Multi-User Privacy Mechanism Design with Non-zero Leakage

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Abstract—A privacy mechanism design problem is studied through the lens of information theory. In this work, an agent observes useful data \( Y = (Y_1, \ldots, Y_N) \) that is correlated with private data \( X = (X_1, \ldots, X_N) \) which is assumed to be accessible by the agent. Here, we consider \( K \) users where user \( i \) demands a sub-vector of \( Y \), denoted by \( C_i \). The agent wishes to disclose \( C_i \) to user \( i \). A privacy mechanism is designed to generate disclosed data \( U \) which maximizes a linear combination of the users utilities while satisfying a bounded privacy constraint in terms of mutual information. In a similar work it has been assumed that \( X_i \) is a deterministic function of \( Y_i \), however in this work we let \( X \) and \( Y \) be arbitrarily correlated.

First, an upper bound on the privacy-utility trade-off is obtained using a specific transformation, Functional Representation Lemma and Strong Functional Representation Lemma, then we show that the upper bound can be decomposed into \( N \) parallel problems. Next, lower bounds on privacy-utility trade-off are obtained using Functional Representation Lemma and Strong Functional Representation Lemma. The upper bound is tight within a constant and the lower bounds assert that the disclosed data is independent of all \( \{X_i\}_{i=1}^N \) except one which we allocate the maximum allowed leakage to it. Finally, the obtained bounds are studied in special cases.

I. INTRODUCTION

Recently, the privacy mechanism design problem through the lens of information theory is receiving increased attention [1]–[14]. Specifically, fundamental limits of the privacy utility trade-off measuring the leakage using estimation-theoretic guarantees are studied in [1]. A related source coding problem with secrecy is studied in [2]. The concept of maximal leakage has been introduced in [3] and some bounds on the privacy utility trade-off have been derived. The concept of privacy funnel is introduced in [4], where the privacy utility trade-off has been studied considering the log-loss as privacy measure and a distortion measure for utility.

The privacy-utility trade-offs considering equivocation and expected distortion as measures of privacy and utility are studied in both [2] and [5]. In [6], the problem of privacy-utility trade-off considering mutual information both as measures of utility and privacy given the Markov chain \( X \rightarrow Y \rightarrow U \) is studied. It is shown that under the perfect privacy assumption, i.e., no leakages are allowed, the privacy mechanism design problem can be obtained by a linear program. Moreover, in [6], it has been shown that information can be only revealed if the kernel (leakage matrix) between useful data and private data is not invertible. In [7], we generalize [6] by relaxing the perfect privacy assumption allowing some small bounded leakage. More specifically, we design privacy mechanisms with a per-letter privacy criterion considering an invertible kernel where a small leakage is allowed. We generalized this result to a non-invertible leakage matrix in [8].

In [9], the problem of secrecy by design is studied and bounds on privacy-utility trade-off for two scenarios where the private data is hidden or observable are derived by using the Functional Representation Lemma. These results are derived under the perfect secrecy assumption, i.e., no leakages are allowed. In [12], we generalized the privacy problems considered in [9] by relaxing the perfect secrecy constraint and allowing some leakages. Furthermore, in the special case of perfect privacy we derived a new upper bound for the perfect privacy function and it has been shown that this new bound generalizes the bound in [9]. Moreover, it has been shown that the bound is tight when \( |X| = 2 \).

In [13], we considered the privacy-utility trade-off with two different per-letter privacy constraints in two scenarios where the private data is hidden or observable. Upper and lower bounds are derived and it has been shown that the bounds in the first scenario, where the private data is hidden to the agent, are asymptotically optimal when the private data is a deterministic function of useful data.

Our problem here is closely related to [14], where fundamental limits of private data disclosure are studied. The goal is to minimize the leakage under the utility constraints with non-specific tasks. It has been shown that under the assumption that the private data is an element-wise deterministic function of useful data, the main problem can be reduced to multiple privacy funnel (PF) problems. Moreover, the exact solution to each problem has been obtained.

