Symmetry Groups of $A_n$ Hypergeometric Series

Yasushi KAJIHARA

Department of Mathematics, Kobe University, Rokko-dai, Kobe 657-8501, Japan
E-mail: kajihara@math.kobe-u.ac.jp

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Abstract. Structures of symmetries of transformations for Holman–Biedenharn–Louck $A_n$ hypergeometric series: $A_n$ terminating balanced $\binom{4}{3}$ series and $A_n$ elliptic $\binom{10}{9}$ series are discussed. Namely the description of the invariance groups and the classification all of possible transformations for each types of $A_n$ hypergeometric series are given. Among them, a “periodic” affine Coxeter group which seems to be new in the literature arises as an invariance group for a class of $A_n \binom{4}{3}$ series.

Key words: multivariate hypergeometric series; elliptic hypergeometric series; Coxeter groups

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Dedicated to Professors Anatol N. Kirillov and Tetsuji Miwa for their 65th birthday

1 Introduction

In this paper, we discuss structures of symmetries of transformations for two classes of $A_n$ hypergeometric series: $A_n$ terminating balanced $\binom{4}{3}$ series and $A_n$ elliptic $\binom{10}{9}$ series. Namely we give descriptions of the invariance groups and classification all of possible transformations for each type of $A_n$ hypergeometric series group-theoretically. Among them, a “periodic” affine Coxeter group which seems to be new in the literature arises as an invariance group for a class of $A_n \binom{4}{3}$ series.

The hypergeometric series $r+1 F_r$ is defined by

$$r+1 F_r \left[ \begin{array}{c} a_0, a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{array}; z \right] := \sum_{k \in \mathbb{N}} \frac{[a_0,a_1,\ldots,a_r]_k}{[b_1,\ldots,b_r]_k} \frac{z^k}{k!},$$

where $[c]_k = c(c+1) \cdots (c+k-1)$ is Pochhammer symbol and $[d_1,\ldots,d_r]_k = [d_1]_k \cdots [d_r]_k$.

Investigations of the symmetry of the hypergeometric series goes back to 19th century in the case of $\binom{3}{2}$ series. Thomae [33] has considered the following $\binom{3}{2}$ transformation formula

$$\binom{3}{2} \left[ \begin{array}{c} a, b, c \\ d, e \end{array}; 1 \right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} \binom{3}{2} \left[ \begin{array}{c} a, d-b, d-c \\ d, d+e-b-c \end{array}; 1 \right],$$

where $\Gamma(x)$ is the Euler gamma function. Later, Hardy [8] formulated this case as follows, where we give a refined form (see also Whipple [36]):

**Theorem** (Hardy). Let $s = s(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + x_3 - x_4 - x_5$. The function

$$\frac{1}{\Gamma(s)\Gamma(2x_4)\Gamma(2x_5)} \binom{3}{2} \left[ \begin{array}{c} 2x_1-s, 2x_2-s, 2x_3-s \\ 2x_4, 2x_5 \end{array}; 1 \right]$$

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is a symmetric function of the 5 variables $x_1, x_2, x_3, x_4, x_5$. Thus the $3F_2$ series have a symmetry of the symmetric group $S_5$ of degree 5.

Motivated by quantum mechanics and representation theory, the symmetry of hypergeometric series has been investigated by many authors including physicists. It is also related to highest weight representations of the unitary group $SU(2)$. (for a expository in this direction, we refer to the paper by Krattenthaler and Srinivasa Rao [22]). The Clebsch–Gordan coefficients can be expressed in terms of $3F_2$ series, in other words, Hahn polynomials and by using Hardy’s result, one finds non-trivial zeros of the coefficients. That is, one can clarify the structure of the highest weight representations. The Racah coefficients can be expressed in terms of terminating balanced $4F_3$ series, in other words, Racah polynomials. The corresponding results for the $4F_3$ series has been given by Beyer, Louck and Stein [3] (see also Section 2.2). The results of the groups of symmetry for hypergeometric series have been generalized for each types of hypergeometric series (see [23, 34, 35]).

We also mention that recently, number-theorists have investigated in this direction: Formicella, Green and Stade [5] and Mishev [26] discussed in the case of non-terminating (but) balanced $4F_3$ series with a connection with Fourier coefficients of $GL_n$ automorphic form. In [21], Krattenthaler and Rivoal presented a different but considerably interesting approach related to their investigations regarding odd values for Riemann zeta functions.

Elliptic hypergeometric series has first introduced by Frenkel and Turaev [6] in the context of elliptic $6j$-symbol. They obtained transformation and summation formulas for elliptic hypergeometric series by using invariants of links which extends the works by A.N. Kirillov and N.Yu. Reshetikhin [20] (see also [19]).

In 1970’s, Holman, Biedenharn and Louck [10] and Holman [9] has introduced a class of multivariate generalization of hypergeometric series which is nowadays called as $A_n$ hypergeometric series (or hypergeometric series in $SU(n+1)$) for explicit expressions of Clebsch–Gordan and Racah coefficients of the higher dimensional unitary group $SU(n+1)$. It includes $A_n$ $4F_3$ series which we discuss in Section 2. Results of transformation and summation formulas for $A_n$ hypergeometric series including basic and elliptic generalization and extension to other (classical) root systems has known by many authors (for summary, see an excellent exposition by S.C. Milne [24]).

Among them, we obtained a number of transformation formulas for (mainly basic) hypergeometric series of type $A$ with different dimensions in [14] (see also [13] and [15]). In the joint work with M. Noumi [17], we showed the results can be extended in the case of balanced series and proposed the notion of duality transformation formula. In [14] and [17], we have obtained our results by starting from the Cauchy kernels and their action of $(q)$-difference operators of Macdonald type. The class of hypergeometric transformations of type $A$ with different dimensions in our previous works can be considered to involve some of previously known $A_n$ hypergeometric transformation formulas in 20th century (see [24]). In [16] (see also [13] and [17]), we proved a number of their results by combining some special cases (hypergeometric transformations between $A_n$ hypergeometric series and one-dimensional ($A_1$) hypergeometric series). This paper can be considered to be a continuation of [16].

In this paper, we discuss the symmetry of some classes of $A_n$ hypergeometric series including $n = 1$ case. Namely we investigate the invariance forms and the groups describing the symmetry of each type of hypergeometric series. For $n \geq 2$, the symmetry of the $A_n$ hypergeometric series is more restricted than $n = 1$ case if we fix the symmetry corresponding to the dimension of the summation. So, the groups of symmetry are subgroups of that in the case of $n = 1$. Furthermore, we classify all the hypergeometric transformations which can be obtained by the combinations of possible permutations of the parameters and the hypergeometric transformations without trivial transformations in each cases. The classifications are given by double coset decomposition of the corresponding groups.
In Section 2, we discuss symmetries of $A_n$ terminating balanced $4F_3$ series. Among these, a "periodic affine" Weyl group that is periodic with respect to the translations arises in a class of $A_n$ $4F_3$ series. It seems not to have previously appeared in the literature as a Coxeter group (see [4, 11] and the paper by Iwahori and Matsumoto [12] regarding the Weyl groups with translations). In Section 3, we discuss symmetries of $A_n$ elliptic hypergeometric series. What is remarkable in this case is a subgroup structure.

It would be interesting if the discussions and results does work for future works not only for multivariate hypergeometric transformations themselves, but also for deeper investigations to the structure of irreducible decompositions of the tensor products of certain representations of higher dimensional unitary group $SU(n+1)$ and elliptic quantum groups of $SU(n+1)$, the original problem to introducing $A_n$ and elliptic hypergeometric series.

On the other hand, Kajiwara et al. [18] found that elliptic hypergeometric series $10E_9$ arises as a class of solutions of the elliptic Painlevé equation associated to the affine Weyl group $W(E_8^{(1)})$ which is the one of the family of the Painlevé equations introduced by Sakai [31] from the geometry of rational surfaces. We also mention the work of Rains [27] on relations between elliptic hypergeometric integrals and tau functions of elliptic Painlevé equations (see also [28] and [34]). It would be an interesting problem to give geometric interpretation of the symmetries of classes of $A_n$ hypergeometric series in terms of certain rational surfaces.

2 Symmetry groups of $A_n$ $4F_3$ series

2.1 Preliminaries on $A_n$ hypergeometric series

Here, we note the conventions for naming series as $A_n$ (ordinary) hypergeometric series (or hypergeometric series in $SU(n+1)$). Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ be a multi-index. We denote

$$\Delta(x) := \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad \text{and} \quad \Delta(x + \gamma) := \prod_{1 \leq i < j \leq n} (x_i + \gamma_i - x_j - \gamma_j),$$

as the Vandermonde determinant for the sets of variables $x = (x_1, \ldots, x_n)$ and $x + \gamma = (x_1 + \gamma_1, \ldots, x_n + \gamma_n)$ respectively. In this paper we refer multiple series of the form

$$\sum_{\gamma \in \mathbb{N}^n} \frac{\Delta(x + \gamma)}{\Delta(x)} H(\gamma)$$

which reduce to hypergeometric series $r+1F_r$ for a nonnegative integer $r$ when $n = 1$ and symmetric with respect to the subscript $1 \leq i \leq n$ as $A_n$ hypergeometric series. We call such a series balanced if it reduces to a balanced series when $n = 1$. Terminating, balanced and so on are defined similarly. The subscript $n$ in the label $A_n$ attached to the series is the dimension of the multiple series (2.1).

Before beginning our discussion, we summarize $q \to 1$ results of $A_n$ Sears transformation from [16] which we discuss in this paper. For the procedure of $q \to 1$ limit, one can find in the book by Gasper–Rahman [7] (see also [14]).

We introduce the notation for $A_n$ $4F_3$ series as follows

$$4F_3 \left( \begin{array}{l|cc|c|} \{b_1\}_n & a_1, a_2 & c \\ \{x_i\}_n & e_1, e_2 & d \end{array} \right)$$

$$:= \sum_{\gamma \in \mathbb{N}^n} \frac{\Delta(x + \gamma)}{\Delta(x)} \prod_{1 \leq i,j \leq n} \frac{[b_j + x_i - x_j]_{\gamma_i}}{[1 + x_i - x_j]_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{[c + x_i]_{\gamma_i}}{[d + x_i]_{\gamma_i}} [a_1, a_2]_{|\gamma|},$$

where $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ is a length of the multi-index $\gamma$. 

...
Here we give two $A_n$ Whipple transformations which discuss in this paper.

**Rectangular version** (the $q \to 1$ limit of $A_n$ Sears transformation formula, Corollary 4.5 in [16])

\[
\begin{align*}
&\quad \quad 4F_3^n\left(\begin{array}{c}
\{a_1, a_2\} \\
\{x_i\}
\end{array}\right| \begin{array}{cc}
\{c\} & 1 \\
\{d\}
\end{array} \right) = \frac{[d + e_1 - a_2 - c]_n}{[d + e - a_2 - c]_n} \prod_{1 \leq i \leq n} \frac{[d - a_1 + x_i]_n}{[d + x_i]_n} \\
&\times 4F_3^n\left(\begin{array}{c}
\{a_1, a_2\} \\
\{\tilde{x}_i\}
\end{array}\right| \begin{array}{cc}
\{e_1 - c\} & 1 \\
\{e_1 + e_2 - a_2 - c\}
\end{array} \right),
\end{align*}
\]

where $|M| = m_1 + m_2 + \cdots + m_n$ and $\tilde{x}_i = -m_i + |M| - x_i$ for $1 \leq i \leq n$. The balancing condition in this case is

\[
a_1 + a_2 + c + 1 - |M| = d + e_1 + e_2.
\]

Note that \(4F_3^n\left(\begin{array}{c}
\{a_1, a_2\} \\
\{x_i\}
\end{array}\right| \begin{array}{cc}
\{c\} & 1 \\
\{d\}
\end{array} \right)\) series terminates with respect to a multi-index.

