Non-trivial checks of novel consistency relations

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Abstract. Single-field perturbations satisfy an infinite number of consistency relations constraining the squeezed limit of correlation functions at each order in the soft momentum. These can be understood as Ward identities for an infinite set of residual global symmetries, or equivalently as Slavnov-Taylor identities for spatial diffeomorphisms. In this paper, we perform a number of novel, non-trivial checks of the identities in the context of single field inflationary models with arbitrary sound speed. We focus for concreteness on identities involving 3-point functions with a soft external mode, and consider all possible scalar and tensor combinations for the hard-momentum modes. In all these cases, we check the consistency relations up to and including cubic order in the soft momentum. For this purpose, we compute for the first time the 3-point functions involving 2 scalars and 1 tensor, as well as 2 tensors and 1 scalar, for arbitrary sound speed.

Keywords: inflation, primordial gravitational waves (theory), non-gaussianity, quantum field theory on curved space

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1 Introduction

The consistency relations for adiabatic modes [1–10] serve as a powerful discriminating principle among various classes of inflationary models. For perturbations on a spatially-flat, Friedmann-Robertson-Walker (FRW) background, the consistency relations take the schematic form

$$\lim_{\vec{q} \to 0} M_n \frac{\partial^n}{\partial q^n} \left( \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) O(\vec{k}_1, \ldots, \vec{k}_N) \rangle + \frac{1}{P_\gamma(q)} \langle \gamma(\vec{q}) O(\vec{k}_1, \ldots, \vec{k}_N) \rangle \right) \sim -M_n \frac{\partial^n}{\partial k^n} \langle O(\vec{k}_1, \ldots, \vec{k}_N) \rangle,$$

(1.1)

where $M_n$ is a suitable projector (with indices suppressed), and $O(\vec{p}_1, \ldots, \vec{p}_N)$ represents an arbitrary equal-time product of scalar $\zeta$ and tensor $\gamma_{ij}$ perturbations, with momenta $\vec{k}_1, \ldots, \vec{k}_N$. At each integer order $n$, these identities constrain — completely for $n = 0, 1$, and partially for $n \geq 2$ — the $q^n$ behavior of an $N + 1$-point correlation function with a soft scalar or tensor mode in terms of an $N$-point function. These relations are powerful probes of the inflationary era, since they hold in all models of ‘single-clock’ inflation in which the background is an attractor. Conversely, they can be violated if multiple fields contribute to density perturbations and/or $\zeta$ grows outside the horizon [11–15]. Observationally, the consistency relations can be tested with the cosmic microwave background, as well as with the large scale structure [16–23].

Just like the soft-pion theorems of the strong interactions, the cosmological consistency relations are the consequence of Ward identities for spontaneously broken symmetries [7, 8, 24]. The symmetries in this case are global, gauge-preserving spatial coordinate transformations. They map field configurations which fall off at infinity into those which do not. Nevertheless, certain linear combinations of these transformations can be smoothly...
extended to physical configurations which fall off at infinity, and as such constitute adiabatic modes [25].

Recently, it has been shown that the consistency relations (1.1) all derive from a single, master identity, which follows from the Slavnov-Taylor identity for spatial diffeomorphisms [9]. (See [10] for an alternative derivation using the wavefunction.) The approach underscores that the consistency conditions are the consequence of the underlying diffeomorphism invariance, despite the fact that the general coordinate invariance is not the symmetry of the action upon introduction of the gauge-fixing term for the graviton. The master identity is valid at any value of $q$ and therefore goes beyond the soft limit. By differentiating it $n$ times with respect to $q$ and translating to correlation functions, one recovers (1.1) at each order [9].

For instance, the identity for soft 3-point vertices with hard scalar modes takes the form [9]

\[ q^j \left( \frac{1}{3} \delta_{ij} \Gamma^{\zeta\zeta\zeta}(\vec{q},\vec{k},-\vec{q}-\vec{k}) + 2 \Gamma^{\zeta\gamma\zeta}_{ij}(\vec{q},\vec{k},-\vec{q}-\vec{k}) \right) = q_i \Gamma^\zeta(k) - k_i \left( \Gamma^\zeta(|\vec{q}+\vec{k}|) - \Gamma^\zeta(k) \right), \tag{1.2} \]

where $\Gamma^{\zeta\zeta\zeta}$ and $\Gamma^{\zeta\gamma\zeta}_{ij}$ are respectively the cubic vertex functions for 3 scalars, and for 2 scalars−1 tensor, each without the momentum-conserving delta function, while $\Gamma^\zeta$ is the inverse scalar propagator. The solution for the vertex functions can be obtained as a power series around $q = 0$, up to an arbitrary symmetric, transverse matrix $A_{ij}$. This arbitrary term is model-dependent, and hence contains physical information about the underlying theory. It stems from the fact that (1.2) only constrains the longitudinal components of the vertex functions. The key assumption underlying the consistency relations is that $A_{ij}$ is analytic in $q$, specifically that it starts at $O(q^2)$. For standard inflationary scenarios, this is equivalent to the usual assumption of constant asymptotic solutions for the mode functions. For more exotic examples, such as khrnon inflation [26], the analyticity criterion is the unambiguous one.

In this paper, we perform a number of novel, non-trivial checks of the identities (1.1) in the context of single field inflationary models with arbitrary sound speed $c_s \neq 1$. The lowest-order identities, $n = 0$ and $n = 1$, have been checked with various examples elsewhere [4]. Our primary interest lies in the higher-order $(n \geq 2)$ identities. For concreteness, we focus on soft 3-point function relations, involving as hard modes all possible combinations of $\zeta$’s and $\gamma$’s. In all these cases, we check the consistency relations up to and including $n = 3$. For this purpose, the correlation functions $\langle \zeta^2 \zeta \rangle$ and $\langle \gamma^2 \gamma \rangle$ with $c_s \neq 1$ are computed here for the first time, using the technology of the effective field theory of inflation [27].

Aside from the obvious upshot of establishing the validity of the identities, we are motivated by two observations:

1. At order $q^2$ and higher, only part of the correlation functions are constrained by the identities, while the remainder represents a model-dependent (i.e., physical) piece. For instance, the $q^2$ contribution in $\langle \zeta^2 \zeta \rangle$ corresponds to spatial curvature [28]. Therefore, when checking the identities, the projector $M_n$ in (1.1) plays a crucial role in extracting the relevant part of the correlators.

2. To date, no background-wave argument has been formulated for the $n \geq 2$ identities. In particular, at order $q^3$ the mode function of the long mode includes an imaginary

\footnote{Some simple checks of $n = 2$ and $n = 3$ identities were performed in [7].}
piece,
\[ \zeta(q, \tau) = \frac{H}{\sqrt{4\epsilon M^2_{\text{Pl}}} q^3} (1 - iq \tau) e^{iq\tau} \simeq \frac{H}{\sqrt{4\epsilon M^2_{\text{Pl}}} q^3} \left( 1 + \frac{(q\tau)^2}{2} + i\frac{(q\tau)^3}{3} + \ldots \right), \quad (1.3) \]
and hence cannot be treated as completely classical. \(^2\) The fact that the consistency relation holds at this order, as verified, tells us that at least part of the long mode is classical and can be removed by a suitable coordinate transformation.

From a technical point of view, the checks performed are highly non-trivial because of the parametric dependence. Let us focus, for instance, on the hard modes being scalars, i.e., \( \zeta \zeta \). In this case, the right-hand side of the identity (1.1) is proportional to
\[ P_{\zeta} \sim \frac{1}{c_s \epsilon}. \quad (1.4) \]
On the left-hand side of (1.1), meanwhile, the 3-point functions schematically have the following parametric dependence, to leading order in slow-roll parameters:
\[ \frac{\langle \zeta \zeta \zeta \rangle}{P_{\zeta}} \sim \frac{1 - c_s^2}{\epsilon c_s^3} q^2 + \ldots; \]
\[ \frac{\langle \gamma \zeta \zeta \rangle}{P_{\gamma}} \sim \frac{1}{\epsilon c_s} \left( 1 + Aq + Bq^2 + Cq^3 \ldots \right), \quad (1.5) \]
where \( A, B \) and \( C \) are constants. We make the following observations:

- At lowest order in \( q \), namely \( n = 0 \) and \( n = 1 \), the identity (1.1) constrains each 3-point function separately in terms of \( P_{\zeta} \). Because \( \langle \zeta \zeta \zeta \rangle \) starts at order \( q^2 \), both sides of the \( n = 0 \) and \( n = 1 \) scalar relations must vanish identically, which is indeed the case [4]. The soft tensor relations, on the other hand, are non-trivial. As can be seen the parametric dependence of \( \langle \gamma \zeta \zeta \rangle \) at order \( q^0 \) and \( q \) matches that of \( P_{\zeta} \), and the consistency relations are satisfied.

- For \( n \geq 2 \), however, the 3-point correlators on the left-hand side of (1.5) contain terms proportional to \( 1/c_s^3 \), which cannot be matched by \( P_{\zeta} \sim 1/c_s \) on the right-hand side. However, the identity (1.1) only constrains a particular linear combination of the 3-point correlators, and we will find indeed that the \( 1/c_s^3 \) cancel out, leaving a \( 1/c_s \) remainder that matches the right-hand side.

- In the limit \( c_s \to 1 \), the \( \langle \zeta \zeta \zeta \rangle \) correlator vanishes (i.e., becomes subleading in slow-roll parameters). As a result, to leading order in \( \epsilon \), the identity (1.1) must be satisfied solely due to \( \langle \gamma \zeta \zeta \rangle \). Our explicit calculations confirm this expectation.

The paper is organized as follows. We first derive the non-linear symmetries governing cosmological perturbations in \( \zeta \)-gauge (section 2) and briefly review the derivation of the corresponding Ward identities (section 3). In section 4 we calculate the various 3-point functions for arbitrary sound speed \( c_s = \text{constant} \), to leading order in slow-roll parameters. We then turn to explicit checks of the Ward identities up to and including \( q^3 \) order, with \( \zeta \zeta \) (section 5), \( \zeta \gamma \) (section 6) and \( \gamma \gamma \) (section 7) as hard mode insertions. We summarize our results in section 8.

\(^2\)We thank Paolo Creminelli and Leonardo Senatore for discussions on this point.
2 Gauge-preserving coordinate transformations

In \( \zeta \)-gauge, the scalar field is unperturbed, \( \phi = \tilde{\phi}(t) \), and the spatial metric takes the form
\[
h_{ij} = a^2(t) e^{2\zeta} (e^\gamma)_{ij} ; \quad \gamma^i{}_i = 0 ; \quad \partial^i \gamma_{ij} = 0 .
\]
(2.1)
Scalar perturbations are captured by the conformal mode \( \zeta \); tensor modes are encoded in \( \gamma_{ij} \).

This choice completely fixes the gauge, at least for diffeomorphisms that fall off sufficiently fast at spatial infinity. However, there is an infinite number of residual, global coordinate transformations that diverge at infinity [5, 7]. These hold on any spatially-flat FRW background and do not rely on any slow-roll or quasi-de Sitter approximation.

Let us briefly review the derivation of these residual symmetries. Consider a general time-dependent spatial diffeormorphism, \( x^i \rightarrow x^i - \xi^i(t, \vec{x}) \). Being purely spatial, this transformation clearly leaves the choice \( \phi = \phi(t) \) invariant. The question is: under what conditions does it also leave (2.1) invariant?

At linear order in \( \xi \), the spatial metric (2.1) transforms as
\[
h'_{ij} = a^2 e^{2\zeta'} (e^{\gamma'})_{ij} = h_{ij} + \partial_i \xi^k h_{kj} + \partial_j \xi^k h_{ki} + \xi^k \partial_k h_{ij} + \mathcal{O}(\xi^2) ,
\]
or, explicitly,
\[
e^{2\zeta'} (e^{\gamma'+\delta\gamma})_{ij} = (1 + 2 \xi^k \partial_k \zeta) (e^\gamma)_{ij} + \partial_i \xi^k (e^\gamma)_{kj} + \partial_j \xi^k (e^\gamma)_{ki} + \xi^k \partial_k (e^\gamma)_{ij} ,
\]
where we have defined \( \delta\zeta = \zeta' - \zeta \) and \( \delta\gamma = \gamma' - \gamma \). Notice that even though \( \zeta \) and \( \gamma \) are not small, their variation \( \delta\zeta, \delta\gamma \sim \mathcal{O}(\xi) \) is small.\(^3\) Multiplying both sides from the right by \( (e^{-\gamma})_{jm} \) gives
\[
e^{2\zeta'} (e^{\gamma'+\delta\gamma})_{ij} (e^{-\gamma})_{jm} = (1 + 2 \xi^k \partial_k \zeta) \delta_{im} + \partial_i \xi^m (e^{-\gamma})_{jm} + \partial_j \xi^m (e^{-\gamma})_{ki} + \xi^k \partial_k (e^{-\gamma})_{ij} (e^{-\gamma})_{jm} .
\]
(2.3)

We can solve for \( \delta\zeta \) by tracing with \( \delta^i{}^j \) and applying the Hadamard lemma
\[
e^{X+\delta X} e^{-X} = 1 + \delta X + \frac{1}{2} [X, \delta X] + \frac{1}{3!} [X, [X, \delta X]] + \ldots
\]
\[
\left( \frac{d}{dt} e^{X(t)} \right) e^{-X} = \frac{d}{dt} X(t) + \frac{1}{2} \left[ X, \frac{d}{dt} X(t) \right] + \frac{1}{3!} \left[ X, \left[ X, \frac{d}{dt} X(t) \right] \right] + \ldots
\]
The result is
\[
\delta\zeta = \frac{1}{3} \partial_i \xi^i + \xi^k \partial_k \zeta .
\]
(2.5)
Substituting \( \delta\zeta \) back into (2.4), we can then solve for \( \delta\gamma \). Following [7], we do so perturbatively in powers of \( \gamma \), expanding the tensor variation and the diffeomorphism as follows:
\[
\delta\gamma_{ij} = \delta\gamma^{(0)}_{ij} + \delta\gamma^{(1)}_{ij} + \ldots
\]
\[
\xi_i = \xi_i^{(0)} + \xi_i^{(1)} + \ldots
\]
(2.6)
\(^3\)Throughout this calculation we keep terms up to order \( \mathcal{O}(\xi) \).
At zeroth-order in $\gamma$, the result is

$$\delta\gamma^{(0)}_{ij} = \partial_i \xi^{(0)}_j + \partial_j \xi^{(0)}_i - \frac{2}{3} \partial^k \xi^{(0)}_k \delta_{ij}, \quad (2.7)$$

which is manifestly traceless. To ensure $\delta\gamma^{(0)}$ is also transverse, $\xi^{(0)}$ must satisfy

$$\nabla^2 \xi^{(0)}_i + \frac{1}{3} \partial_i \partial^k \xi^{(0)}_k = 0. \quad (2.8)$$

As a check, note that if we assume that $\xi^{(0)}$ falls off at spatial infinity, then it is easy to see that the unique solution is $\xi^{(0)} = 0$. This confirms that the gauge is completely fixed with respect to these boundary conditions.