In this paper, \( Y = (Y_1, \ldots, Y_N) \) denotes the useful data where \( Y_1, \ldots, Y_N \) are mutually independent random variables (RV). The useful data is correlated with the private data denoted by \( X = (X_1, \ldots, X_N) \) where \( X_1, \ldots, X_N \) are mutually independent RVs. As shown in Fig. 1, user \( i, i \in \{1, \ldots, K\} \), demands an arbitrary sub-vector of \( Y \) denoted by \( C_i \) and an agent wishes to design disclosed data denoted by \( U \) which maximizes a linear combination of the utilities (weighted sum-utility) while satisfying the bounded leakage constraint, i.e., \( I(X; U) \leq \epsilon \). Utility of user \( i \) is

![Fig. 1. In this work the agent has access to \( Y \) and \( X \). Each user demands a sub-vector of \( Y \) and the agent releases message \( U \) which maximizes a linear combination of utilities.](image-url)
measured by the mutual information between $C_i$ and $U$, i.e., $I(C_i, U)$. The problem considered in this paper is motivated by the dual problem of the privacy design problem studied in [14]. The assumption that the private data is a deterministic function of useful data is very restrictive, hence, in this work we generalize the assumption by letting $X_i$ and $Y_i$ be arbitrarily correlated.

We first find upper bounds on the privacy-utility trade-off by using the same transformation as used in [14, Th. 1] and [15, Th. 1]. We then show that the upper bound can be decomposed into $N$ parallel privacy design problems. Moreover, lower bounds on privacy-utility trade-off are obtained by using Functional Representation Lemma (FRL) and Strong Functional Representation Lemma (SFRL). The lower bound provides us a privacy mechanism design which is based on the randomization used in [12]. Furthermore, a similar randomization technique has been used in [16]. We show that the upper bound is tight within a constant term and the lower bound is optimal when the private data is an element-wise deterministic function of the useful data. Finally, we study the bounds in different scenarios.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Let $P_{XY}(x, y) = P_{X_1,...,X_N,Y_1,...,Y_N}(x_1, ..., x_N, y_1, ..., y_N)$ denote the joint distribution of discrete random variables $X$ and $Y$ defined on finite alphabets $\mathcal{X}$ and $\mathcal{Y}$. Here, we assume that $\{(X_i, Y_i)\}_{i=1}^N$ are mutually independent, hence, $P_{XY}(x, y) = \prod_{i=1}^N P_{X_iY_i}(x_i, y_i)$, where $X_i$ and $Y_i$ are arbitrarily correlated. The demand of user $i$, i.e., $C_i$, $i \in \{1, ..., K\}$, is a discrete random variable with finite alphabet $\mathcal{C}_i$. As we mentioned earlier the agent has access to both $X$ and $Y$ and design disclosed data $\tilde{U}$ defined on $\mathcal{U}$ which maximizes a linear combination of the utilities while satisfying a privacy leakage constraint. The relation between $X$ and $Y$ is given by $P_{X|Y}$ where we represent the leakage mechanism map $P_{X|Y}$ by a matrix defined on $\mathbb{R}^{|X| \times |Y|}$. The relation between $U$ and the pair $(Y, X)$ is described by the conditional distribution $P_{U|Y,X}(u|x,y)$. In this work, RVs $X$ and $Y$ denote the private data and the useful data and $U$ describes the disclosed data.

The privacy mechanism design problem in this scenario can be stated as follows

$$h^K(\mathcal{P}_{XY}) = \sup_{P_{U|Y,X}:I(U;X)\leq \epsilon} \sum_{i=1}^K \lambda_i I(C_i; U),$$  \hspace{1cm} (1)

where for $i \in \{1, ..., K\}$, $\lambda_i \geq 0$ is fixed.

In the following, we study the case where $0 \leq \epsilon < I(X; Y)$, otherwise the optimal solution of $h^K(\mathcal{P}_{XY})$ is $\sum_{i=1}^K \lambda_i H(C_i)$ achieved by $U = Y$.

