Non-Markovianity degree for random unitary evolution

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We analyze the non-Markovianity degree for random unitary evolution of d-level quantum systems. It is shown how non-Markovianity degree is characterized in terms of local decoherence rates. In particular we derive a sufficient condition for vanishing of the backflow of information.

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Recently, much effort was devoted to the analysis of non-Markovian quantum evolution \[1\]–\[26\] (see also \[27\] for the recent review). The two most popular approaches are based on divisibility of the corresponding dynamical map \[3\]–\[7\] and distinguishability of states \[8\]. Other approaches use quantum entanglement \[9\], quantum Fisher information \[10\], fidelity \[11\], mutual information \[11\]–\[12\], channel capacity \[13\]–\[23\], geometry of the set of accessible states \[14\], non-Markovianity degree \[24\] and the quantum regression theorem \[24\]–\[25\]. There is also an alternative approach based on divisibility of the corresponding dynamical map \[26\] but we do not consider it in this paper.

In what follows we analyze non-Markovianity degree of random unitary quantum evolution of d-level quantum system. Let us briefly recall the notion of non-Markovianity degree \[20\]: if Λt is a dynamical map then it is called k-divisible iff the corresponding propagator \(V_{t,s} \equiv \Lambda_t \Lambda_s\) defined \(V_{t,s} \equiv \Lambda_t \Lambda_s\) (\(t \geq s\)) defines \(k\)-positive map \[28\]. Hence, if the system Hilbert space is \(d\)-dimensional, then \(k \in \{1,2,\ldots,d\}\). Map which is \(d\)-divisible we call CP-divisible (the corresponding propagator is completely positive (CP)) and 1-divisible we call P-divisible (the corresponding propagator is positive (P)). The evolution is Markovian iff the corresponding dynamical map is CP-divisible. Note that if \(\Lambda_t\) is \(k\)-divisible, then is necessarily \(l\)-divisible for all \(l < k\). Maps which are even not \(P\)-divisible we call essentially non-Markovian. Having defined the notion of \(k\)-divisibility one assign the non-Markovianity degree as follows: if \(\Lambda_t\) is \(k\)-divisible (but not \((k+1)\)-divisible), then its non-Markovianity degree \(\text{NMD}\left[\Lambda_t\right] = d - k\). Clearly, if \(\Lambda_t\) is Markovian, then \(\text{NMD}\left[\Lambda_t\right] = 0\) and if \(\Lambda_t\) is essentially non-Markovian, then \(\text{NMD}\left[\Lambda_t\right] = d\).

Let us recall that a quantum channel \(\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\) is called random unitary if its Kraus representation is given by

\[
\mathcal{E}(X) = \sum_k p_k U_k X U_k^\dagger, \quad (1)
\]

where \(U_k\) is a collection of unitary operators and \(p_k\) stands for a probability distribution. The characteristic feature of such channels is unitarity, that is, \(\mathcal{E}(1) = 1\). Actually, for qubits (\(\dim \mathcal{H} = 2\)), it turns out \[29\] that any unital channel is random unitary. However, for higher level systems it is no longer true. A random unitary dynamics is represented by a dynamical map \(\Lambda_t\) such that for all \(t > 0\) the channel \(\Lambda_t\) is random unitary.

Consider the following set of unitary generalized spin (or Weyl) operators in \(\mathbb{C}^d\) defined by

\[
U_{kl} = \sum_{k,l=0}^{d-1} \omega^{kl}|m\rangle\langle m + l|, \quad (2)
\]

with \(\omega = e^{2\pi i/d}\). They satisfy well known relations

\[
U_{kl}U_{rs} = \omega^{ks}U_{k+r,l+s}, \quad U_{kl}^\dagger = \omega^{kl}U_{-k,-l}. \quad (3)
\]

