FINITE TEMPERATURE FIELD THEORIES ON THE LIGHT-FRONT

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Abstract
In this talk, we describe our present understanding of thermal field theories on the light-front with an application to Schwinger model.

1. INTRODUCTION
Light-front field theories [1] have been of great interest for a variety of reasons and more recently, there have been a lot of interest in the study of such theories at finite temperature [2,3,4]. At first sight, the study of this question would appear to be quite straightforward. Namely, since in the light-front formalism,

\[ p^+ = \frac{1}{2} \left( p_0 + p_3 \right); \]

plays the role of the Hamiltonian, it would seem natural to define the finite temperature partition function of the system to correspond to

\[ Z(\beta) = \text{Tr} e^{p^+}; \]

where \( \beta \) represents the inverse temperature in units of the Boltzmann constant. As we will see, such a generalization leads to problems and in this talk we would discuss an alternate generalization of the partition function as well as applications following from it.

The organization of the presentation is as follows. In section 2, we would discuss the problems associated with the naive generalization of the partition function in (2) through a simple example and propose an alternate generalization of the partition function for light-front field theories. In section 3, we will describe the systematic derivation of such a generalization. In section 4, we will apply the alternate definition of the partition function to study various questions at finite temperature associated with the Schwinger model such as the anomaly, condensates and bound states. We will close with a short conclusion in section 5.

2. DIFFICULTIES WITH THE NAIVE GENERALIZATION
Let us, for completeness, note that in the light-front formalism, one defines

\[ x = \frac{1}{2} (x^0 \quad x^3); \]

where \( x^+ \) is identified with the time coordinate, \( x \) the longitudinal coordinate, while the transverse coordinates remain unchanged. Correspondingly, we have

\[ p^+ = \frac{1}{2} (p_0 \quad p_3); \]

where \( p^+ \) can be identified with energy (Hamiltonian). In the light-front coordinates, the metric is off-diagonal with the form

\[ + + = 1; \quad ij = i\bar{j}; \quad ++ = 0; \quad i; j = 1; 2: \]

We note that since conventionally, for a system with Hamiltonian \( \mathcal{H} \), the density matrix is given by

\[ (\rho) = e^{-\beta \mathcal{H}}; \]
it is natural to consider the density matrix for light-front field theories to be given by

\[
(\rho_{\text{LF}}) = e^{\rho_{\text{LF}} p^+} = e^{\rho_{\text{LF}} p^+};
\]

(7)

where \(\rho_{\text{LF}}\) denotes the inverse of the temperature in the light-front frame. We will see in the next section how to identify this temperature, but for the moment, let us assume for simplicity that

\[
\rho_{\text{LF}} = \frac{1}{T};
\]

(8)

Using Eqs. (7) and (8), we can now define the propagators in any theory. For example, for a massive scalar theory the propagator in the imaginary time formalism will have the form

\[
G(p) = \frac{1}{2p + i\rho_{\text{LF}}};
\]

(9)

where \(p \equiv 2i nT\) and \(\rho_{\text{LF}}^2 = p^2 + m^2\) with \(n\) an integer. On the other hand, in the real time formalism, the relevant component of the propagator (from one loop point of view) would have the form

\[
G_{++}(\rho) = \frac{1}{2p^+ + i\rho_{\text{LF}}};
\]

(10)

where \(n_B\) denotes the bosonic distribution function

\[
n_B(\rho) = \frac{1}{e^{p/\rho_{\text{LF}}} - 1};
\]

(11)

The sign of trouble in the naive generalization of the density matrix (and, therefore, the partition function) is already manifest in (11). Namely, the distribution function is not damped at \(p^+ \to 0\). We are, of course, used to such a divergent behavior in massless theories, but here we are considering a massive scalar theory.

