Non-integrability of some Hamiltonians with rational potentials

Primitivo Acosta-Humánez & David Blázquez-Sanz

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Abstract

In this work we compute the families of classical Hamiltonians in two degrees of freedom in which the Normal Variational Equation around an invariant plane falls in Schrödinger type with polynomial or trigonometric potential. We analyze the integrability of Normal Variational Equation in Liouvillian sense using the Kovacic’s algorithm. We compute all Galois groups of Schrödinger type equations with polynomial potential. We also introduce a method of algebrization that transforms equations with transcendental coefficients in equations with rational coefficients without changing essentially the Galoisian structure of the equation. We obtain Galoisian obstructions to existence of a rational first integral of the original Hamiltonian via Morales-Ramis theory.

Key Words: Picard-Vessiot Theory, Hamiltonian systems, Integrability, Kovacic’s Algorithm.

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1 Introduction

In [Mo-Si], C. Simó and J. Morales-Ruiz find the complete list of two degrees of freedom classical hamiltonians, with an invariant plane $\Gamma = \{ x_2 = y_2 = 0 \}$ with Normal Variational Equation of Lamé type along generic curves in $\Gamma$. In those computations they use systematically differential equations satisfied by coefficients of Lamé equation. It allow us to develop a method to find the families of hamiltonians with invariant plane $\Gamma$, with the property of having NVE satisfying certain conditions. It allow us to find the families of hamiltonians with generic NVE of type Mathieu, Shrödinger with polynomial potential of odd degree, quantum harmonic oscillator, and some other simpler examples.

Then, we analyze those families of linear differential equations in the context of Picard-Vessiot theory. For equations with rational coefficients, we use Kovacic’s Algorithm. In particular we give a complete description of Galois groups of Schrödinger type equations with polynomial potential (theorem 4). For equations with transcendental coefficients, we develop a method of algebraization. We characterize equations that can be algebraized through change of variables of certain type (hamiltonian). We prove the following result.

**Algebraization algorithm.** The differential equation $\ddot{y} = r(t)y$ is algebraizable through a hamiltonian change of variable $x = x(t)$ if and only if there exists $f, \alpha$ such that $\frac{\alpha'}{\alpha}, \frac{L}{\alpha} \in \mathbb{C}(x)$, where $f(x(t)) = r(t), \quad \alpha(x) = 2(H - V(x)) = \dot{x}^2$. Furthermore, the algebraic form of the equation $\ddot{y} = r(t)y$ is

$$y'' + \frac{1}{\alpha} \frac{\alpha'}{\alpha} y' - \frac{f}{\alpha} y = 0.$$  

Once we get the complete analysis of linearized equations, we apply a theorem of Morales-Ramis, obtaining the following results on the non integrability of those hamiltonians for generic values of the parameters.

**Non-integrability Results.** *Hamiltonians,*

1. $\frac{y^2 + y^2}{2} + \frac{\lambda}{(\lambda_3 + 2\lambda_2 x_1)} + \lambda_0 - \lambda_1 x_2^2 - \lambda_2 x_1 x_2^2 - \lambda_3 x_1^2 x_2^2 + \beta(x_1, x_2) x_2^3$, with $\lambda_3 \neq 0$;
2. \( \frac{y^2 + y_2^2}{2} + \lambda_0 + Q(x_1)x_2^2 + \beta(x_1, x_2)x_3^2 \), where \( Q(x_1) \) is a non-constant polynomial;

3. \( \frac{y_1^2 + y_2^2}{2} + \mu_0 + \mu_1 x_1 + \omega x_2^2 - \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 + \beta(x_1, x_2)x_3^2 \), with \( \omega \neq 0 \).

4. \( \frac{y_1^2 + y_2^2}{2} + \mu_0 + \left( \frac{\mu_1}{(\lambda_1 + 2\lambda_2 x_1 x_2)} \right) + \lambda_1 \omega x_1 + \omega x_2^2 - \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 - \lambda_2 x_1 x_2^2 + \beta(x_1, x_2)x_3^2 \), with \( \omega \neq 0 \), and \( \lambda_2 \neq 0 \).

where \( \beta(x_1, x_2) \) is any analytical function around \( \Gamma = \{ x_2 = y_2 = 0 \} \), do not admit any additional rational first integral.

**Corollary.** Every integrable (by rational functions) polynomial potentials with invariant plane \( \Gamma = \{ x_2 = y_2 = 0 \} \) can be written in the following form

\[
V = Q_1(x_1, x_2)x_3^3 + \lambda_1 x_2^3 + \lambda_0, \quad \lambda_0, \lambda_1 \in \mathbb{C}.
\]

Note that for \( \beta(x_1, x_2) \) polynomial and some values of the parameters we fall in the case of homogeneous polynomial potentials. Those cases had been analyzed deeply in \[Ma-Pr, \ Na-Yo\].

### 1.1 Picard-Vessiot theory

The Picard-Vessiot theory is the Galois theory of linear differential equations. In the classical Galois theory, the main object is a group of permutations of the roots, while in the Picard-Vessiot theory is a linear algebraic group. For polynomial equations we want a solution in terms of radicals. From classical Galois theory it is well known that this is possible if and only if the Galois group is solvable.

An analogous situation holds for linear homogeneous differential equations. For more details see \[Va-Si\]. The following definition is true for matrices \( n \times n \), but for simplicity we are restricting to matrices \( 2 \times 2 \).

**Definition 1** An algebraic group of matrices \( 2 \times 2 \) is a subgroup \( G \subset GL(2, \mathbb{C}) \), defined by algebraic equations in its matrix elements. That is, there exists a set of polynomials

\[
\{ P_i(x_{11}, x_{12}, x_{21}, x_{22}) \}_{i \in I},
\]

such that

\[
\begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{pmatrix} \in G \iff \forall i \in I, P_i(x_{11}, x_{12}, x_{21}, x_{22}) = 0.
\]

In this case we say that \( G \) is an algebraic manifold provided of a structure of group. In the remainder of this paper we only work, as particular case, with linear differential equations of second order

\[
y'' + ay' + by = 0, \quad a, b \in \mathbb{C}(x).
\]
Suppose that $y_1, y_2$ is a fundamental system of solutions of the differential equation. This means that $y_1, y_2$ are linearly independent over $\mathbb{C}$ and every solution is a linear combination of these two. Let $L = \mathbb{C}(x)(y_1, y_2) = \mathbb{C}(x)(y_1, y_2, y_1', y_2')$, that is the smallest differential field containing to $\mathbb{C}(x)$ and $\{y_1, y_2\}$.

**Definition 2 (Differential Galois Group)** The group of all differential automorphisms of $L$ over $\mathbb{C}(x)$ is called the Galois group of $L$ over $\mathbb{C}(x)$ and denoted by $\text{Gal}(L/\mathbb{C}(x))$ or also by $\text{Gal}_{\mathbb{C}(x)}^L$. This means that for $\sigma: L \rightarrow L$, $\sigma(a') = \sigma'(a)$ and $\forall a \in \mathbb{C}(x)$, $\sigma(a) = a$.

If $\sigma \in \text{Gal}(L/\mathbb{C}(x))$ then $\sigma y_1, \sigma y_2$ is another fundamental system of solutions of the linear differential equation. Hence there exists a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sigma y_1 \\ \sigma y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$ 

This defines a faithful representation $\text{Gal}(L/\mathbb{C}(x)) \rightarrow \text{GL}(2, \mathbb{C})$ and it is possible to consider $\text{Gal}(L/\mathbb{C}(x))$ as a subgroup of $\text{GL}(2, \mathbb{C})$. It depends on the choice of fundamental system $y_1, y_2$, but only up to conjugate.

One of the fundamental results of the Picard-Vessiot theory is the following theorem.

**Theorem 1** The Galois group $G = \text{Gal}(L/\mathbb{C}(x))$ is an algebraic subgroup of $\text{GL}(2, \mathbb{C})$.

Now we are interested in the differential equation

$$\xi'' = r \xi, \quad r \in \mathbb{C}(x). \quad (1)$$

We recall that equation (1) can be obtained from the general second order linear differential equation

$$y'' + ay' + by = 0, \quad a, b \in \mathbb{C}(x),$$

through the change of variable

$$y = e^{-\frac{1}{2} \int a \xi}, \quad r = \frac{a^2}{4} + \frac{a'}{2} - b.$$ 

On the other hand, through the change of variable $v = \xi'/\xi$ we get the associated Riccatti equation to equation (1)

$$v' = r - v^2, \quad v = \frac{\xi'}{\xi}. \quad (2)$$
For the differential equation \( y'' + ay' + by = 0 \), \( a, b \in F \), \( G = Gal(G/C(x)) \) is an algebraic subgroup \( SL(2,C) \).

Recall that an algebraic group \( G \) has a unique connected normal algebraic subgroup \( G^0 \) of finite index. This means that the identity component \( G^0 \) is the biggest connected algebraic subgroup of \( G \) containing the identity.

**Definition 3** Let \( F \) be a differential extension of \( C(x) \), and let be \( \eta \) solution of the differential equation

\[
y'' + ay' + by = 0
\]

1. \( \eta \) is algebraic over \( F \) if \( \eta \) satisfies a polynomial equation with coefficients in \( F \), i.e. \( \eta \) is an algebraic function of one variable.

2. \( \eta \) is primitive over \( F \) if \( \eta' \in F \), i.e. \( \eta = \int f \) for some \( f \in F \).

3. \( \eta \) is exponential over \( F \) if \( \eta'/\eta \in F \), i.e. \( \eta = e^{\int f} \) for some \( f \in F \).

**Definition 4** A solution \( \eta \) of the previous differential equation is said to be Liouvillian over \( F \) if there is a tower of differential fields

\[
F = F_0 \subset F_1 \subset \ldots \subset F_m = L,
\]

with \( \eta \in L \) and for each \( i = 1, \ldots, m \), \( F_i = F_{i-1}(\eta_i) \) with \( \eta_i \) either algebraic, primitive, or exponential over \( F_{i-1} \). In this case we say that the differential equation is integrable.

Thus a Liouvillian solution is built up using algebraic functions, integrals and exponentials. In the case \( F = C(x) \) we get, for instance logarithmic, trigonometric functions, but not special functions such that the Airy functions.

