Quantum mechanics using two auxiliary inner products

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Abstract

The current applications of non-Hermitian but $\mathcal{PT}$–symmetric Hamiltonians $H$ cover several, mutually not too closely connected subdomains of quantum physics. Mathematically, the split between the open and closed systems can be characterized by the respective triviality and non-triviality of an auxiliary inner-product metric $\Theta = \Theta(H)$. With our attention restricted to the latter, mathematically more interesting unitary-evolution case we show that the intuitive but technically decisive simplification of the theory achieved via an “additional” $\mathcal{PCT}$–symmetry constraint upon $H$ can be given a deeper mathematical meaning via introduction of a certain second auxiliary inner product.

Keywords

formulations of quantum mechanics;
Schrödinger representation;
non-Hermitian Hamiltonians;
$\mathcal{PT}$– and $\mathcal{PCT}$–symmetry;
physical and auxiliary inner products;
1 Introduction

In the textbooks on quantum mechanics the basic features of bound states $|\psi\rangle$ are best illustrated via ordinary differential Hamiltonian operators

$$H = -\frac{d^2}{dx^2} + V(x)$$

(1)

containing various real and confining one-dimensional potentials $V(x)$ [1]. Operators [1] are assumed acting in the most common physical Hilbert space $L^2(\mathbb{R})$ of complex square-integrable functions. In 1998, Bender with Boettcher [2] attracted attention of the physics community to certain less conventional Hamiltonians [1] in which the potential was complex. The Hamiltonian became non-Hermitian but its spectrum remained real, discrete and bounded from below, i.e., compatible with the possible unitarity of the evolution of the system in question, in principle at least. After a replacement of the requirements of Hermiticity by the intuitively more acceptable conditions of $\mathcal{PT}$—symmetry

$$H \mathcal{PT} = \mathcal{PT} H$$

(2)

plus $\mathcal{PCT}$—symmetry

$$H \mathcal{PCT} = \mathcal{PCT} H$$

(3)

(where $\mathcal{P}$ is parity and $\mathcal{C}$ is charge while $\mathcal{T}$ stands for the time reversal) the unusual choice of the specific non-Hermitian Hamiltonians has very soon been shown fully compatible with the unitarity of dynamics. A new, antilinear-symmetry-based formulation of quantum mechanics in Schrödinger representation has been born [3, 4, 5].

Currently, the family of innovative non-Hermitian models is increasingly popular in several branches of physics [6, 7]. In our present letter we intend to support this trend by a comment on some interesting mathematical inner-product structures emerging behind the new paradigm. For introduction we have to add a brief but important remark on the terminology. The point (or rather an unexpected difficulty) is that even during the very short history of the new paradigm, its scope and range were already split into two different sub-paradigms. In the literature, unfortunately, both of them share the same name of $\mathcal{PT}$—symmetric quantum mechanics. As long as such a terminological confluence became a source of multiple misunderstandings, we feel urged to avoid the potential confusion by distinguishing, strictly, between the open-system $\mathcal{PT}$—symmetric quantum theory (OST) and the closed-system $\mathcal{PT}$—symmetric quantum theory (CST). Exclusively, our attention will be paid to the latter approach.

Another, closely connected introductory remark should be added emphasizing the deep difference between the physics, background, motivation and impact of the respective OST and CST studies. Indeed, the most characteristic phenomenological characteristics of the OST approach lies in its interest in the description of various non-unitary forms of quantum evolution covering the
resonant and/or dissipative processes. In some sense, the OST approach is both mathematically more straightforward (working just with a unique, single inner product) and, phenomenologically, much more traditional (with one of its roots being the Feshbach’s effective, model-subspace description of systems living in a larger Hilbert space). In contrast, the CST approach can be perceived as a much younger part of quantum physics (according to review [8], hardly taken too seriously before the publication of the pioneering letter [2] in 1998). At the same time, the CST formalism is currently a true challenge even for mathematicians (the mathematics-oriented book [4] can be recalled for the first reading). For both of these reasons, our present letter will exclusively be devoted to the latter, CST theory. Our marginal supportive argument is that only this form of the theory is fundamental, dealing with a complete information about the dynamics, and concerning just the description of the unitarily evolving quantum systems.

