ON THE SIZE OF MINIMAL SEPARATORS FOR TREEDEPTH DECOMPOSITION

A PREPRINT

Zijian Xu, Vorapong Suppakitpaisarn
The University of Tokyo
xuzijian@ms.k.u-tokyo.ac.jp,
vorapong@is.s.u-tokyo.ac.jp

ABSTRACT

We give a conjecture that we still have an optimal treedepth decomposition even we use only minimal separators no larger than the treewidth of input graphs. Then, we give some theoretical results for the conjecture. Treedepth decomposition has several practical applications and can be used to speed up many parameterized algorithms. There are several works aiming to give a scalable and exact method for the decomposition. Those include works based on a set of all minimal separators. In our experiments with several graphs, when we consider only minimal separators which are no larger than treewidth, we can significantly speed up the decomposition while still have an optimal treedepth decomposition. By the conjecture, we solve more instances than any algorithm submitted to PACE 2020. Theoretically, we prove that the conjecture for many classes of graphs. We also prove a relaxed version of the conjecture that the size of the separators should not be larger than $2 \cdot tw + 1$ when $tw$ is the treewidth.\footnote{A preliminary version of this paper will be published as a solver description of team xuzijian629 at PACE 2020 competition. However, this paper is not mainly focused on the solver but theoretical results obtained after the competition.}

1 Introduction

Treedepth decomposition, also known as vertex ranking number, cycle rank, or minimum height of elimination tree, is an important combinatorial optimization problem because of its applications to VLSI design\cite{20,23} and numerical algorithms\cite{21}. For a particular classes of graphs, when we have an optimal treedepth decomposition of the graphs, we can have a faster algorithm for classical problems such as maximum matching, negative cycle detection, minimum weight cycle, or weighted vertex cover\cite{8,15}.

Solving treedepth decomposition problem is NP-hard\cite{7}, but there are many works aiming to propose algorithms with small computational complexity. Those include an exact exponential algorithm which the computation time is $O(1.9602^n)$ in\cite{9}, an algorithm based on tree decompositions of input graphs in\cite{22}, and an algorithm based on vertex cover solutions of input graphs in\cite{18}.

In addition to algorithms with small complexity, algorithms which can solve treedepth decomposition exactly, and are scalable in experiments are also proposed in many recent works. For example, an algorithm based on SAT solver is proposed in\cite{10,27}. At a competition called as PACE 2020\cite{1}, participants are asked to submit exact and scalable algorithms for the problem. The development of algorithms for treedepth decomposition has been significantly advanced there. The most scalable software could solve the problem only when the input graph has no more than 30 nodes before, while many solvers can solve up to 100 nodes at the competition.

Many algorithms submitted to the PACE 2020 competition are based on a collection of all input graphs’ minimal separators. The idea is obtained from one of the most scalable algorithms for tree decomposition\cite{26}. Unfortunately, none of the submissions based on the set of separators is the most scalable algorithm at the competition. That is because, while we need the collection of separators only for the input graph in tree decomposition, we also need the collections for many of its subgraphs in treedepth decomposition. In the worst case, we may have to enumerate the separators for exponential number of subgraphs in those algorithms.
1.1 Our Contribution and Paper Organization

We give a conjecture that, we do not need a set of “all” minimal separators. Only separators which size no larger than treewidth of the input graph is enough to have an optimal decomposition. We will formally discuss our conjecture at Section 3 of this paper.

We cannot prove the conjecture in this paper, but we can give the following theoretical results.

1. When \( tw \) is treewidth of the input graph, at Section 4, we prove that only separators which size no larger than \( 2 \cdot tw + 1 \) is enough to have an optimal decomposition.
2. At Section 5, we prove the conjecture for some classes of graphs including chordal graphs, outerplanar graphs and cographs.
3. At Section 3, we give an input graph that the set of separators of size no larger than \( tw - 1 \) is not enough to have an optimal decomposition.

At Section 6, we show by experiments that, by using only separators which size no larger than treewidth of the input graph, we can significantly speed up the computation while always have an exact solution. While the winner of PACE 2020, which does not use the set of minimal separators, can solve 81 private instances (78 instances at the judge environment at the competition), we can solve 82 instances there. We observe that, compared to previous solvers, our solver perform particularly well in instances of which treewidth is much smaller than treedepth.

2 Preliminary

2.1 Notation

In this paper, \( G \) denotes undirected unweighted graph. \( V(G) \) or simply \( V \) denote the vertex set. We use \( n \) and \( m \) for the number of nodes and edges, respectively.

For a vertex set \( S \subseteq V \), \( G[S] \) is the subgraph induced by \( S \). We use \( G \setminus S \) for the graph obtained from \( G \) by removing \( S \), that is, \( G \setminus S = G[V \setminus S] \). When \( S = \{v\} \), we simply write \( G \setminus v \) for short. Lastly, we write \( C(G) \) to denote the set of connected components of (possibly connected) graph \( G \).

