INDUCTIVE LIMITS OF K-THEORETIC COMPLEXES
WITH TORSION COEFFICIENTS

SØREN EILERS AND ANDREW S. TOMS

Abstract. We present the first range result for the total K-theory of $C^*$-algebras. This invariant has been used successfully to classify certain separable, nuclear $C^*$-algebras of real rank zero. Our results complete the classification of the so-called AD algebras of real rank zero.

1. Introduction

A theorem which classifies the objects of a category up to some notion of equivalence via an invariant begs naturally the question of range for the classifying invariant. In the classification theory of $C^*$-algebras by K-theoretic invariants the fundamental range-of-invariant result is the theorem of Effros, Handelman, and Shen (9), which states that the ordered groups arising as $K_0$-groups of AF algebras are exactly the dimension groups studied first by Riesz (22) and Fuchs (17). Post AF classification results for $C^*$-algebras have required invariants more complex than $K_0$ alone, yet it has typically been possible to pair such results with an Effros-Handelman-Shen type theorem establishing the range of the classifying invariant. Notable examples include (24) and (15).

The aim of the present paper is to give the first range-of-invariant result associated to the classification of certain $C^*$-algebras of real rank zero, completed by Dadarlat and Gong in (6). To obtain complete invariants for this class of non-simple $C^*$-algebras, the ordered $K_*$-group $K_0(-) \oplus K_1(-)$ must be augmented: the addition of ordered K-groups with torsion coefficients and certain natural homomorphisms between them is required. While the completeness of this invariant has been established for almost a decade in a number of cases (11, 8, 6), there have been no range results available until now.

The situation is complicated by the intricate nature of the augmented K-theory. Following (6), this invariant associates to each $C^*$-algebra $\mathfrak{A}$

2000 Mathematics Subject Classification. Primary 46L35, Secondary 46L80.
Key words and phrases. $C^*$-algebras, K-theory with coefficients, classification.
a family of groups

\[ K_0(\mathfrak{A}), K_1(\mathfrak{A}), K_0(\mathfrak{A}; \mathbb{Z}/n), K_1(\mathfrak{A}; \mathbb{Z}/n) \]

with \( n \) ranging over \{2, 3, \ldots \}, as well as an order structure on

\[ K_0(\mathfrak{A}) \oplus K_1(\mathfrak{A}) \oplus \bigoplus_{n \in \{2, 3, 4, \ldots \}} [K_0(\mathfrak{A}; \mathbb{Z}/n) \oplus K_1(\mathfrak{A}; \mathbb{Z}/n)] \]

and families of group homomorphisms

\[ \rho_n^i : K_i(\mathfrak{A}) \rightarrow K_i(\mathfrak{A}; \mathbb{Z}/n) \]
\[ \beta_n^i : K_i(\mathfrak{A}; \mathbb{Z}/n) \rightarrow K_{i+1}(\mathfrak{A}) \]
\[ \kappa_{n,m}^i : K_i(\mathfrak{A}; \mathbb{Z}/n) \rightarrow K_i(\mathfrak{A}; \mathbb{Z}/m). \]

Thus, an isomorphism of invariants amounts to a family

\[ \phi_i : K_i(\mathfrak{A}) \rightarrow K_i(\mathfrak{B}) \]
\[ \psi_i : K_i(\mathfrak{A}; \mathbb{Z}/n) \rightarrow K_i(\mathfrak{B}; \mathbb{Z}/n) \]

which preserves the order structure and intertwines all morphisms \( \rho, \beta, \kappa. \)

To keep technicalities to a minimum while staying in a class where, by the counterexamples given in \[7\] and \[5\], the full force of such an invariant is really needed, we shall concentrate on the class of so-called AD algebras of real rank zero. Recall that an AD algebra is an inductive limit of finite direct sums of matrix algebras over elements of

\[ D := \{ \mathbb{C}, C(S^1), I_{\sim 2}, I_{\sim 3}, I_{\sim 4}, \ldots \}, \]

where \( I_{\sim n} \) is the dimension drop algebra

\[ \{ f \in C([0, 1], M_n(\mathbb{C})) \mid f(0), f(1) \in \mathbb{C}1 \}. \]

Such \( C^* \)-algebras may be classified by a more manageable invariant of the form

\[ K_0(\mathfrak{A}) \otimes \mathbb{Q} \rightarrow K_0(\mathfrak{A}; \mathbb{Q}/\mathbb{Z}) \rightarrow K_1(\mathfrak{A}) \]

provided that they are of real rank zero, cf. \[4\]. Here, as we shall recall below, \( K_0(\mathfrak{A}; \mathbb{Q}/\mathbb{Z}) \) should be thought of as a kind of conglomerate of \( K_0(\mathfrak{A}; \mathbb{Z}/n) \) for all \( n \). If the torsion part of \( K_1(\mathfrak{A}) \) is annihilated by a fixed integer \( n \), then the even more manageable invariant

\[ \overline{K}_n(\mathfrak{A}) \]
\[ K_0(\mathfrak{A}) \rightarrow K_0(\mathfrak{A}; \mathbb{Z}/n) \rightarrow K_1(\mathfrak{A}) \]

suffices.

Our strategy is to establish a range result in the latter case first, then use this to derive the general result. The key technical element of the proof is a decomposition result for refinement monoids attributed to Tarski by Wehrung (\[25\]).

To illustrate our results we revisit examples of AD algebras originally considered by Dadarlat and Loring, which showed that such algebras could have isomorphic ordered \( K_* \)-groups without having isomorphic
augmented K-theory. Using our main result, we parametrize the real rank zero AD algebras with $K_*$-groups as considered in [7], and show that there are uncountably many non-isomorphic such algebras.

## 2. BUILDING BLOCKS

In this section we introduce the notion of an $n$-coefficient complex. This type of object is meant to abstract the characteristics of certain augmented K-theoretic invariants for AD algebras of real rank zero — invariants which will be reviewed in detail in section 3. We begin with some preliminaries and notation.

A graded ordered group is a graded group $G_0 \oplus G_1$ in which the $G_0$-component dominates the order in the sense that

$$
(x, y) \geq 0 \quad \text{and} \quad (x, y') \geq 0 \quad \implies \quad (x, y \pm y') \geq 0
$$

For any group $G$, we denote by $G[n]$ the subgroup of elements of $G$ annihilated by $n \in \mathbb{N}$. When $G$ is an ordered group, we denote by $I(x)$ the order ideal containing $x$. Recall the notions of unperforated and weakly unperforated groups from [20].

Let $G$ be an ordered abelian group, $H$ an abelian group, and $f : G \to H$ a surjective group homomorphism. Say that $h \in H$ is positive if it is the image of a positive element in $G$. The important and obvious feature of the order on $H$ thus defined (the so-called quotient order) is that every positive element in $H$ lifts to a positive element in $G$ ([20]).

**Definition 2.1.** Let $n \in \{2, 3, \ldots \}$. An $n$-coefficient complex $\mathcal{G}$ is an exact sequence

$$
G_0 \xrightarrow{\rho} G_n \xrightarrow{\beta} G_1
$$

of abelian groups which, setting

$$
G_* := G_0 \oplus G_1, \quad G_0^\mathbb{N} := G_0 \oplus G_n,
$$

has the following properties:

(i) $nG_n = 0$

(ii) $\ker \rho = nG_0, \text{ im } \beta = G_1[n]$.

(iii) $G_*$ and $G_0^\mathbb{N}$ are graded ordered groups restricting to the same order on $G_0$

(iv) $G_*$ has the Riesz interpolation property.

(v) $G_0 \oplus \rho(G_0)$ has the quotient order coming from $\text{id}_{G_0} \oplus \rho$

(vi) $G_0 \oplus \beta(G_n)$ has the quotient order coming from $\text{id}_{G_0} \oplus \beta$

(vii) $G_0$ is unperforated and $G_*$ is weakly unperforated.
We say that an element \((x, y, z)\) is positive in \(\mathcal{G}\) if and only if
\[
gd \ni (x, y) \geq 0 \text{ and } G_* \ni (x, z) \geq 0.
\]

A morphism \(\theta : \mathcal{G} \to \mathcal{H}\) of \(n\)-coefficient complexes is a positive ordered triple of linear maps \((\theta_0, \theta_n, \theta_1)\) such that
\[
\theta_0 : G_0 \to H_0, \; \theta_n : G_n \to H_n, \; \theta_1 : G_1 \to H_1,
\]
and the maps commute with \(\rho\) or \(\beta\) as appropriate.