In this paper, we call such series as rectangular and the multiple series which terminates with respect to the length of multi-indices as triangular.

**Triangular version** (the $q \to 1$ limit of $A_n$ Sears transformation formula, Proposition 4.5 in [16])

\[
\begin{align*}
&\quad \quad 4F_3^n\left(\begin{array}{c}
\{b_1\} \\
\{x_i\}
\end{array}\right| \begin{array}{cc}
-N, a & c \\
-e_1, e_2
\end{array} \right) = \frac{[d + e_1 - a]_n}{[d + e - a - B - c]_n} \prod_{1 \leq i \leq n} \frac{[d - b_i + x_i]_n}{[d + x_i]_n} \\
&\times 4F_3^n\left(\begin{array}{c}
\{b_1\} \\
\{\tilde{x}_i\}
\end{array}\right| \begin{array}{cc}
-N, e_1 - a & e_1 - c \\
-e_1 + e_2 - a - c
\end{array} \right),
\end{align*}
\]

where $|B| = b_1 + b_2 + \cdots + b_n$ and $\tilde{x}_i = b_i - B - x_i$ for $1 \leq i \leq n$. The balancing condition in this case is

\[
a + B + c + 1 - N = d + e_1 + e_2.
\]

**Remark 2.1.** In the case when $n = 1$ and $x_1 = 0$, (2.2) and (2.3) reduce to the Whipple transformation formula for terminating balanced $4F_3$ series

\[
\begin{align*}
&\quad \quad 4F_3\left(\begin{array}{c}
-N, a_1, a_2, a_3 \\
-d_1, d_2, d_3
\end{array}; 1 \right) = \frac{[d_2 - a_1, d_1 + d_2 - a_2 - a_3]_N}{[d_2 + d_2 - a_2 - a_3]_N} \\
&\quad \quad \quad \times 4F_3\left(\begin{array}{c}
-N, a_1, a_2, a_3 \\
-d_1, d_1 + d_3 - a_2 - a_3, d_1 + d_2 - a_2 - a_3
\end{array}; 1 \right) \cdot (2.4)
\end{align*}
\]

Note that identity above (2.4) is valid if the balancing condition

\[
a_1 + a_2 + a_3 + 1 - N = d_1 + d_2 + d_3
\]

holds.

### 2.2 Symmetries of $4F_3$ transformations ($A_1$ case)

Here, we discuss the symmetry for terminating balanced $4F_3$ series, namely the $A_1$ case:

\[
\begin{align*}
&\quad \quad 4F_3\left(\begin{array}{c}
-N, a_1, a_2, a_3 \\
-d_1, d_2, d_3
\end{array}; 1 \right) \quad (2.5)
\end{align*}
\]
with the balancing condition
\[ a_1 + a_2 + a_3 + 1 - N = d_1 + d_2 + d_3. \] (2.6)

Though most of all the results were originally obtained in [3] with a different formulation, we continue our discussion in order to give a correspondence to the results in Section 2.5.

The action of the parameters \( a_i, d_i, i = 1, 2, 3 \), for the Whipple transformation (2.4) is given as follows

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
d_1 \\
d_2 \\
d_3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a_1 \\
d_1 - a_3 \\
d_1 - a_2 \\
d_1 \\
d_1 + d_2 - a_2 - a_3 \\
d_1 + d_3 - a_2 - a_3
\end{pmatrix}.
\]

One can consider it as a linear transformation acting on the vector \( \vec{v}_1 = (a_1, a_2, a_3, d_1, d_2, d_3) \).

The matrix realization \( S \) for transformation \( s \) is given as follows

\[
S :=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 1 & 1 & 0 \\
0 & -1 & -1 & 1 & 0 & 1
\end{bmatrix}.
\]

It is easy to see that the \( _4F_3 \) series is invariant under the action of the permutation in the two sets of parameters \( \{a_1, a_2, a_3\} \) and \( \{d_1, d_2, d_3\} \). For \( i = 1, 2 \), let \( r_i \) be the permutation of \( a_i \) and \( a_{i+1} \) and let \( t_i \) be the permutation of \( d_i \) and \( d_{i+1} \). The matrix realizations \( R_i \) (resp. \( T_i \)) of \( r_i \) (resp. \( t_i \)) is given by its action on the vector \( \vec{v}_1 \). For example,

\[
R_1 :=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\text{ and } T_1 :=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

We have the invariant form for the terminating balanced \( _4F_3 \) series:

**Proposition 2.1** (invariance for terminating balanced \( _4F_3 \) series).

\[
_4F_3[\vec{v}_1] := [d_1, d_2, d_3]_N _4F_3 [-N, a_1, a_2, a_3; 1]
\]

is invariant under all of the actions \( r_i, t_i, i = 1, 2 \), and \( s \).

Obviously, the transformations \( r_1 \) and \( r_2 \) enjoy the braid relation \( r_1 r_2 r_1 = r_2 r_1 r_2 \) and \( r_i^2 = id \) for \( i = 1, 2 \). So do \( t_1 \) and \( t_2 \). The relations among the element \( s \) and others are summarized as follows:

**Lemma 2.1.** We have \((st_1)^3 = (sr_1)^3 = id \) and \((st_2)^2 = (sr_2)^2 = id \).
We define the mapping $\pi_1$ as

$$s_1 \rightarrow \sigma_3, \quad r_i \rightarrow \sigma_{3-i}, \quad t_i \rightarrow \sigma_{3+i}, \quad i = 1, 2.$$ 

Then, by braid relations among $r_i$ and $t_i$ and the lemma above, we see that the following relation holds:

$$\begin{cases} 
\sigma_i \neq \text{id}, & \sigma_i^2 = \text{id}, \quad i = 1, 2, 3, 4, 5, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, 2, 3, 4, \\
\sigma_i \sigma_j = \sigma_j \sigma_i, & |i - j| \geq 2.
\end{cases}$$

Thus we have the following.

**Proposition 2.2.** The set generated by $r_i$, $t_i$ for $i = 1, 2$ and $s$ forms a Coxeter group. Furthermore, the group is isomorphic to $\mathfrak{S}_6$.

Here we classify the possible transformations for terminating balanced $4F_3$ series of the form (2.5). Recall that $4F_3$ series is invariant under the action $\sigma_k$ for $k = 1, 2, 4, 5$. Thus our problem reduces to give an orbit decomposition of the double coset $H \setminus G/H$, where $G := \{\sigma_i \mid i = 1, 2, 3, 4, 5\}$, $G_1 := \{\sigma_i \mid i = 1, 2\}$, $G_2 := \{\sigma_i \mid i = 4, 5\}$ and $H = G_1 \times G_2$. The representatives of orbits of $H \setminus G/H$ is given by

$$\begin{align*}
(i) & \quad \omega_0 = \text{id}, \\
(ii) & \quad \omega_1 = \sigma_3, \\
(iii) & \quad \omega_2 = \sigma_3 \sigma_4 \sigma_2 \omega_1 = \sigma_3 \sigma_4 \sigma_2 \sigma_3, \\
(iv) & \quad \omega_3 = \sigma_3 \sigma_4 \sigma_5 \sigma_2 \sigma_1 \omega_2 = \sigma_3 \sigma_4 \sigma_5 \sigma_2 \sigma_1 \sigma_3 \sigma_4 \sigma_2 \sigma_3.
\end{align*}$$

Thus we are ready to present a list of the $4F_3$ transformations according to the representatives above. We frequently make the simplification of the product factor by using the balancing condition (2.6).

The transformation associated with $(i)$ is identical. The second one $(ii)$ is the Whipple transformation (2.4) itself. The third one $(iii)$ is given by

$$4F_3 \left[ \begin{array}{c} -N, a_1, a_2, a_3 \\ d_1, d_2, d_3 \end{array} ; 1 \right] = \frac{[d_1 + d_2 - a_1 - a_3, d_1 + d_2 - a_2 - a_3, a_3]_N}{[d_1, d_2, d_1 + d_2 - a_1 - a_2 - a_3]_N} \times 4F_3 \left[ \begin{array}{c} -N, d_1 - a_3, d_1 - a_3, d_1 + d_2 - a_1 - a_2 - a_3 \\ d_1 + d_2 - a_2 - a_3, d_1 + d_2 - a_1 - a_3, d_1 + d_2 + d_3 - a_1 - a_2 - 2a_3 ; 1 \end{array} \right]. \quad (2.7)$$

The forth one $(iv)$ is

$$4F_3 \left[ \begin{array}{c} -N, a_1, a_2, a_3 \\ d_1, d_2, d_3 \end{array} ; 1 \right] = \frac{[a_1, a_2, a_3]_N}{[d_1, d_2, d_1 + d_2 - a_1 - a_2 - a_3]_N} \times 4F_3 \left[ \begin{array}{c} -N, d_1 + d_2 - a_1 - a_2 - a_3, d_1 + d_3 - a_1 - a_2 - a_3, \\ d_1 + d_2 + d_3 - a_1 - a_2 - 2a_3, d_1 + d_2 + d_3 - a_1 - 2a_2 - a_3, \\ d_2 + d_3 - a_1 - a_2 - a_3, \\ d_1 + d_2 + d_3 - 2a_1 - a_2 - a_3 ; 1 \end{array} \right]. \quad (2.8)$$

The transformation (2.8) has an alternative expression

$$4F_3 \left[ \begin{array}{c} -N, a_1, a_2, a_3 \\ d_1, d_2, d_3 \end{array} ; 1 \right] = (-1)^N \frac{[a_1, a_2, a_3]_N}{[d_1, d_2, d_3]_N} \times 4F_3 \left[ \begin{array}{c} -N, 1 - N - d_1, 1 - N - d_2, 1 - N - d_3 \\ 1 - N - a_1, 1 - N - a_2, 1 - N - a_3 ; 1 \end{array} \right]. \quad (2.9)$$

Note also that (2.9) is an inversion of the order of the summation in the $4F_3$ series.
2.3 Symmetry of $A_n$ $4F_3$ series of rectangular type

Here, we describe the invariance group for $A_n$ Whipple transformation of rectangular type (2.2), namely the series of the form

$$
4F_3^n (\{ -m_i \}_n, a_1, a_2, c | d, 1),
$$

(2.10)

with the balancing condition

$$
a_1 + a_2 + c + 1 - |M| = d + e_1 + e_2.
$$

(2.11)

Suppose that $n \geq 2$ till stated otherwise. Hereafter, we also fix the symmetry of the dimension of the summation in the multiple series.

Recall that the transformation of coordinates in the right hand side of (2.2) is given as follows

$$
s: \begin{pmatrix}
a_1 \\
a_2 \\
c \\
d \\
e_1 \\
e_2
\end{pmatrix} \mapsto \begin{pmatrix}
a_1 \\
a_2 \\
c \\
d \\
e_1 \\
e_2
\end{pmatrix} = \begin{pmatrix}
a_1 \\
e_1 - a_2 \\
e_1 - c \\
e_1 + e_2 - a_2 - c \\
e_1 \\
e_1 + d - a_2 - c
\end{pmatrix}.
$$

It is easy to see that this transformation of coordinates is linear for $a_1, a_2, c, d, e_1$ and $e_2$. Thus we give a $6 \times 6$ matrix realization for transformation for $s$ acting on the vector $\vec{v} = [a_1, a_2, c, d, e_1, e_2]$ as follows

$$
S : \begin{pmatrix}
a_1 \\
a_2 \\
c \\
d \\
e_1 \\
e_2
\end{pmatrix} = S_1 \vec{v}, \quad S_1 := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 1 & 1 & 1 & 0
\end{pmatrix}.
$$

Note that the series (2.10) is symmetric with respect to the two sets of parameters $\{a_1, a_2\}$ and $\{e_1, e_2\}$. Let $s_0$ be a permutation of $a_1$ and $a_2$ and let $s_2$ be a permutation of $e_1$ and $e_2$. The matrix realization $S_0$ (resp. $S_2$) of $s_0$ (resp. $s_2$) is given by

$$
S_0 := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad S_2 := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

For the action on the variables $x_i$, we set to be $s_1 \cdot x_i = \tilde{x}_i = -m_i + |M| - x_i$ and otherwise to be identical.