2.2 Treewidth

Treewidth of \( G \), denoted by \( tw(G) \) or \( tw \), is number to shown how much \( G \) is close to being a tree. The number \( tw(G) \) is one when \( G \) is a tree and it is as large as \( |V| - 1 \) when \( G \) is a completed graph.

Before giving a definition of treewidth, we define tree decomposition in the following definition.

**Definition 2.1 (Tree decomposition).** A tree decomposition of a graph \( G \) can be defined as \((T, f)\) where \( T \) is a tree and \( f \) is a function from \( V(T) \) to \( 2^{V(G)} \) with the following properties:

1. \( \bigcup_{\tau \in T} f(\tau) = V(G) \);
2. For each edge \( u, v \) of \( G \), there is a node \( \tau \in V(T) \) such that \( u, v \subseteq f(\tau) \);
3. For each \( v \in V(G) \), if \( T_v \) is a subtree of \( T \) induced by the set of nodes \( \{\tau \in V(T) : v \in f(\tau)\} \), then \( T_v \) is connected.

For each \( \tau \in V(T) \), we call the node set \( f(\tau) \) as a bag of \( T \). We denote the maximum bag size of a tree decomposition \((T, f)\) by \( b(T, f) := \max_{\tau \in V(T)} |f(\tau)| \). Treewidth of \( G \) is then can be defined as in the following definition:

**Definition 2.2 (Treewidth).** A tree decomposition \((T^*, f^*)\) is an optimal tree decomposition of \( G \) if, for any tree decomposition \((T, f)\), \( b(T, f) \geq b(T^*, f^*) \). Treewidth of \( G \) or \( tw(G) \) is the maximum bag size of \((T^*, f^*)\), i.e. \( tw(G) := b(T^*, f^*) \).

Calculating \( tw(G) \) is NP-hard [16]. However, experimental algorithms for the calculation are proposed in many works [5]. Currently, the algorithm proposed in [26] is known to be the fastest. It is based on the enumeration of all input graphs’ minimal separators.
2.3 Treedepth

The main challenge of this work is to calculate an optimal treedepth decomposition of the input graph $G$. A treedepth decomposition of $G$ is defined as follows:

**Definition 2.3 (Treedepth decomposition).** A rooted tree $T$ is called a treedepth decomposition of $G$ if

1. $V(T) = V(G)$.
2. For any $(u, v) \in E(G)$, $u$ and $v$ satisfies ancestor-descendant condition in $T$, that is, $u$ is an ancestor of $v$ or $v$ is an ancestor of $u$ in $T$.

By the definition of treedepth decomposition, we can define the treedepth of graph $G$ in the following definition.

**Definition 2.4 (Treedepth).** The treedepth of graph $G$, denoted by $td(G)$ is the minimum height among all treedepth decompositions of $G$.

A treedepth decomposition of $G$ is called optimal if its height is equal to the treedepth of the graph.

It is known that, for any graph $G$, $td(G) \geq tw(G) + 1$ (See [2, 9] for detailed explanation).

2.4 Separators and Minimal Separators

A node set $S \subseteq V$ is called an $a$-$b$ separator if $a, b \in V$ are not connected in $G - S$. A minimal $a$-$b$ separator $S$ is called a minimal separator for some $a, b \in V$.

A graph may have an exponential number of minimal separators even when the treewidth is small (See Figure 1 for example). In [25] an algorithm for enumerating all separators are proposed. The running time of the algorithm is in $O(n^3m)$ per separator. The algorithm is later modified in [26] to enumerate only minimal separators with bounded size. The time complexity is no longer $O(n^3m)$ per separator anymore, but the modified algorithm is practically fast and is used in many software [19, 26].

2.5 Calculating Treedepth Using Minimal Separators

It is discussed in [6] that we can compute an optimal treedepth decomposition by determining separators in a top-down way. Let $\Delta_G$ be a collection of all minimal separators of $G$. The authors show the following equation:

$$td(G) = \begin{cases} |V| & \text{if } G \text{ is complete} \\ \min_{S \in \Delta_G} \left( |S| + \max_{H \in C(G \setminus S)} td(H) \right) & \text{otherwise} \end{cases}$$

To calculate an optimal treedepth decomposition from Equation (1), the authors begin by finding the set $\Delta_G$. Then, for each minimal separator $S \in \Delta_G$ and for each connected component $H \in C(G \setminus S)$, they recursively calculate $td(H)$.

