A CLASSIFICATION OF BISYMMETRIC POLYNOMIAL FUNCTIONS OVER INTEGRAL DOMAINS OF CHARACTERISTIC ZERO

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Abstract. We describe the class of $n$-variable polynomial functions that satisfy Aczél’s bisymmetry property over an arbitrary integral domain of characteristic zero with identity.

1. Introduction

Let $\mathcal{R}$ be an integral domain of characteristic zero (hence $\mathcal{R}$ is infinite) with identity and let $n \geq 1$ be an integer. In this paper we provide a complete description of all the $n$-variable polynomial functions over $\mathcal{R}$ that satisfy the (Aczél) bisymmetry property. Recall that a function $f: \mathcal{R}^n \to \mathcal{R}$ is bisymmetric if the $n^2$-variable mapping

$$(x_{11}, \ldots, x_{1n}; \ldots; x_{n1}, \ldots, x_{nn}) \mapsto f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn}))$$

does not change if we replace every $x_{ij}$ by $x_{ji}$.

The bisymmetry property for $n$-variable real functions goes back to Aczél [1, 2]. It has been investigated since then in the theory of functional equations by several authors, especially in characterizations of mean functions and some of their extensions (see, e.g., [3, 5–7]). This property is also studied in algebra where it is called mediality. For instance, an algebra $(A, f)$ where $f$ is a bisymmetric binary operation is called a medial groupoid (see, e.g., [8, 9, 11]).

We now state our main result, which provides a description of the possible bisymmetric polynomial functions from $\mathcal{R}^n$ to $\mathcal{R}$. Let $\text{Frac}(\mathcal{R})$ denote the fraction field of $\mathcal{R}$ and let $\mathbb{N}$ be the set of nonnegative integers. For any $n$-tuple $x = (x_1, \ldots, x_n)$, we set $|x| = \sum_{i=1}^{n} x_i$.

Main Theorem. A polynomial function $P: \mathcal{R}^n \to \mathcal{R}$ is bisymmetric if and only if it is

(i) univariate, or
(ii) of degree $\leq 1$, that is, of the form

$$P(x) = a_0 + \sum_{i=1}^{n} a_i x_i,$$

where $a_i \in \mathcal{R}$ for $i = 0, \ldots, n$, or

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(iii) of the form

$$P(x) = a \prod_{i=1}^{n}(x_i + b)^{\alpha_i} - b,$$

where $a \in \mathbb{R}$, $b \in \text{Frac}(\mathbb{R})$, and $\alpha \in \mathbb{N}^n$ satisfy $ab^k \in \mathbb{R}$ for $k = 1, \ldots, |\alpha| - 1$ and $ab^{\alpha} - b \in \mathbb{R}$. 

The following example, borrowed from [10], gives a polynomial function of class (iii) for which $b \notin \mathbb{R}$.

**Example 1.** The third-degree polynomial function $P: \mathbb{Z}^3 \to \mathbb{Z}$ defined on the ring $\mathbb{Z}$ of integers by

$$P(x_1, x_2, x_3) = 9x_1x_2x_3 + 3(x_1x_2 + x_2x_3 + x_3x_1) + x_1 + x_2 + x_3$$

is bisymmetric since it is the restriction to $\mathbb{Z}$ of the bisymmetric polynomial function $Q: \mathbb{Q}^3 \to \mathbb{Q}$ defined on the field $\mathbb{Q}$ of rationals by

$$Q(x_1, x_2, x_3) = 9 \prod_{i=1}^{3}(x_i + \frac{1}{3}) - \frac{1}{3}.$$

Since polynomial functions usually constitute the most basic functions, the problem of describing the class of bisymmetric polynomial functions is quite natural. On this subject it is noteworthy that a description of the class of bisymmetric lattice polynomial functions over bounded chains and more generally over distributive lattices has been recently obtained [4, 5] (there bisymmetry is called self-commutation), where a lattice polynomial function is a function representable by combinations of variables and constants using the fundamental lattice operations $\land$ and $\lor$.

