Consistency of equations of motion in conformal frames

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Four dimensional scalar-tensor theory is considered within two conformal frames, the Jor-
dan frame (JF) and the Einstein frame (EF). The actions for the theory are equivalent and
equations of motion can be obtained from each action. It is found that the JF equations of
motion, expressed in terms of EF variables, translate directly into and agree with the EF
equations of motion obtained from the EF action, provided that certain simple consistency
conditions are satisfied, which is always the case. The implication is that a solution set
obtained in one conformal frame can be reliably translated into a solution set for the other
frame, and therefore the two frames are, at least, mathematically equivalent.

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1. INTRODUCTION

The “frames issue” has long been in existence and has stirred much debate as to whether the
Jordan frame and Einstein frames are mathematically and physically equivalent, i.e., simply two
equivalent representations of the same theory or not. (See, for example, [1]-[4] and references
therein.) In the Jordan frame (JF) of a scalar-tensor theory a “dilaton” scalar field, say \( \varphi \), couples
nonminimally to the Ricci scalar \( \tilde{R}[\tilde{g}_{\mu\nu}] \) in the action, whereas a conformal transformation to the
Einstein frame (EF) results in an action where the dilaton decouples from the new Ricci scalar
\( R[g_{\mu\nu}] \), but becomes coupled to the matter sector \([5],[6]\). In addition, the lagrangian for the dilaton
changes form under the conformal transformation (see, e.g., [1]-[8]). The classic 1962 paper by
Dicke [9], for example, convincingly argues that, at least at the classical level, these are simply
two different representations of the same theory. This conclusion is based upon the
equivalence of the actions in the two different frames. This point of view has often been exploited to choose a
convenient frame where an analysis of the system can be done, with the reasonable assumption that
the solutions of the equations of motion can be reliably translated into the other frame. However,
a physical equivalence of the two frames has often been questioned or challenged (see, e.g., [2],[10]).

Here, we take a look at a four dimensional scalar-tensor theory of a fairly general form, similar
to that used by Flanagan [3] and Damour and Esposito-Farèse [6], and consider the action in both
frames. The JF and EF actions are equivalent, up to discarded surface terms. From each action
the equations of motion can be obtained. However, what is new here is that we then use the same
conformal transformation to translate one set of equations of motion into the variables of the other
frame. For example, we take the JF equations of motion, expressed in terms of the JF variables,
say \( \tilde{g}_{\mu\nu} \), \( \tilde{R}[\tilde{g}_{\mu\nu}] \), \( \tilde{R}_{\mu\nu} \), and \( \varphi \), and rewrite these JF equations of motion in terms of the EF variables
\( g_{\mu\nu} \), \( R[g_{\mu\nu}] \), \( R_{\mu\nu} \), and \( \varphi \). A comparison is then made between the set of translated equations of
motion and the action-based equations of motion, both expressed with the same variables. What
we find is that these sets of equations are the same, i.e., one set maps smoothly into the other, provided that simple consistency conditions are satisfied, which, as another new result, we show is always the case. So, along with the actions for the two frames being equivalent, up to discarded surface terms (i.e., the lagrangians are equivalent up to discarded total divergences), we find that the equations of motion are also equivalent. It is felt that these results further fortify the conclusion that the two frames are, at least, mathematically equivalent. Therefore, mathematical solutions in one frame can be reliably translated into the solutions of the other frame. Whether this implies a physical equivalence of frames, as well, has been debated (e.g., [1],[2],[3],[9]).

The present study is motivated by occasional questions of whether the JF and EF action-based equations of motion for a theory do in fact map into one another under a conformal transformation. In other words, does an equivalence of actions necessarily imply an equivalence of equations of motion in different frames? After all, a conformal transformation of the JF Ricci scalar term $\sqrt{\tilde{g}}\tilde{R}[\tilde{g}_{\mu\nu}]$ produces the EF term $\sqrt{g}R[g_{\mu\nu}]$ in the action, along with kinetic terms for $\varphi$, and total divergences in the lagrangian which are dropped. So, even though the actions are equivalent, one may wonder if the equations of motion, and their solutions, are always equivalent, given the mixing of the metric and dilaton fields.

