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Tome 67, no 4 (2017), p. 1725-1738.

<http://aif.cedram.org/item?id=AIF_2017__67_4_1725_0>

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GAPS IN SUMSETS OF \textit{s} PSEUDO \textit{s}-TH POWERS

by Javier CILLERUELO\textsuperscript{†} & Jean-Marc DESHOUILLERS (*)

Abstract. — We study the length of the gaps between consecutive members in the sumset \( sA \) when \( A \) is a pseudo \( s \)-th power sequence, with \( s \geq 2 \). We show that, almost surely, \( \lim \sup (b_{n+1} - b_n)/\log b_n = s^s!/\Gamma^s(1/s) \), where \( b_n \) are the elements of \( sA \).

Résumé. — On étudie la taille des différences entre les termes consécutifs de la suite \( sA \) où \( A \) est une suite de pseudo-puissances \( s \)-ièmes avec \( s \geq 2 \). On montre qu'on a presque sûrement \( \lim \sup (b_{n+1} - b_n)/\log b_n = s^s!/\Gamma^s(1/s) \), où les \( b_n \) sont les éléments de la suite \( sA \).

1. Introduction

Erdős and Rényi [3] proposed in 1960 a probabilistic model for sequences \( A \) growing like the \( s \)-th powers: they build a probability space \((\mathcal{U}, \mathcal{T}, P)\) and a sequence of independent random variables \((\xi_n)_{n \in \mathbb{N}}\) with values in \{0, 1\} and \( P(\xi_n = 1) = \frac{1}{s} n^{-1+1/s} \); to any \( u \in \mathcal{U} \), they associate the sequence of positive integers \( A = A_u \) such that \( n \in A_u \) if and only if \( \xi_n(u) = 1 \). In short, the events \( \{n \in A\} \) are independent and \( P(n \in A) = \frac{1}{s} n^{-1+1/s} \). The counting function of these random sequences \( A \) satisfies almost surely the asymptotic relation \( |A \cap [1, x]| \sim x^{1/s} \), whence the terminology pseudo \( s \)-th powers. Erdős and Rényi studied the random variable \( r_s(A, n) \) which counts the number of representations of \( n \) in the form \( n = a_1 + \cdots + a_s \),

Keywords: Additive Number Theory, Pseudo \( s \)-th powers, Probabilistic method.
Math. classification: 11B83.

(*) The first author was supported by MINECO project MTM2014-56350-P and ICMAT Severo Ochoa project SEV-2015-0554.
The second author has been partly supported by the Indo-French Centre for the Promotion of Advanced Research - CEFIPRA, project No 5401-1.
Both authors are thankful to Ecole Polytechnique which made their collaboration easier.

Javier Cilleruelo untimely passed away on May 15th, 2016. I express my deep sorrow for the loss of a brilliant collaborator and a friend. J-M. D.
For the simplest case $s = 2$ they proved that $r_2(A, n)$ converges to a Poisson distribution with parameter $\pi/8$, when $n \to \infty$. They also claimed the analogous result for $s > 2$ but their analysis did not take into account the dependence of some events. J. H. Goguel [4] proved indeed that for each integer $d$, the sequence of the integers $n$ such that $r_s(A, n) = d$ has almost surely the density $\lambda^d e^{-\lambda_s}/d!$, where $\lambda_s = \Gamma^s(1/s)/(s^s s!)$. B. Landreau [5] gave a proof of this result based on correlation inequalities and also showed that the sequence of random variables $(r_s(A, n))_n$ converges in law towards the Poisson distribution with parameter $\lambda_s$.

In particular, both the sets of the integers belonging, or not belonging, to $sA = \{a_1 + \cdots + a_s : a_i \in A\}$ have almost surely a positive density and it makes sense to study the length of the gaps in $sA$. The aim of the paper is to obtain a precise estimate for the maximal length of such gaps.

