Results of investigations of non-isothermal turbulent flows based on stochastic equations of the continuum and equivalence of measures

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Abstract. The review of main results of new stochastic theory of turbulence for non-isothermal flows is presented. Based on sets of the stochastic equations for the laws of conservation and for the equivalence of measures between deterministic process and random process, the review of results of analytical expressions is presented for main parameters of non-isothermal turbulent flow in the tube depending on the initial fluctuation in the non-turbulent medium. It is shown that the derived formulas for the critical Reynolds numbers and critical point, for the velocity and temperature profiles, for the second-order correlation and the turbulent Prandtl number, and heat transfer coefficients are in satisfactory agreement with the classical experimental data for these flow characteristics.

1. Introduction

Searching for the strong solution of the Navier-Stokes equations or the creation of new theories and systems of equations, which allow to predict the phenomenon onset of turbulence out of deterministic movement are in the constant attention[1–11]. As known, the impact of initial fluctuations on the result of equations solution, which are used to describe turbulent processes, is significant. This is especially important for numerical modeling and for developing computational codes. However till today, numerous numerical solutions do not allow to define the essence of the phenomenon of turbulence even together with experimental results. Also these results do not allow to provide the answer, in which proposes to determine the physical regular pattern about the reason of onset of turbulence and present the mathematical method to describe the phenomenon on the basis of this physical regular pattern. In articles [12–25] the physical regularity of the equivalence of measures and the systems of the stochastic equations for the determination onset of turbulence in isothermal and non-isothermal and flows were presented. Moreover in the article [12-25], based on the definition of the equivalence of measures between the deterministic process and random process, the systems of stochastic equations of energy, mass and momentum were defined for next space-time areas:

1) the beginning of the generation of turbulence;
2) the generation of turbulence;
3) the diffusion;
4) the dissipation of turbulence.

2. Sets of Equations

According to [1], the physical process is represented as a non-equilibrium thermodynamic system with i-subsets, which is characterized by the values of the energy \( U_i(E_i) \), the momentum \( U_i(M_i) \), and the mass \( U_i(M_i) \). Here, \( U_i \) is the speed, \( U_i(E_i) \) is the stochastic-field energy (index \( g_m \)); and \( U_i(E_i) \) is the fraction of the field energy, its deterministic component (index \( c_m \)) having the zero stochastic component of measure; and \( U_i(E_i) \) is the fraction of the field energy, which is actually the stochastic field component (index \( st \)). Similarly, components of the momentum and the mass (\( \rho \)-the density) are determined. Stochastic equations of conservation, which were derived in [1-5] for non-isothermal conditions, take the form:

the equation of mass (continuity)

\[
\frac{d(\rho)_{col}}{d\tau} = \frac{-\rho_{st}}{\tau_{cor}} \frac{d(\rho)_{st}}{d\tau},
\]

the momentum equation

\[
\frac{d(\rho u_i)_{col}}{d\tau} = \text{div}(\tau_i, j)_{col} + \text{div}(\tau_i, j)_{st} - \frac{(\rho U)_{st}}{\tau_{cor}} \frac{d(\rho U)_{st}}{d\tau} + F_{col} + F_{st},
\]

and the energy equation

\[
\frac{dE_{col}}{d\tau} = \text{div}(\lambda \frac{\partial T}{\partial x_j} + u_i \tau_i, j)_{col} + \text{div}(\lambda \frac{\partial T}{\partial x_j} + u_i \tau_i, j)_{st} - \left( \frac{E_{st}}{\tau_{cor}} \right) \frac{dE_{st}}{d\tau} + (a_i F)_{col} + (a_i F)_{st}.
\]

Here, \( \rho, \dot{U}, u_i, \dot{u}_i, \mu, \tau, \tau_{ij}, \) are the density; the velocity vector; the velocity components in directions \( x_i, x_j, x_l \); the dynamic viscosity; the time; and stress tensor \( \tau_{ij} = P + \sigma_{ij}, \delta_{ij} = 1 \) if \( i=j, \delta_{ij} = 0 \) for \( i \neq j \). \( P \) is the pressure of liquid or gas; \( \lambda \) is the thermal conductivity; \( c_p \) and \( c_v \) are the specific heat at constant pressure and volume, respectively; \( F \) is the external force, and

\[
\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \delta_{ij} \left( \frac{\varepsilon - 2}{3} \right) \frac{\partial u_i}{\partial x_i}.
\]

