A first-quantized formalism for cosmological particle production

To cite this article: Alex Hamilton et al JHEP07(2004)024

View the article online for updates and enhancements.

Related content
- Decoupling in an expanding universe boundary RG-flow affects initial conditions for inflation
  Koenraad Schalm, Gary Shiu and Jan Pieter van der Schaar
- Holography, diffeomorphisms, and scaling violations in the CMB
  Finn Larsen and Robert McNees
- Clean Time-Dependent String Backgrounds from Bubble Baths
  Ofer Aharony, Michal Fabinger, Gary T. Horowitz et al.

Recent citations
- Hawking-de Sitter Thermalization of Quasi-Minkowskian Massless Scalaron Production, Energy Density Content and Back-Reaction
  Ana-Camelia Pîrghie et al
- Quantum noise, scaling, and domain formation in a spinor Bose-Einstein condensate
  Nigel R. Cooper et al
- Reduced Hamiltonian for intersecting shells
  Francesco Fiamberti and Pietro Menotti
A first-quantized formalism for cosmological particle production

Alex Hamilton, Daniel Kabat and Maulik Parikh

Department of Physics, Columbia University
New York, NY 10027, U.S.A.

Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855, U.S.A.

E-mail: hamilton@phys.columbia.edu, kabat@phys.columbia.edu, mkp@phys.columbia.edu

ABSTRACT: Given suitable boundary conditions, we show that the initial state and the amount of particle production in a cosmological spacetime are encoded in the Feynman propagator. The propagator can be represented in terms of a particle path integral in an auxiliary spacetime, and particle production can be extracted from the auxiliary propagator. This provides a first-quantized formalism for computing cosmological particle production which, unlike conventional Bogolubov transformations, may be amenable to a string-theoretic generalization.

KEYWORDS: Bosonic Strings, Space-Time Symmetries, Cosmology of Theories beyond the SM.
1. Introduction

The standard approach to cosmological particle production [1, 2] involves finding two sets of solutions to the wave equation, with pair production given by a Bogolubov transformation between the two sets of modes. Although this method has a long and proven record of success, there are reasons to search for alternative techniques. In particular, the second-quantized formalism underlying Bogolubov transformations does not carry over to string theory, whose standard perturbative formulation is first-quantized. In order to estimate, say, the spectrum of density fluctuations coming from stringy effects [3–8], one would like to have a first-quantized technique in which one could systematically compute $\alpha'$ corrections.

In this paper we develop a first-quantized formalism for computing pair-production in an arbitrary cosmological background. The method is based on the Feynman propagator. It turns out to be convenient to work in terms of the propagator in an auxiliary ‘reflected’ spacetime. We show that pair production is encoded in this auxiliary propagator. Moreover the auxiliary propagator admits a straightforward path integral representation, so our formalism allows us to compute particle production without using Bogolubov transformations. Note that first-quantized methods have been used to study pair creation from horizons [9, 10], where again Bogolubov transformations are usually used. First-quantized techniques for particle production have also been developed in [11, 12].
An outline of this paper is as follows. In section 2 we give a brief review of the standard treatment of a scalar field in an FRW universe. In section 3 we consider the Feynman propagator. We discuss the way in which initial conditions and particle production are encoded in the propagator. We then give a path integral representation of the auxiliary propagator. Next, we present some compact formulas for the total amount of particle production. Finally, we show that the propagator in the original spacetime can be constructed from the auxiliary propagator using the method of images. In section 4 we apply our techniques to the case of a cosmology with a power-law scale factor. We conclude in section 5.

2. Scalar field quantization

Consider a free minimally-coupled real scalar field $\phi(x)$ in a Robertson-Walker universe with metric

$$ds^2 = a^2(\eta) \left( -d\eta^2 + h_{ij} dx^i dx^j \right).$$

Here $a(\eta)$ is the scale factor, $\eta$ is conformal time, and $h_{ij}$ is the metric of a maximally-symmetric $n$-dimensional space. We make the field redefinition

$$\hat{\phi}(x) = a^{(1-n)/2}(\eta) \tilde{\phi}(x),$$

and write $\tilde{\phi}(x)$ as a sum over momentum modes:

$$\tilde{\phi}(x) = \sum_k \left( \hat{a}_k f_k(\vec{x}) u_k(\eta) + \hat{a}_k^\dagger f_k^*(\vec{x}) u_k^*(\eta) \right).$$

The corresponding mode functions $u_k(\eta)$ satisfy

$$u_k'' + V_k u_k = 0$$

where

$$V_k(\eta) = k^2 + a^2 m^2 - \frac{1}{2} (n-1) a'' \frac{a'}{a} - \frac{1}{4} (n-1)(n-3) \left( \frac{a'}{a} \right)^2.$$  

Here $'$ indicates a derivative with respect to $\eta$. If one thinks of $\eta$ as position, then this is a one-dimensional Schrödinger equation with potential $-\frac{1}{2} V_k(\eta)$.

