The finite-state character of physical dynamics

Norman Margolus
Massachusetts Institute of Technology, Cambridge MA 02139. nhm@mit.edu

Finite physical systems have only a finite amount of distinct state. This finiteness is fundamental in statistical mechanics, where the maximum number of distinct states compatible with macroscopic constraints defines entropy. Here we show that finiteness of distinct state is similarly fundamental in ordinary mechanics: energy and momentum are defined by the maximum number of distinct states possible in a given time or distance. More generally, any moment of energy or momentum bounds distinct states in time or space. These results generalise both the Nyquist bandwidth-bound on distinct values in classical signals, and quantum uncertainty bounds. The new certainty bounds are achieved by finite-bandwidth evolutions in which time and space are effectively discrete, including quantum evolutions that are effectively classical. Since energy and momentum count distinct states, they are defined in classical finite-state dynamics, and they relate classical relativity to finite-state evolution.

We live in a world that, like a digital photograph, has only finite resolution. This was first recognised in statistical mechanics, when Planck introduced a finite grain-size \( \hbar \) to get a realistic counting of distinct states.\(^1\) Once it was understood that \( \hbar \) relates all energy and momentum to waves, finite resolution was explained as a property of waves: a tradeoff between range of frequencies superposed, and maximum localisation in time or space.\(^2\)\(^3\)

There is also a tradeoff, in superpositions of waves, between frequency range and average localisation. This is known in communications theory as the Nyquist bound: a finite bandwidth signal can carry only a finite number of distinct values per unit length. This holds because a finite number of Fourier components can add up to chosen values at only a finite number of places.\(^2\)\(^3\)

In this paper, we combine and generalise these tradeoffs. We count how many quantum states can be distinguished from each other with certainty, in a finite time or distance, given average constraints on wavefunction bandwidth. These certainty bounds redefine energy and momentum as maximum counts, and challenge the distinction between continuous and discrete in physics.

To illustrate the connection between bandwidth and distinct quantum states, consider a free particle moving in one dimension, in a periodic space of length \( L \). Momentum eigenstates must have a whole number of oscillations in period \( L \), so allowed spatial frequencies \( p_n / \hbar \) are \( 1/L \) apart. A superposition using \( N \) different spatial frequencies must have at least \( N - 1 \) times this minimal separation, between minimum and maximum frequencies:

\[
\frac{p_{\text{max}} - p_{\text{min}}}{\hbar} \geq \frac{N - 1}{L} . \tag{1}
\]

This is a bandwidth bound for a superposition of \( N \) distinct momentum states. For large \( N \), possible positions and momenta define a region with at least area \( \hbar \) per distinct state. The number of distinct states that can occur as the particle moves is also bounded by \( \lceil \frac{L}{\hbar} \rceil \), for this cannot exceed the number of energy-momentum eigenstates in the superposition. Thus if \( \lambda = L/N \), the average separation in space between \( N \geq 2 \) distinct states of the motion, \( (p_{\text{max}} - p_{\text{min}})\lambda \geq \hbar / 2 \). Similar arguments apply to evolution periodic in time rather than space.\(^2\)\(^3\)

More generally, any absolute moment of energy or momentum is an average measure of the frequency-width of the wavefunction, and can play the role that momentum-bandwidth does in a definite count of distinct states for any portion of any evolution with that moment. For \( N = 2 \) these tradeoffs become minimum uncertainty relations—some previously known.\(^7\)\(^8\)\(^9\)\(^10\)\(^11\)\(^12\)\(^13\)\(^14\)

To achieve the maximum count, an evolution must use a finite range of frequencies. All points in space or time are then determined by a discrete sample of them.\(^20\)\(^31\)

Perhaps the most interesting moment is average energy above the minimum possible.\(^15\) What we call energy classically, counts how many distinct states can occur in a unit of time. How much change. How many computational steps. We can also count just the distinct states due to overall motion, by comparing energy counts in rest and non-rest frames. Surprisingly, motional change is bounded not by the kinetic energy \( E - mc^2 \), but by \( pv \) instead. This difference makes the classical action a count of possible distinct states. It also defines an ideal momentum in finite-state dynamics.\(^20\) Of course, energy also bounds what can be distinguished experimentally. For example, using optics, with \( n \) photons of the same frequency there are at most \( 2n + 1 \) distinct phases within one cycle of oscillation, according to \( \lceil n \rceil \).

Below, we first establish energy bounds on the maximum number of states distinguishable-with-certainty that can occur in a given time. We then establish related certainty bounds on overall motion, and discuss distinctness in classical dynamics. The arguments used are elementary, and the results are verified numerically.

