STOCHASTIC DYNAMICS OF 2D FRACTIONAL GINZBURG-LANDAU EQUATION WITH MUTLIPlicative NOISE

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Abstract. In this work, we analyze the stochastic fractional Ginzburg-Landau equation with multiplicative noise in two spatial dimensions with a particular interest in the asymptotic behavior of its solutions. To get started, we first transfer the stochastic fractional Ginzburg-Landau equation into a random equation whose solutions generate a random dynamical system. The existence of a random attractor for the resulting random dynamical system is explored, and the Hausdorff dimension of the random attractor is estimated.

1. Introduction. Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation operator. The last few decades have seen an enormous growth in the development of related dynamical concepts and in the applicability of nonlinear fractional models in a rather diverse scientific fields, such as fluid mechanics [23], biology [8], kinetic theories of systems with chaotic dynamics [18], pseudochaotic dynamics [25], dynamics in a complex or porous media [12, 14], and quantum theory [9]. As described by Wheatcraft and Meerschaert [23], a fractional conservation of mass equation is needed to model fluid flow when the control volume is not large enough compared to the scale of heterogeneity and when the flux within the control volume is non-linear. In complex media, the propagation of acoustical waves, e.g. biological tissue, commonly implies attenuation obeying a frequency power-law. This kind of phenomenon has been described by a causal wave equation which incorporates fractional time derivatives [8]. Quite many other types of nonlinear fractional models have also been proposed and studied in physics and engineering, which include the fractional Schrödinger equation [4, 6], fractional Landau-Lifshitz equation [7], fractional Landau-Lifshitz-Maxwell equation [16], and fractional Ginzburg-Landau equation [21] etc. However, in some physical and biological phenomena, it is commonly believed that perturbations may be neglected in
the derivation of the ideal model such as molecular collisions in gases, liquids and electric fluctuations in resistors [5]. When considering the perturbations of each microscopic unit to the model, which will lead to a large complex system, people usually represent the micro effects by random perturbations in the dynamics of the macro observable. Recently, Stochastic and fractional partial differential equations and its applications have attracted continuous attention in the fields of mathematical physics and mathematical biology. One of the most interesting problems of stochastic partial differential equations is the asymptotic behavior of random dynamical systems. Some fundamental properties have been established, for example, by Crauel, Debussche and Flandoli [1, 2] who developed the theory for the existence of random attractors for stochastic systems that closely parallels the deterministic theory [22], and by Debussche [3] who proved that the Hausdorff dimension of the random attractor could be estimated by using the global Lyapunov exponents.

For fractional partial differential equations, much attention has been devoted to the well-posedness of solutions [7, 16] and the existence of positive solutions [20]. However, for stochastic fractional partial differential equations, very little has been undertaken as far as we know. This motivates us to start our study by considering the asymptotic behavior of solutions for an interesting stochastic fractional Ginzburg-Landau equation. As we know, the fractional Ginzburg-Landau equation was initially derived from the variational Euler-Lagrange equation for the fractal media, which was used to describe the dynamical processes in a medium with fractal dispersion [21]. In [17], Pu and Guo considered a one-dimensional fractional complex Ginzburg-Landau equation

\[
\dot{u} + (1 + iv)(-\Delta)^{\alpha}u + (1 + i\mu)|u|^{2\sigma}u = \rho u.
\]

The well-posedness of solutions was obtained by applying the semigroup method under the condition

\[
\frac{1}{2} \leq \sigma \leq \frac{1}{\sqrt{1 + \mu^2} - 1}.
\]

The existence of global attractors in \(L^2\) was also presented under the condition \(\sigma = 1\). In [10], we once studied the well-posedness and asymptotic behaviors of solutions of the fractional complex Ginzburg-Landau equation with the initial and periodic boundary conditions in two spatial dimensions. Estimates of the upper bounds of Hausdorff and fractal dimensions for the global attractor were presented.

In this paper, we consider the two-dimensional stochastic fractional Ginzburg-Landau equation with multiplicative noise

\[
d\dot{u} + \left[(1 + iv)(-\Delta)^\alpha u + (1 + i\mu)|u|^{2\sigma}u\right] dt = \rho dt + \beta dW(t), \quad x \in \mathbb{R}^2, \quad t > 0 \tag{1.1}
\]

under the initial and periodic boundary conditions

\[
\begin{align*}
\dot{u}(x, 0) &= u_0, & x \in \mathbb{R}^2, \quad (1.2) \\
u(x + 2\pi e_i, t) &= u(x, t), & i = 1, 2, & x \in \mathbb{R}^2, \quad t > 0, \quad (1.3)
\end{align*}
\]

where \(u(x, t)\) is a complex-valued function on \(\mathbb{R}^2 \times [0, +\infty)\). In (1.1), \(i\) is the imaginary unit, \(\rho > 0, \sigma > 0, \beta > 0, v\) and \(\mu\) are real constants, and \(\alpha \in (1/2, 1)\). The white noise described by a two-sided Wiener process \(W(t)\) on a complete probability space results from the fact that small irregularity has to be taken account into some circumstances. The study in this paper is a continuation of the previous work by Lu and Lü [11] in which asymptotic behaviors of solutions to equation (1.1) were discussed in the case of \(\sigma = 1\) in one spatial dimension.
As we see, for dynamical properties of random attractors, discussions are usually carried out in $L^2$ and do not include the estimate of the Hausdorff dimension of random attractors. In this study, we investigate the existence of the random attractor in $H^1_p$ for the stochastic fractional Ginzburg-Landau equation with multiplicative noise in two spatial dimensions, and furthermore estimate the Hausdorff dimension of the random attractor by analyzing the corresponding random dynamical system and the Lyapunov exponents.

The rest of this paper is organized as follows. In Section 2, some preliminaries and notations for random dynamical systems are introduced. In Section 3, we present a continuous random dynamical system for the stochastic fractional Ginzburg-Landau equation. In Section 4, the existence of the random attractor for the stochastic Ginzburg-Landau equation is established. Section 5 is dedicated to the upper bound estimates of the Hausdorff dimension of the random attractor.

2. Preliminaries and notations. In this section, we introduce some basic concepts related to random attractors of stochastic dynamical systems. For the detailed information and related applications, we refer to [1, 2] and the references therein.

Let $(X, ||\cdot||_X)$ be a separable Hilbert space with the Borel $\sigma$-algebra $B(x)$, and $\{\theta_t : \Omega \rightarrow \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$ such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = I$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$. Thus, $(\theta_t)_{t \in \mathbb{R}}$ is a flow, and $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a (measurable) dynamical system. We also denote the mapping $S(t, s; \omega) : X \rightarrow X, -\infty < s < t < \infty$, with the explicit dependence on $\omega \in \Omega$.

