PARTITION UNIVERSALITY FOR GRAPHS OF BOUNDED DEGENERACY AND DEGREE

PEETER ALLEN AND JULIA BÖTTCHER

ABSTRACT. We prove asymptotically optimal bounds on the number of edges a graph $G$ must have in order that any $r$-colouring of $E(G)$ has a colour class which contains every $D$-degenerate graph on $n$ vertices with bounded maximum degree. We also improve the upper bounds on the number of edges $G$ must have in order that any $r$-colouring of $E(G)$ has a colour class which contains every $n$-vertex graph with maximum degree $\Delta$, for each $\Delta \geq 4$. In both cases, we show that a binomial random graph with $Cn$ vertices and a suitable edge probability is likely to provide the desired $G$.

1. INTRODUCTION

Ramsey theory, an important and lively branch of combinatorics, concerns the appearance of structure in large and potentially unstructured objects. It was initiated by Ramsey’s theorem, which implies that for every graph $F$ and every number $r$, if $N$ is large enough then any $r$-colouring of the edges of the complete graph $K_N$ on $N$ vertices admits a monochromatic copy of $F$. Two natural questions then are how big $N$ needs to be for a given $F$, and if the complete graph $K_N$ can be replaced by a sparser graph. Erdős, Faudree, Rousseau and Schelp [13] combined both questions by defining the following parameter and encouraging its study.

Given a graph $F$, the size-Ramsey number $\hat{r}_r(F)$ is defined to be the minimum number of edges in a graph $G$ such that any $r$-colouring of the edges $E(G)$ of $G$ admits a monochromatic copy of $F$. In this paper we shall concentrate on graphs $F$ with bounded maximum degree (it is easy to come up with examples of $F$ without bounded maximum degree with very large size-Ramsey number).

From Ramsey’s theorem [24], it follows that the size-Ramsey number always exists. By a result of Chvátal, Rödl, Szemerédi and Trotter [6], for any fixed $\Delta$, the Ramsey number in $r$ colours of any $n$-vertex $F$ with maximum degree at most $\Delta$ is $O(n)$, and so an upper bound for these graphs is $\hat{r}_r(F) = O(n^2)$. A trivial lower bound on the other hand is $\hat{r}_r(F) \geq e(F)$, where the number of edges $e(F)$ of $F$ is at most $\Delta n$. There has been considerable interest in improving these two bounds in recent years. We now give a brief overview of progress in this direction.

Much research focused on determining graph classes for which the trivial lower bound gives the right order of magnitude for the size-Ramsey number, that is, graph classes whose graphs have size-Ramsey number linear in the number of their vertices. This was first established for paths in the 1980s by Beck [4], answering a question of Erdős. For trees this follows from a famous result of Friedman and Pippenger [14], and for cycles this was proved by Haxell, Kohayakawa and Łuczak [17]. Much more recently, linear bounds were obtained for powers of paths by Clemens, Jenssen, Kohayakawa, Morrison, Mota, Reding, and Roberts [7] (see Date: November 30, 2022.)
also [16]) and for graphs of bounded maximum degree and bounded treewidth by Berger, Kohayakawa, Maesaka, Martins, Mendonça, Mota and Parczyk [5] and by Kamačev, Liebenau, Wood and Yepremyan [19]. Another class of graphs that possibly have linear size Ramsey number are $\sqrt{n} \times \sqrt{n}$-grid graphs, though here currently the best known bound is $O(n^{5/4})$, which was established by Conlon, Nenadov, and Trujić [10] and improves on bounds by Clemens, Mirałaei, Reding, Schacht, and Taraz [8].

Returning from special graph classes to the family of all graphs with bounded maximum degree, what is known in more generality about improvements to the easy bounds mentioned above? Rödl and Szemerédi [25] showed that not all such graphs have linear size-Ramsey number $F$ of maximum degree three on $n$ vertices with $\hat{r}_2(F) = \Omega(n(\log n)^{1/60})$. Only very recently this was improved by Tikhomirov [28], who gave a construction which improves the bound to $\Omega(n \exp(c \sqrt{\log n}))$ for some constant $c$.

Improving the simple upper bound, on the other hand, Kohayakawa, Rödl, Schacht and Szemerédi [21] proved that $\hat{r}_r(F) = O(n^{2-1/\Delta}(\log n)^{1/\Delta})$ for any $F$ with $n$ vertices and maximum degree $\Delta$. Prior to this paper, this remains the best general bound. Improvements were obtained in some special cases though. For graphs $F$ which are $K_{\Delta+1}$-free Nenadov [23] showed that $\hat{r}_r(F) = \tilde{O}(n^{2-1/\Delta - 0.5})$. For $\Delta = 3$, the general upper bound on $\hat{r}_r(F)$ was improved, first by Conlon, Nenadov and Trujić [11] to $O(n^{8/5})$, and then very recently by Draganić and Petrova [12] to $O(n^{3/2 + o(1)})$.

In many of the above positive results, something stronger is actually proved, namely so-called partition universality. A graph is called universal for a class $\mathcal{G}$ of graphs if it contains copies of all graphs in $\mathcal{G}$, and $r$-partition universal for $\mathcal{G}$ if whenever its edges are coloured with $r$ colours, there is a colour class which contains copies of all graphs in $\mathcal{G}$. Let $\mathcal{G}(\Delta, n)$ denote the class of all graphs on $n$ vertices with maximum degree at most $\Delta$. The above-mentioned upper bounds on the size-Ramsey of graphs in $\mathcal{G}(\Delta, n)$ are actually obtained by constructing graphs which are $r$-partition universal for the class. In particular, the result of [21] states that there is an $r$-partition universal graph for $\mathcal{G}(\Delta, n)$ with $e(G) = O(n^{2-1/\Delta}(\log n)^{1/\Delta})$ edges. For partition universality we have much better lower bounds than for size-Ramsey numbers: A simple counting argument shows that any universal graph (and so certainly any $r$-partition universal graph) must have a reasonably large number of edges. For $\mathcal{G}(\Delta, n)$, this argument is due to Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [2] and shows that any graph $G$ which is universal for $\mathcal{G}(\Delta, n)$ has $e(G) = \Omega(n^{2-2/\Delta})$ edges.

Which graph $G$ does [21] (and many other results) use for establishing partition universality? This is the random graph $G(N, p)$ for suitable $p$, which is the graph on $N$ vertices obtained by including each of the potential $\binom{N}{2}$ edges independently at random with probability $p = \mu(N)$. Hence, the result of [21] really is a partition universality statement for $G(N, p)$. This is also true for the result of Conlon, Nenadov and Trujić [11], and in this setting the bound they obtain is sharp: when $p$ is of smaller order than the edge probability they are using, then $G(N, p)$ ceases to be partition universal for $G(3, n)$. To improve on their bounds when it comes to size-Ramsey numbers, Draganić and Petrova [12] provide an ingenious construction of a graph which has random elements, but also contains large cliques.

In this paper we also study partition universality of the random graph $G(N, p)$. Our first result gives a polynomially improved bound for the class of bounded degree $\mathcal{G}(\Delta, n)$ for arbitrary $\Delta \geq 4$.

**Theorem 1.** For all $r, \Delta \in \mathbb{N}$ with $\Delta \geq 3$ and $\mu > 0$ there is $c > 0$ such that the following holds. If $\Delta = 3$ set $p = c^{-1}N^{-2/5}$. Otherwise set $p = N^{-\mu - 1/(\Delta - 1)}$. Then the random graph
$G(N,p)$ a.a.s. is $r$-partition universal for $G(\Delta, cn)$. Furthermore, if $p = N^{-1/2+\mu}$ then a.a.s. $G(N,p)$ is $r$-partition universal for the subset of $K_4$-free graphs in $G(3,cN)$.

This theorem is sharp in random graphs for $\Delta = 3$ (re-proving the earlier result of [11]), since for $p < N^{-2/3}$ there are a.a.s. 2-colourings of $G(N,p)$ which contain no monochromatic copy of $K_4$. For $\Delta = 4$, similarly we need $p \gg N^{-1/3}$ and the exponent is asymptotically correct. For $\Delta \geq 5$ the best lower bound we know is $p \gg N^{-2/(\Delta+2)}$, by considering $K_{\Delta+1}$, and we tend to suspect the lower bound is more likely correct than our upper bound. As explained above this result gives an immediate improvement on the size-Ramsey number for graphs in $G(\Delta,n)$.

**Corollary 2.** For $\Delta > 3$ and $\mu > 0$ any $F \in G(\Delta,n)$ has $\hat{r}_r(F) = O(n^{2+\mu-1/(\Delta-1)})$.

We also study how this bound can be improved for graphs with degeneracy. A graph $F$ has degeneracy at most $D$ if its vertices can be ordered in such a way so that every vertex has at most $D$ neighbours among its predecessors. Let $G(D,\Delta,n)$ denote the class of $n$-vertex graphs with maximum degree at most $\Delta$ and degeneracy at most $D$ (where we usually want to think about $D$ being much smaller than $\Delta$). Important classes of graphs have bounded degeneracy. For example, it follows from a theorem proved independently by Kostochka [22] and Thomason [27] that this is true for the graphs of any non-trivial minor closed graph class. Our second main theorem implies that for graphs $F \in G(D,\Delta,n)$ we have $\hat{r}_r(F) = O(n^{2+\mu-1/D})$.

**Theorem 3.** For all $r,D,\Delta \in \mathbb{N}$ and $\mu > 0$ there is $c > 0$ such that the random graph $G(N,p)$ with $p = N^{-1/D+\mu}$ a.a.s. is $r$-partition universal for $G(\Delta, cn)$.

This bound is asymptotically sharp in two ways. First, if we reduce $p$ to $o(N^{-1/D})$, then a simple first moment argument shows that for most graphs $F$ in $G(D,\Delta,cN)$, it is not likely that $F$ is a subgraph of $G(N,p)$, let alone that $G(N,p)$ is universal for the class after adversarial colouring. Second, the following result shows that for $\Delta = 2D + 1$ any universal graph for $G(D,\Delta,cN)$ necessarily contains $\Omega(N^{2-1/D})$ edges, so that the random graph is an asymptotically optimal $r$-partition universal graph for $G(D,\Delta,cN)$.

**Theorem 4** ([1]). Given $D \geq 1$, suppose that $n$ is sufficiently large and that the graph $G$ is $G(D,2D + 1,n)$-universal. Then $e(G) \geq \frac{1}{100}n^{2-1/D}$.

**Organisation.** The remainder of this paper is organised as follows. We start with notation, some probabilistic tools, and some results concerning densities and partitions of certain graphs in Section 2. We then outline the proofs of our main theorems and state the main lemmas used in these proofs in Section 3. In Section 4 we provide the proofs of the main theorems. The three main lemmas that are used in these proofs are established in the subsequent sections: Lemma 19 in Section 5, Lemma 21 in Section 6, and Lemma 20 in Section 7. As we shall explain, one further tool that we shall need is a strengthened sparse regularity lemma (Lemma 24) and a corresponding counting lemma (Lemma 25). Their proofs use standard machinery; we provide the details in Appendix A.

2. Preliminaries

2.1. **Notation.** Let $F$ be a graph on $n$ vertices. For vertex sets $X, Y \subseteq V(F)$ we write $e_F(X)$ for the number of edges of $F$ with both endpoints in $X$, and $e_F(X,Y)$ for the number of edges with one endpoint in $X$ and one endpoint in $Y$. The distance $\text{dist}(x,y)$ between two vertices
\(x, y \in V(F)\) is the length of a shortest \(x, y\)-path in \(F\); if such a path does not exist we set \(\text{dist}(x, y) = \infty\).

We shall often consider a graph \(F\) given with some order on the vertices. In this case we shall usually simply assume that the vertex set of \(F\) in \([n]\), and the given order is the natural order of \([n]\). We then denote by \(N_F(x)\) the left neighbourhood of \(x\), that is, the set of all neighbours \(y\) of \(x\) with \(y < x\). The left degree of \(x\) is \(\deg_F(x) = |N_F(x)|\). Further, we say that such an order of the vertices of \(F\) is \(D\)-degenerate if \(\deg_F(x) \leq D\) for all vertices \(x\).

We write \(F[S]\) for the subgraph of \(F\) induced by \(S\). The \(F\)-neighbourhood of \(x\) intersected with a set \(X \subseteq V(F)\) is denoted by \(N_F(x; X)\). The joint neighbourhood of the vertices in \(S\) is denoted by \(N_F(S)\), and this joint neighbourhood intersected with \(X\) by \(N_F(S; X)\). We omit the subscript \(F\) in this notation when it is clear from the context.

2.2. Probabilistic tools, and properties of \(G(N, p)\). We shall use the following Chernoff bounds for binomially distributed random variables.

**Theorem 5** (Corollary 2.4 from [18]). Suppose \(X\) is a random variable which is the sum of a collection of independent Bernoulli random variables. Then we have for \(\delta \in (0, 3/2)\)

\[
\Pr \left( X > (1 + \delta)\mathbb{E}X \right) < e^{-\delta^2 \mathbb{E}X/3} \quad \text{and} \quad \Pr \left( X < (1 - \delta)\mathbb{E}X \right) < e^{-\delta^2 \mathbb{E}X/3}.
\]

We shall next define a few properties that we shall need from the random graph. We shall start with the neighbourhood property, which states that joint neighbourhoods are as expected, and the star property, which generalises this to the union of several independent joint neighbourhoods.

**Definition 6** (neighbourhood property, star property). We say that a graph \(\Gamma\) on \(N\) has the \((D, \varepsilon, p)\)-neighbourhood property if for every \(k \leq D\) and for every collection \(x_1, \ldots, x_k\) of distinct vertices of \(\Gamma\), we have \(|N_{\Gamma}(x_1, \ldots, x_k)| = (1 \pm \varepsilon)p^k N\).

We say that \(\Gamma\) has the \((D, \varepsilon, p)\)-star property if for each \(1 \leq k \leq D\) and each collection \(\mathcal{B}\) of at most \(\varepsilon p^{-k}\) pairwise disjoint \(k\)-sets in \(V(\Gamma)\), we have \(|\bigcup_{B \in \mathcal{B}} N_{\Gamma}(B)| = (1 \pm \varepsilon)p^k N|\mathcal{B}|\).

Observe that if a graph has the \((D, \varepsilon, p)\)-star property, it also has the \((D, \varepsilon, p)\)-neighbourhood property. The following lemma is proved as part of [23, Proposition 2.5].

**Lemma 7.** For every \(D\) and \(\varepsilon > 0\) there exists \(C\) such that for \(p \geq \left(\frac{C \log N}{N}\right)^{1/D}\) the random graph \(G(N, p)\) a.a.s. has the \((D, \varepsilon, p)\)-star property, and hence also the \((D, \varepsilon, p)\)-neighbourhood property.

Next, we shall be interested in the count of subgraphs in the random graph which extend a given set of pre-embedded root vertices. Given graphs \(H\) and \(\Gamma\), let \(R \subseteq V(H)\) be a set of vertices, which we also call the roots, and assume that we fix an injection \(\pi\) from \(R\) to \(V(\Gamma)\). We say that an injection \(\phi : V(H) \to V(\Gamma)\) is an \(H\)-extension of \(\pi\) in \(\Gamma\) if \(\phi|_R = \pi\) and if for every \(xy \in E(H)\) with \(|\{x, y\} \cap R| \leq 1\) we have \(\phi(x)\phi(y) \in E(\Gamma)\). In other words, \(\phi\) extends \(\pi\) and is an embedding of \(H\) to \(\Gamma\) apart from possibly ignoring edges within the root set \(R\).

In \(\Gamma = G(N, p)\), the expected number of \(H\)-extensions of \(\pi\) is roughly

\[
(1) \quad \mathbb{E}_{\pi} = N^{\left|V(H)\right| - R} \cdot p^{|E(H)\setminus R| + |E(H)\setminus V(R)|}.
\]

The following theorem of Spencer [26] provides a condition on \(p\) for the actual number of \(H\)-extensions in \(G(n, p)\) being concentrated around this expectation. This condition uses the following definition.
Definition 8 (Spencer density). Let $H$ be a graph and $R$ be a set of vertices in $H$. Then the Spencer density $m_S(H, R)$ of $(H, R)$ is

$$m_S(H, R) = \max_{X \subseteq V(H) \setminus R} \frac{e_H(X) + e_H(X, R)}{|X|}.$$ 

Theorem 9 (Spencer [26]). Let $H$ be a graph, $R$ be a set of vertices in $H$, and $\varepsilon > 0$. There exist constants $K$ and $t$ such that every injection $\pi: R \to V(\Gamma)$ the number $\text{Ext}_\pi$ of $H$-extensions of $\pi$ in $\Gamma$ satisfies $\text{Ext}_\pi = (1 \pm \varepsilon)\mathbb{E}_\pi$.

In our applications of this theorem we shall work with the edge probability $p = n^{-\frac{1}{D} + \mu}$, where $D$ is the degeneracy of the graph we seek to embed and $\mu$ is an arbitrarily small constant. For this probability Theorem 9 applies to $H$ and $R$ which satisfy the following condition (see Corollary 11).

Definition 10 ($(D, \mu)$-Spencer). Given a graph $H$ and a set $R$ of vertices in $H$, we say that $(H, R)$ is $(D, \mu)$-Spencer if for every set of vertices $X$ in $V(H) \setminus R$, we have

$$e_H(X) + e_H(X, R) \leq (D + \mu)|X|.$$ 

The following is an immediate corollary of Theorem 9 and the previous definition.

Corollary 11. Let $D \geq 1$ be a natural number and $\mu, \varepsilon > 0$. Let $H$ be a graph and $R \subseteq V(H)$ such that $(H, R)$ is $(D, \mu)$-Spencer. The random graph $\Gamma = G(N, p)$ with $p \geq N^{-\frac{1}{D} + \mu}$ a.a.s. satisfies that for every injection $\pi: R \to V(\Gamma)$ the number $\text{Ext}_\pi$ of $H$-extensions of $\pi$ in $\Gamma$ satisfies $\text{Ext}_\pi = (1 \pm \varepsilon)\mathbb{E}_\pi$.

Proof. Indeed, as $H$ is $(D, \mu)$-Spencer, by definition we have $m_S(H, R) \leq D + \mu$. Since $-\mu + D^2 \mu + D \mu^2 > 0$ we have $-(D + \mu) + (D + \mu)D\mu > -D$ and dividing by $D(D + \mu)$ gives

$$\frac{-1}{D + \mu} > \frac{1}{D + \mu}.$$ 

This implies $N^{-\frac{1}{D} + \mu} \geq K \frac{(\log N)^t}{N^{1/m_S(H, R)}}$ for any $K$ and $t$ and for $N$ sufficiently large in terms of $K$ and $t$. Hence, we get $\mathbb{E}_\pi = (1 \pm \varepsilon)\mathbb{E}_\pi$ from Theorem 9.\hfill \square

2.3. Finding roots. For applying Corollary 11 we will need to find a suitable set of roots. To this end, we show next that for any $D$-degenerate $H$ and initial segment $I$, for any set $T$ there is a not-too-large superset $T'$ such that $(H, I \cup T')$ is $(D, \mu)$-Spencer.

Lemma 12. Given any $D \geq 1$ and $\mu > 0$, let $H$ be $D$-degenerate with a given $D$-degeneracy order, let $I$ be the vertices of an initial segment of this order, and let $T$ be a set of vertices of $H$. Then there is a superset $T'$ of $T$, with $|T'| \leq D^2 |T|^2 \mu^{-1}$, such that every vertex of $T'$ is connected in $H[T']$ to some member of $T$, and such that $(H, I \cup T')$ is $(D, \mu)$-Spencer.