A brief remark should be finally added emphasizing the innovative aspects of the CST-related physics. The continuous emergence of open questions offered also a basic motivation of our present study. Two of its aspects have to be emphasized for introduction. First, in contrast to the “natural” [3], quickly and widely accepted $\mathcal{PT}$—symmetry assumption (2), the process of acceptance of the second, formally equally important $\mathcal{PCT}$—symmetry assumption (3) appeared to be much slower [8]. Moreover, it always looked, in the context of phenomenology, too ad hoc and slightly suspicious (see, e.g., a general theory in [9], or a characteristic sample of criticism in [10]). In this sense, our present approach seems to offer a mathematically as well as phenomenologically very persuasive new inner-product background for the introduction of the charge $C$. Secondly, our results will imply that the standard version and interpretation of the role of the charge $C$ can be modified and generalized. In this sense, indeed, our present letter may be read as an immediate and strong motivation for the consideration of larger classes of quantum models using non-Hermitian though not necessarily parity-times-time-reversal symmetric Hamiltonians with real spectra.

2 Quasi-Hermiticity

The assumption of the non-Hermiticity of Hamiltonians seemed to be, initially, in a sharp contradiction with Stone theorem [11], i.e., with the well known correspondence between the unitarity of the evolution and the properties of the underlying closed-system Hamiltonian. Fortunately, the puzzle found an almost immediate resolution. In essence (cf. review [3]) it has been concluded that the Hamiltonians in question are Hermitian in a “better-chosen”, amended Hilbert space $\mathcal{H}$. The manifest non-Hermiticity of $H$ in $L^2(\mathbb{R})$ has been declared inessential, connected just with the auxiliary role of the most common inner product $\langle \psi_1 | \psi_2 \rangle$ in $L^2(\mathbb{R})$.

For any “false but favored” Hilbert spaces of the latter type we will use, in what follows, the dedicated symbol $\mathcal{F}$. One should add that even without any reference to the above-mentioned
antilinear symmetries the simultaneous use of the two different inner products (i.e., of the auxiliary, friendly one in $\mathcal{F}$ and of the correct, more complicated but physical one in $\mathcal{H}$) was already a part of an older modified version of quantum mechanics called quasi-Hermitian (see, e.g., its compact 1992 review [9]). In this specific formulation of quantum theory the “correct Hilbert space” $\mathcal{H}$ appeared “hidden”. Represented, in $\mathcal{F}$, via an explicit formula
\[
\langle \psi_1 | \psi_2 \rangle_{\mathcal{H}} = \langle \psi_1 | \Theta | \psi_2 \rangle_{\mathcal{F}} \tag{4}
\]
which defines the correct physical inner product in $\mathcal{H}$ via its simpler partner in $\mathcal{F}$.

The correspondence (and non-equivalence) between Hilbert spaces $\mathcal{F}$ and $\mathcal{H}$ is characterized by the inner-product-metric operator $\Theta$ which must be, i.a. [9], positive definite and self-adjoint,
\[
\Theta = \Theta^\dagger \quad \text{(in $\mathcal{F}$)}. \tag{5}
\]
As a consequence of the use of the two spaces $\mathcal{F}$ and $\mathcal{H}$ there is no problem with the coexistence of the non-Hermiticity $H \neq H^\dagger$ in the auxiliary space $\mathcal{F}$ and the “hidden” Hermiticity of $H$ in $\mathcal{H}$. In order to avoid confusion it is sufficient to mark the latter property by another superscript, say, as follows,
\[
H = H^\sharp \quad \text{(in $\mathcal{H}$)}. \tag{6}
\]
By construction, the latter, unitarity-guaranteeing relation becomes equivalent to formula
\[
H^\dagger \Theta = \Theta H \quad \text{(in $\mathcal{F}$)} \tag{7}
\]
i.e., to the $\Theta$-pseudo-Hermiticity alias quasi-Hermiticity of $H$ in $\mathcal{F}$. One can conclude that by relations $\langle 4 \rangle$ and $\langle 7 \rangle$, the picture of physics is fully transferred from $\mathcal{H}$ to auxiliary $\mathcal{F}$.

