Cosmic strings in hidden sectors: 1. Radiation of standard model particles

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Abstract. In hidden sector models with an extra U(1) gauge group, new fields can interact with the Standard Model only through gauge kinetic mixing and the Higgs portal. After the U(1) is spontaneously broken, these interactions couple the resultant cosmic strings to Standard Model particles. We calculate the spectrum of radiation emitted by these “dark strings” in the form of Higgs bosons, Z bosons, and Standard Model fermions assuming that string tension is above the TeV scale. We also calculate the scattering cross sections of Standard Model fermions on dark strings due to the Aharonov-Bohm interaction. These radiation and scattering calculations will be applied in a subsequent paper to study the cosmological evolution and observational signatures of dark strings.

Keywords: Cosmic strings, domain walls, monopoles, particle physics - cosmology connection

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1 Introduction

In the Standard Model of particle physics, all of the matter fields are charged under the gauge group of the theory, and consequently all of the particles participate in the gauge interactions. It is natural to ask whether there can be new particles that do not participate in any of the Standard Model gauge interactions, and whose fields are singlets under the Standard Model gauge group. Such fields would be sequestered in a “hidden sector” where they participate in their own gauge interactions under which the SM fields are singlets. Despite their minimal nature, hidden sector models admit a rich phenomenology; they have been well-studied in the context of collider physics [1–5] as well as dark matter [6–11]. In refs. [12, 13] we have pointed out that these models may also contain cosmic string solutions, called “dark strings”, that have novel interactions with Standard Model fields. The aim of the present paper is to derive the radiative and scattering properties of these strings. In a subsequent paper we will use these properties to study potential astrophysical and cosmological signatures of dark strings.

The Lagrangian for the hidden sector model under consideration is of the form

\[ \mathcal{L} = \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{HS}} + \mathcal{L}_{\text{int}} . \]

The first term, $\mathcal{L}_{\text{SM}}$, is the Standard Model (SM) Lagrangian; the second term,

\[ \mathcal{L}_{\text{HS}} = |D_\mu S|^2 - \frac{1}{4} \hat{X}_{\mu\nu}^2 - \kappa (S^* S - \sigma^2)^2 , \]

is the hidden sector (HS) Lagrangian with $S$ a complex scalar field charged under a U(1)$_X$ gauge group that has $\hat{X}_\mu$ as its gauge potential, $D_\mu = \partial_\mu - ig_x \hat{X}_\mu$; and the third term,

\[ \mathcal{L}_{\text{int}} = -\alpha (\Phi^1 \Phi - \eta^2) (S^* S - \sigma^2) - \frac{\sin \epsilon}{2} \hat{X}_{\mu\nu} Y^{\mu\nu} , \]

is the interaction Lagrangian with $\Phi$ the Higgs doublet and $Y_\mu$ the hypercharge gauge field. The mass scale of the hidden sector fields is set by the parameter $\sigma$, and $\eta = 174 $ GeV is the vacuum expectation value (VEV) of the Higgs field. The two terms in $\mathcal{L}_{\text{int}}$ are called the Higgs portal (HP) term [14] and the gauge-kinetic mixing (GKM) term [15, 16], respectively. For $\sigma \lesssim \text{TeV}$, the HP and GKM couplings are well-constrained, $|\alpha|, |\sin \epsilon| \ll 1$ [17, 18], but if $\sigma$ is above the TeV scale, making HS particles inaccessible at laboratory energies, the hidden sector model is (as yet) unconstrained. In principle the hidden sector can be extended to include additional fields and interactions; we retain only the minimal degrees of freedom necessary to study radiation of SM particles from the cosmic string.
The VEV $\langle S \rangle = \sigma$ spontaneously breaks the $U(1)_X$ completely. Consequently the model admits topological (cosmic) string solutions [19]. The string tension is set by the symmetry breaking mass scale $\mu \approx \sigma^2$, and we will use $M \equiv \sqrt{\mu} \approx \sigma$ through the text. In ref. [13] we studied these “dark string” solutions, which were found to contain a non-trivial structure in the dark sector fields, $S$ and $\hat{X}_\mu$, as well as in the SM fields, $\Phi$ and $Y_\mu$ (see also [20] for the case when $\sin \epsilon = 0$). In the decoupling limit, $\sigma \gg \eta$, the dark fields form a thin core of thickness on the order of $\sigma^{-1}$, and the SM fields form a wide dressing with thickness $\eta^{-1}$. The dressing arises because the string core sources the SM Higgs and Z boson fields, $\phi_H$ and $Z_{\mu}$. In the limit $\sigma \gg \eta$ we can integrate out the heavy HS fields leaving only the zero thickness string core. In ref. [13] we found the effective interaction of the string core with the light SM fields to be

$$S^{(1)}_{\text{int}} = g_{\text{str}}^H \eta \int d^2 \sigma \sqrt{-\gamma} \phi_H(\hat{X}^\mu) + \frac{g_{\text{str}}^Z}{2} \left( \frac{\eta}{\sigma} \right)^2 \int d\sigma^{\mu\nu} Z_{\mu\nu}(\hat{X}^\mu) \quad (1.4)$$

where $\hat{X}^\mu$ denotes the location of the zero thickness string core, and the rest of the notation is defined in appendix A. The coupling constants $g_{\text{str}}^H$ and $g_{\text{str}}^Z$ have been derived in ref. [13] in terms of $\alpha$, $\sin \epsilon$, and other Lagrangian coupling constants. We shall treat them as free parameters in the present paper. Note that the interaction in eq. (1.4) is valid for $\sigma \gg \eta$, when the string core is much thinner than the SM dressing. If the core and dressing widths are comparable, the effective interaction formalism breaks down and the full field theory equations must be solved to evaluate string-particle interactions.

The linear interactions given above arise because the Higgs gets a VEV, and the string acts as a source that modifies the VEV. In addition, the string also couples to the SM fields through the more generic quadratic interactions. Upon integrating out the heavy hidden sector fields, the effective quadratic interactions for the Higgs and Z boson are

$$S^{(2)}_{\text{int}} = g_{\text{str}}^H \int d^2 \sigma \sqrt{-\gamma} \phi_H^{2} H(\hat{X}) + g_{\text{str}}^Z \left( \frac{\eta}{\sigma} \right)^4 \int d^2 \sigma \sqrt{-\gamma} Z_{\mu}(\hat{X}) Z^{\mu}(\hat{X}) \quad \text{.} \quad (1.5)$$

The quadratic Higgs interaction derives directly from the HP term in eq. (1.3), and we can estimate $g_{\text{str}}^H \approx \alpha$ up to order one factors related to integrals of the profile functions. The quadratic $Z$ boson interaction results from the mixing of the $Z$ boson with the heavy $\hat{X}^\mu$ field. The mixing angle goes like $(\sin \epsilon)(\eta/\sigma)^2$ [13], and therefore we obtain the quadratic interaction in eq. (1.5) with $g_{\text{str}}^Z \approx \sin^2 \epsilon$. The W bosons will have a coupling similar to the Z boson coupling in eq. (1.5), and our results for the Z bosons carry over to the other weak bosons as well. The remaining bosonic SM fields, the gluons and the photons, do not couple to the string worldsheet at leading order [13].

In addition to interactions with $\phi_H$ and $Z_{\mu}$, the string also couples to the SM fermions due to an Aharonov-Bohm (AB) interaction [21]. Upon circumnavigating the string on a length scale larger than the width of the SM dressing fields, the fermion wavefunction picks up a phase that is $2\pi$ times [13]

$$\theta_q = -\frac{2 \cos \theta_W \sin \epsilon}{g_X} q. \quad (1.6)$$

where $q$ is the electromagnetic charge on the fermion, and $\theta_W$ is the weak mixing angle. The AB interaction is topological, insensitive to the details of the structure of the string, and in particular, does not assume $\sigma \gg \eta$. 

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By virtue of the interactions in eqs. (1.4) and (1.5), dynamical dark strings will emit Higgs and Z bosons, and it will emit SM fermions through the AB interaction. In the following sections, we calculate the spectrum of radiation in the form of Higgs and Z bosons that is emitted from cusps, kinks, and kink collisions on cosmic string loops (see figure 1). The scalar boson radiation channels have been derived previously [22–29]. We refine these calculations by carefully estimating all dimensionless coefficients, and in some cases also correcting errors. Most importantly, we find that the calculation of ref. [22] underestimates the scalar radiation by a factor of $\sqrt{ML} \gg 1$, which arises because the radiation from the cusp is highly boosted. The vector boson channels have not been worked out previously, and we present them here for the first time. We also estimate radiation from the Aharonov-Bohm interaction by drawing on results from the literature. Our results, it should be emphasized, are not unique to the dark string model; instead, the spectra derived here apply to any model with effective interactions of the form in eqs. (1.4) and (1.5).

Particle radiation is expected to play an important role in the evolution of light cosmic string for which gravitational radiation is suppressed. Specifically, we find that Higgs boson emission is the dominant energy loss mechanism for light dark strings. The emission of SM particles may also lead to observational signatures of dark strings through astrophysics or cosmology, and we will explore this possibility in a companion paper [30].

2 Radiation of Standard Model particles

The interactions in eqs. (1.4) and (1.5) allow a dark string to emit Higgs and Z bosons, and SM fermions are radiated by virtue of the non-local Aharonov-Bohm interaction. In the subsections below we first present the spectrum of Higgs and Z boson radiation from a general string configuration, and we then specify to the cases of cusps, kinks, and kink-kink collisions as these are expected to be the three most copious sources of particle radiation. We leave the details of these calculations to the appendices.

1Quanta of the string-forming fields, $S$ and $X$, are too heavy to be radiated via the perturbative interactions that we study here, namely where the effective couplings are derived by linearizing the fields around the string background. Numerical simulations suggest that there may still be significant radiation through non-perturbative effects [31] though also see [32, 33].
2.1 Higgs boson emission via linear coupling

The physical Higgs field, \( \phi_H(x) \), couples to the dark string through the effective interaction

\[
S_{\text{int}}^H = g_{str}^H \eta \int d^2 \sigma \sqrt{-\gamma} \phi_H(\chi). \tag{2.1}
\]

Since this term is linear in \( \phi_H \) it acts as a classical source term for the Higgs field and leads to radiation from the string. Note that the dimensional prefactor, \( \eta \approx 174 \text{ GeV} \), is the vacuum expectation value of the Higgs field; this interaction would not be present if not for electroweak symmetry breaking. In appendix B we calculate the spectrum of Higgs boson radiation for a string loop. Taking \( A = g_{str}^H \eta \) in eq. (B.7) we find

\[
dN_H = (g_{str}^H \eta)^2 |I(k)|^2 |k| d\omega d\Omega, \tag{2.2}
\]

where the integral

\[
I(k) = \int d^2 \sigma \sqrt{-\gamma} e^{ik\cdot\chi}, \tag{2.3}
\]

is a functional of the string worldsheet, \( \chi^\mu(\tau, \sigma) \), that describes the motion of the string loop. The kinematical variables are defined by \( k^\mu = (\omega, \mathbf{k}) \) with \( \omega = (m_H^2 + |k|^2)^{1/2} \) and \( m_H \) the Higgs boson mass. In the following subsections we specify \( \chi^\mu \) so as to evaluate the spectrum and total power of Higgs boson emission from cusps, kinks, and kink-kink collisions.

2.1.1 Higgs emission from a cusp

A cusp occurs when there is a point on the worldsheet where \( \partial_\sigma \chi = 0 \). At this point the velocity of the string segment approaches the speed of light, and the radiation is highly boosted. In the rest frame of the loop, the momentum of the emitted radiation cannot exceed the inverse string thickness, i.e. \( |k| < M \) where \( M = \sqrt{\mu} \), else the point-like interaction in eq. (2.1) is inapplicable, and the radiation is suppressed. However, due to the large boost factor, \( \gamma_{\text{boost}} \sim \sqrt{ML} \), the radiation does not cut off until \( |k| \approx M\sqrt{ML} \) (see appendix E).

Inserting the scalar integral from eq. (D.12) into the spectrum in eq. (2.2) we find

\[
dN_{H}^{(\text{cusp})} = (g_{str}^H \eta)^2 S^{(\text{cusp})} L^{4/3} |k|^{5/3} \frac{d\omega d\Omega}{2(2\pi)^3}, \tag{2.4}
\]

where \( \psi \approx 0.01 \) (see eq. (C.18)), \( \Theta \approx 0.1 \) (below eq. (C.13)), and \( 0.2 \lesssim S^{(\text{cusp})} \lesssim 10 \) (see below eq. (D.12)). As explained above, the spectrum is cutoff in the UV by the (boosted) string thickness, and it cuts off in the IR due to a destructive interference that is manifest in the breakdown of the saddle point approximation. Since typically \( m_H L \gg 1 \), the radiation is ultra-relativistic and we can approximate \( |k| \approx \omega \) and \( d|k| \approx d\omega \).

The radiation is emitted into a cone that has an opening angle \( \Theta(|k|L)^{-1/3} \). Integrating over the solid angle, we find the spectrum to be

\[
dN_{H}^{(\text{cusp})} \approx (g_{str}^H \eta)^2 \Theta^2 S^{(\text{cusp})} L^{2/3} \frac{d|k|}{|k|^{7/3}}, \quad \psi m_H \sqrt{m_H L} < |k| < M\sqrt{ML}. \tag{2.5}
\]
The total energy emitted from a cusp is

\[ E^{(\text{cusp})}_H = \int \omega \, dN^{(\text{cusp})}_H = \frac{3(g_{\text{str}}^H \eta)^2}{4(2\pi)^2} \psi^{-1/3} \Theta^2 S^{(\text{cusp})} \sqrt{\frac{L}{m_H}} \left( 1 - \psi^{1/3} \sqrt{\frac{m_H}{M}} \right). \tag{2.6} \]

Since we are interested in heavy strings, \( M \gg m_H \), we can neglect the second term in the parenthesis. If cusps appear on a loop with frequency \( f_c/T \) where \( T = L/2 \) is the loop oscillation period, then the average power emitted per oscillation is \( P_H = 2E_H f_c/L \), or

\[ P^{(\text{cusp})}_H = \Gamma^{(\text{cusp})}_H \frac{(g_{\text{str}}^H \eta)^2}{\sqrt{m_H L}} \tag{2.7} \]

where \( \Gamma^{(\text{cusp})}_H \equiv \frac{3}{2(2\pi)^2} \psi^{-1/3} \Theta^2 \eta f_c S^{(\text{cusp})} \). Assuming \( f_c \approx 1 \), the dimensionless coefficient takes values in the range \( 10^{-4} \lesssim \Gamma^{(\text{cusp})}_H \lesssim 10^{-1} \). This result agrees with previous calculations in the literature [24, 25, 27].

### 2.1.2 Higgs emission from a kink

A kink occurs where there is a discontinuity in the derivative of the string worldsheet \( \partial_x \Sigma \). We obtain the spectrum of Higgs radiation emitted from a single kink over the course of one loop oscillation period by evaluating eq. (2.2) with eq. (D.14), and we find

\[ dN^{(\text{kink})}_H = \frac{(g_{\text{str}}^H \eta)^2}{2(2\pi)^3} S^{(\text{kink})} \frac{L^{2/3}}{|k|^{7/3}} \psi m_H \sqrt{m_H L} \, |d| \, d\Omega, \quad \psi m_H \sqrt{m_H L} < |k| < M, \quad \theta < \Theta (|k| L)^{-1/3} \text{ (band)} \tag{2.8} \]

where the dimensionless coefficient is typically in the range \( 0.05 \lesssim S^{(\text{kink})} \lesssim 10 \). Here the upper bound on \( k \) is \( M \), rather than \( M/\sqrt{M L} \) as for the cusp, since the string velocity at the kink is not highly boosted in the loop’s rest frame. The lower bound on \( k \) is the same as in the case of the cusp as it arises from our use of the saddle point approximation in one of the worldsheet integrals \( I_{\pm} \) (see appendix C). Unless the loop is very small, \( L < M^2/m_L^3 \), the lower cutoff will exceed the upper cutoff; in this case, there is no Higgs radiation from the kink within our approximations. This argument is in contrast with the calculation of ref. [29], where scalar radiation from the kink was also studied.

