Centralizers and pseudo-degree functions

Johan Richter *

December 2, 2013

Abstract

We generalize some results by Hellström and Silvestrov, concerning centralizers in graded algebras. We show how to apply their methods to certain non-graded algebras, possessing a notion giving the degree of elements in it.

1 Introduction

To state the results that has motivated this paper we shall recall a definition.

Definition 1.1. Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $\delta$ an additive function, $R \to R$, satisfying

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

for all $a, b \in R$. (Such $\delta$-s are known as $\sigma$-derivations.) The Ore extension $R[x; \sigma, \delta]$ is the over-ring of $R$ satisfying $xr = \sigma(r)x + \delta(r)$ for all $r \in R$ and such that every element of $R[x; \sigma, \delta]$ can be written $\sum a_ix^i$ with $a_i \in R$. If $\sigma = \text{id}$ then $R[x; \text{id}_R, \delta]$ is called a differential operator ring.

In a paper by Amitsur [1] one can find the following theorem.

Theorem 1.2. Let $K$ be a field of characteristic zero with a derivation $\delta$. Let $F$ denote the subfield of constants. Form the differential operator ring $S = k[x; \text{id}, \delta]$, and let $P$ be an element of $S$ of degree $n > 0$. Set $F[P] = \{\sum_{j=0}^m b_jP^j \mid b_j \in F \}$, the ring of polynomials in $P$ with constant coefficients. Then the centralizer of $P$ is a commutative subring of $S$ and a free $F[P]$-module of rank at most $n$.

Later authors have found other contexts where Amitsur’s method of proof can be made to work. We mention an article by Goodearl and Carlson [6] and one by Goodearl alone [7] that both generalize Amitsur’s result to a wider class of rings. The proof has also been generalized by Bavula [2], Mazorchouk [9] and Tang [10], among other authors.

This paper is most directly inspired by a paper by Hellström and Silvestrov [8], however. Hellström and Silvestrov study graded algebras satisfying a condition they call $l$-BDHC (short for ‘Bounded-Dimension Homogeneous Centralizers’). Hellström and Silvestrov follow the ideas of Amitsur’s proof. The details need modification however, especially to handle the case when $l > 1$.

*Centre for Mathematical Sciences, Lund University, Box 118, 22100 Lund, Sweden, E-mail: johanr@maths.lth.se
Definition 1.3. Let $K$ be a field, $l$ a positive integer and $A$ a $\mathbb{Z}$-graded $K$-algebra. The homogeneous components of the gradation are denoted $A_m$ for $m \in \mathbb{Z}$. Let $\text{Cen}(n, a)$, for $n \in \mathbb{Z}$ and $a \in S$, denote the homogeneous elements in the $n$-component of the gradation that commute with $a$. We then say that $A$ has $l$-BDHC if for all $n \in \mathbb{Z}$, nonzero $m \in \mathbb{Z}$ and nonzero $a \in S$ it holds that $\dim_K \text{Cen}(n, a) \leq l$.

To explain the results further we introduce some more of their notation. Denote by $\pi_n$ the projection from $A$ to $A_n$. Hellström and Silvestrov define a function $\bar{\chi}: A \setminus \{0\} \rightarrow \mathbb{Z}$ by

$$\bar{\chi}(a) = \max \{n \in \mathbb{Z} | \pi_n(a) \neq 0\},$$

and set $\bar{\chi}(0) = -\infty$. (They also define an analogue with max instead of max.) Set further $\bar{\pi}(a) = \pi_{\bar{\chi}(a)}(a)$.

Now we have introduced enough notation to state the relevant results. The following result is the main part of Lemma 2.4 in their paper.

Theorem 1.4. Assume $A$ has $l$-BDHC and that there are no zero divisors in $A$. If $a \in A \setminus A_0$ has $\bar{\chi}(a) = m > 0$ and $\bar{\pi}(a)$ is not invertible in $A$ then there exists a finite $K[a]$-module basis $\{b_1, \ldots, b_k\}$ for the centralizer of $a$. Furthermore $k \leq ml$.

The reason they refer to it as a lemma is that their main interest is in the following corollary of this result. (Which is proved the same way as Corollary 2.5 in this paper.)

