SOME ABSTRACT WEGNER ESTIMATES WITH APPLICATIONS

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ABSTRACT. We prove some abstract Wegner bounds for random self-adjoint operators. Applications include elementary proofs of Wegner estimates for discrete and continuous Anderson Hamiltonians with possibly sparse potentials, as well as Wegner bounds for quantum graphs with random edge length or random vertex coupling. We allow the coupling constants describing the randomness to be correlated and to have quite general distributions.

1. INTRODUCTION

Background. Wegner estimates for random Schrödinger operators have been the subject of active research for the last three decades. Given a random self-adjoint operator $A(\omega)$ with a discrete spectrum $\{E_j(\omega)\}$ and a fixed interval $I$, the aim is to obtain good bounds on the average number of $E_j(\omega)$ in $I$. Such estimates can be used in a proof of Anderson localization via multiscale analysis, or in the study of the continuity of the integrated density of states. These estimates are named after Wegner’s work [46].

The aim of this paper is to derive some abstract Wegner bounds for some random self-adjoint operators on a Hilbert space, and to apply them afterwards for specific models. This approach proves to be rewarding, if only because it considerably shortens the proof of a Wegner bound for the model at hand. This is not the first attempt to provide abstract bounds; see [9] for a previous one.

Results. The abstract Wegner estimates are stated in Section 2 and applied in Section 3. We first obtain optimal bounds on the lattice and non-optimal bounds in the continuum. We allow the potential to be sparse, i.e. make no covering assumption. This includes models with surface and Delone potentials. We then give Wegner bounds for quantum graphs with random edge lengths or random vertex couplings. We allow the coupling constants entering the randomness to be correlated and only assume that their distributions have no atoms. A comparison with previous results is provided for each application. We conclude the paper with an appendix describing the spectra of Anderson models with half-space potentials. This illustrates the non-triviality of some of our bounds.

Notations. We assume the probability space has the form $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = B^I$ for some Borel set $B \subseteq \mathbb{R}$ and some countable index set $I$. Here $\mathbb{P}$ is a probability measure on $\Omega$ and $\mathcal{F} = \otimes_{\alpha \in I} \mathcal{B}$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of $B$. By definition, $\mathcal{F}$ is generated by cylinder sets of the form $\{\omega = (\omega_\alpha) : \omega_\alpha \in A_1, \ldots, \omega_\alpha \in A_n\}$, with $\alpha \in I$ and $A_j \in \mathcal{B}$. Any product space $B^I$ is assumed to be endowed with the $\sigma$-algebra $\mathcal{F} = \otimes_{\alpha \in I} \mathcal{B}$, which we shall often omit.

Fix $\alpha \in I$, let $Y_\alpha := B^I \backslash \{\alpha\}$, $\mathcal{F}_\alpha := \otimes_{\beta \neq \alpha} \mathcal{B}$ and denote $\hat{\omega}_\alpha := (\omega_\beta)_{\beta \neq \alpha}$. Define $\tau_\alpha : \Omega \to B \times Y_\alpha$ by $\omega \mapsto (\omega_\alpha, \hat{\omega}_\alpha)$. Then applying [1, Corollary 10.4.15] to $(B \times Y_\alpha, \mathcal{B} \otimes \mathcal{Y}_\alpha, \mathbb{P} \circ \tau_\alpha^{-1})$, we may find for each $\hat{\omega}_\alpha \in Y_\alpha$ a probability measure $\mu_{\hat{\omega}_\alpha}$ on $(B, \mathcal{B})$ such that, if $A \in \mathcal{F}$ and $A_{\hat{\omega}_\alpha} := \{\omega_\alpha : (\omega_\alpha, \hat{\omega}_\alpha) \in \tau_\alpha(A)\}$, then the map $\hat{\omega}_\alpha \mapsto \mu_{\hat{\omega}_\alpha}(A_{\hat{\omega}_\alpha})$ is $\mathcal{Y}_\alpha$-measurable and $\mathbb{P}(A) = \int_{Y_\alpha} \mu_{\hat{\omega}_\alpha}(A_{\hat{\omega}_\alpha}) d\mathbb{P}_{Y_\alpha}(\hat{\omega}_\alpha)$. Here $\mathbb{P}_{Y_\alpha} := \mathbb{P} \circ \pi_1^{-1}$, where $\pi_{Y_\alpha} : \omega \mapsto \omega_\alpha$.

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\(\hat{\omega}_\alpha\). The measures \(\mu_{\hat{\omega}_\alpha}(B)\), for \(B \subseteq \mathbb{R}\), are essentially regular versions of the conditional probability \(\mathbb{P}\{\omega_\alpha \in B|\hat{\omega}_\alpha\}\). We will usually omit the identification map \(\tau_\alpha\) and simply regard elements of \(\Omega\) as ordered pairs \((\omega_\alpha, \hat{\omega}_\alpha)\), so that \(A_{\hat{\omega}_\alpha}\) is a section of \(A\), \(A_{\hat{\omega}_\alpha} = \{\omega_\alpha : (\omega_\alpha, \hat{\omega}_\alpha) \in A\}\).

Now fix a finite set \(\mathcal{I}_F \subseteq \mathcal{I}\) (e.g., \(\mathcal{I} = \mathbb{Z}^d\) and \(\mathcal{I}_F \subset \mathbb{Z}^d\) a cube). We express our Wegner bounds in terms of the following modulus of continuity

\[
s_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \mathbb{E}_{\omega_\alpha}\left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_\alpha}(E, E + \varepsilon) \right\}.
\]

We show in Section 5.2 that for any probability measure \(\mu\) on \(\mathbb{R}\),

\[
\sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) = \sup_{E \in \mathbb{Q}} \mu(E, E + \varepsilon).
\]

In particular, \(\hat{\omega}_\alpha \mapsto \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_\alpha}(E, E + \varepsilon)\) is \(Y_\alpha\)-measurable, so \(s_F(\mathbb{P}, \varepsilon)\) is well defined. We also verify that in the special case where \(\mathbb{P} = \otimes_{\alpha \in \mathcal{I}} \mu_\alpha\) for some probability measures \(\mu_\alpha\) on \(\mathbb{R}\), we have \(s_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \sup_{E \in \mathbb{R}} \mu_\alpha(E, E + \varepsilon)\).

**Remark.** Our bounds are useful if the probability measure \(\mathbb{P}\) is continuous. If \(\mathbb{P} = \otimes_{\alpha \in \mathcal{I}} \mu_\alpha\), it is sufficient for localization to have \(\mu\) Hölder (or even log-Hölder) continuous. This, of course, encompasses the case where \(\mu\) has a bounded density \(\rho(\lambda)d\lambda\).

We will not treat here random Schrödinger operators with sign-indefinite single-site potentials. The reader can find some Wegner estimates for such models in [28], [17], [45], [14], [36] and [32], assuming the distribution \(\mu\) of the \(\omega_\alpha\) has a density. See also the recent survey [12]. For sign-indefinite models on the lattice, the density assumption on \(\mu\) can be relaxed if the disorder is large; see [13] Theorem 1.2 and [14] Proposition 5.1] for a related result. For sign-indefinite models in the continuum however, there are to the best of our knowledge no Wegner bounds without the hypothesis that \(\mu\) has a density.

## 2. Abstract Theorems

In the following we give three abstract Wegner estimates. Theorem 2.2 is optimal, but is only valid for finite dimensional spaces. It can be applied for example to discrete Schrödinger operators on finite cubes \(\Lambda\), acting on \(\ell^2(\Lambda)\). Theorems 2.3 and 2.4 on the other hand are valid in an arbitrary separable Hilbert space, but they are not optimal.

### 2.1. Finite dimensional Hilbert spaces.

**Hypotheses \(A\).**

1. We fix a probability space \((\Omega, \mathfrak{F}, \mathbb{P})\) with \(\Omega = B^\mathbb{Z}\) for some Borel set \(B \subseteq \mathbb{R}\), some countable index set \(\mathcal{I}\), and fix a finite-dimensional Hilbert space \(\mathcal{H}\).
2. \(H(\omega)\) is a self-adjoint operator on \(\mathcal{H}\) for each \(\omega \in \Omega\).
3. Fix a bounded interval \(\mathcal{I}\). There exist a constant \(\gamma > 0\) and a self-adjoint operator \(W\) such that \(\mathbb{P}\)-almost surely,

\[
\chi_\mathcal{I}(H(\omega))W\chi_\mathcal{I}(H(\omega)) \geq \gamma \chi_\mathcal{I}(H(\omega)).
\]
4. The operator \(W\) takes the form

\[
W = \sum_{\alpha \in \mathcal{I}_F} U_\alpha,
\]

for some finite set \(\mathcal{I}_F \subseteq \mathcal{I}\), where the \(U_\alpha\) are self-adjoint operators.
5. Fix an orthonormal basis \(\{e_j\}_{j \in \mathcal{J}}\) for \(\mathcal{H}\). We define \(\mathcal{I}_j := \{\alpha \in \mathcal{I}_F : U_\alpha e_j \neq 0\}\),

\[
C_{\infty} := \max_{\alpha \in \mathcal{J}}|I_{\alpha}|\] and \(J_{\text{eff}} := \{j \in \mathcal{J} : U_{\alpha} e_j \neq 0\}\) for some \(\alpha \in \mathcal{I}_F\).

Note that one may take \(\mathcal{I} = \mathcal{I}_F = \mathcal{J}\) and \(W = \sum_{j \in \mathcal{J}} P_j = \text{Id}\), where \(P_j f := \langle j, e_j \rangle e_j\), in which case conditions 3 and 4 hold trivially on any interval with \(\gamma = 1\) and \(C_{\infty} = 1\).

For random Schrödinger operators, the \(U_\alpha\) can be the single-site potentials. Condition 3
is sometimes called an uncertainty principle, and an efficient criterion to check its validity was established in [3]. The constant $\gamma$ often depends on $I$.

The following proposition is the key idea for obtaining optimal Wegner bounds without covering assumptions. It decomposes the trace into local contributions of the $U_\alpha$. The proof is given in Section 4.1.

**Proposition 2.1.** Suppose that $H(\omega)$ satisfies Hypotheses (A) in the interval $I$. Then $\mathbb{P}$-almost surely,

$$\text{tr}[\chi_I(H(\omega))] \leq \gamma^{-2}C_{\text{fin}} \sum_{j \in J_{\text{eff}}} \sum_{\alpha \in I_j} \langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle.$$ 

For our first Wegner bound, we need one more hypothesis:

**Hypothesis (B).** $H(\omega)$ satisfies Hypotheses (A). Moreover, given $\omega = (\omega_\alpha)_{\alpha \in I} \in \Omega$, $H(\omega)$ has the form

$$H(\omega) = H_1 + \sum_{\alpha \in I_F} \omega_\alpha U_\alpha,$$

where $H_1$ is self-adjoint, $U_\alpha \geq 0$ and $\|U_\alpha\| \leq C_U$ for all $\alpha$.

Hence, randomness must appear as an additive perturbation and the $U_\alpha$ must be positive operators. The proof of the next theorem is given in Section 4.2.

**Theorem 2.2.** Suppose that $H(\omega)$ satisfies Hypotheses (A) and (B) in the interval $I$. Then $\text{tr}[\chi_I(H(\omega))]$ is measurable and

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\text{eff}}| \cdot s_F(\mathbb{P}, |I|),$$

where $C_W := 6\gamma^{-2}C_{\text{fin}}^2$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (1.1).

The fact that uncertainty principles imply Wegner bounds was first realized in [7] and [8]. There however, the authors considered the spectral projectors $\chi_I(H_1)$. It was later noticed in [43] that the arguments become simpler if one considers $\chi_I(H(\omega))$, and this idea was used again in [26] and [11].

It is worthwhile to note that if $H_1$ has the special form $H_1 = \sum_{\alpha \in I_F} c_\alpha U_\alpha$, that is, if $H(\omega) = \sum_{\alpha \in I_F} (c_\alpha + \omega_\alpha) U_\alpha$ for some bounded non-random constants $c_\alpha$, then analogs of Proposition 2.1 and Theorem 2.2 hold for intervals not containing 0, without the need for an uncertainty principle. Such models arise when studying discrete acoustic operators on $\ell^2(\mathbb{Z}^d)$. We refer the reader to [25] for details.