**Distinguishability in time**

For any evolution with period \( T \), passing through \( N \) distinct (mutually orthogonal) states at a constant rate,\(^14\)

\[
\frac{2(E - E_0)}{\hbar} \geq \frac{N - 1}{T} , \tag{2}
\]

where \( E_0 \) is the lowest energy eigenvalue used in constructing the system’s state. The left side is, as in \( (1) \), a measure of the width of an eigenfrequency distribution: twice the average half-width. The right side is, again,
the minimum frequency width for \( N \) distinct states. We show that (2) holds even if the time intervals between distinct states are unconstrained. Letting \( \tau \) be the average time separating consecutive distinct states, (2) becomes

\[
(E - E_0) \tau \geq \frac{N - 1}{N - 2} h \, .
\]

(3)

We show (3) also holds for a portion of an evolution, comprising \( N \) distinct states with average separation \( \tau \). For \( N = 2 \) this becomes the minimum separation bound (14). We provide similar bounds for other moments of energy.

**We formalise our problem as a minimisation.** Consider a finite-sized isolated system with a time evolution expressed as a superposition of energy eigenstates:

\[
|\psi(t)\rangle = \sum_n a_n e^{-2\pi i \nu_n t} |E_n\rangle ,
\]

(4)

with \( \nu_n = E_n / h \). We define a set of average frequency widths (moments) about a frequency \( \alpha \):

\[
\langle \nu - \alpha \rangle_M = \left( \sum_n |a_n|^2 |\nu_n - \alpha|^M \right)^{1/M} ,
\]

(5)

with \( M > 0 \). If evolution (4) passes through a series of mutually orthogonal states \( |\psi(t_k)\rangle \) at times \( t_k \), then

\[
\langle \psi(t_m)|\psi(t_k)\rangle = \sum_n |a_n|^2 e^{2\pi i \nu_n (t_m - t_k)} = \delta_{mk} .
\]

(6)

We seek the minimum frequency widths (5) of states satisfying the constraints (6) for any sequence of \( N \) distinct states within a time interval of length \( T_N \).

We assume, without loss of generality, that all \( \nu_n \) are distinct (in both (5) and (6), coefficients for a repeated \( \nu_n \) can be consolidated), and that overall evolution is periodic with some recurrence-time \( T \) that may be much longer than \( T_N \). Then the discrete spectrum, bounded from below (15) includes at most all of the frequencies

\[
\nu_n = \nu_0 + n/T ,
\]

(7)

with \( n \) a non-negative integer. These are all of the possible eigenfrequencies of energy eigenstates that cycle with period \( T \), up to an overall phase. This spectrum restricts the maximum period to be \( T \), but evolution can repeat more than once in this time. For \( T \) sufficiently large (7) approaches a continuous spectrum, allowing us to minimise over the union of all possible discrete spectra.

**We first consider an evolution with a constant rate of distinct change.** If \( N > 1 \) distinct states have equal separations \( \tau \) within period \( T = N \tau \), then \( t_m = m \tau \) and from (6) and (7),

\[
\langle \psi(t_{k+m})|\psi(t_k)\rangle = e^{2\pi i \nu_0 m} \sum_n |a_n|^2 e^{2\pi i \nu n m/N} .
\]

(8)

There are only \( N \) distinct phases in the sum (8), so we can minimise all \( \langle \nu - \alpha \rangle_M \) for a given \( \alpha \) by using a set of \( N \) consecutive \( \nu_n \)’s, centred as closely as possible on \( \alpha \): we get the same orthogonality times in (8) with smaller width \( |a_n|^2 \) outside the set to 0, and transferring its weight to the equivalent phase within the set. Then, since \( \langle \psi(t_{k+m})|\psi(t_k)\rangle = \delta_{mk} \), the \( N \) consecutive non-zero \( |a_n|^2 \) are just the discrete Fourier transform of a Kronecker delta impulse, and so they all equal 1/N. Thus all \( \langle \nu - \alpha \rangle_M \) are minimised by an equal superposition with minimum bandwidth for \( N \) distinct states (illustrated in Figure 1), so the dimensionless product

\[
\tau \langle \nu - \alpha \rangle_M \geq f_\alpha(M,N)
\]

(9)

for some \( f_\alpha(M,N) \) defined by the minimising state. For example, if \( \alpha = \nu_0 \), the closest to centring a minimum bandwidth state on \( \alpha \) is for \( \nu_0 \) to be the lowest frequency. Then, given (4), equality in (9) requires

\[
f_{\nu_0}(M,N) = N^{-1} \left( \sum_{n=0}^{N-1} n^M \right)^{1/M} .
\]

(10)

For \( M \geq 1 \) this ranges from 1/4 to 1. \( f_{\nu_0}(1,N) \) gives (3).

