Multi-orientable group field theory

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Abstract

Group field theories (GFTs) are quantum field theories over group manifolds; they can be seen as a generalization of matrix models. GFT Feynman graphs are tensor graphs generalizing ribbon graphs (or combinatorial maps); these graphs are not always dual to manifolds. In order to simplify the topological structure of these various singularities, colored GFT was recently introduced and intensively studied. We propose here a different simplification of GFT, which we call multi-orientable GFT. We study the relation between multi-orientable GFT Feynman graphs and colorable graphs. We prove that tadfaces and some generalized tadpoles are absent. Some Feynman amplitude computations are performed. A few remarks on the renormalizability of both multi-orientable and colorable GFT are made. A generalization from three-dimensional to four-dimensional theories is also proposed.

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(Some figures may appear in colour only in the online journal)

1. Introduction and motivation

Graph theory [1] is known to play a fundamental role in describing the combinatorics of quantum field theory (QFT) [2–4]. A natural generalization of graphs, the combinatorial maps [5] (or ribbon graphs) play the same role for matrix models, known to be related to non-commutative QFT [6] or to two-dimensional quantum gravity [7, 8].

The generalization of these models to three- and four-dimensional space is known as group field theory (GFT) [9]. Feynman amplitudes of GFT can be equivalently expressed as spin-foam models [10], which together with string theory, represent one of the most developed approaches today for a fundamental theory of quantum gravity [11].

Nowadays, GFT is allocated a great deal of interest by the mathematical physics community (see [12] and references within). The role that combinatorial maps play for
matrix models is now played by tensor graphs. (These graphs have three strands per edge in three-dimensional models and four strands per edge in four-dimensional models.)

Nevertheless, these tensor graphs are not always dual to manifolds; the topological structure of their singularities is indeed very complicated. This can be seen as a drawback or, in contrast, as a mathematical richness (for example, in QFT models on the non-commutative Moyal space, it is the ‘non-planar’ tadpole-like graphs—definitely more complicated from a topological point of view than the planar ones—that give rise to the celebrated phenomenon of ultraviolet/infrared mixing [13]; getting rid of these ‘non-planar’ graphs would eliminate this new, physically rich phenomenon).

For the sake of completeness, let us mention here that other ways of implementing noncommutative QFT exist in the literature, ways which do not lead to the ultraviolet/infrared mixing (see for example [14]). Taking yet another point of view, one can use braided QFT and this also leads to no ultraviolet/infrared mixing [15].

Coming back to GFT, in order to eliminate a whole class of ‘wrapping singularities’, a restricted class of models was introduced—the colored GFT [16]. The idea is to have each of the $D$ edges incoming/outgoing to a vertex colored with a distinct color ($D$ being the dimension of spacetime) and to have two types of vertices. From a graph theoretical point of view, this last part means that one restricts the set of graphs to bipartite graphs (also known as bigraphs—see for example [17]). Related to this topic of colored GFT, let us also mention here the recent literature on manifold crystallization [18]. Moreover, it is this notion of bigraphs which plays a key role when proving that the color in colored GFT guarantees orientability of the piecewise linear pseudo-manifold associated with each graph of the perturbative expansion [19].

Colored GFT has lately received a great deal of interest within GFT; several achievements concerning these colored models have been made (see [20–24] and references within).

In this paper, we propose a simplification of the GFT tensor graph class which is different from the coloring one; we call this new simplification multi-orientability. The idea behind this is to introduce a notion of orientability at the level of the GFT vertex. This was already successfully implemented for ribbon graphs in QFT on Moyal space; the respective non-commutative models were intensively studied in various papers of mathematical physics (see [25, 26] and references within).

We introduce this new class of GFT models both in three and in four dimensions. We also analyze the relation between the multi-orientable GFT graphs and the colorable ones. We prove that tadfaces and some class of generalized tadpoles are not allowed within this multi-orientable framework. Our proofs also allow us to recover the known result that tadfaces are also forbidden within the colorable framework. Furthermore, a particular class of multi-orientable GFT graphs is identified. The associated Feynman amplitudes for the BF-theories are computed, both in three (the Boulatov model [27]) and four dimensions (the Ooguri model [28]).

