On the Breiman conjecture

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Abstract

Let $Y_1, Y_2, \ldots$ be positive, nondegenerate, i.i.d. $G$ random variables, and independently let $X_1, X_2, \ldots$ be i.i.d. $F$ random variables. In this note we show that whenever $\sum X_i Y_i / \sum Y_i$ converges in distribution to nondegenerate limit for some $F \in \mathcal{F}$, in a specified class of distributions $\mathcal{F}$, then $G$ necessarily belongs to the domain of attraction of a stable law with index less than 1. The class $\mathcal{F}$ contains those nondegenerate $X$ with a finite second moment and those $X$ in the domain of attraction of a stable law with index $1 < \alpha < 2$.

1 Introduction and results

Let $Y, Y_1, \ldots$ be positive, nondegenerate, i.i.d. random variables with distribution function [df] $G$, and independently let $X, X_1, \ldots$ be i.i.d. nondegenerate random variables with df $F$. Let $\phi_X$ denote the characteristic function [cf] of $X$. We shall use the notation $Y \in D(\beta)$ to mean that $Y$ is in the domain of attraction of a stable law of index $0 < \beta < 1$, and $Y \in D(0)$ will denote that $1 - G$ is slowly varying at infinity. Furthermore $\mathcal{R}\mathcal{V}_\infty(\rho)$ will signify the class of positive measurable functions regularly varying at infinity with index $\rho$, and $\mathcal{R}\mathcal{V}_0(\rho)$ the class of positive measurable functions regularly varying at zero with index $\rho$. In particular, using this notation $Y \in D(\beta)$, with $0 \leq \beta < 1$, if and only if $1 - G \in \mathcal{R}\mathcal{V}_\infty(-\beta)$.

For each integer $n \geq 1$ set

$$T_n = \sum_{i=1}^{n} \frac{X_i Y_i}{\sum_{i=1}^{n} Y_i}. \quad (1)$$

Notice that $\mathbb{E}|X| < \infty$ implies that $T_n$ is stochastically bounded. Theorem 4 of Breiman [2] says that $T_n$ converges in distribution along the full sequence $\{n\}$ for every $X$ with finite expectation, and with at least one limit law being nondegenerate if and only if

$$Y \in D(\beta), \text{ with } 0 \leq \beta < 1. \quad (2)$$

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Let $X$ denote the class of nondegenerate random variables $X$ with $E|X| < \infty$ and let $X_0$ denote those $X \in X$ such that $E X = 0$. At the end of his paper Breiman conjectured that if for some $X \in X$, $T_n$ converges in distribution to some nondegenerate random variable $T$, written

$$T_n \to_d T, \text{ as } n \to \infty, \text{ with } T \text{ nondegenerate}, \quad (3)$$

then (2) holds. By Proposition 2 and Theorem 3 of [2], for any $X \in X$, (2) implies (3), in which case $T$ has a distribution related to the arcsine law. Using this fact, we see that his conjecture can be restated to be: for any $X \in X$, (2) is equivalent to (3).

It has proved to be surprisingly challenging to resolve. Mason and Zinn [8] partially verified Breiman’s conjecture. They established that whenever $X$ is nondegenerate and satisfies $E|X|^p < \infty$ for some $p > 2$, then (2) is equivalent to (3). In this note we further extend this result.

**Theorem** Assume that for some $X \in X_0$, $1 < \alpha \leq 2$, positive slowly varying function $L$ at zero and $c > 0$,

$$\frac{-\log (\text{Re} \phi_X(t))}{|t|^\alpha L(|t|)} \to c, \text{ as } t \to 0, \quad (4)$$

(in the case $\alpha = 2$ we assume that $\liminf_{t \to 0} L(t) > 0$). Whenever (3) holds then $Y \in D(\beta)$ for some $\beta \in [0, 1)$.

Let $F$ denote the class of random variables that satisfy the conditions of the theorem. Applying our theorem in combination with Proposition 2 and Theorem 3 of [2] we get the following corollary.

**Corollary** Whenever $X - E X \in F$, (2) is equivalent to (3).

**Remark 1** It can be inferred from Theorem 8.1.10 of Bingham et al. [1] that for $X \in X_0$, (4) holds for some $1 < \alpha < 2$, positive slowly varying function $L$ at zero and $c > 0$ if and only if $X$ satisfies $P \{|X| > x\} \sim L(1/x)x^{-\alpha}c\Gamma(\alpha)^2 \sin \left(\frac{\pi \alpha}{2}\right)$. Note that a random variable $X \in X_0$ in the domain of attraction of a stable law of index $1 < \alpha < 2$ satisfies (4). Also a random variable $X \in X_0$ with variance $0 < \sigma^2 < \infty$ fulfills (4) with $\alpha = 2$, $L = 1$ and $c = \sigma^2/2$.