Radiation is emitted into a band that has an angular width \( \Theta (|k| L)^{-1/3} \) and angular length \( \sim 2\pi \). Integrating over the sold angle \( \Delta \Omega \approx 2\pi \Theta (|k| L)^{-1/3} \) gives the spectrum

\[ dN^{(\text{kink})}_H = \frac{(g_{\text{str}}^H \eta)^2}{2(2\pi)^3} \Theta S^{(\text{kink})} \frac{L^{1/3}}{|k|^{8/3}} \psi m_H \sqrt{m_H L} < |k| < M. \tag{2.9} \]

The total energy emitted by the kink into this channel during one loop oscillation is

\[ E^{(\text{kink})}_H = \int \omega \, dN^{(\text{kink})}_H = \frac{3(g_{\text{str}}^H \eta)^2}{4(2\pi)^2} \psi^{-2/3} \Theta S^{(\text{kink})} \frac{1}{m_H} \left( 1 - \psi^{2/3} \frac{m_H L^{1/3}}{M^{2/3}} \right). \tag{2.10} \]

Note that the energy is logarithmically sensitive to both the upper and lower cutoffs of the spectrum. If the loop carries \( N_k \) kinks, then the average power radiated during one loop oscillation period, \( T = L/2 \), is given by

\[ P^{(\text{kink})}_H = \Gamma^{(\text{kink})}_H \frac{(g_{\text{str}}^H \eta)^2}{m_H L} \left( 1 - \psi^{2/3} \frac{m_H L^{1/3}}{M^{2/3}} \right). \tag{2.11} \]
with $\Gamma^{(\text{kink})}_H = \frac{3}{2(2\pi)^2} N_k \psi^{-2/3} \Theta S^{(\text{kink})}$. Taking $N_k \approx 1$ the dimensionless prefactor is estimated to be $10^{-3} \lesssim \Gamma^{(\text{kink})}_H \lesssim 1$. This result disagrees with a previous calculation \cite{29} of Higgs radiation from a kink, as explained in appendix D.2.

### 2.1.3 Higgs emission from a kink-kink collision

A kink-kink collision occurs when two kinks momentarily overlap at the same point on the string worldsheet. We find the spectrum of Higgs radiation at the collision using eq. (2.2) along with the scalar integral in eq. (D.16):

$$dN^{(k-k)}_H = \frac{(g_{\text{str}}^H)^2}{(2\pi)^2} \frac{|k|}{\omega^4} d\omega d\Omega, \quad m_H < \omega < M \quad (2.12)$$

where $0.05 < S^{(k-k)} < 200$. The bound $\omega > m_H$ subsumes the bound $\omega > L^{-1}$ in eq. (D.16) assuming $m_H L \gg 1$.

The radiation is emitted approximately isotropically (no beaming), and the angular integration gives

$$dN^{(k-k)}_H = \frac{(g_{\text{str}}^H)^2}{(2\pi)^2} \frac{|k|}{\omega^4} d\omega, \quad m_H < \omega < M \quad (2.13)$$

The total energy emitted by a kink-kink collision is found to be

$$E^{(k-k)}_H = \int \omega dN^{(k-k)}_H = \frac{(g_{\text{str}}^H)^2}{(2\pi)^2} \frac{1}{m_H} S^{(k-k)} \quad (2.14)$$

Defining $N_{kk}$ as the number of kink-kink collisions during one loop oscillation period, $T = L/2$, we can express the average power radiated by

$$P^{(k-k)}_H = \frac{\Gamma^{(k-k)}_H g_{\text{str}}^H}{m_H L} \quad (2.15)$$

with $\Gamma^{(k-k)}_H = \frac{2}{(2\pi)^2} N_{kk} S^{(k-k)}$. We can estimate the number of collisions per loop oscillation period as $N_{kk} \approx N_k^2$, where $N_k$ is the number of kinks on the loop. Estimating $N_{kk} \approx 1$ we obtain a range $10^{-2} < \Gamma^{(k-k)}_H < 10$ for the dimensionless prefactor.

### 2.2 Higgs boson emission via quadratic coupling

The radial component of the Higgs field also couples to the dark string through the quadratic interaction

$$S^{HH}_{\text{int}} = g_{\text{str}}^H \int d^2 \sigma \sqrt{-\gamma} \phi^2_H(X) \quad (2.16)$$

Unlike in the linear type coupling discussed above, this interaction is not proportional to the Higgs field VEV, and it would exist even if the electroweak symmetry were unbroken. This quadratic interaction with the string produces two Higgs bosons, and thus the final state contains two different momenta, $k$ and $\bar{k}$. The spectrum of radiation is given by eq. (B.12) with $C = g_{\text{str}}^H$:

$$dN_{HH} = \frac{(g_{\text{str}}^H)^2}{(2\pi)^3} \frac{|k| d\omega d\Omega |\bar{k}| d\bar{\omega} d\bar{\Omega} |Z(k + \bar{k})|^2}{2(\pi)^3} \quad (2.17)$$

where $k^\mu = \{\omega, k\}$ with $\omega = (m_H^2 + |k|^2)^{1/2}$ and $m_H$ the Higgs boson mass. The barred quantities are defined similarly.
2.2.1 Higgs-Higgs emission from a cusp

Before we can evaluate the spectrum in eq. (2.17) we must know the value of the scalar integral $I(k + \bar{k})$ for a cusp configuration. In eq. (D.12) we found that this integral evaluates to

$$ |I^{(\text{cusp})}(k)|^2 = \mathcal{S}^{(\text{cusp})} \frac{L^{4/3}}{|k|^{8/3}}, \quad \psi^2 m_H \sqrt{m_H L} < |k|, \quad \theta < \Theta(|k| L)^{-1/3} \tag{2.18} $$

when its argument is the approximately null 4-vector momentum $k^2 = m_H^2 \ll |k|^2$. If the argument of the integral is a time-like vector, as in eq. (2.17), the derivation still leads to eq. (2.18), but the saddle point approximation gives an additional bound on the angle between $k$ and $\bar{k}$. In order to justify the saddle point approximation, we were forced to impose the bound in eq. (C.17). Since the argument of the integral in eq. (2.17) is $k + \bar{k}$, we must generalize eq. (C.17) by replacing $\omega \rightarrow \omega + \bar{\omega}$ and $|k| \rightarrow |k + \bar{k}| = \sqrt{|k|^2 + |\bar{k}|^2 + 2|k||\bar{k}| \cos \theta_{k\bar{k}}}$ where $\theta_{k\bar{k}}$ is the angle between $k$ and $\bar{k}$. The bound becomes

$$ \frac{\Theta^2}{4\pi} L^{2/3} (\omega + \bar{\omega} - \sqrt{|k|^2 + |\bar{k}|^2 + 2|k||\bar{k}| \cos \theta_{k\bar{k}}}) < (|k|^2 + |\bar{k}|^2 + 2|k||\bar{k}| \cos \theta_{k\bar{k}})^{1/6}. \tag{2.19} $$

It is useful to consider two limiting cases. If $\theta_{k\bar{k}} = 0$ then the inequality translates into a lower bound on the momentum,

$$ \psi^2 m_H \sqrt{m_H L} < \frac{(|k||\bar{k}|)^{3/4}}{\sqrt{|k| + |\bar{k}|}}, \tag{2.20} $$

and we have used $\psi = (\Theta/8\pi)^{3/4}$. When $|k| \approx |\bar{k}|$ we regain the original bound $\psi^2 m_H \sqrt{m_H L} < |k|$. The inequality also imposes an upper bound on $\theta_{k\bar{k}}$. Approximating $\cos \theta_{k\bar{k}} \approx 1 - \theta^2_{k\bar{k}}/2$ and using $\omega \approx |k|$ and $\bar{\omega} \approx |\bar{k}|$, we can resolve the inequality as

$$ \theta_{k\bar{k}} < \psi^{-2/3} \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{\sqrt{|k||\bar{k}|}}. \tag{2.21} $$

For $|k| \approx |\bar{k}|$ this becomes $\theta_{k\bar{k}} < (2/\psi)^{2/3} (|k| L)^{-1/3}$, which agrees with a similar estimate in ref. 22.

From the arguments above, we obtain the cusp integral to be

$$ |I^{(\text{cusp})}(k + \bar{k})|^2 = \mathcal{S}^{(\text{cusp})} \frac{L^{4/3}}{|k|^{8/3}} (|k| + |\bar{k}|)^{8/3}, \quad \psi^2 m_H \sqrt{m_H L} < |k|, |\bar{k}| \leq M \sqrt{ML}, \quad \theta_{k\bar{k}} < \psi^{-2/3} \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{\sqrt{|k||\bar{k}|}} \quad (\text{cone}), \quad \theta_{k\bar{k}} < \Theta(|k| + |\bar{k}|)^{-1/3} L^{-1/3} \quad (\text{cone}) \tag{2.22} $$

with $0.2 \lesssim \mathcal{S}^{(\text{cusp})} \lesssim 10$. We have also used $\theta_{k\bar{k}} \ll 1$ to approximate $|k + \bar{k}| \approx |k| + |\bar{k}|$. Inserting eq. (2.22) into eq. (2.17) we obtain the spectrum

$$ dN^\text{cusp}_{HH} = \frac{(\theta_{\text{str}})^2}{4(2\pi)^6} \mathcal{S}^{(\text{cusp})} \frac{L^{4/3} |k| |\bar{k}|}{(|k| + |\bar{k}|)^{8/3}} \frac{d|k| d|\bar{k}| d\Omega d\bar{\Omega}}{d\Omega}, \quad \psi^2 m_H \sqrt{m_H L} < |k|, |\bar{k}| \leq M \sqrt{ML}, \quad \theta_{k\bar{k}} < \psi^{-2/3} \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{\sqrt{|k||\bar{k}|}} \quad (\text{cone}), \quad \theta_{k\bar{k}} < \Theta(|k| + |\bar{k}|)^{-1/3} L^{-1/3} \quad (\text{cone}). \tag{2.23} $$
The upper bound on $\theta_{kk}$ implies that $\mathbf{k}$ and $\mathbf{\bar{k}}$ are approximately parallel to one another, and the upper bound on $\theta_{k+k}$ implies that their sum points along the direction of the cusp. The geometry is such that the radiation is emitted into a pair of overlapping cones, and the angular integrations yield

$$\int d\Omega d\Omega \approx \frac{(2\pi)^2}{4} \psi^{-4/3} \Theta^2 \frac{(|\mathbf{k}| + |\mathbf{\bar{k}}|)^{2/3}}{|\mathbf{k}||\mathbf{\bar{k}}|} L^{-4/3} ,$$

and the spectrum becomes

$$dN^{(\text{cusp})}_{HH} = \frac{(g_{\text{str}}^{\text{hh}})^2}{16(2\pi)^4} \psi^{-4/3} \Theta^2 S^{(\text{cusp})} \frac{d|\mathbf{k}| d|\mathbf{\bar{k}}|}{(|\mathbf{k}| + |\mathbf{\bar{k}}|)^2} , \quad \psi m_H \sqrt{m_H L} < |\mathbf{k}|, |\mathbf{\bar{k}}| < M \sqrt{ML} .$$

The total energy emitted from a cusp is given by

$$E^{(\text{cusp})}_{HH} = \int (\omega + \bar{\omega}) dN^{(\text{cusp})}_{HH} \approx \frac{(g_{\text{str}}^{\text{hh}})^2}{16(2\pi)^4} \psi^{-4/3} \Theta^2 S^{(\text{cusp})} M \sqrt{ML} .$$

If the frequency of cusp appearance is $f_{\text{cusp}} = f_c/T$ with $T = L/2$ is the loop oscillation period, then the average power emitted is

$$P^{(\text{cusp})}_{HH} = \Gamma^{(\text{cusp})}_{\text{int}} \frac{(g_{\text{str}}^{\text{hh}} M)^2}{\sqrt{ML}} ,$$

where $\Gamma^{(\text{cusp})}_{\text{int}} \equiv \frac{1}{8(2\pi)^4} \int f_c \psi^{-4/3} \Theta^2 S^{(\text{cusp})}$. Estimating $f_c \approx 1$ gives $10^{-5} < \Gamma^{(\text{cusp})}_{\text{int}} < 10^{-2}$.

Scalar boson pair radiation from a cusp has been calculated previously by ref. [22]. Our calculation matches the UV-sensitive spectrum, eq. (2.25), of the earlier reference. In calculating the total power, we integrate up to an energy of $M \sqrt{ML}$, where $1/M$ is the string thickness and $\sqrt{ML}$ is the boost factor that translates between the cusp and loop rest frames (see section 2.1.1). This boost factor was overlooked in the previous calculations, and the power was found to be $O(M/L)$, typically a significant underestimate compared to eq. (2.27).

2.2.2 Higgs-Higgs emission from a kink

We calculate the spectrum of Higgs boson radiation from the kink by evaluating the spectrum in eq. (2.17) using the scalar integral in eq. (D.14). After also generalizing the saddle point criterion, as discussed in section 2.2.1, we obtain

$$dN^{(\text{kink})}_{HH} = \frac{(g_{\text{str}}^{\text{hh}})^2}{4(2\pi)^6} S^{(\text{kink})} \frac{L^{4/3} |\mathbf{k}| |\mathbf{\bar{k}}|}{(|\mathbf{k}| + |\mathbf{\bar{k}}|)^{8/3}} d|\mathbf{k}| d|\mathbf{\bar{k}}| d\Omega d\Omega , \quad \psi m_H \sqrt{m_H L} < |\mathbf{k}|, |\mathbf{\bar{k}}| < M ,$$

$$\theta_{kk} < \psi^{-2/3} \frac{(|\mathbf{k}| + |\mathbf{\bar{k}}|)^{2/3} L^{-1/3}}{\sqrt{|\mathbf{k}||\mathbf{\bar{k}}|}} \Theta (\text{cone}), \quad \theta_{k+k} < \Theta (|\mathbf{k}| + |\mathbf{\bar{k}}|)^{-1/3} L^{-1/3} \text{ (band)} .$$

where $0.05 < S^{(\text{kink})} < 10$. The momenta $\mathbf{k}$ and $\mathbf{\bar{k}}$ are separated by an angle $\theta_{kk}$, and their sum is oriented in a band of angular with $\Theta(|\mathbf{k}| + |\mathbf{\bar{k}}|)^{-1/3} L^{-1/3}$. Performing the angular integrations we obtain

$$dN^{(\text{kink})}_{HH} = \frac{(g_{\text{str}}^{\text{hh}})^2}{8(2\pi)^3} \psi^{-4/3} \Theta S^{(\text{kink})} L^{1/3} \frac{d|\mathbf{k}| d|\mathbf{\bar{k}}|}{(|\mathbf{k}| + |\mathbf{\bar{k}}|)^{5/3}} , \quad \psi m_H \sqrt{m_H L} < |\mathbf{k}|, |\mathbf{\bar{k}}| < M .$$
The spectrum is UV-sensitive, which allows us to neglect the lower limit, and upon integrating we find the total energy output to be

\[
E^{(kink)}_{HH} = \int (\omega + \bar{\omega}) dN^{(kink)}_{HH} \approx \frac{9 (g_{HH}^m)^2}{16 (2\pi)^4} \psi^{-4/3} \Theta S^{(kink)} L^{1/3} M^{4/3} \left( 1 - 5 \psi m_H \sqrt{m_H L} M \right)
\]

(2.30)

where we have used \(4/\left[3(2^{1/3} - 1)\right] \approx 5\) in the second term. If the loop contains \(N_k\) kinks, then the average power output during one loop oscillation period \((T = L/2)\) is given by

\[
P^{(kink)}_{HH} = \Gamma^{(kink)}_{HH} \frac{(g_{str}^m)^2}{(ML)^2/3} \left( 1 - 5 \psi m_H \sqrt{m_H L} M \right).
\]

(2.31)

where \(\Gamma^{(kink)}_{HH} = \frac{9}{8(2\pi)^2} N_k \psi^{-4/3} \Theta S^{(kink)}\). Estimating \(N_k \approx 1\) and using the range for \(S^{(kink)}\) given above, the dimensionless prefactor can be estimated as \(10^{-4} < \Gamma^{(kink)}_{HH} < 10^{-1}\).

### 2.2.3 Higgs-Higgs emission from a kink-kink collision

To calculate the spectrum of Higgs boson radiation from a kink-kink collision we use the scalar integral from eq. (D.16) in the spectrum from eq. (2.17) to obtain

\[
dN^{(k-k)}_{HH} = \frac{(g_{str}^m)^2}{4(2\pi)^6} S^{(k-k)} \frac{|k| |\bar{k}|}{(\omega + \bar{\omega})^4} d\omega d\bar{\omega} d\Omega, \quad m_H < \omega, \bar{\omega} < M
\]

(2.32)

where \(0.05 < S^{(k-k)} < 200\). The radiation can be emitted isotropically; performing the angular integration gives a factor of \((4\pi)^2\) and leaves

\[
dN^{(k-k)}_{HH} = \frac{(g_{str}^m)^2}{(2\pi)^4} S^{(k-k)} \frac{|k| |\bar{k}|}{(\omega + \bar{\omega})^4} d\omega d\bar{\omega}, \quad m_H < \omega, \bar{\omega} < M.
\]

(2.33)

The total energy output of a kink-kink collision is calculated as

\[
E^{(k-k)}_{HH} = \int (\omega + \bar{\omega}) dN^{(k-k)}_{HH} = \frac{(g_{str}^m)^2}{4(2\pi)^4} S^{(k-k)} M.
\]

(2.34)

If there are \(N_{kk}\) kink-kink collisions during a loop oscillation period \(T = L/2\) then the average power is found to be

\[
P^{(k-k)}_{HH} = \Gamma^{(k-k)}_{HH} \frac{(g_{str}^m)^2}{ML}.
\]

(2.35)

where \(\Gamma^{(k-k)}_{HH} = \frac{1}{2(2\pi)^2} N_{kk} S^{(k-k)}\). For \(N_{kk} \approx 1\) we can estimate \(10^{-4} < \Gamma^{(k-k)}_{HH} < 10^{-1}\) using the range for \(S^{(k-k)}\) given above.