By that, they can obtain a set $S^* \in \arg \min_{S \in \Delta_G} \left( |S| + \max_{H \in C(G \setminus S)} td(H) \right)$. An optimal treedepth decomposition obtained from the algorithm is a tree which:

1. the top of the tree is a simple path consisting of all nodes in $S^*$;
2. the bottom end of the simple path have several branches, each of the branches is connected to the root of an optimal treedepth decomposition for \( H \in \mathcal{C}(G \setminus S) \), which can be computed recursively by the same algorithm.

We illustrate the above algorithm in Figure 2. Later, we will introduce Equation (2) and (3) which are similar to Equation (1). A treedepth decomposition from those equation can be obtained using the same idea as in the above algorithm.

While only \( \Delta_G \) is enough to calculate an optimal tree decomposition, we also need \( \Delta_H \) for several subgraph \( H \) of \( G \) to have an optimal treedepth decomposition. The number of subgraphs which we have to consider could be exponential of \( n \). Because of that, enumerating all minimal separators is a bottleneck for all softwares which are based on Equation (1). We aim to speed up the enumeration in this paper.

3 Main Conjecture

Let \( \Delta^p_G \) be a set of separators no larger than \( p \), i.e. \( \Delta^p_G := \{ S \in \Delta : |S| \leq p \} \). Define \( td'(G) \) as follows:

\[
td'(G) = \begin{cases} 
|V| & \text{if } G \text{ is complete} \\
\min_{S \in \Delta^{|tw(G)}_G} \left( |S| + \max_{H \in \mathcal{C}(G \setminus S)} td(H) \right) & \text{otherwise}
\end{cases}
\]  

(2)

Our main conjecture is as follows:

Conjecture 3.1. For any graph \( G \), \( td'(G) = td(G) \).

The definition of \( td'(G) \) is very similar to \( td(G) \). The only difference is, for \( td(G) \), we select \( S \) from the set of all separators, but, for \( td'(G) \), we select the separator only from the set of separators no larger than \( tw(G) \). We can have a much faster algorithm for treedepth decomposition by limiting the selection only from the smaller set, as there is a scalable algorithm to list all separators in \( \Delta^p_G \) for any \( p \) [26].

We cannot prove the conjecture. However, in the following sections, we can give several theoretical results for the conjecture.

The following property shows that the conjecture is already tight.

Proposition 3.1 (Tightness of Conjecture 3.1). For any \( f : G \rightarrow \mathbb{N} \) such that \( f(G) < tw(G) \) for all \( G \), there exists a graph \( G' \) such that none of the graph separators is smaller than or equal to \( f(G') \).

The proposition can be shown using the cycle of length 4. The treewidth of the graph is 2, and none of the graph separator has size equal to 1. It implies that, if we change \( tw(G) \) in Equation 2 to some function just slightly smaller than that (such as \( tw(G) - 1 \)), we may have \( \Delta^f_G \) being empty and the algorithm is not anymore valid for all \( G \).
Relaxed Version of the Conjecture

As stated in the previous section, we cannot prove the main conjecture. However, we can prove a relaxed version of it. The relaxed version significantly contributes to experimental algorithms of treedepth decomposition. We discuss the version in this section.

Let \( g(G) := 2 \cdot tw(G) + 1 \). Define \( td''(G) \) as follows:

\[
td''(G) = \begin{cases} 
|V| & \text{if } G \text{ is complete} \\
\min_{S \in \Delta_G(G')} \left( |S| + \max_{H \in \mathcal{C}(G \setminus S)} td(H) \right) & \text{otherwise}
\end{cases}
\]  

The relaxed version of our conjecture can be stated as follows:

**Theorem 4.1.** For any graph \( G \), \( td(G) = td''(G) \).

Although we strongly believe that our conjecture in the previous section is correct, those who use only algorithms that has a correctness proof can consider this theorem instead. We can imply from the above theorem that, instead of considering all separators, it is sufficient to consider only separators with size no larger than \( 2 \cdot tw(G) + 1 \). Considering only a set of small separators still give a much better performance than considering all.

The proof of our theorem uses the following lemma.

**Lemma 4.1** (Lemma 7.19 of [4]). Let \( G \) be a graph and let \( U \subseteq V \). Then, there exists a separator \( S \) of \( G \), such that

- \( |S| \leq tw + 1 \)
- \( G[U \setminus S] \) has more than one connected components
- the size of each connected component in \( G[U \setminus S] \) is at most \( |U|/2 \)

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** For any separator \( S \in \Delta_G \), let \( d(S) := |S| + \max_{H \in \mathcal{C}(G) \setminus S} td(H) \). Consider a separator \( S' \) with size larger than \( 2 \cdot tw(G) + 1 \). By Lemma 4.1, there is a separator \( S'' \) of \( G \) with size no larger than \( tw(G) + 1 \), such that each connected component \( C \) of \( G \setminus S'' \) has \( |V(C) \cap S''| \leq |S''|/2 \). To prove this theorem, we will show that \( d(S'') \leq d(S') \), and, hence, there is always a separator with size no larger than \( 2 \cdot tw(G) + 1 \) that is a member of the set \( \arg \min_{S \in \Delta_G} \left( |S| + \max_{H \in \mathcal{C}(G \setminus S)} td(H) \right) \).

