PERI-ABELIAN CATEGORIES AND
THE UNIVERSAL CENTRAL EXTENSION CONDITION

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Abstract. We study the relation between Bourn’s notion of peri-abelian category and conditions involving the coincidence of the Smith, Huq and Higgins commutators. In particular we show that a semi-abelian category is peri-abelian if and only if for each normal subobject $K \leq X$, the Higgins commutator of $K$ with itself coincides with the normalisation of the Smith commutator of the denormalisation of $K$ with itself. We show that if a category is peri-abelian, then the condition (UCE), which was introduced and studied by Casas and the second author, holds for that category. In addition we show, using amongst other things a result by Cigoli, that all categories of interest in the sense of Orzech are peri-abelian and therefore satisfy the condition (UCE).

Introduction

Using Janelidze and Kelly’s general notion of central extension [26], the classical theory of universal central extensions valid for groups and Lie algebras—see, for instance, [31, 57]—may be generalised to the context of semi-abelian categories [27, 2] with enough projectives. As explained in [14], most of this generalisation is entirely straightforward. Somewhat surprisingly though, there is a difficulty in obtaining a general version of the standard recognition theorem for universal central extensions, which characterises universality of a central extension in terms of properties of its domain. In the case of groups, this result says that a central extension $u: U \to Y$ of groups is universal if and only if $H_1(U, Z) = H_2(U, Z) = 0$ or, equivalently, if and only if $U$ is perfect and every central extension of the group $U$ splits [31].

As it turns out, this general theory of universal central extensions works well when the underlying semi-abelian category satisfies an additional requirement, called the universal central extension condition or (UCE) in [14], that is, if $B$ is a perfect object and $f: A \to B$ and $g: B \to C$ are central extensions in $\mathcal{X}$, then the extension $g \cdot f$ is also central. Indeed, for any perfect object $U$ of $\mathcal{X}$, the statements

(i) each central extension $u: U \to Y$ is universal;
(ii) each central extensions of $U$ splits;
(iii) each universal central extensions of $U$ splits;
(iv) $H_2(U) = 0$

are equivalent if and only if (UCE) holds. Here we assume that $\mathcal{X}$ has enough projectives; furthermore, centrality, perfectness and homology are all defined with respect to the Birkhoff subcategory $\text{Ab}(\mathcal{X})$ of abelian objects of $\mathcal{X}$. The condition (UCE) clearly holds for groups and Lie algebras; on the other hand, the category of non-associative algebras over a field is semi-abelian but does not satisfy (UCE), which shows that this condition does not hold in an arbitrary semi-abelian category.

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The aim of the present paper is to understand how the condition (UCE) is related to other conditions occurring in categorical algebra, in particular which conditions it follows from. We analyse it in terms of basic commutator conditions, proving that it is closely related to the notion of peri-abelian category with appeared in recent work by Bourn [7]. For any semi-abelian category there is a natural notion of action [10]—generalising that of a group $G$ acting on a group—as well as Beck’s notion of $G$-module [1]. The category of groups being peri-abelian amounts to the fact that the universal way to make a $G$-action on a group $X$ into a $G$-module is to abelianise $X$. For semi-abelian categories this becomes a condition which may or may not hold. We show in Proposition 2.5 that a semi-abelian category is peri-abelian (satisfies condition (PA)) if and only if there is a partial coincidence of the Higgins and Smith commutators in it: the Higgins [29] commutator $[K,K]$ of any normal subobject $K \triangleleft X$ in it is the normalisation (= zero-class, see [5, 2]) of the Smith [36, 34] commutator $[R,R]^S$, where $R$ is the equivalence relation corresponding to $K$ (= its denormalisation). As a consequence, combining results in [15], [32] and [12], we see that any category of interest in the sense of Orzech [33] is peri-abelian. It is not a coincidence that categories of non-associative rings need not be such [15, Example 5.3.7]. Indeed—this is Theorem 3.12—a semi-abelian category which is peri-abelian will always satisfy (UCE), as explained in Section 3.

We start with a revision of some basic commutator theory in Section 1. In Section 2 we reformulate the concept of peri-abelian category in the language of commutators, which gives us the equivalent conditions of Proposition 2.5. The final Section 3 leads towards our main Theorem 3.12: (PA) implies (UCE).

1. Preliminaries

Throughout the text we assume that $\mathcal{X}$ is a semi-abelian category. In this section we recall the definitions and some basic properties of the Huq, Smith and Higgins commutators. Before doing so let us introduce some terminology and notation. We will call a diagram

$$
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow s
\end{array}
$$

where $fs = 1_Y$ a point in $\mathcal{X}$, and a diagram

$$
\begin{array}{c}
\begin{array}{c}
X' \xrightarrow{f'} Y' \\
\downarrow \theta
\end{array} \\
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \phi
\end{array}
\end{array}
$$

where the top and bottom rows are points and $\phi f' = f \theta$ and $\theta s' = s \phi$, a morphism of points. We will denote by $\text{Pt}(\mathcal{X})$ the category of points and by $\text{Pt}_Y(\mathcal{X})$ the fiber above $Y$ of the fibration sending a point to the codomain of its split epimorphism. We will call a diagram

$$
\begin{array}{c}
K \xrightarrow{k} X \xrightarrow{f} Y \\
\downarrow s
\end{array}
$$

where $fs = 1_Y$ and $k$ is a kernel of $f$ a point with chosen kernel, and a diagram

$$
\begin{array}{c}
\begin{array}{c}
K' \xrightarrow{k'} X' \xrightarrow{f'} Y' \\
\downarrow \theta
\end{array} \\
\begin{array}{c}
K \xrightarrow{k} X \xrightarrow{f} Y \\
\downarrow s
\end{array}
\end{array}
$$
where the top and bottom rows are points with chosen kernels and \( \theta k' = k \nu, \phi f' = f \theta \) and \( \theta s' = s \phi \), a morphism of points with chosen kernel.