We conclude this section by introducing three types of \(n\)-coefficient complexes — our so-called building blocks.

**Definition 2.2.** The

(i) \((\mathbb{C}, n)\) complex
\[
\begin{align*}
\mathbb{Z} & \xrightarrow{\rho} \mathbb{Z}/n \xrightarrow{\beta} 0 \\
\rho : 1 & \mapsto (1, \beta : 1 \mapsto 0),
\end{align*}
\]

(ii) \((\mathbb{I}_m, n)\) complex
\[
\begin{align*}
\mathbb{Z} & \xrightarrow{\rho} \mathbb{Z}/n \oplus \mathbb{Z}/m \xrightarrow{\beta} \mathbb{Z}/m \\
\rho : 1 & \mapsto (1, 0); \beta : (1, 0) \mapsto 0, (0, 1) \mapsto 1
\end{align*}
\]
and

(iii) \((C(S^1), n)\) complex
\[
\begin{align*}
\mathbb{Z} & \xrightarrow{\rho} \mathbb{Z}/n \xrightarrow{\beta} \mathbb{Z} \\
\rho : 1 & \mapsto 1, \beta : 1 \mapsto 0,
\end{align*}
\]
where \(G_0\) and \(G_*\) have the strict order coming from the first direct summand, are \(n\)-coefficient complexes.

The motivation for these defining these objects, hinted at by their very names, will be made clear in the following section.

3. K-theory with Coefficients

In this section we collect a suite of known results which together prove that \(n\)-coefficient complexes appear as the K-theory of certain \(C^*\)-algebras.

The following definitions, originating in the work of Dadarlat, Gong, and the first named author (see [26, 2.3]) are based on the observation (from [26, 2.3]) that the lattices of order ideals of \(K_0(\mathfrak{A})\) and of ideals of \(\mathfrak{A}\) are naturally isomorphic for \(C^*\)-algebras with minimal ranks.
Definition 3.1. Let $\mathfrak{A}$ be a $C^*$-algebra of real rank zero and stable rank one. When $I$ is an order ideal of $K_0(\mathfrak{A})$ we define

$$[I] = \iota_*(K_1(\mathfrak{I}))$$ and $$[J] = \iota_*(K_0(\mathfrak{I}; \mathbb{Z}/n)),$$

where $\mathfrak{I}$ is the unique ideal of $\mathfrak{A}$ with $I = \iota_*(K_0(\mathfrak{I}))$ and $\iota : \mathfrak{I} \hookrightarrow \mathfrak{A}$ is the inclusion map.

We now equip $K^*_n(\mathfrak{A}) = K_0(\mathfrak{A}) \oplus K_1(\mathfrak{A})$ and $K^0_n(\mathfrak{A}) = K_0(\mathfrak{A}) \oplus K_0(\mathfrak{A}; \mathbb{Z}/n)$ with the orders given by

$$(x, y) \geq 0 \Rightarrow \begin{cases} x \geq 0 \\ y \in [I(x)] \end{cases}$$

and

$$(x, z) \geq 0 \Rightarrow \begin{cases} x \geq 0 \\ z \in [I(x)] \end{cases},$$

respectively.

It is well known (cf. [14]) that the order thus defined on $K_0(\mathfrak{A})$ will coincide with the standard order on $K_0(\mathfrak{A})$ derived from the isomorphism

$$K_0(\mathfrak{A}) \cong K_0(\mathfrak{A} \otimes C(S^1))$$

In general, the order on $K^0_n(\mathfrak{A})$ will not be the one similarly derived from the isomorphism

$$K^0_n(\mathfrak{A}) \cong K_0(\mathfrak{A} \otimes \mathbb{I}^+_n) \cong K_0(\mathfrak{A} \otimes \mathbb{I}^+_n) \cong K_0(\mathfrak{A} \otimes \mathbb{I}^+_n).$$

But since, as seen in [4], these two order structures allow the same positive group isomorphisms for a large class of $C^*$-algebras including the $AD$ algebras, the choice of order structure for the invariant has no influence on the associated classification results.

Proposition 3.2. Let $\mathfrak{A}$ be a $C^*$-algebra of real rank zero and stable rank one. Assume that $K_*(\mathfrak{A})$ is weakly unperforated, and that $K_0(\mathfrak{A})$ is unperforated. For any $n \in \{2, 3, \ldots\}$,

$$K_n(\mathfrak{A}) : K_0(\mathfrak{A}) \longrightarrow K_0(\mathfrak{A}; \mathbb{Z}/n) \longrightarrow K_1(\mathfrak{A})$$

is an $n$-coefficient complex.

Proof: We verify properties (i) – (vii) from Definition 2.1.

The purely algebraic properties (i) and (ii) hold true for any such sequence, cf. [23]. Furthermore, it is clear from our definition of the order on $K_*(\mathfrak{A})$ and $K^0_n(\mathfrak{A})$ that they are graded order groups based on the same order on $K_0(\mathfrak{A})$. Since we have noted that we are in fact working with the standard order on $K_*(\mathfrak{A})$, [3] or [14] show that condition (iv) is met.
Inspection of the diagram
\[
\begin{array}{cccccc}
K_0(\mathcal{J}) & \overset{\rho}{\longrightarrow} & K_0(\mathcal{J};\mathbb{Z}/n) & \overset{\beta}{\longrightarrow} & K_1(\mathcal{J}) \\
\downarrow \iota_* & & \downarrow \iota_* & & \downarrow \iota_* \\
K_0(\mathcal{A}) & \overset{\rho}{\longrightarrow} & K_0(\mathcal{A};\mathbb{Z}/n) & \overset{\beta}{\longrightarrow} & K_1(\mathcal{A})
\end{array}
\]

when $\mathcal{J}$ is the ideal of $\mathcal{A}$ corresponding to the order ideal $I(x)$ shows that if $G_0 \ni x \geq 0$, then $G_n \ni (x, \rho(x)) \geq 0$, and, similarly, that if $G_n \ni (x, \beta(y)) \geq 0$. To prove property (v) we look again at the diagram (1). By assumption, $y \in K_0(\mathcal{A};\mathbb{Z}/n)$ is in the image of both maps with target $K_0(\mathcal{A};\mathbb{Z}/n)$, so an easy diagram chase gives the desired result whenever $\iota_* : K_1(\mathcal{J}) \rightarrow K_1(\mathcal{A})$ is injective. But $\iota_*$ is always injective by \cite{21}. Combining this fact with the observation of the preceding paragraph, we have property (v).

For (vi), we do a similar diagram chase. Finally, we have explicitly required the properties in (vii). □

4. Decomposition Lemmas

In this section we establish some decomposition results in the spirit of Riesz for $n$-coefficient complexes. These lemmas will allow us to prove an Effros-Handelman-Shen-type result for these complexes, realising them as inductive limits of our building blocks.

We shall rely heavily on results in \cite{15} pertaining to the family of ordered $K_*(\mathcal{A})$-groups of AH algebras with real rank zero. These groups have the following property:

**Definition 4.1.** (Cf. Goodearl \cite[Lemma 8.1]{19} and Elliott \cite{15}) An ordered group $G$ is said to be weakly unperforated if

(i) whenever $mx \in G_+$ there exists $t \in \text{tor}(G)$ with $x + t \in G_+$ and $mt = 0$;

(ii) whenever $y \in G_+$, $t \in \text{tor}(G)$, and $ny + t \in G_+$ for some $n \in \mathbb{N}$, then $y + t \in G_+$.

Note that property (ii) is automatic in our case since all torsion is localized in the odd part of a graded ordered group. Although we do not apply the next observation in the sequel, we nevertheless record it for possible future use: all of the results in this section hold true if the condition of unperforation in $G_0$ in Definition 2.1(vii) is relaxed to weak unperforation.
Lemma 4.2 (Elliott ([15, Corollary 6.6])). Let \( G_* = G_0 \oplus G_1 \) be a weakly unperforated graded ordered group with the Riesz decomposition property. If \( s_1, \ldots, s_m \leq g \) where \( g \in G_0^+ \) and \( s_1, \ldots, s_m \in G_1 \), then \( g = g_1 + \cdots + g_m \) with \( g_1, \ldots, g_m \in G_0^+ \) and \( s_i \leq g_i \).