### 2.2. Separable Hilbert spaces.

We now work in the general setting.

Given $C \subseteq \mathbb{R}$, we say that $f : C_I \to \mathbb{R}$ is monotone increasing (resp. monotone decreasing) if $\nu_\alpha \leq \nu_\alpha'$ for all $\alpha \in I$ implies $f(\nu) \leq f(\nu')$ (resp. $f(\nu) \geq f(\nu')$).

**Hypotheses (C).**

1) We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = C_I$ for some interval $C \subseteq \mathbb{R}$ and some countable index set $I$. We assume $\mathbb{P}$ has no atoms, more precisely $s_F(\mathbb{P}, \varepsilon) \to 0$ as $\varepsilon \to 0$. We also fix a separable Hilbert space $\mathcal{H}$.

2) $H(\omega)$ is a self-adjoint operator on $\mathcal{H}$ for each $\omega \in \Omega$. It is bounded from below and has an orthonormal basis of eigenvectors, with eigenvalues $\lambda_1(\omega) \leq \lambda_2(\omega) \leq \ldots$.

3) Fix an open interval $I$. There exists a number $K$ independent of $\omega$ such that

$$n > K \implies \lambda_n(\omega) \notin I.$$

4) Either $\mathcal{D} := D(H(\omega))$ or $\mathcal{D} := D(\mathcal{H}^o)$ is independent of $\omega$, where $\mathcal{H}^o$ is the form associated with $H(\omega)$. In the first case we define $f_u(\omega) := \langle H(\omega) u, u \rangle$, in the second case we define $f_u(\omega) := \mathcal{H}^o[u]$, for $u \in \mathcal{D}$. 

There exists a finite set $\mathcal{I}_F \subseteq \mathcal{I}$ such that $f_u(\omega)$ only depends on $(\omega_{\alpha})_{\alpha \in \mathcal{I}_F}$. We denote by $1_F$ the element $1_F := (x_\alpha) \in \mathbb{R}^\mathcal{I}$ with $x_\alpha = 1$ for $\alpha \in \mathcal{I}_F$ and $x_\alpha = 0$ otherwise.

We also assume that there exists $\gamma > 0$ such that for every $\omega \in \mathcal{D}$, $f_u(\omega)$ satisfies one of the following properties for every $\omega \in \Omega$ and $t \geq 0$ such that $\omega - t \cdot 1_F \in \Omega$:

a. $f_u$ is monotone increasing and $f_u(\omega) - f_u(\omega - t \cdot 1_F) \geq \gamma \|u\|^2$.

b. $f_u \in C^1(\Omega)$, $\frac{\partial f_u(\omega)}{\partial \omega_\alpha} \geq 0 \text{ } \forall \omega \in \mathcal{I}_F$ and $\sum_{\alpha \in \mathcal{I}_F} \frac{\partial f_u(\omega)}{\partial \omega_\alpha} \geq \gamma \|u\|^2$.

c. $f_u$ is monotone decreasing and $f_u(\omega) - f_u(\omega - t \cdot 1_F) \leq -\gamma \|u\|^2$.

d. $f_u \in C^1(\Omega)$, $\frac{\partial f_u(\omega)}{\partial \omega_\alpha} \leq 0 \text{ } \forall \omega \in \mathcal{I}_F$ and $\sum_{\alpha \in \mathcal{I}_F} \frac{\partial f_u(\omega)}{\partial \omega_\alpha} \leq -\gamma \|u\|^2$.

Here $f_u \in C^1(\Omega)$ if $f_u$ is continuous on $\Omega$ and continuously differentiable on $\hat{\Omega}$.

Note that a Wegner bound formulated in terms of $s_F(\mathbb{P}, \varepsilon)$ is useful precisely when $\mathbb{P}$ has no atoms. We need this assumption for technical reasons in Section 5.2. Condition 3 typically holds for any bounded $I \subseteq \mathbb{R}$. If $\mathcal{H}$ is finite dimensional, it is satisfied with $K = \dim \mathcal{H}$ (since there is no eigenvalue with $n > K$). For infinite dimensional spaces, it is satisfied if $H(\omega)$ is bounded from below by a non random operator $H_0$ with a compact resolvent. In this case, $K$ usually depends on $I$. The only “real” conditions are 4 and 5. Condition 5 assumes monotonicity and “diagonal covering” for $H(\omega)$. We remove the latter restriction in Theorem 2.4.

In the applications, it will be convenient that $H(\omega)$ is not supposed to have the form $H(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}} \omega_\alpha U_\alpha$, and that Hypothesis (C.5.b) is still sufficient to conclude. Let us state the theorem, see Section 4.3 for a proof, which is based on ideas from [12].

**Theorem 2.3.** Suppose that $H(\omega)$ satisfies Hypotheses (C) in the interval $I$. Then $\text{tr}[\chi_I(H(\omega))]$ is $\mathcal{F}_\mathbb{P}$-measurable, where $\mathcal{F}_\mathbb{P}$ denotes the $\mathbb{P}$-completion of $\mathcal{F}$, and

$$\mathbb{E}(\text{tr}[\chi_I(H(\omega))]) \leq 2K \cdot |\mathcal{I}_F| \cdot s_F(\mathbb{P}, |I|),$$

where $\mathbb{E}$ denotes the extension of $\mathbb{E}$ to $\mathcal{F}_\mathbb{P}$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (1.1).

Note that we have $s_F(\mathbb{P}, \varepsilon)$ in the RHS; the quantity $s_F(\mathbb{P}, \varepsilon)$ has not been defined. For the applications, classic arguments from [21] show that $\text{tr}[\chi_I(H(\omega))]$ is actually $\mathcal{F}$-measurable, so that $\mathcal{E}$ reduces to $\mathbb{E}$ in the LHS.

For a random Schrödinger operator restricted to a cube $\Lambda$, the constant $K$ comes e.g. from a Weyl law and takes the form $C \cdot |\Lambda|$. The term $|\mathcal{I}_F|$ measures the contribution of the random potential in $\Lambda$, and will be approximately $|\Lambda|$ for standard single-particle systems. Hence, the upper bound is not linear in $|\Lambda|$.

There are mainly two applications for Wegner estimates: the first to prove localization via multiscale analysis, the second to study the continuity of the integrated density of states (IDS) of $H(\omega)$. For the first purpose, Theorem 2.3 is satisfactory because the term $s_F(\mathbb{P}, |I|)$ will be very small assuming $\mathbb{P} = \otimes \mu$ with $\mu$ (log-) Hölder continuous, so it will completely overweight the terms $K$ and $|\mathcal{I}_F|$. For the study of the IDS however, this theorem is not satisfactory.

It seems the “bad” term here is $K$. Indeed, for discrete models with sparse potentials supported in a set $G$, one expects $|\Lambda \cap G|$ in the upper bound (see Section 3.1), and this is precisely the term $|\mathcal{I}_F|$ in this case, not $K$ which arguably will be $|\Lambda|$.

Theorem 2.2 also has the advantage of avoiding the diagonal cover in Hypothesis (C.5) by means of the uncertainty principle. As we show in Section 4.3, there is a related counterpart of this idea for Theorem 2.3. Namely, if one can prove that the eigenvalues $\lambda_n(\omega)$ of $H(\omega)$ are monotone increasing and satisfy

$$(\lambda_n(\omega) - \lambda_n(\omega - t \cdot 1_F)) \chi_I(\lambda_n(\omega)) \geq t \gamma \cdot \chi_I(\lambda_n(\omega)),$$

then Theorem 2.3 is still valid if we only assume Hypotheses (C.1) to (C.3).
We finally give our last abstract theorem, which is probably the most original result of this section.

Theorem 2.4. Suppose that $H(\omega)$ satisfies Hypotheses (C.1) to (C.4) in the interval $I = (E_1, E_2)$, where $E_2 < 0$. Assume moreover that $\Omega = [q_-, q_+]$, fix $q > q_+$, and suppose that there exists $\xi \neq 0$ such that for any $u \in \mathcal{D}$,

\begin{equation}
 f_u(\omega) = a(u) - \sum_{\alpha \in \mathcal{I}_F} (q - \omega_\alpha) b_\alpha(u)
\end{equation}

for some finite set $\mathcal{I}_F \subseteq \mathcal{I}$ and some constants $a(u) \geq 0$ and $b_\alpha(u) \geq 0$. Then $\text{tr}[\chi_I(H(\omega))]$ is $\mathcal{F}_\omega$-measurable and

\[ \mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq 2K \cdot |\mathcal{I}_F| \cdot s_F \left( \mathbb{P}, (q - q_-) \left( 1 + \frac{|I|}{|E_2|} \right)^{\frac{1}{N}} - 1 \right). \]

In particular, if $\xi = \pm 1$, we have

\[ \mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq 2K \cdot |\mathcal{I}_F| \cdot s_F \left( \mathbb{P}, \frac{q - q_-}{|E_2|} |I| \right). \]

The proof is given in Section 4.4. It uses an idea from [35], who roughly considered the case $a(u) \equiv 0$ and $\mathbb{P} = \otimes \mu_\alpha$ with $\mu_\alpha = \rho_\alpha(\lambda)d\lambda$. Both hypotheses were important to their proof, and we overcome this difficulty by generalizing ideas from [12]. Note however that we need (C.4), the argument of [35] holds under a weaker assumption on $D(H(\omega))$.

Of course, the main advantage here in comparison with Theorem 2.3 is that we only suppose $b_\alpha(u) \geq 0$. Theorem 2.3 would need a condition like $\sum_{\alpha \in \mathcal{I}_F} b_\alpha(u) \geq \gamma \|u\|^2$ for all $u$. The price to pay is that the bound only holds for specific intervals.

In the applications we shall only need the case $\xi = 1$. However the greater generality does not require additional effort, and we believe it could be useful for models not considered here. For example, the case $\xi = -2$ appears in the model of [35].

3. Applications

3.1. Discrete multi-particle models. Consider the Hilbert space $\ell^2(\mathbb{Z}^d)$, where $n \in \mathbb{N}^*$ represents the number of particles living in $\mathbb{Z}^d$. Let $\mathcal{B} \subseteq \mathbb{R}$ be a Borel set and consider a probability space $(\Omega, \mathbb{P})$, where $\Omega = \mathcal{B}\mathbb{Z}^d$. Given $\omega = (\omega_\alpha) \in \Omega$, let

\[ H(\omega) := H_0 + V^\omega, \quad H_0 := -\Delta + V_0, \]

where $-\Delta$ is the discrete Laplace operator on $\ell^2(\mathbb{Z}^d)$, $V_0$ is a real non-random potential (possibly an interaction) and for $x = (x_1, \ldots, x_n) \in (\mathbb{Z}^d)^n = \mathbb{Z}^{nd}$,

\[ V^\omega(x) = \sum_{1 \leq i \leq n} V^\omega(x_i) = \sum_{1 \leq i \leq n} \sum_{\alpha \in \mathbb{Z}^d} \omega_\alpha u_\alpha(x_i). \]

Since $\mathbb{P}$ is arbitrary, the $(\omega_\alpha)$ are allowed to be correlated. We assume $V_0$ is bounded and the $u_\alpha : \mathbb{Z}^d \to \mathbb{R}$ satisfy $0 \leq u_\alpha \leq C_\alpha$ for some uniform $C_\alpha \geq 0$. We also assume the $u_\alpha$ are compactly supported, that is, if for $j \in \mathbb{Z}^d$, $j \in \mathbb{Z}^{nd}$ and $L \in \mathbb{N}$ we define the cubes

\[ \Lambda_{L}^{(1)}(j) := \{ x \in \mathbb{Z}^d : \|x - j\|_\infty \leq L \}, \quad \Lambda_{L}^{(n)}(j) := \{ x \in \mathbb{Z}^{nd} : \|x - j\|_\infty \leq L \}, \]

then we assume there exists an $R \geq 0$ such that $u_\alpha(j) = 0$ for all $j \notin \Lambda_{R}^{(1)}(\alpha)$.

