How many singlets are needed to create a bipartite state using LOCC?

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One-shot entanglement dilution is the protocol by which two distant parties convert a finite number of singlets, which they initially share, to a given bipartite target state, using local operations and classical communication. We refer to the minimum number of singlets needed to achieve this conversion as the one-shot entanglement cost of the target state. In this paper we evaluate this quantity in the most general case where dilution is achieved with a fixed accuracy. For the case of exact dilution, the expression for the cost reduces to a simple form analogous to the entanglement of formation. Extending our result to the asymptotic scenario yields an alternative proof of the equivalence between the asymptotic entanglement cost and the regularized entanglement of formation.

I. INTRODUCTION

While the characterization of entangled states is given as an algebraic condition, namely, entangled states are those states that cannot be decomposed into a convex combination of product states, the characterization of non-local states is much more complicated, as it is usually given in terms of the usefulness of a given state as a resource for some non-local task. For example, non-locality of a state could be quantified by the degree of violation of a set of Bell’s inequalities when local measurements are performed on it [1]. Alternatively, non-locality of a set of states could be described by the possibility of discriminating between them with local measurements [2], or copying them by local operations [3]. Hence, there exist many different notions of non-locality, depending on the operational criteria chosen to define them. However, irrespective of the way in which non-locality of a state or a set of states is defined, it seems that the notions of entanglement and non-locality are not equivalent (see e.g. [4, 5]), as there exist entangled states admitting a local hidden variables model [1], and sets of local states requiring entangled measurements for their discrimination [2].

Such an inequivalence has, however, only been established from a qualitative point of view. In fact, while the study of non-locality has been mainly considered in the finite scenario, entanglement theory, on the contrary, has been developed almost exclusively in the asymptotic regime. A fundamental property, common to both entanglement and non-locality, is that they cannot be created or increased by local operations and classical communication (LOCC) alone. Hence, one step towards improving the understanding of the relationship between entanglement and non-locality is to quantify this point of view, and to develop a theory of entanglement for finite resources, using LOCC. The results we present in this paper are a step towards the establishment of such a theory.

Entanglement is considered to be a fungible resource in quantum information processing tasks, and hence it is important to study how it can be converted from one form to another, under LOCC. Entanglement dilution is the process by which a maximally entangled state, shared between two parties (who are in different locations), can be converted into a given partially entangled bipartite state, using LOCC. In [6] the conversion of $m_n$ copies of singlets to $n$ copies of a two-qubit pure target state $\rho_{AB}$ was studied. The more general case of a mixed bipartite state was considered in [7]. However, since these conversions cannot in general be achieved perfectly for a finite $n$, only the case of asymptotically ($n \to \infty$) vanishing error was considered. The optimal rate of this entanglement dilution, referred to as the (asymptotic) entanglement cost of $\rho_{AB}$, was defined to be the ratio $m_n/n$ in this limit. This quantity is often heuristically interpreted as the minimum number of singlets needed to create a copy of the state $\rho_{AB}$. Note, however, that the entanglement cost, by its very definition, is not equal to the amount of entanglement needed to create only one copy of the target state $\rho_{AB}$. The true characterization of this amount would be an important step in developing a finite entanglement theory under LOCC.

In addition to the relevance of this research in investigating the relationship between non-locality and entanglement, there is another fundamental motivation behind it. In fact, the consideration of the asymptotic limit in evaluating the entanglement cost is not meaningful from the practical point of view, since it is unrealistic to consider the two parties to initially share an infinite amount of entanglement. Instead, it is more realistic to consider the conversion of a finite number of shared singlets (or equivalently, a maximally entangled state of finite rank) to a given bipartite target state, with finite accuracy, using LOCC.

In this paper we evaluate the minimum number of shared singlets required to create a single copy of a given bipartite target state $\rho_{AB}$, under LOCC. We refer to the protocol used as one-shot entanglement dilution under LOCC. The corresponding minimum number of singlets needed is referred to as the one-shot entanglement cost of $\rho_{AB}$. In analogy with the asymptotic scenario, we also
refer to this quantity as the optimal rate of one-shot entanglement dilution. For sake of generality, we evaluate the cost in the situation in which the conversion can be achieved with a fixed accuracy. The results of this paper rely on a key lemma, proved in [3], which gives an expression for the fidelity of entanglement dilution for finite resources (see also [3, 10]).

The optimal rate of one-shot entanglement dilution, not under LOCC, but under a larger class of maps, was evaluated in [11]. This class of maps comprise those that generate negligible or no entanglement, and contains LOCC as a subclass. However, these maps can in general be non-local. Hence, the corresponding entanglement cost does not yield a measure of the non-locality of the target state.

Finally, we would like to remark that Hastings’ counterexample [12] to the additivity conjecture has revived interest in the entanglement cost, in that it implicitly proves the inequivalence between entanglement cost and the entanglement of formation [13].