2.1 Final state

Throughout this paper we assume that the scale factor changes adiabatically in the future. Equivalently, we assume that the WKB approximation becomes valid as $\eta \to +\infty$, so we can take the modes to satisfy

$$u_k(\eta) \equiv u_k^{\text{out}}(\eta) \sim e^{-i \int^\eta d\eta' \sqrt{V(\eta')}}$$

as $\eta \to +\infty$.

That is, we take $u_k(\eta)$ to be positive-frequency in the future. Thus the operators $\hat{a}_k$ annihilate the preferred future, or out, vacuum:

$$\hat{a}_k|\text{out}\rangle = 0 \quad \forall k.$$
2.2 Initial state

We also need to specify the initial state of the field. First choose a conformal time $\eta_0$ at which to fix initial conditions. Then for each momentum $k$ choose a complex parameter $\omega_k$. Where no physical principle dictates the choice of initial state, the $\omega_k$ are just arbitrary parameters, aside from the fact that (for reasons given below) $\text{Re} \omega_k > 0$ and $\omega_k = \omega_{-k}$.

Now let $\hat{\chi}_k(\eta)$ be a Fourier component of the field,

$$\hat{\chi}_k(\eta) = \int dx f_k(\vec{x}) \hat{\chi}(\eta, \vec{x}) = \hat{a}_k u_k(\eta) + \hat{a}_{-k}^\dagger u_{-k}^*(\eta).$$

(2.8)

To specify the initial state of the field we define the operators

$$\hat{b}_k \equiv \sqrt{\frac{\omega_k |\omega_k|}{2 \text{Re} \, \omega_k}} \left( \hat{\chi}_k(\eta_0) + \frac{i \hat{\pi}_k(\eta_0)}{\omega_k} \right).$$

(2.9)

Here the conjugate momentum $\hat{\pi}_k(\eta) = \partial_\eta \hat{\chi}_k$ satisfies $i[\hat{\pi}_k, \hat{\chi}_{k'}] = \delta_{kk'}$ as well as $\hat{\chi}_k = \hat{\chi}_{-k}$, $\hat{\pi}_k = \hat{\pi}_{-k}$. The normalization of $\hat{b}_k$ and the condition $\omega_k = \omega_{-k}$ ensure that $[\hat{b}_k, \hat{b}_{k'}] = \delta_{kk'}$, $[\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger] = 0$. We take the initial state to satisfy $\hat{b}_k |\text{in}\rangle = 0$, or equivalently

$$\partial_\eta \hat{\chi}_k(\eta_0) |\text{in}\rangle = i \omega_k \hat{\chi}_k(\eta_0) |\text{in}\rangle.$$  

(2.10)

For example, in Minkowski space, the standard Poincaré-invariant vacuum satisfies this relation with $\omega_k = \sqrt{k^2 + m^2}$. Other choices are possible: for example the Rindler vacuum is captured by one complex parameter.

We do not assume that the universe is adiabatic at early times. Thus (unlike the operators $\hat{a}_k$ and $\hat{a}_{k}^\dagger$) there may be no useful sense in which the operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ actually annihilate and create particles. Rather we just use these operators to specify the initial state of the field. An equivalent way to characterize the initial state is to note that, as a consequence of (2.10), the initial wavefunctional for the field has a general gaussian form, of the sort familiar in Schrödinger picture field theory [13]:

$$\Psi[\chi]_{\eta=\eta_0} \sim \exp \left(-\frac{1}{2} \sum_k \omega_k |\chi_k|^2 \right).$$

(2.11)

The widths of the gaussian are determined by the adjustable parameters $\omega_k$; we must have $\text{Re} \omega_k > 0$ so that the wavefunctional is normalizable.

The space of initial states we can consider in this manner is quite large: the values of the parameters $\omega_k$ are essentially arbitrary, so we can specify one complex parameter per momentum mode. Thus we can sweep out the entire space of states with wavefunctions of the form (2.11).\footnote{Such states were referred to as ‘Fock vacua’ in [3].} Equivalently we can sweep out the entire space of states that are related by Bogolubov transformations.\footnote{Although two complex parameters $\alpha_k$ and $\beta_k$ appear in the Bogolubov transformations (2.12), they are constrained to satisfy $|\alpha_k|^2 - |\beta_k|^2 = 1$. Also a Bogolubov transformation which merely multiplies $\hat{b}_k$ by a phase is redundant since it leaves $|\text{in}\rangle$ invariant. Hence the physical content of a Bogolubov transformation is captured by one complex parameter.} Moreover, our formalism can easily be extended to non-vacuum initial states of the form $(\hat{b}_k^\dagger)^n |\text{in}\rangle$, by following the analysis in [14].
2.3 Bogolubov transformations

