Mathematical Programming Models for Mean Computation in Dynamic Time Warping Spaces

Vincent Froese\(^1\) and Christoph Hansknecht\(^2\)

\(^1\)Technische Universität Berlin, Faculty IV, Institute of Software Engineering and Theoretical Computer Science, Algorithmics and Computational Complexity, vincent.froese@tu-berlin.de

\(^2\)Institute for Mathematical Optimization, TU Braunschweig, c.hansknecht@tu-braunschweig.de

December 6, 2019

Abstract

The dynamic time warping (dtw) distance is an established tool for mining time series data. The DTW-MEAN problem consists of computing a series which minimizes the so-called Fréchet function, that is, the sum of squared dtw-distances to a given sample of time series. DTW-MEAN is NP-hard and intractable in practice. So far, this challenging problem has been solved by various heuristic approaches without any performance guarantees.

We give a polynomial-time algorithm yielding lower bounds on the domain of a mean time series which translate into lower bounds on the Fréchet function. We then formulate the problem as a discrete nonlinear optimization problem based on network flows. We introduce several mixed-integer nonlinear programming (MINLP) formulations in order to solve DTW-MEAN optimally. Our formulations are based on techniques such as outer approximations and nonlinear reformulations of the well-known big \(M\) indicator constraints.

Finally, we conduct several computational experiments to compare the different formulations on several instances derived from the UCR Time Series Classification Archive. While in general DTW-MEAN remains quite challenging, our formulations yield good results in several important specialized problem settings.

**Keywords:** time series averaging, mixed integer nonlinear programming, upper and lower bounds

1 Introduction

Dynamic time warping (dtw) is a widely used distance measure for distance-based time series mining [3, 1]. It allows to cope with temporal variation in
the data via nonlinear alignments between two input time series (see Section 2 for details). Averaging a sample of time series under the dtw-distance is a challenging optimization problem in dtw-based time series mining. Given samples $s_1, \ldots, s_k$, the formal problem is to find a time series $z$ with minimum Fréchet variance

$$F(z) = \frac{1}{k} \sum_{i=1}^{k} \text{dtw}(z, s_i)^2,$$

where $\text{dtw}(z, s_i)$ denotes the dtw-distance between $z$ and $s_i$. A mean is any time series minimizing $F$.

**Related Work.** It is known that a mean (of length at most $nk$, where $n$ is the maximum length of any input series) always exists [20]. Even for binary input series, the problem is known to be NP-hard, W[1]-hard with respect to the number $k$ of samples (that is, presumably not solvable in $f(k) \cdot n^\mathcal{O}(1)$ time for any function $f$) and even not solvable in $n^{o(k)} \cdot f(k)$ time assuming the *Exponential Time Hypothesis*\(^1\) [8]. The currently fastest exact algorithm uses dynamic programming and runs in $\mathcal{O}(n^{2k+2^k})$ time [7]. Over the past decade, various heuristic approaches have been developed [17, 24, 11, 26, 23]. However, they all come without any theoretical performance guarantees and have been shown to yield poor results in practice [7].

2 Preliminaries

**Time Series.** For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, \ldots, n\}$. We consider finite univariate rational time series, which we will simply denote as time series. A time series is a sequence $s \in \mathbb{Q}^n$ for some $n \in \mathbb{N}$. We let $S_n$ be the set of all time series of length $n$ and $\mathcal{S} := \bigcup_{n \in \mathbb{N}} S_n$ be the set of all time series.

**Dynamic Time Warping.** The diagonal grid graph $D(m, n)$ for $m, n \in \mathbb{N}$ is the directed graph with vertices

$$V(D(m, n)) := [m] \times [n]$$

and arcs

$$A(D(m, n)) := \{((i, j), (i + 1, j + 1)) \mid i \in [m - 1], j \in [n - 1]\} \cup \{((i, j), (i + 1, j)) \mid i \in [m - 1], j \in [n]\} \cup \{((i, j), (i, j + 1)) \mid i \in [m], j \in [n - 1]\}.$$\(^1\)

\(^1\)An assumption in complexity theory asserting that 3-SAT is not solvable in time $\mathcal{O}(2^n)$ for some constant $c > 0$, where $n$ is the number of variables [18].
Figure 1: An alignment between two time series (left) together with its warping path (right).

The origin $s_{m,n}$ of $D(m,n)$ is defined as the vertex $(1,1)$. The set of destinations is given as

$$T(D(m,n)) := \{(m,j) \mid j \in [n]\} \subseteq V(D(m,n)).$$

The destination of $D(m,n)$ is the vertex $t_{m,n} := (m,n)$. A warping path of order $m \times n$ is an $s_{m,n}$-$t_{m,n}$-path through $D(m,n)$. We let $\mathcal{P}_{m,n}$ be the set of all warping paths of order $m \times n$.

Let $s, s' \in \mathcal{S}$ be two time series with lengths $m$ and $n$ respectively. The cost $C_P(s,s')$ of a warping path in $P \in \mathcal{P}_{m,n}$ is given by

$$C_P(s,s') := \sum_{(i,j) \in V(P)} (s_i - s'_j)^2.$$

We say that a warping path aligns elements of $s$ and $s'$, where $s_i$ and $s'_j$ are aligned by $P$ if $(i,j) \in V(P)$. An alignment between two time series as well as the corresponding warping path is depicted in Figure 1.

The dtw-distance between $s$ and $s'$ is the minimum cost of any warping path of order $m \times n$, that is,

$$\text{dtw}(s, s') := \min_{P \in \mathcal{P}_{m,n}} \sqrt{C_P(s,s')}.$$

A warping path $P$ with $\sqrt{C_P(s,s')} = \text{dtw}(s,s')$ is called an optimal warping path. An optimal warping path between two time series can be found in quadratic time using dynamic programming [25]. Note that the dtw-distance is not a metric.

**Global Warping Path Constraints.** The definition of warping paths between time series allows alignments straying far from the diagonal (vertices
of \( D_{m,n} \) with equal coordinates, that is, \((i,i) \in V(D_{m,n})\). This leads to optimal warping paths aligning elements which are far apart in time. Since this behavior is generally considered undesired, global constraints restricting the set of warping paths are added in practice \([5, 10]\).

Formally, a **global constraint** is a relation \( R \subseteq [m] \times [n] \) such that the preimage of \( j \) under \( R \), defined as \( R^{-1}[j] := \{ j \in [n] \mid (i,j) \in R \} \) corresponds to an **interval** of values, i.e., to a set

\[
[a, b] := \{a, \ldots, b\} \subseteq [m]
\]
determined by endpoints \( a, b \in \mathbb{N} \) with \( a \leq b \). A warping path \( P \in \mathcal{P}_{m,n} \) satisfies the global constraint \( R \) iff all vertices \((i,j)\) \( \in V(P) \) are contained in \( R \). The definition of the dtw-distance is easily amended to the constrained case.

