Better Bounds for Incremental Frequency Allocation
in Bipartite Graphs

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Abstract

We study frequency allocation in wireless networks. A wireless network is modeled by an
undirected graph, with vertices corresponding to cells. In each vertex we have a certain number
of requests, and each of those requests must be assigned a different frequency. Edges represent
conflicts between cells, meaning that frequencies in adjacent vertices must be different as well.
The objective is to minimize the total number of used frequencies.

The offline version of the problem is known to be \( \text{NP}-\text{hard} \). In the incremental version,
requests for frequencies arrive over time and the algorithm is required to assign a frequency
to a request as soon as it arrives. Competitive incremental algorithms have been studied for
several classes of graphs. For paths, the optimal (asymptotic) ratio is known to be \( 4/3 \), while for
hexagonal-cell graphs it is between \( 1.5 \) and \( 1.9126 \). For \( \xi \)-colorable graphs, the ratio of \((\xi + 1)/2\)
can be achieved.

In this paper, we prove nearly tight bounds on the asymptotic competitive ratio for bipartite
graphs, showing that it is between \( 1.428 \) and \( 1.433 \). This improves the previous lower bound of
\( 4/3 \) and upper bound of \( 1.5 \). Our proofs are based on reducing the incremental problem to a
purely combinatorial (equivalent) problem of constructing set families with certain intersection
properties.

1 Introduction

Static frequency allocation. In the frequency allocation problem, we are given a wireless net-
work and a collection of requests for frequencies. The network is modeled by a (possibly infinite)
undirected graph \( G \), whose vertices correspond to the network’s cells. Each request is associated
with a vertex, and requests in the same vertex must be assigned different frequencies. Edges repre-
sent conflicts between cells, meaning that frequencies in adjacent vertices must be different as well.
The objective is to minimize the total number of used frequencies. We will refer to this model as
static, as it corresponds to the scenario where the set of requests in each vertex does not change
over time.

A more rigorous formulation of this static frequency allocation problem is as follows: Denote
by \( \ell_v \) the load at a vertex \( v \) of \( G \), that is the number of frequency requests at \( v \). A frequency
allocation is a function that assigns a set $L_v$ of frequencies (represented, say, by positive integers) to each vertex $v$ and satisfies the following two conditions: (i) $|L_v| = \ell_v$ for each vertex $v$, and (ii) $L_v \cap L_w = \emptyset$ for each edge $(v, w)$. The total number of frequencies used is $|\bigcup_{v \in G} L_v|$, and this is the quantity we wish to minimize. We will use notation $\text{opt}(G, \vec{\ell})$ to denote the minimum number of frequencies for a graph $G$ and a demand vector $\vec{\ell}$.

If one request is issued per node, then $\text{opt}(G, \vec{\ell})$ is equal to the chromatic number of $G$, which immediately implies that the frequency allocation problem is NP-hard. In fact, McDiarmid and Reed [7] show that the problem remains NP-hard for the graph representing the network whose cells are regular hexagons in the plane, which is a commonly studied abstraction of wireless networks. (See, for example, the surveys in [8, 1]). Polynomial-time $\frac{4}{3}$-approximation algorithms for this case appeared in [7] and [9].

**Incremental frequency allocation.** In the incremental version of frequency allocation, requests arrive over time and an incremental algorithm is required to assign frequencies to requests as soon as they arrive. An incremental algorithm $A$ is called asymptotically $R$-competitive if, for any graph $G$ and load vector $\vec{\ell}$, the total number of frequencies used by $A$ is at most $R \cdot \text{opt}(G, \vec{\ell}) + \lambda$, where $\lambda$ is a constant independent of $\vec{\ell}$. We allow $\lambda$ to depend on the class of graphs under consideration, in which case we say that $A$ is $R$-competitive for this class. We refer to $R$ as the asymptotic competitive ratio of $A$. As in this paper we focus only on the asymptotic ratio, we will skip the word “asymptotic” (unless ambiguity can arise), and simply use terms “$R$-competitive” and “competitive ratio” instead. Following the terminology in the literature (see [2, 3], for example), we will say that the competitive ratio is absolute when the additive constant $\lambda$ is equal 0.

Naturally, research in this area is concerned with designing algorithms with small competitive ratios for various classes of graphs, as well as proving lower bounds. For hexagonal-cells graphs, Chan et al. [2, 3] give an incremental algorithm with competitive ratio 1.9216 and prove that no ratio better than 1.5 is possible. A lower bound of $4/3$ for paths was given in [4], and later Chrobak and Sgall [6] gave an incremental algorithm with the same ratio. Paths are in fact the only non-trivial graphs for which tight asymptotic ratios are known. As pointed out earlier, there is a strong connection between frequency allocation and graph coloring, so one would expect that the competitive ratio can be bounded in terms of the chromatic number. Indeed, for $\xi$-colorable graphs Chan et al. [2, 3] give an incremental algorithm with competitive ratio of $(\xi + 1)/2$. (This ratio is in fact absolute.) On the other hand, the best known lower bounds on the competitive ratio, 1.5 in the asymptotic and 2 in the absolute case [2, 3], hold for hexagonal-cell graphs, but no stronger bounds are known for graphs of higher chromatic number.

