Multigrid finite element method in stress analysis of three-dimensional elastic bodies of heterogeneous structure

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Abstract. To calculate the three-dimensional elastic body of heterogeneous structure under static loading, a method of multigrid finite element is provided, when implemented on the basis of algorithms of finite element method (FEM), using homogeneous and composite three-dimensional multigrid finite elements (MFE). Peculiarities and differences of MFE from the currently available finite elements (FE) are to develop composite MFE (without increasing their dimensions), arbitrarily small basic partition of composite solids consisting of single-grid homogeneous FE of the first order can be used, i.e. in fact, to use micro approach in finite element form. These small partitions allow one to take into account in MFE, i.e. in the basic discrete models of composite solids, complex heterogeneous and microscopically inhomogeneous structure, shape, the complex nature of the loading and fixation and describe arbitrarily closely the stress and stain state by the equations of three-dimensional elastic theory without any additional simplifying hypotheses. When building the \(m\)-grid FE, \(m\) of nested grids is used. The fine grid is generated by a basic partition of MFE, the other \(m-1\) large grids are applied to reduce MFE dimensionality, when \(m\) is increased, MFE dimensionality becomes smaller. The procedures of developing MFE of rectangular parallelepiped, irregular shape, plate and beam types are given. MFE generate the small dimensional discrete models and numerical solutions with a high accuracy. An example of calculating the laminated plate, using three-dimensional 3-grid FE and the reference discrete model is given, with that having 2.2 milliards of FEM nodal unknowns.

1. Introduction

In current paper, to calculate the elastic bodies with heterogeneous (microscopically inhomogeneous) structure and static loading, a method of multigrid finite elements using three-dimensional composite MFE is proposed. The essence of MFE is as follows. At a basic partition (on the fine grid) of \(m\)-grid FE, \(m \geq 2\), which consists of the 1st order single-grid homogeneous FE and takes into account its heterogeneous structure according to the micro approach [1], the total potential energy as a function of many variables depending on the fine grid nodal displacements has been determined. On the other \(m-1\) large grids (enclosed in the fine one), the displacement functions used to reduce the dimension of the \(F\) function are built by the FEM algorithms [2–4] that allows one to develop MFE of small dimension. To approximate the MFE displacement functions, power-law and Lagrangian polynomials of various orders and the equations of three-dimensional problem of elasticity [5], written in local Cartesian coordinate systems without any simplifying hypotheses are used [6, 7]. Lagrange...
polynomials are effectively applied in the construction of plate and beam MFE. The mixed multigrid discrete models used to calculate the composite solids of irregular shape have been considered.

Advantages of MFE are to take into account the heterogeneous structure of three-dimensional bodies according to the micro approach, to form the discrete models, whose dimensions are $10^4 \times 10^4$ less than the ones of the basic models, to generate computational solutions with a high rate of convergence to the exact solutions that allows one to find solutions with a small error. The calculations have shown that realization of the FEM for multigrid discrete models requires $10^4 \times 10^8$ times less of computer memory than the base models at low time cost. 3-grid FE have been developed and computationally investigated to calculate the sandwich plate.

2. The first procedure of developing composite 2-grid FE

The main principals of the given procedure (without loss of statement generality) are shown by the example of constructing a composite 2-grid FE (2gFE) $V_a$ (of 2nd order) of a $6h \times 6h \times 6h$ cube (figure 1), $Oxyz$ - local Cartesian system coordinate. The connections between the heterogeneous structure components of 2gFE are considered to be ideal. The components are homogeneous isotropic bodies.

![Figure 1. 2gFE $V_a, V_S$](image)

