Geodesic rigidity of Levi-Civita connections admitting essential projective vector fields

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Abstract
In this paper, it is proved that a connected 3-dimensional Riemannian manifold or a closed connected semi-Riemannian manifold $M^n (n > 1)$ admitting a projective vector field with a non-linearizable singularity is projectively flat.

1 Introduction
Let $\nabla$ be a torsion-free affine connection on a manifold $M^n$. The projective class $[\nabla]$ for $\nabla$ consists of the torsion-free affine connections on $M$ having the same un-parametrized geodesics defined by $\nabla$. It is well known that:

$$\nabla \in [\nabla] \iff \nabla = \nabla + \eta \otimes Id + Id \otimes \eta, \quad \eta \in \Gamma(T^*M)$$

Let $X$ be a vector field on $M$. Let $\phi^t$ be the flow generated by $X$. Then $X$ is a projective vector field for $\nabla$ if $\phi^t$ preserves the unparametrized geodesics defined by $\nabla$. If one denotes $L_X \nabla$ the Lie derivative of $\nabla$ with respect to $X$, this is equivalent to:

**Trace free part** ($L_X \nabla$) = 0

The projective vector field $X$ is *affine* for $\nabla$ if $L_X \nabla = 0$. It is *essential* if it is not affine for any connection in $[\nabla]$.

It is a classical topic to work with projective structures induced by Levi-Civita connections. Some classical results have been obtained by mathematicians like Dini, Levi-Civita, Weyl, and Solodovnikov. One can refer to Theorems 7-10 from [4] for their results. The local description of projectively equivalent metrics is well understood by Bolsinov and Matveev in [10], and [8] in terms of BM structures. Kobayashi and Nagano give a concrete description of projective structures in terms of Cartan geometries in [1].

Given a projective structure $[\nabla]$ on some manifold $M^n$, how its projective transformation group or Lie algebra determines the projective structure $[\nabla]$ has been an interesting topic. For example, one may ask what additional assumption on the projective transformation group or algebra is necessary to deduce that the
projective structure is flat on the manifold or some special subsets. Sometimes it turns out $|\nabla|$ is determined by assumptions less than expected. Such results are referred as the geodesic rigidity of the manifold $(M^n, \nabla)$. For the global theory of projective structures, one has the following projective Lichnerowicz-Obata conjecture:

**Conjecture 1.** Let $G$ be a connected Lie group acting on a complete connected or closed connected manifold $(M^n, g)$ by projective transformations. Then either $G$ acts on $M$ by affine transformations, or $(M^n, g)$ is Riemannian with positive constant sectional curvature.

The conjecture above implies non-affine projective vector fields cannot exist for non-flat projective structures induced by closed or complete connected $(M^n, g)$. The open case for this conjecture is when $g$ is an indefinite metric, and $D(M^n, g)$ is precisely two, where $D(M^n, g)$ is the degree of mobility of $g$ on $M$ defined in Definition 2.4. (This conjecture has recently been proved for the case $(M, g)$ is a closed connected Lorentzian manifold, see [16].) One may refer to the main theorems in [3], [4] and [16] for details. In the local theory of projective structures, whether there is a result analogous to the conjecture above for locally defined metrizable projective structures is still open in general.

Let $|\nabla|$ be a metrizable projective structure admitting a projective vector field $X$ with a non-linearizable singularity $x$. This means $X$ cannot be conjugated into a linear vector field on any neighbourhood of $x$. The rigidity of such projective structures arises from the generalization of results obtained in projective geometries in [2] by Nagano and Ochiai, and analogous results in conformal geometries by Frances and Melnick in [12] and [13]. In [2], the following theorem is proved.

**Theorem 1.1** (Nagano, Ochiai[2]). Let $X$ be a projective vector field for a closed connected Riemannian manifold $(M^n, g)$. Suppose there exists $x \in M$ such that the order $O(X, x)$ of $X$ at $x$ is exactly 2. It follows that $(M^n, g)$ is either $\mathbb{S}^n$ or $\mathbb{R}P^n$.

To generalize this theorem, one expects an analogous result to hold for semi-Riemannian closed connected manifolds, with the weaker assumption that $X$ is non-linearizable at $x$. Obviously, this generalization of Theorem 1.1 follows from Lichnerowicz-Obata conjecture.

The dynamics of a projective vector field near its singularity can lead to theorems on the rigidity of projective structures. For example, since the Weyl curvature is invariant under projective maps, Nagano and Ochiai obtained the following proposition (See Lemma 5.6 of [2] for details), which is the main ingredient to prove Theorem 1.1. If an affine connection $\nabla$ admits a projective vector field $X$ such that $O(X, x) = 2$ at some point $x$, then $|\nabla|$ is projectively flat near $x$. Suppose that $x$ is a non-linearizable singularity of a projective vector field
X. Then on some special subsets containing x, the flow $\phi^t$ generated by X admits dynamics similar to the case $O(X,x) = 2$. This may imply X admits a non-linearizable singularity at x is a good substitution for the assumption $O(X,x) = 2$.

Projective and conformal structures are both $|1|$-graded parabolic geometries in terms of Cartan geometries. In conformal geometries one has the following result from [13].

**Theorem 1.2** (C.Frances, K.Melnick[13]). Let X be a conformal vector field for a semi-Riemannian manifold $(M^n, g)$ with $n \geq 3$ with a singularity x. If the 1-parameter group $\{ (D\phi_X)^t_x : t \in \mathbb{R} \}$ is bounded, one of the following is true:

- There exists a neighbourhood V of x on which X is complete and generates a bounded flow. In this case, it is linearizable.
- There is an open set $U_0 \subset M$, with $x \in \overline{U_0}$ such that g is conformally flat on $U_0$.

In terms of the local theory of projective structures, one can expect a statement analogous to Theorem 1.2 to hold in projective geometries. Let X be a projective vector field for $[\nabla]$ vanishing at x. The minimal conditions for the projective class $[\nabla]$ being flat near x are still open. The only known case is when the manifold is 2-dimensional. The results from [14] and [15] show in this case a metrizable projective structure near a non-linearizable singularity x of a projective vector field has to be flat near x.

In this paper, the following theorem on projective geometries for Riemannian manifolds is proved.

**Theorem 1.3.** Let $(M^n, g)$ with $n \geq 3$ be a connected Riemannian manifold admitting a projective vector field X. Suppose X vanishes at $o \in M$, and X is not linearizable at $o$. One has $D(M^n, g)$ is at least 3. When $n = 3$, this implies g has constant sectional curvature.

For closed and connected manifolds, the following generalization of Theorem 1.1 by Nagano and Ochiai is proved.

**Theorem 1.4.** Let $(M^n, g)$ with $n > 1$ be a closed connected pseudo-Riemannian manifold. Suppose X is a projective vector field for $(M, g)$ vanishing at $o \in M$. If X is not linearizable at $o$, then g is Riemannian with constant positive sectional curvature.

After deriving the proof, we recently discovered that the part of the proof of Theorem 1.4 in Section 4.2 is analogous to Section 9.2 of [5].
2 Preliminaries and Backgrounds

2.1 General theory for projective structures in the view of Cartan geometries

The definitions of a Cartan geometry are listed below since it is important for this paper. Let $\hat{G}$ be a Lie group, and $G'$ is a closed subgroup of $\hat{G}$. Denote $\mathfrak{g}, \mathfrak{g}'$ their Lie algebras, respectively. One has the following definitions from Cartan geometries.

Definition 2.1. A Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{g}')$ is a triple $(M, B, \omega)$. Here $B$ is a $G'$ principle bundle over $M$, and the Cartan connection $\omega$ is a $\mathfrak{g}$ valued 1-form. In addition, it satisfies the following conditions:

- $\forall b \in B$, one has $\omega_b : T_b B \to \mathfrak{g}$ is an isomorphism.
- $\forall g \in G'$, $R^* g \omega = Ad(g^{-1}) \omega$
- $\forall b \in B$, $\forall \tilde{g} \in \mathfrak{g}'$, one has $\omega \left( \frac{d}{dt} |_{t=0} \exp(t \tilde{g}) \right) = \tilde{g}$

Here $\omega$ is the Cartan connection, and $\kappa = d\omega + \frac{1}{2} [\omega, \omega]$ is the curvature of this Cartan geometry.

In addition, one has the definition of exponential maps in Cartan geometries.

Definition 2.2. Suppose $(M, B, \omega)$ is a Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{g}')$. Given any $v \in \mathfrak{g}$, one has $\omega^{-1}(v)$ is a vector field on $B$. Denote $\Phi_v$ the flow generated by $\omega^{-1}(v)$. The exponential map of $\omega$ at $b \in B$ is defined as $\exp_b(v) = \Phi_v(1, b)$, wherever it is well defined. Thus $\exp_b$ gives a local diffeomorphism between a neighbourhood of $0$ of $\mathfrak{g}$ and a neighbourhood of $b$.