There are two commonly used constraints regarding admissible warping paths: The Sakoe-Chiba band \([25]\) of width \( r \in \mathbb{N} \) restricts path to vertices \( v = (i,j) \) such that \(|i-j| \leq r\). Similarly, the Itakura parallelogram \([19]\) of slope \( \sigma \geq 1 \) restricts the the warping path to vertices \((i,j)\) such that

\[
\frac{1}{\sigma} \leq \frac{j}{i} \leq \sigma \quad \text{and} \quad \frac{1}{\sigma} \leq \frac{n-j+1}{m-i+1} \leq \sigma.
\]  

Both restrictions have the additional advantage of decreasing the running time needed to compute the dtw-distance. Note that both restrictions require \( s \) and \( s' \) to be compatible with respect to their lengths \( m \) and \( n \). Specifically, whenever \(|m-n| > r\), no path in \( \mathcal{P}_{m,n} \) is contained in the Sakoe-Chiba band. The Itakura parallelogram in turn requires that \( \frac{1}{\sigma} \leq \frac{m}{n} \leq \sigma \) to allow for any feasible path.

**Fréchet Mean.** Consider a finite sample \( \mathcal{X} := \{s^1, \ldots, s^k\} \) of time series. The **Fréchet function** \( F \) measures the average squared dtw-distance of a time series \( z \in \mathcal{S} \) to the set \( \mathcal{X} \) and is defined as

\[
F(z) := \frac{1}{k} \sum_{i=1}^{k} \text{dtw}(z, s^i)^2.
\]

The DTW-Mean problem is to find a mean time series, that is, a time series \( z \) minimizing \( F(z) \). The decision problem is defined as follows.

**DTW-Mean**

**Input:** A list of \( k \) time series \( s_1, \ldots, s_k \) and \( c \in \mathbb{Q} \).

**Question:** Is there a time series \( z \) such that \( F(z) \leq c \)?

It is known that a mean always exists (not necessarily unique) \([20]\). In fact, there always exists a mean of length bounded linearly in the input size.
Theorem 2.1 ([20]). Let $s^1, \ldots, s^k$ be time series with lengths $m_1, \ldots, m_k$. There exists a mean $z \in \mathcal{S}$ of length at most
\[
\sum_{l=1}^{k} m_l - 2(k - 1).
\] (2)

Moreover, it is known that the optimal warping paths between a mean and the input time series determine the value of a mean element to be the arithmetic mean of the input values aligned to it.

Lemma 2.2 ([26]). Let $z = (z_1, \ldots, z_L) \in \mathcal{S}$ be a mean of the time series $s^1, \ldots, s^k$ and let $P_1, \ldots, P_k$ be the corresponding optimal warping paths. Then, for $j \in [L]$, it holds that
\[
z_j = \frac{\sum_{l \in [k]} \sum_{(i,j) \in V(P_l)} s^l_i}{\sum_{l \in [k]} |\{(i,j) \in V(P_l)\}|}.
\] (3)

Note that the global constraints (Sakoe-Chiba band and Itakura parallelogram) mentioned above can be added to the DTW-Mean problem by restricting the warping paths between $z$ and $s^l$ using global constraints $\mathcal{R}_l \subseteq [m_l] \times [n]$ for all $l \in [k]$.

3 Bounding the Mean Domain

In order to obtain tight mathematical programming formulations for DTW-Mean we proceed to bound not only the length of a mean but also the individual values. In the following, we focus on upper bounds for the mean values, the case of lower bounds is symmetric.

Let $s^1, \ldots, s^k$ be time series with lengths $m_1, \ldots, m_k$. A simple upper bound for the value of a mean element is given by the maximum value occurring in the input
\[
ub^{\text{sim}} := \max_{l \in [k]} \max_{i \in [m_l]} s^l_i.
\]
While this bound is easily computed in linear time, it does not translate into a nontrivial bound on the Fréchet function $F$.

Regarding improvement, note that, by definition of a warping path, every mean element is aligned with a consecutive subseries of each input series $s^l$ defined by an interval $I_l = [a_l, b_l] \subseteq [m_l]$. Lemma 2.2 yields the improved bound
\[
ub^{\text{imp}} := \max_{I_1, \ldots, I_k} \frac{\sum_{l \in [k]} \sum_{i \in I_l} s^l_i}{\sum_{l \in [k]} |I_l|}.
\] (4)

In order to compute (4) in polynomial time, we follow an approach by Eppstein and Hirschberg [14]. Observe that for $K \in \mathbb{Q}$, we have that
\[
\text{ub}^{\text{imp}} \leq K \iff \sum_{l \in [k]} \sum_{i \in I_l} s_{l_i}^l \leq \sum_{l \in [k]} |I_l| \leq K \quad \forall I_1, \ldots, I_k
\]
\[
\iff \sum_{l \in [k]} \left( \sum_{i \in I_l} s_{l_i}^l - K|I_l| \right) \leq 0 \quad \forall I_1, \ldots, I_k
\]
\[
\iff \max_{I_1, \ldots, I_k} \sum_{l \in [k]} \left( \sum_{i \in I_l} s_{l_i}^l - K|I_l| \right) \leq 0
\]
\[
\iff \sum_{l \in [k]} \max_{I_l = [a_l, b_l]} 1 \leq a_l \leq b_l \leq m_l \left( \sum_{i \in I_l} s_{l_i}^l - K|I_l| \right) \leq 0.
\]

Define the function \( f : \mathbb{Q} \to \mathbb{Q} \) as follows
\[
f(K) := \sum_{l \in [k]} \max_{I_l = [a_l, b_l]} 1 \leq a_l \leq b_l \leq m_l \left( \sum_{i \in I_l} s_{l_i}^l - K|I_l| \right). \quad (6)
\]

Then, \( \text{ub}^{\text{imp}} \leq K \) if and only if \( f(K) \leq 0 \). Note that \( f \) is a piecewise linear decreasing function since it is a sum of \( k \) piecewise linear decreasing functions (maxima of linear decreasing functions). Thus, computing \( \text{ub}^{\text{imp}} \) corresponds to finding the root of \( f \). Note that we can evaluate \( f \) by enumerating all intervals of all of input time series in \( \mathcal{O}(\sum_{l \in [k]} m_l^2) \) time. In order to compute an (approximate) root of \( f \), we employ a binary search. Note that
\[
\frac{1}{k} \sum_{l \in [k]} \max_{i \in m_l} s_{l_i}^l \leq \text{ub}^{\text{imp}} \leq \text{ub}^{\text{sim}}.
\]

We can therefore approximate \( \text{ub}^{\text{imp}} \) in polynomial time.

