A non-trivial ghost kernel for the equivariant stable cohomotopy of projective spaces

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Abstract

It is shown that the ghost kernel for certain equivariant stable cohomotopy groups of projective spaces is non-trivial. The proof is based on the Borel cohomology Adams spectral sequence and the calculations with the Steenrod algebra afforded by it.

1 Introduction

In equivariant stable homotopy theory, see [6] and [7] for background, there are many results which compare the equivariant stable homotopy category to the usual, non-equivariant, stable homotopy category. For example, it is well-known that $G$-equivalences $f$ can be detected by their $H$-fixed points $f^H$ for the various subgroups $H \leq G$. In contrast, the analogous statement is known to be false for the class of $G$-null maps, and this will be amplified here.

For easy book-keeping, let us fix a prime $p$, and let $G$ be a finite group of order $p$. These groups have precisely two subgroups, and it will turn out that even this simple case is interesting enough for a start. The group $[X,Y]^G$ of equivariant stable homotopy classes between (pointed) $G$-CW-complexes $X$ and $Y$ can be studied by means of the ghost map

$$[X,Y]^G \longrightarrow H^0(G;[X,Y]) \oplus [X^G,Y^G].$$
which sends a stable $G$-map $f$ to the pair $(f, f^G)$. Clearly, the kernel of a ghost map is a place to look for genuinely equivariant phenomena. See [2] for a conceptual framework for related matters. The following examples show that ghost kernels can be non-trivial, so that, contrary to the case of equivalences, maps which are essential cannot necessarily be detected by their ghosts.

**Example 1.1.** The target of the ghost map for $[S^1, S^0]^G$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if both spheres have the trivial $G$-action. However, the splitting theorem shows that the group $[S^1, S^0]^G$ has an extra summand $[S^1, BG] \cong G$ for the present choice of $G$.

**Example 1.2.** The ‘boundary map’ of the cofibration sequence $EG_+ \to S^0 \to \tilde{E}G$ is in the ghost kernel of $[\tilde{E}G, \Sigma EG_+]^G$: the target of the ghost map is zero because both spaces $\tilde{E}G$ and $(EG_+)^G$ are contractible. However, if that map were zero, this would split $S^0$ into $EG_+ \vee \tilde{E}G$, but there is no non-trivial idempotent in the Burnside ring $[S^0, S^0]^G$.

The main aim of this text is to display a new family of elements in the ghost kernel for another naturally occurring situation: the equivariant stable cohomotopy of projective spaces. Apart from the intrinsic interest in these spaces [5] and maps like this as explained above, this is also used in [10] to show that the equivariant Bauer-Furuta invariants of Galois covers of smooth 4-manifolds are not determined by non-equivariant data. It has been that application which originally led to the work presented here.

Let us now investigate in more detail the $G$-spaces $X$ and $Y$ which will be considered here, and why they are of interest also from the point of view of pure equivariant homotopy theory. If $V$ is a complex $G$-representation, then $\mathbb{C}P(V)_+$ denotes the complex projective space of $V$ with the induced $G$-action, and with a disjoint $G$-fixed base point added. The other fixed points of $\mathbb{C}P(V)_+$ are the $G$-invariant lines $L$ in $V$. In the language of representation theory, the fixed point sets are the projective spaces of the isotypical components of $V$ which correspond to 1-dimensional irreducible representations of $G$. If $G$ is abelian, as in our case, all irreducible representations are 1-dimensional. This shows that the fixed point sets all have the same dimension if and only if $V$ is a multiple of the regular representation $CG$. That singles out this case, and we shall therefore assume it from now on.
on. Also, the $G$-spaces $\mathbb{C}P(V)$ appear as skeleta of the classifying space for $G$-line bundles.

At a fixed point $L$ of $\mathbb{C}P(V)$, the tangent $G$-representation is $\text{Hom}(L, V/L)$. (For typographical reasons, a notation such as $V/L$ may refer to the (orthogonal) complement of $L$ in $V$ in the following.) If $V$ is a multiple of the regular representation, these tangent representations are all isomorphic, namely to $W = V/\mathbb{C}$. This is another fact that makes the present choice of $V$ special, and it also shows that $S^W$ is a natural target in this situation. The corresponding collapse maps $\mathbb{C}P(V)_+ \to S^W$ to the one-point-compactification $S^W$ of $W$ generate the group

$$[\mathbb{C}P(V)_+, S^W]^G,$$

see [8], where it is shown that the ghost map is injective in that case. As it will turn out, we can change this if we replace $W$ by $W/\mathbb{R}G$, and this is situation studied here. Note that $\mathbb{R}G$ can be embedded in $a\mathbb{C}G/\mathbb{C}$ if and only if $a \geq 2$. This discussion motivates the following notation which will be used throughout the entire text.

Choose an integer $a \geq 2$ and let

$$(1.1) \quad V_a = a\mathbb{C}G \quad \text{and} \quad W_a = a\mathbb{C}G/(\mathbb{C} \oplus \mathbb{R}G),$$

considered as a complex and real $G$-representation, respectively. Then the main result proved in this text is the following.

**Theorem 1.3.** If $p \geq 5$, then the ghost kernel for $[\mathbb{C}P(V_a)_+, S^W_a]^G$ is non-trivial.

It is easily seen that the target of the ghost map is zero in this case, so that it needs to be shown that the group is non-zero, which is non-obvious. In fact, slightly more will be proven, see Theorem 8.2 For all $p \geq 3$ the $p$-power torsion of this group is an elementary abelian $p$-group of rank $r$ for some $1 \leq r \leq (p + 1)/2$. While this not only shows the non-triviality of the group, it also gives an upper bound on its structure. I have reasons to conjecture that the group turns out to be $\mathbb{Z}/p$ in all cases; see the remarks at the end of Section 8. Furthermore, it is
tempting to relate the groups for various \( a \geq 2 \), and this problem is discussed in Section 9.

The proof of Theorem 8.2 relies on the Adams spectral sequence based on Borel cohomology \([4]\) and on the computations done with it in \([9]\). In Sections 2 and 8 we will recall the necessary facts about that spectral sequence and about the Borel cohomology of projective spaces, respectively. The filtration of these by projective subspaces will later (in Section 7) be used to feed in the computations of \([9]\). Before that, in Sections 4, 5, and 6 a new method is introduced to algebraically calculate the groups of homomorphisms of modules over the Steenrod algebra in the relevant cases. The results obtained there are the other, new ingredient which is needed as an input for the computation. Section 8 combines all this to give a proof of the main theorem. There and in the concluding Section 9 we also discuss some related open problems.

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2 Borel cohomology and its Adams spectral sequence

Let \( p \) be an odd prime number, and let \( G \) be the cyclic group of order \( p \). The notation \( H^* \) will be used for (reduced) ordinary cohomology with coefficients in the field \( \mathbb{F} \) with \( p \) elements. For a finite (pointed) \( G \)-CW-complex \( X \), the Borel cohomology is defined as

\[
b^*X = H^*(EG_+ \wedge GX).
\]

Therefore, the coefficient ring \( b^* = b^*S^0 = H^*(BG_+) \) is the mod \( p \) cohomology ring of the group. Since \( p \) is odd, this is the tensor product of an exterior algebra
on a generator $\sigma$ in degree 1 and a polynomial algebra on a generator $\tau$ in degree 2.

If $X$ and $Y$ are finite $G$-CW-complexes, the Borel cohomology Adams spectral sequence takes the form

$$
E_2^{s,t} = \text{Ext}_{b^*b}^{s,t}(b^*Y, b^*X) \Rightarrow ([X, Y]^G_{t-s})_p.
$$

The convergence to the indicated target has been established by Greenlees, see [4].

