Analysis of 3D-reinforced composite annular plate subjected to concentrated forces

B S Sarbayev and D T Bregvadze

Bauman Moscow State Technical University, 2-ya Baumanskaya ul. 5, str. 1, Moscow, 105005, Russia.

E-mail: Znatok-david-bregv@yandex.ru

Abstract. This paper is dedicated to the solution of the problem about 3D-reinforced composite annular plate compressed by concentrated forces. 3D-reinforced composite materials, including high-temperature composites, are widely used in modern space industry, while their properties are insufficiently studied. Analytical solution of considered problem about annular plate compression will be a foundation for a method of determination of anisotropic composite materials stress-strain properties needed when conducting structural analysis: elastic and shear moduli, and Poisson’s ratio. Transverse shear deformations and changes in annular plate metrics during deformation are taken into account in the process of finding analytical solution of the problem. The analytical solution of the problem about annular plate compression was obtained by the authors. This solution is used to find relations between required properties of annular plate and radial displacements of annular plate characteristic points.

Keywords: composite materials, annular plate, 3D-reinforcement, stress-strain properties, anisotropic materials, analytical solution

1. Introduction

3D-reinforced composite materials, including high-temperature composite materials engineered to work under temperatures higher than 1000 °C are widely used in modern space industry. They are present in numerous applications in high-heat elements of rocket engines, such as combustion chambers and nozzle heads, as well as in body parts of rockets and hypersonic spacecrafts, that are subject to significant aerodynamical and thermal loads under their operation, such as payload fairings and leading edges. This topic is more fully studied in papers [1], [2] and [3].

At the same time one should notice that physical and mechanical properties of high-temperature anisotropic composites, such as carbon-carbon composite materials (CCCM) and carbon-ceramic materials, are not yet fully studied. A major part of existing data on properties of such materials is obtained solely by experiments, like in extensive studies of high-temperature ceramics [4] and [5]. The same applies to articles on CCCM tensile and shear strength [6] and [7], as well as to papers [8] [9] [10] [11] [12] [13] [14]. In all of those cases obtained data is relevant only for a specific material or composition, and can’t be generalized for a corresponding class of materials, that is apparently a substantial drawback. Therefore the problem of development of methods for physical and mechanical properties determination of high-temperature composites (such as elastic and shear moduli, Poisson’s...
ratio) is highly relevant. Radial compression experiments are often carried out to solve this problem, which is studied in [15] and [16]. Apart from that, blade-shaped specimens are often used for tensile tests [17] [18].

Method of physical and mechanical properties determination for high-temperature anisotropic materials, offered by authors of this paper, is based on the problem solution about an annular plate subjected to compressing forces with consideration of transverse shear deformations. One should notice that analytical solution of a similar problem about diametrally compressed thin annular disk is present in paper [19]. At the same time it’s highly complicated to obtain the analytical solution for a problem studied by authors for anisotropic, in particular, orthotropic, materials because of equation system difficulty. Analytical solution of the problem about an annular plate subjected to compression is obtained by the authors of this paper. This solution can be used to find relations between required elasticity characteristics and radial displacements of annular plate characteristic points.

2. Statement of the problem

Orthogonally reinforced annular plate with inner radius \( r_1 \), outer radius \( r_2 \) and thickness \( h \) is studied in this paper. Annular plate is subjected to compressing forces \( F \), as shown on figure 1.

![Figure 1. Annular plate subjected to external load.](image)

Annular plate behavior under applied load is governed by following equations:

A) Differential equations of equilibrium.

Let’s highlight an annular plate elementary sector of width \( d\varphi \) for an arbitrary section \( \varphi \) and examine it’s equilibrium (figure 2). The following loads (per unit of width) act in annular plate section: axial force \( N \), transverse force \( Q \) and bending moment \( M \).