**Our contribution.** In this paper, we prove nearly tight bounds on the optimal competitive ratio of incremental algorithms for bipartite graphs, showing that it is between $10/7 \approx 1.428$ and $(18 - \sqrt{5})/11 \approx 1.433$. This improves the lower and upper bounds for this version of frequency allocation. The best previously known lower bound was $4/3$, which holds in fact even for paths [4, 6]. The best upper bound of 1.5 was shown in [2, 3] and it holds even in the absolute case.

Our proofs are based on reducing the incremental problem to a purely combinatorial (equivalent) problem of constructing set families, which we call F-systems, with certain intersection properties. A rather surprising consequence of this reduction is that the optimal competitive ratio can be achieved by an algorithm that is topology-independent; it assigns a frequency to each vertex $v$ based only on the current optimum value, the number of requests to $v$, and the partition of the
vertex \( v \); that is, independently of the actual frequencies already assigned to the neighbors of \( v \).

To achieve a competitive ratio below 2 for bipartite graphs, we need to use frequencies that are shared between the two partitions of the graph. The challenge is then to assign these shared frequencies to the requests in different partitions so as to avoid collisions – in essence, to break the symmetry. In our construction, we develop a symmetry-breaking method based on the concepts of “collisions with the past” and “collisions with the future”, which allows us to derive frequency sets in a systematic fashion. We believe that these two ideas – the concept of F-systems and our symmetry-breaking method – can be extended to frequency assignment problems in other types of graphs.

**Other related work.** Determining optimal absolute ratios is usually easier than for asymptotic ratios and it has been accomplished for various classes of graphs, including paths [4] and bipartite graphs in general [2,3], and hexagonal-cell graphs and 3-colorable graphs in general [2,3]. The asymptotic ratio model, however, is more relevant to practical scenarios where the number of frequencies is typically very large, so the additive constant can be neglected.

In the dynamic version of frequency allocation each request has an arrival and departure time. At each time, any two requests that have already arrived but not departed and are in the same or adjacent nodes must be assigned different frequencies. As before, we wish to minimize the total number of used frequencies. As shown by Chrobak and Sgall [6], this dynamic version is \( NP \)-hard even for the special case when the input graph is a path.

It is natural to study the online version of this problem, where we introduce the notion of “time” that progresses in discrete steps, and at each time step some requests may arrive and some previously arrived requests may depart. This corresponds to real-life wireless networks where customers enter and leave a network’s cells over time, in an unpredictable fashion. An online algorithm needs to assign frequencies to requests as soon as they arrive. The competitive ratio is defined analogously to the incremental case. (The incremental static version can be thought of as a special case in which all departure times are infinite.) This model has been well studied in the context of job scheduling, where it is sometimes referred to as time-online. Very little is known about this online dynamic case. Even for paths the optimal ratio is not known; it is only known that it is between \( \frac{14}{9} \approx 1.571 \) [6] and \( \frac{5}{3} \approx 1.667 \) [4].

## 2 Preliminaries

For concreteness, we will assume that frequencies are identified by positive integers, although it does not really matter. Recall that we use the number of frequencies as the performance measure. In some literature [4,5,3], authors used the maximum-numbered frequency instead. It is not hard to show (see [6], for example, which does however involve a transformation of the algorithm that makes it not topology independent) that these two approaches are equivalent.

For a bipartite graph \( G = (A, B, E) \), it is easy to characterize the optimum value. As observed in [4,6], in this case the optimum number of frequencies is

\[
\text{opt}(G, \ell) = \max_{(u,v) \in E} \{ \ell_u + \ell_v \}. \tag{1}
\]

For completeness, we include a simple proof: Trivially, \( \text{opt}(G, \ell) \geq \ell_u + \ell_v \) for each edge \((u, v)\). On the other hand, denoting by \( \omega \) the right-hand side of (4), we can assign frequencies to nodes...
as follows: for \( u \in A \), assign to \( u \) frequencies \( 1, 2, \ldots, \ell_u \), and for \( u \in B \) assign to \( u \) frequencies \( \omega - \ell_u + 1, \omega - \ell_u + 2, \ldots, \omega \). This way each vertex \( u \) is assigned \( \ell_u \) frequencies and no two adjacent nodes share the same frequency.

Throughout the paper, we will use the convention that if \( c \in \{A, B\} \), then \( c' \) denotes the partition other than \( c \), that is \( \{c, c'\} = \{A, B\} \).

