Computability on the countable ordinals and the Hausdorff-Kuratowski theorem

Arno Pauly
Clare College
University of Cambridge, United Kingdom
Arno.Pauly@cl.cam.ac.uk

In this note, we explore various potential representations of the set of countable ordinals. An equivalence class of representations is then suggested as a standard, as it offers the desired closure properties. With a decent notion of computability on the space of countable ordinals in place, we can then state and prove a computable uniform version of the Hausdorff-Kuratowski theorem.

1 Introduction

This note continues a research programme to investigate concepts from descriptive set theory in the very general setting of represented spaces, and in a fashion that produces both classical and effective results simultaneously. A survey of this approach is given in [31]. One of the first theorems studied in this way is the Jayne-Rogers theorem ([18], simplified proof in [26]); a computable version holding also in some non-Hausdorff spaces was proven by the author and de Brecht in [33] using results about Weihrauch reducibility in [1].

Our goal for this note is to state and prove a corresponding version of the Hausdorff-Kuratowski theorem. For this, we require a notion of computability on the space of countable ordinals – and such a theory would be foundational for several further results in the research programme. Apart from some initial investigations in [23], there is no established definition of a computability structure on the countable ordinals. We will investigate some promising candidates, and suggest one equivalence class as the standard to be adopted. With this in place, we can then fulfill our original goal.

1.1 Represented spaces

We shall briefly introduce the notion of a represented space, which underlies computable analysis [40]. For a more detailed presentation we refer to [30]. A represented space is a pair $X = (X, \delta_X)$ of a set $X$ and a partial surjection $\delta_X : \subseteq \mathbb{N}^\mathbb{N} \to X$ (the representation). A represented space is called complete, if its representation is a total function.

A multi-valued function between represented spaces is a multi-valued function between the underlying sets. For $f : \subseteq X \Rightarrow Y$ and $F : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, we call $F$ a realizer of $f$ (notation

1There is, of course, a well-established notion of what a computable ordinal is, however, this does not suffice for our purposes. Also, the attempts to extend some sense of effectivity beyond $\omega^{CK}$ using the machinery of local computability as done e.g. in [9] do not help our quest.
Countable ordinals and Hausdorff-Kuratowski

\[ F \vdash f \), iff \( \delta_Y(F(p)) \in f(\delta_X(p)) \) for all \( p \in \text{dom}(f\delta_X) \).

\[
\begin{array}{ccc}
\mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\
\downarrow{\delta_X} & & \downarrow{\delta_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

A map between represented spaces is called computable (continuous), iff it has a computable (continuous) realizer. Similarly, we call a point \( x \in X \) computable, iff there is some computable \( p \in \mathbb{N}^\mathbb{N} \) with \( \delta_X(p) = x \).

Given two represented spaces \( X, Y \) we obtain a third represented space \( C(X,Y) \) of functions from \( X \) to \( Y \) by letting \( 0^p1p \) be a \( [\delta_X \rightarrow \delta_Y] \)-name for \( f \), if the \( n \)-th Turing machine equipped with the oracle \( p \) computes a realizer for \( f \). As a consequence of the UTM theorem, \( C(-,-) \) is the exponential in the category of continuous maps between represented spaces, and the evaluation map is even computable (as are the other canonic maps, e.g. currying).

Based on the function space construction, we can obtain the hyperspaces of open \( O \), closed \( A \), overt \( V \) and compact \( K \) subsets of a given represented space using the ideas of synthetic topology \[8\].

1.2 Weihrauch reducibility

Several of our results are negative, i.e. show that certain operations are not computable. We prefer to be more precise, and not to merely state failure of computability. Instead, we give lower bounds for Weihrauch reducibility. The reader not interested in distinguishes degrees of non-computability may skip the remainder of the subsection, and in the rest of the paper (with the exception of Section 9), read any statement involving Weihrauch reducibility \( \leq_W, \equiv_W, <_W \) as indicating the non-computability of the maps involved.

**Definition 1** (Weihrauch reducibility). Let \( f, g \) be multi-valued functions on represented spaces. Then \( f \) is said to be Weihrauch reducible to \( g \), in symbols \( f \leq_W g \), if there are computable functions \( K, H : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \) such that \( K(\text{id}, GH) \vdash f \) for all \( G \vdash g \).

The relation \( \leq_W \) is reflexive and transitive. We use \( \equiv_W \) to denote equivalence regarding \( \leq_W \), and by \( <_W \) we denote strict reducibility. By \( \mathcal{W} \) we refer to the partially ordered set of equivalence classes. As shown in \[29, 8\], \( \mathcal{W} \) is a distributive lattice. The algebraic structure on \( \mathcal{W} \) has been investigated in further detail in \[17, 5\].

A prototypic non-computable function is \( \text{LPO} : \mathbb{N}^\mathbb{N} \rightarrow \{0,1\} \) defined via \( \text{LPO}(0^N) = 1 \) and \( \text{LPO}(p) = 0 \) for \( p \neq 0^N \). The degree of this function was already studied by \textsc{Weihrauch} \[39\].

A few years ago several authors (\textsc{Gherardi} and \textsc{Marcone} \[10\], \textsc{P.} \[29, 28\], \textsc{Brattka} and \textsc{Gherardi} \[2\]) noticed that Weihrauch reducibility would provide a very interesting setting for a metamathematical inquiry into the computational content of mathematical theorems. The fundamental research programme was outlined in \[2\], and the introduction in \[4\] may serve as a recent survey.

2 Representations of the space of countable ordinals

We shall investigate several representations of the set of all countable ordinals (to be denoted by \( \text{CO}r\)), and identity their equivalence classes up to computable translations. Along the way,
we shall see how the representations of the countable ordinals restrict to the finite ordinals, and compare to established representations of the natural numbers. Theorem 16 will establish a number of candidates as equivalent, and we shall tentatively propose to consider these the standard representations of \( \text{COrd} \). An investigation of which operations on the countable ordinals are computable is postponed until Section 3.

Our first candidate is a straightforward adaption of Kleene’s notation 20 of the recursive ordinals to a representation of the countable ordinals.

