A Taylor expansion of the square root matrix function

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Abstract
This short note provides an explicit description of the Fréchet derivatives of the principal square root matrix function at any order. We present an original formulation that allows to compute sequentially the Fréchet derivatives of the matrix square root at any order starting from the first order derivative. A Taylor expansion at any order with an integral remainder term is also provided, yielding the first result of this type for this class of matrix function.

Keywords: Fréchet derivative, square root matrices, Taylor expansion, Sylvester equation, spectral and Frobenius norms, matrix exponential.

Mathematics Subject Classification: 15A60, 15B48, 15A24.

1 Introduction
The computation of matrix square roots arise in a variety of application domains, including in physics, signal processing, optimal control theory, and many others. The literature abounds with numerical techniques for computing matrix square roots, see for instance \[1, 12, 13, 17, 18, 19, 22\]. Perturbative techniques often resume to Lipschitz type estimates \[4\] or on the refined analysis of the first order Fréchet derivative of the principal square root matrix function; see for instance \[11, 23\], as well as chapter X in the seminal book by R. Bhatia \[5\] and references therein. We also refer to the article \[10\] for a first order analysis of more general matrix \(n\)-th roots. For further details on the \(n\)-th roots of matrices we refer to \[23\].

The purpose of this article is to derive an explicit description of the Fréchet derivatives of the principal square root matrix function at any order. We also provide a non asymptotic Taylor expansion at any order, with computable estimates of the integral remainder terms. These expansions provide a perturbation computation of the square root \(\sqrt{A + H}\) of a positive definite matrix \(A\) perturbed by some symmetric matrix \(H\), as soon as \(A + \epsilon H\) is positive semidefinite for any \(\epsilon \in [0, 1]\).

We underline that the perturbation analysis developed in this article differs from Taylor expansion techniques often used to define functions on the spectrum of diagonalizable matrices via Jordan canonical forms. This Sylvester’s formulation of matrices are related to the Sylvester matrix theorem (a.k.a. Lagrange-Sylvester interpolation) which allows to express an analytic function of a matrix in terms of its eigenvalues and eigenvectors. For a more thorough discussion on these interpolation techniques we refer to the first chapter in the seminal book by N. J. Higham \[16\].

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This study has been motivated by applications in signal processing and more particularly in the analysis of Ensemble Kalman-Bucy filters [7]. In this context, the sample and interacting covariance matrices satisfy a stochastic matrix Riccati diffusion. The diffusion term depends on the matrix square root of the sample covariance. The perturbation analysis developed in this article is used to derive non asymptotic Taylor-type expansions of stochastic matrix Riccati flows w.r.t. some perturbation parameter.

We denote by $S_r$ the space of symmetric $(r \times r)$-matrices $A$ with real entries equipped with the $L_2$-norm $\|A\| = \|A\|_2 = \sqrt{\lambda_{\text{max}}(A^2)}$ or the Frobenius norm $\|A\| = \|A\|_F = \sqrt{\text{Tr}(A^2)}$. We recall that these norms are equivalent and $\|A\|_2 \leq \|A\|_F \leq \sqrt{r} \|A\|_2$.

We let $S_r^0 \subset S_r$ be the closed convex cone of positive semi-definite matrices, and its interior $S_r^+ \subset S_r^0$ which resumes to the open subset of positive definite matrices. We consider the principal square root function

$$\varphi : Q \in S_r^+ \mapsto \varphi(Q) = Q^{1/2} \in S_r^+$$

For any $Q_1, Q_2 \in S_r^+$ we have the Ando-Hemmen inequality

$$\|\varphi(Q_1) - \varphi(Q_2)\| \leq \left[\lambda_{\text{min}}^{1/2}(Q_1) + \lambda_{\text{min}}^{1/2}(Q_2)\right]^{-1} \|Q_1 - Q_2\| \quad (1)$$

for any unitary invariant matrix norm $\|\cdot\|$. See for instance Theorem 6.2 on page 135 in [16], as well as Proposition 3.2 in [4]. For a more thorough discussion on the geometric properties of positive semidefinite matrices and square roots we refer to [21].