The projective classes on $M$ can be described in terms of Cartan geometries by the following. One has $G = PGL(n + 1, \mathbb{R})$ acting on $\mathbb{R}P^n$ transitively. Choose $e_0 = [1, 0, \cdots, 0] \in \mathbb{R}P^n$, and let $H$ be its stabilizer. Denote $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of $G$ and $H$, respectively. Then one has the following identification (see Page 2 of [2]):

$$\mathfrak{sl}(n + 1, \mathbb{R}) = \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathbb{R}^n \oplus GL(n, \mathbb{R}) \oplus (\mathbb{R}^n)^*, \quad \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

Note that the standard Euclidean metric gives an identification $\mathbb{R}^n \simeq (\mathbb{R}^n)^*$.

The identification is given by:

$$u \oplus A \oplus v^* \mapsto \begin{bmatrix} \frac{-1}{n+1} Tr(A) & v^T \\ u & A - \frac{1}{n+1} Tr(A) \cdot Id \end{bmatrix} \in \mathfrak{sl}(n + 1, \mathbb{R}) \tag{2}$$

The following is the standard chart of $\mathbb{R}P^n$ near $e_0$:

$$i_0 : [x_0, \cdots, x_n] \mapsto \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)$$
In this chart $i_0$, any $h \in H$ is a local diffeomorphism at $0 \in \mathbb{R}^n$ with $h(0) = 0$. If $f$ is a local diffeomorphism at $0 \in \mathbb{R}^n$ with $f(0) = 0$, let $J^k(f)(0)$ be its $k$-jet at the origin. One defines $G_k^n$ as the $k$-jet at $0$ of all such functions. Clearly elements in $G_k^n$ form a group. Since every $h \in H$ is such a diffeomorphism in the standard chart $i_0$, define the subgroup $H^2(n)$ of $G^2(n)$:

$$H^2(n) = \{ J^2(h)(0) : h \in H \}$$

The above in fact gives an identification $H \cong H^2(n)$. One also has $G^1(n) \cong GL(n, \mathbb{R})$. Let $F^2(M)$ be the 2-jet frame bundle of $M$, then it is a $G^2(n)$ principle bundle. One can take $F^2(M)$ as a sub-bundle of $F^1(F^1(M))$. Denote $\theta$ the canonical form on $F^1(F^1(M))$, then it is a $\mathfrak{gl}_n(\mathbb{R}) \oplus \mathbb{R}^n$ valued 1-form. It follows that $\theta|_{F^2(M)}$ has the following decomposition:

$$\theta = \theta^i + \theta_j^i, \quad \theta^i \in \Gamma(\text{Hom}(T(F^2 M), \mathbb{R}^n)), \quad \theta_j^i \in \Gamma(\text{Hom}(T(F^2 M), \mathfrak{gl}_n(\mathbb{R})))$$

Here $\theta = \theta^i + \theta_j^i$ is the canonical form on $F^2(M)$. One can refer to Page 9 of [1] for a more precise definition.

A projective Cartan geometry on $M$ is a Cartan geometry $(M, B, \omega)$ modelled on the pair $(\mathfrak{g}, \mathfrak{h})$. It is normal if the components of the curvature $\kappa$ satisfies Equation (2) and (3) from [1]. Under the identification given by Equations (1) and (2), one has by Proposition 3 of [1], on any $H^2(n)$ sub-bundle $P$ of $F^2(M)$, there is a unique normal projective Cartan connection $\omega = \omega_i^i + \omega_j^i + \omega^i$ with $\omega_i = \theta_i$, and $\omega_j^i = \theta_j^i$. One calls this connection the normal projective Cartan connection associated to $P$.

Let $F^2_1(M) = F^2(M)/GL_n(\mathbb{R})$, and $\pi_1^2 : F^2(M) \to F^1(M)$ be the canonical projection. Notice that every section $\Gamma$ of $F^2_1(M)$ induces a unique natural bundle inclusion:

$$\gamma_\Gamma : F^1(M) \to F^2(M), \quad \pi_1^2 \circ \gamma_\Gamma = id$$

Given a torsion-free affine connection $\nabla$, $\forall x \in M$, the exponential map of $\nabla$ at $x$, denoted as $\exp_{\nabla}^x$ is a map:

$$\exp_{\nabla}^x : U \subset T_x M \to M, \ 0 \mapsto x$$

Here $U$ is an open set of $T_x M$ containing the origin.

One can define a bundle inclusion $i_{\nabla} : F^1(M) \to F^2(M)$ as follows. Any $p \in F^1(M)$ in the fibre of $x$ can be uniquely identified with a linear map $\tilde{p} : \mathbb{R}^n \to T_x M$. Then one defines:

$$i_{\nabla}(p) = J^2(\exp_{\nabla}^x \circ \tilde{p})(0), \quad \forall p \in F^1(M)$$

Let $P(\nabla)$ be the $H^2(n)$ sub-bundle of $F^2(M)$ induced by $i_{\nabla}$. One has from Proposition 12 of [1], $P(\nabla) = P(\nabla)$ if and only if $\nabla$ and $\nabla$ are projectively
equivalent.

In the proof of Proposition 10 of [1], the local coordinates of \( F^2(M) \) defined there gives a 1-1 correspondence between the sections of \( F^2(M) \) and symmetric affine connections on \( M \). One has the following by summarizing Proposition 10 and 11 of [1]:

**Theorem 2.1** (Nagano,Kobayashi[1]). Let \( \theta = \theta_i + \theta^i_j \) be the canonical form on \( F^2(M) \) as usual. There is a 1-1 correspondence between sections of \( F^2(M) \) and symmetric affine connections on \( M \). For a symmetric connection \( \nabla \), denote \( \Gamma \) the corresponding section of \( F^2(M) \), then the following hold:

- The natural bundle inclusion \( \gamma \Gamma \) is exactly \( (i_\nabla)^* \theta^i \).
- \( (i_\nabla)^* \theta^i \) is the canonical form on \( F^1(M) \).
- \( (i_\nabla)^* \theta^i_j \) is the connection form for \( \nabla \).

### 2.2 Dynamics of projective vector fields near singularities

Every projective vector field \( X \) on \( M \) for \( \nabla \) can be uniquely lifted to a vector field \( \tilde{X} \) on \( P = P(\nabla) \) such that \( \mathcal{L}_{\tilde{X}} \omega = 0 \).

**Definition 2.3.** Let \( (M,B,\omega) \) be a Cartan bundle. If \( \tilde{X} \in \chi(B) \) is the lift of some vector field \( X \) on \( M \) such that \( \mathcal{L}_{\tilde{X}} \omega = 0 \), then \( \tilde{X} \) is called an infinitesimal automorphism of the Cartan bundle.

For the flat model \( (\mathbb{RP}^n,G,\omega_G) \), the infinitesimal automorphisms are just right invariant vector fields on \( G \).

Given any torsion-free connection \( \nabla \) on \( M^n \), set \( P = P(\nabla) \), and let \( \omega \) be the normal projective Cartan connection associated to \( P \). Denote \( \pi : P \to M \) the standard projection. If \( X \) vanishes at \( o \in M \), one has \( \forall p \in \pi^{-1}(o) \), \( \omega(\tilde{X})(p) \in \mathfrak{o} \).

One can prove the following local result:

**Proposition 2.1.** Let \( X \) be a projective vector field for \( (M,\nabla) \). Assume \( X_o = 0 \) for some \( o \in M \). Then the following are equivalent:

- \( X \) is linearizable at \( o \).
- There exist a neighbourhood \( U \) of \( o \) and a torsion-free affine connection \( \nabla' \in [\nabla]|_U \) such that \( X \) is an affine vector field for \( \nabla' \).

To prove the proposition above, one needs the following. Denote \( \omega \) the normal projective Cartan connection associated to \( P = P(\nabla) \) as before. Fix any \( p \) in the fibre of \( o \), and let \( \exp_p \) be the exponential map of \( \omega \) at \( p \). Then there is a small neighbourhood \( U \) of \( 0 \in \mathfrak{g} \) such that \( \sigma_p = \pi \circ \exp_p : U \to M \) gives a local coordinate of \( M \) at \( o \). One calls such coordinates the normal coordinates for \( P(\nabla) \). The \( GL_n \) sub-bundle given by \( \sigma_p \) induces an affine connection \( \nabla_U \in [\nabla]|_U \) near \( o \). By Theorem 2.1, \( \sigma_p \) is also the normal coordinate for the affine connection \( \nabla_U \in [\nabla]|_U \) at \( o \).
Lemma 2.1. Suppose $X$ is a projective vector field for $\nabla$ s.t. $X_o = 0$. Let $P = P(\nabla)$, and define $\omega$ on $P$ as before. Choose any $p \in \pi^{-1}(o)$, then in the normal coordinate $\sigma_p$ for $P$ at $p$, the form of $\phi^i$ in the local coordinate $\sigma_p$ is determined by $\omega(\tilde{X})(p)$, regardless of the specific $\omega$ chosen.

