A DIFFERENT APPROACH TO ENDPOINT WEAK-TYPE ESTIMATES FOR CALDERÓN-ZYGMUND OPERATORS

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Abstract. The purpose of this article is to provide a different proof of the classical weak-type \((1, 1)\) estimate for Calderón-Zygmund operators. This is a simplification of a proof given by Nazarov, Treil, and Volberg addressing the nonhomogeneous setting. An application of this technique in the setting of weighted Lebesgue spaces is also given.

Keywords: singular integrals; weak-type estimates; weighted inequalities.

1. Introduction

Throughout this paper, \(T\) will denote a Calderón-Zygmund operator. It is a fundamental result in harmonic analysis that \(T\) satisfies the following weak-type \((1, 1)\) estimate.

**Theorem 1.** If \(f \in L^1(\mathbb{R}^n)\), then

\[
\|Tf\|_{L^1, \infty(\mathbb{R}^n)} := \sup_{t > 0} t \# \{|Tf| > t\} \lesssim \|f\|_{L^1(\mathbb{R}^n)}.
\]

Theorem 1 was originally proved using the Calderón-Zygmund decomposition, see [3, 12]. The Calderón-Zygmund decomposition crucially relies on the doubling property: a measure \(\mu\) has the doubling property if

\[
\mu(B(x, 2r)) \lesssim \mu(B(x, r))
\]

for all \(r > 0\) and all \(x\) in the space.

In [8], Nazarov, Treil, and Volberg gave a new proof of the weak-type \((1, 1)\) estimate to develop the Calderón-Zygmund theory in nonhomogeneous settings. A nonhomogeneous space is a space where the underlying measure fails to possess the doubling property, but instead satisfies the polynomial growth condition

\[
\mu(B(x, r)) \lesssim r^n.
\]

Since the doubling property is not available in this setting, their proof avoids the Calderón-Zygmund decomposition. Of course, since Lebesgue measure on \(\mathbb{R}^n\) satisfies the polynomial growth condition, the Nazarov-Treil-Volberg technique immediately gives a new proof of Theorem 1. We are interested if this proof can be simplified if one again assumes the doubling condition. Indeed, the main result of Section 3 in this paper is a new proof of the classical Theorem 1 using a simplified version of this idea.

The key idea in the Nazarov-Treil-Volberg technique in the nonhomogeneous setting is to approximate an \(L^1\) function by point mass measures, and then prove a weak-type estimate on finite linear combinations of point masses. Proving the weak-type estimate on point mass measures involves approximating the measures by appropriately constructed Borel sets. It is then left to control a final term using the size condition of the Calderón-Zygmund kernel, a Guy-David type lemma, and the Stein-Weiss duality trick involving the adjoint of \(T\).
When we allow the doubling property, the Nazarov-Treil-Volberg proof technique simplifies significantly—the weak-type estimate on point mass measures, the size condition of the kernel, the Guy-David lemma, and the Stein-Weiss duality trick are no longer needed. Instead, we may directly approximate the $L^1(\mathbb{R}^n)$ function by appropriately constructed Borel sets and, due to the doubling property of Lebesgue measure, we finish with a term that is easily controlled.

The Nazarov-Treil-Volberg technique can be adapted to handle more general situations. See, for example, the alternative proof of the weak-type $(1,\ldots,1;\frac{1}{m})$ estimate for multilinear Calderón-Zygmund operators given by the author and B. Wick in [14]. Another application is given in Section 4 of this paper where a weighted weak-type $(1,1)$ estimate is proved. We say that $w$ is an $A_p$ weight if $w$ is locally integrable, positive almost everywhere, and satisfies

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty;$$

when $p = 1$, the quantity $\left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1}$ is interpreted as $(\inf_Q w)^{-1}$.

We prove the following quantitative estimate.

**Theorem 2.** If $1 \leq p < \infty$, $w \in A_p$, and $f \in L^1(w)$, then

$$\|T(fw)^{w^{-1}}\|_{L^{1,\infty}(w)} \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} \|f\|_{L^1(w)}.$$  

Theorem 2 was proved in [10] using the Calderón-Zygmund decomposition by Ombrosi, Pérez, and Recchi. In Section 4, the Nazarov-Treil-Volberg technique is used to give a different proof of Theorem 2. Weighted weak-type inequalities have come up in related areas, further motivating their interest. See [1,2,6,9] for other results.