Remark 1. The problem defined in (1) is motivated by the dual to the problem considered in [14, Eq. (8)], however the Markov chain $(X, C_1, ..., C_K) - Y - U$ is removed due to the assumption that the agent has access to both $X$ and $Y$ and $C_i$ is a sub-vector of $Y$. Here, instead of minimizing the leakage we maximize the linear combination of the utilities (weighted sum-utility) under the bounded privacy leakage constraint.

Remark 2. The weights $\lambda_1, ..., \lambda_K$ can correspond to the Lagrange multipliers of [14, Eq. (8)]. Furthermore, they can correspond to different priorities for different users. For instance, if the utility of user $i$ is more important than the utility of user $j$ we let $\lambda_i$ be larger than $\lambda_j$.

Remark 3. In case of perfect privacy, i.e., $\epsilon = 0$, and $N = K = 1$, (1) leads to the secret-dependent perfect privacy function $h_0(\mathcal{P}_{XY})$, studied in [9], where upper and lower bounds on $h_0(\mathcal{P}_{XY})$ have been derived. In [12], we have strengthened these bounds.

III. MAIN RESULTS

In this section, we derive upper and lower bounds for $h^K(\mathcal{P}_{XY})$. Next, we study the bounds under the perfect privacy assumption, moreover, consider the case where the private data is an element-wise deterministic function of the useful data.

Upper bounds on $h^K(\mathcal{P}_{XY})$:

In this section, we first derive a similar result as [14, Th. 1] which results in upper bounds on the privacy-utility trade-off defined in (1). To do so we introduce a random variable $\bar{U} = (\bar{U}_1, ..., \bar{U}_N)$ which is constructed based on [15, Th. 1] and study the corresponding properties. We then decompose the upper bound into $N$ parallel problems. First, let us recall the FRL [9, Lemma 1] and SFRL [17, Theorem 1] for discrete $X$ and $Y$.

Lemma 1. (Functional Representation Lemma [9, Lemma 1]): For any pair of RVs $(X,Y)$ distributed according to $\mathcal{P}_{XY}$, there exists a RV $U$ such that

$$I(U; X) = 0, \hspace{1cm} (2)$$

and $|U| \leq |X|(|Y| - 1) + 1$.

Lemma 2. (Strong Functional Representation Lemma [17, Theorem 1]): For any pair of RVs $(X,Y)$ distributed according to $\mathcal{P}_{XY}$ with $I(X,Y) < \infty$, there exists a RV $U$ such that

$$I(U; X) = 0, \hspace{1cm} H(Y|U,X) = 0, \hspace{1cm} I(X; U|Y) \leq \log(I(X; Y) + 1) + 4,$$

and $|U| \leq |X|(|Y| - 1) + 2$.

In the next lemma we introduce $\bar{U} = (\bar{U}_1, ..., \bar{U}_N)$ and provide some properties.

Lemma 3. Let $\bar{U} = (\bar{U}_1, ..., \bar{U}_N)$ where for all $i \in \{1, ..., N\}$, $\bar{U}_i$ is a discrete RV defined on the alphabet $\bar{U}_i = U \times X_i \times ... \times X_{i-1}$. For any feasible $U$ in (1), let $(\bar{U}, X, Y, U)$ have the following joint distribution

$$P_{\bar{U},X,Y,U}(\bar{u}, x, y, u) = P_{X,Y,U}(x, y, u) \prod_{i=1}^N P_{\bar{U}_i|X_i}(\bar{u}_i|x_i).$$ \hspace{1cm} (4)

Furthermore, for all $i \in \{1, ..., N\}$ let

$$P_{\bar{U}_i|X_i}(\bar{u}_i|x_i) = P_{X_1,...,X_{i-1},U|X_i}(x_1^{(i)},...,x_{i-1}^{(i)},u^{(i)}|x_i),$$ \hspace{1cm} (5)

where $\bar{u}_i = (x_1^{(i)},...,x_{i-1}^{(i)},u^{(i)})$. Then, we have

i. $\bar{U} - X - (Y, U)$ forms a Markov chain.
ii. $\{(\bar{U}_i, Y_i, X_i)\}_{i=1}^N$ are mutually independent.
Proof: The proof of (i) is similar as proof of [14, Th. 1]. Furthermore, for proving (ii) note that by assumption we have $P_{XY}(x,y) = \prod_{i=1}^{N} P_{X_i|Y_i}(x_i,y_i)$ so that the same proof as [14, Th. 1] works.

