TRIANGULATIONS, SUBDIVISIONS, AND COVERS FOR CONTROL
OF AFFINE HYPERSURFACE SYSTEMS ON POLYTOPES

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ABSTRACT. This paper studies the problem for an affine hypersurface system to reach a
polytopic target set starting from inside a polytope in the state space. We present an
exhaustive solution which begins with a characterization of states which can reach the
target by open-loop control and concludes with a systematic procedure to synthesize a
feedback control. Our emphasis is on methods of subdivision, triangulation, and covers
which explicitly account for the capabilities of the control system. In contrast with previous
literature, the partition methods are guaranteed to yield a correct feedback synthesis,
assuming the problem is solvable by open-loop control.

1. Introduction

Problems of reachability for dynamical systems have been extensively studied in the control
literature for a long time. These problems have attracted renewed interest due to the emer-
gence of new paradigms for switched and piecewise linear systems. This paper studies the
problem for an affine system to reach a polytopic target set starting from inside a polytope
in the state space. Promising new ideas have appeared in the last five years in this area,
and these ideas have stimulated deeper study of the many open questions that remain. An
important gap in the literature is an exhaustive solution which covers the following sub-
problems: explicit conditions for and an analysis of all states which can reach the target
by open-loop control; a method to approximate the open-loop reachable states when there
are control constraints; a systematic method to form a subdivision of the polytope into a
set of reachable states and a set of failure states; and finally a systematic procedure to syn-
thesize a feedback control. While parts of this research program have been studied under
various assumptions, no overall end-to-end solution has been presented. The reason is that
the problem is generally extremely difficult and for certain steps, almost nothing is known
about systematic procedures. In order to tackle this problem, rather than scoping back the
problem specification as has typically been done before, we retain the complete problem
statement but work with a specific class of systems: affine hypersurface systems which are
n-dimensional affine systems with \((n - 1)\) inputs. This class is our focus of study for the
following reasons: (1) The problem of developing a systematic methodology to synthesize
controllers for reachability specifications is essentially open, and beginning with a specific
class of models provides much needed insight which can be built upon for generalization.
The outcome of our study is that we are able to provide a complete solution for affine hyper-
surface systems. In so doing we introduce new techniques for triangulation and subdivision
which can be adapted to the general problem. (2) Hypersurface systems include as a spe-
cial case second-order mechanical systems, which are an important benchmark class for new

Date: December 21, 2009.
control design methods. More generally, second-order systems have attracted extensive theoretical study due to their strong geometric properties. (See [7] for a recent example). (3) Hypersurface systems have particularly simple reachable sets. By studying these systems, we separate the challenges inherent in dealing with complex reachable sets from the other challenges presented by dealing with control synthesis on a state space which is a polytope. The contributions of the paper are therefore squarely in the area of triangulations, subdivisions, and covers. What this suggests for future investigations is very important: if the designer is willing to relax the requirement to find the largest set of states in the polytope that can reach the target and instead he works with approximations which have reasonable properties, and where importantly, reasonable properties are determined not based on traditional interpretations of hard and easy reachability computations, but based on how easily one can find triangulations and subdivisions to solve the synthesis problem, then one has a hope to develop systematic procedures which are provably successful.

We will now outline the sub-problems which are addressed in this paper. Some of these sub-problems simply involve packaging known results in an appropriate way. Other results are novel and have never been studied before. The latter are especially in the area of forming triangulations and subdivisions adapted to a given control synthesis problem. The first sub-problem we address is: given a polytopic state space \( \mathcal{P} \) and a polytopic target set \( \mathcal{F} \) in the boundary of \( \mathcal{P} \), characterize explicitly the set of states in \( \mathcal{P} \) which can reach the target by open-loop control. This result relies on well-known properties of the reachable sets of hypersurface systems. While it is a stepping stone to later results, its importance stems from the fact that knowing explicitly if a particular reachability problem is solvable by open-loop control gives a concrete metric against which to test our results: our synthesis methods should apply to any problem for which is a solution by open-loop control exists.

The next sub-problem is to develop an algorithm which “cuts off” the failure states in a systematic way, so that the remaining set \( \mathcal{P}' \) contains the original target set \( \mathcal{F} \) and is a polytope for which the open-loop reachability problem is solvable. It is shown that a systematic method to cut off the failure regions can be done with only two techniques based on the system structure. This algorithm can be easily adapted to include bounds on the control input. Once it is known that for a polytopic subset of the state space all points can reach the target by open-loop control, one then addresses questions of control synthesis, and this is the heart of the paper.

We develop a set of triangulation and subdivision procedures which are organized hierarchically. By a hierarchical organization we mean the following. At level one of the hierarchy is a subdivision method which solves the given reachability problem on the polytope. When that subdivision method is applied, it leads to sub-reachability problems on sub-polytopes which are solved by subdivision methods at level two of the hierarchy, and so forth. What is important is that there are a finite number of levels and one can prove that the refinement by new subdivisions terminates. This contrasts with the view that one simply refines by arbitrary subdivisions selected by a computer program - a method that has no guarantee to terminate even if the problem is solvable by open-loop control.

One of the challenges in developing these subdivision methods is to determine the simplest set of methods which can completely solve the problem. We present such a set, though it is by no means unique. We propose a five-level hierarchy:

1. For linear and affine systems, problems of reachability are closely tied to existence of equilibria. Therefore, the first level is a subdivision along the hyperplane \( \mathcal{O} \) of the
possible equilibria of the system; namely those points in the state space for which there exists a control input to make the vector field exactly zero.

(2) The second level of the hierarchy is a subdivision of a polytope which has two possible target sets: a neighboring polytope and a target facet. The subdivision is in fact a cover to be discussed further below, and is with respect to the boundary of the two reachability sets.

(3) The third level entails a subdivision with respect to $F$ the target set. In particular, it applies to the case when $F$ is not a facet of $P$ (motivation for this is discussed in Section 1.2). Two techniques are presented: one is a subdivision and the other is a cover.

(4) The fourth subdivision is a triangulation within a polytope whose interior does not intersect $O$ and it has a single target facet. The triangulation is determined using information about the dynamics on that polytope.

(5) The fifth subdivision is a triangulation within a simplex whose interior does not intersect $O$ and it has a single target facet.

In the remainder of the introduction, we review the relevant literature on control synthesis for reachability problems on simplices and polytopes. In Section 1.2 we give the context of the problem and from this arises the motivation and characteristics of our solution. Section 1.3 presents notation and the organization of the remainder of the paper.

1.1. Historical Overview and Related Literature. While control problems of reaching target sets in the state space have been studied since the 1960’s, our formulation and approach arise from more recent investigations on affine and piecewise affine systems defined on simplices and polytopes. The first problem to be studied of this type was by Luc Habets and Jan van Schuppen [11] in which they formulated the so-called control-to-facet problem. Further results were given in [12]. Given an affine system, the problem is to synthesize an affine control to reach an exit facet of a simplex in finite time. Necessary conditions called invariance conditions in the form of linear inequalities defined at the vertices of the simplex were presented which restrict the closed-loop vector field to point inside appropriately defined tangent cones of the simplex. A sufficient condition was also presented to ensure that all closed-loop trajectories exit the simplex. Based on the invariance conditions, an elegant synthesis method was proposed to obtain an affine control $u = Kx + g$ to solve the problem. In [13] the control-to-facet problem using affine controls for affine systems defined on simplices was improved (to allow that trajectories need not exit the simplex at the first time they reach the exit facet), and more concise necessary and sufficient conditions were obtained. The new conditions consist of the original invariance conditions of [11] [12] combined with a flow condition which guarantees that all trajectories exit the simplex, or equivalently that the closed-loop system has no equilibria in the simplex. The control-to-facet problem for a polytope as well as hybrid systems was also studied in [13]. The proposed method is to partition the state space into simplices, to form a discrete graph capturing the adjacency of simplices, and then to solve, via a dynamic programming algorithm, a sequence of control-to-facet problems. When the algorithm terminates successfully it is guaranteed to provide a piecewise affine controller solving the reachability problem.

The problem of reachability with state constraints is related to the viability/capturability problem in viability theory, especially characterizing viability kernels with a target and viable-capture basins for differential inclusions. The concept of viability kernel with a target by a Lipschitz set-valued map has been introduce and studied in [18]: This is the subset of
initial states in a constrained set from which at least one solution remains in the constrained set (i.e., is viable) forever or reaches (i.e., captures) a target in finite time before possibly violating the constraints. The set of initial states satisfying only the latter condition is called the viable-capture basin of the target. Some abstract properties and characterizations of viability kernels and viable-capture basins of a target are further studied and provided in [2].

A number of methods to construct piecewise affine feedbacks on polytopes for various control specifications such as stabilization, optimal control, and set invariance have already been developed. The recent text [5] presents an overview of methods for set invariance, which can be viewed as the dual to the problem of reachability. Piecewise affine systems have been the subject of a large number of papers. A small sampling of recent papers includes [3, 4, 6, 9, 19]. Several interesting applications of piecewise affine modeling have recently been explored. See for example [8].

1.2. Context and Motivation. This paper considers the problem of reaching a target $X_f$ with state constraint in a set $X$, denoted as $X \rightarrow X_f$. The motivation for this fundamental problem arises from a family of related reachability problems. Two sample reachability problems are as follows.

(1) *Reach - Avoid problem.* Starting at any initial state in a bounded set $Q$, reach a target set $Q_1$ while avoiding an unsafe region $Q_u$. The problem can be formulated as $X \rightarrow Q_1$, where $X = Q - Q_u$. A typical example of the problem is motion planning of multiple vehicles.

(2) *Temporal Logic Controller Synthesis.* Consider, for example, three areas of interest denoted by $Q_0, Q_1, Q_2$ such that $Q_1, Q_2 \subset Q_0$ (see Figure 1) and the temporal logic specification $\Box Q_0 \land \Diamond (Q_1 \land (Q_1 U Q_2))$, which is interpreted in natural language as: “Stay always in $Q_0$ and visit $Q_1$, then stay in $Q_1$ until it visits $Q_2$ eventually.” The problem can be thought of as two reachability problems $Q_0 \rightarrow Q_2, Q_0 \rightarrow Q_1 \rightarrow Q_2$.

This family of reachability problems motivates the particular features of the problem studied in the paper, in which each $Q_i$ is a polytope and each target is a polytope in the boundary of $Q_i$. The dynamics in each $Q_i$ may or may not be the same (although we do not study the hybrid problem here). Sub-reachability problems are sequenced in order to achieve a global specification. It may happen that a certain reachability problem fails for a particular polytope and one must restrict the polytope by cutting off failure regions. Such restrictions would propagate to neighboring polytopes and reduce their feasible target sets. It would be extremely tedious to leave these interventions to the designer, and rather, an automated
algorithm should resolve these failures. This justifies our choice to solve the reachability problem when the target is not a facet of a polytope.

1.3. Notation and Organization. We use the following notation. Let \( \text{rank}(B) \) and \( \text{Im}(B) \) denote the rank and the image of a matrix \( B \). Let \( A \) be a set. \( \hat{A}, \text{conv}(A), \text{vert}(A), \) and \( \text{aff}(A) \) denote the interior of \( A \), the convex hull of \( A \), the vertices of \( A \), and the smallest affine space containing \( A \), respectively. Let \( B \) be another set. \( A \setminus B \) expresses the set difference. Moreover, \( \text{dist}(x, A) \) expresses the distance from a point \( x \) to the set \( A \). Finally, we let \( \mu[C] \) be the volume of an \( n \)-dimensional set \( C \). If \( C \) is of dimension less than \( n \) then \( \mu[C] = 0 \).

The paper is organized as follows. In Section 2 we formulate the problems to be solved. In Section 3 we characterize the set of states which can reach the target \( F \) starting in a polytope \( P \). Then in Sections 4 and 5 we show how to synthesize piecewise affine controls on simplices and polytopes, respectively, assuming the problem is solvable by open loop control. In Section 6 we show how to subdivide a polytope with respect to a target set which is not a facet, and in Section 7 we show how to subdivide a polytope with respect to the set of possible equilibria of the system.

2. Problem Formulation

Let \( P \) be an \( n \)-dimensional polytope in \( \mathbb{R}^n \) and consider an affine control system on \( P \),

\[
\Sigma : \dot{x} = Ax + a + Bu =: f(x, u), \quad x \in P,
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, a \in \mathbb{R}^n \) and the control \( u \in \mathbb{R}^m \) lives in the space of piecewise continuous functions. Assuming that \( \text{rank}(B) = n - 1 \), we call \( \Sigma \) an affine hypersurface system. Given a piecewise continuous function \( u : t \mapsto u(t) \) and an initial state \( x_0 \in P \), let \( \phi^u_t(x_0) \) denote the unique solution of \( \Sigma \) starting from \( x_0 \).

