NEF VECTOR BUNDLES ON A PROJECTIVE SPACE WITH FIRST CHERN CLASS 3 AND SECOND CHERN CLASS 8

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Abstract. We describe nef vector bundles on a projective space with first Chern class three and second Chern class eight over an algebraically closed field of characteristic zero by giving them a minimal resolution in terms of a full strong exceptional collection of line bundles.

1. Introduction

This paper is a continuation of [Ohn16]. Throughout this paper, as in [Ohn16], we work over an algebraically closed field $K$ of characteristic zero. Let $E$ be a nef vector bundle of rank $r$ on a projective space $\mathbb{P}^n$ with first Chern class $c_1$ and second Chern class $c_2$. In [Ohn16, Theorem 1.1], we classified such $E$’s in case $c_1 = 3$ and $c_2 < 8$, and in [Ohn16, Proposition 1.2], we also gave an example of such $E$’s on a projective plane with $c_1 = 3$ and $c_2 = 8$. In this paper, we complete the classification of such $E$’s with $c_1 = 3$ and $c_2 = 8$ by giving them a minimal resolution in terms of a full strong exceptional collection of line bundles. The precise statement is as follows.

Theorem 1.1. Let $E$ be as above. Suppose that $c_1 = 3$ and that $c_2 = 8$. Then $n = 2$ and $E$ fits in an exact sequence

$$0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) \to E \to 0.$$ 

This implies that the example given in [Ohn16, Proposition 1.2] is nothing but the unique type of nef vector bundles with $c_1 = 3$ and $c_2 = 8$.

Note that, for a nef vector bundle $E$ with $c_1 = 3$, the anti-canonical bundle on $\mathbb{P}(E)$ is ample if $n \geq 3$ and nef if $n \geq 2$. Moreover, if $n = 2$, it is big if and only if $c_2 \leq 8$. So we can say that we have classified, except for the case (11) of [Ohn16, Theorem 1.1], weak Fano manifolds of the form $\mathbb{P}(E)$ where $E$ is a vector bundle on a projective space $\mathbb{P}^n$ under the assumption that $E$ is nef and $c_1 = 3$. Recall here that a projective manifold $M$ is called weak Fano if its anti-canonical bundle is nef and big, and that a vector bundle $\mathcal{F}$ is called a weak Fano bundle if $\mathbb{P}(\mathcal{F})$ is a weak Fano manifold. We hope that the theorem above together with [Ohn16, Theorem 1.1] would be useful for some part of the classification of weak Fano bundles.

This paper is organized as follows. We first concentrate our attention to the case $n = 2$. In §2 we recall and summarize results obtained in [Ohn16] by taking into account that we only consider nef vector bundles with $c_1 = 3$ and $c_2 = 8$. In §3 we show that $E$ does not contain $\mathcal{O}(1)$ as a subsheaf. In §4 we first observe...
that $E$ must fit in the exact sequence given in [Ohn16, Proposition 1.2] and then show that $E$ fits in the exact sequence in the theorem above. Finally, in § 5 we show that the case $n \geq 3$ does not happen.

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1.1. Notation and conventions. Basically we follow the standard notation and terminology in algebraic geometry. For a vector bundle $E$, $\mathbb{P}(E)$ denotes $\text{Proj} S(E)$, where $S(E)$ denotes the symmetric algebra of $E$. For a coherent sheaf $F$ on a smooth projective variety $X$, we denote by $c_i(F)$ the $i$-th Chern class of $F$. For coherent sheaves $F$ and $G$ on $X$, $h^q(F)$ denotes $\dim H^q(F)$. Finally we refer to [Laz04] for the definition and basic properties of nef vector bundles.

2. Set-up for the two-dimensional case

In the following, let $E$ be a nef vector bundle on a projective space $\mathbb{P}^n$ with $c_1 = 3$ and $c_2 = 8$. In this section, we assume that $n = 2$. It follows from [Ohn16, (3.10), (3.11) and (3.12)] that

\begin{align}
(2.1) & \quad h^1(E(-2)) = 5, \\
(2.2) & \quad h^0(E(-1)) - h^1(E(-1)) = -2, \\
(2.3) & \quad h^0(E) = r + 1.
\end{align}

Note here that, for a nef vector bundle $E'$ in general, unlike the case of globally generated vector bundles, an inequality $h^0(E') \geq r - 1$ does not necessarily imply that $E'$ fits in an exact sequence of the form

$$0 \to \mathcal{O}^{\text{Br}-1} \to E' \to I_Z \otimes \det E' \to 0$$

for some closed subscheme $Z$ of $\mathbb{P}^2$, where $I_Z$ denotes the ideal sheaf of $Z$ (see [Ohn16] §13 for some examples). Set $e_{0,1} = h^0(E(-1))$.