From the Main Theorem we can derive the following test to determine whether a non-univariate polynomial function $P: \mathbb{R}^n \to \mathbb{R}$ of degree $p \geq 2$ is bisymmetric. For $k \in \{p-1, p\}$, let $P_k$ be the homogenous polynomial function obtained from $P$ by considering the terms of degree $k$ only. Then $P$ is bisymmetric if and only if $P_k$ is a monomial and $P_p(x) = P(x-b1) + b$, where $1 = (1, \ldots, 1)$ and $b = P_{p-1}(1)/(pP_p(1))$.

Note that the Main Theorem does not hold for an infinite integral domain $\mathbb{R}$ of characteristic $r > 0$. As a counterexample, the bivariate polynomial function $P(x_1, x_2) = x_1^r + x_2^r$ is bisymmetric.

In the next section we provide the proof of the Main Theorem, assuming first that $\mathbb{R}$ is a field and then an integral domain.

2. **Technicalities and proof of the Main Theorem**

We observe that the definition of $\mathbb{R}$ enables us to identify the ring $\mathbb{R}[x_1, \ldots, x_n]$ of polynomials of $n$ indeterminates over $\mathbb{R}$ with the ring of polynomial functions of $n$ variables from $\mathbb{R}^n$ to $\mathbb{R}$.

It is a straightforward exercise to show that the $n$-variable polynomial functions given in the Main Theorem are bisymmetric. We now show that no other $n$-variable polynomial function is bisymmetric. We first consider the special case when $\mathbb{R}$ is a field. We then prove the Main Theorem in the general case (i.e., when $\mathbb{R}$ is an integral domain of characteristic zero with identity).

Unless stated otherwise, we henceforth assume that $\mathbb{R}$ is a field of characteristic zero. Let $p \in \mathbb{N}$ and let $P: \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree $p$. Thus $P$
can be written in the form
\[ P(x) = \sum_{|\alpha| \leq p} c_\alpha x^\alpha, \quad \text{with } x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}, \]
where the sum is taken over all \( \alpha \in \mathbb{N}^n \) such that \(|\alpha| \leq p\).

The following lemma, which makes use of formal derivatives of polynomial functions, will be useful as we continue.

**Lemma 2.** For every polynomial function \( B: \mathcal{R}^n \to \mathcal{R} \) of degree \( p \) and every \( x_0, y_0 \in \mathcal{R}^n \), we have
\[ B(x_0 + y_0) = \sum_{|\alpha| \leq p} \frac{y_0^\alpha}{\alpha!} (\partial_x^\alpha B)(x_0), \]
where \( \partial^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \) and \( \alpha! = \alpha_1! \cdots \alpha_n! \).

**Proof.** It is enough to prove the result for monomial functions since both sides of \((1)\) are additive on the function \( B \). We then observe that for a monomial function \( B(x) = c x^d \) the identity \((1)\) reduces to the multi-binomial theorem. \( \square \)

As we will see, it is useful to decompose \( P \) into its homogeneous components, that is, \( P = \sum_{k=0}^{p} P_k \), where
\[ P_k(x) = \sum_{|\alpha| = k} c_\alpha x^\alpha \]
is the unique homogeneous component of degree \( k \) of \( P \). In this paper the homogeneous component of degree \( k \) of a polynomial function \( R \) will often be denoted by \( [R]_k \).

Since \( P_p \neq 0 \), the polynomial function \( Q = P - P_p \), that is
\[ Q(x) = \sum_{|\alpha| < p} c_\alpha x^\alpha, \]
is of degree \( q < p \) and its homogeneous component \([Q]_q\) of degree \( q \) is \( P_q \).

