ON THE COMMON ZEROS OF QUASI-MODULAR FORMS FOR $\Gamma_0^+(N)$
OF LEVEL $N = 1, 2, 3$

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Abstract. In this paper, we study common zeros of the iterated derivatives of the Eisenstein series for $\Gamma_0^+(N)$ of level $N = 1, 2$ and $3$, which are quasi-modular forms. More precisely, we investigate the common zeros of quasi-modular forms, and prove that all the zeros of the iterated derivatives of the Eisenstein series $d^m E_N^k(\tau)$ of weight $k = 2, 4, 6$ for $\Gamma_0^+(N)$ of level $N = 2, 3$ are simple by generalizing the results of Meher [15] and Gun and Oesterlé [10] for $SL_2(\mathbb{Z})$.

1. Introduction and the main results

Let $\Gamma$ denote a Fuchsian group of the first kind, and for a positive integer $N$, let $\Gamma_0^+(N)$ be a subgroup of $SL_2(\mathbb{R})$ generated by the Hecke congruence group $\Gamma_0(N)$ and the Fricke involution $w_N := \begin{pmatrix} 0 & -\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$, and $\mathbb{H}$ be the complex upper half-plane.

First, we recall the definition of a quasi-modular form for $\Gamma$ which Kaneko and Zagier [13] have introduced.

Definition 1.1. For a positive integer $k$ and a non-negative $\ell$, a quasi-modular form of weight $k$ and depth $\ell$ for $\Gamma$ is a holomorphic function $f$ on $\mathbb{H}$ satisfying the following conditions:

(i) There exist holomorphic functions $Q_i(f)$ for $i = 0, 1, \ldots, \ell$ that satisfy

$$f[\gamma]_k = \sum_{i=0}^{\ell} Q_i(f) X(\gamma)^i, \quad \text{with } Q_\ell(f) \neq 0,$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

where the operator $[\gamma]_k$ is defined by

$$f[\gamma]_k(\tau) = (c\tau + d)^{-k} f(\gamma \tau),$$

and the function $X(\gamma)$ is defined by

$$X(\gamma)(\tau) = \frac{\tau}{c\tau + d}.$$

(ii) $f$ is polynomially bounded, i.e., there exists a constant $\alpha > 0$ such that $f(\tau) = O((1 + |\tau|^2)^{\alpha})$ as $\nu \to \infty$ and $\nu \to 0$, where $v = 3(\tau)$.

One may define the quasi-modular form using the notion of almost holomorphic modular forms, that is, the constant term $F_0(\tau)$ in the variable $v$ of an almost holomorphic modular form $F(v) := \sum_{i=0}^{\ell} F_i(\tau)(-4\pi v)^{-i}$. In fact, the notions of quasi-modular forms and almost
holomorphic modular forms coincide since if $f = F_0$ is a quasi-modular form inherited from an almost holomorphic modular form $F(z) := \sum_{i=0}^{\ell} F_i(\tau)(-4\pi v)^{-i}$ then $Q_i(f) = F_i$. See \cite[Section 5.3]{2} for more details.

The following proposition shows the structure of the space of quasi-modular forms.

**Proposition 1.2.** \cite[Proposition 20]{2} Let $\ell$ be a non-negative integer and $\Gamma$ be a non-cocompact Fuchsian group of the first kind. For a non-negative integer $k$, denote by $M_k(\Gamma)$ the space of modular forms of weight $k$ for $\Gamma$ and by $\hat{M}_k^{(\leq \ell)}(\Gamma)$ the space of quasi-modular forms of weight $k$ and depth $\leq \ell$ for $\Gamma$. Let $\theta := \frac{d}{d\tau}$ and $\phi$ be a quasi-modular form of weight 2 for $\Gamma$ which is not modular. Then we have the following:

(i) $\theta(\hat{M}_k^{(\leq \ell)}(\Gamma)) \subseteq \hat{M}_k^{(\leq \ell+1)}(\Gamma)$.

(ii) $\hat{M}_k^{(\leq \ell)}(\Gamma) = \bigoplus_{r=0}^{\ell} M_{k-2r}(\Gamma) \cdot \phi^r$.

Let $j_2$ and $j_3$ be the Hauptmodul for $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$, respectively, defined by

\[
    j_2(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24 + 2^{12} \left( \frac{\eta(2\tau)}{\eta(\tau)} \right)^{24},
\]

\[
    j_3(\tau) = \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{12} + 12 + 3^6 \left( \frac{\eta(3\tau)}{\eta(\tau)} \right)^{12},
\]

where $\eta(\tau)$ is the Dedekind eta function,

\[
    \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]

For each positive even integer $k$, let

\[
    E_k^{(N)}(\tau) = \frac{1}{1 + N^{k/2}}(E_k(\tau) + N^{k/2}E_k(N\tau)), \quad \text{for } N = 1, 2, 3.
\]

We note that $E_k^{(N)}$ is a modular form of weight $k$ for $\Gamma_0^+(N)$ if $k \geq 4$, and in particular for $N = 1$,

\[
    E_k^{(1)}(\tau) := E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\tau^n, \quad \text{where } \sigma_{k-1}(n) = \sum_{d|n} d^{k-1},
\]

is the standard Eisenstein series of weight $k$ for $\text{SL}_2(\mathbb{Z})$. One can express $j_N$ as a rational function in $E_k^{(N)}$ and $E_6^{(N)}$ (see \cite[12]{1}).

For $N = 2, 3$ and for a non-negative integer $\ell \in \mathbb{Z}_{\geq 0}$, Proposition \cite[12]{1} implies that every quasi-modular form $f$ of weight $k$ and depth $\leq \ell$ for $\Gamma_0^+(N)$ can be written as

\[
    f(\tau) = f_0(\tau) + f_1(\tau)E_2^{(N)}(\tau) + \cdots + f_\ell(\tau)(E_2^{(N)}(\tau))^\ell,
\]

where for $0 \leq i \leq \ell$, $f_i$ is a modular form of weight $k - 2i$ for $\Gamma_0(N)$.

The zeros of certain modular forms for $\text{SL}_2(\mathbb{Z})$ have been studied actively. It dates back to Rankin and Swinnerton-Dyer’s celebrated result \cite{21}, which has proved Wohlfahrt’s conjecture \cite{24}, that is, they have proved that all the zeros of the Eisenstein series $E_k$ for $k \geq 4$ lies on the unit circle $|\tau| = 1$ in the standard fundamental domain $\{ \tau \in \mathbb{H} : -\frac{1}{2} \leq \Re \tau \leq \frac{1}{2}, |\tau| \geq 1 \}$. Since then, extensive and various studies on the zeros of modular and quasi-modular forms
have been conducted in this regard. Getz [8] has generalized Rankin and Swinnerton-Dyer’s result [20] for the modular form of arbitrary weight \( k \geq 4 \) for \( \text{SL}_2(\mathbb{Z}) \). They proved that if a holomorphic modular form \( f(\tau) \) has the Fourier expansion of the form \( f(\tau) = 1 + cq^{\dim S_k + 1} + \ldots \) for some \( c \in \mathbb{C} \), where \( q := e^{2\pi i \tau} \) and \( S_k \) is the space of cusp forms of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \), then the zeros of \( f(z) \) has the same property. On the other hand, Rankin [19] has extended Rankin and Swinnerton’s results [20] to the Poincaré series whose order is a rational function with real coefficients. Gun [4] has generalized the argument of Rankin [19] to prove a similar result of [8] for certain cusp forms. Kohnen [14] has given the explicit formula of zeros of \( \text{Eis} \) lying on the unit circle in the right half plane.

Also, for the aspect of our interest, we note that Saber and Sebbar [23] have studied the zeros of \( f' \) of a modular form \( f \) for \( \text{SL}_2(\mathbb{Z}) \). For an even integer \( k \geq 4 \), Balasubramanin and Gun [3] have paid attention to the Eisenstein series \( \text{Eis} \) for \( \text{SL}_2(\mathbb{Z}) \) and their common divisors. However, contrary to the case \( k < 2 \), the common zeros of quasi-modular forms may be reduced to the problem on certain polynomials and their common divisors. However, contrary to the case \( N = 1 \), the Ramanujan identities for \( N = 2, 3 \) (see [4] and [5] in Section 3) are not of the polynomial forms anymore. This is a serious obstruction to resolving the problem. In Section 4, we show how to overcome

**Theorem 1.3.** For \( N = 2, 3 \), all zeros of \( \frac{d^j \text{Eis}_2^{(N)}(\tau)}{d\tau^j}, \frac{d^j \text{Eis}_4^{(N)}(\tau)}{d\tau^j}, \frac{d^j \text{Eis}_6^{(N)}(\tau)}{d\tau^j} \) are simple for all integers \( j \geq 0 \).

We give the proof of Theorem 1.3 in Section 4 and Appendix A. The idea of the proof is based on the method of Gun and Oesterlé given in [10] which states that the problem on the common zeros of quasi-modular forms may be reduced to the problem on certain polynomials and their common divisors. However, contrary to the case \( N = 1 \), the Ramanujan identities for \( N = 2, 3 \) (see [4] and [5] in Section 3) are not of the polynomial forms anymore. This is a serious obstruction to resolving the problem. In Section 4, we show how to overcome...
for \( \tau \) and \( \tau^k \) is a quasi-modular form of weight \( \text{Lemma 2.2}. \) We also prove that if \( j_m \) for any positive integer \( r \), rational Fourier coefficients for \( \text{Proposition 3.9}. \) We also recall them briefly, first.

this obstruction by a delicate analysis of the arithmetic nature of coefficients that appear in certain polynomials associated to the iterated derivatives of the Eisenstein series for \( \Gamma_0^+(N) \) for \( N = 2, 3 \). We note that some calculations for \( \Gamma_0^+(3) \) is postponed to Appendix A since the main idea is same with the case for \( \Gamma_0^+(2) \), except that the required calculations are more complicated for \( \Gamma_0^+(3) \). We also treat the case for \( N = 1 \) for reader’s convenience.

In Section 3 we investigate the common zeros of various quasi-modular forms for \( \Gamma_0^+(N) \) of level \( N = 1, 2, 3 \). Especially, we prove that there are no common zeros of \( E_2^{(N)}(\tau) \) and \( \frac{d^n E_2^{(N)}(\tau)}{d \tau^n} \) for any positive integer \( m \). This can be shown by two different methods (see Proposition 3.5 and Proposition 3.9). We also prove that if \( j_N(\alpha) \) is an algebraic number, then \( \alpha \) can never be a zero of \( \frac{d^n E_2^{(N)}(\tau)}{d \tau^n} \) for \( n \geq 0 \) (see Proposition 3.6). These are the generalization of all of Meher’s results in \([15]\), and the proofs are similar. We also give various pairs of quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \) which have no common zero. We show that there is no common zero of any holomorphic modular form and a quasi-modular form of maximal depth. Also we deduce that there is no common zero of \( \frac{d^n E_2^{(N)}}{d \tau^n} \) and arbitrary non-zero holomorphic modular form with rational Fourier coefficients for \( \text{SL}_2(\mathbb{Z}) \), for weight \( k = 4 \) and \( 6 \).