Then

$$h^1(E(-1)) = e_{0,1} + 2 \geq 2.$$

It follows from [Ohn16] (3.13)] that $5 \geq h^1(E(-1))$. Therefore

$$0 \leq e_{0,1} \leq 3.$$

We apply to $E$ the Bondal spectral sequence [OT14, Theorem 1]

$$E_2^{p,q} = \text{Tor}^A_{-p}(\text{Ext}^q(G, E), G) \Rightarrow E^{p+q} = \begin{cases} E & \text{if} \quad p + q = 0 \\ 0 & \text{if} \quad p + q \neq 0. \end{cases}$$

As we have seen in [Ohn16] §3.1 and Lemma 5.1], $E_2^{p,q}$ vanishes unless $(p, q) = (-2, 1), (-1, 1)$ or $(0, 0)$, and $E_2^{-2,1}$ and $E_2^{-1,1}$ fit in an exact sequence of coherent sheaves

$$0 \to E_2^{-2,1} \to \mathcal{O}(-3) \xrightarrow{\varphi_3} \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}} \to E_2^{-1,1} \to k(w) \to 0$$

for some point $w$ in $\mathbb{P}^2$, where $k(w)$ denotes the residue field of $w$. Note that this exact sequence is a consequence of the vanishing $H^1(E) = 0$, and recall
that $H^1(\mathcal{E})$ vanishes by the Kawamata-Viehweg vanishing theorem since $c_2 < 9$. Moreover we have the following exact sequences

\begin{align}
(2.6) & \quad 0 \to E_{2}^{-2,1} \to E_{2}^{0,0} \to E_{3}^{0,0} \to 0, \\
(2.7) & \quad 0 \to E_{3}^{0,0} \to \mathcal{E} \to E_{2}^{-1,1} \to 0, \\
(2.8) & \quad 0 \to \mathcal{O}^{\oplus e_{0,1}} \to \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r+1} \to E_{2}^{0,0} \to 0.
\end{align}

We shall divide the proof according to the value of $e_{0,1}$.

3. The case $n = 2$ and $e_{0,1} > 0$

Suppose that $n = 2$ and $e_{0,1} > 0$. Since $e_{0,1} > 0$ and $h^0(\mathcal{E}(-2)) = 0$ by the argument in [Ohn16, §3], we have an exact sequence

\[0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{F} \to 0\]

where $\mathcal{F}$ is a torsion-free sheaf with $c_1(\mathcal{F}) = 2$, $c_2(\mathcal{F}) = 6$ and $h^0(\mathcal{F}(-1)) = e_{0,1} - 1$. Denote by $\mathcal{F}^{\vee\vee}$ the double dual of $\mathcal{F}$, and consider the quotient $Q$ of the inclusion $\mathcal{F} \subset \mathcal{F}^{\vee\vee}$:

\[0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to Q \to 0.\]

The support of $Q$ has dimension zero, and its length is equal to $-c_2(Q)$. By [Ohn16, Lemma 12.1], $\mathcal{F}^{\vee\vee}$ is a nef vector bundle of rank $r - 1$ with $c_1(\mathcal{F}^{\vee\vee}) = 2$, $c_2(\mathcal{F}^{\vee\vee}) = 6 + c_2(Q)$ and $h^0(\mathcal{F}^{\vee\vee}(-1)) \geq e_{0,1} - 1$.