3 \hspace{1em} $\mathcal{PT}$ – and $\mathcal{PCT}$ – symmetries

The amendment and transfer of the older quasi-Hermitian quantum mechanics of Ref. [9] to its innovated version was inspired by the 1998 letter [2]. The essence of the success of the innovation can be seen in a simplification of the technicalities due to the assumption that the Hamiltonians of the form $\langle 1 \rangle$ were special, viz., $\mathcal{PT}$ – and $\mathcal{PCT}$ – symmetric. In this context, what was truly essential was, first of all, the introduction of the charge $\mathcal{C}$. This made the theory mathematically consistent and, in the unbroken dynamical symmetry regime, fully compatible with standard textbooks. Simultaneously, the introduction of the concept of charge also helped to circumvent one of the main weaknesses of the quasi-Hermitian quantum mechanics in which, in the words of review [9], the metric $\Theta = \Theta(H)$ “is, in general, not unique”. For both of these reasons, the growth of popularity of the $\mathcal{PT}$ – and $\mathcal{PCT}$ – symmetric quantum mechanics was truly impressive: A concise outline of the whole story can be found, e.g., in reviews [4, 8].
New open questions also emerged: we will mention some of them in section 4 below. We will emphasize there, once more, that the success of the upgraded formalism (called, usually, just $\mathcal{PT}$—symmetric quantum mechanics) was mainly given by the surprising simplicity of its technical aspects. The main reason was that the fathers-founders of $\mathcal{PT}$—symmetric quantum mechanics \[2\] complemented the fundamental requirement of the reality of the spectrum of $H$ by an apparently redundant $\mathcal{PT}$—symmetry \(2\) alias parity-pseudo-Hermiticity \[4\] assumption

$$H^\dagger \mathcal{P} = \mathcal{P} H.$$ \hspace{1cm} (8)

This was a fortunate decision. The appeal and influence of such an assumption (where $\mathcal{P}$ may but need not denote the operator of parity) were not only technical (cf. the mathematically oriented reviews \[4, 5\]) but also intuitive and inspiring (at present, the concept has applications far beyond its original scope \[6, 7\]). Indeed, relation (8) can be re-read not only as the condition of self-adjointness of $H$ in the Krein space endowed with the indefinite (pseudo)metric $\mathcal{P}$ but also as the property of an antilinear symmetry $H \mathcal{PT} = \mathcal{PT} H$ of $H$, with the antilinear involution operator $\mathcal{T}$ carrying, as we already mentioned, the physical meaning of time reversal \[2, 3, 4\].

In spite of such a rich phenomenological background of the $\mathcal{PT}$—symmetry alias $\mathcal{P}$—pseudo-Hermiticity of $H$, by far the most important (albeit not always emphasized) feature of the $\mathcal{PT}$—symmetric quantum theory of unitary systems has to be seen in the role played by the other, independent antilinear symmetry $H \mathcal{PCT} = \mathcal{PCT} H$ of $H$ were the operator $\mathcal{C}$ may but need not represent a charge \[4\]. In any case, the product $\mathcal{PC}$ can be reinterpreted as one of the most interesting metric-operator solutions $\Theta = \Theta(H)$ of Eq. (7). In other words, the knowledge of the charge leads to the relation

$$H^\dagger \mathcal{PC} = \mathcal{PC} H$$ \hspace{1cm} (9)

which guarantees the unitarity of the evolution of the system \[3, 4, 9, 10\].

The basic idea of our present note is that in the area of physics using non-Hermitian operators the introduction of the two inner products \[1\] was in fact motivated mathematically. One of the reasons lied in the complicated nature of Hermitian conjugation of some Hamiltonians (or of some other relevant operators $\Lambda$) in $\mathcal{H}$. Firstly, such a conjugation [i.e., in the light of convention used in Eq. \[6\], the map $\Lambda \rightarrow \Lambda\sharp$] is inner-product dependent. For this reason it makes sense to characterize it by a subscripted antilinear operator $\mathcal{T}_\mathcal{H}$ which can easily be distinguished from its partner $\mathcal{T}_\mathcal{F}$ acting in $\mathcal{F}$. Secondly, after the abbreviation of the action of $\mathcal{T}_\mathcal{H}$ or $\mathcal{T}_\mathcal{F}$ by the respective dedicated superscripts $\sharp$ and $\dagger$, the pull-down of the conjugation from $\mathcal{H}$ to $\mathcal{F}$ [sampled by the replacement of Eq. (6) by Eq. (7)] acquired an explicit form,

$$\Lambda\sharp = \Theta^{-1} \Lambda\dagger \Theta.$$ \hspace{1cm} (10)

In multiple applications the fully consistent quasi-Hermitian model-building recipe as described in review \[9\] was successful. A simplification of Eq. \[6\] (representing the Hermiticity of $H$ in
\( \mathcal{H} \) has been achieved via its reduction to Eq. (7) in \( \mathcal{F} \). Briefly, one can say that the initial Hamiltonian-Hermiticity difficulty was weakened.

