Topological Constraints on Long-Distance Neutrino Mixtures
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Abstract
A new internal description of fundamental fermions (quarks and leptons), based on a matrix-generalization (F) of the scalar fermion-number f, predicts that only three families of quarks and leptons, and their associated neutrinos (ν_e, ν_μ and ν_τ), exist. Moreover, this description places important topological constraints on neutrino mixing. For example, with respect to F, the topology of the ν_e (ν_μ or ν_τ) is that of a cylinder (Möbius strip). Assuming that a change in topology during neutrino-neutrino transitions is suppressed (e.g., one cannot continuously deform a donut into a sphere), while neutrino-neutrino transitions without topology-change are (relatively) enhanced, one may have an explanation for recent short-distance experimental observations of (nearly) maximal ν_μ-ν_τ mixing at the Super Kamiokande. To test this idea, I was able to use simple topological arguments to deduce a matrix describing long-distance neutrino mixtures, which is identical to that proposed by Georgi and Glashow on different grounds. Experimental confirmation of this prediction would strongly support the new description of fundamental fermions, which requires, among other things, that the ν_e and (ν_μ or ν_τ) neutrinos start life as topologically-distinct quantum objects.
1.0 Introduction

Except where explicitly prevented by some “absolute” conservation law (e.g., the conservation of electric charge or spin angular momentum), quantum mechanics generally permits transitions between states having different topologies [1]. While a change in topology may be energetically (or otherwise) inhibited, unavoidable quantum fluctuations are expected to catalyze such processes. Hence, there is always the possibility of mixing between otherwise similar states having distinct topologies [2].

Recently, a new internal description of fundamental fermions (quarks and leptons) was proposed [3]. The new description is based on a matrix-generalization $F$ of the scalar fermion number $f$. One of the main predictions of the new description is that only three families of quarks and leptons exist—hence that there are only three low-mass neutrino flavors, namely, the $\nu_e$, $\nu_\mu$, and $\nu_\tau$ neutrinos. Moreover, because of the way fundamental fermions are represented by certain geometric objects in the space on which the matrix transformation $F$ acts [Ref. 3, p. 57 and pp. 85–87, Ref. 4, 5 and Ref. 6, pp. 244–255], the $\nu_e$ and ($\nu_\mu$ or $\nu_\tau$) neutrinos are found to have different topologies with respect to $F$. This fact turns out to have important implications for neutrino mixing.

With respect to the matrix transformation $F$, the topology of both the $\nu_\mu$ and $\nu_\tau$ is found to be that of a Möbius strip [Ref. 6, p. 143]. By contrast, the topology of the $\nu_e$ (with respect $F$) is that of a cylinder. And because a change in topology during transitions tends to be suppressed (e.g., one cannot continuously deform a donut into a sphere) the foregoing topological distinctions between neutrinos may help explain recent experimental observations of (nearly) maximal $\nu_\mu$-$\nu_\tau$ mixing at the Super Kamiokande facility [7]. The purpose of this paper is to determine the constraints that these topological distinctions place on long-distance neutrino mixtures, and by implication, short-distance neutrino mixtures as well.

2.0 Topological Constraints on Neutrino Mixtures

In all conventional weak transitions, including neutral-current weak transitions, the $e$-$\mu$ and $\tau$ quantum numbers are additively conserved. Although the observed neutrino mixing [7] violates these conservation laws, all transi-
tions of neutrinos from one flavor to another (i.e., \( \nu_l \to \nu_{l'}, l \neq l' \)) will tend to be suppressed. In this paper we implicitly assume that this degree, and kind, of suppression is the same for all \( \nu_l \to \nu_{l'} \), and therefore, that this effect can be completely ignored when discussing (relative) topological constraints placed on long-distance neutrino mixtures.

### 2.1 Additional constraints on the matrix \( M \)

Given that the \( \nu_e \) (\( \nu_\mu \) or \( \nu_\tau \)) neutrino has the topology of a cylinder (Möbius strip) with respect to the internal transformation \( F \), and assuming that topological constraints are the primary determinants of the matrix \( M \) describing long-distance neutrino mixtures, the form of \( M \) is easily determined [8]. Moreover, these same topological constraints can be used to place numerical bounds on the components of \( M \). To accomplish these results we need only apply the following very general principle to neutrino-neutrino transitions:

*All other things being equal, any neutrino flavor \( \nu_l \), (i.e., \( \nu_e \), \( \nu_\mu \) or \( \nu_\tau \)) that undergoes neutrino-neutrino transitions involving a change in topology, will tend to be suppressed, while neutrino-neutrino transitions involving no change in topology will tend to be (relatively) enhanced.*

To this principle we add the following corollary,

*All other things being equal, because the \( \nu_\mu \) and \( \nu_\tau \) have the same topology, they will act the same way in all neutrino-neutrino transitions.*

Given these principles, we immediately have the following two constraints on long-distance neutrino mixtures:

A. No matter what neutrino flavor (\( \nu_l \)) and topology one starts with at some distant source (say a supernova), by the time the neutrino mixture reaches its “equilibrium” state, it should contain equal fractions of \( \nu_\mu \) and \( \nu_\tau \), because these neutrinos have the same topology.

B. Because the \( \nu_\mu \) and \( \nu_\tau \) have the same topology, if one starts out with either a pure \( \nu_\mu \) or a pure \( \nu_\tau \) source, one should end up with the same long-distance equilibrium mixture of \( \nu_e \), \( \nu_\mu \) and \( \nu_\tau \).

To describe A) and B) in conventional terms [9], let \( D_l \) be the number of detected neutrinos of type \( \nu_l \) and \( B_l \) be their number at “birth” at some distant source. Then

\[
\{D_e, D_\mu, D_\tau\} = M\{B_e, B_\mu, B_\tau\},
\]  

(1)
where $M$ is a $3 \times 3$ matrix of fractional coefficients describing the long-distance neutrino mixture, and $\{ \ldots \}$ are column “vectors”. By definition, all rows and columns of $M$ must sum to unity (100%), which means that $D_l$ and $B_l$ must satisfy the constraint

$$D_e + D_\mu + D_\tau = B_e + B_\mu + B_\tau. \quad (2)$$

### 2.2 The form of the matrix $M$

Constraints A) and B), together with Eqs. (1) and (2), dictate that the matrix $M$ describing long-distance neutrino mixtures must have the symmetrical form

$$
\begin{pmatrix}
D_e \\
D_\mu \\
D_\tau
\end{pmatrix} =
\begin{pmatrix}
a & b & b \\
b & c & c \\
b & c & c
\end{pmatrix}
\begin{pmatrix}
B_e \\
B_\mu \\
B_\tau
\end{pmatrix}.
\quad (3)
$$

While topological constraints alone cannot determine the exact numerical values of the matrix elements $a$, $b$ and $c$, they can place numerical bounds on these quantities. For example, because the $\nu_e$ is inhibited by its topology from turning into a $\nu_\mu$ or $\nu_\tau$, we naturally expect that (See Fig. 2.1)

$$a \geq b. \quad (4)$$

Similarly, because the $\nu_\mu$ and/or the $\nu_\tau$ are inhibited by their topology from turning into a $\nu_e$, we naturally expect that (See Fig. 2.1)

$$c \geq b. \quad (5)$$

Equations (4) and (5), together with the requirement that all rows and columns of the matrix $M$ in (1) and (3) sum to unity (i.e., $a+2b = b+2c = 1$), further leads to the following bounds on $a$, $b$ and $c$:

$$\left(\frac{1}{3} \leq a \leq 1, \quad 0 \leq b \leq \frac{1}{3}, \quad \frac{1}{3} \leq c \leq \frac{1}{2}\right). \quad (6)$$

Moreover, since $a+2b = b+2c$, the arithmetic mean of $a$ and $b$ is $c$, i.e., $(a + b)/2 = c$, which means that $c$ lies half-way between $a$ and $b$. Hence, the matrix elements $a$, $b$ and $c$ are subject to the constraint

$$a \geq c \geq b. \quad (7)$$
Transitions *without* topology change
(5 matrix elements)

Transitions *with* topology change
(4 matrix elements)

Figure 2.1. Matrix elements $a$, $b$ and $c$ describing long-distance neutrino mixtures [see Eq. (3)]. Neutrino-neutrino transitions without topology change are “preferred” relative to neutrino-neutrino transitions involving a change in topology. That is, the matrix elements $a$ and $c$ are both greater than $b$. The symbol $\otimes$ (\cmidrule{1-1} $\otimes$) for a cylinder (Möbius strip) is used to represent the topology of the $\nu_e$ ($\nu_\mu$ or $\nu_\tau$) with respect to the internal transformation $F$. Note that the matrix elements $b$ and $c$ are unaffected by an exchange of the subscripts $\mu$ and $\tau$.

