SOLVABILITY FOR PARABOLIC POINCARÉ PROBLEM

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain and $\mathcal{L}$ be an elliptic second order differential operator defined in $\Omega$. Consider a vector field $\ell(x)$ defined on $\partial \Omega$. It generates a first order boundary operator $\mathcal{B}$ through oblique derivative with respect to $\ell(x)$. The couple $(\mathcal{L}, \mathcal{B})$ defines an elliptic oblique derivative problem (ODP). It is well known that a boundary value problem for any elliptic (and parabolic) differential operator is well-posed if it satisfies the Shapiro-Lopatinskii (S-L) complementary condition. In a case of a second order operator the ODP is regular (or it satisfies the mentioned condition) when the generating vector field is nowhere tangential to the boundary. Otherwise the problem is called degenerate or tangential. An exception of this rule is the two dimensional case when the S-L condition always holds, i.e. the oblique derivative problem in the plane is always regular even when $\ell(x)$ is tangential in some points of the boundary. The theory of regular elliptic and parabolic ODP elaborated in Hölder and Sobolev spaces could be found in [1], [5], [7], [8] and many others.

For a first time a tangential problem for the Laplace operator $(\Delta, \mathcal{B})$ was posed by Poincaré in his study of the theory of tides, but he did not solve it. Later it became an object of study through modern mathematical techniques as the theory of pseudo differential operators, the Fourier integral operators and others. The Poincaré problem still provokes interest because of different effects that occur near to the set of tangency and the lack of unify approach in its study.

The qualitative properties of the solution strongly depend on the behavior of the $\ell$-integral curves near to the set of tangency (order of contact, direction of integral curves, etc.). In this context we can distinguish three types of contact: neutral — the $\ell$-curves always enter into or leave $\Omega$ (i.e. $\ell$ preserves its sign near to the set of tangency); emergent — the $\ell$-curves first enter into $\Omega$ and after the contact go out of the domain ($\ell$ changes its sign from $-\to+$ near to the set of tangency); submergent — al contrary of the previous case $\ell$ changes its sign from $+\to-$ (under a sign of $\ell$ we always mean the sign of the scalar product $\ell \cdot \nu$ and $\nu$ is the unit outer normal to $\partial \Omega$). A various results about elliptic and parabolic Poincaré problems in Hölder and $H^s$ spaces are presented in [10], [12], [11] [13], [16], [17], [18], [9].

In the present work we consider degenerate ODP in a bounded cylinder $Q = \Omega \times (0, T)$ for a linear second-order uniformly parabolic operator $\mathcal{P}$ with
coefficients allowing discontinuity in \( t \). The vector field \( \ell(x,t) \) generating \( \mathcal{B} \) is defined on \( S = \partial\Omega \times (0, T) \) and is tangential to it in some subset \( E \). The kind of contact is of neutral type and we suppose that \( \gamma(x,t) = (\ell(x,t) \cdot \nu(x)) \geq 0 \) on \( S \). It means that the boundary value problem under consideration is of Fredholm type, i.e. both the kernel and cokernel are of finite dimension.

We are interested of strong solvability of our problem in \( W^{2,1}_p(Q), p \in (1, \infty) \). Because of the loss of regularity of the solution near to the set of tangency \( E \) we impose higher regularity in \( \nu \). The study is based on the original Winzel’s idea to extend \( \ell \) into \( \Omega \) such that to obtain explicit representation of the solution through the integral curves of that extension. Thus the problem is reduced to obtaining of suitable a priori estimates for the solution and its derivatives on an expanding family of cylinders. Further, the solvability is proved using regularization technique which, roughly speaking, means to perturb the vector field \( \ell \) by adding small \( \epsilon \) times \( \nu \), to solve the such obtained regular \( \text{ODP} \) and then pass to limit as \( \epsilon \to 0 \). The perturbed problem regards linear uniformly parabolic operator \( \mathcal{P} \) with VMO coefficients and boundary operator \( \mathcal{B} \) with \( (\ell \cdot \nu) > 0 \). In this case we dispose of unique solvability result in \( W^{2,1}_p(Q), p \in (1, \infty) \) supposing \( \mathcal{P}u \in L^p(Q) \) and initial and boundary data belonging to the corresponding Besov spaces (see [15], [8]).

Poincaré problem for linear uniformly parabolic operators with Hölder continuous coefficients is studied in [11] (see also [13]) where unique solvability in the corresponding Hölder spaces is obtained. Moreover, the linear results were applied to the study of semilinear parabolic problem in Hölder spaces. A tangential ODP for second-order uniformly elliptic operators with Lipschitz continuous coefficients was studied in [9] (see also [8]). It is obtained strong solvability in \( W^{2,p}(\Omega) \) but for \( p > n/2 \).

In our case the parabolic structure of the equation permits to obtain an a priori estimate for the solution in \( W^{2,1}_p(Q) \) only through the data of the problem. Thus we are able to prove unique solvability for all \( p \in (1, \infty) \) avoiding the use of maximum principle and omitting any additional conditions on the vector field.

