A note on the Faddeev-Popov determinant and Chern-Simons perturbation theory

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Abstract

A refined expression for the Faddeev-Popov determinant is derived for gauge theories quantised around a reducible classical solution. We apply this result to Chern-Simons perturbation theory on compact spacetime 3-manifolds with quantisation around an arbitrary flat gauge field isolated up to gauge transformations, pointing out that previous results on the finiteness and formal metric-independence of perturbative expansions of the partition function continue to hold.
Introduction

The Faddeev-Popov gauge-fixing procedure [1], used to rewrite the functional integrals arising in non-abelian gauge theories in a form to which perturbative techniques can be applied, can be formulated in the general setting where the theory is quantised around a general classical solution [2]. However, when the classical solution is a reducible gauge field (e.g. the zero-instanton in Yang-Mills theory on $S^4$ or an arbitrary flat gauge field in Chern-Simons theory on $S^3$ or lens spaces) the Faddeev-Popov determinant obtained by the standard derivation in this setting is degenerate, and the perturbative techniques fail because the ghost propagator is ill-defined. To avoid these and related problems the considerations have been restricted to irreducible gauge fields on many previous occasions in the literature (e.g. [3], [4]). In this note we show that these problems can be avoided by a more careful derivation of the Faddeev-Popov determinant, taking account of certain gauge-fixing ambiguities which arise when the classical solution is reducible. The result is the following:

Instead of the usual Faddeev-Popov determinant

$$\det(\nabla_A^* \nabla_A),$$ \hspace{1cm} (1)

where $A^c$ denotes the classical solution around which the theory is quantised, we obtain

$$V(H_{A^c})^{-1} \det\left(\nabla_A^* \nabla_A \bigg|_{\ker(\nabla_A^c)^\perp}\right)$$ \hspace{1cm} (2)

Here $\nabla_A : \Gamma_0 \to \Gamma_1$ is the covariant derivative map determined by a gauge field $A$; the gauge fields are identified with the connection 1-forms on a principal fibre bundle $P$ over the compactified spacetime manifold $M$ and $\Gamma_q$ denotes the space of q-forms on $M$ with values in the bundle $P \times_G \mathfrak{g}$, where the compact, semisimple gauge group $G$ acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation. The vector spaces $\Gamma_q$ have inner products determined by a riemannian metric on $M$ and invariant inner product in $\mathfrak{g}$; these determine the adjoint map $\nabla_A^*$ in (1)-(2) and the orthogonal complement $\ker(\nabla_A^c)^\perp$ of the nullspace $\ker(\nabla_A^c)$ of $\nabla_A^c$. Note that the determinant in (2) makes sense at the formal level since $\text{Im}(\nabla_A^* \nabla_A) \subseteq \text{Im}(\nabla_A^*) = \ker(\nabla_A^c)^\perp$. In (2) $V(H_{A^c})$
denotes the (finite) volume of the isotropy group $H_{A^c}$ of $A^c$, i.e. the subgroup of the group $\mathcal{G}$ of gauge transformations which leave $A^c$ unchanged.

We go on to apply the result (2) to the Chern-Simons perturbation theory on compact spacetime 3-manifolds developed by S. Axelrod and I. Singer in [4], pointing out that their results on the finiteness and formal metric-independence of the perturbative expansions of the partition function derived for $A^c$ an irreducible flat gauge field continue to hold for reducible $A^c$ when (2) is used. As in [4] we still require $A^c$ to be isolated modulo gauge transformations though. The case of reducible $A^c$ isolated up to gauge transformations is an important special case since it applies for all the flat gauge fields on a number of basic 3-manifolds, e.g. $S^3$ and the lens spaces, when $G = SU(2)$.