In the next lemma we show that for any feasible $U$ in (1), $\bar{U}$ achieves the same privacy leakage as $U$.

**Lemma 4.** For any feasible $U$ in (1), let $\bar{U}$ be the RV that is constructed as in Lemma 3. We have

$$I(X;U) = I(X;\bar{U}).$$

**Proof:** The proof is based on (5) and the fact that $H(X|U_i) = H(X|X_{i-1},...,X_1,U_i)$. By checking the proof of privacy leakage as in [14, Th. 1], the assumption that $X$ is an element-wise deterministic function of $Y$ has not been used. Thus, the same proof can be used here.

Next result is an extension of [14, Th. 1] which helps us to derive upper bounds on the privacy-utility trade-off defined in (1).

**Theorem 1.** For any feasible $U$ in (1), there exists RV $U^* = (U_1^*,...,U_K^*)$ with conditional distribution $P_{U_i^*|Y_i}(u_i^*|y_i) = \prod_{i=1}^{N} P_{U_i^*|Y_i}(u_i^*|y_i)$ that obtains the same leakage as $U$, i.e., we have

$$I(X;U^*) = I(X;U),$$

and for all $i \in \{1,...,K\}$ it bounds the utility achieved by $U$ as follows

$$I(C_i;U) \leq I(C_i;U^*) + \min\{\Delta_1^i, \Delta_2^i\},$$

where

$$\Delta_1^i = I(X;C_i) + \sum_{j:Y_j \in C_i} H(X_j|Y_j)$$

$$\Delta_2^i = I(X;C_i) + \sum_{j:Y_j \in \bar{C}_i} \{\log(I(X_j;Y_j) + 1) + 4\}$$

**Sketch of the proof:** The complete proof is provided in [18, Appendix A]. For the first upper bound on $I(C_i;U)$ we construct $U^* = (U_1^*,...,U_K^*)$ as follows: Let $\bar{U} = (\bar{U}_1,..,\bar{U}_N)$ be the RV constructed in Lemma 3 and $\bar{U} = (U_1,..,U_N)$ where $\bar{U}_i$ is the RV found by the FRL (Lemma 2) using $X \leftrightarrow (\bar{U}_i,X_i)$ and $Y \leftrightarrow Y_i$. Now let $U_i^* = (U_i^*,\bar{U}_i)$. We first show that $I(X;U^*) = I(X;U)$. To prove the upper bound we use the relation between $I(C_i;U)$ and $I(C_i;U^*)$ which can be found based on key expressions for $I(C_i;U)$ and $I(C_i;U^*)$. We have

$$I(C_i;U^*) = I(C_i;X) + H(C_i|X) = H(C_i|X) - I(X;U^*|C_i)$$

and

$$I(C_i;U) = I(C_i;X) + H(C_i) = H(C_i|X) - I(X;U|C_i).$$

Using $I(X;U^*|C_i) = I(X;U|C_i)$ we have

$$I(C_i;U) = I(C_i;U^*) + H(C_i) = H(C_i|X) + I(X;U^*|C_i)$$

$$- I(X;U|C_i).$$

We can build $\bar{U}_i$ so that $(\bar{U}_i,\bar{U}_i,\bar{X}_i,Y_i)$ is mutually independent. The upper bound is derived using (9). To derive the second bound we replace $\bar{U}_i$ by $U_i^*$ which is the RV found by the SFRL (Lemma 2) using $X \leftrightarrow (U_i^*,X_i)$ and $Y \leftrightarrow Y_i$. Then, we let $U_i^* = (U_i^*,\bar{U}_i)$ and by using (9) the upper bound is derived.