### 2. Statement of the problem and main results

Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \) be a bounded domain with \( \partial\Omega \in C^{2,1} \) and \( Q = \Omega \times (0,T) \) be a cylinder in \( \mathbb{R}^{n+1} \). Set \( \ell(x,t) = (\ell^1(x,t), \ldots, \ell^n(x,t),0) \) for a unit vector field defined on the lateral boundary \( S = \partial\Omega \times (0,T) \). We consider the following oblique derivative problem

\[
\begin{align*}
\mathcal{P}u &\equiv u_t - a^{ij}(x,t)D_{ij} u = f(x,t) \quad \text{in } Q, \\
\mathcal{I}u &\equiv u(x,0) = \psi(x) \quad \text{on } \Omega, \\
\mathcal{B}u &\equiv \frac{\partial u}{\partial \ell} = \ell^i(x,t)D_i u = \varphi(x,t) \quad \text{on } S.
\end{align*}
\]

Denote by \( \nu(x) = (\nu^1(x), \ldots, \nu^n(x)) \) the unit outward normal to \( \partial\Omega \). Then we can write \( \ell(x,t) = \tau(x,t) + \gamma(x,t)\nu(x) \) where \( \tau(x,t) \) is tangential projection
of \( \ell \) on \( S \) and \( \gamma(x, t) = (\ell(x, t) \cdot \nu(x)) \geq 0 \). Let \( E \subset S \) be the set of tangency and \( E \cap \partial \Omega = E_0 \).

The set of tangency has the form \( E = E_0 \times (0, T) \) where \( E_0 \subset \partial \Omega \). By \( \Sigma \) we denote a cylinder with a base \( \Sigma_0 \subset \Omega \) being small neighborhood of \( E_0 \), \( S_\Sigma = \partial \Sigma \cap S \) such that \( E \subset S_\Sigma \). Thus \( \Sigma = \Sigma_0 \times (0, T) \) is a domain where we shall impose more restrictive conditions on the data of (2.1), while in \( Q \setminus \Sigma \) we can take the same conditions as in the regular case. Precisely:

(i) \( \mathcal{P} \) is a uniformly parabolic operator: \( \exists \lambda > 0 \) such that

\[
\begin{aligned}
&\Lambda |\xi|^2 \leq a^{ij}(x, t)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \text{a.a.} \ (x, t) \in Q, \ \forall \xi \in \mathbb{R}^n, \\
&a^{ij} \in V^1_\infty(\Sigma) \cap VMO(Q), \quad a^{ij} = a^{ji} \quad (\implies a^{ij} \in L^\infty(Q)).
\end{aligned}
\]

Here \( VMO \) is the Sarason class of function with vanishing mean oscillation (see [14]) and \( V^1_\infty(\Sigma) := \{ v, D_x v \in L^\infty(\Sigma) \} \).

(ii) \( \mathcal{B} \) is a degenerate oblique derivative operator defined through a tangential vector field of neutral type:

\[
\begin{aligned}
&\gamma(x, t) = (\ell(x, t) \cdot \nu(x)) \geq 0 \text{ on } S, \quad \gamma(x, t) = 0 \text{ on } E \subset S, \\
&\ell'(x, t) \in \text{Lip}(S) \cap W^2_\infty(S_\Sigma)
\end{aligned}
\]

where the space \( W^2_\infty(S_\Sigma) \) consists of functions having \( L^\infty(S_\Sigma) \) derivatives in \( x \) up to order 2 and in \( t \) of order 1.

(iii) Regularity of the data: for all \( p \in (1, \infty) \)

\[
\begin{aligned}
(iii_a) \quad &f(x, t) \in V_p(Q) := L^p(Q) \cap V^1_p(\Sigma), \\
&\|f\|_{V_p(Q)} = \|f\|_{p, Q} + \|f\|_{V^1_p(\Sigma)} = \|f\|_{p, Q} + \|D_x f\|_{p, \Sigma}; \\
(iii_b) \quad &\psi(x) \in W^{3-2/p}_p(\Omega), \quad \|\psi\|_{(3-2/p), \Omega} = \sum_{s=0}^{[3-2/p]} \sum_{(s)} \|D^s_x \psi\|_{p, \Omega} + \ll \psi \gg_{(3-2/p), \Omega}
\end{aligned}
\]

\[
\ll \psi \gg_{(3-2/p)} = \sum_{(s=[3-2/p])} \left( \int_\Omega dx \int_\Omega |D^s_x \psi(x) - D^s_y \psi(y)|^p \frac{dy}{|x-y|^{n+\alpha}} \right)^{1/p}
\]

where \( \alpha = 3 - 2/p - [3 - 2/p] \).

\[
(iii_c) \quad \varphi(x, t) \in W_p(S) := W^{1-1/p, 1/2-1/2p}_p(S) \cap W^{2-1/p, 1-2/p}_p(S_\Sigma), \\
&\|\varphi\|_{W_p(S)} = \|\varphi\|_{1-1/p, \Sigma} + \|\varphi\|_{2-1/p, \Sigma}, \text{ where} \\
&\|\varphi\|_{1-1/p, \Sigma} = \sum_{0 \leq 2r+s \leq [l]} \|D^s_t D^r_x \varphi\|_{p, \Sigma} + \ll \varphi \gg_{p, \Sigma} \\
&\ll \varphi \gg_{p, \Sigma} = \sum_{2r+s=[l]} \ll D^s_t D^r_x \varphi \gg_{p, \Sigma} + \sum_{0 \leq l-2r-s < 2} \ll D^s_t D^r_x \varphi \gg_{p, \Sigma}
\]

and for \( 0 < \alpha < 1 \) we define

\[
\ll v \gg^\alpha_{p, x, S} = \left( \int_0^T dt \int_{\partial \Omega} dx \int_{\partial \Omega} |v(x, t) - v(y, t)|^p \frac{dy}{|x-y|^{n+1+\alpha}} \right)^{1/p},
\]

\[
\ll \varphi \gg^\alpha_{p, t, S} = \left( \int_{\partial \Omega} dx \int_0^T dt \int_0^T |\varphi(x, t) - \varphi(x, \tau)|^p \frac{d\tau}{|t-\tau|^{1+\alpha}} \right)^{1/p}.
\]
(iii) Compatibility condition on $\partial \Omega$:

$$B\psi(x) = \varphi(x, 0) \quad \text{for} \quad \begin{cases} x \in E_0 & \text{and} \quad p > 3/2, \\ x \in \partial \Omega \setminus E_0 & \text{and} \quad p > 3. \end{cases}$$

(iv) The integral curves of $\ell$ on $E$ are non closed and of finite length.

