COMPLETE ORDER EQUIVALENCE OF SPIN UNITARIES

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ABSTRACT. This paper is a study of linear spaces of matrices and linear maps on matrix algebras that arise from spin systems, or spin unitaries, which are finite sets $S$ of selfadjoint unitary matrices such that any two unitaries in $S$ anticommute. We are especially interested in linear isomorphisms between these linear spaces of matrices such that the matricial order within these spaces is preserved; such isomorphisms are called complete order isomorphisms, which might be viewed as weaker notion of unitary similarity. The main result of this paper shows that all $m$-tuples of anticommuting selfadjoint unitary matrices are equivalent in this sense, meaning that there exists a unital complete order isomorphism between the unital linear subspaces that these tuples generate. We also show that the $C^*$-envelope of any operator system generated by a spin system of cardinality $2^k$ or $2^k + 1$ is the simple matrix algebra $M_{2^k}(C)$. As an application of the main result, we show that the free spectrahedra determined by spin unitaries depend only upon the number of the unitaries, not upon the particular choice unitaries, and we give a new, direct proof of the fact [11] that the spin ball $B_m^{\text{spin}}$ and max ball $B_m^{\text{max}}$ coincide as matrix convex sets in the cases $m = 1, 2$.

1. INTRODUCTION

This paper is a study of linear spaces of matrices and linear maps on matrix algebras that arise from spin systems, or spin unitaries, which are finite sets $S$ of self-adjoint unitary matrices such that any two unitaries in $S$ anticommute. In addition to their interest from the perspective of linear algebra, these linear spaces and linear maps are commonly studied in operator algebra theory and in applications that include quantum information theory. We are especially interested in linear isomorphisms between these linear spaces of matrices such that the matricial order within these spaces is preserved; such isomorphisms are called complete order isomorphisms.

We denote the algebra of $d \times d$ matrices over the field $\mathbb{C}$ of complex numbers by $\mathcal{M}_d(\mathbb{C})$, the $\mathbb{R}$-vector space of selfadjoint complex $d \times d$ matrices by $\mathcal{M}_d(\mathbb{C})_{\text{sa}}$, and the cone of positive semidefinite $d \times d$ matrices by $\mathcal{M}_d(\mathbb{C})_+$. The unitary group in $\mathcal{M}_d(\mathbb{C})$ is denoted by $\mathcal{U}_d$, and $\text{Tr}$ denotes the canonical trace on $\mathcal{M}_d(\mathbb{C})$.

Definition 1.1. A subset $S \subseteq \mathcal{U}_d$ of unitary matrices is a spin system of unitaries, or simply a spin system, if

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If a spin system \( S \) is called a pair of spin unitaries \( u, v \) yields \( \text{Tr}(uv) = -\text{Tr}(vu) \) for every pair of distinct elements \( u, v \in S \).

The third of the three conditions in the definition of a spin system \( S \) indicates that no element of \( S \) is the identity matrix \( 1 \) or its negative \(-1\). Hence, each \( u \in S \) has spectrum \((-1, 1)\) and, by the Spectral Theorem, can be expressed as a difference \( u = p - q \), where \( p, q \in M_d(\mathbb{C}) \) are projections such that \( pq = qp = 0 \) and \( p + q = 1 \).

The most basic example of a spin system of unitaries is afforded by the Pauli matrices:

\[
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Definition 1.2. The subset \( \mathcal{P} = \{\sigma_x, \sigma_y, \sigma_z\} \subset M_2(\mathbb{C}) \) is called the Pauli spin system.

The following elementary (and, surely, well-known) result establishes some basic linear-algebraic facts about spin systems.

Proposition 1.3. If \( S \subset U_d \) is a spin system, then

1. \( d \) is an even integer,
2. \( \text{Tr}(u) = 0 \), for every \( u \in S \),
3. \( \text{Tr}(uv) = 0 \) for all \( u, v \in S \) with \( v \neq u \), and
4. the elements of \( S \) are linearly independent.

Proof. Select \( u \in S \). By definition, there is an element \( v \in U \) with \( v \neq u \). Because \( uv = -vu \), we deduce from

\[
\det(u)\det(v) = \det(uv) = \det(-1uv) = (-1)^d \det(uv)
\]

that \((-1)^d = 1\), and so \( d \) is even. Likewise, \( uv = -vu \) implies that \( vuv = -u \), and so

\[
-\text{Tr}(u) = \text{Tr}(-u) = \text{Tr}(vuv) = \text{Tr}(uv^2) = \text{Tr}(u)
\]

yields \( \text{Tr}(u) = 0 \).

Likewise, if \( u, v \in S \) are distinct, then \( uv = -vu \) implies that \( \text{Tr}(uv) = -\text{Tr}(vu) \); since \( \text{Tr}(ab) = \text{tr}(ba) \) for all \( a, b \), we deduce that \( \text{Tr}(uv) = -\text{Tr}(uv) \) and, thus, \( \text{Tr}(uv) = 0 \).

Lastly, suppose that \( u_1, \ldots, u_m \in S \) and that \( \sum_{j=1}^{m} \alpha_j u_j = 0 \). Fix \( i \in \{1, \ldots, m\} \) and note that the trace property \( \text{Tr}(ab) = \text{Tr}(ba) \) along with the anticommutation relations \( u_iu_j = -u_ju_i (\text{for } j \neq i) \) yield

\[
0 = \text{Tr}(0 \cdot u_i) = \text{Tr} \left( \sum_{j=1}^{m} \alpha_j u_j u_i \right) = \alpha_i \text{d} + \sum_{j \neq i} \alpha_j \text{Tr}(u_j u_i)
\]

and so \( \alpha_i = 0 \). As the choice of \( i \) is arbitrary, each \( \alpha_j = 0 \), showing that \( u_1, \ldots, u_m \) are linearly independent. \( \square \)
A key concept in this paper is that of an operator system of matrices, and its matricial cone of positive semidefinite matrices.

**Definition 1.4.** An operator system of matrices, or more simply an operator system, is a linear subspace $\mathcal{R}$ of $M_d(\mathbb{C})$ and a sequence of convex cones $(M_n(\mathcal{R}_+))_{n \in \mathbb{N}}$ in the matrix algebras $M_n(M_d(\mathbb{C}))$ such that

1. $\mathcal{R}$ contains the identity matrix $1$ (sometimes denoted by $1_d$),
2. $\mathcal{R}$ contains the adjoint $x^*$ of each matrix $x \in \mathcal{R}$, and
3. $X \in M_n(\mathcal{R}_+)$ if and only if $X$ is an $n \times n$ positive semidefinite matrix with entries from $\mathcal{R}$.

The definition of operator system given above applies only to matrices, but it is generally sufficient for our purposes in this paper. A somewhat more general definition is that an operator system is a unital linear subspace $\mathcal{R}$ of a unital $C^*$-algebra $A$ such that $x^* \in \mathcal{R}$ for every $x \in \mathcal{R}$. Most general of all is the axiomatic definition of an operator system as a matrix-ordered $*$-vector space possessing an Archimedean order unit. [16].

**Definition 1.5.** If $\mathcal{R}$ and $\mathcal{T}$ are operator systems, then a linear transformation $\phi : \mathcal{R} \to \mathcal{T}$ is $n$-positive if $\phi^n(X) \in M_n(\mathcal{T}_+)$ for every $X \in M_n(\mathcal{R}_+)$, where $\phi^n : M_n(\mathcal{R}) \to M_n(\mathcal{T})$ is the linear map defined by

$$\phi^n \left( \begin{bmatrix} r_{ij} \end{bmatrix}_{i,j=1}^n \right) = \left[ \phi(r_{ij}) \right]_{i,j=1}^n.$$

Further:

1. $\phi$ is unital, if $\phi(1_\mathcal{R}) = 1_\mathcal{T}$ (i.e., $\phi$ maps the identity to the identity);
2. $\phi$ is positive, if $\phi$ is $n$-positive for $n = 1$; and
3. $\phi$ is completely positive, if $\phi$ is $n$-positive for every $n \in \mathbb{N}$.

We turn next to the notion of isomorphism in the category of operator systems and unital completely positive linear maps.

**Definition 1.6.** If $\mathcal{R}$ and $\mathcal{T}$ are operator systems, then a linear transformation $\phi : \mathcal{R} \to \mathcal{T}$ is

1. a unital complete order embedding if $\phi$ is a unital, linear, completely positive, and injective map, and
2. a unital complete order isomorphism if $\phi$ is a unital, linear bijection in which both $\phi$ and $\phi^{-1}$ are completely positive.

To illustrate the notion of complete order isomorphism, suppose that $w \in U_d$ is any unitary matrix and define the linear map $\phi_w : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ by $\phi_w(x) = w^*xw$, for all $x \in M_d(\mathbb{C})$. Thus, $\phi_w$ is a unital complete order automorphism of $M_d(\mathbb{C})$. In fact, every unital complete automorphism of $M_d(\mathbb{C})$ arises from a unitary $w$ in this way; however, if $\mathcal{R}$ and $\mathcal{T}$ are operator subsystems of $M_d(\mathbb{C})$, then there may exist unital complete order isomorphisms of $\mathcal{R}$ and $\mathcal{T}$ that do not arise from a unitary similarity transformation $\phi_w$.