Having written the equilibrium equations \( \sum F_y = 0; \sum F_c = 0; \sum M_A = 0 \) in local coordinate system \( yOz \), we obtain the following:

\[
\frac{dN}{d\varphi} = -Q; \frac{dQ}{d\varphi} = N; \frac{dM}{d\varphi} = QR
\]  

(1)

B) Kinematic (deformation) equations (the Cauchy equations).

Determine relations between generalized strains \( \varepsilon_{22}, \kappa_2, \Theta_{23} \), and displacements \( v, w, \theta_{23} \) (circumferential, radial and angular respectively) taking into account compatibility relations [20]:

\[
\varepsilon_{22} = \frac{1}{R} \left( \frac{dv}{d\varphi} + w \right); \kappa_2 = \frac{d\theta_{23}}{Rd\varphi}; \Theta_{23} = \frac{1}{R} \left( R\theta_{23} + \frac{dv}{d\varphi} - v \right)
\]  

(2)

C) Physical relations (Hooke’s law).

Determine relations between strains and stresses in an arbitrary section of annular plate. Let’s write them in matrix representation in global and local coordinate systems:

\[
\{\varepsilon\} = [S]\{\sigma\}; \{\varepsilon\} = [S]\{\sigma\}
\]  

(3)
Here the elastic compliances matrix in global coordinate system:

\[
\begin{bmatrix}
 s_{22}' & s_{23}' & s_{26}' \\
 s_{32}' & s_{33}' & s_{36}' \\
 s_{62}' & s_{63}' & s_{66}'
\end{bmatrix} = \begin{bmatrix}
 \frac{1}{E_2} & -\frac{v_{23}}{E_2} & 0 \\
 -\frac{v_{23}}{E_2} & \frac{1}{E_3} & 0 \\
 0 & 0 & \frac{1}{G_{23}}
\end{bmatrix}
\] (4)

Using the following relations of annular plate stress-strain state parameters in global and local coordinate systems:

\[
\{\sigma\} = [T_1]\{\sigma'\}; \{\epsilon\} = [T_2]\{\epsilon'\}
\] (5)

Where matrices of rotation:

\[
[T_1] = [T_1](\varphi) = \begin{bmatrix}
 \cos^2 \varphi & \sin^2 \varphi & \sin 2\varphi \\
 \sin^2 \varphi & \cos^2 \varphi & -\sin 2\varphi \\
 -\frac{\sin 2\varphi}{2} & \frac{\sin 2\varphi}{2} & \cos 2\varphi
\end{bmatrix}; [T_2] = [T_2](\varphi) = \begin{bmatrix}
 \cos^2 \varphi & \sin^2 \varphi & \sin 2\varphi \\
 \sin^2 \varphi & \cos^2 \varphi & -\frac{\sin 2\varphi}{2} \\
 -\sin 2\varphi & \sin 2\varphi & \cos 2\varphi
\end{bmatrix}
\] (6)

Through mathematical manipulations we obtain relations for the elastic compliances matrix components in local coordinate system:

\[
s_{22} = I_1 + I_2 \cos 2\varphi + I_3 \cos 4\varphi; s_{33} = I_4 - I_2 \cos 2\varphi + I_3 \cos 4\varphi; s_{23} = s_{32} = I_1 - \frac{I_4}{2} - I_3 \cos 4\varphi
\]

\[
s_{26} = s_{62} = -\left(2I_2 \sin 4\varphi + I_2 \sin 2\varphi\right); s_{36} = s_{63} = 2I_3 \sin 4\varphi - I_2 \sin 2\varphi; s_{66} = I_4 - 4I_3 \cos 4\varphi
\] (7)

Where:

\[
I_1 = \frac{3s_{22} + 2s_{23} + 3s_{33} + s_{66}}{8}; I_2 = \frac{s_{22} - s_{33}}{2}; I_3 = \frac{s_{22} - 2s_{23} + s_{33} - s_{66}}{8}; I_4 = \frac{s_{22} - 2s_{23} + s_{33} + s_{66}}{2}
\] (8)