Similarly, let \( \alpha \) be the midpoint of \( N \) consecutive frequencies \( \nu_n \). Then \( \alpha = \tilde{\nu} \), the mean frequency of the minimising state, and for \( M \geq 1 \) no other \( \alpha \) makes \( \langle \nu - \alpha \rangle_M \) smaller. Thus

\[
f_\nu(M,N) = N^{-1} \left( \sum_{n=0}^{N-1} \left| n - \frac{N-1}{2} \right|^M \right)^{1/M} ,
\]

(11)

which ranges from 2/9 to 1/2. This bounds \( \tau \langle \nu - \alpha \rangle_M \) regardless of \( \alpha \), for \( M \geq 1 \). For \( M < 1 \), letting \( \alpha \) be the \( \nu_n \) nearest the midpoint is optimal. Excluding \( f_\nu \) for \( M < 2 \), both \( f_{\nu_0} \) and \( f_\nu \) strictly increase with \( N \).

For spectra that include \( N \) evenly-spaced energy eigenvalues, these bounds are all achievable. For states that achieve them, continuous evolution is an interpolation of a discrete one (19) given \( \nu_0 \), we can Fourier transform \( N \) distinct \( |\psi(t_k)\rangle \) to recover \( |\psi(t)\rangle \) as a superposition of \( N \) energy eigenstates with time-dependent phases.

FIG. 1. | A periodic evolution with \( N \) distinct states \( \tau \) apart (solid real, dashed imaginary, depicted for \( N = 10 \)). Only a discrete set of frequencies fit the period: all are allowed in the minimisation. An equally weighted superposition \( |\psi(t)\rangle \) of \( N \) consecutive frequencies is the narrowest that gives \( N \) distinct states in time. Centred on \( \alpha \), it minimises all \( \tau \langle \nu - \alpha \rangle_M \).
Now consider an evolution with a constant rate portion. Suppose there are \( N \) distinct states, spaced \( \tau \) apart, within an interval \( T_N \). To find the minimum of \( \tau \langle \nu - \nu_0 \rangle \), we assume evolution outside of \( T_N \) puts no constraints on the minimisation problem: it adds no orthogonality constraints, and the maximum period \( T \) is unbounded so all frequencies are allowed in \( \mathbb{Z} \).

We find, in general, that the optimal evolution containing \( T_N \) repeats with period \( N \tau \), and so the bounds are again \( f_\alpha(M,N) \). We see this in the example of Figure 2, plotting the minimum of \( \tau \langle \nu - \nu_0 \rangle \) with \( N = 5 \) for different \( T \) (see Numerical Methods in online content). The global minimum recurs whenever \( T \) is an integer multiple of \( N \tau \), so the bound for \( T = N \tau \) holds for \( T \to \infty \). An exception occurs when \( f_\alpha \) decreases with \( N \): the period \( N \tau \) is not optimal if a longer period gives a lower bound. This only happens for \( f_\nu \) with \( M < 2 \) and \( N \) even. In these cases, we can determine minimising states and corresponding achievable bounds numerically.

As in the constant rate evolution, it is easy to see that only a finite bandwidth is relevant to the minimisation: since \( t_m = m \tau \), whenever \( T/\tau \) is an integer there are only \( T/\tau \) distinct phases in \( [0,1] \), hence in the limit \( T \to \infty \) only a bandwidth \( 1/\tau \) is relevant. This is slightly larger than the minimum possible bandwidth of \( (N - 1)/N \tau \), achievable only in evolutions that repeat with period \( N \tau \) (otherwise there are too many constraints \( \nu \) to satisfy). For large \( N \), the difference of \( 1/N \tau \) becomes negligible. For large \( M \), minimum bandwidth minimises \( \tau \langle \nu - \nu_0 \rangle_M \), since \( \tau \langle \nu - \nu_0 \rangle_M \) is the (dimensionless) bandwidth. For small \( M \) and \( N \), we surveyed ten thousand cases numerically (some illustrated in Extended Data Figures 1–4).

In all cases the shortest constant rate period was optimal, except for \( f_\nu \) with \( M < 2 \) and \( N \) even.

Finally, if equal separation is optimal, constant rate bounds hold with \( \tau \) the average separation. We expect equal separations within \( T_N \) to minimise \( \langle \nu - \alpha \rangle_M \), since this puts the fewest constraints \( \nu \) on the minimisation problem. Equality is also required for minimum bandwidth, hence at least minimises \( \langle \nu - \alpha \rangle_M \).

Of course equal separation is optimal for \( N = 2 \), since there is only one separation. Then \( f_\alpha(M,2) \) bounds the minimum separation time \( \tau_{\text{min}} \) for \( \alpha = \nu_0 \) and \( M > 0 \), and for \( \alpha = \bar{\nu} \) and \( M \geq 2 \):

\[
\tau_{\text{min}} \langle \nu - \alpha \rangle_M \geq f_\alpha(M,2) .
\]

This agrees with known \( \nu_0 \) bounds \( \langle \nu - \alpha \rangle_M \) for \( M > 0 \), and with the Mandelstam-Tamm \( \bar{\nu} \) bound \( \langle \nu - \alpha \rangle_M \). Similar bounds hold for other cases, and can be determined numerically.

