A novel mathematical construct for the family of leptonic mixing patterns

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Abstract

Inspired by the idea in quantum mechanics, we construct a novel mathematical object to induce a family of mixing patterns which accommodate the experimental data. We give an example with the hybrid of two elements from the group $S_4$. This example shows that infinite finite groups could give the mixing pattern satisfying the experimental constraint.

PACS numbers: 14.60.St, 14.60.Pq.

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How to explain the mixing pattern of leptons is a challenging question in neutrino physics. One of the most popular strategies is resorting to a discrete flavor group \([1-9]\). A special group \(G_f\) is chosen to design the Lagrangian of the general theory. Then after the spontaneous symmetry breaking of \(G_f\), the residual symmetries in the leptonic sector could determine the mixing pattern of leptons \([10]\). However, the mixing pattern accommodating the experimental data is not unique. There are a set of patterns which could not be discriminated by experiments. Furthermore, considering the results of the scan of finite groups, the order of the flavor group which gives a viable pattern is large \([11, 12]\). Accordingly, the dynamic model based on these groups is complex in techniques. So we confront an important question whether there is a simple mathematical construct to induce a family of mixing patterns which satisfy the experimental constraints. In this Letter, we propose a mathematical object inspired by the idea in quantum mechanics (QM) to address this question. As we know, a general state of a system in QM could be decomposed with the eigenstates of some observables. With the free coefficients of superposition, we could obtain a complete set of the states. Similarly, we speculate that the mathematical construct which gives a family of mixing patterns could be composed of simple group elements which predict special patterns. Its expression is as follow

\[
X(\theta) = \cos \theta A + i \sin \theta B,
\]

where \(A, B\) are elements of a small finite group in the 3-dimensional unitary representation and \(\theta\) is a parameter to mark the hybrid of them, \(i\) is the imaginary factor. To keep that \(X(\theta)\) is unitary, \(A, B\) satisfy the condition

\[
AB^+ = BA^+, \quad A^+ B = B^+ A.
\]

This condition is equivalent to the following one

\[
A = C_1 B, \quad A = BC_2, \quad \text{with} \quad C_1^2 = C_2^2 = I,
\]

where \(I\) is the identity matrix.

We suppose that the neutrino sector satisfies the symmetry expressed by \(X(\theta)\). So the mass matrix of neutrinos follows the relation

\[
X^T(\theta)M_\nu X(\theta) = M_\nu,
\]

for Majorana neutrinos or

\[
X^*(\theta)M_\nu^* M_\nu X(\theta) = M_\nu^* M_\nu,
\]
for Dirac neutrinos. Then employing the leptonic mixing matrix, \( X(\theta) \) could be diagonalized as

\[
U^+ X(\theta) U = \text{diag}(\pm 1, \, \pm 1, \, \pm 1), \quad \text{for Majorana neutrinos,} \tag{6}
\]

\[
U^+ X(\theta) U = \text{diag}(e^{i\alpha}, \, e^{i\beta}, \, e^{i\gamma}), \quad \text{for Dirac neutrinos,} \tag{7}
\]

where \( U \) is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix. Here we work in the basis where the mass matrix of charged leptons is diagonal. In other words, the symmetry in this sector is trivial which could only affect the mixing matrix through the permutation of rows and nonphysical phases. We note that \( X(\theta) \) is an element of the group \( Z_2 \) in the case of Majorana neutrinos. So its power in inducing general mixing patterns is weak. We focus on the case of Dirac neutrinos in this Letter.

Once the group elements \( A, B \) are given, we could obtain a set of mixing patterns with the eigenvectors of the symmetry \( X(\theta) \). As an illustrative example, we choose \( A, B \) from the group \( S_4 \) generated by 3 elements which observe the following relations \([13, 14]\):

\[
S^2 = V^2 = (SV)^2 = (TV)^2 = E, \quad T^3 = (ST)^3 = E, \quad (STV)^4 = E,
\]

where \( E \) is the identity element. The 3-dimensional representations of the generators are expressed as \([13, 14]\)

\[
S = \frac{1}{3} \begin{pmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{pmatrix}, \quad T = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{pmatrix}, \quad V = \mp \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \tag{9}
\]

where \( \omega = e^{i2\pi/3} \), the sign \( \mp \) denotes the representation \( 3 \) and \( 3' \) respectively. Considering the unitary condition in the form of Eq. (2) or Eq. (3), a viable realization of \( A, B \) reads \( A = TV, \quad B = STV \). With the minus sign for the representation of \( V \), \( X(\theta) \) is expressed as

\[
X(\theta) = \begin{pmatrix}
\frac{i}{3} \sin \theta - \cos \theta & \frac{2}{3} e^{i\pi/6} \sin \theta & -\frac{2}{3} e^{-i\pi/6} \sin \theta \\
-\frac{2i}{3} \sin \theta & \frac{2}{3} e^{i\pi/6} \sin \theta & \frac{1}{3} e^{-i\pi/6} \sin \theta - e^{-i2\pi/3} \cos \theta \\
-\frac{2i}{3} \sin \theta & -\frac{1}{3} e^{i\pi/6} \sin \theta - e^{i2\pi/3} \cos \theta & -\frac{2}{3} e^{-i\pi/6} \sin \theta
\end{pmatrix}. \tag{10}
\]