### 3 Competitive F-Systems

In this section we show that finding an \( R \)-competitive algorithm for bipartite graphs can be reduced to an equivalent problem of constructing certain families of sets that we call F-systems.

Suppose that for any \( c \in \{A, B\} \) and any integers \( t, k \) such that \( 0 < k \leq t \), we are given a set \( F_{t,k}^c \) of positive integers (frequencies). Denote by \( \mathcal{F} = \{F_{t,k}^c\} \) the family of those sets. Then \( \mathcal{F} \) is called an F-system if

\[
\text{(F1)} \quad \vert F_{t,k}^c \vert \geq k \quad \text{for all} \quad c, t, k, \quad \text{and}
\]

\[
\text{(F2)} \quad F_{t,k}^A \cap F_{t,k'}^B = \emptyset \quad \text{for all} \quad k, k', t, t' \quad \text{such that} \quad k + k' \leq \max(t, t').
\]

An F-system is called \( R \)-competitive if for all \( t \) we have

\[
\left\lfloor \sum_{c=A,B} \sum_{k \leq t} F_{T,k}^c \right\rfloor \leq R \cdot t + \lambda,
\]

where \( \lambda \) is a constant independent of \( t \). The competitive ratio of \( \mathcal{F} \) is the smallest \( R \) for which \( \mathcal{F} \) is \( R \)-competitive.

**Lemma 3.1.** For any \( R \geq 1 \), there is an \( R \)-competitive incremental algorithm for frequency allocation in bipartite graphs if and only if there is an \( R \)-competitive F-system.

**Proof.** (\( \Rightarrow \)) Let \( A \) be an \( R \)-competitive incremental algorithm. To prove this implication, we define a “universal” infinite bipartite graph \( H = (A, B, E) \) and we will issue requests to this graph. For \( c \in \{A, B\} \), the vertices in \( c \) have the form \((t, k)_c\), where \( k \leq t \). Two vertices \((t, k)_A \) and \((t', k')_B \) are connected by an edge if \( k + k' \leq \max(t, t') \).

The requests are issued in phases numbered \( t = 1, 2, \ldots \). In phase \( t \), for each node \((t, k)_c\), we issue \( k \) requests to this node. Let \( F_{t,k}^c \) be the set of frequencies that \( A \) assigns to \((t, k)_c\). After phase \( t \), by the definition of \( H \) and by [1], the optimum number of frequencies is \( t \), so \( A \) uses at most \( Rt + \lambda \) frequencies, for some \( \lambda \). In other words, [2] holds. Thus \( \mathcal{F} = \{F_{t,k}^c\} \) is an \( R \)-competitive F-system.

(\( \Leftarrow \)) Let \( \mathcal{F} \) be an \( R \)-competitive F-system. We use \( \mathcal{F} \) to define an incremental algorithm \( A \) that works as follows. Let \( G = (A, B, E) \) be the given bipartite graph. Consider one step of the computation in which a new request arrives at a vertex \( u \in c \), where \( c \in \{A, B\} \). Denote by \( t \) the current optimum number of frequencies, that is \( t = \max_{(v, w) \in E}(\ell_v + \ell_w) \). Choose any frequency \( f \in F_{t,k}^c \), for \( k = \ell_u \), that is not yet used on \( u \) and assign \( f \) to this request. Such \( f \) exists, because by property (F1) we have \( \vert F_{t,k}^c \vert \geq k \) and the number of frequencies assigned so far to \( u \) is \( k - 1 \).

Trivially, all frequencies assigned by \( A \) to one node are different. We claim that adjacent nodes will be assigned different frequencies as well. Consider again a step where a frequency \( f \) is assigned to a \( k \)th request to a vertex \( u \), when the optimum value is \( t \), as described above. So \( k = \ell_u \).
Without loss of generality, assume $u \in A$. For an edge $(u, v) \in E$, let $k' = \ell_v$ be the current load at $v$. If $g$ is any frequency assigned by $A$ to $v$ then, by the definition of $A$, we have that $g \in F^{B}_{t,k''}$ for some $t' \leq t$ and $k'' \leq \min(t', t')$. Thus $k + k'' \leq k + k' \leq t$, by the definition of $t$. Using condition (F2), we now get that $F^{A}_{t,k} \cap F^{B}_{t',k'} = \emptyset$, and therefore $f \neq g$.

Finally, when the optimum value is $t$, then any frequency used by $A$ is from some set $F^c_{\tau,\kappa}$ for $\kappa \leq \tau \leq t$. Therefore $A$ is $R$-competitive, by the property (2) of $F$.

\[ \blacksquare \]

4 An Upper Bound

In this section we prove that there is an $R_0$-competitive incremental algorithm, for $R_0 = (18 - \sqrt{5})/11 \approx 1.433$. Using Lemma 3.1 it is sufficient to design an $R_0$-competitive $F$-system.