3.1. The case $e_{0,1} > 1$. Suppose that $e_{0,1} > 1$. Then it follows from [Ohn14, Theorem 6.5] that $\mathcal{F}^{\vee\vee}$ is isomorphic to either $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$ or $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-3}$, or $\mathcal{F}^{\vee\vee}$ fits in an exact sequence

\[0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1} \to \mathcal{F}^{\vee\vee} \to 0.\]

Suppose that $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$. Since $c_2(\mathcal{F}^{\vee\vee}) = 0$, the length of $Q$ is 6. Let $\mathcal{G}$ be the image of the composite of the inclusion $\mathcal{F} \to \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2}$ and the projection $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{O}^{\oplus r-2}$. Note that the kernel of the surjection $\mathcal{F} \to \mathcal{G}$ is a subsheaf of $\mathcal{O}(2)$. Hence it can be written as $\mathcal{I}_Z(2)$ where $\mathcal{I}_Z$ is the ideal sheaf of some closed subscheme $Z$ of $\mathbb{P}^2$. Now we have the following commutative diagram with exact lows and columns

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_Z(2) & \longrightarrow & \mathcal{O}(2) & \longrightarrow & \mathcal{O}_Z(2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-2} & \longrightarrow & Q & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}^{\oplus r-2} & \longrightarrow & Q_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 0 & 0 & & 0 & & & & \\
\end{array}
\]
where \( \mathcal{Q}_1 \) is defined by the diagram above. Since \( \mathcal{O}_Z(2) \to \mathcal{Q} \) is injective, we see that \( \dim \mathcal{Z} \leq 0 \), and thus \( \mathcal{O}_Z(2) \cong \mathcal{O}_Z \). If \( \mathcal{Q}_1 \neq 0 \), then take a line \( L \) intersecting with the support of \( \mathcal{Q}_1 \). Then the kernel of the surjection \( \mathcal{O}_L^\mathcal{E}_{-1} \to \mathcal{Q}_1|_L \) has a negative degree line bundle as a direct summand, which implies that some negative degree line bundle is a quotient of \( \mathcal{G}|_L \), \( \mathcal{F}|_L \) and \( \mathcal{E}|_L \). This contradicts that \( \mathcal{E} \) is nef. Hence \( \mathcal{Q}_1 = 0 \). Thus \( \mathcal{G} \cong \mathcal{O}^{\mathcal{E}_{-r-2}}_Z \), \( \mathcal{Q}_Z \cong \mathcal{Q} \), and \( \mathcal{O}_Z \) has length 6. Since \( h^0(\mathcal{G}(-1)) = 0 \), we infer that \( h^0(\mathcal{I}_Z(1)) = e_{0,1} - 1 > 0 \). Hence there exists a line \( L \) passing through \( Z \). Since length \( \mathcal{O}_Z = 6 \), this implies that the kernel of the restriction \( \mathcal{O}_L(2) \to \mathcal{O}_Z \) to the line \( L \) of the surjection \( \mathcal{O}(2) \to \mathcal{O}_Z \) is isomorphic to \( \mathcal{O}_L(-4) \). By restricting the diagram above to the line \( L \), we see that \( \mathcal{F}|_L \) has a negative degree line bundle as a quotient; this is a contradiction. Hence \( \mathcal{F}^{\mathcal{E}} \) cannot be isomorphic to \( \mathcal{O}(2) \oplus \mathcal{O}^{\mathcal{E}_{-r-2}} \).

Suppose that \( \mathcal{F}^{\mathcal{E}} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\mathcal{E}_{-r-2}} \). Since \( c_2(\mathcal{F}^{\mathcal{E}}) = 1 \), the length of \( \mathcal{Q} \) is 5. Let \( \mathcal{G} \) be the image of the composite of the inclusion \( \mathcal{F} \to \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\mathcal{E}_{-r-3}} \) and the projection \( \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\mathcal{E}_{-r-3}} \to \mathcal{O}^{\mathcal{E}_{-r-3}} \), and \( \mathcal{Q}_1 \) the cokernel of the inclusion \( \mathcal{G} \to \mathcal{O}^{\mathcal{E}_{-r-3}} \). Then there exists a surjection \( \mathcal{Q} \to \mathcal{Q}_1 \), and thus the support of \( \mathcal{Q}_1 \) has dimension \( \leq 0 \). If \( \mathcal{Q}_1 \neq 0 \), we get a contradiction by the same argument as above. Therefore we may assume that \( \mathcal{Q}_1 = 0 \); thus \( \mathcal{G} \cong \mathcal{O}^{\mathcal{E}_{-r-3}} \). Let \( \mathcal{H} \) be the kernel of the surjection \( \mathcal{F} \to \mathcal{O}^{\mathcal{E}_{-r-3}} \). Then we have the following commutative diagram with exact rows and columns.