Equations (1) and (3) determine the general form of $M$, while (6) places bounds on its components, whose relative magnitudes are further constrained by (7).

Finally, because rows and columns of $M$ sum to unity, $M$ may be expressed in terms of the single parameter $a$ as

$$M(a) = \frac{1}{4} \begin{pmatrix} 4a, & 2(1 - a), & 2(1 - a) \\ 2(1 - a), & (1 + a), & (1 + a) \\ 2(1 - a), & (1 + a), & (1 + a) \end{pmatrix}. \quad (8)$$

### 2.3 The Georgi-Glashow matrix $M(\frac{1}{2})$

The form of $M$ specified in (3) and (8), together with the constraints provided by (6) and (7), are completely consistent with the matrix proposed by Georgi.
and Glashow [9] to describe long-distance neutrino mixtures, namely,

\[ M(\frac{1}{2}) = \frac{1}{8} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}. \]  

(9)

This matrix can be viewed in two different, but apparently complementary, ways as being

1) The direct result of an assumed form for the neutrino mass-matrix (and associated mixing parameters), and other factors dictated by requiring consistency with the standard-model description of neutrinos, including all known experimental properties of neutrinos. This is the point of view espoused in [9], and/or

2) The result of a balance between opposing topological and quantum “forces” acting on neutrinos. This is the point of view espoused in the present paper.

If 1) and 2) above are to be compatible points of view, then it follows that such things as the neutrino mass-matrix is, in some sense, the result of an interplay between opposing topological and quantum “forces” acting on neutrinos. This conclusion would gain support if it were possible to derive the matrix \( M \) without reference to the approach taken in [9]. Such a derivation is presented in Section 3.0. But, before turning to the derivation of \( M \), we must first examine the boundary conditions on this matrix.

### 2.4 Boundary conditions on \( M(a) \)

Consider the general matrix \( M(a) \) defined on the boundaries of the physically accessible region \((\frac{1}{3} \leq a \leq 1)\). When \( a = a_{\frac{1}{3}} = \frac{1}{3} \) (See Eq. 8 and Ref. 10)

\[ M(\frac{1}{3}) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]  

(10)

In this case, topological constraints are effectively nonexistent—topology change in neutrino-neutrino transitions is completely unrestricted \((a = b = \frac{1}{3})\).
“Forces” acting to oppose topology-change

(No topology change)

M(a_1)

M(a_0)

M(a_{1/3})

a_{1/3} = \frac{1}{3}

(Unrestricted topology change)

(“Forces” catalyzing topology-change, i.e., quantum fluctuations)

Figure 2.2. The balance between opposing quantum and topological “forces”. The parameter \( a \) varies from \( a_{1/3} = \frac{1}{3} \) (no effective topological constraints on neutrino mixing) to \( a_1 = 1 \) (maximal topological-constraints in neutrino mixing). The shaded regions are physically inaccessible. Owing to unavoidable quantum-fluctuations, the “forces” acting to oppose topology-change are, themselves opposed, by quantum “forces” acting to catalyze topology-change. These two opposing “forces” must seek a stable “equilibrium” at some point \( a = a_0 \) located between \( a_{1/3} \) and \( a_1 \). Thus the matrix \( M(a_0) \) represents a “balance” between the extreme conditions \( M(a_{1/3}) \) and \( M(a_1) \).

\( c = \frac{1}{3} \). That is, quantum fluctuations, which act to catalyze topology-change, are completely dominant when \( a = \frac{1}{3} \). Figure 2.2 illustrates the situation.

When \( a = a_1 = 1 \) (See Eq. 8 and Ref. 10)

\[
M(1) = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.
\]

(11)

In this case, topological constraints are maximal—topology-change in neutrino-neutrino transitions is completely prevented \( (a = 1, b = 0, c = \frac{1}{2}) \). That is, quantum fluctuations, which would otherwise catalyze topology-change, are effectively overwhelmed by topological energy (or other topological) “barriers” when \( a = 1 \). Figure 2.2 illustrates the situation.
Now, because of (8, 10 and 11), any matrix $M(a)$ defined on the interval $(\frac{1}{3} \leq a \leq 1)$ can be written as a function of the single parameter $a$ as

$$M(a) = \frac{3}{2}(1 - a)M(\frac{1}{3}) + \frac{3}{2}(a - \frac{1}{3})M(1) \quad (12)$$

where

$$\frac{3}{2}(1 - a) + \frac{3}{2}(a - \frac{1}{3}) = 1. \quad (13)$$

Clearly, the fractions $\frac{3}{2}(1 - a)$ and $\frac{3}{2}(a - \frac{1}{3})$, determine the fraction of $M(a)$, which depends on the topology-changing piece $M(\frac{1}{3})$ and the topology-maintaining piece $M(1)$, respectively.