We let $\mathcal{L}(S_r, S_r)$ be the set of bounded linear functions from $S_r$ into itself. Let $O_r \subset S_r$ be a non empty open and convex subset of $S_r$. We recall that a mapping $\Upsilon : O_r \mapsto S_r$ defined in some domain $O_r$ is Fréchet differentiable at some $A \in O_r$ if there exists a continuous linear function $\nabla \Upsilon(A) \in \mathcal{L}(S_r, S_r)$ such that

$$\lim_{\|H\| \to 0} \|H\|^{-1} \|\Upsilon(A + H) - \Upsilon(A) - \nabla \Upsilon(A) \cdot H\| = 0$$

In other words, for any given $A \in O_r$ and $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\|H\| \leq \delta \implies A + H \in O_r \quad \text{and} \quad \|\Upsilon(A + H) - \Upsilon(A) - \nabla \Upsilon(A) \cdot H\| \leq \epsilon \|H\|$$

The l.h.s. condition is met for $O_r = S_r^+ \ni A$. We check this claim using Weyl’s inequality

$$\lambda_{\text{min}}(A + H) \geq \lambda_{\text{min}}(A) + \lambda_{\text{min}}(H) \geq \lambda_{\text{min}}(A) - \|H\|_2$$

This shows that

$$\|H\|_2 < \lambda_{\text{min}}(A) \implies A + H \in S_r^+$$

The function $\Upsilon$ is said to be Fréchet differentiable on $O_r$ when the mapping

$$\nabla \Upsilon : A \in O_r \mapsto \nabla \Upsilon(A) \in \mathcal{L}(S_r, S_r)$$

is continuous. Higher Fréchet derivatives are defined in a similar way. For instance, the mapping $\Upsilon$ is twice Fréchet differentiable at $A \in O_r$ when the mapping $\nabla^2 \Upsilon$ is also Fréchet differentiable at $A \in O_r$. Identifying $\mathcal{L}(S_r, \mathcal{L}(S_r, S_r))$ with the set $\mathcal{L}(S_r \times S_r, S_r)$ of continuous bilinear maps from $(S_r \times S_r)$ into $S_r$, the second derivative

$$\nabla^2 \Upsilon : A \in O_r \mapsto \nabla^2 \Upsilon(A) \in \mathcal{L}(S_r \times S_r, S_r)$$
is defined by a continuous and symmetric bilinear map $\nabla^2 \Upsilon(A)$ such that the limit
\[
\lim_{\|H_2\| \to 0} \|H_2\|^{-1} \|\nabla \Upsilon(A + H_2) \cdot H_1 - \nabla \Upsilon(A) \cdot H_1 - \nabla^2 \Upsilon(A) \cdot (H_1, H_2)\| = 0
\]
extists uniformly w.r.t. $H_1 \in \mathcal{S}_r$ in bounded sets. The polarization formula
\[
\nabla^2 \Upsilon(A) \cdot (H_1, H_2) = \frac{1}{4} \left[ \nabla^2 \Upsilon(A) \cdot (H_1 + H_2, H_1 + H_2) - \nabla^2 \Upsilon(A) \cdot (H_1 - H_2, H_1 - H_2) \right]
\]
shows that it suffices to compute the second order derivatives $\nabla^2 \Upsilon(A) \cdot (H, H)$ in the same direction $H = H_1 = H_2$. Identifying $\mathcal{L}(\mathcal{S}_r \times \mathcal{S}_r, \mathcal{S}_r)$ with $\mathcal{L}(\mathcal{S}_r \otimes \mathcal{S}_r, \mathcal{S}_r)$ sometimes we set $\nabla^2 \Upsilon(A) \cdot H^\otimes 2$ instead of $\nabla^2 \Upsilon(A) \cdot (H, H)$. For a more detailed discussion on these tensor product identifications of symmetric multilinear maps we refer to chapter 5 in [15].

Higher Fréchet derivatives $\nabla^n \Upsilon(A) \cdot H^\otimes n$ of order $n$ are defined recursively in a similar manner. For a more thorough discussion on higher Fréchet derivatives, we refer the reader to the seminal books of Cartan [11] and Dieudonné [14], as well as section 5.2 in the book by Dudley and Norvaisa [15] and the article by Higham and Relton [20]. The latter addresses the general case, as well as the matrix exponential and inverse.

