Small curvature laminations in hyperbolic 3–manifolds

WILLIAM BRESLIN

We show that if \( L \) is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \( L \) are all in the interval \((-\delta, \delta)\) for a fixed \( \delta \in [0, 1) \) and no complementary region of \( L \) is an interval bundle over a surface, then each boundary leaf of \( L \) has a nontrivial fundamental group. We also prove existence of a fixed constant \( \delta_0 > 0 \) such that if \( L \) is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \( L \) are all in the interval \((-\delta_0, \delta_0)\) and no complementary region of \( L \) is an interval bundle over a surface, then each boundary leaf of \( L \) has a noncyclic fundamental group.

1 Introduction

In [9], Zeghib proved that any totally geodesic codimension-one lamination in a closed hyperbolic 3–manifold is a finite union of disjoint closed surfaces. In this paper we investigate whether a similar result holds for codimension-one laminations with small principal curvatures. We will prove the following theorems:

**Theorem 1** Let \( \delta \in [0, 1) \). If \( L \) is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \( L \) are everywhere in \((-\delta, \delta)\) for a fixed constant \( \delta \in [0, 1) \) and no complementary region of \( L \) is an interval bundle over a surface, then each boundary leaf of \( L \) has a nontrivial fundamental group.

**Theorem 2** There exists a fixed constant \( \delta_0 > 0 \) such that if \( L \) is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \( L \) are everywhere in \((-\delta_0, \delta_0)\) and no complementary region is an interval bundle over a surface, then each boundary leaf of \( L \) has a noncyclic fundamental group.
2 Examples

Let \( \mathcal{L} \) be a codimension-one lamination in a complete hyperbolic 3–manifold \( M \). Let \( L \) be a leaf of \( \mathcal{L} \) and endow it with the path metric induced from \( M \). Let \( \tilde{L} \) be the universal cover of \( L \) and lift the inclusion \( i_L: L \to M \) to a map \( \tilde{i}_L: \tilde{L} \to \mathbb{H}^3 \). A map \( f: X \to Y \) from a metric space \( X \) to a metric space \( Y \) is a \( k; c \)-quasi-isometry if
\[
\frac{1}{k} d_X(a, b) - c \leq d_Y(f(a), f(b)) \leq k d_X(a, b) + c.
\]
The leaf \( L \) is quasi-isometric if the map \( \tilde{i}_L \) is a \( (k, c) \)-quasi-isometry for some \( k, c \). The lamination \( \mathcal{L} \) is quasi-isometric if each leaf of \( \mathcal{L} \) is quasi-isometric for the same fixed constants \( k, c \).

Let \( \delta \in (0, 1) \). If the principal curvatures of \( \tilde{i}_L(\tilde{L}) \) are everywhere in \((-\delta, \delta)\), then the map \( \tilde{i}_L \) is a \( (k, c) \)-quasi-isometry for constants \( k, c \) depending only on \( \delta \) (see Thurston [8]). Also see Leininger [6] for an elementary proof.

The constant \( \delta_0 \) in Theorem 2 is less than 1, so a lamination satisfying the hypotheses of Theorem 1 or Theorem 2 is necessarily quasi-isometric. Thus it makes sense to ask whether these results hold for general quasi-isometric laminations.

**Quasi-isometric laminations with no compact leaves** Cannon and Thurston [3] proved that the stable and unstable laminations of the suspension of a pseudo-Anosov homeomorphism of a closed surface are quasi-isometric, and each leaf is a plane or annulus in this case. In addition to these examples, Fenley [5] produced infinitely many examples of closed hyperbolic 3–manifolds with quasi-isometric laminations in which each leaf is an annulus, a mobius band, or a plane. Note that Theorem 2 implies that the examples of Cannon–Thurston and Fenley cannot have principal curvatures everywhere in the interval \((-\delta_0, \delta_0)\).

One can also ask if we need to require that no complementary region is an interval bundle over a surface.