We now assume that \( P \) is a bisymmetric polynomial function. This means that the polynomial identity
\[ P(P(r_1), \ldots, P(r_n)) - P(P(c_1), \ldots, P(c_n)) = 0 \]
holds for every \( n \times n \) matrix
\[ X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathcal{R}_n^n, \]
where \( r_i = (x_{i1}, \ldots, x_{in}) \) and \( c_j = (x_{1j}, \ldots, x_{nj}) \) denote its \( i \)th row and \( j \)th column, respectively. Since all the polynomial functions of degree \( \leq 1 \) are bisymmetric, we may (and henceforth do) assume that \( p \geq 2 \).

From the decomposition \( P = P_p + Q \) it follows that
\[ P(P(r_1), \ldots, P(r_n)) = P_p(P(r_1), \ldots, P(r_n)) + Q(P(r_1), \ldots, P(r_n)), \]
where \( Q(P(r_1), \ldots, P(r_n)) \) is of degree \( pq \).

To obtain necessary conditions for \( P \) to be bisymmetric, we will equate the homogeneous components of the same degree \( > pq \) of both sides of \((2)\). By the previous observation this amounts to considering the equation
\[ [P_p(P(r_1), \ldots, P(r_n)) - P_p(P(c_1), \ldots, P(c_n))]_d = 0, \quad \text{for } pq < d \leq p^2. \]
By applying (1) to the polynomial function $P_p$ and the $n$-tuples
$$x_0 = (P_p(r_1), \ldots, P_p(r_n)) \quad \text{and} \quad y_0 = (Q(r_1), \ldots, Q(r_n)),$$
we obtain
$$P_p(P(r_1), \ldots, P(r_n)) = \sum_{|\alpha| \leq p} \frac{y_0^\alpha}{\alpha!} \partial_\alpha^p P_p(x_0)$$
and similarly for $P_p(P(c_1), \ldots, P(c_n))$. We then observe that in the sum of (5) the term corresponding to a fixed $\alpha$ is either zero or of degree
$$q|\alpha| + (p - |\alpha|) p = p^2 - (p - q)|\alpha|$$
and its homogeneous component of highest degree is obtained by ignoring the components of degrees $< q$ in $Q$, that is, by replacing $y_0$ by $(P_q(r_1), \ldots, P_q(r_n))$.

Using (4) with $d = p^2$, which leads us to consider the terms in (5) for which $|\alpha| = 0$, we obtain
$$P_p(P_p(r_1), \ldots, P_p(r_n)) - P_p(P_p(c_1), \ldots, P_p(c_n)) = 0.$$
Thus, we have proved the following claim.

**Claim 3.** The polynomial function $P_p$ is bisymmetric.

We now show that $P_p$ is a monomial function.

**Proposition 4.** Let $H: \mathcal{R}^n \to \mathcal{R}$ be a bisymmetric polynomial function of degree $p \geq 2$. If $H$ is homogeneous, then it is a monomial function.

**Proof.** Consider a bisymmetric homogeneous polynomial $H: \mathcal{R}^n \to \mathcal{R}$ of degree $p \geq 2$. There is nothing to prove if $H$ depends on one variable only. Otherwise, assume for the sake of a contradiction that $H$ is the sum of at least two monomials of degree $p$, that is,
$$H(x) = a x^\alpha + b x^\beta + \sum_{|\gamma| = p} c_\gamma x^\gamma,$$
where $a, b \neq 0$ and $|\alpha| = |\beta| = p$. Using the lexicographic order $\prec$ over $\mathbb{N}^n$, we can assume that $\alpha \succ \beta \succ \gamma$. Applying the bisymmetry property of $H$ to the $n \times n$ matrix whose $(i, j)$-entry is $x_i y_j$, we obtain
$$H(x)^p H(y^p) = H(y^p)^p H(x^p),$$
where $x^p = (x_{p1}, \ldots, x_{pn})$. Regarding this equality as a polynomial identity in $y$ and then equating the coefficients of its monomial terms with exponent $p \alpha$, we obtain
$$H(x)^p = a^{p-1} H(x^p).$$
Since $\mathcal{R}$ is of characteristic zero, there is a nonzero monomial term with exponent $(p-1) \alpha + \beta$ in the left-hand side of (7) while there is no such term in the right-hand side since $p \alpha > (p-1) \alpha + \beta > p \beta$ (since $p \geq 2$). Hence a contradiction. \hfill $\Box$

The next claim follows immediately from Proposition 4.