The Calderón-Zygmund decomposition and Nazarov-Treil-Volberg proofs follow similar schemes. We describe the proofs in the context of Lebesgue measure on $\mathbb{R}^n$. To prove Theorem 1, one shows

$$\{|\{Tf\} > t\| \lesssim t^{-1}\|f\|_{L^1(\mathbb{R}^n)}$$

for all $t > 0$ and all $f \in L^1(\mathbb{R}^n)$. Both proof techniques involve decomposing $f \in L^1(\mathbb{R}^n)$ into summands,

$$f = g + b = g + \sum_{i=1}^{\infty} b_i$$

where $g$ is “good” and $b$ is “bad,” and then controlling

$$\{|\{Tf\} > t\| \leq \left| \{Tg > \frac{t}{2}\} \right| + \left| \{Tb > \frac{t}{2}\} \right|.$$  

In both proofs, the term with the operator applied to $g$ is handled by using Chebyshev’s inequality, the boundedness of $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and the $L^\infty(\mathbb{R}^n)$ norms of $g$ to obtain the desired estimate. Since the decompositions of $f$ into “good” and “bad” pieces are different in the two proofs, the terms with the operator applied to $b$ are estimated differently.

Much of the effort in the Calderón-Zygmund decomposition method is spent in carefully decomposing $f$ into its “good” and “bad” parts so that certain desirable properties hold. In this case, the functions $b_i$ have mean value zero, are supported on cubes of appropriate
measure, and have useful $L^1(\mathbb{R}^n)$ control. The decomposition also produces an exceptional set, $\Omega^*$, which is related to the support of $b$ and has appropriate measure. One then estimates

$$\left| \left\{ |Tb| > \frac{t}{2} \right\} \right| \leq |\Omega^*| + \left| \left\{ x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \frac{t}{2} \right\} \right|.$$ 

The first term is controlled by the construction in the Calderón-Zygmund decomposition. To control the final term, one uses the cancellation involved in the $b_i$ to introduce a term with the kernel evaluated at the center of the cube on which $b_i$ is supported. One can then use the smoothness assumption of the kernel and the $L^1(\mathbb{R}^n)$ control of the $b_i$ to recover the estimate.

Using the Nazarov-Treil-Volberg method, the decomposition of $f$ into its “good” and “bad” parts is more straightforward. The exceptional set, $G$, is defined explicitly at the start, and $g$ and $b$ are defined as restrictions of $f$ to $\mathbb{R}^n \setminus G$ and $G$ respectively. To remedy the fact that there is no cancellation in $b$, we apply a Whitney decomposition to write $G$ as a union of dyadic cubes with disjoint interiors and restrict $b$ to each cube, writing $b = \sum_{i=1}^{\infty} b_i$.

To introduce cancellation in the $b_i$, we construct a Borel set, $E_i$, with the same center as $\text{supp}(b_i)$ and with appropriate measure; a related set, $E^*$, is included in the exceptional set. Adding and subtracting a function defined using characteristic functions of the $E_i$, $\sigma$, allows for the estimate

$$\left| \left\{ |Tb| > \frac{t}{2} \right\} \right| \leq |E^* \cup G| + \left| \left\{ x \in \mathbb{R}^n \setminus (E^* \cup G) : |Tb(x) - t\sigma(x)| > \frac{t}{4} \right\} \right| + \left| \left\{ |\sigma| > \frac{1}{4} \right\} \right|.$$ 

The first term is controlled by construction of the exceptional sets (the control of $|E^*|$ is the only place where the doubling property is used). The second term is controlled since $\sigma$ introduces cancellation that allows for the use of the smoothness assumption of the Calderón-Zygmund kernel. The final term is controlled in a way similar to the the term with the “good” function.

The Nazarov-Treil-Volberg proof has some benefits over the Calderón-Zygmund decomposition proof. For example, the decomposition used to write $f = g + b$ in this argument does not require a stopping time argument, the Hardy-Littlewood maximal function, or the doubling property. The doubling property is only used later in the proof to gain extra separation in the exceptional set. Also, this proof shows that the $L^1$ control of the $b_i$ given in the Calderón-Zygmund decomposition is not necessary for the weak-type $(1, 1)$ estimate.

