Symmetries in the third Painlevé equation arising from the modified Pohlmeyer-Lund-Regge hierarchy

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Abstract

We propose a modification of the AKNS hierarchy that includes the “modified” Pohlmeyer-Lund-Regge (mPLR) equation. Similarity reductions of this hierarchy give the second, third, and fourth Painlevé equations. Especially, we present a new Lax representation and a complete description of the symmetry of the third Painlevé equation through the similarity reduction. We also show the relation between the tau-function of the mPLR hierarchy and Painlevé equations.

1 Introduction

Painlevé equations and their higher order analogues are obtained by certain reductions of infinite-dimensional integrable systems. This clarifies the origin of various aspects of Painlevé equations, such as the affine Weyl group symmetry, bilinear relations for \(\tau\)-functions, Lax formalism, the solutions described by special polynomials.

In [15] Noumi and Yamada proposed a systematic description of nonlinear differential equations comprising the second, fourth and fifth Painlevé equations which possess \(A_n(1)\) affine Weyl group symmetry. These systems are obtained by similarity reductions from the \(A_n(1)\) Drinfeld-Sokolov hierarchy which is an infinite-dimensional integrable system characterized by affine Lie algebras and their Heisenberg subalgebras [2], [5]. Also, Fuji and Suzuki derived dynamical systems, including the sixth Painlevé equation, by similarity reductions from the \(D_{2n}^{(1)}\) Drinfeld-Sokolov hierarchy [3]. Therefore, the affine Weyl group symmetries in the Painlevé II, IV, V, VI equations can all be derived from integrable systems associated with affine Lie algebra.

In this paper we provide a natural and intrinsic description of the symmetry of the third Painlevé equation (P\(_{III}\))

\[
\frac{d^2 f}{dx^2} = \frac{1}{f} \left( \frac{df}{dx} \right)^2 - \frac{1}{x} \frac{df}{dx} + \frac{1}{x} (c_1 f^2 + c_2) + c_3 f^3 + c_4 f
\]  

(1.1)

in terms of a hierarchy of soliton equations. P\(_{III}\) (1.1) has an equivalent equation Painlevé III’ (P\(_{III’}\)) [10]:

\[
\frac{d^2 y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{y^2}{4s^2} (c_3 y + c_1) + \frac{c_2}{4s} + \frac{c_4}{4y},
\]  

(1.2)

which is obtained by the change of the variables \(s = x^2, y = xf\). In what follows we assume that \(c_3 c_4 \neq 0\) (\(D_6^{(1)}\)-type), in which case the number of parameters contained in (1.2) is two by means of a suitable change of scales for \(y\) and \(s\). It is known that the transformation group of solutions of (1.2) is isomorphic to the affine Weyl group of type \(A_1^{(1)} \times A_1^{(1)}\) (or \(B_2^{(1)}\)) [10].
There are several works concerning the third Painlevé equation based on the theory of integrable systems [8], [9], [13], [18], [21]. However, there have not been any satisfactory theories presented so far which could explain the relationship between Okamoto’s theory, especially the symmetry of the affine Weyl group based on a Hamiltonian equation and the $\tau$-function, and the soliton equations realized as representations of affine Lie algebras. So we develop the theory of the “modified” Pohlmeyer-Lund-Regge (mPLR) hierarchy that includes the derivative nonlinear Schrödinger hierarchy studied by Kakei and the author [10], [11], in the same way as the sine-Gordon hierarchy includes the mKdV hierarchy. The similarity reduction of the mPLR hierarchy gives the third Painlevé equation and its symmetry.

The “modified” Pohlmeyer-Lund-Regge (mPLR) equation is, by definition, the following system of equations:

\[
\begin{align*}
\frac{\partial^2 q}{\partial t_1 \partial \bar{t}_1} &= 4q + 4qr \frac{\partial q}{\partial \bar{t}_1}, \\
\frac{\partial^2 r}{\partial t_1 \partial \bar{t}_1} &= 4r - 4qr \frac{\partial r}{\partial \bar{t}_1},
\end{align*}
\]

where \(q = q(t_1, \bar{t}_1), \ r = r(t_1, \bar{t}_1)\). The zero-curvature representation of (1.3) is

\[
\frac{\partial B_1}{\partial t_1} - \frac{\partial \bar{B}_1}{\partial \bar{t}_1} = \bar{B}_1B_1 - B_1\bar{B}_1,
\]

where

\[
\begin{align*}
B_1 &= \begin{bmatrix} 2qr & -2q \\ 0 & -2qr \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2r & -1 \end{bmatrix} \zeta, \\
\bar{B}_1 &= \begin{bmatrix} 1 & -2\bar{q}e^{2\phi} \\ 0 & -1 \end{bmatrix} \zeta^{-1} + \begin{bmatrix} 0 & 0 \\ 2\bar{r}e^{-2\phi} & 0 \end{bmatrix}.
\end{align*}
\]

Here \(\zeta \in \mathbb{C}\) is a parameter and the variables \(\bar{q} = \bar{q}(t_1, \bar{t}_1), \ \bar{r} = \bar{r}(t_1, \bar{t}_1)\) and \(\phi = \phi(t_1, \bar{t}_1)\) satisfy the following equations:

\[
\begin{align*}
\frac{\partial q}{\partial t_1} &= -2\bar{q}e^{2\phi}, & \frac{\partial \phi}{\partial t_1} &= 2qr, & \frac{\partial \bar{q}}{\partial t_1} &= -2qe^{-2\phi}, \\
\frac{\partial r}{\partial t_1} &= 2\bar{r}e^{-2\phi}, & \frac{\partial \phi}{\partial t_1} &= -2\bar{r}, & \frac{\partial \bar{r}}{\partial t_1} &= 2re^{2\phi}.
\end{align*}
\]

Originally, the integrable Pohlmeyer-Lund-Regge system was derived in a study of the dynamics of relativistic vortices by Lund and Regge [12], and independently in an investigation of the nonlinear sigma model in field theory by Pohlmeyer [17]. In [8], Jimbo and Miwa showed that the third Painlevé equation is obtained through a similarity reduction from the Pohlmeyer-Lund-Regge equation. We will show later that the equations (1.7) are obtained by Miura transformation from the equations in [8].

This article is organized as follows. In Section 2, we formulate the mPLR hierarchy based on the Sato theory by using the affine Lie algebra and group of type \(A_1^{(1)}\). Then in Section 3, we introduce the Bäcklund transformations for the mPLR hierarchy and show that the group of these transformations provides a realization of an extended affine Weyl group. In Section 4, we study a similarity reduction of the mPLR hierarchy and give the action of the transformations defined in Section 3 on the parameters of the similarity condition. In Section 5 the third Painlevé equation and its symmetry are derived from the mPLR hierarchies. In Section 6, we recall the similarity reduction to the fourth Painlevé equation and show that the second Painlevé equation can also be derived from the mPLR hierarchy.
2 Modified Pohlmeyer-Lund-Regge hierarchy

The zero-curvature equation for the mPLR (1.4) is a compatibility condition for the linear equations

\[ \frac{\partial Y}{\partial t_1} = B_1 Y, \quad \frac{\partial Y}{\partial \bar{t}_1} = \bar{B}_1 Y. \] (2.1)

In this section we construct formal solutions of these equations based on the theory of the affine Lie algebra \( \mathfrak{sl}_2 \) and its group \([1, 6]\). This is equivalent to a modification of the 2-component Toda lattice hierarchy \([20]\).

2.1 Gauss decomposition and \( \tau \)-functions

Let \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \) and \( \hat{\mathfrak{g}} = \mathfrak{sl}_2 \otimes \mathbb{C}[\zeta, \zeta^{-1}] \otimes \mathbb{C} + \mathbb{C}d \) the associated affine Lie algebra of type \( A_1^{(1)} \). \( \hat{\mathfrak{g}} \) is generated by the Chevalley generators \( h_i, e_i^\pm (i = 0, 1) \) and \( d \) with the defining relations

\[
\begin{align*}
[h_i, h_j] &= 0, \quad [h_i, e_j^\pm] = \pm a_{ij} e_j^\pm, \quad [e_i^+, e_j^-] = \delta_{ij} h_i, \quad (ade_i^\pm)^{1-a_{ij}} e_j^\pm = 0, \\
[d, h_i] &= 0, \quad [d, e_i^+] = \delta_{i,0} e_i^+, \quad [d, e_i^-] = -\delta_{i,0} e_i^-.
\end{align*}
\]

Here \( A = (a_{ij})_{i,j=0,1} = \left[ \begin{array}{cc} 2 & -2 \\ -2 & 2 \end{array} \right] \) is the Cartan matrix of type \( A_1^{(1)} \). \( \hat{\mathfrak{g}} \) has the following triangular decomposition:

\[ \hat{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \] (2.2)

where \( \mathfrak{n}_- \) denotes the subalgebra of \( \hat{\mathfrak{g}} \) generated by \( e_+ \) and \( \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathbb{C}d \) is the Cartan subalgebra of \( \hat{\mathfrak{g}} \). Note that \( h_0 + h_1 = c \). The homogeneous Heisenberg subalgebra \( \mathfrak{h} \) of \( \hat{\mathfrak{g}} \) is given by \( \mathfrak{h} = \oplus_{n \in \mathbb{Z}} \mathbb{C}H_n \oplus \mathbb{C}c \) with the commutation relations \([H_m, H_n] = m\delta_{m+n,0}c\).

Let \((\pi_j, L_j)\), \( j = 0, 1 \), be the basic representation with the highest weight vector \( |v_j \rangle \). The action of the Chevalley generators is given by

\[ \pi_j (e_i^\pm) |v_j \rangle = 0, \quad \pi_j (h_i) |v_j \rangle = \delta_{ij} |v_j \rangle, \quad \pi_j (e_i^-)^{d_{ij}+1} |v_j \rangle = 0. \] (2.3)

We will denote by \((\pi_j^*, L_j^*)\) the dual representation of \((\pi_j, L_j)\), with the highest weight vector \( \langle v_j | \) and the action of the Chevalley generators is given by

\[ \pi_j^* (e_i^\pm) = \pi_j (e_i^\mp), \quad \pi_j^* (h_i) = \pi_j (h_i), \quad \langle v_i |^* = \langle v_i |. \]

It is known that \( L_j \) has the structure \( L_j \cong \mathbb{C}[x_1, x_2, \ldots] \otimes \mathbb{C}[Q] \), where \( Q \) is the root lattice of \( \mathfrak{sl}_2(\mathbb{C}) \) and \( \mathbb{C}[Q] \) is the group algebra of \( Q \), and the action of \( e_i^\pm \) is given in terms of vertex operators. But we will not need this realization in this paper.

Let \((\pi, L) = (\pi_0 \oplus \pi_1, L_1 \oplus L_2)\). Since \((\pi, L)\) is an integrable representation, we can define the actions of \( \exp \pi (e_i^\pm) \) \( (i = 0, 1) \) on \( L \). Set

\[ S_i := \exp \pi (e_i^-) \exp (-\pi (e_i^+)) \exp \pi (e_i^-) \quad (i = 0, 1). \] (2.4)

\( S_0 \) and \( S_1 \) generate the affine Weyl group of type \( A_1^{(1)} \).