Definition 2.1. Given $t \in \mathbb{R}$ and $\omega \in \Omega$, $K(t, \omega) \subset X$ is called an attracting set, if for all bounded sets $B \subset X$,

$$d(S(t, s; \omega)B, K(t, \omega)) \rightarrow 0 \text{ as } s \rightarrow -\infty,$$

where $d(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} ||y - z||_X$, for any $Y, Z \subseteq X$.

Definition 2.2. A set-valued map $A(\omega) : \Omega \rightarrow 2^X$ taking values in the closed subsets of $X$ is called to be measurable, if for all $x \in X$ the mapping $\omega \mapsto d(A(\omega), x)$ is measurable.

Definition 2.3. The random omega limit set of a bounded set $B \subset X$ at time $t$ is given by

$$A(B, t, \omega) = \bigcap_{T \leq t} \bigcup_{s \leq T} S(t, s; \omega)B.$$

Definition 2.4. Let $S(t, s; \omega)_{t \geq s, \omega \in \Omega}$ be a stochastic dynamical system. A random set $A(\omega)$ is called a random attractor if the following conditions are satisfied for $P$-a.e. $\omega \in \Omega$.

- It is the minimal closed set such that for $t \in \mathbb{R}$ and $B \subset X$ it holds
  $$d(S(t, s; \omega)B, A(\omega)) \rightarrow 0 \text{ as } s \rightarrow -\infty.$$

  Namely, $A(\omega)$ attracts $B$ ($B$ is a deterministic set).

- $A(\omega)$ is the largest compact measurable set, which is invariant in the sense that
  $$S(t, s; \omega)A(\theta_s \omega) = A(\theta_t \omega), \quad s < t.$$

Following [2], we have the result regarding the existence of random attractors.
Theorem 2.1. Let \( S(t,s;\omega)_{t \geq s, \omega \in \Omega} \) be a stochastic dynamical system satisfying the following conditions:

1. \( S(t,r;\omega)S(r,s;\omega) = S(t,s;\omega)x \) for all \( s \leq r \leq t \) and \( x \in X \);
2. \( S(t,s;\omega) \) is continuous in \( X \), for all \( s \leq t \);
3. for all \( s < t \) and \( x \in X \), the mapping \( \omega \mapsto S(t,s;\omega)x \) from \( (\Omega,\mathcal{F}) \) to \( (X,\mathcal{B}(x)) \) is measurable;
4. for all \( t,x \in X \) and \( \omega \in \Omega \), the mapping \( s \mapsto S(t,s;\omega)x \) is right continuous at any point.

Assume that there exists a group \( \theta_t, t \in \mathbb{R} \), of measure preserving mapping, such that

\[
S(t,s;\omega)x = S(t,s,0;\omega)x
\]

holds for \( \omega \in \Omega \). There exists a compact attracting set \( K(\omega) \) at time 0 for \( \omega \in \Omega \). We set \( A(\omega) = \bigcup_{B \in X} A(B,\omega) \), where the union is taken over all the bounded subsets of \( X \), and \( A(B,\omega) \) is given by

\[
A(B,t,\omega) = \bigcap_{T \leq 0} \bigcup_{s \leq T} S(t,s;\omega)B.
\]

Then, \( A(\omega) \) is the random attractor.

Although the random attractor is not uniformly bounded, it is expected that the theory on the Hausdorff dimension of a global attractor of a deterministic dynamical system can be generalized to the stochastic case under some assumptions. Due to [3], we have

Theorem 2.2. Let \( \mathcal{A}(\omega) \) be a compact measurable set which is invariant under a random map \( S(\omega), \omega \in \Omega \), for some ergodic metric dynamical system \( (\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t \in \mathbb{R}}) \). Assume that the following conditions are satisfied.

1. \( S(\omega) \) is almost surely uniformly differentiable on \( \mathcal{A}(\omega) \), that is, for every \( u, u + h \in \mathcal{A}(\omega) \) there exists \( DS(\omega,u) \) in \( \mathcal{L}(X) \), the space of the bounded linear operators from \( X \) to \( X \), such that

\[
\|S(\omega)(u + h) - S(\omega)u - DS(\omega,u)h\| \leq \tilde{k}(\omega)\|h\|^{1+\delta},
\]

where \( \delta > 0 \) and \( \tilde{k}(\omega) \) is a random variable satisfying \( \tilde{k}(\omega) \geq 1 \) and \( E(\ln \tilde{k}) < \infty \).

2. \( \omega_d(DS(\omega,u)) \leq \bar{\omega}_d(\omega) \) holds when \( u \in \mathcal{A}(\omega) \) and there is some random variable \( \bar{\omega}_d(\omega) \) satisfying \( E(\ln(\bar{\omega}_d)) < 0 \), where

\[
\omega_d(DS(\omega,u)) = \alpha_1(DS(\omega,u)) \cdots \alpha_d(DS(\omega,u)),
\]

\[
\alpha_d(DS(\omega,u)) = \sup_{G \subseteq X, \dim G \leq d, \|\varphi\|_X = 1} \inf_{\varphi \in G} \|DS(\omega,u)\varphi\|.
\]

3. \( \alpha_1(DS(\omega,u)) \leq \bar{\alpha}_1(\omega) \) holds when \( u \in \mathcal{A}(\omega) \) and there is a random variable \( \bar{\alpha}_1(\omega) \geq 1 \) with \( E(\ln \bar{\alpha}_1) < \infty \).

Then the Hausdorff dimension \( d_H(\mathcal{A}(\omega)) \) of \( \mathcal{A}(\omega) \) is less than \( d \) almost surely.

For convenience, we redefine some notations related to fractional derivative equations and fractional Sobolev spaces. Firstly, we present the definition and properties of \((-\Delta)^\alpha\) through the Fourier series [7]. Since \( u \) is a periodic function, it can be
expressed by a Fourier series $u = \sum_{k \in \mathbb{Z}^2} u_k e^{ik \cdot x}$, and $u_{x_i} = \sum_{k \in \mathbb{Z}^2} ik_i u_k e^{ik \cdot x}$ ($i = 1, 2$). So $(\triangle)^\alpha$ can be defined as

$$(\triangle)^\alpha u = \sum_{k \in \mathbb{Z}^2} |k|^{2\alpha} u_k e^{ik \cdot x},$$

where $\triangle = \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2$.