**Remark 2** Consult Kevei and Mason [7] for a fairly exhaustive study of the asymptotic distributions of $T_n$ along subsequences, along with revelations of their unexpected properties.

The theorem follows from the two propositions below. First we need more notation. For any $\alpha \in (1, 2]$ define for $n \geq 1$

$$S_n(\alpha) = \sum_{i=1}^{n} Y_i^\alpha / (\sum_{i=1}^{n} Y_i)^\alpha. \quad (5)$$

**Proposition 1** Assume that the assumptions of the theorem hold. Then for some $0 < \gamma \leq 1$

$$E S_n(\alpha) \to \gamma, \text{ as } n \to \infty. \quad (6)$$

The next proposition is interesting in its own right. It is an extension of Theorem 5.3 by Fuchs et al. [4], where $\alpha = 2$ (see also Proposition 3 of [8]).
Proposition 2 If (6) holds with some \( \gamma \in (0,1) \) then \( Y \in D(\beta) \), for some \( \beta \in [0,1) \), where \(-\beta \in (-1,0]\) is the unique solution of
\[
\text{Beta}(\alpha - 1, \beta + 1) = \frac{\Gamma(\alpha - 1)\Gamma(1 + \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.
\]

In particular, \( Y \in D(0) \) for \( \gamma = 1 \).

Conversely, if \( G \in D(\beta) \), \( 0 \leq \beta < 1 \), then (6) holds with
\[
\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 + \beta)} = \frac{1}{(\alpha - 1)\text{Beta}(\alpha - 1, \beta + 1)}.
\]

2 Proofs

Set for each \( n \geq 1 \), \( R_i = Y_i / \sum_{i=1}^{n} Y_i \), for \( i = 1, \ldots, n \). For notational ease we drop the dependence of \( R_i \) on \( n \geq 1 \). Consider the sequence of strictly decreasing continuous functions \( \{\varphi_n\}_{n \geq 1} \) on \([1, \infty)\) defined by \( \varphi_n(y) = \mathbb{E}(\sum_{i=1}^{n} R_i^y), y \in [1, \infty) \). Note that each function \( \varphi_n \) satisfies \( \varphi_n(1) = 1 \).

By a diagonal selection procedure for each subsequence of \( \{n\}_{n \geq 1} \) there is a further subsequence \( \{n_k\}_{k \geq 1} \) and a right continuous nonincreasing function \( \psi \) such that \( \varphi_{n_k} \) converges to \( \psi \) at each continuity point of \( \psi \).

Lemma 1 Each such function \( \psi \) is continuous on \((1, \infty)\).

Proof Choose any subsequence \( \{n_k\}_{k \geq 1} \) and a right continuous nonincreasing function \( \psi \) such that \( \varphi_{n_k} \) converges to \( \psi \) at each continuity point of \( \psi \) in \((1, \infty)\). Select any \( x > 1 \) and continuity points \( x_1, x_2 \in (1, \infty) \) of \( \psi \) such that \( 1 < x_1 < x < x_2 < \infty \). Set \( \rho_1 = x_1 - 1 \) and \( \rho_2 = x_2 - 1 \). Since \( \rho_2 / \rho_1 > 1 \) we get by Hölder’s inequality
\[
\sum_{i=1}^{n_k} R_i^{x_1} = \sum_{i=1}^{n_k} R_i^{\rho_1} R_i \leq \left( \sum_{i=1}^{n_k} R_i^{\rho_2} R_i \right)^{\rho_1 / \rho_2} = \left( \sum_{i=1}^{n_k} R_i^{x_2} \right)^{\rho_1 / \rho_2}.
\]

Thus by taking expectations and using Jensen’s inequality we get \( \varphi_{n_k}(x_1) \leq (\varphi_{n_k}(x_2))^{\rho_1 / \rho_2} \). Letting \( n_k \to \infty \), we have \( \psi(x_1) \leq (\psi(x_2))^{\rho_1 / \rho_2} \). Since \( x_1 < x \) and \( x_2 > x \) can be chosen arbitrarily close to \( x \) we conclude by right continuity of \( \psi \) at \( x \) that \( \psi(x-) = \psi(x+) = \psi(x) \). \( \square \)