To see further signs of trouble, let us calculate the one loop self-energy in the 4 theory in \(D\) dimensions. The finite temperature part is easily analyzed in the real time formalism and has the form

\[
i\Gamma_{++}(\rho) = \frac{1}{2(2\pi)^D} \int d^D k n_B(k) \left(2k^+ + i\rho_{\text{LF}}^2\right);
\]

(12)

We note the peculiar feature in (12) that under the redefinition

\[
k^+ \to k^+; \quad k^+ \to k^+;
\]

(13)

the integral becomes independent of temperature, even though this is the finite temperature part of the self-energy. This is again a signal of the divergent structure of the integrand. In fact, one can evaluate this integral in a regulated manner as

\[
i\Gamma_{++}(\rho) = \frac{1}{2(2\pi)^D} \int d^D k n_B(k) \left(2k^+ + i\rho_{\text{LF}}^2\right);
\]

(14)

In deriving this result, we have set a reference mass scale to unity for convenience.
There are several things to note from the form of the amplitude in (14). First, it is exact in the sense that we have not made any high or low $T$ expansion. Second, it is divergent and the divergence structure is independent of temperature. This is a particularly serious problem in that it would suggest that one would need new zero temperature counterterms at finite temperature. This is completely counter intuitive in that the interaction with a thermal medium takes place on-shell (which is rather clear in the real time formalism of (10)) and would render the question of renormalization intractable in such theories. We note that the temperature dependence of the amplitude has the same logarithmic behavior independent of the number of dimensions which is not consistent with our experience [5] in thermal field theories. Note also that the temperature dependent part of the amplitude in (14) diverges for any $D$ other than $D = 2$ thereby necessitating finite temperature counterterms. Finally, let us note that in the limit of zero temperature ($\Gamma = 1$), this amplitude does not vanish even though this is supposed to represent the finite temperature part.

The source of the problem is not hard to see. The density matrix in (6) is defined in a frame where the heat bath is at rest. On the other hand, on the light-front, we cannot have a heat bath at rest. Consequently, our starting point for generalization is not quite correct. An alternate generalization for the density matrix can be given as [2]

$$\rho = e^{p_0}.$$ (15)

The physical basis for such a generalization will be discussed in the next section. For the moment, let us note that it seems like we are dealing with the conventional density matrix, but one must remember that, in light-front field theories, the time ordering in correlation functions is with respect to $x^+$. With the alternative generalization, the propagator for the massive scalar theory will have the form (in imaginary time formalism)

$$G(p) = \frac{1}{2p^+ p^0};$$ (16)

with $p = 2p^+ nT$. The corresponding component of the real time propagator will have the form

$$G_{++}(\varphi) = \frac{1}{2p^+ p^0} \left( \frac{1}{2} p^+ + p^0 \right),$$ (17)

We note that the distribution function, in this case, has proper damping. We can now evaluate the one-loop scalar self-energy in the $^4$ theory and the thermal part of the amplitude now can be evaluated in the high temperature limit (in $D = 4$) to have the behavior

$$\Gamma^{(1)}_{++}(\varphi) = \frac{i}{24} \frac{Z}{Z^2} \frac{d^4k}{k^4} \left( \frac{1}{n_B} \frac{1}{k^+} + \frac{2}{2k^+} \right),$$ (18)

which is well behaved and coincides with the result obtained in the conventional thermal field theories [5]. The thermal self-energy can also be calculated in the $^3$ theory [2] and although at first sight, the result looks quite different from the conventional one, Weldon has shown [3] through a clever change of coordinates that they are indeed identical.