We recall that a group \( G \) is called solvable if and only if there exists a chain of normal subgroups

\[
e = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G
\]

such that the quotient \( G_i/G_j \) is abelian for all \( n \geq i \geq j \geq 0 \).

**Theorem 2** The equation \( y'' + ay' + by = 0 \) is integrable (has Liouvillian solutions) if and only if for \( G = Gal(L/C(x)) \), the identity component \( G^0 \) is solvable.

Using the theorem Kovacic in 1986 introduced an algorithm to solve the differential \( y'' + ay' + by = 0 \) and show that \( y'' + ay' + by = 0 \) is integrable if and only if the solution of the equation is a rational function (case 1), is a root of polynomial of degree two (case 2) or is a root of polynomial of degree 4, 6, or 12 (case 3) (see Appendix A). Based in the Kovacic’s algorithm we have the following key result.

The Galois group of differential equation \( y'' + ay' + by = 0 \) with \( r = Q_k(x) \) a polynomial of degree \( k > 0 \) is a non-abelian connected group.

For complete result, details and proof see Appendix A.
1.2 Morales-Ramis theory

Morales-Ramis theory [Mo-Ra1, Mo-Ra2], see also [Mo1], relates the integrability of hamiltonian systems to the integrability of linear differential equations. In this approach analyze the linearization (variational equations) of hamiltonian systems along some known particular solution. If the hamiltonian system is integrable, then we expect that the linearized equation has good properties in the sense of Picard-Vessiot theory. Exactly, for integrable hamiltonian systems, the Galois group of the linearized equation must be virtually abelian. It gives us the best non-integrability criterion known for hamiltonian systems. This approach has been extended to higher order variational equations in [M-R-S].

1.2.1 Integrability of hamiltonian systems

A symplectic manifold (real or complex), $M_{2n}$, is a $2n$-dimensional manifold, provided with a closed 2-form $\omega$. This closed 2-form gives us a natural isomorphism between vector bundles, $\flat: TM \to T^*M$. Given a function $H$ on $M$, there is an unique vector field $X_H$ such that,

$$\flat(X_H) = dH$$

this is the hamiltonian vector field of $H$. Furthermore, it gives an structure of Poisson algebra in the ring of differentiable functions of $M_{2n}$ by defining:

$$\{H, F\} = X_H F.$$

We say that $H$ and $F$ are in involution if $\{H, F\} = 0$. From our definition, it is obvious that $F$ is a first integral of $X_H$ if and only if $H$ and $F$ are in involution. In particular $H$ is always a first integral of $X_H$. Moreover, if $H$ and $F$ are in involution, then their flows commute.

The equations of the flow of $X_H$, in a system of canonical coordinates, $p_1, \ldots, p_n, q_1, \ldots, q_n$ (i.e. such that $\omega_2 = \sum_{i=1}^n p_i \land q_i$), are written

$$\dot{q} = \frac{\partial H}{\partial p} (= \{H, q\}), \quad \dot{p} = -\frac{\partial H}{\partial q} (= \{H, p\}),$$

and they are known as Hamilton equations.

**Theorem 3 (Liouville-Arnold)** Let $X_H$ be a hamiltonian defined on a real symplectic manifold $M_{2n}$. Assume that there are $n$ functionally independent first integrals $F_1, \ldots, F_n$ in involution. Let $M_a$ be a non-singular (i.e. $dF_1, \ldots, dF_n$ are independent on very point of $M_a$) level manifold,

$$M_a = \{ p: F_1(p) = a_1, \ldots, F_n(p) = a_n \}.$$

Then,

1. If $M_a$ is compact and connected, then it is a torus $M_a \simeq \mathbb{R}^n/\mathbb{Z}^n$. 

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2. In a neighborhood of the torus $M_a$ there are functions $I_1, \ldots, I_n, \phi_1, \ldots, \phi_n$ such that
\[ \omega_2 = \sum_{i=1}^{n} dI_i \wedge d\phi_i, \]
and $\{H, I_j\} = 0$ for $j = 1, \ldots, n$.

From now on, we will consider $\mathbb{C}^{2n}$ as a complex symplectic manifold. Liouville-Arnold theorem gives us a notion of integrability for Hamiltonian systems. A Hamiltonian $H$ in $\mathbb{C}^{2n}$ is called integrable in the sense of Liouville if there exist $n$ independent first integrals of $X_H$ in involution. We will say that $H$ is integrable by terms of rational functions if we can find those first integrals between rational functions, or whatever.

### 1.2.2 Variational equations

We want to relate integrability of Hamiltonian systems with Picard-Vessiot theory. We deal with non-linear Hamiltonian systems. But, given a Hamiltonian $H$ in $\mathbb{C}^{2n}$, and $\Gamma$ an integral curve of $X_H$, we can consider the first variational equation (VE), as
\[ \mathcal{L}_{X_H} \xi = 0, \]
in which the linear equation induced in the tangent bundle ($\xi$ represents a vector field supported on $\Gamma$).

Let $\Gamma$ be parameterized by $\gamma: t \mapsto (x(t), y(t))$ in such way that
\[
\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}(\gamma(t)), \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}(\gamma(t)).
\]

Then the VE along $\Gamma$ is the linear system,
\[
\left( \begin{array}{c} \xi_i \\ \eta_i \end{array} \right) = \left( \begin{array}{cc} \frac{\partial^2 H}{\partial y_i \partial x_j}(\gamma(t)) & \frac{\partial^2 H}{\partial y_i \partial y_j}(\gamma(t)) \\ -\frac{\partial^2 H}{\partial x_i \partial x_j}(\gamma(t)) & -\frac{\partial^2 H}{\partial x_i \partial y_j}(\gamma(t)) \end{array} \right) \left( \begin{array}{c} \xi_i \\ \eta_i \end{array} \right).
\]

From the definition of Lie derivative, comes that
\[
\xi_i(t) = \frac{\partial H}{\partial y_i}(\gamma(t)), \quad \eta_i(t) = -\frac{\partial H}{\partial x_i}(\gamma(t)),
\]
is a solution of the VE. We can use it to reduce, using some generalization of D’Alambert method \cite{Mo-Ra1, Mo-Ra2} (see also \cite{Mo1}) our equation, obtaining the Normal Variational Equation (NVE), which is a linear system of rank $2(n - 1)$. In the case of 2-degrees of freedom Hamiltonian systems, those NVE can be seen as second order linear homogeneous differential equation.

### 1.2.3 Non-integrability tools

Morales-Ramis theory, concerns several results that relate the existence of first integrals of $H$ with the Galois group of the variational equations. Along this paper we will use systematically the following one:
Theorem 4 ([Mo-Ra1]) Let $H$ be a Hamiltonian in $\mathbb{C}^{2n}$, and $\gamma$ a particular solution such that the NVE has irregular singularities at the points of $\gamma$ at infinity. Then, if $H$ is completely integrable by terms of rational functions, then the Identity component of Galois Group of the NVE is abelian.

Remark 1 Here, the field of coefficients of the NVE is the field meromorphic functions on $\gamma$.

2 Some results on linear differential equations

2.1 Algebrization of Linear Differential Equations

For some differential equations it is useful, if is possible, to replace the original differential equation over a compact Riemann surface, by a new differential equation over the Riemann sphere $\mathbb{P}^1$ (i.e., with rational coefficients) by a change of the independent variable. This equation on $\mathbb{P}^1$ is called the algebraic form or algebrization of the original equation. Kovacic’s algorithm can be applied over the algebraic form to solve the original equation. In a more general way we will consider the effect of a finite ramified covering on the Galois group of the original differential equation. In [Mo1, Mo-Ra1] the following theorem is given.

Theorem 5 (Morales-Ramis [Mo1, Mo-Ra1]) Let $X$ be a (connected) Riemann surface. Let $f : X' \to X$ be a finite ramified covering of $X$ by a Riemann surface $X'$. Let $\nabla$ be a meromorphic connection on $X$. We set $\nabla' = f^*\nabla$. Then we have a natural injective homomorphism $\text{Gal}(\nabla') \to \text{Gal}(\nabla)$ of differential Galois groups which induces an isomorphism between their Lie algebras.

Remark 2 We observe that, in terms of the differential Galois groups, this theorem means that the identity component of the differential Galois group is invariant by the covering. In other words, if the original differential equation over the Riemann surface $X'$ is transformed by a change of the independent variable in a differential equation over the Riemann surface $X$, then both equations have the same identity component of the differential Galois group.

Proposition 1 (Change of the independent variable) Let us consider the following equation, with coefficients in $\mathbb{C}(x)$:

$$y'' + a(x)y' + b(x)y = 0, \quad y' = \frac{dy}{dx} \quad (3)$$

and $\mathbb{C}(x) \hookrightarrow L$ the corresponding Picard-Vessiot extension. Let $(K, \delta)$ be a differential field with $\mathbb{C}$ as field of constants. Let $\xi \in K$ be a non-constat. Then, by the change of variable $x = \xi$, $\mathbb{C}$ is transformed in,

$$\dot{\hat{y}} + \left(a(\xi)\dot{\xi} - \frac{\xi}{\eta}\right)\dot{\hat{\xi}} + b(\xi)(\dot{\xi})^2\hat{y} = 0, \quad \hat{z} = \delta \hat{z}. \quad (4)$$
Let, $K_0 \subset K$ be the smallest differential field containing $\xi$ and $C$. Then (4) is a differential equation with coefficients in $K_0$. Let $K_0 \hookrightarrow L_0$ be the corresponding Picard-Vessiot extension. Assume that

$$C(x) \to K_0, \quad x \mapsto \xi$$

is an algebraic extension, then

$$\text{Gal}(L_0/K_0)^0 = \text{Gal}(L/C(x))^0.$$ 

**Proof.** By the chain rule we have

$$\frac{d}{dx} = \frac{1}{\xi},$$

and,

$$\frac{d^2}{dx^2} = \frac{1}{(\xi)^2} \dot{\xi}^2 - \frac{\ddot{\xi}}{(\xi)^3} \delta,$$

now, changing $y'$, $y''$ in (3) and making monic this equation we have (4)

$$\ddot{y} + \left( a(\xi) \dot{\xi} - \frac{\ddot{\xi}}{\xi} \right) \dot{y} + b(\xi)(\dot{\xi})^2 y = 0.$$ 

In the same way we can obtain (3) through (4).

