Robust Assessment on the Developments of Three Extended Exponential Models with Some New Properties and Applications

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1. Introduction

In statistics and probability, one of the ever active and fragile activities is to propose mathematical models having short names such as distributions, models, and/or probability models. This activity is novel and is based on motivations and logical reasoning; see, for example, Pearson, Pareto, Burr, Johnson, and Tukey families and many more models and families of distributions published in the literature. Furthermore, many notes and letters to editors had been published in many peer-reviewed statistical and mathematical journals pointing out typos and mistakes in articles related to the model and its properties and other related issues. See, for example, some selected references [1–4]:

(i) Nadarajah [5] developed only explicit expressions for the moments of modified Weibull which was actually proposed by Xie et al. [6] as the XTG model.

(ii) Nadarajah and Kotz [7] stated that some of newly proposed modified Weibull models in the literature were not new but rather can arise from Gurvich et al. [8] generalized family.

(iii) Nadarajah and Kotz [9], the model proposed by Wu et al. [10], which exhibits a bathtub-shaped hazard rate, is in fact not new but originally due to Chen [11]. Furthermore, the model arises from Gurvich et al. [8] generalized family.

(iv) Bidram [12] pointed out that the proposed model “complementary exponential geometric” is not new.

(v) Lee and Tsai [13] found several typos in mathematical equations and in the incorrect proof of
(v)OkorieandAkpana[15]revisitedthedataprocurementanddiscussedtheinadequacyofthemodel
“transmuted generalized inverted exponential distribution” (TGIED) developed by Elbatal [16] and
empirically investigated by Khan [17]. The authors suggested that TGIED is not a good model to study
survival time data of 50 devices reported by Aarset [18], which was applied by Khan [17].

(vi) Nadarajah and Okorie [19] reported a very minor
correction in the likelihood function of the
“Gumbel–Burr XII model” introduced by Osatho-
hammwen et al. [20] and then suggested the cor-
rected version.

(vii) Nadarajah and Zhang [21] showed that a one-pa-

ter model can perform better as compared to
the three-parameter model “transmuted inverse
Weibull distribution” by Khan et al. [22] if applied to
the datasets used by these authors.

(ix) Nadarajah and Chan [23] stated that the moments’
and incomplete moments’ expressions developed
by Mazucheli et al. [24] in their developed model
“one-parameter unit-Lindley distribution” are ei-
ther incorrect or not in the closed form. Then,
authors proposed closed-form expressions for
moments and incomplete moments of the one-
parameter unit-Lindley distribution.

We may find such short communications as a part of
improvement(s), additional work, additional properties(s),
more characterization, correction(s) in the properties,
justification of the methodology or citations and criticism
e.g., starting with “A note on . . .”, “A comment on . . .”, “A
short note on the . . .”, “On the moments . . .”, “On the
distribution of . . .”, “On the alternative to . . .”, “Com-
ment(s) on . . .”, “Correspondence: Letter to the Editor” etc.
But our main and specific concern here and will remain
always be “why some old models were published as new one
without citation.”

The exponential, gamma, Weibull, Lomax, Burr, and
log-logistic models are basic motivating models for re-
searchers and practitioners to think about extensions and
modifications. If T is a lifetime random variable, then the
cumulative distribution functions (cdfs) and probability
density functions (pdfs) of exponential, gamma, Weibull,
Lomax, Burr, and log-logistic models are, respectively, given by

\[
F_E(t) = 1 - \exp[-\mu t] \text{ and } f_E(t) = \mu \exp[-\mu t], \quad t > 0, \mu > 0,
\]

\[
F_{Ga}(t) = \frac{\gamma(\delta, \lambda t)}{\Gamma(\delta)} \text{ and } f_{Ga}(t) = \Gamma(\delta)^{-1} \lambda^\delta t^{\delta-1} \exp[-\lambda t], \quad t > 0, \gamma, \delta > 0,
\]

\[
F_W(t) = 1 - \exp[-\lambda t^\beta] \text{ and } f_W(t) = \lambda \beta t^{\beta-1} \exp[-\lambda t^\beta], \quad t > 0, \lambda, \beta > 0,
\]

\[
F_{Le}(t) = 1 - (1 + \lambda t)^{-a} \text{ and } f_{Le}(t) = \lambda (1 + \lambda t)^{-a-1}, \quad t > 0, \lambda, \alpha > 0,
\]

\[
F_{Be}(t) = 1 - \left(1 + \lambda t^\beta\right)^{-a} \text{ and } f_{Be}(t) = \lambda a \beta t^{\beta-1} \left(1 + \lambda t^\beta\right)^{-a-1}, \quad t > 0, \lambda, \alpha, \beta > 0,
\]

\[
F_{Li}(t) = 1 - \left(1 + \lambda t^\beta\right)^{-2} \text{ and } f_{Li}(t) = \lambda \beta t^{\beta-1} \left(1 + \lambda t^\beta\right)^{-2}, \quad t > 0, \lambda, \beta > 0,
\]

where \( \gamma(p, qx) = \int_0^\infty x^{p-1} \exp[-qx] dx \) and \( \Gamma p = \int_0^\infty x^{p-1} \exp[-x] dx \) are lower incomplete gamma and
complete gamma functions, respectively.

In the recent past, many authors have proposed exten-
sions and generalizations of the exponential model to increase
its flexibility by adding parameters or modifying the func-
tional form. Some extensions of exponential distributions
other than Nadarajah–Haghighi (NH) (Nadarajah and
Haghighi [1]) are linear exponential (LE) (or linear failure
rate) (Bain [25]), generalized exponential (GE) (Gupta and
Kundu [26]), extended exponential of type 1 (ExtE1) (Mir-
hossaini and Dolati [27]), extended exponential of type 2
(ExtE2) (Çelebioglu [28]), extended exponential of type 3
(ExtE3) (Olapade [29]), and extended exponential of type 4
(ExtE4) (Gámé et al. [30]). The cdfs of LE, GE, ExtE1, ExtE2,
ExtE3, and ExtE4 distributions are, respectively, given by

\[
F_E(t) = 1 - \exp[-\mu t] \text{ and } f_E(t) = \mu \exp[-\mu t], \quad t > 0, \mu > 0,
\]

\[
F_{Ga}(t) = \frac{\gamma(\delta, \lambda t)}{\Gamma(\delta)} \text{ and } f_{Ga}(t) = \Gamma(\delta)^{-1} \lambda^\delta t^{\delta-1} \exp[-\lambda t], \quad t > 0, \gamma, \delta > 0,
\]

\[
F_W(t) = 1 - \exp[-\lambda t^\beta] \text{ and } f_W(t) = \lambda \beta t^{\beta-1} \exp[-\lambda t^\beta], \quad t > 0, \lambda, \beta > 0,
\]

\[
F_{Le}(t) = 1 - (1 + \lambda t)^{-a} \text{ and } f_{Le}(t) = \lambda (1 + \lambda t)^{-a-1}, \quad t > 0, \lambda, \alpha > 0,
\]

\[
F_{Be}(t) = 1 - \left(1 + \lambda t^\beta\right)^{-a} \text{ and } f_{Be}(t) = \lambda a \beta t^{\beta-1} \left(1 + \lambda t^\beta\right)^{-a-1}, \quad t > 0, \lambda, \alpha, \beta > 0,
\]

\[
F_{Li}(t) = 1 - \left(1 + \lambda t^\beta\right)^{-2} \text{ and } f_{Li}(t) = \lambda \beta t^{\beta-1} \left(1 + \lambda t^\beta\right)^{-2}, \quad t > 0, \lambda, \beta > 0,
\]
where $\lambda > 0$, $\mu > 0$, $\delta > 0$, and $-1 < \omega < 1$ are scale parameters, while $\theta > 0$ and $\alpha > 0$ are power (or shape) parameters.