**Theorem 1.1.** — For any $s \geq 2$ the sequence $sA = (b_n)_n$, sum of $s$ copies of a pseudo $s$-th power sequence $A$, satisfies almost surely

\[
\limsup_{n \to \infty} \frac{b_{n+1} - b_n}{\log b_n} = \frac{s^s s!}{\Gamma^s(1/s)}.
\]

We remark that this result is heuristically consistent with the fact that for a random sequence $S$ with $P(n \in S) = 1 - e^{-\lambda}$, we have $\limsup(s_{m+1} - s_m)/\log s_m = 1/\lambda$, an exercise on Borel–Cantelli Lemma.

## 2. Notation, hint of the proof and general lemmas

### 2.1. Notation

We retain the notation of the introduction, for the probability space $(\mathcal{U}, \mathcal{T}, P)$ and the definition of the random sequences $A = A_u$, where the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s} n^{-1+1/s}$. We further use the following notation.

1. We write $\omega$ to denote a set of distinct positive integers and we denote by $E_\omega$ and $E_\omega^c$ the events

\[
E_\omega = \{\omega \subset A\} \quad \text{and} \quad E_\omega^c = \{\omega \not\subset A\}
\]

respectively. We write $\omega \sim \omega'$ to mean that $\omega \cap \omega' \neq \emptyset$ but $\omega \neq \omega'$; we remark that $\omega \sim \omega'$ if and only if the events $E_\omega$ and $E_{\omega'}$ are distinct and dependent.
If \( \omega = \{x_1, \ldots, x_r\} \) we write
\[
\sigma(\omega) = \{a_1 x_1 + \cdots + a_r x_r : a_1 + \cdots + a_r = s, \ a_i \geq 1\}
\]
for the set of all integers which can be written as a sum of \( s \) integers using all the integers \( x_1, \ldots, x_r \). For an integer \( z \) we let
\[
\Omega_z = \{\omega : z \in \sigma(\omega)\}.
\]
(2) Given \( \alpha > 0 \), we denote by \( I_i \) the interval \([i, i + \alpha \log i]\) and we denote by \( F_i^{(\alpha)} \), or simply \( F_i \) when the context is clear, the event
\[
F_i = F_i^{(\alpha)} = \{sA \cap I_i = \emptyset\}.
\]
We denote by \( \Omega_{I_i} \) the family of sets
\[
\Omega_{I_i} = \{\omega : \sigma(\omega) \cap I_i \neq \emptyset\}.
\]
(3) We let \( \lambda_s = \frac{\Gamma(s)(1/s)}{s!s^s} \).
(4) We use Vinogradov’s notation \( \ll \), where \( f \ll g \) is equivalent to Landau’s notation \( f = O(g) \).

2.2. Hints for the proof to Theorem 1.1

We wish to prove that for \( \alpha > \lambda_s^{-1} \), the event \( F_i^{(\alpha)} \), defined in Section 2.1, occurs — almost surely — for only finitely many \( i \)’s and that for \( \alpha < \lambda_s^{-1} \) it occurs — almost surely — for infinitely many \( i \)’s. There is a flavour of Borel–Cantelli and indeed a key point in the proof is Lemma 3.5 which asserts relation (3.2), namely
\[
(2.1) \quad P(F_i^{(\alpha)}) = i^{-\alpha \lambda_s + o(1)}.
\]
Let us first see how we can obtain that relation. Here \( \alpha \) is fixed and we do not mention it anylonger. By the definition, the event \( F_i \) occurs if and only if for any family of \( s \) non necessarily distinct integers which sum up to an integer in \( I_i \), at least one of them is not in \( A \); with our notation, this leads to
\[
F_i = \bigcap_{\omega \in \Omega_{I_i}} E_{\omega}^c.
\]
If the \( \omega \)’s which are involved had pairwise empty intersections, the events \( E_{\omega}^c \) would be independent and we would have
\[
P(F_i) = \prod_{\omega \in \Omega_{I_i}} P(E_{\omega}^c).
\]
Although this is not the case, the structure of the events $E_\omega$, which are finite intersections of events taken from from an independent family, permits us to use Harris’ inequality (or FKG inequality, cf. Theorem 2.2 below) to get the lower bound

$$P(F_i) \geq \prod_{\omega \in \Omega_{I_i}} P(E^c_\omega).$$

It also permits us, thanks to Janson’s Correlation Inequality (cf. Theorem 2.2 below), to get the upper bound

$$P(F_i) \leq \prod_{\omega \in \Omega_{I_i}} P(E^c_\omega) \times \exp \left(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})\right),$$

where the notation $\omega \sim \omega'$ is defined in Section 2.1. It is then a matter of computation, based on Lemma 2.3, to get the central inequality (2.1).