The paper is organized as follows. In Section II we introduce the necessary notations and definitions. Section III contains our main result, stated as Theorem [1]. This theorem is proved in Section IV. Finally, in Section V we show how Theorem III can be used to yield an alternative proof of the equivalence of the asymptotic entanglement cost and the regularised entanglement of formation [14]. A study of entanglement distillation in the finite regime will be presented in a forthcoming paper [15].

II. NOTATIONS AND DEFINITIONS

Let $\mathcal{G}(H)$ denote the set of states (or density operators, i.e. positive operators of unit trace) acting on the Hilbert space $H$. Given a state $\rho \in \mathcal{G}(H)$, we denote by $\Pi_\rho$ the projector onto its support, and, for a pure state $|\phi\rangle$, we denote the projector $|\phi\rangle \langle \phi|$ simply as $\phi$. Throughout this paper we consider finite dimensional Hilbert spaces and take the logarithm to base 2.

For given orthonormal bases $\{|i_A\rangle\}_{i=1}^{d_A}$ and $\{|i_B\rangle\}_{i=1}^{d_B}$ in Hilbert spaces $H_A$ and $H_B$ of dimensions $d_A$ and $d_B$ respectively, we define the canonical maximally entangled state (MES) of rank $M \leq \min\{d_A, d_B\}$ to be

$$|\Phi^+_M\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |i_A\rangle \otimes |i_B\rangle.$$ (1)

The trace distance between two operators $A$ and $B$ is given by

$$\|A - B\|_1 := \text{Tr}[A - B],$$

where $|X| := \sqrt{X^\dagger X}$. The fidelity of two states $\rho$ and $\sigma$ is defined as

$$F(\rho, \sigma) := \text{Tr}\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} = \|\sqrt{\rho} \sqrt{\sigma}\|_1.$$ (2)

We use the following lemma [16, 17]:

**Lemma 1 (Gentle measurement lemma)** For a state $\rho \in \mathcal{G}(H)$ and an operator $0 \leq P \leq 1$, if $\text{Tr}[\rho^\dagger P] \geq 1 - \delta$, then

$$\|\rho - \sqrt{P}\rho\sqrt{P}\|_1 \leq 2\sqrt{\delta}.$$ (3)

The same holds if $\rho$ is a subnormalized density operator.

Our main result is expressed in terms of the following entropies:

**Definition 1 (Zero-Rényi entropies)** The relative Rényi entropy of order zero between a state $\rho$ and a positive operator $\sigma$ is defined as

$$S_0(\rho|\sigma) := -\log \text{Tr}[\Pi_\rho \sigma].$$ (4)

If $\Pi_\rho \Pi_\sigma = 0$, $S_0(\rho|\sigma) := +\infty$. From this, given a bipartite state $\rho_{AB}$, we define the conditional zero-entropy of $\rho_{AB}$ given $\sigma_B \in \mathcal{G}(H_B)$ as

$$H_0(\rho_{AB}|\sigma_B) := -S_0(\rho_{AB}||_A \otimes \sigma_B),$$ (5)

and

$$H_0(\rho_{AB}|B) := \max_{\sigma_B \in \mathcal{G}(H_B)} H_0(\rho_{AB}|\sigma_B).$$ (6)

For any given decomposition of a bipartite state $\rho_{AB}$ into the pure-state ensemble $E = \{\rho_i, \phi_{AB}^i\}$, we introduce the tripartite classical-quantum (c-q) state

$$\rho_{RAB}^\varepsilon := \sum_i \rho_i |i\rangle \langle i| \otimes \phi_{AB}^i,$$ (7)

where $R$ denotes an auxiliary classical system represented by the fixed orthonormal basis $\{|i_R\rangle\}$. The set of all such tripartite c-q extensions of $\rho_{AB}$—which are in one-to-one correspondence with its pure-state decompositions—is denoted in the following by $\mathcal{D}_{c-q}(\rho_{AB})$. From now on, the superscript $E$ in $\rho_{RAB}^\varepsilon$ will be dropped for notational simplicity.

In analogy with [18], we define the following quantity:

**Definition 2 (c-q smoothing)** For any $\varepsilon \geq 0$, we define the c-q-smoothed conditional zero-Rényi entropy of a c-q state, $\rho_{RA} := \sum_i \rho_i |i\rangle \langle i| \otimes \rho_A^i$, given $R$ as:

$$H_0^\varepsilon(\rho_{RA}|R) := \min_{\omega_{RA} \in B_{c-q}(\rho_{RA})} H_0(\omega_{RA}|R),$$ (8)

where the minimum is taken over classical-quantum operators belonging to the set $B_{c-q}(\rho_{RA})$ defined as follows:

$$B_{c-q}(\rho_{RA}) := \left\{ \omega_{RA} \geq 0 \mid \omega_{RA} = \sum_i |i\rangle \langle i| \otimes \omega_A^i \quad & \|\omega_{RA} - \rho_{RA}\|_1 \leq \varepsilon \right\}.$$ (9)