Displacement, stress and deformation functions of the components satisfy the Hooke’s law, and Cauchy relations corresponding to the three-dimensional problem of elasticity theory [5], i.e. in the entire area of 2gFE, three-dimensional stress state without simplifying hypotheses is realized. 2gFE is fiber-reinforced, parallel to the axis $Ox$; fiber sections shown in figure 1 are shaded. To construct 2gFE, two nested grids: fine $h_a$ and coarse $H_a$ are used. The grid $h_a$ is generated by a basic partition $R_a$ of 2gFE, which consists of a homogeneous FE $V_e$ of the 1st order in the form of cube with a side $h$ [2–4], and takes into account the heterogeneous structure of 2gFE, its shape, loading and fixing $e = 1, \ldots, M$; $M$ is a total number of FE $V_e$. Uniform grid $h_a$ with a step $h$ is shown in figure 1. The uniform coarse grid $H_a$ (has the $n_1 \times n_2 \times n_3$ dimension) with a step $3h$ whose nodes are depicted by the dots is identified on the grid $h_a$ (in figure 1 $n_1 = n_2 = n_3 = 3$). The total potential energy $\Pi_a$ of the 2gFE base partition is given as follows

$$\Pi_a = \sum_{e=1}^{M} \left( \frac{1}{2} \mathbf{q}_{e}^{T} \mathbf{K}_{e} \mathbf{q}_{e} - \mathbf{q}_{e}^{T} \mathbf{P}_{e} \right),$$

(1)

where $[K_e]$ – stiffness matrix, $\mathbf{P}_{e}$ – nodal force vector, $\mathbf{q}_{e}$ – nodal displacement vector of FE $V_e$, $\mathbf{T}$ – transposition.

Using the Lagrange second order polynomials [3] on the grid $H_a$, displacement functions $u_a$, $v_a$, $w_a$ of 2gFE are determined and written as follows

$$u_a = \sum_{\beta=1}^{n} N_{\beta} u_{\beta}, \quad v_a = \sum_{\beta=1}^{n} N_{\beta} v_{\beta}, \quad w_a = \sum_{\beta=1}^{n} N_{\beta} w_{\beta}.$$  

(2)
where \( u_\beta, v_\beta, w_\beta, N_\beta \) - displacements and shape function of \( \beta \) - node of the grid \( H_a \), \( n \) - a total number of grid nodes \( H_a \), \( n = n_1 n_2 n_3 \) \( (n = 27 \) for 2gFE \( V_a \) of the second kind, figure 1).

Let \( \mathbf{q}_a = \{u_1, ..., u_n, v_1, ..., v_n, w_1, ..., w_n\}^T \) is the vector of nodal displacements of the coarse grid \( H_a \), i.e. vector of nodal displacements of 2gFE. Using (2), \( \mathbf{q}_e \) vector is expressed through \( \mathbf{q}_a \) vector; as a result, the following relation is obtained

\[
\mathbf{q}_e = [A_e] \mathbf{q}_a, \tag{3}
\]

where \([A_e]\) - rectangular matrix, \( e = 1, ..., M \).

Using (3) in (1), provided that \( \partial \Pi_a / \partial \mathbf{q}_a = 0 \), the relation \([K_a] \mathbf{q}_a = \mathbf{F}_a\) for 2gFE \( V_a \) corresponding to its state of equilibrium is obtained, where

\[
[K_a] = \sum_{e=1}^{M} [A_e]^T [K_e] [A_e], \quad \mathbf{F}_a = \sum_{e=1}^{M} [A_e]^T \mathbf{P}_e, \tag{4}
\]

\([K_a], \mathbf{F}_a\) - stiffness matrix and nodal force vector of 2gFE \( V_a \).

Note that the order of 2gFE \( V_a \) equals \( n_a \) when \( p = n_1 = n_2 = n_3 \), where \( n_a = p - 1 \).

**Commentary 1.** The solution corresponding to \( H_a \) grid of 2gFE \( V_a \) using the formula (3) is projected onto the fine grid \( h_a \) of the 2gFE basic partition, providing the stress calculation in either FE \( V_e \) of the basic partition of 2gFE \( V_a \), i.e. to determine stresses in any component of heterogeneous structure of three-dimensional body.