In the case of the Georgi-Glashow matrix in (9) we find

$$M(\frac{1}{2}) = \frac{3}{4}M(\frac{1}{3}) + \frac{1}{4}M(1). \quad (14)$$

Presumably, this particular three-to-one “mixture” of $M(\frac{1}{3})$ and $M(1)$, respectively, represents the equilibrium condition between opposing topology-changing and topology-maintaining “forces”, respectively.

### 3.0 A Simple Calculation of $a$, $b$ and $c$.

According to (3) and (8) the matrix elements $a$, $b$ and $c$ are given by

$$\begin{align*}
  a &= a \\
  b &= \frac{1}{2}(1 - a) \\
  c &= \frac{1}{4}(1 + a)
\end{align*} \quad \text{(15)}$$

Focus attention on the pair of matrix elements $a$ and $b$, which *bracket* $c$. Now, topology-changing “forces” (quantum fluctuations) constantly “attempt” to maximize $b$ at the expense of $a$ (and $c$), while at the same time, topology-maintaining “forces” (energy barriers?) constantly “attempt” to maximize $a$ (and $c$) at the expense of $b$ (See Figures 2.1 and 2.2).

Since $a$ and $b$ are both functions of $a$, a “compromise” between these opposing “forces” should be reached when the product $f(a) = ab$ is *maximized* [11] with respect to variations of $a$. From (15) one has

$$f(a) = \frac{1}{2}a(1 - a), \quad (16)$$
which possesses a maximum at a value of \( a \) equal to the harmonic mean \(^{12}\) of the boundary values \( a_{\frac{1}{3}} \) and \( a_1 \) (See Fig. 2.2), namely,

\[
a_0 = \frac{2a_{\frac{1}{3}} \cdot a_1}{a_{\frac{1}{3}} + a_1} = \frac{1}{2}.
\]

(17)

And, this value of \( a \) is precisely the value required to yield the Georgi-Glashow matrix \( M(\frac{1}{2}) \) of (9). However, one other conceivable solution for \( a_0 \) exists.

The function \( g(a) = abc \) also possesses a maximum, which occurs at a value of \( a \) equal to the geometric mean of the boundary values \( a_{\frac{1}{3}} \) and \( a_1 \) (See Fig. 2), namely,

\[
a_0 = \sqrt[3]{a_{\frac{1}{3}} \cdot a_1} = \frac{\sqrt{3}}{3} = 0.577.
\]

(18)

Equations (17) and (18) represent the only physically acceptable solutions involving extremal functions that depend on simple products of the matrix elements \( a, b \) and \( c \) (i.e., \( ab, bc, ac \) and \( abc \)).

For example, while the function \( h(a) = b \cdot c \) possesses a maximum, this occurs at a value of \( a \) equal to zero, which point lies outside the physically acceptable region \( (a_{\frac{1}{3}} \leq a \leq a_1) \). And, the function \( i(a) = a \cdot c \) possesses a minimum instead of a maximum. Moreover, this occurs at an “unphysical” value of \( a \), namely, \( a_0 = -\frac{1}{2} \).

In summary, without further constraints, (17) and (18) would seem to be equally likely solutions. Is there some way to decide which solution nature should choose?

### 3.1 Discussion

Three reasons can be cited in favor of (17) and \( M(\frac{1}{2}) \). These are

1. From the standpoint of simplicity and parsimony the quadratic \( f(a) = ab \) is obviously simpler and more parsimonious than the cubic \( g(a) = a \cdot b \cdot c \). For example, the quadratic is symmetrical about its maximum, while the cubic is asymmetrical about its maximum. The function \( f(a) \) is also simpler and more parsimonious than \( g(a) \), in the sense that \( g(a) \) has two factors \( a \) and \( c \) that describe the effect of topology-maintaining “forces”, while \( f(a) \) has only one such factor \( a \). Hence, for reasons of simplicity and parsimony, (17) and \( M(\frac{1}{2}) \) are favored over (18) and \( M(0.577) \), respectively.
2. From [9] the matrix $M(\frac{1}{2})$ is known to be consistent with a reasonable form for the neutrino mass-matrix and associated mixing-parameters. The choice of (18) and $M(0.577)$ would require changes in these quantities that may, or may not, be consistent with the presently known experimental properties of neutrinos. Hence, (17) and $M(\frac{1}{2})$ appear to be favored for this reason over (18) and $M(0.577)$, respectively.