Section 2 describes the notation and definitions, as well as a lemma that will be used in the proof of the main results. Section 3 includes the new proof of Theorem 1. Section 4 contains the new proof of Theorem 2.

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2. PRELIMINARIES

Throughout the paper we will use the notation $A \lesssim B$ if there exists $C > 0$, possibly depending on $n$ or $T$, so that $A \leq CB$. The Lebesgue measure of $A \subseteq \mathbb{R}^n$ is represented by $|A|$. For a weight $w$, the quantity $\int_A w(x)dx$ is represented by $w(A)$. The cube with center $x \in \mathbb{R}^n$ and side length $r$ is denoted $Q(x, r)$. If $Q$ is a cube, then $rQ$ denotes the cube with the same center as $Q$ and side length equal to $r$ times the side length of $Q$.

We say $K : \mathbb{R}^n \rightarrow \mathbb{C}$ is a Calderón-Zygmund kernel if there exists $\delta > 0$ such that the following conditions hold:
(1) (size) 
\[ |K(x, y)| \lesssim \frac{1}{|x - y|^n} \]
for all \( x, y \in \mathbb{R}^n \) with \( x \neq y \),

(2) (smoothness) 
\[ |K(x, y) - K(x', y)| \lesssim \frac{|x - x'|^\delta}{|x - y|^{n+\delta}} \]
whenever \( |x - x'| \leq \frac{1}{2} |x - y| \), and
\[ |K(x, y) - K(x, y')| \lesssim \frac{|y - y'|^\delta}{|x - y|^{n+\delta}} \]
whenever \( |y - y'| \leq \frac{1}{2} |x - y| \).

Let \( \mathcal{S}(\mathbb{R}^n) \) denote the space of Schwartz functions on \( \mathbb{R}^n \) and \( \mathcal{S}'(\mathbb{R}^n) \) the space of tempered distributions on \( \mathbb{R}^n \). We say a linear operator \( T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) is a Calderón-Zygmund operator with kernel \( K \) if \( K \) is a Calderón-Zygmund kernel, \( T \) extends to a bounded operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \), and
\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy
\]
for almost every \( x \in \mathbb{R}^n \setminus \text{supp}(f) \).

**Lemma 1.** If \( f : \mathbb{R}^n \to \mathbb{C} \) is such that \( \text{supp}(f) \subseteq Q(x, r) \) and \( \int_{Q(x, r)} f(y) dy = 0 \), then
\[
\int_{\mathbb{R}^n \setminus Q(x, 2\sqrt{n}r)} |Tf(y)| dy \lesssim \|f\|_{L^1(\mathbb{R}^n)}.
\]

**Proof.** First, notice that since \( \int_{Q(x, r)} f(y) dy = 0 \) and \( \text{supp}(f) \subseteq Q(x, r) \),
\[
|Tf(y)| = \left| \int_{Q(x, r)} K(y, z) f(z) dz \right| = \left| \int_{Q(x, r)} (K(y, z) - K(y, x)) f(z) dz \right|.
\]
Therefore, using the Fubini theorem and the smoothness estimate of \( K \), we see
\[
\int_{\mathbb{R}^n \setminus Q(x, 2\sqrt{n}r)} |Tf(y)| dy \leq \int_{\mathbb{R}^n \setminus Q(x, 2\sqrt{n}r)} \int_{Q(x, r)} |K(y, z) - K(y, x)| |f(z)| dz dy
\]
\[
= \int_{Q(x, r)} |f(z)| \int_{\mathbb{R}^n \setminus Q(x, 2\sqrt{n}r)} |K(y, z) - K(y, x)| dy dz
\]
\[
\leq \int_{Q(x, r)} |f(z)| \int_{d(x, y) \geq 2d(x, z)} |K(y, z) - K(y, x)| dy dz
\]
\[
\lesssim \|f\|_{L^1(\mathbb{R}^n)}.
\]

\[ \Box \]
3. Unweighted Estimate

We give a proof of the following classical theorem. This is a simplification of the proof given in [8].