We consider the following treedepth decomposition \( T \) (See Figure 3 for inexact but helpful illustration).

- The top of \( T \) is a path of nodes in \( S'' \).
- The bottom node of the path has \( k'' := |\mathcal{C}(G \setminus S'')| \) branches. We denote components in \( \mathcal{C}(G \setminus S'') \) by \( C_1, \ldots, C_{k''} \). A tree rooted at branch \( i \in \{1, \ldots, k''\} \), denoted by \( T_i \), will soon be a treedepth decomposition of \( C_i \).
- If \( V(C_i) \cap S' = \emptyset \), \( T_i \) is an arbitrary optimal treedepth decomposition of \( C_i \).
We know that no treedepth decomposition which has 
\( S \) branches. Each of the branches is an arbitrary optimal treedepth decomposition of each component in 
\( C(C_i \setminus S') \).

It is straightforward to check that \( T \) is a valid treedepth decomposition of \( G \). Indeed, \( T_i \) is actually a treedepth decomposition of \( C_i \), and, for \( i \neq j \), \( V(T_i) \cap V(T_j) = \emptyset \).

We know that no treedepth decomposition which has \( S'' \) at the top has depth more than \( d(S'') \). As \( T \) has \( S'' \) at its top, we know that the depth of \( T \), denoted by \( depth(T) \), is no less than \( d(S'') \), i.e. \( d(S'') \leq depth(T) \). In the next paragraph, we will show that \( depth(T) \leq d(S') \). By that, we would have \( d(S') \leq d(S'') \) and complete the proof.

We have
\[
\begin{align*}
\text{depth}(T) &= |S''| + \max_i \text{depth}(T_i) \\
&= |S''| + \max_i \left[ |V(C_i) \cap S'| + \max_{C \in C(C_i \setminus S')} \text{td}(C) \right] \\
&\leq |S''| + \max_i \left[ |S'| / 2 + \max_{C \in C(C_i \setminus S')} \text{td}(C) \right] \\
&\leq |S''| + |S'| / 2 + \max_i \left[ \max_{C \in C(C_i \setminus S')} \text{td}(C) \right] \\
&\leq |S''| + |S'| / 2 + \max_i [\text{td}(C_i)].
\end{align*}
\]

Since \( |S'| \geq 2 \cdot \text{tw}(G) + 2 \) and \( |S''| \leq \text{tw}(G) + 1 \), we have \( |S''| \leq |S'| / 2 \). Hence,
\[
\begin{align*}
\text{depth}(T) &\leq |S'| / 2 + |S'| / 2 + \max_i [\text{td}(C_i)] \\
&= |S'| + \max_i \left[ \text{td}(C_i) \right] = d(S').
\end{align*}
\]

\[\square\]

5 Proof for Some Graph Classes

In this section, we prove the main conjecture stated in Section 3 for graph classes.

5.1 Chordal Graphs

A cycle is chordless if there are two nodes in the cycle that are not adjacent to each others. A graph \( G \) is chordal if there is no chordless cycle of length four or more.

To handle maximal cliques and minimal separators, we introduce clique trees.

Definition 5.1 (Clique tree). Let \( G \) be a graph, and let \( \mathcal{V} \) be the set of all maximal cliques in \( G \). A clique tree is a tree \( T = (V, E) \) such that, for every vertex \( v \) in \( G \), the set of maximal cliques containing \( v \) induces a connected subtree of \( T \).

Following lemmas states the important characteristics of chordal graphs.

Lemma 5.1 ([11]). A graph \( G \) is chordal if and only if there exists a clique tree of \( G \).

Lemma 5.2 ([14]). Let \( S \) be a minimal separator of chordal graph \( G \), and let \( T = (V, E) \) be a clique tree of \( G \). Then, there exist two cliques \( C, C' \in \mathcal{V} \) such that \( \{C, C'\} \in E \) and \( C_i \cap C_j = S \).

Lemma 5.3 ([3]). Let \( G \) be a chordal graph, and let \( \mathcal{V} \) be the set of all maximal cliques in \( G \). Then, we have
\[
\text{tw}(G) = \max_{C \in \mathcal{V}} |C| - 1.
\]

We can then prove our conjecture for chordal graphs.

Theorem 5.1. Conjecture 3.1 is true for chordal graphs.
We can conclude from above that any separator in $\Delta$.

We then have the following theorem, which is quite straightforward from the lemma.