Proof of Proposition 1 For a complex \( z \), we use the notation for the principal branch of the logarithm, \( \log(z) = \log |z| + i \arg z \), where \(-\pi < \arg z \leq \pi \), i.e. \( z = |z| \exp(i \arg z) \). We see that for all \( t \)
\[
\mathbb{E} \exp(itT_n) = \mathbb{E} \left( \prod_{j=1}^{n} \phi_X(tR_j) \right) = \mathbb{E} \left( \prod_{j=1}^{n} \exp(\log \phi_X(tR_j)) \right).
\]
Since $EX = 0$ we have $\Re \phi_X(u) = 1 - o_+(u)$, where $o_+(u) \geq 0$, and $o_+(u)$ and $o_+(u)/u \to 0$ as $u \to 0$; and $\Im \phi_X(u) = o(u)$. This when combined with

$$(\arctan \theta)' = \frac{1}{1 + \theta^2}$$

gives as $u \to 0$,

$$\arg \phi_X(u) = \arctan \left( \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right) = o(u).$$

Note that for all $|u| > 0$ sufficiently small so that $\Re \phi_X(u) > 0$

$$Log \phi_X(u) = Log(\Re \phi_X(u) + i\Im \phi_X(u)) = \log \Re \phi_X(u) + Log \left(1 + \frac{i\Im \phi_X(u)}{\Re \phi_X(u)}\right),$$

where for the second term

$$\Re Log \left(1 + \frac{i\Im \phi_X(u)}{\Re \phi_X(u)}\right) = \frac{1}{2} \left(\frac{\Im \phi_X(u)}{\Re \phi_X(u)}\right)^2 (1 + o(u)), \text{ as } u \to 0.$$

Thus for every $\varepsilon > 0$ for all $|t| > 0$ sufficiently small and independent of $n \geq 1$ and $R_1, \ldots, R_n$

$$1 - \varepsilon^2 t^2 \leq \cos(\varepsilon t) \leq \Re \left(\exp \left(\sum_{j=1}^n Log \left(1 + \frac{i\Im \phi_X(tR_j)}{\Re \phi_X(tR_j)}\right)\right)\right) \leq e^{2-1 \varepsilon t^2} \leq 1 + \varepsilon t^2.$$

Thus we obtain

$$\mathbb{E} \exp \left\{\sum_{j=1}^n \log \Re \phi_X(tR_j)\right\} (1 - \varepsilon^2 t^2) \leq \mathbb{E} (\Re \exp (itT_n))$$

$$= \Re \mathbb{E} \exp (itT_n)$$

$$\leq \mathbb{E} \exp \left\{\sum_{j=1}^n \log \Re \phi_X(tR_j)\right\} (1 + \varepsilon t^2).$$

We shall show (4) implies that (3) holds for some $0 < \gamma \leq 1$. Now using (4) we get for any $0 < \delta < c$ and all $|t|$ small enough independent of $n \geq 1$,

$$-\varepsilon t^2 + \log \mathbb{E} \exp \left( - (c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L (|t| R_i)\right)\right) \leq \log (\Re \mathbb{E} \exp (itT_n))$$

$$\leq \varepsilon t^2 + \log \mathbb{E} \exp \left( - (c - \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L (|t| R_i)\right)\right).$$

Next since $

\log s/(1 - s) \to -1 \text{ as } s \nearrow 1, \text{ for all } |t| \text{ small enough independent of } n \geq 1 \text{ and } R_1, \ldots, R_n, \text{ (keeping mind that } \sum_{i=1}^n R_i = 1 \text{ and } 1 < \alpha \leq 2) \n
log \mathbb{E} \exp \left( - (c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L (|t| R_i)\right)\right)$$

$$\geq - \left(1 + \frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left( - (c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L (|t| R_i)\right)\right)\right)$$

$$4
and
\[
\log \mathbb{E} \exp \left( - (c - \delta) |t|^\alpha \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right) \right) \\
\leq - \left( 1 - \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( - (c - \delta) |t|^\alpha \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right) \right) \right).
\]