**Theorem 1.** If $f \in L^1(\mathbb{R}^n)$, then

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}.$$

**Proof of Theorem 1.** Let $t > 0$ be given. We wish to show

$$|\{|Tf| > t\}| \lesssim t^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

By density, we may assume $f$ is a continuous function with compact support. Set $G := \{|f| > t\}$.

Apply a Whitney decomposition to write

$$G = \bigcup_{i=1}^{\infty} Q_i,$$

a disjoint union of dyadic cubes where

$$2\sqrt{n}l(Q_i) \leq d(Q_i, \mathbb{R}^n \setminus G).$$

Put

$$g := f1_{\mathbb{R}^n \setminus G}, \quad b := f1_G, \quad \text{and} \quad b_i := f1_{Q_i}.$$

Then

$$f = g + b = g + \sum_{i=1}^{\infty} b_i,$$

and

$$|\{|Tf| > t\}| \leq \left|\{|Tg| > \frac{t}{2}\}\right| + \left|\{|Tb| > \frac{t}{2}\}\right|$$

$$= I + II.$$

To control $I$, use the Chebyshev inequality, the boundedness of $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and the fact that $\|g\|_{L^\infty(\mathbb{R}^n)} \leq t$ to estimate

$$I \lesssim t^{-2} \int_{\mathbb{R}^n} |Tg(x)|^2 dx \lesssim t^{-2} \int_{\mathbb{R}^n} |g(x)|^2 dx \lesssim t^{-1} \int_{\mathbb{R}^n} |g(x)| dx \leq t^{-1} \|f\|_{L^1(\mathbb{R}^n)}.$$

We now control $II$. Let $c_i$ denote the center of $Q_i$ and let $a_i := \int_{Q_i} b_i(x) dx$. Set

$$E_1 := Q(c_1, r_1)$$

where $r_1 > 0$ is chosen so that $|E_1| = \frac{|a_1|}{t}$. In general, for $i = 2, 3, \ldots$, set