**Definition 2.** We define \( \delta_K : \subseteq \mathbb{N}^N \to \text{COrd} \) inductively via:

1. \( \delta_K(0p) = 0 \)
2. \( \delta_K(1p) = \delta_K(p) + 1 \)
3. \( \delta_K(2\langle p_0, p_1, p_2, \ldots \rangle) = \sup_{i \in \mathbb{N}} \delta_K(p_i), \) provided that \( \forall i \in \mathbb{N} \delta_K(p_i) < \delta_K(p_{i+1}). \)

A potential modification of the preceding definition that immediately comes to mind would be to drop the restriction of sup’s to increasing sequences. We thus arrive at:

**Definition 3.** We define \( \delta_{nk} : \subseteq \mathbb{N}^N \to \text{COrd} \) inductively via:

1. \( \delta_{nk}(0p) = 0 \)
2. \( \delta_{nk}(1p) = \delta_{nk}(p) + 1 \)
3. \( \delta_{nk}(2\langle p_0, p_1, p_2, \ldots \rangle) = \sup_{i \in \mathbb{N}} \delta_{nk}(p_i). \)

A third definition proceeding along similar lines can be extracted from Moschovakis’ definition of the Borel codes in [24, 12]:

**Definition 4.** We define \( \delta_M : \subseteq \mathbb{N}^N \to \text{COrd} \) inductively via:

1. \( \delta_M(0p) = 0 \)
2. \( \delta_M(1\langle p_0, p_1, p_2, \ldots \rangle) = \sup_{i \in \mathbb{N}} (\delta_M(p_i) + 1). \)

Another scheme to obtain representations of the countable ordinals starts with the view of countable ordinals as the heights of countable wellfounded relations. A countable relation is given by two sets \( A \subseteq \mathbb{N} \) and \( R \subseteq \mathbb{N} \times \mathbb{N}, \) where \( A \) denotes which points are present, and then \( R \) provides the order relation. There are three common spaces of subsets of \( \mathbb{N}, \) the open subsets \( \mathcal{O}(\mathbb{N}), \) the closed subsets \( \mathcal{A}(\mathbb{N}) \) or the clopens \( \mathcal{O}(\mathbb{N}) \land \mathcal{A}(\mathbb{N}). \) The computable points in these spaces are the recursively enumerable, the co-recursively enumerable and the decidable subsets of \( \mathbb{N} \) respectively. Thus, we arrive at a number of representations:

**Definition 5.** Let \( X, Y \in \{ \mathcal{O}(\mathbb{N}), \mathcal{A}(\mathbb{N}), \mathcal{O}(\mathbb{N}) \land \mathcal{A}(\mathbb{N}) \}. \) We define a representation \( \delta^{X,Y}_R : \subseteq \mathbb{N}^N \to \text{COrd} \) by \( \delta^{X,Y}_R(\langle p, q \rangle) = \alpha, \) iff \( \alpha \) is the order-type of the structure \( (A, \prec), \) where \( p \) is an \( X \)-name for \( A, q \) an \( Y \)-name for \( R, \) and \( \forall i, j \in A \ (i \prec j \iff \langle i, j \rangle \in R). \)

Potentially, it would appear to be more appropriate to consider countable ordinals as order types of countable wellorders, rather than just heights of wellfounded orders. This is the approach taken by Hamkins and Li [23].

**Definition 6.** Let \( X, Y \in \{ \mathcal{O}(\mathbb{N}), \mathcal{A}(\mathbb{N}), \mathcal{O}(\mathbb{N}) \land \mathcal{A}(\mathbb{N}) \}. \) Let \( \delta^{X,Y}_{wR} : \subseteq \mathbb{N}^N \to \text{COrd} \) be the restriction of \( \delta^{X,Y}_R \) to those \( \langle p, q \rangle \) where \( q \) encodes a wellorder.

Finally, we introduce a representation tailor-made for the formulation and proof of a computable Hausdorff-Kuratowski theorem below. Let a nice relation be a well-founded quasi-order \( \preceq \) on \( \mathbb{N}, \) such that \( \forall n \ n \preceq 0, \) and whenever \( n \prec m, \) then \( n > m. \)
Definition 7. We define a representation $\delta_{nR} : \subseteq \{0, 1\}^N \to \text{COrd}$ by $\delta_{nR}(p) = \alpha$, iff the relation $\preceq_p$ defined via $n \preceq_p m$ iff $p(n, m) = 1$ is a nice relation with order type $\alpha + 1$ (the order type of any nice relation is a countable successor ordinal, and every successor ordinal should arise as such an order type).

To obtain some initial understanding of how the various representations work, we shall consider what happens to the finite ordinals. Besides the usual natural numbers $\mathbb{N}$, also the spaces $\mathbb{N}_<$, $\mathbb{N}_>$ and $\mathbb{N}^\triangledown$, where a number $n$ is represented by an increasing, respectively decreasing, respectively arbitrary sequence of integers which eventually converge to $n$.

Observation 8. $(\text{id} : \mathbb{N}^\triangledown \to \mathbb{N}) \equiv W (\text{id} : \mathbb{N}_< \to \mathbb{N}) \equiv W C\mathbb{N}$; $(\text{id} : \mathbb{N}_> \to \mathbb{N}) \equiv W LPO^*$ and $LPO \leq W (\text{id} : \mathbb{N}_> \to \mathbb{N}_<)$.

Proposition 9.
1. $(\text{COrd}, \delta_K)|_N \cong \mathbb{N}$
2. $(\text{COrd}, \delta_{nK})|_N \cong \mathbb{N}_<$
3. $(\text{COrd}, \delta_M)|_{\{n \in \mathbb{N} : n > 0\}} \cong (\mathbb{N}_<)|_{\{n \in \mathbb{N} : n > 0\}}$
4. $(\text{COrd}, \delta_{A(N), Y}^w)|_N \cong (\text{COrd}, \delta_{A(N), Y}^R)|_N \cong \mathbb{N}^\triangledown$ (regardless of the choice of $Y \in \{O(\mathbb{N}), A(\mathbb{N}), O(\mathbb{N}) \land A(\mathbb{N})\}$)

Proof. 1. The third rule of the definition cannot be used for finite ordinals. The first two rules just define the usual natural numbers.

2. For finite ordinals, the nested occurrences of the second and the third rules can be computably reshuffled such that a single sup is on the outside, and all ordinals the scope there are built by the first two rules.

3. Straightforward.

4. The identity from $(\text{COrd}, \delta_{A(N), Y}^A)|_N$ to $(\text{COrd}, \delta_{A(N), Y}^R)|_N$ is trivially computable. To get from $(\text{COrd}, \delta_{A(N), Y}^R)|_N$ to $\mathbb{N}^\triangledown$, note that we can extract candidates for the actual value of the ordinal from its finite prefixes, and if the ordinal is indeed finite, this process will stabilize eventually. Finally, to move from $\mathbb{N}^\triangledown$ to $(\text{COrd}, \delta_{A(N), Y}^A)|_N$, we first point out that this clearly works with $\mathbb{N}$ in place of $\mathbb{N}^\triangledown$. Now, whenever the current approximation of the number in $\mathbb{N}^\triangledown$ changes, we use the fact that the domain of the structure is given as a closed set in order to erase the current relation, and restart afresh.

We can extend Proposition 9 (3) to:

Lemma 10. $(\text{COrd}, \delta_M)|_{\{\alpha > 0\}} \cong (\text{COrd}, \delta_{nK})|_{\{\alpha > 0\}}$

Proof. The translation from left to right is straightforward. W.l.o.g., we may assume that any $\delta_{nK}$-name is using a sup as outmost operation. If the represented ordinal is non-zero, we can furthermore assume that the next layer is always using the successor operation (by merging nested supers into the top level, and by ignoring 0s). But this then corresponds to the repeated construction in a $\delta_M$-name.

The same argument used in proving Proposition 9 (4) can also be used to establish:
Proposition 11. \((\text{COnd}, \delta_{\omega}^X, Y) \cong \bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y)\) and \((\text{COnd}, \delta_{\omega}^X, Y) \cong \bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y)\).