1.1. The Huq commutator. A cospan of monomorphisms in \( \mathcal{X} \) as on left

\[
\begin{tikzcd}
K \ar{r}{k} \ar{d}{l} & X \ar{d}{l} \\
L \ar{r}{k} & \ar{u}{1_{K,0}} K
\end{tikzcd}
\]

is said to \((Huq-)commute\) \([9, 24]\) when there exists a (necessarily unique) morphism \( \varphi \) making the diagram on the right commute. The Huq commutator of \( k \) and \( l \) \([8, 2]\) is defined to be the smallest normal subobject \( [k, l]_X: [K, L]_X \to X \), making the images of \( k \) and \( l \) commute in the quotient \( X/[K, L] \). In this context it can be shown that the Huq commutator always exists, and can be constructed as as the kernel \( [K, L]_X \) of the (normal epi)morphism \( X \to Q \), where \( Q \) is the colimit of the outer square above.

1.2. The Smith commutator. Given a pair of equivalence relations \( (R, S) \) on a common object \( X \) of \( \mathcal{X} \) as on the left

\[
\begin{tikzcd}
R \ar{r}{r_1} & X \ar{d}{s_1} \ar{r}{s_2} & S \\
R \ar{r}{r_2} & X \ar{u}{s_R} \ar{r}{s_1} & S
\end{tikzcd}
\]

consider the induced pullback of \( r_2 \) and \( s_1 \) in the middle. The equivalence relations \( R \) and \( S \) are said to centralise each other or to \((Smith-)commute\) \([36, 34, 9]\) when there exists a (necessarily unique) morphism \( \theta \) making the diagram on the right commute. In a similar way as for the Huq commutator, the Smith commutator is defined to be the smallest equivalence relation \( [R, S]^S \) on \( X \), making the images of \( R \) and \( S \) in the quotient \( X/[R, S]^S \) commute. In this context it can be shown to always exist, since it admits a construction similar to the Huq commutator’s. It follows that \( R \) and \( S \) commute if and only if \( [R, S]^S = \Delta_X \), where \( \Delta_X \) denotes the smallest equivalence relation on \( X \).

We say that \( R \) is a central equivalence relation when it commutes with \( \nabla_X \), the largest equivalence relation on \( X \), so that \([R, \nabla_X]^S = \Delta_X \). A central extension is a regular epimorphism \( f: X \to Y \) whose kernel pair \( Eq(f) \) is a central equivalence relation.

Smith commutators characterise internal groupoids \([33]\); a reflexive graph

\[
\begin{tikzcd}
X \ar{d}{d} & Y \\
X \ar{r}{f} & Y
\end{tikzcd}
\]
in \( \mathcal{X} \) will be a groupoid if and only if \([Eq(d), Eq(c)]^S = \Delta_X \). In particular, it characterises Beck modules \([1]\), since any Beck module in \( \mathcal{X} \), which is an abelian object \((f, s): X \to Y \) in the category \( \text{Pty}(\mathcal{X}) \) of points over \( Y \), so in particular a split extension \( f \) with chosen splitting \( s \), may be seen as an internal groupoid of the form

\[
\begin{tikzcd}
X \ar{r}{f} & Y
\end{tikzcd}
\]
1.3. The coincidence of the Smith and Huq commutators. It is well known, and easily verified, that if the Smith commutator \([R, S]^S\) of two equivalence relations \(R\) and \(S\) is trivial, then the Huq commutator \([K, L]_X\) of their normalisations \(K\) and \(L\) is also trivial \([9]\). It is also well known that, in general, the converse is false; there are counterexamples in the category of digroups \([2, 6]\), which is a variety of \(\Omega\)-groups \([23]\) (and hence semi-abelian), and in the semi-abelian variety of loops \([22]\). The requirement that the two commutators vanish together is known as the condition (SH). As explained in \([30, 22]\), it is important in the study of internal crossed modules \([25]\). The condition (SH) holds for all action accessible categories \([12]\), hence, in particular \([32]\), for any category of interest in the sense of Orzech \([33]\).

In order to simplify our notations, we shall write \([K, L]^S\) for the normalisation of the Smith commutator \([R, S]^S\) (which coincides with the Ursini commutator of \(K\) and \(L\) defined and studied in \([28]\)). The condition (SH) for a semi-abelian category \(\mathcal{X}\) then amounts to the equality \([K, L]^S = [K, L]_X\) for all \(K, L \subseteq X\) in \(\mathcal{X}\).

The special case of central extensions is worth mentioning. A regular epimorphism \(f : X \to Y\) with kernel \(K\) is central in the above sense if and only if \([K, X]^S = 0\). It was shown in \([19]\) that always \([K, X]^S = [K, X]_X\), so centrality of \(f\) may be expressed as the vanishing of a Huq commutator, that is, as the condition \([K, X]_X = 0\). Via the analysis in \([5]\), this concept of central extension is also an instance of the notion coming from categorical Galois theory \([26]\), namely the special case where one considers the Galois structure determined by the abelianisation functor.