The Faddeev-Popov determinant

Recall that the Faddeev-Popov procedure for rewriting the functional integrals of the form

$$\int \mathcal{D}A \, f(A) \, e^{-\frac{1}{\alpha^2}S(A)}$$

(3)

(where $S(A)$ is the action functional of the theory, $f(A)$ is a gauge-invariant functional and $\alpha$ is the coupling parameter) involves inserting $1 = P_{A^c}(A) / P_{A^c}(A)$ in the integrand, where

$$P_{A^c}(A) = \int_{\mathcal{G}_0} \mathcal{D}\phi \, \delta(\nabla_{A^c}^*(\phi \cdot A - A^c))$$

(4)

is the Faddeev-Popov functional associated with the gauge-fixing condition

$$\nabla_{A^c}^*(A - A^c) = 0$$

(5)

Following [2], to avoid problems with the Gribov ambiguity, we have taken the domain of the formal integration in (4) to be the subgroup $\mathcal{G}_0$ of topologically trivial gauge

\footnote{In Chern-Simons gauge theory we replace $e^{-\frac{\alpha}{2}S(A)}$ in (3) by $e^{ikS(A)}$ where $S(A)$ is the Chern-Simons action functional.}
transformations. Using the $G_0$-invariance of $S(A)$, $f(A)$ and $P_{A^c}(A)$ the resulting expression for $\mathcal{R}$ is

$$V(G_0) \int DA f(A) e^{-\frac{1}{\alpha} S(A)} P_{A^c}(A)^{-1} \delta(\nabla_{A^c}^*(A - A^c)) \tag{6}$$

The standard evaluation of $P_{A^c}(A)^{-1}$ in $\mathcal{R}$ leads to the Faddeev-Popov determinant $\mathcal{P}$. We will show that a more careful evaluation of $P_{A^c}(A)^{-1}$ leads to the new expression $\mathcal{Q}$. The gauge-fixing condition of $\mathcal{R}$ has ambiguities coming from $H_{A^c}$, i.e.

$$\nabla_{A^c}^*(A - A^c) = 0 \Rightarrow \nabla_{A^c}^*(\phi, A - A^c) = 0 \quad \forall \phi \in H_{A^c}. \tag{7}$$

To take this into account in the evaluation of $P_{A^c}(A)^{-1}$ we introduce the map

$$Q : \text{Lie}(H_{A^c})^\perp \times H_{A^c} \to G_0, \quad Q(v, \phi) := \exp(v)\phi \tag{8}$$

where $\text{Lie}(H_{A^c})^\perp \subseteq \text{Lie}(G_0) = \Gamma_0$. The differential (i.e. ‘Jacobi matrix’) of $Q$ at $(0, \phi)$,

$$\mathcal{D}_{(0,\phi)}Q : \text{Lie}(H_{A^c})^\perp \oplus T_{\phi}H_{A^c} \to T_{\phi}G_0 \tag{9}$$

is an isometry, so formally

$$|\det(\mathcal{D}_{(0,\phi)}Q)| = 1 \quad \forall \phi \in H_{A^c}. \tag{10}$$

(To see that $\mathcal{D}$ is an isometry consider for fixed $\phi \in H_{A^c}$ the composition of maps

$$\text{Lie}(G_0) = \text{Lie}(H_{A^c})^\perp \oplus \text{Lie}(H_{A^c}) \xrightarrow{\cong} \text{Lie}(H_{A^c})^\perp \oplus T_{\phi}H_{A^c} \xrightarrow{\mathcal{D}_{(0,\phi)}Q} T_{\phi}G_0 \xrightarrow{\cong} \text{Lie}(G_0)$$

where the first map is the isometry given by $(w, a) \mapsto (w, \frac{d}{dt}|_{t=0} e^{ta} \phi)$ and the last map is the inverse of the isometry $\text{Lie}(G_0) \xrightarrow{\cong} T_{\phi}G_0$ given by $v \mapsto \frac{d}{dt}|_{t=0} e^{tv} \phi$. It is easy to see that this composition of maps is the identity on $\text{Lie}(G_0)$. It follows that $\mathcal{D}_{(0,\phi)}Q$ must be an isometry since all the other maps are isometries.) We now use the change of variables formula to calculate

$$P_{A^c}(A) = \int_{G_0} \mathcal{D} \phi \delta(\nabla_{A^c}^*(\phi, A - A^c))$$

$$= \int_{H_{A^c} \times \text{Lie}(H_{A^c})^\perp} \mathcal{D} \phi \mathcal{D} v |\det(\mathcal{D}_{(v,\phi)}Q)| \delta(\nabla_{A^c}^* e^{v} \phi, A - A^c)$$
\[ \begin{align*}
&= \int_{H_{A^c}} \mathcal{D}\phi \left| \det \left( \nabla_{A^c}^* \nabla_{\phi, A} \right|_{\text{Lie}(H_{A^c})^\perp} \right|^{-1} \\
&= \int_{H_{A^c}} \mathcal{D}\phi \left| \det \left( \nabla_{A^c}^* \nabla_A \right|_{\text{Lie}(H_{A^c})^\perp} \right|^{-1} \\
&= V(H_{A^c}) \left| \det \left( \nabla_{A^c}^* \nabla_A \right|_{\text{Lie}(H_{A^c})^\perp} \right|^{-1} \\
&\text{(11)}
\end{align*} \]

