COHOMOLOGY OF ALGEBRAS OVER WEAK HOPF ALGEBRAS

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Abstract. In this paper we present the Sweedler cohomology for a cocommutative weak Hopf algebra $H$. We show that the second cohomology group classifies completely weak crossed products, having a common preunit, of $H$ with a commutative left $H$-module algebra $A$.

Introduction

In [26] Sweedler introduced the cohomology of a cocommutative Hopf algebra $H$ with coefficients in a commutative $H$-module algebra $A$. We will denote it as Sweedler cohomology $H_{\varphi_A}(H^*, A)$ where $\varphi_A$ is a fixed action of $H$ over $A$. If $H$ is the group algebra $kG$ of a group $G$ and $A$ is an admissible $kG$-module, the Sweedler cohomology $H^i_{\varphi_A}(kG, A)$ is canonically isomorphic to the group cohomology of $G$ in the multiplicative group of invertible elements of $A$. If $H$ is the enveloping algebra $UL$ of a Lie algebra $L$, for $i > 1$, the Sweedler cohomology $H^i_{\varphi_A}(UL, A)$ is canonically isomorphic to the Lie cohomology of $L$ in the underlying vector space of $A$. Also, in [26] we can find an interesting interpretation of $H^2_{\varphi_A}(H, A)$ in terms of extensions: This cohomology group classifies the group of equivalence classes of cleft extensions, i.e., classes of equivalent crossed products determined by a 2-cocycle. This result was extended by Doi [15] proving that, in the non commutative case, there exists a bijection between the isomorphism classes of $H$-cleft extensions $B$ of $A$ and equivalence classes of crossed systems for $H$ over $A$. If $H$ is cocommutative the equivalence is described by $H^2_{\varphi_Z(A)}(H, Z(A))$ where $Z(A)$ is the center of $A$. Subsequently, Schauenburg in [25] extended the cohomological results about extensions including a theory of abstract kernels and their obstructions. The dual Sweedler theory was investigated by Doi and Takeuchi in [13] giving a general formulation of a cohomology for comodule coalgebras for a commutative Hopf algebra as for example the coordinate ring of an affine algebraic group.

With the recent arise of weak Hopf algebras, introduced by Böhm, Nill and Szlachányi in [9], the notion of crossed product can be adapted to the weak setting. In the Hopf algebra world, crossed products appear as a generalization of semi-direct products of groups to the context of Hopf algebras [23, 7], and are closely connected with cleft extensions and Galois extensions of
Hopf algebras [8, 14]. In [11] Brzeziński gave an interesting approach that generalizes several types of crossed products, even the ones given for braided Hopf algebras by Majid [22] and Guccione and Guccione in [18]. On the other hand, in [19] we can find a general and categorical theory, the theory of wreath products, that contains as a particular instance the crossed structures presented by Brzeziński.

The key to extend the crossed product constructions presented in the previous paragraph to the weak setting is the use of idempotent morphisms combined with the ideas in [11]. In [6] the authors defined a product on $A \otimes V$, for an algebra $A$ and an object $V$ both living in a strict monoidal category $C$ where every idempotent splits. In order to obtain that product we must consider two morphisms $\psi^A_V : V \otimes A \to A \otimes V$ and $\sigma^A_V : V \otimes V \to A \otimes V$ that satisfy some twisted-like and cocycle-like conditions. Associated to these morphisms it is possible to define an idempotent morphism $\nabla_{A \otimes V} : A \otimes V \to A \otimes V$ and the image of $\nabla_{A \otimes V}$ inherits the associative product from $A \otimes V$. In order to define a unit for $\text{Im}(\nabla_{A \otimes V})$, and hence to obtain an algebra structure, we require the existence of a preunit $\nu : K \to A \otimes V$. In [16] we can find a characterization of weak crossed products with a preunit as associative products on $A \otimes V$ that are morphisms of left $A$-modules with preunit. Finally, it is convenient to observe that, if the preunit is an unit, the idempotent becomes the identity and we recover the classical examples of the Hopf algebra setting. The theory presented in [6, 16] contains as a particular instance the one developed by Brzeziński in [11]. There are many other examples of this theory like the weak smash product given by Caenepeel and De Groot in [12], the theory of wreath products presented in [19] and the weak crossed products for weak bialgebras given in [24]. Recently, G. Böhm showed in [10] that a monad in the weak version of the Lack and Street’s 2-category of monads in a 2-category is identical to a crossed product system in the sense of [6] and also in [17] we can find that unified crossed products [1] and partial crossed products [21] are particular instances of weak crossed products.

Then, if in the Hopf algebra setting the second cohomology group classifies crossed products of $H$ with a commutative left $H$-module algebra $A$, what about the weak setting? The answer to this question is the main motivation of this paper. More precisely, we show that if $H$ is a cocommutative weak Hopf algebra and $A$ is a commutative left $H$-module algebra, all the weak crossed products defined in $A \otimes H$ with a common preunit can be described by the second cohomology group of a new cohomology that we call the Sweedler cohomology of a weak Hopf algebra with coefficients in $A$.

The paper is organized as follows: In Section 1 after recalling the basic properties of weak Hopf algebras, we introduce the notion of weak $H$-module algebra and define the cosimplicial complex $\text{Reg}_{\varphi,A}(H^\bullet, A)$ for a cocommutative weak Hopf algebra $H$ and a commutative left $H$-module algebra $A$. Then, we introduce the Sweedler cohomology of $H$ with coefficients in
A as the one defined by the associated cochain complex. In the second section we present the results about the characterization of weak crossed products induced by morphisms \( \sigma \in \text{Reg}_{\mathcal{F}}(H^2, A) \) proving that the twisted condition, and the cocycle condition of the general theory of weak crossed products can be reduced to twisted an 2-cocycle for the action and \( \sigma \). Also, in this section we introduce the normal condition that permits to obtain a preunit in the weak crossed product induced by the morphism \( \sigma \). Finally, in the third section we characterize the equivalence between two weak crossed products obtaining the main result of this paper that assures the following: There is a bijective correspondence between \( H^2_{\mathcal{F}}(H, A) \) and the equivalence classes of weak crossed products of \( A \otimes_\alpha H \) where \( \alpha : H \otimes H \rightarrow A \) satisfies the 2-cocycle and the normal conditions.

1. The Sweedler cohomology in a weak setting

From now on \( C \) denotes a strict symmetric category with tensor product denoted by \( \otimes \) and unit object \( K \). With \( c \) we will denote the natural isomorphism of symmetry and we also assume that \( C \) has equalizers. Then, under these conditions, every idempotent morphism \( q : Y \rightarrow Y \) splits, i.e., there exist an object \( Z \) and morphisms \( i : Z \rightarrow Y \) and \( p : Y \rightarrow Z \) such that \( q = i \circ p \) and \( p \circ i = \text{id}_Z \). We denote the class of objects of \( C \) by \( |C| \) and for each object \( M \in |C| \), the identity morphism by \( \text{id}_M : M \rightarrow M \). For simplicity of notation, given objects \( M, N, P \) in \( C \) and a morphism \( f : M \rightarrow N \), we write \( P \otimes f \) for \( id_P \otimes f \) and \( f \otimes P \) for \( f \otimes id_P \).

An algebra in \( C \) is a triple \( A = (A, \eta_A, \mu_A) \) where \( A \) is an object in \( C \) and \( \eta_A : K \rightarrow A \) (unit), \( \mu_A : A \otimes A \rightarrow A \) (product) are morphisms in \( C \) such that \( \mu_A \circ (A \otimes \eta_A) = \text{id}_A = \mu_A \circ (\eta_A \otimes A) \). We will say that an algebra \( A \) is commutative if \( \mu_A \circ c_{A,A} = \mu_A \).

Given two algebras \( A = (A, \eta_A, \mu_A) \) and \( B = (B, \eta_B, \mu_B) \), \( f : A \rightarrow B \) is an algebra morphism if \( \mu_B \circ (f \otimes f) = f \circ \mu_A \) and \( f \circ \eta_A = \eta_B \).

If \( A, B \) are algebras in \( C \), the object \( A \otimes B \) is an algebra in \( C \) where \( \eta_{A \otimes B} = \eta_A \otimes \eta_B \) and

\[
(1) \quad \mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B).
\]

A coalgebra in \( C \) is a triple \( D = (D, \varepsilon_D, \delta_D) \) where \( D \) is an object in \( C \) and \( \varepsilon_D : D \rightarrow K \) (counit), \( \delta_D : D \rightarrow D \otimes D \) (coproduct) are morphisms in \( C \) such that \( (\varepsilon_D \otimes D) \circ \delta_D = \text{id}_D = (D \otimes \varepsilon_D) \circ \delta_D \). We will say that \( D \) is cocommutative if \( c_{D,D} \circ \delta_D = \delta_D \) holds.

If \( D = (D, \varepsilon_D, \delta_D) \) and \( E = (E, \varepsilon_E, \delta_E) \) are coalgebras, \( f : D \rightarrow E \) is a coalgebra morphism if \( (f \otimes f) \circ \delta_D = \delta_E \circ f \) and \( \varepsilon_E \circ f = \varepsilon_D \).

When \( D, E \) are coalgebras in \( C \), \( D \otimes E \) is a coalgebra in \( C \) where \( \varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E \) and

\[
(2) \quad \delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E).
\]
If \( A \) is an algebra, \( B \) is a coalgebra and \( \alpha : B \to A, \beta : B \to A \) are morphisms, we define the convolution product by

\[ \alpha \otimes \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B. \]

Let \( A \) be an algebra. The pair \((M, \phi_M)\) is a right \( A \)-module if \( M \) is an object in \( C \) and \( \phi_M : M \otimes A \to M \) is a morphism in \( C \) satisfying \( \phi_M \circ (M \otimes \eta_A) = \text{id}_M, \phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A) \). Given two right \( A \)-modules \((M, \phi_M)\) and \((N, \phi_N)\), \( f : M \to N \) is a morphism of right \( A \)-modules if \( \phi_N \circ (f \otimes A) = f \circ \phi_M \). In a similar way we can define the notions of left \( A \)-module and morphism of left \( A \)-modules. In this case we denote the left action by \( \varphi_M \).

By weak Hopf algebras we understand the objects introduced in [9], as a generalization of ordinary Hopf algebras. Here we recall the definition of these objects in the symmetric monoidal setting.

**Definition 1.1.** A weak Hopf algebra \( H \) is an object in \( C \) with an algebra structure \((H, \eta_H, \mu_H)\) and a coalgebra structure \((H, \varepsilon_H, \delta_H)\) such that the following axioms hold:

1. \( \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_H \otimes H \),
2. \( \varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_H,H \otimes \delta_H) \otimes H) \),
3. \( (\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) = (H \otimes (\mu_H \otimes c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) \).
4. There exists a morphism \( \lambda_H : H \to H \) in \( C \) (called the antipode of \( H \)) satisfying:
   1. \( \text{id}_H \land \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \)
   2. \( \lambda_H \land \text{id}_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \)
   3. \( \lambda_H \land \text{id}_H \land \lambda_H = \lambda_H. \)

**1.2.** If \( H \) is a weak Hopf algebra in \( C \), the antipode \( \lambda_H \) is unique, antimultiplicative, antico-unit invariant and leaves the unit and the counit invariant:

\[ \lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}; \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H; \]
\[ \lambda_H \circ \eta_H = \eta_H; \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \]

If we define the morphisms \( \Pi_H^L \) (target), \( \Pi_H^R \) (source), \( \Pi_H^L \) and \( \Pi_H^R \) by

\[ \Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \]

\[ \Pi_H^R = (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)), \]

then

\[ \Pi_H^L = \Pi_H^R. \]
\[ \Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \]
\[ \Pi_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H), \]
\[ \Pi_H^{LR} = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \]

it is straightforward to show (see [9]) that they are idempotent and \( \Pi_H^L, \Pi_H^R \) satisfy the equalities
\[ (5) \quad \Pi_H^L = id_H \land \lambda_H; \quad \Pi_H^R = \lambda_H \land id_H, \]

and then
\[ (6) \quad \Pi_H^L \land \Pi_H^L = \Pi_H^L, \quad \Pi_H^R \land \Pi_H^R = \Pi_H^R. \]

Moreover, we have that
\[ (7) \quad \Pi_H^L \circ \Pi_H^L = \Pi_H^L; \quad \Pi_H^L \circ \Pi_H^R = \Pi_H^R; \quad \Pi_H^R \circ \Pi_H^L = \Pi_H^L; \quad \Pi_H^R \circ \Pi_H^R = \Pi_H^R; \]
\[ (8) \quad \Pi_H^L \circ \Pi_H^L = \Pi_H^L; \quad \Pi_H^L \circ \Pi_H^R = \Pi_H^R; \quad \Pi_H^R \circ \Pi_H^L = \Pi_H^L; \quad \Pi_H^R \circ \Pi_H^R = \Pi_H^R. \]

For the morphisms target an source we have the following identities:
\[ (9) \quad \Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H, \quad \Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H, \]
\[ (10) \quad (H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \quad (\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R, \]
\[ (11) \quad \mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \]
\[ (12) \quad (H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \]
\[ (13) \quad \mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H), \]
\[ (14) \quad (\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) \]

and
\[ (15) \quad \mu_H \circ (\Pi_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \]
\[ (16) \quad \mu_H \circ (H \otimes \Pi_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \]
\[ (17) \quad (\Pi_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \]
\[ (18) \quad (H \otimes \Pi_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \]

Finally, if \( H \) is (co)commutative we have that \( \lambda_H \) is an isomorphism and \( \lambda_H^{-1} = \lambda_H \).
Example 1.3. As group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras. Recall that a groupoid $G$ is simply a category in which every morphism is an isomorphism. In this example, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of $G$ will be denoted by $G_0$ and the set of morphisms by $G_1$. The identity morphism on $x \in G_0$ will also be denoted by $id_x$ and for a morphism $\sigma : x \rightarrow y$ in $G_1$, we write $s(\sigma)$ and $t(\sigma)$, respectively for the source and the target of $\sigma$.

Let $G$ be a groupoid, and $R$ a commutative ring. The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma \in G_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise, i.e. $\sigma \tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma \tau = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. Then $RG$ is a cocommutative weak Hopf algebra, with coproduct $\delta_{RG}$, counit $\epsilon_{RG}$ and antipode $\lambda_{RG}$ given by the formulas $\delta_{RG}(\sigma) = \sigma \otimes \sigma$, $\epsilon_{RG}(\sigma) = 1$, and $\lambda_{RG}(\sigma) = \sigma^{-1}$. For the weak Hopf algebra $RG$ the morphisms target and source are respectively,

$$\Pi^L_{RG}(\sigma) = id_{t(\sigma)}$$, \hspace{1em} $$\Pi^R_{RG}(\sigma) = id_{s(\sigma)}$$.

Definition 1.4. Let $H$ be a weak Hopf algebra. We will say that $A$ is a weak left $H$-module algebra if there exists a morphism $\varphi_A : H \otimes A \rightarrow A$ satisfying:

(b1) $\varphi_A \circ (\eta_H \otimes A) = id_A$.
(b2) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$.
(b3) $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$.

and any of the following equivalent conditions holds:

(b4) $\varphi_A \circ (\Pi^L_H \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes A)$.
(b5) $\varphi_A \circ (\Pi^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
(b6) $\varphi_A \circ (\Pi^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
(b7) $\varphi_A \circ (\Pi^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
(b8) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varphi_A \circ (H \otimes \eta_A)) \otimes (\varphi_\varepsilon \circ \mu_H)) \circ (\delta_H \otimes H)$.
(b9) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varphi_\varepsilon \circ \mu_H) \otimes (\varphi_\varepsilon \circ (H \otimes \eta_A))) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$.

If we replace (b3) by

(b3-1) $\varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \circ \varphi_A)$

we will say that $(A, \varphi_A)$ is a left $H$-module algebra.

