Finite groups with $\mathbb{P}$-subnormal and strongly permutable subgroups

V. S. Monakhov, I. L. Sokhor

Abstract: Let $H$ be a subgroup of a group $G$. The permutizer $P_G(H)$ is the subgroup generated by all cyclic subgroups of $G$ which permute with $H$. A subgroup $H$ of a group $G$ is strongly permutable in $G$ if $P_U(H) = U$ for every subgroup $U$ of $G$ such that $H \leq U \leq G$. We investigate groups with $\mathbb{P}$-subnormal or strongly permutable Sylow and primary cyclic subgroups. In particular, we prove that groups with all strongly permutable primary cyclic subgroups are supersoluble.

Keywords: finite group, permutizer, $\mathbb{P}$-subnormality, simple group, supersoluble group.

1 Introduction

All groups in this paper are finite. A group of prime power order is called a primary group.

Let $H$ be a subgroup of a group $G$. The permutizer of $H$ in $G$ is the subgroup generated by all cyclic subgroups of $G$ which permute with $H$, i.e.

$$P_G(H) = \langle x \in G \mid \langle x \rangle H = H \langle x \rangle \rangle.$$ 

The permutizer $P_G(H)$ contains the normalizer $N_G(H)$, see [1, p. 26]. X. Liu and Y. Wang [2] proved that a group $G$ has a Sylow tower of supersoluble type if $P_G(X) = G$ for every Sylow subgroup $X$ of $G$. A. F. Vasil’ev, V. A. Vasil’ev and T. I. Vasil’eva [3] described the structure of a group $G$ in which $P_Y(X) = Y$ for every Sylow subgroup $X$ of $G$ and every subgroup $Y \geq X$. They proposed the following notation.

Definition 1. A subgroup $H$ of a group $G$ is

(1) permutable in $G$ if $P_G(H) = G$;
(2) strongly permutable in $G$ if $P_U(H) = U$ for every subgroup $U$ of $G$ such that $H \leq U \leq G$.

We note that a quasinormal subgroup is strongly permutable. In the symmetric group $S_n$, $n \in \{3, 4, 6\}$, a Sylow 2-subgroup is strongly permutable, but it is not quasinormal.

A. F Vasil’ev, V. A. Vasil’ev and V. N. Tyutyanov [4] proposed the following notation.

Definition 2. Let $\mathbb{P}$ be the set of all primes. A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$ if there is a subgroup chain

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

such that $|H_i : H_{i-1}| \in \mathbb{P} \cup \{1\}$ for every $i$.

The class of all groups with $\mathbb{P}$-subnormal Sylow subgroups is denoted by $w\mathfrak{A}$ and the class of all groups with $\mathbb{P}$-subnormal primary cyclic subgroups is denoted by $v\mathfrak{A}$. These classes are quite well studied [4, 5]. In particular, these classes are subgroup-closed saturated formations.

In a soluble group, a $\mathbb{P}$-subnormal Hall subgroup (in particular, a Sylow subgroup) is strongly permutable [3, 3.8]. We prove that in a soluble group, the converse is true, see Proposition [1]. As a result we obtain new criteria for the supersolubility of a group and also [7, Theorem A]: a group $G \in w\mathfrak{A}$ if and only if every Sylow subgroup is $\mathbb{P}$-subnormal or strongly permutable in $G$.

In the following theorem, we enumerate all simple non-abelian groups with a $\mathbb{P}$-subnormal or strongly permutable Sylow subgroup.

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Theorem A. Let $G$ be a simple non-abelian group and let $R$ be a Sylow $r$-subgroup of $G$. 

(1) If $R$ is $\mathbb{P}$-subnormal in $G$, then $r = 2$ and $G$ is isomorphic to $L_2(7)$, $L_2(11)$ or $L_2(2^m)$ and $2^m + 1$ is a prime.

(2) If $R$ is strongly permutable and $\mathbb{P}$-subnormal in $G$, then $r = 2$ and $G \cong L_2(7)$.

For a primary cyclic subgroups we prove two following theorems.

Theorem B. If all primary cyclic subgroups of a group $G$ are strongly permutable, then $G$ is supersoluble.

Theorem C. If every primary cyclic subgroup of a group $G$ is $\mathbb{P}$-subnormal or strongly permutable, then $G \in \mathfrak{U}$.

2 Preliminaries

Let $G$ be a group. We use $\pi(G)$ to denote the set of all prime devisors of $|G|$. If $r$ is a maximal element of $\pi(G)$, then we write $r = \max \pi(G)$. By $H \leq G$ ($H < G$, $H \triangleleft G$, $H \triangleleft G$) we denote a (proper, maximal, normal) subgroup $H$ of $G$.

We use the GAP system [8] to build examples. Note that GAP package Permut [9] is especially useful for testing subgroups permutability.

Lemma 1. Let $H$ and $L$ be subgroups of a group $G$ and let $N$ be a normal subgroup of $G$.

(1) If $H$ is $\mathbb{P}$-subnormal in $G$, then $H \cap N$ is $\mathbb{P}$-subnormal in $N$ and $HN/N$ is $\mathbb{P}$-subnormal in $G/N$ [3, Lemma 3].

(2) If $H$ is $\mathbb{P}$-subnormal in a soluble group $G$ and $U \leq G$, then $H \cap U$ is $\mathbb{P}$-subnormal in $U$ [5, Lemma 4 (1)].

(3) If $H \leq L$, $H$ is $\mathbb{P}$-subnormal in $L$ and $L$ is $\mathbb{P}$-subnormal in $G$, then $H$ is $\mathbb{P}$-subnormal in $G$ [5, Lemma 3].

Lemma 2 ([5, Lemma 2 (2)]). Every subgroup of a supersoluble group is $\mathbb{P}$-subnormal.

Lemma 3. Let $H$ be a $\mathbb{P}$-subnormal subgroup of a group $G$. If $r = \max \pi(G)$, then $O_r(H) \leq O_r(G)$.