Further since \((1 - \exp(-y))/y \to 1\) as \(y \to 0\), for all \(|t|\) small enough independent of \(n \geq 1\),
\[
- \left( 1 + \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( - (c + \delta) |t|^\alpha \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right) \right) \right) \\
\geq - (1 + \delta) (c + \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right)
\]
and
\[
- \left( 1 - \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( - (c - \delta) |t|^\alpha \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right) \right) \right) \\
\leq - (1 - \delta) (c - \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right).
\]

Therefore for all \(|t|\) small enough independent of \(n\),
\[
- \varepsilon t^2 - (1 + \delta) (c + \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right) \\
\leq \log (9 \mathbb{E} \exp (\varepsilon t T_n)) \\
\leq \varepsilon t^2 - (1 - \delta) (c - \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^{n} R_i^\alpha L(|t| R_i) \right).
\]

By the Potter’s bound, Theorem 1.5.6 (i) in [1], for all \(A > 1\) and \(1 < \alpha_1 < \alpha < \alpha_2\), for all \(t > 0\) small enough independent of \(n \geq 1\),
\[
A^{-1} \sum_{i=1}^{n} R_i^{\alpha_2} \leq \sum_{i=1}^{n} R_i^\alpha L(|t| R_i)/L(|t|) \leq A \sum_{i=1}^{n} R_i^{\alpha_1}. \tag{7}
\]

We see now that for all \(n \geq 1\) and \(0 < 4\varepsilon < c\), appropriate \(1 < \alpha_1 < \alpha < \alpha_2\) and all \(|t|\) small,
enough independent of \( n \),

\[-\varepsilon t^2 - (1 + \varepsilon) (c + 2\varepsilon) |t|^\alpha L (|t|) \mathbb{E} S_n (\alpha_2)\]

\[= -\varepsilon t^2 - (1 + \varepsilon) (c + 2\varepsilon) |t|^\alpha L (|t|) \mathbb{E} \left( \sum_{i=1}^n R_{i2}^{\alpha} \right)\]

\[\leq \log (\Re \mathbb{E} \exp (\imath t T_n)) \]

\[\leq \varepsilon t^2 - (1 - \varepsilon) (c - 2\varepsilon) |t|^\alpha L (|t|) \mathbb{E} \left( \sum_{i=1}^n R_{i1}^{\alpha_1} \right)\]

\[= \varepsilon t^2 - (1 - \varepsilon) (c - 2\varepsilon) |t|^\alpha L (|t|) \mathbb{E} S_n (\alpha_1).\]

Choose any subsequence \( \{n_k\}_{k \geq 1} \) and a right continuous nonincreasing function \( \psi \) such that \( \varphi_{n_k} \) converges to \( \psi \) at each continuity point of \( \psi \), which by Lemma 1 above is all \((1, \infty)\). We see that \( \mathbb{E} S_{n_k} (\alpha) \rightarrow \psi (\alpha) \), \( \mathbb{E} S_{n_k} (\alpha_1) \rightarrow \psi (\alpha_1) \) and \( \mathbb{E} S_{n_k} (\alpha_2) \rightarrow \psi (\alpha_2) \), where necessarily \( 0 < \psi (\alpha_2) \leq \psi (\alpha) \leq \psi (\alpha_1) \leq 1 \). (The case \( \psi (\alpha_1) = 0 \) cannot happen, since this would imply that \( T \) is degenerate.) We see that for all \( |t| \) sufficiently small independent of \( n_k \geq 1 \),

\[-\varepsilon - (1 + \varepsilon) (c + 3\varepsilon) \psi (\alpha_2) \leq \log (\Re \mathbb{E} \exp (\imath t T_{n_k})) / (|t|^\alpha L (|t|)) \leq \varepsilon - (1 - \varepsilon) (c - 3\varepsilon) \psi (\alpha_1),\]

where for \( \alpha = 2 \) we use the assumption that in this case \( \lim \inf_{t \searrow 0} L (t) > 0 \). Since \( 0 < 4\varepsilon < c \) can be made arbitrarily small and \( 0 \leq \psi (\alpha_1) - \psi (\alpha_2) \) can be made as close to zero as desired, by letting \( n_k \rightarrow \infty \), we get that for all \( |t| \) sufficiently small

\[-\varepsilon - (1 + \varepsilon) (c + 4\varepsilon) \psi (\alpha) \leq \log (\Re \mathbb{E} \exp (\imath t T)) / (|t|^\alpha L (|t|)) \leq \varepsilon - (1 - \varepsilon) (c - 4\varepsilon) \psi (\alpha),\]

which can happen only if \( \psi (\alpha) \) does not depend on \( \{n_k\} \). Thus (6) holds for some \( 0 < \gamma \leq 1 \), namely \( \gamma = \psi (\alpha) \). \(\Box\)