We are interested in solvability of the problem (2.1) in the Sobolev space

$$W_p^{2,1}(Q) = \left\{ u \in L^p(0, T; W^{2,p}(\Omega)), u_t \in L^p(Q), p \in (1, \infty) \right\}$$
endowed by the norm

$$\|u\|_{W_p^{2,1}(Q)} = \sum_{j=0}^{2} \ll u \gg_{p,Q}^{(j)} = \sum_{j=0}^{2} \sum_{2r+s=j} \|D_r D_s u\|_{p,Q}.$$  

Under a strong solution to (2.1) we mean a function $u \in W_p^{2,1}(Q)$ satisfying $Pu = f$ almost everywhere in $Q$ and the initial and boundary conditions hold in trace sense.

**Theorem 1** (A priori estimate). Suppose conditions (i) – (iv) to be fulfilled, and $u \in W_p^{2,1}(Q)$ for $1 < p < \infty$. Let $Pu \in \mathcal{V}_p(Q)$, $Iu \in W^{3-2/p}_p(\Omega)$ and $Bu \in \mathcal{W}_p(S)$ then

$$\|u\|_{W_p^{2,1}(Q)} \leq C \left( \|Pu\|_{\mathcal{V}_p(Q)} + \|Iu\|_{W^{3-2/p}_p(\Omega)} + \|Bu\|_{\mathcal{W}_p(S)} \right)$$  

where the constant depends on $n, p, \lambda, T, \partial \Omega, \ell, \|a^{ij}\|_{L^\infty(\Sigma)}$, and the VMO-moduli of the coefficients.

**Theorem 2** (Unique strong solvability). Assume (i) – (iv) to be fulfilled. Then for all $f \in \mathcal{V}_p(Q)$, $\psi \in W^{3-2/p}_p(\Omega)$ and $\varphi \in \mathcal{W}_p(S)$, the problem (2.1) admits a unique solution $u \in W_p^{2,1}(Q)$ for all $p \in (1, \infty)$.

As in the case of regular ODP, the embedding result [6] gives Hölder continuity of the solution to (2.1) for appropriate values of $p$.

**Corollary 3.** Let $u \in W_p^{2,1}(Q)$ be a solution of (2.1). Then

i) $u \in C^{0,\alpha}(\bar{Q})$ with $\alpha < 2 - (n+2)/p$ if $p \in ((n+2)/2, n+2]$;

ii) $D_x u \in C^{0,\beta}(\bar{Q})$ with $\beta < 1 - (n+2)/p$ if $p > n+2$. 

3. Auxiliary results

The following assertion gives some geometrical properties of \( \ell \) (see [18], [13]).

**Proposition 4.** Let \( \ell \) and \( \Omega \) satisfy the assumptions listed above. There exists a finite upper bound \( \kappa_0 \) of the arclength of \( \ell \)-integral curves lying on the set \( E \). Moreover, there exist extensions \( L(x,t) \in W^{2,1}_\infty(\Sigma) \) and \( \bar{\nu}(x) \in C^{1,1}(\Sigma_0) \) of \( \ell(x,t) \) and \( \nu(x) \), and a cylinder \( Q_0 = \Omega_0 \times (0,T) \) with the following properties:

The base \( \Omega_0 \subset \Omega, \partial \Omega_0 \subset C^{1,1} \) is such that \( E_0 \subset \partial \Omega \setminus \partial \Omega_0 \). Denote by \( \partial Q_0 \) the lateral boundary of \( Q_0 \) and \( S_0 := \partial Q_0 \setminus S \). The extension \( L(x,t) \) is strictly transversal to \( S_0 \) and each point of \( S_{\Sigma} \) can be reached from \( S_0 \) along an \( L \)-integral curve of length at most \( \kappa' > \kappa_0 \).

Define \( Q_\tau = Q_0 \cup \{e^sL(x,t) \in Q : (x,t) \in S_0, 0 \leq s \leq \tau, \tau > 0\} \).

Under \( \partial Q_\tau \) we understand the lateral boundary of \( Q_\tau \), and \( S_\tau = \partial Q_\tau \setminus S = \{e^sL(x,t), (x,t) \in S_0\} \) The family \( \{Q_\tau\}_{\tau \geq 0} \) is non-decreasing and for every \( \delta > 0 \) there exists \( \theta = \theta(\delta) > 0 \), independent of \( \tau \), such that dist \((S_\tau, S_{\tau+\delta}) \geq \theta \) whenever \( Q \setminus Q_{\tau+\delta} \neq \emptyset \). The field \( L \) is strictly transversal to \( S_\tau \in W^{2,1}_\infty \) uniformly in \( \tau \).

Let us note that in the geometrical construction above we want that \( E_0 \subset \partial \Omega \setminus \partial \Omega_0 \) although the set \( \Omega_0 \cap \Sigma_0 \) could not be empty. In fact in the cylinder \( Q_0 = \Omega \times (0,T) \) we have regular ODP with boundary vector field \( L \) which coincides with \( \ell \) on \( \partial Q_0 \cap S \).