Based on the upper bound \( \text{ub}^{\text{imp}} \) and its counterpart \( \text{lb}^{\text{imp}} \) we can bound the dtw-distance between a mean \( z \) and input series \( s^l \) as follows
\[
dtw(s^l, z)^2 \geq \sum_{i \in [m_l]} \begin{cases} 
(s_{l_i}^l - \text{ub}^{\text{imp}})^2 & \text{if } s_{l_i}^l > \text{ub}^{\text{imp}}, \\
(s_{l_i}^l - \text{lb}^{\text{imp}})^2 & \text{if } s_{l_i}^l < \text{lb}^{\text{imp}}, \\
0 & \text{otherwise}.
\end{cases} \quad (7)
\]

This bound on \( dtw(s^l, z)^2 \) translates into a nontrivial lower bound on the Fréchet function \( F \) analogously to the well-known LB Keogh bound for the dtw-distance [21].

Global constraints like the Sakoe-Chiba band or the Itakura parallelogram (Section 2) can be combined with the upper bound (4) by restricting
Two sample time series consisting of 129 data points each from the “TwoPatterns” instance from [12].

The improved bounds \( \text{lb}^{\text{imp}} \) and \( \text{ub}^{\text{imp}} \) in case of absence of global constraints as well as with respect to Itakura parallelograms of slopes 1.5 and 1.1 (encompassing the shaded areas).

Figure 2: An illustration of the improved lower and upper bounds on the mean values.

The set of intervals \( I_l \). In this case, the bound \( \text{ub}^{\text{imp}} \) becomes dependent upon the index \( j \) of the mean element \( z_j \) under consideration. Specifically, we let

\[
\text{ub}^{\text{imp}}_j := \max_{I_l=\[a_l,b_l]\subseteq \mathbb{R}, |I_l|} \frac{\sum_{l \in [k]} \sum_{i \in I_l} s_i^l}{|I_l|}.
\]

Clearly, the computation of these bounds does not differ much from the computation of the original bound \( \text{ub}^{\text{imp}} \). Analogously, we let \( \text{lb}^{\text{imp}}_j \) be the corresponding lower bound value (see Figure 2 for an example). The bound on the dtw-distance can be generalized to the constrained case in a similar fashion:

\[
dtw(s^l, z)^2 \geq \sum_{i \in [m_l]} \min_{j \in [n]} \begin{cases} 
\left(s_i^l - \text{ub}^{\text{imp}}_j\right)^2 & \text{if } s_i^l > \text{ub}^{\text{imp}}_j, \\
\left(s_i^l - \text{lb}^{\text{imp}}_j\right)^2 & \text{if } s_i^l < \text{lb}^{\text{imp}}_j, \\
0 & \text{otherwise.}
\end{cases}
\]
4 Formulations

In the following, we will give multiple mixed integer nonlinear programming (MINLP) formulations of the DTW-Mean problem. To this end, we consider the $k$ diagonal grid graphs $D_1, \ldots, D_k$, where $D_l := D(m_l, N)$ for $N$ being the upper bound on the mean length given by (2). Since we do not a priori know the exact length of a mean $z$, we model the mean length using binary variables. Thus, we can solve the DTW-Mean problem using a single (albeit large) MINLP. For notational convenience, we let

$$V_l := V(D_l), \quad A_l := A(D_l),$$
$$s_l := s(m_l, N), \quad T_l := T(D_l).$$

(8)

4.1 A Vertex-Based Formulation

We begin by introducing a formulation based on binary variables denoting whether or not a vertex in $V_l$ is part of the warping path $P_l$ aligning $s_l$ and a mean $z$. Since we do not know the length of $z$, we include variables $x_j$ determining the length:

$$\sum_{j=1}^{N} x_j = 1$$
$$x_j \in \{0, 1\} \quad \forall j \in [N].$$

(9)

The membership of vertices from $V_l$ in $P_l$ is determined by binary variables $y_{lv}$ for $v \in V_l$. It is clear that the source $s_l$ of $D_l$ must be contained in $P_l$ as well as one of the vertices in $T_l$. Furthermore, if $u \in P_l$, then either $u = (m_l, j)$ for $j$ being the mean length, or one of the out-neighbors of $u$ must be in $P_l$ as well. Thus, the set of vertices of the $k$ warping paths can be described as follows:

$$y_{lv} \in \{0, 1\} \quad \forall l \in [k], v \in V_l$$
$$y_{lv} = 1 \quad \forall l \in [k]$$
$$y_{lu} \leq \sum_{(u,v) \in A_l} y_{lv} \quad \forall l \in [k], u \in V_l \setminus T_l$$
$$y_{lu} \leq \sum_{(u,v) \in A_l} y_{lv} + x_j \quad \forall l \in [k], u = (m_l, j) \in T_l.$$

(10)

Lastly, the distance between input elements $s^l_i$ and mean elements $z_j$ must be included in order to model the objective function (the Fréchet function $F$). However, not all $s^l_i$ and $z_j$ are necessarily aligned in an optimal solution. Let $d^l_{uv}$ for $v = (i, j)$ be the variable denoting the cost contribution of $v$ to the cost of $P_l$. We note that $d^l_{uv} \geq 0$ (clearly, if $v \notin P_l$, then $d^l_{uv} = 0$) and, more importantly, that $d^l_{uv} \leq (M^l_{uv})^2$, where

$$M^l_{uv} := \max(|s^l_i - \text{lb}^\text{imp}_j|, |\text{ub}^\text{imp}_j - s^l_i|).$$
We can therefore include the mean values $z_j$ and the corresponding distances via
\[ d_v^l \geq (z_j - s_i^l)^2 - (M_v^l)^2 \left(1 - y_v^l\right) \quad \forall l \in [k], v = (i, j) \in V_l \]
\[ 0 \leq d_v^l \leq (M_v^l)^2 \quad \forall l \in [k], v \in V_l \]
\[ z_j \in [lb_{imp}^j, ub_{imp}^j] \quad \forall j \in [N]. \]

The objective can then be expressed solely in terms of the variables $d_v^l$, yielding the complete formulation:
\[
\min \frac{1}{k} \sum_{l \in [k]} \sum_{v \in V_l} d_v^l \quad \text{(DTW-V)}
\]
\[ \text{s.t. (9), (10), and (11).} \]

