Analytical Results to Fuel-optimal Spacecraft Formation Maneuver

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Abstract: In this study, analytical results are obtained for fuel-optimal spacecraft formation maneuver, including initialization and reconfiguration. The method consists of two stages. The first is parameterization of the problem and a new form of the problem is developed for simplicity. Second, the out-of-plane and in-plane maneuver are studied with the help of a useful inequality introduced and proved, the lower bound of the fuel consumption and the corresponding constraints of the control forces are derived and obtained. The result shows that the minimum total fuel consumption depends on the relative size parameters, and that the existence of optimal control algorithms depends upon the initial conditions. Numerical simulation of application proved the validity and efficiency of the proposed method, which can also be utilized in the maneuver of PCF.

1. Introduction

Design of fuel saving control algorithms for spacecraft formation flying maneuver, including initialization and reconfiguration¹¹,¹²,¹³ has attracted worldwide attention recently. This research falls into two categories according to the different models used for the design of the algorithms: linear model or orbit element differences.

The design of the fuel saving control algorithms using linear model has appeared in many literatures, such as Hill’s equations⁴, also known as Clohessy-Wiltshire equations⁵ in circular orbits, Lawden equations⁶ and Tschauner-Hempel equations⁷ in elliptic Keplerian orbits. Approaches used for optimization can be summarized as linear programming⁸, minimum sliding mode error feedback controller⁹, hybrid multi-agent optimization architecture and genetic algorithm¹⁰, particle swarm optimization¹¹ niched evolutionary algorithm¹², Hamilton-Jacobian-Bellman optimality¹³,¹⁴, genetic algorithm and primer vector theory¹⁵,¹⁶, hybrid linear/nonlinear controller¹⁷, analytical solutions¹⁸, pseudo spectral homotopy algorithm¹⁹, quadratic homotopy method²⁰, indirect method and successive convex programming²¹ and the use of nonlinear trajectory generation software package²². Other research aspects include reachability and optimal phasing analysis for formation reconfiguration²³, a general method for optimal guidance of formations by optimizing the orbit design, open-time minimum-fuel problem with impulsive control²⁴.

Other noteworthy approaches for modeling spacecraft formations and designing control algorithms are orbit element differences²⁵,²⁶ and Theona theory²⁷. Achievements in this field include analytical, two-impulse solution to achieve the expected orbital-elemental differences²⁸,²⁹, maneuver guidance with analytical performance by mapping relative orbital elements into a fuel equivalent space³⁰, feedback control law with guaranteed neighboring fuel-optimality³¹, fuel-optimal using Gauss pseudospectral method³²,³³, and fuel-optimal maneuver using low-thrust propulsion³⁴.
However, the methods mentioned above have lots of problems, strong constrains and obvious shortcomings. Due to the lack of reliable analytical methods, some researchers have to rely on numerical methods. The high numerical sensitivity and nonlinearity corresponding to the bang-bang control are existed in numerical methods. Almost all methods use merely impulse or continuous thrust, which limit flexibility in formation maneuvering. Some methods even cannot guarantee the optimality of solutions.

This work obtains analytical solutions to the optimal spacecraft formation maneuver problem whose cost function expressed by characteristic velocity, with the help of the use of configuration parameters. The in-plane and out-of-plane maneuver are studied. The minimum fuel consumption and the corresponding constraints of control forces are derived. Application of orbital maneuvers is discussed and numerical simulation is illustrated.