$$E_i := Q(c_i, r_i) \setminus \bigcup_{k=1}^{i-1} E_k$$

and

$$|\{|Tb_i| > \frac{t}{2}\}| \leq \frac{C}{t} \|b_i\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{t} |Q_i| \leq \frac{C}{t}.$$
where \( r_i > 0 \) is chosen so that \(|E_i| = \frac{|a_i|}{t}\). Note that such \( E_i \) exist since the function \( r \mapsto |Q(x, r)| \) is continuous for each \( x \in \mathbb{R}^n \). Define
\[
E^*_1 := Q(c_1, 2\sqrt{n}r_1),
\]
and, for general \( i = 2, 3, \ldots \), set
\[
E^*_i := Q(c_i, 2\sqrt{n}r_i) \setminus \bigcup_{k=1}^{i-1} E^*_k.
\]
Set
\[
E := \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad E^* := \bigcup_{i=1}^{\infty} E^*_i.
\]
Let \( 1 \leq j_1, j_2, \ldots \) be the indices where \( a_{j_i} \geq 0 \) and \( 1 \leq k_1, k_2, \ldots \) be the indices where \( a_{k_i} < 0 \). Set
\[
E^+ := \bigcup_{i=1}^{\infty} E_{j_i} \quad \text{and} \quad E^- := \bigcup_{i=1}^{\infty} E_{k_i}.
\]
Define
\[
\sigma := T(\mathbb{1}_{E^+} - \mathbb{1}_{E^-}).
\]
Then
\[
\Pi \leq |E^* \cup G| + \left| \left\{ \mathbb{R}^n \setminus (E^* \cup G) : |Tb - t\sigma| > \frac{t}{4} \right\} \right| + \left| \left\{ |\sigma| > \frac{1}{4} \right\} \right|
\]
\[
= \Pi_1 + \Pi_2 + \Pi_3.
\]
We first control \( \Pi_1 \). Use the doubling property of Lebesgue measure and the Chebyshev inequality to estimate
\[
\Pi_1 \leq |G| + \sum_{i=1}^{\infty} |E^*_i|
\]
\[
\lesssim |G| + \sum_{i=1}^{\infty} |E_i|
\]
\[
\leq t^{-1}\|f\|_{L^1(\mathbb{R}^n)} + t^{-1} \sum_{i=1}^{\infty} \|b_i\|_{L^1(\mathbb{R}^n)}
\]
\[
\lesssim t^{-1}\|f\|_{L^1(\mathbb{R}^n)}
\]
For \( \Pi_2 \), use the Chebyshev inequality, and Lemma 1, which applies since
\[
\text{supp}(b_{j_i} - t\mathbb{1}_{E_{j_i}}) \subseteq Q_{j_i} \cup Q(c_{j_i}, r_{j_i}), \quad \int_{\mathbb{R}^n} b_{j_i}(x) - t\mathbb{1}_{E_{j_i}}(x)dx = 0, \quad \text{and}
\]
\[
2\sqrt{n}Q_{j_i} \cup Q(c_{j_i}, 2\sqrt{n}r_{j_i}) = 2\sqrt{n}Q_{j_i} \cup E^*_{j_i} \subseteq E^* \cup G;
\]
and similarly since
\[
\text{supp}(b_{k_i} + t\mathbb{1}_{E_{k_i}}) \subseteq Q_{k_i} \cup Q(c_{k_i}, r_{k_i}), \quad \int_{\mathbb{R}^n} b_{k_i}(x) + t\mathbb{1}_{E_{k_i}}(x)dx = 0, \quad \text{and}
\]
\[
2\sqrt{n}Q_{k_i} \cup Q(c_{k_i}, 2\sqrt{n}r_{k_i}) = 2\sqrt{n}Q_{k_i} \cup E^*_{k_i} \subseteq E^* \cup G,
\]
to estimate
\[ II_2 \lesssim t^{-1} \int_{\mathbb{R}^n \setminus (E^* \cup G)} |Tb(x) - t\sigma(x)| \, dx \]
\[ \leq t^{-1} \sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus (E^* \cup G)} |T(b_{j_i} - t1_{E_{j_i}})(x)| \, dx + t^{-1} \sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus (E^* \cup G)} |T(b_{k_i} + t1_{E_{k_i}})(x)| \, dx \]
\[ \lesssim t^{-1} \sum_{i=1}^{\infty} \|b_{j_i} - t1_{E_{j_i}}\|_{L^1(\mathbb{R}^n)} + t^{-1} \sum_{i=1}^{\infty} \|b_{k_i} + t1_{E_{k_i}}\|_{L^1(\mathbb{R}^n)} \]
\[ \leq t^{-1} \sum_{i=1}^{\infty} (\|b_i\|_{L^1(\mathbb{R}^n)} + \|t1_{E_i}\|_{L^1(\mathbb{R}^n)}) \]
\[ \leq t^{-1} \|f\|_{L^1(\mathbb{R}^n)}. \]

We will control \( II_3 \) in the same way that we controlled \( I \). Use the Chebyshev inequality, the boundedness of \( T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \), and the fact that \( |E| \leq t^{-1} \|f\|_{L^1(\mathbb{R}^n)} \) to estimate
\[ II_3 \lesssim \int_{\mathbb{R}^n} |T(1_{E^+} - 1_{E^-})(x)|^2 \, dx \]
\[ \lesssim \int_{\mathbb{R}^n} |1_{E^+}(x) - 1_{E^-}(x)|^2 \, dx \]
\[ = \int_{\mathbb{R}^n} 1_{E}(x)dx \]
\[ \leq t^{-1} \|f\|_{L^1(\mathbb{R}^n)}. \]

Putting the estimates for \( I, II_1, II_2, \) and \( II_3 \) together, we get
\[ |\{Tf > t\}| \leq I + II_1 + II_2 + II_3 \lesssim t^{-1} \|f\|_{L^1(\mathbb{R}^n)}. \]
This completes the proof. \( \square \)

4. Weighted Estimate

The main difficulty in adapting the proof of Section 3 in the unweighted setting to the weighted setting is controlling the term with the “good” function. Hytönen’s solution of the \( A_2 \) conjecture in [4] is used to handle this term.

**Theorem 3.** If \( 1 < p < \infty \) and \( w \in A_p \), then \( T \) is bounded from \( L^p(w) \) to \( L^p(w) \) and
\[ \|T\|_{L^p(w) \to L^p(w)} \lesssim pp'[w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}. \]

We now give the new proof of Theorem 2. This is an adaptation of the proof given in Section 3.

