On the generalized wormhole in the Eddington-inspired Born–Infeld gravity

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Received 27 May 2015, revised 28 September 2015
Accepted for publication 12 October 2015
Published 18 November 2015

Abstract
In this paper, we wish to investigate certain observable effects in the recently obtained wormhole solution of the Eddington-inspired Born–Infeld (EiBI) theory, which generalizes the zero-mass Ellis–Bronnikov wormhole of general relativity. The solutions of EiBI theory contain an extra parameter $\kappa$ having the inverse dimension of the cosmological constant $\Lambda$, and which is expected to modify various general relativistic observables such as the masses of wormhole mouths, tidal forces and light deflection. A remarkable result is that a non-zero $\kappa$ could prevent the tidal forces in the geodesic orthonormal frame from becoming arbitrarily large near a small throat radius ($r_0 \sim 0$) contrary to what happens near a small Schwarzschild horizon radius ($M \sim 0$). The role of $\kappa$ in the flare-out and energy conditions is also analyzed, which reveals that the energy conditions are violated. We show that the exotic matter in the EiBI wormhole cannot be interpreted as a phantom ($\omega = \frac{\rho}{p_r} < -1$) or ghost field $\phi$ of general relativity due to the fact that both $\rho$ and $p_r$ are negative for all $\kappa$.

Keywords: wormhole, EiBI gravity, tidal forces

(Some figures may appear in colour only in the online journal)
1. Introduction

One of the fundamental discoveries in astrophysics in recent times is that the universe is currently accelerating [1, 2]. A possible explanation for the late-time cosmic acceleration could be due to the infra-red modifications [3] of Einstein’s General Relativity (GR) theory. Such alternative theories of gravity involve more general combinations of curvature invariants than the pure Einstein–Hilbert term. One such modified theory is the Eddington-inspired Born–Infeld (EiBI) gravity. This is a prototype of theories that could be termed as the ‘gravitational avatar of non-linear electrodynamics’ [4].

To be more specific, note that Eddington’s original gravitational action is incomplete in the sense that it does not contain matter. Bañados and Ferreira [5] resurrected Eddington’s proposal for the gravitational action in the presence of a cosmological constant, extending it to include matter fields in the form of a Born–Infeld-like structure [6] of non-linear electrodynamics. The outcome is the modern form of EiBI gravity, which provides an alternative theory of the Big Bang with a novel, non-singular description of the universe. The EiBI model is currently extensively applied in the literature to many other astrophysical scenarios such as the solar system, structure of neutron stars or dark matter, etc [7–14]. Astrophysical scenarios today also include wormholes as an integral part, and we shall be dealing with one such solution here.

The solutions of the EiBI theory contain an extra parameter $\kappa$ having the inverse dimension of the cosmological constant $\Lambda$, that is, $(\text{length})^2$. The theory is ideologically relatively new and very different from GR, except in the limit $\kappa \to 0$. Thus, the true EiBI theory must always have $\kappa \neq 0$, and this parameter is expected to modify different GR physical observables. In the same spirit, we wish to investigate the effect of $\kappa$ on the observable quantities associated with a wormhole in EiBI theory. Such a wormhole has in fact been recently derived by Harko et al [15], and could be regarded as a $\kappa \neq 0$ generalization of the original ‘zero total mass’ Ellis–Bronnikov (EB) wormhole of the Einstein minimally coupled scalar field theory with a negative kinetic term$^6$. Assuming that the EB wormhole has a standard coordinate throat radius $r_0$, what we mean by zero total mass here is that the individual masses in suitable units of the two mouths ($+r_0/2$ and $-r_0/2$) add exactly to zero, when $\kappa = 0$. The new generalized wormhole ($\kappa \neq 0$) derived by Harko et al [15] is being extensively cited in the literature [18]. Thus, it is of interest to find out what corrections $\kappa$ contributes to the observables of the zero-mass general relativistic EB wormhole.

The purpose of the present article is to derive several useful results that can be stated as follows: (i) The zero total mass behavior is preserved even when $\kappa \neq 0$. (ii) A non-zero $\kappa$ prevents the tidal forces in the geodesic orthonormal frame from becoming arbitrarily large near $r_0 \sim 0$, contrary to what happens near the small Schwarzschild horizon radius, $M \sim 0$. (iii) A non-zero $\kappa$ also influences light bending, which provides a possibility to estimate $\kappa$ through gravitational lensing observations. (iv) A non-zero $\kappa$ has a role in the flare-out and energy conditions. (v) Finally, in the Appendix, we point out the reasons why one cannot interpret the EiBI exotic matter either as a phantom or as a ghost field as considered in GR.

In section 2, we give a brief outline of the EiBI gravity to make the topic more transparent. Then, in section 3, we briefly describe the wormhole under investigation and calculate the masses of its two mouths. After a brief review of tidal forces in a Lorentz-boosted frame in

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$^6$ Recall that the 1973 Ellis ‘drainhole’ solution [16] was independently discovered also by Bronnikov [17]. The term ‘wormhole’ was seemingly not in vogue in 1973. Hence our current nomenclature EB wormhole, which belongs to general relativity, and hence corresponds to $\kappa = 0$. The EiBI wormhole derived in [15] can be called its EiBI generalization due to the presence of the parameter $\kappa \neq 0$. 

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In section 4, we calculate in section 5 the excess tidal forces experienced by a traveler in geodesic motion near the throat of the wormhole. We devote section 6 to a discussion of the role of $\kappa$ in the flare-out and energy conditions. In section 7, we calculate the effect of $\kappa$ on the bending of light passing by the positive mass mouth. The final section (section 8) concludes the paper, followed by an appendix. We take units so that $G = 1$, $c = 1$.

2. EiBI field equations

In 1924, Eddington [19] suggested that at least in free, de Sitter space, the fundamental dynamical variable should be the affine connection $\Gamma$ and proposed a gravitational action $S_{\text{Edd}} = 2\kappa \int d^4x \sqrt{\det |R_{\mu\nu}(\Gamma)|}$, where $\kappa$ is a constant with inverse dimension of $\Lambda$. Varying the affine connection $\Gamma$, one obtains the field equations $\nabla_\alpha (2\kappa \sqrt{|R|} R^{\mu\nu}) = 0$, where $\nabla_\alpha$ is the covariant derivative defined by $\Gamma$ and $R_{\mu\nu}$ is the contravariant Ricci tensor. Eddington’s field equations can be solved if we define a new tensor $q_{\mu\nu}$ such that $\nabla_\alpha (\sqrt{|q|} q^{\mu\nu}) = 0$. The field equations then become $2\kappa \sqrt{|R|} R^{\mu\nu} = \sqrt{|q|} q^{\mu\nu}$, which reduce to Einstein–de Sitter field equations if we identify $g^{\mu\nu} = q^{\mu\nu}$ and $\kappa = \Lambda^{-1}$. Thus Eddington’s proposal is a good motivation for building a more general action alternative to Einstein’s gravity. However, Eddington’s theory does not include matter. Therefore, Bañados and Ferreira [5] included matter, a metric $g_{\mu\nu}$, a Born–Infeld [6]-like structure replacing the pure affine Eddington action by a new action that gave birth to what is now called EiBI theory in the literature (for details, see [5]).

