Static and semistatic hedging as contrarian or conformist bets

Svetlana Boyarchenko1 | Sergei Levendorskii2

1Department of Economics, The University of Texas at Austin, Austin, Texas
2Calico Science Consulting, Austin, Texas

Correspondence
Sergei Levendorskii, Calico Science Consulting, 2708 Bolton Street, Austin, TX 78748.
Email: levendorskii@gmail.com

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Abstract
In this paper, we argue that, once the costs of maintaining the hedging portfolio are properly taken into account, semistatic portfolios should more properly be thought of as separate classes of derivatives, with nontrivial, model-dependent payoff structures. We derive new integral representations for payoffs of exotic European options in terms of payoffs of vanillas, different from the Carr–Madan representation, and suggest approximations of the idealized static hedging/replicating portfolio using vanillas available in the market. We study the dependence of the hedging error on a model used for pricing and show that the variance of the hedging errors of static hedging portfolios can be sizably larger than the errors of variance-minimizing portfolios. We explain why the exact semistatic hedging of barrier options is impossible for processes with jumps, and derive general formulas for variance-minimizing semistatic portfolios. We show that hedging using vanillas only leads to larger errors than hedging using vanillas and first touch digital. In all cases, efficient calculations of the weights of the hedging portfolios are in the dual space using new efficient numerical methods for calculation of the Wiener–Hopf factors and Laplace–Fourier inversion.

KEYWORDS
barrier options, exotic European options, Fourier–Laplace inversion, Lévy processes, semistatic hedging, sinh-acceleration, static hedging, Wiener–Hopf factorization
INTRODUCTION

There is a large literature studying static hedging and replication of exotic European options, and semistatic hedging and replication of barrier and other types of options. What this literature ignores, however, is the cost of maintaining the hedging position, which can drive the payoff of the overall portfolio negative. In this paper, we argue that, once the costs of maintaining the hedging portfolio are properly taken into account, semistatic portfolios should more properly be thought of as separate classes of derivatives, with nontrivial, model-dependent payoff structures. Depending on the structure of the option being hedged and the model, the semistatic hedging portfolio may either function as a contrarian bet—small losses with high probability and large gains with low probability—or as a conformist bet—small gains with high probability and large losses with low probability.

We suggest new versions of static and semistatic hedging, provide qualitative analysis of errors of different static and semistatic procedures, explain why in the jump-diffusion case the exact replication of barrier options by European options, hence, the model-independent replication, is impossible, and produce numerical examples to demonstrate how different sources of hedging errors depend on a model. We consider European and down-and-in barrier options in Lévy models; the approach of the paper can be generalized and extended to cover options of other types, in more complicated models. Pricing barrier options and the calculation of the variance of the hedging portfolio at expiry are based on new efficient numerical procedures for calculation of the Wiener–Hopf factors and Laplace–Fourier inversion. These procedures can be useful in other applications as well.

The underlying idea of the static hedge (Carr & Madan, 2001) of European options with exotic payoffs is simple. One replicates the payoff of an exotic European option by a linear combination of payoffs of the underlying stock and vanillas, and uses the portfolio of the stock and options to replicate or hedge the exotic option. In Section 2, we start with the derivation of integral representations for an exact static hedging portfolio. Contrary to Carr and Madan (2001), we work in the dual space, and derive a representation in terms of vanillas only; this representation if different from the one in Carr and Madan (2001). By construction, the portfolios we construct and the portfolio in Carr and Madan (2001) are model-independent, which looks very attractive. However, the continuum of vanillas does not exist, and even if it did, the integral portfolio would have been impossible to construct anyway. Hence, one has to approximate each integral by a finite sum. The hedging error of the approximation is inevitably model dependent. We design simple constructions of approximate hedging portfolios and study the dependence of the static hedging error on a model using a portfolio of available vanillas. We derive an approximation in an almost $C(\mathbb{R})$-norm, and then calculate the weights of the variance-minimizing hedging portfolio.

In Section 3, we outline the general structure of the semistatic variance-minimizing hedging of barrier options; in the paper, we consider the down-and-out and down-and-in options. The initial version of the semistatic hedging portfolio for barrier options was suggested in Derman, Ergener, and Kani (1995): put options with strikes equal to the barrier, with different expiry dates, are added to the portfolio in such a way that the portfolio value is zero at the barrier. Assuming that, at the moment the barrier is breached, the underlying is exactly at the barrier, the weights of portfolio can be calculated backward. It is clear that if the underlying can cross the barrier with a jump, the procedure cannot be exact, and the implicit error is inevitably model dependent. A different semistatic hedging of barrier options is developed in Carr and Chou (1997), Carr, Ellis, and Gupta (1998), and Carr and Lee (2009), but the underlying assumption is the same as in Derman et al. (1995).

For a given barrier option, an exotic European payoff $G_{ex}$ is constructed so that, at maturity or at the time of early expiry (the case of “out” options) or activation (the case of “in” options), the price of the
hedging portfolio for barrier option equals the price of the European option. As the barrier is reached (the presumption is that it cannot be crossed by a jump), the portfolio is liquidated. The European option being exotic, an approximate static hedging portfolio for the latter is presumed to be used. Hence, in the presence of jumps, the hedging errors are model dependent even if one believes that an auxiliary exotic option can be hedged exactly using a portfolio of vanillas, and the question of the interaction of two types of errors naturally arises. The option with the payoff $G_{c\xi}$ is more exotic than the usual exotic options (the structure of the payoff is more complicated), and the more exotic the option is, the larger the hedging errors are. Even in diffusion models, the errors are sizable, and the approximation is justified under a certain rather restrictive symmetry condition on the parameters of the model (see Nalholm & Paulsen, 2006).

The paper (Kirkby & Deng, 2019) uses the approximate semistatic hedging of Carr and Lee (2009) and an approximation of the exotic European option that approximates the barrier option; this leads to at least two sources of model-dependent errors, which can be large if the jump component is sizable; in addition, the symmetry condition is more restrictive than in the case of diffusion models. In the introduction of Kirkby and Deng (2019), it is claimed that Carr and Lee (2009) rigorously justified the semistatic procedure for jump-diffusion models; the picture is more complicated. In Section 3.1, we explain that the standard semistatic construction has numerous sources of errors, and even an approximation can be justified under additional rather restrictive conditions only. In particular, in the presence of jumps, the semistatic procedure is never exact.

The variance-minimizing hedging portfolio has certain advantages. We can directly construct the hedging portfolio using the securities traded in the market provided that a pricing model is chosen, and one can calculate the option prices $V_j$ in the portfolio and products of the prices as functions of $(t, x)$, $0 \leq t \leq T$, where $T$ is time to maturity. Accurate and fast calculations are possible for wide classes of options (barrier options, lookbacks, American options, Asians, etc.), and many popular pricing methods working in the state space can be applied. However, to calculate the weights of the hedging portfolio, we need to calculate the expectations of the products of the discounted prices at time $\tau \wedge T$, where $\tau$ is the first entrance time into the early exercise region. Hence, one needs to approximate the products of prices by functions that are amenable to application of efficient option pricing techniques, which are, typically, based on the Laplace–Fourier transform. In the paper, we suggest and use new efficient methods for the numerical Fourier–Laplace inversion and calculation of the Wiener–Hopf factors; these methods are of a general interest.

We work in the dual space; calculations in the dual space are also necessary to accurately address the following practically important effect. There is an additional source of errors of hedging portfolios consisting of vanillas only. In all popular models used in finance, the prices of vanilla options are infinitely smooth before the maturity date and up to the boundary but prices of barrier options in Lévy models are not smooth at the boundary, the exceptions being double jump diffusion model, hyper-exponential jump diffusion model, and other models with rational characteristic functions. For wide classes of purely jump models, it is proved in Boyarchenko and Levendorskiı (2002b) and Boyarchenko, de Innocentis, and Levendorskiı (2011) that the price of an “out” barrier option near the barrier behaves as $c(T)|x - h|^{\kappa}$, where $\kappa \in [0, 1)$ is independent of time to maturity $T$, $c(T) > 0$, and $|x - h|$ is the log-distance from the barrier. For finite variation processes with the drift pointing from the boundary, $\kappa = 0$, and the limit of the price at the barrier is positive. Similarly, the price of the first touch digital behaves as $1 - c_1(T)|x - h|^{\gamma}$. Even if the diffusion component is present, the prices of the barrier and first touch options are not differentiable at the barrier (Boyarchenko & Levendorskiı, 2002b), and if the diffusion component is small, then essentially the same irregular behavior of the price will be observed outside a very small vicinity of the barrier. Calculations in the state space are based on approximations by fairly regular functions, hence, cannot reproduce these effects sufficiently accurately. See
examples in Levendorskii (2014), where it is demonstrated that Carr’s randomization method (Boyarchenko & Levendorskii, 2009), which relies on the time randomization and interpolation in the state space, underprices barrier options in a small vicinity of the barrier. From the point of view of the qualitative composition of the hedging portfolio, one should expect that an accurate hedging of barrier options is impossible unless the corresponding first touch digitals are included. Figure 1 clearly shows that the first touch digital is much closer to the down-and-in option than a put option, and the first-touch options with the payoffs \((S/H)^\gamma, \gamma > 0\), would be even better hedging instruments.

We calculate hedging portfolios consisting of vanillas only and of vanillas and the first-touch option in Sections 5 and 6 using the Wiener–Hopf factorization technique. We recall the latter in Section 4, and introduce the new efficient method for the calculation of the Wiener–Hopf factors based on the sinh-acceleration technique (Boyarchenko & Levendorskii, 2019). The numerical examples for static hedging and calculation of the Wiener–Hopf factors and expectations related to barrier options are discussed at the end of the corresponding Sections; a numerical example for hedging of barrier options is in Section 7. In Section 8, we summarize the results of the paper and outline natural extensions.

2 | STATIC HEDGING OF EUROPEAN OPTIONS

2.1 | Static hedging in the ideal world

Let \(X\) be the Lévy process, and let \(G(X_T) = G(\ln X_T)\) be the payoff at maturity. Our standing assumption is

**Assumption (G).** \(G\) is a piece-wise differentiable function satisfying the following conditions:

1. \(dG'\) is a signed measure, without the singular component.
2. \(G'\) has only a finite number of points of discontinuity.
3. The measure \(dG'_{at} = \sum_{Y>0}(G'(Y + 0) - G'(Y - 0))\delta_Y\) is finite.
   
   If \(\beta < -1\), define
   
   \[
   G_1(S) = G(S) - \sum_{Y>0}(G'(Y + 0) - G'(Y - 0))(S - Y)_+, 
   \]
   
   and if \(\beta > 0\), set
   
   \[
   G_1(S) = G(S) - \sum_{Y>0}(G'(Y + 0) - G'(Y - 0))(Y - S)_+. 
   \]

   Clearly, the measure \(dG_1\) is absolutely continuous, and \(dG' = dG'_{at} + d(G_1)'\). To Assumption (G), we add the following condition:

4. \(\exists \beta \in (-\infty, -1) \cup (0, +\infty)\) s.t.
   
   - functions \(S \mapsto S^\beta G_1(S), S \mapsto S^{\beta+1}(G_1)'(S)\) tend to 0 as \(S \to 0, +\infty\);
   - functions \(S \mapsto S^{\beta-1}G_1(S), S \mapsto S^\beta(G_1)'(S)\) are of the class \(L_1(\mathbb{R}_+)\).

**Theorem 2.1.** Let Assumption (G) hold. Then, if \(\beta < -1\),

\[
G(S) = \int_0^{+\infty} (S - K)_+ dG'(K),
\]
and if $\beta > 0$,

$$G(S) = \int_{0}^{+\infty} (K - S)_{+} d\mathcal{G}'(K).$$

**Proof.** In view of (1)–(2), it suffices to consider the case $d\mathcal{G}' = d\mathcal{G}'_1$. Set $G_1(x) = G_1(e^x)$, and let $\beta > 0$. Set $k = \ln K$, and calculate the Fourier transform of the right-hand side (RHS) of (4) with respect to $x := \ln S$, for $\xi$ on the line \{Im $\xi = \beta$\}. We use Fubini’s theorem, integrate by parts twice, and use the fact that $K^{1-i\xi}G'(K) \to 0$ and $K^{1-i\xi}G'(K) \to 0$ as $K \to 0, +\infty$ because Re$(-i\xi) = \beta$:

$$\int_{\mathbb{R}} e^{-ix\xi} \int_{\mathbb{R}_+} (K - e^x)_{+} d(G_1)'(K) dx = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-ix\xi}(K - e^x)_{+} dx d(G_1)'(K)$$

$$= \int_{\mathbb{R}_+} \frac{K^{1-i\xi}}{i\xi(i\xi - 1)} d(G_1)'(K) = \int_{\mathbb{R}_+} \frac{K^{-i\xi}}{i\xi}(G_1)'(K) dK$$

$$= \int_{\mathbb{R}_+} \frac{K^{-i\xi}}{i\xi} dG_1(K) = \int_{\mathbb{R}_+} K^{-1-i\xi}G_1(K) dK$$

$$= \hat{G}_1(\xi).$$
Thus, in the case $G = G_1$, the Fourier transforms of the left-hand side (LHS) and RHS of (4) with respect to $x = \ln S$ coincide on the line $\text{Im} \xi = \beta$, which proves (4). If $\beta < -1$, then, in the proof above, we replace $(K - S)_+$ with $(S - K)_+$, and modify all the steps accordingly.

**Remark 2.2.** If $dG_1$ has the compact support and no atoms, then both representations, in terms of puts and calls, are valid, with integration with respect to the same measure. This (mildly surprising) fact can be verified using the put-call parity and the following calculation:

$$
\int_0^{+\infty} (S - K)dG'(K) = G'(K)(S - K)|_{0}^{+\infty} + \int_0^{+\infty} G'(K)dK
$$

$$=(G'(K)(S - K) + G(K))|_{0}^{+\infty} = 0 - 0 = 0.
$$

**Example 2.3.** The payoff function of the powered call of order $\beta > 1$ with the strike $K_0$ is $G(S) = (S - K_0)_+^\beta$. As $\beta > 1$, there are no kinks, $dG'(K)$ has no atoms, and

$$((S - K_0)_+)^\beta = \beta(\beta - 1) \int_{K_0}^{+\infty} dK (K - K_0)^{\beta - 2}(S - K)_+.
$$

**Example 2.4.** Consider the down-and-in call option with barrier $H$ and strike $K_0 > H$. A popular semistatic replicating portfolio for this option is a European option with the payoff

$$
G_{ex}(S_T) = (G(S_T) + (S_T / H)^\beta G(H^2 / S_T))\mathbf{1}_{S_T \leq H}
$$

(see Appendix and Carr & Madan, 2001). As $K_0 > H$, the payoff simplifies $G_{ex}(S) = (S / H)^\beta (H^2 / S - K_0)_+\mathbf{1}_{(0,H]}(S) = H^{-\beta} K_0 G_{ex,1}(S)$, where $G_{ex,1}(S) = S^{\beta - 1}(K_1 - S)_+$, and $K_1 = H^2 / K_0 < H$. Clearly, it suffices to construct the hedging portfolio for the option with the payoff function $G_{ex,1}$, which vanishes above $K_1$. At $K_1$, $G_{ex,1}$ has a kink, and $G'_{ex,1}(K_1 - 0) = -K_1^{\beta - 1}$. On $(0, K_1)$, $G_{ex,1}$ is infinitely smooth, and $G'_{ex,1}(S) = K_1(\beta - 1)S^{\beta - 2} - \beta S^{\beta - 1}, G''_{ex,1}(S) = K_1(\beta - 1)(\beta - 2)S^{\beta - 3} - \beta(\beta - 1)S^{\beta - 2} = (\beta - 1)S^{\beta - 2}(\beta - 2)K_1 / S - \beta)$. Hence,

$$
G_{ex}(S) = H^{-\beta} K_0 \left(K_1^{\beta - 1}(K_1 - S)_+ + (\beta - 1) \int_0^{K_1} dK K^{\beta - 2}(\beta - 2)K_1 / K - \beta)(K - S)_+ \right).
$$

**2.2 Approximate static hedging**

In the real world, only finite number of options are available, hence, one has to approximate the measure $dG'(K)$ using an atomic measure, typically, with a not very large number of atoms. For instance, it is documented in Carr and Wu (2014) that static hedging with three to five options produces good results. Hence, the static hedging will be approximate, and the quality of the approximation depends (naturally) on the model and choice of the approximation procedure. We will approximate the payoff of an exotic option by linear combinations of payoffs of vanillas, in the norm of a Sobolev space with an exponential weight. To be more specific, we minimize the difference between the payoff of an exotic option and the portfolio payoff $G(x) := G(e^x)$ in the norm of the Sobolev space $H^{s,0}(\mathbb{R})$ of order $s$, with an appropriate exponential weight $e^{ox}$ (also known as damping factor).

By one of the Sobolev embedding theorems (see, e.g., Theorem 4.3 in Eskin, 1981), if $s > 1/2$, $H^{s}(\mathbb{R})$ is continuously embedded into $C_0(\mathbb{R})$, the space of uniformly bounded continuous functions vanishing at infinity, with $L_\infty$-norm. Hence, we can estimate the error in the $C$-norm (with the corresponding weight), which is natural for the approximate static hedge: if the error 0 is not achievable, we control the maximal error. Let $\omega, s \in \mathbb{R}$. The Sobolev space $H^{s,0}(\mathbb{R})$ of order $s$, with weight $e^{ox}$, is
the space of the generalized functions $u$ such that $u_\omega := e^{i\omega} u \in H^s(\mathbb{R})$. The scalar product in $H^{s,\omega}(\mathbb{R})$ is defined by $(u, v)_{s,\omega} = (u_\omega, v_\omega)_{H^s(\mathbb{R})}$. Thus,

$$(u, v)_{H^{s,\omega}(\mathbb{R})} = \int_{\text{Im}\xi = \omega} (1 + (\xi - i\omega)^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi.$$ 

For any $\omega \in (-1, 1)$ and $s > 1/2$, an approximation in the $H^{s,\omega}$-topology gives a uniform approximation over any fixed compact.