Let $H^p_\alpha(D)$ denote the complete Sobolev space of the order $\alpha$ under the norm:

$$\|u\|_{H^p_\alpha(D)} = \left( \sum_{k \in \mathbb{Z}^2} |k|m^\alpha |u_k|^2 + \sum_{k \in \mathbb{Z}^2} \|u_k\|_p^2 \right)^{\frac{1}{2}} = \left( \|(-\triangle)^\alpha u\|^2 + \|u\|^2 \right)^{\frac{1}{2}}.$$

By virtue of the definition of $(\triangle)^\alpha$ and integration by parts [7], we have

**Lemma 2.1.** If $f, g \in H^{2\alpha}_p(D)$, then there holds

$$\int_D (-\triangle)^\alpha f \cdot g dx = \int_D (-\triangle)^{\alpha_1} f \cdot (-\triangle)^{\alpha_2} g dx,$$

(2.4)

where $\alpha_1, \alpha_2$ are nonnegative constants and satisfy $\alpha_1 + \alpha_2 = \alpha$.

In addition, the following Gagliardo-Nirenberg inequality [15] will be frequently used.

**Lemma 2.2.** Suppose that $\Lambda \subset \mathbb{R}^n$ is a bounded domain and its boundary is smooth. Let $u$ belong to $L^q(\Lambda)$ and its derivatives of the order $m$, $D^m u$, belong to $L^r(\Lambda)$, where $1 \leq q$ and $r \leq \infty$. For the derivatives $D^j u$, $0 \leq j < m$, it holds

$$\|D^j u\|_{L^p} \leq c\|u\|_{W^{m,r}}^{\frac{j}{m}} \|u\|_{L^q}^{1-\frac{j}{m}},$$

(2.5)

where

$$\frac{1}{p} = \frac{j}{n} + \theta \left( \frac{1}{r} - \frac{m}{n} \right) + (1-\theta) \frac{1}{q},$$

for all $\theta$ in the interval

$$\frac{j}{m} \leq \theta \leq 1.$$

Here the constant $c$ depends only on $n, m, j, q, r$ and $\theta$, with the two exceptional cases:

1. If $j = 0$, $rm < n$ and $q = \infty$, then we make the additional assumption that either $u$ tends to zero at infinity or $u \in L^{\tilde{q}}$ for some $\tilde{q} > 0$.

2. If $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, then inequality (2.5) holds only for $j/m \leq \theta < 1$.

Let $D = [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^2$. Throughout the whole paper, we denote by $(\cdot, \cdot)$ the usual inner product of $L^2(D)$, by $\|\cdot\|_{H^m}$ the norm of the Sobolev space $H^m(D)$, and $\|\cdot\|_m = \|\cdot\|_{L^\infty(D)} (m = 1, 2, \ldots, \infty)$. Let $L^2_i(D) = \{ \varphi \in L^2(D) | \varphi(x + 2\pi e_i) = \varphi(x), i = 1, 2 \}$ with the norm defined as that of $L^2(D)$, and $H^m_i(D) = \{ \varphi \in H^m(D) | \varphi(x + 2\pi e_i) = \varphi(x), i = 1, 2 \}$ with the norm defined as that of $H^m(D)$.

In the forthcoming discussions, we use $c$ and $c_j$ ($j = 1, 2, \ldots$) to denote different positive constants which depend only on the constants $\rho, \nu, \mu, \alpha$ and $\sigma$. Moreover, we denote $\int_D f dx$ by the notation $\int f$ for simplicity.
3. Stochastic fractional Ginzburg-Landau equation. In this section, we present the existence of a continuous random dynamical system for the stochastic fractional complex Ginzburg-Landau equation perturbed by a multiplicative white noise in the Itô sense. Thanks to the special linear multiplicative noise, the stochastic fractional Ginzburg-Landau equation can be reduced to an equation with random coefficients by a suitable transform of variables. Let us consider a set of continuous functions with the value 0 at 0:

$$\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}.$$ 

Let $W(t, \omega) = \omega(t)$. Defined the time shift by

$$\theta_s \omega(t) = \omega(t + s) - \omega(t), \quad s, t \in \mathbb{R}.$$

The process $z(t) = e^{-\beta W(t)}$ satisfies the stochastic differential equation:

$$dz(t) = \frac{1}{2} \beta^2 z dt - \beta z dW(t).$$

Furthermore, for any $t$ and $s$ it has

$$z(t, \theta_s \omega) = z(t + s, \omega), \quad \text{P-a.s.}$$

Here the exceptional set may be a priori depending on $t$ and $s$. In fact, we suppose that $z$ has a continuous modification and, once this modification is chosen, the exceptional set is independent of $t$.

We rewrite the unknown $v(t)$ as $v(t) = z(t) u(t)$ to obtain the following random differential equation

$$v_t = -(1 + i \nu)(-\Delta)^{\alpha} v + \left( \rho + \frac{1}{2} \beta^2 \right) v - (1 + i \mu) z^{-2\sigma} |v|^{2\sigma} v, \quad t > s, \quad (3.6)$$

with the initial data at time $s$

$$v(x, s) = v_s, \quad x \in \mathbb{R}^2 \quad (3.7)$$

and the periodic boundary condition

$$v(x + 2\pi e_i, t) = v(x, t), \quad x \in \mathbb{R}^2. \quad (3.8)$$

We now construct a random dynamical system modeling the stochastic fractional Ginzburg-Landau equation. The existence and uniqueness of the solution of the problem (3.6)-(3.8) can be obtained [17], which defines a stochastic dynamical system $(S(t, s; w))_{t \geq s, \omega \in \Omega}$ by

$$S(t, s; w) u_s = u(t, \omega; s, u_s) = v(t, \omega; s, u_s z(s, \omega)) z(t, \omega).$$

4. A priori estimates and existence of a random attractor. In this section, we discuss a priori estimates of the solution of equation (3.6), which can be used to prove the existence of a compact absorbing set. By virtue of Theorem 2.1, the existence of random attractor can be established.

**Lemma 4.1.** There exists a random radius $r_0(\omega) > 0$ such that, for any given $R > 0$, there exists $\delta_1(\omega) \leq -2$ such that for all $s \leq \delta_1(\omega)$ and $u_s \in L^2_p(D)$ satisfying $\|u_s\| \leq R$, the following inequality holds for P-a.e. $\omega \in \Omega$,

$$\|v(t)\|^2 + e^{-\rho t} \int_s^t e^{\rho \tau} \|v(\tau)\|^2 d\tau \leq r_0^2(\omega), \quad \forall t \in [-2, 0], \quad (4.9)$$
where \( r_0(\omega) \) is to be determined.