Moreover, we illustrate the following phenomenon. Within the framework of colorable GFT, divergent two- and four-point graphs (for the three-dimensional case) represent quantum corrections of types not present in the bare action of the model. Within the multi-orientable framework, these graphs represent quantum corrections of type already existing in the bare action.

The paper is organized as follows. In section 2, we present the definition of colorable Boulatov three-dimensional model and give the definition of multi-orientable GFT. The following section deals with graph theoretical consequences of this definition. We look closely at the issues of tadpoles and tadfaces and establish the relation between colorable and multi-orientable three-dimensional GFT graphs. In section 4, we compute some Feynman...
amplitudes of non-colorable but multi-orientable, colorable (and multi-orientable) and also non-multi-orientable (and non-colorable) graphs. Section 5 presents some considerations with respect to the renormalizability of the colorable and of the multi-orientable models. Section 6 proposes a generalization of multi-orientability to the four-dimensional case. The last section presents some perspectives for future developments of these new types of models.

2. Multi-orientable GFT in three dimensions

In this section, we first recall the definition of the colorable models and then give the definition of the multi-orientable ones.

The GFT models (colorable or non-colorable) usually studied in the literature are orientable: this means that the simplicial complexes dual to these graphs are orientable. We keep this simplification in this paper.

The field $\phi$ of the three-dimensional topological Boulatov model is a map $\phi : G^3 \rightarrow \mathbb{R}$, where the group $G$ is the $SU(2)$ one. For such an orientable model, the action is written

$$S[\phi] = \frac{1}{2} \int dg_1 \, dg_2 \, dg_3 \phi(g_1, g_2, g_3) \phi(g_1, g_2, g_3) + \frac{\lambda}{4!} \int dg_1 \cdot \cdot \cdot dg_6 \, dg_1 \cdot \cdot \cdot dg_6 \phi(g_1, g_2, g_3) \times \phi(g_3, g_4, g_5) \phi(g_5, g_6, g_6) \phi(g_6, g_4, g_1) \delta(g_1 g_4^{-1}) \cdots \delta(g_6 g_6^{-1}) \tag{2.1}$$

which is equivalent to the simplified form

$$S[\phi] = \frac{1}{2} \int dg_1 \, dg_2 \, dg_3 \phi(g_1, g_2, g_3) \phi(g_1, g_2, g_3) + \frac{\lambda}{4!} \int dg_1 \cdot \cdot \cdot dg_6 \phi(g_1, g_2, g_3) \phi(g_3, g_4, g_5) \phi(g_5, g_6, g_6) \phi(g_6, g_4, g_1) \tag{2.2}$$

Note that the integration over the group is done with the invariant Haar measure, both in (2.1) and (2.2).

Let us make the following remark. The integral kernel represented by the product of $\delta$ functions in (2.1) can be interpreted as a natural generalization, within the GFT formalism, of the crucial notion of locality of QFT. Thus, if one takes the example of the local $\Phi^4$ model, the non-quadratic part (the interaction) of the action in configuration (or direct) space is written

$$\int dx \, dx_1 \cdots dx_4 \delta(x - x_1) \cdots \delta(x - x_4) \Phi(x_1) \cdots \Phi(x_4) = \int dx \Phi^4(x) \tag{2.3}$$

It is this form of the vertex which is generalized in formulas (2.1) and (2.2). Later on, within the framework of QFT on the non-commutative Moyal space, this notion of locality was replaced by a more adapted notion of 'Moyality' (see again [6] and references within). We thus advocate to keep the general form (2.1) for the vertex of any GFT model, topological or not, in three or in four dimensions, in order to have a natural generalization of the notion of locality (and of 'Moyality'). Let us stress that this has been also defended for example in [29].

Nevertheless, a different point of view is usually taken in the GFT literature, a point of view inspired from the spin-foam experience in quantum gravity. Thus, for a non-topological model, the interaction is the one which is modified (and not the propagator), because in this way one has a clear geometrical meaning of the added constraints, already at the level of the action. However, in order to perform renormalizability studies of GFT, it seems natural to us to switch to the QFT point of view advocated here for the reasons explained above. This idea can also be defended by the fact that, in theoretical physics, renormalization is done within some QFT framework.