### 2.3 Z-Boson emission via linear coupling

The interaction

\[
S_{int}^Z = \frac{g_{Z\sigma}}{2} \left( \frac{\eta}{\sigma} \right)^2 \int d\sigma^\mu Z_{\mu\nu}(X)
\]

(2.36)
allows Z bosons to be radiated from the string. The radiation calculation is carried out in appendix B. The spectrum is given by eq. (B.21) after replacing $C = g_{\text{str}}^2 (\eta/\sigma)^2$: \begin{equation}
 dN_Z = (g_{\text{str}}^2)^2 \frac{\eta^4}{\sigma^4} \frac{4 |k| d\omega d\Omega}{(2\pi)^3} m_Z^2 \Pi(k). \tag{2.37}
 \end{equation}

In this expression $\omega = (|k|^2 + m_Z^2)^{1/2}$ with $m_Z$ the Z boson mass and $\Pi(k)$ is a functional of the string worldsheet, given by eq. (B.22). In the following subsections we calculate the spectrum and total power in Z boson emission from cusps, kinks, and kink-kink collisions.

### 2.3.1 Z emission from a cusp

The spectrum of Z boson emission from a cusp is calculated using eq. (2.37) with the integral in eq. (D.18). Combining these formulae we obtain
\begin{equation}
 dN_{Z}^{(\text{cusp})} = \frac{(g_{\text{str}}^2)^2}{4(2\pi)^3} \Theta^2 \left( \frac{\eta}{\sigma} \right)^4 T^{(\text{cusp})} \frac{L^{1/3}}{|k|^{5/3}} m_Z^2 d|k| d\Omega, \quad \psi m_Z \sqrt{m_Z L} < |k| < M \sqrt{M L}, \quad \theta < \Theta(|k|L)^{-1/3} \ (\text{cone}). \tag{2.38}
\end{equation}

where the dimensionless coefficient takes values $0.5 \lesssim T^{(\text{cusp})} \lesssim 50$. The direction of the outgoing Z boson lies within a cone centered at the cusp and has an opening angle $\Theta(|k|L)^{-1/3}$. We integrate over the solid angle to obtain the spectrum
\begin{equation}
 E_{Z}^{(\text{cusp})} = \int \omega dN_{Z}^{(\text{cusp})} = \frac{3(g_{\text{str}}^2)^2}{4(2\pi)^3} \psi^{-1/3} \Theta^2 \left( \frac{\eta}{\sigma} \right)^4 T^{(\text{cusp})} m_Z \sqrt{m_Z L} \tag{2.40}
\end{equation}

and if cusps arise with a frequency $f_c/T$ where $T = L/2$ is the loop oscillation period, then the average power per loop oscillation is found to be
\begin{equation}
 P_{Z}^{(\text{cusp})} = \Gamma_{Z}^{(\text{cusp})} \left( \frac{\eta}{\sigma} \right)^4 \frac{(g_{\text{str}}^2 m_Z)^2}{\sqrt{m_Z L}} \tag{2.41}
\end{equation}

where the power coefficient is $\Gamma_{Z}^{(\text{cusp})} \equiv \frac{3}{4(2\pi)^3} T^{(\text{cusp})} f_c \psi^{-1/3} \Theta^2$. Assuming $f_c \approx 1$ we estimate $10^{-4} \lesssim \Gamma_{Z}^{(\text{cusp})} \lesssim 10^{-1}$.

### 2.3.2 Z emission from a kink

To calculate the spectrum of Z boson emission from a single kink, we use the expression eq. (2.37) along with the expression eq. (D.20) for $\Pi(k)$ for a kink to find
\begin{equation}
 dN_{Z}^{(\text{kink})} = \frac{(g_{\text{str}}^2)^2}{2(2\pi)^3} \left( \frac{\eta}{\sigma} \right)^4 T^{(\text{kink})} \frac{L^{2/3}}{|k|^{7/3}} m_Z^2 d|k| d\Omega, \quad \psi m_Z \sqrt{m_Z L} < |k| < M, \quad \theta < \Theta(|k|L)^{-1/3} \ (\text{band}). \tag{2.42}
\end{equation}
where $0.5 < T^{(\text{kink})} < 100$. Radiation is emitted in a band with angular width $\Theta(|k|L)^{-1/3}$, and we integrate over the solid angle to find

$$
dN^\text{kink}_{Z} = \frac{(g^z_{\text{str}})^2}{2(2\pi)^3} \Theta \left( \frac{\eta}{\sigma} \right)^4 T^{(\text{kink})} \frac{L^{1/3}}{|k|^{8/3}} m^2_Z d|k|, \quad \psi m_Z \sqrt{m_Z L} < |k| < M
$$

(2.43)

The total energy emitted by a kink during one loop oscillation is

$$
E^{(\text{kink})}_{Z} = \int \omega dN^{\text{kink}}_{Z} = \frac{3(g^z_{\text{str}})^2}{4(2\pi)^2} \psi^{-2/3} \Theta \left( \frac{\eta}{\sigma} \right)^4 T^{(\text{kink})} m_Z \left( 1 - \psi^{-2/3} \frac{m_Z L^{1/3}}{M^{2/3}} \right),
$$

(2.44)

and if there are $N_k$ kinks on the loop then the average radiated power during one loop oscillation period ($T = L/2$) is

$$
P^{(\text{kink})}_{Z} = \Gamma^{(\text{kink})}_{Z} \left( \frac{\eta}{\sigma} \right)^4 \frac{(g^z_{\text{str}} m_Z)^2}{m_Z L} \left( 1 - \psi^{-2/3} \frac{m_Z L^{1/3}}{M^{2/3}} \right)
$$

(2.45)

with $\Gamma^{(\text{kink})}_{Z} = \frac{3}{2(2\pi)^2} \psi^{-2/3} \Theta T^{(\text{kink})} N_k$. Estimating $N_k \approx 1$ gives $10^{-2} < \Gamma^{(\text{kink})}_{Z} < 10$.

### 2.3.3 Z emission from a kink-kink collision

Inserting eq. (D.23) into eq. (2.37) we obtain the spectrum of Z boson emission from a collision of kinks to be

$$
dN^{(k-k)}_{Z} = \frac{(g^z_{\text{str}})^2}{2(2\pi)^3} \left( \frac{\eta}{\sigma} \right)^4 T^{(k-k)} \frac{|k| m^2_Z}{\omega^4} d\omega d\Omega, \quad m_Z < \omega < M
$$

(2.46)

with the constant $0.1 < T^{(k-k)} < 50$. The emission is isotropic, and after performing the angular integration we obtain

$$
dN^{(k-k)}_{Z} \approx \frac{(g^z_{\text{str}})^2}{4(2\pi)^2} \left( \frac{\eta}{\sigma} \right)^4 T^{(k-k)} \frac{|k| m^2_Z}{\omega^4} d\omega, \quad m_Z < \omega < M
$$

(2.47)

The total energy emitted by a kink-kink collision is found to be

$$
E^{(k-k)}_{Z} = \int \omega dN^{(k-k)}_{Z} = \frac{(g^z_{\text{str}})^2}{4(2\pi)^2} \left( \frac{\eta}{\sigma} \right)^4 T^{(k-k)} m_Z .
$$

(2.48)

If $N_{kk}$ such collisions occur during one loop oscillation period, $T = L/2$, then the average power is

$$
P^{(k-k)}_{Z} = \Gamma^{(k-k)}_{Z} \left( \frac{\eta}{\sigma} \right)^4 \frac{(g^z_{\text{str}} m_Z)^2}{m_Z L}
$$

(2.49)

with $\Gamma^{(k-k)}_{Z} = \frac{1}{2(2\pi)^2} N_{kk} T^{(k-k)}$. Estimating $N_{kk} \approx 1$ gives $10^{-3} < \Gamma^{(k-k)}_{Z} < 1$.

### 2.4 Z boson emission via quadratic coupling

An interaction of the form

$$
S^{zz}_{\text{int}} = g^{zz}_{\text{str}} \left( \frac{\eta}{\sigma} \right)^4 \int d^2 \sigma \sqrt{-\gamma} Z_\mu(\mathcal{X}) Z^\mu(\mathcal{X})
$$

(2.50)
also allows $Z$ bosons to be radiated from the string. For heavy strings, the coefficient $(\eta/\sigma)^4$ is very small, and this radiation channel is negligible. However, we present the calculation of the radiation spectra for completeness. The spectrum is given by eq. (B.28) after replacing $C = g_{str}^{zz} (\eta/\sigma)^4$,
\[
\frac{dN_{ZZ}}{k} = 4(g_{str}^{zz})^2 \left( \frac{\eta}{\sigma} \right)^8 \frac{|k| d\omega d\Omega}{2(2\pi)^3} \frac{|k| d\bar{\omega} d\bar{\Omega}}{2(2\pi)^3} |I(k + \bar{k})|^2 ,
\]
where $k^\mu = \{ \omega, k \}$ and $\omega = (m_Z^2 + |k|^2)^{1/2}$ with similar definitions for the barred quantities. Note the similarity between eq. (2.51) and the spectrum of Higgs boson pair radiation given by eq. (2.17). Since both spectra depend on the same scalar integral, $I(k + \bar{k})$, we can simply carry over all the results from section 2.2. We need only to make the replacement $(g_{str}^{zz})^2 \rightarrow 4(g_{str}^{zz})^2 (\eta/\sigma)^8$.

### 2.4.1 $Z$-$Z$ emission from a cusp

We calculate the spectrum of $Z$ boson radiation from a cusp following section 2.2.1. We find the spectrum
\[
dN_{ZZ}^{(cusp)} = \frac{(g_{str}^{zz})^2}{4(2\pi)^4} \psi^{-4/3} \Theta^2 \left( \frac{\eta}{\sigma} \right)^8 \frac{d|k| d|\bar{k}|}{(|k| + |\bar{k}|)^2} , \quad \psi m_Z \sqrt{m_Z L} < |k|, |\bar{k}| < M \sqrt{ML} ,
\]
the energy radiated per cusp event
\[
E_{ZZ}^{(cusp)} = \frac{(g_{str}^{zz})^2}{4(2\pi)^4} \psi^{-4/3} \Theta^2 \left( \frac{\eta}{\sigma} \right)^8 S^{(cusp)} M \sqrt{ML} ,
\]
and the average power output if cusps arise with frequency $2f_c/L$
\[
P_{ZZ}^{(cusp)} = \Gamma_{ZZ}^{(cusp)} \left( \frac{\eta}{\sigma} \right)^8 \frac{(g_{str}^{zz} M)^2}{\sqrt{ML}} .
\]
The dimensionless coefficient is defined as $\Gamma_{ZZ}^{(cusp)} = \frac{1}{2(2\pi)^4} \int f_c \psi^{-4/3} \Theta^2 S^{(cusp)}$ and it may be estimated as $10^{-4} < \Gamma_{ZZ}^{(cusp)} < 10^{-2}$.

### 2.4.2 $Z$-$Z$ emission from a kink

We calculate the spectrum of $Z$ boson radiation from a kink following section 2.2.2. We find the spectrum
\[
dN_{ZZ}^{(kink)} = \frac{(g_{str}^{zz})^2}{2(2\pi)^4} \psi^{-4/3} \Theta \left( \frac{\eta}{\sigma} \right)^8 S^{(kink)} L^{1/3} \frac{d|k| d|\bar{k}|}{(|k| + |\bar{k}|)^{5/3}} , \quad \psi m_Z \sqrt{m_Z L} < |k|, |\bar{k}| < M ,
\]
the energy radiated per kink during one loop oscillation
\[
E_{ZZ}^{(kink)} \approx \frac{9(g_{str}^{zz})^2}{4(2\pi)^4} \psi^{-4/3} \Theta \left( \frac{\eta}{\sigma} \right)^8 S^{(kink)} L^{1/3} M^{4/3} \left( 1 - 5\psi \sqrt{m_Z L} \frac{M}{M} \right) ,
\]
and the average power emitted from a loop containing $N_k$ kinks
\[
P_{ZZ}^{(kink)} = \Gamma_{ZZ}^{(kink)} \left( \frac{\eta}{\sigma} \right)^8 \frac{(g_{str}^{zz} M)^2}{(ML)^{2/3}} \left( 1 - 5\psi \sqrt{m_Z L} \frac{M}{M} \right) .
\]
The dimensionless coefficient is defined by $\Gamma_{ZZ}^{(kink)} = \frac{9}{2(2\pi)^4} N_k \psi^{-4/3} \Theta S^{(kink)}$ and it can be estimated as $10^{-3} < \Gamma_{ZZ}^{(kink)} < 10^{-1}$. 

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\[\text{JCAP09(2014)030}\]
2.4.3 Z-Z emission from a kink-kink collision

We calculate the spectrum of Z boson radiation from a collision of two kinks following section 2.2.2. We find the spectrum

\[ dN_{ZZ}^{(k-k)} = \left( \frac{g_{zz \text{str}}^2}{16(2\pi)^3} \right)^2 \left( \frac{\eta}{\sigma} \right)^8 \left( \frac{|k| \bar{|k|}}{\omega + \bar{\omega}} \right)^4 d\omega d\bar{\omega}, \quad m_Z < \omega, \bar{\omega} < M, \]  

(2.58)

the energy radiated during the collision

\[ E_{ZZ}^{(k-k)} = \left( \frac{g_{zz \text{str}}^2}{16(2\pi)^3} \right)^2 \left( \frac{\eta}{\sigma} \right)^8 \left( \frac{g_{zz \text{str}}^2 M}{\sigma} \right)^2, \]  

(2.59)

and the average power radiated from a loop that experiences \( N_{kk} \) collisions during one loop oscillation period

\[ P_{ZZ}^{(k-k)} = \Gamma_{zz}^{(k-k)} \left( \frac{\eta}{\sigma} \right)^8 \left( \frac{g_{zz \text{str}} M}{\sigma} \right)^2. \]  

(2.60)

The dimensionless coefficient is defined by \( \Gamma_{zz}^{(k-k)} \equiv \frac{1}{8(2\pi)^3} N_{kk} S^{(k-k)} \), and we can estimate \( 10^{-5} < \Gamma_{zz}^{(k-k)} < 10^{-2} \).

2.5 Fermion emission via Aharonov-Bohm coupling

The cosmic string can radiate fermions through a direct coupling, such as the ones we have been studying for the Higgs and Z bosons, or through a non-local AB interaction. SM fermions couple directly to the string worldsheet through interactions of the form

\[ S_{\text{int}}^{(\psi)} = \frac{g_{\psi \text{str}}^2}{M} \left( \frac{\eta}{\sigma} \right)^2 \int d^2\sigma \sqrt{-\gamma} \bar{\Psi}(X^\mu) \Psi(X^\mu), \]  

(2.61)

where \( g_{\psi \text{str}}^2 \) is a dimensionless coupling constant, and the factor of \( (\eta/\sigma)^2 \) arises from the mixing between the Higgs field and the HS scalar field [13]. Note that dimensional analysis requires the string mass scale to appear in the denominator. The radiation calculation with \( S_{\text{int}}^{(\psi)} \) is very similar to the case of Higgs radiation via the quadratic interaction, see section B.5. We find the spectrum of \( \psi \) radiation to be

\[ dN_{\psi \psi} = 4 \left( \frac{\eta}{\sigma} \right)^4 \left( \frac{g_{\psi \text{str}}^2}{g_{\text{str}}^2} \right)^2 \frac{k \cdot \bar{k} - m_\psi^2}{M^2} dN_{HH}, \]  

(2.62)

where \( dN_{HH} \) is the spectrum of Higgs radiation, given by eq. (2.17). Because of the mixing angle factor, \( (\eta/\sigma)^4 \ll 1 \), this radiation channel is inefficient.

The non-local AB interaction provides an additional channel for particle production from the cosmic string [21]. Refs. [34–36] studied the AB radiation of scalars, fermions, and vectors from a string. In these calculations, the authors assumed that the string carries only one kind of magnetic flux, which is usually the case. The structure of the dark string, however, is more complex. As we saw in ref. [13], the string core contains flux of the HS \( X^\mu \) field and the dressing contains flux of the SM \( Z^\mu \) field. When a fermion travels around the perimeter of the string, outside of both the core and the dressing, its wavefunction picks up an AB phase due to both fluxes, and the overall phase is given by \( 2\pi \theta_q \), where \( \theta_q \) is defined...
in eq. (1.6). On the other hand, when the fermion makes a loop around the core by passing through the region of space containing the dressing fields, it will acquire a different AB phase.

In order to setup the radiation calculation we must know the effective AB interaction of the fermions with the string. The discussion above is intended to illustrate that this interaction will be scale dependent. At energies below the inverse dressing width, \( \sim \frac{1}{\eta} \), the core plus dressing can be treated together as a zero width string. In this limit the structure of the string is unimportant, and the AB interaction can be derived following refs. [34–36] with the AB phase given by \( \theta_q \). At higher energies the Compton wavelength of the radiation drops below the dressing thickness. Here the effective coupling will presumably decrease as the particle “sees” less and less of the flux carried by the dressing. This behavior is in contrast with the Higgs and Z boson radiation channels that we considered previously. In those cases, the light SM fields coupled directly to the string core itself, and the dressing was neglected.

