Time-Periodic Solutions to the Navier-Stokes Equations

Giovanni P. Galdi and Mads Kyed

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The work of G.P. Galdi was partially supported by the NSF grant DMS-1614011

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© Springer International Publishing AG 2016
Y. Giga, A. Novotny (eds.), Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, DOI 10.1007/978-3-319-10151-4_10-1
Abstract

The Navier-Stokes equations with time-periodic data are investigated with respect to solutions of the same period. In the physical terms, such a system models the flow of a viscous liquid under the influence of a time-periodic force. The three most relevant types of flow domains, from a physical point of view, are considered: a bounded domain, an exterior domain, and an infinite pipe. Methods to show existence of both weak and strong solutions are introduced. Moreover, questions regarding regularity, uniqueness, and asymptotic structure at spatial infinity of solutions are addressed.

1 Introduction

An object performing a time-periodic interaction with a viscous liquid is one of the most frequently occurring mechanical systems in nature. In such systems, the object exerts a time-periodic force on the liquid, and it can be expected that the resulting flow undergoes a motion of the same period. The mathematical analysis of such motions naturally leads to the study of time-periodic Navier-Stokes equations.

If the region of flow is a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the Navier-Stokes equations governing the motion of a liquid subjected to a body force $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ can be written as

$$\begin{cases}
\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f & \text{in } \mathbb{R} \times \Omega, \\
\text{div } u = 0 & \text{in } \mathbb{R} \times \Omega,
\end{cases}$$

where $u : \mathbb{R} \times \Omega \to \mathbb{R}^n$ denotes the Eulerian velocity field, $p : \mathbb{R} \times \Omega \to \mathbb{R}$ the pressure field, and $\nu$ the constant coefficient of kinematic viscosity of the liquid. As is natural for time-periodic problems, the time axis is taken to be the whole of $\mathbb{R}$. If the domain $\Omega$ has a boundary, that is, $\Omega \neq \mathbb{R}^n$, a boundary condition

$$u = u_* \quad \text{on } \mathbb{R} \times \partial \Omega$$

is added to the system. In case $\Omega$ is unbounded, an asymptotic value of the velocity field at spatial infinity

$$\lim_{|x| \to \infty} u(t, x) + u_\infty = 0$$

is prescribed. If the data $f$, $u_*$, and $u_\infty$ are all time-periodic, say of period $T > 0$, the condition that the liquid undergoes a motion of the same period

$$\forall (t, x) \in \mathbb{R} \times \Omega : u(t + T, x) = u(t, x)$$

completes the system. In the following, a mathematical analysis of (1), (2), (3), and (4) will be carried out for a fixed period $T$. 
Interaction with a moving boundary is often the driving force in a fluid flow. As typical examples, one may think of an object performing a motion in a fluid or the motion of a fluid in a container with the flow driven by momentum flux on the fluid-structure boundary due to an alteration of the surface geometry. In both cases, the domain $\Omega$ changes in time. In all relevant applications, a body force in the fluid is typically absent or, at the most, is potential-like and, therefore, can be absorbed in the pressure term in (1). In order to mathematically investigate the corresponding equations of motion, however, such a system is usually rewritten in a fixed (in time) domain. In the new system, an artificial body force then appears. If the motion of the boundary, and hence the domain, is time-periodic, the artificial body force will be too. Consequently, the mathematical analysis of (1), (2), (3), and (4) in a fixed domain $\Omega$ is imperative for understanding more natural time-periodic fluid flows.

In this article, basic questions concerning existence, regularity, uniqueness, and asymptotic properties of solutions to (1), (2), (3), and (4) will be addressed. The aim is to comprehensively introduce the most basic methods that can deliver answers to these questions. Focus will be on the following types of spatial domains:

(i) $\Omega$ a bounded domain (Sect. 3).
(ii) $\Omega$ an exterior domain, in which case a time-periodic velocity $u_{\infty}(t)$ is prescribed at spatial infinity (Sect. 4).
(iii) $\Omega$ a pipe, in which case a Poiseuille flow is prescribed at spatial infinity of each outlet (Sect. 5).

Models based on bounded domains cover a large class of important physical systems. The motion of a liquid in a container with a time-periodic inflow (and outflow) is a classic example. The flow of a liquid in the gap between two rotating concentric spheres or cylinders is another. In fact, many intrinsic properties of viscous fluid flows are traditionally observed and studied in the setting of a time-periodic flow in a bounded domain. The exterior domain case is just as interesting, as (1), (2), (3), and (4) in this case models a fluid flow past an object moving with time-periodic velocity $u_{\infty}$ through a liquid. Also highly relevant from a physical point of view is the time-periodic fluid flow in a pipe. The cardiovascular system, for example, is essentially a fluid flow in a piping system with a prescribed time-periodic flow rate. Generally, time-periodic fluid flows occurring in nature fall into one of the categories (i)–(iii).

The investigation of time-periodic Navier-Stokes equations was initiated in a short note by Serrin [43], who suggested to study (1) as a dynamical system and identify a time-periodic solution as a periodic orbit. Serrin made the proposition that for time-periodic data $f$ and any initial value, the solution $u(t, x)$ to the corresponding initial-value problem tends to a periodic orbit as $t \to \infty$. However, as remarked by Serrin himself, the assumptions he makes are very stringent and, certainly at the time when the paper appeared, not sustained by any known result. The first complete results on existence of time-periodic solutions are due to Prodi [39] and Yudovich [53]. These authors considered the Poincaré map that takes an initial value into the state described by the solution to the initial-value problem.
at time $T$. A time-periodic solution can then be obtained via a fixed point of this map. The two originally proposed methods were subsequently used by several authors to build a mathematical foundation for the time-periodic Navier-Stokes equations. Of particular importance is the work [41] of Prouse, in which the celebrated method introduced by Hopf in [21] for the initial-value Navier-Stokes problem was adapted to the time-periodic setting for the first time. Although Prouse focused on weak solutions in bounded domains, the approach of Hopf based on a Galerkin approximation can be employed, as will be shown in the following, more broadly. The first to treat strong solutions were Kaniel and Shinbrot [23], followed shortly after by Takeshita [46]. Time-dependent domains were treated by Morimoto [38] and Miyakawa and Teramoto [37]. The first results for unbounded domains are due to Maremonti [34, 35], who treated the half- and whole-space problem. More general unbounded domains were investigated by Maremonti and Padula [36] by combining the Galerkin approximation with the “invading domain” technique. An important contribution was given by Kozono and Nakao [25], who, for the first time, proposed a direct representation formula for a time-periodic solution. Yamazaki [52] employed this formula to treat the case of a three-dimensional exterior domain. Further results for exterior domains were obtained by Galdi and Sohr [7] and Taniuchi [47]. Another direct representation formula was introduced by Kyed [27, 28] based on the Fourier transform on the locally compact abelian group $\mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n$. This idea further lead to the concept of a time-periodic fundamental solution [30], maximal $L^p$ regularity in $\mathbb{R}^n$ [29] for the linearization of (1), and an analysis by Lemarié-Rieusset [32] of time-periodic whole-space Navier-Stokes equations in critical spaces. Maximal $L^p$ regularity was established in the two- and three-dimensional exterior domain by Galdi [11, 12] and Galdi and Kyed [14], respectively. Investigation of time-periodic Navier-Stokes equations in cylindrical domains was initiated by da Veiga [1] and continued by Galdi [9].

After the basic questions of existence, regularity, and uniqueness of solutions have been addressed, another critical issue emerges in the case of unbounded domains, namely, the inquiry into the asymptotic structure of solutions at spatial infinity. In the exterior domain case in particular, where $u_\infty(t)$ in (3) describes the velocity of an object moving through a liquid, does the asymptotic structure of a solution reveal important physical properties. For $u_\infty = 0$, the leading term in an asymptotic expansion was identified by Kang, Miura, and Tsai [22] to be the same as the one found for the corresponding steady-state equations, namely, the Landau solution. Also for $u_\infty \neq 0$, it has been shown [12, 14, 15, 27] that the leading term in the time-periodic case coincides with leading term found in the steady-state case. Other results concerning the asymptotic properties of time-periodic Navier-Stokes flow include a technique developed by Baalen and Wittwer [48] and the investigation by Silvestre [44] of flows with finite kinetic energy.

The references given above concern results directly related to (1), (2), (3), and (4). Over the years, related systems have been investigated. Although out of the scope of this article, the system of equations governing the flow past a rigid body moving freely in a liquid under the action of a time-periodic force deserves mentioning. In order to mathematically investigate this problem, it is necessary to
rewrite the equations of motion for the liquid, that is, the Navier-Stokes system, in a frame of reference attached to the body. The result is a system of time-periodic Navier-Stokes equations in a frame of reference that is not necessarily an inertial frame. This special type of time-periodic Navier-Stokes problem was investigated by Galdi and Silvestre [16,17], who extended a famous result of Weinberger [50,51] and Serre [42] to the time-periodic case.

A general approach to time-periodic fluid flow problems was developed recently by Geissert, Hieber, and Nguyen [18]. A comprehensive treatment of time-periodic partial differential equations, including the Navier-Stokes equations, can be found in the books of Vejvoda [49] and Lions [33].

### Notation

In the following, \( \Omega \subset \mathbb{R}^n \) will always denote a domain, namely, an open connected set. Points in \( \mathbb{R} \times \Omega \) are generally denoted by \( (t, x) \), with \( t \) being referred to as time and \( x \) as the spatial variable. Differential operators act only in the spatial variable unless otherwise indicated. In particular, \( \partial_j = \partial x_j \) for \( j = 1, \ldots, n \).

The notation \( B_R \) refers to a ball in \( \mathbb{R}^n \) centered at 0 with radius \( R > 0 \). Moreover, \( B^R := \mathbb{R}^n \setminus B_R \) and \( B_{R_1,R_2} := B_{R_2} \setminus B_{R_1} \). Additionally, \( \Omega_R := \Omega \cap B_R, \Omega_{R_1,R_2} := \Omega \cap B_{R_1,R_2}, \) and \( \Omega^R := \Omega \cap B^R \).

Einstein’s summation convention, that is, implicit summation over all repeated indices, is employed throughout.

Constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

For vector fields \( f, g \) and second-order tensor fields \( F, G \) on a domain \( \Omega \subset \mathbb{R}^n \), the notation

\[
(f, g) := \int_{\Omega} f \cdot g \, dx = \int_{\Omega} f_i \cdot g_i \, dx, \quad (F, G) := \int_{\Omega} F \cdot G \, dx = \int_{\Omega} F_{ij} : G_{ij} \, dx
\]

is used to denote their inner products.

Classical Lebesgue spaces with respect to spatial domains are denoted by \( L^q(\Omega) \) and with respect to time-space domains by \( L^q((0, T) \times \Omega) \). If no confusion can arise, the norm is denoted by \( \| \cdot \|_p \) in both cases. The norm in \( L^q((0, T) \times \Omega) \) is normalized so that

\[
\|f\|_{L^q((0,T)\times\Omega)} := \left( \frac{1}{T} \int_0^T \int_{\Omega} |f(t,x)|^q \, dx \, dt \right)^{\frac{1}{q}}.
\]

The Lebesgue space \( L^q(\Omega) \) is treated as the subspace of functions in \( L^q((0, T) \times \Omega) \) that are time independent. This identification is made without further notification. With the normalization above, the \( L^q(\Omega) \) norm coincides with the \( L^q((0, T) \times \Omega) \) norm on the subspace of time-independent functions.
Classical Sobolev spaces are denoted by $W^{m,q}(\Omega)$ and their norms by $\| \cdot \|_{m,q}$. The subspaces of Sobolev functions vanishing on the boundary are denoted by $W^{m,q}_0(\Omega) := C_0^\infty(\Omega)\| \cdot \|_{m,q}$. The dual space of the latter is denoted by $W^{-m,q}_0(\Omega) := (W^{m,q}_0(\Omega))^*$ and its norm by $\| \cdot \|_{-1,2}$. Sobolev spaces over time-space domains are introduced for $j,k \in \mathbb{N}_0$ and $q \in [1, \infty)$ in the form

$$W^{j,k,q}(0, T) \times \Omega := \{ f \in L^1_{loc}(0, T) \times \Omega) \mid \| f \|_{j,k,q} < \infty \},$$

$$\| f \|_{j,k,q} := \left( \sum_{|\alpha| \leq j} \| \partial_\alpha^j f \|_{L^q_0(0, T) \times \Omega}^q + \sum_{0 < \beta \leq k} \| \partial_\beta^j f \|_{L^q(0, T) \times \Omega}^q \right)^{\frac{1}{q}}. \quad (5)$$

In order to introduce the appropriate function spaces of $T$-time-periodic functions, the space

$$C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega) := \{ f \in C^\infty(\mathbb{R} \times \Omega) \mid f(t + T, x) = f(t, x), \quad f \in C^\infty([0, T] \times \Omega) \},$$

of smooth time-periodic functions with compact support in the spatial variable is defined. Clearly, $\| \cdot \|_{j,k,q}$ is a norm on $C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)$, and one can thus define (anisotropic) Sobolev spaces of time-periodic functions as

$$W^{j,k,q}_{0,\text{per}}(\mathbb{R} \times \Omega) := C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)\| \cdot \|_{j,k,q}, \quad W^{j,k,q}_{\text{per}}(\mathbb{R} \times \Omega) := C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)\| \cdot \|_{j,k,q}.$$

In the special case $j = k = 0$, the Lebesgue space

$$L^q_{\text{per}}(\mathbb{R} \times \Omega) := C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)\| \cdot \|_q$$

is obtained. Clearly, $L^q_{\text{per}}(\mathbb{R} \times \Omega)$ can be identified with the space obtained by periodically extending functions in the classical Lebesgue space $L^q((0, T) \times \Omega)$. In this spirit, for an arbitrary Banach space $X$ the Lebesgue space

$$L^q_{\text{per}}(\mathbb{R}; X) := \{ f \in L^1_{loc}(\mathbb{R}; X) \mid f(\cdot + T) = f(\cdot) \text{ a.e.}, \quad \| f \|_{L^q((0, T); X)} < \infty \}$$

of time-periodic vector-valued $L^q$ functions is defined.

Finally, subspaces of solenoidal vector fields are introduced. By $C^\infty_{0,\sigma}(\Omega)$, the subspace of vector fields $u \in C^\infty_0(\Omega)$ with $\text{div} u = 0$, which then leads to the classical Lebesgue spaces $L^q(\Omega) := C^\infty_0(\Omega)\| \cdot \|_q$, Sobolev spaces $W^{1,2}_0(\Omega) := C^\infty_0(\Omega)\| \cdot \|_{1,2}$, and homogeneous Sobolev spaces $D^{1,2}_0(\Omega) := C^\infty_0(\Omega)\| \cdot \|_{2}$ of solenoidal vector fields. Similarly,

$$C^\infty_{0,\sigma,\text{per}}(\mathbb{R} \times \Omega) := \{ u \in C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega) \mid \text{div} u = 0 \}$$
denotes the space of smooth time-periodic solenoidal vector fields of compact support.

3 Bounded Domains

The time-periodic Navier-Stokes equations in a bounded domain $\Omega$ with homogeneous Dirichlet boundary condition, the so-called no slip boundary condition $u_\ast = 0$, take the form

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u &= v \Delta u - \nabla p + f \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
u &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega, \\
u(t + T, x) &= \nu(t, x).
\end{aligned}
\] (6)

In light of the classical results for the corresponding steady-state and initial-value problems, the following questions emerge naturally as the most essential to which affirmative answers can be expected:

1. existence of weak solutions for arbitrarily “large” data,
2. uniqueness of solutions for “small” data,
3. existence of strong solutions for “small” data in the case $n = 3$ and arbitrarily “large” data in the case $n = 2$,
4. regularity of weak solutions satisfying a well-known integrability condition introduced for the corresponding initial-value problem in the pioneering works of Leray,
5. existence of $L^q$ estimates for solutions $(\nu, p)$ to the Stokes linearization of (6) in terms of the data $f$.

3.1 Existence of Weak Solutions

There are a number of different, but essentially equivalent, definitions of weak time-periodic solutions. The standard theory for time-periodic distributions is based on $C_0^{\infty}(\mathbb{R} \times \Omega)$ as the space of test functions. An argument can therefore be made that the definition of weak solutions to (6) should be based on the subspace $C_0^{\infty}(\mathbb{R} \times \Omega) \subset C_0^{\infty}(\mathbb{R} \times \Omega)$ of solenoidal test functions:

Definition 1. Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain. Let $f \in L^2_{\text{per}}(\mathbb{R}; W_0^{-1,2}(\Omega))$. A vector field $\nu \in L^2_{\text{per}}(\mathbb{R}; W_0^{1,2}(\Omega))$ is called a weak time-periodic solution to (6) if
\[
\int_0^T - (u, \partial_t \varphi) + \nu (\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) - (f, \varphi) \, dt = 0 \tag{7}
\]
for all \( \varphi \in C^\infty_{0, \sigma, \text{per}}(\mathbb{R} \times \Omega) \).

As usual, a weak solution is extended to a solution in the sense of distributions by augmenting it with a pressure term \( \rho \) that can be added in (7) to make the equality valid for all test functions \( \varphi \in C^\infty_{0, \sigma, \text{per}}(\mathbb{R} \times \Omega) \).

Just as for the corresponding steady-state and initial-value problem, existence of a weak time-periodic solution can be shown without any restrictions on the “size” of the data. In fact, one can use the same idea based on a Galerkin approximation that was introduced by Hopf [21] in 1951 to treat the initial-value problem. The extension of Hopf’s method to the time-periodic problem was first carried out by Prouse [41]. The theorem below and its proof is credited to him.

**Theorem 1.** Let \( n \geq 2 \) and \( \Omega \subset \mathbb{R}^n \) be any bounded domain. For every vector field \( f \in L^2_{\text{per}}(\mathbb{R}; W^{1,2}_0(\Omega)) \), there is a corresponding weak time-periodic solution \( u \) to (6) that satisfies \( u \in L^\infty_{\text{per}}(\mathbb{R}; L^2(\Omega)) \) and the energy inequality

\[
\int_0^T \int_{\Omega} |\nabla u(t, x)|^2 \, dx \, dt \leq \frac{1}{\nu} \int_0^T \left( f(t), u(t) \right) \, dt. \tag{8}
\]

**Proof.** Let \( \{\Psi_i\}_{i=1}^\infty \) be a basis \( W^{1,2}_{0, \sigma}(\Omega) \), that is, orthonormal in \( L^2(\Omega) \). Put

\[
u_k(t, x) = \sum_{i=1}^k \alpha_{ki}(t) \Psi_i(x) \tag{9}
\]

with \( \alpha_k(t) := (\alpha_{k1}(t), \ldots, \alpha_{kk}(t)) \) a solution to the system of ordinary differential equations

\[
\begin{cases}
\frac{d\alpha_{kj}}{dt} = -\nu \sum_{i=1}^k (\nabla \Psi_i, \nabla \Psi_j) \alpha_{ki} - \sum_{i,l=1}^k (\Psi_i \cdot \nabla \Psi_l, \Psi_j) \alpha_{kl} \alpha_{ki} + \left( f(t), \Psi_j \right), \\
\alpha_{kj}(0) = a_j.
\end{cases} \tag{10}
\]

By standard theory, the above system possesses a unique solution \( \alpha_k \in W^{1,2}(0, T_k) \) on some interval \( (0, T_k) \), where \( T_k \) can be chosen with the property that either \( T_k = \infty \) or \( |\alpha(t)| \to \infty \) as \( t \to T_k \). A blowup of \( \alpha(t) \) in finite time, however, can be excluded. To this end, first multiply (10) with \( \alpha_{kj} \) and sum over \( j \). By
orthonormality of \( \{ \Psi_i \}_{i=1}^{\infty} \) in \( L^2(\Omega) \) and the identity \( (\Psi_i \cdot \nabla \Psi_i, \Psi_i) = 0 \), it then follows that

\[
\frac{d}{dt} \left[ \|u_k(t)\|_2^2 \right] + 2v \| \nabla u_k(t) \|_2^2 = 2 (f(t), u_k(t)).
\]  

(11)

By Poincaré’s inequality, applied on the left-hand side, and Young’s inequality, applied on the right-hand side,

\[
\frac{d}{dt} \left[ \|u_k(t)\|_2^2 \right] + \nu c_0 \| u_k(t) \|_2^2 \leq c_1(v) \| f(t) \|_{-1,2}^2.
\]  

(12)

From this differential inequality, it is deduced for any \( t \in [0, T_k] \) that

\[
\|u_k(t)\|_2^2 \leq e^{-\nu c_0 t} \left( \|u_k(0)\|_2^2 + c_1 \int_0^{T_k} e^{\nu c_0 \tau} \| f(\tau) \|_{-1,2}^2 d\tau \right).
\]  

(13)

Since \( \|u_k(t)\|_2^2 = |\alpha(t)|^2 \), it follows that \( |\alpha(t)| \) does not blow up at any time, whence only \( T_k = \infty \) is possible. This means the map \( J : \mathbb{R}^k \to \mathbb{R}^k \), \( J(a) := \alpha_k(T) \) is well defined, where \( a = (a_1, \ldots, a_k) \) denotes the initial value in (10). Since \( \|u_k(0)\|_2^2 = |a|^2 \), it follows from (13) that \( J \) becomes a self-mapping \( J : \overline{B}_\rho \to \overline{B}_\rho \) for

\[
\rho^2 \geq \frac{c_1 \int_0^T e^{\nu c_0 \tau} \| f(\tau) \|_{-1,2}^2 d\tau e^{-\nu c_0 T}}{1 - e^{-\nu c_0 T}}.
\]  

(14)

Consequently, \( J \) has a fixed point \( \widetilde{a} \in \overline{B}_\rho \). From now on, let \( u_k \) be the vector field defined in (9) corresponding to the solution \( \alpha_k \) to (10) with initial value \( \widetilde{a} \). Then \( u_k(x, 0) = u_k(x, T) \). Integration from 0 to \( T \) in (11) yields

\[
\| \nabla u_k \|_{L^2(0,T;L^2(\Omega))}^2 = \frac{1}{v} \int_0^T \langle f(t), u_k(t) \rangle dt.
\]  

(15)

Moreover, from (13) and (14), it follows that

\[
\|u_k\|_{L^\infty(0,T;L^2(\Omega))} \leq c_2(v, T, \Omega) \left( \rho^2 + \| f \|_{L^2(0,T;H^{-1,2}_0(\Omega))}^2 \right)^{\frac{1}{2}} \leq c_3(v, T, \Omega) \| f \|_{L^2(0,T;H^{-1,2}_0(\Omega))}
\]  

(16)

A periodic extension of \( u_k \) to all \( t \in \mathbb{R} \) is now carried out. In fact, the solution \( \alpha_k(t) \) to (10) is \( T \)-periodic on the whole line \( \mathbb{R} \) at the outset, whence an extension of \( u_k \) is not even necessary. By construction,

\[
(\partial_t u_k(t), \Psi_j) + \nu (\nabla u_k(t), \nabla \Psi_j) + (u_k(t) \cdot \nabla u_k(t), \Psi_j) - \langle f(t), \Psi_j \rangle = 0
\]  

(17)
for all \( j = 1, \ldots, k \). Multiplication by functions \( e^{i 2\pi lt} \) in (17) and subsequent integration from 0 to \( T \) yields for functions

\[
\varphi \in X_h := \left\{ \sum_{j=1, |l|=0}^{h} \beta_{lj} e^{i 2\pi lt} \Psi_j(x) \mid \beta_{lj} \in \mathbb{C} \right\};
\]

with \( h \leq k \) the identity

\[
\int_0^T \left( (u_k, \partial_t \varphi) + \nu (\nabla u_k, \nabla \varphi) + (u_k \cdot \nabla u_k, \varphi) - (f, \varphi) \right) \, dt = 0. \tag{18}
\]

In the next step, passage to the limit \( k \to \infty \) in the equation above is pursued in a meaningful way. For this purpose, deduce from (15) that \( \{u_k\}_{k=1}^\infty \) is bounded in \( L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega)) \) uniformly in \( k \). Thus there is a subsequence, for convenience still denoted by \( \{u_k\}_{k=1}^\infty \), that converges weakly to some \( u \). However, to pass to the limit in the nonlinear term in (18), better convergence of \( \{u_k\}_{k=1}^\infty \) is needed. Strong convergence in \( L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \) is sufficient, and it can be obtained via equicontinuity of the sequence of functions \( \{u_k(t), \Psi_j\}_{k=1}^\infty \). The equicontinuity follows by integrating (17) between two arbitrary times \( s \) and \( t \)

\[
(u_k(t), \Psi_j) - (u_k(s), \Psi_j) = \int_s^t \left( (\nabla u_k, \nabla \Psi_j) - (u_k \cdot \nabla u_k, \Psi_j) + (f(\tau), \Psi_j) \right) \, d\tau
\]

and utilizing the bound of \( \{u_k\}_{k=1}^\infty \) in \( L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega)) \) in combination with Sobolev’s embedding theorem and Cauchy-Schwarz’s inequality. It then follows by Ascoli-Arzela’s theorem that a subsequence of \( \{u_k(t), \Psi_j\}_{k=1}^\infty \) converges uniformly to a continuous function, which must coincide with \( (u(t), \Psi_j) \). By a Cantor diagonalization argument, a single subsequence can be chosen that converges for all \( j \). As a consequence, one easily shows for any \( F \in L^2(\Omega) \) that a subsequence of \( \{u_k(t), F\}_{k=1}^\infty \) converges uniformly to the continuous function \( (u(t), F) \). After passing to this subsequence, it then follows from Friedrich’s inequality (see [10, Lemma II.5.2]) that \( u_k \to u \) strongly in \( L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \). It is now possible to let \( k \to \infty \) in (18) and conclude that \( u \) satisfies (1) for all \( \varphi \in X_h, h \in \mathbb{N} \). By Fourier series expansion, any function in \( W^{1,2}_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega)) \) can be approximated by a sequence \( \{\varphi_h\}_{h=1}^\infty \) of functions with \( \varphi_h \in X_h \). It thus follows that \( u \) satisfies (1) for all \( \varphi \in C_0^\infty(\Omega \times \mathbb{R}) \subset W^{1,2}_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega)) \). Due to the uniform bound (16), it also follows that \( u \in L^\infty_{\text{per}}(\mathbb{R}; L^2(\Omega)) \). Moreover, the energy inequality (8) follows by passing to the limit \( k \to \infty \) in (15).
Remark 1. The weak solution constructed in the proof of Theorem 1 above actually belongs to $C(\mathbb{R}; L^2_{\text{weak}}(\Omega))$, where $L^2_{\text{weak}}(\Omega)$ denotes the space $L^2(\Omega)$ endowed with the weak topology, that is, $\mathbf{u}$ is weakly continuous in $L^2(\Omega)$. In the proof above, this property is obtained by construction. It is also possible, however, to establish the property for an arbitrary weak solution after possibly modifying it on set of measure 0, which was shown in [27, Theorem 6.4.3]. Moreover, it is possible to show for a weak solution, again after possibly modifying it on set of measure 0, for any $s, t \in \mathbb{R}$ and $\psi \in C^1(\Omega)$ the identity

$$\langle \mathbf{u}(t), \psi \rangle - \langle \mathbf{u}(s), \psi \rangle = \int_s^t -\left(\nabla \mathbf{u}, \nabla \psi\right) - \langle \mathbf{u} \cdot \nabla \mathbf{u}, \psi \rangle + \langle f(t), \psi \rangle \, dt. \quad (20)$$

This can be shown directly for the particular solution constructed in the proof of Theorem 1 by simply passing to the limit in (19). See [27, Lemma 6.4.1 and Lemma 6.4.2] for a proof in the general case. On the other hand, it is easy to verify that any $\mathbf{u} \in L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega)) \cap C(\mathbb{R}; L^2_{\text{weak}}(\Omega))$ satisfying (20) and

$$\mathbf{u}(0) = \mathbf{u}(T) \quad (21)$$

is a weak time-periodic solution in the sense of Definition 1. Thus, Definition 1 is equivalent to (20) and (21).

