EXISTENCE RESULTS FOR QUASILINEAR SCHRÖDINGER EQUATIONS WITH A GENERAL NONLINEARITY

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(Communicated by Jaemyoung Byeon)

Abstract. Consider the quasilinear Schrödinger equation

\[-\Delta u + V(x)u - \Delta(u^2)u = h(u) \quad \text{in} \quad \mathbb{R}^N,\]

where \(N \geq 3\), \(V : \mathbb{R}^N \to \mathbb{R}\) and \(h : \mathbb{R} \to \mathbb{R}\) are functions. Under some general assumptions on \(V\) and \(h\), we establish two existence results for problem (A) by using variational methods. The main novelty is that, unlike most other papers on this problem, we do not assume the nonlinear term to be 4-superlinear at infinity.

1. Introduction and main results. Consider the quasilinear elliptic equation

\[-\Delta u + V(x)u - \Delta(u^2)u = |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,\]

where \(N \geq 3\), \(2 < p < 2\cdot 2^*\) with \(2^* = \frac{2N}{N-2}\) being the critical Sobolev exponent, and \(V : \mathbb{R}^N \to \mathbb{R}\) is a continuous function. It is known that, via the ansatz \(\psi(t,x) = e^{-i\omega t}u(x)\), weak solutions of problem (1.1) correspond to stationary waves of the time-dependent quasilinear Schrödinger equation

\[i\partial_t \psi = -\Delta \psi + W(x)\psi - \Delta(|\psi|^2)\psi - |\psi|^{p-2}\psi \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N,\]

where \(W(x) = V(x) + \omega\) is a new potential function. Quasilinear Schrödinger equations like (1.2) arise in plasma physics and condensed matter theory, see [19, 20, 28] and references therein for more details on the physical background of (1.2).

The natural energy functional corresponding to problem (1.1) is given by

\[\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx,\]

2010 Mathematics Subject Classification. Primary: 35J10; Secondary: 35J20, 35J60.

Key words and phrases. Quasilinear Schrödinger equation, well potential, radial potential, general nonlinearity.

H. Liu is supported by National Natural Science Foundation of China (No.11701220, No. 11926334, No.11926335). L. Zhao is supported by National Natural Science Foundation of China (No.11671026, No.11771385) and Beijing Municipal Commission of Education KZ202010028048.

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which is not well defined for all \( u \in H^1(\mathbb{R}^N) \). Due to this fact, the standard variational methods cannot be applied directly to problem (1.1). This difficulty makes problems like (1.1) more interesting and challenging. Indeed, during the last twenty years, a considerable amount of research is devoted to studying (1.1) and related problems. Many existence and multiplicity results were proved by using different approaches, such as minimizations [24, 29], change of variables [8, 11, 22], Nehari method [23] and perturbation method [25, 27]. In [22], by a suitable change of variables, problem (1.1) was reduced to a semilinear elliptic equation and existence results were proved under four different types of potentials in an Orlicz space framework. Using a similar change of variables, Colin and Jeanjean [8] investigated the new energy functional in the usual Sobolev space \( H^1(\mathbb{R}^N) \). They established the existence of positive solutions for problem (1.1) with positive constant potential and a general nonlinearity introduced by Berestycki and Lions. Moreover, under the following variant of the global Ambrosetti-Rabinowitz condition

\[
\text{(AR) there exists } \mu > 4 \text{ such that } 0 < \mu \int_0^t h(s) \, ds \leq h(t)t \text{ for all } t \in \mathbb{R}^+,
\]

an existence result was also obtained for problem (1.1) with well potential and the power nonlinearity \( |u|^{p-2}u \) replaced by \( h \). As is well known, (AR) is used to guarantee the boundedness of Palais-Smale sequences for the energy functional.

It is worth pointing out that most of these aforementioned results are based on the condition \( 4 \leq p < 2 \cdot 2^* \). As observed in [23], the number \( 2 \cdot 2^* \) behaves as a critical exponent for problem (1.1). As a matter of fact, nonexistence result of (1.1) can be proved by using a Pohožaev type identity in the case where \( p \geq 2 \cdot 2^* \) and \( \nabla V(x) \cdot x \geq 0 \) for all \( x \in \mathbb{R}^N \).

To the best of our knowledge, few results are known about problem (1.1) with \( 2 < p < 4 \) and we are only aware of the papers [4, 8, 12, 18, 24, 29, 30]. In [24, 29], an unknown Lagrange multiplier appears in the equation. In [8], an existence result was established for problem (1.1) with positive constant potential. In [12], Gloss dealt with a semiclassical problem related to (1.1) and she proved that there exists a positive semiclassical solution which concentrates at a local minimum of the potential. Recently, Ruiz and Siciliano [30] proved the existence of a positive ground state solution of problem (1.1) with \( 2 < p < 2 \cdot 2^* \). The proof relies on a constrained minimization procedure. We remark that a concavity hypothesis was imposed on the potential, which is technique and important in their arguments. In [4, 18], the authors investigated the quasilinear elliptic equation

\[
-\Delta u - \Delta (u^2)u = \lambda h(x, u) \quad \text{in } \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) and \( \lambda \) is a positive parameter. Existence of multiple solutions were established provided that \( \lambda \) is sufficiently large. For more results related to (1.1), we refer the reader to [1, 3, 9, 10, 13, 14, 21, 26] and references therein.

Motivated by [4, 8, 18, 30], we are interested in the quasilinear Schrödinger equation

\[
-\Delta u + V(x)u - \Delta (u^2)u = hu(u) \quad \text{in } \mathbb{R}^N,
\]

where \( N \geq 3 \) and \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) satisfies the following conditions:

\[
\begin{align*}
(V_1) \, & \, 0 < \alpha := \inf_{x \in \mathbb{R}^N} V(x) \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x) < +\infty \text{ for all } x \in \mathbb{R}^N; \\
(V_2) \, & \, \text{there exists } \sigma \in [1, 2) \text{ such that } \nabla V(x) \cdot x \in L^{2^* - \sigma}(\mathbb{R}^N).
\end{align*}
\]
For the nonlinear term $h$, we assume that:

$h_1$ $h \in C(\mathbb{R}, \mathbb{R})$ and $h(t) = o(t)$ as $t \to 0^+$;

$h_2$ there exists $C > 0$ and $q \in (2, 2 \cdot 2^*)$ such that $|h(t)| \leq C(1 + t^{q-1})$ for all $t \in \mathbb{R}^+$;

$h_3$ $\lim_{t \to +\infty} \frac{h(t)}{t} = +\infty$.

The first result of this paper is the following theorem.

**Theorem 1.1.** Suppose that $(V_1) - (V_2)$ and $(h_1) - (h_3)$ hold. Then problem (1.3) has at least a positive solution.