As $\text{supp}\ u_\alpha$ is compact, we may interchange the sums and write

\[ V^\omega = \sum_{\alpha \in \mathbb{Z}^d} \omega_\alpha U_\alpha, \quad \text{with} \quad U_\alpha(x_1, \ldots, x_n) = \sum_{1 \leq i \leq n} u_\alpha(x_i). \]
Example. A simple and interesting case is when \( n = 1 \), a non-empty set \( G \subset \mathbb{Z}^d \) is given, and \( u_\alpha = \delta_\alpha \) inside \( G \) and \( u_\alpha \equiv 0 \) outside \( G \), where \( \delta_\alpha \) is the characteristic function of \( \{ \alpha \} \). In this case,

\[
H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha.
\]

For instance, we may take \( G = \mathbb{Z}^d \setminus \{0\} \), which gives rise to a non-covering situation. More generally, \( G \) could be a Delone set (i.e. \( \exists K \geq 0 \) such that \( \forall j \in \mathbb{Z}^d \), the cube \( \Lambda^{(1)}_K(j) \) contains at least one point of \( G \)) or a subspace \( \mathbb{Z}^d \times \{0\} \) of \( \mathbb{Z}^d \), in which case one speaks of surface potentials.

Discussion of the results.

- In the case of covering, i.e. \( u_\alpha \geq c \cdot \delta_\alpha \) for all \( \alpha \), we have an optimal Wegner bound in any interval \( I \). This extends \([20, \text{Theorem 2.1}]\) and \([27, \text{Theorem 2.3}]\) (because we neither assume that \( u_\alpha = \delta_\alpha \) nor that \( \mathbb{P} = \otimes \mu \) with \( \mu = \rho(t)dt \)) and improves \([0, \text{Theorem 1}]\) (because our bound is linear in \( |\Lambda| \)). Note that the arguments of \([27]\) actually allow for \( \mathbb{P} \) as general as ours. The multiscale analysis also requires two-volume Wegner bounds (cf. \([27, \text{Corollary 2.4}]\)); we prove these in \([40]\).

- If we have no covering and \( \Omega = [q_-, q_+]^d \) with \( q_- < 0 \), i.e. the perturbation can be negative, we obtain Wegner bounds below \( E_0 := \inf \sigma(\mathbb{H}_0) \). This extends \([23, \text{Theorems 8,13}]\), first because we make no regularity assumption on \( \mathbb{P} \), second because our bound is optimal and valid for multi-particles. Our result also extends the optimal bound \([24, \text{Theorem 4.1}]\) because we allow for general \( u_\alpha \) and \( n \).

But is there any spectrum below \( E_0 \)? We show in Section 5.1 that if \( n = 1 \), if \( G \subset \mathbb{Z}^d \) contains a half-space, if \( V_0 \) is periodic and if \( H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha \), then \( H(\omega) \) has a spectral interval below \( E_0 \) almost surely, provided that \( \mathbb{P} = \otimes \mu_\alpha \) and supp \( \mu_\alpha \subset [a,b] \), \( a < b \leq 0 \). This illustrates that our bound is indeed non-trivial. The advantage here compared to the first item is, of course, the fact that we allow \( G \neq \mathbb{Z}^d \).

- If we have no covering and \( \Omega = [q_-, q_+]^d \) with \( 0 < q_- \), i.e. the perturbation is positive, we obtain optimal Wegner bounds below \( E_q := \inf \sigma(\mathbb{H}_0 + qW) \) for any \( q > q_- \), where \( W := \sum_{\alpha} U_\alpha \). But again, is there any spectrum below \( E_q \)?

The recent preprints \([11]\) and \([38]\) have the advantage of giving a complete Wegner bound for some operators in this situation. Namely, the paper \([11]\) assumes that \( n = 1 \), \( H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha \), where \( G \subseteq \mathbb{Z}^d \) is a Delone set, \( \mathbb{P} = \otimes \mu_\alpha \) and supp \( \mu_\alpha \subset [0,M] \). Under some condition (cf. \([11, \text{Eq.(1.13)}]\)), the authors establish Wegner bounds for intervals near \( E_0 \), and in contrast to our result, they show that these intervals contain some spectrum of \( H(\omega) \) almost surely. A different proof for this Delone Wegner bound can be found in \([38]\), in the special case where \( V_0 \equiv 0 \).

To conclude, let us mention that we can actually use the results of \([11]\) to illustrate that our Wegner bound for positive perturbations is indeed interesting. Namely, if we take \( t := q \) sufficiently large, then \([11, \text{Theorem 1.3}]\) combined with \([11, \text{Proposition 1.5}]\) assert that the above Delone operator has \( E_q > E_0 \) and some spectrum in \([E_0, E_q]\) almost surely. Our Wegner bound is thus nontrivial for \( I \subseteq [E_0, E_q] \).

Boundary conditions. Let \( \Lambda := \Lambda^{(n)}_K(\mathbf{x}) \subset \mathbb{Z}^{nd} \) be a cube. The simple boundary conditions are obtained by restricting the matrix of \( H(\omega) \) to \( \Lambda \), i.e. if \((\mathbf{e}_i)_{\mathbf{i} \in \mathbb{Z}^{nd}}\) is the standard basis of \( L^2(\mathbb{Z}^{nd}) \), then \( H^{\Lambda}_N(\omega)(\mathbf{i}, \mathbf{j}) = \langle H(\omega)e_\mathbf{i}, e_\mathbf{j} \rangle \) if both \( \mathbf{i}, \mathbf{j} \in \Lambda \) and \( H^{\Lambda}_N(\omega)(\mathbf{i}, \mathbf{j}) = 0 \) otherwise. Next, there are the Dirichlet \( H^{\mathbb{D}}(\omega) \) and Neumann \( H^{\mathbb{N}}(\omega) \) boundary conditions, which were introduced in \([41]\) to provide analogs for the lattice of the Dirichlet-Neumann

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1Note that a single nonzero \( \omega_\alpha u_\alpha \) actually suffices to create a spectral point below \( E_0 \) if \( q_- < q_* \), \( q_* = q_*([H_0], u_\alpha) \). So our Wegner bound is also useful when the perturbation is highly negative and the spectral bottom is not an isolated point. This is likely to be the case if the operator is ergodic, e.g. \( n = 1 \) and \( H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha \), with \( G = (M\mathbb{Z})^d \) and \( V_0 \) \( M \)-periodic.
bracketing; see [19] Section 5.2 for details. The only identity we need is [19] Eq. (5.42)],
which asserts that if $H = -\Delta + V$, then $H \leq H^D + H^{V_d}$. In particular, if $E_0 := \inf \sigma(H)$,
$E_0^D := \inf \mathcal{H}(A^D)$ and $g \in \mathcal{E}(\Lambda^D)$ is identically zero, then
\[
E^D_0 = \inf_{f \in \mathcal{E}(\Lambda^D), \|f\| = 1} \langle H^D, f \rangle + \langle H^{V_d}, g \rangle \geq \inf_{\varphi \in \mathcal{E}(\mathbb{Z}^d), \|\varphi\| = 1} \langle H \varphi, \varphi \rangle = E_0,
\]
i.e. Dirichlet boundary conditions shift the spectrum up.

The result. Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}^d$ and $\Lambda_L^{(n)}(x) \subset \mathbb{Z}^d$. Consider the Hilbert
space $\mathcal{H} := \ell^2(\Lambda_L^{(n)}(x))$ with standard basis $(e_j)_{j \in \Lambda_L^{(n)}(x)}$ and the operator $H^\bullet_{\Lambda_L^{(n)}(x)}(\omega)$,
where $\bullet = S, D$ or $N$. Let
\[
W := \sum_{\alpha \in \mathbb{Z}^d} U_\alpha, \quad \text{ and } \quad W_{\Lambda_L^{(n)}(x)} := \sum_{\alpha \in \Lambda_F} U_\alpha, \quad \text{where } \Lambda_F := \bigcup_{i=1}^n \Lambda_L^{(1)}(x_i).
\]

We first show in Theorem [3.1] that uncertainty principles imply Wegner bounds, then we give in Lemma [3.2] concrete cases in which the uncertainty principle holds.

**Theorem 3.1.** Let $\Lambda := \Lambda_L^{(n)}(x)$ be a cube and suppose $H^\bullet_{\Lambda L}^{(n)}(\omega)$ satisfies
\[
(3-1) \quad \chi_I(H^\bullet_{\Lambda L}^{(n)}(\omega)) W_{\Lambda L} \chi_I(H^\bullet_{\Lambda L}^{(n)}(\omega)) \geq \gamma \chi_I(H^\bullet_{\Lambda L}^{(n)}(\omega)) \quad \mathbb{P}-a.s.
\]
in an interval $I$, for some $\gamma > 0$. Then
\[
\mathbb{E}\{\text{tr}[\chi_I(H^\bullet_{\Lambda L}^{(n)}(\omega))]\} \leq C_{W} \cdot \|\Lambda_L^{(n)}\| \cdot s_F(\mathbb{P}, |I|),
\]
where $C_{W} = 6n^4 \gamma^{-2} C^2_2(2R + 1)^{2d}$ and $\Lambda_L^{(n)} := \{j \in \Lambda_L^{(n)}(x) : U_\alpha e_j \neq 0 \text{ for some } \alpha \in \Lambda_F\}$.

If $u_\alpha = c_\alpha \delta_\alpha$ with $c_\alpha \geq 0$, then $C_u = \sup_{\alpha \in \mathcal{F}} c_\alpha$ and $R = 0$. If, moreover $n = 1$ and
$H(\omega) = H_0 + \sum_{\alpha \in \mathcal{G}} \omega_\alpha \delta_\alpha$, then $\Lambda_L^{(1)} = \Lambda_L^{(1)}(x) \cap \mathcal{G}$.

**Proof.** $H^\bullet_{\Lambda L}(\omega)$ is a self-adjoint operator given by $H^\bullet_{\Lambda L}(\omega) = H_1 + \sum_{\alpha \in \mathcal{F}} \omega_\alpha U_\alpha$, with $H_1 = H^\bullet_{\Lambda L}$ self-adjoint. Moreover, $U_\alpha \geq 0$, $\|U_\alpha\| \leq C_U := nC_u$ and $\mathcal{J}_I := \{\alpha : U_\alpha e_j \neq 0\} \subseteq \bigcup_{k=1}^n \Lambda_R^{(1)}(j_k)$, hence $C_{\text{fin}} := \max |\mathcal{J}_I| \leq n(2R + 1)^d$. The claim now follows from Theorem 2.2.

**Lemma 3.2.** Fix $\eta > 0$. The uncertainty principle [3.1] holds in any interval
\begin{enumerate}
\item[(1)] $I \subset \mathbb{R}$, if $\exists c > 0$ such that $u_\alpha \geq c \cdot \delta_\alpha$ for all $\alpha$, with $\gamma = nc$.
\item[(2)] $I \subset (-\infty, q_0 - \eta]$, if $\Omega = [-q, q_0]^d$, $q > q_-$ and $E_\eta := \inf \sigma(H_0 + q W)$, for the
Dirichlet restriction $H^D_{\Lambda}$, with $\gamma \geq \frac{n}{q_- - \eta}$.
\end{enumerate}

Theorem [3.1] combined with Lemma [3.2] thus provide a Wegner bound in either situation. If $q_- < 0$, we may take $q = 0$ and obtain a Wegner bound below $E_0 := \inf \sigma(H_0)$. Otherwise, $0 \leq q_- < q$, and the bound is interesting if $E_\eta > E_0$, for $I \subseteq (E_0, E_\eta)$. $\square$

**Proof.** For (1), note that if $u_\alpha \geq c \cdot \delta_\alpha$, then for any $y \in \Lambda_L^{(n)}(x)$,
\[
W_{\Lambda L}(y) \geq c \sum_{1 \leq i \leq n} \sum_{\alpha \in \mathcal{F}} \delta_\alpha(y_i) = c \sum_{1 \leq i \leq n} 1 = nc,
\]
so that $W_{\Lambda L} \geq nc$ and [3.1] holds trivially in any interval with $\gamma = nc$.

