Classical communication with indefinite causal orders for $N$ completely depolarizing channels

Sk Sazim, Michal Sedlak, Kratveer Singh, and Arun Kumar Pati

RCQI, Institute of Physics, Slovak Academy of Sciences, 845 11 Bratislava, Slovakia
Indian Institute of Science Education and Research, Bypass Road, Bhaouri, Bhopal 462066 India
QIC Group, Harish-Chandra Research Institute, HBNI, Allahabad, 211019, India

(Dated: April 29, 2021)

If two identical copies of a completely depolarizing channel are put into a superposition of their possible causal orders, they can transmit non-zero classical information. Here, we study how well we can transmit classical information with $N$ depolarizing channels put in superposition of $M$ causal orders via quantum SWITCH. We calculate Holevo quantity if the superposition uses only cyclic permutations of channels and find that it increases with $M$ and it is independent of $N$. For a qubit it never reaches 1 if we are increasing $M$. On the other hand, the classical capacity decreases with the dimension $d$ of the message system. Further, for $N = 3$ and $N = 4$ we studied superposition of all causal orders and uniformly superposed causal orders belonging to different cosets created by cyclic permutation subgroup.

I. INTRODUCTION

In classical information theory, it is assumed that the information carries are deterministic, and the transmission lines are used in a definite configuration in space as well as in fixed time [1, 2]. However, physical systems obey principles of quantum theory and they offer resources which are not available in its classical counterpart. These unique resources can be harnessed to achieve communication protocols which are impossible in classical information theory [3–5]. These findings led to a complete revolution in quantum information theory [6, 7]. Still, quantum information theory assumes that the channels maintain a specific order in space and time. However, quantum theory allows the configurations where channels themselves are in superposition [8, 9]. Moreover, recently, it was realised that the superposition can exist also in the order of channels in time, in a scenario known as Indefinite Causal Order or quantum SWITCH [10–14]. In quantum SWITCH, the relative order of the two channels is indefinite, and gives rise to quantum advantages in reducing communication complexity [15, 16], improving channel discrimination [17, 18] and quantum computing [19]. Moreover, several proposals for an experimental realization of a quantum SWITCH has been actually build and tested [20, 21], suggesting that the notion is not just a theoretical possibility.

Recently, in Ref.[22], authors showed that one may achieve non-zero classical communication rates using two completely depolarising channels (CDCs) inserted into the quantum SWITCH, which has also been experimentally demonstrated later in [23]. In the same note, it is reported that using two completely entanglement breaking channels in SWITCH, one may achieve perfect quantum communication [24–26]. After these findings, several applications of quantum SWITCH have been discovered in quantum metrology, quantum thermometry and quantum information as well [27–34].

Extension of such settings beyond superposition of two channels is an immediate and interesting generalization to make to see whether it provides bigger communication advantage. However, such generalization comes with a serious concern whether it is not out of the experimental scope. In Ref.[35], authors showed that there is almost twofold increase in communication rate if causal superposition of 3 channels is used instead of 2 channel causal superposition. On the other hand, the number of relevant configurations jumps from 2 to $3! = 6$. This makes experimental implementation very cumbersome, but nevertheless, possible. Furthermore, their numerical results suggests that usage of 3 channels in three cyclic causal orders gives similar gain as all 3! causal orders. This bolsters the idea that considering $N$ causal orders for $N$ channels should be efficient. An extension to $N$ channels with $N!$ causal orders has been proffered in Ref.[36], however, they used numerical approach to find the communication rates which might suffer from numerical errors. An analytical approach is in demand to delve deeper into these matters and to answer the following open questions: a) Can $N$ channels in quantum SWITCH allow perfect transmission of classical information?, and b) Can we achieve substantial gain in classical communication rates with optimal number of causal orders in a quantum SWITCH? We answer these questions in detail in this paper.

Cyclic permutations of $N$ elements form a subgroup of all $N$ element permutations. In this paper, we find that all cosets of permutation group factorized with respect to cyclic permutations behave equivalently, when they determine used casual orders of channels from more than one coset as non-cyclic causal superposition. On the other hand, the number of relevant configurations jumps from 2 to $3! = 6$. We refer to causal orders of channels from a single coset as cyclic causal orders. Similarly, we derive some results for $M \in [2, N]$ non-cyclic causal orders, when $N = 3$ and $N = 4$. We find that for the cyclic case classical communication rate depends on $M \leq N$, the number of superposed cyclic permutations, but does not depend on $N$, the number of CDCs in the quantum SWITCH. If we keep on increasing $M$ (and necessarily increasing also $N$), we observed that the communication rate increases rapidly.

* sk.sazimq49@gmail.com
† michal.sedlak@savba.sk
with the increase in the number of causal orders, but it never reaches noiseless transmission. For example, it saturates at 0.311 bits for qubit systems. For non-cyclic case, the increase of the classical communication rate is not directly linked to $M$. On the other hand, we uncover that the classical communication rate decreases almost exponentially with the dimension of the message state, which seems a bit counter-intuitive at first sight.

The rest of the paper is organized as follows. In the next section, we will briefly introduce the quantum Switch formalism. In Sec. III we present results for $N$ completely depolarizing channels in a quantum Switch, while superposing only cyclic causal orders. Sec. IV contains the detailed analysis of $N$ CDCs with arbitrary non-cyclic causal orders in a quantum SWITCH with special emphasis on $N = 3$ and 4. Finally, we conclude in Sec. V.

II. QUANTUM SWITCH AND QUANTUM CHANNELS IN SUPERPOSITION OF DIFFERENT CAUSAL ORDERS

Quantum communication devices can be modelled as quantum channels, i.e., a completely positive and trace preserving linear maps, $\Lambda : L(H) \rightarrow L(H)$. Any such map admits Kraus decomposition, i.e., $\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$, where $\{K_i\}$ is a set of Kraus operators with $\sum_i K_i^\dagger K_i = 1$, and $\rho \in L(H)$.

In this work, we are considering a scenario where $N$-channels are put into a coherent superposition of their differently ordered concatenations. Originally, quantum SWITCH was used to construct a superposition of $N = 2$ causal orders [19]. In this case, quantum SWITCH is a higher order map which takes two channels as input, and outputs the superposition of their orders based on the state of the control qubit. Mathematically, quantum SWITCH transforms two input channels $\Lambda_1$ and $\Lambda_2$, with Kraus decomposition $\{K_1^{(1)}\}$ and $\{K_2^{(2)}\}$ respectively, into the overall channel

$$S(\Lambda_1, \Lambda_2)(\cdot) = \sum_{ij} W_{ij}(\cdot) W_{ij}^\dagger,$$

whose Kraus operators $W_{ij}$ are defined as

$$W_{ij} = |0\rangle \langle 0| \otimes K_j^{(1)} K_i^{(2)} + |1\rangle \langle 1| \otimes K_i^{(2)} K_j^{(1)}.$$

Note that though $W_{ij}$ depends on the specific Kraus decomposition of channels $\Lambda_1$ and $\Lambda_2$, the effective quantum channel $S(\Lambda_1, \Lambda_2)$ depends only on the input channels, allowing SWITCH to be a valid higher order map [19]. We can extend the SWITCH formalism for more than two inputs, i.e., for $N > 2$ [36, 37]. In this case, the extended SWITCH is a higher order map which takes $N$ channels as input, and outputs the superposition of orders based on the state of the control system that must have sufficiently high dimensionality. Then for $N$-channels, $\{\Lambda_p\}$, with Kraus representations $\{K_p^{(j)}\}$, the extended SWITCH will output an effective channel of the form

$$S(\Lambda_1, \Lambda_2, ..., \Lambda_N)(\cdot) = \sum_{ij...\eta} W_{ij...\eta}(\cdot) W_{ij...\eta}^\dagger,$$

whose Kraus operators $W_{ij...\eta}$ are defined as

$$W_{ij...\eta} = \sum_{\ell=0}^{M-1} \prod_{k=0}^{\ell} \mathcal{P}_\ell(K_i^{(1)}), K_j^{(2)}, ..., K_\eta^{(N)}),$$

where $M \in [2, N!]$ and $\mathcal{P}_\ell \in S_N$ represents concatenation of $N$ operators reordered according to the permutation $\ell$, e.g., $\mathcal{P}_0(K_1^{(1)}, K_2^{(2)}, ..., K_N^{(N)}) = K_1^{(1)} K_2^{(2)} ... K_N^{(N)}$. For brevity, we will drop the upper index ‘$\ell$‘ in the rest of the paper.

To have a simple, but sufficient picture suitable for further considerations let us consider two unitary channels, $U_1$ and $U_2$ and the control qubit in the state $|\psi\rangle_c = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. For the pure state $|\Phi\rangle$ of the message system (we often call it a target state as well), the output of the SWITCH will be a pure state $S(U_1, U_2)(|\psi\rangle_c |\psi| \otimes |\Phi\rangle) = |\xi\rangle \langle \xi|$, where $|\xi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes U_1 U_2 |\Phi\rangle + |1\rangle \otimes U_2 U_1 |\Phi\rangle)$. To see the interference phenomenon, one needs to measure the control qubit in the Fourier basis, i.e., $\{|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}$, then the resulting target state will take the form $|\Psi_+^f\rangle = \frac{1}{\sqrt{2}}(U_1 U_2 \pm U_2 U_1) |\Phi\rangle$.