**Small curvature laminations with simply connected boundary leaves** Let \( S \) be a closed totally geodesic embedded surface in a closed hyperbolic 3–manifold \( M \). Let \( N(S) = S \times [0, 1] \) be a closed embedded neighborhood of \( S \) in \( M \). If the neighborhood \( N(S) \) is small then the surfaces \( S \times t \) will have small principal curvatures. Since \( \pi_1(S) \) is left-orderable, there exist faithful representations \( \rho: \pi_1(S) \to \text{Homeo}([0, 1]) \) such that some points have trivial stabilizers (see Calegari [2]) The foliated bundle whose holonomy is \( \rho \) has a leaf which is simply connected. Replace \( N(S) \) with this foliated bundle. We can blow up the simply connected leaf and remove the interior to get a lamination which is \( C^\infty \) close to the original (so that the leaves have small principal curvatures) and such that some boundary leaf is simply connected. See Calegari [1] to see why the foliated bundle can be embedded in \( M \) so that the leaves are smooth. Note that this lamination has a complementary region which is an interval bundle over a surface.
Small curvature laminations with no compact leaves  One may also construct small curvature laminations in closed hyperbolic 3–manifolds with no compact leaves. The author would like to thank Chris Leininger for describing the following construction. The idea is to construct a small curvature branched surface in a closed hyperbolic 3–manifold which has an irrational point in the space of projective classes of measured laminations carried by the branched surface. A lamination corresponding to this irrational point will contain no compact leaves. There are totally geodesic immersed closed surfaces in the figure-eight knot complement $M_8$ arbitrarily close to any plane in the tangent bundle (see Reid [7]). Using this and the fact that $\pi_1(M_8)$ is LERF, one can find two such surfaces which lift to embedded surfaces $S_1$ and $S_2$ in a finite cover $M$ of $M_8$ which intersect in a nonseparating (in both surfaces) simple closed geodesic $l$. Flatten out the intersection to get a branched surface with small principal curvatures in which $S_1$ connects one side of $S_2$ to the other side. The branched surface has three branch sectors (an annulus, $S_1 \setminus l$, and $S_2 \setminus l$) and one branch equation ($x_1 = x_2 + x_3$). A solution to the branch equation in which two coordinates are not rationally related (eg, $x_1 = 1/2, x_2 = 1/\pi, x_3 = 1/2 - 1/\pi$) will correspond to a lamination with no compact leaves which can be isotoped to have small principal curvatures. Since the leaves do not have any cusps, we can fill the cusps of $M$ to get a small curvature lamination in a closed hyperbolic 3–manifold with no compact leaves.

3  Proof of Theorem 1

Let $\epsilon > 0$ be so small that if $P_1, P_2, P_3$ are three disjoint smoothly embedded planes in hyperbolic 3–space with principal curvatures in $(-1, 1)$ which intersect the same $\epsilon$–ball, then one of the $P_i$ separates the other two.

Let $\mathcal{L}$ be a codimension-one lamination in a finite volume hyperbolic 3–manifold $M$ such that the principal curvatures of each leaf are everywhere in the interval $(-\delta, \delta)$ for some $\delta \in (0, 1)$. Assume that no complementary region of $\mathcal{L}$ is an interval bundle over a surface. Let $\tilde{\mathcal{L}}$ be the lift of $\mathcal{L}$ to $\mathbb{H}^3$. Since every leaf of $\mathcal{L}$ has principal curvatures everywhere in $(-\delta, \delta)$, the lamination $\tilde{\mathcal{L}}$ is a quasi-isometric lamination, and cannot be a foliation of $M$ by Fenley [4].

Let $L_0$ be a boundary leaf of $\mathcal{L}$. Suppose, for contradiction, that $\pi_1(L_0)$ is trivial, which implies that $L_0$ has infinite area. Since $M$ is closed, $L_0$ must intersect some fixed compact ball in $M$ infinitely many times. Thus given any integer $k$, we can find a point $y_k$ in $L_0$ such that the next leaf over on the boundary side of $L_0$ is within $1/k$ of $y_k$. 