**Proof of Proposition 2** To begin with, we note that whenever (6) holds, necessarily \( \mathbb{E} Y = \infty \). To see this, write \( D_{n1} = \max_{1 \leq i \leq n} Y_i / (\sum_{i=1}^n Y_i) \) and observe that

\[
\left( D_{n1} \right)^\alpha = \max_{1 \leq i \leq n} \frac{Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} \leq S_n (\alpha)
\]

\[\leq \max_{1 \leq i \leq n} \frac{Y_i^{\alpha - 1}}{(\sum_{i=1}^n Y_i)^{\alpha - 1}} = \left( D_{n1} \right)^{\alpha - 1}.\]

From these inequalities it is easy to prove that \( \mathbb{E} S_n (\alpha) \rightarrow 0 \), \( n \rightarrow \infty \), if and only if

\[D_{n1} \rightarrow P \ 0, \ n \rightarrow \infty. \quad (8)\]

Proposition 1 of Breiman [2] says that (8) holds if and only there exists a sequence of positive constants \( B_n \) converging to infinity such that

\[\sum_{i=1}^n Y_i / B_n \rightarrow P \ 1, \ n \rightarrow \infty. \quad (9)\]
Since $\mathbb{E}Y < \infty$ obviously implies (9), it readily follows that $\mathbb{E}S_n(\alpha) \to 0$, $n \to \infty$, and thus (6) cannot hold.

We shall first prove the first part of Proposition 2. Following similar steps as in [8] we have that

$$
\mathbb{E} \frac{\sum_{i=1}^{n} Y_i^\alpha}{\left(\sum_{i=1}^{n} Y_i\right)^\alpha} = \frac{n}{\Gamma(\alpha)} \int_{0}^{\infty} Y_1^\alpha e^{-t \sum_{i=1}^{n} Y_i t^\alpha-1} dt
$$

$$
= \frac{n}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \mathbb{E} \left( e^{-t Y_1} \right) (\mathbb{E} e^{-t Y_1})^{n-1} dt
$$

$$
= \frac{n}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \phi_\alpha(t) \phi_0(t)^{n-1} dt.
$$

Next, assuming (6) and arguing as in the proof of Theorem 3 in [2] we get

$$
s \int_{0}^{\infty} t^{\alpha-1} \phi_\alpha(t) e^{s \log \phi_0(t)} dt \to \gamma \Gamma(\alpha), \quad s \to \infty. \tag{10}
$$

For $y \geq 0$, let $q(y)$ denote the inverse of $- \log \varphi_0(t)$. Changing the variables to $y = - \log \varphi_0(t)$ and $t = q(y)$, we get from (10) that

$$
s \int_{0}^{\infty} (q(y))^{\alpha-1} \phi_\alpha(q(y)) \exp(-sy) dq(y) \to \gamma \Gamma(\alpha), \quad s \to \infty.
$$

By Karamata’s Tauberian theorem, see Theorem 1.7.1’ on page 38 of [1], we conclude that

$$
v^{-1} \int_{0}^{v} (q(x))^{\alpha-1} \phi_\alpha(q(x)) dq(x) \to \gamma \Gamma(\alpha), \quad v \downarrow 0,
$$

which, in turn, by the change of variable $y = q(x)$ gives

$$
\lim_{t \to 0} \frac{\int_{0}^{t} y^{\alpha-1} \phi_\alpha(y) dy}{- \log \phi_0(t)} \to \gamma \Gamma(\alpha), \quad t \downarrow 0.
$$

Now using that $- \log \phi_0(t) \sim 1 - \phi_0(t)$ as $t \to 0$, we end up with

$$
\lim_{t \to 0} \frac{\int_{0}^{t} y^{\alpha-1} \phi_\alpha(y) dy}{1 - \phi_0(t)} = \gamma \Gamma(\alpha).
$$

Since $\phi_\alpha(y) = \int_{0}^{\infty} e^{-uy} u^\alpha G(du)$, by Fubini’s theorem

$$
\int_{0}^{t} y^{\alpha-1} \phi_\alpha(y) dy = \int_{0}^{\infty} u^\alpha G(du) \int_{0}^{t} y^{\alpha-1} e^{-uy} dy
$$

$$
= \int_{0}^{\infty} G(du) \int_{0}^{ut} z^{\alpha-1} e^{-z} dz
$$

$$
= \int_{0}^{\infty} \mathcal{G}(z/t) z^{\alpha-1} e^{-z} dz
$$

$$
= \int_{0}^{\infty} \mathcal{G}(u) u^{\alpha-1} e^{-u} du.
$$
A partial integration gives

\[ 1 - \phi_0(t) = t \int_0^\infty \overline{G}(u)e^{-ut}du. \]