Now, each bound \( \langle \nu - \alpha \rangle_M \) applies to any evolution, allowing a smallest frequency width when the minimum separation between any two distinct states is as large as possible. For any \( T_N \) this occurs when the interval is divided up evenly, since not all separations can be above average. Similar arguments apply to the minimum time to go through a sequence of \( N' < N \) distinct states, since the smallest \( \langle \nu - \alpha \rangle_M \) must scale inversely with that as well. Equal times are optimal at all scales \( \langle \nu - \alpha \rangle_M \).

Figure 3 shows a numerical test that finds the minimum average energy for 12,000 different sets of separations, verifying that for a periodic evolution with period \( T \), the bound \( \langle E - E_{\text{bound}} \rangle \) is only achievable if the number \( N_{\text{different}} \) of different separation lengths in the evolution is one. The dashed line is approached by almost-equal separations. As long as the separations aren’t exactly equal the bound is altered by a discrete jump for each additional length: requiring \( \langle \psi(t)\psi(0) \rangle = 0 \) for times arbitrarily close to equal separation essentially adds a slope \( = 0 \) constraint at equal separations (which results in the dashed line), so average energy is greater. Other moments, and portions of evolutions, behave similarly (see Extended Data Figures 5–8).
Distinguishability in space

For an isolated system in motion, some distinct states can be attributed to the motion. We can determine how many by comparing with the same evolution seen in its rest frame: any extra distinct states when moving must be due to the motion. Energy bounds the number of distinct states in each frame, yielding a bound on motion:

\[ p \lambda \geq \frac{N - 1}{N} \frac{h}{2} . \]  

(13)

Here \( p \) is the magnitude of the system’s average momentum, and \( \lambda \) is the average separation in space within a sequence of \( N \) states that are distinct due to the motion. Similar bounds hold for other moments of momentum.

We first count macroscopically, in two frames. Assuming \( N \gg 1 \), set \( h = 2 \) and take the energy of flat empty space to be \( E_0 = 0 \) in both frames.\(^{[10,16]} \) Then (2) becomes \( 1/\tau \leq E \), and energy is the maximum average rate of distinct state change physically possible.

In the laboratory frame, in a time interval \( \Delta t \), an isolated system evolves through at most \( E \Delta t \) distinct states. Meanwhile, moving at speed \( v \), it travels a distance \( \Delta x = v \Delta t \). In the corresponding rest frame evolution, at most \( E_\tau \Delta t_\tau \) states are distinct. The difference, which is a familiar relativistic quantity

\[ E \Delta t - E_\tau \Delta t_\tau = p \Delta x , \]  

(14)

counts the extra distinct states possible in the frame where there is overall motion (Figure 4). Thus \( p \) is the extra per unit distance, agreeing with \(^{[13]}\) for \( N \gg 1 \).

Dividing (13) by \( \Delta t \), we see that \( E - E_\tau /\gamma = vp \) bounds the average rate of motional state change, even at low velocities. This is slightly surprising, since conventionally the smaller quantity \( E - E_\tau \) is taken as the energy of motion. Indeed, if we model the motion of a free particle by treating its rest energy \( E_r \) as its minimum possible energy \( E_0 \), then (3) gives \( E - E_r \) as the maximum average rate of motional state change, for \( N \gg 1 \). In general, though, \( E_r \) is the average energy of a rest frame dynamics, so \( E - E_r \) is the difference of maximum rates in two different frames—which is not a rate in either.

To find precise momentum bounds, consider a massless particle. If all energy in the superposition moves in the same direction, it can all contribute to overall motion. Then \( E = cp \), and since \( \lambda = ct \) is the average distance between distinct states, \(^{[3]}\) becomes (13).

Other energy bounds similarly turn into momentum bounds. If all energy of a massless particle is moving in the +x direction, we can take \( H = cp_\mu \), so \( H\tau = p_\mu \lambda \) in \(^{[9]}\). Letting \( \mu = p_\mu /h \) be the spatial frequency operator along the direction of motion, we get

\[ \lambda \langle \mu - \alpha \rangle_M \geq f_\alpha (M, N) . \]  

(15)

This is the general shift-in-space counterpart of the shift-in-time bound. In effect, we attribute the extra distinctness possible in a non-rest frame to the steady shifting motion of the frame: if the entire dynamics were \( H = vp_\mu \), we would again get (15), with average energy \( vp \).

To bound only non-rest-frame distinctness, we apply (15) to a wavefunction that describes only non-rest-frame dynamics: one that uses a non-negative momentum spectrum along the direction of motion, so no momenta cancel and contribute to rest-frame average energy. This bound is consistent with \(^{[1]}\) with \(^{[13]}\), with Luo’s bound\(^{[10]}\) on \( \langle |p| \rangle \), and with Yu’s bound\(^{[12]}\) of \( \Delta p \lambda_{\min} \geq h/4 \).