The basis $\{|i_R\rangle\}$ used in the above definition is the same as that appearing in $\rho_{RA}$. 
Since, for a c-q operator $\omega_{RA}$, $\Pi_{RA} = \sum_i |i\rangle\langle i| \otimes \Pi_{\omega_A}$, note that we can equivalently write

$$
H_0^{\epsilon}(\rho_{RA}|R) = \min_{\omega_{RA} \in B_{\epsilon}(\rho_{RA})} \max_{\sigma_R \in C(\mathcal{H}_R)} \{ -S_0(\omega_{RA}\|\sigma_R \otimes 1_A) \}
$$

$$= \min_{\omega_{RA} \in B_{\epsilon}(\rho_{RA})} \max_{\sigma_R \in C(\mathcal{H}_R)} \{ \log \text{Tr}[\Pi_{\omega_{RA}} (\sigma_R \otimes 1_A)] \}
$$

$$= \min_{\omega_{RA} \in B_{\epsilon}(\rho_{RA})} \{ \log \lambda_{\max} \left( \sum_i |i\rangle\langle i|_R \text{Tr}[\Pi_{\omega_A}] \right) \}
$$

$$= \min_{\omega_{RA} \in B_{\epsilon}(\rho_{RA})} \{ \max \log \text{Tr} [\Pi_{\omega_A}] \},
$$

(10)

where the notation $\lambda_{\max}(X)$ denotes the maximum eigenvalue of the self-adjoint operator $X$.

### III. ONE-SHOT ENTANGLEMENT DILUTION AND ITS OPTIMAL RATE

Two parties, Alice and Bob, share a single copy of a maximally entangled state $|\Psi_M^+\rangle$ of Schmidt rank $M$, and wish to convert it into a given bipartite target state $\rho_{AB}$ using an LOCC map $\Lambda$. We refer to the protocol used for this conversion as one-shot entanglement dilution. For sake of generality, we consider the situation where the final state of the protocol is $\epsilon$-close to the target state with respect to a suitable distance measure, for any given $\epsilon \geq 0$. More precisely, defining the fidelity of the protocol to be $F^2(\Lambda(\Psi_M^+), \rho_{AB})$, we require $F^2(\Lambda(\Psi_M^+), \rho_{AB}) \geq 1 - \epsilon$. Further, we denote the optimal fidelity of one-shot entanglement dilution as

$$F_{\text{dil}}(\rho_{AB}, M) := \max_{\Lambda \in \text{LOCC}} F^2(\Lambda(\Psi_M^+), \rho_{AB}).
$$

**Definition 3** ($\epsilon$-achievable dilution rates) For any given $\epsilon \geq 0$, a real number $R \geq 0$ is said to be an $\epsilon$-achievable rate for dilution, if there exists an integer $M$, with $\log M \leq R$, such that $F_{\text{dil}}(\rho_{AB}, M) \geq 1 - \epsilon$.

**Definition 4** (One-shot entanglement cost) For any given $\epsilon \geq 0$, the one-shot entanglement cost $E_{\text{C}}^{(1)}(\rho_{AB}; \epsilon)$ is the minimum of all $\epsilon$-achievable dilution rates.

The main result contained in this paper is summarized in the following theorem:

**Theorem 1** (Main result) For any $\epsilon \geq 0$, the one-shot entanglement cost under LOCC, corresponding to an error less than or equal to $\epsilon$, satisfies the following bounds:

$$
\min_{\mathcal{D}_{cq}(\rho_{AB})} H_0^{2\epsilon}(\rho_{RA}|R) \leq E_{\text{C}}^{(1)}(\rho_{AB}; \epsilon) \leq \min_{\mathcal{D}_{cq}(\rho_{AB})} H_0^{\epsilon/2}(\rho_{RA}|R),
$$

(12)

where the minimum is taken over all tripartite extensions $\rho_{RAB}$ defined in $\mathcal{D}$ and $\rho_{RA} = \text{Tr}_B[\rho_{RAB}]$.

In particular, for the case of exact (i.e., zero error) entanglement dilution, we obtain the corresponding cost to be given by

$$E_{\text{C}}^{(1)}(\rho_{AB}) = \min_{\mathcal{D}_{cq}(\rho_{AB})} H_0(\rho_{RA}|R).$$

(13)

**Remark.** Notice that, for any given $\epsilon \geq 0$, eq. (12) essentially identifies $\min_{\mathcal{D}_{cq}(\rho_{AB})} H_0^{\epsilon}(\rho_{RA}|R)$ with the one-shot entanglement cost $E_{\text{C}}^{(1)}(\rho_{AB}; \epsilon)$.