![FIG. 1. Illustration of a Quantum SWITCH.](image)

Extending this idea to $N$ channels is possible [36], and the number of possible causal orders increases to $M \in [2, N!]$. For the brevity of explanation, let’s consider $N$ unitary channels $\{U_j\}$ and the control state $\frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} |j\rangle$, then the final target state after the measurement of the control system will be $|\Psi_+^f\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{2\pi i j k} |j\rangle \mathcal{P}_j(U_1, U_2, ..., U_N) |\Phi\rangle$, where $\{|e_k\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{2\pi i j k} |j\rangle\}$ is the Fourier basis of $\{|j\rangle\}$. 
III. USING CYCLIC ORDERS OF N COMPLETELY DEPOLARIZING CHANNELS IN QUANTUM SWITCH

A completely depolarizing channel in $d$-dimensions can be described by

$$\Lambda(X) = \frac{1}{d^2} \sum_{i=1}^{d^2} U_i X U_i^\dagger = \frac{1}{d} \text{Tr}[X] \mathbb{I}_d,$$

(3)

where $\{U_i; \; i = 1, 2, \ldots, d^2\}$ are $d \times d$ unitary operators satisfying $\text{Tr}[U_i^\dagger U_j] = \delta_{ij}$, $\mathbb{I}_d$ is identity operator of order $d$, and $X$ is an arbitrary linear operator on $d$-dimensional Hilbert space. Direct transmission of information through single or several concatenated CDCs necessarily results in zero classical communication rate. In contrast, it was shown that given two identical CDCs labeled as $\Lambda_1$ and $\Lambda_2$ and a control qubit state, $|\psi\rangle_c = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, there is a possibility of non-zero classical communication using quantum SWITCH [22].

Here, we will generalize the scheme represented in [22] to $N$ CDCs $\{\Lambda_i\}$. We will be considering first only $M \in [2, N]$ possible cyclic orders. Accordingly, the state of control qubit is $|\psi\rangle_c = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} |j\rangle$. Therefore, the Kraus operators of a channel resulting from $N$ CDCs in extended quantum SWITCH can be written as (see Eq.(2))

$$K_{ij\ldots n} = \frac{1}{d^{N-1}} \sum_{l=0}^{M-1} |l\rangle \otimes P_l^{(c)}(U_i, U_j, \ldots, U_n),$$

(4)

where $P_l^{(c)}(U_i, U_j, \ldots, U_n)$ defines the cyclic permutations of unitaries. For example, for $N = 3$ the cyclic permutations are $P_0^{(c)}(U_i, U_j, U_k) = U_i U_j U_k$, $P_1^{(c)}(U_i, U_j, U_k) = U_j U_k U_i$ and $P_2^{(c)}(U_i, U_j, U_k) = U_k U_i U_j$. If the sender prepared the target system in the state $\psi$, then the receiver will receive the output from the Quantum SWITCH as

$$\rho_M := S(\Lambda_1, \Lambda_2, \ldots, \Lambda_N)(\rho_c \otimes \rho)$$

$$= \frac{1}{Md^{2N}} \sum_{i,j,\ldots, n} \sum_{l=0}^{M-1} \sum_{l'=0}^{M-1} |l\rangle \langle l'| \otimes$$

$$\otimes P_l^{(c)}(U_i, U_j, \ldots, U_n) \rho \left(P_{l'}^{(c)}(U_i, U_j, \ldots, U_n)\right)^\dagger$$

(5)

where $\rho_c = |\psi\rangle \langle \psi|$. For cyclic permutations of orders we will see below that only two types of contributions are present in the final output state: i) diagonals ($l = l'$) which are proportional to $\mathbb{I}$ and ii) the off-diagonals ($l \neq l'$) which are proportional to $\rho$. All the diagonal terms are equivalent to the following prototype form

$$\frac{1}{d^{2N}} \sum_{i,j,\ldots, n}^{d^2} \int_U \cdots \int_U (\rho) U_i^\dagger \cdots U_n^\dagger U_j \cdots U_i$$

$$= \frac{1}{d^{2N-2}} \text{Tr}(\rho) \sum_{j,\ldots, n=1}^{d^2} \int_U \cdots \int_U \frac{U_j^\dagger \cdots U_n^\dagger}{d} U_j \cdots U_n$$

$$= \frac{1}{d^{2N-2}} d^{2(N-1)} \frac{\mathbb{I}}{d} = \frac{\mathbb{I}}{d},$$

(6)

and all off-diagonal terms are equivalent to the term below

$$\frac{1}{d^{2N-2}} \sum_{j,\ldots, n=1}^{d^2} \int_U \cdots \int_U (\rho) \frac{U_j^\dagger \cdots U_n^\dagger}{d} U_j \cdots U_n$$

$$= \frac{1}{d^{2N-2}} \sum_{j,\ldots, n=1}^{d^2} \text{Tr}(\rho) \int_U \cdots \int_U \frac{U_j^\dagger \cdots U_n^\dagger}{d} U_j \cdots U_n$$

$$= \frac{1}{d^{2N-2}} d^{2(N-1)} \frac{\mathbb{I}}{d} = \frac{\mathbb{I}}{d},$$

(7)

where $k \in \{1, \ldots, N-1\}$, we used $\sum_{i=1}^{d^2} \frac{1}{d} \text{Tr}(U_i^\dagger U_j) U_i = X$ and Eq. (3) multiple times. Note that the index $k$ represents all possible cyclic $k$-shifts. Therefore, for $N$ CDCs in SWITCH the final output state for $M \in [2, N]$ causal orders is

$$\rho_M = \frac{1}{M} \left( I \otimes \mathbb{I} + \sum_{i \neq j} \frac{|i\rangle \langle j| \otimes \rho}{d^2} \right)$$

(8)

We note that the output density matrix does not depend on $N$ and only its off-diagonal entries depend on $\rho$. Further, if the control system is measured in the Fourier basis and the outcome is known, the target system will regain dependence on $\rho$ and one can extract information about it [22]. Therefore, there is a possibility of nonzero classical communication according to HSW theorem [38, 39]. In what follows, we will quantitatively investigate the above scheme.

A. Method

Classical capacity of quantum communication channel $\Lambda$ is characterized by the Holevo quantity, which is defined as

$$\chi(\Lambda) = \max_{\{\rho_i\}} \left[ H(\Lambda(\rho)) - \sum_i p_i H(\Lambda(\rho_i)) \right].$$

(9)

In the Ref.[22], authors found how to evaluate Holevo quantity for two quantum channels when the information is send through a pair of quantum channels processed by a quantum SWITCH. Since, this method works with the channel induced by the SWITCH on the control plus target system after the insertion of the depolarizing channels, it automatically work also for $N$-channels in the generalized SWITCH.

Therefore, the Holevo quantity of $N$ CDCs in the SWITCH is given by (we refer readers to the supplementary material of Ref.[22])

$$\chi(M) = \log d + H(\hat{\rho}_{c}(M)) - H_{\text{min}}(\rho_M), \forall M \in [2, N];$$

(10)
where \( \tilde{\rho}_c(M) \) is reduced state of control qubit after evolution (see Eq. (8)), i.e., \( \tilde{\rho}_c(M) = \frac{1}{d} \left( \sum_i |i\rangle \langle i | + \sum_{i \neq j} |i\rangle \langle j | \right) \) and \( H_{\min} \) is min-entropy, i.e., \( H_{\min}(\rho_M) = \min_{\rho} H(\rho_M) \) with \( H(\cdot) \) being the von Neumann entropy. However, the main difficulty will be to calculate the eigenvalues of the \( Md \times Md \) matrix, \( \rho_M \) to evaluate \( H_{\min}(\rho_M) \). Fortunately, we are able to use the method given in Ref. [40, 41] to find its eigenvalues. The determinant of the matrix, \( \rho_M \) is given by

\[
\text{Det}(\rho_M) = \text{Det} \left( \frac{1}{M} \left[ I - \rho + \frac{\rho}{d^2} \right] \right)^{M-1} \times \text{Det} \left( \frac{1}{M} \left[ I - \frac{\rho}{d} + (M-1) \frac{\rho}{d^2} \right] \right),
\]

(11)

where \( \text{Det} \times (M-1) \) denotes that there are \( M-1 \) products of same determinant (Full details of the calculations are given in Appendix A). This beautiful simplified form of determinant of actual \( Md \times Md \) matrix tells us that finding the eigenvalues of actual matrix has reduced to finding the eigenvalues of small matrices, i.e., \( \frac{1}{M} \left[ \frac{d}{I} - \frac{\rho}{d^2} \right] \) with degeneracy \( M-1 \) and \( \frac{1}{M} \left[ \frac{d}{I} + (M-1) \frac{\rho}{d^2} \right] \) with degeneracy one. As \( \|, \tilde{\rho} = 0 \), the eigenvalues of the actual matrix will be the union of the eigenvalues of these two smaller matrices with their appropriate degeneracy. Let \( \{\lambda_i^+\}_{i=1}^d \) and \( \{\lambda_i^-\}_{i=1}^d \) be the eigenvalues of \( \frac{1}{M} \left[ \frac{d}{I} + (M-1) \frac{\rho}{d^2} \right] \) respectively, then

\[
\lambda_i^+ = \frac{1}{Md} + \frac{M-1}{Md^2} \lambda_i^+; \quad \lambda_i^- = \frac{1}{Md} - \frac{1}{Md^2} \lambda_i^-;
\]

where \( \{\lambda_i^+\}_{i=1}^d \) are eigenvalues of \( \rho \). As \( H_{\min}(\rho_M) = \min_{\rho} H(\rho_M) \), certainly the minima will be ascertained if \( \lambda_i^+ = 1 \) and \( \lambda_i^- = 0 \) with \( i \neq j \). Therefore, with the constraint that \( \sum_i \lambda_i^+ = 1 \), we can find that

\[
H_{\min}(\rho_M) = -\left\{ \frac{d + (M-1)}{Md^2} \log \frac{d + (M-1)}{Md^2} + \frac{(M-1)(d-1)}{Md^2} \log \frac{(d-1)}{d} - \frac{1}{Md} \right\}.
\]

(12)

Now, the remaining task is to find the expression for \( H(\tilde{\rho}_c(M)) \), which is given by

\[
H(\tilde{\rho}_c(M)) = -\left\{ \frac{d^2 - 1}{Md^2} \log \frac{d^2 - 1}{Md^2} + (M-1) \frac{d^2 - 1}{Md^2} \log \frac{d^2 - 1}{Md^2} \right\}.
\]

(13)

With these expressions, we can evaluate the classical communication rate, \( \chi^{(M)} \) for \( N \) CDS with SWITCH from Eq.(10) for cyclic causal orders \( M \in [2, N] \). Notice that for \( M = 2 \), it reduces to the result for \( N = 2 \) scenario as discussed in Ref.[22]. This observation tells us that the gain in classical communication depends only on the number of superposed causal orders, \( M \).