We introduce the normalized $4F_3^n$ series $\bar{4F}_3^n ((\vec{v}, x))$ as

$$
\bar{4F}_3^n ((\vec{v}, x)) := [e_1, e_2]_{|M|} \prod_{1 \leq i \leq n} [d + x_i m_i]_{4F_3^n} (\{ -m_i \}_n, a_1, a_2 | c, 1),
$$

(2.12)

under the balancing condition (2.11).
Proposition 2.3 (invariance form for multiple series of type (2.10)). \( 4\tilde{F}_3^m ((\vec{v}, x)) \) is invariant under the action of \( s_0, s_1 \) and \( s_2 \).

In the course of the proof of this proposition, we use the following lemma.

Lemma 2.2. If (2.11) holds, then we have the following

1) \([b]|_M = (-1)^|M|[d + e_1 + e_2 - a_1 - a_2 - c - b]|_M|,\)

2) \([f + x_i]|_m = (-1)^m[d + e_1 + e_2 - a_1 - a_2 - c + x_i - f]|_m|.\)

One can prove this lemma by using an elementary manipulation of shifted factorials \([z]_m = (-1)^m[1 - z - m]_m\) and the balancing condition (2.11).

Proof of Proposition 2.3. Since the \( 4\tilde{F}_3^m ((\vec{v}, x)) \) is symmetric with respect to the sets of parameters \( \{a_1, a_2\} \) and \( \{e_1, e_2\} \), it is obvious in the case of \( s_0 \) and \( s_2 \). For the case of \( s_1 \), by using the transformation formula (2.2) and Lemma 2.2

\[
4\tilde{F}_3^m (s_1(\vec{v}, x)) = 4\tilde{F}_3^m ((S_1\vec{v}, \vec{x})) = [e_1, d + e_1 - a_2 - c]|_M| \prod_{1 \leq i \leq n} [d - a_1 + x_i]|_m|
\times 4\tilde{F}_3^m \left( \begin{array}{c|cc|}
\{-m_i\}_n & a_1, e_1 - a_2 & c \\
\{x_i\}_n & e_1, d + e_1 - a_2 - c & d \\
\end{array} \right) \bigg|_1
\]

\[
= [e_1, d + e_1 - a_2 - c]|_M| \prod_{1 \leq i \leq n} [d - a_1 + x_i]|_m|
\times \frac{[d + e_1 - a_1 - a_2 - c]|_M|}{[d + e_2 - a_1 - a_2 - c]|_M|} \prod_{1 \leq i \leq n} [d - a_1 + x_i]|_m|
4\tilde{F}_3^m \left( \begin{array}{c|cc|}
\{-m_i\}_n & a_1, a_2 & c \\
\{x_i\}_n & e_1, e_2 & d \\
\end{array} \right) \bigg|_1
\]

\[
= 4\tilde{F}_3^m ((\vec{v}, x)).
\]

Thus we complete the proof of the proposition. \(\blacksquare\)

The set \( \{s_0, s_1, s_2\} \) form a Coxeter group. Let \( G \) be the group generated by \( s_0, s_1 \) and \( s_2 \). The relations can be summarized as follows:

Lemma 2.3. The generators \( s_0, s_1, s_2 \) of the group \( G \) satisfy the following relations:

1) \( s_0^2 = s_1^2 = s_2^2 = \text{id}, \) \hspace{1cm} (2.13)

2) \( (s_0s_2)^2 = \text{id}, \quad (s_0s_1)^4 = (s_1s_2)^4 = \text{id}, \) \hspace{1cm} (2.14)

3) \( (s_2s_1s_0s_1)^3 = (s_1s_2s_1s_0)^3 = \text{id}. \) \hspace{1cm} (2.15)

Proof. One can check by direct computation using the matrix realization given above. We shall leave to readers. \(\blacksquare\)

Remark 2.2. The relations (2.13) and (2.14) in Lemma 2.3 are the relations same as that of the affine Weyl group \( W(\tilde{C}_2) \):

\[
\begin{array}{ccc}
\tilde{s}_0 & \tilde{s}_1 & \tilde{s}_2 \\
\hline
\end{array}
\]

Dynkin diagram of \( \tilde{C}_2 \)
We utilize properties of affine Weyl group $W(C_2)$, especially translations in $W(C_2)$ to describe the structure of the group $G$ (For properties of affine Weyl groups, see Iwahori–Matsumoto [12] and Humphreys’ book [11]). Here we follow the notation of [11].

In general, it is well known that affine Weyl group is a semidirect product of a Weyl group of the corresponding finite root system and the translation group corresponding to the coroot lattice. We define the root vectors in the two dimensional Euclidean space $V$ for the root system $C_2$ as the following picture:

![Roots of root system $C_2$](image)

The null root $\alpha_0$ for $C_2$ is given by $-2\alpha_1 - \alpha_2$. For a root $\alpha$, we denote by the corresponding coroot $\alpha^\vee$ given by $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, where $(\cdot, \cdot)$ is the Killing form. In this case, the generators of the Weyl group of the root system $C_2$ be given by $s_1$ and $s_2$. We denote $L^\vee$ by the coroot lattice of the root system $C_2$. For $d \in V$, let $t(d)$ be the translation which sends $\lambda \in V$ to $d+\lambda$.

**Lemma 2.4.** The group $G$ is of order 72. Furthermore, $G$ is isomorphic to a semidirect product of $W(C_2)$ and $L^\vee/3L^\vee$.

**Proof.** Note that $s_2s_1s_0s_1$ is the translation $t(\alpha_2^\vee)$ of minimum length in $V$. It is obvious to see that $s_1t(\alpha_2^\vee)s_1 = s_1s_2s_1s_0$ is $t(s_1\alpha_2^\vee) = t(\alpha_1^\vee + \alpha_2^\vee)$. Thus the group $G$ is a subgroup of the group $W(C_2) \ltimes L^\vee/3L^\vee$. In order to see $G$ is isomorphic to $W(C_2) \ltimes L^\vee/3L^\vee$, it suffices to check that $t(\alpha_2^\vee) \neq \text{id}$ and $t(\alpha_1^\vee + \alpha_2^\vee) \neq \text{id}$. Both of them can be done by direct computation using the matrix realization.

**Remark 2.3.** The Coxeter group $G$ can be considered as a “periodic” affine Weyl group. In particular, the relation (2.15) implies “periodicity” with respect to translations for the coroot lattice. David Bessis informed us that the group $G$ is not one of complex reflection groups [32]. He proved this by calculating the character of the group $G$.

We are going to classify possible and non-trivial transformation for the $A_n$ $4F_3$ series of rectangular type (2.10).

Let $H$ be the subgroup of the group $G$ generated by $s_0$ and $s_2$. Recall that the $4F_3^n$ series of type (2.10) is invariant under the action of $s_0$ and $s_2$. Then our problem reduces to give an orbit decomposition of the double coset $H \setminus G/H$. The representatives of this decomposition are given by

\[
(i) \; \text{id}, \quad (ii) \; s_1, \quad (iii) \; s_1s_2s_1, \quad (iv) \; s_1s_0s_1, \quad (v) \; s_1s_0s_2s_1,
\]
\[
(vi) \; s_1s_2s_1s_0s_1, \quad (vii) \; s_1s_0s_1s_2s_1, \quad (viii) \; s_1s_0s_2s_1s_0s_2s_1. \quad (2.16)
\]
We are ready to exhibit a complete list of possible transformations for the series of form (2.10) according to the representative (2.16) of each orbit in \(G\). We use Lemma 2.2 frequently without stated otherwise in simplifying the factors.

The first one \((i)\) in (2.16) is identical. The second is (2.2). The transformation corresponding \((iii)\) is

\[
\begin{aligned}
4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, a_2 & c & 1 \\ \{x\}_n & e_1, e_2 & d & 1 \\ \end{array} \right) &= \frac{[d + e_1 - a_2 - c, e_1 - a_1 |M]}{[d + e - a_1 - a_2 - c, e_1 |M]} \\
&\times 4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, d - c & 1 \\ \{x\}_n & d + e_2 - a_2 - c, d + e_1 - a_2 - c & d - a_2 & 1 \\ \end{array} \right),
\end{aligned}
\tag{2.17}
\]

which is equivalent to \(q \to 1\) limit of the first \(A_n\) Sears transformation (4.23) of [16]. The one \((iv)\) is

\[
\begin{aligned}
4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, a_2 & c & 1 \\ \{x\}_n & e_1, e_2 & d & 1 \\ \end{array} \right) &= \frac{[d - c |M]}{[d + e - a_1 - a_2 - c |M]} \prod_{1 \leq i \leq n} \frac{[d + e_1 - a_1 - a_2 + x_i |m_i]}{[d + x_i |m_i]}
\times 4F_3^m \left( \begin{array}{c|cc|c} -m_i & e_1 - a_1, e_1 - a_2 & c & 1 \\ \{x\}_n & e_1 + e_2 - a_1 - a_2, e_1 & d + e_1 - a_1 - a_2 & 1 \\ \end{array} \right).
\tag{2.18}
\end{aligned}
\]

The fifth one \((v)\) is

\[
\begin{aligned}
4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, a_2 & c & 1 \\ \{x\}_n & e_1, e_2 & d & 1 \\ \end{array} \right) &= \frac{[d + e_1 - a_2 - c, a_2 |M]}{[d + e - a_1 - a_2 - c, e_1 |M]} \prod_{1 \leq i \leq n} \frac{[d + e_1 - a_1 - a_2 + x_i |m_i]}{[d + x_i |m_i]}
\times 4F_3^m \left( \begin{array}{c|cc|c} -m_i & e_1 - a_2, d + e_1 - a_1 - a_2 - c & 1 \\ \{x\}_n & d + e_1 + e_2 - a_1 - 2a_2 - c, d + e_1 - a_2 - c & d - a_2 & 1 \\ \end{array} \right).
\tag{2.19}
\end{aligned}
\]

The one \((vi)\) is

\[
\begin{aligned}
4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, a_2 & c & 1 \\ \{x\}_n & e_1, e_2 & d & 1 \\ \end{array} \right) &= \frac{[d - c, d + e_1 + e_2 - a_1 - 2a_2 - c |M]}{[d + e_1 - a_1 - a_2 - c |M]} \prod_{1 \leq i \leq n} \frac{[d - a_1 + x_i |m_i]}{[d + x_i |m_i]}
\times 4F_3^m \left( \begin{array}{c|cc|c} -m_i & e_1 - a_2, e_2 - a_2 & 1 \\ \{x\}_n & e_1 + e_2 - a_1 - a_2, d + e_1 + e_2 - a_1 - 2a_2 - c & e_1 + e_2 - a_1 - a_2 - c & 1 \\ \end{array} \right).
\tag{2.20}
\end{aligned}
\]