This paper is organized as follows: the main concern regarding the actual proposal of the extended exponential model is addressed in Section 2. A useful development procedure for Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas models is described in Section 3. A brief review of the literature on Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas models is given in Section 4. In Section 5, some corrections in the moments of the Nadarajah–Haghighi model and an alternate method for the moments of Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas are developed. The final Section 6 concludes the paper listing further G-classes that can be developed from Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas models.

### 2. The Main Objective of Research

In this section, we consider the three related models, namely, Nadarajah–Haghighi, power generalized Weibull, and Dimitrakopoulou, Adamidis, and Loukas distributions, and then discuss the development of these models to prove “to whom the credit actually must go naturally as the pioneer.”

Bagdonavičius and Nikulin ([2], p. 110) first introduced a generalized Weibull family which exhibits all possible hazard rate shapes constant, increasing and decreasing (monotone), and bathtub and upside-down bathtub (nonmonotone) and later called it as the generalized power Weibull (GPW) and power generalized Weibull (PGW) distribution by Nikulin and Haghighi [31, 32], respectively. The GW and GPW are nowadays popular as the PGW model, which is actually an extension of exponential, Weibull, and extended exponential (NH) models. The cdf of PGW (due to Bagdonavičius and Nikulin [2] and Nikulin and Haghighi [31, 32]) is given by

$$F_{GW}(x) = 1 - \exp \left\{ 1 - \left( \frac{x}{\lambda} \right)^{\theta} \right\}, \quad x > 0,$$

where $\lambda > 0$ is the scale parameter, while $\gamma > 0$ and $\beta > 0$ are shape parameters.

Dimitrakopoulou, Adamidis, and Loukas [4] (which we acronym here as DAL based on the last names of the authors) proposed another extension of the exponential and/or Weibull distribution. The cdf and pdf of the DAL distribution are, respectively, given by

$$F_{DAL}(x) = 1 - \exp\left\{ 1 - \left( 1 + \lambda x^{\beta} \right)^{a} \right\}, \quad x > 0,$$

$$f_{DAL}(x) = \lambda a \beta \left( 1 + \lambda x^{\beta} \right)^{a-1} \exp\left\{ 1 - \left( 1 + \lambda x^{\beta} \right)^{a} \right\}, \quad x > 0,$$

where $\lambda > 0$ is the scale parameter, while $\alpha > 0$ and $\beta > 0$ are shape parameters.

**Note 1.** Although there seems some difference in the cdfs of DAL and PGW, in general (after reparametrization), both models are similar (see Peña-Ramírez et al. [33]).

Nadarajah and Haghighi [1] introduced the extended version of the exponential distribution and called it extended exponential, which is well known as the Nadarajah–Haghighi (NH) distribution. The cdf and pdf of the NH model are, respectively, given by

$$F_{NH}(x) = 1 - \exp\left\{ 1 - \left( 1 + \lambda x^{a} \right)^{\gamma} \right\}, \quad x > 0,$$

$$f_{NH}(x) = \lambda a \left( 1 + \lambda x^{a} \right)^{a-1} \exp\left\{ 1 - \left( 1 + \lambda x^{a} \right)^{a} \right\}, \quad x > 0,$$

where $\lambda > 0$ is the scale parameter, while $\alpha > 0$ is the shape parameter.

**Note 2.** There is one special case of the NH model, that is, the exponential distribution if $\alpha = 1$, but there are three special cases of the PGW or DAL model: (i) if $\beta = 1$, the PGW or DAL reduces to the NH distribution, (ii) if $\alpha = 1$, PGW or DAL reduces to the Weibull distribution, and (iii) if $\alpha = \beta = 1$, PGW or DAL reduces to the exponential distribution. Furthermore, more special cases can be generated with the help of variable transformations (see, for example, Dimitrakopoulou et al. [4], pp. 308-9).
2.1. Comparing Developments of PGW vs. DAL vs. NH Models. The PGW appeared first in the book of Bagdonavičius and Nikulin ([2], p. 110) under the title “Accelerated Life Models” published by a well-known publisher, the Chapman & Hall/CRC, London, which may be read by every statistician and researcher interested in distribution theory, reliability analysis, and lifetime modelling. Furthermore, books or monographs of Chapman & Hall/CRC and Wiley publishers are available in most of libraries of every country and are in easy access of the students, teachers, and practitioners. The idea of flexible hazard rate (all possible hazard rate shapes) was reconsidered in an article by Nikulin and Haghighi [31] while proposing a $\chi^2$-test for the PGW family, and then the PGW model was empirically investigated in the presence of type-II censoring for “Head and Neck Cancer Data.” It is evident that Mr. Nikulin has expertise in $\chi^2$-testing and had coauthored a book with P. Greenwood under Wiley publisher titled “A Guide to Chi-square Testing.”

Detail properties of PGW such as quantile, analytical shapes of the density and hazard rate, moments, mode, parameter estimation by the maximum likelihood method, simulation study, and empirical investigation through “Head and Neck Cancer Data” were studied by Nikulin and Haghighi [32].

The DAL model (same as PGW) was published in 2007 by Dimitrakopoulou, Adamidis, and Loukas (which they actually submitted in the IEEE journal in 2006) in which they took the same plea on proposing a model with flexible hazard rate shapes, stated relations with submodels (by using transformations), presented motivation in the risk scenario, and investigated the model properties such as quantile function, moments, hazard rate behaviour, and parameter estimation, but did not cite the three earlier works.

The NH model which appeared online in 2010 (16 March 2010) and was published in 2011 did not consider citing the previous four key references despite the fact that one of the coauthors was well aware of the development of PGW (a more extended model than NH). It was better if the authors of the NH model had given a credit to deserving ones.

Evidently, the three articles on PGW (2009), DAL (2007), and NH (2011) are useful extensions of the exponential or Weibull models and apparently look similar up to some extent (model formulation) but differ with respect to the type and number of parameters, motivation, content, and presentation.

Finally, after thorough consideration, we may be able to conclude that the model proposed by Nadarajah and Haghighi [1] was not new but in fact a special case of the PGW or DAL model as an extended Weibull model or extended exponential model pioneered by Bagdonavičius and Nikulin [2] and Nikulin and Haghighi [31] and in some way Dimitrakopoulou et al. [4]. We find it very difficult to admit that the four references related to PGW or DAL models were slipped from the attention of NH model’s authors.

3. A Useful Development Procedure of NH, PGW, or DAL Models

If $G(x)$ and $\overline{G}(x) = 1 - G(x)$ are the cdf and survival function (sf) of the baseline model, then odd ratio is defined as $G(x)/\overline{G}(x)$. Following the T-X criterion (Alzaatreh et al. [34]), the cdf of the odd exponential-G (OEG) class is defined by

$$F_{OEG}(x) = \int_0^{G(x)/\overline{G}(x)} \mu e^{-\mu t} dt = 1 - \exp\left\{-\mu \left[ \frac{G(x)}{\overline{G}(x)} \right]\right\}.$$  

(8)

Many new composite models can be generated from the OEG class. Table 1 lists some baseline models, their odd ratios $G(x)/\overline{G}(x)$, and published models of some well-known distributions generated from the OEG class.