1.4. The Higgins commutator. Central extensions may also be characterised in terms of the Higgins commutator \([23, 21, 29]\), which is defined through a co-smash product. Given two objects \(K\) and \(L\) of \(\mathcal{X}\), their co-smash product \([13]\)

\[
K \circ L = \text{Ker} \left( \begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix} : K + L \to K \times L \right)
\]

behaves as a kind of “formal commutator” of \(K\) and \(L\). In fact it is the Huq commutator of the two coproduct inclusions; see \([21]\) and \([29]\). If \(k : K \to X\) and \(l : L \to X\) are subobjects of an object \(X\), the Higgins commutator \([K, L] \subseteq X\) is the subobject of \(X\) given by the image of the induced composite morphism

\[
K \circ L \xrightarrow{K \circ L} K + L \\
\uparrow \psi \\
[K, L] \xrightarrow{\psi} X.
\]

If \(K\) and \(L\) are normal subobjects of \(X\) and \(K \vee L = X\), then it turns out that the Higgins commutator \([K, L]\) is normal in \(X\) and coincides with the Huq commutator. In particular, \([X, X] = [X, X]_X\). More generally, we always have \([K, X] = [K, X]_X\), so that a regular epimorphism \(f : X \to Y\) is a central extension when either one of the three commutators \([K, X] = [K, X]_X = [K, X]^S\) vanishes. In general the Huq commutator \([K, L]_X\) is the normal closure in \(X\) of the Higgins commutator \([K, L]\). So, \([K, L] \subseteq [K, L]_X\) and \([K, L] = 0\) if and only if \([K, L]_X = 0\). An example in \([13]\) shows that in the category of non-associative rings the two commutators generally need not coincide. Thus the coincidence \([K, L] = [K, L]_X\) for all \(K, L \subseteq X\) becomes a basic condition which a semi-abelian category may or may not satisfy; this condition, which we will denote by \((NH)\), was introduced by Cigoli.
in his Ph.D. thesis [15] and was studied further, by Cigoli together with the present authors, in [16].

1.5. The ternary commutator. The Higgins commutator does not preserve joins in general, but the defect may be measured precisely—it is a ternary commutator which can be computed by means of a ternary co-smash product. Let us extend the definition above: given a third subobject \( m: M \rightarrow X \) of the object \( X \), the ternary Higgins commutator \([K, L, M] \leq X\) is the image of the composite

\[
K \circ L \circ M \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\epsilon_m} X
\]

where \( \iota_{K,L,M} \) is the kernel of

\[
\langle i_K, i_K, 0 \rangle \quad \langle i_L, 0, i_L \rangle \quad \langle i_M, 1, 1 \rangle
\]

\( i_k, i_l \) and \( i_m \) denote the injection morphisms. The object \( K \circ L \circ M \) is called the ternary co-smash product of \( K, L \) and \( M \). Note that higher-order co-smash products and their associated commutators have been defined, but since we shall only use binary and ternary Higgins commutators, these will not be needed in this paper. Higgins commutators have good stability properties:

**Proposition 1.6.** [21][22] For all \( X_1, X_2, X_3 \leq X \) and \( M \leq X \) in \( \mathcal{V} \), \( n \in \{2, 3\} \) and \( \sigma \in S_n \) we have the following:

- Symmetry: \([X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)}] = [X_1, \ldots, X_n]\).
- Join decomposition: \([X_1, X_2 \vee X_3] = [X_1, X_2] \vee [X_1, X_3] \vee [X_2, X_3]\).
- Monotonicity: \([M, X_2, \ldots, X_n] \leq [X_1, \ldots, X_n]\).
- Removal of brackets: \([X_1, X_2, X_3] \leq [X_1, X_2, X_3]\).
- Removal of duplicates: if \( X_2 = X_3 \) then \([X_1, X_2, X_3] = [X_1, X_2] \).

1.7. Two lemmas. We end this preliminary section with two known lemmas

**Lemma 1.8.** For each \( K \leq X \) and \( S \leq X \) in \( \mathcal{V} \), the join \( K \vee S \leq X \) can be constructed as the preimage of \( S/(K \wedge S) \leq X/K \) along \( X \rightarrow X/K \) as in the diagram

\[
\begin{array}{ccc}
K \wedge S & \rightarrow & S \\
\downarrow & & \downarrow \\
K & \rightarrow & S/(K \wedge S) \\
\end{array}
\]

Moreover, when \( S \) is normal in \( X \), the join \( K \vee S \), being the preimage of the image of a normal subobject, is normal in \( X \). \( \square \)

**Lemma 1.9.** [16] Lemma 2.6] In a semi-abelian category, consider a point with chosen kernel as in bottom row of the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & K' \rightarrow & X' & \rightarrow & Y \rightarrow & 0 \\
\downarrow & & \downarrow & \Leftrightarrow & \downarrow & \Leftrightarrow & \downarrow \\
0 & \rightarrow & K & \rightarrow & X & \rightarrow & Y \rightarrow & 0
\end{array}
\]
such that \( k \circ \kappa \) is normal. Then this point lifts along \( \kappa : K' \to K \) to yield a morphism of points with chosen kernels.