**Claim 5.** $P_p$ is a monomial function.

By Claim 5 we can (and henceforth do) assume that there exist $c \in \mathcal{R} \setminus \{0\}$ and $\gamma \in \mathbb{N}^n$, with $|\gamma| = p$, such that
$$P_p(x) = c x^\gamma.$$
A polynomial function $F: \mathbb{R}^n \to \mathbb{R}$ is said to depend on its $i$th variable $x_i$ (or $x_i$ is essential in $F$) if $\partial_{x_i} F \neq 0$. The following claim shows that $P_p$ determines the essential variables of $P$.

**Claim 6.** If $P_p$ does not depend on the variable $x_j$, then $P$ does not depend on $x_j$.

*Proof.* Suppose that $\partial_{x_i} P_p = 0$ and fix $i \in \{1, \ldots, n\}$, $i \neq j$, such that $\partial_{x_j} P_p \neq 0$. By taking the derivative of both sides of \[ \text{(2)} \] with respect to $x_{ij}$, the $(i, j)$-entry of the matrix $X$ in \[ \text{(3)} \], we obtain

\[
(\partial_{x_i} P_p)(P_p(1), \ldots, P_p(n))(\partial_{x_j} P_p)(r_i) = (\partial_{x_i} P_p)(P_p(1), \ldots, P_p(n))(\partial_{x_j} P_p)(c_j).
\]

Suppose for the sake of a contradiction that $\partial_{x_j} P_p \neq 0$. Then, neither side of \[ \text{(9)} \] is the zero polynomial. Let $R_j$ be the homogeneous component of $\partial_{x_j} P_p$ of highest degree and denote its degree by $r$. Since $P_p$ does not depend on $x_j$, we must have $r < p - 1$. Then the homogeneous component of highest degree of the left-hand side in \[ \text{(9)} \] is given by

\[
(\partial_{x_i} P_p)(P_p(1), \ldots, P_p(n)) R_j(r_i)
\]

and is of degree $p(p - 1) + r$. But the right-hand side in \[ \text{(9)} \] is of degree at most $rp + p - 1 = (r + 1)p - 1 + r < p(p - 1) + r$, since $r < p - 1$ and $p \geq 2$. Hence a contradiction. Therefore $\partial_{x_j} P_p = 0$. \[ \square \]

We now give an explicit expression for $P_q = [P - P_p]_q$ in terms of $P_p$. We first present an equation that is satisfied by $P_q$.

**Claim 7.** $P_q$ satisfies the equation

\[
\sum_{i=1}^{n} P_q(r_i)(\partial_{x_i} P_p)(P_p(1), \ldots, P_p(n)) = \sum_{i=1}^{n} P_q(c_i)(\partial_{x_i} P_p)(P_p(1), \ldots, P_p(n))
\]

for every matrix $X$ as defined in \[ \text{(3)} \].

*Proof.* By \[ \text{(6)} \] and \[ \text{(8)} \] we see that the left-hand side of \[ \text{(10)} \] for $d = p^2$ is zero. Therefore, the highest degree terms in the sum of \[ \text{(3)} \] are of degree $p^2 - (p - q) > p q$ (because $(p - 1)(p - q) > 0$) and correspond to those $\alpha \in \mathbb{N}^n$ for which $|\alpha| = 1$. Collecting these terms and then considering only the homogeneous component of highest degree (that is, replacing $Q$ by $P_q$), we see that the identity \[ \text{(4)} \] for $d = p^2 - (p - q)$ is precisely \[ \text{(10)} \]. \[ \square \]

**Claim 8.** We have

\[
P_q(x) = \frac{P_q(1)}{c p} P_p(x) \sum_{j=1}^{n} \frac{\gamma_j}{x_j^{p-q}}.
\]

Moreover, $P_q = 0$ if there exists $j \in \{1, \ldots, n\}$ such that $0 < \gamma_j < p - q$.