4. Literature Review on NH and DAL Models

In this section, we present a needful review to NH, exponentiated NH, inverted NH, and PGW (or DAL) models. Details on extended or generalized NH and PGW (or DAL) models are out of scope of this article, and the interested readers may read the referred (or cited) articles directly.

4.1. NH Model. Nadarajah and Haghighi [1] proposed three motivations for the NH model, reported some useful mathematical properties such as quantile, analytical shapes of the density and hazard rate, moments (complete and incomplete), L-moments, Bonferroni and Lorenz curves, Rényi entropy, and order statistics, and also dealt parameter estimation. The NH density offers reversed-J and right-skewed shapes, while the hazard rate shapes could be increasing, decreasing, and constant (not very attractive from the plotted graphs).

4.1.1. Order Statistics and Records. Kumar et al. [36] established recurrence relations for the single and product moments of order statistics from the NH distribution. MirMostafae et al. [37] obtained some recurrence relations for the single and product moments of upper records from the NH model. Selim [38] dealt estimation of NH model parameters through the maximum likelihood and Bayes method based on record values and also considered point and interval predictions of the future record values. Khan and Sharma [39] obtained exact expressions for the Shannon entropy of the NH model based on generalized order statistics. Sana and Faizan [40] considered maximum likelihood and Bayesian estimation of the NH model based on upper records and obtained Bayes estimates under squared error loss, balanced squared error, and general entropy loss functions.

4.1.2. Life Testing under Censoring Schemes. Haghighi [41] introduced a simple step-stress accelerated life test and derived an optimum plan for the NH model. Haghighi [42] proposed a design for the step-stress accelerated life test for the NH distribution in the presence of type-I censoring and then estimated model parameters for such circumstances. El-Din et al. [43] also proposed a simple step-stress accelerated life test for the NH model under type-II progressive censoring, obtained maximum likelihood and Bayes...
estimates for NH model parameters, and also derived approximate, bootstrap, and credible intervals for the estimators. El-Din et al. [44] considered the progressive stress accelerated life test under progressive type-II censoring and obtained parameter estimates of the NH model through maximum likelihood and Bayes methods of estimation along with Bayes credible intervals. Singh et al. [45] considered parameter estimation using classical and Bayesian methods for NH model parameters under progressive type-II censoring under binomial removal. El-Raheem [46] considered the optimal allocation problem in multiple constant-stress censoring.

4.1.3. Discrete NH Versions. Kumar et al. [47] proposed a discrete version of the NH model, which they called count extended exponential model \( C_{n}(t) \) using the following formula: 
\[
C_{n}(t) = F_{n}(t) - F_{n-1}(t), \quad n = 0, 1, \ldots,
\]
and investigated some mathematical properties. Recently, Ali et al. [48] suggested the bivariate discrete NH model and reported some useful mathematical properties along with the estimation of model parameters with the help of seven well-known methods.

4.1.4. T-X Family for the NH Model. Recently, Nasiru et al. [49] proposed the \( T\)-NH[Y] family based on the quantile function approach pioneered by Alzaatreh et al. [34] and Aljarrah et al. [50] having cdf
\[
F_{X}(x) = \int_{0}^{1-\exp[1-(1+\lambda x)^{\alpha}]} f_{T}(t) dt = F_{T}(Q_{T} \{ 1 - \exp[1 - (1 + \lambda x)^{\alpha}] \}), \quad x, \theta, \lambda > 0,
\]
where \( F_{R}(x) = [1 - \exp[1 - (1 + \lambda x)^{\alpha}]] \). Nasiru et al. [49] also investigated mathematical properties of the \( T\)-NH[Y] family such as mode, quantile, moments, and Shannon entropy along with the estimation of parameters, simulation study, and empirical investigation.

4.1.5. Truncated (Unit) NH Version. Recently, Nasiru et al. [51] developed the truncated version of the NH model for bounded unit interval \((0, 1)\) based on the left truncation criterion and then proposed the unit Nadarajah–Haghighi (UNH) model and unit Nadarajah–Haghighi generalized (UNH-G) family having cdf, respectively, as follows:
\[
F_{\text{UNH}}(x) = \frac{1 - \exp[1 - (1 + \lambda x)^{\alpha}]}{1 - \exp[1 - (1 + \lambda)^{\alpha}]}, \quad x \in (0, 1), \lambda, \alpha > 0,
\]
\[
(10)
\]
\[
F_{\text{UNHG}}(x) = \frac{1 - \exp[1 - (1 + \lambda G(x; \xi)^{\alpha})]}{1 - \exp[1 - (1 + \lambda)^{\alpha}]}, \quad x, \lambda, \alpha > 0.
\]
\[
(11)
\]

They investigated some useful properties of the UNH-G family and performed simulation studies for the two special models UNH-Weibull and UNH-log-logistic along with empirical investigation. It is pertinent to mention here that recently, Alzaatreh et al. [52] proposed and studied right-truncated and left-truncated \(T\)-\(X\) families of distributions, a more generalized concept and formulation.

4.1.6. Methods of Estimation. Singh et al. [53] considered the classical and Bayesian estimation of the NH model parameters and reliability characteristics. Dey et al. [54] investigated the estimation of NH model parameters by using methods of the maximum likelihood, moment, percentile, least squares and weighted least squares, and Bayesian and compared them using a simulation study. Minić [55] estimated NH model parameters using simple random sampling following the maximum likelihood method, moment method, and modified maximum likelihood method and also using ranked set sampling under imperfect and perfect ranking. Through simulation study, Minić showed that the estimators obtained through ranked set sampling using perfect or imperfect ranking are better in performance as compared to estimators obtained through simple random sampling.

4.1.7. Miscellaneous Contributions to the NH Model. El-Damcese and Ramadan [56] proposed and studied the modified version of NH, called modified Nadarajah–Haghighi, having cdf
\[
F_{\text{MNH}}(x) = 1 - \exp\left[1 - \left(1 + \lambda x + \delta x^{2}\right)^{\alpha}\right], \quad x, \lambda, \alpha, \delta > 0,
\]
\[
(12)
\]
where \( \lambda \) and \( \delta \) are scale parameters, while \( \alpha \) is the shape parameter.

Khan et al. [57] introduced a weighted version of the NH model by defining the cdf as.
and studied a very few basic properties.

Peña-Ramírez et al. [58] proposed a compounded NH-Lindley model for components of a system arranged in series, having a new survival function (product of two sfs) as

\[ F_{\text{NH}}(x) = \frac{1 - \exp\{1 - (1 + \lambda x)^\theta\}}{1 + \exp\{1 - (1 + \lambda x)^\theta\}} \quad x, \lambda, \theta > 0, \]  

and clearly stated in page 1336 that Bagdonavičios and Nikulin [2] were the first who introduced the GPW family of distributions. Dimitrakopoulou, Adamidis, and Loukas [4] introduced the GPW model claiming flexible hazard rate shapes, motivated the model in the risk scenario, and obtained some mathematical properties such as quantile function, moments, and hazard rate behaviour along with parameter estimation. Nikulin and Haghighi [32] obtained mathematical properties of PGW such as quantile, analytical shapes of the density and hazard rate, moments, and mode along with parameter estimation by the maximum likelihood method, simulation study, and empirical investigation.