We now recall some standard results [1, 2]. Bogolubov transformations express the operators $\hat{a}_k$, which annihilate the out-vacuum, in terms of $\hat{b}_k$, and vice versa:

$$
\hat{a}_k = \sum_{k'} \left( \alpha_{k'k} \hat{b}_{k'} + \beta^*_{k'k} \hat{a}_{k'} \right), \quad \hat{b}_k = \sum_{k'} \left( \alpha^*_{k'k} \hat{a}_{k'} - \beta_{k'k} \hat{b}_{k'} \right).
$$

(2.12)

Equivalently, they express the positive-frequency out-modes in terms of the positive- and negative-frequency in-modes, and vice versa:

$$
\begin{align*}
\hat{u}^{\text{out}}_k &= \sum_{k'} \left( \alpha^*_{k'k} \hat{u}^{\text{in}}_{k'} - \beta_{k'k} \hat{u}^{\text{in}}_{k'} \right), \\
\hat{u}^{\text{in}}_k &= \sum_{k'} \left( \alpha_{k'k} \hat{u}^{\text{out}}_{k'} + \beta^*_{k'k} \hat{u}^{\text{out}}_{k'} \right).
\end{align*}
$$

(2.13)

The spatial modes are orthonormal, so the Bogolubov coefficients are diagonal in $k$:

$$
\begin{align*}
\alpha_{kk} &= \alpha_k^2 - \beta_k \beta^*_{-k}, \\
\beta_{kk} &= \beta_k^* \beta^*_{-k}.
\end{align*}
$$

(2.14)

The initial state can be written as a squeezed state [15, 16],

$$
|\text{in}\rangle = \prod_k C_k \exp \left( -\frac{1}{2} \gamma_k \hat{a}^+_k \hat{a}^-_k \right) |\text{out}\rangle,
$$

(2.15)

where $C_k = (1 - |\gamma_k|^2)^{1/4}$ so that $\langle \text{in}|\text{in}\rangle = 1$. Requiring that $|\text{in}\rangle$ be annihilated by the $\hat{b}_k$ implies (using (2.12) and the commutation algebra) that $\gamma_k = -\beta_k^*/\alpha_k^*$. Note that, from (2.13),

$$
\gamma_k = -\frac{\text{out}|\hat{a}^-_k \hat{a}^+_k|\text{in}\rangle}{\text{out}|\text{in}\rangle}.
$$

(2.16)

We see that $\gamma_k$ is related to pair creation; indeed, it is the normalized probability amplitude for the initial state to evolve at late times into a state that has two particles in it, $\hat{a}^+_k \hat{a}^+_k |\text{out}\rangle$. The average number of particles produced is given by the expectation value of the number operator,

$$
\langle \text{in}|\hat{N}_k|\text{in}\rangle = |\beta_k|^2 = \frac{|\gamma_k|^2}{1 - |\gamma_k|^2},
$$

(2.17)

where we have used the normalization condition $|\alpha_k|^2 - |\beta_k|^2 = 1$.

3. Particle production from the propagator

In this section we present our main results. We begin by discussing the way in which initial conditions and particle production are encoded in the Feynman propagator. Then we give a path integral representation of the Feynman propagator of an auxiliary spacetime. Along with the parameter $\omega$ that encodes the choice of initial state, this auxiliary propagator neatly yields the particle production for the given cosmology, as we then show. Finally, we use the method of images to give a first-quantized representation of the actual propagator in that cosmology.

\footnote{By positive-frequency in-mode we just mean the coefficient of $\hat{b}_k$ in the mode expansion of the field.}
3.1 The Feynman propagator

The Feynman propagator is defined as the vacuum expectation value of the time-ordered product of field operators. However, the vacua appearing on the left and right of the operator product can in general be different:

\[ iG_F(k, \eta_1; k', \eta_2) = \frac{\langle \text{out} | T \hat{\chi}_k(\eta_1) \hat{\chi}_{-k'}(\eta_2) | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} . \] (3.1)

The amount of particle production gets encoded in the propagator. To see this, we write \( \hat{\chi}_k \) in terms of modes that are positive-frequency at late times, \( \hat{\chi}_k = \hat{a}_k u_k(\eta) + \hat{a}^\dagger_{-k} u^*(\eta) \), and find that (for \( \eta_1 > \eta_2 \))

\[ iG_F(k, \eta_1; k', \eta_2) = \frac{\langle \text{out} | \hat{a}_k \hat{a}^\dagger_{k'} | \text{in} \rangle u_k(\eta_1) u_{k'}(\eta_2)}{\langle \text{out} | \text{in} \rangle} + \frac{\langle \text{out} | \hat{a}_k \hat{a}^\dagger_{-k'} | \text{in} \rangle u_k(\eta_1) u_{-k'}(\eta_2)}{\langle \text{out} | \text{in} \rangle} \]
\[ = (u_k(\eta_1) u_k^*(\eta_2) - \gamma_k u_k(\eta_1) u_{-k}(\eta_2)) \delta_{kk'} . \] (3.2)

To obtain the second term in the second line we used (2.16). Since we can take \( u_k(\eta) = u_{-k}(\eta) \) we will suppress all the \( k \) indices henceforth. Then in general the Feynman propagator, expressed in terms of the out-modes, is simply

\[ iG_F(\eta_1, \eta_2) = u(\eta_1) u^*(\eta_2) \theta(\eta_{12}) + u(\eta_2) u^*(\eta_1) \theta(\eta_{21}) - \gamma u(\eta_1) u(\eta_2) . \] (3.3)

The first two terms are standard but the third term, being proportional to \( \gamma \), is a signal of pair creation; see (2.17).

