Brane worlds with bolts

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Abstract: We construct a family of \((p + 3)\)-dimensional brane worlds in which the brane has one compact extra dimension, the bulk has two extra dimensions, and the bulk closes regularly at codimension two submanifolds known as bolts. The \((p + 1)\)-dimensional low energy spacetime \(M_{\text{low}}\) may be any Einstein space with an arbitrary cosmological constant, the value of the bulk cosmological constant is arbitrary, and the only fields are the metric and a bulk Maxwell field. The brane can be chosen to have positive tension, and the closure of the bulk provides a singularity-free boundary condition for solutions that contain black holes or gravitational waves in \(M_{\text{low}}\). The spacetimes admit a nonlinear gravitational wave whose properties suggest that the Newtonian gravitational potential on a flat \(M_{\text{low}}\) will behave essentially as the static potential of a massless minimally coupled scalar field with Neumann boundary conditions. When \(M_{\text{low}}\) is \((p + 1)\)-dimensional Minkowski with \(p \geq 3\) and the bulk cosmological constant vanishes, this static scalar potential is shown to have the long distance behaviour characteristic of \(p\) spatial dimensions.

Keywords: Brane world, Randall-Sundrum, Melvin, brane waves

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1. Introduction

The idea that our universe might be a 3–brane embedded in a higher–dimensional spacetime with large bulk dimensions [1]–[4] has become the focus of intense research over the past two years. A pivotal observation was Randall and Sundrum’s discovery [5] that a flat positive tension 3–brane can be embedded in 5–dimensional anti–de Sitter space in such a way that the static gravitational potential on the brane reduces in the large distance limit to the (3+1)-dimensional Newtonian potential with polynomial corrections [5]–[8]. The subject has been developed into several directions, including black holes on branes [9]–[15], cosmological brane worlds (see for example [16] and the references therein), and brane worlds with more than one extra dimensions [17]–[31].

In the Randall–Sundrum single-brane model [5] the unperturbed bulk spacetime is locally anti–de Sitter. One appealing consequence is the possibility to analyse brane worlds in the context of M-theoretic AdS/CFT correspondence [7, 25, 32, 33, 34, 35]. What is less appealing is the presence of an anti–de Sitter Killing horizon in the bulk. This horizon tends to develop curvature singularities upon addition of perturbations, as has been found with black holes [9, 13] and gravitational waves [35, 36]. This raises questions about boundary conditions that may need to be imposed in the bulk and about their consequences on the brane. For example, in the higher-codimension brane world scenarios of [22, 23], the treatment of a bulk singularity affects the corrections to Newton’s law on the brane [25].

In this paper we introduce a family of brane world spacetimes in which the anti–de Sitter horizons of the Randall–Sundrum model are replaced by bolts [37]: totally geodesic codimension two submanifolds at which a rotational Killing vector field vanishes. The bulk closes regularly at the bolts, and this closing provides a topological, singularity-free boundary condition for gravitational waves or any other perturbations one may wish to consider. The effective low energy spacetime $M_{\text{low}}$ may be any Einstein space, with any value of the low energy cosmological constant. The brane is the product of $M_{\text{low}}$ and one compact extra dimension, while the bulk has two extra dimensions and solves the electrovacuum Einstein equations with an arbitrary value of the bulk cosmological constant. Solutions with positive brane tension exist for any values of the low energy cosmological constant and the bulk cosmological constant, while for certain values of these constants there also exist solutions with a negative brane tension.

Related brane worlds with more than one extra dimension have been presented in several contexts [17]–[31], and the general constraints on these types of constructions are discussed in [29]. Our main observation is that a pure Einstein-Maxwell theory succeeds in regularly closing the bulk for any values of the low energy cosmological constant and the bulk cosmological constant.

As the extra dimensions in our model are compact, one expects that the effective gravitational dynamics in $M_{\text{low}}$ reduces to Einstein’s equations at length scales that are large compared with the characteristic scales of the bulk. In particular, when $M_{\text{low}}$ is $(p + 1)$–dimensional Minkowski spacetime with $p \geq 3$, one expects the Newtonian gravitational potential in $M_{\text{low}}$ to be proportional to $-|\mathbf{x} - \mathbf{x}'|^{2-p}$, with corrections that vanish exponentially when $|\mathbf{x} - \mathbf{x}'| \to \infty$. We shall not attempt to verify this from a full linearized
perturbation analysis, but we shall show that when the bulk cosmological constant vanishes, the corresponding result does hold for the static potential of a massless minimally coupled scalar field with Neumann boundary conditions on the brane. We shall also present an exact nonlinear gravitational wave solution on our brane world background and use it to argue that the scalar field with Neumann boundary conditions is likely to capture the essentials of the linearized gravitational field. This is known to be the case for linearized perturbations of the Randall–Sundrum brane world [5]–[8].

We begin in section 2 with a brief review of the Einstein-Maxwell thick brane world model introduced by Gibbons and Wiltshire in 1987 [4], on which our construction is based. These thick branes are smooth codimension two submanifolds, with a number of attractive features in their own right, including localized chiral fermion modes on the brane, a mass gap for scalar and tensor modes in the presence of a negative bulk cosmological constant, and a mass gap for these modes even without a bulk cosmological constant if the thick brane has positive curvature. However, the major deficiency of the model is the lack of a well-defined graviton zero mode that would produce an effective low-dimensional Newton’s law on the brane.

In section 3 we reinterpret the thick brane of Gibbons and Wiltshire as a bolt at which the bulk spacetime closes, introduce a thin brane between two such thick branes and interpret the thin brane as the product of the low energy spacetime $M_{\text{low}}$ with one compact extra dimension. We show that the field equations can be satisfied with a pure brane tension term on the thin brane and analyse the constraints on the the brane tension and the bulk magnetic field for given values of the low energy cosmological constant and the bulk cosmological constant. In particular, we show that when the low energy cosmological constant vanishes, there is necessarily a nonvanishing bulk magnetic field.

In section 4 we present an exact nonlinear gravitational wave on the brane world background and show that the field equations for this gravitational wave reduce to a massless minimally coupled scalar field equation with Neumann boundary conditions on the thin brane. In section 5 we analyse the static potential of a massless minimally coupled scalar field when $M_{\text{low}}$ is $(p + 1)$–dimensional Minkowski spacetime with $p \geq 3$ and the bulk cosmological constant vanishes. Section 6 is devoted to brief concluding remarks.

Our metric signature is $(−+⋯)$. The spacetime dimension is $p + 3$ with $p \geq 1$.

2. The thick brane world

In this section we briefly review the thick brane worlds of Gibbons and Wiltshire [4]. The idea is to make the low energy spacetime a codimension two warped product submanifold of a higher–dimensional curved spacetime with matter in the bulk, such that the extra dimensions have infinite volume$^1$.