Voinov et al. [68] proposed modified goodness-of-fit tests based on the maximum likelihood of PGW parameters and, through Monte Carlo simulation, showed that power of the tests of the PGW model (cited reference of Bagdonavičios and Nikulin [2]) is better than two-parameter Weibull, three-parameter Weibull, and generalized Weibull models. Kumar and Dey [69] developed the recurrence relation for the single and product moments of order statistics from the PGW model and stated that this model is actually due to Bagdonavičios and Nikulin [2]. Kumar and Jain [70] obtained explicit expressions for the recurrence relation for the single, product, and conditional moments of order statistics from the PGW model and stated that it is due to Bagdonavičios and Nikulin [2]. Pandey and Kumari [71] used the Bayesian estimation approach for the parameter estimation of GPW while considering Lindley’s approximation and Markov chain Monte Carlo under type-II censoring. Sabry et al. [72] used double-ranked set sampling (DRSS) and general double-ranked set sampling (GDRSS) approaches for the parameter estimation of the GPW model and proved through simulation study that the GDRSS approach yields more efficient results as compared to ranked set sampling, extreme ranked set sampling, and DRSS schemes. Almetwaly and Almomy [73] dealt parameters’ estimation of the GPW model through classical and Bayesian methods for the complete sample and censored samples (type-II censoring and type-II progressive censoring schemes). El-Din et al. [74] considered step-stress accelerated life testing for testing the lifetime of GPW and used maximum likelihood and Bayesian methods for the estimation of model parameters under type-II progressive censoring. Jones et al. [75] developed the bivariate version of GPW, defined its copula presentation, and then investigated bivariate shared frailty of adaptive PGW and bivariate shared frailty of PGW models.

4.2. Exponentiated NH Model. Lemonte [59] proposed and studied a simple extension of NH by using Lehmann alternative 1 (see Gupta et al. [60]) called exponentiated NH (ENH) distribution. The induction of one additional shape parameter into NH (ENH) resulted in flexible shapes of the density and hazard rate. The ENH density can exhibit reversed-J, symmetrical, and right-skewed, while the hazard rate shapes are increasing, decreasing, bathtub, and upside-down bathtub. Lemonte obtained useful mathematical properties such as quantile function, analytical shapes of the density and hazard rate, moments, MacGillivray’s skewness measure, entropies and Kullback–Leibler divergence, stress-strength reliability parameter, and estimation of model parameters. Sira et al. [61] developed an extended ENH model called beta-ENH which offers flexible shapes of the density and hazard rate. Alhussain and Ahmed [63] considered classical and Bayesian methods of the estimation of ENH model parameters under progressive type-II censoring.

4.3. Inverted NH (INH) Model. Tahir et al. [64] introduced the inverted version of the NH model, obtained some mathematical properties, and compared INH model parameters through a simulation study by using different methods of estimation such as maximum likelihood, least squares and weighted least squares, maximum product spacing, Cramér–von Mises, Anderson–Darling, right-tail Anderson–Darling, and Bayesian. Recently, Raffiq et al. [65] proposed the Marshall–Olkin INH distribution and obtained a very few basic mathematical properties.

4.4. PGW (or DAL) Model. Bagdonavičios and Nikulin [2], p. 110) first proposed the PGW model as the family of scale-shape distributions, as an alternate to the Weibull, log-logistic, and lognormal distribution, which exhibits all possible shapes hazard rate. The cdf is given by

\[ F(x) = \frac{1 - \exp\{1 - (1 + \lambda x)^\beta\}}{1 + \exp\{1 - (1 + \lambda x)^\beta\}} \quad x, \lambda, \alpha, \beta > 0. \]  

and and clearly stated in page 1336 that Bagdonavičios and Nikulin [2] were the first who introduced the GPW family of distributions. Dimitrakopoulou, Adamidis, and Loukas [4] introduced the GPW model claiming flexible hazard rate shapes, motivated the model in the risk scenario, and obtained some mathematical properties such as quantile function, moments, and hazard rate behaviour along with parameter estimation. Nikulin and Haghighi [32] obtained mathematical properties of PGW such as quantile, analytical shapes of the density and hazard rate, moments, and mode along with parameter estimation by the maximum likelihood method, simulation study, and empirical investigation.
5. G-Classes from NH and DAL (or PGW) Models

Alzaatreh et al. [34] proposed a general method for constructing G-classes by using the transformed-transformer (T-X) approach. Let \( r(t) \) be the pdf and \( R(t) \) be the cdf of a rv \( T \in [a, b] \) for \( -\infty < a < b < \infty \), and let \( W[G(x)] \) be a function of the cdf \( G(x) \) of any baseline rv \( W(\cdot) \) is known as the generator) such that \( W[G(x)] \) satisfies three conditions:

(i) \( W[G(x)] \in [a, b] \)

(ii) \( W[G(x)] \) is differentiable and monotonically nondecreasing

(iii) \( \lim_{x \to -\infty} W[G(x)] = a \) and \( \lim_{x \to \infty} W[G(x)] = b \)

The cdf of the T-X family is

\[
F_{TX}(x) = \int_a^{W[G(x)]} r(t)dt = R(W[G(x)]),
\]

where \( W[G(x)] \) satisfies conditions (i)–(iii).

The pdf corresponding to equation (15) is

\[
f_{TX}(x) = r(W[G(x)]) \frac{d}{dx} W[G(x)].
\]

For \( T \in [0, \infty) \), the following generators \( W[G(\cdot)] \) have been reported so far, which can be used to define NH G-classes for rv \( T \): (i) \( -\log G(x) \) (Alzaatreh et al. [34]), (ii) \( G(x)/G(x) \) (odds) (Bourguignon et al. [82]), and (iii) \( [-\log G(x)]/G(x) \) (Ahmad et al. [91]).
For \( T \in [0, \infty) \), the following generators \( W[G(x)] \) have been reported so far, which can be used to define DALG-classes for rv \( T \): (i) \(-\log G(x)\) (Zografos and Balakrishnan [76]; Alzaatreh et al. [34]), (ii) \( G(x)/G(x) \) (odds) (Gleaton and Lynch [89]; Bourguignon et al. [82]), and (iii) \([-\log G(x)]/G(x)\) (Ahmad et al. [91]).

6. Corrected Moments of the NH Model
Following (5), the corrected \( r \)-th moment expression will be

\[
\mathbb{E}(T^k) = \frac{e}{\lambda} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \Gamma \left( \frac{i}{\alpha} + 1, 1 \right),
\]

and the first four moments of \( T \) are given by

\[
\begin{align*}
\mathbb{E}(T) &= \frac{e}{\lambda} \left[ -1 + \Gamma \left( \frac{1}{\alpha} + 1, 1 \right) \right], \\
\mathbb{E}(T^2) &= \frac{e}{\lambda^2} \left[ 1 - 2 \Gamma \left( \frac{1}{\alpha} + 1, 1 \right) + \Gamma \left( \frac{2}{\alpha} + 1, 1 \right) \right], \\
\mathbb{E}(T^3) &= \frac{e}{\lambda^3} \left[ -1 + 3 \Gamma \left( \frac{1}{\alpha} + 1, 1 \right) - 3 \Gamma \left( \frac{2}{\alpha} + 1, 1 \right) + \Gamma \left( \frac{3}{\alpha} + 1, 1 \right) \right], \\
\mathbb{E}(T^4) &= \frac{e}{\lambda^4} \left[ 1 - 4 \Gamma \left( \frac{1}{\alpha} + 1, 1 \right) + 6 \Gamma \left( \frac{2}{\alpha} + 1, 1 \right) - 4 \Gamma \left( \frac{3}{\alpha} + 1, 1 \right) + \Gamma \left( \frac{4}{\alpha} + 1, 1 \right) \right].
\end{align*}
\]
\[ \mathbb{E}(T^k | T > t) = \frac{e}{\lambda^k} \sum_{i=0}^{k} \binom{k}{i} \left(-1\right)^{k-i} \Gamma \left( \frac{i}{\alpha} + 1, \omega \right), \]  