D) Boundary conditions

Let’s write down boundary conditions for annular plate neutral axis, taking symmetry into account:

\[
v(0) = 0; \theta_{23}(0) = 0; Q(0) = 0; v\left(\frac{\pi}{2}\right) = 0; \theta_{23}\left(\frac{\pi}{2}\right) = 0; Q\left(\frac{\pi}{2}\right) = \frac{F}{2h}
\] (9)

3. Solution of the problem

Because of symmetrical geometry and load it’s convenient to analyze only 1/4 of annular plate in solving. The problem is solved in polar coordinate system \(rO\varphi\) (figure 1). Dependence of elastic properties of annular plate section on a corresponding section angle \(\varphi\) is a specialty of featured problem.

Let’s assume \(\sigma_{33} = 0\) due to small value of interlaminar stresses. Then, having written (3) in expanded form, we obtain through manipulations:

\[
\sigma_{22} = d_{22} e_{22} + d_{26} e_{26}; \sigma_{23} = d_{26} e_{22} + d_{66} e_{23}
\] (10)

Where:

\[
E_2 = d_{22} = \frac{1}{I_1 + I_2 \cos 2\varphi + I_3 \cos 4\varphi}; G_{23} = d_{66} = \frac{1}{I_4 - 4I_3 \cos 4\varphi}; \eta_{26} = \frac{d_{26}}{d_{66}} = \frac{I_2 \sin 2\varphi + 2I_2 \sin 4\varphi}{I_4 - 4I_3 \cos 4\varphi}
\] (11)
Because we take into account changes in annular plate metrics (its curvature radius change), we have before and after deformation (Figure 2):

\[ CD = ds \rightarrow AB = ds \left(1 + \frac{z}{R}\right); C'D' = ds \left(1 + \varepsilon_{22}\right) \rightarrow A'B' = ds \left(1 + \varepsilon_{22}\right) \left(1 + \frac{z}{R'}\right) \]  (12)

Curvature of annular plate neutral axis before and after deformation, considering (12):

\[
\frac{1}{R} \frac{d\theta_{23}^*}{ds} = \frac{1}{R'} \frac{d\theta_{23}^{**}}{ds} = \frac{d}{ds} \left(\theta_{23}^* + \psi_{23} \right) = \frac{1}{1 + \varepsilon_{22}} \left(\frac{1}{R} + \kappa_2 \right)
\]  (13)

Where \( \theta_{23}^*, \theta_{23}^{**} \) — angles describing position of annular plate section before and after deformation respectively, \( \psi_{23} \) — angle of rotation of annular plate section normal line, \( \kappa_2 = \frac{d\psi_{23}}{ds} \) — annular plate curvature change, \( R = r_1 + b_0 \). As a result circumferential strain \( \varepsilon_{22}^z \) of annular plate layer, separated by \( z \) from its neutral axis, considering (12) and (13):

\[
\varepsilon_{22}^z = \frac{A'B' - AB}{AB} = \frac{ds \left(1 + \varepsilon_{22}\right) \left[1 + \frac{z}{1 + \varepsilon_{22}} \left(1 + \kappa_2\right)\right] - ds \left(1 + \frac{z}{R}\right)}{ds \left(1 + \frac{z}{R}\right)} = \frac{1 + \varepsilon_{22} + \frac{z}{R} + \frac{z\kappa_2}{R} \left(1 + \frac{z}{R}\right)}{1 + \frac{z}{R}} = \frac{1 + \frac{z}{R}}{1 + \frac{z}{R}}
\]  (14)

Let’s assume that annular plate neutral axis matches with its middle surface, i.e. \( R = r_1 + \frac{b}{2} \) (figure 3).
Figure 3. To obtaining of physical relations for annular plate