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Let $\bar{L}_0$ be a lift of $L_0$ to $\mathbb{H}^3$. Lift the points $y_k$ to a fixed fundamental domain of $\bar{L}_0$ and call them $y_k$. Let $\bar{L}_k$ be the next leaf over from $\bar{L}_0$ which is within $1/k$ of $y_k$. We now have a sequence of leaves $\bar{L}_k$ in $\mathcal{L}$ on the boundary side of $\bar{L}_0$ such that for each $k$ the distance from $\bar{L}_k$ to $y_k$ is less than $1/k$, and there is no leaf of $\mathcal{L}$ between $\bar{L}_0$ and $\bar{L}_k$. We also have that $\partial \bar{L}_0 \neq \partial \bar{L}_k$ for all $k$, because otherwise the region between $L_0$ and $L_k$ would be an interval bundle in the complement of $\mathcal{L}$.

Let $k$ be so large that $1/k < \epsilon/8$. Since $\widetilde{L}_k$ eventually diverges from $\widetilde{L}_0$ we can find a point $x_k \in \widetilde{L}_0$ such that the distance from $x_k$ to $\widetilde{L}_k$ is exactly $\epsilon/8$. Let $b_k$ be the $(\epsilon/32)$--ball tangent to $\widetilde{L}_0$ at $x_k$ on the boundary side of $\widetilde{L}_0$.

We will show that infinitely many of the balls $b_k$ are disjointly embedded in $M$, contradicting the fact that $M$ has finite volume. Suppose that $\gamma(b_l) \cap b_k \neq \emptyset$ for some integers $l$, $k$ and some $\gamma$ in $\pi_1(M)$. Note that $\gamma(\tilde{L}_0) \neq \tilde{L}_0$, since $L_0$ has trivial fundamental group. Now $\tilde{L}_0$, $\tilde{L}_k$, and $\gamma(\tilde{L}_0)$ all intersect some $\epsilon$--ball, so we must have that one of them separates the other two. Since there are no leaves of $\mathcal{L}$ between $\tilde{L}_0$ and $\tilde{L}_k$, and $\gamma(\tilde{L}_0)$ is closer to $x_k$ than $\tilde{L}_k$, we must have that $\tilde{L}_0$ separates $\tilde{L}_k$ and $\gamma(\tilde{L}_0)$ (see Figure 1(a)). Also note that $\tilde{L}_0$, $\tilde{L}_k$, and $\gamma(\tilde{L}_l)$ are all on the boundary side of $\gamma(\tilde{L}_0)$ (ie, the side which contains the ball $\gamma(b_l)$).

Now we will show no matter where $\gamma$ sends $\tilde{L}_l$, we get a contradiction. We cannot have $\gamma(\tilde{L}_l) = \tilde{L}_k$, because this would imply that $\gamma^{-1}(\tilde{L}_0)$ separates $\tilde{L}_l$ and $\tilde{L}_0$. Thus we have $\gamma(\tilde{L}_l) \neq \tilde{L}_k$.

Since $\tilde{L}_0$, $\tilde{L}_k$, and $\gamma(\tilde{L}_l)$ all intersect some fixed $\epsilon$--ball, we must have that one of them separates the other two. We cannot have that $\gamma(\tilde{L}_l)$ separates $\tilde{L}_0$ and $\tilde{L}_k$, because there are no leaves of $\mathcal{L}$ between $\tilde{L}_0$ and $\tilde{L}_k$ (See Figure 1(b)). If $\tilde{L}_0$ separates $\tilde{L}_k$ and $\gamma(\tilde{L}_l)$, then $\gamma(\tilde{L}_l)$ is between $\tilde{L}_0$ and $\gamma(\tilde{L}_0)$, so that $d(x_l, \tilde{L}_l) = d(\gamma(x_l), \gamma(\tilde{L}_l)) \leq \epsilon/16$ which is a contradiction (see Figure 1(c)). Thus $\tilde{L}_0$ cannot separate $\tilde{L}_k$ and $\gamma(\tilde{L}_l)$. If $\tilde{L}_k$ separates $\tilde{L}_0$ and $\gamma(\tilde{L}_l)$, then $\gamma^{-1}(\tilde{L}_k)$ separates $\tilde{L}_0$ and $\tilde{L}_l$ which is a contradiction (see Figure 1(d)). Thus $\tilde{L}_k$ cannot separate $\tilde{L}_0$ and $\gamma(\tilde{L}_l)$.