For (2), let $H_q := H_0 + q W$ and given $\omega \in \Omega$, let $\lambda_\omega(t) := \inf \sigma(H^D_\Lambda(\omega) + t W_{\Lambda})$. Then for any $t \geq q_- \text{ we have}
\[
\lambda_\omega(t) = \inf \sigma(H^D_\Lambda + V^\omega + (t - q) W_{\Lambda}) \geq \inf \sigma\left(H_0 + \sum_{\alpha \in \mathbb{Z}^d} (\omega_\alpha + t - q) U_\alpha\right) \geq E_q
\]
where we used the fact that Dirichlet boundary conditions shift the spectrum up. Thus, if $I \subset (-\infty, q_0 - \eta]$, we get $\lambda_\omega(q - q_-) - \max I \geq \eta$. By [3 Theorem 1.1], [3.1] thus holds in $I$ with $\gamma \geq \frac{n}{q_- q_-}$. $\square$
3.2. Continuum multi-particle models. Consider the Hilbert space $L^2(\mathbb{R}^{nd})$, where $n \in \mathbb{N}^*$ represents the number of particles living in $\mathbb{R}^d$. Let $G \subset \mathbb{R}^d$ be a discrete non-empty set such that $\#\{\Lambda \cap G\} < \infty$ for any bounded $\Lambda \subset \mathbb{R}^d$ and consider a probability space $(\Omega, \mathcal{P})$, where $\Omega := \{q, q_+\}^G$ and $\mathcal{P}$ has no atoms. Given $\omega = (\omega_n) \in \Omega$, let

$$H(\omega) = H_0 + V^\omega, \quad H_0 := -\Delta + V_0,$$

where $V_0 \geq v_0$ is a bounded real non-random potential. We can consider more general $H_0$; we only need $H_0, \Lambda$ to satisfy a Weyl law, and this is true for $H_0 = (-i\nabla - A)^2 + V_0$ with weak conditions on $A$ and $V_0$; see [18 Lemma 5]. Given $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \equiv \mathbb{R}^{nd}$,

$$V^\omega(x) = \sum_{1 \leq i \leq n} V^\omega(x_i) = \sum_{1 \leq i \leq n} \sum_{\alpha \in G} \omega_\alpha(x_i).$$

Let $\Lambda^{(n)}_\omega(x) := \{y \in \mathbb{R}^{nd} : \|y - x\|_\infty < L\}$. We assume the $u_\alpha : \mathbb{R}^d \to \mathbb{R}$ satisfy $0 \leq u_\alpha \leq C_u$ for some uniform $C_u > 0$ and $\text{supp} u_\alpha \subset \Lambda^{(1)}(\alpha)$ for some $R > 0$ independent of $\alpha$. This model encompasses sparse potentials such as Delone and surface potentials. Now put

$$U_\alpha(y_1, \ldots, y_n) := \sum_{i=1}^n u_\alpha(y_i) \quad \text{and} \quad W := \sum_{\alpha \in G} U_\alpha.$$

Given $z = (z_1, \ldots, z_n) \in (\mathbb{R}^d)^n$ and a cube $\Lambda^{(n)}_\omega(z) \subset \mathbb{R}^{nd}$, let $H^{\bullet, (n)}_\omega(\mathbb{R}^d)$ be a restriction of $H(\omega)$ acting on $\mathcal{H} := L^2(\Lambda^{(n)}_\omega(z))$, with $\bullet = D, N, \text{per}$. Note that without a growth condition on $G$, $H(\omega)$ may not be self-adjoint (cf. [23]), but here we are only concerned with its restriction, which is self-adjoint.

**Discussion the results.** Our bounds are not linear in $|\Lambda|$, but may be used for localization.

- The covering situation, i.e. when $G = \mathbb{Z}^d$ and $u_\alpha \geq c\chi_\alpha$ for all $\alpha$, where $\chi_\alpha$ is the characteristic function of $[\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]^d$, has already been analyzed in [2] and [31]. There the authors proved Wegner bounds in any interval $I \subset \mathbb{R}$ and for arbitrary $\mathcal{P}$. We do the same here, simply to illustrate Theorem 2.3.

- For negative perturbations, we have a Wegner bound below $E_0 := \inf \sigma(H_0)$. This extends [23] Theorems 8,13 because we do not impose any regularity on $\mathcal{P}$, but [23] has some Wegner bounds which depend linearly on $|\Lambda|$. In the case of surface potentials, i.e. when $G = \mathbb{Z}^{d_1} \times \{0\}$, [23] Theorem 2.1] provides an optimal bound.

As in the lattice, there is the issue of whether $H(\omega)$ has some spectrum below $E_0$. We show in the Appendix (Section 5.1) that continuum single-particle operators with half-space potentials are good examples of operators which have no covering condition and to which we have a non-trivial Wegner bound.

- For positive perturbations, we obtain a Wegner bound below $E_q := \inf \sigma(H_0 + qW)$, for any $q > q_+$. This result is very close in spirit to [3 Theorem 2.1], because both are interesting when $E_0$ is a weak fluctuation boundary, i.e. when $E_0 < E_q$. Besides the fact that we allow for multi-particles\(^2\), note that our proof is quite elementary. On the other hand, [3] builds on the results of [8], which are technically involved, but they provide an optimal bound.

In contrast to the lattice, the question of whether there are interesting operators for which $E_0 < E_q$ is well established in the continuum when $n = 1$. Already in [22] Theorem 2.2, it is shown that if $E(t) := \inf \sigma(H_0 + tW)$, then $E(t) - E_0$ grows linearly in $t$, even if the $u_\alpha$ have small support, provided $G = \mathbb{Z}^d$ and $V_0$ is periodic. It was later shown in [4] Sections 4,5] that $E_q > E_0$ for more general operators with surface or

\(^2\)Let us mention here that there is a work in progress by Hislop and Klopp in which an optimal Wegner estimate is derived for some non-covering multi-particle Hamiltonians.
Delone potentials, assuming $V_0$ is periodic. In the case of Delone potentials, this result was very recently improved in [26] Lemma 4.2, namely, it is shown that $E(t) - E_0$ grows linearly in $t$, and $V_0$ is no longer assumed to be periodic.

Much stronger results are known if $n = 1$, $G$ is a Delone set, each $u_\alpha > 0$ in an open set and $\mathbb{P} = \otimes_{\alpha \in G} \mu_\alpha$. Namely, the Wegner bound of [39], which was improved in [26], is valid for any small interval, not just intervals near the spectral bottom. The result of [26] also extends the one of [8] who considered $G = \mathbb{Z}^d$, but relies on it.

**Theorem 3.3.** For any $I = (E_1, E_2)$, there exists $C_W > 0$ such that for any cube $\Lambda_L^{(n)}(x)$,

1. If $G = \mathbb{Z}^d$ and $\exists \varepsilon > 0$ with $u_\alpha \geq c \cdot \chi_\alpha$ for all $\alpha$, where $\chi_\alpha := x_{\alpha - \frac{1}{2}, \alpha + \frac{1}{2}}$, then

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L}^{(n)}(x))]\} \leq C_W \cdot |\Lambda_L^{(n)}(x)| \cdot |I| \cdot s_F\left(\frac{|I|}{nc}\right),$$

where $I := \left(\bigcup_{j=1}^n \Lambda_{L+R}(x_j)\right) \cap G$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (4.1).

2. In the general case, for any $q > q_+$, if $E_2 < E_q := \inf \sigma(H_0 + qW)$, then

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L}^{(n)}(x))]\} \leq C_W \cdot |\Lambda_L^{(n)}(x)| \cdot |I| \cdot s_F\left(\frac{q - q - 1}{E_2 - E_0 - |I|}\right).$$

Here $C_W = C_W(\eta, E_0, v_0)$ if $q_+ \geq 0$ and $C_W = C_W(\eta, E_0, v_0, v_0, \ldots, v_n)$ otherwise.

If $q_+ < 0$, i.e. the perturbation is negative, we may take $q = 0$ and obtain a Wegner bound below $E_0 := \inf \sigma(H_0)$. Otherwise, $0 \leq q_+ < q$ and $E_0 > E_q$, for many models.

**Proof.** Let $\Lambda := \Lambda_L^{(n)}(x)$. For (1), note that $H_{\Lambda}^{(n)}(\omega)$ is a self-adjoint operator given by $H_{\Lambda}^{(n)}(\omega) = H_1 + \sum_{\alpha \in I_{\Lambda}} \omega_\alpha U_\alpha$, where $H_1 := H_{0,\Lambda}$. Given $u \in D(H_{\Lambda}^{(n)})$, if $f_u(\omega) = (H_{\Lambda}^{(n)}(\omega)u, u)$, then $f_u$ is monotone increasing since $U_\alpha \geq 0$. Moreover, if $y \in \Lambda_L^{(n)}(x)$, then $W_{\Lambda}(y) := \sum_{\alpha \in I_{\Lambda}} U_\alpha(y) \geq \sum_{1 \leq i \leq n} \sum_{\alpha \in I_{\Lambda}} \chi_\alpha(y_\alpha) = nc$. Hence, $f_u(\omega + t \cdot 1_I) - f_u(\omega) = t(W_{\Lambda}u, u) \geq nct\|u\|$. Hypotheses (C) are thus satisfied with $\gamma = nc$, a Weyl constant $K = C|\Lambda|$, and the claim follows from Theorem 2.3.

For (2), let $A(\omega) := H_0^{(n)}(\omega) - E_q$ and $I' = (E_1 - E_q, E_2 - E_q)$. Then $\chi_I(\lambda) = \chi_{I'}(\lambda - E_q)$, hence $\mathbb{E}\{\text{tr}[\chi_I(H_0^{(n)}(\omega))])\} = \mathbb{E}\{\text{tr}[\chi_{I'}(A(\omega))])\}$.

Now $A(\omega)$ is a self-adjoint operator given by $A(\omega) = H_1 + \sum_{\alpha \in I_{\Lambda}} (\omega_\alpha - q)U_\alpha$, where $H_1 := H_{0,\Lambda} + qW_{\Lambda} - E_q$. Since Dirichlet boundary conditions shift the spectrum up, we have $H_1 \geq 0$. Thus, $A(\omega)$ satisfies the hypotheses of Theorem 2.4 in $I'$ with $\zeta = 1$, a Weyl constant $K = C|\Lambda|$, and the claim follows since $|I'| = |I|$. \hfill \Box

### 3.3. Quantum graphs with random edge length

Consider the metric graph $(\mathcal{E}, \mathcal{V})$ with vertex set $\mathcal{V} = \mathbb{Z}^d$ and edge set $\mathcal{E} = \{(m, m + h_j) : m \in \mathbb{Z}^d, j = 1, \ldots, d\}$, where $(h_j)_{j=1}^d$ is the standard basis of $\mathbb{Z}^d$. Each edge $e = (v, v')$ has an initial vertex $ve = v$ and a terminal vertex $ve = v'$. Now fix $0 < l_{\min} < l_{\max} < \infty$ and let $(\Omega, \mathbb{P})$ be a probability space, where $\Omega := [l_{\min}, l_{\max}]^\mathcal{E}$ and $\mathbb{P}$ has no atoms. Given $l^\omega = (l_e^\omega) \in \Omega$, we identify each edge $e$ with $[0, l_e^\omega]$, such that $ve$ and $ve$ correspond to $0$ and $l_e^\omega$, respectively, and consider the Hilbert space $\mathcal{H} := \otimes_{e \in \mathcal{E}} L^2[0, l_e^\omega]$. Fix $\alpha \in \mathbb{R}$ and define the operator

$$H(l^\omega, \alpha) : (f_e) \mapsto (-f_e^\omega).$$

$$D(H(l^\omega, \alpha)) := \left\{ f = (f_e) \in \otimes_{e \in \mathcal{E}} W^{2,2}(0, l_e^\omega) : f \text{ is continuous at each } v \in \mathcal{V} \text{ and } f'(v) = \alpha f(v). \right\}$$

By continuity at $v$, we mean that if $\tau e = vb = v$, then $f_e(l_e^\omega) = f_b(0) =: f(v)$. Here $f'(v) := \sum_{\epsilon \in v} f_\epsilon'(0) - \sum_{\epsilon \in v} f_\epsilon'(l_e^\omega)$.