$^1$This model differs in a number of aspects from previous models with extra dimensions of infinite volume studied in the mid-1980s: the models of Akama [1] and Rubakov and Shaposhnikov [2] were by contrast based on the field theoretic trapping of matter about a surface in a higher–dimensional flat spacetime. The model of Visser [3] was based on a 5–dimensional curved spacetime with a diagonal metric whose time component was multiplied by a warp factor but whose space components were not, resulting in a 4–dimensional submanifold for the physical spacetime which did not possess an exact Lorentz invariance.
The field equations are the \((p+3)\)-dimensional Einstein-Maxwell equations, derivable from the action
\[
S = \int d^{p+3}x \sqrt{-\tilde{g}} \left\{ \frac{1}{2\kappa^2} (\mathcal{R} - 2\Lambda) - \frac{1}{4} F_{ab} F^{ab} \right\},
\]
(2.1)
where \(F_{ab}\) is the field strength of the \(U(1)\) gauge field. The solutions are conveniently written as
\[
ds^2 = r^2 \tilde{g}_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{\Delta} + \Delta d\varphi^2,
\]
(2.2)
\[
F = \frac{p(p+1)^{1/2} B}{\kappa r^{p+1}} dr \wedge d\varphi,
\]
(2.3)
where
\[
\Delta = \tilde{\lambda} + \frac{A}{r^p} - \frac{B^2}{r^{2p}} - \frac{2\Lambda r^2}{(p+1)(p+2)},
\]
(2.4)
\(A\) and \(B\) are real-valued constants and \(\tilde{g}_{\mu\nu}(x)\) is the metric on a \((p+1)\)-dimensional Einstein spacetime of signature \((- + + \cdots)\),
\[
\bar{R}_{\mu\nu} = p\tilde{\lambda} \tilde{g}_{\mu\nu}.
\]
(2.5)
We denote this \((p+1)\)-dimensional Einstein spacetime by \(M_{\text{low}}\). The numerical factor in (2.5) is chosen so that \(\tilde{\lambda}\) is equal to the Gaussian curvature of \(M_{\text{low}}\). Latin indices from the beginning of the alphabet run over all bulk dimensions, \(a, b = 0, 1, \ldots, p+2\), while Greek indices run over the dimensions of \(M_{\text{low}}, \mu, \nu = 0, 1, \ldots, p\).

The bulk metric is thus the warped product of \(M_{\text{low}}\) and a 2-dimensional space, \(\mathcal{D}_2\), and for \(B \neq 0\) there is a magnetic field in the bulk. These solutions are dual by double analytic continuation to solutions that generalize electrically charged Reissner-Nordström spacetimes \[4\].

Among all the various cases – which arise from a classification of the zeros of \(\Delta\) \[4\] – the spacetimes of interest as thick brane worlds are those in which \(\mathcal{D}_2\) is geodesically complete and of infinite total volume\(^2\). These are respectively: (i) \(\Lambda = 0, \tilde{\lambda} = 0\); (ii) \(\Lambda = 0, \tilde{\lambda} > 0\); (iii) \(\Lambda < 0, \tilde{\lambda}\) arbitrary. If the largest positive zero of \(\Delta(r)\) is at \(r = r_0\) and \(\Delta'(r_0) > 0\), where the prime denotes derivative with respect to \(r\), then \(\Delta(r)\) is positive for \(r_0 < r < \infty\). Provided \(\varphi\) has period \(4\pi/\Delta'(r_0)\), the geometry can be regularly extended to the \((p+1)\)-dimensional totally geodesic submanifold \(r = r_0\), which consists of the fixed points of the Killing vector \(\partial/\partial\varphi\) and is known as a “bolt” in the terminology of \[37\]. This submanifold is interpreted as the core of the brane.

These thick brane world are generalizations of Melvin’s magnetic universe \[39, 40\]. This is most readily seen in case (i). Taking \(B \neq 0\) and \(A > 0\) and defining the coordinates \((\tilde{x}^\mu, \rho, \phi)\) by
\[
\tilde{x}^\mu = \left( \frac{B^2}{A} \right)^{1/p} x^\mu,
\]
(2.6)
\(^2\)The case in which \(\mathcal{D}_2\) was non-compact but of finite volume and geodesically incomplete was studied previously by Wetterich \[38\].
\[ \rho = \frac{2}{pA} \left( \frac{B^2}{A} \right)^{1/p} r^p \sqrt{\Delta}, \]  
(2.7)

\[ \phi = \frac{pA^2}{2B^2} \left( \frac{A}{B^2} \right)^{1/p} \phi, \]  
(2.8)

the solution reads

\[ ds^2 = \left( 1 + \frac{\rho^2}{a^2} \right)^{2/p} \left( \eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu + d\rho^2 \right) + \frac{\rho^2 d\phi^2}{\left( 1 + \frac{\rho^2}{a^2} \right)^2}, \]  
(2.9)

\[ F = \frac{2}{\kappa a} \left( \frac{p+1}{p} \right)^{1/2} \frac{\rho d\rho \land d\phi}{\left( 1 + \frac{\rho^2}{a^2} \right)^2}, \]  
(2.10)

where \( \eta_{\mu\nu} = \text{diag}(−1, 1, 1, \cdots) \) and

\[ a = \frac{2B}{pA} \left( \frac{B^2}{A} \right)^{1/p}. \]  
(2.11)

The periodicity of \( \phi \) implies that \( \phi \) has period \( 2\pi \), and the solution is evidently regular at the bolt at \( \rho = 0 \). Melvin’s magnetic universe is recovered for \( p = 1 \).

The solutions are stable [4] for much the same reasons as Melvin’s solution is stable [40]. Essentially one has an equilibrium configuration in which the mutual repulsion of magnetic flux lines is balanced by their gravitational attraction.

Generalizations including a bulk dilaton were subsequently given by Gibbons and Maeda [41]. The properties of the non-compact space \( D_2 \) in these generalizations are similar. Recent generalizations in the context of \( M \)-theory are given in [42, 43, 44, 45].

In [4] the problem of localising gravity and gauge fields on the thick brane was considered in some detail. For case (i), it was shown that the bulk Dirac equation has zero mode solutions that are confined to a neighbourhood of the brane and move within it like massless \((p+1)\)-dimensional chiral fermions. Note that this localization is achieved by gravity and a bulk Abelian gauge field alone. By contrast, the recent mechanisms of [46] for localising fermions on brane worlds invoke matter fields in addition to gravity.

For a massless minimally coupled Klein-Gordon field, cases (ii) and (iii) were found to have a mass gap, but in case (i) the mass spectrum is continuous down to zero and does not give localization on the brane. A perturbation analysis was not carried out for all the gravitational modes, but a detailed analysis of nonlinear radial gravitational perturbations in case (i) yielded a continuous mass spectrum and supported the conclusion that the broad picture for the gravitational modes is similar to that for the scalar modes.