Notation 1.5. Let $H$ be a weak Hopf algebra. For $n \geq 1$, we denote by $H^n$ the $n$-fold tensor power $H \otimes \cdots \otimes H$. By $H^0$ we denote the unit object of $C$, i.e. $H^0 = K$. 
If $n \geq 2$, $m^n_H$ denotes the morphism

$$m^n_H : H^n \to H$$

defined by $m^2_H = \mu_H$ and by

$$m^3_H = m^2_H \circ (H \otimes \mu_H), \ldots, m^n_H = m^{n-1}_H \circ (H^{n-2} \otimes \mu_H)$$

for $k > 2$. Note that by the associativity of $\mu_H$ we have

$$m^n_H = m^{n-1}_H \circ (\mu_H \otimes H^{n-2}).$$

Let $(A, \varphi_A)$ be a weak left $H$-module algebra and $n \geq 1$. With $\varphi^n_A$ we will denote the morphism

$$\varphi^n_A : H^n \otimes A \to A$$

defined as $\varphi^1_A = \varphi_A$ and $\varphi^n_A = \varphi_A \circ (H \otimes \varphi^{n-1}_A)$. If $n > 1$, we have that

$$\varphi_A \circ (m^n_H \otimes \eta_A) = \varphi^{n-1}_A \circ (H^{n-1} \otimes (\varphi_A \circ (H \otimes \eta_A)))$$

holds. In what follows, we denote the morphism $\varphi_A \circ (m^n_H \otimes \eta_A)$ by $u_n$ and the morphism $\varphi_A \circ (H \otimes \eta_A)$ by $u_1$. Note that, by (b3) of Definition \[14\] for $n \geq 2$,

$$u_n = \varphi^{n-1}_A \circ (H^{n-1} \otimes u_1).$$

Finally, with $\delta_H$, we denote the coproduct defined in (2) for the coalgebra $H^n$. Then,

$$\delta_H = \delta_{H^n} \otimes H^{n-k} = \delta_{H^{n-k} \otimes H^k},$$

for $k \in \{1, \ldots, n-1\}$.

**Proposition 1.6.** Let $H$ be a cocommutative weak Hopf algebra. The following identities hold.

(i) $\delta_H \circ \Pi^I_H = (\Pi^I_H \otimes \Pi^I_H) \circ \delta_H$ for $I \in \{L, R\}$.
(ii) $\Pi^I_H \otimes \Pi^J_H \circ \delta_H = (H \otimes \Pi^I_H) \circ \delta_H \circ \Pi^I_H = \delta_H \circ \Pi^I_H$, for $I, J \in \{L, R\}$.
(iii) $\Pi^I_H \otimes \delta_H \circ \mu_H = (\Pi^I_H \otimes \mu_H) \circ (\delta_H \otimes H)$.
(iv) $(H \otimes \Pi^R_H) \circ \delta_H \circ \mu_H = (\mu_H \otimes \Pi^R_H) \circ (H \otimes \delta_H)$.

**Proof.** First note that if $H$ is cocommutative $\Pi^I_H = \Pi^I_H$ for $I \in \{L, R\}$. The proof for (i) with $I = L$ follows by

$$\delta_H \circ \Pi^L_H$$

$$= \mu_H \circ (\delta_H \otimes (\delta_H \otimes \lambda_H)) \circ \delta_H$$

$$= \mu_H \circ (\delta_H \otimes (c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H$$

$$= (\mu_H \otimes \Pi^L_H) \circ (H \otimes c_{H,H}) \circ (c_{H,H} \otimes \lambda_H) \circ (H \otimes \delta_H) \circ \delta_H$$

$$= (\Pi^L_H \otimes \Pi^L_H) \circ \delta_H$$
where the first equality follows by (a1) of Definition 1.1, the second by the antimultiplicative
property of \( \lambda_H \), the third one relies on the naturality of \( c \), the coassociativity of \( \delta_H \) and the
cocommutativity of \( H \). Finally, the last one follows by the cocommutativity of \( H \) and the
naturality of \( c \).

The proof for \( I = R \) is similar.

Note that, by (i) and the idempotent property of \( \Pi^I_H \), we have (ii) for \( I = J \). If \( I = L \) and
\( J = R \), by (7), we have

\[
(\Pi^L_H \otimes H) \circ \delta_H \circ \Pi^R_H = ((\Pi^L_H \circ \Pi^R_H) \otimes \Pi^R_H) \circ \delta_H = ((\Pi^L_H \circ \Pi^R_H) \otimes \Pi^R_H) \circ \delta_H
\]

\[
= (\Pi^R_H \otimes \Pi^R_H) \circ \delta_H = (\Pi^R_H \otimes \Pi^R_H) \circ \delta_H = \delta_H \circ \Pi^R_H.
\]

The proof of the equality (iv) follows a similar pattern and we leave the details to the reader.

**Proposition 1.7.** Let \( H \) be a cocommutative weak Hopf algebra. The following identities hold.

(i) \( \delta_{H^2} \circ \delta_H = (\delta_H \otimes \delta_H) \circ \delta_H \).

(ii) \( \delta_{H^{n+1}} \circ (H^i \otimes \delta_H \otimes H^{n-i-1}) = (H^i \otimes \delta_H \otimes H^{n-1} \otimes \delta_H \otimes H^{n-i-1}) \circ \delta_{H^n} \) for \( n \geq 2 \) and
\( i \in \{0, \cdots, n-1\} \).

(iii) \( \delta_{H^n} \circ (H^i \otimes \Pi^I_H \otimes H^{n-i-1}) = (H^i \otimes \Pi^I_H \otimes H^{n-1} \otimes \Pi^I_H \otimes H^{n-i-1}) \circ \delta_{H^n} \) for \( I \in \{L, R\},
\( n \geq 2 \) and \( i \in \{0, \cdots, n-1\} \).

(iv) \( \delta_{H^{n+1}} \circ (H^i \otimes ((\Pi^I_H \otimes H) \circ \delta_H) \otimes H^{n-i-1}) \)

\[
= (H^i \otimes ((\Pi^I_H \otimes H) \circ \delta_H \otimes H^{n-1} \otimes ((\Pi^I_H \otimes H) \circ \delta_H) \otimes H^{n-i-1}) \circ \delta_{H^n}
\]

for \( I \in \{L, R\}, n \geq 2 \) and \( i \in \{0, \cdots, n-1\} \).

(v) \( \delta_{H^{n+1}} \circ (H^i \otimes ((H \otimes \Pi^I_H) \circ \delta_H) \otimes H^{n-i-1}) \)

\[
= (H^i \otimes ((H \otimes \Pi^I_H) \circ \delta_H \otimes H^{n-1} \otimes ((H \otimes \Pi^I_H) \circ \delta_H) \otimes H^{n-i-1}) \circ \delta_{H^n}
\]

for \( I \in \{L, R\}, n \geq 2 \) and \( i \in \{0, \cdots, n-1\} \).

**Proof.** The assertion (i) follows by the coassociativity of \( \delta_H \) and the cocommutativity of \( H \).

Indeed:

\[
\delta_{H^2} \circ \delta_H = (H \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H = (\delta_H \otimes \delta_H) \circ \delta_H.
\]

The proof for (ii) can be obtained using (i) and mathematical induction. Also, by this
method and Proposition 1.6 we obtain (iii), (iv) and (v).
Remark 1.8. If $H$ is a weak Hopf algebra, we denote by $H_L$ the object such that $p_L \circ i_L = id_{H_L}$ where $i_L, p_L$ are the injection and the projection associated to the target morphism $\Pi^L_H$. If $H$ is cocommutative, by (i) of Proposition 1.6 we have that $H_L$ is a coalgebra and the morphisms $i_L, p_L$ are coalgebra morphisms for $\delta_{H_L} = (p_L \otimes p_L) \circ \delta_H \circ i_L$ and $\varepsilon_{H_L} = \varepsilon_H \circ i_L$. Therefore, $\delta_{H_L} \circ p_L = (p_L \otimes p_L) \circ \delta_H$ and $\varepsilon_{H_L} \circ p_L = \varepsilon_H$.

Proposition 1.9. Let $H$ be a weak Hopf algebra. Then, if $n \geq 3$ the following equality holds.

(22) $(H^{i-1} \otimes \mu_H \otimes H^{n-i-1} \otimes H^{i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ \delta_{H^n} = \delta_{H^{n-1}} \circ (H^{i-1} \otimes \mu_H \otimes H^{n-i-1}),$

for all $i \in \{1, \cdots, n-1\}$.

Proof: First note that, by (a1) of Definition 1.1, we have $(\mu \otimes \mu_H) \circ \delta_{H^2} = \delta_H \circ \mu_H$. Then, using this identity we have:

$$(\mu_H \otimes H^{n-2} \otimes \mu_H \otimes H^{n-2}) \circ \delta_{H^n} = (\mu_H \otimes H^{n-2} \otimes \mu_H \otimes H^{n-2}) \circ (H^2 \otimes c_{H^{n-2} \otimes H^{n-2}}) \circ (\delta_{H^2} \otimes \delta_{H^{n-2}}) = (H \otimes c_{H,H^{n-2}} \otimes H^{n-2}) \circ (((\mu_H \otimes \mu_H) \circ \delta_{H^2}) \circ \delta_{H^{n-2}}) = \delta_{H^{n-1}} \circ (\mu_H \otimes H^{n-2}).$$

Then, as a consequence, we have

$$(H^{i-1} \otimes \mu_H \otimes H^{n-i-1} \otimes H^{i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ \delta_{H^n} = (H^{i-1} \otimes \mu_H \otimes H^{n-i-1} \otimes H^{i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ$$

$$\circ (H^{i-1} \otimes H^{n-i-1} \otimes H^{i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ (\delta_{H^{i-1}} \otimes \delta_{H^{n-1-i-1}}) = (H^{i-1} \otimes \mu_H \otimes H^{n-i-1} \otimes H \otimes H^{n-i-1}) \circ (H^{i-1} \otimes c_{H^{i-1},H^{n-i-1} \otimes H^{n-i-1}}) \circ (\delta_{H^{i-1}} \otimes ((\mu_H \otimes H^{n-i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ \delta_{H^{n-1-i-1}})) = \delta_{H^{n-1}} \circ (H^{i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ (\mu_H \otimes H^{n-i-1})$$

Proposition 1.10. Let $H$ be a weak Hopf algebra. The following identity holds for $n \geq 2$.

(23) $\delta_H \circ m^n_H = (m^n_H \otimes m^n_H) \circ \delta_{H^n}.$

Proof: As in the previous proposition we proceed by induction. Obviously the equality (23) holds for $n = 2$. If we assume that it is true for $n = k$, it is true for $n = k + 1$ because:

$$(m^{k+1}_H \otimes m^{k+1}_H) \circ \delta_{H^{k+1}} = ((\mu_H \circ (m^k_H \otimes H)) \circ (\mu_H \circ (m^k_H \otimes H))) \circ \delta_{H^{k+1}} = \mu_H \circ (m^k_H \otimes H) \circ (m^k_H \otimes H) \circ \delta_H = \delta_H \circ m^{k+1}_H.$$

Proposition 1.11. Let $H$ be a weak Hopf algebra and $(A, \varphi_A)$ be a weak left $H$-module algebra. Then, if $n \geq 1$, the equality

(24) $u_n \wedge u_n = u_n$

holds.
Proposition 1.13. Let \( \mu \) hold for all \( u \) and if \( n = 1 \) the equality follows from
\[
u_1 \wedge u_1 = \mu \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes \eta_A) = \varphi_A \circ (\eta_A \otimes \eta_A)) = u_1.
\]

Definition 1.12. Let \( H \) be a cocommutative weak Hopf algebra and \((A, \varphi_A)\) be a weak left\( H \)-module algebra. For \( n \geq 1 \), with
\[
\text{Reg}_{\varphi_A}(H^n, A)
\]
we will denote the set of morphisms \( \sigma : H^n \to A \) such that there exists a morphism \( \sigma^{-1} : H^n \to A \) (the convolution inverse of \( \sigma \)) satisfying the following equalities:

(c1) \( \sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_n \),

(c2) \( \sigma \wedge \sigma^{-1} \wedge \sigma = \sigma \).

(c3) \( \sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1} \).

By \( \text{Reg}_{\varphi_A}(H_L, A) \) we denote the set of morphisms \( g : H_L \to A \) such that there exists a morphism \( g^{-1} : H_L \to A \) (the convolution inverse of \( g \)) satisfying
\[
g \wedge g^{-1} = g^{-1} \wedge g = u_0, \quad g \wedge g^{-1} \wedge g = g, \quad g^{-1} \wedge g \wedge g^{-1} = g^{-1}
\]
where \( u_0 = u_1 \circ i_L \). Then, by (b7) of the definition of weak \( H \)-module algebra, we have \( u_1 = u_0 \circ p_L \).

Note that the equality
\[
\mu \circ (u_1 \otimes \sigma) \circ (\delta_H \otimes H^{n-1}) = \sigma
\]
holds for all \( \sigma \in \text{Reg}_{\varphi_A}(H^n, A) \). Indeed,
\[
\mu \circ (u_1 \otimes \sigma) \circ (\delta_H \otimes H^{n-1})
= \mu \circ (u_1 \otimes (u_n \wedge \sigma)) \circ (\delta_H \otimes H^{n-1})
= \mu \circ ((\mu \circ (u_1 \otimes u_n)) \otimes \sigma) \circ (H \otimes \delta_{H^n}) \circ (\delta_H \otimes H^{n-1})
= \mu \circ ((\mu \circ (u_1 \otimes u_n) \circ (\delta_H \otimes H^{n-1}) \otimes \sigma) \circ \delta_{H^n}
= u_n \wedge \sigma
= \sigma
\]
because by (b4) and (b2) of Definition 1.4 we have
\[
\mu \circ (u_1 \otimes u_n) \circ (\delta_H \otimes H^{n-1}) = u_n.
\]

Proposition 1.13. Let \( H \) be a cocommutative weak Hopf algebra and let \((A, \varphi_A)\) be a weak left \( H \)-module algebra. Then, for all \( \sigma \in \text{Reg}_{\varphi_A}(H^{n+1}, A) \) the following equalities hold:

(i) \( \sigma \circ (H^i \otimes ((\Pi_H^i \otimes H) \circ \delta_H) \otimes H^{n-i-1}) = \sigma \circ (H^i \otimes \eta_H \otimes H^{n-i}) \) for all \( i \in \{0, \ldots, n-1\} \).
(ii) \( \sigma \circ (H^{n-1} \otimes ((H \otimes \Pi^R_H) \circ \delta_H)) = \sigma \circ (H^n \otimes \eta_H) \)

Proof. First note that if \( \sigma \in \text{Reg}_{\varphi_A}(H^{n+1}, A) \), by (iv) of Proposition 1.7 and the equality \( \Pi^L_H \wedge id_H = id_H \), we obtain that \( \sigma \circ (H^i \otimes ((\Pi^L_H \otimes H) \circ \delta_H) \otimes H^{n-i-1}) \in \text{Reg}_{\varphi_A}(H^n, A) \) with inverse \( \sigma^{-1} \circ (H^i \otimes ((\Pi^L_H \otimes H) \circ \delta_H) \otimes H^{n-i-1}) \).

Moreover, by the naturally of \( c \) and the equality (12), we obtain (i) because:

\[
\begin{align*}
\sigma \circ (H^i \otimes \eta_H & \otimes H^{n-i}) \\
= & \ (u_{n+1} \wedge \sigma) \circ (H^i \otimes \eta_H \otimes H^{n-i}) \\
= & \ \mu_A \circ (u_n \otimes \sigma) \circ (H^i \otimes \mu_H \otimes c_{H^{n-i-1}, H^i} \otimes H \otimes H^{n-i-1}) \\
& \circ (H^i \otimes H \otimes c_{H^{n-i-1}, H} \otimes H \otimes H^{n-i-1}) \\
& \circ (\delta_{H^i} \otimes (\delta_H \circ \eta_H) \otimes \delta_H \otimes \delta_{H^{n-i-1}}) \\
= & \ \mu_A \circ (u_n \otimes \sigma) \circ (H^i \otimes H \otimes c_{H^{n-i-1}, H} \otimes H \otimes H^{n-i-1}) \\
& \circ (H^i \otimes c_{H^{n-i-1}, H} \otimes (\delta_{H^i} \otimes \delta_H \otimes \delta_{H^{n-i-1}})) \\
& \circ (\delta_{H^i} \otimes \delta_H \otimes \delta_{H^{n-i-1}}) \\
= & \ \sigma \circ (H^i \otimes ((\Pi^L_H \otimes H) \circ \delta_H) \otimes H^{n-i-1}) \\
= & \ \sigma \circ (H^i \otimes ((\Pi^L_H \otimes H) \circ \delta_H) \otimes H^{n-i-1}),
\end{align*}
\]

The proof for (ii) is similar using (14) and we leave the details to the reader.

1.14. Let \( H \) be a cocommutative weak Hopf algebra and \((A, \varphi_A)\) be a weak left \( H \)-module algebra. Then, \( u_0 \in \text{Reg}_{\varphi_A}(H_L, A), u_n \in \text{Reg}_{\varphi_A}(H^n, A) \) and \( \text{Reg}_{\varphi_A}(H_L, A), \text{Reg}_{\varphi_A}(H^n, A) \) are groups with neutral elements \( u_0 \) and \( u_n \) respectively. Also, if \( A \) is commutative, we have that \( \text{Reg}_{\varphi_A}(H_L, A), \text{Reg}_{\varphi_A}(H^n, A) \) are abelian groups.