2. Problem Formulation

The first-order approximation of relative motion for formation spacecraft expressed in Local-Vertical-Local-Horizontal (LVLH) frame is the well-known Hill’s equations

\[ X(t) = \begin{bmatrix} 0 & I & 0 \\ A_1 & A_2 & 0 \\ \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 0 \\ \end{bmatrix} U(t) \]

where \( X(t) = [x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)]^T \) are the states of the relative motion, including position and velocity, and \( U(t) = [u_x(t), u_y(t), u_z(t)]^T \) are the control forces. Sub-matrices \( A_1 \) and \( A_2 \) are given by

\[ A_1 = \begin{bmatrix} 3n^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -n^2 \\ \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2n & 0 \\ -2n & 0 & 0 \\ 0 & 0 & 0 \\ \end{bmatrix} \]

where \( n \) denotes the mean motion of the chief orbit. The state transfer matrix of (1) is obtained as

\[ \Phi(t) = e^{At} = \begin{bmatrix} 4 - 3 \cos(nt) & 0 & 0 & - \frac{\sin(nt)}{n} & \frac{2}{n} (1 - \cos(nt)) & 0 \\ 6 \sin(nt) - 6nt & 1 & 0 & \frac{2}{n} (\cos(nt) - 1) & \frac{4 \sin(nt) - 3nt}{n} & 0 \\ 0 & 0 & \cos(nt) & 0 & 0 & \frac{\sin(nt)}{n} \\ 3n \sin(nt) & 0 & 0 & \cos(nt) & 2 \sin(nt) & 0 \\ 6n (\cos(nt) - 1) & 0 & 0 & -2 \sin(nt) & 4 \cos(nt) - 3 & 0 \\ 0 & 0 & -n \sin(nt) & 0 & 0 & \cos(nt) \end{bmatrix} \]

Generally, the formation maneuver can be regarded as a controlled orbit transfer from the initial states \( X_0 \) to terminal free states \( X_f(t_f) \), as follows:

\[ X_f(t_f) = \Phi(t_f) X_0 + \int_0^{t_f} \Phi(t_j - t)BU(t)dt \]

where \( t_f \) denotes the terminal time. Furthermore, the target configuration should be stable, i.e.,

\[ \dot{y}_{\mu\nu} = -2nx_{\mu\nu}. \]

A simple form from Eq. (4) can be expressed as

\[ \int_0^{t_f} \Phi(-t)BU(t)dt = \Delta X_0 \]

where \( \Delta X_0 = X_{f0} - X_0 \) denotes the initial error states.

It is assumed that there are three different thrusters, one for each direction. Fuel consumption due to a single impulse is proportional to the 1-norm of the control force. The optimal maneuver can be formulated as follows:

\[ \min J = \int_0^{t_f} [u_x(t) + u_y(t) + u_z(t)]dt \]

s.t. \[ \int_0^{t_f} \Phi(-t)BU(t)dt = \Delta X_0 \]

where \( J \) is the fuel cost function. By substituting Eq. (3) into Eq. (6), one can obtain
\[ \begin{align*}
\text{min} & \quad J = \int_0^t \left[ \left| u_x(t) \right| + \left| u_y(t) \right| + \left| u_z(t) \right| \right] dt \\
& \quad \text{s.t.} \quad \begin{bmatrix}
-\sin(nt) & 2 - 2\cos(nt) \\
\cos(nt) & -2\sin(nt) \\
2\cos(nt) - 2 & -4\sin(nt) + 3nt \\
2\sin(nt) & 4\cos(nt) - 3
\end{bmatrix} u_x(t) dt = \frac{n\Delta x}{\Delta t} \\
& \quad \begin{bmatrix}
\sin(nt) \\
\cos(nt)
\end{bmatrix} u_x(t) dt = \frac{-n\Delta z}{\Delta t}
\end{align*} \] (7)

It is noted that control forces \( u_x(t), u_y(t), \) and \( u_z(t) \) become delta functions for impulsive formation maneuver.