Proof. Let $T_0 = T$. We now iterate the following procedure for $i = 0, 1, \ldots$. If $(H, I \cup T_i)$ is $(D, \mu)$-Spencer, we stop. Otherwise, we take a minimal set $X$ witnessing that $(H, I \cup T_i)$ is not $(D, \mu)$-Spencer, set $T_{i+1} = T_i \cup X$, and repeat.
We first claim that in each iteration we maintain the desired connectivity. This is trivially true for $T_0$; suppose inductively that it holds for $T_i$. Let $X = T_{i+1}\setminus T_i$. By choice of $X$ we have

$$e(X) + e(X, I \cup T_i) > (D + \mu)|X|.$$ 

Now let $Y \subseteq X$ be the set of vertices which are not connected in $H[T_{i+1}]$ to any member of $T$. By definition of $Y$ there are no edges from $Y$ to $(X \setminus Y) \cup T_i$, and hence $e(X \setminus Y) = e(X) - e(Y)$ and $e(X, I \cup T_i) = e(X, I \cup T_i) - e(Y, I)$. Since $I$ is an initial segment of the degeneracy order each edge from $Y$ to $Y \cup I$ has the endpoint that comes later in the degeneracy order in $Y$, hence there can be at most $D|Y|$ such edges, that is, $e(Y) + e(Y, I) \leq D|Y|$. We conclude that

$$e(X \setminus Y) + e(X \setminus Y, I \cup T_i) = e(X) - e(Y) + e(X, I \cup T_i) - e(Y, I)$$
$$\geq e(X) + e(X, I \cup T_i) - D|Y| > (D + \mu)|X| - D|Y| > (D + \mu)|X \setminus Y|,$$

which is a contradiction to the assumed minimality of $X$ unless $Y = \emptyset$.

Second, we claim that for each $i$ we have

$$D|T_i \setminus T| + i \leq e(T_i \setminus T, I \cup T) \leq D|T_i|.$$ 

The second inequality is easy: as before we observe that any edge counted in the middle sum has the endpoint that comes later in the degeneracy order in $T_i$. The first inequality we prove inductively. It is trivially true for $i = 0$, when the left-hand side and middle sum are equal to zero. Assuming it is true for a given $i \geq 0$, because we obtain $T_{i+1}$ from $T_i$ by adding a witness that $(H, I \cup T_i)$ is not $(D, \mu)$-Spencer, by definition the number of edges we add to the middle sum is strictly larger than $D|T_{i+1} \setminus T_i|$, justifying the first inequality.

We conclude from (2) that $i \leq D|T_i|$, justifying that we stop after $t \leq D|T|$ steps. Moreover, (2) gives $e(T_{i+1} \setminus T) + e(T_{i+1} \setminus T, I \cup T) \leq D|T_{i+1}|$ and $e(T_i \setminus T) + e(T_i \setminus T, I \cup T) \geq D|T_i|$. Since

$$e(T_{i+1} \setminus T) + e(T_{i+1} \setminus T, I \cup T_i) = e(T_{i+1} \setminus T) + e(T_{i+1} \setminus T, I \cup T) - e(T_i \setminus T) - e(T_i \setminus T, I \cup T)$$

we conclude

$$D|T_{i+1} \setminus T_i| + D|T| \geq e(T_{i+1} \setminus T_i) + e(T_{i+1} \setminus T_i, I \cup T_i).$$

Because $T_{i+1} \setminus T_i$ is a witness that $(H, I \cup T_i)$ is not $(D, \mu)$-Spencer, this in turn implies

$$D|T_{i+1} \setminus T_i| + D|T| > (D + \mu)|T_{i+1} \setminus T_i|.$$ 

It follows that $|T_{i+1} \setminus T_i| < D\mu^{-1}|T|$, and so we stop with a set $T_i$ of size at most $D^2|T|^2\mu^{-1}$, as desired. \qed

2.4. **Densities and partitions of graphs.** Given a graph $H$ with at least one edge and three vertices, we define $d_2(H) := \frac{e(H) - 4}{v(H) - 3}$. For convenience, we say that $d_2(K_2) = \frac{1}{2}$ and $d_2(H) = \frac{1}{2}$ for $H$ any graph with no edges. We define $m_2(H)$ to be the maximum of $d_2(H')$ over all subgraphs $H'$ of $H$. This quantity is important in random graph theory in that there is a transference of the Counting Lemma for counting copies of $H$ to subgraphs of $G(n,p)$ if $p \gg n^{-1/m_2(H)}$; we will state this formally and derive one of the main tools needed for this paper in Section 5. For now, we need to upper bound $m_2(H)$ for various graphs $H$.

**Lemma 13.** Let $F$ be a $D$-degenerate graph with $D \geq 1$. Then

(a) $m_2(F) \leq D$, with equality possible only if $D \leq 2$.

Let $H$ be a connected graph with maximum degree $D + 1$ for $D \geq 2$. Then
(b) \( m_2(H) = \frac{5}{2} \) if \( H = K_4 \),

(c) \( m_2(H) \leq 2 \) if \( D = 2 \) and \( H \neq K_4 \), and

(d) \( m_2(H) \leq D \) if \( D \geq 3 \).

Now assume in addition that \( Q \) is an induced cycle or an induced path in \( H \) and let \( H^+ \) be the graph obtained from \( H \) by duplicating \( Q \), that is, by adding a copy \( Q' \) of \( Q \) with new vertices \( \{y': y \in V(Q)\} \) and joining \( y' \) to all vertices \( N_H(y) \setminus V(Q) \). Then

(e) \( m_2(H^+) \leq \max \left( m_2(H), D \right) \) if \( Q \) is an induced cycle with \( \ell \geq 3 \) vertices,

(f) \( m_2(H^+) \leq \max \left( m_2(H), D \frac{\ell^2}{\ell-2} \right) \) if \( Q \) is an induced path with \( \ell \geq 3 \) vertices.

Proof. Obviously (b) holds. We now prove (a). As subgraphs of \( D \)-degenerate graphs are \( D \)-degenerate, it suffices to show that \( d_2(F) \leq D \) for all degenerate \( F \), with equality only possible if \( D = 2 \). We prove this by induction on \( v(F) \), the base case being \( v(F) \leq D + 1 \) when

\[
d_2(F) \leq d_2(K_{D+1}) = \begin{cases} 1 & \text{if } D = 1 \\ \frac{D^2+2}{2} & \text{if } D > 1, \end{cases}
\]

where we have \( d_2(K_{D+1}) = D \) only for \( D \leq 2 \). So assume \( v(F) > D + 1 \). By \( D \)-degeneracy, there is a vertex \( v \) in \( F \) of degree at most \( D \). By induction, we have \( e(F-v) - 1 \leq D(v(F) - 3) \), with equality only if \( D = 2 \). So

\[
e(F) - 1 \leq D(v(F) - 3) + D \leq D(v(F) - 2)
\]

and we conclude \( d_2(F) \leq D \), with equality only if \( D = 2 \).

We next turn to (c) and (d). If we remove any vertex from \( H \), then we get a graph which is \( D \)-degenerate. Thus either we have \( m_2(H) \leq D \) by (a) and are done, or we have \( m_2(H) = d_2(H) \) and \( v(H) \geq D + 2 \). In the latter case, since \( e(H) \leq \frac{1}{2}(D+1)v(H) \), we get

\[
m_2(H) \leq \frac{1}{2}(D+1)v(H) - 1 = \frac{D+1}{2} + \frac{D}{v(H) - 2}.
\]

This expression decreases as \( v(H) \) increases. For \( D = 2 \), if \( H \neq K_4 \) then we can assume \( v(H) \geq 5 \). A 5-vertex graph with maximum degree 3 has at most 7 edges and so 2-density at most \( \frac{7}{5} = 2 \), as required. If on the other hand \( v(G) \geq 6 \), then by the above inequality we have

\[
m_2(H) \leq \frac{3}{2} + \frac{2}{4} = 2. \quad \text{For } D \geq 3, \text{ we have } v(H) \geq D + 2 \text{ and so } m_2(H) \leq \frac{D+3}{2} \leq D.
\]

Finally, we consider \( H^+ \). Let \( X \) be a subset of \( V(H^+) \) which maximises \( d_2(H[X]) \). If \( d_2(H[X]) \leq D \) there is nothing to prove, so suppose \( d_2(H[X]) > D \); in particular we have \( |X| \geq 4 \). Observe that there cannot be \( v \in X \) with less than \( D + 1 \) neighbours in \( X \), otherwise removing \( v \) increases the 2-density. If \( X \) does not intersect both copies of \( Q \) in \( H^+ \), then \( H^+[X] \) is a subgraph of \( H \) and so \( d_2(H[X]) \leq D \), so we can assume \( X \) intersects both copies of \( Q \) in \( H \). But since the vertices in both copies of \( Q \) have degree at most \( D + 1 \) in \( H^+ \), we see that \( X \) has to contain all neighbours of each such vertex, i.e. \( X \) contains both copies of \( Q \) and their neighbours. Since the remaining vertices of \( H^+ \) have degree \( D + 1 \) and \( H \) is connected, we conclude \( X = V(H^+) \).

If \( Q \) is an induced cycle whose vertices have degree at most \( D + 1 \), then there are at most \( D\cdot|Q| \) edges with at least one end in \( Q \); if \( d_2(H^+) > D \) then removing a copy of \( Q \) gives \( d_2(H) > D \), a contradiction.

The remaining case is that \( Q \) is an induced path and \( X = V(H^+) \). Here we have \( e(H^+) \leq \frac{1}{2}(D+1)v(H) + (D-1)v(Q') + 2 + v(Q') - 1 = \frac{1}{2}(D+1)v(H) + Dv(Q) + 1 \), since the vertices
of \( Q' \) each have at most \( D - 1 \) neighbours in \( V(H) \), apart from the end vertices of \( Q' \) which may have \( D \) neighbours, and \( e(Q') = v(Q') - 1 \). We obtain
\[
d_2(H^+) \leq \frac{1}{2}(D + 1)v(H) + Dv(Q) \leq Dv(H) + v(Q) - 2 < D \frac{v(Q) - 2}{v(H) - 2},
\]
where for the second inequality we use \( \frac{1}{2}(D + 1) \leq D \). This gives (f). \( \square \)

Further, we shall need the following result on equipartitions of graphs of bounded maximum degree into large sets of vertices which are far from each other. The Hajnal–Szemerédi Theorem [15] states that every graph with maximum degree \( \Delta \) has a \((\Delta + 1)\)-colouring with colour classes differing in size by at most 1. We call a partition of the vertex set into such sets differing in size by at most 1 an equipartition. The following result is an immediate consequence of the Hajnal–Szemerédi Theorem applied to the \( \ell \)-th power of the graph \( F \), which is obtained from \( F \) by adding all edges between vertices of distance at most \( \ell \), and which has maximum degree at most \( \Delta + \sum_{i=2}^{\ell}(\Delta - 1)^i \leq 2\Delta^\ell - 1 \).

**Theorem 14.** Let \( F \) be a graph with maximum degree at most \( \Delta \geq 2 \) and \( \ell \) be a number. Set \( h = 2\Delta^\ell \). Then there is an equipartition of \( V(F) \) into sets \( X_1, \ldots, X_h \) such that for every \( i \in [h] \) every pair of vertices \( x \neq x' \in V_i \) satisfies \( \text{dist}_F(x, x') > \ell \). \( \square \)

### 3. Proof strategy and main lemmas

In this section we explain our strategy for the proofs of our main theorems, and provide the main lemmas that capture different parts of this strategy. It turns out that almost all the work is to prove Theorem 3, and the proof of Theorem 1 requires only small modifications. We will therefore in this section mainly concentrate on the proof of Theorem 3, but we will also explain the modifications required for Theorem 1 and state our main lemmas in the generality required for applying them in both proofs.

In this section, \( \tilde{F} \) will always be the graph we are trying to embed, \( \Gamma \) will be a typical random graph of appropriate edge density whose edges we \( r \)-colour, and \( G \) will be a subgraph of \( \Gamma \) given by the edges of an appropriately chosen colour \( \chi \), on a carefully chosen subset of vertices, where we want to embed \( \tilde{F} \).

Our strategy is as follows. Firstly, we shall argue that we can reduce the problem of finding an embedding of \( F \) in \( G \) to the problem of ‘robustly’ finding partial homomorphisms from \( F \) to \( G \) satisfying certain properties (see Lemma 21). This technique is taken from [23]. For finding these homomorphisms we then proceed vertex by vertex. Assuming the vertices of \( F \) are given in a degeneracy order, at step \( x \) we extend the homomorphism \( \psi_{x-1} \) from \( F[[x-1]] \) to \( G \) to a homomorphism \( \psi_x \) from \( F[[x]] \) to \( G \).

To do this, we need that we have a large *candidate set* for \( x \), which is the set of common neighbours in \( G \) of the set \( \psi_{x-1}(N^-(x)) \) of vertices of \( G \) hosting already embedded neighbours of \( x \). Observe that \( \psi_{x-1}(N^-(x)) \) is a set of up to \( D \) vertices. We will be able to ensure that most sets of size up to \( D \) have a large common neighbourhood, but a tiny proportion of such sets will be *unpromising*, i.e. they have only few common neighbours. Our strategy will ensure that \( N^-(x) \) is never embedded to an unpromising set. The next paragraph gives the idea of how this works, but omits some technical details.

We look ahead a large constant distance \( \ell_1 \). So, when we want to embed \( x \), we look ahead to each vertex \( y \) which is after \( x \) in the degeneracy order, but whose distance from \( x \) is at most \( \ell_1 \). In order to decide how to embed \( x \), we look at the induced subgraph \( H^1(y) \) of all vertices at distance between 1 and \( \ell_1 \) from \( y \) in \( F \). We shall show (in Lemma 19) that, before
embedding any vertices, only a tiny fraction of embeddings of $H'_1(y)$ are unpromising, i.e. send $N^-(y)$ to an unpromising set. As we embed more and more of $F$, we keep track of the fraction of unpromising extensions of $H'_1(y)$. Here extensions means embeddings of $H'_1(y)$ which are consistent with the current partial homomorphism of $F$. When we embed a vertex outside $H'_1(y)$, by definition the fraction of unpromising extensions does not change. When we embed $x$, on the other hand, which is a vertex of $H'_1(y)$ we will make sure that the fraction of unpromising extensions does not increase too much: we shall cross off from the candidate set of $x$ vertices $v$ which would increase the fraction of unpromising extensions by too large a factor, i.e. we shall not embed $x$ to $v$.

An easy counting argument proves that if $x$ is at distance strictly less than $\ell_1$ from $y$, then we only cross off few vertices to which we could map $x$. If $x$ is at distance exactly $\ell_1$ from $y$, on the other hand, then we do not cross off any vertices: no choice of $v$ can increase the fraction of unpromising embeddings by a large factor (see Lemma 20).

Putting the pieces together, this strategy ensures that the fraction of unpromising extensions is always strictly less than 1. This is in particular true once all vertices in $N^-(y)$ are embedded, which means we do not choose an unpromising set to embed $N^-(y)$ to. From the point of view of embedding $x$ on the other hand, as we said we look ahead to all $y$ within distance $\ell_1$ of $x$; since $\Delta(F) \leq \Delta$ there are a bounded number of these. For each such $y$, we cross off the vertices from the candidate set of $x$ which would make the fraction of unpromising extensions of $H'_1(y)$ increase too much. Since there are few such vertices, and since we inductively ensured that $N^-(x)$ was not embedded to an unpromising set, there remain many un-crossed-off vertices to which we can embed $x$.

In the next subsection, we fill in the technical details missing from the above, and state the various lemmas we need to prove that it works. We should warn the reader that the word ‘extension’ above will turn out not to refer to an embedding of $H'_1(y)$ into $G$, but to something a bit more complicated, the explanation of which we omitted here for simplicity (but which will be provided shortly).

3.1. Main lemmas and definitions. We first fix values for the various constants we need, and formally define the various subgraphs of $F$ that we are interested in. The left distance $\text{dist}_F(y, x)$ of vertices $y > x$ of $F$ is the length $\ell$ of a shortest path $y = x_0, \ldots, x_\ell = x$ such that $x_i > x_{i+1}$ for all $i$; it is infinite if no such path exists. The following Setup mentions a subgraph $H_0(y)$ not discussed in the above sketch: we will explain it immediately afterwards.

**Setup 15** ($F$ and its subgraphs $H_0(y)$ and $H_1(y)$). Given $r, D, \Delta \in \mathbb{N}$ and $\mu > 0$ we define the constants

\begin{equation}
(3) \quad h_0 := 16D^5\mu^{-3}, \quad \ell_1 := D^2h_0^2\mu^{-1} + 20, \quad \text{and} \quad h_1 = 2\Delta \ell_1.
\end{equation}

Let $F$ be a graph in $\mathcal{G}(D, \Delta, n)$ with vertex set $[n]$ such that the natural order on $[n]$ is a $D$-degeneracy order for $F$. For each fixed $y \in V(F)$, let $H_1(y)$ be the subgraph of $F$ induced by the vertices $\{x \in [n] : x \leq y, \text{dist}_F(y, x) \leq \ell_1\}$. We refer to the vertices of $H_1(y)$ at left distance exactly $\ell_1$ from $y$ as the boundary of $H_1(y)$. Let $H_0(y)$ be a connected induced subgraph of $H_1(y)$ with at most $h_0$ vertices that contains $y$ and $N^-(y)$ and is such that $(H_1(y), V(H_0(y)))$ is $(D, \mu)$-Spencer. Let $H'_0(y) := H_0(y) - y$ and $H'_1(y) := H_1(y) - y$. Let $\phi : V(F) \to [h_1]$ be a map which is injective on $V(H_1(y))$ for each vertex $y \in V(F)$.

For the proof of Theorem 1, we assume that a final segment $Q$ of $V(F)$ in degeneracy order is an induced path or cycle. We let $H_1(Q)$ be the subgraph of $F$ induced by the vertices
\{x \in [n] : x \notin Q, \text{dist}_F(y, x) \leq \ell_1 \}. \text{ We define as above boundary vertices and } H_0(Q), \text{ and set } H'_i(Q) = H_i(Q) \text{ for } i = 0, 1, \text{ and we insist of } \phi \text{ in addition that it is injective on } V(H_1(Q)).

Note that this setup explains how \( H'_0(y) \) is chosen but not why such a choice is possible; we will show that such a choice is possible (in the proof of Theorem 3) and also that \( H'_0(y) \) contains none of the boundary vertices of \( H_1(y) \).

The choice of \( H'_0(y) \) satisfies two requirements of our proof, which we now explain. The first is that we will see (Lemma 19 below) that there are very few \( \phi \)-partite copies of \( H'_0(y) \) which map \( N^{-}(y) \) to an unpromising set. Formalising this we make the following definition. After the two definitions below we will state the second requirement of \( H'_0(y) \), which will explain why we consider two different graphs \( H'_0(y) \) and \( H'_1(y) \).

**Definition 16** (promising embeddings). Given a graph \( G \) and disjoint sets \( V_1, \ldots, V_{h_1} \subseteq V(G) \) each of size \( N/K \), let \( H_0 \) be a graph on at most \( h_0 \) vertices, let \( y \in V(H_0) \), and let \( H'_0 = H_0 \setminus y \). Let \( \phi : V(H_0) \rightarrow [h_1] \) be an injection. We say that a \( \phi \)-partite embedding \( \psi \) of \( H'_0 \) into \( G \) is \( dp \)-promising (for \( y \)) if

\[
|N_G(\psi(N_{H_0}(y)) \setminus V_\phi(y))| \geq \tfrac{1}{2} (dp)^{\deg_{N_{H_0}(y)}(y)} \cdot \frac{N}{K}.
\]

and \( dp \)-unpromising (for \( y \)) otherwise.

We write \( dp \) in the above rather than a single letter for consistency with our proof. For proving Theorem 1, we further require an analogous description on embeddings which are not completable embeddings in the following sense.