Also, as an extension of Meher’s result \([15, \text{Theorem 2.4}]\), we prove that there are infinitely many \( \Gamma_0^+(N) \)-inequivalent zeros in \( \mathbb{H} \) of a quasi-modular form of depth 1 in Theorem 2.3 of Section 2.

2. Equivariant forms

El Basraoui and Sebbar have investigated several properties of equivariant forms in \([7]\). Let us recall them briefly, first.

The “double-slash” operator is defined for a meromorphic function \( f \) on \( \mathbb{H} \) by

\[
f[\gamma](\tau) = (c\tau + d)^{-2} f(\gamma \tau) - \frac{c}{(c\tau + d)}, \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).
\]

For a function \( h : \mathbb{H} \to \mathbb{C} \) which is not the identity, we let

\[
h(\tau) := \frac{1}{h(\tau) - \tau}.
\]

A meromorphic function \( h \) on \( \mathbb{H} \) is called an equivariant function for \( \Gamma \) if it satisfies

\[h(\gamma \tau) = \gamma h(\tau), \quad \text{for each} \quad \gamma \in \Gamma.\]

If \( h \) is meromorphic at every cusp of \( \Gamma \), then \( h \) is called an equivariant form for \( \Gamma \). Here we say that \( h \) is meromorphic at the cusp \( s \) if \( h ||[\delta](\tau) \) is meromorphic at \( \infty \), where \( \delta \in \text{SL}_2(\mathbb{R}) \) sends \( \infty \) to \( s \).

**Proposition 2.1.** \([3, \text{Proposition 3.1}]\) Let \( h \) be a meromorphic function on \( \mathbb{H} \). Then, for \( \gamma \in \Gamma \) and \( \tau \in \mathbb{H} \), we have

\[h(\gamma \tau) = \gamma h(\tau) \text{ if and only if } h ||[\gamma](\tau) = h(\tau).\]

**Lemma 2.2.** If \( f = f_0 + f_1 E_2^{(2)}(\tau) \) is a quasi-modular form of weight \( k \) and depth 1 for \( \Gamma_0^+(2) \), then for \( \tau \in \mathbb{H} \), \( h_2(\tau) := \tau + \frac{i}{\pi \tau} h(\tau) \) is an equivariant form for \( \Gamma_0^+(2) \). Similarly, if \( g = g_0 + g_1 E_2^{(3)}(\tau) \) is a quasi-modular form of weight \( k \) and depth 1 for \( \Gamma_0^+(3) \), then \( h_3(\tau) = \tau + \frac{3}{\pi \tau} g(\tau) \) is an equivariant form for \( \Gamma_0^+(3) \).
Proof. Here \( \hat{h}_2(\tau) = \frac{i\pi}{4} \frac{f(\tau)}{f_1(\tau)} \). Let \( \gamma \in \Gamma_0^+(2) \). Applying the double-slash operator \( ||[\gamma] \) on \( \hat{h}_2 \) and using the fact that

\[
E_2^{(2)}(\gamma \tau) = (c\tau + d)^2 E_2^{(2)}(\tau) + \frac{4}{i\pi} c(c\tau + d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \cup \Gamma_0^+(2),
\]

we have

\[
\hat{h}_2 ||[\gamma](\tau) = (c\tau + d)^{-2} \hat{h}_2(\gamma \tau) - c(c\tau + d)^{-1} \\
= (c\tau + d)^{-2} \left( \frac{i\pi}{4} \frac{f_0(\gamma \tau)}{f_1(\gamma \tau)} + \frac{i\pi}{4} E_2^{(2)}(\gamma \tau) \right) - c(c\tau + d)^{-1} \\
= \frac{i\pi}{4} \frac{f_0(\tau)}{f_1(\tau)} + \frac{i\pi}{4} E_2^{(2)}(\tau) = \hat{h}_2(\tau).
\]

By Proposition 2.3, \( h_2(\gamma \tau) = \gamma h_2(\tau) \) for \( \gamma \in \Gamma_0^+(2) \), i.e., \( h_2 \) is an equivariant function for \( \Gamma_0^+(2) \). In the same manner, since

\[
E_2^{(3)}(\gamma \tau) = (c\tau + d)^2 E_2^{(3)}(\tau) + \frac{3}{i\pi} c(c\tau + d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \cup \Gamma_0^+(3),
\]

we also see that \( h_3 \) is an equivariant function for \( \Gamma_0^+(3) \).

Let \( \delta \in \text{SL}_2(\mathbb{R}) \) send \( \infty \) to the cusp \( s \) for \( \Gamma_0^+(2) \). We see that

\[
\hat{h}_2 ||[\delta](\tau) = \frac{i\pi}{4} \frac{f(\delta) k(\tau)}{f_1(\delta) k-2(\tau)} \quad \text{for } \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \cup \Gamma_0^+(2),
\]

where the last equality follows from (11). Note that since the function \( f_0 \) and \( f_1 \) are modular forms of weight \( k \) and \( k-2 \), respectively, for \( \Gamma_0^+(2) \), they are holomorphic at the cusp \( s \), i.e., \( f_0[\delta]k \) and \( f_1[\delta]k-2 \) are holomorphic at \( \infty \). Therefore, \( \hat{h}_2 \) is meromorphic at the cusp \( s \), so \( h \) is an equivariant form for \( \Gamma_0^+(2) \). The same argument holds for \( h_3 \).

**Theorem 2.3.** Let \( N = 2, 3 \). Then for a quasi-modular form \( f \) of weight \( k \) and depth 1 for \( \Gamma_0^+(N) \), there are infinitely many \( \Gamma_0^+(N) \)-inequivalent zeros of \( f \) in \( \mathbb{H} \).

**Proof.** We follow the same argument as in the proof of [22, Theorem 4.1], and we give the details of the proof for \( N = 2 \) for readers. We note that the case \( N = 3 \) can be proved by applying the same argument. Let \( f(\tau) = f_0(\tau) + f_1(\tau) E_2^{(2)} \) as in Lemma 2.2 and let \( h(\tau) = \tau + \frac{4}{i\pi} \frac{h(\tau)}{f(\tau)} \) be an equivariant form for \( \Gamma_0^+(2) \). There is a point \( \tau_0 \in \mathbb{H} \) with \( h(\tau_0) \in \mathbb{R} \). Indeed, if \( h \) has a pole, say \( \tau_1 \), then

\[
h(\gamma \tau) = \gamma h(\tau) = \gamma \infty \in \mathbb{Q}
\]

for some \( \gamma \in \Gamma_0^+(2) \). If \( h \) is a equivariant form that is holomorphic in \( \mathbb{H} \), then by [7, Proposition 3.3, 3.4], \( h(\mathbb{H}) \) is not contained in \( \mathbb{H} \), also not contained in \( \mathbb{H}^- \), so \( h(\mathbb{H}) \cap \mathbb{R} \neq \emptyset \). In either case, there is a \( \tau_0 \in \mathbb{H} \) with \( h(\tau_0) \in \mathbb{R} \), and it is not an elliptic point; otherwise, \( h(\tau_0) \) is equal to \( \tau_0 \) or \( -\tau_0 \), both are not in \( \mathbb{R} \).
Let $V$ be an open set of $\tau_0$ such that there are no elliptic points in $V$, and $U$ be an open set of $\tau_0$ such that there are no poles of $h$ in $U$. Let $D = h(V \cap U)$, which is open in $H$. Since each orbit of a cusp for $\Gamma_0(2)$ is dense in $\mathbb{R}$, there are infinitely many rational points that are $\Gamma_0(2)$-equivalent to the cusp $\infty$. Let $r$ be a such rational number with $r = h(\tau)$ where $\tau \in U \cap V$ and let $\gamma \in \Gamma_0(2)$ such that $\gamma \tau = \infty$. Then $\gamma \tau$ is a pole of $h$ and we can take infinitely many such $r$ and $\tau$, and they are all $\Gamma_0^+(2)$-inequivalent. Therefore the function $f$ has infinitely many inequivalent zeros for $\Gamma_0^+(2)$ in $\mathbb{H}$. 

3. Common zeros of quasi-modular forms for $N = 1, 2, 3$

3.1. Common zeros for $N = 2, 3$. In 1916, Ramanujan proved the following Ramanujan identity for $\text{SL}_2(\mathbb{Z})$:

\[
\begin{align*}
\theta E_2 &= \frac{1}{12}(E_2^2 - E_4), \\
\theta E_4 &= \frac{1}{3}(E_2 E_4 - E_6), \\
\theta E_6 &= \frac{1}{2}(E_2 E_6 - E_4^2).
\end{align*}
\]

(Zudilin [25] showed the analogues of the Ramanujan identities for $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$, that is,

\[
\begin{align*}
\theta E_2^{(2)} &= \frac{1}{8}(E_2^{(2)})^2 - E_4^{(2)}, \\
\theta E_4^{(2)} &= \frac{1}{2}(E_2^{(2)} E_4^{(2)} - E_6^{(2)}), \\
\theta E_6^{(2)} &= \frac{1}{4}(3E_2^{(2)} E_6^{(2)} - 2(E_4^{(2)})^2 - \frac{(E_6^{(2)})^2}{E_4^{(2)}}),
\end{align*}
\]

and

\[
\begin{align*}
\theta E_2^{(3)} &= \frac{1}{8}(E_2^{(3)})^2 - E_4^{(3)}, \\
\theta E_4^{(3)} &= \frac{3}{2}(E_2^{(3)} E_4^{(3)} - E_6^{(3)}), \\
\theta E_6^{(3)} &= \frac{1}{2}(2E_2^{(3)} E_6^{(3)} - (E_4^{(3)})^2 - \frac{(E_6^{(3)})^2}{E_4^{(3)}}).
\end{align*}
\]

In this section, for $N = 2, 3$, using the Ramanujan identities (1) and (5) for $\Gamma_0^+(N)$, we investigate common zeros of quasi-modular forms for $\Gamma_0^+(N)$. We first establish Proposition 3.3 below which let us reduce the question on the common zeros of quasi-modular forms whose Fourier coefficients are algebraic numbers to the question about the greatest common divisor of certain polynomials in $\mathbb{Q}[x, y, z]$, based on the idea of Gun and Oesterlé in [10]. Using Proposition 3.3, we investigate common zeros of various quasi-modular forms, as an extension of Meher’s result in [15].

**Lemma 3.1.** Let $N = 2$ or $3$. For any $\alpha \in \mathbb{H}$, the ideal

\[
P_\alpha := \{ F \in \mathbb{Q}[x, y, z] : F \left( E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha) \right) = 0 \}
\]

is a principal ideal of $\mathbb{Q}[x, y, z]$.

In order to prove Lemma 3.1 we need the following recent result on the algebraic independence for the Eisenstein series, which is a generalization of Nesterenko’s theorem [17] for the Fricke groups.