If $K_0$ is an algebraic extension of $C(x)$, then we can identify $K_0$ with the ring of meromorphic functions on a compact $X$ Riemann surface, and $\xi$, is a finite ramified covering of the Riemann sphere,

$$X \xrightarrow{\xi} S^1.$$ 

If we consider $\nabla$ the meromorphic connection in $S^1$ induced by equation (3), then $\xi^*(\nabla)$, is the meromorphic connection induced by equation (4) in $X$, and finally by theorem (5) we conclude. $\square$

Recently Manuel Bronstein in [B] has implemented an algorithm to solve differential equation over $C(t, e^{\int f(t)})$ without algebraize the equation. As immediate consequence of the proposition (3) we have the following corollary.

**Corollary 1 (Linear differential equation over $C(t, e^{\int f(t)})$)** Let be $f \in C(t)$, the differential equation

$$\ddot{y} - \left( f(t) + \frac{f(t)}{f'(t)} - f(t) e^{\int f(t)} a \left( e^{\int f(t)} \right) \right) \dot{y} + \left( f(t) \left( e^{\int f(t)} \right) \right)^2 b \left( e^{2\int f(t)} \right) y = 0,$$

by the change $x = e^{\int f(t)}$ is algebrizable and its algebraic form is given by

$$y'' + a(x)y' + b(x) y = 0.$$ 

(5)
Remark 3 In this corollary, we have the following cases.

1. $f = n \frac{h'}{h}$, for a rational function $h$, $n \in \mathbb{Z}_+$, we have the trivial case, both equations are over the Riemann sphere and they have the same differential field, so that the equation (5) do not need be algebrized.

2. $f = \frac{1}{n} \frac{h'}{h}$, for a rational function $h$, $n \in \mathbb{Z}_+$, the equation (5) is defined over an algebraic extension of $\mathbb{C}(t)$ and so that this equation is not necessarily over the Riemann sphere.

3. $f \neq q \frac{h'}{h}$, for any rational function $h$, $q \in \mathbb{Q}$, the equation (5) is defined over an transcendental extension of $\mathbb{C}(t)$ and so that the this equation is not over de Riemann sphere.

In first and second case, we can apply proposition taking $K_0 = \mathbb{C}(t, e^{\int f(t)})$, so that the identity component of the algebrized equation es conserved. The conservation of Galois group in the third case, requires further analysis, and will not be discussed here. We just remark that the Galois group corresponding to original equation is a subgroup of the Galois group of the algebrized equation.

Definition 5 (Regular and irregular singularity) The point $x = x_0$ is called regular singular point (or regular singularity) of the equation (3) if and only if

$$(x - x_0)a(x), \quad (x - x_0)^2b(x),$$

are both analytic in $x = x_0$. If $x = x_0$ is not a regular singularity, then is an irregular singularity.

Remark 4 (Change to infinity) To study asymptotic behaviors (or the point $x = \infty$) in (3) we can take $x(t) = \frac{1}{t}$ and to analyze in (4) the behavior in $t = 0$. That is, to study the behavior of the equation (3) in $x = \infty$ we should study the behavior in $t = 0$ of the equation

$$\ddot{y} + \left( \frac{2}{t} - \left( \frac{1}{t^2} \right) a \left( \frac{1}{t} \right) \right) \dot{y} + \frac{1}{t^2} b \left( \frac{1}{t} \right) y = 0. \quad (6)$$

In this way, by the definition we say that $x = \infty$ is a regular singularity of the equation (3) if and only if $t = 0$ is a regular singularity of the equation (4).

To algebrize second order linear differential equations is easier when the term in $\dot{y}$ is absent and the change of variable is hamiltonian, that is, RLDE $\ddot{y} = r(t)y$.

Definition 6 (Hamiltonian change of variable) A change of variable $x = x(t)$ is called hamiltonian if and only if $(x(t), \dot{x}(t))$ is a solution curve of the autonomous hamiltonian system

$$H = H(x, y) = \frac{y^2}{2} + V(x).$$
Assume that we algebrize equation \( H \) through a hamiltonian change of variables, \( x = \xi(t) \). Then, \( K_0 = \mathbb{C}(\xi, \dot{\xi}, \ldots) \), but, we have the algebraic relation,
\[
(\dot{\xi})^2 = 2h - 2V(\xi), \quad h = H(\xi, \dot{\xi}) \in \mathbb{C},
\]
so that \( K_0 = \mathbb{C}(\xi, \dot{\xi}) \) is an algebraic extension of \( \mathbb{C}(x) \). We can apply proposition \( H \) and then the identity component of the Galois group is conserved.

**Proposition 2 (Algebrization algorithm)** The differential equation
\[
\ddot{y} = r(t)y
\]
is algebrizable through a hamiltonian change of variable \( x = x(t) \) if and only if there exists \( f, \alpha \) such that
\[
\frac{\alpha'}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(x), \quad \text{where} \quad f(x(t)) = r(t), \quad \alpha(x) = 2(H - V(x)) = \dot{x}^2.
\]
Furthermore, the algebraic form of the equation \( \ddot{y} = r(t)y \) is
\[
y'' + \frac{1}{2} \frac{\alpha'}{\alpha} y' - \frac{f}{\alpha} y = 0.
\]

**Proof.** Because \( x = x(t) \) is a hamiltonian change of variable for the differential equation \( \ddot{y} = r(t)y \) so that \( \dot{x} = y, \quad \ddot{x} = -V'(x) \) and there exists \( f, \alpha \) such that \( \ddot{y} = f(x(t))y \) and \( \dot{x}^2 = 2(H - V(x)) = \alpha(x) \). By the proposition \( H \) we have \( -f(x) = b(x)\dot{x}^2 \) and \( a(x)\dot{x} - \dot{x}/\dot{x} = 0 \), therefore \( a(x) = \dot{x}/\dot{x}^2 \) and \( b(x) = -f(x)/\alpha(x) \). In this way \( \ddot{y} = r(t)y \) is algebrizable if and only if \( a(x), b(x) \in \mathbb{C}(x) \). As \( 2(H - V(x)) = \alpha(x) \) then \( \alpha'(x) = -2V'(x) = 2\dot{x} \), and therefore \( a(x) = \frac{1}{2} \frac{\alpha'(x)}{\alpha(x)} \). In this way, we obtain the equation \( H \). \( \Box \)

As consequence of the proposition \( H \) we have the following result.

**Corollary 2** Let be \( r(t) = g(x_1, \ldots, x_n) \), where \( x_i = e^{\lambda_i t}, \lambda_i \in \mathbb{C} \). The differential equation \( \ddot{y} = r(t)y \) is algebrizable if and only if
\[
\lambda_i \lambda_j \in \mathbb{Q}, \quad 1 \leq i \neq j \leq n, \quad g \in \mathbb{C}(x).
\]
Furthermore, we have \( \lambda_i = c_i \lambda \), where \( \lambda \in \mathbb{C} \) and \( c_i \in \mathbb{Q} \) and one change of variable is
\[
x = e^{\frac{\lambda}{q_i}}, \quad \text{where} \quad c_i = \frac{p_i}{q_i}, \quad \gcd(p_i, q_i) = 1 \quad \text{and} \quad q = \text{mcm}(q_1, \ldots, q_n).
\]

**Remark 5 (Using the algebrization algorithm)** To algebrize the differential equation \( \ddot{y} = r(t)y \) it should keep in mind the following steps.

**Step 1** Find a hamiltonian change of variable \( x = x(t) \).
Step 2 Find $f$ and $\alpha$ such that $r(t) = f(x(t))$ and $(\dot{x}(t))^2 = \alpha(x(t))$.

Step 3 Write $f(x)$ and $\alpha(x)$.

Step 4 Verify if $f(x)/\alpha(x) \in \mathbb{C}(x)$ and $\alpha'(x)/\alpha(x) \in \mathbb{C}(x)$ to see if the RLDE is algebrizable or not.

Step 5 If the RLDE is algebrizable, write the algebraic form of the original equation such as follows

$$y'' + \frac{1}{2} \frac{\alpha'}{\alpha} y' - \frac{f}{\alpha} y = 0.$$ 

When we have algebrized second order linear differential equation we study its integrability and its Galois groups.

Remark 6 (Monic Polynomials) Let be the reduced linear differential equation (RLDE)

$$\ddot{y} = \left( \sum_{k=0}^{n} c_k x^k \right) y, \quad c_k \in \mathbb{C}, \quad k = 1, \ldots, n.$$ 

By the algebrization algorithm we can take $x = \mu t$, $\mu \in \mathbb{C}$, so that

$$\dot{x} = \mu, \quad \alpha(x) = \mu^2, \quad \alpha'(x) = 0 \text{ and } f(x) = \sum_{k=0}^{n} c_k \left( \frac{x}{\mu} \right)^k, \quad c_k, \mu \in \mathbb{C}.$$ 

Now, by the equation (7) the new differential equation is

$$y'' = \left( \sum_{k=0}^{n} \left( \frac{c_k}{\mu^{k+2}} \right) x^k \right) y, \quad c_k, \mu \in \mathbb{C}.$$ 

In general, for $\mu = n^{\frac{1}{2}} e^{\pi i}$ we can obtain the equation

$$y'' = \left( x^n + \sum_{k=0}^{n-1} d_k x^k \right) y, \quad d_k = \left( \frac{c_k}{\mu^{k+2}} \right), \quad k = 0, \ldots, n - 1.$$ 

Furthermore, we can observe, by definition (8) and equation (9), that the point at $\infty$ is an irregular singularity for the differential equation with non-constant polynomial coefficients because using equation (10) we can see that zero is not an ordinary point and neither is a regular singularity for the differential equation.