The choice of initial state is also encoded in early-time boundary conditions on the propagator. To see this, we set the boundary conditions on \( iG_F \) at some time in the far past, \( \eta_0 \), using (2.11):

\[ \partial_{\eta_2} iG_F(\eta_1, \eta_2)|_{\eta_2=\eta_0} = \frac{\langle \text{out} | \hat{\chi}(\eta_1) (\partial_{\eta_2} \hat{\chi})(\eta_0) | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} = \frac{\langle \text{out} | \hat{\chi}(\eta_1) (i\omega) \hat{\chi}(\eta_0) | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} . \] (3.4)

Hence the choice of initial state fixes the boundary condition

\[ \partial_{\eta_2} iG_F(\eta_1, \eta_2)|_{\eta_2=\eta_0} = (i\omega) iG_F(\eta_1, \eta_0) \quad \text{for} \quad \eta_1 > \eta_0 . \] (3.5)

3.2 Auxiliary path integral

We have seen that the amount of particle production is encoded in the Feynman propagator; this makes it natural to seek a path integral representation for the propagator.

It is tempting to set

\[ iG_F(\eta_1, \eta_2) \overset{\gamma}{=} \langle \eta_1 | \frac{i}{-D^2 + i\epsilon} | \eta_2 \rangle = \int_0^\infty ds e^{-cs} \int_{\eta(0)=\eta_2, \eta(s)=\eta_1} D\eta(\cdot) e^{iS[\eta]} . \] (3.6)

Here \( D^2(\eta) = \partial^2_\eta + V(\eta) \) is the differential (Schrödinger) operator that acts on the modes in (2.4). We have introduced an \( i\epsilon \) prescription and given a path integral representation involving a Schwinger proper-time parameter \( s \) and a worldline action

\[ S[\eta] = \int_0^s dt \left( -\frac{1}{4} (\partial_\eta)^2 - V(\eta) \right) . \] (3.7)
There are two potential difficulties with this expression for the propagator. The first is that the potential \( V(\eta) \) may be singular at early times, which could make the path integral ill-defined. The second is that, even if one can represent a Green’s function in this way, there’s no reason to expect it to satisfy the boundary conditions (3.5).

Rather than discuss \( iG_F \) directly, we avoid these difficulties by introducing an auxiliary spacetime. The propagator in the auxiliary spacetime \( i\tilde{G}_F \) admits a straightforward path integral representation, and as we show in the next section, particle production can be computed from \( i\tilde{G}_F \). We will work out a path integral representation for \( i\tilde{G}_F \) itself in section 3.4.

Thus consider a separate, auxiliary problem in which we reflect the spacetime symmetrically about \( \eta_0 \). We then have a new operator, \( \tilde{D}^2(\eta) \), defined by

\[
\tilde{D}^2(\eta) = \begin{cases} 
D^2(\eta) & \eta \geq \eta_0 \\
D^2(2\eta_0 - \eta) & \eta \leq \eta_0 
\end{cases}
\]  

(3.8)

The corresponding potential \( \tilde{V}(\eta) \) is the reflection of \( V \) about \( \eta_0 \). We define the Green’s function in this reflected problem with an \( i\epsilon \) prescription, so that it can be computed in first-quantized terms.

\[
i\tilde{G}_F(\eta_1, \eta_2) = \langle \eta_1 | \frac{i}{-D^2 + i\epsilon} | \eta_2 \rangle = \int_0^\infty ds e^{-s} \int_0^\infty \mathcal{D}\eta(s) e^{i\tilde{S}[\eta]} 
\]

\[
\tilde{S}[\eta] \equiv \int_0^s dt \left( -\frac{1}{4} (\partial_t \eta)^2 - \tilde{V}(\eta) \right) .
\]

(3.9)

Notice that, as we are throwing away the part of the potential before \( \eta_0 \), any early-time singularity of the potential is irrelevant. The reflected potential may have a cusp at \( \eta_0 \) but this causes no problems. A further advantage of reflecting the potential is that the reflected spacetime is asymptotically adiabatic in the past as well as in the future, so the \( i\epsilon \) prescription automatically selects the preferred in- and out-vacua of the reflected problem. That is, as \( \eta_1, \eta_2 \to \pm\infty \) the reflected Green’s function obeys the adiabatic boundary conditions considered in section 2.1. For example\(^4\)

\[
\partial_{\eta_1} \log i\tilde{G}_F(\eta_1, \eta_2) \simeq i\sqrt{\tilde{V}(\eta_2)} \quad \text{as} \quad \eta_2 \to -\infty .
\]

(3.11)

3.3 Formulas for particle production

We now derive some compact formulas for the amount of particle production. We will find two expressions, one that still refers to mode solutions and one which is expressed entirely in terms of the reflected propagator introduced above.

