Research Article

On the Geodesic Identification of Vertices in Convex Plane Graphs

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A shorted path between two vertices u and v in a connected graph G is a \( u - v \) geodesic. A vertex \( w \) of G performs the geodesic identification for the vertices in a pair \((u, v)\) if either \( v \) belongs to a \( u - w \) geodesic or \( u \) belongs to a \( v - w \) geodesic. The minimum number of vertices performing the geodesic identification for each pair of vertices in G is called the strong metric dimension of G. In this paper, we solve the strong metric dimension problem for three convex plane graphs by performing the geodesic identification of their vertices.

1. Introduction

The identification of vertices of a graph using various graph parameters is a fascinating problem for researchers. In the literature, almost over 1000 articles are contributed to explore the identification of vertices in a remarkable way by using various graph theory concepts including graph coloring, labeling of vertices, domination in graphs, vertex covering, graph automorphisms with symmetry breaking technique, independence of vertices, and by defining the metric on graphs, to name a few.

By defining the metric on a graph, the problem of identification of vertices attracted many researchers due to its significant applications in several extents including verification, security, and discovery of networks [1], the chemistry of pharmaceutics for drug designing [2], mastermind game strategies [3], navigation of robots [4], connected joins in graphs [5], and solution of coin weighing problems [6]. Because of these practical significances of this problem, from the last two decades, numerous researchers identified vertices of a graph by considering the metric-related well-known concept of the metric dimension [2, 7, 8]. Later on, many researchers extended the study of this concept by defining its several variations including the fractional metric dimension [9], the resolving domination [10], the doubly metric dimension [11], the independent metric dimension [12], the weighted metric dimension [13], the \( k \)-metric dimension [14], the solid metric dimension [15], the mixed metric dimension [16], the local metric dimension [17], the simultaneous metric dimension [18], the strong metric dimension [5], and the connected metric dimension [19].

This paper is aimed to identify the vertices of three convex plane graphs by using geodesics between them, which provides the strong metric dimension of these graphs.
2. Geodesic Identification: The Strong Metric Dimension

Let $G$ be a simple connected graph. A shortest path between two vertices $x$ and $y$ in $G$ is known as a $x - y$ geodesic. The geodesic identification of vertices is equivalently the strong resolvability of vertices, which is defined as follows: a vertex $w$ of $G$ performs the geodesic identification for the vertices in a pair $(x, y)$ (i.e., $w$ strongly resolves the vertices in a pair $(x, y)$) if either $x$ belongs to (lies on) a $y - w$ geodesic or $y$ belongs to (lies on) a $x - w$ geodesic. A set $S$ of vertices in $G$ is said to be a strong metric generator for $G$ if for every pair of vertices of $G$, there is always a vertex in $S$ which performs the geodesic identification for the vertices in the pair. The cardinality of such a smallest set $S$ is called the strong metric dimension of $G$, denoted by $sdim(G)$ [5].

Sebő and Tannier initiated the study of geodesic identification for vertices in [5]. Later on, this study was extended by many researchers, and they contributed to the literature with a variety of remarkable research work. To develop the readers interest, we shortly survey the strong metric dimension problem as follows:

(i) The strong metric dimension problem has been solved for Sierpiński graph in [20], for hamming graphs in [21], for some convex polytopes in [22, 23], for wheel related graphs (including $n$-fold wheel, sunflower, helm, and friendship graphs) in [24], for path, cycle, complete, complete bipartite, and tree graphs in [25], for Cayley graphs in [26], for Cartesian sum graphs in [27], for the power graph of a finite group in [28], for distance-hereditary graphs in [29], for generalized butterfly and starbarbell graphs in [30], for antiprism and Möbius ladder graphs in [32], and for crossed prism in [33].

(ii) The strong metric dimension of various products of graphs including Cartesian product, direct product, strong product, lexicographic product, rooted product, and corona product has been supplied through the articles in [26, 27, 30, 31, 33–39].

(iii) The fractional version of the strong metric dimension problem has been introduced in [40] and further studied for various graphs and graph products in [41, 42].

(iv) The technique of the computation of strong metric dimension with the concept of the vertex cover number has been provided in [38] by proposing the construction of strong resolving graph of a connected graph. Furthermore, the article in [25] supplied some fundamental realizations and characterizations of the strong metric dimension problem in connection with the strong resolving graph.

(v) To solve the strong metric dimension problem, the genetic algorithmic approach is used in [43], and the variable neighborhood search method is used in [44].

(vi) The article in [45] supplied the Nordhaus–Gaddum type results whenever the strong metric dimension problem was solved for graphs and their compliments.

(vii) To compute the strong metric dimension of graphs using optimization techniques, the integer linear programming model for the strong metric dimension problem was formulated in [22].

(viii) The complexity and the optimal approximability in the computation of the strong metric dimension problem of graphs have been discussed in [38, 46].

(ix) Furthermore, we refer a survey in [47] and the Ph.D. thesis in [48] to the readers having interest in the computation of the strong metric dimension of graphs.

With this paper, we extend the study of the identification of vertices using the concept of strong metric dimension. We consider three convex plane graphs and investigate their strong metric dimension by performing the geodesic identification of vertices in the graphs.