Note, moreover, the formal analogy between $E_{\text{C}}^{(1)}(\rho_{AB})$ and the entanglement of formation $E_F(\rho_{AB})$, defined as

$$E_F(\rho_{AB}) := \min_{\mathcal{E} = \{\rho_A, \phi_{AB}\}} \sum_i p_i S(\rho_A^i),
$$

(14)

where $\rho_A^i = \text{Tr}_B[\phi_{AB}^i]$ and $S(\rho) := -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy. In fact, using e.g. Lemma 4 in [19], it is possible to write the entanglement of formation as follows

$$E_F(\rho_{AB}) = \min_{\mathcal{D}_{cq}(\rho_{AB})} H(\rho_{RA}|R),
$$

(15)

where $H(\rho_{RA}|R)$ is defined through eqs. (11) and (12), with $S_0(\rho||\sigma)$ replaced by the quantum relative entropy $S(\rho||\sigma)$ defined as

$$S(\rho||\sigma) := \begin{cases} \text{Tr}[\rho \log \rho - \rho \log \sigma], & \text{if } \rho, \sigma \leq \Pi, \\ +\infty, & \text{otherwise}. \end{cases}
$$

(16)

Since $S(\rho||\sigma) \geq S_0(\rho||\sigma)$, we have that $E_F(\rho_{AB}) \leq E_{\text{C}}^{(1)}(\rho_{AB})$, as one would expect.

### IV. PROOF OF THE MAIN RESULT

The proof of our main result relies on the following lemma, proved in [3] (see also [9, 10]):

**Lemma 2** For any given bipartite state $\rho_{AB}$, the optimal dilution fidelity is given by

$$F_{\text{dil}}(\rho_{AB}, M) = \max_{\epsilon} \sum_i p_i \sum_{j=1}^M \lambda_j^{(i)},
$$

(17)

where the maximum is taken over all pure state decompositions $\mathcal{E} = \{p_i, \phi_{AB}^i\}$ of $\rho_{AB}$, and $\{\lambda_j^{(i)}\}$ are the eigenvalues of $\rho_A^i = \text{Tr}_B[\phi_{AB}^i]$, arranged in non-increasing order.

Given an ensemble of pure bipartite states $\mathcal{E} = \{p_i, \phi_{AB}^i\}$ for $\rho_{AB}$, let $\rho_{RA} := \text{Tr}_B[\rho_{RAB}]$, with $\rho_{RAB}$ defined in $\mathcal{D}$. We introduce the following quantity:

$$E_\epsilon(\mathcal{E}) := \min_{\Pi_A} \left\{ \max \log \text{Tr} [\Pi_A] \left| \sum_i p_i \text{Tr}[\Pi_A \rho_A^i] \geq 1 - \epsilon \right. \right\}.
$$

(18)
Lemma 3 For any $\varepsilon \geq 0$, and any choice of the ensemble $\mathcal{E}$ for $\rho_{AB}$, the following holds:

$$H_0^{2\varepsilon}(\rho_{RA}|R) \leq E^\varepsilon(\mathcal{E}) \leq H_0^{\varepsilon/2}(\rho_{RA}|R).$$

Proof. We first prove the bound

$$H_0^{\varepsilon}(\rho_{RA}|R) \geq E^{2\varepsilon}(\mathcal{E}).$$

Let $\omega_{RA} = \sum_i |i\rangle_R \otimes \omega^i_A$, belonging to $B_{cq}(\rho_{RA})$, be the operator achieving the minimum in (10). The projection onto its support is given by $\Pi_{\omega_{RA}} = \sum_i |i\rangle_R \otimes \Pi_{\omega^i_A}$. Hence, $\omega_{RA} \in B_{cq}(\rho_{RA})$ yields a set of projectors $\{\Pi_{\omega^i_A}\}$ for which

$$\sum_i p_i \text{Tr}[\Pi_{\omega^i_A} \rho^i_A] = \text{Tr}[\Pi_{\omega_{RA}} \rho_{RA}] \geq 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon.$$

In the last line we have made use of the fact that $\omega_{RA} \in B_{cq}(\rho_{RA})$, due to which $\text{Tr}[\omega_{RA}] \geq 1 - \varepsilon$. This implies that the set of projectors $\{\Pi_{\omega^i_A}\}$ satisfies the required condition in the definition (13) of $E^{2\varepsilon}(\mathcal{E})$, hence proving (20).

We now prove the lower bound

$$E^{\varepsilon}(\mathcal{E}) \geq H_0^{2\varepsilon}(\rho_{RA}|R).$$

Let $\{\Pi^i_A\}$ be the set of projectors achieving the minimum in eq. (12). Therefore, $\sum_i p_i \text{Tr}[\Pi^i_A \rho^i_A] \geq 1 - \varepsilon$. For later convenience, let us set $\varepsilon_i := 1 - \text{Tr}[\Pi^i_A \rho^i_A]$. Let us define $\omega^i_A := \Pi^i_A \rho^i_A \Pi^i_A$ and $\omega_{RA} := \sum_i p_i |i\rangle_R \otimes \omega^i_A$. Then, by Lemma 1 and the concavity of $x \mapsto \sqrt{x}$,

$$|\omega_{RA} - \rho_{RA}|_1 = \sum_i p_i \|\omega^i_A - \rho^i_A\|_1 \leq \sum_i p_i 2\sqrt{\varepsilon_i} \leq 2 \sqrt{\sum_i p_i \varepsilon_i} = 2\sqrt{\varepsilon},$$

and hence $\omega_{RA} \in B_{cq}^{2\varepsilon}(\rho_{RA})$. Moreover, since by construction $\Pi_{\omega^i_A} \leq \Pi^i_A$ for all $i$, we obtain eq. (22). □

With Lemma 3 in hand, the proof of our main result, Theorem 1 reduces to the proof of the following identity:

$$E^{(1)}_{C}(\rho_{AB};\varepsilon) = \min_{\mathcal{E}} E^{\varepsilon}(\mathcal{E}).$$

We split the proof of this into Lemma 4 and Lemma 5 below.

