Braid Groups of the Sun Graph

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Abstract

In this article we calculate the n-string braid groups of certain non-contractible graphs. We use techniques from the work of A. Abrams, F. Connolly and M. Doig combined with Van Kampen’s Theorem to prove these results.

1 Introduction and Statement of Results

Recently, there has been some interest in the braid groups of a graph $G$ (see for example [1], [2], [3], [4], [5], [6]). These groups, $B_n(G,c)$, were calculated indirectly in [6] in the case where $G$ is a star. Doig critically extends this result in [5] by providing explicit generators for $B_n(G,c)$ for any star $G$. In [4], Connolly and Doig provide a method of calculating $B_n(T,c)$ where $T$ is a linear tree. Their method seems limited to the realm of contractible graphs (trees).

In this paper we exhibit a free basis of order $n$ for $B_n(L_1,c)$ for a certain non-contractible graph $L_1$. We also provide an explicit free basis of order $n + 1$ for $B_2(L_n,c)$ for certain non-contractible graphs $L_n$ with $n$ nodes.

We define $L_n$ and configuration spaces and braid groups as follows:

Definition 1.1. (Sun Graph)

Define $A_j = \{re^{2ij\pi/n} : 1 \leq r \leq 2\}$ for each $j$, $0 \leq j < n$. We then define the $n$-ray sun graph as $L_n = S^1 \cup A_0 \cup A_1 \cup ... \cup A_{n-1}$.

Fig. 1: Example Sun Graphs

\begin{center}
\begin{tabular}{ccc}
\hspace{2cm} & \hspace{2cm} \\
$L_1$ & $L_2$ & $L_8$
\end{tabular}
\end{center}

Definition 1.2. (Configuration Space and Braid Group)

Let $X$ be a space. The $n$-point configuration space of $X$ is
\[
U_n^{top}(X) = \{ c \subset X : |c| = n \},
\]
given the quotient topology from the natural surjection
\[
p : X^n - \Delta \to U_n^{top}(X)
\]
\[
(x_1, x_2, ..., x_n) \mapsto \{ x_1, x_2, ..., x_n \}
\]
where \( \Delta = \{ (x_1, x_2, ..., x_n) \in X^n : x_i = x_j \text{ for some } i \neq j \} \) is the fat diagonal of \( X^n \). The \( n \)-string braid group of \( X \) is
\[
B_n(X, c) = \pi_1(U_n^{top}(X), c)
\]
where \( c \in U_n^{top}(X) \).

In order to establish our results, we will need a version of Van Kampen’s theorem to compute \( \pi_1(A \cup B) \) when \( A \cap B \) is not path connected. This seems to be a well known folk theorem, but no explicit reference is available. For the reader’s convenience, the theorem and a proof are given in the Appendix.

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2 A discussion of generators of \( B_n(S_3) \)

Let \( S_3 \) denote the graph on 4 vertices with one vertex \( v_0 \) of degree 3 and three vertices \( v_1, v_2 \) and \( v_3 \) of degree one. Label the arms of \( S_3 \) as \( A_j = [v_0, v_j] \) for \( j = 1, 2, 3 \).

Fig. 2: \( S_3 \)

\[
\begin{array}{c}
v_3 \\
A_3 \\
v_0 \\
A_2 \\
A_1 \\
v_1
\end{array}
\]

As we later make repeated use of embeddings of \( S_3 \) into \( L_n \), it is not necessary to specify the generators of \( B_n(S_3) \). We later use the images of these generators to give us generators of \( B_n(L_n) \).

In [4], Connolly and Doig construct a one-dimensional deformation retract, \( D_n(S_3) \), of \( U_n^{top}(S_3) \). A vertex \( c \) in \( D_n(S_3) \) is uniquely determined by the
quantities \( a_i = |A_i \cap c| \). Therefore, we may represent each vertex by its 3-tuple, \((a_1, a_2, a_3)\). We classify vertices as type I and type II following Construction 2.2 of [4]. If \( c \) is a type I vertex, then \( a_1 + a_2 + a_3 = n + 2 \) with \( a_i < 0 \forall i \). If \( c \) is a type II vertex, then \( a_1 + a_2 + a_3 = n \) with \( a_i \geq 0 \forall i \). Let \( c^{(n)} \) be the configuration \((n, 1, 1)\).