### 3. Parameterization of the Problem

At first, a configuration parameters vector, denoted as \( P = [p, \phi, s, l, q, \theta]^T \), are defined

\[ p = \sqrt{3n\Delta x_0 + 2\Delta y_0} + \Delta z_0^2 / n \quad \phi = \arctan \frac{\Delta x_0}{3n\Delta x_0 + 2\Delta y_0} \]
\[ s = 4\Delta x_0 + 2\Delta y_0 / n \quad \theta = \arctan \frac{\Delta x_0}{n\Delta y_0} \]
\[ l = \Delta y_0 - 2\Delta x_0 / n \]
\[ q = \sqrt{(n\Delta z_0)^2 + \Delta z_0^2} / n \]

where \( \phi, \theta \in [0, 2\pi) \). Let us assume that \( \phi = 0 \) when \( p = 0 \), and \( \theta = 0 \) when \( q = 0 \). The error states \( \Delta X(t) \) can be formulated with \( P \) as

\[ \Delta X(t) = \Phi(t)\Delta X_0 = X(P, t) \] (9)

The detail expression is

\[ \begin{align*}
\Delta x(t) &= -p\cos(nt + \phi) + s \\
\Delta y(t) &= 2p\sin(nt + \phi) + l - 1.5nt \\
\Delta z(t) &= q\sin(nt + \theta) \\
\Delta x(t) &= np\sin(nt + \phi) \\
\Delta y(t) &= 2np\cos(nt + \phi) - 1.5ns \\
\Delta z(t) &= nq\cos(nt + \theta)
\end{align*} \] (10)

Obviously, parameters \( p \) and \( \phi \) represent the size and orientation of the in-plane motion, \( s \) and \( l \) denote the center offsets, and \( q \) and \( \theta \) describe the size and orientation of the out-of-plane motion, respectively.

With the help of the relationship between \( \Delta X_0 \) and \( P \), the optimization problem Eq. (7) can be developed as

\[ \begin{align*}
\text{min} & \quad J = \int_0^t \left[ \left| u_x(t) \right| + \left| u_y(t) \right| + \left| u_z(t) \right| \right] dt \\
& \quad \text{s.t.} \quad \int_0^t \left[ \cos(nt)u_x(t) - 2\sin(nt)u_y(t) \right] dt = np\sin\phi \\
& \quad \int_0^t \left[ \sin(nt)u_x(t) + 2\cos(nt)u_y(t) \right] dt = np\cos\phi \\
& \quad \int_0^t u_z(t) dt = 0.5ns \\
& \quad \int_0^t \left[ 3ntu_x(t) - 2u_y(t) \right] dt = nl \\
& \quad \int_0^t \left[ \sin(nt)u_x(t) \right] dt = -nq\sin\theta \\
& \quad \int_0^t \left[ \cos(nt)u_y(t) \right] dt = nq\cos\theta
\end{align*} \] (11)

Here, Eq. (11) is called the parameterized problem of the fuel-optimal formation maneuver. When compared with Eq. (7), the integral terms of constraints in Eq. (11) seem much easier to deal with.

### 4. Problem Solving
To discuss and solve the parameterized problem Eq. (11) directly, a useful inequality is introduced first as follows:

\[
\left[ \int_{0}^{t} \sqrt{f'(t) + g'(t)} dt \right]^2 \geq \left[ \int_{0}^{t} f(t) dt \right]^2 + \left[ \int_{0}^{t} g(t) dt \right]^2
\]  

(12)

where both \( f(t) \) and \( g(t) \) are real integrable functions defined on interval \([0, t_f]\).

**Proof:** Let us define a plane curve, denoted by \( L \), with a parametric equation as follows:

\[
L(t) = \begin{cases} 
  x(t) = \int_{0}^{t} f(t) dt \\
  y(t) = \int_{0}^{t} g(t) dt 
\end{cases}; \quad t \in [0, t_f] 
\]

(13)

Obviously, the initial point \( L(0) = (0, 0) \) is with origin \( O \), as illustrated in Fig.1. The length of \( L \) can be derived as

\[
h = \int_{0}^{t_f} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_{0}^{t_f} \sqrt{f'^2(t) + g'^2(t)} dt
\]

(14)

where \( (\cdot)' \) denotes the derivation with respect to \( t \). In plane \( Oxy \), the line segment \( OL_f \) has the minimum distance than that of any curve connected to points \( O \) and \( L_f \), hence

\[
h^2 \geq |OL_f|^2 = x^2(t_f) + y^2(t_f)
\]