**Lemma 4 (Direct part) For any $\varepsilon \geq 0$,**

$$E^{(1)}_{C}(\rho_{AB};\varepsilon) \leq \min_{\mathcal{E}} E^{\varepsilon}(\mathcal{E}).$$

**Proof.** From Lemma 2

$$F_{\text{dil}}(\rho_{AB}, M) = \max_{\mathcal{E}} \sum_{i} p_i \text{Tr}[Q^i_M \rho^i_A],$$

where $Q^i_M$ are the projectors onto the eigenvectors associated with the $M$ largest eigenvalues of $\rho^i_A$. Let us now fix an ensemble decomposition $\mathcal{E} := \{\hat{p}_i, \hat{\rho}_{AB}\}$ for $\rho_{AB}$, and let us choose the integer $M$ such that $\log M = E^{\varepsilon}(\mathcal{E})$. Then, from the definition (13), we know that there exists a set of projectors $\{\Pi^i_A\}$, with rank $\Pi^i_A \leq M$ for all $i$, such that $\sum_i \hat{p}_i \text{Tr}[\Pi^i_A \rho^i_A] \geq 1 - \varepsilon$. This implies that $\log M$ is an $\varepsilon$-achievable rate, since

$$F_{\text{dil}}(\rho_{AB}, M) \geq \sum_i \hat{p}_i \text{Tr}[Q^i_M \rho^i_A] \geq \sum_i \hat{p}_i \text{Tr}[\Pi^i_A \rho^i_A] \geq 1 - \varepsilon,$$

where $Q^i_M$ is the projector onto the $M$ largest eigenvalues of $\rho^i_A$. The second inequality in (27) is due to the fact that, for any projector $\Pi^i_A$ with rank $\Pi^i_A \leq M$, $\text{Tr}[\Pi^i_A \rho^i_A] \leq \text{Tr}[Q^i_M \rho^i_A]$. Hence $E^{\varepsilon}(\mathcal{E})$ is itself an $\varepsilon$-achievable rate for any choice of $\mathcal{E}$, and the statement of the lemma follows. □

**Lemma 5 (Weak converse) For any $\varepsilon \geq 0$,**

$$E^{(1)}_{C}(\rho_{AB};\varepsilon) \geq \min_{\mathcal{E}} E^{\varepsilon}(\mathcal{E}).$$

**Proof.** Let $\log M$ be an $\varepsilon$-achievable rate. This is equivalent to saying that $F_{\text{dil}}(\rho_{AB}, M) \geq 1 - \varepsilon$. In the following, we prove that this implies that

$$\log M \geq \min_{\mathcal{E}} E^{\varepsilon}(\mathcal{E}).$$

Let $\mathcal{E} := \{p_i, \hat{\rho}_{AB}\}$ be the ensemble decomposition of $\rho_{AB}$ achieving $F_{\text{dil}}(\rho_{AB}, M)$ in (17), and consider the Schmidt decomposition $|\phi_{AB}\rangle = \sum_j \sqrt{\lambda_j} |j_A\rangle |j_B\rangle$, where the Schmidt coefficients are arranged in non-increasing order. The optimal dilution fidelity given by eq. (17) can then be expressed as

$$F_{\text{dil}}(\rho_{AB}, M) = \sum_i p_i \text{Tr}[\tilde{\omega}^i_A],$$

where

$$\tilde{\omega}^i_A := \sum_{j=1}^M \lambda_j^{(i)} |j^{(i)}\rangle \langle j^{(i)}|_A.$$
We now proceed by observing that
\[ E^C(\mathcal{E}) \]
\[ \leq \min_{\{\rho_A^i\}} \left\{ \max_i \log \text{Tr}[\Pi_{\rho_A^i}] \, \left| \sum_i p_i \text{Tr}[\rho_A^i] \geq 1 - \varepsilon \right\} \right. \]
\[ \left. \quad \text{for all } \rho_A^i \leq \rho_A^i, \forall i \right\} . \] (32)

This is due to the fact that, for any candidate set \{\rho_A^i\} for the right hand side, the corresponding set of projectors \{\Pi_{\rho_A^i}\} satisfies the conditions in the definition \[\text{20}\] of \( E^C(\mathcal{E}) \). This is because \( 1 - \varepsilon \leq \sum_i p_i \text{Tr}[\Pi_{\rho_A^i}] \leq \sum_i p_i \text{Tr}[\rho_A^i] \). In particular, since \( \omega_A^i \leq \rho_A^i \), for all \( i \), and \( \sum_i p_i \text{Tr}[\omega_A^i] \geq 1 - \varepsilon \), due to \[\text{19}\], we have
\[ \min_{\varepsilon} E^C(\mathcal{E}) \leq E^C(\mathcal{E}) \leq \max_i \log \text{Tr}[\Pi_{\omega_A^i}] \]
\[ \leq \log M, \]
for any \( \varepsilon \)-achievable rate \( \log M \). ■