Note that the sign of \( V \) is not important in this case. The replacement \( V \to -V \) is equivalent to the transformation \( X(\theta) \to -X(\theta) \). The eigenvalues and corresponding eigenvectors of \( X(\theta) \) are

\[
\lambda_1 = -1, \quad \lambda_2 = -e^{-i\theta}, \quad \lambda_3 = 1, \tag{11}
\]

\[
u_1 = \frac{1}{N_1} \begin{pmatrix}
e^{i\pi/3} + 1 - \sqrt{3} e^{-i\pi/6} \\
e^{i\pi/3} - e^{i\theta} \\
1
\end{pmatrix}, \quad \nu_2 = \frac{1}{\sqrt{3}} \begin{pmatrix}
\omega \\
\omega^2 \\
1
\end{pmatrix}, \quad \nu_3 = \frac{1}{N_3} \begin{pmatrix}
e^{i\pi/3} - 1 + \sqrt{3} e^{-i\pi/6} \\
e^{i\pi/3} + e^{-i\theta} \\
1
\end{pmatrix}, \tag{12}
\]
where $N_1$ and $N_3$ are normalization factors. Then the mixing matrix of leptons reads

$$U(\theta) = (u_1 \ u_2 \ u_3)$$  \hspace{1cm} (13)

to permutations of rows or columns and nonphysical phases. We give some comments here:

1. Although $X(\theta)$ could be transformed to the diagonal form, i.e., $\text{diag}(-1, \ e^{-i\theta}, 1)$, in general, it is not equivalent to a $U(1)$ group because $U(\theta)$ depends on the parameter $\theta$. Furthermore, $X(\theta)$ could not denotes an infinite group in general cases, because neither the identity element nor the inverse element of $X(\theta)$ could be written in the form of $X(\theta')$.

2. If $\theta/2\pi$ equals $\pm j/k$, where $i, j$ are two natural numbers coprime to each other, $X(\theta)$ becomes a generator of the group $Z_{2k}$ or $Z_k$ when $k$ is odd or even. Thus, in this special case the hybrid of two elements of a finite group generates a new one. However, we note that if a special value $2(\pm j/k)\pi$ of $\theta$ could make the mixing matrix $U(\theta)$ accommodate the experimental data, infinite ones in the form $2(\pm j/k \pm j'/k')\pi$ could also hold when $j'/k'$ is small enough. In other words, if a finite group survives in the experimental constraints, so do infinite ones.

3. Because the translation $\theta \rightarrow \theta + \pi$ is equivalent to the replacement $X(\theta) \rightarrow -X(\theta)$, so the independent range of the parameter $\theta$ is $[0, \pi]$.

Now we go no to give the phenomenological results of $X(\theta)$ in the form of Eq. (10). Compared with the standard parametrization of the mixing matrix, i.e.,

$$U = 
\begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\
    s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}
\end{pmatrix},$$  \hspace{1cm} (14)

where $s_{ij} \equiv \sin \theta_{ij}$, $c_{ij} \equiv \cos \theta_{ij}$, $\delta$ is the Dirac CP-violating phase, we could obtain the mixing angles with the following relations:

$$\sin^2 \theta_{13} = |U_{e3}|^2, \quad \sin^2 \theta_{23} = |U_{\mu 3}|^2/(1 - |U_{e3}|^2), \quad \sin^2 \theta_{12} = |U_{e2}|^2/(1 - |U_{e3}|^2).$$  \hspace{1cm} (15)

Employing the $\chi^2$ function defined as

$$\chi^2 = \sum_{ij=13,23,12} \frac{(\sin^2 \theta_{ij} - (\sin^2 \theta_{ij})_{\exp})^2}{\sigma_{ij}^2},$$  \hspace{1cm} (16)

where $(\sin^2 \theta_{ij})_{\exp}$ are best global fit values from Ref. [15]. $\sigma_{ij}$ are $1\sigma$ errors, the best fit data of the parameter $\theta$ and mixing angles are shown in Table 1. Note that because of the freedom of permutations of rows or columns of the mixing matrix in Eq. (13), the best fit value of $\theta$ in the
| Ordering   | $\chi^2_{\text{min}}$ | $\theta_{bf1}$ | $(\sin^2 \theta_{13})_{bf}$ | $(\sin^2 \theta_{23})_{bf}$ | $(\sin^2 \theta_{12})_{bf}$ |
|-----------|------------------------|----------------|-------------------------------|-------------------------------|-------------------------------|
| Normal    | 11.9396                | 0.11496$\pi$   | 0.021504                      | 0.39575                      | 0.34066                      |
| Inverted  | 9.0379                 | 0.11546$\pi$   | 0.02169                       | 0.6047                       | 0.34072                      |

range $[0, \pi]$ is not unique. The complete set of the best fit values for either mass ordering is listed as

$$
\theta_{bf1}, \theta_{bf2} = \pi/3 - \theta_{bf1}, \theta_{bf3} = \pi/3 + \theta_{bf1}, \\
\theta_{bf4} = 2\pi/3 - \theta_{bf1}, \theta_{bf5} = 2\pi/3 + \theta_{bf1}, \theta_{bf6} = \pi - \theta_{bf1}.
$$

