Beyond worst-case analysis in private singular vector computation

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Abstract

We consider differentially private approximate singular vector computation. Known worst-case lower bounds show that the error of any differentially private algorithm must scale polynomially with the dimension of the singular vector. We are able to replace this dependence on the dimension by a natural parameter known as the coherence of the matrix that is often observed to be significantly smaller than the dimension both theoretically and empirically. We also prove a matching lower bound showing that our guarantee is nearly optimal for every setting of the coherence parameter. Notably, we achieve our bounds by giving a robust analysis of the well-known power iteration algorithm, which may be of independent interest. Our algorithm also leads to improvements in worst-case settings and to better low-rank approximations in the spectral norm.

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1 Introduction

Spectral analysis of graphs and matrices is one of the most fundamental tools in data mining. The singular vectors of data matrices are used for spectral clustering, principal component analysis, latent semantic indexing, manifold learning, multi-dimensional scaling, low rank matrix approximation, collaborative filtering, and matrix completion. They provide a means of avoiding the curse of dimensionality by discovering an (approximate) low-dimensional representation of seemingly very high dimensional data. Unfortunately, many of the datasets for which spectral methods are ideal are composed of sensitive user information: browsing histories, friendship networks, movie reviews, and other data collected from private user interactions. The Netflix prize dataset is a perfect example of this phenomenon: a dataset of supposedly "anonymized" user records was released for the Netflix Prize Challenge, which was a matrix of user/movie review pairs. The goal of the competition was to predict user/movie review pairs missing from the matrix. Unfortunately, the ad-hoc anonymization of this dataset proved to be insufficient, and Narayanan and Shmatikov [NS08] were able to re-identify many of the users. Because of the privacy concerns that the attack brought to light, the second proposed Netflix challenge was canceled.

In the past decade, a rigorous formulation of privacy known as differential privacy has been developed, along with a collection of powerful theoretical results. With very few exceptions, existing algorithms come with utility guarantees that hold in the worst case over the choice of the private data. As a result, these utility bounds can sometimes be too weak to be meaningful on particular data sets of interest.

Several algorithms are known for computing approximate top singular vectors of a matrix under differential privacy. In fact, nearly optimal error bounds are known in the worst case. Unfortunately, differential privacy unavoidably forces these bounds to degrade with the dimension of the data. More concretely, given an $n \times n$ matrix $A$, any differentially private algorithm must in the worst case output a vector $x$ such that $\|Ax\|_2 \leq \sigma_1(A) - O(\sqrt{n})$, where $\sigma_1(A)$ denotes the top singular value of $A$. If the matrix $A$ has bounded entries and is sparse as is very common, the dependence on $n$ in the error term can easily overwhelm the signal. This dependence on $n$ is discouraging, because one of the most compelling goals of tools such as PCA is to overcome the “curse of dimensionality” inherent in the analysis of very high dimensional data. We therefore ask the question: Can we hope to achieve a nearly dimension-free bound under a reasonable assumption on the input matrix?

We answer this question in the affirmative. Specifically, we give an algorithm to compute an approximate singular vector that achieves error $O(\sqrt{\mu(A)} \log(n))$. Here, $\mu(A)$ denotes the coherence of the input matrix. The coherence varies between 1 and $n$. We say that $A$ has low coherence if $\mu(A)$ is significantly smaller than $n$. Roughly, a matrix has low coherence if none of its singular vectors have any large coordinates. Low coherence is a widely observed property of large matrices. Random models exhibit low coherence as well as many real-world matrices. Indeed, many recent results in matrix completion, Robust Principal Component Analysis and Low-rank approximation rely crucially on the assumption that the input matrix has low coherence. The error of our algorithm depends essentially only on the square root of the coherence of the data matrix. Moreover, we show that the exact dependence on the coherence that we achieve is best possible: Specifically, for each value of the coherence parameter, we give a family of matrices for which no differentially private algorithm can get a better approximation to the top singular vector than our algorithm does, up to logarithmic
factors.

Our algorithm is also highly efficient and can be implemented using a nearly linear number of vector inner product computations. In particular, our running time is nearly linear in the number of nonzeros of the matrix. In fact, our algorithm is a new variant of the classical power iteration method that has long been the basis of many practical eigenvalue solvers.

1.1 Our Results

We say that a matrix \( A \in \mathbb{R}^{m \times n} \) with singular value decomposition \( A = U \Sigma V^T \) has coherence

\[
\mu(A) \overset{\text{def}}{=} \{m\|U\|_2^2, n\|V\|_2^2\}.
\]

For now we assume that \( m = n \), but all of our results apply to general matrices. Note that \( \mu(A) \in [1, n] \). We give a simple \((\varepsilon, \delta)\)-differentially private algorithm which achieves the following guarantee.

**Theorem 1.1** (Informal, some parameters hidden). *For any matrix \( A \) that satisfies a mild assumption on the decay of its singular values, Private Power Iteration returns a vector \( x \) such that with high probability*

\[
\frac{\|Ax\|}{\|x\|} \geq \sigma_1(A) - O\left(\varepsilon^{-1} \sqrt{\mu(A) \log(1/\delta) \log n}\right).
\]

We also show a nearly matching lower bound:

**Theorem 1.2** (Informal). *For any coherence parameter \( c \in [2, \ldots, n] \), there exists a family of matrices \( A \) such that for each \( A \in \mathcal{A} \), \( \mu(A) = c \), and such that for every \((\varepsilon, \delta)\)-differentially private algorithm \( M \) with \( \delta = \Omega(1/n) \) there is a matrix \( A \in \mathcal{A} \) so that with high probability, \( M(A) \) outputs a vector \( x \) such that*

\[
\frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_1(A) - \Omega\left(\varepsilon^{-1} \sqrt{\mu(A)}\right).
\]

Note that in addition to showing that our dependence on \( \mu(A) \) is tight, this theorem shows that the error of any data-independent guarantee must be at least \( \Omega(\varepsilon^{-1} \sqrt{n}) \).

Finally, we show how our algorithm can be used to compute accurate rank \( k \)-approximations to the private matrix \( A \) in the spectral norm, for any \( k \). For \( k = 1 \), the quality of our approximation is optimal. For \( k \geq 2 \), as in previous work [HR12], our bounds depend on \( r \), where \( r \) is the rank of \( A \). Note that these bounds still improve on the best worst-case bounds when \( A \) is low rank.

**Theorem 1.3** (Informal, some parameters hidden). *There is an \((\varepsilon, \delta)\)-differentially private algorithm such that for any matrix \( A \) that satisfies a mild assumption on the decay of its singular values, it returns a rank-1 matrix \( A_1 \) such that with high probability*

\[
\|A - A_1\|_2 \leq \sigma_2(A) + O\left(\varepsilon^{-1} \sqrt{\mu(A) \log(1/\delta) \log n}\right)
\]

*Moreover, there is an \((\varepsilon, \delta)\)-differentially private algorithm such that for any rank \( r \) matrix \( A \) that satisfies a mild assumption on the decay of its singular values, it returns a rank-\( k \) matrix \( A_k \) such that with high probability:*

\[
\|A - A_k\|_2 \leq \sigma_{k+1}(A) + O\left(\varepsilon^{-1} k^2 \sqrt{(r \cdot \mu(A) + k \log n) \log(1/\delta) \log n}\right).
\]
1.2 More efficient and improved worst-case bounds

Our robust power iteration analysis can also be applied easily to worst-case settings without any incoherence assumptions. For example, we resolve multiple questions asked by Kapralov and Talwar [KT13]. Specifically, we improve the running time of their algorithm by large polynomial factors, give a much simpler algorithm and improve the error dependence on k. In the main body of the paper we study differential privacy under changes of single entries. Here, we consider unit changes in spectral norm as proposed by [KT13]. Our algorithm easily adapts to this definition and gives the following corollary.

**Corollary 1.4.** There is an algorithm such that for every matrix \( A \) that satisfies a mild assumption on the decay of its singular values, it returns a rank-\( k \) matrix \( A_k \) such that with high probability,

\[
\|A - A_k\|_2 \leq \sigma_{k+1}(A) + O\left(\varepsilon^{-1} k^2 \sqrt{n \log(1/\delta) \log n}\right).
\]

Moreover, the algorithm satisfies \((\varepsilon, \delta)\)-differential privacy under unit spectral perturbations. For \((\varepsilon, 0)\)-differential privacy the error bound satisfies

\[
\|A - A_k\|_2 \leq \sigma_{k+1}(A) + O\left(\varepsilon^{-1} k^2 n \log n\right).
\]

We stress that Equation 1 is the first bound for \((\varepsilon, \delta)\)-differential privacy under unit spectral norm perturbations. The dependence on \( n \) matches the error achieved by randomized response for single entry changes.

1.3 Our Techniques

Our main technical contribution includes a novel “robust” analysis of the classical power iteration algorithm for computing the top eigenvector of a matrix, which may be of independent interest. Specifically, we analyze power iteration in which an arbitrary sequence of perturbations \( g_1, \ldots, g_t \) may be added to the matrix vector products at each round \( 1, \ldots, T \). We give simple conditions on the perturbation vectors \( g_1, \ldots, g_t \) such that under these conditions, perturbed powering of a matrix \( A \in \mathbb{R}^{n \times n} \) for \( O(\log n) \) rounds results in a vector \( x \) such that:

\[
\frac{\|Ax\|}{\|x\|} \geq (1 - \beta) \sigma_1(A) \text{ where } \sigma_1(A) \text{ is the top singular value of } A.
\]

Using this general analysis, we are then free to choose the perturbations appropriately to guarantee differential privacy. The accuracy bounds we obtain are a function of the scale of the noise that is necessary for privacy. It is immediate that the magnitude of the perturbation that must be used to guarantee differential privacy (of the matrix) when computing a matrix vector product is proportional to the magnitude of the largest coordinate in the vector. To prove our accuracy guarantees, therefore, it suffices to bound the maximum magnitude of any coefficient of any of the vectors \( x_1, \ldots, x_T \) that emerge during the steps of power iteration. Of course, if the matrix is incoherent, then each \( x_t \) can be written as a linear combination of basis vectors that each have small coordinates \( x_t = \sum_{i=1}^n \alpha_i v_i \). Unfortunately this does not suffice to guarantee that \( x_t \) will have small coordinates without incurring a blow-up that depends on the number of nonzero coefficients. However, we show that at each round, \( \text{sign}(\alpha_1), \ldots, \text{sign}(\alpha_n) \) are independent, unbiased \([-1, 1]\) random variables. This, together with the incoherence assumption, is enough to complete the analysis.
Finding a unit vector $x$ such that $\|Ax\| \geq (1 - \beta)\sigma_1(A)$ is sufficient to compute an accurate rank-1 approximation to $A$ in spectral norm. If $x$ was exactly equal to the top singular vector of $A$, we could then recurse, and compute the top singular vector of $A' = A - \sigma_1 xx^T$, from which we could compute an optimal rank 2 approximation to $A$. Unfortunately, $x$ is only an approximation to the top singular vector. Therefore, in order to be able to usefully recurse on $A' = A - \tilde{\sigma}_1 xx^T$, we require two conditions: (1) That $\|A'\|_2 \approx \sigma_2(A)$, and (2) that $A'$ is nearly as incoherent as $A$. Condition (1) has already been shown by Kapralov and Talwar [KT13]. Therefore, it remains for us to show condition (2). We show that indeed the incoherence of the matrix cannot increase by more than a factor of $\sqrt{r}$, where $r$ is the rank of $A$, during any number of “deflation” steps. However, we do not know whether this factor of $\sqrt{r}$ is necessary, or is merely an artifact of our analysis. We leave removing this factor of $\sqrt{r}$ from our approximation factor for computing rank-$k$ approximations when $k \geq 2$ as an intriguing open problem.