**Commentary 2.** When developing the approximative functions of 2gFE displacements on the \( H_a \) grid the known polynomials of the 1st, 2nd and 3d orders [2–4] can be used. It is appropriate to apply the following \( P_3 \) polynomial of the third kind [4]

\[
P_3 = a_0 + a_1 x + a_2 y + a_3 z + a_4 x y + a_5 x z + a_6 y z + a_7 x^2 + a_8 y^2 + a_9 z^2 + a_{10} x^2 y + a_{11} x^2 z + a_{12} y^2 z + a_{13} x y^2 + a_{14} x z^2 + a_{15} y z^2 + a_{16} x^3 + a_{17} y^3 + a_{18} z^3 + a_{19} x y z + a_{20} x y^2 + a_{21} x z^2 + a_{22} y^2 z + a_{23} y^3 + a_{24} z^3 + a_{25} x^2 y + a_{26} x^2 z + a_{27} y^2 z + a_{28} x y^2 z + a_{29} y^2 x z + a_{30} x^2 z^2 + a_{31} y^3 + a_{32} z^3, \tag{5}
\]

3. **The second procedure of developing composite 2gFE**

The second procedure is briefly considered (without loss of statement generality) by producing 2gFE \( V_S \) of a \( 6h \times 6h \times 6h \) cubic form having a heterogeneous structure and located in a local Cartesian coordinate system \( Oxyz \), as a 2gFE \( V_a \) (figure 1). Fine grid \( h_a \) and the basic partition \( R_a \) of 2gFE \( V_a \) are used. Superelement \( G \) is built at a basic partition \( R_a \) of 2gFE \( V_S \) using a condensation method [3]. The total potential energy \( \Pi_g \) of the superelement \( G \) is written as

\[
\Pi_g = \frac{1}{2} \mathbf{q}_g^T [K_g] \mathbf{q}_g - \mathbf{q}_g^T \mathbf{P}_g, \tag{6}
\]

where \([K_g]\) - stiffness matrix; \( \mathbf{P}_g, \mathbf{q}_g \) - nodal force vector and nodal displacement vector of the superelement \( G \).

The coarse grid \( H_S \) nested in the fine grid \( h_a \) is determined at the boundary of the superelement \( G \). Let \( \mathbf{q}_S \) is the node displacement vector of the grid \( H_S \), i.e. the nodal displacements of 2gFE \( V_S \).

Using the displacement function, built on a coarse grid \( H_S \), the relation between the vectors \( \mathbf{q}_S, \mathbf{q}_g \) is as follows

\[
\mathbf{q}_g = [A_g] \mathbf{q}_S, \tag{7}
\]
where \([A_g^S]\) – a rectangular matrix.

Using (7) in (6) provided that \(\frac{\partial \Pi}{\partial q} = 0\), the relation \([K_S]\ q = F_S\) corresponding its equilibrium state is obtained for 2gFE \(V_S\), where \(F_S = [A_g^S]^T P_S\), \([K_S] = [A_g^S]^T [K_g] [A_g^S]\), \([K_S]\) – stiffness matrix, \(F_S\) – nodal force vector of 2gFE \(V_S\).

Composite elastic 2–dimensional 2gFE of rectangular shape are projected by the procedures that are similar to those in section 2, 3 [6, 7]. When developing the composite 2gFE of triangular shape (tetragonal ones), the 1st, 2nd and 3rd order polynomials have been used [2–4].

**Commentary 3.** Calculations have demonstrated that 2gFE \(V_S\) (based on the second procedure) generate more exact solutions than 2gFE \(V_a\) (based on the first procedure). However, the second procedure involves the development of \(G\) superelement that is associated with the high-order matrix inversion. This increases the span time required to build 2gFE \(V_S\).

**Commentary 4.** With the help of the step ratio varying of fine and coarse grids of 2gFE \(V_a, V_S\), the solution error projected for double grid discrete models is regulated.

**Commentary 5.** According to (3) the vector dimension \(\delta_a\) (2gFE \(V_a\) dimension) doesn’t depend on the number \(M\) i.e. on the dimension of \(R_a\) partition. Therefore, arbitrarily small basic partitions \(R_a\) comprised of FE \(V_e\) can be used to take into account the heterogeneous and microinhomogeneous structure of 2gFE \(V_a\) (see section 2). In this case, the three-dimensional stress state in FE \(V_e\) is arbitrarily closely described (without any simplifying hypotheses). With a sharp increase in the dimensions of basic 2gFE partitions, i.e. \(M\) number (dimension of superelement \(G\), see section 3), the time spending needed to develop 2gFE \(V_a, V_S\) is dramatically increased. Therefore, it is worth using a 3-grid FE requiring less time costs to be constructed and giving rise to the discrete models of three-dimensional bodies of smaller dimension than 2gFE.