\[
\begin{array}{cccccc}
0 & & 0 & & & \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{H} & \to & \mathcal{O}(1)^{\oplus 2} & \to & \mathcal{Q} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{F} & \to & \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\mathcal{E}_{-r-3}} & \to & \mathcal{Q} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}^{\mathcal{E}_{-r-3}} & & = & & \mathcal{O}^{\mathcal{E}_{-r-3}} & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

Since \( h^0(\mathcal{O}^{\mathcal{E}_{-r-3}}(-1)) = 0 \), we infer that \( h^0(\mathcal{H}(-1)) = e_{0,1} - 1 > 0 \). Since \( \mathcal{H}(-1) \) is a subsheaf of \( \mathcal{O}^{\mathcal{E}_{-2}} \), this implies that \( \mathcal{H}(-1) \cong \mathcal{I}_Z \oplus \mathcal{O} \) and \( \mathcal{Q}(-1) \cong \mathcal{O}_Z \) for some 0-dimensional closed subscheme \( Z \) of length 5 in \( \mathbb{P}^2 \). Now take a line \( L \) that intersect with \( Z \) in length \( l \geq 2 \). Then the kernel of \( \mathcal{O}_L(1)^{\oplus 2} \to \mathcal{O}_{Z\cap L}(1) \) is of the form \( \mathcal{O}_L(1-l) \oplus \mathcal{O}_L(1) \). This implies that \( \mathcal{F}|_L \) has a negative degree line bundle as a quotient, which is a contradiction. Hence \( \mathcal{F}^{\mathcal{E}} \) cannot be isomorphic to \( \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\mathcal{E}_{-r-3}} \) either.

Suppose that \( \mathcal{F}^{\mathcal{E}} \) fits in the exact sequence \( (3.1) \). Since \( c_2(\mathcal{F}^{\mathcal{E}}) = 2 \), the length of \( \mathcal{Q} \) is 4. Define a torsion-free sheaf \( \mathcal{F}_0 \) as a quotient of \( \mathcal{F}^{\mathcal{E}} \) by an injection \( \mathcal{O}(1) \to \mathcal{F}^{\mathcal{E}} \). Then \( \mathcal{F}_0 \) fits in an exact sequence \( 0 \to \mathcal{O}(-1) \to \mathcal{O}^{\mathcal{E}_{-r-1}} \to \mathcal{F}_0 \to 0 \).

Let \( \mathcal{G} \) be the image of the composite of the inclusion \( \mathcal{F} \to \mathcal{F}^{\mathcal{E}} \) and the projection \( \mathcal{F}^{\mathcal{E}} \to \mathcal{F}_0 \). Since \( h^0(\mathcal{F}_0(-1)) = 0 \), we see that \( h^0(\mathcal{G}(-1)) = 0 \). Let \( \mathcal{H} \) be
the kernel of the surjection $\mathcal{F} \to \mathcal{G}$. Then we have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \mathcal{H} & \mathcal{O}(1) & \mathcal{Q}_2 & \mathcal{Q}_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{F} & \mathcal{F}^{\oplus} & \mathcal{Q} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{G} & \mathcal{F}_0 & \mathcal{Q}_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 
\end{array}
$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined by the diagram above. Since $h^0(\mathcal{G}(-1)) = 0$, we see that $h^0(\mathcal{H}(-1)) = e_{0,1} - 1 > 0$. Since $\mathcal{H}(-1)$ is a subsheaf of $\mathcal{O}$, this implies that $\mathcal{H}(-1)$ is $\mathcal{O}$ itself; thus $\mathcal{Q}_2 = 0$, $\mathcal{Q} \cong \mathcal{Q}_1$ and $\mathcal{Q}_1$ has length 4. As we have seen in the proof of [Ohm13], $\mathcal{F}_0$ is locally free outside at most one point, and if $\mathcal{F}_0$ is not locally free at a point $z$, then $\mathcal{F}_0$ is isomorphic to $\mathfrak{m}_z(1) \oplus \mathcal{O}^{\oplus r-3}$, where $\mathfrak{m}_z$ is the ideal sheaf of $z$, since $n = 2$. Suppose that $\mathcal{F}_0$ is not locally free. Then take a line $L$ passing through $z$ and meeting the support of $\mathcal{Q}_1$. We see that the surjection $\mathcal{F}_0 \to \mathcal{Q}_1$ induces a surjection $\mathcal{O}_L^{\oplus r-2} \to \mathcal{Q}_1|_L$, whose kernel has a negative degree line bundle as a quotient, and thus so does $\mathcal{G}_L$, $\mathcal{F}_L$ and $\mathcal{E}_L$. This is a contradiction. Suppose that $\mathcal{F}_0$ is locally free. Then take a line $L$ which intersects with $\mathcal{Q}_1$ in length $l \geq 2$. Since $\mathcal{F}_0|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}^{\oplus r-3}$, we see that $\mathcal{G}_L$ admits a negative degree line bundle as a quotient; this is a contradiction. Hence $\mathcal{F}^{\oplus}$ cannot fit in the exact sequence (3.1).