3. In Appendix A we present a crude “derivation” of the matrix $M$ based on a harmonic-oscillator description of the balance between topological and quantum “forces.” This description also seems to favor (17) and $M(\frac{1}{2})$ over (18) and $M(0.577)$, respectively.

It is remarkable that, in the absence of an in-depth understanding of topology-maintaining and topology-changing “forces” (See Fig. 2.2), it has nevertheless proven possible to obtain an exact calculation (see qualifying remarks above) of the matrix elements $a$, $b$, and $c$, which describe long-distance neutrino mixtures. This apparent success strongly supports the main premise of this paper, which is that topological constraints play a major role in describing both short- and long-distance neutrino mixtures.

### 4.0 Conclusions

Topological constraints appear to play a major role in determining the nature of both short- and long-distance neutrino mixtures. Given that the $\nu_e$ and ($\nu_\mu$ or $\nu_\tau$) neutrinos have distinct topologies, and assuming that topology changes are suppressed in neutrino-neutrino transitions, while neutrino-neutrino transitions without topology-change are relatively enhanced—one easily determines both the form of the matrix describing long-distance neutrino mixtures, and even fixes the numerical magnitudes of the individual matrix elements. If the predicted matrix (9), or one close to it [i.e., $M(0.577)$], is eventually confirmed by observations of neutrinos from distant astronomical sources, this will strongly support the new description of fundamental fermions, which requires, among other things, that the $\nu_e$ and ($\nu_\mu$ or $\nu_\tau$) neutrinos start life as topologically-distinct quantum objects [3–5].
5.0 Appendix A. A Toy Model for Calculating \( M(a_0) \).

Imagine that the parameter \( a \) describes a dynamical “system” associated with neutrino-neutrino mixtures. In particular, this system has a single degree of freedom (is constrained to move on a straight line), which is further bounded by \( a_1 \) and \( a_1^\dagger \) \( (a_1^\dagger \leq a \leq a_1) \). Treating \( a \) as a kind of “generalized coordinate” we are led to consider the “generalized forces” acting on the “system” and to determine its equilibrium “position” \( a_0 \) (See Fig. 2.2). Quantum fluctuations, in effect, “force” the “system,” as described by \( a \) from the value \( a = a_1 \) toward \( a = a_1^\dagger \). At the same time, topological energy (or other topological) “barriers” effectively “force” the system as described by \( a \), from the value \( a = a_1^\dagger \) toward \( a = a_1 \). These two opposing “forces” must reach a compromise, i.e., stable equilibrium, at some value \( a \) (call it \( a_0 \)), which lies somewhere between \( a_1^\dagger \) and \( a_1 \). This value \( (a_0) \) we take to describe long-distance neutrino mixtures via the matrix \( M(a_0) \).

To calculate \( 0 \), let us model the “system” using a simple (classical) harmonic-oscillator “potential” (See Ref. 13 and Figure A.1),

\[
V(x) = \frac{k}{2}x^2, \tag{A1}
\]

where the Hooke’s-law “restoring force” is

\[
F = -kx = -\frac{\partial V(x)}{\partial x}, \tag{A2}
\]

and \( x = (a - a_0) \) is the “displacement” of the system from its (stable) equilibrium “position” \( a = a_0 \).

From (A1) we have

\[
\frac{\partial V(x)}{\partial x} = k(a - a_0). \tag{A3}
\]

Clearly, if \( k(a - a_0) \) were known on each boundary, then \( k \) and \( a_0 \) could be calculated. To do this, note first that \( \frac{\partial V(x)}{\partial x} \) must be a number with the “units” of \( a \) (assuming \( k \) is dimensionless). And, since the only number available to describe the system on the boundary of \( a \), is \( a \) itself, we make the following assumptions

\[
\left. \frac{\partial V(x)}{\partial x} \right|_{a_1} = k(a_1 - a_0) = +a_1; \quad a_1 > a_0 \tag{A4}
\]
Figure A1. The “potential” $V(a)$ describing the opposing quantum and topological “forces”. These “forces” balance one another at the (stable) equilibrium position $a_0$. Here, $k = 2$, $V(x) = \frac{k}{2} x^2 = (a - a_0)^2$, and the “restoring force” is $F(a) = -2(a - a_0)$.