**Proof.** Taking the inner product of equation (3.6) with \( v \) in \( L^2 \) and considering the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \|(-\triangle)^{\frac{3}{4}}v\|^2 + z^{-2\sigma} \int |v|^{2\sigma+2} = \left( \rho + \frac{1}{2} \beta^2 \right) \|v\|^2. \tag{4.10}
\]

By Young’s inequality, we have

\[
(\beta^2 + 3\rho + 1)\|v\|^2 = (\beta^2 + 3\rho + 1) \int |v|^2 \\
\leq z^{-2\sigma} \int |v|^{2\sigma+2} + C'_0 z^2,
\]

where \( C'_0 = \sigma|\Omega| \left( \frac{\beta^2+3\rho+1}{\sigma+1} \right)^{\frac{\sigma+1}{\sigma}} \). Rewrite (4.10) as

\[
\frac{d}{dt} \|v\|^2 + \rho \|v\|^2 + 2\|(-\triangle)^{\frac{3}{4}}v\|^2 + z^{-2\sigma} \int |v|^{2\sigma+2} + \|v\|^2 \leq C'_0 z^2, \tag{4.11}
\]

which implies that for any \( t > s \) it holds

\[
\|v(t)\|^2 + e^{-\rho t} \int_s^t e^{\rho \tau} \|v(\tau)\|^2 d\tau \leq e^{-\rho t} \left( e^{\rho s} \|v_s\|^2 + C'_0 t \int_s^t e^{\rho \tau} z^2(\tau) d\tau \right) \\
= e^{-\rho t} \left( e^{\rho s} z^2(s) \|u_s\|^2 + C'_0 t \int_s^t e^{\rho \tau} z^2(\tau) d\tau \right).
\]

Now, we estimate the last two terms in the above inequality. First, thanks to

\[
\lim_{s \to -\infty} \frac{W(s)}{s} = 0, \quad \text{P-a.s..} \tag{4.12}
\]

We infer that for \( \varepsilon = \frac{\rho}{43} \), there exists \( s'_1(\omega) \leq -2 \) such that

\[
\left| \frac{W(s)}{s} \right| < \varepsilon, \quad \text{as } s < s'_1(\omega),
\]

which implies that for \( \forall s < s'_1(\omega) \) there holds

\[
e^{\rho s} z^2(s) = e^{\rho s} e^{-2\beta W(s)} = e^{s(\rho-2\beta W(s)/s)} < e^{\frac{2}{2}s}.
\]

So the last term can be bounded as

\[
C'_0 \int_s^t e^{\rho \tau} z^2(\tau) d\tau \leq C'_0 \int_s^0 e^{\rho \tau} e^{-2\beta W(\tau)} d\tau \\
\leq C'_0 \left( \int_s^{s'_1(\omega)} e^{\rho \tau} e^{-2\beta W(\tau)} d\tau + \int_{-\infty}^{s'_1(\omega)} e^{\frac{2}{2} \tau} d\tau \right).
\]

Next, from (4.12) it is easy to check that

\[
e^{\rho s} z^2(s) = e^{\rho s} e^{-2\beta W(s)} \to 0, \quad \text{P-a.s. as } s \to -\infty. \tag{4.13}
\]

So we see that, for any \( u_s \in L^2_p(D) \) with \( \|u_s\| \leq R \), there exists a time \( s_1(\omega) \leq s'_1(\omega) \) such that

\[
e^{\rho s} z^2(s) \|u_s\|^2 \leq e^{\rho s} z^2(s) R^2 \leq 1, \quad \text{P-a.s., for } \forall s \leq s_1(\omega).
\]
For any $t \in [-2, 0]$ we deduce that
\[
\|v(t)\|^2 + e^{-pt}\int_s^t e^\rho\|v(\tau)\|^2 d\tau \leq e^{2\rho} \cdot \left[1 + C_0\left(\int_{s}^{0} e^{\rho\tau - 2\beta W(\tau)} d\tau + \int_{-\infty}^{s} e^{\rho\tau - 2\beta\varepsilon \tau} d\tau\right)\right]
\]
\[
\triangleq r_0^2(\omega). \tag{4.14}
\]
Consequently, we see that the proof is completed. \hfill \Box

**Lemma 4.2.** Suppose that $\sigma$ and $\mu$ satisfy the condition
\[
\sigma \leq \frac{1}{\sqrt{1 + \mu^2} - 1}. \tag{4.15}
\]
There exists a random radius $r_1(\omega) > 0$ such that, for any given $R > 0$, there exists $s_2(\omega) \leq -2$ such that for all $s \leq s_2(\omega)$ and $u_s \in H^1_0(D)$ satisfying $\|u_s\|_{H^1_0(D)} \leq R$, the following inequality holds for $P$-a.e. $\omega \in \Omega$,
\[
\|\nabla v(t)\|^2 + e^{-pt}\int_s^t e^\rho\|\nabla v(\tau)\|^2 d\tau \leq e^{2\rho} + c_1 \cdot r_0^2(\omega),
\]
where
\[
r_0^2(\omega) = e^{2\rho} + c_1 \cdot r_0^2(\omega).
\]
Proof. Taking the inner product of equation (3.6) with $-\Delta v$ in $L^2$ and considering the real part, we obtain
\[
\frac{d}{dt}\|\nabla v\|^2 + 2\|(-\Delta)^{\frac{\sigma + 1}{2\sigma}} v\|^2 - 2z^{-2\sigma}\text{Re}(1 + i\mu)(|v|^{2\sigma} v, \Delta v) \tag{4.17}
\]
Estimating the third term on the left-hand side by integration by parts gives
\[
-\text{Re}(1 + i\mu)(|v|^{2\sigma} v, \Delta v)
\]
\[
= \text{Re}(1 + i\mu) \int \left(\sigma + 1\right)|v|^{2\sigma} |\nabla v|^2 + \sigma|v|^{2(\sigma - 1)}(v \nabla \bar{v})^2\Bigg)\]
\[
= \frac{1}{2} \int |v|^{2(\sigma - 1)} (2(\sigma + 1)|v|^2 |\nabla v|^2 + \sigma(1 + i\mu)(v \nabla \bar{v})^2 + \sigma(1 - i\mu)(\bar{v} \nabla v)^2)\]
\[
= \frac{1}{2} \int |v|^{2(\sigma - 1)}YM^H,
\]
where
\[
Y = \begin{pmatrix} \bar{v} \nabla v \\ v \nabla \bar{v} \end{pmatrix}^H, \quad M = \begin{pmatrix} \sigma + 1 & \sigma(1 - \mu) \\ \sigma(-1 + \mu) & \sigma + 1 \end{pmatrix},
\]
and $Y^H$ represents the conjugate transpose of the matrix $Y$. We observe that condition (4.15) implies that the matrix $M$ is nonnegative definite. Thus, we have
\[
-2z^{-2\sigma}\text{Re}(1 + i\mu)(|v|^{2\sigma} v, \Delta v) \geq 0. \tag{4.18}
\]
On the other hand, by the Gagliardo-Nirenberg inequality we deduce that
\[
(3\rho + \beta^2)\|\nabla v\|^2 \leq \|(-\Delta)^{\frac{\sigma + 1}{4\sigma}} v\|^2 + c_1\|v\|^2, \tag{4.19}
\]
where
\[
c_1 = c_0\left(\frac{3\rho + \beta^2}{\alpha + 1}\right)^{\frac{1+\alpha}{\alpha}} + 1.
\]
Substituting (4.18) and (4.19) into (4.17) yields
$$\frac{d}{dt} \|\nabla v\|^2 + \left\| (-\Delta)^{\frac{\alpha+1}{2}} v \right\|^2 + \rho \|\nabla v\|^2 \leq c_1 \|v\|^2. \quad (4.20)$$