Consider an exotic option whose payoff vanishes below $K$; we normalize $K = 1$. For practical purposes, we may assume that the strikes of European options used for hedging are close to 1, and the spot is close to 1; hence, the log-spot $x$ is close to 0, and if $\omega$ is not large in modulus, the differences among the hedging weights for different omegas are not large. Likewise, if $\hat{u}(\xi)$ decay fairly fast at infinity, the norms of $u$ in $H^{s,\omega}(\mathbb{R})$ will be close if $s \in [1/2, s_0]$ and $s_0$ is close to 1/2. Assume that $\beta < -1$. Thus, we have a call-like option, which is hedged using a portfolio of call options. We fix $\omega \leq \beta$ as discussed above, $s \geq 1/2$, and the set of call options with the payoff functions $G_j := G(K_j, \cdot)$. Set $G_0 = G$. We look for the set of weights $n = (n_1, \ldots, n_N)$ (numbers of call options in the portfolio) that minimizes $\text{StHG}(n) := -G_0 + \sum_{j=1}^N n_j G_j$ in the $H^{s,\omega}(\mathbb{R})$ norm. Denote $G^{s,\omega}_{jk} = (G_j, G_k)_{s,\omega}$; these scalar products can be easily calculated with a sufficiently high precision because the integrands in the formula for $G^{s,\omega}_{jk}$ decay as $|\xi|^{-s}$.

Furthermore, if $\hat{G}_0(\xi)$ is of the form $e^{-ik_0\xi}\hat{G}_0(\xi)$, where $\hat{G}_0(\xi)$ is a rational function, and $k_0 \in \mathbb{R}$, the integrands in the formulas for $(G_j, G_k)_{s,\omega}$ are of the form $e^{-ik_1\xi}\hat{G}_{jk,0}(\xi)(1 + (\xi - i\omega)^2)^s$, where $k_{jk} \in \mathbb{R}$, and $\hat{G}_{jk,0}(\xi)$ are rational functions. Hence, the scalar products can be calculated with almost machine precision and very fast using the sinh-acceleration technique (Boyarchenko & Levendorskiï, 2019). After an appropriate change of variables of the form $\xi = i\omega_1 + b \sinh(\omega_1 + y)$, the simplified trapezoid rule with a dozen of terms typically sufficient to satisfy the error tolerance of the order of $10^{-10}$ and smaller. The minimizer $n^{s,\omega}$ of $F^{s,\omega}(n) := \|\text{StHG}(n)\|^2$ is given by

$$n^{s,\omega} = \frac{1}{2} A(G; s, \omega)^{-1} G^{s,\omega}_0,$$

where $G^{s,\omega} = [G^{s,\omega}_{01}, \ldots, G^{s,\omega}_{0N}]^t$ is a vector-column, and $A(G; s, \omega) = [G^{s,\omega}_{jk}]_{j,k=1,\ldots,N}.

2.3 Variance-minimizing hedging portfolio

The hedging error is the random variable

$$\text{Err}(n; x, X_T) = e^{-rT} \left(-G_0(X_T \mid X_0 = x) + \sum_{j=1}^N n_j G_j(X_T \mid X_0 = x)\right).$$

Set $V_j(T, x) = e^{-rT} \mathbb{E}^x[G_j(X_T)]$ (the expectations are under an EMM $\mathbb{Q}$ chosen for pricing). Assuming that $\hat{G}_j, j = 0, 1, \ldots, N$, are calculated, calculation of the mean hedging error

$$\mathbb{E}\left[\text{Err}(n; x, X_T)\right] = -V_0(T, x) + \sum_{j=1}^N n_j V_j(T, x)$$

is reducible to the Fourier inversion. As it is explained in Boyarchenko and Levendorskiï (2014), Levendorskiï (2016), and Boyarchenko and Levendorskiï (2019), if (a) $\hat{G}_j$ is of the form $\hat{G}_j(\xi) = e^{-ik_j\xi}\hat{G}_{j0}(\xi)$, where $k_j \in \mathbb{R}$ and $\hat{G}_{j0}$ is a rational functions, and (b) $\psi$ is of the form $\psi(\xi) = -i\mu\xi +$
\(\psi^0(\xi),\) where \(\mu \in \mathbb{R}\) and \(\text{Re} \ \psi^0(\xi) \rightarrow +\infty\) as \(\xi \rightarrow \infty\) remaining in a cone around the real axis, then it is advantageous to represent \(V_j(T, x)\) in the form

\[
V_j(T; x) = \frac{1}{2\pi} \int_{\text{Im} \xi = \omega} e^{ix'\xi - T(\tau + u^0(\xi))} \hat{G}_{j\xi}(\xi) d\xi,
\]  
(7)

where the set of admissible \(\omega \in \mathbb{R}\) depends on \(\hat{G}_{j\xi}\), and \(x' = x + \mu T - k_j\). Then we use a conformal deformation of the contour of integration in (7) and the corresponding change of variables, and apply the simplified trapezoid rule. The most efficient change of variables (the sinh-acceleration) suggested in Boyarchenko and Levendorski (2019) is of the form \(\xi = i\omega + b \sinh(i\omega' + y)\), where \(\omega'\) is of the same sign as \(x'\); the upper bound on admissible \(|\omega'|\) depends on \(\psi^0\) and \(\hat{G}_{j\xi}\). The variance can be calculated using the equality \(e^{-2\tau T}E[(G - \mathbb{E}[G])^2] = e^{-2\tau T}(\mathbb{E}[G^2] - \mathbb{E}[G]^2)\). To calculate \(\mathbb{E}^s[Err(n; x, X_T)]\), we need to calculate \(V_{\xi\ell}(T, x) := e^{-2\tau T} \mathbb{E}^s[G_j(S_T)\hat{G}_{\xi\ell}(S_T)]\), \(j, \ell = 0, 1, \ldots, N\), which is the price of the European option with the payoff \(G_j(S_T)G_k(S_T)\) at maturity \(T\). We calculate the Fourier transform \(\hat{G}_{j\xi}(\xi)\) of the product \(G_j\hat{G}_{\xi\ell}\), multiply by the characteristic function \(e^{-T\psi(\xi)}\), and apply the inverse Fourier transform. For typical exotic options and vanillas, \(\hat{G}_{j\xi}(\xi)\) is of the form \(e^{-ik_j\xi} \hat{G}_{j\xi\ell}(\xi)\), where \(\hat{G}_{j\xi\ell}(\xi)\) is a rational function, and \(k_{j\xi} \in \mathbb{R}\). Hence,

\[
V_{\xi\ell}(T; X) = \frac{1}{2\pi} \int_{\text{Im} \xi = \omega} e^{ix'\xi - T(\tau + u^0(\xi))} \hat{G}_{j\xi\ell}(\xi) d\xi,
\]
(8)

where \(x' = x + \mu T - k_{j\xi}\). The numbers of options in the variance-minimizing portfolio are

\[
n(T; x) = \frac{1}{2} A(T; x)^{-1} V^0(T; x),
\]
(9)

where \(V^0(T; x) = ([V_{0j}(T; x) - V_j(T, x)]_{j=1}^N)\) is a vector-column, and \(A(T; x) = [V_{\xi\ell}(T; x) - V_{\ell\xi}(T; x)V_{\xi\ell}(T; x)]_{j,\ell=1,\ldots,N}\) is a matrix.

### 2.4 An example: Static hedging and variance-minimizing hedging of the exotic option with the payoff function \(G_{ex}\) given by (5)

The case \(G(S) = (S - K)_+\). Let \(k := \ln K > h := \ln H\). Then \(G(x)1_{(-\infty, h]}(x) = 0\), and \(2h - k = h - (k - h) < h\). Direct calculations show that the Fourier transform of \(\hat{G}_0(\xi) = G_{ex}(e^{\xi})\) is given by

\[
\hat{G}_0(\xi) = H^{-\beta} K^{1-\beta} \frac{e^{-i\xi(2h-k)}}{(-i\xi + \beta - 1)(-i\xi + \beta)}. \quad \text{Im} \xi > 1 - \beta.
\]
(10)

In the hedging portfolio, we use put options with strikes \(K_j \leq K_0 = H^2/K, \ j = 1, 2, \ldots, N\). Set \(G_j(x) = (K_j - e^{\xi})_+; \) then \(\hat{G}_j(\xi) = K_j^{1-\beta}/(i\xi(i\xi^2 - 1))\) is analytic throughout \{\text{Im} \xi > 0\}.

### 2.4.1 Construction of an approximate static hedging portfolio

We take \(\omega > (1 - \beta)_+\). Typically, \(\beta < 0, \) hence, \(\omega > 1\). We have

\[
(G_0, G_0)_{s,\omega} = H^{4\omega - 2\beta} K^{2(1-\beta-\omega)} \int_{\mathbb{R}} \frac{(1 + \xi^2)^\delta d\xi}{(\xi^2 + (\omega + \beta - 1)^2)(\xi^2 + (\omega + \beta)^2)},
\]

where \(k = \ln K\). We can calculate the integral accurately and fast making the simplest sinh-change of variables \(\xi = b \sinh y\), and applying the simplified trapezoid rule. See Boyarchenko and Levendorskiī
(2019) for explicit recommendations for the choice of $b$ and the parameters $\zeta, N$ of the simplified trapezoid rule.

Next, for $j = 1, 2, \ldots, N$, we set $k_j = \ln K_j$ and calculate

$$(G_0, G_j)_{s, \omega} = H^{2\omega - \beta} K^{1 - \beta - \omega} K_j^{1 + \omega} \int_{\mathbb{R}} e^{-i\xi(2h - k)} (1 + \xi^2)^{\omega} \frac{d\xi}{(i\xi + \omega + \beta - 1)(-i\xi + \omega + \beta)(i\xi + \omega + 1)(i\xi + \omega)}.$$  

If $k_j = 2h - k$, we make the sinh-change of variables of the same form as above (the choice of the parameters $b, \zeta, N$ is slightly different). If $2h - k - k_j < 0$ (respectively, $> 0$), then it is advantageous to deform the contour of integration so that the wings of the deformed contour point up (respectively, down). Hence, we make the change of variables $\xi = i\omega_0 + b \sinh(i\omega + y)$, where $\omega' \in (0, \pi/2)$ (respectively, $\omega \in (-\pi/2, 0)$) and set $\omega_0 = -b \sin \omega'$. The parameters $b, \zeta, N$ are chosen as explained in Boyarchenko and Levendorskii (2019). Finally, for $j, \ell = 1, 2, \ldots, N$, and $\omega > 0$,

$$(G_j, G_\ell)_{s, \omega} = (K_j K_\ell)^{1 + \omega} \int_{\mathbb{R}} e^{-i\xi(k_j - k_\ell)} (1 + \xi^2)^{\omega} \frac{d\xi}{(\xi^2 + \omega^2)(\xi^2 + (\omega + 1)^2)}.$$  

If $k_j > k_\ell$, we deform the wings of the contour down, equivalently, use the sinh-acceleration with $\omega' \in (-\pi/2, 0)$. If $k_j < k_\ell$, we use $\omega' \in (0, \pi/2)$. Finally, if $k_j = k_\ell$, we may use any $\omega' \in [-\pi/2, \pi/2]$; the choice $\omega' = 0$ is the best one. After the scalar products are calculated, we apply (6) to find the approximate static hedging portfolio. In our numerical experiments, we will use a modification of this scheme when the hedging portfolio has the fixed amount $H^{-\beta} K_0 K_1^{1 - \beta} = (H/K_0)^{\beta - 2}$ of put options with strike $K_1$, and the weights of the other put options in the hedging portfolio are calculated mimizing the hedging error.

**2.4.2 Construction of the variance-minimizing hedging portfolio**

Calculating the integral on the RHS of (7), we use (10) for $j = 0$; for $j = 1, 2, \ldots, N$, $\hat{G}_j(\xi) = K_j^{1 - \xi} / (i\xi(i\xi - 1))$. To calculate the integral on the RHS of (8), we need to calculate the Fourier transforms of the products of the payoff functions. The straightforward calculations give

1. $$(\hat{G}_0)^2(\xi) = e^{-i\xi(2h - k)} \hat{G}_0(\xi),$$ for $\xi$ in the half-plane $\{\text{Im}\xi > 2(1 - \beta), \}$. where

$$\hat{G}_0(\xi) = \frac{2H^2\beta K^{2(1 - \beta)}}{(-i\xi + 2\beta - 2)(-i\xi + 2\beta - 1)(-i\xi + 2\beta)}.$$  

2. For $j = 1, 2, \ldots, N$ and $\xi$ in the half-plane $\text{Im}\xi > (1 - \beta), \hat{G}_j(\xi) = e^{-ik_j\xi} \hat{G}_0(\xi)$, where

$$\hat{G}_0(\xi) = \frac{(K_j / H)^\beta (H^2 / K)}{-i\xi + \beta} \left[ \frac{H^2 / K}{-i\xi + \beta - 1} - \frac{K_j}{-i\xi + \beta + 1} \right].$$  

3. If $K_j \leq K_\ell$, then the Fourier transform of $G_{jk}(x) = (K_j - e^x)(K_\ell - e^x)$, where

$$\hat{G}_{jk}(\xi) = K_j e^{-ik_j\xi} (K_\ell / i\xi - 1) \left( \frac{K_\ell}{i\xi} - \frac{K_j}{i\xi} - 2 \right),$$  

is well defined in the half-plane $\{\text{Im}\xi > 0\}$. 
2.4.3 | Numerical experiments

In Tables 1 and 2 (see the Appendix), we study the relative performance of the static hedging and variance-minimizing hedging. We consider the exotic European option with the payoff $C_{ex}(S) = (S/H)^\beta (H^2/S - K_0)$; the hedging portfolios consist of put options with strikes $K_j = H^2/K_0 - (j - 1)0.02, j = 1, \ldots, #K$, where #K = 3, 5. For different variants of hedging, we list numbers $n_j$ of the options with strikes $K_j$ in the hedging portfolio. Static portfolios are constructed minimizing the hedging error in the $H^\infty$ norm; the results are essentially the same for $\omega = 2(1 - \beta)_+, 0.1, \omega = 2(1 - \beta)_+ + 0.2$, and weakly depend on $s = 0.5, 0.55$. We study the dependence of the (normalized by the price $V_{ex}(T, x)$ of the exotic option) standard deviations $nStd$ of the static hedging portfolio and variance-minimizing portfolio on the process, time to maturity $T$ and $x := \log(S/K_1) \in [-0.03, 0.03]$. The static portfolios are independent of time to maturity $T$ and the process but $nStd$ does depend on both as well as on the spot $S$. For the static hedging portfolios, for each process and time to maturity, we show the range of $nStd$ as the function of $x \in [-0.03, 0.03]$. In the case of the variance-minimizing hedging, $n_j$ depend on $x$ by construction, and we show $n_j$ and $nStd$ for each $x$ in a table. We considered two variants of the variance-minimizing portfolios: VM1 means that $n_1$ (the same as for static
TABLE 2 KoBoL close to BM, with a sizably asymmetric jump density, and positive “drift”: $m_2 = 0.1, \nu = 1.95, \lambda_- = -12, \lambda_+ = 8, \mu = 0.15, \sigma = 0, c = 0.00288$ (rounded), $\beta = 4, r = 0.100$ (rounded)

| $T = 0.01$ | $\# K = 3$ | $\# K = 5$ |
|---|---|---|
| Static | $n_1$ | $n_2$ | $n_3$ | $nStd$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_4$ | $nStd$ |
| $s = 0.5$ | 0.961 | -0.867 | -0.080 | 0.50–1.12 | 0.961 | -0.870 | 0.006 | 0.001 | -0.086 | 0.46–1.07 |
| $s = 0.55$ | 0.961 | -0.896 | -0.049 | 0.51–1.15 | 0.961 | -0.898 | 0.004 | 0.001 | -0.054 | 0.48–1.12 |

VM1

- $x' = -0.03$ 0.961 -0.163 -0.192 0.032 0.961 -0.207 -0.059 -0.081 -0.194 0.016
- $x' = -0.01$ 0.961 -0.186 -0.157 0.043 0.961 -0.217 -0.042 -0.082 -0.242 0.025
- $x' = 0.00$ 0.961 -0.193 -0.152 0.054 0.961 -0.223 -0.033 -0.082 -0.270 0.033
- $x' = 0.01$ 0.961 -0.197 -0.158 0.071 0.961 -0.228 -0.024 -0.078 -0.298 0.044
- $x' = 0.03$ 0.961 -0.195 -0.208 0.140 0.961 -0.236 -0.010 -0.047 -0.367 0.086

$T = 0.1$

| Static | 0.73–1.08 | 0.70–1.04 |
|---|---|---|
| VM1 | 0.72–0.88 | 0.70–0.85 |

$T = 0.5$

| Static | 0.72–0.88 | 0.70–0.85 |
|---|---|---|
| VM1 | 0.72–0.88 | 0.70–0.85 |

portfolios) is fixed, VM2 means that all $n_j$ may vary. As the variances of the portfolios of both types are close, in the tables, we show the variances of type-VM1 portfolios only.