Proof. By the hypothesis, there is a subgroup chain

$$H = H_0 \leq H_1 < H_2 < \ldots < H_n = G$$

such that for every $i$, $|H_i : H_{i-1}| \in \mathbb{P}$. Assume that $O_r(H) \leq O_r(H_{i-1})$, we prove $O_r(H) \leq O_r(H_i)$. Let $H_{i-1} = A$ and $H_i = B$. If $A$ is normal in $B$, then $O_r(A)$ is subnormal in $B$, and $O_r(H) \leq O_r(A) \leq O_r(B)$. If $A$ is not normal in $B$, then $|B : A| = q \in \mathbb{P}$ and $A = N_B(A)$. Consider the representation of $B$ by permutations on the right cosets of $A$ [10] 1.6.2. Note that $B/A_B$ is isomorphic to a subgroup of the symmetric group $S_q$ and $|B/A_B : A/A_B| = q$. Since $|S_q| = q! = (q-1)!q$ and $|B/A_B|$ divides $|S_q|$, we get $|A/A_B|$ divides $(q-1)!$. As $q \in \pi(B) \subseteq \pi(G)$ and $r = \max \pi(G)$, we have $q \leq r$ and $A/A_B$ is an $r'$-group. Therefore $O_r(A) \leq A_B$. Since $O_r(A)$ is normal in $A_B$, we get $O_r(A)$ is subnormal in $B$ and $O_r(H) \leq O_r(A) \leq O_r(B)$. Hence $O_r(H) \leq O_r(G)$ by induction. \qed

The following lemma contains permutable and strongly permutable subgroups properties we need.
Lemma 4. Let $H$ be a subgroup of a group $G$ and let $N$ be a normal subgroup of $G$.

1. If $H$ is (strongly) permutable in $G$, then $HN/N$ is (strongly) permutable in $G/N$ [3, lemma 3.2. (1),(4)].

2. If $N \leq H$, then $H$ is (strongly) permutable in $G$ if and only if $H/N$ is (strongly) permutable in $G/N$.

3. If $H$ is strongly permutable in $G$ and $H \leq U$, then $H$ is strongly permutable in $U$.

Proof. (2) This statement is known for permutable subgroups [3, lemma 3.2 (3)]. We prove it for strongly permutable subgroups. If $H$ is a strongly permutable subgroup of $G$, then in view of Statement (1), $H/N$ is strongly permutable in $G/N$. Conversely, let $H/N$ be strongly permutable in $G/N$ and let $A$ be a subgroup of $G$ containing $H$. Then $P_{A/N}(H/N) = A/N$. In view of [3, Lemma 3.6], $P_{A/N}(H/N) = P_A(H)/N$ and $P_A(H) = A$, hence $H$ is strongly permutable in $G$.

(3) This is evident in view of Definition 1(2). $\square$

Lemma 5. Let $r = \max \pi(G)$ and let $R$ be a Sylow $r$-subgroup of a group $G$. Then $N_G(R) = P_G(R)$. In particular, if $R$ is permutable in $G$, then $R$ is normal in $G$.

Proof. Let $x \in G$ and $R(x) = \langle x \rangle R$. It is clear $\langle x \rangle = \langle x_1 \rangle \times \langle x_2 \rangle$, where $\langle x_1 \rangle$ is a Sylow $r$-subgroup of $\langle x \rangle$ and $\langle x_2 \rangle$ is a Hall $r'$-subgroup of $\langle x \rangle$. In view of [10, VI.4.7], $R = R(x_1)$, therefore $R(x) = R(x_2)$. Now, all Sylow $r'$-subgroups of $R(x)$ are cyclic. As $r = \max \pi(R(x))$, it implies that $R$ is normal in $R(x)$ by [10, IV.2.7], and $\langle x \rangle \leq N_G(R)$. Since $P_G(R)$ is generated by elements $x$ such that $R(x) = \langle x \rangle R$, we conclude $P_G(R) = N_G(R)$. $\square$

Lemma 6. If every Sylow subgroup of a group $G$ is a subgroup of $G$ and let $R$ be a Sylow $r$-subgroup of a group $G$. Then $N_G(R) = P_G(R)$. In particular, if $R$ is permutable in $G$, then $R$ is normal in $G$. If $P_G(R)$ is generated by elements $x$ such that $R(x) = \langle x \rangle R$, we conclude $P_G(R) = N_G(R)$. $\square$

Lemma 7 ( [11, Lemma 2.1]). Let $M$ be a maximal subgroup of a soluble group $G$, and assume that $G = MC$ for a cyclic subgroup $C$. Then $|G : M|$ is a prime or 4. Also, if $|G : M| = 4$, then $G/M_G = S_4$.

We will also repeatedly use the following statement.

Lemma 8 ( [5, Lemma 2.2]). Let $\mathfrak{F}$ be a saturated formation and let $G$ be a group. Suppose that $G \notin \mathfrak{F}$ but $G/N \in \mathfrak{F}$ for any normal subgroup $N$ of $G$, $N \neq 1$. Then $G$ is a primitive group.

3 Groups with permutable and $\mathbb{P}$-subnormal Sylow subgroups

Proposition 1. Let $G$ be a soluble group and let $H$ be a Hall subgroup. Then $H$ is $\mathbb{P}$-subnormal in $G$ if and only if $H$ is strongly permutable in $G$. 