The major deficiency of this thick brane model is the absence of a normalizable graviton zero mode that would produce linearized \((p+1)\)-dimensional Einstein gravity at the core of the thick brane, and in particular produce the \((p+1)\)-dimensional Newtonian limit when \( M_{\text{low}} \) is flat. In the next section we will change the viewpoint and use the solution (2.2)-(2.5) to build a thin brane world in which a normalizable graviton zero mode does exist.
3. New brane world by brane surgery

We now add to the \((p + 3)\)-dimensional action (2.1) the term

\[
S_{\text{brane}} = -\lambda \int_{\text{brane}} d^{p+2}x \sqrt{-\det (g_{\hat{\mu}\hat{\nu}})},
\]

(3.1)

where the (new) brane is a timelike hypersurface of codimension one, \(g_{\hat{\mu}\hat{\nu}}\) is the induced metric on this brane, and the brane tension \(\lambda\) is a nonvanishing constant. The metric and \(F_{ab}\) are assumed continuous across the brane but their first derivatives may be discontinuous. Einstein’s equations at the brane amount to the Israel junction conditions [47, 48, 49, 50]

\[
K^+_{\hat{\mu}\hat{\nu}} - K^-_{\hat{\mu}\hat{\nu}} = -\frac{\kappa^2 \lambda}{p+1}g_{\hat{\mu}\hat{\nu}},
\]

(3.2)

where \(K^\pm_{\hat{\mu}\hat{\nu}}\) are the extrinsic curvatures of the brane in the geometries on the two sides, with respect to the normal that points from the “−” side to the “+” side. The hatted Greek indices run over the \(p+2\) dimensions on the brane.

To construct the solution with a thin brane, we begin with the bulk solution (2.2)–(2.5). Let \(r = r_0\) be a positive zero of \(\Delta\), such that \(\Delta'(r_0) \neq 0\). We make \(\varphi\) periodic with period \(4\pi/|\Delta'(r_0)|\). If \(\Delta'(r_0) > 0\), we choose a constant \(r_* > r_0\) such that \(\Delta > 0\) for \(r_0 < r \leq r_*\). If \(\Delta'(r_0) < 0\), we similarly choose a constant, also denoted by \(r_*\), so that \(0 < r_* < r_0\) and \(\Delta > 0\) for \(r_* \leq r < r_0\). We denote the spacetime in which \(r\) ranges from \(r_0\) to \(r_*\) by \(M_-\). \(r = r_0\) is a bolt in the interior of \(M_-\), and the geometry is regular there, while \(r = r_*\) is a boundary of \(M_-\), consisting of \(M_{\text{low}}\) (with metric multiplied by \(r_*^2\)) and a spacelike circle with circumference \(4\pi \Delta^{1/2}(r_*)/|\Delta'(r_0)|\).

Let \(M_+\) be a copy of \(M_-\). We glue \(M_-\) and \(M_+\) together at \(r = r_*\), using the identity gluing in \(M_{\text{low}}\) and choosing the gluing in \(\varphi\) so that \(F_{ab}\) is continuous. An embedding diagram of the resulting spacetime \(M\), with \(M_{\text{low}}\) suppressed, is shown in Figure 1.

![Figure 1: An embedding of the \(r\) and \(\varphi\) dimensions of \(M\) into \(\mathbb{R}^3\).](image)

We wish to make \(M\) into a solution with the thin brane action (3.1), such that the thin brane is at the junction \(r = r_*\). It is easily verified that Maxwell’s equations are satisfied

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\(^3\)We follow the convention of [43, 44] for the sign of the extrinsic curvature.
at the junction. As the extrinsic curvature of a constant \( r \) hypersurface in the metric (2.2) is \( \frac{1}{2} \Delta^{1/2} \partial_r g_{\mu\nu} \), with respect to the normal pointing towards increasing \( r \), the gravitational junction conditions (3.2) amount to the pair

\[
\sqrt{\Delta(r_*)} = \frac{\epsilon \kappa^2 \lambda r_*}{2(p+1)},
\]

(3.3)

\[
\Delta'(r_*) = \frac{\epsilon \kappa^2 \lambda \sqrt{\Delta(r_*)}}{p+1},
\]

(3.4)

where \( \epsilon = \text{sgn}(r_* - r_0) \). Eliminating \( \lambda \) yields

\[
\frac{\partial}{\partial r} \left( \frac{\Delta}{r^2} \right) \bigg|_{r=r_*} = 0.
\]

(3.5)

The solutions are thus obtained by solving (3.3), subject to the conditions on \( r_0 \) and \( r_* \) stated above, and then determining \( \lambda \) from (3.3) or (3.4). Note that \( \lambda \) has the same sign as \( r_* - r_0 \).

An elementary analysis shows that solutions with positive brane tension exist for any \( \bar{\lambda} \) and \( \Lambda \). The restrictions on \( B \) and the sign of \( \lambda \) for given \( \bar{\lambda} \) and \( \Lambda \) are collected in Table 1.

| \( \bar{\lambda} \) | \( \lambda < 0 \) | \( \lambda = 0 \) | \( \lambda > 0 \) |
|---|---|---|---|
| \( \lambda < 0 \) | \( B \neq 0 \) | \( B \neq 0 \) | \( B \neq 0 \) |
| \( \lambda = 0 \) | \( \lambda > 0 \), \( B \neq 0 \) | \( \lambda > 0 \), \( B \neq 0 \) | \( B \neq 0 \) |
| \( \lambda > 0 \) | \( \lambda > 0 \) | \( \lambda > 0 \) | none |

Table 1: Restrictions on \( B \) and the sign of \( \lambda \) for given \( \bar{\lambda} \) and \( \Lambda \).

We note in passing that if \( \mathcal{M}_- \) and \( \mathcal{M}_+ \) are glued together so that \( F_{ab} \) changes sign at the junction, we obtain as above solutions to the Einstein-Maxwell equations with an external current proportional to \( \partial/\partial \varphi \) on the brane. A variational principle producing this external current is obtained by including the “cosmological current” term

\[
\int d^{p+3}x \, \tilde{J}^a A_a,
\]

(3.6)

where \( \tilde{J}^a \) is a prescribed conserved current density with support only on the brane. However, these solutions have the unphysical feature that the brane’s stress-energy tensor, which is unaffected by the term (3.6), contains no contribution from charge carriers on the brane.\(^4\)

We seek to interpret \( \mathcal{M} \) as a brane world in which the spacetime of low energy physics is the \( (p+1) \)-dimensional Einstein spacetime \( M_{\text{low}} \) on the thin brane. The brane thus has one extra dimension, which is compact, and the two extra dimensions in the bulk close regularly at two thick branes of codimension two. For this interpretation to be viable, the Newtonian potential between sources concentrated on the thin brane and constant in \( \varphi \) needs to reduce to the \( (p+1) \)-dimensional Newton’s law in the Newtonian limit. The next two sections will present evidence that this is likely to be the case.

\(^4\)We thank Eric Poisson for correspondence on this point.
4. Non–linear gravitational waves on the brane world

Chamblin and Gibbons [36] have constructed plane polarized nonlinear gravitational waves on the Randall–Sundrum spacetime and its thick brane generalizations. The construction utilizes the solution-generating technique of Garfinkle and Vachaspati [51], which works for any electrovacuum spacetime that admits a hypersurface-orthogonal null Killing vector. We now show how this technique can be applied to the spacetimes of sections 2 and 3, assuming that \( M_{\text{low}} \) admits a hypersurface-orthogonal null Killing vector.

Let \( k^\mu \) be a hypersurface-orthogonal null Killing vector on \( M_{\text{low}} \). We raise and lower the index on \( k^\mu \) with \( g_{\mu \nu} \). As discussed in [51], there then exists on \( M_{\text{low}} \) at least locally a scalar \( \tilde{f} \) such that \( \partial_{[\mu}k_{\nu]} = k_{[\nu}\partial_{\mu]}\tilde{f} \) and \( k^\mu \partial_\mu \tilde{f} = 0 \).