Now we define the \( \tau \)-functions of the mPLR hierarchy. Let \( \hat{G} \) be the affine Lie group generated by \( \exp \mathfrak{g} \), \( i = 0, 1 \), \( s \in \mathbb{C} \) and put \( g(0) \in \hat{G} \) as the initial value of the hierarchy. We introduce the time-evolution of \( g(0) \) with respect to time variables \( t = (t_1, t_2, \ldots) \) and \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \) by

\[ g(t, \bar{t}) = \exp \left( \sum_n t_n H_n \right) g(0) \exp \left( -\sum_n \bar{t}_n H_{-n} \right). \] (2.5)

In the definition \( (2.5) \) there are infinitely many independent variables but in the following we shall take almost all \( t_i \) and \( \bar{t}_i \)'s to be zero. We mention that our definition \( (2.5) \) differs from the previous paper \([11]\). Here we use not only the variable \( t_1 \) but also \( \bar{t}_i \).
The \( \tau \)-functions of the mPLR hierarchy are defined by
\[
\tau_{m,n}^{(i)}(t, \bar{t}) := \langle v_i | T^{-m} g(t, \bar{t}) T^n | v_i \rangle, \quad (i = 0, 1, \ m, n \in \mathbb{Z}) \tag{2.6}
\]
where \( T := S_0 S_1 \).

We will establish the relation between the \( \tau \)-function and the components of \( g \). We require that the element \( g = g(t, \bar{t}) \) \text{(2.5)} can be written as
\[
g = g_{<0}^{-1} g_0 g_{>0}, \tag{2.7}
\]
where \( g_{<0} \in \exp \pi(\hat{\mathfrak{n}}_-), \ g_{>0} \in \exp \pi(\hat{\mathfrak{n}}_+) \) and \( g_0 \in \exp \pi(\hat{h}') \). The factorization \text{(2.7)} is called the Gauss decomposition. It is known that if an element \( g \in \hat{G} \) belongs to the dense open subset of \( \hat{G} \) called a big cell, then \( g \) is written as \text{(2.7)}. Put
\[
g_{<0}(t, \bar{t}) = \exp \left( q(t, \bar{t}) e_0^- + r(t, \bar{t}) e_1^- + \cdots \right), \tag{2.8}
\]
\[
g_0(t, \bar{t}) = \exp(\phi_0(t, \bar{t}) h_0 + \phi_1(t, \bar{t}) h_1), \tag{2.9}
\]
\[
g_{>0}(t, \bar{t}) = \exp \left( q(t, \bar{t}) e_1^+ + r(t, \bar{t}) e_0^+ + \cdots \right). \tag{2.10}
\]

Since \( \exp \pi(\hat{\mathfrak{n}}_-) \) and \( \exp \pi(\hat{\mathfrak{n}}_+) \) stabilize \( |v_i\rangle \) and \( \langle v_i| \) \((i = 0, 1)\) respectively, by substituting \text{(2.10)} into \text{(2.6)} we have
\[
\tau_{m,n}^{(i)}(t, \bar{t}) = \langle v_i | g(t, \bar{t}) | v_i \rangle = \langle v_i | g_0(t, \bar{t}) | v_i \rangle
= \langle v_i | \exp(\phi_0(t, \bar{t}) h_0 + \phi_1(t, \bar{t}) h_1) | v_i \rangle = e^{\phi_i(t, \bar{t})}. \tag{2.11}
\]

Next we give the relation to the components of \( g_{<0} \text{(2.8)}, g_{>0} \text{(2.11)} \) and the \( \tau \)-functions. By the relations \text{(2.3)}, the action of \( g = g(t, \bar{t}) \) on the vector \( |v\rangle := |v_0\rangle + |v_1\rangle \) is computed as follows:
\[
g|v\rangle = g_{<0}^{-1} g_0 g_{>0} |v\rangle = \tau_{00}^{(0)}(1 - q e_0^- + \cdots) |v_0\rangle + \tau_{01}^{(1)}(1 - r e_1^- + \cdots) |v_1\rangle.
\]

Similarly, by calculating \( \langle v| g \), we obtain
\[
q = -\frac{\langle v_0 | e_0^+ g | v_0 \rangle}{\tau_{00}^{(0)}}, \quad \bar{q} = \frac{\langle v_1 | e_1^- g | v_1 \rangle}{\tau_{00}^{(1)}},
\]
\[
r = -\frac{\langle v_1 | e_1^- g | v_1 \rangle}{\tau_{00}^{(1)}}, \quad \bar{r} = \frac{\langle v_0 | e_0^- g | v_0 \rangle}{\tau_{00}^{(0)}}.
\]

On the other hand, we have the action of \( T^m \) on \( |v\rangle \) and \( |v| \), for example, \( T |v_0\rangle = S_0 S_1 |v_0\rangle = S_0 |v_0\rangle = e_0^- |v_0\rangle \). By comparing these results, we have
\[
q = -\frac{\tau_{1,0}^{(0)}}{\tau_{0,0}^{(0)}}, \quad r = \frac{\tau_{1,0}^{(1)}}{\tau_{0,0}^{(1)}}, \quad \bar{q} = -\frac{\tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)}}, \quad \bar{r} = \frac{\tau_{0,1}^{(0)}}{\tau_{0,0}^{(0)}}. \tag{2.12}
\]

### 2.2 Sato-Wilson equations and Zero-curvature equations

From now on, we consider the following level-0 realization of \( \hat{g} \):
\[
e_0^+ \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_1^- \mapsto \begin{bmatrix} 0 & 0 \\ \zeta & 0 \end{bmatrix},
\]
\[
e_0^- \mapsto \begin{bmatrix} 0 & \zeta^{-1} \\ 0 & 0 \end{bmatrix}, \quad e_1^+ \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.13}
\]
$c \mapsto 0$ and $d \mapsto \zeta \frac{d}{dc}$. Then the generators of the homogeneous Heisenberg subalgebra are realized by $H_n \mapsto \begin{bmatrix} \zeta^n & 0 \\ 0 & -\zeta^n \end{bmatrix}$. The elements of the affine Lie group $\hat{G}$ can be written by means of the following 2 × 2 matrices:

\begin{align*}
g_{<0}(t, \bar{t}) &= I + W_1(t, \bar{t}) \Lambda^{-1} + W_2(t, \bar{t}) \Lambda^{-2} + \cdots, \\
g_0(t, \bar{t}) &= \exp \Phi(t, \bar{t}), \\
g_{>0}(t, \bar{t}) &= I + \bar{W}_1(t, \bar{t}) \Lambda + \bar{W}_2(t, \bar{t}) \Lambda^2 + \cdots,
\end{align*}

where $\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\Phi(t, \bar{t}), W_i(t, \bar{t}), \bar{W}_i(t, \bar{t})$ ($i = 1, 2, \ldots$) are 2 × 2 diagonal matrices. Especially, we see that

\begin{align*}
W_1(t, \bar{t}) &= \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}, & \Phi(t, \bar{t}) &= \begin{bmatrix} \phi & 0 \\ 0 & -\phi \end{bmatrix} \quad \text{and} \quad W_1(t, \bar{t}) &= \begin{bmatrix} \bar{q} & 0 \\ 0 & \bar{r} \end{bmatrix},
\end{align*}

where

\begin{equation}
e^\phi = e^{\phi_{1\rightarrow 0}} = \frac{(1)}{0,0},
\end{equation}

By the defining relation (2.5), $g = g(t, \bar{t})$ satisfies the equation

\begin{equation}
\frac{\partial g}{\partial t_n} = H_n g, \quad \frac{\partial g}{\partial t_{-n}} = -g H_{-n}.
\end{equation}

Using the Gauss decomposition (2.7), $g_{<0} = g_{<0}(t, \bar{t})$ and $g_{>0} := g_0(t, \bar{t}) g_{>0}(t, \bar{t})$ satisfy the following linear equations, the so-called Sato-Wilson equations:

\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial g_{<0}}{\partial t_n} = -(g_{<0} H_n g_{<0})_{<0} g_{<0}, \\
\frac{\partial g_{>0}}{\partial t_n} = (g_{<0} H_n g_{<0})_{>0} g_{>0},
\end{array} \right.
\end{align*}

\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial g_{<0}}{\partial t_n} = (g_{>0} H_n g_{>0})_{<0} g_{<0}, \\
\frac{\partial g_{>0}}{\partial t_n} = -(g_{>0} H_n g_{>0})_{>0} g_{>0},
\end{array} \right.
\end{align*}

where $(\cdot)_{<0}$ and $(\cdot)_{>0}$ denote the negative and nonnegative power parts of $\Lambda$ (not of $\zeta$), i.e., for a matrix $A = \sum_{i \in \mathbb{Z}} A_i A^i$, we put $A_{<0} = \sum_{i < 0} A_i A^i$ and $A_{>0} = \sum_{i > 0} A_i A^i$. The equations (2.20) for $n = 1$ give the modified Pohlmeyer-Lund-Regge equations (1.7). Also, from the formulas (2.19) we obtain the relations

\begin{align*}
w_{21} &= -\frac{1}{2} \frac{\partial}{\partial t_1} \log \tau_{0,0}^{(1)}, & \bar{w}_{21} &= \frac{1}{2} \frac{\partial}{\partial t_1} \log \tau_{0,0}^{(1)}, \\
w_{22} &= \frac{1}{2} \frac{\partial}{\partial t_1} \log \tau_{0,0}^{(1)}, & \bar{w}_{22} &= \frac{1}{2} \frac{\partial}{\partial t_1} \log \tau_{0,0}.
\end{align*}

Here $W_2 := \text{diag}(w_{21}, w_{22})$ and $\bar{W}_2 := \text{diag}(\bar{w}_{21}, \bar{w}_{22})$. Notice that $H_n = \Lambda^{2n}$.

Next we will construct formal solutions to the linear equation (2.1). Let $\alpha, \beta \in \mathbb{C}$ be constants. We define the wave functions $\Psi(\zeta; \alpha, t, \bar{t})$ and $\hat{\Psi}(\zeta; \beta, t, \bar{t})$ by

\begin{align*}
\Psi(\zeta; \alpha, t, \bar{t}) &= g_{<0}^{\alpha H_0} \Psi_0(\zeta; t), \\
\hat{\Psi}(\zeta; \beta, t, \bar{t}) &= g_{>0}^{\beta H_0} \Psi_0(\zeta; \bar{t}),
\end{align*}

where

\begin{align*}
\Psi_0(\zeta; t) &= \exp \left( \sum_n t_n H_n \right) = \begin{bmatrix} e^{t_1 \zeta + t_2 \zeta^2 + \cdots} & 0 \\ 0 & e^{-(t_1 \zeta + t_2 \zeta^2 + \cdots)} \end{bmatrix}, \\
\hat{\Psi}_0(\zeta; \bar{t}) &= \exp \left( \sum_n t_n H_{-n} \right) = \begin{bmatrix} e^{t_1 \zeta^{-1} + t_2 \zeta^{-2} + \cdots} & 0 \\ 0 & e^{-(t_1 \zeta^{-1} + t_2 \zeta^{-2} + \cdots)} \end{bmatrix}.
\end{align*}
Equations of hierarchy (2.27), (2.26) contain an infinite number of dependent variables (components of $W_i$ and $\bar{W}_i$, $i = 1, 2, 3, \ldots$). However, we will show that the coefficients of $B_n$, $\bar{B}_n$ are differential polynomial of ($q$, $r$, $\bar{q}$, $\bar{r}$) with respect to $t_1$, $\bar{t}_1$ respectively. Set

$$L := g_{<0} H_0 g_{<0}^{-1} = H_0 + U_1 \Lambda^{-1} + U_2 \Lambda^{-2} + \cdots,$$

$$\bar{L} := g_{\geq0} H_0 g_{\geq0}^{-1} = g_0 (H_0 + \bar{U}_1 \Lambda + \bar{U}_2 \Lambda^2 + \cdots) g_0^{-1},$$

where $U_i = \begin{bmatrix} u_{i1} & 0 \\ 0 & u_{i2} \end{bmatrix}$, $\bar{U}_i = \begin{bmatrix} \bar{u}_{i1} & 0 \\ 0 & \bar{u}_{i2} \end{bmatrix}$. By the Sato-Wilson equations (2.20) we have the following relations:

$$U_{2n} = \frac{\partial \Phi}{\partial t_n}, \quad \bar{U}_{2n} = -\frac{\partial \Phi}{\partial \bar{t}_n},$$

and

$$U_{2n+1} = -\frac{\partial \bar{W}_1}{\partial t_n}, \quad \bar{U}_{2n+1} = -\frac{\partial \bar{W}_1}{\partial \bar{t}_n}.$$ 