\(^4\)The proof is as follows. The Feynman propagator \( i\tilde{G}_F(\eta_1, \eta_2) = \langle \eta_1 | \frac{1}{-\tilde{D}^2 + i\epsilon} | \eta_2 \rangle \). Insert a complete set of energy eigenstates \( \tilde{D}^2(\omega) = E(\omega)|\omega\rangle \langle \omega| \) and let \( \psi_\omega(\eta) = |\eta|\langle \omega| \). Adiabaticity implies that for \( \eta \approx \eta_2 \) we can take \( \psi_\omega(\eta) \approx \exp(-i\omega_\eta) \) with \( E(\omega) \approx -\omega^2 + \tilde{V}(\eta_2) \). Then

\[
i\tilde{G}_F(\eta_1, \eta_2) \approx \int_{-\infty}^{\infty} d\omega \rho(\omega) \frac{i}{\omega^2 - \tilde{V}(\eta_2) + i\epsilon} \psi_\omega(\eta_1)e^{i\omega\eta_2} \]

(3.10)

where \( \rho(\omega) \) is the density of states. The usual contour deformation picks up the pole at \( \omega = \sqrt{\tilde{V}(\eta_2)} - i\epsilon \), and (3.11) follows. Our approximations become exact as \( \eta_2 \to -\infty \).
What we would like is an expression for $|\gamma|^2$; see (2.17). As we have seen, this quantity appears in the propagator. Inserting the expression for the propagator (3.3) into the boundary conditions (3.5), we find that

$$u(\eta_1)u^*(\eta_0) - \gamma u(\eta_1)u'(\eta_0) = i\omega (u(\eta_1)u^*(\eta_0) - \gamma u(\eta_1)u(\eta_0)) \tag{3.12}$$

which implies

$$\gamma = \frac{\omega + i\partial \ln u^*(\eta_0)}{\omega + i\partial \ln u(\eta_0)} \cdot \frac{u^*(\eta_0)}{u(\eta_0)}. \tag{3.13}$$

We can simplify this expression by defining

$$\bar{\omega} \equiv i\partial_\eta \ln u(\eta)|_{\eta=\eta_0}. \tag{3.14}$$

The amount of pair creation is determined by $|\gamma|^2$, which is given by

$$|\gamma|^2 = \left| \frac{\omega - \bar{\omega}}{\omega + \bar{\omega}} \right|^2. \tag{3.15}$$

This is the equation we are after. It tells us that particle production is completely determined by $\omega$ and $\bar{\omega}$. The initial state only enters through the parameter $\omega$, so it’s very easy to see how particle production depends on the choice of initial state. The final state, and the dynamics of the scale factor, only enter in determining $\bar{\omega}$. Thus $\bar{\omega}$ is the only thing one actually has to compute. It can be obtained from (3.14) without using the in-modes, by differentiating the out-modes at the point $\eta_0$ where initial conditions are specified.

Next we show that $\bar{\omega}$ can also be determined simply by taking a derivative of the reflected propagator, without knowing either the in-modes or the Bogolubov transformations. To establish this, first note that the reflected Green’s function can be expressed as

$$i\bar{G}_F(\eta_1, \eta_2) = \bar{u}(\eta_1)\bar{u}^*(\eta_2)\theta(\eta_2) + \bar{u}(\eta_2)\bar{u}^*(\eta_1)\theta(\eta_2) - \bar{\gamma}\bar{u}(\eta_1)\bar{u}(\eta_2). \tag{3.17}$$

This is the analog of (3.3) for the reflected problem, where $\bar{u}(\eta)$ is a positive-frequency out-mode in the reflected problem, and where $\bar{\gamma}$ indicates particle production in the reflected spacetime. For $\eta \geq \eta_0$ we can set $\bar{u}(\eta) = u(\eta)$ so that

$$\partial_{\eta_2} \ln i\bar{G}_F(\eta_1, \eta_2)|_{\eta_1=\eta_2=\eta_0} = \partial_\eta \ln (u^*(\eta) - \bar{\gamma}u(\eta))|_{\eta=\eta_0}. \tag{3.18}$$

Next note that a positive-frequency in-mode in the reflected problem is given by

$$\bar{u}^{\text{in}}(\eta) = \begin{cases} u^*(2\eta_0 - \eta) & \eta \leq \eta_0 \\ \bar{\alpha}u(\eta) + \bar{\beta}u^*(\eta) & \eta \geq \eta_0 \end{cases} \tag{3.19}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are Bogolubov coefficients in the reflected problem (see (2.13)). The in-mode