**Remark 4.** Theorem 1 is an extension of [14, Th. 1] for correlated $X_i$ and $Y_i$, $\forall i \in \{1,...,N\}$.

Let $\delta_i = \min\{\Delta_1^i, \Delta_2^i\}$. By using Theorem 1 we can decompose $I(U_i;Y_i)$ and $I(U_i;X_i)$ into N parts and derive an upper bound on (1) as follows. We show that without loss of optimality we can replace the constraint $\sum_{i=1}^{N} I(X_i;U_i) \leq \epsilon$ by N individual constraints using auxiliary variables $\{\epsilon_i\}_{i=1}^{N}$.

**Lemma 5.** For any $(X,Y)$ distributed according to $P_{XY}(x,y) = \prod_{i=1}^{N} P_{X_i,Y_i}(x_i,y_i)$ and any $\epsilon < I(X;Y)$ we have

$$h_i^K(P_{XY}) < \sup_{\{P_{U_i|x_i,y_i}\}_{i=1}^{N}} \sum_{i=1}^{N} \left( \sum_{j:Y_j \in C_j} \lambda_j \right) I(U_i;Y_i) + \delta_i.$$

**Proof:** Proof of (a) directly follows from Theorem 1. The proof of equivalency, i.e., step (b), is provided in [18, Lemma 5].

Note that we can maximize (10) in two phases. First, we fix $\epsilon_i$ and decompose (10) into N problems and find an upper bound for each of them. We then show that the upper bound can be written as a linear program over $\{\epsilon_i\}_{i=1}^{N}$ and we derive the final upper bound. For fixed $\epsilon_i$, $i \in \{1,...,N\}$, (10) can be decomposed into N parallel privacy problems, where each privacy problem can be stated as follows

$$h_i^K(P_{XY}) \leq \sup_{\{P_{U_i|x_i,y_i}\}_{i=1}^{N}} I(Y_i;U_i).$$

The last argument holds since the terms $\sum_{j:Y_j \in C_i} \lambda_j$ and $\sum_{j:Y_j \in \bar{C}_i} \lambda_j$ are constants. The problem (11) has been studied in [12] and [9], where an upper bound has been derived. Using [12, Lemma 8] we have

$$h_i^K(P_{XY}) \leq H(Y_i|X_i) + \epsilon_i, \forall i \in \{1,...,N\}.$$  

As shown in [12, Corollary 2] the upper bound in (12) is tight if $X_i$ is a deterministic function of $Y_i$. In the next result, we derive the final upper bound by using Lemma 5 and (12).

**Theorem 2.** For any $(X,Y)$ distributed according to $P_{XY}(x,y) = \prod_{i=1}^{N} P_{X_i,Y_i}(x_i,y_i)$ and any $\epsilon < I(X;Y)$ we have

$$h_i^K(P_{XY}) \leq$$

$$\epsilon \max_i \left\{ \sum_{j:Y_j \in C_j} \lambda_j + \sum_{i=1}^{N} \left( \sum_{j:Y_j \in C_j} \lambda_j \right) (H(Y_i|X_i) + \delta_i) \right\},$$

where $\delta_i = \min\{\Delta_1^i, \Delta_2^i\}$, $\Delta_1^i$ and $\Delta_2^i$ are defined in Theorem 1.
Proof: The proof directly follows from Theorem 1, Lemma 5 and (12). We have

\[ h^K_\epsilon(P_{XY}) \leq \sup_{\epsilon_i \leq \epsilon, \sum_i \epsilon_i \leq \epsilon} \left( \sum_{j \in C_i} \lambda_j \right) \left( H(Y_i|X_i) + \epsilon_i + \delta_i \right) \]

\[ = \epsilon \max_i \left( \sum_{j \in C_i} \lambda_j \right) \left( \sum_{j \in C_i} \lambda_j \right) \left( H(Y_i|X_i) + \delta_i \right). \]

The last line follows since the terms \( \sum_{j \in C_i} \lambda_j \), \( \sum_{i=1}^{N} \sum_{j \in C_i} \lambda_j \delta_i \) and \( \sum_{i=1}^{N} \sum_{j \in C_i} \lambda_j H(X_i|Y_i) \) are constants, thus, the maximum occurs at \( \epsilon_i = \epsilon \) where \( i = \arg \max_{i} \left( \sum_{j \in C_i} \lambda_j \right). \)

