On a classical limit of $q$-deformed Whittaker functions

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Abstract. We provide a derivation of the Givental integral representation of the classical $\mathfrak{gl}_{\ell+1}$ Whittaker function as a limit $q \to 1$ of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function represented as a sum over the Gelfand-Zetlin patterns.

Introduction

The $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions can be defined as eigenfunctions of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Toda chains [Ru], [Et]. Among various eigenfunctions there exists a special class of eigenfunctions with the support in the positive Weyl chamber. By analogy with the classical case we call such functions the class one $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions. In [GLO1] an explicit representation of the class one $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a sum over the Gelfand-Zetlin patterns was proposed. This representation has remarkable integrality and positivity properties. Precisely each term in the sum is a positive integer multiplied by a weight factor $q^{wt}$ and a character of the torus $U^{\ell+1}_1$. This allows to represent the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a character of a $\mathbb{C}^* \times U^{\ell+1}_1$-module (i.e. it allows a categorification). The interpretation of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a character shall be considered as a $q$-version of the Shintani-Casselman-Shalika formula [Sh], [CS]. Indeed in the limit $q \to 0$ the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function can be identified with the non-Archimedean Whittaker function and the representation of the $q$-deformed Whittaker function as a character reduces to the standard Shintani-Casselman-Shalika formula for non-Archimedean Whittaker function [Sh], [CS].

In the limit $q \to 1$ the $q$-Whittaker functions reproduces the classical Whittaker functions. It was pointed out in [GLO1] that in this limit an explicit sum type representation of the class one $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function turns into the Givental integral representation for the class one $\mathfrak{gl}_{\ell+1}$-Whittaker function [Gi] (see also [GKLO]). Thus the Givental integral representation shall be considered as the Archimedean counterpart of the Shintani-Casselman-Shalika formula (for more details on this interpretation see [GLO2], [GLO3], [G]). In this note we provide a precise description of the $q \to 1$ limit reducing the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function to its classical analog and explicitly demonstrate that the Givental integral representation arises as a limit of the sum representation of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function. This result is given by Theorem 3.1. The established relation between a sum over the Gelfand-Zetlin patterns for $\mathfrak{gl}_{\ell+1}$ and the Givental integrals for $\mathfrak{gl}_{\ell+1}$ is a special case of a general relation between the Gelfand-Zetlin patterns and the Givental type integrals for classical series of Lie algebras [GLO4]. This relation elucidates the identification of the Givental and the Gelfand-Zetlin graphs noticed in [GLO4]. The relation between the Gelfand-Zetlin and Givental constructions described in this note should be also compared with the duality type relation introduced in [GLO5]. We are going to discuss the
general form of the relation between the Gelfand-Zetlin and the Givental constructions for classical Lie algebras elsewhere.

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1 \(q\)-deformed \(gl_{\ell+1}\)-Whittaker function

In this Section we recall the explicit construction of the class one \(q\)-deformed \(gl_{\ell+1}\)-Whittaker functions derived in [GLO]. Quantum \(q\)-deformed \(gl_{\ell+1}\)-Toda chain (see e.g. [Rut], [E]) is defined by a set of \(\ell + 1\) mutually commuting functionally independent quantum Hamiltonians \(\mathcal{H}_{r}^{\ell+1}\), \(r = 1, \ldots, \ell + 1\):

\[
\mathcal{H}_{r}^{\ell+1}(P_{\ell+1}) = \sum_{I_{r}} \left( \tilde{X}_{i_{1}}^{1-\delta_{i_{2}-i_{1}+1}} \cdots \tilde{X}_{i_{r}+1}^{1-\delta_{i_{r}-i_{r-1}+1}} \cdot \tilde{X}_{i_{r}}^{1-\delta_{i_{r+1}-i_{r}}+1} \right) T_{i_{1}} \cdots T_{i_{r}},
\]

where \(r = 1, \ldots, \ell + 1\) and \(i_{r+1} = \ell + 2\). The summation in (1.1) goes over all ordered subsets \(I_{r} = \{i_{1} < i_{2} < \cdots < i_{r}\}\) of \(\{1, 2, \ldots, \ell + 1\}\). Here we use the notations

\[
T_{i}f(P_{\ell+1}) = f(\tilde{P}_{\ell+1}), \quad \tilde{p}_{\ell+1,k} = pt_{\ell+1,k} + \delta_{k,i}, \quad \tilde{X}_{i} = 1 - q^{pt_{\ell+1,i} - pt_{\ell+1,i+1} + 1}, \quad i = 1, \ldots, \ell, \quad \tilde{X}_{\ell+1} = 1.
\]

