Quantum information approach to the description of quantum phase transitions

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Abstract. The fidelity, linear entropy and entanglement entropy, concepts of information theory, are used to determine the localization of quantum phase transitions of the Lipkin-Meshkov-Glick (LMG) and the Dicke models for a finite number of particles. Another concept, the so-called fidelity susceptibility, gives us information of the scaling behavior of the quantum phase transitions of the models. In particular universal curves are presented for the specific susceptibility. It is shown that the control parameters of both models have a similar dependence in the number of particles when the systems approach to the thermodynamic limit.

1. Introduction

In the last thirty years the fields of quantum information and quantum computation have grown tremendously, and the community has been using quantum mechanics as a powerful framework to help the development of information technologies. Since the establishment of the Landauer principle in 1961, it is well understood that information has physical manifestations and, due to improvements in the storing of information, in ever smaller devices, this has led the science community to assume naturally that the physics that will rule such devices will be quantum mechanics [1, 2].

Recently, there have been remarkable discoveries that have called the attention of the physicist and mathematician communities, they are quantum teleportation [3], the use of quantum cryptography in commercial business [4] and, equally important, the appearance of Shor’s factorization algorithm [5].

The purpose of this contribution is to show how basic concepts of quantum information can be used in many body physics. As examples we consider two Hamiltonian models used in different fields of physics, which are simple enough to avoid being lost in the calculations. A review of the LMG and Dicke Hamiltonian models [6, 7] is given, and the definitions of concepts such as the fidelity, and the fidelity susceptibility are introduced. We present briefly which are the values of the control parameters of the considered models that lead to quantum phase transitions. This is done by calculating the expectation value of the Hamiltonian with respect to appropriate coherent states which define a energy surface. This is a function depending on variables and control parameters, which is characterized by its singularities. The theory of singularities or catastrophes has been applied in many different fields [8]. For each of the two models considered, we have found their corresponding separatrix, that is the region of points in the control parameter space where there are sudden changes in the structure of the ground
state of the systems [9, 10, 11]. Afterwards, the fidelity and fidelity susceptibility observables are calculated. We show that the minimum value of the fidelity or the maximum of the fidelity susceptibility as a function of the control parameter can be used to determine the localization of the quantum phase transitions for a finite number of particles. We are considering that there is a quantum phase transition when there is a sudden change in the ground state properties of a finite system. When the number of particles of the system is very large the size of the changes grows to infinity [12]. Additionally, the values of the parameter inducing the transition move to its thermodynamic limit. The fidelity susceptibility can be used to get the scaling behavior of the maximum susceptibility in terms of the number of particles \( N \), and it can be used to determine how the coupling parameter \( \gamma_m \) of maximum susceptibility goes to its thermodynamic limit. Additionally, we use the definition of the specific susceptibility introduced in [13], which lets find universal curves in both models. Finally, we discuss the meaning of entanglement in a composite system, and its importance in quantum information. To measure the degree of entanglement of a bipartite system, we use the linear and von Neumann entropies. We calculate both measures of entanglement for the LMG and Dicke models and find that their maximum values can be used to localize finite quantum phase transitions.

This work is dedicated with appreciation to the seventieth birthday of Jerry P. Draayer for his great contributions in the fields of nuclear and computational physics.

2. Hamiltonian Models

The LMG model has been used in different contexts. It was introduced in nuclear physics as a toy model to test all kinds of many body theories because it is an exactly solvable model. The model assumes that the nucleus is a system of fermions which can occupy two levels with the same degeneracy \( \Omega \), separated by an energy \( \hbar \omega \). One of the levels is just above the Fermi energy and the other also just below. There are residual interactions which scatter pairs of particles between the two levels without changing the total number of particles occupying the shells [6, 14]. Thus the LMG Hamiltonian can be written in terms of the angular momentum operators, using the quasispin formalism, as

\[
H_{\text{LMG}} = \omega J_z + \gamma_x J_x^2 + \gamma_y J_y^2,
\]

with \( \Omega = 2j + 1 \equiv N + 1 \), where \( N \) denotes the number of particles.