Remark 2. It is possible to associate to the weak solution $\mathbf{u}$ from Theorem 1 a pressure $p$ such that $(\mathbf{u}, p)$ is a solution in the sense of time-periodic distributions $\mathcal{D}'_{\text{per}}(\mathbb{R} \times \Omega)$. In [27, Proof of Theorem 6.5.1], construction of such a pressure term was demonstrated by expansion of $\mathbf{u} \in L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega))$ into a Fourier series with coefficients $\{\mathbf{u}_k\}_{k \in \mathbb{Z}} \in \ell^2(W^{1,2}_{0,\sigma}(\Omega))$ and $\mathbf{u} \cdot \nabla \mathbf{u} \in L^1(\mathbb{R}; W^{-1,2}_0(\Omega))$ into a Fourier series with coefficients $\{H_k\}_{k \in \mathbb{Z}} \in \ell^\infty(W^{-1,2}_0(\Omega))$, which then yields the systems of equations

$$\begin{cases}
-ik \frac{2\pi}{T} \mathbf{u}_k - \nu \Delta \mathbf{u}_k + H_k + \nabla p_k = f_k & \text{in } \Omega, \\
\nabla \cdot \mathbf{u}_k = 0 & \text{in } \Omega, \\
\mathbf{u}_k = 0 & \text{on } \partial \Omega
\end{cases}$$

satisfied, in the weak sense, by each Fourier coefficient $\mathbf{u}_k$. A pressure term $p_k$ can then be constructed by standard methods for each $k$. The Fourier series of the sequence of coefficients $\{\frac{1}{k} p_k\}_{k \in \mathbb{Z}}$ can then be identified with a function $P \in L^2_{\text{per}}(\mathbb{R}; L^2_{\text{loc}}(\Omega))$ such that $p := \partial_t P$ satisfies the required properties of a pressure corresponding to $\mathbf{u}$. 
3.2 Uniqueness

Uniqueness of time-periodic solutions can be established in the class of weak solutions satisfying the energy inequality, provided one such solution exists in the space $L^\infty_{\text{per}}(\mathbb{R}; L^n(\Omega))$ and its norm in this space is sufficiently small. A similar result, which includes the smallness requirement, is well known for the corresponding steady-state problem. As the steady-state problem is a particular case of the time-periodic problem, one cannot expect a better uniqueness result. This should be contrasted with an analogous result for the initial-value problem, where uniqueness holds in the same class, but without smallness assumption; see [26].

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be a bounded domain. There is a positive constant $\varepsilon_0(\Omega, v, T) > 0$ such that if $u$ is a weak time-periodic solution to (6) in the sense of Definition 1 that satisfies

$$\|u\|_{L^\infty_{\text{per}}(\mathbb{R}; L^n(\Omega))} \leq \varepsilon_0,$$

then any other weak solution to (6) that satisfies the energy inequality (8) must coincide with $u$.

**Proof.** Let $w$ be another weak solution to (6) that satisfies (8). The main idea is to employ $w$ as a test function in the weak formulation for $u$ and then $u$ as a test function in the weak formulation for $w$. Since $w$ and $u$ do not possess enough regularity to serve as test functions directly, regularizations will be considered instead. Consider for this purpose an approximate identity \{${j}_h$\}$_{h=1}^\infty$ on the torus (circle group) $\mathbb{R}/T\mathbb{Z}$ consisting of smooth functions $j_h \in C^\infty(\mathbb{R}/T\mathbb{Z})$, that is, $j_h \circ q \in C^\infty(\mathbb{R})$ with $q : \mathbb{R} \to \mathbb{R}/T\mathbb{Z}$ denoting the quotient mapping. One may, for example, take $j_h$ to be the Fejér kernel. For any Banach space $X$, $q \in [1, \infty)$ and $f \in L^q(\mathbb{R}/T\mathbb{Z}; X)$, the convolution $f * j_h$ then belongs to $C^\infty(\mathbb{R}/T\mathbb{Z}; X)$ and $\lim_{h \to \infty} f * j_h = f$ in $L^q(\mathbb{R}/T\mathbb{Z}; X)$. Clearly, the spaces $L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega))$ and $L^q(\mathbb{R}/T\mathbb{Z}; W^{1,2}_{0,\sigma}(\Omega))$ as well as $C^\infty_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega))$ and $C^\infty(\mathbb{R}/T\mathbb{Z}; W^{1,2}_{0,\sigma}(\Omega))$ can be identified with each other, respectively, whence $u_h := u * j_h$ is well-defined as an element in the space $C^\infty_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega))$ with $\lim_{h \to \infty} u_h = u$ in $L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega))$. In addition to the approximate identity, consider also an orthonormal basis $\{\Phi_r\}_{r=1}^\infty$ in $W^{1,2}_{0,\sigma}(\Omega)$ constituted by smooth solenoidal vector fields $\Phi_r \in C^\infty_{0,\sigma}(\Omega)$. Consider the orthogonal projections

$$u_{k,h} := \sum_{r=1}^{k} (u_h, \Phi_r)_{W^{1,2}_{0,\sigma}(\Omega)} \Phi_r,$$

$$w_{k,h} := \sum_{r=1}^{k} (w_h, \Phi_r)_{W^{1,2}_{0,\sigma}(\Omega)} \Phi_r.$$
Both $u_{k,h}$ and $w_{k,h}$ belong to $C^{0,\infty}_{0,\text{per}}(\mathbb{R} \times \Omega)$ and can be therefore be used as test functions in the weak formulation (7). It follows that

$$\int_0^T -\langle w, \partial_t u_{k,h} \rangle + v(\nabla w, \nabla u_{k,h}) + (w \cdot \nabla w, u_{k,h}) - \langle f, u_{k,h} \rangle \, dt = 0,$$

\hspace{1cm} (23)

and similarly

$$\int_0^T -\langle u, \partial_t w_{k,h} \rangle + v(\nabla u, \nabla w_{k,h}) - (u \cdot \nabla w_{k,h}, u) - \langle f, w_{k,h} \rangle \, dt = 0,$$

\hspace{1cm} (24)

where integration by parts has been carried out in the nonlinear term in the latter equation (24). By construction, \( \lim_{k \to \infty} w_{k,h} = w_h \) and \( \lim_{k \to \infty} u_{k,h} = u_h \) in \( L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,0}(\Omega)) \), on the strength of which a passage to the limit \( k \to \infty \) in (23) furnishes

$$\int_0^T -\langle w, \partial_t u_h \rangle + v(\nabla w, \nabla u_h) + (w \cdot \nabla w, u_h) - \langle f, u_h \rangle \, dt = 0.$$

\hspace{1cm} (25)

Since \( u \in L^\infty_{\text{per}}(\mathbb{R}; L^n(\Omega)) \), a passage to the limit \( k \to \infty \) is also accomplished in (23) with the result that

$$\int_0^T -\langle u, \partial_t w_h \rangle + v(\nabla u, \nabla w_h) - (u \cdot \nabla w_h, u) - \langle f, w_h \rangle \, dt = 0.$$

\hspace{1cm} (26)

By Fubini’s theorem and integration by parts, it follows that

$$\int_0^T (w, \partial_t u_h) \, dt = -\int_0^T (u, \partial_t w_h) \, dt.$$

Addition of (25) and (25) thus yields

$$\int_0^T v(\nabla w, \nabla u_h) + v(\nabla u, \nabla w_h) + (w \cdot \nabla w, u_h) - (u \cdot \nabla w_h, u) \, dt$$

$$= \int_0^T \langle f, u_h \rangle + \langle f, w_h \rangle \, dt.$$

The absence of time derivatives allows for a passage to the limit \( h \to \infty \) above, which together with the elementary identities

$$\int_0^T (u \cdot \nabla u, u) \, dt = 0, \quad \int_0^T (w \cdot \nabla u, u) \, dt = 0$$
implies
\[ \int_0^T 2\nu (\nabla w, \nabla u) + ((w - u) \cdot \nabla (w - u), u) \, dt = \int_0^T (f, u) + (f, w) \, dt. \] (27)

By assumption, \( w \) satisfies the energy inequality (8). Based on the integrability assumption \( u \in L^\infty_{\text{per}}(\mathbb{R}; L^n(\Omega)) \), it can be shown that also \( u \) satisfies the energy inequality; see Remark 3 below. A utilization of the energy inequality (8) for both \( u \) and \( w \) in combination with (27) and the simple identity
\[ \int_0^T \int_{\Omega} |\nabla u - \nabla w|^2 \, dx \, dt = \int_0^T \int_{\Omega} |\nabla u|^2 + |\nabla w|^2 - 2 \nabla u \cdot \nabla w \, dx \, dt, \] (28)
now yields
\[ \int_0^T \int_{\Omega} |\nabla u - \nabla w|^2 \, dx \, dt \leq -c_0 \int_0^T ((w - u) \cdot \nabla (w - u), u) \, dt. \]

In the case \( n = 3 \), Hölder’s inequality in combination with the classical Sobolev inequality then furnishes
\[ \int_0^T \int_{\Omega} |\nabla u - \nabla w|^2 \, dx \, dt \leq \int_0^T \|w - u\|_{6} \|\nabla (w - u)\|_{2} \|u\|_{3} \, dt \]
\[ \leq c_1 \|u\|_{L^\infty(0,T;L^3(\Omega))} \int_0^T \|\nabla (w - u)\|_{2}^{2} \, dt. \]

From (22), it then follows that \( u = w \), provided \( \varepsilon_0 \) is chosen sufficiently small. The case \( n = 2 \) is concluded in a similar manner.

Remark 3. Weak solutions in \( L^\infty_{\text{per}}(\mathbb{R}; L^n(\Omega)) \) automatically satisfy the energy equality, that is, (8) with equality. In fact, since a weak time-periodic solution can be viewed as a solution to an initial-value problem with initial value in \( L^2_0(\Omega) \), recall Remark 1 and in particular (20) and (21), this property follows from a well-known result for the Navier-Stokes initial-value problem. More specifically, the energy equality can be established for weak solutions in \( L^4_{\text{per}}(\mathbb{R}; L^4(\Omega)) \); see [8, Theorem 4.1]. In the case \( n = 2 \), one readily verifies that every weak solution belongs to this class and thus satisfies the energy equality. In the case \( n = 3 \), it is an open question whether a weak solution obeys the energy equality. However, when \( n = 3 \), it follows by Sobolev embedding that a weak solution in the class
\[ L^r_{\text{per}}(\mathbb{R}; L^s(\Omega)), \quad \frac{3}{s} + \frac{2}{r} = 1, \quad s \in [3, \infty] \] (29)
also belongs to $L^4_{\text{per}}(\mathbb{R}; L^4(\Omega))$ and thus satisfies the energy equality; see [8, Remark 4.3].

### 3.3 Existence of Strong Solutions

If the data and domain are more regular, existence of a strong time-periodic solution, that is, a solution for which all derivatives appearing in the equation (1) exist as integrable functions with a suitable amount of summability, can be shown. In the three-dimensional case, such a result requires a restriction on the “size” of the data. In the two-dimensional case, no such restriction is necessary. The situation concerning existence of strong solutions is therefore the same as for the corresponding initial-value problem. In fact, just as for the initial-value problem, one can prove the result by modifying the Galerkin approximation employed in the proof of Theorem 1 according to an idea due to Prodi [40] of choosing as orthogonal basis a sequence of eigenfunctions to the Stokes operator. It is well known that Prodi’s idea can be applied successfully to both the steady-state and the initial-value problem. In [16], it was shown that his method also works for the time-periodic problem.

The proof of existence relies decisively on a certain Sobolev inequality, which explains the sensitivity to the dimension of the domain. The three-dimensional case is covered by:

**Theorem 3.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^2$. There is a positive constant $\varepsilon_1(\Omega, v, T) > 0$ such that for any $f \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))$ with $\|f\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))} \leq \varepsilon_1$, there is a time-periodic solution $(u, p)$ to (1) that satisfies

$$u, \nabla u \in L^\infty(\mathbb{R}; L^2(\Omega)), \quad \partial_t u, \nabla^2 u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)), \quad p \in L^2_{\text{per}}(\mathbb{R}; W^{1,2}(\Omega)).$$

(30)

Moreover,

$$\|u, \nabla u\|_{L^\infty(\mathbb{R}; L^2(\Omega))} + \|\partial_t u, \nabla^2 u\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))} \leq C_1(\Omega, v, T, \|f\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))})$$

(31)

with $C_1(\Omega, v, T, \|f\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))}) \to 0$ as $\|f\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))} \to 0$.

**Proof.** Let $P_H : L^2(\Omega)^3 \to L^2(\Omega)$ denote the Helmholtz-Weyl projection. It is well known that the Stokes operator $P_H \Delta : W^{1,2}_{\sigma}(\Omega) \cap W^{2,2}(\Omega) \to L^2(\Omega)$ is a homeomorphism; see, for example, [10, Theorem IV.6.1]. Since $\Omega$ is a bounded domain, the inverse $(-P_H \Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact. Since $(-P_H \Delta)^{-1}$ is also positive and symmetric, there is an orthogonal basis in $L^2(\Omega)$ of eigenfunctions $\{\Psi_i\}_{i=1}^\infty$ to $(-P_H \Delta)^{-1}$ with positive eigenvalues. Thus $P_H \Delta \Psi_i =$
\[ \lambda_i \Psi_j \text{ with } \lambda_i > 0. \] It is easy to verify that \( \{ \Psi_i \}_{i=1}^{\infty} \) is also an orthogonal basis in \( W_{0,\sigma}^{1,2}(\Omega) \). It can therefore be used as the basis in the Galerkin approximation in the proof of Theorem 1, in which case multiplication of both sides in (10) with \( \lambda_j c_{kj} \) and summation over \( j \) yields

\[ \frac{1}{2} \frac{d}{dt} \| \nabla u_k(t) \|_2^2 + \nu \| \nabla u_k(t) \|_2^2 = (u_k(t) \cdot \nabla u_k(t), P_H \Delta u_k(t)) - (f, P_H \Delta u_k(t)). \] (32)

Recall at this point the following standard elliptic estimate for the Stokes problem (see, e.g., [10, Theorem IV.6.1]):

\[ \| \nabla^2 u_k(t) \|_2 \leq c_0 \| P_H \Delta u_k(t) \|_2. \] (33)

Utilization of the Gagliardo-Nirenberg inequality for three-dimensional domains (see, e.g., [10, Lemma II.3.3 and Exercise II.3.12]) therefore implies

\[ \| \nabla u_k(t) \|_3 \leq c_1 \| \nabla u_k \|_{L^1,2}^{\frac{1}{3}} \| \nabla u_k \|_2^{\frac{2}{3}} \leq c_2 \left( \| P_H \Delta u_k(t) \|_2^{\frac{1}{2}} + \| \nabla u_k \|_2 \right) \| \nabla u_k \|_2^{\frac{1}{2}}. \] (34)

The nonlinear term \((u_k(t) \cdot \nabla u_k(t), P_H \Delta u_k(t))\) on the right-hand side in (32) can now be estimated by employing first Hölder’s inequality with the parameters 6, 3, and 2, then (34), and finally Young’s inequality to obtain

\[ \frac{d}{dt} \| \nabla u_k(t) \|_2^2 + \nu \| P_H \Delta u_k(t) \|_2^2 \leq c_3 \left( \| \nabla u_k(t) \|_2^2 + \| \nabla u_k(t) \|_2^5 + \| f(t) \|_2^2 \right). \] (35)

Since (11) implies \( \| \nabla u_k \|_{L^2(0,T;L^2(\Omega))} \leq c_4 \| f \|_{L^2(0,T;L^2(\Omega))} \), the mean value theorem for continuous functions secures the existence of \( s_0 \in [0,T] \) such that

\[ \| \nabla u_k(s_0) \|_2^2 = \frac{1}{T} \int_0^T \| \nabla u_k(t) \|_2^2 \, dt \leq c_5 \| f \|_{L^2(0,T;L^2(\Omega))}^2. \] (36)

Integration from \( s_0 \) to \( t \) in (35) then yields

\[ \| \nabla u_k(t) \|_2^2 \leq c_6 \left( \| f \|_{L^2(0,T;L^2(\Omega))}^2 + \int_{s_0}^t \| \nabla u_k(s) \|_2^4 + \| \nabla u_k(s) \|_2^6 \, ds \right). \] (37)

A first \( t \in [s_0,2T] \) such that \( \| \nabla u_k(t) \|_2^2 = 2c_6 \| f \|_{L^2(0,T;L^2(\Omega))}^2 \) therefore invokes a contradiction if \( \| f \|_{L^2(0,T;L^2(\Omega))} \) is sufficiently small. Consequently, with such a restriction imposed on the size of \( \| f \|_{L^2(0,T;L^2(\Omega))} \), the estimate
\[ \forall t \in [s, 2T] : \| \nabla u_k(t) \|_2^2 < 2c_6 \| f \|_{L^2(0,T;L^2(\Omega))}^2 \]

follows. By \( T \)-periodicity of \( u_k \), the same estimate must hold for all \( t \in [0, T] \). With this pointwise bound on \( \| \nabla u_k(t) \|_2 \), a return to (35) implies, after integration from 0 to \( T \), a similar bound on \( \| P_H \Delta u_k \|_{L^2(0,T;L^2(\Omega))} \). Due to (33), passage to the limit \( k \to \infty \) as in the proof of Theorem 1 therefore yields

\[ \| \nabla \mathbf{u} \|_{L^\infty_t(R;L^2(\Omega))} + \| \nabla^2 \mathbf{u} \|_{L^2_t(R;L^2(\Omega))} \leq c_7 \| f \|_{L^2_t(R;L^2(\Omega))}, \]

By similar arguments, \( \partial_t u_k(t) \) can be analyzed. Multiplying both sides in (10) with \( \frac{d}{dt} c_{kj} \) and summing over \( j \), one obtains

\[ \| \partial_t u_k(t) \|_2^2 + \frac{1}{2} \frac{d}{dt} \| \nabla u_k(t) \|_2^2 = (u_k(t) \cdot \nabla u_k(t), \partial_t u_k(t)) + (f, \partial_t u_k(t)). \]

It is easy to estimate the right-hand side of this equation and argue as above to conclude

\[ \| \partial_t \mathbf{u} \|_{L^2_t(R;L^2(\Omega))} \leq c_8 \| f \|_{L^2_t(R;L^2(\Omega))}. \]

With the additional regularity obtained for \( \mathbf{u} \), arguments from the well-known analysis of the corresponding initial-value problem, or alternatively by the approach described in Remark 2, yield a pressure \( p \in L^2_t(R;W^{1,2}(\Omega)) \) such that \((\mathbf{u}, p)\) is a solution to (1).

The same argument as in the proof above can be used in the two-dimensional case to show existence of a strong solution without any restrictions on the “size” of the data. The improvement in this case is due to the Sobolev inequality employed in the first estimate of (34) taking different parameters in the two-dimensional case, which means that the following periodic version of Gronwall’s inequality can be employed:

**Lemma 1.** Let \( f, g \in L^1_{\text{per}}(\mathbb{R}) \) be nonnegative \( T \)-periodic functions. If \( y \in C^1_{\text{per}}(\mathbb{R}) \) is a nonnegative \( T \)-periodic function satisfying

\[ y'(t) \leq g(t) \cdot y(t) + f(t), \]

then

\[ y(t) \leq e^{2 \int_0^T g(s) \, ds} \left( \frac{1}{T} \int_0^T y(s) \, ds + 2 \int_0^T f(s) \, ds \right). \] (38)

**Proof.** By the mean value theorem for continuous functions, there is an \( s_0 \in [0, T] \) such that \( y(s_0) = \frac{1}{T} \int_0^T y(t) \, dt \). The classical Gronwall inequality yields (38) on the interval \([s_0, 2T]\). The \( T \)-periodicity of \( y \) extends the estimate to all of \( \mathbb{R} \). \( \Box \)

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded \( C^2 \)-domain. For any \( f \in L^2_{\text{per}}(\mathbb{R};L^2(\Omega)) \), there is a time-periodic solution \((\mathbf{u}, p)\) to (1) that satisfies (30) and (31).
Proof. One can proceed exactly as in the proof of Theorem 3 until (34) is reached. At this point, the Sobolev inequality
\[ \|v\|_4 \leq c_0 \|v\|_2^{\frac{1}{2}} \cdot \|v\|_{1,2}^{\frac{1}{2}} \]  
(39)
can be employed, which is valid for all \( v \in W^{1,2}(\Omega) \) since \( \Omega \) is a two-dimensional domain; see, for example, [10, Lemma II.3.3 and Exercise II.3.12]. The nonlinear term on the right-hand side in (32) can now be estimated by Hölder’s inequality with parameters 4, 4, and 2 followed by two applications of (39) to obtain
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla u_k(t)\|_2^2 + v \|P_H \Delta u_k(t)\|_2^2 &
\leq c_1 \left( \|u_k(t)\|_2^{\frac{1}{2}} \|\nabla u_k(t)\|_2 \|\nabla u_k(t)\|_2^{\frac{1}{2}} \|P_H \Delta u_k(t)\|_2 + \|f(t)\|_2^2 \right).
\end{aligned}
\]
Standard elliptic estimate for the Stokes problem (see again [10, Theorem IV.6.1]) imply \( \|\nabla u_k\|_{1,2} \leq c_2 \|P_H \Delta u_k\|_2 \). Since, by (16), \( \|u_k\|_{L^\infty(0,T;L^2(\Omega))} \) is uniformly bounded, Young’s inequality with parameters 4 and \( \frac{4}{3} \) implies
\[
\frac{d}{dt} \|\nabla u_k(t)\|_2^2 + v \|P_H \Delta u_k(t)\|_2^2 \leq c_3 \left( g_k(t) \|\nabla u_k(t)\|_2^2 + \|f(t)\|_2^2 \right),
\]  
(40)
with \( g_k(t) := \|\nabla u_k(t)\|_2^2 \). From (15), it follows that \( \int_0^T g_k(t) \, dt \) is uniformly bounded in \( k \) by a multiple of \( \|f\|_{L^2(0,T;L^2(\Omega))} \), whence the differential inequality for \( \|\nabla u_k(t)\|_2^2 \) obtained by ignoring the term \( v \|P_H \Delta u_k(t)\|_2^2 \) on the left-hand side above implies, by Lemma 1, the bound \( \|\nabla u_k(t)\|_2^2 \leq c_4(f) \) with \( c_4(f) \to 0 \) as \( \|f\|_{L^2(0,T;L^2(\Omega))} \to 0 \). With this pointwise bound, a return to (40) yields, after integration from 0 to \( T \), a similar uniform bound on \( \|P_H \Delta u_k(t)\|_2^2 \). The rest of the proof follows the proof of Theorem 3.

Remark 4. By classical Sobolev embedding, the strong solution from Theorem 3 \((n = 3)\) and Theorem 4 \((n = 2)\) satisfies
\[ \|u\|_{L^\infty_{\text{per}}(\mathbb{R};L^6(\Omega))} \leq \|\nabla u\|_{L^\infty_{\text{per}}(\mathbb{R};L^2(\Omega))}. \]
It therefore follows from (31) that the strong solution satisfies the uniqueness criteria (22) in Theorem 2 for sufficiently “small” data \( f \). In other words, the strong solution is unique in the class of weak solutions satisfying the energy inequality, provided the magnitude of the data is sufficiently small.
3.4 Regularity

While the pursuit of a minimal set of requirements that yield strong regularity for weak solutions has been an outstanding problem for the corresponding initial-value problem for decades, the time-periodic problem has attracted less attention over the years. Nevertheless, it is possible to establish virtually all the regularity results known for the initial-value problem also in the time-periodic case. The integrability criteria below is an example of one well-known result for the initial-value problem that is also valid in the time-periodic case.

**Theorem 5.** Let \( \Omega \subset \mathbb{R}^n \) \( (n = 2, 3) \) be a bounded domain of class \( C^2 \). Moreover, let \( f \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \). If a weak solution \( w \) to (1) satisfies \( w \in L^\infty_{\text{per}}(\mathbb{R}; L^2(\Omega)) \) and

\[
 w \in L^r_{\text{per}}(\mathbb{R}; L^s(\Omega)) \quad \text{for some} \quad r \in (2, \infty) \quad \text{and} \quad s \in (n, \infty] \quad \text{with} \quad \frac{n}{s} + \frac{2}{r} = 1,
\]

then \( w \) is a strong solution in the sense that \( (w, \pi) \) is a solution to (6) in the class (30) for some pressure \( \pi \).