The corresponding eigenvalue problem can be written in the following form:

\[
\mathcal{H}_{r}^{\ell+1}(P_{\ell+1}) \Psi_{z_{1},\ldots,z_{\ell+1}}^{\ell+1}(P_{\ell+1}) = \left( \sum_{I_{r}} \prod_{i \in I_{r}} z_{i} \right) \Psi_{z_{1},\ldots,z_{\ell+1}}^{\ell+1}(P_{\ell+1}),
\]

and the first nontrivial Hamiltonian is given by

\[
\mathcal{H}_{1}^{\ell+1}(P_{\ell+1}) = \sum_{i=1}^{\ell} (1 - q^{pt_{\ell+1,i} - pt_{\ell+1,i+1} + 1}) T_{i} + T_{\ell+1}.
\]

One of the main results of [GLO] now can be formulated as follows. Given \(P_{\ell+1} = (p_{t_{\ell+1,1}}, \ldots, p_{t_{\ell+1,\ell+1}})\) let us denote by \(P^{(\ell+1)}(P_{\ell+1})\) a set of collections of the integer parameters \(p_{k,i}, k = 1, \ldots, \ell, i = 1, \ldots, k\) satisfying the Gelfand-Zetlin conditions \(p_{k+1,i} \geq p_{k,i} \geq p_{k+1,i+1}\). Let \(P_{t_{\ell+1}}(P_{\ell+1})\) be a set of \(p_{t_{\ell},\ell} = (p_{t_{\ell,1}}, \ldots, p_{t_{\ell,\ell}})\), \(p_{t,\ell} \in \mathbb{Z}\), satisfying the conditions \(p_{t+1,i} \geq p_{t,i} \geq p_{t+1,i+1}\).

**Theorem 1.1** A common solution of the eigenvalue problem (1.2) can be written in the following form. For \(P_{\ell+1}\) being the dominant domain \(p_{t_{\ell+1,1}} \geq \ldots \geq p_{t_{\ell+1,\ell+1}}\), the solution is given by

\[
\Psi_{z_{1},\ldots,z_{\ell+1}}^{\ell+1}(P_{\ell+1}) = \sum_{p_{k,i} \in P^{(\ell+1)}(P_{\ell+1})} \prod_{k=1}^{\ell+1} \frac{\sum_{i_{1}=0}^{k-1} \prod_{k=1}^{\ell+1} \prod_{i=1}^{k} (p_{k,i} - p_{k,i+1}) q^{i_{1}}}{\prod_{k=1}^{\ell+1} \prod_{i=1}^{k} (p_{k+1,i} - p_{k,i}) q^{i_{1}}}.
\]
where we use the notation \((n)_q! = (1 - q)...(1 - q^n)\). When \(p_{\ell+1} \) is outside the dominant domain we set

\[
\Psi^{g_{\ell+1}}_{p_1, ..., p_{\ell+1}}(p_{\ell+1,1}, ..., p_{\ell+1,\ell+1}) = 0.
\]

**Example 1.1** Let \( g = gl_2 \), \( p_{2,1} := p_1 \in \mathbb{Z} \), \( p_{2,2} := p_2 \in \mathbb{Z} \) and \( p_{1,1} := p \in \mathbb{Z} \). The function

\[
\Psi^{gl_2}_{z_1, z_2}(p_1, p_2) = \sum_{p_2 \leq p \leq p_1} \frac{p_{p_1+p_2-p} z_1^{p_1+p_2-p}}{(p_1 - p)_q!(p - p_2)_q!}, \quad p_1 \geq p_2,
\]

\[
\Psi^{gl_2}_{z_1, z_2}(p_1, p_2) = 0, \quad p_1 < p_2,
\]
is a common eigenfunction of mutually commuting Hamiltonians

\[
\mathcal{H}^{gl_2}_1 = (1 - q^{p_1-p_2})T_1 + T_2, \quad \mathcal{H}^{gl_2}_2 = T_1T_2.
\]

The formula (1.4) can be easily rewritten in the recursive form.