The seventh one \((vii)\) is

\[
\begin{aligned}
4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, a_2 & c & 1 \\ \{x\}_n & e_1, e_2 & d & 1 \\ \end{array} \right) &= \frac{[d + e_1 - a_2 - c, d + e_1 - a_1 - c |M]}{[d + e - a_1 - a_2 - c, e_1 |M]} \prod_{1 \leq i \leq n} \frac{[c + x_i |m_i]}{[dx_i |m_i]}
\times 4F_3^m \left( \begin{array}{c|cc|c} -m_i & e_1 - a_1 - a_2 - c & 1 \\ \{x\}_n & d + e_1 - a_1 - c, d + e_1 - a_2 - c & d + e_1 + e_2 - a_1 - a_2 - 2c & 1 \\ \end{array} \right).
\tag{2.21}
\end{aligned}
\]

The one \((viii)\) is

\[
\begin{aligned}
4F_3^m \left( \begin{array}{c|cc|c} -m_i & a_1, a_2 & c & 1 \\ \{x\}_n & e_1, e_2 & d & 1 \\ \end{array} \right) &= \frac{[a_1, a_2 |M]}{[e_1, d + e_1 - a_1 - a_2 - c |M]} \prod_{1 \leq i \leq n} \frac{[c + x_i |m_i]}{[d + x_i |m_i]}
\times 4F_3^m \left( \begin{array}{c|cc|c} -m_i & e_1 - a_2 - c & 1 \\ \{x\}_n & d + e_1 - a_1 - c, d + e_1 - a_2 - c & d + e_1 + e_2 - a_1 - a_2 - 2c & 1 \\ \end{array} \right).
\end{aligned}
\]
We describe the invariance group for triangular transformation (2.3) is given by
\[
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & -1 & 1 & 1 & 0
\end{pmatrix}.
\]

(2.22) has an alternative expression:
\[
{4F}_3^n \left( \begin{array}{cccc}
-m_1 & n \\
x_i & n
\end{array} \right) = \begin{pmatrix} a_1, a_2, c \\
e_1, e_2, d \\
1 \\
\end{pmatrix} = \begin{pmatrix} a_1, a_2 \\
e_1, e_2 \\
1 \\
\end{pmatrix} \prod_{1 \leq i \leq n} \frac{[c + x_i, m_i]}{[d + x_i, m_i]}
\times {4F}_3^n \left( \begin{array}{cccc}
-1 - |M| - e_1, 1 - |M| - e_2 \\
1 - |M| - a_1, 1 - |M| - a_2 \\
1 - |M| - c, 1 \\
\end{array} \right).
\]

Note that this expression of the formula implies the reversing the order of the summation as \(4F_3^n\) series of the form (2.10).

2.4 \(A_n\) \(4F_3\) series of triangular type

We describe the invariance group for triangular \(A_n\) Whipple transformation (2.3), namely the series of the form
\[
4F_3^n \left( \begin{array}{cccc}
b_i \\
x_i \\
1 \\
\end{array} \right) = \begin{pmatrix} b \\
- N, a, c \\
e_1, e_2, d \\
1 \\
\end{pmatrix}
\]

with the balancing condition
\[
a + B + c + 1 - N = d + e_1 + e_2.
\]

Note that, on contrast to the case of (2.10), the action of the permutation \(s_0\) in Section 2.3 is not valid. So what we are to consider is the action of the permutation \(s_2\) of the parameters \(e_1\) and \(e_2\) and the transformation of the parameters in (2.3). The action of each parameters of the transformation (2.3) is given by
\[
s_1: \begin{pmatrix} b \\
- N, a, c \\
e_1, e_2, d \\
1 \\
\end{pmatrix} = S_1 \begin{pmatrix} b \\
o \\
1 \\
\end{pmatrix}, \quad S_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

Let \(G_t\) be the group generated by the transformations \(s_1\) and \(s_2\). The relations between \(s_1\) and \(s_2\) are completely same as that between \(s_1\) and \(s_2\) in Section 2.3. Namely,
\[
s_1^2 = s_2^2 = \text{id}, \quad (s_1 s_2)^4 = \text{id}.
\]

It follows that the group \(G_t\) is isomorphic to \(W(C_2)\), the Weyl group associated to the root system \(C_2\).

To classify possible and non-trivial transformation formula, what is going to see is to give an orbit decomposition of the double coset \(H \setminus G_t / H\), where \(H\) is a subgroup of \(G_t\) generated by \(s_2\). Note that \(H\) is isomorphic to \(\mathfrak{S}_2\). The representatives of each orbits associated to this decomposition are given by \((i)\) id, \((ii)\) \(s_1\) and \((iii)\) \(s_1 s_2 s_1\).
We now present the corresponding $A_n \, 4F_3$ transformations attached to each representative given above. The transformation for $(i)$ is identical and the second one $(ii)$ is (2.3). The third one $(iii)$ is

$$4F_3^m \left( \begin{array}{c|cc|c} \{b_i\}_n & -N, a & c \\ \{x_i\}_n & e_1, e_2 & d \\ \hline & 1 \end{array} \right) = \left[ \frac{e_1 - B, d + e_1 - a - c}{e, d + e_1 - a - B - c} \right]_N$$

$$\times 4F_3^m \left( \begin{array}{c|cc|c} \{b_i\}_n & -N, d - c \\ \{x_i\}_n & d + e_2 - a - c, d + e_1 - a - c & d - a \\ \hline & d \end{array} \right),$$

(2.24)

which is the $q \to 1$ limit case of $A_n$ Sears transformation ((4.2) in [13]).

2.5 Remarks on the results of Section 2

Finally, we close the present paper to give remarks on the structure of the corresponding groups of the $A_n$ hypergeometric series of each cases.

Remark 2.4 (the case when $n = 1$ in $A_n \, 4F_3$ series). In the case when $n = 1$, all the transformations (2.17), (2.2) and (2.18) of rectangular type and (2.24) of triangular type reduce to the Whipple transformation formula (2.4). All the transformations (2.20), (2.19), and (2.21) of rectangular type reduce to (2.7). The transformation (2.23) of rectangular type reduces to (2.9) and implies the reversing of the order of the summation in the $A_n \, 4F_3$ series of rectangular type.

Remark 2.5 (correspondence of the groups in Sections 2.2 and 2.3). By direct manipulation of the matrix realization in Section 2.3, we have $s = \sigma_4 \sigma_3 \sigma_1 \sigma_5 \sigma_4$. Thus we find that the group $G$ is isomorphic to the subgroup of $\mathfrak{S}_6$ generated by $\sigma_2$, $\sigma_5$ and $s$.

Remark 2.6. Except for Hardy type invariant form (2.12) for $4F_3^m$ series of the form (2.10), all other results are valid in the basic case and one can obtain in the same line as the discussion in this section. For Hardy type invariant form for terminating balanced $4\phi_3$ series have already appeared in Van der Jeugt and Srinivasa Rao [35].

3 Symmetry groups of $A_n$ elliptic hypergeometric series

3.1 Preliminaries on $A_n$ elliptic hypergeometric series

Here, we give notations for (multiple) elliptic hypergeometric series and recall the results of our previous paper with M. Noumi [17].

Let $[[x]]$ be a non-zero and homomorphic odd function in $\mathbb{C}$ which satisfies the Riemann relation:

1) $[[-x]] = -[[x]],$
2) $[[x + y]] [[x - y]] [[u + v]] [[u - v]]$
   $$= [[x + u]] [[x - u]] [[y + v]] [[y - v]] - [[x + v]] [[x - v]] [[y + u]] [[y - u]].$$

(3.1)

There are following three classes of such functions:

- $\sigma(x; \omega_1, \omega_2)$: Weierstrass sigma function with the periods $(\omega_1, \omega_2)$ (elliptic),
- $\sin(\pi x)$: the sine function (trigonometric),
- $x$: rational.
It is classically known [37] that all function \([x]\) satisfy the condition (3.1) are obtained from above three functions by transformation of the form \(e^{ax^2+bx}\) for complex numbers \(a, b, c \in \mathbb{C}\).

Fix a generic constant \(\delta \in \mathbb{C}\) so that for all integer \(k \in \mathbb{Z}\), \([k\delta]\) does not equal to zero. In the case when \([x]\) is Weierstrass sigma function \(\sigma(x; \omega_1, \omega_2)\) (the elliptic case for short), the condition for \(\delta\) is given by \(\delta \notin \mathbb{Q}\omega_1 + \mathbb{Q}\omega_2\).

Throughout the present paper, we consider the function \([x]\) as the elliptic case unless otherwise stated.

Next a shifted factorial \([x]\)_k is defined by
\[
[x]_k := [x][x + \delta] \cdots [x + (k - 1)\delta], \quad k = 0, 1, 2, \ldots.
\]

Further, we denote
\[
[x_1, \ldots, x_r]_k := [x_1]_k \cdots [x_r]_k.
\]

The elliptic hypergeometric series \(r + 3E_{r+2}\) is defined as follows
\[
r + 3E_{r+2}(s; \{u_k\}_r) = r + 3E_{r+2}(s; u_1, \ldots, u_r) := \sum_{m \in \mathbb{N}} \frac{[s + 2m\delta]}{[s]} \frac{[s]_m}{[\delta]_m} \prod_{1 \leq i \leq r} \frac{[u_i]_m}{[\delta + s - u_i]_m}.
\]

In the case when \([x]\) is a trigonometric function \(\sin x\), this series reduces to the basic very well-poised hypergeometric series \(r + 3W_{r+2}\). Note that \(r + 3E_{r+2}\) series are also symmetric with respect to the parameter \(u_k\) for \(1 \leq k \leq r\).

All the \(r + 3E_{r+2}\) series discussed in this paper is balanced, namely we assume
\[
u_1 + \cdots + \nu_r = \frac{r - 1}{2} s + \frac{r - 3}{2}.
\]

Now, we note the conventions for naming series as \(A_n\) elliptic hypergeometric series (or referred as elliptic hypergeometric series in \(SU(n + 1)\)). Let \(\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n\) be a multi-index. We denote generalizations of the Vandermonde determinant
\[
\Delta[x] := \prod_{1 \leq i < j \leq n} [x_i - x_j] \quad \text{and} \quad \Delta[x + \gamma\delta] := \prod_{1 \leq i < j \leq n} [x_i + \gamma_i\delta - x_j - \gamma_j\delta],
\]
for the sets of variables \(x = (x_1, \ldots, x_n)\) and \(x + \gamma\delta = (x_1 + \gamma_1\delta, \ldots, x_n + \gamma_n\delta)\) respectively. In this paper we refer multiple series of the form
\[
\sum_{\gamma \in \mathbb{N}^n} \frac{\Delta[x + \gamma\delta]}{\Delta[x]} H(\gamma)
\]
which reduce to elliptic hypergeometric series \(r + 1E_r\) for a nonnegative integer \(r\) when \(n = 1\) and symmetric with respect to the subscript \(1 \leq i \leq n\) as \(A_n\) elliptic hypergeometric series. Other terminology are similar to the case of \(A_n\) (ordinary) hypergeometric series. The subscript \(n\) in the label \(A_n\) attached to the series is the dimension of the multiple series (3.2).

We are going to introduce the multiple elliptic hypergeometric series \(E^{n,m}\) which is defined by
\[
E^{n,m} \left( \frac{a_i}{x_i} \right)_{n; \{u_k\}_m; \{v_k\}_m} := \sum_{\gamma \in \mathbb{N}^n} \frac{\Delta[x + \gamma\delta]}{\Delta[x]} \prod_{1 \leq i \leq n} \frac{[\gamma_i\delta + s + x_i]}{[s + x_i]}
\times \prod_{1 \leq j \leq n} \frac{[s + x_j]_{\gamma_i}}{[\delta + s - a_j + x_j]_{\gamma_i}} \left( \prod_{1 \leq i \leq n} \frac{[a_j + x_i - x_j]_{\gamma_i}}{[\delta + x_i - x_j]_{\gamma_i}} \right).
\]
In the case when \( n = 1 \), \( E^{1,m} \) series reduces to (one dimensional) elliptic hypergeometric series \( 2m+4E_{2m+3} (s; \{ u_k \}_m, \{ v_k \}_m) \).