We say that a family \( H_1, \ldots, H_n \) of subgroups of a given group \( G \) is independent if
\[
\sum_{i=1}^{n} x_i = 0, x_i \in H_i \implies x_1 = \cdots = x_n = 0
\]

Lemma 4.3 (Elliott ([15, Corollary 6.3])). Let \( G_* = G_0 \oplus G_1 \) be a weakly unperforated graded ordered group with the Riesz decomposition property. Suppose \( x \leq \sum_{j=1}^{k} g_j \), where \( x \in G_1 \) and \( g_j \in G_0^+ \). Then, there exist an independent family \( H_j, j \in \{1, \ldots, k\} \), of finitely generated subgroups of \( G_1 \) such that \( H_j \leq g_j \) and a decomposition \( x = \sum_{j=1}^{k} x_j \) such that \( x_j \in H_j \).

Note that if \( x \) is as in Lemma 4.3 and has order \( m \), then each \( x_j \) has order at most \( m \) by the independence of the \( x_j \). Thus, by property \((ii)\) of Definition 2.1 if the \( G_* \) of Lemma 4.3 is in fact \( G_* \) for some \( n \)-coefficient complex \( G \) and \( x \) is in the image of \( \beta \), then so too are the \( x_j \).

Wehrung attributes the following observation to Tarski:

Lemma 4.4 (Cf. Wehrung ([25, Lemma 1.9])). Let \( G_0 \) be an ordered group with the Riesz interpolation property, and let \( a, b \in G_0 \) satisfy \( a, b \geq 0 \) and \( a \leq nb, n \in \mathbb{N} \). Then, there exist \( b_0, \ldots, b_n \geq 0 \) such that \( b = \sum_{i=0}^{n} b_i \) and \( a = \sum_{i=1}^{n} ib_i \).

Lemma 4.5. Let \( n \in \{2, 3, \ldots\} \) and let \( \overline{G} \) be an \( n \)-coefficient complex. Let \( (e, f, g) \in \overline{G} \) be a positive element and let there be given a decomposition \( e = \sum_{j=1}^{k} e_j, e_j \in G_0^+ \). Then, there exist elements
\[
g_1, g_2, \ldots, g_k \in G_1 \quad \text{and} \quad f_1, f_2, \ldots, f_k \in G_n
\]
such that
\[
(e, f, g) = \sum_{j=1}^{k} (e_j, f_j, g_j)
\]
and \( (e_j, f_j, g_j) \) is positive in \( \overline{G} \) for each \( j \in \{1, \ldots, k\} \).

Proof: Since \( \beta(f), g \leq \sum_{j=1}^{k} e_j \), there exist elements
\[
g_1, \ldots, g_k, l_1, \ldots, l_k \in G_1
\]
such that $\beta(f) = \sum_{j=1}^{k} l_j$ and $g = \sum_{j=1}^{k} g_j$ with $l_j, g_j \leq e_j$ (Lemma 4.3). As noted in the comment following that lemma, we may assume that $l_i \in \text{im} \beta$, so by $(vi)$ of Definition 2.1, the $l_j$ have $\beta$-lifts $\widetilde{l_j}$ such that $\widetilde{l_j} \leq e_j$. Thus, both $f$ and $\sum_{j=1}^{k} \widetilde{l_j}$ are majorised by $e$ and have the same image under $\beta$. We conclude that the difference $f - \sum_{j=1}^{k} \widetilde{l_j}$ is in the image of $\rho$, and is majorised by $e$. By property $(v)$ of Definition 2.1, we may choose $c = \sum_{j=1}^{k} \widetilde{l_j} + \rho(c)$, $c \in I(e)$. Since $I(e) = I(e_1) + \cdots + I(e_k)$, there is a decomposition $c = c_1 + \cdots + c_k$, $c_j \in I(e_j)$. Put $f_j = \widetilde{l_j} + \rho(c_j)$, so that $f = \sum_{j=1}^{k} f_j$. By construction, we have $f_j, g_j \leq e_j$, so that $(e_j, f_j, g_j)$ is positive in the $n$-coefficient complex for each $j \in \{1, \ldots, k\}$. \hfill $\square$

Note that the lemma above holds even when one specifies the $g_j \leq e_j$ a priori.

In the following, we will use the term refinement of a collection of elements $x_1, \ldots, x_s$ to denote a new collection of elements $\tilde{x}_1, \ldots, \tilde{x}_t$ with the property that $\{1, \ldots, t\}$ can be partitioned into $s$ subsets, such that the sum of the elements corresponding to the indices in the $j$th subset is exactly $x_j$.

**Lemma 4.6.** Fix $n \in \{2, 3, \ldots\}$ and let $\overline{G}$ be an $n$-coefficient complex. Let $(e_i, f_i, 0)$, $i \in \{1, \ldots, k\}$, be positive in $\overline{G}$. Let there be given elements $x_1, \ldots, x_r \in G_0^+$ and $z_1, \ldots, z_r \in G_1$ such that $z_j \leq x_j$, and non-negative integers $\lambda_{ij}, \delta_{ij}$, $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, r\}$, such that

$$e_i = \sum_{j=1}^{r} \lambda_{ij} x_j, \quad \beta(f_i) = \sum_{j=1}^{r} \delta_{ij} z_j.$$

Then, there exist refinements $\tilde{x}_1, \ldots, \tilde{x}_s$ of $x_1, \ldots, x_r$ and $\tilde{z}_1, \cdots, \tilde{z}_a$ of $z_1, \ldots, z_r$, and lifts $\tilde{y}_l \in G_n$ of the $\tilde{z}_l$ with $\tilde{y}_l \leq \tilde{x}_l$, having the following property: there are non-negative integers $\gamma_{il}, \kappa_{il}$, and $n_{il}$, $i \in \{1, \ldots, k\}$ and $l \in \{1, \ldots, s\}$, such that

$$e_i = \sum_{l=1}^{s} \gamma_{il} \tilde{x}_l$$

and

$$f_i = \sum_{l=1}^{s} \kappa_{il} \tilde{y}_l + n_{il} \rho(\tilde{x}_l).$$

Furthermore, $\gamma_{il} \neq 0$ whenever $n_{il} \neq 0$.

**Proof:** Following the proof of Lemma 4.5, we may assume that we have lifts $y_j$ of each $z_j$, and positive elements $c_{ij} \in I(x_j)$ such that
\[ f_i = \sum_{j=1}^{r} \delta_{ij} y_i + \rho(c_{ij}) \quad \text{and} \quad \delta_{ij} y_j + \rho(c_{ij}) \leq x_j, \quad 1 \leq j \leq r, \quad 1 \leq i \leq k. \]  

Fix \( j \). By Lemma 4.4 there is, for each \( i \in \{1, \ldots, k\} \), a decomposition \( x_j = x_{j,1} + \cdots + x_{j,k_i}, \quad k_i \in \mathbb{N}, \) such that \( c_{ij} \) is in the non-negative integral linear span of \( \{x_{j,1}, \ldots, x_{j,k_i}\} \). Choose by the Riesz property in \( G_0 \) a decomposition \( x_j = x_{j,1} + \cdots + x_{j,m_j} \), some \( m_j \in \mathbb{N}, \) which simultaneously refines all of the \( x_j = x_{j,1} + \cdots + x_{j,k_i} \) decompositions, \( 1 \leq i \leq k \). Then, there exist non-negative integers \( n_{j,1}, \ldots, n_{j,m_j}, \quad 1 \leq i \leq k \), such that \( c_{ij} = n_{j,1} x_{j,1} + \cdots + n_{j,m_j} x_{j,m_j} \). Since \( \delta_{ij} y_j + \rho(c_{ij}) \leq x_j \) we have that \( \delta_{ij} y_j \leq x_j \) (property (v) of Definition 2.1), whence \( y_j \leq x_j = x_{j,1} + \cdots + x_{j,m_j} \) (Lemma 4.2). By Lemma 4.5 there is a decomposition \( y_j = y_{j,1} + \cdots + y_{j,m_j} \) such that \( y_{j,l} \leq x_{j,l}, \quad 1 \leq l \leq m_j \). Thus,

\[ \delta_{ij} y_j + \rho(c_{ij}) = \sum_{p=1}^{m_j} \delta_{ij} y_{j,p} + n_{j,p} \rho(x_{j,p}). \]

Define \( \{\tilde{x}_1, \ldots, \tilde{x}_s\} := \cup_{j \leq r} \{x_{j,1}, \ldots, x_{j,m_j}\}, \) \( \{\tilde{y}_1, \ldots, \tilde{y}_s\} := \cup_{j \leq r} \{y_{j,1}, \ldots, y_{j,m_j}\}, \) and \( \tilde{z}_l := \beta(\tilde{y}_l) \). The lemma follows. \( \square \)

### 5. Bounded Torsion

#### 5.1. A Local Property

In this section we establish that \( n \)-coefficient complexes satisfy a local property such as the one whose importance was realized by Shen (cf. [9]) in the setting of classical dimension groups.