Then for the non-isothermal motion of the medium, using the definition of equivalency measures between deterministic and random process in the critical point, the sets of stochastic equations of energy, momentum, and mass are defined for the next space-time areas: 1) the beginning of the generation (index 1,0 or 1); 2) generation (index 1,1); 3) diffusion (1,1,1) and 4) the dissipation of the turbulent fields. These results provide an opportunity to introduce the concept of the correlator, which is defined for the potential physical quantities and combinations (N, M). This correlator, in its structure, will determine the possible range of motion in space depending on the different combinations of (M, N) and the corresponding values that determine the correlation interval of space-time. According to [12,13], the correlator in space-time is

\[
\lim_{m_l \rightarrow m_c, r_l \rightarrow r_c, \Delta \tau_l \rightarrow \tau_c} D_{N,M} (m_l \rightarrow m_c, r_l \rightarrow r_c, \Delta \tau_l \rightarrow \tau_c) = 0,
\]

\( \tau_c \)
\[ D_{N,M}(m_i; r; \tau_c) = \sum_{i} \lim_{m_i \to m_c} \lim_{r_i \to r_c} \lim_{\Delta \tau_c \to r_c} \left\{ m\left[T^M Z^* \cap T^N Y^*\right] - R_{TY}^m Z^* Y^* m\left[T^M Z^*\right] \right\}. \quad (5) \]

Subscript \( j \) denotes the parameters \( m_j \) (\( j = 3 \) means mass, momentum, and energy). For the case of the binary intersections, it was written that \( X = Y + Z + W \). Here subscripts \( \langle cr \rangle \) or \( \langle c \rangle \) refer to critical point \( r(x_c, \tau_c) \) or \( r_c \): the space-time point of the beginning of the interaction between the deterministic field and random field that leads to turbulence. In addition, subsets \( Y, Z, W \) are called extended in \( X \), if the measures \( m(Y), m(Z), m(W) \) have the properties:

\[ m(Y) = m(Y') = m(T^n Y) + \bigcup_{k=0}^{k=n-1} m(T^k (G_{1}^{n-k}))) \quad \text{and wandering subsets} \quad \bigcup_{k=0}^{k=n-1} m(T^k (G_{2}^{n-k})) \subset Y; \]
\[ m(Z) = m(Z') = m(T^n Z^*) + \bigcup_{k=0}^{k=n-1} m(T^k (G_{2}^{n-k}))) \quad \text{and wandering subsets} \quad \bigcup_{k=0}^{k=n-1} m(T^k (G_{2}^{n-k})) \subset Z; \]
\[ m(W) = m(W^*) = m(T^n W^*) + \bigcup_{k=0}^{k=n-1} m(T^k (G_{3}^{n-k}))) \quad \text{and wandering subset} \quad \bigcup_{k=0}^{k=n-1} m(T^k (G_{3}^{n-k})) \subset W. \]

The correlation function produces a set of equations that determine the equivalence of measures:

\[ m(T^n Z) = (R_{n Z T Y}^m)_{n,a} \left| m(T^n Y) \right|, \quad 0 \leq (R_{n Z T Y}^m) \leq 1. \]

Here, \( R_{n Z T Y}^m \) is the fractal correlation function assumed to be equal to unity to obtain analytical solutions. Thus, \( m(Z) = (R_{Z T Y}^m)_{n,a} \left| m(T Y) \right| \) for the pair \( (N, M) = (1,0) \), and \( m(Z) = (R_{Z T Y}^m)_{n,a} \left| m(T Y) \right| \) for \( (N, M) = (1,1) \). Here, \( T^n \) is the conservative transformation of \( X \) for all \( n \), such that there is \( n > n_d \) when \( T^n \) is the dissipative transformation for \( Y^n \subset X \) and \( Z^n \subset X \).