Note that \( E^{n,m} \) series is symmetric within two sets of parameters \( \{ u_k \}_m \) and \( \{ v_k \}_m \) respectively. This fact will be a key of the latter discussion of the symmetry for the \( E^{n,m} \) series.

Here, we present the balanced duality transformation formula for multiple elliptic hypergeometric series from [17].

Under the balancing condition
\[
c_1 + c_2 + d_1 + \sum_{1 \leq i \leq n} a_i + \sum_{1 \leq k \leq m} (u_k + v_k) = (m + N + 1)\delta + (m + 2)s.
\]

We have the following transformation formula between \( E^{m+2} (A_m 2m+8E_{2m+7}) \) series and \( E^{m,n+2} (A_m 2m+8E_{2m+7}) \) series ((3.17) in [17]):
\[
E^{m,m+2} \left( \{ a_i \}_n \bigg\{ x_i \}_n \bigg| s; c_1, c_2, \{ u_k \}_m, \{ v_k \}_m, d_1, -N\delta, \{ w_i \}_n \right) \\
= \frac{[[\delta + s - c_1 - d_1, \delta + s - c_2 - d_1]]_N}{[[\delta + s - c_1, \delta + s - c_2]]_N} \prod_{1 \leq k \leq m} \frac{[[u_k, \delta + s - u_k - d_1]]_N}{[[\delta + s - u_k, v_k - d_1]]_N} \\
\times \prod_{1 \leq i \leq n} \frac{[[\delta + s + x_i, \delta + s + x_i - a_i - d_1]]_N}{[[\delta + s + x_i - a_i, \delta + s + x_i - d_1]]_N} \\
\times E^{m,n+2} \left( \{ b_k \}_m \bigg\{ y_k \}_m \bigg| t; c_1, -c_2, \{ z_i \}_n, d_1, -N\delta, \{ w_i \}_n \right),
\]
(3.3)

where
\[
t = d_1 + d_2 - s - \delta, \quad b_k = \delta + s - u_k - v_k, \quad y_k = \delta + s - v_k, \quad k = 1, \ldots, m,
\]
\[
z_i = x_i - a_i, \quad w_i = d_1 + d_2 - s - x_i, \quad i = 1, \ldots, n.
\]

Remark 3.1. In [30], Rosengren also obtained (3.3) by a different way from [17]. In the case when \( m = n = 1 \) and \( x_1 = y_1 = 0 \), (3.3) reduces to the following transformation formula for terminating balanced \( 10E_9 \) series:
\[
10E_9 (s; c_0, c_1, c_2, c_3, d_0, d_1, -N\delta) = \frac{[[d_0, \delta + s]]_N}{[[d_0 - d_1, \delta + s - d_1]]_N} \\
\times \prod_{0 \leq k \leq 3} \frac{[[\delta + s - c_k - d_1]]_N}{[[\delta + s - c_k]]_N} 10E_9 (\tilde{s}; \tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{d}_0, d_1, -N\delta),
\]
(3.4)

where
\[
\tilde{s} = d_1 + d_2 - d_0, \quad \tilde{d}_0 = d_1 + d_2 - s, \quad \tilde{c}_k = \delta + s - d_0 - c_k, \quad k = 0, 1, 2, 3.
\]

Note that the \( 10E_9 \) transformation (3.4) can also be obtained by iterating twice in an appropriate manner the (rather well-known) elliptic version of the Bailey transformation [1] (see also [2]) for \( 10E_9 \) series due to Frenkel and Turaev [6]:
\[
10E_9 (s; c_0, c_1, c_2, d_0, d_1, d_2, -N\delta) = \frac{[[\delta + s]]_N}{[[\delta + s - d_0 - d_1 - d_2]]_N}
\]
Here, we describe the symmetry of 1-dimensional elliptic Bailey transformation for \(10E_9\). For the sake of the connection of the results in this section, we have verified that the result here has appeared in the paper by S. Lievens and J. Van der Jeugt [23] in the case of very well-poised basic hypergeometric series \(10E_9\). But our description given here is modified for the sake of the connection of the results in this section.

3.2 Symmetry of \(10E_9\) series (A\(_1\) case)

Here, we describe the symmetry of 1-dimensional elliptic Bailey transformation for \(10E_9\) series (3.5), namely for the \(10E_9\) series of the form

\[
10E_9 (s; c_0, c_1, c_2, c_3, c_4, c_5, -N\delta),
\]

with the balancing condition

\[
c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = (2 + N)\delta + 3s.
\]

The result here has appeared in the paper by S. Lievens and J. Van der Jeugt [23] in the case of very well-poised basic hypergeometric series \(10E_9\). But our description given here is modified for the sake of the connection of the results in this section.

For \(k = 1, \ldots, 5\), let \(s_k\) be the permutation for the parameters \(c_{k-1}\) and \(c_k\). Let \(b\) be the transformation of parameters for the Bailey transformation (3.5). Note that these are affine transformations in 7-dimensional vector space. Here we shall give a \(8 \times 8\) matrix realization acting on the vector \(\vec{v}_1 = [s, c_0, c_1, c_2, c_3, c_4, c_5, \delta]\) for these transformations. The transformation of parameters \(b\) for Bailey transformation (3.5) and its matrix realization are given by

\[
b : \vec{v}_1 = \begin{bmatrix} s \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \delta \end{bmatrix} \mapsto \begin{bmatrix} 2s + \delta - c_0 - c_1 - c_2 \\ s + \delta - c_1 - c_2 \\ s + \delta - c_0 - c_2 \\ s + \delta - c_0 - c_1 \\ c_3 \\ c_4 \\ c_5 \\ \delta \end{bmatrix} = B \cdot \vec{v}_1,
\]
and
\[ B = \begin{bmatrix}
2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \]
respectively. The matrix realization for \( s_1 \) is given by
\[ S_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \]
and so on.

**Proposition 3.1** (Hardy type invariant form for \( 10E_9 \) series with the condition (3.7)). If the balancing condition (3.7) holds,
\[ 10\tilde{E}_9((\vec{v}^i)) := \prod_{0 \leq k \leq 5} \frac{[\delta + s - c_k]_N}{[[s]_N} \]
\[ 10E_9(s;c_0,c_1,c_2,c_3,c_4,c_5,-N\delta) \]
is invariant under the action of \( b \) and \( s_k \) for all \( 1 \leq k \leq 5 \).

Note that \( B^2 = \text{id}_8 \), namely \( b^2 = \text{id} \). By definition, \( \{ s_i | i = 1, 2, 3, 4, 5 \} \) satisfy the relation
\[
\begin{align*}
\{ s_i \neq \text{id}, & \quad s_i^2 = \text{id}, \quad i = 1, 2, 3, 4, 5, \\
\} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & \quad i = 1, 2, 3, 4, \\
\} \quad s_i s_j = s_j s_i, & \quad |i - j| \geq 2.
\end{align*}
\]
Relation (3.8)

The relations between \( b \) and \( s_i, i = 1, 2, 3, 4, 5 \) summarized as the following lemma:

**Lemma 3.1.** We have \( (s_3 b)^3 = \text{id} \). And \( b \) commutes with any other \( s_i, i = 1, 2, 4, 5 \).

The proof of this lemma can be given by using the matrix realization given above.

By combining this lemma and (3.8), we conclude:

**Theorem 3.1.** The group \( G_1 \) generated by the permutations of the parameters \( \{ c_k | k = 0, \ldots, 5 \} \) and the Bailey transformation (3.5) for \( 10E_9 \) series is isomorphic to the Weyl group \( W(E_6) \) associated to the root system \( E_6 \):

![Dynkin diagram of E_6](image-url)
Here we classify the possible non-trivial transformation formulas for $10E_9$ series. Notice again that $10E_9$ series is symmetric for the permutation of parameters $c_0, \ldots, c_5$. That is, it is invariant under the action $s_i$ for all $i = 1, \ldots, 5$. Thus our problem turn out to give an orbit decomposition of the double coset $\mathfrak{S}_6 \backslash W(E_6)/\mathfrak{S}_6$.

We define the mapping $\pi_1$ according to Bourbaki [4] as follows:

$$s_i \mapsto w_1, \quad b \mapsto w_2, \quad s_i \mapsto w_{i+1}, \quad i = 2, 3, 4, 5, 6.$$ 

The representatives of orbits in the double coset $\mathfrak{S}_6 \backslash W(E_6)/\mathfrak{S}_6$ are given as follows:

1) $\tau_1 = \text{id}$,  
2) $\tau_2 = w_2$,  
3) $\tau_3 = w_2w_4w_3w_5w_4w_2$,  
4) $\tau_4 = w_2w_4w_3w_1w_5w_4w_3w_6w_5w_4w_2$,  
5) $\tau_5 = w_2w_4w_3w_1w_5w_4w_3w_4w_5w_6w_5w_4w_2w_3w_1w_4w_3w_5w_4w_2$.

Thus we are ready and we shall exhibit a list of the possible $10E_9$ transformations. We assume that all the $10E_9$ series in the formulas listed here satisfy the balancing condition (3.7).

The transformation corresponding to $\tau_1 = \text{id}$ is identical as $10E_9$ transformation. The transformation corresponding to $\tau_2$ is equivalent to the Bailey transformation due to Frenkel–Turaev (3.5). The third one corresponding to $\tau_3$ is equivalent to (3.4). The forth one ($\tau_4$) is

$$10E_9 \left(s; c_0, c_1, c_2, c_3, c_4, c_5, -N\delta\right) = \frac{[[\delta + s]]_N}{[[3\delta + 4s - c_0 - c_1 - c_2 - 2c_3 - 2c_4 - 2c_5]]_N} \times \prod_{0 \leq k \leq 2} \frac{[[c_{k+3}, \delta + s - c_0 - c_1 - c_2 + c_k]]_N}{[[\delta + s - c_k, \delta + s - c_{k+3}]]_N} 10E_9 \left(s; c_0, c_1, c_2, d_0, d_1, d_2, -N\delta\right), \quad (3.9)$$

$$\hat{s} = (1 - N)\delta + s - c_3 - c_4 - c_5 = 3\delta + 4s - c_0 - c_1 - c_2 - 2c_3 - 2c_4 - 2c_5,$$

$$\hat{c}_k = \delta + s - c_3 - c_4 - c_5 + c_k,$$

$$\hat{c}_{k+3} = -N\delta + c_k - s = 2\delta + 2s - c_0 - c_1 - c_2 - 2c_3 - 4c_4 - 5c_k, \quad k = 0, 1, 2.$$

Finally, the fifth one corresponding to $\tau_5$ is

$$10E_9 \left(s; c_0, c_1, c_2, c_3, c_4, c_5, -N\delta\right) = \frac{[[\delta + s]]_N}{[[4\delta + 5s - 2c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 - 2c_5]]_N} \times \prod_{0 \leq k \leq 5} \frac{[[c_k]]_N}{[[\delta + s - c_k]]_N} 10E_9 \left(s; \hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4, -N\delta\right), \quad (3.10)$$

$$\hat{s} = -2N\delta - s = 4\delta + 5s - 2c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 - 2c_5,$$

$$\hat{c}_k = -N\delta + c_k - s = 2\delta + 2s - c_0 - c_1 - c_2 - c_3 - 4c_4 - 5c_k, \quad k = 0, 1, 2, 3, 4, 5.$$

Note that (3.10) implies reversing order of the summation in $10E_9$ series.