Finally, we give a pointwise lower bound that shows that (up to log factors), our algorithm for privately computing singular vectors is tight for every setting of the coherence parameter. We do this by reducing to reconstruction lower bounds of Dinur and Nissim [DN03]. Specifically, we show, for every coherence parameter $C$, how to construct a matrix with coherence $C$ from some private bit-valued database $D$ such that improving on the performance of our algorithm would imply that an adversary would be able to reconstruct $D$. Since reconstruction attacks are precluded by reasonable values of $\varepsilon$ and $\delta$, a lower bound for all $(\varepsilon, \delta)$ private algorithms follows.

1.4 Related Work

There is by now an extensive literature on a wide variety of differentially private computations, which we do not attempt to survey here. Instead we focus on only the most relevant recent work.

There are several papers that consider the problem of privately approximating the singular vectors of a matrix without any assumptions on the data. Blum et al. [BDMN05] first studied this problem, and gave a simple “input perturbation” algorithm based on adding noise directly to the covariance matrix. Chaudhuri et al [CSS12] and Kapralov and Talwar [KT13] give matching worst-case upper and lower bounds for privately computing the top eigenvector of a matrix under the constraint of $(\varepsilon, 0)$-differential privacy: They achieve additive error $O(n/\varepsilon)$. Both algorithms involve sampling a singular vector from the exponential mechanism. [KT13] also give a polynomial time algorithm for performing this sampling from the exponential mechanism, whereas [CSS12] give a heuristic, but practical implementation using Markov-Chain Monte-Carlo. Our algorithm matches these worst case bounds, and also gives worst case bounds for $(\varepsilon, \delta)$-privacy, with error $O(\sqrt{n}/\varepsilon)$. In the event that the matrix has low coherence, we improve substantially over the worst case bounds. Moreover, we give the first analysis of a natural, efficient algorithm for this problem. Indeed, our algorithm is simply a variant on the classic power iteration method, and runs in time nearly linear in the input sparsity.

Low coherence conditions have been recently studied in a number of papers for a number of matrix problems, and is a commonly satisfied condition on matrices. Recently, Candes and Recht [CR09] and Candes and Tao [CT10] considered the problem of matrix completion. Accurate matrix completion is impossible for arbitrary matrices, but [CR09, CT10] show
the remarkable result that it is possible under low coherence assumptions. Candes and Tao [CT10] also show that almost every matrix satisfies a low coherence condition, in the sense that randomly generated matrices will be low coherence with extremely high probability.

Talwalkar and Rostamizadeh recently used low-coherence assumptions for the problem of (non-private) low-rank matrix approximation [TR10]. They showed that under low-coherence assumptions similar to those of [CR09, CT10], the spectrum of a matrix is in fact well approximated by a small number of randomly sampled columns, and give formal guarantees on the approximation quality of the sampling based Nyström method of low-rank matrix approximation.

Most related to this paper is Hardt and Roth [HR12], which gives an algorithm for giving a rank-$k$ approximation to a private matrix $A$ in the Frobenius norm, where the approximation quality also depends on a (slightly different) notion of matrix coherence. This work differs from [HR12] in several respects. First, a matrix may not have any good approximation in the Frobenius norm (and hence the bounds of [HR12] might be vacuous), but still might have an excellent approximation in the spectral norm. Second, [HR12] does not give any means to actually compute the top singular vector of the private matrix, and hence cannot be easily used for applications (such as PCA, or spectral clustering) that require direct access to the singular vector itself. Moreover, unlike in this paper, [HR12] do not show that their dependence on the coherence is tight—only that their guarantees surpass any data-independent worst case guarantees. The bounds of [HR12] also incur a constant multiplicative error, in addition to an additive error. In this paper, we are able to avoid any multiplicative error. Finally, the bounds of [HR12] depend on the rank of the private matrix $A$, a dependence that we are able to remove when computing the top singular vector of $A$, as well as a rank 1 approximation of $A$.

Related to the problem of approximating the spectrum of a matrix is the problem of approximating cuts in a graph. This problem was first considered by Gupta, Roth, and Ullman [GRU12] who gave methods for efficiently releasing synthetic data for graph cuts with additive error $O(n^{1.5})$. Blocki et al [BBDS12] gave a method which achieves improved error for small cuts, but does not improve the worst-case error. Improving these bounds to the information theoretically optimal bound of $O(n \log n)$ via an efficient algorithm remains an interesting open question. Note that smaller error is efficiently achievable for a polynomial number of cut queries, using private multiplicative weights [HR10] or randomized response.

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2 Preliminaries

We view our dataset as a real valued matrix $A \in \mathbb{R}^{m \times n}$.

Definition 2.1. We say that two matrices $A, A' \in \mathbb{R}^{m \times n}$ are neighboring if $A - A' = \alpha e_s e_t^T$ where $e_s, e_t$ are two standard basis vectors and $\alpha \in [-1, 1]$. In other words $A$ and $A'$ differ in precisely one entry by at most 1 in absolute value.

We use the by now standard privacy solution concept of differential privacy:
Definition 2.2. An algorithm $M : \mathbb{R}^{m \times n} \to R$ (where $R$ is some arbitrary abstract range) is $(\epsilon, \delta)$-differentially private if for all pairs of neighboring databases $A, A' \in \mathbb{R}^{m \times n}$, and for all subsets of the range $S \subseteq R$ we have $P\{M(A) \in S\} \leq \exp(\epsilon) P\{M(A') \in S\} + \delta$.

We make use of the following useful facts about differential privacy.

Fact 2.3. If $M : \mathbb{R}^{m \times n} \to R$ is $(\epsilon, \delta)$-differentially private, and $M' : R \to R'$ is an arbitrary randomized algorithm mapping $R$ to $R'$, then $M'\left(M(\cdot)\right) : \mathbb{R}^{m \times n} \to R'$ is $(\epsilon, \delta)$-differentially private.

The following useful theorem of Dwork, Rothblum, and Vadhan tells us how differential privacy guarantees compose.

Theorem 2.4 (Composition [DRV10]). Let $\epsilon, \delta \in (0,1), \delta' > 0$. If $M_1, ..., M_k$ are each $(\epsilon, \delta)$-differentially private algorithms, then the algorithm $M(A) \equiv (M_1(A), ..., M_k(A))$ releasing the concatenation of the results of each algorithm is $(k\epsilon, k\delta)$-differentially private. It is also $(\epsilon', k\delta + \delta')$-differentially private for $\epsilon' < \sqrt{2k \ln(1/\delta')}\epsilon + 2k\epsilon^2$.

We denote the 1-dimensional Gaussian distribution of mean $\mu$ and variance $\sigma^2$ by $N(\mu, \sigma^2)$. We use $N(\mu, \sigma^2)^d$ to denote the distribution over $d$-dimensional vectors with i.i.d. coordinates sampled from $N(\mu, \sigma^2)$. We write $X \sim D$ to indicate that a variable $X$ is distributed according to a distribution $D$. We note the following useful fact about the Gaussian distribution.

Fact 2.5. If $g_i \sim N(\mu_i, \sigma_i^2)$, then $\sum g_i \sim N \left( \sum_i \mu_i, \sum_i \sigma_i^2 \right)$.

The following theorem is well known folklore.

Theorem 2.6 (Gaussian Mechanism). Let $\epsilon > 0, \delta \in (0, 1/2)$. Let $u,v \in \mathbb{R}^d$ be any two vectors such that $\|u - v\|_2 \leq c$. Put $\sigma = 4\epsilon^{-1}\sqrt{\log(2/\delta)}$. Then, for every measurable set $A \subseteq \mathbb{R}^d$ and $g \sim N(0, \sigma^2)^d$, we have $\exp(-\epsilon) P\{v + g \in A\} - \delta \leq P\{u + g \in A\} \leq \exp(\epsilon) P\{v + g \in A\} + \delta$.

Vector and matrix norms. We denote by $\|\cdot\|_p$ the $\ell_p$-norm of a vector and sometimes use $\|\cdot\|$ as a shorthand for the Euclidean norm. Given a real $m \times n$ matrix $A$, we will work with the spectral norm $\|A\|_2$ and the Frobenius norm $\|A\|_F$ defined as

$$\|A\|_2 \overset{\text{def}}{=} \max_{\|x\|=1} \|Ax\| \quad \text{and} \quad \|A\|_F \overset{\text{def}}{=} \sqrt{\sum_{i,j} a_{ij}^2}.$$ (2)

For any $m \times n$ matrix $A$ of rank $r$ we have $\|A\|_2 \leq \|A\|_F \leq \sqrt{r} \cdot \|A\|_2$.

Singular Value Decomposition. Given a matrix $A \in \mathbb{R}^{m \times n}$, the right singular vectors of $A$ are the eigenvectors of $A^T A$. The left singular vectors of $A$ are the eigenvectors of $AA^T$. The singular values of $A$ are denoted by $\sigma_i(A)$ and defined as the square root of the $i$-th eigenvalue of $A^T A$. The singular value decomposition is any decomposition of $A$ satisfying $A = U \Sigma V^T$ where $U \in \mathbb{R}^{m \times m}, V \in \mathbb{R}^{n \times n}$ are unitary matrices and $\Sigma \in \mathbb{R}^{m \times n}$ satisfies $\Sigma_{ii} = \sigma_i(A)$ and $\Sigma_{ij} = 0$ for $i \neq j$. The columns of $U$ are the left singular vectors of $A$ and the columns of $V$ are the right singular vectors of $A$. 


