Black hole thermodynamics in the presence of a maximal length and minimum measurable in momentum

B. Hamil¹ and B.C. Lütfüoğlu²,³

¹ Département de TC de SNV, Université Hassiba Benbouali, Chlef, Algeria.
² Department of Physics, University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czechia.
³ Department of Physics, Akdeniz University, Campus 07058 Antalya, Turkey.

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Abstract – In this work, incorporating the effect of the minimum measurable in momentum and maximal length, we studied thermodynamics property of Schwarzschild black hole and the Unruh effect. According to this scenario, we see that the black hole temperature cannot be smaller than a certain minimum value of $T_{\text{min}}$. Moreover, we find that black hole mass cannot be larger than a maximum mass value of $M_{\text{max}}$. Considering these findings first we compute the corrected Hawking temperature versus the mass and examine its characteristic behavior. Then, we derive the black hole’s entropy and heat capacity. We find that the black hole is stable when $\frac{M_{\text{max}}}{\sqrt{3}} < M < M_{\text{max}}$. Finally, we examined the modified Unruh effect. We find that the modified Unruh temperature explicitly depends on $\alpha$.

Introduction. – In general, incompatibility between Einstein’s general relativity and quantum field theory is regarded as the most fundamental motivation for the development of quantum gravity. Different approaches which are used to form the quantum theory of gravity, namely string theory [1-5], noncommutative geometry [6], and loop quantum gravity [7], put forward the necessity of a fundamental lower limit length value. One way of defining minimal length value is the generalization of the Heisenberg uncertainty principle (HUP) by modifying the position and momentum operators. It is shown that such a generalization is not unique. [8]. For example, in one of the generalized uncertainty principle (GUP) scenarios, the commutation of the position and momentum operators end up with a term depending on the momentum operator instead of a constant on the Hilbert space [9,11]. Many applications according to various GUP scenarios have been investigated in [12-34]. In particular, in [31], the authors explored the influences of the GUP scenario on the thermodynamics of black holes (BHs). They found that in that scenario the Hawking radiation can stop, so that, remnants of BHs occur. Therefore, the GUP scenarios are considered as appropriate scenarios to resolve the information paradox problem of BHs.

On the other hand, quantum gravity effects are assumed to act also on the large-scale dynamics of the universe. Phenomenologically, the effects of quantum gravity at large distances can be encoded in another kind of generalization scenario, namely in the extended uncertainty principle (EUP) scenario, where the commutation relations of the position and momentum operators yield to a position operator instead of a constant or a momentum operator that appear in the HUP and GUP scenarios, respectively [35]. In the EUP scenario, a minimal measurable momentum value emerges which corresponds to an infrared cutoff [35–41]. Mignemi proved that the EUP scenario can also be extracted from the definition of quantum mechanics on an (A)dS background with an appropriate choice of the parameterization [39]. Recently, Perivolaropoulos proposed a new generalized uncertainty principle to all orders in the Hubble parameter [42, 43]. That scenario implies the presence of a minimum measurable momentum and a maximum measurable length together. The existence of a maximum observable measurable length limit, in other words an infrared cutoff, emerges naturally in the context of cosmological particle’s horizon [27,41] or cosmic topology [10]. Note that these effects are expected to be effective in the early Universe.

In this paper we consider a maximal length and a minimal observable momentum scenario with the modified
Heisenberg algebra and investigate the effects of our choice on the thermodynamics of Schwarzschild black hole. We construct the paper as follows: At first, we introduce the modified algebra. Next, we obtain the thermodynamic functions in the modified algebra. Then, we examine the Unruh temperature and conclude the article with a brief conclusion.

Quantum mechanics in the presence of a maximal length and minimum measurable in momentum.— In this manuscript we take the modified commutation relation that is introduced in \[42,43\] into account.

\[ [X, P] = \frac{i\hbar}{1 - \alpha X^2}. \] (1)

Here, three parameters, namely the Hubble parameter, \(H_0\), speed of light, \(c\), and a dimensionless parameter, \(\alpha_0\), are used to define a parameter in the form of:

\[ \alpha = \left( \frac{\alpha_0 H_0}{c} \right)^2. \]

In order to satisfy Eq. (1), we consider the following position and momentum operators

\[ X = x, \quad P = \frac{\hbar}{i} \frac{1}{1 - \alpha x^2} \frac{d}{dx}. \] (2)