Lower bounds on \( h^K_\epsilon(P_{XY}) \): To find the first lower bound, let us construct \( U = (U_1, ..., U_N) \) as follows: For all \( i \in [1,...,N] \), let \( U_i \) be the RV found by the EFRL in [12, Lemma 4] using \( X \leftarrow X_i \), \( Y \leftarrow Y_i \) and \( \epsilon \leftarrow \epsilon_i \). Thus, we have

\[ I(U_i' ; X_i) = \epsilon_i, \]

\[ H(Y_i|X_i, U_i') = 0, \]

\[ I(X_i; U_i'|Y_i) \leq \alpha_i H(X_i|Y_i) + (1 - \alpha_i) \left( \log(I(X_i; Y_i) + 1) + 4 \right), \]

where \( \alpha_i = \frac{\epsilon_i}{\epsilon} \). The utility attained by \( U_i' \) can be lower bounded as follows

\[ I(U_i' ; Y_i) = \]

\[ I(U_i; X_i) + H(Y_i|X_i) - I(X_i; U_i') - H(Y_i|U_i'), \]

where in the last line we used (14), (15) and \( I(X; U_i'|Y_i) \leq H(X_i|Y_i) \). Furthermore, similar to the upper bounds, we can construct \( U \) so that \( \{ (U_i', X_i, Y_i) \}_{i=1}^{N} \) become mutually independent. Hence, we can find the second lower bound for \( g_c(P_{XY}) \) as in the next lemma.

Lemma 6. For any \((X,Y)\) distributed according \( P_{XY}(x,y) = \prod_{i=1}^{N} P_{X_i,Y_i}(x_i,y_i) \) and any \( \epsilon < I(X;Y) \) we have

\[ h^K_\epsilon(P_{XY}) \geq \epsilon \max_{i} \left( \sum_{j \in C_i} \lambda_j \right) \left( \sum_{j \in C_i} \lambda_j \right) \left( H(Y_i|X_i) - H(X_i|Y_i) + \epsilon_i \right). \]

Proof: The proof follows from (16) and the fact that \( \{ (U_i', X_i, Y_i) \}_{i=1}^{N} \) are independent over different i’s. The maximum over \( \{ \epsilon_i \}_{i=1}^{N} \) occurs at \( \epsilon_i = \epsilon \) where \( i = \arg \max_{i} \left( \sum_{j \in C_i} \lambda_j \right). \)

Remark 5. The privacy mechanism design that attains the lower bound in Lemma 6 asserts that we only release information about \( X_i \) where \( i = \arg \max_{i} \left( \sum_{j \in C_i} \lambda_j \right) \) and the leakage from \( X_i \) is equal to \( \epsilon \) (maximum allowed leakage). In other words, the disclosed data that achieves (17) satisfies \( I(U; X_i) = \epsilon \) and \( I(U; X_j) = 0 \), \( \forall j \neq i \), where \( i \) is defined earlier.

To find the second lower bound, let us construct \( U' = (U'_1, ..., U'_N) \) as follows: For all \( i \in [1,...,N] \), let \( U'_i \) be the RV found by the ESFRL in [12, Lemma 4] using \( X \leftarrow X_i \), \( Y \leftarrow Y_i \) and \( \epsilon \leftarrow \epsilon_i \). Thus, we have

\[ I(U'_i ; X_i) = \epsilon_i, \]

\[ H(Y_i|X_i, U'_i) = 0, \]

\[ I(X_i; U'_i|Y_i) \leq \alpha_i H(X_i|Y_i) + (1 - \alpha_i) \left( \log(I(X_i; Y_i) + 1) + 4 \right) \]

(20)

where \( \alpha_i = \frac{\epsilon_i}{\epsilon} \). The utility attained by \( U'_i \) can be lower bounded as follows

\[ I(U'_i ; Y_i) = \]

\[ I(U'_i; X_i) + H(Y_i|X_i) - I(X_i; U'_i) - H(Y_i|U'_i), \]

where in the last line we used (18), (19) and (20). Furthermore, similar to the upper bounds, we can construct \( U' \) so that \( \{ (U'_i, X_i, Y_i) \}_{i=1}^{N} \) become mutually independent. Hence, we can find the second lower bound for \( g_c(P_{XY}) \) as follows.