Doig [5] and Connolly-Doig [4] prove that \( B_n(S_3, c^{(n)}) \) is a free group on \( \binom{n}{2} \) generators. In [4], Connolly and Doig construct an explicit a maximal tree \( T \) of the deformation retract \( D_n(S_3) \). We now exploit the work of Connolly-Doig to give an explicit set of free generators for \( B_n(S_3, c_3) \).

The edges in \( D_n(S_3) - T \) are those edges beginning at the type I vertex \((a_1, a_2, a_3)\) having \( a_3 > 1 \) and ending at the type II vertex \((a_1 - 1, a_2, a_3 - 1)\). We denote each such directed edge by \( e_{a_1, a_2, a_3} \), where \((a_1, a_2, a_3)\) is its associated type I vertex. Let \( \rho \) and \( \rho' \) be paths in \( T \) from the base vertex \((n, 1, 1) = c^{(n)} \) to the type I and type II vertices of \( e_{a_1, a_2, a_3} \), respectively. Then the edge \( e_{a_1, a_2, a_3} \) determines a loop, \( \sigma_{a_1, a_2, a_3} = \rho \cdot e_{a_1, a_2, a_3} \cdot \rho'^{-1} \). Because \( T \) is a maximal tree in the graph \( D_n(S_3) \), Van Kampen’s Theorem implies that the homotopy classes of these loops, \( \sigma_{a_1, a_2, a_3} \), form a free basis for \( B_n(S_3, c^{(n)}) \).

### 3 The n-string braid group of the 1-ray sun graph

In this section we make use of a result of Abrams to give a free basis for \( B_n(L_1, c) \). In [1], Abrams defines the combinatorial configuration space, \( UC_n(G) \), of a graph \( G \), and proves that this is a deformation retract of \( U_n^{top}(G) \). We define a slightly more general space, \( U_n(G) \).

**Definition 3.1. (Discretized Configuration Space)**

Let \( G \) be a graph with all edges of length \( \kappa \). Then the discretized configuration space, \( U_n(G) \), is defined as the subset of \( U_n^{top}(G) \) consisting of all configurations in which for any two points in the configuration we can find an open edge \( e \) between them.

When the distance between any two essential vertices on \( G \) is at least \( \kappa(n+1) \) and the length of any cycle in \( G \) is at least \( \kappa(n+1) \), Abrams (II) proves that \( UC_n(G) \) is a deformation retract of \( U_n^{top}(G) \). As we change only the standard edge length, this proof also shows that \( U_n(G) \) is a deformation retract of \( U_n^{top}(G) \).

Set \( \kappa = \pi/(2n - 1) \). Let \( L = S^1 \cup [1, 2\kappa(n - 1) + 1] \), a subset of \( C \). Clearly, \( L \approx L_1 \) and therefore \( B_n(L, c) = B_n(L_1, c') \) for any \( c \in U_n^{top}(L) \) and \( c' \in U_n^{top}(L_1) \). We make \( L \) into a graph by dividing \( S^1 \) into \( 4n - 2 \) edges of length \( \kappa \) such that 1 is a vertex. We divide the ray into \( 2n - 2 \) edges of length \( \kappa \).
We define \( I \subset S^1 \) to be the closed interval of length \( \kappa/2 \) centered at -1. We define \( Y = Cl_L(L - I) \). Let \( S_3 \) be as before. However, specify that each arm of \( S_3 \) has length \( 2\kappa(n - 1) \). There is a natural injective, distance preserving map \( i: S_3 \to Y \) such that \( i(v_0) = 1 \) and \( i(A_1) = [1, 2\kappa(n - 1) + 1] \). The map \( i \) induces an injective map \( i_c: U_n(S_3) \to U_n(L) \). We define \( c_n = i_c(c^{(n)}) \). Note that \( i_c(D_n(S_3)) \subset U_n(L) \), where the unit length for \( D_n(S_3) \) is taken to be 2\( \kappa \) (instead of 1 as in [5]).

Let \( \beta_0 \) be the homotopy class of the loop in \( U_n(L) \) at the base point \( c^n \) in which the point at 1 moves once counterclockwise around \( S^1 \) and the other points remain fixed. The map \( i_c \) induces a homomorphism \( i_c^*: B_n(S_3, c^{(n)}) \to B_n(L, c^n) \) of fundamental groups. Let \( \beta_i = i_c^*([\sigma_{i,n-i,2}]) \) for \( 1 \leq i < n \), where \( [\sigma_{i,n-i,2}] \) is a generator of \( B_n(S_3, c^{(n)}) \), as described in section 2.