*Proof.* Considering Eq. \[ \text{(10)} \] for a matrix $X$ such that $r_i = x$ for $i = 1, \ldots, n$, we obtain

\[
c p P_q(x) P_p(x)^{p-1} = P_q(1) \sum_{i=1}^{n} x_i^q (\partial_{x_i} P_p)(c x_1^p, \ldots, c x_n^p).
\]

Since $\partial_{x_i} P_p(x) = \gamma_i P_p(x)/x_i$, the previous equation becomes

\[
c p P_q(x) P_p(x)^{p-1} = P_q(1) P_p(x)^p \sum_{i=1}^{n} \frac{\gamma_i}{x_i^{p-q}}.
\]
from which Eq. (11) follows. Now suppose that \( P_q \neq 0 \) and let \( j \in \{1, \ldots, n\} \).
Comparing the lowest degrees in \( x_j \) of both sides of (12), we obtain
\[
(p - 1) \gamma_j \leq \begin{cases} 
  p \gamma_j - p + q, & \text{if } \gamma_j \neq 0, \\
  p \gamma_j, & \text{if } \gamma_j = 0.
\end{cases}
\]
Therefore, we must have \( \gamma_j = 0 \) or \( \gamma_j \geq p - q \), which ensures that the right-hand side of (11) is a polynomial. \( \Box \)

If \( \varphi: \mathcal{R} \to \mathcal{R} \) is a bijection, we can associate with every function \( f: \mathcal{R}^n \to \mathcal{R} \) its
conjugate \( \varphi(f): \mathcal{R}^n \to \mathcal{R} \) defined by
\[
\varphi(f)(x_1, \ldots, x_n) = \varphi^{-1}\left( f(\varphi(x_1), \ldots, \varphi(x_n)) \right).
\]
It is clear that \( f \) is bisymmetric if and only if so is \( \varphi(f) \). We then have the following fact.

Fact 9. The class of \( n \)-variable bisymmetric functions is stable under the action of conjugation.

Since the Main Theorem involves polynomial functions over a ring, we will only consider conjugations given by translations \( \varphi_b(x) = x + b \).

We now show that it is always possible to conjugate \( P \) with an appropriate
translation \( \varphi_b \) to eliminate the terms of degree \( p - 1 \) of the resulting polynomial function \( \varphi_b(P) \).

Claim 10. There exists \( b \in \mathcal{R} \) such that \( \varphi_b(P) \) has no term of degree \( p - 1 \).

Proof. If \( q < p - 1 \), we take \( b = 0 \). If \( q = p - 1 \), then using (11) with \( y_0 = b1 \), we get
\[
[\varphi_b(P)]_{p-1} = P_{p-1} + b \sum_{i=1}^{n} \partial x_i P_p.
\]
On the other hand, by (11) we have
\[
P_{p-1} = \frac{P_{p-1}(1)}{cp} \sum_{i=1}^{n} \partial x_i P_p.
\]
It is then enough to choose \( b = -P_{p-1}(1)/(cp) \) and the result follows. \( \Box \)

We can now prove the Main Theorem for polynomial functions of degree \( \leq 2 \).

Proposition 11. The Main Theorem is true when \( \mathcal{R} \) is a field of characteristic zero and \( P \) is a polynomial function of degree \( \leq 2 \).