When $\alpha > \lambda_s^{-1}$ the series $\sum_i P(F^{(\alpha)}_i)$ converges and the Borel–Cantelli lemma immediately implies that for such $\alpha$ the events $F^{(\alpha)}_i$ almost surely occur for only finitely many $i$’s.

When $\alpha < \lambda_s^{-1}$ the series $\sum_i P(F_i)$ diverges, but this is not enough to conclude directly since the events $F_i$’s are not independent. However, P. Erdős and A. Rényi proved that a weak dependence among the $F_i$’s is sufficient for obtaining an “inverse Borel–Cantelli” result (cf. Theorem 2.1 below). It is thus important to have a small upper bound for $P(F_i \cap F_j) - P(F_i)P(F_j)$ in average. With our notation, we have

$$F_i \cap F_j = \bigcap_{\omega \in I_i \cup I_j} E^c_\omega,$$

and here again Janson’s inequality will help us to obtain a suitable bound.

### 2.3. Probabilistic results

We use the following generalization of the Borel–Cantelli Lemma, proved indeed by P. Erdős and A. Rényi in 1959 [2].

**Theorem 2.1 (Borel–Cantelli Lemma).** — Let $(F_i)_{i \in \mathbb{N}}$ be a sequence of events and let $Z_n = \sum_{i \leq n} P(F_i)$.

If the sequence $(Z_n)_n$ is bounded, then, with probability 1, only finitely many of the events $F_i$ occur.

If the sequence $(Z_n)_n$ tends to infinity and

$$\lim_{n \to \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

then, with probability 1, infinitely many of the events $F_i$ occur.
The next result, which combines Harris’ inequality and Janson’s Correlation Inequality, can be found in [1].

**Theorem 2.2.** — Let \((E_\omega)_{\omega \in \Omega}\) be a finite collection of events which are intersections of elementary independent events and assume that \(P(E_\omega) \leq \frac{1}{2}\) for any \(\omega \in \Omega\). Then

\[
\prod_{\omega \in \Omega} P(E_\omega^c) \leq P\left( \bigcap_{\omega \in \Omega} E_\omega^c \right) \leq \prod_{\omega \in \Omega} P(E_\omega^c) \times \exp\left(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})\right),
\]

where \(\omega \sim \omega'\) means that the events \(E_\omega\) and \(E_{\omega'}\) are dependent events.

### 2.4. A technical lemma

**Lemma 2.3.** — Given \(1 \leq t \leq s - 1\) and positive integers \(a_1, \ldots, a_t\) we have, as \(z\) tends to infinity:

1. \[
\sum_{\substack{1 \leq x_1, \ldots, x_t \leq z \\ a_1 x_1 + \cdots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+t/s}. 
\]
2. \[
\sum_{\substack{1 \leq x_1, \ldots, x_t \leq z \\ a_1 x_1 + \cdots + a_t x_t < z}} (x_1 \cdots x_t)^{-1+1/s} \left(z - (a_1 x_1 + \cdots + a_t x_t)\right)^{-2t/s} \ll z^{-t/s} \log z.
\]
3. \[
\sum_{\substack{1 \leq x_1 < \cdots < x_s \leq z \\ x_1 + \cdots + x_s = z}} (x_1 \cdots x_s)^{-1+1/s} \sim s^s \lambda_s.
\]

