Towards a Change of Variable Formula for “Hypergeometrization”

Petr Blaschke

Abstract. We are going to study properties of “hypergeometrization” – an operator which acts on analytic functions near the origin by inserting two Pochhammer symbols into their Taylor series. In essence, this operator maps elementary functions into hypergeometric. The main goal is to produce number of “change of variable” formulas for this operator which, in turn, can be used to derive great number of transform for multivariate hypergeometric functions.

1. Introduction

Hypergeometric functions and their multivariate analogs are well studied objects in mathematics. The classical references include Érdelyi [1], Luke [2], Bailey [3], Slater [4] just to mention few. A very nice survey article about multivariate hypergeometric function of “Appell’s type” was written by M. Schlosser in [5].

There are numerous ways how to extend hypergeometric function into higher dimension. There are Appell’s function [6], Functions from the Horn’s list [7], Kampé de Feriét functions [8, 9], Lauricella functions [10], Srivastava function [11], Saran’s functions [12, 13], A-hypergeometric function [14, 15, 16], hypergeometric functions of matrix argument [17, 18], and so on.

These functions appears surprisingly often in all of analysis and have many application, e.g. in quantum field theory, in computing of Feynman integrals (see e.g. [19]), even appear also in chemistry [20]. Recently a Karlsson’s \( {}_1 F_1 \) function [11, 21, 22] appeared in the literature [23] in the context of harmonic Bergman spaces.

The main object of study for these functions are various “transforms” i.e. identities that relates two of them together or one function to itself but with different values of parameters and/or argument(s).

A common feature of all of the mentioned functions (safe for functions of matrix argument) is the presence of a Pochhammer symbol, i.e. the quantity \((a)_k := a(a+1)\cdot\cdot\cdot(a+k-1)\) in their series expansion.

It is therefore only natural to study a linear operator \( \mathcal{H}_c^{a} \) called “hypergeometrization” depending on two complex parameters \( a, c \in \mathbb{C} \) which acts on analytic functions near the origin by inserting two Pochhammer symbols into their Taylor series.

Definition 1. Let \( \mathcal{C}^{\omega} \) denotes a space of functions analytic near the origin, i.e.

\[
f \in \mathcal{C}^{\omega} \iff \exists R > 0 : \quad f(x) = \sum_{n=0}^{\infty} f_n x^n, \quad \forall |x| < R,
\]

for some complex coefficients \( f_n \).

Let \( a, c \in \mathbb{C} \), so that \( 1 - c \not\in \mathbb{N} \). Then the hypergeometrization is the linear operator

\[
\mathcal{H}_c^{a} : \mathcal{C}^{\omega} \rightarrow \mathcal{C}^{\omega},
\]

given by

\[
(1.1) \quad \mathcal{H}_c^{a} f(x) := \sum_{n=0}^{\infty} f_n \frac{(a)_n}{(c)_n} x^n,
\]

where \((a)_n = a(a+1)\cdot\cdot\cdot(a+n-1)\) is the Pochhammer symbol.
Remark 1. Most of the time we will make hypergeometrization with respect to the \( x \) variable, or with respect to a variable which is clear from context. However, in case there is a need to stress the variable in use, we will write it in brackets like so:

\[
\mathcal{H}_c^a \equiv \mathcal{H}_c^a(x).
\]

Application of operator \( \mathcal{H}_c^a \) on elementary functions can produce large number of special functions, particularly (as the name suggests) hypergeometric functions. Concretely, Gauss’s hypergeometric function is trivially given by

\[
\mathcal{H}_c^a(1 - x)^{-b} = 2F_1 \left( \begin{array}{ccc} a & b \\ c & \end{array} ; x \right).
\]

Similarly, we have an expression for the confluent hypergeometric function

\[
\mathcal{H}_c^a e^x = 1F_1 \left( \begin{array}{ccc} a \\ c \end{array} ; x \right),
\]

and Bessel’s function

\[
\mathcal{H}_c^a \cos(2\sqrt{x}) = qF_1 \left( \begin{array}{ccc} - & & \\ c & \end{array} ; -x \right) = \Gamma(c) x^{\frac{1}{2}} J_{c-1}(2\sqrt{x}).
\]

In fact, as we will see in Proposition 2, all the generalized hypergeometric functions \( F_q \) can be constructed from elementary functions (by iterative application of hypergeometrization). We will also show that great number of multivariate analogues of hypergeometric functions are also images of \( \mathcal{H}_c^a \). For instance Appell’s functions \([6, 9]\):

\[
\mathcal{H}_c^a(1 - tx)^{-b_1}(1 - ty)^{-b_2} = 3F_1 \left( \begin{array}{ccc} a & b_1 & b_2 \\ c & tx, ty \end{array} \right).
\]

But we will also deal with functions from the Horn’s list \( G_2, H_4, \Phi_1, \Phi_3, [1] \).

Remark 2. All the claimed identities in this section can be checked following the link above the equality sign.

Our main focus is the question whether there exists a “change of variable formula” for the operator \( \mathcal{H}_c^a \). That is, is there a way how to compute hypergeometrization of a composite function in terms hypergeometrization with respect to the inner function? In symbols, we want to produce formulas of the form

\[
\mathcal{H}_c^a f(y(x)) = F \left( y, \mathcal{H}_c^a(y) \right) f(y),
\]

where \( F \) is some non-commutative expression involving \( y \) and some finite number of hypergeometrization operators \( \mathcal{H}_c^a \) with various parameters.

For some function \( y \) the answer is yes. For instance, it is an easy exercise based on properties of the Pochhammer symbol that the following holds:

\[
\mathcal{H}_c^a \left( \frac{x}{y} \right) = \mathcal{H}_c^a(1 - \frac{x}{y}) \quad y = S_0(x) := ax.
\]

\[
\mathcal{H}_c^a \left( \frac{x}{y} \right)^{\frac{a + 1}{2}} = \mathcal{H}_c^a(y) \mathcal{H}_c^a(y), \quad \quad y = M_2(x) := x^2.
\]
We will show that a change of variable formula holds also for the function \( x/(x - 1) \) which reads:

\[
H(x) = (1 - y)^a H(y) (1 - y)^{-c}, \quad y = P(x) := \frac{x}{x - 1}.
\]

The last identity – which we call “Pfaff property” – seems to be of fundamental importance. Throughout this article we will show that this single formula is all one need to derive surprisingly large numbers of transform of special function, including:

Pfaff transform:

\[
\begin{align*}
&\binom{a}{b} \binom{b}{c} \binom{x}{a-b} = (1 - x)^{-b} F_1 \left( \begin{array}{cc} c-a & b \\ \frac{1}{a} & \frac{1}{a} \end{array} ; \frac{x}{x-1} \right). \\
&F_1 \left( \begin{array}{ccc} a & b & c \\ b_1 & b_2 & x \end{array} \right) = (1 - x)^{-a} F_1 \left( \begin{array}{ccc} a & b & c \\ b_1 & b_2 & x \end{array} ; \frac{x}{x-1} \right).
\end{align*}
\]

Quadratic transform:

\[
\begin{align*}
&\binom{a}{b} \binom{a}{2} \binom{x}{2} = (1 - x)^{-b} F_1 \left( \begin{array}{cc} c-a & b \\ \frac{1}{a} & \frac{1}{a} \end{array} ; \left( \frac{x}{1-x} \right)^2 \right). \\
&F_1 \left( \begin{array}{ccc} a & b & c \\ b_1 & b_2 & x \end{array} \right) = (1 - x)^{-a} F_1 \left( \begin{array}{ccc} a & b & c \\ b_1 & b_2 & x \end{array} ; \frac{x}{x-1} \right).
\end{align*}
\]

Alternative representations for \( F_1 \):

\[
\begin{align*}
&F_1 \left( \begin{array}{ccc} a & b & c \\ b_1 & b_2 & x \end{array} \right) = (1 - x)^{-a} F_1 \left( \begin{array}{cc} a & b \\ c-b_2 & c-b_1 \end{array} ; \frac{x}{x-1} \right), \\
&\text{and many more. Our main result is to give a change of variable formula valid for a one parameter group of functions.}
\end{align*}
\]

**Theorem 1.** Let

\[ y = F_m(x) := 1 - (1 - x)^m, \quad m \in \mathbb{Z}. \]

Then assuming either

1) \( m \in \{-2, -1, 0\} \), \( \forall a, c \in \mathbb{C} \), or

2) \( \forall m \in \mathbb{Z} \setminus \{0\}, \quad a - c \in \mathbb{Z} \),

it holds

\[
\begin{align*}
&\binom{a}{c} \binom{a}{x} = \left( \frac{m x}{y} \right)^{1-c} (1 - y)^{1+a-c} \prod_{j=1}^{m} (1 - y)^{a+c-1} \frac{\binom{c+j}{a+c-1}}{\binom{c+j}{a+c-1}} (y) \left( \frac{m x}{y} \right)^{a-1}.
\end{align*}
\]
Remark 3. The product $\prod_{j=1}^{m}$ in (1.13) is understood to be naturally extended for negative $m$ and zero. Let $\{A_j\}_{j \in \mathbb{Z}}$ be a sequence of invertible linear operators. Then we set

\begin{equation}
0 \leq \prod_{j=1}^{m} A_j := 0, \quad -m \leq \prod_{j=1}^{m} A_j := \prod_{j=1}^{m} A_{-1}^{-1}, \quad \forall m \in \mathbb{N}.
\end{equation}

Remark 4. It is the author believe that Theorem 1 is not in fact limited to parameters $a, c$ which differs by an integer but it holds for all their (permissible) complex values. All the restrictions on $m, a$ and $c$ thus reflect only the author’s inability to prove the theorem in full generality.

Conjecture 1. The formula (1.13) holds for generic values of $a, c \in \mathbb{C}$ and all $m \in \mathbb{Z} \subseteq \{0\}$.

In summary, using Theorem 1 a “change of variable” formula can be obtained for any function $y$ that can be written as a finite composition of

\begin{equation}
s_a(x) = a x, \quad M_n(x) = x^n, \quad F_n(x) = 1 - (1 - x)^n,
\end{equation}

(right now with additional restriction that $a - c \in \mathbb{Z}$). Note that $F_{-1}(x) = x/(x - 1) = P(x)$.

Here is a small sample of identities on can construct from these functions which are valid for all values of $a$ and $c$:

\begin{equation}
(1 - x)^{1-c} \frac{a}{c} H(x) (1-x)^{a-1} = \frac{a+c-1}{c} H(y) (1-y)^{c-y} \frac{a}{c} \frac{H^+}{H}(y), \quad y = 4x(1-x).
\end{equation}

\begin{equation}
(1 - x)^{c+a-1} \frac{a}{c} H(x) (1-x) = \frac{c+a-1}{c} H(y) (1-y)^{c-y} \frac{a}{c} \frac{H^+}{H}(y), \quad y = -4x/(1-x^2).
\end{equation}

\begin{equation}
(1 + x)^{c+a-1} \frac{a}{c} H(x) (1+x) = \frac{c+a-1}{c} H(y) (1-y)^{c-y} \frac{a}{c} \frac{H^+}{H}(y), \quad y = 4x/(1+x^2).
\end{equation}

\begin{equation}
(1 - x)^x \frac{a}{c} H(x) (1-x)^{a-1} \frac{a}{c} H(y) (1-y)^{c-y} \frac{a}{c} \frac{H^+}{H}(y), \quad y = x^2(4x-1).
\end{equation}

\begin{equation}
(1 - x)^{1-c} \frac{a}{c} H(x) (1-x)^{a-1} \frac{a+c-1}{c} H(y) (1-y)^{c-y} \frac{a}{c} \frac{H^+}{H}(y), \quad y = 4x(1-x)/(1-2x)^2.
\end{equation}

\begin{equation}
(1 - x^2)^{1-c} \frac{a}{c} H(x) (1-x^2)^{a-1} \frac{a+c-1}{c} H(y) (1-y)^{c-y} \frac{a}{c} \frac{H^+}{H}(y), \quad y = x^2/(2-2x^2).
\end{equation}

And so on.