**Theorem 3.2.** [11, Theorem 4.6] Let $N = 1, 2, 3$. Then, for any $\alpha \in \mathbb{H}$, at least three of the numbers $e^{2\pi i \alpha}, E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha)$ are algebraically independent over $\mathbb{Q}$, hence over $\mathbb{Q}$.
Proposition 3.3. The idea of the proof follows the proof of [25, Proposition 6], but we give its self-contained proof for readers. Note that a map
\[
\mathbb{Q}[x, y, z] \rightarrow \mathbb{Q}[E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha)]
\]
which sends \(x, y, z\) to \(E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha)\), respectively, induces an isomorphism
\[
\mathbb{Q}[x, y, z]/P_\alpha \cong \mathbb{Q}[E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha)].
\]
By Theorem 3.2, we know that \(\text{trdeg}_{\mathbb{Q}}[E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha)] \geq 2\), so
\[
\text{ht}(P_\alpha) \leq \dim \mathbb{Q}[x, y, z] - \dim \mathbb{Q}[x, y, z]/P_\alpha
= 3 - \text{trdeg}_{\mathbb{Q}}[E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha)] \leq 1.
\]
If \(\text{ht}(P_\alpha) = 0\), then \(P_\alpha = \{0\}\), so it is principal. If \(\text{ht}(P_\alpha) = 1\), then since \(\mathbb{Q}[x, y, z]\) is a Noetherian factorization domain, \(P_\alpha\) is a principal ideal of \(\mathbb{Q}[x, y, z]\).

Let
\[
(6) \quad \rho : \mathbb{Q}(E_2^{(N)}, E_4^{(N)}, E_6^{(N)}) \rightarrow \mathbb{Q}(x, y, z)
\]
be a natural isomorphism which sends \(E_2^{(N)}, E_4^{(N)}, E_6^{(N)}\) to \(x, y, z\), respectively. Such an isomorphism exists since the functions \(E_2^{(N)}, E_4^{(N)}, E_6^{(N)}\) are algebraically independent over \(\mathbb{Q}(e^{2\pi i\tau})\) by [25, Proposition 6].

Proposition 3.3. For \(N = 2, 3\), let \(f\) and \(g\) be quasi-modular forms in \(\mathbb{Q}(E_2^{(N)}, E_4^{(N)}, E_6^{(N)})\) which have a common zero in \(\mathbb{H}\). Write \(\rho(f) = \frac{f_1}{f_0}\), and \(\rho(g) = \frac{g_1}{g_0}\) in reduced forms, where \(f_0, f_1, g_0, g_1 \in \mathbb{Q}[x, y, z]\). Then \(f_1\) and \(g_1\) have a non-unit common divisor.

Proof. Denote the common zero of \(f\) and \(g\) by \(\alpha\). It is obvious that \(f_1, g_1 \in P_\alpha\), so they have a non-unit common divisor, namely, a generator of \(P_\alpha\).

Corollary 3.4. For \(N = 2, 3\), let \(f\) and \(g\) be quasi-modular forms with algebraic Fourier coefficients for \(\Gamma_0^{+}(N)\). Let \(h := \rho^{-1}(\text{ngcd}(\rho(f), \rho(g)))\) where \(\text{ngcd}(F, G)\) is the gcd of the numerators of reduced forms of \(F\) and \(G\) in \(\mathbb{Q}(x, y, z)\). Then
\[
Z(f, g) = Z(h),
\]
where \(Z(f_1, f_2, \ldots, f_r)\) denotes the common zero set of \(f_1, f_2, \ldots, f_r\) in \(\mathbb{H}\).

Since \(x\) and \(c_0(y, z) + c_1(y, z)x + \cdots + c_n(y, z)x^n\) for some \(c_i(y, z) \in \mathbb{Q}[y, z]\) have no non-unit common divisor if \(c_0(y, z) \neq 0\), we get the following proposition.

Proposition 3.5. Let \(N = 2, 3\). Let \(f = f_0 + f_1E_2^{(N)} + \cdots + f_n\left(\frac{E_2^{(N)}}{E_2^{(N)}}\right)^n\) be a quasi-modular form of depth \(n\) for \(\Gamma_0^{+}(N)\) such that \(f_0\) is not identically zero and \(f_i \in \mathbb{Q}(E_2^{(N)}, E_4^{(N)})\) for \(1 \leq i \leq n\). Then, there is no common zero of \(f\) and the Eisenstein series \(E_2^{(N)}\). In particular, there is no common zero of \(\theta^mE_2^{(N)}\) and \(E_2^{(N)}\) for any positive integer \(m\).

Proof. Referring to (6) for \(\rho\), since \(\rho(E_2^{(N)}) = x\) and the numerator of a reduced form of \(\rho(f) = \rho(f_0) + \rho(f_1)x + \cdots + \rho(f_n)x^n\) has no non-unit common divisor, Proposition 3.3 implies that \(E_2^{(N)}\) and \(f\) have no common divisor.
Proposition 3.6. Let $N = 2$ or $3$ and $\alpha \in \mathbb{H}$ be such that $j_{N}(\alpha)$ is an algebraic number. Then $\theta^n E_2^{(N)}(\alpha)$ is transcendental and hence non-zero for any positive integer $n$.

Proof. Note that $j_{N} \in \mathcal{Q}(E_4^{(N)}, E_6^{(N)})$. Suppose $j_{N}(\alpha)$ is an algebraic number, where $\alpha \in \mathbb{H}$. This means $E_4^{(N)}(\alpha)$ and $E_6^{(N)}(\alpha)$ are algebraically dependent over $\mathbb{Q}$. If $e^{2\pi i \alpha}$ and $\theta^n E_2^{(N)}(\alpha)$ are algebraically dependent over $\mathbb{Q}$, then

$$\text{tr deg}_{\mathbb{Q}} \mathbb{Q}\left( e^{2\pi i \alpha}, E_4^{(N)}(\alpha), E_6^{(N)}(\alpha), \theta^n E_2^{(N)}(\alpha) \right) \leq 2.$$ 

Note that $\theta^n E_2^{(N)} = f_0 + f_1 E_2^{(N)} + \cdots + f_{n+1}(E_2^{(N)})^{n+1}$, where $f_i$’s are modular forms of weight $2(n-i+1)$ in $\mathbb{Q}(E_4^{(N)}, E_6^{(N)})$. That is, $f_1$ can be written as

$$f_1 = \frac{\sum_{u,v} \alpha_{uv}(i) (E_4^{(N)})^u (E_6^{(N)})^v}{\sum_{r,s} \beta_{rs}(i) (E_4^{(N)})^r (E_6^{(N)})^s},$$

where $\alpha_{uv}(i)$ and $\beta_{rs}(i)$ are algebraic numbers. Thus $E_2^{(N)}$ is algebraic over $\mathbb{Q}\left( E_4^{(N)}, E_6^{(N)}, \theta^n E_2^{(N)} \right)$ and so

$$\text{tr deg}_{\mathbb{Q}} \mathbb{Q}\left( e^{2\pi i \alpha}, E_2^{(N)}(\alpha), E_4^{(N)}(\alpha), E_6^{(N)}(\alpha) \right) \leq \text{tr deg}_{\mathbb{Q}} \mathbb{Q}\left( e^{2\pi i \alpha}, E_4^{(N)}(\alpha), E_6^{(N)}(\alpha), \theta^n E_2^{(N)}(\alpha) \right) \leq 2,$$

which contradicts Theorem 3.2. Hence, $e^{2\pi i \alpha}$ and $\theta^n E_2^{(N)}(\alpha)$ are algebraically independent over $\mathbb{Q}$, and in particular, $\theta^n E_2^{(N)}(\alpha)$ is a transcendental number. \hfill \Box

Let

$$m = \begin{cases} 3 & \text{if } N = 1, \\ 4 & \text{if } N = 2, \\ 6 & \text{if } N = 3, \end{cases} \quad \text{and} \quad d = \begin{cases} 12 & \text{if } N = 1, \\ 8 & \text{if } N = 2, \\ 6 & \text{if } N = 3. \end{cases}$$

(7)

For $N = 1, 2, 3$, let $f$ be a modular form of weight $k$ for $\Gamma_0^+(N)$. Define

$$\vartheta f := \theta f - \frac{k}{d} E_2^{(N)} f.$$

By the equations (11) and (12) in the proof of Lemma 2.2, the operator $\vartheta$ sends $f$ to a modular form of weight $k + 2$ for $\Gamma_0^+(N)$. This derivation is a kind of the Serre derivative. In general, the Serre derivative of a non-cocompact Fuchsian group $\Gamma$ is defined as follows: Let $\phi$ be a quasi-modular form of weight 2 and depth 1 for $\Gamma$ which is not modular. By multiplying by an appropriate constant, we may assume that $Q_1(\phi) = 1$. Then, the Serre derivative of weight $k$ for $\Gamma$ is

$$\vartheta_{\Gamma} := \theta - \frac{k}{2\pi i} \phi.$$ 

See [2, p.48 and p.62] for more details.

Let $\Delta_N = (E_4^{(N)})^3 - (E_6^{(N)})^2$.

Lemma 3.7. For $N = 1$, 2 or 3, let $f$ be a modular form of weight $k$ for $\Gamma_0^+(N)$, which is expressed as a polynomial in $E_4^{(N)}$ and $E_6^{(N)}$ over $\mathbb{C}$.

(a) If $N = 1$, then $\vartheta f$ is identically zero if and only if $f \in \mathbb{C} \Delta_1^r$ for some $r \geq 1$. 


(b) If \( N = 2 \) or \( 3 \), then \( \vartheta f \) is not identically zero.

**Proof.** (a) Suppose \( \vartheta f \equiv 0 \), i.e., \( \theta f = k/12 \). By the Ramanujan identity (3), we have
\[
D' \rho(f) = \frac{k}{12} x \rho(f),
\]
where
\[
D' = q \frac{\partial}{\partial q} + \frac{1}{12} (x^2 - y) \frac{\partial}{\partial x} + \frac{1}{3} (xy - z) \frac{\partial}{\partial y} + \frac{1}{2} (xz - y^2) \frac{\partial}{\partial z}.
\]
By the proof of [17, Lemma 4.1], \( \rho(f) \) is of the form \( c \Delta r \) for \( r, s \in \mathbb{Z} \geq 0 \) and a constant \( c \in \mathbb{C} \), where \( \Delta = \rho(\Delta_1) = y^3 - z^2 \). Moreover, since \( \rho(f) \in \mathbb{C}[y, z] \), \( \rho(f) \) is of the form \( c \Delta^{r} \).

Conversely, let \( f = c \Delta^{r} \) for \( r \geq 1 \). Then it is easy to see that
\[
\theta f = \theta(c \Delta^{r}) = r c E_2^{(1)} \Delta^{r} = r E_2^{(1)} f.
\]
Since \( \Delta_1 \) has weight 12 for \( \text{SL}_2(\mathbb{Z}) \), \( f \) has weight \( k = 12r \), so \( r = k/12 \). This proves \( \vartheta f \equiv 0 \).