In light of the discussion above, we will proceed as follows. We calculate the spectrum of radiation due to the AB interaction where the coupling is set by the AB phase \( \theta_q \). If the thickness of the string dressing is \( \sim \frac{1}{\eta} \), then this spectrum is valid up to energies \( |k| \approx \eta \sqrt{\eta L} \) for the cusp or \( |k| \approx \eta \) for the kink and kink collision. At higher energies, we suppose that the effective coupling begins to decrease as the fermion radiation begins to penetrate inside of the dressing, and consequently the spectrum drops sharply.

The AB interaction can be treated perturbatively as follows. Let \( V_\mu(x) \) be the appropriate linear combination of the \( X_\mu \) and \( Z_\mu \) gauge fields that couples to the fermions, and let \( g_\psi \) be the coupling constant. Then the interaction is given by

\[
\mathcal{L}_{\text{eff}} = g_\psi V_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x). \tag{2.63}
\]

We treat \( V_\mu \) as a classical background field induced by the flux that the string carries: \( \Phi = (2\pi/g_\psi) \theta_q \). This lets us write (Lorentz gauge, \( \partial_\nu V^\nu = 0 \)) [21]

\[
V_\mu = -\frac{\Phi}{2} \int_{\text{ret.}} \frac{d^4p}{(2\pi)^4} \frac{ip^\nu}{p^2} \int d\sigma_{\mu\nu} e^{-ip \cdot (x-X)} \tag{2.64}
\]

where the integration contour extends above the poles at \( p^0 = \pm |p| \), as in the calculation of a retarded Green’s function. Note that \( V_\mu(x) \) has support outside of the string, unlike the purely local interactions in eqs. (1.4) and (1.5).

The interaction in eq. (2.63) allows the string to radiate fermion pairs with momenta \( k^\mu = \{ \omega = \sqrt{m_\psi^2 + |k|^2}, k \} \) and \( \bar{k}^\mu = \{ \bar{\omega} = \sqrt{m_\psi^2 + |\bar{k}|^2}, \bar{k} \} \). The spectrum is given by eq. (B.40) after replacing \( C = -(2\pi \theta_q)/2 \):

\[
dN_{\psi \bar{\psi}} = \frac{(2\pi \theta_q)^2}{8(2\pi)^6} \Pi(k + \bar{k}) |k|d\omega d\Omega |\bar{k}|d\bar{\omega} d\bar{\Omega} \tag{2.65}
\]

where \( \Pi \) is given by eq. (D.5).
2.5.1 Fermion AB emission from a cusp

We find the spectrum of radiation from a cusp by inserting eq. (D.18) into eq. (2.65):

\[ dN_{\text{AB}}^{(\text{cusp})} = \frac{(2\pi \theta_q)^2}{8(2\pi)^6} T^{(\text{cusp})} \frac{L^{4/3}|k||\bar{k}|}{(|k| + |\bar{k}|)^{8/3}} d|k| d\Omega d|\bar{k}| d\bar{\Omega}, \quad \psi m_\psi \sqrt{m_\psi L} < |k|, |\bar{k}| < \eta \sqrt{\eta L}, \]

\[ \theta_{kk} < \psi^{-2/3} \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{|k|^{1/2}|\bar{k}|^{1/2}} \quad (\text{cone}), \quad \theta_{k+\bar{k}} < \Theta(|k| + |\bar{k}|)^{-1/3} L^{-1/3} \quad (\text{cone}) \]

(2.66)

where \(0.5 \lesssim T^{(\text{cusp})} \lesssim 50\). Recall from the discussion of section 2.2.1 that the momentum sum \(k + \bar{k}\) is oriented within a cone of angle \(\Theta(|k| + |\bar{k}|)^{-1/3} L^{-1/3}\) centered on the cusp, and the angle between \(k\) and \(\bar{k}\) cannot exceed \(\psi^{-2/3} (|k| + |\bar{k}|)^{2/3} L^{-1/3} / \sqrt{|k||\bar{k}|}\). Upon performing the angular integrations as in eq. (2.24), we obtain

\[ dN_{\text{AB}}^{(\text{cusp})} = \frac{(2\pi \theta_q)^2}{32(2\pi)^4} \psi^{-4/3} \theta^2 T^{(\text{cusp})} d|k| d|\bar{k}| \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{|k|^{1/2}|\bar{k}|^{1/2}}, \quad \psi m_\psi \sqrt{m_\psi L} < |k|, |\bar{k}| < \eta \sqrt{\eta L} \]  

(2.67)

We calculate the total energy output as

\[ E^{(\text{cusp})} = \int (\omega + \bar{\omega}) dN_{\text{AB}}^{(\text{cusp})} \approx \frac{(2\pi \theta_q)^2}{32(2\pi)^4} \psi^{-4/3} \theta^2 T^{(\text{cusp})} m_\psi \eta \sqrt{\eta L} \]  

(2.68)

and the average power output per loop oscillation as

\[ \Gamma_{\text{AB}}^{(\text{cusp})} = \frac{(2\pi \theta_q \eta)^2}{\sqrt{\eta L}} \]  

(2.69)

where \(\Gamma_{\text{AB}}^{(\text{cusp})} = \frac{1}{32(2\pi)^4} \psi^{-4/3} \theta^2 f T^{(\text{cusp})}\). Using the range for \(T^{(\text{cusp})}\) given above, we can estimate \(10^{-5} \lesssim \Gamma_{\text{AB}}^{(\text{cusp})} \lesssim 10^{-2}\). This agrees the result of ref. [35] after the boost factor is included.

2.5.2 Fermion AB emission from a kink

We find the spectrum of radiation from a kink by inserting eq. (D.20) into eq. (2.65):

\[ dN_{\text{AB}}^{(\text{kink})} = \frac{(2\pi \theta_q)^2}{8(2\pi)^6} T^{(\text{kink})} \frac{L^{2/3}}{(|k| + |\bar{k}|)^{10/3}} |k||\bar{k}| d|k| d\Omega d|\bar{k}| d\bar{\Omega}, \quad \psi m_\psi \sqrt{m_\psi L} < |k|, |\bar{k}| < \eta, \]

\[ \theta_{kk} < \psi^{-2/3} \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{|k|^{1/2}|\bar{k}|^{1/2}}, \quad \theta_{k+\bar{k}} < \Theta(|k| + |\bar{k}|)^{-1/3} L^{-1/3} \]  

(2.70)

where \(0.5 \lesssim T^{(\text{kink})} \lesssim 100\). Recall that \(k + \bar{k}\) is oriented in a ribbon with angular width \(\theta_{k+\bar{k}}\), and the opening angle between \(k\) and \(\bar{k}\) does not exceed \(\theta_{kk}\). After performing the angular integrations we obtain

\[ dN_{\text{AB}}^{(\text{kink})} = \frac{(2\pi \theta_q)^2}{16(2\pi)^4} \psi^{-4/3} \theta T^{(\text{kink})} d|k| d|\bar{k}| \frac{(|k| + |\bar{k}|)^{2/3} L^{-1/3}}{(|k| + |\bar{k}|)^{7/3} L^{1/3}}, \quad \psi m_\psi \sqrt{m_\psi L} < |k|, |\bar{k}| < \eta . \]

(2.71)
We calculate the total energy output as
\[ E_{\text{AB}}^{(\text{kink})} = \int (\omega + \bar{\omega}) dN_{\text{AB}}^{(\text{kink})} \approx \frac{9(2\pi \theta_q)^2}{16(2\pi)^4} \psi^{-4/3} \Theta \frac{\eta^{2/3}}{L^{1/3}} \left( 1 - \psi^{2/3} \frac{m_\psi L^{1/3}}{\eta^{2/3}} \right) , \quad (2.72) \]
and the average power output from \( N_k \) kinks during one loop oscillation period \((T = L/2)\) as
\[ P_{\text{AB}}^{(\text{kink})} = \Gamma_{\text{AB}}^{(\text{kink})} (2\pi \theta_q \eta)^2 \left( 1 - \psi^{2/3} \frac{m_\psi L^{1/3}}{\eta^{2/3}} \right) \quad (2.73) \]
where \( \Gamma_{\text{AB}}^{(\text{kink})} \equiv \frac{9}{8(2\pi)^2} \psi^{-4/3} \Theta \frac{\eta^{2/3}}{L^{1/3}} N_k \). Using the range for \( \tau^{(\text{kink})} \) given above along with \( N_k \approx 1 \), we can estimate \( 10^{-2} \lesssim \Gamma_{\text{AB}}^{(\text{kink})} \lesssim 1 \).

2.5.3 Fermion AB emission from a kink-kink collision
We find the spectrum radiation from a kink collision by inserting eq. \((D.23)\) into eq. \((2.65)\):
\[ dN_{\text{AB}}^{(k-k)} = \frac{(2\pi \theta_q)^2}{8(2\pi)^6} \frac{1}{\omega + \bar{\omega}} \frac{1}{\omega + \bar{\omega}} |k| d\omega d\Omega |\bar{k}| d\bar{\omega} d\bar{\Omega} , \quad m_\psi < \omega, \bar{\omega} < \eta \quad (2.74) \]
with \(0.1 < \tau^{(k-k)} < 50\). In this case, the emission is isotropic, and we can estimate \((k + \bar{k})^2 \approx 2\omega \bar{\omega}\) up to an \(O(1)\) factor associated with the angle between \(k\) and \(\bar{k}\). The angular integration is trivial, and we find
\[ dN_{\text{AB}}^{(k-k)} = \frac{(2\pi \theta_q)^2}{64(2\pi)^4} \frac{1}{\omega + \bar{\omega}} d\omega d\bar{\omega} , \quad m_\psi < \omega, \bar{\omega} < \eta . \quad (2.75) \]
We calculate the total energy radiated as
\[ E_{\text{AB}}^{(k-k)} = \int (\omega + \bar{\omega}) dN_{\text{AB}}^{(k-k)} = \frac{(2\pi \theta_q)^2}{64(2\pi)^4} \tau^{(k-k)} \eta , \quad (2.76) \]
and the average power emitted from a loop which experiences \(N_{kk}\) collisions during a loop oscillation period \((T = L/2)\) is found to be
\[ P_{\text{AB}}^{(k-k)} = \Gamma_{\text{AB}}^{(k-k)} \frac{(2\pi \theta_q \eta)^2}{\eta L} \quad (2.77) \]
where \( \Gamma_{\text{AB}}^{(k-k)} \equiv \frac{1}{32(2\pi)^2} N_{kk} \tau^{(k-k)} \). Using the parameter ranges given above along with \(N_{kk} \approx 1\), we can estimate \(10^{-6} < \Gamma_{\text{AB}}^{(k-k)} < 10^{-3}\). This agrees the result of ref. [35].

3 Scattering cross sections
The interactions discussed in section 1 allow SM particles to scatter off of the dark string. Interactions of the Higgs and Z bosons with the string, given by eqs. (1.4) and (1.5), will lead to a “hard core” scattering, and the AB phases of the SM fermions, given by eq. (1.6), will lead to a non-local AB scattering. If the couplings are comparable for the direct and the AB interactions, then the latter generally dominates [19], and therefore we focus on AB scattering here. Moreover, in the cosmological context the dark string will scatter from the SM plasma, which consists mostly of electrons and nuclei at late times.
The AB interaction allows fermions to scatter from a cosmic string. The scattering cross section (per length of string) was found to be [21]

\[ \frac{d\sigma_{AB}}{d\theta} = \frac{\sin^2(\pi \theta_q)}{2\pi k_\perp \sin^2(\theta/2)} \]  

(3.1)

where the AB phase for SM fermions, \( \theta_q \), is given in eq. (1.6), and \( k_\perp \) is the magnitude of the momentum transverse to the string. Inserting the expression for \( \theta_q \) and expanding in the \( \theta_q \ll 1 \) limit gives

\[ \frac{d\sigma_{AB}}{d\theta} \approx \frac{2\pi \cos^2 \theta_W \sin^2 \epsilon q^2}{g_\chi^2 k_\perp \sin^2(\theta/2)} \]  

(3.2)

where \( q \) is the electromagnetic charge of the fermion.

To study the motion of strings through the cosmological medium, we are interested in the drag (momentum transfer) experienced by the string. This is calculated in terms of a “transport cross section” (see [19]) given by

\[ \sigma_{AB,t}(k) = \int_0^{2\pi} d\theta \frac{d\sigma_{AB}}{d\theta} (1 - \cos \theta) = \frac{2}{k_\perp} \sin^2(\pi \theta_q) \approx \frac{8\pi^2 \cos^2 \theta_W \sin^2 \epsilon q^2}{g_\chi^2} \]  

(3.3)

To obtain the total drag due to the entire medium, we must sum over the various species with their respective charges \( q \).

The derivation of the AB phase, given by eq. (1.6), assumed that the particle circumnavigates the string on a length scale larger than the width of the SM dressing. In this way, the particle trajectory encloses both the flux carried by the thin HS string core and the thick SM dressing. This length scale is microscopic, \( \Delta x \sim \eta^{-1} \approx 10^{-16} \) cm, and therefore this assumption is well-justified for the cosmological medium at late times, where the inter-particle spacing is much larger than \( \Delta x \).

4 Summary and conclusion

The dark string couples to the SM fields through the local interactions in eqs. (1.4) and (1.5) and through the non-local Aharonov-Bohm interactions of charged fermions. These interactions lead to radiation of Higgs bosons, Z bosons, and fermions from cusps, kinks, and kink collisions on cosmic strings. The total power emitted in each of various channels is summarized as follows. For Higgs emission via a linear coupling

\[ P_{H}^{(\text{cusp})} = \Gamma_H^{(\text{cusp})} \left( \frac{g_{\text{str}}^H \eta}{m_H L} \right)^2 10^{-4} < \Gamma_H^{(\text{cusp})} < 10^{-1} \]  

(4.1a)

\[ P_{H}^{(\text{kink})} = \Gamma_H^{(\text{kink})} \left( \frac{g_{\text{str}}^H \eta}{m_H L} \left( 1 - \psi^{2/3} \frac{m_H L^{1/3}}{M^{2/3}} \right) \right) 10^{-3} < \Gamma_H^{(\text{kink})} < 1 \]  

(4.1b)

\[ P_{H}^{(\text{k-k})} = \Gamma_H^{(\text{k-k})} \left( \frac{g_{\text{str}}^H \eta}{m_H L} \right)^2 10^{-2} < \Gamma_H^{(\text{k-k})} < 10 \]  

(4.1c)
for Higgs emission via a quadratic coupling
\[
P_{HH}^{(cusp)} = \Gamma_{HH}^{(cusp)} \left( \frac{g_{str}^2 m_H^2}{\sqrt{ML}} \right) 10^{-5} < \Gamma_{HH}^{(cusp)} < 10^{-2} \quad (4.2a)
\]
\[
P_{HH}^{(kink)} = \Gamma_{HH}^{(kink)} \left( \frac{g_{str}^2 m_H^2}{ML} \right)^{2/3} \left( 1 - 5\eta \frac{m_H \sqrt{m_H L}}{M} \right) 10^{-4} < \Gamma_{HH}^{(kink)} < 10^{-1} \quad (4.2b)
\]
\[
P_{HH}^{(k-k)} = \Gamma_{HH}^{(k-k)} \left( \frac{g_{str}^2 m_H^2}{ML} \right) 10^{-4} < \Gamma_{HH}^{(k-k)} < 10^{-1} \quad , \quad (4.2c)
\]

for Z boson emission via a linear coupling
\[
P_{Z}^{(cusp)} = \Gamma_{Z}^{(cusp)} \left( \frac{\eta}{\sigma} \right)^4 \left( g_{str}^2 m_Z \right)^2 \frac{\sqrt{m_Z L}}{M} 10^{-4} < \Gamma_{Z}^{(cusp)} < 10^{-1} \quad (4.3a)
\]
\[
P_{Z}^{(kink)} = \Gamma_{Z}^{(kink)} \left( \frac{\eta}{\sigma} \right)^4 \left( g_{str}^2 m_Z \right)^2 \frac{\sqrt{m_Z L}}{M} \left( 1 - \psi^2/3 \frac{m_Z L}{M^{2/3}} \right) 10^{-2} < \Gamma_{Z}^{(kink)} < 10 \quad (4.3b)
\]
\[
P_{Z}^{(k-k)} = \Gamma_{Z}^{(k-k)} \left( \frac{\eta}{\sigma} \right)^4 \left( g_{str}^2 m_Z \right)^2 \frac{\sqrt{m_Z L}}{M} 10^{-3} < \Gamma_{Z}^{(k-k)} < 1 \quad , \quad (4.3c)
\]

for Z boson emission via a quadratic coupling
\[
P_{ZZ}^{(cusp)} = \Gamma_{ZZ}^{(cusp)} \left( \frac{\eta}{\sigma} \right)^8 \left( g_{str}^2 m_Z \right)^2 \frac{\sqrt{ML}}{M} 10^{-4} < \Gamma_{ZZ}^{(cusp)} < 10^{-2} \quad (4.4a)
\]
\[
P_{ZZ}^{(kink)} = \Gamma_{ZZ}^{(kink)} \left( \frac{\eta}{\sigma} \right)^8 \left( g_{str}^2 m_Z \right)^2 \left( 1 - \psi^2/3 \frac{m_Z \sqrt{m_Z L}}{M} \right) 10^{-3} < \Gamma_{ZZ}^{(kink)} < 10^{-1} \quad (4.4b)
\]
\[
P_{ZZ}^{(k-k)} = \Gamma_{ZZ}^{(k-k)} \left( \frac{\eta}{\sigma} \right)^8 \left( g_{str}^2 m_Z \right)^2 \frac{\sqrt{ML}}{M} 10^{-5} < \Gamma_{ZZ}^{(k-k)} < 10^{-2} \quad , \quad (4.4c)
\]

and for fermion emission via the AB interaction
\[
P_{AB}^{(cusp)} = \Gamma_{AB}^{(cusp)} \left( \frac{2\pi \theta q \eta}{\sqrt{\eta L}} \right)^2 10^{-5} < \Gamma_{AB}^{(cusp)} < 10^{-2} \quad (4.5a)
\]
\[
P_{AB}^{(kink)} = \Gamma_{AB}^{(kink)} \left( \frac{2\pi \theta q \eta}{(\eta L)^{1/3}} \right) \left( 1 - \psi^2/3 \frac{m_q \sqrt{m_q L}}{M^{2/3}} \right) 10^{-2} < \Gamma_{AB}^{(kink)} < 1 \quad (4.5b)
\]
\[
P_{AB}^{(k-k)} = \Gamma_{AB}^{(k-k)} \left( \frac{2\pi \theta q \eta}{\sqrt{\eta L}} \right)^2 10^{-6} < \Gamma_{AB}^{(k-k)} < 10^{-3} \quad . \quad (4.5c)
\]

Here \( \psi \approx 0.1 \) [see eq. (C.18)] and the other dimensionless coefficients (\( \Gamma \) factors) depend on undetermined parameters that characterize the radiating string segment, e.g., the curvature nearby to the cusp or the sharpness of the kink. We quantify our ignorance of these parameters, as described in appendix D, and this leads to the ranges shown above. The kink expressions are only valid for small \( L \) where the power is positive.

