9. Local reciprocity cycles

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In this section we introduce a description of totally ramified Galois extensions of a local field with finite residue field (extensions have to satisfy certain arithmetical restrictions if they are infinite) in terms of subquotients of formal power series $\mathbb{F}_p^{\text{sep}}[[X]]^\ast$. This description can be viewed as a non-commutative local reciprocity map (which is not in general a homomorphism but a cocycle) which directly describes the Galois group in terms of certain objects related to the ground field. Abelian class field theory as well as metabelian theory of Koch and de Shalit [K], [KdS] (see subsection 9.4) are partial cases of this theory.

9.1. Group $U_{N(L/F)}^\otimes$

Let $F$ be a local field with finite residue field. Denote by $\varphi \in G_F$ a lifting of the Frobenius automorphism of $F_{\text{ur}}/F$.

Let $F^\varphi$ be the fixed field of $\varphi$. The extension $F^\varphi/F$ is totally ramified.

Lemma ([KdS, Lemma 0.2]). There is a unique norm compatible sequence of prime elements $\pi_E$ in finite subextensions $E/F$ of $F^\varphi/F$.

Proof. Uniqueness follows from abelian local class field theory, existence follows from the compactness of the group of units.

In what follows we fix $F^\varphi$ and consider Galois subextensions $L/F$ of $F^\varphi/F$. Assume that $L/F$ is arithmetically profinite, ie for every $x$ the ramification group $\text{Gal}(L/F)^x$ is open in $\text{Gal}(L/F)$ (see also subsection 6.3 of Part II). For instance, a totally ramified $p$-adic Lie extension is arithmetically profinite.

For an arithmetically profinite extension $L/F$ define its Hasse–Herbrand function $h_{L/F}: [0, \infty) \to [0, \infty)$ as $h_{L/F}(x) = \lim h_{M/F}(x)$ where $M/F$ runs over finite subextensions of $L/F$ (cf. [FV, Ch. III §5]).
If $L/F$ is infinite let $N(L/F)$ be the field of norms of $L/F$. It can be identified with $k_F((\Pi))$ where $\Pi$ corresponds to the norm compatible sequence $\pi_E$ (see subsection 6.3 of Part II, [W], [FV, Ch.III §5]).

Denote by $\varphi$ the automorphism of $N(L/F)_{ur}$ and of its completion $N(L/F)$ corresponding to the Frobenius automorphism of $F_{ur}/F$.

**Definition.** Denote by $U^{\circ}_{\overline{N(L/F)}}$ the subgroup of the group $U_{\overline{N(L/F)}}$ of those elements whose $\widehat{F}$-component belongs to $U_F$. An element of $U^{\circ}_{\overline{N(L/F)}}$ such that its $\widehat{F}$-component is $\varepsilon \in U_F$ will be called a lifting of $\varepsilon$.

The group $U^{\circ}_{\overline{N(L/F)}}/U_{N(L/F)}$ is a direct product of a quotient group of the group of multiplicative representatives of the residue field $k_F$ of $F$, a cyclic group $\mathbb{Z}/p^a$ and a free topological $\mathbb{Z}_p$-module. The Galois group $\text{Gal}(L/F)$ acts naturally on $U^{\circ}_{\overline{N(L/F)}}/U_{N(L/F)}$.

**9.2. Reciprocity map $N_{L/F}$**

To motivate the next definition we interpret the map $\Upsilon_{L/F}$ (defined in 10.1 and 16.1) for a finite Galois totally ramified extension $L/F$ in the following way. Since in this case both $\pi_\Sigma$ and $\pi_L$ are prime elements of $L_{ur}$, there is $\varepsilon \in U_{L_{ur}}$ such that $\pi_\Sigma = \pi_L \varepsilon$. We can take $\sigma = \sigma \varphi$. Then $\pi_{\Sigma}^{\sigma - 1} = \varepsilon^{1 - \sigma \varphi}$. Let $\eta \in U_{\overline{L}}$ be such that $\eta^{\sigma - 1} = \varepsilon$. Since $(\eta^{\sigma - 1} \varepsilon^{-1})^{\varphi - 1} = (\eta^{\sigma - 1})^{\varphi - 1}$, we deduce that $\varepsilon = \eta^{\sigma - 1} \eta^{(1 - \sigma) \varphi} \rho$ with $\rho \in U_L$.

