Upper tail for homomorphism counts in constrained sparse random graphs

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Abstract
Consider the upper tail probability that the homomorphism count of a fixed graph \( H \) within a large sparse random graph \( G_n \) exceeds its expected value by a fixed factor \( 1 + \delta \). Going beyond the Erdős–Rényi model, we establish here explicit, sharp upper tail decay rates for sparse random \( d_n \)-regular graphs (provided \( H \) has a regular 2-core), and for sparse uniform random graphs. We further deal with joint upper tail probabilities for homomorphism counts of multiple graphs \( H_1, \ldots, H_k \) (extending the known results for \( k = 1 \)), and for inhomogeneous graph ensembles (such as the stochastic block model), we bound the upper tail probability by a variational problem analogous to the one that determines its decay rate in the case of sparse Erdős–Rényi graphs.

KEYWORDS
d-regular graphs, graph homomorphism, large deviations, sparse random graph

1 | INTRODUCTION

Let \( \text{Hom}(H, G) \) denote the number of copies of a connected graph \( H = (V = [v(H)], E) \) present within some other graph \( G \) of \( n \) vertices, which in terms of the adjacency matrix \( A_G \) of \( G \), is

\[
\text{Hom}(H, G) := \sum_{\phi: [v(H)] \to [n]} \prod_{(k,l) \in E} A_G(\phi(k), \phi(l)).
\]  

(1.1)

The upper tail homomorphism problem for a given nonrandom, connected \( H, \delta > 0 \) fixed and \( G = G_n \) drawn from an ensemble of random graphs on \( n \) vertices with law \( \mathbb{P} \), is to estimate the tail probability rate

\[
\text{UT}(H, n, \delta) := -\log \mathbb{P}(\text{Hom}(H, G_n) \geq (1 + \delta)\mathbb{E}(\text{Hom}(H, G_n))).
\]  

(1.2)
This question has been extensively studied in the context of Erdős–Rényi (ER) binomial random graphs $\mathcal{G}(n, p)$ (namely, when each edge independently selected with probability $p$). First, the growth rate of $UT(H, n, \delta)$ as $n \to \infty$, was established after considerable effort in the sparse regime, provided $p = p(n) \to 0$ at a slow enough rate (cf. [8,14,15,23–25,28,33] and related questions in the texts [7,26]).

Then, relying on regularity and compactness properties of the cut-metric, Chatterjee and Varadhan [11,12] proved the large deviation principle (LDP) in the dense regime, where $p \in (0, 1)$ is fixed. In particular, they estimate $UT(H, n, \delta)$ by a variational problem over the space of graphons, within $1 + o(1)$ relative error as $n \to \infty$. Following [11,12], a region where a constant graphon is optimal for such variational problems is characterized in [31] (and recently [17] establishes the LDP for uniform graphs of prescribed degrees $\{d_i\}$, in the dense regime where $d_i = O(n)$).

To describe what is known in the sparse regime $p \to 0$, for $X = (x_{ij})$ from the collection $\mathcal{X}_n$ of symmetric, $n \times n$ matrices with $[0, 1]$-valued entries and zero main diagonal (i.e., $x_{ii} \equiv 0$), we let

$$\text{hom}(H, X) := n^{-v(H)}p^{-e(H)} \sum_{\phi : [v(H)] \to [n]} \prod_{(k,l) \in E(H)} X(\phi(k), \phi(l)),$$

denote the normalized weighted count of copies of a given connected $H$ having $v(H) = |V(H)|$ vertices and $e(H) = |E(H)|$ edges, further using

$$I_p(X) := \sum_{1 \leq i \neq j \leq n} I_p(x_{ij}), \quad \text{where} \quad I_p(x) := x \log \frac{x}{p} + (1 - x) \log \frac{1 - x}{1 - p},$$

for the relative entropy of such $X \in \mathcal{X}_n$ WRT the parameter $p$. The sparse regime poses extra difficulties, as graphon theory is no longer applicable. In lieu of that, a general scheme is introduced in [10] for approximating the partition function of a Gibbs measure on the hypercube, whose potential has a low complexity gradient. Utilizing this approach, [10] show that for $\mathcal{G}(n, p)$ under certain modest polynomial decay of $p(n)$, the upper tail rate $UT(H, n, t - 1)$ is for $t > 1$ within $1 + o(1)$ relative error of

$$\Phi_{n,p}(H, t) := \frac{1}{2} \inf \{I_p(X) : X \in \mathcal{X}_n, \text{hom}(H, X) \geq t\} \quad (1.3)$$

(c.f. [9]). Invoking stochastic analysis tools, Eldan [18] obtains general conditions for approximating such Gibbs measures by a mixture of products, and as a result relaxes somewhat the restriction of [10] on the decay of $p(n)$. Beyond functions on the hyper-cube, [34] adapts the approach of [10] to general Banach spaces, whereas Austin [2] utilizes information inequalities to extend Eldan’s results to arbitrary product spaces. Taking different, more direct approaches, [13] and Augeri [1], independently establish large deviation results for a host of spectral and geometric functionals on the hypercube, and in particular extend the $1 + o(1)$ relative error between (1.2) and (1.3), to a much larger sparsity regime. For a specific sub-class of graphs $H$, even smaller $p(n)$ is allowed in [22] which develops for this a method of entropic stability, and in its followup [3] which extends these results to a larger sub-class of graphs $H$.

While the analysis in all these works relies on the independence and homogeneity inherent of $\mathcal{G}(n, p)$, as well as the simpler geometry of the tail event when only one $H$ is considered, we dispense here from most of these restrictions. Specifically, Theorem 1.5 expands our understanding of the upper tail problem, by considering random graphs $\mathcal{G}_n^{(m)}$ chosen uniformly among all graphs of $n$ vertices and $m$ edges, as well as uniformly chosen random regular graphs $\mathcal{G}_n^d$ on $n$ vertices, each having the same degree $d = d_n$. Similarly to the ER-model, for both $\mathcal{G}_n^d$ and $\mathcal{G}_n^{(m)}$ the relevant large deviation events correspond to planting specific small structures within $G_n$. However, the degree constraints sometimes
prohibit the planting strategy optimal for the ER-case, requiring us to achieve the excess count by planting multiple disjoint small structures and to develop in the proof of Theorem 1.1 new tools for the study of the relevant variational problem. Turning back to edge independence, Proposition 1.7 shows that also for an inhomogeneous setting (such as the stochastic block model), the upper tail probability decay rate boils down to a suitable variational problem, while within the ER-model $G(n, p)$, as well as for the uniformly random graphs $G_n^{(m)}$, Theorem 1.12 provides the complete solution of the upper tail problem for joint counts of graphs $\{H, i = 1, \ldots, k\}$.

Throughout we adjust across different ensembles for equivalent sparsity. That is, we parameterize $G_n^d$ via $p = d/n$ and likewise parameterize $G_n^{(m)}$ via $p = m/\binom{n}{2}$. Denoting by $\Delta = \Delta(H)$ the maximal degree in $H$, and writing hereafter $a_n \sim b_n$ whenever $a_n/b_n = 1 + o(1)$, recall that for any $p = p(n) \gg n^{-1/\Delta}$ one has that $\mathbb{E}[\text{Hom}(H, G_n)] \sim n^{\Delta}p^{\rho(H)}$ in the ER-model. It is easy to see that this applies also for $G_n^{(m)}$ in such regime of $p(n)$, while [21, Corollary 2.2] establishes the same conclusion for $G_n^d$. Thus, using the normalized $\text{hom}(H, G)$ for a random graph $G$ on $n$ vertices from either of our ensembles, and setting $\mathbb{P}^{(m)}$ and $\mathbb{P}^d$ for the laws of $G_n^{(m)}$ and $G_n^d$, we have in analogy with (1.2), the upper tails

$$
\begin{align*}
\text{UT}^{(m)}(H, n, \delta) & := - \log \mathbb{P}^{(m)}(\text{hom}(H, G_n^{(m)}) \geq 1 + \delta), \\
\text{UT}^d(H, n, \delta) & := - \log \mathbb{P}^d(\text{hom}(H, G_n^d) \geq 1 + \delta).
\end{align*}
$$

(1.4)

Recall the collection $\mathcal{X}_n$ of adjacency matrices for $[0, 1]$-weighted simple graphs on $n$ vertices, while

$$
\mathcal{X}_n^{(m)} := \left\{ (x_{ij}) \in \mathcal{X}_n : \sum_{i,j=1}^{n} x_{ij} = m \right\}, \quad \mathcal{X}_n^d := \left\{ (x_{ij}) \in \mathcal{X}_n : \sum_{i=1}^{n} x_{ij} = d, \quad 1 \leq j \leq n \right\},
$$

indicate such matrices for graphs of a given total weight, or of a given constant vertex weight, respectively. Indeed, we show in the sequel that the corresponding rate functions for homomorphism counts within uniform random graphs and within random $d$-regular graphs are:

$$
\begin{align*}
\Phi_n^{(m)}(H, t) & := \frac{1}{2} \inf \{ I_p(X) : X \in \mathcal{X}_n^{(m)}, \ \text{hom}(H, X) \geq t \}, \\
\Phi_n^d(H, t) & := \frac{1}{2} \inf \{ I_p(X) : X \in \mathcal{X}_n^d, \ \text{hom}(H, X) \geq t \}
\end{align*}
$$

(1.5) (1.6)

(bounds on $\text{UT}^{(m)}(H, n, \delta)$ are given in [23, Thm. 4.1], with $\Phi_n^{(m)}(H, t)$ appearing in [16, Prop. 3.3], where the asymptotic of $\text{UT}^{(m)}(H, n, \delta)$ is established for the very slow decay $p(n) \gg (\log n)^{-1/2\rho(H)}$).

Recall that, the $k$-core of a graph is the maximal subgraph within which every vertex has degree at least $k$. It is not hard to check that the rate of growth of each of the variational problems (1.3) and (1.5) is $a_{n,p} := n^2 p^\Delta \log(1/p)$ (which is also the rate for (1.6) when the 2-core of $H$ is $\Delta$-regular). More precisely, it is shown in [5] that for any $\delta > 0$, connected graph $H$ of maximal degree $\Delta \geq 2$ and $n^{-1/\Delta} \ll p = o(1)$, for the normalized variational problem $\phi_{n,p}(\cdot) := a_{n,p}^{-1} \Phi_n^{(m)}(\cdot)$ one has that

$$
\lim_{n \to \infty} \phi_{n,p}(H, 1 + \delta) = c(H, \delta) := \begin{cases} 
\min \left\{ \theta, \frac{1}{2} \delta^{2/\rho(H)} \right\}, & \text{for regular } H, \\
\theta, & \text{otherwise}
\end{cases}
$$

(1.7)

(with triangle counts, namely $H = C_3$, settled earlier in [32]). Here $\theta = \theta(H, \delta)$ is the unique positive solution of $P_{H^*}(\theta) = 1 + \delta$, for the independence polynomial $P_{H^*}(\cdot)$ of the sub-graph $H^* = H[V^*]$
induced by $H$ on its set of vertices $V^* \subset V$ of degree $\Delta$. The two expressions on the RHS of (1.7) correspond to planting a relatively small clique, at rate $\delta^{2/\nu(H)}$, or hub (=anti-clique), at rate $\theta$. This interpretation is further detailed in Remark 1.11, where for joint $k \geq 2$ homorphism counts one often gets a clique + hub planting as the optimal solution. We further note in passing that such a variational problem for lower tails is addressed in [35], with [6] and [4] studying analogous variation problems for arithmetic progressions on random sets and for the upper tail of edge eigenvalues in case of the ER-model.