### 4 Another inner product and another quasi-Hermiticity

During the growth of popularity of the new paradigm it has been revealed that the construction of the positive definite metric \( \Theta \) restricted by the Hermiticity constraint (5) may be often the most difficult task in applications [12]. The techniques used in this setting may range from the direct reconstruction of \( \Theta = \Theta(H) \) using relation (7) up to the sophisticated application of the vielbein formalism as discussed in the very recent preprint [13]. A recommended thorough review of these techniques is provided by chapter 4 of review [4]. In principle, the desirable result may even be just an indefinite pseudometric like \( \mathcal{P} \) in Eq. (2) (cf., e.g., formula number 78 in chapter 3 of review [4]).

The latter list of techniques can also be read as an inspiration and starting point of our present considerations. Our basic idea is that one can feel guided by the traditional quasi-Hermitian construction pattern of Ref. [9]. The replacement of a single inner product (i.e., of a single Hilbert space \( \mathcal{H} \)) by the pair of inner products (i.e., by a pair of non-equivalent Hilbert spaces \( \mathcal{H} \) and \( \mathcal{F} \)) can be applied not only in the study of Hamiltonian but also during the construction of the metric.

In the context of physics the most promising aspect of the latter idea is that in the traditional \( \mathcal{P}\mathcal{T} \)–symmetric formalism as outlined in preceding section one simplifies the mathematical construction of the metric and, subsequently, the evaluation of the probabilistic predictions at an expense of narrowing the scope of the physical model-building. For this reason, any enhancement of flexibility of the formalism is desirable. One has to keep in mind that the conventional choice of the Hamiltonian and charge forms just a very specific set of the eligible quasi-Hermitian operators of observables (see, e.g., [14] or the fairly general discussion of this point in [9]). Also, in a way emphasized in [4], even the traditional intuitive \( \mathcal{P}\mathcal{T} \)–symmetry itself is a restriction which may simplify some technicalities but which certainly narrows the range of applicability of the non-Hermitian Hamiltonians with real spectra.

By Mostafazadeh [4] the generalized though still positive-indefinite analogues of parity were denoted by the symbol \( \eta \). In proceedings [15] we replaced \( \eta \) by \( Q \) and used the operator in the role of an inner-product metric in an auxiliary Pontryagin space. Another form of simplicity of the model has been achieved, in Ref. [16], with parity \( \mathcal{P} \) replaced by its positive definite alternative \( \mathcal{P}^+ \). The merits of the latter, extraordinary choice (yielding in fact another auxiliary Hilbert space) were illustrated by its relevance in the study of an N-site-lattice Legendre-oscillator toy-model Hamiltonian. Last but not least, the latter choice of \( \mathcal{P}^+ \) has been shown, in [17], to play a
key role in a consistent non-Hermitian (i.e., CST) description of scattering.

Whenever needed, the generic standard or non-standard realization of the (possibly, generalized) parity will be denoted either by symbol $P_g$ or, for simplicity, by $P$. The parallel generalization of the charge will be written, analogously, either as $C_g$ or as $C$. Using this convention we are now prepared to realize the replacement of a single inner product by its two alternatives. In other words, the two older CST approaches based on the use of a single Hilbert space $\mathcal{H}$, or of a pair of non-equivalent Hilbert spaces $\mathcal{H}$ and $\mathcal{F}$ will be generalized via a transition from the Hilbert-space doublet $[\mathcal{H}, \mathcal{F}]$ to an inner-product-space triplet denoted as $[\mathcal{H}, \mathcal{R}, \mathcal{F}]$.

For the sake of brevity, let us now restrict attention just to the Hilbert-space setup postponing, temporarily, the account of the above-mentioned Krein-space alternative to the next section. This is a shortcut which enables us to split the triplet of spaces into two doublets. In the first one, the intermediate space $\mathcal{R}$ will play the role of a substitute for $\mathcal{F}$ with respect to $\mathcal{H}$. In this case, we will have to replace metric $\Theta$ in $\mathcal{F}$ by the special positive charge $C_+$ in $\mathcal{R}$. In the second scenario, $\mathcal{R}$ will be a substitute for $\mathcal{H}$ with respect to $\mathcal{F}$. Then we will replace $\Theta$ by $P_+$. Summarizing, we are now able to upgrade Eq. (4) as follows,

$$
\langle \psi_1 | \psi_2 \rangle_{\mathcal{H}} = \langle \psi_1 | C_+ | \psi_2 \rangle_{\mathcal{R}}, \quad \langle \psi_1 | \psi_2 \rangle_{\mathcal{R}} = \langle \psi_1 | P_+ | \psi_2 \rangle_{\mathcal{F}}.
$$

Due to our temporary assumptions, $C_+$ does not represent a charge, and also $P_+$ is not parity.