In [12,23-25], for the transfer of substantial values \( F \) (the mass (the density\( \rho \)), the momentum \( \rho U \)), the energy \( E \) of deterministic (laminar) motion into random (turbulent) motion (for the area 1), the beginning of the generation of turbulence, the pair \( (N, M) = (1,0) \), the equivalence of measures was written as

\[ \left( d\Phi_{col, st} \right)_{1,0} = -R_{1,0} \left( \Phi_{st} \right), \quad \left( \frac{d\Phi_{col, st}}{d\tau} \right)_{1,0} = -R_{1,0} \left( \frac{\Phi_{st}}{\tau_{cor}} \right). \quad (6) \]

In addition, for \( \langle \text{correlator} \rangle \) \( D_{N,M}(r; m_i; \tau_c) = D_{1,1}(r; m_i; \tau_c) \), the pair \( (N,M) = (1,1) \) and for the space-time area 2) the generation of turbulence \( r_1 \left(x_i + \Delta x_i + \Delta x_i; \tau_c + \Delta \tau_c + \Delta \tau_i\right) - r_0 \), the author of [12, 25] wrote that

\[ \left( d\Phi_{col, st} \right)_{1,1} = -R_{1,1} \left( d\Phi_{st} \right), \quad \left( \frac{d\Phi_{col, st}}{d\tau} \right)_{1,1} = -R_{1,1} \left( \frac{d\Phi_{st}}{d\tau} \right). \quad (7) \]

\( R_{1,0}, R_{1,1} \) are fractal coefficients. For example, to obtain the new analytical dependences, these coefficients are taken equal unit. For the integration of the set, we can write an expression for time interval \( \tau_{cor} \). Expressions for non-isothermal flows are

\[ \left( \frac{\rho}{\ell} \right)_{1,0,0, \rho} = \left( \frac{T}{\ell} \right)^{\rho}, \quad \left( \frac{\rho}{\ell} \right)_{1, \rho, \rho} = \left( \frac{T}{\ell} \right)^{\rho}, \quad \left( \frac{\rho}{\ell} \right)_{1,0, \rho} = \left( \frac{T}{\ell} \right)^{\rho}. \quad (8) \]
For the correlation time we have the following representations for the case of non-isothermal flow: Further, \( L = L_u, P = L_4 \) is the scale of turbulence. Indexes \((\bar{u}, \bar{p})\) and \((\bar{\tau})\) refer to the velocity field and index \((\bar{\tau})\) refers to the temperature field. \( L_y \) on \( x_2 = y \), or \( L_x, x_1 = x \). Here, \( x_1 \) and \( x_2 \) are coordinates along and normal to the wall. Here \( L_T = \frac{L}{Pr} \), \( \rho = \frac{\rho \cdot v \cdot C}{\lambda} \) is the Prandtl number.

3. The critical Reynolds number
The space-time area 1) is the area of the beginning of the generation of turbulence. On the basis of equivalence of measures, let us define the expressions for the critical Reynolds number, the velocity profiles, temperature, and the second-order correlations for the non-isothermal processes. For the area 1) \( r_0(x_i + \Delta y_i; \tau + \Delta \tau_i) - r_i \), referring the pair \((N, M) = (1,0)\), we have a set of equations of mass, momentum, and energy:

\[
\left\{\begin{array}{l}
\frac{d(\rho U)}{d\tau} = -\frac{\rho U}{\tau_{cor}} \\
\frac{d(E)}{d\tau} = \frac{E}{\tau_{cor}} \\
\text{div}(\tau_{i,j}) = \frac{\rho U}{\tau_{cor}} \\
\text{div}(\tau_{i,j}) + u_i \tau_{i,j} = \frac{E}{\tau_{cor}} \\
\end{array}\right.
\]  

