A 1d lattice realization of chiral fermions with a non-Hermitian Hamiltonian

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ABSTRACT: Quantum Hall state is characterized by a 2-dimensional(2D) bulk insulator and a chiral fermion edge state, which gives quantized Hall conductance. It is generally believed that the corresponding 1D chiral fermion state can not exist by itself without its parent surface system. This is closely related to the quantum anomaly of chiral fermion in quantum field theory. It has been recognized that such an anomalous state implies the existence of a topologically nontrivial bulk. Here we show that such a 1D chiral fermion state can be realized without its corresponding surface system in a non-Hermitian Hamiltonian. Our model possesses correct chiral anomaly and gravitational anomaly, which ensures the quantization of the Hall conductance.
1 Introduction

As the first discovered topological state, the integer quantum Hall effect (IQHE) [1–3] is characterized by a two-dimensional (2D) insulating bulk and a chiral fermion edge state. The quantization of the electronic transport in IQHE is tracked by the Hall conductance, which relates to the Chern number of the bulk state and/or the chirality of the edge state. Such a relation between physical quantities of the bulk state and edge states is the so-called bulk-boundary correspondence. It has long been believed that such a chiral fermion state cannot exist by itself without a bulk, as proved by the famous Nielsen-Ninomiya theorem [4]. With the further study, it was realized that the reason behind this phenomenon is the quantum anomaly, the chiral anomaly [5, 6] and gravitational anomaly [7], of the 1D chiral fermion. In recent years, many similar states have been proposed as D-dimensional boundary states of corresponding topological phases in D+1 dimensional space, but cannot be realized in a lattice model in the D-spatial dimension. For example, the states with odd numbers of Dirac cones on the surface of 3D topological insulator [8, 9] and the anomalous symmetry enriched topological order on the surface of 3D symmetry protected topological state [10, 11]. The existence of such nontrivial boundary states has been considered as one of the key signatures of a topological phase. The reason to prohibit the lattice realization of these states in the same space dimension is the quantum anomaly. In fact, it has been proposed that quantum anomaly in a theory is equivalent to saying that the theory has a topologically nontrivial bulk, hence a system with quantum anomaly in D-dimension can only be realized in D+1 dimensional lattice model [1, 12].

In the past few years, the study of non-Hermitian systems has attracted a lot of attention [13–24]. In this paper, we use 1D chiral fermion as an example to show that a theory with quantum anomaly in D-dimension can be realized in a non-Hermitian lattice model in
the same spatial dimension. At the beginning of the paper, we construct a non-Hermitian 1D lattice model for a 1D left-moving chiral fermion. Then we demonstrate that the model realizes the edge state of an integer quantum hall state by studying the chirality and quantum anomaly of the model, which corresponds to the quantization of the Hall conductance in the 2D bulk, in various approaches. At last, we check the stability of the chiral fermions against perturbations. Our model elaborates how to realize quantum anomaly in lattice model in the same spatial dimension with the help of non-Hermiticity. Our study may also provide an alternative possibility, besides the spontaneous symmetry breaking, for the origin of some phenomena with nonzero chirality in nature.

2 The Model

A 1D left(right) moving chiral fermion field $\Psi_{L(R)}(x)$ can be achieved by projecting the Dirac spinor with the projection operator $P_{L,R} = (1 \mp \gamma_5)/2$. Their chiral charges, -1 for $\Psi_L(x)$ and +1 for $\Psi_R(x)$, are given by the eigenvalue of $\gamma_5$. The action of a free left moving fermion is given by

$$S_{\text{Weyl}} = \int dt dx \left\{ -i \dot{\Psi}_L^\dagger(x) \Psi_L(x) + \Psi_L^\dagger(x) \left[-i \partial_x + A_1(x)\right] \Psi_L(x) \right\}, \quad (2.1)$$

where we set the velocity of fermion $v_F = 1$ and choose the gauge $A_0 = 0$, $A_1 = A_1(t)$. The corresponding Hamiltonian reads

$$H = \Psi_L^\dagger(x) [i \partial_x - A_1] \Psi_L(x). \quad (2.2)$$

In general, a lattice realization of a field theory is constructed by replacing the differential operator $\partial_\mu$ with some kind of difference operator on the lattice. Different choices of the difference operators may lead to different lattice models. In this paper, we consider a 1D discrete spatial lattice with continuous time, a natural choice of the difference operator is

$$\partial_x \Psi(x) \to \frac{1}{2a} [c_{n+1} - c_{n-1}], \quad (2.3)$$

where $a$ is lattice constant, and $c_n$ is the fermion annihilation operator at lattice site $n$. However, the corresponding lattice model has the well-known fermion doubling problem, i.e. besides the left-moving fermion at $k \sim 0$, there is a spurious right-moving fermion at $k \sim \pi/a$ as shown in fig. 1(a). This is consistent with the Nielsen-Ninomiya theorem as discussed in the introduction. In the following, we refer to the above difference operator and the corresponding model as symmetric lattice operator and symmetric lattice model.