We extend \( k^\mu \) into a vector field \( l^a \) on the spacetime \((2.2) - (2.5)\) by the natural extension, \( l^a = (k^\mu, 0, 0) \). We raise and lower the index on \( l^a \) with the full spacetime metric \((2.2)\). It is straightforward to verify that \( l^a \) is a nonvanishing hypersurface-orthogonal null Killing vector. Further, \( l^a \) satisfies \( \partial_{[a}l_{b]} = l_{[a}\partial_{b]}(f + 2\ln r) \) and \( \ell^a \partial_a(f + 2\ln r) = 0 \), where \( f \) is the pull-back of \( \tilde{f} \) to \((2.2)\).

As \( l^a F_{ab} = 0 \) and \( L_i F_{ab} = 0 \), \( l^a \) satisfies the conditions of the solution-generating technique of [51]. Thus let us take \( H \) to be a scalar function on the spacetime \((2.2) - (2.3)\), satisfying \( l^a \partial_a H = 0 \) and \( \nabla_a \nabla^a H = 0 \), where \( \nabla_a \) is the covariant derivative in the metric \((2.3)\). It follows [51] that adding to \((2.2)\) the term

\[
2He^{-f} k_{\mu}k_{\nu}dx^\mu dx^\nu \tag{4.1}
\]

gives a solution to the Einstein-Maxwell equations. This solution can be interpreted as a nonlinear gravitational wave on the spacetime \((2.2) - (2.5)\), travelling in the direction of \( l^a \).

To include the wave term \((4.1)\) in the thick brane worlds of section 3, one needs to check that the geometry remains regular at the bolt. This is equivalent to the requirement of regularity of \( H \) as a scalar field on the spacetime without the term \((4.1)\). To include the wave term in the thin brane worlds of section 3, one needs in addition to check that the junction conditions \((3.3)\) are satisfied at the thin brane. As the only \( r \)-dependence in \((4.1)\) is in the factor \( r^2H \), the junction conditions consist of \((3.3), (3.4)\) and

\[
\sqrt{\Delta(r_s)} \partial_r(r^2H) \bigg|_{r=r_s} = \frac{ek^2 \lambda r_s^2 H(r_s)}{p + 1}. \tag{4.2}
\]

Using \((3.3), (4.2)\) reduces to \( \partial_r H \big|_{r=r_s} = 0 \). When viewed as a scalar field on the thin brane spacetime without the term \((4.1)\), \( H \) thus obeys the Neumann boundary condition at the thin brane.

As an example, set \( g_{\mu \nu} = \eta_{\mu \nu} \). Adopting double null coordinates \((u, v, x^k)\), where \( k = 2, \ldots, p \), and choosing \( k^\mu = (\partial_r)^\mu \), the solution reads

\[
ds^2 = r^2 \left[-du dv + H(u, x^k, r, \phi)du^2 + \delta_{ij}dx_i^k dx^j \right] + \frac{dr^2}{\Delta} + \Delta d\phi^2, \tag{4.3}
\]

where \( \Delta(r) \) is given by \((2.4)\) with \( \lambda = 0 \). The construction is thus global on \( M_{\text{low}} \). The scalar wave equation for \( H \) reads explicitly

\[
H_{rr} + \left(\frac{p + 1}{r} + \frac{\Delta_{rr}}{\Delta}\right)H_{r} + \frac{H_{\phi\phi}}{\Delta^2} + \frac{\delta_{ij} H_{,ij}}{r^2 \Delta} = 0. \tag{4.4}
\]
Note that $H$ does not depend on $v$ but its dependence on $u$ is arbitrary. The linearized limit of the solution can be discussed as in [36]. In particular, $H = h_{ij}(u)x^i_\perp x^j_\perp$ is clearly a solution, it satisfies $\delta^{ij}H_{,ij} = 0$, and its linearized limit is analogous to the famous normalizable massless mode in the Randall–Sundrum spacetime [5]. This is the normalizable graviton zero mode we promised at the end of section [2].

We note that in the context of the thick brane worlds of Gibbons and Wiltshire, the scalar field spectral analysis of [5] applies directly to the nonlinear gravitational wave (4.3). To see this explicitly, we decompose $H$ as

$$H = \mathcal{F}(r)e^{i\tilde{n}\varphi}e^{ik_jx^j_\perp},$$

(4.5)

where $k_j$ may depend on $u$. As $\varphi$ has period $4\pi/\Delta'(r_0)$, regularity at the bolt requires $\tilde{n} = \frac{1}{2}\Delta'(r_0)n$, $n \in \mathbb{Z}$. For modes that do not exhibit faster-than-light propagation on the thick brane, we need $\delta^{ij}H_{,ij} = m^2H$, where $m^2 \geq 0$: this is achieved if $k_j$ is purely imaginary. The radial function $\mathcal{F}(r)$ then obeys

$$-\frac{d}{dr}\left[r^{p+1}\Delta\frac{d\mathcal{F}}{dr}\right] - m^2r^{p-1}\mathcal{F} + \tilde{n}^2r^{p+1}\frac{\mathcal{F}}{\Delta} = 0,$$

(4.6)

which is identical to Eq. (6.1c) in [4]. The results of [4] imply that if $\Lambda < 0$, the requirement that $H$ vanishes at $r \to \infty$ makes the spectrum of $m^2$ discrete and positive. If $\Lambda = 0$, on the other hand, the spectrum of $m^2$ for $\tilde{n} = 0$ is the positive real line.

As a second example, we take $\bar{g}_{\mu\nu}$ to be the metric of anti–de Sitter space. Adopting double null horospherical coordinates $(z, u, v, x^k_\perp)$, where $z > 0$ and $k = 3, \ldots, p$, and choosing $k^\mu = (\partial_u)^\mu$, the solution reads

$$ds^2 = \frac{r^2}{z^2} \left[ dz^2 - du dv + H(u, x^k_\perp, \varphi)du^2 + \delta_{ij}dx^i_\perp dx^j_\perp \right] + \frac{dr^2}{\Delta} + \Delta d\varphi^2,$$

(4.7)

where $\Delta(r)$ is given by (2.4) with $\lambda = -1$. However, the horospherical coordinates are not global on anti–de Sitter space (see for example [52]), and generic solutions for $H$ develop singularities on the Killing horizons at $z \to 0$ [35, 36, 53].

5. Massless scalar field on the brane world

In this section we analyse the static potential for a massless minimally coupled scalar field on the brane world of section [3] with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\Lambda = 0$. As $\lambda = 0$, Table [1] shows that the magnetic field is necessarily nonzero. We work in the coordinates in which $\mathcal{M}_\pm$ are given by (2.9) and (2.10). The bolt on each side is at $\rho = 0$, and the brane is at $\rho = \rho_*$, where

$$\rho_* = \left(\frac{p}{p+2}\right)^{1/2}|a|.$$

(5.1)

However, the Killing vector $\partial_0$ may be continued into an everywhere nonvanishing null Killing vector on anti–de Sitter space. In the embedding of anti–de Sitter space as the hyperboloid $-\alpha^2 = \eta_{AB}X^AX^B$ in flat space with the metric $ds^2 = \eta_{AB}dX^AdX^B$, where $\eta_{AB} = \text{diag}(-1, -1, 1, \ldots)$, $\partial_0$ can be written as $(m_A^2 - \eta_A m^B)X^A\partial_B$, where $m^A$ and $n^A$ are two constant orthogonal null vectors. We thank John Barrett for this observation.
From now on we drop the tildes from the coordinates $\tilde{x}^\mu$ in (2.3). For simplicity of presentation, we consider a scalar field on $\mathcal{M}_-$ with Neumann boundary conditions at $\rho = \rho_*$, and we evaluate the static potential of this field between two points at $\rho = \rho_*$. The static potential on the thin brane of section 3 is obtained by multiplying this result by two. Our technique closely follows that of [7].