Proof. Let \( P \) be a bisymmetric polynomial function of degree \( p \leq 2 \). If \( p \leq 1 \), then
\( P \) is in class (ii) of the Main Theorem. If \( p = 2 \), then by Claim 10 we see that \( P \)
is (up to conjugation) of the form \( P(x) = c_2 x_i x_j + c_0 \). If \( i = j \), then by Claim 8 we see that \( P \) is a univariate polynomial function, which corresponds to the class (i). If \( i \neq j \), then by Claim 8 we have \( c_0 = 0 \) and hence \( P \) is a monomial (up to conjugation).

By Proposition 11 we can henceforth assume that \( p \geq 3 \). We also assume that
\( P \) is a bivariate polynomial function. The general case will be proved by induction on the number of essential variables of \( P \).

Proposition 12. The Main theorem is true when \( \mathcal{R} \) is a field of characteristic zero and \( P \) is a bivariate polynomial function.
**Proof.** Let $P$ be a bisymmetric bivariate polynomial function of degree $p \geq 3$. We know that $P_p$ is of the form $P_p(x, y) = c x^{\gamma_1} y^{\gamma_2}$. If $\gamma_1 \gamma_2 = 0$, then by Claim 6 we see that $P$ is a univariate polynomial function, which corresponds to the class (i).

Conjugating $P$, if necessary, we may assume that $P_{p-1} = 0$ (by Claim 10) and it is then enough to prove that $P = P_p$ (i.e., $P_q = 0$). If $\gamma_1 = 1$ or $\gamma_2 = 1$, then the result follows immediately from Claim 8 since $p - q \geq 2$. We may therefore assume that $\gamma_1 \geq 2$ and $\gamma_2 \geq 2$. We now prove that $P = P_p$ in three steps.

**Step 1.** $P(x, y)$ is of degree $\leq \gamma_1$ in $x$ and of degree $\leq \gamma_2$ in $y$.

Proof. We prove by induction on $r \in \{0, \ldots, p-1\}$ that $P_{p-r}(x, y)$ is of degree $\leq \gamma_1$ in $x$ and of degree $\leq \gamma_2$ in $y$. The result is true by our assumptions for $r = 0$ and $r = 1$ and is obvious for $r = p$. Considering Eq. (4) for $d = p^2 - r > pq$, with $r_1 = r_2 = (x, y)$, we obtain

$$[P(x, y)^r]_{p^2-r} = [P(x, x)^{\gamma_1} P(y, y)^{\gamma_2}]_{p^2-r}.$$  \hfill (13)

Clearly, the right-hand side of (13) is a polynomial function of degree $\leq p \gamma_1$ in $x$ and $\leq p \gamma_2$ in $y$. Using the multinomial theorem, the left-hand side of (13) becomes

$$[P(x, y)^r]_{p^2-r} = \left[ \sum_{k=0}^{p} P_{p-k}(x, y)^p \right]_{p^2-r} = \sum_{\alpha \in A_{p-r}} \left( \begin{array}{c} p \\ k \end{array} \right) \prod_{k=0}^{p} P_{p-k}(x, y)^{\alpha_k},$$

where

$$A_{p-r} = \left\{ \alpha = (\alpha_0, \ldots, \alpha_p) \in \mathbb{N}^{p+1} : \sum_{k=0}^{p} k \alpha_k = r, |\alpha| = p \right\}.$$  \hfill (14)

Observing that for every $\alpha \in A_{p-r}$ we have $\alpha_k = 0$ for $k > r$ and $\alpha_r \neq 0$ only if $\alpha_r = 1$ and $\alpha_0 = p - 1$, we can rewrite (13) as

$$p P_p(x, y)^{p-1} P_{p-r}(x, y) = [P(x, x)^{\gamma_1} P(y, y)^{\gamma_2}]_{p^2-r} - \sum_{\alpha \in A_{p-r}} \left( \begin{array}{c} p \\ k \end{array} \right) \prod_{k=0}^{p} P_{p-k}(x, y)^{\alpha_k}. $$

