hankl: A lightweight Python implementation of the FFTLog algorithm for Cosmology

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Abstract

We introduce hankl, a lightweight Python implementation of the FFTLog algorithm for Cosmology. The FFTLog algorithm is an extension of the Fast Fourier Transform (FFT) for logarithmically spaced periodic sequences. It can be used to efficiently compute Hankel transformations, which are paramount for many modern cosmological analyses that are based on the power spectrum or the 2-point correlation function multipoles. The code is well-tested, open source, and publicly available at https://github.com/minaskar/hankl.

1 Introduction

The Hankel transform (also known as the Fourier-Bessel transform) is an integral transformation whose kernel is a Bessel function. The Hankel transform appears very often in physical problems with spherical or cylindrical symmetry as it emerges when one writes the usual Fourier transform in spherical coordinates. The Hankel transform finds application in a wide range of scientific fields, namely cosmology, astrophysics, geophysics, and fluid mechanics.

As an example, in modern cosmology, the large-scale clustering of galaxies in the observable universe is often described by means of the configuration-space 2-point correlation function and its Fourier space counterpart, the power spectrum (Peebles, 2020). Due to the statistical isotropy of the universe these two quantities are related by a Hankel transformation. In the 3D case, the pair is related by a sine transform which can be considered as a special case of the Hankel transform. The ability to perform such transformations in a fast and accurate manner is of paramount importance for studies of the large-scale structure of the universe.

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However, the implementation of the Hankel transform poses some serious numerical challenges. Most importantly, the Bessel function kernel is a highly oscillatory function and any naive implementation of the quadrature numerical integration methods could lead to grossly inaccurate results. To successfully overcome this issue, Talman (1978), and later Hamilton (2000), introduced the FFTLog algorithm, which can be thought of as the Fast Fourier Transform of a logarithmically spaced periodic sequence.

The Hankel transform pair, most commonly appearing in the cosmological literature, has the form (Hamilton, 2000)

\[ f(k) = \int_0^\infty F(r)(kr)q J_\mu(kr) rdr, \]  
\[ F(r) = \int_0^\infty f(k)(kr)^{-q} J_\mu(kr) rdk. \]  

If the substitution \( F(r) = g(r) r^{-q} \) and \( f(k) = G(k) y^q \) is made, then the aforementioned transform pair reduces to the standard form of the Hankel transform

\[ g(k) = \int_0^\infty G(r) J_\mu(kr) rdr, \]  
\[ G(r) = \int_0^\infty f(k) J_\mu(kr) rdk. \]  

In the continuous case, the transform pairs of equations 1 and 2 are equivalent. The transformations are different when they are made discrete in periodic sequences. The FFTLog algorithm computes such discrete transforms with arbitrary power–law bias \((kr)^\pm q\) of the form of equations 1. In terms of the \(\ln r\) and \(\ln k\) variables, the above integral transformations become a convolution which can be evaluated as a multiplication in Fourier space. This is the main idea behind Fast Hankel Transforms (Siegman, 1977) and FFTLog (Talman, 1978).

## 2 Implementation

\texttt{hankl} is a lightweight \texttt{Python} implementation of the FFTLog algorithm that particularly focuses on cosmological applications. \texttt{hankl} relies on the \texttt{NumPy} and \texttt{SciPy} libraries in order to provide fast and accurate Hankel transforms with minimal com-
putational overhead. \texttt{hankl} is well suited for scientific applications that require a simple and modular \texttt{Python} interface along with C-level performance.

\texttt{hankl} is designed to perform general Hankel transformations of the form seen in equations 1 as well as the more cosmologically relevant transformations between the power spectrum and the 2-point correlation function, such as

$$\xi^{(n)}(r) = i^\ell \int_0^\infty k^2 \frac{dk}{2\pi^2} (kr)^{-n} P^{(n)}(k) j_\ell(k s),$$  \tag{3}$$

and the inverse

$$P^{(n)}(k) = 4\pi (-i)^\ell \int_0^\infty r^2 dr (kr)^n \xi^{(n)}(r) j_\ell(k r).$$  \tag{4}$$

One example, with analytically tractable Hankel transformation, is shown in figure 1. The original function is

$$f(r) = r^{\mu+1} \exp\left(-\frac{r^2}{2}\right),$$  \tag{5}$$

with its Hankel transform given by

$$g(k) = k^{\mu+1} \exp\left(-\frac{k^2}{2}\right).$$  \tag{6}$$

The figure shows the case of $\mu = 0$. The absolute difference between calculated and analytic results is less than $4 \times 10^{-5}$ and it can be reduced further, according to the needs of the analysis, by extending the integration range and increasing the sampling resolution.

As discussed in the introduction, a very common application of the Hankel transform in cosmology is the transformation of the power spectrum multipoles $P_\ell(k)$ from Fourier space to 2-point correlation function multipoles $\xi_\ell(s)$ in configuration space. To demonstrate this transformation using \texttt{hankl} we transformed the monopole $P_0(k)$ and the quadrupole $P_2(k)$ to their respective correlation function multipoles $\xi_0$ and $\xi_2$. The power spectrum multipoles are computed using the Legendre expansion formula

$$P_\ell(k) = \frac{2\ell + 1}{2} \int_{-1}^1 d\mu P(k, \mu) \mathcal{L}_\ell(\mu),$$  \tag{7}$$

where $\mathcal{L}_\ell$ is the Legendre polynomial of order $\ell$, and $\mu$ is the cosine of the angle.
Figure 1: The plot shows the Hankel transform $g(k)$ (middle panel) of $f(r)$ (top panel) as well as the absolute difference (bottom panel) of the result estimated with FFTLog (orange) and the exact analytical result $g_0(k)$ (blue).
between the line-of-sight vector and the Fourier mode $k$. The 2D power spectrum is computed using the Kaiser formula (Kaiser, 1987)

$$P(k, \mu) = (b + f \mu^2)^2 P_m(k), \quad (8)$$

where $b$ is the linear bias, $f$ is the logarithmic growth rate, and $P_m(k)$ is the linear real-space power spectrum computed using CLASS (Blas et al., 2011). Figure 2 shows the input power spectra and the resulted correlation function multipoles.

## 3 Conclusion

The aim of this project was to develop a tool that could compute the Hankel transform efficiently and accurately as it is required by modern cosmological analyses. To this end we introduced hankl, a lightweight Python implementation of the FFT-Log algorithm, that combines a modular and user-friendly Python interface with
C-level performance. We hope that hankl will prove useful to the cosmological community by facilitating modern analyses. The code is publicly available at https://github.com/minaskar/hankl with detailed documentation and examples that can be found at https://hankl.readthedocs.io.

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