(15)

where \( L_f = L(t_f) \) is the terminal point of \( L \). By substituting Eqs. (13) and (14) into Eq. (15), we can obtain

\[
\left[ \int_{0}^{t_f} \sqrt{f'^2(t) + g'^2(t)} dt \right]^2 \geq \left[ \int_{0}^{t_f} f(t) dt \right]^2 + \left[ \int_{0}^{t_f} g(t) dt \right]^2
\]

(16)

When the following equation

\[
\left[ \int_{0}^{t_f} \sqrt{f'^2(t) + g'^2(t)} dt \right]^2 = \left[ \int_{0}^{t_f} f(t) dt \right]^2 + \left[ \int_{0}^{t_f} g(t) dt \right]^2
\]

(17)

holds, \( L(t) \) becomes a beeline \( OL_f \), and its property can be obtained as

\[
k(t) = \frac{y'(t)}{x'(t)} = \frac{f(t)}{g(t)} = \frac{\int_{0}^{t} f(t) dt}{\int_{0}^{t} g(t) dt} = \text{const}
\]

(18)

where \( k(t) \) denotes the slope of \( L \). Equation (18) implies that the terminal point \( L_f \) determinates the nature of curve \( L \), which satisfies Eq. (17).

![Fig.1 Curve L](image)

4.1 *Out-of-plane*

The optimization problem of out-of-plane motion can be obtained from Eq. (11)
4.2 In-plane

According to Eq. (11), the optimization problem of in-plane motion can be written as

$$\sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - T_{m}) \left( \begin{array}{l} \sum_{m}^{m} \delta(t - 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\[
\begin{align*}
\min J_{xy} &= \int_0^T \left[ |u_x(t)| + |u_y(t)| \right] dt \\
&\quad + \int_0^T \left[ \cos(nt)u_x(t) - 2\sin(nt)u_y(t) \right] dt = np\sin \phi \\
\text{s.t.} \quad &\quad \int_0^T \left[ \sin(nt)u_x(t) + 2\cos(nt)u_y(t) \right] dt = np\cos \phi \\
&\quad \int_0^T u_x(t) dt = 0.5ns \\
&\quad \int_0^T (3ntu_y(t) - 2u_x(t)) dt = nl
\end{align*}
\] (25)

Considering inequality Eq. (12) and the first two constraints in Eq. (25), the cost function \( J_{xy} \) becomes

\[
J_{xy} = \int_0^T \left[ |u_x(t)| + |u_y(t)| \right] dt \\
\geq 0.5\int_0^T \left[ u_x(t) + 4u_y(t) \right] dt \\
\geq 0.5\int_0^T \left[ \cos(nt)u_x(t) - 2\sin(nt)u_y(t) \right] dt \\
\geq 0.5\int_0^T \left[ \sin(nt)u_x(t) + 2\cos(nt)u_y(t) \right] dt \\
= 0.5np
\] (26)

It has been shown that \( 0.5np \) is a lower bound of \( J_{xy} \). When \( J_{xy} = 0.5np \), the cost function reaches its minimum value, and the corresponding optimal control forces satisfy

\[
\begin{align*}
u_x'(t) &= 0 \\
u_x'(t)\sin(nt) &= -p\sin \phi = p\sin(-\phi) \\
u_x'(t)\cos(nt) &= p\cos \phi = p\cos(-\phi) \\
u_x'(t) psin(n - \phi) &\geq 0 \\
u_x'(t) pcos(\phi) &\geq 0
\end{align*}
\] (27)

