Interplay between strong and weak measurement

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Weak values are traditionally obtained using a weak interaction between the measured system and a pointer state. In this paper, we show that weak values can also be measured using strong interaction accompanied by either a suitably prepared pointer state or quantum erasure. Presented theoretical derivations prove analytical equivalence of these approaches. Moreover, we have performed an experimental verification of our model on a linear-optical controlled phase gate. Our results open new ways of performing non-invasive quantum measurements without collapsing the measured system.

I. INTRODUCTION

The concept of weak values was proposed by Aharonov, Albert and Vaidman in 1988 [1]. Although a subject of some controversy [2, 3] and vivid scientific debate [4–14], weak values allow acquiring information on a quantum state without causing its collapse. With the usage of weak interaction, we are able to measure qubit without destroying it [15], directly measure the quantum wave function [16], observe trajectory in quantum systems in the semiclassical regime [17], study three-box paradox [18], determine the past of photons passing through an interferometer [14] and even amplify the nonlinear effect [19]. Typically, weak value measurement requires an infinitesimally weak interaction between the measured system and a pointer state [20–23]. Such interaction, however, might not always be available or practical. For instance, until the introduction of tunable controlled phase gate designed by Lanyon et al. [24], weak controlled-phase interaction was not available on the linear-optical platform and even now requires interferometric setup. It is therefore interesting to replace the weak interaction with another one. Quite surprisingly, weak interaction can be replaced e.g. by quantum cloning [25] as shown by Hofmann in 2012.

In this paper, we show how to replace weak interaction by a strong one accompanied by a suitable strategy. The first strategy is to use a pointer state weakly sensitive to the interaction. An operationally similar measurement was presented by Pryde et al. in 2005 [26]. In this experiment, however, the authors claimed to achieve weak interaction and followed the standard formalism of weak value measurement. We, on the other hand, present a detailed theoretical analysis of this procedure and show its mathematical equivalence to the standard weak interaction approach. The second strategy is to apply a quantum erasure to the pointer state after the strong interaction. Again, we prove its mathematical equivalence to the standard weak value measurement. Moreover, we experimentally compare our approaches to the approach based on a genuine weak interaction to verify their working principle.

II. THEORETICAL FRAMEWORK

In this section we prove that performing weak value measurement is possible by applying two seemingly contradictory strategies, i.e., weak and strong interactions. We demonstrate this on an example of two interacting spin-$\frac{1}{2}$ particles (two qubits). For a single spin-$\frac{1}{2}$ particle we can express all possible unitary operations by means of generators of the associated unitary group, i.e., Pauli operators, labelled $X = |+\rangle \langle +| - |−\rangle \langle −|$, $Y = |L\rangle \langle L| - |R\rangle \langle R|$, and $Z = |0\rangle \langle 0| - |1\rangle \langle 1|$. The eigenstates of $Z$ operator, labelled $|0\rangle$ and $|1\rangle$, correspond to various physical states depending on the particular two-level system. The eigenstates of the remaining operators are related in the following way: $|±\rangle = (|0\rangle ± |1\rangle)/\sqrt{2}$, $|L\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$, and $|R\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$.

A. Weak measurement by weak interaction

We start by reviewing the typical formalism of weak value measurement by means of a weak interaction. Let us assume that $|\Psi_i\rangle$ is the overall input state

$$|\Psi_i\rangle = |\psi_i\rangle |\phi\rangle$$

(1)

consisting of the measured system $|\psi_i\rangle$ and a pointer state $|\phi\rangle$. Let the interaction Hamiltonian between these two subsystems be specified as

$$H = A \otimes Z,$$

(2)
where $A$ and $Z$ act on the measured system and the probe respectively. The evolution induced by time-independent Hamiltonian can be represented as $e^{-itH}$. We now assume that the interaction between the measured system and the probe is sufficiently weak so that the evolution operator can be approximated as

$$U_w = e^{-it\frac{\mathcal{B}}{\hbar}} \approx 1 - \frac{itH}{\hbar}. \quad (3)$$

Let us consider post-selection on the final state of the weakly measured system

$$|\psi_f\rangle \langle \psi_f| e^{-it\frac{\mathcal{B}}{\hbar}} |\psi_i\rangle \approx |\psi_f\rangle \langle \psi_f| \left(1 - \frac{itH}{\hbar}\right) |\psi_i\rangle \equiv |\psi_i\rangle e^{-it\frac{\mathcal{B}}{\hbar}}. \quad (4)$$