**Theorem 5.2.** Conjecture 3.1 is true for outerplanar graphs.

**Theorem 5.3.** Conjecture 3.1 is true for cographs.

To prove our conjecture for cographs, we use the following lemma:

**Lemma 5.6 (Full Component).** Let $G$ be a graph and let $S$ be a separator. A connected component of $G\backslash S$ is called a full component associated with $S$ if $N(C) = S$, where $N(C)$ is the open neighbors of $C$ in graph $G$.

The following lemma characterizes minimal separators by full components.

**Lemma 5.5 (Full Component).** Let $G$ be a graph and let $S$ be a separator. A connected component of $G\backslash S$ such that both $C_1$ and $C_2$ are full components associated with $S$.

Now we can prove our conjecture for outerplanar graphs, which is formally stated as follows.

**Theorem 5.2.** Conjecture 3.1 is true for outerplanar graphs.

**Proof.** Let $G$ be an outerplanar graph. It is well known that the treewidth of outerplanar graph is at most 2. When the treewidth is 1, the graph is a tree. We know that all minimal separators of the graph tree have size 1. We therefore have $\Delta = \Delta^1 = \Delta^{tw(G)}$ and prove the conjecture.

From now, we consider the outerplanar graphs with treewidth exactly 2. We will show that our conjecture is true for outerplanar graphs, by showing that every minimal separators of an outerplanar graph have size at most 2.

Let $S$ be a minimal separator of $G$. Then, by Lemma 5.5, we have at least two full components $C_1$ and $C_2$ associated with $S$. Suppose $|S| \geq 3$. Let $v_1 \in C_1$ and $v_2 \in C_2$. Consider a graph $G'$ that is obtained from $G$ by contracting all edges in $C_1$ and $C_2$. In $G'$, $v_1$ and $v_2$ are connected to all nodes in $S$ because $N(C_1) = N(C_2) = S$ in $G$. Then, $G'$ contains a complete bipartite graph between $U = \{v_1, v_2\}$ and $V = S$ as a subgraph. Since $|S| \geq 3$, $G'$ contains $K_{2,3}$ as a minor. That contradicts Lemma 5.4. Therefore, for any minimal separator $S$, we have $|S| \leq 2$. Again, we have $\Delta = \Delta^2 = \Delta^{tw(G)}$ and prove the conjecture.

**5.3 Cographs**

A graph $G$ is cograph if any of its subgraph with size 4 is not a simple path.

To prove our conjecture for cographs, we use the following lemma:

**Lemma 5.6 (Full Component).** Let $G$ be a graph and let $S$ be a separator. A connected component of $G\backslash S$ such that both $C_1$ and $C_2$ are full components associated with $S$.

Now we can prove our conjecture for outerplanar graphs, which is formally stated as follows.

**Theorem 5.2.** Conjecture 3.1 is true for outerplanar graphs.

**Proof.** Let $G$ be an outerplanar graph. It is well known that the treewidth of outerplanar graph is at most 2. When the treewidth is 1, the graph is a tree. We know that all minimal separators of the graph tree have size 1. We therefore have $\Delta = \Delta^1 = \Delta^{tw(G)}$ and prove the conjecture.

From now, we consider the outerplanar graphs with treewidth exactly 2. We will show that our conjecture is true for outerplanar graphs, by showing that every minimal separators of an outerplanar graph have size at most 2.

Let $S$ be a minimal separator of $G$. Then, by Lemma 5.5, we have at least two full components $C_1$ and $C_2$ associated with $S$. Suppose $|S| \geq 3$. Let $v_1 \in C_1$ and $v_2 \in C_2$. Consider a graph $G'$ that is obtained from $G$ by contracting all edges in $C_1$ and $C_2$. In $G'$, $v_1$ and $v_2$ are connected to all nodes in $S$ because $N(C_1) = N(C_2) = S$ in $G$. Then, $G'$ contains a complete bipartite graph between $U = \{v_1, v_2\}$ and $V = S$ as a subgraph. Since $|S| \geq 3$, $G'$ contains $K_{2,3}$ as a minor. That contradicts Lemma 5.4. Therefore, for any minimal separator $S$, we have $|S| \leq 2$. Again, we have $\Delta = \Delta^2 = \Delta^{tw(G)}$ and prove the conjecture.

**5.3 Cographs**

A graph $G$ is cograph if any of its subgraph with size 4 is not a simple path.

To prove our conjecture for cographs, we use the following lemma:

**Lemma 5.6 (Full Component).** Let $G$ be a graph and let $S$ be a separator. A connected component of $G\backslash S$ such that both $C_1$ and $C_2$ are full components associated with $S$.

Now we can prove our conjecture for outerplanar graphs, which is formally stated as follows.

**Theorem 5.2.** Conjecture 3.1 is true for outerplanar graphs.

**Proof.** Let $G$ be an outerplanar graph. It is well known that the treewidth of outerplanar graph is at most 2. When the treewidth is 1, the graph is a tree. We know that all minimal separators of the graph tree have size 1. We therefore have $\Delta = \Delta^1 = \Delta^{tw(G)}$ and prove the conjecture.