If \((A, \varphi_A)\) is a left \( H \)-module algebra, the groups \( \text{Reg}_{\varphi_A}(H_L, A), \text{Reg}_{\varphi_A}(H^n, A) \), \( n \geq 1 \) are the objects of a cosimplicial complex of groups with coface operators defined by

\[
\partial_{0,i} : \text{Reg}_{\varphi_A}(H_L, A) \rightarrow \text{Reg}_{\varphi_A}(H, A), \ i \in \{0, 1\}
\]
\[
\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi^R_H)) \circ \delta_H, \ \partial_{0,1}(g) = g \circ p_L
\]

\[
\partial_{1,i} : \text{Reg}_{\varphi_A}(H, A) \rightarrow \text{Reg}_{\varphi_A}(H^2, A), \ i \in \{0, 1, 2\}
\]
\[
\partial_{1,0}(h) = \varphi_A \circ (H \otimes h), \ \partial_{1,1}(h) = h \circ \mu_H, \ \partial_{1,2}(h) = h \circ \mu_H \circ (H \otimes \Pi^L_H);
\]

\[
\partial_{k-1,i} : \text{Reg}_{\varphi_A}(H^{k-1}, A) \rightarrow \text{Reg}_{\varphi_A}(H^k, A), \ k > 2, \ i \in \{0, 1, \cdots, k\}
\]

\[
\partial_{k-1,0}(\sigma) = \varphi_A \circ (H \otimes \sigma),
\]

\[
\partial_{k-1,i}(\sigma) = \begin{cases} 
\partial_{k-1,i}(\sigma) = \sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i-1}), & i \in \{1, \cdots, k - 1\} \\
\partial_{k-1,k}(\sigma) = \sigma \circ (H^{k-2} \otimes (\mu_H \circ (H \otimes \Pi^L_H)))) & \end{cases}
\]
and codegeneracy operators are defined by

\[
s_{1,0} : \text{Reg}_{\varphi_A}(H, A) \to \text{Reg}_{\varphi_A}(H_L, A),
\]

\[
s_{1,0}(h) = h \circ i_L,
\]

\[
s_{2,i} : \text{Reg}_{\varphi_A}(H^2, A) \to \text{Reg}_{\varphi_A}(H, A), \quad i \in \{0, 1\}
\]

\[
s_{2,0}(\sigma) = \sigma \circ (\eta_H \otimes H), \quad s_{2,1}(\sigma) = \sigma \circ (H \otimes \eta_H),
\]

and

\[
s_{k+1,i} : \text{Reg}_{\varphi_A}(H^{k+1}, A) \to \text{Reg}_{\varphi_A}(H^k, A), \quad k \geq 2, \quad i \in \{0, 1, \cdots, k\}
\]

\[
s_{k+1,0}(\sigma) = \sigma \circ (\eta_H \otimes H^k),
\]

\[
s_{k+1,i}(\sigma) = \sigma \circ (H^i \otimes \eta_H \otimes H^{k-i}), \quad i \in \{1, \cdots, k-1\}
\]

\[
s_{k+1,k}(\sigma) = \sigma \circ (H^k \otimes \eta_H).
\]

The morphism \(\partial_{0,0}\) is a well defined group morphism because:

\[
\partial_{0,0}(g) \otimes \partial_{0,0}(f) = \mu_A \circ ((\varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R))) \otimes (\varphi_A \circ (H \otimes (f \circ p_L \circ \Pi_H^R)))) \circ \delta_H^2 \circ \delta_H
\]

\[
= \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H, A} \otimes A) \circ (\delta_H \otimes (((g \circ p_L \circ \Pi_H^R) \otimes (f \circ p_L \circ \Pi_H^R)) \circ \delta_H) \circ \delta_H
\]

\[
= \varphi_A \circ (H \otimes (((g \circ p_L) \wedge (f \circ p_L)) \circ \Pi_H^R)) \circ \delta_H.
\]

\[
= \partial_{0,0}(g \otimes f).
\]

where the first equality follows by (i) of Proposition 1.7, the second one by the naturality of \(c\), the third one by (b2) of Definition 1.3, the fourth one by (i) of Proposition 1.6 and in the last one was used that \(p_L\) is a coalgebra morphism (see Remark 1.8).

Using \(p_L\) is a coalgebra morphism, we obtain that \(\partial_{0,1}\) is a group morphism. Moreover, by (b2) of Definition 1.3 (a1) of Definition 1.1 Proposition 1.9 and (i) of Proposition 1.6 we have that \(\partial_{k-1,i}\) are well defined group morphisms for \(k \geq 1\).

On the other hand, by (i) of Proposition 1.6 we have that \(s_{1,0}\) is a group morphism and by Propositions 1.6 and 1.13 we obtain that \(s_{k+1,i}\) are well defined group morphisms for \(k \geq 0\).

We have the cosimplicial identities from the following: For \(j = 1\), by (iv) of Proposition 1.6 and the condition of left \(H\)-module algebra for \(A\), we have

\[
\partial_{1,1}(\partial_{0,0}(g)) = \varphi_A \circ (\mu_H \otimes (g \circ p_L \circ \Pi_H^R)) \circ (H \otimes \delta_H) = \partial_{1,0}(\partial_{0,0}(g)).
\]

Moreover, if \(H\) is cocommutative, \(\Pi_H^L = \Pi_H^R\) and as a consequence \(\Pi_H^L \circ \Pi_H^R = \Pi_H^R\). Then, by (i) and (iv) of Proposition 1.6 and the properties of left \(H\)-module algebra we get

\[
\partial_{1,2}(\partial_{0,0}(g))
\]
be the coboundary morphisms of the cochain complex

\[ \partial_s (\varphi_A \circ (\mu_H \otimes (g \circ p_L \circ \Pi^R_H))) \circ (H \otimes (\delta_H \circ \Pi^L_H)) \]

\[ = \varphi_A \circ (\mu_H \otimes (H \otimes (\varphi_A \circ (H \otimes (g \circ p_L \circ \Pi^R_H)) \circ \delta_H))) \circ (H \otimes \Pi^L_H) \]

\[ = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes (g \circ p_L \circ \Pi^R_H)) \circ \delta_H))) \circ \delta_H) \]

\[ = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes (H \otimes (u_0 \circ p_L) \circ (g \circ p_L) \circ \delta_H))) \circ \delta_H) \]

\[ = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes (H \otimes (u_0 \circ p_L) \circ p_L) \circ \delta_H))) \circ \delta_H) \]

\[ = \varphi_A \circ (H \otimes (H \otimes (u_0 \circ p_L) \circ p_L) \circ \delta_H) \]

\[ = \varphi_A \circ (H \otimes (H \otimes (u_0 \circ p_L) \circ p_L) \circ \delta_H) \]

\[ = \varphi_A \circ (H \otimes (H \otimes (u_0 \circ p_L) \circ p_L) \circ \delta_H) \]

\[ = \partial_{1,0}(\partial_{0,1}(g)). \]

Also, by \([\Pi]\) we obtain that \(\partial_{1,2}(\partial_{0,1}(g)) = \partial_{1,1}(\partial_{1,0}(g))\). In a similar way, by the associativity of \(\mu_H\),

\[ \partial_{k,j} \circ \partial_{k-1,i} = \partial_{k,i} \circ \partial_{k-1,j-1}, \quad j > i \]

for \(k > 1\).

On the other hand, trivially

\[ s_{k-1,j} \circ s_{k,i} = s_{k-1,i} \circ s_{k,j+1}, \quad j \geq i. \]

Moreover, \(s_{1,0}(\partial_{0,0}(g)) = \varphi_A \circ ((\Pi^L_H \circ i_L) \otimes (g \circ p_L \circ \Pi^R_H \circ i_L)) \circ \delta_H = \varphi_A \circ (u_0 \otimes g = g, \text{ and} )\)

\(s_{1,0}(\partial_{0,1}(g)) = g. \) Also, \(s_{2,0}(\partial_{1,0}(h)) = h = s_{2,0}(\partial_{1,1}(h)), \) \(s_{2,0}(\partial_{1,2}(h)) = h = \partial_{1,1}(s_{1,0}(h)), \)

\(s_{2,1}(\partial_{1,0}(h)) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes (\Pi^R_H \circ \Pi^R_H) \circ \delta_H = \varphi_A \circ (H \otimes (\Pi^R_H \circ \Pi^R_H) \circ \delta_H = \partial_{0,0}(s_{1,0}(h)) \)

and \(s_{2,1}(\partial_{1,1}(h)) = h = s_{2,1}(\partial_{1,2}(h))\) because \(\Pi^L_H \circ \eta_H = \eta_H. \)

Finally, for \(k > 2, \) the identities

\[ s_{k+1,j} \circ \partial_{k,i} = \begin{cases} 
\partial_{k-1,i} \circ s_{k,j-1}, & i < j \\
\text{id}_{\text{Reg}_{\varphi_A}(H^k, A)}, & i = j, \ i = j + 1 \\
\partial_{k-1,i-1} \circ s_{k,j}, & i > j + 1 
\end{cases} \]

follow as in the Hopf algebra setting.

Let

\[ D^k_{\varphi_A} = \partial_{k,0} \wedge \partial_{k,1}^{-1} \wedge \cdots \wedge \partial_{k,k+1}^{(-1)^{k+1}} \]

be the coboundary morphisms of the cochain complex

\( \text{Reg}_{\varphi_A}(H_L, A) \xrightarrow{D^0_{\varphi_A}} \text{Reg}_{\varphi_A}(H, A) \xrightarrow{D^1_{\varphi_A}} \text{Reg}_{\varphi_A}(H^2, A) \xrightarrow{D^2_{\varphi_A}} \cdots \)

\( \cdots \xrightarrow{D^{k-1}_{\varphi_A}} \text{Reg}_{\varphi_A}(H^k, A) \xrightarrow{D^k_{\varphi_A}} \text{Reg}_{\varphi_A}(H^{k+1}, A) \xrightarrow{D^{k+1}_{\varphi_A}} \cdots \)

associated to the cosimplicial complex \(\text{Reg}_{\varphi_A}(H^*, A).\)
Then, when \((A, \varphi_A)\) is a commutative left \(H\)-module algebra, \((\text{Reg}_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})\) gives the Sweedler cohomology of \(H\) in \((A, \varphi_A)\). Therefore, the \(k\)th group will be defined by
\[
\frac{\text{Ker}(D^k_{\varphi_A})}{\text{Im}(D^{k-1}_{\varphi_A})}
\]
for \(k \geq 1\) and \(\text{Ker}(D^0_{\varphi_A})\) for \(k = 0\). We will denote it by \(H^k_{\varphi_A}(H, A)\).

The normalized cochain subcomplex of \((\text{Reg}_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})\) is defined by
\[
\text{Reg}^+_{\varphi_A}(H^{k+1}, A) = \bigcap_{i=0}^k \text{Ker}(s_{k+1,i}),
\]
\[
\text{Reg}^+_{\varphi_A}(H_L, A) = \{ g \in \text{Reg}_{\varphi_A}(H_L, A) : g \circ p_L \circ \eta_H = \eta_A \}
\]
and \(D^{k+}_A\) the restriction of \(D^k_{\varphi_A}\) to \(\text{Reg}^+_{\varphi_A}(H^\bullet, A)\).

We have that \((\text{Reg}^+_{\varphi_A}(H^\bullet, A), D^+_{\varphi_A})\), is a subcomplex of \((\text{Reg}_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})\) and the injection map induces an isomorphism of cohomology (see [20] for the dual result). Then,
\[
H^2_{\varphi_A}(H, A) \simeq H^2_{\varphi_A}(H, A) = \frac{\text{Ker}(D^{2+}_{\varphi_A})}{\text{Im}(D^{1+}_{\varphi_A})}.
\]

Note that
\[
\text{Reg}^+_{\varphi_A}(H, A) = \text{Ker}(s_{1,0}) = \{ h \in \text{Reg}_{\varphi_A}(H, A) : h \circ i_L = u_0 \},
\]
and
\[
\text{Reg}^+_{\varphi_A}(H^2, A) = \text{Ker}(s_{2,0}) \cap \text{Ker}(s_{2,1})
\]
\[
= \{ \sigma \in \text{Reg}_{\varphi_A}(H^2, A) : \sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = u_1 \}.
\]

The following proposition give a different characterization of the morphisms in \(\text{Reg}^+_{\varphi_A}(H, A)\).

**Proposition 1.15.** Let \(H\) be a weak Hopf algebra and \((A, \varphi_A)\) be a weak left \(H\)-module algebra. Let \(h : H \to A\) be a morphism satisfying
\[
h \wedge h^{-1} = h^{-1} \wedge h = u_1, \ h \wedge h^{-1} \wedge h = h, \ h^{-1} \wedge h \wedge h^{-1} = h^{-1}.
\]
The following equalities are equivalent
\[
\begin{align*}
(i) & \ h \circ \eta_H = \eta_A. \\
(ii) & \ h \circ \Pi^L_H = u_1. \\
(iii) & \ h \circ \Pi^L_H = u_1.
\end{align*}
\]

**Proof:** The assertion (ii) \(\Rightarrow\) (i) follows by
\[
h \circ \eta_H = h \circ \Pi^L_H \circ \eta_H = u_1 \circ \eta_H = \eta_A.
\]

Now we get (i) \(\Rightarrow\) (ii):
\[
h \circ \Pi^L_H
\]
\[(u_1 \wedge h) \circ \Pi^L_H = (\varepsilon_H \circ \mu_H) \otimes (\mu_A \circ (u_1 \otimes h)) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \circ c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) = \mu_A \circ (u_2 \otimes h) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes \Pi^L_H) = \mu_A \circ (\varphi_A \otimes h) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes (u_1 \circ \Pi^L_H)) = \mu_A \circ (\varphi_A \otimes h) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes u_1) = \mu_A \circ (\varphi_A \otimes h) \circ (\Pi^L_H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes u_1) = \mu_A \circ ((\mu_A \circ c_{A,A} \circ (u_1 \otimes A)) \otimes h) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes u_1) = \mu_A \circ c_{A,A} \circ (((u_1 \wedge h) \circ \eta_H) \otimes u_1) = \eta_A.

The first equality follows by the properties of \(h\), the second one by the naturality of \(c\) and the coassociativity of \(\delta_H\), the third one by (11), the fourth one by (b3) of Definition 1.4, the fifth one by (b6) of Definition 1.4, the sixth one by (17), the seventh one by (b5) of Definition 1.4, the eight one by the naturality of \(c\) and the associativity of \(\mu_A\), the ninth one the by the properties of \(h\) and the last one by (ii).

The assertion (iii) \(\Rightarrow\) (i) follows because
\[h \circ \eta_H = h \circ \Pi^L_H \circ \eta_H = u_1 \circ \eta_H = \eta_A.

The proof for (i) \(\Rightarrow\) (iii) is the following:
\[h \circ \Pi^L_H = (u_1 \wedge h) \circ \Pi^L_H = \mu_A \circ (h \otimes ((u_1 \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)) \circ ((\delta_H \circ \eta_H) \otimes H) = \mu_A \circ (h \otimes (u_1 \circ \mu_H \circ (\Pi^L_H \otimes H))) \circ ((\delta_H \circ \eta_H) \otimes H) = \mu_A \circ (h \otimes \varphi_A) \circ ((\delta_H \circ \eta_H) \otimes u_1) = \mu_A \circ (h \otimes (\varphi_A \circ (\Pi^L_H \otimes A))) \circ ((\delta_H \circ \eta_H) \otimes u_1) = \mu_A \circ (h \otimes (\varphi_A \circ (u_1 \otimes A))) \circ ((\delta_H \circ \eta_H) \otimes u_1) = \mu_A \circ (((h \otimes u_1 \circ \eta_H) \otimes u_1) = \mu_A \circ (((h \wedge u_1) \circ \eta_H) \otimes u_1) = u_1.

The first equality follows by the properties of \(h\), the second one by the coassociativity of \(\delta_H\), the third one by (16), the fourth one by (b3) of Definition 1.4, the fifth one by (b7) of Definition 1.4 and (12), the sixth one by (b4) of Definition 1.4, the seventh one by the associativity of \(\mu_H\), the eight one by the properties of \(h\) and the last one by (iii).