**Theorem 2.** If \( 1 \leq p < \infty \), \( w \in A_p \), and \( f \in L^1(w) \), then
\[ \|T(fw)w^{-1}\|_{L^{1,\infty}(w)} \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} \|f\|_{L^1(w)}. \]
Proof of Theorem 2. Let \( t > 0 \) be given. We wish to show
\[
w(\{|T(fw)|w^{-1} > t\}) \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} t^{-1} \|f\|_{L^1(w)}.
\]
Assume that \( \int_{\mathbb{R}^n} w(x)dx > t^{-1} \|f\|_{L^1(w)} \) (otherwise there is nothing to prove). By density, we may assume \( f \) is a continuous function with compact support. Set
\[
G := \{|f| > t\}.
\]
Apply a Whitney decomposition to write
\[
G = \bigcup_{i=1}^{\infty} Q_i,
\]
a disjoint union of dyadic cubes where
\[
2\sqrt{n}l(Q_i) \leq d(Q_i, \mathbb{R}^n \setminus G).
\]
Put
\[
g := f \mathbb{1}_{\mathbb{R}^n \setminus G}, \quad b := f \mathbb{1}_G, \quad \text{and} \quad b_i := f \mathbb{1}_{Q_i}.
\]
Then
\[
f = g + b = g + \sum_{i=1}^{\infty} b_i,
\]
and
\[
w(\{|T(fw)|w^{-1} > t\}) \leq w \left( \left\{|T(gw)|w^{-1} > \frac{t}{2}\right\} \right) + w \left( \left\{|T(bw)|w^{-1} > \frac{t}{2}\right\} \right) = I + II.
\]
To control \( I \), let \( r > p \) be a constant to be chosen later (we will actually choose \( r \) so that \( r > 2 \) as well). Then \( w \in A_w, [w]_{A_r} \leq [w]_{A_p}, w^{1-r'} \in A_r, \) and \([w^{1-r'}]_{A_r} = [w]_{A_r}^{r'-1} \). Use the Chebyshev inequality, Theorem 3, the fact that \( \|g\|_{L^\infty(\mathbb{R}^n)} \leq t \), and the facts listed above to estimate
\[
I \leq 2^r t^{-r'} \int_{\mathbb{R}^n} |T(gw)(x)|^{r'} w(x)^{1-r'} dx \leq \left( r^r \left[\frac{1}{w^{1-r'}}\right]_{A_r}^{\max\{1, \frac{1}{r'-1}\}} \right) t^{-r'} \int_{\mathbb{R}^n} |g(x)|^{r'} w(x) dx \leq r^r [w]_{A_r}^{r'} \int_{\mathbb{R}^n} |g(x)| w(x) dx \leq r^r [w]_{A_p}^{r'} t^{-1} \|f\|_{L^1(w)}.
\]
We need to address the factors \( r^r \) and \([w]_{A_p}^{r'} \). First consider \( r^r \). Let \( h(x) = \frac{1}{x}(1 + x)^{1 + \frac{1}{x}} \). Note that \( h(1) = 4 \) and that \( h'(x) = -\frac{1}{x^2}(1 + x)^{1 + \frac{1}{x}} \log(1 + x) \leq 0 \) for all \( x \in [1, \infty) \). Thus \( h(x) \leq 4 \) for all \( x \in [1, \infty) \). In particular, letting
\[
r = 1 + \max\{p, \log(e + [w]_{A_p})\} > 2
\]
and computing
\[
r' = 1 + \frac{1}{\max\{p, \log(e + [w]_{A_p})\}} < 2,
\]
we have
\[
\frac{r^{r'}}{\max\{p, \log(e + [w]_{A_p})\}} = h(\max\{p, \log(e + [w]_{A_p})\}) \leq 4.
\]
Thus
\[
r' \leq 4 \max\{p, \log(e + [w]_{A_p})\}.
\]

Now consider \([w]_{A_p}^{r'}\). Set \(k(x) = x^{\frac{1}{\log(e+x)}}\). Notice that \(k(1) = 1, \lim_{x \to \infty} k(x) = e\), and \(k'(x) = x^{\frac{1}{\log(e+x)}} \left(\frac{1}{x^{\log(e+x)}} - \frac{\log(x)}{(e+x)(\log(e+x))^2}\right) \geq 0\) for all \(x \in [1, \infty)\). Thus \(1 \leq k(x) \leq e\) for all \(x \in [1, \infty)\). In particular,
\[
[w]_{A_p}^{r' - 1} = [w]_{A_p}^{\frac{1}{\max\{p, \log(e + [w]_{A_p})\}}} \leq [w]_{A_p}^{\log(e + [w]_{A_p})} = k([w]_{A_p}) \leq e.
\]