We may escape from the Nielsen-Ninomiya theorem by giving up the Hermiticity [25, 26]. This can be understood through a proof of the Nielsen-Ninomiya theorem based on the Poincare-Hopf theorem [26]. According to the Poincare-Hopf theorem, the sum of indices of all the isolated zero modes of a 1D lattice model of chiral fermion is zero. For a local, Hermitian, translation invariant model, the index of an isolated zero mode is just its chirality
\( \chi = \pm 1 \). Thus one must have equal numbers of left-movers and right-movers. However, in the non-Hermitian case, the index of an isolated zero mode is in general 0. Therefore, it may be possible to have only one left-movers in a non-Hermitian lattice model.

The non-Hermitian model can be constructed by considering a different difference operator, the oriented lattice operator,

\[
\partial_x \Psi(x) \rightarrow \frac{1}{a} [c_{n+1} - c_n],
\]

which leads to a non-Hermitian lattice model,

\[
H_- = \sum_{j=1}^{N-1} \frac{i}{a} e^{iaA_1} (c^+_j c_{j+1} + c^+_n c_1) - \sum_{j=1}^{N} \frac{i}{a} c^+_j c_j.
\]

This model describes a free left moving fermion and will be referred as the oriented lattice model in the following. A Hamiltonian describing a free right moving fermion can be simply obtained by acting a parity transformation on (2.5):

\[
H_+ = \sum_{j=2}^{N} \frac{i}{a} e^{iaA_1} (c^+_j c_{j-1} + c^+_1 c_n) - \sum_{j=1}^{N} \frac{i}{a} c^+_j c_j.
\]

3 The spectrum

In general, a non-Hermitian model may still have a real spectrum. For example, a non-Hermitian Hamiltonian with PT symmetry has only real eigenvalues. If we consider a difference operator in the form \( D_\lambda = \frac{1}{a} [\lambda (c_{n+1} - c_n) + (1 - \lambda)(c_n - c_{n-1})] \), the resultant Hamiltonian is PT symmetric if and only if \( \lambda = 1/2 \), which corresponds to the symmetric lattice operator (2.3). Therefore, the PT symmetric Hamiltonian must be Hermitian and can not describe a chiral fermion according to the Nielsen-Ninomiya theorem.

The Hamiltonian (2.5) explicitly breaks the PT symmetry. Thus it is not surprising to find that it has a complex spectrum, which is given by

\[
E_k = \frac{i}{a} \left( e^{iak} - 1 \right),
\]

Figure 1. The energy dispersion of the symmetric lattice model (a) and the oriented lattice model (b) with periodic boundary conditions and \( A_1 = 0 \). Blue line: the real part of the energy; red line: the imaginary part of the energy.
Figure 2. Illustration of spectral flow of the band driven by varying $A_1$ from $-\pi/L$ to $\pi/L$ adiabatically are shown in (a) for the symmetric lattice model and (b) for the oriented lattice model. Solid line means state is right-handed and dash line indicates state is left-handed. Strength of the line implies the observation probability of the state. Here, we set $a = 1$, $N = 100$.

in the case with periodic boundary conditions in the absence of gauge field. The propagator of the fermion with energy $E_k$ suggests that the real part $\text{Re}E_k$ can be understood as the ordinary energy of the fermion. Note the imaginary part $\text{Im}E_k$ is negative for every $k$, it can be regarded as a loss rate or inverse of the lifetime of the fermion due to the coupling with some kinds of environments. As shown in fig. 1(b), $\text{Re}E_k$ is similar to the one of the symmetric lattice model shown in fig. 1 (a). Thus we still have a left-moving fermion at $k \sim 0$ and a right-moving fermion at $k \sim \pi/a$. However, these two fermions have different lifetimes. For the left-moving fermion, we have $E_k \approx -k - iak^2/2$, while for the right-moving fermion, $E_{k'} \approx k' - 2i/a$ with $k' = k - \pi/a$. Thus the lifetime of left-movers is much larger than the one of right-movers provided $a$ is small enough. And in the continuum limit $a \to 0$, the left-mover has infinite lifetime, while the lifetime of right-mover vanishes. This indicates that the lattice Hamiltonian will reduce to the Hamiltonian (2.2) in the continuum limit. And also there is an emergent PT symmetry in the continuum limit, though PT symmetry is explicitly broken in the lattice model.