Remark 1.16. Note that as a consequence of Proposition 1.15
\[\text{Reg}^+_\varphi(H,A) = \{h \in \text{Reg}_\varphi(H,A) ; h \circ \eta_H = \eta_A\},\]
and by Proposition 1.13 we have
\[ \text{Reg}_{\varphi, A}^+(H^2, A) = \{ \sigma \in \text{Reg}_{\varphi, A}(H^2, A) : \sigma \circ (\Pi_H^L \otimes H) \circ \delta_H = \sigma \circ (H \otimes \Pi_H^R) \circ \delta_H = u_1 \}. \]

2. Weak crossed products for weak Hopf algebras

In the first paragraphs of this section we resume some basic facts about the general theory of weak crossed products in $\mathcal{C}$ introduced in [16] particularized for a weak Hopf algebra $H$.

Let $A$ be an algebra and let $H$ be a weak Hopf algebra in $\mathcal{C}$. Suppose that there exists a morphism
\[ \psi^A_H : H \otimes A \to A \otimes H \]
such that the following equality holds
\[ (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\psi^A_H \otimes A) = \psi^A_H \circ (H \otimes \mu_A). \]

As a consequence of (27), the morphism $\nabla_{A \otimes H} : A \otimes H \to A \otimes H$ defined by
\[ \nabla_{A \otimes H} = (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (A \otimes H \otimes \eta_A) \]
is an idempotent. Moreover, it satisfies that
\[ \nabla_{A \otimes H} \circ (\mu_A \otimes H) = (\mu_A \otimes H) \circ (A \otimes \nabla_{A \otimes H}), \]
that is, $\nabla_{A \otimes H}$ is a left $A$-module morphism (see Lemma 3.1 of [16]) for the regular action $\varphi_{A \otimes H} = \mu_A \otimes H$. With $A \times H$, $i_{A \otimes H} : A \times H \to A \otimes H$ and $p_{A \otimes H} : A \otimes H \to A \times H$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A \otimes H}$. Finally, if $\psi^A_H$ satisfies (27), the following identities hold
\[ (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\nabla_{A \otimes H} \otimes A) = (\mu_A \otimes H) \circ (A \otimes \psi^A_H) = \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \psi^A_H). \]

From now on we consider quadruples $\mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H)$ where $A$ is an algebra, $H$ an object, $\psi^A_H : H \otimes A \to A \otimes H$ a morphism satisfying (27) and $\sigma^A_H : H \otimes H \to A \otimes H$ a morphism in $\mathcal{C}$.

We say that $\mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H)$ satisfies the twisted condition if
\[ (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\sigma^A_H \otimes A) = (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes \psi^A_H) \]
and the cocycle condition holds if
\[ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\sigma^A_H \otimes H) = (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes \sigma^A_H). \]

Note that, if $\mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H)$ satisfies the twisted condition, in Proposition 3.4 of [16] we prove that the following equalities hold:
\[ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes \nabla_{A \otimes H}) = \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H), \]
\[ \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\nabla_{A \otimes H} \otimes H) = \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H). \]
Then, if \( \nabla_{A \otimes H} \circ \sigma^A_H = \sigma^A_H \) we obtain

\[
\begin{align*}
(34) \quad (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes \nabla_{A \otimes H}) &= (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H), \\
(35) \quad (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\nabla_{A \otimes H} \otimes H) &= (\mu_A \otimes H) \circ (A \otimes \sigma^A_H).
\end{align*}
\]

By virtue of (30) and (31) we will consider from now on, and without loss of generality, that

\[
(36) \quad \nabla_{A \otimes H} \circ \sigma^A_H = \sigma^A_H
\]

holds for all quadruples \( \mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H) \) (see Proposition 3.7 of [16]).

For \( \mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H) \) define the product

\[
(37) \quad \mu_{A \otimes H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma^A_H) \circ (A \otimes \psi^A_H \otimes H)
\]

and let \( \mu_{A \times H} \) be the restriction of \( \mu_{A \otimes H} \) to \( A \times H \), i.e.

\[
(38) \quad \mu_{A \times H} = p_{A \otimes H} \circ \mu_{A \otimes H} \circ (i_{A \otimes H} \otimes i_{A \otimes H}).
\]

If the twisted and the cocycle conditions hold, the product \( \mu_{A \otimes H} \) is associative and normalized with respect to \( \nabla_{A \otimes H} \) (i.e. \( \nabla_{A \otimes H} \circ \mu_{A \otimes H} = \mu_{A \otimes H} = \mu_{A \otimes H} \circ (\nabla_{A \otimes H} \otimes \nabla_{A \otimes H}) \) and, by the definition of \( \mu_{A \otimes H} \),

\[
(39) \quad \mu_{A \otimes H} \circ (\nabla_{A \otimes H} \otimes A \otimes H) = \mu_{A \otimes H}
\]

holds and therefore

\[
(40) \quad \mu_{A \otimes H} \circ (A \otimes H \otimes \nabla_{A \otimes H}) = \mu_{A \otimes H}.
\]

Due to the normality condition, \( \mu_{A \times H} \) is associative as well (Proposition 2.5 of [16]). Hence we define:

**Definition 2.1.** If \( \mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H) \) satisfies (30) and (31) we say that \( (A \otimes H, \mu_{A \otimes H}) \) is a weak crossed product.

The next natural question that arises is if it is possible to endow \( A \times H \) with a unit, and hence with an algebra structure. As we recall in [16], in order to do that, we need to use the notion of preunit to obtain an unit in \( A \times H \). In our setting, if \( A \) is an algebra, \( H \) an object in \( \mathcal{C} \) and \( m_{A \otimes H} \) is an associative product defined in \( A \otimes H \) a preunit \( \nu : K \rightarrow A \otimes H \) is a morphism satisfying

\[
(41) \quad m_{A \otimes H} \circ (A \otimes H \otimes \nu) = m_{A \otimes H} \circ (\nu \otimes A \otimes H) = m_{A \otimes H} \circ (A \otimes H \otimes (m_{A \otimes H} \circ (\nu \otimes \nu))).
\]

Associated to a preunit we obtain an idempotent morphism

\[
\nabla^\nu_{A \otimes H} = m_{A \otimes H} \circ (A \otimes H \otimes \nu) : A \otimes H \rightarrow A \otimes H.
\]
Take $A \times^\nu H$ the image of this idempotent, $p^\nu_A \otimes H$ the projection and $i^\nu_A \otimes H$ the injection. It is possible to endow $A \times^\nu H$ with an algebra structure whose product is
\[ m_{A \times^\nu H} = p^\nu_A \otimes H \circ m_A \otimes H \circ (i^\nu_A \otimes H \otimes i^\nu_A \otimes H) \]
and whose unit is $\eta_{A \times^\nu H} = p^\nu_A \otimes H \circ \nu$ (see Proposition 2.5 of [16]). If moreover, $\mu_A \otimes H$ is left $A$-linear for the actions $\varphi_{A \otimes H} = \mu_A \otimes H$, $\varphi_{A \otimes H \otimes A \otimes H} = \varphi_{A \otimes H} \otimes A \otimes H$ and normalized with respect to $\nabla^\nu_{A \otimes H}$, the morphism
\[ \beta_\nu : A \to A \otimes H, \quad \beta_\nu = (\mu_A \otimes H) \circ (A \otimes \nu) \]
is multiplicative and left $A$-linear for $\varphi_A = \mu_A$.

Although $\beta_\nu$ is not an algebra morphism, because $A \otimes H$ is not an algebra, we have that $\beta_\nu \circ \eta_A = \nu$, and thus the morphism $\tilde{\beta}_\nu = p^\nu_A \otimes H \circ \beta_\nu : A \to A \times^\nu H$ is an algebra morphism.

In light of the considerations made in the last paragraphs, and using the twisted and the cocycle conditions, in [16] we characterize weak crossed products with a preunit, and moreover we obtain an algebra structure on $A \times H$. These assertions are a consequence of the following theorem proved in [16].

**Theorem 2.2.** Let $A$ be an algebra, $H$ a weak Hopf algebra and $m_{A \otimes H} : A \otimes H \otimes A \otimes H \to A \otimes H$ a morphism of left $A$-modules for the actions $\varphi_{A \otimes H} = \mu_A \otimes H$, $\varphi_{A \otimes H \otimes A \otimes H} = \varphi_{A \otimes H} \otimes A \otimes H$.

Then the following statements are equivalent:

(i) The product $m_{A \otimes H}$ is associative with preunit $\nu$ and normalized with respect to $\nabla^\nu_{A \otimes H}$.

(ii) There exist morphisms $\psi^A_H : H \otimes A \to A \otimes V$, $\sigma^A_H : H \otimes H \to A \otimes H$ and $\nu : k \to A \otimes H$ such that if $\mu_A \otimes H$ is the product defined in [37], the pair $(A \otimes H, \mu_A \otimes H)$ is a weak crossed product with $m_{A \otimes H} = \mu_A \otimes H$ and satisfying:

\[ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes \nu) = \nabla_{A \otimes H} \circ (\eta_A \otimes H), \]
\[ (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\nu \otimes H) = \nabla_{A \otimes H} \circ (\eta_A \otimes H), \]
\[ (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\nu \otimes A) = \beta_\nu, \]

where $\beta_\nu$ is the morphism defined in (42). In this case $\nu$ is a preunit for $\mu_A \otimes H$, the idempotent morphism of the weak crossed product $\nabla_{A \otimes H}$ is the idempotent $\nabla^\nu_{A \otimes H}$, and we say that the pair $(A \otimes H, \mu_A \otimes H)$ is a weak crossed product with preunit $\nu$.

**Remark 2.3.** Note that in the proof of the previous Theorem for $(i) \Rightarrow (ii)$ we define $\psi^A_H$ and $\sigma^A_H$ as

\[ \psi^A_H = m_{A \otimes H} \circ (\eta_A \otimes H \otimes \beta_\nu), \]
\[ \sigma^A_H = m_{A \otimes H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H). \]
Corollary 2.4. If \((A \otimes H, \mu_{A\otimes H})\) is a weak crossed product with preunit \(\nu\), then \(A \times H\) is an algebra with the product defined in (38) and unit \(\eta_{A \times H} = p_{A\otimes H} \circ \nu\).

Remark 2.5. As a consequence of the previous corollary we obtain that if a weak crossed algebra with the product defined in (38) and unit \(\eta\) admits two preunits \(\nu_1, \nu_2\), as in (ii) of Theorem 2.2 we have \(p_{A\otimes H} \circ \nu_1 = p_{A\otimes H} \circ \nu_2\) and then

\[
\nu_1 = \nabla_{A \otimes H} \circ \nu_1 = \nabla_{A \otimes H} \circ \nu_2 = \nu_2.
\]

Definition 2.6. Let \(H\) be a weak Hopf algebra, \((A, \varphi_A)\) a weak left \(H\)-module algebra and \(\sigma : H \otimes H \to A\) a morphism. We define the morphisms

\[
\psi^A_H : H \otimes A \to A \otimes H, \quad \sigma^A_H : H \otimes H \to A \otimes H,
\]

by

\[
(48) \quad \psi^A_H = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)
\]

and

\[
(49) \quad \sigma^A_H = (\sigma \otimes \mu_H) \circ \delta_{H^2}.
\]

Proposition 2.7. Let \(H\) be a weak Hopf algebra and \((A, \varphi_A)\) a weak left \(H\)-module algebra. The morphism \(\psi^A_H\) defined above satisfies (27). As a consequence the morphism \(\nabla_{A \otimes H}\), defined in (28), is an idempotent and the following equalities hold:

\[
(50) \quad \nabla_{A \otimes H} = ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H),
\]

\[
(51) \quad \mu_A \circ (u_1 \otimes \varphi_A) \circ (\delta_H \otimes A) = \varphi_A,
\]

\[
(52) \quad (\mu_A \otimes H) \circ (u_1 \otimes \psi^A_H) \circ (\delta_H \otimes A) = \psi^A_H,
\]

\[
(53) \quad (A \otimes \varphi_H) \circ \psi^A_H \circ (H \otimes \eta_A) = u_1,
\]

\[
(54) \quad (\mu_A \otimes H) \circ (u_1 \otimes c_{H,A}) \circ (\delta_H \otimes A) = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\psi^A_H \circ (H \otimes \eta_A)) \otimes A),
\]

\[
(55) \quad (A \otimes \varphi_H) \circ \nabla_{A \otimes H} = \mu_A \circ (A \otimes u_1).
\]

\[
(56) \quad (A \otimes \delta_H) \circ \nabla_{A \otimes H} = (\nabla_{A \otimes H} \otimes H) \circ (A \otimes \delta_H).
\]

Proof. For the morphism \(\psi^A_H\) we have

\[
(57) \quad (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\psi^A_H \otimes A)
= (\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)) \otimes H) \circ (H \otimes A \otimes c_{H,A}) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)
\]

\[
(58) \quad (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\psi^A_H \otimes A)
= (\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)) \otimes H) \circ (H \otimes A \otimes c_{H,A}) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)
\]

Also, (45) implies that \(\nabla_{A \otimes H} \circ \nu = \nu\).
\[ = (\varphi_A \circ (H \otimes \mu_A)) \otimes H) \circ (H \otimes A \otimes c_{H,A}) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) \]

where the first equality follows by the naturality of \(c\) and the coassociativity of \(\delta_H\), the second one by (b2) of Definition 1.4 and the third one by the naturality of \(c\). Thus, \(\psi^A_H\) satisfies (27).

As a consequence, \(\nabla_{A \otimes H}\) is an idempotent and (50), (53), (55) follow easily from the definition of \(\psi^A_H\). On the other hand, (51) follows by (50) and (b2) of Definition 1.4 because:

\[
\mu_A \circ (u_1 \otimes \varphi_A) \circ (\delta_H \otimes A) \\
= \mu_A \circ (A \otimes \varphi_A) \circ ((\nabla_{A \otimes H} \circ (\eta_A \otimes H)) \otimes A) \\
= \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes \eta_A \otimes A) \\
= \varphi_A.
\]

Analogously, by (b2) of Definition 1.4, we obtain (52). Finally, the equality (56) follows from (50) and the coassociativity of \(\delta_H\), and (54) is an easy consequence of the naturality of \(c\).

**Proposition 2.8.** Let \(H\) be a weak Hopf algebra, \((A, \varphi_A)\) a weak left \(H\)-module algebra and \(\sigma : H \otimes H \to A\) a morphism. The morphism \(\sigma^A_H\) introduced in Definition 2.6 satisfies the following identity:

\[ (A \otimes \delta_H) \circ \sigma^A_H = (\sigma^A_H \otimes \mu_H) \circ \delta_{H^2}. \]

**Proof:** The proof is the following:

\[
(A \otimes \delta_H) \circ \sigma^A_H \\
= (A \otimes \mu_H \otimes \mu_H) \circ (\sigma \otimes \delta_{H^2}) \circ \delta_{H^2} \\
= (\sigma \otimes \mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ \\
(((\delta_H \otimes H) \circ \delta_H) \otimes ((\delta_H \otimes H) \circ \delta_H)) \\
= (\sigma^A_H \otimes \mu_H) \circ \delta_{H^2}.
\]

The first equality follows by (a1) of Definition 1.1, the second one by the naturality of \(c\) and the last one by the coassociativity of \(\delta_H\) and the naturality of \(c\).

**Proposition 2.9.** Let \(H\) be a cocommutative weak Hopf algebra, \((A, \varphi_A)\) a weak left \(H\)-module algebra and \(\sigma \in \text{Reg}_{\varphi_A}(H^2, A)\). The morphism \(\sigma^A_H\) introduced in Definition 2.6 satisfies the following identities:

(i) \(\nabla_{A \otimes H} \circ \sigma^A_H = \sigma^A_H\).

(ii) \((A \otimes \varepsilon_H) \circ \sigma^A_H = \sigma\).

**Proof:** By Proposition 2.8 and the properties of \(\sigma\) we have that

\[
\nabla_{A \otimes H} \circ \sigma^A_H \\
= ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H) \circ \sigma^A_H \\
= ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (\sigma^A_H \otimes \mu_H) \circ \delta_{H^2}
\]
Remark 2.10. Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. Note that, by Propositions 2.7, 2.8 and 2.9, we have a quadruple $\mathcal{A}_H = (A, H, \psi^A_H, \sigma^A_H)$ such that $\psi^A_H$ satisfies (27) and $\nabla_{A\otimes H} \circ \sigma^A_H = \sigma^A_H$.