Thus, for $\xi = \eta^{\sigma - 1}$

$$\Upsilon_{L/F}(\sigma) \equiv N_{\Sigma/F} \pi_{\Sigma} \equiv N_{\Sigma/F} \pi_{\Sigma}^{\varphi - 1} \equiv N_{L/F} \xi \equiv N_{L/F} \xi^{1 - \varphi} \equiv \pi_{L,F}^{\sigma - 1}. $$

**Definition.** For a $\sigma \in \text{Gal}(L/F)$ let $U_{\sigma} \in U_{\overline{N(L/F)}}$ be a solution of the equation

$$U^{1 - \varphi} = \Pi^{\sigma - 1}$$

(recall that id $- \varphi: U_{\overline{N(L/F)}} \to U_{\overline{N(L/F)}}$ is surjective). Put

$$N_{L/F}: \text{Gal}(L/F) \to U^{\circ}_{\overline{N(L/F)}}/U_{N(L/F)}, \quad N_{L/F}(\sigma) = U_{\sigma} \mod U_{N(L/F)}.$$

**Remark.** Compare the definition with Fontaine-Herr’s complex defined in subsection 6.4 of Part II.

**Properties.**

1. $N_{L/F} \in Z^1(\text{Gal}(L/F), U^{\circ}_{\overline{N(L/F)}}/U_{N(L/F)})$ is injective.
Let $U$.

(2) For a finite extension $L/F$ the $\hat{F}$-component of $N_{L/F}(\sigma)$ is equal to the value $Y_{L/F}(\sigma)$ of the abelian reciprocity map $Y_{L/F}$ (see the beginning of 9.2).

(3) Let $M/F$ be a Galois subextension of $L/F$ and $E/F$ be a finite subextension of $L/F$. Then the following diagrams of maps are commutative:

$$\begin{align*}
\text{Gal}(L/E) &\rightarrow U_{N_{L/E}}/U_{N_{N_{L/E}}} & \text{Gal}(L/F) &\rightarrow U_{N_{L/F}}/U_{N_{N_{L/F}}} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Gal}(L/F) &\rightarrow U_{N_{L/F}}/U_{N_{N_{L/F}}} & \text{Gal}(M/F) &\rightarrow U_{N_{M/F}}/U_{N_{N_{M/F}}}.
\end{align*}$$

(4) Let $U_{n,N_{L/F}}$ be the filtration induced from the filtration $U_{n,N_{L/F}}$ on the field of norms. For an infinite arithmetically profinite extension $L/F$ with the Hasse–Herbrand function $h_{L/F}$ put $\text{Gal}(L/F)_n = \text{Gal}(L/F)_{h_{L/F}(n)}$. Then $N_{L/F}$ maps $\text{Gal}(L/F)_n \setminus \text{Gal}(L/F)_{n+1}$ into $U_{n,N_{L/F}} - U_{N_{N_{L/F}}} \setminus U_{n+1}$. Then clearly $\text{im}(N_{L/F})$ is a group isomorphic to $\text{Gal}(L/F)$.

**Problem.** What is $\text{im}(N_{L/F})$?

One method to solve the problem is described below.

### 9.3. Reciprocity map $H_{L/F}$

**Definition.** Fix a tower of subfields $F = E_0 - E_1 - E_2 - \ldots$, such that $L = \cup E_i$, $E_i/F$ is a Galois extension, and $E_i/E_{i-1}$ is cyclic of prime degree. We can assume that $|E_{i+1} : E_i| = p$ for all $i \geq i_0$ and $|E_{i_0} : E_0|$ is relatively prime to $p$.