In other fields like quantum optics and condensate matter it has been used to describe two level atoms in a cavity interacting with a classical electromagnetic field, It describes also a system of \( N \) particles with spin \( s = 1/2 \) mutually interacting with a magnetic field transverse to the interaction directions \( X, Y \), or as a two mode Bose Einstein condensate by means of the Schwinger realization of the angular momentum operators [15, 16, 17, 18]. Notice that the unitary transformation \( e^{-i\pi J_z} \) changes the operators \( J_x \rightarrow -J_x \) and \( J_y \rightarrow -J_y \) and so it leaves the Hamiltonian invariant. While the LMG model was initially used as a toy model, now it has a more realistic role because of the experimental and technological advances.

The Dicke Hamiltonian, which describes a cloud of cold two-level atoms interacting with a one mode electromagnetic field in an optical cavity in the dipolar approximation, is given by [7, 16]

\[
H_D = \hat{a}^\dagger \hat{a} + \omega_A \hat{J}_z + \frac{\gamma}{\sqrt{N}} \left( \hat{a}^\dagger + \hat{a} \right) \left( \hat{J}_+ + \hat{J}_- \right).
\]

If we consider again the completely symmetric configuration then \( N = 2j \). This Hamiltonian is invariant under the transformation \( \exp(i\pi\Lambda) \), where the operator \( \Lambda \equiv \hat{a}^\dagger \hat{a} + J_z + \sqrt{J^2 + 1/4} - 1/2 \) denotes the total number of excitations of the system and is a conserved quantity within the Tavis-Cummings model [11]. For the Dicke model, the eigenvalue of \( \Lambda \) is denoted by \( \lambda \) and the
eigenvectors must have definite parity [19, 20]. This Hamiltonian can be also interpreted as describing a one mode interacting boson-fermion model.

For these two models, we have followed a procedure established by Gilmore [8] which considers the algebraic structures of the models, constructed the associated coherent states as variational states, and by evaluating the energy surface of the model determine their separatrix, which is the place in the control parameter space where there is a quantum phase transition [9, 10].

In both cases, as it was mentioned previously, there is a residual symmetry which can be used to get a better agreement with the exact solution of the Hamiltonian models for a finite number of particles. This is done by taking linear symmetry adapted coherent states to the point symmetry of the cyclic group \( C_2 \) [18, 19, 20].

### 3. Fidelity and Fidelity Susceptibility

The fidelity is a basic concept in classical information theory, which measures the accuracy of a transmission, and which usually presents imperfection due mainly to the limitation of the resources and the random noise [21]. For the quantum case, it is known that for two pure states described by the density operators \( \rho_1 = |\chi\rangle\langle\chi| \) and \( \rho_2 = |\phi\rangle\langle\phi| \), the fidelity is defined by

\[
F(|\chi\rangle\langle\chi|, |\phi\rangle\langle\phi|) = |\langle\chi|\phi\rangle|^2.
\]

It is a measure of the distance between the states. Then it has values in the range \( 0 \leq F \leq 1 \), with \( F = 1 \) indicating that the states are the same up to a global phase and \( F = 0 \) when they are orthogonal. Thus, it means also the transition probability from one state to another and its geometric interpretation is the closeness of states [2].

For one pure state \( |\chi\rangle \) and other mixed state \( \rho_2 \), one has

\[
F(|\chi\rangle\langle\chi|, \rho_2) = \langle\chi|\rho_2|\chi\rangle,
\]

that denotes the probability to be a pure state. It is used for example to measure how reliable is the transmission of quantum information for a given communication protocol.

In general, for mixed states, it has been proved that [21, 22] the definition

\[
F(\rho_1, \rho_2) = \left\{ \text{Tr}\left( \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right) \right\}^2, \tag{3}
\]

of the fidelity satisfies the properties:

\[
0 \leq F(\rho_1, \rho_2) \leq 1, \\
F(\rho_1, \rho_2) = F(\rho_2, \rho_1), \\
F(U\rho_1, U\rho_2) = F(\rho_1, \rho_2). \tag{4}
\]