As a vector space, the algebra $b^*b$ is the tensor product $A^* \otimes b^*$, where $A^*$ is the mod $p$ Steenrod algebra. The algebra $A^*$ is generated by the Bockstein element $\beta$ in degree 1, and the Steenrod powers $P^i$ for $i \geq 1$ in degree $2i(p - 1)$. By convention, $P^0$ is the unit of the Steenrod algebra. Often the total power operation

$$
P = \sum_{i=0}^{\infty} P^i
$$

will be used, which acts multiplicatively on cohomology algebras. As an example, the $A^*$-action on the coefficient ring $b^* = b^*S^0$ is given by

$$
\beta(\sigma) = \tau, \quad \beta(\tau) = 0, \quad P(\sigma) = \sigma, \quad \text{and} \quad P(\tau) = \tau + \tau^p.
$$

The multiplication in $b^*b = A^* \otimes b^*$ is the twisted product, the twist being given by the $A^*$-action on $b^*$.

3 Projective spaces and their Borel cohomology

Let $V$ be a complex $G$-representation. In order to describe the Borel cohomology of the projective spaces $\mathbb{C}P(V)_+$, a few basic facts about Chern classes of representations [11] will have to be recalled.

By definition, the Borel cohomology of $\mathbb{C}P(V)_+$ is the ordinary cohomology of the space $EG_+ \wedge_G \mathbb{C}P(V)_+ = (EG \times_G \mathbb{C}P(V))_+$. The space $EG \times_G \mathbb{C}P(V)$ is nothing but the projective bundle associated to the vector bundle $EG \times_G V$ over $BG$. There is a tautological line bundle over the space $EG \times_G \mathbb{C}P(V)$. Write $\xi_V$ for
its first Chern class. The same symbol will be used for the reduction modulo $p$. Let $n$ be the dimension of $V$. It follows from the Leray-Hirsch theorem that the $b^*$-module $b^*\mathbb{C}P(V)_+$ is free of rank $n$ with basis $1, \xi_V, \xi_V^2, \ldots, \xi_V^{n-1}$. The relation

$$\sum_{j=0}^{n} (-1)^j c_j(V) \xi_V^{n-j} = 0$$

(3.1)

can be used as the definition of the Chern classes $c_j(V)$ of the vector bundle $EG \times_G V$ over $BG$. If $V = V_1 \oplus V_2$ is a direct sum, then the total Chern class $c(V) = \sum_{j=0}^{n} c_j(V)$ equals the product of the total Chern classes of the summands: $c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2)$.

Since $G$ is cyclic, the representations are easy to describe. Given an integer $\alpha$, let $\mathbb{C}(\alpha)$ be the $G$-representation where a chosen generator of $G$ acts as multiplication by $\exp(2\pi i \alpha / p)$. We may define $\tau$ to be the first Chern class of the $G$-representation $\mathbb{C}(1)$. Since multiplication of irreducible 1-dimensional representations corresponds to addition in cohomology, the first Chern class of the representation $\mathbb{C}(\alpha)$ of $G$ is $\alpha \tau$. If

$$V = \bigoplus_{j=1}^{n} \mathbb{C}(\alpha_j),$$

consider the polynomial

$$f(V) = \prod_{j=1}^{n} (x - \alpha_j \tau)$$

in $b^*[x]$. For example, if $V$ is the complex regular representation $\mathbb{C}G$,

$$f(\mathbb{C}G) = \prod_{\alpha=1}^{p} (x - \alpha \tau) = x^p - \tau^{p-1}x = x(x^{p-1} - \tau^{p-1}).$$

(3.2)

This polynomial will become prominent in the following Section 4. By (3.1), the map from $b^*[x]$ to $b^*\mathbb{C}P(V)_+$ which sends $x$ to $\xi_V$ induces an isomorphism

$$b^*[x]/f(V) \cong b^*\mathbb{C}P(V)_+.$$  

(3.3)

The structure of $b^*\mathbb{C}P(V)_+$ as a $b^*$-module is clear by the Leray-Hirsch theorem. As for the action of the Steenrod algebra $A^*$, it suffices to study that on
the generators of \( b^*\mathbb{C}P(V)_+ \) as a \( b^* \)-module and, by multiplicativity, on \( \xi_V \). But this element has degree 2, so the action of the total Steenrod power \( P \) on it is clear: \( P(\xi_V) = \xi_V + \xi_V^p \). Since \( \xi_V \) is an integral class, \( \beta(\xi_V) = 0 \), so \( \beta \) acts trivially on all elements of even degree.

If \( U \subseteq V \) is a subrepresentation, the inclusion of \( \mathbb{C}P(U)_+ \) into \( \mathbb{C}P(V)_+ \) induces a surjection in Borel cohomology. In the following, the notation \( V/U \) will often denote an (orthogonal) complement of \( U \) in \( V \), for typographical reasons. If \( V \) contains a \( G \)-line \( L \), the cofibre sequence

\[
\mathbb{C}P(V/L)_+ \longrightarrow \mathbb{C}P(V)_+ \longrightarrow S^{\text{Hom}(L,V/L)}
\]

induces a short exact sequence

\[
0 \longleftarrow b^*\mathbb{C}P(V/L)_+ \longleftarrow b^*\mathbb{C}P(V/L \oplus L)_+ \longleftarrow b^*S^{\text{Hom}(L,V/L)} \longleftarrow 0
\]

of \( b^*b \)-modules, which in turn induces long exact sequences

\[
\cdots \longleftarrow \text{Ext}^{s+1,t}_{b^*b}(b^*Y, b^*S^{\text{Hom}(L,V/L)}) \longleftarrow \text{Ext}^{s,t}_{b^*b}(b^*Y, b^*\mathbb{C}P(V/L)_+) \longleftarrow \cdots
\]

for any \( Y \). These will be used frequently later on. The long exact sequences converge to the long exact sequences

\[
\cdots \longleftarrow [\Sigma^{-1}S^{\text{Hom}(L,V/L)}, Y]_{f-s}^G \longleftarrow [\mathbb{C}P(V/L)_+, Y]_{f-s}^G \longleftarrow \cdots
\]

induced by the cofibre sequence \((3.4)\).

### 4 From topology to algebra

One of our main ingredients for the later calculations with the Adams spectral sequence is the vector space \( \text{Hom}_{b^*b}(b^*S^W, b^*\mathbb{C}P(V_0)_+) \). The purpose of this section is to prove Proposition \([4.5]\) which establishes an isomorphism between that vector space and another one which is defined purely in terms of polynomial algebra.
**Definition 4.1.** Let $M_a$ be the vector space of all elements $\mu$ in $b^*\mathbb{C}P(V_a)_+$ of degree $(2a-1)p - 3$ for which the equation

$$P(\mu) = (1 + \tau^{p-1})^{\frac{2a-1}{2}} \mu$$

(4.1)

describes the action of the total Steenrod operation.

**Lemma 4.2.** Evaluation on a generator of $b^*S^W_a$ gives an isomorphism between the vector space $\text{Hom}_{b^*}(b^*S^W_a, b^*\mathbb{C}P(V_a)_+)$ and $M_a$.