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Proof. Let $H$ be $\mathbb{P}$-subnormal in $G$. According to \cite[3.8]{}, $H$ is strongly permutable in $G$. For completeness, we give the proof of this statement. We use induction on the order of $G$. Since $H$ is $\mathbb{P}$-subnormal in $G$, there is a maximal subgroup $M$ of $G$ such that $H \leq M$, $|G : M| \notin \mathbb{P}$ and $H$ is $\mathbb{P}$-subnormal in $M$. By induction, $H$ is strongly permutable in $M$ and $M = P_M(H) \leq P_G(H)$. Since $M$ is a maximal subgroup of $G$, we assume $P_G(H) = M$. Suppose that $M_G \neq 1$ and $L$ is a minimal normal subgroup of $G$ that is contained in $M_G$. According to Lemma \cite[(1)]{}, $HL/L$ is $\mathbb{P}$-subnormal in $G/L$, and by induction, $HL/L$ is permutable in $G/L$. Hence $HL$ is permutable in $G$ in view of Lemma \cite[(3)]{}. Since $G$ is soluble, we conclude $L$ is an elementary abelian $q$-group for some $q \in \pi(G)$. If $q \in \pi(H)$, then $HL = H$ and $H$ is permutable in $G$, a contradiction. Therefore we can assume that $q \notin \pi(H)$. Since $HL$ is permutable in $G$, then $P_G(HL) = G$ and there is $x \in G \setminus M$ such that $\langle x \rangle HL = HL(x) = A$. Suppose that $A$ is a proper subgroup of $G$. As $H$ is $\mathbb{P}$-subnormal in $G$, by Lemma \cite[(2)]{}, it follows that $H$ is $\mathbb{P}$-subnormal in $A$, and by induction, $H$ is permutable in $A$. Therefore $A = P_A(H) \leq P_G(H) = M$ and $x \in M$, a contradiction. Hence $G = \langle x \rangle HL$. If $L \leq \Phi(G)$, then $G = \langle x \rangle H$ and $x \in P_G(H) = M$, a contradiction. Consequently, $L$ is not contained in $\Phi(G)$ and there is a maximal subgroup $K$ of $G$ that does not contain $L$. In that case, $G = LK$ and we can assume $H \leq K$. By induction, $H$ is permutable in $K$ and $K = P_K(H) \leq P_G(H) = M$. Hence we get $M = K$ and $L \leq K$, a contradiction. Thus, $M_G = 1$ and $G$ is a primitive group. Consequently, $G = N \rtimes M$, where $N = F(G)$ is a unique minimal normal subgroup of $G$. Since $|G : M| \notin \mathbb{P}$, we deduce $N$ is a cyclic subgroup and $N \leq P_G(H) = M$, a contradiction. Thus $H$ is permutable in $G$, and in view of Lemma \cite[(2)]{}, $H$ is strongly permutable in $G$.

Conversely, let $H$ be a Hall strongly permutable subgroup of a soluble group $G$. Using induction on the order of $G$ we prove that $H$ is $\mathbb{P}$-subnormal in $G$. Let $H \leq M < G$. By Lemma \cite[(3)]{}, $H$ is strongly permutable in $M$, and by induction, $H$ is $\mathbb{P}$-subnormal in $M$. If $|G : M| \notin \mathbb{P}$, then $H$ is $\mathbb{P}$-subnormal in $G$ by Lemma \cite[(3)]{}. Hence we can assume that $|G : M| \notin \mathbb{P}$, in particular, $M$ is not normal in $G$. According to Lemma \cite[(1)]{}, $G/M_G$ contains a strongly permutable Hall subgroup $HM_G/M_G$. As $HM_G \leq M$, we obtain that $H$ is $\mathbb{P}$-subnormal in $HM_G$ by induction. If $M_G \neq 1$, then $HM_G/M_G$ is $\mathbb{P}$-subnormal in $G/M_G$ by induction. By Lemma \cite[(1)]{}, $HM_G$ is $\mathbb{P}$-subnormal in $G$, and $H$ is $\mathbb{P}$-subnormal in $G$ in view of Lemma \cite[(3)]{}.

Therefore we can assume that $M_G = 1$. Since $G$ is soluble, we get $G = N \rtimes M$, $N = F(G) = C_G(N) = O_p(G)$ is a unique minimal normal subgroup in $G$. Let $HN < G$. By induction, $HN/N$ is $\mathbb{P}$-subnormal in $G/N$, and $H$ is $\mathbb{P}$-subnormal in $G$. Finally, we consider the case, when $H = M$ is a Hall subgroup. By the hypothesis, there is $x \in G \setminus H$ such that $\langle x \rangle H = H(x) = G$. Let $\langle x \rangle = \langle x_1 \rangle \times \langle x_2 \rangle$, where $\langle x_1 \rangle$ is a $p$-subgroup and $\langle x_2 \rangle$ is a $p'$-subgroup. Since $N$ is a normal Sylow $p$-subgroup of $G$, we conclude $\langle x_1 \rangle \leq N$ and $|x_1| = p$. According to \cite[VI.4.6]{}, $H = \langle x_2 \rangle H$. Now, $G = \langle x \rangle H = \langle x_1 \rangle H$ and $|G : H| = p$.  

**Corollary 1.1.** \cite{Theorem A} If every Sylow subgroup of a group $G$ is $\mathbb{P}$-subnormal or strongly permutable, then $G \in \mathfrak{w1}$. Conversely, if $G \in \mathfrak{w1}$, then every Sylow subgroup is $\mathbb{P}$-subnormal and strongly permutable in $G$.

**Proof.** Assume that every Sylow subgroup of $G$ is $\mathbb{P}$-subnormal or strongly permutable. By Lemma \cite{} $G$ has a Sylow tower of supersoluble type, which means that $G$ is soluble. Hence by Proposition \cite{} every strongly permutable Sylow subgroup of $G$ is $\mathbb{P}$-subnormal and $G \in \mathfrak{w1}$.

Conversely, let $G \in \mathfrak{w1}$. By the definition of $\mathfrak{w1}$, every Sylow subgroup is $\mathbb{P}$-subnormal in $G$. Since $G$ is soluble, according to Proposition \cite{} every Sylow subgroup is strongly permutable in $G$.  

**Corollary 1.2.** Let $G$ be a group. The following statements are equivalent.

1. $G$ is supersoluble.
2. Every Hall subgroup of $G$ is $\mathbb{P}$-subnormal or strongly permutable.
3. Every Hall subgroup of $G$ is $\mathbb{P}$-subnormal or permutable.
According to properties of primitive groups, \( \pi \) for subgroup. If \( M \) normal subgroup in \( G \), then \( P \subseteq \pi(G) \). In view of Lemma 7, then \( P \) is a Hall subgroup-closed saturated formation, we deduce \( P \) is a Hall subgroup in \( G \). From Lemma 4 (1), thus the hypothesis holds for \( G \) by Lemma 8.