It is well known (see [4]) that there exists a neighborhood \( \mathcal{N} \) of \( \partial \Omega \) such that for any \( x \in \mathcal{N} \) there exists unique closest point \( y(x) \in \partial \Omega \) and \( x = y(x) - \nu(y)d(x), y, d \in C^{1,1}(\Omega) \). Let \( \Omega \setminus \Omega_0 \subset \mathcal{N} \), then \( d, y \in C^{1,1}(\Omega \setminus \Omega_0) \) and we set

\[
\bar{\nu}(x) = \nu(y(x)), \quad L(x,t) = \ell(y(x),t) + d(x)\bar{\nu}(x) \quad \forall x \in \Omega \setminus \Omega_0, t \in (0,T).
\]

The regularity of \( L \) follows by the regularity properties of \( \ell \) and \( d \). The rest of the proof repeats the arguments in [18, Proposition 3.2] and [13, Proposition 3.2.5].

We need also of the following variant of the Gronwall inequality (see [18]).

**Proposition 5.** Let \( \zeta(\tau) \) be continuous, bounded and positive function, defined on \([0, \infty)\). Suppose there exist positive constants \( \delta, A \) and \( C \) such that

\[
\zeta(\tau) \leq A + C \int_0^\tau \zeta(s + \delta)ds \quad \text{for all} \quad \tau > 0.
\]
Then $\zeta(\tau) \leq A \left(1 + (2\pi)^{-1/2}(1 - C\delta e)^{-1}e^{\tau/\delta}\right)$ for $C\delta e < 1$.

4. Cauchy problem

In the present section we discuss the solvability of the Cauchy problem and the reducing of (2.1) to one with homogeneous initial data. Consider

\begin{equation}
\begin{cases}
  v_t - \Delta v = 0 & (x, t) \in \mathbb{R}^n \times (0, T) \\
  v(x, 0) = \overline{\psi}(x) & x \in \mathbb{R}^n
\end{cases}
\end{equation}

where $\overline{\psi} \in W^{3-2/p}_p(\mathbb{R}^n)$ is an extension of $\psi$ as zero for $x \notin \bar{\Omega}$. The solution of (4.4) is given by the potential

$$v(x, t) = (\Gamma \ast_1 \overline{\psi}) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \overline{\psi}(y)dy = \frac{1}{(4\pi t)^{n/2}} \int_{\Omega} e^{-\frac{|x-y|^2}{4t}} \psi(y)dy$$

and it can be considered as an extension in $t > 0$ of the initial data preserving its regularity. In our case we have supposed higher regularity of $I$ looking for a solution of (4.4) possessing higher regularity in $x$. Having in mind the estimates of the heat potential in Sobolev spaces (see [6, Ch IV, § 3, 4]) it follows that $v \in W^{2,1}_p(\mathbb{R}^n \times (0, T))$ and corresponding a priori estimate holds.

Moreover, since $D_x v = D_x (\Gamma \ast_1 \overline{\psi}) = (\Gamma \ast_1 D_x \overline{\psi})$ we obtain also that

$$\ll (\Gamma \ast_1 D_x \overline{\psi}) \gg \ll (2)_{p,\mathbb{R}^n \times \mathbb{R}^+} \leq C \ll D_x \overline{\psi} \gg \ll (2-2/p)_{p,\mathbb{R}^n} \ll \overline{\psi} \gg (3-2/p)_{p,\mathbb{R}^n}.$$

On the other hand according to (2.2) and (iii.a) we can write

$$\ll (\Gamma \ast_1 D_x \overline{\psi}) \gg \ll (2)_{p,\mathbb{R}^n \times \mathbb{R}^+} = \|D^2_x(\Gamma \ast_1 D_x \overline{\psi})\|_{p,\mathbb{R}^n \times \mathbb{R}^+} + \|D_t(\Gamma \ast_1 D_x \overline{\psi})\|_{p,\mathbb{R}^n \times \mathbb{R}^+} = \|D^2_xv\|_{p,\mathbb{R}^n \times \mathbb{R}^+} + \|D_tD_xv\|_{p,\mathbb{R}^n \times \mathbb{R}^+} \leq C \ll \overline{\psi} \gg (3-2/p)_{p,\mathbb{R}^n} \leq C\|\psi\|_{(3-2/p)_{p,\Omega}}.$$

The function $w = u - v$ satisfies

$$\begin{cases}
  \mathcal{P}w \equiv w_t - a^{ij}D_{ij}w = f(x, t) + (a^{ij} - \delta^{ij})D_{ij}v := \tilde{f}(x, t) & \text{in } Q, \\
  \mathcal{I}w \equiv w(x, 0) = 0 & \text{on } \Omega, \\
  \mathcal{B}w \equiv \frac{\partial w}{\partial \ell} = \varphi(x, t) + \ell^i(x, t)D_i v := \tilde{\varphi}(x, t) & \text{on } S
\end{cases}$$

where $\tilde{f} \in \mathcal{V}_p(Q)$ and $\tilde{\varphi} \in W_p(S)$ according to (iii) and the regularity of $v$.

5. A priori estimates

In our further considerations we always suppose that $\mathcal{I}u = 0$. Our main goal is to obtain an a priori estimate for the solution only through the data that allows proving of unique solvability of (2.1).