Utilizing the same normalization, we turn to the explicit solution of (1.6), noting first that for any $X \in \mathcal{X}^d_n$ the value of $\text{hom}(H, X)$ is invariant to removal from $H$ any vertex of degree one, since summing over the index of such a vertex merely gives a multiplicative factor $d = np$. In particular, if $H$ is a tree then $\text{UT}^d(H, n, \delta) = -\Phi^d_n(H, 1 + \delta) = -\infty$ for any $\delta > 0$, while replacing any other $H$ by its nonempty 2-core changes neither $\text{UT}^d(H, n, \delta)$ nor $\Phi^d_n(H, t)$. Thus, whenever we consider $G^d_n$, we assume WLOG that the minimal degree of $H$ is at least two.

**Theorem 1.1.** For $\delta > 0$, a connected graph $H$ having no vertices of degree one and maximal degree $\Delta \geq 2$, and for any $n^{-\frac{1}{2}} \ll p := \frac{d}{n} = o(1),$

$$
\lim_{n \to \infty} \Phi^d_n(H, 1 + \delta) = c^d(H, \delta) := \begin{cases} \lfloor \delta \rfloor + \lfloor \delta \rfloor^{2/\nu(H)}, & \text{if } \Delta \geq 2, \\
\delta^{2/\nu(H)}, & \text{if } \Delta \text{-regular } H, \Delta \geq 3, \\
\infty, & \text{otherwise},
\end{cases}
$$

where $\Phi^d_n(\cdot) := \alpha^{-1}_{n, h}d^d(\cdot)$ is the normalized value of (1.6) and $\lfloor \delta \rfloor$ denotes the fractional part of $\delta$.

**Remark 1.2.** The degree constraints of $G^d_n$ rule out any hub and further limit the allowed planted clique size. Hence our result in Theorem 1.1, corresponding for regular $H$ to the planting of $\lfloor \delta \rfloor$ disjoint cliques when $\Delta = 2$ (higher values of $\Delta$ require smaller clique size, so our size limit no longer affects the solution, see $X^*_n$ of (2.2) versus (2.9)). When $H$ as in Theorem 1.1 is irregular with $\Delta \geq 3$, as shown already in (1.8) even the growth rate of $\Phi^d_n(H, 1 + \delta)$ differs from that for the ER-model. While our paper was under review, [20, Thm 2.7] established the growth rate of $\Phi^d_n(H, 1 + \delta)$ for a large class of irregular graphs $H$.

As promised before, we next show that the variational problems we solved in Theorem 1.1 control the asymptotic rates of $\text{UT}^d(H, n, \delta)$, whereas $\text{UT}^m(H, n, \delta)$ follow the same asymptotic as $\text{UT}(H, n, \delta)$ (indeed, the relatively small structures which dominate the upper tail variational problems for small $p(n)$, are unaffected by a global edge constraint).

**Proposition 1.3.** For a graph $H = (V, E)$ of maximal degree $\Delta \geq 2$, set

$$
\Delta_*(H) := \frac{1}{2} \max_{\{v_1, v_2\} \in E(H)} \{\deg_H(v_1) + \deg_H(v_2) \} \geq 1.
$$

Then, denoting by $C_t$ a cycle of length $t \geq 3$, for fixed $t > 1$ and any

$$
1 \gg p \gg \begin{cases} \max \left( \frac{n^{\frac{1}{2} - \Delta_*}}{(\log n)^{\frac{1}{2} - \Delta_*}}, \frac{\log n}{\sqrt{n}} \right), & H = C_t, \\
n^{-\frac{1}{4} \Delta_*(H)}(\log n)^{1/(2 \Delta_*(H))}, & \text{otherwise},
\end{cases}
$$

(1.9)
one has that for some $\kappa_n \uparrow \infty$,
\[
\begin{align*}
a_n^{-1} \log \mathbb{P}^\text{in}(\text{hom}(H, G_n^\text{in}) \geq t) & \leq -\phi_{n,p}(H, t - o(1)) + o(1), \quad (1.10) \\
a_n^{-1} \log \mathbb{P}^\text{en}(\text{hom}(H, G_n^\text{en}) \geq t) & \leq -(\phi_n^H(H, t - o(1)) - o(1)) \wedge \kappa_n. \quad (1.11)
\end{align*}
\]

**Remark 1.4.** Similarly to Theorem 1.1, we replace $H$ in (1.11) by its 2-core before setting $\Delta$, $a_n$, and the allowed range (1.9) for $p(n)$. Since we get (1.10) by a direct comparison with the ER-model, for a $\Delta$-regular $H$ one has that (1.10) holds up to $p \gg n^{-1/\Delta}$, by relying on [22, Thm 1.5] or [3, Thm. 1.2] instead of [13]. A similar improvement may likewise hold in (1.11) when the 2-core of $H$ is $\Delta$-regular.

Building on Theorem 1.1 and Proposition 1.3, we get the following.

**Theorem 1.5.** Fix $\delta > 0$ and connected graph $H$.

(a) Replacing $H$ by its 2-core of maximal degree $\Delta \geq 2$, for any $p = \frac{d}{n}$ as in (1.9),
\[
\lim_{n \to \infty} a_n^{-1} \log \mathbb{P}^\text{en}(\text{hom}(H, G_n^\text{en}) \geq t) = c^\text{en}(H, \delta). \quad (1.12)
\]

(b) Assuming $\Delta(H) = \Delta \geq 2$, for any $p = m/\left(\frac{n}{2}\right)$ as in (1.9),
\[
\lim_{n \to \infty} a_n^{-1} \log \mathbb{P}^\text{in}(\text{hom}(H, G_n^\text{in}) \geq t) = c(H, \delta). \quad (1.13)
\]

The stated lower bounds on the limits in (1.12)-(1.13) are immediate from Theorem 1.1 and Proposition 1.3. We attain the complementary upper bounds by planting cliques or a hub according to the explicit optimal strategies $X_n^\star$ we use in Proposition 2.1 or those used in proving (1.7), as a by product of which we further deduce that for any $n^{-\frac{1}{2}} \ll p := m/\left(\frac{n}{2}\right) = o(1)$,
\[
\lim_{n \to \infty} a_n^{-1} \log \mathbb{P}^\text{en}(\text{hom}(H, G_n^\text{en}) \geq t) = c(H, \delta), \quad (1.14)
\]

for $c(H, \delta)$ given on the RHS of (1.7).

**Remark 1.6.** Having only the limiting upper tail rate, as in Theorem 1.5, is not enough for precise information about the law of the (rare) graphs $G_n$ for which $\text{hom}(H, G_n)$ exceeds its mean by factor $1 + \delta$. Nevertheless, our results provide additional evidence that such graphs be typically close to a sample from the original ensemble with an added structure of suitable $o(n)$-size that mimic the explicit optimizers we use in the proof of that theorem.

Next, consider the inhomogeneous ER setting, where given probability vectors $\{\alpha^{(n)}\}$ of length $\ell$ each, the vertices of $G_n = G_n^{(\ell)}$ are split to $\ell$ blocks, having sizes $\alpha_r^{(n)} n$ for $1 \leq r \leq \ell$, and the edges between vertices within the $r$-th and $r'$-th blocks are formed independently, with probability $c_{rr'}^{(n)} p$. Assuming that $\{c_{rr'}^{(n)}\}$ are uniformly bounded, $\{\alpha_r^{(n)}\}$ and $\{\zeta_r^{(n)}\}$ are bounded away from zero, while $p = p(n) \to 0$ at a suitable rate, we denote by $\mathbb{P}^{(\ell)}$ the law of the resulting random graph $G_n^{(\ell)}$, parameterized by the symmetric $n \times n$ matrix $p = (p_{ij})$ of entry values $\{c_{rr'}^{(n)} p(n)\}$ as above, and for $X \in \mathcal{X}_n$, set
\[
I_p(X) := \sum_{1 \leq i < j \leq n} I_{p_{ij}}(X_{ij}).
\]
Analogously to Proposition 1.3, we next show that the upper tail event for $G_n^{[r]}$ is characterized by

$$
\Phi_{n,p}^{[r]}(H, t) := \frac{1}{2} \inf \{ I_p(X) : X \in \mathcal{X}_n, \hom(H, X) \geq t b_H \}.
$$

(1.15)

The constant $b_H = b_H^{(n)} \sim \mathbb{E}^{[r]}[\hom(H, G_n)]$ denotes the following sum over partitions $\{S_r\}$ of $V(H)$ to $r$ parts (possibly empty),

$$
b_H := \sum_{\{ S_r \}} \prod_{r=1}^\ell \alpha_r^{|S_r|} \prod_{1 \leq r < r' \leq \ell} e(H[S_r \rightarrow S_{r'}]),
$$

(1.16)

where $e(H[S_r \rightarrow S_{r'}])$ count edges between blocks $S_r$ and $S_{r'}$ (and $\{ \alpha_r^{(n)}, c_r^{(n)} \}$ may depend on $n$).

**Proposition 1.7.** For fixed $t > 1$, a graph $H$ of maximal degree $\Delta \geq 2$ and $p = p(n)$ of (1.9),

$$
a_n^{-1} \log \mathbb{P}^{[r]}(\hom(H, G_n^{[r]}) \geq t b_H) \leq -\Phi_{n,p}^{[r]}(H, t - o(1)) + o(1).
$$

(1.17)

For $1 \gg p \gg n^{-1/(2\Delta_r(H))(\log n)^1/2\Delta_r(H)}$ we have in addition that

$$
a_n^{-1} \log \mathbb{P}^{[r]}(\hom(H, G_n^{[r]}) \geq t b_H) \geq -\Phi_{n,p}^{[r]}(H, t + o(1)) - o(1).
$$

(1.18)

**Remark 1.8.** Additional work should yield that (1.18) holds for $p(n)$ in the full range of (1.9).

Turning to the joint upper tail for the vector $\hom(H, G) := (\hom(H_1, G), \ldots, \hom(H_k, G))$ corresponding to a given collection $H := (H_1, \ldots, H_k)$ of connected graphs $\{H_i\}$, we endow $\mathbb{R}^k$ with the usual coordinate-wise partial orders $\geq$ and $>$. As we show next, for ER-model $G(n, p)$, whose law we denote hereafter by $\mathbb{P}_p$, the rate function at $t = (t_1, \ldots, t_k) > 1 := (1, \ldots, 1) \in \mathbb{R}^k_+$ is then

$$
\Phi_{n,p}^k(H, t) := \frac{1}{2} \inf \{ I_p(X) : X \in \mathcal{X}_n, \hom(H, X) \geq t \}
$$

(1.19)

(compare with (1.3) which corresponds to $k = 1$).

**Proposition 1.9.** For $k' \geq 1$ let $\Delta := \min_{i \in [k']} \{ \Delta(H_i) \} \geq 2$ denote the minimal value among the maximal degrees of given connected graphs $\{H_j, j \in [k']\}$. Assume WLOG that $\Delta(H_i) = \Delta$ iff $i \in [k]$ for some $k \in [k']$ and set $\phi_{n, p}^k(\cdot) := a_n^{-1} \Phi_{n,p}^k(\cdot)$ for such $k$ and the scaling $a_n, p$ induced by $\Delta$. Then, for any $t \in [1, \infty)^{k'}$ and $p = p(n)$ in the intersection of ranges (1.9) applicable to $H_i$, $i \in [k]$,

$$
a_n^{-1} \log \mathbb{P}_p(\hom(H_i, G_n) \geq t) \leq -\phi_{n, p}^k(\pi_k(H), \pi_k(t) - o(1)) + o(1),
$$

(1.20)

where $\pi_k$ denotes the restriction to the first $k$ coordinates (both for $H$ and on $[1, \infty)^{k'}$).

We complement Proposition 1.9 by the following explicit solution of the variational problem (1.19).

**Proposition 1.10.** Fix $k \geq 1$, $s \geq 0$ and suppose the connected graphs $\{H_i, i \in [k]\}$ have the same maximal degrees $\Delta(H_i) = \Delta \geq 2$ and $H_1$ is $\Delta$-regular iff $i \leq s$. Then, for any $\delta \in \mathbb{R}^k_+$ and $n^{-1/\Delta} \ll p = o(1)$

$$
\lim_{n \rightarrow \infty} \phi_{n, p}^k(H, 1 + \delta) = c(H, \delta)
$$

:= \min \{ x + \frac{1}{2} y^2 : \mathbb{P}_{H_i}(x) + \mathbb{P}_{H_i}(y) \geq 1 + \delta_i, i \leq k \}.