In what follows, and to demonstrate the technique, we are going to use hypergeometrization to derive many known identities involving special functions. There are, however, three identities which are possibly new (or at least the author is unable to find them in the literature). These are:

- A quadratic transform for $F_1$ function: Let $\beta := \frac{a+c-1}{2}$. Then

\begin{equation}
F_1\left(\frac{a}{c} \beta, \beta; \tau \right) \equiv (1-x)^{-\beta} F_1\left(\beta; \frac{a+c}{c} - \frac{a}{c} \frac{1}{1+x^2} \frac{1}{\beta} \right),
\end{equation}

where

\begin{equation}
\tau \equiv 2 \left(2t-1\right)^2 \pm \sqrt{4t(t-1)}.
\end{equation}

- A semi-cubic reduction of $F_1$ to $2F_1$:

\begin{equation}
(1-x)^{-2a} F_1\left(\frac{a}{c} \frac{2a}{3} + 1; \frac{x}{x+1}\right) \equiv \frac{x}{4x(x-1), 3x(x-1)}.
\end{equation}
• $G_2$ to $F_2$ conversion:

$G_2\left( a \quad c; \quad b_1 - b_2 \quad ; x, y \right)$

\begin{equation}
\left( 1 + x \right)^{b_1} \left( 1 + y \right)^{b_2} F_2 \left( 1 - c - a \quad 1 - c \quad 1 - a; \quad \frac{x}{x+1} \quad \frac{y}{y+1} \right).
\end{equation}

Particularly, it does not seem to be possible to derive the first formula (1.22) from Carlson’s results about quadratic transforms of $F_1$ function given in [24].

The structure if the paper is as follows: Basic properties of hypergeometrization operator are discussed in Section 2. In Section 3 the methodology of representing a special functions via hypergeometrization is described. Section 4 introduces the Pfaff property. Its consequences are discussed in Section 5. Treatment of the change of variable formula is done in Section 6. Finally, in Section 7 we prove Theorem 1 and provide some supporting evidence for Conjecture 1.

REMARK 5. The concept of hypergeometrization was introduced by the present author in [25] and was also mentioned in [26]. It can be understood as a Hadamard product (or a convolution)

$\mathcal{H}_c f(x) = 2F_1 \left( a \quad c; \quad x \right) \ast f(x),
$ where the Hadamard product of the two formal power series $g(x) = \sum_{k \geq 0} g_k x^k$, $h(x) = \sum_{k \geq 0} h_k x^k$ is defined

$g(x) \ast h(x) := \sum_{k=0}^{\infty} g_k h_k x^k.$

Before [25], a linear operator which brings a function to its Hadamard product with some hypergeometric function (i.e. to its hypergeometrization) appeared also in [27] and elsewhere. But hypergeometrization is a special case of Hadamard product, and – as we will endeavor to show – has many properties the general Hadamard product does not posses.

2. Basic properties

An important property of hypergeometrization is that (generically) it does not change the radius of convergence.

**Proposition 1.** Let $R > 0$ be a radius of convergence of the following power series:

$f(x) = \sum_{n=0}^{\infty} f_n x^n, \quad |x| < R.$

Let $1 - a, 1 - c \notin \mathbb{N}$. Then

$\mathcal{H}_c f(x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} f_n x^n,$

converges for all $|x| < R$.

**Proof.** It is a standard result for $\Gamma$ function that

$\lim_{n \to \infty} n^{c-a} \frac{(a)_n}{(c)_n} = \lim_{n \to \infty} n^{c-a} \frac{\Gamma(a+n)\Gamma(c)}{\Gamma(a)\Gamma(c+n)} = \frac{\Gamma(c)}{\Gamma(a)}.$

and thus the introduced factor $(a)_n/(c)_n$ grows only polynomially in $n$ and is therefore negligible comparing to the exponential behavior of $x^n$ term. \hfill \Box

Another crucial observation for our purposes is that when the parameters $a, c$ differ by an integer, the hypergeometrization reduces to a differential operator.

$\mathcal{H}_a^0 (x) = \frac{(a + x \partial_x)_n}{(a)_n}.$

The proof is straightforward.
Some additional elementary properties of hypergeometrization includes:

\begin{align}
\text{(2.2)} & \quad \frac{a}{c} \mathcal{H}(\alpha f + \beta g) = \alpha \frac{a}{c} f + \beta \frac{a}{c} g, \quad \text{linearity,} \\
\text{(2.3)} & \quad \frac{a}{c} \frac{b}{d} \mathcal{H} = \frac{b}{d} \frac{a}{c} \mathcal{H}, \quad \text{commutativity,} \\
\text{(2.4)} & \quad \frac{a}{c} \frac{b}{d} \mathcal{H} = \frac{a}{c} \frac{b}{d} \mathcal{H} = \frac{b}{d} \frac{a}{c} \mathcal{H}, \quad \text{parameter exchange,} \\
\text{(2.5)} & \quad \left(\frac{a}{c}\right)^{-1} = \mathcal{H}, \quad \text{inverse,} \\
\text{(2.6)} & \quad \frac{a}{c} x^n = \frac{a}{c} \frac{(a)_{n}}{(c)_{n}} x^{a+n} \mathcal{H}^n, \quad \text{shift,} \\
\text{(2.7)} & \quad \left(\partial_x\right)^n \mathcal{H} = \frac{(a)_{n}}{(c)_{n}} \mathcal{H}^n \left(\partial_x\right)^n, \quad \text{dual shift,} \\
\text{(2.8)} & \quad \frac{a}{c} x^n = \frac{a}{c} (x^n), \quad \text{argument scaling,} \\
\text{(2.9)} & \quad \frac{a}{c} (x^n) = \frac{a}{c} \left(\frac{x^2}{x}\right)^{\frac{n}{2}} \mathcal{H} \left(\frac{x^2}{x}\right) \mathcal{H}, \quad \text{argument square,} \\
\text{(2.10)} & \quad \frac{a}{c} (x^n) = \frac{a}{c} \mathcal{H}^n (x^n), \quad \text{n-th power,} \\
\text{(2.11)} & \quad c \mathcal{H} - a + (a-c) \frac{a}{c} \mathcal{H} = 0, \quad \text{contiguous relation,} \\
\text{(2.12)} & \quad \frac{a}{c} \frac{a+1}{a+1-a} \mathcal{H} = \frac{1}{2} \frac{a}{a+1} + \frac{1}{2} \frac{1}{1-a} \mathcal{H}, \quad \text{per partes.}
\end{align}

Here the function $f, g$ are analytic near the origin, $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{N}$. Parameters $a, b, c, d$ can be arbitrary complex numbers with the possible restriction on the lower parameters $1-c \not\in \mathbb{N}$.

**Proof.** Since we are working on function analytic near origin, it is actually sufficient to verify all these claims only on monomials $x^n$ which is – mostly – straightforward and are left to the reader as an stimulating exercise. Identities (2.9), (2.10) are based on the following property of Pochhammer symbols:

\begin{equation}
\forall n, k \in \mathbb{N} : \quad (a)_{nk} = \frac{(a)}{n}_{k} \frac{(a+1)}{n}_{k} \cdots \frac{(a+n-1)}{n}_{k} n^{nk}.
\end{equation}

A property that perhaps deserves some comment is the very last one. It too can be very easily checked on monomials as follows:

\begin{align}
\text{(2.13)} & \quad \frac{a}{c} \frac{a+1}{a+1-a} \mathcal{H} x^n = \frac{(a)n(-a)n}{(a+1)n(1-a)n} x^n = \frac{-a^2}{(a+n)(n-a)} x^n = \frac{a}{2(n+a)} x^n - \frac{a}{2(n-a)} x^n = \frac{1}{2} \mathcal{H} x^n + \frac{1}{2} \mathcal{H} x^n.
\end{align}

But why is it called “per partes”?

Remember that from (2.1) when the upper parameter differs from the lower one by 1, the hypergeometrization reduces to:

\begin{equation}
\frac{a+1}{a} \mathcal{H} = \frac{a + x \partial_x}{a} = \frac{1}{a} x^{1-a} \partial_x x^a.
\end{equation}

Thus its inverse is an integral operator

\begin{equation}
\frac{a}{a+1} \mathcal{H} = \left(\frac{a+1}{a}\right)^{-1} = \frac{1}{a} x^{1-a} \partial_x x^a \mathcal{H}^{-1} = a x^{-a} \int dx x^{a-1},
\end{equation}

modulo integration constant, of course. Hence

\begin{equation}
\frac{a}{a+1} \mathcal{H} a x^{-a} \int dx x^{a-1} (-a) x^a \int dx x^{-a-1} = -a^2 x^{-a} \int dx x^{2a-1} \int dx x^{-a-1}
\end{equation}
\[ -a^2 x^{-a} \left( \frac{x^{2a}}{2a} \int dx x^{-a-1} - \frac{a}{2} x^{-a} \int dx \frac{x^{2a}}{2a} \partial_x \int dx x^{-a-1} \right) \]

\[ = -\frac{a}{2} x^a \int dx x^{-a-1} + \frac{a}{2} x^{-a} \int dx x^{a-1} = \frac{1}{2} \frac{a}{1-a} + \frac{a}{2} \frac{x}{1+x} . \]

Here we have used “integration per partes” in the operator notation:

\[ \int dx x^{2a-1} = \frac{x^{2a}}{2a} - \int dx \frac{x^{2a}}{2a} \partial_x . \]

\[ \square \]

3. Special function representation

3.1. Generalized hypergeometric functions. Remember:

**Definition 2.** Generalized hypergeometric functions \( pF_q \) are defined as follows:

\[ (3.1) \quad pF_q \left( \begin{array}{c} a_1 \ldots a_p \\ c_1 \ldots c_q \end{array} ; x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(c_1)_k \ldots (c_q)_k} \frac{x^k}{k!} , \quad 1 - c_k \not\in \mathbb{N}, \forall k. \]

The series converges in the entire complex plane if \( p \leq q \). For \( p = q + 1 \) it converges in the unit disc \(|x| < 1\) and for \( p > q + 1 \) it is generally divergent unless one of the upper parameters is a negative integer, in which case the series terminates and the resulting hypergeometric function is actually a polynomial.

**Proposition 2.** For \( n \in \mathbb{N} \) let

\[ (3.2) \quad f_n(x) := \frac{1}{n} \left( e^{nz_0} \sqrt{x} + e^{nz_1} \sqrt{x} + \ldots + e^{nz_{n-1}} \sqrt{x} \right) = \sum_{k=0}^{\infty} \frac{n^{nk} x^k}{(nk)!} , \quad z_j := i \frac{2j}{\pi}. \]

In particular

\[ f_1 = e^x, \]
\[ f_2 = \frac{1}{2} (e^{\sqrt{x}} + e^{-\sqrt{x}}) = \cosh(2\sqrt{x}), \]
\[ f_3 = \frac{1}{3} (e^{3\sqrt{x}} + 2e^{-\frac{3}{2}\sqrt{x}} \cos \left( \frac{3\sqrt{x}}{2} \right)) , \]
\[ \vdots \]

Then for any complex numbers \( a_1, \ldots, a_m \) and \( c_1, \ldots, c_{m+n-1} \in \mathbb{C} \), such that \( 1 - c_i \not\in \mathbb{N} \forall i \) it holds:

\[ mF_{m+n-1} \left( \begin{array}{c} a_1 \ldots a_m \\ c_1 \ldots c_{n+m-1} \end{array} ; x \right) = \frac{1}{H} \frac{1}{H} \ldots \frac{1}{H} H \ldots H \ldots H \frac{a_1}{c_1} \ldots \frac{a_m}{c_{n+m-1}} f_n(x) . \]

**Proof.** From (2.13) it follows that:

\[ (nk)! = (1)_{nk} = \binom{1}{k} \binom{2}{n-k} \cdots \binom{n-1}{n-k} \frac{k!}{n^k} . \]

Thus

\[ f_n = \sum_{k=0}^{\infty} \frac{n^{nk} x^k}{(nk)!} = \frac{1}{n} F_{n-1} \left( \begin{array}{c} 1 \1 \ldots \frac{1}{n} \\ 2 \ldots n-n \end{array} ; x \right) . \]

The result is obtained by successive application of definition (1.1). \( \square \)

The one advantage of this approach is that it makes questions of convergence clear. Since, evidently, the hypergeometrization does not change the region of convergence, we can see at once that the series \( q+1F_q \) converges in the unit disk (since those functions originated from \((1-x)^{-b}\)) and the rest \( pF_q \ (p \leq q) \) converges everywhere since they are constructed from entire functions like \( e^x, \cosh(2\sqrt{x}) \) etc.
3.2. Appell’s functions. Appell’s functions are defined by the following double series:

(3.3) \[ F_1 \left( \frac{a}{c} ; b_1, b_2 ; x, y \right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k} (b_1)_{j} (b_2)_{k}}{(c)_{j+k} j! k!} x^j y^k, \]

(3.4) \[ F_2 \left( a ; b_1, b_2, c_1, c_2 ; x \right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k} (b_1)_j (b_2)_k}{(c_1)_j (c_2)_k j! k!} x^j y^k, \]

(3.5) \[ F_3 \left( -a, -b_1, -b_2, c ; x, y \right) := \sum_{j,k=0}^{\infty} \frac{(a_1)_j (b_1)_j (a_2)_k (b_2)_k}{(c)_j (d)_k j! k!} x^j y^k, \]

(3.6) \[ F_4 \left( a ; b, c, d ; x, y \right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k} (b)_{j+k}}{j! k! (c)_j (d)_k} x^j y^k. \]

All of these functions can be as well represented as a hypergeometricization of some elementary function:

**Proposition 3.**

**Appell’s F₁ function:**

(3.7) \[ \frac{a}{c} H(t) (1 - tx)^{-b_1} (1 - ty)^{-b_2} = F_1 \left( \frac{a}{c} ; b_1, b_2 ; tx, ty \right). \]

**Appell’s F₂ function:**

(3.8) \[ \frac{b_1}{c_1} H(x) H(y) (1 - x - y)^{-a} = F_2 \left( a ; b_1, b_2, c_1, c_2 ; x, y \right). \]

**Appell’s F₃ function:**

(3.9) \[ \frac{a_1}{c_1} H(x) H(y) (1 - x - y)^{-a} = F_3 \left( \frac{a_1}{c_1} ; a_1 - a_2, c_2, d ; tx, ty \right). \]

**Appell’s F₄ function:**

(3.10) \[ \frac{1}{d} H(x) H(y) \frac{a}{c} H(t) \frac{1 - t(x + y)}{1 - 2t(x + y) + t^2(x + y)^2} = F_4 \left( -b ; a, b, c, d ; tx, ty \right). \]

**Proof.** The proof amounts to show that

\[
\begin{align*}
(1 - tx)^{-b_1} (1 - ty)^{-b_2} &= F_1 \left( \frac{c}{c} ; b_1, b_2 ; tx, ty \right), \\
(1 - x - y)^{-a} &= F_2 \left( a ; b_1, b_2, c_1, c_2 ; x, y \right), \\
\frac{\arctan \sqrt{txy - tx - ty}}{\sqrt{t^2 xy - tx - ty}} &= F_3 \left( \frac{1}{2} ; 1 - \frac{1}{2}, 1 - \frac{1}{2}, 1 - \frac{1}{2} ; tx, ty \right), \\
\frac{1 - t(x + y)}{1 - 2t(x + y) + t^2(x + y)^2} &= F_4 \left( 1 - \frac{1}{2} ; a - \frac{1}{2}, b - \frac{1}{2} ; tx, ty \right),
\end{align*}
\]

which is left to the reader as an easy exercise. \(\square\)

Once again we can retrieve the information about the regions of convergence for Appell’s series from their elementary origins. Since the hypergeometrization does not change the radius of convergence, we can deduce from the fact that

\[
(1 - x)^{-b_1} (1 - y)^{-b_2} = \sum_{j,k=0}^{\infty} \frac{(b_1)_j (b_2)_k}{j! k!} x^j y^k < \infty \iff |x| < 1, |y| < 1,
\]

that the same is true for \(F_1\) function.
This trick is, essentially, Horn’s principle (3.11) (1
implies a representation of Appell’s $F$
and so on. See [on the specific values of parameters – safe for some exceptional pathological values, like negative integers and so on. See [7].)

**Example 1.** The approach of hypergeometrization helps to understand some of the various transforms valid for these functions. For example, equating $x = y = 1, t = 1$ in the formula for $F_1$ function (3.7) we obtain

$$F_1\left( \frac{a}{c}: \frac{b_1}{-} = \frac{b_2}{x}: x, x \right) = 2F_1\left( \frac{a}{c} + \frac{b_1}{b_2}: x \right),$$

since

$$(1 - x)^{-b_1}(1 - x)^{-b_2} = (1 - x)^{-(b_1 + b_2)}.$$

**Example 2.** Similarly, from the fact that

$$(1 - x)^{-b}(1 + x)^{-b} = (1 - x^2)^{-b},$$

we can easily deduce using (2.13)

$$F_1\left( \frac{a}{c}: \frac{b}{-} = \frac{b}{x}: x, -x \right) = 3F_2\left( \frac{a}{c} + \frac{b}{2} + \frac{1}{2}: x^2 \right).$$

**Example 3.** The following elementary identity

$$(3.11) \quad (1 - x - y)^{-a} = (1 - x)^{-a} \left( 1 - \frac{y}{1 - x} \right)^{-a},$$

implies a representation of Appell’s $F_2$ function in the form

$$(3.12) \quad F_2\left( \frac{a}{-}: \frac{b_1}{c_1} = \frac{b_2}{c_2}: x, y \right) = \begin{pmatrix} b_1 \end{pmatrix}_{c_1} \begin{pmatrix} b_2 \end{pmatrix}_{c_2} (1 - x - y)^{-a} \begin{pmatrix} b_1 \end{pmatrix}_{c_1} \begin{pmatrix} b_2 \end{pmatrix}_{c_2} (1 - x)^{-a} \left( 1 - \frac{y}{1 - x} \right)^{-a}.$$ 

The argument is as follows:

$$F_2\left( \frac{a}{-}: \frac{b_1}{c_1} = \frac{b_2}{c_2}: x, y \right) \overset{(3.8)}{=} \begin{pmatrix} b_1 \end{pmatrix}_{c_1} \begin{pmatrix} b_2 \end{pmatrix}_{c_2} (1 - x - y)^{-a} \overset{(3.11)}{=} \begin{pmatrix} b_1 \end{pmatrix}_{c_1} \begin{pmatrix} b_2 \end{pmatrix}_{c_2} \begin{pmatrix} b_1 \end{pmatrix}_{c_1} \begin{pmatrix} b_2 \end{pmatrix}_{c_2} (1 - x)^{-a} \left( 1 - \frac{y}{1 - x} \right)^{-a}.$$ 

The question of when (3.12) holds is not trivial. But it perhaps worth noting that, in some sense, the equality (3.12) should be valid whenever (3.11) is. We will not endeavor to make this statement precise.
EXAMPLE 4. Likewise, we can ask what relation of special functions is induced by the following elementary identity

\[(1 - x)^{-b_1}(1 - y)^{-b_2} = \left(\frac{y}{x}\right)^{-b_2} (1 - x)^{-b_1} \left(1 - \frac{1 - \frac{y}{x}}{1 - x}\right)^{-b_2}.\]

Changing the variables to \(x \to tx, y \to ty\) we obtain

\[(1 - tx)^{-b_1}(1 - ty)^{-b_2} = \left(\frac{y}{x}\right)^{-b_2} (1 - tx)^{-b_1} \left(1 - \frac{1 - \frac{y}{x}}{1 - tx}\right)^{-b_2}.\]

Applying \(H^c(t)\) to both sides yields

\[F_1\left(\frac{a}{c}; b_1, b_2; tx, ty\right) = \left(\frac{y}{x}\right)^{-b_2} \frac{H^c(t)(1-tx)^{-b_1}}{H^c(t)(1-ty)^{-b_2}} F_1\left(b_1 + b_2, b_2, 1 - \frac{y}{x} ; \frac{b_1 + b_2}{tx}, 1 - \frac{x}{y}\right).\]

Altogether we find the following known relationship between Appell’s \(F_1\) and \(F_2\) function:

\[F_1\left(\frac{a}{c}; b_1, b_2; x, y\right) = \left(\frac{y}{x}\right)^{-b_2} F_2\left(b_1 + b_2, b_2, \frac{a}{c}; b_1 + b_2, 1 - \frac{x}{y}\right).\]

3.3. Horn’s functions. Similarly, we can deal with other multi-variable hypergeometric functions. Including the Appell’s functions there are altogether 28 function on Horn’s list (see [1]). \(G\)-family of functions is defined as follows:

\[G_1\left(\frac{a}{c}; b_1, b_2; x, y\right) = \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}}{j!k!} (b_1)^j (b_2)^k x^j y^k,\]

\[G_2\left(\frac{a}{c}; b_1, b_2; x, y\right) = \sum_{j=0}^{\infty} \frac{(a)_{j-k}}{j!k!} (b_1)^j (b_2)^k x^j y^k,\]

\[G_3\left(\frac{a}{c}; x, y\right) = \sum_{j,k=0}^{\infty} \frac{(a)_{2j-k}}{j!k!} x^j y^k.\]

We are able to give a representation for \(G_2\):

PROPOSITION 4. For generic values of \(a, c, b_1, b_2 \in \mathbb{C}\) it holds:

\[G_2\left(\frac{a}{c}; b_1, b_2; x, y\right) = \frac{b_1}{\mathcal{H}_c(x)} \frac{b_2}{\mathcal{H}_c(y)} (1 + y)^{-c} (1 + x)^{-a} (1 - xy)^{c+a-1}.\]

Therefore the double sum \(G_2\) converges for

\[|x| < 1, \quad |y| < 1, \quad |xy| < 1,\]

Proof. To prove hypergeometric representation of \(G_2\) and also its region of convergence, all we have to do is to show that

\[G_2\left(\frac{a}{c}; 1 - c, 1 - a; x, y\right) = (1 + y)^{-c} (1 + x)^{-a} (1 - xy)^{c+a-1}.\]

Starting with

\[G_2\left(\frac{a}{c}; 1 - c, 1 - a; x, y\right) = \sum_{j,k=0}^{\infty} \frac{(a)_{j-k}}{j!k!} (1 - c)^j (1 - a)^k x^j y^k,\]
and using the identities
\[ (a)_{j-k} = \frac{(a)_j}{(1 - a - j)_k} (-1)^{j-k}, \quad (c)_{k-j} = \frac{(-1)^j (c-j)_k}{(1 - c)_j}, \]
we obtain
\[ = \sum_{j,k=0}^{\infty} \frac{(a)_j (c-j)_k (1 - a)_k}{j! (1 - a - j)_k k!} (-x)^j (-y)^k = \sum_{j=0}^{\infty} \frac{(a)_j}{j!} (-x)^j_2 F_1 \left( \begin{array}{c} c - j - 1 - a \n 1 - a - j \end{array} ; -y \right) \]
\[ = (1+y)^{-c} \sum_{j=0}^{\infty} \frac{(a)_j}{j!} (-x)^j_2 F_1 \left( \begin{array}{c} 1 - a - c - j \n 1 - a - j \end{array} ; -y \right) = (1+y)^{-c} \sum_{j,k=0}^{\infty} \frac{(a)_j}{(j+k)!} (-x)^j (1 - a - c)_k k! y^k \]
rearranging the terms \( j \to j + k \) we obtain
\[ = (1+y)^{-c} \sum_{j,k=0}^{\infty} \frac{(a)_j}{j!} (-x)^{j+k} (1 - a - c)_k k! (y)^k = (1+y)^{-c} (1+x)^{-a} (1-xy)^{a+c-1}. \]

The function \( G_1 \) can be represented via the following link with the \( F_4 \) function:

**Proposition 5.**
\[ G_1 \left( \begin{array}{c} a \n b_1 \ b_2 ; x, y \end{array} \right) = (1 + x + y)^{-a} F_4 \left( \begin{array}{c} a \n b_1 - b_2 \ n 1 - b_1 \ b_2 ; 1 - b_1 \ 1 - b_2 ; 1 + x + y \end{array} ; \frac{x}{1 + x + y} \right), \]

which we state without proof only as a curiosity. At the moment the author is not aware of any simple representation of the \( G_3 \) functions.