The matrices $L$ and $\bar{L}$ satisfy the following system of equations (Lax equations):

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad \frac{\partial \bar{L}}{\partial t_n} = [\bar{B}_n, L], \quad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L].$$

2.3 Lax equations and a conserved density

It seems that the equations of hierarchy (2.27), (2.26) contain an infinite number of dependent variables (components of $W_i$ and $\bar{W}_i$, $i = 1, 2, 3, \ldots$). However, we will show that the coefficients of $B_n$, $\bar{B}_n$ are differential polynomial of ($q$, $r$, $\bar{q}$, $\bar{r}$) with respect to $t_1$, $\bar{t}_1$ respectively. Set

$$L := g_{<0} H_0 g_{<0}^{-1} = H_0 + U_1 \Lambda^{-1} + U_2 \Lambda^{-2} + \cdots,$$

$$\bar{L} := g_{\geq0} H_0 g_{\geq0}^{-1} = g_0 (H_0 + \bar{U}_1 \Lambda + \bar{U}_2 \Lambda^2 + \cdots) g_0^{-1},$$

where $U_i = \begin{bmatrix} u_{i1} & 0 \\ 0 & u_{i2} \end{bmatrix}$, $\bar{U}_i = \begin{bmatrix} \bar{u}_{i1} & 0 \\ 0 & \bar{u}_{i2} \end{bmatrix}$. By the Sato-Wilson equations (2.20) we have the following relations:

$$U_{2n} = \frac{\partial \Phi}{\partial t_n}, \quad \bar{U}_{2n} = -\frac{\partial \Phi}{\partial \bar{t}_n},$$

and

$$U_{2n+1} = -\frac{\partial \bar{W}_1}{\partial t_n}, \quad \bar{U}_{2n+1} = -\frac{\partial \bar{W}_1}{\partial \bar{t}_n}.$$ 

The matrices $L$ and $\bar{L}$ satisfy the following system of equations (Lax equations):

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad \frac{\partial \bar{L}}{\partial t_n} = [\bar{B}_n, L], \quad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L].$$

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Proposition 1. The components of $U_n$ (2.29) can be described by differential polynomials of $q$ and $r$ with respect to $t_1$, and $\bar{U}_n$ by differential polynomials of $\bar{q}$ and $\bar{r}$ with respect to $\bar{t}_1$.

Proof. By the definition of $L$ (2.29), $L^2 = g_{<0}H_0^2 g_{<0}^{-1} = I$ and hence

$$H_0U_j + U_j\Lambda^{-j}H_0\Lambda^j + \sum_{k=1}^{j-1} U_k\Lambda^{-k}U_{j-k}\Lambda^k = 0 \quad (j = 1, 2, \ldots). \quad (2.33)$$

On the other hand, the derivation of $L$ with respect to $t_1$ in (2.32) is expressed by

$$\frac{\partial L}{\partial t_1} = [L, (\zeta L)_{\leq 0}] = [L_{\geq -2}, (\zeta L)_{\leq 0}]
\quad = [H_0 + U_1\Lambda^{-1} + U_2\Lambda^{-2}, U_3\Lambda^{-3} + U_4\Lambda^{-4} + \cdots].$$

By comparing the coefficients of $\Lambda^{2k}$ and $\Lambda^{2k+1}$ we have

$$\frac{\partial U_{2k}}{\partial t_1} = [U_1\Lambda, U_{2k+1}\Lambda^{-1}],
\frac{\partial U_{2k+1}}{\partial t_1} \Lambda = [H_0, U_{2k+3}\Lambda] + [U_1\Lambda, U_{2k+2}] + [U_2, U_{2k+1}\Lambda]$$

(k = 1, 2, 3, \ldots). Together with the relation (2.33), the $U_j$ (j ≥ 2) are recursively determined as differential polynomials of $q$ and $r$ with respect to $t_1$. $\bar{L}$ can be treated in the same way.

We give some examples of $u_{n1}$ and $u_{n2}$ for small $n$.

$$\begin{align*}
  u_{31} &= -q', & u_{41} &= q'r - qr' - 2q^2 r^2, & u_{51} &= -\frac{q''}{2} + 2q^2 r' + 4q^3 r^2, \\
  u_{32} &= -r', & u_{42} &= -u_{41}, & u_{52} &= \frac{r''}{2} + 2q^2 r' - 4q^3 r^3, \\
  u_{61} &= \frac{1}{2}(q''r + qr'' - q'r') - 4q^3 r^3, & u_{71} &= -\frac{q'''}{4} + 3qq'r' + 6q^2 r^2 q', \\
  u_{62} &= -u_{61}, & u_{72} &= -\frac{r'''}{4} + 3qr'r' + 6q^2 r^2 r'.
\end{align*} \quad (2.35) \quad (2.36)$$

where $f'$ means $\frac{\partial f}{\partial t_1}$. By Proposition 1, the equations (2.31) can be regarded as nonlinear differential equations for the variables $(q, r)$ or $(\bar{q}, \bar{r})$. For example, we have

$$\begin{align*}
  \frac{\partial q}{\partial t_2} &= \frac{q''}{2} - 2q^2 r' - 4q^3 r^2, & \frac{\partial q}{\partial t_3} &= \frac{q'''}{4} - 3qq'r' - 6q^2 r^2 q', \\
  \frac{\partial r}{\partial t_2} &= \frac{r''}{2} - 2r^2 q' + 4q^2 r^3, & \frac{\partial r}{\partial t_3} &= \frac{r'''}{4} - 3qq'r' - 6q^2 r^2 r'.
\end{align*} \quad (2.37)$$

The first couple of equations is the derivative nonlinear Schrödinger equation we have studied in [10], [11] and the second one is a modification of the coupled modified KdV equation.

Other important functions are $W_2$ in (2.11) and $W_2$ in (2.10), which are written as in (2.24) by means of $\tau$-functions. They are not differential polynomials of $q, r$ or $\bar{q}, \bar{r}$. By the Sato-Wilson equations (2.20) they satisfy the relations

$$\begin{align*}
  \frac{\partial W_2}{\partial t_n} &= -U_{2n+2} - U_{2n+1}(\Lambda W_1\Lambda^{-1}), \\
  \frac{\partial W_2}{\partial t_n} &= U_{2n-2} + e^{2g} U_{2n-1}(\Lambda W_1\Lambda^{-1}) \quad (\bar{U}_0 := H_0), \quad (2.38)
\end{align*}$$

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and
\[\begin{align*}
\frac{\partial \tilde{W}_2}{\partial t_n} &= U_{2n-2} + e^{-2\Phi} U_{2n-1} (\Lambda \tilde{W}_1 \Lambda^{-1}) \quad (U_0 := H_0), \\
\frac{\partial W_2}{\partial t_n} &= -\tilde{U}_{2n+2} - \tilde{U}_{2n+1} (\Lambda \tilde{W}_1 \Lambda^{-1}).
\end{align*}\] (2.39)

So, the trivial identity \(\frac{\partial^2 W_2}{\partial t_n \partial t_m} = \frac{\partial^2 W_2}{\partial t_m \partial t_n}\) gives us the \(t_n\)-conserved density \(F_n = \frac{\partial W_2}{\partial t_n}\), i.e., for all \(m\) there exists a \(t_n\)-differential polynomial \(G_m\) such that \(\frac{\partial F_n}{\partial t_m} = \frac{\partial G_m}{\partial t_n}\). For instance,
\[\begin{align*}
\frac{\partial w_{21}}{\partial t_1} &= -2\hat{q}_r, \\
\frac{\partial w_{22}}{\partial t_1} &= 2\hat{q}_r,
\end{align*}\] (2.40)

Here we put
\[\hat{q} := \frac{1}{2} \frac{\partial q}{\partial t_1} - q^2 r, \quad \hat{r} := -\frac{1}{2} \frac{\partial r}{\partial t_1} - qr^2.\] (2.41)

### 2.4 Miura-type transformation to the AKNS hierarchy

Here we describe the Miura-type transformations of the mPLR hierarchy to the PLR hierarchy. Miura-type transformations are defined by
\[g_{<0} \mapsto \tilde{g}_{<0} := \begin{bmatrix} 1 & 0 \\ -r & 1 \end{bmatrix} g_{<0} + \cdots + \begin{bmatrix} w_{21} & q \\ \hat{r} & w_{22} - qr \end{bmatrix} \zeta^{-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\] (2.42)
\[g_0 \mapsto \tilde{g}_0 := \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} g_0 = \begin{bmatrix} \tilde{g} \\ c & d \end{bmatrix},\] (2.43)
\[g_{>0} \mapsto \tilde{g}_{>0} := \begin{bmatrix} 1 & -\hat{q} \\ 0 & 1 \end{bmatrix} g_{>0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} w_{21} - \hat{q} \hat{r} \\ \hat{r} & w_{22} \end{bmatrix} \zeta + \cdots.\] (2.44)

where \(\hat{r}\) is defined by (2.41) and we also put
\[\hat{q} := \frac{1}{2} \frac{\partial \tilde{q}}{\partial t_1} - \tilde{q}^2 \hat{r}, \quad \hat{r} := -\frac{1}{2} \frac{\partial \tilde{r}}{\partial t_1} - \tilde{q}\tilde{r}^2.\] (2.45)

We remark that the expressions of \(\hat{r}\) in (2.42) and \(\hat{q}\) in (2.44) are proved by using the relations (2.29) and (2.35). The variables \((a, b, c, d, q, \hat{r}, \hat{q}, \hat{r})\) obtained as above, satisfy the following relations:
\[\frac{\partial}{\partial t_1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -2\hat{q} \\ 2\hat{r} & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \frac{\partial}{\partial t_1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 2\hat{q} \\ -2\hat{r} & 0 \end{bmatrix},\] (2.46)
\[\frac{\partial q}{\partial t_1} = -2ab, \quad \frac{\partial \tilde{r}}{\partial t_1} = 2cd, \quad \frac{\partial \tilde{q}}{\partial t_1} = 2bd, \quad \frac{\partial \tilde{r}}{\partial t_1} = -2ac.\] (2.47)

This system of equations is the Pohlmeyer-Lund-Regge equation discussed in [8]. Furthermore, we have
\[\begin{align*}
\frac{\partial q}{\partial t_2} &= \frac{1}{2} \frac{\partial^2 q}{\partial t_1^2} + 4\hat{q}^2 \hat{r}, \\
\frac{\partial \tilde{r}}{\partial t_2} &= \frac{1}{2} \frac{\partial^2 \tilde{r}}{\partial t_1^2} - 4\hat{q}\tilde{r}^2,
\end{align*}\] (2.48)

They are the nonlinear Schrödinger equation and the coupled modified KdV equation.
The Miura-type transformation of the wave functions (2.22), i.e., \( Y = \tilde{g}_{<0} \zeta^n \Phi_0 \) or \( \tilde{g}_{>0} \zeta^b \tilde{\Phi}_0 \) satisfy the linear equations
\[
\frac{\partial Y}{\partial n} = \tilde{B}_n Y, \quad \frac{\partial Y}{\partial n} = \tilde{B}_n Y,
\]
where \( \tilde{B}_n = (\tilde{g}_{<0} H_n \tilde{g}_{<0})^{-1}, \tilde{B}_n = (\tilde{g}_{>0} H_{-n} \tilde{g}_{>0})^{-1} \tilde{g}_{<0} \). For example,
\[
\tilde{B}_1 = \begin{bmatrix} 0 & -2q \\ 2\bar{r} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \zeta, \\
\tilde{\bar{B}}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \zeta^{-1} & 0 \\ 0 & -\zeta^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}.
\]
Compatibility conditions for these equations give the AKNS hierarchy.