3. Basic Works

Let $G$ be a simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote the adjacent vertices $u$ and $v$ by $u \sim v$ and nonadjacent vertices by $u \not\sim v$ in $G$. The neighborhood of a vertex $v$ in $G$ is $N(v) = \{u \in V(G) : u \sim v \text{ in } G\}$. The number of vertices adjacent with a vertex $v$ is called its degree and is denoted by $d(v)$. The metric on $G$ is a mapping $d : V(G) \times V(G) \rightarrow \mathbb{Z}^+ \cup \{0\}$ defined by $d(x, y) = l$, where $l$ is the length of the (number of edges in) a $x - y$ geodesic. Accordingly, a vertex $w$ of $G$ performs the geodesic identification for the vertices in a pair $(x, y)$ if and only if either $d(x, w) = d(x, y) + d(y, w)$ or $d(y, w) = d(y, x) + d(x, w)$.

Kratika et al. in [22] supplied the following two results, which are useful tools for the geodesic identification of vertices.

Lemma 1 (see [22]). Let $(u, v)$ be a pair of distinct vertices in a connected graph $G$ such that

$$
\begin{align*}
    d(u, v) &\geq d(w, v), & \forall w \in N(u), \\
    d(u, v) &\geq d(w, u), & \forall w \in N(v).
\end{align*}
$$

Then, there is no vertex in $V(G) - \{u, v\}$ which performs the geodesic identification for the pair $(u, v)$.

Proposition 1 (see [22]). If $S$ is a strong metric generator for a connected graphs $G$, then for every pair $(u, v)$ of distinct vertices in $G$ satisfying both the conditions in (1), either $u \in S$ or $v \in S$. 
For a connected graph $G$, the number \( \text{diam}(G) = \max\{\text{ecc}(v) = \max_{v \in V(G)} d(v,u); v \in V(G)\} \) is called the diameter of $G$, where ecc($v$) is the eccentricity of a vertex $v$. The following result, provided by Kratica et al. in [22], is also a useful tool for the geodesic identification of vertices.

**Proposition 2** (see [22]). If $S$ is a strong metric generator for a connected graphs $G$, then for every pair $(u, v)$ of distinct vertices in $G$ such that $d(u, v) = \text{diam}(G)$, either $u \in S$ or $v \in S$.

A vertex $v$ of $G$ for which $d(v) = 1$ is known as a leaf. The following result describes the geodesic identification of leaves in a connected graph.

**Lemma 2.** If $S$ is a strong metric generator for a connected graph $G$ and $(u,v)$ is a pair of distinct leaves in $G$, then either $u \in S$ or $v \in S$.

**Proof.** Since $u$ and $v$ are leaves, $d(u) = 1 = d(v)$. Without loss of generality, let $N(u) = \{x\}$ and $N(v) = \{y\}$. Now, if $x = y$, then

\[
\begin{align*}
d(u, v) &= 2 > d(x, v), \\
d(u, v) &= 2 > d(x, u).
\end{align*}
\]

If $x \neq y$, then

\[
\begin{align*}
d(u,v) &= d(u,x) + d(x,y) + d(y,v) = d(x,y) + 2, \\
d(x,v) &= d(x,y) + d(y,v) = d(x,y) + 1 < d(u,v), \\
d(y,u) &= d(y,x) + d(x,u) = d(x,y) + 1 < d(u,v).
\end{align*}
\]

The identities (2)–(5) provide that the pair $(u,v)$ satisfies both the conditions in (1), and hence, Proposition 1 yields the required result.

Lemma 2 supplies the following straightforward proposition.

**Proposition 3.** If $S$ is a strong metric generator for a connected graph $G$ and $L$ is the set of $m$ leaves in $G$, then $S$ must contain at least $m - 1$ leaves from $L$.

### 4. Convex Plane Graphs

A graph $G$ is convex if a straight line joining any two vertices of $G$ entirely lies within the region occupied by $G$. A graph $G$ is a plane graph if it is free from crossing of edges. Throughout this section, it should be helpful to keep in mind the following points:

(i) The right arrow (\(\rightarrow\)) in the supper script of the notation of geodesics will indicate forward nature of geodesics (see Figure 1).

(ii) The left arrow (\(\leftarrow\)) in the supper script of the notation of geodesics will indicate backward nature of geodesics (see Figure 1).

![Figure 1: Illustrating forward and backward natures of $a_i - a_{i+1}$ and $a_i - a_{i+1}$ geodesics.](image)

(iii) The indices greater than $n$ or less than 1 will be taken modulo $n$.

In the next sections, we investigate the strong metric dimension of three convex plane graphs.

#### 4.1. Convex Plane Graph $S_n^p$.

The graph of a convex polytope $S_n$ is defined in [49] for $n \geq 3$. The vertex set of $S_n$ is $V(S_n) = \{a_i, b_i, c_i, d_i; 1 \leq i \leq n\}$, and its edge set is

\[
E(S_n) = \{a_i \sim a_{i+1}, a_i \sim b_i, b_i \sim b_{i+1}, b_i \sim a_{i+1}; 1 \leq i \leq n\} \\
\cup \{b_i \sim c_i, c_i \sim c_{i+1}, c_i \sim d_i, d_i \sim d_{i+1}; 1 \leq i \leq n\}.
\]

The convex plane graph $S_n^p$ ($p$ for pendant) can be obtained from the graph of a convex polytope $S_n$ by attaching one pendant vertex (leaf) $e_i$ to the vertex $d_i$ of $S_n$ for each $1 \leq i \leq n$ (see Figure 2) [50]. Thus, the vertex and edge sets of $S_n^p$ are

\[
\begin{align*}
V(S_n^p) &= V(S_n) \cup \{e_i; 1 \leq i \leq n\}, \\
E(S_n^p) &= E(S_n) \cup \{d_i \sim e_i; 1 \leq i \leq n\}.
\end{align*}
\]

We investigate the strong metric dimension of $S_n^p$ by supplying the following main result.