### B. Results

In order to illustrate the behaviour of the Holevo quantity \( \chi^{(M)} \) with respect to dimension, \( d \), of the input state \( \rho \) and the number of causal orders we prepared a plot in Fig. 2.

We find that the classical communication capacity increases as we increase \( M \), however, it decreases almost-exponentially as \( d \) increases. Fig.3 shows the communication rates for different choice of \( (M, d) \). It is clear from the contour plot that the higher values of communication rates are achieved with smaller \( d \) values as well as higher \( M \) values. This means that using quantum SWITCH with \( M \) causal orders in \( d = 2 \) will offer maximum classical communication rate. However, we find that the communication rate saturates with the increase of \( M \), hinting that it is not possible to reach perfect communication in the asymptotic limit, i.e., \( \chi^{(\infty)} \neq 1 \) (see Fig.4).

To prove this claim, we write down the expression for Holevo quantity for \( d = 2 \), i.e.,

\[
\chi^{(M)}_{d=2} = 1 + \frac{1}{4} \left( \log_2 \frac{4}{27} + (1 + \frac{1}{M}) \log_2 \left( 1 + \frac{1}{M} \right) \right)
\]

\[
+ \frac{1}{M} \log_2 \frac{27}{M^2} - (1 + \frac{3}{M}) \log_2 \left( 1 + \frac{3}{M} \right).
\]

Therefore, at \( M \to \infty \), the Holevo quantity, \( \chi^{(\infty)}_{d=2} \cong 1 + \frac{1}{4} \log_2 \frac{1}{27} \approx 0.311 \text{bits/transaction} \).

### IV. Generalization to Various Combinations of Cyclic and Non-Cyclic Causal Orders

There are \( N! \) possible permutations of \( N \) elements, thus there exists \( N! \) causal orders of \( N \) channels. However, above, we have considered only \( M \leq N \) cyclic causal orders of channels, because the increase of communication rate is substantial to notice and it becomes problematic to implement superposition of \( N! \) causal orders in an experiment.

In this section, we will discuss whether the extension from \( N \) cyclic causal orders to all \( N! \) orders for \( N \) CDCs is provided significant improvement in the communication rates. In a recent work Ref.[36], authors showed that communication rates do not increase evenly with the increase of the number
of causal orders. We think that this behavior can be attributed to inevitable mixing of cyclic and non-cyclic causal orders, which hinders the potential benefit due to appearance of input state independent terms (see Appendix B). This is why we considered the cyclic orders separately.

By considering any order of action of \( N \) channels as the zeroth element and then performing cyclic permutations of this element, we can form a coset of cyclic causal orders (coset of permutation group with respect to the subgroup of cyclic permutations of \( N \) elements) which contains \( N \) elements. In this way, we can identify \( \frac{N!}{M} = (N - 1)! \) such cosets in the set of all \( N! \) permutations. We refer to these individual cosets as to cyclic causal orders, and each of them yields the same classical communication rate, which we already evaluated in the previous section.

However, if we consider superposition of orders of elements from different cosets the situation becomes a bit demanding. Off-diagonal contributions to \( \rho_M \) that map between control system states belonging to a single coset act on the message system as \( \frac{\partial}{\partial \tau} \) (see Eq. (7)), however, this is no longer the case if the control system states are not from the same coset.

Let us note that as the number of channels increases the number of types of off-diagonals contributions increases and will include the terms presented in the following Table I. For derivation of the terms appearing in the table we refer the reader to Appendix B. In accordance with Table I for \( N \geq 3 \) we found that lot of cross-coset off-diagonal terms are proportional to \( I \) and the terms which are proportional to \( \rho \) have a decreasing weight factor as \( N \) increases. To illustrate the impact of the above findings we present a case study for \( N = 3 \) and \( N = 4 \).

### Table I. Entries for off-diagonal blocks in the output density matrix

| \( N \) | Terms in off-diagonal block |
|---|---|
| 2 | \( \frac{\partial}{\partial \tau} \rho \) |
| 3 | \( \frac{\partial}{\partial \tau} \rho, \frac{\partial}{\partial \tau} \mathbb{I} \) |
| 4 | \( \frac{\partial}{\partial \tau} \rho, \frac{\partial}{\partial \tau} \mathbb{I}, \frac{\partial}{\partial \tau} \rho \) |
| \( \ldots \) | \( \ldots \) |
| \( N = 2k \) | \( \frac{\partial}{\partial \tau} \rho, \frac{\partial}{\partial \tau} \mathbb{I}, \ldots, \frac{\partial}{\partial \tau} \rho, \frac{\partial}{\partial \tau} \mathbb{I} \) |
| \( N = 2k + 1 \) | \( \frac{\partial}{\partial \tau} \rho, \frac{\partial}{\partial \tau} \mathbb{I}, \ldots, \frac{\partial}{\partial \tau} \rho, \frac{\partial}{\partial \tau} \mathbb{I}, \frac{\partial}{\partial \tau} \mathbb{I} \) |

A. Case study for \( N = 3 \)

For three CDCs, there will be two cosets of cyclic causal orders. The Kraus operator for three CDCs with SWITCH can be expressed as \( d^3 K_{ijk} = \sum_{\ell=0}^{M-1} \langle \ell | \otimes P_\ell (U_1, U_2, U_3) \rangle \), where \( 2 \leq M \leq 6 \). For the message qubit prepared in \( \rho \), the evolved state at the output of quantum SWITCH is \( \rho_{M_1, M_2} = S(\Lambda_1, \Lambda_2, \Lambda_3) (\rho \otimes \rho) \), where \( M_1, M_2 \) denotes the number of causal orders from two cosets respectively, and \( M_1 + M_2 = M \). In appendix B 1 we find that

\[
\rho_{M_1, M_2} = \frac{1}{M} \left\{ I_c \otimes \frac{I}{d} + L_{M_1, M_2} \otimes \frac{\rho}{d^2} + B_{M_1, M_2} \otimes \frac{I}{d^3} \right\},
\]

(14)
where $2 \leq M \leq 6$. The matrix $L$ and $B$ are $M \times M$ matrices with the following form,

$$
L = \left( \begin{array}{cc} S_{M \times M} & 0_{M \times M} \\ 0_{M \times M} & S_{M \times M} \end{array} \right), \quad B = \left( \begin{array}{cc} 0_{M \times M} & 1_{M \times M} \\ 1_{M \times M} & 0_{M \times M} \end{array} \right),
$$

where the matrix $S = \sum_{i \neq j=0}^{M-1} |i\rangle \langle j|$. $0$ is a null matrix and $1$ is a matrix with all entries equal to one. For such scenario, the general expression for the classical communication for three CDCs with $M \in [2, 6]$ causal orders can efficiently be written as before

$$
\chi^{(M_1,M_2)} = \log d + H(\bar{\rho}_c(M)) - H_{\min}(\rho_{M_1,M_2}),
$$

(15)

where $\bar{\rho}_c(M)$ is reduced state of the control system after evolution, i.e., $\bar{\rho}_c(M) = \frac{1}{d^2} (I + \frac{1}{d} \sum_{i \neq j} |i\rangle \langle j|)$.

To evaluate $H(\bar{\rho}_c(M))$ and $H_{\min}(\rho_{M_1,M_2})$, we need to diagonalize the matrices, $\bar{\rho}_c(M)$ and $\rho_{M_1,M_2}$ respectively. We know that the expression for the $H(\bar{\rho}_c(M))$ is given in Eq. (13). However, diagonalizing $\rho_{M_1,M_2}$ is much more complicated. Analytical diagonalization is done in Appendix B 1 using the method in Appendix C. It follows that for $M_1 = M_2$, $[L, B] = 0$ and therefore, the matrices $L$ and $B$ are simultaneously diagonalizable. Consequently, $H_{\min}(\rho_{M_1,M_1})$ is given by

$$
-H_{\min}(\rho_{M_1,M_1}) = \lambda_+ \log \lambda_+ + \lambda_- \log \lambda_-
$$

$$
+ (d - 1) \left[ \lambda_0^0 \log \lambda_0^0 + \lambda_0^d \log \lambda_0^0 \right]
$$

$$
+ (M - 2) \left[ \frac{1}{Md^2} \log \frac{(d - 1)}{Md^2} + \frac{1}{Md} \log \frac{1}{Md} \right],
$$

where $\lambda_\pm = \frac{d^2 + (M_1-1)d + M_1}{Md^2}$ and $\lambda_0^{d} = \frac{d^2 + M_1}{Md^2}$ with $M = 2M_1$. We show in Appendix B 1 that the output states for $M_1 \neq M_2$ can also be diagonalized using the method presented in Appendix C.