Given $L \in \mathbb{N}^*$, let $\Lambda_L := \{e \in \mathcal{E} : \|e\|_\infty \leq L \text{ or } \|e\|_\infty \leq L\}$ be a cube and put $\mathcal{V}_\Lambda := \{ve : e \in \Lambda_L\} \cup \{ ve : e \in \Lambda_L\}$. This yields a graph $(\Lambda_L, \mathcal{V}_\Lambda)$ and a corresponding operator $H_{\Lambda_L}(l^\omega, \alpha)$. We denote $H_{\Lambda_L}(l^\omega, \alpha) := H_{\Lambda_L}(l^\omega, \alpha)$. 
Theorem 3.4. Let $I \subset (0, \infty)$ be an interval such that $I \cap D_0 = \emptyset$, where $D_0 := \bigcup_{k \in \mathbb{Z}} [\pi k^2 \tau_{\text{max}}, \pi k^2 \tau_{\text{min}}]$. Then there exists $c_1 = c_1(d)$ and $c_2 = c_2(I) > 0$ such that for any interval $J \subset I$ and any cube $\Lambda$ we have

$$\mathbb{P}\{ \sigma(H^\omega_{\alpha}(\alpha)) \cap J \neq \emptyset \} \leq c_1 \cdot |\Lambda|^2 \cdot s_F(\mathbb{P}, c_2|J|),$$

where $|\Lambda|$ is the number of edges in $\mathcal{I}_F := \Lambda$ and $s_F(\mathbb{P}, \varepsilon)$ is as in (1–1).

Previous estimates appeared in [35] and [30], both assumed that $\mathbb{P} = \otimes_{e \in E} \mu_e$, with $\mu_e = h_e(\lambda) d\lambda$, but their bounds were linear in $|\Lambda|$. Our proof heavily relies on the analysis of [30]. Our point here is twofold: first, if one makes use of the black box Theorem 2.3, then a large part of the proof of [30] can be omitted; second, this allows to extend their localization results in case $\alpha > 0$ to $\mu_e$ which are (log-)Hölder continuous.

Proof. It is proved in [30] Eq. (9)-(14)], by spectral analytic arguments and without any assumption on $\mathbb{P}$, that if $E_J$ is the midpoint of $J$, then there exists a discrete random self-adjoint operator $M_{\Lambda}(l^e, E_J)$ acting on $\ell^2(\mathcal{V}_\Lambda)$ and $b > 0$ such that

$$\mathbb{P}\{ \sigma(H^\omega_{\alpha}(\alpha)) \cap J \neq \emptyset \} \leq \mathbb{P}\{ \text{dist} (\sigma(M_{\Lambda}(l^e, E_J)), \alpha) \leq b|J| \}.$$

Moreover, given $u \in \ell^2(\mathcal{V}_\Lambda)$, the map $I^e \mapsto f_u(l^e) := \langle M_{\Lambda}(l^e, E_J)u, u \rangle$ is in $C^1(\Omega)$, only depends on $(l^e)_{e \in \Lambda}$ and there exists $\beta > 0$ such that

\begin{itemize}
  \item $\frac{\partial M_{\Lambda}(l^e, E_J)}{\partial l^e} \geq \beta \cdot I^e$ for all $e \in \Lambda$, where $I^e f(v) = f(v)$ if $v \in \{u, \tau e\}$ and $I^e f(v) = 0$ otherwise,
  \item and $\sum_{e \in \Lambda} \frac{\partial M_{\Lambda}(l^e, E_J)}{\partial l^e} \geq \beta \cdot \sum_{e \in \Lambda} I^e \geq \beta \cdot \text{Id}_{ell^2(\mathcal{V}_\Lambda)}$.
\end{itemize}

Thus, $\frac{\partial f_u(l^e)}{\partial l^e} \geq \beta (|u|_e)^2 + |u(\tau e)|^2 \geq 0$ for $e \in \Lambda$ and $\sum_{e \in \Lambda} \frac{\partial f_u(l^e)}{\partial l^e} \geq \beta \cdot \|u\|^2$. Hence $M_{\Lambda}(l^e, E_J)$ satisfies Hypothesis (C.5.b). Since $\ell^2(\mathcal{V}_\Lambda)$ is finite dimensional, the rest of Hypotheses (C) are clearly satisfied with $\mathcal{I}_F = \Lambda$ and $K = |\mathcal{V}_\Lambda| \leq c_d|\Lambda|$. We may thus apply Theorem 2.3 and Markov inequality to get

$$\mathbb{P}\{ \text{dist} (\sigma(M_{\Lambda}(l^e, E_J)), \alpha) \leq b|J| \} = \mathbb{P}\{ \text{tr} \chi_{\alpha - b|J|, \alpha + b|J|} (M_{\Lambda}(l^e, E_J)) \geq 1 \} \leq 2c_d|\Lambda|^2 s_F(\mathbb{P}, \frac{2b}{\beta} |J|). \quad \blacksquare$$

3.4. Quantum graphs with random vertex coupling. We finally show that Theorem 2.7 can tackle random vertex coupling models without any analytic effort. It seems there are no previous Wegner estimates for such models.

For simplicity consider the graph $(\mathcal{E}, \mathcal{V})$ given by $\mathcal{V} = \mathbb{Z}^d$ and $\mathcal{E}$ the set of segments $e = (v, v')$ between the vertices, assigned lengths $l_e$ with $l_{\text{min}} \leq l_e \leq l_{\text{max}}$. More general structures can be treated similarly. Given $e = (v, v')$, we put $ue = v$ and $re = v'$.

Fix $\alpha_- < \alpha_+ \in \mathbb{R}$, $\alpha_- < \alpha_+ < \alpha_+$ and $\emptyset \neq G \subset \mathcal{V}$. Let $(\Omega, \mathbb{P})$ be a probability space, where $\Omega = [\alpha_-, \alpha_+]^G$, $\mathbb{P}$ has no atoms and let $\mathcal{H} = \otimes_{e \in \mathcal{E}} L^2[0, l_e]$. Let $V = (V_e)$ be a bounded real potential, $V \geq c_0$, and given $\alpha^\omega = (\alpha^\omega_e) \in \Omega$, consider the operator

$$H(\alpha^\omega) : (f_e) \mapsto (-f''_e + V_e f_e),$$

acting on $(f_e) \in \otimes_{e \in \mathcal{E}} W^{2,2}(0, l_e)$ which are continuous at all vertices, i.e. $f_e(l_e) = f_b(0) =: f(v)$ if $\tau e = -b = v$, and which satisfy

$$f'(v) = \sum_{e : e = -v} f'_e(0) - \sum_{e : e = +v} f'_e(l_e) = \begin{cases} \alpha^\omega_e f(v) & \text{if } v \in G \\ 0 & \text{otherwise}. \end{cases}$$

The authors in [29] studied the case $G = \mathcal{V}$ and established localization for high disorder and near spectral edges using the fractional moments method (which does not rely on Wegner bounds). Their idea was to reduce the problem to one on $\ell^2(\mathcal{V})$, for an associated discrete operator. Below we prove a direct Wegner bound instead.
Given $\Lambda \subseteq E$, let $\mathcal{V}_\Lambda := \{ e : e \in \Lambda \} \cup \{ \tau e : e \in \Lambda \}$ and $\partial \Lambda := \mathcal{V}_\Lambda \cap \mathcal{V}_{\Lambda^c}$. Consider the form

$$\mathfrak{h}_\Lambda^{\omega D}[f] = \sum_{e \in \Lambda} \left( ||f'||^2_{L^2([0,L_e])} + (V_{eff}, f_e)_{L^2(0,L_e)} \right) + \sum_{v \in G \cap \mathcal{V}_\Lambda} \alpha_v^{\omega} |f(v)|^2$$

acting on $(f_e) \in \oplus_{e \in \Lambda} W^{1,2}(0,L_e)$ which are continuous at $v \in \mathcal{V}_\Lambda \setminus \partial \Lambda$ and vanish at $v \in \partial \Lambda$. Note that $\partial \Lambda$ is empty if $\Lambda = E$. It is known (see \cite{33} or \cite{35} Lemma 4.1) that $\mathfrak{h}_\Lambda^{\omega D}$ is closed and bounded from below, and thus corresponds to a self-adjoint operator $H_\Lambda^{\omega D}(\alpha^\omega)$. Moreover, $H(\alpha^\omega) = H_\Lambda^{\omega D}(\alpha^\omega)$, so we denote $\mathfrak{h}^\omega := \mathfrak{h}_E^{\omega D}$.

**Lemma 3.5.** For any $\Lambda \subseteq E$, $H(\alpha^\omega) \leq H_\Lambda^{\omega D}(\alpha^\omega) + H_{\Lambda^c}^{\omega D}(\alpha^\omega)$. If $\Lambda$ is finite, $H_\Lambda^{\omega D}(\alpha^\omega)$ has a compact resolvent. Its eigenvalues, denoted $E_j^{\omega D}$, counting multiplicity, satisfy the following Weyl law: for any $S \in \mathbb{R}$, there exists a non-random $C = C(S, c_0, \alpha_-, l_{\text{min}}, l_{\text{max}})$ such that $E_j^{\omega D} > S$ if $j > C \cdot |\Lambda|$, where $|\Lambda|$ is the number of edges in $\Lambda$.

**Proof.** The bracketing result follows \cite{35} Lemma 4.2, namely, $D(\mathfrak{h}_\Lambda^{\omega D}) \supseteq D(\mathfrak{h}_{\Lambda^c}^{\omega D})$ since a function in $D(\mathfrak{h}_\Lambda^{\omega D}) \supseteq D(\mathfrak{h}_{\Lambda^c}^{\omega D})$ is automatically continuous at all $v$. Moreover, if $f = f_1 \oplus f_2 \in D(\mathfrak{h}_{\Lambda^c}^{\omega D})$, then $\mathfrak{h}_\Lambda^{\omega D}[f_1] + \mathfrak{h}_{\Lambda^c}^{\omega D}[f_2] = \mathfrak{h}^\omega[f]$ because $f(v) = 0$ on boundary vertices. Thus, $H \leq H_\Lambda^{\omega D} + H_{\Lambda^c}^{\omega D}$.

Now suppose $\Lambda$ is finite and as in \cite{15}, consider the Neumann-decoupled Laplacian $-\Delta_N^{\text{dec}}$ defined via the form $\mathfrak{h}[f] = \sum_{e \in \Lambda} ||f'||^2_{L^2([0,L_e])}$ with $D(\mathfrak{h}) = \oplus_{e \in \Lambda} W^{1,2}(0,L_e)$. Then $D(\mathfrak{h}_\Lambda^{\omega D}) \subset D(\mathfrak{h})$ and $\mathfrak{h}_\Lambda^{\omega D}[f] \geq \mathfrak{h}[f] + c_0 ||f||^2 + \alpha - \sum_{v \in G \cap \mathcal{V}_\Lambda} |f(v)|^2 \geq \frac{1}{2}(t + C)[f]$ for some $C = C(l_{\text{min}}, l_{\text{max}}, \ldots, c_0)$ by standard trace estimates, see e.g. \cite{33} Lemma 8. Thus, $H_\Lambda^{\omega D}(\alpha^\omega) \geq \frac{1}{2}(\Delta_N^{\text{dec}} + C)$. But since $-\Delta_N^{\text{dec}} = \oplus_{e \in \Lambda} -\Delta_{(0,L_e)}$, its eigenvalues $E_j^{\omega D}$ are just the eigenvalues $E_k(-\Delta_{(0,L_e)}) = \frac{\pi^2 k^2}{4L_e^2}$ with multiplicity $|\Lambda|$. In particular, $E_j^{\omega D} \to \infty$ as $j \to \infty$, hence $E_j^{\omega D} \to \infty$ as $j \to \infty$ and $H_\Lambda^{\omega D}(\alpha^\omega)$ has a compact resolvent by \cite{37} Theorem XIII.64. Moreover, we have $E_j^{\omega D} \geq \frac{1}{2}(t + C)$. By the explicit form of $E_j^{\omega D}$, we know that $E_j^{\omega D} > 2S - C$ if $j > C|\Lambda|$ for some $C = C(l_{\text{min}}, l_{\text{max}}, S, C)$. Thus, $E_j^{\omega D} > S$ if $j > C|\Lambda|$ and we are done.

We may now state our Wegner bound. Fix $q > \alpha_+$ and let $H_0, H_\gamma$ be the operators corresponding to $\mathfrak{h}_0[f] = \sum_{e \in \mathcal{E}} \left( ||f'||^2_{L^2([0,L_e])} + (V_{eff}, f_e)_{L^2(0,L_e)} \right)$ and $\mathfrak{h}_\gamma[f] = \mathfrak{h}_0[f] + q \sum_{v \in G \cap \mathcal{V}_\Lambda} |f(v)|^2$ respectively, with $D(\mathfrak{h}_0) = D(\mathfrak{h}_\gamma) = D(\mathfrak{h}^\omega)$. Let $\mathcal{I}_F := G \cap \mathcal{V}_\Lambda$ and $s_F(\mathbb{P}, \varepsilon)$ as in \cite{11}.

