Expanded Empirical Formula for Coulomb Final State Interaction in the Presence of Lévy Sources

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Abstract—Measurements of momentum space correlations in heavy ion reactions represent unique tools to investigate the properties of the created medium. However, these analyses require the careful handling of the final state interactions such as the Coulomb repulsion of the involved particles. In small systems such as $e^+e^-$ or $p+p$ the well-known Gamow factor gives an acceptable description but in the case of extended sources like those created in heavy ion collisions, a more sophisticated approach has to be developed. In this paper we expand our previous work on the investigation of the Coulomb final state interaction in the presence of a Lévy source. Such sources were shown to be a statistically acceptable assumption to describe the quantum statistical correlation functions in high energy heavy ion reactions.

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1. INTRODUCTION

In high energy reactions, the space-time characteristics of the particle emitting source can be examined via quantum statistical correlations. The shape of the source, hence the shape of the correlation functions are traditionally assumed to be Gaussian, but recently Lévy distribution got much interest as a statistically acceptable description of such correlation functions in 1 and 3 dimension (for details see [1, 2]). The Lévy type source function restores the Gaussian as well as the Cauchy source function in special cases, and its parameters may carry information about underlying physical processes, such as the type of the phase transition, partial coherence or in-medium mass modifications, see e.g. [1, 3, 4]. Thus, the investigation of the parameters of correlation functions is crucial which implies that the precise determination of the final state interactions is desired.

The most important final state interaction is the Coulomb repulsion which can be handled with the well-known Gamow correction in small systems but in case of extended source, such as those created in heavy ion collisions, it overestimates the effect. The usual approach for large systems is to take the source averaged Coulomb wave function, see e.g. [5, 6]. Except some special cases, the average cannot be calculated analytically. In this paper we present our extended results on the final state Coulomb interaction in the presence of a Lévy source. For the previous results see [7].

2. COULOMB EFFECT IN BOSE–EINSTEIN CORRELATIONS

In the hydrodynamical picture of high energy collisions, the basic quantity is the source function which characterizes the particle emitting source. The one- and two-particle momentum distribution can be expressed [8] with this function as

\begin{equation}
N_1(p) = \int d^3 r S(r, p) |\psi_p(r)|^2
\end{equation}

and

\begin{equation}
N_2(p_1, p_2) = \int d^3 r_1 d^3 r_2 S(r_1, p_1) S(r_2, p_2) |\psi_{p_1 p_2}(r_1, r_2)|^2,
\end{equation}

where $\psi_{p_1 p_2}$ is the two-particle wave function which must be symmetric in the spatial variables for bosons. Basically, this symmetrization effect is the origin of the (Bose–Einstein) correlations. We can introduce the pair distribution function in the following form:

\begin{equation}
D(r, p_1, p_2) = \int d^3 R S(R + \frac{r}{2}) S(R - \frac{r}{2}).
\end{equation}

Let us introduce the average and relative momentum variables as $K = 0.5(p_1 + p_2)$ and $q = p_1 - p_2$. With these variables, the two-particle correlation function can be written as

\begin{equation}
C_2(q, K) = \frac{\int d^3 r D(r, K) |\psi_{p_1 p_2}(r)|^2}{\int d^3 r D(r, K)}.
\end{equation}
Since the \( q \) dependence of the correlation function is usually more rapid than the \( K \) dependence, it is convenient to measure correlations as a function of the relative momentum at a given \( K \) and thus the parameters will depend on \( K \).

In the core–halo picture \([9]\), the source is divided into parts: the core contains the promptly produced particles, the halo consists of product of resonance decays. The ratio of these parts can be characterized by the correlation strength parameter:

\[
\lambda = \frac{N_{\text{core}}}{N_{\text{core}} + N_{\text{halo}}},
\]

Assuming a source defined in Eq. (2), this parameter can be introduced into the definition of the correlation function given in Eq. (3) as

\[
C_2(q, K) = 1 - \lambda + \lambda \int d^3 r D_{\text{core}}(r, K) \left| \psi_q^{(2)}(r) \right|^2.
\]  

(5)

This is the well-known Bowler–Sinyukov formula \([5, 6]\). One can see that the value of the correlation function evaluated at zero relative momentum is \(1 + \lambda\). Although the free case corresponds to \( \lambda = 1 \), the experimental observations do not support this value. The core-halo model gives a natural explanation for this behavior of the \( \lambda \) intercept parameter. Moreover, this parameter might carry information about underlying processes, such as in-medium mass modification or partial coherence, see e.g. \([1, 10]\).