Classical distinctness

Although bounds on certainty are usually regarded as quintessentially quantum mechanical, the finite distinctness of finite-energy physical dynamics is evident even in the classical realm.

Macroscopic systems obey the energy and momentum bounds. Unlike typical small systems\(^{[13]}\) macroscopic systems traverse a succession of almost perfectly distinct states as they explore their enormous state spaces: the expected overlap of two randomly chosen \( N \)-dimensional states is \( 1/\sqrt{N} \). We can model finite maximum entropy as classical finite state, and finite energy and momentum as discreteness in time and space\(^{[25,26]}\). Historically, modelling finite entropy has led to classical finite-state models being of fundamental interest in statistical mechanics\(^{[12]}\) but such models have had a lower status in dynamics, as mere computational treatments of the “real” continuum dynamics.\(^{[43,44]}\) An exception has been finite-state lattice models isomorphic to continuum models sampled at integer times\(^{[26,16,10]}\). These are similar to bandlimited quantum models\(^{[13,19]}\).

Since energy and momentum count distinct states, so does classical action: \( pv \) counts possible distinct changes per unit time due to overall motion, so \( -L = H - \sum p_i v_i \) counts possible changes \textit{not} due to particle motion. Moreover, the energy count has a combinatorial character in classical finite-state lattice models: there is a smallest unit of energy, so total energy determines how many places could possibly change in a given time\(^{[13,26]}\).

Classical finite-state systems obey the certainty bounds. From the viewpoint of quantum computation, classical reversible computation is a special case of what a quantum evolution can do\(^{[23]}\). Classical mechanics doesn’t have this status, because it has an infinite rate of distinct
state change. Only classical finite-state dynamics can be recast as finite-energy quantum dynamics, with distinct classical configurations identified with distinct quantum states. If we find the least-energetic realisation mathematically possible, no physical implementation of the finite-state dynamics can do better.

A realistic quantum realisation is constrained both by certainty bounds and by relativity. For example, if a particle in a finite-state model travels at speed $v$ through a long sequence of distinct positions $\lambda$ apart, its minimum possible momentum is $p = h/(2\lambda)$, and the energy required by distinct motion is $pv$. If $v < c$ though, total energy must be larger, since relativistically $E = pv/(v/c)^2 > pv$.

It might seem surprising that it is, in fact, possible to recast a classical finite-state dynamics with perfect locality and determinism, as a quantum Hamiltonian dynamics with continuous space and time. In this case, finite-distinctness is encoded in the finiteness of the energy and momentum of the initial state. The desired finite-state evolution constitutes a finite set of distinct sample values, which are continuously interpolated in space and time. Quantum bounds on certainty simply reflect finite distinctness in a continuous description.

Classical signals obey a version of the bounds. A classical signal is like the wavefunction of a scalar particle evolving under a one-dimensional shift dynamics, $H = v p_x$. Any finite frequency-moment bounds the number $N$ of distinct states in an interval of the quantum evolution, hence at most $N$ points in the interval can have values specified independently, by superposing the distinct states. This generalises the Nyquist rate from a bandwidth bound to an any-frequency-moment bound.

Conclusions

The quantum revolution began with the recognition that a finite thermodynamic system has a finite number of distinguishable states. This finite character is shared by mechanics, and can similarly be captured by counting the maximum number of states distinguishable-with-certainty—in a time evolution. For macroscopic systems, we can equate total relativistic energy with the maximum number of distinct states physically possible in a unit of time. The magnitude of momentum is then the maximum number, per unit distance, allowed by overall motion. If a portion of the dynamics is isolated from the rest, its portion of the energy bounds its distinct state change. These and related certainty-bounds constrain what can be measured. They also redefine basic quantities of classical mechanics as local measures of quantum distinguishability in space and time, and so might help bridge the gap between the classical theory of gravity, where entropy is already a dynamical quantity, and the quantum (cf). They certainly provide a bridge between discrete and continuous models of the physical world.

Online Content

Numerical Methods, along with Extended Data Figures, are available in the online version of the paper.
Margolus, N. The ideal energy of classical lattice dynamics. *Lect. Notes Comput. Sc.* **9099**, 169 (2015).

Succi, S., Benzi, R. Lattice Boltzmann equation for quantum mechanics. *Physica D* **69**, 327 (1993).

Yepez, J. Quantum lattice gas model of Dirac particles in 1+1 dimensions. arXiv:1307.3595v1 (2013).

Yepez, J. Time dilation relation for a Dirac hamiltonian generator on a qubit array. arXiv:1512.02550v2 (2015).