Introducing a single index \(\alpha \equiv (m,n)\) via \(\alpha = md + n\) \((\alpha = 0, \ldots, d^2 - 1)\). One has \(U_0 = \mathbb{I}\) and \(\text{Tr}[U_{\alpha\beta}U_{\alpha'}^\dagger] = d\delta_{\alpha\beta}\) for \(\alpha, \beta = 0,1,\ldots,d^2-1\). In this paper we consider a random unitary evolution defined by the following dynamical map

\[
\Lambda_t(X) = \sum_{\alpha=0}^{d^2-1} p_\alpha(t) U_{\alpha} X U_{\alpha}^\dagger, \quad (4)
\]

with time-dependent probability distribution \(p_\alpha(t)\) satisfying \(p_0(0) = 1\). Assuming time-local Master Equation

\[
\dot{\Lambda}_t = L_t\Lambda_t, \quad (5)
\]

it is well known that \(\Lambda_t\) is CP-divisible iff \(L_t\) has the standard Lindblad form for all \(t \geq 0\). To find the time-local generator \(L_t\) let us observe that

\[
\Lambda_t(U_\alpha) = \lambda_\alpha(t) U_\alpha, \quad (6)
\]

where the eigenvalues \(\lambda_\alpha(t)\) read as follows

\[
\lambda_\alpha(t) = \sum_{\alpha,\beta=0}^{d^2-1} H_{\alpha\beta} p_\beta(t), \quad (7)
\]

with \(H\) being \(d^2 \times d^2\) Hadamard matrix defined by

\[
H_{ij,kl} = \omega^{-i l + j k}. \quad (8)
\]

This definition implies that \(H_{\alpha\beta}\) is a Hermitian matrix. Simple algebra gives

\[
L_t(X) = \sum_{k=1}^{d^2-1} \gamma_k(t) [U_k X U_k^\dagger - X], \quad (8)
\]
where the local decoherence rates read
\[ \gamma_\alpha(t) = \frac{1}{d^2} \sum_{\beta=0}^{d^2-1} H_{\alpha\beta} \mu_\beta(t) , \] 
with \( c > 0 \). One finds \( p_3(t) = 0 \) and
\[ p_1(t) = p_2(t) = \frac{1}{4} (1 - e^{-ct}) , \]
and hence the corresponding dynamical map reads
\[ \Lambda_t(\rho) = \frac{1 + e^{-ct}}{2} \rho + \frac{1 - e^{-ct}}{4} (\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2) . \] 
Interestingly \( \Lambda_t \) is a convex combination of two Markovian semigroups \( \Lambda_t^{(1)} \) and \( \Lambda_t^{(2)} \) generated by
\[ L_t^{(k)}(\rho) = \frac{c}{2} [\sigma_k \rho \sigma_k - \rho] ; \ k = 1, 2. \]
One finds \( \Lambda_t = \frac{1}{t} (\Lambda_t^{(1)} + \Lambda_t^{(2)}) \).

Example 2 This construction may be easily generalized for \( d = 3 \). Let us assume that
\[ \gamma_k(t) = \frac{c}{3} , \] 
for \( k \neq 4, 8 \).

Note, that \([U_4, U_8] = 0 \) (see Appendix for the list of \( U_k \)). We look for \( \gamma(t) := \gamma_4(t) = \gamma_8(t) \) such that \( p_4(t) = p_8(t) = 0 \). One easily finds
\[ \gamma(t) = - \frac{2c}{3} e^{2ct} - e^{-ct} , \]
which proves that \( \gamma(t) < 0 \) for \( t > 0 \). Note that \( p_k(t) = p(t) \) (\( k \neq 4, 8 \)) with
\[ p(t) = \frac{1}{9} (1 - e^{-ct/3}) . \]

Similarly as for \( d = 2 \) this evolution may be represented as a convex combination of three Markovian semigroups \( \Lambda_t^{(1)}, \Lambda_t^{(2)}, \Lambda_t^{(3)} \) generated by
\[ L_t^{(1)}(\rho) = c[U_1 \rho U_1^+, U_2 \rho U_2^+ - 2\rho] , \]
\[ L_t^{(2)}(\rho) = c[U_3 \rho U_3^+, U_6 \rho U_6^+ - 2\rho] , \]
\[ L_t^{(3)}(\rho) = c[U_5 \rho U_5^+, U_7 \rho U_7^+ - 2\rho] . \]

Note, that \([U_1, U_2] = [U_3, U_6] = [U_5, U_7] = 0 \). One finds \( \Lambda_t = \frac{1}{t} (\Lambda_t^{(1)} + \Lambda_t^{(2)} + \Lambda_t^{(3)}) \). Again, \( \Gamma_4(t) = \Gamma_8(t) < 0 \) but the evolution \( \Lambda_t \) is well defined. It is clear that one may generalize this example for arbitrary \( d \).