3. A SYSTEMATIC STUDY

In this section, we will describe following [3] a systematic way of studying thermal field theories in different frames, which will also clarify the meaning of the density matrix in (15).
Let us consider a physical system in a generalized coordinate system related to the conventional
one through a linear invertible transformation of the form
\[ x = L x ; \quad x = L x : \]  
(19)
It follows from (19) that
\[ L L = ; \quad L L = : \]  
(20)
An arbitrary vector will transform under this redefinition as
\[ V = L V ; \quad V = V L : \]  
(21)
It follows from (20) and (21) that a scalar will remain invariant under such a redefinition while the metric
tensor will transform as
\[ g = L L g ; \quad g = L L g ; \quad g g = : \]  
(22)
It is worth emphasizing here that such a coordinate redefinition does not necessarily represent a Lorentz
transformation and as a result, the form of the metric can be different in different frames. However, the
volume element will remain the same in the two frames, namely,
\[ \frac{P}{g d^4 x} = \frac{P}{g d^4 x} : \]  
(23)
Let us next see how we can describe statistical mechanics in the new coordinate system. First, let
us recall that when we have a system interacting with a heat bath which is moving with a normalized
velocity \( u \) (with \( u u = 1 \)), the density matrix is given by
\[ ( ) = e^ u p : \]  
(24)
In the transformed coordinate (since scalars do not transform), the density matrix, therefore, will have
the form
\[ ( ) = e^ u p : \]  
(25)
In the rest frame of the heat bath in the transformed coordinates, the four velocity will have the form
\[ \boldsymbol{u}_{\text{rest}} = p \frac{1}{g_{00}} ; 0 ; 0 ; 0 : \]  
(26)
so that in the rest frame of the heat bath in the transformed coordinates, the density matrix will have the form
\[ = e^ p : \]  
(27)
This identifies the inverse temperature in the transformed coordinate system to be
\[ : \]  
(28)
The density matrix can, in fact, be checked to satisfy the KMS condition with the periodicity given by .
Let us now specialize to light-front field theories. In this case, the transformations can be easily
constructed, but most important is the observation that in these coordinates, the metric has the form
(suppressing the transverse degrees of freedom) as given in (5), namely,
\[ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \]  
(29)
so that, in this case, we have $g_{00} = 0$. As a result, statistical description fails for conventional light-front theories in the sense that any finite temperature is mapped to zero temperature. The reason for this failure, as we have noted earlier, can be seen from (26) to be the absence of a rest frame for the heat bath.

To proceed further, let us note that light-front quantization only requires that we identify $x^+$ with time and quantize on equal $x^+$ surfaces. It does not say anything about how the other coordinates should be chosen. We can try to take advantage of this and see if a suitable light-front coordinate system can be defined where statistical description is possible. We note that conventional light-front quantization has several attractive features (such as the linearity of dispersion relation, the large number of kinematic generators etc) and choosing a new coordinate system, it would be desirable to maintain such features as much as is possible. Taking into account various considerations (as well as the desire to identify $L_F = \cdots$), it turns out that the redefinition

$$x^0 = x^0 + x^3; \quad x^3 = x^3;$$

(30)

has all the desired properties and yet allows a statistical description. We note that under the redefinition,

$$p_0 = p_0; \quad p_3 = p_0 + p_3;$$

(31)

and the transformed metric has the form (suppressing the transverse degrees)

$$g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \quad P \overline{g} = 1.$$  

(32)

As a result, the density matrix has the form

$$(\rho) = e^{p_0} = e^{p_0};$$

(33)

which can be compared with the alternate generalization in (15).

4. SCHWINGER MODEL

With the density matrix in (15) (or (33)), one can now define the propagators for various theories both in the imaginary time as well as the real time formalisms [4] and can carry out calculations. In this section, we will discuss some of the results obtained from an application of this formalism to light-front Schwinger model at finite temperature.