From the second equation of system (10) we obtain:

$$\gamma_{23} = \frac{\sigma_{23}}{d_{66}} - \frac{d_{26}}{d_{66}} \varepsilon_{22}$$ (15)

Where $\varepsilon_{22}, \gamma_{23}$ – circumferential and shear strains of annular plate neutral axis. Then, considering (14) and (15), we obtain transverse shear deformation averaged to section height [21]:

$$\Theta_{23} = \frac{1}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma_{23} \left( 1 + \frac{z}{R} \right) dz = \frac{1}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left( \frac{\sigma_{23}}{d_{66}} - \frac{d_{26}}{d_{66}} \left( \varepsilon_{22} + \frac{z \kappa_2}{1 + \frac{z}{R}} \right) \right) \left( 1 + \frac{z}{R} \right) dz = \frac{Q}{d_{66}b} - \frac{d_{26}}{d_{66}} \varepsilon_{22}$$ (16)

Where transverse force per unit of width $Q = Q_{23} = \sigma_{23}b$, then from (16):

$$Q = bd_{26}\varepsilon_{22} + d_{66}\Theta_{23} = C_{23}\varepsilon_{22} + K_2\Theta_{23}$$ (17)

Having inserted (10), (15) and (17) into integral relations for loads per unit of width [22], we obtain:

$$N = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{22}dz = B_{22}\varepsilon_{22} + C_{22}\kappa_2 + C_{23}\Theta_{23}, M = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{22}z dz = C_{22}\varepsilon_{22} + D_{22}\kappa_2N$$ (18)

Where:

$$B_{22} = \left( r_i + \frac{b}{2} \right) \ln \left( \frac{b}{r_i} + 1 \right) \left[ \frac{d_{22}}{d_{66}} - \frac{d_{26}}{d_{66}} \right] + b \frac{d_{26}}{d_{66}}$$ – tensile stiffness. (19)

$$C_{22} = \left( r_i + \frac{b}{2} \right) \left( \frac{b}{r_i} + 1 \right) - b \left[ \frac{d_{22}}{d_{66}} - \frac{d_{26}}{d_{66}} \right]$$ – combined stiffness. (20)

$$D_{22} = \left( r_i + \frac{b}{2} \right)^2 \left( \frac{b}{r_i} + 1 \right) - b \left[ \frac{d_{22}}{d_{66}} - \frac{d_{26}}{d_{66}} \right]^2$$ – bending stiffness. (21)

Having written (17) and (18) combined in matrix representation, we obtain:
\( \{F\} = [B]\{\varepsilon\} \)  

(22)  

Where:

\[
\{F\} = \begin{bmatrix} N \\ Q \end{bmatrix}; \quad [B] = \begin{bmatrix} B_{22} & C_{22} & C_{23} \\ C_{22} & D_{22} & 0 \\ C_{23} & 0 & K_2 \end{bmatrix}; \quad \{\varepsilon\} = \begin{bmatrix} \varepsilon_{22} \\ \varepsilon_{23} \end{bmatrix} 
\]  

(23) combined with (1) and (2) form a resulting equations system of our problem, which is resolved related to unknown forces and displacements considering boundary conditions (9). Having inserted (2) in (23), we obtain a system of 6 equations with 6 unknowns \( (N, M, Q, v, w, \theta_{23}) \). Analytical solution of the problem was obtained by the authors of this paper through mathematical manipulations.