### 3.3 Symmetry of $A_n$ Bailey transformations of rectangular type

Here we discuss the symmetry for two $A_n$ elliptic Bailey transformation formulas (3.13) and (3.14). The corresponding series is $A_n$ elliptic hypergeometric series of rectangular type, which the multiple series terminates with respect to a multi-index. Namely the $E^{n,3}$ series of the form

$$E^{n,3} \left(\{ -m_i \delta \}_n \mid s; c_0, c_1, c_2, d_0, d_1, d_2 \right) \quad (3.11)$$
with the balancing condition
\[(c_0 + c_1 + c_2) + (d_0 + d_1 + d_2) = (2 + |M|)\delta + 3s. \tag{3.12}\]

In [17], we obtained several $A_n$ generalizations of the elliptic Bailey transformation formula (3.5) for $E^{n,3}$ series by iterating (3.6) twice. Among these, here we give two transformations which $E^{n,3}$ series of rectangular type which we discuss here. These can be obtained in a similar way as in Section 3.2.

**$A_n$ Bailey transformation for $E^{n,3}$ of rectangular type (3.11)** (Theorem 4.2 in [17]). Suppose that $a_i = -m_i\delta$, $m_i \in \mathbb{N}$ for all $i = 1, \ldots, n$. For $c_k$, $d_k$, $k = 0, 1, 2$, suppose that the balancing condition (3.12). Then we have two types of $A_n$ Bailey transformation formula.

**$A_n$ Bailey I (Milne–Newcomb type)**

\[
E^{n,3}\left( \frac{\{-m_i\delta\}_n}{\{x_i\}_n} \mid s; c_0, c_1, c_2; d_0, d_1, d_2 \right) = \frac{[[\delta + s - c_1 - d_0, \delta + s - c_2 - d_0]|M]}{[[\delta + s - c_1, \delta + s - c_2]|M]}
\times \prod_{1 \leq i \leq n} \frac{[[\delta + s + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i]|m_i]}{[[\delta + s - d_0 + x_i, 2\delta + 2s - c_0 - d_0 - d_2 + x_i]|m_i]}
\times E^{n,3}\left( \frac{\{-m_i\delta\}_n}{\{x_i\}_n} \mid \tilde{s}; \tilde{c}_0, \tilde{c}_1, \tilde{c}_2; d_0, \tilde{d}_1, \tilde{d}_2 \right),
\tag{3.13}
\]

where
\[
\tilde{s} = \delta + 2s - c_0 - d_1 - d_2, \quad \tilde{c}_0 = \delta + s - d_1 - d_2, \quad \tilde{d}_1 = \delta + s - c_0 - d_2, \quad \tilde{d}_2 = \delta + s - c_0 - d_1.
\]

**$A_n$ Bailey II (Kajihara–Noumi type)**

\[
E^{n,3}\left( \frac{\{-m_i\delta\}_n}{\{x_i\}_n} \mid s; c_0, c_1, c_2; d_0, d_1, d_2 \right) = \prod_{1 \leq i \leq n} \frac{[[\delta + s + x_i, \delta + s - d_0 - d_1 + x_i]|m_i]}{[[\delta + s - d_0 + x_i, \delta + s - d_1 + x_i]|m_i]}
\times \frac{[[\delta + s - d_0 - d_2 + x_i, \delta + s - d_1 - d_2 + x_i]|m_i]}{[[\delta + s - d_2 + x_i, \delta + s - d_0 - d_1 + x_i]|m_i]}
\times E^{n,3}\left( \frac{\{-m_i\delta\}_n}{\{x_i\}_n} \mid \tilde{s}; \tilde{c}_0, \tilde{c}_1, \tilde{c}_2; d_0, d_1, d_2 \right),
\tag{3.14}
\]

where
\[
\tilde{s} = \delta + 2s - c_0 - c_1 - c_2, \quad \tilde{c}_0 = \delta + s - c_1 - c_2, \quad \tilde{c}_1 = \delta + s - c_0 - c_2, \quad \tilde{c}_2 = \delta + s - c_0 - c_1, \quad \tilde{x}_i = -m_i\delta - x_i + |M|\delta, \quad i = 1, \ldots, n.
\]

**Remark 3.2.** In the case when $n = 1$, $x_1 = 0$, both (3.13) and (3.14) reduce to the elliptic Bailey transformation for $10E_9$ series (3.5). Bailey I (3.13) is originally due to Rosengren [29] which is an elliptic version of $A_n$ Bailey transformation formula by Milne–Newcomb [25]. Bailey II (3.14) has originally appeared in our previous work [17] together with the basic case.

First, we shall show the invariance property for the transformations for $E^{n,3}$ series of rectangular type (3.11). Suppose that $n \geq 2$ till we will state otherwise. Recall that the transformations of coordinates in the right hand side of the Bailey I (3.13) and Bailey II (3.14) are described as
follows

\[
\begin{bmatrix}
  s \\
  c_0 \\
  c_1 \\
  c_2 \\
  d_0 \\
  d_1 \\
  d_2 \\
  \delta
\end{bmatrix} \mapsto
\begin{bmatrix}
  s^{1.0} \\
  c_0^{1.0} \\
  c_1^{1.0} \\
  c_2^{1.0} \\
  d_0^{1.0} \\
  d_1^{1.0} \\
  d_2^{1.0} \\
  \delta
\end{bmatrix} =
\begin{bmatrix}
  2s + \delta - c_0 - d_1 - d_2 \\
  s + \delta - d_1 - d_2 \\
  c_1 \\
  c_2 \\
  d_0 \\
  s + \delta - c_0 - d_2 \\
  s + \delta - c_0 - d_1
\end{bmatrix},
\]

\[
\begin{bmatrix}
  s \\
  c_0 \\
  c_1 \\
  c_2 \\
  d_0 \\
  d_1 \\
  d_2 \\
  \delta
\end{bmatrix} \mapsto
\begin{bmatrix}
  s^{0.1} \\
  c_0^{0.1} \\
  c_1^{0.1} \\
  c_2^{0.1} \\
  d_0^{0.1} \\
  d_1^{0.1} \\
  d_2^{0.1} \\
  \delta
\end{bmatrix} =
\begin{bmatrix}
  2s + \delta - c_0 - c_1 - c_2 \\
  s + \delta - c_1 - c_2 \\
  s + \delta - c_0 - c_2 \\
  s + \delta - c_0 - c_1 \\
  d_0 \\
  d_1 \\
  d_2
\end{bmatrix}.
\]

Note that these are compositions of linear transformations for parameters \(s, c_0, c_1, c_2, d_0, d_1, d_2\) and shift by \(\delta\), namely affine transformations of 7-dimensional vector space. Thus we give a realization for these transformations in terms of \(8 \times 8\) matrices acting on the vector \(\vec{v} = [s, c_0, c_1, c_2, d_0, d_1, d_2, \delta]\) as follows

\[
\begin{bmatrix}
  s^{1.0} \\
  c_0^{1.0} \\
  c_1^{1.0} \\
  c_2^{1.0} \\
  d_0^{1.0} \\
  d_1^{1.0} \\
  d_2^{1.0} \\
  \delta
\end{bmatrix} = B_1 \cdot \vec{v},
\]

\[
\begin{bmatrix}
  s^{0.1} \\
  c_0^{0.1} \\
  c_1^{0.1} \\
  c_2^{0.1} \\
  d_0^{0.1} \\
  d_1^{0.1} \\
  d_2^{0.1} \\
  \delta
\end{bmatrix} = B_2 \cdot \vec{v},
\]

where the matrix \(B_1\) is given by

\[
B_1 =
\begin{bmatrix}
  2 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \\
  1 & 0 & 0 & 0 & -1 & -1 & 1 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  1 & -1 & 0 & 0 & 0 & -1 & 1 \\
  1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

and \(B_2\) is given by

\[
B_2 =
\begin{bmatrix}
  2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
  1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
  1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
  1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
Proposition 3.2.

Since \( (3.12) \), one can check (3.17) in a similar fashion. ■

For the action on the variable \( x_i \), we set to be \( b_2 \cdot x_i = \tilde{x}_i = -m_i \delta - x_i + |M| \delta \) and identical otherwise. We denote \( I_n \) as the unit \( n \times n \) matrix.

We introduce the normalized elliptic hypergeometric series \( \tilde{E}^{n,3}((\vec{v}, x)) \) as follows

\[
\tilde{E}^{n,3}((\vec{v}, x)) := \frac{\prod_{0 \leq k \leq 2} ([\delta + s - c_k]|M|) \left( \prod_{1 \leq i \leq n} ([\delta + s - d_k + x_i]|m_i) \right)}{\prod_{1 \leq i \leq n} ([\delta + s + x_i]|m_i)} \times E^{n,3}\left( \{ -m_i \delta \}_{n} \big| s; c_0, c_1, c_2; d_0, d_1, d_2 \right). \tag{3.15}
\]

Now, we show an invariance property for \( E^{n,3} \) series of type (3.11).

**Proposition 3.2 (Hardy type invariant form for \( E^{n,3} \) series of type (3.11)).** Under the balancing condition (3.12), \( \tilde{E}^{n,3}((\vec{v}, x)) \) is invariant under the action of \( b_1, b_2, s_0, s_1, t_0 \) and \( t_1 \).

In the course of the proof of this proposition, we use the following lemma.

**Lemma 3.2.** If (3.12) holds, we have the following

1) \( [[a]|M] = (1)^{|M|}[[3s + 3\delta - c_0 - c_1 - c_2 - d_0 - d_1 - d_2 - a]|M|], \tag{3.16} \)

2) \( [[a + x_i]|m_i] = (1)^{|m_i|}[[3s + 3\delta - c_0 - c_1 - c_2 - d_0 - d_1 - d_2 - a + \tilde{x}_i]|m_i]. \tag{3.17} \)

**Proof.** Since \([a]|x\) is a odd function of \( x \),

\[
[[a]|M] = [[a]] [[a + \delta] \cdots [a + (|M| - 1)\delta]] = (1)^{|M|} [[-a + (1 - |M|)\delta] \cdots [-a]] = (1)^{|M|} [[-a + (1 - |M|)\delta]|M|].
\]

By the balancing condition (3.12), we have

\[-a + (1 - |M|)\delta = 3\delta + 3s - c_0 - c_1 - c_2 - d_0 - d_1 - d_2 - a.\]

Thus we have (3.16). Further, one can check (3.17) in a similar fashion. ■

**Proof of Proposition 3.2.** It is not hard to see in the case of \( s_0, s_1, t_0, t_1 \) since \( \tilde{E}^{n,3} \) (3.15) is symmetric with respect to the subscript \( k \). For the case of \( b_2 \),

\[
\tilde{E}^{n,3}(b_2 \cdot (\vec{v}, x)) = \tilde{E}^{n,3}((B_2\vec{v}, \tilde{x}))
\]
Remark 3.3. Recall that the variables $\tilde{x}_i$ for $b_i$ are defined as transformations arising from compositions of the generators of a Coxeter group. Let $\bar{\tilde{x}}_i = m_i - |M| - x_i$. Thus, $\bar{\tilde{x}}_i = m_i - |M| - x_i$. It is easy to see that by iterating twice,

$$\bar{\tilde{x}}_i = m_i - |M| - \bar{\tilde{x}}_i = x_i.$$ 

Here we used the transformation (3.14) and Lemma 3.2. For $b_1$, one can check similarly.

Here we shall investigate the compositions of the transformations $b_1$, $b_2$, $s_0$, $s_1$, $t_0$, and $t_1$.

**Lemma 3.3.** The relations $b_1^2 = b_2^2 = s_0^2 = s_1^2 = t_0^2 = t_1^2 = \text{id}$ hold (where id stands for identical as a transformation). Thus the set of the transformations $\{b_1, b_2, s_0, s_1, t_0, t_1\}$ constitutes the generators of a Coxeter group by compositions.