Intuitions. Our construction below may appear rather mysterious, so we begin by gradually introducing its main ideas. We will distinguish between two types of frequencies: private and shared. A-private frequencies will be used only in sets $F^A_{t,k}$, $B$-private frequencies will be used only in sets $F^B_{t,k}$, while shared frequencies can be used in some sets from both partitions $A$ and $B$.

Competitive ratio 2 can be easily achieved using only private frequencies. For each $c \in \{A, B\}$, let $P^c$ denote an infinite pool of $c$-private frequencies, with $P^A$ and $P^B$ disjoint. We simply let $F^c_{t,k}$ consist of the first $k$ frequencies from $P^c$. Conditions (F1) and (F2) are trivially true. For any given $t$, the set on the left-hand side of inequality (2) contains $2t$ frequencies, so (2) holds for $R = 2$.

We now show how to improve the ratio to 1.5. To accomplish this, we must use some shared frequencies. Let $S$ denote an infinite pool of shared frequencies, where $S$ is disjoint with $P^A \cup P^B$. To avoid collisions (that is, violations of (F2)), we need to use these shared frequencies judiciously. The main idea is this: for any given $c, t, k$, $F^c_{t,k}$ will only contain some of the first $t/2$ $c$-private frequencies and some of the first $t/2$ shared frequencies. (For simplicity, we temporarily ignore the fact that $t/2$ may not be integer.) This will guarantee that we will use at most $1.5t$ frequencies for all sets $F^c_{\tau,\kappa}$ with $\tau \leq t$. If $k \leq t/2$, then we have enough $c$-private frequencies to completely fill $F^c_{t,k}$. Otherwise, for $k > t/2$, in addition to the first $t/2$ $c$-private frequencies, $F^c_{t,k}$ we use $k - t/2$ last shared frequencies with indices at most $t/2$. So these frequencies will be indexed between $t/2 - (k - t/2) = t - k$ and $t/2$. Clearly, $F^c_{t,k}$ has at least $k$ frequencies, so (F1) holds. The intuition behind (F2) is this: Suppose $t' \leq t$. Then $F^A_{t,k}$ conflicts with each $F^B_{t',k'}$ for $k' \leq t - k$. As $k' \leq t - k$, the “worst” such conflict is with $F^B_{t-k,t-k}$, which is disjoint with $F^A_{t,k}$, by our choice of shared frequencies.

To make it more precise, for any real number $x \geq 0$ let

\[
S_x = \text{the first } \lfloor x \rfloor \text{ frequencies in } S,
\]

\[
P^c_x = \text{the first } \lfloor x \rfloor \text{ frequencies in } P^c, \text{ for } c \in \{A, B\}.
\]

We now let $\mathcal{F} = \{F^c_{t,k}\}$, where for $c \in \{A, B\}$ and $k \leq t$ we have

\[
F^c_{t,k} = P^c_{t/2+1} \cup (S_{t/2} \setminus S_{t-k}).
\]

We claim that $\mathcal{F}$ is a 1.5-competitive $F$-system. If $k \leq \lfloor t/2 \rfloor + 1$, then $|F^c_{t,k}| \geq k$ is trivial. If $k \geq \lfloor t/2 \rfloor + 2$, then $t - k \leq t - \lfloor t/2 \rfloor - 2 \leq t/2$, so $S_{t-k} \subseteq S_{t/2}$ and thus $|F^c_{t,k}| \geq \lfloor t/2 \rfloor + 1 + (\lfloor t/2 \rfloor - \lfloor t - k \rfloor) \geq k$. So (F1) holds.
To verify (F2), pick any two pairs $k \leq t$ and $k' \leq t'$ with $k + k' \leq \max(t, t')$. Without loss of generality, assume $t' \leq t$ and $c = A$. If $k' \leq \lfloor t'/2 \rfloor + 1$, then $F_{c,t',k'}^B \subseteq P^B$, so (F2) is trivial. If $k' \geq \lfloor t'/2 \rfloor + 2$, then $t'/2 \leq k' \leq t - k$, so $F_{c',t',k'}^B \subseteq P^B \cup S_{t'/2} \subseteq P^B \cup S_{t-k}$, which implies (F2) as well.

Finally, for any $c \in \{A, B\}$ and $\kappa \leq \tau \leq t$, we have $F_{c,\tau,\kappa} \subseteq P_{t/2+1}^{A} \cup P_{t/2+1}^{B} \cup S_{t/2}$, so the inequality \([2]\) holds with $R = 1.5$ and $\lambda = 2$. We can thus conclude that this $\mathcal{F}$ is 1.5-competitive.

A geometric interpretation of the used sets of frequencies is is shown in Figure 1. For $k > t/2$, set $F_{c,t,k}^\tau$ conflicts with $F_{c',\tau}^\tau$ for $\tau = t - k$ and $F_{c',\tau}^\tau$ uses shared frequencies numbered at most $\lfloor \tau'/2 \rfloor = (t - k)/2$. Thus all shared frequencies that “conflict with the past” are within the region below the line $x = (t - k)/2$. This region is disjoint with the shaded region assigned to $F_{c',t,k}^\tau$, whose boundary is the line $x = t - k$.