2.1 Matrix coherence

We will work with the following standard notion of coherence throughout the paper.

**Definition 2.7 (µ-Coherence).** Let $A \in \mathbb{R}^{m \times n}$ with $m \leq n$ be a symmetric real matrix with a given singular value decomposition $A = U\Sigma V^T$. We define the $\mu$-coherence of $A$ with respect to $U$ and $V$ as

$$\mu(A) \overset{\text{def}}{=} \max \{m\|U\|_\infty^2, n\|V\|_\infty^2\}.$$  

Note that $1 \leq \mu(A) \leq n$.

We remark that the coherence of $A$ is defined with respect to a particular singular value decomposition since the SVD is in general not unique.

2.2 Reduction to symmetric matrices

Throughout our work we will restrict our attention real symmetric $n \times n$ matrices. All of our results apply, however, more generally to asymmetric matrices. Indeed, given $A \in \mathbb{R}^{m \times n}$ with SVD $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$ and rank $r$, we can instead consider the symmetric $(m+n) \times (m+n)$ matrix

$$B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}.$$  

The next fact summarizes all properties of $B$ that we will need.

**Fact 2.8.** The matrix $B$ has the following properties: $B$ has a rank $2r$ and singular values $\sigma_1, \ldots, \sigma_r$ each occurring with multiplicity two. The singular vectors corresponding to a singular value $\sigma$ are spanned by the vectors $\{(u_i, 0), (0, v_i) : \sigma_i = \sigma\}$. An entry change in $A$ corresponds to two entry changes in $B$. Furthermore, $\mu(B) = \mu(A)$.

In particular, this fact implies that an algorithm to find the singular vectors of $B$ will also recover the singular vectors of $A$ up to small loss in the parameters. Moreover, an algorithm that achieves $(\epsilon/2, \delta/2)$-differential privacy on $B$ is also $(\epsilon, \delta)$-differentially private with respect to $A$.

3 Robust convergence of power iteration

In this section we analyze a generic variant of power iteration in which a perturbation is added to the computation at each step. The noise vector can be chosen adaptively and adversarially in each round. We will derive general conditions under which power iteration converges.

**Lemma 3.1 (Robust Convergence).** Let $A$ be a matrix such that $\sigma_{k+1}(A) \leq (1 - \gamma)\sigma_k(A)$ for some $k < n$ and $\gamma > 0$. Let $U$ be the space spanned by the top $k$ singular vector of $A$, let $V$ be the space spanned by the last $n - k$ singular vectors. Further assume that there are numbers $\Delta, \Delta_U, \Delta_V > 0$ such that the following conditions are met:

1. For all $t$, $\|g_t\| \leq \Delta$, $\|P_U g_t\| \leq \Delta_U$ and $\|P_V g_t\| \leq \Delta_V$.

2. $\|P_U x_0\| \geq \frac{8\Delta_U}{\gamma \sigma_k(A)}$ and $\|P_V x_0\| \geq \frac{8\Delta_V}{\gamma \sigma_k(A)}$.  

Input: Matrix $A \in \mathbb{R}^{n \times n}$, number of iterations $T \in \mathbb{N}$, parameter $\beta \in (0,1)$,

1. Let $x_0$ be unit vector.
2. For $t = 1$ to $T$:
   (a) Let $g_t$ be an arbitrary perturbation.
   (b) Let $x_t' = Ax_{t-1} + g_t$,
   (c) If $\|x_t'\| \geq (1-\beta)\sigma_1$, then terminate and output $x_{t-1}$.
   (d) Otherwise let $x_t = \frac{x_t'}{\|x_t'\|}$, and continue.

Output: Vector $x_T \in \mathbb{R}^n$ unless the algorithm terminated previously.

Figure 1: Power iteration with adversarial noise

3. $\sigma_1(A) \geq 9\Delta/\beta \gamma$, for some $0 < \beta < 1$.

Then, for $T = 4\log(\sigma_1(A))$, the algorithm outputs a vector $x \in \mathbb{R}^n$ such that

$$\frac{\|Ax\|}{\|x\|} \geq (1-\beta)\sigma_1(A).$$

Proof. Put $\sigma = \sigma_1(A)$ and note that by assumption $\sigma_{k+1} \geq (1-\gamma)\sigma_k$ for some $\gamma > 0$. We will consider the potential function

$$\Psi_t = \frac{\|P_V x_t\|}{\|P_U x_t\|}$$

Suppose that in some round $t$, we have

$$\sigma \|P_V x_{t-1}\| \geq \frac{8\Delta_V}{\gamma} \quad \text{and} \quad \sigma \|P_U x_{t-1}\| \geq \frac{8\Delta_U}{\gamma}.$$  \hfill (3)

We note that by our assumption on the matrix, these conditions are met in the first round $t = 1$ as a consequence of Item 2. Let us derive an expression for the potential drop in round $t$ under the above assumption. We have, using Item 1,

$$\frac{\|P_V x_t\|}{\|P_U x_t\|} = \frac{\|P_V (Ax_{t-1} + g_t)\|}{\|P_U (Ax_{t-1} + g_t)\|} \leq \frac{\|P_V Ax_{t-1}\| + \|P_V g_t\|}{\|P_U Ax_{t-1}\| - \|P_U g_t\|} \leq \frac{(1-\gamma)\sigma \|P_V x_{t-1}\| + \Delta_V}{\sigma \|P_U x_{t-1}\| - \Delta_U}.$$  

By the assumption in Equation 3, we have

$$\frac{(1-\gamma)\sigma \|P_V x_{t-1}\| + \Delta_V}{\sigma \|P_U x_{t-1}\| - \Delta_U} \leq \frac{(1-7\gamma/8)\sigma \|P_U x_{t-1}\|}{(1-\gamma/8)\sigma \|P_U x_{t-1}\|} \leq \left(1 - \frac{\gamma}{2}\right)\|P_V x_{t-1}\| = \left(1 - \frac{\gamma}{2}\right)\Psi_{t-1}.$$  

We furthermore claim that if the conditions in Equation 3 hold true in round $t$, then we must have $\|P_U x_t\| \geq \|P_U x_{t-1}\|$. This follows from our previous analysis, because $\Psi_t \leq \Psi_{t-1}$ but

$$1 = \|x_t\| = \sqrt{\|P_U x_t\|^2 + \|P_V x_t\|^2}.$$  

This in particular means that if the conditions are true in round $t$, then the second condition in Equation 3 continues to be true in round $t+1$, and only the first condition can fail. At this point we distinguish two cases.
**Case 1.** Suppose there is a round where the \( t \leq T \), where the first condition fails to hold. Let \( t^* \) be the smallest such round and put \( x = x_{t^*-1} \). By the previous argument, in this round we must have

\[
1 = \|x\|^2 = \|P_V x\|^2 + \|P_U x\|^2 \leq \left( \frac{8\Delta_V}{\gamma \sigma} \right)^2 + \|P_U x\|^2. \tag{4}
\]

From this we conclude that \( \|P_U x\| \geq \sqrt{1 - (8\Delta_V / \gamma \sigma)^2} \geq 1 - 8\Delta_V / \gamma \sigma \). Hence,

\[
\|Ax_{t^*-1} + g_{t^*}\| \geq \|AP_U x\| - \|g_{t^*}\| \geq \left( 1 - \frac{8\Delta_V}{\gamma \sigma} \right) \sigma - \Delta \geq \left( 1 - \frac{8\Delta}{\gamma \sigma} - \frac{\Delta}{\sigma} \right) \sigma \geq \left( 1 - 9\Delta / \gamma \sigma \right) \sigma
\]

Here we used that \( \Delta_V \leq \Delta \) which is without loss of generality. Therefore, using Item 3,

\[
\|x'_{t^*}\| \geq \left( 1 - \frac{9\Delta}{\gamma \sigma} \right) \sigma \geq (1 - \beta)\sigma_1,
\]

This means that the algorithm terminates in round \( t^* \) and outputs \( x_{t^*-1} \), which satisfies the conclusion of the lemma.

**Case 2.** Suppose there is no round \( t \leq T \), where Equation 3 fails. By our potential argument and the choice of \( T \), this means that

\[
\Psi_T \leq \left( 1 - \frac{\gamma}{2} \right)^T \Psi_0 \leq \frac{\exp(-\gamma T/2)}{\|P_U x_0\|} \leq \frac{\gamma \sigma}{8\Delta_U} \exp(-\gamma T/2) = \frac{\gamma}{8\Delta_U \sigma} \leq \beta
\]

In particular, \( x + T \) satisfies \( \|P_U x_T\| \leq \beta \|P_U x_T\| \leq \beta \). Thus, \( \|P_U x_T\| \geq \sqrt{1 - \beta^2} \) and \( \|Ax_T\| \geq (1 - \beta)\sigma \). This show that \( x_T \) satisfies the conclusion of the lemma. \(\square\)

The next corollary states a variant of Lemma 3.1 where we express all conditions in terms of \( \sigma_1(A) \) rather than \( \sigma_k(A) \).

**Corollary 3.2.** Let \( \alpha \in (0,1) \). Let \( A \) be a matrix such that \( \sigma_{k+1}(A) \leq (1 - \alpha / 2) \sigma_1(A) \) for some \( k < n \). Let \( U \) be the space spanned by the top \( k \) singular vector of \( A \), let \( V \) be the space spanned by last \( n - k \) singular vectors. Further assume that there are numbers \( \Delta, \Delta_U, \Delta_V > 0 \) such that the following conditions are met:

1. For all \( t \), \( \|g_t\| \leq \Delta, \|P_U g_t\| \leq \Delta_U \) and \( \|P_V g_t\| \leq \Delta_V \).
2. \( \|P_U x_0\| \geq \frac{32k \Delta_U}{\gamma (1 - \gamma) \sigma_1(A)} \) and \( \|P_V x_0\| \geq \frac{32k \Delta_V}{\gamma (1 - \gamma) \sigma_1(A)} \).
3. \( \sigma_1(A) \geq \frac{72 \Delta}{\beta \gamma (1 - \gamma)} \).