In order to precisely formulate our problem, we begin by defining two concepts. The first is that of reaching a target with constraint in a set, which is the analogue to the notion of capturability in viability theory [2]. Second, we define \( \Omega \)-invariant sets. In viability theory such a set is called locally invariant relative to \( \Omega \) [2].

**Definition 2.1.** Let \( \Omega \) and \( \Omega_f \) be closed sets satisfying \( \Omega \cap \Omega_f \neq \emptyset \).

- (a) A point \( x \in \Omega \) can reach \( \Omega_f \) with constraint in \( \Omega \), denoted by \( x \xrightarrow{\Omega} \Omega_f \), if there exists a piecewise continuous control \( u : t \mapsto u(t) \) and \( T \geq 0 \) satisfying \( \phi^u_T(x) \in \Omega_f \) and \( \phi^u_t(x) \in \Omega \) for all \( t \in [0, T] \). Otherwise, we say \( x \) cannot reach \( \Omega_f \) with constraint in \( \Omega \), denoted by \( x \not\xrightarrow{\Omega} \Omega_f \).

- (b) A set \( \Omega' \subseteq \Omega \) can reach \( \Omega_f \) with constraint in \( \Omega \), denoted by \( \Omega' \xrightarrow{\Omega} \Omega_f \), if \( x \xrightarrow{\Omega} \Omega_f \) for every \( x \in \Omega' \).

The maximal reachable set of \( \Omega_f \) in \( \Omega \) will be denoted by \( \text{Reach}(\Omega, \Omega_f) \).

**Definition 2.2.** For a closed set \( \Omega \), a set \( A \subseteq \Omega \) is called \( \Omega \)-invariant if for all \( x_0 \in A \) and for all piecewise continuous functions \( u : t \mapsto u(t) \), every trajectory \( \phi^u_T(x_0) \) in \( \Omega \) on an interval \( [0, T] \) with \( T < \infty \) or \( [0, \infty) \) is in \( A \) on the same time interval.
The definition means that the trajectories cannot leave \( A \) before leaving \( \Omega \). The following are elementary properties of \( \Omega \)-invariant sets which can be obtained directly from the definition.

**Lemma 2.1.** The union and intersection of two \( \Omega \)-invariant sets are also \( \Omega \)-invariant sets. The union of all points \( x_0 \in A \) for which each trajectory segment \( \phi_t^{u}(x_0) \) is in \( A \) for the same time it is in \( \Omega \) is the maximal \( \Omega \)-invariant set in \( A \).

The following result relates the set of states that can reach \( \Omega_f \) with constraint in \( \Omega \) to \( \Omega \)-invariant sets. The proof is in the Appendix.

**Proposition 2.2.** \( \Omega \arr \Omega_f \) if and only if no \( \Omega \)-invariant set is in \( \Omega \setminus \Omega_f \).

Now we introduce the assumptions on \( \Sigma \). Let \( B \) denote the \((n - 1)\)-dimensional subspace spanned by the column vectors of \( B \) (namely, \( B = \text{Im}(B) \), the image of \( B \)). Define

\[
\mathcal{O} := \{x \in \mathbb{R}^n : Ax + a \in B\}.
\]

When the pair \((A, B)\) is controllable it can be shown that \( \mathcal{O} \) is an affine space (see also [17]). Notice that \( f(x, u) \) on \( \mathcal{O} \) can vanish for an appropriate choice of \( u \), so \( \mathcal{O} \) is the set of all possible equilibrium points of the system. We make the following standing assumptions until Section 7.

**Assumption 2.1.**

(A1) \( \text{rank}(B) = n - 1 \).

(A2) The pair \((A, B)\) is controllable.

(A3) \( \mathring{P} \cap \mathcal{O} = \emptyset \).

(A4) The target set \( \mathcal{F} \) is an \((n - 1)\)-dimensional polytope on the boundary of \( P \).

**Problem 2.1.** We are given \( \Sigma \) such that Assumption 2.1 holds.

(a) Find necessary and sufficient conditions such that \( P \arr \mathcal{F} \).

(b) Find \( \text{Reach}(P, \mathcal{F}) \), the maximal reachable set of \( \mathcal{F} \) in \( P \).

(c) Find a triangulation \( T \) and a piecewise affine feedback such that \( \text{Reach}(P, \mathcal{F}) \arr \mathcal{F} \).

3. Reachability on Polytopes

In this section, we focus on the open-loop reachability problem and the first aim is to find necessary and sufficient conditions for \( P \arr \mathcal{F} \). The strategy is to isolate all \( P \)-invariant sets in \( P \setminus \mathcal{F} \). Note that proofs of lemmas for this part can either be found in the Appendix or we have omitted them in case they were direct logic arguments not adding insight for the reader.

Denote by \( B_x \) the hyperplane parallel to \( B \) and going through a point \( x \). Let \( \beta \) be the unit normal vector to \( B \) satisfying \( \beta^T(Ax + a) \leq 0 \) for all \( x \in P \). Such \( \beta \) always exists by our assumption that \( \mathring{P} \cap \mathcal{O} = \emptyset \). Let \( v^- \) be a point in \( \text{arg min}\{\beta^T x : x \in \mathcal{F}\} \) and \( v^+ \) a point in \( \text{arg max}\{\beta^T x : x \in \mathcal{F}\} \). Define the closed half-spaces in \( P \)

\[
\mathcal{H}^- := \{x \in P \mid \beta^T x \leq \beta^Tv^-\},
\]

\[
\mathcal{H}^+ := \{x \in P \mid \beta^T x \geq \beta^Tv^+\}.
\]
Also, for any \( z \in \mathbb{R}^n \), define \( \mathcal{H}^-(z) := \{ x \in \mathcal{P} \mid \beta^T x \leq \beta^T z \} \) and \( \mathcal{H}^+(z) := \{ x \in \mathcal{P} \mid \beta^T x \geq \beta^T z \} \). Finally, we introduce the set \( \mathcal{P}^+ := \arg \max \{ \beta^T x \mid x \in \mathcal{P} \} \).

Because \( \mathcal{P} \cap \mathcal{O} = \emptyset \), we know that for each initial condition in \( \mathcal{P} \), all trajectories will only flow in one direction relative to hyperplane \( \mathcal{B} \). In particular, the \( \beta \)-component of any trajectory, \( \beta \cdot \phi_u(t) \), is always non-increasing by the convention that \( \beta \cdot (Ax + a) \leq 0 \), \( \forall x \in \mathcal{P} \). Now the points \( v^- \) and \( v^+ \) mark the points in \( \mathcal{F} \) with minimum and maximum \( \beta \) components. It is clear that if there is any \( x_0 \in \mathcal{P} \) with a \( \beta \) component smaller than \( v^- \), then no \( \phi_u(t) \) can reach \( \mathcal{F} \). The following lemma confirms this intuition by showing that \( \mathcal{H}^- \) and \( \mathcal{H}^+ \) are \( \mathcal{P} \)-invariant sets.

**Lemma 3.1.** Let \( z \) be a point in \( \mathcal{P} \). The sets \( \mathcal{H}^-(z) \) and \( \mathcal{H}^+(z) \) are \( \mathcal{P} \)-invariant.

The previous discussion suggests that a first necessary condition for \( \mathcal{P} \overset{P}{\rightarrow} \mathcal{F} \) is that \( \mathcal{H}^- \setminus \mathcal{F} \subset \mathcal{P} \setminus \mathcal{F} \) is empty. This is not quite right. There can be points \( x \in B_v^- \cap \mathcal{O} \) which can still reach \( \mathcal{F} \). This is the content of the next lemma.

**Lemma 3.2.** Let \( y, z \) be distinct points in \( \mathcal{P} \) and let \( l \) be the line segment joining them. If \( z, y \in \mathcal{O} \) and \( y \in B_z \), then \( z \overset{l}{\rightarrow} y \).

In light of this, we define the following set:

\[
\mathcal{B}^- := \begin{cases} 
B_{v^-} \cap \mathcal{O} & \text{if } B_{v^-} \cap \mathcal{O} \cap \mathcal{F} \neq \emptyset \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Lemma 3.2 says that points in \( \mathcal{B}^- \) can reach \( \mathcal{F} \), so these points should not be included in a candidate failure set. Thus we arrive at our first necessary condition for \( \mathcal{P} \overset{P}{\rightarrow} \mathcal{F} \): \( \mathcal{H}^- \setminus (\mathcal{F} \cup \mathcal{B}^-) = \emptyset \). Figure 2 shows a shaded region corresponding to failure of this condition.

Another failure leading to a second necessary condition is as follows. If \( x_0 \in \mathcal{O} \), then it is on the boundary of \( \mathcal{P} \) and the instantaneous motion from this point is only along \( \mathcal{B} \). If \( \mathcal{B} \) does not intersect the tangent cone of \( \mathcal{P} \) at \( x_0 \), then the only way to avoid a trajectory leaving \( \mathcal{P} \) immediately is to place an equilibrium point at \( x_0 \). The following lemma captures this situation by showing how \( \mathcal{P} \)-invariant sets arise along \( \mathcal{O} \). See the right side of Figure 2.
Lemma 3.3. Let $z$ be a point in $\mathcal{P}$. If $B_z \cap \mathcal{P} \subset \mathcal{O}$, then $B_z \cap \mathcal{P}$ and $\mathcal{P} \setminus B_z$ are $\mathcal{P}$-invariant.

A more subtle argument is needed to show that our proposed conditions are also sufficient to solve the reachability problem. Sufficiency relies on two properties: the system is controllable, so it has sufficient maneuverability on $\mathcal{O}$, and the following lemma which provides the required maneuverability off of $\mathcal{O}$.

Lemma 3.4. Let $y \neq z \in \mathcal{P}$ and let $l$ be the line segment joining them. If $z, y \notin \mathcal{O}$ and $y \in \overline{H^-(z)}$, then $z \xrightarrow{l} y$.

Theorem 3.5. $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ if and only if (a) $H^- \setminus (\mathcal{F} \cup B^-) = \emptyset$, and (b) $\mathcal{P}^+ \notin \mathcal{O} \cap \overline{H^+}$.

Proof of Theorem 3.5 (\implies) First, suppose (a) does not hold. If $\overline{H^-} \neq \emptyset$, then $\overline{H^-}$ is a $\mathcal{P}$-invariant set by Lemma 3.1 and it is in $\mathcal{P} \setminus \mathcal{F}$, so the conclusion follows from Proposition 2.2. Instead, if $\overline{H^-} = \emptyset$, then $\overline{H^-} = B^- \cap \mathcal{P}$. Since (a) does not hold, there is $z \in (B^- \cap \mathcal{P}) \setminus (\mathcal{F} \cup B^-)$. For this point $\beta^T(Az + a) < 0$, so for any $u$, $\beta^T(Az + a + Bu) < 0$. This implies any trajectory starting at $z$ immediately leaves $\mathcal{P}$. Hence, $z \xrightarrow{\mathcal{P}} \mathcal{F}$.

Second, suppose (b) does not hold, i.e. $\mathcal{P}^+ \subset \mathcal{O} \cap \overline{H^+}$. Let $z \in \mathcal{P}^+$ and notice that $\mathcal{P}^+ = B_z \cap \mathcal{P}$. Thus, $B_z \cap \mathcal{P} \subset \mathcal{O}$. By Lemma 3.3 it follows that $\mathcal{P}^+$ is $\mathcal{P}$-invariant. In addition, from the assumption that $\mathcal{P}^+ \subset \overline{H^+}$, $\mathcal{P}^+ \subset \mathcal{P} \setminus \mathcal{F}$. The conclusion follows from Proposition 2.2.

(\impliedby) Suppose conditions (a) and (b) hold. For a point $x \in \mathcal{F}$, it is trivial that $x \xrightarrow{\mathcal{P}} \mathcal{F}$.

Let $x \in \mathcal{P} \setminus (\mathcal{F} \cup \mathcal{O})$. By assumption (a) $x \notin \overline{H^-}$ or equivalently $v^- \notin \overline{H^-}(x)$. Consequently, there is a point $y \in \mathcal{N}(v^-) \cap \mathcal{F}$ satisfying $y \in \overline{H^-}(x)$, where $\mathcal{N}(v^-)$ is a sufficiently small neighborhood of $v^-$. If $\mathcal{F}$ is not in $\mathcal{O}$, such a point $y$ can be chosen not in $\mathcal{O}$. Then these two points $x$ and $y$ satisfy the assumption in Lemma 3.4, so $x \xrightarrow{l} y$, where $l$ is the line segment joining $x$ and $y$. Clearly, $l$ is in $\mathcal{P}$ as $\mathcal{P}$ is convex. Hence, $x \xrightarrow{\mathcal{P}} \mathcal{F}$. Otherwise, suppose $\mathcal{F} \subset \mathcal{O}$. It is easy to show that because $(A, B)$ is controllable, $B$ is not parallel to $\mathcal{O}$. It means we can select a control $u$ so that $f(y, u)$ points outside of $\mathcal{P}$. Thus, there is a sufficiently small $\epsilon > 0$ such that $\phi^\epsilon(y, t) \notin \overline{H^-}(x)$. Note that $\phi^\epsilon(t)$ is continuous and $y \in \overline{H^-}(x)$, so there is a point $z \in \phi^\epsilon(y, t) \in (-\epsilon, 0)$ satisfying $z \in \overline{H^-}(x)$ and therefore $z \in \mathcal{F}$. Thus, $\beta^T(Az + a) < 0$ by assumption. Applying Lemma 3.4 for the two points $x$ and $z$ leads to $x \xrightarrow{l} z$, where $l$ is the line segment in $\mathcal{P}$ joining $x$ and $z$. Considering $z \xrightarrow{\mathcal{P}} y \in \mathcal{F}$, we then have $x \xrightarrow{\mathcal{P}} \mathcal{F}$.