Lemma 7. Let \( \beta_i = H(X_i|Y_i) - \alpha_i H(X_i|Y_i) + \epsilon_i - (1 - \alpha_i) \left( \log(I(X_i; Y_i) + 1) + 4 \right) \) and \( \alpha_i = \frac{\epsilon_i}{\epsilon} \). For any \((X,Y)\) distributed according \( P_{XY}(x,y) = \prod_{i=1}^{N} P_{X_i,Y_i}(x_i,y_i) \) and any \( \epsilon < I(X;Y) \) we have

\[ h^K_\epsilon(P_{XY}) \geq \epsilon \max_{i} \left( \sum_{j \in C_i} \lambda_j \right) \left( \sum_{j \in C_i} \lambda_j \right) \left( H(Y_i|X_i) - H(X_i|Y_i) + \epsilon_i \right). \]

(22)

Proof: The proof follows from (21) and the fact that \( \{ (U_i', X_i, Y_i) \}_{i=1}^{N} \) are independent over different i’s. Note that the right hand side of (22) is a linear program which can be rewritten as follows: Let \( \gamma_i = 1 - \frac{H(X_i|Y_i)}{H(X_i)} + \frac{\log(I(X_i; Y_i) + 1) + 4}{H(X_i)} \), and \( \mu_i = \sum_{j \in C_i} \lambda_j \). We have

\[ \frac{N}{\sum_{i=1}^{N} \left( \sum_{j \in C_i} \lambda_j \right) \beta_i} \]

\[ = \sum_{i=1}^{N} \left( \sum_{j \in C_i} \lambda_j \right) \beta_i \]

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(21)

(22)

In step (a) we used the fact that the maximum over \( \{ \epsilon_i \}_{i=1}^{N} \) occurs at \( \epsilon_i = \epsilon \) where \( i = \arg \max_{i} \left( \sum_{j \in C_i} \lambda_j \right). \)

Theorem 3. Let \( \gamma_i = 1 - \frac{H(X_i|Y_i)}{H(X_i)} + \frac{\log(I(X_i; Y_i) + 1) + 4}{H(X_i)} \), and \( \mu_i = \sum_{j \in C_i} \lambda_j \). For any \((X,Y)\) distributed according \( P_{XY}(x,y) = \prod_{i=1}^{N} P_{X_i,Y_i}(x_i,y_i) \) and any \( \epsilon < I(X;Y) \) we have

\[ h^K_\epsilon(P_{XY}) \geq \max\{L_1^e, L_2^e\}, \]

(23)
where

\[ L_1^e = \epsilon \max_i \{ \mu_i \} + \sum_{i=1}^N \mu_i \left( H(Y_i|X_i) - H(X_i|Y_i) \right), \]
\[ L_2^e = \sum_{i=1}^N \mu_i \left( H(Y_i|X_i) - (\log(I(X_i;Y_i)) + 1)4 \right) + \max_i \{ \mu_i \gamma_i \} \epsilon. \]

**Proof:** The proof directly follows from Lemma 6 and (23).

**Remark 6.** Similar to Remark 5, the disclosed data that attains the lower bound (23) only releases the information of \( X_i \) with leakage equal to \( \epsilon \), where \( \epsilon = \arg \max_{\gamma_m} \{ \gamma_m^* \} = \arg \max_{\gamma_m} \left\{ \left( \sum_{j=1}^N \gamma_j \right) \gamma_m \right\} \) and \( \gamma_m = 1 - \frac{H(X_i|Y_i)}{H(X_i)} \). In this case, we have \( I(U;X_i) = \epsilon \) and \( U \) is independent of \( X_j \) for \( j \neq i \).