Therefore it gives a measure of how similar are two probability densities. It is currently used to establish the accuracy of quantum computations in the presence of errors induced by noise effects due to interaction with the environment or imperfections in the hardware. It has been used in the experimental generation of entanglement because the entangled state should be deterministic [23], that is, there must be a very large fidelity between the generated state and a Bell state. The fidelity has been also used as a measure of the accuracy with which the initial quantum state can be recovered by inverting the dynamics with a perturbed Hamiltonian, and in that case it is called Loschmidt echo, providing a criterion for stability of a quantum motion under changes in the initial conditions. A review of the fidelity approach to the characterization of quantum phase transitions is given in [24].
Here we are going to show that the fidelity [25] can be used to determine when the ground state of a quantum system presents a sudden change as function of a control parameter. If we denote that parameter by \( \lambda \) one has

\[
F(\lambda, \lambda + \delta \lambda) = |\langle \psi(\lambda) | \psi(\lambda + \delta \lambda) \rangle|^2,
\]

and its minimum value determines where the system is changing drastically, then it can be used to characterize the quantum phase transitions. For a first order quantum phase transition associated to a discontinuity on the first derivative of the energy of the system the states in the different sides of the transition are orthogonal and so the fidelity goes to zero.

From the expression of the fidelity, a Taylor series expansion can be done around the critical parameter \( \lambda_c \),

\[
F(\lambda_c, \lambda_c + \delta \lambda) = F(\lambda_c, \lambda_c) + \delta \lambda \frac{dF}{d\lambda} \bigg|_{\lambda=\lambda_c} + \frac{1}{2!} \delta \lambda^2 \frac{d^2F}{d\lambda^2} \bigg|_{\lambda=\lambda_c} (\delta \lambda)^2 + \cdots,
\]

and as we are looking for the minimum value, the first derivative of the fidelity is zero. Thus one can define the fidelity susceptibility by [26]

\[
\chi_F = \frac{21 - F(\lambda_c, \lambda_c + \delta \lambda)}{(\delta \lambda)^2}.
\]

It is dependent on the Hamiltonian term that causes the phase transition.

If a Hamiltonian can be written in the form \( H = H_0 + \lambda H_I \), then the fidelity susceptibility can also be calculated by means of the first order perturbation expansion

\[
\chi_F(\lambda) = \sum_{k \neq 0} \frac{|\langle \Psi_k(\lambda) | H_I | \Psi_0(\lambda) \rangle|^2}{(E_k(\lambda) - E_0(\lambda))^2},
\]

where \( |\Psi_0(\lambda)\rangle \) and \( |\Psi_k(\lambda)\rangle \) denote the ground state and the \( k \) eigenstate of the system.

### 4. Separatrix

#### 4.1. Lipkin Model

The expectation value of the Hamiltonian per particle with respect to the SU(2) coherent states is given by [9, 10]

\[
\varepsilon(\theta, \phi) = -2 \cos \theta + \gamma_x \sin^2 \theta \cos^2 \phi + \gamma_y \sin^2 \theta \sin^2 \phi,
\]

where \((\theta, \phi)\) denote a point on the Bloch sphere. We use \( \omega = 2, \tilde{\gamma}_x = \frac{2}{N+1} \gamma_x, \tilde{\gamma}_y = \frac{2}{N+1} \gamma_y, \) and \( \varepsilon(\theta, \phi) = \langle H_{LMG}/N \rangle \) except for constant terms. This energy surface allows to determine the critical points, bifurcation sets, and Maxwell sets. These sets organize all the critical points according to their stability within the control parameter space.

The minima of the energy surface can be denoted

\[
\varepsilon(0, 0) = -2, \quad \varepsilon(\theta_c, \phi_c) = \Gamma + \frac{1}{\Gamma},
\]

where we have defined

\[
\Gamma = \begin{cases} 
\gamma_x, & \text{for } \gamma_x < -1, \text{ and } \gamma_x < \gamma_y, \\
\gamma_y, & \text{for } \gamma_y < -1, \text{ and } \gamma_y < \gamma_x, \\
\tilde{\gamma}, & \text{for } \tilde{\gamma} < -1,
\end{cases}
\]
where in the last expression we have $\gamma_x = \gamma_y = \bar{\gamma}$. The Maxwell sets are the loci in parameter space for which the classical energies at two or more critical points are equal. When a Maxwell set is crossed, the energy surface jumps from one critical branch to another, and phase transitions take place for $|\Gamma| \geq 1$. Phase transitions for the minimum values of the classical energy surface can only happen for $\Gamma \leq -1$. They correspond to the regions where $\gamma_x \leq -1$ and $\gamma_y \leq -1$. Bifurcation sets are the loci in parameter space where the function $\varepsilon$ changes because equilibria points are either created or destroyed. They are obtained from the vanishing of the determinant of the matrix of second derivatives of $\varepsilon$ evaluated at the critical points, that is, \( \det \varepsilon_{ij} = 0 \), with $i, j = \theta, \phi$. The parameter values that satisfy these conditions are

$$\gamma_x = \pm 1 ; \quad \gamma_y = \pm 1 ; \quad \gamma_x = \gamma_y \quad \text{with} \quad |\gamma_x| \geq 1 . \quad (8)$$