Overall, both the number of variables and the number of constraints are in $O(k^2n^2)$, where $n = \max(m_1, \ldots, m_k)$. Note that the constrains are linear except for the quadratic distance constraints on $d_v^l$ in (11). It is straightforward to replace these constraints by linear ones:
\[ u_v^l \geq (z_j - s_i^l) - M_v^l \left(1 - y_v^l\right) \quad \forall l \in [k], v = (i, j) \in V_l \]
\[ u_v^l \geq (s_i^l - z_j) - M_v^l \left(1 - y_v^l\right) \quad \forall l \in [k], v = (i, j) \in V_l \]
\[ 0 \leq u_v^l \leq M_v^l \quad \forall l \in [k], v \in V_l \]
\[ z_j \in [lb_{imp}^j, ub_{imp}^j] \quad \forall j \in [N]. \]

and minimize $\sum_{l \in [k]} \sum_{v \in V_l} (u_v^l)^2$ instead, thus modeling DTW-MEAN as a mixed-integer quadratic program (MIQP). In any case, the distance constraints are of the so-called big-$M$ type, known to be numerically more challenging and to yield poor lower bounds in general.

Remark 1 (Size). A notable disadvantage of formulation (12) is its size in terms of number of variables: For a set of $k$ time series of a uniform length of $m$, formulation (12) consists of $\Theta(km^2)$ many variables, making it challenging to solve. The large size stems from the fact that both the mean length and the alignments are entirely unknown beforehand.

Interestingly, the addition of global constraints alleviates both problems, greatly facilitating the practical tractability of the problem: Firstly, the Sakoe-Chiba band and the Itakura parallelogram restrict the maximum mean length $N$ to be relatively close to the input length $m$. Secondly, many alignments are excluded beforehand, eliminating the corresponding variables and constraints. Thus, the addition of global constraints is not only advantageous with respect to qualitative considerations, but also makes the problem computationally more tractable.
4.2 An Arc-Based Formulation

Recall that the \( y \)-variables in the vertex-based formulation (DTW-V) model warping paths through the digraphs \( D_l \). Conventionally, paths through networks are described in terms of unit network flows (see [22, pp. 173]). We will therefore proceed to give an arc-based formulation in addition to the vertex-based formulation introduced above. We will then discuss the merits of this formulation (in Section 5 we also conduct several computational experiments).

Formally, we formulate the problem of finding a set of \( k \) warping paths through the graphs \( D_l \) in terms of binary arc variables forming a set of unit flows, i.e.,

\[
f^l_a \in \{0, 1\} \quad \forall l \in [k], a \in A_l, \]

\[
f^l(\delta^+(v)) - f^l(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s_l, \\ -x_j & \text{if } v = (m_l, j) \in T_l, \\ 0 & \text{otherwise.} \end{cases} \quad \forall l \in [k], v \in V_l. \tag{13}\]

where \( \delta^+(u) \) and \( \delta^-(u) \) denote the outgoing and incoming arcs of a vertex \( u \) respectively. This definition of warping paths in terms of flows enables us to express the variables \( y^l_v \) in terms of the corresponding flow variables by means of the following coupling constraints

\[
y^l_v = \begin{cases} f^l(\delta^+(v)) + x_j & \text{if } v = (m_l, j) \in T_l, \\ f^l(\delta^+(v)) & \text{otherwise} \end{cases} \quad \forall l \in [k], v \in V_l. \tag{14}\]

The resulting flow-based formulation of DTW-MEAN is given as

\[
\min \frac{1}{k} \sum_{l \in [k]} \sum_{v \in V_l} d^l_v \quad \text{s.t.} \quad (9), (11), (13), \text{ and (14)}. \tag{DTW-A}\]

In practice, we can use the constraints (14) in order to entirely eliminate the \( y \)-variables.

As for the differences between the formulations: While the digraphs \( D_l \) are relatively sparse, we still roughly triple the number of variables required to model all warping paths in (DTW-A). Since the size of the formulations is already significant (see Remark 1), a further increase in size seems undesirable.

However, we can also judge different formulations in terms of their tightness. Specifically, let NLP(\( \cdot \)) be the NLP relaxation of a formulation, i.e., the nonlinear program (NLP) obtained by dropping integrality requirements from a MINLP: A formulation \( A \) is said to be tighter than \( B \) iff NLP(\( A \) \( \subseteq \)


NLP(B). A tighter formulation yields stronger bounds, making branch-and-bound procedures more efficient. It is easy to show that \((\text{DTW-A})\) is tighter than \((\text{DTW-V})\):

**Lemma 4.1.** \(\text{proj}_{y,d}(\text{NLP(\text{DTW-A})}) \subseteq \text{NLP(\text{DTW-V})}\).

**Proof.** Let \((x,f,y,d)\) be a solution of NLP(\text{DTW-A}), i.e., satisfying all constraints of (DTW-A) except for the integrality condition \(f_a^l \in \{0,1\}\) which is relaxed to \(0 \leq f_a^l \leq 1\) for all \(l \in [k], a \in A_l\). In order to prove \((x,y,d)(\text{DTW-V})\) it is sufficient to show that \(y\) satisfies the relaxation of the constraints (10).

Consider a vertex \(u \in V_l\) for some \(l \in [k]\). Based on the coupling constraints (14), \(y_a^l\) must be non-negative. The flow constraints (13) in turn imply that \(y_a^l\) is less than or equal to one, where equality holds for \(y_a^l\). If \(u \notin T_l\) it holds that

\[
y_a^l = f^l(\delta^+(u)) = \sum_{(u,v) \in A_l} f^l(u,v) \leq \sum_{(u,v) \in A_l} f^l(\delta^-(v))
\]

\[
= \sum_{(u,v) \in A_l: v \in V_l \setminus T_l} f^l(\delta^-(v)) + \sum_{(u,v) \in A_l: v = (m_l,j) \in T_l} f^l(\delta^-(v)) + x_j
\]

The case of \(u \in T_l\) can be treated analogously. \(\square\)

### 4.3 Distance Formulations

Recall that the distance constraints involving the variables \(d_v^l\) are given by

\[
d_v^l \geq (z_j - s_l^l)^2 - \left( M_v^l \right)^2 \left( 1 - y_v^l \right) \quad \forall l \in [k], v = (i,j) \in V_l,
\]

where \(y_v^l\) denotes whether or not a vertex \(v\) is contained in a warping path \(P_l\). Determining the optimal warping paths, while constituting the key difference between the arc-based (DTW-A) and vertex-based (DTW-V) formulations above, is independent of how the distances are modeled exactly. We can therefore study different formulations of the objective independently of the underlying graph model. As mentioned before, the constraints (16) are big \(M\) constraints switched on and off by the \(y\)-variables. Since these constraints generally yield poor relaxations, it is worth investigating alternative modeling techniques.