Similarly, we also find:

Proposition 12. \((\bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y) \cong \bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y)\)\)

Proof. As the relation is given as a closed set, it is always possible to make all numbers encountered so far incomparable. As the relation is not required to be total, and by assumption, the ordinal is non-zero, the presence of these numbers will not impact the value of the represented ordinal.

Merely requiring the domain of the structure to be enumerable, rather than decidable, does not impact the representation at all though. For not necessarily wellordered relations, the same applies to the relation itself.

Lemma 13. \((\text{COnd}, \delta_{\omega}^X, Y) \cong \bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y)\) and \((\text{COnd}, \delta_{\omega}^X, Y) \cong \bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y)\)

Proof. The natural numbers (as used to interpret ordinals) are indistinguishable for this purpose. We simulate an enumeration procedure while deciding that the natural numbers we encounter are not part of the domain of the structure. Once a number is enumerated into the structure, we decide that the next unencountered number is part of the structure, and replace the former with the latter in the relation.

Lemma 14. \((\text{COnd}, \delta_{\omega}^X, Y) \cong \bigwedge_{\{\alpha > 0\}} (\text{COnd}, \delta_{\omega}^X, Y)\)

Proof. Essentially, whenever new information about the relationship between two already settled points occurs, one can create a fresh copy of everything encountered so far. As smaller relations (w.r.t subset inclusion) have smaller order type, the extra copy does not impact the ordinal represented thus.

Theorem 15. Let \(\delta\) be a representation of \text{COnd} such that the maps

1. \(0 : 1 \to (\text{COnd}, \delta)\)
2. \(+1 : (\text{COnd}, \delta) \to (\text{COnd}, \delta)\)
3. \(\sup : \mathcal{C}(\mathbb{N}, (\text{COnd}, \delta)) \to (\text{COnd}, \delta)\)

are computable. Then \(\text{id} : (\text{COnd}, \delta_{\omega}) \to (\text{COnd}, \delta)\) is computable.

Proof. Induction along the definition of \(\delta_{\omega}\).

Theorem 16. The following representations are equivalent:

1. \(\delta_{\omega}\)
2. \(\delta_{\omega}^X\)
3. \(\delta_{\omega}^{X, O(X)}\)
4. \(\delta_{\omega}^{O(X)}\)
5. \(\delta_{\omega}^{X, O(X)}\)
6. \(\delta_{\omega}^{O(X)}\)
Theorem 17. 1. (id : (COrd, δK) → (COrd, δnK)) is computable, but C_n ≤_W (id : (COrd, δ_nK) → (COrd, δ_K)).
2. (id : (COrd, δ_M) → (COrd, δ_nK)) is computable, but LPO ∋_W (id : (COrd, δ_nK) → (COrd, δ_M)).
3. (id : (COrd, δ_nK) → (COrd, δ_R^X,A(N))) is computable, but LPO^* ≤_W (id : (COrd, δ_R^X,A(N)) → (COrd, δ_K)).
4. (id : (COrd, δ_nK) → (COrd, δ_R^A(N),Y)) is computable, but LPO^* ≤_W (id : (COrd, δ_R^A(N),Y) → (COrd, δ_K))
5. (id : (COrd, δ_{nK}^{C,O}) → (COrd, δ_nK)) is computable, but (id : (COrd, δ_nK) → (COrd, δ_{wR}^{A,O,A,O})).

Proof. 1. ⇒ 2. We will invoke Theorem [15]. We see that 0 ∈ (COrd, δ_{nR}) is computable using the relation identifying all numbers. The successor operation is realized by shifting all numbers by 1, and letting 0 be above all other numbers. Finally, the countable supremum is realized by taking the disjoint union (via some tupling function), and then identifying all 0’s in the individual slices.
2. ⇒ 3. Just remove the equivalence class of 0, as well as all duplicate points.
3. ⇔ 4. Lemma [13]
4. ⇔ 5. Lemma [14]
5. ⇔ 6. Lemma [14]
6. ⇒ 1. Given some open set A ⊆ N, some open relation R on A and n ∈ A, we can compute A_n := \{ i ∈ A | (i, n) ∈ R \} ∈ O(N). Let α_n be the height of R restricted to A_n, and α be the height of R on A. Then α = sup_{n ∈ A}(α_n + 1). By interspersing 0’s for when the next n ∈ A has not been found yet, this can be extended to an inductive translation from δ_{R^O(N),O(N)} to δ_{nK}.

Proof. 1. The first translation is realized by the identity. For the lower bound of the reverse direction, consider its restriction to the finite ordinals. By Proposition [9] (1,2), it becomes (id : N < → N), and Observation [8] yields the claim.
2. The first translation is straightforward; combining rules 2 and 3 in Definition [4]. To see that LPO →_W (id : (COrd, δ_{nK}) → (COrd, δ_M)), note it suffices to make a case-distinction between 0 and non-zero ordinals due to Lemma [10]. A δ_{nK}-name denotes a non-zero ordinal iff it uses the successor operation somewhere, thus, LPO suffices to make the case distinction. For the reduction in the other direction, note that ν : S → (COrd, δ_{nK}) with ν(⊥) = 0 and ν(⊤) = 1 is computable, and so is ν^(-1) : (COrd, δ_M) → 2^C. Composing these maps with the translation shows that LPO ≡_W (id : S → 2) ≤_W (id : (COrd, δ_{nK}) → (COrd, δ_M)).
3. The first translation becomes a simple removal of additional information taking into consideration the equivalence in Theorem [16] (7,8,9) respectively. For the lower bound in the other direction, note that Proposition [9] together with Proposition [12] give us (id : N^V → N_<) ≤_W (id : (COrd, δ_{R^X,A(N)}) → (COrd, δ_K)). Then note (id : N_> → N) ≤_W (id : N^V → N_<), and use Observation [8].
4. As in (3), just with Proposition [12] replaced by Proposition [11].
5. The translation in the first direction follows from Theorem 16. If the other translation were computable, too, then the binary supremum would be computable with respect to the representation $\delta_{wR}^{A\land O, A\land O}$. This however would contradict [23, Corollary 24].

\[ \square \]

**Definition 18.** We will consider the equivalence class of $\delta_{nK}$ identified in Theorem 16 as the standard representation of $C_{Ord}$, and thus abbreviate $C_{Ord} := (C_{Ord}, \delta_{nK})$.

Besides $C_{Ord}$, we will also consider $C_{Ord}^M := (C_{Ord}, \delta_{M})$, $C_{Ord}^K := (C_{Ord}, \delta_{K})$ and $C_{Ord}^{HL} := (C_{Ord}, \delta_{wR}^{A\land O, A\land O})$. The representations using well-founded structures given as closed sets would seem to be too weak to be of much interest, following Propositions 11, 12, and thus will no longer be considered.

### 3 Computability on $C_{Ord}$

In order to justify the stance that the represented space $C_{Ord}$ really is the space of countable ordinals, we shall investigate the computable operations on it and related properties.