Finally, let $x \in (\mathcal{P} \cap \mathcal{O}) \setminus \mathcal{F}$. Clearly, $x$ is on the boundary of $\mathcal{P}$. If $B_x \cap \mathcal{P} \notin \mathcal{O}$, then select a point $y \in (B_x \cap \mathcal{P}) \setminus \mathcal{O}$ and let $u$ be chosen such that $f(x, u) = Ax + a + Bu = y - x$, which is possible because both $(Ax + a)$ and $(y - x)$ are in $\text{Im}(B)$. Note that this vector field $f(x, u)$ points inside the polytope $\mathcal{P}$. This implies the trajectory instantaneously enters the interior of $\mathcal{P}$, which is not in $\mathcal{O}$ any more. Then by the previous argument, it can be driven to reach $\mathcal{F}$ through a line. Otherwise, if $B_x \cap \mathcal{P} \subset \mathcal{O}$, then the whole set $B_x \cap \mathcal{P}$ is on the boundary of $\mathcal{P}$, and moreover it comprises either $\mathcal{P}^+$ or $\arg \min \{\beta^T x : x \in \mathcal{F} \}$. From condition (b), $\mathcal{P}^+ \subset \mathcal{H}^-(v^+)$ and this implies $\mathcal{P}^+ \cap \mathcal{F} \neq \emptyset$. From condition (a), $\mathcal{H}^- = \emptyset$ so $\arg \min \{\beta^T x : x \in \mathcal{F} \} \subset \mathcal{H}^+(v^-)$, which implies $\arg \min \{\beta^T x : x \in \mathcal{F} \} \cap \mathcal{F} \neq \emptyset$. For both
cases, we get \( B_x \cap \mathcal{F} \neq \emptyset \). Now we select a point \( y \in B_x \cap \mathcal{F} \). Then these two points satisfy the assumption in Lemma 3.2. Thus, it follows that \( x \xrightarrow{l} y \), where \( l \) is the line segment joining from \( x \) to \( y \), and so \( x \xrightarrow{p} \mathcal{F} \). \( \square \)

Theorem 3.5 gives necessary and sufficient conditions for the reachability problem \( \mathcal{P} \xrightarrow{P} \mathcal{F} \). This result, in turn, can be tied to failure sets, apropos Proposition 2.2, which are the \( \mathcal{P} \)-invariant sets in \( \mathcal{P} \setminus \mathcal{F} \):

\[
\mathcal{A}^- = \mathcal{H}^- \setminus (\mathcal{F} \cup B^-) \\
\mathcal{A}^+ = \begin{cases} 
\mathcal{P}^+ & \text{if } \mathcal{P}^+ \subset \mathcal{O} \cap \mathcal{H}^+ \\
\emptyset & \text{otherwise.} 
\end{cases}
\]

Corollary 3.6. Let \( \mathcal{A} = \mathcal{A}^- \cup \mathcal{A}^+ \). Then Reach(\( \mathcal{P}, \mathcal{F} \)) = \( \mathcal{P} \setminus \mathcal{A} \). Moreover, Reach(\( \mathcal{P}, \mathcal{F} \)) \xrightarrow{Reach(\( \mathcal{P}, \mathcal{F} \))} \mathcal{F} \) and \( \mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{F} \).

We have identified Reach(\( \mathcal{P}, \mathcal{F} \)), the maximal reachable set of \( \mathcal{F} \) in \( \mathcal{P} \). This set, in general, is not closed. This leads to difficulties with unbounded control effort and unbounded time to reach \( \mathcal{F} \). Consequently, once failure sets have been identified, it is desirable to remove them via a procedure that both well-approximates the maximal reachable set and also yields a closed \( n \)-dimensional polytope that can reach \( \mathcal{F} \). The approach is to “cut off” failure sets from \( \mathcal{P} \) by one of two procedures. One procedure is for removing the failure \( \mathcal{A}^- \) by cutting along a hyperplane which is parallel to a slightly shifted version of \( B \). The second procedure is for removing \( \mathcal{A}^+ \) by cutting exactly along a hyperplane parallel to \( B \). These cuts are chosen arbitrarily close to the failure sets and so that the remaining polytope has no failure sets. It should be noted that the following procedure can be easily adapted to convert explicit bounds on the controls to an appropriate \( \epsilon \).

Algorithm 1. (Let \( \epsilon > 0 \) be sufficiently small.)

1. If \( \mathcal{A}^- \neq \emptyset \), select affinely independent points \( z_1, \ldots, z_k \) in \( \mathcal{F} \cap B^- \) and also in the relative boundary of the facet containing \( \mathcal{F} \), and then select points \( z_{k+1}, \ldots, z_n \) in \( \mathcal{P} \cap \mathcal{H}^+ \) such that \( \mathcal{G} := \text{aff}\{z_1, \ldots, z_n\} \) is of dimension \( n-1 \) and \( \max_{x \in \mathcal{P} \cap \mathcal{G}} \text{dist}(x, B^-) = \epsilon \).

Then divide \( \mathcal{P} \) along \( \mathcal{G} \).

2. If \( \mathcal{A}^+ \neq \emptyset \), select a point \( z \in \mathcal{P} \) such that \( \max_{x \in \mathcal{A}^+} \text{dist}(x, B_z) = \epsilon \). Then divide \( \mathcal{P} \) along \( B_z \).

Let \( \mathcal{A}_{\epsilon^-}, \mathcal{A}_{\epsilon^+}, \) and Reach(\( \mathcal{P}, \mathcal{F} \)) be the collection of sets after the application of the division rules in Algorithm 1 where \( \mathcal{A}_{\epsilon^-} \) contains \( \mathcal{A}^- \), \( \mathcal{A}_{\epsilon^+} \) contains \( \mathcal{A}^+ \), and Reach(\( \mathcal{P}, \mathcal{F} \)) is the remainder. Clearly, these three sets (if not empty) are \( n \)-dimensional polytopes and \( \mathcal{F} \subset \text{Reach}(\mathcal{P}, \mathcal{F}) \subset \text{Reach}(\mathcal{P}, \mathcal{F}) \). Then we have the following corollary which follows directly from Algorithm 1 and Theorem 3.5.

Corollary 3.7.

(a) Reach(\( \mathcal{P}, \mathcal{F} \)) Reach(\( \mathcal{P}, \mathcal{F} \)) \xrightarrow{\mathcal{F}} \mathcal{F}.

(b) \( \lim_{\epsilon \to 0} \mu [\text{Reach}(\mathcal{P}, \mathcal{F}) \setminus \text{Reach}(\mathcal{P}, \mathcal{F})] = 0 \).

(c) For any \( x \in \mathcal{P} \), if \( x \xrightarrow{\mathcal{F}} \mathcal{F} \) there exists an \( \epsilon > 0 \) such that \( x \in \text{Reach}(\mathcal{P}, \mathcal{F}) \).
Example 3.1. Consider the example in Figure 3 which illustrates the first step of Algorithm 1. Suppose that there is a failure set to reach $F$ along the red segment. If one were to cut only along points in $F \cap B_v$ which corresponds to the blue plane, then the failure set would not be cut off. Instead, if points in the relative boundary of the facet containing $F$ can be used, then this failure set can be removed.

Example 3.2. A simple example is presented to illustrate the possible failure sets and how Algorithm 1 cuts them off. Consider the system

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u.
\end{align*}
$$

It can be easily verified that $O = \{(x_1, x_2) : x_2 = 0\}$, the $x_1$ axis, and that $B$ is just the $x_2$ axis. Suppose that the polytope $P$ and the target set $F$ are as shown in Figure 4. The hyperplane $O$ touches the polytope $P$ but has empty intersection with its interior. We get $A^- = H^- \setminus F$ which is the patterned region in Figure 4 and $A^+$ is just a point. $\text{Reach}(P, F)$ is the set $P'$ not including the boundary of $A^-$ and $A^+$. This set is not closed. Moreover, if an initial state $x_0 \in P'$ approaches the boundary of $A^-$, the control input $u(x_0)$ tends to infinity in order to reach $F$ with constraint in $P$. Also, if $x_0 \in P'$ approaches the boundary of $A^+$, the time to reach $F$ tends to infinity. Applying Algorithm 1 a good closed $\epsilon$-approximation $\text{Reach}_\epsilon(P, F)$ of $P'$ is given on the right of Figure 4.
Consider an $n$-dimensional simplex $S$ with vertices $v_0, v_1, \ldots, v_n$ and facets $F_0, \ldots, F_n$ (the facet is indexed by the vertex not contained). Let $I := \{1, \ldots, n\}$.

**Problem 4.1.** Consider system (2.1) defined on $S$. Find an affine feedback control $u = Fx + g$ such that for every $x_0 \in S$ there exist $T \geq 0$ and $\epsilon > 0$ satisfying:

(i) $\phi_t^u(x_0) \in S$ for all $t \in [0, T]$;
(ii) $\phi_T^u(x_0) \in F_0$;
(iii) $\phi_t^u(x_0) \notin S$ for all $t \in (T, T + \epsilon)$.

Condition (iii) is interpreted to mean that the closed-loop dynamics on $S$ are extended to a neighborhood of $S$.

**Definition 4.1.** The invariance conditions for $S$ require that there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that:

$$h_j \cdot (Av_i + a + Bu_i) \leq 0, \quad i \in \{0, \ldots, n\}, \quad j \in I \setminus \{i\}. \quad (4.1)$$

**Theorem 4.1.** [13, 20] Given the system (2.1) and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \ldots, u_n = u(v_n)$, the closed-loop system satisfies $S \xrightarrow{S} F_0$ if and only if

(a) The invariance conditions (4.1) hold.
(b) There is no equilibrium in $S$.

Theorem 4.1 cannot be used directly for our present work because it enforces that affine feedbacks be used. Unfortunately, this class is not large enough if solvability of RCP by open-loop control is the starting point. The next result shows that for hypersurface systems on simplices, one sufficiently rich feedback class is piecewise affine feedbacks. The proof is in the Appendix.

**Theorem 4.2.** [15] If $S \xrightarrow{S} F_0$ then there exists a piecewise affine state feedback $u = F_\sigma x + g_\sigma$, $\sigma \in \{1, 2\}$ that also achieves $S \xrightarrow{S} F_0$.

5. **Control Synthesis on Polytopes**

We now begin our investigation of state feedback synthesis on polytopes. We want to show that if $P \xrightarrow{P} F$ using open-loop control then there exists a piecewise affine feedback solving the reachability problem. The idea is to triangulate the polytope, transform the reachability problem within a polytope into a set of reachability problems for simplices, and then devise appropriate piecewise affine controllers on each simplex using Proposition 4.2 of the previous section. The triangulation must be performed properly otherwise the procedure may fail. First we present a lemma that aids in finding a proper triangulation.

**Lemma 5.1.** If $P \xrightarrow{P} F$, then there exists a vertex $v_*$ of $P$ in $P^+$ such that either $v_* \notin \mathcal{O}$ or $v_* \in F$.

**Proof.** Suppose by contradiction that for any vertex $v \in P^+$ we have $v \in \mathcal{O}$ and $v \notin F$. Note that $v \notin F$ for all vertices $v \in P^+$ implies, by convexity, $P^+ \subseteq \mathcal{H}^+$. Moreover, since $v \in \mathcal{O}$
for all \( v \in \mathcal{P}^+ \), it follows from the convexity of \( \mathcal{O} \) that \( \mathcal{P}^+ \subset \mathcal{O} \). Hence, by Theorem 3.5, this contradicts \( \mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F} \).

We review some concepts on triangulation [10]. Suppose \( V \) is a finite set of points such that \( \text{conv}(V) \) is \( n \)-dimensional. A subdivision of \( V \) is a finite collection \( \mathcal{P} = \{ \mathcal{P}_1, \ldots, \mathcal{P}_m \} \) of \( n \)-dimensional polytopes such that the vertices of each \( \mathcal{P}_i \) are drawn from \( V \); \( \text{conv}(V) \) is the union of \( \mathcal{P}_1, \ldots, \mathcal{P}_m \); and \( \mathcal{P}_i \cap \mathcal{P}_j (i \neq j) \) is a common (possibly empty) face of \( \mathcal{P}_i \) and \( \mathcal{P}_j \). A triangulation of \( V \) is a subdivision in which each \( \mathcal{P}_i \) is a simplex. In the following we assume that \( \mathcal{F} \) is a facet of \( \mathcal{P} \).