where we have used (10) in the second line and \( \nabla_{A^c}^* \nabla_{\phi, A} = (\phi^\cdot) \nabla_{A^c}^* \nabla_A (\phi^\cdot)^{-1} \) for \( \phi \in H_{A^c} \) in the third line. Since \( \nabla_{A^c}^* \nabla_{A^c} \) is a positive operator we can discard the numerical signs in (11) in the relevant case where \( A \) is close to \( A^c \) in \( \mathcal{A} \) and arrive at the new expression (2) for \( P_{A^c}(A)^{-1} \) as promised.

The gauge-fixed functional integral (6) can now be written in a form which can be perturbatively expanded. We expand the action functional around the classical solution \( A^c \) as a polynomial in \( B \in \Gamma_1 \):

\[ S(A^c + B) = S(A^c) + \langle B , D_{A^c}B \rangle + S_{A^c}^I(B) \tag{12} \]

where \( D_{A^c} \) is a uniquely determined selfadjoint operator on \( \Gamma_1 \). Substituting the expression (2) for \( P_{A^c}(A)^{-1} \) in (11) and writing the determinant as a formal integral over independent anticommuting variables \( C, \bar{C} \in \ker(\nabla_{A^c})^\perp \) leads to the following expression for the gauge-fixed functional integral:

\[ \begin{align*}
V(\mathcal{G}_0) & V(H_{A^c})^{-1} \det(\hat{\Box}_{A^c})^{-1/2} e^{-\frac{1}{\alpha^2} S(A^c)} \\
\times & \int \text{Im}(\nabla_{A^c})^\perp \otimes \ker(\nabla_{A^c})^\perp \otimes \ker(\nabla_{A^c})^\perp \mathcal{D}(\alpha B) \mathcal{D}\bar{C} \mathcal{D}C f(A^c + \alpha B) \exp \left\{ \right. \\
& - \langle B , D_{A^c}B \rangle - \langle \bar{C} , \Box_{A^c}C \rangle - \frac{1}{\alpha^2} S_{A^c}^I(\alpha B) - \alpha < \bar{C} , \nabla_{A^c}[B,C] > \left. \right\} \\
&\text{(13)}
\end{align*} \]

where \( \Box_{A^c} = \nabla_{A^c}^* \nabla_{A^c} \) and \( \hat{\Box}_{A^c} \) denotes its restriction to the orthogonal complement of its nullspace. In the Yang-Mills- and Chern-Simons gauge theories \( \det(\hat{\Box}_{A^c}) \) can be given well-defined meaning via zeta-regularisation. The ghost propagator \( \Box_{A^c}^{-1} \) is well-defined on \( \ker(\nabla_{A^c})^\perp \). The gauge field propagator \( D_{A^c}^{-1} \) on \( \text{Im}(\nabla_{A^c})^\perp \) is well-defined provided that \( A^c \) is isolated up to gauge transformations in the space \( \mathcal{C} \) of critical points for \( S \). This follows from \( \ker(D_{A^c}) = T_{A^c}\mathcal{C} \) (which can be shown by a general argument when the moduli space of \( \mathcal{C} \) is smooth at the point represented by
$A^c$) since in this case $T_A^c C = T_A^c (G \cdot A^c) = \text{Im}(\nabla A^c)$. Thus when $A^c$ is isolated up to gauge transformations the standard perturbative techniques may be applied to (13), leading to a perturbative expansion of the form

$$V(G_0) Z_{sc}(\alpha; A^c) \sum_{V=0}^{\infty} \alpha^V I_V(A^c)$$

where $I_0(A^c) = f(A^c)$ and

$$Z_{sc}(\alpha; A^c) = V(H_{A^c})^{-1} \text{det} \left( \frac{1}{\pi \alpha^2} \tilde{D}_{A^c} \right)^{-1/2} \text{det} \left( \tilde{\square}_{A^c} \right)^{1/2} e^{-\frac{1}{\alpha^2} S(A^c)}$$