**Theorem 1.** For $n \geq 3$, let $S_n^p$ be a convex plane graph. Then,

\[
s\dim(S_n^p) = \begin{cases} 
n, & \text{when} \ n \text{ is odd,} \\
\frac{3n}{2}, & \text{when} \ n \text{ is even.}
\end{cases}
\]

The next four lemmas will lead the proof of Theorem 1.
Proof. Let $n = 2k + 1$ with $k \geq 1$, and consider Table 1 for $1 \leq i \leq n$.

In $S_n^p$, note the following points:

P1: $L = \{e_i; 1 \leq i \leq n\}$ is the set of leaves.

P2: for each $1 \leq i \leq n$, $d(a_i, e_{i+k}) = k + 4$. It follows, according to Table 1, that the pair $(a_i, e_{i+k})$ satisfies both the conditions in (1).

P3: $\text{diam}(S_n^p) = k + 4 = d(a_i, e_{i+k})$ for each $1 \leq i \leq n$.

From P1, without loss of generality, we suppose that $e_2, e_3, \ldots, e_n \in S$ by Proposition 3. Then, since the pair $(a_{k+1}, e_1)$ satisfies P2 and P3, we have either $a_{k+1} \in S$ or $e_1 \in S$, by Propositions 1 and 2. Hence, $|S| \geq n - 1 + 1 = n$.

Lemma 4. For even values of $n \geq 4$, if $S$ is a strong metric generator for $S_n^p$, then $|S| \geq (3n/2)$.

Proof. Let $n = 2k$ with $k \geq 2$, and consider Table 2 for $1 \leq i \leq n$.

In $S_n^p$, the following points hold:

\begin{align*}
\overline{P_1}: & a_i \sim a_{i+1} \sim a_{i+2} \sim \cdots \sim a_{i+k} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_2}: & a_i \sim b_i \sim b_{i+1} \sim b_{i+2} \sim \cdots \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_3}: & a_i \sim b_i \sim c_i \sim c_{i+1} \sim c_{i+2} \sim \cdots \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_4}: & a_i \sim b_i \sim c_i \sim d_i \sim d_{i+1} \sim d_{i+2} \sim \cdots \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_5}: & a_i \sim a_{i-1} \sim a_{i-2} \sim \cdots \sim c_{i-k} = a_{i+k+1} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_6}: & a_i \sim b_{i-1} \sim b_{i-2} \sim \cdots \sim b_{i+k} = b_{i+k+1} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_7}: & a_i \sim b_{i-1} \sim c_{i-1} \sim c_{i-2} \sim \cdots \sim c_{i-k} = c_{i+k+1} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overline{P_8}: & a_i \sim b_{i-1} \sim c_{i-1} \sim d_{i-1} \sim d_{i-2} \sim \cdots \sim d_{i+k} = d_{i+k+1} \sim d_{i+k} \sim e_{i+k}.
\end{align*}

Table 1: Vertices willing to perform the geodesic identification in $S_n^p$.

| Vertex $v$ | $N(v)$ | $d(w, v)$ for all $w \in N(v)$ |
|------------|--------|----------------------------------|
| $a_i$      | $\{a_i, a_{i+1}, b_{i+1}, b_i\}$ | $d(w, e_{i+k}) = k + 3$ |
| $e_{i+k}$  | $\{d_{i+k}\}$                              | $d(w, a_i) = k + 3$ |

P1: $L = \{e_i; 1 \leq i \leq n\}$ is the set of leaves.
P2: for each $1 \leq i \leq n$, $d(a_i, e_{i+k-1}) = d(a_i, e_{i+k}) = d(b_i, e_{i+k}) = k + 3$. Therefore, Table 2 yields that the pairs $(a_i, e_{i+k-1})$, $(a_i, e_{i+k})$ and $(b_i, e_{i+k})$ satisfy both the conditions in (1) for each $1 \leq i \leq n$. Table 2 further implies that the pair $(a_i, a_{j+k})$ satisfies both the conditions in (1) because $d(a_i, a_{j+k}) = k$ for each $1 \leq i \leq n$.
P3: $\text{diam}(S_n^p) = k + 3 = d(a_i, e_{i+k-1}) = d(a_i, e_{i+k}) = d(b_i, e_{i+k})$ for each $1 \leq i \leq n$.

Due to P1, without loss of generality, we let $e_2, e_3, \ldots, e_n \in S$, by Proposition 3. Then, as the pairs $(a_{k+1}, e_1)$, $(a_{k+1}, e_1)$ and $(b_{k+1}, e_1)$ satisfy P2 and P3, we must have either $a_{k+1} \in S$ or $b_{k+1} \in S$ or $e_1 \in S$, by Propositions 1 and 2. For the geodesic identification of these three pairs, it is enough to consider $e_1 \in S$. Furthermore, for each $1 \leq j \leq k$, either $a_j \in S$ or $a_j \in S$, due to P2 and Proposition 1. Hence, $|S| \geq n - 1 + 1 = k = (3n/2)$.

Lemma 5. If $n = 2k + 1$ with $k \geq 1$, then the set $S = \{e_1, e_2, \ldots, e_n\} \subset V(S_n^p)$ is a strong metric generator for $S_n^p$.

Proof. For any $s \in S$ and any $u \in V(S_n^p) - \{s\}$, since the vertex $s$ performs the geodesic identification for the pair $(s, u)$, we have to perform the geodesic identification for every pair $(x, y)$ of distinct vertices with $x, y \in V(S_n^p) - \{s\}$. For each $1 \leq i \leq n$, consider the following $a_i \sim e_{i+k}$ geodesics of length $k + 4$:
Lemma 6. If \( n = 2k \) with \( k \geq 2 \), then the set \( S = \{ a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_n \} \subset V(S^n_0) \) is a strong metric generator for \( S^n_0 \).