**Proof.** First let us translate the grading into a suspension, so that we then have to deal with the group $\text{Hom}_{b^*}(\Sigma^{-1}b^*S^W_a, b^*\mathbb{C}P(V_a)_+)$. A $b^*$-linear map from $\Sigma^{-1}b^*S^W_a$ into $b^*\mathbb{C}P(V_a)_+$ is just a $b^*$-linear map which is also $A^*$-linear. A $b^*$-linear map like that is the same as an element $\mu$ in $b^*\mathbb{C}P(V_a)_+$ of degree one less than the degree of the generator of $b^*S^W_a$. This is to say that $\mu$ is to have degree $(2a-1)p - 3$. In other symbols:

$$\text{Hom}_{b^*}(\Sigma^{-1}b^*S^W_a, b^*\mathbb{C}P(V_a)_+) \cong b^{(2a-1)p-3}\mathbb{C}P(V_a)_+.$$ 

Such an element $\mu$ in $b^*\mathbb{C}P(V_a)_+$ corresponds to an $A^*$-linear map if and only if the Steenrod algebra acts on $\mu$ as on the generator of $b^*S^W_a$. In order to proceed, one has to know the $A^*$-action on the Borel cohomology of $S^W_a$. The (real) dimension of $W_a$ is $(2a-1)p - 2$ and the (real) dimension of its fixed point set is $(2a-1) - 2$. The difference is $(2a-1)(p-1)$. Thus $A^*$ acts on a generator of $b^*S^W_a$ as on $\tau^{(2a-1)p-1}$ in $b^*$. This means that $\beta$ acts trivially, and $P$ acts by multiplication with

$$(1 + \tau^{p-1})^{\frac{2a-1}{2}}.$$ 

Note that this element is not homogeneous, since $P$ is not homogeneous. This can now be compared to the action of the Steenrod algebra on $\mu$. Since the degree of $\mu$ is even, $\beta$ acts trivially. Thus, the only condition is on the power operations. One has to require that

$$P(\mu) = (1 + \tau^{p-1})^{\frac{2a-1}{2}} \mu$$

in $b^*\mathbb{C}P(V_{a-1})_+$ for $\mu$ to represent a $b^*b$-linear map. \qed
Motivated by the lemma, let us write
\[
\varepsilon_a = (2a - 1)\frac{p - 1}{2} = (p - 1)a - \frac{p - 1}{2}
\]
and
\[
h_a = (1 + \tau^{p-1}) \varepsilon_a
\]
for short. Thus, equation (4.1) now reads \(P(\mu) = h_a \mu\).

A comment on the degrees might be appropriate. Until now, all the degrees have come from the usual topologist’s grading of cohomology groups. Since our problem will eventually be reduced to polynomial algebra, it will be convenient to use algebraic degrees. Then the elements \(x\) and \(\tau\) of topological degree 2 will have algebraic degree 1. The algebraic degree of \(\mu\) will be written
\[
\delta_a = \frac{1}{2}((2a - 1)p - 3) = pa - \frac{p + 3}{2}.
\]
From now on, the degrees used will be algebraic unless stated otherwise.

Recall from (3.3) that one has an isomorphism \(b^* \mathbb{C}P(V) + \cong b^*[x]/f(V)\) for any \(G\)-representation \(V\). This will be used to identify the two rings. In particular, we have \(b^* \mathbb{C}P(V_a) + = b^*[x]/f(V_a)\). If we set \(r = f(\mathbb{C}G)\), then
\[
r = \prod_{\lambda \in \mathbb{F}}(x - \lambda \tau) = x^p - x\tau^{p-1} = x(x^{p-1} - \tau^{p-1})
\]
as in (3.2), and \(f(V_a) = f(\mathbb{C}G) = r^a\). Note that the algebraic degree of \(r^a\) is \(ap\).

**Definition 4.3.** Let \(M_a\) be the vector space of polynomials \(m\) in the subring \(\mathbb{F}[\tau, x]\) of \(b^*[x]\) such that the algebraic degree of \(m\) is \(\delta_a\) and \(r^a\) divides \(P(m) - h_a m\).

**Lemma 4.4.** There is an isomorphism between \(M_a\) and \(\overline{M}_a\) which is the identity on representatives.

**Proof.** With the notation already established, we see that \(\overline{M}_a\) is the vector space of all elements \(\mu\) in \(b^{2\delta_a} \mathbb{C}P(V_a) +\) for which the equation \(P(\mu) = h_a \mu\) holds.
First note that the algebraic degree of \( r^a \), which is \( ap \), is larger than the algebraic degree \( \delta_a \) of the \( \mu \). This shows that there are as many elements \( \mu \) of that degree in \( b^* \text{CP}(V_a)_+ \) as in \( \mathbb{F}[\tau, x] \). In other words, the map

\[
\mathbb{F}[\tau, x] \subset b^*[x] \longrightarrow b^*[x]/f(V_a) = b^* \text{CP}(V_a)_+
\]

which is the identity on representatives is an isomorphism in this degree. In the polynomial ring, the condition \( P(m) \equiv h_a m \) modulo \( r^a \) ensures that the image \( \mu \) of \( m \) satisfies \( P(\mu) = h_a \mu \).

Taken together, the two previous lemmas imply the following result, which yields the translation of our problem into polynomial algebra.

**Proposition 4.5.** The vector space \( \text{Hom}^1_{b^*b^*}(b^*S^w_{b^*}, b^* \text{CP}(V_a)_+) \) is isomorphic to the vector space \( M_a \) of polynomials \( m \) in the polynomial ring \( \mathbb{F}[\tau, x] \) such that the algebraic degree of \( m \) is \( \delta_a \) and \( r^a \) divides \( P(m) - h_a m \). An isomorphism is given by associating to a map the representative of the evaluation at the generator.

**Proof.** By Lemma 4.2, evaluation on a generator of \( b^*S^w_{b^*} \) gives an isomorphism between the vector space \( \text{Hom}^1_{b^*b^*}(b^*S^w_{b^*}, b^* \text{CP}(V_a)_+) \) and \( M_a \). By Lemma 4.4, there is an isomorphism between \( M_a \) and \( M_a \) which is the identity on representatives.

**5 The algebra in the case \( a = 2 \)**

As a special case of Definition 4.3, \( M_2 \subset \mathbb{F}[\tau, x] \) is the space of polynomials of degree \( 3(p - 1)/2 \) that satisfy \( P(m) \equiv h_2 m \) modulo \( r^2 \). The purpose of this section is to prove the following estimate.

**Proposition 5.1.** The dimension of \( M_2 \) is at least \( (p + 1)/2 \).

This will be achieved by first producing enough elements in \( M_2 \), and then noting that they are linearly independent.
Lemma 5.2. For every integer $k$ such that $0 \leq k \leq (p - 1)/2$, the polynomial
\[
\tau^{p-1}x^k(kx^{p-1} + (1 - k)\tau^{p-1})
\]
is in $M_2$.

Proof. The cases $k = 1$ and $k = 0$ can easily be dealt with directly. We will only deal with the case $k \geq 2$.

Let $m$ be the polynomial displayed in the proposition. Since it has the correct degree, it remains to be shown that $r^2$ divides $P(m) - h_2m$. Let us use the notation
\[
E_\tau = (1 + \tau^{p-1}) \quad \text{and} \quad E_x = (1 + x^{p-1}).
\]
Then we have $P(\tau) = \tau E_\tau$, $P(x) = xE_x$, $r = x(E_x - E_\tau)$ and $h_2 = E_\tau^{3(p-1)/2}$. Rearranging the terms, one sees that $P(m) - h_2m$ equals $\tau^{\frac{p-1}{2} - k} x^k$ times the following term.