To further elucidate our findings here, we compute the Holevo quantity for all $(M_1, M_2)$ values in the Table II for $d = 2$. We find that the Holevo quantity is higher for the case when either $M_1 = 0$ or $M_2 = 0$ compared to $M_1 \neq M_2$. However, we find that the Holevo quantity reaches its maximum for $M_1 = M_2 = 3$.

| $M_2$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| 0     | - | 0 | 0.0488 | 0.0817 |
| 1     | 0 | 0 | 0.0334 | 0.0640 |
| 2     | 0.0488 | 0.0334 | 0.0524 | 0.0767 |
| 3     | 0.0817 | 0.0640 | 0.0767 | 0.0981 |

TABLE II. Table shows the communication rates $\chi^{(M_1,M_2)}$ for three CDCs in superposition of $M = M_1 + M_2$ causal orders from two cosets. Here, we consider message state to be a qubit ($d = 2$). Table shows that the Holevo quantity is higher for the cases when either $M_1$ or $M_2$ is zero.

We also plot the Holevo quantity for different $(M_1, M_2)$ values with respect to the dimension of the target state $d$ in Fig.5, and find that the Holevo quantity decreases with $d$ in an exponential-like fashion. In Fig.6, we plot the Holevo quantity of $(N = 3, M = 6)$ (all cosets) as well as $(N = 3, M = 3)$ (cyclic orders) against the dimension $d$. The figure shows that both scenarios yield the same communication rates except for $d = 2$, where the former dominates. This plot indicates that it might be efficient to consider only cyclic permutations (one coset).

B. Case study for $N = 4$

For four channels there are six $(4-1)! = 6$ cosets of cyclic causal orders. Let us denote $M_\eta$ as the number of causal orders in a coset where $M_\eta \in [0, 3]$ for each $\eta \in [1, 6]$. Let us consider that the target state is $\rho$. Now, if we consider $M = (\sum_\eta M_\eta)$ causal orders in a quantum SWITCH, the output state will have – a) within each cyclic coset, the diagonal terms are proportional to $\frac{1}{d}$ and off-diagonal terms are proportional to $\frac{d^2}{d^4}$; and b) cross-coset off-diagonal terms are proportional to $\frac{1}{d^2}$ as well as $\frac{d^2}{d^4}$ (for detailed calculation see Ap-
pendix B). We will use a particular way of listing causal orders for each coset. Namely, the first entry of each coset is linked with the order $\Lambda_1A_2A_3A_4$ by some permutation of the last three labels. In this way, the first element in each coset is starting with $\Lambda_1$. And rest is constructed using cyclic permutations from first element of corresponding coset. For $\eta = 1, 2, ..., 6$, the zeroth elements are respectively $\Lambda_1A_2A_3A_4, \Lambda_1A_2A_4A_3, \Lambda_1A_3A_2A_4, \Lambda_1A_3A_4A_2, \Lambda_1A_4A_2A_3$ and $\Lambda_1A_4A_3A_2$. Using this setup, we find that the output state after evolution driven by the SWITCH is given by

$$\rho_{M\eta} = \frac{1}{M}\left\{\mathbb{I}_c \otimes \mathbb{I}_d + \left(\frac{L_{M\eta}}{d^2} + \frac{Q_{M\eta}}{d^4}\right) \otimes \rho + B_{M\eta} \otimes \frac{\mathbb{I}}{d^3}\right\},$$

(16)

where the matrices, $L$, $B$ and $Q$ are specified in Appendix B. Also, in this case, the general expression for the classical communication for four CDCs with $2 \leq M \leq 24$ causal orders can efficiently be written as

$$\chi^{(M\eta)} = \log d + H(\tilde{\rho}_c(M\eta)) - H_{\min}(\rho_{M\eta}),$$

(17)

where $\tilde{\rho}_c(M\eta)$ is reduced state of control qubit after evolution, i.e., $\tilde{\rho}_c(M\eta) = \frac{1}{d^2}(I + \frac{1}{d^2}\{L_{M\eta} + B_{M\eta}\} + \frac{1}{d^4}Q_{M\eta})$.

It is very hard to evaluate Holevo quantity for arbitrary $M\eta$, as diagonalizing $\rho_{M\eta}$ is usually hard analytically. Therefore, we mostly resort to numerical approach. However, there are specific cases where it is possible to analytically diagonalize $\rho_{M\eta}$, e.g., the scenario with $(M_3 = M_2 = 4; M = 8)$ (We refer readers to Appendix B 2 for complete analysis).

To graphically illustrate our findings for $N = 4$ we present two plots. In Fig.7 we plot dependence of Holevo quantity on $d$ for different $M$ values, i.e., $M = \{6, 8, 12, 16, 24\}$. For $M = 6$, we consider two scenarios - a) $(M_1 = 4, M_2 = 2$) and b) $M\eta = 1 \forall \eta$ (see red and blue lines (with circular points) respectively in Fig.7). In former case, we consider a situation such that only two cross-coset off-diagonal terms are dependent on $\rho$. However, in later case, we consider maximum number of $\rho$ dependent cross-coset off-diagonal terms (see Appendix B 2). Fig.7 shows that the Holevo quantity is decreasing in exponential-like fashion with $d$. Unlike Fig.5, here the Holevo quantity for all the plotted $M\eta$ are very close to each other except for $d = 2$. This is due to the presence of a big number of cross-coset off-diagonal terms $\frac{\mathbb{I}}{d^3}$ in these scenarios. In Fig.8 we plot the Holevo quantity for $(N = 4, M = 24)$ and its cyclic counter part $(N = 4, M = 4)$ with respect to message system dimension $d$. Analogously to Fig. 6, also here the plot shows that both scenarios provide the same communication rates except for $d = 2$. Comparing Figs.6 and 8 we uncover that the gap between Holevo quantity for $(N, M = N!)$ (all cosets) and $(N, M = N)$ (one coset) increases for $d = 2$ as we increase $N$ from three to four.

V. CONCLUSIONS

Completely depolarizing channel erases all information about its input state and always prepares a completely mixed state. Thus, sequential application of two such channels on the same system in any fixed order must have zero classical (or quantum) communication capacity. It was a rather surprising finding of Ebler, Salek and Chiribella [22] that processing of two depolarizing channels by Quantum SWITCH followed by suitable control system measurement enables nonzero classical communication rate.

In this paper, we studied a generalization of this scenario to $N$ completely depolarizing channels inserted into (generalized) quantum SWITCH. Quickly growing number of possible causal orders ($N!$) might become a roadblock for experimental realization of the SWITCH, thus one might wonder if less demanding superpositions of causal orders could provide similar advantages. As previous numerical results for $N = 3$ show [35], already $M = N$ superposed causal orders can provide almost the fully achievable communication rate. We provide analytical results for transmission of classical information via superpositions of $M$ cyclically permuted Completely depolarizing channels. We find that the Holevo quantity is increasing with $M$ and is independent of $N$. Surprisingly, the classical capacity decreases with the dimension $d$ of the message system. We found that the classical commu-
communication rate for a qubit never reaches 1 if we are increasing \( M \) (and inevitably also \( N \)). It saturates at around 0.311 bits per transmission. Out of \( N! \) possible causal orders for \( N \) channels there are \( N \) cyclic permutations forming a subgroup. Factorizing the permutation group with respect to it we obtain \((N-1)!\) cosets, each of which is shown to be equally usable for the investigated task. However, for general \( N \) we did not consider all possible causal orders together as the cross-coset off-diagonal terms are mostly independent of the message state. Instead, for \( N = 3, 4 \) we studied separately cyclic, all causal orders and various superpositions of causal orders consisting of different number of terms from different cosets. For the causal orders from arbitrary number of cosets, i.e., non-cyclic case, we find that with growing \( N \) the cross-coset off-diagonal terms have smaller scaling factors and are either proportional to Identity or to the message state \( \rho \). Therefore, we find that the Holevo quantity doesn’t always increase as we increase the number of superposed causal orders. Our findings support the belief that considering superpositions of cyclic causal orders might yield almost optimal classical communication rates for \( N \) CDCs in the SWITCH.

**Acknowledgement:** SS acknowledges the financial support through the Štefan Schwarz stipend from Slovak Academy of Sciences, Bratislava. MS and SS acknowledge the financial support through the project OPTIQUE (APVV-18-0518) and HOQIT (VEGA 2/0161/19). MS was also supported by Grant No. 61466 from the John Templeton Foundation, as part of the The Quantum Information Structure of Spacetime (QISS) Project (qiss.fr). The opinions expressed in this publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation. We thank Mario Ziman for valuable discussions. SS thanks Nidhin for a small yet crucial help.

**Note added:** After publishing first version of our work in arXiv, we noticed a similar work by Chiribella et al. [43] appeared in arXiv which independently derives one of our results. They also claim that the classical capacity of \( N \)-channels with superposition of cyclic orders is exactly equal to the Holevo quantity, which will also strengthen our results in Section.III.

**Appendix A: Calculations for \( N \) channels with cyclic orders in a quantum SWITCH**

Initially, we calculated the determinant of the matrix from Eq. (8) for small values of \( M \) using the Eq. (C1). It also allowed us to anticipate its form for general \( M \). However, the following lemma will be proved in a simpler way using the properties of block circulant matrix [42].

**Lemma 1.** For hermitian matrix \( \rho \) determinant of \( Md \times Md \) matrix \( \rho_M \) defined in Eq. (8) is

\[
\text{Det}(\rho_M) = \text{Det}\left(\frac{1}{M}\left[\mathbb{I} + (M - 1)\frac{\rho}{d^2}\right]\right) \times \text{Det}\left(\frac{1}{M}\left[\mathbb{I} - \frac{\rho}{d^2}\right]\right) \times (M-1),
\]

where \( \text{Det}(\cdot)^{(M-1)} \) denotes that there are \( M - 1 \) products of same determinant.