**Theorem 3.6.** Let $I = (E_1, E_2)$ be an open interval.

There exists $C_W = C_W(E_2, c_0, \alpha_-, l_{\text{min}}, l_{\text{max}}) > 0$ such that for any finite $\Lambda \subseteq E$ and any $q > \alpha_+$, if $E_2 < E_\gamma := \inf(\sigma(H_\gamma))$, then

$$\mathbb{E}\{ |\mathfrak{I}(\lambda(H_\Lambda^{\omega D}(\alpha^\omega)))| \} \leq C_W \cdot |\Lambda| \cdot |\mathcal{I}_F| \cdot s_F \left( \mathbb{P}, \frac{q - \alpha_+}{E_\gamma - E_2}, I \right).$$

If $\alpha_+ < 0$, we may take $q = 0$ and obtain a Wegner bound below $E_0 := \inf(\sigma(H_0))$. This result is non-trivial at least when $G = \mathbb{Z}^d$ and the disorder is high, because $H(\alpha^\omega)$ will have some spectrum below $E_0$ in this case almost surely; see \cite{29} Theorem 12] and the remark thereafter. If $\alpha_+ = 0$, the non-triviality will be ensured if $E_0 < E_\gamma$.

**Proof.** Let $A(\omega) := H_\Lambda^{\omega D}(\alpha^\omega) - E_0$ and $I' = (E_1 - E_\gamma, E_2 - E_\gamma)$. Then $\chi(\lambda) = \chi_I'(\lambda - E_0)$, hence $\mathbb{E}\{ |\mathfrak{I}(\lambda(H_\Lambda^{\omega D}(\alpha^\omega)))| \} = \mathbb{E}\{ |\mathfrak{I}(\lambda(A(\omega)))| \}$. Moreover, $A(\omega)$ corresponds to the form $\mathfrak{a}^\omega[f] = (\mathfrak{h}_\Lambda^{\omega D} - E_\gamma)[f]$ with $\mathcal{D} := \mathfrak{a}^\omega = D(\mathfrak{h}_\Lambda^{\omega D})$ non-random, and we have $\mathfrak{a}^\omega[f] = b_1[f] + \sum_{v \in G \cap \mathcal{V}_\Lambda} (\alpha_v^\omega - q) |f(v)|^2$, where $b_1 := \mathfrak{h}_\Lambda^{\omega D} - E_\gamma$. By the bracketing in Lemma 3.5 we have $b_1 \geq 0$. Thus, $A(\omega)$ satisfies Hypotheses (C.1) to (C.4) in $I'$, with a Weyl constant $K = C|\Lambda|$ from Lemma 3.5 and the claim follows from Theorem 2.4.
4. Proofs of the general theorems

4.1. Proof of Proposition 2.1

Proof. Put $\chi_I := \chi_I(H(\omega))$. By hypothesis, for a.e. $\omega$,
\begin{equation}
\langle \chi_I W e_j, e_j \rangle = \langle \chi_I W e_j, \chi_I e_j \rangle \leq \|\chi_I W e_j\| \|\chi_I e_j\|
\leq \frac{c}{2} \|\chi_I W e_j\|^2 + \frac{1}{2c} \|\chi_I e_j\|^2 = \frac{c}{2} \langle W \chi_I W e_j, e_j \rangle + \frac{1}{2c} \langle \chi_I e_j, e_j \rangle
\end{equation}
for any $c > 0$. Summing over $j \in J$ and choosing $c = \gamma^{-1}$ we get by (4-1)
\begin{equation}
\text{tr} [\chi_I] \leq \gamma^{-1} \text{tr} [W \chi_I W] + \frac{1}{2\gamma^{-1}} \text{tr} [\chi_I].
\end{equation}
Thus,
\begin{align*}
\text{tr} [\chi_I] & \leq \gamma^{-2} \text{tr} [W \chi_I W] \\
& = \gamma^{-2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_F} \langle U_a \chi_I U_{a'} e_j, e_j \rangle \\
& = \gamma^{-2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_F} \langle \chi_I U_{a'} e_j, \chi_I U_a e_j \rangle \\
& \leq \gamma^{-2} \frac{1}{2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_F} (\|U_{a'} e_j\|^2 + \|\chi_I U_a e_j\|^2) \\
& \leq \gamma^{-2} C_{\text{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_F} \|\chi_I U_a e_j\|^2.
\end{align*}
This completes the proof, since $\|\chi_I U_a e_j\|^2 = \langle U_a \chi_I U_a e_j, e_j \rangle$, and the terms with $j \notin J_{\text{eff}}$ are zero. \hfill \Box

4.2. Proof of Theorem 2.2. We first recall [43, Theorem 3.2]:

Spectral Averaging. Let $\mu$ be a probability measure on $\mathbb{R}$ and $H$ a Hilbert space. If $A$ is a self-adjoint operator and $0 \leq B$ is a bounded operator on $H$, then for any interval $I$ and any $\phi \in H$ we have
\begin{equation}
\int_{\mathbb{R}} (B^{1/2} \chi_I (A + tB)B^{1/2} \phi, \phi) \, dt \leq 6 \|B\| \|\phi\|^2 s(\mu, |I|),
\end{equation}
where $s(\mu, \varepsilon) := \sup_{E \in \mathbb{R}} \mu (E, E + \varepsilon)$.

Note that we could use instead the spectral averaging of [8]; in this case the upper bound should be replaced by $4 \|B\| (1 + \|B\|) \|\phi\|^2 s(\mu, |I|)$.

The proof in [43] actually gives $s(\mu, \varepsilon) = \sup_{E \in \mathbb{R}} \mu (E, E + \varepsilon)$, but since
\begin{equation}
\sup_{E \in \mathbb{R}} \mu (E, E + \varepsilon) = \sup_{E \in \mathbb{R}} \mu (E, E + \varepsilon) = \sup_{E \in \mathbb{R}} \mu (E, E + \varepsilon)
\end{equation}
(see Section 3.2), the above bound holds.

Proof of Theorem 2.2. To show that $\chi_I(H(\omega))$ is weakly measurable, it suffices to show that $H(\omega)$ is weakly measurable; see [21]. Let $\varphi, \psi \in H$ and let $g(\omega) = \langle H(\omega) \varphi, \psi \rangle = \langle H_0 \varphi, \psi \rangle + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha (U_\alpha \varphi, \psi)$. Then $g$ only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$, i.e. $\{\omega : g(\omega) \geq a\} = A \times B^{\mathcal{I}_F}$ for some $A \subseteq B^{\mathcal{I}_F}$, so by definition of 3, it suffices to show that $A \subseteq \otimes_{\alpha \in \mathcal{I}_F} B^\mathcal{I}_F$. In turn, it suffices to show that the map $g_0 : B^{\mathcal{I}_F} \rightarrow \mathbb{R}$ given by $g_0 : (\omega_\alpha)_{\alpha \in \mathcal{I}_F} \mapsto \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha \varphi_\alpha$ is weakly measurable.
\( (H_0\varphi, \psi) + \sum_{\alpha \in \mathcal{I}_\alpha} \omega_{\alpha}(U_{\alpha}\varphi, \psi) \) is Borel measurable, but this is obvious since it is affine. Hence, \( \chi_I(H(\omega)) \) is weakly measurable and \( \text{tr}[\chi_I(H(\omega))] \) is measurable.

We may thus integrate in Proposition 2.1 to get
\[
\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq \gamma^{-2} C_{\text{fin}} \sum_{j \in \mathcal{J}_{\text{eff}}} \sum_{\alpha \in \mathcal{I}_j} \mathbb{E}\{(U_{\alpha}\chi_I(H(\omega))U_{\alpha}e_j, e_j)\}.
\]

Fix \( j \in \mathcal{J}_{\text{eff}}, \alpha \in \mathcal{I}_j \) and put \( \phi := U_{\alpha}^{1/2}e_j \). Then by [10] Theorem 10.2.1,
\[
\mathbb{E}\{(U_{\alpha}\chi_I(H(\omega))U_{\alpha}e_j, e_j)\} = \mathbb{E}_Y \left\{ \int_{R} (U_{\alpha}^{1/2}\chi_I(H(\omega))U_{\alpha}^{1/2}\phi, \phi) d\mu_{\omega_{\alpha}}(\omega_{\alpha}) \right\}.
\]
Using the spectral averaging with \( A = H_1 + \sum_{j \neq \alpha} \omega_{\beta}U_{\beta}, B = U_{\alpha} \) and \( t = \omega_{\alpha} \), we get
\[
\mathbb{E}\{(U_{\alpha}\chi_I(H(\omega))U_{\alpha}e_j, e_j)\} \leq 6\|U_{\alpha}\|\|U_{\alpha}^{1/2}e_j\|^2 \mathbb{E}_Y \{s(\mu_{\omega_{\alpha}}, |I|)\} \leq 6C_{U}^{2} \mathbb{E}_Y \{s(\mu_{\omega_{\alpha}}, |I|)\}.
\]
Since \( \mathbb{E}_Y \{s(\mu_{\omega_{\alpha}}, |I|)\} \leq s_F(\mathbb{P}, |I|), \) the proof is complete. \( \square \)

4.3. Proof of Theorem 2.3. Through this subsection \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space with \( \Omega := C^{\mathbb{I}}, \) where \( C \subseteq \mathbb{R} \) is an interval and \( \mathcal{I} \) is a countable index set. \( \mathcal{F} \) denotes the \( \mathbb{P} \)-

completeness of \( \mathcal{F} \). We fix a finite set \( \mathcal{I}_F \subseteq \mathcal{I} \) and denote by \( 1_F \) the element \( 1_F = (x_\alpha) \in \mathbb{R}^{\mathcal{I}} \) such that \( x_\alpha = 1 \) if \( \alpha \in I_F \) and \( x_\alpha = 0 \) otherwise.

We will use the fact that monotone functions \( \varphi : \Omega \to \mathbb{R} \) which depend on finitely many \( \omega_{\alpha} \) are \( \mathcal{F}_F \)-measurable; this is proved in Lemma 5.2. Note that for any fixed \( x \in \mathbb{R}^{\mathcal{I}}, \) the map \( \varphi(x - \omega) \) is also monotone, hence \( \mathcal{F}_F \)-measurable. We may thus state the following lemma, which is a basic idea from [22], see also [5] and [2].

**Lemma 4.1.** Suppose \( \varphi : \Omega \to \mathbb{R} \) is monotone increasing and depends on finitely many \( \omega_{\alpha} \).

Given \( c \in \mathbb{R} \) and \( \eta > 0 \), define \( A := \{\omega : \varphi(\omega) \leq c\}, A^0 := \{\omega : \omega - \eta \cdot 1_F \in \Omega \text{ and } \varphi(\omega - \eta \cdot 1_F) \leq c\}, B := \{\omega : \varphi(\omega) \geq c\} \) and \( B^0 := \{\omega : \omega + \eta \cdot 1_F \in \Omega \text{ and } \varphi(\omega + \eta \cdot 1_F) \geq c\} \).

Then
\[
\mathbb{P}(A^0 \setminus A) \leq |I_F| \cdot s_F(\mathbb{P}, \eta) \quad \text{and} \quad \mathbb{P}(B^0 \setminus B) \leq |I_F| \cdot s_F(\mathbb{P}, \eta),
\]
where \( \mathbb{P} \) denotes the extension of \( \mathbb{P} \) to \( \mathcal{F}_F \) and \( s_F(\mathbb{P}, \eta) \) is as in (1.1).