(1.21)
Remark 1.11. On the RHS of (1.21) we have the normalized size of a planted hub (= $x$) and a planted clique (= $y$), in the limiting $n \to \infty$ solution of (1.19). As shown in [5], for $k = 1$ such optimum is always attained for $x = 0$ or $y = 0$, yielding (1.7). In contrast, for $k \geq 2$ the optimum in general has both $x > 0$ and $y > 0$, corresponding as mentioned before, to simultaneously planting both a clique and a hub (e.g., for $H_1 = K_3$ and $H_2 = K_{1,2}$ we are to minimize $x + \frac{1}{2}y^2$ in (1.21) subject to $1 + 3x + y^3 \geq 1 + \delta_1$ and $1 + x \geq 1 + \delta_2$. The latter constraint rules out $x = 0$, and for $\delta_1 - 3\delta_2 > 27/8$, taking $(x, y) = (\delta_2, (\delta_1 - 3\delta_2)^{1/3})$ is better than the hub solution $(x, y) = (\delta_1/3, 0)$).

Building on Propositions 1.9 and 1.10 we establish the following sharp joint upper tail asymptotic.

**Theorem 1.12.** With $c(H, \delta)$ given by (1.21), we have in the setting of Proposition 1.9, that

$$
\lim_{n \to \infty} a_{n,p}^{-1} \log \mathbb{P}_p(\text{Hom}(H, G_n) \geq 1 + \delta) = -c(\pi_k(H), \pi_k(\delta)). \tag{1.22}
$$

Furthermore, the same applies for the law $\mathbb{P}^{(m)}$ of the uniformly random graph $G_n^{(m)}$.

Remark 1.13. We believe that the analog of (1.22) holds for $G_n^{(m)}$, provided $c(H, \delta)$ of (1.21) is replaced by $\max_{i=1}^{\infty} \{ c^d(H_i, \delta_i) \}$. Indeed, our proof of Theorem 1.1 extends to show that if in addition $\Delta(H) = \Delta \geq 2$ for all $i \leq k$, then for any $\delta_\Delta = p$ as in Proposition 1.10,

$$
\lim_{n \to \infty} a_{n,p}^{-1} \inf \left\{ \frac{1}{2} I_p(X) : X \in \mathcal{X}_n^d, \; \text{hom}(H, X) \geq 1 + \delta \right\} = \max_{i=1}^{k} \{ c^d(H_i, \delta_i) \}. \tag{1.23}
$$

The bulk of this paper is Section 2, where we settle Theorem 1.1, Proposition 1.3 and Theorem 1.5 on the upper tail problem for random $d$-regular and uniformly random graphs. The short Section 3 then establishes Proposition 1.7 about the inhomogeneous random graph $G_n^{[\ell]}$, while Section 4 deals with joint homomorphism counts, proving Propositions 1.9 and 1.10 and Theorem 1.12.

### 2 | Uniform Random and Random Regular Graphs

#### 2.1 | Proof of Theorem 1.1

We start by exhibiting in **Step 1**, that is, in Proposition 2.1, an optimal strategy for $\Delta$-regular $H$, thereby upper bounding $\lim_n \phi^d_n(H, 1 + \delta)$. The technically challenging lower bounds are then separately proved when $\Delta = 2$ (in **Step 2**), and when $\Delta \geq 3$ (in **Step 3**).

**Proposition 2.1.** Fixing $\delta > 0$ and connected $\Delta$-regular $H$, $\Delta \geq 2$, if $p = \frac{d}{\Delta} \to 0$, $d \to \infty$, then

$$
\Phi^d_n(H, 1 + \delta) \leq c^d(H, \delta)a_{n,p}(1 + o(1)). \tag{2.1}
$$

**Proof.** Consider first $\Delta = 2$, for which $H = C_l$ must be a cycle of length $l \geq 3$. In this case, our candidate for (1.6) is the block adjacency matrix $X^*_n \in \mathcal{X}_n^{d}$, of the form

$$
X^*_n := \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & r \\
0 & 0 & \ldots & r & q
\end{bmatrix}. \tag{2.2}
$$
where we have \([\delta]\) principal blocks of ones, denoted by \(1\), the first \([\delta]\) of which are of the maximal size \(d + 1\) each, while the last block is of a size \(s_1\) such that \(s_1 \sim \{\delta\}^{1/d}\). Here \(q\) denotes a block matrix with zeros on the diagonal and \(r\) on the off-diagonals, and \(r\) denotes the matrix with all elements \(r\). Setting \(s := \lfloor \delta \rfloor (d + 1) + s_1\), the row-sum constraint of \(X_n^d\) is satisfied by \(X_n^\star\), provided \(r\) and \(q\) are such that

\[
d = s_1 - 1 + (n - s)r = rs_1 + q(n - s - 1).
\]

Since \(d = np\), this results with

\[
r = \frac{np - s_1 + 1}{n - s}, \quad q = \frac{np - s_1 r}{n - s - 1}.
\]

As \(p = p(n) \to 0\), it follows from (2.3) that eventually \(r \leq p\) and furthermore \(q/p \to 1\). We denote the homomorphism density of \(H = ([v(H)], E)\) in \(X \in \mathcal{X}_n\), by

\[
t(H, X) := n^{-v(H)} \text{Hom}(H, X) = n^{-v(H)} \sum_{1 \leq i_1, \ldots, i_\ell(H) \leq n} \prod_{(i, j) \in E} x_{i_k, j_l}.
\]

Now, with \(e(C_i) = v(C_i)\), upon considering only contributions when all vertices of \(C_i\) are in the same principal block of \(X_n^\star\), we find that

\[
t(C_i, X_n^\star) \geq [\delta] \left( \frac{d}{n} \right)^{v(C_i)} + \left( \frac{s_1 - 1}{n} \right)^{v(C_i)} + \left( \frac{n - s - 1}{n} \right)^{v(C_i)} q^{e(C_i)}
\]

\[
\sim ([\delta] + \{\delta\}) p^{v(C_i)} + p^{e(C_i)} = (1 + \delta) p^{e(C_i)},
\]

as required in (1.6). As for the entropy of \(X_n^\star\), clearly

\[
I_p(X_n^\star) \leq ([\delta] + \{\delta\}) 2^{[\Delta]} n^2 p^2 I_p(1) + [\delta] n^2 p I_p(0) + 2ns I_p(r) + n^2 I_p(q).
\]

With \(I_p(1) = \log(1/p)\), the first term on the RHS is precisely \(2e^d(H, \delta) a_{n,p}\) (see the RHS of (1.8) for \(\Delta = 2\)). Since \(I_p(0) = o(p \log(1/p))\), the second term on the RHS is \(o(a_{n,p})\). Recall that eventually \(r \leq p\), hence \(I_p(r) \leq I_p(0)\) and with \(s_1 \leq np\), the third term is similarly \(o(a_{n,p})\). As for the last term, note that \(I_p(x) = \log \left\{ \frac{x(1-p)}{p(1-x)} \right\} \) is uniformly bounded over \(x \in [p/2, 2p]\) and \(p \leq 1/3\). Furthermore, from (2.3) we have that

\[
q - p = \frac{p(s - 1) - rs_1}{n - s - 1} = O(p^\Delta).
\]

In particular, as \(p \to 0\), eventually \(q \in [p/2, 2p]\). With \(I_p(0) = 0\), we then have that

\[
I_p(q) \leq |q - p| \sup_{x \in \left[\frac{p}{2}, 2p\right]} |I_p'(x)| = O(p^\Delta) = o(p^\Delta \log(1/p))
\]

so the last term on the RHS of (2.6) is also \(o(a_{n,p})\).

In case \(\Delta \geq 3\) it suffices to plant a single clique. Specifically, consider \(X_n^\star\) as in (2.2), except for having now only its single, last block of ones, namely, set an integer \(s = s_1 \sim \delta^{1/v(H)} np^{\Delta/2}\) and

\[
X_n^\star = \begin{bmatrix} 1 & r \\ r & q \end{bmatrix}.
\]


Here $\Delta/2 > 1$ and $p \to 0$, so $s = s_1 = o(d)$ regardless of the fixed value of $\delta$, allowing us to set $r, q \in [0, 1]$ per (2.3), provided $p$ is small enough, to guarantee that $X_n^* \in \mathcal{X}^d_n$. In contrast, for $\Delta = 2$ this would have given $s = O(d)$, hence the need for the more elaborate construction (2.2). Next, here $e(H) = \Delta v(H)/2$ and considering contributions when all vertices of $H$ are in the same principal block of $X_n^*$, we find similarly to (2.5), that

$$r(H, X_n^*) \geq \left( \frac{s_1 - 1}{n} \right)^{v(H)} + \left( \frac{n - s_1 - 1}{n} \right)^{v(H)} \delta p^{\Delta v(H)/2} + p^{\delta(H)} = (1 + \delta)p^{\delta(H)}, \quad (2.10)$$

as required in (1.6). Furthermore, similarly to (2.6), we now have

$$I_p(X_n^*) \leq s_1^2 I_p(1) + 2ns_1 I_p(r) + n^2 I_p(q),$$

where the first term on the RHS is $2c^d(H, \delta)a_{n,p}(1 + o(1))$. It is easy to see that here

$$p - r = \frac{(1 - p)s_1 - 1}{n - s_1} = O(p^{\Delta/2}), \quad q - p = \frac{(p - r)s_1 - 1}{n - s_1 - 1} = O(p^\Delta). \quad (2.11)$$

With $q/p \to 1$ satisfying (2.7), as argued in case $\Delta = 2$, here again $n^2 I_p(q) = O(n^2 p^\Delta) = o(a_{n,p})$. Now also $r/p \to 1$, so by the same reasoning $I_p(r) = O(p^{\Delta/2})$. With the corresponding term in our bound on $I_p(X_n^*)$ being $o(a_{n,p})$, this completes the proof of the proposition.

**Step 2** (Lower bound $\Delta = 2$): If $\Delta = 2$ then $H = C_l$ for some $l \geq 3$, and our starting point in bounding below

$$\Phi_p^l(C_l, 1 + \delta) = \frac{1}{2} \inf \{I_p(X), X \in \mathcal{X}^d_n, \text{ hom}(C_l, X) \geq 1 + \delta \},$$

is the inequality

$$I_p(x) \geq (1 - o(1))(x - p)^2 I_p(1), \quad \forall x \in [0, 1] \quad (2.12)$$

which applies for any $p = o(1)$. Indeed, we get (2.12) by combining the elementary inequality $I_p(p - x) \geq I_p(p + x)$ for $0 \leq x \leq p \leq \frac{1}{2}$ (cf. [4, Lemma 3.3]), with the bound

$$\lim_{p \to 0} \inf_{x \in (0, 1 - p)} \left\{ \frac{I_p(p + x)}{x^2 I_p(1)} \right\} = 1$$

of [32, Corollary 3.5]. Recall that $X \in \mathcal{X}^d_n$ is symmetric, of nonnegative entries with $\sum_{i=1}^n X_{ij} = d$ and $X_{ij} = 0$ for all $j$. In particular, all eigenvalues $\{ \lambda_i \}$ of $X \in \mathcal{X}^d_n$, are in $[-d, d]$, whereas with $d = np$ and $a_{n,p} = n^2 p^2 I_p(1)$, we deduce from (2.12) that

$$I_p(X) \geq (1 - o(1)) \sum_{1 \leq i \neq j \leq n} (X_{ij} - p)^2 I_p(1)$$

$$\geq (1 - o(1)) \left( \frac{n}{d^2} \sum_{i,j=1}^n X_{ij}^2 - d^2 \right) I_p(1) = (1 - o(1))a_{n,p} \left( \sum_{i=1}^n (\lambda_i/d)^2 - 1 \right). \quad (2.13)$$
Furthermore, hom\((G_t, X) = \(np\)^{-l} \sum_{i=1}^{n} \lambda_i^l\) for any \(X \in \mathcal{X}_n\) and \(l \geq 3\), so re-scaling \(\eta_l := \lambda_i/(np)\), it follows from (2.13) that

\[
\phi_n^d(C_t, 1 + \delta) \geq \frac{1}{2} \left(1 - o(1)\right) \inf \left\{ \sum_{i=1}^{n} \eta_l^2 - 1 : |\eta_l| \leq 1, \sum_{i=1}^{n} \eta_l^\prime \geq 1 + \delta \right\}.
\] (2.14)

The optimal \(\{\eta_l\}\) in (2.14) are nonnegative, so the desired bound \(\phi_n^d(C_t, 1 + \delta) \geq (1 - o(1))c^d(C_t, \delta)\) follows from considering our next lemma at \(x_l = \eta_l^i \in [0, 1]\), \(\beta = 2/\ell\) and \(\theta = 1 + \delta\).