The $V$’th term in (14) is the contribution from all the Feynman diagrams of order $V$. The weak coupling limit of (14), $V(G_0) Z_{sc}(\alpha; A^c) f(A^c)$, coincides with the contribution from $A^c$ to the semiclassical approximation for (3) obtained from the invariant integration method of A. Schwarz [5, App. II (9)]. This is reassuring since Schwarz’s method does not use gauge fixing, unlike ours. This also indicates that (2) will allow the relationship between the Faddeev-Popov determinant and the natural metric on the orbit space of gauge fields, pointed out by O. Babelon and C.-M. Viallet [3] when the considerations are restricted to irreducible gauge fields, to be extended to the reducible case, although we will not pursue this here.

**Application to Chern-Simons perturbation theory on compact 3-manifolds**

Following [4] we consider perturbative expansion of the Chern-Simons partition function

$$Z(M, k) = \frac{1}{V(G_0)} \int \mathcal{D} A e^{ikS(A)}$$

where

$$S(A) = \frac{1}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

Here the spacetime $M$ is a closed oriented 3-manifold, and for simplicity $P$ is assumed trivial so the gauge fields $A$ are $\mathfrak{g}$-valued 1-forms on $M$. Let $A^c$ be an arbitrary flat gauge field on $M$ which is isolated up to gauge transformations, with flat covariant derivative $d = d^{A^c} = \oplus_{q=0}^{3} d_q^{A^c}$ on the space $\Omega = \Omega(M, \mathfrak{g}) = \oplus_{q=0}^{3} \Omega^q(M, \mathfrak{g})$ of $\mathfrak{g}$-valued
differential forms on $M$ (so $\nabla A^c = d_A^c$), and let $H(A^c) = \bigoplus_{q=0}^{3} H^q(A^c)$ denote its cohomology. The requirement that $A^c$ be isolated up to gauge transformations is equivalent to $H^1(A^c) = 0$.

We apply the gauge-fixing procedure of the preceding section to (1.6). In this case we obtain the perturbative expansion

$$Z(M, k, A^c) = Z_{sc}(M, k, A^c) \sum_{V=0,2,4,...} \left( \frac{1}{\sqrt{k}} \right)^V I_V(M, A^c)$$

(18)

where $I_0(M, k, A^c) = 1$ and

$$Z_{sc}(M, k, A^c) = e^{-\frac{\pi}{4} (s d_1)} \left( \frac{4 \pi \lambda g}{k} \right)^{\text{dim} H^0(A^c)/2} V(H_{A^c})^{-1} \tau(M, A^c)^{1/2}$$

(19)

In obtaining (19) we have used the results of [6]. Here $< a, b >_g = -\lambda g \text{Tr}(ab)$ and $\tau(M, A^c)$ is the Ray-Singer torsion of $A^c$. The modulus of (19) is metric-independent [7, §5]. The metric-dependent phase factor is discussed in [8, §2]. It has been verified [9] for wide classes of 3-manifolds that the expression for the semiclassical approximation for $Z(M, k)$ obtained from (19) coincides with the weak coupling (i.e. large $k$) limit of the expressions for $Z(M, k)$ obtained from Witten’s non-perturbative prescription [8].

In [4] the perturbative expansion (18) was considered for irreducible $A^c$. The results of the preceding section allow these considerations to be extended to reducible $A^c$ isolated up to gauge transformations (after suitable changes of variables in (13)) and the expressions for the coefficients $I_V(M, A^c)$ derived in [4] continue to hold: $I_V(M, A^c) = 0$ for $V$ odd, and for $V$ even [4, (3.54)–(3.55)]

$$I_V(M, A^c) = c_V \prod_{i=1}^{V} \left[ \int_{M_{x_i}} f_{a^ib^c} \frac{\partial}{\partial j^{a^i}(i)} \frac{\partial}{\partial j^{b^i}(i)} \frac{\partial}{\partial j^{c^i}(i)} \right] L_{tot}(x_1, \ldots, x_V)^{\frac{3V}{2}}$$

$$= c_V \int_{M^V} \text{Tr} \left( L_{tot}(x_1, \ldots, x_V)^{\frac{3V}{2}} \right)$$