2. Peri-abelian categories and the condition (WNH)

In this section we consider a weakening of the condition (NH) from Subsection 1.4. Instead of requiring that Higgins commutators of pairs \( K, L < X \) of normal subobjects are normal, we require this only in the special case where \( K = L \). This condition is closely related to the concept of peri-abelian category introduced in [7].

Following [16], we write \( K \times X \) if \( K \times X \) is the kernel of a split extension as in

\[
0 \to K \to X \xrightarrow{f} Y \to 0
\]

and call \( K \) a protosplit normal subobject of \( X \).

Proposition 2.1. For a semi-abelian category, the following are equivalent, and determine a condition which we shall denote by (WNH):

(i) if \( K < X \) then \([K, K] < X\);
(ii) if \( K < X \) then \([K, K] = [K, K]_X\);
(iii) if \( K < X \) then \([K, K]_K < X\);
(iv) if \( K \times X \) then \([K, K]_K < X\);
(v) each point with chosen kernel \( \bullet \) lifts to a morphism of points with chosen kernels

\[
0 \to [K, K]_K \to X' \xrightarrow{f'} Y \to 0
\]

\[
0 \to K \to X \xrightarrow{f} Y \to 0;
\]

(vi) each point with chosen kernel \( \bullet \) induces a morphism of points with chosen kernels

\[
0 \to K \to X \xrightarrow{f} Y \to 0
\]

\[
0 \to K/[K, K]_K \to \tilde{X} \xrightarrow{\tilde{f}} Y \to 0;
\]

(vii) each action on \( K \) restricts to an action on \([K, K]_K\);
(viii) each action on \( K \) induces an action on \([K, K]_K\).

Proof. The equivalence of (i), (ii) and (iii) follows from the fact that the Huq commutator is always the normal closure of the Higgins commutator, and that the Higgins commutator of normal subobjects is normal in the join of those subobjects. (iv) is a special case of (iii). The fact that the conditions (v) and (vii) are equivalent and the conditions (vi) and (viii) are equivalent follows from the equivalence of categories between actions and points from [3]. The implication (vi) \( \Rightarrow \) (v) follows from the fact that the functor \( \text{Ker} : \text{Pt}_{\mathcal{V}}(\mathcal{X}) \to \mathcal{X} \) preserves limits and in particular kernels. We will show that (iv) \( \Rightarrow \) (vi). Suppose that (iv) holds and that \( \bullet \) is a point with chosen kernel. From Lemma 1.9 we obtain the morphism of points as in (v) where \( k \circ \mu_K \) is normal in \( X \) and so \( x \) is normal in \( \text{Pt}_{\mathcal{V}}(\mathcal{X}) \). The induced morphism of points as in (vi) is obtained by taking the cokernel of \( x \) in \( \text{Pt}_{\mathcal{V}}(\mathcal{X}) \). Finally, the last implication (vii) \( \Rightarrow \) (iii) follows from the fact that a subobject \( S \leq X \) is normal in \( X \) if and only if the conjugation on \( X \) restricts to \( S \). Indeed,
the conjugation action on \( X \) restricts to an action on \( K \) which by assumption restricts further to an action on \([K, K]_K\) as required. □

Note that for the category of groups, Condition (vii) of Proposition 2.1 corresponds to the fact that the commutator \([K, K]_K\) is a characteristic subgroup of \( K\): being invariant under automorphisms means precisely that every action on \( K \) restricts to an action on \([K, K]_K\). This observation can be extended to arbitrary semi-abelian categories when the definition of characteristic subobject from [17] is used; see [16] for the proof under the condition (NH).

**Proposition 2.2.** Every arithmetical category satisfies (WNH).

**Proof.** This is due to the fact that in an arithmetical category for each \( K \triangleleft X \) the commutator \([K, K]_X = K \cap K = K\), which follows essentially from [2, Corollary 1.11.13]. □

**Proposition 2.3.** For a semi-abelian category \( X \) the conditions (WNH) and strong protomodularity are independent.

**Proof.** Let us consider the category \( X \) whose objects are set equipped with the structure of an abelian group with operations denoted by \(0, +, -\), together with a binary operation \( \cdot \) satisfying:

\[
x + x = 0, \quad x \cdot x = x, \quad x \cdot y = y \cdot x, \quad \text{and} \quad x \cdot 0 = 0.
\]

The morphisms in \( X \) are (as usual) the structure preserving maps. According to Proposition 2.9.2, Proposition 2.9.11 and Definition 2.9.13 in [2], to show that \( X \) is arithmetical it is sufficient to find a ternary operation \( p \) which satisfies:

\[
p(x, y, y) = x, \quad p(x, x, y) = y \quad \text{and} \quad p(x, y, x) = x.
\]