There are more functions from the Horn’s list that have very nice representation, namely the \( H_4 \) function and functions \( \Phi_1, \Phi_2, \Phi_3 \) defined as

\[ H_4 \left( \begin{array}{c} a \n b \ c \ d ; x, y \end{array} \right) := \sum_{j,k=0}^{\infty} \frac{(a)_{2j+k}}{j! k!} \frac{(b)_k}{(c)_{j+k} (d)_k} x^j y^k, \]
\[ \Phi_1 \left( \begin{array}{c} a \n b \ c ; x, y \end{array} \right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}}{j! (c)_{j+k} k!} x^j y^k, \]
\[ \Phi_2 \left( \begin{array}{c} b_1 \ n b_2 \ c ; x, y \end{array} \right) := \sum_{j,k=0}^{\infty} \frac{(b_1)_j (b_2)_k}{(c)_{j+k} j! k!} x^j y^k, \]
\[ \Phi_3 \left( \begin{array}{c} b \ n c \ d ; x, y \end{array} \right) := \sum_{j,k=0}^{\infty} \frac{(b)_j}{(c)_{j+k} k!} x^j y^k. \]

**Proposition 6.** For generic values of parameters it holds:

\[ \frac{b}{c} \mathcal{H}(y) \mathcal{H}(x) (1 - y)^2 - 4x)^{-\frac{b}{c}} = H_4 \left( \begin{array}{c} a \n b \ c \ d ; x, y \end{array} \right), \]
\[ \mathcal{H}(t) \mathcal{H}(x) (1 - ty)^{-b} = \Phi_1 \left( \begin{array}{c} a \n b \ c ; x, y \end{array} \right), \]
\[ \mathcal{H}(t) \mathcal{H}(x) e^{(c-x-y)} = e^{-ty} \Phi_2 \left( \begin{array}{c} a \n b_1 \ b_2 \ c ; x, y \end{array} \right), \]
\[ \mathcal{H}(t) \mathcal{H}(x) \cosh(2\sqrt{y})e^{-tx} = e^{-tx} \Phi_3 \left( \begin{array}{c} a \n b \ c ; x, y \end{array} \right). \]

**Proof.** For the first three representations it suffices to establish the following special cases:
\[ H_4 \left( \begin{array}{c} a \n b + 1 \ c + 1 \ d ; x, y \end{array} \right) = ((1 - y)^2 - 4x)^{-\frac{b}{c}}. \]
Clearly, it is enough to check the claim on monomials.

\[ e^{-ty} \Phi_2 \left( \frac{b_1 - b_2}{b_1 + b_2} ; tx, ty \right) = \Phi_1 \left( \frac{b_1}{b_1 + b_2} ; t(x - y) \right), \]

which are left to the reader. The last representation can be proved as follows:

\[ e^{-tx} \Phi_3 \left( \frac{b}{c} ; tx, ty \right) = e^{-tx} \sum_{j, k} \frac{(by)^j (ty)^k}{(c)_{j+k} j! k!} = e^{-tx} \sum_k \frac{(ty)^k}{(c)_{k} k!} H_1 \left( \frac{b}{c + k} ; xt \right). \]

Using the well known Kummer transform

\[ 1F_1 \left( \frac{a}{c} ; x \right) = e^x 1F_1 \left( \frac{c-a}{c} ; -x \right), \]

we obtain

\[ e^{-tx} \Phi_3 \left( \frac{b}{c} ; tx, ty \right) = e^{-tx} \sum_k \frac{(ty)^k}{(c)_{k} k!} H_1 \left( \frac{c-b+k}{c+k} ; xt \right) = e^{-tx} \sum_k \frac{H_1 \left( \frac{c-b+k}{c+k} ; xt \right)}{(c)_{k} k!} \]

\[ = e^{-tx} \sum_k \frac{(ty)^k}{(c)_{k} k!} e^{-xt} = \frac{H_1 \left( \frac{c-b}{c} ; ty \right)}{c-b} \sum_k \frac{(ty)^k}{(c)_{k} k!} e^{-xt} \]

\[ = \frac{H_1 \left( \frac{c-b}{c} ; ty \right)}{c-b} \cosh (2\sqrt{yt}) e^{-xt}. \]

\[ \square \]

### 4. Pfaff Property

**Proposition 7.** Let

\[ y(x) := \frac{x}{x - 1}. \]

Then

\[ (1 - x)^{a+b} \frac{H_c(x)}{H_c(-x)} (1 - x)^{-c} = \frac{a}{c}(y). \]

**Proof.** Clearly, it is enough to check the claim on monomials.

\[ (1 - x)^{a+b} \frac{H_c(x)}{H_c(-x)} y(x)^n = (1 - x)^{a+b} \frac{H_c(-x)}{H_c(-x)} (1 - x)^{-c-n} (2.6) \]

\[ = (1 - x)^{a+b} \frac{H_c(x)}{H_c(-x)} y(x)^n = (1 - x)^{a+b} \frac{H_c(x)}{H_c(-x)} (1 - x)^{-c-n} \]

\[ = (1 - x)^{a+b} \frac{H_c(x)}{H_c(-x)} y(x)^n = (1 - x)^{a+b} \frac{H_c(x)}{H_c(-x)} y(x)^n. \]

\[ \square \]

**Example 5.** A consequence of the following elementary identity

\[ (1 - x)^{-b} = (1 - x)^{-c} \left( 1 + \frac{x}{1 - x} \right)^{b-c} = (1 - x)^{-c} (1 - y)^{b-c}, \quad y := \frac{x}{x - 1}, \]

is a well known identity called “Pfaff transform” [28, 15.8.1]:

\[ 2F_1 \left( \frac{a}{c} ; x \right) = (1 - x)^{-a} 2F_1 \left( \frac{a-b}{c} ; \frac{x}{x - 1} \right). \]  

(Pfaff transform.)

The argument is as follows:

\[ 2F_1 \left( \frac{a}{c} ; x \right) \stackrel{(1.2)}{=} \frac{H_c(x)}{H_c(-x)} \frac{a}{c}(1 - x)^{-b} = \frac{a}{c}(1 - x)^{-c} \left( 1 + \frac{x}{1 - x} \right)^{b-c} \]
\[ (1 - x)^{-a} \frac{a}{c} (1 - y)^{b-c} \]
\[ = (1 - x)^{-a} \, _2F_1 \left( \frac{a}{c}, \frac{c - b}{c} ; \frac{x}{x-1} \right). \]

Notice that this transform applied twice lead back to the original function. In other words, the Pfaff transform is an involution. There is an additional obvious involution related to the fact that the function \( _2F_1 \) is symmetrical with respect to the upper parameters \( a, b \):

\[ (4.3) \quad _2F_1 \left( \frac{a}{c}, \frac{b}{c} ; x \right) = _2F_1 \left( \frac{b}{c}, \frac{a}{c} ; x \right). \quad \text{(Parameter swap.)} \]

If we combine these – i.e. we first perform Pfaff transforms, then swap the upper parameters and then Pfaff transform again, we discover new identity, called “Euler transform” [28, 15.8.1]:

\[ (4.4) \quad _2F_1 \left( \frac{a}{c}, \frac{b}{c} ; x \right) = (1 - x)^{c-a-b} \, _2F_1 \left( \frac{c-a}{c}, \frac{c-b}{c} ; x \right). \quad \text{(Euler transform.)} \]

**Example 6.** The same argument can be used to derive similar transform for the \( _1F_1 \) Appell’s function. Starting from the identity

\[ (1 - tx)^{-b_1} (1 - ty)^{-b_2} = (1 - tx)^{-c} \left( 1 - \frac{tx}{tx - 1} \right)^{b_1 + b_2 - c} \left( 1 - \frac{tx}{tx - 1} \cdot \frac{x - y}{x} \right)^{-b_2}, \]

we apply \( \mathcal{H}_c^a(t) \) on both sides to get:

\[ (LHS) = \frac{a}{c} \mathcal{H}(t)(1 - tx)^{-b_1} (1 - ty)^{-b_2} \]
\[ = _1F_1 \left( \frac{a}{c} ; \frac{b_1}{c} \right) ; tx, ty \right). \]

\[ (RHS) = \mathcal{H}(t)(1 - tx)^{-c} \left( 1 - \frac{tx}{tx - 1} \right)^{b_1 + b_2 - c} \left( 1 - \frac{tx}{tx - 1} \cdot \frac{x - y}{x} \right)^{-b_2} \]
\[ = \frac{a}{c} \mathcal{H}(tx)(1 - tx)^{-c} \left( 1 - \frac{tx}{tx - 1} \right)^{b_1 + b_2 - c} \left( 1 - \frac{tx}{tx - 1} \cdot \frac{x - y}{x} \right)^{-b_2} \]
\[ = (1 - tx)^{-a} \mathcal{H}(t) \left( \frac{a}{c} ; \frac{c - b_1 - b_2}{c} \right) ; z, \frac{x - y}{x} \right). \]

Putting \( t = 1 \) we thus obtain:

\[ (4.5) \quad _1F_1 \left( \frac{a}{c} ; \frac{b_1}{c} \right) \cdot \frac{x, x}{x, x-1} \right) = (1 - x)^{-a} _1F_1 \left( \frac{a}{c} ; \frac{c - b_1 - b_2}{c} \right) \cdot \frac{x, x}{x, x-1} \right). \]

**Example 7.** Generally, we can use the identity

\[ \prod_{i=1}^{n}(1 - tx_i)^{-b_i} = (1 - tx_1)^{-c} \left( 1 - \frac{tx_1}{tx_1 - 1} \right)^{b_1 + \ldots + b_n - c} \prod_{i=2}^{n} \left( 1 - \frac{tx_1}{tx_1 - 1} \cdot \frac{x_i - x_{i-1}}{x_i - 1} \right)^{-b_i}, \]

to obtain

\[ (4.6) \quad _1F_1 \left( \frac{a}{c} ; \frac{b}{c} ; tx \right) = (1 - x)^{-a} _1F_1 \left( \frac{a}{c} ; \frac{c - \sum b_i}{c} \right) \cdot \frac{x_1}{x_1 - 1} \cdot \frac{x_1 - x_2}{x_1 - 1} \ldots \frac{x_1 - x_n}{x_1 - 1}, \]

where the \( _1F_1 \) function is the multivariate generalization of \( _1F_1 \) Appell’s function defined by

\[ F_1 \left( \frac{a}{c} ; \frac{b}{c} ; tx \right) := \frac{a}{c} \mathcal{H}(t)(1 - tx)^{-b_1} \ldots (1 - tx_n)^{-b_n}, \]
where \( b, x \in \mathbb{R}^n \) such that \( b := (b_1, \ldots, b_n), \ x := (x_1, \ldots, x_n) \). Notice that \( n = 1 \) corresponds to Gauss’s hypergeometric function \( _2 F_1 \) and \( n = 2 \) corresponds to \( _3 F_1 \) Appell’s function. Details are left to the reader.

**Example 8.** Perhaps surprisingly, we can also derive a quadratic transform for \( _2 F_1 \). Using

\[
(1 - 2x)^{-b} = (1 - x)^{-2b} \left( 1 - \left( \frac{x}{1 - x} \right)^2 \right)^{-b},
\]

we have

\[
_2 F_1 \left( \begin{array}{c} a \ b \\ 2b \end{array} ; 2x \right) = \frac{(1 - x)^{-a} H_b^a(y) (1 - y^2)^{-b}}{2} = \frac{(1 - x)^{-a} H_b^a(y^2)}{(1 - y^2)^{-b}} \left( 1 - \frac{x}{1 - x} \right)^{2b},
\]

\[
F \left( \begin{array}{c} 1 \ b \\ 2b \end{array} ; 2x \right) = \frac{(1 - x)^{-a} H_b^a(y) (1 - y^2)^{-b}}{2} = \frac{(1 - x)^{-a} H_b^a(y^2)}{(1 - y^2)^{-b}},
\]

\[
_2 F_1 \left( \begin{array}{c} a + 1 \ 2b + 1 \\ 2b + 1 \end{array} ; y^2 \right) = \frac{1}{2} (1 - x)^{-a} H_b^a(y^2) \left( 1 - y^2 \right)^{-a}. 
\]

Thus we obtained a well-known identity:

\[
_2 F_1 \left( \begin{array}{c} a \ b \\ 2b \end{array} ; 2x \right) = (1 - x)^{-a} _2 F_1 \left( \begin{array}{c} a + 1 \ b + \frac{1}{2} \\ 2b + 1 \end{array} ; \left( \frac{x}{1 - x} \right)^2 \right). 
\]

**Example 9.** A similar elementary identity for the third power, i.e.

\[
(1 - zx)^{-b} = (1 - x)^{-3b} \left( 1 + \left( \frac{x}{1 - x} \right)^3 \right)^{-b}, \quad z + \bar{z} = 3, \ z\bar{z} = 3,
\]

does not give us a cubic transform of \( _2 F_1 \) but \( _3 F_1 \) to \( _3 F_2 \) reduction, i.e. taking \( H_{3b}^a \) of both sides we get:

\[
F_3 \left( \begin{array}{c} a \ b \\ 3b \end{array} ; zx, \bar{z}x \right) = (1 - x)^{-a} _3 F_2 \left( \begin{array}{c} a + 1 \ b + 1 \\ 3b + 1 \end{array} ; \left( \frac{x}{1 - x} \right)^3 \right).
\]

Again, the details are left to the reader.