Remark 7 (Extended Mathieu) The Extended Mathieu differential equation is

$$\ddot{y} = (a + b \sin t + c \cos t) y, \quad (8)$$

in particular, when $b = 0$ or $c = 0$ and $|a| + |b| \neq 0$, we have the so called Mathieu equation. Applying the corollary (2) and the steps of the algorithm we have $x = e^{it}$, $\dot{x} = ix$, therefore

$$f(x) = \frac{(b+c)x^2 + 2ax + c-b}{2x}, \quad \alpha(x) = -x^2, \quad \alpha'(x) = -2x,$$
so that the algebraic form of the equation (8) is
\[ y'' + \frac{1}{x} y' + \frac{(b + c)x^2 + 2ax + c - b}{2x^3} y = 0, \] (9)
Making the change \( x = 1/z \) in the equation (9) we obtain
\[ \ddot{\zeta} + \left( \frac{1}{z} \right) \dot{\zeta} + \left( \frac{(c - b)z^2 + 2az + (b + c)}{2z^3} \right) \zeta = 0. \] (10)
We can observe, by definition 8 and equation (6), that \( z = 0 \) is an irregular singularity for the equation (10) and therefore \( x = \infty \) is an irregular singularity for the equations (9) and (8).

Now, we compute the Galois group and the integrability in equation (9). So that, the RLDE is given by
\[ \xi'' = - \left( \frac{(b + c)x^2 + (2a + 1)x + c - b}{2x^3} \right) \xi. \] (11)
Applying the Kovacic’s algorithm, see Appendix A, we can see that for \( b \neq -c \) this equation falls in case 2: \((c_3, \infty_3), E_0 = \{3\}, E_\infty = \{1\}\) and so that \( D = \emptyset \) because \( m = -1 \notin \mathbb{Z}_+ \). In this way we have that the equation (11) is not integrable, the Galois Group is the connected group \( SL(2, \mathbb{C}) \), and finally, by the theorem 8 the identity component of the Galois group for equation (8) is exactly \( SL(2, \mathbb{C}) \) that is not an abelian group. In the same way, for \( b = -c \) we have the equations
\[ \ddot{y} = (a - be^{-it})y, \quad y'' + \frac{1}{x} y' + \frac{2ax - 2b}{2x^3} y = 0, \]
in which \( \infty \) continues being an irregular singularity. Now, Its RLDE is given by
\[ \xi'' = - \left( \frac{(2a + 1)x - 2b}{2x^3} \right) \xi, \]
applying Kovacic’s algorithm we can see that this equation falls in case 2: \((c_3, \infty_2), E_0 = \{3\}, E_\infty = \{0, 2, 4\}\), so that \( D = \emptyset \) because \( 1/2(e_\infty - e_0) \notin \mathbb{Z}_+ \). This means that the Galois group continues being \( SL(2, \mathbb{C}) \). Using this result, taking \( \epsilon \) instead of \( i \), we can say that in the case of harmonic oscillator with exponential waste
\[ \ddot{y} = (a + be^{-\epsilon t})y, \quad \epsilon > 0, \]
the point at \( \infty \) is an irregular singularity and the identity component of the Galois group is \( SL(2, \mathbb{C}) \).

2.2 Galois groups of Schrödinger equations with polynomial potential

Let be the reduced linear differential equation (RLDE)
\[ \ddot{\xi} = \sum_{k=0}^{n} c_k t^k \xi, \quad a_k \in \mathbb{C}. \]
By the remark 6, through the change of variable $x = \mu t$, $\mu = n + \sqrt{\frac{c}{k}}$, this equation becomes

$$\xi'' = P_n(x)\xi, \quad P_n(x) \text{ is a monic polynomial of degree } n.$$  

Kovacic in [Ko] remarked that for $n = 2k + 1$ there are not liouvillian solutions and therefore the Galois group of the RLDE is $SL(2, \mathbb{C})$. Using the Kovacic’s algorithm (see Appendix), we can see that for $n = 2k$ the equation falls in case 1, specifically in $c_0$ (because it has not poles) and $\infty_3$ (because $\circ r_\infty = -2k$), that is $\{c_0, \infty_3\}$.

**Lemma 1 (Completing Squares)** Every monic polynomial of degree even can be written in one only way completing squares, that is

$$Q_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} q_k x^k = \left( x^n + \sum_{k=0}^{n-1} a_k x^k \right)^2 + \sum_{k=0}^{n-1} b_k x^k. $$

**Proof.** Firstly, we can see that

$$\left( x^n + \sum_{k=0}^{n-1} a_k x^k \right)^2 = x^{2n} + 2 \sum_{k=0}^{n-1} a_k x^{n+k} + \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} a_k a_j x^{k+j},$$

so that by indeterminate coefficients we have

$$a_{n-1} = \frac{q_{2n-1}}{2}, \quad a_{n-2} = \frac{q_{2n-2} - a_{n-1}^2}{2}, \quad a_{n-3} = \frac{q_{2n-3} - 2a_{n-1} a_{n-2}}{2}, \ldots,$$

$$a_0 = \frac{q_n - 2a_1 a_{n-1} - 2a_2 a_{n-2} - \cdots}{2}, \quad b_0 = q_0 - a_0^2, \quad b_1 = q_1 - 2a_0 a_1, \quad \ldots,$$

$$b_{n-1} = q_{n-1} - 2a_0 a_{n-1} - 2a_1 a_{n-2} - \cdots,$$

therefore, we have proven the lemma. □

By case 1, $\{c_0, \infty_3\}$, of the Kovacic’s algorithm, remark [12] [15] (See Appendix), remark [13] and by lemma [14] we have proven the following theorem.

**Theorem 6 (Galois groups in polynomial case)** Let us consider the equation,

$$\xi = Q(x)\xi,$$

with $Q(x)$ a polynomial of degree $k > 0$. Then, it falls in one of the following cases:
1. $k = 2n$ is even, $\pm b_{n-1} - n = 2m$, $m \in \mathbb{Z}_+$, and there exist a monic polynomial $P_m$ of degree $m$ satisfying

$$P_m'' + 2 \left( x^n + \sum_{k=0}^{n-1} a_k x^k \right) \frac{d}{dx} \left( n x^{n-1} + \sum_{k=0}^{n-2} (k+1)a_{k+1} x^k + \sum_{k=0}^{n-1} b_k x^k \right) P_m = 0,$$

the solutions are given by

$$\xi_1 = P_m e^{x^{n+1} + \sum_{k=0}^{n-1} a_k x^k},$$

$$\xi_2 = P_m e^{x^{n+1} + \sum_{k=0}^{n-1} a_k x^k} \int \frac{dx}{P_m^2 e^{2 \left( x^{n+1} + \sum_{k=0}^{n-1} a_k x^k \right)}}.$$

and the Galois group is $\mathbb{C}^* \ltimes \mathbb{C}$ (non-abelian, resoluble, connected group).

2. The equation has not liouvillian solutions and its Galois group is $\text{SL}(2, \mathbb{C})$.

**Remark 8 (Quadratic case)** consider the case where $r$ is a polynomial of degree two:

$$\dddot{y} = (At^2 + Bt + C) y.$$

There are no poles and the order at $\infty$, or $r_{\infty}$, is $-2$, so we need to follow case 1 in $\{c_0, \infty_3\}$ of the algorithm. Now, by remark 6 and by lemma 1 we have

$$y'' = ((x+a)^2 + b) y.$$

We find that

$$\sqrt{r}_{\infty} = x + a$$

$$\alpha_{\infty}^{\pm} = \frac{1}{2} (\pm b - 1)$$

$$m = \alpha_{\infty}^+ \text{ or } \alpha_{\infty}^-.$$

If $b$ is not an odd integer then $m$ cannot be an integer so case 1 cannot hold so the RLDE in $x$ has no Liouvillian solutions. If $b$ is an odd integer than we can complete steps 2 and 3 and actually (only) find a solution it which is

$$y = P_m e^{\frac{x^2}{2} + ax}.$$

In particular, the quantum harmonic oscillator

$$y'' = (x^2 + \lambda) y$$

is integrable when $\lambda$ is an odd integer and the only one solution obtained by means of kovacic’s algorithm is given by

$$y = H_m e^{\frac{x^2}{2}},$$

where $H_m$ denotes the classical Hermite’s polynomials.

For another approach to this problem see Vidunas [11] and Zoladek in [20].
3 Determining families of Hamiltonians with specific NVE

Let us consider a two degrees of freedom classical Hamiltonian,

$$H = \frac{y_1^2 + y_2^2}{2} + V(x_1, x_2).$$

$V$ is the potential function, and it is assumed to be analytical in some open subset of $\mathbb{C}^2$. The evolution of the system is determined by Hamilton equations:

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -\frac{\partial V}{\partial x_1}, \quad \dot{x}_2 = y_2, \quad \dot{y}_2 = -\frac{\partial V}{\partial x_2}.$$

Let us assume that the plane $\Gamma = \{x_2 = 0, y_2 = 0\}$ is an invariant manifold of the Hamiltonian. We keep in mind that the family of integral curves lying on $\Gamma$ is parameterized by the energy $h = H|_{\Gamma}$, but we do not need to use it explicitly. We are interested in studying the linear approximation of the system near $\Gamma$. Since $\Gamma$ is an invariant manifold, we have

$$\left.\frac{\partial V}{\partial x_2}\right|_{\Gamma} = 0,$$

so that the Normal Variational Equation for a particular solution

$$t \mapsto \gamma(t) = (x_1(t), y_1 = \dot{x}_1(t), x_2 = 0, y_2 = 0),$$

is written,

$$\ddot{\xi} = \eta, \quad \dot{\eta} = -\left[\frac{\partial^2 V}{\partial x_2^2}(x_1(t), 0)\right] \xi.$$

Let us define,

$$\phi(x_1) = V(x_1, 0), \quad \alpha(x_1) = \frac{\partial^2 V}{\partial x_2^2}(x_1, 0),$$

and then we write the second order Taylor series in $x_2$ for $V$, obtaining the following expression for $H$

$$H = \frac{y_1^2 + y_2^2}{2} + \phi(x_1) - \alpha(x_1)\frac{x_2^2}{2} + \beta(x_1, x_2)x_3^3,$$    \hspace{1cm} (12)

which is the general form of a classical a Hamiltonian, with invariant plane $\Gamma$. The NVE associated to any integral curve lying on $\Gamma$ is,

$$\ddot{\xi} = \alpha(x_1(t))\xi.$$    \hspace{1cm} (13)
3.1 General Method

We are interested in compute Hamiltonians of the family (12), such that its NVE (13) belongs to a specific family of Linear Differential Equations. Then we can apply our results about the integrability of this LDE, and Morales-Ramis theorem to obtain information about the non-integrability of such Hamiltonians.