In the further development of this article, by symmetry arguments we only consider differentials in a given direction $H$. To simplify the presentation, sometimes we write $\nabla^n \Upsilon(A) \cdot H$ instead of $\nabla^n \Upsilon(A) \cdot H^\otimes n$. The $n$-th derivatives $\nabla^n \Upsilon(A) \cdot (H_1, \ldots, H_n)$ in $n$ different directions are defined as above by polarization of $n$-linear symmetric operators, see for instance theorem 5.6 in [15]. The symmetry property of the $n$-linear mappings $\nabla^n \Upsilon(A)$ is a consequence of Schwarz theorem (see for instance theorem 5.27 in [15]).

Let $\Upsilon: \mathcal{O}_r \to \mathcal{S}_r$ be a Fréchet differentiable mapping at any order at some $A \in \mathcal{O}_r$. Given some $H \in \mathcal{S}_r$ s.t. $A + H$ is included in $\mathcal{O}_r$, we have
\[
\Upsilon(A + H) = \Upsilon(A) + \sum_{1 \leq k \leq n} \frac{1}{k!} \nabla^k \Upsilon(A) \cdot H + \nabla^{n+1} \Upsilon[A, H]
\]
with the $(n + 1)$-th order remainder function in the Taylor expansion given
\[
\nabla^{n+1} \Upsilon[A, H] := \frac{1}{n!} \int_0^1 (1 - \epsilon)^n \nabla^{n+1} \Upsilon(A + \epsilon H) \cdot H \, d\epsilon
\]
Using the convexity of the set $\mathcal{O}_r$, we underline that the line segment $\epsilon \in [0, 1] \mapsto A + \epsilon H$ joining $A \in \mathcal{O}_r$ to $A + H \in \mathcal{O}_r$ is included in $\mathcal{O}_r$; that is, we have that
\[
A \in \mathcal{O}_r \quad \text{and} \quad B = A + H \in \mathcal{O}_+ \implies \forall \epsilon \in [0, 1] \quad A + \epsilon H = (1 - \epsilon) A + \epsilon B \in \mathcal{O}_r
\]
For a more detailed account on Taylor’s formulae with integral remainders for smooth functions on open convex subsets of Banach spaces with values in another Banach space we refer the reader to section 5.3 in the book by Dudley and Norvaisa [15].

We also consider the multi-linear operator norm
\[
\|\nabla^n \Upsilon(P)\| = \sup_{\|H\| = 1} \|\nabla^n \Upsilon(P) \cdot H\|
\]
In the further development of this article $C_n = \frac{1}{n+1} \binom{2n}{n}$ stands for the Catalan number.
Theorem 1.1. The square root function \( \varphi : Q \subset S_r^+ \mapsto \varphi(Q) = Q^{1/2} \in S_r^+ \) is Fréchet differentiable at any order on \( S_r^+ \) with the first order derivative given for any \((A,H) \in (S_r^+ \times S_r)\) by the formula

\[
\nabla \varphi(A) \cdot H = \int_0^{\infty} e^{-t\varphi(A)} H \, e^{-t\varphi(A)} \, dt
\]

(4)

The higher order derivatives are defined inductively for any \( n \geq 2 \) by the formula

\[
\nabla^n \varphi(A) \cdot H = -\nabla \varphi(A) \cdot \left[ \sum_{p+q=n-2} \frac{n!}{(p+1)!(q+1)!} [\nabla^{p+1} \varphi(A) \cdot H] \left[ \nabla^{q+1} \varphi(A) \cdot H \right] \right]
\]

(5)

In the above display, the summation is taken over all integers \( p, q \geq 0 \) s.t. \( p + q = n - 2 \). Assume that \( A \) and \( A + H \in S_r^+ \). In this situation the function \( \varphi \) has a Taylor expansion \((2)\) at any order. In addition, for any \( n \geq 0 \) we have the estimates

\[
\| \nabla^{n+1} \varphi(A) \| \leq K^n (n + 1)! C_n 2^{-(2n+1)} \lambda_{\min}(A)^{-(n+1/2)}
\]

(6)

\[
\| \nabla^{n+1} \varphi(A) \cdot H \| \leq K^n (n + 1) C_n 2^{-2n} \lambda_{\min}(A)^{-(n+1/2)} \| H \|^{n+1}
\]

where \( K = \sqrt{r} \) for the Frobenius norm, and \( K = 1 \) for the \( \| \cdot \|_2 \)-norm.