5.1 The propagator

The action of a massless minimally coupled scalar field $\Phi$ on $\mathcal{M}_-$ reads

$$S_\Phi = -\frac{i}{2} \int d^{p+3}x \, (-g)^{1/2} (\nabla_a \Phi)(\nabla^a \Phi).$$

(5.2)

The Green’s function, $\Delta_{p+3}$, obeys

$$\nabla_a \nabla^a \Delta_{p+3} = \frac{\delta^{p+1}(x-x') \delta(\rho - \rho') \delta(\phi - \phi')}{\sqrt{-g}} = \frac{\delta^{p+1}(x-x') \delta(\rho - \rho') \delta(\phi - \phi')}{\rho \left(1 + \frac{\rho^2}{a^2}\right)^{2/p}}.$$  

(5.3)

The Neumann boundary condition at $\rho = \rho_*$ is

$$\partial_\rho \Delta_{p+3}|_{\rho = \rho_*} = 0.$$  

(5.4)

In order to solve (5.3), we Fourier decompose $\Delta_{p+3}$ as

$$\Delta_{p+3}(x, \rho, \phi; x', \rho', \phi') = \int \frac{d^{p+1}k}{(2\pi)^{p+2}} e^{ik_\mu(x^\mu - x'^\mu)} \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi')} \Delta_{k,n}(\rho, \rho')$$  

(5.5)

and substitute in (5.3) to obtain

$$\left[\frac{1}{\rho} \partial_\rho (\rho \partial_\rho) + q^2 - \frac{n^2}{\rho^2} \left(1 + \frac{\rho^2}{a^2}\right)^{2(1+p)/p}\right] \Delta_{k,n}(\rho, \rho') = \frac{\delta(\rho - \rho')}{\rho},$$  

(5.6)

where $q^2 = -k_\mu k^\mu$. The index on $x^\mu$ and $k^\mu$ is raised and lowered with $\eta_{\mu\nu}$. The boundary condition (5.4) reads

$$\partial_\rho \Delta_{k,n}(\rho, \rho')|_{\rho = \rho_*} = 0.$$  

(5.7)

For $\rho \neq \rho'$, Eq. (5.6) is recognized as a Sturm-Liouville equation that reduces to Bessel’s equation as $\rho \to 0$. For each $n$, we choose for the homogeneous equation linearly independent solutions $X_n(\rho)$ and $Y_n(\rho)$ that at $\rho \to 0$ have the asymptotic behaviour

$$X_n(\rho) \sim J_n(q\rho),$$  

(5.8)

$$Y_n(\rho) \sim N_n(q\rho),$$  

(5.9)

where $J_n$ and $N_n$ are respectively Bessel functions of the first and second kind. The Wronskian of $X_n(\rho)$ and $Y_n(\rho)$ then satisfies

$$W[X_n(\rho), Y_n(\rho)] \equiv X_n \partial_\rho Y_n - Y_n \partial_\rho X_n = \frac{2}{\pi\rho}.$$  

(5.10)
As the homogeneous equation for \( n = 0 \) is exactly Bessel’s equation, we have
\[
X_0(\rho) = J_0(q\rho), \quad (5.11)
\]
\[
Y_0(\rho) = N_0(q\rho). \quad (5.12)
\]

We denote the solution to (5.3) by \( \Delta_<(\rho, \rho') \) for \( \rho < \rho' \) and \( \Delta_>(\rho, \rho') \) for \( \rho > \rho' \). \( \Delta_< \) and \( \Delta_\rangle \) are each linear combinations of \( X_n(\rho) \) and \( Y_n(\rho) \), determined by the matching conditions arising from (5.6) and the boundary conditions at \( \rho = 0 \) and \( \rho = \rho_* \). The boundary condition at \( \rho = 0 \) is regularity of the solution as a function on the spacetime at the bolt, which excludes \( Y_n(\rho) \), and the boundary condition at \( \rho = \rho_* \) is (5.7). These boundary conditions imply
\[
\Delta_<(\rho, \rho') = A(\rho') X_n(\rho), \quad (5.13)
\]
\[
\Delta_>(\rho, \rho') = B(\rho') [Y_{sn} X_n(\rho) - X_{sn} Y_n(\rho)], \quad (5.14)
\]
where
\[
X_{sn} \equiv \left. \frac{dX_n}{d\rho} \right|_{\rho=\rho_*}, \quad Y_{sn} \equiv \left. \frac{dY_n}{d\rho} \right|_{\rho=\rho_*}. \quad (5.15)
\]
The matching conditions from (5.6) read
\[
\Delta_\langle |_{\rho=\rho'} = \Delta_\rangle |_{\rho=\rho'} \quad (5.16)
\]
\[
\partial_\rho (\Delta_\rangle - \Delta_\langle) |_{\rho=\rho'} = \frac{1}{\rho'}, \quad (5.17)
\]
and when applied to (5.13) and (5.14) they imply
\[
A(\rho') X_n(\rho') = B(\rho') \left[ Y_{sn} X_n(\rho') - X_{sn} Y_n(\rho') \right], \quad (5.18)
\]
\[
B(\rho') \left[ Y_{sn} \partial_\rho X_n(\rho') - X_{sn} \partial_\rho Y_n(\rho') \right] - A(\rho') \partial_\rho X_n(\rho') = \frac{1}{\rho'}. \quad (5.19)
\]
With the help of (5.10), equations (5.18) and (5.19) can be solved for \( A(\rho') \) and \( B(\rho') \) provided \( X_{sn} \neq 0 \), with the result
\[
\Delta_{k,n} = -\frac{\pi X_n(\rho_<)}{2X_{sn}} [Y_{sn} X_n(\rho_\rangle) - X_{sn} Y_n(\rho_\rangle)], \quad (5.20)
\]
where \( \rho_\rangle \) (\( \rho_< \)) denotes the greater (lesser) of \( \rho \) and \( \rho' \). We thus have
\[
\Delta_{p+3}(x, \rho, \phi; x', \rho', \phi') = \int \frac{d^{p+1}k}{4(2\pi)^{p+1}} e^{ik_n(x^n-x'^n)} \times
\]
\[
\times \sum_{n=-\infty}^{n=\infty} e^{i\phi - \phi'} X_n(\rho_<) \left[ Y_n(\rho_\rangle) - \frac{Y_{sn}}{X_{sn}} X_n(\rho_\rangle) \right]. \quad (5.21)
\]

The choice of the Green’s function can be specified in (5.21) by a prescription at the zeros of \( X_{sn} \), where the poles of the integrand are. The retarded Green’s function is obtained by the prescription \( q^2 \to (k^0 + i\epsilon)^2 - k^2, \epsilon \to 0_+ \).
5.2 Static potential on the brane

We wish to recover from (5.21) the static potential at \( \rho = \rho' = \rho^* \). We assume from now on that \( p \geq 3 \).