**Proof.** Only the case \( n = 3 \) is treated. The case \( n = 2 \) follows by similar arguments. Consider the linearization

\[
\begin{cases}
\partial_t u + w \cdot \nabla u = \nu \Delta u - \nabla p + f & \text{in} \ \mathbb{R} \times \Omega, \\
\text{div} \ u = 0 & \text{in} \ \mathbb{R} \times \Omega, \\
u \ u = 0 & \text{on} \ \mathbb{R} \times \partial \Omega, \\
 u(t + T, x) = u(t, x)
\end{cases}
\]

of (6) around \( w \). Existence of a solution \( u \) to (42) is shown via a Galerkin approximation based on the same basis of eigenfunctions \( \{ \Psi_i \}_{i=1}^\infty \) for the Stokes operator \(- \mathcal{P}_H \Delta \) that was used in the proof of Theorem 3. To this end, define \( u_k \) by (9) with coefficients \( \alpha_{ki} \) chosen as a solution to the system of ordinary differential equations

\[
\begin{cases}
\frac{d\alpha_{kj}}{dt} = -\nu \sum_{i=1}^k (\nabla \Psi_i, \nabla \Psi_j) \alpha_{ki} - \sum_{i=1}^k (w \cdot \nabla \Psi_i, \Psi_j) \alpha_{ki} + (f(t), \Psi_j), \\
\alpha_{kj}(0) = a_j.
\end{cases}
\]
Proceeding as in the proof of Theorem 3, one obtains the identities

$$\| \nabla u_k \|_{L^2(0,T;L^2(\Omega))}^2 = \int_0^T \langle f(t), u_k(t) \rangle \, dt,$$  \hspace{1cm} (44)

$$\frac{1}{2} \frac{d}{dt} \| \nabla u_k(t) \|_2^2 + v \| P_H \Delta u_k(t) \|_2^2 = (w(t) \cdot \nabla u_k(t), P_H \Delta u_k(t)) - (f, P_H \Delta u_k(t)).$$  \hspace{1cm} (45)

$$\| \partial_t u_k(t) \|_2^2 + \frac{1}{2} \frac{d}{dt} \| \nabla u_k(t) \|_2^2 = (w(t) \cdot \nabla u_k(t), \partial_t u_k(t)) + (f, \partial_t u_k(t)).$$  \hspace{1cm} (46)

In order to estimate the nonlinear terms on the right-hand side of (45) and (46), the following observation is made for an arbitrary vector field $a$:

$$| (w \cdot \nabla u_k, a) | \leq \| w \|_s \| \nabla u_k \|_{\frac{2s}{2s-1}} \| a \|_2 \leq c_0 \| w \|_s \| \nabla u_k \|_2^{\frac{s-1}{s}} \| \nabla u_k \|_1^{\frac{1}{2}} \| a \|_2,$$  \hspace{1cm} (47)

which follows by first applying Hölder’s inequality and then a Sobolev inequality \cite[Lemma II.3.3 and Exercise II.3.12]{10}. By the standard elliptic estimate for the Stokes problem (see, e.g., \cite[Theorem IV.6.1]{10}), \( \| u_k \|_{L^2(T;L^2(\Omega))} \leq c_1 \| P_H \Delta u_k \|_2 \). One can thus employ Young’s inequality in (47) with parameters \( \frac{2s}{s-1}, \frac{2s}{s}, \) and 2 to deduce

$$| (w \cdot \nabla u_k, a) | \leq c_2(\varepsilon) \| w \|_s^s \| \nabla u_k \|_2^2 + \varepsilon \| P_H \Delta u_k \|_2^2 + \varepsilon \| a \|_2^2$$  \hspace{1cm} (48)

for any \( \varepsilon > 0 \). By a utilization of (48) with \( a = P_H \Delta u_k(t) \), it follows from (45) that

$$\frac{d}{dt} \| \nabla u_k(t) \|_2^2 + v \| P_H \Delta u_k(t) \|_2^2 \leq c_3 \| w \|_s^s \| \nabla u_k \|_2^2 + c_4 \| f \|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))}.$$  \hspace{1cm} (49)

Thus \( \| \nabla u_k(t) \|_2^2 \) satisfies a differential inequality, which by Lemma 1 in combination with (44) and the assumption that \( \int_0^T \| w(\tau) \|_s^2 \, d\tau < \infty \) implies a uniform bound \( \| \nabla u_k(t) \|_2^2 \leq c_5(f) \). With this estimate, one can return to (49), integrate from 0 to \( T \), and thereby obtain a similar bound on \( \| P_H \Delta u_k \|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))} \). By (33), the estimate \( \| \nabla^2 u_k \|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))} \leq c_6(f) \) follows. Finally, a uniform estimate on \( \| \partial_t u_k \|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))} \) is obtained by letting \( a = \partial_t u_k \) in (47) and using the resulting estimate in (46). Based on the uniform estimates in \( k \), one can let \( k \to \infty \) as in the proof of Theorem 1 and obtain a strong solution \( u = \lim_{k \to \infty} u_k \) to (42). The theorem can now be concluded by showing \( u = w \). The approach from the proof of Theorem 2 can be used for this purpose. More specifically, first multiply the first
equation in (42), which is satisfied by the strong solution \( u \) pointwise, a.e., with the weak solution \( w \), and integrate in space and time. After an integration by parts,

\[
\int_0^T \int_{\Omega} \partial_t u \cdot \mathbf{w} - (\mathbf{w} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} \, dx \, dt = \int_0^T \int_{\Omega} -\nu \nabla \mathbf{u} : \nabla \mathbf{w} + \mathbf{f} \cdot \mathbf{w} \, dx \, dt \tag{50}
\]

follows. On the other hand, by approximating \( u \) by test functions from \( C^0_0(\mathbb{R} \times \Omega) \) it follows from the regularity of a strong solution that \( u \) itself can be inserted as a test function in the weak formulation (7) for \( w \). This yields

\[
\int_0^T \int_{\Omega} -\mathbf{w} \cdot \partial_t \mathbf{u} - (\mathbf{w} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} \, dx \, dt = \int_0^T \int_{\Omega} -\nu \nabla \mathbf{u} : \nabla \mathbf{w} + \mathbf{f} \cdot \mathbf{u} \, dx \, dt. \tag{51}
\]

One can verify that both a strong solution \( u \) and a weak solution \( w \) satisfying (41) obey the energy inequality (8); recall Remark 3. As in the proof of Theorem 2, it therefore follows from the identity (28) and addition of (50) and (51) that

\[
\int_0^T \int_{\Omega} \left| \nabla \mathbf{u} - \nabla \mathbf{w} \right| \, dx \, dt \leq 0.
\]

Consequently \( u = w \), which concludes the theorem.

**Remark 5.** Further regularity properties of strong time-periodic solutions can be obtained from the well-known regularity theory for the initial-value problem. Indeed, once it has been established that a time-periodic solution \( (u, p) \) to (6) is in the class (30), it can be treated on any time-interval \([t_0, t_1]\) as a solution to the initial-value Navier-Stokes problem with initial value \( u(t_0) \in \mathcal{W}^{1,2} \). By well-known regularity theory for the initial-value problem, it is then possible to show that \( (u, p) \) is as regular as the data \( f \) allows for. In particular, for smooth data \( f \in C^0_{\text{per}}(\mathbb{R} \times \Omega) \), it follows that also \( u \in C^0_{\text{per}}(\mathbb{R} \times \Omega) \).

### 3.5 \( L^q \) Estimates for the Linearized Problem

Another way to study strong solutions to (6), both their existence and basic properties, is to treat the nonlinear terms as perturbations to the linearization of (6). For this purpose, \( L^q \) estimates of solutions to the linearization of (6) are essential. Consider therefore the linearization, also referred to as the time-periodic Stokes problem,

\[
\begin{aligned}
\partial_t \mathbf{u} &= \nu \Delta \mathbf{u} - \nabla p + \mathbf{f} \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } \mathbf{u} &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
\mathbf{u} &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega, \\
\mathbf{u}(t + T, x) &= \mathbf{u}(t, x).
\end{aligned} \tag{52}
\]
Existence of a solution $u$ to (52) with a certain amount of regularity can be shown with the same Galerkin approximation that was used in the proof of Theorem 1. In order to establish $L^q$ estimates for this solution, a representation formula of some sort with respect to the data is typically needed. One such representation can be obtained by identifying $u$ as the solution to the initial-value Stokes problem

$$
\begin{cases}
\partial_t u = \nu \Delta u - \nabla p + f & \text{in } (t_0, \infty) \times \Omega, \\
\text{div } u = 0 & \text{in } (t_0, \infty) \times \Omega, \\
u = 0 & \text{on } (t_0, \infty) \times \partial \Omega, \\
u(t_0, \cdot) = u(t_0, \cdot)
\end{cases}
$$

for some $t_0 \in \mathbb{R}$. A representation formula and corresponding $L^q$ estimates are well known for the solution to this initial-value problem. However, these $L^q$ estimates necessarily involve the initial value $u(t_0, \cdot)$ on the right-hand side, which is clearly not admissible in a proper $L^q$ estimate for the time-periodic problem. To nevertheless take advantage of the estimates for the initial-value problem, the idea was introduced in [13] to identify the solution to (53) as a sum $u = v + w$ of solutions to

$$
\begin{cases}
\partial_t v = \nu \Delta v - \nabla p & \text{in } (t_0, \infty) \times \Omega, \\
\text{div } v = 0 & \text{in } (t_0, \infty) \times \Omega, \\
v = 0 & \text{on } (t_0, \infty) \times \partial \Omega, \\
v(t_0, \cdot) = u(t_0, \cdot)
\end{cases}
$$

and

$$
\begin{cases}
\partial_t w = \nu \Delta w - \nabla \pi + f & \text{in } (t_0, \infty) \times \Omega, \\
\text{div } w = 0 & \text{in } (t_0, \infty) \times \Omega, \\
w = 0 & \text{on } (t_0, \infty) \times \partial \Omega, \\
w(t_0, \cdot) = 0 & \text{in } \Omega,
\end{cases}
$$

respectively, for an appropriately chosen initial time $t_0$. By showing that $v$ tends to 0 in the relevant norm as $t \to \infty$, time-periodic $L^q$ estimates for $u$ can be established from $L^q$ estimates of $w$.

**Theorem 6.** Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ be a bounded domain of class $C^2$. Let $q \in (1, \infty)$. For any $f \in L^q_{\text{per}}(\mathbb{R} \times \Omega)$, there is a solution $(u, p) \in W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) \times W^{0,1,q}_{\text{per}}(\mathbb{R} \times \Omega)$ to (52) which satisfies

$$
\|u\|_{1,2,q} + \|p\|_{0,1,q} \leq C_2 \|f\|_q,
$$

respectively.
with $C_2 = C_2(\Omega, v, q)$. Moreover, if $r \in (1, \infty)$ and $(\mathbf{w}, \pi) \in W^{1,2,r}_{\text{per}}(\mathbb{R} \times \Omega) \times W^{0,1,r}_{\text{per}}(\mathbb{R} \times \Omega)$ is another solution, then $\mathbf{w} = \mathbf{w}$ and $\pi = \pi + d(t)$ for some $T$-periodic function $d : \mathbb{R} \to \mathbb{R}$.

**Proof.** Only the case $n = 3$ is treated. Similar arguments can be used to establish the result for $n = 2$. By density of $C_0^{\infty}_{\text{per}}(\mathbb{R} \times \Omega)$ in $L^q_{\text{per}}(\mathbb{R} \times \Omega)$, it suffices to consider $f \in C_0^{\infty}_{\text{per}}(\mathbb{R} \times \Omega)$. As mentioned above, a solution to (52) can be obtained by a Galerkin approximation as in the proofs of Theorems 1 and 3. Due to the absence of a nonlinear term in this case, the arguments from the proof of Theorem 3 can even be applied without any smallness assumption on the “size” of the data to establish a strong solution to (52) with $u \in L^2_{\text{per}}(\mathbb{R}; W^{3,2}(\Omega))$, $\partial_t u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))$, and $p \in L^2_{\text{per}}(\mathbb{R}; W^{1,2}(\Omega))$. By Sobolev’s embedding theorem, it may be assumed that

$$
\forall t > t_0 : \quad \| \partial_t v(t) \|_q + \| A v(t) \|_q \leq c_0 (t - t_0)^{-1}.
$$

(57)

Also by classical results (see, e.g., [20, Theorem 2.8]), there exists a solution $(\mathbf{w}, \pi) \in W^{1,2,q}((t_0, \infty) \times \Omega) \times W^{0,1,q}((t_0, \infty) \times \Omega)$ to (55) that is continuous in the sense that $u \in C_{\text{per}}([t_0, \infty); L^2_{\omega}(\Omega))$ and satisfies

$$
\forall \tau \in (t_0, \infty) : \quad \| \mathbf{w} \|_{W^{1,2,q}((t_0, \tau) \times \Omega)} + \| \pi \|_{W^{0,1,q}((t_0, \tau) \times \Omega)} \leq c_1 \| f \|_{L^q((t_0, \tau) \times \Omega)}
$$

(58)

with $c_1$ independent on $\tau$. Since both $u$ and $v + w$ solve (53), a standard uniqueness argument implies $u = v + w$. Due to the $T$-periodicity of $u$ and $f$, it follows for all $m \in \mathbb{N}$ that

$$
\int_0^T \| \partial_t u(t) \|_q^q + \| A u(t) \|_q^q \, dt = \frac{1}{m} \int_{2T}^{(m+2)T} \| \partial_t u(t) \|_q^q + \| A u(t) \|_q^q \, dt
$$

$$
\leq c_2 \frac{1}{m} \int_T^\infty t^{-q} \, dt + c_3 \frac{1}{m} \| f \|_{L^q((0,(m+1)T) \times \Omega)}^q
$$

$$
\leq c_4 \frac{1}{m^{q-1}} T^{1-q} + c_3 \frac{m + 1}{m} \| f \|_{L^q((0,T) \times \Omega)}^q.
$$

(59)

Now let $m \to \infty$ to conclude

$$
\| \partial_t u \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} + \| A u \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} \leq c_5 \| f \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)}.
$$
The estimate $\| \nabla^2 u \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} \leq c_6 \| Au \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)}$ is a consequence of well-known $L^q$ theory for the Stokes problem in bounded domains; see, for example, [10, Theorem IV.6.1]. Consequently, the estimate $\| u \|_{1,2,q} \leq c_7 \| f \|_{q}$ follows by employing Poincaré’s inequality. Now modify the pressure $p$ by adding a function depending only on $t$ such that $\int_{\Omega} p(t,x) \, dx = 0$, which ensures the validity of Poincaré’s inequality for $p$. A similar estimate is then obtained for $p$ by isolating $\nabla p$ in (52). This establishes (56). To show the statements of uniqueness, a duality argument can be employed. For this purpose, let $\varphi \in C_{0,\text{per}}(\mathbb{R} \times \Omega)$ and let $(\psi, \eta)$ be a solution to the problem

$$
\begin{aligned}
\begin{cases}
\partial_t \psi = -v \Delta \psi - \nabla p + \varphi & \text{in } \mathbb{R} \times \Omega, \\
div \psi = 0 & \text{in } \mathbb{R} \times \Omega, \\
\psi = 0 & \text{on } \mathbb{R} \times \partial \Omega, \\
\psi(t + T, x) = \psi(t, x)
\end{cases}
\end{aligned}
$$

adjoint to (52). The existence of a solution $(\psi, \eta)$ follows by the same arguments that yield a solution to (52). Since $\varphi \in C_{0,\text{per}}(\mathbb{R} \times \Omega)$, the solution satisfies $(\psi, \eta) \in W^{1,2}_\text{per}(\mathbb{R} \times \Omega) \times W^{1,s}_\text{per}(\mathbb{R} \times \Omega)$ for all $s \in (1, \infty)$. The regularity of $(\psi, \eta)$ ensures validity of the following computation:

$$
\int_0^T \int_\Omega (w - \bar{w}) \cdot \varphi \, dx \, dt = \int_0^T \int_\Omega (w - \bar{w}) \cdot (\partial_t \psi + v \Delta \psi + \nabla p) \, dx \, dt
\begin{equation}
= \int_0^T \int_\Omega (\partial_t [w - \bar{w}] - v \Delta [w - \bar{w}] + \nabla [\pi - \bar{\pi}]) \cdot \psi \, dx \, dt = 0.
\end{equation}
$$

Since $\varphi \in C_{0,\text{per}}(\mathbb{R} \times \Omega)$ was arbitrary, $\bar{w} - w = 0$ follows. In turn, $\nabla \pi = \nabla \bar{\pi}$ and thus $\bar{\pi} = \pi + d(t)$ follows. 

4 Exterior Domains

Investigation of the Navier-Stokes equations in an exterior domain is of particular interest, as the system in this case describes a fluid flow around an object. Consider therefore an exterior domain $\Omega \subset \mathbb{R}^n$, that is, the complement of some closed simply connected set $B$, which one may consider as a physical object. In this case, the Navier-Stokes equations in $\Omega$ describe a fluid flow around $B$. An asymptotic value $-u_\infty$ is prescribed for the velocity at spatial infinity, that is, in addition to the Navier-Stokes equations (1) and the boundary condition (2), also (3) is taken into consideration. In physical terms, $u_\infty \in \mathbb{R}^n$ describes the velocity of the object $B$. Consequently, $u_\infty = 0$ models an object at rest, whereas $u_\infty \neq 0$ models an object moving in the fluid. In the time-periodic setting, a time-periodic velocity $u_\infty(t)$ is considered.
For convenience, in this section, the time-periodic Navier-Stokes equations (1) (2), and (3) shall be expressed in terms of the relative velocity \( u + u_\infty \), which results in the following problem:

\[
\begin{aligned}
\frac{\partial_t}{\partial t} u + (u - u_\infty) \cdot \nabla u &= \nu \Delta u - \nabla p + f & \text{in } \mathbb{R} \times \Omega, \\
\text{div } u &= 0 & \text{in } \mathbb{R} \times \Omega, \\
u u &= u_\infty & \text{on } \mathbb{R} \times \partial \Omega, \\
u u(t + T, x) &= u(t, x), \\
\lim_{|x| \to \infty} u(t, x) &= 0.
\end{aligned}
\] (61)

Questions regarding existence and regularity of solutions to (61) shall be addressed. Concerning regularity not only local regularity but also the rate and type of convergence with which the limit \( \lim_{|x| \to \infty} u(t, x) = 0 \) is attained is an issue. Several mathematical and physical properties of the solution can be derived from the asymptotic structure of \( u(t, x) \) as \( |x| \to \infty \).

4.1 Sobolev Spaces

In order to capture decay properties at spatial infinity on the level of function spaces, additional Sobolev spaces are introduced. Although the notation \( \Omega \subset \mathbb{R}^n \) is used throughout Sect. 4 to denote an exterior domain, in this subsection, \( \Omega \) may also denote the whole-space \( \mathbb{R}^n \).

A key to the exploration of decay properties lies in two simple complementary projections \( \mathcal{P} \) and \( \mathcal{P}_\perp \) defined for sufficiently regular functions by

\[
\mathcal{P} u(t, x) := \frac{1}{T} \int_0^T u(s, x) \, ds \quad \text{and} \quad \mathcal{P}_\perp u(t, x) := u(t, x) - \mathcal{P} u(t, x). \quad (62)
\]

Note that \( \mathcal{P} \) and \( \mathcal{P}_\perp \) decompose a time-periodic vector field \( u \) into a time-independent part \( \mathcal{P} u \) and a time-periodic part \( \mathcal{P}_\perp u \) with vanishing time-average over the period. Vector fields of the former type are referred to as steady states, and vector fields of the latter type as oscillatory. The projections are further used to decompose Lebesgue and Sobolev spaces of \( T \)-time-periodic functions. In the following, time-independent vector fields defined on \( \mathbb{R} \times \Omega \) are implicitly identified as functions over \( \Omega \). The identification \( \mathcal{P} L^q_{\text{per}}(\mathbb{R} \times \Omega) = L^q(\Omega) \) can thus be made. The projection \( \mathcal{P}_\perp \) induces subspaces of oscillatory functions:

\[
L^q_{\text{per}, \perp}(\mathbb{R} \times \Omega) := \mathcal{P}_\perp L^q_{\text{per}}(\mathbb{R} \times \Omega) = \{ f \in L^q_{\text{per}}(\mathbb{R} \times \Omega) \mid \mathcal{P} f = 0 \},
\]

\[
W^{1,2,q}_{\text{per}, \perp}(\mathbb{R} \times \Omega) := \mathcal{P}_\perp W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) = \{ u \in W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) \mid \mathcal{P} u = 0 \}.
\]
As one readily verifies, $\mathcal{P}$ and $\mathcal{P}_{\perp}$ are continuous projections on both $L^q_{\text{per}}(\mathbb{R} \times \Omega)$ and $W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega)$. Consequently, the subspaces introduced above are Banach spaces.

Investigation of (61) involves a decomposition via $\mathcal{P}$ and $\mathcal{P}_{\perp}$ of the system into a steady-state and an oscillatory problem, respectively. Recall for this purpose the following Sobolev-type space from the theory for the linearized steady-state Navier-Stokes equations in exterior domains ($\lambda > 0$):

$$
\|v\|^{\lambda}_{X(q)} := \begin{cases} 
\lambda^{\frac{1}{2}}\|v\|_{L^q_{\text{loc}}} + \lambda^{\frac{1}{2}}\|\nabla v\|_{L^q_{\text{loc}}} + \lambda\|\partial_1 v\|_q + \|\nabla^2 v\|_q & \text{if } n = 3 \text{ and } q \in (1, 2), \\
\lambda^{\frac{2}{3}}\|v\|_{L^q_{\text{loc}}} + \lambda^{\frac{1}{3}}\|\nabla v\|_{L^q_{\text{loc}}} + \lambda\|\partial_1 v\|_q + \|\nabla^2 v\|_q & \text{if } n = 2 \text{ and } q \in \left(1, \frac{3}{2}\right).
\end{cases}
$$

$X(q) := \{v \in L^q_{\text{loc}}(\Omega) \mid \|v\|^{\lambda}_{X(q)} < \infty\}$.

Also homogeneous Sobolev spaces are needed (in addition to the space $D^{1,2}_{0,\text{per}}(\Omega)$ defined in Sect. 2). For this purpose, fix a constant $R_0 > 0$ such that $\mathbb{R}^n \setminus \Omega \subset \subset B_{R_0}$, and recall for $q \in [1, \infty)$ that

$$
\langle p \rangle_{1,q} := \|\nabla p\|_q + \left|\int_{\Omega \setminus B_{R_0}} p(x) \, dx\right|, \quad \langle p \rangle_{0,1,q} := \|\nabla p\|_q + \int_0^T \left|\int_{\Omega \setminus B_{R_0}} p(t,x) \, dx\right| \, dt
$$

defines a norm on $C^\infty_0(\Omega)$ and $C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)$, respectively. Classical and time-periodic homogeneous Sobolev spaces can thus be introduced as the Banach spaces

$$
D^{1,q}(\Omega) := \overline{C^\infty_0(\Omega)}_{L^q(\Omega)}, \quad D^{1,q}_{\text{per}}(\mathbb{R} \times \Omega) := \overline{C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)}_{L^q(\Omega)},
$$

respectively. It is easy to verify that the latter space coincides with the canonical $T$-periodic extension of functions in $L^q((0,T); D^{1,q}(\Omega))$. Both characterizations, as well as the subspace

$$
D^{1,q}_{\text{per},\perp}(\mathbb{R} \times \Omega) := \mathcal{P}_{\perp} D^{1,q}_{\text{per}}(\mathbb{R} \times \Omega),
$$

are used in the following.

### 4.2 Existence of Weak Solutions

Existence of weak solutions to (61) in the case $\Omega$ is a three-dimensional exterior domain can be established without any restrictions on the “size” of the data. The approach employed below is sometimes referred to as the *invading domain technique*. It is based the Galerkin approximation used in the proof of Theorem 1.
Firstly, a definition of a weak solution to (61) is needed. Compared to Definition 1 in the case a bounded domain $\Omega$, the concept of a weak solution in the exterior domain case has to incorporate also the decay property $\lim_{|x| \to \infty} u(t, x) = 0$ of the solution. The definition below is a simple extension of a similar definition in the steady-state case [10, Definition X.1.1].

**Definition 2.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Let $f \in L^2_{\text{per}}(\mathbb{R}; D_0^{-1,2}(\Omega))$. A vector field $u \in L^2_{\text{per}}(\mathbb{R}; D^{1,2}(\Omega))$ is called a weak time-periodic solution to (61) if $\text{div} u = 0$, $u = u_\infty$ on $\mathbb{R} \times \partial \Omega$ in the trace sense, the identity

$$
\int_0^T -\langle u, \partial_t \varphi \rangle + \nu \langle \nabla u, \nabla \varphi \rangle + \langle (u - u_\infty) \cdot \nabla u, \varphi \rangle - \langle f, \varphi \rangle \, dt = 0
$$

holds for all $\varphi \in C^\infty(\mathbb{R} \times \Omega)$, and $u$ satisfies for almost all $t \in \mathbb{R}$ the decay property

$$
\lim_{R \to \infty} \frac{1}{|\partial B_R|} \int_{\partial B_R} |u(t, x)| \, d\sigma(x) = 0.
$$

In order to employ a Galerkin approximation to establish existence of a weak solution, it is necessary to first “lift” the non-homogeneous boundary values in (61), that is, find a suitable extension of the boundary values and subtract it from $u$ to obtain an equivalent system with homogeneous boundary data. It is critical that the terms in which the extension appears in the new system can be suitably estimated in each step of the Galerkin approximation. A well-known method, which goes back to Leray and Hopf, can be modified to construct an appropriate extension.

**Lemma 2.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^{0,1}$. Let $u_\infty \in W^{1,2}_{\text{per}}(\mathbb{R})$. For every $\varepsilon > 0$, there is a vector field $\mathcal{W}_\varepsilon \in W^{1,2}_{\text{per}}(\mathbb{R}; W^{2,2}(\Omega))$ satisfying $\mathcal{W}_\varepsilon = u_\infty$ on $\partial \Omega$, $\text{div} \mathcal{W}_\varepsilon = 0$ in $\Omega$,

$$
\| \mathcal{W}_\varepsilon \|_{W^{1,2}_{\text{per}}(\mathbb{R}; W^{2,2}(\Omega))} \leq C_3 \| u_\infty \|_{1,2},
$$

and

$$
\forall \varphi \in C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega) : \int_{\Omega} (\varphi(t, x) \cdot \nabla \mathcal{W}_\varepsilon(t, x), \varphi(t, x)) \, dx \leq \varepsilon \| \nabla \varphi(t) \|_2^2.
$$

**Proof.** Let $\psi_\varepsilon \in C^\infty(\Omega; \mathbb{R})$ be a “cutoff” function with $\psi_\varepsilon = 1$ in a neighborhood of $\partial \Omega$. It follows directly that

$$
\mathcal{W}_\varepsilon(t, x) := \nabla \times \begin{pmatrix}
\psi_\varepsilon(x) \cdot x_3 \cdot u_{\infty 2}(t) \\
\psi_\varepsilon(x) \cdot x_1 \cdot u_{\infty 3}(t) \\
\psi_\varepsilon(x) \cdot x_2 \cdot u_{\infty 1}(t)
\end{pmatrix}
$$
satisfies all the desired properties except (67). To obtain (67), \( \psi_\varepsilon \) needs to choose in a certain way. For example, one may choose \( \psi_\varepsilon \) as in [10, Lemma III.6.2].

**Remark 6.** Lemma 2 can be extended to yield solenoidal extensions for a much larger class of time-periodic boundary values \( u_\ast(t, x), (t, x) \in \mathbb{R} \times \partial \Omega \). For example, [10, Lemma X.4.1] can be modified to include time-periodic boundary values, which would then produce such a class. All results in this section continue to hold if the boundary condition \( u = u_\infty \) in (61) is replaced with \( u = u_\ast \) for a time-periodic vector field \( u_\ast \) that has a solenoidal extension with the properties from Lemma 2.

With the lemma above, existence of a weak solution to (61) can be shown with the *invading domain* technique. The main idea is to apply a Galerkin approximation to show existence of a solution on the bounded domain \( \Omega \cap B_R(0) \). After securing a priori estimates independent on \( R \), a weak solution is found by a limiting process \( R \to \infty \).

**Theorem 7.** Let \( \Omega \subset \mathbb{R}^3 \) be an exterior domain of class \( C^{0,1} \). If \( u_\infty \in W^{1,2}_{\text{per}}(\mathbb{R}) \) and \( f \in L^2_{\text{per}}(\mathbb{R}; D_0^{1,2}(\Omega)) \), then there is a weak time-periodic solution \( u \in L^2_{\text{per}}(\mathbb{R}; D^{1,2}(\Omega)) \) to (61) in the sense of Definition 2.