Dalmonte, M., Montangero, S. Lattice gauge theories simulations in the quantum information era. arXiv:1602.03776 (2016).

’t Hooft, G. The cellular automaton interpretation of quantum mechanics. arXiv:1405.1548v3 (2015).

Kok, P., Braunstein, S., Dowling, J. Quantum lithography, entanglement and Heisenberg-limited parameter estimation. *J. Opt. B* **6**, S811 (2004).

Giovannetti, V., Lloyd, S., Maccone, L. Quantum-enhanced measurements. *Science* **306**, 1330 (2004).

Giovannetti, V., Lloyd, S., Maccone, L. Quantum measurement bounds beyond the uncertainty relations. *Phys. Rev. Lett.* **108**, 260405 (2012).

Hall, M. J. W., Berry, D. W., Zwierz, M., Wiseman, H. W. Universality of the Heisenberg limit for estimates of random phase shifts. *Phys. Rev. A* **85**, 041802 (2012).

Zwierz, M., Pérez-Delgado, C. A., Kok, P. Ultimate limits to quantum metrology and the meaning of the Heisenberg limit. *Phys. Rev. A* **85**, 042112 (2012).

Bocchieri, P., Loinger, A. Quantum recurrence theorem. *Phys. Rev.* **107**, 337 (1957).

Verch, R. On generalizations of the spectrum condition.

**Acknowledgements** I thank Gerald Sussman, Jeffrey Yepez, Samuel Braunstein, Lorenzo Maccone, Charles Bennett, Lev Levitin and Tom Toffoli for their comments.
NUMERICAL METHODS

A Mathematica Notebook, which is published online at http://arxiv.org/src/1109.4994 contains code and results for numerical experiments that confirm and extend the energy-bound analysis above, and that generate the graphs in the figures above and below.

The fundamental minimisation problem outlined above requires determination of non-negative coefficients $|a_n|^2$ that minimise (5) while satisfying (4) for a given set of separations in time between distinct states, using the spectrum (7). Both the objective function (5) (raised to the $\tau$) and the constraints (6) are linear combinations of the coefficients, so given a set of separations between distinct states, we can find the global minimum to arbitrary accuracy using linear optimisation (linear programming). We take separations to be integers, allowing us to deal with only a finite number of $|a_n|^2$ in our minimisations: if both the total period $T$ and the time $t$ are integers, then $T$ is the number of distinct phases possible in the constraints (6). Using more than $T$ consecutive $|a_n|^2$ with a given $\alpha$ would increase the frequency moments (5) without allowing any new constraints. Large integer $T$ allows as much resolution in $t/T$ as desired.

In surveying evolutions similar to Figure 2 with a portion constrained to go through $N$ distinct states with equal separations $\tau$ (see Extended Data Figures 7 through 9), the number of consecutive $|a_n|^2$ needed for large $T$ is only about $T/\tau$, rather than $T$. This is the asymptotically relevant bandwidth $1/\tau$ (discussed earlier), divided by the spacing $1/T$ between allowed frequencies. Neglecting the smallest possible values of $T$, which give minimum moments too large to appear on our graphs, we find that in our tests, enough $|a_n|^2$ for the largest $T$ is sufficient for all $T$. Our choice of $\tau$ sets the horizontal resolution of the graphs—these examples use $\tau = 43$. For moments about a mean, the position of the mean frequency relative to the other frequency components makes a difference, so minimisation for each choice of total period $T$ involves searching a range of width $1/T$ for the $\alpha$ that minimises $\langle \nu - \alpha \rangle_M$. For $M \geq 1$, the $\alpha$ found is always the mean $\bar{\nu}$ of the minimising state—except for $M = 1$ with $T = N\tau$ and $N$ even, in which case all the $\alpha$ give the same minimum. For $0 < M < 1$ we must add a constraint to each optimisation problem, that the mean equals the $\alpha$ being tried. Behaviour similar to Figure 2 is seen for $\tau (\nu - \alpha)_{\bar{\nu}}$ for almost all $M$ (tested for $M$ up to 1000 and for $M = \infty$) and $N$ (tested up to $N = 30$). The only exceptions are moments about $\bar{\nu}$ with $M < 2$ and $N$ even: in some of these cases the intervals between the deepest local minima are larger than $N\tau$, and in some cases the pattern of minima is less regular. Of course an estimate of the global minimum can always be obtained by simply minimising any case with large $T$. In our tests (see Extended Data Figure 4), the difference between local maxima and the global minimum falls as $T^{-2}$ asymptotically for finite $M$, and as $T^{-4}$ for $M = \infty$. The latter result is implied by a large-$T$ bandwidth bound of $1/\tau + 1/T$: we need to round up the asymptotically relevant bandwidth $1/\tau$ to an integer multiple of $1/T$. The $M = \infty$ graphs in all of these figures are obtained from the bandwidths (or bandwidths above the mean) of states that minimise $\tau (\nu - \alpha)_{\bar{\nu}}$ for finite $M$. For all data shown, the minimising bandwidths are found to be independent of $M$ for $\nu_0$ and, for $M \geq 30$, for $\bar{\nu}$.