Here,

$$\langle a_w \rangle = \frac{\langle \psi_f|A|\psi_i \rangle}{\langle \psi_f|\psi_i \rangle} \quad (5)$$

represents the weak value and $e^{-it\frac{\mathcal{B}}{\hbar}} |\phi\rangle$ is the output pointer state. If we take

$$|\phi\rangle = |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad (6)$$

as the initial pointer state, after some time we obtain

$$|\phi\rangle \rightarrow |\phi'\rangle = (1 - \frac{it}{\hbar} \langle a_w \rangle Z) |\phi\rangle, \quad (7)$$

which leads to

$$|\phi'\rangle = |+\rangle - \frac{it}{\hbar} \langle a_w \rangle |-\rangle. \quad (8)$$

The final pointer state is weakly shifted with respect to its initial state (see Fig. 1). The magnitude of the shift is dependent on the force of interaction. This shifted state is illustrated in Fig. 1 and can be found experimentally. If we measure the pointer $|\phi'\rangle$ for observable $X$, we obtain

$$\langle X \rangle_{\phi'} = 1 - |\frac{t}{\hbar} \langle a_w \rangle|^2. \quad (9)$$

Alternatively, if we measure observable $Z$, the outcome is

$$\langle Z \rangle_{\phi'} = 2i \text{Im} \left( \frac{t}{\hbar} \langle a_w \rangle \right) = \text{Re} \left( \frac{t}{\hbar} \langle a_w \rangle \right). \quad (10)$$

Similarly, the real part of $\frac{t}{\hbar} \langle a_w \rangle$ can be estimated by measuring $Y$.

**B. Weak measurement by strong interaction and insensitive pointer**

Here, we demonstrate our first strategy to weak value measurement even with strong interaction (i.e., controlled-sign gate), given as

$$U_s = e^{-it\frac{\mathcal{A}}{\hbar}} = |a_0\rangle \langle a_0| \otimes I + |a_1\rangle \langle a_1| \otimes Z, \quad (11)$$

where $|a_{0,1}\rangle$ are eigenvectors of observable $\mathcal{A} = |a_0\rangle \langle a_0| - |a_1\rangle \langle a_1|$ of the measured system which is strongly coupled to observable $Z$ of the pointer. These states form a basis for a spin-$\frac{1}{2}$ system, i.e., $I = |a_0\rangle \langle a_0| + |a_1\rangle \langle a_1|$. Here, we can successfully realize weak measurement, if we use quantum erasure properly. Similarly to the case of weak interaction, we apply the unitary operator $U_s$ to the measured system and probe states and subsequently post-select the measured system onto $|\psi_f\rangle$

$$|\psi_f\rangle \langle \psi_f| e^{-it\frac{\mathcal{B}}{\hbar}} |\psi_i\rangle \equiv |\psi_i\rangle e^{-it\frac{\mathcal{B}}{\hbar}}. \quad (12)$$

For convenience, let us introduce operator $B$ defined as $B = \langle \psi_f|U_s|\psi_i \rangle$. One can show by direct calculations that

$$B = \langle \psi_f|\mathcal{A}|\psi_i \rangle + \langle \psi_f|a_1\rangle \langle a_1|\psi_i \rangle \frac{I}{\langle \psi_f|\psi_i \rangle} - \langle \psi_f|\mathcal{A}|\psi_i \rangle - \langle \psi_f|a_0\rangle \langle a_0|\psi_i \rangle \frac{Z}{\langle \psi_f|\psi_i \rangle}. \quad (13)$$

Equation (13) can be further simplified to obtain

$$2B = (1 + \langle a_w \rangle)I + (1 - \langle a_w \rangle)Z, \quad (14)$$

where

$$\langle a_w \rangle = \frac{\langle \psi_f|\mathcal{A}|\psi_i \rangle}{\langle \psi_f|\psi_i \rangle} \quad (15)$$

is the weak value. We now use this results to rewrite Eq. (12) as

$$|\psi_f\rangle B|\phi\rangle = \frac{1}{2} |\psi_f\rangle [(1 + \langle a_w \rangle)I + (1 - \langle a_w \rangle)Z]|\phi\rangle. \quad (16)$$

The shift in ancilla state is then

$$|\phi'\rangle = \frac{1}{2} [(1 + \langle a_w \rangle)I + (1 - \langle a_w \rangle)Z]|\phi\rangle. \quad (17)$$