Theorem 1.5. Assume $A$ has $l$-BDHC and that there are no zero divisors in $A$. If $a \in A \setminus A_0$ and $b \in A$ are such that $ab = ba$, $\bar{\chi}(a) > 0$ and $\bar{\pi}(a)$ is not invertible in $A$ then there exists a nonzero polynomial $P$ in two commuting variables with coefficients from $K$ such that $P(a, b) = 0$.

This type of result, asserting the existence of a polynomial annihilating two commuting elements, was first proved by Burchnall and Chaundy for ordinary differential operators [3, 4, 5].

They also have result asserting that certain centralizers are commutative, applicable in the case when $A$ has 1-BDHC.

Theorem 1.6. Assume $A$ has 1-BDHC and that there are no zero divisors in $A$. If $a \in A \setminus A_0$ has $\bar{\chi}(a) = m > 0$ and $\bar{\pi}(a)$ is not invertible in $A$ then there exists a finite $K[a]$-module basis $\{b_1, \ldots, b_k\}$ for the centralizer of $a$, where the cardinality, $k$, of the basis divides $m$. Furthermore the centralizer of $a$ is commutative.

It shall be the goal of this paper to generalize the results we have cited from [5].

1.1 Notation and conventions

$\mathbb{R}$ will denote the field of real numbers, $\mathbb{C}$ the field of complex numbers. $\mathbb{Z}$ will denote the integers.

If $R$ is a ring then $R[x_1, x_2, \ldots, x_n]$ denotes the ring of polynomials over $R$ in central indeterminates $x_1, x_2, \ldots, x_n$. 

2
By a ring we will always mean an associative and unital ring. All morphisms
between rings are assumed to map the multiplicative identity element to the
multiplicative identity element.

By an ideal we shall mean a two-sided ideal.

Let $R$ be a commutative ring and $S$ an $R$-algebra. Two commuting elements,
$p, q \in S$, are said to be \textit{algebraically dependent} (over $R$) if there is a non-zero
polynomial, $f(s, t) \in R[s, t]$, such that $f(p, q) = 0$, in which case $f$ is called an
annihilating polynomial.

If $S$ is a ring and $a$ is an element in $S$, the \textit{centralizer} of $a$, denoted $C_S(a)$,
is the set of all elements in $S$ that commute with $a$.

2 Centralizers in algebras with degree functions

Upon reading the proofs in [8] closely it turns out that their proof relies mainly
upon the properties of the function $\bar{\chi}$ they define. It has properties very similar
to the degree of function for polynomials. We shall axiomatize the properties
that are needed to make their proof work.

\textbf{Definition 2.1.} Let $K$ be a field and let $S$ be a $K$-algebra. A function, $\chi$
from $S$ to $\mathbb{Z} \cup \{-\infty\}$ is called a pseudo-degree function if it satisfies the follow-
ing conditions:

\begin{itemize}
  \item $\chi(a) = -\infty$ iff $a = 0$,
  \item $\chi(ab) = \chi(a) + \chi(b)$ for all $a, b \in S$,
  \item $\chi(a + b) \leq \max(\chi(a), \chi(b))$,
  \item $\chi(a + b) = \chi(a)$ if $\chi(b) < \chi(a)$.
\end{itemize}

We also need a condition that can replace l-BDHC. We formulate it next.

\textbf{Definition 2.2.} Let $B$ be a subalgebra of a $K$-algebra $S$ with a pseudo-degree
function, $\chi$, and let $l$ be a positive integer. $B$ is said to satisfy condition $D(l)$
if $\chi(b) \geq 0$ for all non-zero $b \in B$ and if, whenever we have $l + 1$ elements
$b_1, \ldots, b_{l+1} \in B$, all mapped to the same integer by $\chi$, there exist $\alpha_1, \ldots, \alpha_{l+1} \in K$, not all zero, such that $\chi(\sum_{i=1}^{l+1} \alpha_i b_i) < \chi(b_1)$.