Multiplying this inequality by $e^{\rho t}$, integrating it over $(s, t)$ with $s < t$, and using (4.9), we deduce that
$$\|\nabla v(t)\|^2 + e^{-\rho t} \int_s^t e^{\rho \tau} \left\| (-\Delta)^{\frac{\alpha+1}{2}} v(\tau) \right\|^2 d\tau$$
$$\leq e^{-\rho t} \left( e^{\rho s} \|\nabla u_s\|^2 + c_1 \int_s^t e^{\rho \tau} \|v(\tau)\|^2 d\tau \right)$$
$$= e^{-\rho t} \left( e^{\rho s} \|\nabla u_s\|^2 + c_1 \int_s^t e^{\rho \tau} \|v(\tau)\|^2 d\tau \right)$$
$$\leq e^{-\rho t} \cdot e^{\rho s} z^2(s) \|\nabla u_s\|^2 + c_1 r_0^2(\omega).$$

From (4.13), we see that for any $u_s \in H^1_p(D)$ with $\|u_s\|_{H^1_p(D)} \leq R$, there exists a time $\bar{s}_2(\omega) \leq \bar{s}(\omega)$ such that
$$e^{\rho s} z^2(s) \|\nabla u_s\|^2 \leq e^{\rho s} z^2(s) R^2 \leq 1, \quad \text{P-a.s., for } s \leq \bar{s}_2(\omega).$$

For all $t \in [-2, 0]$, we derive that
$$\|\nabla v(t)\|^2 + e^{-\rho t} \int_s^t e^{\rho \tau} \left\| (-\Delta)^{\frac{\alpha+1}{2}} v(\tau) \right\|^2 d\tau$$
$$\leq e^{2\rho t} + c_1 \cdot r_0^2(\omega)$$
$$\triangleq r_1^2(\omega).$$

Hence, we complete the proof. \hfill \Box

**Lemma 4.3.** Assume that
$$\frac{1}{2} \leq \sigma \leq \frac{1}{\sqrt{1+\mu^2} - 1}. \quad (4.21)$$

Then there exists a random radius $r_1^2(\omega) > 0$ such that, for any given $R > 0$, there exists $\bar{s}_2(\omega) \leq -2$ such that for all $s \leq \bar{s}_2(\omega)$ and $u_s \in H^1_p(D)$ satisfying $\|u_s\|_{H^1_p(D)} \leq R$, the following inequality holds P-a.s.,
$$\|(-\Delta)^{\frac{\sigma}{2}} v(t)\|^2 + e^{-\rho t} \int_s^t e^{\rho \tau} \|\Delta v(\tau)\|^2 d\tau \leq r_1^2(\omega), \quad \forall \ t \in [-1, 0], \quad (4.22)$$

where
$$r_1^2(\omega) = (c + c_2) \cdot r_0^2(\omega) + r_1^2(\omega) + c \cdot r_1^2(\omega).$$

**Proof.** Taking the inner product of equation (3.6) with $(-\Delta)^{2-\alpha} v$ in $L^2$ and considering the real part, we obtain
$$\frac{d}{dt} \left\| (-\Delta)^{\frac{\sigma}{2}} v \right\|^2 + 2\|\Delta v\|^2$$
$$= -2z^{-2\sigma} \text{Re}(1 + i\mu) \left( \|v^{2\sigma} v, (-\Delta)^{2-\alpha} v \| + (2\rho + \beta^2) \left\| (-\Delta)^{\frac{\sigma}{2}} v \right\|^2 \right). \quad (4.23)$$

We estimate the two terms on the right-hand side of equation (4.23), respectively. When $\sigma \geq \frac{1}{2}$, using the Gagliardo-Nirenberg inequality, Young’s inequality, as well
as integration by parts, we derive that
\[-2z^{-2}\Re(1 + i\mu)(|v|^{2\sigma} v, (-\Delta)^{2-\alpha} v)\]
\[\leq c \cdot |z^{-2}| \int \left( |v|^{2\sigma-1} |\nabla v|^2 (-\Delta)^{1-\alpha} v \right)^2 + |v|^{2\sigma} |\nabla v| (-\Delta)^{1-\alpha} v)\]
\[\leq c \cdot |z^{-2}| \left( \|(-\Delta)^{1-\alpha} v\|^2 H^\alpha \|\nabla v\|^\frac{3}{2} \|\nabla v\|^\frac{4-\alpha}{2\sigma} + \|\nabla v\|^2 \|\Delta v\| \right)\]
\[\leq c \cdot |z^{-2}| \left( \|v\|^\alpha H^{\alpha-1} \|\nabla v\|^\frac{3}{2} \|\nabla v\|^\frac{4-\alpha}{2\sigma} \right)\]
\[\leq \frac{1}{2} \|\Delta v\|^2 + c \cdot E_0(t), \quad (4.24)\]
where
\[E_0(t) = \left| z^{-\frac{2\alpha}{H^\alpha}} \right| \cdot \|v\|^{\frac{2(2\sigma-1)}{H^\alpha}} \|v\|^{\frac{2(2\alpha+1)}{2\sigma}} + \left| z^{-\frac{4\alpha}{H^\alpha}} \right| \cdot \|v\|^{\frac{4(2\alpha-1)}{H^\alpha}} \|v\|^{\frac{4(2\alpha+1)}{4\alpha-1}} + \|v\|^2.\]

In addition, applying the Gagliardo-Nirenberg inequality again, we have
\[(3\rho + \beta^2) \|(-\Delta)^{1-\frac{\alpha}{2}} v\|^2 \leq \frac{1}{2} \|\Delta v\|^2 + c_2 \|v\|^2, \quad (4.25)\]
where
\[c_2 = \frac{c(3\rho + \beta^2)^2 + 1}{2}.\]
Substituting (4.24) and (4.25) into equation (4.23) yields
\[\frac{d}{dt} \|(-\Delta)^{1-\frac{\alpha}{2}} v\|^2 + \|\Delta v\|^2 \leq c \cdot E_0 + c_2 \|v\|^2. \quad (4.26)\]