Let us highlight the important features of these calculations:

1. This system is characterized by three hierarchical length scales, the string thickness, the inverse particle mass, and the string loop length: \( 1/M \ll 1/m \ll L \). The radiation calculation is not amenable to dimensional analysis, because it is always possible to form dimensionless combinations that are far from order one, e.g., \( ML \gg 1 \) or \( m/M \ll 1 \). Additionally, some of the spectra are UV sensitive \( dN = d|k|/|k|^n \) with \( n \leq 2 \) while others are IR sensitive \( n > 2 \), and as a result some of the power formulae depend on the UV mass scale, \( M \), while others depend on the IR mass scale, \( \eta, m_H, \) or \( m_Z \).
2. In the physically relevant parameter regime, $ML \gg m_H L \gg 1$, the dominant radiation channel is Higgs emission from cuspy loops via the quadratic interaction, see $P_{HH}^{(cusp)}$ in eq. (4.2a).

3. The string loop also radiates gravitational waves from cusps, kinks, and kink collisions. The power output into this channel is well-known: $P_{\text{grav}} = \Gamma_g G M^4$ where $\mu = M^2$ is the string tension, $\Gamma_g \approx 100$, and $G$ is Newton’s constant [19]. For comparison, $P_{HH}^{(cusp)} \sim M^{3/2}/L^{1/2}$. If the string mass scale is large, then string loops will primarily radiate in the form of gravitational waves, as originally observed by ref. [22]. However, it is important to emphasize that particle emission will dominate if the scale of symmetry breaking is low, e.g., for a TeV scale string. For instance, taking $L \approx 40 \text{Gly}$ to be the size of the horizon today we find $P_{HH}^{(cusp)}/P_{\text{grav}} \approx 10^4 (M/\text{TeV})^{-5/2}$. Moreover, in general Higgs emission dominates over gravitational emission for small loops: $L < (\Gamma_{\text{HH}} g_{\text{str}}^2 (M/m_H)^{2/3}/G^2 M^5)$. 

4. Comparing Higgs emission from a cuspy loop via the linear and quadratic interactions, we find $P_{HH}^{(cusp)}/P_{HH}^{(cusp)} \approx (\Gamma_{\text{HH}}^{(cusp)}/\Gamma_{\text{HH}}^{(cusp)}) (g_{\text{str}}^H g_{\text{str}}^H)^2 (M/m_H)^{3/2}$ where we have approximated $\eta \approx m_H$. Typically $(\Gamma_{\text{HH}}^{(cusp)}/\Gamma_{\text{HH}}^{(cusp)}) \approx 10^{-1}$ and $(g_{\text{str}}^H g_{\text{str}}^H) \approx 1$ and $(M/m_H) \gg 1$, and we find that the quadratic interaction is a much more efficient radiation channel. Note that the dimensionless coefficients (the $\Gamma$ factors) for the quadratic interactions are typically smaller than the corresponding coefficient for the linear interaction; this is a result of the additional phase space suppression (factors of $2\pi$).

5. In Fig. 2 we show the six Higgs boson radiation channels. We use eqs. (4.1a) to (4.2c) taking $g_{\text{str}}^H = g_{\text{str}}^H = 1$ and choosing the largest allowed values for the dimensionless prefactors. In the first panel, the line representing gravitational emission is off the scale of the plot at approximately $10^{-20}$. For the largest loops, $m_H L \gg 1$, gravitational emission dominates (pink, dot-dashed). For the smallest loops, $m_H L \approx 1$, the dominant radiation channel is either pair emission from a cusp (red, solid) or pair emission from a kink (blue, solid). There is no radiation from kinks on large loops, $L \gg M^2/m_H^3$, since the spectrum is bounded as $m_H \sqrt{m_H L} < |k| < M$. 

Figure 2. The total power emitted in Higgs radiation from a cusp (red), a kink (blue), and a kink collision (green) due to the linear (dashed) and quadratic (solid) interactions of the Higgs field with the string worldsheet. We vary the loop length, $L$, and show three different string mass scales $M$. For reference, $m_H L = 10^{19}$ corresponds to a loop length of $L = 1 \text{ km}$, and $L = 40 \text{Gly}$ corresponds to $m_H L = 10^{15}$. Note that the scale is different in the left panel.
6. The spectrum of radiation from kinks extends over the range \( m \sqrt{mL} < |k| < M \) where \( m \) is the particle mass and \( M \) is the string mass scale. For momenta below the IR cutoff, a destructive interference from different segments of the string loop leads to a suppression of radiation. (In the language of appendix C.1, the saddle point approximation fails.) For momenta above the UV cutoff, the Compton wavelength of the radiated particle is smaller than the string thickness, \( 1/M \), and the radiation is once again suppressed. (By comparison, the UV cutoff at a cusp is raised to \( M \sqrt{ML} \) due to the large boost factor.) Thus only kinks on small loops, \( L < M^2/m^3 \), give appreciable radiation.

7. The Z boson radiation channels are suppressed compared to the corresponding Higgs radiation channels by the fourth or eighth power of \( (\eta/\sigma) \ll 1 \), and this makes Z boson emission negligible. The factor of \( (\eta/\sigma) \) entered the calculation directly in the coupling of the Z boson field to the string, see eq. (1.4). For the dark string, the Z boson radiation is only possible by virtue of the gauge-kinetic mixing, and the mixing angle vanishes in the decoupling limit where \( (\eta/\sigma) \ll 1 \) [13]. For a different model in which this coupling is unsuppressed, the vector boson radiation will be comparable to the Higgs boson radiation, compare eqs. (4.2a) and (4.4a).

Throughout this analysis we have assumed that the light SM fields are coupled to the zero thickness dark string core, which is composed of the heavy HS fields. As we found in ref. [13], the dark string has a much richer structure: the thin core is surrounded by a wide dressing made up of the SM Higgs and Z boson fields. The presence of this dressing could lead to a backreaction that was neglected in our particle production calculations, and this deserves further investigation. Additionally, as with most calculations of radiation from cosmic strings, we neglect the more familiar backreaction effect: a reduction in radiation power as cusps and kinks are gradually smoothed as a result of energy loss in the form of particle and gravitational radiation [37, 38].

The particle production calculations that we have presented here play a central role in the study of astrophysical and cosmological signatures of cosmic strings. For instance, Higgs bosons emitted from the string at late times will decay and produce cosmic rays that are potentially observable on Earth [25]. In our followup paper [30], we will study the cosmological evolution of the network of dark strings and assess the prospects for their detection.

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A Worldsheet formalism

In this appendix we review the string worldsheet formalism (see, e.g., [19]). Let \( \tau \) and \( \sigma \) be the time-like and space-like worldsheet coordinates, and let \( X^\mu(\tau, \sigma) \) be the string worldsheet. Then \( d^2\sigma = d\tau d\sigma \) is the worldsheet volume element and \( d\sigma^{\mu\nu} = d\tau d\sigma \epsilon^{\mu\nu\alpha\beta} \partial_\alpha X_\alpha \partial_\beta X_\beta \) is the worldsheet area element. Repeated Greek indices are summed from 0 to 4 and Latin indices from 0 to 1 with \( \partial_\alpha X^\mu = \partial_\tau X^\mu = \dot{X}^\mu \) and \( \partial_\sigma X^\nu = \partial_\tau X^\nu = \ddot{X}^\nu \). We define the pullback of the metric as \( \gamma_{ab} \equiv g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \) and \( \sqrt{-\gamma} \equiv \sqrt{-\det \gamma} = \sqrt{- \frac{1}{2} \epsilon^{ab} \epsilon^{cd} \gamma_{ab} \gamma_{cd}} \).
We now specify to the conformal gauge by imposing
\[ \dot{X} \cdot X' = 0 \quad \text{and} \quad \dot{X} \cdot \dot{X} + X' \cdot X' = 0. \] (A.1)

Then we have
\[ d\sigma^{\mu\nu} = d\tau d\sigma \epsilon^{\mu\alpha\beta} (\dot{X}_\alpha X'_\beta - \dot{X}_\beta X'_\alpha) \] (A.2)
\[ \sqrt{-\gamma} = |\dot{X} \cdot \dot{X}| = \dot{X} \cdot \dot{X} \] (A.3)

where in the last equality we have used the fact that \( \dot{X}^\mu \) is not spacelike. We are also free to choose \( \tau = t \).

Solutions of the equation of motion for a free string, \( \ddot{X} = X'' \), can be written as
\[ X^\mu(t, \sigma) = \frac{1}{2} [a^\mu(\sigma_-) + b^\mu(\sigma_+)] \] (A.4)

where we have introduced the right- and left-movers, \( a^\mu(\sigma_-) \) and \( b^\mu(\sigma_+) \), which are functions of \( \sigma_\pm \equiv (\sigma \pm t) \). For regularly oscillating string loops, these functions obey the periodicity conditions
\[ a^\mu(L + \sigma_-) = a^\mu(\sigma_-) \quad \text{and} \quad b^\mu(L + \sigma_+) = b^\mu(\sigma_+) \] (A.5)

in the center of mass frame of the loop. The derivatives are
\[ \dot{X}(t, \sigma) = \frac{1}{2} [-a'(\sigma_-) + b'(\sigma_+)] \]
\[ \dot{X}'(t, \sigma) = \frac{1}{2} [a'(\sigma_-) + b'(\sigma_+)] \] (A.6)
\[ \ddot{X} = \dddot{X} = \frac{1}{2} [a''(\sigma_-) + b''(\sigma_+)] \cdot \]

We can use the residual gauge freedom to choose
\[ (a)^\mu = \{-\sigma_-, a(\sigma_-)\}, \quad (b)^\mu = \{\sigma_+, b(\sigma_+)\} \]
\[ (a')^\mu = \{-1, a'(\sigma_-)\}, \quad (b')^\mu = \{1, b'(\sigma_+)\} \]
\[ (a'')^\mu = \{0, a''(\sigma_-)\}, \quad (b'')^\mu = \{0, b''(\sigma_+)\} \] (A.7)

along with the condition that \( a' \) and \( b' \) should be null, which implies
\[ |a'(\sigma_-)|^2 = |b'(\sigma_+)|^2 = 1 \]
\[ a' \cdot a'' = b' \cdot b'' = a' \cdot a'' = b' \cdot b'' = 0 \] (A.8)
\[ a' \cdot a''' + a'' \cdot a'' = b' \cdot b''' + b'' \cdot b'' = 0. \]

This parametrization lets us write
\[ d^2\sigma = d\tau d\sigma = \frac{1}{2} d\sigma_+ d\sigma_- \]
\[ \sqrt{-\gamma} = \frac{a' \cdot b'}{2} \]
\[ d\sigma^{\mu\nu} = d\tau d\sigma \epsilon^{\mu\alpha\beta} b'_\alpha a'_\beta. \] (A.9)

Note that \( a' \cdot b' = -1 - a' \cdot b' \leq 0 \) and therefore \( \sqrt{-\gamma} \geq 0 \) as it should be.
B Calculation of particle radiation from the string

In this appendix, we calculate the spectrum of scalar and vector boson emission due to a coupling with a cosmic string of the linear or quadratic form. We also derive the spectrum of fermions emitted due to a direct coupling and an Aharonov-Bohm coupling. The results we obtain are not unique to the dark string model; they apply to any model that has couplings of the form considered here.

We use the matrix element formalism to perform these calculations \cite{22}. Since the linear coupling gives rise to a classical source for the scalar or vector field, the radiation in these cases can also be calculated by solving the classical field equation \cite{25, 26}. We have verified that both approaches give identical spectra. We also retain all factors of 2 and \(\pi\), which were neglected in the previous calculations.

B.1 Scalar radiation via linear coupling

Consider a real scalar field \(\phi(x)\) of mass \(m\) that is coupled to the string worldsheet \(X^\mu(\tau, \sigma)\) through the effective interaction

\[
L_{\text{eff}} = A \phi(x) \int d^2\sigma \sqrt{-\gamma(4)} \delta^{(4)}(x - X) \quad (B.1)
\]

where \(A\) is an arbitrary real parameter with mass dimension one. We calculate the amplitude for particle production by making a perturbative expansion in \(A\). Then to leading order we have\(^2\)

\[
A = i \int d^4x \langle k | L_{\text{eff}}(x) | 0 \rangle \quad (B.2)
\]

where \(|k\rangle\) is a one-particle state of momentum \(k\). The action of the field operator on the one-particle state is simply

\[
\phi(x)|k\rangle = e^{-ik \cdot x}|0\rangle \quad \text{and} \quad \langle k|\phi(x) = e^{ik \cdot x}|0\rangle \quad (B.3)
\]

where \(k^\mu = \{\omega, k\}\) with \(\omega = \sqrt{m^2 + |k|^2}\). Then upon inserting eq. (B.1) into eq. (B.2) we obtain

\[
A(k) = i A \int d^4x e^{ik \cdot x} \int d^2\sigma \sqrt{-\gamma} \delta^{(4)}(x - X) = i A I(k) \quad (B.4)
\]

where

\[
I(k) \equiv \int d^2\sigma \sqrt{-\gamma} e^{ik \cdot X}. \quad (B.5)
\]

In appendix D we calculate this integral for various string configurations, as specified by \(X^\mu(\tau, \sigma)\).

For a given \(X\) we calculate the number of scalar bosons emitted into a phase space volume \(d^3k = |k|^2 d|k| d\Omega\) as

\[
dN = \frac{d^3k}{(2\pi)^3 2\omega} |A(k)|^2. \quad (B.6)
\]

\(^2\)More accurately, the initial state is not vacuum, but it is a state containing the string, \(|S\rangle\), and the final state contains a deformation of the initial string state, \(|S'\rangle\). Provided that the radiation has a negligible backreaction on the string state, one can neglect the deformation and then \(\langle S'|S\rangle \approx \langle S|S\rangle = \langle 0|0\rangle\) \cite{22}.
Using eq. (B.4) in eq. (B.6) we obtain the final spectrum
\[ dN = A^2 \frac{d^3k}{(2\pi)^3 2\omega} |I(k)|^2 \] (B.7)
where the dimensionful coefficient is equal to \( A = g_{\text{str}}^H \eta \) for the dark string.