In our numerical experiments, $H = 1, K_0 = 1.02$, the underlying $S_t = e^{X_t}$ pays no dividends, $X$ is KoBoL (see Section 3.3) or the Brownian motion (BM) with the embedded KoBoL component. The second instantaneous moment is $m_2 = 0.1$ or $0.15$, and $c$ is determined by $m_2, \lambda_+, \lambda_-$ and $\sigma$. The riskless rate is found from the EMM condition $\psi(-i) + r = 0$; $\beta$ from $\psi(\xi) = \psi(-\xi - i\beta)$. If $X$ is KoBoL, then $\beta$ exists only if $\mu = 0$, and then $\beta = -\lambda_+ - \lambda_-$. If the BM component is present, then we choose $\sigma$ and $\mu$ so that $\beta = -\lambda_+ - \lambda_- = -\mu/(2\sigma^2)$.

In our experiments, we observed the following general effects, which we illustrate with two tables only for the sake of brevity (additional tables are available on request):

1. If the number of vanillas in a static hedging portfolio is sufficiently large, the portfolio provides uniform (approximate) hedging over wide stretches of spots and strikes. Hence, if the jump density decays slowly, one expects that, far from the spot, the static hedging portfolio will outperform the variance-minimizing portfolio. Even in cases when the rate of decay of the jump density is only moderately small, and the process is not very far from the BM, the variance of the static portfolio differs from the variance of the variance-minimizing portfolios (constructed separately for each spot from a moderate range, and using the information about the characteristics of the process) by several percent only; if the jump density decays slower and/or process is farther from the BM, the relative difference is smaller. Hence, if the rate of decay of the jump density is not large and the
density is approximately symmetric, the static portfolio is competitive for hedging risks of small fluctuations. It is clear that the hedging performance of the static portfolio in the tails must be better still.

2. However, if the jump density decays moderately fast, then the variance of the static portfolio can be sizably larger than the one of the variance-minimizing portfolios, and if a moderate BM component is added, then the relative difference becomes large.

3. If \( X \) is a pure jump processes with a moderately asymmetric density of jumps, the variance of the static portfolio is much larger than the variance of variance-minimizing portfolios.

4. The quality of variance-minimizing portfolios VM1 and VM2 is essentially the same in almost all cases when five vanillas are used although the portfolio weights can be rather different. Hence, as a rule of thumb, one can recommend to use vanillas associated with the atomic part of the measure in the integral representation of the ideal static portfolio—provided these vanillas are available in the market.

The implication of observations (1)–(2) for semistatic hedging of barrier options is as follows. If the variance of the BM component makes a nonnegligible contribution to the instantaneous variance of the process, the ideal semistatic hedging using a continuum of options improves but the quality of an approximation of the integral of options by a finite sum decreases. Hence, one should expect that the variance-minimizing hedging of barrier options is significantly better than an approximate semistatic hedging in most cases.

3 | HEDGING DOWN-AND-IN AND DOWN-AND-OUT OPTIONS

3.1 | Semistatic hedging

Carr and Lee (2009) formulate several equivalent conditions on a positive martingale \( M \) under the reference measure \( \mathbb{P} \), call the class of these martingales PCS processes, and design semistatic replication strategies for various types of barrier options. As the proof of Theorem 5.10 in Carr and Lee (2009) strongly relies on the assumption that the underlying is exactly at the barrier at the random time \( \tau \) when the barrier \( H \) is breached, in Remark 5.11 in Carr and Lee (2009), the authors state that these strategies replicate the options in question for all PCS processes, including those with jumps, provided that the jumps cannot cross the barrier. However, one of the equivalent conditions that define PCS processes is the equality of the distributions of \( M_T / M_0 \) under \( \mathbb{P} \) and \( M_0 / M_T \) under \( \mathbb{M} \), where \( d\mathbb{M} / d\mathbb{P} = M_T / M_0 \). This condition implies that \( M \) has positive jumps if and only if \( M \) has negative jumps. Hence, if jumps in \( M \) does not cross the barrier, equivalently, there are no jumps in the direction of the barrier, then there are no jumps in the opposite direction, and, therefore, \( M \) has no jumps. Further relaxing PCS conditions, Carr and Lee (2009) generalize to various asymmetric dynamics, but the property that jumps in one direction are impossible means that the results for semistatic replication of barrier options under these asymmetric dynamics are valid only if there are no jumps. Carr and Lee (2009) give additional conditions that ensure the super-replication property of the semistatic portfolio for “in” options; but the corresponding portfolio for “out” options recommended in Carr and Lee (2009) under-replicates the option.

For an additional clarification of these issues, in Section A, we derive the generalized symmetry condition for the case of a Lévy process \( X \) in terms of the characteristic function \( \psi: \exists \beta \in \mathbb{R} \text{ s.t. } \psi(\xi) = \psi(-\xi - i\beta) \) for all \( \xi \) in the domain of \( \psi \), and show that this condition implies that either \( X \) is the BM
and the riskless rate equals the dividend rate or there are jumps in both directions, and the asymmetry of the jump component is uniquely defined by the volatility $\sigma$ and drift $\mu$. Furthermore, if $\sigma = 0$, then $\mu = 0$ as well. For the case of the down-and-in option with the payoff $G(X_T) = G(e^{X_T})$ and barrier $H$, we rederive the formula (5) for the payoff of the exotic European option, which, in the presence of jumps, replicates the barrier option only approximately. The numerical examples above show that the variance of the hedging error of the static portfolio for the exotic option with the payoff (5) is close to the variance of the variance-minimizing portfolio if the BM component is 0, the jump density does not decrease fast and is approximately symmetric; if the BM component is sizable and/or the density of jumps is either asymmetric or fast decaying, then the variance of the static portfolio is significantly larger than the variance of the variance-minimizing portfolio. Hence, the static portfolio is a good (even best) choice in cases when the idealized semistatic replicated exotic option is a bad approximation to the barrier option.

3.2 General scheme of variance-minimizing hedging

We consider one-factor models. The underlying is $e^{X_T}$, there are no dividends, and the riskless rate $r$ is constant. Let $V^0(t, X_t)$ be the price of the contingent claim to be hedged, of maturity $T$, under an EMM $\mathbb{Q}$ chosen for pricing. Let $V^j(t, X_t)$, $j = 1, 2, \ldots, N$, be the prices of the options used for hedging. We assume that the latter options do not expire before $\tau \wedge T$, where $\tau = \tau_h^-$ is the first entrance time into the activation region $U$ of the down-and-in option (in the early expiry region of the down-and-out option).

As in the papers on the semistatic hedging, we assume that, at time $\tau \wedge T$, the hedging portfolio is liquidated. Let $(-1, n_1, \ldots, n_N)$ be the vector of numbers of securities in the hedging portfolio. The portfolio at the liquidation date $\tau \wedge T$ is the random variable

$$P_0(\tau \wedge T, X_{\tau\wedge T}) = -V^0(\tau \wedge T, X_{\tau\wedge T}) + \sum_{j=1}^{N} n_j V^j(\tau \wedge T, X_{\tau\wedge T}),$$

and the discounted portfolio at the liquidation date is $P(\tau \wedge T, X_{\tau\wedge T}) = e^{-rt\wedge T} P_0(\tau \wedge T, X_{\tau\wedge T})$. One can consider the variance minimization problem for either $P_0(\tau \wedge T, X_{\tau\wedge T})$ or $P(\tau \wedge T, X_{\tau\wedge T})$, and we can calculate the variance under either the EMM $\mathbb{Q}$ used for pricing or the historic measure $\mathbb{P}$. We consider the minimization of the variance of $P(\tau \wedge T, X_{\tau\wedge T})$ under $\mathbb{Q}$.

Let $x = X_0$, and, for $j, k = 0, 1, \ldots, N$, set $C^0_{j\ell}(x) = V^j(0, x)V^\ell(0, x)$ and

$$C_{j\ell}(x) = \mathbb{E}^x \left[ e^{-2rt\wedge T} V^j(\tau \wedge T, X_{\tau\wedge T})V^\ell(\tau \wedge T, X_{\tau\wedge T}) \right]. \quad (14)$$

Using $\mathbb{E}[(U - \mathbb{E}[U])^2] = \mathbb{E}[U^2] - \mathbb{E}[U]^2$, we represent the variance of $P(\tau \wedge T, X_{\tau\wedge T})$ in the form

$$\operatorname{Var}_P(x) = C_{00}(x) - C^0_{00}(x) - 2n_j \sum_{j=1}^{N} \left( C_{0j}(x) - C^0_{0j}(x) \right)$$

$$+ \sum_{j=1}^{N} w_j^2 \left( C_{jj}(x) - C^0_{jj}(x) \right) + \sum_{j \neq \ell, j, \ell = 1, 2, \ldots, N} n_j n_\ell \left( C_{j\ell}(x) - C^0_{j\ell}(x) \right),$$

and find the minimizing $n = n(x)$ as

$$n(x) = A(x)^{-1} B(x), \quad (16)$$
where \( B(x) = [C_{0j}(x) - C_{0j}^0(x)]_{j=1,...,N} \) is a column vector, and \( A = [C_{j,\ell}(x) - C_{j,\ell}^0(x)]_{j,\ell=1,...,N} \), \( A \) is an invertible square matrix if the random variables \( V^j(\tau \land T, X_{\tau,T}) \), \( j = 1, 2, \ldots, N \), are uncorrelated.

To calculate \( C_{j,\ell}(x) \), it suffices to calculate \( V^j(0, x) \) and \( V^\ell(0, x) \). We decompose \( V^j(0, x) \) into the sum of the first-touch option \( V_{f,t}^j(x) \) with the payoff \( V^j(T, X_T)1_{\tau>T} \), and no-touch option \( V_{n,t}^j(x) \) with the payoff \( V^j(T, X_T)1_{\tau\leq T} \). Given a model for \( X \), we can calculate the prices of the no-touch and first touch options. Similarly, we can decompose \( C_{j,\ell}(x) \) into the sum of the first-touch option \( C_{f,t;f,\ell}(x) \) with the payoff \( V^j(\tau, X_\tau)V^\ell(\tau, X_\tau)1_{\tau\leq T} \), and no-touch option \( C_{n,t;f,\ell}(x) \) with the payoff \( V^j(\tau, X_\tau)V^\ell(\tau, X_\tau)1_{\tau>T} \).

The no-touch options can be efficiently calculated if the Fourier transforms of the payoff functions of options \( V^j \) and \( V^\ell \) can be explicitly calculated; then several methods based of the Fourier inversion can be applied. In the present paper, we directly apply the general formulas for the double Fourier/Laplace inversion. These formulas are the same as the ones in Boyarchenko and Levendorskiï (2013) in the case of no-touch options, with the following improvement: instead of the fractional-parabolic changes of variables, the sinh-acceleration is used. In the case of the first touch options, an additional generalization is needed because, contrary to the cases considered in Boyarchenko and Levendorskiï (2013), the payoff depends on \( (t, X_t) \) rather than on \( X_t \) only.

One can use other methods that use approximations in the state space. Any such method has several sources of errors, which are not easy to control. Even in the case of pricing European and barrier options, serious errors may result (see, e.g., Boyarchenko & Levendorskiï, 2014, 2015; Levendorskiï, 2016), and, typically, very long and fine grids in the state space are needed. The recommendations in Kirkby (2015) for the choice of the truncation parameter rely on the ad hoc recommendation for the truncation parameter used in a series of papers (Fang & Oosterlee, 2008, 2009; Fang, Jönsson, Oosterlee, & Schoutens, 2010). As examples in Boyarchenko and Levendorskiï (2014) and de Innocentis and Levendorskiï (2014) demonstrate, this ad hoc recommendation can be unreliable even if applied only once. In the hedging framework suggested in the present paper, the truncation needs to be applied many times, for each \( t_j \) used in the time-discretization of the initial problem, hence, the error control becomes almost impossible.

### 3.3 Conditions on processes and payoff functions

We consider the down-and-out case; \( H = e^h \) is the barrier, \( T \) is the maturity date, and \( G(X_T) \) is the payoff at maturity. The most efficient realizations of the pricing/hedging formulas are possible if the characteristic exponent admits analytic continuation to the union of a strip and cone and behaves sufficiently regularly at infinity. For the general definition of the corresponding class of Lévy processes (called \textit{SINH-regular processes}) and applications to pricing European options in Lévy models and affine models, see Boyarchenko and Levendorskiï (2019). In the present paper, for simplicity, we assume that the characteristic exponent admits analytic continuation to the complex plane with two cuts.

**Assumption \( (X) \).**

1. \( X \) is a Lévy process with the characteristic exponent \( \psi \) admitting analytic continuation to the complex plane with the cuts \( i(-\infty, \lambda_-), i[\lambda_+, +\infty) \).
2. \( \psi \) admits the representation \( \psi(\xi) = -i\mu \xi + \psi^0(\xi) \), where \( \mu \in \mathbb{R} \), and \( \psi^0(\xi) \) has the following asymptotics as \( \xi \to \infty \): for any \( \varphi \in (-\pi/2, \pi/2) \):
   \[
   \psi^0(\rho e^{i\varphi}) = c^0_\infty e^{iv\rho} + O(\rho^{-1}), \quad \rho \to +\infty, \tag{17}
   \]
where \( \nu \in (0, 2], \nu_1 < \nu \) are independent of \( \varphi \), and \( c^0_\infty > 0 \).

It follows that

\[
\text{Re} \, c^0_\infty e^{i\nu \varphi} > 0, \quad |\varphi| < \pi/(2\nu).
\]

(18)

This implies that if \( \nu \in (0, 1) \), then \( |\varphi| \geq \pi/2 \) can be admissible. For standard classes of Lévy processes used in finance, this is possible if we consider analytic continuation to an appropriate Riemann surface (see Boyarchenko & Levendorskiï, 2014; Levendorskiï, 2014 for details). We will not use analytic continuation to Riemann surfaces in the present paper.

**Example 3.1.** Essentially all Lévy processes used in quantitative finance are elliptic SINH-regular Lévy processes: BM; Merton model (Merton, 1976); normal inverse Gaussian (NIG) model (Barndorff-Nielsen, 1998); hyperbolic processes (Eberlein & Keller, 1995); double-exponential jump-diffusion (DEJD) model (Kou, 2002; Lipton, 2002) its generalization: hyperexponential jump-diffusion (HEJD) model, introduced in Levendorskiï (2002) and Lipton (2002) and studied in detail in Levendorskiï (2002, 2004); the majority of processes of the \( \beta \)-class (Kuznetsov, 2010); and the generalized Koponen’s family (Boyarchenko & Levendorskiï 2000; Cont, Bouchaud, & Potters 1997) and its subclass KoBoL (Boyarchenko & Levendorskiï, 2002b). A subclass of KoBoL (known as the CGMY model—see Carr, Geman, Madan, & Yor, 2002) is given by the characteristic exponent

\[
\psi(\xi) = -i\mu \xi + c\Gamma(-\nu)(\lambda_+^{\nu} - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu),
\]

(19)

where \( \nu \in (0, 2), \nu \neq 1 \) (in the case \( \nu = 1 \), the analytical expression is different, see Boyarchenko & Levendorskiï, 2000, 2002b). A different (less convenient for an analytical study of the properties of option prices and numerical calculations) representation for \( \psi \) was given in Cont et al. (1997); the restriction \( \nu \neq 1 \) was missing. The characteristic exponents of NTS processes constructed in Barndorff-Nielsen and Levendorskiï (2001) are given by

\[
\psi(\xi) = -i\mu \xi + \delta[\alpha^2 + (\xi + i\beta)^2]^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2},
\]

(20)

where \( \nu \in (0, 2), \delta > 0, |\beta| < \alpha \). If \( X \) is BM, DEJD, and HEJD, the asymptotic coefficient \( c^0_\infty = \sigma^2/2 \); if \( X \) is given by (19), \( c^0_\infty = -2c\Gamma(-\nu)\cos(\pi\nu/2) \).

**Assumption (G).** \( G \) is a measurable function admitting a bound

\[
|G(x)| \leq C(1 + e^{\beta x}),
\]

(21)

where \( C > 0 \) and \( \beta \in [0, -\lambda_-) \) are independent of \( x \in \mathbb{R} \), and either (i) \( G(x) = e^{\beta x}, x \in \mathbb{R} \), or (ii) \( \hat{G}(\xi) \) is well defined on the half-space \( \{\text{Im} \, \xi < -\beta\} \), and admits the representation \( \hat{G}(\xi) = e^{-ia\xi}\hat{G}_0(\xi) \), where \( a \in \mathbb{R} \) and \( \hat{G}_0(\xi) \) is a rational function decaying at infinity.

Note that only the values \( G(x), x > h \), matter, hence, we may replace \( G \) with \( 1_{(h, +\infty)} G \), and then there is no need to consider the case (i) separately.

**Example 3.2.**

(a) \( G = 1_{(h, +\infty)} \): The payoff function of the no-touch option and the square of this payoff.

(b) \( G(x) = e^{\xi} \): The value of the underlying at the maturity date \( T \) and \( X_T = x \), and the product of the latter and the payoff of the no-touch digital.

(c) \( G(x) = e^{2\xi} \): The square of the underlying at the maturity date \( T \) and \( X_T = x \).
(d) \( G(x) = (e^x - K)_+ \): The payoff function of the call option. \( \hat{G}(\xi) = e^{-ik\xi} \hat{G}_0(\xi) \), where \( k = \ln K \), and \( \hat{G}_0 = \frac{K}{(i\xi(i\xi - 1)} \) has two simple poles at 0 and \(-i\).

(e) \( G(x) = (e^x - K_j)_+ (e^x - K_\rho)_+ \): The product of the payoff functions of two call options. With \( K_j = K_\rho \), we have the payoff function of a powered call option. \( \hat{G}_0 \) has three simple poles at 0, \(-i\) and \(-2i\). If \( K_j \geq K_\rho \), \( \hat{G}(\xi) \) is given by (13).