**Definition 1.7.** Suppose that $S$ is a spin system of unitaries.

1. The spin operator system generated by $S$ is the linear space $O_S \subseteq M_d(\mathbb{C})$ defined by

$$O_S = \text{Span} \{ 1, u \mid u \in S \}.$$
(2) The spin operator algebra generated by $S$ is the $C^*$-subalgebra $A_S \subseteq M_d(\mathbb{C})$ defined by

$$A_S = \text{Alg}(O_S).$$

Because the elements of a spin system $S$ have trace zero, so does every linear combination of such elements; hence, the identity matrix $1$ is linearly independent of $S$. Further, the Pauli system $P$ determines a 4-dimensional operator system in the 4-dimensional matrix algebra $M_2(\mathbb{C})$, implying that

$$O_P = A_P = M_2(\mathbb{C}).$$

Criteria for when two matrices are unitarily equivalent were given by a classical result of Specht [19], as well as by others subsequently (see [18] for a good survey). For operator systems, the concept of unital complete order isomorphism is weaker than the notion of unitary equivalence, which makes the following definition of interest.

**Definition 1.8.** A $k$-tuple $x = (x_1, \ldots, x_k)$ of matrices $x_i \in M_d(\mathbb{C})$ is completely order equivalent to a $k$-tuple $y = (y_1, \ldots, y_k)$ of matrices $y_i \in M_\ell(\mathbb{C})$ if there exists a unital complete order isomorphism $\phi: O_x \to O_y$, where

$$O_x = \text{Span}\{1_d, x_1, x_1^*, \ldots, x_k, x_k^*\} \quad \text{and} \quad O_y = \text{Span}\{1_\ell, y_1, y_1^*, \ldots, y_k, y_k^*\}.$$  

The definition of complete order equivalence of $k$-tuples of matrices is related to that of interpolation by completely positive maps [1, 13]; however, a key difference in our interpretation is that we require the interpolating maps to be complete order isomorphisms, not just completely positive maps.

We use the notation

$$x \simeq_{\text{ord}} y$$

to indicate that the $k$-tuples $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are completely order equivalent, and the notation

$$x \simeq_{\text{U}} y$$

to denote that the $k$-tuples $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are completely order equivalent via a unitary similarity transformation $x_j \to w^*x_jw$ for some unitary matrix $w$.

More generally, if $\mathcal{R}$ and $\mathcal{T}$ are operator systems of matrices in $M_d(\mathbb{C})$ and $M_\ell(\mathbb{C})$, respectively, then the notation $\mathcal{R} \simeq_{\text{ord}} \mathcal{T}$ denotes the existence of a unital complete order isomorphism $\phi: \mathcal{R} \to \mathcal{T}$, while the notation $\mathcal{R} \simeq_{\text{U}} \mathcal{T}$ indicates that $\mathcal{R}$ and $\mathcal{T}$ are unitarily equivalent (i.e., $\ell = d$ and there exists a unital complete order isomorphism $\phi: \mathcal{R} \to \mathcal{T}$ of the form $\phi(x) = w^*xw$, for some unitary $w \in U_d$).

The following example, whose details we defer to the next section of this paper, is a good illustration of the information captured by these notions of equivalence.

**Example 1.9.** If $u \in U_d$, and $v \in U_d$, are selfadjoint unitary matrices such that neither of them is a scalar multiple of the identity, then

1. $u \simeq_{\text{ord}} v$, and
2. $u \simeq_{\text{U}} v$ if and only if $d_1 = d_2$. 
The properties of anticommuting selfadjoint unitaries are not particular to matrices; indeed, these properties may be present in arbitrary complex associative unital algebras with a positive involution *. With this understanding in mind, it is useful to consider the most abstract form of a spin system: the universal algebra [5, §II.8.3] that is defined purely from algebraic (rather than spatial) relations.

Definition 1.10. Let $\mathcal{S} = \{u_1, \ldots, u_m\}$ be a set of symbols and let $\Omega$ be the set of relations
\[
\Omega = \{u_j^2 = u_j \text{ and } u_j u_i + u_i u_j = 0, \forall j \neq i\}.
\]

1. The universal C*-algebra $A_{\text{spin}(m)}$ generated by $\mathcal{S}$ subject to the relations $\Omega$ is called the universal algebra of $m$ spin unitaries.
2. The operator subsystem $O_{\text{spin}(m)}$ generated by $\mathcal{S}$ is called the universal operator system of $m$ spin unitaries.

Universality leads immediately to the following result.

Theorem 1.11. If $\mathcal{S} = \{u_1, \ldots, u_m\} \subset \mathcal{U}_d$ is a spin system, then there exists a unital completely positive linear map $\phi : O_{\text{spin}(m)} \to O_{\mathcal{S}}$ such that $\phi(u_j) = u_j$, for every $j = 1, \ldots, m$.

Proof. As a universal C*-algebra, the algebra $A_{\text{spin}(m)}$ has the property that, whenever $\mathcal{S} = \{u_1, \ldots, u_m\} \subset \mathcal{U}_d$ are spin unitaries, there exists a C*-algebra homomorphism $\pi : A_{\text{spin}(m)} \to M_d(\mathbb{C})$ such that $\pi(u_j) = u_j$, for $j = 1, \ldots, m$. Consequently, the linear map $\phi = \pi|_{O_{\text{spin}(m)}}$ is a unital completely positive linear map of $O_{\text{spin}(m)}$ onto $O_{\mathcal{S}}$ such that $\phi(u_j) = u_j$, for every $j = 1, \ldots, m$.

Universal C*-algebras are normally large objects; however, $A_{\text{spin}(m)}$ is a finite-dimensional C*-algebra. Indeed, $A_{\text{spin}(m)}$ is spanned by the identity 1 and all products (of which there are finitely many) of the form $u_{j_1} u_{j_2} \cdots u_{j_\ell}$, where $\ell \leq m$ and $j_1 < j_2 < \cdots < j_\ell$ [17, Chapter 3].

Our main results of the paper are the following theorem and its corollaries.

Theorem 1.12. If $\mathcal{S} \subset \mathcal{U}_d$ is a spin system of cardinality $m$, then
\[
O_{\mathcal{S}} \simeq_{\text{ord}} O_{\text{spin}(m)}.
\]

More precisely, the unital completely positive linear map $\phi : O_{\text{spin}(m)} \to O_{\mathcal{S}}$ in Proposition 1.11 is a unital complete order isomorphism.

Corollary 1.13. If $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ are $m$-tuples of spin unitary matrices $u_j \in \mathcal{U}_d$, $v_k \in \mathcal{U}_d$, then $u \preceq_{\text{ord}} v$.

We say that a tuple $x = (x_1, \ldots, x_m)$ of $d \times d$ matrices is irreducible if the only $d \times d$ matrices that commute with each $x_j$ are scalar multiples of the identity matrix.

Corollary 1.14. If $M_d(\mathbb{C})$ contains an irreducible $m$-tuple $u = (u_1, \ldots, u_m)$ of spin unitaries, then every $m$-tuple $v = (v_1, \ldots, v_m)$ of $d \times d$ spin unitaries is also irreducible and $u \preceq_{\text{ord}} v$.

Corollary 1.15. If $u, v, w \in \mathcal{U}_d$ are anticommuting selfadjoint unitary matrices, then there exists a $k \in \mathbb{N}$ and subspace $\mathcal{L} \subseteq \mathbb{C}^2 \otimes \mathbb{C}^k$ such that $u$, $v$, and $w$ are compressions to $\mathcal{L}$ of, respectively, $\sigma_x \otimes 1_k$, $\sigma_y \otimes 1_k$, and $\sigma_z \otimes 1_k$.

If one has an operator system of matrices $\mathcal{R} \subset M_d(\mathbb{C})$ and a unital complete order embedding $\phi : \mathcal{R} \to \mathcal{A}$ into some unital C*-algebra $A$, then the C*-subalgebra
$C^*$ of $\mathcal{A}$ generated by $\phi(\mathcal{R})$ need not be isomorphic to the $C^*$-subalgebra $C^*(\mathcal{R})$ of $\mathcal{M}_d(\mathbb{C})$ generated by $\mathcal{R}$. Because, in the category of operator systems, we do not distinguish between $\mathcal{R}$ and any unital complete order isomorphic copy of $\mathcal{R}$, a single operator system $\mathcal{R}$ can, in principle, generate many non-isomorphic $C^*$-algebras. However, there always exists “smallest” such algebra, which is known as the $C^*$-envelope of $\mathcal{R}$.

Theorem 1.16 (Hamana). ([10, 16]) If $\mathcal{R}$ is an operator system, then there exists a unital $C^*$-algebra $\mathcal{A}_e$ and unital complete order embedding $\iota_e: \mathcal{R} \to \mathcal{A}_e$ such that $\iota_e(\mathcal{R})$ generates $\mathcal{A}_e$ and such that, if $\phi: \mathcal{R} \to \mathcal{A}$ is any unital complete order embedding of $\mathcal{R}$ into a unital $C^*$-algebra $\mathcal{A}$, then there is a surjective $C^*$-algebra homomorphism $\pi: \mathcal{A} \to \mathcal{A}_e$ such that $\iota_e = \pi \circ \phi$.