**Example 10.** Once again, we can attempt to generalize this result to multivariate \( F_1 \) function. From:

\[
\prod_{i=1}^{n-1} (1 - (z_i x))^{-b} = (1 - x)^{-nb} \left( 1 - \left( \frac{x}{x - 1} \right)^n \right)^{-b}, \quad z_k := e^{\frac{\pi i}{n}},
\]

we get the following identity:

\[
F_1 \left( \begin{array}{c} a \ b \\ nb \end{array} ; (1 - z_1 x), \ldots, (1 - z_{n-1} x) \right) = (1 - x)^{-a} _n F_{n-1} \left( \begin{array}{c} a + \cdots + \frac{n-1}{n} \ b + \cdots + \frac{n-1}{n} \\ b + \cdots + \frac{n-1}{n} \end{array} ; \left( \frac{x}{x - 1} \right)^n \right).
\]

**Example 11.** Furthermore, with the aid of the Pfaff property (4.1) we can establish an alternative representation for \( F_1 \) function involving only single use of hypergeometrization.

\[
F_1 \left( \begin{array}{c} a \ b \ c \\ c \ b \ end{array} ; x, y \right) = \frac{b_1}{c - b_2} H_{c - b_2}(x) (1 - x)^{-a} _2 F_1 \left( \begin{array}{c} a \ b_2 \\ c \end{array} ; \left( \frac{y - x}{1 - x} \right) \right).
\]
The argument is as follows:

\[ F_1 \left( \frac{a}{c} ; \frac{c-b_2}{b_2}; x, y \right) \overset{(4.5)}{=} (1-x)^{-a} F_1 \left( \frac{a}{c} ; \frac{0}{c-b_2}; \frac{x}{x-1}, \frac{y-x}{1-x} \right) = (1-x)^{-a} F_2 \left( \frac{a}{c} ; \frac{y-x}{1-x} \right). \]

Now just apply \( H_{b_2}^{c-b_2}(x) \) to both sides.

This representation allows us, for instance, to easily see that the following identity holds:

\[ \text{(4.11)} \quad F_1 \left( \frac{a}{c} ; \frac{b_1}{c-b_2}; x, 1 \right) = \frac{\Gamma(c) \Gamma(c-a-b_2)}{\Gamma(c-a) \Gamma(c-b_2)} \frac{\Gamma(c-a-b_2)}{\Gamma(c-b_2)} \quad \text{Re}(c-a-b_2) > 0. \]

Just put \( y = 1 \) and use the well known Gauss’s summation formula (see [30, 15.4.20])!

\[ 2F_1 \left( \frac{a}{c} ; \frac{b}{c}; 1 \right) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad \text{Re}(c-a-b) > 0. \]

**Example 12.** We can also obtain some transform for \( F_2 \) function. Take the following identity

\[ (1-x-y)^{-a} = (1-x)^{-a}(1-y)^{-a} \left( 1 - \frac{xy}{(1-x)(1-y)} \right)^{-a}, \]

and apply operators \( H_a^{b_1}(x) H_a^{b_2}(y) \) on both sides.

\[ (LHS) = H_a^{b_1}(x) H_a^{b_2}(y)(1-x-y)^{-a} \overset{(3.8)}{=} F_2 \left( \frac{a}{c} ; \frac{b_1}{a}; \frac{b_1}{a}; x, y \right). \]

\[ (RHS) = H_a^{b_1}(x) H_a^{b_2}(y) \left( 1 - \frac{xy}{(1-x)(1-y)} \right)^{-a} \]

\[ \overset{(2.2)}{=} H_a^{b_1}(x) H_a^{b_2}(y)(1-x)^{-a} \left( 1 - \frac{xy}{(1-x)(1-y)} \right)^{-a} \]

\[ \overset{(4.1)}{=} (1-x)^{-b_1}(1-y)^{-b_2} F_1 \left( \frac{b_1}{a} ; \frac{b_2}{a}; x \tilde{y} \right). \]

Altogether we have

\[ F_2 \left( \frac{a}{c} ; \frac{b_1}{a}; \frac{b_1}{a}; x, y \right) = (1-x)^{-b_1}(1-y)^{-b_2} F_1 \left( \frac{b_1}{a} ; \frac{b_2}{a}; \frac{x}{x-1} \frac{y}{y-1} \right). \]

**Example 13.** We will now compute the following link between \( G_2 \) and \( F_2 \) functions:

\[ \text{(4.14)} \quad G_2 \left( \frac{a}{c} ; \frac{b_1}{c-b_2}; x, y \right) = (1+x)^{-b_1}(1+y)^{-b_2} F_2 \left( \frac{1-c-a}{c} ; \frac{b_1}{1-c}; \frac{b_2}{1-a}; \frac{x}{x+1}, \frac{y}{y+1} \right). \]

Once again, there is an elementary identity in behind the transform:

\[ (1+y)^{-c}(1+x)^{-a}(1-xy)^{c+a-1} = (1+y)^{a-1}(1+x)^{c-1} \left( 1 - \frac{y}{x+1} \frac{x}{y+1} \right)^{c+a-1}. \]

To prove (4.14) simply apply \( H_a^{b_1}(x) H_a^{b_2}(y) \) on both sides of (4.15) and use the Pfaff property when appropriate.
5. Euler property

Remember that Euler transform (4.4) of $2F_1$ function can be obtained by applying the Pfaff transform (4.2) twice (with a swaping of parameters). The same procedure can be also applied on the level of hypergeoenthrization:

**Proposition 8.** Let $a, b, c \in \mathbb{C}$, such that $1 - c \not\in \mathbb{N}$. Then on functions analytic near origin it holds:

\begin{equation}
(1 - x)^{a+b-c} \frac{a}{c} \mathcal{H}(1-x)^{-b} = \frac{c-b}{c} \mathcal{H}(1-x)^{-(c-a)} \frac{a}{c-b}.
\end{equation}

_Proof._

\[ \frac{a}{c}(x) = (1 - y)^{a} \frac{a}{c}(y)(1 - y)^{-c}, \quad y := \frac{x}{x - 1}, \]

\[ (1 - x)^{-a} \frac{b}{c} \mathcal{H}(y)(1 - x)^{c} \]

\[ (1 - x)^{a+b} \frac{b}{c}(x)(1 - x)^{-c+a} \frac{a}{c}(x)(1 - x)^{c-b}. \]

This is what we want just in different form. \hfill \square

**Example 14.** Applying (5.1) on the constant function 1 we get

\[ LHS = (1 - x)^{a+b-c} \frac{a}{c} \mathcal{H}(1-x)^{-b} 1 = (1 - x)^{a+b-c} 2F_1 \left( \begin{array}{cc} a & b \\ c & \end{array} ; x \right), \]

\[ RHS = \frac{c-b}{c} \mathcal{H}(1-x)^{-(c-a)} \frac{a}{c-b} 1 = \frac{c-b}{c} \mathcal{H} (1-x)^{-(c-a)} = 2F_1 \left( \begin{array}{cc} c-b & c-a \\ c & \end{array} ; x \right), \]

which is exactly Euler transform (4.4). \hfill \square

**Example 15.** We can also derive Euler-like transform for $3F_2$ function in the form

\begin{equation}
3F_2 \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ c_1 & c_2 & \end{array} ; x \right) = (1 - x)^{\sigma} \frac{\mathcal{H}}{c_1} (1-x)^{-(c_1-a_1)} 3F_2 \left( \begin{array}{ccc} a_1 & c_2-a_2 & c_2-a_3 \\ \sigma+a_1 & c_2 & \end{array} ; x \right),
\end{equation}

where the so-called _parameter excess_ $\sigma$ is $\sigma := c_1 + c_2 - a_1 - a_2 - a_3$. Proof is done by the following argument:

\[ 3F_2 \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ c_1 & c_2 & \end{array} ; x \right) = \frac{a_1}{c_1} 2F_1 \left( \begin{array}{cc} a_2 & a_3 \\ c_2 & \end{array} ; x \right) \]

\[ = \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ c_1 & c_2 & \end{array} ; x \right) \]

\[ = \left( \begin{array}{ccc} a_1 & c_2-a_2 & c_2-a_3 \\ \sigma+a_1 & c_2 & \end{array} ; x \right) \]

\[ = (1 - x)^{\sigma} \frac{\mathcal{H}}{c_1} (1-x)^{-(c_1-a_1)} 3F_2 \left( \begin{array}{ccc} a_1 & c_2-a_2 & c_2-a_3 \\ \sigma+a_1 & c_2 & \end{array} ; x \right). \]

\hfill \square

**Corollary 1.** Let $\{c_j\}_{j \in \mathbb{Z}}, \{a_j\}_{j \in \mathbb{Z}}$ are given sequences of complex numbers. Then for any $n \in \mathbb{Z}$ it holds:

\begin{equation}
\prod_{j=1}^{n} (1 - x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} \mathcal{H}(x) = (1 - x)^{c_1-\tilde{c}_1} \left( \prod_{j=1}^{n} (1 - x)^{\tilde{c}_j} \frac{\tilde{a}_j}{\tilde{a}_{j-1}} \mathcal{H}(x) \right) (1 - x)^{a_n-\tilde{a}_{n-1}} \frac{a_n}{\tilde{a}_n} \mathcal{H}(x),
\end{equation}

An important corollary that will be useful later on is the following:

**Corollary 2.** Let $\{c_j\}_{j \in \mathbb{Z}}, \{a_j\}_{j \in \mathbb{Z}}$ are given sequences of complex numbers. Then for any $n \in \mathbb{Z}$ it holds:

\begin{equation}
\prod_{j=1}^{n} (1 - x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} \mathcal{H}(x) = (1 - x)^{c_1-\tilde{c}_1} \left( \prod_{j=1}^{n} (1 - x)^{\tilde{c}_j} \frac{\tilde{a}_j}{\tilde{a}_{j-1}} \mathcal{H}(x) \right) (1 - x)^{a_n-\tilde{a}_{n-1}} \frac{a_n}{\tilde{a}_n} \mathcal{H}(x),
\end{equation}

An important corollary that will be useful later on is the following:
and
\begin{equation}
\prod_{j=1}^{n} (1-x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} H(x) = (1-x)^{\tilde{c}_1 - c_1} \left( \prod_{j=1}^{n} (1-x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} H(x) \right) \frac{\tilde{a}_n}{a_n} (1-x)^{\tilde{a}_{n-1} - a_{n-1}},
\end{equation}
where
\[ \tilde{a}_j := a_0 + \sum_{k=1}^{j} c_k, \quad \tilde{c}_j := c_j + c_{j-1} - a_{j-1} + a_{j-2}. \]

**Remark 6.** We claim that equations (5.3), (5.4) are valid even for negative \( n \). In that case, concerned products must be interpreted as in (1.14) and, in the same way, we define
\begin{equation}
\tilde{a}_0 := a_0, \quad \tilde{a}_{-j} := a_0 - \sum_{k=1}^{j} c_{1-k}, \quad j \in \mathbb{N}.
\end{equation}

**Proof.** We are going to prove (5.3) only. The second identity (5.4) is just its inverse. There are two cases to consider.