**Lemma 2.2.** For any \(\beta \in (0, 1)\) and \(x \in [0, 1]^\mathbb{N}\), let \(f_\beta(x) := \sum_i x_i^\beta\). Then, for any \(\theta \geq 0\),

\[
f_\beta(x) \geq \theta \Rightarrow f_\beta(x) \geq [\theta] + \{\theta\}^\beta.
\]

**Proof.** Since \(f_\beta(x)\) is increasing in each coordinate, its infimum over \(K_{\geq \theta} := [0, 1]^\mathbb{N} \cap \{x : f_\beta(x) \geq \theta\}\), is attained at the convex set \(K_{= \theta}\). Furthermore, with \(\phi_\beta(\cdot)\) a strictly concave function (as \(\beta \in (0, 1)\), its infimum over \(K_{= \theta}\) is attained at an extreme point of \(K_{= \theta}\), namely when all but at most one of the coordinates of \(x\) are \(\{0, 1\}\)-valued, and the stated lower bound immediately follows. 

**Step 3** (Lower bound \(\Delta \geq 3\)): We lower bound (1.6) by viewing \(X^d_n\) as a subset of the collection \(\mathcal{W}\) of all graphons (i.e., symmetric measurable \(W : [0, 1]^2 \to [0, 1]\)), via the map \(W_X(s, t) = X_{[ns], [nt]}\). Indeed, doing so yields the bound \(\Phi^d(H, u) \geq n^2 \Phi^d(H, u)\) for the continuous problem

\[
\Phi^d(H, u) := \frac{1}{2} \inf \{I_p(W) : W \in \mathcal{W}, p^{-e(H)} t(H, W) \geq u, \int_0^1 W(x, y)dy = p, \forall x \in [0, 1]\},
\] (2.15)

where \(I_p(W) := \int \int I_p(W(t, s))dtds\) denotes the entropy of graphon \(W\) and

\[
t(H, W) := \int_{[0,1]^\mathbb{N}} \prod_{(i,j) \in E(H)} W(x_i, x_j)^{v(H)} d_{x_i}
\] (2.16)

its homomorphism density. Next, as in [5], we change variables to \(U := W - p \in [-p, 1 - p]\), so our extra linear constraint translates to

\[
d(x) := \int_0^1 U(x, y)dy = 0, \quad \forall x \in [0, 1].
\] (2.17)

Using the standard notation \(a_p \leq b_p\) whenever \(a_p/b_p\) is bounded above as \(p \to 0\), in view of our scale \(a_n, b\), it suffices for the lower bound on \(\Phi^d(H, 1 + \delta)\) to consider only \(U\) such that

\[
I_p(p + |U|) \leq I_p(p + U) \leq p^\Delta \log(1/p),
\] (2.18)

with our next lemma thus the key to the lower bound.

**Lemma 2.3.** Suppose \(H\) is a connected graph with maximal degree \(\Delta \geq 3\). If for \(p \to 0\) the symmetric \(U : [0, 1]^2 \to [-p, 1 - p]\) satisfy (2.17), then

\[
p^{-e(H)} t(H, p + U) = 1 + p^{-e(H)} t(H, U) + o(1)
\] (2.19)
and for irregular \( H \) also

\[
p^{-e(H)} t(H, U) = o(1). \tag{2.20}
\]

For irregular \( H \) we have from Lemma 2.3 that \( t(H, p + U) = (1 + o(1))p^{-e(H)} \) whenever \( U \) satisfies (2.17) and hence, that \( \Phi^d(H, 1 + \delta) \to \infty \) as \( p \to 0 \). It follows that \( \Phi^d(H, 1 + \delta)/(p\Delta \log(1/p)) \to \infty \), so with \( \Phi^d_0(H, u) \geq n^2\Phi^d(H, u) \) and rescaling, we deduce that \( \phi^d_0(H, 1 + \delta) \to \infty \) as \( p(n) \to 0 \) (with \( \delta > 0 \) fixed). Turning to deal with \( \Delta \)-regular \( H \), we denote by \( \| \cdot \|_q \) the \( L^q([0, 1]^2) \)-norms and recall that for \( |U| \leq 1 \) and graph \( F \) of maximal degree \( \Delta(F) \geq 2 \), the generalized Hölder’s inequality of [19, Theorem 2.1] for \( v(F) \) variables and power \( \Delta(F) \) at each \( e \in E(F) \), yields that

\[
|t(F, U)| \leq \| U \|_{\Delta/(\Delta(F))} \leq \| U \|_{2}^{2\Delta/(\Delta(F))} \tag{2.21}
\]

(see also [31, Corollary 3.2]). Thus, combining (2.21) and Lemma 2.3 we see that for any \( W = p + U \) which is relevant for the RHS of (2.15) at \( \theta = 1 + \delta \), we must have

\[
\| U \|_{2}^{2} \geq |t(H, U)|^{\Delta/(\Delta(H))} \geq (\delta - o(1))p^{e(H)} \Delta/e(H).
\]

This, together with (2.12), having \( a_{n,p} = n^2p\Delta I_p(1) \) and \( \Delta v(H) = 2e(H) \) (for \( \Delta \)-regular \( H \)), yield the required lower bound

\[
n^2\Phi^d(H, 1 + \delta) \geq \frac{1}{2} \delta^{2/e(H)}(1 + o(1))a_{n,p}.
\]

We thereby proceed to complete the proof of Theorem 1.1, by proving Lemma 2.3.

**Proof of Lemma 2.3.** Our starting point is the decomposition [5, (6.1)],

\[
p^{-e(H)} t(H, W) - 1 = p^{-e(H)}[t(H, p + U) - t(H, p)] = \sum_F \mathcal{N}(F, H)p^{-e(F)}t(F, U), \tag{2.22}
\]

over nonempty sub-graphs \( F \subseteq H \), up to isomorphism, with \( \mathcal{N}(F, H) \) counting the number of sub-graphs of \( H \) isomorphic to \( F \). Furthermore, recall [5, (4.5)] that (2.18) implies in turn

\[
\| U \|_{2}^{2} \leq p^\Delta, \tag{2.23}
\]

whereas for \( \Delta \geq 3 \) and \( U \geq 0 \) it is shown in [5, Corollary 6.2] that under (2.23), any contribution to the RHS of (2.22) which is nonnegligible when \( p \to 0 \),

\[
F_H := \{ F \subset H, \text{ with minimum vertex cover size } \tau(F) = e(F)/\Delta \}, \tag{2.24}
\]

or from \( F = H \) a \( \Delta \)-regular graph. We remove the restriction to \( U \geq 0 \), by noting that for \( \Delta \geq 3 \) the proof of [5, Lemma 6.4] applies to \( |U| \) and since \( t(F, U) \leq t(F, |U|) \), it follows that for \( F \notin F_H \) which is not \( \Delta \)-regular,

\[
t(F, U) = o(p^{e(F)}). \tag{2.25}
\]
We thus complete the proof of the lemma by showing that (2.17) extends the scope of (2.25) to every $F \in \mathcal{F}_H$ which is not $\Delta$-regular. Specifically, splitting $U = U^+ - U^-$ to its positive and negative parts $U^+ \in [0, 1 - p]$, $U^- \in [0, p]$ induces the split $d(x) = d^+(x) - d^-(x)$, where thanks to (2.17),

$$d^+(x) := \int_0^1 U^+(x,y)dy = d^-(x) := \int_0^1 U^-(x,y)dy$$

are in $[0, p]$ for all $x \in [0, 1]$.

In particular, by Jensen’s inequality and [32, Lemma 3.4], we have for $p \leq p_o$ and $b \in [p, 1/3]$, that

$$I_p(p + U) \geq I_p(p + |U|) \geq \int_0^1 I_p(p + 2d^+(x))dx \geq \frac{I_p(p + 2b)}{b^2} \int_0^1 d^+(x)^2dx. \quad (2.26)$$

Recall that $I_p(p + 2b) \sim 2b \log(2b/p)$ when $p^\alpha \leq b \to 0$ with $\alpha < 1$ fixed, and deduce from (2.18) and (2.26) that then

$$\int_0^1 d^+(x)^2dx \leq \frac{b^2 I_p(p + U)}{I_p(p + 2b)} \lesssim p^\Delta b. \quad (2.27)$$

Let $S \subset \{\pm\}^{E(F)}$ enumerate those $s \in \{\pm\}^{E(F)}$, with even number of minus entries. Then, setting

$$U_\pm(x|F) := \prod_{e=(e_1,e_2) \in E(F)} U_{\pm e}(x_{e_1}, x_{e_2}),$$

we have that for any graph $F$

$$t(F, U) = t(F, U^+ - U^-) \leq \sum_{s \in S} t(F, U_s^s), \quad t(F, U) := \int_{[0,1]^{v(F)}} U_s^s(x|F)\prod_{i=1}^{v(F)} dx_i. \quad (2.28)$$

Adapting [5, Lemma 7.4] to our setting, we next show that

$$t(F, U_s^s) = o(p^{\theta(F)}), \quad (2.29)$$

for any $s \in \{\pm\}^{E(F)}$ and every connected irregular bipartite $F$ of maximal degree $\Delta \geq 3$ such that $\tau(F) = \theta(F)/\Delta$. Indeed, [5, Lemma 7.1] shows that such $F$ contains a sub-graph $M$ of $\theta(M) = 2\tau(F)$ edges, whose connected components $M_1, \ldots, M_k$ are path or even cycles, with at least $M_1$ being a path of length $l \geq 1$. Since $\Delta(M_i) = 2$ and the bound (2.23) applies also for $U^\pm$, it follows by (2.21) that for any choice of $s$,

$$|t(M_i, U_s^s)| \lesssim \|U^\pm\|^{\theta(M_i)}_{L_2} \lesssim (p^\Delta)^{\theta(M_i)/2}, \quad i = 1, \ldots, k.$$  

Clearly, then

$$t(F, U_s^s) \leq t(M_i, U_s^s) = \prod_{i=1}^k t(M_i, U_s^s) \lesssim p^{\Delta \theta(M_i)/2} = p^{\Delta \tau(F)} = p^{\theta(F)}.$$  