Therefore we conclude that the case $e_{0,1} > 1$ does not happen.

3.2. The case $e_{0,1} = 1$. Suppose that $e_{0,1} = 1$. If the morphism $\nu_2$ in (2.5) is zero, then $E_2^{2,1}|_L \cong \Omega_{\mathbb{P}^2}(1)|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$ for a line $L$ not containing $w$. By (2.7), this implies that $\mathcal{E}|_L$ has $\mathcal{O}_L(-1)$ as a quotient; this is a contradiction. Hence $\nu_2 \neq 0$, and thus $E_2^{2,1} = 0$, $E_2^{0,0} \cong E_3^{0,0}$ by (2.6), and $E_2^{1,1}$ fits in a central sequence

(3.2) $0 \to \mathcal{O}(-3) \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1) \to E_2^{-1,1} \to k(w) \to 0.$

We see that $E_2^{-1,1}$ is a coherent sheaf of rank one. Since $E_3^{0,0}$ is torsion-free by (2.7), so is $E_2^{0,0}$, and thus $E_2^{0,0}$ has $\mathcal{O}(1)$ as a subsheaf and consequently is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2}$ by (2.8). Hence the exact sequence (2.7) becomes an exact sequence

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2} \xrightarrow{\nu_2} \mathcal{E} \to E_2^{-1,1} \to 0.$$ 

By taking the dual of $\varphi$ and $(r-1)$-th wedge product of the dual, we obtain a morphism $\wedge^{r-1}\mathcal{E}^\vee \to \mathcal{O}(-1)$. Let $\mathcal{I}_Z(-1)$ be the image of this morphism, where $\mathcal{I}_Z$ is the ideal sheaf of a closed subscheme $Z$ of $\mathbb{P}^2$ of dimension $\leq 1$. Note that
$Z$ is the degeneracy locus of $\varphi$ and that if we denote by $\psi$ the induced surjection $\mathcal{E} \cong \wedge^{-1} \mathcal{E}^\vee \otimes \det \mathcal{E} \to \mathcal{I}_Z(-1) \otimes \det \mathcal{E} \cong \mathcal{I}_Z(2)$ then $\psi \circ \varphi = 0$.

Suppose that the degeneracy locus $Z$ of $\varphi$ has codimension $\geq 2$. Then $E_2^{-1,1}$ is torsion-free. This implies that $E_2^{-1,1} \cong \mathcal{I}_Z(2)$ and that $\mathcal{E}$ fits in an exact sequence

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2} \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{I}_Z(2) \to 0.$$  

Note that length $Z = 6$. Since $\mathcal{E}$ is nef, length$(Z \cap L) \leq 2$ for any line $L$ in $\mathbb{P}^2$; let us call this the basic property of $Z$. Let $p$ be any point in $Z$. We may assume that $Z$ is in an affine open subscheme $\text{Spec } K[x, y]$ and that $p = (0,0)$. The local ring $\mathcal{O}_{Z,p}$ can be written as $A/I$, where $A = \mathcal{O}_{\mathbb{P}^2,p} = K[[x, y]]$ and $I$ the ideal of $Z$ in the local ring $A$. Observe here that if length$(A/I) \leq 4$ and thus the support of $Z$ contains another point $q \neq p$, then the basic property of $Z$ implies $I \nsubseteq m^2$, where $m$ denotes the maximal ideal of $A$. Based on this observation, we can deduce from the basic property of $Z$ that $I \nsubseteq m^2$ without any assumption on length$(A/I)$. Now that $Z$ is curvilinear, after changing coordinates $(x, y)$ if necessary, we may assume that $I = \langle y - \varphi(x), x^3 \rangle$, where $\varphi(x) = a_2 x^2 + a_3 x^3 + \cdots \in K[[x]]$ $(a_2 \neq 0)$ and $l = \text{length}(A/I)$. Local computation then shows that there exists a smooth conic $C$ such that length$(Z \cap C) \geq 5$; e.g., if $l \geq 3$, we can take a defining equation of $C$ to be $y = a_2 x^2 + dx y + ey^2$ for some $d, e \in K$. However this again contradicts that $\mathcal{E}$ is nef. Therefore this case cannot happen.