**Lemma 6.** Let $v \in W^{2,1}_p(Q)$, $p \in (1, \infty)$, $v(x, 0) = 0$ and (i) hold. Let $\Omega_1 \subset \Omega_2 \subset \Omega$ with $\partial \Omega_i \in C^{1,1}$, $Q_i = \Omega_i \times (0, T)$, $S_i = \partial Q_i \cap S$, $i = 1, 2$ such
that \( \text{dist} (Q_1, Q \setminus Q_2) \geq \theta > 0 \). Then there exist constants \( C' \) and \( C''(\theta) \) such that
\[
\|v\|_{W^{2,1}_p(Q_1)} \leq C' \left( \|Pv\|_{p,Q_2} + \|v\|_{p,S_2}^{(2-1/p)} \right) + C''(\theta) \left( \|v\|_{V_p^2(Q_2)} + \|v\|_{p,S_2}^{(1-1/p)} \right).
\]

**Proof.** Take a cutoff function \( \eta(x) \in C_0^\infty(\mathbb{R}^n) \) such that \( \eta(x) = 1 \) for \( x \in \bar{\Omega}_1 \), \( \eta(x) = 0 \) for \( x \in \bar{\Omega} \setminus \bar{\Omega}_1 \) and \( \max_{\bar{\Omega}} |D^\alpha \eta(x)| \leq C_\alpha \theta^{-|\alpha|} \). For \( v \in W^{2,1}_p(Q) \) it is easy to see that \( \mathcal{P}(v\eta) \) belongs to \( L^p(Q) \). Having in mind the a priori estimate for homogeneous Cauchy-Dirichlet problem (see [2]), it is a standard procedure to obtain an analogous estimate for non homogeneous one
\[
\|v\|_{W^{2,1}_p(Q_1)} \leq \|\eta v\|_{W^{2,1}_p(Q_2)} \leq \left( \|\mathcal{P}(\eta v)\|_{p,Q_2} + \|\eta v\|_{p,\partial Q_2}^{(2-1/p)} \right)
\]
\[
\leq C_0 \left( \|\mathcal{P}v\|_{p,Q_2} + \|v\|_{p,S_2}^{(2-1/p)} \right) + \frac{C_1}{\theta} \left( \|Dv\|_{p,Q_2} + \|v\|_{p,S_2}^{(1-1/p)} \right) + \frac{C_2}{\theta^2} \|v\|_{p,Q_2}
\]
\[
\leq C' \left( \|\mathcal{P}v\|_{p,Q_2} + \|v\|_{p,S_2}^{(2-1/p)} \right) + C''(\theta) \left( \|v\|_{V_p^2(Q_2)} + \|v\|_{p,S_2}^{(1-1/p)} \right).
\]

**Proof of Theorem 1.** The derivative \( \partial u/\partial L \) satisfies in \( \Sigma \) the following Cauchy-Dirichlet problem
\[
\begin{align*}
D_t \left( \frac{\partial u}{\partial L} \right) - a^{ij} D_{ij} \left( \frac{\partial u}{\partial L} \right) & = \frac{\partial f}{\partial L} + \frac{\partial a^{ij}}{\partial L} D_{ij} u + D_k u D_i L^k \\
- a^{ij} \left( 2 D_{ik} u D_j L^k + D_k u D_{ij} L^k \right) & = F(x,t) \quad \text{in } \Sigma \\
\frac{\partial u(x,0)}{\partial L} & = 0 \quad \text{on } \Sigma_0 \\
\frac{\partial u(x,t)}{\partial L} & = \varphi(x,t) \quad \text{on } S_\Sigma.
\end{align*}
\]

Let \( \Sigma_0'' \subset \Sigma_0 \) and consider a cut-off function \( \eta(x) \in C_0^\infty(\mathbb{R}^n) \) such that \( \eta(x) = 1 \) for \( x \in \Sigma_0'' \), \( \eta(x) = 0 \) for \( x \in \bar{\Omega} \setminus \bar{\Sigma}_0 \) and \( \sup_{\Sigma_0} |D^\alpha \eta(x)| \leq C d_0^{-|\alpha|} \) where \( d_0 \) is the distance between \( \Sigma_0'' \) and \( \partial \Sigma_0 \setminus \partial \Omega \). Then \( V = \frac{\partial u}{\partial L} \eta \) is a solution of
\[
\begin{align*}
D_t V - a^{ij} (x,t) D_{ij} V & = F(x,t) \eta(x) \\
- a^{ij} (x,t) \left( 2 D_{ik} \eta D_j \left( \frac{\partial u}{\partial L} \right) + \frac{\partial u}{\partial L} D_{ij} \eta \right) & =: F_1(x,t) \quad \text{in } \Sigma \\
V(x,0) & = 0 \quad \text{on } \Sigma_0 \\
V(x,t) & = \varphi(x,t) \quad \text{on } S_\Sigma.
\end{align*}
\]

As in Lemma 6 we have an a priori estimate for the solution of the above problem
\[
\|V\|_{W^{2,1}_p(\Sigma)} \leq C \left( \|F_1\|_{p,\Sigma} + \|\varphi\|_{p,\Sigma}^{(2-1/p)} \right) \leq C \left( \|u\|_{W^{2,1}_p(\Sigma)} + \|f\|_{V_p^2(\Sigma)} + \|\varphi\|_{p,\Sigma}^{(2-1/p)} \right).
\]
and the constant depends on the $\infty$-norms of $a^i_j$, $\ell$ and $\text{diam} \Sigma$ (through the extension $L$). Having in mind that $V = \partial u/\partial L$ in $\Sigma''$ we obtain

\[(5.6) \quad \left\| \frac{\partial u}{\partial L} \right\|_{W^{2,1}_{\infty}(\Sigma')} \leq C \left( \|f\|_{V_p(Q)} + \|\varphi\|_{W_p(Q)} + \|u\|_{W^{2,1}_{\ell}(Q)} \right).\]

Considering in analogous way the regular ODP in the cylinder $Q \setminus \Sigma''$ and applying [15, Theorem 1] we obtain

\[(5.7) \quad \|u\|_{W^{2,1}_{\infty}(Q \setminus \Sigma'')} \leq C \left( \|f\|_{V_p(Q)} + \|\varphi\|_{W_p(S)} + \|u\|_{V^1_p(\Sigma')} \right).\]