**Theorem 19.** The following operations are computable:

1. $+$ : $C_{Ord} \times C_{Ord} \rightarrow C_{Ord}$
2. $\times$ : $C_{Ord} \times C_{Ord} \rightarrow C_{Ord}$
3. $\sup$ : $C_{Ord}^N \rightarrow C_{Ord}$
4. $(-1)$ : $C_{Ord} \rightarrow C_{Ord}$, where $(-1)(\alpha + 1) = \alpha$ and for limit ordinals $\gamma$, $(-1)(\gamma) = \gamma$
5. Smaller : $C_{Ord} \nrightarrow C_{Ord}^N$ where $(\alpha_i)_{i \in \mathbb{N}} \in \text{Smaller}(\alpha)$ iff $\{0\} \cup \{\beta \in C_{Ord} | \beta < \alpha\} = \{\alpha_i | i \in \mathbb{N}\}$

**Proof.**

1. Using $\delta_{nK}$ and induction on the second argument: $\alpha + 0 = \alpha$, $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, $\alpha + (\sup_{i \in \mathbb{N}} \beta_i) = \sup_{i \in \mathbb{N}} (\alpha + \beta_i)$.
2. Using $\delta_{R}^{O(N), O(N)}$: We can easily compute the product relation, and its height will be product of the heights of the arguments.
3. Obvious when using $\delta_{nK}$.
4. Once more, we use $\delta_{nK}$ and induction: $(-1)(0) = 0$, $(-1)(\alpha + 1) = \alpha$, $(-1)(\sup_{i \in \mathbb{N}} \alpha_i) = \sup_{i \in \mathbb{N}} ((-1)(\alpha_i))$.
5. Similar to parts of the proof in Theorem 16. We use the representation $\delta_{R}^{O(N), O(N)}$. Given some open set $A \subseteq \mathbb{N}$, some open relation $R$ on $A$ and $n \in A$, we can compute $A_n := \{i \in A | (i, n) \in R\} \in O(N)$. Let $\alpha_n$ be the height of $R$ restricted to $A_n$, and $\alpha$ be the height of $R$ on $A$. Then $\{\alpha_n | n \in \mathbb{N}\} = \{\beta \in C_{Ord} | \beta < \alpha\}$. We can enumerate the $\alpha_n$ by interspersing 0’s as required.

\[ \square \]

**Open Question 20.** Is $(\alpha, \beta) \mapsto \alpha^\beta : C_{Ord} \times C_{Ord} \rightarrow C_{Ord}$ computable?
Proposition 21. \( \text{LPO}^* \leq_W (\rightarrow : \text{COrd} \times \text{COrd} \rightarrow \text{COrd}) \)

Proof. For any fixed \( N \in \mathbb{N} \) we could use \( \rightarrow \) to compute \( \text{id} : \{0, \ldots, N\} \rightarrow \{0, \ldots, N\} \) using the inclusion from Proposition 9. \( \square \)

Next, we shall characterize the open and compact subsets of \( \text{COrd} \). For this we need the spaces \( \mathbb{N}_< \) and \( \mathbb{N}_> \), which are obtained by adjoining \( \infty \) to \( \mathbb{N}_< \) and \( \mathbb{N}_> \) in the appropriate ways. We will understand \( \alpha < \infty \) for all countable ordinals \( \alpha \).

Proposition 22. The map \( n \mapsto \{ \alpha \in \text{COrd} \mid \alpha \geq n \} : \mathbb{N}_> \rightarrow \mathcal{O}(\text{COrd}) \) is a computable isomorphism.

Proof. First we show that the map is computable. Given a natural number \( n \in \mathbb{N} \) and an ordinal \( \alpha \in \text{COrd} \), we can recognize \( n \geq \alpha \) – for this, there have to be \( n - 1 \) nested occurrences of the successor operation in the name of \( \alpha \), and these are contained in some finite prefix. Having \( n \) given as the limit of a decreasing sequence instead does not cause problems, as any premature acceptances stay valid.

Next, we shall argue that the inverse is computable, too. This means to argue that \( \text{min} : \mathcal{O}(\text{COrd}) \rightarrow \mathbb{N}_> \). Given an open set \( U \in \mathcal{O}(\text{COrd}) \), we can test for all natural numbers simultaneously whether \( n \in U \). Any positive answer gives an upper bound for the minimal number in \( U \).

Finally we need to show that the map is surjective. As \( \text{sup} : \text{COrd}^\mathbb{N} \rightarrow \text{COrd} \) is computable, we see that any open set has to be upwards closed. So the only thing left to argue is that any non-empty open set contains a finite ordinal, i.e. that the finite ordinals are dense. Let us assume that an open set \( U \) accepts some ordinal \( \alpha \) after having read a finite prefix of its name. If every hitherto unencountered subterm in the name of \( \alpha \) is replaced by 0, the result is some finite ordinal also accepted by \( U \).

Proposition 23. The map \( n \mapsto \{ \alpha \in \text{COrd} \mid \alpha \geq n \} : \mathbb{N}_< \rightarrow \mathcal{K}(\text{COrd}) \) is a computable isomorphism.

Proof. This follows from basic properties of \( \mathbb{N}_< \). \( \square \)

4 Computability on \( \text{COrd}_K \)

In order to define the concept of a computable ordinal, Kleene’s definition resulting in the space \( \text{COrd}_K \) seems to be the typical choice. A strong reason to reject \( \text{COrd}_K \) as the natural candidate for computability on the countable ordinals nonetheless, lies in the following result:

Proposition 24. \( \text{LPO} \leq_W (\text{max} : \text{COrd}_K \times \text{COrd}_K \rightarrow \text{COrd}_K) \)

Proof. We start with the observation that both \( \iota : \mathbb{S} \rightarrow \text{COrd}_K \) defined via \( \iota(\bot) = \omega \) and \( \iota(\top) = 2\omega \) as well as the constant \( \omega + 1 \) are computable. Then note that being a limit ordinal is decidable on \( \text{COrd}_K \), yielding a computable function \( \text{IsLimit} : \text{COrd}_K \rightarrow 2 \). Now \( t \mapsto \text{IsLimit}(\text{max}(\iota(t), \omega + 1)) \) is identical to \( \text{id} : \mathbb{S} \rightarrow 2 \). \( \square \)
The reason that calling the computable elements in $\text{COrd}_K$ the computable ordinals is justified regardless of $\text{COrd}_K$ not being the right space lies in the fact that both $\text{COrd}_K$ and $\text{COrd}$ have the same computable points. This situation is somewhat reminiscent of Turing’s transient mistake of defining the computable real numbers via the decimal expansion at first \[37\] before correcting himself \[38\].