**Basic Triangulation of \( \mathcal{P} \):**

1. Select \( v_\ast \) as in Lemma 5.1
2. Triangulate each facet \( \mathcal{F}_j \) of \( \mathcal{P} \). Denote \( \{ \mathcal{S}'_{\mathcal{F}_j} : i = 1, \ldots, k_j \} \) the triangulation for \( \mathcal{F}_j \).
3. Let \( \mathcal{S} = \{ \mathcal{S}_1, \ldots, \mathcal{S}_q \} := \{ \text{conv}(v_\ast, \mathcal{S}'_{\mathcal{F}_j}) : \mathcal{F}_j \text{ is any facet of } \mathcal{P} \text{ not containing } v_\ast \} \).

**Lemma 5.2.** The collection \( \mathcal{S} \) is a triangulation of \( \text{vert}(\mathcal{P}) \cup \text{vert}(\mathcal{F}) \) such that every simplex in \( \mathcal{S} \) contains \( v_\ast \) as a vertex.

**Proof.** By construction, it is clear that every simplex \( \mathcal{S}_i \in \mathcal{S} \) contains \( v_\ast \) as a vertex, the vertices of \( \mathcal{S}_i \) are drawn from \( \text{vert}(\mathcal{P}) \cup \text{vert}(\mathcal{F}) \), and \( \mathcal{S}_i \cap \mathcal{S}_j (i \neq j) \) is a common (possibly empty) face of \( \mathcal{S}_i \) and \( \mathcal{S}_j \). Next, we show that \( \mathcal{P} \) is the union of \( \mathcal{S}_1, \ldots, \mathcal{S}_q \). Let \( x \) be a point in the union of \( \mathcal{S}_1, \ldots, \mathcal{S}_q \). Then \( x \) must be in a simplex \( \mathcal{S}_i \). Thus, by convexity of \( \mathcal{P} \), \( x \in \mathcal{P} \). On the other hand, let \( x \) be a point in \( \mathcal{P} \). Draw a line through \( v_\ast \) and \( x \). It intersects at a point \( y \) with a facet (say \( \mathcal{F}_j \)) of \( \mathcal{P} \) that does not contain \( v_\ast \). It means there exists a simplex \( \mathcal{S}'_{\mathcal{F}_j} \) containing \( y \). So \( x \in \text{conv}(v_\ast, \mathcal{S}'_{\mathcal{F}_j}) \), one of the simplices in \( \mathcal{S} \). The conclusion follows. \( \square \)

Now suppose we have a triangulation \( \mathcal{S} = \{ \mathcal{S}_1, \ldots, \mathcal{S}_q \} \) as above, and denote \( \mathcal{S}_0 := \mathcal{F} \). We say \( \mathcal{S}_i \) and \( \mathcal{S}_j \) are adjacent (denoted by \( \mathcal{S}_i \sim \mathcal{S}_j \)) if \( \mathcal{F}_{ij} := \mathcal{S}_i \cap \mathcal{S}_j \) is a facet. A sequence \( (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n) \) is called a path to reach \( \mathcal{S}_n \) if \( \mathcal{S}_{ij} \sim \mathcal{S}_{ij+1} \) and \( \mathcal{S}_{ij} \xrightarrow{\mathcal{S}_{ij}} \mathcal{S}_{ij+1} \) for each \( 1 \leq j \leq k \). The length of such a path is \( k \). We propose a greedy algorithm that orders simplices according to minimum \( \beta \) component of exit vertices first. More precisely, at every iteration a pair \( (\mathcal{S}_i, \mathcal{S}_j) \) is selected that minimizes the \( \beta \)-component of any vertex on the exit facet \( \mathcal{F}_{ij} \). If there is more than one pair achieving the minimum, select a pair which has the maximum number of exit vertices achieving the minimum. In the algorithm below \( \mathcal{R}_f \) and \( \mathcal{R}_u \) denote the finished and unfinished set of simplices, respectively, and let

\[
\begin{align*}
w' \in \arg \min \{ & \beta^T x : x \in \bigcup \{ \mathcal{S}_k \cap \mathcal{S}_l : (\mathcal{S}_k, \mathcal{S}_l) \in \mathcal{R}_u \times \mathcal{R}_f \text{ satisfying } \mathcal{S}_k \sim \mathcal{S}_l \} \}. \end{align*}
\]

**Greedy algorithm for path generation in \( \mathcal{S} \):**

1. Initialization: \( \mathcal{R}_f := \{ \mathcal{S}_0 \} \); \( \mathcal{R}_u := \{ \mathcal{S}_1, \ldots, \mathcal{S}_q \} \).
2. While \( \mathcal{R}_u \neq \emptyset \), choose \( (\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f \) such that \( \mathcal{S}_i \sim \mathcal{S}_j \), it achieves \( \min \beta^T x = \beta^T w' \), and \( \mathcal{F}_{ij} \) contains the maximum number of vertices in \( \mathcal{B}_{w'} \). Then move \( \mathcal{S}_i \) from \( \mathcal{R}_u \) to \( \mathcal{R}_f \).
Once the greedy algorithm has generated paths, the synthesis of a piecewise affine control is straightforward. See also [13].

**Piecewise affine synthesis:**

1. Let \( \{ \ldots, (\ldots, S_i, S_j, \ldots), \ldots \} \) be a collection of paths to reach \( S_0 \);
2. Find \( u^i(x) := F_{\sigma_i}(x) + g_{\sigma_i}(x), i = 1, \ldots, q \), that solves \( S_i \xrightarrow{S_i} F_{ij} \), where \( F_{ij} \) is the common facet of \( S_i \) and the next simplex in the path;
3. For all \( x \in S_i \in S \), let \( u(x) = u^i(x) \). If \( x \in P \) belongs to more than one simplex, set \( u(x) = u^j(x) \) where \( j \) is the index of a simplex that has the shortest path to reach \( S_0 \).

**Theorem 5.3.** Suppose that \( F \) is a facet of \( P \). There exists a piecewise affine state feedback that achieves \( P \xrightarrow{P} F \) if and only if \( P \xrightarrow{P} F \) using open-loop control.

The idea of the proof is to show that the path generation algorithm does not terminate until \( R_u = \emptyset \) by showing that for the next selected pair \( (S_i, S_j) \in R_u \times R_f \), the reachability problem \( S_i \xrightarrow{S_i} S_j \) can be solved. This is done by applying Theorem 3.5 and verifying conditions (a) and (b) for the selected pair \( (S_i, S_j) \in R_u \times R_f \). The main effect of our selection of triangulation based on vertex \( v_* \) is that condition (b) holds trivially for any such pair. The fact that condition (a) can be made to hold is the main feature of the greedy strategy with respect to \( \beta \). This strategy guarantees that the vertex \( v_0 \in S_i \) not contained in the exit facet has a strictly larger \( \beta \)-component, and this means that failure set \( A^- = \emptyset \) for \( S_i \). The proof now easily follows from these observations.

**Proof.** \((\Rightarrow)\) Obvious. \((\Leftarrow)\) If the path generation algorithm terminates with \( R_u = \emptyset \), then by straightforward dynamic programming arguments there exists a piecewise affine feedback control that achieves \( P \xrightarrow{P} F \). It is therefore sufficient to show that if \( R_u \neq \emptyset \), there exists a pair \( (S_i, S_j) \in R_u \times R_f \) such that \( S_i \cap S_j =: F_{ij} \) is a facet and \( S_i \xrightarrow{S_i} S_j \).

Consider any pair \( (S_i, S_j) \in R_u \times R_f \). We must verify conditions (a) and (b) of Theorem 3.5 to show \( S_i \xrightarrow{S_i} F_{ij} \). Consider condition (b). We have two observations about \( v_* \). First, from Lemma 5.2 \( v_* \in S_i \), \( \forall i \), and therefore \( v_* \in F_{ij} \). Second, \( v_* \in P^+ \) implies \( v_* \in S_i^+ \). Applying these two facts, condition (b) for \( S_i \xrightarrow{S_i} S_j \) says that \( S_i^+ \cap \{ x \in S_i : \beta^T x > \beta^T v_* \} \), and this is obviously true.

So far we have shown that for any pair \( (S_i, S_j) \in R_u \times R_f \) as above, condition (b) of Theorem 3.5 holds for the problem \( S_i \xrightarrow{S_i} F_{ij} \). Now we will show that for the selected pair \( (S_i, S_j) \), condition (a) holds. Let \( v_0 \) be the vertex of \( S_i \) not in \( F_{ij} \). Let \( w \in F_{ij} \cap B_{w'} \). There are three cases:

1. Suppose \( \beta^T w < \beta^T v_0 \). Then condition (a) holds.
2. Suppose \( \beta^T w > \beta^T v_0 \). Also, we know \( \beta^T v^- \leq \beta^T v_0 < \beta^T w \) from the assumption \( P \xrightarrow{P} F \). By convexity, for every point \( y \) on the line segment joining \( v^- \) and \( v_0 \), \( \beta^T y < \beta^T w \). However, \( v^- \in S_0 \in R_f \) and \( v_0 \in S_i \in R_u \), which means the line segment contains a point \( y \) on the boundary of \( S_{ij} \in R_u \) and \( S_{ij} \in R_f \). This contradicts the choice of the pair \( (S_i, S_j) \) that achieves \( \min_{x \in F_{ij}} \beta^T x = \beta^T w' \).
Example 5.1. Consider again Example 3.2. After applying Algorithm 1 to cut the failure set off, we know from Corollary 3.7 that \( \text{Reach}_\varepsilon (\mathcal{P}, \mathcal{F}) \) is achieved that \( v \in \mathcal{F} \) and \( \mathcal{F} \) is a facet of a larger polytope \( \mathcal{P}' \). However, this approach has two caveats: (1) Suppose \( \beta^T w = \beta^T v_0 \). Let \( \{v_1, \ldots, v_k\} \) be the set of vertices of \( \mathcal{F}_{ij} \) that lie in \( \mathcal{B}_{w'} \). If \( \mathcal{B}_{w'} \cap \mathcal{P} \subset \mathcal{O} \) then condition (a) holds and we are done. If not, it follows from the assumption \( \mathcal{P} \rightarrow \mathcal{F} \) that either \( \mathcal{B}_{w'} \cap \mathcal{P} \subset \mathcal{F} \) or \( \beta^T w' > \beta^T v \). For both cases we claim that \( \mathcal{G} := \text{conv}\{v_0, v_1, \ldots, v_k\} \) belongs to some \( \mathcal{S}_k \in \mathcal{R}_f \). For the former case, it is obvious since \( \mathcal{G} \subset \mathcal{B}_{w'} \cap \mathcal{P} \subset \mathcal{S}_{i} \in \mathcal{R}_f \). For the latter case, suppose not. Say a point \( x \in \mathcal{G} \) does not belong to some \( \mathcal{S}_k \in \mathcal{R}_f \). Then since the union of sets in \( \mathcal{R}_f \) is a closed set, there exists a point \( y \in \mathcal{P} \) near \( x \) satisfying \( \beta^T y < \beta^T w' \), and \( y \) also does not belong to some \( \mathcal{S}_k \in \mathcal{R}_f \). This contradicts the choice of the pair \( (\mathcal{S}_i, \mathcal{S}_j) \) that achieves \( \min_{x \in \mathcal{F}_{ij}} \beta^T x = \beta^T w' \). Therefore \( \mathcal{G} \) belongs to some \( \mathcal{S}_k \in \mathcal{R}_f \) which implies it belongs to some facet \( \mathcal{F}_{i'j'} \neq \mathcal{F}_{ij} \) with \( \mathcal{F}_{i'j'} = \mathcal{S}_i \cap \mathcal{S}_j' \), where \( \mathcal{S}_i \in \mathcal{R}_w, \mathcal{S}_j' \in \mathcal{R}_f \), and \( \mathcal{F}_{i'j'} \) has one more vertex, namely \( v_0 \), in \( \mathcal{B}_{w'} \). This contradicts the choice of \( \mathcal{F}_{ij} \).

\( \square \)

**Example 5.1.** Consider again Example 3.2. After applying Algorithm 1 to cut the failure sets off, we know from Corollary 3.7 that \( \text{Reach}_\varepsilon (\mathcal{P}, \mathcal{F}) \) is achieved using open-loop control. We want to find a piecewise affine state feedback that achieves \( \text{Reach}_\varepsilon (\mathcal{P}, \mathcal{F}) \). Denote the vertices of \( \text{Reach}_\varepsilon (\mathcal{P}, \mathcal{F}) \) by \( v_1, \ldots, v_5 \). It can be easily obtained that \( v_1 \) is the only vertex of \( \text{Reach}_\varepsilon (\mathcal{P}, \mathcal{F}) \) satisfying the property of Lemma 5.1 so \( v_* = v_1 \). Also, \( v_* \notin \mathcal{F} \). By the proposed triangulation method, we obtain a triangulation \( \mathcal{S} = \{ \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \} \) as shown in Figure 5. By applying Theorem 3.5 to each simplex, it can be easily checked that \( \mathcal{S}_3 \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_1 \rightarrow \mathcal{F} \). Thus, we can find a control to solve the reachability problem on each simplex (based on Proposition 4.2) and then we can construct a piecewise affine control which achieves \( \text{Reach}_\varepsilon (\mathcal{P}, \mathcal{F}) \).