(f) \( G(x) = (K - e^x)_+ \): The payoff function of the put option; \( \hat{G}(\xi) = e^{-ik\xi} \hat{G}_0(\xi) \), where \( k = \ln K \), \( \hat{G}_0 = \frac{K}{(i\xi(i\xi - 1)} \) has two simple poles at 0 and \(-i\).

(g) \( G(x) = (e^x - K_j)_+ (e^x - e^\rho)_+ \): The product of the payoff functions of the call and put options. With \( K_j = K_\rho \), we have the payoff function of a powered put option. \( \hat{G}_0 \) has three simple poles at 0, \(-i\) and \(-2i\). If \( K_j \leq K_\rho \), \( \hat{G}(\xi) \) is given by (13).

In (f) (after the multiplication by \( 1_{(h, +\infty)} \)) and (g), \( G \) satisfies condition (21) with any \( \beta \in \mathbb{R} \), in (b) (after the multiplication by \( 1_{(h, +\infty)} \)) and (d)—with \( \beta = 1 \), and in (c) (after the multiplication by \( 1_{(h, +\infty)} \)) and (e)—with \( \beta = 2 \).

3.4 | More general payoff functions and embedded options

In the case of embedded options, the simple structure of \( \hat{G} \) formalized in Assumption (G) is impossible. The following generalization allows us to consider payoffs that are prices of vanillas and some exotic options. For \( \gamma \in (0, \pi/2) \), define cones in the complex plane \( C_\gamma^+ = \{ z \in \mathbb{C} \mid \arg z \in (-\gamma, \gamma) \} \), \( C_\gamma = C_\gamma^+ \cup (-C_\gamma^+) \). For \( \mu_- < \mu_+ \), set \( S_{(\mu_-, \mu_+)} := \{ \xi \mid \text{Im} \xi \in (\mu_-, \mu_+) \} \).

Assumption \((G_{emb})\).

1. \( G \) is a measurable function admitting the bound (21).
2. There exist \( a \in \mathbb{R} \), \( \delta > 0 \) and \( \gamma \in (0, \pi/2) \) such that \( \hat{G}(\xi) = e^{-ia\xi} \hat{G}_0(\xi) \), where \( \hat{G}_0(\xi) \) is meromorphic in \( S_{(\delta, \delta_\lambda)} \cup C_\gamma \), has a finite number of poles and decays as \( |\xi|^{-\delta} \) or faster as \( \xi \to \infty \) remaining in \( S_{(\delta, \delta_\lambda)} \cup C_\gamma \).

4 | WIENER–HOPF FACTORIZATION

4.1 | Wiener–Hopf factorization: Basic facts

Several equivalent versions of general pricing formulas for no-touch and first touch options were derived in Boyarchenko and Levendorskiï (2002a, 2002b, 2009, 2013) and Levendorskiï (2014) in terms of the Wiener–Hopf factors. In this subsection, we list the notation and facts that we use in the present paper.

4.1.1 | Three forms of the Wiener–Hopf factorization

Let \( X \) be a Lévy process with the characteristic exponent \( \psi \). The supremum and infimum process are defined by \( \overline{X} = \sup_{0 \leq s \leq t} X_s \) and \( \underline{X} = \inf_{0 \leq s \leq t} X_s \), respectively. Let \( q > 0 \) and let \( T_q \) be an exponentially distributed random variable of mean \( 1/q \), independent of \( X \). Introduce functions \( \phi^+_q(\xi) = \mathbb{E}[e^{i\xi \overline{X}_{T_q}}], \phi^-_q(\xi) = \mathbb{E}[e^{i\xi \underline{X}_{T_q}}] \), and normalized resolvents (the EPV operators under \( X, \overline{X} \) and
\( X \), respectively

\[
\mathcal{E}_q u(x) = \mathbb{E}\left[u(X_{T_q})\right] = \mathbb{E}^x \left[ \int_0^{+\infty} q e^{-qt} u(X_t) dt \right].
\]  

\( \mathcal{E}_q^+ u(x) = \mathbb{E}\left[u(X_{T_q})\right] = \mathbb{E}^x \left[ \int_0^{+\infty} q e^{-qt} u(X_t) dt \right],
\]

\( \mathcal{E}_q^- u(x) = \mathbb{E}\left[u(X_{T_q})\right] = \mathbb{E}^x \left[ \int_0^{+\infty} q e^{-qt} u(X_t) dt \right].
\]

The Wiener–Hopf factorization formula in the form used in probability (Rogers & Williams, 1994, p. 98) is

\[
\frac{q}{q + \psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi);
\]

its operator analog is \( \mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+ = \mathcal{E}_q^+ \mathcal{E}_q^- \).

4.1.2 Explicit formulas for and properties of the Wiener–Hopf factors

Let \( X \) be a Lévy process with characteristic exponent admitting analytic continuation to a strip \( \{\text{Im} \xi \in (\lambda_-, \lambda_+), \lambda_- < 0 < \lambda_+\} \), around the real axis, and let \( q > 0 \). Then (see, e.g., Boyarchenko & Levendorskiı̆, 2002b, 2002c)

1. There exist \( \sigma_-(q) < 0 < \sigma_+(q) \) such that

\[
q + \psi(\eta) \notin (-\infty, 0], \quad \text{Im} \eta \in (\sigma_-(q), \sigma_+(q)).
\]

2. The Wiener–Hopf factor \( \phi_q^+(\xi) \) admits analytic continuation to the half-plane \( \{\text{Im} \xi > \sigma_-(q)\} \), and can be calculated as follows: for any \( \omega_- \in (\sigma_-(q), \text{Im} \xi) \),

\[
\phi_q^+(\xi) = \exp \left[ \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega_-} \frac{\xi \ln(q + \psi(\eta))}{\eta(\xi - \eta)} d\eta \right].
\]

3. The Wiener–Hopf factor \( \phi_q^-(\xi) \) admits analytic continuation to the half-plane \( \{\text{Im} \xi < \sigma_+(q)\} \), and can be calculated as follows: for any \( \omega_+ \in (\text{Im} \xi, \sigma_+(q)) \),

\[
\phi_q^-(\xi) = \exp \left[ -\frac{1}{2\pi i} \int_{\text{Im} \eta = \omega_+} \frac{\xi \ln(q + \psi(\eta))}{\eta(\xi - \eta)} d\eta \right].
\]

Note that one may use \( 1 + \psi(\eta)/q \) instead of \( q + \psi(\eta) \); the integrals (27)–(28) do no change. Naturally, in this case, we require

\[
1 + \psi(\eta)/q \notin (-\infty, 0], \quad \text{Im} \eta \in (\sigma_-(q), \sigma_+(q)).
\]

Under additional conditions on \( \psi \), there exist more efficient formulas for the Wiener–Hopf factors. See Boyarchenko and Levendorskiı̆ (2013) and Levendorskiı̆ (2014).
Analytic continuation of the Wiener–Hopf factors with respect to $\xi$, for $q$ fixed

Under Assumption (X), $\psi(\eta)$ is analytic in the complex plane with two cuts $i(-\infty, \lambda_-]$ and $i[\lambda_+, +\infty)$, and, for any $[\omega_-, \omega_+] \subset (\lambda_-, \lambda_+)$, $\text{Re}\, \psi(\eta) \to +\infty$ as $\eta \to \infty$ remaining in the strip. Hence, for $[\omega_-, \omega_+] \subset (\lambda_-, \lambda_+)$, there exists $\sigma > 0$ s.t. if $\text{Re} \ q \geq \sigma$, then (26) holds. It follows that, for $q$ in the half-plane $\{\text{Re} \ q \geq \sigma\}$,

1. (27) defines $\phi^+_q(\xi)$ on the half-plane $\{\text{Im} \, \xi > \omega_-\}$, and analytic continuation of $\phi^-_q(\xi)$ to the strip $S_{(-\infty, \omega_+)}$ can be defined by

$$
\phi^-_q(\xi) = \frac{q}{(q + \psi(\xi))\phi^+_q(\xi)}.
$$

(30)

2. (28) defines $\phi^-_q(\xi)$ on the half-plane $\{\text{Im} \, \xi < \omega_+\}$, and analytic continuation of $\phi^+_q(\xi)$ to the strip $S_{(\omega_-, \omega_+)}$ can be defined by

$$
\phi^+_q(\xi) = \frac{q}{(q + \psi(\xi))\phi^-_q(\xi)}.
$$

(31)

Remark 4.1. It follows that $\phi^+_q(\xi)$ (respectively, $\phi^-_q(\xi)$) admits analytic continuation with respect to $\xi$ to $\mathbb{C} \setminus i(-\infty, \omega_-)$ (respectively, $\mathbb{C} \setminus i[\omega_+, +\infty)$).

For $\omega_1 \in \mathbb{R}$, $\omega \in (-\pi/2, \pi/2)$ and $b > 0$, introduce the function

$$
\mathcal{C} \ni y \mapsto \chi(\omega_1, \omega, b; y) = i\omega_1 + b \sinh(i\omega + b) \in \mathbb{C},
$$

and the curve $\mathcal{L}(\omega_1, \omega, b) = \chi(\omega_1, \omega, b; \mathbb{R})$, the image of the real line under $\chi(\omega_1, \omega, b; \cdot)$. If $\omega = 0(> 0, < 0)$, the curve is flat (wings point upward, wings point downward, respectively). Depending on the situation, we will need the Wiener–Hopf factors either on a curve $\mathcal{L}(\omega_1, \omega, b)$ with the wings pointing up ($\omega > 0$), or on a curve $\mathcal{L}(\omega'_1, \omega', b')$ with the wings pointing down ($\omega < 0$). In the former case, we deform the contour of integration (in the formula for the Wiener–Hopf factors that we use) so that the wings of the deformed contour $\mathcal{L}(\omega'_1, \omega', b')$ point down ($\omega' < 0$), and in the latter case—up ($\omega' > 0$). This straightforward requirement is easy to satisfy, as well as the second requirement: the curves do not intersect. Indeed, if $\omega \in (0, \pi/2)$ and $\omega' \in (-\pi/2, 0)$, the curves do not intersect if and only if the point of intersection of the former with the imaginary axis is above the point of the intersection of the latter with the same axis:

$$
\omega_1 + b \sin \omega > \omega'_1 + b' \sin \omega'.
$$

(32)

The last condition on $\mathcal{L}(\omega'_1, \omega', b')$ is that for $q$ of interest, the function $\mathcal{L}(\omega'_1, \omega', b') \ni \eta \mapsto 1 + \psi(\eta)/q \in \mathbb{C}$ (or $\eta \mapsto q + \psi(\eta)$) is well defined, and, in the process of the deformation of the initial line of integration into $\mathcal{L}(\omega'_1, \omega', b')$, the image does not intersect $(-\infty, 0]$. If the parameters of the curve are fixed, this requirement is satisfied if $\text{Re} \ q \geq \sigma$ and $\sigma > 0$ is sufficiently large. For details, see Levendorskiï (2014), where a different family of deformations (fractional-parabolic ones) was used. In cases of the sinh-acceleration and fractional-parabolic deformation, at infinity, the curves stabilize to rays, hence, the analysis in Levendorskiï (2014) can be used to derive the conditions on the deformation parameters if $q > 0$.

If the Gaver–Stehfest method is applied, we need to use the Wiener–Hopf factorization technique for $q > 0$ only, and the analysis in Levendorskiï (2014) suffices. The restrictions on the parameters
of the deformations are similar to the ones in Levendorski (2014), where the fractional-parabolic deformations were used. For positive \( q \), the maximal (in absolute value) \( \sigma_{\pm}(q) \) are easy to find for all popular classes of Lévy processes used in finance. As it is proved in Boyarchenko and Levendorski (2002b) (see also Levendorski, 2014), the equation \( q + \psi(\xi) = 0 \) has either 0 or 1 or two roots in the complex plane with the two cuts \( i(\lambda_-, \infty) \) and \( i(\lambda_+, \infty) \). Each root is purely imaginary, and of the form \( -i\beta_{q+} \), where \( -\beta_{q+} \in (0, \lambda_+) \) and \( -\beta_{q-} \in (\lambda_-, 0) \). If the root \( -i\beta_{q+} \) exists, then \( \sigma_{\pm}(q) = -\beta_{q+} \), otherwise \( \sigma_{\pm}(q) = \lambda_{\pm} \). See Figure 2 for illustration.

4.2 Calculation of the Wiener–Hopf factors using the sinh-acceleration

If for the Laplace inversion the Gaver–Stehfest method or other methods utilizing only positive \( q \) is used, then we can take any \( \omega \in (-\pi/2, \pi/2) \) and \( \omega' = -\omega \). For each \( q \), we use the following versions of (27)–(28): for \( \xi \in \mathcal{L}(\omega_1, \omega, b) \),

\[
\phi_{q+}^+(\xi) = \exp \left[ \frac{b'}{2\pi i} \int_{\mathbb{R}} \frac{\xi \ln \left( q + \psi(\chi'(\omega_1, \omega', b', y)) \right) \cosh \left( i\omega' + y \right) \mathrm{d}y}{\chi(\omega_1', \omega', b', y) \left( \xi - \chi(\omega_1', \omega', b', y) \right)} \right],
\]

(33)
and for $\xi \in \mathcal{L}(\omega', \omega', b')$,

$$
\phi^-_q(\xi) = \exp\left[\frac{-b}{2\pi i} \int_{\mathbb{R}} \frac{\xi \ln(q + \psi(\chi(\omega_1, \omega, b, y)))}{\chi(\omega_1, \omega, b, y)(\xi - \chi(\omega_1, \omega, b, y))} \cosh(i\omega + y)dy\right].
$$

(34)

Each integral is evaluated applying the simplified trapezoid rule.

### 4.3 Numerical Examples

We apply the above scheme to calculate $\phi^\pm_q(\xi)$ in KoBoL model of order $\nu = 1.2$. If the factors are calculated at 30 points, then approximately 1.5 ms is needed to satisfy the error tolerance $\epsilon = 10^{-15}$, and 1.0–1.2 ms are needed to satisfy the error tolerance $\epsilon = 10^{-10}$. The number of terms is in the range 350–385 in the first case, and 159–175 in the second case. To satisfy the error tolerance of the order of $10^{-20}$, about 500 terms would suffice but, naturally, high precision arithmetic would be needed (Table 3).

### 5 Calculation of No-Touch Options and Expectations of No-Touch Products

#### 5.1 General Formulas for No-Touch Options

In Boyarchenko and Levendorski (2009, 2013) and Levendorski (2014), it is proved that the Laplace transform $\tilde{V}_1(G; \tau, x)$ of $V_1(G; \tau, x) = e^{\tau T} V(G; \tau, x)$ with respect to $\tau$ is given by

$$
\tilde{V}_1(G; q, x) = q^{-1} \left( \mathcal{E}_q^{-1} \mathbf{1}_{(h, +\infty)} \mathcal{E}_q^+ G \right)(x).
$$

(35)

The result is proved under conditions more general than Assumptions $(X)$ and $(G_{emb})$.

Let $F$ denote the operator of the Fourier transform, and set $\Pi^+_h := F\mathbf{1}_{(h, +\infty)} F^{-1}$. The operator $\Pi^+_h$ arises systematically in the theory of the Wiener–Hopf factorization and boundary value problems. See Eskin (1981) for the general setting in applications to multidimensional problems. Using $F$ and $\Pi^+_h$ and taking into account that $\mathcal{E}_q^\pm = F^{-1} \phi^\pm_q F$, we rewrite (35) as

$$
\tilde{V}_1(G; q, x) = q^{-1} \left( F^{-1} \phi^+_q \Pi^+_h \phi^+_q F G \right)(x).
$$

(36)

**Lemma 5.1.** Let $f$ be an analytic function in a strip $S_{(\sigma_-, \sigma_+)}$, where $\sigma_- < \sigma_+$, that decays as $|\xi|^{-s}$, $s > 1$, as $\xi \to \infty$ remaining in the strip. Then $(\Pi^+_h f)(\xi)$ is analytic in the half-plane $\{\xi \mid \text{Im} \xi < \sigma_+\}$, and can be defined by any of the following three formulas:

(a) For any $\omega_+ \in (\text{Im} \xi, \sigma_+)$,

$$
\Pi^+_h f(\xi) = \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega_+} \frac{e^{ih(\eta - \xi)} f(\eta)}{\xi - \eta} d\eta.
$$

(37)

(b) For any $\omega_- \in (-\infty, \text{Im} \xi)$,

$$
\Pi^+_h f(\xi) = f(\xi) + \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega_-} \frac{e^{ih(\eta - \xi)} f(\eta)}{\xi - \eta} d\eta.
$$

(38)
TABLE 3  Wiener–Hopf factors $\phi^\pm_q(\xi)$, $q = 1.1$, $\xi = \sinh(-0.15i + y)$, $|y| \leq 15$, for KoBoL close to NIG, with an almost symmetric jump density, and no “drift”: $m_z = 0.1$, $v = 1.2$, $\lambda_- = -11$, $\lambda_+ = 12$, $\mu = 0$, $\sigma = 0$, $c = 0.3026$ (rounded)

| $y$  | $-15$  | $-10$  | $-5$  | $0$  | $5$  |
|------|--------|--------|-------|------|------|
| Re $\phi^+_q$ | $-3.573612313E-07$ | $-4.67991818674E-05$ | $-3.8732615824580E-03$ | $1.03663360053561E-01$ | $-3.8732615824580E-03$ |
| Im $\phi^+_q$ | $-3.3276442219E-06$ | $4.358762795728E-04$ | $-5.8173831101994E-02$ | $2E-19$ | $5.8173831101994E-02$ |
| Re $\phi^-_q$ | $7.6320768431674E-07$ | $1.0003700599534E-04$ | $1.68715279339566E-02$ | $0.972251835483482$ | $1.68715279339566E-02$ |
| Im $\phi^-_q$ | $4.06659819933492E-06$ | $5.326366781895E-04$ | $6.88181160078408E-02$ | $-6E-19$ | $-6.88181160078408E-02$ |
| Err15 | $1.14E-16$ | $9.86E-17$ | $2.71E-16$ | $2.22E-16$ | $3.19E-16$ |
| Err10 | $6.31E-11$ | $5.57E-11$ | $4.99E-11$ | $1.79E-12$ | $4.99E-11$ |

Note. $\phi^\pm_q(\xi)$ is calculated applying the sinh-acceleration with the parameters $\omega_1 = 0$, $b = 3.37$, $\omega = 0.79$ to the integral in (27). $\phi^\pm_q(\xi)$ is calculated applying the sinh-acceleration with the parameters $\omega_1 = -0.26$, $b = 3.96$, $\omega = -0.86$ to the integral in (28). The parameters are chosen using the general recommendations for the sinh-acceleration method. The mesh and number of points are chosen using the general recommendations for the given error tolerance. If $e = 10^{-15}$, then $\zeta_- = 0.0969$, $N_- = 385$ for $\phi^+_q(\xi)$, and $\zeta_+ = 0.107$, $N_+ = 350$ for $\phi^-_q(\xi)$; if $e = 10^{-10}$, then $\zeta_- = 0.1410$, $N_- = 175$ for $\phi^+_q(\xi)$, and $\zeta_+ = 0.1559$, $N_+ = 159$ for $\phi^-_q(\xi)$. The last two lines are absolute differences between $\phi^+_q(\xi)$ calculated using (27) and $\phi^-_q(\xi)$ calculated using (28) and the Wiener–Hopf identity. CPU time for the calculation at 30 points, the average over 10k runs: 1.63 and 1.48 ms if $e = 10^{-15}$, and 1.15 and 1.06 ms if $e = 10^{-10}$. 
\( \Pi^+_h f(\xi) = \frac{1}{2} f(\xi) + \frac{1}{2\pi i} \text{v.p.} \int_{\text{Im} \eta = \text{Im} \xi} e^{i h (\eta - \xi)} f(\eta) \, d\eta, \) \quad (39)

where v.p. denotes the Cauchy principal value of the integral.