The second part of this paper is motivated by the celebrated paper of Berestycki and Lions [6], where the authors proved the existence of a ground state solution for semilinear elliptic equations under some general conditions on the nonlinearity. In particular, a weak subcritical condition is assumed and superlinear condition $(h_3)$ is not required. A natural question is that whether or not we can obtain existence result of positive solutions for problem (1.3) under similar assumptions on the nonlinear term as in [6]. In the next theorem, we give an affirmative answer to such an interesting question. For this, we make the following assumptions:

$(V_3)$ $V$ is radially symmetric and $0 < \alpha := \inf_{x \in \mathbb{R}^N} V(x) \leq \beta := \sup_{x \in \mathbb{R}^N} V(x) < +\infty$;

$(h_4)$ $\lim_{t \to +\infty} \frac{h(t)}{t^2} = 0$;

$(h_5)$ there exists a positive number $\zeta$ such that $H(\zeta) > \frac{\beta}{2} \zeta^2$, where $H(t) = \int_0^t h(s) \, ds$.

The second result of this paper is stated as follows.

**Theorem 1.2.** Suppose that $V$ satisfies $(V_2) - (V_3)$ and $h$ satisfies $(h_1), (h_4) - (h_5)$. Then problem (1.3) has at least a positive radial solution.

To analyze the assumptions in the current paper and to compare Theorems 1.1 and 1.2 with results in the literature, some remarks are in order.

**Remark 1.** (1) In Theorem 1.1, the potential $V$ is assumed to be well-shaped. In this case, we are allowed to use concentration compactness arguments. However, such an argument does not work for Theorem 1.2 since we do not impose similar geometrical hypothesis on the potential $V$. Therefore, a symmetric property of the potential $V$ is required in Theorem 1.2.

(2) Condition $(V_2)$ shall be used to prove the boundedness of a special Palais-Smale sequence for the energy functional. Similar conditions can be found in [5, 17], where semilinear elliptic equations were considered. It should be mentioned that, due to the well properties of the transformation (see Lemma 2.1 and Corollary 1), we only need a weaker condition than that in [17].

(3) As described before, Ambrosetti-Rabinowitz type condition is used to guarantee the boundedness of Palais-Smale sequences for the energy functional. Nevertheless, from the viewpoint of the celebrated paper [6], it seems that Ambrosetti-Rabinowitz type condition is not essential for the existence of a nontrivial solution. In Theorems 1.1 and 1.2, $(h_3)$ and $(h_5)$ are both weaker than Ambrosetti-Rabinowitz type condition. We also remark that, if $V_\infty := \lim_{|x| \to \infty} V(x)$ exists, then $(h_5)$ can be replaced by a weaker condition

$(h'_5)$ there exists a positive number $\zeta$ such that $H(\zeta) > \frac{V_\infty}{2} \zeta^2$. 

Remark 2. (1) In the case $V \equiv V_\infty$, namely when problem (1.3) is autonomous, Theorems 1.1 and 1.2 are covered by [8, Theorem 1.2]. In this paper, we are interested in non-autonomous problem. Therefore, we assume without loss of generality that $V \not\equiv V_\infty$ throughout this paper.

(2) Compared to [4, 18, 30], the current paper investigates problem (1.3) with a more general nonlinearity and conditions on the potential are somewhat different. Especially, there is no parameter in the equation and we do not need a concavity hypothesis on the potential which is essential in [30]. Thus, Theorems 1.1 and 1.2 can be seen as complements of [4, Theorem 4.2], [18, Theorem 1.1] and [30, Theorem 1.1].

(3) Since positive solutions are of particular interest, we always assume without restriction that $h(t) = 0$ for $t \leq 0$ in the current paper.

To prove Theorems 1.1 and 1.2, we will face several difficulties. On one hand, due to the presence of quasilinear term $\Delta(u^2)u$ and growth condition on the nonlinearity, the natural energy functional related to problem (1.3) is not well defined for all functions in $H^1(\mathbb{R}^N)$. Therefore, we can not apply standard variational methods directly. To overcome this difficulty, we will employ an argument developed in [8, 22] and make a change of variables to reformulate the quasilinear problem to a semilinear one.

On the other hand, it will be shown that the functional $I$ associated with equivalent semilinear problem possesses the mountain pass geometry (see Lemmas 3.2 and 4.1) and so there exists a Palais-Smale sequence for $I$. However, the boundedness of Palais-Smale sequence seems hard to verify. Our strategy is applying Jeanjean’s monotonicity method to find a special Palais-Smale sequence for the functional $I$.

The procedure consists of the following three steps. Firstly, we define a family $\{I_\lambda\}_{\lambda \in \mathcal{J}}$ of $C^1$-functionals such that $I_1 = I$. By an abstract result of Jeanjean, for almost every $\lambda \in \mathcal{J}$, there is a bounded Palais-Smale sequence for the functional $I_\lambda$. Secondly, using a version of global compactness lemma due to Adachi and Watanabe when $V$ is a well potential or restricting in the subspace of radially symmetric functions if $V$ is radially symmetric, we obtain a nontrivial critical point $v_\lambda$ of the functional $I_\lambda$ for almost every $\lambda \in \mathcal{J}$. Eventually, choosing a sequence $\{\lambda_n\}$ of numbers satisfying $\lim_{n \to \infty} \lambda_n = 1$, we obtain a sequence $\{v_{\lambda_n}\}$ of functions with $v_{\lambda_n}$ being a nontrivial critical point of the functional $I_{\lambda_n}$. Then, with the help of Pohožaev type identity and condition $(V_2)$, we prove that $\{v_{\lambda_n}\}$ is indeed a bounded Palais-Smale sequence for the functional $I$.

The paper is organized as follows. In Section 2, following the method in [8, 22], we reformulate (1.3) to a semilinear problem. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

In the sequel, we will use the following notations. The letters $C$ and $C_j$ stand for positive constants which may take different values at different places. The standard norms of $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ are denoted by $|\cdot|_p$ and $\|\cdot\|$ respectively. We also set $H^1_\text{rad}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) | u \text{ is radially symmetric}\}$.

2. Preliminaries. The natural energy functional related to problem (1.3) is defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} H(u) \, dx,$$
which is not well defined for all $u \in H^1(\mathbb{R}^N)$ as mentioned before. To apply variational methods, we employ an argument developed in [8, 22] and make a change of variables.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \quad \text{and} \quad f(0) = 0$$

on $[0, +\infty)$ and by $f(t) = -f(-t)$ on $(-\infty, 0]$. Then the function $f$ is uniquely defined, smooth and invertible. In the next lemma, we summarize some important properties of $f$ which have been proved in [8, 10, 22].