Proof. Let $\tilde{X}$ be the lift of $X$ to $P$ such that $\mathcal{L}_{\tilde{X}} \omega = 0$. Because $X_o = 0$, one has $\omega(\tilde{X})(p) = v_h \in \mathfrak{h}$. Define the following identification along fibres over $o$:

$$\Delta : H \rightarrow pH, \quad h \mapsto ph$$

It follows that $\Delta^* \omega|_{\pi^{-1}(o)}$ is the Maurer–Cartan form $\omega_H$ on $H$, by Definition 2.1. Let $X_h$ be a right-invariant vector field on $G$ with $\omega_G(X_h)(1) = v_h$. Note that $\omega_G(X_h)|_H \in \mathfrak{h}$, and $\mathcal{L}_{X_h} \omega_G = 0$. It follows that $\Delta_*(X_h) = X|_{\pi^{-1}(o)}$.

Denote $\Phi$ the flow generated by $\tilde{X}$ on $P$, so $\Phi$ projects to a flow $\phi^i$ on $M$ fixing $o$. One has $\Phi(t, p) = ph(t)$, where the function $h(t) : \mathbb{R} \rightarrow H$ depends only on $v_h$. Fix any $t_0 \in \mathbb{R}$ and $v \in g_{-1} = \mathbb{R}^n$, and define the curve $l(s) = \exp_p(sv)$. Note that $\pi \circ l(s)$ is a geodesic of $[\nabla]$. Because $\mathcal{L}_{\tilde{X}} \omega = 0$, one has the following:

$$l_{t_0}(s) := \Phi(t_0, l(s)) = \exp_{ph(t_0)}(sv)$$

One also obtains:

$$\phi^i_{t_0} \circ \pi \circ l = \pi \circ l_{t_0} = \pi \circ R_{h(t_0)^{-1}} \circ l_{t_0}$$

By the axioms of the Cartan connections, one has that:

$$R_{h(t_0)^{-1}} \circ l_{t_0}(s) = \exp_p(s(Ad(h(t_0))(v)))$$

Define $v' = Ad(h(t_0))(v)$, then $v'$ is totally determined by value of $v$ and $h(t_0)$. One defines the curve:

$$f(s) := R_{h(t_0)^{-1}} \circ l_{t_0}$$

Because $\pi \circ l(s)$ is geodesic of $[\nabla]$, one has $\pi \circ f(s)$ is also a geodesic of $[\nabla]$. Denote $v'_{-1}$ the $g_{-1}$ component of $v'$. One has $\pi \circ f(s)$ and $\pi \circ \exp_p(sv'_{-1})$ are geodesics for $[\nabla]$ with the same initial condition. It follows that on a small interval $I$ containing 0, $f(s) : I \rightarrow P$ can be written in the following form:

$$f(s) = \exp_p(r(s)v'_{-1})g(s), \quad r(s) : I \rightarrow \mathbb{R}, \quad g(s) : I \rightarrow H$$

$$r(0) = 0, \quad g(0) = 1$$

Differentiating the equation, one obtains:

$$v' = \omega(\frac{df}{ds}) = Ad(g(s)^{-1})(r'(s)v'_{-1}) + \omega_H(g'(s))$$

Given a pair of functions $\{r(s), g(s)\}$, whether it is a solution to this equation depends only on $v'$, independent of the connection $\omega$. On the other hand, the
definition of the exponential map implies that the solution \( \{ r(s), g(s) \} \) satisfying the condition \( g(0) = 1 \) and \( r(0) = 0 \) is unique. Note that \( v' \) and \( v'' \) only depend on \( v \) and \( h(t_0) \). It follows from the uniqueness that \( \{ r(s), g(s) \} \) depends only on \( v \) and \( h(t_0) \). In particular, the function \( r(t) \) and \( v' \in \mathbb{R}^n \) depend only on the parameters \( v, v_h, t_0 \), regardless of the connection \( \omega \). Given any two projective connections \( \omega \) and \( \omega' \) on the \( H^2(n) \) bundle \( P \), as long as the parameters \( v, v_h, t_0 \) are the same, one gets the same the function \( r(t) \) and \( v' \in \mathbb{R}^n \). It follows that \( \phi^o \) in the normal coordinate of \( P \) at \( p \) depends only on \( h(t_0) \). This completes the proof.

\[ \square \]

Suppose \( X \) is a projective vector field for \((M, \nabla)\) vanishing at \( o \), and fix any any \( p \in \pi^{-1}(o) \). From above, choose some right invariant vector field \( Y \) on \( G \) such that \( \omega_G(Y)(1) = \omega(X)(p) \in \mathfrak{h} \), and let \( Y \) be the projection of \( \tilde{Y} \) on \( \mathbb{R}P^n \).

Then \( X \) in the normal coordinate of \( P \) at \( p \) has the same form of \( Y \) in the normal coordinate of the flat model at \( 1 \in G \). Note that the projective vector fields have the maximum rank for the flat bundle. Thus by computations on the flat model, one obtains all possible forms of projective vector fields with a singularity at \( o \) in the normal coordinate for \( P \) at \( p \).

**Lemma 2.2.** Let \( X \) be a projective vector field for \((M, \nabla)\) with \( X_o = 0 \). For any \( p \in \pi^{-1}(o) \), \( X \) has the form \( X_x = Ax + \langle w, x \rangle x \) in the normal coordinate of \( P(\nabla) \) at \( p \). In addition, \( X \) is linearizable if and only if \( w \in ImA^T \).

**Proof.** Let \( X \) be a projective vector field for \((M, \nabla)\) such that \( X_o = 0 \), and choose any \( p \in \pi^{-1}(o) \). First one shows \( X \) has the form: \( X_x = Ax + \langle w, x \rangle x \) in the normal coordinate of \( P(\nabla) \) at \( p \). By Lemma 2.1 and the argument in the previous paragraph, one only needs to show for the flat bundle \( P = (\mathbb{R}P^n, G, \omega_G) \), \( X \) is in this form in the normal coordinate at \( p = 1 \in G \). In this case, the exponential map \( \exp_p \) gives the canonical coordinate \( \bar{0}^{-1} \) of \( \mathbb{R}P^n \) near \( e_0 \) defined earlier on Page 4. The projective vector fields fixing \( o = e_0 \in \mathbb{R}P^n \) are induced by linear vector fields in \( \mathbb{R}^{n+1} \) fixing the line \( e_0 \). Projecting these vector fields to \( \mathbb{R}P^n \), one gets \( X \) has the form \( X_x = Ax + \langle w, x \rangle x \) in the normal coordinate at \( p \).

Next one shows \( X \) in this form is linearizable if and only if \( w \in ImA^T \). If \( w \notin ImA^T \), one has \( w = w_k + w' \) with \( w_k \neq 0 \), where \( w_k \in KerA \) and \( w' \in ImA^T \). Denote \( \phi^t \) the flow generated by \( X \) as usual. In the normal coordinate for \( P(\nabla) \) at \( p \), one has for some small interval \( I \) containing 0:

\[
\phi^t(sw_k) = \frac{s}{1 + tas}w_k, \quad s \in I, \quad a \neq 0
\]

Note that \( D\phi^t(w_k) = w_k \neq 0 \). Without loss of generality, one can assume \( a > 0 \). For \( s > 0 \), one has \( \frac{s}{1 + tas} \rightarrow 0 \) as \( t \rightarrow +\infty \). Then \( X \) is not linearizable by Lemma 4.6 of [12]. Conversely, if \( w \in ImA^T \), the calculation in Remark 1 below shows that one can find \( p' \in \pi^{-1}(o) \) such that \( X_{x'} = (A_p)x \) in the normal coordinate at \( p' \). Hence it is linearizable.

\[ \square \]
Proof. Let Corollary 2.1. Suppose that in the normal coordinate \( \sigma \) vanishes at \( o \). One can assume that \( \sigma \) is well defined on \( U \). For each \( a > 0 \), respectively. It follows that \( \sigma \) is a normal coordinate of \( \nabla' \) at \( o \). The converse is trivial as affine vector fields of \( \nabla' \) vanishing at \( o \) are clearly linear in the normal coordinates of \( \nabla \) at \( o \).

\[ J^1(\hat{\sigma})(0) = J^1(\sigma_\hat{p})(0), \quad (\sigma_\hat{p})^{-1}, X_0 = Ax + \langle w, x \rangle x, \ w \in KerA \]

With the results above, one can prove Proposition 2.1.