It is shown that the JF matter field equations, the JF Einstein equation, and the JF dilaton field equation map into the corresponding EF equations, provided that certain consistency conditions are met. It is shown that these conditions are always met.

2. ACTION-BASED EQUATIONS OF MOTION

We assume a scalar-tensor theory action with the following forms (see, e.g., [11] for the case $\tilde{L}_m = 0$):

$$S = \int d^4x \sqrt{\tilde{g}} \left\{ \frac{F(\varphi)}{2\kappa^2} \tilde{R}[\tilde{g}_{\mu\nu}] + \frac{1}{2} \tilde{k}(\varphi)\tilde{g}^{\mu\nu} \partial_{\mu}\varphi \partial_{\nu}\varphi - V(\varphi) + \tilde{h}(\varphi)\tilde{L}_m(\tilde{g}^{\mu\nu}, \varphi, \psi) \right\}$$

$$= \int d^4x \sqrt{g} \left\{ \frac{1}{2\kappa^2} R[g_{\mu\nu}] + \frac{1}{2} k(\varphi)g^{\mu\nu} \partial_{\mu}\varphi \partial_{\nu}\varphi - U(\varphi) + h(\varphi)L_m(g^{\mu\nu}, \varphi, \psi) \right\}$$

where $\kappa = \sqrt{8\pi G}$ is the inverse of the reduced Planck mass, $\tilde{L}_m (L_m)$ is the matter field lagrangian in the JF (EF), and $\tilde{h} (h)$ is a dilatonic coupling function to matter fields. The “matter” fields are those fields other than the metric or dilaton, and are represented collectively by $\psi$. The terms $\tilde{h}\tilde{L}_m$ and $hL_m$ are independent of $\partial_{\mu}\varphi$, but we allow for the possibility that $\varphi$ couples to matter fields through a potential $W(\varphi, \psi)$ in the lagrangians $\tilde{L}_m$ and $L_m$. We also note that $\tilde{L}_m$ and $L_m$ are the same lagrangians, but expressed with different metrics for the kinetic terms. Various choices for the functions $F$, $k$, $V$, and $h$ can accommodate different types of scalar-tensor theory, including generalized Brans-Dicke theories and higher dimensional theories dimensionally reduced to four dimensions. The form (1a) is the JF representation of the action, and (1b) is the EF representation. The conformal transformation connecting the two frames is given by

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = F^{-1} g_{\mu\nu}, \quad F(\varphi) = e^\varphi, \quad \Omega(\varphi) = F^{-1/2}(\varphi) = e^{-\varphi/2}, \quad \ln \Omega = -\frac{1}{2} \varphi$$
where here we have chosen to parametrize $F$ by the exponential of a dimensionless scalar field $\varphi = \ln F$. (We can relate $\varphi$ to a scalar $\phi$ with canonical mass dimension by writing, e.g., $\varphi = \kappa \phi$.) Note that in the JF action $\tilde{g}_{\mu \nu}$ and $\varphi$ are treated as independent fields, but in the EF action $g_{\mu \nu}$ and $\varphi$ are treated as independent fields, and $\mathcal{L}_m(e^{\varphi}g^{\mu \nu}) = \tilde{\mathcal{L}}_m(\tilde{g}^{\mu \nu})$, so the kinetic terms in the matter lagrangian depend on different metrics and have different functional forms in the two frames.