**Definition 17** (completable embeddings). Given a graph \( G \) and disjoint sets \( V_1, \ldots, V_{h_1} \subseteq V(G) \) each of size \( N/K \), let \( H_0 \) be a graph on at most \( h_0 \) vertices, let \( Q \) be an induced subgraph of \( H_0 \), and let \( H'_0 = H_0 \setminus Q \). Let \( Z \subseteq V(G) \) and let \( \phi : V(H_0) \rightarrow [h_1] \) be an injection. We call a \( \phi \)-partite embedding \( \psi \) of \( H'_0 \) into \( G \) a completable embedding if we can extend \( \psi \) to an embedding of \( H_0 \) into \( G \), using only vertices outside \( Z \cup \bigcup_{i \in [h_1]} V_i \) to embed the vertices of \( Q \).

We are now in a position to explain what exactly ‘extension of \( H'_1(y) \)’ means in the proof sketch; this is exactly where the second requirement of \( H'_0(y) \) comes from. In short, given \( H'_1(y) \) and \( \phi \), and a current partial embedding \( \psi \) of \( F \) into \( G \), we look at embeddings \( \psi' \) of \( H'_1(y) \) into \( \Gamma \) which are consistent with this embedding and which restricted to \( H'_0(y) \) give a \( \phi \)-partite embedding to \( G \). We should stress that ‘consistent’ simply means that \( \psi \) and \( \psi' \) must agree on the embedding of vertices they both embed. It can be that there is a vertex \( w \) embedded by \( \psi \) that is not in \( H'_1(y) \), which has a neighbour \( z \) in \( H'_1(y) \) that is not embedded by \( \psi \) (in which case \( z \) is a boundary vertex), but we do not require that there is any edge of \( G \) or \( \Gamma \) between \( \psi(w) \) and \( \psi'(z) \), so the map \( \psi \cup \psi' \) is not necessarily a homomorphism to \( \Gamma \). What \( \psi' \) extends is not \( \psi \) but the restriction \( \psi|_{V(H'_1(y))} \).

**Definition 18** (promising extension, completion extension). Let \( F, \phi, y, H'_0(y), H'_1(y), Q, H'_0(Q), H'_1(Q) \) be as in Setup 15, let \( G \) be a graph, and let \( V_1, \ldots, V_{h_1} \subseteq V(G) \) be disjoint sets of vertices.

Given \( x - 1 \leq \max (N^{-}(y)) \), let \( \psi : [x - 1] \rightarrow V(G) \) be a graph homomorphism from \( F[[x - 1]] \) to \( G \). Suppose that \( \psi' : V(H'_1(y)) \rightarrow V(\Gamma) \) is a graph homomorphism from \( H'_1(y) \) to \( G \) that agrees with \( \psi \) on \( V(H'_1(y)) \cap [x - 1] \), and suppose furthermore that \( \psi'(u) \in V(\phi(u)) \) for all \( u \in V(H'_0(y)) \) and that \( \psi' \) is injective. We say \( \psi' \) is a \( dp \)-promising extension for \( y \) if the restriction \( \psi'|_{V(H'_0(y))} \) is a \( \phi \)-partite embedding of \( H'_0(y) \) into \( G \) which is \( dp \)-promising for \( y \); otherwise we say it is \( dp \)-unpromising.
We define completable extension analogously. For any \( \psi : [x - 1] \to V(G) \), an extension \( \psi' \) of \( \psi|_{V(H'_0(Q))} \) which embeds \( V(H'_0(Q)) \) is a completable extension if the restriction to \( V(H'_0(Q)) \) is a completable embedding, and otherwise we say it is uncompletable.

The reason for this rather strange definition is that Lemma 20 below, which shows that when we want to embed a boundary vertex for \( y \) we do not cross off anything for \( y \), uses Theorem 9, which gives a count of rooted extensions in \( \Gamma \) (but not in subgraphs). The upshot of applying Theorem 9 is that while we can ask for a few vertices near \( y \) to be in the set of roots (and \( H'_0(y) \) will always be) we need most vertices of \( H'_0(y) \) to be embedded unrestrictedly in \( \Gamma \), as Definition 18 states. The fact that \( H'_0(y) \) is connected, small, and that \( (H'_0(y), H'_0(y)) \) is \((D, \mu)\)-Spencer both says that it indeed consists of few vertices near \( y \), and that Theorem 9, together with a strong upper bound on unpromising embeddings of \( H'_0(y) \), lets us deduce that there are very few unpromising extensions to a homomorphism of \( H'_0(y) \) for the empty partial embedding, before we begin to embed \( F \). This is the second property that we need of \( H'_0(y) \): In order to be able to apply Theorem 9 we need that \( (H'_0(y), H'_0(y)) \) is \((D, \mu)\)-Spencer, and for our embedding strategy to work we need that \( H'_0(y) \) consists of few vertices near \( y \).

We now explain what we require of the colour \( \chi \) in which we want to embed our graphs \( F \). Briefly, this is that we can (for some bounded \( K \)) find subsets \( V_1, \ldots, V_{h_1} \) of size \( N/K \) such that for every possible choice of a \( D \)-degenerate \( H_0 \) with at most \( h_0 \) vertices, and \( \phi \) as in Definition 16, there are very few \( d\bar{p}\)-unpromising \( \phi\)-partite embeddings of \( H'_0 \) in colour \( \chi \), where \( H'_0 = H_0 - y \) for the last vertex \( y \) of \( H_0 \) in the degeneracy order. We remark that we will eventually embed all of \( F \) into these sets \( V_1, \ldots, V_{h_1} \). The following Lemma 19 states that we can find such a colour \( \chi \) and vertex sets \( V_1, \ldots, V_{h_1} \). Lemma 19 further allows us to avoid any given small set \( Z \) in our promising embeddings and completable embeddings. This is needed in the proof of Theorem 1 in order to avoid previously embedded components.

**Lemma 19 (promising and completable embeddings).** For every \( r, D, \mu > 0 \), \( f : \mathbb{N} \to \mathbb{R}^+ \), \( h_0, h_1 \) there are \( K_0, C > 0 \) such that for \( d = \frac{1}{2r} \) and \( p \geq N^{-\frac{1}{\mu} + \mu} \) the random graph \( \Gamma = G(N, p) \) a.a.s. satisfies the following. For every \( r \)-colouring of \( E(\Gamma) \) there is an integer \( K \leq K_0 \) and a colour \( \chi \) such that for the subgraph \( G \) of \( \Gamma \) formed by all edges of colour \( \chi \) and for each \( Z \subseteq V(\Gamma) \) with \( |Z| \leq NK_0^{-2} \), there are pairwise disjoint sets \( V_1, \ldots, V_{h_1} \subseteq V(G) \setminus Z \), each of size \( \frac{N}{K} \), with the following properties.

(a) For every \( D \)-degenerate graph \( H_0 \) with at most \( h_0 \) vertices whose final vertex in the degeneracy order is \( y \), for \( H'_0 = H_0 - y \), and for any injection \( \phi : V(H_0) \to [h_1] \), the number of \( \phi\)-partite embeddings of \( H'_0 \) into \( G \) which are \( d\bar{p}\)-unpromising for \( y \) is at most

\[
f(K) \cdot p^{e(H'_0)} \left( \frac{N}{K} \right)^{v(H'_0)}.
\]

(b) For any connected graph \( H_0 \) with at most \( h_0 \) vertices and \( \Delta(H_0) \leq D + 1 \), with an induced subgraph \( Q \) such that either \( Q \) is an induced cycle with at most \( 2\mu^{-1} \) vertices, or \( Q \) is an induced path with exactly \( 2\mu^{-1} \) vertices, for \( H'_0 = H_0 - V(Q) \), and for any injection \( \phi : V(H_0) \to [h_1] \) the following holds. If either \( D = 2 \) and \( H_0 \neq K_4 \), or \( D \geq 3 \), then the number of \( \phi\)-partite embeddings of \( H'_0 \) which are not completable embeddings is at most

\[
f(K) \cdot p^{e(H')} \left( \frac{N}{K} \right)^{v(H')}.
\]

(c) If \( p \geq CN^{-2/5} \), then \( G \) contains a copy of \( K_4 \) whose vertices are not in \( Z \).
We prove this lemma in Section 5. Its proof uses a strengthened version of the sparse regularity lemma and the sparse counting lemma. The former is needed in order that we can obtain $f(K)$ depending arbitrarily on $K$; the usual sparse regularity lemma would not allow this. One should think of $f(K)$ as being very tiny compared to $r^{-1}, D^{-1}, \mu$ and $K$, which we need in our proofs.

In terms of justifying the sketch proof, the lemma just given identifies for us a colour $\chi$ and a $h_1$-partite subgraph of $\Gamma$ of colour $\chi$ edges, which we call $G$, into which we will eventually embed $F$. Furthermore, using this lemma as well as Theorem 9, we can easily obtain the desired statement that before we commence the embedding for every $y \in V(F)$ only a tiny fraction of extensions to a homomorphism of $H'_1(y)$ are unpromising.

We now need to justify that as we embed vertices we can avoid causing the fraction of unpromising extensions of $\psi|_{V(H'_1(y))}$ to increase too much. As mentioned, when we embed a vertex $x$ whose distance from $y$ is bigger than $\ell_1$, by definition the fraction is unchanged. The following lemma deals with vertices $x$ whose distance is exactly $\ell_1$ to $y$, and states that no matter where we embed them, they do not increase the fraction of unpromising embeddings for $y$ by a large factor. We prove this lemma in Section 7.

**Lemma 20** (vertices with left distance $\ell_1$ have few unpromising extensions). Let $F, \phi, y, H_0(y), H_1(y), Q, H'_0(Q), H'_1(Q)$ and $h_1$, and $\ell_1$ be as in Setup 15. Let $I$ denote the first $q$ vertices of $H'_1(y)$.

Let $\Gamma$ be an $N$-vertex graph that has a subgraph $G$ with disjoint vertex sets $V_1, \ldots, V_{h_1} \subseteq V(G)$ each of size $N/K$ such that the following properties are satisfied.

(S1) $\Gamma$ has the $(D, \frac{1}{10}, p)$-neighbourhood property.

(S2) For every $T \subseteq V(H'_1(y))$ and $R = I \cup T$ such that $(H'_1(y), R)$ is $(D, \mu)$-Spencer, and for every injective $\pi : R \to V(\Gamma)$, the number of $H'_1(y)$-extensions of $\pi$ in $\Gamma$ is

$$\left(1 + \frac{1}{10}\right)N^{-\frac{1}{10}}p^{\ell(U,R)+\varepsilon(U,R)},$$

where $U = V(H'_1(y))$ and we count edges in $U \setminus R$ and between $U$ and $R$ in $H'_1(y)$.

Suppose that $x \in V(F)$ is the $q$-th vertex of $H'_1(y)$, that $\text{dist}_F(x, y) = \ell_1$, and that $\psi$ is a $\phi$-partite homomorphism from $F[[x - 1]]$ to $G$. Let $B$ be the number of $dp$-unpromising extensions of $\psi$ for $y$, and let $X = \psi(N_F(x) \cap V(H'_1(y)))$. Then for all $v \in N_G(X) \cap V_{\phi(x)}$, the embedding $\psi \cup \{x \to v\}$ has at most $\frac{3B}{2Np^{\ell_1}}$ extensions that are dp-unpromising for $y$.

Furthermore, the same statement holds if we replace dp-unpromising for $y$ with incomplete and ask that $\text{dist}_G^+(x, Q) = \ell_1$.

Finally, we said that a counting argument deals with vertices $x$ at distance less than $\ell_1$ to $y$. We will fill in the numbers in the proof of Theorem 3, but the idea is the following. Before we embed $x$, we have a candidate set to which we will embed $x$, which is the $G$-common neighbourhood of $\psi(N^-(x))$ in $V_{\phi(x)}$. The extensions of $\psi|_{V(H'_1(y))}$, on the other hand, can map $x$ to any vertex of the $\Gamma$-common neighbourhood of $\psi(N^-(x))$ (and nowhere else). The latter set is only a constant factor bigger than the former. The unpromising extensions of $\psi|_{V(H'_1(y))}$ are partitioned into parts according to where they map $x$. Each crossed-off vertex corresponds to a large part; since we have inductively avoided having many unpromising extensions of $\psi|_{V(H'_1(y))}$, there can only be few crossed-off vertices. If $x$ were a boundary vertex, this argument would cease to be work; every vertex of $\Gamma$ could be the image of $x$ under some extensions of $\psi|_{V(H'_1(y))}$ because the left-neighbours of $x$ are not in $H'_1(y)$, but
the candidate set of \( x \) remains the \( G \)-common neighbourhood of \( \psi(N^-(x)) \) in \( V_{\phi(x)} \). But for boundary vertices \( x \) we can apply Lemma 20 instead.

At this point, we have (modulo the detailed calculations) described how to construct homomorphisms of \( F \) into \( G \). This construction is robust in that at each step, we have many possible choices for the next embedding. Furthermore, the set of choices is locally determined: having fixed \( V_1, \ldots, V_{\ell_1} \) and \( \phi \), in order to know where we could embed any given \( x \), we need to know what the candidate set of \( x \) is (which is determined by the embedding of \( N^-(x) \)) and which vertices are crossed off (which depends only on the embedding of vertices at distance at most \( 2f_1 \) from \( x \)). Our final lemma, given by a method developed by Nenadov [23], states that such a robust and locally determined collection of homomorphisms into a typical random graph necessarily includes an embedding.

Though this lemma is inherent in [23], it is not made explicit there (but rather the underlying method for a longer proof), and so in order to be self-contained we provide a proof in Section 6.

**Lemma 21** (homomorphisms imply embedding). Given integers \( D, L \) and \( \Delta \), and a constant \( \varrho > 0 \), for all sufficiently large \( N \) the following holds. Let \( p \geq \left( \frac{10^6 DL \log^2 N}{\varrho^2 N} \right)^{1/D} \) and let \( n \leq 10^{-6} \varrho^2 N \). Let \( F \) be a graph in \( \mathcal{G}(D, \Delta, n) \) with vertex set \( [n] \) such that the natural order on \( [n] \) is a degeneracy order for \( F \). Let \( \Gamma \) be an \( N \)-vertex graph with the \((D, \frac{1000}{1000}, p)\)-star property. Let \( \Psi_0 \) be the set containing the trivial homomorphism from the empty graph \( F[\emptyset] \) to \( \Gamma \). For each \( x \in [n] \) let \( S_x \subseteq [x-1] \) be a set and let \( \Psi_x \) be a collection of homomorphisms from \( F[\{x\}] \) to \( \Gamma \) and for each \( \psi \in \Psi_x \) let \( W_\psi \) be a set of vertices \( v \) of \( \Gamma \). Suppose that the following holds for each \( x \in [n] \) and each \( \psi, \psi' \in \Psi_{x-1} \).

\[
\begin{align*}
(P1) & \quad \psi \cup \{x \mapsto v\} \in \Psi_x \text{ for each } v \in W_\psi. \\
(P2) & \quad |W_\psi| \geq \varrho^D \deg_F(x) N. \\
(P3) & \quad |S_x| \leq L. \\
(P4) & \quad \text{If } \psi|_{S_x} = \psi'|_{S_x} \text{, then } W_\psi = W_{\psi'}. 
\end{align*}
\]

Then there exists \( \psi \in \Psi_n \) which is injective.

Observe that although this lemma does not mention the coloured subgraph \( G \) of \( \Gamma \), our homomorphism construction in fact creates homomorphisms to \( G \) and so the injective homomorphism the lemma returns is the desired embedding into \( G \).

What remains is to prove the lemmas and formalise the intuition above, which as remarked above we do in the following sections. We give the proof of our main theorems in the following Section 4. In Section 5 we prove Lemma 19; in Section 6 we prove Lemma 21, and in Section 7 we prove Lemma 20.

4. Proof of the main theorems

In this section we first show how the lemmas from the last section together with the tools from Sections 2.2 and 2.3 imply Theorem 3.

**Proof of Theorem 3.** Given \( r, D, \Delta, \mu \), let \( h_0, \ell_1, h_1 \) be as in (3). Let \( L = 2\Delta \ell_1 \cdot h_1 \) and \( d = \frac{1}{2r} \). Choose

\[
f(K) = \left( \frac{d^D}{20h_1K} \right)^{h_1+1},
\]

and let \( K_0 \) be as given by Lemma 19 for \( r, D, \mu, f, h_0, h_1 \) (the constant \( C \) of Lemma 19 is irrelevant for this proof). Let \( \varrho = \frac{d^D}{4K_0} \) and \( c = 10^{-6} \varrho^2 \).
Let $\Gamma = G(N,p)$ be a random graph with $N$ sufficiently large and with $p = N^{-\frac{1}{23} + \mu}$. Asymptotically almost surely $\Gamma$ satisfies

(PR1) the $(D, \frac{1}{10}, p)$-neighbourhood property and the $(D, \frac{\varrho}{1000}, p)$-star property,

(PR2) the conclusion (a) of Lemma 19 applied with $r, D, \mu, f, h_0, h_1$, and

(PR3) the conclusion of Corollary 11 applied with $D, \mu, \frac{1}{10}$, for every graph $H$ on at most $h_1$ vertices and every set $R \subseteq V(H)$ such that $(H, R)$ is $(D, \mu)$-Spencer,

where (PR1) follows from Lemma 7 applied with $\frac{\varrho}{1000}$.

Now assume we are given an $r$-colouring of $\Gamma$. By (PR2), there is an integer $K \leq K_0$ and a colour $\chi$ such that for the subgraph $G$ of $\Gamma$ formed by all edges of colour $\chi$ there are pairwise disjoint sets $V_1, \ldots, V_{h_1} \subseteq V(G)$, each of size $\frac{N}{K}$ such that Lemma 19(a) holds (where we set $Z = \emptyset$).

We set $\nu = \frac{d^D}{2001K}$.

Our goal is to show that each $F \in G(D, \Delta, n)$ can be embedded in $G[\bigcup_i V_i]$, where $n = cN$. So fix one such graph $F$, assume that it has vertex set $[n]$ and that the natural order of this vertex set is $D$-degenerate.

Observe first that since $h_1 = 2\Delta^\ell_1$ we can use Theorem 14 applied with $\ell = \ell_1$ to conclude that there is an equipartition of $V(F)$ into sets $X_1, \ldots, X_{h_1}$ such that $\text{dist}(x, x') > \ell_1$ for each pair of vertices $x \neq x'$ in $X_i$ for each $i$. We shall embed each $X_i$ into $V_i$, that is, we shall only consider $\phi$-partite homomorphisms from $F$ to $G$ where $\phi(x) = i$ for each $x \in X_i$. Further, for each vertex $y \in V(F)$ let $H_1(y)$ and $H'_1(y)$ be the subgraphs of $F$ as defined in Setup 15, that is, $H_1(y)$ is the subgraph of $F$ induced by $y$ and all vertices preceding $y$ of left distance at most $\ell_1$ from $y$. Observe that $v(H_1(y)) \leq h_1$.

We next explain how to choose the subgraph $H'_0(y)$ that also appears in this setup. Indeed, let $T = N^{-\ell_1}$. By Lemma 12 applied with $I = \emptyset$ there is a set $T' \subseteq V(H'_1(y))$ with $T \subseteq T'$ such that $(H'_1(y), T')$ is $(D, \mu)$-Spencer and $|T'| \leq D^2|T|^2 \mu^{-1} \leq D^4 \mu^{-1} \leq h_0$. We let $H'_0(y)$ be the subgraph of $H'_1(y)$ induced by $T'$ and $H_0(y)$ be the subgraph of $H_1(y)$ induced by $T' \cup \{y\}$.

We shall now, inductively, construct the families $\Psi_x$ of $\phi$-partite homomorphisms from $F[\{x\}]$ to $G$ required by Lemma 21, which we seek to apply with constants $D, L, \Delta$ and $\varrho$.