V. ASYMPTOTIC COST AND THE REGULARIZED ENTANGLEMENT OF FORMATION

In \[\text{14}\] it was proved that the asymptotic entanglement cost of preparing a bipartite state \( \rho_{AB} \) is equal to the regularized entanglement of formation, defined as,
\[ E_F^{\infty}(\rho_{AB}) := \lim_{n \to \infty} \frac{1}{n} E_F(\rho_{AB}^n) \]
\[ = \lim_{n \to \infty} \frac{1}{n} \min_{\rho_{RA}^n} \text{H}^n(\rho_{RA}^n|\sigma_{RA}), \] (34)
In this Section we show how the application of our main result, Theorem \[\text{1}\] to the case of multiple \( n \) copies of the bipartite state \( \rho_{AB} \), and consideration of the asymptotic scenario \( n \to \infty \) yields a new proof of the above equivalence.

More precisely defining the asymptotic entanglement cost as
\[ E_C(\rho_{AB}) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} E_C^{(1)}(\rho_{AB}^n; \varepsilon), \] (35)
we prove the following:
\[ E_C(\rho_{AB}) = E_F^{\infty}(\rho_{AB}). \] (36)

Proof. From the upper bound in Theorem \[\text{1}\] we obtain
\[ \frac{1}{n} E_C^{(1)}(\rho_{AB}^n; 2\varepsilon) \leq \frac{1}{n} \min_{\rho_{RA}^n} \text{H}^n(\rho_{RA}^n|\sigma_{RA}), \] (37)
where \( \rho_{RA}^n = \text{Tr}_{AB}^{\otimes n}[\rho_{AB}^n] \), with \( \rho_{RA}^n \in \mathcal{S}_{\text{AB}}(\mathcal{H}_R \otimes \mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} \) denoting a c-q extension of the state \( \rho_{AB}^n \). By taking the appropriate limits \( n \to \infty \) followed by \( \varepsilon \to 0 \) on either side of \[\text{37}\] and employing the results in the Appendix, we arrive at
\[ E_C(\rho_{AB}) \leq E_F^{\infty}(\rho_{AB}). \] (38)

To prove the converse, namely
\[ E_C(\rho_{AB}) \geq E_F^{\infty}(\rho_{AB}), \] (39)
we start with the lower bound in Theorem \[\text{1}\]. This gives
\[ \frac{1}{n} E_C^{(1)}(\rho_{AB}^n; \varepsilon^2) \]
\[ \geq \frac{1}{n} \min_{\sigma_{RA}^n} \text{H}^n(\rho_{RA}^n|\sigma_{RA}), \]
\[ \geq \frac{1}{n} \min_{\rho_{RA}^n} \text{H}^n(\rho_{RA}^n|\sigma_{RA}), \]
\[ \geq \frac{1}{n} \min_{\rho_{RA}^n} \text{H}^n(\rho_{RA}^n|\rho_{RA}^n) \]
\[ = \lambda \frac{1}{n} \min_{\rho_{RA}^n} \text{H}^n(\rho_{RA}^n|\rho_{RA}^n) \]
\[ = \lambda \frac{1}{n} \min_{\rho_{RA}^n} \text{H}^n(\rho_{RA}^n|\rho_{RA}^n) \]
where: in the fourth line, \( \omega_{RA}^n \) is the minimizing state; in the fifth line we use the fact that \( \text{H}^n(\omega_{RA}^n|\rho_{RA}^n) \geq \text{H}^n(\omega_{RA}^n|\sigma_{RA}) \); the sixth line follows from Lemma 4 in \[\text{19}\].

The last approximation comes by applying the Fannes’ inequality to \( \text{H}^n(\omega_{RA}^n|\omega_{RA}^n) = S(\omega_{RA}^n) - S(\omega_{RA}^n) \). Finally, taking the limit \( n \to \infty \) followed by \( \varepsilon \to 0 \), we arrive at eq. \[\text{39}\]. ■

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APPENDIX A: PROOF OF \( E_C(\rho_{AB}) \leq E_F^{\infty}(\rho_{AB}) \)

In order to prove inequality \[\text{38}\], we employ the well-known Quantum Information Spectrum Method \[\text{21, } 22\]. A fundamental quantity used in this approach is the quantum spectral inf-divergence rate, defined as follows \[\text{21, } 22\]:

Definition 5 (Spectral inf-divergence rate) Given a sequence of states \( \hat{\rho} = \{\rho_n\}_{n=1}^\infty \) and a sequence of positive operators \( \hat{\sigma} = \{\sigma_n\}_{n=1}^\infty \), the quantum spectral
inf-divergence rate is defined in terms of the difference operators \( \Delta_n(\gamma) = \rho_n - 2^{n\gamma} \sigma_n \) as

\[
D(\hat{\rho}||\hat{\sigma}) := \sup \left\{ \gamma : \liminf_{n \to \infty} \frac{1}{n} \left[ \Delta_n(\gamma) \geq 0 \right] \Delta_n(\gamma) = 1 \right\},
\]

where the notation \( \{ X \geq 0 \} \), for a self-adjoint operator \( X \), is used to indicate the projector onto the subspace where \( X \geq 0 \).