There are, however, non-trivial solutions of (2.8) with $\xi^{(0)}$ diverging at infinity. In general, the solution is given by a power series:

$$\xi^{(0)}_i = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{i\ell_0\ldots\ell_n} x^{\ell_0} \ldots x^{\ell_n}, \quad (2.9)$$

where the array $M_{i\ell_0\ldots\ell_n}$ is constant and symmetric in its last $n+1$ indices. To satisfy (2.8), the array must obey the trace condition

$$M_{ij\ell_2\ldots\ell_n} = -\frac{1}{3} M_{jij\ell_2\ldots\ell_n} \quad (\text{for all } n \geq 1). \quad (2.10)$$

Such transformations map field configurations that fall off at infinity into those that do not. However, certain linear combinations of them can be smoothly extended to physical configurations with suitable fall-off behavior [7], i.e., these correspond to adiabatic modes [25]. In particular, the spatial diffeomorphisms (2.9) must be time-independent.\footnote{Strictly speaking, it must also be supplemented by a time-dependent transformation, but this has no impact on the Ward identities at the end of the day. See [7] for a detailed discussion.}

A further adiabatic restriction comes from the requirement that the non-linear shift $\delta\gamma^{(0)}_{ij}$ should remain transverse when extended to a physical mode, i.e., with smooth profile around $\tilde{q} = 0$. To ensure that transversality is preserved at finite momentum, $q^i \delta\gamma_{i\ell_0}(\tilde{q}) = 0$, the $M_{i\ell_0\ldots\ell_n}$ coefficients must become $\tilde{q}$-dependent, such that [7]

$$q^i \left( M_{i\ell_0\ell_1\ldots\ell_n}(\tilde{q}) + M_{i\ell_0i\ell_1\ldots\ell_n}(\tilde{q}) - \frac{2}{3} \delta_{i\ell_0} M_{i\ell_1\ldots\ell_n}(\tilde{q}) \right) = 0. \quad (2.11)$$

Moving on to first-order in $\gamma$, the tensor variation is

$$\delta\gamma^{(1)}_{ij} = \partial_i \xi^{(1)}_j + \partial_j \xi^{(1)}_i - \frac{2}{3} \partial^k \xi^{(1)}_k \delta_{ij} + \frac{1}{2} \left( \mathcal{L}_{\xi^{(0)}} \gamma_{ij} + \xi^{(0)}_k \partial_k \gamma_{ij} - \partial_k \xi^{(0)}_i \gamma_{kj} - \partial_k \xi^{(0)}_j \gamma_{ki} \right), \quad (2.12)$$

where $\mathcal{L}_{\xi^{(0)}} \gamma_{ij} \equiv \xi^{(0)}_k \partial_k \gamma_{ij} + \partial_i \xi^{(0)}_k \gamma_{kj} + \partial_j \xi^{(0)}_k \gamma_{ki}$ is the Lie derivative along $\xi^{(0)}$. Imposing transversality of $\delta\gamma^{(1)}$ allows us to solve for $\xi^{(1)}$. Since $\xi^{(1)} \sim \gamma$, and $\gamma$ falls off at infinity, we can invert Laplacians assuming fall-off boundary conditions. The solution is

$$\xi^{(1)}_i = \frac{1}{2} \frac{\partial_j}{\nabla^2} \left( \delta_{ik} - \frac{1}{4} \frac{\partial_k}{\nabla^2} \right) \left( -\mathcal{L}_{\xi^{(0)}} \gamma_{kj} + \partial_m \xi^{(0)}_k \gamma_{mj} + \partial^l \xi^{(0)}_k \gamma_{lj} \right). \quad (2.13)$$

This procedure can be straightforwardly extended to all orders in $\gamma$.\footnote{The constant term in the series, representing an unbroken spatial translation, has been ignored. It is linearly realized and hence does not lead to a soft-pion theorem.}
Combining the above results, the momentum-space field variations to first order $\gamma$ are given by

$$
\delta \zeta(k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{i \ell_1 \ldots \ell_n} \frac{\partial^n}{\partial k_{\ell_1} \ldots \partial k_{\ell_n}} \left( (2\pi)^3 \delta^3 (k) \right) 
$$

$$
\quad - \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{\ell_0 \ldots \ell_n} \left( \delta^{i \ell_0} \frac{\partial^n}{\partial k_{\ell_1} \ldots \partial k_{\ell_n}} + \frac{k^i}{n+1} \frac{\partial^{n+1}}{\partial k_{\ell_0} \ldots \partial k_{\ell_n}} \right) \zeta(k) 
$$

$$
\quad + \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{\ell_0 \ldots \ell_n} \Gamma^{ij}(k) \frac{\partial^n}{\partial k_{\ell_1} \ldots \partial k_{\ell_n}} \gamma_{ij}(k) + \ldots 
$$

$$
\delta \gamma_{ij}(k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left( M_{ij \ell_1 \ldots \ell_n} + M_{ji \ell_1 \ldots \ell_n} - \frac{2}{3} \delta_{ij} M_{\ell \ell_1 \ldots \ell_n} \right) \frac{\partial^n}{\partial k_{\ell_1} \ldots \partial k_{\ell_n}} \left( (2\pi)^3 \delta^3 (k) \right) 
$$

$$
\quad - \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{\ell_0 \ldots \ell_n} \left( \delta^{i \ell_0} \frac{\partial^n}{\partial k_{\ell_1} \ldots \partial k_{\ell_n}} + \frac{k^i}{n+1} \frac{\partial^{n+1}}{\partial k_{\ell_0} \ldots \partial k_{\ell_n}} \right) \gamma_{ij}(k) 
$$

$$
\quad + \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{ab \ell_1 \ldots \ell_n} \Gamma^{abij}(k) \frac{\partial^n}{\partial k_{\ell_1} \ldots \partial k_{\ell_n}} \gamma_{ij}(k) + \ldots 
$$

(2.14)

where the ellipses indicate higher-order terms in $\gamma$, and

$$
\Gamma_{abij}(k) \equiv \frac{1}{4} \delta_{ab} \delta_{ij} - \frac{1}{8} \delta_{a\ell} \delta_{b\ell} \delta_{ij} - \frac{1}{8} \delta_{i\ell} \delta_{j\ell} \delta_{ab} ;
$$

$$
\Gamma_{abij\ell}(k) \equiv - \frac{1}{2} \left( \delta_{ij} + \delta_{\ell \ell} \right) \left( \delta_{ab} \delta_{\ell j} - \frac{1}{2} \delta_{a\ell} \delta_{b\ell} - \frac{1}{2} \delta_{a\ell} \delta_{b\ell} \right) + \delta_{a(i} \delta_{j)\ell} \delta_{ab} - \delta_{a(i} \delta_{j)\ell} \delta_{ab} - \delta_{a(i} \delta_{j)\ell} \delta_{ab} + 2 \delta_{a\ell} \delta_{b\ell} \delta_{ij}.
$$

(2.15)

## 3 Derivation of the consistency conditions

In this section, we briefly review the derivation of the consistency relations as Ward identities for the non-linearly realized symmetries discussed above. The full derivation can be found in [7], but we review it here both for completeness and by means of highlighting the key steps.

The starting point is the in-vacuum expectation value of the action of the conserved charge $Q$ on some operator $\mathcal{O}$:

$$
\langle \Omega | Q | \mathcal{O} | \Omega \rangle = -i \langle \Omega | \delta \mathcal{O} | \Omega \rangle ,
$$

(3.1)

where $\mathcal{O}$ denotes an arbitrary equal-time product of $\zeta$’s and $\gamma$’s, and $| \Omega \rangle$ is the in-vacuum of the interacting theory. The charge can be decomposed as

$$
Q = Q_0 + W ,
$$

(3.2)

where $Q_0$ generates the non-linear part of the transformations (2.14), and $W$ is the remainder. By definition, $Q_0$ is a symmetry of the free theory (i.e., of the quadratic action). Specifically, for the field variations (2.14),

$$
Q_0 = \lim_{\bar{q} \to q} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{\ell_0 \ldots \ell_n} (\bar{q}) \frac{\partial^n}{\partial \bar{q}_{\ell_1} \ldots \partial \bar{q}_{\ell_n}} \left( \frac{1}{3} \delta^{\ell_0} \Pi_{\zeta}(\bar{q}) + 2 \Pi_{\gamma}^{ij}(\bar{q}) \right)
$$

(3.3)

where $\Pi_{\zeta}$ and $\Pi_{\gamma}^{ij}$ are the conjugate momenta.
By definition, the in-vacuum is related to the free vacuum by $|Ω⟩ = Ω(−∞)|0⟩$, where $Ω(−∞) ≡ U^†(−∞, 0)U_0(−∞, 0)$, with $U$ and $U_0$ denoting respectively the full and free time evolution operators. Similarly,

$$Q Ω(−∞) = Ω(−∞)Q_0 .$$  \tag{3.4}$$

It follows that

$$Q|Ω⟩ = Q Ω(−∞)|0⟩ = Ω(−∞)Q_0|0⟩ .$$ \tag{3.5}$$

The action of the free charge $Q_0$ on the free vacuum $|0⟩$ can be computed by inserting a complete set of free-field eigenstates $|Ω, γ⟩$:

$$Q_0|0⟩ = \int DΩDγ_0 |Ω, γ⟩⟨Ω, γ|Q_0|0⟩ = \lim_{q → 0} \sum_{n=0}^{∞} \frac{(-i)^n}{n!} M_{i_0...i_n} \frac{∂^n}{∂q_{i_1}...∂q_{i_n}} \int DΩDγ_0 |Ω, γ⟩⟨Ω, γ| \left( \frac{1}{3} δ_{ij} Π_{i_0}(q) + 2 δ_{ij} Π_{i_0}(q) |0⟩ \right) .$$ \tag{3.6}$$

The free vacuum wavefunctional $⟨Ω, γ|0⟩$ is a Gaussian

$$⟨Ω, γ|0⟩ ∼ \exp \left[ - \int \frac{d^3 k}{(2π)^3} \left( \frac{1}{4} Π_{i_0}(k) P_{i_0}^{-1}(k) Π_{i_0}^{-1}(k) + \frac{1}{8} γ_{0ij}(k) P_{0}^{-1}(k) γ_{0ij}^{-1}(k) \right) \right] , \tag{3.7}$$

up to an irrelevant phase which eventually drops out of the calculation. Here, $P_0$ denotes the power spectrum of the free theory. Substituting into (3.6), we obtain

$$Q_0|0⟩ = - \lim_{q → 0} \sum_{n=0}^{∞} \frac{(-i)^n+1}{2n!} M_{i_0...i_n} \frac{∂^n}{∂q_{i_1}...∂q_{i_n}} \left( \frac{δ_{ij}}{3} Π_{i_0}(q) + \frac{γ_{0ij}(q)}{P_0(q)} \right) |0⟩ .$$ \tag{3.8}$$

At this point comes the assumption that the fields $ζ$ and $γ$ are conserved operators as $q → 0$. In other words, we assume that

$$\lim_{q → 0} ζ(q) Ω(−∞) = Ω(−∞)ζ_0(q) ; \quad \lim_{q → 0} γ_{ij}(q) Ω(−∞) = Ω(−∞)γ_{ij}(q) . \tag{3.9}$$

Combined with (3.4), this allows us to rewrite (3.8) in terms of the operators of the full interacting theory:

$$Q|Ω⟩ = - \lim_{q → 0} \sum_{n=0}^{∞} \frac{(-i)^n+1}{2n!} M_{i_0...i_n}(q) \frac{∂^n}{∂q_{i_1}...∂q_{i_n}} \left( \frac{δ_{ij}}{3} Π_{i_0}(q) + \frac{γ_{ij}(q)}{P_0(q)} \right) |Ω⟩ .$$ \tag{3.10}$$

Notice that, in this expression the power spectra are the ones of the full interacting theory [7]. The left-hand side of the Ward identity (3.1) therefore reduces to

$$⟨Ω||Q, O||Ω⟩ = \lim_{q → 0} \sum_{n=0}^{∞} \frac{(-i)^n+1}{n!} M_{i_0...i_n}(q) \frac{∂^n}{∂q_{i_1}...∂q_{i_n}} \left( \frac{γ_{ij}(q)}{P_0(q)} + \frac{δ_{ij0}}{3} Π_{i_0}(q) \right) . \tag{3.11}$$

\footnote{This statement is quite subtle, since in (3.6) we are instructed to differentiate $ζ(q)$ and $γ(q)$ before sending $q → 0$. We refer the reader to [7] for a rigorous justification of this result.}
Meanwhile, the right-hand side, \(-i\langle \Omega | \delta \mathcal{O} | \Omega \rangle\), is just given by the field variations (2.14) of the various \(\zeta\)'s and \(\gamma\)'s implicit in \(\mathcal{O}\). As argued in [7], the non-linear parts of \(\delta \zeta\) and \(\delta \gamma\), given by derivatives of delta functions in (2.14) contribute to disconnected diagrams in the Ward identity. Focusing on connected correlators, denoted by \(\langle \ldots \rangle_c\), the Ward identities take the form

\[
\lim_{\bar{q} \to 0} M_{i_0 \ldots i_n}(\bar{q}) \frac{\partial^n}{\partial q_{i_1} \cdots \partial q_{i_n}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i_0}(\bar{q}) \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle_c + \frac{\delta^{i_0}}{3P_\gamma(q)} \langle \zeta(\bar{q}) \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle_c \right)
\]

\[
= - \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{i_0 \ldots i_n}(\bar{q}) \left\{ \sum_{a=1}^{N} \left( \langle \delta^{i_0} \partial_{k_{1a}}^n \cdots \partial_{k_{na}}^n \rangle_c + \frac{k_{n}}{n+1} \partial_{k_{1a}}^{n+1} \cdots \partial_{k_{na}}^{n+1} \right) \langle \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle_c 
\right. \\
- \sum_{a=1}^{M} \Gamma^{i_0 j a \nu}(\bar{k}_a) \frac{\partial^n}{\partial k_{1a}^n \cdots \partial k_{na}^n} \langle \mathcal{O}_\zeta(\vec{k}_1, \ldots, \vec{k}_{a-1}, \vec{k}_{a+1}, \ldots, \vec{k}_M) \gamma_{i_0 j a} \langle \mathcal{O}(\vec{k}_{M+1}, \ldots, \vec{k}_N) \rangle_c 
\right. \\
- \sum_{b=M+1}^{N} \Gamma^{i_0 j b \nu}(\bar{k}_b) \frac{\partial^n}{\partial k_{1b}^n \cdots \partial k_{nb}^n} \langle \mathcal{O}_\gamma(\vec{k}_1, \ldots, \vec{k}_M) \gamma_{i_0 j b} \langle \mathcal{O}(\vec{k}_{M+1}, \ldots, \vec{k}_N) \rangle_c 
\right. \\
+ \ldots \\
\]

(3.12)

where, as before, the ellipses indicate higher-order terms in \(\gamma\). Here, \(\mathcal{O}_\zeta\) and \(\mathcal{O}_\gamma\) respectively denote products of \(\zeta\)'s and \(\gamma\)'s.