Suppose that dim $Z = 1$. Then the ideal sheaf $\mathcal{I}_Z$ of $Z$ is decomposed as $\mathcal{I}_Z \cong \mathcal{I}_{Z_d}(-d)$, where $d$ is the degree of the divisor contained in $Z$ and $\mathcal{I}_{Z_d}$ is the ideal sheaf of a 0-dimensional closed subscheme $Z_d$ of $\mathbb{P}^2$. Consider the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(1) & \oplus & \mathcal{O}^{\oplus r-2} & \xrightarrow{\varphi} & \mathcal{E} & \xrightarrow{\psi} & E_2^{-1,1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}(1) & \oplus & \mathcal{O}^{\oplus r-2} & \to & \mathcal{E} & \to & \mathcal{I}_{Z_d}(2-d) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{I}_{Z_d}(2-d) & \xrightarrow{\mathcal{I}_{Z_d}(2-d)} & \mathcal{I}_{Z_d}(2-d) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 0 & & 
\end{array}
$$

where $\mathcal{K}$ and $\mathcal{T}$ are defined by the diagram above. We see that $\mathcal{K}$ is a coherent sheaf of rank $r - 1$ and thus $\mathcal{T}$ is the torsion subsheaf of $E_2^{-1,1}$, and that Supp $Z = \text{Supp } \mathcal{T} \cup \text{Supp } Z_d$. Hence $E_2^{-1,1}$ has an associated point of codimension one. Now recall the exact sequence $(3.2)$ and split this sequence into the following two exact sequences of coherent sheaves

$$(3.3) \quad 0 \to \mathcal{O}(-3) \xrightarrow{\mathcal{J}} \Omega_{\mathbb{P}^2}(1) \to \mathcal{C} \to 0,$$

$$(3.4) \quad 0 \to \mathcal{C} \to E_2^{-1,1} \to k(w) \to 0.$$
Note that $C$ has an associated point of codimension one since so does $E_2^{-1,1}$. Hence $\nu_2$ passes through $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$.

Suppose that $\nu_2$ passes through $\mathcal{O}(-1)$. Then we have the following commutative diagram with exact lows and columns

$$
\begin{array}{ccccccc}
\mathcal{O}(-3) & \mathcal{O}(-3) & & & & & \\
\downarrow & \downarrow & & & & & \\
0 & \mathcal{O}(-1) & \mathcal{O}_{\mathbb{P}^2}(1) & \mathcal{I}_p & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \mathcal{I}_p & \rightarrow & 0 \\
0 & \mathcal{O}_D(-1) & \mathcal{C} & \mathcal{I}_p & \rightarrow & 0 \\
\downarrow & \downarrow & \mathcal{I}_p & \rightarrow & 0 \\
0 & 0 & & & & & \\
\end{array}
$$

where $\mathcal{I}_p$ is the ideal sheaf of a point $p$, and $D$ is a conic in $\mathbb{P}^2$. We also have the following commutative diagram with exact lows and columns

$$
\begin{array}{ccccccc}
\mathcal{O}_D(-1) & \mathcal{O}_D(-1) & & & & & \\
\downarrow & \downarrow & & & & & \\
0 & \mathcal{C} & \mathcal{E}_2^{-1,1} & \mathcal{I}_p & \rightarrow & 0 \\
\downarrow & \downarrow & \mathcal{I}_p & \rightarrow & 0 \\
0 & \mathcal{D} & \mathcal{I}_p & \rightarrow & 0 \\
\end{array}
$$