Let us parametrise the initial probe state by amplitudes $\alpha$ and $\beta$

$$|\phi\rangle = \alpha|+\rangle + \beta|\rangle. \quad (18)$$

Expressing the shifted probe state $|\phi'\rangle$ in terms of eigenstates of the observable $Z$ yields

$$|\phi'\rangle = \frac{\alpha + \beta}{\sqrt{2}}|0\rangle - \frac{\alpha - \beta}{\sqrt{2}}\langle a_w|1\rangle. \quad (19)$$
In the situation when we set $\alpha = \beta$, we deal with complete erasure of information as $(Z)_{\phi'} = 1$ does not provide any information about $|\bar{a}_w\rangle$. On the other hand, if we set $\alpha = 1$ and $\beta = 0$, we obtain

$$|\phi'\rangle = \frac{1}{2}[(1 - |\bar{a}_w\rangle)(+\rangle) - (1 + |\bar{a}_w\rangle)|-\rangle],$$

(20)

which leads to a strong measurement

$$(Z)_{\phi'} = \frac{1}{2}(1 - |\bar{a}_w\rangle)^2.$$  

(21)

Now, let us consider measurement of $X$ on probe state $|\phi'\rangle$. In case of the complete quantum erasure (i.e., $\alpha = \beta$ and $|\phi'\rangle = |0\rangle$) we observe $(X)_{\phi'} = 0$. If $\alpha = 1$, $\beta = 0$, and we use strong measurement the expected value of $X$ in state $|\phi'\rangle$ is

$$(X)_{\phi'} = \frac{1}{2}[(1 - |\bar{a}_w\rangle)^2 - \frac{1}{2}(1 + |\bar{a}_w\rangle)^2 = \text{Re}(\bar{a}_w).$$  

(22)

Finally, let us consider pointer state $|\phi'\rangle \approx |0\rangle$, where

$$\alpha + \beta \approx \sqrt{2}, \alpha - \beta = \delta, |\delta| \ll 1,$$

(23)

which is transformed due to strong interaction into

$$|\phi'\rangle = |0\rangle + \frac{1}{2}(\bar{a}_w)|1\rangle.$$  

(24)

By slightly deviating the pointer from $|0\rangle$ state we made it weakly sensitive to the interacting system.

In this case, if we measure the pointer for observable $Z$, we obtain

$$(Z)_{\phi'} = 1 - \frac{1}{2}|\bar{a}_w|^2.$$  

(25)

Alternatively, if we measure observable $X$, the outcome is

$$(X)_{\phi'} = \text{Re}(\bar{a}_w).$$  

(26)

Similarly, the imaginary part of $\delta(\bar{a}_w)$ can be estimated by measuring $Y$.

C. Strong interaction and subsequent erasure

Consider a pointer prepared in state $|\phi\rangle = |+\rangle$, and system in preselected state $|\psi_i\rangle = \alpha|0\rangle + \beta|1\rangle$. Let the pointer be in control mode of a $c$-phase gate set to strongest interaction (i.e., controlled-sign gate) and let the system be in the target mode. The choice of this particular gate determines the operator $\bar{A} = Z$ of which the weak value is measured. In this configuration, maximally entangled states can be created. Thus, this regime is referred as strong interaction. Next, the pointer state is measured in $|\phi'\rangle = \gamma|0\rangle + \delta|1\rangle$ and $|\phi'_0\rangle = \delta|0\rangle - \gamma|1\rangle$ basis, where $\delta = \epsilon + \gamma \approx 1$ and $|\epsilon| \ll |\gamma|$. Now, the joint state of the control and target modes reads as

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}[(\gamma + \delta I \otimes \bar{A})|\phi'\rangle|\psi_i\rangle + (\gamma I \otimes \bar{A} - \delta)|\phi'_0\rangle\bar{A}|\psi_i\rangle].$$  

(27)

By measuring the pointer we obtain

$$\langle \phi' | \psi_i \rangle = \frac{1}{\sqrt{2}}(\gamma + \delta \bar{A})|\psi_i\rangle,$$  

(28)

$$\langle \phi'_0 | \psi_i \rangle = \frac{1}{\sqrt{2}}(\gamma \bar{A} - \delta)|\psi_i\rangle.$$  

(29)