**Proof.** We will begin here by mentioning some properties of block circulant matrix [42]. A block circulant matrix \( C \) of the form

\[
C = 
\begin{pmatrix}
A_0 & A_1 & \cdots & A_{M-2} & A_{M-1} \\
A_{M-1} & A_0 & \cdots & A_{M-3} & A_{M-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_2 & A_3 & \cdots & A_0 & A_1 \\
A_1 & A_2 & \cdots & A_{M-1} & A_0
\end{pmatrix},
\]

can be written in the following form

\[
C = \sum_{i=0}^{M-1} P_i \otimes A_i,
\]

where \( P_i \) are Permutation matrices. Now, one can Block-diagonalise \( C \) using the properties of \( P_i \) matrices as mentioned in
Ref.[42]. For the matrix $\rho_M$ in Eq.(8), we can decompose it to the following form

$$\rho_M = P_0 \otimes \frac{I}{Md} + \left( \sum_{i=1}^{M-1} P_i \right) \otimes \frac{\rho}{Md^2},$$

$$= P_0 \otimes \frac{I}{Md} + S \otimes \frac{\rho}{Md^2}, \quad (A2)$$

where $S = \sum_{i=1}^{M-1} P_i$ is a symmetric matrix with the entries, $S_{ij} = 1 - \delta_{ij},$

$$S = \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix}_{M \times M} \quad (A3)$$

Therefore the block-diagonalisation of $\rho_M$ will depend on the properties of the matrix $S$ alone as $P_0 = I$. Then, one can define a matrix function $G : \mathcal{X} \to M^2$ such that

$$G(x) = x^0 \otimes \frac{I}{Md} + x^1 \otimes \frac{\rho}{Md^2}, \quad (A4)$$

where each element of an object in $\mathcal{X}$ is mapped to a $d \times d$ matrix. If $x$ is a (complex) number then the symbol '$\otimes$' will just be a product and the zeroth power of $x$ is a number one. Hence, one can show that $\rho_M = G(S)$, since $S^0 = I$.

Next, we diagonalize the matrix $S$. Its characteristic equation is $\text{Det}(S - \lambda I) = 0 \Rightarrow \{\lambda - (M - 1)\}(\lambda + 1)^{M-1} = 0$, i.e., eigenvalues of $S$ are $M - 1$ with degeneracy one and $-1$ with degeneracy $M - 1$. So, one can find an unitary, $T$ which diagonalises $S$, such that $T^\dagger S T = \text{diag}(M - 1, -1, \cdots, -1)$. Consequently, matrix of the form $T = T \otimes I$ block-diagonalises $\rho_M$ as

$$T^\dagger \rho_M T = P_0 \otimes \frac{I}{Md} + T^\dagger S T \otimes \frac{\rho}{Md^2},$$

i.e., one can write $T^\dagger \rho_M T = \text{diag}(G(M - 1), G(-1), \cdots, G(-1))$. Hence, one can conclude validity of Eq.(A1).

Equipped with the above lemma and its proof it is an easy task to show that the characteristic equation for $\rho_M$ is of the form

$$\text{Det}(\rho_M - \lambda I) = 0 \Rightarrow \text{Det}\left( \frac{1}{M} \left[ \frac{I}{d} + (M - 1) \frac{\rho}{d^2} \right] - \lambda I \right) \times \text{Det}\left( \frac{1}{M} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] - \lambda I \right) = 0. \quad (A5)$$

Therefore, the eigenvalues of $\rho_M$ can be obtained as the union of eigenvalues of matrices $\frac{1}{M} \left[ \frac{I}{d} + (M - 1) \frac{\rho}{d^2} \right]$ with degeneracy one and $\frac{1}{M} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right]$ with degeneracy $M - 1$.

**Appendix B: All possible causal orders**

Here we sketch a proof for the entries in the Table I. Below, in Eq. B1, we show that for more than two channels, if we go beyond cyclic causal orders, the off-diagonal terms are also proportional to $\frac{1}{d^2}$ along with the terms proportional to $\frac{\rho}{d^2}$. Also, in Eq. B2, we show that some of the off-diagonal terms for more than three channels are proportional to $\frac{\rho}{d^2}$. Using these observations, we sketch how off-diagonal terms appear for arbitrary $N$. Let us begin with the following term:

$$\frac{1}{d^{2N}} \sum_{i_1 \cdots i_N} U_{i_N} \cdots U_{i_1} U_k U_j U_i \left( \rho U^\dagger_j U^\dagger_k \right) U^\dagger_k U^\dagger_j \cdots U^\dagger_i \frac{1}{d^{2(N-3)}} \text{Tr}[\rho] \frac{I}{d^2} = \frac{\rho}{d^3}. \quad (B1)$$

Next, consider the following off-diagonal term

$$\frac{1}{d^{2N}} \sum_{i_1 \cdots i_N} U_{j_1} U_{i_N} \cdots U_{i_m} U_k U_j U_i \left( \rho U^\dagger_j U^\dagger_k \right) U^\dagger_k U^\dagger_j \cdots U^\dagger_i \frac{1}{d^{2(N-4)}} \rho = \frac{\rho}{d^2}. \quad (B2)$$
It is evident from above calculation that the term $\frac{1}{d^2}$ will not occur if $N < 4$. Now, notice the similarity between two Eqs.(B1, B2). They are quite similar except the positioning of $U_j$. Using this connection, we consider the following term

$$\frac{1}{d^{2N}} \sum_{i,j,\ldots,n} \overbrace{U_\eta \cdots U_n}^{N-5} U_j U_m U_k U_i \left( \rho U_j^\dagger U_m^\dagger U_k^\dagger U_i^\dagger \right) U_i^\dagger U_n^\dagger \cdots U_\eta^\dagger = \frac{1}{d^{2N}} \cdot d^2 \cdot d^2 \cdot d^{2(N-5)} \text{Tr} [\rho] \frac{\mathbb{I}}{d} = \mathbb{I} \cdot \frac{1}{d^5}.$$ 

The above term will not occur when $N < 5$. Therefore, we can infer the terms proportional to $\frac{1}{d^{3N}}$ and $\frac{\rho}{d^{2N}}$, where $k \in \mathbb{Z}^+$;

$$\frac{1}{d^{2N}} \sum_{i,j,\ldots,n} \overbrace{U_\eta \cdots U_n}^{N-(2k+1)} U_j U_m U_k U_i \left( \rho U_j^\dagger U_m^\dagger U_k^\dagger U_i^\dagger \right) U_i^\dagger U_n^\dagger \cdots U_\eta^\dagger = \frac{1}{d^{2N}} \cdot d^2 \cdot d^{2(k-1)} d^{2(N-(2k+1))} \text{Tr} [\rho] \frac{\mathbb{I}}{d} = \mathbb{I} \cdot \frac{1}{d^{2k+1}}.$$ 

Again, we can comment that the above term will not occur if $N < (2k + 1)$. Analogously, we have the term

$$\frac{1}{d^{2N}} \sum_{i,j,\ldots,n} \overbrace{U_\eta \cdots U_n}^{N-2k} U_j U_m U_k U_i \left( \rho U_j^\dagger U_m^\dagger U_k^\dagger U_i^\dagger \right) U_i^\dagger U_n^\dagger \cdots U_\eta^\dagger = \frac{1}{d^{2N}} \cdot d^2 \cdot d^{2(k-1)} \cdot d^{2(N-2k)} \rho = \frac{\rho}{d^{2k}}.$$ 

Thus, we showed existence of the terms presented in Table I.

1. Three channels in a SWITCH

For three CDCs, there are two cosets (6/3 = 2) of cyclic orders. Let us consider the following permutation of channels $\Lambda_2 \Lambda_1 \Lambda_3$. Now applying cyclic permutations to it, we get a coset $\{\Lambda_2 \Lambda_1 \Lambda_3, \Lambda_1 \Lambda_3 \Lambda_2, \Lambda_3 \Lambda_2 \Lambda_1\}$. One can also see that rest of the channel orders forms the other coset. The remaining task is to see how a message state behaves when it is sent through the superposition of $M_1 + M_2 = M$ causal orders. It is easy to see that off diagonal elements (in the sense of Eq.(5)) within a single coset will contribute in the same way as in Eq.(8). Next, we need to investigate the off-diagonal terms between two cosets. By analogous calculation as in Eq.(7), which we summarize below, we found that these are proportional to identity, implying that they will not contribute to the classical communication. In particular, there are two types of terms that can be evaluated as follows:

$$\frac{1}{d^6} \sum_{i,j,k} U_i U_j U_k \rho U_j^\dagger U_m^\dagger U_i^\dagger = \frac{1}{d^4} \sum_i U_i \left( \frac{1}{d^2} \sum_j U_j (U_k \rho) U_j^\dagger \right) U_i^\dagger U_i^\dagger = \frac{1}{d^4} \sum_i U_i \rho U_i^\dagger = \mathbb{I} \cdot \frac{1}{d^3},$$

$$\frac{1}{d^6} \sum_{i,j,k} U_i U_j U_k \rho U_k^\dagger U_j^\dagger U_i^\dagger = \frac{1}{d^4} \sum_i U_i \left( \frac{1}{d^2} \sum_j U_j (U_k \rho U_j^\dagger) U_j^\dagger \right) U_i^\dagger U_i^\dagger = \frac{1}{d^4} \sum_i U_i \left( \text{Tr} \left[ U_i^\dagger (U_k \rho) \right] \right) \frac{\mathbb{I}}{d} U_i^\dagger U_i^\dagger = \frac{1}{d^4} \sum_i U_i \rho U_i^\dagger = \mathbb{I} \cdot \frac{1}{d^3},$$

where we used $\frac{1}{d} \sum_k \left( \text{Tr} \left[ U_k^\dagger U_k \rho \right] \right) U_k^\dagger = \rho$ and $\frac{1}{d} \sum_i \left( \text{Tr} \left[ U_i^\dagger (U_k \rho) \right] \right) U_i^\dagger U_i^\dagger = \frac{\rho}{d^2}$.