4 Chirality and chiral symmetry

As an edge state of the IQHE, the chirality of the chiral fermion corresponds to the quantization of the Hall conductance of the IQHE. Thus, a good lattice realization of the chiral fermion should respect chiral symmetry and have correct chirality. This can be checked with the Atiyah–Singer index theorem, which in 2D continuous Euclidean space-time is given by

$$\text{index}(\tilde{D}_\pm) = n_+ - n_- = \sum_n \langle \varphi_n | \gamma_5 | \varphi_n \rangle = -\frac{1}{2\pi} \int_M F. \quad (4.1)$$

Here $\tilde{D}_\pm = \tilde{D}(1 \pm \gamma_5)/2$ are Weyl operator with chirality $\chi = \pm 1$ respectively, $\tilde{D}$ is the Dirac operator, and $n_\pm$ represents number of zero modes with chirality $\chi = \pm 1$, respectively, $\varphi_n$ is the eigenstate of $\tilde{D}$ with $\tilde{D} | \varphi_n \rangle = \lambda_n | \varphi_n \rangle$. $F$ is the strength tensor of the U(1) gauge field.
To generalize the theorem to a lattice, we introduce a discrete Dirac operator

$$D(n \mid m) = \gamma_1 e^{iaA_1(\tau)} \delta_{n+1,m} - \delta_{n,m} a + \gamma_2 \partial_\tau,$$

(4.2)

where \(n\) and \(m\) are indices of lattice sites. It can be easily proved that the action of the generalized Dirac operator is invariant under an infinitesimal chiral transformation

$$\Psi(n) \rightarrow e^{i\alpha \gamma_5} \Psi(n), \quad \bar{\Psi}(n) \rightarrow \bar{\Psi}(n) e^{i\alpha \gamma_5}. \quad (4.3)$$

Therefore we can define the corresponding Weyl operators with

$$D^\pm = D(1 \pm \gamma_5)/2.$$

Please notice that \(D^-\) is just the difference operator in the action of the lattice model (2.5) (after a Wick rotation). Thus our lattice Hamiltonian also has the chiral symmetry.

Please notice that \(D\) is also a non-Hermitian operator. Therefore its right eigenstates are not orthogonal to each other, and a left eigenstate of \(D\) is not Hermitian conjugate to the corresponding right eigenstate anymore. Thus, we introduce the so-called bi-orthogonal basis, i.e. the right eigenstates \(\varphi_{n,R}\) as the basis of kets \(|\varphi_n\rangle\) and left eigenstates \(\varphi_{n,L}\) as the basis of bras \langle\varphi_n|\). It can be easily proved that \(\langle\varphi_{n,L} | \varphi_{m,R}\rangle = \delta_{nm}\). With the bi-orthogonal basis, the lattice-version index theorem is written as

$$\text{index}(D^+) = n^+ - n^- = \sum_n \langle \varphi_{n,L} | \gamma_5 | \varphi_{n,R}\rangle = -\frac{1}{2\pi} \int_M F,$$

(4.4)

which will be proved in the following.

Mathematically, the first equal sign in equation (4.4) holds for any elliptic operator, hence it holds for our generalized Dirac operator \(D\), because it is still an elliptic operator \[27\]. For the second equal sign, because \(D | \varphi_{n,R}\rangle = \lambda_n | \varphi_{n,R}\rangle, \quad \langle \varphi_{n,L} | D = \lambda_n \langle \varphi_{n,L} |\) and \(\{D, \gamma_5\} = 0\), we have

$$\lambda_n \langle \varphi_{n,L} | \gamma_5 | \varphi_{n,R}\rangle = \langle \varphi_{n,L} | \gamma_5 D | \varphi_{n,R}\rangle = -\langle \varphi_{n,L} | D \gamma_5 | \varphi_{n,R}\rangle = -\lambda_n \langle \varphi_{n,L} | \gamma_5 | \varphi_{n,R}\rangle.$$
So far a lattice-version index theorem is constructed. By integrating over the strength of the gauge field at the right hand side of eqn. (4.4), it can be checked that $\text{index}(\tilde{D}_+)= -1$ for our lattice model. Thus we can conclude that the lattice-version index correctly capture chirality of lattice chiral fermion.