(20)

where $c_V = (2\pi i)^{\frac{V}{2}} ((3!)^V (2!)^{\frac{3V}{2}} V! (\frac{3}{2} V!)^{-1}$. Briefly, the notations are as follows (see [4, §2–3] for the details)[4] $\{ j^a \}$ is an o.n.b. for $g$, $[j^a, j^b] = f_{abc} j^c$, $\frac{\partial}{\partial j^{i}(i)}$ is interior

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3Here and throughout this section all repeated indices are to be summed over.
multiplication by \( j^a_{(i)} \), \( L_{ab}(x, y) \in \Omega^2(M_x \times M_y) \) (singular at \( x = y \)) is given by

\[
(\hat{L}\psi)^a(x) = \int_{M_y} L_{ab}(x, y) \wedge \psi^b(y) \quad \psi = \psi^b j^b \in \Omega(M, g) \tag{21}
\]

where \( \hat{L} : \Omega(M, g) \to \Omega(M, g) \) is given by

\[
d\hat{L} = \pi_d \quad \quad \hat{L}d = \pi_{d^*} \tag{22}
\]

\( \pi_d \) and \( \pi_{d^*} \) are the projections onto the images of \( d \) and \( d^* \), and finally

\[
L_{tot}(x_1, \ldots, x_V) = \sum_{i,k=1}^V L_{ab}(x_i, x_k) \wedge j^a_{(i)} \wedge j^b_{(k)} \quad \in \Gamma(M_{x_1} \times \cdots \times M_{x_V}; \Lambda(\oplus_{i=1}^V (T^* M_{x_i} \oplus g_i))) \tag{23}
\]

We conclude this note by pointing out that the finiteness– and formal metric–independence properties of the perturbative expansion (18) derived in [4] for irreducible \( A^c \) continue to hold in the present context. With the point–splitting regularisation of [4] the expression (20) for \( I_V(M, A^c) \) is finite [4, theorem 4.2]. The argument for this goes through for arbitrary flat \( A^c \) [4, §6 Remark II(i)]. The extension of the metric–independence result of [4] to the present case is less obvious. It was shown in [4, §5] that the expression (20) for \( I_V(M, A^c) \) is formally metric–independent for irreducible \( A^c \). (A subsequent rigorous treatment [4, §5] [10], taking account of the singularities in \( L_{tot} \) in the diagonals \( x_i = x_k \), reveals an anomalous metric–dependent phase). It was known to the authors of [4] that the formal metric–independence of \( I_V(M, A^c) \) continues to hold for reducible \( A^c \) [4, §6 Remark II(i)], but since an argument has not previously been provided in the literature we will give one here. A rigorous treatment of the problem in the very general case where \( A^c \) is only required to belong to a smooth component of the modulispace has recently been announced and outlined by S. Axelrod [11]. However, the powerful new algebraic techniques outlined there are not necessary to show the formal metric–independence in the present case. As we will see, this can be shown by much simpler means.

To show the formal metric-independence of (20) in the present case we need generalisations of the properties of the propagator derived in [4, §3]. The property (PL1)
generalises to
\[ d_{M \times M_y} L_{ab}(x, y) = (d_{M_x} + d_{M_y}) L_{ab}(x, y) = \left( \delta_{ab} \delta(x, y) - \pi_{ab}(x, y) \right) \]
(24)
where \( \pi_{ab}(x, y) \) is obtained from the projection \( \pi \) onto the harmonic forms in the same way that \( L_{ab}(x, y) \) was obtained from \( \hat{L} \) (see (21)). Explicitly
\[
\pi_{ab}(x, y) = \sum_i h^a_i(x) h^b_i(y) V(M)^{-1} (\text{vol}(y) - \text{vol}(x))
\]
(25)
where \( \text{vol}(x) \) and \( \text{vol}(y) \) are the volume forms on \( M_x \) and \( M_y \) respectively, considered as elements in \( \Omega^3(M_x \times M_y) \), and \( \{h_i = h^a_i j^a\} \) is a metric-independent basis for \( \ker(d_0) \) chosen so that \( \langle h_i(x), h_k(x) \rangle^g = \delta_{ik} \forall x \in M \). We decompose
\[
L_{ab}(x, y) = L_{ab}^{0(2)}(x, y) + L_{ab}^{1(1)}(x, y) + L_{ab}^{2(0)}(x, y)
\]
(26)
\[
\pi_{ab}(x, y) = \pi_{ab}^{0(3)}(x, y) + \pi_{ab}^{3(0)}(x, y)
\]
(27)