\[ \frac{\partial V(x)}{\partial x} \bigg|_{a_{\frac{1}{3}}} = k(a_{\frac{1}{3}} - a_0) = -a_{\frac{1}{3}}; \quad a_{\frac{1}{3}} < a_0. \quad \text{(A5)} \]

From (A4) and (A5) we can easily deduce the unknowns $k$ and $a_0$. One finds

\[ k = \frac{(a_1 + a_{\frac{1}{3}})}{(a_1 - a_{\frac{1}{3}})} = 2, \quad \text{(A6)} \]

and

\[ a_0 = \frac{2(a_{\frac{1}{3}} \cdot a_1)}{(a_{\frac{1}{3}} + a_1)} = \frac{1}{2}. \quad \text{(A7)} \]

Both $k$ and $a_0$ depend only on the values of $a$ on the boundaries. In particular, $a_0$ is found to be the harmonic mean of $a_{\frac{1}{3}} = \frac{1}{3}$ and $a_1 = 1$. 

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6.0 References and Footnotes

[1] A. P. Balachandran, “Bringing Up a Quantum Baby,” quant-ph/9702055.

[2] G. Holzwarth, “Formation of Extended Topological-Defects During Symmetry-Breaking Phase Transitions in $O(2)$ and $O(3)$ Models”, hep-ph/9901296. Analogous examples of fluctuation-induced topology-change in macroscopic objects (e.g., destruction of topological objects due to thermal fluctuations at phase transitions) abounds. For example, otherwise persistent (“conserved”) topological defects in crystals can be destroyed by raising the temperature sufficiently (i.e., by melting the crystal). Similarly, otherwise persistent (“conserved”) magnetic flux-tubes in Type II superconductors and/or vortices in a superfluid, can both be destroyed by raising the temperature above the critical temperature $T_c$. And, conversely, topological defects are always created when such macroscopic systems first condense (or crystallize) as the temperature is lowered. We imagine that something roughly similar can happen when quantum-fluctuations (vacuum fluctuations) act on otherwise very similar quantum states (i.e., same charge, spin and nearly identical mass) that also happen to start life as distinct topological-objects. That is, we are assuming that, if not prevented by some conservation law, transitions between such states (e.g., $\nu_e$ and $\nu_\mu$ neutrinos) will be catalyzed by quantum fluctuations.

[3] Gerald L. Fitzpatrick, The Family Problem-New Internal Algebraic and Geometric Regularities, Nova Scientific Press, Issaquah, Washington, 1997. Additional information: http://physicsweb.org/TIPTOP/ or http://www.amazon.com/exec/obidos/ISBN=0965569500. In spite of the many successes of the standard model of particle physics, the observed proliferation of matter-fields, in the form of “replicated” generations or families, is a major unsolved problem. In this book a new organizing principle for fundamental fermions is proposed, i.e., a minimalistic “extension” of the standard model based, in part, on the Cayley-Hamilton theorem for matrices. To introduce (internal) global degrees of freedom that are capable of distinguishing all observed flavors, the Cayley-Hamilton theorem is used to generalize the familiar standard-model concept of scalar fermion-numbers $f$ (i.e., $f_m = +1$ for all fermions and $f_a = -1$ for all antifermions). This theorem states that every (square) matrix satisfies its characteristic equation. Hence, if $f_m$ and $f_a$ are taken to be the eigenvalues of some real matrix $F$ (a “generalized fermion
number”), it follows from this theorem that both $f$ and $F$ are square-roots of unity. Assuming further that the components of both $F$ and its eigenvectors are global charge-like quantum observables, and that $F$ “acts” on a (real) vector 2-space, both the form of $F$ and the 2-space metric are determined. One finds that the 2-space has a non-Euclidean or “Lorentzian” metric, and that various associated 2-scalars serve as global flavor-defining “charges,” which can be identified with charges such as strangeness, charm, baryon and lepton numbers etc.. Hence, these global charges can be used to describe individual flavors (i.e., flavor eigenstates), flavor doublets and families. Moreover, because of the aforementioned non-Euclidean constraints and certain standard-model constraints, one finds that these global charges are effectively- “quantized” in such a way that families are replicated. Finally, because these same constraints dictate that there are only a limited number of values these charges can assume, one finds that families, and their associated neutrinos, always come in “threes.”