We shall focus on the EiBI theory embodied in the Bañados–Ferreira action [5] given by

$$S_{\text{EiBI}} = \frac{1}{16\pi} \kappa \int d^4x \left[ \sqrt{\det |g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|} - \lambda \sqrt{\det |g_{\mu\nu}|} \right] + S_{\text{matter}} [g, \Psi_{\text{matter}}]$$

where $\Psi_{\text{matter}}$ is a generic matter field, $\sqrt{\det |g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|}$ is a Born–Infeld-like structure [6], $\lambda$ is a dimensionless parameter, $g_{\mu\nu}$ is the physical metric tensor, and $R_{\mu\nu}(\Gamma)$ is the symmetric part of the Ricci tensor built solely from the connection $\Gamma$, yet undefined. For small values of $\kappa R$, the action (1) reproduces the Einstein–Hilbert action with a constant $\frac{\lambda - 1}{\kappa}$, identified as the cosmological constant (this will be more transparent from the expansion of the field equations below):

$$\Lambda = \frac{\lambda - 1}{\kappa}. \quad (2)$$

For large values of $\kappa R$, the action approximates to matter-free Eddington action $S_{\text{Edd}}$. To ensure asymptotic flatness of solutions in the EiBI theory ($\kappa = 0$), one must have $\Lambda = 0$, which in turn would entail $\lambda = 1$ from equation (2).

The field equations are based on a Palatini-type formulation, where the metric tensor $g_{\mu\nu}$ and the connection $\Gamma$ are the two independent dynamical variables that are varied in the action (1). Varying with respect to $g_{\mu\nu}$, one obtains ($[X]$ denotes $\det [X_{\mu\nu}]$):

$$\frac{\sqrt{|g + \kappa R|}}{\sqrt{|g|}} [ (g + \kappa R)^{-1} ]^{\mu\nu} - \lambda g^{\mu\nu} = -8\pi \kappa T^{\mu\nu} \quad (3)$$

7 The action was first proposed by Vollick [20], but the matter fields were introduced in a non-conventional way inside the square root, unlike in (1).
where the usual stress tensor $T^{\mu\nu}$ is raised or lowered with $g_{\mu\nu}$. The field equation (3) expands as [5]:

$$R_{\mu\nu} \simeq \left( \frac{\lambda - 1}{\kappa} \right) g_{\mu\nu} + T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T + \kappa \left[ S_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S \right],$$

where $S_{\mu\nu} = T_{\mu\nu} T^{\mu\nu} - \frac{1}{2} T T^{\mu\nu}$ and $\frac{\lambda - 1}{\kappa}$ can be identified with $\Lambda$. Note that Einstein’s GR is recovered as $\kappa \rightarrow 0$.

The variation with respect to $\Gamma$ can be simplified by introducing an auxiliary metric $q_{\mu\nu}$ compatible with $\Gamma$ defined by

$$q_{\mu\nu} = g_{\mu\nu} \Delta \Gamma,$$

where $S_{TT} g T T S_{TT} = -$ and $\Gamma_{\mu\nu} = \Gamma(g)$ and hence EiBI and GR are completely equivalent.

For the specific case of asymptotically flat solutions, $\lambda = 1$, and hence equation (5) can be rewritten as

$$q^{\mu\nu} = \tau (g^{\mu\nu} - 8 \pi \kappa T^{\mu\nu}),$$

where

$$\tau = \sqrt{\frac{|g|}{|q|}}.$$

Harko et al [15] further simplified the equations (4) and (6), combining them into a form that looks much more familiar:

$$R_{\mu} = 8 \pi S_{\mu},$$

$$S_{\mu} = \tau T_{\mu} - \left( \frac{1}{8 \pi \kappa} + \frac{\tau}{2} \right) \delta_{\mu},$$

where $R_{\mu} = q^{\nu\rho} R_{\nu\rho}$, $R = R_{\mu}$, and $T_{\mu} = T^{\nu\rho} g_{\nu\rho}$, $T = T_{\mu}$. Note the roles of $q$ and $g$ metrics—the Ricci tensor on the left-hand side of equation (8) is raised or lowered with $q$, while the right-hand side is done with the metric $g$.

Alternatively, variation with respect to the connection $\Gamma$ leads to the corresponding field equations. By defining $q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}$ (equation (4)), after some manipulations, the field equations take the form

$$\Gamma_{\mu\nu} = \frac{1}{2} q^{\sigma\rho} \left[ \partial_{\sigma} q_{\rho\nu} + \partial_{\rho} q_{\sigma\nu} - \partial_{\nu} q_{\rho\sigma} \right]$$

(see [12] for details). So, equation (4) and this form are exactly equivalent field equations. This explains the genesis of the auxiliary metric $q_{\mu\nu}$: it is coming from the variation of the dynamical variable $\Gamma$. Of course, equation (4) is more illuminating, and we take it. Only in vacuum $q_{\mu\nu} = g_{\mu\nu}$, but inside matter they are different. This is the essence of the EiBI theory.
3. Wormhole solution: masses of the two mouths

The wormhole solution is derived in [15] by solving equation (8) under certain restrictive conditions such as spherical symmetry and asymptotic flatness, the latter requiring \( \lambda = 1 \). These assumptions of course limit the applicability of EiBI theory but make the problem at hand much simpler to handle. One spin-off is that the description of the physical behavior of the wormhole is now controlled by the only remaining parameter \( \kappa \). The physical metric \( g_{\mu \nu} \) and the auxiliary metric \( q_{\mu \nu} \), respectively are taken as

\[
g_{\mu \nu} \, dx^\mu dx^\nu = -e^{\alpha(r)} dr^2 + e^{\beta(r)} dr^2 + f(r)d\Omega^2,
\]

\[
q_{\mu \nu} \, dx^\mu dx^\nu = -e^{\beta(r)} dr^2 + e^{\alpha(r)} dr^2 + r^2 d\Omega^2.
\]

The wormhole is assumed to be threaded by anisotropic matter described by the stress tensor \( T^{\mu \nu} = \rho g^{\mu \nu} + (p_r + \rho) U^{\mu} U^{\nu} + (p_t - p_r) \chi^{\mu} \chi^{\nu} \), where \( \chi^{\mu} \) is the unit spacelike vector in the radial direction, \( \chi^{\mu} = e^{-\alpha(r)/2} \delta^{\mu}_r \), \( p_r \) is the radial pressure, \( p_t \) is the transverse pressure, \( \rho \) is the energy density and \( U^{\mu} \) is the four velocity such that \( g_{\mu \nu} \, U^\mu U^\nu = -1 \). Since geodesics are determined by the metric \( g_{\mu \nu} \), all observable effects connected to geodesics such as light deflection or tidal forces should be calculated only in the physical metric \( g_{\mu \nu} \).