To estimate the Sobolev norm of $u$ in $\Sigma''$ we make use of an explicit formula for the solution through the $L$-integral curves. Construct a cylinder $Q_0 = \Omega_0 \times (0,T)$ (see Proposition 4) such that $\Omega_0 \subset \{ \Omega \setminus \Sigma'' \}$, $S_0$ lies in $\Sigma \setminus \Sigma''$ where $L$ is well defined. Let $(x,t) \in \Sigma''$ and $\psi(\tau; x,t) = e^{\tau L}(x,t)$, $\tau \in [0, \kappa']$ be the parametrisation of the $L$-integral curve passing through that point $(\psi(0; x,t) = (x,t))$. Then there exists unique $\xi(x,t) \in W^{2,1}_{\infty}(\Sigma)$ such that

\[\begin{align*}
\psi(-\xi(x,t); x,t) \in Q_0 \subset Q \setminus \Sigma'' \quad \text{and} \quad u(x,t) &= u \circ \psi(-\xi(x,t); x,t) + \int_{\tau-\xi(x,t)}^{\tau} \frac{\partial u}{\partial L} \circ \psi(s - \tau; x,t)ds.
\end{align*}\]

First we shall estimate the $L^p$-norm of $D^2u$ in $Q_\tau \cap \Sigma''$ where $\{Q_\tau\}_{\tau \geq 0}$ is the expanding family of cylinders defined in Proposition 4 (note $(x,t) \in S_\tau$)

\[\begin{align*}
\|D^2u\|_{p,Q_\tau \cap \Sigma''} \leq \|D^2u\|_{p,Q_\tau} &\leq C \left( \|D^2u\|_{p,Q_\tau \cap \Sigma''} + \|Du\|_{p,Q_\tau \cap \Sigma''} \right. \\
&\quad + \int_0^\tau \left( \|D^2\left( \frac{\partial u}{\partial L} \right)\|_{p,Q_\tau \cap \Sigma''} + \|D\left( \frac{\partial u}{\partial L} \right)\|_{p,Q_\tau \cap \Sigma''} \right) ds + \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma \setminus \Sigma'')} \\
&\leq C \left( \|u\|_{W^{2,1}_{\infty}(Q \setminus \Sigma'')} + \int_0^\tau \|D^2\left( \frac{\partial u}{\partial L} \right)\|_{p,Q_\tau \cap \Sigma''} ds + \|D\left( \frac{\partial u}{\partial L} \right)\|_{p,\Sigma''} \right. \\
&\quad + \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma \setminus \Sigma'')} \\
&\leq C \left( \|u\|_{W^{2,1}_{\infty}(Q \setminus \Sigma'')} + \int_0^\tau \|D^2\left( \frac{\partial u}{\partial L} \right)\|_{p,Q_\tau \cap \Sigma''} ds + \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma)} \right)
\end{align*}\]

We need an upper bound for the norm under the integral in order to apply the Gronwall type inequality.

In our considerations we distinguish two cases. The first one is when $Q \setminus Q_{s+\delta} \neq \emptyset$ and according to Proposition 4, there exists $\theta = \theta(\delta) > 0$ such that $\text{dist} (S_s, S_{s+\delta}) \geq \theta$. Consider the right-hand side in (5.8). The first term is estimated by (5.7). To estimate the second one we apply Lemma 6 to the solution $\partial u/\partial L$ of (5.5) with $Q_s \cap \Sigma''$ and $Q_{s+\delta} \cap \Sigma''$ instead of $Q_1$ and $Q_2$

\[\begin{align*}
\|D^2\left( \frac{\partial u}{\partial L} \right)\|_{p,Q_{s+\delta} \cap \Sigma''} &\leq \left\| \frac{\partial u}{\partial L} \right\|_{W^{2,1}_{\infty}(Q_s \cap \Sigma'')} \leq C' \left( \|F\|_{p,Q_{s+\delta} \cap \Sigma''} + \|\varphi\|_{p,\partial Q_{s+\delta} \cap S}^{(2-1/p)} \right) \\
&\quad + C''(\theta) \left( \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(Q_{s+\delta} \cap \Sigma'')} + \|\varphi\|_{p,\partial Q_{s+\delta} \cap S}^{(1-1/p)} \right) \left. \right. \\
&\leq C \left( \|u\|_{W^{2,1}_{\infty}(Q_{s+\delta} \cap \Sigma'')} + \|f\|_{V_p(Q)} + \|\varphi\|_{W_p(Q)} + C''(\theta) \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(Q_{s+\delta} \cap \Sigma'')} \right),
\end{align*}\]
Let $\Sigma_0' \subset \Sigma''$ and $\Sigma' = \Sigma_0' \times (0, T)$. Thus

$$\left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(Q_{s+b} \cap \Sigma'')} \leq \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma')} + \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma'' \setminus \Sigma')}

\leq \varepsilon \left\| D^2 \left( \frac{\partial u}{\partial L} \right) \right\|_{p, \Sigma'} + C(\varepsilon) \left\| \frac{\partial u}{\partial L} \right\|_{p, \Sigma'} + C\|u\|_{W^{2,1}_p(Q \setminus \Sigma')}

\leq \varepsilon \left\| \frac{\partial u}{\partial L} \right\|_{W^{2,1}_p(\Sigma')} + C(\varepsilon) \left\| \frac{\partial u}{\partial L} \right\|_{p, \Sigma'} + C(\|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)} + \|u\|_{V^1_p(Q)})