The algebra $\mathcal{A}_e$ in Hamana’s Theorem is unique up to isomorphism; thus, we denote $\mathcal{A}_e$ by $C^*_e(\mathcal{R})$ and say that $C^*_e(\mathcal{R})$ is the $C^*$-envelope of $\mathcal{R}$.

Using our main result, Theorem 1.12, we shall also prove the following theorem.

Theorem 1.17. For every $k \in \mathbb{N}$, $C^*_e(O_{\text{spin}}(2k)) \cong C^*_e(O_{\text{spin}}(2k+1)) \cong M_{2k}(\mathbb{C})$.

Lastly, results such as Theorem 1.12 and Corollary 1.13 touch upon free convexity theory [12]. We defer the pertinent definitions and discussion until later, and simply mention here that in the present paper we show that the free spectrahedra determined by spin unitaries depend only upon the number of the unitaries, not upon the particular choice unitaries, and we give a new, direct proof of the following result.

Theorem 1.18. ([11, Corollary 14.15]) The free spectrahedron $\mathcal{B}_2^{\text{spin}}$ and the max ball $\mathcal{B}_2^{\text{max}}$ coincide.

2. Spin Pairs

It is instructive to begin with smallest of all spin systems: those that consist of just two elements.

Definition 2.1. A spin pair is a spin system $S \subset \mathcal{U}_d$ of cardinality 2.

Thus, any two anticommuting selfadjoint unitary matrices form a spin pair.

Proposition 2.2. If $u \in \mathcal{U}_d$ is a selfadjoint unitary matrix of trace zero, then a selfadjoint unitary $v \in \mathcal{U}_d$ anticommutes with $u$ if and only if there exist $y \in \mathcal{U}_d$ and $w \in \mathcal{U}_n$, where $n = d/2$, such that

$$yuy^* = \begin{bmatrix} 1_n & 0_n \\ 0_n & -1_n \end{bmatrix} \quad \text{and} \quad v = y^* \begin{bmatrix} 0_n & w \\ w^* & 0_n \end{bmatrix} y.$$

Proof. By the Spectral Theorem, $u$ is unitarily equivalent to a diagonal matrix in which the first $n$ entries of this diagonal matrix are 1 and the remaining $n$ entries are $-1$. Thus, $yuy^* = \begin{bmatrix} 1_n & 0_n \\ 0_n & -1_n \end{bmatrix}$, for some unitary matrix $y$.

Let $\bar{u} = yuy^*$ and $\bar{v} = yvy^*$; thus, $\bar{u}$ and $\bar{v}$ form a spin pair. Expressing $\bar{v}$ as a $2 \times 2$ matrix of $n \times n$ matrices, $\bar{v}$ has the form $\bar{v} = \begin{bmatrix} z_{11} & w \\ w^* & z_{22} \end{bmatrix}$ for some matrices
Then \( \phi \in M_n(\mathbb{C}) \) in which end, we identify the matrix space \( \phi \) such that
\[
\begin{bmatrix}
z_{11} & w \\
-w^* & -z_{22}
\end{bmatrix} =
\begin{bmatrix}
-z_{11} & w \\
-w^* & z_{22}
\end{bmatrix},
\]
and so \( z_{11} = z_{22} = 0 \). Under this condition on the diagonal blocks of \( \tilde{v} \), we obtain \( \tilde{v}^2 = 1_d \) if and only if \( \tilde{w}w^* = w^*w = 1_n \).

Therefore, \( uv = -vu \) if and only if \( u \) and \( v \) have, via some unitary \( y \in U_d \), the \( 2 \times 2 \) block-matrix structure indicated in the statement of the proposition, for some unitary \( w \in U_n \).

**Corollary 2.3.** If \( u \in U_d \) is a selfadjoint unitary matrix of trace zero, then \( d \) is an even integer and the set of all \( v \in U_d \) that anticommute with \( u \) forms a path-connected topological space homeomorphic to the unitary group \( U_{\frac{d}{2}} \).

**Proof.** By Proposition 2.2 \( d \) is an even integer and there is a unitary \( y \in U_d \) for which \( yuy^* = \begin{bmatrix} 1_n & 0_n \\ 0_n & -1_n \end{bmatrix} \). The function
\[
F : U_{\frac{d}{2}} \to \{ v \in U_d | uv = -vu \}
\]
defined by \( F(w) = y^* \begin{bmatrix} 0_n & w \\ w^* & 0_n \end{bmatrix} y \) is a homeomorphism. Because the unitary group \( U_{\frac{d}{2}} \) is path connected, so is the set \( \{ v \in U_d | uv = -vu \} \).

Even though Example 1.9(1) is a consequence of our main result, Theorem 1.12, it is worthwhile to make note of the following simple and direct proof.

**Proposition 2.4.** If \( u \in U_{d_1} \) and \( v \in U_{d_2} \) are selfadjoint unitary matrices such that neither of them is a scalar multiple of the identity, then

1. \( u \succeq_{ord} v \), and
2. \( u \succeq_U v \) if and only if \( d_1 = d_2 \).

**Proof.** By Proposition 2.2 and the Spectral Theorem, there are unitaries \( w_j \in U_{d_j} \) such that
\[
w_1^*uw_1 = \begin{bmatrix} 1_{n_1} & 0_{n_1} \\ 0_{n_1} & -1_{n_1} \end{bmatrix} \quad \text{and} \quad w_2^*vw_2 = \begin{bmatrix} 1_{n_2} & 0_{n_2} \\ 0_{n_2} & -1_{n_2} \end{bmatrix},
\]
where each \( n_j = \frac{d_j}{2} \). As conjugation by the unitaries \( w_j \) preserves both unitary and complete order equivalence, we may assume without loss of generality that both \( u \) and \( v \) are these indicated \( 2 \times 2 \) matrices of \( n_j \times n_j \) matrices. In this regard, it is clear that \( u \succeq_U v \) if and only if \( d_1 = d_2 \).

More generally, to prove that \( u \succeq_{ord} v \), we must prove that the linear isomorphism
\[
\phi : \text{Span}(1_{d_1}, u) \to \text{Span}(1_{d_2}, v)
\]
in which \( \phi(1_{d_1}) = 1_{d_1} \) and \( \phi(u) = v \) is a complete order isomorphism. To this end, we identify the matrix space \( M_n(\mathbb{C}) \) with the vector space tensor product \( M_n(\mathbb{C}) \otimes \mathcal{R} \), where \( \mathcal{R} \) is an operator system. In particular, if
\[
\mathcal{R}_1 = \text{Span}(1_{d_1}, u) \quad \text{and} \quad \mathcal{R}_2 = \text{Span}(1_{d_2}, v),
\]
then
\[
M_n(\mathcal{R}_1) = \{ a \otimes 1_{d_1} + b \otimes u_j | a, b \in M_n(\mathbb{C}) \},
\]
where \( u_1 = u \) and \( u_2 = v \) above. That is,

\[
\mathcal{M}_n(\mathbb{R}) = \left\{ \begin{pmatrix} (a + b) \otimes 1_{n_1} & 0 \\ 0 & (a - b) \otimes 1_{n_1} \end{pmatrix} \mid a, b \in \mathcal{M}_n(\mathbb{C}) \right\}.
\]

Therefore, to show that \( \phi \) is a unital complete order isomorphism, we much show, for all selfadjoint \( a, b \in \mathcal{M}_n(\mathbb{C}) \) and all \( n \in \mathbb{N} \), that

\[
\begin{pmatrix} (a + b) \otimes 1_{n_1} & 0 \\ 0 & (a - b) \otimes 1_{n_1} \end{pmatrix}
\]

is positive semidefinite if and only if

\[
\begin{pmatrix} (a + b) \otimes 1_{n_2} & 0 \\ 0 & (a - b) \otimes 1_{n_2} \end{pmatrix}
\]

is positive semidefinite. This bi-implication above, which clearly depends only on \( a \) and \( b \) but not upon \( u \) and \( v \), shows that \( \phi \) is a unital complete order isomorphism. \( \square \)

The next theorem is the main result of this section. In preparation for its statement, we recall the definition of the numerical range, or field of values, of a matrix. For reasons that will be apparent in our discussion of matrix convexity, our notation and terminology for the numerical range (below) is slightly different from the traditional notation and terminology.

**Definition 2.5.** The spatial numerical range of a matrix \( x \in \mathcal{M}_d(\mathbb{C}) \) is the set

\[
W_1^s(x) = \left\{ \langle x\xi, \xi \rangle \mid \xi \in \mathbb{C}^d, \|\xi\| = 1 \right\}.
\]

By a simple direct computation, the spatial numerical range of the \( 2 \times 2 \) matrix

\[
\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}
\]

is the closed disc of radius 1, centered at \( 0 \in \mathbb{C} \).

**Theorem 2.6.** Let \( g = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C}) \). If \( u, v \in \mathcal{M}_d(\mathbb{C}) \) are anticommuting selfadjoint unitary matrices and if \( x = u + iv \), then:

1. \( x^2 = 0 \) and \( \|x\| = 2 \);
2. the spatial numerical range of \( x \) is the closed unit disc;
3. \( x \) is completely order equivalent to \( g \), and
4. \( x \) is unitarily equivalent to \( \bigoplus_{i=1}^{n} g \), where \( n = d/2 \).