where \( Q^{(p,q)}(x, y) \in \Omega(M_x \times M_y \dot{\otimes} M_y) \) denotes a form of degree \( p \) on \( M_x \) and degree \( q \) on \( M_y \). Using (24) we find that the generalisation of the key property (PL4) of \cite[§3]{4} is
\[
\delta \delta g L_{ab}^{1(1)}(x, y) = d_{M_x \times M_y} B_{ab}(x, y)
\]
(28)
for some \( B(x, y) \in \Omega^1(M_x \times M_y, g \otimes g) \) of the form
\[
B_{ab}(x, y) = B_{ab}^{0(1)}(x, y) - B_{ba}^{1(0)}(x, y)
\]
(29)
together with
\[
d_{M_x} \left( \delta g L_{ab}^{0(2)}(x, y) \right) = 0, \quad d_{M_y} \left( \delta g L_{ab}^{2(0)}(x, y) \right) = 0.
\]
(30)
Here \( \delta g \) is a variation of the chosen metric \( g \) on \( M \).

Now, repeating the calculation \cite[(5.83)]{4} gives in the present case
\[
\delta \delta g I_V(M, A^c) = - \frac{3}{2} V \left( \frac{3}{2} V - 1 \right) c_V (I^{(1)}_V - I^{(2)}_V) + 3 V c_V I^{(3)}_V
\]
(31)
\footnote{We are following the convention of \cite[(3.53)]{4}.}
where

\[
I^{(1)}_V = \int_{M^V} \text{TR} \left( B^{(0,1)}_{\text{tot}} \delta g_{\text{tot}} (L_{\text{tot}})^{2V-2} \right)
\]

\[
I^{(2)}_V = \int_{M^V} \text{TR} \left( B^{(0,1)}_{\text{tot}} \pi_{\text{tot}} (L_{\text{tot}})^{2V-2} \right)
\]

\[
I^{(3)}_V = \int_{M^V} \text{TR} \left( (\delta g_{L^{(0,2)}_{\text{tot}}} (L_{\text{tot}})^{2V-1} \right)
\]

(The derivation uses Stoke’s theorem, and is therefore formal since \(L_{\text{tot}}(x_1, \ldots, x_V)\) is not smooth on \(M^V\). At all other points here and below we are rigorous). The integral \(I^{(1)}_V\) is the one appearing in the calculation of [4, §5], and vanishes by the argument given there. The integrals \(I^{(2)}_V\) and \(I^{(3)}_V\) are new features of the present, more general situation where \(A^c\) is reducible. The key to showing that they vanish is to note that

\[
\int_M f_{abc} h^a \phi^b \wedge \psi^c = 0 \quad \forall h \in \ker(d_0), \phi \in \ker(d^*), \psi \in \text{Im}(d^*)
\]

and

\[
L_{ab}(x, y) \in \text{Im}(d^*_{M_x}), \quad \quad L_{ab}(x, y) \in \text{Im}(d^*_{M_y})
\]

The formula (35) follows from \(\int_M \phi^a \wedge \psi^a = 0 \quad \forall \phi \in \ker(d^*), \psi \in \text{Im}(d^*)\) together with \([h, \psi] \in \text{Im}(d^*) \quad \forall h \in \ker(d_0), \psi \in \text{Im}(d^*)\), while (36) follows from (21) and (22).

To see that \(I^{(3)}_V\) vanishes note that it can be expanded as a sum of terms where each term involves an integral of the form

\[
\int_{M^y} f_{ace} (\delta g_{L^{(0,2)}_{ab}}(y, x_i)) L_{cd}(y, x_j) L_{ef}(y, x_k)
\]

(There are also terms where \(L_{cd}(y, x_j) L_{ef}(y, x_k)\) is replaced by \(L_{ce}(y, y)\) in (37) but these vanish since the integrand contains no 3-forms in \(y\) in this case). Because of (30) and (36) it follows from (35) that (37) vanishes.