**Case 1.** Suppose \( n \geq 0 \). Using the obvious identity
\begin{equation}
\prod_{j=1}^{n} A_j B_j = A_1 \left( \prod_{j=2}^{n} B_{j-1} A_j \right) B_n,
\end{equation}
which holds for any sequences of linear operators \( A_j, B_j \) (and in fact for any integer \( n \)) we can see that
\[ \prod_{j=1}^{n} (1-x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} H(x) \stackrel{(2.4)}{=} \prod_{j=1}^{n} (1-x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} H(x) H(x) \stackrel{(5.6)}{=} (1-x)^{\tilde{c}_1} \frac{a_1}{a_0} (1-x)^{\tilde{a}_1 - a_1} \frac{a_n}{a_n} H(x).
\]
Note that \( \tilde{a}_j - \tilde{a}_{j-1} = c_j \). Therefore we can use Euler property to obtain:
\[ \prod_{j=1}^{n} (1-x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} H(x) \stackrel{(5.1)}{=} (1-x)^{\tilde{c}_1} \frac{a_1}{a_0} H(x) \left( \prod_{j=2}^{n} (1-x)^{\tilde{a}_j - a_j} \frac{a_j}{a_{j-1}} H(x) \right) (1-x)^{\tilde{a}_{n-1} - a_{n-1}} \frac{a_n}{a_n} H(x),
\]
here we have used the fact that \( \tilde{c}_j = \tilde{a}_j - a_{j-1} + a_{j-2} - \tilde{a}_{j-2} \) since \( \tilde{a}_j - \tilde{a}_{j-2} = c_j + c_{j-1} \). Observe also that \( \tilde{c}_2 = \tilde{a}_2 - a_1 \) and \( \tilde{a}_0 = a_0 \). Thus
\[ = (1-x)^{\tilde{c}_1 - \tilde{c}_1} \left( \prod_{j=1}^{n} (1-x)^{\tilde{c}_j} \frac{a_j}{a_{j-1}} H(x) \right) (1-x)^{a_{n-1} - \tilde{a}_{n-1}} \frac{a_n}{a_n} H(x).
\]
This proves (5.3) for \( n \geq 0 \).

**Case 2.** The case \( n < 0 \) we will prove by induction. Renaming \( n = -n \) and using the definition for “negative” product (1.14) we have to show that
\[ \prod_{j=1}^{n} \frac{a_{-j}}{a_{-1-j}} (1-x)^{-\tilde{c}_{1-j}} = (1-x)^{-\tilde{c}_1} \left( \prod_{j=1}^{n} \frac{a_{-j}}{a_{-1-j}} H(x) \right) (1-x)^{-\tilde{a}_{n-1} - a_{n-1}} \frac{a_n}{a_n} H(x),
\]
for all \( n = 0, 1, 2, \ldots \). The base case \( n = 0 \) is trivial.
For the induction steps
\[ \prod_{j=1}^{n+1} \frac{a_{-j}}{a_{-1-j}} (1-x)^{-\tilde{c}_{1-j}} = \left( \prod_{j=1}^{n} \frac{a_{-j}}{a_{-1-j}} H(x) \right) (1-x)^{-\tilde{c}_1} \frac{a_{n+1}}{a_{n+1}} H(x).
\]
= (1 - x)^{c_i - \tilde{c}_i} \left( \prod_{j=1}^{n} \tilde{a}_{j-1} \tilde{H}_j(x)(1 - x)^{-\tilde{c}_i-1} \right) (1 - x)^{a_{n-1} - \tilde{a}_{n-1} - \tilde{c}_i-1} \tilde{H}_{n-1}(x)(1 - x)^{-c_{n-1}}

(2.4) \Rightarrow (1 - x)^{c_i - \tilde{c}_i} \left( \prod_{j=1}^{n} \tilde{a}_{j-1} \tilde{H}_j(x)(1 - x)^{-\tilde{c}_i-1} \right) (1 - x)^{a_{n-1} - \tilde{a}_{n-1} - \tilde{c}_i-1} \tilde{H}_{n-1}(x)(1 - x)^{-c_{n-1}}

(5.1) \Rightarrow (1 - x)^{c_i - \tilde{c}_i} \left( \prod_{j=1}^{n} \tilde{a}_{j-1} \tilde{H}_j(x)(1 - x)^{-\tilde{c}_i-1} \right) \tilde{a}_{n-1} \tilde{H}_{n-1}(x)(1 - x)^{a_{n-1} - \tilde{a}_{n-1} - \tilde{c}_i-1}

= (1 - x)^{c_i - \tilde{c}_i} \left( \prod_{j=1}^{n+1} \tilde{a}_{j-1} \tilde{H}_j(x)(1 - x)^{-\tilde{c}_i-1} \right) (1 - x)^{-\tilde{c}_n + a_{n-1} - \tilde{a}_{n-1} - \tilde{c}_i-1} \tilde{H}_{n-1}(x)

= (1 - x)^{c_i - \tilde{c}_i} \left( \prod_{j=1}^{n+1} \tilde{a}_{j-1} \tilde{H}_j(x)(1 - x)^{-\tilde{c}_i-1} \right) (1 - x)^{a_{n-2} - \tilde{a}_{n-2} - \tilde{c}_i-1} \tilde{H}_{n-1}(x),

where the last equality stems from the definition of \( \tilde{c}_n \) and \( \tilde{a}_{n-2} \). Which is what we want. Thus we have proven (5.3) for all integer \( n \).

\[ \square \]

**Example 16.** For \( c_j = a_j - a_{j-1} \) it holds

\[ \tilde{c}_j = c_j, \quad \tilde{a}_j = a_j, \]

and equality (5.3) is a simple identity.

\[ \square \]

**Example 17.** If \( c_j = a_j - a_{j-1} + \alpha \) for some fixed \( \alpha \in \mathbb{C} \) we have

\[ \tilde{c}_j = c_j + \alpha, \quad \tilde{a}_j = a_j + \alpha. \]

Notice that \( \tilde{c}_j = \tilde{a}_j - \tilde{a}_{j-1} + \alpha \). We can therefore repeat the process. If we do it \( m \) times we obtain the following identity:

(5.7) \[
\prod_{j=1}^{n} (1 - x)^{c_j} \frac{a_j}{a_{j-1}} = (1 - x)^{-\alpha m} \left( \prod_{j=1}^{n} (1 - x)^{c_j + m \alpha} \frac{a_j + m \alpha}{a_{j-1} + m \alpha (j-1)} \right) \left( \prod_{k=1}^{m} (1 - x)^{-\alpha (n-1) m} \frac{a_n + (m-1) \alpha m}{a_n + (m+1-1) \alpha m} \right).
\]

Now, if we solve for the first product on the right by multiplying by the inverse of the second product from the right and then by the factor \( (1 - x)^{\alpha m} \) from the left and then rename the sequences \( c_j \to c_j - m \alpha \) and \( a_j \to a_j - m \alpha j \), we obtain an inverse expression which reads:

(5.8) \[
\prod_{j=1}^{n} (1 - x)^{c_j} \frac{a_j}{a_{j-1}} = (1 - x)^{-\alpha m} \left( \prod_{j=1}^{n} (1 - x)^{c_j - m \alpha} \frac{a_j - m \alpha j}{a_{j-1} - m \alpha (j-1)} \right) \left( \prod_{k=1}^{m} a_n + (k-1) \alpha m (1 - x)^{\alpha (n-1)} \right).
\]

But observe this is exactly the same formula which we would get if we put \( m = -m \) into (5.7) and interpret the product as usual (see (1.14)).

Therefore the formula (5.7) is in fact true for all integers \( m \in \mathbb{Z} \).

\[ \square \]

6. Change of coordinates

The Pfaff property (4.1) along with scaling of the argument (2.8) and argument’s power law (2.10), i.e. the following list:

(2.8) \[ \frac{a}{c} \tilde{H}(x) = \frac{a}{c} \tilde{H}(y), \quad y = \alpha x. \]

(2.9) \[ \frac{a}{c} \tilde{H}(x) = \frac{a}{c} \tilde{H}(y) \frac{a+1}{c+1} \tilde{H}(y), \quad y = x^2. \]
can be viewed as instances of change of variable \( x \to y \). Are there any more? Obviously, we can produce additional identities just by combining (4.1), (2.8) and (2.10), for example:

\[
\begin{align*}
(6.1) & \quad (1-x^2)^{\frac{a}{c}} \mathcal{H}(x) (1-x)^{-\frac{c}{2}} = \mathcal{H} \left( \frac{y}{2} \right) (1-y)^{-\frac{c}{2}} \mathcal{H} \left( \frac{y}{2} \right), \quad y = \frac{x^2}{4(x-1)}, \\
(6.2) & \quad \left( 1-\frac{x}{2} \right)^{\frac{a}{c}} \mathcal{H}(x) \left( 1-\frac{x}{2} \right)^{-\frac{c}{2}} = \mathcal{H} \left( \frac{y}{2} \right) (1-y)^{-\frac{c}{2}} \mathcal{H} \left( \frac{y}{2} \right), \quad y = \frac{x^2}{2(x-1)}, \\
(6.3) & \quad (1-x^2)^{\frac{a+1}{c}} \mathcal{H}(x) (1-x^2)^{\frac{c}{2}} = \mathcal{H} \left( \frac{y}{2} \right) (1-y)^{-\frac{c}{2}} \mathcal{H} \left( \frac{y}{2} \right), \quad y = \frac{x^2}{x^2-1}.
\end{align*}
\]

For the proof, define the following functions:

\[
\begin{align*}
S_\alpha(x) &= \alpha x, & \text{Scaling}, \\
M_\alpha(x) &= x^\alpha, & \text{Power}, \\
P(x) &= \frac{x}{x-1}, & \text{Pfaff}.
\end{align*}
\]

Their properties are:

\[
\begin{align*}
S_\alpha \circ S_\beta &= S_{\alpha\beta}, & S_1 &= \text{Id}, \\
M_\alpha \circ M_\beta &= M_{\alpha\beta}, & M_1 &= \text{Id}, \\
P \circ P &= \text{Id}.
\end{align*}
\]

We have

\[
\frac{x^2}{x-1} = P \circ M_2(x), \quad \frac{x^2}{4(x-1)} = P \circ M_2 \circ P \circ S_{\frac{x}{4}}(x),
\]

\[
\left( \frac{x}{2-x} \right)^2 = M_2 \circ P \circ S_{\frac{x}{2}}(x).
\]

Thus the identities (6.1), (6.2), (6.3) are direct consequences of already established properties (4.1), (2.8), (2.10).

**Example 18.** Applying the identity (6.1) on the constant function 1 we get:

\[
(6.4) \quad (1-x)^{\frac{a}{c}} \mathcal{H}(x) (1-x)^{-\frac{c}{2}} = \mathcal{H} \left( \frac{y}{2} \right) (1-y)^{-\frac{c}{2}} \mathcal{H} \left( \frac{y}{2} \right),
\]

a quadratic transform for \( \mathcal{H} \) (the identity 15.8.14 in [28]).

Evidently, any composition chain of \( P, S_\alpha, M_\alpha \) functions will lead to a valid change of coordinates. For instance:

\[
\frac{x^2}{ax+b} = S_{-\frac{a}{2}} \circ P \circ M_2 \circ P \circ S_{-\frac{b}{2}}(x).
\]

A function that cannot be obtain by any finite combination of \( P, S_\alpha, M_\alpha \) is

\[
Q(x) := \frac{-4x}{(1-x)^2},
\]

but the corresponding change of variable is the following:

**Proposition 9.** Let \( \beta := \frac{a+c-1}{2} \). Then it holds:

\[
(6.5) \quad (1-x)^{2\beta} \mathcal{H}(x) (1-x)^{-2\beta} = \mathcal{H} \left( \frac{y}{2} \right) (1-y)^{-\frac{c}{2}} \mathcal{H} \left( \frac{y}{2} \right), \quad y := \frac{-4x}{(1-x)^2}.
\]
Proof. As always, it is sufficient to prove the formula (6.5) only on powers of $y$. The proof is based on a "quadratic transform" of $2F_1$ function valid for $|x| < 1$:

$$2F_1 \left( \frac{a}{2} - \frac{b}{2} + 1; \frac{a + b}{2}; -\frac{4x}{(1-x)^2} \right), \quad |x| < 1.$$  

(6.6)

See [28, 15.8.16]. Let $1 + \alpha \in \mathbb{N}$. Then we have

$$LHS = (1 - x)^{2\beta} \frac{G(x)}{-2y^\alpha} = (1 - x)^{2\beta} \frac{G(x)}{(1 - x)^{2\beta}}(1 - x)^{-2\beta}y^\alpha(1 - x)^{-2\beta}$$

$$= 2\beta \left( \frac{a + \alpha}{a + \alpha} \right) \frac{H(x)}{(c + \alpha)}(1 - x)^{-2\beta}2F_1 \left( \frac{a + \alpha}{a + \alpha} - \frac{2\beta}{c + \alpha}; x \right)$$

(1.2)

$$= 2\beta \left( \frac{a + \alpha}{a + \alpha} \right) \frac{H(x)}{(c + \alpha)}(1 - x)^{-2\beta}2F_1 \left( \frac{a + \alpha}{a + \alpha} - \frac{2\beta}{c + \alpha}; y \right)$$

(6.6)

$$= 2\beta \left( \frac{a + \alpha}{a + \alpha} \right) \frac{H(x)}{(c + \alpha)}(1 - y)^{2\beta}y^\alpha(1 - y)^{-2\beta}H(y)(1 - y)^{-2\beta}y^\alpha$$

$$= RHS.$$  

Example 19. Using (6.5) on a constant function 1 we obtain

$$(1 - x)^{c + a - 1}2F_1 \left( \frac{c + a - 1}{c + a - 1}; x \right) = 2F_1 \left( \frac{c + a - 1}{c + a - 1}; x \right),$$

thus we recovered (6.6). (See [28, 15.8.6].)