For $M_1$ a path of length $l = \theta(M_1) \geq 1$, the generalized Hölder’s inequality yields the sharper bound

$$t(M_1, U_s^s) = \int_{[0,1]^l} d^\pm(x_2)d_{x_2} \prod_{i=2}^l U_{s^{(i+1)}}(x_i, x_{i+1})dx_{i+1} \lesssim \|d^\pm\|_2 \|U^\pm\|_{L_2}^{-1} \lesssim (p^\Delta b)^{1/2} p^{\Delta(l-1)/2} = o(p^{\Delta l/2}).$$  


where the last inequality uses (2.23) and (2.27) with \( b \to 0 \). This establishes (2.29), and consequently also (2.25), whenever the irregular bipartite \( F \) of maximal degree \( \Delta \geq 3 \) such that \( \tau(F) = \frac{e(F)}{\Delta} \), is connected. In particular, this applies for any connected \( F \in F_H \) which is not \( \Delta \)-regular (see [5, Section 6.2]). For nonconnected \( F \in F_H \) we first integrate out all isolated vertices of \( F \) without altering the value of \( t(F, U) \), and complete the proof by noting that thereafter each connected component \( F' \) of \( F \) must be in \( F_H \) (where obviously \( F' \neq H \) cannot be \( \Delta \)-regular). Indeed, the nonempty independent sets \( S \) of \( H^* \) are in one-to-one correspondence with \( F \in F_H \) which consists of all edges of \( H \) incident to \( S \) (so vertices of \( S \) have degree \( \Delta \) in \( F \)). Having all isolated vertices removed from \( F \), each connected component \( F' \) of \( F \) must consist of all edges of \( H \) incident to some nonempty subset of \( S \). Hence, \( F' \in F_H \) as claimed. ■

2.2 Proof of Proposition 1.3

Setting hereafter \( n_\varepsilon := \binom{n}{2} \) and \( K_n^{(m)} := \{ e(G_n) = m \} \), recall Pittel's inequality (cf. [26, (1.6)]), that for \( p = m/n_\varepsilon \), any \( n, m \) and event \( A_n \)

\[
\mathbb{P}_p(A_n \cap K_n^{(m)}) = \mathbb{P}_p(K_n^{(m)}) \mathbb{P}_p(A_n) \geq \frac{1}{3m} \mathbb{P}_p(A_n) \tag{2.30}
\]

whereas before, \( \mathbb{P}_p \) denotes the law of the ER-model \( G(n, p) \). Furthermore, under (1.9) we have that \( a_{n,p} \gg \log n \geq \frac{1}{2} \log m \), so we get the bound (1.10) by combining (2.30) for \( A_n = \{ \text{hom}(H, G_n) \geq t \} \) with [13, Thm. 1.2 and Thm. 1.3] (see Remark 1.4 on its improvement when [22, Thm. 1.5] applies).

A similar, but more delicate argument yields (1.11). Specifically, similarly to [13], we view the ER-law \( \mathbb{P}_p \) of \( A_{G_n} \in \mathcal{X}_n \) as the product Bernoulli measure \( \mu_p \) of a random binary vector \( x \in [0, 1]^n \) (namely, the upper-triangular part of \( A_{G_n} \)). Then, by an intersection with a given closed, convex \( K \subset [0, 1]^n \) one easily extends the nonasymptotic bound of [13, Corollary 2.2] to get for any \( h : [0, 1]^n \to \mathbb{R}^k, k \geq 1, p \in [0, 1]^n, t \in \mathbb{R} \) and \( \delta > 0 \) that

\[
\mu_p(h \geq t) \cap K) \leq \|h\|_\infty \exp\left(-\inf_{x \in K} \int h(x) \, d\mu_p(x)\right) + \mu_p(K \cap h \leq t).
\]

(2.31)

provided \( K \cap \{0, 1\}^n \) is covered by a collection \( \{B_i\}_{i \in \mathbb{N}} \) of closed convex subsets of \( [0, 1]^n \) and

\[
\max_{i \in \mathbb{N}} \sup_{x, y \in B_i \cap K} \|h(x) - h(y)\|_\infty \leq \delta.
\]

(2.32)

Recall that for fixed \( t > 1 \) and \( p \) as in (1.9), one arrives, as in [13, Theorem 1.2 and 1.3], at

\[
a_{n, p}^{-1} \log \mathbb{P}_p(\text{hom}(H, G_n) \geq t) \leq -\phi_{n, p}(H, t - o(1)) + o(1), \tag{2.33}
\]

by applying (2.31) for \( p = p1, \mathbb{R}_+ \)-valued \( h(\cdot) = \text{Hom}(H, \cdot) \) and \( K = [0, 1]^n \), after excluding a set \( E = E_{\kappa, \delta} \) of \( \mathbb{P}_p \)-probability \( \exp(-\kappa a_{n,p}) \) for arbitrarily large \( \kappa \) (cf. [13, (3.12), (4.5), (6.13), (6.23)]). The bulk of the work there is a deterministic analysis to exhibit a cover of \( [0, 1]^n \setminus E \) by \( \exp(o(a_{n,p})) \) many closed convex sets \( \{B_i\} \) that satisfies (2.32) for arbitrarily small \( \delta > 0 \). The same reasoning, now with closed, convex \( K_n \) which is the upper-triangular image of \( \mathcal{X}_d^{\Delta} \) for \( d = np \), and using the same \( \{B_i\}, \mathcal{E} \) as in the proof of [13, Thm. 1.1 and 1.2], yields that for \( K_n^{\Delta} := \{ A_{G_n} \in \mathcal{X}_d^{\Delta} \} \) and some \( \kappa_n \uparrow \infty \),

\[
a_{n, p}^{-1} \log \mathbb{P}_p(\text{hom}(H, G_n) \geq t) \leq -\phi_{n, p}^d(H, t - o(1)) - o(1)) \wedge \kappa_n \tag{2.34}
\]
(keeping the \( \kappa_n \) term in the absence of a uniform upper bound such as [13, (6.1)] on \( \phi_d^d(H, t) \)). Hence, applying the well-known identity
\[
\mathbb{P}_p(A_n \cap \mathcal{K}_n^d) = \mathbb{P}_p(\mathcal{K}_n^d)\mathbb{P}_p(A_n),
\]
in the special case of \( A_n = \{ \text{hom}(H, G_n) \geq t \} \), we complete the proof of (1.11) upon showing that for \( d = np \) and \( p(n) \to 0 \) of (1.9),
\[
a_{n,p}^{-1} \log \mathbb{P}_p(\mathcal{K}_n^d) \to 0.
\]

To this end, with \( g_n(d) \) denoting the number of simple graphs \( G_n \) of degrees \( d = (d_1, d_2, \ldots, d_n) \), recall that the \( \mathbb{P}_p \)-probability of producing a graph of such degrees is precisely
\[
g_n(d) p^{\sum d/2}(1 - p)^{n - \sum d/2},
\]
where \( \overline{d} := n^{-1} \sum_{i=1}^n d_i \) (and hereafter \( n\overline{d} \) assumed even). We thus establish (2.36) by utilizing the asymptotic count of [30, Corollary 1.5],
\[
g_n(d) \sim \sqrt{2} \exp \left( \frac{1}{4} - \frac{\gamma^2}{4\mu^2(1 - \mu)^2} \right) \left( \mu^{\mu}(1 - \mu)^{(1 - \mu)} \right)n \prod_{i=1}^n \left( \frac{n - 1}{d_i} \right),
\]
where \( \mu := \overline{d}/(n - 1), \gamma := (n - 1)^{-2} \sum_{i=1}^n (d_i - \overline{d})^2 \), and (2.37) holds whenever \( n(\overline{d} \wedge (n - 1 - \overline{d})) \to \infty \) and \( n^{-\varepsilon} \max_i |d_i - \overline{d}|^2 = o(\overline{d} \wedge (n - 1 - \overline{d})) \) for some fixed \( \varepsilon > 0 \). In particular, this applies for \( d_i = \overline{d} = np \) and \( n^{-1} \ll p \ll 1 \) (as in (1.9)), where \( \mu = d/(n - 1) \) and \( \gamma = 0 \), resulting with
\[
n^{-1} \log \mathbb{P}_p(\mathcal{K}_n^d) \geq d \log \mu + (n - 1 - d) \log(1 - \mu) + \log \left( \frac{n - 1}{d} \right) + o(1)
\]
\[
\sim -\frac{1}{2} \log d
\]
(using Stirling’s formula in the last step). With \( d = np \) and \( n \log(np) = O(a_{n,p}) \) in the range assumed in (1.9), this implies the limit (2.36), thereby completing the proof.

From (2.37) we further deduce the following estimate, which we later use in proving Theorem 1.5.

**Lemma 2.4.** For any \( \varepsilon > 0 \), \( 1 \gg p \geq \varepsilon n^{-1/2} \), integer \( d = np \) and \( n'/n \to 1 \), one has that
\[
\frac{g_{n'}(d \mathbf{1})}{g_n(d \mathbf{1})} \geq p^{d_1(1 + o(1))}.
\]

**Proof.** For \( \mu = \frac{d}{n'-1} = O(p) \), we get from (2.37) by Stirling’s formula, similarly to (2.38), that
\[
\frac{2}{n'} \log g_{n'}(d \mathbf{1}) = d \log(n' - 1) - d \log d + d \log d + O \left( d^2 n^{-1} \right).
\]
The same applies at \( \hat{n} = n' - d - 1 \), resulting with
\[
\log g_{\hat{n}}(d \mathbf{1}) - \log g_{n'}(d \mathbf{1}) = \frac{d}{2} \left[ \hat{n} \log(\hat{n} - 1) - n' \log(n' - 1) + d \log d + O(d) \right]
\]
\[
= d_1 \log(1 + o(1)) = d_1(1 + o(1)) \log p
\]
as claimed (using in the last step that \( \log(1/\mu) = O(d) \) while \( p \to 0 \) and \( n'/n \to 1 \)).

Next, we prove a combinatorial estimate which we later employ in proving Theorem 1.5.
Lemma 2.5. Fixing \( \varepsilon > 0 \), if \( d = np \geq 2 \) and a collection \( E_n \) of edges on \([n]\) has maximal degree \( s_1 < (1 - \varepsilon)d \), then for any \( uw \in E_n \)

\[
\mathbb{P}(E_n \subseteq G_n^d) \geq \varepsilon^2 p (n - 7p) \mathbb{P}(E_n \setminus \{uw\} \subseteq G_n^d).
\]  

(2.40)

Proof. Let \( C_1 \) denote the collection of \( d \)-regular graphs containing all of \( E_n \), with \( C_0 \) the collection of \( d \)-regular graphs containing \( E_n \setminus \{uw\} \), but not the edge \( uw \). Since

\[
\mathbb{P}(E_n \subseteq G_n^d) = \frac{|C_1|}{|C_1| + |C_0|} \mathbb{P}(E_n \setminus \{uw\} \subseteq G_n^d),
\]

it suffices to show that for all \( n \)

\[
\frac{|C_1|}{|C_0|} \geq (d - s_1)(d - s_1 - 1) \varepsilon nd - 6d^2 (nd)^2.
\]  

(2.41)

To this end, recall as in the proof of [21, Lemma 2.3] that for \( G \in C_1 \), any pair of edges \( u_i w_i \in G \setminus E_n \), \( i = 1, 2 \), with disjoint vertices \( \{u, w, u_i, w_i\} \), such that the triplet \( S' := \{w_{u_1}, w_1 u_2, w_2 u\} \) is disjoint of \( G \), defines a forward switching, where replacing \( S := \{uw, u_1 w_1, u_2 w_2\} \) by \( S' \) results with \( G' \in C_0 \). Conversely, per \( G' \in C_0 \), any disjoint \( \{u, w, u_i, w_i\} \) such that \( S' \) is in \( G' \setminus E_n \) while \( S \) is disjoint of \( G' \), provides a reverse switching where replacing \( S' \) by \( S \) recovers a graph \( G \in C_1 \). A double counting argument bounds \( |C_1|/|C_0| \) below by the minimum over \( G, G' \) of the ratio between the number of reverse switching and the number of forward switching. Counting edges WLOG as oriented, a \( d \)-regular graph \( G \) has at most \( nd \) edges, so the number of forward switching never exceeds \( (nd)^2 \). As for the reverse switching, given \( u \neq w \), we have at least \( d - s_1 \) choices for \( u_1 \notin \{u, w\} \) such that \( w_{u_1} \in G' \setminus E_n \) and \( (d - s_1 - 1) \) choices of \( w_2 \notin \{u, w, u_1\} \) such that \( w_2 u \in G' \setminus E_n \). There are further at least \( n - 2 - 2d \) vertices beyond \( \{w, u, u_1, w_2\} \) which are neither connected by \( G' \) to \( u_1 \) nor to \( w_2 \). Within those vertices there are at least \( d(n - 2(2 + 2d)) \) possible edges \( w_1 u_2 \in G' \), at most \( ns_1 < (1 - \varepsilon)nd \) of which are from \( E_n \). With the number of reverse switching per \( G' \) thus being at least \( (d - s_1)(d - s_1 - 1)d(\varepsilon n - 4 - 4d) \), we arrive at (2.41), as claimed.