(5.1) \[
kx^{p-1}(E_\tau^{p-1+k} E_\tau^{\frac{p-1}{2} - k} - E_\tau^{\frac{p-1}{2} - k}) + (1 - k)\tau^{p-1}(E_x E_\tau^{p-1 + k} - E_\tau^{p-1} E_\tau^{\frac{p-1}{2} - k})
\]

Since $k \geq 2$, the polynomial $P(m) - h_2m$ is divisible by $x^2$, and so in this case it remains to be shown that the term (5.1) is divisible by $(E_x - E_\tau)^2$. Now
\[
E_\tau^{p-1+k} E_\tau^{\frac{p-1}{2} - k} - E_\tau^{\frac{p-1}{2} - k} = E_\tau^{\frac{p-1}{2} - k}(E_\tau^{p-1+k} - E_\tau^{p-1+k}) = E_\tau^{\frac{p-1}{2} - k}(E_x - E_\tau) \sum_{j=0}^{p+k-2} E_x^{p+k-2-j} E_\tau^j
\]
and
\[
E_x^{k} E_\tau^{\frac{p-1}{2} - k} - E_\tau^{\frac{p-1}{2} - k} = E_\tau^{\frac{p-1}{2} - k}(E_x - E_\tau) \sum_{j=0}^{k-1} E_x^{k-1-j} E_\tau^j
\]
are both divisible by $E_x - E_\tau$. It remains to be shown that
\[
kx^{p-1} E_\tau^{\frac{p-1}{2} - k} (\sum_{j=0}^{p+k-2} E_x^{p+k-2-j} E_\tau^j) + (1 - k)\tau^{p-1} E_\tau^{\frac{p-1}{2} - k} (\sum_{j=0}^{k-1} E_x^{k-1-j} E_\tau^j)
\]
is divisible by $E_x - E_{\tau}$. But modulo $E_x - E_{\tau}$ we have $E_x \equiv E_{\tau}$, so this is

$$kx^{p-1}E_{\tau}^{p-1}(p + k - 1)E_{\tau}^{p+k-2} + (1-k)\tau^{p-1}E_{\tau}^{3p-1-k}(kE_{\tau}^{k-1})$$

$$= E_{\tau}^{3p-1-k}(k(k-1)x^{p-1} + (1-k)k\tau^{p-1})$$

$$= E_{\tau}^{3p-1-k}(k^2 - k)(x^{p-1} - \tau^{p-1}),$$

and $x^{p-1} - \tau^{p-1} = E_x - E_{\tau} \equiv 0$ modulo $E_x - E_{\tau}$. This finishes the proof. \hfill \Box

Proof of Proposition 5.1 The $(p + 1)/2$ elements in the preceding lemma are linear independent in $\mathbb{F}[\tau, x]$. This follows from an inspection of the matrix which expresses these in terms of the monomial basis: that matrix has full rank. \hfill \Box

6 The algebra in the cases $a > 2$

In this section, we will see that the vector spaces $M_a$ for $a > 2$ relate to $M_2$. Before that, some general remarks are in order.

A linear polynomial in $\mathbb{F}[\tau, x]$ is one of the form $\kappa \tau + \lambda x$ for some $\kappa$ and $\lambda$ in $\mathbb{F}$. A polynomial will be called split if it is a product of linear polynomials. Assume that $m$ is split into linear factors of the form $L_j = x - \lambda_j \tau$:

$$m = \prod_j L_j.$$

Since $P(L_j) = (L_j + L_j^p) = L_j(1 + L_j^{p-1})$ and $P$ is multiplicative, we have

$$P(m) = m \prod_j (1 + L_j^{p-1}).$$

This proves that, if $m$ splits as above, then $m$ divides $P(m)$. Let us write

$$Q(m) = P(m)/m$$
for the quotient in this case, and recall the definitions of $h_a$, $\delta_a$, and $\varepsilon_a$ from (4.3), (4.2), and (4.4). If $m$ splits, then $m$ divides also $P(m) - h_a m$, and the quotient is

\[(6.1) \quad Q(m) - h_a = \left( \prod_j \left(1 + \frac{1}{L_j} \right)^{p-1} \right) - \left(1 + \frac{1}{\tau^{p-1}} \right)^{2a-1} \frac{\varepsilon_a}{\tau} \]

Let us count the number of factors in both of these terms. One the one hand, if $m$ splits, it does so into $\delta_a$ factors. On the other hand, the polynomial $h_a$ has $\varepsilon_a$ factors $(1 + \tau^{p-1})$. The difference between those two numbers is

\[(6.2) \quad \delta_a - \varepsilon_a = a - 2,\]

so the numbers $\delta_a$ and $\varepsilon_a$ agree if and only if $a = 2$.

By definition, see (4.5), the element $r$ in $\mathbb{F}[\tau, x]$ splits. In order to give a nice formula for $Q(r)$, some more preliminaries are necessary.

\textbf{Lemma 6.1.} For any $\kappa$ in $\mathbb{F}$, the equation

\[x^p - \tau x = (x - \kappa \tau) (\tau x - (x - \kappa \tau)^{p-1})\]

holds in $\mathbb{F}[\tau, x]$.

\textit{Proof.} Note that

\[\prod_{\lambda \in \mathbb{F}} (x - \lambda \tau) = \prod_{\lambda \in \mathbb{F}} (x - (\kappa + \lambda) \tau) = \prod_{\lambda \in \mathbb{F}} ((x - \kappa \tau) - \lambda \tau),\]

and therefore

\[x^p - \tau x = \left( x - \kappa \tau \right)^p - \tau x - \kappa \tau = \left( x - \kappa \tau \right)(\tau x - (x - \kappa \tau)^{p-1}),\]

which proves the claim. \hfill \Box
Lemma 6.2. In $\mathbb{F}[\tau,x][K]$ the equation
\[
\prod_{\lambda \in \mathbb{F}} (K + (x - \lambda \tau)^{p-1}) = (x^p - \tau^{p-1}x)^{p-1} + K(K + \tau^{p-1})^{p-1}
\]
holds.

Proof. For any $\kappa$ in $\mathbb{F}$ we can substitute $K = -(x - \kappa \tau)^{p-1}$ into the right hand side of the equation. We get
\[
(x^p - \tau^{p-1}x)^{p-1} - (x - \kappa \tau)^{p-1}(- (x - \kappa \tau)^{p-1} + \tau^{p-1})^{p-1}.
\]
But this is zero by the equation from the previous lemma, raised to the $(p - 1)$-st power. This argument shows that the left hand side of the equation divides the right hand side. The claim follows by comparing the degrees and a coefficient. \(\square\)

One gets the equation
\[
\prod_{\lambda \in \mathbb{F}} (1 + (x - \lambda \tau)^{p-1}) = (x^p - \tau^{p-1}x)^{p-1} + (1 + \tau^{p-1})^{p-1}
\]
by specializing to $K = 1$ in the previous lemma. Using different notation, this equation says the following.

Proposition 6.3. The equation
\[
(6.3) \quad Q(r) = r^{p-1} + (h_{a+1}/h_a)
\]
holds for every $a \geq 2$.

Let $\kappa$ be in $\mathbb{F}$. Then $r$ is clearly divisible by $x - \kappa \tau$, but $h_{a+1}/h_a$ is not. Therefore, the equation (6.3) above shows that $Q(r)$ is not divisible by $x - \kappa \tau$.

We can now relate the cases $a > 2$ to the case $a = 2$. Recall that $M_a$ has been defined to be the set of polynomials $m$ in $\mathbb{F}[t,x]$ of degree $\delta_a$ that satisfy the condition $P(m) \equiv h_a m$ modulo $r^a$.