**Proof.** For $s_0$, $s_1$, $t_0$, and $t_1$, it is obvious since these are the permutation for the coordinates. For $b_1$ and $b_2$, we can check by direct computations of matrices $B_1$ and $B_2$ that $B_1^2 = B_2^2 = I_8$.

**Remark 3.3.** Recall that the variables $x_i$ in Bailey II (3.14) change to $\tilde{x}_i = m_i - |M| - x_i$. It is easy to see that by iterating twice,

$$\bar{\tilde{x}}_i = m_i - |M| - \bar{\tilde{x}}_i = x_i.$$ 

Thus we see that it turn out to be identity as transformation for $E_{n,3}$ series by iterating Bailey II (3.14) twice.

Let $G_r$ be the group generated by $b_1$, $b_2$, $s_0$, $s_1$, $t_0$, and $t_1$. Now we shall give the relations between the generators of the group $G_r$. By definition of $s_0$, $s_1$, $t_0$ and $t_1$, the following two braid relations hold:

$$(s_0s_1)^3 = (t_0t_1)^3 = \text{id}.$$ 

(3.18)

Note also that, for $i, j \in \{0, 1\}$, $s_i$ and $t_j$ mutually commute. Other relations, among $b_1$, $b_2$ and others, can be summarized as follows:

**Lemma 3.4.** The relations

$$(b_1s_0)^3 = (b_1t_0)^3 = \text{id}$$

hold. Other pairs of generators of $G_r$ commute. Especially, $b_2$ commutes with any other generators.
Then, by braid relations (3.18) and two lemmas above, we see that the following relation holds:

\[
\begin{align*}
&\begin{cases}
\sigma_i \neq \text{id}, & \sigma_i^2 = \text{id}, & \text{id} = 1, 2, 3, 4, 5,
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i = 1, 2, 3, 4,
\sigma_i\sigma_j = \sigma_j\sigma_i, & |i - j| \geq 2,
\end{cases}
\end{align*}
\]

and \(\tau^2 = \text{id}\). In other words, \(\{\sigma \}\) and \(\{\tau\}\) is a realization of \(S_6\) and \(G_2\). Thus we have

**Proposition 3.3.** The group \(G_r\) is isomorphic to the direct product of \(S_6\) and \(G_2\).

To summarize the results here, we state the following:

**Theorem 3.2.** Under the balancing condition (3.12), \(E^{m,3}((\vec{v}, \vec{x}))\) is invariant under the action of the direct product of \(S_6\) and \(G_2\) realized by the mapping \(\pi^{-1}\).

We are going to classifying non-trivial transformations for \(E^{m,3}\) of rectangular type (3.11) by using the realization \(\sigma_i\) and \(\tau\).

Proposition 3.3 tells us that the group \(G_r\) of the symmetry of \(E^{m,3}\) of type (3.11) is isomorphic to a direct product of the \(S_6\) and \(G_2\) and is of order \(6! \times 2! = 1440\). Recall again that \(E^{m,3}\) series of rectangular type is symmetric with respect to the \(c_k, k = 0, 1, 2, d_k, k = 0, 1, 2\). Then it is not hard to see that the the right action of \(\sigma_1, \sigma_2, \sigma_4\) and \(\sigma_5\) corresponds to the permutations of the subscript in the sets of parameters \(\{c_0, c_1, c_2\}\) and \(\{d_0, d_1, d_2\}\) and the left action corresponds to the permutation of the location of coordinates. Thus our problem turns out to give an orbit decomposition of the double coset \(H_r \setminus G_r / H_r\), where \(H_r\) is a subgroup generated by \(\sigma_1, \sigma_2, \sigma_4, \sigma_5\), which is isomorphic to a direct product of two \(\mathfrak{S}_3\). The representatives of orbits in \(H_r \setminus G_r / H_r\) are given by the following:

0) \(\omega_0 = \text{id}\),
1) \(\omega_1 = \sigma_3\),
2) \(\omega_2 = \sigma_3\sigma_4\sigma_2\omega_1 = \sigma_3\sigma_4\sigma_2\sigma_3\)
3) \(\omega_3 = \sigma_3\sigma_4\sigma_5\sigma_2\omega_1\omega_2 = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_3\sigma_4\sigma_2\sigma_3\).

Since the element \(\tau\) commutes with \(\sigma_i\) for all \(i = 1, 2, 3, 4, 5\), one can find that \(\tau\) is also commutative with all the representatives \(\omega_i\) for \(i = 0, 1, 2, 3\).

Before going to present a list of non-trivial possible transformations for \(E^{m,3}\) series of type (3.11), we define a transformation \(\pi^{r,t}\) as \(\pi^{r,t} := \tau^t\omega_r\) for \(r = 0, 1, 2, 3\) and \(t = 0, 1\). We call the transformation formula corresponding to \(\pi^{r,t}\) as \(T(r, t)\) and express it as follows

\[
E^{m,3}\left(\left\{\frac{1}{m_i}\delta\right\}_{x_i}^n \mid s; c_0, c_1, c_2; d_0, d_1, d_2\right) = \left\{\frac{1}{m_i}\delta\right\}_{x_i}^n \mid s^{r,t}, c_0^{r,t}, c_1^{r,t}, c_2^{r,t}; d_0^{r,t}, d_1^{r,t}, d_2^{r,t}\right),
\]

where \(s^{r,t}, c_k^{r,t}, d_k^{r,t} (k = 0, 1, 2)\) is parameters associated to the transformation \(\pi^{r,t}\) and \(P^{r,t}(x; s, C; D) = P^{r,t}(x; s, c_0, c_1, c_2; d_0, d_1, d_2)\) is the corresponding product factor. For the variables \(x_i\), we set \(x_i^0 = x_i\) and \(x_i^1 = (|M| - m_i)\delta - x_i\). Note that Bailey I (3.13) and Bailey II (3.14) correspond to \(T(1, 0)\) and \(T(0, 1)\) respectively. Note also that \(T(0, 0)\) is identical.
Here we exhibit a list of the product factors and transformations for parameters in $T(r, t)$. In order to simplify each product factor, we frequently use Lemma 3.2. Note that the expressions of each product factors have ambiguity because of the balancing condition (3.12).

$T(2, 0)$
- Product factor

$$P^{2,0}(x; s; C; D) = (-1)^{|M|} \frac{[d_2, \delta + s - c_2 - d_0, \delta + s - c_2 - d_1][M]}{[[\delta + s - c_0, \delta + s - c_1, \delta + s - c_2]][M]} \times \prod_{1 \leq i \leq n} \left[ \left[ \frac{[\delta + s + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i][M]}{[\delta + s - d_0 + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i][M]} \right] \times \left[ \frac{[2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i][M]}{[3\delta + 3s - c_0 - c_1 - d_0 - d_1 - 2d_2 + x_i][M]} \right] \right].$$

- Parameters

$$\begin{bmatrix} s^{2,0} \\ c_2^{2,0} \\ c_1^{2,0} \\ c_0^{2,0} \\ d_0^{2,0} \\ d_1^{2,0} \\ d_2^{2,0} \end{bmatrix} = \begin{bmatrix} 3s + 2\delta - c_0 - c_1 - d_0 - d_1 - 2d_2 \\ s + \delta - d_0 - d_2 \\ s + \delta - d_1 - d_2 \\ c_2 \\ s + \delta - c_0 - d_2 \\ s + \delta - c_1 - d_2 \\ 2s + 2\delta - c_0 - c_1 - d_0 - d_1 - d_2 \end{bmatrix}.$$

$T(3, 0)$
- Product factor

$$P^{3,0}(x; s; C; D) = (-1)^{|M|} \frac{[d_0, d_1, d_2][M]}{[[\delta + s - c_0, \delta + s - c_1, \delta + s - c_2]][M]} \times \prod_{1 \leq i \leq n} \left[ \left[ \frac{[\delta + s + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i][M]}{[\delta + s - d_0 + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i][M]} \right] \times \left[ \frac{[2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i][M]}{[\delta + s - d_2 + x_i, 4\delta + 4s - c_0 - c_1 - c_2 - 2d_0 - 2d_1 - 2d_2 + x_i][M]} \right] \right].$$

- Parameters

$$\begin{bmatrix} s^{3,0} \\ c_0^{3,0} \\ c_1^{3,0} \\ c_2^{3,0} \\ d_0^{3,0} \\ d_1^{3,0} \\ d_2^{3,0} \end{bmatrix} = \begin{bmatrix} 4s + 3\delta - c_0 - c_1 - c_2 - 2d_0 - 2d_1 - 2d_2 \\ s + \delta - d_0 - d_1 \\ s + \delta - d_0 - d_2 \\ s + \delta - d_1 - d_2 \\ 2s + 2\delta - c_0 - c_1 - d_0 - d_1 - d_2 \\ 2s + 2\delta - c_0 - c_2 - d_0 - d_1 - d_2 \\ 2s + 2\delta - c_1 - c_2 - d_0 - d_1 - d_2 \end{bmatrix}.$$

$T(1, 1)$
- Product factor

$$P^{1,1}(x; s; C; D) = \frac{[[\delta + s - c_1 - d_0, \delta + s - c_2 - d_0][M]]}{[[\delta + s - c_1, \delta + s - c_2]][M]} \times \prod_{1 \leq i \leq n} \left[ \left[ \frac{[\delta + s + x_i, c_0 + x_i][M]}{[\delta + s - d_0 + x_i, c_0 + x_i][M]} \right] \times \left[ \frac{[\delta + s - d_0 - d_2 + x_i, c_0 - d_0 + x_i][M]}{[\delta + s - d_2 + x_i, c_0 - d_0 + x_i][M]} \right] \right].$$
• Parameters

\[
\begin{bmatrix}
    s_{1,1}^{1,1} \\
    c_{0,1}^{1,1} \\
    c_{1,1}^{1,1} \\
    d_{0,1}^{1,1} \\
    d_{1,1}^{1,1}
\end{bmatrix}
\begin{bmatrix}
    3s + 2\delta - 2c_0 - c_1 - c_2 - d_1 - d_2 \\
    2s + 2\delta - c_0 - c_1 - c_2 - d_1 - d_2 \\
    s + \delta - c_0 - c_2 \\
    s + \delta - c_0 - c_1 \\
    d_0 \\
    s + \delta - c_0 - d_2 \\
    s + \delta - c_0 - d_1
\end{bmatrix}
\]

\[T(2, 1)\]

• Product factor

\[
P^{2,1}(x; s; C; D) = (-1)^{|M|} \frac{[\delta + s - c_2 - d_0, d_2, \delta + s - c_2 - d_1]_{|M|}}{[\delta + s - c_0, \delta + s - c_1, \delta + s - c_2]_{|M|}} \prod_{1 \leq i \leq n} \left[ \frac{[\delta + s + x_i, c_1 + x_i]_{m_i}}{[\delta + s - d_0 + x_i, \delta + s - d_1 + x_i]_{m_i}} \right] \frac{[c_0 + x_i, \delta + s - d_0 - d_1 + x_i]_{m_i}}{[\delta + s - d_2 + x_i, -\delta - s + c_0 + c_1 + d_2 + x_i]_{m_i}}.
\]