Finally, it is convenient to express the identity in terms of on-shell correlation functions, obtained by removing the momentum-conserving delta functions:

\[
\langle \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle = (2\pi)^3 \delta^3(\vec{K}_1) \langle \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle',
\]

(3.13)

where \(\vec{K}_1 = \vec{k}_1 + \ldots + \vec{k}_N\) is the total momentum. Removing the delta functions involves some technical subtleties, which are explained in detail in appendix B. Our convention for going on-shell is to express \(\vec{k}_N\) in terms of the \(\vec{k}\)'s, that is, \(\vec{k}_N = -\vec{k}_1 - \ldots - \vec{k}_{N-1}\). We then distinguish two cases for the Ward identities in terms of primed correlators:

- Hard modes containing at least one \(\gamma\) field: in this case, the primed identities are given by:

\[
\lim_{\bar{q} \to 0} M_{i_0 \ldots i_n}(\bar{q}) \frac{\partial^n}{\partial q_{i_1} \cdots \partial q_{i_n}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i_0}(\bar{q}) \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle_c + \frac{\delta^{i_0}}{3P_\gamma(q)} \langle \zeta(\bar{q}) \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle_c \right)
\]

\[
= - M_{i_0 \ldots i_n}(\bar{q}) \left\{ \sum_{a=1}^{N-1} \langle \delta^{i_0} \partial_{k_{1a}}^n \cdots \partial_{k_{na}}^n \rangle_c - \frac{\delta_{a0}}{n+1} \frac{\partial^n}{\partial k_{1a}^n \cdots \partial k_{na}^n} \langle \mathcal{O}(\vec{k}_1, \ldots, \vec{k}_N) \rangle_c 
\right. \\
- \sum_{a=1}^{M} \Gamma^{i_0 j a \nu}(\bar{k}_a) \frac{\partial^n}{\partial k_{1a}^n \cdots \partial k_{na}^n} \langle \mathcal{O}_\zeta(\vec{k}_1, \ldots, \vec{k}_{a-1}, \vec{k}_{a+1}, \ldots, \vec{k}_M) \gamma_{i_0 j a} \langle \mathcal{O}(\vec{k}_{M+1}, \ldots, \vec{k}_N) \rangle_c 
\right. \\
- \sum_{b=M+1}^{N} \Gamma^{i_0 j b \nu}(\bar{k}_b) \frac{\partial^n}{\partial k_{1b}^n \cdots \partial k_{nb}^n} \langle \mathcal{O}_\gamma(\vec{k}_1, \ldots, \vec{k}_M) \gamma_{i_0 j b} \langle \mathcal{O}(\vec{k}_{M+1}, \ldots, \vec{k}_N) \rangle_c 
\right. \\
- (-1)^n \frac{\partial^n}{\partial k_{1a}^n \cdots \partial k_{na}^n} \Gamma^{i_0 j a \nu}(\bar{k}_a) \frac{\partial^n}{\partial k_{1b}^n \cdots \partial k_{nb}^n} \langle \mathcal{O}_\gamma(\vec{k}_1, \ldots, \vec{k}_M) \gamma_{i_0 j b} \langle \mathcal{O}(\vec{k}_{M+1}, \ldots, \vec{k}_N) \rangle_c 
\right. \\
\times \langle \mathcal{O}_\zeta(\vec{k}_1, \ldots, \vec{k}_M) \mathcal{O}_\gamma(\vec{k}_{M+1}, \ldots, \vec{k}_N) \rangle_c + \ldots
\]

(3.14)

where “OS” stands for “on-shell”.

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Hard modes consisting of scalars only: in this case, the primed identities are given by:

\[
\lim_{\hat{q} \to 0} M_{t\ell_0 \ldots \ell_n}(\hat{q}) \frac{\partial^n}{\partial \hat{q}_{i_1} \cdots \partial \hat{q}_{i_n}} \left( \frac{1}{P_{\hat{q}}(\hat{q})} \langle \gamma^{t\ell_0}(\hat{q}) O(\tilde{k}_1, \ldots, \tilde{k}_N) \rangle'_c + \frac{\delta^{t\ell_0}}{3P_{\hat{q}}(\hat{q})} \langle \zeta(\hat{q}) O(\tilde{k}_1, \ldots, \tilde{k}_N) \rangle'_c \right) \\
= -M_{t\ell_0 \ldots \ell_n}(\hat{q}) \sum_{a=1}^{N-1} \left( \delta^{t\ell_0} \frac{\partial^n}{\partial k_{i_1}^a \cdots \partial k_{i_n}^a} - \frac{\delta_{a0}}{N-1} \delta^{t\ell_0} + \frac{k_a^a}{n+1} \frac{\partial^{n+1}}{\partial k_{i_1}^a \cdots \partial k_{i_n}^a} \right) \langle O(\tilde{k}_1, \ldots, \tilde{k}_N) \rangle'_c \\
- \sum_{a=1}^{N-1} \frac{\partial^n}{\partial k_{i_1}^a \cdots \partial k_{i_n}^a} \langle O^c(\tilde{k}_1, \ldots, \tilde{k}_{a-1}, \tilde{k}_{a+1}, \ldots, \tilde{k}_N) \gamma_{i_1 j_a}(\tilde{k}_a) \rangle'_c \\
- (-1)^n \frac{\partial^n}{\partial \tilde{k}_{i_1}^a \cdots \partial \tilde{k}_{i_n}^a} \langle O^c(\tilde{k}_1, \ldots, \tilde{k}_{N-1}) \gamma_{i_1 j_N}(\tilde{k}_N) \rangle'_c \right) + \ldots
\]

(3.15)

Note that the \( \Upsilon \) term replaces every \( \zeta \) in \( O \) with a \( \gamma \) insertion, whereas the \( \Gamma \) term replaces every \( \gamma \) in \( O \) with another \( \gamma \), with suitably contracted indices. The \( \partial \Gamma / \partial k \) terms were missed in [7] and arise from a careful removal of the momentum-conserving delta functions. See appendix B for a detailed discussion of this point.

4 Three point functions

In this work, we are interested in single-clock models of inflation with arbitrary scalar sound speed \( c_s \neq 1 \). These models can be described at once within the framework of the effective theory of inflation [3]. In this approach, the theories at hand are conveniently expressed in terms of \( \pi \), the St"uckelberg field for broken time translational invariance. Ignoring terms involving the extrinsic curvature perturbation \( \delta K^\mu_\nu \) for simplicity, the effective action is given by

\[
S = \int d^4 x \sqrt{-q} \left[ \frac{M_P^2}{2} R - M_P^2 \left( 3H(t + \pi)^2 + \dot{H}(t + \pi) \right) \\
- M_P^2 \dot{H}(t + \pi) Q + \frac{1}{2} M(t + \pi)^4 Q^2 + \frac{1}{6} c_3(t + \pi) M(t + \pi)^4 Q^3 + \ldots \right],
\]

(4.1)

where \( N \) and \( N^i \) denote the lapse function and shift vector of the ADM decomposition, and

\[
Q = \frac{1}{N^2} \left( 1 + \dot{\pi} - N^i \partial_i \pi \right)^2 - h^{ij} \partial_i \pi \partial_j \pi - 1.
\]

(4.2)

Notice that the action (4.1) is invariant under the spatial diffeomorphisms as well as the “diagonal” time diffeomorphism: \( t \to t + \xi^0(x) \), \( \pi \to \pi - \xi^0(x) \). Also, parameters appearing in (4.1) are considered to be a slowly varying functions of time, i.e. the action is assumed to possess an approximate time translation symmetry. Although we have dropped extrinsic curvature contributions, the above effective theory is sufficiently general to perform non-trivial checks of the consistency relations.

Eventually we are interested in 3-point correlators in \( \zeta \)-gauge specified by (2.1). In order to check the Ward identities (3.14), we need to compute the correlators of \( \zeta \) field itself as well as those involving both \( \zeta \) and \( \gamma \) fields, at leading order in slow roll. It turns out that, for this purpose, the computation can be most conveniently performed in spatially-flat gauge, defined by

\[
\phi(x) = \bar{\phi}(t + \pi(x)) \, , \, \, \, \, \, \, h_{ij} = a^2(t) (\epsilon^\gamma)_{ij} \, , \, \, \, \, \, \, \partial_j \bar{\gamma}_{ij} = \bar{\gamma}_{ii} = 0.
\]

(4.3)
Then once we know the correlators of \( \pi \) and \( \dot{\gamma} \), we can translate them into those of \( \zeta \) and \( \gamma \) via
\[
\zeta \simeq -H\pi; \quad \dot{\gamma}_{ij} \simeq \gamma_{ij},
\]
which are valid to leading order in slow roll.

Therefore all we need for our computation is the action up to the cubic order in perturbations. This requires solving the constraint equations for the lapse and shift to linear order \([1, 3]\)
\[
\delta N \equiv N - 1 = \epsilon H\pi; \quad \partial^i N_i = -\frac{\epsilon}{c_s^2} \frac{\partial}{\partial t} (H\pi),
\]
where \( \epsilon \equiv -\dot{H}/H^2 \), and the sound speed \( c_s \) is related to the coefficients in \((4.1)\) by
\[
c_s^{-2} = 1 - \frac{2M^4(t)}{HM_p^2}.
\]

Substituting for \( \delta N \) and \( N^i \) into \((4.1)\), we obtain the following quadratic Lagrangian density
\[
\mathcal{L}^{(2)} = \frac{M_p^2 a^3}{8} \left( \gamma_{ij} \dot{\gamma}_{ij} - \frac{1}{a^2} \partial_t \gamma_{jk} \partial_t \gamma_{jk} \right) + \frac{M_p^2 a^3 H^2 \epsilon}{c_s^2} \left( \gamma^2 - \frac{c_s^2}{a^2} \partial_t \gamma_{ij} \partial_t \pi + H^2 \pi^2 \left( 3 - 2(s + \epsilon - \eta) \right) \right),
\]
where, as usual, other slow roll parameters are defined by \( \eta \equiv H^{-1}\ln \epsilon / dt \) and \( s \equiv H^{-1}\ln c_s / dt \). We also assume that they are of the same order as \( \epsilon \): \( \eta \sim s \sim O(\epsilon). \)

The 2-point functions for \( \zeta \) and \( \gamma \) can be readily obtained from \((4.7)\). To leading order in the slow roll approximation, they are given by
\[
\langle \zeta_{i-k} \rangle' = \frac{H^2}{4\epsilon c_s M_p^2 k^3}; \quad \langle \gamma^{ij}_{k-l} \rangle' = \frac{H^2}{M_p^2 k^3} \Pi^{ijkl}(k),
\]
where
\[
\Pi^{ijkl}(k) \equiv \sum_{s=\pm} c_s^{ij}(k) c_s^{kl}(k) = \Pi_{lk}\Pi_{jk} + \Pi_{lj}\Pi_{jk} - \Pi_{jk}\Pi_{lk}; \quad \Pi_{ij}(k) \equiv \delta_{ij} - \vec{k}_i \vec{k}_j.
\]

The cubic Lagrangian density for the perturbations is given by
\[
\mathcal{L}^{(3)} = \mathcal{L}_{\pi\pi\pi} + \mathcal{L}_{\gamma\pi\pi} + \mathcal{L}_{\gamma\gamma\pi} + \mathcal{L}_{\gamma\gamma\gamma},
\]
with
\[
\mathcal{L}_{\pi\pi\pi} = a^3 \frac{M_p^2 H^2 \epsilon}{c_s^2} \left( C_{\pi 3} \hat{\pi}^3 + C_{\pi(\partial \pi)} \frac{1}{a^2} \hat{\pi} (\partial \pi)^2 + C_{\pi 2 \pi} \hat{\pi}^2 + C_{\pi(\partial \pi)^2} \frac{1}{a^2} \pi (\partial \pi)^2 + C_{NL} \hat{\pi} \partial \pi \partial \pi - \frac{1}{c_s^2} \hat{\pi} \right);
\]
\[
\mathcal{L}_{\gamma\pi\pi} = \frac{M_p^2 a^3}{2} \left( \partial^1 N^i \partial_m \gamma_{ij} N^m + \frac{1}{2} \delta N \gamma_{ij} (\partial_i N^j + \partial_j N^i) \right) + M_p^2 a^3 H^2 \epsilon c_s \gamma_{ij} \partial_i \pi \partial_j \pi;
\]
\[
\mathcal{L}_{\gamma\gamma\pi} = \frac{M_p^2 a^3}{2} \left( -\frac{1}{4a^2} \gamma_{ij} \partial_m \gamma_{ij} N^m - \frac{1}{4} \delta N \gamma_{ij} \gamma_{ij} - \frac{\delta N}{4a^2} \gamma_{ij} \partial_i \gamma_{jk} \partial_j \gamma_{km} \right);
\]
\[
\mathcal{L}_{\gamma\gamma\gamma} = \frac{M_p^2 a^3}{2} \left( \frac{1}{4a^2} \gamma_{im} \partial_i \gamma_{jk} \partial_m \gamma_{jk} - \frac{1}{2a^2} \gamma_{im} \partial_i \gamma_{kj} \partial_j \gamma_{km} \right),
\]