In parallel, due care must be also paid to the transition from the auxiliary antilinear operators $[T_{\mathcal{H}}, T_{\mathcal{F}}]$ to the triplet $[T_{\mathcal{H}}, T_{\mathcal{R}}, T_{\mathcal{F}}]$. By definition they play the role of the operators of Hermitian conjugation in different Hilbert spaces so that in the context of the “upper” two-space subset $[\mathcal{H}, \mathcal{R}]$, the physical unitarity-guaranteeing hidden Hermiticity relation (6) remains unchanged while, once we mark the action of $T_{\mathcal{R}}$ by a new superscript $\dagger$, Eqs. (5) and (7) become upgraded as follows,

$$
C_+ = C_+^\dagger \quad \text{(in $\mathcal{R}$)},
$$

$$
H^\dagger C_+ = C_+ H \quad \text{(in $\mathcal{R}$)}.
$$

Similarly, in the context of the “lower” two-space subset $[\mathcal{R}, \mathcal{F}]$, the metric-Hermiticity relation (12) in $\mathcal{R}$ becomes accompanied by its quasi-Hermiticity equivalent in $\mathcal{F}$,

$$
C_+^\dagger P_+ = P_+ C_+.
$$

Concerning the Hamiltonian itself, a brief calculation leads, finally, to the result

$$
H^\dagger \Theta = \Theta H, \quad \Theta = P_+ C_+
$$

which just reproduces the conventional Eq. (9) above.

In order to avoid confusion it may also be useful to rewrite Eq. (11) in a more precise form. Thus, the conventional product $\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle_{\mathcal{F}}$ (with the well known meaning, say, after the
frequent choice of $\mathcal{F} = L^2(\mathbb{R})$) may be re-introduced via the definition $\mathcal{T}_F : |\psi_1\rangle \rightarrow \langle \psi_1 |$ of the ket-vectors in $\mathcal{F}$. In terms of this convention, the old formula $\mathcal{T}_H : |\psi_1\rangle \rightarrow \langle \langle \psi_1 |$ (say, of Ref. [8]) and the new, ad hoc-notation formula $\mathcal{T}_R : |\psi_1\rangle \rightarrow \langle \langle \langle \psi_1 |$ define the more explicitly characterized ket-vectors in $\mathcal{H}$ and in $\mathcal{R}$, respectively. Ultimately, one can identify $\langle \psi_1 |\psi_2\rangle_{\mathcal{H}} \equiv \langle \langle \psi_1 |\psi_2 \rangle$ and write also $\langle \psi_1 |\psi_2\rangle_{\mathcal{R}} = \langle \langle \langle \psi_1 |\psi_2 \rangle$ in $\mathcal{F}$, therefore (note that these conventions are different from those used, e.g., in [18]).

Table 1: Hamiltonians $H$ in quantum theory using one, two or three Hilbert spaces.

| spaces | operators       | comment       |
|--------|----------------|---------------|
|        | non-Hermitian  | quasi-Hermitian| Hermitian     |
| $\mathcal{H}$ | -              | $H = H^\dagger$ | Ref. [1]     |
| $\mathcal{H}$ | $H \neq H^\dagger$ | $H^\dagger \Theta = \Theta H$ | $\Theta = \Theta^\dagger > 0$ | Ref. [9] |
| $\mathcal{F}$ | $H \neq H^\dagger$ | $H^\dagger C = C H$ | $C = F$ | Eq. [6] |
| $\mathcal{R}$ | $H \neq H^\dagger$ | $C \neq C^\dagger$ | $\Theta = \Theta^\dagger$ | $\Theta = \Theta^\dagger$ | $\Theta = \Theta^\dagger$ |
| $\mathcal{R}$ | $H \neq H^\dagger$ | $H^\dagger \Theta = \Theta H$ | $\Theta = \Theta^\dagger$ | $\Theta = \Theta^\dagger$ | $\Theta = \Theta^\dagger$ |

Now we can return to the compact notation with $\mathcal{C} \rightarrow \mathcal{C}$ and with $\mathcal{P} \rightarrow \mathcal{P}$. The situation is summarized in Table 1. In a way guided by the quasi-Hermitian quantum mechanics of Ref. [9], both of the positive inner-product metrics $\mathcal{C} = \mathcal{C}_g$ and $\mathcal{P} = \mathcal{P}_g$ may be treated as operators which are bounded, invertible and positive definite, with bounded inverses. In such a scenario both of these operators share the properties of their two-Hilbert-space inner-product-metric predecessors. As a consequence, they also share and extend their physical interpretation. A new, three-Hilbert-space reformulation of the conventional unitary quantum mechanics is born.