**4. Developing the composite 3-grid FE**

The main principals of this procedure are considered to be an example of the 3-grid FE (3gFE) \(V_b\) of the third order of \(12h \times 12h \times 12h\) cube located in a local Cartesian coordinate system \(Oxyz\) (figure 2). The area of 3gFE consists of 2gFE \(V_n^a\), \(n = 1, ..., N\), \(N\) - the total number of 2gFE \(V_n^a\) \((6h \times 6h \times 6h\) sizes), \(N = 8\) is shown in figure 2. Coarse grids of 2gFE generate 3gFE fine grid \(h_b\). Coarse grid \(H_b\) of 3gFE is identified at \(h_b\) grid. 27 grid nodes \(H_b\) are point - marked in the figure 2.

![Figure 2. 3gFE \(V_b\)](image)

Displacement functions \(u_b, v_b, w_b\) of 3gFE built on a grid using the Lagrange polynomials [3] can be written as follows
and a coarse one - displacements and shape function of \( \beta \) node of the \( H_b \) grid, \( m = 27 \) is shown in figure 2.

To build 3gFE \( V_b \), three typical grids are used: two grids of 2gFE \( V_n^a \) and a coarse one \( H_b \). Total potential energy \( W_b \) of 3gFE \( V_b \) is presented by the following expression

\[
W_b = \sum_{n=1}^{N} \frac{1}{2} (\boldsymbol{\delta}_n^a)^T \left[ K_n^a \right] \boldsymbol{\delta}_n^a - (\boldsymbol{\delta}_n^a)^T \boldsymbol{p}_n^a, \tag{9}
\]

where \( [K_n^a] \) - stiffness matrix; \( \boldsymbol{p}_n^a \), \( \boldsymbol{\delta}_n^a \) - nodal force and displacements vectors of 2gFE \( V_n^a \).

Let \( \boldsymbol{\delta}_b \) is a nodal displacement vector of the \( H_b \) grid. Using (8), the nodal displacements of the \( \boldsymbol{\delta}_n^a \) vector in terms of the nodal \( \boldsymbol{\delta}_b \) vector displacement of the coarse grid \( H_b \) are expressed. As a result the following relation can be deduced

\[
\boldsymbol{\delta}_n^a = [A_n^b] \boldsymbol{\delta}_b, \tag{10}
\]

where \( [A_n^b] \) - rectangular matrix, \( n = 1, \ldots, N \).

Using (10) in the functional (9) and minimizing it based on the displacements \( \boldsymbol{\delta}_b \), \( [K_b] \boldsymbol{\delta}_b = \boldsymbol{F}_b \) is obtained for 3gFE \( V_b \) which corresponds to its state of equilibrium, where \( [K_b] \) - stiffness matrix, \( \boldsymbol{F}_b \) - nodal force vector of the \( V_b \) 3gFE identified by the following formulas

\[
[K_b] = \sum_{n=1}^{N} [A_n^b]^T [K_n^a] [A_n^b], \quad \boldsymbol{F}_b = \sum_{n=1}^{N} [A_n^b]^T \boldsymbol{p}_n^a. \tag{11}
\]

According to (10), the vector dimension \( \boldsymbol{\delta}_b \) (i.e. the dimension of 3gFE \( V_b \)) doesn’t depend on the total number \( N \) of 2gFE \( V_n^a \), therefore, the partition of 3gFE \( V_b \) into 2gFE \( V_n^a \) can be very small (i.e. the sizes of 2gFE are arbitrarily small). In this case, 3-D stress state with no simplifying hypotheses in basic partitions of 2gFE \( V_n^a \), i.e. in FE \( V_c \) (see section 2, 3), are accurately presented. Determination of the stresses in 3gFE \( V_b \) is as follows. Let vector \( \boldsymbol{\delta}_b \) was build. According to the formula (10), vectors \( \boldsymbol{\delta}_n^a \) of nodal displacements of 2gFE \( V_n^a \) were found. Stresses in FE \( V_c \) of the basic partition of 2gFE \( V_n^a \), \( n = 1, \ldots, N \), are determined based on the algorithms in section 2.