\(^7\)In fact \( \eta, s \) will not appear in the three point functions at leading order in slow roll.
The coefficients in $L_{\pi\pi}$ are related to the parameters of the effective theory, up to next-to-leading order in slow roll, by

\[ C_{k^3} = (1 - c_s^2) \left( 1 + \frac{2}{3}c^3_s \right); \]
\[ C_{(\partial\pi)^2} = -1 + c_s^2; \]
\[ C_{\pi^2} = H \left( -6\epsilon + \eta - 2s + 3\epsilon c_s^2 - 2\epsilon c_s(1 - c_s^2) \right); \]
\[ C_{(\partial\pi)^3} = H (\epsilon - \epsilon c_s^2); \]
\[ C_{\text{NL}} = \frac{2\epsilon H}{c_s^2}. \quad (4.12) \]

Using this cubic action, we can compute various tree-level 3-point functions in the in-in formalism as usual. The results are:

- **Three scalars:**

\[
\langle \zeta_1 \zeta_2 \zeta_3 \rangle' = \frac{H^4(1 - c_s^2) 12 (1 - c_s^2 (1 + \frac{2}{3}c_s)) K_2^6 - 4K K_1^3 K_2^3 - 4K^2 K_1^4 + 11K^3 K_2^3 - 3K^4 K_1^2 + K^6}{32M_{Pl}^4 c_s^2},
\]

where

\[
K \equiv k_1 + k_2 + k_3; \quad K_1 = \sqrt{k_1k_2 + k_1k_3 + k_2k_3}; \quad K_2 = (k_1k_2k_3)^{1/3}.
\]

- **Two scalars and one tensor:**

\[
\langle \gamma^{ij} \zeta_1 \zeta_2 \rangle' = -\frac{H^4}{4M_{Pl}^2 c_s^2} \frac{U(c_s k_1, c_s k_2, p)}{k_1^2 k_2^2 p^2} \Pi^{ij}_{mn}(\hat{p}) k_1^m k_2^n,
\]

where $\Pi_{ijmn}$ was defined in (4.9), and

\[
U(k_1, k_2, p) \equiv \frac{k_1^3 + k_2^3 + p^3 + 2k_1 k_2 p + (2k_1 k_2 + 5 \text{ perms})}{K^2}.
\]

- **One scalar and two tensors:**

\[
\langle \gamma^{ij} \zeta_1 k^k \zeta_2 \rangle' = \frac{H^4}{8M_{Pl}^2 c_s^2} \frac{\Pi^{ij}_{mn}(\hat{p}_1) \Pi^{k\ell}_{mn}(\hat{p}_2)}{k^2 p_1^2 p_2^2} \left( \frac{4p_1^2 p_2^2}{c_s k + p_1 + p_2} - \frac{c_s k (k^2 - p_1^2 - p_2^2)}{2} \right)
\]

- **Three tensors:**

the $\gamma\gamma\gamma$ correlator is of course identical to that in $c_s = 1$ inflation model [29]:

\[
\langle \gamma^{ij} \zeta_1 k^k \zeta_2 \rangle' = \frac{H^4}{8M_{Pl}^2} \frac{U(p_1, p_2, p_3)}{p_1^2 p_2^2 p_3^2} \Pi^{ij}_{mn}(\hat{p}_1) \Pi^{k\ell}_{mn}(\hat{p}_2) \Pi^{c\ell'}_{mn}(\hat{p}_3) t_{abc} t_{a'b'c'},
\]

where

\[
t_{abc} = k^2_a \delta_{bc} + k^b_a \delta_{ac} + k^c_a \delta_{ab}.
\]
The above 3-point functions, given by (4.13), (4.15), (4.17) and (4.18), all agree with [1] for $c_s = 1$, with the exception of a small typo\(^8\) It is important to stress, however, that the $c_s \neq 1$ results given here cannot simply be inferred by rescaling the $k$’s in the $c_s = 1$ correlators, since the cubic Lagrangian involves extra non-trivial $c_s$ dependence as well. Indeed, we can see explicitly that the $\langle \gamma \gamma \zeta \rangle$ correlator (4.17) can not be written as a function of $c_s k$, $p_1$ and $p_2$. (The illusion that $\langle \gamma \zeta \zeta \rangle$ correlator (4.15) can is only an artifact of leading order in the slow roll approximation.)

5 Consistency relations with two scalars

The simplest case is where the two hard modes (with momenta $\vec{k}_1$ and $\vec{k}_2$) are both scalars, i.e., $\mathcal{O}(\vec{k}_1, \vec{k}_2) = \zeta_{\vec{k}_1} \zeta_{\vec{k}_2}$. In this case, the identity (3.14) reduces to

$$\lim_{\vec{q} \to 0} M_{id0 \ldots \epsilon_n}(\vec{q}) \frac{\partial^n}{\partial q_i \ldots \partial q_\epsilon} \left( \frac{\langle \gamma_{t0} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle}{P_\zeta(q)} + \frac{\delta_{t0} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle}{3 P_\zeta(q)} \right)$$

$$= -M_{id0 \ldots \epsilon_n}(\vec{q}) \left( \delta_{t0} \frac{\partial^n}{\partial k_1^{i1} \ldots \partial k_1^{in}} + k_1^{i1} \ldots n + 1 \delta k_1^{j0} \ldots \partial k_1^{jn} \right) P_\zeta(k_1), \quad \text{for all } n \geq 0 . \quad (5.1)$$

Note that the $\Upsilon$ contribution in this case is proportional to $\langle \zeta \rangle$ and hence vanishes identically. The $\Gamma$ contribution is absent since the hard modes are scalars only.

In what follows we will check this identity explicitly for $0 \leq n \leq 3$. For this purpose, we will need the 3-point functions (4.13) and (4.15) expanded to cubic order in the soft momentum. Note that we must first impose the on-shell condition before Taylor expanding. For instance, letting $k_1 \to \vec{q}$ in (4.13), we first set $\vec{k}_3 = \vec{k}_2 - \vec{q}$ and then expand in powers of $q$. The result is

$$\frac{\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle}{P_\zeta(q)} = \frac{H^2 (c_s^2 - 1)}{16 M_{Pl}^2 c_s^3 k^3} \left[ q^2 \left( 8 + (3 + 2 c_3) c_s^2 - 5 \left( \vec{q} \cdot \vec{k} \right)^2 \right) \right. \left. - \frac{q^2}{2 k^2} \left( 3 (3 + (3 + 2 c_3) c_s^2) + 5 \vec{q} \cdot \vec{k} (10 + (3 + 2 c_3) c_s^2) - 35 \left( \vec{q} \cdot \vec{k} \right)^3 \right) + \ldots \right]$$

$$\frac{\langle \gamma_{ij} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle}{P_\gamma(q)} = \frac{H^2}{8 M_{Pl}^2 c_s^3 k^3} \Pi_{mn}^{ij}(\vec{q}) \hat{k}_m \hat{k}_n \left[ 3 - \frac{15 q^2}{2 k} \vec{q} \cdot \vec{k} + \frac{5 q^2}{4 k^2 c_s^2} \left( 1 - 3 c_s^2 + 3 c_s^2 \left( \vec{q} \cdot \vec{k} \right)^2 \right) \right.$$ \left. - \frac{q^2}{8 k^3 c_s^2} \left( 6 + 35 c_s \left( 1 - 4 c_s^2 \right) \vec{q} \cdot \vec{k} + 315 c_s^3 \left( \vec{q} \cdot \vec{k} \right)^3 \right) + \ldots \right], \quad (5.2)$$

where, for simplicity, we have denoted the hard momentum by $\vec{k}$. Since $\langle \zeta \zeta \zeta \rangle / P_\zeta$ starts at order $q^2$, it does not contribute to the $n = 0$ and $n = 1$ relations, as mentioned in the Introduction. The $n = 0$ and $n = 1$ identities therefore boil down to the anisotropic and linear-gradient consistency relations [4].

\(^8\)In eq. (4.13) of [1], given by

$$\langle \zeta_{k_1} \gamma_{k_2}^2 \gamma_{k_3}^2 \rangle \sim \left( -\frac{1}{4} k_1^2 + \frac{1}{2} k_1 \left( k_2^2 + k_3^2 \right) + 4 \frac{k_1^2 k_2^2}{K} \right), \quad (4.20)$$

the $-1/4$ coefficient of the $k_1^2$ term should be $-1/2$. This is crucial checking the consistency relations with one scalar and one tensor insertion.
Notice that the 3-point correlation functions involving a soft tensor mode are proportional to $\Pi_{ijk\ell}(\hat{q})$. This could, in principle, lead to the appearance of the derivatives of $\Pi_{ijk\ell}(\hat{q})$ on the left-hand side of (3.14). However, it was shown in [9] that

$$M_{i\ell_0\ldots\ell_n}(\hat{q}) \frac{\partial^n \Pi_{k\ell}(\hat{q})}{\partial q_{\ell_1} \ldots \partial q_{\ell_n}} = 0,$$

which follows solely from the properties of $M_{i\ell_0\ldots\ell_n}(\hat{q})$. For this reason, we did not expand $\Pi_{ijk\ell}(\hat{q})$ in the Taylor series (5.2).

- **Anisotropic scaling consistency relation** ($n = 0$): the identity in this case reduces to

$$\lim_{\hat{q} \to 0} M_{ij}(\hat{q}) \frac{\langle \gamma_{ij} \zeta \xi_1 \xi_2 \rangle'}{P_\gamma(q)} = -M_{ij}(\hat{q}) k^i \frac{\partial}{\partial k^j} P_\zeta(k),$$

where $M_{ij}$ can be assumed symmetric and traceless, without loss of generality. Thus it can be expressed as the linear combination of the polarization tensors, $M_{ij}(\hat{q}) = a e_{ij}^+(\hat{q}) + b e_{ij}^-(\hat{q})$, and therefore

$$M_{ij}(\hat{q}) \Pi_{ij mn}(\hat{q}) = 2 M_{mn}(\hat{q}).$$

Then the left-hand side of (5.4) gives

$$L.H.S. = \lim_{\hat{q} \to 0} M_{ij}(\hat{q}) \frac{\langle \gamma_{ij} \zeta \xi_1 \xi_2 \rangle'}{P_\gamma(q)} = \frac{3H^2}{4M^2_P c^3 k^3} M_{mn}(\hat{q}) k^m \dot{k}^n.$$  

(5.6)

This agrees with the right-hand side:

$$R.H.S. = -M_{ij}(\hat{q}) k^i \frac{\partial}{\partial k^j} P_\zeta(k) = \frac{3H^2}{4M^2_P c^3 k^3} M_{mn}(\hat{q}) k^m \dot{k}^n.$$  

(5.7)

- **Linear-gradient consistency relation** ($n = 1$): the identity at this order reduces to

$$\lim_{\hat{q} \to 0} M_{ijm_1}(\hat{q}) \frac{\partial}{\partial q_{m_1}} \frac{\langle \gamma_{ij} \zeta \xi_1 \xi_2 \rangle'}{P_\gamma(q)} = -M_{ijm_1}(\hat{q}) \left( \delta^{ij} \frac{\partial}{\partial k^{m_1}} + \frac{k^i}{2} \frac{\partial^2}{\partial k_j \partial k^{m_1}} \right) P_\zeta(k).$$

(5.8)

When taking the derivative on the left-hand side, note that the term proportional to $\partial \Pi_{ij mn}/\partial q$ is projected out by $M_{ijm_1}(\hat{q})$, as mentioned in (5.3). The result is

$$L.H.S. = \lim_{\hat{q} \to 0} M_{ijm_1}(\hat{q}) \frac{\partial}{\partial q_{m_1}} \frac{\langle \gamma_{ij} \zeta \xi_1 \xi_2 \rangle'}{P_\gamma(q)} = \frac{5H^2}{8M^2_P c k^4} \left( M_{jjm_1}(\hat{q}) \dot{k}^{m_1} - 3 M_{mnm_1}(\hat{q}) \dot{k}^m \dot{k}^n \dot{k}^{m_1} \right).$$

(5.9)
This agrees with the right-hand side:

\[
\text{R.H.S.} = -M_{ijm_1}(\hat{q}) \left( \delta^{ij} \frac{\partial}{\partial k^{m_1}} + \frac{k^i}{2} \frac{\partial^2}{\partial k^j \partial k^{m_1}} \right) P_\zeta(k)
\]

\[
= \frac{5H^2}{8M_P^2 c_s\epsilon k^3} \left( M_{jjm_1}(\hat{q}) \hat{k}^{m_1} - 3M_{mnm_1}(\hat{q}) \hat{k}^m \hat{k}^{n m_1} \right). \tag{5.10}
\]

- **Quadratic consistency relation** \((n = 2)\): at this order, we have a new consistency relation

\[
\lim_{\hat{q} \to 0} M_{ijm_1 m_2}(\hat{q}) \frac{\partial^2}{\partial q_m \partial q_{m_2}} \left( \frac{\langle \gamma_q^i \zeta_k^j \zeta_k^m \rangle'}{P_\gamma(q)} \right) = -M_{ijm_1 m_2}(\hat{q}) \left( \delta^{ij} \frac{\partial^2}{\partial k^{m_1} \partial k^{m_2}} + \frac{k^i}{3} \frac{\partial^3}{\partial k^j \partial k^{m_1} \partial k^{m_2}} \right) P_\zeta(k). \tag{5.11}
\]

From (5.2), the second derivatives of the scalar 3-point function is given by

\[
\lim_{\hat{q} \to 0} M_{ijm_1 m_2}(\hat{q}) \frac{\partial^2}{\partial q_m \partial q_{m_2}} \left( \frac{\langle \gamma_q^i \zeta_k^j \zeta_k^m \rangle'}{P_\gamma(q)} \right) = \frac{H^2}{M_P^2} \frac{5(1 - c_s^2)}{24c_s^3 \epsilon k^3} M_{jjm_1 m_2}(\hat{q}) \hat{k}^{m_1} \hat{k}^{m_2}, \tag{5.12}
\]

where we have again used the fact that terms with derivatives on \(\Pi\) are projected out.