The purpose of this section is to prove the following:

**Theorem 3.1.** \( B_n(L, c^n) = F(\beta_0, \beta_1, \ldots, \beta_{n-1}) \), the free group on the \( n \) letters \( \beta_0, \beta_1, \ldots, \beta_{n-1} \).

**Proof.** Define \( *_0 = \{x\} \cup c^{n-1} \) and \( *_1 = \{y\} \cup c^{n-1} \) where \( x \) and \( y \) are the uppermost and lowermost points in \( I \). Define \( A \) and \( B \) as follows:

\[
A = \{ c \in U_n(L) | c \cap I \neq \emptyset \} \\
B = U_n(L) \cap U_n^{top}(Y) \text{ where } Y = Cl_L(L - I)
\]

Notice that \( B \) contains all configurations in \( U_n(L) \) containing \( x \) or \( y \), and that at most one point can be in \( I \) in any given configuration in \( U_n(L) \) because the length of \( I \) is less than \( \kappa \). Note that \( U_n(L) = A \cup B \). We will apply the Generalized Van Kampen’s Theorem (see Appendix) to these sets.

We have \( A \cap B = C \cup D \), where:

\[
C = \{ c \in U_n(L) | x \in c \} \\
D = \{ c \in U_n(L) | y \in c \}
\]

Note that both \( C \) and \( D \) are deformation retracts of \( A \).
We note that $i_c(D_n(S_3))$ is a deformation retract of $B$. Therefore $\pi_1(B, *_0) \cong F(n)$ by the formula given in [5] for the $n$-string braid group of a 3-star.

Let $\tau$ be a path from $*_0$ to $c(n)$. Define $\beta_{i,j,k}$ to be the homotopy class $[\tau \sigma_{i,j,k} \tau^{-1}]$. The generators for $\pi_1(B, *_0)$ are $\beta_{i,j,k}$ where $i + j + k = n + 2$ and $k > 1$.

There a natural map $k : D_n(S_3) \to A$ given by $c \mapsto i_c(c) \cup \{x\}$. This map is a homotopy equivalence. Therefore $\pi_1(A, *_0) \cong B_{n-1}(S_3, c(n-1))$. By the formula given in Doig for the $n-1$-string braid group of a 3-star, $\pi_1(A, *_0) \cong F(n-1)$.

The generators for $\pi_1(A, *_0)$ are $\alpha_{i,j,k} = k_*([\sigma_{i,j,k}])$ where $i + j + k = n + 1$ and $k > 1$.

Since $C$ and $D$ are deformation retracts of $A$, we also have $\pi_1(C, *_0) \cong F(n-1)$ and $\pi_1(D, *_1) \cong F(n-1)$. The generators $\gamma_{i,j,k}$ for $\pi_1(C, *_0)$ and the generators $\delta_{i,j,k}$ for $\pi_1(D, *_1)$ are defined similarly to those of $A$.

Next, we must find paths $t_A$ from $*_0$ to $*_1$ through $A$ and $t_B$ from $*_0$ to $*_1$ through $B$. These paths lead us to the maps

\[
\begin{align*}
j^A_0 : \pi_1(C, *_0) &\to \pi_1(A, *_0) \\
j^A_1 : \pi_1(D, *_1) &\to \pi_1(A, *_0) \\
j^B_0 : \pi_1(C, *_0) &\to \pi_1(B, *_0) \\
j^B_1 : \pi_1(D, *_1) &\to \pi_1(B, *_0)
\end{align*}
\]

Explicitly,

\[
\begin{align*}
j^A_0(\sigma) &= [\sigma] \\
j^A_1(\tau) &= [t_A^{-1}\tau t_A] \\
j^B_0(\sigma) &= [\sigma] \\
j^B_1(\tau) &= [t_B^{-1}\tau t_B]
\end{align*}
\]

Notice that the first two maps are isomorphisms. Clearly, because $C$ and $D$ are deformation retracts of $A$,

\[
\begin{align*}
j^A_0(\gamma_{i,j,k}) &= \alpha_{i,j,k} \\
j^A_1(\delta_{i,j,k}) &= \alpha_{i,j,k}
\end{align*}
\]