**Commentary 6.** Using 3gFE, based on the procedure similar to one in section 4, 4-grid FE are developed, so \( m \) - grid FE are \([8, 9]\), \( m \geq 4 \). Note that the \( m \) grid FE gives rise to the discrete model of a smaller dimension than the \( m - 1 \) grid FE.

**Commentary 7.** When developing plate (beam) type MFE, Lagrange polynomials having along the \( Ox \), \( Oy \) axes (\( Oy \) - direction) a higher order approximation than along the \( Oz \) axis, \( n_1, n_2 > n_3 \) (\( Oz \), \( Ox \) - directions, \( n_2 > n_1, n_3 \), see section 2). To construct homogeneous MFE similar procedures, described in section 2-4, are used.

5. **Mixed multigrid discrete models of composite solids**

Key point of the combined application of the single - and multigrid modeling of three-dimensional inhomogeneous elastic solids of complex shape is as follows. Subregion of the body, which includes the complex boundary or a body binding, is presented by (fine) base partition, which consists of the homogeneous first order single grid FE of a cube form and takes into account the heterogeneous
structure and complex body shape. The rest of the body is represented by a coarse partition containing the composite cuboid MFE. Fine and coarse partitions using the binding MFE are connected and a mixed discrete model with a less dimension than that of the basic model is obtained. Mixed multigrid discrete models are in detail reviewed in [10].

6. Developing MFE of irregular shape

6.1. The first procedure is briefly considered by the example of 2-D composite 2gFE ABCD of the first order given in figure 3, where the 2gFE curved boundary AB coincides with the boundary of body. The coarse grid nodes are point-marked. The 1-st order polynomial is applied to produce the displacement function on a coarse grid of 2gFE, written in local Cartesian coordinate systems $Oxyz$ (figure 3).

![Figure 3. Two-dimensional 2gFE.](image)

2gFE of irregular shape using the base partition is taken into consideration. The base partition grid of 2gFE is shown in figure 3. The power-law polynomials have been used for the displacement functions to be developed on 2gFE coarse grids, with some nodes of coarse grids being on the 2gFE curved boundaries. It should be noted that 2gFE (MFE) application is effective when calculated homogeneous bodies of irregular shape. Using the basic partitions of homogeneous 2gFE (MFE), their irregular shape (figure 3), the complexity of fixing and body loading have been taken into account. In addition, 2gFE (MFE) generate discrete model of homogeneous bodies whose dimensions are smaller than the ones of the basic model.

6.2. Consider the procedure of building composite MFE in the form of a right prism with the bases of irregular shape [11]. Approximating displacement functions of MFE built on coarse grids and presented as a product of power and Lagrange polynomials have been made use of. The key point of the procedure is considered by 2gFE example of triangular right prism shape with a hole shown in figure 4. The hole cross section is shaded; the coarse grid nodes are point-marked (24 nodes).

![Figure 4. 2gFE of triangular right prism shape.](image)

Basic partition of 2gFEs takes into account its heterogeneous structure and irregular shape. In the $Oxz$ plane (figure 4) on the coarse grid nodes of 2gFE (6 nodes), the approximating displacement functions using the 2nd order polynomial $P_2(x, z)$ of the form

$$P_2(x, z) = a_1 + a_2 x + a_3 z + a_4 xz + a_5 x^2 + a_6 z^2.$$  (12)
Basis function $\varphi_\beta$ for $\beta$ - node of 2gFE coarse grid is represented as a product of the power and Lagrange polynomials, i.e.

$$\varphi_\beta(x, y, z) = N_\beta(x, z)L_\beta(y),$$  \hspace{1cm} (13)

where $N_\beta(x, z)$ is a basis function of the polynomial $P_3(x, z)$ of the form (12) and $L_\beta(y)$ is Lagrange polynomial, corresponding to $\beta$ - node of 2gFE grid, $\beta = 1, ..., N$, $N$ is the total number of the coarse grid nodes.

Figure 4 demonstrates $N = 24$, $L_\beta(y)$ is the third order polynomial, for (13) $\beta = 1, ..., 24$.