Thus
\[
[w]_{A_p}^{r'} \leq e[w]_{A_p}.
\]

Substituting this into the previous estimate yields
\[
I \lesssim r' [w]_{A_p}^{r'} t^{-1} \|f\|_{L^1(w)} \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} t^{-1} \|f\|_{L^1(w)}.
\]

We now control II. Let \(c_i\) denote the center of \(Q_i\) and let \(a_i := \int_{Q_i} b_i(x)w(x)dx\). Set
\[
E_1 := Q(c_1, r_1)
\]
where \(r_1 > 0\) is chosen so that \(w(E_1) = \frac{|a_1|}{t}\). In general, for \(i = 2, 3, \ldots\), set
\[
E_i := Q(c_i, r_i) \setminus \bigcup_{k=1}^{i-1} E_k
\]
where \(r_i > 0\) is chosen so that \(w(E_i) = \frac{|a_i|}{t}\). Note that such \(E_i\) exist since the function \(r \mapsto w(Q(x, r))\) increases to \(\int_{\mathbb{R}^n} w(x)dx\) (or to \(\infty\) if \(v\) is not integrable) as \(r \to \infty\), approaches 0 as \(r \to 0\), and is continuous from the right for almost every \(x \in \mathbb{R}^n\). Also define
\[
E_1^* := Q(c_1, 2\sqrt{n}r_1),
\]
and, for general \(i = 2, 3, \ldots\), set
\[
E_i^* := Q(c_i, 2\sqrt{n}r_i) \setminus \bigcup_{k=1}^{i-1} E_k^*.
\]

Set
\[
E := \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad E^* := \bigcup_{i=1}^{\infty} E_i^*.
\]

Let \(1 \leq j_1, j_2, \ldots\) be the indices where \(\int_{Q_{j_i}} b_{j_i}(x)w(x)dx \geq 0\) and \(1 \leq k_1, k_2, \ldots\) be the indices where \(\int_{Q_{k_i}} b_{k_i}(x)w(x)dx < 0\). Set
\[
E^+ := \bigcup_{i=1}^{\infty} E_{j_i} \quad \text{and} \quad E^- := \bigcup_{i=1}^{\infty} E_{k_i}.
\]

Define
\[
\sigma := T(v(1_{E^+} - 1_{E^-})).
\]
Then
\[ \Pi \leq w(E^* \cup G) + w \left( \left\{ \mathbb{R}^n \setminus (E^* \cup G) : |T(bw) - t\sigma|w^{-1} > \frac{t}{4} \right\} \right) \]
\[ + w \left( \left\{ |\sigma|w^{-1} > \frac{1}{4} \right\} \right) \]
\[ = \Pi_1 + \Pi_2 + \Pi_3. \]

We first control \( \Pi_1 \). Use the doubling property of \( w \), the Chebyshev inequality, and the facts that \([w]_{A_p} \geq 1\) and \( \max\{p, \log(e + [w]_{A_p})\} \geq 1\) to estimate
\[ \Pi_1 \leq w(G) + \sum_{i=1}^{\infty} w(E_i^*) \]
\[ \lesssim w(G) + [w]_{A_p} \sum_{j=1}^{\infty} w(E_j) \]
\[ \leq t^{-1}\|f\|_{L^1(w)} + [w]_{A_p} t^{-1} \sum_{i=1}^{\infty} \|b_i\|_{L^1(w)} \]
\[ \leq [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} t^{-1}\|f\|_{L^1(w)}. \]