Observe that we can apply this lemma, since $p = N^{-\frac{1}{23} + \mu}$ is sufficiently large when $N$ is large and since $F$ has $cN = 10^{-6}g^2N$ vertices. As in Lemma 21, we let $\Psi_0$ only contain the trivial homomorphism from the empty graph to $G$. When constructing $\Psi_x$ with $x \geq 1$, again with the setup of Lemma 21 in mind, we shall construct for each $\psi \in \Psi_{x-1}$ an extension set $W_\psi \subseteq V_{\phi(x)}$. To this end, we let

$$W_\psi' = \left\{ v \in V_{\phi(x)} : v \in N_G\left(\psi(N^-_F(x))\right) \right\}. \tag{4}$$

For each $y > x$ with $\text{dist}^-(y, x) \leq \ell_1$ we then “cross off” a set $C_\psi(y)$ of vertices from $W_\psi'$ that would negatively impact our prospects of embedding $y$, that is, we set

$$W_\psi = W_\psi' \setminus \bigcup_{x \in V(H'_1(y))} C_\psi(y). \tag{5}$$

Before we explain how $C_\psi(y)$ is chosen, let us first state the properties we shall inductively maintain for each pair $a, b \in \{0\} \cup V(F)$ with $a < b$ and each $\psi \in \Psi_a$, which are the following.

$(\text{ind}1)_{a,b}$ If $a \geq \max N^-_F(b)$ then $|N_G\left(\psi(N^-_F(b); V_{\phi(b)})\right)| \geq \frac{1}{2}(dp)^{\deg^-_F(b)} \cdot \frac{N}{K}$. 

If \( a < \max N_F(b) \), then let \( v(a, b) \) be the number of vertices of \( H'_1(b) \) not in \([a]\) and let \( e(a, b) \) be the number of edges of \( H'_1(b) \) not entirely in \([a] \). The number of dp-unpromising extensions of \( \psi \) for \( b \) is at most \( \kappa^{v(a, b) + 1} N^{v(a, b), p^e(a, b)} \).

The first of these two properties is chosen because \( (\text{ind}1)_{x-1} \) immediately gives

\[
|W'_\psi| \geq \frac{1}{2}(dp)^{\deg_F^{-}(x)} \cdot \frac{N}{K},
\]

for each \( \psi \in \Psi_{x-1} \). The second of these two properties is chosen so that we can inductively maintain the first by defining for each \( x \in V(H'_1(y)) \) the set \( C_\psi(y) \) as follows. For \( \psi \in \Psi_{x-1} \) we call a vertex \( v \in W'_\psi \) unpromising for \( y \) when extending \( \psi \) if there are more than \( \kappa^{v(x, y) + 1} N^{v(x, y), p^e(x, y)} \) extensions of \( \psi \cup \{ x \to v \} \) which are dp-unpromising for \( y \), where \( v(y, x) \) and \( e(y, x) \) are as in \( (\text{ind}2)_{x,y} \). We set

\[
C_\psi(y) = \{ v \in W'_\psi : v \text{ is unpromising for } y \text{ when extending } \psi \}.
\]

Let us next argue that \( (\text{ind}2)_{x-1,y} \) implies a bound on \( |C_\psi(y)| \) for each \( y \) such that \( x \in V(H'_1(y)) \). Indeed, any vertex \( v \in C_\psi(y) \) gives more than \( \kappa^{v(x, y) + 1} N^{v(x, y), p^e(x, y)} \) extensions of \( \psi \cup \{ x \to v \} \) which are dp-unpromising for \( y \); all these are also extensions of \( \psi \) which are dp-unpromising for \( y \). Therefore, using \( (\text{ind}2)_{x-1,y} \) we conclude

\[
|C_\psi(y)| \leq \kappa^{v(x, y) + 1} N^{v(x, y), p^e(x, y)},
\]

which gives

\[
|C_\psi(y)| \leq \kappa N^{\deg_{H'_1}^{-}(x)}.
\]

If \( x \) is not a boundary vertex of \( H'_1(y) \), then \( \deg_{H'_1}^{-}(x) = \deg_F^{-}(x) \). If \( x \) is a boundary vertex of \( H'_1(y) \), then \( \deg_{H'_1}^{-}(x) \) may be smaller than \( \deg_F^{-}(x) \). However in this second case we can establish the stronger bound \( |C_\psi(y)| = 0 \).

**Claim 22.** If \( (\text{ind}2)_{x-1,y} \) holds and \( x \) is a boundary vertex of \( H'_1(y) \), then \( |C_\psi(y)| = 0 \) for every \( \psi \in \Psi_{x-1} \).

**Proof.** We appeal to Lemma 20 with \( I = [x-1] \cap V(H'_1(y)) \). We can apply this lemma because \( (PR1) \) gives that \( (S1) \) is satisfied, and \( (PR3) \) gives that \( (S2) \) is satisfied. Note that \( \psi \) is a \( \phi \)-partite homomorphism from \( F|[x-1] \) to \( G \) by construction. By \( (\text{ind}2)_{x-1,y} \) the number \( B \) of dp-unpromising extensions of \( \psi \) for \( y \) is at most \( \kappa^{v(x-1,y) + 1} N^{v(x-1,y), p^e(x-1,y)} \).

Let \( X = \psi(N_F^{-}(x) \cap V(H'_1(y))) \). Observe that \( C_\psi(y) \) is a subset of \( N_G(X) \cap V_\phi(x) \).

By Lemma 20, for every \( v \in N_G(X) \cap V_\phi(x) \) the number of extensions of \( \psi \cup \{ x \to v \} \) which are dp-unpromising for \( y \) is at most

\[
\frac{3B}{2Np^{|X|}} \leq \frac{3\kappa^{v(x-1,y) + 1} N^{v(x-1,y), p^e(x-1,y)}}{2Np^{|X|}} \leq \kappa^{v(x,y) + 1} N^{v(x,y), p^e(x,y)},
\]

hence \( v \) is not unpromising for \( y \) when extending \( \psi \). Thus \( C_\psi(y) = \emptyset \). \( \square \)

Putting (7) and Claim 22 together, we see that whether or not \( x \) is a boundary vertex of \( H'_1(y) \) we have

\[
|C_\psi(y)| \leq \kappa N^{\deg_{H'_1}^{-}(x)},
\]
assuming that \((\text{ind}2)_{x-1,y}\) holds.

Now we can argue that this and (6) imply our theorem. Indeed, since there are at most \(h_1\) vertices at distance \(\ell_1\) from \(x\) and hence at most as many \(y\) such that \(x \in V(H'_1(y))\), these two inequalities together with (5) give that

\[
|W_{\psi}| \geq \frac{1}{2}(dp)^{\deg_{\tilde{F}}(x)} \cdot \frac{N}{K} - h_1 \cdot \kappa N p^{\deg_{\tilde{F}}(x)} \geq q p^{\deg_{\tilde{F}}(x)} N, 
\]

because \(q = \frac{d^D}{4h_0}\) and \(\kappa = \frac{d^D}{20h_1K}\). Hence the condition \((P2)\) of Lemma 21 holds. We then set

\[
\Psi_x = \{ \psi \cup \{ x \mapsto v \} : \psi \in \Psi_{x-1}, v \in W_{\psi} \},
\]

so that condition \((P1)\) is satisfied. We next show that \((P3)\) and \((P4)\) hold with

\[
S_x = N_{\tilde{F}}(x) \cup \bigcup_{x \not \in V(H'_1(y))} (V(H'_1(y)) \cap [x - 1]).
\]

Observe that we can only have \(x \in V(H'_1(y))\) if \(\text{dist}_F(x, y) \leq \ell_1\). Hence \(|S_x| \leq 2\Delta \ell_1 \cdot h_1 = L\), giving \((P3)\). Now let \(\psi, \psi' \in \Psi_{x-1}\) with \(\psi|_{S_x} = \psi'|_{S_x}\). By (4) we have \(W'_\psi = W'_{\psi'}\), since \(N_{\tilde{F}}(x) \subseteq S_x\). Now consider any \(y > x\) such that \(x \in V(H'_1(y))\). A vertex \(v \in W'_\psi\) is in the set \(C_{\psi}(y)\) if there are too many extensions of \(\psi \cup \{ x \mapsto v \}\) which are dp-unpromising for \(y\). By Definition 18, whether or not an extension \(\tilde{\psi}\) of \(\psi \cup \{ x \mapsto v \}\) is dp-unpromising for \(y\) only depends on \(\tilde{\psi}|_{V(H'_0(y)) \cap [x]}\). Since \(V(H'_0(y)) \cap [x - 1] \subseteq S_x\) and \(\psi|_{S_x} = \psi'|_{S_x}\) we thus have \(C_{\psi}(y) = C_{\psi'}(y)\), which together with \(W'_\psi = W'_{\psi'}\) and (5) implies \(W_{\psi} = W_{\psi'}\), giving \((P4)\).

Thus, all conditions of Lemma 21 are satisfied and we conclude that we obtain the desired injective embedding \(\tilde{\psi} \in \Psi_n\) of \(F\) in \(G\).

It remains to show that \((\text{ind}1)_{0,y}\) and \((\text{ind}2)_{0,y}\) holds for all \(y\) (the base case), and that, if we assume \((\text{ind}1)_{x-1,y}\) and \((\text{ind}2)_{x-1,y}\) hold for all \(x - 1 < y\) and all \(\psi \in \Psi_{x-1}\), then after our choice of \(\Psi_x\) also \((\text{ind}1)_{x,y}\) and \((\text{ind}2)_{x,y}\) for all \(\psi \in \Psi_x\) and all \(x < y\) hold (the induction step). We start with the base case. The only \(\psi \in \Psi_0\) is the trivial one embedding the empty graph to \(G\). Condition \((\text{ind}1)_{0,y}\) is vacuously true since \(0 \geq \min N_{\tilde{F}}(y)\) only holds when \(y\) does not have any left-neighbours, in which case we are merely requiring that \(|V(\tilde{\psi}(y))| \geq \frac{N}{2K}\), which is the case. For checking condition \((\text{ind}2)_{0,y}\), observe that a dp-unpromising extension for \(y\) of the trivial \(\psi\) is simply an injective \(\phi\)-partite embedding \(\psi'\) of \(H'_1(y)\) in \(G\) such that \(\psi'|_{V(H'_0(y))}\) is dp-unpromising for \(y\), where the fact that the embedding has to be injective follows from the fact that all vertices of \(H'_1(y)\) are in different sets \(X_i\) by the definition of the \(X_i\). Hence we want to count the number \(Y\) of such embeddings of \(H'_1(y)\). To this end, observe that by \((PR2)\) the number of \(\phi\)-partite embeddings of \(H'_0(y)\) in \(G\) which are dp-unpromising is bounded above by

\[
Y_0 = f(K) \cdot p^v(H'_0(y)) \left(\frac{N}{K} \right)^{v(H'_0(y))}.
\]

By definition \((H'_1(y),V(H'_0(y)))\) is \((D,\mu)\)-Spencer. We conclude that for each fixed such unpromising \(H'_0(y)\)-copy, by \((PR3)\) there are at most

\[
Y_1 = 2N^v(H'_1(y)) - v(H'_0(y)) p^v(H'_1(y)) - c(H'_0(y))
\]

extensions of this \(H'_0(y)\)-copy to a \(H'_1(y)\)-copy in \(\Gamma \supseteq G\). Hence, we get

\[
Y \leq Y_0 \cdot Y_1 = f(K) \cdot p^v(H'_0(y)) \left(\frac{N}{K} \right)^{v(H'_0(y))} \cdot 2N^v(H'_1(y)) - v(H'_0(y)) p^v(H'_1(y)) - c(H'_0(y))
\]

\[
= \frac{2f(K)}{K^v(H'_0(y))} N^v(0,y) p^v(0,y) \leq \kappa^v(0,y) + 1 \cdot N^v(0,y) p^v(0,y),
\]

where \(\kappa^v(0,y) = \max \left\{ \frac{2f(K)}{K^v(H'_0(y))} : y \geq 0 \right\} \).
where $v(0,y)$ and $v(0,y)$ are as in $(\text{ind}2)_{0,y}$ and where we use $\kappa = \frac{d\bar{D}}{2\psi_0^1 h_1 k}$. This gives $(\text{ind}2)_{0,y}$.

Let us now turn to the induction step. For this, assume that $(\text{ind}1)_{x-1,y}$ and $(\text{ind}2)_{x-1,y}$ hold for all $x - 1 < y$ and all $\psi \in \Psi_{x-1}$. Let $\Psi_x$ be as chosen in (10), and consider any $\psi' = \psi \cup \{x \mapsto v\} \in \Psi_x$. Let $y > x$. If $x \notin V(H(y))$, then $(\text{ind}1)_{x,y}$ and $(\text{ind}2)_{x,y}$ follow immediately from $(\text{ind}1)_{x-1,y}$ and $(\text{ind}2)_{x-1,y}$, because then $v(x,y) = v(x-1,y)$ and $e(x,y) = e(x-1,y)$ and by construction $\psi \in \Psi_{x-1}$. Hence we can assume that $x \in V(H(y))$.

We only need to check $(\text{ind}1)_{x,y}$ when $x \geq \max N_F(y)$, and if $x > \max N_F(y)$ then $(\text{ind}1)_{x,y}$ follows from $(\text{ind}1)_{x-1,y}$. So assume $x = \max N_F(y)$. In this case $x$ is the last of the left neighbours of $y$ and therefore the last vertex in $H(y)$, hence $v(x,y) = 0$ and $e(x,y) = 0$. By definition of $\Psi_x$, we have that $\psi'$ is such that $v$ is not unpromising for $y$ when extending $\psi$, and hence there are at most

$$\kappa^{v(x,y)+1} N^{v(x,y)} p^{e(x,y)} = \kappa < 1$$

extensions of $\psi'$ which are $dp$-unpromising for $y$. Since all vertices of $H(y)$ are embedded by $\psi'$, this can only happen when $\psi'$ induces a $dp$-promising embedding of $H(y)$ for $y$. Thus $(\text{ind}1)_{x,y}$ holds.

For checking $(\text{ind}1)_{x,y}$ assume that $x < \max N_F(y)$. By definition, we chose $\psi'$ such that $v$ is not unpromising for $y$, and hence there are at most $\kappa^{v(x,y)+1} N^{v(x,y)} p^{e(x,y)}$ extensions of $\psi'$ which are $dp$-unpromising for $y$, which is $(\text{ind}2)_{x,y}$.

The proof of Theorem 1 is an easy modification of the proof of Theorem 3; we sketch the required modification and omit the details.

**Sketch proof of Theorem 1.** We begin the proof of Theorem 1 as for Theorem 3, with $D := \Delta - 1$. We can use the same choice of constants as in that proof; and we require the same properties of $G(N,p)$, except that if $\Delta = 3$ the constant $C$ of Lemma 19 is relevant (we require $p \geq CN^{-2/5}$ for this $C$) and we add to $(PR2)$ that $G(N,p)$ satisfies the properties $(b)$ and $(c)$ of Lemma 19, obtaining

$(PR'2)$ the conclusions $(a)$, $(b)$ and $(c)$ of Lemma 19 applied with $r, D, \mu, f, h_0, h_1$

We suppose that $\Gamma = G(N,p)$ satisfies $(PR1)$–$(PR3)$. Given an $r$-colouring of $E(\Gamma)$, we obtain a colour $\chi$ from Lemma 19 as in the proof of Theorem 3. At this point, we diverge slightly from the strategy of that proof. Given $F \in \mathcal{G}(\Delta, cN)$, we pick a first component $F'$ of $F$ and try to embed it in colour $\chi$. We do this as follows.

We set $Z$ to be the image of the previously embedded components (for the first component, the empty set). We obtain sets $V_1, \ldots, V_{h_1}$ from Lemma 19 as in the proof of Theorem 3. Now, we separate three cases. If $F'$ is not $\Delta$-regular, then because it is connected it is $D$-degenerate, where $D = \Delta - 1$. In this case, we embed $F'$ in $D$-degeneracy order exactly as in the proof of Theorem 3, which in particular means we do not use any vertices of $Z$. If $F = K_4$, we use $(PR'2)$ to find a copy of $K_4$ outside $Z$ and embed $F'$ to it. Finally, suppose $F'$ is $\Delta$-regular and not $K_4$. $F'$ contains a cycle, and so it contains a shortest cycle, which is necessarily an induced cycle. If this cycle has $2\mu^{-1}$ or fewer vertices, let its vertices be $Q$. If not, let $Q$ be a set of $2\mu^{-1}$ consecutive vertices on the cycle, so $F'[Q]$ is a path.

We let $H_1(Q)$ be as defined in Setup 15, and obtain $H_1(Q)$ as follows. Let

$$T = \bigcup_{q \in Q} N_{F'}(Q) \setminus Q$$
be the neighbours of $Q$. We have $|T| \leq (\Delta - 1)|Q| \leq 2\mu^{-1}D$. We apply Lemma 12 with $I = \emptyset$ to find $T' \subseteq V(H'_1(Q))$ with $T \subseteq T'$, with $|T'| \leq D^2|T|^\mu \mu^{-1} \leq 4D^4\mu^{-3} \leq h_0$, and such that $(H'_1(Q), T')$ is $(D, \mu)$-Spencer. This lemma applies since $H'_1(Q)$ is $D$-degenerate: it has maximum degree at most $D + 1$ and all its components contain a vertex of degree $D$ (a neighbour of a vertex of $Q$). We let $H'_0(Q) = F'[T']$.

Now $F' - Q$ is a $D$-degenerate graph. We copy almost exactly the proof of Theorem 3 to embed it in a $D$-degeneracy order (note that the final vertex in this order is necessarily the last neighbour of $Q$). The only difference is that we add one more collection of inductive properties $(\text{ind}1)_{v(F' - Q),Q}$ and $(\text{ind}2)_{a,Q}$, where $0 \leq a < v(F' - Q)$. These are defined as follows:

- $(\text{ind}1)_{v(F' - Q),Q}$ is a property of $\psi \in \Psi_{v(F' - Q)}$, stating that the restriction of $\psi$ to $H'_0(Q)$ is completable.
- $(\text{ind}2)_{a,Q}$ is a property of $\psi \in \Psi_{a}$, stating that the number of uncompletable extensions of $\psi$ is at most $\kappa v(a,Q) + 1 N v(a,Q) e(a,Q)$, where $v(a,Q)$ is as before defined to be the number of vertices of $H'_1(Q)$ not in $[a]$, and $e(a,Q)$ the number of edges of $H'_1(Q)$ not entirely in $[a]$.

Observe that this is only different to the inductive properties in the proof of Theorem 3 in that we replace ‘$dp$-unpromising for $b$’ with ‘uncompletable’. We continue from this point to copy the proof of Theorem 3. We define the set $C_{\psi}(Q)$ analogously to $C_{\psi}(y)$, again replacing ‘unpromising for $y$’ with ‘completable’. We define $W_{\psi}$ similarly to (5), removing in addition $C_{\psi}(Q)$ whenever $x \in V(H'_1(Q))$, and as before doing this implies that we maintain all the inductive properties. The analogous argument as for $|C_{\psi}(y)|$, making the same replacement, shows that $|C_{\psi}(Q)|$ is guaranteed to be small (we obtain the same numerical bounds). We need to account for $C_{\psi}(Q)$ in (9), which we can do by replacing $h_1$ by $h_1 + 1$ in the middle term. The final inequality continues to hold; the rest of the proof goes through as written.

These modifications to the proof of Theorem 3 show that we can embed $F' - Q$ in colour $\chi$, without using vertices of $Z$, and with the additional property $(\text{ind}1)_{v(F' - Q),Q}$. This additional property states that we can extend $\psi$ to an embedding $\psi'$ of $F'$ in colour $\chi$, without using vertices of $Z$. This is what we wanted to prove.