Proof. We have to perform the geodesic identification each pair \((u, v)\) of distinct vertices for \( u, v \in V(S^n_0) - S \) because for any \( s \in S \) and any \( v \in V(S^n_0) - s \), the pair \((s, v)\) possesses the geodesic identification by the vertex \( s \). For each \( 1 \leq i \leq n \), consider the following \( a_i - e_{i+k} \) geodesics and \( a_i - e_{i+k} \) geodesics of length \( k + 3 \):

\[
\begin{align*}
\overrightarrow{P_1}: a_1 &\sim a_{i+1} \sim a_{i+2} \sim \cdots \sim a_{i+k-1} \sim b_{i+k-1} \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1}, \\
\overrightarrow{P_2}: a_i &\sim b_i \sim b_{i+1} \sim b_{i+2} \sim \cdots \sim b_{i+k-1} \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1}, \\
\overrightarrow{P_3}: a_i &\sim b_i \sim c_i \sim c_{i+1} \sim c_{i+2} \sim \cdots \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1}, \\
\overrightarrow{P_4}: a_i &\sim b_i \sim c_i \sim d_i \sim d_{i+1} \sim d_{i+2} \sim \cdots \sim d_{i+k-1} \sim e_{i+k-1}, \\
\overleftarrow{P_1}: a_i &\sim a_{i-1} \sim a_{i-2} \sim \cdots \sim \{a_{i-(k-1)} = a_{i+k+1}\} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overleftarrow{P_2}: a_i &\sim b_{i-1} \sim b_{i-2} \sim \cdots \sim \{b_{i-(k-1)} = b_{i+k+1}\} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overleftarrow{P_3}: a_i &\sim b_{i-1} \sim c_{i-1} \sim c_{i-2} \sim \cdots \sim \{c_{i-(k-1)} = c_{i+k+1}\} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k}, \\
\overleftarrow{P_4}: a_i &\sim b_{i-1} \sim c_{i-1} \sim d_{i-1} \sim d_{i-2} \sim \cdots \sim \{d_{i-(k-1)} = d_{i+k+1}\} \sim d_{i+k} \sim e_{i+k}.
\end{align*}
\]

(17) (18) (19) (20) (21) (22) (23) (24)

Proof (Theorem 1). The lower bound for the strong metric dimension of \( S^n_0 \) is provided by Lemmas 3 and 4, whereas the upper bound is due to Lemmas 5 and 6.

4.2 Convex Plane Graph \( T^n_0 \). The graph of a convex polytope \( T^n_0 \) is defined in [49] for \( n \geq 3 \). The vertex set of \( T^n_0 \) is \( V(T^n_0) = \{a_i, b_i, c_i, d_i; 1 \leq i \leq n\} \), and its edge set is
Lemma 7. For odd values of \( n \geq 3 \), if \( S \) is a strong metric generator for \( T^p_n \), then \( |S| \geq \frac{5n}{2} \).

Proof. Let \( n = 2k + 1 \) with \( k \geq 1 \), and consider Table 3 for \( 1 \leq i \leq n \).

The following points are considerable in \( T^p_n \):

P1: \( L = \{ e_i; 1 \leq i \leq n \} \) is the set of leaves.

P2: \( d(a_i, e_{i-k-1}) = d(a_i, e_{i+k}) = k + 3 \) and \( d(b_i, e_{i+k}) = k + 2 \) for each \( 1 \leq i \leq n \). Hence, Table 3 ensures that pairs of vertices in \( T^p_n \) satisfying both the conditions in (1) are \((a_i, e_{i+k-1}), (a_i, e_{i+k}), (b_i, e_{i+k})\), and \((a_i, c_{i+k})\) for each \( 1 \leq i \leq n \).

P3: \( \text{diam}(T^p_n) = k + 3 = d(a_i, e_{i+k-1}) = d(a_i, e_{i+k}) = d(b_i, e_{i+k}) \) for each \( 1 \leq i \leq n \).

By Proposition 3, let us take \( e_1, e_2, \ldots, e_n \in S \) without loss of generality due to P1. Then, from the remaining pairs \((a_{k+1}, e_1), (a_{k+2}, e_1), \text{and} (b_{k+1}, e_1)\), we must have either \( a_{k+1} \in S \) or \( a_{k+2} \in S \) or \( b_{k+1} \in S \) or \( e_1 \in S \), by Proposition 1 and 2, because these pairs are satisfying P2 and P3. Here, it can be seen, by letting \( e_1 \in S \) only, that all these pairs possess the geodesic identification by the vertex \( e_1 \). Furthermore, for each \( 1 \leq i \leq n \), from the pair \((a_i, c_{i+k})\) satisfying P2, either \( a_i \in S \) or \( c_{i+k} \in S \), by Proposition 1. Hence, \( |S| \geq n - 1 + n = 2n \). \( \square \)

Lemma 8. For even values of \( n \geq 4 \), if \( S \) is a strong metric generator for \( T^p_n \), then \( |S| \geq \frac{5n}{2} \).

Proof. Let \( n = 2k \) with \( k \geq 2 \), and consider Table 4 for \( 1 \leq i \leq n \).

The following points are considerable in \( T^p_n \):

P1: \( L = \{ e_i; 1 \leq i \leq n \} \) is the set of leaves.