[4] The eigenvectors $Q$ of $F$ (i.e., $FQ = fQ$ where $f$ is the scalar fermion-number), together with certain pairs of linearly independent vectors ($U$ and $V$) that resolve $Q$ (i.e., $Q = U + V$), namely, various non-Euclidean vector “triads” $(Q, U, V)$—these are the analogs of Euclidean triangles—serve to represent flavor-doublets in terms of a pair of quark or lepton flavor-eigenstates as follows:

$$| \text{“up”} \rangle = |q_1, u_1, v_1, Q^2, U^2, 2U \cdot V \rangle$$

and

$$| \text{“down”} \rangle = |q_2, u_2, v_2, Q^2, U^2, 2U \cdot V \rangle.$$ 

Here, $Q = \{q_1, q_2\}$, $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$ are column-vectors and their components $q_1, q_2, u_1, u_2, v_1$ and $v_2$, together with the non-Euclidean scalar products $Q^2, U^2$ and $V^2$, are various global mutually-commuting flavor-defining charge-like quantum numbers. When we refer to the “topology” of a particular neutrino flavor-eigenstate (e.g., the $\nu_e$) we are referring to the topology of the corresponding vector triad $(Q, U, V)$, with respect to the internal transformation $F$. And, because $F$ generates the Möbius group $Z_2$ (i.e., $F^2 = I_2$), those vector “triads” that are left unchanged by $F$, have the topology of a cylinder, whereas vector triads that are changed by $F$ (but obviously not changed by $F^2 = I_2$), have the topology of a Möbius strip. And, as it turns out, the neutrino flavor $\nu_e$ ($\nu_\mu$ or $\nu_\tau$) corresponds to a vector triad having the topology of a cylinder (Möbius strip) with respect to $F$. 

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An inexact, but nevertheless very suggestive, analogy can be drawn between our 2D (non-Euclidean) charge-vectors \( \mathbf{Q} = \{q_1, q_2\} \), and the 2D (Euclidean) topological-“charge” vector called a Burgers vector. The quantized Burgers vector of a dislocation or defect in a 2D crystal lattice, is defined as the net number of extra rows and columns one encounters while traversing a closed path around the defect, expressed as a vector \( \{\text{columns}, \text{rows}\} \). In general, defects are discontinuities or “tears” in some order-parameter field. In this particular case, the defect is a topological line-singularity in the crystalline order. Because the topological “charge” (Burgers vector) is conserved as the associated defect migrates through the crystal lattice, the associated defect is also conserved within the lattice. Similarly, our quantized charge-vector \( \mathbf{Q} \) is conserved \( (\Delta \mathbf{Q} = 0) \), as the associated fundamental-fermion (quark or lepton) moves through spacetime. Conservation of the vector \( \mathbf{Q} \) leads to the identification of \( \mathbf{Q}^2 \) with the “conserved” baryon- or lepton-number. Hence, we can think of the vector \( \mathbf{Q} \) as being a topological “charge”, which describes certain aspects of a topological “defect” (fundamental fermion) in some spacetime order-parameter field. The geometric object we call a vector triad, namely \( (\mathbf{Q}, \mathbf{U}, \mathbf{V} \text{ where } \mathbf{Q} = \mathbf{U} + \mathbf{V}) \), while not generally conserved (owing to quantum transitions such as \( \mathbf{U}_1 + \mathbf{V}_1 \leftrightarrow \mathbf{U}_2 + \mathbf{V}_2 = \mathbf{Q} \)), nevertheless, provides a further description of such spacetime “defects”. Finally, the topology of fundamental fermions is said to be the same as that of the associated vector-triad under the internal transformation \( \mathbf{F} \).