**Lemma 2.1.** The function $f$ has the following properties:

1. $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
2. $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
3. $|f(t)| \leq 2^{1/2} |t|^{1/2}$ for all $t \in \mathbb{R}$;
4. $\lim_{t \to -\infty} \frac{f(t)}{t} = 1$;
5. $\lim_{t \to +\infty} \frac{f(t)}{\sqrt{t}} = 2^{1/2}$;
6. $\frac{1}{2} f(t) \leq t f'(t) \leq f(t)$ for all $t \in \mathbb{R}^+$;
7. $\frac{1}{2} f^2(t) \leq f(t) f'(t) t \leq f^2(t)$ for all $t \in \mathbb{R}$;
8. There exists a positive constant $C$ such that

$$|f(t)| \geq \begin{cases} C|t|, & \text{if } |t| \leq 1, \\ C|t|^{1/2}, & \text{if } |t| \geq 1. \end{cases}$$

As a consequence of Lemma 2.1, we have

**Corollary 1.** Let $\sigma \in [1, 2)$, then $|f(t)| \leq 2^{1/2} |t|^{\sigma}$ for all $t \in \mathbb{R}$.

Setting $v = f^{-1}(u)$, we obtain

$$I(v) := J(f(v)) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) f^2(v)) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx,$$

which is well defined in the Sobolev space $H^1(\mathbb{R}^N)$ and belongs to $C^1$ under the assumptions of Theorem 1.1 or Theorem 1.2. It is well known that critical points of the functional $I$ are weak solutions of the semilinear elliptic equation $-\Delta v + V(x) f(v) f'(v) = h(f(v)) f'(v)$ in $\mathbb{R}^N$. Furthermore, if $v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ is a critical point of the functional $I$, then $u = f(v)$ is a classical solution of problem (1.3).

3. **Proof of Theorem 1.1.** Since we do not assume that the nonlinear term is 4-superlinear at infinity, it seems hard to prove the boundedness of Palais-Smale sequences for the functional $I$. We will use the following slight modified version of [15, Theorem 1.1 and Lemma 2.3] (see also [16]) to construct a special Palais-Smale sequence for the functional $I$.

**Theorem 3.1.** Let $X$ be a Banach space equipped with the norm $\| \cdot \|_X$ and let $\mathcal{J} \subset \mathbb{R}^+$ be an interval. Consider a family $\{I_\lambda\}_{\lambda \in \mathcal{J}}$ of $C^1$-functionals defined on $X$ of the form

$$I_\lambda(v) = A(v) - \lambda B(v), \quad \forall \lambda \in \mathcal{J},$$

where $B(v) \geq 0$ for all $v \in X$ and either $A(v) \to +\infty$ or $B(v) \to +\infty$ as $\|v\|_X \to \infty$. Assume that, for any $\lambda \in \mathcal{J}$, we have $I_\lambda(0) = 0$, the set $\Gamma_\lambda =...
\(\{ \gamma \in C([0,1], X) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0 \}\) is nonempty and
\[
c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0.
\]

Then, for almost every \(\lambda \in \mathcal{J}\), there exists a sequence \(\{v_n\} \subset X\) such that
- \(\{v_n\}\) is bounded in \(X\);
- \(\lim_{n \to \infty} I_\lambda(v_n) = c_\lambda\);
- \(I'_\lambda(v_n) \to 0\) in the dual \(X^{-1}\) of \(X\).

Furthermore, the map \(\lambda \mapsto c_\lambda\) is continuous from the left.

By \((h_1)\) and \((h_3)\), there exists a positive constant \(K\) such that \(h(t) + Kt \geq 0\) for all \(t \in \mathbb{R}^+\). For simplicity of notations, we denote \(V_K(x) = V(x) + K\), \(h_K(t) = h(t) + Kt\) and \(H_K(t) = \int_0^t h_K(s) \, ds\) in the following. Then \(H_K(t) = H(t) + \frac{t^2}{2} K^2 \geq 0\) for all \(t \in \mathbb{R}\). Now we consider a family of \(C^1\)-functionals defined on \(H^1(\mathbb{R}^N)\):

\[
I_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_K(x) f^2(v)) \, dx - \lambda \int_{\mathbb{R}^N} H_K(f(v)) \, dx,
\]

where \(\lambda \in [\frac{1}{2}, 1]\). To apply Theorem 3.1, we set

\[
A(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_K(x) f^2(v)) \, dx \quad \text{and} \quad B(v) = \int_{\mathbb{R}^N} H_K(f(v)) \, dx.
\]

Then \(I_\lambda(v) = A(v) - \lambda B(v)\) with \(B\) being nonnegative. Next lemma ensures that \(A\) is coercive and the functional \(I_\lambda\) possesses the mountain pass geometry.

**Lemma 3.2.** The following statements hold:

1. \(A(v) \to +\infty\) as \(\|v\| \to \infty\);
2. \(\Gamma_\lambda = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0 \}\neq \emptyset\) for all \(\lambda \in [\frac{1}{2}, 1]\);
3. \(c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0\) for all \(\lambda \in [\frac{1}{2}, 1]\).

**Proof.** From Lemma 2.1 we deduce that

\[
\|v\|^2 = \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \int_{\{x \mid |v(x)| \leq 1\}} v^2 \, dx + \int_{\{x \mid |v(x)| > 1\}} v^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + C_1 \int_{\{x \mid |v(x)| \leq 1\}} f^2(v) \, dx + \int_{\{x \mid |v(x)| > 1\}} v^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + C_2 \int_{\mathbb{R}^N} V_K(x) f^2(v) \, dx + C_3 \left( \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right)^{\frac{2}{2}}
\]

\[
\leq C_4 (A(v) + A(v)^{\frac{2}{2}}).
\]

This inequality implies that \(A\) is coercive.