**Proof.** The computation

\[
x^2 = u^2 + iuv + iuv + i^2v^2 = 1 + i(uv - uv) - 1 = 0
\]

shows that \( x \) is a nilpotent of order 2, while the equations

\[
x^*x = 2(1 + iuv) \quad \text{and} \quad (uv)^2 = -1
\]

show that the eigenvalues of \( uv \) are \( \pm i \) and thus the eigenvalues of \( x^*x \) are 0 and 4, making the norm of \( x \) (i.e., the square root of the spectral radius of \( x^*x \)) equal to 2.

By Proposition 2.2 there exist \( y \in \mathbb{U}_d \) and \( w \in \mathbb{U}_n \), where \( n = d/2 \), such that

\[
yuy^* = \begin{bmatrix} 1_n & 0_n \\ 0_n & -1_n \end{bmatrix} \quad \text{and} \quad v = y^* \begin{bmatrix} 0_n & w \\ w^* & 0_n \end{bmatrix} y.
\]
Hence, \( x = u + iv \) is unitarily equivalent to \( \bar{x} = \begin{bmatrix} 1 & i w \\ i w^* & -1 \end{bmatrix} \). Because the norm, numerical range, and complete order equivalence are invariant under unitary equivalence, we may assume without loss of generality that \( x = \bar{x} \).

Consider the unitary matrix \( h = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & -i w \\ -1 & -i w \end{bmatrix} \in U_d \), and observe that
\[
\begin{align*}
h^* x h &= \frac{1}{2} \begin{bmatrix} 1 & -i w \\ -1 & -i w \end{bmatrix} \begin{bmatrix} 1 & i w \\ i w^* & -1 \end{bmatrix} \begin{bmatrix} 1 & -i w \\ -1 & -i w \end{bmatrix} \\
&= -2 \begin{bmatrix} 0_n & 1_n \\ 0_n & 0_n \end{bmatrix} \\
&= 1_n \otimes \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \\
&\cong \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

Hence, \( x \) is unitarily equivalent to a direct sum of \( n \) copies of the \( 2 \times 2 \) complex matrix \( g = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \). This implies that \( g \) and \( x \) have the same numerical range—namely, the closed unit disc in the complex plane.

It remains to show that \( x \) and \( g \) are completely order equivalent. This follows immediately from the observation that \( g \) and \( \bigoplus_{i=1}^{n} g \) are completely order equivalent.

As a consequence of Theorem 2.6, we recover a fact observed in [4]:

**Corollary 2.7.** The C*-algebra generated by any spin pair is \( M_2(\mathbb{C}) \).

**Proof.** If \( u, v \in U_d \) form a spin pair, then Theorem 2.6 shows that \( x \cong_U \bigoplus_{i=1}^{n} g \), where \( n = d/2 \), \( x = u + iv \), and \( g = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{C}) \). Thus, the C*-algebra generated by \( x \) is isomorphic to the C*-algebra generated by \( g \), namely \( M_2(\mathbb{C}) \).

On the one hand, because \( x = u + iv \in C^*(u, v) \), we deduce that \( C^*(x) \subseteq C^*(u, v) \). On the other hand, \( u = \frac{1}{2}(x + x^*) \) and \( v = \frac{1}{2i}(x - x^*) \) imply that \( u, v \in C^*(x) \), whence \( C^*(u, v) \subseteq C^*(x) \). \( \square \)

3. **Complete Order Equivalence of Spin Unitaries**

The main result, Theorem 1.12, is restated and proved below.

**Theorem 3.1.** If \( S \subset U_d \) is a spin system of cardinality \( m \), then the unital completely positive linear map \( \phi : O_{\text{spin}}(m) \to O_S \) in Theorem 1.11 is a unital complete order isomorphism and, hence,
\[
O_S \cong_{\text{ord}} O_{\text{spin}}(m).
\]
Proof. Let $\mathcal{S} = \{u_1, \ldots, u_m\}$. Theorem [1.11] asserts that there exists a unital completely positive linear map $\phi : \mathcal{O}_{\text{spin}(m)} \to \mathcal{O}_{\mathcal{S}}$ such that $\phi(u_j) = u_j$, for every $j = 1, \ldots, m$. Because $\phi$ is a surjective linear map of vector spaces of equal finite dimension, it is an invertible linear transformation. Thus, we need only show that its linear inverse, $\phi^{-1}$, is completely positive.

Because the universal C*-algebra $\mathcal{A}_{\text{spin}(m)}$ is finite-dimensional [17, Chapter 3], there exist $n \in \mathbb{N}$ and a unital C*-algebra $A \subseteq \mathcal{M}_n(\mathbb{C})$ such that $\mathcal{A}_{\text{spin}(m)}$ and $A$ are isomorphic C*-algebras [8, §5.4]. Thus, without loss of generality, we may assume that $\mathcal{A}_{\text{spin}(m)}$ is a unital C*-subalgebra of $\mathcal{M}_n(\mathbb{C})$. Hence, the unital completely positive linear map $\phi$, when considered as a map into $\mathcal{M}_d(\mathbb{C})$, has an extension to a completely positive linear map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$, by the Arveson Extension Theorem [16, Theorem 7.5]. Therefore, by the Stinespring-Kraus-Choi Theorem [16, Proposition 4.7], there are $\ell$ linear transformations $a_i : \mathbb{C}^d \to \mathbb{C}^n$ such that

$$\Phi(z) = \sum_{k=1}^{\ell} a_k^* z a_k,$$

for every $z \in \mathcal{M}_n(\mathbb{C})$. In particular,

$$\phi(x) = \sum_{k=1}^{\ell} a_k^* x a_k,$$

(1)

for every $x \in \mathcal{O}_{\text{spin}(m)}$.

Because the canonical trace functional $\text{Tr}$ on the matrix algebra $\mathcal{M}_n(\mathbb{C})$ induces an inner product on $\mathcal{M}_n(\mathbb{C})$, the operator system $\mathcal{O}_{\text{spin}(m)}$ is a Hilbert subspace of $\mathcal{M}_n(\mathbb{C})$. Therefore, via the trace as an inner product, two matrices $y_1, y_2 \in \mathcal{O}_{\text{spin}(m)}$ are equal (i.e., $y_1 = y_2$) if and only if $\text{Tr}(xy_1) = \text{Tr}(xy_2)$ for every matrix $x \in \mathcal{O}_{\text{spin}(m)}$. We shall apply this criterion for the equality of matrices in what follows. To clarify notation, we shall denote the trace function on $\mathcal{M}_k(\mathbb{C})$, for a given $k$, by $\text{Tr}_k$.

Select any $x, y \in \mathcal{A}_{\text{spin}(m)}$. Thus,

$$x = \sum_{j=1}^{m} \alpha_j u_j \quad \text{and} \quad y = \sum_{j=1}^{m} \beta_j u_j$$

for some uniquely determined scalars $\alpha_j$ and $\beta_j$. As matrices in $\mathcal{M}_n(\mathbb{C})$, and by using that facts that each $u_j^2 = 1_n$ and the trace of any pair of anticommuting matrices is 0, we see that

$$\text{Tr}_n(xy) = n \sum_{j=1}^{m} \alpha_j \beta_j.$$

Similarly,

$$\text{Tr}_d(\phi(x)\phi(y)) = d \sum_{j=1}^{m} \alpha_j \beta_j = \frac{d}{n} \text{Tr}_n(xy).$$

(2)

At this point we can invoke [14, Theorem 2.2] (suitably modified for operator systems) to deduce that $\phi^{-1}$ is completely positive; however, owing to the complexities of the proof of that result, it is preferable to argue directly (which we do below) that $\phi^{-1}$ is completely positive.
In using the Stinespring-Kraus-Choi representation of φ in (1), the trace equation (2) becomes

\[
\text{Tr}_n(xy) = \frac{n}{d} \text{Tr}_d (\phi(x)\phi(y)) = \frac{n}{d} \text{Tr}_d \left( \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} a_i^* x a_i y a_j \right)
\]

(3)

\[
= \frac{n}{d} \text{Tr}_n \left( x \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} a_i a_j y a_i^* a_j^* \right).
\]

Fixing y and allowing x to vary through all of \( \mathcal{O}_{\text{spin}(m)} \), equation (3) above implies that

\[
y = \frac{n}{d} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} a_i a_j y a_i^* a_j^*.
\]

Therefore, if \( \tilde{\psi} : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \) is the completely positive linear map

\[
\tilde{\psi}(z) = \frac{n}{d} \sum_{i=1}^{\ell} a_i z a_i^*,
\]

for \( z \in \mathcal{M}_d(\mathbb{C}) \), then \( \psi \circ \phi \) is the identity map on \( \mathcal{O}_{\text{spin}(m)} \). Define

\[
\psi : \mathcal{O}_S \to \mathcal{M}_n(\mathbb{C})
\]

to be the restriction of the completely positive linear map \( \tilde{\psi} \) to the operator system \( \mathcal{O}_S \); thus, \( \psi \) is a completely positive left inverse of \( \phi \). However, because left invertible linear maps between finite-dimensional vector spaces of equal dimension are automatically invertible, we deduce that \( \psi = \phi^{-1} \), implying that \( \phi^{-1} \) is completely positive. Hence, \( \mathcal{O}_{\text{spin}(m)} \approx_{\text{ord}} \mathcal{O}_S \). \( \square \)

Because complete order equivalence is a transitive relation, we immediately obtain:

**Corollary 3.2.** If \( u = (u_1, \ldots, u_m) \) and \( v = (v_1, \ldots, v_m) \) are m-tuples of spin unitary matrices \( u_j \in \mathcal{U}_{d_j} \), \( v_k \in \mathcal{U}_{d_k} \), then \( u \approx_{\text{ord}} v \).