Note that \( \tau \) in equation (7) can be obtained from \( T^{\mu \nu} \) through the expression \( \tau = [\nabla^\mu - 8\pi G T^{\mu \nu} ]^{-1/2} \), which in turn can be expressed in terms of stress quantities

\[
\tau = \left[ (1 + 8\pi G \rho) (1 - 8\pi G p_t) (1 - 8\pi G p_r) \right]^{-1/2}.
\]

The above form suggests arbitrary functions \( a, b \) and \( c \) defined by

\[
a(r) = \sqrt{1 + 8\pi G \rho}, \quad b(r) = \sqrt{1 - 8\pi G p_t}, \quad c(r) = \sqrt{1 - 8\pi G p_r},
\]

that help one write the components of the field equation (8) in manageable forms that finally yield

\[
e^\beta = e^\alpha c^2 a^2, \quad e^\alpha = e^\alpha a^2 c^2, \quad f = \frac{r^2}{ab}.
\]

The specific wormhole solution obtained by Harko et al [15] is based on simplifying assumptions that

\[
a(r) b(r) = 1, \quad \beta = 0.
\]

Then the reduced system of field equation (8) yields

\[
e^\alpha = 1 - \frac{r_0^2}{r^2}, \quad a^2 = \frac{1}{1 + 2\kappa r_0^2/r^2}, \quad c^2 = a^2.
\]

These, together with equation (16), lead to

\[
q_{\mu \nu} : e^{\beta(r)} = 1, \quad e^{\alpha(r)} = \frac{1}{1 - r_0^2/r^2},
\]

\[
g_{\mu \nu} : e^{\alpha(r)} = 1, \quad e^{\alpha(r)} = \frac{1 + 2\kappa r_0^2/r^4}{1 - r_0^2/r^2}.
\]
where \( r_0 \) is an arbitrary constant. Hence we have the metric \( g_{\mu\nu} \) given by (19), viz.,

\[
g_{\mu\nu} dx^\mu dx^\nu = -dr^2 + \left( \frac{1 + 2\kappa r_0^2/r^4}{1 - r_0^2/r^2} \right) dr^2 + r^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right].
\]

(20)

The metric (20) is a symmetric, twice asymptotically flat regular wormhole having asymptotic masses on either side of the throat, where \( r_0 \) has the meaning that it is the standard coordinate throat radius \( r_{th} = r_0 \), \( r_0 < r < +\infty \). In the limit \( \kappa \to 0 \), one recovers the massless EB wormhole of GR [16, 17].

To obtain the asymptotic masses, one needs to cover both sides of the wormhole by a single regular chart defined by

\[
r^2 = \ell^2 + r_0^2,
\]

(21)

which is dictated by dimensional considerations, so the throat is now appearing at \( \ell_{th} = 0 \). Then the charts covering individual sides respectively are \(-\infty < \ell \leq 0 \) and \( 0 \leq \ell < +\infty \), both meeting at the throat. Furthermore, the structure of EiBI theory provides an energy density that can be obtained from equations (13) and (18) as

\[
\rho(r) = \frac{1}{8\pi\kappa} \left[ \frac{1}{\sqrt{1 + 2\kappa r_0^2/r^4}} - 1 \right],
\]

(22)

and the pressures from equations (14), (15) and (18)

\[
p_\ell(r) = \frac{\rho(r)}{1 + 8\pi\kappa\rho(r)}, \quad p_t = -\rho.
\]

(23)

Figure 1 shows that \( \rho(r) < 0 \), \( p_\ell(r) < 0 \) for all values of \( r \) and for all values of \( \kappa \), positive or negative. From equation (22), we can obtain masses on individual sides using the
prescription:\[
M^+ = 4\pi \int_0^\infty \rho r^2 \frac{dr}{df},
\]
(24)
\[
M^- = 4\pi \int_{-\infty}^0 \rho r^2 \frac{dr}{df}.
\]
(25)

As such, the integrals cannot be evaluated in a closed form although the integrand is continuous everywhere including at \( \ell = 0 \) and vanishing at \( \ell \to \pm \infty \). Furthermore, the density function \( \rho \to -\frac{r_0^2}{8\pi r^4} \) as \( \kappa \to 0 \) but this is no surprise since at this limit the EiBI theory reduces to Einstein’s theory. Also, note that \( \rho \to 0 \) as \( \kappa \to \infty \). This is in perfect accordance with the pure Eddington theory \( (\kappa R \to \infty) \) without matter. Thus, the behavior of \( \rho \) shows no pathology anywhere and we can legitimately expand it in powers of \( \kappa \), which yields
\[
\rho = -\frac{r_0^2}{8\pi r^4} + \frac{3\kappa r_0^4}{16\pi r^8} - \frac{5\kappa^2 r_0^6}{16\pi r^{12}} + ...
\]
(26)
The limit \( \kappa \to 0 \) yields the first term that is just the familiar exotic scalar field density \( \rho^0 = -\frac{r_0^2}{8\pi r^4} \) in the massless EB wormhole of GR. The masses can be found by term by term integration
\[
M^+ = \frac{r_0}{2} + \frac{3\kappa}{20r_0} - \frac{5\kappa^2}{36r_0^2} + ...
\]
(27)
\[
M^- = -\frac{r_0}{2} - \frac{3\kappa}{20r_0} + \frac{5\kappa^2}{36r_0^2} - ...
\]
(28)

Note the correction terms due to \( \kappa \). It is evident that the masses are of equal value but of opposite signs. Though either mouth of the wormhole can exhibit gravitational effects such as lensing [22] (caused either by attractive \( M^+ \), or by repulsive \( M^- \)), the total mass of the whole configuration adds exactly to zero, \( M^+ + M^- = 0 \), even when \( \kappa \neq 0 \). Hence, the massless character of the general relativistic EB wormhole is preserved also in the case of its EiBI counterpart (20). In the limit \( \kappa \to 0 \), one recovers the usual EB masses \( +\frac{r_0}{2} \) and \( -\frac{r_0}{2} \), which add to zero, that are made purely of the ghost scalar field \( \phi \) of GR defined by the stress \( T_{\mu\nu} = \epsilon \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} \), with \( \epsilon = -1 \).

It should be noted that the Schwarzschild active gravitational masses are trivially zero due to the fact that \( g_{\mu\nu} = -1 \) in the metric (20), whereas the ‘bare masses’ in equations (24) and (25) are trivially summed to zero because the metric is symmetric under changing \( \ell \to -\ell \) and so the derivatives \( \frac{d}{d\ell} \) in equations (24) and (25) are the opposite of each other. In the above, we showed explicit individual mass values that could be useful for lensing purposes.