Similarly, for the 3-point function involving a soft tensor,

\[
\lim_{\hat{q} \to 0} M_{ijm_1 m_2}(\hat{q}) \frac{\partial^2}{\partial q_m \partial q_{m_2}} \left( \frac{\langle \gamma_q^i \zeta_k^j \zeta_k^m \rangle'}{P_\gamma(q)} \right) = \frac{H^2}{M_P^2} \frac{5k^i k^j k^m}{8c_s^2 \epsilon k^3} \left( M_{kkjj}(\hat{q}) (1 + 11c_s^2) + 14c_s^2 M_{klm_1 m_2}(\hat{q}) \hat{k}^{m_1} \hat{k}^{m_2} \right). \tag{5.13}
\]

As advocated earlier, the \(1/c_s^2\) contributions exactly cancel when combining (5.12) and (5.13), resulting in the left-hand side of (5.11) being proportional to \(1/c_s\):

\[
\text{L.H.S.} = \lim_{\hat{q} \to 0} M_{ijm_1 m_2}(\hat{q}) \frac{\partial^2}{\partial q_m \partial q_{m_2}} \left( \frac{\langle \gamma_q^i \zeta_k^j \zeta_k^m \rangle'}{P_\gamma(q)} \right) + \delta^{ij} \frac{\partial^3}{\partial k^j \partial k^{m_1} \partial k^{m_2}} P_\zeta(k)
\]

\[
= \frac{5H^2}{4M_P^2 c_s \epsilon k^5} \left( 6M_{kkjj}(\hat{q}) + 7M_{klm_1 m_2}(\hat{q}) \hat{k}^{m_1} \hat{k}^{m_2} \right). \tag{5.14}
\]

This exactly matches the right-hand side of (5.11):

\[
\text{R.H.S.} = -M_{ijm_1 m_2}(\hat{q}) \left( \delta^{ij} \frac{\partial^2}{\partial k^{m_1} \partial k^{m_2}} + \frac{k^i}{3} \frac{\partial^3}{\partial k^j \partial k^{m_1} \partial k^{m_2}} \right) P_\zeta(k)
\]

\[
= \frac{5H^2}{4M_P^2 c_s \epsilon k^5} \left( 6M_{kkjj}(\hat{q}) + 7M_{klm_1 m_2}(\hat{q}) \hat{k}^{m_1} \hat{k}^{m_2} \right). \tag{5.15}
\]

Note that the validity of the identity at \(q^2\) order crucially relies on taking a particular linear combination of 3-point correlation functions.
Next we consider the case where the hard modes consist of a scalar and a tensor, i.e.,

6 Consistency relations with one scalar and one tensor

From (5.2), the $q^3$ contribution from $\langle \gamma \zeta \zeta \rangle$ is

\[
\lim_{\hat{q} \to 0} M_{ijm1m2m3}(\hat{q}) \frac{\partial^3}{\partial q_{m1} \partial q_{m2} \partial q_{m3}} \frac{\delta_{ij}}{3P_\gamma(q)} \langle \gamma \zeta \zeta \rangle_P \zeta_k^j \zeta_k^j = - \frac{35H^2}{16MPl^2 c_s^6 k^6} M_{ijm1m2m3}(\hat{q}) \hat{k}_m^j \hat{k}_m^j. \tag{5.19}
\]

As before, the $1/c_s^3$ contributions in (5.17) and (5.19) cancel when taking the linear combination, resulting in the left-hand side of (5.16) being proportional to $1/c_s$:

\[
\text{L.H.S.} = \lim_{\hat{q} \to 0} M_{ijm1m2m3}(\hat{q}) \frac{\partial^3}{\partial q_{m1} \partial q_{m2} \partial q_{m3}} \frac{\delta_{ij}}{3P_\gamma(q)} \langle \gamma \zeta \zeta \rangle_P \zeta_k^j \zeta_k^j = \frac{105H^2}{16MPl^2 c_s k^6} \left( 2M_{ijm1m2m3}(\hat{q}) - 9M_{ijm1m2m3}(\hat{q}) \hat{k}_m^j \hat{k}_m^j \right). \tag{5.20}
\]

This agrees with the right-hand side of (5.16):

\[
\text{R.H.S.} = -M_{ijm1m2m3}(\hat{q}) \left( \frac{\partial^3}{\partial k_{m1} \partial k_{m2} \partial k_{m3}} + \frac{k^j}{4\partial k^j \partial k_{m1} \partial k_{m2} \partial k_{m3}} \right) P_\zeta(k) = \frac{105H^2}{16MPl^2 c_s k^6} \left( 2M_{ijm1m2m3}(\hat{q}) - 9M_{ijm1m2m3}(\hat{q}) \hat{k}_m^j \hat{k}_m^j \right). \tag{5.21}
\]
where \( \Upsilon \) was defined in (2.15). Note that the first and third lines of (3.14), being proportional to \( \langle \zeta \rangle \), dropped out. Only the \( \Upsilon \) contribution survives. In checking these relations, we should consider (consistently) both left- and right-hand sides as a function of \( \vec{k}_2 \) and expand \( \vec{k}_0 = -\vec{k}_2 - \vec{q} \) around \( \vec{q} = 0 \). This is related to the way of removing the momentum-conserving delta function from the consistency relations, when writing (3.14) for primed correlation functions. See appendix B for a detailed treatment.

Once again we expand the correlators in powers of \( q \) up to cubic order, by first imposing the on-shell condition \( \vec{k}_1 = -\vec{k}_2 - \vec{q} \):

\[
\left\langle \frac{\zeta^j_m \zeta^a_k \zeta^b_k}{P_\zeta(q)} \right\rangle = \frac{H^2}{M^2_{Pl} k^3} \Pi_{jmn} \left( \langle \Pi_{abmn} \rangle + \frac{1}{2} q^i q^j \frac{\partial^2}{\partial k^i \partial k^j} \Pi_{abmn} \right) \cdots
\]

\[
\times \left( \frac{q \cdot \vec{k}}{k} - \frac{q^2 c_s^2 + c_s + 4 - 3c_s(c_s + 1)(\vec{k} \cdot \vec{q})^2}{8c_s(c_s + 1)} \right) - \frac{q^3}{k^3} \frac{8 (9c_s^4 + 18c_s^2 + 16)}{16c_s(c_s + 1)^2} \vec{k} \cdot \vec{q} - 15c_s(c_s + 1)^2 (\vec{k} \cdot \vec{q})^3 + \cdots
\]

\[
\left\langle \frac{\zeta^a_k \zeta^b_k}{P_\zeta(q)} \right\rangle = \frac{H^2}{M^2_{Pl} c_s(1 + c_s) k^3} \left[ 1 + c_s + c_s^2 - \frac{q^2}{k} \frac{2 + 4c_s + 6c_s^2 + 3c_s^3}{1 + c_s} (\vec{k} \cdot \vec{q}) + \cdots \right]
\]

\[
\times \Pi_{jmn} \left( \langle \Pi_{abmn} \rangle + \frac{1}{2} q^m q^n \frac{\partial^2}{\partial k^m \partial k^n} \Pi_{jmn} \right) + \cdots.
\]

(6.2)

Notice that we have also Taylor expanded the polarization tensor \( \Pi_{jmn} \) in powers of \( q \), applying the identity

\[
\frac{\partial}{\partial k^m_i} \Pi_{jmn} = -\frac{1}{k} \left( \hat{k}_i \Pi_{i,mjmn} + \hat{k}_j \Pi_{jm1mn} + \hat{k}_m \Pi_{ijmn1} + \hat{k}_n \Pi_{ijmn1} \right)
\]

(6.3)

to simplify the last line of (6.2). Since neither correlator gives at contribution at order \( q^0 \), the \( n = 0 \) consistency relation is trivially satisfied.

- **Linear-gradient relation** (\( n = 1 \)): as seen from (6.2), the expansion of \( \langle \zeta \gamma \zeta \rangle / P_\zeta \) starts at order \( q^2 \) and hence does not contribute to the \( n = 1 \) identity. The latter therefore reduces to

\[
\lim_{\vec{q} \to 0} M_{jmo} \left( \langle \frac{\zeta^m_j \zeta^o_k \zeta^k_k}{P_\gamma(q)} \right) \frac{\partial}{\partial q^m} \frac{\langle \zeta^m_j \zeta^o_k \zeta^k_k \rangle}{P_\gamma(q)} = M_{jmo} \Upsilon_{jmo} \frac{\partial}{\partial \hat{k}_2^m} \left( \langle \zeta^m \zeta^o \zeta^k \rangle \right).
\]

(6.4)

The left-hand side gives

\[
\text{L.H.S.} = \lim_{\vec{q} \to 0} M_{jmo} \left( \langle \frac{\zeta^m_j \zeta^o_k \zeta^k_k}{P_\gamma(q)} \right) \frac{\partial}{\partial q^m} \frac{\langle \zeta^m_j \zeta^o_k \zeta^k_k \rangle}{P_\gamma(q)}
\]

\[
= 4 M^2_{Pl} k^2 \frac{M_{mmmn} \left( \langle \Pi_{abmn} \rangle \hat{k}_2^m \right)}{P_\gamma(q)},
\]

(6.5)

Meanwhile, using the explicit expression for \( \Upsilon \) given in (2.15), the right-hand side of the identity becomes

\[
\text{R.H.S.} = M_{jmo} \left( \langle \frac{\zeta^m_j \zeta^o_k \zeta^k_k}{P_\gamma(q)} \right) \frac{H^2}{4 M^2_{Pl} k^2} \frac{\partial}{\partial \hat{k}_2^m} \left( \langle \Pi_{abmn} \rangle \hat{k}_2^m \right)
\]

\[
= M_{jmo} \langle \Pi_{abmn} \rangle \hat{k}_2^m \frac{H^2}{4 M^2_{Pl} k^2},
\]

(6.6)

which agrees with (6.5).
\begin{itemize}
\item **Quadratic consistency relation** \((n = 2)\): the identity at this order reads

\[
\lim_{\tilde{q} \to 0} M_{jmnm\ell_1m_2}(\tilde{q}) \frac{\partial^2}{\partial q_{m_1} \partial q_{m_2}} \left( \frac{\langle \gamma^j \gamma^m \gamma_{\ell_1}^a \gamma_{\ell_2}^b \rangle' \gamma_{\ell_1}^c \gamma_{\ell_2}^d}{P_{\gamma}(q)} + \frac{\delta^j m_{o} \langle \gamma^c_{\ell_1} \gamma_{\ell_2}^d \rangle' \gamma_{\ell_1}^a \gamma_{\ell_2}^b}{3 P_{\gamma}(q)} \right)
\]

\[
= M_{jmnm\ell_1m_2}(\tilde{q}) Y^{j\ell_1m_1m_2}(\hat{k}_2) \frac{\partial^2}{\partial k_{m_1} \partial k_{m_2}} \langle \gamma_{\ell_2}^a \gamma_{\ell_1}^b \rangle'.
\]  \hspace{1cm} \text{(6.7)}

Using the differentiation properties of the projector \(M\) and the polarization tensor \(\Pi\), the first term on the left-hand side of the identity (6.7) gives

\[
\text{L.H.S. 1st term} = \lim_{\tilde{q} \to 0} M_{jmnm\ell_1m_2}(\tilde{q}) \frac{\partial^2}{\partial q_{m_1} \partial q_{m_2}} \left( \frac{\langle \gamma^j \gamma^m \gamma_{\ell_1}^a \gamma_{\ell_2}^b \rangle'}{P_{\gamma}(q)} \right)
\]

\[
= \frac{H^2}{2M_{PI} k_{12}^2} \Pi_{abmn}(\tilde{k}_2) \left( 4 + c_s + c_s^2 \right) M_{mmnm,1}(\tilde{q}) - 4M_{mmnm,1}(\tilde{q}) \hat{k}_{m_1}^m \hat{k}_{m_2}^m
\]

\[
- M_{m_1m_2mn}(\tilde{q}) \hat{k}_{m_1}^m \hat{k}_{m_2}^m + M_{m_1m_2m}(\tilde{q}) \hat{k}_{m_2}^m \left( \Pi_{mnm,1}(\tilde{k}_2) \hat{k}_{m_1}^b + \Pi_{mnm,1}(\tilde{k}_2) \hat{k}_{m_2}^b \right),
\]  \hspace{1cm} \text{(6.8)}

where we have used the fact that

\[
M_{jmnm\ell_1m_2}(\tilde{q}) Y_{jmnm}(\tilde{q}) = M_{mmnm,1}(\tilde{q}) - M_{mmnm,2}(\tilde{q}) - \frac{2}{3} \delta_{mn} M_{j1m,1m}(\tilde{q}).
\]  \hspace{1cm} \text{(6.9)}

Meanwhile, the second term on the left-hand side of (6.7) gives

\[
\text{L.H.S. 2nd term} = \lim_{\tilde{q} \to 0} M_{jmnm\ell_1m_2}(\tilde{q}) \frac{\partial^2}{\partial q_{m_1} \partial q_{m_2}} \left( \frac{\delta^j m_{o} \langle \gamma^c_{\ell_1} \gamma_{\ell_2}^d \rangle' \gamma_{\ell_1}^a \gamma_{\ell_2}^b}{3 P_{\gamma}(q)} \right)
\]

\[
= \frac{2H^2}{3M_{PI} k_{12}^2} c_s(1 + c_s) M_{jjmm}(\tilde{q}) \Pi_{abmn}(\tilde{k}_2).
\]  \hspace{1cm} \text{(6.10)}

The complicated \(c_s\) dependence in (6.8) and (6.10) simplifies tremendously when combining these terms, leaving us with

\[
\text{L.H.S.} = \frac{H^2}{2M_{PI} k_{12}^2} \left( \Pi_{abmn}(\tilde{k}_2) \left( M_{jjmm}(\tilde{q}) - 4 \hat{k}_{m_2}^m \hat{k}_{m_2}^m M_{mmnm,1}(\tilde{q}) - \hat{k}_{m_1}^m \hat{k}_{m_2}^m M_{m_1m_2mn}(\tilde{q}) \right) \right.
\]

\[
- M_{mmnm,1}(\tilde{q}) \hat{k}_{m_2}^m \left( \Pi_{mnm,1}(\tilde{k}_2) \hat{k}_{m_1}^b + \Pi_{mnm,1}(\tilde{k}_2) \hat{k}_{m_2}^b \right) \Bigg) \Bigg). \hspace{1cm} \text{(6.11)}
\]