Proof of Proposition 2.1. By remark 1, one can always choose some \( p \in \pi^{-1}(o) \) such that in the normal coordinate \( \sigma_p \) of \( P(\nabla) \) at \( p \), \( X \) has the following form:

\[ X_x = Ax + \langle w, x \rangle x, \ w \in KerA \]

If \( X \) is linearizable at \( o \), one has \( w \in ImA^T \) by Lemma 2.2. It follows that \( w = 0 \), then one has \( X \) is linear in \( \sigma_p \). According to Theorem 2.1 by Nagano and Kobayashi, the local section of \( F_\pi^2(M) \) induced by the local section \( exp_p(g_{-1}) \) corresponds to a connection \( \nabla' \) projectively equivalent to \( \nabla \) locally defined near \( o \). From the last statement of Theorem 2.1, it is clear that \( \sigma_p \) is a normal coordinate of \( \nabla' \) at \( o \). The converse is trivial as affine vector fields of \( \nabla' \) vanishing at \( o \) are clearly linear in the normal coordinates of \( \nabla \) at \( o \).

Suppose that \( X \) is a non-linearizable projective vector field for \( (M, \nabla) \) vanishing at \( o \in M \). For each \( a > 0 \), one can choose a neighbourhood \( U_a \) of \( o \) such that \( \phi \) is well defined on \( U_a \) for \( t \in I = [-a, a] \). One has on \( U_a \), \( \nabla_t = \phi^t \nabla \) is projectively equivalent to \( \nabla \) for \( t \in I \). If \( \gamma(s) \) is a geodesic segment for \( \nabla \) contained in \( \phi^t(U_a) \) with \( t_0 \in I \), one has \( \phi^{-1} \gamma(s) \) is a geodesic segment on \( U_a \) for \( \nabla_{t_0} \). This leads to the following:

Corollary 2.1.1. Let \( X \) be a projective vector field for \( (M, \nabla) \) admitting a non-linearizable vanishing point \( o \in M \). One has for each \( t \neq 0 \),

\[ \nabla_t = \nabla + \eta_t \otimes Id + Id \otimes \eta_t, \quad (\eta_t)_{o} \neq 0 \]

Proof. Suppose that \( \eta_{t_0}(o) = 0 \) for some \( t_0 \neq 0 \). The connection \( \nabla \) induces a \( GL_n \) sub-bundle \( P_1 \) of \( P(\nabla) \). Choose \( p \in \pi^{-1}(o) \cap P_1 \). Let \( \nabla_p \) be the connection induced by the local section \( exp_p(g_{-1}) \) at \( p \). One has the type \( (2,1) \)-tensor \( (\nabla_p - \nabla) \) vanishes at \( o \). One can assume that \( \nabla = \nabla_p \) in this proof. In the normal coordinate of \( \nabla \) at \( o \), denote \( \Gamma^k_{i,j} \) and \( \Gamma^k_{i,j} \) the Christoffel symbols of \( \nabla \) and \( \nabla_{t_0} \), respectively. It follows that \( \Gamma^k_{i,j}(o) = \Gamma^k_{i,j}(o) = 0 \), because of \( (\eta_t)_{o} = 0 \).
By calculations of the proof of Theorem 2.1 of Nagano and Kobayashi in [1], one has the exponential maps of $\nabla$ and $\nabla_{t_0}$ at $o$ have the same 2-jet. Denote $\exp^\nabla_o$ the exponential map of $\nabla$ at $o$. Take the normal coordinate $\sigma_p$ for $P(\nabla)$ at $p$, and note that $\sigma_p$ gives an identification $\mathbb{R}^n \simeq T_oM$. One has the following map near $0 \in \mathbb{R}^n$:

$$\sigma_p^{-1} \circ \exp^\nabla_o : 0 \in U \subset T_oM \simeq \mathbb{R}^n \to \mathbb{R}^n$$

The map above is a diffeomorphism fixing $0 \in \mathbb{R}^n$. It has the trivial 2-jet $1 \in H^2(n)$ at 0, since $\sigma_p$ is also a normal coordinate for $\nabla$ at $o$. Because $\phi^{-t_0}$ takes geodesics of $\nabla$ to geodesics of $\nabla_{t_0}$, one has the exponential map of $\nabla_{t_0}$ at $o$ is:

$$\phi^{-t_0} \circ \exp^\nabla_o \circ D\phi^{t_0}(o)$$

Then in the local coordinate $\sigma_p$, one has $J^2(\phi^{-t_0} \circ D\phi^{t_0})(o) = 1 \in H^2(n)$. But $X$ is not linearizable at $o$ implies $X$ has the following form in $\sigma_p$:

$$X_x = Ax + \langle w_k + w_i, x \rangle x, \ 0 \neq w_k \in Ker A$$

The condition $w_k \neq 0$ implies that $\phi^{-t_0} \circ D\phi^{t_0}$ is not identity in $\sigma_p$. Then in $\sigma_p$ it has the form of a non-trivial projective map in $\mathbb{R}^n$ with $J^1(\phi^{-t_0} \circ D\phi^{t_0})(0) = Id$. Hence it has the following form in the coordinate $\sigma_p$:

$$x \mapsto \frac{1}{1 + \langle u, x \rangle} x, \ 0 \neq u \in \mathbb{R}^n$$

It follows that for $t_0 \neq 0$, $J^2(\phi^{-t_0} \circ D\phi^{t_0})(o) \neq 1 \in H^2(n)$. One obtains a contradiction. \hfill \Box

### 2.3 BM-structures and Degree of mobility

In general, there is a 1-1 correspondence between the elements in a given projective class $[\nabla]$ on the manifold $M^n$ and the 1-forms on $M^n$. The latter is an infinite dimension vector space, and is hard to analyse. So our focus is to study the metrizable elements of $[\nabla]$, where $\nabla$ is a Levi-Civita connection. From now on, let $g$ be a semi-Riemannian metric on $M^n$, and denote $\nabla$ its Levi-Civita connection.

Fix a metric $g$ on $M$. Then for any metric $\overline{g}$ on $M$, the $g$-strength of $\overline{g}$ is defined to be the $(1,1)$-tensor $K_\overline{g}$ such that

$$\overline{g}(u, v) = g \left( \frac{K_{\overline{g}}^{-1}}{|\det(K_{\overline{g}})|} \cdot u, v \right)$$

One defines map $\rho(g)$ from the space of metrics on $M$ to the space of non-degenerate $g$-adjoint $(1,1)$-tensors on $M$ as follows:

$$\rho(g)(\overline{g}) = K_{\overline{g}}$$

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Clearly $\rho(g)$ is a bijection from the metrics on $M$ to the non-degenerate $g$-adjoint $(1,1)$-tensors on $M$.

Let $f : M^n \to N^n$ be a smooth embedding. Fix metrics $g_1$ on $M$ and $g_2$ on $N$, respectively. One defines the linear map $\rho^f(g_1, g_2) : T^{1,1}_N \to T^{1,1}_M$ by the following:

$$\rho^f(g_1, g_2)(T) = f_* T \circ (\rho(g_1)(f^* g_2))$$

Analogous to Fact 2.1 of [5], if $g'_2$ is a metric on $N$, one has:

$$\rho^f(g_1, g_2)(\rho(g_2)(g'_2)) = \rho(g_1)(f^* g'_2)$$

The map defined above is multiplicative in the following sense: Let $f_1 : N_1 \to N_2$, and $f_2 : N_2 \to N_3$ be smooth embeddings. Fix metrics $g_i$ on $N_i$, then one has:

$$\rho^{f_1 \circ f_2}(g_1, g_3) = \rho^{f_1}(g_1, g_2) \circ \rho^{f_2}(g_2, g_3)$$

To proceed, one needs the following definition from Section 2 of [4]:

**Definition 2.4.** Suppose $g$ is a metric on $M^n$, the space of BM-structures on $M$ for $g$, denoted as $B(M, g)$, is the space of $g$-adjoint $(1,1)$-tensors on $M$ satisfying the following linear PDE, $\forall u, v, w \in T_x M$, $\forall x \in M$:

$$g((\nabla_w K)u, v) = \frac{1}{2}(d(trK)(u)g(v, w) + d(trK)(v)g(u, w))$$

The degree of mobility of $g$ on $M^n$, denoted as $D(M^n, g)$, is the dimension of the vector space $B(M^n, g)$.

