of the classical radiation, well under the typical energy spread of the electron beam with the order of \(10^{-3}\). But \(P_{fs}^{\text{even}}\) in \[40\] for the Gaussian wave-packet\[A2\] is about \(\gamma\) times the magnitude of \(P_{fs}^{\text{odd}}\), thus, \(10^{-2}\) of the power of the classical synchrotron radiation. This is quite above the background noise and should be observed in the accelerator if it exists.

Nevertheless, because of the presence of those complicated quantum corrections in \[26-28\] for every \(\gamma\), the wave-packet is distorted throughout the accelerating process. If an electron in a uniform magnetic field is accelerated from its ground state, the final configuration in the storage ring is not likely to be a Gaussian wave packet similar to \[A2\]. Rather, a wave-packet giving the lowest quantum corrected synchrotron radiation is expected.

Observing that, if the initial wave-packet is prepared with \(\langle \delta \hat{v}_1^2 \rangle \sim \gamma^{-1}, \langle \delta \hat{v}_2^2 \rangle \sim \gamma\) and \(\langle \delta \hat{v}_0^2 \rangle \lesssim \gamma^0\), then the leading order of the even radiated power

\[
\lim_{v \to 1} P_{f_1}^{\text{even}} + P_{f_2}^{\text{even}} + P_{f_3}^{\text{even}} \approx e^2 \frac{\omega_0^2}{4\pi} \left[ 8\gamma^4 \langle \delta \hat{v}_1^2 \rangle^2 + \frac{4}{3} \gamma^2 \langle \delta \hat{v}_2^2 \rangle \right]
\]

is proportional to \(\gamma^3\), just the same order of magnitude as the odd part of the radiated power\[39\]. Further, from the uncertainty relation\[17\], one has

\[
8\gamma^4 \langle \delta \hat{v}_1^2 \rangle + \frac{4}{3} \gamma^2 \langle \delta \hat{v}_2^2 \rangle \geq 2 \sqrt{\frac{32}{3} \gamma^6 \langle \delta \hat{v}_1^2 \rangle^2 \langle \delta \hat{v}_2^2 \rangle} \geq \frac{\hbar|\omega_0|}{m} \frac{4\sqrt{6}}{3} \gamma^3,
\]

\[42\]
where the equality occurs when $8\gamma^4 \langle \delta v^2 \rangle = 4\gamma^2 \langle \delta v^2 \rangle / 3$ while the wave-packet has the minimal uncertainty. Hence the most probable configuration of the wave-packet in the accelerator is the one with $\langle \delta v^2 \rangle = \hbar |\omega_0|/2\sqrt{6}m\gamma$ and $\langle \delta i^2 \rangle = \hbar \sqrt{6} |\omega_0|\gamma/2m$, which is highly squeezed in the radial direction and looks far from the wave function $\psi(A2)$, the “ground state” wave function in the co-moving frame.

Substituting $\hat{A}$ into $\hat{F}$, one finds that the minimum value of $P_{\text{even}}$ is so close to the $|P_{\text{odd}}|$ in $\hat{A}$ that the total $O(\hbar)$ quantum correction to the synchrotron radiation is not detectable in today’s synchrotron radiation X-ray sources. The lowest total $O(\hbar)$ quantum correction $P_{\text{even}} + P_{\text{odd}}$ is again negative, with the magnitude about one fifth of the well-known result $P_{\text{odd}}$ in $\hat{A}$.

**IV. CONCLUSION**

The $O(\hbar)$ correction to the synchrotron radiation from a Lorentz electron moving in a uniform magnetic field has been calculated in the static limit and the high-speed, weak-field limit. While the conventional calculations are mainly focused on the eigenstates for the electron, we consider the four-dimensional wave-packet state centered at the classical trajectory of the electron. We found that the velocity uncertainty of the wave-packet gives an additional quantum correction to the radiation.

In the static limit, the positive corrections from the velocity uncertainty or the width of the wave-packet cancel the negative corrections from the non-commutativity of the system, so that the radiated energy from the ground state wave-packet vanishes. For the excited eigenstates, the velocity uncertainty of the wave-packet induces a positive radiated energy corresponding to the spontaneous emission.

In the high-speed and weak-field limit, we recovered the well-known negative result as a part of our $O(\hbar)$ correction. We found that the velocity uncertainty of the wave-packet may overpower the conventional result and make the total $O(\hbar)$ correction positive in some cases. This indicates that not only the circular trajectory but also the configuration of the moving wave-packet are unstable in the background of uniform magnetic field. The most stable configuration of the wave-packet has the tangential velocity uncertainty $\langle \delta i^2 \rangle = \hbar |\omega_0|/2\sqrt{6}m\gamma$ and the radial velocity uncertainty $\langle \delta v_2^2 \rangle = \hbar \sqrt{6} |\omega_0|\gamma/2m$, which contributes the lowest $O(\hbar)$ quantum correction to the radiated energy.