Definition 2.11. Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. We say that $\sigma$ satisfies the twisted condition if

$$\mu_A \circ (\varphi_A \circ (H \otimes \varphi_A)) \circ (H \otimes H \otimes c_{A, A}) \circ ((H \otimes H \otimes \sigma) \circ \delta_{H^2}) \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma^A_H \otimes A).$$

If

$$\partial_{2,3}(\sigma) \wedge \partial_{2,1}(\sigma) = \partial_{2,0}(\sigma) \wedge \partial_{2,2}(\sigma)$$

holds, we will say that $\sigma$ satisfies the 2-cocycle condition.

2.12. Let $H$ be a weak Hopf algebra. The morphisms

$$\Omega^L_{H \otimes H} = ((\varepsilon_H \circ \mu_H) \otimes H \otimes H) \circ \delta_{H \otimes H} : H \otimes H \rightarrow H \otimes H$$

and

$$\Omega^R_{H \otimes H} = (H \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H \otimes H} : H \otimes H \rightarrow H \otimes H$$

are idempotent. Indeed: By (11) we have

$$\Omega^L_{H \otimes H} = ((\mu_H \circ (H \otimes \Pi^L_H)) \otimes H) \circ (H \otimes \delta_H).$$

Then, by (62), the coassociativity of $\delta_H$ and (11) we have

$$\Omega^L_{H \otimes H} \circ \Omega^L_{H \otimes H} = ((\mu_H \circ (H \otimes (\Pi^L_H \wedge \Pi^L_H))) \otimes H) \circ (H \otimes \delta_H) = \Omega^L_{H \otimes H}.$$

The proof for $\Omega^R_{H \otimes H}$ is similar, using the identity

$$\Omega^R_{H \otimes H} = (H \otimes (\mu_H \circ (\Pi^R_H \otimes H))) \circ (\delta_H \otimes H),$$

and we left the details to the reader.

By (a1) of Definition 1.11 we obtain that

$$\mu_H \circ \Omega^L_{H \otimes H} = \mu_H \circ \Omega^R_{H \otimes H} = \mu_H$$

and it is easy to show that, if we consider the left-right $H$-module actions and a left-right $H$-comodule coactions

$$\varphi_{H \otimes H} = \mu_H \otimes H, \ \phi_{H \otimes H} = H \otimes \mu_H, \ \eta_{H \otimes H} = \delta_H \otimes H, \ \rho_{H \otimes H} = H \otimes \delta_H$$
on $H \otimes H$, we have that $\Omega^L_{H \otimes H}$ is a morphism of left and right $H$-modules and right $H$-comodules and $\Omega^R_{H \otimes H}$ is a morphism of left and right $H$-modules and left $H$-comodules. Moreover, if $H$ is cocommutative it is an easy exercise to prove that $\Omega^L_{H \otimes H} = \Omega^R_{H \otimes H}$ and the following equalities hold:

\begin{equation}
\delta_{H \otimes H} \circ \Omega^L_{H \otimes H} = (H \otimes H \otimes \Omega^L_{H \otimes H}) \circ \delta_{H \otimes H} = (\Omega^L_{H \otimes H} \otimes H \otimes H) \circ \delta_{H \otimes H}.
\end{equation}

As a consequence, we obtain that

\begin{equation}
\delta_{H \otimes H} \circ \Omega^L_{H \otimes H} = (\Omega^L_{H \otimes H} \otimes \Omega^L_{H \otimes H}) \circ \delta_{H \otimes H}.
\end{equation}

Therefore, if $H$ is cocommutative, we will denote the morphism $\Omega^L_{H \otimes H}$ by $\Omega^2_H$.

**Proposition 2.13.** Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$.

(i) $\sigma \circ \Omega^2_H = \sigma$.

(ii) $\sigma^A_H \circ \Omega^2_H = \sigma^A_H$.

(iii) $(A \otimes \Omega^2_H) \circ (\sigma^A_H \otimes H) = (\sigma^A_H \otimes H) \circ H \otimes \Omega^2_H$.

(iv) $\partial_{2,3}(\sigma) = (\sigma \otimes \varepsilon_H) \circ (H \otimes \Omega^2_H)$.

**Proof:** To prove (i) first we show that $u_2 \circ \Omega^2_H = u_2$. Indeed: By (64) we have

\begin{equation}
u_2 \circ \Omega^2_H = \varphi_A \circ ((\mu_H \circ \Omega^2_H) \otimes \eta_A) = \varphi_A \circ (\mu_H \otimes \eta_A) = u_2.
\end{equation}

Then, (i) holds because, by (65), we obtain:

$$\sigma = \sigma \wedge \sigma^{-1} \wedge \sigma = \mu_A \circ (u_2 \otimes \sigma) \circ \delta_{H^2} = \mu_A \circ ((u_2 \circ \Omega^2_H) \otimes \sigma) \circ \delta_{H^2}$$

$$= \mu_A \circ (u_2 \otimes \sigma) \circ \delta_{H^2} \circ \Omega^2_H = (\sigma \wedge \sigma^{-1} \wedge \sigma) \circ \Omega^2_H = \sigma \circ \Omega^2_H.$$

By (65) and the properties of (i) we have:

$$\sigma^A_H \circ \Omega^2_H = ((\sigma \otimes \Omega^2_H) \otimes \mu_H) \circ \delta_{H^2} = \sigma^A_H.$$

Then (ii) holds.

Using that $\Omega^2_H$ is a morphism of left $H$-comodules and $H$-modules we obtain (iii). Finally, (iv) is a consequence of (62).

**Proposition 2.14.** Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. Then $\sigma$ satisfies 2-cocycle condition if and only if the equality

\begin{equation}
\mu_A \circ (A \otimes \sigma) \circ (\sigma^A_H \otimes H) = \mu_A \circ (A \otimes \sigma) \circ (\psi^A_H \otimes H) \circ (H \otimes \sigma^A_H)
\end{equation}

holds.

**Proof:** The proof follows from the following facts: First note

$$\partial_{2,3}(\sigma) \wedge \partial_{2,1}(\sigma)$$
Remark 2.15. Note that, if \((A, \varphi_A)\) is a commutative left \(H\)-module algebra, the 2-cocycle condition means that \(\sigma \in \text{Ker}(D^2_{\varphi_A})\).

Also, by the cocommutativity of \(H\), we have

\[
\sigma^A_H = c_{A,H} \circ \tau^A_H
\]

where \(\tau^A_H = (\mu_H \otimes \sigma) \circ \delta_{H^2}\). Therefore, if \((A, \varphi_A)\) is a commutative left \(H\)-module algebra the twisted condition holds for all \(\sigma \in \text{Reg}_{\varphi_A}(H^2, A)\).

Theorem 2.16. Let \(H\) be a cocommutative weak Hopf algebra, \((A, \varphi_A)\) a weak left \(H\)-module algebra and \(\sigma \in \text{Reg}_{\varphi_A}(H^2, A)\). The morphism \(\sigma\) satisfies the twisted condition (58) if and only if \(k_H\) satisfies the twisted condition (59).

Proof: If \(k_H\) satisfies the twisted condition (59), composing with \(A \otimes \varepsilon_H\) and using (ii) of Proposition 2.9 we obtain that \(\sigma\) satisfies the twisted condition (58). Conversely, assume that \(\sigma\) satisfies the twisted condition (58). Then

\[
\begin{align*}
= \mu_A \circ ((\sigma \otimes \varepsilon_H) \circ (H \otimes \Omega^2_H)) \circ (\sigma \otimes (\mu_H \otimes H)) \circ \delta_{H^3} \\
= \mu_A \circ (A \otimes \sigma) \circ (\varepsilon^A_H \otimes H) \circ (H \otimes \Omega^2_H) \\
= \mu_A \circ (A \otimes (\sigma \otimes \Omega^2_H)) \circ (\sigma^A_H \otimes H) \\
= \mu_A \circ (A \otimes \sigma) \circ (\sigma^A_H \otimes H)
\end{align*}
\]

where the first equality follows by (iv) of Proposition 2.13, the second one by the properties of \(\varepsilon_H\), the third one by (iii) of Proposition 2.13 and, the last one by (i) of Proposition 2.13.

On the other hand, by the naturality of \(c\) we obtain that

\[
\partial_{2,0}(\sigma) \wedge \partial_{2,2}(\sigma) = \mu_A \circ (A \otimes \sigma) \circ (\psi^A_H \otimes H) \circ (H \otimes \sigma^A_H)
\]

and this finish the proof.
(H ⊗ H ⊗ c_{H,H} ⊗ H ⊗ H ⊗ A) \circ (((H ⊗ \delta_H) \circ \delta_H) \circ ((H ⊗ \delta_H) \circ \delta_H) \circ A)
= (\mu_A \otimes H) \circ (A \otimes \varphi_A \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (\sigma \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H^2}) \otimes A) \circ (\delta_{H^2} \otimes A)
= (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\sigma_H^A \otimes A).

The first and the fifth equalities follow by the naturality of c, the cocommutativity of H and the coassociativity of \delta_H, the second one by the cocommutativity of H and the coassociativity of \delta_H, the third and the sixth ones by the the naturality of c, the fourth one by the twisted condition for \sigma and the last one by (a1) of Definition 1.1.

Therefore, \Lambda_H satisfies the twisted condition (30).

**Theorem 2.17.** Let H be a cocommutative weak Hopf algebra, (A, \varphi_A) a weak left H-module algebra and \sigma \in Reg_{\varphi_A}(H^2, A). The morphism \sigma satisfies the 2-cocycle condition (67) if and only if \Lambda_H satisfies the cocycle condition (31).

**Proof.** If \Lambda_H satisfies the cocycle condition (31), composing with A \otimes \varepsilon_H and using (ii) of Proposition 2.9 we obtain that \sigma satisfies the 2-cocycle condition (67). Conversely, assume that \sigma satisfies the 2-cocycle condition (59). Then

\[
\begin{align*}
(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A) \\
= (\mu_A \otimes H) \circ (A \otimes \sigma \otimes \mu_H) \circ (\psi_H^A \otimes c_{H,H} \otimes H) \circ (H \otimes c_{H,A} \otimes H \otimes H) \circ (\delta_H \circ (((A \otimes \delta_H) \circ \sigma_H^A)) \\
= (\mu_A \otimes H) \circ (A \otimes \sigma \otimes \mu_H) \circ (\psi_H^A \otimes c_{H,H} \otimes H) \circ (H \otimes c_{H,A} \otimes H \otimes H) \circ (\delta_H \circ (((\sigma_H^A \otimes \mu_H) \circ \delta_{H^2})) \\
= ((\mu_A \circ (A \otimes \sigma) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A)) \circ (\mu_H \circ (\mu_H \otimes H))) \circ \delta_{H^3} \\
= ((\mu_A \circ (A \otimes \sigma) \circ (\sigma_H^A \otimes H)) \otimes (\mu_H \circ (H \otimes \mu_H))) \circ \delta_{H^3} \\
= (\mu_A \otimes H) \circ (A \otimes \sigma \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((\sigma_H^A \otimes \mu_H) \circ \delta_{H^2}) \otimes \delta_H) \\
= (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\sigma_H^A \otimes H).
\end{align*}
\]

The first equality follows by the naturality of c and the coassociativity of \delta_H, the second and the sixth ones by Proposition 2.8, the third and the fifth ones by the naturality of c and the associativity of \mu_H, the fourth one by the 2-cocycle condition (67).

**Remark 2.18.** By Theorems 2.16 and 2.17 and applying the general theory of weak crossed products, we have the following: If \sigma \in Reg_{\varphi_A}(H^2, A) satisfies the twisted condition (58) (equivalently (67)) and the 2-cocycle condition (59), the quadruple \Lambda_H defined in Remark 2.10 satisfies the twisted and the cocycle conditions (30), (31) and therefore the induced product is associative. Conversely, by Theorem 2.2 we obtain that, if the product induced by the quadruple \Lambda_H defined in Remark 2.10 is associative, \Lambda_H satisfies the twisted and the cocycle condition and, by Theorems 2.16 and 2.17 \sigma satisfies the twisted condition (58) and the 2-cocycle condition (59) (equivalently (67)).
Definition 2.19. Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. We say that $\sigma$ satisfies the normal condition if

$$\sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = u_1,$$

i.e. $\sigma \in \text{Reg}^+_H(H^2, A)$.

Theorem 2.20. Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. Let $\mathcal{K}_H$ be the quadruple defined in Remark 2.10 and assume that $\mathcal{K}_H$ satisfies the twisted and the cocycle conditions (30) and (31). Then, $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to $\mathcal{K}_H$ if and only if

$$\sigma^A_H \circ (\eta_H \otimes H) = \sigma^A_H \circ (H \otimes \eta_H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H).$$

Proof: By Theorem 2.2 to prove the result, we only need to show that (43), (44) and (45) hold for $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ if and only if

$$\sigma^A_H \circ (\eta_H \otimes H) = \sigma^A_H \circ (H \otimes \eta_H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H).$$

Indeed, $\nu$ satisfies (43) if and only if $\sigma^A_H \circ (H \otimes \eta_H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H)$ because:

$$(\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes \nu)
= (\mu_A \otimes H) \circ (A \otimes \sigma^A_H) \circ (\psi^A_H \otimes H) \circ (H \otimes (\psi^A_H \circ (\eta_H \otimes \eta_A)))
= \nabla_{A\otimes H} \circ \sigma^A_H \circ (H \otimes \eta_H)
= \sigma^A_H \circ (H \otimes \eta_H).$$

The first equality follows by the definition of $\nabla_{A\otimes H}$, the second one by the twisted condition and the last one by (ii) of Proposition 2.9.

Also, $\nu$ satisfies (44) if and only if $\sigma^A_H \circ (\eta_H \otimes H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H)$ because by (35) we have

$$(\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\nu \otimes A) = \psi^A_H \circ (\eta_H \otimes A) = \beta^\nu.$$  

Finally, (45) is always true because, by (29), we obtain

$$(\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (\nu \otimes A) = \psi^A_H \circ (\eta_H \otimes A) = \beta^\nu.$$  

Corollary 2.21. Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. Let $\mathcal{K}_H$ be the quadruple defined in Remark 2.10 and assume that $\mathcal{K}_H$ satisfies the twisted and the cocycle conditions (30) and (31). Then, $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to $\mathcal{K}_H$ if and only if $\sigma$ satisfies the normal condition (69).

Proof: If $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to $\mathcal{K}_H$, by Theorem 2.20 we have (70). Then, composing with $(A \otimes \varepsilon_H)$ and using (ii) of Proposition 2.9, we obtain (69). Conversely, if (69) holds, we have:
\[
\begin{align*}
\sigma_{H}^{A} \circ (\eta_{H} \otimes H) \\
= ((\sigma \circ c_{H,H}) \otimes \mu_{H}) \circ (H \otimes (\delta_{H} \circ \eta_{H}) \otimes H) \circ \delta_{H} \\
= ((\sigma \circ c_{H,H}) \otimes H) \circ (H \otimes ((\Pi_{H}^{L} \otimes H) \circ \delta_{H})) \circ \delta_{H} \\
= ((\sigma \circ c_{H,H} \circ (H \otimes \Pi_{H}^{L}) \circ \delta_{H}) \otimes H) \circ \delta_{H} \\
= (\varepsilon_{1} \otimes H) \circ \delta_{H} \\
= \nabla_{A \otimes H} \circ (\eta_{A} \otimes H).
\end{align*}
\]

The first equality follows by the naturality of \(c\), the second one by (17), the fourth one by the coassociativity of \(\delta_{H}\) and the fourth one by (i) of Proposition 1.13. The last one follows by definition.

On the other hand
\[
\begin{align*}
\sigma_{H}^{A} \circ (H \otimes \eta_{H}) \\
= (\sigma \otimes H) \circ (H \otimes ((\Pi_{H}^{R} \otimes H) \circ \delta_{H})) \circ \delta_{H} \\
= ((\sigma \circ (H \otimes \Pi_{H}^{R}) \circ \delta_{H}) \otimes H) \circ \delta_{H} \\
= (\varepsilon_{1} \otimes H) \circ \delta_{H} \\
= \nabla_{A \otimes H} \circ (\eta_{A} \otimes H).
\end{align*}
\]

The first equality follows by (14), the second one by the coassociativity of \(\delta_{H}\), the third one by (ii) of Proposition 1.13, and the last one by definition.

**Corollary 2.22.** Let \(H\) be a cocommutative weak Hopf algebra, \((A, \varphi_{A})\) a weak left \(H\)-module algebra and \(\sigma \in \text{Reg}_{\varphi_{A}}(H^{2}, A)\). Let \(\mathbb{A}_{H}\) be the quadruple defined in Remark 2.10 and \(\mu_{A \otimes H}\) the associated product defined in (37). Then the following statements are equivalent:

(i) The product \(\mu_{A \otimes H}\) is associative with preunit \(\nu = \nabla_{A \otimes H} \circ (\eta_{A} \otimes \eta_{H})\) and normalized with respect to \(\nabla_{A \otimes H}\).