### B.2 Scalar radiation via quadratic coupling

Consider a real scalar field \( \phi(x) \) of mass \( m \) that is coupled to the string worldsheet \( \mathbb{X}^\mu(\tau, \sigma) \) through the effective interaction
\[ \mathcal{L}_{\text{eff}} = C \phi(x)^2 \int d^2\sigma \sqrt{-\gamma} \delta^{(4)}(x - \mathbb{X}) \] (B.8)
where \( C \) is an arbitrary real parameter with mass dimension zero. We can calculate the radiation of scalar boson pairs using perturbation theory provided that \( C \ll 1 \). Consider the radiation of a boson pair with momenta \( k \) and \( \bar{k} \). We can introduce the 4-vectors \( k^\mu = \{ \omega = \sqrt{k^2 + m^2}, k \} \) and \( \bar{k}^\mu = \{ \bar{\omega} = \sqrt{\bar{k}^2 + m^2}, \bar{k} \} \). To leading order in \( C \) the amplitude for this process is
\[ \mathcal{A} = i \int d^4x \langle k \bar{k} | \mathcal{L}_{\text{eff}}(x) | 0 \rangle . \] (B.9)
Inserting eq. (B.8) into eq. (B.9) and using eq. (B.3) we obtain
\[ \mathcal{A} = iC I(k + \bar{k}) \] (B.10)
where \( I(k) \) was defined in eq. (B.5). The number of scalar bosons emitted into the phase space volume \( d^3k d^3\bar{k} = |k|^2 d|k| d\Omega |\bar{k}|^2 d|\bar{k}| d\bar{\Omega} \) is calculated as
\[ dN = \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3\bar{k}}{(2\pi)^3 2\bar{\omega}} |\mathcal{A}|^2 . \] (B.11)
Using eq. (B.10) this becomes
\[ dN = C^2 \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3\bar{k}}{(2\pi)^3 2\bar{\omega}} |I(k + \bar{k})|^2 \] (B.12)
where \( C = g_{\text{str}}^H \) for the dark string.

### B.3 Vector radiation via linear coupling

Consider a vector field \( A_\mu(x) \) of mass \( m \) that couples to the string worldsheet \( \mathbb{X}^\mu(\tau, \sigma) \), via the linear interaction
\[ \mathcal{L}_{\text{eff}} = \frac{C}{2} F_{\mu\nu}(x) \int da^{\mu\nu} \delta^{(4)}(x - \mathbb{X}) \] (B.13)
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength tensor and \( C \) is a real parameter of mass dimension zero. Recall that the worldsheet area element was defined in eq. (A.2). Since the radiation will be relativistic, we can treat the gauge boson as transversely polarized with
two allowed helicities $\lambda = \pm 1$. We calculate the amplitude to radiate a vector boson with momentum $k$ and helicity $\lambda$ as

$$ A = i \int d^4x \langle k, \lambda | L_{\text{eff}}(x) | 0 \rangle .$$  \hfill (B.14) 

The action of the field operator on the one-particle state is

$$ A_\nu(x) | k, \lambda \rangle = \epsilon_\nu(k, \lambda) e^{-ik \cdot x} | 0 \rangle \quad \text{and} \quad \langle k, \lambda | A_\nu(x) = \epsilon^*_\nu(k, \lambda) e^{ik \cdot x} | 0 \rangle$$  \hfill (B.15) 

where $k^\mu = \{ \omega = \sqrt{m^2 + |k|^2} , k \}$. Inserting eq. (B.13) into eq. (B.14) and using eq. (B.15) gives

$$ A = - C k_\mu \epsilon^*_\nu(k, \lambda) I^{\mu \nu}(k)$$  \hfill (B.16) 

where

$$ I^{\mu \nu}(k) \equiv \int d\sigma^{\mu \nu} e^{ik \cdot X} .$$  \hfill (B.17) 

Then the number of vector bosons emitted into the phase space volume $d^3k = |k|^2 |d| |d\Omega$ is calculated as

$$ dN = \sum_\lambda \frac{d^3k}{(2\pi)^3 2\omega} |A|^2$$  \hfill (B.18) 

where we sum over the two polarization states. Using eq. (B.16) this becomes

$$ dN = C^2 \frac{d^3k}{(2\pi)^3 2\omega} k_\mu k_\alpha I^{\mu \nu}(k) I^{\alpha \beta}(k)^* \sum_\lambda \epsilon_\beta(k, \lambda) \epsilon^*_\nu(k, \lambda) .$$  \hfill (B.19) 

We perform the spin sum using the completeness relationship

$$ \sum_{\lambda = \pm 1} \epsilon_\beta(k, \lambda) \epsilon^*_\nu(k, \lambda) = -g_{\beta \nu} .$$  \hfill (B.20) 

Doing so we find the spectrum to be

$$ dN = C^2 \frac{d^3k}{(2\pi)^3 2\omega} k^2 \Pi(k)$$  \hfill (B.21) 

where the (positive, real) function

$$ \Pi(q) \equiv -g_{\nu \beta} \frac{q_\nu q_\beta}{q^2} I^{\nu \mu}(q) I^{\mu \beta}(q)^*$$  \hfill (B.22) 

has dimensions of length$^4$ and carries the dependence on the string worldsheet. By choosing $C = g^2_{\text{str}} (\eta / \sigma)^2$ we obtain the spectrum of $Z$ boson radiation from the dark string.
Upon inserting eq. (B.23) into eq. (B.24) we obtain

\[ \sqrt{s} \text{spins} \]

For the dark string model we take

\[ x = \text{Dirac field } \Psi(\tau, \sigma) \]

Consider a Dirac field \( \Psi(\tau, \sigma) \) that is coupled to the string worldsheet via the quadratic interaction

\[ \mathcal{L}_{\text{eff}} = C A_{\mu}(x) A^{\mu}(x) \int d^{2}\sigma \sqrt{-\gamma} \delta^{(4)}(x - \mathcal{X}) \]  

where \( C \) is a real parameter of mass dimension zero. The amplitude to radiate a pair of vector bosons with momenta \( k \) and \( k' \) and helicities \( \lambda \) and \( \bar{\lambda} \) is calculated as

\[ A = i \int d^{4}x \left\langle \, k, \lambda; k', \bar{\lambda} \, | \mathcal{L}_{\text{eff}}(x) | 0 \right\rangle . \]  

We can introduce the 4-vectors \( k^{\mu} = \{ \omega = \sqrt{k^{2} + m^{2}} , \mathbf{k} \} \) and \( \bar{k}^{\mu} = \{ \bar{\omega} = \sqrt{k'^{2} + m^{2}} , \bar{k} \} \). Upon inserting eq. (B.23) into eq. (B.24) and using eq. (B.15) we obtain

\[ A = iC \epsilon_{s}^{\alpha}(k, s) \epsilon_{s}^{\alpha}(k', \bar{s}) g^{\mu \nu} \mathcal{I}(k + \bar{k}) \]  

where \( \mathcal{I}(k) \) was defined in eq. (B.5). Then the number of vector bosons emitted into the phase space volume \( d^{3}k \, d^{3}\bar{k} = |k|^{2} d|k| d\Omega \, |\bar{k}|^{2} d|\bar{k}| d\Omega \) is calculated as

\[ dN = \sum_{\lambda} \sum_{\bar{\lambda}} \frac{d^{3}k}{(2\pi)^{3}2\omega} \frac{d^{3}\bar{k}}{(2\pi)^{3}2\omega} |A|^{2} \]  

where we sum over the transverse polarization states \( \lambda, \bar{\lambda} = \pm 1 \). (Since the radiation is highly boosted, we can neglect the longitudinal polarization states.) Using eq. (B.25) this becomes

\[ dN = C^{2} \frac{d^{3}k}{(2\pi)^{3}2\omega} \frac{d^{3}\bar{k}}{(2\pi)^{3}2\omega} |\mathcal{I}(k + \bar{k})|^{2} g^{\mu \nu} g^{\alpha \beta} \sum_{\lambda} \epsilon_{\alpha}(k, \lambda) \epsilon_{\beta}(k', \bar{\lambda}) \sum_{\bar{\lambda}} \epsilon_{\beta}(k, \bar{\lambda}) \epsilon_{\alpha}(k', \lambda) . \]

We evaluate the spin sums using the completeness relation in eq. (B.20) to find

\[ dN = 4C^{2} \frac{d^{3}k}{(2\pi)^{3}2\omega} \frac{d^{3}\bar{k}}{(2\pi)^{3}2\omega} |\mathcal{I}(k + \bar{k})|^{2} . \]

For the dark string model we take \( C = \frac{g^{2}_{\text{eff}}}{M^{2}} (\eta/\sigma)^{4} \).

### B.5 Dirac spinor radiation — direct coupling

Consider a Dirac field \( \Psi(x) \) of mass \( m \) that is coupled to the string worldsheet \( \mathcal{X}^{\mu}(\tau, \sigma) \) through the effective interaction

\[ \mathcal{L}_{\text{eff}} = C \frac{\bar{\Psi}(x)}{M} \Psi(x) \int d^{2}\sigma \sqrt{-\gamma} \delta^{(4)}(x - \mathcal{X}) \]  

where \( C \) is an arbitrary real parameter with mass dimension zero, and \( M \) is the string mass scale. Consider the radiation of a particle / anti-particle pair with momenta \( k \) and \( \bar{k} \) and spins \( s \) and \( \bar{s} \). We can introduce the 4-vectors \( k^{\mu} = \{ \omega = \sqrt{k^{2} + m^{2}} , \mathbf{k} \} \) and \( \bar{k}^{\mu} = \{ \bar{\omega} = \sqrt{k'^{2} + m^{2}} , \bar{k} \} \). To leading order the amplitude for this process is

\[ A = i \int d^{4}x \left\langle \, k, s; \bar{k}, \bar{s} \, | \mathcal{L}_{\text{eff}}(x) | 0 \right\rangle . \]
The action of the field operator on the one-particle state is given by

\[ \langle \mathbf{k}, s | \Psi(x) = \bar{u}(\mathbf{k}, s) e^{i \mathbf{k} \cdot x} | 0 \rangle \] and \[ \langle \mathbf{k}, \bar{s} | \Psi(x) = v(\mathbf{k}, \bar{s}) e^{i \mathbf{k} \cdot x} | 0 \rangle \] (B.31)

Inserting eq. (B.29) into eq. (B.30) we obtain

\[ A = i \frac{C}{M} \bar{u}(\mathbf{k}, s) v(\mathbf{k}, \bar{s}) \mathcal{I}(k + \bar{k}) \] (B.32)

where \( \mathcal{I}(k) \) was defined in eq. (B.5). The number of particle pairs emitted into the phase space volume \( d^3 k \, d^3 \bar{k} = |k|^2 \, d|k| \, d\Omega |\bar{k}|^2 \, d|\bar{k}| \, d\Omega \) is calculated as in eq. (B.26) where now the sum is over spin states \( s, \bar{s} = \pm 1/2 \). We use the completeness relations,

\[ \sum_s u(\mathbf{k}, s) \bar{u}(\mathbf{k}, s) = (k_\mu \gamma^\mu + m) \] and \[ \sum_{\bar{s}} v(\mathbf{k}, \bar{s}) \bar{v}(\mathbf{k}, \bar{s}) = (\bar{k}_\mu \gamma^\mu - m) \] (B.33)

Using the familiar Dirac gamma trace relations, we obtain

\[ dN = 4C^2 \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 \bar{k}}{(2\pi)^3 2\bar{\omega}} \mathcal{I}(k + \bar{k}) \frac{2 \, k \cdot \bar{k} - m^2}{M^2} \] (B.34)

where \( C = g_{\text{str}}^0 (\eta/\sigma)^2 \) for the dark string.

**B.6 Dirac spinor radiation — AB coupling**

Consider a Dirac field \( \Psi(x) \) of mass \( m \) that is coupled to the string worldsheet \( X^\mu(\tau, \sigma) \) through the effective interaction

\[ \mathcal{L}_{\text{eff}} = C \bar{\Psi}(x) \gamma_\mu \Psi(x) \int_{\text{ret}} \frac{d^4 p}{(2\pi)^4} \frac{ip_\mu}{p^2} \mathcal{I}^{\mu\nu}(p) e^{-ip \cdot x} \] (B.35)

where \( C \ll 1 \) is an arbitrary real parameter with mass dimension zero, and \( \mathcal{I}^{\mu\nu}(k) \) was defined in eq. (1.17). In the momentum integral, the integration contour is extended above both poles at \( p^0 = \pm |p| \). Following section B.5 we calculate the amplitude for the radiation of a particle/anti-particle pair:

\[ A = -C \bar{u}(\mathbf{k}, s) \gamma_\mu v(\mathbf{k}, \bar{s}) \frac{q_\mu}{q^2} \mathcal{I}^{\mu\nu}(q) \] (B.36)

where \( q \equiv k + \bar{k} \). The number of particle pairs emitted into the phase space volume \( d^3 k \, d^3 \bar{k} = |k|^2 \, d|k| \, d\Omega |\bar{k}|^2 \, d|\bar{k}| \, d\Omega \) is calculated as in eq. (B.26) where now the sum is over spin states \( s, \bar{s} = \pm 1/2 \). Using eq. (B.36) we obtain

\[ dN = 4C^2 \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 \bar{k}}{(2\pi)^3 2\bar{\omega}} \left( k_\mu \bar{k}_\alpha + \bar{k}_\alpha k_\mu - (m^2 + k \cdot \bar{k}) g_{\mu\alpha} \right) \frac{q_\mu q_\nu}{q^4} \mathcal{I}^{\mu\nu}(q) \mathcal{I}^{\alpha\beta}(q)^* \] (B.37)

Then using the antisymmetry of \( \mathcal{I}^{\mu\nu} \) we find

\[ dN = 2C^2 \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 \bar{k}}{(2\pi)^3 2\bar{\omega}} \left[ \Pi(k + \bar{k}) - |\Upsilon(k, \bar{k})|^2 \right] \] (B.38)

where \( \Pi(q) \) is defined in eq. (B.22) and

\[ \Upsilon(k, \bar{k}) = \frac{2 k_\mu \bar{k}_\nu}{(k + \bar{k})^2} \mathcal{I}^{\mu\nu}(k + \bar{k}) \] (B.39)
In general, the evaluation of eq. (B.38) is very involved and must be done numerically for some choice of loops as in [35]. However, to extract the radiation spectrum, it is sufficient to note that $dN > 0$, and so the term containing $\Upsilon$ is never larger than the term containing $\Pi$ [see also eq. (D.10)]. Hence, to extract scalings, we will take

$$dN \approx \frac{2C^2}{(2\pi)^32\omega} \frac{d^3k}{(2\pi)^32\omega} \Pi(k + \bar{k})$$ \hspace{1cm} (B.40)$$

where for the dark string $C = -(2\pi\theta_q)/2$.

### C Calculation of the worldsheet integrals

In appendix B we encountered the two integrals

$$I(k) = \int d^2\sigma \sqrt{-\gamma} e^{ikX} \hspace{1cm} (C.1)$$

$$I^{\mu\nu}(k) = \int d^2\sigma^{\mu\nu} e^{ikX} \hspace{1cm} (C.2)$$

while calculating the radiation spectra. In this appendix and the next, we will analytically calculate these integrals for the cusp, kink, and kink-kink collision string configurations.

It is convenient to define the integrals

$$I^\alpha_+(b; k) \equiv \int_0^L d\sigma_+ b^\mu e^{ik-b/2} \hspace{1cm} \text{and} \hspace{1cm} I^\alpha_-(a; k) \equiv \int_0^L d\sigma_- a'^\mu e^{ik-a/2} \hspace{1cm} (C.3)$$

where $I_+$ is a functional of $b^\mu(\sigma_+)$ with parameter $k^\mu$, and similarly $I_-$ is a functional of $a^\mu(\sigma_-)$. For a regularly oscillating string loop, the periodicity of $a'$ and $b'$ implies the identities

$$k \cdot I_\pm = 0 . \hspace{1cm} (C.4)$$

Additionally, for such a loop we can factorize the original integrals from eqs. (C.1) and (C.2) in terms of $I_+$ and $I_-$. We use eq. (A.9) to factor the integrands, and we use the periodicity of the loop oscillation to rewrite the domain of integration as $\int_0^L d\sigma \int_0^T d\tau = (1/2) \int_0^L d\sigma_+ \int_0^L d\sigma_- \text{ where } T = L/2$ is the loop oscillation period. Doing so gives

$$I(k) = -\frac{1}{4} g^{\alpha\beta}(I_+(b; k))_\alpha (I_-(a; k))_\beta \hspace{1cm} (C.5)$$

$$I^{\mu\nu}(k) = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta}(I_+(b; k))_\alpha (I_-(a; k))_\beta . \hspace{1cm} (C.6)$$

The problem is now reduced to calculating the two integrals, $I^\mu_+$ and $I^\mu_-$, for a given loop configuration, specified by $a^\mu$ and $b^\mu$.

These integrals cannot be performed analytically for general configurations. We, therefore, focus on the configurations that we expect to maximize the integrals, since this corresponds to maximum particle radiation. It turns out that for these optimum configurations, the saddle point and the discontinuity, the integrals are analytically tractable.