We end this section with some comments on the above theorem.

Firstly, arguing as in \((3)\) the convexity of the set \( S_r^+ \) ensures that the line segment joining the matrix \( A \in S_r^+ \) to any matrix \( B = A + H \in S_r^+ \) is always included in \( S_r^+ \). The terminal state condition \( B = A + H \in S_r^+ \) is met for any \( H \in S_r \) s.t. \( \lambda_{\min}(A) > 0 \lor (-\lambda_{\min}(H)) \).

This condition is also clearly met for any \( H \in S_r^0 \).

The inductive formula \((5)\) allows to compute sequentially the Fréchet derivatives of the matrix square root at any order starting from the first order derivative. For instance the second Fréchet derivative is given by

\[
\nabla^2 \varphi(A) \cdot H = -2 \nabla \varphi(A) \cdot \left[ \nabla \varphi(A) \cdot H \right]^2
\]

In this situation, using \((6)\) for any \( A \in S_r^+ \) and \( B = A + H \in S_r^+ \) we find that

\[
\| \varphi(B) - \varphi(A) - \nabla \varphi(A) \cdot (B - A) + \nabla \varphi(A) \cdot \left[ \nabla \varphi(A) \cdot (B - A) \right]^2 \|_2 \leq \frac{3}{8} \lambda_{\min}(A)^{-5/2} \| B - A \|_2^3
\]

As mentioned in the introduction several alternative representations of the Fréchet derivative of the square root function can be found in the literature. To better connect our work with existing results we end this section around this theme.

As shown in \((6)\), the integral representation of the square root matrix function is given in terms of the resolvent of \(-A\) by the formula

\[
\varphi(A) = \frac{1}{\pi} \int_0^{\infty} A(tI + A)^{-1} t^{-1/2} dt
\]

\[
\implies \nabla \varphi(A) \cdot H = \frac{1}{\pi} \int_0^{\infty} (tI + A)^{-1} H (tI + A)^{-1} t^{1/2} dt
\]
The last assertion is proved using a simple differentiation under the integral sign (invoking the dominated convergence theorem). The article [10] also extends this integral formulae to more general $n$-th roots matrix functions. The article [6] (see formula (15)) also provides an alternative formulation in terms of the exponential matrix of $A$; namely

$$
\nabla \varphi(A) \cdot H = \frac{1}{2\sqrt{\pi}} \int_0^\infty \left[ \int_0^t e^{-sA} H e^{-(t-s)A} ds \right] t^{-3/2} dt
$$

It is well know that the Fréchet derivative $X = \nabla \varphi(A) \cdot H$ given in (4) is the unique solution of the Sylvester equation [26] given by

$$
\psi(A) = A^2 \implies \nabla \psi(\varphi(A)) \cdot X = \varphi(A) \cdot X + X \cdot \varphi(A) = H \quad \text{and} \quad (\nabla \varphi)(\psi(A)) = [\nabla \psi(A)]^{-1}
$$

See for instance, section 6.1 in [16], and the article [6]. The Sylvester equation stated above is a particular case of the algebraic Riccati equation. It can also be regarded as a Lyapunov equation. In this connection there are no surprise that (4) coincides with the rather well know solution of the continuous Lyapunov equation. This integral formulation is closely related to the notion of controllability Gramian of a linear dynamical system with drift matrix $-\varphi(A)$, see for instance [9].

The literature also abounds with numerical techniques for solving of the Sylvester equation, see for instance the recent review by V. Simoncini [25] and references therein.

The formulae (5) for higher terms in the Taylor series for the square root provide a polynomial-type perturbation approximation of the square root at any order. These non asymptotic expansions have been used in [7,8] to analyze the fluctuation as well as the bias of the square root function of Wishart matrices and sample covariance matrices associated with stochastic Riccati equations arising in Ensemble-Kalman-Bucy filter theory.