In the position space representation, with the help of the properties \(\langle X^{2n}\rangle \geq \langle X^4 \rangle^{n/2}\) (with \(n > 0\)), it is straightforward to construct the GUP as follows:

\[ (\Delta X)(\Delta P) \geq \frac{\hbar}{2} \left( \frac{1}{1 - \alpha X^2} \right), \]

\[ \geq \frac{\hbar}{2} \left( 1 + \alpha \langle X^2 \rangle + \alpha^2 \langle X^4 \rangle + \alpha^3 \langle X^6 \rangle + \ldots \right), \]

\[ \geq \frac{\hbar}{2} \left( 1 + \alpha \left( [\Delta X]^2 + \langle X \rangle^2 \right) \right. \]

\[ + \alpha^2 \left( [\Delta X]^2 + \langle X \rangle^2 \right)^2 \]

\[ + \alpha^3 \left( [\Delta X]^2 + \langle X \rangle^2 \right)^3 + \ldots \right), \]

\[ \geq \frac{\hbar}{2} \left( 1 - \alpha \left( [\Delta X]^2 + \langle X \rangle^2 \right) \right). \] (3)

In order to determine the minimum measurable momentum of this deformed algebra, we take only the physical states into account which satisfy \(\langle X \rangle = 0\) condition. Then, we solve the reduced GUP

\[ (\Delta X)(\Delta P) = \frac{\hbar}{2} \left( \frac{1}{1 - \alpha (\Delta X)^2} \right), \] (4)

for \((\Delta P)\). We obtain the following minimum observable momentum value

\[ (\Delta P)_{\text{min}} = \frac{3\sqrt{2}}{4} \hbar \sqrt{\alpha}, \] (5)

and hence, the maximum measurable length value:

\[ (\Delta X)_{\text{max}} = \ell_{\text{max}} = \frac{1}{\sqrt{\alpha}}. \] (6)

In this deformed scenario, the usual completeness and inner product definitions between two states change with the following ones:

\[ 1 = \int_{-\ell_{\text{max}}}^{\ell_{\text{max}}} (1 - \alpha x^2) \langle x \rangle \langle x \rangle dx, \] (7)

\[ \langle \psi | \varphi \rangle = \int_{-\ell_{\text{max}}}^{\ell_{\text{max}}} dx \left( 1 - \alpha x^2 \right) \psi^*(x) \varphi(x). \] (8)

Here, the weight function, \((1 - \alpha x^2)\), is required for the symmetry of the operators \(X\) and \(P\).

Black holes.— In this section we examine the thermodynamics of a Schwarzschild BH under the deformed scenario that is described above. In so doing, we consider the following metric.

\[ ds^2 = - \left( 1 - \frac{2MG}{rc^2} \right) c^2 dt^2 + \left( 1 - \frac{2MG}{rc^2} \right)^{-1} dr^2 + r^2 d\Omega^2. \] (9)

Here, \(M\) denotes the mass of BH, \(\Omega\) represents the solid angle, and \(G\) is the Newton universal gravitational constant. From this line element, the event horizon can be written as

\[ r_s = \frac{2MG}{c^2}. \] (10)

According to near-horizon geometry considerations one can set \((\Delta X) = 2\pi r_s\), thus Eq. (10) leads to a maximum horizon radius and a mass value for the BH in the form of \[42\]

\[ (r_s)_{\text{max}} = 2\pi \ell_{\text{max}} \simeq 10^{26} \text{m}; \quad M_{\text{max}} = \frac{\ell_{\text{max}}^2}{4\pi G} \simeq 10^{52} \text{kg}. \] (11)

Note that the particle horizon is correlated with the length scale of the boundary between the unobservable and the observable regions of the Universe [42]. Then, we employ the temperature expression of any massless quantum particle near the Schwarzschild BH horizon

\[ T = \frac{c}{K_B} \left( \Delta P \right), \] (12)

to estimate a minimal observable temperature value of the BH via the minimum uncertainty in momentum.

\[ T_{\text{min}} = \frac{c}{K_B} \left( \Delta P \right)_{\text{min}} \simeq 10^{-29} \text{K}. \] (13)

Next, we investigate the Hawking temperature of the BH. By substituting Eqs. (10) and (12) into Eq. (4) we derive the modified Hawking temperature in terms of the ordinary one, \(T_0 = \frac{\hbar c^3}{8\pi K_B M_{\text{max}}^2}\), as follows:

\[ T_H = \frac{T_0}{\frac{1 - \frac{M^2}{M_{\text{max}}^2}}{1 - \frac{M^2}{M_{\text{max}}^2}}}. \] (14)