\footnote{There is a danger that there can be cancellations between the $\Pi$ and $|\Upsilon|^2$ terms but we find that our scalings agree with the behavior that was numerically obtained in [35] for similar loops.}
C.1 Saddle point integral

The integrals in eq. (C.3) become analytically tractable if there is a saddle point at which the phase is stationary [39]. For the sake of discussion consider the integral $I_+$. Its phase can be expanded about $\sigma_+ = \sigma_s$ as

$$\frac{k \cdot b(\sigma_+)}{2} = \frac{k \cdot b_s}{2} + \frac{k \cdot b_s'}{2} (\sigma_+ - \sigma_s) + \frac{k \cdot b_s''}{4} (\sigma_+ - \sigma_s)^2 + \frac{k \cdot b_s'''}{12} (\sigma_+ - \sigma_s)^3 + \ldots . \quad (C.7)$$

Subscripts are used to denote evaluation of the function at a particular point, e.g., $b'_s = b'(\sigma_s)$. We say that $\sigma_s$ is a saddle point if the stationary phase criterion, $k \cdot b'_s = 0$, is satisfied. Using eq. (A.7) this can be written as

$$k \cdot b'_s = \omega - k \cdot b'_s = \omega - |k| \cos \theta \quad (C.8)$$

where $\theta$ is the angle between $k$ and $b'_s$. If the particle being radiated is massless, $\omega = |k|$, then the saddle point criterion is satisfied by choosing $k = |k| b'_s$ (i.e., $\theta = 0$). Then it follows from the identity in eq. (A.8) that $k \cdot b''_s = 0$ as well, and the leading term in eq. (C.7) is cubic.

For massive particle radiation the saddle point criterion cannot be satisfied exactly. Instead, we have instead a quasi-saddle point, $\sigma_+ = \sigma_{qsp}$, at which the phase is approximately stationary:

$$k = |k| b'_{qsp} \quad , \quad k \cdot b'_{qsp} = \omega - |k| \quad , \quad k \cdot b''_{qsp} = 0 \quad , \quad k \cdot b'''_{qsp} = |k| |b''_{qsp}|^2 \quad , \quad (C.10)$$

where we have used eq. (A.8). It will be convenient to write

$$b''_{qsp} = \frac{2\pi}{L} \beta_{qsp} \hat{b}''_{qsp} \quad \text{and} \quad a''_{qsp} = \frac{2\pi}{L} \alpha_{qsp} \hat{a}''_{qsp} \quad (C.11)$$

where the hatted quantities are unit vectors. The dimensionless parameters $\alpha_{qsp}$ and $\beta_{qsp}$ are related to the acceleration or curvature of the loop at the quasi-saddle point (recall eq. (A.6)). The stationary phase approximation is still applicable as long as $(k \cdot b''_{qsp})(\sigma_+) \ll (k \cdot b''_{qsp})m(\sigma_+)^3 \ll 2\pi$.

Suppose that we are given a configuration $b'(\sigma_+)$ and a $k^\mu$ such that there exists some point $\sigma_+ = \sigma_s$ where the quasi-saddle point condition, eq. (C.10), is satisfied. Then the integral from eq. (C.3) can be approximated by expanding in $\Delta \sigma = \sigma_+ - \sigma_s$, which gives

$$I^\mu_+(b; k) \approx e^{i \frac{k \cdot b_s}{2}} \int_{-\sigma_s}^{L-\sigma_s} d(\Delta \sigma) \left[(b'_s)^\mu + (b''_s)^\mu \Delta \sigma\right] \exp[i \phi(\Delta \sigma)] . \quad (C.12)$$

The phase is also expanded in powers of $\Delta \sigma/L$ as $\phi(\Delta \sigma) = \phi_1(\Delta \sigma) + \phi_3(\Delta \sigma) + \ldots$ where

$$\phi_1 \equiv \frac{\omega - |k|}{2} \Delta \sigma \quad \text{and} \quad \phi_3 \equiv \frac{2\pi |k|}{L^2} \frac{1}{\Theta^3} (\Delta \sigma)^3 . \quad (C.13)$$

Here we have introduced the dimensionless parameter $\Theta \equiv (6/\pi \beta_s^2)^{1/3}$, and the shape parameter is $\beta_s = L |b''_s|/(2\pi)$ as per eq. (C.11).
As long as \( \phi_1 \) is negligible, the integral is in the stationary phase regime, and it can be evaluated directly with the saddle point approximation. Since the integral is dominated by the saddle point, we can extend the limits of integration to infinity. Doing so we obtain

\[
\mathcal{I}_+(b; k) \approx e^{i \frac{2\pi}{3} \Theta} \int_{-\infty}^{\infty} d(\Delta \sigma) \left[ b'_s + b''_s \Delta \sigma \right] \exp \left[ i \phi_3(\Delta \sigma) \right]
\]

\[
= e^{i \frac{2\pi}{3} \Theta} L \left( A_+ \frac{b'_s}{(|k|L)^{1/3}} + iB_+ \frac{Lb''_s}{(|k|L)^{2/3}} \right)
\]

where

\[
A_+ = \frac{(2\pi)^{2/3}}{3\Gamma(2/3)} \Theta \quad \text{and} \quad B_+ = \frac{\Gamma(2/3)}{\sqrt{3}} \frac{\Theta^2}{(2\pi)^{2/3}},
\]

and \( \Theta = (6/\pi \beta_s^2)^{1/3} \) was defined in the paragraph above.

The left-hand side vanishes in the relativistic limit, and the bound becomes saturated as the momentum is lowered. Approximating \( \omega \approx |k| + m^2/2|k| \) we obtain a lower bound on the momentum [22]

\[
\psi m \sqrt{mL} < |k| \quad \text{with} \quad \psi = \left( \frac{\Theta}{8\pi} \right)^{3/4} = \frac{3^{3/4}}{4\pi \sqrt{\beta_s}}.
\]

We can also translate \( \Delta \sigma_{\text{max}} \) into an upper bound on the angle between \( k \) and \( b'_s \):

\[
\theta_{\text{max}} = \frac{\Delta \sigma_{\text{max}}}{L} = \Theta (|k|L)^{-1/3}.
\]

For the \( \mathcal{I}_- \) integral, the analysis is similar, but the saddle point criterion is replaced with \( k \cdot a'_s = -\omega - k \cdot a'_s = 0 \) implying that \( k = -|k|a'_s \) at the quasi-saddle point. Consequently, in the equations analogous to eq. (C.10) all the signs on the right-hand side are flipped. The results for both integrals can be summarized as

\[
\mathcal{I}_+ \approx A_+ \frac{Lb'_s}{(|k|L)^{1/3}} + iB_+ \frac{L^2b''_s}{(|k|L)^{2/3}}, \quad \psi m \sqrt{mL} < |k|, \quad \theta_{kl'_s} < \Theta (|k|L)^{-1/3}
\]

\[
\mathcal{I}_- \approx A_- \frac{L \alpha'_s}{(|k|L)^{1/3}} + iB_- \frac{L^2 \alpha''_s}{(|k|L)^{2/3}}, \quad \psi m \sqrt{mL} < |k|, \quad \theta_{ka'_s} < \Theta (|k|L)^{-1/3}
\]

where \( \theta_{kl'_s} \) and \( \theta_{ka'_s} \) are the angles between \( k \) and \( b'_s \) or \( a'_s \), respectively. The dimensionless parameters are defined as

\[
A_\pm = \frac{2\pi}{3\Gamma(2/3)} \left( \frac{3}{2 \pi^2} \right)^{1/3} \pm \frac{\Gamma(2/3)}{\sqrt{3}} \left( \frac{3}{2 \pi^2} \right)^{2/3} \quad \text{and} \quad B_\pm = \pm \Gamma(2/3) \left( \frac{3}{2 \pi^2} \right)^{2/3} \frac{\gamma_+ = \beta_s}{\gamma_- = \alpha_s},
\]

and the dimensionless shape parameters, \( \beta_s \) and \( \alpha_s \), are defined as in eq. (C.11). For shorter wavelength radiation, \( |k| < \psi m \sqrt{mL} \), there is no saddle point, and the integral vanishes rapidly. Also note that the approximations to \( \mathcal{I}_\pm \) in eq. (C.20) satisfy the identities in eq. (C.4) for \( k = |k|c'_s \) up to \( O(m^2/|k|^2) \) terms.
C.2 Discontinuity integral

In this appendix we will evaluate the integrals in eq. (C.3) for the case in which the gradient of the string worldsheet, $\partial_\sigma X^\mu$, has a discontinuity [39]. We first suppose that $b^\mu(\sigma_+)$ has $N_k$ discontinuities corresponding to $N_k$ kinks on the string loop. The typical distance between the kinks will be $D = L/N_k$. To calculate the contribution to $I_+$ coming from a single discontinuity located at $\sigma_+ = \sigma_d$ we parametrize

$$b'^\mu(\sigma_+) = \begin{cases} b'^\mu_+ = \{1, \hat{m}_+\} & 0 < \sigma_+ - \sigma_d < D/2 \\ b'^\mu_- = \{1, \hat{m}_-\} & -D/2 < \sigma_+ - \sigma_d < 0 \end{cases}$$

where $\hat{m}_\pm$ are unit vectors and $b_\pm = \{\sigma_+, (\sigma_+ - \sigma_d) \hat{m}_\pm\}$. Inserting eq. (C.22) into eq. (C.3) we approximate the worldsheet integral as

$$I_+ \approx \int_{-D/2}^{D/2} d\sigma_+ b'(\sigma_+) e^{ik'\sigma_+ / 2} \approx \left[ \frac{2}{\omega} \left( \frac{b'_+}{k \cdot b'_+} - \frac{b'_-}{k \cdot b'_-} \right) \right. - \left. \frac{2}{\omega} \left( \frac{b'_+}{k \cdot b'_+} e^{i(k \cdot b'_+) \omega D} - \frac{b'_-}{k \cdot b'_-} e^{-i(k \cdot b'_-) \omega D} \right) \right] e^{i(\omega \sigma_d + \pi)}$$

where $\hat{k}' = k' / \omega = \{1, k/\omega\}$. Upon integrating over the entire loop, the second term cancels among the contributions from different discontinuities (summing all kinks). Then we can drop both this second term and the overall phase to write the contribution from a single discontinuity as

$$I_+ \approx \frac{1}{\omega} \left( \beta_+ b'_+ - \beta_- b'_- \right)$$

with $\beta_\pm \equiv 2/(\hat{k} \cdot b'_\pm) = 2/(1 - \hat{k} \cdot \hat{m}_\pm)$.

To calculate the integral $I_-$ we parametrize $a'(\sigma_-)$ in terms of $a'_\pm$ in analogy with eq. (C.22). We can summarize the results of both calculations as follows

$$I_+(k) \approx \frac{1}{\omega} \left( \beta_+ b'_+ - \beta_- b'_- \right), \quad \beta_\pm = \frac{2}{k \cdot b'_\pm}, \quad L^{-1} < \omega$$

$$I_-(k) \approx \frac{1}{\omega} \left( \alpha_+ a'_+ - \alpha_- a'_- \right), \quad \alpha_\pm = \frac{2}{k \cdot a'_\pm}, \quad L^{-1} < \omega \ .$$

The dimensionless coefficients are bounded as $1 \leq \beta_\pm, \alpha_\pm$. In the limit that $k$ coincides with one of the discontinuity vectors, $b'_\pm$ or $a'_\pm$, one finds that $\beta_\pm$ or $\alpha_\pm \to \infty$. This apparent divergence is an artifact of neglecting the second set of terms in eq. (C.23), and upon retaining these terms one can see that $I_+ \sim D \ll 1/\omega$ in the limit that $(\hat{k} \cdot b'_+) \omega D \ll 1$. Therefore we must restrict ourselves to the regime $\omega > D^{-1} \sim N_k L^{-1}$ and where $k \cdot b'_\pm$ is away from zero; it follows that $1 \leq \beta_\pm, \alpha_\pm \lesssim$ few. To properly treat the case $k \cdot b'_+ = 0$ in which the phase is stationary, one should use the saddle point approximation, as described in section C.1.

D Scalar and tensor integrals for cusps, kinks, and kink collisions

Here we evaluate the scalar and tensor integrals, $I$ and $I^{\mu\nu}$ given by eqs. (C.5) and (C.6), for the cusp, kink, and kink-kink collision string configurations. For the scalar integral, we will...
only be interested in the modulus $|I|^2$. For the tensor integral, we will only be interested in the (positive, real) scalar combinations

$$\Pi(q) = -g_{\nu\beta} \frac{q_\mu q_\alpha}{q^2} I^{\mu\nu}(q) I^{\alpha\beta}(q)^* \quad (D.1)$$

$$\Upsilon(k, \bar{k}) = \frac{2k_\nu k_\mu}{(k + \bar{k})^2} I^{\nu\mu}(k + \bar{k}) \quad (D.2)$$

which were originally defined in eqs. (B.22) and (B.39).

We can simplify the expression for $\Pi(q)$ as follows. Using eq. (C.6) and the identity

$$(-g_{\nu\beta})^{\mu\nu \rho \sigma} = g^{\mu\alpha} g^{\nu\beta} \delta_{\rho \sigma} + g^{\mu\rho} g^{\nu\sigma} \delta_{\alpha \beta} + g^{\mu\sigma} g^{\nu\alpha} \delta_{\rho \beta} - g^{\mu\rho} g^{\nu\alpha} \delta_{\delta \alpha}$$

we can write $\Pi$ as

$$\Pi(q) = -\frac{1}{4} [(I_{+} \cdot I_{+})(I_{-} \cdot I_{-}) - |I_{+} \cdot I_{-}|^2] + \frac{1}{2} \text{Re} \left[(q \cdot I_{+})(q \cdot I_{-})(I_{+} \cdot I_{-})\right]$$

$$- \frac{1}{4} \frac{1}{q^2} \left[(q \cdot I_{+})(q \cdot I_{+})(I_{-} \cdot I_{-}) + (q \cdot I_{-})(q \cdot I_{-})(I_{+} \cdot I_{+})\right]. \quad (D.4)$$

Furthermore, from the periodicity of the string worldsheet, we have the identity $q \cdot I_{\pm}(q) = 0$ [see eq. (C.4)]. Making this simplification we finally obtain

$$\Pi(q) = -\frac{1}{4} [(I_{+} \cdot I_{+})(I_{-} \cdot I_{-}) - |I_{+} \cdot I_{-}|^2] \cdot (D.5)$$

We can simplify the expression for $\Upsilon(k, \bar{k})$ as follows. Let $q = k + \bar{k}$ and $p = k - \bar{k}$. Using eq. (C.6) we can express $\Upsilon$ as

$$2\Upsilon(k, \bar{k}) q^2 = p_\nu q_\nu I_{+\alpha}(q) I_{-\beta}(q) \epsilon^{\mu\nu \alpha \beta}$$

$$= -p^0 q \cdot (I_{+} \times I_{-}) + q^0 p \cdot (I_{+} \times I_{-})$$

$$- I_{+} \cdot p \cdot (q \times I_{-}) + I_0 \cdot p \cdot (q \times I_{+}) \quad. \quad (D.6)$$

Using the identities $q \cdot p = q \cdot I_{+} = q \cdot I_{-} = 0$, this can also be written as [35]

$$\Upsilon(k, \bar{k}) = \frac{1}{2q^0} p \cdot (I_{+} \times I_{-}) \quad. \quad (D.7)$$

To compare $\Upsilon$ with $\Pi$, it is convenient to move to the frame in which $q^\mu = \{q^0, \mathbf{0}\}$. Then the identities $q \cdot p = q \cdot I_{+} = q \cdot I_{-} = 0$ require $p$, $I_{+}$, and $I_{-}$ to have vanishing time-like components. In this frame, we can write

$$\Pi(q) = \left|\frac{I_{+}}{|I_{-}|} \right| \sin^2(\theta_{+\pm}) \quad (D.8)$$

where $\theta_{+\pm}$ is the angle between $I_{+}$ and $I_{-}$. Further denoting $\theta_{p+\pm}$ as the angle between $p$ and $I_{+} \times I_{-}$ we have

$$|\Upsilon(k, \bar{k})|^2 = \left[\left(\frac{|p|}{q^0}\right)^2 \cos^2(\theta_{p+\pm})\right] \frac{|I_{+}|^2 |I_{-}|^2}{4} \sin^2(\theta_{+\pm}) \quad. \quad (D.9)$$
The two expressions are related by

\[ |\Upsilon(k, \bar{k})|^2 = \left[ \frac{(|k - \bar{k}|)^2 \cos^2(\theta_{p+})}{\omega + \bar{\omega}} \right] \Pi(q) \]  

(D.10)

where the quantity is square brackets is always \( \leq 1 \). The inequality is saturated when \( p \) is aligned with \( \mathcal{I}_+ \times \mathcal{I}_- \) (i.e., \( \theta_{p+} \approx 0 \)) and either \( |k| \gg |\bar{k}| \) or \( |k| \gg |\bar{k}| \).