**Proof.**

(a) Applying the definition of \( \Pi^+_h \) and Fubini’s theorem, we obtain

\[
\Pi^+_h f(\xi) = \left( F1_{(h, +\infty)} F^{-1} f \right)(\xi) = \int_{h}^{+\infty} e^{-ix\xi}(2\pi)^{-1} \int_{\text{Im} \eta = \omega_+} e^{i\eta} f(\eta) d\eta d\xi
\]

\[
= \int_{\text{Im} \eta = \omega_+} f(\eta)(2\pi)^{-1} \int_{h}^{+\infty} e^{i(\eta - \xi)x} d\eta d\xi = \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega_+} e^{i h (\eta - \xi)} f(\eta) \, d\eta, \quad (40)
\]

which proves (37).

(b) We push the line of integration in (40) down. On crossing the simple pole at \( \eta = \xi \), we apply the Cauchy residue theorem, and obtain (38).

(c) Let \( \omega := \text{Im} \xi \) and \( a = \text{Re} \xi \). We deform the contour \( \text{Im} \xi = \omega \) into

\[
\mathcal{L}_\epsilon = i\omega + ((-\infty, a - \epsilon) \cup (a + \epsilon, +\infty)) \cup \epsilon \{ e^{i\varphi} \mid 0 \leq \varphi \leq \pi \}
\]

and then pass to the limit \( \epsilon \downarrow 0 \). The result is (39). \( \square \)

**Remark 5.2.** Let \( \omega = h = 0 \). Then (39) can be written in the form \( \Pi^+_0 = \frac{1}{2} I + \frac{1}{2\pi i} \mathcal{H} \), where \( I \) is the identity operator, and \( \mathcal{H} \) is the Hilbert transform. Therefore, a realization of \( \Pi^+_0 \) is, essentially, equivalent to a realization of \( \mathcal{H} \).

The following theorem is a part of the proof of the general theorem for pricing no-touch options in Boyarchenko and LevendorskiĂ (2002a, 2002b); we will outline the proof based on (36).

**Theorem 5.3.** Let Assumptions (\( X \)) and (\( G_{\text{emb}} \)) hold, and let \( \omega_- \in (\lambda_-, \lambda_+) \), \( \omega_+ \in (\omega_-, \omega_+) \) be such that \( e^{io} G \in L_1(\mathbb{R}) \) for any \( o \in [\omega_-, \omega_+] \).

Then there exists \( \sigma > 0 \) such that for all \( q \), \( \text{Re} \ q \geq \sigma \), and all \( x > h \),

\[
\hat{V}_1(G; q, x) = \frac{1}{2\pi q} \int_{\text{Im} \xi = \omega_+} d\xi \ e^{i\xi} \phi_+^q(\xi) \frac{1}{2\pi} \int_{\eta = \omega_-} d\eta \ e^{i h (\eta - \xi)} \frac{\phi_+^q(\eta) \hat{G}(\eta)}{i(\xi - \eta)}
\]

\[
+ \frac{1}{2\pi} \int_{\text{Im} \xi = \omega_+} d\xi \ e^{i\xi} \hat{G}(\xi) \frac{1}{q + \psi(\xi)}, \quad (41)
\]

**Proof.** We can choose \( \sigma \) so that, for a fixed \( q \) in the half-plane \( \{ \text{Re} \ q \geq \sigma \} \), \( \phi_+^q(\eta) \) and \( \hat{G}(\eta) \) are analytic in the strip \( \eta \in S_{[\omega_-, \omega_+]}. \) Hence, the product \( \phi_+^q(\eta) \hat{G}(\eta) \) is analytic in the same strip, and, by Lemma 5.1,
the function \((\Pi_n \phi_q^+ \hat{G})(\xi)\) is analytic in the half-plane \(\{\text{Im} \xi \in (-\infty, \omega_+)\}\). Applying (38), we obtain (41)–(42). In more detail, on the RHS of (42), we obtain, first,

\[
\frac{1}{2\pi q} \int_{\text{Im} \xi = \omega_+} d\xi \ e^{i\eta \xi} \phi_q^-(\xi) \phi_q^+(\xi) \hat{G}(\xi),
\]

and then apply the Wiener–Hopf factorization formula (25). The integrals on the RHS of (41)–(42) absolutely converge (or absolutely converge after integration by parts in the oscillatory integrals with respect to \(\xi\)) and their sum equals \(\bar{V}_1(G; q, x)\) (see Boyarchenko & Levendorskií, 2002a, 2002b; Boyarchenko et al., 2011).

Denote by \(\mathcal{M}\) the set of \(q\)’s in the Gaver–Stehfest formula and by \(Q(q)\) the weights (for explicit formulas, see, e.g., Levendorskií, 2014). We have an approximation

\[
V(G, T; x) \approx e^{-rT} \sum_{q \in \mathcal{M}} Q(q) e^{qT} \bar{V}_1(G; q, x). \tag{43}
\]

It remains to design an efficient numerical procedure for the evaluation of \(\bar{V}_1(G; q, x), q > 0\). As in Levendorskií (2014), in our numerical examples, we use Gaver–Wynn’s algorithm (GWR-method).

### 5.2 Sinh-acceleration in the integrals on the RHS of (41)–(42)

First, we rewrite (41)–(42) as follows: for \(x > h\),

\[
\bar{V}_1(G; q, x) = \frac{1}{2\pi q} \int_{\text{Im} \xi = \omega_+} d\xi \ e^{i\eta (x-h)} \phi_q^- (\xi) \frac{1}{2\pi} \int_{\text{Im} \eta = \omega_-} d\eta \ \frac{e^{-i\eta (a-h)} \phi_q^+(\eta) \hat{G}_0(\eta)}{l(\xi - \eta)}, \tag{44}
\]

\[
+ \frac{1}{2\pi} \int_{\text{Im} \xi = \omega_+} d\xi \ \frac{e^{i\eta (x-a)\xi} \hat{G}_0(\xi)}{q + \psi(\xi)}, \tag{45}
\]

where \(a \geq h\) and \(\hat{G}_0\) satisfy the conditions in Assumption \((G_{emb})\).

As \(\eta \to \infty\), \(\hat{G}_0(\eta) \to 0\). If the order \(v \in [1, 2]\) or \(\mu = 0\), then \(\phi_q^\pm(\xi) \to 0\) as \(\xi \to \infty\) in the domain of analyticity; if \(v < 1\) and \(\mu \neq 0\), then one of the Wiener–Hopf factors stabilizes to constant at infinity, and the other one decays as \(1/|\xi|\). As \(x - h > 0\), it is advantageous to deform the outer contour on the RHS of (44) so that the wings of the deformed contour point upward: \(\mathcal{L}^+ := \mathcal{L}(\omega_1, \omega, b) = \mathcal{C}(\omega_1, \omega, b; \mathbb{R})\), where \(\omega > 0\). We deforms the contour so that no pole of the integrand (if it exists) is crossed. In all cases, in the process of deformations, the curves must remain in \(S(\lambda_-) \cup \mathcal{C}_y\), where \(\mathcal{C}_y\) is the cone in Assumption \((G_{emb})\), and \(S(\lambda_-) = \{\xi \mid \text{Im} \xi \in (\lambda_-, \lambda_+)\}\).

If Assumption \((G)\) holds, then we may take any \(\gamma \in (0, \pi/2)\); in our numerical experiments, we will take \(\gamma = \pi/4\).

The type of the deformation of the inner contour depends on the sign of \(a - h\). If \(a \geq h\), which, in the case of puts and calls means that the strike is above or at the barrier, then we may deform the contour downward. The deformed contour is of the form \(\mathcal{L}^- := \mathcal{L}(\omega_1', \omega', b') = \mathcal{C}(\omega_1', \omega', b'; \mathbb{R})\), where \(\omega' < 0\); as in the case of the deformation of the outer contour, we choose the parameters of the deformation so that, in the process of deformation, no pole of the inner integrand is crossed. The parameters of both contours are chosen, so that \(\mathcal{L}^+\) is strictly above \(\mathcal{L}^-\). The case \(a < h\) (the strike is below the barrier) is reducible to the case \(G(x) = 1_{[h, +\infty)} e^{\beta x}\). We have \(\hat{G}(\xi) = e^{(\beta - i\xi)h} \phi_q^+(i\xi - \beta)\), hence, Assumption \((G)\) is valid with \(a = h\). If \(G\) is the value function of an embedded option, then \(a < h\) is possible (e.g., the strike of the embedded European put or call is below the barrier). The type of
deformation of the contour of integration on the RHS of (45) depends on the sign of \( x - a \). If \( x - a > 0 \), we use \( \mathcal{L}^+ \), and if \( x - a < 0 \), then \( \mathcal{L}^- \). If \( x - a = 0 \), then either deformation can be used. We conclude that, for the majority of applications when embedded options are not involved, we may use deformed contours of the form \( \mathcal{L}^+ \) in the outer integral and of the form \( \mathcal{L}^- \) in the inner integral, and, for \( x > h \), write

\[
\tilde{V}_1(G; q, x) = \frac{1}{2\pi i} \int_{\mathcal{L}^+} d\xi e^{i\xi(x-h)} \phi_q^{-}(\xi) \frac{1}{2\pi} \int_{\mathcal{L}^-} d\eta e^{-i\eta(a-h)} \phi_q^{+}(\eta) \hat{G}_0(\eta) + \frac{1}{2\pi} \int_{\mathcal{L}^-} d\xi e^{i(x-a)\xi} \hat{G}_0(\xi),
\]

where the type of the deformed contour \( \mathcal{L}' \) depends on the sign of \( x - a \): \( \mathcal{L}' = \mathcal{L}^+ \) if \( x - a \geq 0 \), and \( \mathcal{L}' = \mathcal{L}^- \) if \( x - a \leq 0 \). To evaluate the integrals numerically, we make the changes of variables \( \xi = i\omega_1 + b \sinh(i\omega_0 + y) \) and \( \eta = i\omega_1' + b' \sinh(i\omega_0' + y') \), and apply the simplified trapezoid rule with respect to \( y \) and \( y' \).

### 5.3 Crossing poles

The efficiency of calculations can be improved if it is possible to cross poles of the integrand so that after the crossing, the integrand along the new contour is smaller in absolute value than the integrand along the initial contour. See Levendorskiï (2014) for applications of this standard idea to pricing barrier options.

### 5.4 Numerical examples

#### 5.4.1 Pricing no-touch options and first touch digitals, down case

Let \( V_{nt}(T, x) \) and \( V_{ft}(T, x) \) be the prices of the no-touch and first touch digital option, respectively. Set \( V_{nt,1}(T, x) = e^{T} V_{nt}(T, x) \) (Table 4). Then (see, e.g., Boyarchenko & Levendorskiï, 2009; Levendorskiï, 2014), for any \( \omega \in (\sigma_-, 0) \) and \( \omega_+ \in (0, \sigma_+(q)) \),

\[
\tilde{V}_{nt,1}(q, x) = \frac{1}{2\pi q} \int_{\text{Im}\xi = \omega} e^{i(x-h)\xi} \phi_q^{-}(\xi) \frac{1}{-i\xi} d\xi = \frac{1}{q} + \frac{1}{2\pi q} \int_{\text{Im}\xi = \omega_+} e^{i(x-h)\xi} \phi_q^{-}(\xi) \frac{1}{-i\xi} d\xi,
\]

and, for any \( \omega_+ \in (0, \sigma_+(q - r)) \),

\[
\tilde{V}_{ft}(q, x) = -\frac{1}{2\pi(q-r)} \int_{\text{Im}\xi = \omega_+} e^{i(x-h)\xi} \phi_q^{-}(\xi) \frac{1}{-i\xi} d\xi.
\]

In both cases, the line of integration is deformed into a contour \( \mathcal{L}_+ := \mathcal{L}(\omega_1+, \omega_+, b_+) \) in the upper half-plane, with wings pointing up. To evaluate \( \phi_q^{-}(\xi), \xi \in \mathcal{L}_+ \), we use (27) to calculate \( \phi_q^{+}(\xi) \) (the line of integration is deformed into a contour in the lower half-plane with the wings pointing down), and then (30). We calculate prices of no-touch options and first touch digitals in the same KoBoL model as in Table 1. Our numerical experiments show that, for the integrals in the formulas for the Wiener–Hopf factors and the Fourier inversion, the relative errors of the order of \( 10^{-8} \) and smaller (for prices) can be achieved using the general recommendations of Boyarchenko and Levendorskiï (2019) for the error tolerance \( \epsilon = 10^{-10} \); if \( \epsilon = 10^{-15} \) is used, the pricing errors decrease insignificantly. Hence, in this example, the errors of GWR method are of the order of \( 10^{-10} - 10^{-8} \). The CPU time (for seven spots) is less than 10 ms for the error tolerance \( \epsilon = 10^{-6} \), and less than 25 ms for the error tolerance \( \epsilon = 10^{-10} \).
TABLE 4 Pricing no-touch and first-touch options (down case) using Gaver–Wynn–Rho method with \( M = 8 \) and sinh-acceleration

| \( \ln(S/H) \) | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 | 0.11 | 0.13 |
|----------------|------|------|------|------|------|------|------|
| \( V_{nt} \)   | 0.159611796 | 0.349192704 | 0.50261129 | 0.628008257 | 0.727388428 | 0.803409906 | 0.859701515 |
| \( V_{ft} \)   | 0.837533576 | 0.645372195 | 0.490377871 | 0.363952758 | 0.263898333 | 0.187440524 | 0.130864377 |
| (A)            | 1.01E-08 | -4.65E-10 | -1.45E-08 | 6.09E-09 | 7.80E-09 | -6.29E-09 | 3.80E-09 |
|                | -6.32E-09 | -2.76E-09 | 8.74E-10 | -3.19E-09 | -6.68E-09 | -1.41E-08 | -1.30E-09 |
| (B)            | 2.08E-05 | -7.47E-06 | -2.68E-05 | -3.36E-05 | -1.89E-04 | 8.85E-05 | 5.24E-06 |
|                | -2.22E-05 | 7.70E-06 | 2.93E-05 | 4.94E-05 | 2.38E-04 | -8.50E-05 | -5.71E-06 |

Note. Benchmark prices and errors for different choices of the parameters of the sinh-acceleration are rounded. The underlying: KoBoL model as in Table 2. Parameters \( m_2 = 0.1, \nu = 1.2, \lambda_+ = -11, \lambda_- = 12, \mu = 0, \sigma = 0, c = 0.3026 \) (rounded), \( \beta = -1, r = 0.100 \) (rounded). Time to maturity \( T = 0.1, \) barrier \( H = 1, S \) the spot. Curve used in the Fourier inversion \( \mathcal{L}_+ := \mathcal{L}(0, 1, 1.1, 9.6). \) Curve used to calculate \( \phi_\epsilon^\pm(\xi), \xi \in \mathbb{L}_+: \mathbb{L}_- := \mathbb{L}(-1, -1.1, 8) \). \( N_\pm, \zeta_\pm \) are chosen using the universal simplified recommendation in Boyarchenko and Levendorski (2019) for \( \epsilon = 10^{-10} \) (A) and \( \epsilon = 10^{-6} \) (B), with \( \zeta_\pm 30\% \) smaller than recommended. Recommendation based on \( \epsilon = 10^{-15} \) does not lead to a sizable improvement. Hence, the errors of GWR method are of the order of \( 10^{-9} - 10^{-8} \). (A): \( \zeta_\pm = 0.1559, N_\pm = 139, \zeta_\pm = 0.1559, N_\pm = 39; \) (B): \( \zeta_\pm = 0.2449, N_\pm = 51, \zeta_\pm = 0.2449, N_\pm = 23. \) CPU time for calculation of \( V_{nt} \) and \( V_{ft} \) at seven spots: 8.28 ms (A) and 23.7 ms (B). Averages over 1,000 runs.
The bulk of the CPU time is spent on the calculation of the Wiener–Hopf factor at the points of the chosen grids; this calculation can be easily parallelized, and the total CPU time significantly decreased.