From now on, we will write \( a(t) = \alpha(x_1(t)) \), for a generic curve \( \gamma \) lying on \( \Gamma \), parameterized by \( t \). Then, the NVE is written

\[
\ddot{\xi} = a(t)\xi. \tag{14}
\]

**Problem.** Assume that \( a(t) \) is a root of a given differential polynomial \( Q(a, \dot{a}, \ddot{a}, \ldots) \in \mathbb{C}[a, \dot{a}, \ddot{a}, \ldots] \). We want to compute all hamiltonians in (12) satisfying such a condition.

In this section we give a method to compute, for any given \( Q(a, \dot{a}, \ldots) \), the family of classical hamiltonians with invariant plane \( \Gamma \) such that, for any integral curve lying on \( \Gamma \), the coefficient \( a(t) \) of the NVE satisfies,

\[
Q(a, \dot{a}, \ddot{a}, \ldots) = 0, \tag{15}
\]

by solving certain differential equations. This method lies under the calculus done by J. Morales and C. Simó in [Mo-Si].

We should notice that, for a generic integral curve \( \gamma(t) = (x_1(t), y_1 = \dot{x}_1(t)) \) lying on \( \Gamma \), equation (14) depends only of the values of functions \( \alpha \) and \( \phi \). It depends of \( \alpha(x_1) \), since \( a(t) = \alpha(x_1(t)) \). We observe that the curve \( \gamma(t) \) is a solution of the restricted Hamiltonian,

\[
h = \frac{y_1^2}{2} + \phi(x_1) \tag{16}
\]

whose associated Hamiltonian vector field is,

\[
X_h = y_1 \frac{\partial}{\partial x_1} - \frac{d\phi}{dx_1} \frac{\partial}{\partial y_1}, \tag{17}
\]

thus \( x_1(t) \) is a solution of the differential equation, \( \ddot{x}_1 = -\frac{d\phi}{dx_1} \), and then, the relation of \( x_1(t) \) is given by \( \phi \).

Since \( \gamma(t) \) is an integral curve of \( X_h \), for any function \( f(x_1, y_1) \) defined in \( \Gamma \) we have

\[
\frac{d}{dt} \gamma^*(f) = \gamma^*(X_h f),
\]

where \( \gamma^* \) denote the usual pullback of functions. Then, using \( a(t) = \gamma^*(\alpha) \), we have for each \( k \geq 0 \),

\[
\frac{d^k a}{dt^k} = \gamma^*(X_h^k \alpha), \tag{18}
\]

so that,

\[
Q(a, \dot{a}, \ddot{a}, \ldots) = Q(\gamma^*(\alpha), \gamma^*(X_h \alpha), \gamma^*(X_h^2 \alpha), \ldots). \]

There is an integral curve of the Hamiltonian passing through each point of \( \Gamma \), so that we have proven the following.
Proposition 3 Let $H$ be a Hamiltonian of the family (12), and $Q(a, \dot{a}, \ddot{a}, \ldots)$ a differential polynomial with constant coefficients. Then, for each integral curve lying on $\Gamma$, the coefficient $a(t)$ of the NVE (14) verifies $Q(a, \dot{a}, \ddot{a}, \ldots) = 0$, if and only if the function

\[ \hat{Q}(x_1, y_1) = Q(\alpha, X_h\alpha, X_h^2\alpha, \ldots), \]

vanish on $\Gamma$.

Remark 9 In fact the NVE of a integral curve depends on the parameterization. Our criterion does not depend on any choice of parameterizations of the integral curves. This is simple, the NVE corresponding to different parameterizations of the same integral curve are related by a translation of time $t$. We need just observe that a polynomial $Q(a, \dot{a}, \ddot{a}, \ldots)$ with constant coefficients is invariant of the group by translations of time. Then if the coefficient $a(t)$ of the NVE (14) for certain parameterization of an integral curve $\gamma(t)$ satisfied $\{Q = 0\}$, then it is also satisfied for any other right parameterization of the curve.

Next, we will see that $\hat{Q}(x_1, y_1)$ is a polynomial in $y_1$ and its coefficients are differential polynomials in $\alpha, \phi$. If we write down the expressions for successive Lie derivatives of $\alpha$, we obtain

\[ X_h\alpha = y_1 \frac{d\alpha}{dx_1}, \quad (19) \]

\[ X_h^2\alpha = y_1^2 \frac{d^2\alpha}{dx_1^2} - \frac{d\phi}{dx_1} \frac{d\alpha}{dx_i}, \quad (20) \]

\[ X_h^3\alpha = y_1^3 \frac{d^3\alpha}{dx_1^3} - y_1 \left( \frac{d}{dx_1} \left( \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1} \right) + 2 \frac{d\phi}{dx_1} \frac{d^2\alpha}{dx_1^2} \right) \quad (21) \]

\[ X_h^4\alpha = y_1^4 \frac{d^4\alpha}{dx_1^4} - y_1^2 \left( \frac{d}{dx_1} \left( \frac{d}{dx_1} \left( \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1} \right) + 2 \frac{d\phi}{dx_1} \frac{d^2\alpha}{dx_1^2} \right) + 3 \frac{d^3\alpha}{dx_1^3} \frac{d\phi}{dx_1} \right) + \]

\[ + \left( \frac{d}{dx_1} \left( \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1} \right) + 2 \frac{d\phi}{dx_1} \frac{d^2\alpha}{dx_1^2} \right) \frac{d\phi}{dx_i}. \quad (22) \]

In general form we have,

\[ X_h^{n+1}\alpha = y_1 \frac{\partial X_h^n\alpha}{dx_1} - \frac{d\phi}{dx_1} \frac{\partial X_h^n\alpha}{dy_1}, \quad (23) \]

it inductively follows that they all are polynomial in $y_1$ with coefficients differential polynomials in $\alpha, \phi$. If we write it down explicitly,

\[ X_h^n\alpha = \sum_{n \geq k \geq 0} E_{n,k}(\alpha, \phi) y_1^k \quad (24) \]
we can see that the coefficients $E_{n,k}(\alpha, \phi) \in \mathbb{C} \left[ \alpha, \phi, \frac{d\alpha}{dx_1}, \frac{d\phi}{dx_1} \right]$, satisfies the following recurrence law,

$$E_{n+1,k}(\alpha, \phi) = \frac{d}{dx_1} E_{n,k}(\alpha, \phi) - (k+1) E_{n,k+1}(\alpha, \phi) \frac{d\phi}{dx_1}$$  \hspace{1cm} (25)

with initial conditions,

$$E_{1,1}(\alpha, \phi) = \frac{d\alpha}{dx_1}, \quad E_{1,k}(\alpha, \phi) = 0 \quad \forall k \neq 1.$$  \hspace{1cm} (26)

**Remark 10** The recurrence law (25) and initial conditions (26) determine the coefficients $E_{n,k}(\alpha, \phi)$. We can compute the value of some of them easily:

- $E_{n,n}(\alpha, \phi) = \frac{d^n \alpha}{dx_1^n}$ for all $n \geq 1$.
- $E_{n,k}(\alpha, \phi) = 0$ if $n - k$ is odd, or $k < 0$, or $k > n$.

### 3.2 Some Examples

Here we compute families of hamiltonians such give rise to specific NVE. Although, in order to this computations, we need to solve Polynomial Differential Equations, we will see that we can deal with this in a branch of cases. Particularly, when $Q$ is a Differential Linear Operator, we will obtain equations that involve products of few Linear Differential Operator.

**Example 1** *Harmonic oscillator* equation is

$$\ddot{\xi} = c_0 \xi,$$  \hspace{1cm} (27)

with $c_0$ constant. Then, a hamiltonian of type gives such NVE if $\dot{a} = 0$. Looking at formula, it follows that $\frac{d\alpha}{dx_1} = 0$, so that $\alpha$ is a constant. We conclude that the general form of a Hamiltonian which give rise to NVE of the type (27) is,

$$H = \frac{y_1^2 + y_2^2}{2} + \phi(x_1) + \lambda_0 x_2^2 + \beta(x_1, x_2) x_2^3,$$

being $\lambda_0$ a constant, and $\phi, \beta$ arbitrary analytical functions.

**Example 2** In M. Audin notice that the hamiltonian,

$$\frac{y_1^2 + y_2^2}{2} + x_1 x_2$$

gives an example of a simple non-integrable classical hamiltonian, since its NVE along any integral curve in $\Gamma$ is an *Airy equation*. Here we compute the family of classical hamiltonians that have NVE of type Airy for integral curves lying on $\Gamma$. General form of Airy equation is

$$\ddot{\xi} = (c_0 + c_1 t) \xi$$  \hspace{1cm} (28)
with \(c_0, c_1 \neq 0\) two constants. If follows that a hamiltonian gives rise to NVE of this type if \(\ddot{a} = 0\), and \(\dot{a} \neq 0\). The equation \(\ddot{a} = 0\) gives, by proposition 3 as we see in formula (20), the following system:

\[
\frac{d^2 \alpha}{dx_1^2} = 0, \quad \frac{d \phi}{dx_1} \frac{d \alpha}{dx_1} = 0.
\]

It split in two independent systems,

\[
\frac{d \alpha}{dx_1} = 0, \quad \left\{ \begin{array}{l}
\frac{d^2 \alpha}{dx_1^2} = 0 \\
\frac{d \phi}{dx_1} = 0
\end{array} \right.
\]

(30)

Solutions of the first one fall into the previous case of harmonic oscillator. Then, taking the general solution of the second system, we conclude that the general form of a classical hamiltonian of type (12) with Airy NVE is:

\[
H = \frac{y_1^2}{2} + \frac{y_2^2}{2} + \lambda_0 + \lambda_1 x_2^2 + \lambda_2 x_1 x_2^2 + \beta(x_1, x_2)x_2^3,
\]

with \(\lambda_2 \neq 0\).