**Proof.** Let \( \mathcal{W}_\varepsilon \) be the extension field from Lemma 2. Existence of a solution to (61) on the form \( u = w + \mathcal{W}_\varepsilon \) can be obtained by finding a solenoidal vector field \( w \in L^2_{\text{per}}(\mathbb{R}; D^{1,2}_{0,\sigma}(\Omega)) \) satisfying

\[
\int_0^T - (w, \partial_t \phi) + v(\nabla w, \nabla \phi) + (w \cdot \nabla w, \phi) + ((\mathcal{W}_\varepsilon - u_\infty) \cdot \nabla w, \phi) \, dt = \int_0^T - (w \cdot \nabla \mathcal{W}_\varepsilon, \phi) + (\mathcal{f}, \phi) \, dt
\]

(68)

for all \( \phi \in C^\infty_{0,\sigma,\text{per}}(\mathbb{R} \times \Omega) \), where

\[
\langle \mathcal{f}, \phi \rangle := \langle f, \phi \rangle - \langle \partial_t \mathcal{W}_\varepsilon + (\mathcal{W}_\varepsilon - u_\infty) \cdot \nabla \mathcal{W}_\varepsilon, \phi \rangle - v(\nabla \mathcal{W}_\varepsilon, \nabla \phi).
\]

(69)

Let \( \{R_k\}_{k=1}^\infty \subset \mathbb{R} \) be a sequence with \( R_k \to \infty \) as \( k \to \infty \). Put \( \Omega_k := \Omega \cap B_{R_k}(0) \). Since \( \Omega_k \) is bounded, a solution \( w_k \in L^2_{\text{per}}(\mathbb{R}; W^{1,2}_{0,\sigma}(\Omega_k)) \) to (68) with respect to test functions \( \phi \in C^\infty_{0,\sigma,\text{per}}(\mathbb{R} \times \Omega_k) \) can be obtained as in the proof of Theorem 1. For this purpose, choose \( \varepsilon \leq \frac{1}{2} v \) and employ (67) in order to deduce an identity equivalent to (11). By repeating the rest of the argument from the proof of Theorem 1, a solution \( w_k \) that satisfies the equivalent to (15), which due to term \( (w \cdot \nabla \mathcal{W}_\varepsilon, \phi) \) on the right-hand side in (68) becomes an inequality

\[
\|w_k\|_{L^2_{\text{per}}(\mathbb{R}; D^{1,2}_{0,\sigma}(\Omega_k))} \leq \frac{2}{v} \int_0^T \langle \mathcal{f}, w_k \rangle \, dt,
\]
is then obtained. Provided the vector field $w_k$ is extended by 0 on $\mathbb{R}^3 \setminus \Omega_k$, it follows that $\{w_k\}_{k=1}^{\infty}$ is bounded in $L^2_{\text{per}}(\mathbb{R}; D^{1,2}_{0,0}(\Omega))$. Consequently, there is a subsequence of $\{w_k\}_{k=1}^{\infty}$ that converges weakly in this space to a vector field $w \in L^2_{\text{per}}(\mathbb{R}; D^{1,2}_{0,0}(\Omega))$. To ensure that $w$ satisfies (68), it suffices to verify that $w_k$ converges strongly in $L^2_{\text{per}}(\mathbb{R}; L^2(\Omega_R))$ for any $R > R_0$. This will in particular ensure convergence of the nonlinear term $(w_k \cdot \nabla w_k, \nabla \varphi)$. The verification can be done by deducing from (68) that $\partial_t w_k$ is bounded in $L^1_{\text{per}}(\mathbb{R}; D^{-1,2}_{0,0}(\Omega_R))$, and utilize that $w_k$ thereby lies in a space that embeds compactly into $L^2_{\text{per}}(\mathbb{R}; L^2(\Omega_R))$.

Finally, since $\Omega$ is a three-dimensional exterior domain, the decay property (65) follows directly from the fact that $w(t) \in D^{1,2}_{0,0}(\Omega)$ and $\mathcal{W}(t) \in W^{2,2}(\Omega)$; see [10, Lemma II.6.3].

**Remark 7.** A similar result is open in the two-dimensional case $n = 2$. Although the arguments in the proof of Theorem 7 that ensure existence of a vector field satisfying (64) are all valid also when $n = 2$, it is not clear if this field satisfies the decay property (65). The same problem has been open for the corresponding steady-state problem for decades. Recall that the steady-state problem is a special case of the time-periodic problem.

### 4.3 Existence of Strong Solutions

Compared to the bounded domain case, integrability properties are a more delicate matter in unbounded domains such as exterior domains, since they describe not only local regularity but also decay properties as $|x| \to \infty$ of the solution. Consequently, different characterizations of strong solutions transpire. Below, a class (71), (72), and (73) of strong solutions is introduced that emerges from adaptation of the methods from Sect.3.3 to the exterior domain case. By modification of the Galerkin approximation from the proof of Theorem 7, existence of a strong solution of this type to (61) for data $f$ and $u_1$ sufficiently restricted in “size” is established in the three-dimensional case $n = 3$. Later, in Sect. 4.8, a different class of strong solutions based on $L^q$ estimates for the linearization of (61) is treated. In comparison, the method based on $L^q$ theory for the linearized problem yields better decay properties of the solution, while the method based on Galerkin approximation is more versatile when it comes to the admissible structure of $u_\infty$.

**Theorem 8.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^2$. There is a constant $\varepsilon_2(\Omega, v, T) > 0$ such that if $f \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \cap L^2_{\text{per}}(\mathbb{R}; D^{-1,2}_{0,0}(\Omega))$ and $u_\infty \in W^{1,2}_{\text{per}}(\mathbb{R})$ satisfy

$$\|f\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))} + \|f\|_{L^2_{\text{per}}(\mathbb{R}; D^{-1,2}_{0,0}(\Omega))} + \|u_\infty\|_{1,2} \leq \varepsilon_2.$$  

(70)
then there is a solution \((u, p)\) to (61) that satisfies

\[
\begin{align*}
  u &\in L^2_{\text{per}}(\mathbb{R}; D^{1,2}(\Omega)), \quad \nabla u \in L_{\text{per}}^\infty(\mathbb{R}; L^2(\Omega)), \\
  \nabla^2 u &\in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)), \quad \partial_t u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)), \\
  \mathcal{P} u &\in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)), \quad p \in L^2_{\text{per}}(\mathbb{R}; D^{1,2}(\Omega)).
\end{align*}
\]

(71) \hspace{1cm} (72) \hspace{1cm} (73)

**Proof.** Again, the “lifting” field \(\mathcal{W}_\varepsilon\) from Lemma 2 is utilized. If \(w\) is a solution to (68) for all \(\varphi \in C^0_{0,\sigma, \text{per}}(\mathbb{R} \times \Omega)\), then \(u := w + \mathcal{W}_\varepsilon\) is a solution to (61). Since \(\mathcal{W}_\varepsilon\) satisfies (71), (72), and (73), it suffices to verify that also \(w\) satisfies (71), (72), and (73). To obtain a vector field \(w\) with the desired properties, one proceeds as in the proof of Theorem 7 and employs a Galerkin approximation to first solve (68) on an ascending sequence of bounded domains \(\Omega_k = \Omega \cap B_{R_k}, \lim_{k \to \infty} R_k \to \infty\). On each \(\Omega_k\), a solution \(w_k\) with \(\|w_k\|_{L^2_{\text{per}}(\mathbb{R}; D^{1,2}_{0,\sigma}(\Omega))}\) bounded independently on \(k\) is thereby obtained. Furthermore, the argument from the proof of Theorem 3 can be reused to deduce that \(w_k\) is a strong solution. One may verify, using, for example, a simple scaling argument, that the constants \(c_0\) and \(c_1\) in the proof of Theorem 3 are independent on \(k\). By repeating the proof of Theorem 3 up till (37) and taking into consideration the additional terms in (68) containing the “lifting” field \(\mathcal{W}_\varepsilon\), which is not present in the proof of Theorem 3, one obtains the estimate

\[
\|\nabla w_k(t)\|_2^2 \leq c_0 \left( \|\mathbf{f}\|_{L^2(0,T; D^{1,2}_0(\Omega))}^2 + \|\mathbf{f}\|_{L^2(0,T; L^2(\Omega))}^2 \right)
\]

\[
+ \int_{s_0}^t \left( \|\mathcal{W}_\varepsilon(s)\|_{2,2}^2 + |u_\infty(s)|^2 \right) \|\nabla w_k(s)\|_2^2 \, ds
\]

\[
+ \int_{s_0}^t \|\nabla w_k(s)\|_4^4 + \|\nabla w_k(s)\|_6^6 \, ds
\]

(74)

with \(\mathbf{f}\) defined as in (69) and the constant \(c_0\) independent on \(k\). Now take \(\varepsilon := \varepsilon_2\), with \(\varepsilon_2\) still to be chosen. Then (66) and the “smallness” assumption (70) furnish

\[
\|\nabla w_k(t)\|_2^2 \leq c_1 \left( \varepsilon_2^2 + \int_{s_0}^t \varepsilon_2^2 \|\nabla w_k(s)\|_2^2 + \|\nabla w_k(s)\|_2^4 + \|\nabla w_k(s)\|_6^6 \, ds \right)
\]

(75)

again with the constant \(c_1\) independent on \(k\). Based on the inequality (75), the argument following (37) in the proof Theorem 3 yields that \(\|\nabla w_k\|_{L^\infty_\text{per}(\mathbb{R}; L^2(\Omega))}\), \(\|\nabla^2 w_k\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))}\), and \(\|\partial_t w_k\|_{L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))}\) are bounded independently on \(k\), provided \(\varepsilon_2\) is chosen sufficiently small. At this point, it is therefore possible to let \(k \to \infty\). After possibly passing to a subsequence, one finds as a weak limit of \(\{w_k\}_{k=1}^\infty\) in the spaces (71) and (72) a solenoidal vector field \(w\). Clearly, \(u := w + \mathcal{W}_\varepsilon\) satisfies (61) with respect to test functions \(\varphi \in C^\infty_{0,\sigma, \text{per}}(\mathbb{R} \times \Omega)\). By a
standard method from the well-known analysis of the corresponding initial-value problem, or alternatively by the approach described in Remark 2, the existence of a pressure \( p \in L^2_{\text{per}}(\mathbb{R}; D^{1,2}(\Omega)) \) that renders \( (u, p) \) a solution to (61) follows. It remains to show \( P_{\perp} u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \). For this purpose, expand \( u \) into a Fourier series \( u(t) = \sum_{h \in \mathbb{Z}} u_h e^{i \frac{2\pi}{T} ht} \), and deduce from \( \partial_t u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \) that \( \{hu_h\}_{h \in \mathbb{Z}} \in \ell^2(L^2(\Omega)) \). Consequently, \( \{u_h\}_{h \in \mathbb{Z} \setminus \{0\}} \in \ell^2(L^2(\Omega)) \). Since the latter sequence can be recognized as the Fourier coefficients of \( P_{\perp} u \), it follows that \( P_{\perp} u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega)) \).

4.4 New Approach to \( L^q \) Estimates for the Linearized Problem in the Whole Space

While the strong solutions to (61) established in Theorem 8 are locally very regular, little information is revealed about their integrability at spatial infinity, that is, the rate of decay as \( |x| \to \infty \). Only when the asymptotic behavior at spatial infinity is known can it be determined whether the fluid flow described by a solution is meaningful from a physical point of view. At the outset, no such information is available for the solution from Theorem 8. To gain more insight, global \( L^q \) estimates of solutions to an appropriate linearization of (61) in terms of the data are needed. Such estimates can be used to extract information from a solution to (61) directly or to establish, for example, by a fixed point argument, existence of a (strong) solution to (61) for which the decay as \( |x| \to \infty \) is better understood. An \( L^q \) theory for an exterior domain problem is usually derived via \( L^q \) estimates for the corresponding whole-space problem. In the following, the linearized time-periodic Navier-Stokes system in the whole-space

\[
\begin{aligned}
\partial_t u - \nu \Delta u - \lambda \partial_1 u + \nabla p &= F \quad \text{in } \mathbb{R} \times \mathbb{R}^n, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n, \\
u(t + T, x) &= u(t, x), \\
\lim_{|x| \to \infty} u(t, x) &= 0,
\end{aligned}
\]

shall be investigated. Here, \( \lambda \geq 0 \) is a constant.

In the context of \( L^q \) estimates for (76), one has to distinguish between the two cases \( \lambda = 0 \) and \( \lambda \neq 0 \). In the former case, the system is referred to as a time-periodic Stokes problem and in the latter as a time-periodic Oseen problem. It is well known from the corresponding steady-state problem that \( L^q \) estimates for the Stokes and Oseen problem are different. From a physical point of view, the discrepancy is not surprising as the Oseen equations model a fluid flow past a moving body, which therefore should exhibit a wake region behind the body, whereas a Stokes flow describes a flow without a wake region around a stationary body.
A simple but important step toward optimal $L^q$ estimates for (76) lies in the decomposition of (76) by the complementary projections $\mathcal{P}$ and $\mathcal{P}_\perp$ introduced in (62). If the velocity field and pressure term are expressed as

$$u = \mathcal{P}u + \mathcal{P}_\perp u =: v + w, \quad p = \mathcal{P}p + \mathcal{P}_\perp p =: p + \pi,$$

one easily verifies that $(v, p)$ is a solution to the steady-state problem

$$\begin{cases}
- \nu \Delta v - \lambda \partial_1 v + \nabla p = \mathcal{P}F & \text{in } \mathbb{R}^n, \\
\text{div } v = 0 & \text{in } \mathbb{R}^n, \\
& \lim_{|x| \to \infty} v(x) = 0, \quad (77)
\end{cases}$$

and $(w, \pi)$ a solution to the time-periodic problem

$$\begin{cases}
\partial_t w - \nu \Delta w - \lambda \partial_1 w + \nabla \pi = \mathcal{P}_\perp F & \text{in } \mathbb{R} \times \mathbb{R}^n, \\
\text{div } w = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\
w(t + T, x) = w(t, x), \\
& \lim_{|x| \to \infty} w(t, x) = 0. \quad (78)
\end{cases}$$

Since (78) resembles (76), not much insight seems to have been won by this decomposition. It turns out, however, that in the context of $L^q$ estimates, system (78) has some remarkable characteristics. Since both the data $\mathcal{P}_\perp F$ and the solution $(w, \pi)$ have vanishing time average over the period, that is, they are purely oscillatory, an analysis of (78) can be carried out in subspaces of $L^q$ consisting entirely of oscillatory functions. As a result, much better $L^q$ estimates materialize for $(w, \pi)$ than can be shown for a solution to the original problem (76). An optimal $L^q$ theory for the time-periodic problem (76) can be established by combining these estimates for $(w, \pi)$ with well-known $L^q$ estimates for the solution $(v, p)$ to the steady-state problem (77).

Below, it is described how to establish the $L^q$ estimates for $(w, \pi)$ by means of Fourier analysis. Alternatively, one can also establish the estimates by extending the method used in the proof of Theorem 6. In comparison, the approach below is more direct as it is based on a direct representation of the solution in terms of a Fourier multiplier. Moreover, it leads naturally to the concept of a fundamental solution for the time-periodic problem. The main idea is to reformulate (78) as partial differential equation on the locally compact abelian group $G := \mathbb{R} / T\mathbb{Z} \times \mathbb{R}^n$ and analyze the problem with the corresponding Fourier transform $\mathcal{F}_G$.

**Theorem 9.** Let $q \in (1, \infty)$. For every $F \in L^q_{\text{per}, \perp} (\mathbb{R} \times \mathbb{R}^n)$, there is a solution

$$(w, \pi) \in W^{1,2,q}_{\text{per}, \perp} (\mathbb{R} \times \mathbb{R}^n) \times W^{1,2,q}_{\text{per}, \perp} (\mathbb{R} \times \mathbb{R}^n) \times D^{1,q}_{\text{per}, \perp} (\mathbb{R} \times \mathbb{R}^n).$$
to (78). The solution satisfies the estimate
\[ \|w\|_{1,2,q} + \|\nabla \pi\|_q \leq C_4 \lambda(T) \|F\|_q, \]  
(79)
where \(P(\lambda, T)\) is a polynomial in \(\lambda\) and \(T\) and \(C_4 = C_4(v,n,q)\). If for some \(r \in (1, \infty)\) \((\w, \tilde{\pi}) \in W^{1,2,r}_{\text{per,\perp}}(\mathbb{R} \times \mathbb{R}^n) \times D^{1,r}_{\text{per,\perp}}(\mathbb{R} \times \mathbb{R}^n)\) is another solution, then \(w = \tilde{w}\) and \(\pi(t,x) = \tilde{\pi}(t,x) + d(t)\) for some \(T\)-periodic function \(d : \mathbb{R} \to \mathbb{R}\).

Some nomenclature from abstract harmonic analysis is needed to sketch a proof of Theorem 9. A topology and an appropriate differentiable structure on the group \(G := \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n\) are inherited from \(\mathbb{R} \times \mathbb{R}^n\). More precisely, \(G\) becomes a locally compact abelian group when equipped with the quotient topology induced by the canonical quotient mapping
\[ q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n, \quad q(t,x) := ([t], x). \]  
(80)
The restriction \(\Pi := q|_{[0,T) \times \mathbb{R}^n}\) is used to identify \(G\) with the domain \([0, T) \times \mathbb{R}^n\); \(\Pi\) is clearly a (continuous) bijection. Via \(\Pi\), one can identify the Haar measure \(dg\) on \(G\) as the product of the Lebesgue measure on \([0, T)\) and the Lebesgue measure on \(\mathbb{R}^n\). The Haar measure is unique up to a normalization factor, which in the following is chosen such that
\[ \int_G u(g) \, dg = \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} u \circ \Pi(t,x) \, dx \, dt. \]
For the sake of convenience, the symbol \(\Pi\) in integrals of \(G\)-defined functions with respect to \(dx \, dt\) shall be omitted. By
\[ C^\infty(G) := \{u : G \to \mathbb{R} \mid u \circ \Pi \in C^\infty([0, T) \times \mathbb{R}^n)\}, \]  
(81)
the space of smooth functions on \(G\) is defined. For \(u \in C^\infty(G)\), derivatives are defined by
\[ \forall (\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0 : \quad \partial_t^\beta \partial_x^\alpha u := \left[ \partial_t^\beta \partial_x^\alpha (u \circ \Pi) \right] \circ \Pi^{-1}. \]  
(82)
It is easy to verify for \(u \in C^\infty(G)\) that also \(\partial_t^\beta \partial_x^\alpha u \in C^\infty(G)\). The subspace \(C_0^\infty(G)\) denotes the compactly supported smooth functions.

With a differentiable structure available on \(G\), the space of tempered distributions can be defined. For this purpose, recall the Schwartz-Bruhat space of generalized Schwartz functions (see, e.g., [2]) given by
\[ \mathcal{S}(G) := \{u \in C^\infty(G) \mid \forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_0 : \rho_{\alpha,\beta,\gamma}(u) < \infty\}, \]
\[ \rho_{\alpha,\beta,\gamma}(u) := \sup_{(t,x) \in G} |x^\alpha \partial_t^\beta \partial_x^\gamma u(t,x)|. \]
Equipped with the semi-norm topology of the family \( \{ \rho_{\alpha,\beta,\gamma} \mid (\alpha, \beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_0 \} \), \( \mathcal{F}(G) \) becomes a topological vector space. The dual space \( \mathcal{F}'(G) \) equipped with the weak* topology is referred to as the space of tempered distributions on \( G \).

For a tempered distribution \( u \in \mathcal{F}'(G) \), distributional derivatives \( \partial^\gamma \partial^\beta \xi u \in \mathcal{F}'(G) \) are defined by duality as in the classical case. Similarly, tempered distributions on \( G \)'s dual group \( \widehat{G} \) are introduced. By associating each \( \kappa / \mathbb{C} \) with the character \( \chi : G \rightarrow \mathbb{C} \), \( \chi(t, x) := e^{ix \xi + i \frac{2\pi}{\kappa} k t} \) on \( G \) (it is standard to verify that all characters are of this form), one can characterize the dual group as \( \widehat{G} = \mathbb{Z} \times \mathbb{R}^n \). By

\[
C^\infty(\widehat{G}) := \{ w \in C(\widehat{G}) \mid \forall k \in \mathbb{Z} : w(k, \cdot) \in C^\infty(\mathbb{R}^n) \},
\]

the space of smooth functions on \( \widehat{G} \) is introduced. The Schwartz-Bruhat space on the dual group \( \widehat{G} \) is given by

\[
\mathcal{S}(\widehat{G}) := \{ w \in C^\infty(\widehat{G}) \mid \forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_0 : \hat{\rho}_{\alpha,\beta,\gamma}(w) < \infty \},
\]

and equipped with the canonical semi-norm topology.

The Fourier transform associated to the locally compact abelian group \( G \) is denoted by \( \hat{\mathcal{F}}_G \). It is explicitly given by

\[
\hat{\mathcal{F}}_G : L^1(G) \rightarrow C(\widehat{G}), \quad \hat{\mathcal{F}}_G(u)(k, \xi) := \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} u(t, x) e^{-ix \xi - i \frac{2\pi}{\kappa} k t} \, dx \, dt.
\]

Since the Haar measure on \( \widehat{G} \) is the product of the counting measure on \( \mathbb{Z} \) and the Lebesgue measure on \( \mathbb{R}^n \), the inverse Fourier transform is formally defined by

\[
\hat{\mathcal{F}}_G^{-1} : L^1(\widehat{G}) \rightarrow C(G), \quad \hat{\mathcal{F}}_G^{-1}(w)(t, x) := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} w(k, \xi) e^{ix \xi + i \frac{2\pi}{\kappa} k t} \, d\xi.
\]

It is standard to verify that \( \hat{\mathcal{F}}_G : \mathcal{F}(G) \rightarrow \mathcal{F}(\widehat{G}) \) is a homeomorphism with \( \hat{\mathcal{F}}_G^{-1} \) as the actual inverse, provided the Lebesgue measure \( d\xi \) is normalized appropriately. By duality, \( \hat{\mathcal{F}}_G \) extends to a homeomorphism \( \mathcal{F}'(G) \rightarrow \mathcal{F}'(\widehat{G}) \).

Similar to the spaces \( L^q_{\text{per}}(\mathbb{R} \times \mathbb{R}^n) \), \( W_{1,2,q}^{1,2,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n) \), and \( D^{1,q}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n) \), Lebesgue and Sobolev spaces with respect to the domain \( \widehat{G} \) are defined by

\[
L^q(\mathcal{G}) := C^\infty_0(\mathcal{G}) \| \cdot \|_q, \quad W^{1,2,q}(\mathcal{G}) := C^\infty_0(\mathcal{G}) \| \cdot \|_{1,2,q}, \quad \text{and} \quad D^{1,q}(\mathcal{G}) := C^\infty_0(\mathcal{G}) \| \cdot \|_{0,1,q},
\]

where \( \| \cdot \|_q \) denotes the \( L^q \)-norm with respect to the Haar measure \( d\xi \), the norm \( \| \cdot \|_{1,2,q} \) defined as in (5), and the norm \( \| \cdot \|_{0,1,q} \) as in (63). It is standard to verify
that $W^{1,2,q}(G) = \{ u \in L^q(G) \mid \| u \|_{1,2,q} < \infty \}$. As in (62), the time-averaging projection $P$ and its complement $P_\perp$ are introduced on functions defined on $G$. Thus, the subspaces $W^{1,2,q}_\perp(G) := P_\perp W^{1,2,q}(G)$, $D^{1,q}_\perp(G) := \mathcal{P}_\perp D^{1,q}(G)$, and $L^q_\perp(G) := \mathcal{P}_\perp L^q(G)$ can be defined.

It is now possible to formulate (78) as a system of partial differential equations on $G$ in a context that permits utilization of the Fourier transform $\mathcal{F}_G$. A representation formula for the solution in terms of a Fourier multiplier can then be obtained. The proof of Theorem 9 below is based on such a representation.

**Proof of Theorem 9.** Since the topology and differentiable structure on $G$ is inherited from $\mathbb{R} \times \mathbb{R}^n$, the $T$-time-periodic problem (78) can be formulated equivalently as a system of $G$-defined vector fields

\[
\begin{align*}
\partial_t w - \nu \Delta w - \lambda \partial_1 w + \nabla \pi &= \mathcal{P}_\perp F & \text{in } G, \\
\text{div } w &= 0 & \text{in } G
\end{align*}
\]

with unknowns $w : G \to \mathbb{R}^n$, $\pi : G \to \mathbb{R}$, and data $F : G \to \mathbb{R}^n$. In this formulation, the periodicity condition is not needed anymore. Indeed, all functions defined on $G$ are intrinsically $T$-time-periodic. Since (83) can be interpreted as a system of equations in $S^0(G)$, the Fourier transform $\mathcal{F}_G$ can be applied to obtain a formula for $w$. An easy calculation shows that $\mathcal{F}_G[\mathcal{P}_\perp F] = (1 - \delta_Z(k))(\mathcal{F}_G[F])$, where $\delta_Z$ denotes the Dirac delta distribution on $\mathbb{Z}$, which is simply the function $\delta_Z(0) := 1$ and $\delta_Z(k) := 0$ for $k \neq 0$. Formally, application of the Fourier transform $\mathcal{F}_G$ in (83) yields

\[
\begin{align*}
    w &= \mathcal{F}_G^{-1}\left[\frac{1 - \delta_Z(k)}{v |\xi|^2 + i(\frac{2\pi}{T}k - \lambda \xi_1)} (I - \frac{\xi \otimes \xi}{|\xi|^2}) \mathcal{F}_G[F]\right], \\
    \pi &= \mathcal{F}_G^{-1}\left[\frac{i \xi}{|\xi|^2} \cdot \mathcal{F}_G[F]\right] = \mathcal{F}_G^{-1}\left[\frac{i \xi}{|\xi|^2} \cdot \mathcal{F}_{\mathbb{R}^n}[F]\right].
\end{align*}
\]

The Fourier multiplier

\[
M(k, \xi) : \hat{G} \to \mathbb{C}, \quad M(k, \xi) := \frac{1 - \delta_Z(k)}{v |\xi|^2 + i(\frac{2\pi}{T}k - \lambda \xi_1)}
\]

is bounded and smooth, that is, $M \in L^\infty(\hat{G})$ and $M \in C^\infty(\hat{G})$, which can easily be seen by observing that the numerator of $M$ vanishes in a neighborhood of the only zero $(0,0)$ of the denominator. It can be shown that $M$ is an $L^q(G)$ multiplier in the sense that the mapping $f \to \mathcal{F}_G^{-1}[M \mathcal{F}_G[f]]$ extends from a mapping $\mathcal{S}(G) \to \mathcal{S}'(G)$ into a bounded operator $L^q(G) \to L^q(G)$. Since multiplier theorems like the ones of Mikhlin, Lizorkin, or Marcinkiewicz are only available in an Euclidean setting and not in the general setting of group multipliers, a proof has to rely on a so-called transference principle. Originally introduced by de Leeuw
[4], a transference principle for Fourier multipliers on local compact abelian groups makes it possible to study the properties of $M$ via a corresponding multiplier defined on $\mathbb{R}^{n+1}$. The transference principle (see, e.g., [5, Theorem B.2.1]) states that $M$ is an $L^q(G)$ multiplier if there is a continuous homomorphism $\Phi: \hat{G} \to \mathbb{R}^{n+1}$ such that $M = m \circ \Phi$ for some $L^q(\mathbb{R}^{n+1})$ multiplier $m$. Moreover, the norm of the $L^q(G)$ operator corresponding to $M$ coincides with the norm of the $L^q(\mathbb{R}^{n+1})$ operator corresponding to $m$. To identify such an $m$ in the particular case above, let $\chi$ be a “cutoff” function with $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$, $\chi(\eta) = 1$ for $|\eta| \leq \frac{1}{2}$, and $\chi(\eta) = 0$ for $|\eta| \geq 1$. Define

$$m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \quad m(\eta, \xi) := \frac{1 - \chi(\frac{\xi}{2\pi} \eta)}{v|\xi|^2 + i(\eta - \lambda \xi_1)}.$$  

(86)

Further, let $\Phi : \hat{G} \to \mathbb{R}^{n+1}$, $\Phi(k, \xi) := (\frac{2\pi}{\xi} k, \xi)$. Clearly, $\Phi$ is a continuous homomorphism and $M = m \circ \Phi$. To show that $m$ is a $L^q(\mathbb{R}^{n+1})$ multiplier, a standard multiplier theorem can be applied. Since $m$ is a rational function with nonvanishing denominator away from $(0, 0)$, it is easy to verify that all functions of type

$$(\eta, \xi) \mapsto \xi_1^{\xi_1} \cdots \xi_n^{\xi_n} \eta^{\xi_n+1} \partial_1 \cdots \partial_n \partial_\eta^{\xi_n+1} m(\eta, \xi)$$  

(87)

stay bounded as $|(\eta, \xi)| \to \infty$. Consequently, it follows from Marcinkiewicz’s multiplier theorem (see, e.g., [45, Chapter IV, §6]) that $m$ is an $L^q(\mathbb{R}^{n+1})$ multiplier.