**Observation 27** (Gregoriades, Kispéter and P. \[11\]).

1. There exists a $\Sigma^1_1$ relation $\leq_\Sigma \subseteq \mathbb{N}^2 \times \mathbb{N}^2$ such that for all $p \in \text{dom}(\delta_M)$ and all $q \in \mathbb{N}^N$ we have that

\[
[q \in \text{dom}(\delta_M) \& \delta_M(q) \leq \delta_M(p)] \iff q \leq_\Sigma p
\]

2. $\text{dom}(\delta_M)$ is not a Borel subset of $\mathbb{N}^N$

**Proof of (1.)** Note that $\delta_M(\langle q_0, q_1, \ldots \rangle) \leq \delta_M(\langle p_0, p_1, \ldots \rangle)$ iff $\exists t \in \mathbb{N}^N$ s.t. $\forall n \in \mathbb{N} \delta_M(q_n) \leq \delta_M(p_{\langle n \rangle})$, assuming $q_i, p_i \in \text{dom}(\delta_M)$. Building upon this idea, consider the closed relation $R$ defined as the least fixed point of:

\[
R(p, q, (t', (t_0, t_1, \ldots)')) :\iff q(0) = 0 \lor (p = 1 \langle p_0, p_1, \ldots \rangle \& q = 1 \langle q_0, q_1, \ldots \rangle \& \forall n \in \mathbb{N} R(p_n, q_{\langle n \rangle}, t_n))
\]

Now $q \leq_\Sigma p \iff \exists t \in \mathbb{N}^N R(p, q, t)$ is a $\Sigma^1_1$ relation, and satisfies our criterion. \[\square\]

When using $\delta_{nK}$ instead of $\delta_M$ the problem would be that names of 0 could have nested occurrences of sup of arbitrary complexity.

**Theorem 28** (Gregoriades, Kispéter and P. \[11\]). For every continuous (even: every Borel-measurable) function $f : \mathbb{N}^N \rightarrow \text{COrd}_M$ there is some $\alpha \in \text{COrd}$ such that $\forall p \in \mathbb{N}^N f(p) \leq \alpha$.

\[\text{\textsuperscript{2}}\text{The idea behind (1.) can be found in the notion of $\varGamma$-norms, see [24] 4B. The second claim is folklore.}\]

\[\text{\textsuperscript{3}}\text{This result essentially is folklore.}\]
Proof. If this we not the case we would have that
\[ q \in \text{dom}(\delta_M) \iff (\exists p)(q \leq_{\Sigma} f(p)), \]
where \( \leq_{\Sigma} \) is as above. Since \( f \) is Borel measurable the preceding equivalence would imply that the set \( \text{dom}(\delta_M) \) is a \( \Sigma_1^1 \) subset of \( \mathbb{N}^\mathbb{N} \). Hence from the Suslin Theorem (e.g. \cite{25}) it would follow that \( \text{dom}(\delta_M) \) is a Borel set, a contradiction and our claim is proved.

Corollary 29. For every continuous (even: every Borel-measurable) function \( f : \mathbb{N}^\mathbb{N} \to \text{COrd} \) there is some \( \alpha \in \text{COrd} \) such that \( \forall p \in \mathbb{N}^\mathbb{N} \; f(p) \leq \alpha \).

Proof. Using Theorem 28 together with Proposition 25.

Corollary 30. There is no total representation \( \delta : \mathbb{N}^\mathbb{N} \to \text{COrd} \) such that \( \text{id} : (\text{COrd}, \delta) \to \text{COrd} \) could be Borel measurable.

Unfortunately, the proof of Theorem 28 is entirely non-constructive and does not offer a way to extract a bound from a description of the function. As a result of Spector establishes the corresponding version in the computable discrete realm, there seems to be hope for a positive answer to at least the weak version of the following:

Open Question 31. Is the function \( \text{sup} : \mathcal{C}(\mathbb{N}^\mathbb{N}, \text{COrd}) \to \text{COrd} \) computable? Is the multifunction \( \text{UpperBound} : \mathcal{C}(\mathbb{N}^\mathbb{N}, \text{COrd}) \Rightarrow \text{COrd} \) computable?

6 Computability on \( \text{COrd}_{\text{HL}} \)

Computability on the space \( \text{COrd}_{\text{HL}} \) was studied by Joel Hamkins and Zhenhao Li in \cite{23}. We briefly survey some of their results:

Theorem 32 (Hamkins & Li \cite{23}). The following operations are computable:

1. \( + : \text{COrd}_{\text{HL}} \times \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
2. \( \times : \text{COrd}_{\text{HL}} \times \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
3. \( (\alpha, \beta) \mapsto \alpha^\beta : \text{COrd}_{\text{HL}} \times \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
4. \( \alpha + 1 \mapsto \alpha : \subseteq \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
5. \( \omega^{\text{CK}} + \omega \mapsto \omega^{\text{CK}} : \subseteq \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)

As with Proposition 24 for \( \text{COrd}_K \), the first item of the following justifies our rejection of \( \text{COrd}_{\text{HL}} \) as proposed standard computability structure on the countable ordinals. We point out that the technique introduced in \cite{23}, Theorem 16] essentially is a Wadge game relative to the representation, similar to the generalizations of the classical Wadge hierarchy on \( \mathbb{N}^\mathbb{N} \) to represented spaces in \cite{34} by Pequignot and \cite{7} by Duparc and Fournier.

Theorem 33 (Hamkins & Li \cite{23}). The following operations are not computable:

1. \( \text{max} : \text{COrd}_{\text{HL}} \times \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
2. \( \alpha \mapsto \text{max}\{\alpha, \omega + 1\} : \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
3. \( \omega \times \alpha \mapsto \alpha : \subseteq \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \)
4. \( \text{Reduce}_n : \subseteq \text{COrd}_{\text{HL}} \to \text{COrd}_{\text{HL}} \) where \( \text{Reduce}_n(\omega) = n \) and \( \text{Reduce}_n(\omega + \omega) = \omega \).
5. \( D \subseteq \text{COrd}_{\text{HL}} \to \{0, 1\} \) where \( D(\omega) = 0 \) and \( D(\omega + 1) = 1 \)

**Corollary 34.** \( \text{id} : \text{COrd}_{\text{HL}} \to \text{COrd}_K \) is not computable.

An open question raised in [23] is whether the supremum of strictly increasing sequences of ordinals can be computed. This boils down to the following:

**Open Question 35** (Hamkins & Li [23]). Is \( \text{id} : \text{COrd}_K \to \text{COrd}_{\text{HL}} \) computable?

Finally, we point out that the investigations in [23, Section 5] concern the point degree spectrum of \( \text{COrd}_{\text{HL}} \) (without using this terminology, though). Point degree spectra of represented spaces were introduced by Kihara and P. in [19].

### 7 A non-deceiving representation of \( \text{COrd} \)?

The trusted recipe of identifying suitable representations of some structure is to pick an admissible representation whose final topology coincides with some natural topology on the structure\(^4\). However, the usual topology on \( \text{COrd} \) would be the order topology, which is not separable – and every represented space is separable. In this section, we shall explore whether a weaker topological requirement could be imposed on a representation.