By induction hypothesis, the right-hand side is of degree $\leq p \gamma_1$ in $x$ and of degree $\leq p \gamma_2$ in $y$. The result then follows by analyzing the highest degree terms in $x$ and $y$ of the left-hand side. \hfill \Box

**Step 2.** $P(x, y)$ factorizes into a product $P(x, y) = U(x)V(y)$.

Proof. By Step 1 we can write

$$P(x, y) = \sum_{k=0}^{\gamma_1} V_k(y),$$

where $V_k$ is of degree $\leq \gamma_2$ and $V_{\gamma_1}(y) = \sum_{j=0}^{\gamma_2} c_{\gamma_2-j} y^j$, with $c_0 = c \neq 0$ and $c_1 = 0$ (since $P_{p-1} = 0$). Equating the terms of degree $\gamma_1^2$ in $z$ in the identity

$$P(P(z, t), P(x, y)) = P(P(z, x), P(t, y)), $$

we obtain

$$V_{\gamma_1}(t)^{\gamma_1} V_{\gamma_1}(P(x, y)) = V_{\gamma_1}(x)^{\gamma_1} V_{\gamma_1}(P(t, y)).$$

Equating now the terms of degree $\gamma_1 \gamma_2$ in $t$ in the latter identity, we obtain

$$c^{\gamma_1} V_{\gamma_1}(P(x, y)) = c V_{\gamma_1}(x)^{\gamma_1} V_{\gamma_1}(y)^{\gamma_2}. $$

We now show by induction on $r \in \{0, \ldots, \gamma_1 \}$ that every polynomial function $V_{\gamma_1-r}$ is a multiple of $V_{\gamma_1}$ (the case $r = 0$ is trivial), which is enough to prove the result.
To do so, we equate the terms of degree $\gamma_1 \gamma_2 - r$ in $x$ in \eqref{eq:14} (by using the explicit form of $V_{\gamma_1}$ in the left-hand side). Note that terms with such a degree in $x$ can appear in the expansion of $V_{\gamma_1}(P(x, y))$ only when $P(x, y)$ is raised to the highest power $\gamma_2$ (taking into account the condition $c_1 = 0$ when $r = \gamma_1$). Thus, we obtain
\[
\sum_{k=0}^{\gamma_1} \left[ \sum_{j=0}^{\gamma_2} c_{\gamma_2-j} (U(x) V(y))^j \right] = c \left[ V_{\gamma_1}(x)^{\gamma_1} \right] V_{\gamma_1}(y)^{\gamma_2},
\]

(here the notation $[.]_{\gamma_1 \gamma_2 - r}$ concerns only the degree in $x$). By computing the left-hand side (using the multinomial theorem as in the proof of Step 1) and using the induction on $r$, we finally obtain an identity of the form
\[
(a V_{\gamma_1}(y)^{\gamma_2-1} V_{\gamma_1}(y) = a' V_{\gamma_1}(y)^{\gamma_2}, \quad a, a' \in R, a \neq 0,
\]
from which the result immediately follows. \qed

**Step 3.** $U$ and $V$ are monomial functions.

**Proof.** Using \eqref{eq:13} with $P(x, y) = U(x) V(y)$ and $V_{\gamma_1} = V$, we obtain
\[
\sum_{j=0}^{\gamma_1} c_{\gamma_2-j} (U(x) V(y))^j = c V(x)^{\gamma_1} V(y)^{\gamma_2}.
\]

Equating the terms of degree $\gamma_2^2$ in $y$ in \eqref{eq:15}, we obtain
\[
\sum_{j=0}^{\gamma_2^2} c_{\gamma_2-j} (U(x) V(y))^j = 0,
\]

which obviously implies $c_k = 0$ for $k = 1, \ldots, \gamma_2$, which in turn implies $V(x) = c x^{\gamma_2} V(y)$. Finally, from \eqref{eq:16} we obtain $U(x) = x^{\gamma_1}$. \qed

Steps 2 and 3 together show that $P = P_p$, which establishes the proposition.