It follows from the definition of $\beta_{i,j,k}$ that

\[
\begin{align*}
j^B_0(\gamma_{i,j,k}) &= \beta_{i,j,k+1} \\
j^B_1(\delta_{i,j,k}) &= \beta_{i,j+1,k}
\end{align*}
\]
We apply Generalized Van Kampen’s Theorem now to \( U_n(L) = \mathcal{A} \cup \mathcal{B} \). We get
\[
B_n(L, *_0) \cong \frac{\pi_1(A, *_0) \ast \pi_1(B, *_0) \ast F(t)}{N},
\]
where \( N \) is the smallest normal group containing the \( \binom{n-1}{2} \) relations of the form
\[
\alpha_{i,j,k} \beta_{i,j,k+1}^{-1}
\]
and the \( \binom{n-1}{2} \) relations of the form
\[
\alpha_{i,j,k} t \beta_{i,j+1,k}^{-1} t^{-1}.
\]
We will first eliminate all \( \alpha_{i,j,k} \). For a given \( \alpha_{i,j,k} \), we have two relations involving \( \alpha_{i,j,k} \). They are:
\[
\alpha_{i,j,k} \beta_{i,j,k+1}^{-1}
\]
and
\[
\alpha_{i,j,k} t \beta_{i,j+1,k}^{-1} t^{-1}
\]
So we may eliminate each \( \alpha_{i,j,k} \). We are left with \( \binom{n-1}{2} \) relations of the form:
\[
\beta_{i,j,k+1} t \beta_{i,j+1,k}^{-1} t^{-1}
\]
Now eliminate the generators of the form \( \beta_{i,j,k+1} \). This will eliminate exactly \( \binom{n-1}{2} \) generators because \( k \geq 2 \).
We are left with \( n-1 \) generators of the form \( \beta_{i,j,2} \) where \( i + j = n \) and \( i, j \geq 1 \). Therefore \( B_n(L, *_0) \) has a free basis consisting of the \( n \) elements in \( \{ \beta_{i,j,2} | i + j = n \text{ and } i, j \geq 1 \} \cup \{ t \} \).
By a simple change of base point from \( *_0 \) to \( c^n \), we find that \( B_n(L, c^n) = F(t', \beta'_{1,n-1,2}, ..., \beta'_{n-1,1,2}) \) where \( t' = [\tau t \tau^{-1}] \) and \( \beta'_{i,n-i,2} = [\tau \beta_{i,n-i,2} \tau^{-1}] \) where \( \tau \) is some path from \( c^n \) to \( *_0 \).
Clearly, these are exactly the generators specified in our statement of the theorem. \( \square \)

4 The 2-Point Braid Group of the n-ray sun graph

Let \( L_n \) be as before. Let \( \zeta = e^{2\pi i/n} \). For each \( j, 0 \leq j < n \), define
\[
c_j = \{ \zeta^j, 2\zeta^j \}
\]
\[
L^j = S^1 \cup A_j
\]
\[
L^{-j} = Cl_{L_n}(L_n - A_j).
\]
Let $t$ denote the loop given by $t(s) = \{e^{2\pi is}, 2\}$ for $0 \leq s \leq 2\pi$. Finally, define $c_* = \{1, \zeta\}$.

For each $j$, $0 \leq j < n$, there is an injection $i_j : S_3 \to L_n$ such that

\begin{align*}
  i_j(v_0) &= \zeta^j \\
  i_j([v_0, v_1]) &= A_j \\
  i_j([v_0, v_2]) &= [\zeta^j, \zeta^{j+1}] \\
  i_j([v_0, v_3]) &= [\zeta^j, \zeta^{j-1}]
\end{align*}

This map induces a map of configuration spaces $i'_j : U^{\text{top}}_n(S_3) \to U^{\text{top}}_n(L_n)$. From the above discussion, $B_2(S_3, c(2)) = F(\beta_{1,1,2})$. Let $\alpha_j = i'_j(\sigma_{1,1,2})$, where $\sigma_{1,1,2}$ is the loop in $U^{\text{top}}_n(S_3)$ described in section 2. We have the following theorem.

**Theorem 4.1.** Let $\tau_j$ be the obvious (counterclockwise) path from $c_0$ to $c_j$. Then the elements of the form $[\tau_j \alpha_j \tau_j^{-1}]$ together with $[\ell]$ form a free basis for the group $B_2(L_n, c_0)$.