### 6. Triangulation with respect to \( \mathcal{F} \)

In this section we study how the previous results can be extended to solve the control synthesis problem if \( \mathcal{F} \) is not given as a facet of \( \mathcal{P} \). If the designer has flexibility in modifying the given state constraints, then one perform a slight modification (by pulling out \( \mathcal{F} \)) so that \( \mathcal{F} \) is a facet of a larger polytope \( \mathcal{P}' \). However, this approach has two caveats: (1)
The problem $P' \xrightarrow{P} F'$ may not be solvable even if $P \xrightarrow{P} F$ is; (2) If $P$ is part of a larger subdivision of the state space, then possibly other polytopes in the subdivision must be modified. A more desirable procedure is to use a triangulation method that refines the given subdivision of the state space by splitting $P$ so that $F$ becomes a facet of one of the polytopes in the refined subdivision. This approach also has pitfalls, because if one does not refine the subdivision properly, failure sets may emerge even if $P \xrightarrow{P} F$ by open-loop control. In this section we show one method (among several) to obtain a proper triangulation.

Let $\bar{F}$ denote the facet of $P$ containing $F$. First we consider a simple case when $v_*$ of Lemma 5.2 can be selected so that $v_* \notin \bar{F}$. See Figure 6.

**Triangulation of $P$ with respect to $F$:**

(a) Select $v_*$ as in Lemma 5.1 and so that $v_* \notin \bar{F}$.

(b) Make a triangulation of $\text{vert}(\bar{F}) \cup \text{vert}(F)$ such that the interior of each resulting simplex is either entirely in $F$ or not in $F$. For the remaining facets $F_j$ of $P$, make a triangulation of $\text{vert}(F_j)$. Denote $\{S_i^{F_j} : i = 1, \ldots, k_j\}$ the triangulation for $F_j$.

(c) Let $S = \{S_1, \ldots, S_q\} := \{\text{conv}(v_*, S_i^{F_j}) : F_j \text{ is any facet of } P \text{ not containing } v_*\}$.

The first thing we notice is that nothing about the proof of Lemma 5.2 is specific to $F$ being a facet, so the lemma still holds for the new triangulation. Also the proof of Theorem 5.3 is unchanged since the essential property of $v_*$ (namely Lemma 5.2) is still true. Therefore, we have the following direct extension of Theorem 5.3.

**Corollary 6.1.** Suppose that $F$ is not a facet of $P$ and there exists $v_*$ as in Lemma 5.1 such that $v_* \notin \bar{F}$. There exists a piecewise affine state feedback that achieves $P \xrightarrow{P} F$ if and only if $P \xrightarrow{P} F$ using open-loop control.

When there does not exist $v_* \notin \bar{F}$, the problem is more complex because Lemma 5.2 breaks down. Nevertheless, we would like to build upon our previous triangulation and control methods by appropriately subdividing $P$. A natural idea would be to form $P_1 := \text{conv}(F, v \mid v \in V \setminus F)$, a polytope for which $F$ is a facet. There are two problems to be addressed. First, can $P_1$ have failure sets for the problem $P_1 \xrightarrow{P_1} F$ even if $P \xrightarrow{P} F$? Theorem 5.3 tells us that $H^- \setminus (F \cap B^-) = \emptyset$ and we observe that this condition is identical for any polytope with the same exit facet $F$. Therefore, condition (a) holds for $P_1 \xrightarrow{P_1} F$.\[\text{Figure 6. $F$ is not a facet of $P$ but } v_* \notin \bar{F}.\]
Instead, it is condition (b) which is problematic because generally $P_1^+ \neq P^+$ and equilibria can appear on $P_1^+$ when we try to solve $P_1 \xrightarrow{P_1 \cup O} \mathcal{F}$. A more careful approach is needed, and inspiration is provided by the proof of Theorem 5.3: for any $n$-dimensional polytope $\tilde{P} \subset P$ with exit facet $\mathcal{F}$, if $\mathcal{P}^+ \cap \mathcal{F} = \emptyset$, then condition (b) automatically holds. See Figure 7 for an example. Thus, we have the following.

**Proposition 6.2.** Suppose there exists $v_* \in \mathcal{F}$ such that $v_* \in \mathcal{F} \cap P^+$. Let $\tilde{P} \subset P$ be an $n$-dimensional polytope such that $\mathcal{F}$ is a facet of $\tilde{P}$. Then $P \xrightarrow{P} \mathcal{F}$ implies $\tilde{P} \xrightarrow{\tilde{P}} \mathcal{F}$.

**Proof.** Consider condition (b) of Theorem 3.5 for $\tilde{P} \xrightarrow{\tilde{P}} \mathcal{F}$. We have to show that $\tilde{P}^+ \not\subset O \cap \{x \in \tilde{P} \mid \beta^T x > \beta^T v^+\}$. But $v_* \in \mathcal{F} \cap P^+$ implies $\tilde{P}^+ = \{x \in \tilde{P} \mid \beta^T x = \beta^T v^+\}$, so condition (b) is obviously true.

For condition (a), Theorem 3.5 tells us that $H^- \setminus (B^- \cup \mathcal{F}) = \emptyset$ and since $\tilde{H}^- = H^-$ and $\tilde{B}^- = B^-$, condition (a) obviously holds for $\tilde{P} \xrightarrow{\tilde{P}} \mathcal{F}$. \qed

Proposition 6.2 gives some indication of how the polytope $P_1$ which has $\mathcal{F}$ as a facet could be constructed. Now we face the second problem. The set $P \setminus P_1$ is of course not a polytope. How shall it be subdivided and what reachability problems need to be assigned to avoid new failure sets from appearing? The problem is difficult due to the generality of the description of $\mathcal{F}$. However, the following proposition gives some indication of how other polytopes can be constructed which do not have $\mathcal{F}$ as their exit facet.

**Proposition 6.3.** Suppose there exists $v_* \in \mathcal{F}$ such that $v_* \in \mathcal{F} \cap P^+$. Let $\tilde{\mathcal{P}} \subset P$ be an $n$-dimensional polytope and let $\tilde{\mathcal{F}}$ be an $(n-1)$-dimensional polytope which is a facet of $\tilde{\mathcal{P}}$. Suppose there exist $\tilde{v}^- \in \tilde{\mathcal{F}} \cap \arg\min\{\beta^T x \mid x \in P\}$ and $\tilde{v}^+ \in \tilde{\mathcal{F}} \cap P^+$. Then $P \xrightarrow{P} \mathcal{F}$ implies $\tilde{\mathcal{P}} \xrightarrow{\tilde{\mathcal{P}}} \tilde{\mathcal{F}}$.

**Proof.** By the assumption $\tilde{v}^+ \in \tilde{\mathcal{F}} \cap P^+$ and by the same argument as in Proposition 6.2, condition (b) for $\tilde{\mathcal{P}} \xrightarrow{\tilde{\mathcal{P}}} \tilde{\mathcal{F}}$ obviously holds. Consider condition (a) for $P \xrightarrow{P} \mathcal{F}$. It says that $\{x \in P \mid \beta^T x \leq \beta^T v^-\} \setminus (\mathcal{F} \cup B^-) = \emptyset$. Equivalently, $\{x \in P \mid \beta^T x < \beta^T v^+\} = \emptyset$ and $B_{v^+} \cap P \subset O \cup \mathcal{F}$. Because $\tilde{v}^- \in \tilde{\mathcal{F}} \cap \arg\min\{\beta^T x \mid x \in P\}$, this means $\{x \in \tilde{\mathcal{P}} \mid \beta^T x < \beta^T v^+\} = \emptyset$.
Proof. \( \beta^T \bar{v}^- \) = \( \emptyset \) and because \( B_v^- \cap P \subset O \cup \mathcal{F} \), one obtains \( B_v^- \cap \bar{P} \subset O \cup \bar{\mathcal{F}} \). Thus condition (a) for \( \bar{P} \xrightarrow{\mathcal{P}} \bar{\mathcal{F}} \) holds. So the conclusion follows. □

We would like to apply Propositions 6.2 and 6.3 to solve the synthesis problem when \( \mathcal{F} \) is not a facet of \( P \) and there exists a vertex of \( \mathcal{F} \) satisfying \( v_* \in \mathcal{F} \cap P^+ \). We introduce an important new construct for synthesis of piecewise affine controllers. Rather than using a subdivision of \( P \) we begin the design with a cover of \( P \), which later will be refined to a subdivision for control synthesis. A cover of \( V \) is a finite collection \( P = \{ P_1, \ldots, P_m \} \) of \( n \)-dimensional polytopes such that the vertices of each \( P_i \) are drawn from \( V \) and \( \text{conv}(V) \) is the union of \( P_1, \ldots, P_m \). Informally, a cover is a subdivision except that the sub-polytopes can intersect on their interiors.

**Cover of \( P \) with respect to \( \mathcal{F} \):**

1. Select \( v_* \) a vertex of \( \mathcal{F} \) such that \( v_* \in \mathcal{F} \cap P^+ \).
2. Construct any hyperplane that goes through points \( v^- \) and \( v_* \), and partitions \( P \) into two \( n \)-dimensional sub-polytopes \( P_2 \) and \( P_3 \).
3. Define \( P_1 = \text{conv}(\mathcal{F}, P_2 \cap P_3) \).
4. Define the cover \( \mathcal{P} := \{ P_1, P_2, P_3 \} \).

By using this cover, we obtain the following result.

**Theorem 6.4.** Suppose that \( \mathcal{F} \) is not a facet of \( P \) and there exists \( v_* \), a vertex of \( \mathcal{F} \), such that \( v_* \in \mathcal{F} \cap P^+ \). There exists a piecewise affine state feedback that achieves \( P \xrightarrow{\mathcal{P}} \mathcal{F} \) if and only if \( P \xrightarrow{\mathcal{P}} \mathcal{F} \) using open-loop control.

**Proof.** \( P_1 \) is an \( n \)-dimensional polytope in \( P \) for which \( \mathcal{F} \) is a facet. Also, \( v_* \in \mathcal{F} \cap P^+ \), so by Proposition 6.2, \( P_1 \xrightarrow{P} \mathcal{F} \). Next, let \( P_{23} = P_2 \cap P_3 \) and notice that \( v^-_{23} = v^- \) and \( v^+_{23} = v_* \). So by Proposition 6.3, \( P_2 \xrightarrow{P_2} P_{23} \) and \( P_3 \xrightarrow{P_3} P_{23} \).

Theorem 5.3 gives a piecewise affine control \( u(x) = F_{\sigma_1}(x)x + g_{\sigma_1}(x), x \in P_1 \), that achieves \( P_1 \xrightarrow{P_1} \mathcal{F} \), and it gives \( u(x) = F_{\sigma_2}(x)x + g_{\sigma_2}(x), x \in P_2 \) and \( u(x) = F_{\sigma_3}(x)x + g_{\sigma_3}(x), x \in P_3 \), that achieve \( P_2 \xrightarrow{P_2} P_{23} \) and \( P_3 \xrightarrow{P_3} P_{23} \), respectively. Since \( P_{23} \subset P_3 \), it means that the controllers can drive all the states not in \( P_1 \) to \( P_1 \). Thus, the following controller

\[
\begin{align*}
  u(x) &= \begin{cases} \\
  F_{\sigma_1}(x)x + g_{\sigma_1}(x) & x \in P_1 \\
  F_{\sigma_2}(x)x + g_{\sigma_2}(x) & x \in P_2 \setminus P_1 \\
  F_{\sigma_3}(x)x + g_{\sigma_3}(x) & x \in P_3 \setminus P_1 
\end{cases}
\end{align*}
\]

achieves \( P \xrightarrow{P} \mathcal{F} \). □

Finally, we are left with the case when \( \mathcal{F} \) is not a facet of \( P \), all vertices of \( P \) satisfying Lemma 5.1 are in \( \mathcal{F} \) but none of them is in \( \mathcal{F} \), and moreover there are no vertices of \( \mathcal{F} \) in \( P^+ \). See Figure 8. Fortunately, this case can be easily handled by our previous results, by observing that \( \mathcal{F} \) and \( P^+ \) are strongly separated so we can split \( P \) into a sub-polytope which contains \( \mathcal{F} \) and satisfies Theorem 6.4 and another sub-polytope that does not contain \( \mathcal{F} \) but must be able to reach it. We have the following straightforward extension of Theorem 6.4 and main result of this section.
\textbf{Theorem 6.5.} Suppose that $\mathcal{F}$ is not a facet of $\mathcal{P}$. There exists a piecewise affine state feedback that achieves $\mathcal{P} \xrightarrow{P} \mathcal{F}$ if and only if $\mathcal{P} \xrightarrow{P} \mathcal{F}$ using open-loop control.