**Construction of an $R_0$-competitive F-system.** To improve the ratio further, the idea is to use even fewer private frequencies, but to assign shared frequencies more carefully. We will actually have three types of shared frequencies, called A-shared, B-shared and symmetric-shared.

To achieve ratio smaller than 1.5 we need to use some shared frequencies even for $k < t/2$. Obviously, to do this we must break symmetry, as $F_{A,t,k}^c$ and $F_{B,t,k}^c$ cannot use any common shared frequency for $k < t/2$. This is the reason why we introduce A-shared and B-shared frequencies. For sets $F_{c,t,k}^c$, as $k$ increases, we first use $c$-private frequencies, then $c$-shared frequencies, then symmetric-shared frequencies, and finally, if $k$ gets sufficiently large, we also “borrow” $c'$-shared frequencies to include in this set. More precisely, we use some $c$-shared frequencies for any $k > t/\phi^2 \approx 0.382t$, while we use symmetric-shared frequencies for $k > t/2$ and $c'$-shared frequencies only for $k > t/\phi \approx 0.618t$. We remark here that symmetric frequencies are still needed. If we restrict ourselves to only private and $c$-shared frequencies then the best ratio we are able to achieve is $\approx 1.447$. 

Figure 1: The structure of frequency sets in the 1.5-competitive algorithm. In this figure, we fix the value of $t$, and show the frequency sets for each value of $k \leq t$. The horizontal axis represents frequencies, with the first frequencies drawn on the left. The vertical axis represents the values of $k$, for each $k$ the intersection of the corresponding horizontal line with the shaded (green) regions shows the frequencies used by the algorithm. For private frequencies on the left, for $k < t/2$, we do not need to use all of the frequencies, and the choice of them is arbitrary. For shared frequencies on the right, the shaded area corresponds exactly to $S_{t/2} \setminus S_{t-k}$.
Figure 2: The structure of frequency sets in the $R_0$-competitive algorithm. Here we show only the shared frequencies, represented similarly as in Figure 1. In addition, we show by different shading which of the unused frequencies would create conflicts with the past and with the future; the bottom unshaded part would cause both types of conflicts.

A geometric interpretation of the used sets of shared frequencies is is shown in Figure 2. The algorithm with ratio 1.5 used the shaded region shown in Figure 1 to avoid collisions with the past, that is with frequencies already assigned to sets $F_{c,\tau}$ for $\tau < t$. As observed earlier, the line $x = t - k$ is not tight; it can be lowered to $x = (t - k)/2$ without creating conflicts. With this modification, only some of the shared frequencies above this line are needed. However, this modification is not sufficient to reduce the ratio below 1.5, because of symmetry: we still will have conflicts for $F_{c,t/2}$ for $k = t/2$. To avoid such conflicts, we also consider, preemptively, “conflicts in the future”, namely with sets $F_{c,t'}$ for $t' > k$. These conflicts are represented in the figure by the half-plane below the line $x = \gamma k$, for an appropriate $\gamma$ while now the conflicts with the past are represented by the half-plane below the line $\gamma'(t - k)$. The optimization of the parameters for all three types of shared frequencies leads to our new algorithm.

The pools of $c$-shared and symmetric-shared frequencies are denoted $S^c$ and $Q$, respectively. As before, for any real $x \geq 0$ we define

$$S^c_x = \text{the first } \lfloor x \rfloor \text{ frequencies in } S^c, \text{ for } c \in \{A, B\}.$$  
$$Q_x = \text{the first } \lfloor x \rfloor \text{ frequencies in } Q.$$  

Our construction uses three constants, defined as

$$\alpha = R_0 - 1 = \frac{7 - \sqrt{5}}{11} = \frac{2}{\phi + 3} \approx 0.433,$$
$$\beta = \alpha/2 = \frac{7 - \sqrt{5}}{22} = \frac{1}{\phi + 3} \approx 0.217, \text{ and}$$
$$\rho = \beta/\phi = \frac{2\sqrt{5} - 3}{11} = \frac{\phi - 1}{\phi + 3} \approx 0.134,$$
where $\phi = (\sqrt{5} + 1)/2$ is the golden ratio. A useful fact is the identity $2\alpha + 2\beta + \rho = R_0$.

We define $F = \{F^c_{t,k}\}$, where for any $t \geq k \geq 0$ we let

$$F^c_{t,k} = P^c_{\alpha t + 4} \cup (S^c_{\beta \min(t, \phi k)} \setminus S^c_{\beta(t-k)}) \cup (S^\prime_{\beta k} \setminus S^\prime_{\phi \beta(t-k)}) \cup (Q_{\rho \min(t, \phi k)} \setminus Q_{\phi \rho(t-k)}).$$  \hspace{1cm} (3)

We now show that $F$ is an $R_0$-competitive F-system. To this end, we show that $F$ satisfies properties (F1), (F2) and (F3).