It remains to complete the embedding of $F$ by embedding the remaining components. We simply choose another component $F'$ and repeat the above strategy, from the point where we set $Z$ to be the image of the previously embedded components, and continue until $F$ is entirely embedded: we can do this because Lemma 19 allows any choice of $Z$ with $|Z| \leq c N = V(F)$. This completes the proof of Theorem 1.

The reader can quickly check that if $\Delta = 3$, the only place in the above argument where $p \geq N^{-\frac{1}{2} + \mu}$ does not suffice is when we ask for $(c)$ to hold. In turn, we need this property only if $F$ contains a copy of $K_4$. It follows that if $p \geq N^{-\frac{1}{2} + \mu}$ then there exists $c > 0$ such that $G(N, p)$ is a.a.s. $r$-partition universal for the class of $c N$-vertex graphs with maximum degree 3 and no copy of $K_4$. \hfill \Box

5. Regularity and promising embeddings: The proof of Lemma 19

In this section, we prove Lemma 19. The basic idea of the proof is fairly straightforward: given a typical $\Gamma = G(n, p)$ and an $r$-colouring of $E(\Gamma)$, we apply the Sparse Regularity Lemma to the colour graphs, and thus obtain a coloured nearly complete auxiliary graph on the parts of the regularity partition (colouring each regular pair by densest colour). We find a $t$-vertex monochromatic colour $i$ clique in this auxiliary graph. Now an application of the counting
KLR conjecture shows that at most a small fraction of partite embeddings into the colour \( i \) graph on the corresponding parts are unpromising.

However there is a subtlety: with this argument, the ‘small fraction’ cannot be made to depend in any useful way on the regularity constants (it will be too big). In order to deal with this, we in fact apply a ‘strengthened’ version of the Sparse Regularity Lemma, and then show that after discarding a small fraction of each part, the fraction of partite embeddings which are unpromising in what remains can have any desired dependence on the regularity constants. In order to give this argument, we begin by giving the required regularity definitions and the strengthened Sparse Regularity Lemma we want. Our strengthened Sparse Regularity Lemma applies for graphs which are upper regular, in the following sense.

We say an \( n \)-vertex graph \( G \) is \((\eta, p)\)-upper regular if for any disjoint vertex sets \( X, Y \subseteq V(G) \) such that \(|X|, |Y| \geq \eta n\) we have \( e(G[X,Y]) \leq (1+\eta)p|X||Y|\). A standard application of the Chernoff bound shows that if \( \Gamma = G(n,p) \) is a typical random graph, with \( p \gg n^{-1} \), then it is \((\eta, p)\)-upper regular, and so all its subgraphs are also \((\eta, p)\)-upper regular.

**Lemma 23.** Given \( \eta > 0 \) there exists \( C > 0 \) such that for any \( p \geq Cn^{-1} \), w.h.p. \( \Gamma = G(n,p) \) has the property that all \( n \)-vertex subgraphs of \( \Gamma \) are \((\eta, p)\)-upper regular. Furthermore if \( X \) and \( Y \) are any two disjoint vertex sets of \( \Gamma \) with \(|X|, |Y| \geq \eta n\), we have \( e(\Gamma[X,Y]) \geq (1-\eta)p|X||Y|\).

**Proof.** Given \( \eta > 0 \), without loss of generality we may assume \( \eta \leq \frac{1}{2} \). Let \( C = 6\eta^{-4} \). Since for any \( G \subseteq \Gamma \) and any \( X, Y \subseteq V(\Gamma) \) we have \( e_G(X,Y) \leq e_\Gamma(X,Y) \), it suffices to prove that w.h.p. \( G(n,p) \) is \((\eta, p)\)-upper regular.

Given \( X, Y \subseteq [n] \), with \(|X|, |Y| \geq \eta n\), the probability that \((X,Y)\) witnesses a failure of upper regularity of \( G(n,p) \) is, by the Chernoff bound, at most

\[
\exp\left(-\frac{\eta^2 p|X||Y|}{3}\right) \leq \exp(-2n).
\]

Since there are at most \( 2^{2n} \) choices of \( X \) and \( Y \), by the union bound the probability that \( G(n,p) \) fails to be \((\eta, p)\)-upper regular is at most \( 2^{2n}\exp(-2n) \), which tends to zero as \( n \to \infty \). Since the same Chernoff bound applies to the probability of \( e_\Gamma(X,Y) < (1-\eta)p|X||Y| \), we obtain the second conclusion by the same argument. \( \Box \)

Given a graph \( G \), and \( p \in [0,1] \), and disjoint vertex sets \( U, V \subseteq V(G) \), we define the \( p\)-density of \((U,V)\) to be \( d_p(U,V) := \frac{e_G(U,V)}{p|U||V|} \). We say \((U,V)\) is \((\epsilon, d, p)\)-regular in \( G \) if \( d_p(U',V') = d \pm \epsilon \) for every \( U' \subseteq U \) and \( V' \subseteq V \) with \(|U'| \geq \epsilon|U| \) and \(|V'| \geq \epsilon|V| \). We say \((U,V)\) is \((\epsilon, p)\)-regular in \( G \) if there exists \( d \) such that \((U,V)\) is \((\epsilon, d, p)\)-regular, and \((\epsilon, \geq d, p)\)-regular if there is \( d' \geq d \) such that \((U,V)\) is \((\epsilon, d', p)\)-regular.

We say a partition \( \mathcal{P} \) of a vertex set \( V(G) \) is equitable if \(|U| = |V| \pm 1 \) for every \( U, V \in \mathcal{P} \). Given a partition \( \mathcal{P} \) of \( V(G) \), we say a partition \( \mathcal{P}' \) of \( V(G) \) refines \( \mathcal{P} \) if for every \( U \in \mathcal{P} \) and \( V \in \mathcal{P}' \) we have either \( V \subseteq U \) or \( U \cap V = \emptyset \). We say \( \mathcal{P}' \) is an equitable refinement of \( \mathcal{P} \) if there is some \( s \in \mathbb{N} \) such that each part of \( \mathcal{P} \) contains exactly \( s \) parts of \( \mathcal{P}' \), and if \( U \) is any part of \( \mathcal{P} \), and \( V, V' \in \mathcal{P}' \) are subsets of \( U \), then \(|V| = |V'| \pm 1 \).

Given a partition \( \mathcal{P} \) of \( V(G) \), in which \(|U| = |V| \pm 2 \) for each \( U, V \in \mathcal{P} \), we say \( \mathcal{P} \) is \((\epsilon, p)\)-regular for \( G \) if for all but at most \( \epsilon|\mathcal{P}|^2 \) pairs \( U, V \in \mathcal{P} \), the pair \((U,V)\) is \((\epsilon, p)\)-regular in \( G \).

We need the following ‘strengthened’ variant of the Sparse Regularity Lemma. The \( p = 1 \) dense case was proved by Alon, Fischer, Krivelevich and Szegedy [3].

**Lemma 24** (strengthened sparse regularity lemma).

For every \( \epsilon > 0 \), every function \( f: \mathbb{N} \to \mathbb{R}^+ \), and every \( k_0 \in \mathbb{N} \) there are \( K \in \mathbb{N} \) and \( \eta > 0 \) such
that for all sufficiently large $n$ and $p > \frac{1}{mn}$, the following holds. If $G_1, \ldots, G_r$ are edge-disjoint graphs on vertex set $[n]$, which are $(\eta, p)$-upper regular, then there is a coarse partition $\mathcal{P}_c$ and a fine partition $\mathcal{P}_f$ of $[n]$ such that for each $i \in [r]$,

(RL1) $\mathcal{P}_c$ and $\mathcal{P}_f$ have between $k_0$ and $K$ parts, $\mathcal{P}_c$ is equitable, and $\mathcal{P}_f$ is an equitable refinement of $\mathcal{P}_c$,

(RL2) $\mathcal{P}_c$ is an $(\varepsilon, p)$-regular partition for $G_i$,

(RL3) $\mathcal{P}_f$ is an $(f(|\mathcal{P}_c|), p)$-regular partition for $G_i$,

(RL4) $d_p(X', Y'; G_i) = d_p(X, Y; G_i) + \varepsilon$ for all but at most $\varepsilon|\mathcal{P}_f|^2$ pairs $(X', Y')$ in $\mathcal{P}_f$, where $X, Y$ are the unique parts of $\mathcal{P}_c$ such that $X' \subseteq X$ and $Y' \subseteq Y$.

We defer the proof of this lemma, which follows a standard strategy, to Appendix A.

In order to prove Lemma 19, we also need the following counting lemma.

Lemma 25. Given any graph $H$ with $m_2(H) \geq 1$, and any $\delta, d, \mu > 0$, there exists $\varepsilon > 0$ such that the following holds. For every $\eta > 0$ there exists $C > 0$ such that if either $p \geq \eta^{-1/m_2(H)}$ or $p \geq C \eta^{-1/m_2(H)}$ and $H$ is strictly $2$-balanced, then a.a.s. the following holds for $\Gamma = G(N, p)$. Suppose $V_1, \ldots, V_r$ are any disjoint vertex sets, each of size at least $\eta n$, in $V(\Gamma)$, and $\phi : V(H) \to [r]$ is any map such that $\phi(x) \neq \phi(y)$ if $xy \in E(H)$. Suppose that $|V_i| \geq \delta |V_j|$ for any $i, j \in [r]$. Let $G$ be any subgraph of $\Gamma$ such that if $xy \in E(H)$ then $(V_{\phi(x)}, V_{\phi(y)})$ is an $(\varepsilon, d_{xy}, p)$-regular pair in $G$, with $d_{xy} \geq d$. Then the number of graph embeddings from $H$ to $G$ mapping $x$ into $V_{\phi(x)}$ for each $x \in V(H)$ is equal to

$$
(1 \pm \delta) \left( \prod_{x \in V(H)} |V_{\phi(x)}| \right) \cdot \prod_{xy \in E(H)} pd_{xy}.
$$

In the special case that $r = v(H)$, $\phi$ is a bijection, $|V_1| = \cdots = |V_r| = n$ and $e(V_i, V_j) = m \geq dpn^2$ for each $i, j$ such that $\phi^{-1}(i)\phi^{-1}(j) \in E(H)$, this lemma was stated by Conlon, Gowers, Samotij and Schacht, as [9, Lemma 1.6(ii)] for the strictly $2$-balanced case, and [9, Section 5.2] for the general case. We use standard methods to deduce Lemma 25 from this special case in Appendix A.

We now deduce from Lemma 25 the following statement which we will use to count unpromising embeddings; the difference between unpromising and ‘poor’ of the following lemma is that the factor $\frac{1}{2}$ is replaced by $\frac{3}{4}$.

Lemma 26. Given any $D \in \mathbb{N}$, any $D$-degenerate $H$, and any $d, \mu, \varrho > 0$ there exists $\nu > 0$ such that for all $\eta > 0$ the following holds. If $p \geq \eta^{-1/D}$, then a.a.s. $\Gamma = G(n, p)$ has the following property. Given any $G \subseteq \Gamma$ and disjoint vertex sets $(V_x)_{x \in V(H)}$, each of size at least $\eta n$, suppose $(V_x, V_x')$ is $(\nu, d_{xx'}, p)$-regular in $G$ for each $xx' \in E(H)$ and $d_{xx'} \geq d$ for each $xx' \in E(H)$. Let $y$ be the last vertex of $H$ in $D$, and let $H' = H - y$.

We say $\psi$ is a poor embedding of $H'$ if $\psi$ is an embedding of $H'$ into $G$ such that $x \in V_x$ for each $x \in V(H')$ and such that $|N_G(\psi(N^{-}(y)); V_y)| < \frac{3}{4}(dp)\deg^{-}(y)|V_y|$. Then the number of poor embeddings of $H'$ is at most

$$
\varrho \left( \prod_{xx' \in E(H')} d_{xx'}p \right) \cdot \left( \prod_{x \in V(H')} |V_x| \right).
$$

Proof. Given $D \in \mathbb{N}$, a $D$-degenerate graph $H$, and $d, \mu, \varrho > 0$, we choose $\delta > 0$ such that $320\delta(1 + \delta) < \varrho$. Let $H'$ be the graph obtained from $H$ by adding a single vertex $y^+$ to $H$,
whose neighbourhood is $N_H(y)$; we require $\nu > 0$ small enough for each of three applications of Lemma 25, with input $D$, $H'$, $\delta, d, \mu$ and similarly replacing $H'$ with $H$ and with $H^+$. Observe that all three of these graphs are $D$-degenerate so by Lemma 13 have 2-density at most $D$.

Assume now that $p \gg n^{\mu - 1/D}$ and that $\Gamma = G(n, p)$ satisfies the good event of each of these three applications of Lemma 25. Given $G$ and $(V_x)_{x \in V(H)}$, as in the lemma statement, we aim to show that there are few poor embeddings. We prove this by an application of the second moment method, using Lemma 25 with $\delta$ to estimate moments.

Let $S$ denote the set of embeddings of $H'$ into $G$ such that $x \in V_x$ for each $x \in V(H')$. Suppose $\psi$ is chosen uniformly from $S$; then let $X := |N_G(\psi(N^-(y)); V_y)|$.

Let $h'$ denote the number of embeddings of $H'$ to $G$ such that $x \in V_x$ for each $x \in V(H')$, and similarly let $h$ be the number of embeddings of $H$ to $G$ such that $x \in V_x$ for each $x \in V(H)$. Finally let $h^+$ be the number of embeddings of $H^+$ to $G$ such that $x \in V_x$ for each $x \in V(H')$ and $y, y^+ \in V_y$. Then by definition we have $\mathbb{E}[X] = h/h'$ and $\mathbb{E}[X^2] = (h^+ + h)/h'$. Applying the good event of Lemma 25, we conclude

$$\mathbb{E}[X] = \frac{(1 + \delta)(\prod_{x \in V(H)} |V_x|) \cdot \prod_{x \in V(H')} |V_x'|}{(1 + \delta)(\prod_{x \in V(H')} |V_x'|)} = (1 + 3\delta)|V_y| \cdot \prod_{x \in N_H(y)} d_{xy} \rho$$

and similarly

$$\mathbb{E}[X^2] = (1 + 3\delta)|V_y|^2 \cdot \prod_{x \in N_H(y)} d_{xy}^2 \rho^2 + (1 + 3\delta)|V_y| \cdot \prod_{x \in N_H(y)} d_{xy} \rho = (1 + 4\delta)|V_y|^2 \cdot \prod_{x \in N_H(y)} d_{xy}^2 \rho^2.$$  

Now by Chebyshev’s inequality we have

$$\mathbb{P}[X < \frac{3}{4} \mathbb{E}[X]] < \frac{16\mathbb{E}[X^2] - 16\mathbb{E}[X]^2}{\mathbb{E}[X]^2} \leq \frac{16(1 + 4\delta)|V_y|^2 \cdot \prod_{x \in N_H(y)} d_{xy}^2 \rho^2 - 16(1 - 3\delta)^2|V_y|^2 \cdot \prod_{x \in N_H(y)} d_{xy}^2 \rho^2}{(1 - 3\delta)^2|V_y|^2 \cdot \prod_{x \in N_H(y)} d_{xy}^2 \rho^2} = \frac{16(1 + 4\delta) - 16(1 - 3\delta)^2}{(1 - 3\delta)^2} \leq 320\delta,$$

and in particular there are at most $320\delta h'$ embeddings of $H'$ that are poor, which, putting in our estimate of $h'$ and by choice of $\delta$, gives the desired upper bound. \qed

Next, we give a very similar lemma which we will use to count completable embeddings.

**Lemma 27.** Given any $D \geq 2$, any $d, \mu, \rho > 0$, any connected $H$ with maximum degree at most $D + 1$, and any $Q$ which is either an induced cycle in $H$, or an induced path in $H$ with $2\mu^{-1}$ vertices, there exists $\nu > 0$ such that for all $\eta > 0$ the following holds. If $D = 2$ and $p \gg n^{-2/5}$, or $D \geq 3$ and $p \gg n^{\mu - 1/D}$, then a.a.s. $\Gamma = G(n, p)$ has the following property.

Given any $G \subseteq \Gamma$ and disjoint vertex sets $(V_x)_{x \in V(H)}$, each of size at least $\eta n$, suppose $(V_x, V_{x'})$ is $(\nu, d_{xx'}, \rho)$-regular in $G$ for each $xx' \in E(H)$ and $d_{xx'} \geq d$ for each $xx' \in E(H)$. Suppose, if $D = 2$, that $H \neq K_4$.

Let $H' = H - V(Q)$. We say $\psi$ is a noncompletion embedding of $H'$ if $\psi$ is an embedding of $H'$ into $G$ such that $x \in V_x$ for each $x \in V(H')$ which does not extend to any embedding of $H$ to $G$ such that $x \in V_x$ for each $x \in V(H)$. Then the number of noncompletion embeddings
of $H'$ is at most

$$
q \left( \prod_{x x' \in E(H')} d_{x x'} p \right) \cdot \left( \prod_{x \in V(H')} |V_x| \right).
$$

Note that a noncompletion embedding is not quite the same as an embedding which is not cometable: the difference is where the vertices we can choose for the extension are located.

**Proof.** Much as in the proof of Lemma 26, we use the second moment method to obtain our desired bound. To that end, we define $H^+$ to be the graph obtained from $H$ by adding (as before) a second copy $Q'$ of $Q$, with vertices $y'$ for each $y \in Q$ with the same neighbours in $H'$ as $y$, and $H' = H - V(Q)$. The 2-density of each of these graphs is upper bounded by Lemma 13. In addition, for our second moment computation, we need bounds on the number of graphs $H_I$ for each proper subset $I \subseteq V(Q)$, where $H_I$ is the graph obtained from $H^+$ by adding edges from each $y \in I$ to the vertices $N(y')$ and then removing $y'$.

From Lemma 13 we have $m_2(H') \leq D$, since $H'$ is $D$-degenerate. If $D = 2$ we have $H \neq K_4$, so $m_2(H), m_2(H^+) \leq 2 \cdot \frac{2\mu^{-1}}{2\mu - 2} = \frac{2}{2\mu - 2}$, while if $D \geq 3$ we have $m_2(H) \leq D$ and $m_2(H^+) \leq \frac{D}{2}$. Note that the bounds on $m_2(H^+)$ rely on the fact that if $Q$ is an induced path then $v(Q) = 2\mu^{-1}$. In particular, $\frac{1}{m_2(F)} \geq \frac{1}{2} \geq \frac{1}{D} - \frac{\mu}{2}$ for all the graphs $F$ we consider.

Finally, we deal with the graphs $H_I$. The expected number of copies of $H_I$ in $G(n, p)$ is $\Theta(n^{v(H_I)} \cdot n^{v(H_I)})$. Observe that if $v \in V(Q)$ is in $I$, but a neighbour of $v$ in $Q$ is not in $I$, then the graph $H_{I \setminus \{v\}}$ has one more vertex, and at most $D$ more edges, than $H_I$. In particular, if $p \geq \mu^{-1/D}$ for any constant $\mu > 0$ this means that the expected number of copies of $H_{I \setminus \{v\}}$ in $G(n, p)$ is bigger by a factor at least $n^\mu$ than the expected number of copies of $H_I$. Hence, by Markov's inequality for any $I$ which is neither empty nor equal to $V(Q)$, the expected number of copies of $H_I$ in $G(n, p)$ is, with probability at least $1 - n^{-\mu/2}$, not more than $n^{v(H^+) - \mu/2} p_\mu(H^+)$. This simple bound on $H_I$ copies will suffice for the second moment calculation (and so we do not need to estimate the 2-densities of the $H_I$).

Given $d, \mu, \rho > 0$, we choose $\delta > 0$ such that $32\rho \delta (1 + \delta) < \rho$. We require $\nu > 0$ small enough for each of three applications of Lemma 25, with input $D$, $H'$, $\delta, d, \frac{1}{2} \mu$ and similarly replacing $H'$ with $H$ and with $H^+$.