**Corollary 1.1** The following recursive relation holds

\[
\Psi^{g_{\ell+1}}_{z_1, ..., z_{\ell+1}}(p_{\ell+1}) = \sum_{p_1 \in P_{\ell+1}, i(P_{\ell+1})} \Delta(p_1) \sum_{i=1}^{\ell} p_{\ell+1,i} - \sum_{i=1}^{\ell} p_{\ell,i} Q_{\ell+1,i}(p_{\ell+1}, p_i) |q| \Psi^{g_{\ell}}_{z_1, ..., z_\ell}(p_i), \quad (1.5)
\]

where

\[
Q_{\ell+1,i}(p_{\ell+1}, p_i) |q| = \frac{1}{\prod_{i=1}^{\ell} (p_{\ell+1,i} - p_{\ell,i}) q! (p_{\ell,i} - p_{\ell+1,i+1}) q!}, \quad (1.6)
\]

\[
\Delta(p_1) = \prod_{i=1}^{\ell-1} (p_{\ell,i} - p_{\ell,i+1}) q!.
\]

The following representations of the class one \( q \)-deformed \( gl_{\ell+1} \)-Whittaker function are a consequence of the positivity and integrality of the coefficients of the \( q \)-series expansions of each term in the sum (1.4) (see [GLO1] for details).

**Proposition 1.1** (i) There exists a \( \mathbb{C}^* \times GL_{\ell+1}(\mathbb{C}) \) module \( V \) such that the common eigenfunction (1.4) of the \( q \)-deformed Toda chain allows the following representation for \( p_{\ell+1,1} \geq p_{\ell+1,2} \geq \ldots \geq p_{\ell+1,\ell+1} \):

\[
\Psi^{g_{\ell+1}}_{\Delta}(p_{\ell+1}) = \text{Tr}_V q^{L_0} \prod_{i=1}^{\ell+1} q^{\lambda(H_i)}, \quad (1.7)
\]

where \( H_i := E_{i,i}, \quad i = 1, \ldots, \ell+1 \) are Cartan generators of \( gl_{\ell+1} = \text{Lie}(GL_{\ell+1}) \) and \( L_0 \) is a generator of \( \text{Lie}(\mathbb{C}^*) \).

(ii) There exists a finite-dimensional \( \mathbb{C}^* \times GL_{\ell+1}(\mathbb{C}) \) module \( V_f \) such that the following representation holds for \( p_{\ell+1,1} \geq p_{\ell+1,2} \geq \ldots \geq p_{\ell+1,\ell+1} \):

\[
\Psi^{g_{\ell+1}}_{\Delta}(p_{\ell+1}) = \Delta(p_{\ell+1}) \Psi^{g_{\ell+1}}_{\Delta}(p_{\ell+1}) = \text{Tr}_V q^{L_0} \prod_{i=1}^{\ell+1} q^{\lambda(H_i)}. \quad (1.8)
\]

The module \( V \) entering (1.7) and the module \( V_f \) entering (1.8) have a structure of modules under the action of (quantum) affine Lie algebras [GLO1].
2 Classical limit of $q$-deformed Toda chain

In this Section we define a limit $q \to 1$ of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain reproducing the standard $\mathfrak{gl}_{\ell+1}$-Toda chain. We provide an explicit check that the first two generators of the ring of quantum Hamiltonians of $\mathfrak{gl}_{\ell+1}$-Toda chain arise as a limit of some combinations of the following quantum Hamiltonians of the $q$-deformed Toda chain

$$H^q_{\ell+1}(p_{\ell+1}|q) = \sum_{i=1}^{\ell} \left(1 - q^{p_{\ell+1,i} - p_{\ell+1,i+1} + 1} \right) T_i + T_{\ell+1},$$

$$H^q_{\ell+1}(p_{\ell+1}|q) = T_1 T_2 \cdots T_{\ell+1}. $$

Let us introduce the following parametrization:

$$q = e^{-\varepsilon}, \quad p_{\ell+1,k} = (\ell + 2 - 2k)m(\varepsilon) + x_{\ell+1,k}\varepsilon^{-1}. \quad (2.1)$$

Here $m(\varepsilon) \in \mathbb{Z}$ is given by

$$m(\varepsilon) = -\lfloor -\varepsilon \ln \varepsilon \rfloor,$$

and $[x] \in \mathbb{Z}$ is the integer part of $x$.

**Proposition 2.1** The following limiting relations hold:

$$H^q_{\ell+1}(p_{\ell+1}|q) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ H^q_{\ell+1}(p_{\ell+1}(x,\varepsilon)|q(\varepsilon)) - (\ell + 1) \right],$$

$$H^q_{\ell+1}(p_{\ell+1}|q) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left[ H^q_{\ell+1}(p_{\ell+1}(x,\varepsilon)|q(\varepsilon)) - H^q_{\ell+1}(p_{\ell+1}(x,\varepsilon)|q(\varepsilon)) - \ell + \frac{1}{2} \left( H^q_{\ell+1}(p_{\ell+1}(x,\varepsilon)|q(\varepsilon)) - 1 \right)^2 \right]$$

where $H^q_{i}$, $i = 1, 2$ are the standard quantum Hamiltonians of the $\mathfrak{gl}_{\ell+1}$-Toda chain:

$$H^q_{1}(p_{\ell+1}) = \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_{\ell+1,i}},$$

$$H^q_{2}(p_{\ell+1}) = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{\ell+1,i} - x_i}.$$ 