We use a notation where $Q' = \partial Q/\partial \varphi = \partial_\varphi Q$, $Q'' = \partial^2 \varphi$, etc. for some function $Q(\varphi)$, and $(\partial \varphi)^2 = \partial_\mu \varphi \partial^\mu \varphi$, and $(\tilde{\partial} \varphi)^2 = \partial_\mu \varphi \partial^\mu \varphi = \tilde{g}^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi$. The JF functions $\tilde{k}$, $V$, $\tilde{h}$, $\tilde{\mathcal{L}}_m$ are left arbitrary, with forms determined by the particular action being considered, and the parametrization $F = e^\varphi$. Therefore the EF functions $k$, $U$, $h$, and $\mathcal{L}_m$ are also arbitrary. The conformal transformation gives the functions $k(\varphi)$ and $U(\varphi)$ of the EF action to be (3) and (4) such that

$$k = \frac{1}{\kappa^2} \left( \frac{3F''}{2F^2} + \frac{\kappa^2 \tilde{k}}{F} \right), \quad U(\varphi) = V(\varphi) F(\varphi)$$

By (2) we have $F' = F'' = F$ and we define the functions $\tilde{C}(\varphi) = \kappa^2 \tilde{k}(\varphi)$ and $C(\varphi) = \kappa^2 k(\varphi)$, so that (2) and (3) give

$$\tilde{g}_{\mu \nu} = e^{-\varphi} g_{\mu \nu}, \quad \tilde{g}^{\mu \nu} = e^{\varphi} g^{\mu \nu}, \quad C = \left( \frac{3}{2} + \tilde{C} e^{-\varphi} \right), \quad U(\varphi) = e^{-2\varphi} V(\varphi).$$

We can write the action $S$ in the forms $S = \int d^4 x \sqrt{\tilde{g}} \tilde{\mathcal{L}} = \int d^4 x \sqrt{g} \mathcal{L}$ and look at the variation of $S$ with respect to matter fields, the metric, and the scalar dilaton field to obtain the action-based equations of motion. Since $\sqrt{\tilde{g}} \tilde{\mathcal{L}} = \sqrt{g} \mathcal{L}$ we can use the result

$$\tilde{\mathcal{L}} = \sqrt{\tilde{g}} \mathcal{L} = F^2 \mathcal{L}$$

where use has been made of (2) with

$$\tilde{g}_{\mu \nu} = F^{-1} g_{\mu \nu}, \quad \tilde{g}^{\mu \nu} = F g^{\mu \nu}, \quad \sqrt{\tilde{g}} = F^{-2} \sqrt{g}$$

### 2.1. Matter fields

The equation of motion (EoM) for a matter field $\psi$ is obtained by varying $S$ with respect to $\psi$, i.e., $\delta_\psi S = 0$. We then obtain the JF and EF EoM given, respectively, by

$$\tilde{\nabla}_\mu \left[ \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_\mu \psi)} \right] - \frac{\partial \tilde{\mathcal{L}}}{\partial \psi} = 0 \quad (7a)$$
$$\nabla_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right] - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (7b)$$

where $\tilde{\nabla}_\mu$ and $\nabla_\mu$ are covariant derivatives with respect to $\tilde{g}_{\mu \nu}$ and $g_{\mu \nu}$, respectively, and $\nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left[ \sqrt{g} g^{\mu \nu} \nabla_\nu V^\mu \right]$ with a similar expression holding for $\tilde{\nabla}_\mu V^\mu$. 
2.2. Dilaton and Einstein equations

The Jordan frame equations of motion that follow from the action $S$ in (1a) due to variations with respect to $\hat{g}^{\mu\nu}$ and $\phi$ are, for arbitrary functions, $\hat{k}(\phi)$, $V(\phi)$, and $\tilde{h}(\phi)$, given by (see (11), for example, for the case $\tilde{\mathcal{L}}_m \to 0$)

$$\tilde{G}_{\mu\nu} = -\tilde{\nabla}_\mu \partial_\nu \phi - \left(1 + \tilde{C} e^{-\phi}\right) \partial_\mu \phi \partial_\nu \phi$$

$$+ \tilde{g}_{\mu\nu} \left\{ \left(1 + \frac{1}{2} \tilde{C} e^{-\phi}\right) (\partial \phi)^2 + \tilde{\Box} \phi - \kappa^2 e^{-\phi} \tilde{V} \right\} - \kappa^2 e^{-\phi} \tilde{h} \tilde{T}_{\mu\nu}$$  \hspace{1cm} (8)