\leq C(\|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)}) + C(\varepsilon)\|u\|_{V^1_p(Q)} + \varepsilon\|u\|_{W^{2,1}_p(\Sigma')}

$$

after applying the Gagliardo-Nirenberg interpolation inequality with suitable $\varepsilon > 0$, (5.6) and (5.7). Finally, the last term in (5.8) is estimated dividing it in two parts, i.e.

$$\left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma'')} + \left\| \frac{\partial u}{\partial L} \right\|_{V^1_p(\Sigma'' \setminus \Sigma')}

\leq \varepsilon \left\| \frac{\partial u}{\partial L} \right\|_{W^{2,1}_p(Q') + C(\|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)}) + C(\varepsilon)\|u\|_{V^1_p(Q)}$$

(5.9)

using (5.6) and (5.7) in the last step. Substituting the above estimates in (5.8) we obtain

$$\|D^2 u\|_{p, Q_{s+b} \cap \Sigma''} \leq C \left( \|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)} \right) + \varepsilon\|u\|_{W^{2,1}_p(Q)}

+ C \int_0^T \|u\|_{W^{2,1}_p(Q_{s+b} \cap \Sigma'')} ds + C(\theta, \varepsilon)\|u\|_{V^1_p(Q)}$$

(5.10)

From the equation $u_t = a^{ij} D_{ij} u + f$ it follows an analogous estimate also for $u_t$. Hence

$$\|D^2 u\|_{p, Q_{s+b} \cap \Sigma''} + \|u_t\|_{p, Q_{s+b} \cap \Sigma''} \leq C \left( \|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)} \right)

+ \varepsilon\|u\|_{W^{2,1}_p(Q)} + C \int_0^T \left( \|D^2 u\|_{p, Q_{s+b} \cap \Sigma''} + \|u_t\|_{p, Q_{s+b} \cap \Sigma''} \right) ds

+ C(\theta, \varepsilon)\|u\|_{V^1_p(Q)}$$

(5.11)

where we have interpolated the norms of the lower derivatives of $u$ under the integral and use that $\int_0^T \|u\|_{p, Q_{s+b} \cap \Sigma''} ds \leq C\|u\|_{p, Q}$.

In the second case $Q_{s+b}$ covers the whole cylinder and hence $Q \setminus Q_{s+b} = \emptyset$. The difference with the first one is the estimate for the second term in (5.8). Using (5.6) and (5.7) instead of Lemma 6 and having in mind $\Sigma'' = Q_{s+b} \cap \Sigma''$, we get

$$\left\| \frac{\partial u}{\partial L} \right\|_{W^{2,1}_p(Q_{s+b} \cap \Sigma'')} \leq C\left( \|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)} \right)

+ \|u\|_{W^{2,1}_p(Q_{s+b} \cap \Sigma'')} + \|u\|_{W^{2,1}_p(Q \setminus \Sigma'')}

\leq C \left( \|f\|_{V^p_p(Q)} + \|\varphi\|_{W^p_p(Q)} + \|u\|_{V^1_p(Q)} \right)

+ \|u\|_{W^{2,1}_p(Q_{s+b} \cap \Sigma'')}.$$
Finally estimating (5.8) through (5.7), (5.12) and (5.9) we obtain again (5.10) and (5.11).

Now we can apply the Proposition 5 to the function \( \zeta(t) = \| D^2 u \|_{p, Q_t} + \| u_t \|_{p, Q_t} \). Hence for values of \( \tau \) grater than \( \kappa' \) for which \( Q_{\tau} \cap \Sigma'' = \Sigma'' \) and \( \delta > 0 \) small enough we have

\[
\| D^2 u \|_{p, \Sigma''} + \| u_t \|_{p, \Sigma''} \leq C(\| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)}) + \varepsilon \| u \|_{W^2_\infty(Q)}.
\]

(5.13)

Combining (5.7) and (5.13), we obtain

\[
\| u \|_{W^2_\infty(Q)} \leq C(\| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)} + \| u \|_{p, Q}).
\]

(5.14)

To estimate the norm of \( u \) we take into account once again the differential equation (see [7, Ch. VII]). Let \( \tilde{u} \) be an extension of \( u \) as a zero for \( (x, t) \notin Q \). Obviously \( \| u \|_{p, Q} = \| \tilde{u} \|_{p, \mathbb{R}^{n+1}} \) and the same holds also for the time derivative. Further it is easy to see that for any \( \varsigma \in (0, T) \)

\[
\int_{\mathbb{R}^n} |\tilde{u}(x, \varsigma)|^p dx = \int_0^\varsigma \int_{\mathbb{R}^n} \frac{d}{dt} |\tilde{u}(x, t)|^p dt dx \leq p \int_0^\varsigma \int_{\mathbb{R}^n} |\tilde{u}(x, t)|^{p-1} |\tilde{u}_t(x, t)| dt dx
\]

\[
\leq p \left( \int_0^\varsigma \int_{\mathbb{R}^n} |\tilde{u}_t(x, t)|^p dt dx \right)^{1/p} \left( \int_0^\varsigma \int_{\mathbb{R}^n} |\tilde{u}(x, t)|^p dt dx \right)^{(p-1)/p}.
\]

According to the above considerations and (5.14) we can write for the first integral

\[
\| u_t \|_{p, \Omega \times (0, \varsigma)} \leq C \| D^2 u \|_{p, \Omega \times (0, \varsigma)} + \| f \|_{p, Q} \leq C \left( \| u \|_{p, \Omega \times (0, \varsigma)} + \| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)} \right).
\]