Hence, the final output state is

$$\rho_{M_1,M_2} = \frac{1}{M} \left\{ \mathbb{I}_c \otimes \frac{\mathbb{I}}{d} + \left( \sum_{\ell \neq \ell'} \sum_{\mu \neq \mu'} \frac{M_1-1}{M} \left( \ell \langle \ell' | c + \sum_{\mu \neq \mu'} | \mu \rangle \langle \mu' | c \right) \right) \otimes \frac{\rho}{d^2} + \sum_{\ell,\mu} (| \ell \rangle \langle \mu | c + | \mu \rangle \langle \ell | c) \otimes \frac{\mathbb{I}}{d^3} \right\}. \tag{B3}$$

To find the eigenvalues of the matrix in Eq.(B3) for general $M_1$ and $M_2$, we will re-write it in the Block form, i.e.,

$$\rho_{M_1,M_2} = \frac{1}{M} \left\{ \mathbb{I}_c \otimes \frac{\mathbb{I}}{d} + L_{M_1,M_2} \otimes \frac{\rho}{d^2} + B_{M_1,M_2} \otimes \frac{\mathbb{I}}{d^3} \right\}, \tag{B4}$$

where $M \times M$ matrices $L$ and $B$ have the following form
where the matrix $S$ is defined in Eq. (A3), $0$ is a null matrix and $I$ is a matrix with all entries are one. For $M_1 = M_2$ \([L, B] = 0\), so these two matrices are simultaneously diagonalizable using a $M \otimes M$ unitary matrix, $U$, i.e., $U^†LU = \text{Diag}(M_1, M_1 - 1, \cdots, M_1 - 1)$ and $U^†BU = \text{Diag}(−M_1, M_1, 0, \cdots, 0)$. Consequently, the unitary matrix $\hat{U} = U \otimes I$ will Block-diagonalize the matrix $\rho^{M_1, M_1}$, i.e.,

$$\hat{U}^†\rho_{M_1, M_1}\hat{U} = \frac{1}{M} \left\{ I \otimes \frac{I}{d} + U^†LU \otimes \frac{\rho}{d^2} + U^†BU \otimes \frac{I}{d^3} \right\}. \quad (B5)$$

This enables us to find the characteristic equations for above Block diagonal matrix as,

$$\text{Det}(\rho_{M_1, M_1} - \lambda I) = 0 \Rightarrow \text{Det}\left( \frac{1}{M} \left[ \frac{I}{d} + (M_1 - 1) \frac{\rho}{d^2} + M_1 \frac{I}{d^3} \right] - \lambda I \right) \times \text{Det}\left( \frac{1}{M} \left[ \frac{I}{d} + (M_1 - 1) \frac{\rho}{d^2} - M_1 \frac{I}{d^3} \right] - \lambda I \right) \times \text{Det}\left( \frac{1}{M} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] - \lambda I \right)^{\times (M-2)} = 0. \quad (B6)$$

Therefore, the eigenvalues of $\rho_{M_1, M_1}$ is the union of eigenvalues of $\frac{1}{M} \left[ \frac{I}{d} + (M_1 - 1) \frac{\rho}{d^2} \pm M_1 \frac{I}{d^3} \right]$ with degeneracy one and $\frac{1}{M} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right]$ with degeneracy $M - 2$.

However, for $M_1 \neq M_2$, we will resort to the technique described in Appendix C. In this scenario, we have three distinct cases, i.e., $(M_1 = 2, M_2 = 1)$, $(M_1 = 3, M_2 = 1)$ and $(M_1 = 3, M_2 = 2)$. To diagonalize the output matrix $\rho_{M_1, M_2}$ for these cases, we consider the characteristic equation, $\text{Det}(\rho_{M_1, M_2} - \lambda I) = 0$ and solve it to find eigenvalues.

\textbf{A1.} (2, 1): The characteristic equation for this case can be written using Appendix (C),

$$\text{Det}\left( \frac{1}{3} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] - \lambda I \right) \times \text{Det}\left( \frac{1}{9} \left[ a^2 \frac{I}{d^2} + a \frac{\rho}{d^3} - \frac{2}{3} \frac{I}{d^3} \right] \right) = 0, \quad (B7)$$

where $a = 1 - 3d\lambda$.

**Proof.** Using Eq.(C1), the characteristic determinant reduces to $\text{Det}(\rho_{2,1} - \lambda I) = \text{Det} \left[ X_{11}^{(2)} \right] \text{Det} \left[ X_{22}^{(1)} \right] \text{Det} \left[ X_{33}^{(0)} \right]$. Again using Eq.(C2), we find $X_{11}^{(2)} = X_{11}^{(1)} - X_{12}^{(1)} \left( X_{22}^{(1)} \right)^{−1} X_{21}^{(1)}$, then $\text{Det} \left[ X_{11}^{(2)} \right] \text{Det} \left[ X_{22}^{(1)} \right] = \text{Det} \left[ X_{11}^{(1)} X_{22}^{(1)} - X_{12}^{(1)} X_{21}^{(1)} \right]$. Now, we notice that $X_{11}^{(1)} = X_{22}^{(1)}$ and $X_{12}^{(1)} = X_{21}^{(1)}$ which means

$$\text{Det} \left[ X_{11}^{(2)} \right] \text{Det} \left[ X_{22}^{(1)} \right] = \text{Det} \left[ X_{11}^{(1)} - X_{12}^{(1)} \right] \text{Det} \left[ X_{11}^{(1)} + X_{12}^{(1)} \right].$$

At this stage we will consider the product of determinants, $\text{Det} \left[ X_{11}^{(1)} + X_{12}^{(1)} \right] \text{Det} \left[ X_{33}^{(0)} \right]$. Applying Eq.(C1), we can simplify it as following

$$\text{Det} \left[ X_{11}^{(1)} + X_{12}^{(1)} \right] \text{Det} \left[ X_{33}^{(0)} \right] = \text{Det} \left\{ \left[ X_{11}^{(0)} \right]^2 + X_{12}^{(0)} X_{11}^{(0)} - 2 \left[ X_{13}^{(0)} \right]^2 \right\},$$

where we have used the fact that $X_{11}^{(0)} = X_{33}^{(0)} = \frac{a}{3d^2}$ and $X_{13}^{(0)} = X_{31}^{(0)} = X_{32}^{(0)} = \frac{1}{3d^3}$. Noticing that $X_{12}^{(0)} = \frac{\rho}{3d^2}$ and $\text{Det} \left( X_{11}^{(1)} - X_{12}^{(1)} \right) = \text{Det} \left( \frac{1}{3} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] \right)$, we complete the proof. \qed

Therefore, the eigenvalues of the matrix $\rho_{2,1}$ are the union of eigenvalues of matrices, $\frac{1}{3} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right]$ and $\frac{1}{3} \left[ \frac{I}{d} + \frac{\rho}{d^2} \pm \frac{1}{2} \sqrt{\frac{8}{d^3} + \frac{\rho^2}{d^6}} \right]$. Note that the above method and results directly apply also for the case $\rho_{1,2}$ due to symmetry with respect to the exchange of $M_1$ and $M_2$.

\textbf{A2.} (3, 1): Using the method described in Appendix C, we find that the characteristic equation for this case takes the form

$$\text{Det} \left( \frac{1}{4} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] - \lambda I \right)^{\times 2} \times \text{Det} \left( \frac{1}{16} \left[ \frac{b^2}{d^2} + \frac{2b\rho}{d^3} - \frac{3}{d^3} \right] \right) = 0, \quad (B8)$$

where $b = 1 - 4d\lambda$. 

Proof. Using Eq. (C1), the characteristic determinant reduces to 
\[ \text{Det}(\rho_{3,1} - \lambda \mathbb{I}) = \text{Det} \left[ X_{11}^{(3)} \right] \text{Det} \left[ X_{22}^{(2)} \right] \text{Det} \left[ X_{33}^{(1)} \right] \text{Det} \left[ X_{44}^{(0)} \right] \] 
Again using Eq. (C2), we find \( X_{11}^{(3)} = X_{11}^{(2)} - X_{12}^{(2)} (X_{22}^{(2)})^{-1} X_{21}^{(2)} \), 
then \( \text{Det} \left[X_{11}^{(2)} \right] \text{Det} \left[X_{22}^{(2)} \right] = \text{Det} \left[X_{11}^{(2)} X_{22}^{(2)} - X_{12}^{(2)} X_{21}^{(2)} \right] \). Now, we notice that \( X_{11}^{(2)} = X_{22}^{(2)} \) and \( X_{12}^{(2)} = X_{21}^{(2)} \) which means 
\[ \text{Det} \left[X_{11}^{(3)} \right] \text{Det} \left[X_{22}^{(2)} \right] = \text{Det} \left[X_{11}^{(2)} - X_{12}^{(2)} \right] \text{Det} \left[X_{11}^{(2)} + X_{12}^{(2)} \right]. \]

At this stage we will consider the product of determinants, \( \text{Det} \left[X_{11}^{(2)} + X_{12}^{(2)} \right] \text{Det} \left[X_{33}^{(1)} \right] \). Applying Eq. (C1), we can simplify it as following
\[ \text{Det} \left[X_{11}^{(2)} + X_{12}^{(2)} \right] \text{Det} \left[X_{33}^{(1)} \right] = \text{Det} \left[ \left( X_{11}^{(1)} \right)^2 + X_{12}^{(1)} X_{11}^{(1)} - 2 \left( X_{12}^{(1)} \right)^2 \right] \]
where we have used the fact that \( X_{11}^{(1)} = X_{11}^{(1)} \) and \( X_{12}^{(1)} = X_{13}^{(1)} = X_{14}^{(1)} = X_{12}^{(1)} \). Now, the product
\[ \text{Det} \left[X_{11}^{(2)} + X_{12}^{(2)} \right] \text{Det} \left[X_{33}^{(1)} \right] \text{Det} \left[X_{44}^{(0)} \right] \]

where we have used the fact that \( X_{04}^{(0)} = X_{44}^{(0)} = \frac{4}{45} \) and \( X_{14}^{(0)} = X_{14}^{(0)} = \frac{1}{45} \). Noticing that \( X_{14}^{(0)} = \frac{\rho}{45} \) and
\[ \text{Det} \left[X_{11}^{(2)} - X_{12}^{(2)} \right] = \text{Det} \left[X_{11}^{(1)} - X_{12}^{(1)} \right] = \left( \frac{4}{45} - \frac{\rho}{45^2} \right) \), we complete the proof.