The argument for the vanishing of \(I^{(2)}_V\) is slightly more involved. Note that \(\pi_{\text{tot}} = \pi_{\text{tot}}(x_1, \ldots, x_V)\) is given by a sum of terms as in (23), leading to an expression for \(I^{(2)}_V\) as a sum of corresponding terms, each consisting of an integral over \(M^V\). A number of these terms vanish for one of the following reasons:

(i) \(\pi_{ab}(x, x) = 0\). (This follows from (24)).
(ii) The integrand in the integral over $M^V$ (a differential form on $M^V$) is not of degree 3 in $x_i$ for all $i = 1, \ldots, V$. (Then the integral over $M_x$ vanishes).

(iii) The term contains an integral of the form

$$\int_{M^V} f_{abcd} h^a(y) L_{bc}(y, x_i) L_{de}(y, x_j)$$

(38)

which vanishes by (35)–(36). The only terms which do not vanish due to (i), (ii) or (iii) are those of the form

$$\int_{M^V} \{ f_{abcd} f_{bfp} h^a(y) B_{bc}^{(0,1)}(z, y) B_{de}^{(2,0)}(y, x_i) \}
\times L_{fg}(z, x_j) L_{pq}(z, x_k) \Psi_{eq}(x_i, x_j, x_k) \}$$

(39)

or

$$\int_{M^V} \{ f_{ade} f_{bcg} h^a(y) h^b(z) vol(z) B_{ed}^{(0,1)}(z, y) \}
\times L_{vf}^{(2,0)}(y, x_i) L_{gh}(z, x_j) \Phi_{fh}(x_i, x_j) \}$$

(40)

To show that these vanish it suffices to show that

$$\int_{M^V} f_{abcd} h^a(y) B_{bc}^{(0,1)}(z, y) L_{de}^{(2,0)}(y, x_i) \in \ker((d_{M_x})_0)$$

(41)

Then (39)–(40) vanish due to (35)–(36) (note for (40) that $h(z) vol(z) \in \ker(d_{M_x}^*)$).

To show (41) we begin by noting that

$$\int_{M^V} f_{acdi} h^a(y) L_{bc}^{(1,1)}(z, y) L_{de}^{(2,0)}(y, x_i) = 0$$

(42)

for the same reason that (38) vanished in (iii) above. Taking the metric-variation of this gives

$$0 = \int_{M^V} f_{acdi} h^a(y) (\delta_{bg} L_{bc}^{(1,1)}(z, y)) L_{de}^{(2,0)}(y, x_i) + \int_{M^V} f_{acdi} h^a(y) L_{bc}^{(1,1)}(z, y) \delta_{bg} L_{de}^{(2,0)}(y, x_i)
\quad = d_{M_x} \int_{M^V} f_{acdi} h^a(y) B_{bc}^{(0,1)}(z, y) L_{de}^{(2,0)}(y, x_i) + \int_{M^V} f_{acdi} h^a(y) L_{bc}^{(1,1)}(z, y) \delta_{bg} L_{de}^{(2,0)}(y, x_i)$$

(43)

where we have used (28)–(29). The first term in (43) belongs to $\text{Im}(d_{M_x})$ while the second term belongs to $\text{Im}(d_{M_x}^*)$ because of (37). Since $\text{Im}(d^*) = \ker(d)^\perp \subset \text{Im}(d)^\perp$
it follows that both terms in (43) vanish individually; the vanishing of the first term implies (41). This completes the argument for the formal metric–independence of $I_V(M, A^c)$.

Conclusion

We carried out a more careful version of the Faddeev-Popov gauge-fixing procedure for gauge theories quantised around a reducible classical solution $A^c$, finding a new refined expression (2) for the Faddeev-Popov determinant. Unlike the usual expression when $A^c$ is reducible, the ghost propagator associated with this expression is well-defined. When $A^c$ is isolated up to gauge transformations in the space of classical solutions the gauge field propagator is also well-defined, and in this case the standard perturbative techniques can be applied to the gauge-fixed functional integrals. We applied this to the partition function of Chern-Simons gauge theory on a general compact 3-manifold, showing that the previous results of S. Axelrod and I. Singer on the finiteness– and formal metric–independence of the perturbation series continue to hold for reducible $A^c$. This opens up the possibility of carrying out explicit perturbative expansions of the Chern-Simons partition function for $S^3$ and lens spaces. Our expression for the lowest order term is consistent with the previously calculated nonperturbative expressions [9] for the weak coupling (large $k$) limit of the partition function.

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