\footnote{Although we based our derivation on the propagator, this result could have been obtained by other means.}
and its derivative are continuous at \( \eta_0 \), so we must have

\[
\partial_\eta \ln u^*(2\eta_0 - \eta)|_{\eta = \eta_0} = \partial_\eta \ln (\tilde{\alpha} u(\eta) + \tilde{\beta} u^*(\eta))|_{\eta = \eta_0}.
\] (3.20)

Taking the complex conjugate of this equation, and recalling that \( \tilde{\gamma} = -\tilde{\beta}^* / \tilde{\alpha}^* \), we find that (3.15) can be written as

\[
\partial_{\eta_2} \ln i\tilde{G}_F(\eta_1, \eta_2)|_{\eta_1 > \eta_2 = \eta_0} = -\partial_\eta \ln u(\eta)|_{\eta = \eta_0} = i\bar{\omega}
\] (3.21)

where we have used (3.14).

3.4 Method of images

At this point we have achieved our goal of finding a first-quantized formalism for calculating the amount of particle production. Nevertheless, it is amusing and instructive to express the original Green’s function in terms of a sum over particle paths.

Recall that the original propagator was subject to nontrivial boundary conditions. As a first step, let us try to implement these boundary conditions using the method of images. We therefore write the original Green’s function in terms of the reflected Green’s function and an image charge \( q \):

\[
iG_F(\eta_1, \eta_2) = i\tilde{G}_F(\eta_1, \eta_2) + q i\tilde{G}_F(\eta_1, 2\eta_0 - \eta_2).
\] (3.22)

For \( \eta_1, \eta_2 > \eta_0 \) note that \( iG_F \) is indeed a Green’s function for the operator \( D^2(\eta) \). It remains to implement the boundary condition at \( \eta_0 \) by choosing \( q \) appropriately. From (3.5) we require

\[
\partial_{\eta_2} i\tilde{G}_F(\eta_1, \eta_2)|_{\eta_1 > \eta_2 = \eta_0} = (i\omega)iG_F(\eta_1, \eta_0)
\] (3.23)

while from (3.22) we have

\[
iG_F(\eta_1, \eta_0) = (1 + q)i\tilde{G}_F(\eta_1, \eta_0),
\]

\[
\partial_{\eta_2} iG_F(\eta_1, \eta_2)|_{\eta_1 > \eta_2 = \eta_0} = (1 - q)\partial_{\eta_2} i\tilde{G}_F(\eta_1, \eta_2)|_{\eta_1 > \eta_2 = \eta_0}.
\] (3.24)

Thus the image charge is given by

\[
q = \frac{\bar{\omega} - \omega}{\omega + \bar{\omega}}.
\] (3.25)

where we have invoked the definition of \( \bar{\omega} \), (3.10).

The method of images leads to a way of representing the original Feynman propagator in terms of a particle path integral. Use (3.22) to represent \( iG_F \) as a sum of two particle path integrals in the auxiliary reflected potential. In each path integral fold the particle paths across \( \eta = \eta_0 \), to obtain particle worldlines that are restricted to satisfy \( \eta(s) \geq \eta_0 \). Note that this folding leaves the worldline action invariant. By adding the two path integrals one can represent \( iG_F \) as a single path integral, just as in (3.6), but where the particle paths are restricted to satisfy \( \eta(s) \geq \eta_0 \), and where the boundary conditions at \( \eta = \eta_0 \) are enforced by weighting paths according to the rule

\[
e^{iS[\eta]} \quad \text{for paths that touch } \eta_0 \text{ an even number of times}
\]

\[
qe^{iS[\eta]} \quad \text{for paths that touch } \eta_0 \text{ an odd number of times}.
\] (3.26)

Similar constructions to enforce boundary conditions on the path integral have been discussed in [17, 18].
This approach leads to a nice conceptual visualization. The formation of a pair of particles from the vacuum intuitively suggests a U-shaped particle worldline, with the two endpoints of the U marking the two particle positions at late times, and with the bottom of the U marking (heuristically) the time of pair creation. We have made this intuition precise, by giving a prescription (3.26) for obtaining the appropriate Feynman propagator $iG_F$ from an integral over particle paths.

4. Power-law FRW

As an example we consider a massless minimally-coupled scalar field in a 3+1-dimensional Robertson-Walker spacetime with a power-law scale factor $a(t) \sim t^c$. We illustrate the formalism of the preceding section by calculating pair production in two ways, using (3.14) and (3.16).