$$\tilde{C} \tilde{\Box} \phi + \frac{1}{2} \tilde{C}'(\delta \phi)^2 + \kappa^2 \tilde{V}' - \frac{1}{2} \delta \phi \tilde{R} - \kappa^2 \partial \phi (\tilde{h} \tilde{L}_m) = 0$$  \hspace{1cm} (9)

where $\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R}$ is the JF Einstein tensor and we have used $F = e^\phi$ along with the definition $\tilde{C} = \kappa^2 \tilde{k}$, and

$$\tilde{T}_{\mu\nu} = \frac{2}{\sqrt{\tilde{g}}} \frac{\partial (\sqrt{\tilde{g}} \tilde{L}_m)}{\partial \tilde{g}^{\mu\nu}}, \ \ \ T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\partial (\sqrt{g} L_m)}{\partial g^{\mu\nu}}$$  \hspace{1cm} (10)

is the stress-energy tensor of the matter in the JF and EF, respectively. Now, (9) can be rewritten by taking the trace of (8), with $\tilde{R} = -\tilde{G} = -\tilde{g}^{\mu\nu} \tilde{G}_{\mu\nu}$, and inserting this into (9). We then have the JF dilaton EoM

$$\left(3 e^\phi + \tilde{C}\right) \tilde{\Box} \phi + \left[\left(3 e^\phi + \frac{1}{2} (\tilde{C} + \tilde{C}')\right) (\partial \phi)^2 + \kappa^2 e^{2\phi} U' - \kappa^2 \left[\partial \phi (h \tilde{L}_m) + \frac{1}{2} h \tilde{T}\right]\right] = 0$$  \hspace{1cm} (11)

where we have used the fact that $V' - 2V = e^{2\phi} U'$.

We can obtain the Einstein frame equations of motion from the action $S$ in (1b) by making the replacements $F \to \tilde{F} = 1$, $F' \to \tilde{F}' = 0$, $\hat{k} \to k(\phi)$, $\tilde{C} \to C$, $\tilde{h} \to h$, $\tilde{L}_m \to \tilde{L}_m$, $\tilde{g}_{\mu\nu} \to g_{\mu\nu}$, $\tilde{R}_{\mu\nu} \to R_{\mu\nu}$, $\tilde{R} \to R$, and $V \to U$ in the equations of (8) and (9), where $C(\phi)$ and $U(\phi)$ are given by (1a) with $C' = e^{-\phi} (\tilde{C}' - \tilde{C})$. Doing so, we obtain the EF equations of motion:

$$G_{\mu\nu} = \left(3 e^\phi + \tilde{C} e^{-\phi}\right) \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left\{ \frac{3}{2} e^\phi + \tilde{C} e^{-\phi}\right\} (\partial \phi)^2 - \kappa^2 U \right\} - \kappa^2 h T_{\mu\nu}$$  \hspace{1cm} (12)

$$\left(3 e^\phi + \tilde{C} e^{-\phi}\right) \tilde{\Box} \phi + \frac{1}{2} e^{-\phi} \left(\tilde{C}' - \tilde{C}\right) (\partial \phi)^2 + \kappa^2 \tilde{U}' - \kappa^2 \partial \phi (h \tilde{L}_m) = 0$$  \hspace{1cm} (13)

We now have the action-based Jordan frame equations of motion given by (7a), (8), and (11), derived from the action (1a), and the action-based Einstein frame equations of motion given by (12), (13), and (1a) derived from the action (1b). The matter that we now want to investigate is whether equations (7a), (8), and (11) will map directly into equations (12), (13), and (1a) under the conformal transformation.
3. TRANSLATION AND CONSISTENCY OF EQUATIONS OF MOTION

We now wish to use the conformal transformation of (2) and (6) to rewrite the Jordan frame equations of motion in terms of Einstein frame variables and then compare the translated Jordan frame equations to the Einstein frame equations obtained from the Einstein frame action. We point out that we use the signs and conventions of [7], and our metric $g_{\mu\nu}$ has opposite signature to that of [12] and [13], and our Ricci tensor $R_{\mu\nu}$ is minus that of [12] and [13], while our Ricci scalar $R$ has the same sign as that of [12] and [13], with $R_{\mu\nu} = \partial_{\nu}\Gamma_{\mu\lambda} - \partial_{\lambda}\Gamma_{\mu\nu} - \Gamma_{\mu\rho}\Gamma_{\rho\nu} + \Gamma_{\nu\sigma}\Gamma_{\mu\rho}$ and $R = g^{\mu\nu}R_{\mu\nu}$.