(ii) The morphism \(\sigma\) satisfies the twisted condition (58), the 2-cocycle condition (59) (equivalently (67)) and the normal condition (69).

**Proof:** The proof is an easy consequence of Theorems 2.2, 2.16, 2.17 and Corollary 2.21.

**Notation 2.23.** Let \(H\) be a cocommutative weak Hopf algebra and \((A, \varphi_{A})\) a weak left \(H\)-module algebra. From now on we will denote by \(A \otimes_{\tau} H = (A \otimes H, \mu_{A \otimes_{\tau} H})\) the weak crossed product, with preunit \(\nu = \nabla_{A \otimes_{\tau} H} \circ (\eta_{A} \otimes \eta_{H})\), defined by \(\tau \in \text{Reg}_{\varphi_{A}}(H^{2}, A)\) when it satisfies the twisted condition, the 2-cocycle condition and the normal condition. The associated algebra will be denoted by \(A \times_{\tau} H = (A \times H, \eta_{A \times_{\tau} H}, \mu_{A \times_{\tau} H})\).

Finally, the quadruple \(\mathbb{A}_{H}\) defined in Remark 2.10 will be denoted by \(\mathbb{A}_{H,\tau}\) and \(\sigma_{H}^{A}\) by \(\sigma_{H,\tau}^{A}\).

**Remark 2.24.** Let \(H\) be a cocommutative weak Hopf algebra and \((A, \varphi_{A})\) a weak left \(H\)-module algebra. Let \(\sigma \in \text{Reg}_{\varphi_{A}}(H^{2}, A)\) be a morphism satisfying the twisted condition (58),
the 2-cocycle condition (59) and the normal condition (69). Then, the weak crossed product
\[ A \otimes_{\sigma} H = (A \otimes H, \mu_{A \otimes_{\sigma} H}) \] with preunit \( \nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H) \) defined previously is a particular instance of the weak crossed products introduced in [16]. Also is a particular case of the instances used in [24] where these crossed structures were studied in a category of modules over a commutative ring without requiring cocommutativity of \( H \) and using weak measurings (see Definition 3.2 of [24]). To prove this assertion we will show that the conditions presented in Lemma 3.8 and Theorem 3.9 of [24] are completely fulfilled. First, note that, if \((A, \varphi_A)\) a weak left \( H \)-module algebra, we have that \( \varphi_A \) is a weak measuring. The idempotent morphism \( \Omega_{A \otimes H} \) related with the preunit \( \nu \) is the morphism \( \nabla_{A \otimes H} \) because, by (35) and (70), we have
\[ \Omega_{A \otimes H} = \mu_{A \otimes_{\sigma} H} \circ (A \otimes H \otimes \nu) = \mu_{A \otimes_{\sigma} H} \circ (A \otimes H \otimes \eta_A \otimes \eta_H) = (\mu_A \otimes H) \circ (A \otimes \sigma_A^H) \circ (\nabla_{A \otimes H} \otimes \eta_H) \]
\[ = (\mu_A \otimes H) \circ (\nabla_{A \otimes H} \circ (\eta_A \otimes H)) = \nabla_{A \otimes H}. \]
Moreover, in the category of modules over and associative commutative unital ring, the normalized condition implies that \( \text{Im}(\mu_{A \otimes_{\sigma} H}) \subset \text{Im}(\nabla_{A \otimes H}) \).

On the other hand, the left action defined in Lemma 3.8 of [24] is \( \varphi_A. \) Indeed:
\[ (A \otimes \varepsilon_H) \circ \mu_{A \otimes_{\sigma} H} \circ (\eta_A \otimes H \otimes ((\mu_A \otimes H) \circ (A \otimes \nu))) = (\mu_A \otimes \varepsilon_H) \circ (A \otimes \sigma_A^H) \circ (\psi_A^H \otimes H) \circ (H \otimes (\nabla_{A \otimes H} \circ (A \otimes \eta_H))) = (\mu_A \otimes \varepsilon_H) \circ (A \otimes \sigma_A^H) \circ (\psi_A^H \otimes H) \circ (H \otimes A \otimes \eta_H) = \mu_A \circ (A \otimes u_1) \circ \psi_A^H = \varphi_A. \]

where the first equality follows by the unit properties, the second one by (34), the third one by (iii) of Proposition 2.9, the fourth one by (69) and finally the last one (b2) of Definition 1.4.

Also, the morphism defined in Lemma 3.8 of [24] is \( \sigma \) because, by (35) and (iii) of Proposition 2.9, we have
\[ (A \otimes \varepsilon_H) \circ \mu_{A \otimes_{\sigma} H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H) = (\mu_A \otimes \varepsilon_H) \circ (A \otimes \sigma_A^H) \circ ((\nabla_{A \otimes H} \circ (\eta_A \otimes H)) \otimes H) = \mu_A \circ (\eta_A \otimes \sigma_A^H) = \sigma. \]

Then, the equalities (a) and (b) of Lemma 3.8 of [24] hold because the first one is the definition of \( \psi_A^H \) and the second one is a consequence of (35) and the definition of \( \sigma_A^H. \)

Therefore, we have that \( \mu_{A \otimes_{\sigma} H} \) satisfies that
\[ \rho_{A \otimes H} \circ \mu_{A \otimes_{\sigma} H} = (\mu_{A \otimes_{\sigma} H} \otimes H) \circ \rho_{A \otimes H \otimes A \otimes H} \]
where \( \rho_{A \otimes H} = A \otimes \delta_H \) and
\[ \rho_{A \otimes H \otimes A \otimes H} = (A \otimes H \otimes A \otimes H \otimes \mu_H) \circ (A \otimes H \otimes c_{H,A \otimes H} \otimes H) \circ (\rho_{A \otimes H} \otimes \rho_{A \otimes H}). \]
Although that $\rho_{A \otimes H \otimes A \otimes H}$ it is not counital, we say that $\mu_{A \otimes_a H}$ is $H$-colinear as in Lemma 3.8 of [24]. Then we obtain that $\sigma$ satisfies the equality (1) of [24], that is:

$$\sigma \circ ((\mu_H \circ (H \otimes \Pi_H^R)) \otimes H) = \sigma \circ (H \otimes (\mu_H \circ (\Pi_H^R \otimes H))).$$

Finally, for the preunit $\nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H)$, by the equalities (50) and (12),

$$(A \otimes \delta_H) \circ \nu = (A \otimes ((H \otimes \Pi_H^R) \circ \delta_H)) \circ \nu$$

holds (i.e., the equality (4) of [24] is true in our setting).

### 3. Equivalent weak crossed products and $H^2_{\varphi_A}(H, A)$

The aim of this section is to give necessary and sufficient conditions for two weak crossed products $A \otimes_\alpha H$, $A \otimes_\beta H$ to be equivalent in the cocommutative setting. To define a good notion of equivalence we need the definition of right $H$-comodule algebra for a weak Hopf algebra $H$.

**Definition 3.1.** Let $H$ be a weak bialgebra and $(B, \rho_B)$ an algebra which is also a right $H$-comodule such that

$$\mu_{B \otimes H} \circ (\rho_B \otimes \rho_B) = \rho_B \circ \mu_B.$$  

The object $(B, \rho_B)$ is called a right $H$-comodule algebra if one of the following equivalent conditions holds:

1. $(\rho_B \otimes H) \circ \rho_B \circ \eta_B = (B \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\rho_B \circ \eta_B) \otimes (\delta_H \circ \eta_H))$,
2. $(\rho_B \otimes H) \circ \rho_B \circ \eta_B = (B \otimes \mu_H \otimes H) \circ ((\rho_B \circ \eta_B) \otimes (\delta_H \circ \eta_H))$,
3. $(B \otimes \Pi_H^L) \circ \rho_B = (\mu_B \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$,
4. $(B \otimes \Pi_H^R) \circ \rho_B = ((\mu_B \circ c_{B,B}) \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$,
5. $(B \otimes \Pi_H^L) \circ \rho_B \circ \eta_B = \rho_B \circ \eta_B$,
6. $(B \otimes \Pi_H^R) \circ \rho_B \circ \eta_B = \rho_B \circ \eta_B$.

**Proposition 3.2.** Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\alpha \in \text{Reg}^+_{\varphi_A}(H^2, A)$ such that satisfies the twisted condition (58) and the 2-cocycle condition (59) (equivalently (67)). Then, the algebra $A \times_\alpha H = (A \times H, \eta_{A \times_\alpha H}, \mu_{A \times_\alpha H})$ is a right $H$-comodule algebra for the coaction

$$\rho_{A \times_\alpha H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ \gamma_{A \otimes H}.$$  

**Proof:** First note that $(A \times_\alpha H, \rho_{A \times_\alpha H})$ is a right $H$-comodule because

$$(A \times H \otimes \varepsilon_H) \circ \rho_{A \times_\alpha H} = p_{A \otimes H} \circ \gamma_{A \otimes H} = \gamma_{A \times H}.$$
and, by (56) and the coassociativity of \( \delta_H \),
\[
(\rho_{A \times \alpha} H \otimes H) \circ \rho_{A \times \alpha} H = (\rho_{A \otimes H} H \otimes H) \circ (A \otimes ((\delta_H \otimes H) \circ \delta_H)) \circ i_{\delta_H^2} = \rho_{A \times \alpha} H.
\]

On the other hand,
\[
\mu_{(A \times \alpha) H \otimes H} \circ (\rho_{A \times \alpha} H \circ \rho_{A \times \alpha} H) = (\rho_{A \times H} \otimes H) \circ (A \otimes \mu_{A \times H} \otimes H) \circ (A \otimes (\delta_H \otimes H) \circ \delta_H) \circ (i_{A \otimes H} \otimes (A \otimes \delta_H) \circ i_{A \otimes H})
\]
\[
= (\rho_{A \times H} \otimes H) \circ (\mu_{A \otimes H} \otimes H) \circ (A \otimes (\sigma^A_H \otimes H) \circ \delta_H) \circ (i_{A \otimes H} \otimes i_{A \otimes H})
\]
\[
= (\rho_{A \times H} \otimes H) \circ (\mu_{A \otimes H} \otimes H) \circ (A \otimes (\delta_H \otimes \sigma^A_H) \circ (A \otimes \psi^A_H \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H})
\]
\[
= \rho_{A \times H} \circ \mu_{A \times \alpha} H
\]

where the first equality follows by the normalized condition for \( \mu_{A \otimes \alpha} H \), the second one by the naturality of \( c \) and the coassociativity of \( \delta_H \), the third one by (77) and the last one by (56).

Finally, by (56) and (12), we obtain that
\[
(\rho_{A \times H} \otimes H) \circ (\mu_{A \times H} \otimes H) \circ (\beta \otimes H) = \rho_{A \times \alpha} H \circ \eta_{A \times \alpha} H.
\]

**Definition 3.3.** Let \( H \) be a cocommutative weak Hopf algebra, \((A, \varphi_A)\) a weak left \( H \)-module algebra and \( \alpha, \beta \in \text{Reg}_{\mathcal{C}}(H^2, A) \) such that satisfy the twisted condition (58) and the 2-cocycle condition (59) (equivalently (67)). Let \( A \otimes \alpha H, A \otimes \beta H \) be the weak crossed products associated to \( \alpha \) and \( \beta \). We say that \( A \otimes \alpha H, A \otimes \beta H \) are equivalent if there is an isomorphism of left \( A \)-modules and right \( H \)-comodule algebras \( \omega_{\alpha, \beta} : A \times_{\alpha} H \to A \times_{\beta} H \).

**Remark 3.4.** Let \( H \) be a weak Hopf algebra, \((A, \varphi_A)\) a weak left \( H \)-module algebra. Let \( \Gamma : A \otimes H \to A \otimes H \) be a morphism of left \( A \)-modules and right \( H \)-comodules for the regular action \( \varphi_{A \otimes H} = \mu_A \otimes H \) and coaction \( \rho_{A \otimes H} = A \otimes \delta_H \). Then
\[
(\eta_{\alpha} \otimes H) \circ (\varphi_{A \otimes H} \otimes H) \circ (\varphi_{A \otimes H} \otimes H) \circ (\eta_{\alpha} \otimes H) = (f_{\Gamma} \otimes H) \circ \delta_H
\]
where \( f_{\Gamma} = (\mu_A \otimes \varepsilon_H) \circ (\eta_{\alpha} \otimes H) \). As a consequence:
\[
(\Gamma \circ (\varphi_{A \otimes H} \otimes H) = (\varphi_{A \otimes H} \otimes H) \circ \delta_H
\]
\[
\Gamma = (\mu_A \otimes H) \circ (A \otimes (\Gamma \circ (\eta_{\alpha} \otimes H))) = (\mu_A \circ (A \otimes f_{\Gamma})) \otimes H) \circ (A \otimes \delta_H).
\]

If \( f : H \to A \) is a morphism and we define
\[
\Gamma_f : A \otimes H \to A \otimes H
\]

by \( \Gamma_f = (\mu_A \circ (A \otimes f)) \otimes H) \circ (A \otimes \delta_H), \) it is clear that \( \Gamma_f \) is a morphism of left \( A \)-modules and right \( H \)-comodules such that \( f_{\Gamma_f} = f \). Also, \( \Gamma_{f_1} = \Gamma \) and then there is a bijection
\[
\Phi : A \text{Hom}_C^H(A \otimes H, A \otimes H) \to \text{Hom}_C(H, A)
\]
defined by \( \Phi(\Gamma) = f_{\Gamma} \) which inverse \( \Phi^{-1}(f) = \Gamma_f \). Note that \( \Phi^{-1}(u_1) = \Gamma_{u_1} = \nabla_{A \otimes H}. \)
Then, it is easy to show that \( \Gamma, \Gamma' \in \mathcal{A}Hom_C^H(A \otimes H, A \otimes H) \) satisfy
\begin{align*}
(\text{e1}) \quad & \Gamma \circ \Gamma' = \Gamma' \circ \Gamma = \nabla_{A \otimes H}.
(\text{e2}) \quad & \Gamma \circ \Gamma' = \Gamma.
(\text{e3}) \quad & \Gamma' \circ \Gamma = \Gamma'.
\end{align*}
if and only if for the morphism \( f \) there exists a morphism \( f_{\Gamma}^{-1} \) satisfying:
\begin{align*}
(\text{i}) \quad & f_{\Gamma} \land f_{\Gamma}^{-1} = f_{\Gamma}^{-1} \land f_{\Gamma} = u_1.
(\text{ii}) \quad & f_{\Gamma} \land f_{\Gamma}^{-1} = f_{\Gamma}.
(\text{iii}) \quad & f_{\Gamma}^{-1} \land f_{\Gamma} = f_{\Gamma}^{-1}.
\end{align*}
Indeed: If \( \Gamma \) satisfies (e1)-(e3) define \( f_{\Gamma}^{-1} \) by \( f_{\Gamma}^{-1} = f_{\Gamma'} \), and, conversely, if for \( f_{\Gamma} \) there exists a morphism \( f_{\Gamma}^{-1} \) satisfying (i)-(iii), define \( \Gamma' = \Gamma_{f_{\Gamma}^{-1}} \).

As a consequence, if \( H \) is comultiplicative, \( \Gamma \in \mathcal{A}Hom_C^H(A \otimes H, A \otimes H) \) satisfies (e1)-(e3) if and only if \( \Phi(\Gamma) = f_{\Gamma} \in \text{Reg}_{\varphi_A}(H, A) \). Conversely, \( f \in \text{Reg}_{\varphi_A}(H, A) \) if and only if \( \Phi^{-1}(f) = \Gamma_f \) satisfies (e1)-(e3).

**Theorem 3.5.** Let \( H \) be a cocommutative weak Hopf algebra, \( (A, \varphi_A) \) a weak left \( H \)-module algebra and \( \alpha, \beta \in \text{Reg}_{\varphi_A}^H(H^2, A) \) such that satisfy the twisted condition \([58]\) and the 2-cocycle condition \([52]\) (equivalently \([63]\)). The weak crossed products \( A \otimes_{\alpha} H, A \otimes_{\beta} H \) associated to \( \alpha \) and \( \beta \) are equivalent if and only if there exist multiplicative and preunit preserving morphisms \( \Gamma, \Gamma' \in \mathcal{A}Hom_C^H(A \otimes H, A \otimes H) \) satisfying (e1)-(e3).