### 3 Action of the extended affine Weyl groups of type \( A_1^{(1)} \)

#### 3.1 Basic transformations

We now discuss symmetries in the mPLR hierarchy. In this section we construct six “basic” transformations on the variables \( g_{<0}, g_0 \) and \( g_{>0} \). First, we define
\[
\sigma : (t, \bar{t}, W_i(t, \bar{t}), \bar{W}_i(t, \bar{t}), \Phi(t, \bar{t})) \mapsto (\bar{t}, t, W_i(t, \bar{t}), \bar{W}_i(t, \bar{t}), -\Phi(t, \bar{t})),
\]
(\( i = 1, 2, \ldots \)). The correspondence of the coefficients is the following:
\[
\sigma(q)(t, \bar{t}) = \bar{q}(\bar{t}, t), \quad \sigma(r)(t, \bar{t}) = \bar{r}(\bar{t}, t), \quad \sigma(\bar{q})(t, \bar{t}) = q(\bar{t}, t), \quad \sigma(\bar{r})(t, \bar{t}) = r(\bar{t}, t).
\]

Second, we consider the birational transformations for the variables \( \phi, q, r, \bar{q} \) and \( \bar{r} \). In ref. [10], we have constructed two types of extended affine Weyl group symmetries (the “left action” and the “right action”) for the derivative NLS hierarchy. Here we extend these to the mPLR hierarchy. We denote the matrices \( S_0, S_1, S_1 \) by
\[
S_0 := \begin{bmatrix} 0 & \zeta^{-1} \\ -\zeta & 0 \end{bmatrix}, \quad S_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]
(3.3) satisfy the relation \( S_0^3 = S_1^4 = I \). Define the “left action” of operators \( S_0, S_1 \) for \( g_{>0}, g_0 \) and \( g_{<0} \) as follows:
\[
s_i^L(g_{<0})(t, \bar{t}) = G_i^L(-t, \bar{t})g_{<0}(-t, \bar{t})S_i, \\
s_i^L(g_0)(t, \bar{t}) = G_i^L(-t, \bar{t})g_0(-t, \bar{t})G_i^L(-t, \bar{t})^{-1}, \\
s_i^L(g_{>0})(t, \bar{t}) = G_i^L(-t, \bar{t})g_{>0}(-t, \bar{t}),
\]
(3.4) where
\[
G_0^L := \begin{bmatrix} -1/q & 0 \\ \zeta & -q \end{bmatrix}, \quad G_1^L := \begin{bmatrix} -r & 1 \\ 0 & -1/\bar{r} \end{bmatrix},
\]
\[
G_0^L := \begin{bmatrix} 1 & 0 \\ -\zeta e^{2\phi}/q & 1 \end{bmatrix}, \quad G_1^L := \begin{bmatrix} 1/\bar{r}e^{2\phi} & 0 \\ 0 & 1 \end{bmatrix}.
\]
(3.5)

We also define the “right action” \( s_0^R, s_1^R \) by
\[
s_i^R(g_{<0})(t, \bar{t}) = G_i^R(t, \bar{t})g_{<0}(t, \bar{t}), \\
s_i^R(g_0)(t, \bar{t}) = G_i^R(t, \bar{t})g_0(t, \bar{t})G_i^R(t, \bar{t})^{-1}, \\
s_i^R(g_{>0})(t, \bar{t}) = G_i^R(t, \bar{t})g_{>0}(t, \bar{t})S_i,
\]
(3.6)
For example, we have

\begin{align}
G_0^R &= \begin{bmatrix} 1 & -\zeta^{-1}e^{2\phi/\bar{r}} \\ 0 & 1 \end{bmatrix}, & G_1^R &= \begin{bmatrix} 1 & 0 \\ -1/\bar{q}e^{2\phi} & 1 \end{bmatrix}, \\
\bar{G}_0^R &= \begin{bmatrix} \bar{r} & -\zeta^{-1} \\ 0 & 1/\bar{r} \end{bmatrix}, & \bar{G}_1^R &= \begin{bmatrix} 1/\bar{q} & 0 \\ -1 & \bar{q} \end{bmatrix}.
\end{align}

(3.7)

For example, we have

\begin{align}
s_0^L(e^\phi)(t, \bar{t}) &= -\frac{e^{\phi(-t, \bar{t})}}{q(-t, \bar{t})}, & s_0^R(e^\phi)(t, \bar{t}) &= -e^{\phi(-t, \bar{t})}r(-t, \bar{t}), \\
s_1^L(e^\phi)(t, \bar{t}) &= \frac{e^{\phi(-t, \bar{t})}}{r(t, -\bar{t})}, & s_1^R(e^\phi)(t, \bar{t}) &= e^{\phi(t, -\bar{t})}\bar{q}(t, -\bar{t}).
\end{align}

(3.8)

(3.9)

Finally, we construct the Dynkin diagram automorphism \(\pi\) by

\[\pi(g_{<0})(t, \bar{t}) = P^{-1}g_{<0}(-t, -\bar{t})P,\]
\[\pi(g_0)(t, \bar{t}) = P^{-1}g_0(-t, -\bar{t})P,\]
\[\pi(g_{>0})(t, \bar{t}) = P^{-1}g_{>0}(-t, -\bar{t})P,\]

where

\[P = \begin{bmatrix} 0 & \zeta^{-1/2} \\ -\zeta^{1/2} & 0 \end{bmatrix}.\]

This action was also introduced in ref. [10]. The relation of the independent variables are \(\pi(e^\phi(t, \bar{t})) = e^{-\phi(-t, -\bar{t})}\) and

\[
\begin{align*}
\pi(q)(t, \bar{t}) &= -r(-t, -\bar{t}), & \pi(q)(t, \bar{t}) &= -\bar{r}(-t, -\bar{t}), \\
\pi(r)(t, \bar{t}) &= -q(-t, -\bar{t}), & \pi(r)(t, \bar{t}) &= -\bar{q}(-t, -\bar{t}).
\end{align*}
\]

(3.11)

**Theorem 1.** The transformations (3.4), (3.6), (3.8) and (3.10) commute with the Sato-Wilson equations (2.20). For the variables \(q, r, \bar{q}\) and \(\bar{r}\), they satisfy the relations

\[
\sigma^2 = (s_0^L)^2 = (s_1^L)^2 = (s_0^R)^2 = (s_1^R)^2 = \pi^2 = \text{Id},
\]

\[
\pi s_0^L = s_1^L \pi, & \quad \pi s_0^R = s_1^R \pi, \\
\sigma s_0^L = s_1^R \sigma, & \quad \sigma s_1^L = s_0^R \sigma.
\]

(3.12)

and for the variable \(\phi\),

\[
\sigma^2 = (s_0^L)^4 = (s_1^L)^4 = (s_0^R)^4 = (s_1^R)^4 = \pi^2 = \text{Id},
\]

\[
\pi s_0^L = -s_1^L \pi, & \quad \pi s_0^R = -s_1^R \pi, \\
\sigma s_0^L = -s_1^R \sigma, & \quad \sigma s_1^L = -s_0^R \sigma.
\]

(3.13)

Especially, the group \(\bar{W}_L = \langle s_0^L, s_1^L, \pi \rangle\) and \(\bar{W}_R = \langle s_0^R, s_1^R, \pi \rangle\) generated by these transformations on \(q, r, \bar{q}, \bar{r}\) are extended affine Weyl groups of type \(A_1^{(1)}\).

**Proof.** This is checked by a direct calculation. For example, by the relation \(S_1^2 = -I\), we have

\[
(s_1^L)^2(g_{<0})(t, \bar{t}) = (-I)g_{<0}(t, \bar{t})(-I) = g_{<0}(t, \bar{t}),
\]

\[
(s_1^L)^2(g_{>0})(t, \bar{t}) = (-I)g_{>0}(t, \bar{t}),
\]

and \((s_1^L)^2(g_{>0})(t, \bar{t}) = g_{>0}(t, \bar{t}). \]
3.2 Relations among the \( \tau \)-functions on the \( A_1^{(1)} \)-root lattice

By Theorem 11 we automatically obtain discrete integrable systems by composing the basic transformations given in previous section. First we consider the compositions \( T_L := s_0^L s_1^L \) and \( T_R := s_0^R s_1^R \). They correspond to parallel transformations on the \( A_1^{(1)} \)-root lattice.

The functions \( \tau_{m,n}^{(i)}(t, \bar{t}) \) \( (i = 0, 1, m, n \in \mathbb{Z}) \) defined by (2.6) are realized as

\[
\tau_{m,n}^{(i)}(t, \bar{t}) = T_L^{m} T_R^{n} \tau_{0,0}^{(i)}(t, \bar{t}).
\]

(3.14)

Because the transformations \( T_L^k \) and \( T_R^k \) act on the left and right index respectively, e.g.

\[
T_L^1(\tau_{m,n}^{(i)}) = \tau_{m+1,n}^{(i)}, \quad T_R^1(\tau_{m,n}^{(i)}) = \tau_{m,n+1}^{(i)},
\]

we obtain a sequence of \( \tau \)-functions.

\[
\begin{array}{c|c|c|c}
 & T_L & T_R & T_L \\\n\hline
\tau_{-1,1}^{(i)} & \tau_{0,1}^{(i)} & \tau_{1,1}^{(i)} & \tau_{0,1}^{(i)} \\
\hline
T_L & T_R & T_L \end{array}
\]

(3.15)

Proposition 2. The actions of the six basic transformations \( \sigma, s_0^L, s_1^L, s_0^R, s_1^R \) and \( \pi \) on the \( \tau \)-functions \( \tau_{m,n} \) \( (m, n \in \mathbb{Z}) \) are given by the following table:

|       | \( \tau_{m,n}^{(0)}(t, \bar{t}) \) | \( \tau_{m,n}^{(1)}(t, \bar{t}) \) |
|-------|----------------------------------|----------------------------------|
| \( \sigma \) | \( \tau_{-n,-m}^{(1)}(t, \bar{t}) \) | \( \tau_{-n,-m}^{(0)}(t, \bar{t}) \) |
| \( s_0^L \) | \(-1)^m \tau_{-m,n}^{(0)}(-t, \bar{t}) \) | \(-1)^{m+1} \tau_{-m,-n}^{(1)}(-t, \bar{t}) \) |
| \( s_1^L \) | \(-1)^m \tau_{-m,n}^{(0)}(-t, \bar{t}) \) | \(-1)^{m+1} \tau_{-m,-n}^{(1)}(-t, \bar{t}) \) |
| \( s_0^R \) | \(-1)^n \tau_{-m,n}^{(0)}(t, -\bar{t}) \) | \(-1)^n \tau_{-m,-n}^{(1)}(t, -\bar{t}) \) |
| \( s_1^R \) | \(-1)^n \tau_{-m,n}^{(0)}(t, -\bar{t}) \) | \(-1)^{n+1} \tau_{-m,-n}^{(1)}(t, -\bar{t}) \) |
| \( \pi \) | \( \tau_{-m,n}^{(1)}(-t, -\bar{t}) \) | \( \tau_{-m,n}^{(0)}(-t, -\bar{t}) \) |