## 2 Proof of theorem 1.1

Any (symmetric) square roots $\varphi(A)$ and $\varphi(B)$ of matrices $A, B \in \mathcal{S}_r^+$ satisfy the Sylvester equation

$$
\varphi(A) \left( \varphi(A) - \varphi(B) \right) + \left( \varphi(A) - \varphi(B) \right) \varphi(A) = (A - B) + (\varphi(A) - \varphi(B))^2 := C
$$

When $\varphi(A) > 0$ we have

$$
Z_t := e^{-t\varphi(A)} C e^{-t\varphi(A)} \longrightarrow_{t \to \infty} 0
$$

$$
\implies -\partial_t Z_t = \varphi(A) Z_t + Z_t \varphi(A)
$$

$$
\implies Z_0 = C = \varphi(A) \left[ \int_0^\infty Z_t dt \right] + \left[ \int_0^\infty Z_t dt \right] \varphi(A)
$$

This implies that

$$
(\varphi(B) - \varphi(A)) = \int_0^\infty e^{-t\varphi(A)} (B - A) e^{-t\varphi(A)} dt - \int_0^\infty e^{-t\varphi(A)} (\varphi(B) - \varphi(A))^2 e^{-t\varphi(A)} dt \quad (7)
$$
We set $B := A + H$. Under our assumptions $B \in S_\rho^+$. Using \((1)\) we conclude that

$$\nabla \varphi(A) \cdot H = \int_0^\infty e^{-t\varphi(A)} H e^{-t\varphi(A)} \, dt \implies \|\nabla \varphi(A)\| \leq K \, 2^{-1} \, \lambda_{\min}^{-1/2}(A) \tag{8}$$

We check the last assertion using the fact that

$$\varphi(A) > 0 \implies -\lambda_{\min}(\varphi(A)) = \rho(-\varphi(A)) = -\lambda_{\min}^{1/2}(A)$$

This yields the remainder formula

$$\nabla^2 \varphi [A, B - A] = \varphi(B) - \varphi(A) - \nabla \varphi(A) \cdot (B - A) = -\nabla \varphi(A) \cdot (\varphi(B) - \varphi(A))^2 \implies \|\nabla^2 \varphi [A, B - A]\| \leq \|\nabla \varphi(A)\| \|\varphi(B) - \varphi(A)\|^2 \leq K \, 2^{-1} \, \lambda_{\min}^{-3/2}(A) \, \|A - B\|^2 \tag{9}$$

The last assertion is a consequence of the Ando-Hemmen inequality \((1)\) and the estimate \((8)\). This ends the proof of the Taylor expansion at rank $n = 1$. We set

$$T_n(A, H) = \sum_{1 \leq k \leq n} \frac{1}{k!} \, \partial^k \varphi(A) \cdot H$$

with the collection of matrices $\partial^k \varphi(A) \cdot H$ defined by

$$\partial^1 \varphi(A) \cdot H = \nabla \varphi(A) \cdot H \tag{10}$$

and for any $n \geq 2$ by the induction

$$\partial^n \varphi(A) \cdot H := -\nabla \varphi(A) \cdot \left[ \sum_{p+q=n-2} \frac{n!}{(p+1)!(q+1)!} \left[ \partial^{p+1} \varphi(A) \cdot H \right] \left[ \partial^{q+1} \varphi(A) \cdot H \right] \right] \tag{11}$$

In the above display, the summation is taken over all integers $p, q \geq 0$ s.t. $p + q = n - 2$. We prove \((11)\) by induction on the parameter $n$. First, we prove that

$$\forall 1 \leq k \leq n \quad \frac{1}{k!} \|\partial^k \varphi(A)\| \leq \frac{K^{k-1}}{2^{\lambda_{\min}^{1/2}(A)}} \frac{C_{k-1}}{2^{2(k-1)\lambda_{\min}(A)k-1}} \tag{12}$$

By \((8)\) and \((10)\) this assertion is clearly met for $n = 1$. Assume that the above estimates \((12)\) are met for any $1 \leq k < n$. Combining \((8)\) with \((11)\) we find that

$$\frac{1}{n!} \|\partial^n \varphi(A)\| \leq \frac{K}{2^{\lambda_{\min}^{1/2}(A)}} \left[ \sum_{p+q=n, \ p,q \geq 1} \frac{1}{p!} \|\partial^p \varphi(A)\| \frac{1}{q!} \|\partial^q \varphi(A)\| \right]$$