• Parameters

\[
\begin{bmatrix}
    s_{2,1}^{2,1} \\
    c_{0,1}^{2,1} \\
    c_{1,1}^{2,1} \\
    c_{2,1}^{2,1} \\
    d_{0,1}^{2,1} \\
    d_{1,1}^{2,1}
\end{bmatrix}
\begin{bmatrix}
    4s + 3\delta - 2c_0 - 2c_1 - c_2 - d_0 - d_1 - 2d_2 \\
    2s + 2\delta - c_0 - c_1 - c_2 - d_0 - d_2 \\
    2s + 2\delta - c_0 - c_1 - c_2 - d_1 - d_2 \\
    s + \delta - c_0 - c_1 \\
    s + \delta - c_0 - d_2 \\
    s + \delta - c_1 - d_2 \\
    2s + 2\delta - c_0 - c_1 - d_0 - d_1 - d_2
\end{bmatrix}
\]

\[T(3, 1)\]

• Product factor

\[
P^{3,1}(x; s; C; D) = (-1)^{|M|} \frac{[d_0, d_1, d_2]_{|M|}}{[\delta + s - c_0, \delta + s - c_1, \delta + s - c_2]_{|M|}} \prod_{1 \leq i \leq n} \left[ \frac{[\delta + s + x_i]_{m_i}}{[-2\delta - 2s + c_0 + c_1 + c_2 + d_0 + d_1 + d_2 + x_i]_{m_i}} \right] \frac{[c_0 + x_i, c_1 + x_i, c_2 + x_i]_{m_i}}{[\delta + s - d_0 + x_i, \delta + s - d_1 + x_i, \delta + s - d_2 + x_i]_{m_i}}.
\]

• Parameters

\[
\begin{bmatrix}
    s_{3,1}^{3,1} \\
    c_{0,1}^{3,1} \\
    c_{1,1}^{3,1} \\
    c_{2,1}^{3,1} \\
    d_{0,1}^{3,1} \\
    d_{1,1}^{3,1}
\end{bmatrix}
\begin{bmatrix}
    5s + 4\delta - 2c_0 - 2c_1 - 2c_2 - 2d_0 - 2d_1 - 2d_2 \\
    2s + 2\delta - c_0 - c_1 - c_2 - d_0 - d_1 \\
    2s + 2\delta - c_0 - c_1 - c_2 - d_0 - d_2 \\
    2s + 2\delta - c_0 - c_1 - c_2 - d_1 - d_2 \\
    2s + 2\delta - c_0 - c_1 - d_0 - d_1 - d_2 \\
    2s + 2\delta - c_0 - c_1 - d_0 - d_1 - d_2 \\
    2s + 2\delta - c_1 - c_2 - d_0 - d_1 - d_2 \\
    2s + 2\delta - c_1 - c_2 - d_0 - d_1 - d_2 \\
    2s + 2\delta - c_1 - c_2 - d_0 - d_1 - d_2 \\
\end{bmatrix}
\]

Note that by reversing the order of the summation for $E^{n,3}$ series, namely by replacing $\gamma_i \mapsto m_i - \gamma_i$ and simplifying the factors, we also obtain $T(3, 1)$. Note also that (3.14) can be obtained by combining $T(3, 0)$ and $T(3, 1)$. 
3.4 The case of triangular $E^{n,3}$ series

Here, we shall discuss triangular case. That is the case of $E^{n,3}$ series of the form

$$E_{n,3} \left( \begin{array}{c} \{a_i\}_n \\ \{x_i\}_n \end{array} \mid s; c_0, c_1, c_2; -N\delta, d_1, d_2 \right),$$

(3.19)

which terminates with respect to the length of multi-indices and is provided the balancing condition

$$\sum_{1 \leq i \leq n} a_i + c_0 + c_1 + c_2 + d_1 + d_2 = (2 + N)\delta + 3s.$$  

(3.20)

In this case, we have also obtained the following $A_n$ elliptic Bailey transformation formulas for $E^{n,3}$ series for triangular type (3.19) in [17].

$A_n$ Bailey transformations for $E^{n,3}$ series of triangular type (Theorem 4.1 in [17]). Under the balancing condition (3.20), we have two types of $A_n$ Bailey transformation formulas.

$A_n$ Bailey I

$$E_{n,3} \left( \begin{array}{c} \{a_i\}_n \\ \{x_i\}_n \end{array} \mid s; c_0, c_1, c_2; -N\delta, d_1, d_2 \right)$$

$$= \frac{[\delta + \bar{s} - c_1, \delta + \bar{s} - c_2]_N}{[\delta + s - c_1, \delta + s - c_2]_N} \prod_{1 \leq i \leq n} \frac{[\delta + s + x_i, \delta + \bar{s} + x_i - a_i]_N}{[\delta + s + x_i - a_i, \delta + \bar{s} + x_i]_N}$$

$$\times E_{n,3} \left( \begin{array}{c} \{a_i\}_n \\ \{x_i\}_n \end{array} \mid \bar{s}; \bar{c}_0, c_1, c_2; -N\delta, \bar{d}_1, \bar{d}_2 \right),$$

(3.21)

where

$$\bar{s} = \delta + 2s - c_2 - d_0 - d_1, \quad \bar{c}_0 = \delta + s - d_1 - d_2,$$

$$\bar{d}_1 = \delta + s - c_0 - d_2, \quad \bar{d}_2 = \delta + s - c_0 - d_1.$$

$A_n$ Bailey II

$$E_{n,3} \left( \begin{array}{c} \{a_i\}_n \\ \{x_i\}_n \end{array} \mid s; c_0, c_1, c_2; -N\delta, d_1, d_2 \right) = \prod_{1 \leq i \leq n} \frac{[\delta + s + x_i, \delta + s + x_i - d_1 - d_2]_N}{[\delta + s + x_i - a_i, \delta + s + x_i]_N}$$

$$\times \frac{[\delta + s + x_i - a_i - d_1, \delta + s + x_i - a_i - d_2]_N}{[\delta + s + x_i - a_i, \delta + s + x_i - d_1 - d_2]_N}$$

$$\times E_{n,3} \left( \begin{array}{c} \{a_i\}_n \\ \{x_i\}_n \end{array} \mid \bar{s}; \bar{c}_0, \bar{c}_1, \bar{c}_2; -N\delta, \bar{d}_1, \bar{d}_2 \right),$$

(3.22)

where

$$\bar{s} = \delta + 2s - c_0 - c_1 - c_2, \quad \bar{c}_0 = \delta + s - c_1 - c_2, \quad \bar{c}_1 = \delta + s - c_0 - c_2,$$

$$\bar{c}_2 = \delta + s - c_0 - c_1, \quad \bar{z}_i = a_i - x_i - |a|, \quad i = 1, \ldots, m.$$

Note that, in the case when $n = 1, x_1 = 0$, (3.21) and (3.22) reduce to the elliptic Bailey transformation formula (3.5).

Recall that, though the right hand side in Bailey I in rectangular case (3.13) contains two types of $d_j$’s: $\bar{d}_0 = d_0$ fixed and $\bar{d}_j = \delta + s - c_0 - d_1 - d_2 + d_j$, $j = 1, 2$, it consists of only $\bar{d}_j$ ($j = 1, 2$) in triangular case (3.21). Thus we find that, on the contrast to rectangular case, the element $t_0 = \sigma_4$ lacks in this case. Thus we have:
**Proposition 3.4.** The group describing the symmetry for the transformations (3.21) and (3.22) is isomorphic to $\mathcal{G}_4 \times (\mathcal{G}_2)^2$.

It is not hard to see that the composition of (3.21) and (3.22) is the only further non-trivial transformation which can be obtained

$$E^{n,3}\left(\begin{array}{c} \{a_i\}_n \\ \{x_i\}_n \end{array} \bigg| s; c_0, c_1, c_2; -N\delta, d_1, d_2 \right)$$

$$= \frac{[2\delta + s - c_0 - c_1 - d_1 - d_2, 2\delta + s - c_0 - c_2 - d_1 - d_2]_N}{[\delta + s - c_1, \delta + s - c_2]_N}$$

$$\times \prod_{1 \leq i \leq n} \frac{[[\delta + s + x_i, x_i + c_0]\_N}{[[\delta + s + x_i - a_i, x_i + c_0 - a_i]\_N}$$

$$\times \frac{[[\delta + s + x_i - d_1 - a_i, \delta + s + x_i - d_2 - a_i]\_N}{[[\delta + s + x_i - d_1, \delta + s + x_i - d_2]\_N}$$

$$\times E^{n,3}\left(\begin{array}{c} \{a_i\}_n \\ \{z_i\}_n \end{array} \bigg| \hat{s}; \hat{c}_0, \hat{c}_1, \hat{c}_2; -N\delta, \hat{d}_1, \hat{d}_2 \right), \quad (3.23)$$

where

$$\hat{s} = 2\delta + 3s - 2c_0 - c_1 - c_2 - d_1 - d_2, \quad \hat{c}_0 = 2\delta + 2s - c_0 - c_1 - c_2 - d_1 - d_2,$$

$$\hat{c}_1 = \delta + s - c_0 - c_2, \quad \hat{c}_2 = \delta + s - c_0 - c_1, \quad \hat{d}_1 = \delta + s - c_0 - d_2,$$

$$\hat{d}_2 = \delta + s - c_0 - d_1, \quad z_i = a_i - x_i - |a|, \quad i = 1, \ldots, n.$$

To simplify the product factor, we used the following lemma which can be proved just in the same line as in the rectangular case.

**Lemma 3.5.** If the balancing condition (3.20) holds, then we have

$$[b]_N = (-1)^N[[3\delta + 3s - b - (c_0 + c_1 + c_2) - (d_1 + d_2) - |a|]_N.$$

### 3.5 Remarks on results of Section 3

We close this paper to give some remarks.

**Remark 3.4.** The transformation $T(1,1)$ in Section 3.3 has appeared as Corollary 4.3 in Rosengren [30] with a different expression and the transformation (3.23) has appeared as Corollary 4.2 in [30].

**Remark 3.5** (in the case when $n = 1, x_1 = 0$). In this case, $T(2,0)$ and $T(1,1)$ in Section 3.3 and (3.23) in Section 3.4, reduce to (3.4) in Section 3.2. $T(3,0)$ and $T(2,1)$ reduce to (3.9). Finally, $T(3,1)$ reduces to (3.10). Notice that (3.10) can also be obtained by reversing order of the summation in the $10E_9$ series.

**Remark 3.6** (correspondence of the group $G_r$ in Section 3.3 and $G_1$ in Section 3.2). By direct computation using the matrix realization in this paper, one finds that $b_1 = \pi^{-1}(\sigma_3)$ can be expressed as

$$\nu^{-1}w_2\nu, \quad \nu = w_4w_5w_6w_3w_4w_5. \quad (3.24)$$
The correspondence between the generators of the group $G_r$ in Section 3.3. and the elements of the group $G_1$ is summarized as follows:

$$G_1 \cong W(E_6) \quad G_r \cong S_6 \times S_2$$

$$w_1 \leftrightarrow \sigma_2,$$

$$w_3 \leftrightarrow \sigma_1,$$

$$w_5 \leftrightarrow \sigma_4,$$

$$w_6 \leftrightarrow \sigma_5,$$

$$w_2 \leftrightarrow \tau,$$

$$\nu^{-1}w_2 \nu \leftrightarrow \sigma_3.$$  

The correspondence is described diagrammatically as follows:

$$n = 1$$

$$n \geq 2$$

Correspondence of the elements

Thus we find the group generated by $w_1, w_3, w_4, w_5$ and $\nu^{-1}w_2 \nu$ is isomorphic to the symmetric group $S_6$ and all the generators commute with $w_2$.

Furthermore, $r_\beta = \nu^{-1}w_2 \nu \in W(E_6)$ (3.24) is the reflection of the root $\beta = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. Note that $\beta$ is the highest root of the root system $D_5$ whose roots are $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\alpha_6$:

Extended Dynkin diagram of $D_5$

Note also that the expression in (3.24) of $r_\beta$ is reduced.
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