A more careful analysis of the bounds obtained for the functions in (87) shows that the norm of the corresponding operator is bounded by a polynomial $P(\lambda, T)$ in $\lambda$ and $T$. It therefore follows that

$$\forall f \in L^q(G) : \| \mathcal{F}_G^{-1}[M \mathcal{F}_G[f]] \|_q \leq P(\lambda, T) \| f \|_q.$$  

Now return to (84). Observe that $I - \frac{\xi \otimes \xi}{|\xi|^2}$ is the symbol of the Helmholtz-Weyl projection, that is, the projection $P_H : L^q(\mathbb{R}^n)^n \to L^q(\mathbb{R}^n)^n$ onto the subspace $L^q_{s0}(\mathbb{R})$ of solenoidal vector fields. It is well known that $P_H$ is continuous on $L^q(\mathbb{R}^n)$ for any $q \in (1, \infty)$. The projection $P_H$ extends trivially to a continuous projection on $L^q(G)$. Now define $(w, \pi)$ by (84). Clearly, $(w, \pi)$ is a solution in the sense of distributions $\mathcal{S}'(G)$ to (83). Moreover,

$$\| w \|_q = \left\| \mathcal{F}_G^{-1}[M(k, \xi) \mathcal{F}_G[P_H F]] \right\|_q \leq P(\lambda, T) \| P_H F \|_q \leq P(\lambda, T) \| F \|_q.$$  

The argument above can be repeated for $\partial_\xi w$ and $\partial_\xi^2 w$ for $|\alpha| \leq 2$. More precisely, one may verify that also $(k, \xi) \mapsto kM(k, \xi)$ and $(k, \xi) \mapsto \xi^\alpha M(k, \xi)$ are $L^q(G)$ multipliers for $|\alpha| \leq 2$. It thus follows that $\| w \|_{1,2, q} \leq P(\lambda, T) \| F \|_q$. Based on (84), it is standard to show $\| \nabla \pi \|_q \leq c \| F \|_q$. Since $\delta_Z$ is the Fourier symbol of $\mathcal{P}$, it follows directly from (84) that $P_{\perp} w = w$ and $P_{\perp} \pi = \pi$. Thus a solution $(w, \pi) \in$
\[ W_{1,2}^q(G) \times D_{1,2}^q(G) \] to (83) that satisfies (79) is obtained. Via the canonical quotient map \( q \), the spaces \( C_{0,\text{per}}^\infty(\mathbb{R} \times \mathbb{R}^n) \) and \( C_0^\infty(G) \) are isometrically isomorphic in the norm \( \| \cdot \|_{1,2,q} \). By the definition of Sobolev spaces as completions of these spaces in the norm \( \| \cdot \|_{1,2,q} \), it follows that also \( W_{1,2}^q(\mathbb{R} \times \mathbb{R}^n) \) and \( W_{1,2}^q(G) \) are isometrically isomorphic. The same is clearly true for \( D_{1}^{q,\text{per}}(\mathbb{R} \times \mathbb{R}^n) \) and \( D_{1}^{q,\text{per}}(G) \) as well as for \( L^q_{\text{per},\text{per}}(\mathbb{R} \times \mathbb{R}^n) \) and \( L^q(G) \). It follows that \( (w \circ q, \pi \circ q) \) is a solution in \( W_{1,2}^q(\mathbb{R} \times \mathbb{R}^n) \times D_{1}^{q,\text{per}}(\mathbb{R} \times \mathbb{R}^n) \) to (78) that satisfies (79).

It remains to establish the desired uniqueness property. It suffices to do so for the system (83). Assume \((\tilde{w}, \tilde{\pi})\) is another solution in \( W_{1,2,r}^q(G) \times D_{1}^{q,r}(G) \). Employ in (83) first the Helmholtz projection \( P_H \) and then the Fourier transform to deduce that \((i \frac{2\pi}{T} k + \nu |\xi|^2 - \lambda i \xi_1) \mathcal{F}_G[w - \tilde{w}] = 0 \). Since the polynomial \( \nu |\xi|^2 + i(\frac{2\pi}{T} k - \lambda \xi_1) \) vanishes only at \((k, \xi) = (0, 0)\), it follows that supp \( \mathcal{F}_G[w - \tilde{w}] \subset \{0, 0\} \). However, since \( P[w - \tilde{w}] = 0 \) also \( \delta_{\mathcal{H}}(k) \mathcal{F}_G[w - \tilde{w}](k, \xi) = 0 \), whence \((0, 0) \notin \text{supp} \mathcal{F}_G[w - \tilde{w}] \). Consequently, supp \( \mathcal{F}_G[w - \tilde{w}] = \emptyset \). It follows that \( \mathcal{F}_G[w - \tilde{w}] = 0 \) and thus \( w = \tilde{w} \). By (83), \( \pi(t, x) = \tilde{\pi}(t, x) + d(t) \) for some \( T \)-periodic function \( d : \mathbb{R} \to \mathbb{R} \).

To complete the \( L^q \) estimate for a solution to (76), also the steady-state part \((v, p)\) needs to be addressed. However, the \( L^q \) theory for the Stokes/Oseen system (77) is well known; see, for example, [10, Theorem IV.2.1 and VII.4.1]. As the steady-state \( L^q \) estimates for \((v, p)\) are completely decoupled from the time-periodic nature of (76), they are omitted here.

### 4.5 Fundamental Solution

The reformulation of the linear \( T \)-time-periodic system (76) on the group \( G \) carried out in the proof of Theorem 9 motivates the introduction of a time-periodic fundamental solution. In the setting of tempered distributions \( \mathcal{S}'(G) \), a fundamental solution to (76) or rather to the equivalent system on the group \( G \)

\[
\begin{cases}
\partial_t u - \nu \Delta u - \lambda \partial_1 u + \nabla p = F & \text{in } G, \\
\text{div } u = 0 & \text{in } G,
\end{cases}
\]

(88)
can be defined as a tensor field

\[
\Gamma_{\text{TP}} := \begin{pmatrix}
\Gamma_{11}^{\text{TP}} & \cdots & \Gamma_{1n}^{\text{TP}} \\
\vdots & \ddots & \vdots \\
\Gamma_{n1}^{\text{TP}} & \cdots & \Gamma_{nn}^{\text{TP}}
\end{pmatrix} \in \mathcal{S}'(G)^{(n+1) \times n}
\]

(89)
that satisfies

\[
\begin{cases}
\partial_t \Gamma_{ij}^{TP} - v \Delta \Gamma_{ij}^{TP} - \lambda \partial_1 \Gamma_{ij}^{TP} + \partial_j \gamma_j^{TP} = \delta_{ij} \delta_G, \\
\partial_i \Gamma_{ij}^{TP} = 0,
\end{cases}
\]  

(90)

where $\delta_{ij}$ and $\delta_G$ denote the Kronecker delta and delta distribution, respectively. For a sufficiently regular right-hand side, say $F \in \mathcal{S}(G)^n$, a solution to (88) is then given by component-wise convolution over the group $G$ with the fundamental solution:

\[
\left( \begin{array}{c} u \\ p \end{array} \right) := \Gamma_{\text{TP}} F.
\]  

(91)

The ability to identify a solution in terms of such a direct expression offers many advantages. To the extent that pointwise information can be obtained for $\Gamma_{\text{TP}}$, a similar type of information can be obtained for (91). In particular, knowledge of the asymptotic structure of $\Gamma_{\text{TP}}$ at spatial infinity can be used to analyze the pointwise behavior of $u(t, x)$ at as $|x| \to \infty$.

As already observed, the projection $P$ can be expressed as $P f := \mathcal{F}_G^{-1}[\delta_2 \mathcal{F}_G[f]]$. From this expression, it is seen that $P$ can be extended to a projection in the context of distributions $P : \mathcal{S}'(G) \to \mathcal{S}'(G)$. The same is true for $P_L$. Consequently, the idea from the previous section to use $P$ and $P_L$ to decompose (90) into a steady-state and oscillatory part can be reused. As a result, it is possible to identify $\Gamma_{\text{TP}}$ as a sum of a well-known fundamental solution to the corresponding steady-state system and an oscillatory fundamental solution. It turns out that the oscillatory fundamental solution has significantly better decay properties as $|x| \to \infty$. The structure of the fundamental solution differs depending on whether $\lambda = 0$ or $\lambda \neq 0$. This phenomenon is well known from the steady-state case, from which one may recall that a fundamental solution $(\Gamma, \gamma) \in \mathcal{S}'(\mathbb{R}^n)^{n \times n} \times \mathcal{S}'(\mathbb{R}^n)^n$ to

\[
\begin{cases}
-v \Delta \Gamma_{ij} - \lambda \partial_1 \Gamma_{ij} + \partial_j \gamma_j = \delta_{ij} \delta_{\mathbb{R}^n}, \\
\partial_i \Gamma_{ij} = 0,
\end{cases}
\]  

(92)

in the Stokes case $\lambda = 0$ is given by the Stokes fundamental solution $(\Gamma^{\text{Stokes}}, \gamma)$ and in the Oseen case $\lambda \neq 0$ by the Oseen fundamental solution $(\Gamma^{\text{Oseen}}, \gamma)$; the pressure $\gamma$ is the same in two cases. Explicit expressions for both are well known; see, for example, [10, Chapter IV.2] and [10, Chapter VII.3] for the Stokes and Oseen fundamental solution, respectively.

**Theorem 10.** Let $n \geq 2$. Put

\[
\Gamma := \begin{cases}
\Gamma^{\text{Stokes}} & \text{if } \lambda = 0 \quad \text{(Stokes case),} \\
\Gamma^{\text{Oseen}} & \text{if } \lambda \neq 0 \quad \text{(Oseen case).}
\end{cases}
\]
Then
\[ \Gamma_{TP} := \Gamma \otimes 1_{R/TZ} + \Gamma^\perp, \] (93)
\[ \gamma_{TP} := \gamma \otimes \delta_{R/TZ}, \] (94)

with
\[ \Gamma^\perp := \mathcal{F}^{-1}_G \left[ \frac{1 - \delta_2(k)}{|\xi|^2 + i \left( \frac{2\pi}{T} k - \lambda \xi_1 \right)} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right] \in \mathcal{S}'(G)^{n \times n} \] (95)
defines a fundamental solution \( \Gamma_{tp} \in \mathcal{S}'(G)^{(n+1)\times n} \) to \( \text{(88)} \) on the form \( \text{(89)} \) satisfying \( \text{(90)} \) and

\[ \forall q \in \left( 1, \frac{n+2}{n} \right) : \quad \Gamma^\perp \in L^q(G)^{n \times n}, \] (96)
\[ \forall q \in \left[ 1, \frac{n+2}{n+1} \right) : \quad \partial_j \Gamma^\perp \in L^q(G)^{n \times n} \quad (j = 1, \ldots, n), \] (97)
\[ \forall r \in [1, \infty) \forall \varepsilon > 0 \exists C > 0 \forall |x| \geq \varepsilon : \| \Gamma^\perp (\cdot, x) \|_{L^r(R/TZ)} \leq \frac{C}{|x|^r}, \] (98)
\[ \forall r \in [1, \infty) \forall \varepsilon > 0 \exists C > 0 \forall |x| \geq \varepsilon : \| \partial_j \Gamma^\perp (\cdot, x) \|_{L^r(R/TZ)} \leq \frac{C}{|x|^{r+1}}, \] (99)
\[ \forall q \in (1, \infty) \exists C > 0 \forall F \in \mathcal{S}(G)^n : \| \Gamma^\perp * F \|_{W^{1,2,q}(G)} \leq C \| F \|_{L^q(G)}, \] (100)

where \( 1_{R/TZ} \in \mathcal{S}'(R/TZ) \) denotes the constant 1.

**Proof.** Apply in \( \text{(90)} \) first the projections \( \mathcal{P} \) and \( \mathcal{P}^\perp \) and subsequently the Fourier transform to deduce

\[ \mathcal{P} \Gamma_{TP} = \Gamma \otimes 1_{R/TZ}, \quad \mathcal{P}^\perp \Gamma_{TP} = \mathcal{F}^{-1}_G \left[ \frac{1 - \delta_2(k)}{|\xi|^2 + i \left( \frac{2\pi}{T} k - \lambda \xi_1 \right)} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right] \]

and

\[ \gamma_{TP} = \mathcal{F}^{-1}_G \left[ \frac{i\xi}{|\xi|^2} \right] = \mathcal{F}^{-1}_{R^n} \left[ \frac{i\xi}{|\xi|^2} \right] \otimes \delta_{R/TZ}. \]

It thus follows that \( (\Gamma_{TP}, \gamma_{TP}) \) given by \( \text{(93)} \) and \( \text{(94)} \) defines a fundamental solution \( \Gamma_{tp} \in \mathcal{S}'(G)^{(n+1)\times n} \) to \( \text{(88)} \) on the form \( \text{(89)} \). Recall \( \text{(85)} \) and the observation made in the proof of Theorem 9 that \( M \in L^\infty(\tilde{G}) \cap C_0^\infty(\tilde{G}) \) to see that the right-hand side in definition \( \text{(95)} \) of \( \Gamma^\perp \) is well defined. The properties \( \text{(96)}, \text{(97)}, \text{(98)}, \) and \( \text{(99)} \)
can be shown by a lengthy but direct computation and subsequent estimate of the
inverse Fourier transform in (95); see [15, 30]. Finally, (100) is just a reiteration of
the statement in Theorem 9.

Remark 8. It is well known (see again [10, Chapter IV.2 and Chapter VII.3]) that
both \( \Gamma_{\text{Stokes}} \) and \( \Gamma_{\text{Oseen}} \) have a pointwise decay rate as \( |x| \to \infty \) that is slower than
\( |x|^{-n} \). Estimate (98) therefore implies that the oscillatory part \( \Gamma_{\text{tp}} \) decays
class slower than the steady-state part. Consequently, the asymptotic behavior as \( |x| \to \infty \)
of a solution \( u \) to (88) given by (91) will be dominated by the steady-state part \( \mathcal{P}u \).

4.6 Weighted Estimates for the Linearized Problem in the Whole
Space

The properties obtained for the fundamental solution \( \Gamma_{\text{tp}} \) in Theorem 10 can be
used to establish pointwise weighted estimates, with weights of type \( (1 + |x|)^{\beta} \),
of solutions to the linearized system (76). Weighted estimates of this type are well
known for the corresponding steady-state problem; see, for example, [10, Lemma
V.8.2]. Thus, if a time-periodic solution is once again decomposed into a steady-
state and oscillatory part, weighted estimates need only be established for the
oscillatory part. As in the case of the \( L^q \) estimates in Theorem 9, better estimates in
terms of decay at spatial infinity materialize for the oscillatory part.

For \( \beta \in [1, \infty) \), let \( \mathcal{X}_{\beta} \) denote the Banach space

\[
\mathcal{X}_{\beta} := \left\{ \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \mid \|\varphi\|_{\beta} < \infty \right\}, \quad \|\varphi\|_{\beta} := \text{ess sup}_{x \in \mathbb{R}^n} (1 + |x|)^{\beta} |\varphi(x)|.
\]

(101)

It is illustrated below how to obtain estimates for oscillatory \( q \)-generalized solutions
to (76) in spaces \( L^\infty_{\text{per,}\perp}(\mathbb{R}; \mathcal{X}_{\beta}) := \mathcal{P}_{\perp} L^\infty_{\text{per}}(\mathbb{R}; \mathcal{X}_{\beta}) \). These estimates can be
augmented with \( L^q \) estimates. For this purpose, recall the interpretation from
Sect. 4.4 of \( T \)-time-periodic vector fields as functions defined on the group \( G \), and let

\[
W^{1,1,q}_{\perp} (G) := \left\{ w \in L^q(G) \mid \|w\|_{1,1,q} < \infty, \mathcal{P}w = 0 \right\},
\]

\[
\|w\|_{1,1,q} := \left\| \mathcal{P}^{-1} \left[ \left( |k|^{\frac{1}{2}} + |\xi| + 1 \right) \mathcal{P}[w] \right] \right\|_q
\]

denote the Sobolev space of oscillatory functions with “half” a derivative in
time and one derivative in space belonging to \( L^q(G) \). The corresponding space
\( W^{1,1,q}_{\text{per,}\perp}(\mathbb{R} \times \mathbb{R}^n) \) is defined canonically via the quotient mapping \( q \).
Theorem 11. Let $n \geq 2$, $\lambda \in [0, \infty)$ and $\beta \in [1, n+1]$. For every $F = \text{div } G$ with $G = L^\infty_{\text{per}, \perp}(\mathbb{R} ; X_\beta)$, there is a solution $(w, \pi)$ to (76) with $w \in L^\infty_{\text{per}, \perp}(\mathbb{R} ; X_\beta)$ and

$$\forall r \in \left(\max \left(\frac{n}{\beta - 1}, 1\right), \infty\right) : w \in W^{\frac{1}{r},1,r}_{\text{per}, \perp}(\mathbb{R} \times \mathbb{R}^n), \pi \in L^\infty_{\text{per}, \perp}(\mathbb{R} ; L'(\mathbb{R}^n))$$

(102)

which satisfies

$$\|w\|_{L^\infty_{\text{per}}(\mathbb{R}^n ; X_\beta)} \leq C_5 \|G\|_{L^\infty_{\text{per}}(\mathbb{R}^n ; X_\beta)} ,$$

(103)

with $C_5 = C_5(\beta, \nu, \lambda, n)$ and for all $r \in \left(\max \left(\frac{n}{\beta - 1}, 1\right), \infty\right)$

$$\|w\|_{\frac{1}{r},1,r} + \|\pi\|_{L^\infty_{\text{per}}(\mathbb{R} ; L'(\mathbb{R}^n))} \leq C_6 \|G\|_{L^\infty_{\text{per}}(\mathbb{R}^n ; X_\alpha)}$$

(104)

with $C_6 = C_6(r, \nu, \lambda, n)$. If $(\tilde{w}, \tilde{\pi})$ is another solution with $\tilde{w} \in L^\infty_{\text{per}, \perp}(\mathbb{R} ; X_\alpha)$ for some $\alpha \in [1, \infty)$ and $\pi \in L^\infty_{\text{per}, \perp}(\mathbb{R} ; L^s(\mathbb{R}^n))$ for some $s \in (1, \infty)$, then $(w, \pi) = (\tilde{w}, \tilde{\pi})$.

Proof. From the integrability of the fundamental solution $\Gamma_\perp$ stated in Theorem 10, recall (97), and the integrability of $G$, it follows that the convolution integral

$$w_i(t, x) := -\frac{1}{T} \int_0^T \int_{\mathbb{R}^n} \partial_k \Gamma \perp_{ij}(t - s, x - y) G_{kj}(s, y) \, dy \, ds \quad (i = 1, \ldots, n)$$

(105)

is well-defined as an element of, say, $L^\infty_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$. From the definition (95) and the interpretation of $T$-time-periodic vector fields as functions defined on the group $G$, it is clear that $w$ together with

$$\pi := \mathcal{F}_G^{-1}\left[\frac{-\xi_i \xi_j}{|\xi|^2} \mathcal{F}_G[G_{ij}]\right] = \mathcal{F}_{\mathbb{R}^n}^{-1}\left[\frac{-\xi_i \xi_j}{|\xi|^2} \mathcal{F}_{\mathbb{R}^n}[G_{ij}]\right]$$

is a solution to (76). It can be shown with the same approach as in the proof of Theorem 9 that $w \in W^{\frac{1}{r},1,r}_{\text{per}, \perp}(\mathbb{R} \times \mathbb{R}^n)$ and satisfies (104). It follows directly from the definition of $\pi$ above that also $\pi$ satisfies (104). Since $w \in L^\infty_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$, it is enough to establish the weighted estimate (103) for sufficiently large $|x|$. Consider for this purpose an $|x| > 2$ and decompose the integral in (105) as

$$w_i(t, x) = -\frac{1}{T} \int_0^T \int_{B_{|x|/2}} + \int_{B_{|x|/2,|x|}|B_1(x)} + \int_{B_1(x)} + \int_{B_{2|x|}} \partial_k \Gamma \perp_{ij}(t - s, x - y) G_{kj}(s, y) \, dy \, ds$$

$$=: I_1(t, x) + I_2(t, x) + I_3(t, x) + I_4(t, x).$$
On the strength of estimate (99) in Theorem 10, it follows that

\[
|I_1(t, x)| \leq \int_{B_{|x|/2}} \|\partial_k \Gamma_{ij}^{\perp}(\cdot, x - y)\|_{L^2_{\text{per}}(\mathbb{R})} (1 + |y|)^{-\beta} \, dy \|G\|_{L^\infty_{\text{per}}(\mathbb{R}; X_{\beta})} \\
\leq c_0 \int_{B_{|x|/2}} |x - y|^{-(n+1)} (1 + |y|)^{-\beta} \, dy \|G\|_{L^\infty_{\text{per}}(\mathbb{R}; X_{\beta})} \\
\leq c_1 |x|^{-\beta} \|G\|_{L^\infty_{\text{per}}(\mathbb{R}; X_{\beta})},
\]

where the last inequality is valid due to the assumption that $\beta \leq n + 1$. The integrals $I_2$ and $I_4$ are estimated in a similar fashion. The summability property (97) from Theorem 10 is needed to deduce that $\partial_k \Gamma^{\perp}$ is integrable over $B_1$, which then leads to an estimate of $I_3$. It follows from the estimates of $I_1$, $I_2$, $I_3$, and $I_4$ that $|w(t, x)| \leq c_2 |x|^{-\beta} \|G\|_{L^\infty_{\text{per}}(\mathbb{R}; X_{\beta})}$ for $|x| > 2$, which implies (103). The uniqueness property can be obtained as in the proof of Theorem 9.

The solution in Theorem 11 is called $q$-generalized since it possesses, at the outset, only “half” a derivative in time and one derivative in space belonging some $L^q$ space, that is, it belongs to $W^{1,1,q}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$. Weighted estimates for strong solutions in $W^{1,2,q}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$ corresponding to data $F \in L^\infty_{\text{per},\perp}(\mathbb{R}; X_{\beta})$ can be obtained in a similar fashion based on the estimates of the fundamental solution $\Gamma^{\perp}$ available in Theorem 10.

4.7 \hspace{1em} L^q Estimates for the Linearized Problem in Exterior Domains

Consider the following linearization of (61) with homogeneous boundary values:

\[
\begin{aligned}
\partial_t u - \nu \Delta u - \lambda \partial_1 u + \nabla p &= F \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
u \text{ on } \mathbb{R} \times \partial \Omega, \\
\lim_{|x| \to \infty} u(t, x) &= 0,
\end{aligned}
\]

(106)

where $\lambda \geq 0$ is a constant. The $L^q$ estimates established in the whole-space case can be extended to solutions to the exterior domain problem (106). As in the whole-space case, the projections (62) can be used to decompose (106) into a steady-state and oscillatory problem. The following $L^q$ estimate holds for the oscillatory problem:
Theorem 12. Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be an exterior domain of class $C^2$. Let $q \in (1, \infty)$ and $\lambda \in [0, \lambda_0]$. For any vector field $F \in L^q_{\text{per.} \perp} (\mathbb{R} \times \Omega)$, there is a solution $(u, p) \in W^{1,2,q}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \times D^{1,q}_{\text{per.} \perp} (\mathbb{R} \times \Omega)$ to (106) which satisfies

$$\|u\|_{1,2,q} + \|\nabla p\|_q \leq C_7 \|F\|_q,$$

with $C_7 = C_7(q, \Omega, v, \lambda_0)$. If $r \in (1, \infty)$ and $(\widetilde{u}, \widetilde{p}) \in W^{1,2,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \times D^{1,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega)$ is another solution, then $\widetilde{u} = u$ and $\widetilde{p} = p + d(t)$ for some $T$-periodic function $d : \mathbb{R} \rightarrow \mathbb{R}$.

A proof of Theorem 12 is given below. Beforehand, the $L^q$ estimates for (106) that follow by combining Theorem 12 with well-known $L^q$ estimates for the corresponding steady-state problem are manifested. Only the case $\lambda \neq 0$ is included. A similar statement can be made in the case $\lambda = 0$; see also Remark 9.

Corollary 1. Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be an exterior domain of class $C^2$ and $\lambda \in (0, \lambda_0]$. Let $q \in (1, \frac{3}{2})$ if $n = 2$, and $q \in (1, 2)$ if $n = 3$. Define

$$X^q_{\lambda} (\Omega) := \{v \in X^q_{\lambda} (\Omega) \mid \text{div} v = 0, \ v = 0 \text{ on } \partial \Omega\},$$

$$D^{1,q}_{\lambda} (\Omega) := \left\{v \in D^{1,q} (\Omega) \mid \int_{B_{R_0}} p \, dx = 0\right\}$$

as subspaces of $X^q_{\lambda} (\Omega)$ and $D^{1,q} (\Omega)$, defined in Sect. 4.1, respectively. Moreover, let $r \in (1, \infty)$ and

$$W^{1,2,q,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega) := \left\{w \in W^{1,2,q}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \cap W^{1,2,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \mid \text{div} w = 0, \ w = 0 \text{ on } \partial \Omega\right\}$$

equipped with the norm $\|w\|_{1,2,q,r} := \|\cdot\|_{1,2,q} + \|\cdot\|_{1,2,r}$. Additionally let

$$D^{1,q,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega) := \left\{\pi \in D^{1,q}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \cap D^{1,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \mid \int_{B_{R_0}} \pi \, dx = 0\right\}$$

with norm $\langle \cdot \rangle_{0,1,q,r} := \langle \cdot \rangle_{0,1,q} + \langle \cdot \rangle_{0,1,r}$ and

$$L^{q,r}_{\text{per.} \perp} (\mathbb{R} \times \Omega) := L^{q}_{\text{per.} \perp} (\mathbb{R} \times \Omega) \cap L^{r}_{\text{per.} \perp} (\mathbb{R} \times \Omega).$$
with norm \( \| \cdot \|_{q,r} := \| \cdot \|_q + \| \cdot \|_r \). Then the \( T \)-time-periodic Oseen operator

\[
A_{\text{Oseen}} : \mathcal{X}^q_{\lambda} (\Omega) \oplus \mathcal{W}^{1,2,q,r}_{\text{per.},\perp} (\mathbb{R} \times \Omega) \\
\times \mathcal{D}^{1,q} (\Omega) \oplus \mathcal{D}^{1,q,r}_{\text{per.},\perp} (\mathbb{R} \times \Omega) \to L^q (\Omega) \oplus L^q_{\text{per.},\perp} (\mathbb{R} \times \Omega),
\]  
\[ (108) \]

\( A_{\text{Oseen}}(v + w, p + \pi) := \partial_t w - \nu \Delta (v + w) - \lambda \partial_t (v + w) + \nabla (p + \pi) \)

is a homeomorphism with \( \| A_{\text{Oseen}}^{-1} \| \) depending only on \( n, q, r, \Omega, \nu, \) and \( \lambda \). If \( q \in (1, \frac{3}{2}) \) in the case \( n = 3 \) or \( q \in (1, \frac{6}{5}) \) in the case \( n = 2 \), then \( \| A_{\text{Oseen}}^{-1} \| \) depends only on the upper bound \( \lambda_0 \) and not on \( \lambda \) itself.