Let \( N \) be a normal subgroup of \( G \), \( N \neq 1 \), and let \( H \) be a Hall \( \pi \)-subgroup of \( G = G/N \) for \( \pi \subseteq \pi(G) \). Then \( H = HN/N \) for a Hall \( \pi \)-subgroup \( H \) of \( G \). If \( H \) is \( \pi \)-subnormal in \( G \), then \( H \) is \( \pi \)-subnormal in \( G \) by Lemma 1 (1). If \( H \) is permutable in \( G \), then \( H \) is permutable in \( G \) by Lemma 4 (1). Thus the hypothesis holds for \( G \), and by induction, \( G \in \mathfrak{U} \). As \( \mathfrak{U} \) is a subgroup-closed saturated formation, we deduce \( G \) is a primitive group by Lemma 8.

According to properties of primitive groups, \( \Phi(G) = 1 \), \( G = R \rtimes M \), \( R = F(G) \) is a minimal normal subgroup in \( G \), \( |R| > r \), \( M \) is a maximal subgroup in \( G \), \( M \in \mathfrak{U} \). Note that \( M \) is a Hall subgroup. If \( M \) is \( \pi \)-subnormal in \( G \), then \( |G : M| = r = |R| \), a contradiction. Suppose that \( M \) is permutable in \( G \), i.e. \( P_0(M) = G \). In that case, there is \( x \in G \setminus M \) such that \( G = M \langle x \rangle \). In view of Lemma 7, \( |G : M| = 4 \), but \( r = \max \pi(G) \). Hence \( G \) is a 2-group.

Corollary 1.3. If every Sylow subgroup of a biprimary group \( G \) is \( \mathbb{P} \)-subnormal or permutable, then \( G \) is supersolvable. Conversely, in a supersolvable biprimary group every Sylow subgroup is \( \mathbb{P} \)-subnormal and strongly permutable.

According to Proposition 11 for a Hall subgroup of a soluble group, \( \mathbb{P} \)-subnormality and strongly permutability are equivalent. In simple groups, this is not true.

Example 1. In \( L_2(8) \), a Hall \( \{2, 7\} \)-subgroup is strongly permutable, but it is not \( \mathbb{P} \)-subnormal.

Example 2. In \( L_2(9) \), a Sylow 2-subgroup is strongly permutable, but it is not \( \mathbb{P} \)-subnormal.

Example 3. In \( L_2(5) \), a Sylow 2-subgroup is \( \mathbb{P} \)-subnormal, but it is not permutable.

Theorem A. Let \( G \) be a simple non-abelian group and let \( R \) be a Sylow \( r \)-subgroup of \( G \).

1. If \( R \) is \( \mathbb{P} \)-subnormal in \( G \), then \( r = 2 \) and \( G \) is isomorphic to \( L_2(7) \), \( L_2(11) \) or \( L_2(2^m) \) and \( 2^m + 1 \) is a prime.

2. If \( R \) is strongly permutable and \( \mathbb{P} \)-subnormal in \( G \), then \( r = 2 \) and \( G \cong L_2(7) \).

Proof. Since \( R \) is \( \mathbb{P} \)-subnormal in \( G \), there is a subgroup chain

\[ R = H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_{n-1} = H \leq H_n = G \]

such that \( |H_{i+1} : H_i| \in \mathbb{P} \). It is clear that \( R \) is \( \mathbb{P} \)-subnormal in \( H \). Let \( |G : H| = p \). Since \( H_G = 1 \), the representation of \( G \) on the set of left cosets by \( H \) is exactly of degree \( p \) and \( G \) is isomorphic to a subgroup of the symmetric group \( S_p \) of order \( p \) for \( p = \max \pi(G) \), \( H \) is a Hall \( p' \)-subgroup of \( G \). From Lemma 3 it follows that a Sylow \( p \)-subgroup of \( G \) is not \( \mathbb{P} \)-subnormal in \( G \), therefore \( r < p \). Since the unit subgroup is \( \mathbb{P} \)-subnormal in \( R \), the unit subgroup is \( \mathbb{P} \)-subnormal in \( H \) and in \( G \). According to [12], [13, p. 342], \( G \) is isomorphic to one of the following groups

\[ L_2(7), \ L_2(11), \ L_3(3), \ L_3(5), \ L_2(2^m), \ 2^m + 1 \text{ is prime.} \]

Let \( G \cong L_2(7) \). Then \( |G| = 2^3 \cdot 3 \cdot 7 \), \( p = 7 \), \( H \cong S_4 \). In \( S_4 \), a Sylow 2-subgroup is \( \mathbb{P} \)-subnormal, a Sylow 3-subgroup is not \( \mathbb{P} \)-subnormal, therefore \( r = 2 \). In \( L_2(7) \), there are two conjugate classes that are isomorphic to \( S_4 \). Since all Sylow 2-subgroups are conjugate, we get \( R \) is contained in two non-conjugate subgroups \( A \leq G \) and \( B \leq G \) that are isomorphic to \( S_4 \).
Since $A = RC_3$, $B = RC_3^x$, $x \in G$, we obtain $G = \langle A, B \rangle \leq P_G(R)$. So $R$ is permutable in $G$. If $R < U < G$, then $U \cong S_4$, therefore $R$ is strongly permutable in $G$.

Let $G \cong L_2(11)$. Then $|G| = 2^5 \cdot 3 \cdot 5 \cdot 11$, $p = 11$, $H \cong L_2(5)$. In $L_2(5)$, only a Sylow 2-subgroup is $\mathbb{P}$-subnormal, therefore $r = 2$. But $R$ is not permutable in $H \cong L_2(5)$, hence $R$ is not strongly permutable in $G \cong L_2(11)$.

Let $G \cong L_3(3)$. Then $|G| = 2^4 \cdot 3^3 \cdot 13$, $p = 13$ and $H \cong M_9 : S_3 \cong C_3^2 : GL_2(3)$. Since $|H| = 2^4 \cdot 3^3$ and $H$ is not 3-closed, we have $r \neq 3$ by Lemma 3 and $r = 2$. But a Sylow 2-subgroup $R$ is not $\mathbb{P}$-subnormal in $H$ according to [14]. Therefore in $G \cong L_3(3)$, there are no $\mathbb{P}$-subnormal Sylow subgroups.