Let $\sigma_i$ be a generator of $\text{Gal}(E_i/E_{i-1})$. Denote $X_i = U_{E_i}^{\sigma_i-1}$.

The group $X_i$ is a $\mathbb{Z}_p$-module of $U_{1,E_i}$. It is the direct sum of a cyclic torsion group of order $p^n$, $n_i \geq 0$, generated by, say, $\alpha_i$ ($\alpha_i = 1$ if $n_i = 0$) and a free topological $\mathbb{Z}_p$-module $Y_i$.

We shall need a sufficiently “nice” injective map from characteristic zero or $p$ to characteristic $p$

$$f_i: U_{E_i}^{\sigma_i-1} \rightarrow U_{N_{L/E_i}} \rightarrow U_{N_{L/F}}.$$
If $F$ is a local field of characteristic zero containing a non-trivial $p$th root $\zeta$ and $f_i$ is a homomorphism, then $\zeta$ is doomed to go to $1$. Still, from certain injective maps (not homomorphisms) $f_i$ specifically defined below we can obtain a subgroup $\prod f_i(U_{N(L/F)}^{p_{n_i}-1})$ of $U_{N(L/F)}$.

**Definition.** If $n_i = 0$, set $A^{(i)} \in U_{N(L/E_i)}$ to be equal to $1$.

If $n_i > 0$, let $A^{(i)} \in U_{N(L/E_i)}$ be a lifting of $\alpha_i$ with the following restriction: $A^{(i)}_{E_{i+1}}$ is not a root of unity of order a power of $p$ (this condition can always be satisfied, since the kernel of the norm map is uncountable).

**Lemma ([F]).** If $A^{(i)} \neq 1$, then $\beta_{i+1} = A^{(i)}_{E_{i+1}}p_{n_i}$ belongs to $X_{i+1}$.

Note that every $\beta_{i+1}$ when it is defined doesn’t belong to $X_{i+1}$. Indeed, otherwise we would have $A^{(i)}_{E_{i+1}}p_{n_i} = \gamma^p$ for some $\gamma \in X_{i+1}$ and then $A^{(i)}_{E_{i+1}}p_{n_i-1} = \gamma\zeta$ for a root $\zeta$ of order $p$ or $1$. Taking the norm down to $E_i$ we get $\alpha_i^{p_{n_i}-1} = N_{E_{i+1}/E_i}\gamma = 1$, which contradicts the definition of $\alpha_i$.

**Definition.** Let $\beta_{i,j}$, $j \geq 1$ be free topological generators of $Y_i$ which include $\beta_i$ whenever $\beta_i$ is defined. Let $B^{(i,j)} \in U_{N(L/E_i)}$ be a lifting of $\beta_{i,j}$ (i.e. $B^{(i,j)}_{E_i} = \beta_{i,j}$), such that if $\beta_{i,j} = \beta_i$, then $B^{(i,j)}_{E_k} = B^{(i)}_{E_k} = A^{(i-1)}_{E_k}p_{n_i-1}$ for $k \geq i$.

Define a map $X_i \to U_{N(L/E_i)}$ by sending a convergent product $\alpha_i \prod_j \beta_{i,j}^{c_j}$, where $0 \leq c \leq n_i - 1$, $c_j \in \mathbb{Z}_p$, to $A^{(i)c} \prod B^{(i,j)c_j}$ (the latter converges). Hence we get a map

$$f_i: U_{N(L/E_i)}^{p_{n_i}-1} \to U_{N(L/E_i)} \to U_{N(L/F)}$$

which depends on the choice of lifting. Note that $f_i(\alpha)_{E_i} = \alpha$.