3.2.1 NVE quantum harmonic oscillator

Let us consider now equations with \(\frac{d^3 \alpha}{dt^3} = 0\), and \(\frac{d^2 \phi}{dt^2} \neq 0\), it is

\[
\ddot{\xi} = (c_0 + c_1 t + c_2 t^2) \xi
\]

(32)

with \(c_2 \neq 0\). Those equation can be reduced to a quantum harmonic oscillator equation by an affine change of \(t\), and its integrability has been studied using Kovacic’s Algorithm. Using proposition 3 and formula (21), we obtain the following system of differential equations for \(\alpha\) and \(\phi\):

\[
\frac{d^3 \alpha}{dx_1^3} = 0, \quad \frac{d \alpha}{dx_1} \frac{d^2 \phi}{dx_1^2} + 3 \frac{d^2 \alpha}{dx_1^2} \frac{d \phi}{dx_1} = 0.
\]

General solution of the first equation is

\[
\alpha = \frac{\lambda_1}{2} + \frac{\lambda_2}{2} x_1 + \frac{\lambda_3}{2} x_1^2,
\]

and substituting it into the second equation we obtain a Linear Differential Equation for \(\phi\),

\[
\frac{d^2 \phi}{dx_1^2} + 3 \frac{2 \lambda_3}{\lambda_2 + 2 \lambda_3 x_1} \frac{d \phi}{dx_1} = 0,
\]

this equation is integrated by two quadratures, and its general solution is

\[
\phi = \frac{\lambda_4}{(\lambda_2 + 2 \lambda_3 x_1)^2} + \lambda_0.
\]
We conclude that the general formula for hamiltonians of type (12) with NVE for any integral curve lying on $\Gamma$ is

$$H = \frac{y_1^2 + y_2^2}{2} + \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} + \lambda_0 - \lambda_1 x_2^2 - \lambda_2 x_1 x_2^2 - \lambda_3 x_1^2 x_2^2 + \beta(x_1, x_2)x_2^3,$$  \hspace{1cm} (33)

with $\lambda_3 \neq 0$.

**Remark 11** Formula (33) is the first example, in this paper, in which we find non-linear dynamics in the invariant plane $\Gamma$. Notice that this dynamic is continuously deformed to linear dynamics when $\lambda_4$ tends to zero. In general case, for a fixed energy $h$, we have the general integral of the equation:

$$8\lambda_2^2 h^2 (t - t_0)^2 = h(\lambda_2 + 2\lambda_3 x_1)^2 - \lambda_4.$$

### 3.2.2 NVE with polynomial coefficient $a(t)$ of odd degree

Let us consider for $n > 0$ the following differential polynomial,

$$Q_m(a, \dot{a}, \ldots) = \frac{d^m a}{dt^m}.$$

It is obvious that $a(t)$ is polynomial of degree $n$ if and only if $Q_n(a, \dot{a}, \ldots) \neq 0$ and $Q_{n+1}(a, \dot{a}, \ldots) = 0$.

Looking a proposition 3 we see that a hamiltonian (12) has NVE along a generic integral curves lying on $\Gamma$,

$$\ddot{\xi} = P_n(t)\xi,$$  \hspace{1cm} (34)

with $P_n(t)$ of degree $n$, if and only if $X^n_0 \alpha \neq 0$ and $X^{n+1}_0 \alpha$ vanish on $\Gamma$. Let us remember expression (24), $X^{n+1}_0 \alpha$ vanish in $\Gamma$ if and only if $(\alpha, \phi)$ is a solution of the differential system,

$$R_{n+1} = \{E_{n+1,0}(\alpha, \phi) = 0, \ldots, E_{n+1,n+1}(\alpha, \phi) = 0\}.$$

Using the recurrence law (25) defining the differential polynomials $E_{n,k}(\alpha, \phi)$ we are going to compute the family of hamiltonians giving rise to $a(t)$ polynomial of odd degree.

**Lemma 2** Let $(\alpha, \phi)$ be a solution of $R_{2m}$. Then, if $\frac{d\phi}{dx_1} \neq 0$, then $(\alpha, \phi)$ is a solution of $R_{2m-1}$.

**Proof.** By remark 11 $E_{2m-1, 2k}(\alpha, \phi) = 0$ for all $m - 1 \geq k \geq 0$. Then let us proof that $E_{2m-1, 2k+1}(\alpha, \phi)$ for all $m - 2 \geq k \geq 0$.

In the first step of the recurrence law defining $R_{2m}$,

$$0 = E_{2m,0}(\alpha, \phi) = \frac{dE_{2m-1,1}(\alpha, \phi)}{dx_1} - \frac{d\phi}{dx_1} E_{2m-1,1}(\alpha, \phi),$$
we use $\frac{d\phi}{dx_1} \neq 0$, and remark \[ E_{2m-1,-1}(\alpha, \phi) = 0 \] to obtain,

$$E_{2m-1,1}(\alpha, \phi) = 0.$$ 

If we assume $E_{2m-1,2k+1}(\phi, \alpha) = 0$, substituting it in the recurrence law

$$E_{2m,2k+1}(\alpha, \phi) = \frac{dE_{2m-1,2k}}{dx_1}(\alpha, \phi) - 2(k+1)\frac{d\phi}{dx_1} E_{2m-1,2(k+1)}(\alpha, \phi),$$

we obtain that

$$E_{2m-1,2k+1}(\alpha, \phi) = 0,$$

and we conclude by finite induction. $\square$

**Corollary 3** Let $H$ be a classical hamiltonian of type (12), then the following statements are equivalent,

1. The NVE for generic integral curve (14) lying on $\Gamma$ has polynomial coefficient $a(t)$ of degree $2m - 1$.

2. $H$ is written,

$$H = y_1^2 + y_2^2 + \lambda_0 - P_{2m-1}(x_1)x_2^2 + \beta(x_1, x_2)x_3^2,$$

for $\lambda_0$ constant, and $P_{2m-1}(x_1)$ polynomial of degree $2m - 1$.

**Proof.** It is clear that condition 1. is satisfied if and only if $(\alpha, \phi)$ is a solution of $R_{2m}$ and it is not a solution of $R_{2m-1}$. By the previous lemma, it implies $\frac{d\phi}{dx_1} = 0$, and then the system $R_{2m}$ is reduced to $\frac{d^{2m}\alpha}{dx_1^{2m}}$ and then, $\phi$ is a constant and $\alpha$ is a polynomial of degree at most $2m - 1$. $\square$

### 3.2.3 NVE Mathieu extended

This is the standard Mathieu equation,

$$\ddot{\xi} = (c_0 + c_1 \cos(\omega t))\xi, \quad \omega \neq 0.$$ \(36\)

We can not apply our method to compute the family of hamiltonians corresponding to this equation, because $\{c_0 + c_1 \cos(\omega t)\}$ is not the general solution of any differential polynomial with constant coefficients. But, let us consider

$$Q(a) = \frac{d^3 a}{dt^3} + \omega^2 \frac{da}{dt},$$ \(37\)

the general solution of $\{Q(a) = 0\}$ is

$$a(t) = c_0 + c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Just notice that,

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = \sqrt{c_1^2 + c_2^2} \cos\left(\omega t + \arctan \frac{c_2}{c_1}\right),$$

$$\omega \neq 0.$$
thus NVE (14), when $a$ is a solution of (37), is reducible to Mathieu equation (36) by a translation of time.

Using Proposition 3, we find the system of differential equations that determine the family of hamiltonians,

$$\frac{d^3 \alpha}{dx_1^3} = 0, \quad \frac{d\alpha}{dx_1} \frac{d^2 \phi}{dx_1^2} + 3 \frac{d^2 \alpha}{dx_1 dx_1} \frac{d\phi}{dx_1} - \omega^2 \frac{d\alpha}{dx_1} = 0.$$ 

General solution of the first equation is

$$\alpha = \lambda_0 + \lambda_1 x_1 + \lambda_2 x_1^2,$$

substituting it in the second equation, and writing $y = \frac{d\phi}{dx_1}$, we obtain a non homogeneous linear differential equation for $y$,

$$\frac{dy}{dx_1} + \frac{6\lambda_2 y}{\lambda_1 + 2\lambda_2 x_1} = \omega^2. \quad (38)$$

We must distinguish two cases depending on the parameter. If $\lambda_2 = 0$, then we just integrate the equation by trivial quadratures, obtaining

$$\phi = \mu_0 + \mu_1 x_1 + \frac{\omega^2 x_1^2}{2}$$

and then,

$$H = \frac{y_1^2 + y_2^2}{2} + \mu_0 + \mu_1 x_1 + \frac{\omega^2 x_1^2}{2} - \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 + \beta(x_1, x_2)x_2^3, \quad (39)$$

If $\lambda_2 \neq 0$, then we can reduce the equation to separable using

$$u = \frac{6\lambda_2 y}{\lambda_1 + 2\lambda_2 x_1},$$

obtaining

$$\frac{3du}{3\omega^2 - 4u} = \frac{6\lambda_2 dx}{\lambda_1 + 2\lambda_2 x_1}, \quad u = \frac{3\omega^2}{4} + \frac{3\mu_1}{4(\lambda_1 + 2\lambda_2 x_1)^4},$$

and then

$$y = \frac{1}{8\lambda_2} \left( \omega^2 \lambda_1 + 2\omega^2 \lambda_2 x_1 + \frac{\mu_1}{(\lambda_1 + 2\lambda_2 x_1)^3} \right),$$

and finally we integrate it to obtain $\phi$,

$$\phi = \int y dx_1 = \mu_0 - \frac{\mu_1}{32\lambda_2^2 (\lambda_1 + 2\lambda_2 x_1)^2} + \frac{\omega^2 \lambda_1 x_1}{8\lambda_2} + \frac{\omega^2 x_1^2}{8},$$

scaling the parameters adequately we write down the general formula for the hamiltonian,

$$H = \frac{y_1^2 + y_2^2}{2} + \mu_0 + \frac{\mu_1}{(\lambda_1 + 2\lambda_2 x_1)^2} + \frac{\lambda_1 \omega^2 x_1}{8\lambda_2} + \frac{\omega^2 x_1^2}{8} +$$

$$-\lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 - \lambda_2 x_1^2 x_2^2 + \beta(x_1, x_2)x_2^3. \quad (40)$$
3.3 Application of non-integrability criteria

3.3.1 NVE with \( a(t) \) polynomial

According to theorem 6 (see Appendix), the Galois group corresponding to equations,

\[ \ddot{\xi} = P(t)\xi \]

where \( P(t) \) is a non-constant polynomial, is a connected non-abelian group. So that, we can apply theorem 4 and we get the following result.