Let $G \cong L_3(5)$. Then $|G| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$, $p = 31$ and $H \cong C_5^2 : GL_2(5)$. Since $|H| = 2^5 \cdot 3 \cdot 5^3$ and $H$ is not 5-closed, we get $r \neq 5$ by Lemma 3 and $r \in \{2, 3\}$. But a Sylow 2-subgroup and 3-subgroup are not $\mathbb{P}$-subnormal in $H$ according to [14]. Therefore in $G \cong L_3(5)$, there are no $\mathbb{P}$-subnormal Sylow subgroups.

Let $G \cong L_2(2^m)$, where $2^m + 1$ is prime. Then $|G| = 2^m(2^m - 1)(2^m + 1)$, $p = 2^m + 1$ and $H = N_G(Q) \cong C_{2^m}^3 : C_{2^m - 1}$, $Q \cong C_2^m$ is a Sylow 2-subgroup of $G$. Since $Q$ is $\mathbb{P}$-subnormal in $H$, we deduce $Q$ is $\mathbb{P}$-subnormal in $G$. Suppose that $\langle g \rangle Q = Q \langle g \rangle$ for some $g \in G$. Then $\langle g \rangle Q \leq N_G(Q)$ according to [10] II.8.27. Hence $P_G(Q) = N_G(Q)$ and a Sylow 2-subgroup of $G \cong L_2(2^m)$ is $\mathbb{P}$-subnormal in $G$, but it is not permutable in $G$. Suppose that $r \neq 2$. Then $R \leq N_G(Q) = H$ and $R$ is $\mathbb{P}$-subnormal in $RQ$ by Lemma 2, since $R$ is $\mathbb{P}$-subnormal in $H$ and $H$ is soluble. By Lemma 3 $R$ is normal in $RQ$ and $R \leq C_G(Q)$, which is impossible in $G \cong L_2(2^m)$.

**Corollary A.1** ([15] Theorem 2.1]). If a Sylow $r$-subgroup of a group $G$ is $\mathbb{P}$-subnormal and $r > 2$, then $G$ is $r$-soluble.

**Proof.** By Theorem A.1(1), $G$ is not simple. Let $N$ be a normal subgroup of $G$, $1 \neq N \neq G$. Then $R \cap N$ is a Sylow $r$-subgroup of $N$ and $R \cap N$ is $\mathbb{P}$-subnormal in $N$ in view of Lemma 1(1). By induction, $N$ is $r$-soluble. Note that $RN/N$ is a Sylow $r$-subgroup of $G/N$ and $RN/N$ is $\mathbb{P}$-subnormal in $G/N$ in view of Lemma 1(1). By induction, $G/N$ is $r$-soluble. Therefore $G$ is $r$-soluble.

**Corollary A.2** ([15] Corollary 2.1.1]). If a Sylow 3-subgroup and Sylow 5-subgroup of a group $G$ is $\mathbb{P}$-subnormal, then $G$ is soluble.

**Proof.** By Corollary A.1 $G$ is 3-soluble and 5-soluble. Hence $G$ has a normal series, in which factors are 3-groups, 5-groups or $\{3, 5\}$-groups. Since $\{3, 5\}$-groups are soluble [16] Theorem, p.18], we conclude $G$ is soluble.

## 4 Groups with permutable and $\mathbb{P}$-subnormal primary cyclic subgroups

Let $\mathfrak{F}$ be a class of groups. A group $G$ is called a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but every proper subgroup of $G$ belongs to $\mathfrak{F}$. Minimal non-$\mathfrak{N}$-groups are also called Schmidt groups. We remind the properties of Schmidt groups and minimal non-supersoluble groups we need.

**Lemma 9** ([17] Theorem 1.1, 1.2, 1.5, [18] Theorem 3]). Let $S$ be a Schmidt group. Then the following statements hold.

1. $S = P \rtimes Q$, where $P$ is a normal Sylow $p$-subgroup and $Q$ is a non-normal Sylow $q$-subgroup, $p$ and $q$ are different primes and
2. If $P$ is abelian, then $P$ is elementary abelian of order $p^m$, where $m$ is the order of $p$ modulo $q$.
(1.2) if $P$ is not abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^m$;
(1.3) if $p > 2$, then $P$ has the exponent $p$; for $p = 2$, the exponent of $P$ is not more than 4.
(1.4) $Q = \langle y \rangle$ is a cyclic subgroup and $y^q \in Z(S)$.

(2) $G$ has exactly two classes of conjugate maximal subgroups

\[ \{P \times \langle y^q \rangle\}, \{\Phi(P) \times \langle x^{-1}yx \rangle \mid x \in P \setminus \Phi(P)\}. \]

**Lemma 10 (19,20)**. Let $G$ be a minimal non-supersoluble group. Then the following statements hold.

1. $G$ is soluble and $|\pi(G)| \leq 3$;
2. If $G$ is not a Schmidt group, then $G$ has a Sylow tower of supersolvable type;
3. $G$ has a unique normal Sylow subgroup $P$ and $P = G^4$;
4. $|P/\Phi(P)| > p$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;
5. If $\Phi(G) = 1$, then $O_p'(G) = 1$ and $Q$ is either nonabelian of order $q^3$ and exponent $q$, or $Q$ is a cyclic $q$-group, or $Q$ is a $q$-group with a cyclic subgroup of index $q$, or $Q$ is a supersoluble Schmidt group.

**Lemma 11 (21 Lemma 1)**. Let $S = P \times Q$ be a supersoluble Schmidt group. Then $P = \langle x \rangle$ is a normal subgroup of order $p$, $Q = \langle y \rangle$ is a cyclic subgroup of order $q^6$, where $q$ divides $p - 1$.

**Lemma 12**. Let $G = P \times Q$ be a Schmidt group, $Q = \langle y \rangle$. If $x \in G$ and $|y|$ does not divide $|x|$, then $x \in P \times \langle y^q \rangle$.