\textit{Proof.} We only consider the case excluded by Corollary 6.1 and Theorem 6.4 as described above. Consider the hyperplane $\mathcal{B}_{v^+} \mathcal{P}$ that partitions $\mathcal{P}$ into two sub-polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{F} \subset \mathcal{P}_1$ and $v^+$ is a vertex of $\mathcal{F}$ satisfying $v^+ \in \mathcal{F} \cap \mathcal{P}_1^+$ (see Figure 8 for an example). From Theorem 6.4, we have that $\mathcal{P}_1 \xrightarrow{p_1} \mathcal{F}$ and from the assumption $\mathcal{P} \xrightarrow{P} \mathcal{F}$ and Theorem 6.5, it can be verified that $\mathcal{P}_2 \xrightarrow{p_2} \mathcal{B}_{v^+} \mathcal{P}$.

\section{7. Triangulation with respect to $\mathcal{O}$}

So far we have studied reachability problems and control synthesis under the assumption $\mathcal{O} \cap \mathcal{P} = \emptyset$. In order to solve the general problem when $\mathcal{O} \cap \mathcal{P} \neq \emptyset$ we want to partition $\mathcal{P}$ along $\mathcal{O}$ and apply the results of the previous sections. There are two related complications. First, it can happen that when we split $\mathcal{P}$ along $\mathcal{O}$ to form two polytopes, $\mathcal{P}_1$ and $\mathcal{P}_2$, one of the two target sets $\mathcal{P}_i \cap \mathcal{F}$, even if not empty, may no longer be an $(n-1)$-dimensional polytope. Even if for example $\mathcal{P}_i \xrightarrow{p_i} \mathcal{P}_i \cap \mathcal{F}$ with the target of dimension less than $(n-1)$, the control synthesis methods of the previous section do not apply. Second, the same lower-dimensional reachability problem can arise even if we have not already partitioned along $\mathcal{O}$. Therefore, we assume in the following that when we say $\mathcal{P} \xrightarrow{P} \mathcal{F}$, there does not exist a full-dimensional set of states in $\mathcal{P}$ that must reach a lower-dimensional (less than $n-1$) subset in $\mathcal{F}$ in order to achieve $\mathcal{P} \xrightarrow{P} \mathcal{F}$.

Now we would like to propose a partition method which splits $\mathcal{P}$ along $\mathcal{O}$ into two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$. Each subpolytope $\mathcal{P}_i$ will then have two possible target sets. One target is the original facet $\mathcal{F} \cap \mathcal{P}_i$. A second target is $\mathcal{O} \cap \mathcal{P}_i$. This second target captures the idea that some trajectories must cross over from one side of $\mathcal{O}$ to the other before reaching $\mathcal{F}$. This means that a new reachability problem must be investigated which involves two targets. One could try to make a subdivision according to which target the points in $\mathcal{P}_i$ can reach.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{$\mathcal{F} \subset \bar{\mathcal{F}} = \text{conv}\{v_1, v_2, v_3, v_4\}$ and all $v_\ast$ are in $\bar{\mathcal{F}}$ but none of them is in $\mathcal{F}$.}
\end{figure}
However, this approach will generally require new techniques not already developed in the paper. We illustrate with an example.

**Example 7.1.** Consider the 2D example as in Fig. 9. Suppose there are two target sets $F_1$ and $F_2$ where $F_2 \subset O$. It can be checked that $\mathcal{P} \rightarrow F_1 \cup F_2$, but neither $\mathcal{P} \rightarrow F_1$ or $\mathcal{P} \rightarrow F_2$ holds. If we were to apply Algorithm 1 to cut off the failure set for reaching $F_1$, we would obtain the region on the left-side of the (red) dotted line (parallel to $B$). However, the approximate failure set to reach $F_1$ cannot reach $F_2$, no matter how small is $\epsilon$, without crossing into the region that can reach $F_1$. Thus, if one insists on a true subdivision, the reachability problem would not be solvable using our feedback methods. On the other hand, $\text{Reach}_\epsilon(\mathcal{P}, F_1)$ and $\text{Reach}_\epsilon(\mathcal{P}, F_2)$ is a cover for $\mathcal{P}$, where $\text{Reach}_\epsilon(\mathcal{P}, F_1)$ is the right-side of the red line and $\text{Reach}_\epsilon(\mathcal{P}, F_2)$ is the left-side of the green line.

To most efficiently overcome the issue in the above example, we first subdivide $\mathcal{P}$ along $O$ and then use a cover in each subpolytope according to two possible target sets.

**Cover of $\mathcal{P}$ with respect to $O$:** (Let $\epsilon > 0$ be sufficiently small.)

(a) Divide $\mathcal{P}$ along $O$ to obtain $\mathcal{P}_1$ and $\mathcal{P}_2$.
(b) If $\dim(\mathcal{P}_i \cap \mathcal{F}) = n - 1$, compute $Q_{i1} := \text{Reach}_\epsilon(\mathcal{P}_i, \mathcal{P}_i \cap \mathcal{F})$, $i = 1, 2$. Otherwise $Q_{i1} = \emptyset$.
(c) If $\dim(\mathcal{P}_i \cap Q_{j1}) = n - 1$, compute $Q_{i2} := \text{Reach}_\epsilon(\mathcal{P}_i, \mathcal{P}_i \cap Q_{j1})$, $i = 1, 2$, $j \neq i$. Otherwise $Q_{i2} = \emptyset$.
(d) Define the cover $\mathcal{P} := \{Q_{11}, Q_{12}, Q_{21}, Q_{22}\}$.

**Theorem 7.1.** Suppose $\mathcal{P} \cap O \neq \emptyset$. There exists a piecewise affine state feedback that achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ if and only if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ using open-loop control.

The main idea of the result is that when $\mathcal{P}$ is partitioned along $O$, there are only two types of points in each sub-polytope: those that reach $\mathcal{F}$ while remaining in the sub-polytope, or those that cross over to the other polytope to then reach $\mathcal{F}$. The proof requires a technical lemma on reachability of two target sets, whose proof is in the Appendix.

**Lemma 7.2.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $(n-1)$-dimensional polytopes on the boundary of $\mathcal{P}$ but not on a common hyperplane and assume $\mathcal{P} \cap O = \emptyset$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_1 \cup \mathcal{F}_2$, then there exists $\epsilon > 0$ sufficiently small such that $\text{Reach}_\epsilon(\mathcal{P}, \mathcal{F}_1) \cup \text{Reach}_\epsilon(\mathcal{P}, \mathcal{F}_2) = \mathcal{P}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Example 7.1.}
\end{figure}
Proof of Theorem 7.1. (⇒) Obvious. (⇐) We use the notation \( \mathcal{P}_i \xrightarrow{P} \mathcal{P}_i \cap \mathcal{F} \) to mean open loop reachability with an \((n-1)\)-dimensional target. We consider two cases. For the first case, suppose there exists one sub-polytope, say w.l.o.g. \( \mathcal{P}_1 \), satisfying \( \mathcal{P}_1 \xrightarrow{P} \mathcal{P}_1 \cap \mathcal{F} \). If, in addition, \( \mathcal{P}_2 \xrightarrow{P} \mathcal{P}_2 \cap \mathcal{F} \), then we are done. Otherwise, find \( \mathcal{Q}_{21} \) by the method above. Also compute \( \mathcal{Q}_{22} := \text{Reach}_\epsilon(\mathcal{P}_2, \mathcal{P}_2 \cap \mathcal{P}_1) \). Now we know that if \( \dim(\mathcal{P}_2 \cap \mathcal{F}) < n - 1 \), then \( \mathcal{P} \xrightarrow{P} \mathcal{F} \) implies \( \mathcal{Q}_{22} = \mathcal{P}_2 \) by our assumption, and we are done. Instead, if \( \dim(\mathcal{P}_2 \cap \mathcal{F}) = n - 1 \) then by Lemma 7.2, \( \mathcal{Q}_{21} \cup \mathcal{Q}_{22} = \mathcal{P}_2 \).

For the second case, suppose no \( \mathcal{P}_i \in \{ \mathcal{P}_1, \mathcal{P}_2 \} \) satisfies \( \mathcal{P}_i \xrightarrow{P} \mathcal{P}_i \cap \mathcal{F} \). Without loss of generality, suppose \( \dim(\mathcal{P}_1 \cap \mathcal{F}) = n - 1 \) and \( \mathcal{Q}_{11} \neq \emptyset \). Find \( \mathcal{Q}_{21} \), as above. Note that \( \mathcal{Q}_{21} \) may be empty. Because \( \mathcal{P} \xrightarrow{P} \mathcal{F} \), there exists \( \epsilon > 0 \) sufficiently small so that \( \dim(\mathcal{P}_2 \cap \mathcal{Q}_{11}) = n - 1 \) and the states in \( \mathcal{P}_2 \) that cannot reach \( \mathcal{P}_2 \cap \mathcal{F} \) must be able to reach \( \mathcal{P}_2 \cap \mathcal{Q}_{11} \). Therefore, we have \( \mathcal{P}_2 \xrightarrow{P} (\mathcal{P}_2 \cap \mathcal{F}) \cup (\mathcal{P}_2 \cap \mathcal{Q}_{11}) \). Compute \( \mathcal{Q}_{22} \) by the method above. Now we know that if \( \dim(\mathcal{P}_2 \cap \mathcal{F}) < n - 1 \), then \( \mathcal{P}_2 \xrightarrow{P} (\mathcal{P}_2 \cap \mathcal{F}) \cup (\mathcal{P}_2 \cap \mathcal{Q}_{11}) \) implies \( \mathcal{Q}_{22} = \mathcal{P}_2 \) by our assumption. Instead, if \( \dim(\mathcal{P}_2 \cap \mathcal{F}) = n - 1 \) then by Lemma 7.2, \( \mathcal{Q}_{21} \cup \mathcal{Q}_{22} = \mathcal{P}_2 \). Repeating the argument for \( \mathcal{P}_1 \), the result is obtained. \( \square \)

8. Conclusion

We have presented methods of triangulation, subdivision, and covers for reachability and control synthesis for affine hypersurface systems. Some unique features of this work are: (1) We begin with an analysis of open-loop reachability, and we do not impose what class of controls should be used to implement the reachability specifications. Because of the structure of hypersurface systems, we then derive that piecewise affine feedbacks are a sufficiently rich class to solve such problems. (2) We place emphasis on techniques of triangulation and subdivision, guided by the principle that these cannot be performed independently of control synthesis. In particular, we show how the flow conditions of a system provide critical information for triangulation, and this can be used to establish greedy dynamic programming algorithms which are guaranteed to outperform dynamic programming algorithms based on random triangulations of the polytopic state space: our algorithm always finds a solution when one exists via open-loop control. (3) We introduce a technique of covers for forming partitions of the state space. This useful technique overcomes many technical problems with taking subdivisions. Fortunately, it naturally leads to synthesis of piecewise affine feedbacks.

We have concentrated on hypersurface systems because of their simple, well-understood reachable sets. To extend our ideas to general systems, a carefully weighed analysis of the tradeoff between the conservatism of reach set approximations and complexity of the resulting algorithms must be made. Our work points in the direction of keeping the algorithms as simple as possible, by using the simplest possible partition methods which can guarantee successful termination of numerical procedures. Our future work will explore these challenging problems.

Appendix

8.1. Proof of Proposition 2.2. (⇒) Assume that there exists an \( \Omega \)-invariant set, say \( \mathcal{A} \), in \( \Omega \setminus \Omega_f \). For any \( x_0 \in \mathcal{A} \), let \( u : t \mapsto u(t) \) be any piecewise continuous function. Then
by Definition 2.2 every trajectory in $\Omega$ on an interval is also in $\mathcal{A}$ on the same time interval. Furthermore, since $\mathcal{A} \cap \Omega_f = \emptyset$ by assumption, it means $x_0 \not\rightarrow_{\Omega} \Omega_f$.

$(\Leftarrow)$ Assume it is not true that $\Omega \rightarrow_{\Omega} \Omega_f$. Then $\Omega$ can be partitioned into two nonempty sets $\Omega'$ and $\Omega''$, where $\Omega' \rightarrow_{\Omega} \Omega_f$ and $\Omega'' \not\rightarrow_{\Omega} \Omega_f$. It is easily seen that $\Omega''$ is an $\Omega$-invariant set, since otherwise there would exist some trajectory $\phi_u(x_0)$ with $x_0 \in \Omega''$ that reaches $\Omega s \setminus \Omega'' = \Omega'$. Also $\Omega'' \subset \Omega \setminus \Omega_f$. This completes the proof.