**Proof.** — (1) We have

\[
\sum_{\substack{1 \leq x_1, \ldots, x_t \leq z \\ a_1 x_1 + \cdots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} = (a_1 \cdots a_t)^{-1/s} \sum_{\substack{1 \leq x_1, \ldots, x_t \leq z \\ a_1 x_1 + \cdots + a_t x_t = z}} (a_1 x_1 \cdots a_t x_t)^{-1+1/s} \leq (a_1 \cdots a_t)^{-1/s} \sum_{\substack{1 \leq y_1, \ldots, y_t \leq z \\ y_1 + \cdots + y_t = z}} (y_1 \cdots y_t)^{-1+1/s}.
\]
If \( y_1 + \cdots + y_t = z \) then at least one of them, say \( y_t \), is greater than \( z/t \) and is determined by \( y_1, \ldots, y_{t-1} \). Thus,

\[
\sum_{1 \leq x_1, \ldots, x_t} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+1/s} \sum_{1 \leq y_1, \ldots, y_{t-1} < z} (y_1 \cdots y_{t-1})^{-1+1/s}
\]

\[
\ll z^{-1+1/s} \left( \sum_{1 \leq y < z} y^{-1+1/s} \right)^{t-1}
\]

\[
\ll z^{-1+1/s} (z^{1/s})^{t-1}
\]

\[
\ll z^{-1+t/s}.
\]

(2) We have

\[
\sum_{1 \leq x_1, \ldots, x_t} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+1/s} \sum_{a_1 x_1 + \cdots + a_t x_t < z} (z - (a_1 x_1 + \cdots + a_t x_t))^{-2t/s}
\]

\[
= \sum_{1 \leq m < z} (z - m)^{-2t/s} \sum_{1 \leq x_1, \ldots, x_t} (x_1 \cdots x_t)^{-1+1/s}
\]

(by (1)) \( \ll \sum_{1 \leq m < z} (z - m)^{-2t/s} m^{-1+t/s} \)

\[
\ll \sum_{1 \leq m \leq z/2} (z - m)^{-2t/s} m^{-1+t/s} + \sum_{z/2 < m < z} (z - m)^{-2t/s} m^{-1+t/s}
\]

\[
\ll z^{-2t/s} z^{t/s} + z^{-1+t/s} \sum_{z/2 < m < z} (z - m)^{-2t/s}
\]

\[
\ll z^{-t/s} + z^{-1+t/s} \left( z^{1-2t/s} \log z \right)
\]

\[
\ll z^{-t/s} \log z.
\]

Remark 2.4. — Except in the case when \( s = 2 \) and \( t = 1 \), the upper bound in (2) may be replaced by \( z^{-t/s} \).

(3) It follows from Lemma 3 of [5].
3. Proof of Theorem 1.1

3.1. Combinatorial lemmas

**Lemma 3.1.** We have

\[
\sum_{\omega \in \Omega_z} P(E_\omega) \sim \lambda_s
\]

as \( z \to \infty \).

**Proof.** We have

\[
\sum_{\omega \in \Omega_z} P(E_\omega) = \sum_{\omega \in \Omega_z \atop |\omega| = s} P(E_\omega) + \sum_{\omega \in \Omega_z \atop |\omega| \leq s-1} P(E_\omega).
\]

The main contribution comes from the first sum.

\[
\sum_{\omega \in \Omega_z \atop |\omega| = s} P(E_\omega) = \frac{1}{s^s} \sum_{1 \leq x_1 < \cdots < x_s \atop x_1 + \cdots + x_s = z} (x_1 \cdots x_s)^{-1+1/s} \sim \lambda_s
\]

as \( z \to \infty \), by Lemma 2.3(3). For the second sum we have

\[
\sum_{\omega \in \Omega_z \atop |\omega| \leq s-1} P(E_\omega) \leq \sum_{1 \leq r \leq s-1} \sum_{a_1, \ldots, a_r = s \atop a_1 + \cdots + a_r = s} \sum_{1 \leq x_1, \ldots, x_r \atop a_1 x_1 + \cdots + a_r x_r = z} (x_1 \cdots x_r)^{-1+1/s}
\]

(Lem. 2.3(1)) \( \ll \sum_{1 \leq r \leq s-1} z^{r-1} \ll z^{-1/s} \). \( \square \)