Let us observe that \( L_t \) may be rewritten as follows
\[ L_t(X) = \Phi_t(X) + 2\gamma_0(t)X , \] 
where the map \( \Phi_t \) is defined via
\[ \Phi_t(X) = \sum_{k=1}^{d^2-1} \gamma_k(t) U_k X U_k^+ - \gamma_0(t) U_0 X U_0^+ , \] 
and
\[ \gamma_0(t) = - \sum_{k=1}^{d^2-1} \gamma_k(t) . \]
Now, the corresponding solution $V_{t,s} = \exp[\int_s^t L_r dr]$ reads
\[
V_{t,s} = v(t; s) \exp \left[ \int_s^t \Phi_r dr \right],
\]
where the scaling factor $v(t; s)$ is given by
\[
v(t; s) = \exp \left( 2 \int_s^t \gamma_0(\tau) d\tau \right).
\]
It is therefore clear that if the map $\Phi_t$ is $k$-positive for all $t \geq 0$, then $\Lambda_t$ is $k$-divisible.

To check for $k$-divisibility we shall use the following result from \cite{12}: let $\Phi(X) = \sum_{a=0}^{d-1} a \alpha U \alpha U \alpha^\dagger$ with $U \alpha$ being Weyl unitary operators and real parameters $\alpha$. Clearly, if $\alpha \geq 0$, then $\Phi$ is CP. Suppose now that some $\alpha$ are negative, that is,
\[
\Phi(X) = \sum_{i=1}^M b_i U_i X U_i^\dagger - \sum_{j=1}^N c_j U_j X U_j^\dagger,
\]
with $M + N = d^2$ and $b_i, c_j \geq 0$ (a set $\{U_i, U_j\}$ defines a permutation of $\{U_\alpha\}$). It means that $\Phi$ is a difference of two CP maps. Let $k$ be a positive integer such that $kN < d$. One proves \cite{12} that if
\[
b_i \geq \frac{k}{d - kN} \sum_{j=1}^N c_j; \quad i = 1, \ldots, M,
\]
then $\Phi$ is $k$-positive. Moreover, if \cite{24} is violated for at least one $i \in \{1, \ldots, M\}$, then $\Phi$ is not $(k+1)$-positive. Hence, conditions \cite{24} are sufficient for $k$-positivity and necessary for $(k+1)$-positivity.

Note, that if $k = 1$, then $N \leq d - 1$ and hence at each moment of time there are at most $d - 1$ negative rates $\gamma_i(t)$. Let $N = d - 1$ and suppose, that $\gamma_1(t), \ldots, \gamma_{d-1}(t) < 0$. Formula \cite{24} implies
\[
\gamma_k(t) \geq |\gamma_1(t)| + \ldots + |\gamma_{d-1}(t)|,
\]
or equivalently
\[
\gamma_k(t) + \gamma_1(t) + \ldots + \gamma_{d-1}(t) \geq 0,
\]
for $k = d, \ldots, d^2 - 1$. Replacing $\{\gamma_1(t), \ldots, \gamma_{d-1}(t)\}$ by an arbitrary set $\{\gamma_i(t), \ldots, \gamma_{d-1}(t)\}$ one finds that if for any $d$-tuple $\{i_1, \ldots, i_d\} \subset \{1, 2, \ldots, d^2 - 1\}$ the following condition is satisfied
\[
\gamma_{i_1}(t) + \ldots + \gamma_{i_d}(t) \geq 0,
\]
for all $t \geq 0$, then $\Lambda_t$ is $P$-divisible.