**Proof**. Let $\langle x \rangle = \langle x_1 \rangle \times \langle x_2 \rangle$, where $\langle x_1 \rangle$ is a Sylow $p$-subgroup and $\langle x_2 \rangle$ is a Sylow $q$-subgroup of $\langle x \rangle$. Since $P$ is normal in $G$, $\langle x_1 \rangle \leq P$. Let $\langle x_2 \rangle \leq Q^y = \langle y^q \rangle$, $g \in G$. As $|y|$ does not divide $|x_2|$, we conclude $\langle x_2 \rangle < \langle y^q \rangle$ and $x_2 \in \langle (y^q)^q \rangle$. But $y^q \in Z(G)$, therefore

\[ (y^q)^q = \underbrace{y^q \cdot y^q \cdot \ldots \cdot y^q}_{q} = g^{-1}y^qg = y^q \]

and $x_2 \in \langle y^q \rangle$. Consequently, $\langle x \rangle \leq P \times \langle y^q \rangle$. \qed

**Lemma 13**. (1) In a supersoluble Schmidt group, every subgroup is strongly permutable.
   (2) Let $G = P \times Q$ be a non-supersoluble Schmidt group. Then the following statements hold.

   2.1) $Q$ is not permutable and $N_G(Q) = P_C(Q) = \Phi(P) \times Q \leq G$.
   2.2) If $H \leq P$ and $P_C(H) = G$, then either $H = P$ or $H \subseteq \Phi(G)$.
   2.3) Every primary permutable subgroup is normal in $G$, and so it is strongly permutable in $G$.

   (3) In Schmidt group $G$, every subgroup of prime order and every cyclic subgroup of order 4 is strongly permutable if and only if $G$ is supersoluble.

**Proof**. In view of Lemma 9(2), maximal subgroups of $G$ are reduced to $N_G(Q^q) = \Phi(P) \times Q^q$, $g \in G$ and $P \times \langle y^q \rangle \not\leq G$, $\langle y \rangle = Q$.

1. Let $G = P \times Q$ be a supersoluble Schmidt group. Then $|P| = p$ and $q$ divides $p - 1$, where $|Q| = q^1$ by Lemma 11. It is clear that $P$ and $Q$ are strongly permutable in $G$. Let $Q_1 \leq Q$. Note that $P \times Q_1$ is cyclic and normal in $G$. Hence all subgroups of $P \times Q_1$ is normal in $G$ and strongly permutable.

2. Let $G = P \times Q$ be a non-supersoluble Schmidt group.
   2.1) Suppose that there is $x \in G \setminus N_G(Q)$ such that $\langle x \rangle Q = Q(x)$. Let $\langle x \rangle = \langle a \rangle \langle b \rangle$, where $\langle a \rangle$ is a Sylow $p$-subgroup and $\langle b \rangle$ is a Sylow $q$-subgroup of $\langle x \rangle$. Since $\langle b \rangle Q = Q$ according to [10 VI.4.7],

\[ \langle x \rangle Q = \langle a \rangle Q = \langle a \rangle \times Q, \quad a \in N_G(Q) = \Phi(P) \times Q, \]

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If all primary cyclic subgroups of a group $G$ are strongly permutable, then $G$ is supersoluble. Conversely, if $G$ is a supersoluble Schmidt group, then by Statement (1), every subgroup of prime order and every cyclic subgroup of order 4 is strongly permutable.

**Lemma 14.** Let $H$ be a $p$-group of exponent $p$ and $x \not\in Z(H)$. Then $N_H(\langle x \rangle) = P_H(\langle x \rangle)$ and $\langle x \rangle$ is not permutable in $H$.

**Proof.** It is clear that $N_H(\langle x \rangle) \leq P_H(\langle x \rangle)$. Choose $y \in H \setminus \langle x \rangle$ such that $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$. Since $H$ is a $p$-group of exponent $p$, we get $|\langle x \rangle \langle y \rangle| = p^2$. Consequently, $H$ is abelian and $\langle x \rangle \langle y \rangle = \langle x \rangle \times \langle y \rangle$, and so $y \in N_H(\langle x \rangle)$ and $N_H(\langle x \rangle) = P_H(\langle x \rangle)$. As $x \not\in Z(H)$, we have $H \neq N_H(\langle x \rangle) = P_H(\langle x \rangle)$ and $\langle x \rangle$ is not permutable in $H$.

**Theorem B.** If all primary cyclic subgroups of a group $G$ are strongly permutable, then $G$ is supersoluble.

**Proof.** We use induction on the group order. In view of Lemma 4(3) and by induction, all proper subgroups of $G$ are supersoluble. Hence $G = P \rtimes S$ is a minimal non-supersoluble group, $P = G^{ab}$. By Lemma 13(3), $G$ is not a Schmidt group, therefore $G$ has a Sylow tower of supersoluble type and $P$ is a Sylow $p$-subgroup of $G$ for $p = \max \pi(G)$. In particular, $p > 2$ and all nontrivial elements in $P$ are of order $p$ by Lemma 10.

From Lemma 14 it follows that $P$ is an elementary abelian $p$-subgroup, and by Lemma 10 $P$ is a minimal normal subgroup in $G$.

Assume that $N$ is a normal subgroup of $G$, $N \neq 1$, and $V/N$ is a cyclic $t$-subgroup, $t \in \pi(G)$. Let $U$ be a subgroup of least order such that $U \leq V$, $U \cap N = \Phi(U)$, $V/N = U/N \cong U/\langle x \rangle$, and therefore $U$ is a cyclic $t$-subgroup. By the hypothesis, $U$ is strongly permutable in $G$, and by Lemma 4(1), $V/N$ is strongly permutable in $G/N$. By induction, $G/N$ is supersoluble, hence $\Phi(G) = 1$. From Lemma 10(5) it follows that $S$ is either a cyclic primary group or supersoluble Schmidt group, and $O_p(G) = 1$.

Let $S$ be either a cyclic primary group or $|\pi(S)| = 2$. In that case, all Sylow subgroups of $S$ are cyclic. Assume that $A \leq P$, $|A| = p$ and $g \in G \setminus N_G(A)$ such that $\langle g \rangle A \leq G$. Since $G = PS$, we conclude $g = bx$, $b \in P$, $x \in S$ and $\langle g \rangle = \langle b \rangle \times \langle x \rangle$. If $x = 1$, then $g = b \in P \leq N_G(A)$, a contradiction. If $b = 1$, then $g = x$ and $\langle g \rangle A = A \times \langle g \rangle$, since $\langle g \rangle A$ is $p$-closed. So, $g \in N_G(A)$, a contradiction. Thus, $b \neq 1$, $x \neq 1$, $S^b \neq S$, $x^b = x \in S \cap S^b = D \neq 1$.