With a similar analysis in above-mentioned discussions on \( m^+, m^- \), \( u_x'(t) \) is also observed to be the sum of a series of the corresponding delta functions as follows:

\[
u_x'(t) = \sum_{j}^{m^+} u_x^-(j)(t - k_jT + T_0) + \sum_{j}^{m^-} u_x^+(j)(t - k_jT - 0.5T + T_0)
\] (28)

where subscript "+" denotes the positive impulses, while "−" denotes the negative impulses. \( m^+ \) and \( m^- \) denote the counts of the positive and negative impulses, respectively, and \( k_j \in \mathbb{Z} \). It must be noted that there is no force or impulse exerted along the \( x \)-axis. The optimal maneuver of in-plane motion in this case is impulsive control strategy that satisfies

\[
\sum_{j}^{m^+} u_x^-(j)(t) - \sum_{j}^{m^-} u_x^+(j)(t) = 0.5np
\] (29)

Furthermore, according to the third constraint of problem Eq. (25), the cost function also satisfies

\[
J_{xy} = \int_0^T \left[ |u_x(t)| + |u_y(t)| \right] dt \geq \int_0^T u_x(t) dt = 0.5n |s|
\] (30)

Equation (30) shows that \( J_{xy} \) has another lower bound \( 0.5n |s| \). When \( J_{xy} = 0.5n |s| \) holds, the optimal control forces become

\[
u_x'(t) = 0, u_y'(t)s \geq 0; \forall t \in [0, t_f]
\] (31)

It has been shown that the optimal force along the \( x \)-axis is zero, while it is either positive or negative along the \( y \)-axis during formation maneuver.

From the previous arguments, the lower bound of cost function \( J_{xy} \) is

\[
J_{xy}^* = \inf J_{xy} \geq \max \{0.5n |s|, 0.5np\}
\] (32)

If \( p > |s| \), from previous discussions, in this case, the optimal control strategy must be impulsive
control with fuel consumption \(0.5np\). By substituting Eq. (29) into Eq. (25), the constraints of the optimal impulses can be obtained as

\[
\sum_{j=1}^{n_s} u_j^+(j) - \sum_{j=1}^{n_p} u_j^-(j) = 0.5np
\]
\[
\sum_{j=1}^{n_s} u_j^+(j) + \sum_{j=1}^{n_p} u_j^-(j) = 0.5ns
\]
(33)
\[
\sum_{j=1}^{n_s} u_j^+(j) v_j^+(j) + \sum_{j=1}^{n_p} u_j^-(j) v_j^+(j) = l/3
\]

where

\[
t_j^+(j) = k_j^+ T - T_p, \quad t_j^-(j) = (k_j^- + 0.5)T - T_p
\]
(34)

and \(u_j^+(j) \geq 0, \quad u_j^-(j) \leq 0\). Equation (37) can be developed into a new form as follows:

\[
\sum_{j=1}^{n_s} u_j^+(j) = \frac{n(p+s)}{4} > 0
\]
\[
\sum_{j=1}^{n_p} u_j^-(j) = -\frac{n(p-s)}{4} < 0
\]
(35)
\[
\sum_{j=1}^{n_s} u_j^+(j) k_j^+ + \sum_{j=1}^{n_p} u_j^-(j) k_j^- = \frac{4l}{3T} + \frac{nsT_p}{2T} + \frac{n(p-s)}{8}
\]

From Eqs. (38) and (39), there are many feasible optimal solutions to formation maneuver problem when \(p > |s|\). Generally, at least three impulses are needed because of the same number of constraints in Eq. (39). Let us suppose that there are two negative impulses and one positive impulse. Thus, Eqs. (38) and (39) become

\[
t_j^+(1) = k_j^+ T - T_p, \quad u_j^+(1) = 0.25n(p+s)
\]
\[
t_j^-(1) = (k_j^- + 0.5)T - T_p, \quad u_j^-(1) = -0.25an(p-s)
\]
\[
t_j^-(2) = (k_j^- + 0.5)T - T_p, \quad u_j^-(2) = -0.25(1 - \alpha)n(p-s)
\]
(36)

where \(\alpha \in [0, 1]\), which satisfies

\[
(p+s)k_1^- - (p-s)k_2^- + (p-s)(k_2^- - k_1^-)\alpha = \frac{4l}{3\pi} + \frac{2sT_p}{T} + \frac{(p-s)}{2}
\]
(37)

Equation (41) can be called as the basic equation of the optimal three-impulse maneuver, where \(p > |s|\).