(b) Let \( m \) and \( d \) be integers as in (7). Suppose \( \vartheta f \equiv 0 \), i.e., \( \theta f = \frac{k}{d} E_2^{(N)} \). Let
\[
D'_N = qy \frac{\partial}{\partial q} + \frac{m-2}{4m} (x^2 - y) \frac{\partial}{\partial x} + \frac{m-2}{m} (xy - z) \frac{\partial}{\partial y} + \frac{3(m-2)}{2m} \left( xyz - \frac{1}{2} y^3 - \frac{m-3}{m} z^2 \right) \frac{\partial}{\partial z}.
\]
By the Ramanujan identities (4) and (5), we have
\[
D'_N \rho(f) = \frac{k}{d} xy \rho(f).
\]
By the same argument as in the proof of [11, Theorem 3.2], \( f \) must be of the form
\[
f = c \Delta_N^{k/4} \left( E_4^{(N)} \right)^{-2k(m-3)/(m-2)}, \text{ for some } c \in \mathbb{C}.
\]
Since \( \rho(f) \in \mathbb{C}[y, z] \), we have that \( \frac{k}{d} \in \mathbb{Z} \). Thus for \( N = 2 \), we have
\[
f = c \left( \frac{\Delta_2}{E_4^{(2)}} \right)^{k/8},
\]
which is not holomorphic form. Similarly for \( N = 3 \),
\[
f = c \left( \frac{\Delta_3}{E_4^{(3)}} \right)^{k/8},
\]
which is not holomorphic. This contradicts the assumption that \( f \) is a modular form of weight \( k \). Hence \( \vartheta f \) is not identically zero.

\( \square \)

**Proposition 3.8.** For \( N = 1, 2 \) or \( 3 \), let \( f \) be a modular form of weight \( k \) for \( \Gamma_0^+(N) \), which is expressed as a polynomial in \( E_4^{(N)} \) and \( E_6^{(N)} \) over \( \mathbb{C} \). Then,

(a) if \( N = 1 \), then \( f' \) and \( E_2^{(1)} = E_2 \) have infinitely many common \( \text{SL}_2(\mathbb{Z}) \)-inequivalent zeros if and only if \( f \in \mathbb{C} \Delta_1^r \) for some \( r \geq 1 \), and

(b) if \( N = 2 \) or \( 3 \), then \( f' \) and \( E_2^{(N)} \) have finitely many common \( \Gamma_0^+(N) \)-inequivalent zeros.
Proof. (a) Suppose \( f' \) and \( E_2^{(N)} \) have infinitely many common \( \text{SL}_2(\mathbb{Z}) \)-inequivalent zeros. Then \( \vartheta f = \theta f - \frac{k}{12} E_2^{(N)} f \) also has infinitely many \( \text{SL}_2(\mathbb{Z}) \)-inequivalent zeros. Since any non-zero modular form has only finitely many \( \text{SL}_2(\mathbb{Z}) \)-inequivalent zeros, \( \vartheta f \) must be 0. By Proposition 3.7, \( f \in \mathbb{C}\Delta_1^k \). Conversely, if \( f = c\Delta_1^k \) for some \( c \in \mathbb{C} \), then \( \vartheta f = rcE_2^{(1)} \Delta_1^k \), so every inequivalent zero of \( E_2^{(1)} \) is a zero of \( f' \), and there are infinitely many of them.

(b) Suppose that \( f' \) and \( E_2^{(N)} \) have infinitely many common \( \Gamma^+_0(N) \)-inequivalent zeros. By the same argument as in the proof of (a), \( \theta f \) must be 0, which is a contradiction to Lemma 3.7(b).

Proposition 3.9. For \( N = 1, 2 \) or \( 3 \), let \( f \) be a modular form of weight \( k \) for \( \Gamma_0^+(N) \), which is expressed as a polynomial in \( E_4^{(N)} \) and \( E_6^{(N)} \) over \( \overline{\mathbb{Q}} \).

(a) If \( N = 1 \), then \( f' \) and \( E_2^{(N)} = E_2 \) have no common zero if and only if \( f \notin \mathbb{C}\Delta_1^k \) for any \( r \geq 1 \).

(b) If \( N = 2 \) or \( 3 \), then \( f' \) and \( E_2^{(N)} \) have no common zero.

Proof. (a) Suppose \( f \in \mathbb{C}\Delta_1^k \). Then by Proposition 3.8(a), \( f' \) and \( E_2^{(1)} \) have infinitely many common zeros. Suppose \( f \notin \mathbb{C}\Delta_1^k \). Then \( \vartheta f \) is a non-zero modular form, by Lemma 3.7(a). Since \( f \in \overline{\mathbb{Q}} \left[ E_4^{(1)}, E_6^{(1)} \right] \), \( \vartheta f \) must be in \( \overline{\mathbb{Q}} \left[ E_4^{(1)}, E_6^{(1)} \right] \). Assume that \( f' \) and \( E_2^{(1)} \) have a common zero, say \( \alpha \). Then \( E_2^{(1)}(\alpha) = 0 \), so \( E_4^{(1)}(\alpha) \) and \( E_6^{(1)}(\alpha) \) must be algebraically independent by Theorem 3.2. However, \( \vartheta f(\alpha) = 0 \) implies that \( E_4^{(1)}(\alpha) \) and \( E_6^{(1)}(\alpha) \) are algebraically dependent. This is a contradiction, hence \( f' \) and \( E_2^{(1)} \) have no common zero.

(b) Suppose \( f' \) and \( E_2^{(N)} \) have a common zero, say \( \beta \). Then \( \beta \) is a zero of non-zero modular form \( \vartheta f \in \overline{\mathbb{Q}} \left[ E_4^{(N)}, E_6^{(N)} \right] \). This yields a contradiction by the same method as in the proof of (a).

We mention that in [15, Theorem 3.5.(iii)] the condition \( f \notin \mathbb{C}\Delta \) is supposed to be \( f \notin \mathbb{C}\Delta^r \) for any \( r \geq 1 \) as in Proposition 3.9(a).

3.2. Common zeros for \( N = 1 \). In this subsection, we provide further examples of quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \) with algebraic Fourier coefficients which have no common zeros.

Recall that for a non-cocompact Fuchsian group \( \Gamma \), we denote the space of quasi-modular forms of weight \( k \) and depth \( \leq \ell \) for \( \Gamma \) by \( \tilde{M}_k^{(\leq \ell)}(\Gamma) \). Also we denote the space of modular forms of weight \( k \) for \( \Gamma \) by \( M_k(\Gamma) \). By abuse of notation, we write \( M_k := M_k(\Gamma) \) and \( \tilde{M}_k^{(\leq \ell)} := \tilde{M}_k^{(\leq \ell)}(\Gamma) \) if \( \Gamma = \text{SL}_2(\mathbb{Z}) \).

Let \( M_\mathbb{Q} := \left( \bigoplus_{k \geq 0} M_k \right) \cap \rho^{-1}(\mathbb{Q}[y, z]) \) be the ring of modular forms for \( \text{SL}_2(\mathbb{Z}) \) with algebraic Fourier coefficients, and let \( \tilde{M}_\mathbb{Q} := \left( \bigcup_{k \geq 0} \bigoplus_{k \geq 0} \tilde{M}_k^{(\leq \ell)} \right) \cap \rho^{-1}(\mathbb{Q}[x, y, z]) \) be the ring of quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \) with algebraic Fourier coefficients.

Proposition 3.10. Let \( f \) and \( g \) be quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \), both of algebraic Fourier coefficients. Write \( g \) as

\[
g = g_0 + g_1 E_2 + \cdots + g_\ell E_2^\ell
\]
for \( g_i \in M_Q \). If there exists a modular form \( h \in f \tilde{M}_Q + g \tilde{M}_Q \) such that \( h \) is relatively prime to \( g_i \) in the ring of modular forms \( M_Q \) for some \( i \in \{0, 1, \ldots, \ell\} \), then \( f \) and \( g \) have no common zero.

\textbf{Proof.} Let \( \gcd(\rho(f), \rho(g)) = d \). Since \( h \in f \tilde{M}_Q + g \tilde{M}_Q \) and is a modular form, \( \rho(h) \) belongs to the first elimination ideal \( (\rho(f), \rho(g))_1 := (\rho(f), \rho(g)) \cap \mathbb{Q}[y, z] \) of \( (\rho(f), \rho(g)) \). Thus \( \rho(h) \) is divisible by \( d \), but here there exists \( \rho(g_i) \) for some \( i \in \{0, 1, \ldots, \ell\} \) which is relatively prime to \( \rho(h) \), so prime to \( d \). Since \( \rho(h) \in \mathbb{Q}[y, z] \) implies that \( d \in \mathbb{Q}[y, z] \) and since \( \rho(h) \mid \rho(g) \), we have \( d \mid \rho(g_j) \) for any \( j \in \{0, 1, \ldots, \ell\} \). In particular \( d \) divides both of \( \rho(h) \) and \( \rho(g_i) \), therefore \( d \) must be a constant.

\textbf{Corollary 3.11.} Let \( f \) and \( g \) be quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \), both of algebraic Fourier coefficients. Suppose that \( g \) has the \( f \)-expansion, i.e.

\[ g = g_0 + g_1 f + \cdots + g_\ell f^\ell \]

for some \( g_0, g_1, \ldots, g_\ell \in M_Q \). If \( g_0 \) is relatively prime to \( g_i \) in \( M_Q \) for some \( i \in \{1, 2, \ldots, \ell\} \), then \( f \) and \( g \) have no common zero.

\textbf{Proof.} It follows from Proposition 3.10 with \( h = g_0 = g - (g_1 + g_2 f + \cdots + g_\ell f^{\ell-1}) f \).

If we let \( f = E_2 \), the above corollary gives Proposition 3.5 for \( N = 1 \) (or, \[15\), Proposition 3.2).

\textbf{Corollary 3.12.} Let \( f \) be a modular form of arbitrary weight for \( \text{SL}_2(\mathbb{Z}) \), and \( g \) be a quasi-modular form for \( \text{SL}_2(\mathbb{Z}) \), both of algebraic Fourier coefficients. If there is a quasi-modular form \( F \in \tilde{M}_Q \) such that \( g \) has the \( F \)-expansion \( g = g_0 + g_1 F + \cdots + g_\ell F^\ell \) for which \( g_i \)'s are coprime in \( M_Q \), then \( f \) and \( g \) have no common zero.

\textbf{Proof.} It follows from Proposition 3.10 with \( h = f \), then since one of \( g_i \) must be relatively prime to \( h \).

\textbf{Example 3.13.} There is no common zero of any holomorphic modular form (equivalently, a quasi-modular form of the minimal depth \( r = 0 \)) and a quasi-modular form of the maximal depth (that is, a quasi-modular form of weight \( k \) and depth \( r = \frac{k}{2} \)) for \( \text{SL}_2(\mathbb{Z}) \).

\textbf{Example 3.14.} Let \( k = 2, 4, 6 \). For each integer \( n \geq 1 \) there is no common zero of \( \frac{d^n E_k(\tau)}{d\tau^n} \) and any non-zero holomorphic modular form of arbitrary weight for \( \text{SL}_2(\mathbb{Z}) \).