Lemma 6.4. If $a \geq 3$ then every element $m$ in $M_a$ is divisible by $r$. \(\square\)
Proof. Some notation is needed first. We can write
\[ m = \sum_{i+j=\delta} c_{i,j} \tau^i x^j \]
with some coefficients \( c_{i,j} \) in \( \mathbb{F} \). Furthermore, setting \( E_\tau = (1 + \tau^{p-1}) \) and similarly \( E_x = (1 + x^{p-1}) \), we have \( P(\tau) = \tau E_\tau \) as well as \( P(x) = x E_x \). Consequently,
\[ P(m) - h_a m = \sum_{i+j=\delta} c_{i,j} \tau^i x^j (E_\tau^i E_x^j - h_a). \]
By assumption on \( m \), we have \( P(m) - h_a m = r^a s \) for some \( s \) in \( \mathbb{F}[\tau, x] \). Putting these together gives
\begin{equation}
(6.4) \quad r^a s = \sum_{i+j=\delta} c_{i,j} \tau^i x^j (E_\tau^i E_x^j - h_a).
\end{equation}
To prove the claim, it suffices to show that \( x - \kappa \tau \) divides \( m \) for each \( \kappa \) in \( \mathbb{F} \).
First assume \( \kappa \neq 0 \). Modulo \( x - \kappa \tau \), we have \( r \equiv 0 \) and \( E_x \equiv E_\tau \), so that equation (6.4) above shows that
\[ 0 \equiv \left( \sum_{i+j=\delta} c_{i,j} \tau^i x^j \right) (E_\tau^{\delta_a} - h_a) \]
modulo \( x - \kappa \tau \). In other words, the right hand side, which is \( m(E_\tau^{\delta_a} - h_a) \), is divisible by \( x - \kappa \tau \). Both, \( E_\tau^{\delta_a} \) and \( h_a \), are powers of \( 1 + \tau^{p-1} \). By (6.2), the exponents differ by \( a - 2 \), so the assumption on \( a \) implies that \( E_\tau^{\delta_a} \neq h_a \). As a consequence, \( x - \kappa \tau \) does not divide \( E_\tau^{\delta_a} - h_a \), so it must divide \( m \).
It remains to show that \( x \) divides \( m \). But modulo \( x \), the equation (6.4) reads
\[ 0 \equiv c_{\delta_a,0} \tau^{\delta_a} (E_\tau^{\delta_a} - h_a). \]
Using \( E_\tau^{\delta_a} \neq h_a \) again, it follows that \( c_{\delta_a,0} \) is zero. In other words, \( x \) divides \( m \).

Lemma 6.5. If \( 3 \leq a \leq p \), division by \( r \) yields an injection from the vector space \( M_a \) into \( M_{a-1} \).
Proof. The previous result shows that every element of \( M_a \) is divisible by \( r \). Given \( rm \) in \( M_a \), one has to show that \( m \) is in \( M_{a-1} \). First of all, the degree of \( m \) is correct. Since \( rm \) is in \( M_a \), it is true that \( P(rm) - h_ar_m \) is divisible by \( r^a \). By assumption, \( r^a \) divides \( r^p mh_{a-1} \), so that \( r^a \) divides

\[
P(rm) - h_ar_m - r^p mh_{a-1} = P(rm) - (h_a + r^{p-1}h_{a-1})rm
\]
\[
= Q(r)P(m) - Q(r)h_{a-1}rm
\]
\[
= Q(r)(rP(m) - rh_{a-1}m).
\]

Since none of the (linear) factors of \( r \) divides \( Q(r) \), we can deduce that the polynomial \( rP(m) - rh_{a-1}m \) must be divisible by \( r^a \), so that \( r^{a-1} \) divides \( P(m) - h_{a-1}m \), which shows that \( m \) is in \( M_{a-1} \). Thus, the map is well-defined. Since \( r \) is not a zero-divisor, it is injective.

Lemma 6.6. Let \( a \) and \( b \) be integers such that \( 2 \leq a \leq p - 1 \) and \( a \leq b \). Then, multiplication with \( r^{b-a} \) is an injection from \( M_a \) into the group \( M_b \).

Proof. Injectivity is clear, if the map is well-defined at all. But, as will be shown now, it is. Equation (6.3) implies that \( Q(r) \equiv h_{a+1}/h_a \) modulo \( r^{p-1} \). Using that result \( b - a \) times, we get \( Q(r)^{b-a} \equiv h_b/h_a \) modulo \( r^{p-1} \). Hence we know that

\[
h_a Q(r)^{b-a} mr^{b-a} \equiv h_br^{b-a}
\]

modulo \( r^{p-1}r^{b-a} \) and therefore, by the assumption on \( a \), also modulo \( r^b \). Suppose now that \( m \) is in \( M_a \). Then \( mr^{b-a} \) has the required degree. If, furthermore, \( m \) satisfies \( P(m) \equiv h_am \) modulo \( r^a \), we have

\[
P(mr^{b-a}) = P(m)r^{b-a}Q(r)^{b-a}
\]
\[
\equiv h_amr^{b-a}Q(r)^{b-a}
\]
\[
\equiv h_br^{b-a}
\]

modulo \( r^b \), and that is what was to be shown.

The following proposition sums up the preceding three lemmas.
Proposition 6.7. Multiplications by $r$ yield isomorphisms

$$M_2 \cong M_3 \cong \ldots \cong M_{p-1} \cong M_p$$

and injections from these into $M_a$ for every $a \geq 2$.

7 Obstructions

As described in Section 2, the $p$-power torsion in $[\mathbb{CP}(V_a)^+, S_{W_a}]^G$ will be detected by the Borel cohomology Adams spectral sequence

$$\text{Ext}^{s,t}_{b^*b}(b^*S_{W_a}, b^*\mathbb{CP}(V_a)^+) \rightarrow ([\mathbb{CP}(V_a)^+, S_{W_a}]^G)_{t-s}^\wedge_p.$$ 

The $E_2$-page of that spectral sequence is hard to come by directly. Instead, we may filter the projective space $\mathbb{CP}(V_a)^+$ by projective subspaces $\mathbb{CP}(V)^+$ for $V \subseteq V_a$. The filtration quotients are linear $G$-spheres, so we may feed in results from [9] one filtration step at a time. In this section, it will be described how this can be done. For the most part, this is straightforward, and only at the end of it, we will spot the crux of the matter. See Lemma 7.8 which uses notation introduced at the beginning of Section 7.2. The reader may consider to take this for granted for the time being and skip to the next section now to see how it fits into the puzzle.

7.1 The cases $V \subseteq V_{a-1}$

The following result is a consequence of Proposition 5 of [9] and its corollaries.

Lemma 7.1. Let $V$ be a complex $G$-representation, $L \subseteq V$ a complex line, and $W$ be any $G$-representation that contains $\text{Hom}_\mathbb{C}(L, V/L)$ up to isomorphism. Then the inclusion of $\mathbb{CP}(V/L)^+$ into $\mathbb{CP}(V)^+$ induces an isomorphism

$$\text{Ext}^{s,t}_{b^*b}(b^*S^W, b^*\mathbb{CP}(V/L)^+) \cong \text{Ext}^{s,t}_{b^*b}(b^*S^W, b^*\mathbb{CP}(V)^+)$$

for $t - s < \dim_{\mathbb{R}} W^G - \dim_{\mathbb{R}} \text{Hom}_\mathbb{C}(L, V/L)^G$. 

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An induction on the dimension of $V$ now proves the following result.