7. Numerical study results

Model problem of bending the $216h_x \times 432h_y \times 6h_z$ $V_0$ three-dimensional rectangular six-layer plate have been considered in Cartesian reference system $Oxyz$ (figure 5) where $h_x = h_y = 3.5$, $h_z = 6$, $h = 6h_z$ is the plate thickness. For $y = 0$: $u = v = w = 0$. Layers are homogeneous isotropic bodies with the thickness $h_0 = h/6 = 6$. The elastic moduli (Young’s modulus) from the bottom are equal to 10, 25, 40, 55, 70, 85 respectively. The Poisson's ratio is 0.3. For $z = h$, $y \geq 324h_y$, the plate is given a uniform vertical surface load $P_z = 0.0003$.

![Figure 5. Sizes of $V_0$ plate.](image)

The $R^0_n$ base discrete models of the plate consist of the first order homogeneous FE of $h_x^x \times h_y^y \times h_z^z$ size, where $h_n^x = h_x/(2n-1)$, $h_n^y = h_y/(2n-1)$, $h_n^z = h_z/(2n-1)$, $n = 1, ..., 6$. Grid of the $R^0_n$ discrete model has the $m_1^1 \times m_2^1 \times m_3^1$ dimension, where $m_1^1 = 216(2n-1)+1$, $m_2^1 = 432(2n-1)+1$, $m_3 = 6(2n-1)+1$, $n = 1, ..., 6$.

The $R_n$ multigrid discrete models of the $V_0$ plate consist of $108h_n^x \times 108h_n^y \times 6h_n^z$ plate composite 3gFE (see section 3). Coarse grid of 3gFE along the $Ox$, $Oy$, $Oz$ axes has the $36h_n^x$, $36h_n^y$, $6h_n^z$ steps, respectively. 3gFE involve the third order 2gFE of $6h_n^x \times 6h_n^y \times 6h_n^z$ sizes developed by section 2. There are 32 nodes on 2gFE coarse grid, the steps of the coarse grid in $Ox$, $Oy$, $Oz$ - directions equal $2h_n^x$, $2h_n^y$, $2h_n^z$, respectively. To approximate the displacement functions on a 2gFE coarse grid, the 3rd order polynomial of the form (5) has been used. The calculation results are shown in Table. 1, where $w_n^m$, $\sigma_n^m$ - the maximum displacement and equivalent stress of the $R_n$ models, $n = 1, ..., 6$. Parameters $\delta_n^m$, $\Delta_n^m$ have been found by the following formulas

$$\delta_n^m(\%) = 100\% | \sigma_n^m - \sigma_{n-1}^m | / \sigma_n^m, \quad \Delta_n^m(\%) = 100\% | w_n^m - w_{n-1}^m | / w_n^m, \quad n \geq 2.$$  

Stresses $\sigma_n^m$ are determined according to the 4th theory of strength.
Analysis of the results has shown (table 1) the nature of the parameter changes $\Delta^m_n(\%)$, $\sigma^m_n(\%)$ $(n = 1, \ldots, 6)$ demonstrates a high convergence rate of the maximum stresses $\sigma^m_n$ and displacements $w^m_n$ to the exact solutions.

**Table 1.** Maximum displacements and stresses of $R_n$ discrete models.

| $R_n$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| $w^m_n$ | 684.854 | 754.407 | 764.893 | 767.851 | 768.131 | 767.838 |
| $\Delta^m_n(\%)$ | $-9.087$ | 0.385 | 0.036 | 0.038 | | |
| $\sigma^m_n$ | 0.686 | 0.948 | 1.071 | 1.125 | 1.153 | 1.166 |
| $\delta^m_n(\%)$ | $-27.637$ | 11.485 | 4.800 | 2.428 | 1.115 | |

As the $\Delta^m_6 = 0.00038$, $\delta^m_6 = 0.01115$ are small, the displacement $w^m_6 = 767.838$ and stress $\sigma^m_6 = 1.166$ can be considered to be the exact solution. There are 2270396480 nodal unknowns in the discrete base model $R^0_6$ of the plate (i.e. 2.2 milliards of unknown FEM), the bandwidth of equation system (ES) of FEM is equal to 955967. The dimension of $R_6$ multigrid model is 318384; bandwidth of ES of FEM is equal to 14699. FEM implementation for the $R_6$ model requires 463770 times less computer memory than for the $R^0_6$ base model, at small time spanding.

**Acknowledgments**

This work was supported by the Russian Foundation for Basic Research (project No. 14-01-00130).

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