On the other hand, substituting for \(Y\), the right-hand side of the identity (6.7) becomes

\[
\text{R.H.S.} = M_{jmnm\ell_1m_2}(\tilde{q}) \left( \frac{1}{4} \delta^j m_{o} \hat{k}_{2m_1}^d \hat{k}_{2m_2}^d - \frac{1}{4} \delta^j m_{o} \hat{k}_{2m_2}^d \hat{k}_{2m_1}^d \right) \frac{\partial^2}{\partial k_{m_1} \partial k_{m_2}} \left( \Pi_{cdab}(\tilde{k}_2) P_{\gamma}(\tilde{k}_2) \right)
\]

\[
= \frac{H^2}{M_{PI} k_{12}^2} M_{jmnm\ell_1m_2}(\tilde{q}) \left[ -2 M_{jmnm}(\tilde{q}) \hat{k}_{m_2}^m \hat{k}_{m_2}^m + \frac{1}{2} M_{m_1m_2m_2ab}(\tilde{k}_2) \delta_{jm_0} \hat{k}_{m_2}^m \hat{k}_{m_2}^m - \frac{1}{2} M_{jmnm,1}(\tilde{q}) \hat{k}_{m_2}^m \hat{k}_{m_2}^m + \frac{1}{2} M_{jmnm,1}(\tilde{q}) \hat{k}_{m_2}^m \hat{k}_{m_2}^m \right]. \hspace{1cm} \text{(6.12)}
\]
Similarly, the second term on the left-hand side gives
\[ \frac{\partial^3}{\partial q_{m_1} \partial q_{m_2} \partial q_{m_3}} \left( \frac{\langle \gamma q \rangle^\text{i} \gamma_{k_1} \gamma_{k_2} \gamma_{k_3} \rangle^\text{i}}{P_{\gamma}(q)} \right) \]
\[ = M_{j m_{0} m_{1} m_{2} m_{3}}(q) \Upsilon^{j m_{0} a b c d}(k_2) \frac{\partial^3}{\partial k_{m_1} \partial k_{m_2} \partial k_{m_3}} \langle \gamma_{k_1} \gamma_{k_2} \gamma_{k_3} \rangle^\text{i} \].
\text{(6.13)}

The first term on the left-hand side becomes

\[ \text{L.H.S. 1st term} = \lim_{q \to 0} M_{j m_{0} m_{1} m_{2} m_{3}}(q) \frac{\partial^3}{\partial q_{m_1} \partial q_{m_2} \partial q_{m_3}} P_{\gamma}(q) \]
\[ = \frac{3H^2}{4M_{Pl}^2} M_{m n m_{1} m_{2} m_{3}}(q) \left\{ \Pi_{a b m n}(k_2) \left( -\frac{16 + 17c_s + 18c_m^2}{c_s(1 + c_m)^2} k_{m_1} \delta_{m_2 m_3} + 15k_{m_1} k_{m_2} k_{m_3} \right) 
\right.
\[ + 2k_{m_1} \frac{\partial}{\partial k_{m_2}} \Pi_{a b m n}(k_2) \left( \frac{4 + c_s + c_m^2}{c_s(1 + c_m)} \delta_{m_2 m_3} - 3k_{m_2} k_{m_3} \right) 
\left. + \frac{k_{m_1} k_{m_2} k_{m_3}}{k_{m_1} k_{m_2} k_{m_3}} \Pi_{a b m n}(k_2) k_{m_3} \right\}. \text{(6.14)} \]

In deriving this expression we have used the properties of \( M \) given in (A-III)–(A-VI) of appendix A to discard terms involving \( M_{m n m_{1} m_{2} m_{3}}(q) \hat{q}^{m_1} \hat{q}^{m_2} \hat{q}^{m_3} \) and \( M_{m n j m_{1}}(q) \hat{q}^{m_3} \).

Similarly, the second term on the left-hand side gives

\[ \text{L.H.S. 2nd term} = \lim_{q \to 0} M_{j m_{0} m_{1} m_{2} m_{3}}(q) \frac{\partial^3}{\partial q_{m_1} \partial q_{m_2} \partial q_{m_3}} \left( \frac{\delta_{j m_0} \langle \gamma_{k_1} \gamma_{k_2} \rangle^\text{i}^\text{i}}{3 \frac{P_{\gamma}(q)}{P_{\gamma}(q)}} \right) \]
\[ = 6H^2 M_{m n m_{1} j m_{2}}(q) \left( \frac{2 + 4c_s + 6c_m^2 + 3c_s c_m}{c_s(1 + c_m)^2} k_{m_1} \Pi_{a b m n}(k_2) 
\right.
\[ - \frac{1 + c_s + c_m^2}{c_s(1 + c_m)} k_{m_2} \frac{\partial}{\partial k_{m_3}} \Pi_{a b m n}(k_2) \right\}. \text{(6.15)} \]

Once again, the complicated \( c_s \) dependence simplifies dramatically when combining these two terms, leaving us with

\[ \text{L.H.S.} = \lim_{q \to 0} M_{j m_{0} m_{1} m_{2} m_{3}}(q) \frac{\partial^3}{\partial q_{m_1} \partial q_{m_2} \partial q_{m_3}} \left( \frac{\langle \gamma q \rangle^\text{i} \gamma_{k_1} \gamma_{k_2} \gamma_{k_3} \rangle^\text{i}}{P_{\gamma}(q)} \right) \]
\[ = \frac{3H^2}{4M_{Pl}^2} M_{m n m_{1} m_{2} m_{3}}(q) \left\{ \Pi_{a b m n}(k_2) \left( 15k_{m_1} \delta_{m_2 m_3} + 15k_{m_1} k_{m_2} k_{m_3} \right) 
\right.
\[ + k_{m_1} k_{m_2} k_{m_3} \Pi_{a b m n}(k_2) \right\}. \text{(6.16)} \]

By explicitly substituting for \( \Pi \) and using (6.3), we have checked with Mathematica that the result agrees with the right-hand side — the \( n = 3 \) identity checks out!
7 Consistency relations with two tensors

Finally we consider the case where the hard modes consist of two tensors, i.e., \( \mathcal{O}(\vec{k}_1, \vec{k}_2) = \gamma_{\vec{k}_1}^{ab,cd} \). In this case, the identity (3.14) reduces to\(^9\)

\[
\lim_{\vec{q} \to 0} M_{jm_0 \ldots m_n}(\vec{q}) \frac{\partial^n}{\partial q^{m_1 \ldots q^{m_n}}} \left( \frac{\langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \ldots m_n \gamma_{\vec{k}_2}} P_\gamma(q) + \delta_{jm_0} \frac{\langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}}}{3 P_\gamma(q)} \right) = - M_{jm_0 \ldots m_n}(\vec{q}) \left( \delta_{jm_0} + k^j \frac{\partial}{\partial q^{m_0}} \right) \langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}} + M_{jm_0 \ldots m_n}(\vec{q}) \left( \Gamma_{jm_0 \ldots m_n}(\vec{\hat{k}}) \langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}} + \Gamma_{jm_0 \ldots m_n}(\vec{\hat{k}}) \langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}} \right),
\]

(7.1)

where \( \Gamma \) was defined in (2.15). On the left-hand side, an important simplification occurs when noting that \( \langle \gamma \gamma \rangle / P_\gamma \) is subleading in \( \epsilon \) compared to \( \langle \gamma \gamma \rangle / P_\gamma \). In other words, only the latter contributes to leading order in slow-roll.

In order to obtain the soft limit expansion of the correlation function, we impose the on-shell condition \( \vec{k}_2 = -\vec{k}_1 - \vec{q} \) and Taylor-expand in \( q \):

\[
\frac{\langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \ldots m_n \gamma_{\vec{k}_2}}}{P_\gamma(q)} = \frac{2H^2}{4M^2 \Pi k^2} \left( 1 - \frac{5\vec{k} \cdot \vec{q}}{2k} - \frac{5}{6k^2} \left( -7(\vec{k} \cdot \vec{q})^2 + 2q^4 \right) - \frac{105(\vec{k} \cdot \vec{q})^3 - 35q^2 \vec{k} \cdot \vec{q} + q^4}{8k^3} + \ldots \right) \times \Pi_{aa'}(\vec{q}) \Pi_{bb'}(\vec{\hat{k}}) \left( k^a \delta_{bc} - (k + q)^b \delta_{ac} + q^c \delta_{ab} \right) \left( k^a' \delta_{b'c'} - (k + q)^b' \delta_{a'c'} + q^c' \delta_{a'b'} \right) \times \left( \Pi_{mn,cc'}(\vec{\hat{k}}) + q^{f_1} \frac{\partial}{\partial k^{f_1}} \Pi_{mn,cc'}(\vec{\hat{k}}) + \frac{1}{2} q^{f_1} q^{f_2} \frac{\partial^2}{\partial k^{f_1} \partial k^{f_2}} \Pi_{mn,cc'}(\vec{\hat{k}}) + \ldots \right).
\]

(7.2)

\* Lowest-order relation (\( n = 0 \)): the identity in this case reduces to

\[
\lim_{\vec{q} \to 0} M_{jm_0}(\vec{q}) \frac{\langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}}}{P_\gamma(q)} = - M_{jm_0}(\vec{q}) \left( \delta_{jm_0} + k^j \frac{\partial}{\partial q^{m_0}} \right) \langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}} + M_{jm_0}(\vec{q}) \left( \Gamma_{jm_0}(\vec{\hat{k}}) \langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}} + \Gamma_{jm_0}(\vec{\hat{k}}) \langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}} \right).
\]

(7.3)

For dilation, \( M_{jm_0} \propto \delta_{jm_0} \), the relation is trivially satisfied. For anisotropic scaling,\(^10\) we find that

\[
\text{L.H.S.} = \lim_{\vec{q} \to 0} M_{jm_0}(\vec{q}) \frac{\langle \zeta \gamma \rangle_{\vec{k}_1}^{jm_0 \gamma_{\vec{k}_2}}}{P_\gamma(q)} = 3H^2 \frac{M_{mn}(\vec{q}) k^m k^n \Pi_{ab,cd}(\vec{\hat{k}})}{M^2 \Pi k^2}.
\]

(7.4)

With the aid of (6.3) and (2.15), it is straightforward to show that this matches the right-hand side of the identity.

\(^9\)We have used the fact that \( \Gamma_{jm_0 \ldots m_n}(\vec{\hat{k}}) = \Gamma_{jm_0 \ldots m_n}(\vec{k}) \), so that the derivative on \( \Gamma \) with respect to \( \vec{k}_2 \) in (3.14) is now translated to derivative w. r. t. the hard momentum \( \vec{k}_1 = \vec{k} \).

\(^10\)In this case, we can assume that \( M_{jm_0} \) is symmetric, traceless and transverse to \( q \), without loss of generality.
• **Linear-gradient relation** \((n = 1)\): the \(n = 1\) relation reads

\[
\lim_{q \rightarrow 0} \frac{d}{dq} m_0 \left( \frac{1}{P_0(q)} \right) \left( \gamma_{jm_0}^{ab,cd}_k \right) = - \frac{d}{dM_0} \left( \delta_{jk} + \frac{k^j}{2} \frac{d^2}{dM_0^2} \right) \left( \gamma_{jm_0}^{ab,cd}_k \right) + \frac{d}{dM_0} \Gamma_{jm_0 \cdmn} (k) \left( \gamma_{jm_0}^{ab,cd}_k \right). \tag{7.5}
\]

As a result of a lengthy computation, we find that the left-hand side becomes

\[
\text{L.H.S.} = \frac{H^2}{M_0^2} \left[ -3k^m \hat{k}^n \Pi_{abc}(\hat{k}) \hat{k}^d \hat{k}^e - \frac{15}{2} \hat{k}^m \hat{k}^n \hat{k}^d \Pi_{abcd}(\hat{k}) + \frac{3}{2} \hat{k}^d \Pi_{cdmn}(\hat{k}) \Pi_{abmn}(\hat{k}) + \delta_{mn} \left( \frac{5}{2} \hat{k}^d \Pi_{abcd}(\hat{k}) + \hat{k}^d \Pi_{abcd}(\hat{k}) \right) \right]. \tag{7.6}
\]

By plugging the explicit form for \(\Pi\) and its derivatives, it is straightforward, though tedious, to show using Mathematica that this expression matches the right-hand side of (7.5).