If $S$ is abelian, then $D < \langle S, S^b \rangle = G$, $D \leq O_p(G) = 1$, a contradiction. Therefore $S$ is not abelian and $S = Q \rtimes R$ is supersoluble Schmidt group by Lemma 10 in view of $\Phi(G) = 1$.

Now, $|Q| = q$, $R = \langle y \rangle$ is an $r$-subgroup, $r$ divides $q - 1$ and $y^r \in Z(S)$.

If $q$ divides $|D|$, then $Q \leq D$ and $Q \lhd \langle S, S^b \rangle = G$, a contradiction. So, $D$ is an $r$-subgroup. If $y^r \neq 1$, then $D_1 = D \cap \langle y^r \rangle \neq 1$ and $D_1 \lhd \langle S, S^b \rangle = G$, a contradiction. Consequently, $y^r = 1$, $D = \langle y \rangle = R$ and $|R| = r$. Since $D = \langle x \rangle$, we get $\langle x \rangle = R \leq N_G(\langle b \rangle)$.
If \( b \sub P Q \), then \( N_G((b)) \geq \langle P Q, R \rangle = G \) and \( b \sub G \), a contradiction. Hence \( b \) is not normal in \( P Q \) and there is \( u \in P Q \setminus N_{PQ}(b) \) such that \( \langle b \rangle \langle u \rangle \not\subseteq \langle u \rangle \langle b \rangle \subseteq P Q \). Let \( u = cf \), \( c \in P \), \( f \in Q \). If \( c = 1 \), then \( \langle b \rangle \langle u \rangle = \langle b \rangle \times \langle f \rangle \) and \( u = f \in N_{PQ}(b) \), a contradiction. If \( f = 1 \), then \( u = c \in P \leq N_{PQ}(b) \), a contradiction. Consequently, \( c \neq 1 \), \( f \neq 1 \) and \( \langle u \rangle = \langle c \rangle \times \langle f \rangle \), \( \langle f \rangle = Q \). \( \langle c \rangle \subseteq S \cap S^c \) and \( S^c \neq S \), since \( c \in P \), \( c \notin N_G(S) = S \). But now \( Q \not\subseteq \langle S, S^c \rangle = G \), a contradiction.

Finally, we consider the case when \( S = R \) is a noncyclic Sylow \( r \)-subgroup of \( G \). Assume that in \( S \), there is a cyclic subgroup \( Z \) of index \( r \). Let \( A \leq P \) such that \( |A| = p \) and let \( g \in G \setminus N_G(A) \) such that \( \langle g \rangle A \leq G \). Since \( G = P R \), we deduce \( g = bx \), \( b \in P \), \( x \in R \) and \( \langle g \rangle = \langle b \rangle \times \langle x \rangle \). If \( b = 1 \), then \( g = \langle x \rangle \) and \( \langle g \rangle A = A \times \langle g \rangle \), since \( \langle g \rangle A \) is \( p \)-closed and \( A \) is a Sylow \( p \)-subgroup of \( \langle g \rangle A \). Hence \( g \in N_G(A) \), a contradiction. If \( x = 1 \), then \( g = b \in P \leq N_G(A) \). This contradicts with the choice of \( g \). So, \( b \neq 1 \), \( x \neq 1 \), \( R^b \neq R \), \( x^b = x \in R \cap R^b \).

If \( \langle x_1 \rangle = \langle x \rangle \cap Z \neq 1 \), then \( \langle x_1 \rangle \not\subseteq R \). Since \( x_1^b = x_1 \in Z^b \cap R^b \), we get \( \langle x_1 \rangle \not\subseteq R^b \). Therefore \( \langle x_1 \rangle \not\subseteq \langle R, R^b \rangle = G \). This contradicts with \( R_G = 1 \). So, \( \langle x \rangle \cap Z = 1 \) and \( R = Z \times \langle x \rangle \), \( |x| = r \), \( x \in C_G((b)) \leq N_G((b)) \).

If \( \langle b \rangle \subseteq P Z \), then \( \langle b \rangle \subseteq G \), a contradiction. Thus, \( N_{PZ}((b)) \subseteq P Z \) and there is \( u \in P Z \setminus N_{PZ}((b)) \) such that \( \langle u \rangle \langle b \rangle \subseteq G \). Since \( u \in P Z \), we conclude \( \langle u \rangle = \langle c \rangle \times \langle y \rangle \), \( \langle c \rangle \subseteq P \), \( \langle y \rangle \subseteq Z \). Verification shows that \( c \neq 1 \), \( y \neq 1 \). From \( N_G(Z) = R \) it follows that \( Z^c \neq Z \) and \( y^c = y \in Z \cap Z^c \). Now, \( \langle y \rangle \not\subseteq \langle R, R^c \rangle = G \), a contradiction.

If \( S \) is not an abelian group of order \( r^3 \) and exponent \( r \), then by Lemma [14] \( S \) contains a nonpermutable cyclic primary subgroup, which contradicts with the choice of \( G \).

**Theorem C.** If every primary cyclic subgroup of a group \( G \) is \( P \)-subnormal or strongly permutable, then \( G \in vU \).

**Proof.** We use induction on the group order. Let \( N \) be a normal subgroup of a group \( G \), \( N \neq 1 \), and let \( \langle a \rangle \) be a cyclic primary subgroup of \( N \). By the choice of \( G \), \( \langle a \rangle \) is \( P \)-subnormal or strongly permutable in \( G \). If \( \langle a \rangle \) is \( P \)-subnormal in \( G \), then by Lemma [1](1), \( \langle a \rangle \) is \( P \)-subnormal in \( N \). If \( \langle a \rangle \) is strongly permutable in \( G \), then by Lemma [3](3), \( \langle a \rangle \) is strongly permutable in \( N \). Now assume that \( A/N \) is a cyclic \( t \)-subgroup. \( t \in \pi(G) \). Let \( B \) be a subgroup of least order such that \( B \leq A \). \( B N = A \). Then \( B \cap N \leq \Phi(B) \), \( A/N = B N/N \cong B / B \cap N \), hence \( B \) is a cyclic \( t \)-subgroup. By the choice of \( G \), \( B \) is \( P \)-subnormal or strongly permutable in \( G \). As \( A/N = B N/N \), according to Lemma [1](1) and Lemma [3](1), \( A/N \) is \( P \)-subnormal or strongly permutable in \( G \). Thus the hypothesis holds for all normal subgroups of \( G \) and all quotients subgroups.