**Lemma 5.1.** Let \( \mathcal{G} \) be an \( n \)-coefficient complex. Let \( \mathcal{G} = \oplus_{i=1}^{n} \mathcal{G}_i \) be a direct sum of \( n \)-coefficient complex (with the direct sum order structure), where each \( \mathcal{G}_i \) is a \((\mathbb{C}, n), (\mathbb{R}^m, n)\) or \((C(S^1), n)\) complex. Let \( \overline{\theta} : \mathcal{G} \to \mathcal{G} \) be a morphism. Then, there exist an \( n \)-coefficient complex \( \overline{H} = \oplus_{j=1}^{m} \overline{H}_j \) with each \( \overline{H}_j \) a \((\mathbb{C}, n), (\mathbb{R}^m, n)\) or \((C(S^1), n)\) complex, and morphisms \( \overline{\gamma} : \mathcal{G} \to \overline{H} \) and \( \overline{\lambda} : \overline{H} \to \mathcal{G} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\overline{\gamma}} & \overline{H} \\
\downarrow{\overline{\theta}} & & \downarrow{\overline{\lambda}} \\
\mathcal{G} & \xrightarrow{\overline{\gamma}} & \overline{H}
\end{array}
\]

commutes and \( \ker \overline{\gamma} = \ker \overline{\theta} \).
Proof: Suppose that the conclusion above is relaxed to read
\[ \text{ker } \gamma = \text{ker } \theta \pmod{\rho(G_0)}, \]
with all other things being equal. Then, the original conclusion of the lemma follows. Indeed, suppose that \( a \in \text{ker } \theta \) is in the image of \( \rho \), i.e., \( a = \rho(q) \) for some \( q \in G_0^+ \). Then, \( \theta_0(q) = n \cdot q' \) for some \( q' \in G_0^+ \).

Put \( H = H \oplus R \), where \( R \) is a \((\mathbb{C}, n)\) complex, and extend \( \lambda \) to \( H' \) by sending the positive generator of \( R_0 \) to \( q' \). Apply the weakened conclusion above to find a direct sum of building block complexes \( H'' \) and maps \( \gamma'' : H' \to H'' \) and \( \lambda'' : H'' \to G \) such that the diagram commutes. Note that \( \text{ker}(\gamma'' \circ \gamma) = \text{ker } \theta \pmod{\rho(G_0)} \). Furthermore, \( a \in \text{ker}(\gamma'' \circ \gamma) \), since \((\gamma'' \circ \gamma)(q) \) must be \( \text{ord}(\rho(q)) \) times the image of the positive generator of \( R_0 \) under \( q' \). Repeating this procedure for each of the finitely many elements in \( \text{ker } \theta \cap \rho(G_0) \) yields the conclusion of the lemma proper.

It remains to prove that the lemma holds if we only require that \( \text{ker } \gamma = \text{ker } \theta \pmod{\rho(G_0)} \). Let \( G_{0i} = \langle e_i \rangle \) (\( e_i \geq 0 \)), \( G_{1i} = \langle g_i \rangle \), and choose \( f_i \in G_{ni} \) (necessarily \( \leq e_i \)) such that \( G_{ni} = \langle \rho(e_i) \rangle \oplus \langle f_i \rangle \). Note that if \( C_i \) is a \((\mathbb{C}, n)\) or \((C(S^1), n)\) complex, then we may (and do) take \( f_i = 0 \). In the case of a \((I_m, n)\) complex, \( \beta(f_i) = g_i \). Define \( a_i := \theta_0(e_i) \), \( b_i := \theta_n(f_i) \) and \( c_i := \theta_1(g_i) \). By the main Theorem in Section 5 of [15], there is a complex \( H = \oplus_{i=1}^{k} \overline{H_i} \) with each \( \overline{H_i} \) a building block (and elements \( e_i', f_i' \) and \( g_i' \) playing roles analogous to those of the \( e_i, f_i \) and \( g_i \) above), and maps
\[ \gamma_0' : G_0 \to H_0', \quad \gamma_1' : G_1 \to H_1' \]
and
\[ \lambda_0' : H_0' \to G_0, \quad \lambda_1' : H_1' \to G_1 \]

and
\[ \lambda_0' : H_0' \to G_0, \quad \lambda_1' : H_1' \to G_1 \]
such that

\[
\begin{array}{ccccccccc}
G_0 & \xrightarrow{\rho} & G_n & \xrightarrow{\beta} & G_1 \\
\downarrow{\gamma_0} & & \downarrow{\beta} & & \downarrow{\gamma_1} \\
H_0' & \xrightarrow{\rho} & H_n' & \xrightarrow{\beta} & H_1' \\
\downarrow{\lambda_0} & & \downarrow{\beta} & & \downarrow{\lambda_1'} \\
G_0 & \xrightarrow{\rho} & G_n & \xrightarrow{\beta} & G_1
\end{array}
\]

commutes and \( \ker(\theta_0, \theta_1) = \ker(\gamma_0', \gamma_1') \). To be fair, [15] only provides the \( H_0' j \) and \( H_1' j \), but we may clearly associate a building block complex to any pair \((H_0' j, H_1' j) = (\mathbb{Z}, R), R \in \{0, \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/3, \ldots\} \). Of course, this association is token for now, as we have no positive maps \( \gamma_n' : G_n \to H_n' \) and \( \lambda_n' : H_n' \to G_n \) to fill in the diagram above.

We have \( a_i = \sum_{l=1}^{k} \kappa_{i,l} \lambda_0'(e_l), \mathbb{Z} \ni \kappa_{i,l} \geq 0, \) and \( \beta(b_i) = \sum_{l=1}^{k} \phi_{i,l} \lambda_1'(g_l), \phi_{i,l} \in \mathbb{Z} \). By Lemma [14] there exist refinements \( \{\tilde{a}_1, \ldots, \tilde{a}_m\} (\tilde{a}_j \geq 0, 1 \leq j \leq m) \) and \( \{\tilde{c}_1, \ldots, \tilde{c}_m\} \) of

\[
\{\lambda_0'(e_1), \ldots, \lambda_0'(e_k)\} \text{ and } \{\lambda_1'(g_1), \ldots, \lambda_1'(g_k)\},
\]

respectively, and elements \( \tilde{b}_1, \ldots, \tilde{b}_s \in G_n \) such that for some integers \( \zeta_{ij}, \xi_{ij}, \nu_{ij} \) and \( n_{ij}, 1 \leq i \leq k, 1 \leq j \leq m, \) we have

\[
a_i = \sum_{j=1}^{m} \zeta_{ij} \tilde{a}_j,
\]

\[
b_i = \sum_{j=1}^{m} \xi_{ij} \tilde{b}_j + n_{ij} \rho(\tilde{a}_j),
\]

and

\[
c_i = \sum_{j=1}^{m} \zeta_{ij} \tilde{c}_j.
\]

Furthermore, \( \tilde{b}_j \) is a lift of \( \tilde{c}_j \) whenever \( \tilde{c}_j \) is in the image of \( \beta \), and is zero otherwise. Note that the \( \tilde{c}_j \) can be and should be chosen to be in the image of \( \beta \) whenever it is a torsion element. By the main Theorem of Section 5, [15], there is an \( n \)-coefficient complex \( \overline{H} = \bigoplus_{j=1}^{m} \overline{H}_j, \overline{H}_j \) a building block complex for each \( j \), and there are maps

\[
\gamma_0 : G_0 \to H_0, \ \gamma_1 : G_1 \to H_1
\]

and

\[
\lambda_0 : H_0 \to G_0, \ \lambda_1 : H_1 \to G_1
\]
such that (with the dotted arrows representing desired but as yet undefined maps)

\[
\begin{array}{ccccccc}
G_0 & \xrightarrow{\rho} & G_n & \xrightarrow{\beta} & G_1 \\
\downarrow{\gamma_0} & & \downarrow{\gamma_1} & & \\
H_0 & \xrightarrow{\rho} & H_n & \xrightarrow{\beta} & H_1 \\
\downarrow{\lambda_0} & & \downarrow{\lambda_1} & & \\
G_0 & \xrightarrow{\rho} & G_n & \xrightarrow{\beta} & G_1
\end{array}
\]

commutes, and \(\gamma_0\) and \(\gamma_1\) factor through \(H'_0\) and \(H'_1\), respectively. Thus, \(\ker(\theta_0, \theta_1) = \ker(\gamma_0, \gamma_1)\). Let \(\tilde{e}_j, \tilde{f}_j,\) and \(\tilde{g}_j\) play roles in \(H_j\) analogous to the roles of the \(e_i, f_i,\) and \(g_i\) in \(G_i\). Then \(\lambda_0(\tilde{e}_j) = \tilde{a}_j\) and \(\lambda_1(\tilde{g}_j) = \tilde{c}_j\).