In order to prove (2), we consider the functional

\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_K(x) u^2) \, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} H_K(u) \, dx.
\]

Set \(V_{K,\infty} = V_{\infty} + K\) and fix a nonnegative function \(\bar{u} \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\). Then, for \(t > 0\), we have

\[
J_{1/2}(t \bar{u}(x/t)) = \frac{t^N}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 \, dx + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V_K(t x) \bar{u}^2 \, dx
\]

\[
+ \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} \bar{u}^2 |\nabla \bar{u}|^2 \, dx - \frac{t^N}{2} \int_{\mathbb{R}^N} H_K(t \bar{u}) \, dx
\]
for all \( \lambda \) positive number \( C \) from which we see that
\[
\text{Since } \lambda \text{ of bounded Palais-Smale sequences.}
\]

The following lemma from [8, Theorems 3.2 and 3.4] states the existence of least properties:

Furthermore, there exists a path \( \lambda \in \Gamma \) such that
\[
\text{Finally, we prove (3). Define } \tilde{H}(t) = -\frac{\alpha}{4} t^2 + H(f(t)). \text{ Using (h1), (h2) and Lemma 2.1, we obtain } \lim_{t\to 0} \frac{\tilde{H}(t)}{t^2} = -\frac{\alpha}{2} \text{ and } \lim_{t\to\infty} \frac{\tilde{H}(t)}{t^2} = 0. \text{ Thus there exists a positive number } C_5 \text{ such that}
\]
\[
\tilde{H}(t) \leq -\frac{\alpha}{4} t^2 + C_5 |t|^{2^*}, \text{ for all } t \in \mathbb{R}.
\]
Since \( \lambda \leq 1 \) and \( H_K(t) \geq 0 \) for all \( t \in \mathbb{R} \), there holds
\[
\begin{align*}
I_\lambda(v) &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_K(x) f^2(v)) \, dx - \int_{\mathbb{R}^N} H_K(f(v)) \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) f^2(v)) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} \tilde{H}(f(v)) \, dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{\alpha}{4} \int_{\mathbb{R}^N} v^2 \, dx - C_5 \int_{\mathbb{R}^N} |v|^{2^*} \, dx \\
&\geq \min \left\{ \frac{1}{2} \cdot \frac{\alpha}{4} \right\} \|v\|^2 - C_6 \|v\|^{2^*},
\end{align*}
\]
from which we see that \( c_\lambda > 0 \). The proof is complete. \( \Box \)

In order to proceed, we recall some known results of the “limit” functional
\[
I^\infty_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_{K,\infty} f^2(v)) \, dx - \int_{\mathbb{R}^N} H_K(f(v)) \, dx.
\]
Define the minimum energy
\[
m^\infty_\lambda = \inf \{ I^\infty_\lambda(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\}, (I^\infty_\lambda)'(v) = 0 \}.
\]

The following lemma from [8, Theorems 3.2 and 3.4] states the existence of least energy solution for autonomous problem which is crucial to ensure the compactness of bounded Palais-Smale sequences.

**Lemma 3.3.** \( m^\infty_\lambda > 0 \) and it is achieved by a positive function \( w^\infty_\lambda \in H^1(\mathbb{R}^N) \). Furthermore, there exists a path \( \gamma_\lambda \in C([0, 1], H^1(\mathbb{R}^N)) \) satisfying the following properties:

- \( \gamma_\lambda(0) = 0 \) and \( I^\infty_\lambda(\gamma_\lambda(1)) < 0 \);
- \( \gamma_\lambda(t)(x) > 0 \) for all \( x \in \mathbb{R}^N \) and \( t \in (0, 1) \);
- \( w^\infty_\lambda \in \gamma_\lambda([0, 1]) \) and \( I^\infty_\lambda(w^\infty_\lambda) = m^\infty_\lambda = \max_{t \in [0, 1]} I^\infty_\lambda(\gamma_\lambda(t)) \).

**Lemma 3.4.** For any \( \lambda \in [\frac{1}{2}, 1] \), there holds \( c_\lambda < m^\infty_\lambda \).

**Proof.** Let \( w^\infty_\lambda \) and \( \gamma_\lambda \) be as in Lemma 3.3. Then, by \( (V_1) \), we have
\[
I_\lambda(\gamma_\lambda(t)) < I^\infty_\lambda(\gamma_\lambda(t)), \text{ for all } t \in (0, 1],
\]
where we have used the assumption $V \neq V_\infty$ as stated in Remark 2. Using the definition of $c_\lambda$ leads to

$$c_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma_\lambda(t)) < \max_{t \in [0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = m_\lambda^\infty.$$ 

The proof is complete. \qed

The following lemma was essentially proved in [2, Lemma 4.2], which describes the decomposition of bounded Palais-Smale sequences for the functional $I_\lambda$.

**Lemma 3.5.** If $\{v_n\} \subset H^1(\mathbb{R}^N)$ is a bounded Palais-Smale sequence for the functional $I_\lambda$, then there exist a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, a nonnegative integer $l$, $l$ sequences of points $\{y_n^k\} \subset \mathbb{R}^N$ and $l$ functions $\{w_n^k\}_{k=1}^l \subset H^1(\mathbb{R}^N)$ such that, as $n \to \infty$,

1. $|y_n^k| \to \infty$ and $|y_n^k - y_n^{k'}| \to \infty$ for $k \neq k'$,
2. $v_n \to v_\lambda$ in $H^1(\mathbb{R}^N)$ with $I_\lambda'(v_\lambda) = 0$,
3. $w_n^k \neq 0$ and $(I_\lambda^\infty)'(w_n^k) = 0$ for $1 \leq k \leq l$,
4. $\|v_n - v_\lambda - \sum_{k=1}^l w_n^k \cdot y_n^k\| \to 0$,
5. $I_\lambda(v_n) \to I_\lambda(v_\lambda) + \sum_{k=1}^l I_\lambda^\infty(w_n^k),$

where we agree that, in the case $l = 0$, the above conclusion holds without $w_n^k$ and $\{y_n^k\}$.

Using Lemmas 3.4 and 3.5, we can prove

**Lemma 3.6.** If $\{v_n\} \subset H^1(\mathbb{R}^N)$ is a bounded Palais-Smale sequence for the functional $I_\lambda$ satisfying $\limsup_{n \to \infty} I_\lambda(v_n) \leq c_\lambda$ and $\|v_n\| \to 0$ as $n \to \infty$, then there is a subsequence of $\{v_n\}$ which converges weakly to a nontrivial critical point $v_\lambda$ of $I_\lambda$ with $I_\lambda(v_\lambda) \leq c_\lambda$.

**Proof.** By Lemma 3.5, up to a subsequence, there exist a nonnegative integer $l$ and $v_\lambda \in H^1(\mathbb{R}^N)$ such that $v_n \to v_\lambda$ in $H^1(\mathbb{R}^N)$, $I_\lambda'(v_\lambda) = 0$ and

$$I_\lambda(v_n) \to I_\lambda(v_\lambda) + \sum_{k=1}^l I_\lambda^\infty(w_n^k),$$

where $\{w_n^k\}_{k=1}^l$ are nontrivial critical points of the “limit” functional $I_\lambda^\infty$.