5 An alternative approach: Two auxiliary Krein spaces

Strictly speaking, our latter conclusion is temporary and formal but still, it delivers an informal message which is nontrivial. It can be formulated as follows. After a removal of the requirement of positivity, we can still work with the two independent general inner-product metrics or pseudo-metrics $\mathcal{C} = \mathcal{C}_g$ and $\mathcal{P} = \mathcal{P}_g$. The Table reflects just an iterated application of the two-inner-products trick. The physical message delivered by the Table becomes different, deserving a separate attention. The point is that besides the above-outlined three-Hilbert-space interpretation of the relations summarized in Table 1, they may be also given another, Krein-space-related, interpretation.
An implicit emphasis upon such an alternative picture of quantum physics motivated us to make the indicative choice of the notation with symbols $C$ (for the first auxiliary inner-product metric or pseudo-metric in $R$) and $P$ (for the second auxiliary inner-product metric or pseudo-metric in $F$). Such a notation hinted at the possibility of treating both of the spaces either as the Hilbert spaces or as the Krein spaces. From the point of view of quantum theory the latter, alternative point of view seems equally interesting. Due to a relaxation of the metric-positivity constraints one can now return to the narrower, more physics-oriented (viz., charge and parity) interpretations of the respective operators. In this way, the most traditional forms of the parity and Krein-space related $PT$—symmetric quantum mechanics as reviewed in [3] acquire the same formal background and structure as their above-outlined three-Hilbert-space alternative. The shared aspects of this correspondence are summarized in Table 2.

Table 2: The Hilbert- and/or Krein-space interpretation of $PT$—symmetric quantum mechanics.

| quantum theory | space | operator | Hermitian conjugation | abbreviated |
|---------------|-------|----------|-----------------------|-------------|
| Hermitian     | $H$   | $H$      | $H \mathcal{T}_H = \mathcal{T}_H H$ | $H = H^\dagger$ |
| quasi-Hermitian| $H$   | $\Theta$ | $\Theta \mathcal{T}_F = \mathcal{T}_F \Theta$ | $\Theta = \Theta^\dagger$ |
| $PT$—symmetric| $R$   | $C$      | $C \mathcal{T}_R = \mathcal{T}_R C$ | $C = C^\dagger$ |
|               | $F$   | $P$      | $P \mathcal{T}_F = \mathcal{T}_F P$ | $P = P^\dagger$ |

One of the main consequences of both of the latter two interpretations of the triple-inner-product-space formulations of quantum theory is a complete reducibility of the formalism to its double-inner-product predecessor. A return to the observable-quantity status of the positive or indefinite operators $P$ and $C$ renders it possible to treat them simply as preselected dynamical-input factors of the overall physical Hilbert-space metric $\Theta = PC$ in $F$.

On the purely pragmatic level the sets of the operator relations listed in our two Tables indicate that we are free to eliminate the intermediate stage and to skip not only the explicit use of the user-unfriendly physical Hilbert space $H$ (as in the two-space formal regime) but also the use of the less user-friendly upper-auxiliary Hilbert or Krein space $R$. The reason is that the representation of both of them is now made available, via positive definite $\Theta = PC$, in the single and preferable second auxiliary Hilbert space $F$.

From the Hilbert/Krein-space double-interpretation perspective let us add that one of the not quite expected phenomenological consequences of our formal results is that even the generalized charge/metric $C$ still represents an observable quantity. In the reformulated theory the proof is easy: The observability property $C^\dagger \Theta = \Theta C$ of the charge (with respect to the physical metric
Θ = PC) is just an immediate formal consequence of Eq. (14).

In another, last comment let us note that in both the Hilbert- and Krein-space setups, another important consequence of the present discovery of the correspondence between the introduction of the auxiliary space \( \mathcal{R} \) and of the auxiliary observable \( \mathcal{C} \) is that the theory now enables us to start building the models in which many of the relevant operators could be chosen in a less constrained, perceivably more elementary forms. An illustrative example of such an option may be found outlined in the next section: the choice of the illustration has been inspired by the multiple formal difficulties encountered, in papers [19] and [20], in supersymmetric setup.