Returning to the general discussion about the Coulomb-corrected correlation function defined in Eq. (5), it can be seen that the evaluation of the integral requires two ingredients: the two particle source function which is assumed to be a Lévy distribution and the two-particle Coulomb-interacting wave function. The latter can be given by the two-body scattering solution of the Schrödinger equation with Coulomb potential, which solution is known in the center-of-mass system of the outgoing particles (abbreviated as PCMS):

\[
\psi_q^{(2)}(r) = \frac{1}{\sqrt{2}} \frac{\Gamma(1 + i \eta)}{e^{\eta/2} \pi^{1/2}} e^{i kr} F(-i \eta, 1, i(kr - kr))
\]

(6)

Here \( \Gamma() \) is the Gamma function, \( F(\cdot, \cdot, \cdot) \) is the confluent hypergeometric function. The variable \( \eta = \alpha_{\text{EM}} m_\pi c / q \) is the so called Sommerfeld parameter where \( \alpha_{\text{EM}} \) is the fine-structure constant. From now on, we restrict this analysis to pion pairs, so the mass parameter \( m_\pi \) in the formula is the pion mass. Since the terminology seems not to be uniform, let us introduce the Coulomb correction as the ratio of the measured correlation function \( C_2(q) \) and the free-case correlation function \( C_2^{(0)}(q) \). We will denote the Coulomb correction with \( K(q) \) and it can be written up according to the above convention as:

\[
K(q) = \frac{C_2(q)}{C_2^{(0)}(q)} \Rightarrow C_2(q) = C_2^{(0)}(q) \cdot K(q).
\]

(7)

This form can restore the simplest case, the so-called Gamow factor that assumes the source to be a point-like particle when calculating \( K(q) \):

\[
S(r) = \delta^{(3)}(r) \Rightarrow K(q) = K_{\text{Gamow}}(q) = \left| \psi_q^{(2)}(0) \right|^2 = \frac{2 \pi \eta}{e^{2 \eta} - 1}.
\]

(8)

The result of the integral in Eq. (5) cannot be expressed analytically so numerical approaches should be employed. In this paper, we present two ways to handle the Coulomb correction in a specific case of the source function—in the presence of a Lévy source.

3. LOOKUP TABLE FOR THE COULOMB CORRECTION FOR LÉVY SOURCES

Recent experimental results showed that a statistically acceptable assumption for the two-particle Bose–Einstein correlation function is to utilize a Lévy source function. The details can be found elsewhere, see e.g. \([1, 3]\). The (spherically symmetric) Lévy distribution has two parameters, the scale parameter \( \alpha \) and the Lévy index \( \alpha \). This distribution is expressed through a Fourier transformation as

\[
\mathcal{L}(\alpha, R, r) := \int \frac{d^3 q}{(2\pi)^3} \ e^{i q r} \exp \left( - \frac{1}{2} q^2 \ R^2 \right)^{\mu/2}.
\]

(9)

In the \( \alpha = 2 \) case this restores the Gaussian distribution, the \( \alpha = 1 \) corresponds to the Cauchy distribution. For other \( \alpha \) values, no simple analytic expression exists.

The integral in Eq. (5) cannot be evaluated analytically for Lévy sources so it has to be calculated numerically. For experimental purposes, it is suitable to load the results into a binary file as a lookup table and then it can be used in the fitting procedure. We should interpolate between the points where the numerical calculations are actually done to express the intermediate ranges. This interpolation, however, could cause numerical fluctuations in the \( \chi^2 \) landscape and could mislead the fit algorithm, so an iterative procedure should be applied; ours is detailed in \([1]\).

4. PARAMETRIZATION OF THE COULOMB CORRECTION FOR LÉVY SOURCES

In this section we discuss a different approach which is based on the numerical lookup table. As we mentioned, the finite resolution of the numerical table could cause fluctuations in the \( \chi^2 \) landscape and that
This formula has the advantage of having only 1 numerical constant parameter (the 1.26 in the denominator) but it assumes \( \alpha = 1 \) and we are looking for a parametrization of the general Lévy case. We considered this as our starting point for the generalization.

In order to generalize the Eq. (10) parametrization we chose to generalize the \( K_{\text{mod}} \) correction part\(^1\) by replacing \( R \) with \( R/\alpha \) to introduce the \( \alpha \)-dependence and take higher order terms in \( qR/\alpha \hbar c \) into consideration as it is detailed in [7]:

\[
K_{\text{mod}}(q) = 1 + B(\alpha, R) \frac{qR}{\alpha \hbar c} + C(\alpha, R) \left( \frac{qR}{\alpha \hbar c} \right)^2 + D(\alpha, R) \left( \frac{qR}{\alpha \hbar c} \right)^3
\]