It is easy to check that \( p(x, y, z) = x + y + x \cdot y + x \cdot z + y \cdot z \) has the desired properties, and hence by Proposition 2.2 \( X \) satisfies (WNH). It remains only to show that \( X \) is not strongly protomodular. For that let \( X \) be the boolean ring with two elements and consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(1_X, 0)} & X \times X \xrightarrow{\pi_2} X \\
\downarrow{\langle 1_X, 0 \rangle} & & \downarrow{\langle 0, 1_X \rangle} \\
X \times X & \xrightarrow{k} & C \xrightarrow{u \cdot s} X
\end{array}
\]

in \( X \) where \( C \) is the object in \( X \) with underlying abelian group \( X \times X \times X \) and with \( \cdot \) defined by

\[
(x, y, z) \cdot (a, b, c) = \begin{cases} 
(xa, yb, ze) & \text{if } (x, y, z) = (a, b, c), \ y = b = 0, \ z = c = 0, \\
(x, y, z) = (0, 0, 0), \text{ or } (a, b, c) = (0, 0, 0); \\
(1, 1, 0) & \text{otherwise}
\end{cases}
\]

and \( u, k, p \) and \( s \) are defined by

\[
u(x, y) = (x, 0, 0), \quad k(x, y) = (x, y, 0), \quad p(x, y, z) = z \quad \text{and} \quad s(x) = (0, 0, x),
\]

respectively. It is easy to check that the above diagram is a morphism of points with chosen kernels where the morphism between the kernels is a normal monomorphism. To show that \( u \) is not normal as a monomorphism of points it is sufficient to show that the composite \( u \circ \langle 1_X, 0 \rangle \) is not normal (as a monomorphism in \( X \)). [20, 4, 35].

Taking into account that every normal monomorphism is the kernel of its cokernel, since \((1, 0, 0) \cdot (1, 1, 1) = (1, 1, 0) \neq u(\langle 1_X, 0 \rangle(X)) \) but the cokernel of \( u \circ \langle 1_X, 0 \rangle \) will send \((1, 0, 0) \) to \( 0 \) and hence \((1, 0, 0) \cdot (1, 1, 1) \) to \( 0 \), it follows that \( u \circ \langle 1_X, 0 \rangle \) is not normal.
The fact that strong protomodularity does not imply (WNH) follows from the fact that the category of non-associative rings is strongly protomodular but does not satisfy (WNH) \([15, \text{Example 5.3.7}]\) \([16, \text{Example 5.4}]\). □

2.4. Peri-abelian categories. A semi-abelian category \(\mathcal{X}\) is said to be peri-abelian \([7]\) if and only if for any \(f: X \to Y\) in \(\mathcal{X}\), the change of base functor
\[f^*: \text{Pt}_Y(\mathcal{X}) \to \text{Pt}_X(\mathcal{X})\]
commutes with abelianisation. (The original definition of \([7]\) is actually given in a wider context, but we shall only consider the case of semi-abelian categories.)

Proposition 2.5. For a semi-abelian category \(\mathcal{X}\), the following are equivalent:

(i) \(\mathcal{X}\) is peri-abelian;

(ii) for any object \(Y\) of \(\mathcal{X}\), the diagram
\[
\begin{array}{ccc}
\text{Pt}_Y(\mathcal{X}) & \xrightarrow{\text{Ab}} & \text{Ab}(\text{Pt}_Y(\mathcal{X})) \\
\downarrow{\text{Ker}} & & \downarrow{\text{Ker}} \\
\mathcal{X} & \xrightarrow{\text{Ab}} & \text{Ab}(\mathcal{X})
\end{array}
\]
commutes;

(iii) for any object \(Y\) of \(\mathcal{X}\), the square of left adjoint functors
\[
\begin{array}{ccc}
\text{Pt}_Y(\mathcal{X}) & \xrightarrow{\text{Ab}} & \text{Ab}(\text{Pt}_Y(\mathcal{X})) \\
\downarrow{F = Y^*} & \uparrow{G} & \\
\mathcal{X} & \xrightarrow{\text{Ab}} & \text{Ab}(\mathcal{X})
\end{array}
\]
satisfies the Beck–Chevalley condition;

(iv) if \(K \rhd X\) then \([K, K] = [K, K]^0\);

(v) if \(K \rhd X\) then \([K, K, X] \leq [K, K]\);

(vi) if \(K \triangleright X\) then \([K, K] = [K, K]^0\);

(vii) if \(K \triangleright X\) then \([K, K, X] \leq [K, K]\);

(viii) each action \(\zeta: Y \triangleright K \to K\) induces an action \(\theta: Y \triangleright A \to A\) and a morphism of \(Y\)-actions \(\zeta \to \theta\), where \(A = \text{Ab}(K) = K/[K, K]\) is the abelianisation of \(K\), such that the diagram
\[
\begin{array}{ccc}
Y \triangleright (A \times A) & \xrightarrow{Y \triangleright m} & Y \triangleright A \\
\downarrow{\langle \theta \circ Y \triangleright \pi_1, \theta \circ Y \triangleright \pi_2 \rangle} & & \downarrow{\theta} \\
A \times A & \xrightarrow{m} & A,
\end{array}
\]
in which \(m\) is the multiplication of the group \(A\), commutes.

Furthermore, (PA) implies (WNH). When, in addition, \(\mathcal{X}\) satisfies (SH) then (WNH) is equivalent to (PA).