Therefore, the eigenvalues of the matrix \( \rho_{3,1} \) are the union of eigenvalues of matrices, \( \frac{1}{4} \left[ \frac{1}{4} \frac{1}{4} \frac{1}{4} \right] \) with degeneracy 2 and
\( \frac{1}{4} \left[ \frac{1}{4} \frac{1}{4} \frac{1}{4} \right] \). Note that due to symmetry the above applies also to the case of \( \rho_{1,3} \).

**A3. (3, 2):** Using the method described in Appendix C, we find that the characteristic equation for this case takes the form
\[ \text{Det} \left[ \frac{1}{4} \left[ \frac{1}{4} - \frac{\rho}{45^2} \right] - \lambda \mathbb{I} \right] = \left( - \frac{d}{d^2} + \frac{\rho}{45^2} \right) \] 

where \( c = 1 - 5d \).

Proof. Using Eq. (C1), the characteristic determinant reduces to 
\[ \text{Det}(\rho_{3,2} - \lambda \mathbb{I}) = \prod_{k=1}^{5} \text{Det}[X_{k,k}^{(5-k)}] \] 
Again using Eq. (C2), we find \( X_{11}^{(4)} = X_{11}^{(4)} - X_{12}^{(3)} (X_{22}^{(3)})^{-1} X_{21}^{(3)} \), then \( \text{Det} \left[X_{11}^{(4)} \right] \text{Det} \left[X_{22}^{(3)} \right] = \text{Det} \left[X_{11}^{(4)} X_{22}^{(3)} - X_{12}^{(3)} X_{21}^{(3)} \right] \). Now, we notice that \( X_{11}^{(3)} = X_{22}^{(3)} \) and \( X_{12}^{(3)} = X_{21}^{(3)} \) which means 
\[ \text{Det} \left[X_{11}^{(4)} \right] \text{Det} \left[X_{22}^{(3)} \right] = \text{Det} \left[X_{11}^{(3)} - X_{12}^{(3)} \right] \text{Det} \left[X_{11}^{(3)} + X_{12}^{(3)} \right]. \]

At this stage we will consider the product of determinants, \( \text{Det} \left[X_{11}^{(3)} + X_{12}^{(3)} \right] \text{Det} \left[X_{33}^{(1)} \right] \). Applying Eq. (C1), we can simplify it as follows
\[ \text{Det} \left[X_{11}^{(3)} + X_{12}^{(3)} \right] \text{Det} \left[X_{33}^{(1)} \right] = \text{Det} \left[ \left( X_{11}^{(2)} \right)^2 + X_{12}^{(2)} X_{11}^{(2)} - 2 \left( X_{12}^{(2)} \right)^2 \right] \]
where we have used the fact that \( X_{11}^{(2)} = X_{22}^{(2)} \) and \( X_{12}^{(2)} = X_{13}^{(2)} = X_{14}^{(2)} = X_{12}^{(2)} \). Next, the product
\[ \text{Det} \left[X_{11}^{(2)} + X_{12}^{(2)} \right] \text{Det} \left[X_{33}^{(1)} \right] \text{Det} \left[X_{44}^{(0)} \right] \]

where we have used the fact that \( X_{04}^{(0)} = X_{44}^{(0)} = \frac{4}{45} \) and \( X_{14}^{(0)} = X_{14}^{(0)} = \frac{1}{45} \). Noticing that \( X_{14}^{(0)} = \frac{\rho}{45} \) and
\[ \text{Det} \left[X_{11}^{(2)} - X_{12}^{(2)} \right] = \text{Det} \left[X_{11}^{(1)} - X_{12}^{(1)} \right] = \left( \frac{4}{45} - \frac{\rho}{45^2} \right) \), we complete the proof.

Therefore, the eigenvalues of the matrix \( \rho_{3,1} \) are the union of eigenvalues of matrices, \( \frac{1}{4} \left[ \frac{1}{4} - \frac{\rho}{45^2} \right] \) with degeneracy 2 and
\( \frac{1}{4} \left[ \frac{1}{4} \frac{1}{4} \frac{1}{4} \right] \). Note that due to symmetry the above applies also to the case of \( \rho_{1,3} \).
where in the first line we have used the fact that $X_{14}^{(1)} = X_{11}^{(1)} = X_{11}^{(1)}$. We know that $X_{44}^{(0)} = \frac{e_1}{d}, X_{45}^{(0)} = X_{54}^{(0)} = \frac{\rho}{d^2}$ and $X_{14}^{(0)} = X_{15}^{(0)} = \frac{1}{5d^2}$. After few tedious algebraic steps, we find that $X_{11}^{(1)} = \frac{c_1}{d^2} - \frac{1}{5d^2}, X_{12}^{(1)} = \frac{\rho}{d^2} - \frac{1}{5d^2}$, and $X_{14}^{(1)} = \frac{1}{5d^2} - \frac{\rho}{d^2}$. Putting all these in the above equation, we find the following simplification

$$
\text{Det} \left[ X_{11}^{(2)} + 2X_{12}^{(2)} \right] \text{Det} \left[ X_{14}^{(0)} \right] \text{Det} \left[ X_{55}^{(0)} \right] = \text{Det} \left( \frac{1}{5} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] - \lambda I \right) \text{Det} \left( \frac{1}{25} \left[ \frac{c_2}{d^2} + 3\frac{c_1}{d^2} + 2\frac{\rho^2}{d^2} - \frac{6\rho}{d^2} \right] \right),
$$

Noticing that $\text{Det} \left( X_{11}^{(2)} - X_{12}^{(2)} \right) = \text{Det} \left( X_{11}^{(3)} - X_{12}^{(3)} \right) = \text{Det} \left( \frac{c_1}{d^2} - \frac{\rho}{d^2} \right)$, we complete our proof.  

Therefore, the eigenvalues of the matrix $\rho_{3,2}$ are the union of eigenvalues of matrices, \( \frac{1}{5} \left[ \frac{I}{d} - \frac{\rho}{d^2} \right] \) with degeneracy 3 and \( \frac{1}{5} \left[ \frac{1}{d} + \frac{3\rho}{d^2} \pm \frac{1}{2} \sqrt{\frac{24I}{d^2} + \frac{9(9-6d-8\rho)}{d^4}} \right] \). Due to symmetry the above results apply also for the case of $\rho_{2,3}$. 

The reduced density matrix for the control system after the evolution can be calculated using the prescription given in Ref.[22] and is given by

$$
\tilde{\rho}_c(M) = \frac{1}{M} \left\{ \mathbb{I}_c + \frac{1}{d^2} \sum_{i \neq \ell} |i\rangle \langle j| \right\},
$$

(B10)

whose eigenvalues are \( \frac{1}{M} \left( 1 - \frac{1}{d^2} \right) \) with degeneracy \( M - 1 \) and \( \frac{1}{M} \left( 1 + \frac{M-1}{d^2} \right) \) with degeneracy one. 

Note that we can retrieve the special cases of the cyclic orders by replacing either $M_1 = 0$ or $M_2 = 0$.

2. Ne=4 case

To show that \( \frac{\rho}{d^2} \) terms occurs in off-diagonal, we will pick two instances:

$$
\frac{1}{d^8} \sum_{i j k \ell} U_j U_k U_i U_\ell \left( \rho U_j U_\ell \right) U_i U_\ell = \frac{1}{d^8} \sum_{j k} U_j U_k \left( \rho U_j \right) U_k = \frac{1}{d^8} \sum_{j} U_j \text{Tr} \left[ \rho U_j U_j \right] = \frac{\rho}{d^8},
$$

where we have used the fact that \( \frac{1}{d} \sum_k \left( \text{Tr}[\rho U_k U_k] \right) U_k = \rho \). And other off-diagonal term,

$$
\frac{1}{d^8} \sum_{i j k \ell} U_j U_k U_i U_\ell \left( \rho U_j U_\ell U_i U_k \right) = \frac{1}{d^8} \sum_{j k} U_j U_\ell U_k \left( \text{Tr}[\rho U_j U_\ell U_k] \right) = \frac{1}{d^8} \left( \text{Tr}[\rho U_j U_j U_k] \right) = \frac{\rho}{d^8},
$$

where we have used the fact that \( \frac{1}{d} \sum_k \left( \text{Tr}[\rho U_k U_k U_k] \right) U_k = \rho U_j U_j \).