According to [14,15], the motion is determined by a quadratic equation of the velocity profile and the temperature profile as a function of the fourth power of the vertical coordinate assuming constant physical properties of the medium. Then let us define that \( Ec = \frac{U_0^2}{\epsilon_0 (T_0 - T_w)} \) is the Eckert number, \( T_r = \frac{T_w}{(T_0 - T_w)} \cdot T_w = \sqrt{\frac{\sum (u_i^2)}{U_0^2}} \cdot U_0 \). Also \( T_0 \) and \( T_w \) are the temperatures along the pipe axis and on the wall; \( R, U_0, \) and \( u_i \) are the pipe radius and velocities along the axis and along \( x_i \). Thus, the critical point is

\[
\left(\begin{array}{l}
\frac{\epsilon_0}{R} \\
\end{array}\right) = \left[\begin{array}{l}
\frac{1}{4} \left(\frac{\epsilon_0}{U_0^2} \right) \left(\frac{R}{T_w} \right) E_{-Pr} \left(\frac{R}{T_w} \right) \left(\frac{R}{T_0} \right) E_{-Pr} \left(\frac{R}{T_0} \right) \left(1 + \frac{2 T_r}{T_0 - T_w} \right) \end{array}\right]^2 .
\]  

The first critical Reynolds number using (8) – (11) is defined as:
\[ \text{Re}_{st,i} = \left( \frac{1}{2} \right)^{1/3} \left( \frac{U}{\sqrt{E/r}} \right) \left( \frac{L}{R} \right)^{4/3} \left( \frac{1+2T_{\text{Ec}}}{1+1/Pr} \right) \left( \frac{1+1/Pr Ec}{1+1/Pr Ec} \right) \left( \frac{1+1/Pr Ec}{1+1/Pr Ec} \right) \cdot \] (12)

Index (T,U) refers to the non-isothermal flow. The first bracket in (13) is an expression of the critical Reynolds number for an isothermal process. The second bracket determines the effects of the temperature field (Pr, Ec), the turbulence intensities (Tu, Ti) and \((u_{i})_{st}(u_{i})_{st}\) as well on the critical Reynolds number. It is seen that a decrease in \(T_{st}\) and an increase in wall cooling lead to an increase in the critical Reynolds number. Notably, if Eckert number Ec\(^{-1} = c (T_{w} - T_{0})/U_{0}^{2} \approx -0.01\), and \(T_{t} = T_{w} = 0.01 \pm 0.03\), Pr = 0.72, then the critical Reynolds number is increased by a factor of \(\sim 1.5 \pm 1.9\). This fact is in satisfactory agreement with the experimental data presented in [26–30].

4. The velocity and temperature profiles for turbulent flow in the pipe

The space-time area 2) is the area of the generation of turbulence. For the turbulence generation region \(r_{i}(x_{i} + \Delta x_{i} + \Delta r_{i} + \Delta r_{i} - \Delta r_{i}) \subset (N, M) = (1, 1)\), «correlator» \(D_{N:M}(\tau_{i}, m_{i}; \tau_{j}) = D_{i:j}(\tau_{i}, m_{i}; \tau_{j})\), the set of stochastic equations was defined as [15, 16]

\[ \frac{d(p_{col}^{\tau})}{d\tau} = \frac{d(p_{cor}^{\tau})}{d\tau}, \] (13)

\[ \frac{d(U^{\tau})_{col, st}}{d\tau} = \frac{d(U^{\tau})_{st}}{d\tau}; \]

\[ \text{div}(\tau_{i,j})_{\text{col}, st} = \frac{d(\rho U^{\tau})}{d\tau}, \]

\[ \frac{d(E^{\tau})_{col, st}}{d\tau} = \frac{d(E^{\tau})}{d\tau}, \]

\[ \text{div}(\alpha \frac{\partial T}{\partial x_{j}} + u_{j} \tau_{j})_{\text{col}, st} = \frac{d(E^{\tau})}{d\tau}. \]

It is known that the experimental study of the averaged characteristics of fully developed turbulence showed that the velocity profiles have affine similarity. Therefore, in the case of the flow in the pipe [16–18], the final equation for the expression for the index of the velocity profile «\(n\)» has the form

\[ 2 \frac{1-n}{n^{2}} \left( \frac{R}{L_{2}} \right)^{4/3} \left( \frac{R_{2}}{R} \right)^{4/3} \left( \frac{R_{2}}{R} \right)^{2} \left( \text{Re}_{st} - \frac{1}{\text{Re}_{st}} \right)^{0.5} \] (14)