If $I_\lambda(v_\lambda) < 0$, then the proof is complete. If $I_\lambda(v_\lambda) \geq 0$, then we claim that $l = 0$. Otherwise, we have

$$c_\lambda \geq \lim_{n \to \infty} I_\lambda(v_n) = I_\lambda(v_\lambda) + \sum_{k=1}^l I_\lambda^\infty(w_n^k) \geq m_\lambda^\infty,$$

which contradicts the conclusion of Lemma 3.4. Thus $v_n \to v_\lambda$ in $H^1(\mathbb{R}^N)$ and $I_\lambda(v_\lambda) \leq c_\lambda$. Note that $\|v_n\| \to 0$ as $n \to \infty$. Therefore, $v_\lambda$ is a nontrivial critical point of the functional $I_\lambda$. The proof is complete. \qed

**Lemma 3.7.** There exists a positive constant $\delta$, independent of $\lambda \in [\frac{1}{2}, 1]$, such that if $v$ is a nontrivial critical point of the functional $I_\lambda$ then $\|v\| \geq \delta > 0$.

**Proof.** By $(h_1), (h_2)$, the definition of $h_K$ and Lemma 2.1, we have $h_K(f(t))f'(t)t \geq 0$ for all $t \in \mathbb{R}$, and

$$\lim_{t \to 0} \frac{h_K(f(t))f'(t)t}{f(t)^2} = K, \quad \lim_{t \to \infty} \frac{h_K(f(t))f'(t)t}{|t|^2} = 0.$$
Then there exists \( C_1 > 0 \) such that
\[
h_K(f(t)) f'(t) t \leq \left( K + \frac{\alpha}{2} \right) f(t) f'(t) t + C_1 |t|^2, \quad \text{for all} \ t \in \mathbb{R}.
\]
Combining this with \( \langle I'_1(v), v \rangle = 0 \) leads to
\[
\int_{\mathbb{R}^N} (|\nabla v|^2 + (\alpha + K) f(v) f'(v)) \, dx \\
\leq \int_{\mathbb{R}^N} (|\nabla v|^2 + V_K(x) f(v) f'(v)) \, dx \\
= \lambda \int_{\mathbb{R}^N} h_K(f(v)) f'(v) \, dx \\
\leq (K + \frac{\alpha}{2}) \int_{\mathbb{R}^N} f(v) f'(v) \, dx + C_1 \int_{\mathbb{R}^N} |v|^2 \, dx,
\]
which implies that
\[
\min \left\{ 1, \frac{\alpha}{4} \right\} \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)) \, dx \\
\leq \int_{\mathbb{R}^N} (|\nabla v|^2 + \frac{\alpha}{2} f(v) f'(v)) \, dx \leq C_1 \int_{\mathbb{R}^N} |v|^2 \, dx \\
\leq C_2 \left( \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)) \, dx \right)^{\frac{q^*}{2}}.
\]
Since \( v \neq 0 \) and \( |f(t)| \leq |t| \) for all \( t \in \mathbb{R} \), we obtain
\[
||v|| \geq \left( \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)) \, dx \right)^{\frac{1}{2}} \geq \delta
\]
for some positive constant \( \delta \). \( \square \)

In the proof of Theorem 1.1, we also need the following Pohožaev type identity. Because the proof is standard, we omit it and refer the reader to [6, Section 2].

**Lemma 3.8.** If \( v \in H^1(\mathbb{R}^N) \) is a critical point of the functional \( I_\lambda \), then
\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} V_K(x) f^2(v) \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v) \, dx - \lambda N \int_{\mathbb{R}^N} H_K(f(v)) \, dx = 0.
\]

**Proof of Theorem 1.1.** By Lemma 3.2 and Theorem 3.1, there exists \( I_1 \subset [\frac{1}{2}, 1] \) with \( meas(I_1) = 0 \) such that, for any \( \lambda \in \left[ \frac{1}{2}, 1 \right] \setminus I_1 \), there is a bounded sequence \( \{v_n\} \subset H^1(\mathbb{R}^N) \) with the following properties: \( \lim_{n \to \infty} I_\lambda(v_n) = c_\lambda \) and \( I'_\lambda(v_n) \to 0 \)
in \( H^{-1}(\mathbb{R}^N) \). It follows from \( c_\lambda > 0 \) that \( \|v_n\| \to 0 \) as \( n \to \infty \). Using Lemmas 3.6 and 3.7, for any \( \lambda \in \left[ \frac{1}{2}, 1 \right] \setminus I_1 \), we obtain a nontrivial critical point \( v_\lambda \) of \( I_\lambda \) with \( I_\lambda(v_\lambda) \leq c_\lambda \) and \( \|v_\lambda\| \geq \delta > 0 \).

Choosing a sequence of numbers \( \{\lambda_n\} \subset [\frac{1}{2}, 1] \setminus I_1 \) such that \( \lim_{n \to \infty} \lambda_n = 1 \), we obtain a sequence of functions \( \{v_{\lambda_n}\} \subset H^1(\mathbb{R}^N) \) satisfying \( \|v_{\lambda_n}\| \geq \delta > 0 \), \( I_{\lambda_n}(v_{\lambda_n}) \leq c_{\lambda_n} \leq c_{1/2} \) and \( I'_{\lambda_n}(v_{\lambda_n}) = 0 \). Next we show that \( \{v_{\lambda_n}\} \) is indeed a bounded Palais-Smale sequence for the functional \( I_1 = I \). Firstly, we deduce from \( I_{\lambda_n}(v_{\lambda_n}) \leq c_{1/2} \) and Lemma 3.8 that
\[
\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v_{\lambda_n}) \, dx + Nc_{1/2}.
\]
Using Hölder inequality and Sobolev inequality leads to

\[
\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \leq \frac{1}{2} |\nabla V(x) \cdot x|_{2^*} \left( \int_{\mathbb{R}^N} f^{\frac{2^*}{2^* - 2}}(v_{\lambda_n}) \, dx \right)^{\frac{2^*}{2^* - 2}} + N c_{1/2}
\]

\[
= C_1 \left( \int_{\mathbb{R}^N} |v_{\lambda_n}|^2 \, dx \right)^{\frac{2^*}{2}} + N c_{1/2}
\]

\[
\leq C_2 \left( \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \right)^{\frac{2^*}{2}} + N c_{1/2},
\]

where we have used assumption (V_2) and Corollary 1. Since \( \sigma \in [1, 2) \), we see that \( \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \leq C_3 \) for some positive constant \( C_3 \). Secondly, we use the same arguments as in the proof of Lemma 3.7 to obtain

\[
A(v_{\lambda_n}) \leq \frac{\max\{1, V_{K, \infty}\}}{2} \int_{\mathbb{R}^N} (|\nabla v_{\lambda_n}|^2 + f^2(v_{\lambda_n})) \, dx
\]

\[
\leq C_4 \int_{\mathbb{R}^N} |v_{\lambda_n}|^2 \, dx \leq C_5 \left( \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \right)^{\frac{2^*}{2}} \leq C_6.
\]

Then it follows from the coercivity of \( A \) that \( \{v_{\lambda_n}\} \) is bounded in \( H^1(\mathbb{R}^N) \). Thirdly, we have

\[
\limsup_{n \to \infty} I(v_{\lambda_n}) = \limsup_{n \to \infty} \left( I_{\lambda_n}(v_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^N} H_K(f(v_{\lambda_n})) \, dx \right) \leq \lim_{n \to \infty} c_{\lambda_n} = c_1,
\]

and, similarly, \( I'(v_{\lambda_n}) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \). Therefore, \( \{v_{\lambda_n}\} \subset H^1(\mathbb{R}^N) \) is a bounded Palais-Smale sequence for the functional \( I \) satisfying \( \limsup_{n \to \infty} I(v_{\lambda_n}) \leq c_1 \) and \( \|v_{\lambda_n}\| \to 0 \) as \( n \to \infty \).