For \( \Pi_2 \), use the Chebyshev inequality and Lemma 1, which applies since
\[ \text{supp}(b_j, w - tw \mathbb{1}_{E_j}) \subseteq Q_{j_i} \cup Q(c_{j_i}, r_{j_i}), \int_{\mathbb{R}^n} b_j(x)w(x) - tw(x) \mathbb{1}_{E_j}(x) dx = 0, \text{ and} \]
\[ 2\sqrt{n}Q_{j_i} \cup Q(c_{j_i}, 2\sqrt{n}r_{j_i}) = 2\sqrt{n}Q_j \cup E_j^* \subseteq E^* \cup G; \]
and similarly since
\[ \text{supp}(b_k, w + tw \mathbb{1}_{E_k}) \subseteq Q_{k_i} \cup Q(c_{k_i}, r_{k_i}), \int_{\mathbb{R}^n} b_k(x)w(x) + tw(x) \mathbb{1}_{E_k}(x) dx = 0, \text{ and} \]
\[ 2\sqrt{n}Q_{k_i} \cup Q(c_{k_i}, 2\sqrt{n}r_{k_i}) = 2\sqrt{n}Q_k \cup E_k^* \subseteq E^* \cup G, \]
to estimate
\[ \Pi_2 \lesssim t^{-1} \int_{\mathbb{R}^n \setminus (E^* \cup G)} |T(bw - t\sigma)(x)| dx \]
\[ \leq t^{-1} \sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus (E^* \cup G)} |T(b_j, w - tw \mathbb{1}_{E_j})(x)| dx + t^{-1} \sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus (E^* \cup G)} |T(b_k, w + tw \mathbb{1}_{E_k})(x)| dx \]
\[ \lesssim t^{-1} \sum_{i=1}^{\infty} \|b_j - t \mathbb{1}_{E_j}\|_{L^1(w)} + t^{-1} \sum_{i=1}^{\infty} \|b_k + t \mathbb{1}_{E_k}\|_{L^1(w)} \]
\[ \leq t^{-1} \sum_{i=1}^{\infty} \left( \|b_i\|_{L^1(w)} + \|t \mathbb{1}_{E_i}\|_{L^1(w)} \right) \]
\[ \lesssim t^{-1} \sum_{i=1}^{\infty} \|b_i\|_{L^1(w)} \]
\[ \leq [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} t^{-1}\|f\|_{L^1(w)}. \]
We will control $\Pi_3$ in the same way that we controlled $I$. Let
\[ r = 1 + \max\{p, \log(e + [w]_{A_p})\} \]
and compute
\[ r' = 1 + \frac{1}{\max\{p, \log(e + [w]_{A_p})\}}. \]
Then $r > p$, $r > 2$, $w \in A_r$, $[w]_{A_r} \leq [w]_{A_p}$, $w^{1-r'} \in A_{r'}$, and $[w^{1-r'}]_{A_{r'}} = [w]_{A_r}^{r'-1}$. Use the Chebyshev inequality, Theorem 3, and the facts above to estimate
\[ \Pi_3 \leq 4^{r'} \int_{\mathbb{R}^n} |\sigma(x)|^{r'} w(x)^{1-r'} dx \]
\[ \lesssim \int_{\mathbb{R}^n} |T(w(1_{E^+} - 1_{E^-}))|^{r'} w(x)^{1-r'} dx \]
\[ \leq \left( r r' \left[ w^{1-r'} \right]_{A_{r'}}^{\max\{1, \frac{1}{r'}\}} \right)^{r'} \int_{\mathbb{R}^n} |1_{E^+}(x) - 1_{E^-}(x)|^{r'} w(x) dx \]
\[ \lesssim r r' [w]_{A_p}^{r'} \int_{\mathbb{R}^n} 1_E(x) w(x) dx \]
\[ \leq r r' [w]_{A_p}^{r'} t^{-1} \|f\|_{L^1(w)}. \]
As before, $r' \leq 4 \max\{p, \log(e + [w]_{A_p})\}$ and $[w]_{A_p}^{r'} \leq e[w]_{A_p}$, so
\[ \Pi_3 \lesssim r r' [w]_{A_p}^{r'} t^{-1} \|f\|_{L^1(w)} \]
\[ \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} t^{-1} \|f\|_{L^1(w)}. \]

Putting the estimates for $I, \Pi_1, \Pi_2$, and $\Pi_3$ together, we get
\[ w(\{|T(fw)|w^{-1} > t\}) \leq I + \Pi_1 + \Pi_2 + \Pi_3 \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} \|f\|_{L^1(w)}. \]
This completes the proof. \qed

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