(3.16)

Proof. We prove only the case of \( s_1^R \). The corresponding equations for the other elements are obtained similarly. When \( n \geq 0 \), the action of \( s_1^R \) on \( \exp(\phi_{m,n}) := T_L^m T_R^n \exp(\phi_{0,0}) = \tau_{m,n}^{(0)} / \tau_{m,n}^{(0)} \) is

\[
s_1^R(\exp(\phi_{m,n})) = s_1^R \left( s_0^R s_1^R \right)^n (\exp(\phi_{m,n})) = \left( s_1^R s_0^R \right)^n s_1^R(\exp(\phi_{m,n})) = s_1^R(\exp(\phi_{m-n,-n})).
\]

Due to the relation (3.16) we have

\[
s_1^R(\exp(\phi_{m-n,-n})) = (\tau_{m,n}^{(1)} / \tau_{m,n}^{(0)}) \times (-\tau_{m-n-1,-n}^{(1)}/\tau_{m-n}^{(0)}) = -\tau_{m-n-1,n}^{(1)}/\tau_{m-n}^{(0)}.
\]
For $n < 0$, the action of $s_1^R$ on $\exp(\phi_{m,n})$ is given by

$$
\begin{align*}
s_1^R(\exp(\phi_{m,n})) &= s_1^R \left(s_1^R s_0^R\right)^{-n} (\exp(\phi_{m,0})) \\
&= - \left(s_0^R s_1^R\right)^{-n-1} s_0^R (\exp(\phi_{m,0})) = - s_0^R (\exp(\phi_{m,-n-1})).
\end{align*}
$$

By the relation (3.3), we obtain $-s_0^R (\exp(\phi_{m,-n-1})) = -\tau_{m,-n-1}^{(1)}/\tau_{m,-n}^{(0)}$. The signs of the transformed $\tau$-function is determined uniquely by supposing $s_1^R(\tau_{m,0}^{(0)}) = \tau_{m,0}^{(0)}$.

Now we represent the variables (2.41) and (2.45), obtained by the Miura transformations, in terms of the $\tau$-functions. By definition, the actions of $T_L$ and $T_R$ on $g_{<0}$ (2.14) are

$$
\begin{align*}
T_L(g_{<0})(t,\bar{t}) &= R_L(t,\bar{t})g_{<0}(t,\bar{t})S_0 S_1, \\
T_R(g_{<0})(t,\bar{t}) &= R_R(t,\bar{t})g_{<0}(t,\bar{t}),
\end{align*}
$$

where

$$
\begin{align*}
R_L &= \left[\begin{array}{cc}
-\bar{q}/q & -q/q \\
0 & -\bar{q}/\bar{q}
\end{array}\right] + \left[\begin{array}{cc}
1 & 0 \\
1/q & 0
\end{array}\right] \hat{\zeta}, \\
R_R &= \left[\begin{array}{cc}
0 & -e^{2\phi}/\bar{r} \\
0 & -r/\bar{r}
\end{array}\right] \hat{\zeta}^{-1} + \left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right].
\end{align*}
$$

Notice that in these transformations the signs of the dependent variables $t$ and $\bar{t}$ do not change. So, by the relations (2.28) the action $T_L$ can be realized as

$$
\begin{align*}
\frac{\partial}{\partial t_n} - T_L(B_n) &= R_L \left(\frac{\partial}{\partial t_n} - B_n\right) \cdot R_L^{-1} \\
&= \frac{\partial}{\partial t_n} - R_L B_n R_L^{-1} - \frac{\partial R_L}{\partial t_n} R_L^{-1}
\end{align*}
$$

and by replacing $R_L$ to $R_R$, we get $T_R$. For example, the left action is obtained by

$$
T_L(q) = \frac{q''}{4} - \frac{(q')^2}{4q} - \frac{q^2 r'}{2} - q^3 r^2, \quad T_L(r) = \frac{1}{\bar{q}}.
$$

These relations provide the following formulas:

$$
\begin{align*}
\hat{q} &= \frac{\tau_{1,0}^{(1)}}{\tau_{0,0}^{(1)}}, \quad \hat{\bar{q}} = \frac{\tau_{-1,0}^{(0)}}{\tau_{0,0}^{(0)}}, \quad \hat{\bar{q}} = \frac{\tau_{0,-1}^{(0)}}{\tau_{0,0}^{(0)}}, \quad \hat{\bar{r}} = \frac{\tau_{1,0}^{(1)}}{\tau_{0,0}^{(1)}}.
\end{align*}
$$

We now deduce the differential equations that are satisfied by the $\tau$-functions $\tau_{m,n}^{(i)}$. By using the correspondence (2.12), (2.21), (3.20) and the action of $T_L^m$, $T_R^n$, the equations (2.40) can be rewritten as

$$
\begin{align*}
D_{t_1}^2 \tau_{m,n}^{(i)} \cdot \tau_{m+1,n}^{(i)} &= 8 \tau_{m+1,n}^{(i)} \cdot \tau_{m-1,n}^{(i)}, \\
D_{t_1} D_{t_2} \tau_{m,n}^{(i)} \cdot \tau_{m+1,n}^{(i)} &= 4 D_{t_1} \tau_{m+1,n} \cdot \tau_{m-1,n}^{(i)}.
\end{align*}
$$

Here $D_x$ is the Hirota bilinear operator defined for a pair of functions $f$, $g$ and for an arbitrary polynomial $Q$ as follows:

$$
Q(D_x) f(x) \cdot g(x) = Q(\frac{\partial}{\partial y})(f(x + y)g(x - y))|_{y=0}.
$$

Furthermore, we obtain

$$
\begin{align*}
(D_{t_1}^2 + 2 D_{t_2}) \tau_{m,n}^{(i)} \cdot \tau_{m+1,n}^{(i)} &= 0, \\
(D_{t_1}^3 - 4 D_{t_2}) \tau_{m,n}^{(i)} \cdot \tau_{m+1,n}^{(i)} &= 0
\end{align*}
$$

from the equations (2.31). Thanks to the symmetry $\sigma$ (3.1), we have similar equations with respect to $\bar{t}$ and the right index $n$. For example, we have

$$
D_{t_1}^2 \tau_{m,n}^{(i)} \cdot \tau_{m,n}^{(i)} = 8 \tau_{m,n+1}^{(i)} \tau_{m,n-1}^{(i)}.
$$

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3.3 Relations among the $\tau$-functions on the $A_1^{(1)}$-weight lattice

Next we consider the compositions $T_1 := s_0^R s_0^L \pi$ and $T_2 := s_1^R s_1^L \pi$. They correspond to shift operators on the $A_1^{(1)}$-weight lattice \cite{[14]}. By definition,

$$
\begin{align*}
T_1(\tau_{m,n}^{(0)})(t, \bar{t}) &= (-1)^{m+n} \tau_{m,n}^{(1)}(t, \bar{t}), \\
T_1(\tau_{m,n}^{(1)})(t, \bar{t}) &= (-1)^{m+n} \tau_{m+1,n+1}^{(0)}(t, \bar{t}), \\
T_2(\tau_{m,n}^{(0)})(t, \bar{t}) &= (-1)^{m+n-1} \tau_{m,n-1}^{(1)}(t, \bar{t}), \\
T_2(\tau_{m,n}^{(1)})(t, \bar{t}) &= (-1)^{m+n} \tau_{m+1,n}^{(0)}(t, \bar{t})
\end{align*}
$$

(3.25)

(3.26)

hold. So we arrange the family of $\tau$-functions on the following diagram:

\[
\begin{array}{c}
\tau_{-1,1}^{(0)} \quad \rightarrow \quad \tau_{0,1}^{(0)} \quad \rightarrow \quad \tau_{1,1}^{(0)} \\
\tau_{-1,0}^{(0)} \quad \rightarrow \quad \tau_{0,0}^{(0)} \quad \rightarrow \quad \tau_{1,0}^{(0)} \\
\tau_{-1,-1}^{(0)} \quad \rightarrow \quad \tau_{0,-1}^{(0)} \quad \rightarrow \quad \tau_{1,-1}^{(0)}
\end{array}
\]

(3.27)

The structure of the transformations \cite{[3.10]} is thought of as the mirror image of this diagram. In fact, $s_0^R$ and $s_1^L$ are reflections with respect to the vertical lines which link $\tau_{0,0}^{(1)}$ and $\tau_{0,-1}^{(1)}$, $\tau_{0,0}^{(0)}$ and $\tau_{0,1}^{(0)}$ respectively. Also, $s_0^R$ and $s_1^L$ are reflections with respect to the horizontal lines which link $\tau_{0,0}^{(1)}$ and $\tau_{-1,0}^{(1)}$, $\tau_{0,0}^{(0)}$ and $\tau_{0,0}^{(0)}$ respectively. Furthermore, the composition $\pi \sigma$ represents the reflection with respect to the slanted line through $\tau_{0,0}^{(0)}$ and $\tau_{0,0}^{(0)}$.

Below we list the Hirota bilinear equations of mPLR hierarchy. From the differential equations for $\phi$ \cite{[2.30]} we get

$$
D_{t_1} \tau_{m,n}^{(0)} \cdot \tau_{m,n}^{(1)} = 2 \tau_{m-1,n}^{(1)} \tau_{m+1,n}^{(1)},
$$

(3.28)

and

$$
(D_{t_1}^2 + 2D_{t_2}) \tau_{m,n}^{(0)} \cdot \tau_{m,n}^{(1)} = 0,
$$

(3.29)

$$
(D_{t_1}^3 - 4D_{t_2}) \tau_{m,n}^{(1)} \cdot \tau_{m,n}^{(0)} = 24 \tau_{m-1,n}^{(0)} \tau_{m+1,n}^{(1)}.
$$

The differential equations for $q, r, \bar{q}, \bar{r}$ \cite{[2.31]} imply

$$
D_{t_1} \tau_{m,n+1}^{(0)} \cdot \tau_{m,n}^{(0)} = 2 \tau_{m-1,n}^{(1)} \tau_{m+1,n}^{(1)},
$$

(3.30)

and the differential equations for $W_2$ and $\bar{W}_2$ \cite{[2.35]} give rise to

$$
(D_{t_1} D_{t_2} + 4) \tau_{m,n}^{(0)} \cdot \tau_{m,n}^{(0)} = -8 \tau_{m-1,n}^{(1)} \tau_{m,n-1}^{(1)}.
$$

(3.31)

This relation is the 2D Toda equation.
We also have an algebraic equation. The transformation $T_{l}$ for $e^{\phi}$ is

$$T_{l}(e^{\phi}) = s_{0}^{R}s_{0}^{L}(e^{\phi}) = e^{\phi} - qR e^{-\phi}.$$  

Rewriting this as a relation among $\tau$-functions, we have

$$(\tau^{(0)}_{m,n})^{2} = \tau^{(1)}_{m,n} \tau^{(1)}_{m-1,n} - \tau^{(1)}_{m-1,n} \tau^{(1)}_{m,n-1}. \quad (3.32)$$