We start with (F2). For $k \leq \tau \leq t$ and $c \in \{A, B\}$ we have

$$F^c_{\tau, k} \subseteq P^c_{\alpha \tau + 4} \cup S^c_{\beta \tau} \cup S^\prime_{\beta k} \cup Q_{\rho \tau} \subseteq P^c_{\alpha t + 4} \cup S^c_{\beta t} \cup S^\prime_{\beta t} \cup Q_{\rho t} \subseteq P^A_{\alpha t + 4} \cup P^B_{\alpha t + 4} \cup S^A_{\beta t} \cup S^B_{\beta t} \cup Q_{\rho t}.$$  

This last set has cardinality at most $(2\alpha + 2\beta + \rho)t + 8 = R_0 t + 8$, so (F2) holds with $\lambda = 8$.

Next, we show (F2). By symmetry, we can assume that $t' \leq t$ in (F2), so $k' \leq t - k$. Then

$$F^B_{t', k'} \subseteq P^B \cup S^B_{\rho \phi k'} \cup S^A_{\phi k'} \cup Q_{\rho \phi k'} \subseteq P^B \cup S^B_{\phi \beta(t-k)} \cup S^A_{\phi \beta(t-k)} \cup Q_{\phi \rho(t-k)},$$

and this set is disjoint with $F^A_{t,k}$ by definition (3). Thus $F^A_{t,k} \cap F^B_{t', k'} = \emptyset$, as needed.

Finally, we prove (F1), namely that $|F^c_{t,k}| \geq k$. We distinguish two cases.

Case 1: $k > t/\phi$. This implies that $\min(t, \phi k) = t$, so in (3) we have $S^c_{\beta \min(t, \phi k)} = S^c_{\beta t}$ and $Q_{\rho \min(t, \phi k)} = Q_{\rho t}$. Thus

$$|F^c_{t,k}| \geq (\alpha t + 3) + [\beta t - \beta(t-k) - 1] + [\beta k - \phi \beta(t-k) - 1] + [\rho t - \phi \rho(t-k) - 1]$$

$$= (\alpha - \beta - (\phi - 1)\rho)t + (2\beta + \phi \beta + \phi \rho)k = k,$$

using the substitutions $\alpha = 2\beta$ and $\rho = \beta/\phi$. Note that this case is asymptotically tight as the algorithm uses all three types of shared frequencies (and the corresponding terms are non-negative).

Case 2: $k \leq t/\phi$. The case condition implies that $\phi k \leq t$, so $S^c_{\beta \min(t, \phi k)} = S^c_{\phi \beta k}$, $Q_{\rho \min(t, \phi k)} = Q_{\phi \rho k}$, and $S^\prime_{\phi \beta(t-k)} \setminus S^\prime_{\phi \beta(t-k)} = \emptyset$. Therefore

$$|F^c_{t,k}| \geq (\alpha t + 3) + [\phi \beta k - \beta(t-k) - 1] + [\phi \rho k - \phi \rho(t-k) - 1]$$

$$= (\alpha - \beta - \phi \rho)t + ((\phi + 1)\beta + 2\phi \rho)k + 1$$

$$= k + 1.$$

using $\alpha = 2\beta$ and $\rho = \beta/\phi$ again. Note that this case is (asymptotically) tight only for $k > t/2$ when $c$-shared and symmetric-shared frequencies are used. For $k \leq t/2$, no symmetric-shared frequencies are used and the corresponding term is negative.

Summarizing, we conclude that $F$ is indeed an $R_0$-competitive F-system. Therefore, using Lemma 3.1, we get our upper bound:

**Theorem 4.1.** There is an $R_0$-competitive incremental algorithm for frequency allocation on bipartite graphs, where $R_0 = (18 - \sqrt{5})/11 \approx 1.433$. 
5 A Lower Bound

In this section we show that if \( R < 10/7 \), then there is no \( R \)-competitive incremental algorithm for frequency allocation in bipartite graphs. By Lemma 3.1 it is sufficient to show that there is no \( R \)-competitive F-system.

The general intuition behind the proof is that we try to reason about the sets \( Z_t = F_{t/2}^A \cup F_{t/2}^B \) for a suitable constant \( \gamma \). These sets should correspond to the symmetric-shared frequencies from our algorithm, for \( \gamma \) such that no \( \ell \)-shared frequencies are used. If \( Z_t \) is too small, then both partitions use mostly different frequencies and this yields a lower bound on the competitive ratio. If \( Z_t \) is too large, then for a larger \( t \) and suitable case, the frequencies cannot be used for either partition, and hopefully this allows to improve the lower bound. We are not able to do exactly this. Instead, for a variant of \( Z_t \), we show a recurrence essentially saying that if the set is too large, then for some larger \( t \), it must be proportionally even larger, leading to a contradiction.