\textbf{Remark.} Let \( I \subseteq \mathbb{C}[x_1, x_2, x_3] \) be the vanishing ideal of a singleton \( \{(1,1,1)\} \subset \mathbb{A}^3 \), and let \( J \) be a weighted homogeneous ideal contained in \( I \). Then the first elimination ideal \( J_1 := J \cap \mathbb{C}[x_2, x_3] \) of \( J \) is contained in \( (x_2^3-x_3^3)\mathbb{C}[x_2, x_3] \). Indeed, if we pick an arbitrary generator \( F \) of the ideal \( J \), then \( f := \rho^{-1}(F) \) is a quasi-modular form of weight \( 2 \text{wtdeg}(F) \), and is cuspidal. Since \( J \) is weighted homogeneous, so is \( J_1 \), which means that any weighted homogeneous element of \( J_1 \) is also cuspidal, i.e., it is corresponding to a holomorphic cusp form. Note that any cusp form is written as \( \Delta_1 \cdot g \) for some modular form \( g \) and the weight 12 cusp form \( \Delta_1 := E_4^3 - E_6^2 \). Hence any generator of an ideal \( J_1 \) is divided by the polynomial \( x_2^3-x_3^3 \).

4. Simplicity of zeros of quasi-modular forms: Proof of Theorem 1.3

In this section, we show the simplicity of zeros, Theorem 1.3, Proposition 3.3 provides a useful method to prove the non-existence of common zeros of certain quasi-modular forms. For example, one can easily deduce the following proposition from Proposition 3.3.
Proposition 4.1. For $N = 2, 3$, all the zeros of each of 

$$E_2^{(N)}, \theta E_2^{(N)}, \theta^2 E_2^{(N)}, \theta^3 E_2^{(N)}$$

are simple. Moreover, there are no common zeros of any two of 

$$E_2^{(N)}, \theta E_2^{(N)}, \theta^2 E_2^{(N)}, \theta^3 E_2^{(N)}, \theta^4 E_2^{(N)}.$$ 

Proof. By the Ramanujan identity (4) for $\Gamma_0^+(2)$, we have 

$$\theta E_2^{(2)} = \frac{1}{8} \left( (E_2^{(2)})^2 - E_4^{(2)} \right),$$

$$\theta^2 E_2^{(2)} = \frac{1}{32} \left( (E_2^{(2)})^3 - 3E_2^{(2)} E_4^{(2)} + 2E_6^{(2)} \right),$$

$$\theta^3 E_2^{(2)} = \frac{1}{256} \left( 3(E_2^{(2)})^4 - 18(E_2^{(2)})^2 E_4^{(2)} + 24E_2^{(2)} E_6^{(2)} - 5(E_4^{(2)})^2 - 4 \left( \frac{E_6^{(2)}}{E_4^{(2)}} \right)^2 \right),$$

$$\theta^4 E_2^{(2)} = \frac{1}{512} \left( 3(E_2^{(2)})^5 - 30(E_2^{(2)})^3 E_4^{(2)} + 60(E_2^{(2)})^2 E_6^{(2)} - 25E_2^{(2)} E_4^{(2)})^2 + 12E_4^{(2)} E_6^{(2)} - 20 \left( \frac{E_2^{(2)}}{E_4^{(2)}} \right)^2 \right),$$

so the numerators of reduced forms of $\rho(\theta^i E_2^{(2)})$ for $i = 0, 1, 2, 3, 4$ are 

$$x, \quad x^2 - y, \quad x^3 - 3xy + 2z, \quad 3x^4 y - 18x^2 y^2 + 24xyz - 5y^3 - 4z^2, \quad 3x^5 y - 30x^3 y^2 + 60x^2 yz - 25xy^2 + 12y^2 z - 20xz^2,$$

respectively. The proof for the case when $N = 2$ follows from the fact that any two numerators listed above have no non-unit common divisors. Similarly, for $N = 3$, by the Ramanujan identity (5) for $\Gamma_0^+(3)$, we have 

$$\theta E_2^{(3)} = \frac{1}{6} \left( (E_2^{(3)})^2 - E_4^{(3)} \right),$$

$$\theta^2 E_2^{(3)} = \frac{1}{18} \left( (E_2^{(3)})^3 - 3E_2^{(3)} E_4^{(3)} + 2E_6^{(3)} \right),$$

$$\theta^3 E_2^{(3)} = \frac{1}{36} \left( (E_2^{(3)})^4 - 6(E_2^{(3)})^2 E_4^{(3)} - (E_4^{(3)})^2 + 8E_2^{(3)} E_6^{(3)} - 2 \left( \frac{E_6^{(3)}}{E_4^{(3)}} \right)^2 \right),$$

$$\theta^4 E_2^{(3)} = \frac{1}{54} \left( (E_2^{(3)})^5 - 10(E_2^{(3)})^3 E_4^{(3)} - 5E_2^{(3)} (E_4^{(3)})^2 + 20(E_2^{(3)})^2 E_6^{(3)} + 3E_4^{(3)} E_6^{(3)} - 10 \left( \frac{E_2^{(3)}}{E_4^{(3)}} \right)^2 + \left( \frac{E_6^{(3)}}{E_4^{(3)}} \right)^3 \right),$$

so the same argument completes the proof for $N = 3$. \qed
This method can be adopted to show the simplicity of zeros of all derivatives of quasi-modular forms $E^{(N)}_2, E^{(N)}_4, E^{(N)}_6$ by analyzing the rational functions $\rho \left( E^{(N)}_2 \right), \rho \left( E^{(N)}_4 \right), \rho \left( E^{(N)}_6 \right)$ and the derivations $D^{(N)}$ in $\mathbb{Q}(x, y, z)$ corresponding to $\theta$ for $N = 1, 2, 3$.

In [10, Theorem 7], the authors considered $N = 1$ cases and proved that all the zeros of $\frac{d^r E_k^{(1)}(r)}{d \tau^r}$ are simple for all integers $r \geq 1$ and even integers $k \geq 2$. Their method, which relies on the properties of certain polynomial rings, has some obstruction to be generalized to the cases when $N \geq 2$, since the Ramanujan identities (11 and 12) for $N = 2, 3$ are not of polynomial forms in terms of $E_k^{(N)}$’s anymore. So we carry out some careful analysis on this issue to generalize the simplicity result to the cases when $N = 2, 3$. For the readers’ convenience, we convey its complete proof including the case $N = 1$.

With the natural isomorphism $\rho: \mathbb{Q} \left( E_2^{(N)}, E_4^{(N)}, E_6^{(N)} \right) \rightarrow \mathbb{Q}(x, y, z)$ such that $\rho \left( E_2^{(N)} \right) = x, \rho \left( E_4^{(N)} \right) = y, \rho \left( E_6^{(N)} \right) = z$ given in (6), the derivation $\theta$ on the space of quasi-modular forms can be represented as the derivation $D^{(N)}$ on $\mathbb{Q}(x, y, z)$ as follows:

$$D^{(N)} f := \begin{cases} \frac{x^2 - y}{12} \frac{\partial}{\partial x} f + \frac{xy - z}{3} \frac{\partial}{\partial y} f + \frac{xx - y^2}{2} \frac{\partial}{\partial z} f, & \text{if } N = 1, \\ \frac{x^2}{8} f_x + \frac{xy}{2} f_y + \frac{3xz - 2y^2 - z^2}{4} f_z, & \text{if } N = 2, \\ \frac{x^2}{6} f_x + \frac{2(xy - z)}{3} f_y + \frac{2xz - y^2 - z^2}{2} f_z, & \text{if } N = 3, \\ = p^{(N)} f_x + q^{(N)} f_y + r^{(N)} f_z, \end{cases}$$

where

$$p^{(1)} := \frac{x^2 - y}{12}, \quad q^{(1)} := \frac{xy - z}{3}, \quad r^{(1)} := \frac{xx - y^2}{2},$$

$$p^{(2)} := \frac{x^2 - y}{8}, \quad q^{(2)} := \frac{xy - z}{2}, \quad r^{(2)} := \frac{3xz - 2y^2 - z^2}{4},$$

$$p^{(3)} := \frac{x^2 - y}{6}, \quad q^{(3)} := \frac{2(xy - z)}{3}, \quad r^{(3)} := \frac{2xz - y^2 - z^2}{2},$$

referring to (6), (11), and (12). Recalling $m$ and $d$ in (7) depending on $N$, we note that $d = \frac{4m}{m-2}$ for each $N = 1, 2, 3$. Then the above equations can be summarized as

$$D^{(N)} f = \frac{1}{d} \left( (x^2 - y) f_x + 4(xy - z) f_y + \left( 6xz - \frac{2m}{m-2} y^2 - \frac{4(m-3)}{m-2} z^2 / y \right) f_z \right).$$

Note that for $N = 2, 3$, $y D^{(N)}$ are operators in $\mathbb{Q}[x, y, z]$, but $D^{(N)}$ are not. Before further discussion on common zeros, we need to carry out some careful analyses on the degrees of the denominators of $(D^{(2)})^n$ and $(D^{(3)})^n$ at $x, y,$ and $z$, for $n \geq 1$.

4.1. Degree analysis on $(D^{(N)})^n$. This section is dedicated to some necessary analyses of the degrees of $(D^{(N)})^n x, (D^{(N)})^n y,$ and $(D^{(N)})^n z$ for $N = 2, 3$, and especially the degrees of their denominators.

We consider the case when $N = 2$ in detail here. The results for the case when $N = 3$ will be stated in this section, and its proof will be given in Appendix A.

Let $D^{(2)} = d D^{(2)} = 8 D^{(2)}$ so that they have the integral coefficients, i.e.,

$$D^{(2)} = (x^2 - y) \frac{\partial}{\partial x} + (4xy - 4z) \frac{\partial}{\partial y} + (6xz - 4y^2 - 2y^{-1}z^2) \frac{\partial}{\partial z}.$$
Let \( \deg_x, \deg_y, \deg_z \) be the degrees of monomials in \( \mathbb{Q}[x, y, z, 1/y] \) with respect to \( x, y \) and \( z \), respectively. (Note that \( \deg_y \) can be negative.) Also, let

\[
\text{wtdeg} = \deg_x + 2\deg_y + 3\deg_z
\]

be the weighted degree. Note that all monomials in \( (D^{(N)})^n x, (D^{(N)})^n y, (D^{(N)})^n z \) have the same wtdeg, so wtdeg can be extended to them, for \( n \geq 0 \). Also we note that \( \text{wtdeg} (D^{(2)} f) = \text{wtdeg}(f) + 1 \) for a weighted homogeneous \( f \in \mathbb{Q}[x, y, z, 1/y] \).