Multiplying this inequality by $e^{\rho t}$ and then integrating it over $(s, t)$ with $-2 \leq s < t \leq 0$ leads to
\[\|(-\Delta)^{1-\frac{\alpha}{2}} v(t)\|^2 + e^{-\rho t} \int_s^t e^{\rho \tau} \|\Delta v(\tau)\|^2 d\tau \]
\[\leq e^{-\rho t} \left( e^{\rho s} \|(-\Delta)^{1-\frac{\alpha}{2}} v(s)\|^2 + c_2 \int_s^t e^{\rho \tau} \cdot \|v(\tau)\|^2 d\tau \right)\]
\[+ c \cdot e^{-\rho t} \left( \int_s^t e^{\rho \tau} \cdot E_0(\tau) d\tau \right)\]
\[\leq e^{-\rho t} \left( e^{\rho s} \|(-\Delta)^{1-\frac{\alpha}{2}} v(s)\|^2 \right) + c_2 \cdot r_1^2(\omega) + c \cdot \tilde{r}_1^2(\omega), \quad (4.27)\]
where
\[\tilde{r}_1^2(\omega) = \sup_{-2 \leq t \leq 0} \left| z^{-\frac{2\alpha}{H^\alpha}} (t) \cdot \left[ r_0^2(\omega) + r_1^2(\omega) \right] \right|^\frac{2(\sigma-1)}{2\sigma} \left( r_0^2(\omega) + r_1^2(\omega) \right) \]
\[+ \sup_{-2 \leq t \leq 0} \left| z^{-\frac{4\alpha}{H^\alpha}} \cdot \left[ r_0^2(\omega) + r_1^2(\omega) \right] \right|^\frac{2(2\sigma-1)}{4\sigma} \left( r_0^2(\omega) + r_1^2(\omega) \right).\]
After integrating (4.27) with respect to $s$ on $[-2, -1]$ and applying the Gagliardo-Nirenberg inequality, by virtue of Lemma 4.1 and 4.2, for any $-1 \leq t \leq 0$ we derive
that

\[ \left\| (-\Delta)^{1-\frac{p}{2}} v(t) \right\|^2 \]
\[ \leq e^{-pt} \int_{-2}^{t} e^{ps} \left\| (-\Delta)^{1-\frac{p}{2}} v(s) \right\|^2 ds + c_2 \cdot r_0^2(\omega) + c \cdot r_1^2(\omega) \]
\[ \leq e^{-pt} \int_{-2}^{t} e^{ps} \left\| (-\Delta)^{\frac{p}{2}} v(s) \right\|^2 ds + (c + c_2) \cdot r_0^2(\omega) + c \cdot r_1^2(\omega) \]
\[ \leq (c + c_2) \cdot r_0^2(\omega) + r_1^2(\omega) + c \cdot r_1^2(\omega) \]
\[ \triangleq r_1^2(\omega). \] (4.28)

Hence, the proof is completed.

By Lemma 4.3, we deduce that for any given \( R > 0 \) there exists an \( s_2(\omega) \leq -2 \) such that for any \( s \leq s_2(\omega) \), it holds

\[ \left\| (-\Delta)^{1-\frac{p}{2}} v(0) \right\|^2 = \left\| (-\Delta)^{1-\frac{p}{2}} u(0) \right\|^2 \leq r_1^2(\omega), \quad \text{for P-a.e. } \omega \in \Omega. \]

Let \( K(\omega) \) be a ball in \( H^1_p(D) \) with the radius \( r_0(\omega) + r_1(\omega) \). It is shown that for any \( B \) bounded in \( H^1_p(D) \) there exists an \( \bar{s}_2(\omega) \) such that for any \( s \leq \bar{s}_2(\omega) \), there holds

\[ S(0, s; \omega)B \subset K(\omega), \quad \text{P-a.e. } \omega \in \Omega. \]

This implies that \( K(\omega) \) is an attracting set at time 0, since \( H^{2-\alpha}_p(D) \) is compact in \( H^1_p(D) \). Applying Theorem 2.1, we can obtain:

**Theorem 4.4.** Under condition (4.21), the stochastic dynamical system associated with the fractional Ginzburg-Landau equation with multiplicative noise (1.1) has a compact random attractor in \( H^1_p(D) \).

### 5. Hausdorff dimension of the random attractor \( \mathcal{A}(\omega) \)

In this section, we show that the Hausdorff dimension of the maximal attractor \( \mathcal{A} \) is finite. Let

\[ S(\omega) = S(1, 0; \omega), \quad T(\omega) = T(1, 0; \omega) \]

and

\[ u(t) = S(t, 0; \omega)u_0 = e^{\beta W(t)}T(t, 0; \omega)v_0. \]

One can see that if \( T(\omega) \) is almost surely uniformly differentiable with the Fréchet derivative \( DT(w) \), then so is \( S(\omega) \) and \( DS(w) = e^{\beta W(1)}DT(w) \).

**Lemma 5.1.** Suppose that the mapping \( T \) is almost surely uniformly differentiable on \( \mathcal{A}(\omega) \) and there exists a linear operator \( DT(\omega, v) \) in \( L(L^p_p(D)) \), the space of continuous linear operators from \( L^2_p(D) \) to \( L^2_p(D) \), such that if \( v \) and \( v + h \) are in \( \mathcal{A}(\omega) \). Then it holds

\[ \| T(\omega)(v + h) - T(\omega)(v) - DT(\omega, v)h \| \leq K(\omega)\| h \|^{1+\chi}, \] (5.29)

where \( K(\omega) \) is a random variable such that

\[ K(\omega) \geq 1, \quad E(\ln K) < \infty, \quad \omega \in \Omega, \]

and \( \chi > 0 \) is a number. For any \( v_0 \in \mathcal{A}(\omega) \), then \( DT(\omega, v_0)h = V(1) \), where \( V \) is the solution of

\[ \frac{dV}{dt} = F'(t, v)V, \] (5.30)
\[ V(0) = h, \] (5.31)
and
\[ v(t) = e^{-\beta W(t)}S(t,0;\omega)u_0, \]
\[ F'(t,v)V = \left( \rho + \frac{1}{2}\beta^2 \right) V - (1 + i\nu)(-\Delta)^\alpha V - (1 + i\mu)z^{-2\sigma}f'(v)V, \]
\[ f'(v)V = (|u|^{2\sigma}v)' = (1 + |u|^{2\sigma}V + |v|^{2\sigma-2}v^2). \]