Suppose that \( G \) is a simple group. If every primary cyclic subgroup of \( G \) is strongly permutable, then \( G \) is supersoluble by Theorem [3]. Consequently, \( G \) contains a cyclic primary subgroup \( A \) such that \( A \) is \( P \)-subnormal in \( G \). Since the unit subgroup is \( P \)-subnormal in \( A \), then it is \( P \)-subnormal in \( G \). According to [12], [13] p. 342, \( G \) is isomorphic to one of the following groups.

\[
L_2(7), L_2(11), L_3(3), L_3(5), L_2(2^m), 2^m + 1 \text{ is prime.}
\]

In every of these groups, a Sylow \( r \)-subgroup \( R \) is cyclic for \( r = \max \pi(G) \). By the choice of \( G \), \( R \) is \( P \)-subnormal or strongly permutable in \( G \). If \( R \) is \( P \)-subnormal in \( G \), then by Lemma [3] \( R \) is normal in \( G \). If \( R \) is strongly permutable in \( G \), then in view of Lemma [3] \( R \) is normal in \( G \). Consequently, \( R \) is normal in \( G \) and \( G \) is not a simple group, a contradiction.

Thus in \( G \), there is a normal subgroup \( N \), \( N \neq 1 \), and by induction, \( G/N \in vU \) and \( N \notin vU \). Hence \( G \) is soluble. In view of Lemma [1](2) and by induction, every proper subgroup of \( G \) belongs to \( vU \) and \( G \) is a minimal non-\( vU \)-group. According to [3, Theorem B (4)], \( G \) is a biprimary minimal non-supersoluble group in which non-normal Sylow subgroups are cyclic. Hence \( G = R \times Q \) is a group such that a Sylow \( r \)-subgroup \( R \) is normal in \( G \) and a Sylow
$q$-subgroup $Q$ is cyclic and $P$-subnormal or strongly permutable in $G$ by the choice of $G$. By Corollary 1.3, $G \in \mathcal{U} \subseteq \mathcal{V}$.

**Example 4.** In $A_4$, every subgroup of order 2 is $P$-subnormal, but it is not permutable.

**Example 5.** In $L_2(7)$, every subgroup of order 3 is permutable, but it is not $P$-subnormal.

**References**

[1] Weinstein M. Between Nilpotent and Solvable. Polygonal. Passaic, N.J. 1982.

[2] Liu X., Wang Y. Implications of permutizers of some subgroups in finite groups. Comm. Algebra. 2005. Vol. 33. 559–565.

[3] Vasil’ev A. F., Vasil’ev V. A., Vasil’eva T. I. On permuteral subgroups in finite groups. Siberian Math. J. 2014. Vol. 55(2). 230–238.

[4] Vasil’ev A. F., Vasil’eva T. I., Tyutyanov V. N. On the finite groups of supersoluble type. Siberian Math. J. 2010. Vol. 51(6). 1004–1012.

[5] Monakhov V. S., Kniahina V. N. Finite group with $P$-subnormal subgroups. Ricerche Mat. 2013. Vol. 62. 307–323.

[6] Monakhov V. S., Finite groups with abnormal and $\mathcal{U}$-subnormal subgroups. Siberian Math. J. 2016. Vol. 57(2). 352–363.

[7] Chen R., Zhao X., Li X. $P$-subnormal subgroups and the structure of finite groups. Ricerche Mat. 2021. https://doi.org/10.1007/s11587-021-00582-4.

[8] The GAP Group: GAP — Groups, Algorithms, and Programming. Ver. 4.11.0 released on 29-02-2020. http://www.gap-system.org.

[9] Ballester-Bolinches A., Cosme-Ll`{o}pez E., Esteban-Romero R. GAP Package Permut. Ver. 2.0.3 released on 19-08-2018. https://gap-packages.github.io/permut.

[10] Huppert B. Endliche Gruppen I. Berlin, Springer, 1967.

[11] Qiao S., Qian G., Wang Y. Finite groups with the maximal permutizer cindition. J. Algebra Appl. 2013. Vol. 12(5). 1250217 (5 pages).

[12] Kazarin L. S. On groups with factorization. Dokl. Akad. Nauk SSSR. 1981. Vol. 256 (1). 26–29.

[13] Cameron P. J., Solomon R. Chains of Subgroups in Symmetric Groups. J. Algebra. 1989. Vol. 127. 340–352.

[14] Conway J. H., Curtis R. T., Norton S. P., Parker R. A., Wilson R. A. Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups. Oxford, Clarendon Press, 1985.

[15] Kniahina V. N., Monakhov V. S. Finite groups with $P$-subnormal Sylow subgroup. Ukrain’s’kyi Matematychnyi Zhurnal. 2020. Vol. 72(10). 1365–1371.

[16] Gorenstein D. Finite simple groups. An introduction to their classification. New York, Plenum Publ. Corp., 1982.
[17] Monakhov V.S. The Schmidt subgroups, its existence, and some of their applications. Proceedings of the Ukrainian Mathematical Congress-2001. Inst. Mat. NAN Ukrainy, Kyiv, 2002. 81–90.

[18] Ballester-Bolinches A., Esteban-Romero R., Robinson D.J.S. On finite minimal non-nilpotent groups. Proceedings of the American Mathematical Society. 2005. Vol. 133(12). 3455–3462.

[19] Doerk K. Minimal nicht überraflösbare, endliche gruppen. Math. Zeit. 1966. Z. 91. 198–205.

[20] Ballester-Bolinches A., Esteban-Romero R. On minimal non-supersoluble groups. Rev. Mat. Iberoamericana. 2007. Vol. 23(1). 127–142.

[21] Monakhov V.S. Finite groups with a given set of Schmidt subgroups. Math. Notes. 1995. Vol. 58(5). 1183–1186.