Using Lemma 3.6 again, we obtain a nontrivial critical point \( v \) of \( I \). By a standard argument, we can show that \( v(x) > 0 \) for all \( x \in \mathbb{R}^N \).

4. **Proof of Theorem 1.2.** In this section, we consider problem (1.3) with a more general nonlinearity. In order to apply Theorem 3.1, we define \( h_j = \max\{-1, (j+1)h_0\} \) for \( j = 1, 2 \). Then \( h = h_1 - h_2 \). By \((h_5)\), there exists a function \( \tilde{u} \in H^1_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) (see step 1 in the proof of [6, Theorem 2]) satisfying

\[
\int_{\mathbb{R}^N} H_1(\tilde{u}) \, dx - \int_{\mathbb{R}^N} \left( H_2(\tilde{u}) + \frac{\beta}{2} \tilde{u}^2 \right) \, dx = \int_{\mathbb{R}^N} \left( H(\tilde{u}) - \frac{\beta}{2} \tilde{u}^2 \right) \, dx > 0,
\]

where \( H_j(t) = \int_0^t h_j(s) \, ds \) for \( j = 1, 2 \). Thus we can find \( \tilde{\lambda} \in (0, 1) \) such that

\[
\tilde{\lambda} \int_{\mathbb{R}^N} H_1(\tilde{u}) \, dx - \int_{\mathbb{R}^N} \left( H_2(\tilde{u}) + \frac{\beta}{2} \tilde{u}^2 \right) \, dx > 0. \tag{4.1}
\]

As in Section 3, we introduce a family of \( C^1 \)-functionals defined on \( H^1_0(\mathbb{R}^N) \)

\[
I_{\lambda}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx + \int_{\mathbb{R}^N} H_2(f(v)) \, dx - \lambda \int_{\mathbb{R}^N} H_1(f(v)) \, dx,
\]

where \( \lambda \in [\tilde{\lambda}, 1] \). Setting

\[
A(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \, dx + \int_{\mathbb{R}^N} H_2(f(v)) \, dx
\]

and

\[
B(v) = \int_{\mathbb{R}^N} H_1(f(v)) \, dx,
\]

we see that \( I_{\lambda}(v) = A(v) - \lambda B(v) \) with \( B \) being nonnegative. Similar to Lemma 3.2, we have
Lemma 4.1. The following statements hold:

1. \( A(v) \to +\infty \) as \( \|v\| \to \infty \);
2. \( \Gamma_\lambda = \{ \gamma \in C([0,1], H^1_0(\mathbb{R}^N)) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0 \} \neq \emptyset \) for all \( \lambda \in \bar{\lambda}, [1] \);
3. \( c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0 \) for all \( \lambda \in \bar{\lambda}, [1] \).

Proof. The arguments in the proof of Lemma 3.2 also work for the items (1) and (3) if we observe that \( H_j(t) \geq 0 \) for all \( t \in \mathbb{R} \) and \( j = 1, 2 \). Next, we prove item (2).

Setting
\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2 \, dx + \int_{\mathbb{R}^N} H_2(u) \, dx - \lambda \int_{\mathbb{R}^N} H_1(u) \, dx,
\]
we have, for \( t > 0 \),
\[
J_\lambda(\bar{u}(x/t)) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(tx) \bar{u}^2 \, dx
+ t^{N-2} \int_{\mathbb{R}^N} \bar{u}^2|\nabla \bar{u}|^2 \, dx + t^N \int_{\mathbb{R}^N} H_2(\bar{u}) \, dx - \lambda t^N \int_{\mathbb{R}^N} H_1(\bar{u}) \, dx
\leq t^N \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 \, dx + \frac{1}{t^2} \int_{\mathbb{R}^N} \bar{u}^2|\nabla \bar{u}|^2 \, dx
+ \frac{1}{2} \int_{\mathbb{R}^N} V(tx) \bar{u}^2 \, dx + \lambda \int_{\mathbb{R}^N} H_1(\bar{u}) \, dx \right).
\]

From (4.1) we see that \( J_\lambda(\bar{u}(x/t)) < 0 \) for \( t > 0 \) sufficiently large. Choose \( u_0 = \bar{u}(\cdot/t) \) with \( t > 0 \) sufficiently large and define \( v_0 = f^{-1}(u_0) \). Then there holds \( I_\lambda(u_0) = J_\lambda(u_0) \leq J_\lambda(v_0) < 0 \) for all \( \lambda \in \bar{\lambda}, [1] \). Defining \( \gamma_0(t) = tv_0 \), we see that \( \gamma_0 \in \Gamma_\lambda \) for all \( \lambda \in \bar{\lambda}, [1] \). \( \square \)

By Lemma 4.1 and Theorem 3.1, there exists \( J_1 \subset \bar{\lambda}, [1] \) with \( meas(J_1) = 0 \) such that, for any \( \lambda \in \bar{\lambda}, [1] \), there is a bounded Palais-Smale sequence \( \{v_n\} \subset H^1_0(\mathbb{R}^N) \) for the functional \( I_\lambda \) at level \( c_1 \). To show the compactness of \( \{v_n\} \), we need the following variant of Brézis-Lieb lemma.

Lemma 4.2. Let \( 2 < p \leq 2^* \) and let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying
\[
|g(t)| \leq C(|t| + |t|^{p-1}), \quad \text{for all } t \in \mathbb{R}.
\]

If \( \{v_n\} \) is bounded in \( H^1(\mathbb{R}^N) \) and \( v_n \to v \) a.e. in \( \mathbb{R}^N \), then we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left( G(v_n) - G(v_n - v) - G(v) \right) \, dx = 0,
\]
where \( G(t) = \int_0^t g(s) \, ds \).