**Proposition 7.2.** Let $V \subseteq V_{a-1}$ be a complex subrepresentation. Then

$$\text{Ext}_{b^*}^{s,t}(b^*S^a_+, b^*\mathbb{CP}(V)_+) = 0$$

for $t - s \leq 0$. In particular, $\text{Ext}_{b^*}^{s,t}(b^*S^a_+, b^*\mathbb{CP}(V_{a-1})_+) = 0$ for $t - s \leq 0$.

As a consequence, note $[\mathbb{CP}(V_{a-1})_+, S^a_+]^G = 0$. Although it will not be necessary to calculate $[\Sigma \mathbb{CP}(V_{a-1})_+, S^a_+]^G$, it will be good to know one of the groups on the relevant column of the $E_2$-page. These will be described in the following proposition.

**Proposition 7.3.** Let $V$ be a complex subrepresentation of $V_{a-2}$. Then the vector space $\text{Hom}_{b^*}^1(b^*S^a_+, b^*\mathbb{CP}(V)_+)$ is zero. Let $U$ be a complex subrepresentation of $\mathbb{C}G$, and set $V = V_{a-2} \oplus U$, so that $V_{a-2} \subseteq V \subseteq V_{a-1}$. Then

$$\dim \text{Hom}_{b^*}^1(b^*S^a_+, b^*\mathbb{CP}(V)_+) = \dim \mathbb{C} U.$$

In particular, $\dim \text{Hom}_{b^*}^1(b^*S^a_+, b^*\mathbb{CP}(V_{a-1})_+) = p$.

**Proof.** The first part can also be proven by induction on the dimension of $V$. The result is stated in the form in which it will be used later; but the pair $(s, t) = (0, 1)$ could be replaced by any pair $(s, t)$ such that $t - s \leq 2$.

The second part can be proven by induction on $\dim \mathbb{C} U$, using Propositions 5 and 9 of [9] and their corollaries.  

### 7.2 The cases $V_{a-1} \subseteq V$

We may now start filtering $V_a$ beyond $V_{a-1}$. Given an integer $\alpha$, let $\mathbb{C}(\alpha)$ be the $G$-representation where a chosen generator of $G$ acts by multiplication by $\exp(2\pi i \alpha / p)$. For any integer $k$ such that $0 \leq k \leq p$ consider the subrepresentation

$$U_k = \bigoplus_{\alpha=0}^{k-1} \mathbb{C}(\alpha)$$
of \( \mathbb{C}G \). This gives a flag

\[
(7.1) \quad 0 = U_0 \subset U_1 \subset \cdots \subset U_p = \mathbb{C}G
\]

with the property that \( U_1 \) is the trivial \( G \)-line and such that also both \( U_{(p+1)/2}/U_1 \) and \( U_p/U_{(p+1)/2} \) are isomorphic to \( \mathbb{R}G/\mathbb{R} \) as real \( G \)-representations. The flag (7.1) yields a filtration of \( \mathbb{C}P(V_a)_+ \) by projective spaces \( \mathbb{C}P(V_{a-1} \oplus U_k)_+ \). For \( k = 0 \) this is \( \mathbb{C}P(V_{a-1})_+ \); for \( k = p \) it is \( \mathbb{C}P(V_a)_+ \). To compute the relevant part of the \( E_2 \)-page of the spectral sequence for \( [\mathbb{C}P(V_{a-1} \oplus U_k)_+, SW_a]^G \), we will proceed inductively. The case \( k = 0 \) has already been settled in the previous subsection. One may therefore assume that \( k \geq 1 \), and that the \( E_2 \)-page for \( \mathbb{C}P(V_{a-1} \oplus U_{k-1})_+ \) has been studied. For typographical reasons, we will use the notation \( V_{a-1,k-1} \) for \( V_{a-1} \oplus U_{k-1} \) from now on, and similarly in other cases.

Let us first see what the quotient \( G \)-spheres of the filtration are. The cofibre of the inclusion of \( \mathbb{C}P(V_{a-1,k-1})_+ \) into \( \mathbb{C}P(V_{a-1,k})_+ \) is given by \( S^{\Pi(a-1,k)} \) with

\[
\overline{H}(a - 1,k) = \text{Hom}_C(U_k/U_{k-1}, V_{a-1,k-1}) \cong V_{a-1} \oplus \text{Hom}_C(U_k/U_{k-1}, U_{k-1}).
\]

The cofibration sequence induces a short exact sequence in Borel cohomology. Applying the functors \( \text{Ext}^s_{b^*b}(b^*SW_a, ?) \) yields long exact sequences. In order to use them, one will have to know the groups

\[
(7.2) \quad \text{Ext}^s_{b^*b}(b^*SW_a, b^*S^{\Pi(a-1,k)})
\]

for \( t - s = 0, 1 \). These groups can be simplified as follows. Since \( U_k \subseteq \mathbb{C}G \), the representation \( \text{Hom}_C(U_k/U_{k-1}, U_{k-1}) \) has no trivial summand. If we write

\[
H(k) = \text{Hom}_C(U_k/U_{k-1}, U_k) \cong \overline{H}(0,k) \oplus \mathbb{C},
\]

then \( H(k) \cong \text{Hom}_C(U_k/U_{k-1}, U_{k-1}) \oplus \mathbb{C} \) is a complex \( k \)-dimensional representation which has exactly one trivial summand. Because of

\[
\text{Ext}^s_{b^*b}(b^*SW_a, b^*S^{\Pi(a-1,k)})
\]

for \( t - s = 0, 1 \). These groups can be simplified as follows. Since \( U_k \subseteq \mathbb{C}G \), the representation \( \text{Hom}_C(U_k/U_{k-1}, U_{k-1}) \) has no trivial summand. If we write

\[
H(k) = \text{Hom}_C(U_k/U_{k-1}, U_k) \cong \overline{H}(0,k) \oplus \mathbb{C},
\]

then \( H(k) \cong \text{Hom}_C(U_k/U_{k-1}, U_{k-1}) \oplus \mathbb{C} \) is a complex \( k \)-dimensional representation which has exactly one trivial summand. Because of

\[
\text{Ext}^s_{b^*b}(b^*SW_a, b^*S^{\Pi(a-1,k)})
\]
Proposition 3 from [9] implies that the group (7.2) is isomorphic to

\[(7.3) \quad \text{Ext}^{\mathcal{E}}_{b^*b}(b^*S^{RG}, b^*S^{H(k)})\].

Note that this is independent of \(a\).

The following discussion is divided into three cases: First we deal with the cases where \(k \leq (p-1)/2\), then with the case \(k = (p+1)/2\), and finally with the remaining cases where \(k \geq (p+3)/2\).

### 7.3 The cases \(k \leq (p-1)/2\)

Here, \(\text{Ext}^{\mathcal{E}}_{b^*b}(b^*S^{RG}, b^*S^{H(k)})\) as in (7.3) is isomorphic to \(\text{Ext}^{\mathcal{E}}_{b^*b}(b^*S^{W}, b^*S^{2})\) for some subrepresentation \(W\) of \(RG\) which properly contains the trivial subrepresentation.

**Proposition 7.4.** If \(k \leq (p-1)/2\) then

\[
\dim_{\mathbb{F}} \text{Ext}^{\mathcal{E}}_{b^*b}(b^*S^{W_a}, b^*\mathbb{C}P(V_{a-1,k})^+) = \begin{cases} 
0 & t-s \leq -2 \\
k & t-s = -1 \\
0 & t-s = 0 \\
p & (s,t) = (0,1)
\end{cases}
\]

holds. The multiplicative structure is the expected one: In the \(t-s = -1\) column, multiplication with the generator of \(\text{Ext}^{1,1}\) which represents multiplication with \(p\) is injective. In the target, this leads to a free module of rank \(k\) over the \(p\)-adic integers.