A second order phase transition of the Ginzburg-Landau type takes place when the straight lines $\gamma_x = -1$, $\gamma_y = -1$ or the point $(\gamma_x, \gamma_y) = (-1, -1)$ are crossed. The crossing of the straight line $\gamma_y = \gamma_x$ yields a first order transition. Special attention must be given to the crossing of the cusp point $(\gamma_x, \gamma_y) = (-1, -1)$ along the straight line $\gamma_y = -\gamma_x - 2$ because in that case there is a third order phase transition, related to a convergence of second order phase transitions [9, 10].

4.2. Dicke Model

The energy surface constructed by the tensorial product of Heisenberg-Weyl and SU(2) coherent states is given by [19, 20]

$$E(q, p, \theta, \phi) = \frac{1}{4j} \left( p^2 + q^2 \right) - \frac{\omega A}{2} \cos \theta + \frac{1}{\sqrt{j}} j q \sin \theta \cos \phi , \quad (9)$$

where we take the harmonic oscillator realization for the field part $\alpha = \frac{1}{\sqrt{2}} (q + ip)$ and the stereographic projection for the angular momentum part $\zeta = e^{-i \phi} \tan \frac{\theta}{2}$. The order parameters $(q, p)$ correspond to the expectation values of the quadratures of the field, while $(\theta, \phi)$ determine a point on the Bloch sphere.

The minima and degenerate critical points are obtained easily. By calculating the Hessian of the energy surface, we see that when $\gamma^2 = \omega A/4$ the critical points degenerate and, for that value of the field-matter coupling, the phase transition from the normal to the superradiant behaviour of the atoms takes place. This value of the coupling parameter corresponds to the thermodynamic phase transition.

The critical points $(q_c, p_c, \theta_c, \phi_c)$ which minimize $H$ are given by

$$\begin{align*}
(0, 0, 0, \phi), & \quad \text{for} \quad |\gamma| < \gamma_c , \\
(-2 \sqrt{j} \gamma \sqrt{1 - (\gamma_c/\gamma)^4} \cos \phi_c, 0, \arccos(\gamma_c/\gamma)^2, \phi_c), & \quad \text{for} \quad |\gamma| > \gamma_c . \quad (10)
\end{align*}$$

The first row defines the minimum critical points for the normal phase where the variable $\phi$ is undetermined, while the second row describes those for the superradiant regime where one has two possibilities for the order parameter $\phi_c = 0$ or $\phi = \pi$. It is convenient to work with the variable $x = \gamma/\gamma_c$ in terms of which the energy values for the minima just described are

$$E_{\text{normal}} = -2 \gamma_c^2 , \quad E_{\text{superradiant}} = -\gamma_c^2 x^2 \left( 1 + x^{-4} \right) . \quad (11)$$

Next, we are going to show how similar information about the localization of the quantum phase transitions can be obtained by means of the quantum information concepts.
Figure 1. The fidelity and fidelity susceptibility as functions of $\gamma_x$ for the even (dotted blue line) and odd (continuous red line) exact quantum solutions for $N = 2^{14}$. The quantum phase transition is localized at the minimum of the fidelity and the maximum of the fidelity susceptibility. For LMG model with $\gamma_y = -1$.

5. Results

As an example, we determine the fidelity and fidelity susceptibility for the LMG model. These are calculated for a number of atoms $N = 2^{14}$, keeping fixed the control parameter $\gamma_y = -1$. Then, we are going to study how sensitive is the Hamiltonian under the changes in the parameter $\gamma_x$.

The additional symmetry transformation of the LMG Hamiltonian, that is invariance under rotations $R(\pi) = e^{i\pi J_z}$, implies that the matrix Hamiltonian, in the angular momentum basis states $\{|jm\rangle\}$, can be broken into two parts: states with an even number of excited particles $n = j + m$ and states with an odd number.