The final equation for the expression for the index of the temperature profile «\(n_{T}\)» has the form
According to [16], there is agreement between the values obtained using equations (14), (15) $n \approx 7$ and $n_r \approx 8$ with the experimental data [26-30] for the indexes of profiles $n$ and $n_r.$

5. Profiles of turbulent characteristics

The space-time area 3) is area of the diffusion of turbulence. For the critical point

$$q = \left(\sqrt{\frac{E_u}{\rho}}\right)_0^2 \left(Re_{\nu,0} - \frac{1}{Re_{\nu,0}}\right) = \left(\frac{E_u}{\rho}\right)_0 \left(Re_{\nu,0} - \frac{1}{Re_{\nu,0}}\right),$$

where $q$ is the kinetic energy of the turbulence at the critical point. Then quantity $q/(u^*)^2$ is equalled to

$$\frac{q}{(u^*)^2} = 0.144\left(\frac{\sqrt{E_u}}{U_m}\right)^2 \frac{Re_{\nu} \left(Re_{\nu,0} - \frac{1}{Re_{\nu,0}}\right)}{0.144*10*0.0001*8000*10 \approx 11.52}$$

(17)

For the first critical number $Re_{\nu} = \frac{xU_0}{\nu} \approx 3*10^7$ and for the initial turbulent Reynolds number $Re_{\nu,0},$ we determine the value of $Re_{\nu} = \frac{\delta U_0}{\nu}.$ Index 0 corresponds to the initial turbulence. $Re_{\nu}$ is the Reynolds number calculated on the thickness of the boundary layer $\delta.$ Then turbulent stresses $q_{xy}$ are

$$\frac{q_{xy}}{(u^*)^2} = \frac{q}{(u^*)^2} = 0.155*0.144\left(\frac{Q}{U_m}\right)^2 \frac{Re_{\nu} \left(Re_{\nu,0} - \frac{1}{Re_{\nu,0}}\right)}{0.155*11.36 \approx 1.8}$$

(18)

The value in (18) is in satisfactory agreement with the data [25-30]. The range of possible values $(q_{xy}/q) = 0.08 \div 0.2$ are in an agreement with Klebanoff’s experimental values [26-30]. Similarly, for each point of the space, performances of profiles $\frac{q}{(u^*)^2}$ and $\frac{q_n}{(u^*)^2}$ are defined.

6. Spectral and correlation functions depending on initial turbulence. Fractal equation by Landau

The space-time area 3) is area of the diffusion of turbulence. For the space-time area of 3) diffusion, according to [12, 17-19] we have two fractal equations. The first equation is written as

$$\frac{d (E_u)}{d \tau} = -(R_{\tau\tau})_{1,1,0} \frac{(E_u)}{\delta \tau}$$

(19)

Then the solution may be written as
\( (E_{st}^j) = C \exp \left[-\frac{(R_{fT})_j(1,0)}{2} \left( \frac{|r|}{\tau_{cor}} \right)^2 \right] \) \( \tag{20} \)

and the spectral function is equalled to

\[ E(w_j) = \frac{C}{\sqrt{\pi\alpha}} \exp \left[ -\frac{w^2}{4\alpha} \right], \] \( \tag{21} \)

\[ C = (E_{st})^j_{cor} = (E_{st})_{cor} \left| Re_{st} - \frac{1}{Re_{st}} \right|^{\frac{1}{2}}. \]

The second fractal equation is written as

\[ \frac{d (E_{st}^j)}{d\tau} = (R_{fT})_j(1,0) \frac{(E_{st})^j}{\delta\tau}. \] \( \tag{22} \)

This expression is the same as the equation for the square of the amplitude of oscillations \( |A|^2 \) in the theory of the turbulence by Landau:

\[ \frac{d|A|^2}{dt} = 2 \cdot \gamma \cdot |A|^2 - \delta \cdot |A|^4. \] \( \tag{23} \)