**Proof.** Using the fact that $\exp(\varepsilon[2(\varepsilon)^{-1}\ln(\varepsilon)]) = e^2(1 + O(\varepsilon^2/\ln \varepsilon))$ we have

$$H^q_{1}(p_{\ell+1}|q) = (\ell + 1) + \varepsilon \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_{\ell+1,i}} + \varepsilon^2 \left( \frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i}^2} - \sum_{k=1}^{\ell} e^{x_{\ell+1,k} - x_{\ell+1,k}} \right) + O(\varepsilon^3),$$

$$H^q_{\ell+1}(p_{\ell+1}|q) = 1 + \varepsilon \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_{\ell+1,i}} + \frac{1}{2} \varepsilon^2 \sum_{i,j=1}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i} \partial x_{\ell+1,j}} + O(\varepsilon^3).$$


Now the limiting formulas can be straightforwardly verified. We have
\[
H_{1}^{gl_{\ell+1}}(p_{\ell+1}|q) - H_{\ell+1}^{gl_{\ell+1}}(p_{\ell+1}|q) - \ell = \epsilon^{2} \left( - \frac{1}{2} \sum_{i,j=1}^{\ell+1} \frac{\partial^{2}}{\partial x_{\ell+1,i} \partial x_{\ell+1,j}} - \sum_{k=1}^{\ell} e^{x_{\ell+1,k} + x_{\ell+1,k} - x_{\ell+1,k}} \right) + O(\epsilon^{3}),
\]
\[
\frac{1}{2} \left( H_{1}^{gl_{\ell+1}}(p_{\ell+1}|q) - 1 \right)^{2} = \frac{1}{2} \epsilon^{2} \left( \sum_{i,j=1}^{\ell+1} \frac{\partial^{2}}{\partial x_{\ell+1,i} \partial x_{\ell+1,j}} \right) + O(\epsilon^{3}),
\]
and thus
\[
H_{1}^{gl_{\ell+1}}(p_{\ell+1}|q) - H_{\ell+1}^{gl_{\ell+1}}(p_{\ell+1}|q) - \ell + \frac{1}{2} \left( H_{\ell+1}^{gl_{\ell+1}}(p_{\ell+1}|q) - 1 \right)^{2} = \epsilon^{2} \left( \frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^{2}}{\partial x_{\ell+1,i}^{2}} - \sum_{k=1}^{\ell} e^{x_{\ell+1,k} + x_{\ell+1,k} - x_{\ell+1,k}} \right) + O(\epsilon^{3}).
\]

\[\square\]

It is easy to see that the eigenfunction problem \((1.2)\) is transformed into the standard eigenfunction problem if we use the following parametrization of the spectral variables \(z_{i} = e^{x_{i}}\lambda_{i}, i = 1, \ldots, \ell + 1\).

3 Classical limit of class one Whittaker function

In the limit \(q \to 1\) defined in the previous Section the class one solution \((1.4)\) of the \(q\)-deformed \(gl_{\ell+1}\)-Toda chain should go to the class one solution of the classical \(gl_{\ell+1}\)-Toda chain. In the classical setting an integral representation for class one \(gl_{\ell+1}\)-Whittaker function was constructed by Givental [G], (see [GKLO] for a choice of the contour realizing class one condition)

\[
\psi_{\lambda_{1},\ldots,\lambda_{\ell+1}}^{gl_{\ell+1}}(x_{1}, \ldots, x_{\ell+1}) = \int_{C} \prod_{k=1}^{\ell+1} d\mathcal{E}_{k} e^{x^{gl_{\ell+1}}}(x), \tag{3.1}
\]

and the function \(\mathcal{F}^{gl_{\ell+1}}(x)\) is given by

\[
\mathcal{F}^{gl_{\ell+1}}(x) = \sum_{n=1}^{\ell+1} \lambda_{n} \left( \sum_{i=1}^{n} x_{n,i} - \sum_{i=1}^{n-1} x_{n-1,i} \right) - \sum_{k=1}^{\ell+1} \sum_{i=1}^{\ell+1} \left( e^{x_{k,i} - x_{k+1,i}} + e^{x_{k+1,i+1} - x_{k,i}} \right), \tag{3.2}
\]

Here \(C \subset N_{+}\) is a small deformation of the subspace \(\mathbb{R}^{(\ell+1)^{2}} \subset \mathbb{C}^{(\ell+1)^{2}}\) making the integral \((3.2)\) convergent. Besides, we use the following notation: \(\lambda = (\lambda_{1}, \ldots, \lambda_{\ell+1})\); \(x_{i} := x_{\ell+1,i}, i = 1, \ldots, \ell+1\).