4. Results of calculations and their discussion

Analytical solution of the problem is the following:

\[
N = -\frac{F}{2h} \sin \varphi; M = \frac{FR}{h} \left[ \sin \varphi - \frac{\theta_0}{2 \pi I_1 (k_1 + 1)} \right]; Q = \frac{F}{2h} \cos \varphi \\

v = \frac{FR}{2bh} \left[ \theta_0 \left( 1 - \frac{2 \varphi}{\pi} \right) + v_s (\pi - 2 \varphi) \sin \varphi + v_{s2} \sin 2 \varphi + v_{s4} \sin 4 \varphi + v_c \cos \varphi + v_{c3} \cos 3 \varphi + v_{c5} \cos 5 \varphi \right] \\

w = -\frac{FR}{2bh} \left[ w_0 + w_s \sin \varphi + w_{s3} \sin 3 \varphi + w_{s5} \sin 5 \varphi + v_s (\pi - 2 \varphi) \cos \varphi + w_{c2} \cos 2 \varphi + w_{c4} \cos 4 \varphi \right] \\

\theta_{23} = -\frac{F}{2bh} \left[ \theta_0 \left( 1 - \frac{2 \varphi}{\pi} \right) + \theta_{s2} \sin 2 \varphi + \theta_{s4} \sin 4 \varphi + \theta_c \cos \varphi + \theta_{c3} \cos 3 \varphi + \theta_{c5} \cos 5 \varphi \right] 
\]  

(24)

Where:

\[
k_1 = \frac{b}{R \ln \left( \frac{2R + b}{2R - b} \right)}; \quad \theta_0 = \frac{b}{3 \pi I_1} I_1 \left( k_1 - \frac{k_1 - 1}{k_1 + 1} \right) \\
v_s = \frac{k_1 (2 I_1 - I_3) + 2 I_4}{8}; \quad v_{s2} = \frac{\theta_0 I_2}{3 \pi I_1} \left( k_1 - \frac{k_1 - 1}{k_1 + 1} \right); \quad v_{s4} = \frac{\theta_0 I_3}{3 \pi I_1} \left( k_1 - \frac{7}{k_1 + 1} \right) \\
w_s = \frac{-48 I_1 + 15 I_2 + 4 I_3}{48} + \frac{15 I_2 + 2 I_3}{6}; \quad w_{s3} = \frac{k_1 (I_2 - I_3) + 8 I_2 + 4 I_3}{15 \pi I_1} \\
\theta_{s2} = \frac{\theta_0 I_2}{2 \pi I_1} \frac{1}{6}; \quad \theta_{s4} = \frac{\theta_0 I_3}{2 \pi I_1} \frac{1}{6}; \quad \theta_c = \frac{\theta_0 I_2}{2 \pi I_1} \frac{1}{6}; \quad \theta_{c3} = \frac{(k_1 I_2 - I_3) - 12 I_3}{16}; \quad \theta_{c5} = \frac{I_3 (k_1 - 20)}{240}
\]  

(25)

Numerical solution of the problem was also obtained by authors using MathCAD 15.0 software. Apart from that, the problem was solved through finite element analysis (FEA) in MSC Patran + MSC Nastran 2014 on a mesh of 45*14 (radial*circumferential directions respectively) QUAD4 elements. All solutions mentioned were obtained and compared with the analytical solution (24) for the following initial values (corresponding to figure 1):

\[ r_1 = 40 \text{ mm} = 0.04 \text{ m}; r_2 = 60 \text{ mm} = 0.06 \text{ m}; h = 10 \text{ mm} = 0.01 \text{ m}; E_2 = 15 \text{ GPa} = 15 \times 10^9 \text{ Pa}; \]
\[ E_3 = 10 \text{ GPa} = 10 \times 10^9 \text{ Pa}; G_{23} = 5 \text{ GPa} = 5 \times 10^9 \text{ Pa}; v_{23} = 0.1; F = 2 \text{ kN} = 2 \times 10^4 \text{ N} \]  

(26)
Figures 4–6 illustrate how displacements of annular plate change with the angle $\varphi$ [deg]. Here the the analytical solution, corresponding to relations (24) and matching the numerical solution obtained in MathCAD 15.0 software, correspond to markers ○. The FEA results correspond to markers ●.