2.3 Proof of Theorem 1.5

Fix hereafter \( H \) and \( p(n) \) as in Proposition 1.3 (replacing first \( H \) by its 2-core if considering (1.12)). The stated lower bounds on the limits (1.12) and (1.13), then follow by combining Proposition 1.3 with Theorem 1.1 and (1.7), respectively.

To prove (1.12), recall that \( c^d(H, \delta) = \infty \) for irregular \( H \), in which case the complementary upper bound in (1.12) trivially holds. Furthermore, in view of (2.35) we get the stated upper bound for connected \( \Delta \)-regular \( H \), upon showing that for any \( \delta' < \delta \)

\[
\lim_{n \to \infty} a_n^{-1} \log \mathbb{P}_p(\{\text{hom}(H, G_n) \geq 1 + \delta'\} \cap \mathcal{K}_n^d) \geq -c^d(H, \delta),
\]  

(2.42)

where \( \mathcal{K}_n^d = \{A_n \in \mathcal{X}_n^d\} \). We derive (2.42) by a change of measure to a inhomogeneous ER-model, denoted hereafter \( \mathbb{P}_\bullet \), where the edge probabilities are set via \( X_n^\bullet \in \mathcal{X}_n^d \) of a block form. Specifically, we split the proof of (2.42) into Case 1 where \( H = C_i, \Delta = 2 \) with \( X_n^\bullet \) taken as in (2.2), and Case 2 which corresponds to \( \Delta \)-regular \( H, \Delta \geq 3 \), using then \( X_n^\bullet \) of (2.9). Denoting by \( E_n^\bullet \) the collection of edges within the 1-blocks of \( X_n^\bullet \) of (2.2) and (2.9), respectively, in both cases \( \mathbb{P}_\bullet \) is supported on the event \( A_n = \{E_n^\bullet \subseteq G_n\} \). Furthermore, thanks to the specific block structure of \( X_n^\bullet \), the constraint
\(A_n \cap \mathcal{K}_n^d\), imposes the 0-blocks (in case of (2.2)), and a nonrandom number of edges per block in \(A_n\) (which matches that block’s aggregate in \(X_n^*\)). Thereby, at every instance within \(A_n \cap \mathcal{K}_n^d\) we have the Radon-Nikodym derivative

\[
\frac{d\mathbb{P}_p}{d\mathbb{P}_*} = e^{-\frac{1}{2}I_p(X_n^*)},
\tag{2.43}
\]

While proving Proposition 2.1 we saw that \(I_p(X_n^*) \leq 2e^d(H, \delta)a_{n,p} + o(a_{n,p})\) both for (2.2) \((\Delta = 2)\), and for (2.9) \((\Delta \geq 3)\). In conclusion, upon intersecting the event in (2.42) with \(A_n\), followed by such change of measure, it suffices to show that in both cases,

\[
\mathbb{P}_*(\text{hom}(H, G_n) < 1 + \delta') \ll \mathbb{P}_*(\mathcal{A}_n \cap \mathcal{K}_n^d) = e^{-o(a_{n,p})},
\tag{2.44}
\]

and thereby having that

\[
a_{n,p}^{-1} \log \mathbb{P}_*(\{\text{hom}(H, G_n) \geq 1 + \delta'\} \cap \mathcal{K}_n^d \cap A_n) \to 0.
\]

Proceeding to establish (2.44), denoting hereafter by \(\tilde{n} := n - s\) the size of the (bottom) \(q\)-block, recall that in both cases \(\tilde{n} / n \to 1\) and \(q/p \to 1\). In Case 1 the nonrandom contribution to \(\text{hom}(H, G_n)\) from the planted cliques under \(\mathbb{P}_*\) is \(\delta(1 + o(1))\) (see (2.5)). The same applies for the single nonrandom clique planted in Case 2, so in both cases we have the following upper bound for any fixed \(\varepsilon < \delta - \delta'\) and \(n\) large enough

\[
\mathbb{P}_*(\text{hom}(H, G_n) < 1 + \delta') \leq \mathbb{P}_q(\text{hom}(H, G_{\tilde{n}}) < 1 - \varepsilon).
\tag{2.45}
\]

Now, we want to invoke upper bound of [[27], Theorem 3] on the lower tail for homomorphism counts to the RHS of (2.45). To this end, note that \((4 - v(J))e(J) \leq v(J)\) for any graph \(J\) with at least one edge \((e(J) \leq 3\) for \(v(J) = 3\), \(e(J) \leq 1\) for \(v(J) = 2\)). Thus, when \(e(J) \geq 1\) and \(\Delta(J) \leq \Delta\), the bound \(2e(J) \leq \Delta v(J)\) implies that \((e(J) - 1) \leq \Delta v(J) - 2\). In particular, our condition (1.9) of \(p \gg n^{-1/\Delta}\) implies that for any \(J \not\subseteq H\) with \(e(J) \geq 1\)

\[
\mathbb{E}_q[\text{Hom}(J, G_{\tilde{n}})] = \tilde{n}^{v(J)}q^{e(J)} \geq (1 + o(1))n^2p.
\]

Hence, by [[27], Theorem 3], for any such \(J\), \(q/p \to 1\), \(p \gg n^{-1/\Delta}\) and all \(n\) large enough,

\[
\mathbb{P}_q(\text{hom}(J, G_{\tilde{n}}) < 1 - \varepsilon) \leq \exp \left(- \Theta \left(\varepsilon^2 \min_{\Delta(J) \leq \Delta, e(J) \geq 1} \{n^{v(J)}p^e(J)\} \right) \right) \\
\leq \exp \left(- \Theta(\varepsilon^2 n^2p) \right) \ll \exp(-O(a_{n,p})).
\tag{2.46}
\]

Considering \(J = H\), it thus follows from (2.45)-(2.46) that (2.44) holds as soon as

\[
a_{n,p}^{-1} \log \mathbb{P}_*(\mathcal{A}_n \cap \mathcal{K}_n^d) \to 0.
\tag{2.47}
\]

To this end, note that combining (2.35) and (2.43) we arrive at the identity

\[
\mathbb{P}_*(\mathcal{A}_n \cap \mathcal{K}_n^d) = e^{\frac{1}{2}I_p(X_n^*)} \mathbb{P}_p(\mathcal{A}_n \cap \mathcal{K}_n^d) = e^{\frac{1}{2}I_p(X_n^*)} \mathbb{P}_p(\mathcal{K}_n^d) \mathbb{P}(\mathbb{E}_* \subseteq G_n).
\tag{2.48}
\]
Recall that \( a_{n,p} c^d(H, \delta) = |E_n^s| \log (1/p)(1 + o(1)) \). Furthermore, while proving Proposition 2.1 we saw that \( a_{n,p} I_p(X_n^s) \rightarrow 2c^d(H, \delta) \). Thus, by (2.36) and (2.48) we get (2.47) once we show that

\[
\mathbb{P}(E_n^s \subseteq G_n^d) \geq p^{-|E_n^s|(1+o(1))}. \tag{2.49}
\]

For Case 2 which has a single clique \( E_n^s \) of size \( s_1 = o(d) \), we get (2.49) by sequentially peeling its \((s_1)_{e}\) edges and iteratively employing Lemma 2.5 for the relevant subsets of \( E_n^s \). Turning to Case 1, the event \{\( E_n^s \subseteq G_n^d \}\) is then the intersection of \([\delta]\) independent events. These amount to having \([\delta]\) disjoint maximal cliques of size \( d + 1 \) each, and for \( [\delta] \in (0, 1) \) having an additional clique of size \( s_1 < (1 - \epsilon(\delta))d \) within the remaining \((d-regular)\) graph \( G_n^d \), restricting \( \delta \) to \( \frac{1}{\epsilon^3} n = p^{1+o(1)} \), the contribution of the latter \( s_1 \)-sized clique to \( \mathbb{P}(E_n^s \subseteq G_n^d) \) is likewise handled by \((s_1)_{e}\) applications of Lemma 2.6. Furthermore, the contribution of maximal cliques to that probability, is precisely the product of the LHS of (2.39) at \( n' = n - (d + 1)j \) for \( 0 < j < [\delta] \), which for \( \tilde{n} = n - s \gg d \) yields the bound (2.49) also in Case 1.

Turning next to the upper bound in (1.13), considering the identity in (2.30) and the characterization of \( \mathcal{c}(H, \delta) \) via the RHS of (1.21) at \( k = 1 \), this amounts to showing that for \( \mathcal{K}_n^{(m)} = \{ A_{G_n} \in \mathcal{X}_n^{(m)} \} \) and connected \( H \) of maximal degree \( \Delta \geq 2 \),

\[
\liminf_{n \to \infty} a_{n,p}^{-1} \log \mathbb{P}_p \left( \{ \text{hom}(H, G_n) \geq 1 + \delta' \} \cap \mathcal{K}_n^{(m)} \right) \geq -x - \frac{1}{2} y^2, \tag{2.50}
\]

provided \( x, y > 0 \) satisfy

\[
\mathbb{P}_{H_n}^*(x) + y^{\mathcal{c}(H)} \parallel_{H \text{ is } \Delta-regular} \geq 1 + \delta. \tag{2.51}
\]

Fixing \( x, y \geq 0 \) for which (2.51) holds, we choose here as \( \mathbb{P}_* \) the inhomogeneous ER-model whose edge probabilities are set by the matrix \( X_n^* \in \mathcal{X}_n^{(m)} \) of the form

\[
X_n^* = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & q \\
1 & q & q
\end{bmatrix}, \tag{2.52}
\]

with principal blocks of (integer) sizes \( s_1, s-s_1, n-s \), where \( s_1 \sim x \Delta n, s \sim y \Delta^2 n \) (so \( s_1 \ll s \ll n \)), and \( q \) satisfies the global edge constraint

\[
(1 - q)[s_e + (n-s)s_1] + q n_e = p n_e \tag{2.53}
\]

As before, \( \mathbb{P}_* \) is supported on \( A_n \) (now for the edges \( E_n^s \) within the \( I \)-blocks of (2.52)), and thanks to (2.52)-(2.53), the relation (2.43) holds throughout \( A_n \cap \mathcal{K}_n^{(m)} \). The contribution of the \( \frac{1}{2} I_p(X_n^*) \) from the \((x + y^2/2)n^2p^\Delta(1 + o(1))\) entries in the \( I \)-blocks of \( X_n^* \) of (2.52) matches the lower bound in (2.50). Due to the constraint (2.53), the value of all but \( O(n^2p^\Delta) \) entries of the latter matrix, is \( q = p - O(p^\Delta) \), so those (non \( I \)) entries have a cumulative \( o(a_{n,p}) \) effect on \( I_p(X_n^*) \) (see (2.8)). Thus, as in the preceding proof of (1.12), it suffices to show (2.44), albeit with \( \mathcal{K}_n^{(m)} \) instead of \( \mathcal{K}_n^{d} \).