**Lemma 3.2.** For any \( z \leq z' \) we have

\[
\sum_{\omega \sim \omega' \atop \omega \in \Omega_z, \omega' \in \Omega_{z'}} P(E_\omega \cap E_{\omega'}) \ll z^{-1/s} \log z.
\]

**Proof.** If \( \omega \in \Omega_z \) then there exist some \( r \leq s \) and some positive integers \( a_1, \ldots, a_r \) with \( a_1 + \cdots + a_r = s \) such that \( a_1 x_1 + \cdots + a_r x_r = z \). Thus, any pair of sets \( \omega \sim \omega' \) with \( \omega \in \Omega_z, \omega' \in \Omega_{z'}, \) \( z \leq z' \) is of the form

\[
\omega = \{x_1, \ldots, x_t, u_{t+1}, \ldots, u_r\}
\]

\[
\omega' = \{x_1, \ldots, x_t, v_{t+1}, \ldots, v_{r'}\}
\]

with \( 1 \leq t \leq r \leq r' \leq s \) and positive integers \( a_1, \ldots, a_r \) and \( b_1, \ldots, b_{r'} \) with

\[
a_1 x_1 + \cdots + a_t x_t + a_{t+1} u_{t+1} + \cdots + a_r u_r = z
\]

\[
b_1 x_1 + \cdots + b_t x_t + b_{t+1} v_{t+1} + \cdots + b_{r'} v_{r'} = z'.
\]

Of course if \( r = t \) then \( \omega = \{x_1, \ldots, x_r\} \) and \( r' \geq t + 1 \). Otherwise \( \omega = \omega' \).

Similarly, when \( r' = t \), we have \( r \geq t + 1 \).
Given positive integers $z, z', t, r, r', a_1, \ldots, a_r, b_1, \ldots, b_{r'}$ we estimate the sum
\[ \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) \]
where the symbol $\sum^*$ means that the sum is extended to the pairs $\omega \sim \omega'$ satisfying the above conditions. We distinguish several cases according to the values of $r$ and $r'$.

**Case $r \geq t + 1$ and $r' \geq t + 1$.** — We have
\[ \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) \leq \sum_{1 \leq x_1, \ldots, x_t \atop a_1 x_1 + \cdots + a_t x_t < z} (x_1 \cdots x_t)^{-1+1/s} \times \sum_{1 \leq u_{t+1}, \ldots, u_r \atop a_{t+1} u_{t+1} + \cdots + a_r u_r = z - (a_1 x_1 + \cdots + a_t x_t)} (u_{t+1} \cdots u_r)^{-1+1/s} \times \sum_{1 \leq v_{t+1}, \ldots, v_{r'} \atop b_{t+1} v_{t+1} + \cdots + b_{r'} v_{r'} = z' - (b_1 x_1 + \cdots + b_t x_t)} (v_{t+1} \cdots v_{r'})^{-1+1/s}. \]

By Lemma 2.3(1) we have
\[ \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) \ll \sum_{x_1, \ldots, x_t \atop a_1 x_1 + \cdots + a_t x_t < z} (x_1 \cdots x_t)^{-1+1/s} \times \left(z - (a_1 x_1 + \cdots + a_t x_t)\right)^{r-t-s-1} \times \left(z' - (b_1 x_1 + \cdots + b_t x_t)\right)^{r'-t-s-1}. \]

Using the inequality $AB \leq A^2 + B^2$, we get
\[ \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) \leq \sum_{x_1, \ldots, x_t \atop a_1 x_1 + \cdots + a_t x_t < z} (x_1 \cdots x_t)^{-1+1/s} \left(z - (a_1 x_1 + \cdots + a_t x_t)\right)^{2(r-t-s)/s} \]
\[ + \sum_{x_1, \ldots, x_t \atop b_1 x_1 + \cdots + b_t x_t < z'} (x_1 \cdots x_t)^{-1+1/s} \left(z' - (b_1 x_1 + \cdots + b_t x_t)\right)^{2(r'-t-s)/s} \]
\[ (\text{Lem. 2.3(2)}) \]
\[ \ll z^{-t/s} \log z \ll z^{-1/s} \log z. \]
Case $r = t$ and $r' \geq t + 1$. — In this case we have