Since

\[
D^{(2)} (x^a y^b z^c) = (a + 4b + 6c)x^{a+1} y^b z^c - (4b + 2c)x^a y^{b-1} z^{c+1} - 4cx^a y^{b+2} z^{c-1} - ax^a y^{b+1} z^c,
\]

when we represent (some coefficient) \( x^a y^b z^c \) as a point \( (a, b, c)^T \in \mathbb{Z}^3 \) (here, \( T \) stands for the transpose) \( D^{(2)} \) transfers \( (a, b, c)^T \) to (at most) four points as

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} \overset{D^{(2)}}{\mapsto} \begin{pmatrix}
a \\
b \\
c
\end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \left\{ \pm \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} \right\}.
\]

We introduce the matrix,

\[
T = \frac{1}{14} \begin{pmatrix} 13 & -2 & -3 \\ 3 & 6 & -5 \\ 1 & 2 & 3 \end{pmatrix}
\]

with its inverse \( T^{-1} = \begin{pmatrix} 1/2 & 3/2 & 2 \\ -3/2 & 3 & 2 \\ 0 & -1 & 3 \end{pmatrix} \),

which is determined to have the following relation, and transforms the above four candidate points into the unit vectors on a plane:

\[
T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \overset{T D^{(2)} T^{-1}}{\mapsto} T \begin{pmatrix} a \\ b \\ c \end{pmatrix} + T \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \left\{ \pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.
\]

and by this way we can represent the monomial \( x^a y^b z^c \) in \( (D^{(2)})^r (x) \) as \( (\lambda, \nu) \in \mathbb{Z}^2 \) with relation

\[
(a, b, c)^T = (1, r/2, 0)^T + T^{-1}(\lambda, \nu, 0)^T,
\]

i.e., \( (a, b, c) = (\lambda + 1, (-\lambda + 3\nu + r)/2, -\nu) \). After applying \( (D^{(2)})^r \), the monomials of \( (D^{(2)})^r x \) are represented as elements in

\[
\left\{ (1, r/2, 0)^T + T^{-1}(\lambda, \nu, 0)^T : |\lambda| + |\nu| \leq r, \quad \lambda + \nu \equiv r \pmod{2} \right\}.
\]

This set is not necessarily the whole set of all monomials in \( (D^{(2)})^r x \); for example, there are some obvious restrictions like the non-negativeness of \( \deg_x \) and \( \deg_z \), which means \( \lambda \geq -1 \) and \( \nu \leq 0 \). We can prove another condition and find the formula for the minimal \( \deg_y \) of monomials of \( (D^{(2)})^r x \).

Let \( x^a y^b z^c \) be a monomial appearing in \( (D^{(2)})^r x \). If \( D^{(2)} (x^a y^b z^c) \) contains a monomial with \( \deg_y \) less than \( b \), then the monomial with decreased \( \deg_y \) is \( -(4b + 2c)x^a y^{b-1} z^{c+1} \). This term is non-zero only when \( c \neq -2b \). When \( c = -2b \), since \( a + 2b + 3c = \text{wtdeg} \left( (D^{(2)})^r x \right) = r + 1 \), we have \( a = r + 1 - 2b - 3c = r + 1 - 4c \). Referring to [10], the corresponding condition for \( (\lambda, \nu) \) is that \( \nu = \frac{1}{2} \lambda - \frac{r}{2} \), i.e., only the monomials (represented as a lattice point \( (\lambda, \nu) \)) off this line can provide the monomials whose \( \deg_y \) drops by 1 after applying \( D^{(2)} \).
Figure 1 shows the situation for $(D^{(2)})^{10}x$. Each points $(\lambda, \nu)$ on the figure represents each monomial $x^ay^bz^c$ appearing in $(D^{(2)})^{10}x$ by the relation given in (10). The red-filled point at $(\lambda, \nu) = (-1, -5)$ (which represents $y^{-2}z^{5}$-term) is off the dashed line $\nu = \frac{1}{2}(\lambda - r)$. So, $D^{(2)}(y^{-2}z^{5})$ provides a function which contains a monomial whose $\text{deg}_y$ is $-3$; one can check that $D^{(2)}(y^{-2}z^{5}) = 22xz^5y^{-2} - 2z^6y^{-3} - 20z^4$.

From the discussion so far, we conclude the following lemma.

**Lemma 4.2.** Let $x^ay^bz^c$ be some monomial in $(D^{(2)})^r x$ (up to nonzero coefficients over $\mathbb{Q}$).

(a) Its corresponding lattice point $(\lambda, \nu)$ via (10) satisfies that $\nu \geq \frac{1}{2}(\lambda - r)$.

(b) $2b + c \geq 0$ and $b \geq -\frac{r + 1}{4}$.

**Proof.** (a) This holds for the initial case $r = 0$. As $r$ increases, all points keep lying above or on the line $\nu = \frac{1}{2}(\lambda - r)$, since the line itself is shifted by $-1/2$ in $y$ axis direction on each step, and the point on the line does not provide a monomial with decreased $\text{deg}_y$, i.e., decreased $\nu$ coordinate.

(b) Referring to (10), since $a + 2b + 3c = r + 1$, the inequality $\nu \geq \frac{1}{2}(\lambda - r)$ is equivalent to $2b + c \geq 0$, which implies the second inequality since $a \geq 0$ and $a + 2b + 3c = r + 1$.

In the remaining section, we prove that the second inequality of Lemma 4.2 is sharp, in the sense that when $r \equiv 3 \pmod{4}$, there exists a monomial with $\text{deg}_y = -\frac{r + 1}{4}$ in $(D^{(2)})^r x$.

To prove this, we need to show that the coefficient of the $y^{-k}z^{2k+1}$-term in $(D^{(2)})^{4k+2}$ is non-zero for all integers $k \geq 0$. It can be shown by proving that their coefficients are all $1$ modulo $5$.
Let $a_{i,j,k}^{(r)}$ denote the coefficient of the $x^iy^jz^k$-term in $(D_1^{(2)})^r x$. By keeping track of the recurrences for the coefficients, we reduce the degrees as follows:

\[
a_{0,-k,2k+1}^{(4k+2)} = (-4(-k + 1) - 2(2k))a_{0,-k+1,2k}^{(4k+1)} = -2k \equiv a_{0,-k,2k}^{(4k+1)} \pmod{5}
\]

\[
= (-4(-k + 2) - 2(2k - 1))a_{0,-k+2,2k-1}^{(4k)} - a_{1,-k,2k}^{(4k)}
\]

\[
= -6a_{0,-k+2,2k-1}^{(4k)} - a_{1,-k,2k}^{(4k)} \equiv -(a_{0,-k+2,2k-1}^{(4k)} + a_{1,-k,2k}^{(4k)}) \pmod{5}.
\]

Since

\[
\begin{cases}
  a_{0,-k+2,2k-1}^{(4k)} = -8a_{0,-k+3,2k-2}^{(4k-1)} - a_{1,-k+1,2k-1}^{(4k-1)} - 8ra_{0,-k,2k}^{(4k-1)}, \\
  a_{1,-k,2k}^{(4k)} = -2a_{1,-k+1,2k-1}^{(4k-1)} + 8ra_{0,-k,2k}^{(4k-1)}
\end{cases}
\]

we have

\[
a_{0,-k,2k+1}^{(4k+2)} \equiv 3\left(a_{1,-k,2k-1}^{(4k-1)} + a_{0,-k+3,2k-2}^{(4k-1)}\right) \pmod{5}.
\]

Similarly, since

\[
\begin{cases}
  a_{0,-k+3,2k-2}^{(4k)} = (-8k + 4)a_{0,-k+1,2k-1}^{(4k-2)} - 10a_{0,-k+4,2k-3}^{(4k-2)} - a_{1,-k+2,2k-2}^{(4k-2)} \\
  a_{1,-k+1,2k-1}^{(4k)} = -2a_{1,-k+2,2k-2}^{(4k-2)} + (8k - 2)a_{0,-k+1,2k-1}^{(4k-2)}
\end{cases}
\]

we have

\[
a_{0,-k,2k+1}^{(4k+2)} \equiv 3a_{0,-k+3,2k-2}^{(4k)} \pmod{5}.
\]

Since $a_{0,0,1} \not\equiv 0 \pmod{5}$, the coefficient $a_{0,-k,2k+1}^{(4k+2)}$ never vanishes.

We conclude that $y^{\ell_r} \cdot (D_2^{(2)})^r x \in \mathbb{Q}[x, y, z] \setminus y\mathbb{Q}[x, y, z]$, with $\ell_r = [(r+1)/4]$.

The analyses for $(D_1^{(2)})^r y$ and $(D_2^{(2)})^r z$ are exactly the same; in conclusion, we have the following lemma.

**Lemma 4.3.** For each integer $r \geq 0$, if we let $\ell_r = [(r+1)/4]$ then

\[
y^{\ell_r} \cdot (D_1^{(2)})^r x, \quad y^{\ell_r+1} \cdot (D_2^{(2)})^r y, \quad y^{\ell_r+2} \cdot (D_2^{(2)})^r z \in \mathbb{Q}[x, y, z] \setminus y\mathbb{Q}[x, y, z].
\]

We get similar results for the case when $N = 3$ as follows and we postpone their proofs given in Appendix A so that the readers can see the proof of Theorem 1.3 just below right away.

**Lemma 4.4.** Let $x^ay^bz^c$ be some monomial in $(D_1^{(3)})^r x$ (up to some nonzero $\mathbb{Q}$ coefficient).

(a) Its corresponding lattice point $(\lambda, \nu)$ via [10] satisfies that $\nu \geq 4(\lambda - r)$.

(b) $4b + 3c \geq 0$ and $b \geq -\frac{r+1}{2}$.

**Lemma 4.5.** For $r \geq 0$, let $\ell_r = \begin{cases} \lfloor (r+1)/2 \rfloor - 1 \quad \text{if } \ell \equiv 1, 2, 3 \pmod{6}, \\
\lfloor (r+1)/2 \rfloor \quad \text{otherwise}. \end{cases}$ Then,

\[
y^{\ell_r} \cdot (D_1^{(3)})^r x, \quad y^{\ell_r+1} \cdot (D_1^{(3)})^r y, \quad y^{\ell_r+2} \cdot (D_2^{(3)})^r z \in \mathbb{Q}[x, y, z] \setminus y\mathbb{Q}[x, y, z].
\]
4.2. Proof of the Theorem 1.3. With the results in Section 4.1, we prove Theorem 1.3

Recall [8] for the definition of $D^{(N)}$. We define the auxiliary differential operators $D_t^{(N)}$ for $t = x, y, z$, as

$$D_t^{(N)} := p_t^{(N)} f_x + q_t^{(N)} f_y + r_t^{(N)} f_z,$$

for each $N = 1, 2, 3$. Especially $D_x^{(N)}$ plays a special role among others as we see later in this section, thus we let $D^{(N)} := D_x^{(N)}$.

We note that for $N = 1, 2, 3$,

$$\widetilde{D}^{(N)} f = (2xf_x + 4yf_y + 6zf_z)/d,$$

and especially,

$$(11) \quad \widetilde{D}^{(N)} x = (2/d)x, \quad \widetilde{D}^{(N)} y = (4/d)y, \quad \widetilde{D}^{(N)} z = (6/d)z.$$

We generalize [10, Lemma 16, Lemma 17] for the case $N = 1$ and get the following two lemmas for $N = 2, 3$.

**Lemma 4.6.** We have the following relations of $D^{(N)}$, $\widetilde{D}^{(N)}$ and $\partial / \partial x$ for each $N = 1, 2, 3$;

(a) $\partial / \partial x D^{(N)} f = \widetilde{D}^{(N)} f + D^{(N)} (f_x)$ for $f \in \mathbb{Q}[x, y, z, 1/y]$.

(b) $D^{(N)} (D^{(N)})^n = (D^{(N)})^n D^{(N)} + \frac{2n}{\partial} (D^{(N)})^n$ for all $n \geq 1$.