The results for the fidelity and the fidelity susceptibility are shown in Fig. 1, for the ground and first excited states. The value of $\gamma_x$ determined by the localization of the phase transition for the ground state is larger than the thermodynamic value $\gamma_{xc} = -1$, the obtained value for the first excited state is even larger. Then the values for the control parameter $\gamma_x$ for the ground and first excited states are given by

$$\gamma_x = -1.000042, \ F = 0.999517, \ \chi_F = 9.66246 \times 10^8,$$
$$\gamma_x = -1.000101, \ F = 0.999929, \ \chi_F = 1.41458 \times 10^8.$$

Thus, we have obtained the localization of the quantum phase transition for a large but finite number of particles. Now we want to determine how fast this value of $\gamma_{xm}$ approaches the thermodynamic limit when the number of particles changes.

Therefore, we are going to calculate the maximum fidelity susceptibility for different number of particles running from $2^{10}$ to $2^{16}$. In Fig. 2 we show two log-log plots, the first one associated to the maximum value of the susceptibility as a function of the number of particles and the second one is related to $\log_2(\gamma_{xc} - \gamma_{xm})$ again versus $\log_2 N$. In both cases, one can fit straight lines which for the even and odd cases they take the form

$$\chi_{max} \approx 2^{-0.16} N^2, \ \gamma_{xc} - \gamma_{xm} \approx 2^{0.46} N^{-1}, \ \text{for the even case},$$
$$\chi_{max} \approx 2^{-2.95} N^2, \ \gamma_{xc} - \gamma_{xm} \approx 2^{1.71} N^{-1}, \ \text{for the odd case},$$

where it is known that $\gamma_{xc} = -1$. The confidence limits of the fitted straight lines are related with the obtained variance of the order of $10^{-2}$. They are given in the mentioned figure.

In conclusion, the maximum susceptibility of the interaction associated to the control parameter $\gamma_x$ grows quadratically with the number of particles while the difference of the
Figure 2. Log-log plots of the maximum fidelity susceptibility $\chi_{\text{max}}$ (left) and the relative coupling parameter $\gamma_c - \gamma_m$ as functions of the number of particles. We propose $\chi_{\text{max}} = 2^b N^a$ and $\gamma_c - \gamma_m = 2^d N^c$. For the LMG model with $\gamma_y = -1$. At the left, one has for the even case $b = -0.16 \pm 0.01$, and $a = 2.00 \pm 0.01$ while for the odd case $b = -2.95 \pm 0.01$ and $a = 2.00 \pm 0.01$. At the right, one has for the even case $c = -1.00 \pm 0.01$, and $d = 0.46 \pm 0.01$ while for the odd case $c = -1.00 \pm 0.01$ and $d = 1.71 \pm 0.01$.

Figure 3. The specific susceptibility determines Universal curves for the even (blue line) and odd (red line) parity states. In the plots its value is shown for all the number of particles indicated in the text and one notices one curve only. For the LMG model with $\gamma_y = -1$, the thermodynamic value with $\gamma_{x,m}$, the value which yields maximum fidelity susceptibility, is proportional to the inverse of the number of particles. This is true for the even and odd states of the system.

Following the work of Ho-Man Kwok et al [13], one can define a universal quantity called the specific susceptibility as follows

$$\chi^\delta_s = \frac{\chi_{\text{max}}(\gamma_{cm})}{\chi_F(\gamma_x)} - 1,$$

where $\delta = e$ denotes the specific susceptibility for the even case and $\delta = o$ denotes specific susceptibility for the odd case. If this function is plotted as a function of the variable $\gamma_x - \gamma_{x,m}$ times the number of particles (see Fig. 3 and Fig. 5), one finds approximately the same values of the specific susceptibility for a large region of the abscissa. In the mentioned figure, there are plots of the specific susceptibility for the following number of particles

$$2048, 3000, 4096, 6000, 8192, 12000, 16584, 20000, 28000, 32768, 40000, 50000, 65536.$$