It is worth noting that the term in the denominator modifies the standard Hawking temperature. Therefore, we observe a critical mass value,

\[ M_{cr} = M_{\text{max}}. \] (15)
which plays a key role. If it has a finite value, it acts like a cut-off because above this value the BH temperature becomes negative. On the other hand, when it has an infinite value, $M_{\text{max}} \to \infty$, then the Hawking temperature reduces to the usual one \cite{47,49}. In the limit case where the mass term is close to the critical mass value, the modified Hawking temperature reaches very large values compared to usual one. If the mass term is very small than the critical mass value, then one can expand the denominator term to

$$T_H \simeq T_0 \left[ 1 + \left( \frac{M}{M_{\text{max}}} \right)^2 + \left( \frac{M}{M_{\text{max}}} \right)^4 + \ldots \right]. \quad (16)$$

We present the variation of the modified Hawking temperature versus the BH mass for different $M_{\text{max}}^{-2}$ values in Fig. 1. We observe that the temperature is divergent not only as $M \to 0$ but also as $M \to M_{\text{max}}$. In addition, we see the presence of a minimum temperature value, $T_{\text{min}} = \frac{3\sqrt{\alpha} \hbar}{4K_B}$. This value is achieved at $M = \frac{M_{\text{max}}}{\sqrt{3}}$, and below that temperature value solution does not exist.

![Fig. 1: Temperature-mass function for $\hbar = c = G = K_B = 1$.](image)

Next, we determine the BH entropy from the first law of the BH thermodynamics which is defined in the form of:

$$S = c^2 \int \frac{dM}{T}. \quad (17)$$

After substituting Eq. \cite{14} into Eq. \cite{17} and performing the integration, we obtain the GUP-corrected BH entropy as

$$\frac{S_H}{K_B} = \frac{S_0}{K_B} \left( 1 - \frac{S_0}{2S_{\text{max}}} \right). \quad (18)$$

Here, $\frac{S_0}{K_B} = 4\pi \frac{M_{\text{Schw}}^2}{M_P^2}$ is the semi-classical Bekenstein-Hawking entropy for the Schwarzschild BH \cite{17,19} and $S_{\text{max}} = 4\pi \left( \frac{M_{\text{max}}}{M_P} \right)^2$ is the maximum entropy value. We would like to remark that the examined modification reproduces a correction term with a negative sign. Then, we express the entropy \cite{18} in terms of the area of the horizon, $A = 4\pi r^2 = 4\ell_p \frac{3S}{K_B}$. We find that Eq. \cite{18} can be described as

$$\frac{S_H}{K_B} = \frac{A_0}{4\ell_p} \left( 1 - \frac{A_0}{2A_{\text{max}}} \right), \quad (19)$$

where $A_{\text{max}} = 4\ell_p \frac{S_{\text{max}}}{K_B}$. We note that in the limit of $\alpha \to 0$ it reduces to the famous area theorem. In the present framework, we observe that a logarithmic-area correction term does not rise, while in other approaches such as string theory, loop quantum gravity, effective models with GUP \cite{12} and/or modified dispersion relations \cite{50} do. In addition, we find that the entropy gets its maximum value, $S_H = 2\pi \left( \frac{M_{\text{max}}}{M_P} \right)^2$, when the BH mass tends to $M_{\text{max}}$. To have a better knowledge of the characteristic behavior of the GUP-corrected BH entropy we plot the entropy, Eq. \cite{18}, versus the BH mass for different values of $M_{\text{max}}^{-2}$ in Fig. 2. Clear from the figure, in the presence of a maximal length and minimum measurable in momentum, the entropy increases but with a decreasing rate lower than the standard case and it gets its maximum when the BH mass goes to $M_{\text{max}}$. Finally, we proceed to compute the modified heat capacity of the BH. To do that, we employ the following relation

$$C = c^2 \frac{dM}{dT} = \left( \frac{1}{c^2} \frac{dT}{dM} \right)^{-1}. \quad (20)$$

After performing the simple algebra, We arrive at

$$C_H = C_0 \left( 1 - \left( \frac{M}{M_{\text{max}}} \right)^2 \right)^2, \quad (21)$$

where $C_0 = -\frac{8\pi K_B M^2}{M_P^2}$ is the standard expression of the heat capacity. We observe that in the absence of modification, Eq. \cite{21} reduces to the usual form of the BH heat
Finally we observe that BH can be stable only for masses around \( \sqrt{M_{\text{max}}} \). In our case, the remnant mass is equal to the critical mass.