To verify that equal times between distinct states is optimal, we performed experiments with unequal times. For example, for Figure 1 we generated 12,000 sets of separations stochastically; each set dividing a period $T$ into $N \leq 12$ intervals; each set involving $N_{\text{different}} \leq 4$ different interval lengths separating adjacent distinct states. Separations were integers between 1 and 100, except for five sets of separations near 1000, which added five extra points to the graph close to the dashed line. For each set of separations we used the total $T$ of the integer separations as the number of consecutive $|a_n|^2$ to appear in the minimisation. We did a fair sampling for each $N_{\text{different}}$, except that half of the choices of number-of-repetitions of a length favoured fewer lengths, and half of the choices of a length favoured the longer lengths. This helped fill out the cases with lower minima using a short experiment—our original experiment was completely unbiased and required a much larger number of samples. Similar experiments with other moments also verified equal times as optimal.

Since the dashed boundary in Figure 3 is formed by evolutions with almost-equal separations, we investigated those cases extensively (see Extended Data Figures 3 through 5). As unequal separations converge towards equal ones, the requirement that arbitrarily-close points of the overlap must be zero contribute additional slope-constraints on top of the equal-separation constraints, as shown in Extended Data Figure 4. The theoretical curves (dashed lines) in Figure 3 and Extended Data Figure 6 were obtained by minimising the equal separations cases, with slope = 0 constraints added at the equal separations. The triples of data points plotted for each $N$ in Extended Data Figure 6 used separations differing by one part in 10, 100, and 1000. For each minimisation, we let the number of frequency components equal the total integer period $T$. The minimisations about the mean for $N_{\text{different}} = 5$ used 600 decimal digits of precision. As is evident in the figure, the improvement in the minimum from using smaller and smaller relative differences diminishes rapidly. There are similarly diminishing returns from using very large numbers of $|a_n|^2$ with almost-equal separations. A minimisation of $\langle \nu - \nu_0 \rangle T$ (not included in the figures) for $N = 4$ different separations that differ from one another by only one part in $10^6$, using 300,000 consecutive $|a_n|^2$, exceeded the difference $\rightarrow 0$ limiting value by only about two parts in $10^5$. This was mostly accounted for by three very high frequency components. The only other non-zero coefficients were $a_{14}$ and below. Figures 7 and 8 show an example of a portion of evolution with separations that differ from each other by about one part in $10^3$. This computation used only 1280 consecutive frequency components, a number based on the bandwidth needed to accurately capture the behaviour near the largest $T$—using much larger bandwidth didn’t change the shape of the graph.

Finally, Extended Data Figure 9 illustrates the calculation of the absolute minimum ($N = 2$) case of the first moment about the mean for a constant-rate portion of a longer evolution—cases with $M = 1$ and $N$-even differ from (11). As usual, the minimum for any $N$ can be determined to arbitrary accuracy numerically. In this case, the same minimum value can be determined by solving a transcendental equation: $f_2(1,2) \rightarrow u/2\pi$ where $u \approx 1.3801$ is a root of $u \sin(u + \sqrt{u^2 - 1}) = 1$. This follows because only three consecutive frequencies are relevant, and from the orthogonality and normalisation constraints and the sign-symmetry of the absolute moment.
Moments about a minimum frequency for a constant rate portion of evolution.

As in Figure 2 of the paper, the graph at row $M$ and column $N$ shows the minimum value of $\tau (\nu - \nu_0)_M$ for each choice of maximum period $T$ for an evolution that includes $N$ distinct states separated by $N-1$ equal intervals $\tau$, with the horizontal axes labelled with $T/\tau$. For easier comparison, the $T = N\tau$ bound $f_{\nu_0}(M, N)$ is subtracted from each value plotted.
Extended Data Figure 2 | Moments about the mean frequency for a constant rate portion of evolution. The graph at row $M$ and column $N$ shows the minimum values of $\tau \langle \nu - \bar{\nu} \rangle_{ab}$ for each choice of maximum period $T$ for an evolution that includes $N$ distinct states separated by $N - 1$ equal intervals $\tau$, with horizontal axes labelled with $T/\tau$. In each graph the global minimum is subtracted, which is equal to the $T = N\tau$ bound $f_{\nu}(M, N)$ except for some $M < 2$ with $N$ even.
Extended Data Figure 3 | Minimising moments for constant rate portion of evolution, using larger N. Minimum values are computed numerically for $\tau \langle \nu - \nu_0 \rangle_M$ (first row) and $\tau \langle \nu - \bar{\nu} \rangle_M$ (second row) as we vary the maximum period $T$, for an evolution that includes $N$ distinct states separated by $N - 1$ equal intervals $\tau$. The first column has $N = 25$, the others $N = 26$. All global minima agree with the $T = N\tau$ bound $f_\alpha(M,N)$ except for the bottom middle case, as expected: here the smallest $\tau \langle \nu - \bar{\nu} \rangle_1$ agrees with $0.249657 = f_\bar{\nu}(1,27)$, rather than $0.25 = f_\bar{\nu}(1,26)$.

