The rectangular peg problem

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Motivation.

In 1911, Otto Toeplitz posed the following question:

**Problem 1 (The Square Peg Problem)**

*Does every continuous Jordan curve in the Euclidean plane contain four points at the vertices of a square?*

It posits a striking connection between the topology and the geometry of the Euclidean plane. It remains open to this day.
Jordan curves.
Inscribed squares.
Why squares / quadrilaterals?

- Three points are ubiquitous: \( \forall \) triangle \( T \) and \( \forall \) Jordan curve \( \gamma \), \( \gamma \) inscribes a triangle similar to \( T \). (Exercise.)
- Five points are not: dissimilar ellipses inscribe dissimilar pentagons. (Distinct ellipses meet in at most four points.)
- Four is where things get interesting: a recurring theme in low-dimensional topology / geometry.
Early progress.

- Emch (1913) solved the problem for smooth convex curves. (Ideas involving configuration spaces, homology)
- Schnirelman (1929) solved it for smooth Jordan curves. In fact, a generic smooth Jordan curve contains an odd number of “inscribed” squares. (Bordism argument)

Tempting approach to original problem: a limiting argument. Any continuous Jordan curve is a limit of smooth ones, so take a limiting sequence of squares.

Problem: the squares may shrink to points.
Variations.

- Varying regularity condition on curve (e.g. recent work of Feller-Golla, Schwartz, Tao).
- Higher dimensional analogues (e.g. inscribed octahedra in $S^2 \hookrightarrow \mathbb{R}^3$).
- Fenn’s table theorem.
- Kronheimer and son (Peter) on the tripos problem.
- Other inscribed features in Jordan curves.

See, e.g. Matschke, *Notices of the AMS*, 2014.
Theorem 1 (Vaughan 1977)

Every continuous Jordan curve contains four vertices of a rectangle.

(Reference: Meyerson, *Balancing Acts*, 1981.)
Proof:

\[ \text{Sym}^2(\gamma) = \{\{z, w\} : z, w \in \gamma\} : \text{unordered pairs of points on } \gamma \]

It is a Möbius band:

- send \( \{z, w\} \in \text{Sym}^2(S^1) \) to the parallelism class of (tangent) line \( \overrightarrow{zw} \)

- obtain \( \text{Sym}^2(S^1) \to \mathbb{R}P^1 \) as an \( I \)-bundle over \( \mathbb{R}P^1 \)
- connected boundary \( \partial = \{\{z, z\} : z \in \gamma\} \)
Define a continuous map $v : \text{Sym}^2(\gamma) \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$:

$$v(\{z, w\}) = \left( \frac{z + w}{2}, |z - w| \right).$$

The “midpoint, distance” map.

- $\text{im}(v)$ hits $\mathbb{R}^2 \times \{0\}$ in $v(\partial) = \gamma \times \{0\}$
\[ v(\{z, w\}) = v(\{x, y\}) \iff \{z, w\} \text{ and } \{x, y\} \text{ span diagonals of a rectangle} \]

Principle:

\[
\{\text{inscribed rectangles in } \gamma\} \leftrightarrow \{\text{self-intersections of } v\}
\]
reflect im(v) across $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$:

get continuous map $v \cup \bar{v}$ of the Klein bottle to $\mathbb{R}^3$, 1-to-1 at $\gamma \times \{0\}$.

$v$ contains a point of self-intersection $\implies \gamma$ inscribes a rectangle. □

Any map of the Klein bottle to $\mathbb{R}^3$ must contain “a lot” of self-intersection, so there should exist many inscribed rectangles in $\gamma$.

How to quantify?
Problem 2 (The rectangular peg problem)

For every (smooth) Jordan curve and every rectangle in the Euclidean plane, do there exist four points on the curve at the vertices of a rectangle similar to the one given?
Step −1.

Published “solution” in 1991.
Idea: intersection theory / bordism argument.
Each inscribed rectangle in $\gamma$ gets a sign; signed count of
inscribed rectangles in $\gamma$ similar to a given one is 2; hence there
exist at least two.
In 2008, Matschke found a mistake: the signed count is 0.

\begin{center}
\begin{tabular}{|c|c|}
\hline
-1 & +1 \\
\hline
\end{tabular}
\end{center}

It suggests a limit to the efficacy of intersection theory /
bordism arguments.
In 2018, Cole Hugelmeyer recovered some new cases of the rectangular peg problem:

**Theorem 2 (Hugelmeyer 2018)**

*Every smooth Jordan curve contains four points at the vertices of a rectangle with aspect angle equal to an integer multiple of \(\pi/n\), for all \(n \geq 3\). In particular, every smooth Jordan curve inscribes a rectangle of aspect ratio \(\sqrt{3}\).*
Resolve $v$ into a 4D version:

$$h_n : \text{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C},$$

$$h_n(\{z, w\}) = \left(\frac{z + w}{2}, (z - w)^{2n}\right)$$

\[
\left\{
\begin{array}{l}
\text{inscribed rectangles in } \gamma \\
\text{with aspect angle } k\pi/n, k \in \mathbb{Z}
\end{array}
\right\} \leftrightarrow \{\text{self-intersections of } h_n\}
Blow up: $\tilde{h}_n : \text{Sym}^2(\gamma) \to X = \mathbb{C} \times \mathbb{R}_{\geq 0} \times S^1$,

$$\tilde{h}_n(z, w) = \left(\frac{z + w}{2}, |z - w|^{2n}, \frac{(z - w)^{2n}}{|z - w|^{2n}}\right), \quad z \neq w$$

$$\tilde{h}_n(z, z) = (z, 0, u(z)^{2n}), \ u(z) \text{ unit tangent to } \gamma \text{ at } z.$$

$M = \text{im}(\tilde{h}_n)$ hits $\partial X = \mathbb{C} \times \{0\} \times S^1$ in a $(1, 2n)$-curve.

insert $X$ into $S^3 \times \mathbb{R}_{\geq 0}$, matching $\partial X$ with an open solid torus in $S^3 \times \{0\}$ by an axial twist.

$\partial M$ maps onto the torus knot $T(2n, 2n - 1)$.

Batson (2014): $T(2n, 2n - 1)$ does not bound a smoothly embedded Möbius band in $S^3 \times \mathbb{R}_{\geq 0}$ for any $n \geq 3$.

Hence $M$ self-intersects $\implies \exists$ asserted inscribed rectangle. □

(The case of a square does not follow: e.g. $T(4, 3)$ bounds a Möbius band in $B^4$.)
Feller and Golla (2020): recovered Hugelmeyer’s result, and the case of a square, for curves obeying a weaker regularity condition than smoothness.
Proof based on branched covering / intersection form arguments (free of gauge theory / symplectic geometry).
Step 3. Hugelmeyer v2.0.

In 2019, Hugelmeyer recovered 1/3 of the rectangular peg problem:

**Theorem 3 (Hugelmeyer 2019)**

*For every smooth Jordan curve $\gamma$, the set of angles $\phi \in (0, \pi/2]$ such that $\gamma$ contains an inscribed rectangle of aspect angle $\phi$ has Lebesgue measure $\geq (1/3)(\pi/2)$.***
Proof:

Reconsider $h = h_2 : \text{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C}$,

$$h(\{z, w\}) = \left( \frac{z + w}{2}, (z - w)^2 \right)$$

It is a smooth embedding. Write $M = \text{im}(h)$.

For $\phi \in \mathbb{R}$, let $R_\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ denote rotation by $\phi$ in the second coordinate:

$$R_\phi(z, w) = (z, e^{i\phi} \cdot w).$$

\[
\begin{align*}
\text{inscribed rectangles in } \gamma \\
\text{with aspect angle } \phi
\end{align*}
\] \leftrightarrow M \cap R_{2\phi}(\hat{M})

Goal: show non-empty for $\geq 1/3$ of angles $\phi \in (0, \pi/2]$. 

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Blow up as before (γ is smooth).

$M_1, M_2$ - rotations of $M$ with disjoint interiors.

Define a comparison $M_1 \prec M_2$ based on linking number.

Fact 1. $\prec$ is antisymmetric.

(Linking number argument.)

$M_1, M_2, M_3$ - rotations of $M$ with pairwise disjoint interiors.

Fact 2. $\prec$ is transitive on $M_1, M_2, M_3$.

(Milnor triple linking number.)

$\prec$ + additive combinatorics (Kemperman / Cauchy-Davenport) delivers the result. □

In fact $\exists M$ (not derived from any $\gamma$) s.t. $\hat{M} \cap R_\phi(\hat{M}) \neq \emptyset$ for $1/3$ of angles $\phi$.

How to ensure that $\hat{M} \cap R_\phi(\hat{M}) \neq \emptyset$ for all $\phi$, $M = \text{im}(h)$?
Step 4. Shift in perspective: symplectic geometry.

Idea: place a symplectic form on $\mathbb{C} \times \mathbb{C}$ so that $M$ is Lagrangian and $R_\phi$ form a family of Hamiltonian symplectomorphisms.

“Optimistic” Arnold-Givental:

$$|M \cap R_\phi(M)| \geq \dim H_*(M; \mathbb{Z}/2\mathbb{Z}) = 2.$$

Technicality: $M$ is nonorientable and has boundary.
Shortcut: nonembeddability of the Klein bottle.
The rectangular peg problem.