In the interval of \( 0 < M < \sqrt{M_{\text{max}}} \), we see that the heat capacity is negative and its value decreases faster with higher values of \( \alpha \) and \( M \). Contrarily for \( M > \sqrt{M_{\text{max}}} \) the heat capacity is positive and reaches zero when \( M \to M_{\text{max}} \). As it is a well-known fact that if a black hole has a positive valued heat capacity function, then it is assumed to be a stable. Alike, it is presumed to be unstable when it has a negative valued specific heat. Therefore, we conclude that BH can be unstable for \( 0 \leq M \leq \sqrt{M_{\text{max}}} \) and stable for \( \sqrt{M_{\text{max}}} \leq M \leq M_{\text{max}} \).

The collapse of a BH ends as soon as the heat capacity function tends to zero. So that, the mass of the BH remains the same. This mass value is called the remnant mass, \( M_{\text{rem}} \). Its value can be achieved by solving the following equation

\[
C_H = C_0 \left( 1 - \left( \frac{M}{M_{\text{max}}} \right)^2 \right)^2 = 0, \tag{22}
\]

In our case, the remnant mass is equal to the critical mass.

\[
M_{\text{rem}} = M_{\text{max}}. \tag{23}
\]

Before ending this section, it should be noted that the GUP scenario is characterised by the presence of a minimum value for the horizon radius and minimum mass \( M_{\text{min}} \). In addition, the heat capacity of the BH vanishes at the end point of the evaporation process which is characterized by BH remnant of mass \( M_{\text{min}} \). In the present framework the BH can be stable only for masses around \( M_{\text{max}} \).

Unruh effect. – In this section, we examine the Unruh effect in this deformed scenario. We start by substituting \( \Delta P = \frac{\Delta E}{c} \) in Eq. (11). We find

\[
(\Delta E) = \frac{\hbar c}{2(\Delta X)} \frac{1}{1 - \alpha(\Delta X)^2}. \tag{24}
\]

Then, we use the minimal distance \( (\Delta X) \) along which each particle must be accelerated to create \( N \) particle

\[
(\Delta X) = \frac{2Nc^2}{a}. \tag{25}
\]

After recalling the well-known relation, \( (\Delta E) = \frac{3}{2} K_B T \), we find

\[
T = T_U \phi (T_U), \tag{26}
\]

where

\[
\phi (T_U, \alpha) = \frac{1}{1 - \frac{\hbar^2 \alpha^2}{9K_B T_U^2}}, \tag{27}
\]

and \( T_U = \frac{\hbar a}{2\pi K_B} \) is the well-known Unruh temperature, while \( a \) is the acceleration of the frame. Finally, as indicated in [51,52], the geometrical interpretation of quantum mechanics via a quantization model implies the existence of maximal acceleration which is conducted to a modification of the Heisenberg uncertainty principle. In a similar way, the present framework is characterized by presence of a minimum energy value

\[
(\Delta E)_{\text{min}} = \frac{3}{2} K_B T_{\text{min}}. \tag{28}
\]

Via this relation, we get a maximal bound value for the acceleration:

\[
a_{\text{max}} \leq \frac{2\pi K_B T_{\text{min}}}{\hbar}. \tag{29}
\]

This idea and analysis are in agreement with [53–55].

Conclusion. – In this manuscript we consider a generalized uncertainty principle out of Heisenberg uncertainty principle that leads to a maximal length as well as a minimum observable momentum value. Then, we examine the thermodynamic of a Schwarzschild black hole. We find a minimal temperature and a maximum mass value. We derive the modified Hawking temperature in terms of the usual one. After demonstrating the mass-temperature function, we derive the entropy and specific heat functions. We observe that both functions has two different characteristic behaviors in two different interval. We find that the black hole is first unstable until a critical mass value. After the mass exceeds the critical mass value it becomes stable. In addition, we explore the presence of a remnant mass value and find it at \( M = M_{\text{max}} \) value. Finally we obtain the Unruh temperature in the deformed algebra. We show that the modified Unruh temperature has a similar characteristic with the usual one.

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