where $\mathcal{D}$ is defined by the diagram above. Suppose that $\mathcal{D}$ has an associated point other than the generic point. Then it must be $w$, and thus $\mathcal{D} \cong \mathcal{I}_w \oplus k(w)$, which also contradicts that $\mathcal{E}$ is nef. Therefore $\mathcal{D}$ is torsion-free. Since $\mathcal{D}$ has rank one, $c_1(\mathcal{D}) = 0$ and $c_2(\mathcal{D}) = 0$. Moreover we see that $p = w$, that $\mathcal{O}_D(-1)$ is the torsion subsheaf $\mathcal{T}$ of $E_2^{-1,1}$, that $Z_d = \emptyset$, and that $Z = D$. If $h^0(E_2^{-1,1}) \neq 0$, then $E_2^{-1,1} \cong \mathcal{O}_D(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$, which contradicts that $\mathcal{E}$ is nef. Hence $h^0(E_2^{-1,1}) = 0$. Since $h^1(\mathcal{O}_D(-1)) = h^2(\mathcal{O}_{\mathbb{P}^2}(-3)) = 1$, this implies that $H^0(\mathcal{D}) = H^0(\mathcal{O}_{\mathbb{P}^2}) \cong H^1(\mathcal{O}_D(-1))$. Suppose that $D$ is smooth. Consider the pull back $\mathcal{O}_D(-1) \rightarrow E_2^{-1,1}|_D \rightarrow \mathcal{O}_D \rightarrow 0$ of the exact sequence above. Note that $E_2^{-1,1}|_D$ has rank at least two since $D$ is the degeneracy locus of $\varphi$. Hence we obtain an exact sequence

$$
0 \rightarrow \mathcal{O}_D(-1) \rightarrow E_2^{-1,1}|_D \rightarrow \mathcal{O}_D \rightarrow 0.
$$
Note that $D \cong \mathbb{P}^1$ and that $\mathcal{O}_D(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ via this isomorphism. Since the sequence above does not split, this implies that $E_2^{-1,1}|_D \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, which contradicts that $\mathcal{E}$ is nef. Suppose that $D$ is a double line. Then $\mathcal{O}_D(-1) \rightarrow \mathcal{O}_D(-1)$. The similar argument as above shows that there exists an exact sequence

$$0 \rightarrow \mathcal{O}_D(-1) \rightarrow E_2^{-1,1}|_D \rightarrow \mathcal{O}_D(-1) \rightarrow 0.$$ 

Hence $E_2^{-1,1}|_D \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$; this contradicts that $\mathcal{E}$ is nef. Suppose that $D$ is a union of two distinct lines: $D = L_1 + L_2$. Then $E_2^{-1,1}|_{L_i} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ by the similar argument as above, and hence this case does not occur either.

Suppose that $\nu_2$ passes through $\mathcal{O}(-2)$ and does not pass through $\mathcal{O}(-1)$. Then we have the following commutative diagram with exact lows and columns

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{O}(-3) & \mathcal{O}(-3) & \mathcal{O}(-3) & \mathcal{O}(-3) & \mathcal{O}(-3) & \mathcal{O}(-3) & \mathcal{O}(-3) & \mathcal{O}(-3) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}(-2) & \mathcal{O}(-2) & \mathcal{O}(-2) & \mathcal{O}(-2) & \mathcal{O}(-2) & \mathcal{O}(-2) & \mathcal{O}(-2) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

where $\mathcal{I}_W$ is the ideal sheaf of a 0-dimensional locally complete intersection $W$ of length three, and $L$ is a line in $\mathbb{P}^2$. We also have the following commutative diagram with exact lows and columns

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) & \mathcal{O}_L(-2) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) & \mathcal{I}_W(1) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

where $\mathcal{D}$ is defined by the diagram above. If $\mathcal{D}$ is not torsion-free, then $\mathcal{D} \cong \mathcal{I}_w \oplus k(w)$, which contradicts that $\mathcal{E}$ is nef. Therefore $\mathcal{D}$ is a torsion-free coherent sheaf of rank one with $c_1(\mathcal{D}) = 1$ and $c_2(\mathcal{D}) = 2$. Hence $\mathcal{O}_L(-2)$ is the torsion subsheaf $\mathcal{T}$ of $E_2^{-1,1}$, and we infer that $\mathcal{D} \cong \mathcal{I}_{Z_1}(1)$ with length $Z_1 = 2$. This also contradicts that $\mathcal{E}$ is nef.
Therefore we conclude that the case $e_{0,1} = 1$ does not happen.