Theorem 4 (G-Lobb 2020)

For every smooth Jordan curve and rectangle in the Euclidean plane, there exist four points on the curve that form the vertices of a rectangle similar to the one given.
Proof, minus details:

Define \( f : \text{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C}, \)

\[
f(\{z, w\}) = \left( \frac{z + w}{2}, \frac{(z - w)^2}{2\sqrt{2}|z - w|} \right) \quad (z \neq w)
\]

Möbius band \( M = \text{im}(f). \)

\( M \) hits \( \mathbb{C} \times \{0\} \) in \( \partial M = \gamma \times \{0\}. \)

Away from \( \partial, \) \( M \) is smooth and Lagrangian w.r.t. symplectic form \( \omega_{\text{std}} = \frac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w}) \) on \( \mathbb{C}^2. \)

Let \( \phi \in (0, \pi/2]. \)

\[
\left\{ \text{inscribed rectangles in } \gamma \right\} \leftrightarrow M \cap R_{2\phi}(\hat{M})
\]

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$R_\phi$ is a symplectomorphism.  
It fixes $\partial M$.  
Hence $M$ and $R_{2\phi}(M)$ are Möbius bands, smooth and Lagrangian away from their common boundary $\gamma \times \{0\}$, where they meet in a controlled way.

We can smooth $M \cup R_{2\phi}(M)$ nearby $\gamma \times \{0\}$ to get a smoothly mapped, Lagrangian Klein bottle.

**Theorem 5 (Shevchishin, Nemirovski 2007)**

There does not exist a smooth, Lagrangian embedding of the Klein bottle in $(\mathbb{C}^2, \omega)$.

Hence $\hat{M} \cap R_{2\phi}(\hat{M}) \neq \emptyset \implies \exists$ inscribed rectangle in $\gamma$ of aspect angle $\phi$.  □
1. Why is $M$ Lagrangian?

$\gamma \subset \mathbb{C}$ is Lagrangian

$$\implies \gamma \times \gamma \subset \mathbb{C} \times \mathbb{C}$$ is

$$\implies \text{Sym}^2(\gamma) - \Delta \subset \text{Sym}^2(\mathbb{C}) - \Delta$$ is.

The map $f$ is just $\mathbb{C} \times \mathbb{C} \overset{\pi}{\to} \text{Sym}^2(\mathbb{C}) \sim \mathbb{C} \times \mathbb{C}$ written explicitly:

$$f = g \circ l,$$

where $g, l : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$,

$$l(z, w) = \left(\frac{z + w}{2}, \frac{z - w}{2}\right), \quad g(z, r, \theta) = (z, r/\sqrt{2}, 2\theta).$$

$l$ is a diffeomorphism and $l^*(\omega) = \omega/2$.

$g$ is smooth and $g^*(\omega) = \omega$ away from $\mathbb{C} \times \{0\}$.

$M = f(\gamma \times \gamma)$ is Lagrangian (away from $\mathbb{C} \times \{0\}$).
2. Why is the smoothing possible?
Work with Lagrangian tori $L = l(\gamma \times \gamma)$ and $R_\phi(L)$.
They intersect cleanly at $\gamma \times \{0\} \subset \mathbb{C} \times \{0\}$.
They are invariant under $R_\pi$.
Apply equivariant Weinstein theorem à la Poźniak:
$\exists \mathbb{Z}/2$-equivariant symplectomorphism of neighborhood of intersection to $S^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with
• coordinates: $x_1, x_2, y_1, y_2$
• symplectic form: $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$
• $\mathbb{Z}/2$ action: $(x_1, x_2, y_1, y_2) \leftrightarrow (x_1, -x_2, y_1, -y_2)$
• Lagrangians: $S^1 \times \mathbb{R} \times \{0\} \times \{0\}$ and $S^1 \times \{0\} \times \{0\} \times \mathbb{R}$.
smooth the intersection $\mathbb{Z}/2$-equivariantly, then project via $g$
Details.

3. **Nonexistence of Lagrangian Klein bottles in \(\mathbb{C}^2\).**

This had been a question of Givental.

Nemirovski’s proof:

Given smoothly embedded Lagrangian Klein bottle \(K \subset (X, \omega)\), \([K] = 0 \in H_2(X; \mathbb{Z}/2)\), do Luttinger surgery. Get dual Klein bottle \(K' \subset (X', \omega')\), \([K'] \neq 0 \in H_2(X'; \mathbb{Z}/2)\).

\((X - N(K), \omega) \approx (X' - N(K'), \omega')\).

Gromov: any symplectic 4-manifold asymptotic to \((\mathbb{C}^2, \omega_{std})\) at \(\infty\) with \(\pi_2 = 0\) is actually \((\mathbb{C}^2, \omega_{std})\).

So could not have been in \((\mathbb{C}^2, \omega_{std})\) in the first place (else get \(\mathbb{C}^2 = X = X'\) and \([K'] \neq 0 \in H_2(\mathbb{C}^2; \mathbb{Z}/2)\) \(\nabla\).
Beyond.

1. Does every smooth Jordan curve inscribe a rectangle of each aspect ratio whose vertices appear in the same cyclic order around both the curve and the rectangle? ("Yes" for the square: Schwartz.)

2. Does every smooth Jordan curve inscribe every cyclic quadrilateral?

3. Is there an “algorithm” to locate an inscribed square in a smooth Jordan curve? Compare: finding a fixed point of a continuous map from the disk to itself.