From now, we consider the outerplanar graphs with treewidth exactly 2. We will show that our conjecture is true for outerplanar graphs, by showing that every minimal separators of an outerplanar graph have size at most 2.

Let $S$ be a minimal separator of $G$. Then, by Lemma 5.5, we have at least two full components $C_1$ and $C_2$ associated with $S$. Suppose $|S| \geq 3$. Let $v_1 \in C_1$ and $v_2 \in C_2$. Consider a graph $G'$ that is obtained from $G$ by contracting all edges in $C_1$ and $C_2$. In $G'$, $v_1$ and $v_2$ are connected to all nodes in $S$ because $N(C_1) = N(C_2) = S$ in $G$. Then, $G'$ contains a complete bipartite graph between $U = \{v_1, v_2\}$ and $V = S$ as a subgraph. Since $|S| \geq 3$, $G'$ contains $K_{2,3}$ as a minor. That contradicts Lemma 5.4. Therefore, for any minimal separator $S$, we have $|S| \leq 2$. Again, we have $\Delta = \Delta^2 = \Delta^{tw(G)}$ and prove the conjecture.

**5.3 Cographs**

A graph $G$ is cograph if any of its subgraph with size 4 is not a simple path.

To prove our conjecture for cographs, we use the following lemma:

**Lemma 5.6 (Full Component).** Let $G$ be a graph and let $S$ be a separator. A connected component of $G\backslash S$ such that both $C_1$ and $C_2$ are full components associated with $S$.

Now we can prove our conjecture for outerplanar graphs, which is formally stated as follows.

**Theorem 5.2.** Conjecture 3.1 is true for outerplanar graphs.

**Proof.** Let $G$ be an outerplanar graph. It is well known that the treewidth of outerplanar graph is at most 2. When the treewidth is 1, the graph is a tree. We know that all minimal separators of the graph tree have size 1. We therefore have $\Delta = \Delta^1 = \Delta^{tw(G)}$ and prove the conjecture.

From now, we consider the outerplanar graphs with treewidth exactly 2. We will show that our conjecture is true for outerplanar graphs, by showing that every minimal separators of an outerplanar graph have size at most 2.

Let $S$ be a minimal separator of $G$. Then, by Lemma 5.5, we have at least two full components $C_1$ and $C_2$ associated with $S$. Suppose $|S| \geq 3$. Let $v_1 \in C_1$ and $v_2 \in C_2$. Consider a graph $G'$ that is obtained from $G$ by contracting all edges in $C_1$ and $C_2$. In $G'$, $v_1$ and $v_2$ are connected to all nodes in $S$ because $N(C_1) = N(C_2) = S$ in $G$. Then, $G'$ contains a complete bipartite graph between $U = \{v_1, v_2\}$ and $V = S$ as a subgraph. Since $|S| \geq 3$, $G'$ contains $K_{2,3}$ as a minor. That contradicts Lemma 5.4. Therefore, for any minimal separator $S$, we have $|S| \leq 2$. Again, we have $\Delta = \Delta^2 = \Delta^{tw(G)}$ and prove the conjecture.
Proof. The theorem is trivial when the input graph $G$ is complete.

For all $S \in \Delta G$, let $d(S) = |S| + \max_{H \subseteq G \setminus S} td(H)$. When $G$ is not complete, we know that $td(G) = \min_{S \in \Delta G} d(S)$. Suppose that there is a minimal separator $S' \in \arg\min_{S \in \Delta G} d(S)$ such that $|S'| > tw(G)$. We then have

\[
\begin{align*}
td(G) &= d(S') \\
&= |S'| + \max_{H \subseteq G \setminus S'} td(H) \\
&\geq |S'| + 1 \\
&> tw(G) + 1.
\end{align*}
\]

This contradicts Lemma 5.6 that $tw(G) = td(G) - 1$. Because of this, we can conclude that, for any $S' \in \arg\min_{S \in \Delta G} d(S)$, we have $S' \leq tw(G)$. Then, $\arg\min_{S \in \Delta G} d(S) = \arg\min_{S \in \Delta_{tw(G)}} d(S)$ and $td(G) = td'(G)$.

\[\square\]

We can use the same proof argument to show that our conjecture is true for any graph $G$ such that $tw(G) = td(G) - 1$.

6 Experiments

The fastest solver based on Equation (1) is SMS at PACE 2020, which is a second winning at the competition. We show in this section by experimental results that our conjecture can significantly speed up the solver. In fact, after using the conjecture, SMS is very competitive to Bute-Plus [28], which is the winning solver at the competition and is currently the most scalable solver even when we use the relaxed conjecture (Theorem 4.1). We call SMS with our conjecture as Extended SMS or ESMS, and we call SMS with the relaxed conjecture as ESMS'.