The Boulatov field $\phi$, taken to be real-valued, is not assumed here to have specific symmetry properties under the permutations of its arguments. Note that the more involved
case, where not only the identical permutation is associated with such an edge, is analyzed in [30] or [31]. Thus, the edge of the graphs considered here is represented as in figure 1. The propagator of the model is written
\[ \int dh \prod_{i=1}^3 \delta(g_i h g_i^{-1}). \]
(2.4)

The form of the vertex chosen in the action (2.2) corresponds to the one in figure 2.

The colorable three-dimensional model is defined in the following way. The (real-valued) Boulatov field is replaced by the four complex-valued field \( \phi_p \), the index \( p = 0, \ldots, 3 \) being referred to as some color index. Moreover, one now has two types of interactions, \( \phi^4 \) and \( \bar{\phi}^4 \). Furthermore, a clockwise cyclic ordering at one of the types of vertices (and an anticlockwise at the second type of vertex) of the four colors at the vertex is imposed; thus the action is written
\[ S_{col} = \frac{1}{2} \sum_p \int \bar{\phi}_p \phi_p + \frac{\lambda}{4!} \int \phi_0 \cdots \phi_4 + \frac{\lambda}{4!} \int \bar{\phi}_0 \cdots \bar{\phi}_4, \]
(2.5)
where the integrations over the group are left implicit.

From a graph-theoretical point of view, this means that one imposes a four-coloring of the edges and that only bipartite graphs are kept. Furthermore, one has to respect the cyclic ordering of the two types of vertices described above. Nevertheless, let us also remark that this cyclic ordering condition has been recently removed from the definition of colored GFT. However, in this paper we will use the initial definition (2.5) when comparing the multi-orientable models with the colorable ones.
Let us now introduce the announced three-dimensional multi-orientable model. As in the colored case, one has a complex-valued field $\phi$. Nevertheless, we do not need more copies of this field. The Boulatov interaction is restricted to vertices where each corner has a $+$ or a $-$ label. Furthermore, each vertex has two corners labeled with $+$ and two corners labeled with $-$, which are cyclically ordered as shown in figure 3. A field propagates from a $-$ to a $+$ corner of some vertex.

This notion of corners of a vertex is natural within the framework of non-local QFT (as is the Moyal non-commutative QFT, where, as mentioned in the introduction, the idea of multi-orientability has proved to be very useful). Nevertheless, GFT also can be seen in some sense as a non-local QFT on the group manifold on which it lives, since the interaction is not defined on some ‘group point’ $g$ (see (2.1) and (2.2)), unlike the interaction of the local QFT models (see (2.3)).

We propose to call this model multi-orientable, because one has

- on the one hand, the usual orientability of the GFT propagation (see figure 1 above) and
- on the other hand, the orientability of the vertex.

The action of the model is written

$$S[\phi] = \frac{1}{2} \int \bar{\phi} \phi + \frac{\lambda}{4!} \int \bar{\phi} \phi \bar{\phi} \phi,$$

(2.6)

where, as in (2.5), the integrations over the group are left implicit.

At a graph-theoretical level, the model (2.6) generates Feynman graphs built-up of vertices like the one represented in figure 3 and edges like the one represented in figure 1.

For the sake of completeness, let us also mention that combinatorial maps with four-valent vertices like the orientable ones in figure 3 have not been counted [32]; this is not the case for maps (of genus $g$) in general, which are very well analyzed from a combinatorial point of view (see again [5] and references within).

Before ending this section, let us also remark that some kind of hybrid model, where the vertex is kept orientable but in which one does not allow the identical permutation of the three arguments of the edge, can be naturally defined.
3. Multi-orientable graphs: relation with the colorable ones

Let us take a closer look at the graph-theoretical consequences of the definition of the multi-orientable models of the previous section. The colorability discards a highly significant class of graphs, including the so-called wrapping singularities, which correspond to graphs containing tadpoles like figures 4 and 5 (see again [16] for more details).

Let us now prove the following result.

Proposition 3.1. Every GFT graph which is colorable is also multi-orientable.

Proof. Recall that the conditions imposed for a graph to be orientable are the existence of two types of vertices, cyclic ordering (clockwise and anti-clockwise) at these vertices and the fact
that the four edges adjacent to such a label wear a specific label (the color). The conditions to define multi-orientability are actually just a subset of these conditions. Thus, the cyclic ordering is imposed, but no labeling (coloring) of the edges—there is no distinction between the two + or the two − corners of a vertex. Moreover, the proof is completed by the following fact. If the multi-orientability condition is satisfied as some vertex, then it is also satisfied at another vertex connected to the first one by an edge of some generic color.