**Proof.** This is again a straightforward induction on \(k\), using the long exact sequence associated with the extension

\[
0 \leftarrow b^*\mathbb{C}P(V_{a-1,k-1})^+ \leftarrow b^*\mathbb{C}P(V_{a-1,k})^+ \leftarrow b^*S^{\mathcal{I}(a-1,k)} \leftarrow 0,
\]

and the data from Figure 8 in Section 4 of [9].
7.4 The first interesting case: $k = (p + 1)/2$

In this case, the sphere $S^{\mathbb{H}(a, \frac{p+1}{2})}$ is the suspension of $S^W_a$, so that there is an isomorphism $\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* S^{\mathbb{H}(a-1, \frac{p+1}{2})}) \cong \text{Ext}^{s, t}_{b^* b}(b^*, b^* S^1)$.

The long exact sequence (3.5) shows that $\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p+1)/2}))$ is zero as soon as the condition $t - s \leq -2$ is fulfilled. The next case $t - s = -1$ is easy, too, since then the vector spaces $\text{Ext}^{s-1, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p-1)/2}))$ and $\text{Ext}^{s+1, t}_{b^* b}(b^* S^W_a, b^* S^{\mathbb{H}(a-1, \frac{p+1}{2})})$ both vanish. In this way we may therefore deduce that the vector space $\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p+1)/2}))$ is an extension of the vector space $\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p-1)/2}))$ by the vector space $\text{Ext}^{s+1, t}_{b^* b}(b^* S^W_a, b^* S^{\mathbb{H}(a-1, \frac{p+1}{2})})$. Using the data from [9], Figure 4 in Section 1, one obtains the following.

$$\dim \text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p+1)/2})) = \begin{cases} 
(p+1)/2 & s = 0 \\
(p+3)/2 & s \geq 1 
\end{cases}$$

The multiplicative structure is again as expected: In the column $t - s = -1$, multiplication with the generator of $\text{Ext}^{1,1}$ which represents multiplication with $p$ is injective. Starting with $t - s = 0$, the situation becomes more interesting. Then, the map

$$\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p+1)/2})) \rightarrow \text{Ext}^{s+1, t}_{b^* b}(b^* S^W_a, b^* S^{\mathbb{H}(a-1, \frac{p+1}{2})})$$

is surjective, since $\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p-1)/2}))$ is zero. As the right hand side is non-zero only for $(s, t) = (1, 1)$, so is $\text{Ext}^{s, t}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p+1)/2}))$. Thus, it remains to determine $\text{Ext}^{1,1}_{b^* b}(b^* S^W_a, b^* \mathbb{C}(V_{a-1,(p+1)/2}))$.

The vector space $\text{Ext}^{1,1}_{b^* b}(b^* S^W_a, b^* S^{\mathbb{H}(a-1, \frac{p+1}{2})})$ is 1-dimensional. The kernel of the surjection displayed right above is isomorphic – via the boundary homomorphism of the long exact sequence – to the cokernel of the map induced by $\text{Hom}^{1}_{b^* b}(b^* S^W_a, \mathbb{C})$. But that induced map is injective, since the group $\text{Hom}^{1}_{b^* b}(b^* S^W_a, b^* S^{\mathbb{H}(a-1, \frac{p+1}{2})})$ is zero. To summarise:
**Lemma 7.5.** The vector space $\text{Ext}_{b^*a^1}^{1,1}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,(p+1)/2}^+) + )$ is either 1-dimensional or zero, depending on whether the injection

$$\text{Hom}_{b^*a^1}^{1,1}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,(p+1)/2}^+) + )$$

$$\text{is also surjective (and therefore an isomorphism) or not (in which case the cokernel is 1-dimensional).}$$

By Proposition 7.4, the space $\text{Hom}_{b^*a^1}^{1,1}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,(p+1)/2}^+) + )$ has dimension $p$. Thus it would be sufficient to know the dimension of the other vector space $\text{Hom}_{b^*a^1}^{1,1}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,(p+1)/2}^+) + )$ in order to determine the dimension of the space $\text{Ext}_{b^*a^1}^{1,1}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,(p+1)/2}^+) + )$. For the moment, let us leave it like that and see how we can proceed.

### 7.5 The final cases: $k \geq (p + 3)/2$

In these cases the vector spaces $\text{Ext}_{b^*a^1}^{s,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k}^+) + )$ as in (7.3) are isomorphic to $\text{Ext}_{b^*a^1}^{s,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k}^+) + )$ for some subrepresentation $V \subseteq \mathbb{R}G$ properly containing the trivial representation. The calculations summarised in Figure 6 of Section 3 in [9] are relevant here.

To determine the dimension of the vector space $\text{Ext}_{b^*a^1}^{s,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k}^+) + )$, a long exact sequence (3.5) will be invoked again. Some of these groups will be non-zero for $t - s \leq -2$, since this holds for $\text{Ext}_{b^*a^1}^{s-1,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k-1}^+) + )$. One can ignore these, since eventually only the case $t - s = 0$ is of interest. For the latter, it is good to know about the groups with $t - s = -1$. In this case, $\text{Ext}_{b^*a^1}^{s-1,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k-1}^+) + )$ vanish, and so do the groups $\text{Ext}_{b^*a^1}^{s+1,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k-1}^+) + )$. As in the previous case, we get a splittable short exact sequence. By induction, using (7.4),

$$\text{dim}_{\mathbb{F}} \text{Ext}_{b^*a^1}^{s,t}(b^*a^1, b^*a^1)\text{CP}(V_{a-1,k}^+) + ) = \begin{cases} 
(p + 1)/2 & s = 0 \\
 k + 1 & s \geq 1. 
\end{cases}$$
The multiplicative structure is again as expected: In the column $t - s = -1$, multiplication with the generator of $\text{Ext}^{1,1}$ which represents multiplication with $p$ is injective.

Now let us turn to the most interesting situation: $t - s = 0$. If in addition $s \neq 1$, the groups $\text{Ext}_{b^*b}^{s,t}(b^*S^W_a, b^*\text{CP}(V_{a-1,k}+))$ are zero. Using that and the corresponding result from the first interesting case as an input, an induction shows that the groups $\text{Ext}_{b^*b}^{s,t}(b^*S^W_a, b^*\text{CP}(V_{a-1,k}+))$ vanish for $s \neq 1$. For $k = p$ this implies the following results.

**Proposition 7.6.** All $p$-power torsion in $[\text{CP}(V_a)_+, S^W_a]^G$ has order $p$.

Therefore, the $p$-adic completion of that group is elementary abelian of some rank $r$. Eventually, we will prove upper and lower bounds on $r$, see Section

**Proposition 7.7.** The non-trivial elements in $[\text{CP}(V_a)_+, S^W_a]^G$ are detected by their Borel cohomology $e$-invariants.

It remains to discuss the vector spaces $\text{Ext}_{b^*b}^{1,1}(b^*S^W_a, b^*\text{CP}(V_{a-1,k}+))$. This is a bit more complicated than in the first interesting case since this time the vector space $\text{Ext}_{b^*b}^{1,1}(b^*S^W_a, b^*\text{CP}(V_{a-1,k-1}+))$ may already be non-zero. Moreover, the boundary homomorphism

$\text{Ext}_{b^*b}^{2,1}(b^*S^W_a, b^*\text{CP}(V_{a-1,k}+)) \leftarrow \text{Ext}_{b^*b}^{1,1}(b^*S^W_a, b^*\text{CP}(V_{a-1,k-1}+))$

maps into a non-zero group. Nevertheless, it is the zero map. This follows from the multiplicative structure and the fact that the boundary map respects this. Thus, while the argument is a little more complicated than in the first interesting case, the result is the same:
Lemma 7.8. For every integer $k$ such that $(p + 1)/2 \leq k \leq p$, the map

$$\text{Ext}^{1,1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,k-1}^+)) \leftarrow \text{Ext}^{1,1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,k}^+))$$

is a surjection. The kernel of this homomorphism is either 1-dimensional or zero, depending on whether the injection from $\text{Hom}^{1,1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,k}^+))$ into $\text{Hom}^{1,1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,k-1}^+))$ is also surjective (and therefore an isomorphism) or not (in which case the cokernel is 1-dimensional).