We remark that the action of operators $T_{i}, T_{2}$ on the linear operators of mPLR hierarchy are obtained by

$$\frac{\partial}{\partial t_{n}} - T_{i}(B_{n}) = R_{i} \left( \frac{\partial}{\partial t_{n}} - B_{n} \right) R_{i}^{-1},$$

where

$$R_{1} = \begin{bmatrix} 0 & -q \epsilon^{2\phi}/(q\bar{\epsilon} - e^{2\phi}) \\ 0 & \bar{\epsilon} \end{bmatrix} \left[ \zeta^{-1/2} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \zeta^{1/2}, \quad (3.33)$$

$$R_{2} = \begin{bmatrix} q/e^{2\phi} \bar{\epsilon} \bar{q}/q \bar{\epsilon} & -q \epsilon^{2\phi}/(q\bar{\epsilon} - e^{2\phi}) \\ 0 & 0 \end{bmatrix} \left[ \zeta^{-1/2} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \zeta^{1/2}. \quad (3.34)$$

### 4 Similarity reduction

#### 4.1 Similarity conditions for the wave functions

Now we construct a similarity reduction of the mPLR hierarchy. Let $\lambda \in \mathbb{C}$ be a complex parameter and set $t_{\lambda} = (\lambda t_{1}, \lambda^{2} t_{2}, \ldots, \lambda^{n} t_{n}, \ldots), \bar{t} \bar{t} = (\lambda^{-1} t_{1}, \lambda^{-2} t_{2}, \ldots, \lambda^{-n} t_{n}, \ldots)$. We consider a one-parameter transformation for $g_{>0}, g_{<0}$ and $g_{0}$.

**Proposition 3.** If $g_{<0}$ and $g_{>0} = g_{0}g_{>0}$ solve the Sato-Wilson equations (2.20), the functions

$$g_{<0}(\zeta; t, \bar{t}) := \lambda^{2H_{0}} g_{<0}(\lambda^{-1}\zeta; t_{\lambda}, \bar{t}_{\lambda^{-1}}) \lambda^{-\alpha H_{0}},$$

$$g_{0}(t, \bar{t}) := \lambda^{\alpha H_{0}} g_{0}(t_{\lambda}, \bar{t}_{\lambda}) \lambda^{-\beta H_{0}},$$

$$g_{>0}(\zeta; t, \bar{t}) := \lambda^{\beta H_{0}} g_{>0}(\lambda^{-1}\zeta; t_{\lambda}, \bar{t}_{\lambda^{-1}}) \lambda^{-\beta H_{0}} \quad (4.1)$$

also solve (2.20).

This is verified by direct calculation. Proposition 3 shows that $g_{<0}, g_{0}$ and $g_{>0}$ have the scaling symmetry, it thus makes sense to look only at functions with the following similarity conditions:

$$g_{<0}(\zeta, t, \bar{t}) = g_{<0}(\zeta, t, \bar{t}), \quad g_{0}(\zeta, t, \bar{t}) = g_{0}(\zeta, t, \bar{t}), \quad g_{>0}(\zeta, t, \bar{t}) = g_{>0}(\zeta, t, \bar{t}). \quad (4.2)$$

The components of these matrices satisfy, for example, $e^{\phi(t, \bar{t})} = \lambda^{2\alpha} e^{\phi(t, \bar{t})},$ and

$$q(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda^{2\alpha-1} q(t, \bar{t}), \quad w_{21}(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda^{-1} w_{21}(t, \bar{t}),$$

$$r(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda^{2\alpha} r(t, \bar{t}), \quad w_{22}(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda^{-1} w_{22}(t, \bar{t}),$$

$$\bar{q}(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda^{-2\beta} \bar{q}(t, \bar{t}), \quad \bar{w}_{21}(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda \bar{w}_{21}(t, \bar{t}),$$

$$\bar{r}(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda^{2\beta+1} \bar{r}(t, \bar{t}), \quad \bar{w}_{22}(t_{\lambda}, \bar{t}_{\lambda^{-1}}) = \lambda \bar{w}_{22}(t, \bar{t}).$$

The similarity conditions (4.2) can be expressed in infinitesimal form:

$$\zeta \frac{\partial g_{<0}}{\partial \zeta} = [\alpha H_{0}, g_{<0}] + \sum_{n} n t_{n} \frac{\partial g_{<0}}{\partial t_{n}} - \sum_{n} n \bar{t}_{n} \frac{\partial g_{<0}}{\partial \bar{t}_{n}}, \quad (4.3)$$

$$0 = (\alpha H_{0}) g_{0} + \sum_{n} n t_{n} \frac{\partial g_{0}}{\partial t_{n}} - \sum_{n} n \bar{t}_{n} \frac{\partial g_{0}}{\partial \bar{t}_{n}} - g_{0}(\beta H_{0}), \quad (4.4)$$

$$\zeta \frac{\partial g_{>0}}{\partial \zeta} = [\beta H_{0}, g_{>0}] + \sum_{n} n t_{n} \frac{\partial g_{>0}}{\partial t_{n}} - \sum_{n} n \bar{t}_{n} \frac{\partial g_{>0}}{\partial \bar{t}_{n}} \quad (4.5)$$
In particular, the variables $W_1$, $\tilde{W}_1$, $W_2$ and $\tilde{W}_2$ satisfy the following conditions by virtue of the Sato-Wilson equations:

$$
\begin{bmatrix}
2\alpha + 1 & 0 \\
0 & -2\alpha
\end{bmatrix}
W_1 = \sum_n nt_n U_{2n+1} + \sum_n n\tilde{t}_n e^{2\phi} \tilde{U}_{2n-1},
$$

(4.6)

$$
\begin{bmatrix}
-2\beta & 0 \\
0 & 2\beta + 1
\end{bmatrix}
\tilde{W}_1 = \sum_n nt_n e^{-2\phi} U_{2n-1} + \sum n\tilde{t}_n \tilde{U}_{2n+1},
$$

$$
W_2 = \sum_n nt_n (U_{2n+2} + U_{2n+1} \Lambda W_1 \Lambda^{-1}) + \sum_n n\tilde{t}_n (\tilde{U}_{2n-2} + e^{2\phi} \tilde{U}_{2n-1} \Lambda W_1 \Lambda^{-1}),
$$

(4.7)

$$
\tilde{W}_2 = \sum_n nt_n (U_{2n-2} + e^{-2\phi} U_{2n-1} \Lambda W_1 \Lambda^{-1}) + \sum_n n\tilde{t}_n (\tilde{U}_{2n+2} + \tilde{U}_{2n+1} \Lambda \tilde{W}_1 \Lambda^{-1}).
$$

We emphasize that the variables $W_2$ and $\tilde{W}_2$ can be described by differential polynomials of $q, r$ and $\tilde{q}, \tilde{r}$ (see Proposition 1) under the similarity conditions (4.2). Furthermore, from the condition for $g_0$ (4.4), we obtain

$$
\sum_n nt_n U_{2n} + \sum_n n\tilde{t}_n \tilde{U}_{2n} = (\beta - \alpha)H_0.
$$

(4.8)

Under the similarity conditions (4.2), the wave functions $Y = \Psi$ and $\tilde{Y}$ (2.22) satisfy

$$
Y(\lambda \zeta; t, \tilde{t}) = \lambda^{\alpha H_0} Y(\zeta; t_\lambda, \tilde{t}_{\lambda^{-1}}),
$$

(4.9)

because of the relations $\Psi_0(\lambda \zeta; t) = \Psi_0(\zeta; t_\lambda)$, $\tilde{\Psi}_0(\lambda \zeta; \tilde{t}) = \tilde{\Psi}_0(\zeta; \tilde{t}_{\lambda^{-1}})$. Formula (4.9) implies the following linear equation:

$$
\frac{\partial Y}{\partial \zeta} = \alpha H_0 Y + \sum n nt_n \frac{\partial Y}{\partial t_n} - \sum_n n\tilde{t}_n \frac{\partial Y}{\partial \tilde{t}_n}
$$

$$
= \left(\alpha H_0 + \sum_n nt_n B_n - \sum n\tilde{t}_n (g_0 \tilde{B}_n g_0^{-1})\right) Y
$$

(4.10)

for $Y = \Psi, \tilde{Y}$. From the compatibility condition of the linear equation (4.10) and (2.25), we get an isomonodromic deformation equation [4, 7]:

$$
\frac{\partial A}{\partial t_i} = [B_i, A] + \zeta \frac{\partial B}{\partial \zeta},
$$

$$
A = \alpha H_0 + \sum n nt_n B_n - \sum n\tilde{t}_n (g_0 \tilde{B}_n g_0^{-1}).
$$

(4.11)

### 4.2 Affine Weyl group symmetry of the reduction parameters

The affine Weyl group symmetry still exists after the similarity reduction. In addition, we have the following transformation for the parameters $\alpha$ and $\beta$ in the similarity conditions (4.1), (4.2).

**Theorem 2.** The six basic transformations $\sigma, s_1^L, s_1^R (i = 0, 1)$ and $\pi$ induce the following transformations of the parameters $\alpha, \beta$ in the similarity conditions (4.1), (4.2):

$$
\begin{array}{c|cc}
\sigma & \alpha & \beta \\
\hline
s_1^L & -\beta - \frac{\alpha}{2} & -\alpha - \frac{\beta}{2} \\
\sigma & -\alpha - 1 & \beta \\
\hline
s_1^R & -\alpha & \beta \\
\pi & \alpha & -\beta - 1 \\
\end{array}
$$

(4.12)
Proof. We rewrite the similarity condition (4.3) by
\[
\left[ \frac{\partial}{\partial \zeta} + \frac{1}{4} H_0, g_{<0}(t, \bar{t}) \right] = \left[ (\alpha + \frac{1}{4}) H_0, g_{<0}(t, \bar{t}) \right] \\
+ \sum_n n t_n \frac{\partial g_{<0}}{\partial t_n} (t, \bar{t}) - \sum_n n \bar{t}_n \frac{\partial g_{<0}}{\partial \bar{t}_n} (t, \bar{t}).
\] (4.13)

If we let \( \sigma \) act on the formula (4.13), then by using the relation \( \left[ \frac{\partial}{\partial \zeta} + \frac{1}{4} H_0, \Lambda^n \right] = \frac{\partial}{\partial \zeta} \Lambda^n \), we have
\[
\left[ -\frac{\zeta}{\partial \zeta} - \frac{1}{4} H_0, g_{>0}(t, \bar{t}) \right] = \left[ (\sigma(\alpha) + \frac{1}{4}) H_0, g_{>0}(t, \bar{t}) \right] \\
+ \sum_n n t_n \frac{\partial g_{>0}}{\partial t_n} (t, \bar{t}) - \sum_n n \bar{t}_n \frac{\partial g_{>0}}{\partial \bar{t}_n} (t, \bar{t}).
\]

Comparing this equation with (4.3), we have \( \sigma(\alpha) = -\beta - \frac{1}{2} \).