Now, shift the parameters by $a \rightarrow a + b$, $c \rightarrow c - b$ so we have

$$(1 - x)^{c + a - 1}2F_1 \left( \frac{c + a - 1}{c + a - 1}; x \right) = 2F_1 \left( \frac{c + a - 1}{c + a - 1}; y \right),$$

and apply transform again to get

$$(1 - x)^{c + a - 1}3F_2 \left( \frac{c + a - 1}{c + a - 1}; x \right) = 3F_2 \left( \frac{c + a - 1}{c + a - 1}; y \right),$$

a quadratic formula for $3F_2$! (See [29, 15.6.1].)

Example 20. Consider the function

$$g(x) := (1 - yt)^{-\beta}, \quad y := \frac{-4x}{(1-x)^2}, \quad \beta := \frac{a + c - 1}{2}.$$  

Note that

$$1 - yt = \frac{1 - 2x(1 - 2t) + x^2}{(1 - x)^2} = \frac{(1 - \tau_+x)(1 - \tau_-x)}{(1 - x)^2},$$

where $\tau_{\pm}$ are complex numbers such that $\tau_+ + \tau_- = 2 - 4t$, $\tau_+\tau_- = 1$, i.e.

$$\tau_{\pm} := 2 \left( (2t - 1)^2 \pm \sqrt{t(t - 1)} \right).$$

Thus

$$g(x) = (1 - \tau_+x)^{-\beta}(1 - \tau_-x)^{-\beta}(1 - x)^{2\beta}.$$
Applying (6.5) on the function \( g \) we obtain:
\[
RHS = \frac{\beta}{c} \frac{H(y)}{(1-y)^{\frac{\alpha}{2}}} H(y)(1-\alpha y)^{-\beta}
\]
\[
= \frac{(1,2)}{\frac{\beta}{c}} \frac{H(y)}{(1-y)^{\frac{\alpha}{2}}} (1-\alpha y)^{-\beta} = F_1 \left( \frac{\beta}{c} ; \frac{\alpha}{2} - \frac{\alpha}{\beta} ; y, yt \right),
\]
\[
LHS = (1-x)^{2\beta} \frac{a}{c} (x) (1-x)^{-2\beta} g(x) = (1-x)^{2\beta} \frac{a}{c} (x) (1-\tau_\pm x)^{-\beta} (1-\tau_\mp x)^{-\beta}
\]
\[
= F_1 \left( \frac{a}{c} ; \beta ; 1 \right),
\]
Altogether we discover a \textit{quadratic} transform for \( F_1 \):
\[
F_1 \left( \frac{a}{c} ; \frac{a+c-1}{2} - \frac{a+c-1}{2} ; \tau_\pm x, \tau_\mp x \right) = (1-x)^{1-a-c} F_1 \left( \frac{a+c-1}{2} ; \frac{c-a}{\beta} - \frac{a}{\beta} ; -\frac{4x}{(1+x)^2} (1+x)^2 \right),
\]
where
\[
\tau_\pm := 2 \left( (2t-1)^2 \pm \sqrt{t(t-1)} \right).
\]

For more quadratic transforms of Appell’s function see [24].

We can of course consider also combinations of \( Q \) with other functions:

**Proposition 10.** Let \( \beta : \frac{a+c-1}{2} \). For generic values of \( a, c \in \mathbb{C} \) it holds:
\[
(6.8) \quad (1+x)^{2\beta} \frac{a}{c} (x) (1+x)^{-2\beta} = \frac{\beta}{c} \frac{H(y)}{(1-y)^{\frac{\alpha}{2}}} H(y),
\]
\[
y = \frac{4x}{(1+x)^2}.
\]
\[
(6.9) \quad (1-x)^{1-a-c} \frac{a}{c} (x) (1-x)^{a-1} = \frac{\beta}{c} \frac{H(y)}{(1-y)^{\frac{\alpha}{2}}} H(y),
\]
\[
y = 4x(1-x).
\]
\[
(6.10) \quad (1-x)^{1-a-c} \frac{a}{c} (x) (1-x)^{a-1} = (1-y)^{\beta} + \frac{a}{c} \frac{H(y)}{(1-y)^{\frac{\alpha}{2}}} H(y),
\]
\[
y = \frac{4x(x-1)}{(1-2x)^2}.
\]
**Proof.** These identities can be obtained, considering the following compositions:
\[
\frac{4x}{(1+x)^2} = P \circ Q(x) = Q \circ S_{-1}(x), \quad 4x(1-x) = Q \circ P(x)
\]
\[
\frac{4x(x-1)}{(1-2x)^2} = P \circ Q \circ P(x).
\]

**Example 21.** Consider the following elementary identity:
\[
(6.11) \quad (1-x)^{-3a} \left( 1 - \left( \frac{x}{x-1} \right) \right)^{-\alpha} = (1-3x(1-x))^{-\alpha}, \quad \alpha \in \mathbb{C}.
\]
Applying the operator
\[
(1-x)^{a-3a} \frac{a}{c} H_{3a-a+1} (1-x)^{a-1},
\]
on the LHS of (6.11) we get:
\[
\text{LHS of (6.11)} \rightarrow (1-x)^{a-3a} \frac{a}{c} H_{3a-a+1} (1-x)^{a-1-3a} \left( 1 - \left( \frac{x}{x-1} \right) \right)^{-\alpha}
\]
\[
\overset{(4.1)}{=} (1-x)^{-3a} Q_3 \left( \alpha + \frac{a+1}{3} \frac{\frac{a+2}{3}}{\alpha + 2-\frac{a}{3}; \alpha + \frac{3-a}{3}; \left( \frac{x}{x-1} \right)^{3}}.\right.
\]
Applying the same operator also on the RHS of (6.11) yields:

\[ \text{RHS of (6.11)} \rightarrow (1 - x)^{\alpha - 3\alpha} \frac{\partial}{\partial x} \frac{\partial^3}{\partial x^3} (1 - x)^{\alpha - 1} (1 - 3x(1 - x))^{\alpha - \alpha} \]

\[ \overset{(6.9)}{=} \frac{\partial^3}{\partial x^3} \left( y(1 - y)^{-\frac{3\alpha - 2\alpha}{3\alpha} + \frac{\alpha}{4\alpha}} \right) (1 - 3\frac{3}{4}y)^{-\alpha - \alpha} (y = 4x(1 - x)) \]

\[ = \frac{\partial^3}{\partial x^3} \left( y(1 - y)^{-\frac{3\alpha - 2\alpha}{3\alpha} + \frac{\alpha}{4\alpha}} \right) \frac{\partial^3}{\partial x^3} F_1 \left( \frac{a}{3\alpha} \alpha \frac{3}{4} \right), \quad y = 4x(1 - x). \]

**Example 22.** Putting \( 3\alpha = 2\alpha \) in (6.12) and using (3.7) we obtain a semi-cubic transform for \( F_1 \) function:

\[ (1 - x)^{-2\alpha} 2F_1 \left( \frac{\alpha}{3\alpha} \frac{2\alpha}{3} ; \left( \frac{x}{x+1} \right)^3 \right) = F_1 \left( \frac{a}{a+1} ; \frac{1}{2} \frac{2\alpha}{3} ; 4x(1 - x), 3x(1 - x) \right). \]

**Example 23.** Putting \( x = \frac{1}{2} \) into (6.13) we get the following summation formula for \( 2F_1(3/4) \):

\[ 2F_1 \left( \frac{a}{a+1} ; \frac{1}{2} \frac{3}{4} ; -1 \right) = \frac{4\Gamma \left( 1 + \frac{a}{2} \right) \Gamma \left( 2 + \frac{a}{2} \right)}{\Gamma \left( 1 + \frac{a}{2} \right) \Gamma \left( 2 + \frac{a}{2} \right)} \Gamma \left( 1 + \frac{a}{2} \right) \Gamma \left( \frac{a+1}{2} \right). \]

This follows from the identity (4.11) and a well known summation formula

\[ 2F_1 \left( \frac{a}{1 - b + a} ; -1 \right) = \frac{2^{-a} \Gamma(1 + a - b) \Gamma(a + \frac{1}{2})}{\Gamma(1 - b + \frac{a}{2}) \Gamma(a + \frac{1}{2})}. \]

See [30, 15.4.26].

It might be possible to derive the formula (6.14) from the known summation formula for \( 2F_1(-1/3) \) in [1, 2.8.53], but the author is unaware at the moment whether the two are related or not.

In a sense, there is a change of variable formula for generic function \( y \), but only when parameters \( a, c \) differ by an integer.

**Proposition 11.** Let \( y \) be analytic function near the origin such that \( y(0) = 0 \). Then for all \( a \in \mathbb{C} \) and for all \( n \in \mathbb{Z} \) it holds:

\[ \frac{\partial}{\partial a} \frac{\partial^3}{\partial x^3} (1 - x)^{\alpha - 1} (1 - 3x(1 - x))^{\alpha - \alpha} \]

\[ \overset{(6.15)}{=} \frac{\partial^3}{\partial x^3} \left( y(1 - y)^{-\frac{3\alpha - 2\alpha}{3\alpha} + \frac{\alpha}{4\alpha}} \right) \frac{\partial^3}{\partial x^3} F_1 \left( \frac{a}{3\alpha} \alpha \frac{3}{4} \right), \quad y = 4x(1 - x). \]

**Proof.** For \( n \in \mathbb{N} \) this is an easy (though tedious) consequence of the formula (2.1):

\[ \frac{\partial}{\partial a} \frac{\partial^3}{\partial x^3} (1 - x)^{\alpha - 1} (1 - 3x(1 - x))^{\alpha - \alpha} = \frac{(a + x\partial_y)x_n}{(a)_n}. \]

and the “change of variable” formula for derivatives:

\[ x\partial_x = \frac{x}{y} y\partial_y. \]
Once obtain we can invert both sides to get
\begin{equation}
\frac{a}{a+n} (x) = \left( \frac{x}{y} \right)^{1-a-n} \left( \prod_{j=1}^{n} \frac{a+n-j}{a+n+1-j} \left( \frac{1}{y} \right)^{1} \right) \left( \frac{x}{y} \right)^{a-1}.
\end{equation}

Now rename \( a \to a - n \) and we get
\begin{equation}
\frac{a-n}{a} (x) = \left( \frac{x}{y} \right)^{1-a} \left( \prod_{j=1}^{n} \frac{a-j}{a+1-j} \left( \frac{1}{y} \right)^{1} \right) \left( \frac{x}{y} \right)^{a-n-1}.
\end{equation}

This is exactly the formula (6.15) for \( n = -n \) if we interpret the product as in (1.14). Therefore (6.15) holds for every integer \( n \).

7. Proof of the main theorem

We are ready to prove Theorem 1. Let us repeat the statement.

**Theorem 1.** Let
\[ y = F_{m}(x) := 1 - (1 - x)^{m}, \quad m \in \mathbb{Z}. \]

Then assuming either
\begin{enumerate}
\item \( m \in \{-2, -1, 1, 2\}, \forall a, c \in \mathbb{C}, \text{ or } \)
\item \( \forall m \in \mathbb{Z} \setminus \{0\}, a - c \in \mathbb{Z}, \)
\end{enumerate}

it holds
\begin{equation}
\frac{a}{c} (x) = \left( \frac{mx}{y} \right)^{1-c} \left( 1 - y \right)^{1+\frac{c}{m}} \left( \prod_{j=1}^{m} (1 - y)^{-\frac{a+c-j}{m}} \right) \left( \frac{mx}{y} \right)^{a-1}.
\end{equation}

**Proof.** For \( m = 1 \) we have \( F_{1}(x) = x \) and (1.13) trivially holds.

For \( m = -1 \) we have \( F_{-1}(x) = \frac{x}{x-1} = P(x) \) and (1.13) is actually a restatement of the Pfaff property (4.1).