A generalization known in GR and having non-zero masses is the twice asymptotically flat regular massive EB wormhole [16, 17, 23–27], sometimes also called the anti-Fisher

9 In curved space with the metric (10), the volume measure contains \( e^{r/\ell} \), while the measure \( 4\pi r^2dr \) below follows the one in [21] already used for wormholes. However, the latter measure corresponds to calculating the real (Schwarzschild) mass, containing a gravitational mass defect for starlike objects with a regular center. On the other hand, in a wormhole, there is no center at all, and \( \ell = 0 \) corresponds to the coordinate value \( r = r_0 \) of the throat. For the justification of using \( 4\pi r^2dr \) for a centerless object, we would refer the readers to [21]. We thank an anonymous referee for raising this point.
solution, given by
\[
d\tau_{\text{EB}}^2 = -F dr^2 + F^{-1} \left[ d\ell^2 + \left( \ell^2 + r_0^2 \right) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right].
\]
(29)

\[
F = \exp \left[ -\pi \gamma + 2 \gamma \tan^{-1} \left( \frac{\ell}{r_0} \right) \right].
\]
(30)

\[
\phi = \lambda \left[ \pi + 2 \tan^{-1} \left( \frac{\ell}{r_0} \right) \right].
\]
(31)

with the constraint \(2 \lambda^2 = 1 + \gamma^2\). The Schwarzschild masses on either side of the EB wormhole (29)–(31) are \(\gamma r_0\) and \(-\gamma r_0 e^{\pi \gamma}\) as can be seen by expanding the metric tensor [16, 17]. Thus, when \(r_0 = 0\), \(\gamma = 0\), these masses vanish and the solution reduces to massless EB wormhole
\[
d\tau_{\text{EB}}^2 = -dr^2 + d\ell^2 + \left( \ell^2 + r_0^2 \right) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).
\]
(32)
\[
\phi = \frac{1}{\sqrt{2}} \left[ \pi + 2 \tan^{-1} \left( \frac{\ell}{r_0} \right) \right].
\]
(33)

Under the transformation \(r^2 = \ell^2 + r_0^2\), one obtains in standard coordinates
\[
d\tau_{\text{EB}}^2 = -dr^2 + \frac{dr^2}{1 - r_0^2 / r^2} + r^2 \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right].
\]
(34)
\[
\phi = \frac{1}{\sqrt{2}} \left[ \pi + 2 \tan^{-1} \left( \frac{\sqrt{r^2 - r_0^2}}{r_0} \right) \right].
\]
(35)

The metric part (sans scalar field \(\phi\)) of the above solution is the EiBI metric (20) with \(\kappa = 0\). As we see, it is just a special case \((r_0 \neq 0, \gamma = 0)\) of the metric part of the massive EB wormhole (29)–(31).

This situation leads to a natural enquiry\(^\text{10}\); just as the metric (20) is the EiBI generalization of the massless EB metric (34), does there exist a similar EiBI generalization of the massive EB metric (29)? We are not aware of such generalization as yet, but the possibility is certainly not ruled out if, instead of the anisotropic source tensor \(T^{\mu
\nu}\) used by Harko et al [15], one uses a ghost or some other kind of scalar field and solves the EiBI field equation (8) to find a solution. This would be a rewarding task by itself but we do not attempt it here.

4. Tidal forces in a Lorentz-boosted frame

We start with the general form of a static spherically symmetric physical metric:
\[
d\tau^2 = -\frac{F(r)}{G(r)} dr^2 + \frac{dr^2}{F(r)} + R^2(r) \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right].
\]
(36)

For a traveler in a static orthonormal basis, we shall denote the only non-vanishing components of the Riemann curvature tensor as \(R_{0101}, R_{0202}, R_{0303}, R_{1212}, R_{3313},\) and \(R_{3323}\). Radially freely falling observers with conserved energy \(E\) are connected to the static orthonormal frame by a local Lorentz boost with an instantaneous velocity given by

\(^{10}\) We thank an anonymous referee for raising this query.
\[ \mathbf{v} = \left[ 1 - \frac{F}{GE^2} \right]^{1/2}. \]  

Then the non-vanishing Riemann curvature components in the Lorentz-boosted frame \((\hat{\cdot})\) with velocity \(v\) are \((k=2,3)\):

\[ R_{\hat{0}\hat{1}\hat{0}} = R_{0101} \]  
\[ R_{\hat{0}\hat{1}\hat{0}\hat{k}} = R_{010k} + (R_{0k0k} + R_{4k4k}) \sinh^2 \alpha \]  
\[ R_{\hat{1}\hat{k}\hat{1}\hat{k}} = R_{1k1k} + (R_{0k0k} + R_{4k4k}) \sinh^2 \alpha \]  
\[ R_{\hat{0}\hat{k}\hat{1}\hat{k}} = (R_{0k0k} + R_{4k4k}) \sinh \alpha \cosh \alpha, \]

where \(\sinh \alpha = v/\sqrt{1 - v^2}\). The relative tidal acceleration \(\Delta a_j\) between two parts of the traveler’s body in its orthonormal basis is given by

\[ \Delta a_j = -R_{\hat{0}\hat{j}\hat{0}\hat{p}} \zeta^p, \]

where \(\zeta\) is the vector separation between the two parts [28]. Thus the curvature components contributing to tidal force on the traveler in the Lorentz-boosted frame are \(R_{\hat{0}\hat{1}\hat{0}}\), \(R_{0\hat{0}\hat{3}3}\) and \(R_{\hat{0}\hat{3}\hat{0}\hat{3}}\) (Components in the coordinate basis are not required here). In the case of a charged Reissner–Nordström black hole, there occurs a remarkable cancellation, viz. \(R_{0k0k} + R_{4k4k} = 0\) such that the tidal accelerations in the static and moving frame are the same. The same cancellation of course occurs in the Schwarzschild spacetime too, which is only an uncharged special case.

For the purpose of demonstration, consider a Schwarzschild mass \(M\), for which the curvature components of interest are

\[ R_{\hat{0}\hat{1}\hat{0}} = R_{0101} = -\frac{2M}{r^3}, \quad R_{0\hat{0}\hat{3}3} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} = M/r^3, \quad \text{etc.} \]  

Thus, at the horizon of the Schwarzschild black hole, \(r_\text{h} = 2M\), the curvature tensor \(R_{\hat{0}\hat{0}\hat{3}3} \propto 1/\pi^2 \to \infty \) as \(M \to 0\). So the smaller the black hole, the larger the tidal forces near the horizon. We wish to examine a similar situation near the throat of a wormhole since the throat is physically entirely different from a black hole horizon.