P2: \( d(a_i, e_{i-1}) = d(a_i, e_{i+k}) = k + 3 \) and \( d(b_i, e_{i+k}) = k + 1 \), which implies according to Table 4, that the pairs \((a_i, c_{i-1}), (a_i, c_{i+k-1})\) and \((b_i, c_{i+k})\) satisfy both the conditions in (1) for each \( 1 \leq i \leq n \). Moreover, for each \( 1 \leq j \leq k \), \( d(a_j, a_{i+k}) = k \), and so Table 4 concludes that the pair \((a_j, a_{j+k})\) also satisfies both the conditions in (1).

P3: \( \text{diam}(T^p_n) = k + 3 = d(a_i, e_{i+k-1}) \) for all \( 1 \leq i \leq n \).

Now, without loss of generality, let us take \( e_2, e_3, \ldots, e_n \in S \), by Proposition 3 and due to P1. Then, from the remaining pairs \((a_{k+2}, e_1)\) and \((a_{k+1}, e_1)\) satisfying P2 and P3, respectively, we must have either \( a_{k+2} \in S \) or \( e_1 \in S \) by Proposition 2, and either \( a_2 \in S \) or \( e_1 \in S \) by Proposition 1. If we take \( e_1 \in S \), then this vertex performs the geodesic identification for both the remaining pairs. Also, Proposition 1 yields that either \( b_1 \in S \) or \( c_{i+k} \in S \) for each \( 1 \leq i \leq n \) and either \( a_i \in S \) or \( c_{i+k} \in S \) for each \( 1 \leq j \leq k \) because of the point P2. Therefore, \( |S| \geq n - 1 + n + k = \frac{5n}{2} \). \( \square \)

Lemma 9. For \( n = 2k + 1 \) with \( k \geq 1 \), the set \( S = \{ a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n \} \subset V(T^p_n) \) is a strong metric generator for \( T^p_n \).

Proof. Note that every pair \((u, v)\) with \( s \in S \) and \( v \in V(T^p_n) - \{s\} \) possesses the geodesic identification by the vertex \( s \). It follows that we should perform the geodesic identification for every pair \((u, v)\) of distinct vertices by the vertices in \( S \) whenever \( u, v \in V(T^p_n) - S \). For each \( 1 \leq i \leq n \), except the pairs of vertices,

\[(v, b_{i-k}), (v, b_{i+k}), (v, c_{i-k}), (v, e_{i+k}), (v, d_{i-k}), (v, d_{i+k}), \] for \( v \in \{b_i, c_i, d_i\} \),

(29)

All other pairs of vertices possess the geodesic identification by the vertices \( e_{i+k-1} \) and \( e_{i+k} \) using \( a_i - e_{i+k-1} \) geodesics and \( a_i - e_{i+k} \) geodesics of length \( k + 3 \), listed from the following equations:
Now, for each $1 \leq i \leq n$, by considering $b_i - e_{ik}$ geodesics, listed from (38) to (43), the vertex $e_{ik}$ performs the geodesic identification for all the pairs of vertices listed in (29):

\[ Q_{i1} : b_i \sim b_{i+1} \sim \cdots \sim b_{i+k-1} \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1}, \]
\[ Q_{i2} : b_i \sim c_i \sim c_{i+2} \sim \cdots \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1}, \]
\[ Q_{i3} : b_i \sim d_i \sim d_{i+1} \sim \cdots \sim d_{i+k-1} \sim e_{i+k-1}, \]
\[ Q_{i4} : a_i \sim a_{i+2} \sim \cdots \sim a_{i+k-1} \sim b_{i+k-1} \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1}. \]
Lemma 10. If $n = 2k$ with $k \geq 2$, then the set

$$S = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n, e_1, e_2, \ldots, e_n\} \subset V(T^p_n),$$

is a strong metric generator for $T^p_n$.

Proof. Since the vertex $s \in S$ performs the geodesic identification for every pair of vertices including $s$, it implies that we have to perform the geodesic identification by the vertices in $S$ for all the pairs of vertices from $V(T^p_n) - S$. For this, consider the following $a_i - e_{i+1}$ geodesics of length $k + 3$:

$$P_{1i} : a_i \sim a_{i+1} \sim a_{i+2} \sim \cdots \sim a_{i+k-1} \sim b_{i+k-1} \sim e_{i+k-1},$$

$$P_{2i} : a_i \sim b_i \sim c_i \sim e_{i+1} \sim c_{i+2} \sim \cdots \sim c_{i+k-1} \sim d_{i+k-1} \sim e_{i+k-1},$$

$$P_{3i} : a_i \sim b_i \sim c_i \sim d_i \sim \cdots \sim d_{i+k-1} \sim e_{i+k-1},$$

$$P_{4i} : a_i \sim a_{i+1} \sim a_{i+2} \sim \cdots \sim \{a_i = a_{i+k}\} \sim b_{i+k} \sim e_{i+k-1},$$

$$P_{5i} : a_i \sim b_{i-1} \sim c_{i-1} \sim \cdots \sim \{c_{i-1} = c_{i+k-1}\} \sim d_{i+k-1} \sim e_{i+k-1},$$

$$P_{6i} : a_i \sim b_{i-1} \sim c_{i-1} \sim d_{i-2} \sim \cdots \sim d_{i+k-1} \sim e_{i+k-1}.$$

Due to geodesics, listed from (45) to (50), the vertex $e_{i+1}$ performs the geodesic identification for all the pairs of vertices except the pairs $(c_i, c_{i+k})$, $(d_i, c_{i+k})$, $(d_i, d_{i+k})$ for all $1 \leq i \leq n$ and the pair $(a_i, d_{i-1})$ for each $k + 1 \leq j \leq n$. These pairs are identified as follows:

(i) For each $1 \leq i \leq n$, the vertex $b_i$ performs the geodesic identification for the pair $(e_i, e_{i+k})$ due to the geodesic $Q_i : b_i \sim c_i \sim c_{i+1} \sim c_{i+2} \sim \cdots \sim c_{i+k-1} \sim c_{i+k}$.