**A Perspective Reformulation.** In the following, we will derive an alternative formulation of the distance constraints in order to avoid the big-\(M\) constraints present in the original formulation. To this end, we will consider
a fixed index \( l \in [k] \) and vertex \( v \in V_l \). A straightforward reformulation of the distance constraints can be obtained by weighing the quadratic distance between mean elements and input elements with the \( y \)-variables, i.e., requiring that

\[
d_v^l \geq y_v^l \cdot \left( z_j - s^l_i \right)^2 \quad \forall l \in [k], v = (i, j) \in V_l.
\]

Unfortunately, these inequalities are non-convex, making it practically impossible to solve the fractional relaxations. We want the variables \( d_v^l, z_j \), and \( y_v^l \) variables to be contained in the union of the following convex bounded sets

\[
P^0 := \{ (d_v^l, z_j, y_v^l) \mid y_v^l = 0, d_v^l = 0, z_j \in [lb_{imp}^j, ub_{imp}^j] \}, \quad \text{and}
\]
\[
P^1 := \{ (d_v^l, z_j, y_v^l) \mid y_v^l = 1, (M_v^l)^2 \geq d_v^l \geq (z_j - s^l_i)^2, z_j \in [lb_{imp}^j, ub_{imp}^j] \}.
\]

In order to obtain a convex optimization problem we would like to have a description of \( \text{conv}(P^0 \cup P^1) \) in terms of a set of convex inequalities. To this end, we can use the so-called perspective reformulation [15]. The perspective function of a function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) for \( t \in T \) be a set of functions such that the sets

\[
K_t^l := \{ x \in \mathbb{R}^n \mid f_t^l(x) \leq 0 \}
\]

are convex and bounded. Then, \( x \in \text{conv}(\bigcup_{t \in T} K_t^l) \) if and only if

\[
x = \sum_{t \in T} x^t, \quad \sum_{t \in T} \lambda_t = 1, \quad \tilde{f}^l(\lambda_t, x^t) \leq 0, \quad \lambda_t \geq 0 \forall t \in T.
\]

Based on Theorem 4.2, we obtain the desired description including an additional variable \( z_j \):

\[
\begin{align*}
 z_j - \overline{z}_j &\in \left[ (1 - y_v^l) \cdot lb_{imp}^j, (1 - y_v^l) \cdot ub_{imp}^j \right] \\
 d_v^l &\leq y_v^l \cdot \left( M_v^l \right)^2 \quad \text{if } y_v^l > 0 \\
 \overline{z}_j &\in \left[ y_v^l \cdot lb_{imp}^j, y_v^l \cdot ub_{imp}^j \right] \\
 d_v^l &\geq (s^l_i - \overline{z}_j/y_v^l)^2, \quad \text{if } y_v^l > 0.
\end{align*}
\]
(a) The feasible region defined by a quadratic distance constraint. (b) The feasible region of an outer approximation with five supporting points, including $s^1_i$.

Figure 3: An illustration of the outer approximation of a quadratic distance constraint.

Note that the domain of the last inequality cannot easily be extended to include $y^l_v = 0$, which is due to the piecewise definition of the perspective function. Observe that for $y^l_v = 0$ the other inequalities already imply that $d^l_v = \tau_j = 0$. Still, it is well-known [15] that solvers frequently struggle with numerical problems when encountering perspective functions. We therefore propose to begin instead by applying an outer approximation [13] to the quadratic constraint $d^l_v \geq (z_j - s^1_i)^2$. The outer approximation of a set $S_f := \{x \in \mathbb{R}^n \mid f(x) \leq 0\}$ given in terms of a convex differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ is given as the polyhedron defined by

$$\{x \in \mathbb{R}^n \mid f(x^i) + \langle \nabla f(x^i), x - x^i \rangle \leq 0 \forall i \in [k]\} \supseteq S_f$$

based on a set $\{x^1, \ldots, x^k\} \subseteq \mathbb{R}^n$ of supporting points. In our case, a fixed value of $z_j$ yields the inequality

$$(\tau_j)^2 - (s^1_i)^2 \geq 2(\tau_j - s^1_i)z_j - d^l_v.$$ 

See Figure 3 for an example of such an outer approximation. Note that this inequality does not depend on the value $d^l_v$ corresponding to $\tau_j$. For a set $\{z^1_j, \ldots, z^r_j\} \subseteq \mathbb{R}$ of supporting points, we obtain a linear system of inequalities $A_r \cdot (z_j, d^l_v) \leq b_r$, where

$$A_r := \begin{pmatrix} 2(z^1_j - s^1_i) & -1 \\ \vdots & \vdots \\ 2(z^r_j - s^1_i) & -1 \end{pmatrix}, \quad b_r := \begin{pmatrix} (z^1_j)^2 - (s^1_i)^2 \\ \vdots \\ (z^r_j)^2 - (s^1_i)^2 \end{pmatrix}.$$ 

We proceed to apply the perspective reformulation to the sets $P^0$ and

$$Q^1 := \{(d^l_v, z_j) \mid g^l_v = 1, d^l_v \leq (M_v^l)^2, A_r \cdot (z_j, d^l_v) \leq b_r, z_j \in [l^v_j \text{imp}, u^v_j \text{imp}]\}.$$ 

13
Note that since both $P^0$ and $Q^1$ are polytopes, the set $\text{conv}(P^0 \cup Q^1)$ is a polytope as well, alleviating the numerical problems of the perspective reformulation based on nonlinear functions. Indeed, optimizing over the union of polytopes is known as a disjunctive programming problem (see [4]). In our case we obtain the following constraints:

$$z_j - \tau_j \in \left[ (1 - y_v^l) \cdot \text{lb}_j^{\text{imp}}, (1 - y_v^l) \cdot \text{ub}_j^{\text{imp}} \right]$$

$$d_v^l \leq y_v^l \cdot (M_v)$$

$$\tau_j \in \left[ y_v^l \cdot \text{lb}_j^{\text{imp}}, y_v^l \cdot \text{ub}_j^{\text{imp}} \right]$$

$$A_r \cdot (\tau_j, d_v^l) \leq y_v^l \cdot b_r$$

In order to solve the DTW-MEAN problem we still have to settle on a set of supporting points: If the set is too small or unevenly spaced, then the outer approximation is not sufficiently tight, and the error with respect to the actual quadratic function becomes too large. On the other hand, each support point increases the size of the resulting program, slowing down the solution process.