The integral representation \((3.1)\) allows a recursive presentation analogous to \((1.5)\)

\[
\psi_{\lambda_{1},\ldots,\lambda_{\ell+1}}^{gl_{\ell+1}}(x_{1}, \ldots, x_{\ell+1}) = \int_{\mathbb{R}^{\ell}} d\mathcal{E} Q_{\mathcal{F}^{gl_{\ell+1}}}(x_{\ell+1}; \mathcal{E}; \lambda_{\ell+1}) \psi_{\lambda_{1},\ldots,\lambda_{\ell}}^{gl_{\ell}}(x_{\ell}, \ldots, x_{\ell}), \tag{3.3}
\]

where

\[
Q_{gl_{\ell}}(x_{\ell+1}; \mathcal{E}; \lambda_{\ell+1}) = \exp \left\{ \sum_{i=1}^{\ell+1} \lambda_{\ell+1} \left( \sum_{i=1}^{\ell+1} x_{\ell+1,i} - \sum_{i=1}^{\ell} x_{\ell,i} \right) \right. \]
\[
- \sum_{i=1}^{\ell} \left( e^{x_{\ell,i} - x_{\ell+1,i}} + e^{x_{\ell+1,i+1} - x_{\ell,i}} \right) \left\}, \tag{3.4}
\]
and we assume \( Q_{\mathfrak{gl}_1}(x_{11}; \lambda_1) = e^{\lambda_1 x_{11}} \).

In the following we demonstrate that in the previously defined limit \( q \to 1 \) the class one \( q \)-deformed \( \mathfrak{gl}_{\ell+1} \)-Whittaker function given by the sum \((1.4)\) indeed turns into the classical class one \( \mathfrak{gl}_{\ell+1} \)-Whittaker function given by the integral representation \((3.1)\). In particular iterative formula \((1.5)\) turns into \((3.3)\). For this purpose we need the following asymptotic of the \( q \)-factorials entering \((1.4)\).

**Lemma 3.1** Let us introduce the following functions

\[
f_\alpha(y, \epsilon) = (y/\epsilon + \alpha m(\epsilon))q!, \quad \alpha = 1, 2,
\]

where \( m(\epsilon) = -[\epsilon^{-1} \ln \epsilon], \, q = e^{-\epsilon}. \) Then for \( \epsilon \to +0 \) the following expansions hold:

\[
f_1(y, \epsilon) = e^{A(\epsilon) + e^{-y} + O(\epsilon)} ; \quad (3.5)
\]

\[
f_2(y, \epsilon) = e^{A(\epsilon) + O(e^{\alpha-1})} , \quad (3.6)
\]

where \( A(\epsilon) = -\frac{\pi^2}{6} \frac{1}{\epsilon} - \frac{1}{2} \ln \frac{\epsilon}{2\pi} \).

**Proof.** Taking into account the identity

\[
\ln \prod_{n=1}^{N} (1 - q^n) = \sum_{n=1}^{N} \ln (1 - q^n) = - \sum_{n=1}^{N} \sum_{r=1}^{+\infty} \frac{1}{r} q^{nr} = - \sum_{r=1}^{+\infty} \frac{q^r}{r} \left( \frac{1 - q^{N r}}{1 - q^r} \right),
\]

and using the substitution \( q = e^{-\epsilon}, \, N = \epsilon^{-1} y + \alpha m(\epsilon) \) we obtain

\[
\ln f_\alpha(y, \epsilon) = - \sum_{r=1}^{+\infty} \frac{e^{-r \epsilon}}{r} \left( \frac{1 - e^{r \alpha m(\epsilon)} e^{-y \epsilon}}{1 - e^{-r \epsilon}} \right).
\]

Now expanding the denominator over small \( \epsilon \) we have

\[
\ln f_\alpha(y, \epsilon) = - \sum_{r=1}^{+\infty} \frac{e^{-r \epsilon}}{r} \left( 1 - e^{r \alpha m(\epsilon)} e^{-y \epsilon} \right) + \cdots = - \sum_{r=1}^{+\infty} \frac{e^{-r \epsilon}}{r^2 \epsilon} \left( \frac{1 - e^{r \alpha m(\epsilon)} e^{-y \epsilon}}{1 - \frac{1}{2} r \epsilon + \frac{1}{4} r^2 \epsilon^2 + \cdots} \right) + \cdots
\]