Under the induction hypothesis, we have

$$\frac{1}{n!} \|\partial^n \varphi(A)\| \leq \frac{K^{n-1}}{(2^{\lambda_{\min}^{1/2}(A)})^3} \left[ \sum_{p+q=n-2, \ p,q \geq 0} \frac{C_p}{2^{2p\lambda_{\min}(A)^p}} \frac{C_q}{2^{2q\lambda_{\min}(A)^q}} \right]$$
Using the recursive formulation of the Catalan numbers $C_{k+1} = \sum_{p+q=k} C_p C_q$, which is valid for any $k \geq 0$ we conclude that

\[
\frac{1}{n!} \| \partial^n \varphi (A) \| \leq \frac{K^{n-1}}{(2 \lambda_{\text{min}}(A))^{3/2}} \frac{1}{2^{(n-2)} \lambda_{\text{min}}(A)^{n-2}} \sum_{p+q=n-2} C_p C_q
\]

\[
= \frac{K^{n-1}}{2 \lambda_{\text{min}}(A)} \frac{C_{n-1}}{2^{(n-1)} \lambda_{\text{min}}(A)^{n-1}}
\]

This ends the proof of the induction. The proof of (12) is now completed.

We further assume that

\[
\forall 0 \leq k \leq n \quad \nabla^k \varphi (A) \cdot H = \partial^k \varphi (A) \cdot H
\]

for some $n \geq 1$ and we set

\[
\Delta_n (A, H) := \varphi (A + H) - \varphi (A) - T_n (A, H)
\]

Using (9) we have

\[
\Delta_{n+1} (A, H) = \sum_{n+2 \leq k \leq 2n} \frac{1}{k!} \partial^k \varphi (A) \cdot H - \nabla \varphi (A) \cdot (\varphi (A + H) - \varphi (A)) \Delta_n (A, H)
\]

\[
- \nabla \varphi (A) \cdot (\Delta_n (A, H) [\varphi (A + H) - \varphi (A)]) + \nabla \varphi (A) \cdot \Delta_n (A, H)^2
\]

Under the induction hypothesis each term in the r.h.s. is of order at least $\| H \|^n$. This implies that

\[
\nabla^{n+1} \varphi (A) \cdot H = \partial^{n+1} \varphi (A) \cdot H
\]

This yields for any $n \geq 0$ the Taylor series expansions

\[
\varphi (A + H) = \sum_{0 \leq k \leq n} \frac{1}{k!} \nabla^k \varphi (A) \cdot H + \nabla^{n+1} \varphi [A, H]
\]

with the remainder term

\[
\nabla^{n+1} \varphi [A, H] = \frac{1}{n!} \int_0^1 (1 - \epsilon)^n \nabla^{n+1} \varphi \left( A + \epsilon H \right) \cdot H \, d\epsilon
\]

To take the final step we notice that

\[
B = A + H \implies \lambda_{\text{min}} (A + \epsilon H) \geq (1 - \epsilon) \lambda_{\text{min}} (A) + \epsilon \lambda_{\text{min}} (B) \geq (1 - \epsilon) \lambda_{\text{min}} (A)
\]

This implies that

\[
\| \nabla^{n+1} \varphi (A + \epsilon H) \cdot H \| \leq \frac{(n + 1)!}{(1 - \epsilon)^{n+1/2}} \frac{C_n K^n}{2^{n+1} \lambda_{\text{min}}(A)^{n+1/2}} \| H \|^{n+1}
\]

from which we conclude that

\[
\| \nabla^{n+1} \varphi [A, H] \| \leq \frac{(n + 1) C_n K^n}{2^{2n} \lambda_{\text{min}}(A)^{n+1/2}} \| H \|^{n+1}
\]

This ends the proof of the theorem.
References

[1] A. H. Al-Mohy and N. J. Higham, Computing the Fréchet derivative of the matrix exponential, with an application to condition number estimation, SIAM J. Matrix Anal. Appl., 30, pp. 1639–1657 (2009).