It is obvious that the state of the system is perturbed and its either $\langle \phi' | \Psi \rangle$ or $\langle \phi'_0 | \Psi \rangle$, depending on the measurement outcome of the pointer. If we use $\bar{A}^{-1}$ operation on the target mode conditioned on finding the pointer in $|\phi'_0\rangle$ state, we always find the system (up to a global phase factor) in a weakly perturbed state

$$|\psi'_i\rangle = (\gamma + \delta \bar{A})|\psi_i\rangle.$$  

Let us consider what happens when we measure operator $Z = |\phi'\rangle \langle \phi'| - |\phi'_0\rangle \langle \phi'_0|$ on the pointer state. Similarly, we can construct $X$ and $Y$ observables, corresponding to Pauli operators. In this case the system state is postselected as usual. The pointer state is detected in $|\phi'\rangle$ with probability

$$|\langle \psi_f | \psi'_i \rangle|^2 \approx |\langle \psi_f | \psi_i \rangle|^2[1 + 2\text{Re}(\delta(\bar{a}_w))],$$  

(30)

wheras for projecting the pointer on $|\phi'_0\rangle$ and performing feed-forward $\bar{A}^{-1}$ operation on the strongly perturbed system (here $\bar{A} = \bar{A}^{-1}$) we obtain

$$|\langle \psi_f | \psi'_i \rangle|^2 \approx |\langle \psi_f | \psi_i \rangle|^2[1 - 2\text{Re}(\delta(\bar{a}_w))].$$  

(31)

Thus, the mean value of $(Z)_{\phi'}$ observed on the pointer for preselected and postselected states $|\psi'_i\rangle$ and $|\psi_f\rangle$, respectively, is

$$(Z)_{\phi'} = \text{Re}(4\delta(\bar{a}_w)).$$  

(32)

We can observe the imaginary part of the weak value if $Y$ operator is used instead of $Z$.

D. Operational equivalence

When we look at Eqs. (10), (26) and (32), we can conclude that probes carry information about weak values of the measured system. By measuring the deviation of the respective pointer from its initial position we can measure these weak values. This proves that the approaches are operationally equivalent and the values of parameters $\delta$ and $\hbar \kappa$ play the roles of strength controllers in their respective scenarios. Weak value measurement was proposed to acquire small amount of information about the measured system while minimizing inflicted disturbance. Similarly, by using a pointer state insensitive to the interaction (Sec. 11B), we do not perturb the measured system. It can easily be seen, that for $\delta \rightarrow 0$, the probe state becomes the eigenstate of the $Z$ operator. Inserting this eigenstate relation into Eq. (11) factorizes the action onto the measured system and the probe. As a result, regardless of the measurement outcome on the probe, the measured system remains unperturbed. In
the same manner, quantum erasing allows to tune the trade-off between the information gained about the system and the damage caused to it (Sec. II). One can verify that in case of complete erasure $\delta \to 0$ and correct feedback, the measured system is not changed.

We demonstrate the above described theoretical concept on an example of a tunable controlled-phase (c-phase) gate. This gate transforms a pair of qubits according to a prescription

$$|mn\rangle \to e^{i \varphi \delta_{mn}} |mn\rangle,$$

(33)

where $m, n \in \{0, 1\}$ stand for logical states of the qubits, $\varphi$ denotes the introduced phase shift and $\delta_{mn}$ is the Kronecker’s delta. By setting the value $\varphi \to 0$ to maximally strong $\varphi = \pi$. Note that, if we set $\varphi = \delta$ we change the strength of the interaction between the two qubits from infinitesimally weak $\varphi \to 0$ to maximally strong $\varphi = \pi$. We have constructed a tunable linear-optical c-phase gate which is experimentally much more demanding as it consists of three nested interferometers [27]. In our experiment, we encoded logical states $|0\rangle$ and $|1\rangle$ into horizontal $|H\rangle$ and vertical $|V\rangle$ polarization states of individual photons. Pairs of photons at 710 nm were obtained in type I spontaneous parametric down-conversion using a BBO ($\beta$-BaB$_2$O$_3$) crystal. We accumulated the signal for 100 s to acquire about 400 – 3000 two-photon coincidences depending on the particular setting.

The c-phase gate itself consists of an interferometer formed by two beam displacers with a partially polarizing beam splitter (PPBS) acting in one of its arms. The PPBS is manufactured to have 100% transmissivity for 5 nm wide M – mirror, BD – beam displacer, F – spectral filter (5 nm wide) M$_1$ – motorized translation, PPBS – partially polarizing beam splitter, HWP – half-wave plate, QWP – quarter-wave plate, Pol – polarizer.