and for the derivative we obtain

\[
\partial_y \ln f_\alpha(y, \epsilon) = - \sum_{r=1}^{+\infty} \frac{1}{r \epsilon} \left( \frac{e^{r \alpha m(\epsilon)} e^{-y \epsilon} - r \epsilon}{1 - \frac{1}{2} r \epsilon + \frac{1}{4} r^2 \epsilon^2 + \cdots} \right) + \cdots = \sum_{k=-1}^{+\infty} c_k I_{\alpha, k}(y, \epsilon),
\]

where

\[
I_{\alpha, k}(y, \epsilon) = \sum_{r=1}^{+\infty} e^{k + \alpha r} r^k e^{-y \epsilon} = e^{k \sum_{r=1}^{+\infty} r^k}, \quad t = e^{-y \epsilon^{\alpha}},
\]

and \( c_{-1} = -1 \). Let us separately analyze the term \( I_{\alpha, -1} \) and the other terms \( I_{\alpha, k \geq 0} \). We have

\[
I_{\alpha, k \geq 0}(y, \epsilon) = e^{k \left( \frac{\partial}{\partial t} \right)^k} \frac{1}{1 - t} = e^{k \frac{\partial^k}{\partial y^k} \frac{1}{1 - e^{\alpha y}}},
\]

and thus

\[
I_{\alpha, k \geq 0} = e^{k + \alpha} e^{-y} + \cdots, \quad \alpha = 1, 2.
\]
Now consider the case of $k = -1$
\[
c_{-1} I_{\alpha, -1}(y, \epsilon) = -\frac{1}{\epsilon} \sum_{r=1}^{+\infty} \frac{r^r}{r} = -\frac{1}{\epsilon} \ln(1 - t) = -\frac{1}{\epsilon} \ln(1 - e^{\alpha} e^{-y})) = e^{\alpha - 1} e^{-y} + \cdots , \quad t = e^{-y} \epsilon^\alpha.
\]

This gives (3.5), (3.6) with an unknown $A(\epsilon)$. To calculate $A(\epsilon)$ we take $e^{-y} = 0$ and notice that the resulting function does not depend on $\alpha$. Thus we should calculate the asymptotic of the following function:
\[
\ln f_{\alpha}(y, \epsilon) \bigg|_{e^{-y} = 0} = -\sum_{r=1}^{+\infty} \frac{1}{r} \left( 1 - e^{\alpha} e^{-ry} \right) \left( 1 - e^{-r\epsilon} \right) = -\sum_{n=1}^{+\infty} \sum_{r=1}^{+\infty} \frac{1}{r} e^{-n\epsilon} = \ln \prod_{n=1}^{+\infty} (1 - e^{-n\epsilon}).
\]

It can be easily done using the modular properties
\[
\eta(-\tau^{-1}) = \sqrt{-i \pi} \eta(\tau),
\]
of the Dedekind eta function
\[
\eta(\tau) = e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).
\]
Namely, taking $\tau = \frac{i\epsilon}{2\pi}$ we have
\[
f_{\alpha}(y; \epsilon) \bigg|_{e^{-y} = 0} = \sqrt{2\pi \epsilon^{-1}} e^{-\frac{\epsilon^2}{6} \epsilon^{-1}} \prod_{n=1}^{\infty} (1 - e^{-\epsilon^{-1}(2\pi)^2 n}) .
\]

This allows to infer the following result for the leading coefficients in the asymptotic expansion of $\ln f_{\alpha}(y, \epsilon) \bigg|_{e^{-y} = 0}$:
\[
A(\epsilon) = -\frac{\pi^2}{6} \epsilon^{-1} , \quad \epsilon \to +0.
\]

This completes the proof of Lemma. \( \square \)

**Theorem 3.1** Let us use the following parametrization
\[
q = e^{-\epsilon}, \quad p_{\ell+1,k} = (\ell + 2 - 2k)m(\epsilon) + e^{-1} x_{\ell+1,k}, \quad z_k = e^{i \epsilon \lambda_k},
\]
where $k = 1, \ldots, \ell + 1$, $m(\epsilon) = -[e^{-1} \ln \epsilon]$. The integral representation (3.1) of the classical $gl_{\ell+1}$-Whittaker function is given by the following limit of the $q$-deformed class one $gl_{\ell+1}$-Whittaker function represented as a sum (1.4)
\[
\psi_{\lambda_1, \ldots, \lambda_{\ell+1}}^{gl_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \lim_{\epsilon \to +0} \left[ e^{\frac{\ell(\ell+1)}{2}} e^{\frac{\ell(\ell+3)}{2} A(\epsilon)} \psi_{\ell+1, \ldots, \ell+1}(p_{\ell+1}) \right],
\]
where $A(\epsilon) = -\frac{\pi^2}{6} \epsilon^{-1} - \frac{1}{2} \ln \frac{\epsilon}{2\pi}$ and $x_i = x_{\ell+1,i}, i = 1, \ldots, \ell + 1$.  