To avoid the problem of having to select a suitable set beforehand, we separate inequalities as needed: We begin with a support set consisting only of $s_i^l$ and solve the resulting problem, obtaining a solution consisting in part of values for $d_v^l$ and $\tau_j$. If these values violate the quadratic constraint sufficiently much, we add the value of $\tau_j$ to the supporting points (thereby cutting off the solution) and resolve. In practice, all state-of-the-art MINLP solvers offer so-called callback functions in order to support the separation of additional constraints during the solution process.

Still, the approach comes at a price in terms of size: For each distance variable $d_v^l$, we need one additional variable and several additional constraints.

**Implicit Distances.** In the following we will consider the framework introduced by Bertsimas, Cory-Wright, and Pauphilet [6]. The authors observed that the big-$M$ modeling approach can be seen as a regularization of logical constraints in a two-stage problem consisting of an outer binary and an inner continuous part. The problem can be dualized, yielding a convex inner function which can be used to derive valid inequalities based on outer approximations. We will adapt this approach to our formulations.

Consider a feasible solution $(x, y)$ of the system comprised of both (9) and (10), that is, a set of warping paths in the digraphs $D_i$ connecting their respective sources with vertices corresponding to a fixed mean length. Let $P_l$ be the path such that $y_v^l = 1$ if and only if $v \in V(P_l)$. For convenience, we let $\overline{y}_v^l := 1 - y_v^l$ be the inverse of the variable $y_v^l$ denoting the absence of a vertex $v$ from $V(P_l)$. We rewrite the distance constraints (11) by introducing
additional variables $\Delta^l_v$:
\[
d^l_v \geq (z_j - s^l_i)^2 - \Delta^l_v \quad \forall l \in [k], v = (i, j) \in V_l
\]
\[
d^l_v, \Delta^l_v \geq 0 \quad \forall l \in [k], v = (i, j) \in V_l
\]  
(17)
together with the following logical constraint:
\[
\Delta^l_v = 0 \quad \text{if } \pi^l_v = 0 \quad \forall l \in [k], v = (i, j) \in V_l.
\]  
(18)
The problem of finding a mean $z$ corresponding to the solution $(x, y)$ is then given as
\[
f(y) := \begin{cases} 
\min_{d, \Delta, z} & \frac{1}{k} \sum_{l \in [k]} \sum_{v \in V_l} d^l_v \\
\text{s.t.} & (d, \Delta, z) \text{ satisfy (17) and (18)},
\end{cases}
\]  
(19)
where the last constraint ensures that the binary values $\pi$ control the range of the variables $\Delta$. If a vertex $v$ is contained in $P_l$, i.e., $\pi^l_v = 0$, then $\Delta^l_v$ is fixed to zero, which may force $d^l_v$ to a positive value in order to satisfy the first constraint. Conversely, if $\pi^l_v = 1$, then we can set $d^l_v$ to zero save costs. Thus, an optimal solution $(d^*, \Delta^*, z^*)$ of (19) is given by
\[
z^*_j := \frac{\sum_{l \in [k]} \sum_{v = (i, j) \in V_l} s^l_i \cdot y^l_v}{\sum_{l \in [k]} \sum_{v = (i, j) \in V_l} y^l_v},
\]
\[
(d^*)^l_v := \begin{cases} 
(z^*_j) - s^l_i & \text{if } \pi^l_v = 0, \\
0 & \text{otherwise},
\end{cases}
\]  
(20)
\[
(\Delta^*)^l_v := \begin{cases} 
0 & \text{if } \pi^l_v = 0, \\
(z^*_j) - s^l_i & \text{otherwise}.
\end{cases}
\]
In order to use the techniques from [6], we add a regularization term $\Omega(\Delta) := \sum_{l \in [k]} \sum_{v \in V_l} \Omega^l_v(\Delta^l_v)$, where
\[
\Omega^l_v(\alpha) := \begin{cases} 
0 & \text{if } |\alpha| \leq (M^l_v)^2, \\
\infty & \text{otherwise}.
\end{cases}
\]
Furthermore, we introduce a function $g(d, \Delta, z)$ encompassing the objective and parts of the constraints of (19):
\[
g(d, \Delta, z) := \begin{cases} 
\frac{1}{k} \sum_{l \in [k]} \sum_{v \in V_l} d^l_v, & \text{if } (d, \Delta, z) \text{ satisfy (17), and} \\
\infty & \text{otherwise}.
\end{cases}
\]
Thus, the problem becomes
\[
\min_{d, \Delta, z} g(d, \delta, z) + \Omega(\Delta)
\]
\[
\text{s.t. } \Delta \text{ satisfies (18)},
\]
which can, according to [6, Theorem 1], be transformed into the following saddle-point problem involving additional variables $\alpha_i^t$:

$$\max_{\alpha} \min_{d, \delta, z} \sum_{l \in [k]} \sum_{v \in V_l} \frac{1}{k} d^t_{l,v} - \alpha^t_{l,v} d^t_{l,v} - \varphi'_{v} |\alpha^t_{l,v}| \left( M^t_{l,v} \right)^2$$

s.t. $(d, \delta, z)$ satisfy (17).

Note that the variables $\Delta$ disappear as $\Omega(\cdot)$ is replaced by its Fenchel conjugate. In order to solve the saddle point problem, we can make several observations regarding the choice of variables $\alpha$: If $\alpha^t_{l,v} > 0$, then the inner problem becomes unbounded since the objective value is strictly decreasing along increasing values of $\delta^t_{l,v}$. Thus, we can assume that $\alpha^t_{l,v} \leq 0$. We can therefore define $\beta^t_{l,v} := -\alpha^t_{l,v}$ and rewrite the problem as:

$$\max_{\beta \geq 0} \min_{d, \delta, z} \sum_{l \in [k]} \sum_{v \in V_l} \frac{1}{k} d^t_{l,v} + \beta^t_{l,v} \left( \delta^t_{l,v} - \varphi'_{v} \left( M^t_{l,v} \right)^2 \right)$$

s.t. $(d, \delta, z)$ satisfy (17).

Since we already know that the objective values of (19) and (21) must coincide, it only remains to find suitable values of $\beta$. Specifically, if we let

$$(\beta^*)_{l,v} := \begin{cases} \frac{1}{k} \text{ if } \varphi'_{v} = 0, \text{ and} \\ 0 \text{ otherwise}, \end{cases}$$

then $(d^*, \delta^* = \Delta^*, z^*)$ is a solution of the inner optimization problem of (21) having the same objective value as the original (19).