So (10) reads

\[ t^{\alpha-1} \left[ \int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}du \right] \rightarrow \gamma \Gamma(\alpha), \quad \text{as } t \searrow 0. \] (11)

From now on we shall assume that (6) holds with \(0 < \gamma \leq 1\). Let us define the function for \(t > 0\)

\[ f(t) = \int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}du. \] (12)

Clearly, \(f\) is monotone decreasing and since \(EY = \infty\), \(\lim_{t \to 0} f(t) = \infty\). Moreover, showing that \(f\) is regularly varying at zero implies that \(G\) is regularly varying at infinity. We use the identity

\[ u^{1-\alpha}e^{-ut} = \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2}e^{-(y+t)u}dy, \]

which holds for \(u > 0\) and \(\alpha \in (1, 2]\). (This is the Weyl-transform, or Weyl-fractional integral of the function \(e^{-ut}\).) This identity combined with Fubini’s theorem (everything is nonnegative) gives

\[
\frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2}f(t+y)dy = \int_0^\infty \overline{G}(u)u^{\alpha-1}du \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2}e^{-(y+t)u}dy = \int_0^\infty \overline{G}(u)e^{-ut}du.
\]

So we can rewrite (11) as

\[
\lim_{t \searrow 0} \frac{t^{\alpha-1}f(t)}{\int_0^\infty y^{\alpha-2}f(t+y)dy} = \frac{\gamma \Gamma(\alpha)}{\Gamma(\alpha-1)} = \gamma(\alpha-1).
\] (13)

A change of variable gives

\[
\int_0^\infty y^{\alpha-2}f(t+y)dy = t^{\alpha-1} \int_1^\infty (u-1)^{\alpha-2}f(ut)du,
\]

and so we have

\[
\lim_{t \searrow 0} \int_1^\infty (u-1)^{\alpha-2}f(ut)du = [\gamma(\alpha-1)]^{-1}.
\] (14)

We can rewrite \(f\) as

\[ f(t) = \int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}du = \frac{1}{\Gamma(\alpha)} \int_0^\infty \overline{G}(u/t)u^{\alpha-1}e^{-u}du,
\]

from which we see that the function

\[ g(t) = \int_0^\infty \overline{G}(u/t)u^{\alpha-1}e^{-u}du = t^\alpha f(t),
\]
is bounded and nondecreasing. Substituting \( g \) into (14) we obtain
\[
\lim_{t \to 0^+} \int_1^\infty (u-1)^{\alpha-2} u^{-\alpha} g(ut) g(t) \, du = [\gamma(\alpha-1)]^{-1}.
\] (15)

Write \( g_\infty(x) = g(x^{-1}) \), \( x > 0 \). Then (15) has the form
\[
\int_1^\infty (u-1)^{\alpha-2} u^{-\alpha} g_\infty(x/u) \, du = \frac{k * g_\infty(x)}{g_\infty(x)} \to [\gamma(\alpha-1)]^{-1}, \quad \text{as } x \to \infty,
\] (16)

where
\[
k(u) = \begin{cases} (u-1)^{\alpha-2} u^{-\alpha+1}, & u > 1, \\ 0, & 0 < u \leq 1, \end{cases}
\]
and
\[
k^M * h(x) = \int_0^\infty h(x/u) k(u)/u \, du
\]
is the Mellin-convolution of \( h \) and \( k \). Note that the Mellin-transform of \( k \),
\[
\tilde{k}(z) = \int_1^\infty (u-1)^{\alpha-2} u^{-\alpha-z} \, du = \int_0^1 (1-v)^{\alpha-2} v^z \, dv = \frac{\Gamma(\alpha-1) \Gamma(1+z)}{\Gamma(\alpha-z)} = \text{Beta}(\alpha-1, 1+z)
\]
is convergent for \( z > -1 \). We apply a version of the Drasin-Shea theorem (Theorem 5.2.3 on page 273 of [1]). To do this we must verify the following conditions:

1. \( \tilde{k} \) has a maximal convergent strip \( a < \Re z < b \) such that \( a < 0 \) and \( b > 0 \), \( \tilde{k}(a^+) = \infty \) and \( \tilde{k}(b^-) = \infty \) if \( b < \infty \). Our \( \tilde{k} \) satisfies this condition with \( a = -1 \) and \( b = \infty \).