Inspired by a property studied in the context of winning conditions for infinite sequential games in [21] by Le Roux and P., we shall call a function \( f : \subseteq \mathbb{N}^\mathbb{N} \to \text{COrd} \) non-deceiving, if whenever \((p_n)_{n \in \mathbb{N}}\) is a sequence converging to \( p \) in \( \text{dom}(f) \) such that \( \forall n \in \mathbb{N} \) \( f(p_n) < f(p_{n+1}) \), then \( \forall i \in \mathbb{N} \) \( f(p_i) < f(p) \).

**Theorem 36** (Gregoriades\(^5\)). Any non-deceiving function \( f : \subseteq \mathbb{N}^\mathbb{N} \to \text{COrd} \) is bounded by some countable ordinal.

**Proof.** As \( \text{dom}(f) \) (as a subspace of \( \mathbb{N}^\mathbb{N} \)) is countably based, it suffices to show that for any \( p \in \text{dom}(f) \) there is some open neighborhood \( U \) of \( p \) such that \( f|_U \) is bounded (as there are only countably many basic open sets, the supremum of the local bounds is a global bound).

Assume the contrary. Then there is some \( p \in \text{dom}(f) \) such that for each \( n \in \mathbb{N} \) and each \( \alpha \in \text{COrd} \) there is some \( q \in \text{dom}(f) \) with \( d(p, q) < 2^{-n} \) and \( f(q) > \alpha \). By choosing a suitable \( q \) countable many times, we arrive at a sequence \((q_i)_{i \in \mathbb{N}}\) with \( \lim_{i \to \infty} q_i = p \) and \( f(p) < f(q_0) < f(q_1) < \ldots \). But this contradicts the non-deceiving condition.

**Corollary 37.** There is no non-deceiving representation of \( \text{COrd} \).

The preceding corollary presumably destroys any hope to find a suitable representation of \( \text{COrd} \) that is admissible w.r.t. some weak limit space structure in the sense of Schröder [36, 35].

### 8 The computable Hausdorff-Kuratowski theorem

We shall now prepare the formulation of the Hausdorff-Kuratowski theorem in the framework of computable endofunctors on the category of represented spaces as introduced by de Brecht and P. in [33, 32, 6]. The setting closely follows the corresponding section in [6] by de Brecht, where a weaker (and non-effective) version of our desired result was proven.

\(^4\)In fact, it is sometimes claimed that it has to be done like that – the present work ought to disprove this.

\(^5\)This theorem is based on a personal communication by Vassilios Gregoriades.
For any sequence of countable ordinals \((\alpha_i)_{i \in \mathbb{N}}\), we define a function \(L_{(\alpha_i)_{i \in \mathbb{N}}} : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}\). The sequence only impacts the domain, but whenever \(L_{(\alpha_i)_{i \in \mathbb{N}}} (p)\) is defined, then \(2L_{(\alpha_i)_{i \in \mathbb{N}}} (p)(n) = p(\max\{i \in \mathbb{N} \mid p(i) \text{ is odd}\} + 1)\); i.e. \(L_{(\alpha_i)_{i \in \mathbb{N}}} \) takes the maximal tail of its input consisting of only even values, and returns the result of pointwise division by 2. Obviously any sequence in the domain of \(L_{(\alpha_i)_{i \in \mathbb{N}}} \) has to contain only finitely many odd entries; and we additionally demand that for \(p \in \text{dom}(L_{(\alpha_i)_{i \in \mathbb{N}}} )\), if \(n < m\), and \(p(n) = 2k + 1\) and \(p(m) = 2j + 1\), then \(\alpha_k > \alpha_j\).

**Definition 38.** We define a computable endofunctor \(\mathcal{L}_{(\alpha_n)_{n \in \mathbb{N}}} \) by \(\mathcal{L}_{(\alpha_n)_{n \in \mathbb{N}}}(X, \delta) = (X, \delta \circ L_{(\alpha_i)_{i \in \mathbb{N}}} )\) and the straightforward extension to functions.

Each endofunctor \(\mathcal{L}_{(\alpha_n)_{n \in \mathbb{N}}} \) captures a version of computability with finitely many mindchanges (e.g. \([41][42]\)): The regular outputs are encoded as even numbers. Finitely many times, the output can be reset by using an odd number, however, when doing so, one has to count down within the list of ordinals parameterizing the function (which in particular ensures that it happens only finitely many times). We thus find it connected to the level introduced by HERTLING [14], and further studied by him and others in [15] [16] [17] [27] [29] [6].

**Definition 39.** Given a function \(f : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}\), we define the sets \(\mathcal{L}_\alpha(f) \subseteq \mathbb{N}^\mathbb{N}\) inductively via:

1. \(\mathcal{L}_0(f) = \text{dom}(f)\)
2. \(\mathcal{L}_{\alpha+1}(f) = \{ \alpha \in \mathcal{L}_\alpha(f) \mid f|_{\mathcal{L}_\alpha} \text{ is discontinuous at } \alpha \}\)
3. \(\mathcal{L}_\gamma(f) = \bigcap_{\beta < \gamma} \mathcal{L}_\beta(f)\) for limit ordinals \(\gamma\).

Then we say \(\text{Lev}(f) := \min\{ \alpha \mid \mathcal{L}_\alpha(f) = \emptyset \}\).

**Theorem 40.** If \(f : \mathbb{N}^\mathbb{N} \rightarrow \mathcal{L}_{(\alpha_i)_{i \in \mathbb{N}}} \mathbb{N}^\mathbb{N}\) is continuous, then \(\text{Lev}(f) \leq (\sup_{i \in \mathbb{N}} \alpha_i) + 1\).

**Proof.** Let \(F\) be a continuous realization of \(f\). For any \(n \in \mathbb{N}\), the set \(U_n := \{ p \in \mathbb{N}^\mathbb{N} \mid \exists k \ F(p)(k) = 2n + 1 \}\) is open. Let \(n\) be such that \(\alpha_n\) is the smallest ordinal in \((\alpha_i)_{i \in \mathbb{N}}\). Then \(f|_{U_n}\) is continuous, as there cannot be any further mindchanges happening, i.e. \(\mathcal{L}_1(f) \subseteq U_n^c\). Then consider \(m\) such that \(\alpha_m\) is the second smallest ordinal in \((\alpha_i)_{i \in \mathbb{N}}\), and conclude \(\mathcal{L}_2(f) \subseteq (U_n \cup U_m)^c\). Iterating this process yields \(\mathcal{L}_{\sup_{i \in \mathbb{N}} \alpha_i}(f) \subseteq (\bigcup_{i \in \mathbb{N}} U_i)^c\), and we notice that \(f|_{(\bigcup_{i \in \mathbb{N}} U_i)^c}\) does not make any mindchanges, hence is continuous. Thus \(\mathcal{L}_{(\sup_{i \in \mathbb{N}} \alpha_i) + 1}(f) = \emptyset\). \(\square\)

**Proposition 41.** Let \((\alpha_i)_{i \in \mathbb{N}}\) be such that \(\exists \alpha \in \text{COrd}\) with \(\{ \alpha_i \mid i \in \mathbb{N} \} = \{ \beta \in \text{COrd} \mid \beta < \alpha \}\). Then \(\text{Lev}(L_{(\alpha_i)_{i \in \mathbb{N}}}) = \alpha + 1\).