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Proof. We prove \(3.9\) by relating the recursive relation \((1.5)\)

\[
\Psi_{\ell+1}^{\ell} \left( \mathcal{P}_{\ell+1}^* \right) = \sum_{p_\ell \in \mathcal{P}_{\ell+1,\ell}(\mathcal{P}_{\ell+1}^*)} \Delta(p_\ell) \sum_{j=1}^{\ell+1} p_{\ell+1,j} \sum_{i=1}^{\ell} p_{\ell,j} \Delta(p_\ell) \Psi_{\ell+1,\ell}(p_\ell) q \Psi_{\ell}^{\ell},
\]

where

\[
Q_{\ell+1,\ell}(p_{\ell+1}^*, p_\ell q) = \frac{1}{\prod_{i=1}^{\ell} (p_{\ell+1,i} - p_{\ell,i})! (p_{\ell,i} - p_{\ell+1,i+1})!},
\]

\[(3.10)\]

with the recursive relation \((3.3)\) for the classical Whittaker function.

Let us introduce the following parametrization of the elements of the Gelfand-Zetlin patterns \(p_\ell \in \mathcal{P}_{\ell+1,\ell}(\mathcal{P}_{\ell+1}^*)\):

\[
p_{\ell,k} = \epsilon^{-1} x_{\ell,k} + a_k m(\epsilon), \quad m(\epsilon) = -[\epsilon^{-1} \ln \epsilon],
\]

\[(3.11)\]

where \(a_k\) are some constants. The Gelfand-Zetlin conditions on weights \(p_{\ell+1}^*\) read as follows:

\[
p_{\ell+1,k} \geq p_{\ell,k} \geq p_{\ell+1,k+1}, \quad k = 1, \ldots, \ell,
\]

and they lead to

\[
\epsilon^{-1} x_{\ell+1,k} + (\ell + 2 - 2k)m(\epsilon) \geq \epsilon^{-1} x_{\ell,k} + a_k m(\epsilon) \geq \epsilon^{-1} x_{\ell+1,k+1} + (\ell - 2k)m(\epsilon).
\]

The requirement that the limit \(\epsilon \to +0\) preserves the conditions \((3.12)\) implies the following restrictions on the parameters \(a_k\):

\[
\ell - 2k + 2 > a_k > \ell - 2k, \quad k = 1, \ldots, \ell.
\]

\[(3.13)\]

Since \(p_\ell = (p_{\ell,1}, \ldots, p_{\ell,\ell}) \in \mathbb{Z}^\ell\) the only consistent choice in the limit \(\epsilon \to +0\) is \(a_k = \ell + 1 - 2k, k = 1, \ldots, \ell\). Although the variables \(p_{\ell,k}\) are restricted to be in positive Weyl chamber i.e. \(p_{\ell,k} \geq p_{\ell,k+1}\), in the limit \(\epsilon \to +0\) the variables \(x_{\ell,k}\) have no such restrictions. This follows from a simple observation that the limit \(\epsilon \to +0\) the \(\frac{a}{\epsilon} - b[\epsilon^{-1} \ln \epsilon] \to +\infty/ - \infty\) depends only on the sign of non-zero coefficient \(b\). Thus we have

\[
p_{\ell,k} = \epsilon^{-1} x_{\ell,k} + (\ell + 1 - 2k)m(\epsilon).
\]

\[(3.14)\]

Now using Lemma \(3.1\) it is easy to obtain the following limiting formulas:

\[
\lim_{\epsilon \to +0} e^{2A(\epsilon)} Q_{\ell+1,\ell}(p_{\ell+1}^*, p_\ell q) = \lim_{\epsilon \to +0} e^{2A(\epsilon)} \prod_{i=1}^{\ell} f_1(x_{\ell+1,i} - x_{\ell,i}, \epsilon) f_1(x_{\ell,i} - x_{\ell+1,i+1}, \epsilon)
\]

\[
= Q_{\ell+1}^{\ell+1} \left( x_{\ell+1}; x_{\ell}; \lambda_{\ell+1} \right) \big|_{\lambda_{\ell+1}=0};
\]

\[(3.15)\]

\[
\lim_{\epsilon \to +0} e^{(1-\ell)A(\epsilon)} \Delta(p_\ell) = \lim_{\epsilon \to +0} e^{(1-\ell)A(\epsilon)} \prod_{i=1}^{\ell-1} f_2(x_{\ell,i} - x_{\ell,i+1}, \epsilon) = 1,
\]