6.1 Setup

Experiments were done on Ubuntu 18.04.5 LTS. The extended SMS and SMS are written in C++, while Bute-Plus is written in C. Both of them were compiled with GCC version 7.5.0 with O3 option. The CPU was Intel(R) Xeon(R) CPU E5-2640 v3 @ 2.60GHz. The time limit is set to 30 minutes (1800 seconds), and the memory limit is set to 8 GB.

6.2 Instances

To avoid the effect of excessive parameter optimization on public instances, we choose to use the 100 private instances of PACE 2020. The instances are available at official website [1]. They are named as 002, 004, \ldots, 198, 200. The instances are ordered lexicographically by non-decreasing $(n, m)$, where $n$ is the number of nodes and $m$ is the number of edges. First 56 instances have $n \leq 100$, next 32 instances have $101 \leq n \leq 200$ and remaining 12 instances have $201 \leq n \leq 500$.

6.3 Computing Treewidth

Since it is also NP-hard to compute treewidth, we compute its upper bound instead. We implemented two well-known heuristics MINDEGREE and MINFILL [12] and took their minimum as the upper bound of treewidth. We also use this method to compute an upper bound for $2tw + 1$ in ESMS'.

6.4 The Number of Solved Instances

Among 100 test instances, the extended SMS (ESMS) solved 82 and the and the extended SMS with relaxed conjecture (ESMS') solved 81 instances. The original SMS solved 78 instances (77 instances at judge environment at PACE 2020) and Bute-Plus, the winning solver at PACE 2020, solved 81 instances (78 instances at the judge environment).

6.5 Comparison of ESMS and SMS

Table 1 summarizes the computation time for those instances where we obtained the optimal solution and at least one of ESMS and SMS took more than 60 seconds to solve. Since computing treewidth can be an overhead when our conjecture does not reduce so much the size of minimal separator, for some instances, the extended SMS took slightly
Table 1: Comparison between Extended SMS and SMS. $t_w$ is the upper bound of treewidth, which is obtained by taking the minimum of $\text{MinDegree}$ and $\text{MinFill}$ heuristics. ESMS and SMS stand for the computation time in seconds by Extended SMS and original SMS, respectively.

| name | n  | m  | td | $t_w$ | ESMS | SMS | ratio |
|------|----|----|----|-------|------|-----|-------|
| 068  | 60 | 90 | 17 | 12    | 220.0| 408.6| 0.54  |
| 084  | 71 | 103| 16 | 11    | 44.5 | 106.9| 0.42  |
| 094  | 81 | 810| 57 | 54    | 475.6| 462.4| 1.03  |
| 108  | 96 | 153| 16 | 10    | 780.4| 812.5| 0.96  |
| 112  | 100| 1800|83| 85    | 1753.4| -   | <0.97 |
| 120  | 111| 1029|38| 30    | 941.4| 901.2| 1.04  |
| 144  | 138| 493| 19 | 12    | 243.1| 225.6| 1.08  |
| 148  | 145| 2512|60| 51    | 105.2| 98.1 | 1.07  |
| 150  | 148| 198| 14 | 5     | 131.1| 311.5| 0.42  |
| 158  | 163| 278| 19 | 10    | 1266.6| -   | <0.70 |
| 162  | 170| 218| 14 | 6     | 1161.5| -   | <0.65 |
| 172  | 195| 342| 19 | 7     | 390.2| -   | <0.22 |
| 174  | 199| 265| 14 | 6     | 223.7| 565.6| 0.40  |
| 180  | 214| 785| 17 | 11    | 1631.4| 1566.5| 1.04 |
| 182  | 225| 771| 17 | 11    | 86.8 | 80.5 | 1.08  |
| 194  | 450| 1799|8 | 7     | 82.3 | 82.3 | 1.00  |

For the 79 instances that are solved by both ESMS and Bute-Plus, we plotted the computation time in Figure 4. Note that since the algorithm and the preprocessing are so different between two solvers, their performance is very dependent on instances and we cannot simply conclude one solver is better than another. However, we can observe that for most instances that were not instantly solved after preprocessing, the computation time significantly differs by the algorithm and combined with Table 1, the extended SMS performs well when the graph (and many of its subgraphs) has small treewidth.
Figure 4: Comparison of the computation time for the instances solved by both solvers. The axes are logarithmic. For those instances in the upper-left triangle, our proposal (ESMS) has shorter time for computation. Note that Bute-Plus has a preprocessing of 60 seconds, while ESMS prepossesses for 1 second.

6.7 Correctness of ESMS

Although we cannot prove that ESMS always give an optimal treedepth decomposition, ESMS gives us an optimal treedepth decomposition for all 79 instances that both ESMS and Bute-plus can solve.