To this end, recall that, \( \tilde{n}/n \to 1 \) and \( q/p \to 1 \). For \( \Delta \)-regular \( H \), the clique of size \((s-s_1)\) planted in the middle of \( X_n^* \) contributes \( y^{\mathcal{c}(H)}(1 + o(1)) \) to \( \text{hom}(H, G_n) \) (see (2.10), where \( 2e(H) = \Delta v(H) \) for any \( \Delta \)-regular \( H \)). Furthermore, by definition of \( H^* \), restricting \( H \) to \( S' = V(H) \setminus S \) for an independent
set $S$ of $H^*$, yields a sub-graph $H_{S^*}$ of precisely $\varepsilon(H) - \Delta |S|$ edges. From (2.51) and the definition of $P_{H^*}(\cdot)$ we thus deduce that

$$(1 + \delta - y^{x(H)} ) n^{x(H)}^{\partial(H)} \leq \sum_{S \text{ independent set of } H^*} \chi^{[S]} n^{x(H)}^{\partial(H)}$$

$$= (1 + o(1)) \sum_{S \text{ independent set of } H^*} \chi^{[S]} n^{x(H_{S^*})}^{\partial(H_{S^*})},$$

which in turn yields for any fixed $\varepsilon < (\delta - \delta')/(1 + \delta)$ and $n$ large enough the bound

$$\mathbb{P}_q(\text{hom}(H, G_n) < 1 + \delta') \leq \sum_{S \text{ independent set of } H^*; e(H_{S^*}) \geq 1} \mathbb{P}_q(\text{hom}(H_{S^*}, G_n) < 1 - \varepsilon). \quad (2.54)$$

Applying (2.46) for $J = H_{S^*}$ and enumerating over the independent sets $S$ on the RHS of (2.54), we deduce that its LHS is bounded for all $n$ large enough by $\exp(-O(a_{n,p}))$. Thus, as in the proof of (2.44), it remains only to show that similarly to (2.47), we have here that

$$a_{n,p}^{-1} \log \mathbb{P}_q(\mathcal{A}_n \cap \mathcal{K}_n^{(m)}) \to 0. \quad (2.55)$$

Our choice of $q$ in (2.53) is such that under $\mathbb{P}_q$, the requirement to be in $\mathcal{K}_n^{(m)}$ amounts to the number of edges in the $q$-block of size $L_q = O(n^2)$ in $X_n^*$, matching its specified integer valued mean $qL_q$. Furthermore, $q/p \to 1$, hence $qL_q = O(m)$ and (2.55) then holds by Pittel’s inequality and (1.9).

3 | INHOMOGENEOUS GRAPH ENSEMBLES

3.1 | Proof of Proposition 1.7

As in the proof of Proposition 1.3, our starting point towards proving (1.17) is again (2.31), taking now $K = [0, 1]^n$ and $\mathbb{R}_+$-valued $h(\cdot) = \text{Hom}(H, \cdot)$, as in the proof of [[13], Thm. 1.1 and 1.2], while replacing the constant vector $p\mathbf{1}$ with the given $p = (p_{ij})$ that corresponds to $\mathbb{P}_q[\cdot]$. Note that our assumption that $c_{\infty} := \max_{n,p,r} \{c_{ij(r)}^{(n)}\}$ is finite, with $a_0 := \inf_n \{a_1^{(n)}\}$ and $c_0 := \inf_p \{c_{11}^{(n)}\}$ positive, guarantee that both $p_{ij}^{(n)}$ and $\phi_{a,p}(H, t)/\phi_{a,p}(H, t)$ be bounded away from zero and infinity, per fixed $H$ and $t > 1$. In particular, up to some change of universal constants the uniform bounds [[13], (6.1) and (6.2)] apply to $\phi_{a,p}(H, t)$. Thus, using hereafter the same cover of $[0, 1]^n \setminus \mathcal{E}$ by $\exp(o(a_{n,p}))$ many closed convex sets $\{B_t\}$ (which satisfy (2.32) for arbitrarily small $\delta > 0$), as in the proof of [[13], Thm. 1.1 and 1.2], one needs only to verify that under our inhomogeneous ER-law:

(a) For any $\gamma < \infty$ the exceptional sets $\mathcal{E}_0(\kappa_0)$ from [[13], Prop. 3.4] and $\mathcal{L}_F(\kappa_1)$ of [[13], (3.14)] for strictly induced sub-graphs $F < H$, are of probability $\exp(-\gamma a_{n,p})$ when $\kappa_1(\gamma)$ are large enough.

(b) For $\kappa = \kappa_p \gg \log(1/p)$ the event

$$\mathcal{E}_0(\kappa) := G(\sqrt{\kappa(np)}, C' \sqrt{np})^\mathcal{E}, \quad (3.1)$$

from [[13], (4.2)], satisfies the bound [[13], (4.5)].
To this end, we replace the adjacency matrix $J_n$ of the complete graph (see [[13], (1.40)]), by the weighted matrix $J_n = p(n)^{-1}$ of zero main diagonal and uniformly bounded above and below entries (in particular, with $n^{-1}\|J_n\|_{HS}$ uniformly bounded). Then, further examining [[13], the preceding tail probability bounds (a) and (b) are direct consequences of the following analogs of [[13], (4.7)] and [[13], (6.13)], for the inhomogeneous adjacency matrix $X_n = A_{G''_n}$.

**Lemma 3.1.** For some $C'$ finite, $c_\ast > 0$, any $t \geq 0$, $p \geq (c,n)^{-1}\log n$ and all $1 \leq k \leq n$, we have

\[
\mathbb{P}[\mathcal{C}'](\|X_n - pJ_n\|_{op} \leq C'\sqrt{kn}\|p\| + t) \leq 4e^{-t^2/16},
\]

\[
\mathbb{P}[\mathcal{C}'](1^\top X_n 1 > \kappa n^2 p) \leq e^{-c_\ast \kappa n^2 p}.
\]

**Proof.** Following the proof of [[13], Lemma 4.3] it suffices for (3.2) to show that

\[
\mathbb{E}[\mathcal{C}'](\|X_n - pJ_n\|_{op}) \leq \frac{C'}{2}\sqrt{np}.
\]

Recall [[29], Example 4.10], that for some $c < \infty$, if $d_n := \max_i \{\sum_j p_{ij}\} \geq \log n$, then

\[
\mathbb{E}(\|X_n - \mathbb{E}X_n\|_{op}) \leq c\sqrt{d_n}
\]

for any symmetric $n \times n$ matrix $X_n$ of independent Bernoulli$(p_{ij})$ entries. For $X_n = A_{G''_n}$ we have that $\mathbb{E}X_n = pJ_n$ and $(np)^{-1}d_n \in [c_\ast, c_\infty]$, yielding (3.4). Next, (3.3) follows from [[13], (6.13)] upon noting that replacing in $\mathbb{P}[\mathcal{C}']$ the values $\{c^{(n)}_{r'}\}$ by $c_\infty$ may only increase the LHS of (3.3).

 Turning next to the proof of (1.18), note first that upon replacing $I_p(Q)$ by $\frac{1}{2}I_p(Q)$ and $\tilde{C}$ by $\tilde{C}/c_\ast$, the bound [[13], (6.5)] holds in our setting, and thereby so does [[13], (6.8)], provided $p \leq 1/(2c_\infty)$. As $1 \gg p$ and $np^{\Delta(H)} \gg 1$, it follows that for some $\eta_{n,p} = o(a_{n,p})$ and all $X \in \mathcal{X}'$

\[
\mathbb{P}[\mathcal{C}'](B_X) \geq \exp \left(-\frac{1}{2}I_p(X) - \eta_{n,p}\right), \quad B_X := \{Z \in \mathcal{X}' : \|Z - X\|_{op} \leq C_0\sqrt{n}\}.
\]

Fixing $t > 1$, $\epsilon > 0$, upon combining (1.17) with the lower bound of the form of [[13], (6.2)] (which holds for $\Phi_{n|\mathcal{H}|}(H, \cdot)$), the reasoning around [[13], (6.25)-(6.26)] applies here verbatim, allowing us to set $\kappa_1 = \kappa_1(H, t, \epsilon)$ and further restrict the minimization in (1.15) at $t + \epsilon$, to $X$ such that in addition $B_X$ intersects $\mathcal{L}_F(\kappa_1)$ of [[13], (3.14)] for all $F < H$. We proceed to set $\epsilon_0 \in (0, 1]$ as in [[13], Sec. 6.3], apart from replacing $\epsilon$ by $\epsilon \inf_n \{b_H^{(n)}\}$. As $B_X$ satisfies [[13], (3.16)] for any $n \geq n_0$ and all $X \in \mathcal{X}_n$ (thanks to our assumption that $np^{2\Delta(H)} \gg 1$), we deduce from [[13], Prop. 3.7] that

$\hom(H, X) \geq (t + \epsilon)b_H, \quad B_X \cap \mathcal{L}_F(\kappa_1) \neq \emptyset, \quad \forall F < H \Rightarrow \quad B_X \subset \{Z : \hom(H, Z) \geq tb_H\}$.

Considering the maximum of $\mathbb{P}[\mathcal{C}'](B_X)$ over such $X$, we get from (3.6) that for any $n \geq n_0$

\[
\log \mathbb{P}[\mathcal{C}'](\hom(H, G_{n|\mathcal{H}|}) \geq t b_H) \geq -\Phi_{n|\mathcal{H}|}(H, t + \epsilon) - \eta_{n,p}.
\]

Dividing both sides by $a_{n,p}$ and taking $\epsilon = \epsilon_{n,p} \downarrow 0$ slowly enough, results with (1.18).
4 | JOINT UPPER TAILS IN ERDŐS–RÉNYI GRAPHS

4.1 | Proof of Proposition 1.9

By ignoring the requirements imposed on \( \text{hom}(H_i, G_n) \) for \( i > k \), we can and shall assume hereafter WLOG that \( k' = k \), with \( \Delta(H_i) = \Delta \geq 2 \) for all \( i \in [k'] \). Furthermore, in case \( \Delta = 2 \) the range in (1.9) be less stringent for cycles than for path (the only other connected graphs having \( \Delta = 2 \)). Thus, in Step 1 we take all \( H_i = C_{l_i} \) to be cycles and adapt the proof of [[13], (1.20)] for \( p(n) \) determined by \( l_o = \min \{ l_i \} \), whereas in Step 2 we consider the general case, adapting the proof of [[13], Thm. 1.1] with \( p(n) \) determined by \( \Delta_* := \max_i \{ \Delta_*(H_i) \} \).

Step 1 Thanks to [[13], Lemma 4.1 and Prop 4.2], apart from \( A_{G_n} \) in the \( \mathbb{P}_p \)-negligible event \( \mathcal{E}_0(\kappa_p) \) of (3.1), we have that for suitable \( \kappa_p \gg \log(1/p) \) and \( k = k_n \gg (\log n)/(l_o - 2) \) (see [[13], (7.3) and (7.5)]), if \( p(n) \) is in range (1.9) for \( H = C_{l_o} \), then

\[
\| (A_{G_n})_{>k_i} \|_{S_i} \leq \varepsilon n p, \quad \forall n \geq n_0, \quad \forall l \geq l_o.
\]

We apply here (2.31) with \( p = p_1 \), \( \mathbb{R}_+^k \)-valued \( h(\cdot) = \text{hom}(H, \cdot) \), \( K = [0, 1]^n \) and an exceptional set which is determined by the multiple Schatten norms via

\[
\mathcal{E}(\varepsilon) := \bigcup_{l \geq l_o} \{ X \in \mathbb{B}_{HS}(n) : \|X_{>k}\|_{S_i} > \varepsilon n p \}.
\]

For the net \( \mathbb{I} := \Sigma \times \mathcal{V} \) of cardinality \( |\mathbb{I}| = \exp(O(kn \log n)) = \exp(o(a_{n_p})) \) from [[13], Lemma 7.2] and \( \delta' > 0 \) as in the proof of [[13], Thm. 7.1], we enumerate over \( y \in \mathbb{I} \), taking the closed convex sets

\[
\mathbb{B}_y(\varepsilon) := \{ M(y) + W + Z : W \in \mathbb{B}_{HS}(\delta'n), Z \in \mathcal{Z}_y(\varepsilon) \} \cap \mathbb{B}_{HS}(n),
\]

where for each \( \varepsilon > 0 \),

\[
\mathcal{Z}_y(\varepsilon) := \{ Z \in \text{Sym}_n(\mathbb{R}) : \text{Im}(Z) \subseteq \text{Ker}(M(y)), \max_{i=1}^{k} \| Z \|_{S_i} \leq \varepsilon n p \}.
\]

This is a suitable cover, since by [[13], Claim 7.4] any \( X \in \mathbb{B}_{HS}(n) \cap \mathcal{E}(\varepsilon)' \) must be in \( \mathbb{B}_y(\varepsilon) \), whereas [[13], Claim 7.5] yields the fluctuation bound

\[
\max_{y \in \mathbb{I}} \sup_{X \in \mathbb{B}_y(\varepsilon)} \| h(X) - h(M(y)) \|_\infty \leq \varepsilon l_o + o(1),
\]

which as in the proof of [[13], Thm. 7.1], completes our proof of (1.20) (for cycles).