2. Our function of interest is
\[
g_\infty(x) = g(x^{-1}) = \int_0^\infty \overline{G}(ux) u^{\alpha-1} e^{-u} \, du, \quad x > 0,
\]
is certainly positive and locally bounded.

3. Also our function \( g_\infty \) is of bounded decrease, since for \( \lambda > 1 \)
\[
\frac{g_\infty(\lambda x)}{g_\infty(x)} = \lambda^{-\alpha} \frac{(\lambda x)^{\alpha} g(1/(\lambda x))}{x^{\alpha} g(1/x)} = \lambda^{-\alpha} \frac{f(1/(\lambda x))}{f(1/x)} \geq \lambda^{-\alpha},
\]
so its lower Matuszewska index is at least \(-\alpha\).

Therefore by Theorem 5.2.3 of [1], whenever,
\[
\frac{k^M * g_\infty(x)}{g_\infty(x)} \to c, \quad \text{as } x \to \infty,
\] (17)
then \( \tilde{k}(\rho) = c \) for some \( \rho \in (-1, \infty) \). (In our case by (16), \( c = [\gamma(\alpha - 1)]^{-1} \).) Moreover, since \( \tilde{k}(z) \) is strictly decreasing on \((-1, \infty)\) and \( \tilde{k}(0) = \frac{1}{z} \), for any \( 0 < \gamma \leq 1 \) the solution \( \rho \) to \( \tilde{k}(\rho) = [\gamma(\alpha - 1)]^{-1} \) must lie in \((-1, 0)\). Theorem 5.2.3 of [1] also says that \( g_\infty(x) \) is regularly varying at infinity with index \( 0 \geq \rho > -1 \).

Next since \( g_\infty(x) = g(x^{-1}) = x^{-\alpha} f(x^{-1}) \in \mathcal{RV}_\infty(\rho) \), where \( \tilde{k}(\rho) = c, g \in \mathcal{RV}_0(-\rho) \), which implies that \( f \in \mathcal{RV}_0(-\rho - \alpha) \). Recalling that

\[
\int_0^x G(u)u^{\alpha-1}e^{-ut}du,
\]

the Karamata Tauberian theorem now gives that

\[
\int_0^x G(u)u^{\alpha-1}du \in \mathcal{RV}_\infty(\alpha + \rho).
\]

Thus by Lemma 2, \( G(u) \in \mathcal{RV}_\infty(\rho) \).

This says that \( Y \in D(\beta) \), where \( \rho = -\beta \in (-1, 0] \) and \( \beta \) is the unique solution of

\[
\text{Beta}(\alpha - 1, \beta + 1) = \frac{\Gamma(\alpha - 1)\Gamma(1 + \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.
\]

We now turn to the proof of the second part of Proposition 2. First consider the the case \( \beta = 0 \). Let \( 0 \leq D_n^{(n)} \leq \cdots \leq D_n^{(1)} \) denote the order statistics of \( Y_1/\sum_{i=1}^n Y_i, \ldots, Y_n/\sum_{i=1}^n Y_i \). We see that

\[
E\left(D_n^{(1)}\right) \leq ES_n(\alpha) = \sum_{i=1}^n E\left(D_n^{(i)}\right) \leq E\left(D_n^{(1)}\right)^{\alpha-1} \leq 1.
\]

Now \( D_n^{(1)} \rightarrow_P 1 \) if and only if \( Y \in D(0) \). (See Theorem 1 of Haeusler and Mason [5] and their references.) Thus if \( Y \in D(0) \) then (1.6) holds with \( \gamma = 1 \).