One can see that all the corresponding specific susceptibilities approximately fall along the same curve.
We conjecture that the result for $\gamma$ where it is known that $\gamma_\text{c} - \gamma_\text{m}$ is proportional to $N^{-2}$ and that the relative critical coupling parameter $\gamma_\text{c} - \gamma_\text{m}$ is also as a function of $N$, considering the value $\gamma_\text{y} = -0.5$. In both cases, one can fit straight lines which for the even and odd cases they take the form

$$\chi_{\text{max}} \approx 2^{-0.85} N^{1.35}, \quad \gamma_{\text{c}} - \gamma_{\text{m}} \approx 2^{0.87} N^{-0.65}, \quad \text{for the even case,}$$

$$\chi_{\text{max}} \approx 2^{-2.04} N^{1.36}, \quad \gamma_{\text{c}} - \gamma_{\text{m}} \approx 2^{1.26} N^{-0.65}, \quad \text{for the odd case,}$$

where it is known that $\gamma_\text{c} = -1$. The confidence limits of the fitted straight lines are indicated in the mentioned figure.

We expected a similar behavior because they all are second order quantum phase transitions. We conjecture that the result for $\gamma_\text{y}$ for the same set of number of particles, one finds a maximum susceptibility proportional to $N^{4/3}$ and that the relative critical coupling parameter $\gamma_{\text{c}} - \gamma_{\text{m}}$ is proportional to $N^{-2/3}$. As an example, we present in Fig.

For this case, we construct the Universal curves associated to the calculation of the specific susceptibility plotted in Fig.

According to our results one concludes that, in general, the second order quantum phase transitions have the same behavior with the number of particles given by $\chi_{\text{max}} \approx N^{4/3}$ and $(\gamma_{\text{c}} - \gamma_{\text{m}}) \approx N^{-2/3}$. There are special cases, as for example moving towards a triple point, where one finds that $\chi_{\text{max}} \approx N^2$ and $(\gamma_{\text{c}} - \gamma_{\text{max}}) \approx N^{-1}$.

For the Dicke model, we have also an extra symmetry related with the parity of the total number of excitations. Thus we have to consider two cases: the even case related with the ground state of the model and the odd case with the first excited state. However in this contribution, we are going to restrict ourselves to the ground state of the system. We also fix the parameter $\omega_\Lambda = 1$, that is we consider the case of resonance. Then, we change the parameter $\gamma$ in the

![Figure 4](image-url) Log-log plots for the maximum susceptibility as a function of the number of particles and for the transitions have the same behavior with the number of particles given by $\chi_{\text{max}} \approx N^{4/3}$ and $(\gamma_{\text{c}} - \gamma_{\text{m}}) \approx N^{-2/3}$. There are special cases, as for example moving towards a triple point, where one finds that $\chi_{\text{max}} \approx N^2$ and $(\gamma_{\text{c}} - \gamma_{\text{max}}) \approx N^{-1}$. For the Dicke model, we have also an extra symmetry related with the parity of the total number of excitations. Thus we have to consider two cases: the even case related with the ground state of the model and the odd case with the first excited state. However in this contribution, we are going to restrict ourselves to the ground state of the system. We also fix the parameter $\omega_\Lambda = 1$, that is we consider the case of resonance. Then, we change the parameter $\gamma$ in the
Figure 5. The specific susceptibility determines Universal curves for the even (blue line) and odd (red line) parity states. We use all the number of particles mentioned in the text and observe approximately one curve only. For the LMG model with $\gamma_y = -0.5$.

Figure 6. Log-log plots of the difference $\gamma_{max} - \gamma_c$ and $\chi_{max}$ as functions of $j$ are displayed. A fit of a straight line of the form $b + ax$ was done and the results are: at left $a = 1.36 \pm 0.01$, and $b = 4.40 \pm 0.01$; at right $a \to -0.66 \pm 0.01$, and $b \to -1.67 \pm 0.01$. For the Dicke model, interval $0.5 \leq \gamma \leq 0.6$, for a number of particles running from $N = 40$ up to $N = 2^{11}$. The log-log plots of the relative critical coupling matter-field strength $\gamma_{max} - \gamma_c$ and the maximum fidelity susceptibility as a function of $N/2$ are displayed in Fig. 6. Straight lines were fitted, at left with positive slope while at right with negative slope. They are approximately given by

$$\chi_{max} \approx N^{\frac{4}{3}}, \quad (\gamma_{max} - \gamma_c) \approx N^{-\frac{2}{3}}.$$ (13)

In the Dicke model the thermodynamic quantum phase transitions is of second order and it is important to stress that the scaling behavior of the maximum fidelity susceptibility and the approaching to the coupling parameter strength satisfy the same relations that in the LMG model.