Comparison of the obtained results has shown that FEA solution are almost identical to the analytical solution. Maximum error of FEA solution for angular displacements doesn’t exceed 5 %, while maximum error for linear displacements doesn’t exceed 1 %. This fact, along with absolute agreement of analytical solution and numerical solution obtained in MathCAD 15.0, confirms the accuracy of relations (24). The results of comparison described above show that relations (24) allow for determination of unknown elasticity characteristics of annular plate from 3D-reinforced composite material. This might be done by using the relations between required elasticity characteristics and displacements of annular plate characteristic points.

5. Conclusions
The analytical solution of the problem about 3D-reinforced composite annular plate compressed by concentrated forces was obtained. Transverse shear deformations and changes in annular plate metrics during deformation are taken into account in the obtained solution. The problem was also solved numerically using 2 techniques: the FEA in MSC Patran + MSC Nastran software, and analysis in MathCAD 15.0 software. The comparison of obtained results demonstrates the offered method applicability for unknown characteristics determination of 3D-reinforced composite materials.

References
[1] Kostikov V I and Varenkov A N 2003 Ultrahigh-temperature composite materials (Moscow: Intermet Engineering) pp 20–23
[2] Baker A A, Dutton S and Kelly D 2004 Composite materials for aircraft structures (Michigan: American Institute of Aeronautics and Astronautics) pp 449–462
[3] Guniaev G M and Gofin M Ya 2013 Aviat. Mater. Technol. S1 pp 62–90
[4] Neuman E W, Hilmas G E and Fahrenholtz W G 2016 J. Am. Ceram. Soc. 99(2) pp 597–603
[5] Neuman E W, Fahrenholtz W G and Hilmas G E 2019 Int. J. Appl. Ceram. Technol. 16(5) pp 1715–22
[6] Hatta H, Aoi T, Kawahara I, Kogo Y and Shiotai I 2004 J. Compos. Mater. 38(19) pp 1667–84
[7] Goto K, Ohkita H, Hatta H, Iseki H and Kogo Y 2007 Proc. Int. Conf. on Composite Materials (Kyoto) pp 1–6
[8] Guo S 2019 J. Am. Ceram. Soc. 102 pp 3843–48
[9] Hua G, Zhong J, Qi Y and Cheng X 2018 J. Am. Ceram. Soc. 101 pp 5717–31
[10] Nelyub V 2020 IOP Conf. Ser.: Mater. Sci. Eng. 709(2) 022037
[11] Bocharov A, Vigovskiy V and Nelyub V 2019 Materials Today: Proc pp 107–111
[12] Nelyub V A, Borodulin A S, Kobets L P and Malysheva G V 2016 Polym. Sci. Ser. D 9(3) pp 286–289
[13] Malysheva G V, Kobets L P and Borodulin A S 2016 Fibre Chemistry 47(6) pp 482–485
[14] Yanyan C, Nelyub V A and Malysheva G V 2019 Polym. Sci. Ser. D 12(3) pp 296–299
[15] Bakulin A A, Magnitskii I V, Ponomarev K A and Tashchilov S V 2015 Const. Comp. Mater. 138 pp 46–51
[16] Ponomarev K A 2015 Const. Comp. Mater. 146 pp 62–67
[17] Kobayashi H, Goto K, Hatta H, Koyama M and Fukuda H 2009 Proc. Int. Conf. on Composite Materials (Edinburgh) pp 1–8
[18] Smerdov A A Taurova L P Mironikhin A N Tashchilov S V and Magnitskii I V 2011 Const. Comp. Mater. 146 pp 35–50
[19] Tokovyy Y V, Hung K M and Ma C C 2010 J. Mathem. Sci. 165(3) pp 342–354
[20] Vasiliev V V 1988 Mechanics Of Composite Structures (Moscow: Mashinostroenie) p 215
[21] Vasiliev V V 1988 Mechanics Of Composite Structures (Moscow: Mashinostroenie) p 67
[22] Vasiliev V V 1988 Mechanics Of Composite Structures (Moscow: Mashinostroenie) p 69