Because an operator system \( \mathcal{R} \subseteq \mathcal{M}_d(\mathbb{C}) \) is closed under the adjoint operation *, the von Neumann Double Commutant Theorem \([5\text{ Theorem 1.9.1.1}]\) implies that \( \mathcal{R}'' = \mathcal{C}^*(\mathcal{R}) \), where \( \mathcal{X}' \) denotes, for a set \( \mathcal{X} \) of matrices, the commutant of \( \mathcal{X} \) (i.e., the set of all matrices that commute with every matrix in \( \mathcal{X} \)), and \( \mathcal{X}'' \) denotes the commutant of \( \mathcal{X}' \). In particular, if \( \mathcal{R}' = \{ \lambda 1_d \mid \lambda \in \mathbb{C} \} \), then \( \mathcal{C}^*(\mathcal{R}) = \mathcal{M}_d(\mathbb{C}) \).

**Definition 3.3.** A spin system \( \mathcal{S} \subset \mathcal{U}_d \) is irreducible if \( \mathcal{S}' = \{ \lambda 1_d \mid \lambda \in \mathbb{C} \} \).

The following result was stated as Corollary [11.14] in the Introduction.

**Proposition 3.4.** If \( \mathcal{M}_d(\mathbb{C}) \) contains an irreducible m-tuple \( u = (u_1, \ldots, u_m) \) of spin unitaries, then every m-tuple \( v = (v_1, \ldots, v_m) \) of \( d \times d \) spin unitaries is also irreducible and \( u \approx_{\text{ord}} v \).

**Proof.** By hypothesis, there exists an irreducible m-tuple \( u = (u_1, \ldots, u_m) \) of spin unitaries. Therefore, the commutant of the operator system \( \mathcal{O}_u \) is \( \{ \lambda 1_d \mid \lambda \in \mathbb{C} \} \), implying that \( \mathcal{A}_u = \mathcal{C}^*(\mathcal{O}_u) = \mathcal{M}_d(\mathbb{C}) \).

Select any other m-tuple \( v = (v_1, \ldots, v_m) \) of \( d \times d \) spin unitaries. By Theorem \( 3.1 \) there is a unital complete order isomorphism \( \phi : \mathcal{O}_u \to \mathcal{O}_v \) in which \( \phi(u_1) = v_1 \),
for every \( j \). Let \( \psi = \Phi^{-1} \), as a ucp map \( \mathcal{O}_v \to \mathcal{O}_u \), and let \( \Phi, \Psi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C}) \) be ucp extensions of \( \phi \) and \( \psi \), respectively. Therefore, \( \Psi \circ \Phi \) is a ucp extension of \( \psi \circ \phi = \text{id}_{\mathcal{O}_u} \). By Arveson’s Boundary Theorem [3, Theorem 2.11] (see also [9], [12]) if \( \psi \circ \Phi \) is a ucp extension of \( \mathcal{O}_u \) then \( C^*(\mathcal{O}_u) \subseteq \mathcal{M}_d(\mathbb{C}) \). Hence, \( \Psi \circ \Phi = \text{id}_{\mathcal{M}_d(\mathbb{C})} \). In other words, \( \Phi \) is a unital complete positive linear map of \( \mathcal{M}_d(\mathbb{C}) \) with a completely positive inverse, which implies (by many results [20], e.g., Wigner’s Theorem) that \( \Phi \)–and, hence, \( \phi \)—is a unitary equivalence transformation \( x \mapsto w^*xw \), for some \( w \in \mathcal{U}_d \). Consequently, \( \psi \) is also an irreducible m-tuple of spin unitaries and \( u \approx v \).

\[ \text{□} \]

4. The \( C^* \)-Envelope of a Spin System

The following two consequences of Hamana’s Theorem (Theorem [1,16]) are of use to us.

**Proposition 4.1.** Suppose that \( \mathcal{R} \subseteq \mathcal{M}_d(\mathbb{C}) \) is an operator system of matrices.

1. If \( \mathcal{T} \subseteq \mathcal{M}_d(\mathbb{C}) \) is an operator system of matrices such that \( \mathcal{T} \approx_{\text{ord}} \mathcal{R} \), then \( C^*_e(\mathcal{T}) = C^*_e(\mathcal{R}) \).
2. If \( C^*(\mathcal{R}) = \mathcal{M}_d(\mathbb{C}) \), then \( C^*_e(\mathcal{R}) = \mathcal{M}_d(\mathbb{C}) \).

Note that, by the Double Commutant Theorem and Proposition 4.1, if \( \mathcal{S} \) is an irreducible spin system, then \( C^*_e(\mathcal{O}_\mathcal{S}) = \mathcal{M}_d(\mathbb{C}) \). Thus, focusing upon irreducible spin systems is important.

The next result is based on a well known construction, but we do not know of a specific reference with regards to the irreducibility of the construction, and so a (straightforward) proof is given below.

**Lemma 4.2.** If \( \mathcal{S} = \{u_1, \ldots, u_m\} \) is an irreducible spin system of \( d \times d \) unitary matrices, then

\[ \mathcal{Q} = \{u_1 \otimes I_2, u_m \otimes \sigma_X, u_m \otimes \sigma_Y, u_m \otimes \sigma_Z | j = 1, \ldots, m - 1\} \]

is an irreducible spin system in \( \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C}) = \mathcal{M}_{2d}(\mathbb{C}) \).

**Proof.** Consider the set \( \mathcal{Q} \subset \mathcal{U}_{2d} \) defined by

\[ \mathcal{Q} = \{u_1 \otimes I_2, u_m \otimes \sigma_X, u_m \otimes \sigma_Y, u_m \otimes \sigma_Z | j = 1, \ldots, m - 1\} \]

Each element of \( \mathcal{Q} \) is a selfadjoint unitary and any two distinct elements anticommute. Hence, \( \mathcal{Q} \) is a spin system. We now show that \( \mathcal{Q} \) is an irreducible spin system.

Because every element of \( \mathcal{Q} \) is selfadjoint, a matrix \( z \) commutes with each element of \( \mathcal{Q} \) if and only if \( z^* \) commutes with each element of \( \mathcal{Q} \). Therefore, the space of matrices commuting with the elements of \( \mathcal{Q} \) is spanned by selfadjoint matrices. Suppose, therefore, that a selfadjoint matrix \( z \in \mathcal{M}_{2d}(\mathbb{C}) \) commutes with every element of \( \mathcal{Q} \). Identifying \( z \in \mathcal{M}_{2d}(\mathbb{C}) \) with \( \mathcal{M}_2(\mathcal{M}_d(\mathbb{C})) \), the selfadjoint matrix \( z \) can written as

\[ z = \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix} \]

for some \( a_{11}, a_{12}, a_{22} \in \mathcal{M}_d(\mathbb{C}) \). Likewise, \( u_j \otimes I_2 \) and \( u_m \otimes \sigma_Z \), for \( j = 1, \ldots, m - 1 \), are given by

\[ \begin{bmatrix} u_j & 0 \\ 0 & u_j \end{bmatrix} \text{ and } \begin{bmatrix} u_m & 0 \\ 0 & -u_m \end{bmatrix}. \]
The commutation relations \( z(u_j \otimes 1_2) = (u_j \otimes 1_2)z \) and \( z(u_m \otimes \sigma_Z) = (u_m \otimes \sigma_Z)z \) yield

\[
\begin{bmatrix}
    a_{11} u_j & a_{12} u_j \\
    a_{12}^* u_j & a_{22} u_j
\end{bmatrix}
= \begin{bmatrix}
    u_j a_{11} & u_j a_{12} \\
    u_j a_{12}^* & u_j a_{22}
\end{bmatrix}
\]

for \( j = 1, \ldots, m - 1 \), and

\[
\begin{bmatrix}
    a_{11} u_m & -a_{12} u_m \\
    a_{12}^* u_m & -a_{22} u_m
\end{bmatrix}
= \begin{bmatrix}
    u_m a_{11} & u_m a_{12} \\
    -u_m a_{12}^* & -u_m a_{22}
\end{bmatrix}.
\]

Therefore, \( a_{12} \) commutes with \( u_j \) for \( j = 1, \ldots, m - 1 \). Furthermore, \( a_{11} \) and \( a_{22} \) commute with every element of \( S \), which implies that \( a_{jj} = \alpha_{jj} 1_{2n} \) for some \( \alpha_{jj} \in \mathbb{R} \). Hence, \( z \) has the form

\[
z = \begin{bmatrix}
    \alpha_{11} 1_{2n} & a_{12} \\
    a_{12}^* & \alpha_{22} 1_{2n}
\end{bmatrix}.
\]