Cases \( m = \pm 2 \) follows from Proposition 9 since
\[ F_{2}(x) = 1 - (1 - x)^{2} = Q \circ P \circ S_{2}(x), \quad F_{-2}(x) = 1 - \frac{1}{(1 - x)^{2}} = P \circ Q \circ P \circ S_{2}(x). \]

What remains is thus to show that the formula (1.13) holds for all \( m \) when \( a - c \in \mathbb{Z} \). Note that
\[ 1 - y = (1 - x)^{m}, \quad y' = m (1 - x)^{m-1} = m (1 - y)^{1-\frac{1}{m}}. \]

Thus
\begin{equation}
\frac{a+n}{a} (x) = \left( \frac{mx}{y} \right)^{1-a} \left( \prod_{j=1}^{n} (1 - y)^{1-\frac{c+j}{m}} \right) \left( \frac{mx}{y} \right)^{a+n-1}.
\end{equation}

Remember, this holds for all integer \( n \). Not just positive. We must distinguish two cases depending on the sign of \( m \). For \( m > 0 \) we are going to apply the general version of Euler property (5.3) with \( c_{j} = 1 - 1/m, a_{j} = a + j \) altogether \( m - 1 \) times as in (5.7). Note that \( c_{j} = a_{j} - a_{j-1} - 1/m \) so \( \alpha = -1/m \).

We obtain
\begin{equation}
\frac{a+n}{a} (x) = \left( \frac{mx}{y} \right)^{1-a} \left( 1 - y \right)^{-\frac{m-1}{m}} \left( \prod_{j=1}^{m-1} \frac{a+n-j}{a+n+1-j} \right) \left( \prod_{k=1}^{m-1} (1 - y)^{1-\frac{a+j}{m}} \right) \left( \frac{mx}{y} \right)^{a+n-1}.
\end{equation}

\begin{equation}
\left( \frac{mx}{y} \right)^{1-a} \left( 1 - y \right)^{-\frac{m-1}{m}} \frac{a+n}{a} (y) \left( \prod_{j=1}^{m-1} (1 - y)^{1-\frac{a+j}{m}} \right) \left( \frac{mx}{y} \right)^{a+n-1}.
\end{equation}
$$= \left(\frac{m x}{y}\right)^{1-a} (1 - y)^{-\frac{a}{m}} \left(\prod_{j=1}^{m} (1 - y)^{\frac{a}{m} \frac{a + \frac{a j}{m}}{a + \frac{a j}{m}(j-1)}(y)}\right) \left(\frac{m x}{y}\right)^{a+n-1}.$$

Changing the notation $a \to c$ and $n \to a - c$ we can rewrite the final result as follows:

$$\frac{a+n}{a} \mathcal{H}^c (x) = \left(\frac{m x}{y}\right)^{1-c} (1 - y)^{-\frac{a + \frac{a j}{m}}{a + \frac{a j}{m}(j-1)}(y)} \left(\prod_{j=1}^{m} (1 - y)^{\frac{a + \frac{a j}{m}}{a + \frac{a j}{m}(j-1)}(y)}\right) \left(\frac{m x}{y}\right)^{a-1}.$$  

Since the crucial identity (5.7) is valid for all integer $n$, this proves (1.13) for all $a - c \in \mathbb{Z}$ in the case $m > 0$.

For $m < 0$ the proof is completely analogous. Starting again with

$$\frac{a+n}{a} \mathcal{H}^c (x) = \left(\frac{m x}{y}\right)^{1-a} \left(\prod_{j=1}^{m} (1 - y)^{\frac{a}{m} \frac{a + \frac{a j}{m}}{a + \frac{a j}{m}(j-1)}(y)}\right) \left(\frac{m x}{y}\right)^{a+n-1}.$$  

Now we apply the general version of Euler property (5.3) with $c_j = 1 - 1/m, a_j = a + j$ altogether $1 - m$ times as in (5.8). Note that $c_j = a_j - a_j - 1/m$ so $\alpha = -1/m$. We obtain

$$\frac{a+n}{a} \mathcal{H}^c (x) = \left(\frac{m x}{y}\right)^{1-a} \left(\prod_{j=1}^{m} (1 - y)^{\frac{a}{m} \frac{a + \frac{a j}{m}}{a + \frac{a j}{m}(j-1)}(y)}\right) \left(\frac{m x}{y}\right)^{a+n-1}.$$  

$$= \left(\frac{m x}{y}\right)^{1-a} \left(\prod_{j=1}^{m} \frac{a}{m} \frac{a + \frac{a j}{m}}{a + \frac{a j}{m}(j-1)}(y)(1 - y)^{-\frac{a}{m}}\right) \left(\frac{m x}{y}\right)^{a+n-1}.$$  

This is exactly the same result as before but for negative $m$. This therefore proves our result (1.13) for all integer $m$ and for parameters $a, c$ such that $a - c \in \mathbb{Z}$.

\[\square\]

**Example 24.** We can combine the function $F_\alpha$ with $S_\alpha$ and $M_\alpha$ to obtain additional interesting formulas. For instance, we can recover the following “cubic” transform: Let

$$y(x) := F_3 \circ S_{\frac{3}{2}} \circ P \circ S_2 (x) = 1 - \left(\frac{1 - x}{1 + 2x}\right)^3 = \frac{9x(1 - x^3)}{(1 - x)(1 + 2x)^3}.$$  

Then

$$1 + 2x)^{a+3c-3} (1 - x^3)^{1-c} \mathcal{H}^c (x)(1 + 2x)^{3-3a-c} (1 - x^3)^{a-1} =$$

$$= (1 - y)^{1-\frac{a}{a+2a}} \mathcal{H}^c (y)(1 - y)^{-\frac{a}{a+2a}} \mathcal{H}^c (y)(1 - y)^{-\frac{a}{a+2a}} \mathcal{H}^c (y)(1 - y)^{-\frac{a}{a+2a}}.$$
Right now, this formula holds only for $a - c \in \mathbb{Z}$. But granted it is true for all $a - c$, it should be in principle possible to obtain the following well known identity:

$$
\begin{align*}
2F1 \left( \frac{1}{3}, \frac{2}{3}; 1 ; 1 - \left( \frac{1 - x}{1 + 2x} \right)^3 \right) &= (1 + 2x) \frac{2F1}{1} \left( \frac{1}{3}, \frac{2}{3}; x^3 \right),
\end{align*}
$$

i.e. the Ramanujan’s cubic transform [28, 15.8.33]. But the author is currently unable to do so.

**Example 25.** Also let

$$
y(x) := F_3 \circ S_2 \circ P(x) = 1 - \left( \frac{1 - x}{1 + x} \right)^3 = \frac{2x(3 + x^2)}{(1 + x)^3}.
$$

Then

$$
(7.2) \quad (1 - x)^{1-c}(1 + x)^{3c+a-3} \left( 1 + \frac{x^2}{3} \right)^{1-c} \frac{H_c(x)(1 - x)^{a-1}(1 + x)^{3-3a+c} \left( 1 + \frac{x^2}{3} \right)^{-a-1}}{(1 - y)^{1-y^{c-a}},}
$$

and then replacing $a \rightarrow -a$ we obtain

$$
(7.3) \quad 2F1 \left( \frac{a}{2a} ; \frac{2x(3 + x^2)}{(1 + x)^3} \right) = (1 - x)^{1-a}(1 + x)^{-3a-3} \left( 1 + \frac{x^2}{3} \right)^{1-2a} \frac{H_{-2a}(x)(1 - x)^2 \left( 1 + \frac{x^2}{3} \right)^{-a-1}}{(1 - y)^{1-y^{c-a}}}.\]

Expanding the term $(1 - x)^2$ and performing hypergeometrization we do obtain a cubic transform of $2F1$ which can be found in [1, (2.11.39)]. This is therefore a supporting evidence for validity of Conjecture 1.

8. Acknowledgment

The author was supported by the GAČR grant no. 21-27941S and RVO funding 47813059.

**References**

[1] H. Bateman, A. Erdélyi, *Higher transcendental functions*, vol. 1, McGraw-Hill Book Co., New York, 1953.

[2] Y.L. Luke, *The special functions and their approximations*, Academic Press, 1969. MR0241700 (39 # 3039).

[3] Bailey, W.N.: *Generalized hypergeometric series*, second edition. Cambridge Mathematical Tract, No. 32. Cambridge University Press, Cambridge (1964).

[4] Slater, L.J.: *Generalized hypergeometric functions*. Cambridge University Press, Cambridge (1966).

[5] Schlosser, M.J.: *Multiple hypergeometric series: Appell series and beyond*. In: Computer Algebra in Quantum Field Theory, pp. 305–324. Springer, Vienna (2013)

[6] Appell, P.: *Sur les séries hypergéométriques de deux variables et sur des équations différentielles linéaires aux dérivées partielles*. Comptes rendus hebdomadaires des séances de l’Académie des sciences 90, 296–298 & 731–735 (1880)

[7] Horn, J.: *Hypergeometrische Funktionen zweier Veränderlichen*. Math. Ann. 105, 381–407 (1931)

[8] Kampé de Fériet, J.: *La fonction hypergéométrique*. Gauthier-Villars, Paris (1937)

[9] P. Appell, J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Gauthier-Villars, Paris 1926.

[10] Lauricella, Giuseppe (1893). *Sulle funzioni ipergeometriche a più variabili*. Rendiconti del Circolo Matematico di Palermo (in Italian) 7 (S1): 111–158. doi:10.1007/BF03012437. JFM 25.0756.01.

[11] P.W. Karlsson, H.M. Srivastava, *Multiple Gaussian hypergeometric series*, Wiley 1985.

[12] Saran, S. *Transformations of certain hypergeometric functions of three variables*. Acta Math. 1955, 93, 293–312.

[13] Luo, M.; Xu, M.; Raina, R.K. *On Certain Integrals Related to Saran’s Hypergeometric Function FK*. Fractal Fract. 2022, 6, 155. https://doi.org/10.3390/fractals6030155

[14] Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: *Hypergeometric functions and toric manifolds*. Funct. Anal. and its Appl. 23, 94–106 (1989).

[15] Beukers, F.: *Algebraic A-hypergeometric functions*. Invent. Math. 180, 589–610 (2010)

[16] Beukers, F. *Monodromy of A-hypergeometric functions*. Journal für die reine und angewandte Mathematik, vol. 2016, no. 718, 2016, pp. 183-206. https://doi.org/10.1515/crelle-2014-0054
[17] I. G. Macdonald (1995). Symmetric Functions and Hall Polynomials. 2nd edition, The Clarendon Press, Oxford University Press, New York-Oxford.

[18] P. Blaschke: Matrix calculus and related hypergeometric functions, Integral Transforms and Special Functions, 2019, 30:9, 743-773, DOI: 10.1080/10652650.2019.1617290

[19] Shpot, M.A.: A massive Feynman integral and some reduction relations for Appell functions. J. Math. Phys. 48 (12), 123512, 13pp. (2007).

[20] G. Wei and B. E. Eichinger (1993). Asymptotic expansions of some matrix argument hypergeometric functions, with applications to macromolecules. Ann. Inst. Statist. Math. 45 (3), pp. 467-475.

[21] H. Exton. Multiple hypergeometric functions and applications, Wiley, 1976

[22] H. Exton: On a hypergeometric function of four variables with a new aspect of SL-symmetry, Ann. Mat. Pura Appl. 161 (1992), 315–343.

[23] M. Englš, E. Youssi, M-harmonic reproducing kernels on the Ball, preprint. https://arxiv.org/pdf/2208.07358.pdf

[24] Carlson, B.C.: Quadratic transformations of Appell functions. SIAM J. Math. Anal. 7, 291–304 (1976)

[25] P. Blaschke: Berezin transform on harmonic Bergman spaces on the real ball, J. Math. Anal. Appl. 411 (2014), no. 2, 607-630.

[26] P. Blaschke: Hypergeometric form of Fundamental theorem of calculus, preprint, https://arxiv.org/abs/1808.04837

[27] Carlson, B. C., Shaffer, D. B.: Starlike and prestarlike hypergeometric functions. SIAM J. Math. Anal. 159, 737–745 (1984) MR0747433 (85j:30014)

https://dlmf.nist.gov/15.8

https://dlmf.nist.gov/16.6

https://dlmf.nist.gov/15.4

Mathematical Institute, Silesian University in Opava, Na Rybnicku 1, 746 01 Opava, Czech Republic
Email address: Petr.Blaschke@math.slu.cz