**Proof.** It is well known that the steady-state Oseen operator, that is, the Oseen operator from (108) restricted to time-independent functions, is a homeomorphism as a mapping \( A_{\text{Oseen}} : \mathcal{X}^q_{\lambda} (\Omega) \times \mathcal{D}^{1,q} (\Omega) \to L^q (\Omega); \) see, for example, [10, Theorem VII.7.1]. By Theorem 12, it follows that also the time-periodic Oseen operator is a homeomorphism as a mapping \( A_{\text{Oseen}} : \mathcal{W}^{1,2,q,r}_{\text{per.},\perp} (\mathbb{R} \times \Omega) \times \mathcal{D}^{1,q,r}_{\text{per.},\perp} (\mathbb{R} \times \Omega) \to L^q_{\text{per.},\perp} (\mathbb{R} \times \Omega). \) Since clearly \( \mathcal{P} \) and \( \mathcal{P}_{\perp} \) commute with \( A_{\text{Oseen}} \), it further follows that \( A_{\text{Oseen}} \) is a homeomorphism in the setting (108). The dependency of \( \| A_{\text{Oseen}}^{-1} \| \) on the various parameters follows from [10, Theorem VII.7.1] and Theorem 12.

**Remark 9.** In Corollary 1, the function space that yields maximal \( L^q \) regularity for the time-periodic Oseen operator, that is, the function space that is mapped homeomorphically onto \( L^q_{\text{per.}} (\mathbb{R} \times \Omega) \) by \( A_{\text{Oseen}} \), is identified. As will be demonstrated later, this function space constitutes a suitable setting for investigation of the fully nonlinear problem in the Oseen case \( \lambda \neq 0 \). More specifically, it is possible to choose the parameters in (108) in such a way that for any vector field \( u \in \mathcal{X}^q_{\lambda} (\Omega) \oplus \mathcal{W}^{1,2,q,r}_{\text{per.},\perp} (\mathbb{R} \times \Omega) \), the corresponding nonlinear term \( u \cdot \nabla u \) belongs to the range of \( A_{\text{Oseen}} \). The Stokes case \( \lambda = 0 \) is different. Although maximal \( L^q \) regularity for a certain range of exponents \( q \) can be obtained also in this case, one can simply use [10, Theorem V.4.8] to identify the appropriate space \( \mathcal{X}^q_{0} (\Omega) \) that is mapped homeomorphically onto \( L^q_{0} (\Omega) \) by the steady-state Stokes operator and use this space instead of \( \mathcal{X}^q_{\lambda} (\Omega) \) in Corollary 1; the setting based on this space is not well suited for the investigation of the nonlinear problem when \( \lambda = 0 \). The weighted spaces introduced in Sect. 4.6 constitute a better alternative in this case.

It is common practice to establish \( L^q \) estimates for an exterior domain problem by decomposing the solution into a solution to a bounded domain problem and a whole-space problem, respectively, and then employ the \( L^q \) theory available for these simpler cases. The decomposition is typically done by multiplying the solution with a “cutoff” function. In the case of the (linearized) Navier-Stokes system (106), this “cutoff” technique produces zero-order terms for the pressure on the “right-hand side” of the new equations. It is particularly challenging to estimate these terms. For
this purpose, the following lemma, which was established for a two-dimensional and three-dimensional exterior domain in [12, 14], respectively, is needed:

**Lemma 3.** Let \( \Omega \) and \( \lambda \) be as in Theorem 12. Moreover, let \( R_0 > 0 \) be a constant such that \( \mathbb{R}^n \setminus \Omega \subset B_{R_0} \) and \( s \in (1, \infty) \). There is a constant \( C_8 = C_8(R_0, \Omega, s) \) such that if \( (u, p) \in W^{1,2}_\text{per,\perp}(\mathbb{R} \times \Omega) \times D^{1,r}_\text{per,\perp}(\mathbb{R} \times \Omega) \) is a solution to (106) with data \( F \in L^r_\text{per,\perp}(\mathbb{R} \times \Omega) \) for some \( r \in (1, \infty) \) and satisfying \( \int_{\Omega_{R_0}} p \, dx = 0 \), then

\[
\|p(t, \cdot)\|_{s,\Omega_{R_0}} \leq C_8 \left( \|F(t, \cdot)\|_s + \|\nabla u(t, \cdot)\|_{s,\Omega_{R_0}} + \|\nabla u(t, \cdot)\|_{s,\Omega_{R_0}}^{t-1} \|\nabla u(t, \cdot)\|_{\frac{1}{t},s,\Omega_{R_0}} \right) \tag{109}
\]

for, a.e., \( t \in \mathbb{R} \). Moreover, for every \( \rho > 0 \) with \( \mathbb{R}^n \setminus \Omega \subset B_{\rho} \), there is a constant \( C_9 = C_9(\rho, \Omega, s) \) such that

\[
\|\nabla p(t, \cdot)\|_{s,\Omega^\rho} \leq C_9 \left( \|F(t, \cdot)\|_s + \|p(t, \cdot)\|_{s,\Omega^\rho} \right) \tag{110}
\]

for, a.e., \( t \in \mathbb{R} \).

**Proof.** For the sake of simplicity, the \( t \)-dependence of functions is not indicated in the proof. All norms are taken with respect to the spatial variables only. Consider an arbitrary \( \varphi \in C_0^{\infty}(\overline{\Omega}) \). Observe that for any \( \psi \in C_0^{\infty}(\mathbb{R}) \) holds

\[
\int_0^T \int_{\Omega} \partial_t u \cdot \nabla \varphi \, dx \, dt = \int_0^T \int_{\Omega} \text{div} u \cdot \varphi \, dt \, dxdt = 0,
\]

which implies \( \int_{\Omega} \partial_t u \cdot \nabla \varphi \, dx = 0 \) for, a.e., \( t \). Moreover,

\[
\int_{\Omega} \partial_1 u \cdot \nabla \varphi \, dx = - \int_{\Omega} \text{div} u \cdot \partial_1 \varphi \, dx = 0.
\]

Hence it follows from (106) that \( p \) is a solution to the weak Neumann problem for the Laplacian:

\[
\forall \varphi \in C_0^{\infty}(\overline{\Omega}) : \int_{\Omega} \nabla p \cdot \nabla \varphi \, dx = \int_{\Omega} F \cdot \nabla \varphi + \nu \Delta u \cdot \nabla \varphi \, dx.
\]

Recall that

\[
p \in L^1_{\text{loc}}(\Omega) \quad \nabla p \in L^r(\Omega) \quad \int_{\Omega_{R_0}} p \, dx = 0. \tag{111}
\]
It is well known (see, e.g., [10, Section III.1]) that the weak Neumann problem for the Laplacian is uniquely solvable in the class (111). One can thus write $p$ as a sum
\[ p = p_1 + p_2 \]
of two solutions (in the class above) to the weak Neumann problem
\[ \forall \varphi \in C_0^\infty(\overline{\Omega}) : \int_\Omega \nabla p_1 \cdot \nabla \varphi \, dx = \int_\Omega F \cdot \nabla \varphi \, dx \]
and
\[ \forall \varphi \in C_0^\infty(\overline{\Omega}) : \int_\Omega \nabla p_2 \cdot \nabla \varphi \, dx = \int_\Omega \nu \Delta u \cdot \nabla \varphi \, dx, \]
respectively. The a priori estimate
\[ \forall q \in (1, \infty) : \| \nabla p_1 \|_q \leq c_0 \| F \|_q \] (112)
is well known. An estimate of $p_2$ shall now be established. Consider for this purpose an arbitrary function $g \in C_0^\infty(\Omega_{R_0})$ with $\int_{\Omega_{R_0}} g \, dx = 0$. The existence of a vector field $h \in C_0^\infty(\Omega_{R_0})$ with $\text{div} \, h = g$ and
\[ \forall q \in (1, \infty) : \| h \|_{1,q} \leq c_1 \| g \|_q \]
is well known; see, for example, [10, Theorem III.3.3]. Let $\Phi$ be a solution to the weak Neumann problem for the Laplacian:
\[ \forall \varphi \in C_0^\infty(\overline{\Omega}) : \int_\Omega \nabla \Phi \cdot \nabla \varphi \, dx = \int_\Omega h \cdot \nabla \varphi \, dx. \]
By classical theory, such a solution exists with
\[ \forall q \in (1, \infty) : \Phi \in C^\infty(\overline{\Omega}) \land \| \nabla \Phi \|_{1,q} \leq c_2 \| h \|_{1,q} \leq c_3 \| g \|_q. \]
Since $\Phi$ is harmonic in $\mathbb{R}^n \setminus \overline{B_{R_0}}$, an asymptotic expansion of $\Phi$ implies $\nabla \Phi(x) = O(|x|^{-n})$. The regularity and decay of $\Phi$ ensure the validity of a computation (see [14] for the details) that yields
\[ \int_\Omega p_2 \, g \, dx = -\int_{\partial\Omega} \nabla u : (\nabla \Phi \otimes n - n \otimes \nabla \Phi) \, d\sigma. \]
Apply first Hölder’s inequality and then a classical trace inequality [10, Theorem II.4.1] to deduce
\[
\left| \int_\Omega p_2 \, g \, dx \right| \leq c_4 \| \nabla u \|_{s,\partial\Omega} \| \nabla \Phi \|_{s, \frac{n-1}{n-1}, \partial\Omega} \\
\leq c_5 \| \nabla u \|_{s,\partial\Omega} \| \nabla \Phi \|_{1, \frac{n}{n-(n-1)}, \Omega_{R_0}} \leq c_6 \| \nabla u \|_{s,\partial\Omega} \| g \|_{\frac{n}{n-(n-1)}, \Omega_{R_0}}.
\]
Since \( \int_{\Omega_{R_0}} p_2 \, dx = 0 \), it follows that

\[
\|p_2\|_{\frac{\alpha}{\alpha - 1}, \Omega_{R_0}} = \sup_{g \in C^\infty_0(\Omega_{R_0}), \|g\|_{W^{\frac{\alpha}{\alpha - 1}, \infty}} = 1} \left| \int_{\Omega} p_2 \, g \, dx \right| \leq c_7 \|\nabla u\|_{s, \beta \Omega}.
\]

Another application of the trace inequality [10, Theorem II.4.1] now implies

\[
\|p_2\|_{\frac{\alpha}{\alpha - 1}, \Omega_{R_0}} \leq c_8 \left( \|\nabla u\|_{s, \Omega_{R_0}} + \|\nabla u\|_{s, \Omega_{R_0}}^\frac{1}{1 - \frac{\alpha}{\alpha - 1}} \right).
\]

By Sobolev’s embedding theorem and (112), it can finally be concluded that

\[
\|p\|_{\frac{\alpha}{\alpha - 1}, \Omega_{R_0}} \leq c_9 \|\nabla p_1\|_{s, \Omega_{R_0}} + \|p_2\|_{\frac{\alpha}{\alpha - 1}, \Omega_{R_0}} \leq c_{10} \left( \|F\|_{s} + \|\nabla u\|_{s, \Omega_{R_0}} + \|\nabla u\|_{s, \Omega_{R_0}}^\frac{1}{1 - \frac{\alpha}{\alpha - 1}} \right)
\]

and thus (109). To show (110), one can introduce an appropriate “cutoff” function \( \chi \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) and analyze the weak Neumann problem satisfied by \( \pi := \chi p \) in a similar manner; see again [14].

Also needed for the proof of Theorem 12 are the following embedding properties of time-periodic Sobolev spaces:

**Lemma 4.** Let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be an exterior domain of class \( C^1 \) and \( q \in (1, \infty) \). Assume that \( \alpha \in [0, 2] \) and \( p_0, r_0 \in (q, \infty] \) satisfy

\[
\begin{align*}
 r_0 &\leq \frac{2q}{2 - \alpha q} \quad \text{if } \alpha q < 2, \\
r_0 &< \infty \quad \text{if } \alpha q = 2, \\
r_0 &\leq \infty \quad \text{if } \alpha q > 2,
\end{align*}
\]

and that \( \beta \in [0, 1] \) and \( p_1, r_1 \in (q, \infty] \) satisfy

\[
\begin{align*}
 p_0 &\leq \frac{nq}{n - (2 - \alpha)q} \quad \text{if } (2 - \alpha)q < n, \\
p_0 &< \infty \quad \text{if } (2 - \alpha)q = n, \\
p_0 &\leq \infty \quad \text{if } (2 - \alpha)q > n,
\end{align*}
\]

(113)

\[
\begin{align*}
 r_1 &\leq \frac{2q}{2 - \beta q} \quad \text{if } \beta q < 2, \\
r_1 &< \infty \quad \text{if } \beta q = 2, \\
r_1 &\leq \infty \quad \text{if } \beta q > 2,
\end{align*}
\]

and that \( p_1, r_1 \in (q, \infty] \) satisfy

\[
\begin{align*}
 p_1 &\leq \frac{nq}{n - (1 - \beta)q} \quad \text{if } (1 - \beta)q < n, \\
p_1 &< \infty \quad \text{if } (1 - \beta)q = n, \\
p_1 &\leq \infty \quad \text{if } (1 - \beta)q > n.
\end{align*}
\]

(114)
Then
\[ \forall u \in W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) : \quad \|u\|_{L^q_{\text{per}}(\mathbb{R}; L^p(\Omega))} + \|\nabla u\|_{L^p_{\text{per}}(\mathbb{R}; L^1(\Omega))} \leq C \|u\|_{1,2,q}.\]

\[(115)\]

**Proof.** See [14]. \[\square\]

On the strength of the estimates for the pressure in Lemma 3 and the embedding properties in Lemma 4, a proof of Theorem 12 can be provided:

**Proof of Theorem 12.** By density of \( C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega) \) in \( L^q_{\text{per}}(\mathbb{R} \times \Omega) \), it suffices to consider only \( F \in C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega) \). The starting point will be the solution \( (u, p) \in W^{1,2,2}_{\text{per}}(\mathbb{R} \times \Omega) \times D^{1,2}_{\text{per}}(\mathbb{R} \times \Omega) \) from Theorem 8. By adding a function that depends only on time to \( p \), one may assume without loss of generality that \( \int_{\Omega \times \mathbb{R}} p \, dx = 0 \). The solution \( (u, p) \) shall be decomposed by multiplication of a “cutoff” function.

For this purpose, fix three constants \( 0 < R_0 < \rho < R_* \) such that \( R_0 > R_\rho \). For convenience, the notation \( \mathbb{T} \) for the time-domain \( [0, T] \) is used in the scope of the proof.

Two fundamental estimates shall be established. To show the first one, a cutoff function \( \psi \in C^\infty(\mathbb{R}; \mathbb{R}) \) is introduced with \( \psi(x) = 1 \) for \( |x| \geq \rho \) and \( \psi(x) = 0 \) for \( |x| \leq R_* \). Let \( \Gamma_L \) denote the fundamental solution to the Laplace operator in \( \mathbb{R}^n \) and put

\[
V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad V := \nabla \Gamma_L * \mathbb{R}^n (\nabla \psi_1 \cdot u),
\]

\[
P : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad P := \Gamma_L * \mathbb{R}^n ([\partial_t - \Delta - \lambda \partial_1] (\nabla \psi_1 \cdot u)),
\]

\[
w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad w(t,x) := \psi_1(x) u(t,x) - V(t,x),
\]

\[
\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad \pi(t,x) := \psi_1(x) p(t,x) - P(t,x).
\]

Then \((w, \pi)\) is a solution to the whole-space problem

\[
\begin{cases}
\partial_t w - \Delta w + \lambda \partial_1 w + \nabla \pi = \\
\psi_1 F - 2 \nabla \psi_1 \cdot \nabla u - \Delta \psi_1 u + \lambda \partial_1 \psi_1 u + \nabla \psi_1 p & \text{in } \mathbb{R} \times \mathbb{R}^n, \\
\nabla \cdot w = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n.
\end{cases}
\]

\[(117)\]

The precise regularity of \((w, \pi)\) is not important at this point. It is enough to observe that \( w \) and \( \pi \) belong to the space of tempered time-periodic distributions \( \mathcal{S}^\prime_{\text{per}}(\mathbb{R} \times \mathbb{R}^n) \), which is easy to verify from the definition (116) and the regularity of \( u \) and \( p \). It is not difficult to show (see [28, Lemma 5.3]) that a solution \( w \) to (117) is unique in the class of distributions in \( \mathcal{S}^\prime_{\text{per}}(\mathbb{R} \times \mathbb{R}^n) \) satisfying \( \mathcal{P}w = 0 \). Consequently, \( w \) coincides with the solution from Theorem 9 corresponding to the right-hand side in (117) and therefore satisfies
Lemma 3 to deduce

\[ \|w\|_{1,2,s} \leq c_0 \|\psi_1 F - 2\nabla\psi_1 \cdot \nabla u - \Delta \psi_1 u + \lambda \partial_1 \psi_1 u + \nabla \psi_1 p\|_s \]
\[ \leq c_1 \left( \|F\|_s + \|u\|_{s, \Omega^\rho} + \|\nabla u\|_{s, \Omega^\rho} + \|p\|_{s, \Omega^\rho} \right) \]

for all \( s \in (1, \infty) \). Clearly,
\[ \|\nabla V\|_s + \|\nabla^2 V\|_s \leq c_2 (\|u\|_{s, \Omega^\rho} + \|\nabla u\|_{s, \Omega^\rho}). \]

Since \( u = w + V \) for \( x \in \Omega^\rho \), the estimates above imply
\[ \|\nabla u\|_{s, \Omega^\rho} + \|\nabla^2 u\|_{s, \Omega^\rho} \]
\[ \leq c_3 (\|F\|_s + \|u\|_{s, \Omega^\rho} + \|\nabla u\|_{s, \Omega^\rho} + \|p\|_{s, \Omega^\rho}). \]

for all \( s \in (1, \infty) \). For a similar estimate on \( u \) itself, first turn to (106) and apply Lemma 3 to deduce
\[ \|\partial_s u\|_{s, \Omega^\rho} \leq c_4 (\|F\|_s + \|\Delta u\|_{s, \Omega^\rho} + \|\lambda \partial_1 u\|_{s, \Omega^\rho} + \|\nabla p\|_{s, \Omega^\rho}) \]
\[ \leq c_5 (\|F\|_s + \|u\|_{s, \Omega^\rho} + \|\nabla u\|_{s, \Omega^\rho} + \|p\|_{s, \Omega^\rho}). \]

Since \( \mathcal{P}u = 0 \), Poincaré’s inequality yields \( \|u\|_{s, \Omega^\rho} \leq c_6 \|\partial_s u\|_{s, \Omega^\rho} \). It thus follows that
\[ \|u\|_{1,2,s, \Omega^\rho} \leq c_7 (\|F\|_s + \|u\|_{s, \Omega^\rho} + \|\nabla u\|_{s, \Omega^\rho} + \|p\|_{s, \Omega^\rho}) \]
\[ (118) \]

for all \( s \in (1, \infty) \).

Now a similar estimate for \( u \) over the bounded domain \( \mathbb{T} \times \Omega^\rho \) shall be established. To this end, a “cutoff” function \( \psi_2 \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) is introduced with \( \psi_2(x) = 1 \) for \( |x| \leq \rho \) and \( \psi_2(x) = 0 \) for \( |x| \geq R_0 \). Let \( \mathbf{V} \) be a vector field with
\[ \mathbf{V} \in W_{1,2,2}^{\text{per}, \perp} (\mathbb{R} \times \mathbb{R}^n), \quad \text{supp} \mathbf{V} \subset \mathbb{R} \times \Omega^\rho, R_0, \quad \text{div} \mathbf{V} = \nabla \psi_2 \cdot u, \]
\[ \forall s \in (1, \infty) : \|\mathbf{V}\|_{1,2,s} \leq c \left( \|u\|_{s, \Omega^\rho, R_0} + \|\nabla u\|_{s, \Omega^\rho, R_0} + \|\partial_s u\|_{s, \Omega^\rho, R_0} \right). \]
\[ (119) \]

Since
\[ \int_{\Omega \times R_0} \nabla \psi_2 \cdot u \, dx = \int_{\Omega \times R_0} \text{div} (\psi_2 u) \, dx = \int_{\partial \Omega \times R_0} u \cdot n \, ds = 0, \]
the existence of a vector field \( \mathbf{V} \) with the properties above can be established by the same construction as the one used in [10, Theorem III.3.3]; see also [27, Proof of Lemma 3.2.1]. Now let
\[ w: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad w(t, x) := \psi_2(x) u(t, x) - \mathbf{V}(t, x), \]
\[ \pi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad \pi(t, x) := \psi_2(x) p(t, x). \]
\[ (120) \]
Then \((w, \pi) \in W^{1,2,2}_{\text{per}, \perp}(\mathbb{R} \times \Omega_{R_0}) \times D^{1,2}_{\text{per}, \perp}(\mathbb{R} \times \Omega_{R_0})\) is a solution to the problem

\[
\begin{aligned}
\partial_t w - \Delta w + \nabla \pi &= \psi_2 F + \lambda \psi_2 \partial_t u - 2 \nabla \psi_2 \cdot \nabla u - \Delta \psi_2 u \\
&\quad + \nabla \psi_2 p + \partial_t \mathbf{V} - \Delta \mathbf{V} \\
\text{div } w &= 0 \\
w &= 0
\end{aligned}
\]

in \(\mathbb{R} \times \Omega_{R_0}\), \(\mathbb{R} \times \Omega_{R_0}\), and \(\mathbb{R} \times \partial \Omega_{R_0}\), respectively.

By Theorem 6, it follows that

\[
\|w\|_{1,2,s} \leq c_8 \left(\|F\|_s + \|u\|_{s,T \times \Omega_{R_0}} + \|\nabla u\|_{s,T \times \Omega_{R_0}} + \|p\|_{s,T \times \Omega_{R_0}} + \|\mathbf{V}\|_{1,2,s}\right)
\]

for all \(s \in (1, \infty)\). Since \(\Omega_{R_0} \subset \Omega^\rho\), (119) and (118) can be combined to estimate

\[
\|u\|_{1,2,s,T \times \Omega_{R_0}} \leq c_{10} \left(\|F\|_s + \|u\|_{s,T \times \Omega_{R_0}} + \|\nabla u\|_{s,T \times \Omega_{R_0}} + \|p\|_{s,T \times \Omega_{R_0}}\right).
\]

In combination with (118), the estimate above implies

\[
\|u\|_{1,2,s} \leq c \left(\|F\|_s + \|u\|_{s,T \times \Omega_{R_0}} + \|\nabla u\|_{s,T \times \Omega_{R_0}} + \|p\|_{s,T \times \Omega_{R_0}}\right) \tag{121}
\]

for all \(s \in (1, \infty)\).

The estimate (121) has been established for all \(s \in (1, \infty)\), but it is not actually known at this point whether the right-hand side is finite or not. At the outset, it is only known that the right-hand side is finite for \(s \in (1, 2]\). A bootstrap argument can be used to show that it is also finite for \(s \in (2, \infty)\). For this purpose, the embedding properties of \(W^{1,2,s}_{\text{per}}(\mathbb{R} \times \Omega)\) stated in Lemma 4 are employed to show the implication

\[
\forall s \in [2, \infty) : \quad u \in W^{1,2,s}_{\text{per}}(\mathbb{R} \times \Omega) \Rightarrow u, \nabla u \in L^{\frac{n}{s}}_{\text{per}}(\mathbb{R} \times \Omega). \tag{122}
\]

Now turn to estimate (109) of the pressure term. By Hölder’s inequality,

\[
\int_0^T \left(\|\nabla u(t, \cdot)\|_{s,T \times \Omega_{R_0}} \|\nabla u(t, \cdot)\|_{1, s,T \times \Omega_{R_0}}\right)^{\frac{2n}{2n-1}} dt \leq \left(\int_0^T \|\nabla u(t, \cdot)\|_{s,T \times \Omega_{R_0}}^{\frac{n}{s(n-1)}} dt\right)^{\frac{s(n-1)-n}{s(n-1)}} |u|_{1,2,s}^{\frac{n}{2n-1}}.
\]

By Lemma 4, utilized this time with \(\beta = 1\), the right-hand side above is finite for all \(s \in [2, \infty)\) provided \(u \in W^{1,2,s}_{\text{per}}(\mathbb{R} \times \Omega)\). Due to the normalization of the pressure \(p\) carried out in the beginning of the proof, Lemma 3 can be applied to infer from (109) that
\[ \forall s \in [2, \infty) : \quad u \in W^{1,2,s}_{\text{per}}(\mathbb{R} \times \Omega) \Rightarrow p \in L^{\frac{ns}{s}}_{\text{per}}(\mathbb{R} \times \Omega_{R_0}). \quad (123) \]

By (121) and the implications (122) and (123), it follows that

\[ \forall s \in [2, \infty) : \quad u \in W^{1,2,s}_{\text{per}}(\mathbb{R} \times \Omega) \Rightarrow u \in W^{1,2,s}_{\text{per}}^{\infty}(\mathbb{R} \times \Omega). \quad (124) \]

Starting with \( s = 2 \), the implication (124) can be bootstrapped a sufficient number of times to deduce \( u \in W^{1,2,s}_{\text{per}}(\mathbb{R} \times \Omega) \) for any \( s \in (2, \infty) \). It follows that the right-hand side of (121) is finite for all \( s \in (1, \infty) \). At this point, it is now possible to use interpolation and (109) in combination with Young’s inequality in (121) to show \[ \|u\|_{1,2,s} \leq c_{11} (\|F\|_s + \|u\|_{s,T \times \Omega_{R_0}}) \] for all \( s \in (1, \infty) \).

It follows that the right-hand side of (106) is necessarily zero, which was just shown above, a standard contradiction argument (see, e.g., [12, Proof of Proposition 2]) can be used to eliminate the lower-order term on the right-hand side in (125) to conclude

\[ \|u\|_{1,2,q} + \|\nabla p\|_q \leq c_{12} (\|F\|_s + \|u\|_{s,T \times \Omega_{R_0}}) \quad (125) \]

for all \( s \in (1, \infty) \).

Now return to the estimate (125). Owing to the fact that a solution to (106) with homogeneous right hand is necessarily zero, which was just shown above, a standard contradiction argument (see, e.g., [12, Proof of Proposition 2]) can be used to eliminate the lower-order term on the right-hand side in (125) to conclude

\[ \|u\|_{1,2,q} + \|\nabla p\|_q \leq c_{13} \|F\|_q. \quad (127) \]

It is easy to verify that \( C^\infty_{0,\text{per},\perp}(\mathbb{R} \times \Omega) \) is dense in \( L^q_{\text{per},\perp}(\mathbb{R} \times \Omega) \). By a density argument, the existence of a solution \((u, p) \in W^{1,2,q}_{\text{per},\perp}(\mathbb{R} \times \Omega) \times D^{1,q}_{\text{per},\perp}(\mathbb{R} \times \Omega) \) to (106) that satisfies (127) therefore follows for any \( F \in L^q_{\text{per},\perp}(\mathbb{R} \times \Omega) \).

Finally, assume \((\widehat{u}, \widehat{p}) \in W^{1,2,r}_{\text{per},\perp}(\mathbb{R} \times \Omega) \times D^{1,r}_{\text{per},\perp}(\mathbb{R} \times \Omega) \) is another solution to (106) with \( r \in (1, \infty) \). Applied to the difference \((u - \widehat{u}, p - \widehat{p})\), the duality argument used in (126) yields \( u = \widehat{u} \) and \( \nabla p = \nabla \widehat{p} \). The proof of theorem is thereby complete.
4.8 Existence of $L^q$ Strong Solutions

The question now arises as to whether or not existence of a solution to the fully nonlinear problem (61) can be established on the basis of the $L^q$ estimates and corresponding function spaces from Sect. 4.7. Compared to the class of strong solutions introduced in Sect. 4.3, more information on the asymptotic structure at spatial infinity could be derived for such a solution. If the $T$-time-periodic velocity $u_\infty(t) \in \mathbb{R}^n$ is directed along a single axis, say $u_\infty(t) = u_\infty(t)e_1$, and its net motion over a period is non-zero, that is, $\int_0^T u_\infty(t) \, dt \neq 0$, the question can be answered affirmatively. If $\int_0^T u_\infty(t) \, dt = 0$, then the $L^q$ estimates from Sect. 4.7 are not adequate. In this case, the fully nonlinear problem (61) can be treated in weighted function spaces based on the estimates introduced in Sect. 4.6. Below, the case $\int_0^T u_\infty(t) \, dt \neq 0$ is investigated more closely.