(23)

and the first four incomplete moments for \( T > t \) are given by

\[ \mathbb{E}(T^k | T > t) = \frac{e}{\lambda^k} \left[ -1 + \Gamma \left( \frac{1}{\alpha} + 1, \omega \right) \right], \]

\[ \mathbb{E}(T^2 | T > t) = \frac{e}{\lambda^2} \left[ 1 - 2\Gamma \left( \frac{1}{\alpha} + 1, \omega \right) + \Gamma \left( \frac{2}{\alpha} + 1, \omega \right) \right], \]

\[ \mathbb{E}(T^3 | T > t) = \frac{e}{\lambda^3} \left[ -1 + 3\Gamma \left( \frac{1}{\alpha} + 1, \omega \right) - 3\Gamma \left( \frac{2}{\alpha} + 1, \omega \right) + \Gamma \left( \frac{3}{\alpha} + 1, \omega \right) \right], \]

\[ \mathbb{E}(T^4 | T > t) = \frac{e}{\lambda^4} \left[ 1 - 4\Gamma \left( \frac{1}{\alpha} + 1, \omega \right) + 6\Gamma \left( \frac{2}{\alpha} + 1, \omega \right) - 4\Gamma \left( \frac{3}{\alpha} + 1, \omega \right) + \Gamma \left( \frac{4}{\alpha} + 1, \omega \right) \right], \]  

(24)

where \( \exp \{ (1 + \lambda x)^\alpha \} \). Sometimes, the lower incomplete moments are of interest for the researchers to investigate additional properties of the model. For example, the first lower incomplete moment is used to compute Bonferroni

\[ \mathbb{E}(T^k | T < t) = \frac{e}{\lambda^k} \sum_{i=0}^{k} \binom{k}{i} \left(-1\right)^{k-i} \left[ \Gamma \left( \frac{i}{\alpha} + 1, 1 \right) - \Gamma \left( \frac{i}{\alpha} + 1, \omega \right) \right] \]  

(25)

and Lorenz curves and to determine the totality of deviations from the mean and median of \( X \). Therefore, the lower incomplete \( r \)th moment expression is given by

\[ \mathbb{E}(T^k) = \frac{e}{\lambda^k} \int_{0}^{\infty} (-1)^k \left(1 - \omega^{(\alpha / \alpha)}\right)^k \exp[-\omega] d\omega. \]  

(26)

By using the binomial expansion given by Cordeiro and Andrade [97, 98] (for \( |\omega| < \infty \)), we can write

\[ (1 - y^{1/y})^r = 1 + \sum_{j=1}^{\infty} \frac{(-1)^{\gamma_j - 1}}{j!} \prod_{j=0}^{\gamma_j} (j - r) y^{\gamma_j}, \]  

(27)

where \( a_i (r) = (-1)^{y + 1 / \alpha} \Gamma \left( \frac{1}{\alpha} + 1, \omega \right) \).

From the last two results, the alternative \( k \)th ordinary moment expression for NH results in

\[ \mathbb{E}(T^k) = \frac{e}{\lambda^k} \sum_{i=0}^{\infty} a_i (k) \int_{0}^{\infty} \omega^{(\alpha / \alpha)} \exp[-\omega] d\omega = \frac{e}{\lambda^k} \left( -1 \right)^k \sum_{i=0}^{\infty} a_i (k) \Gamma \left( \frac{i}{\alpha} + 1, 1 \right). \]  

(28)

In a similar way, the alternate \( r \)th incomplete moment expressions for NH (\( T > t \) and \( T < t \)) can be deduced and are

\[ \mathbb{E}(T^k | T > t) = \frac{e}{\lambda^k} \left( -1 \right)^k \sum_{i=0}^{\infty} a_i (k) \int_{0}^{\infty} \omega^{(\alpha / \alpha)} \exp[-\omega] d\omega = \frac{e}{\lambda^k} \left( -1 \right)^k \sum_{i=0}^{\infty} a_i (k) \Gamma \left( \frac{i}{\alpha} + 1, \omega \right), \]  

(29)
and

\[
E(T^k | T < t) = \frac{e^k}{\lambda^k} \sum_{i=0}^{\infty} a_i(k) \int_1^{\infty} \omega^{i/\alpha} \exp(-\omega) d\omega = \frac{e^k}{\lambda^k} \sum_{i=0}^{\infty} a_i(k) \left[ \Gamma \left( \frac{i}{\alpha} + 1, 1 \right) - \Gamma \left( \frac{i}{\alpha} + 1, \omega \right) \right].
\]

(30)

8. Moments of the PGW (or DAL) Model

Following (3), the \(r\)th moment expression for PGW will be

\[
E(T^k) = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} \left( \frac{k/\beta}{i} \right) (-1)^{k/\beta + 1} i^{i/\alpha} \Gamma \left( \frac{i}{\alpha} + 1, 1 \right).
\]

(31)

The incomplete \(r\)th moment expression for PGW \((T > t)\) and \((T < t)\) is

\[
E(T^k | T > t) = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} \left( \frac{k/\beta}{i} \right) (-1)^{k/\beta + 1} i^{i/\alpha} \left[ \Gamma \left( \frac{i}{\alpha} + 1, 1 \right) - \Gamma \left( \frac{i}{\alpha} + 1, \omega \right) \right].
\]

(33)

which, after using binomial expansion, results in

\[
E(T^k) = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} a_i(k/\beta) \left( \frac{i}{\alpha} + 1, 1 \right).
\]

(35)

The alternate \(r\)th incomplete moment expressions for PGW \((T > t)\) and \((T < t)\) are

\[
E(T^k | T > t) = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} a_i(k/\beta) \int_1^{\infty} \omega^{i/\alpha} \exp(-\omega) d\omega = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} a_i(k/\beta) \left[ \Gamma \left( \frac{i}{\alpha} + 1, 1 \right) - \Gamma \left( \frac{i}{\alpha} + 1, \omega \right) \right],
\]

(36)

and

\[
E(T^k | T < t) = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} a_i(k/\beta) \int_1^{\infty} \omega^{i/\alpha} \exp(-\omega) d\omega = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} a_i(k/\beta) \left[ \Gamma \left( \frac{i}{\alpha} + 1, 1 \right) - \Gamma \left( \frac{i}{\alpha} + 1, \omega \right) \right].
\]

(37)

9. Alternate Expressions for the \(r\)th Moments of the PGW (or DAL) Model

The \(r\)th moment expression for the PGW model becomes

\[
E(T^k) = \frac{e^k}{\lambda^{(k+1)}} \sum_{i=0}^{\infty} a_i(k/\beta) \left( \frac{k/\beta}{i} \right) i^{i/\alpha} \exp(-\omega) d\omega,
\]

(34)

10. Empirical Investigation

In this section, we empirically show the comparison among three models, described in the paper, which are NH, DAL, and PGW distributions. Two real-life datasets are used to compare and illustrate the potentiality of NH, DAL, and PGW models. The datasets are given as follows.