8 Upper and lower bounds

We are now able to put together the algebraic calculations from Sections 4, 5, 6, and the obstruction theory of the previous section to prove our main result, Theorem 8.2. The following result summarizes Propositions 4.5, 5.1, and 6.7 from the former.

Proposition 8.1. For any integer $a \geq 2$, the dimension of the vector space

$$\text{Hom}^{1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_a^+))$$

is at least $(p + 1)/2$.

As for the latter, in Section 7 we have investigated two chains

\begin{align*}
\text{Ext}^{1,1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,(p-1)/2}^+)) & \quad \text{Hom}^{1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,(p-1)/2}^+)) \\
\text{Hom}^{1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,(p+1)/2}^+)) & \quad \text{Hom}^{1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,(p+1)/2}^+)) \\
& \quad \vdots \\
\text{Ext}^{1,1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,p}^+)) & \quad \text{Hom}^{1}_{b^*b}(b^*S^w_a, b^*\mathbb{CP}(V_{a-1,p}^+))
\end{align*}

Lemma 7.5
of homomorphisms. Lemmas 7.5 and 7.8 show that the homomorphisms on the
left hand side are surjective and each kernel is at most 1-dimensional. The maps
on the right hand side are injections, and each cokernel is at most 1-dimensional.
A map on the left is an isomorphism if and only if the corresponding map on
the right is not. Consequently, for any integer $k$ such that $(p + 1)/2 \leq k \leq p$, the
dimensions of $\text{Ext}^{1,1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_{a-1,k}^+))$ and $\text{Hom}^{1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_{a-1,k}^+))$
add up to $p$. Thus, in order to get information on the former, it would be enough to
know something about the latter, and this is exactly what Proposition 8.1 provides
for. This gives the main result of this text.

**Theorem 8.2.** The $p$-power torsion of the group $[\text{CP}(V_a^+), S^W_a]^G$ is non-zero ele-
mentary abelian of rank $r$ with $1 \leq r \leq (p + 1)/2$.

**Proof.** We know from Proposition 7.6 that the group $[\text{CP}(V_a^+), S^W_a]^G$ is element-
ary abelian.

As for the upper bound on its rank, the length of the chains implies that the vector
space

$$\text{Hom}^{1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_{a-1,p}^+)) = \text{Hom}^{1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_a^+))$$

at the bottom of the right hand side is at least $(p - 1)/2$-dimensional, and that
consequently $\text{Ext}^{1,1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_a^+))$ is at most $(p + 1)/2$-dimensional.

In order to obtain the lower bound on the $p$-torsion of $[\text{CP}(V_a^+), S^W_a]^G$, we can use Proposition 8.1 which states that for any integer $a \geq 2$, the dimension of the vector space

$$\text{Hom}^{1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_a^+))$$

is at least $(p + 1)/2$. This implies that one of the inclusions on the right chain
must be an isomorphism. Therefore, one of the surjections on the left cannot
be an isomorphism. So the group $\text{Ext}^{1,1}_{b^*b}(b^*S^W_a, b^*\text{CP}(V_a^+))$ is non-zero. By
multiplicativity, all the differentials must be zero on that group. Thus, the elements
survive to the $E_\infty$-page. \qed
There are indications that the upper bound given in Theorem 8.2 is far from being sharp. Using methods similar to those of Section 5, it can be improved roughly by a factor of 2. My results in this direction do not seem to justify a detailed account, in particular, as I suspect that the $p$-torsion in $[\mathbb{C}P(V_a)_+, S^W_a]_G$ is isomorphic to $\mathbb{Z}/p$ for all primes $p \geq 3$ and all integers $a \geq 2$. While I do not know how to prove this, it is consistent with computer experiments covering all the cases with $pa \leq 50$.

9 Blind alleys and dead ends

In this section, we will pursue the question of how the groups $[\mathbb{C}P(V_a)_+, S^W_a]_G$ may be related for varying $a \geq 2$. It follows from Propositions 4.5 and 6.7 that there are isomorphisms

$$\text{Hom}^1_{b*}(b^* S^W_2, b^* \mathbb{C}P(V_2)_+) \cong \text{Hom}^1_{b*}(b^* S^W_3, b^* \mathbb{C}P(V_3)_+) \cong \ldots$$

and injections from these into $\text{Hom}^1_{b*}(b^* S^W_a, b^* \mathbb{C}P(V_2)_+)$ for every $a \geq 2$. As in the proof of Theorem 8.2, this implies that there are $p$-local isomorphisms

$$[\mathbb{C}P(V_2)_+, S^W_2]^G \cong [\mathbb{C}P(V_3)_+, S^W_3]^G \cong \ldots \cong [\mathbb{C}P(V_p)_+, S^W_p]^G,$$

and injections from these into $[\mathbb{C}P(V_a)_+, S^W_a]^G$ for every $a \geq 2$. However, the morphisms are given algebraically on the level of Adams spectral sequences by multiplication with a class originating from the regular representation. It would be enlightening to see a more geometric and less computational explanation of the phenomenon. However, there are some blind alleys and dead ends on the way towards such an interpretation, and it seems only fair to disclose three of them here.

First, note that $[\mathbb{C}P(V_a)_+, S^W_a]^G \not\cong [\mathbb{C}P(V_{a+1})_+, S^W_{a+1}]^G$ integrally: just compute the structure of these groups away from $p$. Thus the phenomenon is genuinely $p$-local.
Second, the groups $[\mathbb{C}P(V_a)_+, S^w_a]^G$ for varying $a$ are related as shown in the following commutative diagram, in which all the arrows are induced by inclusions.

$$
\begin{array}{c}
[\mathbb{C}P(V_a)_+, S^w_a]^G \\
\uparrow
\end{array} 
\quad 
\begin{array}{c}
[\mathbb{C}P(V_a)_+, S^w_{a+1}]^G \\
\uparrow
\end{array} 
\quad 
\begin{array}{c}
[\mathbb{C}P(V_{a+1})_+, S^w_a]^G \\
\uparrow
\end{array} 
\quad 
\begin{array}{c}
[\mathbb{C}P(V_{a+1})_+, S^w_{a+1}]^G
\end{array}
$$

However it is not possible to explain the phenomenon from this point of view: The horizontal maps are zero, since they are multiplication with the Euler class of $\mathbb{C}G$, which has non-zero fixed points. The group $[\mathbb{C}P(V_{a+1})_+, S^w_a]^G$ seems to be even more difficult to compute than the other three, whereas the group $[\mathbb{C}P(V_a)_+, S^w_{a+1}]^G$ is zero.

Third, one might wonder whether, after translating the situation into a $(T \times G)$-equivariant setting, a suspension isomorphism could be used to prove Theorem 1.3 or Theorem 8.2. But this is not the case: in the $(T \times G)$-equivariant setting, both sides differ by a $2p$-dimensional sphere, but $T$ acts trivially on one of them and non-trivially on the other.

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