Then, for \( T = 4 \log(\sigma_1(A)) \), the algorithm outputs a vector \( x \in \mathbb{R}^n \) such that

\[
\frac{\|Ax\|}{\|x\|} \geq (1 - \beta) \sigma_1(A).
\]
Proof. We claim that there exists a $k' \leq k$ such that $\sigma_{k'}(A) \leq (1 - \gamma/4k)\sigma_1(A)$. Indeed, if this is not the case then

$$\sigma_k(A) \geq \prod_{i=1}^{k} \left(1 - \frac{\gamma}{4k}\right)\sigma_1(A) > \left(1 - \frac{\gamma}{2}\right)\sigma_1(A),$$

thus violating the assumption of the lemma. Moreover, $k'$ satisfies $\sigma_{k'}(A) \geq (1 - \gamma/2)\sigma_1(A)$. We will thus apply Lemma 3.1 to this $k'$ setting $\gamma' = \gamma/4k$. It is easy to verify that by our assumptions above, the conditions of Lemma 3.1 are satisfied. Hence, the output $x$ of the algorithm satisfies

$$\frac{\|Ax\|}{\|x\|} \geq (1 - \frac{\gamma}{2})\sigma_{k'}(A) \geq (1 - \gamma/2)^2 \sigma_1(A) \geq (1 - \gamma)\sigma_1(A).$$

Remark 3.3. We will typically need $k$ in Corollary 3.2 to be relatively small compared to $n$. We think of this as a mild assumption even when $k$ and $\alpha$ are constant. In particular, it is implied by the assumption that $A$ has a good low-rank approximation for small $k$. Indeed, if $\sigma_{k+1} > (1 - \alpha)\sigma_1$, then the best rank $k$ approximation to $A$ has spectral error $(1 - \alpha)\|A\|_2$.

3.1 Privacy-Preserving Power Iteration

We will next turn the robust power iteration algorithm from the previous section into a privacy-preserving version. The algorithm is outlined below.

**Input:** Matrix $A \in \mathbb{R}^{n \times n}$, number of iterations $T \in \mathbb{N}$, privacy parameters $\varepsilon, \delta > 0$, upper bound on coherence $C > 0$.

1. Let $\sigma = 2\varepsilon^{-1}\sqrt{4T \log(1/\delta)}$.
2. Let $x_0 = g_0 \sim N(0, 1/n)^n$.
3. For $t = 1$ to $T$:
   
   (a) If $\|x_{t-1}\|_\infty^2 > C/n$, terminate and output “fail”.
   
   (b) Let $g_t \sim N\left(0, \frac{C\sigma^2}{n}\right)^n$
   
   (c) Let $x_t' = Ax_{t-1} + g_t$
   
   (d) Put $x_t = \frac{x_t'}{\|x_t'\|_2}$

**Output:** Vector $x_T \in \mathbb{R}^n$

Figure 2: Private power iteration (PPI)

**Lemma 3.4.** The algorithm PPI satisfies $(\varepsilon, \delta)$-differential privacy.

**Proof.** By Theorem 2.6, the algorithm satisfies $(\varepsilon', \delta)$-differential privacy in each round. Here, $\varepsilon'$ was chosen small enough so that Theorem 2.4 implies $(\varepsilon, \delta)$-differential privacy for the algorithm over all. □
The next lemma states the guarantees of the algorithm assuming that it successfully terminates.

**Lemma 3.5.** Let $\alpha > 0$. Let $A$ be a matrix satisfying $\sigma_k \leq (1 - \gamma/2)\sigma_1$ for some $k \geq 1$. Put $T = 4\log(\sigma_1(A))$. Further assume that for some $\beta \geq 0$, $A$ satisfies

$$\|A\|_2 = \frac{\Theta T \sqrt{C\log(n)\log(1/\delta)}}{\varepsilon \gamma \beta}.$$

for some sufficiently large constant $\Theta > 0$. Assume that PPI terminates successfully and outputs $x_T$ on input of $A$, $T$, and $C$. Then, with probability $9/10$,

$$\|Ax_T\| \geq (1 - \beta)\|A\|_2.$$

**Proof.** Our goal is to apply Corollary 3.2. For this we need to verify that $A$ and $g_t$ satisfy various assumptions of the lemma. Put $\Delta = \sqrt{4C\log(n)\sigma}$. With this choice of $\Delta$, we have by basic Gaussian concentration bounds (see Lemma A.2):

1. $\mathbb{P}\{\|g_t\| > \Delta\} \leq 1/n^2$.
2. $\mathbb{P}\{\|P_Ug_t\| > \sqrt{\frac{k}{n}\Delta}\} \leq 1/n^2$.
3. $\mathbb{P}\{\|P_Vg_t\| > \sqrt{\frac{n-k}{n}\Delta}\} \leq 1/n^2$.

Hence, with probability $1 - 1/n$, none of these events occur for any $t \in [T]$. This verifies that the first assumption of Corollary 3.2 holds with high probability for this setting of $\Delta$. Further note that, by Gaussian anti-concentration bounds (as stated in Lemma A.2) the following claims are true:

1. $\mathbb{P}\{\|P_Ux_0\| \geq \sqrt{\frac{k}{50cn}}\} \geq 98/100$
2. $\mathbb{P}\{\|P_Vx_0\| \geq \sqrt{\frac{n-k}{50cn}}\} \geq 98/100$

Hence, both of these events occur with probability $96/100$. On the other hand the second condition of Corollary 3.2 requires that $\|P_Ux_0\| \geq O(k\Delta/U/\gamma \sigma_1(A))$. Assuming the event $\|P_Ux_0\| \geq \sqrt{k/100n}$ occurred this corresponds to a lower bound of the form $\sigma_1(A) \geq O(k\Delta/\gamma)$ which is satisfied by Equation 9. The analogous argument applies to $\|P_Vx_0\|$. Finally, the third condition of Lemma 3.1 follows by comparison with Equation 9. Hence, the lemma follows.

### 4 Power Iteration and Incoherence

We will next establish an important symmetry property of the algorithm. Specifically, we will show that for any of the eigenvectors $u$ of $A$ (assuming $A$ is symmetric), the sign of the correlation between $u$ and any intermediate vector $x_t$, i.e. $\text{sign}(\langle u, x_t \rangle)$ is unbiased and independent $\text{sign}(\langle v, x_t \rangle)$ for any other eigenvector $v$. This property is rather obvious in the noise-free case where $x_t$ is simply proportional to $A^t x_0$. Hence, the sign of $\langle u, x_t \rangle$ is determined by the sign of $\langle u, x_0 \rangle$. Intuitively, the property continues to hold in the noisy case, because the noise that we add is symmetric.
Lemma 4.1 (Sign Symmetry). Let \( A \) be a symmetric matrix given in its eigendecomposition as
\[
A = \sum_{i=1}^{n} \sigma_i u_i u_i^T.
\]
Let \( t \geq 0 \) and put \( X_i = \text{sign}(\langle u_i, x_i \rangle) \) for \( i \in [n] \). Then \( (X_1, \ldots, X_n) \) is uniformly distributed in \([-1, 1]^n\).

Proof. We will establish by induction on \( t \) that the following two conditions hold for every \( t \geq 0 \):

1. \( Y_i(t) = \langle u_i, x_t \rangle \) is a symmetric random variable
2. \( \text{sign}(Y_i(t)) \) is independent of \( Y_j(t) \) for all \( j \neq i \).

Observe that these two conditions imply the statement of the lemma. In the base case notice that \( Y_i(0) \) is just a random Gaussian variable \( N(0, 1/n) \) and hence symmetric. Now, let \( t \geq 1 \) and consider
\[
Y_i(t) = \frac{\langle u_i, Ax_{t-1} + g_i \rangle}{\|Ax_{t-1} + g_i\|} = \frac{\sigma_i \langle u_i, x_{t-1} \rangle + \langle u_i, g_i \rangle}{\|Ax_{t-1} + g_i\|} = \frac{\sigma_i Y_i(t-1) + \langle u_i, g_i \rangle}{\|Ax_{t-1} + g_i\|}.
\]

Let \( D_i = \sigma_i Y_i(t-1) + \langle u_i, g_i \rangle \). Notice that \( D_i \) is a symmetric random variable, since it is the sum of two independent symmetric random variable. Here we used the induction hypothesis on \( Y_i(t-1) \). We can see that \( Y_i(t) \) is a rescaling of a symmetric random variable, but we also need to show that the rescaling is independent of \( \text{sign}(D_i) \). Note that
\[
\|Ax_{t-1} + g_i\| = \left\| \sum_{j=1}^{n} u_j (\sigma_j \langle u_j, x_{t-1} \rangle + \langle u_j, g_i \rangle) \right\| = \sqrt{\sum_{i=1}^{n} D_i^2}.
\]

This shows that the normalization term can be computed from \( D_i^2 \) and \( \sigma_i Y_i(t-1) + \langle u_i, g_i \rangle \) for \( j \neq i \). Note that each of these terms is independent of \( \text{sign}(D_i) \). Here we used the induction hypothesis on \( Y_i(t-1) \) and the fact that \( \langle u_j, g_i \rangle \) are independent Gaussians for all \( j \in [n] \). We conclude that \( Y_i(t) = \frac{D_i}{\sqrt{D_1^2 + \cdots + D_n^2}} \) is a symmetric random variable.

It remains to show that \( \text{sign}(Y_i(t)) \) is independent of \( Y_j(t) \), for all \( j \neq [i] \). We have already shown that the normalization term appearing in \( Y_i(t) \) is statistically independent of \( \text{sign}(Y_i(t)) \). Moreover, by induction hypothesis, the numerator \( \sigma_j Y_j(t-1) + \langle u_j, g_i \rangle \) is statistically independent of \( \text{sign}(Y_i(t-1)) \) and statistically independent of \( \langle u_i, g_i \rangle \). In particular, conditioning on any subset of the variables \( Y_j(t), j \neq i \) leaves the two variables \( \text{sign}(Y_i(t-1)) \) and \( \langle u_i, g_i \rangle \) unbiased. This implies that no matter what the value of \( |Y_i(t-1)| \) and \( |\langle u_i, g_i \rangle| \)

We will use the previous lemma to bound the \( \ell_\infty \)-norm of the intermediate vectors \( x_t \) arising in power iteration in terms of the coherence of the input matrix. We need the following large deviation bound.