For this value of $\beta$, we can derive a cutting plane based on the subgradients of the convex function $f$. One subgradient of $f$ is given by

$$(\nabla f(y))^t_{l,v} := \begin{cases} \frac{(M^t_{l,v})^2}{k} \text{ if } \varphi'_{v} = 0, \text{ and} \\ 0 \text{ otherwise}. \end{cases}$$

Based on this subgradient, from each feasible solution $\hat{y}$ of (10), we obtain a linear inequality $f(y) \geq f(\hat{y}) + \langle \nabla f(\hat{y}), (y - \hat{y}) \rangle$ as

$$f(y) \geq f(\hat{y}) + \sum_{l \in [k]} \sum_{v \in V_l : \hat{y}^t_{l,v} = 1} \frac{(M^t_{l,v})^2}{k} \left( y^t_{l,v} - 1 \right),$$

where $f(y)$ corresponds to the value of the mean derived from the solution $y$. To embed this approach into our formulations, we introduce an additional variable $\eta$ denoting the objective value and require

$$\eta \geq f(\hat{y}) + \sum_{l \in [k]} \sum_{v \in V_l : \hat{y}^t_{l,v} = 1} \frac{(M^t_{l,v})^2}{k} \left( y^t_{l,v} - 1 \right) \quad \forall \hat{y} \text{ satisfying (10).}$$
Thus, we can reformulate (DTW-V) as

\[
\min \eta \\
\text{s.t. (9), (10), and (22)}.
\]

The arc-based formulation (DTW-A) can be adapted in much the same way. We would like to point out that this reformulation of the distance constraints is much smaller, since no variables apart from \( x \), \( y \), and \( z \) are required. The inequalities (22) can be separated whenever a feasible solution is obtained throughout the search in a branch-and-bound tree.

5 Computational Results

All experiments were conducted using an implementation in the C++ programming language compiled using the GNU C++ compiler with the optimizing option -O2. We used version 6.0.2 of the SCIP [2] optimization suite and version 8.1 of GUROBI [16] as underlying LP solver. All measurements were taken on an Intel Core i7-965 processor clocked at 3.2 GHz.

We begin by comparing the formulations across several small instances, generated from the “FiftyWords” data set of the UCR archive [12]. Specifically, we sampled sets of \( k \in \{2, 5\} \) time series (of original length \( N = 270 \)). Each sampled time series was reduced to a uniform length of \( m \in \{10, 20\} \) by averaging successive disjoint blocks of \( \lfloor N/m \rfloor \) values.

We measure the quality of the formulations based on the remaining gap after a time limit of one hour has expired. The gap is given as \((p - d)/d\), where \( p \) is the value of the best-known feasible solution, and \( d \) is the dual bound obtained as the minimal relaxation value across the leaves of the branch-and-bound tree. Measuring the gap provides a good overview over the practical performance of the different formulations, since both the solution times of the relaxations and the dual bounds provided by them influence the resulting gap. To reduce the effect of the random sampling used to generate the instances, we measured the average remaining gap over ten instances for each variant. Furthermore, we included global constraints given by both a wide \((\sigma = 1.5)\) and a narrow \((\sigma = 1.1)\) Itakura parallelogram. From the results, displayed in Table 1, we can make several observations: Firstly, the implicit distances clearly perform worst. This seems to be largely due to their poor relaxation values. Indeed, the inequalities (22) separated while traversing the branch-and-bound tree are often insufficient to obtain nontrivial lower bounds. The quadratic distances performed best for all instances. Secondly, the arc-based formulation (DTW-A) performs considerably better than its vertex-based counterpart (DTW-V). Apparently, the tighter approximation of the arc-based formulation more than compensates for the increase in size.
Table 1: Remaining gap after one hour of computation for different instances and formulations. The distance formulations are denoted as Quadratic, Perspective and Implicit respectively.

| Variant | Arc-based | | Vertex-based | | |
|---------|-----------|---|-------------|---|---|
|         | Q         | P | I | Q | P | I |
| $m = 10, k = 2$ | | | | | | |
| free    | 2.10      | 2.56 | $\infty$  | 3.05 | 4.89 | $\infty$ |
| wide    | 0         | 0   | 0           | 0   | 0   | 0           |
| narrow  | 0         | 0   | 0           | 0   | 0   | 0           |
| $m = 20, k = 2$ | | | | | | |
| free    | 21.79     | 69.46 | $\infty$  | 369.29 | 3001.95 | $\infty$ |
| wide    | 0.02      | 0.03 | $\infty$  | 0.36  | 0.38 | $\infty$ |
| narrow  | 0         | 0   | 0           | 0   | 0   | 0           |
| $m = 10, k = 5$ | | | | | | |
| free    | 3.75      | 15.84 | $\infty$  | 7.92 | $\infty$ | $\infty$ |
| wide    | 0         | 0   | $\infty$  | 0   | 0   | $\infty$ |
| narrow  | 0         | 0   | 0           | 0   | 0   | 0           |
| $m = 20, k = 5$ | | | | | | |
| free    | 15.48     | $\infty$ | $\infty$  | 25.93 | $\infty$ | $\infty$ |
| wide    | 0.44      | 1.78 | $\infty$  | 0.88  | 68.40 | $\infty$ |
| narrow  | 0         | 0   | 0           | 0   | 0   | 0           |

Unsurprisingly, an increased size with respect to both the number $k$ of time series and the length $m$ of the time series results in larger gaps remaining after the time limit. The addition of global constraints greatly improved the solution process, presumably because of the reduction in problem size in terms of the number of variables and constraints.

6 Conclusion and Future Works

In this paper, we gave the first formulation of the DTW-MEAN problem as a nonlinear optimization problem. We derived nontrivial bounds on the mean domain, translating into lower bounds on the value of the Fréchet function also taking into account global constraints such as the Skaoe-Chiba band or the Itakura parallelogram.

We introduced several different nonlinear programming formulations of DTW-MEAN, based on different modeling approaches to the combinatorial structure as well as the nonlinear cost function.

We compared these formulations with respect to their computational efficiency, measured in terms of the remaining gap after one hour of com-
putation, concluding that a quadratic big $M$ distance formulation together with an arc-based model for the warping paths performs best in practice.

Unfortunately, solving the DTW-MEAN problem on large-scale instances still seems out of reach. This is likely due to the fact that the introduced MINLP formulations, while being significant in size, yield poor lower bounds, resulting in enormous gaps relative to the primal solutions obtained throughout the course of optimization. As a result, few if any branches of the branch-and-bound tree are can be discarded, and most of the feasible solutions have to be enumerated. On the other hand, it is straightforward to include global constraints into the different MINLP formulations. The formulations can then take advantage of the reduction in combinatorial complexity and solve the resulting problems more efficiently.