Proof. Let \( \mathcal{E}(t) = v_1(t) - v_2(t) - V(t) \), where \( v_j(t) \) \( (j = 1,2) \) are two solutions of equation (3.6) with \( v_j(0) = v^0_j \), \( V(t) \) satisfies system (5.30)–(5.31) and \( h = v^1_1 - v^0_2 \). Then \( \mathcal{E}(t) \) satisfies the equation
\[
\frac{d\mathcal{E}}{dt} = (\rho + \frac{1}{2}\beta^2)\mathcal{E} - (1 + i\nu)(-\Delta)^\alpha \mathcal{E} - (1 + i\mu)z^{-2\sigma}(f(v_1) - f(v_2) - f'(v_2)(v_1 - v_2 - \mathcal{E})) = (\rho + \frac{1}{2}\beta^2)\mathcal{E} - (1 + i\nu)(-\Delta)^\alpha \mathcal{E} + \Phi + \Psi, \]
(5.32)
where
\[
\Phi = -(1 + i\mu)z^{-2\sigma}(|v_1|^{2\sigma}v_1 - |v_2|^{2\sigma}v_2 - (1 + \sigma)|v_2|^{2\sigma}(v_1 - v_2) - \sigma|v_2|^{2\sigma-2}v_2^2(\bar{v}_1 - \bar{v}_2)),
\]
and
\[
\Psi = -(1 + i\mu)z^{-2\sigma}((1 + \sigma)|v_2|^{2\sigma} \mathcal{E} + \sigma|v_2|^{2\sigma-2}v_2^2 \mathcal{E}).
\]

By a similar discussion as described in Section 4, we have
\[
\|v_j(t)\|_{L_\infty}^2 \leq c \left( \|(-\Delta)\frac{1}{2} v_j(t)\|_{L_\infty}^{2\sigma} + \|v_j(t)\|_{L_\infty}^{2\sigma} \right).
\]
Applying Taylor’s expansion for the function \( G(v_1,\bar{v}_1) = |v_1|^{2\sigma}v_1 \) at the point \( (v_2,\bar{v}_2) \), we get
\[
|\Phi| \leq C(\omega) \sqrt{1 + \mu^2} z^{-2\sigma} |v_1 - v_2|^2.
\]

Taking the inner product of equation (5.32) with \( \mathcal{E}(t) \) in \( L^2 \) and considering the real part, we obtain
\[
\frac{d}{dt}\|\mathcal{E}\|^2 + 2\|(-\Delta)^{\frac{1}{2}} \mathcal{E}\|^2 - (2\rho + \beta^2)\|\mathcal{E}\|^2 = 2\text{Re} \int \Phi \mathcal{E} + 2\text{Re} \int \Psi \mathcal{E}. \]
(5.33)

For the first term on the right-side of equation (5.33), using Hölder’s inequality and Young’s inequality, we derive that
\[
2\text{Re} \int \Phi \mathcal{E} \leq C(\omega) \sqrt{1 + \mu^2} z^{-2\sigma} \|\mathcal{E}\|_{L_\infty}^{\frac{1}{2}} \|v_1 - v_2\|_{L_{\infty}^{2\sigma-1}}^2
\[
\leq C(\omega) \sqrt{1 + \mu^2} z^{-2\sigma} \|\mathcal{E}\|_{H^{\alpha}}^{\frac{1}{2}} \|\mathcal{E}\|_{L_{\infty}^{2\sigma-1}}^2 \|v_1 - v_2\|_{H^{\alpha}} \|v_1 - v_2\|
\[
\leq \|(-\Delta)^{\frac{1}{2}} \mathcal{E}\|^2 + c \|\mathcal{E}\|^2 + C(\omega)(1 + \mu^2) z^{-2\sigma} \|v_1 - v_2\|_{H^{\alpha}}^2 \|v_1 - v_2\|^2.
\]
(5.34)

For the second term on the right-hand side of equation (5.33), applying the Gagliardo-Nirenberg inequality, as well as Lemmas 4.1 and 4.2 yields
\[
2\text{Re} \int \Psi \mathcal{E} \leq 6\sqrt{1 + \mu^2} z^{-2\sigma} \|v_2\|_{L_\infty}^{2\sigma} \|\mathcal{E}\|^2.
\]
(5.35)
Substituting (5.34) and (5.35) into (5.33) leads to
\[
\frac{d}{dt} \| \mathcal{E} \|^2 \leq \left( c + 2(\rho + \beta^2) + 6\sqrt{1 + \mu^2} z^{-2\sigma} \|v_2\|^2_{\infty} \right) \| \mathcal{E} \|^2 + C(\omega)(1 + \mu^2) z^{-4\sigma} \|v_1 - v_2\|^2_{H^\omega} \|v_1 - v_2\|^2. \tag{5.36}
\]

By Gronwall’s inequality, we have
\[
\| \mathcal{E}(t) \|^2 \leq c'(\omega) \int_0^t z^{-4\sigma} \|v_1 - v_2\|^2_{H^\omega} \|v_1 - v_2\|^2 dt, \tag{5.37}
\]
where \( c'(\omega) = C(\omega)(1 + \mu^2) e^{\int_0^1 c_1(\rho + \beta^2) + 6C(\omega) \sqrt{1 + \mu^2} t} z^{-2\sigma}(t) dt \).

Since \( v_j(t) (j = 1, 2) \) are two solutions of equation (3.6) with \( v_j(0) = v_j^0 \), we have
\[
(v_1 - v_2)_t = -(1 + i\nu)(-\Delta)^{\alpha}(v_1 - v_2) + (\rho + \frac{1}{2} \beta^2)(v_1 - v_2)
- (1 + i\mu) z^{-2\sigma} (|v_1|^{2\sigma} v_1 - |v_2|^{2\sigma} v_2). \tag{5.38}
\]

Taking the inner product of equation (5.38) with \((v_1 - v_2)\) in \( L^2 \) and considering the real part yields
\[
\frac{d}{dt} \|v_1 - v_2\|^2 + 2 \|(-\Delta)^{\frac{\alpha}{2}} (v_1 - v_2)\|^2 
\leq -2\text{Re}(1 + i\mu) z^{-2\sigma} \int (|v_1|^{2\sigma} v_1 - |v_2|^{2\sigma} v_2)(\bar{v}_1 - \bar{v}_2) + (2\rho + \beta^2) \|v_1 - v_2\|^2. \tag{5.39}
\]

Using Taylor’s series, we get
\[
|v_1|^{2\sigma} v_1 - |v_2|^{2\sigma} v_2 = (\sigma + 1)|v_2 + \theta(v_1 - v_2)|^{2\sigma} (v_1 - v_2)
+ \sigma |v_2 + \theta(v_1 - v_2)|^{2\sigma - 2}(v_2 + \theta(v_1 - v_2))^2 (\bar{v}_1 - \bar{v}_2)
\leq (2\sigma + 1)|v_2 + \theta(v_1 - v_2)|^{2\sigma} |v_1 - v_2|, \quad \forall \theta \in (0, 1).
\]