The projections $\mathcal{P}$ and $\mathcal{P}_\perp$ introduced in (62) can be employed to decompose $u_\infty(t)$ into a constant $\lambda := \mathcal{P}u_\infty$ and a oscillatory part $\mathcal{P}_\perp u_\infty$. A linearization of (61) around $\lambda$ leads to (106). In the case $\lambda \neq 0$, Corollary 1 in combination with a fixed point argument can be used to show existence of a solution to the fully nonlinear problem (61) for data sufficiently restricted in “size.” For this purpose, it is convenient to put

$$W_{\text{per}, \perp}^{1,2,q, \frac{nq}{n-q}}(\mathbb{R} \times \Omega) := W_{\text{per}, \perp}^{1,2,q}(\mathbb{R} \times \Omega) \cap W_{\text{per}, \perp}^{1,2,\frac{nq}{n-q}}(\mathbb{R} \times \Omega),$$

$$D_{\text{per}, \perp}^{1,q, \frac{nq}{n-q}}(\mathbb{R} \times \Omega) := D_{\text{per}, \perp}^{1,q}(\mathbb{R} \times \Omega) \cap D_{\text{per}, \perp}^{1,\frac{nq}{n-q}}(\mathbb{R} \times \Omega).$$

The resolution of (61) then reads:

**Theorem 13.** Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ be an exterior domain of class $C^2$. Assume $u_\infty(t) = u_\infty(t)e_1$ with $\lambda := \mathcal{P}u_\infty > 0$. If $n = 3$, let $q \in [\frac{6}{5}, \frac{4}{3}]$. If $n = 2$, let $q \in (1, \frac{6}{5})$. There is an $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, there is an $\varepsilon_3 > 0$ such that for all $f \in L^q_{\text{per}}(\mathbb{R} \times \Omega)$ and $u_\infty \in W_{\text{per}, \infty}^{1,\infty}(\mathbb{R})$ satisfying

$$\|f\|_q + \|\mathcal{P}_\perp f\|_{\frac{nq}{n-q}} + \|\mathcal{P}_\perp u_\infty\|_{1, \infty} \leq \varepsilon_3,$$

there is a solution

$$(u, p) \in X^q_\perp(\Omega) \oplus W_{\text{per}, \perp}^{1,2,q, \frac{nq}{n-q}}(\mathbb{R} \times \Omega) \times D^{1,q}(\Omega) \oplus D_{\text{per}, \perp}^{1,q, \frac{nq}{n-q}}(\mathbb{R} \times \Omega)$$

(129)

to (61).

**Proof.** Consider first the case $n = 3$ and $q \in [\frac{6}{5}, \frac{4}{3}]$. In order to “lift” the boundary value $u_\infty$ in (61), that is, rewrite the system as one of homogeneous boundary values, a solution $(\mathcal{W}, \Pi_{\perp}) \in W_{\text{per}, \perp}^{1,2,q, \frac{3nq}{3n-3q}}(\mathbb{R} \times \Omega) \times D_{\text{per}, \perp}^{1,q, \frac{3nq}{3n-3q}}(\mathbb{R} \times \Omega)$ to

$$\begin{align*}
\frac{\partial \mathcal{W}}{\partial t} + (\mathcal{P}_\perp \mathcal{W}) \cdot \nabla \mathcal{W} &= \nabla p + \mathcal{P}u_\infty \quad \text{in } \mathbb{R} \times \Omega, \\
\mathcal{W} &= 0 \quad \text{on } \partial \mathbb{R} \times \Omega, \\
\mathcal{W} &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega.
\end{align*}$$

Then, for $n = 2$, let $q \in (1, \frac{6}{5})$.
\[
\begin{aligned}
&\left\{\begin{array}{ll}
-\nu \Delta \mathcal{W} + \nabla \Pi_\perp = \mathcal{W} & \text{in } \mathbb{R} \times \Omega, \\
\text{div } \mathcal{W} = 0 & \text{in } \mathbb{R} \times \Omega, \\
\mathcal{W} = \mathcal{P}_\perp u_\infty & \text{on } \mathbb{R} \times \partial \Omega,
\end{array}\right.
\end{aligned}
\]

is introduced. One can use standard theory for elliptic systems to solve (130) in $T$-time-periodic function spaces and obtain a solution that satisfies

\[
\forall r \in (1, \infty) : \|\mathcal{W}\|_{1,2,r} + \|\nabla \Pi_\perp\|_r \leq c_0 \|\mathcal{P}_\perp u_\infty\|_{1,\infty},
\]

where $c_0 = c_0(r, q, \Omega, v)$. Furthermore, classical results for the steady-state Oseen problem [10, Theorem VII.7.1] ensure existence of a solution $(\mathcal{V}, \Pi_s) \in X^q_\lambda(\Omega) \times D^{1,q}(\Omega)$ to

\[
\begin{aligned}
&\left\{\begin{array}{ll}
-\nu \Delta \mathcal{V} - \lambda \partial_1 \mathcal{V} + \nabla \Pi_s = 0 & \text{in } \Omega, \\
\text{div } \mathcal{V} = 0 & \text{in } \Omega, \\
\mathcal{V} = \lambda & \text{on } \partial \Omega,
\end{array}\right.
\end{aligned}
\]

which satisfies

\[
\forall r \in (1, 2) : \|\mathcal{V}\|_{X^q_\lambda(\Omega)} + \|\nabla \Pi_s\|_r \leq c_1 \lambda,
\]

where $c_1 = c_1(r, \Omega, v)$. Focus will now be on finding a solution $(u, p)$ to (61) on the form

\[
u = v + \mathcal{V} + w + \mathcal{W}, \quad p = p + \Pi_s + \pi + \Pi_\perp,
\]

where $(v, p) \in X^q_\lambda(\Omega) \times D^{1,q}(\Omega)$ is a solution to the steady-state problem

\[
\begin{aligned}
&\left\{\begin{array}{ll}
-\nu \Delta v - \lambda \partial_1 v + \nabla p = \mathcal{R}_1(v, w, \mathcal{V}, \mathcal{W}) & \text{in } \Omega, \\
\text{div } v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{array}\right.
\end{aligned}
\]

with

\[
\mathcal{R}_1(v, w, \mathcal{V}, \mathcal{W}) := -v \cdot \nabla v - v \cdot \nabla \mathcal{V} - \mathcal{V} \cdot \nabla v - \mathcal{V} \cdot \nabla \mathcal{V} \\
+ \mathcal{P}[\partial_1 w] + \mathcal{P}[\partial_1 \mathcal{W}] + \mathcal{P},
\]
and \((w, \pi) \in W^{1,2,q, \frac{3q}{3q}}_{\text{per.}}(\mathbb{R} \times \Omega) \times D^{1,q, \frac{3q}{3q}}_{\text{per.}}(\mathbb{R} \times \Omega)\) a solution to

\[
\begin{aligned}
\partial_t w - \nu \Delta w - \lambda \partial_1 w + \nabla \pi &= \mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W}) \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } w &= 0 \quad \text{in } \mathbb{R} \times \Omega, \quad \text{(136)} \\
\end{aligned}
\]

with

\[
\mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W}) := -\mathcal{P}_\perp \left[ w \cdot \nabla w - \mathcal{P}_\perp \left[ w \cdot \nabla \mathcal{W} \right] - \mathcal{P}_\perp \left[ \mathcal{W} \cdot \nabla w \right] - \mathcal{P}_\perp \left[ \mathcal{W} \cdot \nabla \mathcal{W} \right] \right] - v \cdot \nabla w - v \cdot \nabla \mathcal{W} - w \cdot \nabla v - w \cdot \nabla \mathcal{V} \\
- \mathcal{V} \cdot \nabla w - \mathcal{V} \cdot \nabla \mathcal{W} - \mathcal{W} \cdot \nabla v - \mathcal{W} \cdot \nabla \mathcal{V} \\
+ \mathcal{P}_\perp u_{\infty} \partial_1 v + \mathcal{P}_\perp u_{\infty} \partial_1 \mathcal{V} \\
+ \mathcal{P}_\perp \left[ \mathcal{P}_\perp u_{\infty} \partial_1 w \right] + \mathcal{P}_\perp \left[ \mathcal{P}_\perp u_{\infty} \partial_1 \mathcal{W} \right] \\
- \partial_1 \mathcal{W} - \mathcal{V} + \lambda \partial_1 \mathcal{W} + \mathcal{P}_\perp f.
\]

The systems (135) and (136) appear as the result of inserting (134) into (61) and subsequently applying first \(\mathcal{P}\) then \(\mathcal{P}_\perp\) to the equations. Recalling the function spaces introduced in Corollary 1 to define the Banach space

\[
\mathcal{K}_\lambda^q(\mathbb{R} \times \Omega) := \mathcal{X}_\lambda^q(\Omega) \oplus W^{1,2,q, \frac{3q}{3q}}_{\text{per.}}(\mathbb{R} \times \Omega) \times D^{1,q, \frac{3q}{3q}}(\mathbb{R} \times \Omega) \oplus D^{1,q, \frac{3q}{3q}}_{\text{per.}}(\mathbb{R} \times \Omega),
\]

one can obtain solutions \((v, p)\) and \((w, \pi)\) to (135) and (136), respectively, as a fixed point of the mapping

\[
\mathcal{N} : \mathcal{K}_\lambda^q(\mathbb{R} \times \Omega) \to \mathcal{K}_\lambda^q(\mathbb{R} \times \Omega),
\]

\[
\mathcal{N}(v + w, p + \pi) := A_{\text{Oseen}}^{-1} \left( \mathcal{R}_1(v, w, \mathcal{V}, \mathcal{W}) + \mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W}) \right);
\]

note that \(\mathcal{R}_1(v, w, \mathcal{V}, \mathcal{W}) \in L^q(\Omega)\) and \(\mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W}) \in L^{q, \frac{3q}{3q}}_{\text{per.}}(\mathbb{R} \times \Omega)\). More specifically, one can show that \(\mathcal{N}\) is a contracting self-mapping on ball of sufficiently small radius. For this purpose, let \(\rho > 0\) and consider some \((v + w, p + \pi) \in \mathcal{K}_\lambda^q \cap B_\rho\). Suitable estimates of \(\mathcal{R}_1\) and \(\mathcal{R}_2\) in combination with a smallness assumption on \(\varepsilon_3\) from (128) are needed to guarantee that \(\mathcal{N}\) has the desired properties. Regarding the estimates, one can employ Hölder’s inequality, Sobolev embedding, and basic interpolation to obtain

\[
\|v \cdot \nabla v\|_q \leq \|v\|_{\frac{2q}{3q}} \cdot \|\nabla v\|_2 \leq c_2 \lambda^{-\frac{3q-1}{q}} \|v\|_{X_0^2}^2 \leq c_2 \lambda^{-\frac{3q-3}{q}} \rho^2.
\]

(138)
This step requires \( q \in \left[ \frac{6}{5}, \frac{4}{3} \right] \). The other terms in the definition of \( \mathcal{R}_1 \) can be estimated in a similar fashion to conclude in combination with assumption (128) that

\[
\| \mathcal{R}_1(v, w, V, W) \|_{L^q(\Omega)} \leq c_3 \left( \lambda^{-\frac{3q-3}{q}} \rho^2 + \lambda^{\frac{1}{2}} \rho + \lambda^{\frac{1}{2}} \rho + \lambda^{\frac{3}{2}} \rho + \rho \epsilon_3 + \epsilon_3^2 + \epsilon_3 \right).
\]

An estimate of \( \mathcal{R}_2 \) is required both in the \( L^q_{\text{per}}(\mathbb{R} \times \Omega) \) and \( L^{\frac{3q}{3q-\gamma}}_{\text{per}}(\mathbb{R} \times \Omega) \) norm. Observe that

\[
\| \mathcal{P}_\perp u_\infty \partial_1 v \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} \leq c_4 \| \mathcal{P}_\perp u_\infty \|_{L^\infty(\Omega)} \| \partial_1 v \|_q \leq c_4 \lambda^{-1} \epsilon_3 \rho.
\]

The other terms in \( \mathcal{R}_2 \) can be estimated, in part with the help of the embedding properties from Lemma 4, to obtain

\[
\| \mathcal{R}_2(v, w, V, W) \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} \leq c_5 \left( \lambda^{-1} \epsilon_3 \rho + \rho^2 + \rho \epsilon_3 + \epsilon_3^2 + \lambda \rho + \lambda \epsilon_3 + \epsilon_3 \right).
\]

Lemma 4 can also be used to establish an \( L^{\frac{3q}{3q-\gamma}}_{\text{per}, \perp}(\mathbb{R} \times \Omega) \) estimate of \( \mathcal{R}_2 \). For example,

\[
\| w \cdot \nabla w \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} \leq c_6 \| w \|_{L_{\text{per}}^q(\mathbb{R}; L^\infty(\Omega))} \| \nabla w \|_{L^\infty(\mathbb{R}; L^{\frac{3q}{3q-\gamma}}(\Omega))} \leq c_7 \rho^2,
\]

where Lemma 4 is utilized with \( \alpha = 0 \) and \( \beta = 1 \) in the last inequality. For this utilization of Lemma 4, it is required that \( q \geq \frac{6}{5} \). Further note that

\[
\| \mathcal{P}_\perp u_\infty \partial_1 v \|_{L^{\frac{3q}{3q-\gamma}}_{\text{per}, \perp}(\mathbb{R} \times \Omega)} \leq \epsilon_3 \| v \|_{L^q_{\perp}(\mathbb{R} \times \Omega)} \leq \epsilon_3 \rho,
\]

which explains the choice of the exponent \( \frac{3q}{3q-\gamma} \) in the setting of the mapping \( \mathcal{N} \). The rest of the terms in \( \mathcal{R}_2 \) can be estimated to conclude

\[
\| \mathcal{R}_2(v, w, V, W) \|_{L^q_{\text{per}}(\mathbb{R} \times \Omega)} \leq c_8 \left( \rho^2 + \rho \epsilon_3 + \epsilon_3^2 + \lambda \epsilon_3 + \epsilon_3 \right).
\]

Now choose \( \lambda_0 \leq 1 \) and deduce by Corollary 1; recall that \( \| A_{\text{Oseen}}^{-1} \| \) does not depend on \( \lambda \), the estimate

\[
\| \mathcal{N}(v + w, p + \pi) \|_{L^q_{\perp}(\mathbb{R} \times \Omega)} \leq \| A_{\text{Oseen}}^{-1} \| \left( \| \mathcal{R}_1 \|_{L^q(\Omega)} + \| \mathcal{R}_2 \|_{L^{\frac{3q}{3q-\gamma}}_{\text{per}}(\mathbb{R} \times \Omega)} \right) \leq C_{10} \left( \lambda^{-\frac{3q-3}{q}} \rho^2 + \lambda^{-1} \epsilon_3 \rho + \lambda^{\frac{1}{2}} \rho + \lambda^{\frac{3}{2}} + \epsilon_3^2 + \epsilon_3 \right).
\]
In particular, $\mathcal{N}$ becomes a self-mapping on $B_\rho$ if
\[
C_{10} \left( \lambda^{-\frac{3q-3}{4}} \rho^2 + \lambda^{-1} \varepsilon_3 \rho + \lambda^\frac{1}{4} \rho + \lambda^\frac{3}{2} + \varepsilon_3^2 + \varepsilon_3 \right) \leq \rho.
\]

One may choose $\varepsilon_3 := \lambda^2$ and $\rho := \lambda$ to find the above inequality satisfied for sufficiently small $\lambda$. For such choice of parameters, one may further verify that $\mathcal{N}$ is also a contraction. By the contraction mapping principle, existence of a fixed point for $\mathcal{N}$ follows. This concludes the proof in the case $n = 3$.

The proof in the case $n = 2$ and $q \in \left(1, \frac{6}{5}\right)$ follows along the same lines. To ensure the existence of a solution to (132) satisfying an estimate like (133) with a constant independent of $\lambda$, one cannot use [10, Theorem VII.7.1], but can instead use [10, Theorem XII.5.1] to obtain a solution to (132) that satisfies
\[
\forall r \in (1, 6/5) : \|\nabla\|_{X_\times^r(\Omega)} + \langle \Pi_{\varepsilon} \rangle_{1,r} \leq c_9 \lambda^\frac{2r-2}{r} |\log \lambda|^{-1} \lambda, \tag{140}
\]
with $c_9 = c_9(r, \Omega, v)$. The estimate corresponding to (138) in the case $n = 2$ reads
\[
\|\psi \cdot \nabla \psi\|_q \leq c_{10} \lambda^{-\frac{3q-3}{4}} \|\psi\|_{X_\times^2}^2 \leq c_{10} \lambda^{-\frac{3q-3}{4}} \rho^2; \tag{141}
\]
see, for example, [10, Lemma XII.5.4]. The rest of the proof in the case $n = 2$ follows by simple adjustments to the proof for $n = 3$ above.

**Remark 10.** It is possible to establish higher-order regularity for the solution in Theorem 13 by a bootstrap argument based on the linear theory from Theorem 12. More specifically, if the data possesses higher-order regularity, say $f \in W_{per,\perp}^{m,q}(\mathbb{R} \times \Omega)$, one can put the nonlinear term $u \cdot \nabla u$ on the right-hand side in (61) and iteratively apply Theorem 12 after taking partial derivatives on both sides. With such an argument, it is possible to establish a degree of regularity for $(u, p)$ corresponding to the regularity of the data $f$, that is, $\partial_\alpha^\beta \tilde{u} \in X_\times^q(\Omega) \oplus W_{per,\perp}^{1,2,q}(\mathbb{R} \times \Omega)$ for $|\alpha| + |\beta| \leq m$. For more details on such a result, see [28, Theorem 2.4]. Alternatively, higher-order regularity can be obtained via regularity theory for the initial-value problem as mentioned in Remark 5.

**Remark 11.** The solution $u$ in Theorem 13 possess enough summability at spatial infinity for an adaptation of the uniqueness argument from the proof of Theorem 2 to be carried out. More precisely, given a weak solution $U$ in the sense of Definition 2, it is possible to insert $u$ as a “test function” in the weak formulation (64) for $U$. In addition, after multiplication by $U$ in the system (61) satisfied by $u$, the summability of both, in particular the latter, is adequate to integrate by parts. These are the two main steps in proof of Theorem 2. Provided therefore that both $u$ and $U$ satisfy an appropriate energy inequality corresponding to (61) and the data is sufficiently restricted in “size,” it can be shown that $u = U$. In other words, the strong solution in Theorem 13 can be shown to be unique in a class of weak solutions satisfying an energy inequality. See [28, Theorem 2.3] for more details on such a result.
4.9 Asymptotic Structure

Important physical properties of a solution \( u \) to (61) are related to its asymptotic structure at spatial infinity. The asymptotic structure is best exposed by an asymptotic expansion \( u(t, x) = \mathcal{A}(t, x) + \mathcal{R}(t, x) \) into an explicitly known leading term \( \mathcal{A} \) and a remainder term \( \mathcal{R} \) that decays faster to 0 as \( |x| \to \infty \) than \( \mathcal{A} \). The task of identifying such an expansion shall now be addressed for \( u_\infty(t) = u_\infty(t)e_1 \), that is, for functions \( u_\infty \) directed along a single axis.

The case \( \mathcal{P}u_\infty \neq 0 \) is considered first. A strong solution in the class (129) is singled out for investigation. Theorem 13 yields existence of a solution in this class, so it is a reasonable starting point. By nature of the function space in (129), the steady-state part \( \mathcal{P}u \) of such a solution enjoys, at the outset, different \( L^q \) summability properties than the oscillatory part \( \mathcal{P}\mathcal{R} \). In fact, since \( \mathcal{P}u_\perp \in \mathcal{W}^{1,q}(\Omega) \) and \( \mathcal{P}_\perp \in \mathcal{W}^{1,2,q}(\mathbb{R} \times \Omega) \), better spatial decay is available for \( \mathcal{P}_\perp u \) in the sense of summability, that is, the range of exponents \( q \) for which \( \mathcal{P}_\perp u(t, \cdot) \in L^q(\Omega) \) is lower than the range of exponents \( q \) for which \( \mathcal{P}u \in L^q(\Omega) \). This suggests that the leading term in an asymptotic expansion of \( u \) is dominated by \( \mathcal{P}u \); a key observation that underpins the analysis below.

Recall that the Oseen fundamental solution \( \Gamma^{Oseen} \) introduced in Sect. 4.5 satisfies

\[
\forall q_0 \in [1, 2]: \quad \Gamma^{Oseen} \notin L^{q_0}(\mathbb{R}^n \setminus B_r) \quad \text{for any } r > 0; \quad (142)
\]

see, for example, [10, Chapter VII.3]. On the strength of this information, the theorem below yields an asymptotic expansion of a solution to (61) with the decay of the remainder term characterized in the sense of summability as described above.

**Theorem 14.** Let \( \Omega \subset \mathbb{R}^n (n = 2, 3) \) be an exterior domain of class \( C^2 \). Assume \( u_\infty(t) = u_\infty(t)e_1 \) with \( \lambda := \mathcal{P}u_\infty > 0 \). Moreover, assume \( f \in C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega) \).

If \( n = 3 \), let \( q \in \left[ \frac{6}{5}, \frac{3}{2} \right] \). If \( n = 2 \), let \( q \in (1, \frac{5}{4}) \). There is an \( \varepsilon_4 > 0 \) such that if \( u_\infty \in C^\infty_{\text{per}}(\mathbb{R}) \) satisfies \( \|\mathcal{P}_\perp u_\infty\|_\infty \leq \varepsilon_5 \), then a solution \( (u, p) \) to (61) in the class (129) satisfies

\[
u(t, x) = \Gamma^{Oseen}(x) \cdot \mathcal{F} + \mathcal{R}(t, x), \quad (143)\]

where

\[
\mathcal{F} := \frac{1}{T} \int_0^T \int_{\partial \Omega} S(u, p) \cdot n \, d\sigma + \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} f(t, x) \, dx \, dt \quad (144)
\]

and

\[
\forall q_0 \in \left( \frac{n}{n+1}, \infty \right): \quad \mathcal{R} \in L^\infty_{\text{per}}(\mathbb{R}; L^{q_0}(\Omega)).\quad (145)
\]
Here, \( S(u, p) \) denotes the Cauchy stress tensor \( S(u, p) := \nu (\nabla u + (\nabla u)^T) - pI \).

**Remark 12.** As previously described, a solution to (61) describes the fluid flow around an object moving with constant velocity \( u_\infty \). In physical terms, the quantity \( F \) in (144) equals the total force exerted upon the fluid by the moving object and the external force \( f \) combined over a full time-period.

**Remark 13.** In light of (142), the decomposition (143) indicates that the velocity field \( u \) possesses the asymptotic structure of the Oseen fundamental solution. From a physical point of view, this implies the existence of a wake behind the moving object; see again [10, Chapter VII.3]. Another interesting observation is that the leading term in the expansion (143) is independent of time, which suggests that oscillatory characteristics of the fluid flow appear only locally around the body.