III. EXPERIMENTAL IMPLEMENTATION

We have constructed a tunable linear-optical c-phase gate based on the design by Lanyon et al. [24]. This experimental setup (see Fig. 2) was chosen for its stability and easily adjustable phase shift $\varphi$. Note that optimal linear-optical tunable c-phase gate is experimentally more demanding as it consists of three nested interferometers [27]. In our experiment, we encoded logical states $|0\rangle$ and $|1\rangle$ into horizontal $|H\rangle$ and vertical $|V\rangle$ polarization states of individual photons. Pairs of photons at 710 nm were obtained in type I spontaneous parametric down-conversion using a BBO ($\beta$-BaB$_2$O$_3$) crystal. We accumulated the signal for 100 s to acquire about 400 – 3000 two-photon coincidences depending on the particular setting.

The c-phase gate itself consists of an interferometer formed by two beam displacers with a partially polarizing beam splitter (PPBS) acting in one of its arms. The PPBS is manufactured to have 100% transmissivity for horizontally polarized light and 33% transmissivity for vertically polarized photons. As a result, it implements a c-phase gate with $\varphi = \pi$ on impinging photons [28]. To obtain phase shifts other than $\varphi = \pi$, one needs to make use of ancillary mode provided by the interferometer. For more details, see Ref [24].

In the first step, we show that our setup is capable of implementing traditional weak value measurement (weak-value measurement by weak interaction). For this purpose, we have tuned the gate to a small but still measurable phase shift $\varphi$. For practical reasons, we have selected $|\psi_f\rangle$ to be $\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. The input signal was prepared in the state

$$|\psi_i\rangle = \cos \gamma |H\rangle + \sin \gamma |V\rangle$$

(40)

parametrized by the angle $\gamma$. According to prescription [6], the initial pointer state was set to $|\phi\rangle = \frac{1}{\sqrt{2}}(|H\rangle +
measured onto required in this regime was simulated by projecting the $1$ was measured in the infinitesimally weak interaction is unavailable or experimentally unfeasible. Our approach can be also applied

Next, we test the weak value measurement in the two strong interaction-based strategies. The second regime, weak value measurement by strong interaction and insensitive pointer state, was implemented by using the PPBS as a c-phase gate with $\varphi = \pi$. The lower interferometer arm was blocked. We use the same set of input signal states $|\psi_i\rangle$ and $|\psi_f\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. The pointer initial state was prepared by the HWP$_2$ set to $3^\circ$ resulting in a state close to $|0\rangle$. Again, the pointer was measured in the $\frac{1}{\sqrt{2}}(|H\rangle \pm |V\rangle)$ basis using HWP$_3$. Weak value ($\bar{a}_w$) was obtained using Eq. (25) and fitting of parameter $\delta \approx 0.21$.

The third, and final, regime was implemented again by using only the upper interferometer arm and PPBS. In this case, the pointer was prepared in the $|\delta\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ state by HVP$_2$ and projected in a basis close to $|H\rangle/|V\rangle$ using HVP$_3$. The feed-forward required in this regime was simulated by projecting the measured system onto $\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $\frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$ when the pointer was projected onto states close to $|H\rangle$ and $|V\rangle$ respectively. Weak value was obtained using Eq. (32) and fitting of parameter $\delta \approx 0.08$. In Fig. (3), we compare experimental results obtained using all methods with the theoretical prediction (41). Note that in case of strong interaction $\langle a_w \rangle$ was calculated from $\langle \bar{a}_w \rangle$ using $\langle a_w \rangle = (1 - \langle \bar{a}_w \rangle)$ as explained in the previous section. Our data reasonably overlaps with theoretical derivations proving the working principle of all approaches.

IV. CONCLUSIONS

We have both theoretically and experimentally demonstrated the counter-intuitive fact that weak values can be measured using a strong interaction between the measured system and the pointer. Analytical derivation proves this procedure to be equivalent to typical weak value measurement with infinitesimally weak interaction. Our experimental data obtained on a linear-optical c-phase gate certifies this equivalence. In this particular case, it was much easier to implement a strong c-phase interaction using the PPBS alone rather than the weak interaction using the entire interferometric setup. From this point of view, our strong interaction-based strategies are more robust. Proposed technique opens new possibilities of weak value measurement even in situations when infinitesimally weak interaction is unavailable or experimentally unfeasible. Our approach can be also applied directly if the interaction between system and pointer is counterfactual, i.e., a counterfactual c-phase gate is applied [29]. Moreover, we also believe that our results contribute to a better understanding of the concept of weak values.
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