Extended Data Figure 4 | Asymptotic behaviour of minima with constant rate portion of evolution. On the left we plot $\tau \langle \nu - \nu_0 \rangle$ minus its global minimum, for periodic evolutions of various lengths $T$ that include $N = 10$ distinct states separated by $\tau$; similarly on the right for $\tau \langle \nu - \nu_0 \rangle_\infty$ with $N = 10$. On the left, the red dashed boundary is the function $3.456(T/\tau)^{-2}$. Other finite moments also fall asymptotically like $T^{-2}$. On the right, the boundary is simply $(T/\tau)^{-1}$. This is true of $\tau \langle \nu - \nu_0 \rangle_\infty$ for all $N$; the boundary for $\tau \langle \nu - \bar{\nu} \rangle_\infty$ falls half as fast.
Extended Data Figure 5 | Almost-equal separations. As in Figure 1, we show the real (solid) and imaginary (dashed) parts of the overlap function \(|\langle \psi(t) | \psi(0) \rangle|\) of Equation (6), for \(|\psi(t)\rangle\) that minimise \(|\langle \nu - \nu_0 \rangle|\), for a periodic evolution of length \(T\) with \(N\) distinct states. Left: All \(N = 3\) separations are of length \(\tau = T/N\), and \(|\langle \nu - \nu_0 \rangle|\) is 1. Middle: Separations differ by one part in \(10^3\), and \(|\langle \nu - \nu_0 \rangle|\) is 2.001. Right: Detail of flat region near \(t/\tau = 1\) from the middle graph. If we make the separations more equal, the oscillation gets narrower and its amplitude smaller. In the limit, only the extra constraint “slope = 0 at \(t/\tau = 1\) and 2” keeps Middle distinct from Left, and \(|\langle \nu - \nu_0 \rangle| T \to 2."

Extended Data Figure 6 | Extra width required by almost-equal separations, above that for equal ones. All evolutions have period \(T\). Vertical axes show the extra minimum-width of \(|\langle \nu - \alpha \rangle|\) for almost-equal separations, above the minimum needed for equal separations; horizontal axes show the number \(N_{\text{different}}\) of different separations. The dashed lines are theoretical curves that minimise the width assuming we impose just the usual equal-separation constraints, along with slope = 0 constraints at the equal separations. Triples of points correspond to separations that differ by one part in 10, 100 or 1000. The theoretical bounds shown seem tight for \(M = 1\) or for \(N_{\text{different}} = 2\).
Extended Data Figure 7 | Unequal separations in a portion of an evolution. Left: Minimum of $\tau \langle \nu - \nu_0 \rangle_1$ for two equal separations between three distinct states. Right: Two almost-equal separations require a larger $\tau \langle \nu - \nu_0 \rangle_1$. The unequal separations used here differ by one part in $10^3$, and the spikes are not numerical artefacts. With equal separations the minimum is $1/3$; with the given unequal separations the minimum is about $.527$. The range shown on the right is only half that on the left.

Extended Data Figure 8 | Satisfying almost-equal orthogonality constraints. For the computation shown in Extended Data Figure 7 (right), we look in detail at a particular value of the maximum period: for $T = 7.8 \tau$ we plot the magnitude of the overlap function $\langle \psi(t) | \psi(0) \rangle$ of Equation (6) using the coefficients $|a_n|^2$ that minimise $\langle \nu - \nu_0 \rangle_1$. Left: Full-scale behaviour. Middle: Detail near $t = \tau$. Right: Detail near $t = 2\tau$. The full scale graph depends strongly on our choice of $T$, but the detail graphs near $\tau$ and $2\tau$ don’t.

Extended Data Figure 9 | First absolute moment about the mean. Left: If two distinct states are separated by an interval $\tau$, the global minimum of $\tau \langle \nu - \bar{\nu} \rangle_1$ is about $.22$, and repeats whenever the maximum period $T$ of the evolution is an integer multiple of approximately $2.7\tau$. Middle: At $T = 2\tau$, a three-frequency-wide state (solid) achieves the same minimum $\tau \langle \nu - \bar{\nu} \rangle_1$ as a minimum-width two-frequency state (dashed). Right: The state that achieves the first global minimum at $T \approx 2.7\tau$ uses three frequencies. Knowing this, we can determine the minimum analytically.