7 Conclusion

In this paper, we introduced a conjecture that we can still have an optimal treedepth decomposition when we limit the size of the minimal separators used in Equation (1). We have proved the conjecture for chordal graphs, outerplanar graphs and cographs and also have proved the relaxed conjecture which limits the size of minimal separators to twice the treewidth plus one. We extended the second winning solver at PACE 2020 which is also based on Equation (1) by our conjecture and solved more instances than the winning solver at PACE 2020.

References

[1] PACE 2020. https://pacechallenge.org/2020/. Accessed: 2020-08-11.
[2] H. L. Bodlaender, J. R. Gilbert, H. Hafsteinsson, and T. Kloks, Approximating treewidth, pathwidth, frontsize, and shortest elimination tree, J. Algorithms, 18 (1995), pp. 238–255.
[3] V. Bouchitté and I. Todinca, Treewidth and minimum fill-in: Grouping the minimal separators, SIAM Journal on Computing, 31 (2001), pp. 212–232.
[4] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, Parameterized algorithms, vol. 4, Springer, 2015.
[5] H. Dell, C. Komusiewicz, N. Talmon, and M. Weller, The pace 2017 parameterized algorithms and computational experiments challenge: The second iteration, in IPEC 2017, 2018.

[6] J. S. Deogun, T. Kloks, D. Kratsch, and H. Müller, On the vertex ranking problem for trapezoid, circular-arc and other graphs, Discrete Applied Mathematics, 98 (1999), pp. 39–63.

[7] D. Dereniowski and A. Nadolski, Vertex rankings of chordal graphs and weighted trees, Information Processing Letters, 98 (2006), pp. 96–100.

[8] H. Falko and K. Stefan, Solving connectivity problems parameterized by treedepth in single-exponential time and polynomial space, arXiv preprint arXiv:2001.05364, (2020).

[9] F. V. Fomin, A. C. Giannopoulou, and M. Pilipczuk, Computing tree-depth faster than $2^n$, Algorithmica, 73 (2015), pp. 202–216.

[10] R. Ganian, N. Lodha, S. Ordyniak, and S. Szeider, SAT-encodings for treecut width and treedepth, in ALENEX 2019, 2019, pp. 117–129.

[11] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, Journal of Combinatorial Theory, Series B, 16 (1974), pp. 47–56.

[12] V. Gogate and R. Dechter, A complete anytime algorithm for treewidth, arXiv preprint arXiv:1207.4109, (2012).

[13] M. C. Golumbic, Algorithmic graph theory and perfect graphs, Elsevier, 2004.

[14] C.-W. Ho and R. C. T. Lee, Counting clique trees and computing perfect elimination schemes in parallel, Information Processing Letters, 31 (1989), pp. 61–68.

[15] Y. Iwata, T. Ogasawara, and N. Ohsaka, On the power of tree-depth for fully polynomial FPT algorithms, arXiv preprint arXiv:1710.04376, (2017).

[16] T. Kloks, Treewidth: computations and approximations, vol. 842, 1994.

[17] T. Kloks, H. Müller, and C. Wong, Vertex ranking of asteroidal triple-free graphs, Information Processing Letters, 68 (1998), pp. 201–206.

[18] Y. Kobayashi and H. Tamaki, Treedepth parameterized by vertex cover number, in IPEC 2016, 2017.

[19] T. Korhonen, SMS in PACE 2020, 2020.

[20] C. E. Leiserson, Area-efficient graph layouts, in FOCS 1980, 1980, pp. 270–281.

[21] J. W. Liu, The role of elimination trees in sparse factorization, SIAM Journal on Matrix Analysis and Applications, 11 (1990), pp. 134–172.

[22] F. Reidl, P. Rossmanith, F. S. Villaamil, and S. Sikdar, A faster parameterized algorithm for treedepth, in ICALP 2014, 2014, pp. 931–942.

[23] A. Sen, H. Deng, and S. Guha, On a graph partition problem with application to VLSI layout, Information Processing Letters, 43 (1992), pp. 87–94.

[24] M. M. Syslo, Characterizations of outerplanar graphs, Discrete Mathematics, 26 (1979), pp. 47–53.

[25] K. Takata, Space-optimal, backtracking algorithms to list the minimal vertex separators of a graph, Discrete Applied Mathematics, 158 (2010), pp. 1660–1667.

[26] H. Tamaki, Computing treewidth via exact and heuristic lists of minimal separators, in SEA 2019, 2019, pp. 219–236.

[27] J. Trimble, An algorithm for the exact treedepth problem, in SEA 2020, 2020, pp. 19:1–19:14.

[28] ———, Bute: A bottom-up exact solver for treedepth (submitted to PACE 2020 under username peaty), 2020.