According to Equation (7)-(9) of [3], the non-degenerate elements of $B(M, g)$ are exactly the $g$-strength of the metrics projectively equivalent to $g$ on $M$. Equation (4) is finite-type by Remark 5 of [3], so the solutions on each connected component are uniquely determined by the $k$-th jet at a single point for some $k \in \mathbb{N}$. Then one always has $D(M^n, g) < \infty$. In fact, according to Section 3 of [7], $\nabla$ defines a linear connection on some vector bundle $VM \simeq \bigoplus^2 TM \oplus TM \oplus C^\infty(M)$. By Theorem 3.1 of [7], solutions to Equation (4) are 1-1 correspondence with parallel sections on $VM$. From Page 1 of [6], if $M^n$ is connected, one always has:

$$D(M^n, g) \leq \frac{(n+1)(n+2)}{2}$$

From now, assume $M$ is connected. Let $U$ an open subset of $M$. One has $\forall K \in B(M, g)$, $K|_U \in B(U, g)$. The following restriction map is injective, since $M$ is connected.

$$R_U : B(M, g) \to B(U, g), \quad K \mapsto K|_U$$

One can view $B(M, g)$ as a linear subspace of $B(U, g)$. Suppose $X$ is a projective vector field for $(M^n, g)$, and denote $\phi^t$ the flow generated by $X$. Further assume
that \( \exists a > 0 \) such that \( \phi^t(x) \) is defined for \( \forall x \in U \), and \( \forall t \in I = [-a,a] \). Then the flow \( \phi^t \) induces a well defined 1-parameter family of maps \( L_t : B(M,g) \to B(U,g) \) for \( t \in I \) as follows. Fix any \( x \in U \) and \( t \in I \). Suppose \( \mathcal{F} \) is a metric defined on some neighbourhood \( V_t \) of \( \phi^t(x) \) such that \( \mathcal{F} \) and \( g \) are projectively equivalent on \( V_t \). One has near \( x \), \( \left( \phi^t \right)^* \mathcal{F} \) is a metric projectively equivalent to \( g \). Denote \( K_t \) the \( g \)-strength of \( \left( \phi^t \right)^* g \), so it is well defined on \( U \). One has near \( x \), \( \rho^g \circ \phi_t = \phi_t^* \mathcal{F} \circ K_t \) is a solution to Equation (4) near \( x \). For any \( y \in M \), one can always choose a neighbourhood \( U_y \) of \( y \) such that \( B(U_y,g) \) has a basis consisting of non-degenerate elements. This implies for each \( t \in I \), \( \rho^g \circ \phi_t \) defines a linear map \( L_t : B(M,g) \to B(U,g) \) by \( L_t(K') = \rho^g(\phi_t^* g)(K') \).

If one further assumes that \( D(U,g) = D(M,g) \), every \( K' \in B(U,g) \) can be uniquely extended to an element in \( B(M,g) \). To simplify the notation, define \( B = B(M,g) \). Then one can take \( L_t \) as a map \( L_t : B \to B \) for each \( t \in I \). A natural question to ask is whether \( L_t \) can be extended to a 1-parameter subgroup of \( GL(B) \). This leads to the following lemma.

**Lemma 2.3.** Let \( (M^n,g) \) be connected with a projective vector field \( X \). Suppose \( X \) vanishes at \( o \in M \). Assume that \( U \) with \( D(U,g) = D(M,g) \) is a connected open set containing \( o \) such that \( \phi^t \) is defined on \( U \) for \( t \in I = [-a,a] \) for some \( a > 0 \). Then the map \( L_t : B \to B \) defined in the previous paragraph satisfies the following:

- \( L_{t+s} = L_t \circ L_s \) for \( t,s,t+s \in I \).
- The representation \( L_t : I \to GL(B) \) is continuous in \( t \).

In other words, \( L_t \) can be extended to a 1-parameter subgroup of \( GL(B) \).

**Proof.** Fix any \( K' \in B = B(M,g) \). For any \( t \in I \), \( L_t(K') \) is the unique element in \( B(M,g) \) such that:

\[
L_t(K')|_U = \phi^t(X)(K') \circ K_t \in B(U,g)
\]

Note that given the embedding \( \phi^t : U \to M \), one has on \( U \):

\[
L_t(K')|_U = \rho^g(\phi_t^* g)(K')
\]

The embedding \( \phi^x : U \to M \) gives:

\[
L_s(L_t(K'))|_U = \rho^g(\phi_t^* g)(L_t(K'))
\]

Because \( X \) vanishes at \( o \), there is some neighbourhood \( U_o \) of \( o \) such that \( \phi^x(U_o) \subset U \). Then one has the sequence of embeddings:

\[
U_o \xrightarrow{\phi^x} U \xrightarrow{\phi^t} M
\]
Because \( t,s,t+s \in I \), by Equation (3), one has on \( U_o \):

\[
L_t(L_t(K'))|_{U_o} = \left( \rho^{o^t}(g,g) \circ \rho^{o^s}(g,g) \right)(K')
\]

(5)

\[
= \rho^{o^{t+s}}(g,g)(K')
\]

(6)

\[
= L_{t+s}(K')|_{U_o}
\]

(7)

Since \( U \) is connected, any BM-structure on \( U \) is uniquely determined by its k-th jet at \( o \) for some \( k \geq 0 \). Then \( L_{t+s}(K') = L_s \circ L_t(K') \) on \( U_o \) implies \( L_{t+s}(K') = L_s \circ L_t(K') \) in \( B \).

Next one shows the representation \( L_t : I \to GL(B) \) is continuous in \( t \). Because \( L_t \) is linear for each \( t \) and \( B \) is a finite dimensional vector space, it is sufficient to show for any fixed \( K' \in B \), \( L_t(K') \) is continuous in \( t \). Fix a compact neighbourhood \( V_o \subset U \) of \( o \) and a basis \( \{ K^i \} \) for \( B \). Then one can write \( L_t(K') = \sum c_i(t) K^i \), where \( c_i : I \to \mathbb{R} \). By the fact Equation (4) is of finite type, \( \{ K^i \} \) are linearly independent over \( V_o \). One has on \( U \supset V_o \), \( L_t(K') = \phi_t^0(K') \circ K_t \). Then for any fixed \( t_0 \in I \), one has as \( t \to t_0 \), \( L_t(K') \to L_{t_0}(K') \) uniformly on \( V_o \). It follows that as \( t \to t_0 \), \( c_i(t) \to c_i(t_0) \) for each \( i \). This proves the continuity of \( L_t : I \to GL(B) \).

The following shows the neighbourhood \( U \) in Lemma 2.3 always exists.

**Lemma 2.4.** Let \((M^n, g)\) be a connected manifold. Suppose \( X \) is a projective vector field for \( g \) vanishing at \( o \in M \). Then there exists a connected open set \( U \) containing \( o \) such that \( D(U, g) = D(M^n, g) \). The open set \( U \) also has the following property: \( \exists a > 0 \) such that \( \phi^t \) is well defined on \( U \) for \( t \in I = [-a,a] \).

**Proof.** Define the following sets:

\[
S_i = \{ x \in M : \phi^t(x) \text{ is well defined for } t \in \left[-\frac{1}{i}, \frac{1}{i}\right] \}
\]

Without loss of generality, one can assume \( o \in Int(S_i) \) for all \( i \). Let \( U_i \) be the component of \( Int(S_i) \) containing \( o \). Since each \( U_i \) is open and connected, it is also path connected. Given any \( x \in U_i \), let \( \gamma_x \) be a curve in \( U_i \) joining \( o \) and \( x \). Then clearly one has \( \gamma_x \subset Int(S_{i+1}) \). It follows that \( U_i \subset U_{i+1} \). Similarly, given any \( x \in M \), one can choose a curve \( \gamma'_x \) in \( M \) joining \( o \) and \( x \). Then there exists \( \epsilon > 0 \) and a neighbourhood \( U_\epsilon \) of \( \gamma'_x \) such that \( \phi^t \) is well defined on \( U_\epsilon \) for \( t \in [-\epsilon, \epsilon] \). It follows that \( x \in U_i \) for some \( i \), hence \( \bigcup_{i=1}^{\infty} U_i = M \). One has an increasing sequence of open sets containing \( o \):

\[
o \in U_1 \subset U_2 \subset \cdots , \bigcup_{i=1}^{\infty} U_i = M
\]

Because each \( U_i \) is connected, the restriction map gives a sequence of injective linear maps:

\[
B(U_1, g) \xrightarrow{\ell_1} B(U_2, g) \xrightarrow{\ell_2} \cdots
\]
One has that $D(U,g) \geq D(M,g)$, and $D(U,g) < \infty$. It follows that there exists some $i_0$ such that $r_j : B(U_{j+1}, g) \to B(U_j, g)$ are linear isomorphisms for all $j \geq i_0$. Then any $\tilde{K} \in B(U_{i_0}, g)$ can be uniquely extended to an element in $B(U_j, g)$ for all $j \geq i_0$. Because a BM-structure on a connected manifold is uniquely determined by its finite jet at some point, one has $\tilde{K}$ can be extended to an element in $B(M,g)$. Hence one has $D(U_{i_0}, g) = D(M, g)$. This completes the proof. □

Let $U$ be constructed by the lemma above. The map $L_t$ can be extended to a 1-parameter subgroup of $GL(B)$, also denoted as $L_t$. By the following, one can see the construction is in fact coherent.