One of the feasible solutions of Eq. (41) can be derived as follows:

After defining \(\bar{k} = k_2^- + (k_1^- - k_2^-)\alpha \geq 1\), \(\sigma = \frac{p+s}{p-s} > 0\), and \(b = \frac{1}{p+s} \left(\frac{4l}{3\pi} + \frac{2sT_p}{T} + \frac{p-s}{2}\right)\), Eq.(41) can be rewritten

\[
k_1^- = \sigma\bar{k} + b
\]
(38)

Considering \(k_1^- \in Z^-\) and \(\bar{k} \geq 1\), \(k_1^-\) can be chosen as

\[
k_1^- = \max([\sigma + b], 1]
\]
(39)

Then

\[
\bar{k} = \frac{k_1^- + b}{\sigma}, \quad k_1^- = [\bar{k}], \quad k_2^- = [\bar{k} + 1], \quad k^- = \bar{k} - [\bar{k}]
\]
(40)

where \([\ ]\) denotes the ceil integer of a real number and \([\ ]\) denotes the floor integer of a real number. By integrating Eqs. (40), (43), and (44), an analytical solution of the optimal three-impulse maneuver can be obtained in this case.

5. Applications: Maneuver of PCF

PCF is a formation for which deputy spacecraft is distributed on a circle, as seen from Earth. The PCF can be described with the radius \(R\) and angular position \(\gamma\). The relative configuration parameters vector is
with

\[
P = [0.5R, \gamma, 0, 0, R, \gamma + 0.5\pi]^T
\]

\[
R = \sqrt{R_0^2 + R_J^2 - 2R_0R_J \cos(\gamma_J - \gamma_0)}
\]

\[
\gamma = \arctan\left(\frac{R_0 \sin \gamma_0 - R_J \sin \gamma_J}{R_0 \cos \gamma_0 - R_J \cos \gamma_J}\right)
\]

where \((R_0, \gamma_0)\) and \((R_J, \gamma_J)\) denote the initial and target PCF, respectively. From previous discussions, the optimal PCF reconfiguration is an impulsive maneuver process. For the out-of-plane motion, the optimal impulse is

\[
t^*_1 = (1.75 - 0.5\gamma / \pi)T, \quad u^*_y = nR
\]

For in-plane motion, consider \(p \geq s\), and a feasible optimal solution is three impulses. According to Eqs.(40), (43), and (44), \(k^*_1 = 2\), \(k^*_2 = 2\), \(k^*_3 = 1\), \(\alpha = 0.5\), and the three impulses are

\[
t^*_1(1) = (2 - 0.5\gamma / \pi)T, \quad u^*_y(1) = nR / 8
\]

\[
t^*_2(1) = (1.5 - 0.5\gamma / \pi)T, \quad u^*_y(1) = -nR / 16
\]

\[
t^*_3(2) = (2.5 - 0.5\gamma / \pi)T, \quad u^*_y(2) = -nR / 16
\]

The whole fuel consumption is

\[
J = 1.5nR = 1.25n\sqrt{R_0^2 + R_J^2 - 2R_0R_J \cos(\gamma_J - \gamma_0)}
\]

It can be observed that the minimum fuel consumption is 1.25 \(n\) times larger than the relative size \(R\).

Let us consider that the target spacecraft runs in a circular orbit with 800 km height, 1000 m (\(\pi\)) initial PCF, and 2000 m (\(\pi\)) target PCF. Then, the relative PCF (1000 m, 0) can be calculated.