Dataset 1. (failure time data). The first real dataset is taken from [99] which represents the 50 observations of load, haul, dump machine-C failure time. The data are as follows: 110, 13, 72, 4, 45, 56, 19, 57, 36, 90, 19, 7, 2, 118, 44, 8, 277, 4, 8, 10, 79, 103, 6, 18, 147, 96, 22, 3, 24, 3, 9, 99, 82, 121, 54, 79, 99, 18, 5, 21, 1, 3, 5, 1, 59, 22, 17, 35, 35, and 29.

Dataset 2. (failure time data). The second dataset is taken from [100] which represents the failure time of 20 components. The data are as follows: 0.072, 4.763, 8.663, 12.089, 0.477, 5.284, 9.511, 13.036, 1.592, 7.709, 10.636, 13.949, 79, 103, 6, 18, 147, 96, 22, 3, 24, 3, 9, 99, 82, 121, 54, 79, 99, 18, 5, 21, 1, 3, 5, 1, 59, 22, 17, 35, 35, and 29.

The AdequacyModel package is used in R-Statistical Computing Environment to compute maximum likelihood estimates (MLEs) and the standard errors (SEs) of the estimates of the proposed and other competitive models. The
Table 4: MLEs and their SEs (in parentheses) for dataset 1.

| Distribution | α       | λ       | β       | γ       |
|--------------|---------|---------|---------|---------|
| NH           | 0.6711  | 0.0444  | —       | —       |
|              | (0.1907)| (0.0248)| —       | —       |
| DAL          | 0.9673  | 0.0392  | 0.8760  | —       |
|              | (0.8683)| (0.0264)| (0.2419)| —       |
| PGW          | —       | 36.0362 | 0.8927  | 1.048   |
|              |         | (51.6939)| (0.2427)| (0.9377)|

Table 5: MLEs and their standard errors (in parentheses) for dataset 2.

| Distribution | α       | λ       | β       | γ       |
|--------------|---------|---------|---------|---------|
| NH           | 17.6456 | 0.0044  | —       | —       |
|              | (18.8952)| (0.0048)| —       | —       |
| DAL          | 9.3155  | 0.0065  | 1.1063  | —       |
|              | (11.5537)| (0.0077)| (0.2211)| —       |
| PGW          | —       | 68.1396 | 1.1391  | 0.1428  |
|              |         | (97.9441)| (0.2167)| (0.2065)|

Table 6: The statistics AIC, BIC, HQIC, AD, CvM, and KS and p value for dataset 1.

| Distribution | $-\hat{\ell}$ | AIC  | BIC  | CAIC | HQIC  | AD    | CvM   | KS    | KS p value |
|--------------|----------------|------|------|------|-------|-------|-------|-------|------------|
| NH           | 240.5352       | 485.0704 | 488.8944 | 485.3257 | 486.5266 | 0.4601 | 0.0681 | 0.0855 | 0.8580     |
| DAL          | 240.4329       | 486.8659 | 492.6020 | 487.3876 | 489.0502 | 0.4389 | 0.0649 | 0.0790 | 0.9140     |
| PGW          | 240.4344       | 486.8689 | 492.6050 | 487.3906 | 489.0532 | 0.4411 | 0.0652 | 0.0802 | 0.9050     |

Table 7: The statistics AIC, BIC, HQIC, AD, CvM, and KS and p value for dataset 2.

| Distribution | $-\hat{\ell}$ | AIC  | BIC  | CAIC | HQIC  | AD    | CvM   | KS    | KS p value |
|--------------|----------------|------|------|------|-------|-------|-------|-------|------------|
| NH           | 59.9218        | 123.8435 | 125.8350 | 124.5494 | 124.2323 | 0.4910 | 0.0681 | 0.0855 | 0.8580     |
| DAL          | 59.8936        | 125.7872 | 128.7744 | 127.2872 | 126.3703 | 0.4282 | 0.0649 | 0.0790 | 0.9140     |
| PGW          | 59.9544        | 125.9089 | 128.8961 | 127.4089 | 126.4920 | 0.4282 | 0.0749 | 0.1795 | 0.4848     |

Figure 1: Plots of the estimated pdf of NH, DAL, and PGW models for datasets (a) 1 and (b) 2.
Figure 2: Plots of the estimated hazard rate of (a) NH, (b) DAL, and (c) PGW models for dataset 1.

Figure 3: Continued.
log-likelihood function is evaluated at the MLEs ($\hat{\ell}$). Some well-established goodness-of-fit (GoF) statistics such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike’s information criterion (CAIC), Hannan–Quinn information criterion (HQIC), Anderson–Darling (AD), Cramér–von Mises (CvM), and Kolmogorov–Smirnov (KS) are used for model comparison purposes. The low values of GoFs and high KS $p$ values indicate good fits.

Tables 4 and 5 list the MLEs and their SEs, and Tables 6 and 7 report the values of the GoFs. For dataset 1, the model NH is better in performance as compared to DAL and PGW while considering GoFs AIC, BIC, CAIC, and HQIC. The model DAL is better as compared to NH and PGW if GoFs AD, CvM, and KS are considered. For dataset 2, the model NH is better in performance as compared to DAL and PGW while considering GoFs AIC, BIC, CAIC, and HQIC. The model PGW is better as compared to NH and DAL if GoFs AD, CvM, and KS are considered. Figures 1–3 show the estimated pdf and hazard rate of NH, DAL, and PGW, which support our results in Tables 6 and 7.

### 11. Concluding Remarks

In this article, we dealt the following: (i) we investigated an unbiased and robust investigation of the development of the three interrelated models and gave due credit to the authors who actually deserve, (ii) we presented an updated review of the literature on NH, PGW, and DAL extended models and their related G-classes, (iii) we pointed out mistakes (or typos) in the moments’ section of the NH paper published in the year 2011, and (iv) we provided corrected and extended moments and moment-generating expressions, and lastly, an empirical investigation was carried out where the three models were compared on two datasets.

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### References

[1] S. Nadarajah and F. Haghighi, “An extension of the exponential distribution,” *Statistics*, vol. 45, no. 6, pp. 543–558, 2011.

[2] V. Bagdonavičius and M. Nikulin, *Accelerated Life Models: Modeling and Statistical Analysis*, Chapman and Hall/CRC, London, UK, 2002.

[3] M. Nikulin and F. Haghighi, “A chi-squared test for the generalized power Weibull family for the head-and-neck cancer censored data,” *Journal of Mathematical Sciences*, vol. 133, no. 3, pp. 1333–1341, 2006.

[4] T. Dimitrakopoulou, K. Adamidis, and S. Loukas, “A lifetime distribution with an upside-down bathtub-shaped hazard function,” *IEEE Transactions on Reliability*, vol. 56, no. 2, pp. 308–311, 2007.

[5] S. Nadarajah, “On the moments of the modified Weibull distribution,” *Reliability Engineering & System Safety*, vol. 90, no. 1, pp. 114–117, 2005.

[6] M. Xie, Y. Tang, and T. N. Goh, “A modified Weibull extension with bathtub-shaped failure rate function,” *Reliability Engineering & System Safety*, vol. 76, no. 3, pp. 279–285, 2002.

[7] S. Nadarajah and S. Kotz, “On some recent modifications of Weibull distribution,” *IEEE Transactions on Reliability*, vol. 54, no. 4, pp. 561-562, 2005.

[8] M. R. Gurvich, A. T. Dibenedetto, and S. V. Ranade, “A new statistical distribution for characterizing the random
strength of brittle materials,” *Journal of Materials Science*, vol. 32, no. 10, pp. 2559–2564, 1997.