\textbf{Remark 2.3.} Suppose that $S$ is a $K$-algebra and $a \in S$ is such that $C_S(a)$
satisfies condition $D(l)$ for some $l$. If $b$ is an invertible element then $\chi(b^{-1}) = -\chi(b)$. So all invertible elements of $C_S(a)$ must be mapped to zero by $\chi$. In
particular the scalars are all mapped to zero by $\chi$.

We now proceed to prove an analogue of Theorem 1.4, using just the existence
of some pseudo-degree function.

\textbf{Theorem 2.4.} Let $K$ be a field and let $S$ be a $K$-algebra. Suppose $S$ has a
pseudo-degree function, $\chi$.

Let $a$ be an element of $S$, with $m = \chi(a) > 0$, such that $C_S(a)$ satisfies
condition $D(l)$ for some positive integer $l$. Then $C_S(a)$ is a free $K[a]$-module of
rank at most $lm$.  

\begin{proof}

By the properties of $\chi$ and $D(l)$, the centralizer $C_S(a)$ is a free $K[a]$-module
of rank at most $lm$.
\end{proof}

\textbf{Definition 2.1.} Let $K$ be a field and let $S$ be a $K$-algebra. A function, $\chi$
from $S$ to $\mathbb{Z} \cup \{-\infty\}$ is called a pseudo-degree function if it satisfies the follow-
ing conditions:

\begin{itemize}
  \item $\chi(a) = -\infty$ iff $a = 0$,
  \item $\chi(ab) = \chi(a) + \chi(b)$ for all $a, b \in S$,
  \item $\chi(a + b) \leq \max(\chi(a), \chi(b))$,
  \item $\chi(a + b) = \chi(a)$ if $\chi(b) < \chi(a)$.
\end{itemize}

We also need a condition that can replace l-BDHC. We formulate it next.

\textbf{Definition 2.2.} Let $B$ be a subalgebra of a $K$-algebra $S$ with a pseudo-degree
function, $\chi$, and let $l$ be a positive integer. $B$ is said to satisfy condition $D(l)$
if $\chi(b) \geq 0$ for all non-zero $b \in B$ and if, whenever we have $l + 1$ elements
$b_1, \ldots, b_{l+1} \in B$, all mapped to the same integer by $\chi$, there exist $\alpha_1, \ldots, \alpha_{l+1} \in K$, not all zero, such that $\chi(\sum_{i=1}^{l+1} \alpha_i b_i) < \chi(b_1)$.

\textbf{Remark 2.3.} Suppose that $S$ is a $K$-algebra and $a \in S$ is such that $C_S(a)$
satisfies condition $D(l)$ for some $l$. If $b$ is an invertible element then $\chi(b^{-1}) = -\chi(b)$. So all invertible elements of $C_S(a)$ must be mapped to zero by $\chi$. In
particular the scalars are all mapped to zero by $\chi$.

We now proceed to prove an analogue of Theorem 1.4, using just the existence
of some pseudo-degree function.

\textbf{Theorem 2.4.} Let $K$ be a field and let $S$ be a $K$-algebra. Suppose $S$ has a
pseudo-degree function, $\chi$.

Let $a$ be an element of $S$, with $m = \chi(a) > 0$, such that $C_S(a)$ satisfies
condition $D(l)$ for some positive integer $l$. Then $C_S(a)$ is a free $K[a]$-module of
rank at most $lm$.  

\begin{proof}

By the properties of $\chi$ and $D(l)$, the centralizer $C_S(a)$ is a free $K[a]$-module
of rank at most $lm$.
\end{proof}
Proof. Construct a sequence \( b_1, b_\ldots \) by setting \( b_1 = 1 \) and choosing \( b_k+1 \in C_S(a) \) such that \( \chi(b_k+1) \) is minimal under the restriction that \( b_k+1 \) does not lie in the \( K[a] \)-linear span of \( \{b_1, \ldots, b_k\} \). We will show later in the proof that such a sequence can only \( lm \) elements.

I first claim that

\[
\chi \left( \sum_{i=1}^{k} \phi_ib_i \right) = \max_{i \leq k}(\chi(\phi_i) + \chi(b_i)),
\]

for any \( \phi_1, \ldots, \phi_k \in K[a] \). We show this by induction on \( n = \max_{i \leq k}(\chi(\phi_i) + \chi(b_i)) \). It is clear that the left-hand side of (1) is never greater than the right-hand side. When \( n = 0 \), which is the least value possible, Equation (1) holds since \( \chi(b) \geq 0 \) for all \( b \in C_S(a) \).