When \( \rho = \rho' = \rho^* \), (5.21) reduces with the help of (5.10) to

\[
\Delta_{p+3}(x, \rho^*, \phi; x', \rho^*; \phi') = \frac{1}{\rho^*} \int \frac{d^{p+1}k}{(2\pi)^{p+2}} e^{ik_{\mu}(x^\mu - x'^\mu)} \left\{ \frac{2}{q^2 \rho^*} - \frac{J_2(q\rho^*)}{qJ_1(q\rho^*)} \right. \\
- \sum_{n \neq 0} e^{i(n(\phi - \phi'))} \frac{X_n(\rho^*)}{X_{*n}} \right\}.
\]

(5.22)

We have isolated in (5.22) the \( n = 0 \) term from the rest and used Bessel function identities to write this term in a way that explicitly shows its pole structure near \( q = 0 \).

The static potential is obtained by integrating the retarded Green’s function over the time difference, \( t - t' \). As the retarded Green’s function is nonzero only for \( t - t' > 0 \), it is convenient to do this in (5.22) under the \( d^{p+1}k \) integral, with the result

\[
V(x, \phi; x', \phi') = \frac{1}{\rho^*} \int \frac{d^{p}k}{(2\pi)^p} \int_{-\infty}^{\infty} \frac{dk^0}{i(k^0 - i\epsilon)} e^{ik_{\mu}(x^\mu - x'^\mu)} \left\{ \frac{2}{q^2 \rho^*} - \frac{J_2(q\rho^*)}{qJ_1(q\rho^*)} \right. \\
- \sum_{n \neq 0} e^{i(n(\phi - \phi'))} \frac{X_n(\rho^*)}{X_{*n}} \right\},
\]

(5.23)

where \( q^2 = (k^0 + i\epsilon)^2 - k^2 \) and \( \epsilon \to 0_+ \).

In (5.23) consider the term in the integrand proportional to \((k^0 - i\epsilon)^{-1}q^{-2}\). We close the \( k^0 \) contour in the upper half-plane and do the integral by residues. As the only pole within the contour is at \( k^0 = i\epsilon \), the contribution to \( V(x, \phi; x', \phi') \) is

\[
V_{0,\text{Mink}}(x; x') = -\frac{1}{\pi \rho^2} \int \frac{d^p k}{(2\pi)^p} \frac{e^{ik_{\mu}(x^\mu - x'^\mu)}}{k^2} \frac{1}{\Gamma(\frac{p}{2} - 1)} \frac{4\pi^{(p/2) + 1} \rho^2 |x - x'|^{p-2}}{\pi^{p-1} \rho^{p-1} |x - x'|^{p-2}},
\]

(5.24)

where \( \Omega_{p-1} \) is the volume of the \((p-1)\)-sphere. As is clear already from (5.22), \( V_{0,\text{Mink}}(x; x') \) is proportional to the static potential of a free massless field in \((p + 1)\)-dimensional Minkowski space. Note from (2.9) and (7.1) that the spatial proper distance at \( \rho = \rho^* \) is \( |2(p + 1)/(p + 2)|^{1/p} |x - x'| \).

Let us then consider the remaining part of the \( n = 0 \) term in (5.23). We now close the \( k^0 \) contour in the lower half-plane. As all the zeros of \( J_1 \) are real, the poles within the contour are at \( q = \pm j_{1,s}/\rho^* \), where \( j_{1,s}, s = 1, 2, \ldots \) are the positive zeros of \( J_1 \). The
contribution to \( V(x, \phi; x', \phi') \) is therefore

\[
V_{0,Y}(x; x') = \frac{1}{\pi \rho_s} \int \frac{d^p k}{(2\pi)^p} e^{i k \cdot (x-x')} \sum_{s=1}^{\infty} \frac{J_2(j_{1,s})}{J_1'(j_{1,s})} \left( \frac{j_{1,s}}{\rho_s |x-x'|} \right)^{(p/2)-1} K_{(p/2)-1}\left( j_{1,s} |x-x'| / \rho_s \right),
\]

(5.25)

where \( K \) is the modified Bessel function of the second kind. At large \( |x-x'| \) the terms in \( V_{0,Y}(x; x') \) decay as \( |x-x'|^{(1-p)/2} \exp (-j_{1,s} |x-x'|/\rho_s) \), and \( V_{0,Y}(x; x') \) is thus of Yukawa type. The dominant term is \( s = 1 \), for which \( j_{1,1} \approx 3.83 \) \cite{54}.

For the \( n \neq 0 \) terms in (5.23), we do not have an expression for \( X_n \) in terms of known functions. However, the differential equation (5.6) and the boundary conditions at \( \rho = 0 \) and \( \rho = \rho_s \) imply that the zeros of \( X_{n\rho s} \) occur precisely when \( q^2 \) is an eigenvalue of a self-adjoint differential operator \cite{54, 55}. Bringing the operator to a standard form shows that the spectrum is purely discrete (Ref. \cite{55}, Theorem XIII.7.17), and a simple estimate using the differential equation (5.6) and the asymptotic expression (5.8) shows that there are no eigenvalues with \( q^2 \leq 0 \). This means that the only zeros of \( X_{n\rho s} \) in the complex \( q \) plane are at discrete positive values of \( q^2 \). If the asymptotic behaviour of \( X_n(\rho_s)/X_{n\rho s} \) at large complex \( q \) is such that closing the \( k^0 \) contour in (5.23) in the lower half-plane can be justified, it follows as with (5.25) that the contribution to \( V(x, \phi; x', \phi') \) is exponentially suppressed as \( |x-x'| \to \infty \).

In conclusion, the static potential \( V(x, \phi; x', \phi') \) consists of the effective \( (p+1) \)-dimensional term \( V_{0,\text{Mink}}(x; x') \) (5.24) plus corrections that, subject to our technical assumptions about the \( n \neq 0 \) modes, are exponentially suppressed as \( |x-x'| \to \infty \).

6. Discussion

We have presented a family of \( (p+3) \)-dimensional brane worlds in which the brane has one compact extra dimension, the bulk has two extra dimensions, and the bulk is compact, closing regularly at bolts where a rotational Killing vector vanishes. The spacetimes solve the \( (p+3) \)-dimensional Einstein-Maxwell equations, with an arbitrary bulk cosmological constant, and the field equations at the brane come from a brane tension that can be chosen positive. The low energy spacetime \( M_{\text{low}} \) may be any \( (p+1) \)-dimensional Einstein space. When \( M_{\text{low}} \) is \( (p+1) \) Minkowski with \( p \geq 3 \) and the bulk cosmological constant vanishes, we showed that a massless minimally coupled scalar field satisfying the Neumann boundary conditions on the brane has a static potential on the brane with the long distance behaviour \(-|x-x'|^{2-p}\), characteristic of \( p \) spatial dimensions. We did not attempt a direct calculation of the Newtonian gravitational potential on the brane, but we presented exact nonlinear gravitational wave solutions whose field equations reduce to those of a massless scalar field with Neumann boundary conditions at the brane. Our scalar field results therefore suggest that when \( M_{\text{low}} \) is Minkowski, the long distance behaviour of the Newtonian potential on the brane should also be characteristic of \( p \) spatial dimensions.
As the extra dimensions are compact, one expects on general grounds that the effects of the extra dimensions will be exponentially suppressed in $M_{\text{low}}$ at scales much longer than all the length scales of the extra dimensions. While there exist parameter ranges where the circumference of the extra dimension on the brane is arbitrarily large compared with the orthogonal distance from the brane to the bolt, our numerical experiments have not uncovered parameter ranges where the distance to the bolt could be made arbitrarily large compared with the circumference on the brane. Our spacetimes are thus likely to produce an observationally interesting correction to Newton’s law only at length scales that are comparable to the circumference of the extra dimension on the brane.