**Corollary 2.1.2.** Let $X$ be a projective vector field for $(M, g)$ vanishing at $o$. Suppose $M$ is connected. Let $U, I, and L_t$ be constructed as above. Given any $t_0 \in \mathbb{R}$, there exists some neighbourhood $V_{t_0}$ of $o$ such that $\phi^t$ is well defined for $|t| \leq |t_0|$. Also on $V_{t_0}$, one has $L_{t_0}(K')|_{V_{t_0}} = \phi_{t_0}^i (K') \circ K_{t_0}$.

**Proof.** Without loss of generality, assume $t_0 > 0$. Let $U, I$ be the same as in Lemma 2.3, and $t_0 = nt_1$ with $t_1 \in I$. Given any $K' \in B \simeq B(U, g)$ and $t \in I$, there is some neighbourhood $V_t$ of $o$ such that $\phi^t(V_t) \subset U$. One has in particular, $L_{t_1}(K')|_{V_t} = \phi_{t_1}^i (K') \circ K_{t_1}$. Assume there is some neighbourhood $V_{mt_1} \subset U$ of $o$ such that $\phi^s(V_{mt_1})$ is defined for $s \in [−mt_1, mt_1]$ such that

$$L_{mt_1}(K')|_{V_{mt_1}} = \phi_{mt_1}^s (K') \circ K_{mt_1}$$

One can choose some $V_{(m+1)t_1}$ such that

$$o \in V_{(m+1)t_1} \subset V_{mt_1} \subset U, \quad \phi^t (V_{(m+1)t_1}) \subset V_{mt_1} \text{ for } t' \in I$$

One has $\phi^s$ is well defined on $V_{(m+1)t_1}$ for $s \in [−(m+1)t_1, (m+1)t_1]$. This implies on $V_{(m+1)t_1}:

$$L_{(m+1)t_1}(K')|_{V_{(m+1)t_1}} = L_{t_1}(L_{mt_1}(K'))|_{V_{(m+1)t_1}} = \phi_{t_1}^i (L_{mt_1}(K')) \circ K_{t_1}$$

$$= \phi_{t_1}^i (\phi_{mt_1}^s (K') \circ K_{mt_1}) \circ K_{t_1}$$

$$= \phi_{(m+1)t_1}^s (K') \circ K_{(m+1)t_1}$$

By induction, one has on $V_{t_0} = V_{mt_1}, L_{t_0}(K')|_{V_{t_0}} = \phi_{t_0}^i (K') \circ K_{t_0}$. □

## 3 Local results and General theory when $D(M, g)$ is 2

Let $(M^n, g)$ be a connected manifold with $D(M, g) = 2$. Let $X$ be a projective vector field for $g$ with a singularity $o$. Denote $\phi^t$ the flow generated by $X$. Suppose $X$ is not linearizable at $o$. Then $L_t$ is a 1-parameter subgroup of
Moreover, the fixed set of the Mobius map  

\[ L_t(K') = \phi_t^s(K') \circ K_t \]

In particular on \( V_t \), one has \( L_t(\text{Id}) = K_t \). By Corollary 2.1.1, one has for any \( t \neq 0, g_t \) and \( g \) are not affine equivalent on any neighbourhood of \( o \). This implies the eigenfunctions of \( K_t \) are not all constant on any neighbourhood of \( o \). Otherwise by Equation (4), one has \( \nabla K_t = 0 \) near \( o \), then \( g_t \) and \( g \) are affine equivalent near \( o \). If \( L_t \) is elliptic, one has \( \exists t_0 \neq 0 \) such that \( K_{t_0} = L_{t_0}(\text{Id}) = \text{Id} \). It follows that \( L_t \) cannot be an elliptic 1-parameter subgroup of \( GL(B) \). One could prove \( L_t \) is indeed parabolic.

**Theorem 3.1.** Let \( (M^n, g) \) be a connected semi-Riemannian manifold with \( D(M, g) = 2 \). Let \( X \) be a projective vector field for \( g \) vanishing at \( o \). Suppose \( X \) is not linearizable at \( o \in M \), then \( L_t \) is a 1-parameter parabolic subgroup of \( GL(B) \).

The idea of the proof of Theorem 3.1 follows from [5] by Zeghib. Before proving the theorem, one makes the following observation. Let \( U, I, L_t \) be as before. Fix any \( t_0 \neq 0 \), one has \( \{L_{t_0}(\text{Id}), \text{Id}\} \) is a basis for \( B \). Write \( \mathcal{K} \) for \( L_{t_0}(\text{Id}) \) for simplicity. Analogous to Equation 4.2.1 of [5], one can write:

\[ L_{t_0}(\mathcal{K}) = \alpha \mathcal{K} + \beta \text{Id}, \quad L_{t_0}(\text{Id}) = \mathcal{K} \quad (12) \]

As in Section 4.2 of [5], one defines the associated Mobius map

\[ T : \mathbb{C} \to \hat{\mathbb{C}}, \quad T(z) = \frac{az + b}{z} \]

Now further assume \( t_0 \in I \), then one has \( \mathcal{K}|_U = K_{t_0} \). One has for \( x \in U \):

\[ (\alpha \mathcal{K} + \beta \text{Id})_x = (L_{t_0}(\mathcal{K}))_x = (\phi_t^{s_0}(\mathcal{K}) \circ K_{t_0})_x = (\phi_t^{s_0}(\mathcal{K}))_x \circ K_x \quad (13) \]

For \( x \in U \), one has \( \det(\mathcal{K}_x) = \det((K_{t_0})_x) \neq 0 \). This gives the following:

\[ (\phi_t^{s_0}(\mathcal{K}))_x = (D\phi_t^{s_0})^{-1} \mathcal{K}_{\phi_t^{s_0}(x)} D\phi_t^{s_0} = (\alpha \text{Id} + \beta \mathcal{K}^{-1})_x \]

Note the right hand side is \( (T(\mathcal{K}))[x] \). It follows that \( \mathcal{K}_{\phi_t^{s_0}(x)} \) and \( (T(\mathcal{K}))[x] \) have the same Jordan form. One gets for \( x \in U \):

\[ T(\text{Spec}(\mathcal{K}_x)) = \text{Spec}(\mathcal{K}_{\phi_t^{s_0}(x)}) \quad (14) \]

To prove Theorem 3.1, one also needs the following lemma.

**Lemma 3.1.** Suppose \( L_t \) is induced by a projective vector field admitting a non-linearizable vanishing point \( o \in M \). Fix any \( t_0 \neq 0 \), and define \( \mathcal{K} \) and \( T \) as before. Note that \( L_t \) defines a non-trivial 1-parameter parabolic or hyperbolic subgroup of \( PGL(B) \) acting on \( \mathbb{P}(B) \). Its fixed set on \( \mathbb{P}(B) \) is exactly the following:

\[ D_o = \{[\mathcal{K} - r\text{Id}] : r \in \text{Spec}(\mathcal{K}_o) \cap \mathbb{R}\} \]

Moreover, the fixed set of the Mobius map \( T \) on \( \hat{\mathbb{C}} \) is exactly \( \text{Spec}(\mathcal{K}_o) \).
Proof. By the argument in the first paragraph of this section, one knows \( L_t \) is either hyperbolic or parabolic. Then for any \( t_0 \neq 0 \), the fixed set of \( L_{t_0} \) on \( \mathbb{P}(B) \) is the fixed set of \( L_t \) on \( \mathbb{P}(B) \). It is clearly non-empty. For any fixed \( t_0 \neq 0 \), one has by Corollary 2.1.2, there is a neighbourhood \( V \) of \( o \) such that

\[
L_{t_0}(K')|_V = \phi^{t_0}_*(K') \circ K_{t_0}, \forall K' \in B
\]

Then \((L_{t_0}(K'))_o\) is degenerate if and only if \((K')_o\) is degenerate. This implies \( L_{t_0} \) takes \( D_o \subset \mathbb{P}(B) \) to itself. Because \( D_o \) is a finite discrete subset of \( \mathbb{P}(B) \), one has \( L_t \) fixes all elements in \( D_o \).

Suppose that there is some \([\overline{K} - r_0 Id] \notin D_o \) fixed by \( L_t \), and one seeks a contradiction. Let \( K^1 = \overline{K} - \epsilon o [Id] \), then one has \( L_t(K^1) = e^{rt}K^1 \), for some \( c \in \mathbb{R} \). One has \( K^1 \) is non-degenerate near \( o \). Hence \( K^1 \) defines a metric \( g_{K^1} \) projectively equivalent to \( g \) on some neighbourhood \( V_o \subset U \) of \( o \). Because \( L_t(K^1)|_U = \phi^t(K^1) \circ K_t \) for \( t \in I \), one has \( X \) is a homothetic vector field for \( g_{K^1} \). This is impossible. Also note that \( L_t \) does not fix the line \([Id]\), otherwise it is a homothetic vector field for \( g \). This proves the fixed set of \( L_t \) on \( \mathbb{P}(B) \) is exactly \( D_o \).