5.4.2 | Pricing down-and-out call option

We consider the same model, and the call options with the strikes \( K = 1.04, 1.1 \). In both cases, the barrier \( H = 1 \), and \( T = 0.1, 0.5 \). Even in cases when the spot is close to the barrier, the error tolerance of the order of \( 10^{-5} \) and smaller can be satisfied at a moderate CPU-time cost: 0.1 ms for the calculations at seven spots (Table 5).

6 | PRICING FIRST-TOUCH OPTIONS AND EXPECTATIONS OF FIRST-TOUCH PRODUCTS

Conditions on processes and payoff functions are the same as in Section 3.3. We consider the down-and-in case; \( H = e^h \) is the barrier, \( T \) is the maturity date, and \( G(\tau, X_\tau) \) is the payoff at the first entrance time \( \tau \) into \((-\infty, h]\). We need to calculate \( V(G; T; x) = \mathbb{E}^x[G(\tau, X_\tau)\mathbf{1}_{\tau<T}] \).

6.1 | The simplest case

Let \( G(\tau, X_\tau) = e^{-r\tau + \beta X_\tau} \), where \( \beta \in [0, -\lambda_-] \). For \( \beta = 0 \), \( V(G; T; x) \) is the time-0 price of the first-touch digital, for \( \beta = 1 \) of the stock that is due at time \( \tau \) if \( \tau < T \). The case of the down-and-in forward obtains by linearity. If we need to calculate the expectation of the product of two payoffs of this form, \( \beta \) may assume value \( \beta = 2 \); in this case, we need to replace \( r \) with \( 2r \).

As \( \beta \in [0, -\lambda_-] \), \( \phi_q^{-}(-i\beta) = \mathbb{E}[e^{\beta X_\tau}] \) is finite for any \( q \) in the right half-plane. Furthermore, for any \( \sigma > 0 \), there exists \( \omega > 0 \) such that, for any \( q \) in the half-plane \( \{\text{Re } q \geq \sigma\} \), \( \phi_q^{-}(\xi) \) admits analytic continuation to \( \{\text{Im } \xi < \omega\} \). For such \( \sigma \) and \( \omega \), the general formula for the down-and-out options derived in Boyarchenko and Levendorski (2002a, 2009) (see also Boyarchenko & Levendorski, 2013; Levendorski, 2014) is applicable: for \( x > h \),

\[
V(G; T; x) = \frac{e^{\beta h - r T}}{2\pi i} \int_{\text{Re } q = \sigma} dq \ e^{q T} (\phi_q^{-}(-i\beta))^{-1} \frac{1}{2\pi} \int_{\text{Im } \xi = \omega} e^{i(x-h)\xi} \frac{\phi_q^{-}(\xi)}{\beta - i\xi} d\xi. \tag{49}
\]

Assuming that we use the Gaver–Stehfest method to evaluate the Bromwich integral (the outer integral on the RHS of (49)), we need to calculate the inner integral for \( q > 0 \). As in the case of no-touch options, we deform the contour upward and cross the simple pole at \(-i\beta_q^{-} \) if \( \beta_q^{-} \) exists and \(-\beta_q^{-} \) is not close to \( \lambda_+ \):

\[
\frac{1}{2\pi} \int_{\text{Im } \xi = \omega} e^{i(x-h)\xi} \frac{\phi_q^{-}(\xi)}{\beta - i\xi} d\xi = \frac{1}{2\pi} \int_{\mathbb{L}^+} e^{i(x-h)\xi} \frac{\phi_q^{-}(\xi)}{\beta - i\xi} d\xi - \frac{qe^{\beta_q^{-}(x-h)}}{\phi_q^{-}(-i\beta_q^{-})} \psi'(-i\beta_q^{-}).
\]

6.2 | General case

Even in a simple case of the down-and-in option that, at time \( \tau \), becomes the call option with strike \( K \) and time \( T - \tau \) to maturity, \( G(\tau, X_\tau) = e^{-r\tau} V_{\text{call}}(K; T - \tau, X_\tau) \), the (discounted) payoff at time \( \tau \), is more involved than the payoff functions in Boyarchenko and Levendorski (2002a, 2009) because their pricing formulas were derived under assumption that the dependence of \( G(\tau, X_\tau) \) on \( \tau \) is of the simplest form \( e^{-r\tau} G_1(X_\tau) \). If we calculate the expectation of the product of a discounted down-and-in call option \( e^{-r\tau} V_{\text{call}}(K; T - \tau, X_\tau) \) and \( e^{-r\tau + \beta X_\tau} \), then \( G(\tau, X_\tau) = e^{-2r\tau + \beta X_\tau} V_{\text{call}}(K; T - \tau, X_\tau) \).

In
## Table 5

Pricing down-and-out call options using Gaver–Wynn–Rho method with \(M = 8\) and sinh-acceleration

| \(\ln(S/H)\) | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 | 0.11 | 0.13 |
|---------------|------|------|------|------|------|------|------|
| \(V_{\text{call}}\) | 0.013292606 | 0.029690703 | 0.045214455 | 0.0613371 | 0.078535419 | 0.096973726 | 0.116654007 |
| \(Err\) | 4.1E-10 | 3.4E-10 | -7.7E-10 | -3.5E-10 | 2.5E-09 | 1.3E-08 | -2.2E-11 |
| \(K = 1.04\) | \(T = 0.1\) |
| \(V_{\text{call}}\) | 0.021574742 | 0.047816254 | 0.071305316 | 0.093913695 | 0.11618586 | 0.138385615 | 0.1606790678 |
| \(Err\) | 6.7E-08 | 4.2E-08 | -1.1E-08 | -1.9E-08 | -4.1E-10 | -9.4E-10 | -1.4E-09 |
| \(K = 1.04\) | \(T = 0.5\) |
| \(V_{\text{call}}\) | 0.006541799 | 0.014745612 | 0.023095024 | 0.032627133 | 0.043822976 | 0.056937082 | 0.07205657 |
| \(Err\) | 8.9E-11 | 1.3E-10 | 7.3E-11 | 2.4E-10 | -1.0E-08 | 4.4E-09 |
| \(K = 1.1\) | \(T = 0.1\) |
| \(V_{\text{call}}\) | 0.017419401 | 0.03862725 | 0.057705653 | 0.076216112 | 0.094638672 | 0.113222686 | 0.1321308674 |
| \(Err\) | 2.3E-09 | 1.4E-09 | 1.3E-09 | 1.0E-09 | 6.3E-10 | 4.7E-10 | 3.5E-10 |

Note. Benchmark prices and errors for different choices of the parameters of the sinh-acceleration are rounded. The underlying: KoBoL model as in Table 2. Parameters \(m_2 = 0.1, \nu = 1.2, \lambda_- = -11, \lambda_+ = 12, \mu = 0, \sigma = 0, \epsilon = 0.3026\) (rounded), \(\beta = -1, r = 0.100\) (rounded). Strike \(K = 1.04, 1.1\), time to maturity \(T = 0.1, 0.5\), barrier \(H = 1\), spot. \(K = 1.04, T = 0.1\): meshes: \(\zeta_- = 0.1339;\) number of terms: \(N_- = 24, 32\) (for the realization of \(\Pi_\lambda\) and 1D integral in the case \(\ln(S/K) < 0\)), \(N_+ = 30, 47\) (for the iFT and 1D integral in the case \(\ln(S/K) \geq 0\)), \(N_- = 114\) and \(N_+ = 113\) (for calculation of the Wiener–Hopf factors); CPU time = 47.8 ms (average over 1,000 runs). \(K = 1.1, T = 0.1\): meshes: \(\zeta_- = 0.1753;\) number of terms: \(N_- = 23, 52\) (for the realization of \(\Pi_\lambda\) and 1D integral in the case \(\ln(S/K) < 0\)), \(N_+ = 33, 51\) (for the iFT and 1D integral in the case \(\ln(S/K) \geq 0\)), \(N_- = 124\) and \(N_+ = 1237\) (for calculation of the Wiener–Hopf factors); CPU time = 54.2 ms (average over 1,000 runs). \(K = 1.1, T = 0.5\): meshes: \(\zeta_- = 0.1378;\) number of terms: \(N_- = 37, 73\) (for the realization of \(\Pi_\lambda\) and 1D integral in the case \(\ln(S/K) < 0\)), \(N_+ = 48, 72\) (for the iFT and 1D integral in the case \(\ln(S/K) \geq 0\)), \(N_- = 205\) and \(N_+ = 203\) (for calculation of the Wiener–Hopf factors); CPU time = 87.6 ms (average over 1,000 runs).
the case of the expectation of the product of discounted prices of European options, the structure of $G(\tau, X_\tau)$ is even more involved.

First, we calculate the expectation $V(G; T; x) = \mathbb{E}^x[1_{\tau < T} G(\tau, X_\tau)]$ in the general form, and then make further steps for the special cases mentioned above. We repeat the main steps of the initial proof in Boyarchenko and Levendorskii (2002a) omitting technical details; they are the same as in Boyarchenko and Levendorskii (2002a, 2002b) and Boyarchenko et al. (2011). Let $L = -\psi(D)$ be the infinitesimal generator of $X$. Recall that the pseudo-differential operator $\psi(D)$ with the symbol $\psi$ is the composition of the Fourier transform, multiplication operator by the function $\psi$, and inverse Fourier transform. If $\tilde{u}$ is well defined and analytic in a strip, $\psi$ admits analytic continuation to the same strip, and the product $\psi(\xi)\tilde{u}(\xi)$ decays sufficiently fast as $\xi \to \infty$ remaining in the strip, an equivalent definition is $(\psi(D)\tilde{u})(\xi) = \psi(\xi)\tilde{u}(\xi)$, for $\xi$ in the strip. For details, see Eskin (1981) and Boyarchenko and Levendorskii (2002a, 2002b). The function $V_1(G; t, x) := V(G; T - t; x)$ is the bounded sufficiently regular solution of the boundary value problem

$$
(\partial_t + \psi(D))V_1(G; t, x) = 0, \quad t > 0, x > h; \quad (50)
$$

$$
V_1(G; t, x) = G(t, x), \quad t > 0, x \leq h; \quad (51)
$$

$$
V_1(G; 0, x) = 0, \quad x \in \mathbb{R}. \quad (52)
$$

Making the Laplace transform with respect to $t$, we obtain that if $\sigma > 0$ is sufficiently large, then, for all $q$ in the half-plane $\{\text{Re } q \geq \sigma\}$, $\tilde{V}_1(G; q, x)$ solves the boundary problem

$$
(q + \psi(D))\tilde{V}_1(G; q, x) = 0, \quad x > h; \quad (53)
$$

$$
\tilde{V}_1(G; q, x) = \tilde{G}(q, x), \quad x \leq h, \quad (54)
$$

in the class of sufficiently regular bounded functions. If $\{\tilde{V}_1(G; q, \cdot)\}_{\text{Re } q \geq \sigma}$ is the (sufficiently regular) solution of the family of boundary problems (53)-(54) on $\mathbb{R}$, then $V_1(G; t, x)$ can be found using the Laplace inversion formula. Finally, $V(G; t, x) = V_1(G; T - t, x)$.

The family of problems (53)-(54) is similar to the one in Boyarchenko and Levendorskii (2002a); the only difference is a more involved dependence of $\tilde{G}$ on $q$: in Boyarchenko and Levendorskii (2002a), $\tilde{G}(q, x) = G(x)/(q - r)$. Hence, we can apply the Wiener–Hopf factorization technique as in Boyarchenko and Levendorskii (2002a) and obtain

$$
\tilde{V}_1(G; q, \cdot) = \mathcal{E}_q^{-1}1_{(-\infty, h]}\left(\mathcal{E}_q^{-1}\right)^{-1}\tilde{G}(q, \cdot). \quad (55)
$$

Let $\omega \in (0, \lambda_+)$. If $\sigma > 0$ is sufficiently large, then, for $q$ in the half-plane $\{\text{Re } q \geq \sigma\}$, $\phi_q^{-}(\xi)$ is analytic in the half-plane $\{\text{Im } \xi \leq \omega\}$. For $\xi$ in this half-plane, the double Laplace–Fourier transform of $V_1(G; t, x)$ with respect to $(t, x)$ is given by

$$
\hat{V}_1(G; q, \xi) = \phi_q^{-}(\xi)1_{(-\infty, h]}\hat{W}(q, \cdot)(\xi), \quad (56)
$$

where $W(q, \cdot) = (\mathcal{E}_q^{-1})^{-1}\tilde{G}(q, \cdot)$, and $1_{(-\infty, h]}\hat{W}(q, \cdot)(\xi)$ is the Fourier transform of $1_{(-\infty, h]}W(q, \cdot)$.
We take $\omega' < -\beta$, and, similarly to the proof of Lemma 5.1, calculate

$$1_{(-\infty,h]} W(q, \cdot)(\xi) = \int_{\text{Im} \eta = \omega'} \phi_q^{-1}(\eta) \hat{G}(q, \eta)(2\pi)^{-1} \int_{-\infty}^{h} e^{i(\eta-\xi)x} \, dx \, d\eta.$$  

The result is: for $(q, \xi)$ s.t. $\text{Re} \, q \geq \sigma$ and $\text{Im} \, \xi > \omega'$,

$$1_{(-\infty,h]} W(q, \cdot)(\xi) = \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega'} \frac{e^{ih(\eta-\xi)} \phi_q^{-1}(\eta) \hat{G}(q, \eta)}{\eta - \xi} \, d\eta. \quad (57)$$

Applying the inverse Fourier transform, we obtain

$$\hat{V}_1(G; q, x) = \frac{1}{2\pi} \int_{\xi_0} d\xi \, e^{i(x-h)\xi} \hat{\sigma}_q^{-1}(\xi) \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega'} \frac{e^{ih\eta} \hat{G}(q, \eta)}{\phi_q^{-1}(\eta)(\eta - \xi)} \, d\eta. \quad (58)$$

As $x - h > 0$, we deform the outer contour upward, the new contour being of the type $\mathcal{L}^+$ (meaning: of the form $\mathcal{L}(\omega_1, \omega, b)$, where $\omega > 0$):

$$\hat{V}_1(G; q, x) = \frac{1}{2\pi} \int_{\mathcal{L}^+} d\xi \, e^{i(x-h)\xi} \hat{\sigma}_q^{-1}(\xi) \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega'} \frac{e^{ih\eta} \hat{G}(q, \eta)}{\phi_q^{-1}(\eta)(\eta - \xi)} \, d\eta. \quad (59)$$

Admissible types of deformation of the inner integral on the RHS of (59) depend on the properties of $e^{ih\eta} \hat{G}(q, \eta)$. If $G$ is the price of a vanilla option or the product of prices of two vanilla options, then an admissible deformation is determined by the relative position of the barrier and the strikes of the options involved. Hence, we are forced to consider several cases.