**Proof.** We prove the second bound; the first is similar. Let \( \mathcal{I}_F = \{\alpha_1, \ldots, \alpha_m\} \) and \( \mathcal{I}_k = \{\alpha_1, \ldots, \alpha_k\} \) for \( 1 \leq k \leq m \). Let \( 1_j = (x_\alpha) \in \mathbb{R}^{\mathcal{I}} \) with \( x_\alpha = 1 \) if \( \alpha \in \mathcal{I}_j \) and \( x_\alpha = 0 \) otherwise, so that \( 1_m = 1_F \). Set
\[
B^0_0 := B \quad \text{and} \quad B^0_j := \{\omega : \omega + \eta \cdot 1_j \in \Omega \text{ and } \varphi(\omega + \eta \cdot 1_j) \geq c\}
\]
for \( 1 \leq j \leq m \). Note that if \( B_0, \ldots, B_m \) is any collection of sets, then one checks by induction that \( B_m \setminus B_0 \subseteq \bigcup_{j=1}^{m} (B_j \setminus B_{j-1}) \), so we have in particular
\[
|B^0_m \setminus B_0| \leq \sum_{j=1}^{m} |B^0_j \setminus B^0_{j-1}|.
\]

Now fix \( j \in \{1, \ldots, m\}, \) let \( \omega_j = (\omega_{\beta})_{\beta \neq \alpha_j} \in C^{\mathcal{I}\setminus\{\alpha_j\}} \) and denote by \( (x, \omega_j) \) the element \( (x_\alpha) \in \mathbb{R}^{\mathcal{I}} \) with \( x_{\alpha_j} = x \) and \( x_{\beta} = \omega_{\beta} \) for \( \beta \neq \alpha_j \). Define the section
\[
C_{\omega_j} := \{x \in C : (x, \omega_j) \in B^0_j \setminus B^0_{j-1}\} = (B^0_j \setminus B^0_{j-1})_{\omega_{\alpha_j}}.
\]

We show that \( C_{\omega_j} \) is contained in an interval of length \( \eta \). If \( C_{\omega_j} = \emptyset \), this is clear, so suppose \( x \in C_{\omega_j} \). Fix \( \delta \geq \eta \). If \( x - \delta \in C_{\omega_j} \), then \( (x - \delta, \omega_j) \in B^0_j \) and thus
\[
\varphi((x - \delta, \omega_j) + \eta \cdot 1_j) \geq c.
\]
But \( \varphi \) is monotone increasing, so \( \varphi((x - \delta, \omega_j) + \eta \cdot 1_j) = \varphi((x - \delta + \eta, \omega_j) + \eta \cdot 1_{j-1}) \leq \varphi((x, \omega_j) + \eta \cdot 1_{j-1}) < c \), since \( (x, \omega_j) \notin B^0_{j-1} \). This contradiction shows that \( x - \delta \notin C_{\omega_j} \) for any \( \delta \geq \eta \), i.e. \( C_{\omega_j} \) is contained in a semi-open

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3Note that if \( \omega + \eta \cdot 1_j \in \Omega \), then \( \omega_{\alpha} + \eta \in C \) for any \( \alpha \in \mathcal{I}_j \), so in particular for any \( \alpha \in \mathcal{I}_{j-1} \) and thus \( \omega_{\alpha} + \eta \cdot 1_{j-1} \in \Omega \).
interval $I_{\omega}$ of length $\eta$. Let $D_j^0$ be the set $B_j^0 \setminus B_{j-1}^0$ with each section $C_{\omega_j}$ replaced by $I_{\omega_j}$. Then $B_j^0 \setminus B_{j-1}^0 \subseteq D_j^0$ and $I_{\omega_j}$ is a Borel set for any $\omega_j$. So applying [11 Corollary 10.4.15] to $D_j^0$, taking $Y_j := C^{\lambda_1(\alpha_j)}$ and using (4-2), we may find $\overline{\omega}_{\omega_j}$ such that

\[(4-4) \quad \mathbb{P}(B_j^0 \setminus B_{j-1}^0) \leq \mathbb{E}_Y_j \{\overline{\omega}_{\omega_j}(I_{\omega_j})\} \leq \mathbb{E}_Y_j \left\{ \sup_{E \in \mathcal{E}} \overline{\omega}_{\omega_j}(E, E + \eta) \right\}.\]

But $\mathbb{E}_Y_j \{\mu_{\omega_j}(E, E + \eta)\} = \mathbb{P}\{\omega_{\alpha_j} \in (E, E + \eta)\} = \mathbb{E}_Y_j \{\overline{\omega}_{\omega_j}(E, E + \eta)\}$, hence $\mu_{\omega_j}(E, E + \eta) = \overline{\omega}_{\omega_j}(E, E + \eta)$ outside a $\mathbb{P}_Y_j$-null set $\Omega_E$. Let $\Omega_\ast = \cup_{E \in \mathcal{E}} \Omega_E$. Then $\mathbb{E}_Y_j(\Omega_\ast) = 0$ and $\sup_{E \in \mathcal{E}} \mu_{\omega_j}(E, E + \eta) = \sup_{E \in \mathcal{E}} \overline{\omega}_{\omega_j}(E, E + \eta)$ for any $\omega_j \notin \Omega_\ast$. So using (1-2),

\[\mathbb{E}_Y_j \left\{ \sup_{E \in \mathcal{E}} \overline{\omega}_{\omega_j}(E, E + \eta) \right\} = \mathbb{E}_Y_j \left\{ \sup_{E \in \mathcal{E}} \overline{\omega}_{\omega_j}(E, E + \eta) \right\} \leq s_F(\mathbb{P}, \eta),\]

and the claim follows by (4-3) and (4-4). \qed

We may now prove a first extension of Stollmann’s Lemma from [42]. Namely, we allow intervals $C$ and relax the diagonal condition by adding cutoffs $\chi_I(\varphi(\omega))$. The inclusion of cutoffs is actually immediate and will not be used in the proof of Theorem 2.3. However, this idea plays a major role in the proof of Theorem 2.4.

**Lemma 4.2.** Let $I \subset \mathbb{R}$ an open interval. Suppose $\varphi : \Omega \to \mathbb{R}$ is monotone increasing, depends on finitely many $\omega_\alpha$ and satisfies

\[(4-5) \quad (\varphi(\omega) - \varphi(\omega - t \cdot 1_F))\chi_I(\varphi(\omega)) \geq t\gamma \cdot \chi_I(\varphi(\omega))\]

for some $\gamma > 0$ and all $t \geq 0$ such that $\omega - t \cdot 1_F \in \Omega$. Then

\[\mathbb{P}\{\varphi(\omega) \in I\} \leq \delta \cdot |I_F| \cdot s_F\left(\mathbb{P}, \frac{|I|}{\gamma}\right), \quad \text{where } \delta = \begin{cases} 1 & \text{if } \inf C = -\infty, \\ 2 & \text{otherwise.} \end{cases}\]

This bound is also true if $\varphi$ is monotone decreasing and satisfies

\[(4-6) \quad (\varphi(\omega) - \varphi(\omega - t \cdot 1_F))\chi_I(\varphi(\omega)) \leq -t\gamma \cdot \chi_I(\varphi(\omega)).\]

**Proof.** Let $I = (a, b)$, $\varepsilon := b - a$ and $\eta := \frac{\varepsilon}{\delta}$. We have

\[\mathbb{P}\{\varphi(\omega) \in I\} \leq \mathbb{P}\{\omega : \varphi(\omega) \in I \text{ and } \omega - \eta \cdot 1_F \in \Omega\} + \mathbb{P}\{\omega - \eta \cdot 1_F \notin \Omega\}.\]

Put $A := \{\omega : \varphi(\omega) \leq a\}$, $A^\eta := \{\omega : \omega - \eta \cdot 1_F \in \Omega \text{ and } \varphi(\omega - \eta \cdot 1_F) \leq a\}$ and let $\omega \in A := \{\omega : \varphi(\omega) \in I \text{ and } \omega - \eta \cdot 1_F \in \Omega\}$. Then by (4-5),

\[\varphi(\omega - \eta \cdot 1_F) \leq \varphi(\omega) - \gamma \eta = \varphi(\omega) - \varepsilon \leq b - \varepsilon = a.\]

Hence $\omega \in A^\eta$. Furthermore, $\varphi(\omega) \in I$ implies $\varphi(\omega) > a$ and thus $\omega \notin A$. Hence $A \subseteq A^\eta \setminus A$ and $\mathbb{P}(A) \leq |I_F| \cdot s_F(\mathbb{P}, \eta)$ by Lemma 4.1.

If $\inf C = -\infty$, then $\mathbb{P}\{\omega - \eta \cdot 1_F \notin \Omega\} = 0$, since $C$ is an interval. Otherwise, let $q_- := \inf C$. If $q_- \in C$, then using (4-2),

\[\mathbb{P}\{\omega - \eta \cdot 1_F \notin \Omega\} = \mathbb{P}\{\omega_\alpha \in [q_- - q_+ + \eta) \text{ for some } \alpha \in I_F\} \leq |I_F| \cdot s_F(\mathbb{P}, \eta)\]

since $\mathbb{P}\{\omega_\alpha \in [q_- - q_+ + \eta)\} = \mathbb{E}_Y_{\alpha} \{\mu_{\omega_\alpha} [q_- - q_+ + \eta]\}$. If $q_- \notin C$, replace $[q_- - q_+ + \eta]$ by $[q_- - q_+ + \eta]$.

Finally, if $\varphi$ is decreasing and satisfies (4-6), then $\psi := -\varphi$ is increasing and $\chi_I(\varphi(\omega)) = \chi_I(\psi(\omega))$, where $I' := (-b, -a)$, hence $\psi$ satisfies (4-5) in $I'$. Applying the first part we obtain $\mathbb{P}\{\varphi(\omega) \in I\} = \mathbb{P}\{\psi(\omega) \in I'\} \leq \delta \cdot |I_F| \cdot s_F\left(\mathbb{P}, \frac{|I'|}{\gamma}\right).$ \qed

**Proof of Theorem 2.3.** Let $\{\varphi_\alpha(\omega)\}$ be an orthonormal basis of eigenvectors of $H(\omega)$ with eigenvalues $\lambda_\alpha(\omega)$. Then $\chi_I(\varphi(\omega))\varphi_\alpha(\omega) = \chi_I(\lambda_\alpha(\omega))$. So using (C.3), we get

\[\text{tr}[\chi_I(\varphi(\omega))] = \sum_n \langle \chi_I(\varphi(\omega))\varphi_n(\omega), \varphi_n(\omega) \rangle = \sum_{n \leq K} \chi_I(\lambda_n(\omega)).\]
By (C.4), (C.5) and min-max, each $\lambda_n : \Omega \to \mathbb{R}$ is monotone (see below for (C.5.b) and (C.5.d)) and only depends on $(\omega_\alpha)_{\alpha \in I_F}$. So by Lemma 5.2, each $\lambda_n$ is $P$-measurable, hence $\chi_I(\lambda_n(\omega)) = \chi_{\lambda_n^{-1}(I)}(\omega)$ is $P$-measurable, and we may integrate to get
\[
\mathbb{E}\{\text{tr}[\chi_I(\mathcal{H}(\omega))]\} = \sum_{n \leq K} \mathbb{E}\{\chi_I(\lambda_n(\omega))\} = \sum_{n \leq K} \mathbb{P}\{\lambda_n(\omega) \in I\}.
\]

Now assume (C.5.a) holds. Then by min-max, $\lambda_n(\omega)$ are monotone increasing and satisfy $\lambda_n(\omega) \geq \lambda_n(\omega - t \cdot 1_F) + t \gamma$ for all $t \geq 0$ such that $\omega - t \cdot 1_F \in \Omega$. So by Lemma 4.2
\[
\mathbb{P}\{\lambda_n(\omega) \in I\} \leq 2 \cdot |I_F| \cdot s_F\left(\frac{|I|}{\gamma}\right),
\]
as asserted. The case (C.5.c) is similar. Let us show that (C.5.b) implies (C.5.a) and (C.5.d) implies (C.5.c).