We first note that, by definitions \( [5] \), \( [6] \) and \( [7] \), we have:

\[
\begin{align*}
\min_{\mathcal{D}_c(\rho^n_{RA}|\rho^0_{RA})} H^n_\sigma(\rho^n_{RA}|R) &= - \max_{\mathcal{D}_c(\rho^n_{AB})} \max_{\omega^n_{RA} \in \mathcal{B}_c(\rho^n_{RA})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A), \\
&= - \max_{\mathcal{D}_c(\rho^n_{AB})} \max_{\omega^n_{RA} \in \mathcal{B}_c(\rho^n_{AB})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A). & (A2)
\end{align*}
\]

Then we prove the following lemma:

**Lemma 6** For any bipartite state \( \rho_{AB} \),

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \max_{\epsilon > 0} \max_{\mathcal{D}_c(\rho^n_{AB})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A) \geq \max_{\mathcal{D}_c(\rho^n_{AB})} \min_{\sigma^n_R} D(\hat{\rho}_{RA}||\hat{\sigma}_R \otimes \mathbb{I}_A), & (A3)
\]

where \( \hat{\rho}_{RA} := \{ \rho^n_{RA} | \rho_{RA} \in \mathcal{D}_c(\rho^n_{AB}) \} \) \( n \geq 1 \),
\( \hat{\sigma}_R := \{ \sigma^n_R \in \mathcal{G}(\mathcal{H}_R^\otimes) \} \) \( n \geq 1 \), and \( \mathbb{I}_A := \{ \mathbb{I}^\otimes_A \} n \geq 1 \).

**Proof.** Let \( \hat{\rho}_{RA} \) be the c-q extension of \( \rho_{AB} \) for which the maximum in the r.h.s. of eq. \( (A3) \) is achieved, and let \( \hat{\rho}_{RA} \) be its reduced state. Note that, for any fixed \( \epsilon > 0 \),

\[
\max_{\mathcal{D}_c(\rho^n_{AB})} \max_{\omega^n_{RA} \in \mathcal{B}_c(\rho^n_{RA})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A) \geq \max_{\mathcal{D}_c(\rho^n_{RA})} \min_{\omega^n_{RA} \in \mathcal{B}_c(\rho^n_{RA})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A) & (A4)
\]

\[
\max_{\mathcal{D}_c(\rho^n_{AB})} \max_{\omega^n_{RA} \in \mathcal{B}_c(\rho^n_{RA})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A) \geq \max_{\mathcal{D}_c(\rho^n_{AB})} \min_{\omega^n_{RA} \in \mathcal{B}_c(\rho^n_{RA})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A). & (A5)
\]

For each \( \sigma^n_R \) and any \( \gamma \in \mathbb{R} \), define the projector

\[
P^n_\gamma \equiv P^n_\gamma(\sigma^n_R) := \{ \omega^n_{RA} \in \mathbb{I}^\otimes_A : S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A) \geq 2^{n\gamma} \}.
\]

Since the operator \( \omega^n_{RA} \) in \( (A5) \) is a c-q operator, it is clear that the minimization over \( \sigma^n_R \) in \( (A3) \) can be restricted to states diagonal in the basis chosen in representing c-q operators. Consequently, also \( P^n_\gamma \) has the same c-q structure.

Next, for any sequence \( \hat{\sigma}_R := \{ \sigma^n_R \}_{n \geq 1} \), fix \( \delta > 0 \) and choose \( \gamma \equiv \gamma(\delta \sigma) := D(\hat{\rho}_{RA}||\hat{\sigma}_R \otimes \mathbb{I}_A) - \delta \). Then it follows from the definition \( (A1) \) that, for \( n \) large enough,

\[
\text{Tr} \left[ P^n_\gamma(\hat{\rho}_{RA}) \right] \geq 1 - \epsilon^2/4,
\]

for any \( \epsilon > 0 \). Further, define \( \omega^n_{RA} \equiv \omega^n_{RA}(\sigma^n_R) := P^n_\gamma(\hat{\rho}_{RA}) P^n_\gamma \), which is clearly in \( B_c(\hat{\rho}_{RA}) \), due to Lemma 1.