5. Effect of \(\kappa\) on tidal forces

We want to calculate the effect of geodesic motion on the tidal forces experienced by a freely falling observer. In this direction, we first note that, in the Lorentz-boosted frame, \(R_{\hat{0}\hat{1}\hat{0}} = R_{0101}\), hence it is unaffected by geodesic motion. Second, because of spherical symmetry, we note that \(R_{\hat{0}\hat{0}\hat{3}3} = R_{0\hat{0}\hat{3}3}\), so it is enough to calculate only \(R_{0\hat{0}\hat{3}3}\). And finally, with \(k=2\), we can rewrite equation (39) for the generic metric (36) as:

\[ R_{0\hat{0}\hat{3}3} = -\frac{1}{R} \left[ R'' (E^2 G - F) + \frac{R'}{2} (E^2 G' - F') \right], \]

where primes denote derivatives with respect to \(r\). The conserved energy \(E\) of the falling observer can be decomposed as

\[ E^2 = \frac{F}{G} + \frac{F}{G} \left( \frac{v^2}{1 - v^2} \right) = E_s^2 + E_{\text{ex}}^2, \]
where $E^2$ represents the value of $E^2$ in the static frame and $E^2_{\text{ex}}$ represents the enhancement in $E^2$ due to geodesic motion. We can now decompose $R_{0202}$ as follows [29]:

$$R_{0202} = -\frac{1}{R} \left[ \frac{R'}{2} \left( E^2 G' - F' \right) \right] - \frac{1}{R} \left( R'' G + \frac{R' G'}{2} \right) E^2_{\text{ex}}$$

$$= R^{(s)}_{0202} + R^{(ex)}_{0202}$$

The first term represents the curvature component in the static frame, while the term $R^{(ex)}_{0202}$ represents overall enhancement in curvature in the Lorentz-boosted frame over that in the static frame. It is this part that needs to be examined as the observer approaches the throat.

Applying the above to the generalized EB wormhole metric (20), we see that

$$F(r) = G(r) = \frac{1}{1 + \frac{r^2}{2\kappa r_0^2}}$$

A little algebra will show that $R^{(ex)}_{0202} = 0$ and

$$\left| R^{(ex)}_{0202} \right| = \left( \frac{r_0^3 \left( r^4 + 4\kappa r^2 - 2\kappa r_0^2 \right)}{\left( r^4 + 2\kappa r_0^2 \right)^2} \right) \left( \frac{v^2}{1 - v^2} \right)$$

which, in the limit $r \to r_0$, gives

$$\left| R^{(ex)}_{0202} \right| = \left( \frac{1}{2\kappa + 2r_0^2} \right) \left( \frac{v^2}{1 - v^2} \right)$$

Now suppose that $\kappa \to 0$ (general relativistic EB wormhole) and of course $v \neq 0$. Then, as $r_0 \to 0$, the excess tidal force in the geodesic frame near the throat becomes arbitrarily large, $\left| R^{(ex)}_{0202} \right| \to \infty$. This behavior is very similar to, but not exactly the same as, the case of small-mass Schwarzschild black holes, as explained in section 4. The only physical difference is that here we are near a narrow throat instead of a small black hole horizon. In contrast, however, depending on the values of non-zero $\kappa$, the tidal forces may become arbitrarily small, $\left| R^{(ex)}_{0202} \right| \to 0$, even when $r_0 \to 0$. This is the novelty of the generalized wormhole (20) brought about by the presence of the parameter $\kappa$.

The comparison with the Schwarzschild black hole as above may not be very appropriate but is still cited here to highlight that the phenomenon of excess curvature in the Lorentz-boosted frame was used to develop what is called the ‘naked black hole’ in [29]. What is of interest here is the possibility of having large or small excesses in curvature by controlling $\kappa$ in the generalized EB wormhole.

6. Flare-out and energy conditions

Defining $e^{\sigma(r)} = 1 - \frac{m(r)}{r}$, where $m(r)$ is the Morris–Thorne (MT) [28] shape function, and assuming that the shape of the axially symmetric embedding surface is $z = z(r)$, the requirement that the wormhole flares out to two asymptotically flat spacetimes is that the geometric condition $\frac{dv}{dT} = \frac{m - \dot{m}}{2m_0} > 0$ be satisfied at or near the throat. This inequality imposes a constraint on the type of source stress tensor $T^{\mu \nu}$, that can be nicely rephrased in terms of the MT dimensionless flare-out function defined by:

$$\zeta = -\frac{\rho + p_r}{\rho} = \frac{2\rho^2}{r_0^2 \left( \frac{dv}{dT} \right)^2} > 0 \Rightarrow \rho + p_r < 0.$$ 

It should be noted that, in spherical symmetry, the throat is simply defined as a regular minimum areal radius and in terms of this minimum it
is easy to obtain violation of the null energy condition (NEC) and weak energy condition (WEC), which is really of utmost importance but here we keep to the MT definition of flare-out.

Harko et al [15] defined an alternative flare-out condition that imposes a constraint on the shape function such that \( H := \sigma' e^{-\sigma} = \frac{m(r - m)}{r^2} < 0 \) and using it obtained the generic inequality

\[
8\pi\kappa (\rho + p_t) < \frac{\kappa b^2 (e')^2}{e^2 e^{-\sigma(r)}}. \tag{49}
\]

At the throat \( r_0 = m(r_0) \), \( e^{-\sigma(r)} = 0 \) and so \( (\rho + p_t) < 0 \). Thus, the NEC is violated, showing that this violation is a necessary condition for the flare-out. But if it so happens that \( \frac{e'}{e} e^{-\sigma(r)} \to K \) as \( r \to r_0 \), then \( 0 < \rho + p_t < K \), and the NEC need not be violated, hence no flare-out. Most importantly, note that \( \rho \) and \( p_t \) here are not derived from the Einstein field equations using MT metric form \( e^1, r, mr \), the form being used here only for notational convenience. Instead, \( \rho \) and \( p_t \) are obtained in equations (22) and (23) using only the EiBI equations. Likewise, \( H \) and the left-hand side of the inequality (49) are expressed in terms of the true EiBI functions given in section 3.

The flare-out condition for the present wormhole (20) turns out to be

\[
H = \sigma' e^{-\sigma} = -2r_0 \left( \frac{r^4 + 4\kappa r^2 - 2\kappa r_0^2}{r^4 + 2\kappa r_0^2} \right), \tag{50}
\]

which, at the throat \( r_0 \), yields

\[
H_0 = \sigma' e^{-\sigma} \bigg|_{r_0} = -\frac{2r_0}{2\kappa + \kappa_0^2} < 0, \tag{51}
\]

implying that the NEC is violated: \( \rho + p_t < 0 \). We can explicitly see from equations (22) and (23) that

\[
\rho + p_t = -\frac{r_0^2}{4\pi r^2 \sqrt{r^4 + 2\kappa r_0^2}} < 0, \tag{52}
\]

showing that the NEC is violated for all positive \( \kappa \). The WEC is also violated for all \( r \) including at the throat. As follows from equation (22)

\[
\rho(r) = \frac{1}{8\pi\kappa} \left[ \frac{1}{\sqrt{1 + 2\kappa r_0^2/r^4}} - 1 \right] < 0, \tag{53}
\]

for all positive \( \kappa \). It is thus clear that the source of (20) does not respect the WEC and NEC, implying that the wormhole is threaded by exotic matter.