**Proof:** Assume that \( A \otimes_{\alpha} H, A \otimes_{\beta} H \) are equivalent. Thus there exists an isomorphism of left \( A \)-modules and right \( H \)-comodule algebras \( \omega_{\alpha,\beta} : A \times_{\alpha} H \to A \times_{\beta} H \). Define \( \Gamma \) and \( \Gamma' \) by
\[ \Gamma = i_{A \otimes H} \circ \omega_{\alpha,\beta} \circ p_{A \otimes H}, \quad \Gamma' = i_{A \otimes H} \circ \omega_{\alpha,\beta}^{-1} \circ p_{A \otimes H}. \]

Then,
\[ \Gamma \circ \Gamma' = i_{A \otimes H} \circ \omega_{\alpha,\beta} \circ p_{A \otimes H} \circ i_{A \otimes H} \circ \omega_{\alpha,\beta}^{-1} \circ p_{A \otimes H} = i_{A \otimes H} \circ \omega_{\alpha,\beta} \circ \omega_{\alpha,\beta}^{-1} \circ p_{A \otimes H} = \nabla_{A \otimes H}, \]
and
\[ \Gamma' \circ \Gamma = i_{A \otimes H} \circ \omega_{\alpha,\beta}^{-1} \circ p_{A \otimes H} \circ i_{A \otimes H} \circ \omega_{\alpha,\beta} \circ p_{A \otimes H} = i_{A \otimes H} \circ \omega_{\alpha,\beta}^{-1} \circ \omega_{\alpha,\beta} \circ p_{A \otimes H} = \nabla_{A \otimes H}. \]

Also,
\[ \Gamma \circ \Gamma' = \nabla_{A \otimes H} \circ \Gamma = \Gamma, \quad \Gamma' \circ \Gamma \circ \Gamma' = \nabla_{A \otimes H} \circ \Gamma' = \Gamma' \]
and therefore (e1)-(e3) hold.

The morphism \( \Gamma \) is multiplicative because \( \omega_{\alpha,\beta} \) is an algebra morphism:
\[ \mu_{A \otimes_{\beta} H} \circ (\Gamma \otimes \Gamma) \]
\[ = \mu_{A \otimes_{\beta} H} \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \circ (\omega_{\alpha,\beta} \otimes \omega_{\alpha,\beta}) \circ (p_{A \otimes H} \otimes p_{A \otimes H}) \]
\[ = i_{A \otimes H} \circ \mu_{A \otimes_{\beta} H} \circ (\omega_{\alpha,\beta} \otimes \omega_{\alpha,\beta}) \circ (p_{A \otimes H} \otimes p_{A \otimes H}) \]
\[ \begin{align*}
&= i_{A \otimes H} \circ \omega_{\alpha,\beta} \circ \mu_{A \times_A H} \circ (p_{A \otimes H} \otimes p_{A \otimes H}) \\
&= \Gamma \circ \mu_{A \otimes_A H}
\end{align*} \]

and in a similar way, using that \( \omega_{\alpha,\beta}^{-1} \) is multiplicative, it is possible to prove that \( \Gamma' \) is multiplicative.

On the other hand, \( \Gamma \) preserve the preunit because:

\[ \begin{align*}
\Gamma \circ \eta = i_{A \otimes H} \circ \omega_{\alpha,\beta} \circ \eta_{A \times_A H} = i_{A \otimes H} \circ \eta_{A \times_A H} = \nu.
\end{align*} \]

By the same arguments we obtain that \( \Gamma' \circ \nu = \Gamma' \).

Using (e1), (e2) and the left \( A \)-linearity of \( \omega_{\alpha,\beta} \) we have

\[ \begin{align*}
\varphi_{A \otimes H} &\circ (A \otimes \Gamma) \\
&= \varphi_{A \otimes H} \circ (A \otimes (\nabla_{A \otimes H} \circ \Gamma)) \\
&= \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \Gamma) \\
&= i_{A \otimes H} \circ \varphi_{A \times_A H} \circ (A \otimes \omega_{\alpha,\beta}) \circ (A \otimes p_{A \otimes H}) \\
&= i_{A \otimes H} \circ \varphi_{A \times_A H} \circ (A \otimes p_{A \otimes H}) \\
&= \Gamma \circ (\mu_A \otimes H) \circ (A \otimes \nabla_{A \otimes H}) \\
&= \Gamma \circ \nabla_{A \otimes H} \circ (\mu_A \otimes H) \\
&= \Gamma \circ \varphi_{A \otimes H}.
\end{align*} \]

Similarly, by (e1), (e3) and the left \( A \)-linearity of \( \omega_{\alpha,\beta}^{-1} \) we obtain that \( \Gamma' \) is a morphism of left \( A \)-modules.

Finally, \( \Gamma \) is a morphism of right \( H \)-comodules by (56) and the right \( H \)-comodule morphism property of \( \omega_{\alpha,\beta} \). Indeed:

\[ \begin{align*}
\rho_{A \otimes H} \circ \Gamma &= (i_{A \otimes H} \otimes H) \circ \rho_{A \times_A H} \circ \omega_{\alpha,\beta} \circ p_{A \otimes H} \\
&= ((i_{A \otimes H} \circ \omega_{\alpha,\beta}) \otimes H) \circ \rho_{A \times_A H} \circ p_{A \otimes H} \\
&= (\Gamma \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \\
&= ((\Gamma \circ \nabla_{A \otimes H}) \otimes H) \circ (A \otimes \delta_H) \\
&= (\Gamma \otimes H) \circ \rho_{A \otimes H}.
\end{align*} \]

By a similar calculus we obtain that \( \Gamma' \) is a morphism of right \( H \)-comodules.

Conversely, assume that there exist multiplicative and preunit preserving morphisms

\[ \Gamma, \Gamma' \in A Hom_C^H(A \otimes H, A \otimes H) \]

satisfying (e1)-(e3) of the previous remark. Define

\[ \omega_{\alpha,\beta} = p_{A \otimes H} \circ \Gamma \circ i_{A \otimes H}, \quad \omega_{\alpha,\beta}^{-1} = p_{A \otimes H} \circ \Gamma' \circ i_{A \otimes H}. \]

Then, by (e1), (e2) and (e3), we have

\[ \omega_{\alpha,\beta}^{-1} \circ \omega_{\alpha,\beta} = p_{A \otimes H} \circ \Gamma' \circ \nabla_{A \otimes H} \circ \Gamma \circ i_{A \otimes H} = p_{A \otimes H} \circ \Gamma' \circ \Gamma \circ i_{A \otimes H} = p_{A \otimes H} \circ \nabla_{A \otimes H} \circ i_{A \otimes H} = id_{A \times H} \]
and

$$\omega_{\alpha,\beta}^{-1} \cdot \omega_{\alpha,\beta} = p_A \otimes H \circ \Gamma \circ \nabla_{A \otimes H} \circ \Gamma' \circ i_{A \otimes H} = p_A \otimes H \circ \Gamma \circ \nabla_{A \otimes H} \circ i_{A \otimes H} = p_A \otimes H \circ \nabla_{A \otimes H} \circ i_{A \otimes H} = id_{A \times H}$$

which proves that $\omega_{\alpha,\beta}$ is an isomorphism.

Moreover, using that $\Gamma$ preserves the preunit $\nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H)$ we have

$$\omega_{\alpha,\beta} \circ \eta_{A \times A} = p_A \otimes H \circ \Gamma \circ \nu = p_A \otimes H \circ \nu = \eta_{A \times A}$$

and, by the multiplicative property of $\Gamma$, we obtain

$$\mu_{A \times A} \circ (\omega_{\alpha,\beta} \otimes \omega_{\alpha,\beta}) = p_A \otimes H \circ \Gamma \circ \nabla_{A \otimes H} \circ (i_{A \otimes H} \otimes i_{A \otimes H}) = p_A \otimes H \circ (\mu_A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H})$$

$$= \omega_{\alpha,\beta} \circ \mu_{A \times A} \circ \Gamma.$$

Therefore, $\omega_{\alpha,\beta}$ is an isomorphism of algebras.

On the other hand, using (e1), (e2) and the property of left $A$-module morphism of $\Gamma$ we have

$$\varphi_{A \times A} \circ (A \otimes \omega_{\alpha,\beta}) = p_A \otimes H \circ (\mu_A \otimes H) \circ (A \otimes (\nabla_{A \otimes H} \circ \Gamma \circ i_{A \otimes H})) = p_A \otimes H \circ (\mu_A \otimes H) \circ (A \otimes (\Gamma \circ i_{A \otimes H}))$$

$$= p_A \otimes H \circ \Gamma \circ (\mu_A \otimes H) \circ (A \otimes i_{A \otimes H}) = p_A \otimes H \circ \Gamma \circ \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes i_{A \otimes H}) = \omega_{\alpha,\beta} \circ \varphi_{A \times A}$$

and this proves that $\omega_{\alpha,\beta}$ is a morphism of left $A$-modules.

Finally, using similar arguments and the property of right $H$-comodule morphism of $\Gamma$ we obtain that $\omega_{\alpha,\beta}$ is a morphism of right $H$-comodules because:

$$\rho_{A \times A} \circ \omega_{\alpha,\beta} = (p_A \otimes H \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ \Gamma \circ i_{A \otimes H} = (p_A \otimes H \otimes H) \circ \rho_{A \otimes H} \circ \Gamma \circ i_{A \otimes H}$$

$$= (((p_A \otimes H \circ \Gamma) \circ H) \circ \rho_{A \otimes A} \circ i_{A \otimes H} = (((p_A \otimes H \circ \nabla_{A \otimes H} \circ H) \circ \rho_{A \otimes A} \circ i_{A \otimes H} = (\omega_{\alpha,\beta} \circ H) \circ \rho_{A \otimes A} \circ H.$$

**Remark 3.6.** By the previous theorem, we obtain that the notion of equivalent crossed products is the one used in [24] in a category of modules over a commutative ring. Following the terminology used in [24], the pair of morphisms $f_{\Gamma}$ and $f_{\Gamma}^{-1}$ is an example of gauge transformation. Also, this notion is a generalization of the one that we can find in the Hopf algebra world (see [15], [18]).

The following results, Theorem 3.7 and Corollary 3.8 will be used in Theorem 3.9 to obtain the meaning of the notion of equivalence between two weak crossed products in terms of morphisms of $Reg_{\varphi_A}(H, A)$. Note that this characterization it is the key to prove the main result of this section, that is, Theorem 3.12.

**Theorem 3.7.** Let $\Gamma$ and $f_{\Gamma}$ as in Remark 3.3 and such that

$$\Gamma \circ \nabla_{A \otimes H} = \nabla_{A \otimes H} \circ \Gamma = \Gamma.$$
Under the hypothesis of Theorem 3.5, \( \Gamma \) is a multiplicative morphism that preserves the preunit 
\( \nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H) \) if and only if the following equalities hold:

\[
p_{A \otimes H} \circ \Gamma \circ \nu = p_{A \otimes H} \circ \nu
\]

\[
\mu_A \circ (A \otimes f_{\Gamma}) \circ \psi_H^A = \mu_A \circ (f_{\Gamma} \otimes \varphi_A) \circ (\delta_H \otimes A)
\]

\[
\mu_A \circ (A \otimes f_{\Gamma}) \circ \sigma_{H,\alpha}^A = \mu_A \circ (\mu_A \otimes \beta) \circ (A \otimes \psi_H^A \otimes H) \circ (((f_{\Gamma} \otimes H) \circ \delta_H) \otimes ((f_{\Gamma} \otimes H) \circ \delta_H))
\]

Moreover, if \( \Gamma \) preserves the preunit, we have that

\[
f_{\Gamma} \circ \eta_H = \eta_A.
\]

**Proof:** Assume that \( \Gamma \) is a multiplicative morphism that preserves the preunit. Then (75) follows easily and, by (74), we have

\[
\Gamma \circ (A \otimes \eta_H) = \nabla_{A \otimes H} \circ (A \otimes \eta_H)
\]

because

\[
\Gamma \circ (A \otimes \eta_H) = (\mu_A \otimes H) \circ (A \otimes (\Gamma \circ (\eta_A \otimes \eta_H))) = (\mu_A \otimes H) \circ (A \otimes (\Gamma \circ \nu)) = (\mu_A \otimes H) \circ (A \otimes \nu)
\]

\[
= \nabla_{A \otimes H} \circ (A \otimes \eta_H).
\]

On the other hand, the multiplicative condition for \( \Gamma \) implies that:

\[
\Gamma \circ \mu_{A \otimes H} \circ (\eta_A \otimes H \otimes A \otimes \eta_H) = \mu_{A \otimes H} \circ (\Gamma \otimes \Gamma) \circ (\eta_A \otimes H \otimes A \otimes \eta_H).
\]

Equivalently

\[
\Gamma \circ (\mu_A \otimes H) \circ (A \otimes (\sigma_{H,\alpha}^A \circ (H \otimes \eta_H))) \circ \psi_H^A
\]

\[
= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H,\beta}^A) \circ (A \otimes \psi_H^A \otimes H) \circ ((\Gamma \circ (\eta_A \otimes H)) \otimes (\Gamma \circ (A \otimes \eta_H))).
\]

By the normal condition for \( \alpha \) we have

\[
\sigma_{H,\alpha}^A \circ (H \otimes \eta_H) = ((\alpha \circ (H \otimes \Pi_{H}^B) \circ \delta_H) \otimes H) \circ \delta_H = (\eta_1 \otimes H) \circ \delta_H = \nabla_{A \otimes H} \circ (\eta_A \otimes H)
\]

and then the upper side of (80) is equal to \( \Gamma \circ \psi_H^A \). For the lower side of (80) the following holds:

\[
(\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H,\beta}^A) \circ (A \otimes \psi_H^A \otimes H) \circ ((\Gamma \circ (\eta_A \otimes H)) \otimes (\Gamma \circ (A \otimes \eta_H))) = (\mu_A \otimes H) \circ (f_{\Gamma} \otimes ((\mu_A \otimes H) \circ (A \otimes \sigma_{H,\beta}^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nabla_{A \otimes H} \circ (A \otimes \eta_H))) \circ (\delta_H \otimes A)
\]

\[
= (\mu_A \otimes H) \circ (f_{\Gamma} \otimes ((\mu_A \otimes H) \circ (A \otimes \sigma_{H,\beta}^A) \circ (\psi_H^A \otimes H) \circ (H \otimes A \otimes \eta_H))) \circ (\delta_H \otimes A)
\]

\[
= (\mu_A \otimes H) \circ (f_{\Gamma} \otimes (\mu_A \otimes H) \circ (A \otimes (\nabla_{A \otimes H} \circ (\eta_A \otimes H))) \circ (H \otimes \psi_H^A) \circ (\delta_H \otimes A)
\]

\[
= (\mu_A \otimes H) \circ (f_{\Gamma} \otimes \psi_H^A) \circ (\delta_H \otimes A)
\]
where the first equality follows by (72) and (79), the second one by (81), the third one by (83) and the fourth one by the properties of $\nabla_{A \otimes H}$.

Thus, (80) is equivalent to

\[(82) \quad \Gamma \circ \psi^A_H = (\mu_A \otimes H) \circ (f_\Gamma \otimes \psi^A_H) \circ (\delta_H \otimes A)\]

and then composing in both sides with $A \otimes \varepsilon_H$ we get (76).

Also, the multiplicative condition for $\Gamma$ implies the following:

$$
\Gamma \circ \mu_{A \otimes a.H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H) = \mu_{A \otimes \beta.H} \circ (\Gamma \otimes \Gamma) \circ (\eta_A \otimes H \otimes \eta_A \otimes H).
$$

Equivalently

\[(83) \quad \Gamma \circ (\mu_A \otimes H) \circ (A \otimes \sigma^A_{H,\alpha}) \circ ((\nabla_{A \otimes H} \circ (\eta_A \otimes H)) \otimes H)\]

Therefore, by (81) and (72) we obtain that (83) is equivalent to

\[(84) \quad \Gamma \circ \sigma^A_{H,\alpha} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma^A_{H,\beta}) \circ (A \otimes \psi^A_H \otimes H) \circ ((f_\Gamma \otimes H) \circ \delta_H) \otimes ((f_\Gamma \otimes H) \circ \delta_H)).\]

Composing in both sides with $A \otimes \varepsilon_H$ and using (iii) of Proposition 2.9 we obtain (77).