For the induction step, assume (1) holds when the right-hand side is strictly less than \( n \). To verify that it holds for \( n \) as well, we can assume without loss of generality that \( \chi(\phi_k) + \chi(b_k) = n \), since if \( \chi(\phi_j b_j) < n \) for some term \( \phi_j b_j \) we can drop it without affecting either side of (1). If \( \phi_k \in K \) then \( \chi(\phi_k) = 0 \), by Remark 2.3, and thus \( \chi(b_k) = n \). By the choice of \( b_k \) it then follows that \( \chi(\sum_{i=1}^{k-1} \phi_i b_i) \geq n \), as otherwise \( \sum_{i=1}^{k} \phi_i b_i \) would have been picked instead of \( b_k \). If \( \phi_k \notin K \), then \( \chi(b_k) < n \) and thus \( \chi(b_j) < n \) for \( i = 1, \ldots, k \). Let \( r_1, \ldots, r_k \in K \) and \( \xi_1, \ldots, \xi_k \in K[a] \) be such that \( \phi_i = a \xi_i + r_i \) for \( i = 1, \ldots, k \). We have

\[
\chi(\sum_{i=1}^{k} r_i b_i) < n \text{ and thus by the assumptions on } \chi \text{ we get}
\]

\[
\chi \left( \sum_{i=1}^{k} \phi_i b_i \right) = \chi \left( \sum_{i=1}^{k} a \xi_i b_i + \sum_{i=1}^{k} r_i b_i \right) = \chi \left( a \sum_{i=1}^{k} \xi_i b_i \right) = m + \chi \left( \sum_{i=1}^{k} \xi_i b_i \right).
\]

We also have that \( \max_{i \leq k}(\chi(\phi_i) + \chi(b_i)) = m + \max_{i \leq k}(\chi(\xi_i) + \chi(b_i)) \). By the induction hypothesis

\[
\chi \left( \sum_{i=1}^{k} \xi_i b_i \right) = \max_{i \leq k}(\chi(\xi_i) + \chi(b_i)),
\]

which completes the induction step.

We now show that if \( \chi(b_i) = \chi(b_j) \) for some \( i \leq j \) then \( j - i < l \). Suppose \( b_1, \ldots, b_{l+1} \) all are mapped to zero by \( \chi \). Then there exists \( \alpha_1, \ldots, \alpha_{l+1} \) such that

\[
\chi \left( \sum_{i=1}^{l+1} \alpha_i b_i \right) < 0,
\]

which is impossible since \( \sum_{i=1}^{l+1} \alpha_i b_i \in C_S(a) \).

Suppose now instead that \( b_j, \ldots, b_{j+t} \) are all mapped to the same positive integer, \( q \), by \( \chi \). Then there exists \( \alpha_j, \ldots, \alpha_{j+t} \in K \), not all zero, such that

\[
\chi \left( \sum_{i=j}^{j+t} \alpha_i b_i \right) < q.
\]

But this is only possible if some \( b_i \) lies in the \( K[a] \)-linear span of the previous terms of the sequence, which contradicts the way it was chosen.
It remains only to show that the sequence \((b_i)\) contains only \(lm\) elements. We will prove that every residue class \((\mod m)\) can only contain at most \(l\) elements. Suppose to the contrary, that we had elements \(c_1, \ldots, c_{l+1}\), belonging to the sequence \((b_i)\) and all satisfying that \(\chi(c_i) = n \pmod{m}\). Set \(k = \max_{1 \le i \le l+1}(\chi(c_i))\) and define \(\gamma_i = a^{\frac{k - \chi(c_i)}{m}}.\) Then \(\chi(\gamma_ic_i) = k\), for all \(i \in [l+1]\), which implies that there exists \(\alpha_1, \ldots, \alpha_{l+1} \in K\), such that
\[
\chi \left( \sum_{i=1}^{l+1} \alpha_i \gamma_ic_i \right) < k.
\]

But this contradicts (1).