### 6.3 Down-and-in call and put options

#### 6.3.1 Call option, the strike is at or above the barrier

Consider first the down-and-in call option. As the strike is at or above the barrier, $a := \ln K \geq h = \ln H$. We must have $\lambda_- < -1$. Let $\omega' \in (\lambda_-, -1)$ and $\sigma > 0$ be such that $\text{Re}(q + \psi(\xi)) > 0$ if $\text{Re} \, q \geq \sigma$ and $\text{Im} \, \xi \in [\omega', -1)$. Then the double Laplace–Fourier transform with respect to $(t, x)$ of the discounted price $G(t, x) = e^{-r(T-t)} V_{\text{call}}(T, K; T-t, x)$ is well defined in the region $\{(q, \eta) \mid \text{Re} \, q \geq \sigma, \text{Im} \, \eta \in [\omega', -1)\}$, and it is given by

$$\hat{G}(q, \eta) = e^{-rT} \int_{0}^{+\infty} e^{-qt} \frac{K^{1-i\eta} e^{-\psi(\eta)}}{i\eta(i\eta - 1)} \, dt.$$ 

Integrating, we obtain $\hat{G}(q, \xi) = e^{-i\omega_0 \xi} \hat{G}_0(q, \xi)$, where $a = \ln K$, and

$$\hat{G}_0(q, \eta) = \frac{Ke^{-rT}}{(q + \psi(\eta))i\eta(i\eta - 1)} = \frac{Ke^{-rT}}{(q + \psi(\eta))\eta(\eta + i)}. \quad (60)$$

Under condition ($X$), for each $q > 0$, $\hat{G}_0(q, \cdot)$ is a meromorphic function in the complex plane with two cuts $i(-\infty, \lambda_-)$, $i(\lambda_+, +\infty)$ and simple poles at $0$, $-i$ (and $-i\beta_q^\pm$, if the latter exist). As $h < a$, we may deform the inner contour of integration on the RHS of (59) down. On the strength of (25),

$$\hat{G}_0(q, \eta) = \frac{Ke^{-rT}}{\phi_q^{-1}(\eta)(q + \psi(\eta))\eta(\eta + i)} = \frac{Ke^{-rT} \phi_q^+(\eta)}{q\eta(\eta + i)}.$$
TABLE 6 Hedging portfolios as bets: payoffs and probabilities (rounded)

| $\tau$  | 0.00–0.05 | 0.015–0.02 | 0.035–0.04 | 0.045–0.05 | 0.06–0.065 | 0.075–0.08 | 0.09–0.095 | >0.1 |
|---------|-----------|------------|------------|------------|------------|------------|-----------|------|
| Prob    | 0.0349    | 0.0554     | 0.0324     | 0.0250     | 0.0179     | 0.0136     | 0.0107    | 0.4339 |
| HPCall  | −2.0340   | −1.5958    | −0.9729    | −0.6423    | −0.1206    | 0.4218     | 0.9027    | 1.0101 |
| SS3     | −0.6275   | −0.6628    | −0.7030    | −0.7149    | −0.7126    | −0.6780    | −0.6298   | −0.6195 |
| SS1     | −0.9378   | −0.8109    | −0.6506    | −0.5755    | −0.4719    | −0.3840    | −0.3215   | −0.3082 |
| HP3     | −2.0075   | −1.5800    | −0.9705    | −0.6459    | −0.1322    | 0.4027     | 0.8746    | 0.9795  |
| HP($K_j$) | −1.9986   | −1.5704    | −0.9625    | −0.6401    | −0.1320    | 0.3958     | 0.8632    | 0.9674  |
| HP(1)   | −1.8971   | −1.4841    | −0.9003    | −0.5923    | −0.1100    | 0.3807     | 0.7799    | 1.0085  |
| HFTP2   | −1.3220   | −0.8995    | −0.2980    | 0.0220     | 0.5287     | 1.0582     | 1.5298    | 0.0862  |
| HFT     | −1.145    | −0.7086    | −0.0881    | 0.2413     | 0.7613     | 1.3020     | 1.7812    | −0.1526 |

Note. Bins: Time intervals when the barrier is breached or not at all (the last bin). Prob: Probability that the barrier is breached during the time interval or not breached at all (the last column). As not all events are shown, the probabilities do not sum up to 1. The other rows: Approximate payoffs of the initial bet $HPCall$ and values of hedging portfolios, in units of $V_{d.in. call}$ at initiation, if $S_\tau$ is close to the barrier. The last column: The values at maturity, if the barrier has not been breached.

$SS_j$: Semistatic portfolio of $j$ put options.

$HP_j$: Variance-minimizing portfolio constructed using put options with strikes $K_{j} = 1/04 - (j - 1) * 0.02, j = 1, 2, 3$.

$HFTP_2$: Variance-minimizing portfolio constructed using the first touch digital and put options with strikes $K_1, K_3$.

$HFT$ and $HP(K)$: Variance-minimizing portfolio constructed using the first touch digital and put option with strike $K$.

Hence, (59) becomes

$$\tilde{V}_1(G; q, x) = -\frac{Ke^{-rT}}{2\pi q} \int_{L^+} d\xi e^{i(x-h)\xi} \phi_q^+(\xi) \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega'} \frac{e^{i(h-a)\eta} \phi_q^+(\eta)}{\eta(\eta + i)(\eta - \xi)} d\eta. \quad (61)$$

6.3.2 | Put option, the strike is at or above the barrier

In the case of the put option, we have (59) with $\omega' \in (0, \omega)$. Hence, deforming the contours of integration with respect to $\eta$ down into $L^{--}$, we cross not only a simple pole at $-i\beta_q^+$ but simple poles $0, -i$ as well. Hence, we need to add the corresponding residue terms. We omit straightforward calculations in order to save space (the details are available upon request).

6.3.3 | Call option, the strike is below the barrier

We start with the contour $\{\text{Im} \eta = \omega'\}$, where $\omega' \in (-\beta_q^+, -1)$. As $a < h$, we need to deform the inner contour of integration on the RHS of (59) up. As $\phi_q^+(\eta)$ is analytic and bounded in the half-plane $\{\text{Im} \eta \geq \omega'\}$, the inner integrand on the RHS of (59) has three simple poles at $\eta = -i, 0, \xi$; after the poles are crossed, we add the residue terms and move the line of integration up to infinity.

6.3.4 | Put option, the strike is below the barrier

Evidently, the price is the same as the one of the European put.

6.4 | The case of the product of two European call or put options

In the case of calls, we have

$$G(t, x) = e^{-2rT} e^{2rt} V_{\text{call}}(T, K_1; T - t, x) V_{\text{call}}(T, K_2; T - t, x),$$
TABLE 7 Semistatic and mean-variance hedging of a down-and-in call option $V_{d,in,call}(T, K_0; S)$, using European put options $V_{put}(T, K_j; S)$, $K_j = 1/(1.04 - (j - 1) \times 0.02, j = 1, 2, 3$, and the first touch digital

| $\ln(S/H)$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 |
|------------|------|------|------|------|------|------|------|
| $nStd$     |      |      |      |      |      |      |      |
| $d.~in~call$ | 2.144 | 2.585 | 3.069 | 3.612 | 4.226 | 4.925 | 5.721 |
| SS3        | 8.173 | 9.067 | 10.32 | 11.84 | 13.58 | 15.67 | 18.05 |
| HP3        | 2.068 | 2.541 | 3.024 | 3.558 | 4.146 | 4.843 | 5.619 |
| HFTP2      | 2.027 | 2.471 | 2.938 | 3.461 | 4.049 | 4.732 | 5.507 |
| HFT        | 2.067 | 2.481 | 2.944 | 3.467 | 4.062 | 4.742 | 5.521 |
| HP($K_1$)  | 2.129 | 2.574 | 3.058 | 3.599 | 4.211 | 4.907 | 5.700 |
| HP(1)      | 2.144 | 2.585 | 3.069 | 3.612 | 4.226 | 4.925 | 5.721 |

Weights

| $w(K_1)$ | 0.080 | 0.088 | 0.095 | 0.101 | 0.106 | 0.112 | 0.117 |
| $w(K_2)$ | -0.658 | -0.720 | -0.768 | -0.811 | -0.852 | -0.891 | -0.931 |
| $w(K_3)$ | 0.926 | 1.006 | 1.066 | 1.119 | 1.170 | 1.219 | 1.269 |
| $w(K_1)$ | 0.105 | 0.114 | 0.121 | 0.127 | 0.132 | 0.138 | 0.143 |
| $w(K_2)$ | -0.160 | -0.175 | -0.186 | -0.197 | -0.207 | -0.216 | -0.226 |
| $w(K_3)$ | 0.095 | 0.098 | 0.101 | 0.105 | 0.109 | 0.113 | 0.117 |

Note. Standard deviations $nStd$ and weights $w$ of options in hedging portfolios are in units of the price of the down-and-in call option. The underlying: KoBoL, parameters $m_2 = 0.1, \nu = 12, \lambda_- = -11, \lambda_+ = 12, \mu = 0, \sigma = 0, \varsigma = 0.3026$ (rounded), $r = 0.100$ (rounded). Time to maturity $T = 0.1$, strike $K_0 = 1.04$, barrier $H = 1, S$ the spot.

where $K_j \geq H$. Set $a_j = \ln K_j$. Assuming that $\lambda_- < -2$, we take $\omega_1, \omega_2 \in (\lambda_- + 1, -1)$, and calculate, first, the Laplace transform of

$$G_1(t, x) := e^{\cdot r t} V_{call}(T, K_1; T - t, x) V_{call}(T, K_2; T - t, x)$$

$$= \frac{1}{(2\pi)^2} \int_{\Im \eta_1 = \omega_1} d\eta_1 e^{ix\eta_1 - t\psi(\eta_1)} \frac{K_1^{1 - i\eta_1}}{i\eta_1(i\eta_1 - 1)} \int_{\Im \eta_2 = \omega_2} d\eta_2 e^{ix\eta_2 - t\psi(\eta_2)} \frac{K_2^{1 - i\eta_2}}{i\eta_2(i\eta_2 - 1)}.$$

Applying Fubini’s theorem, we have

$$\hat{G}_1(q, x) = \frac{K_1 K_2}{(2\pi)^2} \int_{\Im \eta_1 = \omega_1} \int_{\Im \eta_2 = \omega_2} e^{(i\eta_1 + i\eta_2)x} e^{-ia_1\eta_1 - ia_2\eta_2} d\eta_1 d\eta_2$$

$$\left( q + \psi(\eta_1) + \psi(\eta_2) \right) i\eta_1(i\eta_1 - 1)i\eta_2(i\eta_2 - 1).$$
Using \((\mathcal{E}_q)^{-1}e^{i(\eta_1 + \eta_2)x}\), we calculate \((\mathcal{E}_q)^{-1}\tilde{G}_1(q, x)\) as

\[
\frac{K_1 K_2}{(2\pi)^2} \int_{\text{Im}\eta_1 = \omega_1} \int_{\text{Im}\eta_2 = \omega_2} e^{i(\eta_1 + \eta_2)x}e^{-i a_1 \eta_1 - ia_2 \eta_2} d\eta_1 d\eta_2.
\]

Next, the Fourier transform of \(1_{(\sim \infty, h)}(\eta)\) is well defined in the half-plane \(\{\text{Im}\, \xi > -\beta\}\) by \(e^{i(\beta - i\xi)}/(\beta - i\xi)\), therefore, taking \(\omega \in (0, -\beta_q^-)\), we can represent

\[
\tilde{V}_1(G; q, x) := \mathcal{E}_q^{-1}1_{(\sim \infty, h)}(\mathcal{E}_q^{-1}\tilde{G}_1(q, x))
\]

in the form

\[
\tilde{V}_1(G; q, x) = \frac{K_1 K_2}{(2\pi)^2} \int_{\text{Im}\xi = \omega} d\xi \int_{\text{Im}\eta_1 = \omega_1} \int_{\text{Im}\eta_2 = \omega_2} e^{-i(a_1 - \eta_1 - a_2 - \eta_2)\eta_1} \phi_q^{-1}(\eta_1 + \eta_2) d\eta_1 d\eta_2.
\]

where \(\Phi(q, \xi; \eta_1, \eta_2) = \phi_q^{-1}(\eta_1 + \eta_2)i(\eta_1 + \eta_2 - \xi)(q + \psi(\eta_1) + \psi(\eta_2)).\)

Products of two puts and a put and call products are considered similarly; only the lines of integrations change. In some cases, the sinh-acceleration cannot be applied; then we use the flat iFT and simplified trapezoid rule with the following trick.

## 6.5 Summation by parts

If \(a \neq 0\) and \(f'(y)\) decreases faster than \(f(y)\) as \(y \to \pm \infty\) (as is the case in the setting of the present paper), then the finite differences \(\Delta f_j = f_{j+1} - f_j\) decay faster than \(f_j\) as \(j \to \pm \infty\) as well. Hence, choosing \(\zeta\) so that \(e^{-i\alpha\zeta} \neq 1\), the number of terms in the infinite trapezoid significantly decreases after the summation by parts

\[
\zeta \sum_{j \in \mathbb{Z}} e^{-i\alpha\zeta} f_j = \frac{\zeta}{1 - e^{-i\alpha\zeta}} \sum_{j \in \mathbb{Z}} e^{-i\alpha\zeta} \Delta f_j.
\]

If each differentiation increases the rate of decay, then the summation by part procedure can be iterated. In the setting of the present paper, the rate of decay increases by approximately 1 with each differentiation. In the numerical examples, we apply the summation by parts three times, which decreases the number of terms many times, and makes the number comparable to the number of terms when the sinh-acceleration can be applied.

## 7 SEMISTATIC HEDGING VERSUS VARIANCE MINIMIZING HEDGING OF DOWN-AND-IN OPTIONS: A NUMERICAL EXAMPLE AND QUALITATIVE ANALYSIS

In this section, we present and discuss in detail several important observations and practically important conclusions using an example of a down-and-in call option. The process (KoBoL) is the same as in Table 1, maturity is \(T = 0.1\), but the strike \(K = 1.04\) is farther from the barrier than in Table 1. The reason for that is twofold: (a) to show that if the restrictive formal conditions for the semistatic hedging are satisfied, then the semistatic procedure works reasonably well for jump processes with rather slowly-decaying jump densities even if the distance from the barrier to the support of the artificial exotic payoff is sizable (about 8%); (b) if this support is too close to the barrier, then the
FIGURE 3  The value of the approximate semistatic hedging portfolio for 1 short down-and-in call option of maturity $T = 0.1$ and strike $K = 1.04$, in units of $V_{\text{d.in.call}}(T; K; 0, S_0)$, where $S_0 = e^{0.04}$, if the barrier $H = 1$ is breached at time $\tau$, and $S_\tau = S$, without counting the riskless bond component. Hedging instruments: Put options of the same maturity, with strikes $K_1 = 1/1.04$, $K_2 = K_1 - 0.02$, $K_3 = K_1 - 0.04$. Weights $w = [1.305, -0.811, 1.119]$. [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 4  The value of the approximate semistatic hedging portfolio for 1 short down-and-in call option of maturity $T = 0.1$ and strike $K = 1.04$, in units of $V_{\text{d.in.call}}(T; K; 0, S_0)$, where $S_0 = e^{0.04}$, if the barrier $H = 1$ is breached at time $\tau$, and $S_\tau = S$. The riskless bond component is taken into account. Hedging instruments: Put options of the same maturity, with strikes $K_1 = 1/1.04$, $K_2 = K_1 - 0.02$, $K_3 = K_1 - 0.04$. Weights $w = [1.305, -0.811, 1.119]$. [Color figure can be viewed at wileyonlinelibrary.com]
The value of the variance-minimizing hedging portfolio for 1 short down-and-in call option of maturity $T = 0.1$ and strike $K = 1.04$, in units of $V_{d.in.call}(T; K; 0, S_0)$, where $S_0 = e^{0.04}$, if the barrier $H = 1$ is breached at time $\tau$, and $S_\tau = S$. The riskless bond component is taken into account. Hedging instruments: First touch digital of the same maturity, weight $w = 1.151$ [Color figure can be viewed at wileyonlinelibrary.com]

summation-by-part procedure can be insufficiently accurate unless high precision arithmetic is used. In the paper, we do calculation with double precision.

We consider the standard situation: an agent sells the down-and-in call option, and invests the proceeds into the riskless bond. We assume that the spot $S_0 = e^{0.04}$ is almost at the strike. That is, the agent makes the bet that the barrier will not be breached during the lifetime of the option. Using the pricing procedure for the no-touch options with $r = 0$, we find that the probability of this happy outcome is not very large; even the probability that the barrier will be breached before $\tau = 0.05$ is about 42%. However, with the probability about 50%, at the time the portfolio is breached, the portfolio value is positive. This is partially due to the fact the bond component in the portfolio increases fairly fast. Nevertheless, if the barrier is breached at time close to 0, the loss in the portfolio value can be rather sizable. Hence, it is natural for the agent to hedge the bet.

Table 6 illustrates the bet structure implicit in different hedging portfolios, showing the approximate payoffs in the vicinity of the barrier when the barrier is breached and the payoff at maturity if the barrier is not breached. We see that the replacement of the initial bet (the naked short down-and-in call option) with the portfolios based on the semistatic argument may lead to a sizable loss with probability more than 90%. At the same time, there is a small probability of a significant gain, if the barrier is breached by a large jump. In our opinion, Tables 6 and 7 demonstrate that the most efficient hedge is with the first touch digital.

Assume that the agent uses the semistatic hedging constructed in the paper. The standard static and semistatic arguments lead to model- and spot-invariant portfolios, which make it impossible to take into account that a hedging portfolio needs to be financed. If we consider the portfolio of three put options, and ignore the riskless bonds borrowed to finance the position, then, at any time $\tau$ the barrier is breached, and at any level $S_\tau \leq H$, the portfolio value is positive or very close to zero (Figure 3). Thus, the semistatic hedge seems to work very well even if the portfolio consists of only three options. Furthermore, if only one option is used, then the portfolio value decreases except in a relatively small
region far from the barrier and close to maturity (we omit the illustration in order to save space). As the probability of the option expiring in that region is very small, a naive argument would suggest that increasing the number of options in the hedging portfolio would increase the overall hedging performance of the portfolio. Recall, however, that the agent borrows riskless bonds to finance the put option position. If the barrier is breached, this short position in the riskless bond has to be liquidated alongside the other positions in the portfolio, complicating the overall picture. Figure 4 demonstrates that, when $S_\tau$ is close to the barrier (a high probability event conditional on breaching the barrier), the value of the hedging portfolio is negative and large in absolute value. Thus, the hedge is far from perfect. Figure 5 shows that a hedging portfolio with just one first touch digital perform approximately as well as portfolios with three options; if the barrier is not crossed very soon, the former is significantly better than the latter.

8 | CONCLUSION

We developed new methods for constructing static hedging portfolios for European exotics and variance-minimizing hedging portfolios for European exotics and barrier options in Lévy models; in both cases, the hedging portfolios use options available in the market, and no approximation procedures that introduce additional errors are required. In particular, we constructed static portfolios for exotic options approximating an exotic payoff with linear combinations of vanillas available in the market, in the norm of the Sobolev space with an appropriate weight. The order of the Sobolev space is chosen so that the space of continuous functions with the same weight is continuously embedded into the weighted Sobolev space, hence, we obtain an approximation in the $C$-norm. The weighted Sobolev space is the Hilbert space, and the scalar products of the elements of this space can be easily calculated evaluating integrals in the dual space.

We have discussed the limitations of the static hedging/replication of barrier options, and, in applications to Lévy models, listed rather serious restrictions on the parameters of the model under which the approximate replication of a barrier option with an appropriate European exotic option can be justified. We explained why in the presence of jumps a perfect semistatic hedging is impossible, and, using an example of the down-and-in call option in KoBoL model, demonstrated that the use of variance-minimizing portfolios with the first touch digitals has certain advantages, and the first-touch options with the payoffs $(S/H)^\gamma$, $\gamma > 0$, would be even better hedging instruments.