The lattice-version index theorem can be understood with the spectral flow under a slowly change of $A_1$ from $-\pi/L$ at time $t = 0$ to $\pi/L$ at time $t = T$, where $L = Na$, $N$ is total number of lattice sites. $T$ should be large enough for an adiabatic process. The resulting spectral flow for the symmetric and oriented lattice models are depicted in fig. 2 (a) and (b), respectively. For the symmetric lattice model, one can find that the chiral charge is changed by 0 which is consistent with the fermion doubling discussed above. On the other hand, for the oriented lattice model, an additional decay factor $e^{T\text{Im}E}$ appears because of the time evolution. For $T$ is large enough ($v_FT \gg a$) but not too large ($v_FT \ll \frac{l_0^2}{a}$, $l_0$ is some finite cut-off length scale), the spectral flow from left moving part is almost unaffected, while the one from right moving part almost totally vanishes because $\text{Im}E_k \propto 1/a$, as illustrated in fig. 2 (b). This suggests that the chiral charge is changed by just 1, which is consistent with single left-handed chiral fermion in the continuum theory (2.2).

### 5 Quantum anomaly

As discussed in the introduction, the fermion doubling problem is closely related to the quantum anomaly of the chiral fermion theory. However, with the help of non-Hermiticity, we may realize the quantum anomaly with a non-Hermitian lattice model with the same space dimension. This can be understood by taking the one higher dimensional bulk as the environment of the edge, then the effects of the bulk can be tracked or manifested by the non-Hermiticity in the model. In the 1D chiral fermion case, the field theory has the so-called chiral anomaly and gravitational anomaly. According to the lattice version index theorem (4.4) and the spectral flow shown in fig. 2, the chiral anomaly is correctly tracked by our lattice model.

An anomaly indicator of the gravitational anomaly of 1D chiral fermion is the chiral central charge. For our lattice model, if the lattice constant $a$ is small enough, we may consider only the states around $k = 0$ and $k = \pi/a$, whose spectrums are given by

$$E_k^L \sim -k - iak^2/2 \quad \quad E_k^R \sim k - 2i/a,$$

(5.1)

where $L$ and $R$ stand for the left-mover and right-mover, respectively. If we ignore the imaginary part of the energy, the spectrum is the same as the free complex fermion conformal field theory (CFT), where the $E^R$ corresponds to the holomorphic component with $c = 1$ and $E^L$ corresponds to the anti-holomorphic component with $\bar{c} = 1$. As we discussed before, the imaginary part of the energy corresponds to the lifetime of the fermions. Thus in the continuum limit $a \to 0$, only the holomorphic component survives. Therefore it corresponds to a CFT with chiral central charge $\bar{c} - c = 1$, which indicates that our lattice model has the same gravitational anomaly as the 1D left-moving chiral fermion.
To help read out unambiguously gravitational anomaly from our lattice model, we calculate numerically the difference of ground state energy, $\Delta E$, between periodic boundary condition (PBC) and anti-periodic boundary condition (APBC). For a CFT, it is well-known that one can extract the central charge of the CFT from the energy difference from the formula

$$\Delta E = \frac{\pi}{4L} (c + \bar{c}),$$  \hspace{1cm} (5.2)

where $c$ and $\bar{c}$ are the central charge of holomorphic and anti-holomorphic parts, respectively, and $L$ is the spatial length of the system.

In our model, the time-dependent single particle states are

$$|\psi_k(T)\rangle = e^{-iHT}|\psi_k\rangle = e^{-i\text{Re}(E_k)T}e^{i\text{Im}(E_k)T}|\psi_k\rangle$$  \hspace{1cm} (5.3)

The time-dependent ground state energy for both PBC and APBC can be calculated by numerically summing over the energy of all the occupied single particle states

$$E(T) = \sum_{|k|,\text{occ}} \langle \psi_{k,L}(T)|H|\psi_{k,R}(T)\rangle \simeq \sum_{k,\text{occ}} e^{2\text{Im}(E_k)T}\text{Re}(E_k)$$

$$=- \sum_{k,\text{occ}} v_F \frac{\sin ka}{a} e^{-\frac{2\sin ka - 1}{a}v_FT},$$  \hspace{1cm} (5.4)

where $k = \frac{2\pi n}{L}$ for PBC, $k = \frac{2\pi(n+\frac{1}{2})}{L}$ for APBC, $L = Na$, $N$ is the number of sites and $v_F$ is the Fermi velocity. The subscript "occ" in the summation means that we only sum over all the occupied states. We write the approximate equality sign in eqn. (5.4), because only the real part of energy $\text{Re}(E_k)$ is considered. By comparing with the result of CFT (5.2), we can extract the the time evolution of $c + \bar{c}$, and the result is depicted in fig.3.