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**APPENDIX A: GAUSSIAN WAVE-PACKET**

To recover the circular motion of a classical charge in a uniform magnetic field, $F_{\text{in}}^{12} = H$, it is convenient to choose the symmetric gauge such that $A_\mu = (0, -Hy_2/2, Hy_1/2, 0)$ $\hat{A}$. Under this choice the Hamiltonian $\hat{A}$ reads

$$\hat{H} = \frac{1}{2m}p_\mu p^\mu + \frac{\epsilon H}{2m}(p_1 y_2 - p_2 y_1) + \frac{e^2 H^2}{8m}(y_1^2 + y_2^2). \tag{A1}$$

The eigenstates for the charge in a uniform magnetic field in $y^3$-direction are therefore equivalent to those for a two-dimensional simple harmonic oscillator.

The simplest four-dimensional wave-packet moving in the classical trajectory $\langle y^\mu(\tau) \rangle = y_{\tau i}^{\mu} = (\gamma\tau, (\gamma v/\omega_0) \sin \omega_0 \tau, (\gamma v/\omega_0) \cos \omega_0 \tau, 0)$ with the minimal “energy” is the Gaussian wave-packet described by the wave function,

$$\psi(x) = Ne^{-\frac{\pi}{\sigma} e^{i\frac{\pi}{2} p_3 y^3}} \sqrt{\frac{1}{\tau + i \sigma}} \exp \left[ -\frac{im}{2\hbar} \frac{\delta t^2}{\tau + i \sigma} - \frac{i}{\hbar} m\gamma \delta t - \frac{|eH|}{4\hbar} \rho^2 + \frac{im}{2\hbar} \gamma v \rho \cos(\phi + \omega_0 \tau) \right], \tag{A2}$$

where $\delta t \equiv t - \gamma \tau$. $N$ is the normalization factor, $\sigma$ is a constant determined by initial conditions, and the local polar coordinate $(\rho, \phi)$ is defined by

$$\rho e^{i\phi} = x - \frac{\gamma v}{\omega_0} \sin \omega_0 \tau + i \left( y - \frac{\gamma v}{\omega_0} \cos \omega_0 \tau \right). \tag{A3}$$
The constant $E$ in $\psi$ is

$$E = \frac{m}{2} \gamma^2 + \frac{\hbar |\omega_0|}{2},$$  \hfill (A4)$$

where the first term $m\gamma^2/2$ is up to the classical level, and we had set $p_3$ to zero at $\tau = 0$. One can verify that the above wave function $\psi(A2)$ does satisfy the Schrödinger equation $\left(i\hbar \partial_\tau - \mathcal{H}\right)\psi = 0$.

When $v = 0$, $\psi$ is the “ground state” of this harmonic-oscillator system, while for $v > 0$, $\psi$ is a superposition of the excited states for the system, though $\psi$ looks the same as the “ground state” wave function in the co-moving frame.

By noting that $\langle v^\mu \rangle = v^\mu_{cl} = (\gamma, \gamma v \cos \omega_0 \tau, -\gamma v \sin \omega_0 \tau, 0)$, the expectation values of $v^2_1$ and $v^2_2$ for the wave-packet $\psi$ in $(A2)$ can be written as

$$\langle v^2_1 \rangle = \int d^4x \left| \frac{\hbar}{m} \left( -\gamma \partial_x + \frac{eH}{2} y \right) \psi \right|^2 = (v^1_{cl})^2 + \frac{\hbar |\omega_0|}{2m},$$  \hfill (A5)$$

$$\langle v^2_2 \rangle = \int d^4x \left| \frac{\hbar}{m} \left( -\gamma \partial_y - \frac{eH}{2} x \right) \psi \right|^2 = (v^2_{cl})^2 + \frac{\hbar |\omega_0|}{2m},$$  \hfill (A6)$$

for all $\tau$. Hence the velocity uncertainties for the wave-packet at $\tau = 0$ are

$$\langle \delta \hat{v}^2_1 \rangle = \langle \delta \hat{v}^2_2 \rangle = \frac{\hbar |\omega_0|}{2m}. \hfill (A7)$$

Besides, if $\sigma$ is a real number, one has

$$\langle \hat{v}^2_0 \rangle = \int d^4x \left| \frac{\hbar}{m} \partial_t \psi \right|^2 = \gamma^2 + \frac{\hbar}{2m\sigma}, \hfill (A8)$$

such that $\langle \delta \hat{v}^2_0 \rangle = \hbar/2m\sigma$. The value of $\langle \delta \hat{v}^2_0 \rangle$ or $\sigma$ is not important either in the static limit or in the high-speed and weak-field limit.