Denote by $Z_i$ the image of $f_i$. Let

$$Z_{L/F} = Z_{L/F}(\{E_i, f_i\}) = \left\{ \prod z^{(i)} : z^{(i)} \in Z_i \right\},$$

$$Y_{L/F} = \{ y \in U_{N(L/F)} : y^{1-\varphi} \in Z_{L/F} \}.$$

**Lemma.** The product of $z^{(i)}$ in the definition of $Z_{L/F}$ converges. $Z_{L/F}$ is a subgroup of $U_{N(L/F)}$. The subgroup $Y_{L/F}$ contains $U_{N(L/F)}$.
Theorem ([F]). For every \((u_{\tilde{E}_i}) \in U_{N(L/F)}^\circ\) there is a unique automorphism \(\tau\) in the group \(\text{Gal}(L/F)\) satisfying
\[(u_{\tilde{E}_i})^{1-\varphi} \equiv \Pi^{\tau - 1} \mod Z_{L/F}.
If \((u_{\tilde{E}_i}) \in Y_{L/F},\) then \(\tau = 1\).

Hint. Step by step, passing from \(\tilde{E}_i\) to \(\tilde{E}_{i+1}\).

Remark. This theorem can be viewed as a non-commutative generalization for finite \(k\) of exact sequence \((\ast)\) of 16.2.

Corollary. Thus, there is map
\[\mathcal{H}_{L/F}: U_{N(L/F)}^\circ / Y_{L/F} \rightarrow \text{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\tilde{E}_i})) = \tau.
The composite of \(N_{L/F}\) and \(\mathcal{H}_{L/F}\) is the identity map of \(\text{Gal}(L/F)\).

9.4. Main Theorem

Theorem ([F]). Put
\[\mathcal{H}_{L/F}: U_{N(L/F)}^\circ / Y_{L/F} \rightarrow \text{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\tilde{E}_i})) = \tau
where \(\tau\) is the unique automorphism satisfying \((u_{\tilde{E}_i})^{1-\varphi} \equiv \Pi^{\tau - 1} \mod Z_{L/F}\). The injective map \(\mathcal{H}_{L/F}\) is a bijection. The bijection
\[N_{L/F}: \text{Gal}(L/F) \rightarrow U_{N(L/F)}^\circ / Y_{L/F}
induced by \(N_{L/F}\) defined in 9.2 is a 1-cocycle.

Corollary. Denote by \(q\) the cardinality of the residue field of \(F\). Koch and de Shalit [K], [KdS] constructed a sort of metabelian local class field theory which in particular describes totally ramified metabelian extensions of \(F\) (the commutator group of the commutator group is trivial) in terms of the group
\[n(F) = \{(u \in U_F, \xi(X) \in F_p^{\text{sep}}[[X]]^* : \xi(X)^{\varphi - 1} = \{u\}(X)/X\}
with a certain group structure. Here \(\{u\}(X)\) is the residue series in \(F_p^{\text{sep}}[[X]]^*\) of the endomorphism \([u](X) \in O_F[[X]]\) of the formal Lubin–Tate group corresponding to \(\pi_F, q, u\).

Let \(M/F\) be the maximal totally ramified metabelian subextension of \(F_{\varphi}\), then \(M/F\) is arithmetically profinite. Let \(R/F\) be the maximal abelian subextension of \(M/F\). Every coset of \(U_{N(M/F)}^\circ\) modulo \(Y_{M/F}\) has a unique representative in

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im(\mathcal{N}_{M/F}). Send a coset with a representative \((u_Q^{-1}) \in U^\infty_{\mathcal{N}(M/F)} (F \subset Q \subset M), |Q:F| < \infty\) satisfying \((u_Q^{-1})^{1-\varphi} = (\pi_Q)^{\tau - 1}\) with \(\tau \in \text{Gal}(M/F)\) to
\[
(u_F^{-1}, u_E) \in U^\infty_{\mathcal{N}(R/F)} (F \subset E \subset R, |E:F| < \infty).
\]
It belongs to \(n(F)\), so we get a map
\[
g: U^\infty_{\mathcal{N}(M/F)}/Y_{M/F} \to n(F).
\]
This map is a bijection [F] which makes Koch–de Shalit’s theory a corollary of the main results of this section.

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