• **Quadratic consistency relation** \((n = 2)\): the identity at this order reduces to

\[
\lim_{q \rightarrow 0} \frac{d}{dq_{m_1}} \frac{d}{dq_{m_2}} \left( \frac{1}{P_0(q)} \right) \left( \gamma_{jm_0}^{ab,cd}_k \right) = - \frac{d}{dM_0} \left( \delta_{jk} + \frac{k^j}{3} \frac{d^2}{dM_0^2} \right) \left( \gamma_{jm_0}^{ab,cd}_k \right) + \frac{d^2}{dM_0^2} \Gamma_{jm_0 \cdmn}(\hat{k}) \left( \gamma_{jm_0}^{ab,cd}_k \right). \tag{7.7}
\]

Again, the left hand side can be written in the following form

\[
\text{L.H.S.} = \frac{H^2}{M_0^2} \left[ -10 \delta_{jm_0} \Pi_{abcd}(\hat{k}) \hat{k}^{m_1} \hat{k}^{m_2} + 35 \Pi_{abcd}(\hat{k}) \hat{k}^{m_0} \hat{k}^{m_1} \hat{k}^{m_2} + 2 \delta_{jm_0} \Pi_{abmn}(\hat{k}) \Pi_{cdmn}(\hat{k}) - \frac{15}{2} \Pi_{abmn}(\hat{k}) \Pi_{cdmn}(\hat{k}) \hat{k}^{m_0} \hat{k}^{m_2} + \frac{15}{2} \Pi_{abmn}(\hat{k}) \Pi_{cdmn}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} - 3 k^m \hat{k}^n \Pi_{abmn}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} + 3 k^m \hat{k}^n \Pi_{abmn}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} + \frac{5}{2} \delta_{jm_0} \Pi_{abcd}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} - \frac{15}{2} \Pi_{abmn}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} \hat{k}^{m_2} - \frac{5}{2} \delta_{jm_0} \Pi_{abcd}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} + \frac{3}{2} \delta_{jm_0} \Pi_{abcd}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} \hat{k}^{m_2} - \frac{15}{2} \Pi_{abmn}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} + \frac{3}{2} \Pi_{abmn}(\hat{k}) \hat{k}^{m_2} \hat{k}^{m_2} \hat{k}^{m_2} \right]. \tag{7.8}
\]

Despite its complexity, explicit substitution in Mathematica, taking into account the explicit form of \(\Pi\) and its derivatives, shows that this relation matches the right-hand side of the identity.
• **Cubic consistency relation** ($n = 3$): at this order, expressions become quite cumbersome. The cubic contribution to the 3-point function is given by

$$\frac{\langle \gamma q^2 \rangle_{c} \Pi_{mnbc}(\vec{k})}{P_s(q)} \geq \frac{3H^2}{4M_{Pl}^2 k^5} \Pi_{\gamma\gamma}(\vec{q}) \Pi_{kbb'}(\vec{k}')(\vec{k}) \left[ \frac{1}{2} q' q'' (\delta_{\alpha' \alpha'} + q' \delta_{\alpha' \nu'}) \frac{\partial^2}{\partial k^2 \partial k'} \Pi_{mnbc}(\vec{k}) k^a \right. $$

$$+ \frac{1}{6} q'^2 q'' q'' \frac{\partial^2}{\partial k^2 \partial k' \partial k''} \Pi_{mnbb'}(\vec{k}) k^a k^{a'} - \frac{105 (\vec{k} \cdot \vec{q})^2 - 35 k^2 \vec{k} \cdot \vec{q} + 2 k^2 q^2}{8k^6} \Pi_{mnbb'}$$

$$- 5 \left( \frac{7 (\vec{k} \cdot \vec{q})^2 + k^2 q^2}{6k^4} \right) \left( 2 \Pi_{mnbc}(\vec{k}) k^a \left( - q'' \delta_{\alpha' \nu'} + q' \delta_{\alpha' \nu'} \right) + q'^2 \frac{\partial}{\partial k^2} \Pi_{mnbb'}(\vec{k}) k^a \right. $$

$$\left. - \frac{5k \cdot \vec{q}}{2k^2} \left( \Pi_{mcab'}(\vec{k}) \left( - q'' \delta_{\alpha \nu} + q' \delta_{\alpha \nu} \right) \left( - q'' \delta_{\alpha' \nu'} + q' \delta_{\alpha' \nu'} \right) + 2 q'^4 \left( - q'' \delta_{\alpha' \nu'} + q' \delta_{\alpha' \nu'} \right) \frac{\partial}{\partial k^2} \Pi_{mnbc}(\vec{k}) k^a \right. $$

$$\left. + \frac{1}{2} q'^4 q'' \frac{\partial^2}{\partial k^2 \partial k' \partial k''} \Pi_{mnbb'}(\vec{k}) k^a k^{a'} \right].$$

After differentiating this expression with respect to $q$, the tedious calculation using Mathematica shows that the consistency condition holds at this order as well.

### 8 Conclusions

Single-field primordial perturbations satisfy an infinite number of consistency relations constraining at each $n \geq 0$ the $q^n$ behavior of correlation functions in the soft limit $q \to 0$. These are the consequence of Ward identities for an infinite set of residual, global symmetries, which are non-linearly realized on the perturbations [7]. Equivalently, they are also the consequence of the Slavnov-Taylor identity for spatial diffeomorphisms [9].

In this paper, we performed a number of non-trivial checks of the consistency relations up to and including $q^3$ order, focusing for simplicity on identities involving 3-point functions with a soft external leg. We considered all possible scalar and tensor combinations of hard modes, i.e., $\zeta\zeta$, $\zeta\gamma$, and $\gamma\gamma$. Our computation was done in the context of single-field quasi-de Sitter inflation with arbitrary constant sound speed $c_s$. For this purpose, the 3-point functions $\langle \gamma \zeta \zeta \rangle$ and $\langle \zeta \gamma \gamma \rangle$ with $c_s \neq 1$ were computed here for the first time.

The $n = 2$ and $n = 3$ checks are particularly non-trivial and physically interesting. Indeed, at order $q^2$ and higher, part of the information encoded in correlation functions is physical — it represents spatial curvature, which is model-dependent and therefore cannot be constrained in terms of lower-point functions. Nevertheless, certain linear combinations of soft correlators are constrained by consistency relations [7]. This showed up explicitly in our checks through non-trivial $c_s$ dependence — only by taking certain linear combinations dictated by the identities did the complicated $c_s$ dependence of the 3-point functions simplify to match that of the 2-point functions.

The $n \geq 2$ checks are important because the corresponding identities have not yet been derived through standard background-wave arguments. In particular, the mode functions in momentum-space acquire an imaginary part at $n = 3$, and hence cannot fully describe a ‘classical’ background wave. Our checks nevertheless confirm that part of the $q^3$ information can be thought of as classical, and is constrained in a background-wave manner.

It is clear that soft limits of cosmological correlation functions encode a wealth of information about the primordial universe. A number of interesting future avenues naturally suggest themselves:

- The lowest-order ($n = 0, 1$) identities can be violated whenever their underlying assumptions — single field, attractor background, adiabatic vacuum — are not satisfied. In this sense, the identities are not tautological: they are physical statements that can be tested observationally.

We expect the same holds true for the higher-order ($n \geq 2$) identities, but it would be reassuring...
to seek explicit violations, e.g., in multi-field scenarios or with non-adiabatic vacua. This is currently in progress [30].

• As is well-known from pion physics [31], another probe of higher-$q$ dependence is to consider multiple soft limits. This is currently under investigation for the conformal algebra associated with scalar perturbations [32]. It would be enlightening to generalize and include tensor modes as well.

• The higher-order identities were discovered through formal, field-theoretic methods [7–9, 24]. A more physical derivation of the consistency relations using background-wave arguments would undoubtedly offer new insights on their physical origin.

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A Properties of the projectors

In this appendix we present some useful relations for the projection operator $M_{\hat{q}_i\ldots\hat{q}_n}$, which follow directly from its defining properties. By definition the projector satisfies the following transversality condition

$$\hat{q}_i \left( M_{ijkm_1m_2m_3}(\hat{q}) + M_{jim_1m_2m_3}(\hat{q}) - \frac{2}{3} \delta_{ij} M_{kkm_1m_2m_3}(\hat{q}) \right) = 0. \quad (A-I)$$

Contracting (A-I) with $\delta_{m_1m_2}$,\textsuperscript{\footnote{\textsuperscript{\footnote{Notice that we do not get anything new if we contract (A-I) with $\delta_{jm_1}$}}} we get

$$\hat{q}_i M_{ijmm_i}(\hat{q}) + \hat{q}_i M_{jimm_i}(\hat{q}) = 0. \quad (A-II)$$

Given that the projector obeys the trace condition (2.10), we arrive at

$$\hat{q}_i M_{ijkm}(\hat{q}) = \hat{q}_i M_{jkim}(\hat{q}) = 0. \quad (A-III)$$

By contracting both sides of (A-I) with $\hat{q}_j$, we get

$$\hat{q}_i \hat{q}_j M_{ijm_1m_2m_3}(\hat{q}) = \frac{1}{3} M_{kkm_1m_2m_3}(\hat{q}). \quad (A-IV)$$

Also we can multiply (A-I) by $\hat{q}_m$. With the aid of (A-IV), this leads us to obtain

$$\frac{1}{3} M_{kkjm_2m_3}(\hat{q}) + \hat{q}_i \hat{q}_m M_{jm_2m_3m_1}(\hat{q}) - \frac{2}{3} \hat{q}_j \hat{q}_m M_{kkm_2m_3m_1}(\hat{q}) = 0. \quad (A-V)$$

Combining (A-III)–(A-V), we get

$$\hat{q}_i \hat{q}_m \hat{q}_n M_{jm_2m_3m_1}(\hat{q}) = 0. \quad (A-VI)$$
B Stripping off delta functions

In this section, we consider the question about how to remove delta functions in correlation functions and write the Ward identities in terms of on-shell correlators. This was partially done in [7]. We will perform a more systematic analysis here.

Notice that the terms appearing in the original Ward Identities (3.12) take the general form of

$$A_n = \sum_{b=1}^{N} \Theta_b(\vec{k}_b(0)) \frac{\partial}{\partial k_{b}^{m_1}} ... \frac{\partial}{\partial k_{b}^{m_n}} \left( f_b(\vec{k}_1, ..., \vec{k}_N) \delta(\vec{K}_t) \right) ,$$  \hspace{1cm} (B-I)

where we denote by $\vec{K}_t = \vec{k}_1 + ... + \vec{k}_N$ the total momentum, and by $\Theta$ collectively the operators such as $\Upsilon, \Gamma, \ldots$ appearing in (3.12). The first task is to write $A_n$ explicitly in terms of the delta function and its derivatives.

To do that, first define $N-1$ relative momenta $\vec{k}_r^{(i)}$, $i = 1, ..., N-1$. Although the choice of $\vec{k}_r$ is arbitrary, once it is made all the $N$ momenta $\vec{k}_b$ — and hence $\Theta_b(\vec{k}_b)$ and $f_b(\vec{k}_1, ..., \vec{k}_N)$ — can be expressed unambiguously as a function of $\vec{K}_t$ and $\{\vec{k}_r\}_{i=1}^{N-1}$:

$$\vec{k}_r = \vec{k}_r(\vec{K}_t ; \vec{k}_r) .$$ \hspace{1cm} (B-II)

As a result, $A_0$ can be rewritten as

$$A_0 = \sum_{b=1}^{N} \Theta_b(\vec{k}_b(0)) f_b(\vec{k}_1 = 0; \vec{k}_r) \delta(\vec{K}_t) \equiv \tilde{A}_0(\vec{K}_t) .$$ \hspace{1cm} (B-III)

Now let us work out the expression for $A_1$:

$$A_1 = \sum_{b=1}^{N} \Theta_b(\vec{k}_b(0)) \left[ \left( \frac{\partial}{\partial k_{b}^{m_1}} f_b(0; \vec{K}_r) \right) \delta(\vec{K}_t) + f_b(0; \vec{k}_r) \left( \frac{\partial}{\partial k_{b}^{m_1}} \delta(\vec{K}_t) \right) \right]$$

$$= \delta(\vec{K}_t) \sum_{b=1}^{N} \Theta_b(\vec{k}_b(0)) \left( \frac{\partial}{\partial k_{b}^{m_1}} f_b(0; \vec{K}_r) \right) + \tilde{A}_0 \frac{\partial}{\partial K_{t}^{m_1}} \delta(\vec{K}_t)$$

$$+ \left( \frac{\partial}{\partial K_{t}^{m_1}} \delta(\vec{K}_t) \right) \sum_{b=1}^{N} f_b(0; \vec{k}_r) \left[ \Theta_b(\vec{k}_b(\vec{K}_t; \vec{k}_r)) - \Theta_b(\vec{k}_b(0; \vec{K}_r)) \right] .$$ \hspace{1cm} (B-IV)

The last line of the above expression can be further simplified:

$$\text{Last Line} = \sum_{b=1}^{N} f_b(0; \vec{k}_r) \frac{\partial}{\partial K_{t}^{m_1}} \left[ \delta(\vec{K}_t) \left( \Theta_b(\vec{k}_b(\vec{K}_t; \vec{k}_r)) - \Theta_b(\vec{k}_b(0; \vec{K}_r)) \right) \right]$$

$$- \sum_{b=1}^{N} f_b(0; \vec{k}_r) \delta(\vec{K}_t) \frac{\partial}{\partial K_{t}^{m_1}} \Theta_b(\vec{k}_b(\vec{K}_t; \vec{k}_r))$$

$$= - \sum_{b=1}^{N} f_b(0; \vec{k}_r) \delta(\vec{K}_t) \frac{\partial \Theta_b(\vec{k}_b(\vec{K}_t; \vec{k}_r))}{\partial K_{t}^{m_1}} \bigg|_{\vec{K}_t=0} ,$$ \hspace{1cm} (B-V)

where in the last equality we have used the identity about the delta function: $\delta(x)[f(x) - f(0)] = 0$. Therefore we get

$$A_1 = \tilde{A}_1 \delta(\vec{K}_t) + \tilde{A}_0 \frac{\partial}{\partial K_{t}^{m_1}} \delta(\vec{K}_t) ,$$ \hspace{1cm} (B-VI)

where we have defined

$$\tilde{A}_1 = \sum_{b=1}^{N} \left[ \Theta_b(\vec{k}_b(0; \vec{k}_r)) \frac{\partial f_b(0; \vec{k}_r)}{\partial k_{b}^{m_1}} - f_b(0; \vec{k}_r) \frac{\partial \Theta_b(\vec{k}_b(\vec{K}_t; \vec{k}_r))}{\partial K_{t}^{m_1}} \right] \bigg|_{\vec{K}_t=0} .$$ \hspace{1cm} (B-VII)

Now we are in a position to find the expression for $A_n$ for generic $n$. Before we dive into that, let us prove the following lemma about the delta function:
**Lemma.** Let \( f(x) \) be some smooth function which vanishes sufficiently fast at infinity \( |x| \to \infty \). Then

\[
f(x)\delta^{(n)}(x) = \sum_{p=0}^{n} (-1)^p \binom{n}{p} f^{(p)}(0) \delta^{(n-p)}(x),
\]

(B-VIII)

where \( \delta^{(n)}(x) \equiv \frac{d^n}{dx^n} \delta(x) \) and \( f^{(n)}(0) \equiv \lim_{x \to 0} \frac{d^n}{dx^n} f(x) \).

**Proof.** For \( n = 0 \), it follows from the property of the delta function. For \( n = 1 \),

\[
f(x)\delta'(x) = \frac{d}{dx} \left[ (f(x) - f(0)\delta(x)) + f(0)\delta'(x) - f'(x)\delta(x) \right] = f(0)\delta'(x) - f'(x)\delta(x),
\]

(B-IX)

which is precisely the hypothetical equation for \( n = 1 \).