The reciprocal statement is not true. A counterexample is the tadpole graph in figure 4, which is multi-orientable (or, in other words, it can multi-oriented) but is non-colorable. Another example, which is not a tadpole graph, is the two-point graph in figure 6.

All of this can be rephrased as that multi-orientability discards a less important class of graphs than colorability.

3.1. Tadpoles and generalized tadpoles

Before investigating the issue of tadpoles within the multi-orientable framework, let us make the following remark. Since the edges of the models we deal with here (colorable or not, multi-orientable or not) do not allow twists (as already stated in the previous section), one can drop the ‘middle’ strand of any edge and obtain a one-to-one correspondence with some ribbon graph (or combinatorial map). The ribbon graph thus obtained is the jacket introduced in [33] and later generalized in [23]. Furthermore, in [22], it was showed that the jacket graphs represent (Heegaard) splitting surfaces for the triangulation dual to the Feynman graph; this allows us to re-express the Boulatov model as a QFT model on these Riemann surfaces (see again [22]).

One can thus refer to the planarity of the respective tensor GFT graph as to the planarity of the ribbon graph associated in this way. Moreover, one can count the number of faces broken by the external legs; we denote this number by $B$. If $B > 1$, we call the respective graph irregular (see again [6] for details).

The tadpole in figure 4 is thus referred to as a planar tadpole, while the tadpole in figure 5 is referred to as a ‘non-planar’ tadpole (although the terminology ‘non-planar’ usually used in the literature is incorrect, because the respective ribbon graph is planar but it just has a number of broken faces superior to one).

We have seen above that planar tadpoles are allowed by multi-orientable models, while the non-planar ones are not.

Let us now recall from [24] the following definition.

**Definition 3.1.** A generalized tadpole is a graph with one external vertex.
Figure 7. A non-multi-orientable tadface graph. The tadface is represented by the dashed line. One notes that the respective face goes twice through the edge relying on the two tadpoles, once through one strand and once through another strand of the edge.

Figure 8. A tadface in some general graph. One has an edge $E$ which is crossed twice by the respective circuit, once through one strand and once through another strand (nevertheless, the ‘middle’ strand cannot be the one of these two strands). The respective edge $E$ separates two subgraphs $G$ and $G'$ which has to be both irregular, the edge $E$ alone breaking a face for each of these subgraphs.

Planar-generalized tadpoles are allowed by multi-orientability. A ‘non-planar’-generalized tadpole is a graph with two external edges and with $B = 2$. These graphs are not allowed by multi-orientability (this is a result already known from the non-commutative QFT literature; see for example [25]).

3.2. Tadfaces

We recall from [33] the following definition.

**Definition 3.2.** A tadface is a face that goes at least twice through a line.

Let us give some more explanation on this. Such a tadface can be obtained if one goes through the respective edge the first time through a strand and the second time through the second strand. A second-order example of such a graph is given in figure 7 (graph which is to be inserted into a ‘bigger’ graph, such that the usual 1PI condition can be kept). Nevertheless, this graph is not multi-orientable, since it is made up of two non-multi-orientable tadpoles like the ones in figure 5.

We now prove the following result.

**Theorem 3.1.** Tadfaces are not allowed by multi-orientability.

**Proof.** Let us first remark that, in order to prove this, one can forget about the ‘middle’ strand since this strand never hooks to any of the ‘external’ strands of an edge and thus cannot lead to a tadface.

Furthermore, in order to obtain a tadface, one needs some edge $E$ to be crossed twice when obtaining a face (according to definition 3.2)—see figure 8. This means that one can...
identify two irregular subgraphs \( G \) and \( G' \), where the edge \( E \) breaks a certain face of each of these subgraphs, while the other edges break other face(s) of the two subgraphs.

Joining together these two subgraphs through the edge \( E \) leads to the tadface (see again figure 8).

We now prove the following intermediate result.