**Proof.** (a) follows from the direct calculation; for $t = x, y, z$,

$$\frac{\partial}{\partial t} D^{(N)} f = p_t^{(N)} f_x + q_t^{(N)} f_y + r_t^{(N)} f_z + p^{(N)} f_x t + q^{(N)} f_y t + r^{(N)} f_z t = D_t^{(N)} f + D^{(N)} (f_t).$$

In order to prove (b), we can verify the following:

$$A := \begin{pmatrix} p_x^{(N)} & p_y^{(N)} & p_z^{(N)} \\ q_x^{(N)} & q_y^{(N)} & q_z^{(N)} \\ r_x^{(N)} & r_y^{(N)} & r_z^{(N)} \end{pmatrix} = \frac{1}{d} \begin{pmatrix} 2x & -1 & 0 \\ 4y & 4x & -4 \\ 6y & 4(m-3)z^2 & 6x - 8(m-3)z \frac{x}{(m-2)y} \end{pmatrix},$$

$$A \begin{pmatrix} p_x^{(N)} \\ q_x^{(N)} \\ r_x^{(N)} \end{pmatrix} = \begin{pmatrix} \frac{D p_x^{(N)}}{D^{(N)}} \\ \frac{D q_x^{(N)}}{D^{(N)}} \\ \frac{D r_x^{(N)}}{D^{(N)}} \end{pmatrix} = \frac{2}{d} \begin{pmatrix} 2p \\ 3q \\ 4r \end{pmatrix},$$

and

$$B := \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ q_{xx} & q_{xy} & q_{xz} \\ r_{xx} & r_{xy} & r_{xz} \end{pmatrix} = \frac{2}{d} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} D p_x^{(N)} \\ D q_x^{(N)} \\ D r_x^{(N)} \end{pmatrix} = \frac{2}{d} \begin{pmatrix} 2p \\ 2q \\ 2r \end{pmatrix}.$$

With these equations, the direct calculation shows that $\widetilde{D}^{(N)} D^{(N)} f - D^{(N)} \widetilde{D}^{(N)} f = \frac{2}{d} D^{(N)} f$.

With this and (a), we can show (b) inductively. \hfill \Box

**Remark.** The property (b) of Lemma 4.6 is not naturally arising from an arbitrary choice of $p, q$ and $r$. The boilage from tedious calculations of $\widetilde{D}^{(N)} D^{(N)} - D^{(N)} \widetilde{D}^{(N)}$ consists of $D_x^{(N)} (p^{(N)} - D^{(N)} p_x^{(N)})$, $D_x^{(N)} (q^{(N)} - D^{(N)} q_x^{(N)})$ and $D_x^{(N)} (r^{(N)} - D^{(N)} r_x^{(N)})$ terms, which differ from $p, q$ and $r$ only by a scalar multiple (specifically, $2/d$), respectively.
Lemma 4.7. Let $N = 1, 2, 3$. For each integer $n \geq 1$, we have
\begin{align*}
\frac{\partial}{\partial x} \left((D^{(N)^n}x)\right) &= \frac{n(n+1)}{d}(D^{(N)^n}x), \\
\frac{\partial}{\partial x} \left((D^{(N)^n}y)\right) &= \frac{n(n+3)}{d}(D^{(N)^n}y), \\
\frac{\partial}{\partial x} \left((D^{(N)^n}z)\right) &= \frac{n(n+5)}{d}(D^{(N)^n}z),
\end{align*}

i.e., $\frac{\partial}{\partial x} (D^{(N)^n}x)$ is a scalar multiple of $(D^{(N)^n}x)$, and so are those at $y$ and $z$.

Proof. We prove the following by induction on $n$:
\begin{align*}
\frac{\partial}{\partial x}((D^{(N)^n}f)) &= n(D^{(N)^n} - \frac{d}{n}(D^{(N)^n}f)). \\
&= (D^{(N)^n}f) + \frac{n(n-1)}{d}(D^{(N)^n}f).
\end{align*}

For $n = 1$, we have that $\frac{\partial}{\partial x} D^{(N)}f = \frac{\partial}{\partial x} D^{(N)}f$ by Lemma 4.6. To proceed by induction, suppose it holds for $n - 1$. Then by Lemma 4.6, we have that
\begin{align*}
\frac{\partial}{\partial x}((D^{(N)^n}f)) &= \frac{\partial}{\partial x} D^{(N)}((D^{(N)^n}f)) \\
&= (D^{(N)^n}f) + \frac{n(n-1)}{d}(D^{(N)^n}f).
\end{align*}

Substituting $x, y, z$ for $f$ in turn yields the desired result, referring to (11). \hfill \Box

Lemma 4.8. Let $N = 1, 2, 3$. Let $f \in \mathbb{Q}[x, y, z]$ be a prime factor of the numerator of a reduced form of one of $(D^{(N)^n}x)$, $(D^{(N)^n}y)$, and $(D^{(N)^n}z)$, for some integer $n \geq 1$. Then $f$ divides neither $y$ nor the numerator of a reduced form of $D^{(N)}f$, and $f_x \neq 0$.

Proof. Note that the only principal ideals $I$ with property $yD^{(N)}(I) \subseteq I$ are $I = (y)$ or $I = (y^3 - z^2)$. (See [11], Theorem 3.2.)

Let $n \geq 1$ be a given integer. Let $F/y^f$ be a reduced form of $(D^{(N)^n}x)$ in $\mathbb{Q}(x, y, z)$ for some integer $\ell \geq 0$. We claim that $F \not\in y\mathbb{Q}[x, y, z]$. When $N = 1$, since the coefficient of $x^{n+1}$-term in $D^{(n)}x$ is not zero, $F \not\in y\mathbb{Q}[x, y, z]$. For $N = 2, 3$, we have shown that $\ell \geq 1$, i.e., $y \not\mid F$, in Lemma 4.3 and Lemma 4.5 when $n \geq 3$. Since $(D^{(N)^n}x)$ and $(D^{(N)^n}y)$ don’t have $y$ as a factor of each, $F \not\in y\mathbb{Q}[x, y, z]$ for $n \geq 1$.

Let $f$ be a prime factor of $F$. If $f$ divides the numerator of a reduced form of $D^{(N)}f$, then $f \mid yD^{(N)}f$ (in $\mathbb{Q}[x, y, z]$), i.e., $f \in (y)$ or $y \in (y^3 - z^2)$. Since $f \not\in (y)$, $f = k(y^3 - z^2)$ for some $k \in \mathbb{Q}$. Thus, we have $(y^3 - z^2) \mid F$.

Note that there is a non-vanishing $x^{n+1}$-term in $(D^{(N)^n}x)$. Since $y^3 - z^2$ divides $F$, $(D^{(N)^n}x)$ has a non-zero $x^{n+1}y^3z^2$-term with the same (non-zero) coefficient. This is clearly impossible when $N = 1$, and also impossible when $N = 2, 3$, referring to Lemma 4.2 (b) and Lemma 4.4 (b).

Also, since $f \mid F = y^f(D^{(N)^n}x)$ has a $x^{n+1}y^f$-term, we can show that $f_x \neq 0$ by a similar argument.

The proof for the cases of $(D^{(N)^n}y)$ or $(D^{(N)^n}z)$ can be done in a similar manner. \hfill \Box
Now we are ready to prove the following proposition.

**Proposition 4.9.** Let \( N = 1, 2, 3 \). For each integer \( n \geq 1 \), we have

\[
\text{ngcd} \left( (D^{(N)})^n x, (D^{(N)})^{n+1} x \right) = \text{ngcd} \left( (D^{(N)})^{n} y, (D^{(N)})^{n+1} y \right) = \text{ngcd} \left( (D^{(N)})^{n} z, (D^{(N)})^{n+1} z \right) = 1.
\]

Here \( \text{ngcd}(f, g) \) is the gcd of the numerators of reduced forms of \( f \) and \( g \), where \( f, g \in \mathbb{Q}[x, y, z, 1/y] \).

**Proof.** The key idea for the proof is to use the interplaying properties such as Lemma 4.6 and Lemma 4.7, and to consider \( \frac{D}{dx} ((D^{(N)})^{n+1} x) \) and \( (D^{(N)}) ((D^{(N)})^n x) \).

Assume that a prime polynomial \( f \in \mathbb{Q}[x, y, z] \) is a common factor of the numerators of reduced forms of \( (D^{(N)})^n x \) and \( (D^{(N)})^{n+1} x \). Then we write

\[
(D^{(N)})^n x = f^k g / y^\ell, \quad (D^{(N)})^{n+1} x = f^k h / y^\ell'
\]

for some integers \( k \geq 1 \) and \( \ell, \ell' \geq 0 \), and for some \( g, h \in \mathbb{Q}[x, y, z] \) such that \( f \nmid \text{gcd}(g, h) \).

Note that \( \ell \) and \( \ell' \in \{ \ell, \ell + 1 \} \) are integers \( \geq 0 \). Referring to Lemma 4.8, we see that \( y \nmid f \).

Referring to Lemma 4.7, we have

\[
\frac{D}{dx} ((D^{(N)})^{n+1} x) = \left( f^k h_x + k f^{k-1} f_x h \right) / y^{\ell'} = \frac{(n+1)(n+2)}{d} f^k g / y^\ell,
\]

so

\[
\left( \frac{(n+1)(n+2)}{d} y^{\ell' - \ell} - h_x \right) f = kf_x h
\]

in \( \mathbb{Q}[x, y, z] \), thus \( f \mid h \), since \( f \) is a prime and \( f_x \neq 0 \) by Lemma 4.8.

Now we consider

\[
(D^{(N)})^{n+1} x = f^k h / y^{\ell'} = D^{(N)}(f^k \cdot gy^{\ell}) = kf^{k-1}gy^{\ell - \ell}D^{(N)}(f) + kfD^{(N)}(gy^{\ell - \ell}),
\]

i.e.,

\[
kgy^{\ell - \ell}D^{(N)}(f) = fh - fy^{\ell'}D^{(N)}(gy^{\ell}).
\]

Note that \( y^{\ell - \ell}D^{(N)}(f) \) or \( D^{(N)}(gy^{\ell}) \) might have some powers of \( y \) in the denominators of their reduced forms, so we let \( y^{\ell''} \) be the largest power dividing their denominators. Then, we have

\[
kgy^{\ell'' + \ell - \ell}D^{(N)}(f) = \left( hy^{\ell''} - y^{\ell'' + \ell'}D^{(N)}(gy^{\ell - \ell}) \right) f
\]

in \( \mathbb{Q}[x, y, z] \). Thus,

\[
f \mid gy^{\ell'' + \ell - \ell}D(f).
\]

Note that \( f \) does not divide the denominator of a reduced form of \( D(f) \) by Lemma 4.8 and also \( f \nmid y \), therefore \( f \mid g \). This contradicts that \( f \nmid \text{gcd}(g, h) \). This proves \( \text{ngcd} \left( (D^{(N)})^n x, (D^{(N)})^{n+1} x \right) = 1 \).

We apply the same argument with respect to \( y \) and \( z \) to complete the proof.

We have proved all ingredients to prove Theorem 1.3.