6 Differential operators

For illustrative purposes let us consider the most conventional non-Hermitian toy-model Hamiltonian (1) acting in the auxiliary and user-friendly Hilbert space \( \mathcal{F} = L^2(\mathbb{R}) \). For a direct, textbook-like treatment of this Hamiltonian in the correct and physical Hilbert space \( \mathcal{H} \) (i.e., after Hermitization), it would be necessary to construct the metric. Unfortunately, the metric would be represented by a hardly tractable, strongly non-local operator \[12\]. Moreover, whenever one decides to employ the mere single auxiliary inner product, it would be comparably difficult to find a sufficiently elementary representation of the charge. For both of these reasons it makes sense to assume that our Hamiltonian in question is \( \mathcal{P}_g \mathcal{C}_g \mathcal{T} \)–symmetric.

In the corresponding modified relation (3) with the conventional Hermitian conjugation \( \mathcal{T} \), with the generalized parity (i.e., with \( \mathcal{P} \) replaced by \( \mathcal{P}_g \)), and with the generalized charge (i.e., with \( \mathcal{C}_g \) in place of \( \mathcal{C} \)) we may work now with the less restricted families of operators represented, say, by the elementary differential expressions. For the sake of brevity let us choose, for example, the generalized charge in its simplest first-order tentative form

\[
\mathcal{C}_g = \frac{d}{dx} + w(x), \quad w(x) = \sigma(x) + i\alpha(x)
\]  

(16)

with the two real functions of a definite symmetry, \( \sigma(x) = \sigma(-x) \) and \( \alpha(x) = -\alpha(-x) \). In the Hamiltonian the potential

\[
V(x) = S(x) + L(x) + i\Sigma(x) + i\Lambda(x)
\]  

(17)

will be also split in such a way that

\[
S(x) = S(-x), \quad L(x) = -L(-x), \quad \Sigma(x) = \Sigma(-x), \quad \Lambda(x) = -\Lambda(-x).
\]  

(18)

Under such an ansatz the antilinear symmetry relation (3) in its explicit pseudo-Hermiticity version (9) can be treated as a compatibility constraint and as an equation which defines the admissible auxiliary metric \( \mathcal{C}_g \) in terms of a given \( \mathcal{H} \), or vice versa.
The elementary form of both $C_g$ and $H$ renders this illustrative problem solvable in closed form. Indeed, it is entirely straightforward to show that Eq. (9) degenerates, in such a case, to the mere pair of relations

$$S'(x) = 2 \sigma'(x)\sigma(x) - 2 \alpha'(x)\alpha(x), \quad \Lambda'(x) = 2 \sigma'(x)\alpha(x) + 2 \alpha'(x)\sigma(x)$$

(19)

where the primes denote the differentiation with respect to $x$. This system is integrable and yields the closed-form solution

$$S(x) = S(x, \omega) = \sigma^2(x) - \alpha^2(x) + \omega, \quad \Lambda(x) = 2 \sigma(x)\alpha(x)$$

(20)

which contains just a single arbitrary integration constant $\omega$. For an arbitrary input charge (16) the dynamical $PCT-$symmetric alias $P g C g T-$symmetric quantum system will always exist for Hamiltonians (1) with potentials (17) satisfying the two explicit constraints (20) imposed upon the potential.

Due to the not too complicated form of relations (20) one could also change the formulation of the problem and treat the two components $S(x)$ and $\Lambda(x)$ of the potential as an independent dynamical input. Then, in a search for all of the eligible “generalized charges” $C_g$ it becomes sufficient to eliminate, say, $\alpha(x) = \Lambda(x)/[= 2 \sigma(x)]$ and arrive at the single implicit compatibility condition

$$S(x) = \sigma^2(x) + \omega - \Lambda^2(x)/[4\sigma^2(x)]$$

(21)

with the two different real and non-vanishing eligible solutions $\sigma(x)$ such that

$$2 \sigma^2(x) = S(x) + \omega + \sqrt{(S(x) + \omega)^2 + \Lambda^2(x)}.$$  

(22)

What now only remains to be verified is the Hermiticity of the product $P g C g$ but this property can immediately be proved by direct insertion. Incidentally, it appears to be equivalent to the $P g T-$symmetry of $C_g$. This implies that the generalized charge is also $P g C g T-$symmetric, i.e., observable in the physical Hilbert space $H$.

7 Summary

It has long been known that, in some sense, the $PT-$symmetric model-building recipe just weakens the technical difficulties by their transfer from $H$ [controlled by Eq. (6)] to $\Theta$ [with the obligatory Hermiticity controlled by Eq. (5)]. On this background our present message can be summarized as a recommendation of an application of the same simplification strategy also to the latter operator.