8.2. Lipschitz Continuity of Marginal Functions. Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets, $G$ be a set-valued map from $\mathcal{Y}$ to $\mathcal{X}$ and $f$ be a real-valued function defined on $\mathcal{X} \times \mathcal{Y}$. We consider the family of maximization problems

$$
g(y) := \max_{x \in G(y)} f(x, y),$$

which depend upon the parameter $y$. The function $g$ is called the marginal function. A general discussion on continuity properties of marginal functions can be found in [1]. Here we focus on the case of linear affine functions and single out a useful consequence of Lipschitz continuity. Let

$$f(x, y) = a^T x + b \quad \text{and} \quad G(y) = \{x \in \mathcal{P} : c^T x = y\},$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, $c \in \mathbb{R}^n$, and $\mathcal{P}$ is a full dimensional polytope in $\mathbb{R}^n$. (In another form, $\mathcal{P}$ can be written as $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \preceq e\}$, where $A \in \mathbb{R}^{m \times n}$, $e \in \mathbb{R}^m$, and $\preceq$ means less or equal componentwise.) The domain of the marginal function $g$ is given by $\mathcal{D} = \{y \in \mathbb{R} : G(y) \neq \emptyset\}$.

**Lemma 8.1.** The marginal function $g(y)$ is locally Lipschitz on its domain $\mathcal{D}$.

**Proof.** For any $y_1, y_2 \in \mathcal{D}$, it is clear that $G(y_1)$ and $G(y_2)$ are lower-dimensional polytopes in $\mathcal{P}$. Let $v_1, \ldots, v_k$ be the vertices of $G(y_1)$. For each $i = 1, \ldots, k$, let a point start moving from $v_i$ along the edges of $\mathcal{P}$. It first meets the hyperplane $c^T x = y_2$ at a point, denoted by $w_i$. Then, $w_i$ must be a vertex of $G(y_2)$ (note that $w_i$ and $w_j$ may not be distinct). The path that the point goes through from $v_i$ to $w_i$ is composed of either a single edge or joint edges of $\mathcal{P}$ (see Figure 10 for an illustration in 2D). Firstly, if it is a single edge of $\mathcal{P}$, this edge can be algebraically represented by $\{x \in \mathbb{R}^n : A_{i1} x = e_{i1} \text{ and } A_{i2} x < e_{i2}\}$, where

![Figure 10. An illustration of the proof.](image-url)
Consider any initial state $x, y$. Notice that with initial state $y$, of $A_1 \in \mathbb{R}^{(n-1)\times n}$ and $A_2 \in \mathbb{R}^{(m-n+1)\times n}$ are formed by the columns of $A$ with suitable order. Since $v_i$ is on the edge and also on the hyperplane $c^T x = y_1$, it follows that

$$v_i = \left[ \begin{array}{c} A_{i1} \\ c^T \end{array} \right]^{-1} \left[ \begin{array}{c} e_{i1} \\ y_1 \end{array} \right].$$

For the same reason, we have

$$w_i = \left[ \begin{array}{c} A_{i1} \\ c^T \end{array} \right]^{-1} \left[ \begin{array}{c} e_{i1} \\ y_2 \end{array} \right].$$

Hence, $\|v_i - w_i\| \leq L_i \|y_1 - y_2\|$, where $L_i$ only depends on $A$ and $c$. Secondly, if it is composed of joint edges, without loss of generality, say there are two connected edges since it has the same argument for the case with more than two edges. Two edges are connected at a point, say $v$, which lies between the hyperplanes $c^T x = y_1$ and $c^T x = y_2$ (see Figure 10 for an example). Let the parallel hyperplane going through the point $v$ be $c^T x = y'$. Thus, $y' \in [y_1, y_2]$. By the same argument above, it follows that $\|v_i - v\| \leq L_i \|y' - y_1\|$ and $\|v - w_i\| \leq L_i \|y_2 - y'\|$, where $L_i$ and $L_i^2$ depend on $A$ and $c$. Let $L_i = \max(L_i^1, L_i^2)$. Thus, we have

$$\|v_i - w_i\| \leq \|v_i - v\| + \|v - w_i\| \leq L_i \|y' - y_1\| + L_i \|y_2 - y'\| \leq L_i (\|y' - y_1\| + \|y_2 - y'\|) = L_i \|y_2 - y_1\|.$$

Next, we show that $g(y_1) - g(y_2) \leq L \|y_1 - y_2\|$, where $L$ is a constant. We know that for any $y_1 \in \mathcal{D}$, there exists an $x_1 \in G(y_1)$ satisfying $g(y_1) = f(x_1, y_1)$. On the other hand, the point $x_1$ can be written as a convex combination of the vertices of $G(y_1)$, i.e., $x_1 = \sum_{i=1}^{k} \lambda_i v_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Now consider the same convex combination of points $w_i, i = 1, \ldots, k$, which is given by $x_2 = \sum_{i=1}^{k} \lambda_i w_i$. Notice that $w_i, i = 1, \ldots, k$ are vertices of $G(y_2)$ as we showed before, so the point $x_2$ is in $G(y_2)$ and therefore $g(y_2) \geq f(x_2, y_2)$. Then we deduce that

$$g(y_1) - g(y_2) \leq f(x_1, y_1) - f(x_2, y_2) \leq \|a\| \|x_1 - x_2\| \leq \|a\| \|\sum_{i=1}^{k} \lambda_i (v_i - w_i)\| \leq \|a\| \sum_{i=1}^{k} \lambda_i \|v_i - w_i\| \leq \|a\| \sum_{i=1}^{k} \lambda_i \|y_1 - y_2\| \leq \|a\| \max_i (L_i) \|y_1 - y_2\|.$$

Recall that $L_i$ depends only on $A$ and $c$. So there is an upper bound $\bar{L}$ only depending on $A$ and $c$ such that $\bar{L} \geq L_i$ for any $i$. Thus, let $L = \|a\| \bar{L}$ and we obtain $g(y_1) - g(y_2) \leq L \|y_1 - y_2\|$. Hence, it is locally Lipschitz.

8.3. Proof of Lemma 3.3. Let $G : \mathbb{R} \to 2^\mathcal{P}$ be the set-valued map $G(y) = \{x \in \mathcal{P} : \beta^T x = y\}$. Its domain is $\mathcal{D} = \{y \in \mathbb{R} : G(y) \neq \emptyset\}$. We define the real-valued function

$$g(y) := \max\{\beta^T (Ax + a) : x \in G(y)\}, \quad y \in \mathcal{D}.$$

By Lemma 3.1, the function $g(\cdot)$ is locally Lipschitz. Let $\phi_t(y_0)$ be the solution of $\dot{y} = g(y)$ with initial state $y_0$. Since $\beta^T (Ax + a) \leq 0$ for all $x \in \mathcal{P}$ and $\mathcal{B}_c \cap \mathcal{P} = G(y^*)$, for some $y^* \in \mathcal{D}$, we have $g(y^*) \leq 0$. Thus, we know $\phi_t(y_0) \leq y^*$ for all $t \geq 0$ if $y_0 \leq y^*$. Now, consider any initial state $x_0 \in H_c \cap \mathcal{P}$ and any piecewise continuous function $u : t \mapsto u(t)$. Let $x(t), t \in [0, T)$ be the trajectory segment defined in $\mathcal{P}$ with initial condition $x_0$ and control input $u(t)$. Introduce $\xi(t) = \beta^T x(t), t \in [0, T)$. Then we have

$$\dot{\xi}(t) = \beta^T \dot{x}(t) = \beta^T (Ax(t) + a + Bu(t)) = \beta^T (Ax(t) + a).$$

Notice that $x(t) \in \mathcal{P}$ and $\beta^T x(t) = \xi(t)$. It implies that $x(t) \in G(\xi(t))$ for $t \in [0, T]$. Hence, we know $\beta^T (Ax(t) + a) \leq g(\xi(t))$ (or equivalently, $\dot{\xi}(t) \leq g(\xi(t))$) from our construction of $g(\cdot)$. By the Comparison Principle (Theorem 1.4.1, [14]) it follows that $\xi(t) \leq \phi_t(\xi(0))$.
for \( t \in [0, \bar{T}] \). Also, \( x_0 \in \mathcal{H}_z \cap \mathcal{P} \) implies that \( \xi(0) \leq y^* \), so \( \phi_t(\xi(0)) \leq y^* \). Consequently, we obtain \( \xi(t) \leq y^* \), which in turn implies \( x(t) \in \mathcal{H}_z \cap \mathcal{P} \) for all \( t \in [0, \bar{T}] \), meaning that \( \mathcal{H}_z \cap \mathcal{P} \) is \( \mathcal{P} \)-invariant. Following along the same lines, \( \mathcal{H}_z \cap \mathcal{B} \) is \( \mathcal{P} \)-invariant.

8.4. **Proof of Lemma 3.2.** For any \( x \in l \), let \( Ax + a + Bu = \lambda'(y - z) \) where \( \lambda' > 0 \) is any constant. Note that \( y \in \mathcal{B}_z \) implies \( (y - z) \in \text{Im}(B) \), and by convexity we have \( \beta^T(Ax + a) = 0 \), which implies \( (Ax + a) \in \text{Im}(B) \). Therefore, the above linear equation has a unique solution \( u_x \). Then following along the same lines as for Lemma 3.4, it is obtained that \( z \overset{t}{\rightarrow} y \).

8.5. **Proof of Lemma 3.3.** If \( \beta^T(Ax + a) = 0 \) for all \( x \in \mathcal{B}_z \cap \mathcal{P} \), then from Lemma 3.1 it follows that \( \mathcal{H}_z \cap \mathcal{P} \) and \( \mathcal{H}_z \cap \mathcal{B} \) are \( \mathcal{P} \)-invariant. On the other hand, rewrite \( \beta^T(Ax + a) = 0 \) as \( (-\beta)^T(Ax + a) = 0 \). Then \( (-\beta)^T(Ax + a) \leq 0 \) for all \( x \in \mathcal{B}_z \cap \mathcal{P} \), so again from Lemma 3.1 we obtain \( \mathcal{H}_z \cap \mathcal{P} \) and \( \mathcal{H}_z \cap \mathcal{B} \) are \( \mathcal{P} \)-invariant. By Lemma 2.1 \( (\mathcal{H}_z \cap \mathcal{P}) \cap (\mathcal{H}_z \cap \mathcal{B}) \) are \( \mathcal{P} \)-invariant. The former set is exactly \( \mathcal{B}_z \cap \mathcal{P} \) and the latter set is \( \mathcal{P} \setminus \mathcal{B}_z \).

8.6. **Proof of Lemma 3.4.** Since \( y \in \mathcal{H}_z \), one obtains \( \beta^T(y - z) < 0 \). It implies that the stacked matrix \([-B_{(y - z)}] \) is of full rank. Then there is a unique solution \( u_x \) and \( \lambda_x \) to the linear equation \( Ax + a + Bu_x = \lambda_x(y - z) \) for a given point \( x \in l \). Moreover, from the assumption \( z, y \notin \mathcal{O} \), we obtain that \( \beta^T(Az + a) < 0, \beta^T(Ay + a) < 0 \) and then by convexity we have \( \beta^T(Ax + a) < 0, \forall x \in l \). So \( \beta^T(Ax + a + Bu_x) < 0 \). This together with \( \beta^T(y - z) < 0 \) leads to \( \lambda_x > 0, \forall x \in l \). Applying \( u_x \), the resulting closed-loop system is \( \dot{x} = \lambda_x(y - z) \). Thus, the trajectory remains in \( l \). Moreover, in the compact set \( l \), \( \lambda_x \) is bounded away from zero. So \( \beta^T \lambda_x(y - z) < \delta \) for some \( \delta < 0 \), which implies that the trajectory starting from \( z \) reaches \( y \) in finite time.

8.7. **Proof of Proposition 4.2.** We begin by checking the invariance condition. Let \( w_-(w_+) \) be a vertex in \( \arg \min \{ \beta^T v : v \in \text{vert}(\mathcal{F}_0) \} \) (\( \arg \max \{ \beta^T v : v \in \text{vert}(\mathcal{F}_0) \} \), respectively).

First, we consider the case that \( v_i \notin \mathcal{O} \). For this, we discuss two situations depending on \( \beta^T v_i > \beta^T w_- \) or \( \beta^T v_i = \beta^T w_- \). (Note that it is impossible to have \( \beta^T v_i < \beta^T w_- \) by Theorem 3.3(a) since \( \mathcal{S} \rightarrow \mathcal{F}_0 \).)