The function \( U(\varsigma) = \int_{\mathbb{R}^n} |\tilde{u}(x, \varsigma)|^p dx \) satisfies

\[
U(\varsigma) \leq C \left( \int_0^\varsigma U(t) dt \right)^{(p-1)/p} \left( \int_0^\varsigma U(t) dt \right)^{1/p} + \| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)}
\]

\[
\leq C \int_0^\varsigma U(t) dt + C \| u \|_{p, Q}^{p-1} (\| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)}).
\]

Hence the classical Gronwall inequality gives a bound for \( U(\varsigma) \), that is

\[
U(\varsigma) \leq C \| u \|_{p, Q}^{p-1} (\| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)})
\]

and from the definition of \( U \) it follows

\[
\| u \|_{p, Q} \leq C (\| f \|_{V_p(Q)} + \| \varphi \|_{W_p(Q)}).
\]

The last one combining with (5.14) and the a priori estimate for the solution of the Cauchy problem give exactly (2.3). \( \Box \)
6. Unique solvability

Proof of Theorem 2. The uniqueness follows trivially from (2.3). To prove solvability we consider perturbed problem

\begin{equation}
\begin{aligned}
P u^\varepsilon &= u^\varepsilon - a^{ij}(x,t)D_{ij}u^\varepsilon(x,t) = f(x,t) \quad \text{in } Q, \\
I u^\varepsilon &= u^\varepsilon(x,0) = 0 \quad \text{on } \Omega, \\
B_\varepsilon u^\varepsilon &= \frac{\partial u^\varepsilon}{\partial \ell_{\varepsilon}} = \varphi(x,t) \quad \text{on } S
\end{aligned}
\end{equation}

where \( \ell_{\varepsilon}(x,t) = \ell(x,t) + \varepsilon \nu(x) \). Obviously, for \( \varepsilon > 0 \), \( \ell_{\varepsilon} \) is nowhere tangential to \( S \) and whence the above problem, being regular, has unique strong solution \( u^\varepsilon \in W_p^{2,1}(Q) \), \( p \in (1, \infty) \). Moreover, in view of Theorem 1, \( u^\varepsilon \) satisfies the estimate

\begin{equation}
\|u^\varepsilon\|_{W_p^{2,1}(Q)} \leq C \left( \|f\|_{p,Q} + \|\varphi\|_{(1-1/p)}^{(1-1/p)} \right) \leq C \left( \|f\|_{V_p^1(Q)} + \|\varphi\|_{V_p(Q)} \right).
\end{equation}

The second estimate is more restrictive but its constant does not depend on \( \varepsilon \) while the first one depends on it through the norm of \( \ell_{\varepsilon} \) as it could be seen from the a priori estimate in [15].

By virtue of the compactness of the embedding \( W_p^{2,1}(Q) \hookrightarrow V^1_p(Q) \hookrightarrow L^p(Q) \), and the weak compactness of bounded sets in \( W_p^{2,1}(Q) \) there exists subsequence, which we reable as \( \{u^\varepsilon\} \), converging weakly to a function \( u \in W_p^{2,1}(Q) \) and \( \|u^\varepsilon - u\|_{V^1_p(Q)} \to 0 \) as \( \varepsilon \to 0 \). Since

\[ \int_Q fg \, dx \, dt = \int_Q (Pu^\varepsilon)g \, dx \, dt \to \int_Q (Pu)g \, dx \, dt \quad \text{as } \varepsilon \to 0, \quad \forall g \in L^{p/(p-1)}(Q) \]

we must have \( Pu = f \) a.e. in \( Q \). Extending \( \ell_{\varepsilon} \) and \( \varphi \) in \( Q \) preserving their regularity, we get

\[ \|B_\varepsilon u^\varepsilon - B_\varepsilon u\|_{p,Q} \leq C \|D(u^\varepsilon - u)\|_{p,Q} \to 0 \quad \text{as } \varepsilon \to 0. \]

Hence

\[ \|\varphi - Bu\|_{p,Q} = \|B_\varepsilon u^\varepsilon - Bu\|_{p,Q} \leq \|B_\varepsilon u^\varepsilon - B_\varepsilon u\|_{p,Q} \]

\[ + \|B_\varepsilon u - Bu\|_{p,Q} \leq C \|D(u^\varepsilon - u)\|_{p,Q} + \varepsilon \|Du\|_{p,Q} \to 0 \quad \text{as } \varepsilon \to 0. \]

Let us stress now our considerations on the cylinder \( \Sigma \). The solution of the regular problem gains two derivatives in \( x \) from \( f \) and one on from \( \varphi \), i.e. \( u^\varepsilon \in V^3_p(\Sigma) \). It could be seen easy taking the representation formula for the solution of (6.15) (see [15, Lemma 1]) and the estimates for the heat potentials (see [6]) as it was done for the solution of the Cauchy problem in Section 4. Then it holds

\[ \|u^\varepsilon\|_{W_p^{2,1}(\Sigma)} + \|D^2_x u^\varepsilon\|_{p,\Sigma} + \|D_x Du^\varepsilon\|_{p,\Sigma} \leq C \left( \|f\|_{V^1_p(\Sigma)} + \|\varphi\|_{V^2_{2p} \Sigma}^{2-1/p} \right). \]

Repeating the above arguments we get \( \|u^\varepsilon - u\|_{V^2_p(\Sigma)} \to 0 \) as \( \varepsilon \to 0 \). For the extensions of \( \varphi \) and \( \ell_{\varepsilon} \) in \( \Sigma \) we get

\[ \|B_\varepsilon u^\varepsilon - B_\varepsilon u\|_{V^2_p(\Sigma)} \to 0 \quad \text{as } \varepsilon \to 0. \]
and $\| \phi - Bu \|_{V^1_\psi (\Sigma)} \to 0$. Therefore $\phi = Bu$ on $S$ in trace sense.  

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