Solutions of equation (22) were written as \([18, 19]\)

\[ (E_{st}^j) = C_L \exp \left[ (R_{fT})_j(1,0) \left( \frac{|r|}{\tau_{cor}} \right)^2 \right], \] \( \tag{24} \)

\[ (E_{st}^j) = C_L \exp \left[ \frac{(R_{fT})_j(1,0)}{2} \left( \frac{|r|}{\tau_{cor}} \right)^2 \right]. \] \( \tag{25} \)

### 7. The turbulent Prandtl number as a function of initial turbulence

The space-time area 3) is area of the diffusion of turbulence. It is known that the turbulent Prandtl number \( Pr_{T} \) is determined by an expression \([25–34]\), which in the particular case of a forced flow has the form

\[ Pr_{T} = \left( \frac{u_{i}'u_{j}''}{u_iT'c_{p}} \right) \left( \frac{dT/dx_i}{du_i/dx_j} \right) = \left( \frac{u_{i}'u_{j}''}{u_{i}T'} \right) \left( \frac{dT/dx_i}{du_i/dx_j} \right) \] \( \tag{26} \)

Here \( q_{sy} = u_{i}'u_{j}'' \) and \( q_{T} = u_{i}T'c_{p} \) are the turbulent stress and the turbulent heat flux, \( u_{i}', u_{j}', T' \) are fluctuations of values \( u_{i}, u_{j}, T \). For the turbulent flow in the pipe \( Pr_{T} \) is \([18]\)

\[ Pr_{T} = \frac{1}{3.18} \left( \frac{T_{T}}{T'} \right) \left( \frac{n}{n_{T}} \right) \left( \frac{x_2}{R} \right)^{(n-n_{p})/(n_{p})} \] \( \tag{27} \)

and for the turbulent flow on the flat plate, it is
\( \Pr_T = \frac{1}{3.18} \Pr^{0.67} \left( \frac{T_u}{T_T} \right) \left( \frac{n}{n_T} \right) \left( \frac{x_2}{R} \right)^{(n-n_T)/(n-n_T)} \)  

Notably, for values \( n = 7, \ n_T = 8, \ T_T = 0.001—0.01, \ T_u = 0.005—0.04, \ x_2/R = x_2/ \delta = 0.05—0.1, \ Pr_T = 0.05—7.0 \) we have \( \Pr_T = 0.4—1.2. \)

### 8. Heat transfer coefficient

According to \([20, 21]\), using above derived solutions, the Nu number has the form for the water flow under the assumption that \( \Pr_L = 3, \ \Pr_W = 4 \) (the heat transfer from fluid to the wall). Then the equation for Nu was written as

\[ Nu_d = 0.0077 \Pr^{7/12} \Re_d^{7/8} = 0.0145 \Re_d^{7/8} \]  

(29)

Indexes \( L \) and \( W \) refer to the parameters of liquid and at the wall, correspondingly. For the air flow with \( \Pr = 0.7 \), the Nu number was written as

\[ Nu_d = 0.00854 \Pr^{7/12} \Re_d^{7/8} = 0.0069 \Re_d^{7/8} \]  

(30)

Checking the formulas (29) and (30) shows satisfactory agreement with the experimental data \([25–30]\).

### 9. Conclusions

The review of main results of new stochastic theory of turbulence for non-isothermal flows is presented. The sets of stochastic equations for the laws of conservation and for the equivalence of measures between the deterministic process and a random process were derived. The review of results of analytical expressions is presented for the main parameters of a non-isothermal turbulent flow in the tube depending on initial fluctuations in the medium. It is shown that the derived formulas for the critical Reynolds numbers and the critical point, for the velocity and temperature profiles, for the second-order correlation, for the turbulent Prandtl number and for the heat transfer number are in satisfactory agreement with the classical experimental data for these flow characteristics. The derived equations determine not only the nature of turbulent processes but also allow us to explain the scatter of the experimental data due to different initial conditions. In addition, new equations provide a more accurate analysis of numerical studies and experiments.

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