[2] A. H. Al-Mohy and N. J. Higham, The complex step approximation to the Fréchet derivative of a matrix function, Numer. Algorithms, vol. 53, pp. 133–148 (2010).

[3] A. H. Al-Mohy, N. J. Higham and S. D. Relton. Computing the Frechet Derivative of the Matrix Logarithm and Estimating the Condition Number. SIAM Journal on Scientific Computing (2013).

[4] T. Ando and J. L. van Hemmen. An inequality for trace ideals. Commun. Math. Phys., vol. 76, pp. 143–148 (1980).

[5] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, Graduate texts in Math. (1997).

[6] R. Bhatia, M. Uchiyama. The operator equation $\sum_{0\leq i\leq n} A^{n-i} X B^i = Y$, Expo. Math., vol. 27, pp. 251–255 (2009).

[7] A.N. Bishop, P. Del Moral, A. Niclas. A perturbation analysis of stochastic matrix Riccati diffusions. arXiv:1709.05071 (2017).

[8] A.N. Bishop, P. Del Moral, A. Niclas. An introduction to Wishart matrix moments. arXiv:1710.10864 (2017).

[9] R.W. Brockett. Finite dimensional linear systems. Society for Industrial and Applied Mathematics (2015).

[10] J. R. Cardoso, Evaluating the Fréchet derivative of the matrix $p$-th root, Electronic Transactions on Numerical Analysis, vol. 38 pp. 202–217 (2011).

[11] H. Cartan, Calcul différentiel. Paris: Hermann, MR 0223194 (1967).

[12] P. I. Davies and N. J. Higham, A Schur-Parlett algorithm for computing matrix functions, SIAM J. Matrix Anal. Appl., vol. 25, pp. 464–485 (2003).

[13] E. Deadman, N. J. Higham, and R. Ralha. Blocked Schur algorithms for computing the matrix square root, in Applied Parallel and Scientific Computing: 11th International Conference, PARA 2012, Helsinki, Finland, P. Manninen and P. Oster, eds., Lecture Notes in Comput. Sci. 7782, Springer-Verlag, Berlin, pp. 171–182 (2013).

[14] J. Dieudonné. Foundations of Modern Analysis, MA: Academic Press, MR 0349288. Boston (1969).

[15] R. M. Dudley, R. Norvaisa. Concrete functional calculus. Springer Monographs in Mathematics. Springer-Verlag New York (2011).

[16] N. J. Higham. Functions of Matrices : Theory and Computation, SIAM, Philadelphia, PA (2008).
[17] N. J. Higham. Stable iterations for the matrix square root. Numerical Algorithms vol. 15, no. 2 : 227-242 (1997).

[18] N. J. Higham. Computing real square roots of a real matrix. Linear Algebra and its applications, vol. 88, pp. 405–430 (1987).

[19] N. J. Higham and L. Lin, A Schur-Padé algorithm for fractional powers of a matrix, SIAM J. Matrix Anal. Appl., vol. 32, pp. 1056–1078 (2011).

[20] N. J. Higham and S.D. Relton. Higher order Fréchet derivatives of matrix functions and the level-2 condition number, SIAM J. Matrix Anal. Appl. 35 pp.1019–1037 (2014).

[21] J.B. Hiriart-Urruty and J. Malick A Fresh Variational-Analysis Look at the Positive Semidefinite Matrices World. J. Optim. Theory Appl. no. 153, pp.551–577 (2012).

[22] B. Meini. The matrix square root from a new functional perspective: theoretical results and computational issues. SIAM journal on matrix analysis and applications, vol. 26, no. 2, pp. 362–376 (2004).

[23] P.J. Psarrakos. On the nth roots of a complex matrix. The electronic Journal of Linear Algebra 9, pp 32–41 (2002).

[24] B. A. Schmitt. Perturbation bounds for matrix square roots and pythagorean sums. Linear Algebra and its Applications vol. 174, pp. 215–227 (1992).

[25] V. Simoncini. Computational Methods for Linear Matrix Equations, SIAM Review vol. 58 pp. 377–441 (2016).

[26] J. J. Sylvester. Sur les racines des matrices unitaires. Comptes Rendus de l’Académie des Sciences, vol. 94, pp. 396–399 (1882).