So the first term on the right-hand side of inequality (5.39) is bounded by
\[
-2\text{Re}(1 + i\mu) z^{-2\sigma} \int (|v_1|^{2\sigma} v_1 - |v_2|^{2\sigma} v_2)(\bar{v}_1 - \bar{v}_2) \leq C(\omega) \|v_1 - v_2\|^2. \tag{5.40}
\]

Combining (5.39) and (5.40) leads to
\[
\frac{d}{dt} \|v_1 - v_2\|^2 + \|(-\Delta)^{\frac{\alpha}{2}} (v_1 - v_2)\|^2 \leq C(\omega) \|v_1 - v_2\|^2. \tag{5.41}
\]

Applying Gronwall’s inequality gives
\[
\|v_1 - v_2\|^2 + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} (v_1 - v_2)\|^2 dt \leq C(\omega) \|v_1^0 - v_2^0\|^2. \tag{5.42}
\]

Hence, inequality (5.37) can be rewritten as
\[
\| \mathcal{E}(t) \|^2 \leq K_1^2(\omega), \tag{5.43}
\]
where
\[
K_1^2(\omega) = \frac{C(\omega)(1 + \mu^2) e^{c_1(\rho + \beta^2) + 6C(\omega) \sqrt{1 + \mu^2} t} M^2}{M = \sup_{0 \leq t \leq 1} z^{-2\sigma}(t)}.
\]

Choose \( K(\omega) = \max\{K_1(\omega), 1\} \). Then, we have \( E(\ln K(\omega)) < \infty \). \(\square\)
Next, we consider three conditions of Theorem 2.2. From condition (3) and equation (5.30), we have
\[
\frac{d}{dt} \|V\|^2 + 2 \|(-\Delta)^{\frac{\sigma}{2}} V\|^2 = (2\rho + \beta^2)\|V\|^2 - 2\Re(1 + i\mu)z^{-2\sigma} \int f'(v)|V|^2.
\]
The second term of the right-side is bounded as
\[
-2\Re(1 + i\mu)z^{-2\sigma} \int f'(v)|V|^2 \leq 6M\sqrt{1 + \mu^2} \int |v|^{2\sigma} |V|^2 \leq 6MC(\omega)\sqrt{1 + \mu^2} \|V\|^2.
\]
This implies that
\[
\|V(t)\| \leq 6MC(\omega)\sqrt{1 + \mu^2} e^{(2\rho + \beta^2 + 6MC(\omega)\sqrt{1 + \mu^2})t}.
\]
Since \(\alpha_1(D\alpha(\omega, v))\) is equal to the norm of \(D\alpha(\omega, v) \in \mathcal{L}(X)\), we choose
\[
\bar{\alpha}_1(\omega) = \max \left\{ e^{\beta W(1) + \int_0^1 \ReTr(F'(s, v(s)) \circ Q_d(s)) ds}, 1 \right\}.
\]
Then it has
\[
\alpha_1(D\alpha(\omega, v)) \leq \bar{\alpha}_1(\omega),
\]
and
\[
E(\ln \bar{\alpha}_1) < \infty.
\]
From [3], we see that
\[
\omega_d(D\alpha(\omega, u)) = \sup_{\|\eta\| \leq 1, i = 1, 2, \ldots, d} \exp \left\{ \beta W(1) + \int_0^1 \ReTr(F'(s, v(s)) \circ Q_d(s)) ds \right\},
\]
where \(Q_d(s)\) is the orthogonal projector in \(L^2\) onto the space spanned by \(V_1(s), \ldots, V_d(s)\), and \(V_i(s)\) is the solution of equation (5.30) with \(V_i(0) = \eta_i\).

Let \(\psi_i\) (\(i \in \mathbb{N}\)) be an orthonormal basis of \(L^2\) such that
\[
Q_d(s)L^2 = \text{span}\{\psi_1(s), \ldots, \psi_d(s)\}.
\]
Then, we have
\[
\ReTr(F'(s, v(s)) \circ Q_d(s)) = \sum_{i=1}^d \Re(F'(v(s)) \circ Q_d(s)\psi_i(s), \psi_i(s))
\leq \left( \rho + \frac{1}{2} \beta^2 + 3M\sqrt{1 + \mu^2} \|v\|_{2\sigma} \right) \sum_{i=1}^d \|\psi_i\|^2 - \sum_{i=1}^d \|(-\Delta)^{\frac{\sigma}{2}} \psi_i\|^2.
\]
Since \(\{\psi_i\} (i = 1, 2, \ldots, d)\) is an orthonormal basis of \(L^2\), we have
\[
\sum_{i=1}^d \|\psi_i\|^2 = d.
\]
By virtue of the Sobolev-Lieb-Thirring inequality [22], it follows that
\[
\sum_{i=1}^{d} \left\| \left( -\Delta \right)^{\frac{\alpha}{2}} \psi_i \right\|^2 \geq \kappa |\Omega|^\alpha d^{1+\alpha} - d,
\]
where the constant \( \kappa \) is independent of the family \( \psi_i, d \) and all parameters of the equation (1.1). So we further have
\[
\text{Re} \text{Tr} F'(v(s)) \circ Q_d(s) \leq \left( 1 + \rho + \frac{1}{2} \beta^2 + 3M \sqrt{1+\mu^2} \left\| v \right\|_\infty^{2\sigma} \right) d - \kappa |\Omega|^\alpha d^{1+\alpha} = \kappa_1 d - \kappa_2 d^{1+\alpha},
\]
where
\[
\kappa_1 = 1 + \rho + \frac{1}{2} \beta^2 + 3M \sqrt{1+\mu^2} \left\| v \right\|_\infty^{2\sigma}, \quad \kappa_2 = \kappa |\Omega|^\alpha.
\]
Denote
\[
\bar{\omega}_d(\omega) = \exp \left\{ \beta W(1) + \kappa_1 d - \kappa_2 d^{1+\alpha} \right\}
\]
and choose
\[
d = \left[ \left( \frac{\kappa_1}{\kappa_2} \right)^{\frac{1}{\alpha}} \right] + 1.
\]
Then, we have \( \omega_d(DS(\omega, u)) \leq \bar{\omega}_d(\omega) \) and \( E(\ln(\bar{\omega}_d)) < 0 \).

Consequently, by Theorem 2.2, we obtain our main result as follows:

**Theorem 5.2.** Let \( A(\omega) \) be the random attractor of system (1.1)–(1.3) which is invariant under a random map \( S(\omega) \), \( \omega \in \Omega \), for the ergodic metric dynamical system \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \). Then the Hausdorff dimension \( d_H(A(\omega)) \) of \( A(\omega) \) is less than \( d \) almost surely.

**Acknowledgments.** This work is supported by the NSF of China No.10972018 and 11272024.

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Received December 2014; revised May 2015.

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