**Lemma 3.1.** One cannot have an irregular multi-orientable graph where one single edge \( E \) breaks a face.

**Proof.** We suppose this graph exists. Without any loss of generality we also suppose that the respective edge \( E \) leaves from a ‘−’ corner of some vertex. Then, one needs that a circuit leaves from a ‘+’ corner and ends up in the opposite ‘+’ corner of the same vertex (the respective circuit cannot close on a ‘−’ corner because it would then leave inside another half-edge leaving from the missed ‘+’ corner). The respective circuit can go through a certain number of vertices (even or odd); note, however, that the circuit hooks to a vertex through a ‘−’ corner and needs to leave from a ‘+’ corner without leaving half-edges inside. One can now directly check, using the parity constraints of the multi-orientable vertex, that such a circuit cannot be built, thus proving the lemma.

Before getting back to the proof of the main theorem, let us remark that irregular multi-orientable graphs (with four, six etc external edges) do exist, but each broken face is broken by an even number of edges. Using now lemma 3.1 above, one concludes that the required subgraphs \( G \) and \( G' \) are forbidden by multi-orientability. This completes the proof.

One can also obtain the following result, already announced in [33].

**Corollary 3.1.** Tadfaces are not allowed by colorability.

**Proof.** This follows directly from theorem 3.1 and proposition 3.1.

Let us now give a posteriori a physical motivation (leaving aside the pure combinatorial one) for getting interested in the presence or absence of these kind of tadfaces in GFT graphs. This motivation comes from the fact that it was proved in [24] that graphs without tadfaces have better bounds per vertex within the framework of a BF model, like the ones we deal with in this paper.

Furthermore, one has the following.

**Corollary 3.2.** ‘Non-planar’ generalized tadpoles are not allowed by multi-orientability.

**Proof.** A ‘non-planar’ generalized tadpole is a graph with two external edges, each of these edges breaking a different face of the graph. One can now use lemma 3.1 to conclude that this type of graph is forbidden by multi-orientability.

As already announced above, this result was already known from the non-commutative QFT literature [25].

Finally, one has the following.

**Corollary 3.3.** ‘Non-planar’ generalized tadpoles are not allowed by colorability.

**Proof.** This follows directly from corollary 3.2 and proposition 3.1.
Figure 9. An example of a non-colorable, multi-orientable four-point GFT graph. The indices in blue label the two internal edges. The rest of the indices refer to the group elements of the external edges.

We resume all these results in the following table, which compares the colorable and the multi-orientable GFT models:

|                        | Colorable | Multi-orientable |
|------------------------|-----------|------------------|
| Generalized planar tadpoles | Forbidden | Allowed          |
| Generalized ‘non-planar’ tadpoles | Forbidden | Forbidden        |
| Tadface                | Forbidden | Forbidden        |

For the sake of completeness, let us also mention that the definition of bubbles or jackets of a general orientable graph, known from the recent GFT literature (see for example [34] for orientable graphs or [35] for a general graph), is to be kept for the multi-orientable GFT introduced in this paper.

Before ending this section, let us make one more comparison with the non-commutative QFT case. When dealing with models on the Moyal space (see again [6] and references within), orientability of the vertex discards the ‘non-planar’-like tadpoles. As already mentioned in the introduction, it is these two-point graphs which are mainly responsible for the appearance of the phenomenon of ultraviolet/infrared mixing. They have a $1/p^2$-like dependence on the external momenta (in the infrared regime of this one) and, when inserted into ‘bigger’ graphs (which are thus non-planar), these external momenta become internal, need to be integrated and this integration leads to a new type of infrared divergence.

Nevertheless, some four-point graphs which are not discarded by the orientability of the vertex of the Moyal interaction still depend on the external momenta in a way which can lead to new types of divergences. This dependence is logarithmic and it leads indeed to the infrared divergence mentioned above when the respective four-point graph is similarly inserted into some ‘bigger’ graph.

Examples of some GFT graphs are given in figure 9. It is these types of graphs that we will carefully analyze in the following section.

4. Feynman amplitude computations

We first investigate here the behavior of a non-colorable but multi-orientable graph, then that of a colorable (and multi-orientable) graph and, finally, the one of a non-colorable and non-multi-orientable graphs.
4.1. A non-colorable, multi-orientable amplitude

Let us now calculate the Feynman amplitude of the GFT graph in figure 9. We denote by \( h_1 \) and by \( h_2 \) the two group elements associated with the internal edges 1 and 2, respectively.