Recall that in the continuum limit $a \to 0$, the lattice model starts from a non-chiral system and then rapidly evolves into a chiral system with only left-moving modes. The chiral fermion state can last for a long time. As a result, there are two important time scales in our model. First, at $T \gtrsim T_1 = \frac{a}{v_F}$, the right moving modes around $ka = \pi$ are damped out, only left moving modes survive. Note that $T_1$ tends to be 0 in the continuum limit. Another important time scale is $T_2 = \frac{L^2}{a^2v_F}$ at which left moving modes around $ka = 0$ begin to decay, which is the situation we hope to avoid. Fortunately, $T_2$ tends to infinity in the continuum limit. For these physical reasons, when $a$ is finite we should consider only the time region $T_1 \ll T \ll T_2$. It is in this region that the lattice model has same properties as a chiral fermion in the field theory with $c = 0, \bar{c} = 1$. Indeed, our numerical result $c + \bar{c} = 1$, as shown in fig. 3, shows consistency with the continuum limit.

6 Stability under local perturbations

The topological nature of the lattice-version index theorem suggests that the zero mode of our non-hermitian model is robust against local perturbations. To check this, we consider
a low energy effective Hamiltonian with coupling between the two kinds of fermions

\[ H = \sum_k - (k + \frac{k^2 a}{2}) c_{0,k}^\dagger c_{0,k} + (k - \frac{2i}{a}) c_{\pi,k}^\dagger c_{\pi,k}^\dagger + V c_{0,k}^\dagger c_{\pi,k} + V c_{\pi,k}^\dagger c_{0,k}, \]

where \( c_{0,k} \) and \( c_{\pi,k} \) corresponds to the left moving fermion around \( k = 0 \) and the right moving fermion around \( k = \pi/a \), respectively. For small enough \( a \) with \( a k \ll 1, aV \ll 1 \), we have

\[ \epsilon_+ \approx -k - \frac{2V^2 + k^2}{4a}, \quad \epsilon_- \approx k - \frac{2i}{a}. \]

Thus, the perturbation does not open a gap. It merely changes the imaginary part of the energy (see fig. 4 for an example), or the inverse lifetime of the fermions. However, the qualitative behavior of lifetime of the fermions, i.e. the left moving fermion has infinite lifetime in the continuum limit, while the lifetime of the right moving fermion is zero in the continuum limit, are unchanged.

We also check the disorder effect by adding \( \sum_j \alpha_j c_j^\dagger c_j \) terms into the Hamiltonian (2.5) with randomly generated \( \alpha_i \in [0, 0.1] \). Our numerical result shows that such a perturbation does not open a gap either. Thus, the zero mode of the oriented lattice model is much more robust than the one of the symmetric lattice model against the local perturbations.

7 Discussion and Conclusion

Finally, we want to discuss the possible experimental implementation of our non-hermitian model. The non-hermitian lattice model may be realized with ultracold atoms in optical
lattices. By taking the gauge field $A_1 = \pi/a$ in eqn. (2.5), this model matches exactly the eqn. F(4) in Ref. [29] with $\kappa = -2J = 1/a$, which has been proposed to be realizable in a system consists of two parallel fine-tuning optical lattices. Alternatively, the model can also be simulated with electric circuits with diodes that induce left-right asymmetry and electrical inductors for the imaginary chemical potential [30, 31]. The zero modes can be detected via prominent two-point impedance peaks.

In summary, we have constructed a local non-hermitian 1D lattice model with a complex spectrum. We have presented several pieces of evidence that this model describes a left-moving chiral fermion in various approaches, and shown that it is stable against local perturbations. Our results suggest that with the help of non-hermiticity, an anomalous field theory may be realized in a non-hermitian lattice model with the same spacetime dimension. The non-hermiticity tracks the coupling of the lattice model with its nontrivial one higher dimensional bulk. Our theory also indicates that a chiral theory may be realized by coupling to some kinds of environments. This provides an alternative possibility, besides the spontaneous symmetry breaking, for the origin of at least some chiral phenomena in nature.

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