Using the commutation relation \( z(u_m \otimes \sigma_X) = (u_m \otimes \sigma_X)z \) and the identification \( u_m \otimes \sigma_X = \begin{bmatrix} 0 & u_m \\ u_m & 0 \end{bmatrix} \), we obtain

\[
\begin{bmatrix}
    u_m a_{12} & \alpha_{22} u_m \\
    \alpha_{11} u_m & u_m a_{12}
\end{bmatrix}
= \begin{bmatrix}
    a_{12} u_m & \alpha_{11} u_m \\
    \alpha_{22} u_m & a_{12}^* u_m
\end{bmatrix},
\]

which yields \( \alpha_{11} = \alpha_{22} \) and \( (a_{12} + a_{12}^*) u_m = u_m (a_{12} + a_{12}^*) \). Because \( a_{12} + a_{12}^* \) also commutes with every \( u_j \) for \( j = 1, \ldots, m - 1 \), we conclude that \( a_{12} + a_{12}^* = \lambda 1_d \), for some \( \lambda \in \mathbb{R} \). In setting \( \alpha = \alpha_{11} \), the commutation relation \( z(u_m \otimes \sigma_Y) = (u_m \otimes \sigma_Y)z \) yields

\[
\begin{bmatrix}
    -ia_{12} u_m & \alpha u_m \\
    \alpha u_m & -ia_{12}^* u_m
\end{bmatrix}
= \begin{bmatrix}
    -iu_m a_{12} & -\alpha u_m \\
    \alpha u_m & iu_m a_{12}
\end{bmatrix}.
\]

Thus, \( a_{12} - a_{12}^* \) commutes with \( u_m \) and with each \( u_j \) for \( j = 1, \ldots, m - 1 \), we conclude that \( a_{12} - a_{12}^* = \mu 1_d \) for some scalar \( \mu \in \mathbb{R} \), and so \( a_{12} = (\lambda + i\mu) 1_d \), which is a scalar multiple of the identity matrix. Therefore, \( a_{12} \) commutes with every matrix. However, because \( a_{12} \) both commutes and anticommutes with \( u_m \), this scalar must be zero. Hence, \( z \) is a scalar multiple of the identity matrix, which proves that \( \Omega \) is an irreducible spin system. \( \square \)

**Theorem 4.3.** \( C^*_e(\mathbb{O}_{\text{spin}(2k)}) = C^*_e(\mathbb{O}_{\text{spin}(2k+1)}) = \bigotimes_1^k M_2(\mathbb{C}) \), for every \( k \in \mathbb{C} \).

**Proof.** The \( C^* \)-algebra generated by any spin pair is the simple algebra \( M_2(\mathbb{C}) \) (Corollary 4.3), while the operator system spanned by the Pauli matrices is \( M_2(\mathbb{C}) \). As any spin pair or triple is completely order equivalent to the pair \((\sigma_X, \sigma_Y)\) or the triple \((\sigma_X, \sigma_Y, \sigma_Z)\) (by Theorem 1.12), we obtain (from Proposition 4.1) the following algebra equalities:

\[ C^*_e(\mathbb{O}_{\text{spin}(2)}) = C^*_e(\mathbb{O}_{\text{spin}(3)}) = M_2(\mathbb{C}). \]

Using the irreducible spin system \( \Omega_1 = \{\sigma_X, \sigma_Y, \sigma_Z\} \subset M_2(\mathbb{C}) \), the construction of the spin system in Lemma 1.2 produces the following irreducible spin system \( \Omega_2 \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \) of 5 elements:

\[
\Omega_2 = \{\sigma_X \otimes 1, \sigma_Y \otimes 1, \sigma_Z \otimes \sigma_X, \sigma_X \otimes \sigma_Y, \sigma_Z \otimes \sigma_Y, \sigma_Z \otimes \sigma_Z\} = \Omega_{2,-} \cup \{\sigma_Z \otimes \sigma_Z\},
\]

where \( \Omega_{2,-} = \Omega_2 \setminus \{\sigma_Z \otimes \sigma_Z\} \).
Another iteration of the construction in Lemma 4.2 yields an irreducible spin system $Q_3$ of 7 elements:

$$Q_3 = Q_{3-} \cup \{\sigma_X \otimes \sigma_Z \otimes \sigma_Z\},$$

where

$$Q_{3-} = \{\sigma_X \otimes 1 \otimes 1, \sigma_Y \otimes 1 \otimes 1, \sigma_Z \otimes \sigma_X \otimes 1, \sigma_Z \otimes \sigma_Y \otimes 1, \sigma_Z \otimes \sigma_Z \otimes \sigma_X, \sigma_Z \otimes \sigma_Z \otimes \sigma_Y\}.$$ 

In general,

$$Q_k = Q_{k-} \cup \left\{ \bigotimes_{i=1}^k \sigma_Z \right\}.$$ 

The key point to observe is that the elements of $Q_{k-}$ consist of $k$ pairs such that, in the order given by the iterative construction, the product of each pair is a product tensor in which all factors are the identity matrix and one tensor factor is $\sigma_X \sigma_Y$. More specifically, if

$$Q_{k-} = \{w_1, w_2, w_3, w_4, \ldots, w_{2k-1}, w_{2k}\} \subset \bigotimes_{i=1}^k M_2(\mathbb{C}),$$

then

$$w_1w_2 = (\sigma_X \sigma_Y) \otimes 1 \otimes 1 \cdots \otimes 1 = i (\sigma_Z \otimes 1 \otimes 1 \cdots \otimes 1)$$

$$w_3w_4 = 1 \otimes (\sigma_X \sigma_Y) \otimes 1 \cdots \otimes 1 = i (1 \otimes \sigma_Z \otimes 1 \cdots \otimes 1)$$

$$\vdots$$

$$w_{2k-1}w_{2k} = 1 \otimes 1 \otimes 1 \cdots (\sigma_X \sigma_Y) = i (1 \otimes 1 \otimes 1 \cdots \otimes \sigma_Z).$$

Hence,

$$\bigotimes_{i=1}^k \sigma_Z = i^{-k} \prod_{j=1}^k w_{2j-1}w_{2j} \in \text{Alg} (Q_{k-}),$$

which shows that

$$C^*(\mathcal{O}_{Q_{k-}}) = C^*(\mathcal{O}_{Q_k}),$$

for every $k \in \mathbb{N}$. Therefore, because $Q_k$ is an irreducible spin system,

$$C^*(\mathcal{O}_{Q_{k-}}) = C^*(\mathcal{O}_{Q_k}) = \bigotimes_{i=1}^k M_2(\mathbb{C}).$$

Therefore, the $C^*$-envelopes of $\mathcal{O}_{Q_{k-}}$ and $\mathcal{O}_{Q_k}$ are also $\bigotimes_{i=1}^k M_2(\mathbb{C})$.

Therefore, by replacing $\mathcal{O}_{\text{spin}(2k)}$ with $\mathcal{O}_{Q_{k-}}$ and $\mathcal{O}_{\text{spin}(2k+1)}$ with $\mathcal{O}_{Q_k}$ (by Theorem 1.12), we obtain

$$C^*_{\mathbb{C}}(\mathcal{O}_{\text{spin}(2k)}) = C^*_{\mathbb{C}}(\mathcal{O}_{\text{spin}(2k+1)}) = \bigotimes_{i=1}^k M_2(\mathbb{C}).$$

$\square$
5. Free Spectrahedra and Dilations

In this partly expository section, we apply the notions developed in this paper to examine some known results on free spectrahedra and matrix ranges, and also prove a new result regarding the dilation of spin triples.

Definition 5.1. (12) Suppose that \( a = (a_1, \ldots, a_m) \) is an \( m \)-tuple of selfadjoint \( d \times d \) matrices.

1. The monic polynomial \( L_a(t_1, \ldots, t_m) = 1_d - \sum_{j=1}^m t_j a_j \), in variables \( t_1, \ldots, t_m \), evaluated at an \( m \)-tuple \( h = (h_1, \ldots, h_m) \) of \( n \times n \) selfadjoint matrices, is the selfadjoint element \( L_a(h) \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C}) \) defined by

\[
L_a(h) = 1_n \otimes 1_d - \sum_{j=1}^m h_j \otimes a_j.
\]

2. The free spectrahedron determined by \( a \) is the sequence \( \mathcal{D}_a = (\mathcal{D}_{a,n})_{n \in \mathbb{N}} \) of subsets

\[
\mathcal{D}_{a,n} = \{ h = (h_1, \ldots, h_n) \mid \text{each } h_j \in M_n(\mathbb{C})_{sa} \text{ and } L_a(h) \text{ is positive semidefinite} \}.
\]

The first result shows that the free spectrahedra determined by spin systems depends only upon the cardinality of the spin system, not upon the choice of spin unitaries.

Proposition 5.2. If \( u = (u_1, \ldots, u_m) \) and \( v = (v_1, \ldots, v_m) \) are \( m \)-tuples of spin unitaries with \( u_j \in \mathcal{U}_{d_1} \) and \( v_k \in \mathcal{U}_{d_2} \), then \( \mathcal{D}_u = \mathcal{D}_v \).