[9] S. Nadarajah and S. Kotz, “The two-parameter bathtub-shaped lifetime distribution,” *Quality and Reliability Engineering International*, vol. 23, no. 2, pp. 279-280, 2007.

[10] J.-W. Wu, H.-L. Lu, C.-H. Chen, and C.-H. Wu, “Statistical inference about the shape parameter of the new two-parameter bathtub-shaped lifetime distribution,” *Quality and Reliability Engineering International*, vol. 20, no. 6, pp. 607–616, 2004.

[11] Z. Chen, “A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function,” *Statistics & Probability Letters*, vol. 49, no. 2, pp. 155–161, 2000.

[12] H. A. Bidram, M. Roman, and V. G. Cancho, “The complementary exponential geometric distribution: model, properties, and a comparison with its counterpart. *Computational Statistics and Data Analysis* 55:2516–2524,” *Computational Statistics & Data Analysis*, vol. 74, p. 180, 2011.

[13] C.-S. Lee and H.-J. Tsai, “A note on the generalized linear exponential distribution,” *Statistics & Probability Letters*, vol. 124, pp. 49–54, 2017.

[14] M. A. W. Mahmoud and F. M. A. Alam, “The generalized linear exponential distribution,” *Statistics & Probability Letters*, vol. 80, no. 11-12, pp. 1005–1014, 2010.

[15] I. E. Okorie and A. C. Akpanta, “A note on the transmuted generalized inverted exponential distribution with application to reliability data,” *Thailand Statistics*, vol. 17, pp. 118–124, 2019.

[16] I. Elbatal, “Transmuted generalized inverted exponential distribution,” *Stochastics and Quality Control*, vol. 28, pp. 125–133, 2013.

[17] M. S. Khan, “Transmuted generalized inverted exponential distribution with application to reliability data,” *Thailand Statistics*, vol. 16, pp. 14–25, 2018.

[18] M. V. Aarset, “How to identify a bathtub hazard rate,” *IEEE Transactions on Reliability*, vol. 36, no. 1, pp. 106–108, 1987.

[19] S. Nadarajah and I. E. Okorie, “A note on a new member from the T-X family of distributions: the gumbel-burr XII distribution and its properties,” *Sankhya A*, vol. 82, no. 1, pp. 257–259, 2020.

[20] P. Osatohanmwen, F. O. Oyegue, and S. M. Ogbornwan, “A new Member from the T – X family of distributions: the Gumbel-Burr XII distribution and its properties,” *Sankhya A*, vol. 81, no. 2, pp. 298–322, 2019.

[21] S. Nadarajah and Y. Zhang, “A note on the transmuted inverse Weibull distribution,” *Thailand Statistics*, vol. 18, pp. 90–94, 2020.

[22] M. S. Khan, R. King, and I. L. Hudson, “Characteristics of the transmuted inverse Weibull distribution,” *ANZIAM Journal*, vol. 55, pp. C197–C217, 2014.

[23] S. Nadarajah and S. Chan, “On moments of the unit Lindley distribution,” *Journal of Applied Statistics*, vol. 47, no. 5, pp. 947–949, 2020.

[24] J. Mazucheli, A. F. B. Menezes, and S. Chakraborty, “On the one parameter unit-Lindley distribution and its associated regression model for proportion data,” *Journal of Applied Statistics*, vol. 46, no. 4, pp. 700–714, 2019.

[25] L. J. Bain, “Analysis for the linear failure-rate life-testing distribution,” *Technometrics*, vol. 16, no. 4, pp. 500–509, 1974.

[26] R. D. Gupta and D. Kundu, “Generalized exponential distributions,” *Australian New Zealand Journal of Statistics*, vol. 41, no. 2, pp. 173–188, 1999.

[27] S. M. Mirhossaini and A. Dolati, “On a new generalization of the exponential distribution,” *Journal of Mathematical Extension*, vol. 3, pp. 27–42, 2008.

[28] S. Celębioglu, “On the extension of the exponential and Weibull distributions,” *SDU Journal of Science*, vol. 5, pp. 137–146, 2010.

[29] A. Olapade, “On extended generalized exponential distribution,” *British Journal of Mathematics & Computer Science*, vol. 4, no. 9, pp. 1280–1289, 2014.

[30] Y. M. Gómez, H. Bolfarine, and H. W. Gómez, “A new extension of the exponential distribution,” *Revista Colombiana de Estadística*, vol. 37, no. 1, pp. 25–34, 2014.

[31] M. Nikulin and F. Haghighi, “A chi-squared test for the generalized power Weibull family for the head-and-neck cancer censored data,” *Journal of Mathematical Sciences*, vol. 133, no. 3, pp. 1333–1341, 2006.

[32] M. Nikulin and F. Haghighi, “On the power generalized Weibull family: model for cancer censored data,” *Metron*, vol. 67, pp. 75–86, 2009.

[33] F. A. Peña-ramírez, R. R. Guerra, G. M. Cordeiro, and P. R. D. Marinho, “The exponentiated power generalized Weibull: properties and applications,” *Anais da Academia Brasileira de Ciências*, vol. 90, no. 3, pp. 2553–2577, 2018.

[34] A. Alzaatreh, C. Lee, and F. Famoye, “A new method for generating families of continuous distributions,” *Metron*, vol. 71, no. 1, pp. 63–79, 2013.

[35] B. Gompertz, “On the nature of the function expressive of the law of human mortality and on the new mode of determining the value of life contingencies,” *Philosophical Transactions of the Royal Society*, vol. A115, pp. 513–580, 1825.

[36] D. Kumar, S. Dey, and S. Nadarajah, “Extended exponential distribution based on order statistics,” *Communications in Statistics-Theory and Methods*, vol. 46, no. 18, pp. 9166–9184, 2017.

[37] S. M. T. K. MirMostafaee, A. Asgharzadeh, and A. Fallah, “Record values from NH distribution and associated inference,” *Metron*, vol. 74, no. 1, pp. 37–59, 2016.

[38] M. A. Selim, “Estimation and prediction for Nadarajah-Haghighi distribution based on record values,” *Pakistan Journal of Statistics*, vol. 34, pp. 77–90, 2018.

[39] M. J. S. K. Khan, A. Sharma, and A. Sharma, “Shannon entropy and characterization of Nadarajah and Haghighi distribution based on generalized order statistics,” *Journal of Statistics: Advances in Theory and Applications*, vol. 19, no. 1, pp. 43–69, 2018.

[40] M. Sana and M. Faizan, “Bayesian estimation for Nadarajah-Haghighi distribution based on upper record values,” *Pakistan Journal of Statistics and Operation Research*, vol. 20, no. 9, pp. 1280–1289, 2014.

[41] F. Haghighi, “Optimal design of accelerated life tests for an extension of the exponential distribution,” *Reliability Engineering & System Safety*, vol. 131, pp. 251–256, 2014.

[42] F. Haghighi, “Simple step-stress model for an extension of the exponential distribution with type-I censoring,” *International Journal of Quality & Reliability Management*, vol. 32, no. 8, pp. 906–920, 2015.

[43] M. M. M. El-Din, S. E. Abu-Youssef, N. S. A. Ali, and A. M. A. El-Raheem, “Parametric inference on step-stress accelerated life testing for the extension of exponential distribution under progressive type-II censoring,” *Communications for Statistical Applications and Methods*, vol. 23, no. 4, pp. 269–285, 2016.