For every pair \((i, j), 1 \leq i \leq k, 1 \leq j \leq m\), define a partial map \(\gamma_{ij}^n : G_{ni} \rightarrow H_{nj}\) by

\[
\gamma_{ij}^n(f_i) := \xi_{ij} \tilde{f}_j + n_{ij} \rho(\tilde{e}_j).
\]

Let \(\lambda_n : H_n \rightarrow G_n\) be defined by \(\lambda(\tilde{f}_j) := \tilde{b}_j\), and put \(\gamma_n = \bigoplus_{i=1}^m (\sum_{j=1}^k \gamma_{ij}^n)\).

These maps are positive and so complete the morphisms \(\lambda\) and \(\gamma\), establishing the desired weak version of the lemma. \(\square\)

5.2. The Range Result.

**Lemma 5.2.** Fix \(n \in \{2, 3, \ldots \}\), let \(\overline{G}\) be a \(n\)-coefficient complex, and consider a positive element \((e, f, g) \in \overline{G}\). There exists a \(n\)-coefficient complex \(\overline{H}\) which is a finite direct sum of \((\mathbb{C}, n), (\mathbb{I}_n, n)\) and \((C(S^1), n)\) complexes, and a positive morphism \(\overline{\theta} : \overline{H} \rightarrow \overline{G}\) such that

\[(e, f, g) \in \overline{\theta}(\overline{H}_+).\]

**Proof:** First consider the case when \(f = 0\). We define \(\overline{\theta}_g : \overline{H}_g \rightarrow \overline{G}\) by

\[1 \mapsto e; \overline{1} \mapsto \rho(e); 1 \mapsto g\]

on the \((C(S^1), n)\)-complex, and note that \((e, 0, g) = \overline{\theta}(1, 0, 1)\). Similarly, when \(g = 0\), we define \(\overline{\theta}_f : \overline{H}_f \rightarrow \overline{G}\) by

\[1 \mapsto e; (\overline{1}, 0) \mapsto \rho(e), (0, \overline{1}) \mapsto f; \overline{1} \mapsto \beta(f)\]

on the \((\mathbb{I}_n, n)\)-complex, and note that \((e, f, 0) = \overline{\theta}(1, (0, \overline{1}), 0).\)
In the general case, we consider

\[ H_f \oplus H_g \xrightarrow{\varphi_f \oplus \varphi_g} \mathcal{G} \]

with \( \gamma, \lambda \) chosen by Lemma 5.1 above. By assumption,

\[ x = \gamma_0((1, 0)) = \gamma_0((0, 1)) \]

so that \( (x, \gamma_n([0, 0], [0, 1]), \gamma_1([1, 0])) \) is a positive preimage of \((e, f, g)\). □

**Theorem 5.3.** Fix \( n \in \{2, 3, \ldots\} \), and let \( \mathcal{G} \) be an \( n \)-coefficient complex. Then, \( \mathcal{G} = \lim_{i \to \infty} \mathcal{G}_i \), where each \( \mathcal{G}_i \) is a finite direct sum of \((\mathcal{C}, n), (\mathbb{I}_m, n)\) and \((C(S^1), n)\) complexes.

**Proof:** Enumerate the positive elements of \( \mathcal{G} \) as \((e_i, f_i, g_i)\) and apply Lemmas 5.2 and 5.1 alternately to get a diagram

where \((e_i, f_i, g_i)\) is the image under \( \bar{\theta}_i \) of a positive element and \( \ker \bar{\theta}_i = \ker \gamma_i \). The maps will then induce an order isomorphism. □

An inductive system of finite direct sums of building blocks is said to have **large denominators** if all connecting morphisms either are zero on \( K_1 \) or have the \( K_0 \)-component greater than or equal to 2.

**Theorem 5.4.** Let \( n \in \{2, 3, \ldots\} \) and let \( \mathcal{G} \) be a complex. The following are equivalent

(i) \( \mathcal{G} \) is a \( n \)-coefficient complex;
(ii) $G$ is an inductive limit of finite direct sums of $(\mathbb{C}, n)$, $(\mathbb{I}_m, n)$ and $(C(S^1), n)$ complexes, and $G_\ast$ has the Riesz property;

(iii) $G$ is an inductive limit of finite direct sums of $(\mathbb{C}, n)$, $(\mathbb{I}_m, n)$ and $(C(S^1), n)$ complexes, such that the inductive system has large denominators;

(iv) $G \cong K_n(\mathfrak{A})$, where $\mathfrak{A}$ is an AD algebra of real rank zero.

Proof: Note first that $(iv) \implies (i)$ was seen in Proposition 3.2. Theorem 5.3 proves $(i) \implies (ii)$, and since the property of large denominators involves only the groups in $G_\ast$, [16, 8.1] proves $(ii) \implies (iii)$.

By compressing an inductive system such as in $(iii)$ if necessary, we may assume that each morphism among building blocks at level $i$ and level $i + 1$ is either zero on $K_1$ or greater than or equal to $M_i$ on $K_0$, where $M_i$ is the largest number for which there is an $(\mathbb{I}_m, n)$ complex among the building blocks at level $i$. Then by [11] the inductive system can be realized by direct sums of building blocks from the set \{C, C(S^1), \mathbb{I}_\gamma, \mathbb{I}_\delta, \ldots\} and *-homomorphisms among them. Furthermore, [16, 8.1] shows how to arrange for real rank zero in the limit.

By construction, the inductive limit $\mathfrak{A}$ of this $C^*$-inductive system is an AD algebra with the desired invariant

$$K_n(\mathfrak{A}) : K_0(\mathfrak{A}) \longrightarrow K_0(\mathfrak{A}; \mathbb{Z}/n) \longrightarrow K_1(\mathfrak{A}).$$

However, since we have not used – or even defined – an ideal based order on the building blocks $C(S^1)$ and $\mathbb{I}_m$, cf. Definition 3.1, we need to verify that the order on $K_n(\mathfrak{A})$ coincides with the order on $\mathfrak{C}$. Since we have used the strict order on all the algebraic building blocks this would follow directly if we knew that all ideals of $\mathfrak{A}$ arise as inductive limits or direct sums of subcollections of the building blocks in the system. And this in turn is a consequence of the minimal real rank of $\mathfrak{A}$, or directly by the construction yielding this property in [16, 8.1].

It is essential to note at this stage that the ordered complex $K_n(\mathfrak{A})$ is not complete for real rank zero AD algebras unless we know that the torsion of $K_1$ is annihilated by the number $n$. Thus, it is only in this case — covered by [11] — that Theorem 5.4 gives a one-to-one correspondence between a class of $C^*$-algebras and a class of algebraic invariants.

In this case, when the equivalent statements above hold true, we may write the AD algebra as an inductive limit using only the building blocks $C(S^1)$ and $\mathbb{I}_m$, cf. [11].
6. The general case

In the following we shall briefly recall definitions from [4]. Let $\Delta$ denote the ordered set $(\mathbb{N}, \leq)$ where $x \leq y \iff x$ divides $y$.

Note that $\Delta$ is directed, so that we may construct inductive limits over $\Delta$. We will denote these by

$$\lim_{\to \Delta} (G_p, f_{q,p})$$

where $f_{q,p} : G_p \to G_q$ are the bonding maps. When a cofinal subset $\Delta'$ of $\Delta$, is given, we may restrict attention to this, as

$$\lim_{\to \Delta} G_n \cong \lim_{\to \Delta'} G_n.$$

We define graded group homomorphisms

$$\kappa_{mn,m} : K_0(\mathfrak{A}; \mathbb{Z} \oplus \mathbb{Z}/m) \to K_0(\mathfrak{A}; \mathbb{Z} \oplus \mathbb{Z}/mn)$$

by

$$\begin{bmatrix} \chi_{mn,n} & 0 \\ 0 & \kappa_{mn,m} \end{bmatrix},$$

where $\chi_{mn,n}$ is just multiplication by $m$ between the relevant copies of $K_0(\mathfrak{A})$. The maps $\kappa_{mn,m}$ are positive, so we may define:

$$K_0(\mathfrak{A}; \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}) = \lim_{\to \Delta} (K_0(\mathfrak{A}; \mathbb{Z} \oplus \mathbb{Z}/n), \kappa_{mn,n});$$

$$K_0(\mathfrak{A}; \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})^+ = \lim_{\to \Delta} (K_0(\mathfrak{A}; \mathbb{Z} \oplus \mathbb{Z}/n)^+, \kappa_{mn,n}).$$

This gives the limit groups the structure of graded ordered groups. The even parts are naturally isomorphic to $K_0(\mathfrak{A}) \otimes \mathbb{Q}$ since

$$\lim_{\to \Delta} (G, \chi_{mn,n}) \cong \lim_{\to \Delta} (G \otimes \mathbb{Z}, \text{id} \otimes \chi_{mn,n}) \cong G \otimes \mathbb{Z} \otimes (\lim_{\to \Delta} (\mathbb{Z}/\mathbb{Z}, \chi_{mn,n})) \cong G \otimes \mathbb{Q}$$

naturally. We shall invoke this isomorphism tacitly in section 7.