Simulation results are shown in Fig.2. In Fig.2(a), the solid line denotes the maneuver process from initial PCF (dashed point line) to target PCF (dash line). In Fig.2(b), the solid lines represent the parameters of maneuver process. It must be noted that the phase parameters \(\phi\) and \(\theta\) are constant during reconfiguration maneuver, as shown in Fig.2(b). This indicates that the impulses are exerted at times when \(\Delta t(t) = 0\) or \(\Delta z(t) = 0\), according to Eqs. (8) and (9).

6. Conclusions

Based on the well-known Hill’s equations, this paper has discussed the fuel-optimal formation maneuver in the first-order approximation. It has been shown that the low bound of the fuel consumption only depends on the relative size parameters. The initial conditions affect the existence of the optimal solution of the in-plane maneuver. In the application of formation reconfiguration, the optimal solution must be impulsive control because both the initial and target configurations are stable, and a good choice is a set of impulses. The method proposed in this paper can also be employed in PCF. Further work will focus on nonlinear optimal maneuver under various constraints. And problems will be investigated later in terms of T-H equations.
References
[1]. Scharf D.P, Hadaegh F.Y, Ploen S.R. (2003) A survey of spacecraft formation flying guidance and control (part 1): guidance. In: Proc. of the 2003 American Control Conference. Dayton. pp. 1733–1739.
[2]. Yang G, Kapila V, Wong H. (2003) Fuel optimal initialization of a spacecraft formation. In: Proc. of the 42Nd IEEE Conference on Decision and Control. Hawaii. pp. 3591-3596.
[3]. Ichimura Y, Ichikawa A. (2008) Optimal impulsive relative orbit transfer along a circular orbit, Journal of Guidance Control and Dynamics. 31(4): 1014-1027.
[4]. Hill G.W. (1878) Researches in lunar theory, American Journal of Mathematics. 1(1): 5–26.
[5]. Clohessy W.H, Wiltshire R.S. (1960) Terminal guidance system for satellite rendezvous. Journal of Aerospace Sciences. 27(9): 653–658.
[6]. Lawden D. (1954) Fundamentals of space navigation. Journal of the British Interplanetary Society, 13(2): 87–101.
[7]. Tschauner J. (1967) Elliptic orbit rendezvous. AIAA Journal, 5(6): 1110–1113.
[8]. Tillerson M, How J.P. (2002) Advanced guidance algorithms for spacecraft formation-keeping. In: Proc. of the American Control Conference, Anchorage.
[9]. Lu S.Z, Lu C, Zhang X.Y, Qin X.Y. (2016) Fuel-optimal Lorentz-augmented Spacecraft Formations Using Novelty Minimum Sliding Mode Error Feedback Controller. In: Joint International Conference on Artificial Intelligence and Computer Engineering and International Conference on Network and Communication Security.
[10]. Yang G, Yang Q.S, Kapila V, Palmer D. (2001) Fuel optimal maneuvers for multiple spacecraft formation reconfiguration using multi-agent optimization. In: Proc. of the 40th IEEE Conference on Decision and Control. pp: 1083-1088.
[11]. Prince E.R, Carr R.W, Cobb R.G. (2017) Fuel Optimal, Finite Thrust Guidance Methods to Circumnavigate with Lighting Constraints. In: Advanced Maui Optical and Space Surveillance Technologies Conference.
[12]. Wang S, Zheng C, Wang Y. (2007) A time-fuel optimal for spacecraft formation reconfiguration. In: Proc. of IEEE Congress on Evolutionary Computation. pp: 994-998.
[13]. Campbell M.E. (2003) Planning Algorithm for Large Satellite Clusters. Journal of Guidance, Control, and Dynamics, 26 (5): 770–780.
[14]. Zanon D.J, Campbell M.E. (2006) Optimal Planner for Spacecraft Formations in Elliptical Orbits, Journal of Guidance, Control, and Dynamics, 29 (1): 161-171.
[15]. Mailhe L.M, Guzman J.J. (2004) Initialization and resizing of formation flying using global and local optimization methods. In: Process of IEEE Aerospace Conference. Big Sky. pp: 547–556.
[16]. Kim D.Y, Woo B, Park S.Y, Choi K.H. (2009) Hybrid optimization for multiple-impulse reconfiguration trajectories of satellite formation flying. Advances in Space Research, 44(11): 1257–1269.
[17]. Kumar B.S, Ng A. (2009) Time-optimal low-thrust formation maneuvering using a hybrid linear/nonlinear controller. Journal of Guidance Control and Dynamics, 32(1): 343-347.
[18]. Li J. (2018) Analytical fuel-optimal impulsive reconfiguration of formation-flying satellites: A revisit and new results. Optimal Control Applications and Methods, 39:1243–1261.
[19]. Yue X.K, Duan X. (2016) Design of Formation Flying Mission Based on Pseudo Spectral Homotopy Algorithm. Aerospace Shanghai, 33(6): 44-52.
[20]. Pan B.F, Pan X, Ma Y.Y. (2018) A Quadratic Homotopy Method for Fuel-Optimal Low-Thrust Trajectory Design. Proceedings of the Institution of Mechanical Engineers Part G Journal of Aerospace Engineering.
[21]. Tang G, Jiang F.H, Li J.F. (2018) Fuel-Optimal Low-Thrust Trajectory Optimization Using Indirect Method and Successive Convex Programming. IEEE Transactions on Aerospace and Electronic Systems,54(4) :2053-2066.
[22]. Milam M.B, Petit N, Murray R.M. (2013) Constrained Trajectory Generation for Micro-Satellite
Formation Flying. In: AIAA Guidance, Navigation & Control Conference.