\(\square\)

We can also prove a result on the existence of annihilating polynomials for pairs of commuting elements.

**Corollary 2.5.** Let \(S\) be a \(K\)-algebra with a pseudo-degree function, \(\chi\). Let \(a \in S\) be such that \(C_S(a)\) satisfies Condition \(D(l)\) for some \(l > 0\). Let \(b\) be any element in \(C_S(a)\). Then there exists a nonzero polynomial \(P(s, t) \in K[s, t]\) such that \(K(a, b) = 0\). (Note that \(K(a, b)\) is well-defined when \(a, b\) commute.)

**Proof.** Since \(C_S(a)\) has finite rank as a \(K[a]\)-module the elements \(b, b^2, \ldots\) can not all be linearly independent over \(K[a]\). Thus there exists \(f_1(x), \ldots, f_k(x) \in K[x]\), not all zero, such that \(\sum_{i=0}^k f_i(a)b^i = 0\). Then \(P(s, t) = \sum_{i=0}^k f(s)t^i = 0\) is a polynomial with the desired property.

\(\square\)

We can also prove a result asserting that certain centralizers are commutative, though for that we need to assume that \(C_S(a)\) satisfies condition \(D(1)\).

**Theorem 2.6.** Let \(K\) be a field and suppose \(S\) is a \(K\)-algebra. Let \(S\) have a pseudo-degree function, \(\chi\). If \(a \in S\) satisfies \(\chi(a) = m > 0\) and \(C_S(a)\) satisfies condition \(D(1)\) then:

1. \(C_S(a)\) has a finite basis as a \(K[a]\)-module, the cardinality of which divides \(m\).

2. \(C_S(a)\) is a commutative algebra.

**Proof.** By Theorem 2.4 it is clear that there is a subset \(H\) of \([m]\) and elements \((b_i) \in H\) such that the \(b_i\) form a basis for \(C_S(a)\). By the proof of the theorem it is also clear that \(\chi(b_i) \neq \chi(b_j)\) if \(i \neq j\). Without loss of generality we can assume \(\chi(b_i) = i\) for all \(i \in H\). We can map \(H\) into \(\mathbb{Z}_m\) in a natural way. Denote the image by \(G\) We want to show \(G\) is a subgroup, for which it is enough to show that it is closed under addition.

Suppose \(g, h \in G\). There exists \(i, j \in H\), with \(i \equiv g \pmod{m}\) and \(j \equiv h \pmod{m}\). We can write \(b_ib_j = \sum_{k \in H} \phi_kb_k\), for some \(\{b_k\}\). It follows that
\[
g + h \equiv i + j = \chi(b_ib_j) = \max(\chi(\phi_k) + \chi(b_k)) \equiv \chi(b_k) = k \pmod{m}
\]
for some \(k \in H\).

Since \(G\) is a subgroup of \(\mathbb{Z}_m\) it is clear that the cardinality of \(G\), which is also the cardinality of \(H\), must divide \(m\).
$G$ is cyclic. Let $g$ be a generator of $G$. Consider the algebra generated by $b_i$ and $a_i$, where $i \equiv g \pmod{m}$. It is a commutative algebra and a sub-$K$-vector space of $C_S(a_i)$. Denote it by $E$. If $c$ is any element of $C_S(a_i)$ we can write $c = e + f$, where $e \in E$ and $\chi(f) < m$, since if $\chi(c) \geq m$ then there exists $k \leq m$ and $j \in \mathbb{N}$ such that $\chi(a^j b_i^k) = \chi(c)$ and thus there exists $\alpha \in K$ such that $\chi(c - \alpha a^j b_i^k) < \chi(c)$.

Thus the quotient $C_S(a)/E$ is finite-dimensional. Each $f \in K[a]$ gives rise to an endomorphism on $C_S(a)/E$, by the action of multiplication by $f$. Since $K[a]$ is infinite-dimensional and the endomorphism ring of $C_S(a)/E$ is finite-dimensional, there is some nonzero $\phi \in K[a]$ that induces the zero endomorphism. But this means that $\phi c \in E$ for any $c \in C_S(a)$.