(17)

In the case of normal mass ordering, these best fit data correspond to the following mixing matrix respectively:

$$
U_1 = S_{23}U, \ U_2 = S_{23}S_{12}US_{13}, \ U_3 = S_{12}US_{13}, \ U_4 = S_{13}S_{12}U, \ U_5 = S_{13}U, \ U_6 = US_{13}, \quad (18)
$$

where $U$ is written as Eq (13), the permutation matrices $S_{ij}$ are expressed as

$$
S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

(19)

The mixing matrices in the case of inverted mass ordering are $U_j' = S_{23}U_j$, with $j = 1, 2, 3, 4, 5, 6$. Either in the case of normal or inverted mass ordering, the six best fit values of $\theta$ correspond to the same $\chi^2_{\text{min}}$ and the same mixing angles. This observation could seen from the function $\chi^2$ shown in Figure 1. Some comments are given as follows:

1. Begin with the curve of the function $\chi^2$ of $U_1$ or $U_j'$, other curves of $\chi^2$ could be obtained by the reflection about the axe $\theta = k\pi/6$ step by step, with $k = 1, 2, 3, 4, 5$. Accordingly, the range of the parameter $\theta$ at the level of $3\sigma$ of any mixing matrix could be derived from that of a special mixing matrix. For example, the range of $\theta$ at $3\sigma$ level for $U_3$ and $U_3'$ is $[0.4422\pi, 0.4545\pi], [0.4422\pi, 0.4546\pi]$ respectively. So that for $U_2$ and $U_2'$ is $[(2/3 - 0.4545)\pi, (2/3 - 0.4522)\pi], [(2/3 - 0.4546)\pi, (2/3 - 0.4522)\pi]$ respectively. The range of $\theta$ at $3\sigma$ level for any mixing matrix is narrow, which results from the stringent experimental constraint of $\sin \theta_{13}$. This observation could seen from the the correlation of two mixing angels such as those of $U_3$. See Figure 2 for example, where the mixing angle $\theta_{13}$ covers its full range at
FIG. 1:  The function $\chi^2$ for the viable mixing matrices. The left panel is for the case of normal mass ordering and the right one is for the inverted case. The black dashed curve is the function $\chi^2$ of $U_1(U')$, the blue curve is for $\chi^2$ of $U_2(U')$, the blue dashed curve is for $\chi^2$ of $U_3(U')$, the red dashed curve is for $\chi^2$ of $U_4(U')$, the red curve is for $\chi^2$ of $U_5(U')$, the black curve is for $\chi^2$ of $U_6(U')$.

3σ level, while $\theta_{23}$ or $\theta_{12}$ covers only several percents of the measure of its 3σ range.

2. As the parameter $\theta$ varies in the range at 3σ level, we could obtain a family of mixing patterns. Specially, the values which could be expressed as $2(j/k)\pi$ are in the range of 3σ level for any mixing matrix. For example, $4\pi/9$ is in the range of $\theta$ for the matrix $U_3$ or $U'_3$. $X(\theta = 4\pi/9)$ is a generator of the group $Z_{18}$. So according to the aforementioned observation, there are infinite finite groups $Z_n$ which could accommodate the experimental data. Furthermore, as an interesting phenomenon, the value $\theta = \pi/3$ which is a axe of the reflection of two curves of the function $\chi^2$ corresponds to the well known tri-bi-maximal mixing pattern, i.e., $U(\theta = \pi/3) = U_{TBM}$.

3. The Dirac CP phase could obtained through the Jarlskog invariant expressed as \[ J_{CP} \equiv \text{Im}[U_{22}U_{23}^*U_{32}^*U_{33}] = \frac{1}{8} \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_{12} \cos \theta_{13} \sin \delta. \] (20)

Substitute $J_{CP}$ with the elements of $U(\theta)$, we find it equals 0 independent of the parameter $\theta$. Hence the Dirac CP phase is trivial, i.e., $\sin \delta = 0$.

Conclusion: Beyond the application of a finite group to derive the mixing pattern of leptons, we construct a novel mathematical object with the hybrid of two elements of a discrete group. This construct with a parameter could induce a set of mixing patterns which accommodate the experimental data. On the base of a specific example made from the group $S_4$, we shows that infinite finite groups could give mixing patterns at 3σ level.

Acknowledgments This work was supported by the National Natural Science Foundation of
FIG. 2: Correlations of the mixing angles of the mixing matrix $U_3$.

China under the Grant No. 11405101 and the research foundation of Shaanxi University of Technology under the Grant No. SLGQD-13-10.

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