We now proceed with the proof. For \( c \in \{ A, B \} \), let \( F_t^c = \bigcup_{\kappa < T \leq t} F_{t, \kappa}^c \). Towards contradiction, suppose that an F-system \( F \) is \( R \)-competitive for some \( R < 10/7 \). Then \( F \) satisfies the definition of competitiveness \( \{2\} \) for some positive integer \( \lambda \). Choose a sufficiently large integer \( \theta \) for which \( R < 10/7 - 1/\theta \).

We first identify shared frequencies in \( F \). Recall that \( F_t^c = \bigcup_{\kappa < T \leq t} F_{t, \kappa}^c \), for \( c \in \{ A, B \} \). Thus the definition of \( R \)-competitiveness says that \( \mid F_t^A \cup F_t^B \mid \leq Rt + \lambda \). The set of level-\( t \) shared frequencies is defined as \( S_t = F_t^A \cap F_t^B \).

**Lemma 5.1.** For any \( t \), we have \( \mid S_t \mid \geq (2 - R)t - \lambda \).

**Proof.** This is quite straightforward. By (F1) we have \( \mid F_t^c \mid \geq t \) for each \( c \), so \( \mid S_t \mid = \mid F_t^A \mid + \mid F_t^B \mid - \mid F_t^A \cup F_t^B \mid \geq 2t - (Rt + \lambda) = (2 - R)t - \lambda \). \( \square \)

Now, let \( S_{2t,t} = S_{2t} \cap (F_{2t,t}^A \cup F_{2t,t}^B) \) be the level-\( 2t \) shared frequencies that are used in \( F_{2t,t}^A \) or \( F_{2t,t}^B \). Each such frequency can only be in one of these sets because \( F_{2t,t}^A \cap F_{2t,t}^B = \emptyset \).

**Lemma 5.2.** For any \( t \), we have \( \mid S_{2t,t} \mid \geq (6 - 4R)t - 2\lambda \).

**Proof.** Observe that \( F_{2t,t}^A \cup F_{2t,t}^B \cup S_{2t} \subseteq F_{2t,t}^A \cup F_{2t,t}^B \) by definition, and thus \( \{2\} \) implies
\[
2Rt + \lambda \geq \mid F_{2t,t}^A \cup F_{2t,t}^B \cup S_{2t} \mid = \mid F_{2t,t}^A \cup F_{2t,t}^B \mid + \mid S_{2t} \mid - ((F_{2t,t}^A \cup F_{2t,t}^B) \cap S_{2t}) = \mid F_{2t,t}^A \mid + \mid F_{2t,t}^B \mid + \mid S_{2t} \mid - \mid S_{2t,t} \mid , \]

where the identities follow from the inclusion-exclusion principle, disjointness of \( F_{2t,t}^A \) and \( F_{2t,t}^B \), and the definition of \( S_{2t,t} \).

Transforming this inequality, we get
\[
\mid S_{2t,t} \mid \geq \mid F_{2t,t}^A \mid + \mid F_{2t,t}^B \mid + \mid S_{2t} \mid - (2Rt + \lambda) \geq (6 - 4R)t - 2\lambda ,
\]
as claimed, by property (F1) and Lemma 5.1. \( \square \)

For any even \( t \) define \( Z_{3t/2,t} = F_{3t/2,t}^A \cap F_{3t/2,t}^B \). In the rest of the lower-bound proof we will set up a recurrence relation for the cardinality of sets \( S_t \cup Z_{3t/2,t} \). The next step is the following lemma.
Figure 3: Illustration of Lemma 5.3.

**Lemma 5.3.** For any even $t$, we have $|S_{2t} \setminus Z_{3t,2t}| \geq |S_t \cup Z_{3t/2,t}| + |S_{2t,t}|$.

*Proof.* From the definition, the two sets $S_t \cup Z_{3t/2,t}$ and $S_{2t,t}$ are disjoint and they are subsets of $S_{2t} - Z_{3t,2t}$. (See Figure 3 for illustration.) This immediately implies the lemma. 