Assume now that $p \geq n^{\mu^{-1/D}}$, that $\Gamma = G(n, p)$ satisfies the good event of each of these three applications of Lemma 25, and in addition that for each proper $I \subseteq V(Q)$ the number of copies of $H_I$ in $\Gamma$ is at most $n^{v(H^+) - \mu/2} p_\mu(H^+)$. Given $G$ and $(V_x)_{x \in V(H)}$ as in the lemma statement, we aim to show that there are few partite embeddings that are noncompletion.

Let $S$ denote the set of embeddings of $H'$ into $G$ such that $x \in V_x$ for each $x \in V(H)$. Suppose $\psi$ is chosen uniformly from $S$; then let $X$ count the number of extensions of $\psi$ to embeddings of $H$ such that $x \in V_x$ for each $x \in V(H)$.

Let $h'$ denote the number of embeddings of $H'$ to $G$ such that $x \in V_x$ for each $x \in V(H')$, and similarly let $h$ be the number of embeddings of $H$ to $G$ such that $x \in V_x$ for each $x \in V(H)$. Finally let $h^+$ be the number of embeddings of $H^+$ to $G$ such that $x \in V_x$ for each $x \in V(H')$ and $y, y' \in V_y$ for each $y \in V(Q)$. Then by definition we have $\mathbb{E}[X] = h/h'$ and

$$
\mathbb{E}[X^2] \leq \frac{h^+ + h + 2^{v(Q)} n^{v(H^+) - \mu/2} p_\mu(H^+)}{h'} \leq \frac{(1 + n^{-\mu/4})h^+ + h}{h'}.
$$
where the final inequality uses the good event of Lemma 25 to see $h^+ = \Theta(n^{\Theta(H^+)}p^{\Theta(H^+)})$. Using the good event again to give precise values for $h', h^+$, we conclude

$$
\mathbb{E}[X] = \frac{(1 + \delta)(\prod_{x \in V(H)} |V_x|) \cdot \prod_{x' \in E(H)} d_{x'p}}{(1 + \delta)(\prod_{x \in V(H')} |V_x|) \cdot \prod_{x' \in E(H')} d_{x'p}}
$$

$$
= (1 + 3\delta) \left( \prod_{x \in V(Q)} |V_x| \right) \prod_{e \in E(H), E(H')} d_{ep}
$$

and similarly

$$
\mathbb{E}[X^2] = (1 + 5\delta) \left( \prod_{x \in V(Q)} |V_x|^2 \right) \prod_{e \in E(H), E(H')} d_{ep}^2 p^2.
$$

Now by Chebyshev’s inequality we have

$$
\mathbb{P}[X = 0] \leq \frac{\mathbb{E}[X^2] - \mathbb{E}[X]^2}{\mathbb{E}[X]^2}
$$

$$
= \frac{(1 + 5\delta) - (1 - 3\delta)^2}{(1 - 5\delta)^2} \leq 20\delta,
$$

and in particular there are at most $20\delta h'$ partite embeddings of $H'$ which are noncompletion embeddings, which by the good event of Lemma 25 and choice of $\delta$ gives the desired upper bound. \qed

We now briefly explain how the proof of Lemma 19 goes. We will apply Lemma 24 to the coloured $G(N, p)$, and find within it a collection of $h_1$ parts of the coarse partition and a colour $\chi$ such that, for each pair of coarse parts, most of the pairs of fine parts contained within these coarse parts are good, i.e.

very regular and dense in colour $\chi$. We say a given fine part is good if most of the pairs of fine parts it is in (with fine parts in any of the other $h_1 - 1$ coarse parts) are good, and otherwise bad. We throw away the exceptional fine parts which are bad and the vertices of $Z$, and what remains gives the sets $V_1, \ldots, V_{h_1}$ of the lemma statement.

Given $H_0$ and $\phi$, we suppose for a contradiction that there are many unpromising $\phi$-partite embeddings in colour $\chi$ of $H_0'$. By averaging, there exist fine parts $V_x \subseteq V_{\phi(x)}$ for each $x$ which contain at least the average number of these unpromising embeddings. For each $x \in V(H_0')$, the fraction of fine parts $V_y \subseteq V_{\phi(y)}$ which do not form a dense regular pair in colour $\chi$ with $V_x$ is small by construction, so most choices of $V_y$ form a dense regular pair with every chosen $V_x$. Now, in order for an embedding of $H_0'$ into the parts $(V_x)_{x \in V(H_0')}$ to be unpromising, it necessarily has to be poor with respect to a significant fraction of these $V_y$. By averaging, there is a choice of $V_y$ such that many embeddings of $H_0'$ into the parts $(V_x)_{x \in V(H_0')}$ are poor with respect to $V_y$. Now, we observe that all these poor embeddings in colour $\chi$ are also embeddings into $\Gamma$. Considering the graph $G'$ with edges all of $\Gamma$ among the $V_x$, and only those of colour $\chi$ between $V_y$ and the $V_x$ (so that all pairs are dense and very regular), we have a contradiction to Lemma 26.

To avoid a profusion of subscripts, we write $H$ and $H'$ in the following proof rather than $H_0$ and $H_0'$.

**Proof of Lemma 19.** Given $r, D \geq 2$, $\mu > 0$ and $f : \mathbb{N} \to \mathbb{R}^+$, and $h_0, h_1 \in \mathbb{N}$, we set as in the lemma statement $d = \frac{1}{2}r$, and we set

$$
\varepsilon = D^{-42^{-4}r^{-4}h_1 r^{-4}h_1}.
$$
We choose $g : \mathbb{R}^+ \to \mathbb{R}^+$ strictly decreasing such that for any given $k$, and any $D$-degenerate $H$ with at most $h_0$ vertices, Lemma 26, with input $\delta = \frac{1}{100} f(k)$, $d$, and $\mu$, returns a regularity parameter at least $g(k)$. Suppose in addition that $g(k)$ is also small enough to be the regularity parameter for Lemma 27, with input $\delta = \frac{1}{100} f(k)$, $d$, and $\mu$ and any $H$ with at most $h_0$ vertices and maximum degree at most $D + 1$, and induced path or cycle $Q$. Suppose that Lemma 25, with the graph $K_4$ (whose 2-density is $5/2$ and which is strictly 2-balanced) and with input $\delta = 1/2$ and $d$, returns a regularity parameter at least $g(1)$. Now suppose Lemma 24, for input $\epsilon$, the function $k \to \frac{1}{2} g(2k)$ and $k_0 = \epsilon^{-1}$, returns $\frac{1}{4} K_0$ and $\eta > 0$; without loss of generality we may assume $\eta < \frac{1}{2K_0 g(K_0)}$. We further suppose $C$ is sufficiently large for Lemma 25 with input as above and $\eta$.

Given $p \geq N^{\mu - 1/D}$, let $\Gamma = G(N, p)$, and suppose that $\Gamma$ satisfies the good event of Lemma 23 for input $\eta$, and satisfies the good event of Lemma 26 for input $\frac{1}{100} f(k)$, $d$, $\mu$, $\eta$, for each $D$-degenerate $H$ with $v(H) \leq h_0$ and each $1 \leq k \leq 2K_0$. If in addition $D = 2$ and $p \geq CN^{-2/5}$, or $D \geq 3$, suppose that $\Gamma$ satisfies the good event of Lemma 27 for input $\frac{1}{100} f(k)$, $d$, $\mu$, $\eta$, for each $1 \leq k \leq 2K_0$, each connected graph $H$ of maximum degree at most $D + 1$, and each induced path of $2\mu^{-1}$ vertices or cycle $Q$, and the good event of Lemma 25 for input as above. Now let an $r$-colouring of $E(\Gamma)$ be given.

The following claim finds the colour $\chi$ and sets $V_1, \ldots, V_{h_1}$ of the lemma.

**Claim 28.** There is a colour $\chi$ such that the following holds. For any $Z \subseteq V(\Gamma)$ with $|Z| \leq N K_0^{-2}$, there exist $K \leq 2K_0$, pairwise disjoint sets $V_1, \ldots, V_{h_1}$ in $V(\Gamma)$ of size $tq = \frac{N}{K}$ and an equipartition $\mathcal{P}_f$ refining $\{V_1, \ldots, V_{h_1}\}$ with $q$ sets of size $t$ in each $V_i$. For each part $U$ of $\mathcal{P}_f$ in $V_i$ and each $j \in [h_1] \setminus \{i\}$, for all but at most $10\sqrt{q} \log(q)$ choices of $V \in \mathcal{P}_f$ with $V \subseteq V_j$ the pair $(U, V)$ is $(g(K), d_{UV}, p)$-regular in colour $\chi$ for some $d_{UV} \geq d$.

**Proof.** We apply Lemma 24, with inputs as above, to the colour graphs $G_1, \ldots, G_r$, and obtain partitions $\mathcal{P}_c$ and $\mathcal{P}_f$ of $V(\Gamma)$ satisfying the conclusions of Lemma 24. We now mark as bad any pair $(U, V)$ of parts of $\mathcal{P}_c$ for which there is some colour $i$ such that the following holds. More than $\varepsilon^{1/2}|\mathcal{P}_f|^2|\mathcal{P}_c|^{-2}$ subpairs $(U', V')$ of $(U, V)$ in $\mathcal{P}_f$ are either not $(\frac{1}{2} g(2|\mathcal{P}_c|), p)$-regular for $G_i$, or $d_p(U', V'; G_i) \neq d_p(U, V; G_i) \pm \varepsilon$.

Suppose that more than $4r \varepsilon^{1/2}|\mathcal{P}_c|^2$ pairs are marked as bad. Then there is some colour $i$ such that in total at least $2 \varepsilon |\mathcal{P}_f|^2$ pairs of parts in $\mathcal{P}_f$ either fail to be $(\frac{1}{2} g(2|\mathcal{P}_c|), p)$-regular for $G_i$, or have the wrong density. Since $\frac{1}{2} g(2|\mathcal{P}_c|) < \varepsilon$, this contradicts either (RL3) or (RL4).

By Turán’s theorem, there is a set $S$ of at least $100\varepsilon^{-1/2} \geq r h_1$ parts which contains no bad pairs. Colouring each pair in $S$ with a majority colour, by Ramsey’s theorem we find a colour $\chi$ and a subset $\{X_1, \ldots, X_{h_1}\}$ of $h_1$ parts between each of which the majority colour is $\chi$. In particular we have $d_p(X_i, X_j; G_\chi) \geq \frac{3}{4} \xi$ for each $i \neq j$. This $\chi$ is the colour of the claim.

Given now a set $Z$ with $|Z| \leq N K_0^{-2}$, we construct the sets $V_1, \ldots, V_{h_1}$ of the claim. Given a pair $(X_i, X_j)$, we mark as bad each part $U$ of $\mathcal{P}_f$ in $X_i$ which is in more than $\sqrt{\varepsilon}|\mathcal{P}_f||\mathcal{P}_c|$ pairs $(U, V)$ with $V \subseteq X_j$ a member of $\mathcal{P}_f'$ such that either $(U, V)$ is not $(\frac{1}{2} g(2|\mathcal{P}_c|), p)$-regular, or $d_p(U, V) \leq \frac{3}{8} r$. In addition, we mark as bad each part $U$ of $\mathcal{P}_f'$ with $|U \cap Z| \geq \frac{|Z|}{20}$. The total number of bad $U$ for a given $(X_i, X_j)$ is at most $\sqrt{\varepsilon}|\mathcal{P}_f'||\mathcal{P}_c| + \frac{20}{K_0}|\mathcal{P}_f'||\mathcal{P}_c|$, and so by choice of $\varepsilon$ and $K_0$, for each $i \in [h_1]$, at most a quarter of the parts $U \subseteq X_i$ with $U \in \mathcal{P}_f'$ are marked as bad for some $X_i$. We now construct a collection $\mathcal{P}_f$ of pairwise disjoint sets by taking, for each set $U \in \mathcal{P}_f'$ that is not bad, a part of size $\frac{n}{100}|U| \pm 2$ which contains no vertices of $Z$; this is possible by definition of ‘bad’. We choose the ‘$\pm 2$’ such that all parts have the
same size; since $\mathcal{P}_c$ is equitable no two parts of $\mathcal{P}_c$ have sizes differing by more than 1; since $\mathcal{P}'_f$ is an equitable refinement of $\mathcal{P}_c$ no two of its parts differ in size by more than 2, so that this is possible. We think of the sets of $\mathcal{P}_f$ as being in correspondence with the non-bad sets of $\mathcal{P}'_f$.

Suppose that $q$ is maximal such that for each $i \in [h_1]$, the set $X_i$ contains at least $2q$ parts of $\mathcal{P}'_f$ which are not bad. For each $i \in [h_1]$, consider all the parts of $\mathcal{P}'_f$ which are contained in $X_i$ and which are not bad. We select some $q$ of these, and let the union of the corresponding parts of $\mathcal{P}_f$ be $V_i$. Thus $\mathcal{P}_f$ is an equipartition of $\bigcup_{i=1}^{h_1} V_i$ refining $\{V_1, \ldots, V_{h_1}\}$, each of which is split into $q$ equal parts of size $t$.

We set $K$ such that each $V_i$ has $N/K = tq$ vertices, and observe that $K \leq 2|\mathcal{P}_c| \leq 2K_0$ by construction. Furthermore, none of the $V_i$ contain vertices of $Z$, and for any part $U$ of $\mathcal{P}_f$ contained in $V_i$, and any $j \in [h_1]\{i\}$, we have that $(U, V)$ is a $(g(2|\mathcal{P}_c|), p)$-regular pair with $d_p(U, V) \geq d$ in $G_\chi$ for all but at most $\sqrt[\ell]{|\mathcal{P}_f|}/|\mathcal{P}_c| \leq 10^{\sqrt[\ell]{\sqrt{q}}}$ choices of $V \in \mathcal{P}_f$ with $V \subseteq V_j$. Since $g$ is monotone decreasing, $g(2|\mathcal{P}_c|) \leq g(K)$, completing the proof of the claim. 

What we still need to do is to verify the upper bound on unpromising embeddings. To that end, let $H$ be a $D$-degenerate graph with $v(H) \leq h_0$ and final vertex in the degeneracy order $y$, let $H' = H - y$, and let $\phi : V(H') \rightarrow [h_1]$ be an injection. We need to prove there are few $dp$-unpromising $\phi$-partite embeddings of $H'$ into $\Gamma$. Let $\ell$ be the degree of $y$ in $H$.

Consider the auxiliary partite graph $G'$ with parts $V_{\phi(x) : x \in V(H)}$, where we put between $V_{\phi(x)}$ and $V_{\phi(y)}$ all the edges of $G_\chi$ for each $x \in N(y)$, and between other pairs of parts we put all the edges of $\Gamma$. A $dp$-unpromising $\phi$-partite embedding $\psi$ of $H'$ into $\Gamma$ is by definition the same thing as a $\phi$-partite embedding of $H'$ into $G'$ which is contained in fewer than $\frac{1}{2}(dp)^{\ell} \cdot \frac{N}{K}$ $\phi$-partite embeddings of $H$ into $G'$.

Fix, for each $x \in V(H')$, a part $V_x$ of $\mathcal{P}_f$ which is contained in $V_{\phi(x)}$. It is enough to show that the number of $dp$-unpromising $\phi$-partite embeddings $\psi$ where each $x$ is in $V_x$, is at most

$$f(K)p^{e(H')}t^{v(H')}$$

where $t$ is the number of vertices in each part of $\mathcal{P}_f$, since then summing over choices of the $V_x$ we obtain the lemma statement. The good event of Lemma 23 holding implies that each pair $(V_x, V_x')$ with $xx' \in E(H)$ is $(g(K), p)$-regular with density $d_p(V_x, V_x') = 1 \pm g(K)$.

We now double count the number of pairs $(\psi, V_y)$ where $V_y$ is a part of $\mathcal{P}_f$ in $V_{\phi(y)}$ which forms a $(g(|\mathcal{P}_c|), dx_y, p)$-regular pair, where $dx_y \geq d$, with each $V_x$ such that $xy \in E(H)$, and $\psi$ extends to fewer than $\frac{2}{3}(dp)^{\ell} t$ embeddings of $H$ with $y$ mapped to $V_y$ (i.e. $\psi$ is a poor embedding with respect to $V_y$). On the one hand, there are $q$ parts of $\mathcal{P}_f$ in $V_{\phi(y)}$, so $q$ choices of $V_y$. For each $V_y$ which forms a suitably regular pair with each $V_x$ such that $xy \in E(H)$, by Lemma 26 there are at most $\frac{1}{2}f(K)p^{e(H')}t^{v(H')}q$ choices of $\psi$ which are poor with respect to $V_y$. We see that the number of pairs is at most $\frac{1}{2}f(K)p^{e(H')}t^{v(H')}$. 

On the other hand, suppose $\psi$ is a $dp$-unpromising embedding. Since $2D \cdot 10^{\sqrt{\sqrt{q}}} \leq \frac{1}{10}$ by choice of $\epsilon$, there are at least $\frac{1}{10}q$ choices of $V_y$ which form $(g(|\mathcal{P}_c|), dx_y, p)$-regular pairs (with $dx_y \geq d$) with each $V_x$ such that $xy \in E(H)$. Since $\frac{2}{3} \cdot \frac{3}{4} > \frac{1}{2}$, it follows that $\psi$ is poor with respect to at least $\frac{1}{10}q$ of these $V_y$. Thus each $dp$-unpromising embedding gives us at least $\frac{1}{10}q$ pairs. Putting this together with the upper bound above, we see that the number of $dp$-unpromising embeddings is at most $f(K)p^{e(H')}t^{v(H')}$, as desired for (a).

We now prove (b). Recall that for each $i \in [h_1]$ the set $X_i$ contains at least $\ell$ parts of $\mathcal{P}_f$ which were not included in $V_i$; we use these parts to show most $\phi$-partite embeddings are completable. To that end, suppose that $H$ is a connected graph with at most $h_0$ vertices and
\[ \Delta(H) \leq D + 1, \] and that \( Q \) is either an induced cycle with at most \( 2p^{-1} \) vertices, or an induced path with \( 2p^{-1} \) vertices, in \( H \). If \( D = 2 \), we suppose \( H \neq K_4 \), and let \( \phi : V(H) \rightarrow [h_1] \) be an injection. Let \( H' = H - V(Q) \).

In order to show that most partite embeddings of \( H' \) are completable, as in the previous part it is enough to find an upper bound on the number of \( \phi \)-partite embeddings \( \psi \) of \( H' \) into \( \Gamma \) such that we cannot extend \( \psi \) to an embedding of \( H \), where the edges \( E(H) \setminus E(H') \) are required to be in \( G \) and the vertices \( V(H) \setminus V(H') \) are required to be outside \( Z \cup \bigcup_{i \in [h_1]} V_i \). In turn, since \( \phi \) is injective, it is enough to show the corresponding bound holds for embeddings to a given collection of fine parts \( V_y \in \mathcal{P}_f \) where \( V_y \subseteq V_{\phi(y)} \) for each \( y \in V(H') \). This is a little simpler than the previous part, since we need only one extension rather than many.

We choose for each \( z \in V(H) \setminus V(H') \) in succession a part \( V_z \) of \( \mathcal{P}_f \) which is contained in \( X_{\phi(z)} \setminus V_{\phi(z)} \). Note that we have at least \( q \) such parts for each \( z \). We make these choices such that for each pair \( yz \in E(H) \setminus e(H') \), the pair \( (V_y, V_z) \) is \( \left( \frac{1}{2} |S_c|, p \right) \)-regular and has \( d_p(V_y, V_z) \geq d \) in \( G_\chi \). This is possible since at most \( (D + 1) \cdot 10 \sqrt{eq} < q \) of the available \( q \) fine parts can fail this condition for each \( z \).