[23]. Palmer P. (2007) Reachability and optimal phasing for reconfiguration in near-circular orbit formations. Journal of Guidance Control and Dynamics, 30(5):1542-1546.

[24]. Hughes, S.P. (2008) General method for optimal guidance of spacecraft formations. Journal of Guidance Control and Dynamics, 31(2): 414-423.

[25]. Schaub H, Vadali S.R, Junkins J.L, Alfriend K.T. (2000) Spacecraft formation control using mean orbit elements. Journal of the Astronautical Sciences, 48(1): 69–87.

[26]. Schaub H. (2004) Relative orbit geometry through classical orbit element differences. Journal of Guidance, Control and Dynamics, 27(5): 839–848.

[27]. Golikov A. (2003) Evolution of formation flying satellite relative motion: analysis based on the Theona satellite theory. In: Proc. of the 17th International Symposium on Space Flight Dynamics, Moscow.

[28]. Vaddi S.S, Alfriend K.T, Vadali S.R, Sengupta P. (2005) Formation establishment and reconfiguration using impulsive control, Journal of Guidance Control and Dynamics, 28(2): 262-268.

[29]. Schaub H, Alfriend K.T. (2001) Impulsive Feedback Control to Establish Specific Mean Orbit Elements of Spacecraft Formations. Journal of Guidance, Control, and Dynamics, 24(4): 739–745.

[30]. Hamel, J.F, Lafontaine J. (2008) Fuel-equivalent relative orbit element space. Journal of Guidance Control and Dynamics, 31(1): 238-244.

[31]. Hamel J.F, Lafontaine J. (2009) Neighboring optimum feedback control law for Earth-orbiting formation-flying spacecraft. Journal of Guidance Control and Dynamics, 32(1): 290-299.

[32]. Huntington G.T, Rao A.V. (2008) Optimal reconfiguration of spacecraft formations using the Gauss pseudospectral method. Journal of Guidance Control and Dynamics, 31(3): 689-698.

[33]. Huntington G.T, Benson D.A, Rao A.V. (2007) Design of Optimal Tetrahedral Spacecraft Formations, Journal of the Astronautical Sciences, 55(2): 141–169.

[34]. Wu B.L, Wang D.W, Poh E.K, Xu G.Y. (2009) Nonlinear optimization of low-thrust trajectory for satellite formation: Legendre pseudospectral approach. Journal of Guidance Control and Dynamics, 32(4): 1371-1381.