**Translation.** From (1), with the use of (5), we find that the matter equation (7a) directly yields (7b). On the other hand, some algebra (see Appendix) yields the translated form of (11) as

$$\left(\frac{3}{2} + \tilde{C}e^{-\varphi}\right)\Box\varphi + \frac{1}{2}e^{-\varphi}(\tilde{C}' - \tilde{C})(\partial\varphi)^2 + \kappa^2 U' - \kappa^2 e^{-2\varphi}\left[\partial_{\varphi}(\tilde{h}\tilde{L}_m) + \frac{1}{2}\tilde{h}\tilde{T}\right] = 0$$

and the translated form of (8) is

$$G_{\mu\nu} = -\left(\frac{3}{2} + \tilde{C}e^{-\varphi}\right)\partial_{\mu}\varphi\partial_{\nu}\varphi + \frac{1}{2}g_{\mu\nu}\left(\frac{3}{2} + \tilde{C}e^{-\varphi}\right)(\partial\varphi)^2 - g_{\mu\nu}\kappa^2 U - \kappa^2 e^{-2\varphi}\tilde{h}\tilde{T}_{\mu\nu}$$

These are the translated JF equation of motion for the dilaton $\varphi$ and the translated JF Einstein equation written in terms of the EF metric $g_{\mu\nu}$. Upon comparing (11) to (14) and (8) to (15), we see that the EF equations of motion coincide with the translated JF ones provided that

$$\partial_{\varphi}(h\tilde{L}_m) = e^{-2\varphi}\left[\partial_{\varphi}(\tilde{h}\tilde{L}_m) + \frac{1}{2}\tilde{h}\tilde{T}\right]$$

and

$$e^{-\varphi}\tilde{h}\tilde{T}_{\mu\nu} = hT_{\mu\nu}$$

Therefore, (16) and (17) serve as consistency conditions for the action-based JF and EF equations of motion to coincide. These conditions are found to always be satisfied.

**Consistency.** Let us first look at (16). Here we note that in obtaining the JF EoM for $\varphi$ we treat $\varphi$ as independent of $\tilde{g}_{\mu\nu}$, whereas in obtaining the EF EoM for $\varphi$ we treat $\varphi$ as independent of $g_{\mu\nu}$. In other words, in evaluating $\partial_{\varphi}\tilde{L}_m = \tilde{L}_m'$ and $\partial_{\varphi}L_m = L'_m$ we mean

$$\tilde{L}_m' = \partial_{\varphi}\tilde{L}_m(\tilde{g}^{\mu\nu}, \varphi)|_{\tilde{g}} = -W'(\varphi)$$

$$L'_m = \partial_{\varphi}L_m(e^{\varphi}g^{\mu\nu}, \varphi)|_g = \frac{\partial L_m}{\partial g^{\mu\nu}}\frac{\partial(e^{\varphi}g^{\mu\nu})}{\partial \varphi} - W'(\varphi) = \tilde{g}^{\mu\nu}\frac{\partial L_m}{\partial \tilde{g}^{\mu\nu}} - W'$$

where $|_{\tilde{g}}$ means evaluate at constant $\tilde{g}^{\mu\nu}$ and $|_g$ means evaluate at constant $g^{\mu\nu}$. Now we can rewrite the right hand side of (16) in terms of EF variables. To do this, we compare the dilaton-matter part of the action in (1a) and (1b) where we see that $\sqrt{\tilde{g}h} = \sqrt{\tilde{g}h}$, using the fact that
\[ \tilde{\mathcal{L}}_m(\tilde{g}^{\mu\nu}) = \mathcal{L}_m(e^{\phi}g^{\mu\nu}), \text{ and therefore} \]
\[ \tilde{h} = \frac{\sqrt{\tilde{g}}}{{\sqrt g}} h = e^{2\phi} h, \quad \tilde{h}\tilde{\mathcal{L}}_m = e^{2\phi} h \mathcal{L}_m \quad (19) \]