$$\sum_{\omega \sim \omega'} s \, P(E_\omega \cap E_{\omega'})$$

$$\leq \sum_{1 \leq x_1, \ldots, x_t} (x_1 \cdots x_t)^{-1+1/s}$$

$$\times \sum_{1 \leq v_{t+1}, \ldots, v_{r'}} (v_{t+1} \cdots v_{r'})^{-1+1/s}$$

$$(\text{Lem. 2.3(1)}) \leq \sum_{1 \leq x_1, \ldots, x_t} (x_1 \cdots x_t)^{-1+1/s} \left( z' - (b_1 x_1 + \cdots + b_t x_t) \right)^{r' - t - 1}$$

$$\leq \sum_{1 \leq x_1, \ldots, x_t} (x_1 \cdots x_t)^{-1+1/s}$$

$$\leq z^{t - 1} \leq z^{-1/s}.$$

Case $r' = t$ and $r \geq t + 1$. — This case is similar to the previous one. □

**Lemma 3.3.** — Let $\alpha > 0$ and the let $I_i$ be the interval $[i, i + \alpha \log i]$. For any $i \leq j$ we have

$$\sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s}(\log i)^2(\log j).$$

**Proof.** — We have

$$\sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \leq \sum_{z \in I_i, z' \in I_j} \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})$$

$$(\text{Lem. 3.2}) \ll \sum_{z \in I_i, z' \in I_j} z^{-1/s} \log z$$

$$\ll (\log i)^2(\log j)i^{-1/s}. □$$

**Lemma 3.4.** — We have

$$\prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) = i^{-\alpha \lambda_s + o(1)}.$$
Proof. — We observe that
\[
\prod_{z \in I_i} \prod_{\omega \in \Omega_z} P(E^c_{\omega}) \leq \prod_{\omega \in \Omega_{I_i}} P(E^c_{\omega}) \leq \prod_{z \in I_i} \prod_{\omega \in \Omega_z} P(E^c_{\omega}).
\]

Writing \( P(E^c_{\omega}) = 1 - P(E_{\omega}) \) and taking logarithms we have
\[
\log \left( \prod_{z \in I_i} \prod_{\omega \in \Omega_z} P(E^c_{\omega}) \right) = \sum_{z \in I_i} \sum_{\omega \in \Omega_z} \log(1 - P(E_{\omega})) \sim - \sum_{z \in I_i} \sum_{\omega \in \Omega_z} P(E_{\omega})
\]
\[
\sim - \sum_{z \in I_i} \lambda_s \sim - \alpha \lambda_s \log i.
\]

On the other hand,
\[
\log \left( \prod_{z \in I_i, \omega \in \Omega_z} P(E^c_{\omega}) \right) = \sum_{z \in I_i, \omega \in \Omega_z} \log(1 - P(E_{\omega})) \sim - \sum_{z \in I_i, \omega \in \Omega_z} P(E_{\omega})
\]
\[
= - \sum_{z \in I_i} \sum_{\omega \in \Omega_z} \frac{1}{s^s (x_1 \cdots x_s)^{1+1/s}}
\]
\[
\sim - \lambda_s \alpha \log i. \quad \square
\]

Lemma 3.5. — We have
\[(3.2) \quad P(F_i) = i^{- \alpha \lambda_s + o(1)}. \]

Proof. — As we noticed it in Section 2.2, we have
\[
F_i = \bigcap_{\omega \in \Omega_{I_i}} E^c_{\omega}.
\]

Since \( P(E_{\omega}) \leq 1/2 \) for any \( \omega \), Theorem 2.2 applies and we have
\[
\prod_{\omega \in \Omega_{I_i}} P(E^c_{\omega}) \leq P(F_i) \leq \prod_{\omega \in \Omega_{I_i}} P(E^c_{\omega}) \times \exp \left( 2 \sum_{\omega \sim \omega'} P(E_{\omega} \cap E_{\omega'}) \right).
\]

After Lemma 3.4 we only need to prove
\[
\sum_{\omega \sim \omega'} P(E_{\omega} \cap E_{\omega'}) = o(1).
\]