One has

\[
\int dh_1 dh_2 \delta(g_1 h_1 h_2^{-1} g_5) \delta(g_1 h_1 g_5^{-1}) \delta(g_2 h_2 g_2^{-1}) \delta(g_2^{-1} h_1 h_2 g_5^{-1}).
\]  

(4.1)

Performing the integral on \( h_2 \) using the third \( \delta \) function in (4.1) and performing the integral on \( h_1 \) using the second \( \delta \) function in (4.1) lead to the following result:

\[
\delta(g_1 g_4^{-1} g_2^{-1} g_2 g_5) \delta(g_1^{-1} g_4 g_2^{-1} g_2^{-1} g_2 g_5^{-1}).
\]  

(4.2)

As expected, the Feynman amplitude (4.1) is not divergent (this could have been directly stated from the fact that there is no internal bubble of the GFT graph).

Nevertheless, an interesting phenomenon of some kind of ‘ultraviolet/infrared’ mixing on the group manifold takes place. Thus, for

\[
g_1 = g_5^{-1} g_2^{-1} g_2 g_4 g_2^{-1} g_1^{-1} \quad \text{or} \quad g_1 = g_4 g_5^{-1} g_2^{-1} g_2 g_5^{-1}
\]  

(4.3)

the Feynman amplitude (4.2) becomes divergent. This comes from the fact that one has a non-trivial dependence of the amplitude on the external group momenta.

4.2. A colorable, multi-orientable amplitude

However, this phenomenon is not specific to the multi-orientable, non-colorable graphs. In the case of the graph in figure 10, which is colorable (and multi-orientable), the Feynman amplitude is written

\[
\int dh_1 dh_2 \delta(g_1 h_1 g_1^{-1}) \delta(g_1 h_1 g_1^{-1}) \delta(h_1^{-1} h_2) \delta(g_2 h_2 g_2^{-1}) \delta(g_2 h_2 g_2^{-1}).
\]  

(4.4)

As above, we integrate on \( h_2 \) using the third \( \delta \) function in (4.4) and then integrate on \( h_1 \) using the first of the \( \delta \) functions in (4.4). The result can be written as

\[
\delta(g_1 g_4^{-1} g_2^{-1} g_1 g_1^{-1}) \delta(g_2 g_4^{-1} g_1 g_2^{-1}) \delta(g_1 g_1^{-1} g_1 g_2^{-1}).
\]  

(4.5)

The group ‘ultraviolet/infrared’ mixing described above is still present, with several independent directions in the group

\[
g_4 = g_4 g_1^{-1} g_1 \quad \text{or} \quad g_2 = g_2 g_4^{-1} g_1 \quad \text{or} \quad g_5 = g_5 g_4^{-1} g_1.
\]  

(4.6)

turning the product (4.5) divergent.
4.3. A non-colorable, non-multi-orientable amplitude

For the sake of completeness, we end this section by analyzing an associated non-colorable, non-multi-orientable GFT graph, like the one in figure 11. The Feynman amplitude of this graph is written

\[ \int dh_1 dh_2 \delta(g_1 h_1 g_1^{-1}) \delta(g_2 h_1 g_2 h_2 g_4) \delta(g_4 h_2 g_4) \delta(g_5 h_2 g_5^{-1}). \] (4.7)

Integrating first on \( h_1 \) using the first of the \( \delta \) functions in (4.7) and then on \( h_2 \) using the last of the \( \delta \) function in (4.7) leads to the result

\[ \delta(g_6 g_6^{-1} g_6^{-1} g_6 g_1 g_1) \delta(g_4 g_5^{-1} g_4 g_5^{-1} g_4 g_4 g_1 g_1). \] (4.8)

This product of \( \delta \) functions on external group elements can become infinite for any of the following independent group directions:

\[ g_6 = g_4^{-1} g_5^{-1} g_4 g_5^{-1} g_1 \text{ or } g_4 = g_4^{-1} g_5^{-1} g_4 g_5^{-1} g_1 \text{ or } g_2 = g_4^{-1} g_1^{-1} g_1 g_5. \] (4.9)

The same type of phenomenon takes place when computing the Feynman amplitudes of the tadpoles of figures 4 and 5, which are non-colorable but multi-orientable and non-colorable, non-multi-orientable, respectively.