Step 2 By a union bound we deduce from [[13], (6.22)] that for any large enough \( \kappa_1 \geq \kappa_1(H, l) \) and all \( l \gg p \gg \left( \frac{1}{n} \log n \right)^{1/(2\Delta_*)} \), the event

\[
\mathcal{E}_{H}(\kappa_1) := \left\{ X \in \mathcal{X}_n : \max_{i \in [k]} \max_{F \sim H_i} \{ \text{hom}(F, X) \} > \kappa_1 \right\},
\]

has a negligible \( \mathbb{P}_p \)-probability. Furthermore, from [[13], Prop. 3.7] and our choice of \( \Delta_* \), it follows that for \( h(\cdot) = \text{hom}(H, \cdot) \), some \( f_*, \kappa_1 > 0 \), all convex \( B \subset \mathcal{E}_{H}(\kappa_1)' \) and \( \varepsilon \in [0, 1] \),

\[
\max_{X, Y \in B} \| X - Y \|_{op} \leq \varepsilon f_* \kappa_1^{-1}n^{\Delta_*} \Rightarrow \sup_{X, Y \in B} \| h(X) - h(Y) \|_\infty \leq \varepsilon .
\]
Fixing $\epsilon < 1$, we adapt the proof of [[13], Thm. 1.1], applying again (2.31), now for the $\mathbb{P}_p$-negligible events $\mathcal{E} = \mathcal{E}_0(k_0) \cup \mathcal{E}^*_H(k_1)$. We also take for $B_j = C_j$ the closed convex hull of sets from the net $\bar{I}$ constructed in [[13], Prop. 3.4] for $\delta_0 = \epsilon_4 p^\Delta / (4C_\star k_1)$ and $k_p = \lfloor k_0 (p^\Delta / \delta_0^2) \log(1/p) \rfloor$, as in [[13], (6.14)-(6.15)]. Thanks to (1.9), its cardinality is $|\bar{I}| = \exp(O(k_p n \log n)) = \exp(o(a_{n,p}))$, while combining [[13], (3.13)] with (4.2) yields maximal fluctuation $\epsilon$ of $h(\cdot)$ on each $C_j$. Thus, considering $\epsilon = \epsilon_{n,p} 0$ slowly enough, concludes the proof of Proposition 1.9.

### 4.2 Proof of Proposition 1.10

We start with an asymptotically tight upper bound on the value of $\phi^k_{n_p}(H, 1 + \delta)$, analogously to Proposition 2.1.

**Proposition 4.1.** For connected graphs $\{H_i, i \in [k]\}$, all of whom having maximal degree $\Delta \geq 2$. Fixing $\delta \in \mathbb{R}_+^k$ and $x, y \geq 0$ such that (2.51) holds simultaneously for the pairs $(H_i, \delta_i), i \in [k]$, one has that for any $n^{-1/\Delta} \ll p = o(1)$,

$$\limsup_{n \to \infty} \phi^k_{n_p}(H, 1 + \delta) \leq x + \frac{1}{2} y^2. \quad (4.3)$$

**Proof.** By continuity, it suffices to consider $x, y > 0$, for which our candidate be the weighted adjacency matrix $X^+_n$ of (2.52), with principal blocks of sizes $s_1 \sim xp^\Delta n$, $s - s_1$ for $s \sim yp^{\Delta/2} n$ and $\tilde{n} : = n - s$. Indeed, in the course of proving (2.50), we have shown that

$$\frac{1}{2} I_p(X^+_n) = \left( x + \frac{1}{2} y^2 + o(1) \right) a_{n,p} \quad (4.4)$$

and consequently, it suffices to show that for any connected $H$ of maximal degree $\Delta \geq 2$,

$$t(H, X^+_n) \geq (yp^{\Delta/2} s(H)) + P_{H^*}(x)p^{s(H)} + o(p^{s(H)}). \quad (4.5)$$

To this end, note that the contribution from having all vertices of $H$ in the middle block of $X^+_n$ is $(s_2/n)^{s(H)} \sim (yp^{\Delta/2} s(H))$. Proceeding to consider the contribution when no vertex of $H$ is within the middle block of $X^+_n$, recall that $e(H_{\delta'}) = e(H) - \Delta|S|$ for any independent set $S$ of $H^*$. Enumerating over the possible independent sets of $H^*$, the contribution to $t(H, X^+_n)$ from having $S$ within the $s_1$-sized top principal block, is at least

$$\sum_S \left( \frac{s_1 - 1}{n} \right)^{|S|} \left( \frac{n - 1}{\tilde{n}} \right)^{v(H) - |S|} q^{e(H_{\delta'})} = (1 + o(1)) \sum_S \left( \frac{s_1 - 1}{n} \right)^{|S|} \left( \frac{n - 1}{\tilde{n}} \right)^{v(H) - |S|} q^{e(H_{\delta'})} \geq (1 + o(1))P_{H^*}(x)p^{e(H)}.$$

This implies our claim (4.5) and thereby completes the proof.

Given Proposition 4.1, similarly to Step 3 in proving Theorem 1.1, it suffices to show that as $p \to 0$,

$$\phi^k_p(H, \delta) : = \frac{1}{2} \inf \left\{ \frac{I_p(W)}{p^\Delta I_p(1)} : W \in \mathcal{W}, p^{-e(H)} t(H, W) \geq 1 + \delta, \forall i \in [k] \right\} \geq c(H, \delta) - o(1) \quad (4.6)$$
Proof of Theorem 1.12

Recall (2.12) that for any $b$, $\delta \in (0,1)$, $\pi_k(H, \delta)$ is at least $\frac{1}{2}I_p(p + U) \geq (1 - o(1))(x_b + \frac{1}{2}y_b) + Ip(1)$. Furthermore, from [[5], (6.7)–(6.8)] we know that if $I_p(p + U) = O(p^\Delta Ip(1))$ and $p^{-e(H)}I(H, p + U) \geq 1 + \delta$, then for any $b \to 0$ slowly enough in terms of $p$,

$$\mathbb{P}_{H^\star}(x_b) + \gamma_b^{\nu(H)} I_{\{H \text{ is } \Delta\text{-regular}\}} \geq 1 + \delta - o(1).$$

The same choice of $b$ applies for multiple connected $H_i$ of maximal degree $\Delta \geq 2$, thereby bounding below $\phi_b^k(\cdot, \cdot)$ as in (4.6) and completing the proof of Proposition 1.10.

4.3 Proof of Theorem 1.12

Starting with the ER model, from Propositions 1.9 and 1.10 we get

$$\lim sup_{n \to \infty} a_{n,p}^{-1} \log \mathbb{P}_p(\text{hom}(H, G_n) \geq 1 + \delta) \leq -c(\pi_k(H), \pi_k(\delta)).$$

(4.7)

Pittel’s inequality (2.30) with $A_n = \{\text{hom}(H, G_n) \geq 1 + \delta\}$, yields for any $n$ and $m$,

$$\mathbb{P}^{(m)}(\text{hom}(H, G_n) \geq 1 + \delta) \leq 3\sqrt{m} \mathbb{P}(\text{hom}(H, G_n) \geq 1 + \delta),$$

hence in view of (1.9), the upper bound (4.7) applies also for the law $\mathbb{P}(\cdot)$. Turning to the complementary lower bound, note that the probability of $\{\text{hom}(H, G_n) \geq 1 + \delta\}$ under both $\mathbb{P}(\cdot)$ and $\mathbb{P}^{(m)}(\cdot)$ is at least

$$\mathbb{P}(\{\text{hom}(H, G_n) \geq 1 + \delta\} \cap \mathcal{K}^{(m)}_n),$$

and thereby it suffices to lower bound the rate of decay of the latter. That is, to show that per fixed $\delta > \delta t \in \mathbb{R}_+^k$ and $x, y > 0$ which satisfy (2.51) simultaneously for $(H_i, \delta_i), i \in [k]$, one has the bound

$$\lim inf_{n \to \infty} a_{n,p}^{-1} \log \mathbb{P}_p(\{\text{hom}(H, G_n) \geq 1 + \delta'\} \cap \mathcal{K}^{(m)}_n) \geq -x - \frac{1}{2}y^2.$$  (4.9)

The proof of (4.9) proceeds precisely as the derivation of (2.50), by first making a change of the measure to the planted ER-model $\mathbb{P}_*$ that corresponds to $X_n^*$ of (2.52), after which it remains only to show that (2.44) holds simultaneously for all $(H_i, \delta_i), i \leq k'$. In case $i \leq k$ the same argument as in the proof of (2.44) applies here, thanks to our assumption that (2.51) holds for $H_i$ and some $\delta_i > \delta_i'$. Next, fixing $i \in (k, k']$, for $H = H_i$ let $S^c$ be any maximal subset of $V(H)$, such that $\Delta(H_{S^c}) \leq \Delta$. Clearly $S \neq \emptyset$ (for $t(H, W)$ as in (2.16)). Since $x \to I_p(x)$ is nonincreasing on $[0, p]$, we may and will set $W = p + U$ in (4.6), with $U \in [0, 1 - p]$. Then, following [5] we define for any $b \in [0, 1]$ and such $U$,

$$B_b = B_b(U) := \{x \in [0, 1] : \int_0^1 U(x, y) dy \geq b\}, \quad \overline{B}_b := [0, 1] \setminus B_b,$$

$$x_b := p^{-\Delta} \int_{B_b \times \overline{B}_b} U^2(t, s) dt ds, \quad y_b^2 := p^{-\Delta} \int_{B_b \times \overline{B}_b} U^2(t, s) dt ds.$$
since $\Delta(H_i) > \Delta$, while the maximality of $S'$ implies that $\mathfrak{e}(H) \geq \mathfrak{e}(H_{S'}) + (\Delta + 1)|S|$. Consequently, the contribution under $\mathbb{P}_q$ to $\text{hom}(H, G_n)$ from homomorphisms with $S$ in the hub of size $s_1$ of $X_n^*$ and $S'$ in its $q$-block of size $\tilde{n}$, is at least $\text{hom}(H_{S'}, G_{\tilde{n}})$ times

$$\left(\frac{s_1 - 1}{n}\right)^{|S|} \left(\frac{\tilde{n} - 1}{\tilde{n}}\right)^{|S'|} q^{\mathfrak{e}(H_{S'})} p^{\mathfrak{e}(H)} \geq x^{\Delta |S|} p^{-\Delta (1 - o(1))}.$$ 

Since the latter (nonrandom) factor diverges for any $x > 0$ fixed and $p = p(n) \to 0$, the required bound (2.44) holds for $H = H_i$ and any fixed $\delta' < \infty$, provided

$$\liminf_{n \to \infty} a_{n,p}^{-1} \log \mathbb{P}_q(\text{hom}(H_{S'}, G_{\tilde{n}}) < 1 - \varepsilon) < 0, \quad \forall \varepsilon > 0, \quad 1 \gg p(n) \gg n^{-1/\Delta}. \quad (4.10)$$

By construction $\Delta(H_{S'}) \leq \Delta$, in which case we have already proved (4.10) (in the course of proving Theorem 1.5). In conclusion, (2.44) holds for all $H_i, i \in [k']$, hence (4.9) holds as well, completing the proof of the theorem.

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