But it is a consequence of Lemma 3.3 with \( j = i \).
\[
\sum_{\omega \sim \omega'} P(E_{\omega} \cap E_{\omega'}) \ll i^{-1/s + o(1)}. \quad \square
\]
Lemma 3.6. — If \(i < j\) and \(I_i \cap I_j = \emptyset\) then
\[
\prod_{\omega \in \Omega_i \cup \Omega_j} P(E_{\omega}^c) \leq P(F_i)P(F_j)(1 + O(j^{-1/s} \log j)).
\]

Proof. — It is clear that
\[
\prod_{\omega \in \Omega_i \cup \Omega_j} P(E_{\omega}^c) = \left( \prod_{\omega \in \Omega_i} P(E_{\omega}^c) \right) \left( \prod_{\omega \in \Omega_j} P(E_{\omega}^c) \right) \cdot \left( \prod_{\omega \in \Omega_i \cap \Omega_j} P(E_{\omega}^c) \right)^{-1}.
\]
Harris’ inequality, applied to the first two products, gives
\[
\prod_{\omega \in \Omega_i \cup \Omega_j} P(E_{\omega}^c) \leq P(F_i)P(F_j) \left( \prod_{\omega \in \Omega_i \cap \Omega_j} P(E_{\omega}^c) \right)^{-1}.
\]
The logarithm of the last factor is
\[
- \sum_{\omega \in \Omega_i \cap \Omega_j} \log(1 - P(E_{\omega})) \sim \sum_{\omega \in \Omega_i \cap \Omega_j} P(E_{\omega})
\]
Since \(I_i \cap I_j = \emptyset\), if \(\omega \in \Omega_i \cap \Omega_j\) then \(|\omega| \leq s - 1\). Thus,
\[
\sum_{\omega \in \Omega_i \cap \Omega_j} P(E_{\omega}) \leq \sum_{|\omega| \leq s - 1} P(E_{\omega})
\]
\[
\leq \sum_{z \in I_j} \sum_{1 \leq r \leq s - 1} \sum_{a_1 + \cdots + a_r = s} \left( x_1 \cdots x_r \right)^{-1 + 1/s}.
\]
(Lem. 2.3(1)) \(\ll j^{-1/s} \log j\).
Thus,
\[
\left( \prod_{\omega \in \Omega_i \cap \Omega_j} P(E_{\omega}^c) \right)^{-1} \leq 1 + O(j^{-1/s} \log j)
\]
which ends the proof of the Lemma.

3.2. End of the proof

After those Lemmas we are ready to finish the proof of Theorem 1.1. If \(\alpha > 1/\lambda_s\) then
\[
\sum_i P(F_i) = \sum_i i^{-\alpha\lambda_s + o(1)} < \infty
\]
and Theorem 2.1 implies that with probability 1 only finitely many events $F_i$ occur. This proves that
\[
\limsup_{k \to \infty} \frac{b_{k+1} - b_k}{\log b_k} \leq 1/\lambda_s.
\]

If $\alpha < 1/\lambda_s$ then
\[
Z_n = \sum_{i \leq n} P(F_i) = \sum_{i \leq n} i^{-\alpha \lambda_s + o(1)} = n^{1-\alpha \lambda_s + o(1)} \to \infty.
\]

If in addition
\[
(3.3) \quad \lim_{n \to \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,
\]
Theorem 2.1 implies that with probability 1 infinitely many events $F_i$ occur and
\[
\limsup_{k \to \infty} \frac{b_{k+1} - b_k}{\log b_k} \geq 1/\lambda_s.
\]

Note that $P(F_i \cap F_j) \geq P(F_i)P(F_j)$ for all $1 \leq i < j \leq n$, so the limit (3.3) is not negative.