For any fixed \( t_0 \neq 0 \), the associated Mobius map is of the form \( T(z) = \frac{az + \beta}{z} \).

Under the basis \( \{\overline{K}, Id\} \), \( L_{t_0} \) has the following matrix representation:

\[
\begin{bmatrix}
\alpha & 1 \\
\beta & 0
\end{bmatrix}
\]

Denote \( F(T) \) the fixed set of \( T \) on \( \widehat{\mathbb{C}} \). Then \( L_{t_0} \) fixes exactly \( D_o \) implies \( F(T) \cap \mathbb{R} \) is exactly \( \text{Spec}(\overline{K}) \cap \mathbb{R} \). Then the equation \( z^2 = \alpha z + \beta \) has 1 or 2 distinct real root. In either case, one has \( F(T) \) has to be a subset of \( \mathbb{R} \), so one gets \( F(T) = \text{Spec}(\overline{K}) \cap \mathbb{R} \). In addition, the finite subsets of \( \widehat{\mathbb{C}} \) preserved by \( T \) are subsets of \( F(T) \). According to Equation (14), one has \( \text{Spec}(\overline{K}_o) \) is a finite set fixed by \( T \). It follows that \( F(T) = \text{Spec}(\overline{K}) \). This completes the proof.  

Now one can prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.1, \( L_t \) is either hyperbolic or parabolic. Suppose \( L_t \) is hyperbolic. Choose \( 0 \neq t_0 \in I \), then one has \( K_{t_0} \) is the \( g \)-strength of \( g_{t_0} \) on \( U \). Denote \( \nabla \) the Levi-Civita connection for \( g \). Let \( P = P(\nabla) \) be the projective Cartan bundle for \( \nabla \). Then \( \nabla \) induces a \( GL_n \) sub-bundle \( \Gamma \) of \( P \). Choose \( p \in \Gamma \cap \pi^{-1}(o) \). The section given by \( \exp_{p}(g_{-1}) \) locally defines a symmetric affine connection \( \overline{\nabla} \in [\nabla|_V] \) on some neighbourhood \( V \) of \( o \). Let \( \sigma_p \) be the normal coordinate of \( P \) at \( p \). Clearly by Theorem 2.1, \( \sigma_p \) is a normal coordinate of \( \overline{\nabla} \) at \( o \). Because \( X \) is not linearizable at \( o \), one has by Lemma 2.2, \( (\sigma_p)^{-1}X \) has the following form:

\[
X_x = Ax + \langle w, x \rangle x, \quad w \notin \text{Im}(A^T)
\]
Choose \( v \in \text{Ker}A \) such that \( \langle w, v \rangle \neq 0 \). One has in the local coordinate \( \sigma_p \), there exists \( a \neq 0 \) and \( \epsilon > 0 \) such that

\[
\phi^t(\gamma v) = \left( \frac{y}{1 + tay} \right) v, \; y \in (-\epsilon, \epsilon), \; t \in I
\] (15)

Let \( \gamma(s) \) and \( \gamma(s(y)) \) be geodesics with initial vector \( (\sigma_p)_* v \) for \( \nabla \) and \( \nabla \), respectively. Denote \( E : T_oM \to M \) and \( \overline{E} : T_oM \to M \) the exponential maps for \( \nabla \) and \( \nabla \) at \( o \), respectively. From Theorem 2.1 by Nagano and Kobayashi, one has \( J^2(E)(0) = J^2(\overline{E})(0) \), because one has \( p \in \Gamma \cap \pi^{-1}(o) \). Then one has:

\[
\frac{ds}{dy}(0) = 1, \quad \frac{d^2s}{dy^2}(0) = 0
\] (16)

Note that \( \phi^t \) preserves the unparametrized geodesic given by \( \gamma \). Then for small \( s \), one can define a parametrized family of functions \( \tau_t \) with \( \tau_t(0) = 0 \) for \( t \in I \) by the following:

\[
\phi^t \circ \gamma(s) = \gamma(\tau_t(s))
\]

Let \( \tau = \tau_0 \) for simplicity. From Equation (15), one also has \( \frac{d\tau}{ds}(0) = 1 \). As in Equation (5) of [3], one defines the function:

\[
\psi(s) = \left( 1 - \frac{1}{2} \log(\det(K_{t0})) \right)(\gamma(s))
\]

Then for small \( s \), one has by Equation (2) and (3) of [3]:

\[
\frac{d\psi}{ds} = \frac{1}{2} \frac{d}{ds}(\log(\frac{d\tau}{ds}))
\]

It follows that \( \frac{d\psi}{ds}(0) = \frac{1}{2} \frac{d^2\tau}{ds^2}(0) \). According to Lemma 3.1, \( \text{Spec}((K_{t0})_o) = \{\lambda_u, \lambda_b\} \subset \mathbb{R} \). Here \( \lambda_u, \lambda_b \) are the unstable and stable fixed point of the associated Mobius map \( T(z) = \frac{az + \beta}{2} \), respectively. One can apply the Splitting Lemma from [8]. On some neighbourhood \( V' \subset V \) of \( o \), there is a smooth local coordinate in which \( K_{t0} \) can be written in the following block-diagonal form:

\[
K_{t0} = \begin{bmatrix} K_u & 0 \\ 0 & K_b \end{bmatrix}, \quad \text{Spec}((K_u)_o) = \{\lambda_u\}, \; \text{Spec}((K_b)_o) = \{\lambda_b\}
\]

One may choose \( V' \) small enough so that \( \text{Spec}(K_u)|_{V'} \subset D_u \), and \( \text{Spec}(K_b)|_{V'} \subset D_b \). Here \( D_u, D_b \) are 2 disjoint disks in \( \mathbb{C} \) centered at \( \lambda_u, \lambda_b \), respectively. It follows that:

\[
\frac{d\psi}{ds} = \frac{1}{2} \log(\det(K_u))(\gamma(s)) + \frac{1}{2} \log(\det(K_b))(\gamma(s))
\] (17)

Define \( f_u(s) = \det(K_u)(\gamma(s)) \), and \( f_b(s) = \det(K_b)(\gamma(s)) \). Without loss of generality, let us assume \( t_0a > 0 \). From Equation (15), for small \( s > 0 \), one has
τ(s) < s, and \( \phi^{m_0}(\gamma(s)) \to \circ \) as \( m \to +\infty \). One chooses the eigenfunctions of \( K_u \) and \( K_b \) to be continuous on \( V' \), then one can show the eigenfunctions of \( K_u \) have to be constant on \( \gamma(s) \) for small \( s > 0 \). Suppose this is not the case. Let \( \tilde{k}_u \) be an eigenfunction of \( K_u \), and write \( k_u(s) = \tilde{k}_u(\gamma(s)) \). Then there is some \( s_0 > 0 \) such that \( \gamma([0, s_0]) \subset V \), \( k_u(s_0) \neq \lambda_u \).

One chooses the eigenfunctions of \( K_u \) and \( K_b \) to be continuous on \( V' \), then one can show the eigenfunctions of \( K_u \) have to be constant on \( \gamma(s) \) for small \( s > 0 \). Suppose this is not the case. Let \( \tilde{k}_u \) be an eigenfunction of \( K_u \), and write \( k_u(s) = \tilde{k}_u(\gamma(s)) \). Then there is some \( s_0 > 0 \) such that \( \gamma([0, s_0]) \subset V \), \( k_u(s_0) \neq \lambda_u \).

One has \( T \) is a continuous map on \( \hat{C} \). Therefore, \( T^m \circ k_u : [0, s_0] \to \hat{C} \) is a continuous map for each \( m \). For large \( m \), one has \( T^m(k_u(s_0)) \in D_b \). On the other hand, for any \( s' \in [0, s_0] \) one has:

\[ T^m(k_u(s')) \in \text{Spec}(\phi^{m_0} \circ \gamma(s')) \subset D_u \cup D_b \]

Because \( T^m(k_u(0)) = \lambda_u \) for all \( m \), one has \( T^m \circ k_u([0, s_0]) \) is not connected for large \( m \). This contradicts the continuity.