**Proposition 4** Hamiltonians,

\[ H = \frac{y_1^2 + y_2^2}{2} + x^2 Q(x_1) + \beta(x_1, x_2)x_1^2 \]

where \( Q(x_1) \) is a non-constant polynomial, and \( \beta(x_1, x_2) \) analytic function around \( \Gamma \), do not admit any additional rational first integral.

**Corollary 4** Every integrable (by rational functions) polynomial potential with invariant plane \( \Gamma = \{ x_2 = y_2 = 0 \} \) is written in the following form

\[ V = Q_1(x_1, x_2)x_2^3 + \lambda_1 x_2^2 + \lambda_0, \quad \lambda_0, \lambda_1 \in \mathbb{C}. \]

3.3.2 NVE reducible to quantum harmonic oscillator

We have seen, that hamiltonians (33), has generic NVE along curves in \( \Gamma \) of type (32). Once again, we apply theorem 6.

**Proposition 5** Hamiltonians of the family (33) do not admit any additional rational first integral.

We can also discuss, the Picard-Vessiot integrability of those equations (32). First, we shall notice that by just an scaling of \( t \),

\[ t = \frac{\tau}{\sqrt{c_2}} - \frac{c_1}{2c_2} \]

we reduce it to an quantum harmonic oscillator equation,

\[ \frac{d^2 \xi}{d\tau^2} = (\tau^2 - E)\xi, \quad E = \frac{c_1^2 - 4c_0 c_1}{4\sqrt{c_2^4}}. \tag{41} \]

In appendix (remark 8) we analyze this equation. It is Picard-Vessiot integrable if and only if \( E \) is an odd positive number. Then, let us compute de parameter \( E \) associated to NVE of integral curves of Hamiltonians (33).

Let us keep in mind that the family of those curves is parameterized by

\[ h = \frac{y_1^2}{2} + \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} \]
In order to fix the parameterization of those curves, let us assume that time \( t = 0 \) corresponds to \( x_1 = 0 \). The NVE corresponding to a curve, depending on energy \( h \), is written:

\[
\ddot{\xi} = (c_0(h) + c_1(h)t + c_2(h)t^2)\xi.
\]

We compute these coefficients \( c_i(h) \) using,

\[
y_1 = \sqrt{2h(\lambda_2 + 2\lambda_3x_1)^2 - 2\lambda_4} \quad \text{as } t \to 0 \quad \Rightarrow \quad \frac{\sqrt{2h\lambda_2^2 - 2\lambda_4}}{\lambda_2},
\]

and then, by applying the hamiltonian field,

\[
c_0(h) = \frac{\lambda_1}{2}, \quad c_1(h) = \sqrt{\frac{h\lambda_2^2 - \lambda_4}{2}}, \quad c_2(h) = \lambda_3 h,
\]

and then, it is reducible to equation (11) with parameter,

\[
E = \frac{1}{8\sqrt{\lambda_3^3}} \left( \frac{\lambda_2^2 - 4\lambda_1\lambda_3}{\sqrt{h}} - \frac{\lambda_4}{\sqrt{h}^3} \right).
\]

If \( \lambda_2 = 4\lambda_1\lambda_3 \) and \( \lambda_4 = 0 \), then parameter \( E \) vanish for every integral curve in \( \Gamma \). For any other case, \( E \) is a non-constant analytical function of \( h \).

We have proven that those NVE are, generically not Picard-Vessiot integrable for any hamiltonian of the (33) family.

### 3.3.3 NVE Mathieu

In order to apply theorem 4, we just need to make some remarks on the field of coefficients. Let \( \gamma \) be a generic integral curve of (39), or (40). Those curves are, in general, Riemann spheres. The field of coefficients \( \mathcal{M}_\gamma \), is generated by \( x_1, y_1 \), so that it is \( C(x_1, \dot{x}_1) \). We also have,

\[
a = \lambda_0 + \lambda_1 x_1 + \lambda_2 x_1^2, \quad \dot{a} = \dot{x}_1 (\lambda_1 + 2\lambda_2 x_1),
\]

so that, for \( \lambda_2 = 0 \), we have

\[
\mathcal{M}_\gamma = (a, \dot{a}) = C(\sin t, \cos t) = C(e^{it}).
\]

and, for \( \lambda_2 \neq 0 \),

\[
C(e^{it}) \leftrightarrow \mathcal{M}_\gamma
\]

is an algebraic extension.

So that, in our algebrization algorithm, the field of coefficients of Mathieu equation is taken, \( C(e^{it}) \). For \( \lambda_2 = 0 \), we can apply directly theorem 4 and for \( \lambda_2 \neq 0 \), we can apply theorem 5 and then theorem 4.

Non trivial equations of type Mathieu, with field of coefficients \( C(e^{it}) \), analyzed in remark 7, have Galois group \( SL(2, \mathbb{C}) \). Thus for computed families of hamiltonians with NVE of type Mathieu, we get:

**Proposition 6** Hamiltonians of the families (39) and (40) if \( \lambda_1 \neq 0 \) and \( (\lambda_1, \lambda_2) \neq (0, 0) \) respectively, do not admit any additional rational first integral.
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Appendix

A Kovacic’s Algorithm

This algorithm is devoted to solve the reduced linear differential equation (RLDE) \( \xi'' = r \xi \) and is based on the algebraic subgroups of \( SL(2, \mathbb{C}) \). For more details see [Ko]. Improvements for this algorithm is given in [Ul-We], in where is not necessary to reduce the equation. Here, we follows the original version given by Kovacic.

**Theorem 7** Let \( G \) be an algebraic subgroup of \( SL(2, \mathbb{C}) \). Then one of the following four cases can occur.

1. \( G \) is triangularizable.
2. \( G \) is conjugate to a subgroup of infinite dihedral group (also called metabelian group) and case 1 does not hold.
3. Up to conjugation \( G \) is either of following finite groups: Tetrahedral group, Octahedral group or Icosahedral group, and cases 1 and 2 do not hold.
4. \( G = SL(2, \mathbb{C}) \).

Each case in the Kovacic algorithm is related with each one of the algebraic subgroups of \( SL(2, \mathbb{C}) \) and the associated Riccatti equation

\[
\theta' = r - \theta^2 = (\sqrt{r} - \theta) (\sqrt{r} + \theta), \quad \theta = \frac{\xi'}{\xi}.
\]

According to theorem 7 there are four cases in the Kovacic algorithm. Only for cases 1, 2 and 3 we can solve the differential equation RLDE, but for the case 4 we have not Liouvillian solutions for RLDE. Is possible that the Kovacic algorithm only can provide us only one solution \( (y_1) \), so that we can obtain the second solution \( (y_2) \) through

\[
y_2 = y_1 \int \frac{dx}{y_1^2}.
\] (42)
Notations. For the equation RLDE with
\[ r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x] \]
we use the following notations.

1. Denote by \( \Gamma' \) be the set of (finite) poles of \( r \), \( \Gamma' = \{ c \in \mathbb{C} : t(c) = 0 \} \).
2. Denote by \( \Gamma = \Gamma' \cup \{ \infty \} \).
3. By the order of \( r \) at \( c \in \Gamma' \), \( \circ(r) \), we mean the multiplicity of \( c \).
4. By the order of \( r \) at \( \infty \), \( \circ(r) \), we mean the order of \( \infty \) as a zero of \( r \). That is \( \circ(r) = \deg(t) - \deg(s) \).
5. By the order of \( r \) at \( c \in \Gamma' \), \( \circ(r) \), we mean the multiplicity of \( c \).

A.1 The four cases

Case 1. In this case \( \sqrt[r]{c} \) and \( \sqrt[r]{\infty} \) means the Laurent series of \( \sqrt[r]{} \) at \( c \) and the Laurent series of \( \sqrt[r]{} \) at \( \infty \) respectively. Furthermore, we define \( \varepsilon(p) \) as follows: if \( p \in \Gamma \), then \( \varepsilon(p) \in \{ +, - \} \). Finally, the complex numbers \( \alpha_c^\pm, \alpha_\infty^\pm, \alpha_\infty^- \) will be defined in the first step. If the differential equation has not poles it only can fall in this case.

Step 1. Search for each \( c \in \Gamma' \) and for \( \infty \) the corresponding situation such as follows:

\( (c_0) \) If \( \circ(r_c) = 0 \), then \( \sqrt[r]{c} = 0, \quad \alpha_c^\pm = 0 \).

\( (c_1) \) If \( \circ(r_c) = 1 \), then \( \sqrt[r]{c} = 0, \quad \alpha_c^\pm = 1 \).

\( (c_2) \) If \( \circ(r_c) = 2 \), and

\[ r = \cdots + b(x-c)^{-2} + \cdots, \quad \text{then} \]

\[ \sqrt[r]{c} = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}. \]

\( (c_3) \) If \( \circ(r_c) = 2v \geq 4 \), and

\[ r = (a(x-c)^{-v} + \cdots + d(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots, \quad \text{then} \]

\[ \sqrt[r]{c} = a(x-c)^{-v} + \cdots + d(x-c)^{-2}, \quad \alpha_c^\pm = \frac{1}{2} \left( \pm \frac{b}{a} + v \right). \]
(∞₁) If \( \circ (r_\infty) > 2 \), then
\[
[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1
\]

(∞₂) If \( \circ (r_\infty) = 2 \), and \( r = \cdots + bx^2 + \cdots \), then
\[
[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = \frac{1 \pm \sqrt{1 + 4b}}{2}
\]

(∞₃) If \( \circ (r_\infty) = -2v \leq 0 \), and
\[
r = (ax^v + \cdots + d)^2 + bx^{v-1} + \cdots,
\]
then
\[
[\sqrt{r}]_\infty = ax^v + \cdots + d, \quad \text{and} \quad \alpha_\infty^\pm = \frac{1}{2} \left( \pm \frac{b}{a} - v \right).
\]

Step 2. Find \( D \neq \emptyset \) defined by
\[
D = \left\{ m \in \mathbb{Z}_+ : m = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c \in \Gamma'} \alpha_c^{\varepsilon(c)}, \forall (\varepsilon(p))_{p \in \Gamma} \right\}.
\]

If \( D = \emptyset \), then we should start with the case 2. Now, if \( \#D > 0 \), then for each \( m \in D \) we search \( \omega \in \mathbb{C}(x) \) such that
\[
\omega = \varepsilon(\infty) [\sqrt{r}]_\infty + \sum_{c \in \Gamma'} \left( \varepsilon(c) [\sqrt{r}]_c + \alpha_c^{\varepsilon(c)} (x - c)^{-1} \right).
\]

Step 3. For each \( m \in D \), search for a monic polynomial \( P_m \) of degree \( m \) with
\[
P_m'' + 2\omega P_m' + (\omega' + \omega^2 - r) P_m = 0.
\]

If success is achieved then \( \xi_1 = P_m e^{\int \omega} \) is a solution of the differential equation RLDE. If not, then case 1 cannot hold.