(ii) For each $1 \leq i \leq n$, the vertex $b_{i+k}$ performs the geodesic identification for the pair $(d_i, c_{i+k})$ due to the geodesic $R_i : d_i \sim d_{i+1} \sim \cdots \sim d_{i+k-1} \sim c_{i+k} \sim b_{i+k}$.

(iii) For each $1 \leq i \leq n$, the vertex $e_{i+k}$ performs the geodesic identification for the pair $(d_i, d_{i+k})$ due to the geodesic $S_i : d_i \sim d_{i+1} \sim \cdots \sim d_{i+k-1} \sim d_{i+k} \sim e_{i+k}$.

(iv) For each $k + 1 \leq j \leq n$, the vertex $e_{j-1}$ performs the geodesic identification for the pair $(a_j, d_{j-1})$ due to the geodesic $T_j : a_j \sim b_{j-1} \sim d_{j-2} \sim \cdots \sim d_{j+k-1} \sim e_{j-1}$.

Hence, we conclude that $S$ is a strong metric generator for $T^p_n$.

Proof. (Theorem 2). The lower bound for the strong metric dimension of $T^p_n$ is provided by Lemmas 7 and 8, whereas the upper bound is due to Lemmas 9 and 10.

4.3. Convex Plane Graph $U^p_n$. The graph of a convex polytope $U_n$ is defined in [49] for $n \geq 3$. The vertex set of $U_n$ is $V(U_n) = \{a_i, b_i, c_i, d_i, e_i: 1 \leq i \leq n\}$, and its edge set is

$$E(U_n) = \{a_i \sim a_{i+1}, a_i \sim b_i, b_i \sim b_{i+1}, b_i \sim c_i: 1 \leq i \leq n\} \cup \{c_i \sim d_i, c_i \sim d_{i+1}, d_i \sim e_i, e_i \sim e_{i+1}: 1 \leq i \leq n\}.$$

The convex plane graph $U^p_n$ ($p$ for pendant) is obtained from the graph of a convex polytope $U_n$ by attaching one pendant vertex (leaf) $f_i$ to the vertex $e_{i}$ of $U_n$ for each $1 \leq i \leq n$ (see Figure 4) [50]. Thus, we have the followings vertex and edge sets for $U^p_n$:

$$V(U^p_n) = V(U_n) \cup \{f_i: 1 \leq i \leq n\},$$

$$E(U^p_n) = E(U_n) \cup \{e_i \sim f_i: 1 \leq i \leq n\}.$$

The following main result provides the strong metric dimension of $U^p_n$.
The proof of Theorem 3 will be followed after the next four lemmas.

Lemma 11. For odd values of $n \geq 3$, if $S$ is a strong metric generator for $U_n^p$, then $|S| \geq 2n$.

Proof. Let $n = 2k + 1$ with $k \geq 1$, and consider Table 5 for $1 \leq i \leq n$.

Note the following points in $U_n^p$:

P1: $L = \{f_i; 1 \leq i \leq n\}$ is the set of leaves.

P2: for each $1 \leq i \leq n$, the pairs of vertices $(a_i, f_{i+1})$ and $(c_i, d_{i+1})$ satisfy both the conditions in (1) because of Table 5 and since $d(a_i, f_{i+1}) = k + 5$, $d(c_i, d_{i+1}) = k + 3$.

P3: $\text{diam}(U_n^p) = k + 5 = d(a_i, f_{i+1})$ for each $1 \leq i \leq n$.

Due to P1 and by Proposition 3, we let $f_2, f_3, \ldots, f_n \in S$ without loss of generality. Then, the remaining pair $(a_{k+2}, f_1)$ satisfies both P2 and P3. So, we must have either $a_{k+2} \in S$ or $f_1 \in S$, by Propositions 1 and 2. Furthermore, Proposition 1 yields that, from the pair $(c_i, d_{i+1})$, either $c_i \in S$ or $d_{i+1} \in S$ for all $1 \leq i \leq n$ due to P2. Hence, $|S| \geq n - 1 + 1 + n = 2n$. \hfill \square

Lemma 12. For even values of $n \geq 4$, if $S$ is a strong metric generator for $U_n^p$, then $|S| \geq (5n/2)$.

Proof. Let $n = 2k$ with $k \geq 2$, and consider Table 6 for $1 \leq i \leq n$.

The following points hold in $U_n^p$:

(i) If $a_i \in S$ and $c_i \in S$ for all $1 \leq i \leq n$, then we must have either $d_j \in S$ or $d_{j+1} \in S$ for all $1 \leq j \leq k$, by the point P2 and Proposition 1. For otherwise, we get the pair of vertices $(d_j, d_{j+k})$, for all $1 \leq j \leq k$, which is left unidentified by the elements of $S$, but we have $|S| \geq 2n - 1 + n + n = k = (7n/2)$.

(ii) If $a_i \in S$ and $d_{i+k} \in S$ for all $1 \leq i \leq n$, then we must have either $c_j \in S$ or $c_{j+k} \in S$ for all $1 \leq j \leq k$, by the point P2 and Proposition 1. Otherwise, we get the pair of vertices $(c_j, c_{j+k})$, for all $1 \leq j \leq k$, which is left unidentified by the elements of $S$, but we have $|S| \geq 2n - 1 + n + n = k = (7n/2)$.