Proof. Condition (ii) is the special case of (i) where \(f: Y \to 0\); it is explained in \([7]\) that this is sufficient. Condition (iii) is a reformulation of (ii). Using the equivalence of categories between \(\text{Pt}_Y(\mathcal{X})\) and \(\mathcal{X}^{Y^*}\) from \([3]\), it can be seen that (ii) and (viii) are equivalent. Indeed, the commutativity of the diagram in (viii) amounts to saying that \(\theta\) is an abelian object; so if (ii) holds, then we can just take \(\theta = \text{Ab}(\zeta)\). Conversely, the morphism \(\zeta \to \theta\) provided by (viii) induces a morphism \(\text{Ab}(\zeta) \to \theta\) which is both an isomorphism and a monomorphism, so that \(A\) is indeed a kernel of \(\text{Ab}(\zeta)\), and thus (ii) holds.
We now prove (ii) $\Leftrightarrow$ (iv). We recalled (in Subsection 1.2 above) that a point 
$(f,s): X \rightrightarrows Y$ is abelian if and only if the kernel pair of $f$ commutes with itself, 
and that the abelianisation of $(f,s)$ is obtained through the quotient $X/[K,K]^S$. 
The kernel of this split extension is $K/[K,K]^S$. On the other hand, by definition, 
$\mathcal{C}$ is peri-abelian precisely when the kernel of the abelianisation is $K/[K,K]_K = K/[K,K]$, which happens if and only if $[K,K] = [K,K]^S$. This proves the equivalence between (ii) and (iv).

By [22] Theorem 5.2, $[K,K]^S = [K,K] \vee [K,K,X]$, which gives us (iv) $\Leftrightarrow$ (v) 
and (vi) $\Leftrightarrow$ (vii). It is clear that (vi) implies (iv). We still have to prove (iv) $\Rightarrow$ (vi), 
but here the proof of Theorem 2.3 in [30] may be repeated.

Since for $K \lhd X$, the Smith commutator $[K,K]^S$ is normal in $X$, it follows 
that condition (iv), which tells us that if $K \lhd X$, then $[K,K] = [K,K]^S$, gives 
Condition (i) of Proposition 2.1. This proves that (PA) implies (WNH).

It remains to show that when the condition (SH) holds we can also obtain the 
converse: (WNH) implies (PA). Indeed, by Theorem 4.6 in [22], the inequality 
$[K,K,X] \leq [K,K]^S$ follows from (SH). But if $[K,K]_X = [K,K]^S$, which follows 
from Condition (ii) of Proposition 2.1, this gives us condition (vii).

Corollary 2.6. All categories of interest are peri-abelian, while categories of non-
associative algebras, and the categories of loops and of digroups, are not. In particular, 
strong protomodularity [14] does not imply (PA).

Proof. Theorem 5.3.6 in Cigoli’s thesis [15] shows that any category of interest 
satisfies (WNH), while those categories satisfy (SH) by action accessibility [12] 
combined with [32]. It is known that the condition (SH) fails for the categories 
of loops and digroups [22] [13] [2] [6], and that the condition (WNH) fails for non-
associative rings [15] Example 5.3.7 [16] Example 5.4]. Note the that the latter 
category is strongly protomodular.

3. THE UNIVERSAL CENTRAL EXTENSION CONDITION

Definition 3.1. [14] We say that a semi-abelian category $\mathcal{C}$ satisfies the condition 
(UCE) when: for each pair of composable central extensions $f: A \rightarrow B$ and 
g: $B \rightarrow C$, if $B$ is perfect, then the composite $g \circ f$ is a central extension.

Example 3.2. The variety of non-associative algebras over a field $K$ is semi-abelian, 
even strongly protomodular, but need not satisfy (UCE) [14].

Our aim is to prove that $\mathcal{C}$ satisfies (UCE) as soon as (PA) holds. This implies 
in particular that (UCE) holds for any category in which the Smith commutator 
and the Higgins commutator coincide. As a consequence of Corollary 2.6, then all 
categories of interest [33] satisfy (UCE). Our argument is essentially a categorical 
version of the proof for groups given in [31].

Lemma 3.3. Let $K$ be the kernel of an extension $f: A \rightarrow B$ with a perfect 
codomain $B$. Then $A = K \vee [A,A]$.

Proof. It follows from Lemma 1.8 that the join $K \vee [A,A]$ is the preimage along of 
the image of $[A,A] \triangleleft A$ along $f$. Since the image of $[A,A] \triangleleft A$ along $f$ is $[B,B] \triangleleft B$, 
and $[B,B] = B$ because $B$ is perfect, it follows that $K \vee [A,A] = A$ as required. □

Lemma 3.4. If $X$, $Y$, $Z \leq A$ are subobjects of $A$ such that $X \vee Y = A$ and 
$[X,A] = 0$, then $[A,Z] = [Y,Z]$.

Proof. Since by Proposition 1.6 $[X,Y,Z] \leq [X,A,A] \leq [X,A] = 0$, applying the 
same proposition, we see that 
$[A,Z] = [X \vee Y,Z] = [X,Z] \vee [Y,Z] \vee [X,Y,Z] = [Y,Z]$. □
Lemma 3.5. If \( f: A \to B \) is a central extension of a perfect object \( B \), then the composite \( f \circ \mu_A: [A, A] \to B \) is a central extension with perfect domain.

Proof. It follows from Lemma 3.3 and Lemma 3.4 that \([A, A]\) is perfect. Since subobjects of central extensions are central extensions it follows that \( f \circ \mu_A \) is a central extension.

The following lemma appeared in [14]. It also easily follows from Theorem 2.1 (1) \( \iff \) (8) in [8], which generalises Theorem 5.2 (i) \( \iff \) (viii) in [20]; we repeat the proof to make the paper more self-contained.

Lemma 3.6. Suppose that \( f: A \to B \) is a central extensions and \( P \) is a perfect object. If \( p_1, p_2: P \to A \) are parallel morphisms such that \( f \circ p_1 = f \circ p_2 \), then \( p_1 = p_2 \).