It is to be noted\(^{11}\) that \( \kappa \) can also be negative [8, 12], say \( \kappa = -\kappa', \kappa' > 0 \). Then

\[
H_0 = -\frac{2r_0}{r_0^2 - 2\kappa'}, \tag{54}
\]

which implies that \( H_0 < 0 \) imposes a condition on the throat radius: \( r_0^2 > 2\kappa' \). Precisely the same condition is required for the WEC and NEC violations as well. From equations (52) and (53), we have at the throat

\(^{11}\) We thank another anonymous referee for pointing this out.
\[ \rho|_{r_0} = -\frac{1}{8\pi\kappa}\left[ -1 + \frac{r_0}{\sqrt{r_0^2 - 2\kappa^2}} \right] < 0, \quad (\rho + p_t)|_{r_0} = -\frac{1}{4\pi r_0\sqrt{r_0^2 - 2\kappa^2}} < 0, \quad (55) \]

both hold only if the reality condition \( r_0^2 > 2\kappa^2 \) holds. This suggests that the value \( \sqrt{2} |\kappa| \) provides a lower bound on the size of the throat \( r_0 \), when \( \kappa < 0 \).

Note that \( H \sim (\text{length})^{-1} \), while \( \rho + p_t \sim (\text{length})^{-2} \), by definition. Hence we find a difference between the equation (54) and the second of equation (55), but they qualitatively mean the same physical behavior—flare-out and the concomitant NEC violation, respectively. The influence of \( \kappa \) on the energy conditions and the flare-out condition is evident from the above equations (50)–(55). The individual plots of \( \rho(r) \) and \( p_t(r) \) exhibit similar behavior to that of \( \rho + p_t \) and hence only the representative plots of \( \rho + p_t \) are given in figure 2 for several values of \( \kappa \).

Since the wormhole (20) is threaded by exotic matter (WEC and NEC are both violated), it would be quite reasonable to enquire whether EiBI exotic matter could somehow be connected to the phantom energy or ghost scalar field \( \phi \) within the framework of GR. Unfortunately, this connection seems unlikely at the level of either field equations or solutions, when \( \kappa = 0 \) (see appendix). The reason is that the EiBI paradigm (\( \kappa \sim 0 \)) is very different from that of GR (\( \kappa \to 0 \)). Specifically, in the EiBI field equation (8), the left-hand side is made entirely of the auxiliary metric \( g_{\mu\nu} \), while the right-hand source term \( S_f^\mu \) is a combination of \( g_{\mu\nu} \) and \( q_{\mu\nu} \) (via \( \tau = \sqrt{|g|/|q|} \)). The GR limit implies through equations (6) and (7) that \( \tau = 1 \), when \( g_{\mu\nu} \) and \( q_{\mu\nu} \) become identical, and only then from equation (8) we end up with Einstein’s field equations.

The above notwithstanding, one might be curious to try, at the solution level, to embed \( e^{-\sigma(r)} = 1 - \frac{m(r)}{r} \) and \( \nu = 0 \) \( \Rightarrow \) redshift function \( \Phi = 0 \) from (19) into the Einstein field equations, and use the reverse technique of MT [28] to find the GR version of the EiBI exotic matter:

\[ \rho_{GR} = \frac{1}{8\pi r^2} \frac{dm}{dr} = \frac{r_0^2 (6\kappa^2 r_0^2 + 4\kappa^2 r_0^2 - 6\kappa r^4 - r^6)}{8\pi \left( r^5 + 2\kappa r_0^2 \right)^2}, \quad (56) \]

\[ (\rho + p_t)_{GR} = \frac{1}{8\pi} \left( \frac{1}{r^2} \frac{dm}{dr} - \frac{m}{r^3} \right) \frac{r_0^2 (2\kappa^2 r_0^2 - 4\kappa r^2 - r^4)}{4\pi \left( r^4 + 2\kappa r_0^2 \right)^2}. \quad (57) \]

These are evidently very different from the corresponding EiBI equations (52) and (53). However, when \( \kappa \to 0 \), both EiBI equation (52) and the GR equation (57) converge to the same EB value at the throat as expected, viz. \( (\rho + p_t)|_{r_0} = -1/4\pi r_0^2 \). The plots of equation (57) in figure 3 are given for \( r_0 = 1 \) and several values of \( \kappa \) [that is, fixing the values of masses; see equations (27) and (28)]. For values of \( \kappa \approx 0 \), figures 2 and 3 exhibit different behavior. The difference is pronounced for large negative values of \( \kappa \). As an example, for \( \kappa = -4 \), equations (56) and (57) give \( \rho_{GR} > 0 \), \( (\rho + p_t)_{GR} > 0 \) in the neighborhood of the throat \( r \sim r_0 = 1 \), i.e., no violation of WEC and NEC, which is in contradiction to the EiBI plots in figure 2. Nonetheless, values of \( r_0 \) and \( \kappa \) may be suitably adjusted so that \( \rho_{GR} < 0 \), \( (\rho + p_t)_{GR} < 0 \) can also be achieved (lower plots in figure 3). But this GR version of EiBI exotic matter corresponds to neither phantom nor ghost scalar field, as will be shown in the appendix.
7. Light deflection

The light path equation in the equatorial plane, to the second order in \( r_0^2 \), where \( \kappa \) appears first, is given by \( \left( u = \frac{1}{r} \right) \):

\[
\frac{d^2 u}{d\varphi^2} + u = -\left[ \frac{u}{b^2} + 2 \left( 1 - \frac{2\kappa}{b^2} \right) u^3 + 6\kappa u^5 \right] r_0^2,
\]  

(58)

where \( b \) is the impact parameter. The exact light path equation for the \( \kappa = 0 \) case, derived earlier by Bhattacharya and Potapov [30], can be recovered from the above. The minimum of \( r \), or the maximum of \( u \), denoted \( u_{\max} \), is the turning point of the motion. This occurs where \( du/d\varphi = 0 \) giving

\[
b = 1/u_{\max} = R_0.
\]  