The closing of the bulk at the bolts has two main appealing consequences. First, as the extra dimensions are compact, the bulk has no horizons that could develop singularities upon addition of gravitational waves or black holes. For example, the nonlinear gravitational wave of section 4 is manifestly regular in the bulk. Second, the bolts make the brane world possible without negative tension branes, whereas with a periodic transverse dimension there would need to be at least one negative tension brane [28, 57]. One could put our brane at an orbifold by identifying the two sides of the bulk, provided such an orbifold is considered an acceptable classical solution, but there seems to be a motivation to do so only in the cases where the brane has negative tension. It might be interesting to investigate whether our spacetimes with a negative tension brane on an orbifold exhibit nonlinear instabilities similar to those found in [25].

As $M_{\text{low}}$ may be any Einstein space, our construction includes the Schwarzschild black hole on the brane, as well as dynamical vacuum black hole solutions on the brane. Such solutions offer an arena for examining how the extra dimensions affect nonlinear gravitational effects and black hole thermodynamics. For example, one could ask whether bulk geodesics around black holes can form halos near the brane [14].

Low energy phenomenological matter could be added by a Lagrangian that is confined to the brane [3]. The bulk Maxwell field need a priori not couple to any such low energy matter. However, one might wish to investigate whether coupling the bulk Maxwell field to charges on the brane induces an effective $(p + 1)$–dimensional Maxwell theory on $M_{\text{low}}$.

The dilatonic thick brane solutions of Gibbons and Maeda [41] suggest possible generalizations of our brane worlds to include a bulk dilaton field. As supergravity theories give rise to dilatonic bosonic sectors, this raises the possibility of recovering bolt-brane-bolt solutions to $M$-theory or to one of its supergravity limits. One may hope to analyse such solutions in terms of $M$-theoretic (A)dS/CFT correspondence [59, 60, 61, 62]. Since the extra dimensions form a two-dimensional submanifold the problem of realising the solutions in a supergravity context may prove to be simpler than the analogous problem in the Randall–Sundrum model.

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References

[1] K. Akama, “Pregeometry”, in K. Kikkawa, N. Nakanishi and H. Nariai (eds), “Gauge Theory and Gravitation”, Lect. Notes Phys. 176 (1982) 267 [hep-th/0001113].

[2] V. A. Rubakov and M. E. Shaposhnikov, “Do we live inside a domain wall?”, Phys. Lett. B 125 (1983) 136.

[3] M. Visser, “An exotic class of Kaluza-Klein models”, Phys. Lett. B 159 (1985) 22 [hep-th/910093]; E. J. Squires, “Dimensional reduction caused by a cosmological constant”, Phys. Lett. B 167 (1986) 286.

[4] G. W. Gibbons and D. L. Wiltshire, “Spacetime as a membrane in higher dimensions”, Nucl. Phys. B 287 (1987) 717 [hep-th/910093].

[5] L. Randall and R. Sundrum, “An alternative to compactification”, Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].

[6] J. Garriga and T. Tanaka, “Gravity in the brane-world”, Phys. Rev. Lett. 84 (2000) 2778 [hep-th/9911058].

[7] S. B. Giddings, E. Katz and L. Randall, “Linearized gravity in brane backgrounds”, J. High Energy Phys. 0003 (2000) 023 [hep-th/0002091].

[8] Z. Kakushadze, “Gravity in Randall-Sundrum brane-world revisited”, Phys. Lett. B 497 (2001) 125 [hep-th/0008128].

[9] A. Chamblin, S.W. Hawking and H.S. Reall, “Brane-world black holes”, Phys. Rev. D 61 (2000) 065007 [hep-th/9909203].

[10] R. Emparan, G. T. Horowitz and R. C. Myers, “Exact description of black holes on branes”, J. High Energy Phys. 0001 (2000) 007 [hep-th/9911043].

[11] R. Emparan, G. T. Horowitz and R. C. Myers, “Exact description of black holes on branes. 2. Comparison with BTZ black holes and black strings”, J. High Energy Phys. 0001 (2000) 021 [hep-th/9912135].

[12] R. Emparan, G. T. Horowitz and R. C. Myers, “Black holes radiate mainly on the brane”, Phys. Rev. Lett. 85 (2000) 499 [hep-th/0003118].

[13] A. Chamblin, H. S. Reall H. Shinkai, and T. Shiromizu, “Charged brane world black holes”, Phys. Rev. D 63 (2001) 064015 [hep-th/0008177].

[14] A. Chamblin, “Capture of bulk geodesics by brane world black holes”, Class. and Quant. Grav. 18 (2001) L17 [hep-th/0011123].

[15] R. Emparan, R. Gregory and C. Santos, “Black holes on thick branes”, Phys. Rev. D 63 (2001) 104022 [hep-th/0012100].
[16] H. A. Bridgman, K. A. Malik and D. Wands, “Cosmological perturbations in the bulk and on the brane”, Phys. Rev. D 65 (2002) 043502 [astro-ph/0107245].

[17] A. Chodos and E. Poppitz, “Warp factors and extended sources in two transverse dimensions”, Phys. Lett. B 471 (1999) 113 [hep-th/9909193].

[18] R. Gregory, “Nonsingular global string compactifications”, Phys. Rev. Lett. 84 (2000) 2564 [hep-th/9911015].

[19] Z. Chacko and A. E. Nelson, “A solution to the hierarchy problem with an infinitely large extra dimension and moduli stabilization”, Phys. Rev. D 62 (2000) 085006 [hep-th/9912186].

[20] J.-W. Chen, M. A. Luty and E. Ponton, “A critical cosmological constant from millimeter extra dimensions”, J. High Energy Phys. 0009 (2000) 012 [hep-th/0003067].

[21] I. Olasagasti and A. Vilenkin, “Gravity of higher-dimensional global defects”, Phys. Rev. D 62 (2000) 044014 [hep-th/0003300]; I. Olasagasti, “Gravitating global defects: the gravitational field and compactification”, Phys. Rev. D 63 (2001) 124016 [hep-th/0101203].

[22] T. Gherghetta and M. Shaposhnikov, “Localizing gravity on a string-like defect in six dimensions”, Phys. Rev. Lett. 85 (2000) 240 [hep-th/0004014].