**Lemma 5.4.** For any even $t$, we have $|Z_{3t,2t}| \geq |S_t \cup Z_{3t/2,t}| - (3R - 4)t - \lambda$.

*Proof.* As $F^A_{3t,2t}$, $F^B_{3t,2t}$, $S_t$ and $Z_{3t/2,t}$ are all subsets of $F^A_{3t}$ and $F^B_{3t}$, inequality (2) implies
\[
3Rt + \lambda \geq |F^A_{3t,2t} \cup F^B_{3t,2t} \cup S_t \cup Z_{3t/2,t}|
= |F^A_{3t,2t} \cup F^B_{3t,2t}| + |S_t \cup Z_{3t/2,t}|
= |F^A_{3t,2t}| + |F^B_{3t,2t}| - |F^A_{3t,2t} \cap F^B_{3t,2t}| + |S_t \cup Z_{3t/2,t}|
= |F^A_{3t,2t}| + |F^B_{3t,2t}| - |Z_{3t,2t}| + |S_t \cup Z_{3t/2,t}|
\]
where the identities follow from the inclusion-exclusion principle, the fact that $F^A_{3t,2t} \cup F^B_{3t,2t}$ and $S_t \cup Z_{3t/2,t}$ are disjoint, and the definition of $Z_{3t,2t}$.

Transforming this inequality, we get
\[
|Z_{3t,2t}| \geq |F^A_{3t,2t}| + |F^B_{3t,2t}| + |S_t \cup Z_{3t/2,t}| - (3Rt + \lambda)
\geq |S_t \cup Z_{3t/2,t}| - (3R - 4)t - \lambda
\]
as claimed, by property (F1). 

We are now ready to derive our recurrence. By adding the inequalities in Lemma 5.3 and Lemma 5.4, taking into account that $|S_{2t} \setminus Z_{3t,2t}| + |Z_{3t,2t}| = |S_{2t} \cup Z_{3t,2t}|$, and then applying Lemma 5.2 for any even $t$ we get
\[
|S_{2t} \cup Z_{3t,2t}| \geq 2 \cdot |S_t \cup Z_{3t/2,t}| + |S_{2t,t}| - (3R - 4)t - \lambda
\geq 2 \cdot |S_t \cup Z_{3t/2,t}| + (10 - 7R)t - 3\lambda.
\]
For $i = 0, 1, \ldots, \theta$, define $t_i = 6\theta \lambda^2 i$ and $\gamma_i = |S_{t_i} \cup Z_{3t_i/2,t_i}|/t_i$. (Note that each $t_i$ is even.) Since $S_{t_i} \cup Z_{3t_i/2,t_i} \subseteq S_{2t_i}$, we have that $\gamma_i \leq |S_{2t_i}|/t_i \leq 2R + 1/(6\theta) < 3$. Dividing recurrence (4) by $t_{i+1} = 2t_i$, we obtain, for $i = 0, 1, \ldots, \theta - 1$,

$$
\gamma_{i+1} \geq \gamma_i + 5 - 7R/2 - 3\lambda/(2t_i)
\geq \gamma_i + 7/(2\theta) - 1/(4\theta) \geq \gamma_i + 3/\theta.
$$

But then we have $\gamma_\theta \geq \gamma_0 + 3 \geq 3$, which contradicts our earlier bound $\gamma_i < 3$, completing the proof. Thus we have proved the following.

**Theorem 5.5.** If $A$ is an $R$-competitive incremental algorithm for frequency allocation on bipartite graphs, then $R \geq 10/7 \approx 1.428$.

As a final remark we observe that our lower bound works even if the additive constant $\lambda$ is allowed to depend on the actual graph. I.e., for every $R < 10/7$ we can construct a single finite graph $G$ so that no algorithm is $R$-competitive on this graph. In our lower bound, we can restrict our attention to sets $F_{c,t_i}, F_{2t_i,t_i}, F_{3t_i,2t_i}$, and $F_{3t_i/2,t_i}$, for $i = 0, 1, \ldots, \theta$ and $c = A, B$. Then, in the construction from the proof of Lemma 3.1 for the lower bound sequence we obtain a finite graph together with a request sequence. However, for a fixed $\theta$, the graphs for different values of $\lambda$ are isomorphic, as all the indices scale linearly with $\lambda$. So, instead of using different isomorphic graphs, we can use different sequences corresponding to different values of $\lambda$ on a single graph $G$.

### 6 Final Comments

We proved that the competitive ratio for incremental frequency allocation on bipartite graphs is between 1.428 and 1.433, improving the previous bounds of 1.33 and 1.5. Closing the remaining gap, small as it is, remains an intriguing open problem. Besides completing the analysis of this special case, the solution is likely to involve sophisticated techniques that may be of its own interest.

The two other obvious directions of study are to prove better bounds for the dynamic case and for $k$-partite graphs. The general idea of distinguishing “collisions with the past” and “collisions with the future”, that we use to define our frequency sets, should be useful to derive upper bounds for these problems. Our concept of F-systems can be extended in a natural way to $k$-partite graphs, but with a caveat: for $k \geq 3$ the maximum load on a $k$-clique is only a lower bound on the optimum (unlike for $k = 2$, where the equality holds). Therefore in Lemma 3.1 only one direction holds. This lemma is still sufficient though to establish upper bounds on the competitive ratio. It is also conceivable that a lower bound can be proved using graphs where the optimum number of frequencies is equal to the maximum load of a $k$-clique.

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