Proof. We use similar arguments as in the proof of [7, Theorem 1]. First of all, using (4.2) and Fatou’s lemma yields that
\[
\int_{\mathbb{R}^N} G(v) \, dx \leq C_1 \int_{\mathbb{R}^N} (v^2 + |v|^p) \, dx \leq C_1 \liminf_{n \to \infty} \int_{\mathbb{R}^N} (v_n^2 + |v_n|^p) \, dx < +\infty.
\]

Next we claim that, given \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
|G(a + b) - G(a)| \leq \varepsilon(a^2 + |a|^p) + C_\varepsilon (b^2 + |b|^p)
\]
for all \(a, b \in \mathbb{R}\). In fact, using (4.2) and Young inequality, we have
\[
|G(a + b) - G(a)| = \left| \int_0^1 g(a + \theta b) b d\theta \right|
\leq \int_0^1 C(|a + \theta b| + |a + \theta b|^{p - 1}) |b| d\theta
\leq \int_0^1 \left( \varepsilon |a + \theta b|^2 + \varepsilon |a + \theta b|^p + C'_\varepsilon (b^2 + |b|^p) \right) d\theta
\leq \varepsilon (a^2 + b^2) + \varepsilon (|a|^p + |b|^p) + C'_\varepsilon (b^2 + |b|^p)
= \varepsilon (a^2 + |a|^p) + C'_\varepsilon (b^2 + |b|^p),
\]
where \(C'_\varepsilon = C'_\varepsilon + \varepsilon\).

Taking \(a = v_n - v\) and \(b = v\) in (4.3) and using (4.2) again leads to
\[
|G(v_n) - G(v_n - v) - G(v)| \leq \varepsilon ((v_n - v)^2 + |v_n - v|^p) + C''_\varepsilon (v^2 + |v|^p).
\]

Then, defining
\[
f^*_n = \max \{|G(v_n) - G(v_n - v) - G(v)| - \varepsilon ((v_n - v)^2 + |v_n - v|^p), 0\},
\]
we have \(0 \leq f^*_n \leq C''_\varepsilon (v^2 + |v|^p) \in L^1(\mathbb{R}^N)\). By Lebesgue’s dominated convergence theorem, there holds
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} f^*_n dx = 0.
\]
It follows from
\[
|G(v_n) - G(v_n - v) - G(v)| \leq f^*_n + \varepsilon ((v_n - v)^2 + |v_n - v|^p)
\]
that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |G(v_n) - G(v_n - v) - G(v)| dx \leq C_2 \varepsilon.
\]
Letting \(\varepsilon \to 0\), we finish the proof. \(\square\)

**Lemma 4.3.** From the proof we see that, under the assumptions of Lemma 4.2, there also holds
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) (G(v_n) - G(v_n - v) - G(v)) dx = 0,
\]
where \(K(x) \in L^\infty(\mathbb{R}^N)\).

Next we recall a compactness lemma, which is essentially due to Strauss (see [31, Lemma 2] or [6, Theorem A.1]).

**Lemma 4.4.** Let \(P\) and \(Q : \mathbb{R} \to \mathbb{R}\) be continuous functions satisfying
\[
\lim_{t \to \infty} \frac{P(t)}{Q(t)} = 0,
\]
\(\{v_n\}, v\) and \(\varphi\) be measurable functions from \(\mathbb{R}^N\) to \(\mathbb{R}\), with \(\varphi\) bounded, such that
\[
\sup_n \int_{\mathbb{R}^N} |Q(v_n)\varphi| dx < +\infty
\]
and
\[
P(v_n) \to v \text{ a.e. in } \mathbb{R}^N,
\]
then \(\lim_{n \to \infty} \int_\Omega |(P(v_n) - v)\varphi| dx = 0\) for any bounded Borel set \(\Omega\).

Moreover, if we assume also that
\[
\lim_{t \to 0} \frac{P(t)}{Q(t)} = 0
\]
and \( \lim_{|x| \to \infty} \sup_n |v_n(x)| = 0 \), then \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |(P(v_n) - v)\varphi| \, dx \to 0 \).

**Lemma 4.5.** Let \( \lambda \in [\lambda, 1] \) be fixed. If \( \{v_n\} \subset H_1^1(\mathbb{R}^N) \) is a bounded Palais-Smale sequence for the functional \( I_\lambda \) at level \( c_\lambda \), then, up to a subsequence, \( \{v_n\} \) converges to a positive critical point \( v_\lambda \) of \( I_\lambda \) with \( I_\lambda(v_\lambda) = c_\lambda \).

**Proof.** Since \( \{v_n\} \subset H_1^1(\mathbb{R}^N) \) is bounded, we may assume up to a subsequence that \( v_n \to v_\lambda \) in \( H_1^1(\mathbb{R}^N) \) and \( v_n \to v_\lambda \) a.e. in \( \mathbb{R}^N \). By Lebesgue’s dominated convergence theorem and Lemma 4.4 with \( P(t) = h_j(f(t))f'(t) \), \( Q(t) = |t|^{2^* - 1} \) and \( \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}) \), we conclude that \( I_\lambda'(v_\lambda) = 0 \).

Set \( w_n = v_n - v_\lambda \) and define
\[
A_1(w_n) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)f^2(w_n)) \, dx.
\]
We claim that \( A_1(w_n) \to 0 \) as \( n \to \infty \). Indeed, it follows from Remark 4.3 that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)(f(v_n)f'(v_n)v_n - f(v_\lambda)f'(v_\lambda)v_\lambda) \, dx = 0. \tag{4.4}
\]
Using the well known Strauss radial lemma (see [31, Lemma 1]) and Lemma 4.4 with \( P(t) = h_j(f(t))f'(t) \), \( Q(t) = t^2 + |t|^{2^*} \) and \( \varphi = 1 \) yields that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (h_j(f(v_n))f'(v_n)v_n - h_j(f(v_\lambda))f'(v_\lambda)v_\lambda) \, dx = 0 \tag{4.5}
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} h_j(f(w_n))f'(w_n)w_n \, dx = 0. \tag{4.6}
\]
Combining (4.4)–(4.6) leads to \( \lim_{n \to \infty} \langle I_\lambda'(v_n), v_n \rangle - \langle I_\lambda'(v_\lambda), v_\lambda \rangle - \langle I_\lambda'(w_n), w_n \rangle = 0 \) and then \( \lim_{n \to \infty} \langle I_\lambda'(w_n), w_n \rangle = 0 \). Therefore, using (4.6) and Lemma 2.1, we have
\[
\limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(w_n) \, dx \right)
\leq \limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx + \int_{\mathbb{R}^N} V(x)f(w_n)f'(w_n)w_n \, dx \right)
= \limsup_{n \to \infty} \left( \lambda \int_{\mathbb{R}^N} h_1(f(w_n))f'(w_n)w_n \, dx - \int_{\mathbb{R}^N} h_2(f(w_n))f'(w_n)w_n \, dx \right)
= 0,
\]
which implies that \( A_1(w_n) \to 0 \) as \( n \to \infty \).