4.1 Mode solution method

The line element reads

$$ds^2 = -dt^2 + a^2(t) d\Sigma_3^2 = \text{const} \eta^\nu (-d\eta^2 + d\Sigma_3^2),$$

where $0 < \eta < \infty$. We take $0 < c < 1$ so the expansion is decelerating. The mode equation (2.4) reads

$$\left( \partial_\eta^2 + k^2 - \frac{\nu^2 - 1/4}{\eta^2} \right) u_k(\eta) = 0, \quad \nu = \frac{3c - 1}{2(1-c)}.$$  

We specify the initial state as in section 2.2, by setting

$$\hat{\pi}_k|\text{in}\rangle = i\omega_k \hat{\chi}_k|\text{in}\rangle \quad \text{at} \quad \eta = \eta_0.$$  

There is a preferred out-vacuum since the field evolves adiabatically at late times. The late-time positive-frequency modes are

$$u_k(\eta) = N \sqrt{\eta} H^{(2)}_\nu(k\eta),$$

so that, from (3.14), we find

$$i\omega = - \partial_\eta \ln \sqrt{\eta} H^{(2)}_\nu(k\eta) \bigg|_{\eta = \eta_0}.$$  

This can be inserted into (3.15) to find the particle production parameter,

$$|\gamma|^2 = \left| \frac{\omega - \frac{\omega^*}{\omega + \omega^*} \omega^*}{\omega + \omega^*} \right|^2.$$  

Note that when $\nu^2 = 1/4$ the mode functions reduce to plane waves, and the expression for $|\gamma|$ reduces to

$$|\gamma| = \frac{\omega - k}{\omega + k}.$$
4.2 First-quantized approach

In this approach we first need to find the propagator in the reflected potential. This satisfies the differential equation

\[ \mathcal{D}^2(\eta)\tilde{G}_F(\eta_1, \eta_2) = -\delta(\eta_1 - \eta_2), \tag{4.8} \]

where

\[ \mathcal{D}^2(\eta) = \partial^2_\eta + k^2 - \begin{cases} \nu^2 - 1/4 & \eta \geq \eta_0 \\ \nu_0^2 - 1/4 - (2\eta_0 - \eta)^2 & \eta \leq \eta_0. \end{cases} \tag{4.9} \]

The argument of the differential operator can be either \( \eta_1 \) or \( \eta_2 \).

We can solve this equation by holding \( \eta_2 \) fixed and solving the homogeneous equation for \( \eta_1 \) in the two regions which are separated by the delta function. The final Green’s function can then be determined by matching across the delta function. This gives

\[ \tilde{G}_F(\eta_1, \eta_2) = \frac{i\pi}{4c_1} \chi(\eta_<) \left( \sqrt{\eta_-} H^{(2)}_{\nu_{\nu'}}(k\eta) \right), \tag{4.10} \]

where \( \eta_<(\eta_>) \) is the lesser (greater) of \( \eta_1 \) and \( \eta_2 \). Here \( \chi(\eta) \) is defined such that

\[ \chi(\eta) = \begin{cases} \sqrt{\eta} \left[ c_1 H^{(1)}_{\nu_{\nu'}}(k\eta) + c_2 H^{(2)}_{\nu_{\nu'}}(k\eta) \right] & \eta > \eta_0 \\ \sqrt{2\eta_0 - \eta} H^{(2)}_{\nu_0}(k(2\eta_0 - \eta)) & \eta < \eta_0, \end{cases} \tag{4.11} \]

with the coefficients \( c_1 \) and \( c_2 \) determined by the continuity of \( \chi(\eta) \) and its first derivative at \( \eta_0 \).

As in section 3.3, we can easily calculate \( \bar{\bar{\omega}} \) as

\[
\begin{align*}
   i\bar{\bar{\omega}} &= \partial_{\eta_2} \ln i\tilde{G}_F(\eta_1, \eta_2) \bigg|_{\eta_1 > \eta_2 = \eta_0} \\
   &= \partial_{\eta} \ln \chi(\eta) \bigg|_{\eta_0} \\
   &= \partial_{\eta} \ln \left( \sqrt{2\eta_0 - \eta} H^{(2)}_{\nu_{\nu'}}(k(2\eta_0 - \eta)) \right) \bigg|_{\eta_0} \\
   &= -\partial_{\eta} \ln \sqrt{\eta} H^{(2)}_{\nu_{\nu'}}(k\eta) \bigg|_{\eta_0}.
\end{align*}
\]

This is precisely what was found in (4.5) from the mode solutions. Moreover the original Green’s function can be expressed in terms of (4.10), by using the method of images as in section 4.4.