Now we assume the eq. (B-VIII) holds for all \( n \leq K - 1, K \in \mathbb{Z} \cap [1, \infty) \). For \( n = K \),

\[
f(x)\delta^{(K)}(x) = \frac{d}{dx} \left[ f(x)\delta^{(K-1)}(x) \right] - f'(x)\delta^{(K-1)}(x)
\]

\[
= \sum_{p=0}^{K-1} (-1)^p \binom{K-1}{p} f^{(p)}(0)\delta^{(K-p)}(x) - \sum_{p=0}^{K-1} (-1)^p \binom{K-1}{p} f^{(p+1)}(0)\delta^{(K-1-p)}(x)
\]

\[
= \sum_{p=0}^{K-1} (-1)^p \binom{K-1}{p} f^{(p)}(0)\delta^{(K-p)}(x) + \sum_{p=1}^{K} (-1)^p \binom{K-1}{p-1} f^{(p)}(0)\delta^{(K-p)}(x)
\]

\[
= \sum_{p=0}^{K} (-1)^p \binom{K}{p} f^{(p)}(0)\delta^{(K-p)}(x).
\]

(B-X)

Therefore we can conclude that the hypothetical equation (B-VIII) holds for all integers \( n \).

Now let us go back to \( A_n \). Using the Leibniz and the above lemma, we get that

\[
A_n = \sum_{b=1}^{N} \Theta_b(\vec{K}_b) \sum_{p=0}^{n} \binom{n}{p} \left[ \frac{\partial^p}{\partial k_b^m_1 \ldots \partial k_b^m_p} f_b(0; \vec{K}_b) \right] \delta^{(n-p)}(\vec{K}_b)
\]

\[
= \sum_{b=1}^{N} \sum_{p=0}^{n} \binom{n}{p} \frac{\partial^p}{\partial k_b^m_1 \ldots \partial k_b^m_p} f_b(0; \vec{K}_b) \binom{n-p}{q} \frac{\partial^q \Theta_b(\vec{K}_b; \vec{K}_b)}{\partial K_b^{m+1} \ldots \partial K_b^{m+q}} \bigg|_{\vec{K}_b=0} \delta^{(n-p-q)}(\vec{K}_b).
\]

(B-XI)

Changing the dummy variable \( q \) to \( L \equiv q + p \) and noticing that

\[
\binom{n}{p} \binom{n-p}{q} = \binom{n}{p+q} \binom{p+q}{p}
\]

(B-XII)

we can rewrite the sum in the equation above as

\[
A_n = \sum_{L=0}^{n} \binom{n}{L} \delta^{(n-L)}(\vec{K}_b) \sum_{b=1}^{N} \sum_{p=0}^{L} (-1)^{L-p} \binom{L}{p} \frac{\partial^p}{\partial k_b^m_1 \ldots \partial k_b^m_p} f_b(0; \vec{K}_b) \frac{\partial^{L-p} \Theta_b(\vec{K}_b; \vec{K}_b)}{\partial K_b^{m+1} \ldots \partial K_b^{m+L}} \bigg|_{\vec{K}_b=0}
\]

\[
= \sum_{L=0}^{n} \binom{n}{L} \delta^{(n-L)}(\vec{K}_b) A_L.
\]

(B-XIII)

\(^{12}\text{Keep in mind that all the free index } m_1, \ldots, m_n \text{ here are contracted with the symmetric part of the projector } M_{m_1 \ldots m_n}(q).\)
with

$$\tilde{A}_L = \sum_{b=1}^{N} \sum_{p=0}^{L} (-1)^{L-p} \binom{L}{p} \frac{\partial^p}{\partial k_b^{m_1} \cdots \partial k_b^{m_p}} f_b(0; \tilde{K}_r) \frac{\partial^{L-p} \Theta_b(\tilde{K}_r)}{\partial K_t^{m_{p+1}} \cdots \partial K_t^{m_L}} \bigg|_{\tilde{K}_t=0}.$$  \hfill (B-XIV)

The formidable expression for $\tilde{A}_L$ given above can be greatly simplified if we specify the on-shell condition to be

$$\tilde{K}_r^{(1)} = \tilde{k}_1, \; \ldots, \; \tilde{K}_r^{(N-1)} = \tilde{k}_{N-1}, \; \tilde{K}_T = \tilde{k}_1 + \cdots + \tilde{k}_N,$$  \hfill (B-XV)

which can also be written reversely as

$$\tilde{k}_1 = \tilde{K}_r^{(1)}, \; \ldots, \; \tilde{k}_{N-1} = \tilde{K}_r^{(N-1)}, \; \tilde{k}_N = \tilde{K}_T - \tilde{k}_1 - \cdots - \tilde{k}_{N-1}.$$  \hfill (B-XVI)

Then it is easy to see that

$$\frac{\partial^f}{\partial K_t^{m_1} \cdots \partial K_t^{m_p}} \Theta_b(\tilde{K}_r(0; \tilde{K}_r)) = \sum_{m_{i_1}, \ldots, m_{i_N}} \frac{\partial k_b^{m_{i_1}}}{\partial K_t^{m_1}} \cdots \frac{\partial k_b^{m_{i_N}}}{\partial K_t^{m_N}} \frac{\partial^f}{\partial k_b^{m_1} \cdots \partial k_b^{m_p}} \Theta_b(\tilde{k}_b) = 0, \text{ for } \ell \geq 1 \text{ and } b \neq N,$$

$$\frac{\partial^f}{\partial K_t^{m_1} \cdots \partial K_t^{m_N}} f_N(0; \tilde{K}_r) = \frac{\partial^f}{\partial K_t^{m_1} \cdots \partial K_t^{m_N}} f_N(\tilde{k}_1, \ldots, \tilde{k}_{N-1}) = 0, \text{ for } \ell \geq 1.$$  \hfill (B-XVII)

Therefore the only terms that survive in $\tilde{A}_L$ are

$$\tilde{A}_L = \sum_{b=1}^{N-1} \left[ \frac{\partial^L}{\partial k_b^{m_1} \cdots \partial k_b^{m_L}} f_b(0; \tilde{K}_r) \Theta_b(\tilde{k}_b(0; \tilde{K}_r)) + (-1)^L f_N(0; \tilde{K}_r) \frac{\partial^L \Theta_N(\tilde{k}_N(\tilde{K}_r; \tilde{K}_r))}{\partial K_t^{m_{p+1}} \cdots \partial K_t^{m_L}} \bigg|_{\tilde{K}_t=0} \right]$$

$$= \sum_{b=1}^{N-1} \left[ \frac{\partial^L}{\partial k_b^{m_1} \cdots \partial k_b^{m_L}} f_b(0; \tilde{K}_r) \Theta_b(\tilde{k}_b(0; \tilde{K}_r)) + (-1)^L f_N(0; \tilde{K}_r) \frac{\partial^L \Theta_N(\tilde{k}_N)}{\partial K_t^{m_1} \cdots \partial K_t^{m_L}} \bigg|_{\tilde{k}_N=\tilde{k}_1 - \cdots - \tilde{k}_{N-1}} \right].$$  \hfill (B-XVIII)

### B.1 Ward identities without the delta function

We have mentioned that all the terms in the original Ward identities take the form of (B-I), so (3.12) can be written abstractly as

$$I_n = \sum_i A_n^{(i)} + \frac{1}{n+1} B_{n+1} = 0,$$  \hfill (B-XIX)

where we have used $B_{n+1}$ unambiguously to denote the term $\sum_{a=1}^{N} k_a^{m_{n+1}} \frac{\partial^m \Theta_a}{\partial k_a^{m_{n+1}}} (\mathcal{O}) \delta(\tilde{K}_L)$ and $i$ to label to other terms in (3.12). Using (B-XIII), we find that

$$0 = I_n = \sum_{L=0}^{n} \binom{n}{L} \delta^{(n-L)}(\tilde{K}_L) \left[ \sum_i \tilde{A}_n^{(i)} \right] + \frac{1}{n+1} \sum_{L=0}^{n+1} \binom{n+1}{L} \delta^{(n+1-L)}(\tilde{K}_L) \tilde{B}_L$$

$$= \sum_{L=0}^{n} \binom{n}{L} \delta^{(n-L)}(\tilde{K}_L) \left[ \sum_i \tilde{A}_n^{(i)} + \frac{1}{L+1} \tilde{B}_{L+1} \right],$$  \hfill (B-XX)

where in the last equality, we have changed the dummy indice $L$ to $L' = L - 1$ and made used the fact that

$$\tilde{B}_0 = \sum_{a=1}^{N} k_a^{m_{n+1}} (\mathcal{O}) \bigg|_{\tilde{K}_r=0} = 0.$$  \hfill (B-XXI)
Therefore, the $n = 0$ Ward Identity is equivalent to\textsuperscript{13} \[ \sum_{i} \hat{A}_i^{(i)} + \hat{B}_1 = 0. \] For $n \geq 1$, the new information contained in each higher order identity is

\[ \sum_{i} \hat{A}_i^{(i)} + \frac{1}{n+1} \mathcal{B}_{n+1} = 0. \] (B-XXII)

Since the delta function has been removed from $\hat{A}_n$ and $\mathcal{B}_n$, (B-XXII) is in the form that we are after.

It remains to work out explicitly each term in the Ward identities in this delta-function-removed form, by applying (B-XVIII). For instance, for the term on the left-hand side of (3.12), effectively correlators, we have

\[ \langle \cdots \rangle = \sum_{M} \frac{1}{M} \int_{\mathcal{R}_M} \mathcal{I}_{M}^{i_0 j_0} (\vec{k}_a) \mathcal{Q}^{i_0} (\vec{q}_a) \mathcal{Q}^{j_0} (\vec{q}_a) \mathcal{Q}^{i_1} (\vec{q}_a) \mathcal{Q}^{j_1} (\vec{q}_a) \ldots \mathcal{Q}^{i_N} (\vec{q}_a) \mathcal{Q}^{j_N} (\vec{q}_a) \mathcal{I}_{N}^{i_0 j_0} (\vec{k}_a) \mathcal{I}_{N}^{i_1 j_1} (\vec{k}_a) \ldots \] (B-XXIII)

where we have denoted by $\langle \cdots \rangle'$ the correlation function with delta function removed and with on-shell condition imposed. We can play a similar trick on the terms on the right hand side of (3.12). Finally we reach the Ward identities expressed in terms of primed correlators:

For $\mathcal{O}$ containing at least one $\gamma$ field:

\[ \text{L. H. S.} \rightarrow \frac{\partial^n}{\partial q^{m_1} \ldots \partial q^{m_n}} \left( \langle \gamma(q) \mathcal{O} \rangle_{\langle \gamma \rangle} + \frac{\langle z(q) \mathcal{O} \rangle_{\langle z \rangle}}{3} \right) \]

\[ = \frac{\partial^n}{\partial q^{m_1} \ldots \partial q^{m_n}} \left( \langle \gamma(q) \mathcal{O} \rangle'_{\langle \gamma \rangle} + \frac{\langle z(q) \mathcal{O} \rangle'_{\langle z \rangle}}{3P_z(q)} \right), \] (B-XXIII)

\[ \lim_{\vec{q} \to 0} M_{d(a) \ldots e_n}(\vec{q}) \frac{\partial^n}{\partial q^{t_1} \ldots \partial q^{t_n}} \left( \frac{1}{P_z(q)} \langle \gamma^{i_0} (\vec{q}) \mathcal{O} (\vec{k}_1, \ldots, \vec{k}_N) \rangle_{\langle \gamma \rangle} + \frac{\delta^{i_0}}{3P_z(q)} \langle z (\vec{q}) \mathcal{O} (\vec{k}_1, \ldots, \vec{k}_N) \rangle_{\langle z \rangle} \right) \]

\[ = -M_{d(a) \ldots e_n}(\vec{q}) \left\{ \sum_{a=1}^{N-1} \frac{\partial^n}{\partial k^{t_1}_a \ldots \partial k^{t_n}_a} \left( \mathcal{O}^i (\vec{k}_1, \ldots, \vec{k}_a-1, \vec{k}_a+1, \ldots, \vec{k}_M) \gamma_{i_{a+1}} (\vec{k}_a) \mathcal{O}_j (\vec{k}_a+1, \ldots, \vec{k}_N) \right) \right. \]

\[ - \sum_{a=1}^{M} \Gamma^{i_0 j_0} (\vec{k}_a) \left. \mathcal{Q}^{i_0} (\vec{q}_a) \mathcal{Q}^{j_0} (\vec{q}_a) \ldots \mathcal{Q}^{i_{a+1}} (\vec{q}_a) \mathcal{Q}^{j_{a+1}} (\vec{q}_a) \ldots \right\} \]

\[ + \ldots, \] (B-XXIV)

\textsuperscript{13} Notice that $\hat{A}_L$ and $\hat{B}_L$ are already computed on shell.
For $O = O^c$: 

$$
\lim_{\vec{q} \to 0} M_{\ell_0 \ldots \ell_n}(\vec{q}) \left( \partial^n \frac{1}{P_0(q)} \langle \delta^{\ell_0}(\vec{q}) O(\vec{k}_1 , \ldots , \vec{k}_N) \rangle_c + \frac{\delta^{\ell_0}}{3P_0(q)} \langle \gamma(\vec{q}) O(\vec{k}_1 , \ldots , \vec{k}_N) \rangle_c \right) 
$$

$$
= -M_{\ell_0 \ldots \ell_n}(\vec{q}) \left\{ \sum_{a=1}^{N-1} \left( \delta^{\ell_0} \partial_{\vec{k}^2_{a}} \ldots \partial_{\vec{k}^2_{N}} \right)^n \delta \frac{\partial^n}{N-1} \frac{\partial^{n+1}}{n+1} \langle O(\vec{k}_1 , \ldots , \vec{k}_N) \rangle_c 
$$

$$
- \sum_{a=1}^{N-1} \mathbb{Y}_{\ell_0 \ldots \ell_n}^{\ell_a} \left( \partial_{\vec{k}^2_{a}} \ldots \partial_{\vec{k}^2_{N}} \right)^n \langle O^c(\vec{k}_1 , \ldots , \vec{k}_{a-1}, \vec{k}_{a+1}, \ldots , \vec{k}_N) \gamma_{i_a j_a}(\vec{k}_a) \rangle_c 
$$

$$
- (-1)^n \left[ \frac{\partial^n}{\partial_{\vec{k}^2_{1}} \ldots \partial_{\vec{k}^2_{N}}} \mathbb{Y}_{\ell_0 \ldots \ell_n}^{\ell_{N-1} j_{N}} \langle \vec{k}_N \rangle \right]_{OS} \langle O^c(\vec{k}_1 , \ldots , \vec{k}_{N-1}) \gamma_{i_{N} j_{N}}(\vec{k}_N) \rangle_c 
$$

$$
+ \ldots ,
$$

(B-XXV)

where the on-shell condition is specified explicitly by $\vec{k}_N = -\vec{k}_1 - \ldots - \vec{k}_{N-1}$.

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