\[(3.16)\]
where $Q_{gl}^{\ell+1}(x_{\ell+1}; z; \lambda_{\ell+1})$ is given by (3.4). This implies the following identity:

$$
\lim_{\epsilon \to +0} \left\{ e^{\ell} \sum_{p_j \in P_{\ell+1,\ell}(P_{\ell+1})} e_{p_{\ell+1,\ell}} \left[ e^{(\ell+1)A(\epsilon)} Q_{\ell+1,\ell}(p_{\ell+1}; q) \Delta(p_j) \right] \right\}
$$

$$
\times e^{-\frac{(\ell-1)}{2} A(\epsilon)} e^{-\frac{(\ell-1)(\ell+1)}{2} A(\epsilon)} \Psi_{z_1,\ldots,z_{\ell}}(p_j) \right\}
$$

$$
\int_{\mathbb{R}^\ell} \frac{d\tilde{\omega}}{\epsilon} \exp \left\{ i\lambda_{\ell+1} \left( \sum_{i=1}^{\ell+1} x_{\ell+1,i} - \sum_{j=1}^{\ell} x_{\ell,j} \right) \right\}
$$

$$
\times \lim_{\epsilon \to +0} \left[ e^{(\ell+1)A(\epsilon)} Q_{\ell+1,\ell}(\tilde{p}_{\ell+1}; q(\epsilon)) \Delta(x_{\ell+1}, \epsilon) \right]
$$

$$
\times \lim_{\epsilon \to +0} \left[ e^{-\frac{(\ell-1)}{2} A(\epsilon)} e^{-\frac{(\ell-1)(\ell+1)}{2} A(\epsilon)} \Psi_{z_1,\ldots,z_{\ell}}(\tilde{p}_{\ell+1}) \right].
$$

Thus we recover the recursive relations (3.3) for the Givental integrals directly leading to the integral representation (3.1) for the classical $gl_{\ell+1}$-Whittaker function. Using (3.17) iteratively over $\ell$ we obtain (3.3). □

**Example 3.1** For $\ell = 1$ we have

$$
\Psi^{gl}_{z_1,z_2}(p_{2,1}, p_{2,2}) = \sum_{p_{2,2} \leq p_{1,1} \leq p_{2,1}} \frac{p_{1,1}^{p_{2,1}+p_{2,2}-p_{1,1}}}{(p_{1,1}-p_{2,2})! (p_{2,1}-p_{1,1})!}, \quad p_{2,2} \leq p_{2,1},
$$

$$
\Psi_{z_1,z_2}(p_{2,1}, p_{2,2}) = 0, \quad p_{2,2} > p_{2,1}.
$$

Using the parametrization

$$
q = e^{-\epsilon}, \quad p_{21} = m(\epsilon) + x_{21} e^{-1} \quad p_{22} = -m(\epsilon) + x_{21} e^{-1} \quad z_i = e^{\epsilon \lambda_i}, \quad i = 1, 2,
$$

with $m(\epsilon) = -[\epsilon^{-1} \ln \epsilon]$ we obtain

$$
\Psi^{gl}_{z_1,z_2}(p_{21}, p_{22}) = \sum_{x_{22} - m(\epsilon) \leq x_{11} \leq x_{21}, \epsilon m(\epsilon)} e^{\lambda_1 x_{11} + \lambda_2 (x_{21} + x_{22} - x_{11})} \frac{((x_{11} - x_{22})/\epsilon + m(\epsilon)) q!}{((x_{21} - x_{11})/\epsilon + m(\epsilon)) q!},
$$

where we use the notations $p_{11} = x_{11}/\epsilon$. Taking into account

$$
\frac{1}{(y/\epsilon + m(\epsilon)) q!} = e^{\frac{y^2}{6} + \frac{1}{2} \ln \frac{y}{\epsilon} - \ln \frac{\pi}{y} - y + O(\epsilon)},
$$

we obtain

$$
\Psi^{gl}_{\lambda_1,\lambda_2}(x_1, x_2) = \lim_{\epsilon \to +0} e^{-\frac{y^2}{6} + \frac{1}{2} \ln \frac{y}{\epsilon}} \Psi^{gl}_{z_1,z_2}(p_{21}, p_{22})
$$

$$
= \int_{\mathbb{R}} dx_{11} e^{\lambda_1 x_{11}} e^{\lambda_2 (x_{21} + x_{22} - x_{11})} e^{-e^{x_{11}} - x_{21} - e^{x_{22}} - x_{11}}.
$$
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