Let us end this section with the following remark. The presence of the ‘middle’ strand of these GFT edges (which makes the difference with respect to the combinatorial maps or ribbon maps of non-commutative QFT) is necessary for defining the bubbles, as it was already stated in [36]. Moreover, this third strand is also required for the computation of Feynman amplitudes (which are related to the new concept of bubbles), as we have seen in detail in this section.

5. Remark on multi-orientable GFT and renormalizability; comparison with the colorable GFT

In this section, we give an example of a four-point GFT graph which within the framework of the colorable models represents a quantum correction of a type not present in the bare action; within the framework of the multi-orientable models, this type of graph represents a quantum correction of a type already existent in the bare action. We then show a more general result regarding the quantum corrections of the two- and four-point functions within the multi-orientable framework.
Let us investigate the GFT behavior of the tensor graph in figure 12. Denoting the group elements associated with the four external edges by $g_1, \ldots, g_{12}$ and the group elements associated with the 14 internal edges by $h_1, \ldots, h_{14}$ as indicated in figure 12, the Feynman amplitude is written
\[
\int dh_1 \cdots dh_{14} \delta(g_1 h_1 g_4^{-1}) \delta(h_1^{-1} h_2 h_{14}^{-1} h_3) \delta(h_1 h_4 h_{14} h_3) \delta(g_3 h_2 h_{11} h_6 g_{12}^{-1}) \delta(h_2 h_3) \\
\delta(g_5 h_3^{-1} h_{11} h_7 g_{11}^{-1}) \delta(g_5^{-1} h_3 h_{13}^{-1} h_8 g_8) \delta(h_4 h_5) \delta(g_6 h_5^{-1} h_{13}^{-1} h_9^{-1} g_7) \delta(h_{14} h_{11} h_{12} h_{13}) \\
\delta(h_6 h_{10}^{-1} h_9 h_{14}^{-1}) \delta(h_6 h_7) \delta(h_7 h_{12} h_8 h_{10}) \delta(h_8 h_9) \delta(g_9 h_{10} g_{10}^{-1}).
\] (5.1)

One remarks that we have a total of nine internal $\delta$ functions (i.e. $\delta$ functions only in the internal parameters $h_1, \ldots, h_{14}$) which correspond to the nine internal faces of the tensor graph in figure 12. Upon direct inspection, one can then check that this graph is divergent as we did in the previous section (we recall that the Feynman amplitude computation is identical for Boulatov, colorable Boulatov or multi-orientable Boulatov models). Within the framework of the multi-orientable model (2.6), this represents a quantum correction of type $\phi\phi\phi\phi$, a term which is present in the bare action. If one now colors the edges of this graph following the recipe indicated in section 2, the quantum correction here is for a term of type $\phi_p\phi_p\phi_p\phi_p$ (where $p$ indicates a specific color), which is not present in the bare colored action (2.5).

Let us now prove the following result.

**Theorem 5.1.**

- No two-point function of type $\phi\phi$ or $\bar{\phi}\bar{\phi}$ is permitted by multi-orientability.
- No four-point function of a type distinct of $\phi\phi\phi\phi$ is permitted by multi-orientability.

**Proof.** Let us first prove the first point above. If one has two external edges, then $B \leq 2$. If $B = 2$, these types of graphs are known not to be permitted by multi-orientability (see subsection 3.1). If $B = 1$, this means that one has the two external edges breaking the same
Figure 13. Four-dimensional vertex for GFT models, topological or not.

Let us now prove the last item. As above, we suppose that a graph $G$ of type $\phi\bar{\phi}\bar{\phi}\bar{\phi}$ exists. Let us remark that this particular choice does not restrict the generality of the statement. This means that one can identify a subgraph $G'$ of the original four-point graph $G$ such that two external edges of type $\phi$ and $\bar{\phi}$ follow. This is, however, excluded and thus concludes the proof.

□

Let us emphasize that the result above is valid at any order in perturbation theory. These phenomena appear as some kind of consequence of the fact that the restrictivity of the colorability condition is more significant than the restrictivity of the multi-orientability condition.

6. Generalization to four-dimensional GFT models

The definition of the multi-orientable models to GFT in even dimension is not straightforward. This comes from the fact that the interaction $\phi^{D+1}$ is odd; one thus has two inequivalent choices of distribution of the $-$ and $+$ signs on the corners of the vertex. For the four-dimensional case (where the vertex is given in figure 13), the case of interest for quantum gravity, these two possibilities of interactions are given in figures 14 and 15.