Proof. Suppose \( f: A \to B \) is a central extension, and \( p_1, p_2: P \to A \) are parallel morphisms with perfect domain, such that \( f \circ p_1 = f \circ p_2 \). Let \( k: K \to A \) be the kernel of \( f \), and let \( \varphi: K \times A \to A \) be the morphism showing that \( k \) and \( 1_A \) commute. It is well known that the kernel pair of \( f \) can be presented as \((\varphi, \pi_2): K \times A \to A\). Since \( f \circ p_1 = f \circ p_2 \) it follows by the universal property of the kernel pair that there exists a morphism \( q: P \to K \times A \) such that \( \pi_2 \circ q = p_2 \) and \( \varphi \circ q = p_1 \). Hence \( q = (d, p_2) \) for some morphism \( d: P \to K \). Since \( P \) is perfect and \( K \) is abelian it follows that \( d \) is the zero morphism. We have \( p_1 = \varphi \circ q = \varphi \circ (0, p_2) = \varphi \circ (0, 1_A) \circ p_2 = 1_A \circ p_2 = p_2 \), as required.

For a composite of central extensions, using the above lemma and induction, we obtain the following lemma.

Lemma 3.7. Suppose that \( f: A \to B \) is a composite of central extensions and \( P \) is a perfect object. If \( p_1, p_2: P \to A \) are parallel morphisms such that \( f \circ p_1 = f \circ p_2 \), then \( p_1 = p_2 \).

Proposition 3.8. Let \( \mathcal{X} \) be a semi-abelian category satisfying (PA). Consider objects \( B \) and \( P \) in \( \mathcal{X} \). If \( P \) is perfect then also \( B \circ P \) is perfect.

Proof. Let us write \( F: \mathcal{X} \to \mathcal{Pt}_B(\mathcal{X}) \) and \( G: \text{Ab}(\mathcal{X}) \to \text{Ab}(\mathcal{Pt}_B(\mathcal{X})) \) for the left adjoints of the kernel functors as in Proposition 2.5. Then

\[
\text{Ab}(B \circ P) = \text{Ab}(\text{Ker}(F(P))) = \text{Ker}(\text{Ab}(F(P))) = \text{Ker}(\text{Ab}(G(P))) = \text{Ker}(G(0)) = 0,
\]

so \( B \circ P \) is perfect.

Remark 3.9. It is well known that any non-abelian simple group is perfect (and trivially the corresponding statement is true in any semi-abelian category). It is natural to ask whether the above proposition holds for non-abelian simple objects. That is, is it true that for any objects \( B \) and \( X \) in a peri-abelian category, if \( X \) is non-abelian and simple, then \( B \circ X \) is non-abelian and simple? This turns out to be false even for the category of groups. In fact for any non-trivial groups \( G \) and \( X \), the group \( G \circ X \) is a proper normal subobject of \( G \circ X \): for non-trivial \( g \) and \( x \) in \( G \) and in \( X \), respectively, the word \( gxg^{-1}x^{-1} \) is in \( G \circ X \) but not in \( G \circ X \), while the word \( gxg^{-1}x^{-1} \) is in both.

Lemma 3.10. Let \( \mathcal{X} \) be a semi-abelian category and let \( \kappa: K' \to K \) be a composite of central extensions with perfect domain. If \( \mathcal{X} \) satisfies (PA), then
(i) for each action \( \zeta : B \triangleright K \to K \) there exists at most one action \( \theta \) making the diagram
\[
\begin{array}{ccc}
B \triangleright K' & \overset{B \triangleright \kappa}{\longrightarrow} & B \triangleright K \\
\theta \downarrow & & \downarrow \zeta \\
K' & \overset{\kappa}{\longrightarrow} & K
\end{array}
\]
commute;

(ii) for each point with chosen kernel
\[
0 \longrightarrow K \overset{k}{\longrightarrow} A \overset{\phi}{\longrightarrow} B \longrightarrow 0,
\]
there exists, up to isomorphism, at most one lifting
\[
0 \longrightarrow K' \overset{k'}{\longrightarrow} A' \overset{\phi'}{\longrightarrow} B \longrightarrow 0.
\]

Proof. Condition (ii) follows from (i) by the equivalence between actions and points. We prove that (i) follows from (PA). Suppose \( \theta : B \triangleright K' \to K' \) and \( \phi : B \triangleright K' \to K' \) are two actions making the diagram in Condition (i) commute. Since, by Proposition \[3.8\] the object \( B \triangleright K' \) is perfect, and since \( \kappa \circ \theta = \zeta \circ (B \triangleright \kappa) = \kappa \circ \phi \) and \( \kappa \) is a composite of central extensions, it follows by Lemma \[3.7\] that \( \theta \) equals \( \phi \). \( \square \)

**Lemma 3.11.** Let \( \mathcal{X} \) be a semi-abelian category satisfying (PA) and let \( f : A \to B \) and \( g : B \to C \) be central extensions in \( \mathcal{X} \). If \( B \) is perfect, then the morphism \( \ker (g \circ f) : \ker (g \circ f) \to A \) commutes with \( \mu_A : [A, A] \to A \).