(i) If \( \beta^T v_i > \beta^T w_- \), then there is a point \( p \in \mathcal{S} \) such that \( \beta^T v_i > \beta^T p \) (or equivalently \( \beta^T(p - v_i) < 0 \)). Let \( Av_i + a + Bu_i = \lambda(p - v_i) \),

\[
Av_i + a + Bu_i = \lambda(p - v_i), \tag{8.1}
\]

where \( \lambda \) is a scalar to be determined. Writing in a compact form, we have

\[
\begin{bmatrix}
-\mathcal{B} & (p - v_i)
\end{bmatrix}
\begin{bmatrix}
\lambda \\
u_i
\end{bmatrix}
= Av_i + a.
\]

Note that \( (p - v_i) \notin \mathcal{B} \), so the matrix \([-\mathcal{B} (p - v_i)] \) is of full rank and therefore the above equation has a unique solution \( u_i \) and \( \lambda \). Also, notice that \( \beta^T(Av_i + a + Bu_i) = \beta^T(Av_i + a) < 0 \) and that \( \beta^T(p - v_i) < 0 \). Thus, we have \( \lambda > 0 \) from (8.1). From the definition of simplices,
it follows that $h_j \cdot v_i = c_j$ and $h_j \cdot p < c_j$ for any $j \neq i$, where $c_j$ is a constant. This leads to $h_j \cdot (p - v_i) < 0$, which further implies that there exists an $u_i$ attained from (8.1) satisfying

$$h_j \cdot (Av_i + a + Bu_i) = \lambda h_j \cdot (p - v_i) < 0 \text{ for any } j \neq i. \quad (8.2)$$

(ii) If $\beta^T v_i = \beta^T w_-$, then $v_i \neq v_0$ since otherwise it contradicts to $(H_{w_-} \cap S) \setminus (F_0 \cup O) = \emptyset$ inferred from $S \rightarrow S \rightarrow F_0$ by Theorem 3.3. Then for this, we claim that $\beta^T v_0 > \beta^T w_-$. (To see this, assume in contrast that $\beta^T v_0 = \beta^T w_-$. Since $v_i \notin O$, there is a point $y$ on the line segment joining $v_0$ and $v_i$ and also in a small neighborhood of $v_i$, satisfying $y \notin O$ and $y \in (H_{w_-} \cap S) \setminus F_0$. It contradicts to $(H_{w_-} \cap S) \setminus (F_0 \cup O) = \emptyset$ again.) Consequently, there is a point $p \in S$ such that $\beta^T v_0 > \beta^T p$. Let

$$Av_i + a + Bu_i = \lambda(p - v_0), \quad (8.3)$$

where $\lambda$ is a scalar to be determined. Following along the same lines as above, there exists a $u_i$ attained from (8.3) satisfying

$$h_j \cdot (Av_i + a + Bu_i) = \lambda h_j \cdot (p - v_0) < 0 \text{ for any } j \neq 0. \quad (8.4)$$

Second, we consider the case that $v_i \in O$. For this we discuss two situations depending on $v_0$ (namely, $\beta^T w_+ \geq \beta^T v_0 \geq \beta^T w_-$ and $\beta^T v_0 > \beta^T w_+$).

(i) If $\beta^T w_+ \geq \beta^T v_0 \geq \beta^T w_-$, then there is a $p' \in S \setminus \{v_0\}$ such that $\beta^T v_0 = \beta^T p'$. Let

$$Av_i + a + Bu_i = \lambda'(p' - v_0), \quad (8.5)$$

where $\lambda' > 0$ is an arbitrary constant. Note that $(p' - v_0) \in B$ by this choice and that $Av_i + a \in B$ (due to $v_i \in O$), so there is a $u_i$ satisfying the equation above. On the other hand, from the definition of simplices, it follows that $h_j \cdot (p' - v_0) \leq 0$ for any $j \neq 0$. Thus, there exists a $u_i$ attained from (8.5) satisfying

$$h_j \cdot (Av_i + a + Bu_i) = \lambda h_j \cdot (p' - v_0) \leq 0 \text{ for any } j \neq 0. \quad (8.6)$$

In addition, for this $p'$, there has to be a facet $F_k$ not containing $p'$, where $k \in \{1, \ldots, n\}$. Thus, we have $h_k \cdot (p' - v_0) < 0$ and therefore

$$h_k \cdot (Av_i + a + Bu_i) = \lambda h_k \cdot (p' - v_0) < 0 \text{ for every } v_i \in O. \quad (8.7)$$

(ii) If $\beta^T v_0 > \beta^T w_+$, then we claim that $\beta$ together any $n - 1$ vectors from $h_1, \ldots, h_n$ are linearly independent. (To this end, assume it is not true. Without loss of generality, we suppose that $\beta$ and $h_1, \ldots, h_{n-1}$ are linearly dependent. Then $\beta$ can be written as $\beta = \lambda_1 h_1 + \cdots + \lambda_{n-1} h_{n-1}$. Thus,

$$\beta \cdot (v_n - v_0) = \lambda_1 h_1 \cdot (v_n - v_0) + \cdots + \lambda_{n-1} h_{n-1} \cdot (v_n - v_0).$$

Note that $h_j \cdot (v_n - v_0) = 0$ for any $j = 1, \ldots, n - 1$, so $\beta \cdot (v_n - v_0) = 0$ and $\beta^T v_n = \beta^T v_0$, a contradiction.) Since $S \rightarrow S \rightarrow F_0$ and $\partial S_{\text{max}} = \{v_0\}$ in this case, from Theorem 3.3 we have $v_0 \notin O$. Let $\delta < 0$ be a scalar. Since $\beta$ and $h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n$ are linear independent, there is a unique solution to the following linear equation

$$\beta \cdot y = 0, \quad h_j \cdot y = \delta, \quad j = 1, \ldots, i - 1, i + 1, \ldots, n$$

Moreover, note that $\beta \cdot y = 0$ implies $y \in B$ and that $v_i \in O$ implies $Av_i + a \in B$. So there exists a $u_i$ satisfying $Av_i + a + Bu_i = y$, which further implies that

$$h_j \cdot (Av_i + a + Bu_i) = h_j \cdot y = \delta < 0 \text{ for any } j \neq i. \quad (8.8)$$
Thus, it is proved that the invariance condition holds at every vertex, from (8.2), (8.4), (8.6), and (8.8).

Note that once the control inputs $u_0, \ldots, u_n$ at corresponding vertices $v_0, \ldots, v_n$ are found, an affine control $u = Fx + g$ can be uniquely constructed by solving the equation

$$\begin{bmatrix} u_0 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} F & g \end{bmatrix} \begin{bmatrix} v_0 \\ 1 \\ \vdots \\ v_n \end{bmatrix}.$$  \hfill (8.9)

Now we examine three cases to synthesize the feedback.

First, consider the case when $\beta^T w_+ \geq \beta^T v_0 \geq \beta^T w_-$. Select the control inputs $u_0, \ldots, u_n$ satisfying the invariance condition and construct the affine control $u(x) = Fx + g$ from (8.9).

With this choice of control, we have shown that (8.7) also holds for every vertex $v_i \in O$. Since $O \cap S = \emptyset$, one obtains that $O \cap S$ is the convex hull of these vertices in $O$. By convexity, it follows from (8.7) that $h_k \cdot (Ax + a + Bu(x)) < 0$ for any $x$ in $O \cap S$. Recall that the possible equilibria of the closed-loop system lie in $O$. So it implies that no equilibrium of the closed-loop system is in $S$. Therefore, by Theorem 4.1, the affine control $u(x) = Fx + g$ solves Problem 4.1 and therefore achieves $S \xrightarrow{S} F_0$.

Second, consider the case when $\beta^T v_0 > \beta^T w_+$ and $F_0 \not\subset O$. Select the control inputs $u_0, \ldots, u_n$ satisfying the invariance condition and construct the affine control $u(x) = Fx + g$ from (8.9). We know $v_0 \not\in O$ and there is a vertex $v_k \in F_0$ not in $O$. From (8.8), we have $h_k \cdot (Av_i + a + Bu(v_i)) < 0$ for every $v_i \in O$ since $k \neq i$. Following along the same lines as above, by Theorem 4.1 an affine control $u(x) = Fx + g$ solves Problem 4.1 and therefore achieves $S \xrightarrow{S} F_0$.

Finally, consider the case when $\beta^T v_0 > \beta^T w_+$ and $F_0 \subset O$. By Theorem 3.5 $B$ is not parallel to $O$, which implies $\beta^T w_+ > \beta^T w_-$. So we can pick a point $v'$ on the line segment joining $w_-$ and $v_0$ satisfying $\beta^T w_+ > \beta^T v' > \beta^T w_-$. The simplex $S$ is then partitioned into two simplices, $S_1$ and $S_2$, along the hyperplane containing $v'$ and the vertices in vert($F_0$) \{w_\}. See Figure 11 for an example. Note that in this case $O$ is the hyperplane containing $F_0$, so $v'$ and $v_0$ are not in $O$. Let $F_0'$ be the common facet of $S_1$ and $S_2$. For $S_1$, we know $\beta^T v_0 > \max\{\beta^T v_i : v_i \in F_0'\}$ and $F_0'$ is not in $O$. Hence, from the second case above,
there exists an affine feedback $u = F_1 x + g_1$ that achieves $S_1 \overset{S_1}{\rightarrow} F'_0$. For $S_2$, we have $\beta^T w_+ > \beta^T v' > \beta^T w_-$. So from the first case above, there exists an affine feedback $u = F_2 x + g_2$ that achieves $S_2 \overset{S_2}{\rightarrow} F_0$. In total, the feedback

$$u = \begin{cases} F_1 x + g_1 & \text{if } x \in S_1 \setminus S_2, \\ F_2 x + g_2 & \text{if } x \in S_2 \end{cases}$$

achieves $S \overset{S}{\rightarrow} F_0$.

8.8. Proof of Lemma 7.2. For $i = 1, 2$, let $A_i^−$ and $A_i^+$ be the possible failure sets to reach $F_i$ defined in (3.1) and (3.2), respectively.

We first claim that if $A_i^+ \neq \emptyset$ then $A_i^+ = \emptyset$ for $j \neq i$. To see this, suppose that $A_i^+ \neq \emptyset$. Then it follows from (3.2) that $A_i^+ = \mathcal{P}^+$. Also for same reason, $A_i^+ \neq \emptyset$ implies $A_i^+ = \mathcal{P}^+$.

It means $\mathcal{P}^+$ is a failure set to reach $F_1 \cup F_2$, a contradiction to $\mathcal{P} \not\rightarrow F_1 \cup F_2$.

Second, we claim that if $A_i^− \neq \emptyset$ then $A_j^− = \emptyset$ for $j \neq i$. Suppose instead that both sets are not empty. Then from (3.1), there is a point $p \in \arg \min \{\beta^T x : x \in \mathcal{P}\}$ that belongs to both $A_i^−$ and $A_j^−$. So this point cannot reach $F_1 \cup F_2$, a contradiction, too.

Third, we claim that $A_i^− \cap A_j^+ = \emptyset$ for $i \neq j$. Note that $A_j^+ \subseteq \mathcal{P}^+$, so if there is a $p \in \arg \min \{\beta^T x : x \in F_i\}$ such that $p \notin \mathcal{P}^+$, then it is clear from (3.1) that $A_i^− \cap A_j^+ = \emptyset$. Instead if for all $p \in \arg \min \{\beta^T x : x \in F_i\}$, $p \in \mathcal{P}^+$, then we know $F_i \subset \mathcal{P}^+$. So $\mathcal{P}^+$ is of $(n-1)$-dimension that is clearly parallel to $\mathcal{B}$. Notice that $\mathcal{O}$ is not parallel to $\mathcal{B}$ from the controllability assumption. Hence, $\mathcal{P}^+ \not\subset \mathcal{O} \cap \mathcal{H}^+$, which implies $A_j^+ = \emptyset$ from (3.2). So the conclusion follows.

Now we come to prove that $\text{Reach}_i(\mathcal{P}, F_1) \cup \text{Reach}_i(\mathcal{P}, F_2) = \mathcal{P}$. Let $A_i^−$ and $A_i^+$ ($i = 1, 2$) be the over-approximations of $A_i^−$ and $A_i^+$ obtained by applying Algorithm 1. Consider a point $x \in \mathcal{P} \setminus \text{Reach}_i(\mathcal{P}, F_1)$ if it exists. Then $x$ is either in $A_i^− \subset A_i^1$. Consider the first case when $x \in A_i^1$. That means, $A_i^− \neq \emptyset$. Thus by our first claim, we get $A_2^+ = \emptyset$. Moreover, by our third claim that $A_i^− \cap A_j^+ = \emptyset$, we know for sufficiently small $\epsilon$, $A_i^1 \cap A_j^2 = \emptyset$. Thus $x \notin A_j^2$ and it must be in $\text{Reach}_i(\mathcal{P}, F_2)$. Consider now the second case when $x \in A_i^1$. That means, $A_i^+ \neq \emptyset$. Then by our second claim, we obtain that $A_i^+ = \emptyset$. Moreover, since $A_i^+ \cap A_j^− = \emptyset$, then by the same argument as above, we know the point $x$ has to be in $\text{Reach}_i(\mathcal{P}, F_2)$.

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