**Proof.** The base case when $n = 1$ has already been established in the previous section. Now suppose that the theorem is true for all $L_k$ with $k \leq n$. For the time being, we will work with the base configuration $c_*$. Define the following two sets:

\begin{align*}
  A &= U^{\text{top}}_2(L^{-0}) \cup U^{\text{top}}_2(L^{-1}) \\
  B &= \{ c \in U^{\text{top}}_2(L_n) | c \cap A_0 \neq \emptyset \text{ and } c \cap A_1 \neq \emptyset \}
\end{align*}

The sets $B$ and $A \cap B$ are contractible subsets of $U^{\text{top}}_2(L_n)$. In fact, $B$ is homeomorphic to $[0,1] \times [0,1]$, and $A \cap B$ is homeomorphic to $[0,1]$. By Van Kampen's theorem, it follows that $B_2(L_n, c_0) \cong \pi_1(A, c_0)$. Since $A$ is path-connected, it is enough to compute $\pi_1(A, c_0)$.

Define the following sets:

\begin{align*}
  D &= U^{\text{top}}_2(L^{-0}) \\
  E &= U^{\text{top}}_2(L^{-1}).
\end{align*}

By the induction hypothesis applied to $L_{n-1}$, we know the free generators for $B_2(L_{n-1}, c_0)$ are of the form $[\tau_j \alpha_j \tau_j^{-1}]$ for $0 \leq j \leq n-1$ together with $[\ell]$. There
is a homeomorphism of \( L_{n-1} \) onto \( L^{-0} \) which maps \( A_j \subset L_{n-1} \) to \( A_{j+1} \subset L^{-0} \) for all \( j \). This homeomorphism induces a homeomorphism of \( U_2^{\text{top}}(L_{n-1}) \) onto \( D \), which in turn induces an isomorphism of fundamental groups \( B_2(L_{n-1}, c_0) \to B_2(L^{-0}, c_0) \). The isomorphism maps the generator \([\tau_j \alpha_j \tau_j^{-1}]\) in \( B_2(L_{n-1}, c_0) \) to \( \delta_{j+1} = [\tau_{j+1} \alpha_{j+1} \tau_{j+1}^{-1}] \) in \( B_2(L^{-0}, c_0) \). Similarly, there is a homeomorphism of \( L_{n-1} \) onto \( L^{-1} \) which maps \( A_j \subset L_{n-1} \) to \( A_{j+1} \subset L^{-1} \) for \( 1 \leq j < n \) and maps \( A_0 \subset L_{n-1} \) to \( A_0 \subset L^{-1} \). This homeomorphism induces a homeomorphism of \( U_2^{\text{top}}(L_{n-1}) \) onto \( E \), which in turn induces an isomorphism of fundamental groups \( B_2(L_{n-1}, c_0) \to B_2(L^{-1}, c_0) \). The isomorphism maps the generator \([\tau_j \alpha_j \tau_j^{-1}]\) in \( B_2(L_{n-1}, c_0) \) to \( \epsilon_{j+1} = [\tau_{j+1} \alpha_{j+1} \tau_{j+1}^{-1}] \) for \( j > 0 \) and to \( \epsilon_0 = [\tau_0 \alpha_0 \tau_0^{-1}] \) for \( j = 0 \). Let the image of \([t]\) under these isomorphisms be denoted by \( \delta \in B_2(L^{-0}, c_0) \) and \( \epsilon \in B_2(L^{-1}, c_0) \).

Now consider \( D \cap E \). There is a homeomorphism between this space and \( U_2^{\text{top}}(L_{n-2}) \) which maps \( A_j \subset L_{n-1} \) to \( A_{j+2} \subset L^{-0} \) for all \( j \). The induced isomorphism of fundamental groups will send \([\tau_j \alpha_j \tau_j^{-1}]\) in \( B_2(L_{n-2}, c_0) \) to \( \gamma_{j+2} = [\tau_{j+2} \alpha_{j+2} \tau_{j+2}^{-1}] \) in \( B_2(L^0 \cap L^1, c_0) \). Let \( \gamma \) denote the image of \([t]\) under this isomorphism. Since

\[
\begin{align*}
\iota^D(\gamma_j) &= \delta_j \\
\iota^E(\gamma_j) &= \epsilon_j \\
\iota^D(\gamma) &= \delta \\
\iota^E(\gamma) &= \epsilon,
\end{align*}
\]

it follows by (Classical) Van Kampen’s Theorem that \( B_2(L_n, c_0) = F(\gamma_0, \delta_1, \delta_2, \ldots, \delta_{n-1}, t) \).