(59)
The perturbative solution is taken as
\[ u = u_0 + u_1 \]
so that the linearized equations are
\[
\begin{align*}
\frac{d^2 u_0}{d\varphi^2} + u_0 &= 0 \Rightarrow u_0 = \frac{\cos \varphi}{R}, \\
\frac{d^2 u_1}{d\varphi^2} + u_1 &= -\left[ \frac{u_0}{b^2} + 2 \left( 1 - \frac{2\kappa}{b^2} \right) u_0^3 + 6\kappa u_0^5 \right] r_0^2,
\end{align*}
\]
where \( R \) is a constant. The remaining equation (62) can be integrated so that the solution \( u \) becomes:
\[
\begin{align*}
u &= \frac{\cos \varphi}{R} - \frac{r_0^2}{64b^2 R^2} \left\{ 56\kappa R^2 + 16R^4 - 2b^2 \left( 33\kappa + 14R^2 \right) \right\} \cos \varphi \\
&+ \left\{ b^2 \left( 15\kappa + 4R^2 \right) - 8\kappa R^2 \right\} \cos 3\varphi + b^2 \kappa \cos 5\varphi \\
&- \left\{ 120b^2 \kappa + 48b^2 R^2 - 96\kappa R^2 - 32R^4 \right\} \varphi \sin \varphi \right\},
\end{align*}
\]
After changing \( \varphi \to \pi/2 + \delta \) in equation (63), and assuming small \( \delta \) such that \( \sin \delta \approx \delta, \cos \delta \approx 1 \), and expanding to the order \( r_0^2 \), we find, following Bodenner and Will [31], that
\[
\delta \approx \frac{\pi r_0^2}{8R^2} - \frac{1}{4b^2} + \frac{15\kappa}{16R^2} - \frac{3\kappa}{4b^2 R^2}.
\]
We now have to find the minimum value of \( R \), which is the closest approach distance \( R_0 \). The minimum of \( R \) is the maximum \( u_{\text{max}} \), which can be shown by differentiation to occur at \( \varphi = 0 \). Putting \( \varphi = 0 \) in equation (63), setting \( u_{\text{max}} = 1/R_0 \), and inverting, we get,
\[
\frac{1}{R} \approx \frac{1}{R_0} + O \left( \frac{1}{R_0^3} \right) \Rightarrow R \approx R_0 = b.
\]
Using this in equation (64), we get the two-way deflection \( \epsilon \) as
\[
\epsilon = 2\delta \approx \frac{\pi r_0^2}{4R_0^2} + \frac{3\pi R_0^2}{8R_0^2}.
\]
The first term coincides exactly with that obtained in [30], while the second term explicitly reveals the effect of \( \kappa \).

8. Conclusions

The work reported here is an extension of the work by Harko et al[15], wherein they derived a wormhole solution that could be described either as an EiBI wormhole or as a generalized massless EB wormhole of GR. To make the paper readable and understandable, we attempted to present the EiBI basics maintaining clarity and brevity, leading the readers from the motivation all the way to the EiBI wormhole (20) that contains a crucial parameter \( \kappa \). The value of \( \kappa \) away from zero signifies departure from general relativistic effects and has been shown in the literature to depend on the chosen astrophysical scenarios [7–14, 32, 33]. In the same spirit, we have found in the foregoing the correction terms due to \( \kappa \) contributing to various observables in the massless EB wormhole.
We showed in section 3 that the massless character is preserved also in the generalized EB wormhole (20), where $\kappa \approx 0$. In section 5, we found a remarkable result in that the tidal forces can be arbitrarily small or finite even at a small throat radius ($r_0 \sim 0$) for non-zero values of $\kappa$. This result is in contradiction to that in general relativity, where the tidal forces become arbitrarily large in the limit of the small Schwarzschild horizon radius ($M \sim 0$), as argued in the previous section 4. Then we discussed in section 6 the inter-relations among $\kappa$, the flare-out and energy conditions in EiBI showing that the source of (20) does not respect the WEC and NEC for $\kappa > 0$. For $\kappa < 0$, the throat radius has a lower bound $2\sqrt{\kappa}$ for $\rho$ and $\rho + p_1$ to be real, but the energy conditions are still not respected. Posing the EiBI wormhole as a general relativistic one, we find that energy conditions may or may not be respected depending on the choices of $r_0$ and $\kappa$. This is more of a curious GR exercise as we show in the appendix that the EiBI wormhole cannot be fitted into the GR framework with a phantom or ghost source scalar field $\phi$ even with a potential $V(\phi)$. In section 7, we have shown that the two-way light deflection on the positive side of the mouth has a correction term proportional to $\kappa$.

Some immediate tasks remain: the gravitational lensing by the general relativistic ($\kappa = 0$) EB wormhole has already been investigated by Abe [22]. Hence it would be of interest to study the influence of $\kappa \neq 0$ on the lensing observables in the generalized metric (20) taking into account our correction term to light deflection obtained in equation (66).

Another important question is the issue of stability. It is already shown within the framework of GR that the $\kappa = 0$ case is unstable both under linear and non-linear perturbations [23–25] only if the EB wormhole has a phantom scalar as a source. The same metric can be obtained with another source, an exotic fluid, and then the dynamics are quite different, and the equation of state of this fluid can be chosen in such a way that this wormhole will be stable. All this is explicitly shown in [34]. Stability of the generalized wormhole (20) has to be studied within the framework of EiBI theory for which $\kappa = 0$ and it is yet to be determined whether the presence of non-zero $\kappa$ can allow stability.

Acknowledgments

This work was supported in part by an internal grant from M. Akmullah Bashkir State Pedagogical University in the field of natural sciences. We thank two anonymous referees for their insightful comments that have led to a considerable improvement of the paper.

Appendix

We shall show that the exotic matter threading the EiBI or generalized EB wormhole (20) ($\kappa \neq 0$) is neither phantom nor ghost in the GR framework. For phantom matter, the equation of state parameter should be $\omega = \frac{p}{\rho} < -1$. On the other hand, we have from equations (22) and (23)

$$\omega = \frac{p}{\rho} = \sqrt{1 + 2\kappa r_0^2/r^4} > 0, \forall \kappa, r \quad (A1)$$

including at the throat $r = r_0$. The EiBI exotic matter therefore cannot be a phantom anywhere in the spacetime regardless of whether $\kappa$ is positive or negative.

However, for $\kappa = 0$, it is well known that the EB wormhole (34) is threaded by matter made purely of a minimally coupled ghost scalar field in GR. The question we then ask is: can
we find in GR a similar minimally coupled scalar field $\phi$ with an arbitrary potential $V(\phi)$ for the $\kappa \neq 0$ EiBI wormhole (20)? The answer, unfortunately, seems to be in the negative.