The maps

$$\kappa_{mn,m} : K_0(\mathfrak{A}; \mathbb{Z} \oplus \mathbb{Z}/m) \to K_0(\mathfrak{A}; \mathbb{Z} \oplus \mathbb{Z}/mn)$$

can be described explicitly when $\mathfrak{A} \in \{ C, C(S^1), \mathbb{I}_2^\times, \mathbb{I}_3^\times, \ldots \}$ (10).

In this section we study exact sequences of abelian groups

$$G_0 \to G_0 \otimes \mathbb{Q} \to G_n \to G_1$$

which are meant to represent the natural K-theoretic invariants for AD algebras of real rank zero having unbounded torsion in $K_1$. We
begin by listing the properties that such an abstract sequence should have before one may even consider whether the sequence arises as the invariant

\[
\begin{align*}
\overline{K}_n : & \quad K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A}) \otimes \mathbb{Q} \longrightarrow K_0(\mathcal{A}/\mathbb{Z}) \longrightarrow K_1(\mathcal{A})
\end{align*}
\]

for some AD algebra \( \mathcal{A} \) of real rank zero. We shall denote the invariant consisting of two graded ordered groups and two group homomorphisms as in (2) by \( \overline{K}_n(\mathcal{A}) \) and will see that the conditions in the definition below are sufficient to ensure that the sequence

\[
\mathcal{G}_0 \longrightarrow \mathcal{G}_0 \otimes \mathbb{Q} \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{G}_1
\]

does indeed arise in such a manner.

The next definition should be compared with Definition 2.1.

**Definition 6.1.** An exact sequence

\[
\mathcal{G}_0 \longrightarrow \mathcal{G}_0 \otimes \mathbb{Q} \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{G}_1
\]

(which we denote by \( \overline{\mathcal{G}} \)) of countably generated abelian groups is an \( n \)-coefficient complex if

(i) \( \mathcal{G}_n \) is pure torsion

(ii) \( \text{im } \beta = \text{tor } \mathcal{G}_1 \), and every element \( x \in \mathcal{G}_1 \) of order \( l \) has a \( \beta \)-lift of order \( l \)

(iii) \( \mathcal{G}_* := \mathcal{G}_0 \oplus \mathcal{G}_1 \) and \( \mathcal{G}_\mathbb{Q} := \mathcal{G}_0 \otimes \mathbb{Q} \oplus \mathcal{G}_n \) are graded ordered groups inducing the same order on \( \mathcal{G}_0 \)

(iv) \( \mathcal{G}_* \) has the Riesz interpolation property.

(v) \( \mathcal{G}_0 \otimes \mathbb{Q} \oplus \rho(\mathcal{G}_0) \) has the quotient order coming from \( \text{id}_{\mathcal{G}_0 \otimes \mathbb{Q}} \oplus \rho \)

(vi) \( \mathcal{G}_0 \oplus \beta(\mathcal{G}_n) \) has the quotient order coming from \( \text{id}_{\mathcal{G}_0 \oplus \beta} \)

(vii) \( \mathcal{G}_0 \) is unperforated and \( \mathcal{G}_* \) is weakly unperforated.

Note that since \( \mathcal{G}_0 \) is torsion free, it is determined by \( \mathcal{G}_0 \otimes \mathbb{Q} \) in this setup. In the proofs below we may hence concentrate our work on the rightmost three groups in the complex.

**Proposition 6.2.** Let

\[
\overline{\mathcal{G}} : \quad \mathcal{G}_0 \otimes \mathbb{Q} \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{G}_1
\]

be an \( n \)-coefficient complex. Then, there exist a strictly increasing sequence of natural numbers \( \langle n_i \rangle \) with \( n_i | n_{i+1} \), a \( n_i \)-coefficient complex \( \overline{\mathcal{G}}^i \) for each \( i \), and positive morphisms \( \theta_i : \overline{\mathcal{G}}^i \to \overline{\mathcal{G}}^{i+1} \) such that

\[
\overline{\mathcal{G}} \simeq (\overline{\mathcal{G}}^i, \theta_i).
\]
Proof: Let \( \langle n_i \rangle \) be a strictly increasing sequence of natural numbers with the property that every natural number divides some \( n_i \). By the main theorem of section 5 of [15], the graded ordered group \( (G_0 \otimes \mathbb{Q}, G_1) \) in Definition 6.1 is the limit of an inductive sequence

\[
( (G_0^i, G_1^i), (\phi_0^i, \phi_1^i) )
\]

of graded ordered groups \( (G_0^i, G_1^i) \). Furthermore, each \( G_0^i, G_1^i \) consists of the first and third groups of an \( n_i \)-coefficient complex which is a direct sum of \( (\mathbb{C}, n_i), (\mathbb{I}_m^\sim, n_i) \) and \( (C(S^1), n_i) \) complexes. Let

\[
\phi_0^{\infty} : G_0^i \to G_0, \quad \phi_1^{\infty} : G_1^i \to G_1,
\]

be the canonical maps. Assume that the \( n_i \) have been chosen large enough for the inclusions \( G_0^i \subseteq \rho^i(G_n[n_i]) \) and \( \text{tor}(G_1^i) \subseteq \beta(G_n[n_i]) \) to hold. We then have

\[
(G_0 \otimes \mathbb{Q}, G_1) = \lim_{i \to \infty} \left( (\rho^i(G_n[n_i]), \beta(G_n[n_i]) \cup \phi_1^{\infty}(G_1^i)), \iota \right),
\]

where \( \iota \) is the inclusion map. Note that

\[
(\rho^{-1}(G_n[n_i]), \beta(G_n[n_i]) \cup \phi_1^{\infty}(G_1^i))
\]

is a graded ordered group for each natural number \( i \) — it is an order hereditary subgroup of \( (G_0 \otimes \mathbb{Q}, G_1) \). One can then verify that the complex

\[
\rho^{-1}(G_n[n_i]) \xrightarrow{\rho} G_n[n_i] \xrightarrow{\beta} \beta(G_n[n_i]) \cup \phi_1^{\infty}(G_1^i)
\]

is an \( n_i \)-coefficient complex. (The only subtle point is the exactness of the sequence, which follows from the second half of property iii of Definition 6.1.) The limit of the inductive sequence

\[
\left( \rho^{-1}(G_n[n_i]) \xrightarrow{\rho} G_n[n_i] \xrightarrow{\beta} \beta(G_n[n_i]) \cup \phi_1^{\infty}(G_1^i), \kappa_{i,i+1} \right)
\]

— \( \kappa_{i,i+1} \) is the inclusion map — is then

\[
G_0 \otimes \mathbb{Q} \xrightarrow{\rho} G_n \xrightarrow{\beta} G_1
\]

by construction. □

The following theorem is the generalised integer coefficient version of Theorem 5.3.

**Theorem 6.3.** Let \( \mathcal{G} \) be an \( n \)-coefficient complex. Then,

\[
\mathcal{G} = \lim_{i \to \infty} \langle \overline{H}_i, \overline{\gamma}_i \rangle,
\]

where each \( \overline{H}_i \) is a direct sum of \( (\mathbb{C}, n_i), (\mathbb{I}_m^\sim, n_i) \) and \( (C(S^1), n_i) \) complexes, some \( n_i \in \mathbb{N} \).
**Proof:** Assume the inductive sequence decomposition of $\mathcal{G}$ from Lemma 7.2. For brevity, write

$$H^{n_i} = \rho^{-1}(G_\mathcal{G}[n_i]) \xrightarrow{\rho} G_n[n_i] \xrightarrow{\beta} \beta(G_n[n_i]) \cup \phi_\mathcal{G}^{i,\infty}(G_\mathcal{G})$$

Put $\kappa_{l,m} = \kappa_{m-1,m} \circ \cdots \circ \kappa_{l,l+1}$. By Theorem 5.3, each $H^{n_i}$ is the limit of an inductive system $(H_\mathcal{G}[n_i], \theta^{n_i}_{k,k+1})$, where each $H_\mathcal{G}[n_i]$ is a direct sum of $(\mathbb{C}, n_i)$, $(I_{m,n_i})$ and $(S^1, n_i)$ complexes.