[44] M. Mohie El-Din, Z. E. Abu-Youssef, S. N. S. Ali, and A. M. Abd El-Raheem, “Classical and Bayesian inference on
Mathematical Problems in Engineering

progressive-stress accelerated life testing for the extension of the exponential distribution under progressive type-II censoring,” Quality and Reliability Engineering International, vol. 33, no. 8, pp. 2483–2496, 2017.

[45] S. K. Singh, U. Singh, M. Kumar, and P. K. Vishwakarma, “Classical and Bayesian inference for an extension of the exponential distribution under progressive type-II censored data with binomial removals,” Journal of Statistics Applications & Probability Letters, vol. 1, no. 3, pp. 75–86, 2014.

[46] A. M. Abd El-Raheem, “Optimal design of multiple constant-stress accelerated life testing for the extension of the exponential distribution under type-II censoring,” Journal of Computational and Applied Mathematics, vol. 382, Article ID 113094, 2021.

[47] M. Kumar, S. K. Singh, and U. Singh, “Extension of exponential count model and its application to emissions of beta particles from a nuclear reaction,” Journal of Advanced Statistics, vol. 1, pp. 136–145, 2016.

[48] S. Ali, M. Shafqat, I. Shah, and S. Dey, “Bivariate discrete Nadarajah and Haghighi distribution: properties and different methods of estimation,” Filomat, vol. 33, no. 17, pp. 5589–5610, 2019.

[49] S. Nasiru, A. G. Abubakari, and J. Abonongo, “Quantile generated Nadarajah-Haghighi family of distributions,” Annals of Data Science, 2020a.

[50] M. A. Aljarrah, C. Lee, and F. Famoye, “On generating T-X family of distributions using quantile functions,” Journal of Statistical Distributions and Applications, vol. 1, 2014.

[51] S. Nasiru, A. G. Abubakari, and J. Abonongo, “Unit Nadarajah-Haghighi generated family of distributions: properties and applications,” Sankhya Series A: The Indian Journal of Statistics, vol. 115, 2020.

[52] A. Alzaatreh, M. A. Aljarrah, M. Smithson et al., “Truncated family of distributions with applications to time and cost to start a business,” Methodology and Computing in Applied Probability, vol. 23, 2020.

[53] S. K. Singh, U. Singh, and A. S. Yadav, “Reliability estimation and prediction for extension of exponential distribution using informative and non-informative priors,” International Journal of System Assurance Engineering and Management, vol. 6, no. 4, pp. 466–478, 2015.

[54] S. Dey, C. Zhang, A. Asgharzadeh, and M. Ghorbanezhad, “Comparisons of methods of estimation for the NH distribution,” Annals of Data Science, vol. 4, no. 4, pp. 441–455, 2017.

[55] M. Minic, “Estimation of parameters of Nadarajah-Haghighi extension of the exponential distribution using perfect and imperfect ranked set sample,” Yugoslav Journal of Operations Research, vol. 30, no. 2, pp. 177–198, 2020.

[56] M. A. El-Damcese, “Studies on properties and estimation problems for modified extension of exponential distribution,” International Journal of Computer Applications, vol. 125, no. 4, pp. 21–28, 2015.

[57] M. N. Khan, A. Saboor, G. M. Cordeiro, M. Nazir, and R. R. Pescim, “A weighted Nadarajah-Haghighi distribution,” UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, vol. 80, pp. 133–140, 2018.

[58] F. A. Peña-Ramirez, R. R. Guerra, and G. M. Cordeiro, “The nadarajah-haghighi Lindley distribution,” Anais da Academia Brasileira de Ciências, vol. 91, Article ID e20170856, 2019.

[59] A. J. Lemonte, “A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function,” Computational Statistics & Data Analysis, vol. 62, pp. 149–170, 2013.

[60] R. C. Gupta, P. L. Gupta, and R. D. Gupta, “Modeling failure time data by Lehman alternatives,” Communications in Statistics - Theory and Methods, vol. 27, no. 4, pp. 887–904, 1998.

[61] D. Sira KA, G. O. Orwa, and O. Ngesa, “Exponentiated Nadarajah-Haghighi Poisson distribution,” International Journal of Statistics and Probability, vol. 8, no. 5, pp. 34–48, 2019.

[62] A. Saboor, M. N. Khan, G. M. Cordeiro, I. Elbatal, and R. R. Pescim, “The beta exponentiated Nadarajah-Haghighi distribution: theory, regression model and application,” Mathematica Slovaca, vol. 69, no. 4, pp. 939–952, 2019.

[63] Z. A. Allhussain and E. A. Ahmed, “Estimation of exponentiated Nadarajah-Haghighi distribution under progressively type-II censored sample with application to bladder cancer data,” Indian Journal of Pure and Applied Mathematics, vol. 51, no. 2, pp. 631–657, 2020.

[64] M. H. Tahir, G. M. Cordeiro, S. Ali, S. Dey, and A. Manzoor, “The inversed Nadarajah-Haghighi distribution: estimation methods and applications,” Journal of Statistical Computation and Simulation, vol. 88, no. 4, pp. 2775–2798, 2018.

[65] G. Rathi, I. S. Dar, M. A. U. Haq, and E. Ramos, “The marshall-olkin inversed nadarajah-haghighi distribution: estimation and applications,” Annals of Data Science, vol. 79, 2020.

[66] G. S. Mudholkar and D. K. Srivastava, “Exponentiated Weibull family for analyzing bathtub failure-rate data,” IEEE Transactions on Reliability, vol. 42, no. 2, pp. 299–302, 1993.

[67] G. S. Mudholkar, D. K. Srivastava, and M. Freimer, “The exponentiated Weibull family: a reanalysis of the bus-motor-failure data,” Technometrics, vol. 37, no. 4, pp. 436–445, 1995.

[68] V. Voinov, N. Pya, N. Shapakov, and Y. Voinov, “Goodness-of-fit tests for the power-generalized Weibull probability distribution,” Communications in Statistics - Simulation and Computation, vol. 42, no. 5, pp. 1003–1012, 2013.

[69] D. Kumar and S. Dey, “Power generalized Weibull distribution based on order Statistics,” Journal of Statistical Research, vol. 51, no. 1, pp. 61–78, 2017.

[70] D. Kumar and N. Jain, “Power generalized Weibull based on generalized order statistics,” J. Data Sci, vol. 16, pp. 621–646, 2018.

[71] R. Pandey and N. Kumari, “Bayesian analysis of power generalized Weibull distribution,” International Journal of Applied and Computational Mathematics, vol. 4, 2018.

[72] M. A. Sabry, H. Z. Muhammed, A. Nabih, and M. Shaaban, “Parameter estimation for the power generalized Weibull distribution based on one- and two-stage ranked set sampling designs,” Journal of Applied Probability and Statistics, vol. 8, pp. 113–128, 2019.

[73] E. M. Almetwaly and H. M. Almomy, “Estimation of the generalized power Weibull distribution parameters using progressive censoring schemes,” International Journal of Statistics and Probability, vol. 7, pp. 51–61, 2018.

[74] M. M. M. El-Din, A. M. A. El-Raheem, and O. S. A. El-Azeem, “On step-stress accelerated life testing for power generalized Weibull distribution under progressive type-II censoring,” Annals of Data Science, vol. 8, 2020.

[75] M. Jones, A. Noufaily, and K. Burke, “A bivariate power generalized Weibull distribution: a flexible parametric model for survival analysis,” Statistical Methods in Medical Research, vol. 29, no. 8, pp. 2295–2306, 2019.