5. Conclusions

To summarize: given a cosmological background and an initial state characterized by some value of \( \omega \), our recipe is to compute the Green’s function, \( i\tilde{G}_F \), in the reflected potential. Particle production is then encoded in

\[ i\bar{\bar{\omega}} = \partial_{\eta_2} \ln i\tilde{G}_F(\eta_1, \eta_2) \bigg|_{\eta_1 > \eta_2 = \eta_0}. \tag{5.1} \]

Since \( i\tilde{G}_F \) and hence \( \bar{\bar{\omega}} \) can be computed without reference to mode solutions, we have achieved our goal of finding a first-quantized formalism for computing cosmological particle
production. The amount of particle production can be directly expressed in terms of $\omega$ and $\bar{\omega}$, via

$$\langle N \rangle = \frac{|\gamma|^2}{1-|\gamma|^2} \quad \text{where} \quad |\gamma|^2 = \left| \frac{\omega - \bar{\omega}^*}{\omega + \bar{\omega}} \right|^2. \quad (5.2)$$

Furthermore, the original Green’s function can also be cast in terms of an integral over particle worldlines using the method of images.

Finally, let us mention some directions for future work. First and foremost, it would be an interesting challenge to extend our first-quantized formalism to string theory. At a very naive level, the generalization is immediate: merely replace “particle worldlines” with “string worldsheets.” In practice, the main difficulty seems to be finding a reflected spacetime which is a legitimate string background. As another potential application, the image charge construction of the propagator could be used to study back-reaction. It would be particularly interesting to study back-reaction in de Sitter space, where the image charge construction of alpha vacua is well known [19, 20].

Acknowledgments

We are very grateful to Nori Iizuka for collaboration in the early stages of this project. We thank Brian Greene, Gilad Lifschytz, and Govindan Rajesh for valuable discussions. D.K. and M.P. are grateful to the organizers of the Amsterdam and Aspen workshops where part of this work was completed, and D.K. is grateful to the Rutgers theory group for its hospitality. A.H. is supported by DOE grant DE-FG02-92ER40699, D.K. is supported by DOE grant DE-FG02-92ER40699 and by US–Israel Bi-national Science Foundation grant #200359, and M.P. is supported in part by DOE grant DF-FCO2-94ER40818.

References

[1] N.D. Birrell and P.C.W. Davies, Quantum fields in curved space, Cambridge University Press, Cambridge 1982.
[2] T. Jacobson, Introduction to quantum fields in curved spacetime and the Hawking effect, gr-qc/0308048.
[3] R.H. Brandenberger, Inflationary cosmology: progress and problems, hep-ph/9910410.
[4] J.C. Niemeyer, Inflation with a high frequency cutoff, Phys. Rev. D 63 (2001) 123502 astro-ph/0005533.
[5] A. Kempf, Mode generating mechanism in inflation with cutoff, Phys. Rev. D 63 (2001) 083514 astro-ph/0009209.
[6] R. Easther, B.R. Greene, W.H. Kinney and G. Shiu, Imprints of short distance physics on inflationary cosmology, Phys. Rev. D 67 (2003) 063508 hep-th/0110226.
[7] N. Kaloper, M. Kleban, A.E. Lawrence and S. Shenker, Signatures of short distance physics in the cosmic microwave background, Phys. Rev. D 66 (2002) 123510 hep-th/0201158.
[8] R. Easther, B.R. Greene, W.H. Kinney and G. Shiu, A generic estimate of trans-planckian modifications to the primordial power spectrum in inflation, Phys. Rev. D 66 (2002) 023518 hep-th/0204129.
[9] M.K. Parikh and F. Wilczek, *Hawking radiation as tunneling*, Phys. Rev. Lett. **85** (2000) 5042 [hep-th/9907001].

[10] M.K. Parikh, *New coordinates for de Sitter space and de Sitter radiation*, Phys. Lett. B **546** (2002) 189 [hep-th/0204107].

[11] S.D. Mathur, *Is the polyakov path integral prescription too restrictive?*, [hep-th/9306090]

[12] S.D. Mathur, *Real time propagator in the first quantized formalism*, [hep-th/9311025]

[13] R. Jackiw, *Analysis on infinite dimensional manifolds: Schroedinger representation for quantized fields*, in proceedings of 5th Jorge Andre Swieca Summer School, O. Eboli, M. Gomes and A. Santoro eds., World Scientific, Singapore 1990.

[14] N.D. Birrell and J.G. Taylor, *Analysis of interacting quantum field theory in curved space-time*, J. Math. Phys. **21** (1980) 1740.

[15] B.L. Hu, G. Kang and A. Matacz, *Squeezed vacua and the quantum statistics of cosmological particle creation*, Int. J. Mod. Phys. A **9** (1994) 991 [gr-qc/9312014].

[16] M.B. Einhorn and F. Larsen, *Squeezed states in the de Sitter vacuum*, Phys. Rev. D **68** (2003) 064002 [hep-th/0305056].

[17] L.S. Schulman, *Techniques and applications of path integration* Wiley, 1981.

[18] E. Farhi and S. Gutmann, *The functional integral on the half line*, Int. J. Mod. Phys. A **5** (1990) 3023.

[19] E. Mottola, *Particle creation in de Sitter space*, Phys. Rev. D **31** (1985) 754.

[20] B. Allen, *Vacuum states in de Sitter space*, Phys. Rev. D **32** (1985) 3136.