Next we consider the actions of \( s^L_0 \) and \( s^R_0 \). Since the generator of permutation \( S_0 = S_0(\zeta) \) satisfies \( S_0(\zeta) = \lambda^{-H_0} S_0(\lambda^{-1} \zeta) \), under the similarity conditions, the matrices in the definition of the left action (3.5) and the right action (3.7) satisfy
\[
\begin{align*}
G_0^L(\zeta; t, \bar{t}) &= \lambda^{-\alpha H_0} G_0^L(\lambda^{-1} \zeta; t, \bar{t}), \\
G_0^L(\zeta; t, \bar{t}) &= \lambda^{-\alpha H_0} G_0^L(\lambda^{-1} \zeta; t, \bar{t}), \\
G_1^L(t, \bar{t}) &= \lambda^{-\alpha H_0} G_1^L(t, \bar{t}), \\
G_1^L(t, \bar{t}) &= \lambda^{-\alpha H_0} G_1^L(t, \bar{t}), \\
G_0^R(\zeta; t, \bar{t}) &= \lambda^{-\beta H_0} G_0^L(\lambda^{-1} \zeta; t, \bar{t}), \\
G_0^R(\zeta; t, \bar{t}) &= \lambda^{-\beta H_0} G_0^L(\lambda^{-1} \zeta; t, \bar{t}), \\
G_1^R(t, \bar{t}) &= \lambda^{-\beta H_0} G_1^L(t, \bar{t}), \\
G_1^R(t, \bar{t}) &= \lambda^{-\beta H_0} G_1^L(t, \bar{t}).
\end{align*}
\]

Then, for example, the actions of \( s^L_0 \) on the matrices \( g_{<0}, g_0 \) and \( g_{>0} \) with the similarity condition (4.12) are
\[
\begin{align*}
\lambda^L_0(g_{<0})(\zeta; t, \bar{t}) &= \lambda^{-\alpha H_0} \lambda^L_0(g_{<0})(\lambda^{-1} \zeta; t, \bar{t}), \\
\lambda^L_0(g_0)(t, \bar{t}) &= \lambda^{-\alpha H_0} \lambda^L_0(g_0)(t, \bar{t}), \\
\lambda^L_0(g_{>0})(\zeta; t, \bar{t}) &= \lambda^{-\beta H_0} \lambda^L_0(g_{>0})(\lambda^{-1} \zeta; t, \bar{t}).
\end{align*}
\] (4.14)

This relation imposes the action of \( s^L_0 \) on \( \alpha \) and \( \beta \). Other transformations can be computed in the same way.

The action of \( \pi \) is obtained by the relation \( P(\lambda \zeta) = \lambda^{-\frac{1}{2} H_0} P(\zeta) = P(\zeta) \lambda^{\frac{1}{2} H_0} \).

**Corollary 1.** The shift operators on the \( A^{(1)}_1 \) root- and weight-lattice act on the parameters \( \alpha, \beta \) by the following table:

| \( s^L_0 \) | \( s^R_0 \) | \( T_L \) | \( T_R \) | \( T_1 \) | \( T_2 \) |
|---|---|---|---|---|---|
| \( s^L_0 \) | \( s^R_0 \) | \( s^L_0 \) | \( s^R_0 \) | \( s^L_0 \) | \( s^R_0 \) |
| \( \alpha \) | \( \beta \) | \( \alpha + 1 \) | \( \beta \) | \( \alpha + \frac{1}{2} \) | \( \beta + \frac{1}{2} \) | \( \alpha + \frac{1}{2} \) | \( \beta - \frac{1}{2} \) |
5 The third Painlevé equation

5.1 Deriving the third Painlevé equation

In this section, we use only the independent variables $t_1$, $\bar{t}_1$ and the dependent variables $\bar{q}, \bar{r}, e$. So from the mPLR equations (1.7) we eliminate $q, r, e$ and the variable $y$.

When we put $\bar{y}$ in (1.2) with the parameters $c_2 = 4(2\alpha + 1)$, $c_3 = 4$, $c_4 = -4$. We remark that if we express the Hamiltonian $h$ (5.6) in the variable of the mPLR hierarchy, we get

$$h = 2t_1 q^2 r^2 + 2oqr + 2t_1 (q^2 e^{-2\phi} + \bar{q}r e^{2\phi})$$

$$= -w_{21} + \bar{t}_1 = \frac{1}{2} \frac{\partial}{\partial \bar{t}_1} \log r_{00}^{(0)} + \bar{t}_1$$
by the similarity condition (4.7).

The system of equations (5.13) is obtained by the compatibility condition of the linear equations

$$\frac{\partial \Psi}{\partial \zeta} = A \Psi, \quad \frac{\partial \Psi}{\partial t_1} = B_1 \Psi,$$

with the coefficient matrices

$$A = -\alpha H_0 + t_1 B_1 - \bar{t}_1 \bar{B}_1$$

and

$$\begin{bmatrix} -t_1 & 2t_1 \bar{q} e^{2\phi} \\ 0 & \bar{t}_1 \end{bmatrix}^{-1} + \begin{bmatrix} -\alpha + 2t_1 q r & -2t_1 q \\ -2t_1 \bar{r} e^{-2\phi} & \alpha - 2t_1 q r \end{bmatrix} + \begin{bmatrix} t_1 & 0 \\ 2t_1 r & -t_1 \end{bmatrix} \zeta,$$

(5.8)

and $B_1 \tau_1$, see (4.14). We believe this is a new Lax formation of the third Painlevé equation. Notice that the relations

$$y = -2t_1 \frac{q}{\bar{q} e^{2\phi}}, \quad z = \bar{q} r e^{2\phi}, \quad yz = -2t_1 q r = 2t_1 \bar{q} r + \alpha - \beta$$

(5.9)

hold.

### 5.2 Bilinear equations

The canonical coordinates of the third Painlevé equation $y$ and $z$ are described in the tau function as follows:

$$y = t_1 \frac{\partial}{\partial t_1} \log \frac{\tau_0 (1)_{0,0,1}}{\tau_0 (1)_{0,0,1}}, \quad z = -\frac{\tau_0 (1)_{0,0,1}}{\tau_0 (1)_{0,0,1}},$$

(5.10)

**Proposition 4.** The four $\tau$-functions $\tau^{(0)}_{0,0}, \tau^{(1)}_{1,0}, \tau^{(1)}_{0,1}$, and $\tau^{(1)}_{0,-1}$ (see the diagram (3.27) in (5.10)) with the similarity condition satisfy the following relations:

$$t_1 D_t \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = 2t_1 \tau^{(1)}_{0,0} \tau^{(1)}_{0,1} + (2^\alpha + 1) \tau^{(0)}_{0,0} \tau^{(0)}_{1,0},$$

(5.11)

$$t_1^2 D_t \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = -2t_1 \tau^{(0)}_{0,0} \tau^{(0)}_{1,0} + 4(\alpha + 1) \bar{t}_1 \tau^{(1)}_{0,0} \tau^{(1)}_{1,0},$$

(5.12)

$$t_1 D_t \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = 2t_1 \tau^{(0)}_{0,0} \tau^{(0)}_{1,0},$$

(5.13)

$$t_1^2 D_t \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = -2t_1 \tau^{(0)}_{0,0} \tau^{(0)}_{1,0} + 2(2 \beta + 1) t_1 \tau^{(0)}_{0,0} \tau^{(0)}_{1,0}.$$  

(5.14)

**Proof.** The equation (5.13) is nothing but (3.30) (apply $T_2$). From the similarity condition (4.6) we have

$$t_1 \frac{\partial q}{\partial t_1} - \bar{t}_1 \frac{\partial q}{\partial t_1} = -(2\alpha + 1)q.$$  

(5.15)

Translating this into the $\tau$-functions, we get (5.11).

The equations (5.12) and (5.14) are obtained by differentiating the equation (5.16) with respect to $t_1$, together with the equation

$$2t_1 \tau^{(0)}_{1,0} \tau^{(1)}_{0,-1} + 2\bar{t}_1 \tau^{(0)}_{1,0} \tau^{(1)}_{0,1} = (\alpha - \beta) \tau^{(0)}_{0,0} \tau^{(0)}_{1,0},$$

(5.16)

which is obtained from the similarity condition for $e^\phi$ (4.8), and

$$\frac{D_{t_1}^2 \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0}}{(\tau^{(0)}_{0,0})^2} = \frac{(\tau^{(0)}_{0,0})^2}{t_1} \tau^{(0)}_{0,0} \tau^{(0)}_{1,0} - \frac{2 \bar{t}_1 \tau^{(1)}_{0,0} \tau^{(1)}_{0,1}}{t_1} - \frac{8 \bar{t}_1 \tau^{(1)}_{0,0} \tau^{(1)}_{1,0}}{t_1} - \frac{4 \bar{t}_1}{t_1},$$

(5.17)

which is obtained from the similarity condition for $W_2$ (4.4).
The bilinear form for the third Painlevé equation (5.7)

\[
\begin{aligned}
&\frac{t_1^2}{2} \left( \frac{D^2\tau_{0,0}^{(1)} \cdot \tau_{0,1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{1,0}^{(1)}} - \left( \frac{D\tau_{0,0}^{(1)} \cdot \tau_{0,1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{1,0}^{(1)}} \right)^2 - \frac{D^2\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}} + \left( \frac{D\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}} \right)^2 \right) \\
&\quad + t_1 \left( \frac{D\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}} + \frac{D\tau_{0,0}^{(1)} \cdot \tau_{0,1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,1}^{(1)}} \right) \\
&= 4t_1^2 \left( \frac{\tau_{0,0}^{(1)} \cdot \tau_{1,0}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{1,0}^{(1)}} \right)^2 - 4\beta t_1 \frac{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}} - 2(2\alpha + 1)t_1 \frac{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}} - 4t_1^2 \left( \frac{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}}{\tau_{0,0}^{(1)} \cdot \tau_{0,-1}^{(1)}} \right)^2 \\
\end{aligned}
\]

(5.18)
can be derived from Proposition \[4\].

### 5.3 Affine Weyl group symmetry of P\(_\text{III}'\)

The action of the six basic transformations on the parameters \(\alpha\) and \(\beta\) is given by \[11,12\]. We now consider the action on the variables \(y\) and \(z\) \[5.3\], \[5.9\]. The similarity condition discussed above leads to the following assumption for the solutions of the mPLR equation:

\[
\bar{q}(t_1, \bar{t}_1) = (2t_1)^{2\beta} \bar{Q}(\xi), \quad \bar{r}(t_1, \bar{t}_1) = (2\bar{t}_1)^{-2\beta - 1} \bar{R}(\xi),
\]

(5.19)

where \(\xi = 2t_1\bar{t}_1\). The functions \(\bar{Q}(\xi)\) and \(\bar{R}(\xi)\) satisfy

\[
\begin{aligned}
\frac{d^2 \bar{Q}}{d\xi^2} = \frac{\bar{Q}}{\xi} - \frac{2\alpha + 1}{\xi} \frac{d\bar{Q}}{d\xi} + 2 \left( \frac{d\bar{Q}}{d\xi} \right)^2 \frac{d\bar{R}}{d\xi}, \\
\frac{d^2 \bar{R}}{d\xi^2} = \frac{\bar{R}}{\xi} + 2\beta \frac{d\bar{R}}{d\xi} - 2 \left( \frac{d\bar{Q}}{d\xi} \right)^2 \frac{d\bar{R}}{d\xi}.
\end{aligned}
\]

(5.20)

Equations (5.20) are a rewriting of the equations (5.3). Then the variables \(y\) and \(z\) can be expressed

\[
y(t_1, \bar{t}_1) = \frac{2t_1 \bar{t}_1}{Q(2t_1\bar{t}_1)} \frac{d\bar{Q}}{d\xi}(2t_1\bar{t}_1), \quad z(t_1, \bar{t}_1) = \bar{Q}(2t_1\bar{t}_1) \frac{d\bar{R}}{d\xi}(2t_1\bar{t}_1).
\]

(5.21)

Hereafter we regard the variables \(y\) and \(z\) as a function of \(\xi = 2t_1t_2\).