4. THE CASE $n = 2$ AND $e_{0,1} = 0$

Suppose that $e_{0,1} = 0$. Then $E_{-2,1}^2 \cong \mathcal{O}(-3)$ and $E_{-1,1}^2 \cong k(w)$ by (2.5) and (3.8). Thus we have the following two exact sequences by (2.6) and (2.7)

\begin{align*}
(4.1) & \quad 0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r+1} \to E_{-3}^{0,0} \to 0, \\
(4.2) & \quad 0 \to E_{-3}^{0,0} \to \mathcal{E} \to k(w) \to 0.
\end{align*}

These two exact sequences show that $\mathcal{E}$ must fit in the exact sequence given in [Ohn16, Proposition 1.2]. We shall show that $\mathcal{E}$ has a resolution in terms of a full strong exceptional sequence of line bundles as in Theorem 1.1 in accordance with the framework given in [Ohn14].

Since $h^1(E_{-3}^{0,0}(1)) = 0$, we have the following commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}(-3) & \mathcal{J} & \mathcal{I}_w(-1) \to 0 \\
\downarrow & \downarrow & g & \downarrow \\
0 & \mathcal{O}^{\oplus r+1} & \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) & \mathcal{O}(-1) \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & E_{-3}^{0,0} & \mathcal{E} & k(w) \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

where $\mathcal{I}_w$ is the ideal sheaf of $w$, and $\mathcal{J}$ and $g$ are defined by the diagram above. We also have the following commutative diagram with exact rows and columns

\[
\begin{array}{cc}
0 & 0 \\
\downarrow & \downarrow \\
\mathcal{O}(-3) & \mathcal{O}(-3) \\
\downarrow & \downarrow \\
0 & \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-2)^{\oplus 2} \to 0 \\
\downarrow & \downarrow \\
0 & \mathcal{O}(-3) \to \mathcal{J} \to \mathcal{I}_w(-1) \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

where $f$ is defined by the diagram above. We claim here that the composite of $f$ and the projection $\mathcal{O}(-3) \oplus \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-3)$ is non-zero. Suppose, to the
contrary, that the composite is zero. Then $J \cong \mathcal{O}(−3) \oplus \mathcal{I}_w(−1)$. By taking the double dual, the composite of the inclusion $\mathcal{I}_w(−1) \to J$ and $g$ extends to a splitting injection of the projection $\mathcal{O}^{\mathbb{P}r+1} \oplus \mathcal{O}(−1) \to \mathcal{O}(−1)$; we obtain the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{I}_w(−1) & \longrightarrow & \mathcal{O}(−1) & \longrightarrow & k(w) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \xrightarrow{g} & \mathcal{O}^{\mathbb{P}r+1} \oplus \mathcal{O}(−1) & \longrightarrow & \mathcal{E} & \longrightarrow & 0.
\end{array}
\]

Since the induced morphism $k(w) \to \mathcal{E}$ is a splitting injection of the surjection $\mathcal{E} \to k(w)$, we have an isomorphism $\mathcal{E} \cong \mathcal{E}_\mathbb{P} \oplus k(w)$, which is absurd. Hence the claim holds; thus $J \cong \text{Coker}(f) \cong \mathcal{O}(−2)\mathbb{P}^3$. Therefore we obtain the desired exact sequence

\[
0 \to \mathcal{O}(−2)\mathbb{P}^2 \to \mathcal{O}^{\mathbb{P}r+1} \oplus \mathcal{O}(−1) \to \mathcal{E} \to 0.
\]

5. THE CASE $n \geq 3$

In this section, we shall show that the case $n \geq 3$ does not happen. By considering the restriction $\mathcal{E}|_{L^3}$ to a 3-dimensional linear subspace $L^3 \subseteq \mathbb{P}^n$, we may assume that $n = 3$. We have

\[
\chi(\mathcal{E}(−1)) = \frac{c_3}{2} - 10
\]

by [Ohn16] (3.20)]. In particular, $c_3$ is even. We also have

\[
c_3 \geq 21
\]

by [Ohn16] (3.23)]. Since the equality in $c_3 \geq 21$ does not hold, we infer that $H(\mathcal{E})$ is big, and thus $h^q(\mathcal{E}(−1)) = 0$ for all $q > 0$ by [Ohn16] (3.3)]. Therefore $h^0(\mathcal{E}(−1)) \geq 1$. On the other hand, $H^0(\mathcal{E}(−2)) = 0$ by the argument in [Ohn16], §3], and $h^0(\mathcal{E}|_H(−1)) = 0$ for any plane $H \subset \mathbb{P}^3$ as is shown in §3. Hence $h^0(\mathcal{E}(−1)) = 0$, which is a contradiction. Therefore the case $n \geq 3$ does not happen.

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