Consider the action with a minimally coupled scalar field $\phi$ and a potential $V(\phi)$ given by

$$ S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ R - \epsilon (\nabla \phi)^2 - 2V(\phi) \right], \quad (A2) $$

where, notationally, $(\nabla \phi)^2 \equiv g^{\mu\nu} \dot{\phi}_\mu \dot{\phi}_\nu$, $\dot{\phi}_\mu \equiv \partial \dot{\phi}/\partial x^\mu$ and $\epsilon = \pm 1$. Variation with respect to the metric $g_{\mu\nu}$ and $\phi$ respectively gives the field equations

$$ R_{\mu\nu} = \epsilon \dot{\phi}_\mu \dot{\phi}_\nu + g_{\mu\nu} V, \quad (A3) $$

and

$$ \dot{\phi}^{\alpha} = - \frac{\partial V}{\partial \phi}. \quad (A4) $$

The value $\epsilon = -1$ corresponds to what is called a ghost scalar field $\phi$. We choose the metric ansatz

$$ dr^2 = -B(r)dr^2 + A(r)r^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right]. \quad (A5) $$

From the left-hand side of the field equation (A3), since $\dot{A} = 0$, it follows that $R_\nu = R_\tau = \frac{\dot{A}}{2A} = 0$, so we get $\dot{\phi} \ddot{\phi} = 0$, where prime denotes differentiation with respect to $r$ and dot denotes differentiation with respect to $t$. So we can either have $\ddot{\phi} = 0$ or $\dot{\phi} = 0$. We choose the latter and assume $\dot{\phi} = \phi(r)$ so that we get from the equations (A3):

$$ \frac{B'}{2A} - \frac{B'}{4A} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{B'}{rA} = V, \quad (A6) $$

$$ -\frac{B''}{2B} + \frac{B'}{4B} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{A'}{rA} = \epsilon \phi'^2 + AV, \quad (A7) $$

$$ 1 - \frac{1}{A} + \frac{rA'}{2A^2} - \frac{rB'}{2AB} = r^2V. \quad (A8) $$

For the EiBI metric (20), we have

$$ B(r) = 1, \quad A(r) = \frac{1 + 2\kappa r_0^2/r^4}{1 - r_0^2/r^2}. \quad (A9) $$

Putting them in (A6), we have $V = 0$ but the difficulty is that the field equation (A8), viz.,

$$ 1 - \frac{1}{A} + \frac{rA'}{2A^2} = 0 \quad (A10) $$

is not satisfied by the function $A(r)$ unless $\kappa = 0$. This lack of self-consistency indicates that the exotic source matter in (20) is unlikely to be represented by a GR ghost scalar field. Note that although the GR equations (56) and (57) yield (for suitable values of $r_0$ and $\kappa$) exotic source matter obtained via the reverse MT [29] method, unless we are able to derive them from some kind of exotic scalar field via action of the type (A2), we cannot connect the solution (20) with $\kappa \neq 0$ to a GR solution with a coupled scalar field $\phi$ typical of the EB solutions.

However, there is always the possibility to introduce ghost or phantom or some other scalar field into the EiBI theory itself by including them in the action (1) from the start and analyzing the corresponding solutions, if any. That would be a separate task by itself and is left for the future. Having said that, we point out that Deser and Gibbons [4] considered the
EiBI type of Lagrangian and took the usual Christoffel connection $\Gamma(g)$ [instead of $\Gamma(q)$] and treated $g_{\mu
u}$ as the only dynamical variable. The resulting field equations were fourth order with ghosts [20]. But the EB solutions result from second-order field equations with ghost source, and are thus different from the one considered in [4].

References

[1] Riess A et al 1998 Astron. J. 116 1009
[2] Perlmutter S et al 1999 Astrophys. J. 517 565
[3] Carroll S M et al 2004 Phys. Rev. D 70 043528
[4] Deser S and Gibbons G W 1998 Class. Quantum Grav. 15 L35
[5] Bañados M and Ferreira P G 2010 Phys. Rev. Lett. 105 011101
[6] Born M and Infeld L 1934 Proc. Roy. Soc. A 144 425
[7] Casanellas J, Pani P, Lopes I and Cardoso V 2012 Astrophys. J. 745 15
[8] Pani P, Cardoso V and Delsate T 2011 Phys. Rev. Lett. 107 031101
[9] Harko T et al 2013 Phys. Rev. D 88 044032
[10] Scargill J H C, Bañados M and Ferreira P G 2012 Phys. Rev. D 86 103533
[11] Avetisian P P and Ferreira R Z 2012 Phys. Rev. D 86 044001
[12] Pani P and Sotiriou T P 2012 Phys. Rev. Lett. 109 251102
[13] Fu Q-M et al 2014 Phys. Rev. D 90 104007
[14] Du X-L et al 2014 Phys. Rev. D 90 044054
[15] Harko T et al 2015 Mod. Phys. Lett. A 30 1550190
[16] Ellis H G 1973 J. Math. Phys. 14 104
[17] Ellis H G 1974 J. Math. Phys. 15 520 (erratum)
[18] Bronnikov K A 1973 Acta Phys. Polon. B 4 251
[19] Odintsov S D et al 2014 Phys. Rev. D 90 044003
[20] Lagos M et al 2014 Phys. Rev. D 89 024034
[21] Harko T et al 2014 Galaxies 2 496
[22] Makarenko A N et al 2014 Phys. Rev. 90 024066
[23] Bouhmadi-López M et al 2014 J. Cosmol. Astropart. Phys. JCAP11(2014)007
[24] Bouhmadi-López M et al 2014 Phys. Rev. D 90 123518
[25] Bouhmadi-López M et al 2015 Eur. Phys. J. C 75 90
[26] Wei S-W et al 2015 Eur. Phys. J. C 75 253
[27] Shaikh R 2015 Phys. Rev. D 92 024015
[28] Berti E et al 2015 arXiv:1501.07274 [gr-qc]
[29] Eddington A S 1924 The Mathematical Theory of Relativity (Cambridge, UK: Cambridge University Press)
[30] Vollick D N 2004 Phys. Rev. D 69 064030
[31] Vollick D N 2005 Phys. Rev. D 72 084026
[32] Visser M, Kar S and Dadhich N 2003 Phys. Rev. Lett. 90 201102
[33] Kar S, Dadhich N and Visser M 2004 Pramana 63 859
[34] Abe F 2010 Astrophys. J. 725 787
[35] González J A, Guzmán F S and Sarbach O 2009 Class. Quantum Grav. 26 015010
[36] González J A, Guzmán F S and Sarbach O 2009 Class. Quantum Grav. 26 015011
[37] Bronnikov K A, Fabris J C and Zhidenko A 2011 Euro. Phys. J 71 1791
[38] Bronnikov K A, Konoplya R A and Zhidenko A 2012 Phys. Rev. D 86 024028
[39] Bronnikov K A and Grinyok S 2004 Grav. Cosmol. 10 237
[40] Bronnikov K A and Starobinsky A A 2007 JETP Lett. 85 1
[41] Morris M S and Thorne K S 1988 Am. J. Phys. 56 395
[42] Horowitz G T and Ross S F 1997 Phys. Rev. D 56 2180
[43] Bhattacharya A and Potapov A A 2010 Mod. Phys. Lett. A 29 2399
[44] Bodenner J and Will C M 2003 Am. J. Phys. 71 770
[45] Izmaïlov R et al 2015 Mod. Phys. Lett. A 30 1550056
[46] Potapov A A et al 2015 J. Cosmol. Astropart. Phys. JCAP07(2015)018
[47] Bronnikov K A, Lipatova L N, Novikov I D and Shatskiy A A 2013 Grav. Cosmol. 19 269