Lemma 4.2. Let \( \alpha_1, \ldots, \alpha_n \) be scalars such that \( \sum_{i=1}^{n} \alpha_i^2 = 1 \) and \( u_1, \ldots, u_n \) are unit vectors in \( \mathbb{R}^n \). Put \( B = \max_{i=1}^{n} |u_i|_\infty \). Further let \( (s_1, \ldots, s_n) \) be chosen uniformly at random in \([-1, 1]^n\). Then,
\[
\mathbb{P} \left( \left\| \sum_{i=1}^{n} s_i \alpha_i u_i \right\|_\infty > 4B \sqrt{\log n} \right) \leq 1/n^3.
\]
Proof. Let $X = \sum_{i=1}^{n} X_i$ where $X_i = s_i \alpha_i u_i$. We will bound the deviation of $X$ in each entry and then take a union bound over all entries. Consider $Z = \sum_{i=1}^{n} Z_i$ where $Z_i$ is the first entry of $X_i$. The argument is identical for all other entries of $X$. We have $\mathbb{E} Z = 0$ and $\mathbb{E} Z^2 = \sum_{i=1}^{n} \mathbb{E} Z_i^2 \leq B^2 \sum_{i=1}^{n} \alpha_i^2 = B^2$. Hence, by Theorem A.3 (Chernoff bound),

$$\mathbb{P}\{ |Z| > 4B \sqrt{\log(n)} \} \leq \exp\left( -\frac{16B^2 \log(n)}{4B^2} \right) \leq \exp(-4\log(n)) = \frac{1}{n^4}.$$  

The claim follows by taking a union bound over all $n$ entries of $X$. \hfill \blacksquare

**Lemma 4.3.** Let $A \in \mathbb{R}^{n \times n}$. Suppose PPI is invoked on $A$, $T \leq n$, and $C \geq 16\mu(A) \log(n)$ and any choice of $\epsilon, \delta > 0$. Then, with probability $1 - 1/n$, the algorithm terminates successfully after round $T$.

**Proof.** The only way for the algorithm to terminate prematurely in step $t + 1$ is if the vector $x_t$ satisfies $\|x_t\|_\infty > 4\sqrt{\mu(A)} \log(n)/n$. We will argue that this happens only with probability $1/n^2$. Hence, by taking a union bound over all rounds $T \leq n$, we conclude that the algorithm must terminate with probability $1 - 1/n$.

Indeed, let $A = \sum_{i=1}^{n} \sigma_i u_i u_i^T$ be given in its eigendecomposition. Note that $B = \max_{i=1}^{n} \|u_i\|_\infty \leq \sqrt{\mu(A)/n}$. On the other hand, we can write $x_t = \sum_{i=1}^{n} s_i \alpha_i u_i$ where $\alpha_i$ are non-negative scalars such that $\sum_{i=1}^{n} \alpha_i^2 = 1$, and $s_i \in \{-1, 1\}$. Notice that $s_i = \text{sign}(\langle x_t, u_i \rangle)$. Hence, by Lemma 4.1, the signs $(s_1, \ldots, s_n)$ are distributed uniformly at random in $\{-1, 1\}^n$. Hence, by Lemma 4.2, it follows that

$$\mathbb{P}\{ \|x_t\|_\infty > 4\sqrt{\log(n)} \} \leq 1/n^3.$$  

Hence, a union bound over all $t \in [T]$ completes the proof. \hfill \blacksquare

Finally, we can combine Lemma 3.5 and Lemma 4.3 to conclude that private power iteration converges does not terminate prematurely and the output vector gives the desired error bound.

**Theorem 4.4.** Let $\gamma, \beta > 0$. Let $A$ be a matrix satisfying $\sigma_k \leq (1 - \gamma/2)\sigma_1$ for some $k \geq 1$. Put $T = 4\log(\sigma_1(A))$. Further assume $A$ satisfies

$$\|A\|_2 = \frac{\Theta T k \sqrt{\mu(A)} \log(1/\delta) \log(n)}{\epsilon \gamma \beta}.$$  

for some sufficiently large constant $\Theta > 0$. Then, with probability $8/10$, on input of $A$, $T$, $(\epsilon, \delta)$ and $C \geq 16\mu(A) \log(n)$, the algorithm PPI outputs a vector $x$, such that

$$\frac{\|Ax\|}{\|x\|} \geq (1 - \beta)\|A\|_2.$$  

Equivalently:

$$\|Ax_T\| \geq \sigma_1(A) - \frac{\Theta T k \sqrt{C \log(n) \log(1/\delta)}}{\epsilon \gamma}.$$  

**Proof.** The proof follows directly by combining Lemma 3.5 applied with $C = 16\mu(A) \log(n)$ with Lemma 4.3. The latter lemma implies that for this setting the algorithm terminates with probability $1 - 1/n$. The former lemma implies that the stated error bound holds in this case with probability $9/10$. Both event occur simultaneously with probability $9/10 - o(1)$. \hfill \blacksquare
Remark 4.5 (On choosing $T$ and $C$). As stated Theorem 4.4 requires the input to the algorithm to depend on two sensitive quantities, i.e., $\sigma_1(A)$ and $\mu(A)$. It is easy to get rid of this using standard techniques. We can upper bound $\sigma_1(A)$ by $||A||_1 = \sum |A_{ij}|$ which can be computed efficiently and privately (as it is 1-sensitive). Since the dependence on $\sigma_1(A)$ in the choice of $T$ is only logarithmic, this can change the error bounds only by constant factors. To get rid of $\mu(A)$, we can try all choices of $C = 2^i, i \in \{0, 1, \ldots, \log(n)\}$. Since $\mu(A) \leq n$, this process will eventually find a setting of $C$ that gives the right upper bound up to an overestimate of at most a factor 2. As we need to scale down $(\epsilon, \delta)$ by a $\log(n)$ factor in each execution, the error bounds deteriorate by an $O(\log(n))$-factor. This loss can be replaced by $O(\log \log n)$ using the exponential mechanism [MT07]. We omit the details as they are standard.

5 Rank $k$ approximations and Deflation

In this section, we show how to successively call our algorithm for obtaining rank 1 approximations to obtain a rank $k$ approximation. To do this, we need to argue two things. First, we must argue that approximately optimal rank 1 approximations to successively ‘deflated’ versions of our original matrix can be combined to yield an approximately optimal rank $k$ approximation. Second, we must argue that incoherence is propagated throughout the deflation process, so that we can in fact obtain good rank 1 approximations to the deflated matrices.

| Input: | Matrix $A \in \mathbb{R}^{n \times n}$, target rank $k$, number of iterations $T \in \mathbb{N}$, privacy parameters $\epsilon, \delta > 0$, upper bound on coherence $C > 0$. |
|--------|--------------------------------------------------------------------------------------------------|
| 1.     | Let $\epsilon' = \epsilon/\sqrt{4k \ln(1/\delta)}$, $\delta' = \delta/k$.                      |
| 2.     | Let $A_0 \leftarrow A$, $B_0 \leftarrow 0$.                                                       |
| 3.     | For $i = 1$ to $k$:
|        | (a) Let $v_i \leftarrow PPI(A_{i-1}, T, \epsilon', \delta', C)$                               |
|        | (b) Let $\hat{\sigma}_i = ||A_{i-1}v_i||_2 + \text{Lap}(1/\epsilon')$                       |
|        | (c) Let $A_i \leftarrow A_{i-1} - \hat{\sigma}_i v_i v_i^T$, $B_i \leftarrow B_{i-1} + \hat{\sigma}_i v_i v_i^T$. |
| Output:| Matrix $A_k$                                                                                     |

Figure 3: Rank $k$ approximation (rank-k).

Our analysis will be based on a useful lemma of Kapralov and Talwar, that shows that the standard “matrix deflation” method can be applied even given only approximate eigenvectors. The lemma here is actually an easy modification of the lemma from [KT13]. The details can be found in Appendix B.

Lemma 5.1 (Deflation Lemma [KT13]). Let $A$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. There exists a constant $C > 0$ so that the following holds. Let $x$ be any unit vector such that $||Ax|| \geq (1 - \alpha/C)\lambda_1$, where $\alpha \in (0, 1)$. Let $A' = A - tv \cdot v^T$, where $t \in (1 \pm \alpha/C)||Ax||$. Denote the eigenvalues of $A'$ by $\lambda'_1 \geq \ldots \geq \lambda'_n$. 

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1. \( \lambda_k \leq \lambda'_{k-1} \leq \min(\lambda_{k-1}, \lambda_k + \alpha \lambda_1) \) for each \( k \in \{1, \ldots, n\} \).

We now argue that deflation preserves incoherence. Here, we make use of two lemmas from [HR12].

**Definition 5.2** \((\mu_0\)-coherence\). Let \( U \) be an \( n \times r \) matrix with orthonormal columns and \( r \leq n \). Recall, that \( P_U = U U^T \). The \( \mu_0\)-coherence of \( U \) is defined as

\[
\mu_0(U) = \frac{n}{r} \max_{1 \leq j \leq r} \|P_U e_j\|^2 = \frac{n}{r} \max_{1 \leq j \leq r} \|U(j)\|^2.
\]

Here, \( e_j \) denotes the \( j \)-th \( n \)-dimensional standard basis vector and \( U(j) \) denotes the \( j \)-th row of \( U \). The \( \mu_0\)-coherence of an \( n \times n \) matrix \( A \) of rank \( r \) given in its singular value decomposition \( U \Sigma V^T \) where \( U \in \mathbb{R}^{n \times r} \) is defined as \( \mu_0(U) \).

Observe that we always have \( \mu_0(A) \leq \mu(A) \).

**Lemma 5.3** ([HR12]). Let \( u_1, \ldots, u_r \in \mathbb{R}^n \) be orthonormal vectors. Pick unit vectors \( n_1, \ldots, n_k \in S^{n-1} \) uniformly at random. Assume that

\[
n \geq c_0 k (r + k) \log(r + k)
\]

where \( c_0 \) is a sufficiently large constant. Then, there exists a set of orthonormal vectors \( v_1, \ldots, v_{r+k} \in \mathbb{R}^n \) such that \( \text{span}\{v_1, \ldots, v_{r+k}\} = \text{span}\{u_1, \ldots, u_r, n_1, \ldots, n_k\} \) and furthermore, with probability 99/100,

\[
\mu_0([v_1 | \cdots | v_{r+k}]) \leq 2\mu_0([u_1 | \cdots | u_k]) + O\left(\frac{k \log n}{r}\right)
\]

**Lemma 5.4** ([HR12]). Let \( U \) be an orthonormal \( n \times r \) matrix. Suppose \( w \in \text{range}(U) \) and \( \|w\| = 1 \). Then,

\[
\|w\|_\infty^2 \leq \frac{r}{n} \cdot \mu_0(U).
\]

**Lemma 5.5.** Let \( A \in \mathbb{R}^{n \times n} \) be a matrix. Define a set of vectors \( s_1, \ldots, s_k \) and matrices \( A'_1, \ldots, A'_k \) as follows. Let \( A'_0 = A \). For each \( i, s_i = A'_{i-1} t_i + c_i n_i \), where \( t_i \in \mathbb{R}^n \) is an arbitrary vector, \( c_i \) is an arbitrary real number, and \( n_i \in S^{n-1} \) is selected uniformly at random. Let \( A'_i = A'_{i-1} - d_i s_i s_i^T \), where \( d_i \) is an arbitrary real number. Then for all \( i \):

\[
\mu(A'_i) \leq 2 r \mu(A) + O(i \log n)
\]

**Proof.** We write \( A = \sum_{j=1}^{r} \sigma_j u_j v_j^T \). The proof will follow easily from **Lemma 5.3** and the following claim.