The above implies \( f_u(s) \) is constant for small \( s \geq 0 \). Similarly, one can prove \( f_b(s) \) is constant for small \( s \leq 0 \). From Equation (17), one has \( \frac{d\psi}{ds}(0) = 0 \). It follows that:

\[ \frac{d^2\tau}{ds^2}(0) = 0 \]

Define the Mobius map \( \hat{T}(y) = \frac{y}{1 + t_0ay} \). From Equation (15), one has near 0:

\[ \tau \circ s = s \circ \hat{T}(y) \]

By Equation (16), one has \( J^2(\tau)(0) = J^2(\hat{T})(0) \). This gives \( \frac{d^2}{dy^2}(\hat{T})(0) = 0 \), which is clear impossible because \( t_0a \neq 0 \). One obtains a contradiction. Hence \( L_t \) can only be a 1-parameter parabolic subgroup of \( GL(B) \).

4 Global results when \( (M^n, g) \) is closed or Riemannian

4.1 Result for the case \( g \) is Riemannian, proof of Theorem 1.3

In this section, one gives the proof of Theorem 1.3 stated in the introduction.

Before prove the theorem, one makes the following observations. Let \( (M^n, g) \) with \( n \geq 3 \) be a connected Riemannian manifold. One has that \( \forall K' \in B(M, g) \), \( K' \) is real diagonalizable, because it is a self-adjoint operator for the Riemannian metric \( g \). Let \( U, I, L_t \) be as before. Fix any \( 0 \neq t_0 \in I \), by Lemma 3.1, \( (K_{t_0})_o \) has only 1 real eigenvalue \( \lambda > 0 \). One has \( (K_{t_0})_o = \lambda Id \). Because \( X \) is not
linearizable at $o$, one has by Lemma 2.2, $(D\phi^t)_o$ fixes some non-zero $v \in T_oM$. It follows that:

$$g(v, v) = g_{o}(v, v) = \frac{1}{\det((K_{t_o})_o)}g((K_{t_o})_o^{-1}v, v)$$

One has $\lambda = 1$, and $(K_{t_o})_o = Id$. By Lemma 3.1, the associated Mobius map for $L_{t_0}$ is $T(z) = \frac{2z-1}{z}$.

Now one is ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** First one proves $D(M^n, g) \geq 3$. Suppose $D(M, g) = 2$, and one tries to obtain a contradiction.

Let $U, I, L_t$ be constructed as before. Fix some $0 < t_0 \in I$. One has $(\phi^t)^*g(o) = g(o)$ for all $t \in I$. This implies $(D\phi^t)_o$ is a 1-parameter subgroup of $SO(g)$ at $o$. By Remark 1, one can choose $p \in \pi^{-1}(o)$ such that in the normal coordinate $\sigma_p$ for $P = P(\nabla)$ at $p$, $X$ has the following form:

$$X_x = Ax + \langle w, x \rangle x, \quad A \in \mathfrak{so}(n), \ w = -e_1 \in Ker A$$

Then in this local coordinate $\sigma_p$, the flow $\phi^t$ of $X$ has the following form:

$$\phi^t(x) = \frac{1}{1 + tx_1} (e^{tx}x), \quad x = (x_1, \ldots, x_n) \quad \text{(18)}$$

Choose a convex neighbourhood $C$ of $o$ which lies in the image of the local coordinate $\sigma_p$. By Corollary 3 of [4], for all $i \in \{1, \ldots, n-1\}$, the eigenfunctions $\lambda_i$ of $K_{t_0}$ are globally ordered on $C$ in the following sense:

- $\lambda_i(x) \leq \lambda_{i+1}(y)$ for all $x, y \in C$.
- If $\exists x \in C$ such that $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(y) < \lambda_{i+1}(y)$ for almost all $y \in C$.

At $o$, one has $\lambda_i(o) = 1$ for all $i$. Note that $n \geq 3$ implies $\lambda_2 = \cdots = \lambda_{n-1} \equiv 1$ on $C$. It follows that for $n \geq 3$, $\lambda_1(x) \leq \lambda_2(x) = 1$, and $\lambda_n(x) \geq \lambda_{n-1}(x) = 1$ for all $x \in C$. One can show all eigenfunctions $\lambda_i$ have to be constant on $C$. In the coordinate $\sigma_p$, define the following subsets of $C$:

$$C^+ = \{x \in C : x_1 > 0\}, \ C^- = \{x \in C : x_1 < 0\}$$

If $\exists x_1 \in C$ such that $\lambda_1(x_1) < 1$, one can find $x_0 \in C^+$ such that $\lambda_1(x_0) < 1$, and $\phi^t(x_0) \in C^+$ for all $t \geq 0$. Denote $D$ the closure of the integral curve of $\phi^t(x_0)$ for $t \geq 0$, then clearly $D \subset C$. From Equation (18), one can see $D$ is compact and connected. Hence one has $\lambda_1(D)$ is an interval $I_1 = [d, 1]$ with $d < 1$. The eigenfunctions of $K_{t_0}$ are all positive on $U$, so one has $0 < d < 1$ and $0 < \lambda_1(x) \leq 1 \forall x \in D$. Because $T(z) = \frac{2z-1}{z}$ is monotonically increasing on $\mathbb{R}^+$, one has $T(\lambda_1(x)) = \lambda_1(\phi_0(x))$ for all $x \in D$. It follows that:

$$T([d, 1]) = T(\lambda_1(D)) = \lambda_1(\phi_0(D)) \subset \lambda_1(D) = [d, 1], \ 0 < d < 1$$
This is clearly impossible for the Mobius map \( T(z) = \frac{2z - 1}{z} \) as \( T(d) < d \) for \( 0 < d < 1 \). Hence \( \lambda_1 \equiv 1 \) on \( C \). Replacing \( C^+ \) with \( C^- \), and \( T \) with \( T^{-1} \), respectively, one can show \( \lambda_n \equiv 1 \) on \( C \). It follows that all eigenfunctions of \( K_{t_0} \) are constant on \( C \).

If all eigenfunctions of \( K_{t_0} \) are constant on \( C \), one has \( \phi^{t_0}g \) and \( g \) are affine equivalent on \( C \). This is clearly impossible by Corollary 2.1.1. It follows that \( D(M, g) \neq 2 \).

Since \( X \) is a projective vector field for \((M^n, g)\), according to Section 2.1 of [4], one has:

\[
K' = g^{-1}L_Xg - \frac{1}{n+1}Tr(g^{-1}L_Xg) \cdot 1d \in B(M, g)
\]

Then \( D(M, g) = 1 \) implies that \( X \) is a homothetic vector field for \( g \), which is impossible. Hence one has \( D(M, g) \geq 3 \).

When \( n = 3 \), one has by Section 1.2 of [9], the maximum degree of mobility of a 3-dimensional connected Riemannian manifold with non-constant curvature is 2. This completes the proof. \( \square \)

**Remark 2.** The conditions \( n \geq 3 \), and \( g \) is Riemannian are necessary in the proof. If \( n = 2 \), one may end up with \( \lambda_1 = 1 \) on \( C^+ \), \( \lambda_1 < 1 \) on \( C^- \), together with \( \lambda_2 = 1 \) on \( C^- \), \( \lambda_2 > 1 \) on \( C^+ \). If \( g \) is not Riemannian, \( (K_{t_0})_o \) may not be the identity matrix. Besides, the global ordering of eigenfunctions of BM-structures can only be applied for Riemannian metrics.

### 4.2 Global results when \((M^n, g)\) is closed, proof of Theorem 1.4

In this section, one gives the proof of Theorem 1.4 stated at the end of the introduction.

**Proof of Theorem 1.4.** Since \( X \) is not linearizable at \( o \), one has \( D(M, g) \geq 2 \). First suppose \( D(M, g) = 2 \), then \( L_t \) is a 1-parameter parabolic subgroup by Theorem 3.1. This is in fact impossible by the following (One recently discovered that the argument below is analogous to part of Section 9.2 of [5]).

Because \( L_t \) is parabolic, there exists \( K \in B = B(M, g) \) such that

\[
L_t(Id) = e^{tb}(tK + Id), \ b \in \mathbb{R}
\]

\( X \) is complete because \( M \) is compact. Just fix \( t = 1 \), then \( L_1(Id) = e^{b}(K + Id) \) is the \( g \)-strength of \( \phi^1 \) on \( M \). Because \( M \) is closed and connected, according to Theorem 6 of [10], all non-real eigenfunctions of \( L_1(Id) \) are constant. It follows that all non-real eigenfunctions of \( K \) are constant on \( M \). On the
other hand, one has all real eigenfunctions of $K$ are identically zero. Otherwise, \( \exists t_0 \in \mathbb{R} \) such that \( L_{t_0}(Id) = K_{t_0} \) is degenerate. Then all eigenfunctions of $K$ are constant. This implies \( g_t \) and \( g \) are affine equivalent for all \( t \in \mathbb{R} \), which is impossible.

From above one has \( D(M, g) \geq 3 \). According to Corollary 5.2 of [11], one has \( g \) is Riemannian with positive constant sectional curvature.

\[ \square \]

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