Conversely, assume that (75), (76) and (77) hold. Then,

$$
\Gamma \circ \nu = \nabla_{A \otimes H} \circ \Gamma \circ \nu = \nabla_{A \otimes H} \circ \nu = \nu
$$

and $\Gamma$ preserves the preunit. Moreover, to prove that $\Gamma$ is multiplicative first we show that, if (76) holds, then (82) holds and similarly for (77) and (84). Indeed:

$$
\Gamma \circ \psi^A_H = ((\mu_A \circ (A \otimes f_\Gamma)) \otimes H) \circ (A \otimes \delta_H) \circ \psi^A_H
$$

The first equality follows by (73), the second and the last ones by the coassociativity of $\delta_H$ and the third one by (76).

$$
\Gamma \circ \sigma^A_{H,\alpha} = ((\mu_A \circ (A \otimes f_\Gamma)) \otimes H) \circ (A \otimes \delta_H) \circ \sigma^A_{H,\alpha}
$$

The first equality follows by (73), the second one by (57), the third one by (77) and the last one by the definition of $\psi^A_H$, the naturality of $c$ and the coassociativity of $\delta_H$.

Then,
\( \Gamma \circ \mu_{A^{\otimes} \alpha, H} \)

\[
= ((\mu_A \circ (A \otimes f_\Gamma)) \otimes H) \circ (\mu_A \otimes \delta_H) \circ (\mu_A \otimes \sigma_{H, \alpha}^A) \circ (A \otimes \psi_{H}^A \otimes H)
= (\mu_A \otimes H) \circ (\mu_A \otimes (\Gamma \circ \sigma_{H, \alpha}^A)) \circ (A \otimes \psi_{H}^A \otimes H)
= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H, \beta}^A) \circ (\mu_A \otimes ((\mu_A \otimes H) \circ (f_\Gamma \otimes \psi_{H}^A) \circ (\delta_H \otimes A)) \otimes H)\circ (A \otimes \psi_{H}^A \otimes ((f_\Gamma \otimes H) \circ \delta_H))
= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H, \beta}^A) \circ (A \otimes (\Gamma \circ (\mu_A \otimes H) \circ (A \otimes \psi_{H}^A) \circ (\psi_{H}^A \otimes A)))) \otimes H)\circ (A \otimes \psi_{H}^A \otimes ((f_\Gamma \otimes H) \circ \delta_H))
= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H, \beta}^A) \circ (A \otimes (\mu_A \otimes H) \circ (A \otimes \sigma_{H, \beta}^A) \circ (A \otimes (\Gamma \circ (\mu_A \otimes H) \circ (f_\Gamma \otimes \psi_{H}^A) \circ (\delta_H \otimes A) \otimes H) \circ (A \otimes H \otimes A \otimes ((f_\Gamma \otimes H) \circ \delta_H))
= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H, \beta}^A) \circ (A \otimes \sigma_{H, \beta}^A) \circ (A \otimes (\mu_A \otimes H) \circ (f_\Gamma \otimes \psi_{H}^A) \circ (\delta_H \otimes A) \otimes H) \circ (A \otimes H \otimes \Gamma)
= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H, \beta}^A) \circ (A \otimes \sigma_{H, \beta}^A) \circ (A \otimes (\mu_A \otimes H) \circ (f_\Gamma \otimes \psi_{H}^A) \circ (\delta_H \otimes A) \otimes H) \circ (A \otimes H \otimes \Gamma)

The first equality follows by (73), the second and the last ones by the associativity of \( \mu_A \), the third one by (81), the fourth and the sixth ones by (82) and the left \( A \)-linearity of \( \Gamma \), the fifth one by (27).

Finally, (78) follows by:

\[
\begin{align*}
\Gamma \circ f_\Gamma \circ \eta_H &= (A \otimes \varepsilon_H) \circ \Gamma \circ (\eta_A \otimes \eta_H) = (A \otimes \varepsilon_H) \circ \Gamma \circ \nabla_{A^{\otimes} H} \circ (\eta_A \otimes \eta_H) \\
&= (A \otimes \varepsilon_H) \circ \Gamma \circ \nu = (A \otimes \varepsilon_H) \circ \nu = \eta_A.
\end{align*}
\]

**Corollary 3.8.** Under the hypothesis of Theorem 3.7, if (76) holds, (77) is equivalent to

\[
(85) \quad \mu_A \circ (A \otimes f_\Gamma) \circ \sigma_{H, \alpha}^A = [\mu_A \circ ((\varphi_A \circ (H \otimes f_\Gamma)) \otimes f_\Gamma) \circ (H \otimes c_{H, H} \circ (\delta_H \otimes H))] \wedge \beta.
\]

Then, if \( f_\Gamma \in \text{Reg}_{\varphi_A}(H, A) \), we obtain that (77) is equivalent to

\[
\begin{align*}
\alpha \wedge \partial_{1,1}(f_\Gamma) &= \partial_{1,0}(f_\Gamma) \wedge \partial_{1,2}(f_\Gamma) \wedge \beta.
\end{align*}
\]

**Proof.** If (85) holds, by (76), the naturality of \( c \) and the coassociativity of \( \delta_H \), we obtain (77):

\[
\begin{align*}
\mu_A \circ (A \otimes f_\Gamma) \circ \sigma_{H, \alpha}^A \\
= [\mu_A \circ ((\varphi_A \circ (H \otimes f_\Gamma)) \otimes f_\Gamma) \circ (H \otimes c_{H, H} \circ (\delta_H \otimes H))] \wedge \beta \\
= [\mu_A \circ ((\mu_A \circ (A \otimes f_\Gamma) \circ \psi_{H}^A) \otimes \beta) \circ (H \otimes c_{H, A} \otimes H) \circ (\delta_H \otimes ((f_\Gamma \otimes H) \circ \delta_H))] \\
= [\mu_A \circ ((\mu_A \circ f_\Gamma \circ \varphi_A) \otimes (\delta_H \otimes A)) \otimes \beta) \circ (H \otimes c_{H, A} \otimes H) \circ (\delta_H \otimes ((f_\Gamma \otimes H) \circ \delta_H))] \\
= [\mu_A \circ (\mu_A \otimes \beta) \circ (A \otimes \psi_{H}^A \otimes H) \circ (((f_\Gamma \otimes H) \circ \delta_H) \otimes ((f_\Gamma \otimes H) \circ \delta_H))]
\end{align*}
\]

On the other hand, (76) holds we have that (82) holds and then if we assume (77), using (73), the definition of \( \psi_{H}^A \), the naturality of \( c \) and the coassociativity of \( \delta_H \), we obtain:

\[
\begin{align*}
\mu_A \circ (A \otimes f_\Gamma) \circ \sigma_{H, \alpha}^A \\
= [\mu_A \circ (\mu_A \otimes \beta) \circ (A \otimes \psi_{H}^A \otimes H) \circ (((f_\Gamma \otimes H) \circ \delta_H) \otimes ((f_\Gamma \otimes H) \circ \delta_H))] \\
= [\mu_A \circ (\mu_A \otimes \beta) \circ (\mu_A \otimes \sigma_{H, \alpha}^A) \circ (A \otimes \psi_{H}^A \otimes H) \circ ((\Gamma \circ \psi_{H}^A \otimes H) \circ (H \otimes ((f_\Gamma \otimes H) \circ \delta_H))]
\end{align*}
\]

\[
\begin{align*}
\mu_A \circ (A \otimes f_\Gamma) \circ \sigma_{H, \alpha}^A \\
= [\mu_A \circ (\mu_A \otimes \beta) \circ (A \otimes \psi_{H}^A \otimes H) \circ ((\Gamma \circ \psi_{H}^A \otimes H) \circ (H \otimes ((f_\Gamma \otimes H) \circ \delta_H)))]
\end{align*}
\]
\[ \mu_A \circ (A \otimes f_T) \circ \sigma^A_{H,\alpha} = \alpha \wedge \partial_{1,1}(f_T) \]

and, by (65) and \( \beta \circ \Omega^2_H = \beta \) we have

\[ \partial_{1,0}(f_T) \wedge \partial_{1,2}(f_T) \wedge \beta = \]

\[ [\mu_A \circ ((\varphi_A \circ (H \otimes f_T)) \otimes f_T) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)] \wedge \beta. \]

**Theorem 3.9.** Under the hypothesis of Theorem 3.5, the weak crossed products \( A \otimes_\alpha H, A \otimes_\beta H \) associated to \( \alpha \) and \( \beta \) are equivalent if and only if there exists \( f \in \text{Reg}^T_{\varphi_A}(H,A) \) such that the equalities (76) and (86) hold.

**Proof:** If the weak crossed products \( A \otimes_\alpha H, A \otimes_\beta H \) are equivalent, by Theorem 3.5 there exist multiplicative and preunit preserving morphisms \( \Gamma, \Gamma' \in \text{AHom}_C^H(A \otimes H, A \otimes H) \) satisfying (e1)-(e3). Then, by Remark 3.4 \( f_T \in \text{Reg}_{\varphi_A}(H,A) \), and by Theorem 3.7 the equalities (76) and \( f_T \circ \eta_H = \eta_A \) hold. Finally, by Corollary 3.8 we get (80). Conversely, let \( f \in \text{Reg}^T_{\varphi_A}(H,A) \), with inverse \( f^{-1} \). Then, \( \Gamma_f \) and \( \Gamma_{f^{-1}} \) are morphisms of left \( A \)-modules and right \( H \)-comodules satisfying (e1)-(e3) and preserving the preunit \( \nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H) \). Indeed: By (17) and (iii) of Proposition 1.6 we have

\[ \Gamma_f \circ \nu = (f \otimes H) \circ \delta_H \circ \eta_H = ((f \otimes H') \otimes H) \circ \delta_H \circ \eta_H = (u_1 \otimes H) \circ \delta_H \circ \eta_H = \nu. \]

Similarly, \( \Gamma_{f^{-1}} \circ \nu = \nu \). By Theorem 3.7 and Corollary 3.8 \( \Gamma_f \) is multiplicative and

\[ \omega_{\alpha,\beta} = p_{A \otimes H} \circ \Gamma_f \circ i_{A \otimes H} \]

is an \( H \)-comodule algebra isomorphism with inverse \( \omega^{-1}_{\alpha,\beta} = p_{A \otimes H} \circ \Gamma_{f^{-1}} \circ i_{A \otimes H} \). Then, \( \Gamma_{f^{-1}} \) is multiplicative and, by Theorem 3.5 we obtain that \( A \otimes_\alpha H, A \otimes_\beta H \) are equivalent.

**Remark 3.10.** Note that, by the previous Theorem, if \( A \otimes_\alpha H, A \otimes_\beta H \) are equivalent and \( f \in \text{Reg}^T_{\varphi_A}(H,A) \) is the morphism inducing the equivalence, by Theorem 3.7 and Corollary 3.8 we also have

\[ \mu_A \circ (A \otimes f^{-1}) \circ \psi^A_H = \mu_A \circ (f^{-1} \otimes \varphi_A) \circ (\delta_H \otimes A), \]

\[ \Gamma_{f^{-1}} \circ \psi^A_H = (\mu_A \otimes H) \circ (f^{-1} \otimes \psi^A_H) \circ (\delta_H \otimes A) \]

\[ \mu_A \circ (A \otimes f^{-1}) \circ \sigma^A_{H,\beta} = \mu_A \circ (\mu_A \otimes \alpha) \circ (A \otimes \psi^A_H \otimes H) \circ ((f^{-1} \otimes H) \circ \delta_H) \otimes ((f^{-1} \otimes H) \circ \delta_H), \]

\[ \Gamma_{f^{-1}} \circ \sigma^A_{H,\beta} = (\mu_A \otimes H) \circ (\mu_A \otimes f^{-1}) \circ (A \otimes \psi^A_H \otimes H) \circ ((f^{-1} \otimes H) \circ \delta_H) \otimes ((f^{-1} \otimes H) \circ \delta_H), \]
crossed products

(93) \( \mu_A \circ (A \otimes f^{-1}) \circ \sigma_{H, \beta}^A = [\mu_A \circ ((\varphi_A \circ (H \otimes f^{-1})) \otimes f^{-1}) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)] \wedge \alpha, \)

and

(94) \( \beta \wedge \partial_{1,1}(f^{-1}) = \partial_{1,0}(f^{-1}) \wedge \partial_{1,2}(f^{-1}) \wedge \alpha. \)

**Remark 3.11.** Note that, if \( H \) is a cocommutative weak Hopf algebra, \((A, \varphi_A)\) is a weak left \( H \)-module algebra and \( f : H \to A \) is a morphism, the equality \((76)\) it is always true if \( A \) is commutative, because:

\[
\mu_A \circ (A \otimes f) \circ \psi_H^A = \mu_A \circ (\varphi_A \otimes f) \circ (H \otimes c_{H,A}) \circ ((c_{H,H} \circ \delta_H) \otimes A) = \mu_A \circ c_{A,A} \circ (f \otimes \varphi_A) \circ (\delta_H \otimes A)
\]

Then, \( \mu_A \circ (A \otimes f) \circ \psi_H^A = \mu_A \circ (f \otimes \varphi_A) \circ (\delta_H \otimes A). \)

Then, if \((A, \varphi_A)\) is a commutative left \( H \)-module algebra, the equivalence between two weak crossed products \( A \otimes_{\alpha} H, A \otimes_{\beta} H \) is determined by the inclusion of \( f \) in \( \text{Reg}_{\varphi_A}^+ (H, A) \) and the equality \((86)\). In this case \((86)\) is equivalent to say that \( \alpha \wedge \beta^{-1} \in \text{Im}(D_{\varphi_A}^{1+}). \)

**Theorem 3.12.** Let \( H \) be a cocommutative weak Hopf algebra and \((A, \varphi_A)\) a commutative left \( H \)-module algebra. Then there is a bijective correspondence between \( H_{\varphi_A}^2 (H, A) \) and the equivalence classes of weak crossed products of \( A \otimes_{\alpha} H \) where \( \alpha : H \otimes H \to A \) satisfy the 2-cocycle condition \((59)\) (equivalently \((67)\)) and the normal condition \((69)\).

**Proof:** First note that \( H_{\varphi_A}^2 (H, A) \) is isomorphic to \( H_{\varphi_A}^2 (H, A) \). Then, it is suffices to prove the result for \( H_{\varphi_A}^2 (H, A) \). Let \( \alpha, \beta \in \text{Reg}_{\varphi_A}^+ (H^2, A) \) such that satisfies the 2-cocycle condition \((59)\) (in the commutative case the twisted condition it is always true). If \( A \otimes_{\alpha} H, A \otimes_{\beta} H \) are equivalent, by the previous remark, we have that there exists \( f \in \text{Reg}_{\varphi_A}^+ (H, A) \) such that \( \alpha \wedge \beta^{-1} \in \text{Im}(D_{\varphi_A}^{1+}). \) Then, \( \alpha \) and \( \beta \) are in the same class in \( H_{\varphi_A}^2 (H, A) \). Conversely, if \([\alpha] = [\beta]\) in \( H_{\varphi_A}^2 (H, A) \), \( \alpha \) and \( \beta \) satisfies \((86)\), i.e. \( \alpha \wedge \beta^{-1} = D_{\varphi_A}^{1+} (f) \), for \( f \in \text{Reg}_{\varphi_A}^+ (H, A) \). Then, if \( \Gamma_f \) is the morphism defined in Remark \(3.4\) we have that \( \Gamma_f \) satisfies \((75)\), because, using that \( f \in \text{Reg}_{\varphi_A}^+ (H, A) \), we obtain

\[
p_{A \otimes H} \circ \Gamma_f \circ \nu = p_{A \otimes H} \circ (f \otimes H) \circ \delta_H \circ \eta_H = p_{A \otimes H} \circ ((f \circ \Pi_H^L) \otimes H) \circ \delta_H \circ \eta_H
\]

\[
= p_{A \otimes H} \circ ((f \circ \Pi_H^L) \otimes H) \circ \delta_H \circ \eta_H = p_{A \otimes H} \circ (u_1 \otimes H) \circ \delta_H \circ \eta_H = p_{A \otimes H} \circ \nu.
\]

In a similar way, \( \beta \wedge \alpha^{-1} = D_{\varphi_A}^{1+} (f^{-1}) \) and \( \Gamma_{f^{-1}} \) satisfies \((75)\). Then, by Theorem \(3.7\), \( \Gamma_f \) and \( \Gamma_{f^{-1}} \) are multiplicative morphisms of left \( A \)-modules and right \( H \)-comodules preserving the preunit and satisfying \((e1)-(e3)\). Therefore, by Theorem \(3.5\) we obtain that \( A \otimes_{\alpha} H, A \otimes_{\beta} H \) are equivalent weak crossed products.

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