**Claim 5.6.** For \( i \in \{0, \ldots, k\} \), let \( w_1, \ldots, w_{r+i} \) denote the left singular vectors of \( A'_i \). Then, \( w_1, \ldots, w_{r+i} \in \text{span}\{u_1, \ldots, u_r, n_1, \ldots, n_i\} \).

**Proof.** We prove this by induction. The claim is immediate when \( i = 0 \), which forms the base case. For the inductive case, consider \( A'_i = A'_{i-1} - d_i s_i s_i^T \). Write the singular value
decomposition of $A'_{i-1}$ as: $A'_{i-1} = \sum_{j=1}^{r+i-1} \sigma_j y_j z_j$. And write the singular value decomposition of $A'_j$ as: $A'_j = \sum_{j=1}^{r+i} \sigma' y'_j z_j$. We can also write $A'_j = \sum_{j=1}^{r+i-1} \sigma' y'_j z_j - d_i s_i s_i^T$. Therefore, for all $j$,

$$
\lambda_j w_j = A'_j x_j = \sum_{\ell=1}^{r+i-1} \left( \sigma'_j \langle x_j, z_\ell \rangle \right) y_\ell - \left( d_i \langle x_j, s_i \rangle \right) s_i
$$

Therefore, $w_j \in \text{span}(y_1, \ldots, y_{r+i-1}, s_i)$. But $s_i = A'_{i-1} t_1 + c_i n_i$, so $s_i \in \text{span}(y_1, \ldots, y_{r+i-1}, n_i)$, and by our inductive assumption, $y_1, \ldots, y_{r+i-1} \in \text{span}(u_1, \ldots, u_r, n_1, \ldots, n_{i-1})$. Therefore, we can conclude that $w_j \in \text{span}(u_1, \ldots, u_r, n_1, \ldots, n_i)$ for all $j$. \hfill \blacksquare

By Lemma 5.3, we can conclude that for all $j$, $w_j \in \text{span}(v'_1, \ldots, v'_{r+i})$ such that $v'_1, \ldots, v'_{r+i}$ are orthonormal with:

$$
\mu_0(v'_1 | \ldots | v'_{r+i}) \leq 2 \mu_0(A) + O \left( \frac{i \log n}{r} \right) \leq 2 \mu(A) + O \left( \frac{i \log n}{r} \right).
$$

Therefore, we have:

$$
\mu(A'_j) = n \cdot \max_{j= [r+i]} \|w_j\|_\infty^2 \leq r \mu_0(v'_1 | \ldots | v'_{r+i}) \leq 2 r \mu(A) + O(i \log n)
$$

where the first inequality follows from Lemma 5.4. \hfill \blacksquare

We are now ready to state our results for obtaining good rank $k$ approximations in the spectral norm. First, we translate our bounds from Section 4 into a statement about rank-1 matrix approximation.

**Theorem 5.7.** Let $\gamma, \beta > 0$. Let $A$ be a matrix satisfying $\sigma_c \leq (1 - \gamma/2) \sigma_1$ for some $c \geq 1$. Put $T = 4 \log(\sigma_1(A))$. Further assume $A$ satisfies

$$
\|A\|_2 = \frac{\Theta T c \sqrt{\mu(A) \log(1/\delta)} \log(n)}{\epsilon \gamma \beta}.
$$

for some sufficiently large constant $\Theta > 0$. Then, with probability $7/10$, on input of $A$, $T$, $(\epsilon, \delta)$ and $C \geq 9 \mu(A) \log(n)$, the algorithm rank-$k(A, T, \epsilon, \delta, 1)$ outputs a rank 1 matrix $A_1$ such that:

$$
\|A - A_1\|_2 \leq \sigma_2(A) + \beta \sigma_1(A).
$$

**Proof.** This follows directly from Theorem 4.4, together with Corollary B.4, and the observation that:

$$
\mathbb{P} \{ |\tilde{\sigma}_1 - \sigma_1| \geq c \cdot \beta \sigma_1 \} = \exp(-\epsilon' \beta \sigma_1) = O \left( n^{-c' \sqrt{\mu(A)/\gamma}} \right) = o(1)
$$

Therefore, with probability at least $7/10$, both of the hypotheses of Corollary B.4 are satisfied. \hfill \blacksquare

Our rank $k$ approximation result follows similarly, but we lose a factor of $\sqrt{r}$, where $r$ is the rank of the initial matrix to be approximated, due to the potential degradation in matrix coherence during the deflation process. It is not clear whether this factor of $r$ is necessary, or whether it is an artifact of our analysis.
Theorem 5.8. Let $\gamma, \beta > 0$. Fix a $A$ be a rank $r$ matrix such that there exist indices $c_1, \ldots, c_k$ such that for each $i$: $\sigma_i \leq (1 - \gamma/2)(\sigma_i - (i - 1)\beta\sigma_1)$. Put $T = 4\log(\sigma_1(A))$. Further assume $A$ satisfies

$$\Theta T c_k \sqrt{(r\mu(A) + \sqrt{k}\log(n)\log(k/\delta)\log(n)}$$

for each $i \in [k]$, for some sufficiently large constant $\Theta > 0$. Then, with probability $7/10$, on input of $A, T, (\varepsilon, \delta)$ and $C \geq 9\mu(A)\log(n)$, the algorithm rank-$k(A, k, T, \varepsilon, \delta)$ outputs a rank $k$ matrix $A_k$ such that:

$$\|A - A_k\|_2 \leq \sigma_{k+1}(A) + k\beta\sigma_1(A).$$

Proof. This follows directly from Theorem 4.4, together with Corollary B.4, and our bound on the degradation of the coherence of $A$ under deflation, Lemma 5.5.

6 Lower Bound

In this section, we prove a lower bound showing that our dependence on the coherence $\mu$ is tight. For every value of $\mu \in [2, n]$, there is a family of $n \times n$ matrices such that no $\varepsilon$-differentially private algorithm $A$ can compute a vector $A(M) = v$ with the guarantee that $\|Av\| \geq \sigma_1 - \Theta(\frac{\sqrt{n}}{\varepsilon})$.

Theorem 6.1. For every value of $C \in [2, n]$, there is a family of $n \times n$ matrices $M_C$ such that:

1. For every $M \in M_C$, $\mu(M) = C$

2. For any $\delta = \Omega(1/n)$, no $(\varepsilon, \delta)$-differentially private algorithm $A : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ has the guarantee that for every $M \in M_C$, with constant probability, $A(M) = v$ such that $\|Mv\|/\|v\| \geq \sigma_1(M) - \Theta(\frac{\sqrt{n}}{\varepsilon})$

Remark 6.2. Note that this theorem shows that our upper bound for computing rank 1 approximations to incoherent matrices is tight along the entire curve of values $\mu$, up to logarithmic factors.

Proof. For each $C \in [2, n]$, we define our family of matrices $M_C$ as follows. Let $D \subset \mathbb{R}^n$ be the set of boolean valued vectors with exactly $n/2$ non-zero entries: $D = \{D \in \{0, 1\}^n : \|D\|_0 = n/2\}$. We will intuitively think of each $D \in D$ as a private bit-valued database, whose entries we are protecting with a guarantee of differential privacy. For each $D \in D$, let $\hat{D} = D/\|D\|_2$ be the rescaling of $D$ to a unit vector. Note that $\hat{D} \in [0, \sqrt{2}/\sqrt{n}]^n$. Define $s(C) = n/C$, and $u \in \mathbb{R}^d$ to be the vector such that $u_i = 1/\sqrt{s(C)}$ for $i \in \{1, \ldots, s(C)\}$ and $u_i = 0$ for $i > s(C)$. Finally, we define our class of matrices $M_C$ to be:

$$M_C = \{M_D : M_D = (\sqrt{ns(C)}u \cdot \hat{D}^T : D \in D\}$$

Note that each $M_D \in M$ is a matrix in which the first $s(C)$ rows are identical copies of the database $D \in \{0, 1\}^n$, and the remaining $n - s(C)$ rows are the zero vector. Moreover, for each $M \in M_C$, we have

$$\sigma_1(M) = \left(\frac{\sqrt{ns(C)}}{\sqrt{2}}\right) = \left(\frac{n}{\sqrt{2}C}\right), \quad \mu(M) = n \cdot \max\left(\frac{1}{s(C)} \cdot \frac{2}{n}\right) = \frac{n}{s(C)} = C$$
Now consider any unit vector $v$. For each $M_D \in \mathcal{M}_C$, we have:

$$\|M_Dv\|_2 = \sqrt{\frac{\sigma_1(M_D)^2 \cdot (\bar{D}, v)^2}{s(C)}} \cdot s(C) = \sigma_1(M_D) \cdot (\bar{D}, v)$$  \hfill (11)

Therefore, any unit vector $v$ such that $\|M_Dv\|_2 \geq \frac{999}{1000} \sigma_1(M_D)$ must be such that $(\bar{D}, v) \geq \frac{999}{1000}$. However, if we view $\bar{D}$ as being a private database, it is clear that it is not possible to privately approximate it well:

**Lemma 6.3.** For $\delta \leq 1/5$, Let $B: \mathbb{R}^n \to \mathbb{R}$ be a $(1, \delta)$-differentially private algorithm with respect to the entries of its input vector. Let $D \in \mathcal{D}$ be chosen uniformly at random. Then with probability $\geq 1/2$ $B(D) = v$ such that $(\bar{D}, v/\|v\|_2) \leq 1 - \frac{2}{1000}$.