Using the ordering within cosets described in the main text, we find that the output state after evolution is given by

$$
\rho_{M_n} = \frac{1}{M} \left\{ \mathbb{I}_c \otimes \mathbb{I}_d + L_{M_n} \otimes \frac{\rho}{d^2} + B_{M_n} \otimes \frac{\rho}{d^2} + Q_{M_n} \otimes \frac{\rho}{d^2} \right\},
$$

(B11)

where the matrices, $L$, $B$ and $Q$ are given by

$$
L = \begin{pmatrix}
S_{M_1 \times M_1} & 0_{M_1 \times M_2} & 0_{M_1 \times M_3} & \cdots & 0_{M_1 \times M_6} \\
0_{M_2 \times M_1} & S_{M_2 \times M_2} & 0_{M_2 \times M_3} & \cdots & 0_{M_2 \times M_6} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{M_6 \times M_1} & 0_{M_6 \times M_2} & \cdots & S_{M_6 \times M_6} & 0_{M_6 \times M_6} \\
0_{M_6 \times M_1} & 0_{M_6 \times M_2} & \cdots & 0_{M_6 \times M_6} & S_{M_6 \times M_6} \\
\end{pmatrix}.
$$
and for sake of simplicity we are writing the form of $B$ and $Q$ when $M = 4!$, i.e.,
\[
B = \begin{pmatrix}
0 & B_1 & B_2 & B_3 & B_4 & B_7 \\
B_1 & 0 & B_3 & B_6 & B_2 & B_4 \\
B_2 & B_3 & 0 & B_4 & B_5 & B_1 \\
B_3 & B_6 & B_3 & 0 & B_1 & B_2 \\
B_4 & B_2 & B_5 & B_1 & 0 & B_3 \\
B_7 & B_4 & B_1 & B_2 & B_3 & 0 \\
\end{pmatrix},
\]
\[
Q = \begin{pmatrix}
0 & Q_1 & Q_2 & Q_3 & Q_4 & Q_7 \\
Q_1 & 0 & Q_3 & Q_6 & Q_2 & Q_4 \\
Q_2 & Q_3 & 0 & Q_4 & Q_5 & Q_1 \\
Q_3 & Q_6 & Q_4 & 0 & Q_1 & Q_2 \\
Q_4 & Q_2 & Q_3 & Q_1 & 0 & Q_3 \\
Q_7 & Q_4 & Q_1 & Q_2 & Q_3 & 0 \\
\end{pmatrix},
\]
where $B_i$ and $Q_i$ are defined below,
\[
B_1 = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
\end{pmatrix},
B_5 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
\end{pmatrix};
Q_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix},
Q_5 = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{pmatrix},
\]
and $B_p = \pi_p[B_1]$ for $p = 2, 3, 4$ and $B_p = \pi_p[B_5]$ for $p = 6, 7$; and similarly, $Q_p = \pi_p[Q_1]$ for $p = 2, 3, 4$ and $Q_p = \pi_p[Q_5]$ for $p = 6, 7$, where $\pi_p$ are permutation operators which permutes among the columns as well as rows of the target matrix. For our scenario, $\pi_2 = \pi_{1\leftrightarrow 2} \cdot \pi_{1\leftrightarrow 3}$, $\pi_3 = \pi_{1\leftrightarrow 4} \cdot \pi_{2\leftrightarrow 3}$, and $\pi_4 = \pi_5 = \pi_{1\leftrightarrow 4} \cdot \pi_{2\leftrightarrow 3}$, such that we get the following output state
\[
\rho_{\pi_2}(\rho) = \frac{1}{M} \left( l_c + \frac{1}{d^2} (L_{M_q} + B_{M_q}) + \frac{1}{d^4} Q_{M_q} \right),
\]
where $M = 8$; We are considering a scenario where most of the off-diagonal elements are proportional to $\rho$. Such a situation occurs when $M_3 = M_5 = 4$ and other $M_q = 0$. Therefore, the output density matrix is given by
\[
\rho_{4,4,6} = \frac{1}{8} \left( l_c \otimes \frac{\mathbb{I}}{d} + L_{4,4} \otimes \frac{\rho}{d} + B_{4,4} \otimes \frac{1}{d^2} + Q_{4,4} \otimes \frac{\rho}{d^4} \right),
\]
where the coefficient matrices $(L, B, Q)$ are
\[
L_{4,4} = \begin{pmatrix}
S_{4\times 4} & 0_{4\times 4} \\
0_{4\times 4} & S_{4\times 4} \\
\end{pmatrix},
B_{4,4} = \begin{pmatrix}
0_{4\times 4} & B_5 \\
B_5 & 0_{4\times 4} \\
\end{pmatrix},
\]
\[
Q_{4,4} = \begin{pmatrix}
0_{4\times 4} & Q_5 \\
Q_5 & 0_{4\times 4} \\
\end{pmatrix},
\]
As $[L_{4,4}, Q_{4,4}] = 0$, the matrices $L, B$ and $Q$ are simultaneously diagonalizable, i.e., there exists a unitary matrix $U_{4,4}$ such that $U_{4,4} L_{4,4} U_{4,4}^\dagger = \text{Diag}(3, 3, -1, \cdots, -1)$ and $U_{4,4} B_{4,4} U_{4,4}^\dagger = \text{Diag}(-2, 2, 0, \cdots, 0)$.

**A2.** $M = 6$: We can choose many scenarios here with different $M_q$. The most trivial case can occur when $M_q = 1 \forall q$. However, there exists two interesting scenarios: 1) $(M_1 = 3, M_4 = 3)$ and 2) $(M_3 = 4, M_5 = 2)$.

**Example 1:** We consider those causal order for $(M_1 = 3, M_4 = 3)$ such that the output state is exactly that of the $(N = 3, M = 6)$ given in Eq.(B4). And we know that the output state is exactly diagonalizable using simultaneously diagonalisation method.

**Example 2:** Here we choose those causal orders for $(M_3 = 4, M_5 = 2)$ such that we get the following output state
\[
\rho_{4,2,6} = \frac{1}{6} \left( l_c \otimes \frac{\mathbb{I}}{d} + L_{4,2} \otimes \frac{\rho}{d} + B_{4,2} \otimes \frac{1}{d^2} + Q_{4,2} \otimes \frac{\rho}{d^4} \right),
\]
where the coefficient matrices $(L, B, Q)$ are
\[
L_{4,4} = \begin{pmatrix}
S_{4\times 4} & 0_{4\times 2} \\
0_{2\times 4} & S_{2\times 2} \\
\end{pmatrix},
B_{4,4} = \begin{pmatrix}
0_{4\times 4} & \hat{B}_5^T \\
\hat{B}_5 & 0_{2\times 2} \\
\end{pmatrix},
\]
\[
Q_{4,4} = \begin{pmatrix}
0_{4\times 4} & \hat{Q}_5^T \\
\hat{Q}_5 & 0_{2\times 2} \\
\end{pmatrix},
\]
with $\tilde{B}_5 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ and $\tilde{Q}_5 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$. The output state in this case can be diagonalized using the method given in Appendix C.

Example 3: The trivial scenario can occur when $M_\eta = 1$ for all $\eta$ where all the off-diagonal terms are proportional to $I$. However, we consider another scenario where not all off-diagonal elements are proportional to $I$ and the output state is given by

$$\rho_1 = \frac{1}{6} \begin{pmatrix} \frac{1}{\alpha} & \rho & \rho & \rho & \rho & \rho \\ \rho & \frac{1}{\alpha} & \rho & \rho & \rho & \rho \\ \rho & \rho & \frac{1}{\alpha} & \rho & \rho & \rho \\ \rho & \rho & \rho & \frac{1}{\alpha} & \rho & \rho \\ \rho & \rho & \rho & \rho & \frac{1}{\alpha} & \rho \\ \rho & \rho & \rho & \rho & \rho & \frac{1}{\alpha} \end{pmatrix}.$$  \hspace{1cm} (B14)

This matrix can be diagonalized using the method presented in Appendix C.

Appendix C: Determinant of block matrices

In order to find eigenvalues of the $dM \times dM$ block matrix, $\rho_{\text{out}}$, in the main text we need to find how its determinant factorizes into determinant of small matrices as discussed in the Ref.[40, 41]. We will state the Lemma from the Ref.[40, 41] below.

**Lemma 2.** Let $A$ be an $pN \times pN$ complex matrix partitioned into $N^2$-Blocks, each of size $p \times p$, i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}_{pN \times pN},$$

then its determinant is given by

$$\det[A] = \prod_{k=1}^{N} \det[X^{(N-k)}_{kk}],$$  \hspace{1cm} (C1)

where $X^{(i)}$ are defined as

$$X^{(0)}_{ij} = A_{ij},$$

$$X^{(k)}_{ij} = A_{ij} - \tilde{b}_{i,N-k+1} \tilde{A}_k^{-1} \tilde{a}_{N-k+1,j}, \hspace{1cm} k \geq 1,$$

with $\tilde{a}_{ij} = (A_{ij}, A_{i+1,j}, \cdots, A_{Nj})^T$, $\tilde{b}_{i} = (A_{ij}, A_{i,j+1}, \cdots, A_{iN})$, and $\tilde{A}_k$ being the $k \times k$ block matrix formed from the lower right corner of $A$. The author in Ref.[41] also notices that

$$X^{(k+1)}_{ij} = X^{(k)}_{ij} - X^{(k)}_{i,N-k} \left(X^{(k)}_{N-k,N-k} \right)^{-1} X^{(k)}_{N-k,j}.$$  \hspace{1cm} (C2)

Equipped with the above Lemma, we will try to find the eigenvalues of matrix $\rho_M$. 

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