(iii) If $c_{i+k} \in S$, then of course $c_i \in S$ for all $1 \leq i \leq n$. Moreover, we must have either $d_j \in S$ or $d_{j+k} \in S$ for $1 \leq j \leq k$. Therefore, Table 6 again yields that the pairs of vertices $(c_j, c_{j+k})$ and $(d_j, d_{j+k})$ satisfy both the conditions in (1) for all $1 \leq j \leq k$.
all $1 \leq j \leq k$, by the point P2 and Proposition 1. For otherwise, we get the pair of vertices $(d_{i,j}, d_{j+1})$, for all $1 \leq j \leq k$, which is left unidentified by the elements of $S$. Then, we have $|S| \geq n - 1 + 1 + n + k = (5n/2)$.

(iv) If $c_{i+j} \in S$ and $d_{i+j} \in S$ for all $1 \leq i \leq n$, then we have $|S| \geq n - 1 + 1 + n + n = 3n$.

It can be concluded from these four cases that the most suitable choice to construct a strong metric generator $S$ with minimum cardinality is found in the Case 3, which yields that $|S| \geq (5n/2)$.

\[ \bar{P}_1: a_i \sim a_{i+1} \sim a_{i+2} \sim \cdots \sim a_{i+k} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_2: a_i \sim b_i \sim b_{i+1} \sim b_{i+2} \sim \cdots \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_3: a_i \sim b_i \sim c_i \sim d_i \sim e_i \sim e_{i+1} \sim e_{i+2} \sim \cdots \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_4: a_i \sim a_{i-1} \sim a_{i-2} \sim \cdots \sim a_{i-k} = a_{i+k+1} \sim b_{i+k+1} \sim c_{i+k+1} \sim d_{i+k+1} \sim e_{i+k+1} \sim f_{i+k+1}, \]  

\[ \bar{P}_5: a_i \sim b_i \sim c_1 \sim d_1 \sim e_1 \sim e_{i+1} \sim e_{i+2} \sim \cdots \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_6: c_i \sim d_i \sim c_{i+1} \sim d_{i+1}, \]  

\[ \bar{P}_7: c_i \sim d_i \sim c_{i+1} \sim d_{i+1}, \]  

\[ \bar{P}_8: c_i \sim d_i \sim e_i \sim e_{i+1} \sim e_{i+2} \sim \cdots \sim e_{i+j-1} \sim e_{i+j} \sim d_{i+j}. \]  

Lemma 13. If $n = 2k + 1$ with $k \geq 1$, then the set $S = \{c_1, c_2, \ldots, c_n, f_1, f_2, \ldots, f_n\} \subset V(U^p_n)$ is a strong metric generator for $U^p_n$.

Proof. It is enough to perform the geodesic identification for those pairs of vertices of $U^p_n$ having no element from the set $S$ because every pair of vertices having one element $s$ from $S$ possesses the geodesic identification by the element $s$. Let us consider the following $a_i - f_{i+k}$ geodesics of length $k + 5$, for each $1 \leq i \leq n$:

\[ \bar{P}_{1i}: a_i \sim a_{i+1} \sim a_{i+2} \sim \cdots \sim a_{i+k} \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_{2i}: a_i \sim b_i \sim b_{i+1} \sim b_{i+2} \sim \cdots \sim b_{i+k} \sim c_{i+k} \sim d_{i+k} \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_{3i}: a_i \sim b_i \sim c_i \sim d_i \sim e_i \sim e_{i+1} \sim e_{i+2} \sim \cdots \sim e_{i+k-1} \sim f_{i+k-1}, \]  

\[ \bar{P}_{4i}: a_i \sim a_{i-1} \sim a_{i-2} \sim \cdots \sim a_{i-k} = a_{i+k+1} \sim b_{i+k+1} \sim c_{i+k+1} \sim d_{i+k+1} \sim e_{i+k+1} \sim f_{i+k+1}, \]  

\[ \bar{P}_{5i}: a_i \sim b_i \sim c_i \sim d_i \sim e_i \sim e_{i+1} \sim e_{i+2} \sim \cdots \sim e_{i+k} \sim f_{i+k}, \]  

\[ \bar{P}_{6i}: c_i \sim d_i \sim c_{i+1} \sim d_{i+1} \sim e_i \sim e_{i-1} \sim e_{i-2} \sim \cdots \sim e_{i-(j-1)} \sim e_{i-j} \sim d_{i-j}. \]  

Hence, we conclude that $S$ is a strong metric generator for $U^p_n$.

Lemma 14. If $n = 2k$ with $k \geq 2$, then the set $S = \{c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_k, f_1, f_2, \ldots, f_n\} \subset V(U^p_n)$

is a strong metric generator for $U^p_n$.

Proof. For $s \in S$, a pair of vertices $(s, s)$ with $v \in V(U^p_n) - \{s\}$ possesses the geodesic identification by the vertex $s$. Therefore, we should perform the geodesic identification for each pair $(x, y)$ of vertices of $U^p_n$ by the vertices in $S$ for both $x, y \in S$. For each $1 \leq i \leq n$, the following $a_i - f_{i+k-1}$ and $a_i - f_{i+k}$ geodesics of length $k + 4$ are useful for our purpose:
The geodesic identification for the pairs given in (60) and (61).

| Pairs | Vertex performing the geodesic identification | Due to the geodesic in |
|-------|----------------------------------------------|-----------------------|
| (d_i, d_{i-1}) | c_i | (62) |
| (d_i, d_{i+1}) | c_{i+1} | (63) |
| 2 \leq j \leq k | c_i | (64) |
| (d_i, d_{i+1}) | c_{i+1} | (65) |

Table 7: Geodesic identification for the pairs given in (60) and (61).