**Proof.** Let $D \in \mathcal{D}$ be a randomly chosen database $D \in \mathbb{R}^n$ with $\|D\|_0 = n/2$ entries. Let $\bar{D} = D/\|D\|$ be its normalization to a unit vector. Suppose that $v \in \mathbb{R}^n$ is a unit vector such that $(\bar{D}, v) \geq 1 - \alpha$. We may therefore write:

$$v = (1 - \alpha)\bar{D} + \sqrt{1 - (1 - \alpha)^2} \bar{D}^\perp$$

where $\bar{D}^\perp$ is some unit vector orthogonal to $\bar{D}$. We therefore have:

$$\|\sqrt{\frac{n}{2}}v - D\|_1 = \|\alpha D + \sqrt{\frac{n}{2}} \sqrt{1 - (1 - \alpha)^2} \bar{D}^\perp\|_1$$

$$\leq \alpha \|D\|_1 + \sqrt{\frac{n}{2}} \sqrt{1 - (1 - \alpha)^2} \|\bar{D}^\perp\|_1$$

$$\leq \alpha \|D\|_1 + \sqrt{\frac{n}{2}} \sqrt{2\alpha} \|\bar{D}^\perp\|_1$$

$$\leq \alpha \left(\frac{n}{2}\right) + \sqrt{2\alpha} \left(\frac{n}{\sqrt{2}}\right)$$

$$\leq \frac{n}{2} \left(3\sqrt{\alpha}\right)$$

Let $D^*$ denote the vector that results from setting $D^*_i = 1$ in each coordinate $i$ in which $\sqrt{\frac{n}{2}}v_i > 1/2$, and setting $D^*_i = 0$ in all other coordinates. Since $\|\sqrt{\frac{n}{2}}v - D\|_1 \leq \frac{n}{2} (3\sqrt{\alpha})$, it follows that $\|D^* - D\|_0 \leq \frac{n}{4} (6\sqrt{\alpha})$. Now consider the probability that a randomly chosen index $i \in \{i : D_i > 0\}$ is such that $D^*_i > 0$. This occurs with probability at least $1 - (6\sqrt{\alpha})$. On the other hand, consider the probability that a randomly chosen index $j \in \{i : D_i = 0\}$ is such that $D^*_j > 0$. This occurs with probability at most $(6\sqrt{\alpha})$ Note also that because (over the random choice of $D$), each index $i \in D$ is set to 1 uniformly at random, $i$ and $j$ are drawn from the same marginal distribution. Finally, consider the neighboring database $D' = D - \{i\} + \{j\}$, and note that $D'$ is also uniformly distributed among the set of databases $\mathcal{D}$. We therefore have that by differential privacy: $(1 - (6\sqrt{\alpha})) \leq e \cdot (6\sqrt{\alpha}) + \delta$. If $\alpha < \frac{1}{1000}$ and $\delta < 1/5$, this is a contradiction.
It remains to observe that changing a single entry of \( D \) results in changing \( s(C) = n/C \) entries of \( M_D \). Therefore, by the composition properties of differential privacy, any algorithm \( A : \mathbb{R}^{n \times n} \) which is \((\varepsilon, \delta)\)-differentially private with respect to entry changes in its input is \(( (n/C)\varepsilon, (n/C)\delta)\)-differentially private with respect to entry changes in \( D \) when given \( M_D \) as input. Therefore, Lemma 6.3 taken together with equation 11 implies that if \( \varepsilon \leq C/n \) and \( \delta \leq (C/(5n)) \), then no \((\varepsilon, \delta)\)-differentially private algorithm, when given as input a uniformly randomly chosen matrix \( M_D \in M_C \) can with probability greater than \( 1/2 \) return a vector \( A(M_D) = v \) such that

\[
\|M_Dv\|_2 \geq \sigma_1(1 - 1/1000) = \sigma_1 - \left(\frac{n}{1000\sqrt{2C}}\right)
\]

Finally, for point of contradiction, suppose that there was an \( \varepsilon \)-differentially private algorithm that for every matrix \( M \), with probability greater than \( 1/2 \) returned a vector \( A(M) = v \) such that \( \|Mv\| \geq \sigma_1(M) - o\left(\frac{\sqrt{n}}{\varepsilon}\right) \). Letting \( \varepsilon = C/n \), and letting \( M = M_D \) be chosen from \( M_C \), we would have that:

\[
\|Mv\|_2 \geq \sigma_1(M) - o\left(\frac{n}{\sqrt{C}}\right)
\]

which is a direct contradiction. This completes the proof.

\[\blacksquare\]

7 Conclusions and Open Problems

We have shown nearly optimal data dependent bounds for privately computing the top singular vector of a matrix, in terms of its \( \mu \)-coherence. We conclude with several specific open problems, as well as a general research agenda.

Specifically, it would be nice to resolve the following technical questions:

1. We have shown that our dependence on \( \mu \)-coherence is tight, but it remains possible that there might be a weaker notion of coherence that this or other algorithms could take advantage of. One candidate is \( \mu_k \)-coherence, which only bounds the magnitude of the entries in the top \( k \) singular vectors, and leaves the others unconstrained. We do not know how to show that \( \mu_k \)-coherence is sufficient to bound the quality of the approximation to the top singular vector. However, as evidence of this conjecture, in Appendix C, we show that the local sensitivity of the powering operation can be bounded in terms of \( \mu_k \) coherence.

2. When we “deflate” \( A \) so as to recurse and compute an approximation to the higher singular vectors, we lose a \( 1 \)-time factor of \( \sqrt{r} \) in the coherence, where \( r \) is the rank of the matrix. As a result, our bounds for rank \( k \) approximation for \( k \geq 2 \) have a dependence on the matrix rank. Can this factor of \( \sqrt{r} \) be removed, or is it inherent?

More generally, this paper is an instance of a broader research agenda: overcoming worst-case lower bounds in differential privacy by giving data-dependent accuracy bounds. In many settings (especially if the data set is small), the worst case bounds necessary to achieve
differential privacy can be prohibitive. However, natural data sets tend to have structural properties (like low coherence) that can potentially be taken advantage of in a variety of settings. It would be interesting to understand the relevant features of the data that allow more accurate private analyses in domains other than spectral analysis.

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A Deviation bounds

**Lemma A.1** (Gaussian Anti-Concentration). Let $\xi_i \sim N(0, 1)$ and let $a_i \geq 0$ for $1 \leq i \leq n$. Then, for every $\gamma > 0$,

$$
P \left\{ \sum_{i=1}^{n} a_i \xi_i^2 \leq \gamma \sum_{i=1}^{n} a_i \right\} \leq \sqrt{e\gamma}.
$$

We thank George Lowther for pointing out the following proof.

**Proof.** We may assume without loss of generality that $\sum_{i=1}^{n} a_i = 1$ and $\gamma < 1$. For every $\lambda > 0$,

$$
P \left\{ \sum_{i=1}^{n} a_i \xi_i^2 \leq \gamma \sum_{i=1}^{n} a_i \right\} \leq \mathbb{E} e^{\lambda (\gamma - \sum_{i=1}^{n} a_i \xi_i^2)} = e^{\lambda \gamma} \prod_{i=1}^{n} \mathbb{E} e^{a_i \xi_i^2} = e^{\lambda \gamma} \prod_{i=1}^{n} (1 + 2\lambda a_i)^{-\frac{1}{2}} \leq e^{\lambda \gamma} (1 + 2\lambda)^{-\frac{1}{2}}.
$$

The claim follows by setting $\lambda = (\gamma^{-1} - 1)/2$. ■

The following direct consequence was needed earlier.

**Lemma A.2.** Let $U$ be a $k$-dimensional subspace of $\mathbb{R}^n$ and let $g \sim N(0, 1)^n$. Then,

1. for every $\gamma > 0$, $P \left\{ \|P_U g\| \leq \sqrt{k} \gamma \right\} \leq \sqrt{e\gamma}$.
2. for every $t \geq 1$, we have $P \left\{ \|P_U g\| > \sqrt{tk} \right\} \leq \exp(-t)$.

**Proof.** The first claim follows directly by Lemma A.1. The second can be verified by direct computation. ■

**Theorem A.3** (Chernoff bound). Let the random variables $X_1, \ldots, X_m$ be independent random variables. Let $X = \sum_{i=1}^{m} X_i$ and let $\sigma^2 = \mathbb{V} X$. Then, for any $t > 0$,

$$
P \left\{ |X - \mathbb{E} X| > t \right\} \leq \exp \left( -\frac{t^2}{4\sigma^2} \right).
$$
B Proofs from Section 5

Lemma B.1 (Deflation Lemma [KT13]). Let $A$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. There exists a constant $C > 0$ so that the following holds. Let $x$ be any unit vector such that $x^T Ax \geq (1 - \alpha/C)\lambda_1$, where $\alpha \in (0,1)$. Let $A' = A - tv \cdot v^T$, where $t \in (1 - \alpha/C)x^T Ax$. Denote the eigenvalues of $A'$ by $\lambda_1' \geq \ldots \geq \lambda_n'$.

1. $\lambda_k \leq \lambda_{k-1}' \leq \min(\lambda_{k-1}, \lambda_k + \alpha \lambda_1)$ for each $k \in \{1, \ldots, n\}$.

Because our algorithm returns a vector $v$ with a guarantee on the quantity $\|Av\|$, rather than on the Rayleigh Quotient $v^T Av$, we must relate these two quantities, which we do presently.

Lemma B.2. For any unit vector $x$, $|x^T Ax| \leq \|Ax\|$

Proof. By the Cauchy-Schwarz inequality, $|x^T Ax| = |\langle x, Ax \rangle| \leq \|x\| \cdot \|Ax\| = \|Ax\|$. ■

We now prove a partial converse, for vectors $x$ such that $\|Ax\|$ is large.

Lemma B.3. For any $0 \leq \alpha \leq 1/4$ and for any unit vector $x$ such that $\|Ax\| \geq (1 - \alpha)\lambda_1$:

$$x^T Ax \geq (1 - 5\alpha)\lambda_1$$

Proof. Let $v_1, \ldots, v_n$ denote the eigenvectors of $A$ in order from largest to smallest eigenvalue: $|\lambda_1| \geq \ldots \geq |\lambda_n|$. Then we may write $x = \sum_{i=1}^n \alpha_i \cdot v_i$. Likewise $Ax = \sum_{i=1}^n \alpha_i \lambda_i \cdot v_i$, where $\sum_{i=1}^n \alpha_i^2 = 1$, since $x$ is a unit vector. Hence, $\|Ax\| = \sqrt{\sum_{i=1}^n \alpha_i^2 \lambda_i^2}$ and $x^T Ax = \langle x, Ax \rangle = \sum_{i=1}^n \alpha_i^2 \lambda_i$.