The action of the proposed model is thus written

$$S[\phi] = \frac{1}{2} \int \bar{\phi}\phi + \frac{\lambda_1}{5!} \int \bar{\phi}\phi\bar{\phi}\phi\bar{\phi} + \frac{\lambda_2}{5!} \int \phi\bar{\phi}\phi\bar{\phi}\phi.$$

(6.1)
Figure 14. The first possibility of an orientable vertex for four-dimensional GFT.

Figure 15. The second possibility of an orientable vertex for four-dimensional GFT.
where, as in (2.5) or (2.6), the integrations over the group are left implicit. Moreover, in order to
keep generality, we choose to have two distinct coupling constants $\lambda_1$ and $\lambda_2$; they are \textit{a priori} free to flow under the renormalization group.

The considerations of the previous sections extend for this four-dimensional multi-
orientable GFT model.

7. Conclusions and perspectives

We have introduced multi-orientable GFT in this paper as a way of simplifying the topology
and combinatorics of GFT; this simplification is different from the one proposed within the
colorability of GFT. An analysis of the differences between the classes of tensor graphs
discarded by these two types of models has been done and some Feynman amplitude
computations have been performed. Moreover, we have given explicit examples of two- and
four-point graphs, which in the colorable framework lead to quantum corrections of types not
present in the bare action, while in the multi-orientable framework lead to quantum corrections
of types already presented in the bare action. Finally, a generalization from three-dimensional
to four-dimensional models has been proposed.

Let us now give more details on the general usefulness of such a concept of multi-
orientable GFT. As in the case of colored GFT, the first motivation comes from the fact that
GFTs are known to have an extremely involved combinatorial and topological structure [31].
Multi-orientable GFT discards (via the indicated QFT recipe) some of these graphs and this
means that explicit mathematical manipulation is more likely to be pushed further in this
simpler case rather then in the general case. One can thus see multi-orientable (or colorable)
GFT (at least) as a laboratory for testing various QFT tools before tackling the same problems
in the general case.

Moreover, the results of section 5 show that multi-orientable GFTs have an encouraging
behavior when one has in mind renormalizability studies. Thus, these models are shown to
comport better than colorable models, where one has quantum corrections of types not present
in the bare action.

Finally, let us also recall a general motivation for colored GFT (see again [16]), motivation
which also applies for multi-orientable GFT. This motivation comes from an analogy with
matrix models, where only identically distributed models are topological in some scaling limit
[7]. Nevertheless, it is topology which governs the power counting of more involved models
(the renormalizable ones, [37] and [38]). It is this mathematical feature that the multi-orientable
(and colorable) GFT is exploiting.

Since this paper gives a proposal for a new type of GFT models, the perspectives for the
future work on this subject appear to us as particularly vast. Thus, it would be interesting to
check whether or not the various achievements obtained within the framework of colorable
GFT models (see again [20–24] and references within) can be adapted (and under what
conditions) to multi-orientable GFT.

A first perspective may be the investigation of the issue of orientability of the piecewise
linear pseudo-manifold associated with the graph of the perturbative expansion of multi-
orientable GFT. One needs to check whether or not the proof of the paper [19] for the colored
Boulatov models adapts to the multi-orientable framework described here.

Among other possible perspectives, we can list here the definition of some computable
cellular homology, the (celebrated) $1/N$-expansion, the investigation of scaling behavior, of
Borel summability or of (local and global) gauge transformations.

Regarding this last point of gauge symmetries, let us stress the following issue which
appears to be of crucial importance. In an ‘ordinary’ gauge theory, for example in QED, one
can redundantly integrate over a continuous infinity of physically equivalent configurations. This type of problem can be fixed using Fadeev–Popov-like tricks (see again textbooks like [2]). This situation can also arise in GFT (colorable or not, multi-orientable or not) and it seems that one needs to make sure if this indeed happens or not in order to correctly interpret the divergences of the various models [39].

More precisely, in [21] a global diffeomorphism symmetry was identified for the colored Boulatov model. The first step in the project mentioned above is thus to look for a local form of such a symmetry and then to check whether or not one can factorize (a part of) the colored Boulatov divergences. The same can be proposed for multi-orientable models. In this case, however, one has to look first for an equivalent of the diffeomorphism symmetry identified in [21].

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