We have shown that $\tilde{L}_l$ has nowhere to go under the map $\gamma$, so that $\gamma(b_l) \cap \gamma(b_k) = \emptyset$ for any integers $l$, $k$ and any $\gamma \in \pi_1(M)$. This implies that $M$ contains infinitely many disjoint $(\epsilon/32)$--balls, contradicting the fact that $M$ has finite volume.

\[\square\]

4 Proof of Theorem 2

Let $\epsilon > 0$ be so small that if $P_1$, $P_2$, $P_3$ are three disjoint smoothly embedded planes in hyperbolic $3$--space with principal curvatures in $(-1, 1)$ which intersect the same
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Figure 1: (a) \( \bar{L}_0 \) separates \( \bar{L}_k \) and \( \gamma(\bar{L}_0) \). (b) \( \gamma(\bar{L}_l) \) cannot separate \( \bar{L}_0 \) and \( \bar{L}_k \). (c) \( \bar{L}_0 \) cannot separate \( \bar{L}_k \) and \( \gamma(\bar{L}_l) \). (d) \( \bar{L}_k \) cannot separate \( \bar{L}_0 \) and \( \gamma(\bar{L}_l) \).

Let \( \mathcal{L} \) be a codimension-one lamination in a finite volume hyperbolic 3–manifold \( M \) such that the principal curvatures of each leaf are everywhere in the interval \( (\delta_0, \frac{1}{2}) \). Assume that no complementary region of \( \mathcal{L} \) is an interval bundle over a surface. Let \( \mathcal{L}_0 \) be a boundary leaf of \( \mathcal{L} \). Suppose, for contradiction, that \( \pi_1(\mathcal{L}_0) \) is cyclic, which implies that \( \mathcal{L}_0 \) has infinite area. Since \( M \) is closed, \( \mathcal{L}_0 \) must intersect some fixed compact ball in \( M \) infinitely many times. Also, by Theorem 1, we know that \( \pi_1(\mathcal{L}_0) \) is nontrivial, so that \( \pi_1(\mathcal{L}_0) \approx \mathbb{Z} \).

Let \( \bar{L}_0 \) be a lift of \( \mathcal{L}_0 \) to \( \mathbb{H}^3 \). Since \( \mathcal{L}_0 \) intersects a fixed compact ball in \( M \) infinitely many times, we can find a sequence of points \( y_k \) in \( \bar{L}_0 \) such that the closest leaf of \( \bar{L}_0 \) to \( y_k \) on the boundary side of \( \bar{L}_0 \) is within \( 1/k \) of \( y_k \). Let \( \bar{L}_k \) be the leaf which is closest to \( y_k \) on the boundary side of \( \bar{L}_0 \). Note that there is no leaf of \( \bar{L}_0 \) between \( \bar{L}_0 \) and \( \bar{L}_k \). We have \( \partial \bar{L}_0 \neq \partial \bar{L}_k \) for all \( k \), because the complement of \( \mathcal{L} \) contains no interval bundle components. We may assume that all \( y_k \) are contained in a fixed fundamental domain \( D \) of \( \bar{L}_0 \), and that \( y_k \) converge to a point \( y_{\infty} \in \partial \bar{L}_0 \).

For \( k \) large enough we have \( \partial \bar{L}_0 \neq \partial \bar{L}_k \) and \( d(y_k, \bar{L}_k) \leq \epsilon/8 \), so that we can find a point \( x_k \) such that \( d(x_k, \bar{L}_k) = \epsilon/8 \).
Case 1 We can choose the sequence of points $x_k \in \tilde{L}_0$ to be contained in a fixed fundamental domain $D$ of $\tilde{L}_0$ such that $x_k$ exit an end of $D$ whose projection to $M$ has infinite area.