We next prove (3.3). We observe that
\[
F_i \cap F_j = \bigcap_{\omega \in \Omega_i \cup \Omega_j} E_\omega^c,
\]
so we can use Janson inequality to get
\[
P(F_i \cap F_j) \leq \prod_{\omega \in \Omega_i \cup \Omega_j} P(E_\omega^c) \times \exp \left(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \right).
\]

Observe that
\[
\sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \leq \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) + \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})
\]
\[
+ \sum_{\omega \in \Omega_i, \omega' \in \Omega_j} P(E_\omega \cap E_{\omega'}).
\]

Applying Lemma 3.3 to the three sums we have
\[
\sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s}(\log i)^3 + j^{-1/s}(\log j)^3 + i^{-1/s}(\log i)^2(\log j),
\]
\[
\omega, \omega' \in \Omega_i \cup \Omega_j
\]
and so
\[
\exp \left( 2 \sum_{\omega \sim \omega' \in \Omega_i \cup \Omega_j \atop \omega, \omega' \in \Omega_i \cup \Omega_j} P(E_\omega \cap E_{\omega'}) \right) \leq 1 + O \left( i^{-1/s} (\log i)^2 (\log j) \right).
\]

Thus,
\[
P(F_i \cap F_j) \leq \prod_{\omega \in \Omega_i \cup \Omega_j} P(E_\omega^c) \times \left( 1 + O(i^{-1/s} (\log i)^2 (\log j)) \right).
\]

Since \( \alpha < \lambda_s \), the number \( \beta = (1 - \alpha \lambda_s)/2 \) is positive. Now we split the sum in (3.3) into three sums:

\[
\Delta_{1n} = \sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)
\]

\[
\Delta_{2n} = \sum_{1 \leq i < j \leq n \atop i \leq n^\beta} P(F_i \cap F_j) - P(F_i)P(F_j)
\]

\[
\Delta_{3n} = \sum_{1 \leq i < j \leq n \atop j - \log j \leq i \leq j} P(F_i \cap F_j) - P(F_i)P(F_j)
\]

(1) Estimate of \( \Delta_{1n} \). Since in this case we have \( I_i \cap I_j = \emptyset \), we can apply Lemma 3.6 to (3.5) to get
\[
\prod_{\omega \in \Omega_i \cup \Omega_j} P(E_\omega^c) \leq P(F_i)P(F_j)(1 + O(j^{-1/s} \log j)).
\]

This inequality and (3.5) gives
\[
P(F_i \cap F_j) \leq P(F_i)P(F_j) \times \left( 1 + O(i^{-1/s} (\log i)^2 (\log j)) \right),
\]

so
\[
P(F_i \cap F_j) - P(F_i)P(F_j) \ll P(F_i)P(F_j)i^{-1/s} (\log i)^2 (\log j)
\]

\[
\ll n^{-\beta/s + o(1)} P(F_i)P(F_j).
\]

Thus
\[
\Delta_{1n} \ll n^{-\beta/s + o(1)} \sum_{i,j \leq n} P(F_i)P(F_j) \ll n^{-\beta/s + o(1)} Z_n^2.
\]
(2) Estimate of $\Delta_{2n}$. In this case we use the crude estimate

$$\tag{3.7} P(F_i \cap F_j) - P(F_i)P(F_j) \leq P(F_j).$$

We have

$$\tag{3.8} \Delta_{2n} \leq \sum_{j \leq n} \sum_{i \leq j} P(F_j) \leq \sum_{j \leq n} j^\beta P(F_j) \leq n^\beta Z_n \leq n^{-\beta+o(1)} Z_n^2,$$

since $Z_n = n^{1-\alpha \lambda_s+o(1)} = n^{2\beta+o(1)}$.

(3) Estimate of $\Delta_{3n}$. Again we use (3.7) and we have

$$\tag{3.9} \Delta_{3n} \leq \sum_{j \leq n} \sum_{j-\alpha \log j \leq i \leq j} P(F_j) \leq \alpha \log n \sum_{j \leq n} P(F_j) \leq n^{-2\beta+o(1)} Z_n^2.$$

Finally, using the estimates in (3.6), (3.8) and (3.9) we have

$$\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j) \frac{Z_n^2}{n^2} \ll n^{-\beta/s+o(1)} + n^{-\beta+o(1)} + n^{-2\beta+o(1)} \to 0.$$ 

This ends the proof of (3.3) and hence that of Theorem 1.1. □

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