Again, we let \( G' \) denote the auxiliary subgraph of \( \Gamma \) obtained by taking the edges of \( \Gamma \) between parts \( V_y \) and \( V_z \) where both \( y, z \in V(H') \), and otherwise the edges of \( G \). Now any embedding which is not completable is a noncompletion embedding with respect to these choices of the \( V_z \) for \( z \in V(H) \setminus V(H') \), and Lemma 27 gives the required bound on the number of partite embeddings of \( H' \) which are noncompletion.

Finally, for (c), we need to show that if \( p \geq CN^{-2/5} \) then there is a copy of \( K_4 \) in \( G_\chi \) outside \( Z \). This follows from the good event of Lemma 25 holding, by choosing any four parts of \( \mathcal{P}_f \) contained in \( V_1, \ldots, V_4 \) respectively which are pairwise regular and dense in colour \( \chi \), and have small intersection with \( Z \).

6. Embeddings from homomorphisms: The proof of Lemma 21

In this section we prove Lemma 21. For this we need a standard concentration inequality for the hypergeometric distribution, see e.g. [18].

Lemma 29 (Hypergeometric inequality). Given \( S \subseteq [n] \) and \( 0 \leq \ell \leq n \), let \( X \) be a uniform random \( \ell \)-set in \([n]\). Then for each \( 0 < \delta < \frac{1}{2} \) we have

\[
\Pr \left[ |S \cap X| \neq (1 \pm \delta)|S| \frac{\ell}{n} \right] < 2 \exp \left( -\frac{\delta^2 |S| \ell}{2n} \right).
\]

The idea of the proof is as follows. We will split \( V(\Gamma) \) randomly into \( \log N \) sets, the first of which has \( \frac{1}{2} N \) vertices, and the rest of which are all the same size. We will construct an embedding of \( H \) one vertex at a time, ensuring that at each step \( i \) we choose an injective member of \( \Psi_i \) by choosing a new vertex \( v_i \), to which we embed \( i \), which lies in the first possible \( V_j \). We argue that the number of vertices we embed to each \( V_j \) is upper bounded by a geometrically decreasing sequence in \( j \), which guarantees that there are always vertices in \( V_{\log N} \) to which we can embed if needed.

Proof. Suppose graphs \( \Gamma, \ F, \) sets of homomorphisms \( (\Psi_x)_{x \in \{0\} \cup [n]} \), and sets \( W_\psi \) for \( \psi \in \bigcup_{x \in [n]} \Psi_{x-1} \) and \( (S_x)_{x \in [n]} \), satisfying the conditions of the lemma, are given. We begin by choosing a partition of \( V(\Gamma) \) into \( \ell = [\log N] \) parts \( V_1, \ldots, V_\ell \), with \( |V_1| = \frac{1}{2} N \) and \( |V_2| = \cdots = |V_\ell| = \left[ \frac{N}{2^{\ell-2}} \right] \), uniformly at random. We now argue that all the sets \( W_\psi \), and the vertex sets described by the star property, are well distributed over the partition. For the former,
we use the fact that \( W_\psi \) is determined by the embedding of the bounded-size set \( S_x \) to argue that we only need a union bound over a polynomial number of sets \( W_\psi \).

**Claim 30.** With high probability, the following two statements hold. For each \( j \in [\ell] \), each \( x \in [n] \) and each \( \psi \in \Psi_{x-1} \) we have \( |V_j \cap W_\psi| \geq \frac{1}{2} \deg_\psi(x)|V_j| \). For each \( j \in [\ell] \), each \( 1 \leq d \leq D \) and each collection \( B \) of at most \( \frac{d}{1000}p^{-d} \) pairwise disjoint \( d \)-sets in \( V(\Gamma) \), we have

\[
\left| \bigcup_{B \in B} N_\Gamma(B) \cap V_j \right| = \left(1 - \frac{e^\delta}{100}\right)p^d|V_j||B|.
\]

**Proof.** For the first statement, to begin with fix \( j \in [\ell] \), \( x \in [n] \) and \( \psi \in \Psi_{x-1} \). By (P2) we have \( |W_\psi| \geq \deg_\psi(x)n \), so by Lemma 29, with \( \delta = \frac{1}{2} \) we have

\[
\Pr\left[|V_j \cap W_\psi| < \frac{1}{2} \deg_\psi(x)|V_j|\right] < 2\exp\left(-\frac{\deg_\psi(x)|V_j|}{2}\right) < 2^{-L-3},
\]

where the final inequality follows since \( \deg_\psi(x)|V_j| \geq p^D|V_j| \geq 10^7DLp^{-2}\log N \). We now take a union bound over the choices of \( j \in [\ell] \), \( x \in [n] \) and distinct sets \( W_\psi \) as \( \psi \) ranges over \( \Psi_{x-1} \). By (P4), the set \( W_\psi \) is decided by the partial map \( \psi|_{S_x} \). Since \( |S_x| \leq L \) by (P3), there are only at most \( N^L \) distinct choices for \( \psi|_{S_x} \) and hence at most \( N^L \) distinct sets \( W_\psi \) as \( \psi \) ranges over \( \Psi_{x-1} \). It follows that the first statement holds with probability at least \( 1 - (1 - e^{-L})^\ell \), which tends to one as \( N \) tends to infinity.

For the second statement, again to begin with we fix \( j \in [\ell] \), \( 1 \leq d \leq D \) and a collection \( B \) of at most \( \frac{d}{1000}p^{-d} \) pairwise disjoint \( d \)-sets in \( V(\Gamma) \). Since \( \Gamma \) has the \((D, p, \frac{d}{1000})\)-star property, we have

\[
\left| \bigcup_{B \in B} N_\Gamma(B) \cap V_j \right| = \left(1 - \frac{e^\delta}{100}\right)p^d|V_j||B|,
\]

and by Lemma 29, with \( \delta = \frac{d}{400} \), we have

\[
\Pr\left[\left| \bigcup_{B \in B} N_\Gamma(B) \cap V_j \right| < (1 - \frac{e^\delta}{400})^2p^d|V_j||B|\right] < 2\exp\left(-\frac{\deg_\psi(x)|V_j||B|}{10^6}\right).
\]

Note that \( (1 - \frac{e^\delta}{400})^2 \) is contained in the range \( 1 \pm \frac{e^\delta}{400} \). Since

\[
p^d|V_j| \geq p^D|V_j| \geq 10^7DLp^{-2}\log N,
\]

this failure probability is at most \( 2\exp\left(-10D|B|\log N\right) = 2N^{-10D|B|} \). Taking the union bound over the choices of \( j \), \( 1 \leq d \leq D \), the size \( 1 \leq b < N \) of \( |B| \), and the at most \( N^{Db} \) choices of \( B \) with \( |B| = b \), we see that the probability of the second statement failing is at most \( N^2 \sum_{b=1}^{N} N^{Db}N^{-10D_b} \), which tends to zero as \( N \) tends to infinity.

Suppose that the chosen partition of \( V(\Gamma) \) satisfies the two high-probability statements of Claim 30. We now construct injective maps as follows. Let \( \psi_0 \in \Psi_0 \) be the trivial map \( \psi_0 : \emptyset \to V(\Gamma) \). Now, for each \( x \in [n] \) in succession, let \( j \in [\ell] \) be minimal such that \( W_{\psi_{x-1}} \cap V_j \) is contained in the range \( 1 \pm \frac{e^\delta}{400} \). If no such \( j \) exists, we say the construction fails at step \( x \). Choose \( v_x \in W_{\psi_{x-1}} \cap V_j \) and set \( \psi_x := \psi_{x-1} \cup \{x \to v_x\} \).

By definition of the sets \( W_\psi \), if at step \( x \) the construction does not fail then \( \psi_x \) is a member of \( \Psi_x \) and it is injective. In particular, if the construction does not fail at all then \( \psi_n \in \Psi_n \) is injective, as desired. So we only need to show that the construction does not fail. To that end, we set \( k_1 := 10^{-6}g^2N \) and for each \( 2 \leq j \leq \ell \) we set \( k_j := \frac{16d\Delta^D}{g^2|V_{j-1}|k_{j-1}} \).
The following claim states that for each $j$, at most $k_j$ vertices are embedded to $V_j$. Note that for $j = 1$ this is automatic, since $n \leq k_1$. The idea here is the following. When, for some $j \geq 2$, a vertex $x \in V(F)$ is embedded to $V_j$, the reason is that we cannot embed it to $V_{j-1}$, i.e., the many (by Claim 30) vertices of $W_{\psi_{j-1}} \cap V_{j-1}$ are all contained in the at most $k_{j-1}$ vertices $X$ of $V_{j-1}$ which are in the image of $\psi_{j-1}$. In particular, $N^-(x)$ forms the set of leaves of a large collection of stars whose centres are in a small set $X$. Now if there is a large set $Q$ of such vertices, we can choose a large subset of vertices $Q'$ of vertices all of which have $d$ left neighbours (for some $1 \leq d \leq D$) and whose left neighbours are pairwise disjoint $d$-sets. This gives us a collection of pairwise disjoint $d$-sets, each of which has a large number of common neighbours within a set $X$ of only $k_{j-1}$ vertices. If $Q'$, and hence $Q$, is too large we obtain a contradiction to Claim 30.

**Claim 31.** For each $x \in [n]$ and each $j \in [\ell]$, provided the construction does not fail at or before step $x$, we have $|\text{im}(\psi_x) \cap V_j| \leq k_j$.

**Proof.** Suppose for a contradiction that the claim statement does not hold. Let $x^* \in [n]$ be the first step at which the claim statement does not hold, and let $j^* \in [\ell]$ be minimal such that $|\text{im}(\psi_{j^*}) \cap V_{j^*}| > k_{j^*}$. Since $V(H) \leq k_1$ we have $j^* \geq 2$.

Let $Q = \{x : v_x \in V_{j^*}\}$. By assumption, we have $|Q| > k_{j^*}$. For each $x \in Q$, because in the construction of $\psi_x$ we choose $j$ minimal, we have $W_{\psi_{j^*-1}} \cap V_{j^*-1} \subseteq \text{im}(\psi_{j^*-1}) \cap V_{j^*-1} \subseteq \text{im}(\psi_{\nu - 1}) \cap V_{\nu - 1} =: X$.

Let for each $x \in Q$ the set $B_x := \{v_y : y < x, xy \in E(H)\}$. Since the construction has not failed, $\psi_x$ is injective and hence the set $B_x$ has $\deg^-\Gamma(x)$ members. By definition of $W_{\psi}$, for each $x \in Q$ the set $W_{\psi_{j^*-1}}$ is contained in $N^-\Gamma(B_x)$, so by Claim 30, $B_x$ is a set of $\deg^-\Gamma(x)$ vertices of $\Gamma$ with at least $\frac{1}{2}p\deg^-\Gamma(x)|V_{j^*-1}|$ common neighbours in $X$. In particular, since $\frac{1}{2}p|V_{j^*}| > k_1$ we have $\deg^-\Gamma(x) \geq 1$.

We now choose a set $Q' \subseteq Q$ as follows. First, let $d$ be a most common value of $\deg^-\Gamma(x)$ for $x \in Q$; as observed, we have $1 \leq d \leq D$. We choose greedily $x \in Q$ such that $\deg^-\Gamma(x) = d$ and such that $B_x$ is disjoint from $B_{x'}$ for each previously chosen $x'$, stopping once we have chosen a set of size $|Q'| = \left\lceil \frac{k_{j^*}}{\alpha \Delta D}p^{D-d} \right\rceil$. To see that this is possible, observe that there are at least $\frac{|Q|}{\alpha \Delta D} \geq \frac{k_{j^*}}{\alpha \Delta D}$ vertices $x \in Q$ such that $\deg^-\Gamma(x) = d$. If at some stage we have chosen a set $Q''$ with $|Q''| < \frac{k_{j^*}}{\alpha \Delta D}$, then there are in total $d|Q''|$ vertices in $\{B_{x'} : x' \in Q''\}$, which have in total at most $d\Delta|Q''| < k_{j^*}\frac{\alpha \Delta}{\alpha \Delta D} \leq \frac{\alpha \Delta}{\alpha \Delta D}$ neighbours. If we cannot choose any further vertices of $Q$, then each vertex $x$ of $Q$ with $\deg^-\Gamma(x) = d$ must be adjacent to at least one member of $\{B_{x'} : x' \in Q''\}$, a contradiction.

Finally, the sets $\{B_x : x \in Q\}$ are a collection of $\left\lceil \frac{k_{j^*}}{\alpha \Delta D}p^{D-d} \right\rceil$ pairwise disjoint sets, each of size $d$, each of which has at least $\frac{1}{2}p\deg^-\Gamma(x)|V_{j^*-1}|$ common neighbours in the set $X$. By minimality of $j^*$, we have $|X| \leq k_{j^*-1}$. Let $Z := \bigcup_{x \in Q'} N^-\Gamma(B_x) \cap V_{j^*-1}$. Now we have

$$\left\lceil \frac{k_{j^*}}{\alpha \Delta D}p^{D-d} \right\rceil \leq \frac{2k_{j^*}}{\alpha \Delta D}p^{D-d} = \frac{2k_{j^*}}{\alpha|V_{j^*-1}|}p^{D-d} \leq \frac{\alpha |V_{j^*-1}|}{1000p^{D-d}}$$

by choice of $k_1$ and definition of $k_j$ for $j \geq 2$, so by Claim 30 we have

$$|Z| \geq (1 - \frac{\alpha}{1000})p^d|V_{j^*-1}|Q'.$$

But for each $x \in Q'$ the set $B_x$ has at least $\frac{1}{2}p\deg^-\Gamma(x)|V_{j^*-1}|$ common neighbours in $X$, and, using Claim 30 with $B = \{B_x\}$, at most $(1 + \frac{\alpha}{1000} - \frac{1}{4})p^d|V_{j^*-1}|$ common neighbours in $V_{j^*-1} \setminus X$. 

From the latter, we conclude
\[ |Z| \leq (1 + \frac{\ell}{100} - \frac{\ell}{4})p^d|V_{j*} - 1||Q'| + |X| \leq (1 - \frac{\ell}{4})p^d|V_{j*} - 1||Q'| + k_{j* - 1}. \]

Putting these two bounds on \( |Z| \) together, we have
\[ k_{j* - 1} \geq \frac{\ell}{4}p^d|V_{j*} - 1||Q'| \geq \frac{\ell}{80} \Delta D p^D|V_{j*} - 1|k_{j*} . \]

By definition we have
\[ k_{j* - 1} = \frac{\ell}{16 \Delta D} p^D |V_{j*} - 1|k_{j*} , \]

which contradiction completes the proof. \( \square \)

Finally, we show Claim 31 implies that the construction does not fail. Trivially, the construction does not fail at step 0. Let \( x \in [n] \). Provided that the construction has not failed at step \( x - 1 \), we have
\[ |W_{\psi_{x-1}} \cap V_{\ell} \setminus \text{im}\, \psi_{x-1}| \geq |W_{\psi_{x-1}} \cap V_{\ell}| - |\text{im}\, \psi_{x-1} \cap V_{\ell}| \geq \frac{1}{2} \mu p^\Delta (x)|V_{\ell}| - k_{\ell} > 0 , \]

where the second inequality uses Claims 30 and 31. The final inequality holds for sufficiently large \( N \) since we have
\[ k_{\ell} = (\frac{16 \Delta D}{\mu p^\Delta (x)})^{\ell-1}k_1 < 4^{-\ell+1}N < 1 \quad \text{and} \quad \frac{1}{2} \mu p^\Delta (x)|V_{\ell}| \geq \frac{1}{2} \mu p^D |V_{\ell}| > \log N . \]

So the construction also does not fail at step \( x \). By induction, we conclude the construction does not fail, as desired. \( \square \)

7. Vertices with Large Left Distance: The Proof of Lemma 20

The proof here goes as follows. We use Lemma 12 to obtain \( H^* \) containing \( H_0' \) such that \( (H_1', I \cup V(H^*)) \) is \( (D, \mu) \)-Spencer. Importantly, \( x \) is neither in \( H^* \) nor does it send an edge to \( H^* \). Given a homomorphism \( \psi \) from \([x - 1]\) to \( G \), we look at the collection of extensions of \( \psi \) which embed \( H_1' \). For a given copy \( h \) of \( H^* \) in \( G \), either all the extensions of \( \psi \) which embed \( H^* \) to \( h \) are promising (if \( h \) contains a promising copy of \( H_0' \)) or none of them are. Furthermore, for any choice of \( h \) and any \( v \) not in \( h \), by \((S2)\), the number of extensions of \( \psi \) which send \( x \) to \( v \) and \( H^* \) to \( h \) is roughly the same for every choice of \( v \) and \( h \). Putting these observations together a short calculation shows that the unpromising extensions of \( \psi \) are roughly equally distributed over the vertices \( v \) to which \( x \) can be mapped, which is what we want to prove.

Proof of Lemma 20. Let \( \mu, F, \phi, y, H_0(y), H_1(y), h_1, \ell_1 \) be as in Setup 15; let \( I \) denote the first \( q \) vertices of \( H_1(y) \), and \( x \) be the \( q \)th vertex of \( H_1(y) \). Suppose that \( \text{dist}_{H}(x, y) = \ell_1 \).

Note that every vertex of \( H_0'(y) \) is within distance \( h_0 \) of \( y \), and \( |H_0'(y)| \leq h_0 \). By choice of \( \ell_1 \), we have \( \ell_1 > D^2 h_0^2 \mu^{-1} + 10 \).

Let \( \Gamma \) and \( G \) be as in the lemma statement. We apply Lemma 12 with input \( D \) and \( \mu > 0 \), to the graph \( H_1'(y) \) with \( I \) the initial segment of degeneracy order of \( H_1'(y) \) and with \( T = V(H_1'(y)) \). We obtain a superset \( T' \) of \( T \) with \( |T'| \leq D^2 |T|^2 \mu^{-1} \), and let \( H^* = H_1'(y)[T'] \); we have that every vertex of \( H^* \) is connected to \( H^* \) to some vertex of \( T \). In particular by choice of \( \ell_1 \), no vertex of \( H^* \) is at distance \( \ell_1 \) or \( \ell_1 - 1 \) from \( y \), so \( x \) is neither in nor adjacent to any vertex of \( H^* \). Furthermore \((F, I \cup T') \) is \( (D, \mu) \)-Spencer. Let \( X = \psi(N_{H_1'(y)}(x) \cap I) \).

Given \( \psi \) a \( \phi \)-partite partial homomorphism from \( F[[x - 1]] \) to \( G \), we partition the set of unpromising extensions of \( \psi \) for \( y \) into sets \( S_{v,h} \), where \( v \) is a vertex of \( \Gamma \) and \( h \) is a \( \phi \)-partite embedding of \( H^* \) to \( \Gamma \). Observe that if \( \psi' \) is a unpromising extension of \( \psi \) for \( y \) in the set \( S_{v,h} \), then necessarily \( v \) is in \( N_\Gamma(X) \), \( v \) is not in the image of \( h \), and \( h|_H \) is a \( dp \)-unpromising
embedding of $H'_0(y)$, and $ψ ∪ h$ is a partial homomorphism from $F$ to $Γ$. Conversely, if $v$ and $h$ satisfy these conditions, then any embedding of $H'_1(y)$ to $Γ$ which agrees with $ψ ∪ h$ on their common domain is a $dp$-unpromising extension of $ψ$ for $y$.