We define:

$$i^* \triangleq \max\{1 \leq i \leq n : |\lambda_i| \geq \lambda_1(1 - 4\alpha)\}$$

to be the largest index such that the $i^*$th eigenvalue has magnitude at least $(1 - 4\alpha)\lambda_1$. Now define the quantities:

$$S_1 \triangleq \sum_{i=1}^{i^*} \alpha_i^2 \quad S_2 \triangleq \sum_{i=i^*+1}^n \alpha_i^2$$

and note that $S_2 = 1 - S_1$. We can calculate:

$$(1 - \alpha)\lambda_1 \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \lambda_i^2} \leq \sqrt{S_1 \lambda_1^2 + S_2 (1 - 4\alpha)\lambda_1^2} \leq \lambda_1 \left(\sqrt{S_1} + \sqrt{1 - S_1} \sqrt{1 - 4\alpha}\right) \leq \lambda_1 \left(\sqrt{S_1} + \sqrt{1 - S_1 (1 - 2\alpha)}\right)$$

Solving for $S_1$, we find:

$$S_1 \geq \frac{1 - 4\alpha + 8\alpha^2 - 10\alpha^3 + 6\alpha^4 + (-1 + \alpha)(-1 + 2\alpha)\sqrt{1 + \alpha(-2 + 3\alpha)}}{2(1 + 2(-1 + \alpha)\alpha)^2} \geq 1 - 4\alpha^2.$$
Therefore, we also have \( S_2 \leq 4\alpha^2 \). Finally, we may calculate:

\[
x^T Ax = \sum_{i=1}^{n} \alpha_i^2 \lambda_i \geq S_1 (1 - 4\alpha) \lambda_1 - S_2 (1 - 4\alpha) \lambda_1 \geq (1 - 8\alpha^2) (1 - 4\alpha) \lambda_1 \geq (1 - 5\alpha) \lambda_1
\]

where the last inequality holds since \( \alpha \leq 1/4 \). 

As a corollary, we get a modified version of the deflation lemma of [KT13]

**Corollary B.4 (Modified Deflation Lemma [KT13]).** Let \( A \) be a symmetric matrix with eigenvalues \( \lambda_1 \geq \ldots \geq \lambda_n \). There exists a constant \( C > 0 \) so that the following holds. Let \( x \) be any unit vector such that \( \|Ax\| \geq (1 - \alpha/C) \lambda_1 \), where \( \alpha \in (0, 1) \). Let \( A' = A - tv \cdot v^T \), where \( t \in (1 \pm \alpha/C) \|Ax\| \).

1. \( \lambda_k \leq \lambda'_{k-1} \leq \min(\lambda_{k-1}, \lambda_k + \alpha \lambda_1) \) for each \( k \in \{1, \ldots, n\} \).

### C Perturbation bounds for matrix powers

In this section we prove a perturbation bound for matrix powers. The result be seen as bounding the so-called local sensitivity of power iteration. Notably, we can use following notion weaker form of \( \mu \)-coherence that depends only on the top few singular singular vectors.

**Definition C.1.** Let \( M \) be a real valued \( m \times n \) matrix with singular value decomposition \( M = \sum_{i=1}^{n} \sigma_i u_i v_i^T \), where \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0 \). We define the top-\( k \) coherence of \( M \) as

\[
\mu_k(M) = \max \max \left\{ m \|u_i\|_\infty^2, n \|v_i\|_\infty^2 \right\}.
\]

Note that \( 1 \leq \mu_k(M) \leq \max\{m, n\} \).

**Theorem C.2.** Let \( q \geq 1 \) be a number. Let \( M \) be a real valued \( n \times n \) matrix with singular value decomposition \( M = \sum_{i=1}^{n} \sigma_i u_i v_i^T \), where \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0 \). Assume that \( \sigma_1 \geq 4q \), \( \sigma_{k+1} \leq \sigma_1/2 \) and \( q \geq \log n + 1 \). Then, with \( g \sim N(0, 1)^n \),

\[
\mathbb{E} \|((M + E)^q - M^q) g\|_2 \leq 9 \min \left\{ 1, \sqrt{\frac{k \cdot \mu_k(M)}{n}} \right\} q \sigma_1^{q-1}.
\]

**Remark C.3.** We note that the above bound could easily be turned into a high probability guarantee.

The next lemma will be helpful in simplifying some expressions arising in the proof of the theorem.

**Lemma C.4.** Let \( E = e_s e_t^T \) and \( \alpha \geq 1 \), Then,

- \( E^2 = E \) if \( s = t \) and \( E^2 = 0 \) otherwise.
- \( EM^\alpha E = C_\alpha E \), where \( C_\alpha = \sum_{i=1}^{n} \sigma_i^\alpha \langle u_i, e_s \rangle \langle v_i, e_t \rangle \).
Proof. Both claims are immediate.

In the following we will use $C = \mu_k(M)$ as a shorthand.

**Lemma C.5.** Let $\delta = \min\{\sqrt{kC/n}, 1\}$. Then, $C_\alpha \leq \delta^2 \sigma_1^\alpha + \frac{\sigma_1^\alpha}{2\alpha}$.

**Proof.** Appealing to the definition of $C_\alpha$ from **Lemma C.4**, we have

$$C_\alpha = \sum_{i=1}^{n} \sigma_i^\alpha \langle u_i, e_s \rangle \langle u_i, e_t \rangle = \sum_{i=1}^{k} \sigma_i^\alpha \langle u_i, e_s \rangle \langle v_i, e_t \rangle + \sum_{i=k+1}^{n} \sigma_i^\alpha \langle u_i, e_s \rangle \langle v_i, e_t \rangle$$

$$\leq \delta^2 \sigma_1^\alpha + \frac{\sigma_1^\alpha}{2\alpha} \sum_{i=k+1}^{n} \langle u_i, e_s \rangle \langle v_i, e_t \rangle$$

$$\leq \delta^2 \sigma_1^\alpha + \frac{\sigma_1^\alpha}{2\alpha} \parallel e_s \parallel \parallel e_t \parallel = \delta^2 \sigma_1^\alpha + \frac{\sigma_1^\alpha}{2\alpha}.$$

**Lemma C.6.** Recall that $\delta = \min\{\sqrt{kC/n}, 1\}$. We have,

$$\mathbb{E} \parallel M_\alpha EM_\beta g \parallel \leq \left( \delta \sigma_1^\alpha + \frac{\sigma_1^\beta}{2\alpha} \right) \left( \delta \sigma_1^\beta + \frac{\sigma_1^\alpha}{2\alpha} \right).$$

**Proof.** First note that

$$\mathbb{E} \parallel M_\alpha EM_\beta g \parallel^2 = \mathbb{E} \parallel M_\alpha e_s \parallel^2 \mathbb{E} \langle e_T^T M_\beta, g \rangle^2 = \mathbb{E} \parallel M_\alpha e_s \parallel \parallel e_T^T M_\beta \parallel^2,$$

where we used that $g \sim N(0, 1)^n$. Hence, by Jensen's inequality,

$$\mathbb{E} \parallel M_\alpha EM_\beta g \parallel \leq \sqrt{\mathbb{E} \parallel M_\alpha EM_\beta g \parallel^2} = \parallel M_\alpha e_s \parallel \cdot \parallel e_T^T M_\beta \parallel.$$

It remains to bound the right hand side of the previous inequality. Indeed,

$$\parallel M_\alpha e_s \parallel = \sqrt{\sum_{i=1}^{n} \sigma_i^{2\alpha} \langle u_i, e_s \rangle^2} \leq \frac{Ck}{n} \sigma_1^{\alpha} + \frac{\sigma_1^{\alpha}}{2\alpha} \sum_{i=1}^{k} \langle u_i, e_s \rangle^2 \leq \sqrt{\frac{Ck}{n} \sigma_1^{\alpha} + \frac{\sigma_1^{\alpha}}{2\alpha}}.$$

We can bound $\parallel e_T^T M_\beta \parallel$ with the same reasoning.

**Lemma C.7.** Let $A = M_\alpha^1 EM_\alpha^2 \cdot EM_\alpha^{\ell-1} EM_\alpha^\ell$ with $\alpha_i \geq 0$. Then,

$$\mathbb{E} \parallel Ag \parallel \leq \left( \frac{\sigma_1^\alpha}{2} \right)^\sum_{i=1}^{\ell} \alpha_i + \delta (1 + \delta) \sigma_1^{\sum_{i=1}^{\ell} \alpha_i}.$$

**Proof.** First we apply **Lemma C.4** to all intermediate terms $EM_\alpha^i E$ where $i \in \{2, \ldots, \ell - 1\}$. Then we apply **Lemma C.5** and **Lemma C.6** to the remaining term. Noting that $\delta^2 \leq \delta$, since $0 \leq \delta \leq 1$, we have established that

$$\mathbb{E} \parallel Ag \parallel \leq \sigma_1^{\sum_{i=1}^{\ell} \alpha_i} \prod_{i=1}^{\ell} \left( \delta + \frac{1}{2\alpha_i} \right).$$
On the other hand, it is not hard to see that the following inequality holds,

\[
\prod_{i=1}^{\ell} \left( \delta + \frac{1}{2\sigma_i} \right) \leq \left( \frac{1}{2} \right)^{\sum_{i=1}^{\ell} \alpha_i} + \delta(1 + \delta)^\ell.
\]

We are now ready to prove Theorem C.2.

Proof of Theorem C.2. Observe that the matrix \((M + E)q - Mq\) equals the sum of \(2^q - 1\) matrices that are either zero or of the form \(A = M^{\alpha_1}EM^{\alpha_2}E \cdots EM^{\alpha_{\ell-1}}EM^{\alpha_{\ell}}\) as described in Lemma C.7. Let us say that \(\sum_{i=1}^{\ell} \alpha_i\) is the “order” of the matrix \(A\). Clearly, the order of the matrix \(A\) is at most \(q - \ell + 1\). Furthermore, there are at most \(\binom{q}{\ell}\) matrices of order \(z\).

Using the fact that \(\binom{q}{\ell} \leq q^{q-z}\) and the assumption that \(q \leq \sigma_1/4\), we will apply Lemma C.7 to each such matrix and sum over the resulting error terms:

\[
\sum_{z=0}^{q-1} \binom{q}{z} \sigma_1^z \left( \frac{1}{2} + \delta(1 + \delta)^{q-z} \right) \leq q \sum_{z=0}^{q-1} \binom{\sigma_1/4}{z}^{q-z-1} \sigma_1^z \left( \frac{1}{2} + \delta 2^{q-z+1} \right)
\]
\[
\leq q \sigma_1^{q-1} \sum_{z=0}^{q-1} \left( \frac{1}{2^{q-1}} + \frac{4\delta}{2^{q-z-1}} \right)
\]
\[
\leq q \sigma_1^{q-1} \left( 8\delta + \frac{q}{2^{q-1}} \right)
\]
\[
\leq 9\delta q \sigma_1^{q-1}.
\]

In the last step we used that \(\delta \geq \sqrt{1/n}\) and the assumption that \(q \geq \log(n) + 1\).

The theorem follows now straightforwardly. By the previous argument and linearity of expectation, we have

\[
\|((M + E)^q - Mq)g\| \leq 9\delta q \sigma_1^{q-1} = 9\min \left\{ 1, \sqrt{\frac{Ck}{n}} \right\} q \sigma_1^{q-1}.
\]