Then, using the fact that \( \Pi_{\omega^n_{RA}} \leq P^n_\gamma \), and Lemma 2 of \( 23 \), we have, for any fixed \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \left( A3 \right) \geq \lim_{n \to \infty} \frac{1}{n} \min_{\mathcal{D}_c(\rho^n_{AB})} \min_{\sigma^n_R} S_0(\omega^n_{RA}||\sigma^n_R \otimes \mathbb{I}^\otimes_A).
\]

Since this holds for any arbitrary \( \delta > 0 \), it yields the required inequality \( (A3) \) in the limit \( \epsilon \to 0 \).

From \( (37) \), \( (A2) \) and Lemma 6 it follows that

\[
E(\rho_{AB}) \leq - \max_{\mathcal{D}_c(\rho^n_{AB})} \min_{\sigma^n_R} D(\hat{\rho}_{RA}||\hat{\sigma}_R \otimes \mathbb{I}_A), & (A9)
\]

with \( \hat{\rho}_{RA} = \{ \rho^n_{RA} \}_{n \geq 1} \). Further, from the Generalized Stein’s Lemma \( 24 \) and Lemma 4 in \( 19 \), the lemma below follows:

**Lemma 7** For any given bipartite state \( \rho_{RA} \),

\[
\min_{\sigma^n_R} D(\hat{\rho}_{RA}||\hat{\sigma}_R \otimes \mathbb{I}_A) = S(\rho_{RA}||\rho_{RA} \otimes \mathbb{I}_A), & (A10)
\]

where \( \hat{\rho}_{RA} = \{ \rho^n_{RA} \}_{n \geq 1} \), \( \hat{\sigma}_R := \{ \sigma^n_R \in \mathcal{G}(\mathcal{H}^\otimes_R) \} \) \( n \geq 1 \), and \( \mathbb{I}_A := \{ \mathbb{I}^\otimes_A \} n \geq 1 \).

**Proof.** Consider the family of sets \( M := \{ M_n \}_{n \geq 1} \)
\( M_n := \{ \sigma^n_R \otimes \mathbb{I}_A \in \mathcal{G}(\mathcal{H}_R^\otimes \otimes \mathcal{H}_A^\otimes) \} \),

such that \( \tau^n_A := \mathbb{I}^\otimes_A / d^n_A \). For this family, the Generalized Stein’s Lemma (Proposition III.1 of \( 24 \)) holds.

More precisely, for a given bipartite state \( \rho_{RA} \), let us define

\[
S_{\infty}(\rho_{RA}) := \lim_{n \to \infty} \frac{1}{n} S_{M_n}(\rho^n_{RA}), & (A12)
\]

with \( S_{M_n}(\rho^n_{RA}) := \min_{\omega^n_{RA} \in M_n} S_0(\omega^n_{RA}||\omega^n_{RA}) \), and \( \Delta_n(\gamma) = \mathbb{I}^\otimes_A - 2^{n\gamma} \omega^n_{RA} \). From the Generalized Stein’s Lemma \( 24 \) it follows that, for \( \gamma > S^{\infty}(\rho_{RA}) \),

\[
\lim_{n \to \infty} \min_{\omega^n_{RA} \in M_n} \text{Tr} \left[ \{ \omega^n_{RA} \geq 0 \} \Delta_n(\gamma) \right] = 0, & (A13)
\]

implying that \( \min_{\omega^n_{RA} \in M_n} D(\hat{\rho}_{RA}||\omega^n_{RA}) \leq S^{\infty}(\rho_{RA}) \). On the other hand, for \( \gamma < S^{\infty}(\rho_{RA}) \),

\[
\lim_{n \to \infty} \min_{\omega^n_{RA} \in M_n} \text{Tr} \left[ \{ \omega^n_{RA} \geq 0 \} \Delta_n(\gamma) \right] = 1, & (A14)
\]
implying that \( \min_{\omega_{RA} \in M} D(\hat{\rho}_{RA} \| \hat{\omega}_{RA}) \geq S_{\infty}^M(\rho_{RA}) \). Hence

\[
\min_{\omega_{RA} \in M} D(\hat{\rho}_{RA} \| \hat{\omega}_{RA}) = S_{\infty}^M(\rho_{RA}).
\] (A11)

Finally, by noticing that, due to the definition of \( \mathcal{M} \),

\[
\min_{\omega_{RA} \in M} D(\hat{\rho}_{RA} \| \hat{\omega}_{RA}) = \min_{\sigma_R} D(\hat{\rho}_{RA} \| \hat{\sigma}_R \otimes \hat{1}_A) - \log d_A,
\] (A15)

and that, due to Lemma 4 in [19],

\[
S_{\infty}^M(\rho_{RA}) = S(\rho_{RA} \| \rho_R \otimes \hat{1}_A) - \log d_A,
\] (A16)

we obtain the statement of the lemma. ■

From (A16) and Lemma 7 we arrive at

\[
E_C(\rho_{AB}) \leq \min_{\rho_{RAD} \in \mathcal{D}_{\infty}(\rho_{AB})} \left\{ -S(\rho_{RA} \| \rho_R \otimes \hat{1}_A) \right\}
\] (A17)

Inequality (38) is finally obtained by employing standard blocking arguments.

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