Proof. The canonical linear bases of \( \mathcal{O}_u \) and \( \mathcal{O}_v \) are, respectively, \( \{1_{d_1}, u_1, \ldots, u_m\} \) and \( \{1_{d_2}, v_1, \ldots, v_m\} \). In particular, by identifying \( M_n(\mathcal{O}_u) \) with \( M_n(\mathbb{C}) \otimes \mathcal{O}_u \), a selfadjoint matrix \( y \in M_n(\mathcal{O}_u) \) is expressed as

\[
y = b_0 \otimes 1_{d_1} + \sum_{j=1}^m b_j \otimes u_j,
\]

for some (uniquely determined) selfadjoint matrices \( b_0, b_1, \ldots, b_m \in M_n(\mathbb{C}) \). Likewise, the element

\[
\tilde{y} = b_0 \otimes 1_{d_2} + \sum_{j=1}^m b_j \otimes v_j
\]

is a selfadjoint elements of \( M_n(\mathbb{C}) \otimes \mathcal{O}_v = M_n(\mathcal{O}_v) \).

Corollary 1.13 asserts that the linear map \( \phi : \mathcal{O}_u \to \mathcal{O}_v \) in which \( \phi(1_{d_1}) = 1_{d_2} \) and \( \phi(u_j) = v_j \), for each \( j \), is a unital complete order isomorphism of \( \mathcal{O}_u \) and \( \mathcal{O}_v \). Thus, the equation \( \tilde{y} = (\text{id}_{M_n(\mathbb{C})} \otimes \phi)[y] \) shows that

\[
b_0 \otimes 1_{d_1} + \sum_{j=1}^m b_j \otimes u_j \text{ is positive semidefinite}
\]

if and only if

\[
b_0 \otimes 1_{d_2} + \sum_{j=1}^m b_j \otimes v_j \text{ is positive semidefinite}.
\]
In particular, given an \( m \)-tuples \( h \) of selfadjoint matrices \( h_j \in M_n(\mathbb{C}) \), \( L_u(h) \) is positive semidefinite if and only if \( L_v(h) \) is positive semidefinite. Hence, \( D_{u, n} = D_{v, n} \), for every \( n \in \mathbb{N} \).

In [11], the spin ball \( B^\text{spin}_m \) is defined to be the free spectrahedron determined by a spin system constructed iteratively from the Pauli matrices, as in the proof of Theorem 4.3. In light of Proposition 5.2, the spin ball can be defined unambiguously as follows.

**Definition 5.3 (Spin Ball).** The spin ball \( B^\text{spin}_m \) is the free spectrahedron \( D_u \) for any \( m \)-tuple \( u \) of spin unitaries \( u_j \in U_d \).

Free spectrahedra are easily seen to be matrix convex. Before defining matrix convexity below, note that the Cartesian product \( \prod_1^m M_n(\mathbb{C}) \) is a unital C\(^*\)-algebra, which makes the consideration of completely positive linear maps between such spaces of interest. In particular, if \( \gamma : \mathbb{C}^n \to \mathbb{C}^k \) is a linear transformation, then we have an induced completely positive linear map

\[
\Gamma : \prod_1^m M_n(\mathbb{C}) \to \prod_1^m M_k(\mathbb{C})
\]

defined by \( \Gamma(x) = \gamma^* \cdot x \cdot \gamma \), for all \( x = (x_1, \ldots, x_m) \in \prod_1^m M_n(\mathbb{C}) \), where

\[
\gamma^* \cdot x \cdot \gamma = (\gamma^* x_1 \gamma, \ldots, \gamma^* x_m \gamma).
\]

**Definition 5.4.** (II2) For a fixed \( m \in \mathbb{N} \), suppose that \( \mathcal{K} = (\mathcal{K}_n)_{n \in \mathbb{N}} \) is a sequence of subsets \( \mathcal{K}_n \subseteq \prod_1^m M_n(\mathbb{C}) \). If the sequence \( \mathcal{K} \) has the property that

\[
\sum_{t=1}^1 \gamma_t^* \cdot \Lambda \cdot \gamma_t \in \mathcal{K}_n,
\]

for all \( t \in \mathbb{N} \), all \( \Lambda \in \mathcal{K}_{n,t} \), and all linear transformations \( \gamma_t : \mathbb{C}^n \to \mathbb{C}^{n_t} \) such that

\[
\sum_{t=1}^t \gamma_t^* \gamma_t = 1_n,
\]

the \( \mathcal{K} \) is said to be matrix convex.

In addition to free spectrahedra, matrix ranges form another class of matrix convex sets.

**Definition 5.5 (Matrix Range).** If \( x = (x_1, \ldots, x_m) \) is an \( m \)-tuple of matrices \( x_j \in M_d(\mathbb{C}) \), then the matrix range of \( x \) is the sequence \( W(x) = (W^n(x))_{n \in \mathbb{N}} \) in which each \( W^n(x) \) is defined by

\[
W^n(x) = \{ \phi(x) \mid \phi : \mathcal{O}_x \to M_n(\mathbb{C}) \text{ is a ucp map} \},
\]

where \( \mathcal{O}_x \) is the operator subsystem of \( M_d(\mathbb{C}) \) generated by \( x \) and where \( \phi(x) \) is the \( m \)-tuple of elements in \( M_n \) given by

\[
\phi(x) = (\phi(x_1), \ldots, \phi(x_m)).
\]
The relevance of matrix ranges to complete order equivalence and unitary equivalence originates in the work of Arveson [3].

Theorem 5.6. (3) The following statements are equivalent for tuples $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ of matrices $x_j \in \mathcal{M}_d(\mathbb{C})$ and $y_k \in \mathcal{M}_d(\mathbb{C})$:

1. $x \simeq_{ord} y$;
2. $W(x) = W(y)$.

There are many examples of matrix ranges in the literature. The following example, which is relevant to the subject of the present paper, can be deduced as a special case of [7, Example 1].

Example 5.7. Let $u \in \mathcal{U}_d$ be non-scalar selfadjoint unitary matrices. Then, for every $n \in \mathbb{N}$, $h \in W^n(u)$ if and only if there exist $a, b \in \mathcal{M}_n(\mathbb{C})$, such that $a + b = 1_n$ and $a − b = h$.

There is also a spatial version of the matrix range.

Definition 5.8. If $x = (x_1, \ldots, x_m)$ is an $m$-tuple of matrices $x_j \in \mathcal{M}_d(\mathbb{C})$, then the spatial matrix range of $x$ is the finite sequence $W_s(x) = (W^n(x))_{n=1}^d$ in which each $W^n_s(x)$ is defined by

$$W^n_s(x) = \{\langle \gamma^* \cdot x \cdot \gamma \rangle : \mathbb{C}^n \to \mathbb{C}^d \text{ is a linear isometry} \}.$$ 

Note that $W^n_s(x) \subseteq W^n(x)$ for every $n = 1, \ldots, d$. However, the spatial matrix range lacks the strong feature of matrix convexity; indeed, $W^n_s(x)$ is the unitary orbit of $x$, and therefore fails to contain any line segments (in the classical sense) whatsoever.

In the case of $n = 1$, the spatial matrix range $W^1_s(x)$ is better known in linear algebra as the (joint) numerical range of $x = (x_1, \ldots, x_m)$. The following elementary calculation is well known to many linear algebraists.

Example 5.9. If $\sigma = (\sigma_X, \sigma_Y, \sigma_Z)$, then

1. the spatial numerical range $W_s^1(\sigma)$ is the unit Euclidean sphere in $\mathbb{R}^3$, and
2. the numerical range $W^1(\sigma)$ is the closed unit Euclidean ball in $\mathbb{R}^3$.

Proof. Every unital positive linear functional $\phi : \mathcal{M}_2(\mathbb{C}) \to \mathbb{C}$ is a convex combination of linear functionals of the form $\omega_\xi(x) = \langle x\xi, \xi \rangle$, for a unit vector $\xi \in \mathbb{C}^2$. Thus, $W^1(\sigma)$ is the convex hull of

$$W^1_s(\sigma) = \{\langle \sigma_X \xi, \xi \rangle, \langle \sigma_Y \xi, \xi \rangle, \langle \sigma_Z \xi, \xi \rangle \mid \xi \in \mathbb{C}^2, \langle \xi, \xi \rangle = 1 \}.$$ 

Thus, statement (2) follows by showing statement (1) holds. To obtain (1), note that because the inner product on $\mathbb{C}^2$ is not bilinear but sesquilinear, it is enough to compute the spatial numerical range using unit vectors of the form $\xi = (\cos \theta) e_1 + e^{i\theta}(\sin \theta) e_2$, for all $\theta, \delta \in \mathbb{R}$. Since

$$\langle \sigma_X \xi, \xi \rangle = 2\Re(e^{i\theta} \sin \theta \cos \theta) = \cos \delta \sin(2\theta),$$
$$\langle \sigma_Y \xi, \xi \rangle = 2\Re(e^{-i\theta} \sin \theta \cos \theta) = \sin \delta \sin(2\theta),$$
$$\langle \sigma_Z \xi, \xi \rangle = \cos^2 \theta - \sin^2 \theta = \cos(2\theta),$$

we obtain the spherical coordinates for the unit Euclidean sphere $S^2$ in $\mathbb{R}^3$. Hence, $W^1_s(\sigma) = S^2$.

For each $m \in \mathbb{N}$, let $\mathbb{B}_m$ denote the closed unit Euclidean ball of $\mathbb{R}^m$.  