The geodesic identification for the pairs given in (73)–(76).

| Pairs | Vertex performing the geodesic identification | Due to the geodesic in |
|-------|----------------------------------------------|-----------------------|
| (a_i, a_{i+k}), (a_i, b_{i+k}) | c_{i+k} | (77) |
| (b_i, b_{i+k}) | c_{i+k} | (78) |
| (e_j, d_{j+k}), (e_j, e_{j+k}) | d_j | (79) |
| (d_i, d_{i+1}) | c_r | (80) |
| 2 \leq m \leq k - 1 | c_i | (81) |
| (d_i, d_{i+m}) | k + 1 \leq l \leq n - 2 | |

It can be seen that, for some pairs of vertices, the vertex \( f_{i+k} \) performs the geodesic identification due to geodesics in (67)–(69), and for some pairs of vertices, the vertex \( f_{i+k} \) performs the geodesic identification due to geodesics in (70)–(72). The pairs of vertices, which are left unidentified, are as follows:

\[ (a_i, a_{i+k}), (a_i, b_{i+k}), (b_i, b_{i+k}), \text{ for } 1 \leq i \leq k, \]  
\[ (e_j, d_{j+k}), (e_j, e_{j+k}), \text{ for } 1 \leq j \leq k, \]  
\[ (d_i, d_{i+1}), \text{ for } k + 1 \leq r \leq n - 1, \]  
\[ (d_i, d_{i+m}), \text{ for } k + 1 \leq l \leq n - 2, 2 \leq m \leq k - 1. \]  

All these pairs of vertices possess the geodesic identification, as described in Table 8, with the help of the following geodesics, for \( 1 \leq i \leq k, 1 \leq j \leq k, k + 1 \leq r \leq n - 1, k + 1 \leq l \leq n - 2, \) and \( 2 \leq m \leq k - 1:\)

\[ P_{i+1}: \ a_i \sim a_{i-1} \sim a_{i-2} \sim \cdots \sim a_{i-(k-1)} \sim b_{i-(k-1)} \sim c_{i-(k-1)} \sim \{d_{i-k} = d_{i+k}\} \sim e_{i+k} \sim f_{i+k}, \]  
\[ P_{2i}: \ a_i \sim b_i \sim b_{i-1} \sim b_{i-2} \sim \cdots \sim b_{i-(k-1)} \sim c_{i-(k-1)} \sim \{d_{i-k} = d_{i+k}\} \sim e_{i+k} \sim f_{i+k}, \]  
\[ P_{3i}: \ a_i \sim b_i \sim c_i \sim d_{i-1} \sim e_{i-1} \sim e_{i-2} \sim \cdots \sim e_{i-(k-1)} \sim \{e_{i-k} = e_{i+k}\} \sim f_{i+k}. \]  
\[ Q_{i+1}: \ a_i \sim a_{i+1} \sim a_{i+2} \sim \cdots \sim a_{i+k} \sim b_{i+k} \sim c_{i+k}, \]  
\[ Q_{2i}: \ a_i \sim b_i \sim b_{i+1} \sim b_{i+2} \sim \cdots \sim b_{i+k} \sim c_{i+k}, \]  
\[ Q_{3i}: \ d_j \sim e_j \sim e_{j+1} \sim e_{j+2} \sim \cdots \sim e_{j+k-1} \sim e_{j+k} \sim d_{j+k}, \]  
\[ S_j: \ c_r \sim d_r \sim c_{r+1}, \]  
\[ T_m: \ c_l \sim e_l \sim e_{l+1} \sim e_{l+2} \sim \cdots \sim e_{l+m-1} \sim e_{l+m} \sim d_{l+m}. \]

Thus, \( S \) is a strong metric generator for \( U_n^p. \)

**Proof** (Theorem 3). The lower bound for the strong metric dimension of \( U_n^p \) is provided by Lemmas 11 and 12, whereas the upper bound is due to Lemmas 13 and 14.
5. Concluding Remarks

The geodesic identification (strong resolvability) for each pair of vertices has been performed in three families $S_{ni}^p$, $T_{ni}^p$, and $U_{ni}^p$ of convex plane graphs for all $n \geq 3$. In fact, we extended the study of the identification of vertices by solving the strong metric dimension problem for three more families of graphs. The problem is solved with the following investigations:

(i) In $S_{ni}^p$, total $\left( \frac{5n}{2} \right)$ pairs of vertices are identified with $n$ vertices by using $8n$ geodesics when $n$ is odd and with $(3n/2)$ vertices by using $8n + 1$ geodesics when $n$ is even.

(ii) In $T_{ni}^p$, total $\left( \frac{5n}{2} \right)$ pairs of vertices are identified with $2n$ vertices by using $14n$ geodesics when $n$ is odd and with $(5n/2)$ vertices by using $(19n/2)$ geodesics when $n$ is even.

(iii) In $U_{ni}^p$, total $\left( \frac{6n}{2} \right)$ pairs of vertices are identified with $2n$ vertices by using $(n(n + 3))$ geodesics when $n$ is odd and with $(5n/2)$ vertices by using $(n^2/4) + 6n + 3$ geodesics when $n$ is even.

Data Availability

The data used to support the findings of this study are available from the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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