[5] H. Umezawa, *Advanced Field Theory-Micro, Macro and Thermal Physics*, Am. Inst. of Physics, New York, 1993, pp. 128–129. An inexact, but nevertheless very suggestive, analogy can be drawn between our 2D (non-Euclidean) charge-vectors \( \mathbf{Q} = \{q_1, q_2\} \), and the 2D (Euclidean) topological-“charge” vector called a Burgers vector. The quantized Burgers vector of a dislocation or defect in a 2D crystal lattice, is defined as the net number of extra rows and columns one encounters while traversing a closed path around the defect, expressed as a vector \( \{\text{columns}, \text{rows}\} \). In general, defects are discontinuities or “tears” in some order-parameter field. In this particular case, the defect is a topological line-singularity in the crystalline order. Because the topological “charge” (Burgers vector) is conserved as the associated defect migrates through the crystal lattice, the associated defect is also conserved within the lattice. Similarly, our quantized charge-vector \( \mathbf{Q} \) is conserved \( (\Delta \mathbf{Q} = 0) \), as the associated fundamental-fermion (quark or lepton) moves through spacetime. Conservation of the vector \( \mathbf{Q} \) leads to the identification of \( \mathbf{Q}^2 \) with the “conserved” baryon- or lepton-number. Hence, we can think of the vector \( \mathbf{Q} \) as being a topological “charge”, which describes certain aspects of a topological “defect” (fundamental fermion) in some spacetime order-parameter field. The geometric object we call a vector triad, namely \( (\mathbf{Q}, \mathbf{U}, \mathbf{V} \text{ where } \mathbf{Q} = \mathbf{U} + \mathbf{V}) \), while not generally conserved (owing to quantum transitions such as \( \mathbf{U}_1 + \mathbf{V}_1 \leftrightarrow \mathbf{U}_2 + \mathbf{V}_2 = \mathbf{Q} \)), nevertheless, provides a further description of such spacetime “defects”. Finally, the topology of fundamental fermions is said to be the same as that of the associated vector-triad under the internal transformation \( \mathbf{F} \).

[6] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, New York, 1983.

[7] T. Kajita, for the Super-Kamiokande, Kamiokande collaboration. [hep-ex/9810001].

[8] M. Gronau, *Patterns of Fermion Masses, Mixing Angles and CP Violation*, in The Fourth Family of Quarks and Leptons, First International Symposium, edited by: D.B. Cline and Amarjit Soni, Annals of The New York Academy of Sciences, New York, New York, Volume 518, 1987, p. 190.

In the case of strongly-interacting quarks, topological constraints of the kind considered here can, at most, play a minor role in determining such things as KM-type matrix elements. For example, it is well known that the angle \( \theta_c \), \( V_{12} \) and the \( d, s \) quark masses \( m_d \) and \( m_s \), respectively, are related via \( V_{12} = \sin \theta_c = \sqrt{\frac{m_d}{m_s}} \), where \( m_s > m_d \neq 0 \), even though the \( d \) and \( s \) quarks
are characterized, like the $\nu_e$ and $(\nu_\mu$ or $\nu_\tau$) neutrinos, respectively, by distinct topologies. Somehow, the underlying topology plays a major role in mixing, in the case of the weakly-interacting neutrinos, but not in the case of the strongly-interacting quarks. The small value, and near degeneracy, of neutrino masses is probably a major factor in explaining such differences between quarks and neutrinos.

[9] H. Georgi and S. L. Glashow, “Neutrinos on Earth and in the Heavens,” hep-ph/9808293, page 5, Equation 20.

[10] If $M(a) = M(\frac{1}{3})$ any initial neutrino “mixture” at birth would turn into the mixture $(\frac{1}{3}\nu_e + \frac{1}{3}\nu_\mu + \frac{1}{3}\nu_\tau)$ when detected at a great distance. If $M(a) = M(1)$ any initial neutrino “mixture” at birth (e.g., $\alpha\nu_e + \beta\nu_\mu + \gamma\nu_\tau; \alpha + \beta + \gamma = 1$) would turn into the mixture $[\alpha\nu_e + (\frac{\beta + \gamma}{2})\nu_\mu + (\frac{\beta + \gamma}{2})\nu_\tau]$ when detected at great distance.

[11] Ordinarily one thinks of a maximization process as describing a condition of unstable equilibrium. However, in the present situation, two different kinds of “forces” (topological and quantum) are in opposition. Taken together, these “forces” are constantly attempting to maintain the product $f(a) = a \cdot b$ at its maximum value. That is, the “system” described by $f(a)$ should be in a state of stable equilibrium.

[12] To put it another way, $a_0^{-1}$ is the arithmetic mean of $a_{\frac{1}{3}}^{-1}$ and $a_1^{-1}$.

[13] In Section 3.0, the matrix elements $a$, $b$ and $c$ were calculated without an in-depth understanding of the underlying dynamics. The present description, based on a simple harmonic-oscillator “potential” $V(x)$, represents a very crude first-attempt to remedy this situation. However, strictly speaking, to reach equilibrium at $a = a_0$, it would also be necessary to include “damping” of some sort. This has not been done.