Furthermore, using \( \sqrt{\tilde{g}}\tilde{h}\tilde{\mathcal{L}}_m(\tilde{g}^{\mu\nu}) = \sqrt{\tilde{g}}h\mathcal{L}_m(e^{\phi}g^{\mu\nu}) \) and the definitions of \( \tilde{T}_{\mu\nu} \) and \( T_{\mu\nu} \) in (10), it follows that
\[ \tilde{h}\tilde{T}_{\mu\nu} = \frac{2\tilde{h}}{\sqrt{\tilde{g}}} \partial(\sqrt{\tilde{g}}\tilde{\mathcal{L}}_m) = \frac{2he^{2\phi}}{\sqrt{g}} \partial(\sqrt{g}\mathcal{L}_m) = \frac{2he^{2\phi}}{\sqrt{g}} \partial(\sqrt{g}\mathcal{L}_m) = e^{2\phi} h T_{\mu\nu} \quad (20) \]

We therefore have the results
\[ \tilde{h}\tilde{T}_{\mu\nu} = e^{2\phi} h T_{\mu\nu}, \quad \tilde{h}\bar{T} = e^{2\phi} h \bar{T} \quad (21) \]

where \( \tilde{g}^{\mu\nu} = e^{\phi} g^{\mu\nu} \) has been used. We note that (21) guarantees that the consistency condition (17) is satisfied.

Next, with the help of (18), (19), and (21), we can rewrite the consistency condition (16) as
\[ \mathcal{L}'_{m\mid g} = \left(2\mathcal{L}_m + \frac{T}{2}\right) + \tilde{\mathcal{L}}'_{m\mid \tilde{g}} \implies \tilde{g}^{\mu\nu} \partial\mathcal{L}_m \partial g^{\mu\nu} - W' = \left(2\mathcal{L}_m + \frac{T}{2}\right) - W'(\phi) \quad (22) \]

where \( \mathcal{L}'_m = \partial_\phi \mathcal{L}_m \) and (18) has been used. If the condition (22) is satisfied, then the EoM for \( \phi \) in the JF will be faithfully translated into the action-based EF EoM for \( \phi \). Also note that this condition is independent of the dilaton coupling functions \( \tilde{h} \) and \( h \), and depends only upon the matter lagrangian and the trace of the stress-energy tensor derived from it.

In fact, we can see that the condition (22) holds, in general. This follows from the definition of the stress-energy tensor (10) along with (19) and (21). We note that
\[ \mathcal{L}'_{m\mid g} = \frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}} \frac{\partial(e^{\phi}g^{\mu\nu})}{\partial \phi} - W' = \tilde{g}^{\mu\nu} \frac{\partial\mathcal{L}_m}{\partial \tilde{g}^{\mu\nu}} - W' \\
= \left(\frac{1}{2} \bar{T} + 2\mathcal{L}_m\right) - W' = \left(\frac{1}{2} T + 2\mathcal{L}_m\right) - W' \quad (23) \]

where we have used \( \tilde{T}_{\mu\nu} = 2\frac{\partial\mathcal{L}_m}{\partial \tilde{g}^{\mu\nu}} - \tilde{g}_{\mu\nu} \mathcal{L}_m \) and \( \bar{T} = 2\tilde{g}^{\mu\nu} \frac{\partial\mathcal{L}_m}{\partial \tilde{g}^{\mu\nu}} - 4\mathcal{L}_m \). From (19) \( \tilde{h} = e^{2\phi} h \), which when combined with (21) yields \( \bar{T} = T \). Therefore, the consistency condition (16), or equivalently (22), is satisfied, in general, for any matter lagrangian (that does not depend upon derivatives of the metric). We conclude that the consistency conditions (16) and (17) hold, in general, so that the translation of the JF EoM, derived from the JF action (1a), under the conformal transformation (2), will coincide with the EF EoM derived from the EF action (1b).
4. SUMMARY