**Proof of Theorem 14.** Only the case \( n = 3 \) is considered. The case \( n = 2 \) can be treated by a similar approach. Since the data \( f \) and \( u_\infty \) are assumed to be smooth, recall from Remark 10 that also \( u_2 \in C^1_\text{per}(\mathbb{R} \times \Omega) \) and \( p \in C^\infty_\text{per}(\mathbb{R} \times \Omega) \). Although not strictly necessary, the smoothness of the solution simplifies the proof as local regularity becomes a nonissue. Let \( \chi \in C^\infty_0(\mathbb{R}^2; \mathbb{R}) \) be a “cutoff” function with \( \chi = 0 \) on \( B_1 \) and \( \chi = 1 \) on \( \mathbb{R}^2 \setminus B_2 \). Put \( \chi_R(x) := \chi \left( \frac{x}{R} \right) \) for some \( R > R_0 \). Since

\[
\hat{\int}_{B_{R,2R}} \nabla \chi_R(x) \cdot u(t, x) \, dx = \int_{\Omega_{2R}} \nabla \chi_R(x) \cdot u(t, x) \, dx = - \int_{\Omega_{2R}} \chi_R(x) \cdot \text{div} u(t, x) \, dx = 0,
\]

there is a vector field \( U \in C^\infty_\text{per}(\mathbb{R} \times \mathbb{R}^3) \) satisfying both \( \text{supp} U \subset \mathbb{R} \times B_{R,2R} \) and \( \text{div} U = \nabla \chi_R \cdot u \). The construction of \( U \) is well known in the case of time-independent vector fields; see [10, Theorem III.3.3]. The same construction can be applied in the above case of time-periodic vector fields; see [27]. Let \( \tilde{u} := \chi_R u - U \) and \( \tilde{p} := \chi_R p \). Then

\[
\left\{ \begin{array}{ll}
\partial_t \tilde{u} - \nu \Delta \tilde{u} - \lambda \partial_i \tilde{u} + \nabla \tilde{p} = \mathcal{P} u_\infty \partial_1 \tilde{u} - \tilde{u} \cdot \nabla \tilde{u} + \chi_R f + h & \text{in } \mathbb{R} \times \mathbb{R}^3, \\
\text{div} \tilde{u} = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3, \\
\tilde{u}(t + T, x) = \tilde{u}(t, x) & \text{in } \mathbb{R} \times \mathbb{R}^3, \end{array} \right.
\]

(146)

with \( h \in C^\infty_\text{per}(\mathbb{R} \times \mathbb{R}^3) \) of bounded support \( \text{supp} h \subset \mathbb{R} \times B_{2R} \). Put \( v := \mathcal{P} \tilde{u} \) and \( w := \mathcal{P} \tilde{p} \). At the outset

\[
(v + w, p) \in X_\lambda^q(\mathbb{R}^3) \oplus W^{1,2,q,\frac{3q}{q-3}}_\text{per.\perp}(\mathbb{R} \times \mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3) \oplus D^{1,q,\frac{3q}{q-3}}_\text{per.\perp}(\mathbb{R} \times \mathbb{R}^3).
\]

(147)
However, it is possible to show that

$$
\forall q_0 \in (1, q] : \quad v \in X^{q_0}_\lambda (\mathbb{R}^3), \ w \in W^{1,2,q_0}_\text{per,\perp} (\mathbb{R} \times \mathbb{R}^3). \quad (148)
$$

For this purpose, put

$$
M_R := \lambda \nabla \hat{u}
$$

and identify \((\hat{u}, \hat{p})\) as a fixed point of the mapping

$$
\mathcal{N} : K^q_\lambda (\mathbb{R} \times \mathbb{R}^3) \to K^q_\lambda (\mathbb{R} \times \mathbb{R}^3),
$$

$$
\mathcal{N}(\hat{u}, \hat{p}) := A_{\text{Oseen}}^{-1}(\mathcal{P}_\perp u_\infty \partial_1 \hat{u} - M_R \hat{u} + \nabla f + h),
$$

where \(K^q_\lambda\) is defined as in (137). Provided \(R\) is chosen sufficiently large and \(\|\mathcal{P}_\perp u_\infty\|_\infty\) sufficiently small, it can be shown, by estimates very similar to those made in the proof of Theorem 13, that \(\mathcal{N}\) as mapping

$$
\mathcal{N} : K^{q_0}_\lambda (\mathbb{R} \times \mathbb{R}^3) \cap K^q_\lambda (\mathbb{R} \times \mathbb{R}^3) \to K^{q_0}_\lambda (\mathbb{R} \times \mathbb{R}^3) \cap K^q_\lambda (\mathbb{R} \times \mathbb{R}^3)
$$

is a contraction for all \(q_0 \in (1, q]\). By the contraction mapping principle, \(\mathcal{N}\) therefore has a unique fixed point. Uniqueness of the fixed point implies (148). From the embedding properties in Lemma 4, it immediately follows from (148) that

$$
\forall q_0 \in (1, \infty) : \quad w \in L^\infty_{\text{per}}(\mathbb{R}; L^{q_0}(\mathbb{R}^3)). \quad (149)
$$

Focus is now shifted to \(v\). Applying \(\mathcal{P}\) to the system (146), one finds that \(v\) is a solution to the steady-state problem

$$
\begin{cases}
- v \Delta v - \lambda \partial_1 v + \nabla p \\
= \mathcal{P}\left[\mathcal{P}_\perp u_\infty \partial_1 w - v \cdot \nabla v - \mathcal{P}[w \cdot \nabla w] + \mathcal{P}[\chi_R f + h]\right] \quad \text{in } \mathbb{R}^3,
\end{cases}
$$

$$
\text{div } v = 0 \quad \text{in } \mathbb{R}^3.
$$

(150)

It is therefore standard to show that \(v\) satisfies

$$
v = \Gamma^{\text{Oseen}} \ast \left[\mathcal{P}\left[\mathcal{P}_\perp u_\infty \partial_1 w - v \cdot \nabla v - \mathcal{P}[w \cdot \nabla w] + \mathcal{P}[\chi_R f + h]\right]\right]. \quad (151)
$$

More precisely, since

$$
\begin{align*}
\forall q_0 \in (2, \infty) : \ & \Gamma^{\text{Oseen}} \in L^{q_0}(\mathbb{R}^3 \setminus B_r) \quad \text{for any } r > 0, \\
\forall q_0 \in [1, 3) : \ & \Gamma^{\text{Oseen}} \in L^{q_0}_{\text{loc}}(\mathbb{R}^3)
\end{align*}
$$

(152)

(see, e.g., [10, Chapter VII.3]), it follows as a consequence of (148) that the convolution on the right-hand side in (151) is well defined and thus a solution to
A standard uniqueness argument implies that this solution coincides with $v$. An asymptotic expansion of $v$ can be derived from (151). To this end, recall (see again [10, Chapter VII.3]) that
\[ 8 \frac{q_0}{2} \leq 3; \]
for any $r > 0$.
\[ \forall q_0 \in (4/3, \infty) : \quad \nabla \Gamma^{Oseen} \in L^q_0(\mathbb{R}^3 \setminus B_r) \quad \text{for any } r > 0, \]
\[ \forall q_0 \in [1, 3/2) : \quad \nabla \Gamma^{Oseen} \in L^q_{loc}(\mathbb{R}^3). \]
(153)
Consequently, the summability properties from (148) in combination with Young’s inequality imply
\[ \left( \Gamma^{Oseen} * \left[ \mathcal{P}[\mathcal{P}_\perp u_\infty \partial_1 w] - v \cdot \nabla v - \mathcal{P}[w \cdot \nabla w] \right] \right)_i \]
\[ = \left( \Gamma^{Oseen} * \text{div} \left[ \mathcal{P}[\mathcal{P}_\perp u_\infty w \otimes e_1] - v \otimes v - \mathcal{P}[w \otimes w] \right] \right)_i \]
\[ = \partial_k \Gamma^{Oseen}_{ij} * \left[ \mathcal{P}[\mathcal{P}_\perp u_\infty w \otimes e_1] - v \otimes v - \mathcal{P}[w \otimes w] \right]_{jk} \in L^q_0(\mathbb{R}^3) \]
(154)
for all $q_0 \in \left( \frac{4}{3}, \infty \right)$. Moreover, due to (153), it is standard to show
\[ \Gamma^{Oseen} * \mathcal{P}[\chi_R f + h] - \Gamma^{Oseen} \cdot \int_{\mathbb{R}^3} \mathcal{P}[\chi_R f + h] \, dx \in L^q_0(\mathbb{R}^3) \]
(155)
for all $q_0 \in \left( \frac{4}{3}, \infty \right)$. It remains to compute the integral in (155). For this purpose, isolate $\mathcal{P}[\chi_R f + h]$ in (146), and compute
\[ \int_{\mathbb{R}^3} \mathcal{P}[\chi_R f + h] \, dx = \lim_{R \to \infty} \frac{1}{T} \int_0^T \int_{B_R} \partial_1 \tilde{u} - v \Delta \tilde{u} - \lambda \partial_1 \tilde{u} + \nabla \tilde{p} - \mathcal{P}_\perp u_\infty \partial_1 \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} \, dx \, dt \]
\[ = \lim_{R \to \infty} \frac{1}{T} \int_0^T \int_{B_R} -\text{div} \mathcal{S}(\tilde{u}, \tilde{p}) + \text{div} \left[ (\tilde{u} - u_\infty) \otimes \tilde{u} \right] \, dx \, dt \]
\[ = \lim_{R \to \infty} \frac{1}{T} \int_0^T \int_{\partial B_R} -\mathcal{S}(u, p) \cdot n + \left[ (u - u_\infty) \otimes u \right] \cdot n \, d\sigma \, dt \]
\[ = \frac{1}{T} \int_0^T \int_{\Omega} \mathcal{S}(u, p) \cdot n \, d\sigma \, dt + \frac{1}{T} \int_0^T \int_{\Omega} f \, dx \, dt, \]
(156)
where the change of sign in the last integral is due to $n$ denoting the outer normal on $\partial \Omega$. The identity above together with (149), (154), and (155) concludes the proof. \qed
Under certain conditions, the asymptotic expansion in Theorem 14 can also be established with a pointwise decay estimate of the remainder term. Below, a sketch of a proof is given in the case $\mathcal{P}u_\infty \neq 0$ and $\mathcal{P}_\perp u_\infty = 0$. The proof is based on the pointwise estimates of the fundamental solution in Sect. 4.5. The assumption $\mathcal{P}_\perp u_\infty = 0$ is equivalent to the requirement that $u_\infty$ is constant. Only the case of a three-dimensional exterior domain is included. A similar result is not available in the two-dimensional case.

**Theorem 15.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^2$. Assume $u_\infty = \lambda e_1$ with $\lambda > 0$ a constant. Let $f \in C_0^{\infty}(\mathbb{R} \times \Omega)$. Then a solution $(u, p)$ to (61) in the class (129) satisfies (143) with

$$\forall \varepsilon > 0 \exists C > 0 \forall |x| > 1 : \|R(\cdot, x)\|_\infty \leq C|x|^{-\frac{1}{2}+\varepsilon}.$$  \hspace{1cm} (157)

**Proof.** As in the proof of Theorem 14, $\tilde{u} := \chi_R u - \mathcal{U}$ and $\tilde{p} := \chi_R p$ are introduced and the system (146) investigated. As in the proof of Theorem 9, the interpretation of (146) as a system of partial differential equations on the group $G$ is employed. One can proceed as in the proof of Theorem 14 to deduce (148), which implies that the convolution of the time-periodic fundamental solution $T$ from Sect. 4.5 with the right-hand side in (146) is well defined in the classical sense and thus constitutes a solution to (146). By a uniqueness argument similar to the one made in the proof of Theorem 9, it therefore follows that

$$\tilde{u} = \Gamma^{TP} \ast \left[ -\tilde{u} \cdot \nabla \tilde{u} + \chi_R f + h \right].$$

Compared to the right-hand side in (146), the term $\mathcal{P}_\perp u_\infty \partial_1 \tilde{u}$ is missing above since $u_\infty = \lambda e_1$ implies $\mathcal{P}_\perp u_\infty = 0$. The structure of $\Gamma^{TP}$ identified in Theorem 10 now implies for $v := \mathcal{P}\tilde{u}$ and $w := \mathcal{P}_\perp \tilde{u}$ the identities

$$v = \Gamma^{Oseen} \ast \left[ -v \cdot \nabla v - \mathcal{P}[w \cdot \nabla w] + \mathcal{P}[\chi_R f + h] \right],$$  \hspace{1cm} (158)

$$w = \Gamma^{\perp} \ast \left[ -v \cdot \nabla w - w \cdot \nabla v - \mathcal{P}_\perp[w \cdot \nabla w] + \mathcal{P}_\perp[\chi_R f + h] \right],$$  \hspace{1cm} (159)

where in the first identity, it is used that $\Gamma^{Oseen} \otimes 1_{R/TZ} \ast F = \Gamma^{Oseen} \ast \mathcal{P}F$ and in the second the property $\Gamma^{\perp} \ast \mathcal{P}_\perp F = \Gamma^{\perp} \ast F$ for all sufficiently regular $F$. Inspired by the proof in [10, Chapter X.8] of the asymptotic expansion of a solution to the corresponding steady-state problem, one can show

$$\forall \varepsilon > 0 : \int_{\mathbb{R}^3 \setminus B_R} |\nabla v|^2 \, dx + \frac{1}{T} \int_0^T \int_{\mathbb{R}^3 \setminus B_R} |\nabla w|^2 \, dx \, dt \leq CR^{-1+\varepsilon}.$$  \hspace{1cm} (160)
The above estimate is crucial. Together with the pointwise estimate (98) of \( \Gamma \), it delivers the foundation for a pointwise estimate of all terms on the right-hand side of (158) and (159). The resulting estimates for \( v \) and \( w \) (see [15] for the details) are

\[
\forall \varepsilon > 0 : \quad |v(x)| \leq C |x|^{-1+\varepsilon}, \tag{161}
\]

\[
\forall \varepsilon > 0 : \quad |w(t, x)| \leq C \left( |x|^{-\frac{1}{2}} + |x|^{-\frac{1}{2}+\varepsilon} (\|v\|_{L^\infty(B|x|^{1/2})} + \|w\|_{L^\infty(\mathbb{R} \times B|x|^{1/2})}) \right). \tag{162}
\]

Now insert (161) into (162). After two bootstrap iterations, it follows that

\[
|w(t, x)| \leq C |x|^{-\frac{1}{2}+\varepsilon}. \tag{163}
\]

see again [15] for the details. With this decay estimate for \( w \), one can return to (158) and deduce as in [10, Theorem X.8.1] that

\[
\forall \varepsilon > 0 : \quad |w(t, x)| \leq C |x|^{-\frac{1}{2}+\varepsilon}; \tag{164}
\]

The theorem follows from (163), (164), and the computation made in (156).

The asymptotic structure of solutions to (61) changes significantly when \( \mathcal{P} u = 0 \). The canonical leading term in this case is no longer expressed in terms of the fundamental solution to the corresponding steady-state equation. In contrast to the case \( \mathcal{P} u \not\equiv 0 \), where the leading term, as was seen in Theorem 14 and Theorem 15, is expressed in terms of the Oseen fundamental solution \( \Gamma_{\text{Oseen}} \), the canonical leading term in the case \( \mathcal{P} u \equiv 0 \) is not expressed in terms of the Stokes fundamental solution \( \Gamma_{\text{Stokes}} \). Instead, it is identified, in the three-dimensional case, as a so-called Landau solution; see below. In the purely steady-state case, the identification was made for the first time by Korolev and Sverak [24]. An extension of their result to the time-periodic case with \( u = 0 \) was made by Kang, Miura, and Tsai [22]. Both results hold for solutions \( u \) that are small in the norm of the weighted space \( X_1(\Omega) \) introduced in (101).

The Landau solution \( (U_{\text{Landau}}^b, P^b) \), corresponding to a parameter \( b \in \mathbb{R}^3 \), is a solution in \( \mathcal{D}'(\mathbb{R}^3) \) to

\[
\begin{cases}
-\Delta U_{\text{Landau}}^b + U_{\text{Landau}}^b \cdot \nabla U_{\text{Landau}}^b + \nabla P^b = b \delta, \\
\text{div} U_{\text{Landau}}^b = 0
\end{cases} \tag{165}
\]

that is axially symmetric about the axis \( b \mathbb{R} \) and \((-1\)-)homogeneous. Here, \( \delta \) denotes the delta distribution. The Landau solution can be given explicitly. Assume for simplicity that \( b = k e_1, k \in \mathbb{R} \), then
\begin{equation}
U^b_{\text{Landau}}(x) = \frac{2}{|x|} \left( \frac{c \frac{x_1}{|x|} - 1}{c - \frac{x_1}{|x|}} \frac{x}{|x|^2} + \frac{1}{c - \frac{x_1}{|x|}} e_3 \right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},
\end{equation}
(166)

\begin{equation}
P^b = \frac{4}{|x|^2} \left( \frac{c \frac{x_1}{|x|} - 1}{c - \frac{x_1}{|x|}} \right)^2 \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},
\end{equation}

where

\begin{equation}
k = \frac{8\pi c}{3(c^2 - 1)} \left( 2 + 6c^2 - 3c(c^2 - 1) \log \frac{c + 1}{c - 1} \right). \quad (167)
\end{equation}

As one may easily verify, for each \( k \in \mathbb{R} \setminus \{0\} \), there exists a unique \( c \in \mathbb{R} \) with \(|c| > 1\) so that \((k, c)\) satisfies (167). Hence, for each \( b \in \mathbb{R}^3 \setminus \{0\} \), a Landau solution \((U^b_{\text{Landau}}, P^b)\) to (165) is given by the expression above. Moreover, \( b = k e_1 \to 0 \) as \(|c| \to \infty\). The Landau solution was originally constructed by Landau [31]. See [3] for the explicit computation of the expressions above.

The asymptotic expansion at spatial infinity of a solution to (61) in the case \( u_\infty = 0 \) established by Kang, Miura, and Tsai [22] is given below. The isotropically weighted function spaces from Sect. 4.6 are employed to express the decay of the remainder term. For simplicity, the result is stated for a strong solution in \( W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) \) that is small in the norm of \( L^\infty_{\text{per}}(\mathbb{R}; X_1(\Omega)) \). While the assumption on the regularity of the solution can easily be reduced, the smallness assumption is critical.

**Theorem 16.** Let \( \Omega \subset \mathbb{R}^3 \) be an exterior domain of class \( C^2 \). Assume \( u_\infty = 0 \) and \( f \in C_{0,\text{per}}(\mathbb{R} \times \Omega) \). There is an \( \varepsilon_6 > 0 \) such that a solution \((u, p)\) to (61) in \( W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) \times D^{1,q}_{\text{per}}(\mathbb{R} \times \Omega) \), for some \( q \in (1, \infty) \), with \( \|u\|_{L^\infty_{\text{per}}(\mathbb{R}; X_1(\Omega))} \leq \varepsilon_6 \) satisfies

\begin{equation}
u(t, x) = U^F_{\text{Landau}}(x) + \mathcal{R}(t, x),
\end{equation}

where \( F \) is given by (144) and \( \mathcal{R} \in L^\infty_{\text{per}}(\mathbb{R}; X_\beta(\Omega)) \) for all \( \beta \in (1, 2) \).

**Proof.** As in the proof of Theorem 14, a cutoff function is applied to reduce the problem to that of determining the asymptotic expansion of a solution \( \tilde{u} \) to the whole-space problem (146). Also as in the proof of Theorem 14, the velocity is decomposed into a steady-state part \( \nu := \mathcal{P}\tilde{u} \) and oscillatory part \( \mathcal{w} := \mathcal{P}_\perp \tilde{u} \). Since \( \nu \) satisfies the steady-state problem (150) with \( \lambda = 0 \), the expansion (168) can be shown for \( \nu \) as in [24]. The assumption \( \|u\|_{L^\infty_{\text{per}}(\mathbb{R}; X_1(\Omega))} \leq \varepsilon_6 \) is needed for this step. It remains to show that \( \mathcal{w} \in L^\infty_{\text{per}}(\mathbb{R}; X_\beta(\Omega)) \) for all \( \beta \in (1, 2) \). As in the proof of Theorem 15 (recall (159)), \( \mathcal{w} \) can be expressed as the convolution
of $\Gamma^\perp$ with a vector field of type $\text{div} \, G + g$, where $g$ has compact support and $G \in L^\infty_{\text{per}, \perp}(\mathbb{R}; \mathcal{X}_2)^{n \times n}$ by assumption. Estimates as in the proof of Theorem 15, based on the properties of $\Gamma^\perp$ from Theorem 10, can then be carried out to conclude that $w \in L^\infty_{\text{per}}(\mathbb{R}; \mathcal{X}_2)$. \hfill \Box

## 5 Flow in a Pipe

Also highly relevant from a physical point of view are the equations governing time-periodic Navier-Stokes flows in infinite pipes. In many models, not least the model of a cardiovascular system, a time-periodic flow rate is prescribed. Below it is outlined how to adapt the methods from Sect. 3 to establish existence and uniqueness of strong time-periodic solutions for this particular model.

For simplicity, a piping system with only two outlets whose cross sections are described by bounded domains $S_1 \subset \mathbb{R}^2$ and $S_2 \subset \mathbb{R}^2$, respectively, is considered. The region of flow can then be written as a disjoint union

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_0$$

with $\Omega_0 \subset \mathbb{R}^3$ bounded and

$$\Omega_1 := \{x \in \mathbb{R}^3 \mid x_1 < -1, \ (x_2, x_3) \in S_1\},$$
$$\Omega_2 := \{x \in \mathbb{R}^3 \mid 1 < x_1, \ (x_2, x_3) \in S_2\}.$$

A time-periodic Poiseuille flow shall be imposed at spatial infinity of each outlet.

**Proposition 1.** Let $S \subset \mathbb{R}^2$ be a bounded domain and $\Phi \in W^{1,2}_{\text{per}}(\mathbb{R})$. The problem

$$\begin{cases}
\partial_t V = \Delta V + G & \text{in } \mathbb{R} \times S, \\
V = 0 & \text{on } \mathbb{R} \times \partial S, \\
\int_S V(t, x) \, dx = \Phi(t)
\end{cases}$$

admits a unique solution $(V, G)$ in the class

$$V \in L^2_{\text{per}}(\mathbb{R}; W^{2,2}(S)), \ \partial_t V \in L^2_{\text{per}}(\mathbb{R}; L^2(S)),
V \in C^1_{\text{per}}(\mathbb{R}; W^{1,2}_0(S)), \ G \in L^2_{\text{per}}(S),$$

which satisfies
\[ \|V\|_{L^2_{\text{per}}(\mathbb{R}; W^{2,2}(S))} + \|\partial_t V\|_{L^2_{\text{per}}(\mathbb{R}; L^2(S))} + \|V\|_{L^\infty_{\text{per}}(\mathbb{R}; W^{1,2}_0(S))} + \|G\|_2 \leq C_{11}\|\Phi\|_{1,2}, \]

(171)

with \(C_{11} = C_{11}(S)\).

**Proof.** See [9, Theorem 1.2].

**Definition 3.** Let \((V, G)\) be the solution in Proposition 1. Then \((v_p, p_p)\) defined by

\[
\begin{align*}
v_p : \mathbb{R} \times S &\to \mathbb{R}^3, \quad v_p(t, x_1, x_2, x_3) := V(x_2, x_3)e_1, \\
p_p : \mathbb{R} \times S &\to \mathbb{R}, \quad p_p(t, x_1, x_2, x_3) := -G(t)x_1
\end{align*}
\]

is called the time-periodic Poiseuille flow corresponding to flow rate \(\Phi\) and cross section \(S\).

A flow described by the velocity field of a Poiseuille flow is also called fully developed. It is the natural asymptotic value, from a physical point of view, to impose at spatial infinity for a flow through a pipe. Given a flow rate \(\Phi\), the two time-periodic Poiseuille flows \(v_{p_1}\) and \(v_{p_2}\) corresponding to \(\Phi\) and cross sections \(S_1\) and \(S_2\), respectively, are introduced. The full time-periodic Navier-Stokes system in the pipe \(\Omega\) can then be written as

\[
\begin{cases}
\partial_t u + u \cdot \nabla u = v\Delta u - \nabla p & \text{in } \mathbb{R} \times \Omega, \\
\text{div } u = 0 & \text{in } \mathbb{R} \times \Omega, \\
u = 0 & \text{on } \mathbb{R} \times \partial \Omega, \\
\int_{S_i} u(t, x) \cdot n \, d\sigma(x_2, x_3) = \Phi(t) & \text{for } [i = 1 \land x_1 < -1] \text{ and } [i = 2 \land 1 < x_1], \\
\lim_{|x| \to \infty} (u(t, x) - v_{p_i}(t, x)) = 0 & (i = 1, 2), \quad u(t, x) = u(t, x).
\end{cases}
\]

(172)

In order to adapt the approach from Sect. 3 based on a Galerkin approximation to (172), the system must be rewritten as one of homogeneous boundary values and flow rate. For this purpose, a vector field that can “lift” those values is needed. This leads to the introduction of the following so-called flow-rate carrier:

**Proposition 2.** Let \(\Omega \subset \mathbb{R}^3\) be a (pipe) domain of type (169) with a \(C^2\)-smooth boundary and \(\Phi \in W^{1,2}_{\text{per}}(\mathbb{R})\). There is a \(T\)-time-periodic flow-rate carrier \(A\) with

\[
\begin{align*}
\forall \omega \subset \Omega \text{ bounded} : A &\in L^2_{\text{per}}(\mathbb{R}; W^{1,2}(\omega)), \quad \partial_t A \in L^2_{\text{per}}(\mathbb{R}; L^2(\omega)), \\
\text{div } A &= 0, \quad A = 0 \text{ on } \mathbb{R} \times \partial \Omega,
\end{align*}
\]

(173)

(174)
\[
\int_{S_i} \mathbf{A}(t, x) \cdot n \, d\sigma(x_2, x_3) = \Phi(t) \quad \text{for } [i = 1 \land x_1 < -1] \text{ and } [i = 2 \land 1 < x_1],
\]
(175)

\[A = v_{p_i} \text{ in } \Omega_i,\]
(176)

\[\forall \omega \subset \Omega \text{ bounded } \exists C_{12} > 0 : \]
\[
\|A\|_{L^\infty_\text{per}(\mathbb{R}; W^{1,2}(\omega))} + \|A\|_{L^2_\text{per}(\mathbb{R}; W^{2,2}(\omega))} + \|\partial_t A\|_{L^2_\text{per}(\mathbb{R}; L^2(\omega))} \leq C_{12}\|\Phi\|_{1.2}.
\]
(177)

**Proof.** See [9, Section 1.3.2]. \qed

With the necessary “lifting” field at hand, one can show:

**Theorem 17.** Let \(\Omega \subset \mathbb{R}^3\) be a (pipe) domain of type (169) with a \(C^2\)-smooth boundary. Moreover, let \(\Phi \in W^{1,2}_\text{per}(\mathbb{R})\) be a \(T\)-time-periodic flow rate; \(v_{p_1}\) and \(v_{p_2}\) Poiseuille flows corresponding to \(\Phi\) and cross sections \(S_1\) and \(S_2\), respectively; and \(A\) a \(T\)-time-periodic flow-rate carrier satisfying (173), (174), (175), (176), and (177). There is a constant \(\varepsilon_7 > 0\) such that if

\[
\|\Phi\|_{1,2} \leq \varepsilon_7,
\]
(178)

then there is \((w, p)\) with

\[
w \in L^\infty_\text{per}(\mathbb{R}; D^{1,2}_0(\Omega)) \cap L^2_\text{per}(\mathbb{R}; W^{2,2}(\Omega)), \quad \partial_t w \in L^2_\text{per}(\mathbb{R}; L^2(\Omega)),
\]
\[p \in L^2_\text{per}(\mathbb{R}; D^{1,2}(\Omega))
\]
(179)

such that \((u, p)\) with \(u := A + w\) is a solution to (172). Moreover, this solution satisfies

\[
\|w\|_{L^\infty_\text{per}(\mathbb{R}; W^{1,2}(\omega))} + \|w\|_{L^2_\text{per}(\mathbb{R}; W^{2,2}(\omega))} + \|\partial_t w\|_{L^2_\text{per}(\mathbb{R}; L^2(\omega))} \leq C_{13}\|\Phi\|_{1,2},
\]
(180)

where \(C_{13} = C_{13}(\Omega)\). In addition,

\[
\lim_{r \to \infty} \|u(t, x) - v_{p_i}(t, x)\|_{L^\infty(\Omega_r)} \quad (i = 1, 2)
\]
(181)

with \(\Omega_{1r} := \{x \in \Omega_1 \mid x_1 < -r\}\) and \(\Omega_{2r} := \{x \in \Omega_2 \mid r < x_1\}\). Finally, if \((\tilde{u}, \tilde{p})\) is another solution in the class (179), then \(u = \tilde{u}\) and \(\nabla p = \nabla \tilde{p}\).

**Proof.** One looks for a solution \(u\) on the form \(u = w + A\) with \(w\) satisfying
\[
\begin{aligned}
\partial_t w - \nu \Delta w + \nabla p + w \cdot \nabla w + A \cdot \nabla w + w \cdot \nabla A &= f \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } w &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
|w| &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega, \\
\int_{S_i} w(t, x) \cdot n \, d\sigma(x_2, x_3) &= 0 \quad \text{for } [i = 1 \land x_1 < -1] \text{ and } [i = 2 \land 1 < x_1], \\
\lim_{|x| \to \infty} w(t, x) &= 0, \quad w(t, x) = w(t, x).
\end{aligned}
\]

(182)

Since (182) is a system with homogeneous boundary values, flow-rate and asymptotic value at spatial infinity, one can initiate a Galerkin approximation. The same technique that was used in the proof of Theorem 3 can also be used here to first solve the problem on bounded domains of type \( \Omega_k := \Omega \cap \{x \in \mathbb{R}^3 \mid |x_1| < k\} \).

The desired solution is then found as the limit \( k \to \infty \) of the resulting sequence in the same manner as in the proof of Theorem 8. For the details, see [9, Proof of Theorem 1.6].

6 Conclusion

The time-periodic Navier-Stokes problem can be viewed as a generalization of the steady-state Navier-Stokes problem. Indeed, time-independent data and solutions are trivially time periodic. On the other hand, time-periodic solutions may also be viewed as solutions to an initial-value Navier-Stokes problem for some unspecified initial value. Both perspectives can be used to extend a number of fundamental results to the time-periodic problem. Without restrictions on the “size” of the data, existence of a weak solution can be shown. Uniqueness of this solution is guaranteed when smallness in the space \( L^\infty_{\text{per}}(\mathbb{R}; L^2(\Omega)) \) is assumed. Existence and uniqueness of a strong solution can be established when the magnitude of the data is sufficiently restricted. In the two-dimensional case, existence of a strong solution even follows without this restriction on the data. Moreover, a classical integrability condition ensures regularity of weak solutions. An appropriate Fourier transform can be utilized to show \( L^q \) estimates for the linearized equations in the whole space based on Fourier multipliers. These estimates can be extended to the cases of bounded and exterior domains. Finally, a fundamental solution can be identified and used to analyze the asymptotic structure of solutions.

Cross-References

- Stationary Navier-Stokes Flow in Exterior Domains and Landau Solutions
- Steady-State Navier-Stokes Flow Around a Moving Body
- Stokes Equation in the Lp-Setting: Well-Posedness and Regularity Properties
- Stokes Semigroup, Strong, Weak and Very Weak Solutions for General Domains
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