The Schwinger model describes massless QED in $1 + 1$ dimensions. Under a redefinition of coordinates in (30), it is straightforward to see that the Dirac matrices transform as

$$\gamma^0 = \gamma^0 + \gamma^3; \quad \gamma^3 = \gamma^3.$$  

(34)

In terms of the transformed Dirac matrices, let us define the projection operators,

$$P = \frac{1}{2} \gamma^0 \gamma^1 = \frac{1}{2} (1_5); \quad P^+ = \frac{1}{2} \gamma^1 \gamma^0 = \frac{1}{2} (1 + \gamma_5);$$

(35)

which can be thought of as chirality projection operators and satisfy

$$PP = P; \quad PP = 0; \quad P^+ P = 1.$$  

(36)

With the projection operators in (35), let us define the chiral components

$$\rho = P.$$  

(37)
The fermion part of the Lagrangian density for the Schwinger model, in terms of these chiral components, takes the simple form

$$ L = i \gamma \theta_1 + i \gamma (2 \theta_0 + \theta_1) + e \gamma A_1 + e \gamma (2A_0 + A_1); \quad (38) $$

It is clear from the form of the Lagrangian density in (38) that the two fermion components are decoupled and one of them, namely, becomes non-dynamical. As a result, it does not thermalize and the finite temperature propagator for the fermions (in the real time formalism) takes the form

$$ iS(p) = \frac{i}{p_1}; \quad iS_+(p) = \frac{i}{2p_0 + p_1} 2 \text{sgn} (p_1) n_F (2p_0 + p_1); \quad (39) $$

where “sgn” stands for the sign function and $n_F$ represents the fermion distribution function

$$ n_F (x) = \frac{1}{e^x + 1}; \quad (40) $$

The non-thermalization of the non-dynamical component is particularly easy to see in the imaginary time formalism.

Given the fermion propagator in (39), we can now calculate the thermal correction to the anomaly (coming from the fermion loop). There are two contributions - one linear and the other quadratic in the distribution function. The linear term in $n_F$ has the form (after factoring out the Lorentz structure)

$$ 4ie^2 \frac{Z}{(2 \pi)^2} \text{sgn} (k_1) n_F (2k_0 + k_1) = 0; \quad (41) $$

since the integrand is odd. The term quadratic in $n_F$ vanishes algebraically because of the identity

$$ (2p_0 + p_1) (2k_0 + k_1) = 0; \quad (42) $$

so that there is no thermal correction to the anomaly as is the case in conventionally quantized theories.

However, when one calculates the thermal corrections to the self-energy (as well as higher point functions), they disagree with the conventional calculations off-shell. Namely, on the mass-shell, such thermal corrections vanish thus coinciding with the conventional results. However, off-shell, the thermal amplitudes in the light-front have contribution only from one of the chiral components while in the conventional calculation, both the chirality components thermalize and contribute, leading to a different structure.

The Schwinger model is a simple theory where even bound state questions can be studied [6]. This is best studied in the bosonized form of the theory which corresponds to a free massive boson with the mass related to the anomaly in the Schwinger model. Since the anomaly is unaffected by temperature, it follows that the bound state equation as well as the solution are independent of temperature. Finally, one can also calculate the fermion condensate in the theory which is more conveniently calculated using bosonization. Here, the temperature dependent condensate takes the forms

$$ i_T = \left\{ \begin{array}{cl}
  \frac{1}{2} \frac{T}{m_{ph}} e^{\frac{n_{ph}}{T}} & \text{for } T \text{ small}; \\
  2T e^{\frac{n_{ph}}{T}} & \text{for } T \text{ large};
\end{array} \right. \quad (43) $$

This coincides with the results calculated with conventional quantization [7].
5. CONCLUSION

In summary, we have shown in this talk how the naive generalization of the density matrix to light-front field theories runs into serious problems. We have identified the source of the difficulty to be the absence of a rest frame for the heat bath on the light-front. We have proposed an alternate generalization of the density matrix [2] which is free from the divergence problems and has a natural meaning [3] when analyzed systematically as a density matrix in a transformed coordinate system. We have applied this formalism to the Schwinger model and studied both perturbative as well as bound state questions [4]. We have shown that the physical results coincide with the ones calculated in conventional quantization. However, the structures of the amplitudes off-shell do not quite coincide.

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