[23] T. Gherghetta, E. Roessl and M. Shaposhnikov, “Living inside a hedgehog: higher dimensional solutions that localize gravity”, Phys. Lett. B 491 (2000) 353 [hep-th/0006251].

[24] S. Hayakawa and K. I. Izawa, “Warped compactification with an abelian gauge theory”, Phys. Rev. Lett. B 493 (2000) 380 [hep-th/0008111].

[25] E. Ponton and E. Poppitz, “Gravity localization on string-like defects in codimension two and the AdS/CFT correspondence”, J. High Energy Phys. 0102 (2001) 042 [hep-th/0012033].

[26] O. Corradini and Z. Kakushadze, “A solitonic three-brane in 6-D bulk”, Phys. Lett. B 506 (2001) 167 [hep-th/0103031].

[27] P. Kanti, R. Madden and K. A. Olive, “A 6-D brane world model”, Phys. Rev. D 64 (2001) 044021 [hep-th/0104177].

[28] S. Hayakawa and K. I. Izawa, “Warped compactification with a four-brane”, Prog. Theor. Phys. 106 (2001) 641 [hep-th/0106101].

[29] F. Leblond, R. C. Myers and D. J. Winters, “Consistency conditions for brane worlds in arbitrary dimensions”, J. High Energy Phys. 0107 (2001) 031 [hep-th/0106140]; “Brane world sum rules and the AdS soliton”, hep-th/0107034.

[30] I. I. Kogan, S. Mouslopoulos, A. Papazoglou and G. G. Ross, “Multigravity in six dimensions: generating bounces with flat positive tension branes”, Phys. Rev. D 64 (2001) 124014 [hep-th/0107086].

[31] O. Corradini, A. Iglesias, Z. Kakushadze and P. Langfelder, “Gravity on a 3-brane in 6D Bulk”, Phys. Lett. B 521 (2001) 96 [hep-th/0108055].

[32] S. S. Gubser, “AdS/CFT and gravity”, Phys. Rev. D 63 (2001) 084017 [hep-th/9912001].
S. Nojiri, S. D. Odintsov and S. Zerbini, “Quantum (in)stability of dilatonic AdS backgrounds and holographic renormalization group with gravity”, *Phys. Rev. D* **62** (2000) 064006 [hep-th/0001192];
S. Nojiri and S. D. Odintsov, “Brane world inflation induced by quantum effects”, *Phys. Lett. B* **484** (2000) 119 [hep-th/0004097];
S. Nojiri, O. Obregon and S. D. Odintsov, “(Non)-singular brane-world cosmology induced by quantum effects in d5 dilatonic gravity”, *Phys. Rev. D* **62** (2000) 104003 [hep-th/0005127].

S. W. Hawking, T. Hertog and H. S. Reall, “Brane New World”, *Phys. Rev. D* **62** (2000) 043501 [hep-th/0003052].

D. Brecher, A. Chamblin and H. S. Reall, “AdS/CFT in the infinite momentum frame”, *Nucl. Phys. B* **607** (2001) 155 [hep-th/0012078].

A. Chamblin and G. W. Gibbons, “Nonlinear supergravity on a brane without compactification”, *Phys. Rev. Lett.* **84** (2000) 1090 [hep-th/9909133].

G. W. Gibbons and S. W. Hawking, “Classification of gravitational instanton symmetries”, *Commun. Math. Phys.* **66** (1979) 291.

C. Wetterich, “The cosmological constant and noncompact internal spaces in Kaluza-Klein theories”, *Nucl. Phys. B* **255** (1985) 480.

M. A. Melvin, “Pure magnetic and electric geons”, *Phys. Lett. B* **8** (1963) 65;
“Dynamics of cylindrical electromagnetic universes”, *Phys. Rev. 139* (1965) B223.

K. S. Thorne, “Energy of infinitely long, cylindrically symmetric systems in general relativity”, *Phys. Rev. 138* (1965) B251;
“Absolute stability of Melvin’s magnetic universe”, *Phys. Rev. 139* (1965) B244.

G. W. Gibbons and K. Maeda, “Black holes and membranes in higher dimensional theories with dilaton fields”, *Nucl. Phys. B* **298** (1988) 741.

P. M. Saffin, “Gravitating fluxbranes”, *Phys. Rev. D* **64** (2001) 024014 [gr-qc/0104014].

M. Gutperle and A. Strominger, “Fluxbranes in string theory”, *J. High Energy Phys.* **0106** (2001) 033 [hep-th/0104138].

M. S. Costa, C. A. R. Herdeiro and L. Cornalba, “Fluxbranes and the dielectric effect in string theory”, *Nucl. Phys. B* **619** (2001) 155 [hep-th/0105023].

A. M. Uranga, “Wrapped fluxbranes”, [hep-th/0108196].

B. Bajc and G. Gabadadze, “Localization of matter and cosmological constant on a brane in anti–de Sitter space”, *Phys. Lett. B* **474** (2000) 282 [hep-th/9912232];
I. Oda, “Localization of matters on a string–like defect”, *Phys. Lett. B* **496** (2000) 113 [hep-th/0006203];
S. Randjbar-Daemi and M. Shaposhnikov, “Fermion zero-modes on brane-worlds”, *Phys. Lett. B* **492** (2000) 361 [hep-th/0008079].

W. Israel, “Singular hypersurfaces and thin shells in general relativity”, *Nuovo Cim. B* **44** (1966) 3 (erratum: *ibid.* B48 (1966) 463).

C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

C. Barrabes and W. Israel, “Thin shells in general relativity and cosmology: The lightlike limit”, *Phys. Rev. D* **43** (1991) 1120.
[50] S. Mukohyama, “Perturbation of junction condition and doubly gauge-invariant variables”, Class. and Quant. Grav. 17 (2000) 4777 [hep-th/0006146].

[51] D. Garfinkle and T. Vachaspati, “Cosmic string traveling waves”, Phys. Rev. D 42 (1990) 1960.

[52] V. Balasubramanian, P. Kraus and A. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter spacetime”, Phys. Rev. D 59 (1999) 046003 [hep-th/9805171].

[53] J. Podolsky, “Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe”, Class. and Quant. Grav. 15 (1998) 719 [gr-qc/9801052].

[54] M. Abramowitz and I. A. Stegun (editors), Handbook of Mathematical Functions (Dover, New York, 1965).

[55] N. Dunford and J. S. Schwartz, Linear Operators (Interscience, New York, 1963), Vol. II.

[56] M. Reed and B. Simon, Methods of Modern Mathematical Physics (Academic, New York, 1975), Vol. II.

[57] G. Gibbons, R. Kallosh and A. Linde, “Brane world sum rules”, J. High Energy Phys. 0101 (2001) 023 [hep-th/0011223].

[58] D. Marolf and M. Trodden, “Black Holes and Instabilities of Negative Tension Branes”, Phys. Rev. D 64 (2001) 065019 [hep-th/0102135].

[59] J. Maldacena, “The large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2 (1998) 231 [hep-th/9711200].

[60] E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. 2 (1998) 253 [hep-th/9802150].

[61] E. Witten, “Quantum gravity in de Sitter space”, [hep-th/0106103].

[62] A. Strominger, “The dS/CFT correspondence”, J. High Energy Phys. 0110 (2001) 034 [hep-th/0106113].