5 Appendix

Theorem 5.1. Generalized Van Kampen’s Theorem:

Let \( X \) be a polyhedron. Let \( A \) and \( B \) be subpolyhedra of \( X \) such that \( A \cup B = X \) and \( A \cap B = \bigcup_{i=0}^n C_i \) where each \( C_i \) is path connected and nonempty. Choose a base point \( c_i \in C_i \) for each \( C_i \). Find paths \( t_1^A, \ldots, t_n^A, t_1^B, \ldots, t_n^B \) such that \( t_i^A \) goes from \( c_0 \) to \( c_i \) in \( A \) and \( t_i^B \) goes from \( c_0 \) to \( c_i \) in \( B \). Let \( t_0^A = t_0^B \) be the constant path. The following maps of fundamental groups are induced by the inclusion of \( C_i \) into \( A \) and \( B \) followed by a base point change:

\[
\begin{align*}
j_i^A : \pi_1(C_i, c_i) &\to \pi_1(A, c_0) \\
&[\sigma] \mapsto [(t_i^A)^{-1} \sigma t_i^A] \\
j_i^B : \pi_1(C_i, c_i) &\to \pi_1(B, c_0) \\
&[\sigma] \mapsto [(t_i^B)^{-1} \sigma t_i^B]
\end{align*}
\]
Then there exists an isomorphism

\[ \Phi : \frac{\pi_1(A, c_0) \ast \pi_1(B, c_0) \ast F(t_1, \ldots, t_n)}{N} \to \pi_1(X, c_0) \]

where the \( t_i \)'s are indeterminates and \( N \) is the smallest normal subgroup containing all words of the form \( j^A([\sigma]) (t_j j^B([\sigma]) t_i^{-1})^{-1} \) for \( 0 \leq i \leq n \). \( t_0 \) is defined as the identity element of \( \pi_1(A, c_0) \ast \pi_1(B, c_0) \).

The map \( \Phi \) is specified by

\[ \begin{align*}
\Phi(\alpha) &= i^A \alpha \text{ for all } \alpha \in \pi_1(A, c_0) \\
\Phi(\beta) &= i^B \beta \text{ for all } \beta \in \pi_1(B, c_0) \\
\Phi(t_i) &= [t_i^A(t_i^B)^{-1}] \text{ for } 1 \leq i \leq n
\end{align*} \]

where \( i^A \) and \( i^B \) are the homomorphisms induced by the inclusions of \( A \) and \( B \) into \( X \).

**Proof.** Let \( X \) be a polyhedron with subpolyhedra \( A \) and \( B \) such that \( X = A \cup B \) and \( A \cap B = \bigcup_{i=0}^n C_i \) where each \( C_i \) is path connected and nonempty. Let \( c_i \in C_i \) be base points.

Recall that an arc is a homeomorphic copy of the unit interval. Let \( t_i^A \subset A \) be arcs connecting \( c_0 \) to \( c_i \) in \( A \) and \( t_i^B \subset B \) be arcs connecting \( c_0 \) to \( c_i \) in \( B \) for \( 1 \leq i \leq n \) such that \( t_i^A \cup t_i^B \approx S^1 \). We also require that \( A \cap (t_i^A \cup t_i^B) \) and \( B \cap (t_i^A \cup t_i^B) \) are arcs.

We will work by induction. Assume that the theorem holds for some \( n \geq 1 \). The base case \((n = 0)\) is obtained from Van Kampen’s Theorem.

Set

\[ \begin{align*}
A' &= A \cup t_1^B \\
B' &= B \cup t_1^A
\end{align*} \]

Now \( A' \cup B' = X \), but \( A' \cap B' = C_1' \cup C_2 \cup \ldots \cup C_n \) where \( C_1' = C_0 \cup C_1 \cup t_1^A \cup t_1^B \). Note that \( A' \cap B' \) has one less component than \( A \cap B \).

We know that

\[ \begin{align*}
\pi_1(A', c_0) &= \pi_1(A, c_0) \ast F(t) \\
\pi_1(B', c_0) &= \pi_1(B, c_0) \ast F(t) \\
\pi_1(C', c_0) &= \pi_1(C_0, c_0) \ast \pi_1(C_1 \cup t_1^A \cup t_1^B, c_0)
\end{align*} \]

Then we apply the inductive hypothesis to \( X \) as \( A' \cup B' \). The theorem follows. \( \square \)
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