D.1 Scalar integral — cusp

A cusp occurs when both integrals \( \mathcal{I}_+(b; q) \) and \( \mathcal{I}_-(a; q) \) have a saddle point at the same value of \( q^\mu \) [see eq. (C.10)]. This requires \( q = |q| b'_c = -|q| a'_c \) or equivalently

\[ a'_c = -b'_c. \]  

(D.11)

The scalar integral, \( \mathcal{I} \) from eq. (C.5), is evaluated using the expressions for \( \mathcal{I}_\pm \) in eq. (C.20). Using the identities from eq. (A.8) most of the four-vector contractions vanish. The surviving term is proportional to \( a''_c \cdot b''_c = (2\pi/L)^2 \alpha_c \beta_c \hat{a}'_c \cdot \hat{b}'_c \) where we have used shape shape parameters, introduced in eq. (C.11). Then, the squared integral evaluates to

\[ |\mathcal{I}^{(\text{cusp})}(q)|^2 = S^{(\text{cusp})} \frac{L^{4/3}}{|q|^{8/3}}, \quad \psi m\sqrt{mL} < |q|, \quad \theta < \Theta (|q|L)^{-1/3} \]  

(10.1)

where \( S^{(\text{cusp})} = \pi^4 \alpha_c^2 \beta_c^2 B^2 \bar{B}^2 \cos^2 \theta_{ab} \) with \( B_\pm \) defined in eq. (C.21), and where \( \theta_{ab} \) is the angle between \( a''_c \) and \( b''_c \). The angle between \( k \) and \( b'_c = -a'_c \) is bounded above by the saddle point criterion, and therefore \( k \) falls within a cone of opening angle \( \Theta(|q|L)^{-1/3} \) centered on the cusp.

The dimensionless prefactor, \( S^{(\text{cusp})} \), may be estimated using the expressions for \( B_\pm \) in eq. (C.21). The shape parameters, \( \alpha_c \) and \( \beta_c \), are expected to be \( O(1) \), but their precise values cannot be determined without greater knowledge of the nature of the cusp. In order to track how this uncertainty in the magnitude of the shape parameter feeds into the particle production calculation, we will consider a fiducial range of values for \( \alpha_c \) and \( \beta_c \). Estimating \( 1/5 \lesssim \alpha_c, \beta_c \lesssim 5 \) and \( \cos \theta_{ab} \approx 1 \) we find \( 0.2 \lesssim S^{(\text{cusp})} \lesssim 10 \). The dimensionless parameters \( \psi \) and \( \Theta \), given by eqs. (C.18) and (C.19), are less sensitive to the uncertainty in the shape parameters. Typically \( \psi \approx 0.01 \) and \( \Theta \approx 0.1 \).

D.2 Scalar integral — kink

A kink occurs when the derivative of one of the functions \( b^\mu(\sigma_+) \) or \( a^\mu(\sigma_-) \) appearing in \( \mathcal{I}_+(b; q) \) or \( \mathcal{I}_-(a; q) \) has a discontinuity, and the other integral has a saddle point. For the sake of discussion we suppose that \( \mathcal{I}_+ \) contains the saddle point and \( \mathcal{I}_- \) the discontinuity. We calculate \( \mathcal{I} \) by inserting eqs. (C.20) and (C.25) in eq. (C.5). From eq. (C.20) we see that the leading order term in \( \mathcal{I}_+ \) is proportional to \( b'_c \) and the subleading term is proportional to \( b''_c \). Upon contracting with \( \mathcal{I}_- \) the leading order term is negligible: we have the identity \( q \cdot \mathcal{I}_- = 0 \) [eq. (C.4)] and the saddle point criterion \( q = |q| b'_c \) from which it follows that \( b'_c \cdot \mathcal{I}_- = -(q^0/|q| - 1) L_0^0 \approx -(m^2/2|q|^2) L_0^0 \), which is negligible (at \( |q| > m\sqrt{mL} \)) compared to the terms that we keep.\(^4\)

\(^4\)The result of ref. [29] is derived using this “leading order” term, \( \mathcal{I}_+ \sim b'_c \).
The calculation described above yields

$$I^{(\text{kink})}(q) = -i \frac{B_+ L^{4/3}}{4|q|^{5/3}} [\alpha_+ (b_+'' \cdot a_+') - \alpha_- (b_-'' \cdot a_-')]$$  \hfill (D.13)

Here we have used $q^0 \approx |q|$ since the saddle point condition requires $m\sqrt{mL} < |q|$ and $mL \gg 1$ for typical size loops. For the same reason, the bound on the discontinuity integral, $L^{-1} < |q|$, is subsumed by the bound on the saddle point integral, $m\sqrt{mL} < |q|$. The squared integral becomes

$$|I^{(\text{kink})}(q)|^2 = S^{(\text{kink})} \frac{L^{2/3}}{|q|^{10/3}}, \quad \psi m\sqrt{mL} < |q|, \quad \theta < \Theta(|q|L)^{-1/3} \text{ (band)}$$  \hfill (D.14)

where $S^{(\text{kink})} \equiv \frac{(2\pi)^2}{16} B_+^2 \beta_s^2 (\alpha_+ (b_+'' \cdot a_+') - \alpha_- (b_-'' \cdot a_-'))^2$ and we have used the shape parameters, introduced in eq. (C.11). The saddle point criterion requires $q$ to be aligned with $b'_s$. Consequently, the radiation is emitted into a band (whose orientation is determined by $b'_s(\sigma_+)$) of angular width $\Theta(|q|L)^{-1/3}$ and angular length $\sim 2\pi$.

We can estimate a range of uncertainty for $S^{(\text{kink})}$ as we did in appendix D.1. Recall that $B_+$ was given by eq. (C.21). Following the convention established in appendix D.1, we estimate the shape parameter as $1/5 \lesssim \beta_s \lesssim 5$. We also take $1 \lesssim \alpha_\pm \lesssim 5$, as per the discussion below eq. (C.25). Together this lets us estimate $0.1 \lesssim S^{(\text{kink})} \lesssim 20$.

### D.3 Scalar integral — kink collision

For the case of a kink-kink collision both integrals, $I_+$ and $I_-$, have discontinuities and are given by eq. (C.25). The scalar integral is evaluated from eq. (C.5) to be

$$I^{(\text{k}-\text{k})}(q) = -\frac{1}{4\omega} \left[ (b_+'' \cdot a_+')(\beta_+ \alpha_+) - (b_+'' \cdot a_-')(\beta_- \alpha_-) \right. $$

$$\left. - (b_-'' \cdot a_+')(\beta_- \alpha_+) + (b_-'' \cdot a_-')(\beta_- \alpha_-) \right]$$  \hfill (D.15)

where $\omega = q^0$. The square is

$$|I^{(\text{k}-\text{k})}|^2 = \frac{S^{(\text{k}-\text{k})}}{\omega^4}, \quad L^{-1} < \omega.$$  \hfill (D.16)

We have defined $S^{(\text{k}-\text{k})} \equiv \frac{1}{16} \left[ \sum \pm (1 + b_\pm'' \cdot a_\pm') \beta_\pm \alpha_\pm \right]^2$ where the sum runs over all possible combinations of $+$ and $-$ as given by eq. (D.15). For the case of a discontinuity, the worldsheet integrals, $I_{\pm}$, are insensitive to the orientation of $k$ (see section C.2) and the corresponding radiation is emitted approximately isotropically.

Recall that $\alpha_\pm$ and $\beta_\pm$ were given by eq. (C.25), and following the conventions established in appendix D.2, we estimate $1 \lesssim \beta_\pm, \alpha_\pm \lesssim 5$. This yields the estimate $1 \lesssim S^{(\text{k}-\text{k})} \lesssim 500$.

### D.4 Tensor integral — cusp

If both $I_\pm$ contain a saddle point, then we evaluate the tensor integral by inserting eq. (C.20) into eq. (D.5). After making use of the identities in eq. (A.8), many of the terms vanish leaving only

$$\Pi(q) = \frac{B^2 \cdot B^2 \cdot L^8}{4(|q|L)^{8/3}} \left[ (b_\pm'' \cdot b_\pm')(a_\pm'' \cdot a_\pm') - (a_\pm'' \cdot b_\pm')^2 \right]$$  \hfill (D.17)
We extract the factors of $L$ from the bracketed quantities by using the parametrization in eq. \eqref{eq:shape_param}. Doing so gives

$$
\Pi(q) = \mathcal{T} \frac{L^{4/3}}{|q|^{10/3}}, \quad \psi m \sqrt{mL} < |q|, \quad \theta < \Theta (|q|/L)^{-1/3} \quad (\text{cone}) \quad (D.18)
$$

where $\mathcal{T} \equiv \frac{(2\pi)^4}{2} \beta_s \left[ \alpha_+ a_+ b_s \left( 1 - a'_+ \cdot a'_- \right) + \left( b'_s \cdot a'_+ \right) \right]$. The angle between $q$ and $b'_c = -a'_c$ is bounded above by $\Theta(|q|/L)^{-1/3}$, and consequently $q$ is oriented within a cone centered at the cusp.

We can estimate the dimensionless coefficient by making the same estimates as in appendix \ref{sec:shape_param}. Assuming that the shape parameters fall into the range $1/5 < \alpha_c, \beta_c < 5$ and approximating $(1 - \cos^2 \theta_{ab}) \approx 1$ we obtain $0.5 \lesssim \mathcal{T} \lesssim 50$.

\subsection*{D.5 Tensor integral — kink}

If $\mathcal{I}_+$ contains a saddle point and $\mathcal{I}_-$ contains a discontinuity, then we evaluate the tensor integral by inserting eqs. \eqref{eq:shape_param} into eq. \eqref{eq:tensor_integral}. Some of the contractions vanish upon using the identities in eq. \eqref{eq:identities}. As we discussed in section \ref{sec:kink}, the leading order term in $\mathcal{I}_+$ is negligible because the contraction $b'_s \cdot \mathcal{I}_-$ is suppressed by $m^2/|q|^2 \ll 1$. Making these substitutions we are left with

$$
\Pi(q) = \frac{1}{4} \left[ \frac{B_+^2 L^4 (b'_s \cdot b'_s) \left( 1 - a'_+ \cdot a'_- \right)}{|q|^2} - \frac{1}{|q|^2} \frac{B_+^2 L^2 (b'_s \cdot A')^2}{(|q|/L)^{10/3}} \right] \quad (D.19)
$$

where $A' = \alpha_+ a'_+ - \alpha_- a'_-$. We relate $b''_s$ to $b_s$ using the parametrization in eq. \eqref{eq:shape_param}. Then

$$
\Pi(q) = \mathcal{T} \frac{L^{2/3}}{|q|^{10/3}}, \quad \psi m \sqrt{mL} < |q|, \quad \theta < \Theta (|q|/L)^{-1/3} \quad (\text{band}) \quad (D.20)
$$

where

$$
\mathcal{T} \equiv \frac{(2\pi)^2}{2} B_+^2 \beta_s \left[ \alpha_+ a_+ \left( 1 - a'_+ \cdot a'_- \right) + \left( b'_s \cdot a'_+ \right) \right]
$$

$$
- \frac{\alpha_+^2}{2} \left( b'_s \cdot a'_+ \right)^2 - \frac{\alpha_-^2}{2} \left( b'_s \cdot a'_- \right)^2 \quad (D.21)
$$

The momentum $k$ is constrained to fall within a band of angular width $\Theta(|k|L)^{-1/3}$.

Following the conventions from the previous sections, we estimate $1/5 \lesssim \beta_s \lesssim 5$ and determine $B_+$ from eq. \eqref{eq:B_plus}. We estimate the parenthetical factor as simply $|\alpha_+ \alpha_-|$ and take $1 \lesssim \alpha_+ \lesssim 5$ as before. Then together we find $1 \lesssim \mathcal{T} \lesssim 200$.

\subsection*{D.6 Tensor integral — kink collision}

If both $\mathcal{I}_\pm$ possess a discontinuity point, then we evaluate the tensor integral by inserting eq. \eqref{eq:shape_param} into eq. \eqref{eq:tensor_integral}. This gives

$$
\Pi(q) = \frac{1}{4(|q|^2)^4} \left[ (B' \cdot B')(A' \cdot A') - (B' \cdot A')^2 \right] \quad (D.22)
$$
where $B' \equiv \beta_+ b'_+ - \beta_- b'_- \text{ and } A' \equiv \alpha_+ a'_+ - \alpha_- a'_-$. This can be written as

$$\Pi(k)\big|^{(k-k)} = T^{(k-k)} \frac{1}{(q^0)^4}, \quad L^{-1} < q^0$$

(D.23)

where $T^{(k-k)} \equiv [\beta_+ \beta_- \alpha_+ \alpha_- (b'_+ \cdot b'_-)(a'_+ \cdot a'_-) - \frac{1}{4}((\beta_+ b'_+ - \beta_- b'_-) \cdot (\alpha_+ a'_+ - \alpha_- a'_-))^2]$, and we have used that $a'_+$ and $b'_+$ are null vectors.

We can estimate $T^{(k-k)}$ following the conventions established in appendix D.3. We take $1 \lesssim \alpha_\pm, \beta_\pm \lesssim 5$ and approximate $b'_+ \cdot b'_- \approx a'_+ \cdot a'_- \approx b'_+ \cdot a'_- \approx 1$. This allows us to estimate the range $0.2 \lesssim T^{(k-k)} \lesssim 200$ for the dimensionless coefficient.

E  Cusp boost factor and UV sensitivity

It was recognized in ref. [40] (see also [25]) that particle radiation from a cusp will be highly boosted since the cusp tip moves at the speed of light in the rest frame of the loop. (By contrast, the gravitational radiation spectrum is IR sensitive, and the boost factor is not relevant.) At a given point on the string loop, the boost factor is given by

$$\gamma_{\text{boost}}(\tau, \sigma) = \frac{1}{\sqrt{\mathcal{X}^\mu(\tau, \sigma) \mathcal{X}_\mu(\tau, \sigma)}} = \sqrt{\frac{-2}{a^\prime(\sigma - \tau) \cdot b^\prime(\sigma + \tau)}}$$

(E.1)

where we have used the formulae in appendix A. Expanding both $a^\mu$ and $b^\mu$ as in eq. (C.7) and using eq. (A.8) gives

$$\gamma_{\text{boost}}(\Delta \sigma) = \frac{(L/\Delta \sigma)}{\pi \sqrt{\alpha_s^2 + \beta_s^2}}$$

(E.2)

where $\Delta \sigma$ is the distance from the tip of the cusp. The dimensionless shape parameters, $\beta_s$ and $\alpha_s$, were defined in eq. (C.11). The boost factor grows with decreasing $\Delta \sigma$ as one investigates radiation coming from closer and closer to the tip of the cusp. For a ideal string of zero thickness, we can take $\Delta \sigma \to 0$ and $\gamma_{\text{boost}} \to \infty$. In reality, the finite thickness string overlaps with itself at the cusp tip, and a segment of string with length $\Delta \sigma_{\text{min}} \sim \sqrt{L/M}$ will evaporate into particle radiation [40, 41]. This leads to an upper bound on the boost factor,

$$\gamma_{\text{boost}} \lesssim \frac{\sqrt{ML}}{\pi \sqrt{\alpha_s^2 + \beta_s^2}}$$

(E.3)

The radiation spectra that we calculate should drop off when the momentum of the radiated particle exceeds the inverse string thickness. In the rest frame of the radiating string segment this condition is $|\mathbf{k}|_{\text{cusp-frame}} < M$, but in the rest frame of the loop this condition is $|\mathbf{k}|_{\text{loop-frame}} < M \gamma_{\text{boost}}$. For radiation from a cusp this becomes $|\mathbf{k}|_{\text{loop-frame}} < M \sqrt{ML}$ whereas for radiation from a (non-relativistic) kink this becomes $|\mathbf{k}|_{\text{loop-frame}} < M$.

Since $|\mathbf{k}|$ exceeds $M$ for radiation from a cusp, one may worry that the effective field theory assumption has broken down. Upon “integrating out” the heavy string degrees of freedom, $S$ and $X^\mu$, we dropped the infinite tower of higher-order, non-renormalizable interactions between the light SM fields and the string worldsheet. For example, in eq. (B.29) we consider the non-renormalizable interaction of SM fermions with the string worldsheet, but we do not treat higher-order operators such as $\frac{1}{M^4} [\bar{\Psi}(x)\Psi(x)]^2 \int d^2 \sigma \sqrt{-\gamma} \delta^{(4)}(x - \mathcal{X}).$
These operators will also contribute to the radiation spectrum, but their contribution will be proportional to some power of the ratio \( k^2/M^2 = \bar{k}^2/M^2 = m^2/M^2 \) or \((k \cdot \bar{k})/M^2\). Then as long as these ratios are small compared to one, we can neglect the higher order operators. Throughout the paper, we have assumed that \( m_H \sim m_Z \sim m_{\psi} \ll M^2 \), and the first condition is satisfied. The second ratio can be written as \( k \cdot \bar{k}/M^2 \approx (|\mathbf{k}|/M^2)(1 - \cos \theta_{kk}) \) where \( \theta_{kk} \) is the angle between \( \mathbf{k} \) and \( \bar{k} \). Assuming \( |\mathbf{k}| \approx |\bar{k}| \) and \( \theta_{kk} \ll 1 \) this bound becomes \( |\mathbf{k}|\theta_{kk} \ll M \). For isotropic radiation, as in the case of a kink, we need \( |\mathbf{k}| \ll M \). On the other hand, if the radiation is beamed into a cone, then we can have \( |\mathbf{k}| \gg M \) (in the rest frame of the loop) without invalidating the effective field theory analysis. Specifically, for radiation from a cusp we have \( \theta_{kk} < \Theta(|\mathbf{k}|L)^{-1/3} \) and \( |\mathbf{k}| < M\sqrt{ML} \), which together satisfy \( |\mathbf{k}|\theta_{kk} < M \).

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