**Proof of Theorem 1.3** It follows from Theorem 1.3 and Proposition 4.9.
Appendix A. Degree analysis on \((D^{(3)})^n\): Proofs of Lemma 4.4 and Lemma 4.5

In this appendix, we give an analysis on the degrees of denominators of \((D^{(3)})^n\) at \(x, y, z\), and we prove Lemma 4.4 and Lemma 4.5 which have been postponed from Subsection 4.1 since they can be obtained in a similar manner as done for \((D^{(2)})^n\) in Section 4.1. We would like to note that the case for \((D^{(3)})^n\) is more complicated for us to deal with as you can see below.

In order to prove Lemma 4.4 and Lemma 4.5, we will give some conditions for the monomials appearing in each of \((D^{(3)})^n x\), \((D^{(3)})^n y\), \((D^{(3)})^n z\), and show that each of the minimal degrees with respect to \(y\) among the monomials of \((D^{(3)})^n (x)\), \((D^{(3)})^n (y)\), and \((D^{(3)})^n (z)\) decreases by 3 as \(n\) increases by 6, for \(n \geq 0\).

Let \(D^{(3)} := dD^{(3)} = 6D^{(3)}\) so that they have the integral coefficients. Then,

\[
D^{(3)} = (x^2 - y) \frac{\partial}{\partial x} + (4xy - 4z) \frac{\partial}{\partial y} + (6xz - 3y^2 - 3y^{-1}z^2) \frac{\partial}{\partial z},
\]

and

\[
D^{(3)} (x^ay^bz^c) = (a + 4b + 6c)x^{a+1}y^{b+2}z^{c-1} - 3cx^{a}y^{b+2}z^{c-1} - (4b + 3c)x^{a}y^{b}z^{c+1} - ax^{a-1}y^{b+1}z^c.
\]

We choose the same \(T\) as defined in (9), and let an integral lattice point \((\lambda, \nu)\) represent the monomial \(x^ay^bz^c\) via relation (10) as it is done for \(N = 2\) case, i.e.,

\[(a, b, c)^T = (1, r/2, 0)^T + T^{-1}(\lambda, \nu, 0)^T.
\]

Proof of Lemma 4.4 Let \(x^ay^bz^c\) be a monomial in \((D^{(3)})^r\) \(x\). If \(D^{(3)} (x^ay^bz^c)\) contains a monomial whose \(\text{deg}_y\) is less than \(b\), then such a monomial is \(-(4b+3c)x^ay^{b-1}z^{c+1}\), which is non-zero only when \(4b + 3c \neq 0\). Since the condition \(4b + 3c = 0\) is equivalent to \(\nu = \frac{2}{3}(\lambda - r)\) via (10), the points on this line doesn’t produce a monomial with decreased \(\text{deg}_y\) through the differential operator.

(a) holds for the initial case \(r = 0\). As \(r\) increases, all points keep lying above or on the line \(\nu = \frac{2}{3}(\lambda - r)\); so we have \(\nu \geq \frac{2}{3}(\lambda - r)\) for all \(r\). This proves (a).

The inequality from (a) is equivalent to \(4b + 3c \geq 0\) by (10), and we conclude (b) since \(4b + 3c \geq 0\), \(a \geq 0\), and \(a + 2b + 3c = r + 1\).

Proof of Lemma 4.5 Let \(a^{(r)}_{i,j,k}\) denote the coefficient of \(x^iy^jz^k\) in \((D^{(3)})^r\) \(x\).

We prove the lemma by showing that the minimal value among \(\text{deg}_y\) of the monomials in \((D^{(3)})^r (x)\) decreases exactly when applying \(D^{(3)}\) to each of \((D^{(3)})^{(6k+2)} (x)\), \((D^{(3)})^{(6k+3)} (x)\) and \((D^{(3)})^{(6k+4)} (x)\). It is enough to show that the coefficients \(a^{(6k+3)}_{0,-3k-1,4k+2}\), \(a^{(6k+2)}_{0,-3k,4k+1}\), \(a^{(6k+4)}_{0,-3k-2,4k+3}\) never vanish for \(k \geq 0\). We show by the non-triviality of those coefficients modulo 7.

By some tedious calculations, we get;

\[
A := a^{(6k+3)}_{0,-3k-1,4k+2} = -3a^{(6k+2)}_{0,-3k-1,4k+1}, \quad \text{and} \quad a^{(6k+2)}_{0,-3k,4k+1} = -4a^{(6k+1)}_{0,-3k+1,4k+1},
\]

therefore, \(A = 12a^{(6k+1)}_{0,-3k+1,4k}\). Similarly, since \(a^{(6k+1)}_{0,-3k+1,4k} = -5a^{(6k)}_{0,-3k+2,4k-1} - a^{(6k)}_{1,-3k,4k}\), we have

\[
A = -12 \left( 5a^{(6k)}_{0,-3k+2,4k-1} + a^{(6k)}_{1,-3k,4k} \right).
\]
Since
\[
\begin{align*}
\begin{cases}
a_{0,-3k+2,4k-1}^{(6k-1)} &= -6a_{0,0,-3k+3,4k-2}^{(6k-1)} - 12ka_{0,-3k,4k}^{(6k-1)} - a_{1,-3k+1,4k-1}^{(6k-1)}, \\
a_{1,-3k,4k}^{(6k-1)} &= 12ka_{0,-3k,4k}^{(6k-1)} - a_{1,-3k+1,4k-1}^{(6k-1)}
\end{cases}
\end{align*}
\]
we have
\[
A = 12 \times 6 \left( 5a_{0,0,-3k+3,4k-2}^{(6k-1)} + 8ka_{0,-3k,4k}^{(6k-1)} + a_{1,-3k+1,4k-1}^{(6k-1)} \right).
\]
Since
\[
\begin{align*}
\begin{cases}
a_{0,0,-3k+3,4k-2}^{(6k-1)} &= -7a_{0,0,-3k+4,4k-3}^{(6k-2)} + (-12k + 3)a_{0,0,-3k+1,4k-1}^{(6k-2)} - a_{1,-3k+2,4k-2}^{(6k-2)}, \\
a_{0,-3k,4k}^{(6k-1)} &= -a_{0,0,-3k+1,4k-1}^{(6k-1)}, \\
a_{1,-3k+1,4k-1}^{(6k-1)} &= (12k - 2)a_{0,0,-3k+1,4k-1}^{(6k-2)} - 2a_{1,-3k+2,4k-2}^{(6k-2)}
\end{cases}
\end{align*}
\]
we have
\[
A = 12 \times 6 \left( -35a_{0,0,-3k+4,4k-3}^{(6k-2)} + (-56k + 13)a_{0,0,-3k+1,4k-1}^{(6k-2)} - 7a_{1,-3k+2,4k-2}^{(6k-2)} \right) \equiv -2a_{0,0,-3k+1,4k-1}^{(6k-2)} \pmod{7}.
\]
Finally, since \(a_{0,-3k+1,4k-1}^{(6k-2)} = -a_{0,-3k+1,4k-2}^{(6k-3)}\), we also have
\[
A = a_{0,0,-3k-1,4k+2}^{(6k+3)} \equiv 4a_{0,0,-3k+1,4k+2}^{(6k-1)+3} \pmod{7}.
\]
Recalling that \(D^{(3)}(x) = 6x^4 - 36x^2y + 48xz - 6y^2 - \frac{12z^2}{y}\), since \(a_{0,-1,2}^{(3)} = -12 \not\equiv 0 \pmod{7}\), the coefficient \(a_{0,-3k-1,4k+2}^{(6k-3)}\) never vanishes. Many parts of the calculations for the recurrences of the coefficients are recyclable for other two cases. In short, we get
\[
\begin{align*}
a_{0,-3k,4k+1}^{(6k+2)} &= -4a_{0,0,-3k+1,4k+1}^{(6k+1)} = 4 \left( 5a_{0,0,-3k+2,4k+1}^{(6k)} + a_{1,-3k,4k}^{(6k)} \right) \\
&= -24 \left( 5a_{0,0,-3k+3,4k-2}^{(6k-1)} + 8ka_{0,-3k,4k}^{(6k-1)} + a_{1,-3k+1,4k-1}^{(6k-1)} \right) \\
&= -24 \left( -35a_{0,0,-3k+4,4k-3}^{(6k-2)} + (-56k + 13)a_{0,0,-3k+1,4k-1}^{(6k-2)} - 7a_{1,-3k+2,4k-2}^{(6k-2)} \right) \\
&\equiv 3a_{0,0,-3k+1,4k-1}^{(6k-3)} \equiv 3a_{0,0,-3k+1,4k-1}^{(6k-4)} \equiv 4a_{0,0,-3k+3,4k-3}^{(6k-4)} \pmod{7},
\end{align*}
\]
and
\[
\begin{align*}
a_{0,0,-3k-2,4k+3}^{(6k+3)} &= -2a_{0,0,-3k-1,4k+2}^{(6k+2)} = 6a_{0,0,-3k,4k+1}^{(6k)} = -24a_{0,0,-3k+1,4k+1}^{(6k+1)} \\
&= 24 \left( 5a_{0,0,-3k+2,4k-1}^{(6k)} + a_{1,-3k,4k}^{(6k)} \right) \\
&= -24 \times 6 \left( 5a_{0,0,-3k+3,4k-2}^{(6k-1)} + 8ka_{0,-3k,4k}^{(6k-1)} + a_{1,-3k+1,4k-1}^{(6k-1)} \right) \\
&\equiv 4a_{0,0,-3k+1,4k-1}^{(6k-2)} \pmod{7},
\end{align*}
\]
thus
\[
\begin{align*}
a_{0,0,-3k,4k+1}^{(6k+2)} &\equiv 4a_{0,0,-3k+1,4k+1}^{(6k-1)+2} \pmod{7}, \\
a_{0,0,-3k-2,4k+3}^{(6k+4)} &\equiv 4a_{0,0,-3k+1,4k+1}^{(6k-1)+4} \pmod{7}.
\end{align*}
\]
Since \(a_{0,0,1}^{(2)} = 4 \not\equiv 0 \pmod{7}\) and \(a_{0,-2,3}^{(4)} = 24 \not\equiv 0 \pmod{7}\), we conclude that the coefficients \(a_{0,0,-3k-1,4k+2}^{(6k+3)}, a_{0,0,-3k,4k+1}^{(6k+2)}, a_{0,0,-3k-2,4k+3}^{(6k+4)}\) never vanish for \(k \geq 0\), i.e., they provide non-zero monomials with decreased degree after applying \(D^{(3)}\). With Lemma 4.4(b), the minimal degree among monomials appearing in \((D^{(3)})^{r+1}(x)\) drops by 1 than those for \((D^{(3)})^{r}(x)\), only when \(r \equiv 2, 3, 4 \pmod{6}\) as shown in (15), (16) and (17). Therefore we conclude that
\[
y^r \cdot (D^{(3)})^r x \in \mathbb{Q}[x,y,z] \setminus y\mathbb{Q}[x,y,z],
\]
with \(\ell_1, \ldots, \ell_6 = 0, 0, 1, 2, 3, 3\) and \(\ell_{r+6} = \ell_r + 3\) for all \(r \geq 1\).

The analyses for \((D^{(3)})^r y\) and \((D^{(3)})^r z\) can be done in the same way, and this completes the proof. \(\square\)
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