**Proof.** (Sketch): In this situation, the set inclusions in the proof of Theorem 40 are tight. \(\square\)

The computable Hausdorff-Kuratowski theorem has at its heart a dependent sum type; namely the construction \(\sum_{(\alpha_i)_{i \in \text{COrd}}} \left( C(X, \mathcal{L}_{(\alpha_i)_{i \in \text{COrd}}} Y) \right)\) for some represented spaces \(X, Y\). A point in this space is a pair, consisting of a sequence of countable ordinals and a function \(f : X \rightarrow Y\), the latter given only in a \(\mathcal{L}_{(\alpha_i)_{i \in \text{COrd}}}-\)continuous way.

**Theorem 42** (Computable Hausdorff-Kuratowski theorem). Let \(X, Y\) be represented spaces, and \(X\) be complete. Then the map \(\text{HK} : C(X,Y) \rightarrow \sum_{(\alpha_i)_{i \in \text{COrd}}} \left( C(X, \mathcal{L}_{(\alpha_i)_{i \in \text{COrd}}} Y) \right)\) where \(((\alpha_i)_{i \in \mathbb{N}}, g) \in \text{HK}(f)\) iff \(g = f\), is computable.
Proof. The general case reduces to the situation where \( X = Y = \mathbb{N}^\mathbb{N} \): As a complete represented space, \( X \) has a total representation \( \delta_X : \mathbb{N}^\mathbb{N} \to X \). We can then operate on a realizer of the original \( f \), as all involved endofunctors are derived from jump operators.

That we have \( f \in C(\mathbb{N}^\mathbb{N}, (\mathbb{N}^\mathbb{N})^\vee) \) means we may evaluate \( f \) with finitely many mindchanges. Any such mindchange occurs after a finite prefix of the input has been read. Thus, we may identify countably many mindchange occurrences. Using \( 0 \in \mathbb{N} \) to denote no mindchange at all, we can proceed to obtain a relation \( \preceq \) and a numbering of the mindchange occurrences, such that if mindchange \( n \) occurs after mindchange \( m \), then \( m \prec n \). If we are not aware of any not-yet-numbered mindchange occurrences, we just allocate the next natural number to the non-mindchange at 0 again.

We adjust the realizer for \( f \) in a way such that any regular output \( n \) is replaced by \( 2n \), and a mindchange symbol corresponding to the \( m \)-th mindchange is replaced by \( 2m + 1 \).

As \( f \) is total, we find that any decreasing chain through \((\mathbb{N}, \prec)\) corresponds to the mindchanges made for some input to \( f \). Thus, the relation \( \prec \) is well-founded, and the other properties of a nice relation (cf. Definition 7) follow from the construction. Using the algorithm underlying Theorem 19 (5), we can identify for each \( n \in \mathbb{N} \) the corresponding ordinal of its height in the relation, yielding the sequence \((\alpha_i)_{i \in \mathbb{N}}\).

Corollary 43. Let \( f : X \to Y \) be computable with finitely many mindchanges, and \( X \) be complete. Then \( \text{Lev}(f) \) exists and is a computable ordinal.

Proof. Combine Theorem 42 with Theorem 40 and Theorem 19 (3).

The result of the preceding corollary was also announced by Selivanov at CCA 2014.

9 An application to \( \Delta_2^0 \)-determinacy

We can use the computable Hausdorff-Kuratowski theorem to classify the strength of \( \Delta_2^0 \)-determinacy in the Weihrauch lattice. This extends (and uses) the results in [22] by Le Roux and P. on the strength of determinacy for the levels of the difference hierarchy. We refer to [22] for detailed definitions.

The problem we want to consider is the multivalued function \( \text{Det}_{\Delta_2^0} \) that takes a Gale-Stewart game on \( \{0, 1\}^\mathbb{N} \) with a \( \Delta_2^0 \)-winning set as input, and produces a pair of strategies for the two players, such that one of the strategies is a winning strategy. The upper bound we want to provide will make use of an iteration of a function\(^6\) over all countable ordinals.

Definition 44. Fix a standard enumeration \((\Phi_n)_{n \in \mathbb{N}}\) of the computable functions \( \Phi_n : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \). Given a partial function \( f : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \), we define \( f^\dagger : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) as follows:

1. \( f^\dagger((0p, q)) = q \)
2. \( f^\dagger((1p, nuq)) = (f(\Phi_n(f^\dagger((p, q)))), f^\dagger((p, q))) \)
3. \( f^\dagger((2\langle p_0, p_1, \ldots \rangle, n\langle q_0, q_1, \ldots \rangle)) = (f(\Phi_n(f^\dagger((p_0, q_0)), f^\dagger((p_1, q_1), \ldots))), f^\dagger((p_0, q_0)), f^\dagger((p_1, q_1), \ldots)) \)

\(^6\) An extension to multivalued functions should be possible, but would complicate notation even more, and thus is left for later.
Essentially, the first argument of $f^+$ is a $\delta_{nK}$-name for some ordinal (that is iterated over); while the second argument provides the actual inputs to the function $f$. In particular, $\text{dom}(f^+) \subseteq \langle \text{dom}(\delta_{nK}), N^N \rangle$. Thus, we can define $f^{+,\alpha}$ for some ordinal $\alpha$ as the restriction of $f^+$ to $\langle \delta_{nK}^{-1}(\{\beta \mid \beta < \alpha\}), N^N \rangle$. Thus, trivially, if $\beta \leq \alpha$, then $f^{+,\beta} \leq_W f^{+,\alpha}$. Additionally, we point out that $f^{+,0} \equiv_W \text{id}$, $f^{+,1} \equiv_W (f + \text{id})$ and more generally, $f^{+,n} \equiv_W (f + \text{id})^n$.

**Proposition 45.** If $(f + \text{id}) \star (f + \text{id}) \equiv_W f$, then $f^+ \equiv_W f$.

**Proof sketch.** By induction over the first parameter. The first case uses $\text{id} \leq_W f$, the second case uses $f \equiv_W f \star f$, and the third case uses $f \equiv_W f \star \hat{f}$. 

**Open Question 46.** Is $f^+ \equiv_W (f^+)^+$ true? What about $f^{+,\alpha} \star f^{+,\beta} \equiv_W f^{+,\alpha \times \beta}$?

The results in [22, Section 3] together with the computable Hausdorff-Kuratowski theorem immediately imply:

**Corollary 47.** $\text{Det}_{\Delta_0^2} \leq_W \text{lim}^+$

**Conjecture 48.** $\text{Det}_{\Delta_0^2} \equiv_W \text{lim}^+$

**References**

[1] Vasco Brattka, Matthew de Brecht & Arno Pauly (2012): *Closed Choice and a Uniform Low Basis Theorem*. Annals of Pure and Applied Logic 163(8), pp. 968–1008.

[2] Vasco Brattka & Guido Gherardi (2011): *Effective Choice and Boundedness Principles in Computable Analysis*. Bulletin of Symbolic Logic 1, pp. 73 – 117. ArXiv:0905.4685.