We have considered a scalar-tensor theory of a general form given by (1a) and (1b). The Jordan frame and Einstein frame actions are equivalent, and a complete set of equations of motion can be derived from the action in each frame. Using the conformal transformation, the set of equations of motion can be transcribed from the JF to the EF, i.e., the JF equations of motion are written in terms of EF variables. It was shown that the JF field equations for the matter fields map onto the EF field equations for those fields, and that the JF Einstein equation and JF dilaton equation map onto the corresponding EF equations, provided that consistency conditions, given by (16) and (17), are satisfied. It was shown that these conditions are always satisfied, so that a solution set obtained in one conformal frame can be translated, with confidence, via the conformal transformation, to the other frame.

Appendix A

Here we list some useful relations for the translation of equations of motion. We start by writing the Jordan frame and Einstein frame covariant derivatives, respectively,

$$\tilde{\nabla}_\mu B_\nu = \partial_\mu B_\nu - \tilde{\Gamma}^\lambda_{\mu\nu} B_\lambda, \quad \nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma^\lambda_{\mu\nu} B_\lambda$$

(A1)

with $B_\nu = \partial_\nu \varphi$. The JF connection $\tilde{\Gamma}^\lambda_{\mu\nu}$, under the conformal transformation $\tilde{\Omega}^2 g_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega = e^{-\varphi/2}$, is related to the EF connection $\Gamma^\lambda_{\mu\nu}$ by

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \Omega^{-1} \left[ \delta^\lambda_\mu \partial_\nu \varphi + \delta^\lambda_\nu \partial_\mu \varphi - g_{\mu\nu} \partial^\lambda \varphi \right]$$

(A2)

Then (A1) and (A2) give

$$\tilde{\nabla}_\mu \partial_\nu \varphi = \partial_\mu \partial_\nu \varphi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \varphi + \frac{1}{2} \left[ \delta^\lambda_\mu \partial_\nu \varphi + \delta^\lambda_\nu \partial_\mu \varphi - g_{\mu\nu} \partial^\lambda \varphi \right] \partial_\lambda \varphi$$

$$= \left( \partial_\mu \partial_\nu \varphi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \varphi \right) + \frac{1}{2} \left[ 2 \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \partial^\lambda \varphi \partial_\lambda \varphi \right]$$

$$= \nabla_\mu \partial_\nu \varphi + \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} (\partial \varphi)^2$$

(A3)

We also have

$$\Box \varphi = \frac{1}{\sqrt{\tilde{g}}} \partial_\mu \left[ \sqrt{\tilde{g}} g^{\mu\nu} \partial_\nu \varphi \right] = e^\varphi [\Box \varphi - (\partial \varphi)^2], \quad (\partial \varphi)^2 = \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = e^\varphi (\partial \varphi)^2$$

(A4)

Also

$$\tilde{R}_{\mu\nu} = \tilde{R}_{\mu\nu} + 2 \nabla_\mu \partial_\nu \ln \Omega + g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \partial_\beta \ln \Omega$$

$$- 2 (\partial_\mu \ln \Omega) \partial_\nu \ln \Omega + 2 g_{\mu\nu} g^{\alpha\beta} (\partial_\alpha \ln \Omega) \partial_\beta \ln \Omega$$

(A5)
or, since $\ln \Omega = -\frac{1}{2} \varphi$,

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \nabla_\mu \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \Box \varphi - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} g_{\mu\nu} (\partial \varphi)^2$$  \hspace{1cm} (A6)

which leads to

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - \nabla_\mu \partial_\nu \varphi + g_{\mu\nu} \Box \varphi - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4} g_{\mu\nu} (\partial \varphi)^2$$  \hspace{1cm} (A7)