It will suffice to define a sequence of positive morphisms

$$\gamma_{i,i+1} : H^{n_i}_i \to H^{n_{i+1}}_{i+1}$$

making the diagram

\[ \begin{array}{ccc}
H_1 & \xrightarrow{\theta_{1,2}^{n_i}} & H_2 \\
\downarrow{\gamma_{1,2}} & & \downarrow{\gamma_{2,3}} \\
\cdots & & \cdots \\
H_1 & \xrightarrow{\theta_{1,2}^{n_i}} & H_2 \\
\downarrow{\gamma_{1,2}} & & \downarrow{\gamma_{2,3}} \\
\cdots & & \cdots \\
\end{array} \]

commute; by compressing the sequence for $H^{n_i}$ one can ensure that every positive element in $\mathcal{G}$ is the image under $\gamma_{i,\infty}$ of a positive element in $H^{n_i}_i$ for some $i \in \mathbb{N}$.

Let $\theta_{1,\infty} : H^{n_1}_1 \to H^{n_1}_1$ be the canonical morphism. Let $M$ be a minimal set of positive generators for $H^{n_1}_1$. Find, by compressing the inductive sequence for $H^{n_2}$ if necessary, a set $\tilde{M}$ of positive pre-images via $\theta_{1,\infty}^{n_2}$ of the elements of $\kappa_{1,2} \circ \theta_{1,\infty}(M)$ in $H^{n_2}_2$. Note for future reference that each element of $\kappa_{1,2} \circ \theta_{1,\infty}(M)$ is divisible by $n_{i+1}/n_i$ inside $H^{n_2}_2$, so we may assume that $m(n_{i+1}/n_i) \in H^{n_2}_2$ whenever $m \in \tilde{M}$. Define $\gamma_{i,2}$ by sending an element $m \in M$ to the corresponding pre-image of $\kappa_{1,2} \circ \theta_{1,\infty}(m)$ in $\tilde{M}$.

$\square$
The following theorem is the generalised integer coefficients version of Theorem 5.4.

**Theorem 6.4.** The following are equivalent:

(i) \( \mathcal{G} \) is a \( n \)-coefficient complex;

(ii) \( \mathcal{G} \) is an inductive limit of finite direct sums of \((\mathbb{C}, n), (\mathbb{I}_{m}, n)\) and \((C(S^1), n)\) complexes, where \( n \) ranges over the natural numbers, and \( \mathcal{G}_* \) is a Riesz group;

(iii) \( \mathcal{G} \) is an inductive limit of finite direct sums of \((\mathbb{C}, n), (\mathbb{I}_{m}, n)\) and \((C(S^1), n)\) complexes where \( n \) ranges over the natural numbers, and such that the inductive system has large denominators;

(iv) \( \mathcal{G} \cong K_n(\mathfrak{A}) \), where \( \mathfrak{A} \) is an AD algebra of real rank zero.

**Proof:** The proof follows the proof of Theorem 5.4 with the exception that in \((iii) \implies (iv)\), one needs to realize maps from \( K_n(\mathfrak{A}_i) \) to \( K_{n+1}(\mathfrak{A}_i) \) by a triple of maps of the form

\[
(\chi_{n+1,n+1} \circ f_*, \kappa_{n+1,n+1} \circ f_*, f_*)
\]

rather than directly by a \( * \)-homomorphism. However, as noted at the end of the proof of Theorem 6.4, the maps in question will have a \( K_0 \)-component which is divisible by \( n_{i+1}/n_i \), so this may be arranged as in the proof of Theorem 5.4. \( \square \)

7. **The example of Dadarlat and Loring**

It follows from the work of Bödigheimer ([1], [2]) that the unspliced short exact sequence

\[
0 \longrightarrow K_0(\mathfrak{A})/n \longrightarrow K_0(\mathfrak{A}; \mathbb{Z}/n) \longrightarrow K_1(\mathfrak{A})[n] \longrightarrow 0
\]

will always split. This has been useful in the analysis of other aspects of this object ([3], [4]) but we have not been able to employ the fact in the proofs leading to Theorem 5.4 since the splitting map is unnatural, it is difficult to use it when trying to establish the range of the invariant. By contrast, it is a useful result when trying to describe the amount of freedom one has in the choice of equipping \( \mathcal{C} \) as an \( n \)-coefficient complex when \( G_* \) is fixed, as we shall see below.

However, since Theorem 5.4 combines with the results mentioned above to prove that every \( n \)-coefficient complex will split when unspliced, it is perhaps worthwhile to note that this follows already from properties (i), (ii) and the fact (contained in (vii)) that \( G_0 \) is torsion free. One proves this by first establishing that \( \text{im} \rho \) is a pure subgroup of \( G_* \) and then appealing to [18].
Remark 7.1. For $G_0$ not necessarily torsion free the properties (i)-(vi), (viii)-(ix) would not imply splitness, as the example

$$\mathbb{Z}[\frac{1}{n}] \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2$$

shows. When this is equipped with the strict order induced by the standard order on $\mathbb{Z}[\frac{1}{n}]$ it has all the properties of our 4-coefficient complexes except unperforation, but could not be the augmented K-theory of a C$^*$-algebra. Thus to extend range results beyond the case considered above, one would have to impose an extra condition; for instance that $\text{im} \rho$ was a pure subgroup of $G_n$.

Using the splitness, we may, up to isomorphism, for any $n$-coefficient complex write $G_0 = R \oplus B$ such that $R = G_0/n$, $B = G_1[n]$ and

$$\rho(x) = (x + nG_0, 0) \quad \beta(r, b) = b$$

The properties (i)-(ix) simplify accordingly.

Remark 7.2. Note that graded ordered every dimension group with torsion $G_*$ can be extended to an $n$-coefficient complex when $G_0$ is unperforated, simply by ordering $G_0 \oplus (R \oplus B)$ by

$$(x, (r, b)) \iff \begin{cases} (x, (r, 0)) \geq 0 \\ (x, b) \geq 0 \end{cases}$$

where the quotient order of $G_0 \oplus G_0$ and the order on $G_*$, respectively, are used to determine whether $(x, (r, 0))$ and $(x, b)$ are positive.

Fix $G_*$. We have seen that up to isomorphism, every $n$-coefficient complex is of the form

$$G_0 \longrightarrow R \oplus B \longrightarrow G_1,$$

so determining how many $n$-coefficient complexes with this particular $G_*$ are possible comes out to determining which order structures on $G_0$ will satisfy properties (iii), (v), (vi), (viii) and (ix).

As an example this process, and an application of Theorem 5.4, let us return to the example of Dadarlat and Loring which originally established the need for ordered K-theory with coefficients. They considered the $G_*$-group given by

$$G_0 = \left\{ (x, y_i) \in \mathbb{Z}[\frac{1}{n+1}] \oplus \prod_{i=-\infty}^{\infty} \mathbb{Z} \mid y_i = x(n+1)^{|i|}, \text{a.e.}(i) \right\}$$

$$G_1 = \mathbb{Z}/n$$
equipped with the standard order on $G_0$ and the strict order herefrom on $G_*$. In \cite{7} examples were given to show, in effect, that there were two different ways to complete $G_*$ to an $n$-coefficient complex. For convenience, let $n = 2$. With the notation above we have

$$R = \left\{(a, b_i) \in \mathbb{Z}/2 \oplus \prod_{-\infty}^{\infty} \mathbb{Z}/2 \mid b_i = a, \text{a.e.}(i)\right\}$$

$$B = \{c \in \mathbb{Z}/2\}$$

where

$$\rho(x, y_i) = (\overline{a}, \overline{y_i}) \text{ with } x = \frac{a}{3^i}.$$ 

In the proof below we need elements $\delta_j, \Delta_N \in (\mathbb{Z}/2)^\mathbb{Z}$ defined by

$$(\delta_j)_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (\Delta_N)_i = \begin{cases} 1 & |i| > N \\ 0 & |i| \leq N \end{cases}$$

**Proposition 7.3.** Up to isomorphism every 2-coefficient complex completing $G_*$ defined in \cite{3} will order $G_0^\oplus \mathbb{Z}$ by

$$(x, y_i, (a, b_i, c)) \geq 0 \iff \begin{cases} x \vdash a \\ y_i \vdash b_i + \epsilon_i c \\ x \vdash c \end{cases}$$

where $(\epsilon_i)_{i \in \mathbb{Z}}$ is a sequence in $(\mathbb{Z}/2)^\mathbb{Z}$.