**Remark 1** It is easy to show that random unitary evolution is $P$-divisible if it satisfies the well known BLP condition \cite{3}:
\[
\frac{d}{dt} \|\Lambda_t(\rho_1 - \rho_2)\|_{tr} \leq 0,
\]
for any pair of initial states $\rho_1$ and $\rho_2$. Hence, \cite{27} implies \cite{25}.

**Remark 2** Interestingly, if the random unitary evolution is $P$-divisible, then
\[
\frac{d}{dt} S(\Lambda_t(\rho)) \geq 0,
\]
where $S$ denotes the von Neumann entropy. It shows that whenever the inequality \cite{24} is violated the evolution is essentially non-Markovian.

**Remark 3** Authors of \cite{14} introduced the geometric measure of non-Markovianity via
\[
\mathcal{N}[\Lambda_t] = \frac{1}{V(0)} \int dV(t) \frac{dV(t)}{dt} dt,
\]
where $V(t)$ denotes the volume of admissible states at time $t$. It is clear that for Markovian evolution one has $\frac{d}{dt} V(t) \leq 0$. Note, that
\[
\sum_{k=1}^{d^2-1} \gamma_k(t) = -\gamma_0(t) \geq 0,
\]
guaranties $\mathcal{N}[\Lambda_t] = 0$. The geometric condition \cite{31} is much weaker than condition for $P$-divisibility \cite{27}.

**Example 3** For $d = 2$ conditions \cite{24} give
\[
\gamma_1(t) + \gamma_2(t) \geq 0, \gamma_1(t) + \gamma_3(t) \geq 0, \gamma_2(t) + \gamma_3(t) \geq 0.
\]
Actually, it was shown \cite{34} that these conditions are also necessary for $P$-divisibility. Note, that $\gamma_k(t)$ defined in \cite{17} satisfy these conditions and hence the corresponding dynamics is $P$-divisible (but not CP-divisible since $\gamma_3(t) < 0$).

**Example 4** For $d = 3$ conditions \cite{24} give
\[
\gamma_{i_1}(t) + \gamma_{i_2}(t) + \gamma_{i_3}(t) \geq 0,
\]
for all triples $\{i_1, i_2, i_3\} \subset \{1, \ldots, 8\}$. Conditions \cite{25} are sufficient (but not necessary) for $P$-divisibility. For $k = 2$ one has $N \leq 1$ and hence taking $N = 1$ the formula \cite{24} implies: if
\[
\gamma_{i_1}(t) + 2\gamma_{i_2}(t) \geq 0,
\]
for all pairs $\{i_1, i_2\} \subset \{1, \ldots, 8\}$, then the evolution is $2$-divisible. Note, that conditions \cite{25} are sufficient for $P$-divisibility and necessary for $2$-divisibility whereas \cite{25} are sufficient for $2$-divisibility. It is clear that \cite{25} are much stronger than \cite{24}. Hence, if all $\gamma_k(t) \geq 0$ the evolution is Markovian and $\text{NMD}[\Lambda_t] = 0$. If $\gamma_k(t) \geq 0$ but condition \cite{25} is satisfied then $\text{NMD}[\Lambda_t] = 1$, that is, the evolution is non-Markovian but still $2$-divisible. Finally, if \cite{24} is violated but \cite{25} is satisfied then $\text{NMD}[\Lambda_t] = 2$, that is, the evolution is non-Markovian but still $P$-divisible. However, the violation \cite{24} does not necessarily mean that $\Lambda_t$ is essentially non-Markovian. Actually, we conjecture that this evolution is $P$-divisible.

To summarize: we derived a hierarchy of conditions which guarantee $k$-divisibility of the random unitary evolution of $d$-level quantum system. It is shown how these conditions are related to well known BLP condition \cite{3} and the geometric condition \cite{14}. 
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Appendix

Weyl matrices for $d = 3$: $U_0 = I_3$ and

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix},$$

$U_5 = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}$, $U_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$, $U_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}$, $U_8 = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & 0 & \omega \end{pmatrix}$,

with $\omega = e^{2\pi i/3}$ and $\omega^2 = \omega^* = e^{-2\pi i/3}$.

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