Definition 5.10 (Max Ball). ([11 §14.2.2]) For each \( m \in \mathbb{N} \), the max ball \( \mathcal{B}_{m}^{\max} \) is the sequence \( \mathcal{B}_{m}^{\max} = (B_{m,n})_{n \in \mathbb{N}} \), whereby an \( m \)-tuple \( h = (h_{1}, \ldots, h_{n}) \) of selfadjoint \( n \times n \) matrices belongs to \( B_{m,n} \) if and only if, for every \( m \)-tuple \( a = (a_{1}, \ldots, a_{m}) \) of selfadjoint \( d \times d \) matrices and for every \( d \), the \( m \)-tuple \( h \) is necessarily an element \( \mathcal{D}_{a,n} \) if \( \mathcal{D}_{a,1} \) contains the Euclidean ball \( B_{m} \).

Combining results from [11 §14.2.2], one obtains the following criterion for membership in the max ball.

Proposition 5.11. An \( m \)-tuple \( h = (h_{1}, \ldots, h_{n}) \) of selfadjoint \( n \times n \) matrices belongs to (some element of the sequence) \( \mathcal{B}_{m}^{\max} \) if and only if

\[
1_{n} \otimes 1_{d} - \sum_{j=1}^{m} h_{j} \otimes a_{j} \text{ is positive semidefinite}
\]

for every \( m \)-tuple \( a = (a_{1}, \ldots, a_{m}) \) of selfadjoint \( d \times d \) matrices, and for every \( d \), in which

\[ W_{i}(a) \subseteq B_{m}. \]

The max ball is not, at first glance, a free spectrahedron because the defining conditions for membership in \( \mathcal{B}_{m}^{\max} \) involve, in principle, infinitely many monic polynomials \( L_{a}(t_{1}, \ldots, t_{m}) \). However, in low dimensions, the max ball is a free spectrahedron, as shown by the following theorem of Helton, Klep, McCullough, and Schweighöfer [11 Corollary 14.15]. The authors of [11] give two proofs of this result using dilation. We offer a third alternative below.

Theorem 5.12. \( \mathcal{B}_{1}^{\spin} = \mathcal{B}_{1}^{\max} \) and \( \mathcal{B}_{2}^{\spin} = \mathcal{B}_{2}^{\max} \).

Proof. By Proposition 5.2 the spin ball \( \mathcal{B}_{2}^{\spin} = \mathcal{B}_{2}^{\max} \) is the free spectrahedron determined by the Pauli matrices \( (\sigma_{X}, \sigma_{Y}) \). Because \( \sigma_{X} + i\sigma_{Y} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \), the spatial numerical range \( W_{i}(\sigma_{X}, \sigma_{Y}) \) is equal to the closed Euclidean disc \( B_{2} \) in \( \mathbb{R}^{2} \). Hence, if a selfadjoint pair \( (h_{1}, h_{2}) \in \mathcal{M}_{n}(\mathbb{C}) \times \mathcal{M}_{n}(\mathbb{C}) \) belongs to \( \mathcal{B}_{1}^{\max} \), then by definition

\[
1_{n} \otimes 1_{2} - (h_{1} \otimes \sigma_{X} + h_{2} \otimes \sigma_{Y})
\]
is positive semidefinite. Consequently, \( (h_{1}, h_{2}) \) belongs to \( \mathcal{B}_{2}^{\spin} \), showing that \( \mathcal{B}_{2}^{\max} \subseteq \mathcal{B}_{2}^{\spin} \).

Conversely, assume that a selfadjoint pair \( (h_{1}, h_{2}) \in \mathcal{M}_{n}(\mathbb{C}) \times \mathcal{M}_{n}(\mathbb{C}) \) belongs to \( \mathcal{B}_{2}^{\spin} \). Thus, the matrix in equation (5) above is positive semidefinite. Select any pair \( (a_{1}, a_{2}) \) of \( \ell \times \ell \) selfadjoint matrices in which \( W_{i}(a_{1}, a_{2}) \subseteq B_{2}. \) As the numerical radius of \( y = \frac{1}{2}(a_{1} + ia_{2}) \) is at most 1, there exists, by Ando’s Theorem [2], a positive semidefinite contraction \( b \in \mathcal{M}_{\ell}(\mathbb{C}) \) such that

\[
c = \begin{bmatrix} 1 & y \\ y^{*} & 1_{\ell} - b \end{bmatrix}
\]
is positive semidefinite. Let \( \psi : \mathcal{M}_{2}(\mathbb{C}) \rightarrow \mathcal{M}_{\ell}(\mathbb{C}) \) be the unital linear map in which \( \psi(e_{11}) = b, \psi(e_{21}) = y, \psi(e_{12}) = y^{*}, \) and \( \psi(e_{22}) = 1_{\ell} - b. \) The matrix \( c \) above is the Choi matrix for \( \psi; \) therefore, by Choi’s Criterion [16 Theorem 3.14],
ψ is completely positive. Now since \(a_1 + ia_2 = \psi(\sigma_X + i\sigma_Y)\), equating real and imaginary parts yields \(a_1 = \psi(\sigma_X)\) and \(a_2 = \psi(\sigma_Y)\). Hence, the positivity of

\[
1_n \otimes 1_2 - (h_1 \otimes \sigma_X + h_2 \otimes \sigma_Y)
\]

implies the positivity of

\[
(id_{M_n(\mathbb{C})} \otimes \psi)[1_n \otimes 1_2 - (h_1 \otimes \sigma_X + h_2 \otimes \sigma_Y)] = 1_\ell \otimes 1_2 - (h_1 \otimes a_1 + h_2 \otimes a_2),
\]

That is, \((h_1, h_2)\) belongs to \(\mathcal{B}_1^{\text{spin}}\), proving that \(\mathcal{B}_2^{\text{spin}} \subseteq \mathcal{B}_2^{\text{max}}\).

The proof that \(\mathcal{B}_1^{\text{spin}} = \mathcal{B}_1^{\text{max}}\) is more straightforward, and is left to the interested reader.

It would, of course, be very interesting to know whether Theorem 5.12 extends to higher dimensions. Some evidence that this might be so is presented in [15].

**Definition 5.13.** An \(m\)-tuple \(y = (y_1, \ldots, y_m)\) of matrices \(y_j \in M_{d_j}(\mathbb{C})\) is a dilation of an \(m\)-tuple \(x = (x_1, \ldots, x_m)\) of matrices \(x_j \in M_{d_j}(\mathbb{C})\) if there exists a linear isometry \(w : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}\) such that \(x_j = w^* y_j w\), for every \(j\).

Put differently, \(y = (y_1, \ldots, y_m)\) is a dilation of \(x = (x_1, \ldots, x_m)\) if there is a unitary \(z \in \mathcal{U}_{d_j}\) such that

\[
z^* y_j z = \begin{bmatrix} x_j & \ast \\ \ast & \ast \end{bmatrix},
\]

for every \(j\).

A reinterpretation of Corollary [13] leads to the following dilation result for triples of spin unitaries.

**Proposition 5.14.** For every spin triple \(u, v, w \in \mathcal{U}_{d_3}\), there exists a \(k \in \mathbb{N}\) such that \((\sigma_X \otimes 1_k, \sigma_Y \otimes 1_k, \sigma_Z \otimes 1_k)\) is a dilation of \((u, v, w)\), and there exists a \(\ell \in \mathbb{N}\) such that \((u \otimes 1_\ell, v \otimes 1_\ell, w \otimes 1_\ell)\) is a dilation of \((\sigma_X, \sigma_Y, \sigma_Z)\).

**Proof.** By Corollary [13] \((u, v, w) \approx_{\text{ord}} (\sigma_X, \sigma_Y, \sigma_Z)\). Therefore, the triples \((u, v, w)\) and \((\sigma_X, \sigma_Y, \sigma_Z)\) have identical matrix ranges, by Theorem [6]. In particular,

\[
(u, v, w) \in W^d(\sigma_X, \sigma_Y, \sigma_Z).
\]

The operator system generated by the Pauli matrices is the \(C^*\)-algebra \(M_2(\mathbb{C})\); therefore, the inclusion above indicates that \(u = \phi(\sigma_X), v = \phi(\sigma_Y),\) and \(w = \phi(\sigma_Z)\), for some unitary completely positive linear map \(\phi : M_2(\mathbb{C}) \to M_d(\mathbb{C})\). By the Stinespring Theorem [16], \(\phi\) has the form \(\phi(y) = \gamma^* \pi(y) \gamma\), for some unital representation \(\pi\) of \(M_2(\mathbb{C})\) on which the representing Hilbert space has finite dimension. In other words, \(\pi(y)\) is a dilation of \(\phi(y)\), for every \(y \in M_2(\mathbb{C})\). Because every representation of a full matrix algebra is unitarily equivalent to a direct sum of the identity representation, we may assume that \(\gamma\) and \(\pi\) are so chosen so that

\[
\pi(y) = \bigoplus_{i=1}^k y_i = y \otimes 1_k,
\]

thereby implying that \((\sigma_X \otimes 1_k, \sigma_Y \otimes 1_k, \sigma_Z \otimes 1_k)\) is a dilation of \((u, v, w)\).

The second statement is argued in the same manner by interchanging the roles of the spin triples.
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