Let $b_k$ be the $(\epsilon/32)$–ball tangent to $\tilde{L}_0$ at $x_k$ on the boundary side of $\tilde{L}_0$. For $k$ large enough, say all $k$, the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ moves the center of $b_k$ a distance of at least $\epsilon$. Thus we can assume that $\gamma(b_l) \cap b_k = \emptyset$ for any integers $l, k$ and any $\gamma \in \pi_1(M)$.

We may now proceed as in the proof of Theorem 1 to show that $\gamma(b_l) \cap b_k = \emptyset$ for any integers $l, k$ and any $\gamma \in \pi_1(M)$. This again contradicts the fact that $M$ has finite volume.

Case 2 We cannot choose the sequence of points $x_k$ as in Case 1.

If infinitely many of the leaves $\tilde{L}_k$ were distinct, then we would be able to find a sequence of points as described in Case 1. Thus $\tilde{L}_k = \tilde{L}_+ \kappa$ for some fixed leaf $\tilde{L}_+ \in \tilde{L}$.

Let $U$ be the component of the complement in $\partial \tilde{L}_0$ of the fixed point(s) of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ which contains the point $y_{\infty}$. We will now show that $\partial \tilde{L}_+ \cap \partial \tilde{L}_0$ must contain $U$.

Suppose that $\partial \tilde{L}_+ \cap \partial \tilde{L}_0$ does not contain $U$. Since $d(y_k, \tilde{L}_+) < 1/k$ and $\partial \tilde{L}_+ \cap \partial \tilde{L}_0$ does not contain $U$, we can find a sequence of points $x_k$ in $\tilde{L}_0$ which converge to a point $x_{\infty} \in U$ with $d(x_k, \tilde{L}_+) = \epsilon/8$. Since the point $x_{\infty}$ cannot be a fixed point of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$, a tail of the sequence $x_k$ must be contained in a fixed fundamental domain of $\tilde{L}_0$. This contradicts the fact that we are in Case 2. Thus $\partial \tilde{L}_+ \cap \partial \tilde{L}_0$ must contain $U$, hence must contain the fixed point(s) of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$.

If the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ is parabolic, then it has only one fixed point. This implies that $\partial \tilde{L}_+ = \partial \tilde{L}_0$, giving us a contradiction.

If the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ is loxodromic, then we can argue as above to find a leaf $\partial \tilde{L}_- \kappa$ of $\tilde{L}$ which contains the other component of complement in $\partial \tilde{L}_0$ of the fixed points of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$. So $\partial \tilde{L}_+$ and $\partial \tilde{L}_-$ both contain the endpoints of the axis of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$. Since the principal curvatures of $\tilde{L}_0$, $\tilde{L}_+$, and $\tilde{L}_-$ are all in the interval $(-\delta_0, \delta_0)$, and $\partial \tilde{L}_0$, $\partial \tilde{L}_+$, $\partial \tilde{L}_-$ all contain the endpoints of the axis of the generator of $\text{stab}_{\pi_1(M)}$, we must have that $\tilde{L}_0$, $\tilde{L}_+$, and $\tilde{L}_-$ all intersect some fixed $\epsilon$–ball. Thus one of the three separates the other two. This gives us a contradiction since $\tilde{L}_+$ and $\tilde{L}_-$ are on the same side of $\tilde{L}_0$ (i.e, the boundary side) and there are no leaves of $L$ between $\tilde{L}_0$ and $\tilde{L}_+$ or between $\tilde{L}_0$ and $\tilde{L}_-$.

\[\Box\]
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Department of Mathematics, University of Michigan
530 Church Street, Ann Arbor 48109-1043, United States
breslin@umich.edu
http://www.math.lsa.umich.edu/people/facultyDetail.php?uniqname=breslin

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