The actions of \(\sigma\) and \(\pi\) are given by

\[
\begin{aligned}
\sigma(y) &= \frac{2\xi}{y}, \\
\sigma(z) &= \frac{y(\alpha - \beta - yz)}{2\xi}, \\
\pi(y) &= \frac{2\xi}{\beta - \alpha + yz}, \\
\pi(z) &= \frac{y(\alpha - \beta - yz)}{2\xi},
\end{aligned}
\]

(5.22)

because of the definition and the relations (5.9). We note that the composition \(\pi \sigma\) is simpler than \(\pi\):

\[
\begin{aligned}
\pi \sigma(y) &= y + \frac{\beta - \alpha}{z}, \\
\pi \sigma(z) &= z.
\end{aligned}
\]
The other transformations act on \( y \) and \( z \) as follows:

\[
\begin{align*}
\mathbf{s}^L_0(y)(\xi) &= y(-\xi), \\
\mathbf{s}^R_0(y)(\xi) &= y(-\xi) - \frac{2\alpha + 1}{y(-\xi)} + \frac{2\xi}{y^2(-\xi)}, \\
\mathbf{s}^L_1(y)(\xi) &= \frac{2\xi z(-\xi)(1 - z(-\xi))}{\beta - \alpha + 2\alpha z(-\xi) - y(-\xi)z(-\xi) - y(-\xi)z^2(-\xi)}, \\
\mathbf{s}^R_1(y)(\xi) &= 1 - z(-\xi), \\
\mathbf{s}^R_0(z)(\xi) &= \frac{2\xi y(-\xi)(\beta - \alpha + y(-\xi)z(-\xi))}{\beta - \alpha + y(-\xi)z(-\xi) - (2\beta + 1) + \frac{2\xi}{\beta - \alpha + y(-\xi)z(-\xi)}}, \\
\mathbf{s}^R_1(z)(\xi) &= \frac{(2\beta + 1)z(-\xi)}{\beta - \alpha + y(-\xi)z(-\xi)} + 2\xi \left( \frac{z(-\xi)}{\beta - \alpha + y(-\xi)z(-\xi)} \right)^2, \\
\mathbf{s}^R_1(y)(\xi) &= -y(-\xi), \\
\mathbf{s}^R_1(z)(\xi) &= 1 - z(-\xi).
\end{align*}
\]

These are shown by the relation (5.9) and the similarity conditions. For instance,

\[
s^L_0(r) = -\dot{q} = \frac{1}{2} \frac{\partial q}{\partial t_1} - q^2 r = q^2 r + \frac{1}{t_1} \left( \alpha + \frac{1}{2} \right) q + \frac{t_1}{t_1} e^{2q}
\]

by the similarity condition (5.15).

Here we give a correspondence between the birational transformations discussed in [19] and in this paper. In [19], the \( A_1^{(1)} \) root systems \( \{\alpha_0, \alpha_1\} \) and \( \{\beta_0, \beta_1\} \) are expressed as

\[
(\alpha_0, \alpha_1) = (1 + \alpha - \beta, -\alpha + \beta), \quad (\beta_0, \beta_1) = (1 + \alpha + \beta, -\alpha - \beta).
\]

So the action of the extended affine Weyl group on these root systems are realized as follows:

| Transformation | \( \alpha_0 \) | \( \alpha_1 \) | \( \beta_0 \) | \( \beta_1 \) |
|---------------|----------------|----------------|----------------|----------------|
| \( s^L_0 s^R_0 \) | \( -\alpha_0 \) | \( \alpha_1 + 2\alpha_0 \) | \( \beta_0 \) | \( \beta_1 \) |
| \( \pi \sigma \) | \( \alpha_0 + 2\alpha_1 \) | \( -\alpha_1 \) | \( \beta_0 \) | \( \beta_1 \) |
| \( s^R_0 s^L_0 \pi \sigma \) | \( \alpha_0 \) | \( \alpha_1 \) | \( -\beta_0 \) | \( \beta_1 + 2\beta_0 \) |
| \( s^L_1 s^R_1 \pi \sigma \) | \( \alpha_0 \) | \( \alpha_1 \) | \( \beta_0 + 2\beta_1 \) | \( -\beta_0 \) |
| \( s^R_1 s^L_1 \pi \sigma \) | \( \alpha_1 \) | \( \alpha_0 \) | \( \beta_0 \) | \( \beta_1 \) |
| \( \sigma \) | \( \alpha_0 \) | \( \alpha_1 \) | \( \beta_0 \) | \( \beta_1 \) |
| \( s^L_1 \) | \( \beta_0 \) | \( \beta_1 \) | \( \alpha_0 \) | \( \alpha_1 \) |
| \( \pi \) | \( \beta_1 \) | \( \beta_0 \) | \( \alpha_1 \) | \( \alpha_0 \) |

The action on the variable \( y \) and \( z \) are also given by calculating these compositions.

### 6 The fourth and the second Painlevé equations

Finally, we discuss the relation between the mPLR hierarchy and the fourth and the second Painlevé equations.

#### 6.1 Painlevé IV

As we have shown in [10], if we use the variables \( t_1 \) and \( t_2 \), we can obtain the fourth Painlevé equation. Here we give additional information about \( \tau \)-functions and Hamiltonian functions. In
this case, the similarity conditions for $W_1$ and $g_0$ are
\[
\begin{align*}
t_1q + 2t_2 \left( \frac{q''}{2} - 2q^2r' - 4q^3r^2 \right) &= -(2\alpha + 1)q, \\
2t_1qr + 2t_2 \left( q'r - qr' - 2q^2r^2 \right) &= \beta - \alpha.
\end{align*}
\]
So, the bilinear equations of these relations are
\[
\begin{align*}
\left\{ \begin{array}{l}
(t_1D_1 - t_2D^2_{r_1} - 2\alpha - 1)\tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = 0, \\
(t_1D_1 - t_2D^2_{r_1} + \alpha - \beta)\tau^{(1)}_{0,0} \cdot \tau^{(1)}_{1,0} = 0.
\end{array} \right.
\end{align*}
\]
(6.1)
When we put $t_2 = 1/2$ and
\[
y = 2qr = -\frac{\tau^{(0)}_{1,0} \tau^{(1)}_{0,0}}{\tau^{(0)}_{0,0} \tau^{(1)}_{0,0}} = \frac{(\tau^{(1)}_{0,0})'}{\tau^{(0)}_{0,0}} - \frac{(\tau^{(0)}_{0,0})'}{\tau^{(0)}_{0,0}}, \\
z = 2t_1 + \frac{q'}{q} = 2t_1 + \frac{(\tau^{(1)}_{0,0})'}{\tau^{(0)}_{0,0}} - \frac{(\tau^{(0)}_{0,0})'}{\tau^{(0)}_{0,0}},
\]
we obtain
\[
\begin{align*}
y' &= y(2z - y - 2t_1) + 2(\alpha - \beta), \\
z' &= z(2y - z + 2t_1) - 4\beta.
\end{align*}
\]
This system of equations is a Hamiltonian system with polynomial Hamiltonian
\[
H = yz(z - 2t_1) + 2(\beta - \alpha)z + 4\beta y.
\]
By the similarity condition for $W_2$, this Hamiltonian function is expressed by
\[
H = 4t_1(\alpha - \beta) - 8w_2t.
\]

6.2 Painlevé II

We believe this result is new. There are two steps to get the second Painlevé equation. First, we consider the similarity condition with respect to $t_1$ and $t_3$. The $t_3$-derivatives of $\tilde{q}$ (not $q$) and $\phi$ are
\[
\begin{align*}
\frac{\partial \tilde{q}}{\partial t_3} &= e^{-2\phi}u_{51} = e^{-2\phi} \left( \frac{q''}{2} - 2q^2r' - 4q^3r^2 \right), \\
\frac{\partial \phi}{\partial t_3} &= u_{61} = \frac{1}{2}(q''r + qr'' - q'r') - 4q^3r^3.
\end{align*}
\]
So, the similarity conditions for $\tilde{W}_1$ and $g_0$ are
\[
\begin{align*}
t_1\frac{\partial \tilde{q}}{\partial t_1} + 3t_3\frac{\partial \tilde{q}}{\partial t_3} &= -e^{-2\phi} \left( 2t_1q + 3t_3 \left( \frac{q''}{2} - 2q^2r' - 4q^3r^2 \right) \right) = -2\beta \tilde{q}, \\
t_1\frac{\partial \phi}{\partial t_1} + 3t_3\frac{\partial \phi}{\partial t_3} &= 2t_1qr + 3t_3 \left( \frac{1}{2}(q''r + qr'' - q'r') - 4q^3r^3 \right) = \beta - \alpha.
\end{align*}
\]
(6.2)
(6.3)
Hence, if we put $t_3 = 1/3$ and
\[
\begin{align*}
y &= -\frac{q'}{2q} = \frac{1}{2} \left( \frac{(\tau^{(0)}_{0,0})'}{\tau^{(0)}_{0,0}} - \frac{(\tau^{(1)}_{0,0})'}{\tau^{(1)}_{0,0}} \right), \\
z &= 2q\tilde{r} = -qr' - 2q^2r^2 = 2\frac{\tau^{(0)}_{1,0} \tau^{(1)}_{0,0}}{(\tau^{(0)}_{0,0})^2},
\end{align*}
\]
(6.4)
then the equations (6.2) and (6.3) can be rewritten as

\[
\begin{align*}
    y' &= 2y^2 + 2z + 2t_1 - 2\beta \frac{e^{2\varphi} \bar{q}}{q}, \\
    z' &= -4yz + 2\alpha + 2\beta \left(2e^{2\varphi} \bar{q}r - 1\right). 
\end{align*}
\] (6.5)

Secondly, if we put \(\beta = 0\) and \(s = 2t_1\), we have

\[
\begin{align*}
    \frac{dy}{ds} &= y^2 + z + \frac{s}{2}, \\
    \frac{dz}{ds} &= -2yz + \alpha.
\end{align*}
\] (6.6)

This is a Hamiltonian form of the second Painlevé equation with Hamiltonian

\[
H = \frac{z^2}{2} + y^2z + \frac{t}{2}z - \alpha y.
\]

We remark that the equation (6.2) can be described as

\[
\left(2t_1 + \frac{3}{2}t_3 D_{t_1}^2\right) \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = 2\beta \tau^{(1)}_{0,0} \tau^{(1)}_{0,-1},
\] (6.7)

and the similarity condition for \(q\):

\[
t_1 \frac{\partial q}{\partial t_1} + 3t_3 \frac{\partial q}{\partial t_3} = t_1 q' + 3t_3 \left(\frac{q''}{4} - 3qq'r' - 6q^2r^2 q'\right) = -(2\alpha + 1)q
\]

is expressed as

\[
\left(t_1 D_{t_1} + \frac{3}{4}t_3 D^2_{t_1} - 2\alpha - 1\right) \tau^{(0)}_{0,0} \cdot \tau^{(0)}_{1,0} = 0.
\] (6.8)

The system of equations (6.7), (6.8) with \(\beta = 0\) is a bilinear form of the second Painlevé equation.

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