[3] Vasco Brattka & Guido Gherardi (2011): *Weihrauch Degrees, Omniscience Principles and Weak Computability*. Journal of Symbolic Logic 76, pp. 143 – 176. ArXiv:0905.4679.

[4] Vasco Brattka, Guido Gherardi & Rupert Hölzl (2013). *Probabilistic Computability and Choice*. arXiv 1312.7305. Available at [http://arxiv.org/abs/1312.7305](http://arxiv.org/abs/1312.7305).

[5] Vasco Brattka & Arno Pauly. *On the algebraic structure of Weihrauch degrees*. forthcoming.

[6] Matthew de Brecht (2013). *Levels of discontinuity, limit-computability, and jump operators*. arXiv 1312.0697.

[7] Jacques Duparc & Kevin Fournier. *Reductions by relatively continuous relations on $\mathbb{N}^{\leq \omega}$.* unpublished notes.

[8] Martin Escardó (2004): *Synthetic topology of datatypes and classical spaces*. Electronic Notes in Theoretical Computer Science 87.

[9] Johanna N.Y. Franklin, Asher M. Kach, Russell Miller & Reed Solomon (2013): *Local Computability for Ordinals*. In: Paola Bonizzoni, Vasco Brattka & Benedikt Löwe, editors: The Nature of Computation. Logic, Algorithms, Applications, LNCS 7921, Springer, pp. 161–170.

[10] Guido Gherardi & Alberto Marcone (2009): *How incomputable is the separable Hahn-Banach theorem?* Notre Dame Journal of Formal Logic 50(4), pp. 393–425.

[11] Vassilios Gregoriades, Tamás Kispéter & Arno Pauly (2014). *A comparison of concepts from computable analysis and effective descriptive set theory*. arXiv:1401.3325.

[12] Vassilios Gregoriades & Yiannis N. Moschovakis. *Notes on effective descriptive set theory*. notes in preparation.

[13] Peter Hertling (1996): *Topological Complexity with Continuous Operations*. Journal of Complexity 12(4), pp. 315–338.
A. Pauly

[14] Peter Hertling (1996): *Unstetigkeitsgrade von Funktionen in der effektiven Analysis*. Ph.D. thesis, Fernuniversität, Gesamthochschule in Hagen.

[15] Peter Hertling (2002): *Topological Complexity of Zero Finding with Algebraic Operations*. Journal of Complexity 18, pp. 912–942.

[16] Peter Hertling & Klaus Weihrauch (1994): *Levels of degeneracy and exact lower complexity bounds*. In: 6th Canadian Conference on Computational Geometry, pp. 237–242.

[17] Kojiro Higuchi & Arno Pauly (2013): *The degree-structure of Weihrauch-reducibility*. Logical Methods in Computer Science 9(2).

[18] J.E. Jayne & C.A. Rogers (1982): *First level Borel functions and isomorphisms*. Journal de Mathématiques Pures et Appliqués 61, pp. 177–205.

[19] Takayuki Kihara & Arno Pauly (2014). *Point degree spectra of represented spaces*. arXiv:1405.6866.

[20] S. C. Kleene (1938): *On notation for ordinal numbers*. Journal of Symbolic Logic 3, pp. 150–155. Available at http://journals.cambridge.org/article_S002248120003574X.

[21] Stéphane Le Roux & Arno Pauly (2014). *Infinite sequential games with real-valued payoffs*. arXiv:1401.3325.

[22] Stéphane Le Roux & Arno Pauly (2014). *Weihrauch degrees of finding equilibria in sequential games*. arXiv:1407.5587.

[23] Zhenhao Li & Joel D. Hamkins. *On effectiveness of operations on countable ordinals*. unpublished notes.

[24] Yiannis N. Moschovakis (1980): *Descriptive Set Theory*, Studies in Logic and the Foundations of Mathematics 100. North-Holland.

[25] Yiannis N. Moschovakis (2010): *Classical descriptive set theory as a refinement of effective descriptive set theory*. Annals of Pure and Applied Logic 162, pp. 243–255.

[26] Luca Motto Ros & Brian Semmes (2009): *A New Proof of a Theorem of Jayne and Rogers*. Real Analysis Exchange 35(1), pp. 195–204.

[27] Arno Pauly (2007): *Methoden zum Vergleich der Unstetigkeit von Funktionen*. Masters thesis, FernUniversität Hagen.

[28] Arno Pauly (2010): *How Incomputable is Finding Nash Equilibria?* Journal of Universal Computer Science 16(18), pp. 2686–2710.

[29] Arno Pauly (2010): *On the (semi)lattices induced by continuous reducibilities*. Mathematical Logic Quarterly 56(5), pp. 488–502.

[30] Arno Pauly (2012). *A new introduction to the theory of represented spaces*. http://arxiv.org/abs/1204.3763.

[31] Arno Pauly (2014). *The descriptive theory of represented spaces*. arXiv:1408.5329.

[32] Arno Pauly & Matthew de Brecht. *Towards Synthetic Descriptive Set Theory: An instantiation with represented spaces*. arXiv 1307.1850.

[33] Arno Pauly & Matthew de Brecht (2014): *Non-deterministic Computation and the Jayne Rogers Theorem*. Electronic Proceedings in Theoretical Computer Science 143. DCM 2012.

[34] Yann Pequignot. *A Wadge hierarchy for second countable spaces*. unpublished notes.

[35] Matthias Schröder (2002): *Admissible Representations for Continuous Computations*. Ph.D. thesis, FernUniversität Hagen.

[36] Matthias Schröder (2002): *A Natural Weak Limit Space with Admissible Representation which is not a Limit Space*. ENTCS 66(1), pp. 165–175.

[37] Alan Turing (1936): *On computable numbers, with an application to the Entscheidungsproblem*. Proceedings of the LMS 2(42), pp. 230–265.
[38] Alan Turing (1937): *On computable numbers, with an application to the Entscheidungsproblem: Corrections*. Proceedings of the LMS 2(43), pp. 544–546.

[39] Klaus Weihrauch (1992): *The TTE-interpretation of three hierarchies of omniscience principles*. Informatik Berichte 130, FernUniversität Hagen, Hagen.

[40] Klaus Weihrauch (2000): *Computable Analysis*. Springer-Verlag.

[41] Martin Ziegler (2007): *Real Hypercomputation and Continuity*. Theory of Computing Systems 41, pp. 177 – 206.

[42] Martin Ziegler (2007): *Revising Type-2 Computation and Degrees of Discontinuity*. Electronic Notes in Theoretical Computer Science 167, pp. 255–274.

Acknowledgements

I am grateful to Victor Selivanov for sparking my interest in a computable version of the Hausdorff Kuratowski theorem and to Vasco Brattka and Matthew de Brecht for various discussions on this question. The comparison of the various representations of the countable ordinals started with a discussion with Vassilios Gregoriades.

This work benefited from the Royal Society International Exchange Grant IE111233 and the Marie Curie International Research Staff Exchange Scheme *Computable Analysis*, PIRSES-GA-2011- 294962.