**Corollary 1.** The upper bound in (13) is tight within at most \( \sum_{i=1}^N \left( \sum_{j=1}^N \gamma_j \right) \delta_i \) $H(X_i|Y_i)$, nats, since by using Theorem 2 and Theorem 3 the distance between the lower bound \( L_1^e \) and the upper bound in (13) is \( \sum_{i=1}^N \left( \sum_{j=1}^N \gamma_j \right) \delta_i \) $H(X_i|Y_i)$, nats.

**Special case \( \epsilon = 0 \) (Independent \( X \) and \( U \))**:

In this section we derive lower and upper bounds for \( h_0^K(P_{X,Y}) \). When \( \epsilon = 0 \), (10) is decomposed into \( N \) parallel problems as follows.

\[ h_0^k(P_{X,Y}) = \mu_i \delta_i + \mu_i \sup_{I(U;Y_i)} \frac{I(U;Y_i)}{I(X_i;Y_i)} \]

In this case we obtain the next result.

**Corollary 2.** In case of perfect privacy, i.e., zero leakage, let \( \epsilon = 0 \) in Theorem 2 and Theorem 3, then we have

\[ \max_i \{ \mu_i \} \leq h_0^K(P_{X,Y}) \leq \sum_i \mu_i \left( H(Y_i|X_i) + \delta_i \right), \]

where

\[ L_1^0 = \sum_{i=1}^N \mu_i \left( H(Y_i|X_i) - H(X_i|Y_i) \right), \quad L_2^0 = \sum_{i=1}^N \mu_i \left( H(Y_i|X_i) - (\log(I(X_i;Y_i)) + 1)4 \right) \delta_i \]

and \( \delta_i \) are defined in Theorem 2 and Theorem 3, respectively.

A new upper bound on \( h_0^K(P_{X,Y}) \) can be derived by strengthening (11) with zero leakage. Using [12, Theorem 4] we have

\[ h_0^K(P_{X,Y}) \leq \min_i \{ U_0^1, U_0^2 \} \]

where

\[ U_0^1 = H(Y_i|X_i) \text{ and } U_0^2 = H(Y_i|X_i) + \sum_{Y_i \in Y_i} \left( \sum_{X_i \in X_i} P_{X_i|Y_i}[Y_i|X_i] \right) \geq t \]

Furthermore, if \( |Y_i| = 2 \), then \( U_0^2 \) is tight. Combining (10) and (26) we have

\[ h_0^K(P_{X,Y}) \leq \sum_{i=1}^N \mu_i \left( \min_{i=1}^N \{ U_0^1, U_0^2 \} + \delta_i \right). \]

**Special case \( X_i = f_i(Y_i) \ (H(X_i|Y_i) = 0) \)**:

For all \( i \in \{1, \ldots, N\} \), let \( X_i \) be a deterministic function of \( Y_i \), i.e., \( X_i = f_i(Y_i) \). Using Corollary 2 the gap between the upper bound and lower bound for \( h_0^K(P_{X,Y}) \) is lower than or equal to \( \sum_i \mu_i \delta_i \), where \( \delta_i = \Delta_1 = \sum_{Y_i \in Y_i} H(X_i) \). However, in this case we can modify the proof of Theorem 1 as has been done in [14, Theorem 1] which results in \( \delta_i = 0 \). Thus, by using [14, Theorem 1] and combining it with (10), the gap between upper and lower bound on \( h_0^K(P_{X,Y}) \) becomes zero and we have

\[ h_0^K(P_{X,Y}) = \epsilon \max_{i=1}^N \left\{ \sum_{j=1}^N \lambda_j \right\} + \sum_{i=1}^N \left( \sum_{j=1}^N \lambda_j \right) H(Y_i|X_i). \]

In this case the lower bound attained by Lemma 6 is larger than or equal to the lower bound attained by (23), i.e., \( L_1^e \geq L_2^e \), since we have \( H(Y_i|X_i) + \epsilon_i \geq H(Y_i|X_i) + \epsilon_i - (1 - \alpha_i)(\log(H(X_i|Y_i) + 1) + 4) \).

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