Finally, we want to discuss our results for the entanglement between the matter and field within the Dicke model. The entanglement concept was introduced by Schrödinger in the middle of the thirties and is closely related to the EPR paradox and Bell’s theorem. It is considered as a resource that can be used in the same form that the energy to do, for example, quantum teleportation or send secure information[2]. For the Dicke model [27] and other algebraic models in molecular physics, as the vibron model [28] for example, the Renyi entropy has been calculated.

For a bipartite system, one can have complete information of the composite system but also
complete ignorance about the parts of the system. Examples are the so called Bell states

\[ |\phi\pm\rangle = \frac{1}{\sqrt{2}} (|+, +\rangle \pm |-, -\rangle), \]
\[ |\psi\pm\rangle = \frac{1}{\sqrt{2}} (|+, -\rangle \pm |-, +\rangle), \]

where the states can represent a system of 2 qubits, each of which has two possible eigenvalues, or it can represent a system of two particle with spin \( s = 1/2 \). For these states the linear and von Neumann entropies have maximum values. To illustrate how they can be calculated first we give their definitions: the linear entropy is defined by \( S_L = 1 - \text{Tr}(\rho^2) \) and the entanglement entropy by \( S_{\text{VN}} = -\sum_k \lambda_k \ln \lambda_k \).

For one of the Bell states of two qubits, the system is described by the density operator \[ \rho = \frac{1}{2} \left( |+, +\rangle\langle +, +| + |+, +\rangle\langle -, -| + |-, -\rangle\langle +, +| + |-, -\rangle\langle -, -| \right). \]

Tracing over the first subsystem one gets the reduced density operator

\[ \rho_2 = \frac{1}{2} \left( |+, +\rangle\langle + | + |-, -\rangle\langle -, -| \right), \]

and from it one can immediately calculate that \( S_L = 1/4 \). In the expression of the von Neumann entropy appear the eigenvalues of the reduced density matrix of the subsystem 2. Then, it is straightforward to evaluate that \( S_{\text{VN}} = \ln 2 = 0.693 \).

We calculate the entanglement of the matter with the one mode electromagnetic field for the model, and look numerically for its maximum values. The results are shown in Fig.7, where at left the positions of the maxima are given for the following set:

\[ (N, \gamma) = \{(20, 0.572), (40, 0.543), (100, 0.523), (200, 0.514), (400, 0.509), (1000, 0.505)\}, \]

while at the right one has

\[ (N, \gamma) = \{(20, 0.571), (40, 0.544), (100, 0.524), (200, 0.515), (400, 0.509), (1000, 0.505)\}. \]

This coincides with the calculation of the minimum values of the fidelity to within the accuracy of the numerical calculation. The latter are given by

\[ (N, \gamma) = \{(20, 0.568), (40, 0.543), (100, 0.524), (200, 0.515), (400, 0.509), (1000, 0.505)\}. \]
6. Conclusions

We have shown that the fidelity, the fidelity susceptibility, and the entanglement entropy concepts can be used to localize the position of finite quantum phase transitions in terms of the parameters of the different interaction terms of the Hamiltonian. This was done for the LMG and Dicke models. The agreement of the results with those previously obtained using the Ginzburg Landau formalism is complete.

For a finite number of atoms the localization of the quantum phase transitions is different from its value at the thermodynamic limit, and the $N$ dependence of the maximum fidelity susceptibility was obtained. For a special crossing of the triple point of the LMG model we obtain

$$\chi_{\text{max}} \approx N^2, \quad (\gamma_{xc} - \gamma_{\text{max}}) \approx N^{-1}.$$  

For other crossings of second order phase transitions one gets

$$\chi_{\text{max}} \approx N^{4/3}, \quad (\gamma_{xc} - \gamma_{\text{max}}) \approx N^{-2/3}.$$  

A similar result for the second order quantum phase transition of the Dicke model was obtained by means of the calculation of the von Neumann entropy. The calculated specific fidelity susceptibility, for both models, shows a universal character for all the number of particles.

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