**Proof:** It is easy to see that this defines a 2-coefficient complex. In the other direction, first note that since the order of $G_0 \oplus (R \oplus 0)$ is induced by the graded ordered group $G_0^\oplus G_0$ we have

$$(x, y_i, (a, b_i, 0)) \geq 0 \iff \begin{cases} x \vdash a \\ y_i \vdash b_i \end{cases}.$$ 

Lift the element $((1, 3^{|i|}(1 - \delta_j)), 1) \in G_*$ to a positive element $((1, 3^{|i|}(1 - \delta_j)), (a, b_i, 1)) \in G_*$. If $a = 1$, note that also

$$(1, 3^{|i|}(1 - \delta_j)), (a + 1, b_i + 1 - \delta_j, 1 + 0))$$

is positive, so that we may without loss of generality assume that $a = 0$. Similarly, we may assume that $b_i = 0$ for all $i \neq j$.

We have hence seen that at least one of

$$(1, 3^{|i|}(1 - \delta_j)), (0, \delta_j, 1)) \quad ((1, 3^{|i|}(1 - \delta_j)), (0, 0, 1))$$

is positive in $G_*$. We will define $\epsilon_j$ accordingly such that

$$(1, 3^{|i|}(1 - \delta_j)), (0, \epsilon_j \delta_j, 1)) \geq 0.$$
If \(((x, y_i), (a, b_i, 1)) \geq 0\) then \(x > 0\) because \((x, 1) \geq 0\) in \(G_\ast\). Further, if \(y_j = 0\) then since
\[
((1 + x, 3^{|i|} (1 - \delta_j) + y_i), (a, b_i + \epsilon_i \delta_j, 1 + 1)) \geq 0
\]
we have that \(b_j + \epsilon_j = 0\). We conclude that \(x \vdash a, c\) and \(y_j \vdash b_j + \epsilon_j c_j\).

In the other direction, assume that \(x \vdash a, c\) and \(y_j \vdash b_j + \epsilon_j c_j\). We already know that \(((x, y_i), (a, b_i, c)) \geq 0\) when \(c = 0\), so we can focus on the case \(c = 1\). In this case we will have \(x > 0\) and hence \(y_i > 0\) for \(|j| \geq N\) for some \(N \in \mathbb{N}\). We lift \(((x, y_i), 1)\) to some positive element \(((x, y_i), (a', b'_i, 1))\). If \(a \neq a'\), we note that \(((x, y_i), (1, \Delta_N, 0)) \geq 0\) whence also
\[
((x, y_i), (a' + 1, b'_i + \Delta_N, 1)) \geq 0
\]
so that we without loss of generality may assume that \(a = a'\) and hence that \(b_i = b'_i\) for all but finitely many \(i\). For the remaining \(i\), if \(y_i > 0\), we can adjust to get \(b_j = b'_j\) in a similar fashion. And if \(y_i = 0\) then since we have
\[
((x, y_i)(a + a', b_j + b'_j, 1 + 1)) = ((x, y_i)(0, b_j + b'_j, 0)) \geq 0
\]
we get that \(b_j = b'_j\).

Let “\(\sim\)" be the finest equivalence relation on \((\mathbb{Z}/2)^\mathbb{Z}\) such that
\[
(\epsilon_j) \sim (\epsilon_{j+k}) \sim (1 + \epsilon_j) \sim (\epsilon_j \Delta_N(j)) \sim (\epsilon_{(-1)^{r_j}|j|})
\]
for any \(k \in \mathbb{Z}, N \in \mathbb{N}\) and any \(r_j \in (\mathbb{Z}/2)^{\mathbb{N}\cup\{0\}}\). Using methods from \[\text{[7]}\] one can prove that the augmentations associated to \((\epsilon_i)\) and \((\eta_i)\) in \((\mathbb{Z}/2)^\mathbb{Z}\) are isomorphic precisely when \((\epsilon_i) \sim (\eta_i)\). The examples given in \[\text{[7]}\] correspond to \((\epsilon_i) = (0)\) and \((\eta_i)\) given by 1 on positive entries and 0 on negative ones.

One sees easily that there are uncountably many nonisomorphic 2-coefficient complexes in this case — even though we have only added one bit of information to \(G_0 \oplus (R \oplus 0)\) the amount of freedom in choosing the order structure is immense.

References

[1] C.F. Bödigheimer, Splitting the Künneth sequence in \(K\)-theory, Math. Ann. 242 (1979), 159–171.
[2] , Splitting the Künneth sequence in \(K\)-theory, II, Math. Ann. 251 (1980), 249–252.
[3] L.G. Brown, The Riesz interpolation property for \(K_0(A) \oplus K_1(A)\), preprint.
[4] M. Dădărlat and S. Eilers, Compressing coefficients while preserving ideals in the \(K\)-theory for \(C^*\)-algebras, K-Theory 14 (1998), 281–304.
[5] , The Bockstein map is necessary, Canad. Math. Bull. 42 (1999), no. 3, 274–284. MR 2000d:46070
[6] M. Dădărlat and G. Gong, A classification result for approximately homogeneous C*-algebras of real rank zero, Geom. Funct. Anal. 7 (1997), no. 4, 646–711.

[7] M. Dădărlat and T.A. Loring, Classifying C*-algebras via ordered, mod-p K-theory, Math. Ann. 305 (1996), no. 4, 601–616.

[8] M. Dădărlat and T.A. Loring, A universal multicoefficient theorem for the Kasparov groups, Duke Math. J. 84 (1996), no. 2, 355–377.

[9] E.G. Effros, D.E. Handelman, and C.L. Shen, Dimension groups and their affine representations, Amer. J. Math. 102 (1980), no. 2, 385–407. MR 83g:46061

[10] S. Eilers, Invariants for AD algebras, Ph.D. thesis, Copenhagen University, November 1995.

[11] , A complete invariant for AD algebras with real rank zero and bounded torsion in K₁, J. Funct. Anal. 139 (1996), 325–348.

[12] S. Eilers, Künneth splittings and classification of C*-algebras with finitely many ideals, Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 81–90.

[13] , Approximate homogeneity of C*-algebras with finitely many ideals, Math. Proc. R. Ir. Acad. 101A (2001), no. 2, 149–162. MR 1 925 347

[14] S. Eilers and G.A. Elliott, The Riesz property for the K₁ₙ-group of a C*-algebra of minimal stable and real rank, C. R. Math. Acad. Sci. Soc. R. Can. 25 (2003), no. 4, 108–113. MR 2 013 159

[15] G.A. Elliott, Dimension groups with torsion, Internat. J. Math. 1 (1990), no. 4, 361–380.

[16] , On the classification of C*-algebras of real rank zero, J. Reine Angew. Math. 443 (1993), 179–219.

[17] L. Fuchs, Riesz groups, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 19 (1965), 1–34.

[18] , Infinite abelian groups I, Academic Press, New York, San Francisco, London, 1970.

[19] K. R. Goodearl, K₀ of multiplier algebras of C*-algebras with real rank zero, K-Theory 10 (1996), no. 5, 419–489.

[20] K.R. Goodearl, Partially ordered abelian groups with interpolation, American Mathematical Society, Providence, R.I., 1986.

[21] H. Lin and M. Rørdam, Extensions of inductive limits of circle algebras, J. London Math. Soc. 51 (1995), no. 2, 603–613.

[22] F. Riesz, Sur quelques notions fondamentales dans la théorie générale des opérations linéaires, Ann. of Math. 41 (1940), 174–206. MR 1,147d

[23] C. Schochet, Topological methods for C*-algebras IV: Mod p homology, Pacific J. Math. 114 (1984), 447–468.

[24] J. Villadsen, The range of the elliott invariant of the simple ah-algebras with slow dimension growth, K-theory 15 (1998), no. 1, 1–12.

[25] F. Wehrung, Injective positively ordered monoids. I, J. Pure Appl. Algebra 83 (1992), no. 1, 43–82. MR 93k:06023

[26] S. Zhang, A Riesz decomposition property and ideal structure of multiplier algebras, J. Operator Theory 24 (1990), 209–225.