We have shown that a detailed realization of such a concept is not entirely straightforward. Its core has been found to lie in an introduction of a second auxiliary inner-product space denoted
by a dedicated symbol $R$ and lying, in some sense, in between the “correct physical” $H$ and the “computation friendly” $F$. After an ad hoc amendment of the notation conventions, the specific strength of the resulting non-Hermitian model-building strategy has been shown to lie in a weakening and replacement of the Hermiticity of the metric [cf. Eq. (12)] by its quasi-Hermiticity or pseudo-Hermiticity [sampled by Eq. (14)].

We pointed out that the applicability of the new formalism necessitates the reality of the spectrum of a given diagonalizable candidate $H$ for the Hamiltonian. At the same time, its users encounter an ambiguity problem due to which the set of the eligible Hermitizing Hilbert-space metrics [i.e., of the self-adjoint solutions $\Theta(H)$ of Eq. (7)] is too rich in general. In the conventional $\mathcal{PT}$–symmetric quantum mechanics the uniqueness of the physical metric $\Theta(H) = \mathcal{PC}$ is being based, therefore, on the choice of a second ad hoc observable called charge. Although this choice can be perceived as somewhat arbitrary and artificial (see also its alternative as proposed in [10]) our present considerations showed that such a choice can in fact be given a fairly deep mathematical meaning.

In this context, our observations and basic equations were summarized in two Tables. Their contents emphasize that the conventional formulation of quantum mechanics in Schrödinger picture (using just a single Hilbert space $\mathcal{H}$) may be treated as formally equivalent to the quasi-Hermitian formulation (in which a pair of Hilbert spaces $\mathcal{H}$ and $\mathcal{F}$ is used) as well as to the $\mathcal{PT}$–symmetric formulation. Although the latter formulation is usually characterized just by a specific choice of the metric $\Theta$, the Tables offer an alternative, more satisfactory picture. In it one employs a triplet of spaces $[\mathcal{H}, R, \mathcal{F}]$ endowed with a triplet of different Hermitian conjugations. In this approach, the symbols $C$ and $P$ may represent either the positive definite Hilbert-space metrics [cf. Eq. (11)] or the indefinite Krein-space pseudometrics (i.e., in the most popular special cases, the charge and parity, respectively). In both of these realizations of the idea, fortunately, the formal outcome becomes almost independent of the technical subtleties because the pair of operators $P$ and $C$ only enters the condition of unitarity (9) in the form of their positive definite and self-adjoint product $\Theta = PC$.

Concerning the possible applicability of the present metric-factorization idea in a broader quantum theoretical context we have to admit that our attention has exclusively been paid here to the mere unitary evolution studied in the hiddenly Hermitian Schrödinger picture. A decisive technical advantage of such a restriction has been found in the fact that the underlying metric $\Theta$ must necessarily remain stationary (or, more strictly speaking, quasi-stationary, see the reasons given in [4]). This made its factorization technically straightforward as well as phenomenologically useful. We believe, nevertheless, that an extension of the theory based on the present “relegation of Hermiticity” could also successfully proceed in several other directions.

In this sense we are particularly optimistic in the case of the hiddenly Hermitian Heisenberg picture of Refs. [21]. We expect that the progress might be particularly quick there because in
this theory the metric necessarily remains time-independent as well. In the second, technically more ambitious direction of research based on the hiddenly Hermitian version of the Dirac’s interaction picture \[22, 23\] the situation remains unclear at present. In the latter setting we would remain more sceptical. The success seems to be an open question, indeed. Among the most serious discouraging conceptual problems one finds that the use of time-dependent metrics leads to the necessity of introduction of the time-dependent “Hamiltonians” which cease to be observable \[4, 18, 23, 24\]. This might make all of the constructive “relegation of Hermiticity” considerations much more difficult \[13, 25\]. Still, on positive side one also finds important results like, e.g., the observation that the time-dependence of the metrics \(\Theta = \Theta(t)\) can be controlled via suitable operator differential equations [see, e.g., equation number 4 in \[18\], or the examples of its solvability in \[23, 24\]]. What is only missing here is the sharing of notation conventions: Typically, the time-dependent “Hamiltonians” may be found denoted as \(H_{\text{gen}}(t)\) (in \[22\]) or as \(G(t)\) (in reviews \[23, 25\]) or, in many papers, simply as \(H(t)\) (e.g., in \[18, 24\]), etc. Thus, what remains most encouraging is only the current steady progress in the field.


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