Now let $c_1, c_2$ be two arbitrary elements of $C_S(a)$. Since $E$ is commutative, and everything in $C_S(a)$ commutes with $\phi$, it follows that

$$\phi^2 c_1 c_2 = \phi c_1 \cdot \phi c_2 = \phi c_2 \cdot \phi c_1 = \phi^2 c_2 c_1.$$ 

Since $C_S(a)$ is a domain it follows that $c_1 c_2 = c_2 c_1$ and thus that $C_S(a)$ is commutative.

3 Examples

Theorems 1.4, 1.5, and 1.6 follow from our results combined with Lemma 2.2 and Lemma 2.4 in [8]. But our results can also be applied in certain situations that are not covered by the results in [8].

Proposition 3.1. Let $K$ be a field. Set $R = K[y]$, let $\sigma$ be an endomorphism of $R$ such that $s = \deg(\sigma(y)) > 1$ and let $\delta$ be a $\sigma$-derivation. Form the Ore extension $S = K[x; \sigma, \delta]$. If $a \in S \setminus K$ then $C_S(a)$ is a free $K[a]$-module of finite rank and a commutative subalgebra of $S$.

Proof. If $a \in K[y] \setminus K$ then $C_S(a) = K[y]$ and the claim is true. So suppose that $a \notin K[y]$. We shall apply Theorem 2.6. To do so we need a pseudo-degree function.

There is an obvious notion of the degree of an element with respect to $x$ in $S$. Denote it by $\chi$. It is easy to see that it satisfies all the requirement to be a pseudo-degree function. We proceed to show that $C_S(a)$ satisfies condition $D(1)$. Certainly it is true that $\chi(b) \geq 0$ for all nonzero $b \in C_S(a)$.

Let $b$ be a nonzero element of $S$ that commutes with $a$, such that $\chi(b) = n$. Suppose $\chi(a) = m$. By equating the highest order coefficient of $ab$ and $ba$ we find that

$$a_m \sigma^m(b_n) = b_n \sigma^m(a_m),$$

(2)

where $a_m$ and $b_n$ denote the highest order coefficients of $a$ and $b$, respectively. (Recall that these are polynomials in $y$.) We equate the degree in $y$ of both sides of (2) and find that

$$\deg_y(a_m) + s^m \deg_y(b_n) = \deg_y(b_n) + s^m \deg_y(a_m),$$

which determines the degree of $b_m$ uniquely. It follows that the solutions of (2) form $K$-sub space of $K[y]$ that is at most one-dimensional. This in turn implies that condition $D(1)$ is fulfilled.

We have now verified all the hypothesis necessary to apply Theorem 2.6.

\[\boxempty\]
References

[1] S. A. Amitsur, *Commutative linear differential operators*, Pacific J. Math. 8 (1958), 1–10.

[2] V. V. Bavula, *Dixmier’s problem 6 for the Weyl algebra (the generic type problem)*, Comm. Algebra 34 (2006), no. 4, 1381–1406.

[3] J.L. Burchnall and T.W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. (3) 21 (1923), 420–440.

[4] __________, *Commutative ordinary differential operators*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 118(780) (1928), 557–583.

[5] __________, *Commutative ordinary differential operators. II. the identity $P^n = Q^m$*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 134(824) (1931), 471–485.

[6] R. C. Carlson and K. R. Goodearl, *Commutants of ordinary differential operators*, J. Differential Equations 35 (1980), no. 3, 339–365.

[7] K. R. Goodearl, *Centralizers in differential, pseudodifferential, and fractional differential operator rings*, Rocky Mountain J. Math. 13 (1983), no. 4, 573–618.

[8] L. Hellström and S. Silvestrov, *Ergodipotent maps and commutativity of elements in noncommutative rings and algebras with twisted intertwining*, J. Algebra 314 (2007), no. 1, 17–41.

[9] Volodymyr Mazorchuk, *A note on centralizers in q-deformed Heisenberg algebras*, AMA Algebra Montp. Announc. (2001), Paper 2, 6 pp. (electronic).

[10] Xin Tang, *Maximal commutative subalgebras of certain skew polynomial rings*, Available at http://faculty.uncfsu.edu/xtang/maxsubalgebras.pdf, 2005.