This formula simplifies to Eq. (10) if \( \alpha = 1 \) and \( C = D = 0 \), but could follow the weak \( \alpha \) dependence of the Coulomb correction (see Fig. 1). The next step was to find suitable \( A(\alpha, R), B(\alpha, R), C(\alpha, R), D(\alpha, R) \) functions that yield acceptable approximations of the results of the numerical integration. Utilizing the previously constructed binary table, we could obtain the values of the \( A, B, C, D \) functions for various values of \( R \) and \( \alpha \) and parametrized them with empirically found, suitable 2 dimensional functions:

\[
A(\alpha, R) = (a_A \alpha + a_b) + (a_c R + a_d) + a_e (\alpha R + 1)^3
\]

\[
B(\alpha, R) = \frac{1 + b_A R^{\alpha d_b} - \alpha^{b_c}}{\alpha^2 (\alpha^{b_d} + b_E R^{\alpha d_e})}
\]

\[
C(\alpha, R) = \frac{c_A \alpha^{c_d} + c_R R^{\alpha c_f}}{R^{c_E}}
\]

\[
D(\alpha, R) = d_A + \frac{R^{d_b}}{\alpha^{d_e}}
\]

The parameters in these functions are:

\[
\begin{align*}
a_A &= 0.26984 & a_B &= -0.49123 & a_C &= 0.03523 \\
b_A &= 2.37267 & b_B &= 0.58631 & b_C &= 2.24867 \\
c_A &= -4.30347 & c_B &= 0.00001 & c_C &= 3.30346 \\
d_A &= 0.00057 & d_B &= -0.80527 & d_C &= -0.19261
\end{align*}
\]

\[
\begin{align*}
a_D &= -1.31628 & a_E &= 0.00359 \\
b_D &= -1.43278 & b_E &= -0.05216 & b_F &= 0.72943 \\
c_D &= 0.00001 & c_E &= 0.00003 & c_F &= 1.68883 \\
d_D &= 2.77504 & d_E &= 2.02951 & d_F &= 1.07906.
\end{align*}
\]

\( ^1 \)The other way could be to find a completely new parametrization.
where the $A(\alpha, R)$ and $B(\alpha, R)$ functions have a form as

$$A(\alpha, R) = A_a + A_b \alpha + A_c R + A_d \alpha R + A_e R^2 + A_f (\alpha R)^2,$$

$$B(\alpha, R) = B_a + B_b \alpha + B_c R + B_d \alpha R + B_e R^2 + B_f (\alpha R)^2.$$  \hspace{1cm} (18)

The parameters were chosen based on a fit to numerically calculated Coulomb correction at the $0.1 < q < 0.2$ GeV/c range:

$$K(q, \alpha, R)^{-1} = F(q) \times K_{\text{Gamow}}^{-1}(q) \times K_{\text{mod}}^{-1}(q; \alpha, R)$$

$$+ (1 - F(q)) \times E(q) \hspace{1cm} (20)$$

and the Coulomb corrected correlation function which can be fitted to data, according to the Bowler–Sinyukov method, can be written in the form of

$$C_2(q; \alpha, R) = \left[ 1 - \lambda + K(q; \alpha, R) \lambda \left[ 1 + \exp \left[ q R^2 \right] \right] \right] \times (\text{assumed background}). \hspace{1cm} (21)$$

We used this formula to reproduce previous PHENIX results\textsuperscript{2} from Fig. 3. of [1]; this can be seen on Fig. 2. The two fits are compatible with each other. An example code calculating the Coulomb correction as defined in Eq. (20) can be found in [12].

We investigated the parametrization by looking at its relative deviation from the lookup table also in the case when $\alpha = 1.2$ with different $R$ values and with a two-dimensional histogram of the relative differences

\textsuperscript{2}The data of the shown PHENIX correlation function result can be found at https://www.phenix.bnl.gov/phenix/WWW/info/data/ppg194_data.html.
Fig. 3. On the left hand side we present the relative deviation of the parametrization and the table measured in % for $\alpha = 1.2$ with various $R$ values. On the right hand side the $q$-averaged relative difference is presented. (Averaging was done on $0.01 < q < 0.1$ GeV/c.)

in Fig. 3. The maximum of these relative differences is around 0.07%.

5. CONCLUSIONS

We presented our results on the effect of the Coulomb repulsion to Bose–Einstein correlations in high energy heavy ion reactions in the presence of a Lévy source. We investigated two equivalent methods which could be used in experimental practice. One of them is to fill the values of the numerical integral of the Coulomb correction into a binary table, the other one is a parametrization based on the table. The described parametrization is valid for $R = 2–12$ fm and $\alpha = 0.8–2.0$ ranges, and shown to be compatible with previous results and with the numerical table. Thus, our parametrization can be used in quantum statistical correlation measurements effectively that assume Cauchy, Gaussian or the more general Lévy sources.

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