The results the paper suggest that it might be natural to regard semistatic hedging portfolios as separate classes of derivative securities, with nontrivial payoff structures, which are model dependent. The numerical examples in the paper demonstrate that the properties of the payoffs of semistatic hedging portfolios for short down-and-out options are fundamentally different from the properties of semistatic hedging portfolios for short down-and-in options. For the process and down-options considered in the paper, the semistatic hedging portfolios for down-and-out options do have properties close to the properties of good hedging portfolios: with high probability, the payoff at expiry (or maturity) is positive although with small probability the payoff is negative. However, for the down-and-in options, the properties are the opposite: small losses with high probability and large gains with small probability (essentially, contrarian bets). For up-options under the same process, the semistatic hedging portfolios for “in” options are good but the ones for “out” options are contrarian bets. The properties change with the change of the sign of the drift as well. For double barrier options, the picture becomes even more involved.

To conclude, the formal semistatic argument is applicable only under rather serious restrictions, cannot be exact in the presence of jumps, may lead to very risky portfolios, and the losses of the
variance-minimizing portfolios can be sizable as well (although, in about 50% cases, much smaller than the errors of the formal semistatic hedging portfolios). The deficiencies of both types of the hedging portfolios stem from the fact that both are constructed once and for all, and liquidated at $\tau \land T$. If we agree that a model-independent hedging is seriously flawed (the semistatic hedging disregards the cost of hedging), and the variance-minimizing one does not take into account explicitly the payoff of the portfolio at the time of breaching, then a natural alternative is a hedging portfolio that is rebalanced after reasonable short time intervals so that the profile of the payoff at an uncertain moment of breaching is approximately equal to the profile at the moment of the next rebalancing. The portfolio can be calculated using the approximation in the Sobolev space norm, and the following versions seems to be natural:

1. Myopic hedging, when the hedging instruments expire at the moment of the next rebalancing.
2. Quasi-static hedging, when the first hedging portfolio is constructed using the options of the same maturity as the barrier option, and the other options are added to the existing portfolio at each rebalancing moment.
3. Hedging using first touch digitals only. Possible versions: each period, buy the digital that expires at the end of the rebalancing period; each period, buy first touch digitals that expire at the maturity date.

We leave the study of these versions of hedging to the future.

The construction of the variance-minimizing portfolio for barrier options is based on the novel numerical methods for evaluation of the Wiener–Hopf factors and pricing barrier options that are of a more general interest than applications to hedging. In particular, the Wiener–Hopf factors can be evaluated with the relative error less than E-12 and barrier options with the relative error less than E-08 in a fraction of a millisecond and several milliseconds, respectively. The efficient evaluation of the Wiener–Hopf factors, and, in many cases, the numerical realization of the inverse Fourier transform in the option pricing formulas, are based on the sinh-acceleration method of evaluation of integrals of wide classes, highly oscillatory ones especially, developed in Boyarchenko and Levendorskiı (2019), and the summation-by-parts in the infinite trapezoid rule.

ACKNOWLEDGMENTS

The authors are grateful to Nina Boyarchenko for the comments on the qualitative effects of semistatic hedging described in the paper. The first author is grateful to the participants of Mini workshop on Mathematical Finance, May 20, 2019, Adam Smith Business School, University of Glasgow, Glasgow, UK and to Peter Carr especially for valuable discussions. The second author is grateful to the participants of SIAMFM 2019, June 4–7, 2019, Toronto, and AMaMeF, June 11–14, Paris, for useful comments. The authors are grateful to an anonymous referee and associated editor for the suggestion to streamline and shorten the paper. The remaining errors are ours.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

ENDNOTES

1 See Derman et al. (1995), Carr and Chou (1997), Carr et al. (1998), Carr and Madan (2001), Andersen, Andreasen, and Eliezer (2002), Hirsa, Courtadon, and Madan (2002), Tompkins (2002), Albrecher, Dhaene, Goovaerts, and Schoutens
Recall that even a small asymmetry of the jump component requires the riskless rate be sizable in order that the semistatic procedure be formally justified.

REFERENCES

Albrecher, H., Dhaene, J., Goovaerts, M., & Schoutens, W. (2005). Static hedging of Asian options under Lévy models. *Journal of Derivatives*, 12(3), 63–72.

Andersen, L., Andreasen, J., & Eliezer, D. (2002). Static replication of barrier options: Some general results. *Journal of Computational Finance*, 5(1), 1–25.

Barndorff-Nielsen, O. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2, 41–68.

Barndorff-Nielsen, O., & Levendorski, S. (2001). Feller processes of normal inverse Gaussian type. *Quantitative Finance*, 1, 318–331.

Boyarchenko, M., de Innocentis, M., & Levendorski, S. (2011). Prices of barrier and first-touch digital options in Lévy-driven models, near barrier. *International Journal of Theoretical and Applied Finance*, 14(7), 1045–1090.

Boyarchenko, M., & Levendorski, S. (2009). Prices and sensitivities of barrier and first-touch digital options in Lévy-driven models. *International Journal of Theoretical and Applied Finance*, 12(8), 1125–1170.

Boyarchenko, M., & Levendorski, S. (2015). Ghost calibration and pricing barrier options and credit default swaps in spectrally one-sided Lévy models: The parabolic laplace inversion method. *Quantitative Finance*, 15(3), 421–441.

Boyarchenko, S., & Levendorski, S. (2000). Option pricing for truncated Lévy processes. *International Journal of Theoretical and Applied Finance*, 3(3), 549–552.

Boyarchenko, S., & Levendorski, S. (2002a). Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Annals of Applied Probability*, 12(4), 1261–1298.

Boyarchenko, S., & Levendorski, S. (2002b). *Non-Gaussian Merton-Black-Scholes theory*, Vol. 9. Advanced Series on Statistical Science & Applied Probability. River Edge, NJ: World Scientific Publishing Co.

Boyarchenko, S., & Levendorski, S. (2002c). Perpetual American options under Lévy processes. *SIAM Journal on Control and Optimization*, 40(6), 1663–1696.

Boyarchenko, S., & Levendorski, S. (2013). Efficient Laplace inversion, Wiener–Hopf factorization and pricing lookbacks. *International Journal of Theoretical and Applied Finance*, 16(3). https://doi.org/10.1142/S0219024913500118.

Boyarchenko, S., & Levendorski, S. (2014). Efficient variations of Fourier transform in applications to option pricing. *Journal of Computational Finance*, 18(2), 57–90.

Boyarchenko, S., & Levendorski, S. (2019). Sinh-acceleration: Efficient evaluation of probability distributions, option pricing, and Monte-Carlo simulations. *International Journal of Theoretical and Applied Finance*, 22(3). http://doi.org/10.1142/S0219024919500110.

Carr, P., & Chou, A. (1997). Breaking barriers. *Risk Magazine*, 10(9), 139–145.

Carr, P., Ellis, K., & Gupta, V. (1998). Static hedging of exotic options. *Journal of Finance*, 53(3), 1165–1190.

Carr, P., Geman, H., Madan, D., & Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business*, 75, 305–332.

Carr, P., & Lee, R. (2009). Put call symmetry: Extensions and applications. *Mathematical Finance*, 19(4), 523–560.

Carr, P., & Madan, D. (2001). Optimal positioning in derivative securities. *Quantitative Finance*, 1(1), 19–37.

Carr, P., & Wu, L. (2014). Static hedging of standard options. *Journal of Financial Econometrics*, 12(1), 3–46.

Cont, R., Bouchaud, J.-P., & Potters, M. (1997). Scaling in financial data: Stable laws and beyond. In B. Dubrulle, F. Graner, & D. Sornette (Eds.), *Scale invariance and beyond* (pp. 75–85). Berlin: Springer.

de Innocentis, M., & Levendorski, S. (2014). Pricing discrete barrier options and credit default swaps under Lévy processes. *Quantitative Finance*, 14(8), 1337–1365. http://doi.org/10.1080/14697688.2013.826814.

Derman, E., Ergener, D., & Kani, I. (1995). Static options replication. *Journal of Derivatives*, 2(1), 78–95.

Eberlein, E., & Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli*, 1, 281–299.

Eskin, G. (1981). *Boundary value problems for elliptic pseudodifferential equations*, Vol. 9. Translations of Mathematical Monographs. Providence, RI: American Mathematical Society.
Fang, F., Jönsson, H., Oosterlee, C., & Schoutens, W. (2010). Fast valuation and calibration of credit default swaps under Lévy dynamics. *Journal of Computational Finance, 14*(2), 57–86.

Fang, F., & Oosterlee, C. (2008). A novel pricing method for European options based on Fourier-Cosine series expansions. *SIAM Journal on Scientific Computing, 31*(2), 826–848.

Fang, F., & Oosterlee, C. (2009). Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions. *Numerische Mathematik, 114*(1), 27–62.

Hirsa, A., Courtadon, G., & Madan, D. (2002). The effect of model risk on the valuation of barrier options. *Journal of Risk Finance, 4*, 47–55.

Kirkby, J. (2015). Efficient option pricing by frame duality with the fast Fourier transform. *SIAM Journal on Financial Mathematics, 6*(1), 713–747.

Kirkby, J., & Deng, S. (2019). Static hedging and pricing of exotic options with payoff frames. *Mathematical Finance, 29*(2), 612–658.

Kou, S. (2002). A jump-diffusion model for option pricing. *Management Science, 48*(8), 1086–1101.

Kuznetsov, A. (2010). Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. *Annals of Applied Probability, 20*(5), 1801–1830.

Levendorskiĭ, S. (2002). Pricing of the American put under Lévy processes. Research Report, MaPhySto, Aarhus. Retrieved from http://www.maphysto.dk/publications/MPS-RR/2002/44.pdf, http://www.maphysto.dk/cgi-bin/gp.cgi?publ=441.

Levendorskiĭ, S. (2004). Pricing of the American put under Lévy processes. *International Journal of Theoretical and Applied Finance, 7*(3), 303–335.

Levendorskiĭ, S. (2014). Method of paired contours and pricing barrier options and CDS of long maturities. *International Journal of Theoretical and Applied Finance, 17*(5), 1–58.

Levendorskiĭ, S. (2016). Pitfalls of the Fourier transform method in Affine models, and remedies. *Applied Mathematical Finance, 23*(2), 81–134.

Lipton, A. (2002). Assets with jumps. *Risk, 15*(9), 149–153.

Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics, 3*, 125–144.

Nalholm, M., & Paulsen, R. (2006). Static hedging and model risk for barrier options. *Journal of Futures Markets, 26*(5), 449–463.

Rogers, L., & Williams, D. (1994). *Diffusions, Markov processes, and martingales*, Vol. 1 (2nd ed.). Foundations. Chichester: John Wiley & Sons.

Tompkins, R. (2002). Static versus dynamic hedging of exotic options: An evaluation of hedge performance via simulation. *Journal of Risk Finance, 3*(2), 6–34.

---

**How to cite this article:** Boyarchenko S, Levendorskiĭ S. Static and semistatic hedging as contrarian or conformist bets. *Mathematical Finance.* 2020;30:921–960. https://doi.org/10.1111/mafi.12240

---

**APPENDIX: STANDARD SEMISTATIC HEDGING UNDER LÉVY PROCESSES**

We consider the down-and-in option the payoff function $G$ at maturity date $T$, with the barrier $H = e^h$. Let $\tau := \tau_h$ be the first entrance time by the Lévy process $X$ with the characteristic exponent $\psi$ into $(-\infty, h]$. If $\tau < T$, then, at time $\tau$, the option becomes the European option with the payoff $G(X_\tau)$ at maturity date $T$. Consider the European option $V(G_{ex}; t, X_t)$ of maturity $T$, with the payoff function

$$G_{ex}(x) = (G(x) + e^{\beta(x-h)}G(2h-x))1_{(-\infty,h]}(x). \quad (A.1)$$
Lemma A.1. Let there exists $\beta \in \mathbb{R}$ such that, for any stopping time $\tau$,

$$
\mathbb{E}_\tau \left[ G(X_T)1_{X_T > h} \right] = \mathbb{E}_\tau \left[ e^{\beta(X_T - X_\tau)} G(2X_T - X_\tau)1_{2X_\tau - X_T > h} \right],
$$

(A.2)

and let $X_{\tau_h} = h$. Then the values of the down-and-in and exotic options coincide.

Proof. If $\tau_h > T$, the down-and-in option and the European option expire worthless. Hence, it suffices to prove the equality of the options values at $\tau_h \leq T$. As $X_{\tau_h} = h$, $1_{2X_{\tau_h} - X_T > h} = 1_{X_T < h}$, and using (A.2), we obtain

$$
\mathbb{E}_{\tau_h} \left[ G_{ex}(X_T) \right] = \mathbb{E}_{\tau_h} \left[ (G(X_T) + e^{\beta(X_T - X_\tau)} G(2X_\tau - X_T))1_{X_T < h} \right]
$$

$$
= \mathbb{E}_{\tau_h} \left[ G(X_T)1_{X_T < h} \right] + \mathbb{E}_{\tau_h} \left[ e^{\beta(X_T - X_\tau)} G(2X_\tau - X_T)1_{2X_\tau - X_T > h} \right]
$$

$$
= \mathbb{E}_{\tau_h} \left[ G(X_T)1_{X_T < h} \right] + \mathbb{E}_{\tau_h} \left[ G(X_T)1_{X_T > h} \right] = \mathbb{E}_{\tau_h} \left[ G(X_T) \right].
$$

Note that the assumption $X_\tau = h$ means that there are no jumps down. In a moment, we will show that then there are no jumps up as well. Let $G(x) = (e^x - K)_+$ or, more generally, let $\hat{G}(\xi)$ be well defined in the half-plane $\{\text{Im} \xi < -1\}$ and decay as $|\xi|^{-2}$ as $\xi \to \infty$ remaining in this half-plane. Let $\psi$ be analytic in a strip $S(\lambda_-, \lambda_+)$, where $\lambda_- < -1$. Take $\omega \in (\lambda_-, -1)$, denote $x = X_\tau$, and represent the LHS of (A.2) in the form

$$
\mathbb{E}_\tau \left[ G(X_T)1_{X_T > h} \right] = \frac{1}{2\pi} \int_{\text{Im} \xi = \omega} e^{i\xi x - (r + \psi(\xi))(T - \tau)} G(1_{h, +\infty})(\xi) d\xi.
$$

(A.3)

The RHS of (A.2) can be represented as the repeated integral

$$
\frac{1}{2\pi} \int_{\text{Im} \xi = \omega'} d\xi' e^{i\xi' x - (r + \psi(\xi'))(T - \tau)} \int_{\mathbb{R}} dy e^{-iy\xi'} e^{\beta(y - x)} G(1_{2x - y > h})
$$

if $\omega' \in \mathbb{R}$ can be chosen so that the repeated integral converges (this imposes an additional condition on $\psi$, which will be made explicit in a moment). Changing the variable $2x - y = y'$, and then $-\xi' - i\beta = \xi''$, we see that the RHS of (A.2) equals

$$
\frac{1}{2\pi} \int_{\text{Im} \xi'' = -\omega' - \beta} e^{i\xi'' x - (r + \psi(-\xi'') - i\beta)(T - \tau)} G(1_{h, +\infty})(\xi'') d\xi''.
$$

Comparing with (A.3), we have in order that (A.2) hold for any $G$, $\omega'$, and $\beta$ must satisfy $-\omega' - \beta \in (\lambda_-, -1)$, and the characteristic exponent $\psi$ and $\beta$ must satisfy

$$
\psi(\xi) = \psi(-\xi - i\beta).
$$

(A.4)

If $X$ is the BM with drift $\mu$ and volatility $\sigma$, then (A.4) is equivalent to $\beta = -\mu/(2\sigma^2)$. If $\beta = -1$, $\psi(-i) = 0$ (the stock is a martingale), hence, $\delta = r$ (the dividend rate equals the riskless rate). In the case of the BM with embedded KoBoL component:

$$
\psi(\xi) = \frac{\sigma^2}{2} \xi^2 - i\mu\xi + c_+ \Gamma(-\nu_+) \left( \lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+} \right) + c_- \Gamma(-\nu_-)(-\lambda_-)^{-\nu_-} - (-\lambda_- - i\xi)^{-\nu_-},
$$

the conditions become: (a) $c_+ = c_-$ (hence, either there are no jumps or there are jumps in both directions), and $\nu_+ = \nu_-;$ (b) $\beta = -\mu/(2\sigma^2) = -\lambda_+ - \lambda_-;$ and (c) $\psi(-i) + r - \delta = 0$. We see that if there is no diffusion components, then the “drift” $\mu = 0$, and if there is a diffusion component, and $\mu > 0$
(respectively, \( \mu < 0 \)), then \( \lambda_+ > -\lambda_- \) (respectively, \( \lambda_+ < -\lambda_- \)), which means that the density of jumps decays slower in the direction of the drift. We finish this section with a discussion of a possible size of hedging errors induced by the assumption that the process does not cross the barrier by a jump when, in fact, it does. In the case of the down-and-in options the expected size of the overshoot decreases when \( \lambda_+ \) increases; in the case of up-and-in options, \( -\lambda_- \) increases in absolute value. Hence, if the diffusion component is sizable: \( \sigma^2 > 0 \) is not small, then the condition \( \beta = -\mu/(2\sigma^2) = -\lambda_+ - \lambda_- \) implies a strong symmetry \( \lambda_+ \approx -\lambda_- \) of the positive and negative jump components.