From the proof of Lemma 3.2, we see that
\[
\|w_n\|^2 \leq C \left( A_1(w_n) + A_1(w_n)^{\frac{2}{2^*}} \right).
\]
Since \( \lim_{n \to \infty} A_1(w_n) = 0 \), we have \( w_n \to 0 \) in \( H_1^1(\mathbb{R}^N) \) and then \( v_n \to v_\lambda \) in \( H_1^1(\mathbb{R}^N) \). Hence \( v_\lambda \) is a nontrivial critical point of \( I_\lambda \) with \( I_\lambda(v_\lambda) = c_\lambda \). Using standard arguments, we can prove that \( v_\lambda(x) > 0 \) for all \( x \in \mathbb{R}^N \).

At this point, for almost every \( \lambda \in [\lambda, 1] \), we obtain a positive critical point \( v_\lambda \) of the functional \( I_\lambda \). In general, we do not known whether this conclusion holds for \( \lambda = 1 \). However, we have

**Lemma 4.6.** There exist a sequence of numbers \( \{\lambda_n\} \subset [\lambda, 1] \) and a sequence of positive functions \( \{v_{\lambda_n}\} \subset H_1^1(\mathbb{R}^N) \) such that \( \lim_{n \to \infty} \lambda_n = 1 \), \( I_{\lambda_n}(v_{\lambda_n}) = c_{\lambda_n} \leq c_\lambda \) and \( I_{\lambda_n}'(v_{\lambda_n}) = 0 \).
Next we prove that the sequence \( \{v_{\lambda_n}\} \) obtained in Lemma 4.6 is bounded in \( H^1_\lambda(\mathbb{R}^N) \). For this purpose, we need the following Pohožaev type identity, which is similar to Lemma 3.8.

**Lemma 4.7.** If \( v \in H^1(\mathbb{R}^N) \) is a critical point of the functional \( I_\lambda \), then

\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x) f^2(v) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v) \, dx + N \int_{\mathbb{R}^N} H_2(f(v)) \, dx - \lambda N \int_{\mathbb{R}^N} H_1(f(v)) \, dx = 0.
\]

**Lemma 4.8.** The sequence \( \{v_{\lambda_n}\} \) obtained in Lemma 4.6 is bounded in \( H^1_\lambda(\mathbb{R}^N) \).

**Proof.** From the proof of Lemma 3.2, we see that

\[
|v_{\lambda_n}|^2 \leq C \left( A_1(v_{\lambda_n}) + A_1(v_{\lambda_n})^{2^*} \right),
\]

where

\[
A_1(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x) f^2(v) \right) \, dx.
\]

Thus it suffices to prove that \( \{A_1(v_{\lambda_n})\} \) is bounded. Invoking Lemma 4.7 and using the same arguments as in the proof of Theorem 1.1 leads to

\[
\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \leq C_1
\]

for some positive constant \( C_1 \). By \((h_1), (h_2)\), the definition of \( h_j \) and Lemma 2.1, we have \( h_j(f(t)) f'(t) t \geq 0 \) for all \( t \in \mathbb{R} \), and

\[
\lim_{t \to 0} \frac{h_1(f(t)) f'(t) t}{f^2(t)} = 0, \quad \lim_{t \to \infty} \frac{h_1(f(t)) f'(t) t}{|t|^{2^*}} = 0.
\]

Then there exists \( C_2 > 0 \) such that

\[
h_1(f(t)) f'(t) t \leq \frac{\alpha}{4} f^2(t) + C_2|t|^{2^*}, \quad \text{for all } t \in \mathbb{R}.
\]

Using \( \langle \partial_{\lambda_n} I_1(v_{\lambda_n}), v_{\lambda_n} \rangle = 0 \) and Lemma 2.1 yields

\[
\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_{\lambda_n}) \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx + \int_{\mathbb{R}^N} V(x) f(v_{\lambda_n}) f'(v_{\lambda_n}) v_{\lambda_n} \, dx
\]

\[
= - \int_{\mathbb{R}^N} h_2(f(v_{\lambda_n})) f'(v_{\lambda_n}) v_{\lambda_n} \, dx + \lambda_n \int_{\mathbb{R}^N} h_1(f(v_{\lambda_n})) f'(v_{\lambda_n}) v_{\lambda_n} \, dx
\]

\[
\leq \frac{\alpha}{4} \int_{\mathbb{R}^N} f^2(v_{\lambda_n}) \, dx + C_2 \int_{\mathbb{R}^N} |v_{\lambda_n}|^{2^*} \, dx
\]

\[
\leq \frac{1}{4} \int_{\mathbb{R}^N} V(x) f^2(v_{\lambda_n}) \, dx + C_3 \left( \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \right)^{2^*},
\]

which implies that

\[
A_1(v_{\lambda_n}) \leq C_4 \left( \int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \, dx \right)^{2^*} \leq C_5.
\]

The proof is complete. \( \square \)
Proof of Theorem 1.2 (completed). By Lemma 4.6, there exist a sequence of numbers \( \{\lambda_n\} \subset [\bar{\lambda}, 1] \) and a sequence of positive functions \( \{v_{\lambda_n}\} \subset H^1_{\omega}(\mathbb{R}^N) \) such that \( \lim_{n \to \infty} \lambda_n = 1 \), \( I_{\lambda_n}(v_{\lambda_n}) = c_{\lambda_n} \leq c_1 \) and \( I'_{\lambda_n}(v_{\lambda_n}) = 0 \). It follows from Lemma 4.8 that \( \{v_{\lambda_n}\} \) is bounded in \( H^1_{\omega}(\mathbb{R}^N) \). Observe that \( I_1 = I \). Then, since the map \( \lambda \mapsto c_\lambda \) is continuous from the left, we have

\[
\lim_{n \to \infty} I(v_{\lambda_n}) = \lim_{n \to \infty} \left(I_{\lambda_n}(v_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^N} H_1(f(v_{\lambda_n})) \, dx\right) = \lim_{n \to \infty} c_{\lambda_n} = c_1.
\]

Similarly, there holds \( I'(v_{\lambda_n}) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \). Therefore, \( \{v_{\lambda_n}\} \) is a bounded Palais-Smale sequence for the functional \( I \) at level \( c_1 \). Using Lemma 4.5 again, we obtain a positive critical point \( v \) of the functional \( I \).

Acknowledgments. We should like to thank the anonymous referee for his/her careful readings of our manuscript and the useful comments made for its improvement. The revision of this paper was finished during the first author’s stay at Utah State University as a visiting scholar. He is grateful to Professor Zhi-Qiang Wang for his invitation and hospitality.

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Received January 2019; revised July 2019.

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