Case 2. Search for each \( c \in \Gamma' \) and for \( \infty \) the corresponding situation such as follows:

Step 1. Search for each \( c \in \Gamma' \) and \( \infty \) the sets \( E_c \neq \emptyset \) and \( E_\infty \neq \emptyset \). For each \( c \in \Gamma' \) and for \( \infty \) it is define \( E_c \subset \mathbb{Z} \) and \( E_\infty \subset \mathbb{Z} \) as follows:

\((c_1)\) If \( \circ (r_c) = 1 \), then \( E_c = \{4\} \)

\((c_2)\) If \( \circ (r_c) = 2 \), and \( r = \cdots + b(x - c)^{-2} + \cdots \), then
\[
E_c = \left\{ 2 + k \sqrt{1 + 4b} : k = 0, \pm 2 \right\}.
\]
(c₃) If \( \circ (r_c) = v > 2 \), then \( E_c = \{ v \} \)

(∞₁) If \( \circ (r_\infty) > 2 \), then \( E_\infty = \{ 0, 2, 4 \} \)

(∞₂) If \( \circ (r_\infty) = 2 \), and \( r = \cdots + bx^2 + \cdots \), then
\[
E_\infty = \left\{ 2 + k\sqrt{1 + 4b} : k = 0, \pm 2 \right\}.
\]

(∞₃) If \( \circ (r_\infty) = v < 2 \), then \( E_\infty = \{ v \} \)

**Step 2.** Find \( D \neq \emptyset \) defined by
\[
D = \left\{ m \in \mathbb{Z}_+ : \ m = \frac{1}{2} \left( e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, \ p \in \Gamma \right\}.
\]
If \( D = \emptyset \), then we should start the case 3. Now, if \( \#D > 0 \), then for each \( m \in D \) we search a rational function \( \theta \) defined by
\[
\theta = \frac{1}{2} \sum_{c \in \Gamma'} \frac{e_c}{x - c}.
\]

**Step 3.** For each \( m \in D \), search a monic polynomial \( P_m \) of degree \( m \), such that
\[
P_m'' + 3\theta P_m' + (3\theta' + 3\theta^2 - 4r)P_m' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta + 2r^2)P_m = 0.
\]
If \( P_m \) there is not exists, then the case 2 cannot hold. If such a polynomial is found, set \( \phi = \theta + P'/P \) and let \( \omega \) be a solution of
\[
\omega^2 + \phi\omega + \frac{1}{2} (\phi' + \phi^2 - 2r) = 0.
\]
Then \( \xi_1 = e^{T \omega} \) is a solution of the differential equation RLDE.

**Case 3.** Search for each \( c \in \Gamma' \) and for \( \infty \) the corresponding situation such as follows:

**Step 1.** Search for each \( c \in \Gamma' \) and \( \infty \) the sets \( E_c \neq \emptyset \) and \( E_\infty \neq \emptyset \). For each \( c \in \Gamma' \) and for \( \infty \) it is define \( E_c \subset \mathbb{Z} \) and \( E_\infty \subset \mathbb{Z} \) as follows:

(c₁) If \( \circ (r_c) = 1 \), then \( E_c = \{ 12 \} \)

(c₂) If \( \circ (r_c) = 2 \), and \( r = \cdots + b(x - c)^2 + \cdots \), then
\[
E_c = \left\{ 6 + k\sqrt{1 + 4b} : k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \right\}.
\]
(∞) If \( r(x) = v \geq 2 \), and \( r = \cdots + bx^2 + \cdots \), then
\[
E_\infty = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} \right\} : \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \}, \quad n \in \{4, 6, 12\}.
\]

**Step 2.** Find \( D \neq \emptyset \) defined by
\[
D = \left\{ m \in \mathbb{Z}^+ : \quad m = \frac{n}{12} \left( e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, \ p \in \Gamma \right\}.
\]
In this case we start with \( n = 4 \) to obtain the solution, afterwards \( n = 6 \) and finally \( n = 12 \). If \( D = \emptyset \), then the differential equation has not Liouvillian solution because falls in the case 4. Now, if \( \#D > 0 \), then for each \( m \in D \) with its respective \( n \), it is search a rational function
\[
\theta = \frac{n}{12} \sum_{c \in \Gamma'} \frac{e_c}{x - c}
\]
and a polynomial \( S \) defined as
\[
S = \prod_{c \in \Gamma'} (x - c).
\]

**Step 3.** Search for each \( m \in D \), with its respective \( n \), a monic polynomial \( P_m = P \) of degree \( m \), such that its coefficients can be determined recursively by
\[
P_{-1} = 0, \quad P_n = -P,
\]
\[
P_{i-1} = -SP_i' - ((n - i) S' - S\theta) P_i + (n - i) (i + 1) S^2 r P_{i+1},
\]
where \( i \in \{0, 1, \ldots, n-1, n\} \). If \( P \) there is not exists, then the differential equation has not Liouvillian solution because falls in the case 4. Now, if \( P \) there exists, it is search \( \omega \) such that
\[
\sum_{i=0}^{n} \frac{S^i P}{(n-i)!} \omega^i = 0,
\]
then a solution of the differential equation RLDE is given by
\[
\xi = e^{f \omega},
\]
where \( \omega \) is solution of the previous polynomial of degree \( n \).
A.2 Some remarks on Kovacic’s algorithm

Along of this section we assume that RLDE falls only in one of the four cases.

Remark 12 (Case 1) If RLDE fall in case 1, then its Galois group is given by any of the following groups:

I1 $e$ when the algorithm provides two rational solutions or only one rational solution and the second solution obtained by equation (42) has not logarithmic term.

$$e = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

this group is connected and abelian.

I2 $G_k$ when the algorithm provides only one algebraic solution $\xi$ such that $\xi^k \in C(x)$ and $\xi^{k-1} \notin C(x)$.

$$G_k = \left\{ \begin{pmatrix} \lambda & d \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \text{ is a } k\text{-root of the unity, } d \in C \right\},$$

this group is disconnected and its identity component is abelian.

I3 $C^*$ when the algorithm provides two non algebraic solutions.

$$C^* = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in C^* \right\},$$

this group is connected and abelian.

I4 $C^+$ when the algorithm provides one rational solution and the second solution is not algebraic.

$$C^+ = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} : d \in C \right\}, \quad \xi \in C(x),$$

this group is connected and abelian.

I5 $C^* \ltimes C^+$ when the algorithm only provides one solution $\xi$ such that $\xi$ and its square are not rational functions.

$$C^* \ltimes C^+ = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} : c \in C^*, d \in C \right\}, \quad \xi \notin C(x), \quad \xi^2 \notin C(x).$$

This group is connected and non-abelian.

I6 $SL(2, C)$ if the algorithm do not provides any solution. This group is connected and non-abelian.

Remark 13 (Case 2) If RLDE fall in case 2, then the Kovacic’s Algorithm can provides one or two solutions. This depends of $r$ such as follows:
if \( r \) is given by
\[
    r = \frac{2\phi' + 2\phi - \phi^2}{4},
\]
then there exists only one solution,

II2 if \( r \) is given by
\[
    r \neq \frac{2\phi' + 2\phi - \phi^2}{4},
\]
then there exists two solutions.

II3 The identity component of the Galois group for this case is abelian.

Remark 14 (Case 3) If RLDE falls in case 3, then its Galois group is given by any of the following groups:

III1 Tetrahedral group when \( \omega \) is obtained with \( n = 4 \). This group of order 24 is generated by
\[
    \begin{pmatrix}
        e^{\frac{k\pi i}{3}} & 0 \\
        0 & e^{-\frac{k\pi i}{3}}
    \end{pmatrix}, \quad \frac{1}{3} \left(2 e^{\frac{k\pi i}{3}} - 1\right) \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad k \in \mathbb{Z}.
\]

III2 Octahedral group when \( \omega \) is obtained with \( n = 6 \). This group of order 48 is generated by
\[
    \begin{pmatrix}
        e^{\frac{k\pi i}{4}} & 0 \\
        0 & e^{-\frac{k\pi i}{4}}
    \end{pmatrix}, \quad \frac{1}{2} e^{\frac{k\pi i}{4}} (e^{\frac{k\pi i}{4}} + 1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad k \in \mathbb{Z}.
\]

III3 Icosahedral group when \( \omega \) is obtained with \( n = 12 \). This group of order 120 is generated by
\[
    \begin{pmatrix}
        e^{\frac{k\pi i}{5}} & 0 \\
        0 & e^{-\frac{k\pi i}{5}}
    \end{pmatrix}, \quad \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}, \quad k \in \mathbb{Z},
\]

being \( \phi \) and \( \psi \) defined as
\[
    \phi = \frac{1}{5} \left( e^{\frac{4k\pi i}{5}} - e^{\frac{2k\pi i}{5}} + 4e^{\frac{k\pi i}{5}} - 2 \right), \quad \psi = \frac{1}{5} \left( e^{\frac{4k\pi i}{5}} + 3e^{\frac{2k\pi i}{5}} - 2e^{\frac{k\pi i}{5}} + 1 \right)
\]

III4 The identity component of the Galois group for this case is abelian.

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