Proof. Let \( k : K \to A \) denote the kernel of \( g \circ f \). Consider the diagram
\[
\begin{array}{ccc}
0 \longrightarrow [A, A] & \overset{k}{\longrightarrow} & A' \overset{f'}{\longrightarrow} K \\
\mu_A \downarrow & & \downarrow \pi_2 \\
A & \overset{(1_A, 0)}{\longrightarrow} & A \times K \overset{\pi_2}{\longrightarrow} K \longrightarrow 0 \\
g \circ f \downarrow & & \downarrow \pi_2 \\
[1_C, 0] \longrightarrow & C \times K & \overset{\pi_2}{\longrightarrow} K \longrightarrow 0 \\
\end{array}
\]
where we use Proposition \[2.1\] (via Proposition \[2.5\]) to obtain the dotted lifting. The composite \( f \circ \mu_A \) is a central extension with perfect domain by Lemma \[3.5\].

As a consequence, the morphism \( g \circ f \circ \mu_A : [A, A] \to C \) is a composite of central extensions with perfect domain. Lemma \[3.10\] now tells us that the lifting obtained above is unique up to isomorphism. However, the diagram
\[
\begin{array}{ccc}
0 \longrightarrow [A, A] & \overset{(1_{[A, A]}, 0)}{\longrightarrow} & [A, A] \times K \overset{\pi_2}{\longrightarrow} K \longrightarrow 0 \\
g \circ f \circ \mu_A \downarrow & & \downarrow (g \circ f \circ \mu_A) \times 1_K \\
[1_C, 0] \longrightarrow & C \times K & \overset{\pi_2}{\longrightarrow} K \longrightarrow 0 \\
\end{array}
\]
is another lifting. Thus we obtain the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & [A, A] & \xrightarrow{\langle 1_{[A, A]}, 0 \rangle} & [A, A] \times K & \xrightarrow{\pi_2} & K & \rightarrow & 0 \\
& & \downarrow{\mu_A} & & \downarrow{\varphi} & & & & \\
0 & \rightarrow & A & \xrightarrow{\langle 1_A, 0 \rangle} & A \times K & \xrightarrow{\pi_2} & K & \rightarrow & 0 \\
& & \downarrow{(g \circ f)} & & \downarrow{(g \circ f) \times 1_K} & & & & \\
0 & \rightarrow & C & \xrightarrow{\langle 1_C, 0 \rangle} & C \times K & \xrightarrow{\pi_2} & K & \rightarrow & 0
\end{array}
\]

which is a composite of morphisms of points with chosen kernels. It follows that the composite \( \pi_1 \circ \varphi \) makes the diagram

\[
[A, A] \xrightarrow{\langle 1_{[A, A]}, 0 \rangle} [A, A] \times K \xrightarrow{\langle 0, 1_K \rangle} K
\]

commute, proving that \( \mu_A \) and \( k \) commute as required.

\[\square\]

**Theorem 3.12.** If a semi-abelian category satisfies (PA), then it satisfies (UCE). In particular, (SH) + (WNH) \(\Rightarrow\) (UCE).

**Proof.** Let \( B \) be a perfect object and let \( f: A \rightarrow B \) and \( g: B \rightarrow C \) be central extensions. It follows from Lemma 3.3 that \( A = [A, A] \vee \text{Ker}(f) \) and therefore from Lemma 3.4 and Lemma 3.11 that \([A, \text{Ker}(g; f)] = [[A, A], \text{Ker}(g; f)] = 0\). \[\square\]

**Corollary 3.13.** All categories of interest satisfy (UCE). In particular, the category of groups, the category of rings, and the categories of associative, Lie or Liebniz algebras over a ring all satisfy (UCE).

We include the following lemma to show that semi-abelian categories satisfying (UCE), together with other mild assumptions (valid in all categories of interest, for instance) seem to provide the necessary tools to capture certain aspects of the theory of central extensions valid for groups which are not immediately given by categorical Galois theory. For an object \( X \) we will denote by \( Z(X) \subseteq X \) the largest subobject of \( X \) such that \([X, Z(X)] = 0\). This subobject will be called the centre of \( X \). Note that the centre of an object is always a normal subobject.

**Lemma 3.14** (Gr"{u}n’s Lemma). Let \( \mathcal{X} \) be a semi-abelian category admitting centres and satisfying (UCE)—for instance, \( \mathcal{X} \) may be any category of interest. If \( P \) is perfect, then the quotient \( P/Z(P) \) has trivial centre.

**Proof.** Consider the morphisms

\[
P \xrightarrow{p} P/Z(P) \xrightarrow{q} (P/(Z(P)))/(Z(P/Z(P)))
\]

which are the quotients of \( Z(P) \) and \( Z(P/Z(P)) \) in \( P \) and in \( P/Z(P) \), respectively. By taking kernels we obtain the exact sequence

\[
0 \rightarrow Z(P) = \text{Ker}(p) \rightarrow \text{Ker}(q; p) \rightarrow \text{Ker}(q) \rightarrow 0.
\]

Since \( P \) is perfect and \( p \) is a regular epimorphism it follows that \( P/Z(P) \) is perfect. Therefore, since \( p \) and \( q \) are central extensions, it follows from (UCE) that \( q; p \) is a central extension, and so \( \text{Ker}(q; p) \subseteq Z(P) \). It follows that \( Z(P) = \text{Ker}(q; p) \) and so \( \text{Ker}(q) = Z(P/Z(P)) = 0 \) as required. \[\square\]
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