There are several directions in which this work can be extended. On the one hand, the formulations introduced here yield relaxations of insufficient quality. In order to strengthen the formulations, it might be necessary to derive families of valid inequalities. On the other hand, any a priori bounds on the length of the mean series would aid computations. Conversely, the inclusion of relaxation-based heuristics tailored specifically to the DTW-MEAN problem could increase the practical performance of the formulations as well.

References

[1] Amaia Abanda, Usue Mori, and Jose A. Lozano. “A review on distance based time series classification”. In: Data Mining and Knowledge Discovery 33.2 (2019), pp. 378–412. DOI: 10.1007/s10618-018-0596-4.

[2] Tobias Achterberg. “SCIP: solving constraint integer programs”. In: Mathematical Programming Computation 1.1 (2009), pp. 1–41. DOI: 10.1007/s12532-008-0001-1.

[3] Anthony Bagnall et al. “The great time series classification bake off: a review and experimental evaluation of recent algorithmic advances”. In: Data Mining and Knowledge Discovery 31.3 (2017), pp. 606–660. DOI: 10.1007/s10618-016-0483-9.

[4] Egon Balas. “Disjunctive programming”. In: Annals of Discrete Mathematics. Vol. 5. Elsevier, 1979, pp. 3–51. DOI: 10.1016/S0167-5060(08)70342-X.

[5] Donald J Berndt and James Clifford. “Using dynamic time warping to find patterns in time series”. In: Proceedings of the 3rd International Conference on Knowledge Discovery and Data Mining. 1994, pp. 359–370.
[6] Dimitris Bertsimas, Ryan Cory-Wright, and Jean Pauphilet. “A unified approach to mixed-integer optimization: Nonlinear formulations and scalable algorithms”. In: arXiv e-prints (2019). arXiv: 1907.02109v1 [math.OC].

[7] Markus Brill et al. “Exact mean computation in dynamic time warping spaces”. In: Data Mining and Knowledge Discovery 33.1 (2019), pp. 252–291. doi: 10.1007/s10618-018-0604-8.

[8] Laurent Bulteau, Vincent Froese, and Rolf Niedermeier. “Tight Hardness Results for Consensus Problems on Circular Strings and Time Series”. In: arXiv e-prints (2019). arXiv: 1804.02854v4 [cs.DM].

[9] Sebastián Ceria and João Soares. “Convex programming for disjunctive convex optimization”. In: Mathematical Programming 86.3 (1999), pp. 595–614. doi: 10.1007/s101070050106.

[10] Selina Chu et al. “Iterative deepening dynamic time warping for time series”. In: Proceedings of the 2002 SIAM International Conference on Data Mining (SDM ’02). 2002, pp. 195–212. doi: 10.1137/1.9781611972726.12.

[11] Marco Cuturi and Mathieu Blondel. “Soft-DTW: a Differentiable Loss Function for Time-Series”. In: Proceedings of the 34th International Conference on Machine Learning (ICML ’17). 2017, pp. 894–903.

[12] Hoang Anh Dau et al. The UCR Time Series Classification Archive. https://www.cs.ucr.edu/~eamonn/time_series_data_2018/. 2018.

[13] Marco A. Duran and Ignacio E. Grossmann. “An outer-approximation algorithm for a class of mixed-integer nonlinear programs”. In: Mathematical Programming 36.3 (1986), pp. 307–339. ISSN: 1436-4646. doi: 10.1007/BF02592064.

[14] David Eppstein and Daniel S. Hirschberg. “Choosing Subsets with Maximum Weighted Average”. In: Journal of Algorithms 24.1 (1997), pp. 177–193. doi: 10.1006/jagm.1996.0849.

[15] Oktay Günlük and Jeff Linderoth. “Perspective reformulation and applications”. In: Mixed Integer Nonlinear Programming. Springer, 2012, pp. 61–89. doi: 10.1007/978-1-4614-1927-3_3.

[16] LLC Gurobi Optimization. Gurobi Optimizer Reference Manual. 2018. URL: http://www.gurobi.com.
[17] V. Hautamaki, P. Nykanen, and P. Franti. “Time-series clustering by approximate prototypes”. In: 19th International Conference on Pattern Recognition. 2008, pp. 1–4. DOI: 10.1109/ICPR.2008.4761105.

[18] Russell Impagliazzo and Ramamohan Paturi. “On the Complexity of k-SAT”. In: Journal of Computer and System Sciences 62.2 (2001), pp. 367–375. DOI: 10.1006/jcss.2000.1727.

[19] Fumitada Itakura. “Minimum prediction residual principle applied to speech recognition”. In: IEEE Transactions on Acoustics, Speech, and Signal Processing 23.1 (1975), pp. 67–72. DOI: 10.1109/TASSP.1975.1162641.

[20] Brijnesh J. Jain and David Schultz. “On the Existence of a Sample Mean in Dynamic Time Warping Spaces”. In: arXiv e-prints (2018). arXiv: 1610.04460v3 [cs.CV].

[21] Eamonn Keogh and Chotirat Ann Ratanamahatana. “Exact indexing of dynamic time warping”. In: Knowledge and information systems 7.3 (2005), pp. 358–386. DOI: 10.1007/s10115-004-0154-9.

[22] Bernhard Korte and Jens Vygen. Combinatorial Optimization: Theory and Algorithms. 5th. 2012. ISBN: 978-3-642-24488-9. DOI: 10.1007/978-3-642-24488-9.

[23] Y. Liu, Y. Zhang, and M. Zeng. “Adaptive Global Time Sequence Averaging Method Using Dynamic Time Warping”. In: IEEE Transactions on Signal Processing 67.8 (2019), pp. 2129–2142. DOI: 10.1109/TSP.2019.2897958.

[24] François Petitjean, Alain Ketterlin, and Pierre Gançarski. “A global averaging method for dynamic time warping, with applications to clustering”. In: Pattern Recognition 44.3 (2011), pp. 678–693. DOI: 10.1016/j.patcog.2010.09.013.

[25] Hiroaki Sakoe and Seibi Chiba. “Dynamic programming algorithm optimization for spoken word recognition”. In: IEEE transactions on acoustics, speech, and signal processing 26.1 (1978), pp. 43–49. DOI: 10.1109/TASSP.1978.1163055.

[26] David Schultz and Brijnesh J. Jain. “Nonsmooth analysis and subgradient methods for averaging in dynamic time warping spaces”. In: Pattern Recognition 74 (2018), pp. 340–358. DOI: 10.1016/j.patcog.2017.08.012.