Recall that the action of the symmetric group $S_n$ on functions from $R^n$ to $R$ is defined by
\[
\sigma(f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.
\]

It is clear that $f$ is bisymmetric if and only if so is $\sigma(f)$. We then have the following fact.

**Fact 13.** The class of $n$-variable bisymmetric functions is stable under the action of the symmetric group $S_n$.

Consider also the following action of identification of variables. For $f : R^n \to R$ and $i < j \in [n]$ we define the function $I_{i,j} f : R^{n-1} \to R$ as
\[
(I_{i,j} f)(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{j-1}, x_i, x_j, \ldots, x_{n-1}).
\]

This action amounts to considering the restriction of $f$ to the “subspace of equation $x_i = x_j$” and then relabeling the variables. By Fact 13 it is enough to consider the identification of the first and second variables, that is,
\[
(I_{1,2} f)(x_1, \ldots, x_{n-1}) = f(x_1, x_1, x_2, \ldots, x_{n-1}).
\]

**Proposition 14.** The class of $n$-variable bisymmetric functions is stable under identification of variables.
Proof. To see that $I_{1,2}f$ is bisymmetric, it is enough to apply the bisymmetry of $f$ to the $n \times n$ matrix

\[
\begin{pmatrix}
  x_{1,1} & x_{1,1} & \cdots & x_{1,n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n-1,1} & x_{n-1,1} & \cdots & x_{n-1,n-1}
\end{pmatrix}.
\]

To see that $I_{i,j}f$ is bisymmetric, we can similarly consider the matrix whose rows $i$ and $j$ are identical and the same for the columns (or use Fact 13). □

We now prove the Main Theorem by using both a simple induction on the number of essential variables of $P$ and the action of identification of variables.

Proof of the Main Theorem when $\mathcal{R}$ is a field. We proceed by induction on the number of essential variables of $P$. By Proposition 12 the result holds when $P$ depends on 1 or 2 variables only. Let us assume that the result also holds when $P$ depends on $n-1$ variables ($n-1 \geq 2$) and let us prove that it still holds when $P$ depends on $n$ variables. By Proposition 11 we may assume that $P$ is of degree $p \geq 3$. We know that $P_p(x) = cx_1^{\gamma_1}$ where $c \neq 0$ and $\gamma_i > 0$ for $i = 1, \ldots, n$ (cf. Claim 6). Up to a conjugation we may assume that $P_{p-1} = 0$ (cf. Claim 10). Therefore, we only need to prove that $P = P_p$. Suppose on the contrary that $P - P_p$ has a polynomial function $P_q \neq 0$ as the homogeneous component of highest degree. Then the polynomial function $I_{1,2}P$ has $n-1$ essential variables, is bisymmetric (by Proposition 14), has $I_{1,2}P_p$ as the homogeneous component of highest degree (of degree $p \geq 3$), and has no component of degree $p-1$. By induction hypothesis, $I_{1,2}P$ is in class (iii) of the Main Theorem with $b = 0$ (since it has no term of degree $p-1$) and hence it should be a monomial function. However, the polynomial function $[I_{1,2}P]_q = I_{1,2}P_q$ is not zero by (11), hence a contradiction. □

Proof of the Main Theorem when $\mathcal{R}$ is an integral domain. Using the identification of polynomials and polynomial functions, we can extend every bisymmetric polynomial function over an integral domain $\mathcal{R}$ with identity to a polynomial function on $\text{Frac}(\mathcal{R})$. The latter function is still bisymmetric since the bisymmetry property for polynomial functions is defined by a set of polynomial equations on the coefficients of the polynomial functions. Therefore, every bisymmetric polynomial function over $\mathcal{R}$ is the restriction to $\mathcal{R}$ of a bisymmetric polynomial function over $\text{Frac}(\mathcal{R})$. We then conclude by using the Main Theorem for such functions. □

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