Now assume that \( Y \in D(\beta) \), \( 0 < \beta < 1 \). In this case, there exists a sequence of positive constants \( \{a_n\}_{n \geq 1} \), such that \( a_n^{-1} \sum_{i=1}^n Y_i \rightarrow_d U \), where \( U \) is a \( \beta \)-stable random variable, with characteristic function

\[
Ee^{itU} = \exp\left\{ \beta \int_0^\infty (e^{tu} - 1)u^{-\beta-1}du \right\}.
\]

Moreover, \( Y^\alpha \in D(\beta/\alpha) \), and it is easy to check that \( a_n^{-\alpha} \sum_{i=1}^n Y_i^\alpha \rightarrow_d V \), where \( V \) is a \( \beta/\alpha \)-stable random variable, with cf

\[
Ee^{itV} = \exp\left\{ \frac{\beta}{\alpha} \int_0^\infty (e^{tu} - 1)u^{-\beta/\alpha-1}du \right\}.
\]

Since

\[
\lim_{n \to \infty} n\mathbb{P}\{Y > a_n u, Y^\alpha > a_n^\alpha v\} = \lim_{n \to \infty} n\mathbb{G}(a_n(u \lor v^{1/\alpha})) = u^{-\beta} \land v^{-\beta/\alpha} =: \Pi((u, \infty) \times (v, \infty)),
\]
for \( u, v \geq 0, u + v > 0 \), using Corollary 15.16 of Kallenberg \[6\] one can show that the joint convergence also holds, and the limiting bivariate Lévy measure is \( \Pi \). That is

\[
\left( a_n^{-1} \sum_{i=1}^{n} Y_i, a_n^{-\alpha} \sum_{i=1}^{n} Y_i^\alpha \right) \to_d (U, V),
\]

where the limiting bivariate random vector has cf

\[
\mathbb{E} e^{i(sU + tV)} = \exp \left\{ \int_{[0,\infty]^2} \left( e^{i(su + tv)} - 1 \right) \Pi(du, dv) \right\} = \exp \left\{ \beta \int_{0}^{\infty} \left( e^{i(su + tv)} - 1 \right) u^{-\beta-1} du \right\}.
\]

Since \( \mathbb{P} \{ U > 0 \} = \mathbb{P} \{ V > 0 \} = 1 \), we obtain

\[
S_n(\alpha) \to_d V \frac{U}{a^{\alpha}}.
\]

Thus since \( \mathbb{E} S_n(\alpha) \leq 1 \) for all \( n \geq 1 \)

\[
\mathbb{E} S_n(\alpha) \to \mathbb{E} \left( \frac{V}{U^{\alpha}} \right).
\]

Clearly \( \mathbb{P} \{ U < \infty \} = 1 \), which implies that \( 0 < \mathbb{E} \left( \frac{V}{U^{\alpha}} \right) \leq 1 \), and thus by the first part of Proposition 2

\[
0 < \gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 + \beta)} < 1.
\]

**Lemma 2** Suppose that for some \( \alpha \geq 1, \rho > -1 \) and slowly varying function \( L \) at infinity

\[
U(x) := \int_{0}^{x} G(u) u^{a-1} du = L(x) x^{\alpha + \rho}, \quad x > 0,
\]

then

\[
G(u) \sim (\alpha + \rho) L(u) u^\rho, \quad \text{as} \ u \to \infty.
\]

**Proof** We shall follow closely the proof the lemma on page 446 of Feller \[3\]. Choose any \( 0 < a < b < \infty \). We see that

\[
\frac{U(tb) - U(ta)}{U(t)} = \int_{a}^{b} \frac{G(ut)(ut)^{a-1}}{U(t)} du = \int_{a}^{b} \frac{G(ut)}{L(t) t^{a+\rho}} du = \int_{a}^{b} \frac{G(ut) u^{a-1}}{L(t) t^\rho} du.
\]

Since \( G \) is nonincreasing and \( G(ut) / (L(t) t^\rho) \) is necessarily bounded for each \( u > 0 \) as \( t \to \infty \), just as in Feller one can apply the Helly-Bray theorem to find a positive sequence \( t_k \to \infty \) such that for a measurable function \( \psi \) on \( [0,\infty) \), \( G(ut_k)/L(t_k) t_k^\rho \to \psi(u) \), for all continuity points \( u \) of \( \psi \). This implies that for all \( 0 < a < b < \infty \)

\[
\frac{U(t_k b) - U(t_k a)}{U(t_k)} \to b^{a+\rho} - a^{a+\rho} = \int_{a}^{b} \psi(u) u^{a-1} du.
\]

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This forces $\psi(u)u^{\alpha-1} = (\alpha + \rho)u^{\alpha+\rho-1}$, and since $\psi$ is independent of any particular positive sequence $t_k \to \infty$ defining it,

$$\overline{G}(ut)/(L(ut)^\rho) \to \alpha + \rho, \text{ as } t \to \infty.$$ 

\[\square\]

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