Given $v$ and $h$ such that $v$ is in $N_Γ(X)$, $v$ is not in the image of $h$, and $h|_{H'_0}$ is a $dp$-unpromising embedding of $H'_0(y)$, and $ψ ∪ h$ is a partial homomorphism from $F$ to $Γ$, the set $S_{v,h}$ is precisely the set of rooted copies of $H'_1(y)$ in $Γ$, with roots $I ∪ T'$ embedded as: $I \setminus \{x\}$ is embedded by $ψ$, $x$ is embedded to $v$, and $T'$ is embedded by $h$. Note that some vertices of $T''$ may be in $I$, but we already insisted that $ψ$ and $h$ agree on such vertices. Since $(H'_1(y), I ∪ \{x\} ∪ T')$ is $(D, µ)$-Spencer, letting $U = V(H'_1(y))$ and $R = I ∪ T'$, by (S2) we have

$$\|S_{v,h}\| = (1 + \frac{1}{10}) N^{\lvert U \setminus R \rvert} p^{e(H'_1(y)(U \setminus R) + e_{H'_1(y)(U,R)})}.$$  

By definition, we have

$$B = \sum_h \sum_{v \notin \im h} |S_{v,h}|.$$  

For each $h$ which is compatible with $ψ$ and such that $h|_{H'_0}$ is unpromising, by (S1), there are at least $\frac{9}{10} p^{|X|N - |H^*|}$ choices of $v$ such that $x → v$ extends $ψ$, and such that $v \notin \im h$, namely $N_Γ(X) \setminus \im h$. For each of these, $|S_{h,v}|$ is lower bounded as in (11). Other choices of $h$ do not contribute to $B$. It follows that there are at least

$$B \geq \frac{5B}{4p^{|X|N}} N^{-\lvert U \setminus R \rvert} p^{-e(H'_1(y)(U \setminus R) - e_{H'_1(y)(U,R)})} \frac{1}{10} N^{\lvert U \setminus R \rvert} p^{e(H'_1(y)(U \setminus R) + e_{H'_1(y)(U,R)})} \leq \frac{55B}{40p^{|X|N}},$$

choices of $h$ compatible with $ψ$ and such that $h|_{H'_0}$ is unpromising. Here we use $p^{|X|N} \geq N^\mu$ and that $N$ is sufficiently large.

Now for a given choice of $v$, the number of $dp$-unpromising extensions of $ψ ∪ \{x → v\}$ is $\sum_h |S_{v,h}|$. At most all of the choices of $h$ compatible with $ψ$ and such that $h|_{H'_0}$ is unpromising contribute to this sum, and they contribute at most the upper bound of (11). This gives the upper bound

$$\frac{5B}{4p^{|X|N}} N^{-\lvert U \setminus R \rvert} p^{-e(H'_1(y)(U \setminus R) - e_{H'_1(y)(U,R)})} \frac{1}{10} N^{\lvert U \setminus R \rvert} p^{e(H'_1(y)(U \setminus R) + e_{H'_1(y)(U,R)})} \leq \frac{55B}{40p^{|X|N}},$$

which is what we wanted to prove.

The proof for uncompletable extensions goes through exactly as above, replacing $dp$-unpromising for $y$ with uncompletable, and $y$ with $Q$. □

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Appendix A. Sparse regularity and counting

In this appendix we prove our strengthened sparse regularity lemma, Lemma 24, and the
counting lemma, Lemma 25.

The proof of our Lemma 24, similarly to the dense version in [3], iterates the usual Sparse
Regularity Lemma of Kohayakawa [20] and Rödl, which we now state. We should note that
the version we state is slightly different to than that in [20], in that we allow our part sizes to
differ but do not have an ‘exceptional class’. However the statement there implies this one;

\[ P \text{ PARTITION UNIVERSALITY FOR GRAPHS OF BOUNDED DEGENERACY AND DEGREE 31 } \]

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one can redistribute the vertices of the exceptional class, and it is straightforward to check that this does not substantially affect regularity.

**Lemma 32** (sparse regularity lemma). Given $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exist $T \in \mathbb{N}$ and $\eta > 0$ such that the following holds for any $p > 0$. If $G_1, \ldots, G_r$ are edge-disjoint graphs on $[n]$ which are $(\eta, p)$-upper regular, and $V_1, \ldots, V_s$ is any equitable partition of $[n]$ with $s \leq t_0$, then there is an equitable partition $V_1', \ldots, V_t'$ with $t \leq T$ with respect to which each $G_i$ is $(\varepsilon, p)$-regular.

We also need the following standard equality.

**Lemma 33.** Given $\lambda_1, \ldots, \lambda_s \geq 0$ such that $\sum_{i=1}^s \lambda_i = 1$, and any $d, g_1, \ldots, g_s \in \mathbb{R}$ such that $\sum_{i=1}^s g_i \lambda_i = 0$, we have

$$\sum_{i=1}^s \lambda_i (d + g_i)^2 = d^2 + \sum_{i=1}^s g_i^2. $$

**Proof.** We have

$$\sum_{i=1}^s \lambda_i (d + g_i)^2 = \sum_{i=1}^s \left(d^2 \lambda_i + 2d g_i \lambda_i + g_i^2 \lambda_i \right)$$

$$= d^2 \sum_{i=1}^s \lambda_i + 2d \sum_{i=1}^s g_i \lambda_i + \sum_{i=1}^s g_i^2 \lambda_i$$

$$= d^2 + \sum_{i=1}^s g_i^2 \lambda_i$$

as desired, where the final equality uses the two summation assumptions of the lemma. \hfill \Box

We can now prove Lemma 24.

**Proof of Lemma 24.** Given $r \in \mathbb{N}$, $\varepsilon > 0$ and $f : \mathbb{N} \to \mathbb{R}^+$, and $k_0 \in \mathbb{N}$, without loss of generality we assume $f$ is monotone decreasing. Let $t = 16 \varepsilon^{-3}$. We define $k_1, \ldots, k_t$ as follows. For each $i \geq 1$, given $k_{i-1}$, we let $k_i$ be the $T$ obtained from Lemma 32 with input $r$, min $(\frac{1}{2} \varepsilon, f(k_{i-1}))$ and $k_{i-1}$. We set $K = k_t$. We require $\eta > 0$ to be smaller than all $\eta$ values given in these calls to Lemma 32, and in addition $\eta \leq \frac{1}{300 k_t \varepsilon^3}$.

We begin by taking an arbitrary equipartition $\mathcal{P}_0$ of $[n]$ into $k_0$ parts, and for each $i = 1, \ldots, t - 1$ in successsion do the following. We let $\mathcal{P}_i$ be obtained by applying Lemma 32, with input $r$, $\frac{1}{2} \min (\varepsilon, f(k_{i-1}))$ and $k_{i-1}$, to the graphs $G_1, \ldots, G_r$ with the partition $\mathcal{P}_{i-1}$. If $i \geq 2$, then by construction, $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$ satisfy conditions (RL1)–(RL3). If they in addition satisfy (RL4) we have the desired partitions and stop. Otherwise (if $i = 1$ or if (RL4) is not satisfied), we obtain $\mathcal{P}_i$ from $\mathcal{P}_i'$ by moving a minimum number of vertices from one part to another in order to obtain an equitable partition. Observe that since no two clusters of $\mathcal{P}_i'$ differ in size by more than two, we change (by adding or removing) at most one vertex per cluster. We then increment $i$ and repeat this procedure.

We define the energy of a partition $\mathcal{P}$ to be

$$\mathcal{E}(\mathcal{P}) := \sum_{U \neq V \in \mathcal{P}} |U||V| \sum_{i \in [r]} d_p(U, V; G_i)^2 / n^2.$$

By choice of $\eta$, provided all parts of $\mathcal{P}$ have at least $\eta n$ vertices, we have $\mathcal{E}(\mathcal{P}) \leq (1 + \eta)^2 r$, and trivially we have $\mathcal{E}(\mathcal{P}_0) \geq 0$. 
Suppose that in the above procedure we do not stop at step \( i \geq 2 \). Then the reason is that \( \mathcal{P}_{i-1} \) and \( \mathcal{P}'_i \) do not satisfy (RL4). We use this to show \( \mathcal{E}(\mathcal{P}'_i) \) is significantly larger than \( \mathcal{E}(\mathcal{P}_{i-1}) \). Specifically, consider some pair \((U, V)\) of parts of \( \mathcal{P}_{i-1} \). Suppose that \( d_p(U, V; G_j) = d \). This pair is split as \( s := |\mathcal{P}'_i|^2|\mathcal{P}_{i-1}|^{-2} \) subpairs in \( \mathcal{P}'_i \), where we can write the \( i \)th density as \( d + g_i \) for some \( g_i \), and size (counting pairs of vertices in the subpair as a fraction of \(|U||V|\)) we can denote by \( \lambda_i \). Then the contribution of \((U, V)\) to \( \mathcal{E}(\mathcal{P}_{i-1}) \) is \( d^2 \frac{|U||V|}{n^2} \), and the total contribution of the subpairs of \((U, V)\) to \( \mathcal{E}(\mathcal{P}'_i) \) is, by Lemma 33, \( (d^2 + \sum_{i=1}^{s} \lambda_i g_i^2) \frac{|U||V|}{n^2} \). Since \( \mathcal{P}_{i-1} \) is equitable and \( \mathcal{P}'_i \) is an equitable refinement with at most \( k_i \) parts, and since \( n \) is sufficiently large, it follows that, writing \( g_{U'V'} \) for the difference \( d_p(U', V'; G_j) - d_p(U, V; G_j) \) for each pair \((U', V')\) of \( \mathcal{P}'_i \) contained in the pair \((U, V)\) of \( \mathcal{P}_{i-1} \), and letting \( g_{U'V'} = 0 \) otherwise, we have

\[
\mathcal{E}(\mathcal{P}'_i) - \mathcal{E}(\mathcal{P}_{i-1}) \geq \sum_{U' \neq V' \in \mathcal{P}'_i} \frac{1}{2 \varepsilon} g_{U'V'}^2 \cdot \frac{1}{2|\mathcal{P}_{i-1}|^2}.
\]

Because (RL4) fails, in some \( G_j \) at least \( \varepsilon|\mathcal{P}'_i|^2 \) of the \( g_{U'V'} \) values are at least \( \varepsilon \), so

\[
\mathcal{E}(\mathcal{P}'_i) - \mathcal{E}(\mathcal{P}_{i-1}) \geq \varepsilon|\mathcal{P}'_i|^2 \cdot \frac{1}{2\varepsilon^2} \cdot \frac{1}{2^2|\mathcal{P}_{i-1}|^2} = \frac{1}{4} \varepsilon^3.
\]

Now, moving vertices to obtain an equipartition \( \mathcal{P}_i \) can change the \( p \)-densities in any \( G_j \) between the (in the obvious way) corresponding pairs by at most

\[
\frac{2n|\mathcal{P}_i|^{-1} + 2}{pm^2|\mathcal{P}_i|^{-2}} \leq \frac{1}{10r} \varepsilon^3,
\]

and hence \( \mathcal{E}(\mathcal{P}'_i) - \mathcal{E}(\mathcal{P}_i) \leq \frac{1}{8} \varepsilon^3 \), so \( \mathcal{E}(\mathcal{P}_i) - \mathcal{E}(\mathcal{P}_{i-1}) \geq \frac{1}{8} \varepsilon^3 \). In particular, this implies that the procedure must stop, with the desired two partitions, before reaching \( \mathcal{P}_i \), which would otherwise have an energy exceeding \((1 + \eta)^2 r \).

We now prove Lemma 25. The idea is as follows: we identify three special cases, Cases A, B and C. Case A will be the statement from Conlon, Gowers, Samotij and Schacht [9]. Case B generalises this, and Case C further. We prove that Case A implies Case B, from this we prove Case C, and finally the lemma in full generality.

**Proof of Lemma 25.** Note that the lemma is straightforward (it reduces to the usual counting lemma for dense graphs) if \( p \geq \frac{1}{10} \), and so we in what follows assume \( p < \frac{1}{10} \).

The first special case A we shall consider is that \( r = v(H) \) and \( \phi \) is a bijection, \( |V_1| = \cdots = |V_{v(H)}| = n \) and \( e(V_i, V_j) = m \geq dp \) for each \( ij \) such that \( \phi^{-1}(i) \phi^{-1}(j) \in E(H) \). That the lemma is true in this case is stated in [9, Section 5.2]. Case B, which generalises Case A, is that \( r = v(H) \) and \( \phi \) is a bijection, and \( |V_1| = \cdots = |V_{v(H)}| = n \). Case C, which generalises this even further, is that \( r = v(H) \) and \( \phi \) is a bijection.

We now prove Case B assuming Case A. We will use the obvious coupling of \( G(N, p) \) as a subgraph of \( G(N, 2p) \). We apply Case A with \( \delta' = \frac{1}{4} \delta, \frac{4}{7} \eta \), and obtain \( \varepsilon' > 0 \). We set \( \varepsilon = \min \left( \frac{\varepsilon'}{\lambda_{v(H)}} \cdot \frac{1}{\lambda_{v(H)}} \delta' \right) \). Given \( \eta > 0 \), we choose \( \iota = 10^{-6} \varepsilon^4 \eta^3 \), and let \( C \) be returned by Case A with input \( \delta', \frac{4}{7} \eta \) and sufficiently large for the following assumption to hold with high probability. Given \( p \geq CN^{-1/m_2(H)} \), suppose that the good event of Case A holds for \( \Gamma' = G(N, 2p) \), and in addition that \( e_Y(X, Y) = (2 \pm \iota)p|X||Y| \) and \( e_Y(X, Y) = (1 \pm \iota)p|X||Y| \), for any two disjoint vertex sets \( X, Y \subseteq [n] \) with \( |X|, |Y| \geq \iota N \), where \( \Gamma \) is the coupled subgraph of \( \Gamma' \) distributed as \( G(N, p) \).
Suppose that $G$ is any subgraph of $\Gamma$ satisfying the conditions of Case B, with the sets $V_{\phi(x)}$ for each $x \in V(H)$ given. We first create a supergraph $G^*$ of $G$ as follows: for each $xy \in E(H)$, we take all the edges of $\Gamma$ between each $V_{\phi(x)}$ and $V_{\phi(y)}$, and we add randomly selected edges of $\Gamma \setminus \Gamma$ until we have exactly $m^* = \lceil \frac{3}{2}pn^2 \rceil$ edges. Note that our assumptions on density in $\Gamma$ and $\Gamma'$ imply that we do need to add a positive number of edges to do this, and enough such edges exist in $\Gamma'$. Furthermore, with high probability each pair $(V_{\phi(x)}, V_{\phi(y)})$ is $(\varepsilon, \frac{3}{2}, p)$-regular in $G^*$, and by the good event of Case A we conclude that the number of embeddings of $H$ into $G^*$ such that $x$ is mapped to $V_{\phi(x)}$ is at most $(2p)^{e(H)}n^{v(H)}$.

Let $G'$ be a random subgraph of $G$ obtained as follows. For each edge $xy$ of $H$, we select exactly $d'pn^2 := \lceil \frac{d}{2}pn^2 \rceil$ edges between $V_{\phi(x)}$ and $V_{\phi(y)}$ of $G$ uniformly at random to $G'$. Observe that the probability that a given embedding of $H$ into $G$, which maps each $x \in V(H)$ to $V_{\phi(x)}$, is also an embedding into $G'$, is exactly

$$
\prod_{xy \in E(H)} \frac{d'pn^2}{\varepsilon \delta(V_{\phi(x)}, V_{\phi(y)})},
$$

and so the expected number of embeddings of $H$ into $G'$ which map each $x \in V(H)$ to $V_{\phi(x)}$ is by linearity of expectation the above fraction times the number of such embeddings into $G$.

On the other hand, with probability at least $1 - \eta$, for each $xy \in E(H)$ the pair $(V_{\phi(x)}, V_{\phi(y)})$ is $(\varepsilon', d', p)$-regular in $G'$, and when this event occurs, by Case A the actual number of such embeddings is $(1 + \delta')(d'p)^{e(H)}n^{v(H)}$. When this event does not occur, the actual number of such embeddings is between 0 and the total number into $G^*$, i.e. at most $(2p)^{e(H)}n^{v(H)}$. By choice of $\delta'$, the expected number of such embeddings is $(1 + 2\delta')(d'p)^{e(H)}n^{v(H)}$.

Putting these two expectation calculations together, we conclude that the number of embeddings of $H$ into $G$ such that each $x$ is mapped to $V_{\phi(x)}$ is

$$(1 + 2\delta')(d'p)^{e(H)}n^{v(H)} \prod_{xy \in E(H)} \frac{\varepsilon \delta(V_{\phi(x)}, V_{\phi(y)})}{d'pn^2} = (1 + 2\delta')(1 + \varepsilon)^{e(H)}n^{v(H)} \prod_{xy \in E(H)} d_{xy}p,$$

which is by choice of $\delta'$ and $\varepsilon$ as desired.

Next, we use Case B to prove Case C. The idea is similar to the previous, but rather simpler; we skip the details. For our parameter choices for Case C, we take the same choices as for Case B, except that we return $\varepsilon = \min((\varepsilon')^2, \delta^2)$ when Case B gives $\varepsilon'$. We assume the good event of Case B with these parameter choices holds for $\Gamma = G(N, p)$. Given $G$ in which we want to count embeddings of $H$, we obtain $G'$ by sampling randomly $V'_i$ to be exactly $n = \min_{i \in [p]} |V_i|$ vertices from each $V_i$. By definition of $(\varepsilon, d, p)$-regularity, each pair $(V'_{\phi(x)}, V'_{\phi(y)})$ is $(\varepsilon', d', p)$-regular, and by Case B we obtain bounds on the count of $H$-embeddings in $G'$, which (since they hold with probability 1) are also bounds on the expected count of $H$-embeddings in $G'$. On the other hand, the probability that any given $H$-embedding in $G$ survives in $G'$ is given by $\prod_{x \in V(H)} V_{\phi(x)}^n$, so by linearity of expectation we obtain the expected count of $H$-embeddings in $G'$ in terms of the number of $H$-embeddings in $G$. Putting these two calculations together the result follows.

We now prove the full lemma statement from Case C. We apply Case C with $\delta' = \frac{\delta}{2} + \frac{d}{2}$ and obtain $\varepsilon' > 0$. We set $\varepsilon = \frac{\varepsilon'}{2^{e(H)}}$. Given $\eta$ we set $\eta' = \frac{\eta}{2^{e(H)}}$, and let $C$ be large enough for Case C with input $\delta', \frac{d}{2}$ and $\eta'$. Given $p \geq CN^{-1/\mu_2(H)}$, we ask for the good event of Case C with the given parameter choices to hold for $\Gamma = G(N, p)$. Part 1 of the universal partition theorem follows.
The strategy is again rather similar to the previous, and we omit most of the details. We partition each set $V_i$ into $\phi^{-1}(i)$ equal parts (up to $\pm 1$), one for each vertex of $H$ mapped to $i$, and let $G'$ be the $v(H)$-partite graph, with parts $V'_i$ for $i \in [v(H)]$, with vertices of $H$ assigned one to a part, that we obtain in this way. Now Case C applies to $G'$, and we obtain bounds on the expected count of $H$-embeddings in $G'$.

On the other hand, for any given $H$-embedding in $G$, the probability that the vertices of $H$ are embedded to the correct parts of $G'$ is given by the product over $i \in [r]$ of the probability of partitioning $V_i$ into $|\phi^{-1}(i)|$ ordered parts and finding that each part contains its required image vertex; this is easily seen to approach $\left(\frac{1}{|\phi^{-1}(i)|}\right)^{|\phi^{-1}(i)|}$ as $N$ tends to infinity. By linearity of expectation we obtain the expected count of $H$-embeddings in $G'$ in terms of the number of $H$-embeddings in $G$, and the desired result follows.

The argument above also gives the improved probability bound for $H$ being strictly 2-balanced. We simply need to replace Case A as stated with [9, Theorem 1.6(ii)], which requires $H$ to be strictly 2-balanced but allows $p \geq Cn^{-1/mz(H)}$. It is otherwise identical to Case A as stated. □

(PA) London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, UK
Email address: p.d.allen@lse.ac.uk

(JB) London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, UK
Email address: j.boettcher@lse.ac.uk