Let $f \in C^1(\Omega)$. Given $v, w \in \Omega$, we have $v + t(w - v) \in \Omega$ for any $t \in [0, 1]$. Moreover, $t \mapsto f(v + t(w - v))$ is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$, hence
\[
f(w) - f(v) = \int_0^1 \frac{d}{dt} f(v + t(w - v)) \, dt = \int_0^1 \sum_{\alpha \in I}(w_\alpha - v_\alpha) \frac{\partial f}{\partial \omega_\alpha}(v + t(w - v)) \, dt.
\]
If $f$ only depends on $(\omega_\alpha)_{\alpha \in I_F}$, then the sum reduces to $I_F$. If moreover $\frac{\partial f}{\partial \omega_\alpha} \geq 0$ on $\Omega$ for all $\alpha \in I_F$ and $w_\alpha \geq v_\alpha$, then $f(w) - f(v) \geq 0$, i.e. $f$ is monotone increasing. Similarly, if $\frac{\partial f}{\partial \omega_\alpha} \leq 0$ on $\Omega$, then $f$ is monotone decreasing. Finally, for $w = \omega$ and $v = \omega - \nu \cdot 1_F$ we get
\[
f(\omega) - f(\omega - \nu \cdot 1_F) = \nu \int_0^1 \sum_{\alpha \in I_F} \frac{\partial f}{\partial \omega_\alpha}(\omega - \nu \cdot 1_F + t(\nu \cdot 1_F)) \, dt,
\]
hence $\sum_{\alpha \in I_F} \frac{\partial f}{\partial \omega_\alpha} \geq c$ on $\Omega$ implies $f(w) - f(\omega - \nu \cdot 1_F) \geq \nu c$ and $\sum_{\alpha \in I_F} \frac{\partial f}{\partial \omega_\alpha} \leq -c$ on $\Omega$ implies $f(\omega) - f(\omega - \nu \cdot 1_F) \leq -\nu c$. \hfill \Box

4.4. Proof of Theorem 2.4

The proof of Theorem 2.4 uses two ideas: the first one is roughly to consider the change of variables $v_\alpha = \ln \omega_\alpha$, so that $\mathbb{E}\{f(\omega)\} = \int f(\omega) \mathbb{P}(d\omega) = \int f(e^{v_\alpha}) \mathbb{P}(dv)$. This idea was used before in [35 Theorem 2.9]. The new measure $\tilde{P}$ is easily described if $P$ is a product measure; the general case is given in Lemma 4.3. The second idea is to generalize Stollmann’s lemma to include cutoffs $\chi_I(\varphi(\omega))$ (as we did in Lemma 4.2) and also extend the diagonal growth condition. This is done in Lemma 4.4.

Lemma 4.3. Let $\Omega = [q_-, q_+]^T$, fix $q > q_+$ and let $\tilde{\Omega} := [v_-, v_+]^T$, where $v_- = \ln(q - q_+)$ and $v_+ = \ln(q - q_-)$. Define $T : \Omega \to \tilde{\Omega}$ by $T : (\omega_\alpha) \mapsto (\ln(q - \omega_\alpha))$ and let $\tilde{P} := P \circ T^{-1}$. Then
\[
s_F(\tilde{P}, \varepsilon) \leq s_F(P, (q - q_-)(e^\varepsilon) - 1)).
\]

Here $s_F(\tilde{P}, \varepsilon)$ is defined as before, i.e. if $Z_{\alpha} := [v_-, v_+]^{T\langle \alpha \rangle}$, $\pi_{Z_{\alpha}} : \tilde{\Omega} \to Z_{\alpha}$ is defined by $\pi_{Z_{\alpha}} : v \mapsto \tilde{v}_\alpha$ and if $\tilde{P}_{Z_{\alpha}} = \tilde{P} \circ \pi_{Z_{\alpha}}^{-1}$, then $s_F(P, \varepsilon) = \max_{\alpha \in I_F} \mathbb{E}_{Z_{\alpha}} \{ \sup_{E \in \mathbb{R}} \tilde{\mu}_{\tilde{v}_\alpha}(E, E + \varepsilon)\}$.\hfill \Box

Proof. First recall that by [11 Theorem 4.1.11], if $T : (X, \mathcal{X}, P) \to (Y, \mathcal{Y})$ is any measurable map, and if $T^T = P \circ T^{-1}$, then for any measurable $g : Y \to \mathbb{R}$, we have
\[
E_T\{g(y)\} = E\{(g \circ T)(x)\},
\]
whenever either side exists. Fix $\alpha \in I_F$ and let $G := \{v_\alpha \in (E, E + \varepsilon)\}$. Then
\[
\mathbb{E}_{Z_{\alpha}} \{ \tilde{\mu}_{\tilde{v}_\alpha}(E, E + \varepsilon) \} = \tilde{P}(G) = P(T(\omega) \in G) = \mathbb{P}\{\ln(q - \omega_\alpha) \in (E, E + \varepsilon)\}
\]
\[
= \mathbb{P}\{\omega_\alpha \in (q - e^{E + \varepsilon}, q - e^E)\} = \mathbb{E}_{Y_\alpha}\{\mu_{\omega_\alpha}(q - e^{E + \varepsilon}, q - e^E)\},
\]
where \( Y_\alpha := [q_-, q_+]^{\mathbb{C}(\alpha)} \). Define \( \hat{T}_2 : Z_\alpha \to Y_\alpha \) by \( \hat{T}_2 : (v_\alpha) \mapsto (q - e^{v_\alpha}) \). Then \( \hat{T}_2 \circ \pi_{Z_\alpha} \circ T = \pi_{Y_\alpha} \), so \( \hat{F}_2^\alpha = F_{\hat{\alpha}} \) and using (4-7) we get \( \mathbb{E}_{Y_\alpha} \{ \mu_{\hat{\alpha}}(q - e^{v_\alpha}, q - e^F) \} = \mathbb{E}_{Z_\alpha} \{ \mu_{\hat{T}_2(\hat{\alpha})}(q - e^{v_\alpha}, q - e^F) \} \). Hence \( \mu_{\hat{\alpha}}(E, E + \varepsilon) = \mu_{\hat{T}_2(\hat{\alpha})}(q - e^{v_\alpha}, q - e^F) \) outside a \( \mathbb{P}_{Z_\alpha} \)-null set \( \Omega_E \). Let \( \Omega_* = \cup E \in \Omega_{Q} \Omega_E \). Then \( \mathbb{P}_{Z_\alpha}(\Omega_*) = 0 \) and \( \sup_{E \in \Omega_{Q} \hat{\alpha} \in \Omega}(E, E + \varepsilon) = \sup_{E \in \Omega_{Q} \hat{T}_2(\hat{\alpha})}(q - e^{v_\alpha}, q - e^F) \) for any \( \hat{\alpha} \neq \Omega_\alpha \). So using (1-2) and (4-7),

\[
\mathbb{E}_{Z_\alpha} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\alpha}}(E, E + \varepsilon) \right\} = \mathbb{E}_{Z_\alpha} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{T}_2(\hat{\alpha})}(q - e^{v_\alpha}, q - e^F) \right\} = \mathbb{E}_{Y_\alpha} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\alpha}}(q - e^{v_\alpha}, q - e^F) \right\}.
\]

If \( q - e^F < q_- \), the RHS is zero, since \( \mu_{\hat{\alpha}} \) is supported in \([q_-, q_+] \). So suppose \( e^F \leq q_- \). Then \( (q - e^F) - (q - e^{v_\alpha}) = e^F(e^\varepsilon - 1) \leq (q - q_-)(e^\varepsilon - 1) \). This completes the proof. \( \square \)

**Lemma 4.4.** Let \((\Omega, \mathbb{P})\) be a probability space, \( \Omega = [c_-, c_+]^\mathbb{T} \) and \( I \subset \mathbb{R} \) an open interval. Suppose \( \varphi : \Omega \to \mathbb{R} \) is monotone increasing, depends on finitely many \( \omega_\alpha \) and satisfies

\[
(\varphi(w + t \cdot 1_F) - \varphi(w)) \chi_I(\varphi(w)) \geq \gamma(e^{\xi t} - 1) \cdot \chi_I(\varphi(w))
\]

for some \( \zeta > 0, \gamma > 0 \) and all \( t \geq 0 \) such that \( w + t \cdot 1_F \in \Omega \). Then

\[
\mathbb{P}\{\varphi(w) \in I\} \leq 2 \cdot |I|F \cdot s_F \left( \mathbb{P}, \frac{1}{\zeta} \ln(1 + \frac{|I|}{\gamma}) \right).
\]

This bound is also true if \( \varphi \) is monotone decreasing and satisfies for all \( t \geq 0 \) such that \( w - t \cdot 1_F \in \Omega \) the bound

\[
(\varphi(w) - \varphi(w - t \cdot 1_F)) \chi_I(\varphi(w)) \leq \gamma(1 - e^{\xi t}) \cdot \chi_I(\varphi(w)).
\]

**Proof.** Let \( I = (a, b) \), \( \varepsilon := b - a \) and \( \eta := \frac{1}{\zeta} \ln(1 + \frac{1}{\zeta}) \). Suppose first that \( \varphi \) is monotone increasing and satisfies (4-8). We have

\[
\mathbb{P}\{\varphi(w) \in I\} \leq \mathbb{P}\{\varphi(w) \in I \text{ and } w + \eta \cdot 1_F \in \Omega\} + \mathbb{P}\{w + \eta \cdot 1_F \notin \Omega\}.
\]

For the first term, let \( w \in A := \{ \varphi(w) \in I \text{ and } w + \eta \cdot 1_F \in \Omega\} \). Then by (4-8),

\[
\varphi(w + \eta \cdot 1_F) \geq \varphi(w) + \gamma(e^{\xi \eta} - 1) = \varphi(w) + \varepsilon \geq a + \varepsilon = b,
\]

hence if \( B^\eta := \{ w : w + \eta \cdot 1_F \in \Omega \text{ and } \varphi(w + \eta \cdot 1_F) \geq b \} \), we have \( w \in B^\eta \). Moreover, \( \varphi(w) \in I \) implies \( \varphi(w) < b \) and thus \( w \notin B := \{ w : \varphi(w) \geq b \} \). Hence, \( A \subseteq B^\eta \setminus B \) and

\[
\mathbb{P}(A) \leq |I|F \cdot s_F(\mathbb{P}, \eta) \text{ by Lemma 4.1}.
\]

For the second term, \( \mathbb{P}\{w + \eta \cdot 1_F \notin \Omega\} = \mathbb{P}\{\omega_\alpha \in (c_+ - \eta, c_+) \text{ for some } \alpha \in I_F\} \leq |I_F| \cdot s_F(\mathbb{P}, \eta) \text{ by (4-2)} \). This proves the first claim.

Now suppose \( \varphi \) is decreasing and satisfies (4-9). Again, the result holds.

\[
\mathbb{P}\{\varphi(w) \in I\} \leq \mathbb{P}\{\varphi(w) \in I \text{ and } w - \eta \cdot 1_F \in \Omega\} + \mathbb{P}\{w - \eta \cdot 1_F \notin \Omega\}.
\]

The second term is assessed as before. For the first term, let \( \psi(w) := -\varphi(w) \) and put \( A := \{ w : \psi(w) \leq -b \}, A^0 := \{ w : w - \eta \cdot 1_F \in \Omega \text{ and } \varphi(w - \eta \cdot 1_F) \leq -b \} \) and let \( w \in A' := \{ \psi(w) \in I \text{ and } w - \eta \cdot 1_F \in \Omega\} \). Then by (4-9),

\[
\varphi(w - \eta \cdot 1_F) \geq \varphi(w) - \gamma(1 - e^{\xi \eta}) = \varphi(w) + \varepsilon \geq a + \varepsilon = b,
\]

hence \( \psi(w - \eta \cdot 1_F) \leq -b \) and \( w \in A^0 \). Moreover, \( \varphi(w) \in I \) implies \( \varphi(w) < b \), i.e. \( \psi(w) > -b \) and thus \( w \notin A \). Hence, \( A' \subseteq A^0 \setminus A \) and the claim follows from Lemma 4.1. \( \square \)

**Proof of Theorem 2.3.** Let \( A(\omega) = -H(\omega) \) and \( I' := (-E_2, -E_1) \). Then \( \text{tr}[\chi_I(A(\omega))] = \text{tr}[\chi_{I'}(A(\omega))] \). Moreover, if \( r_\alpha(\omega) = -f_\alpha(\omega) = -a(u) + \sum_{\alpha \in I_F} (q - \omega_\alpha) \cdot b_\alpha(u) \), then using min-max for \( H(\omega) \), we obtain the formula

\[
\mu_n(\omega) = \inf_{\varphi_1, \ldots, \varphi_{n-1}} \sup_{u \in D, \|u\| = 1} \left( r_\alpha(\omega) \right)
\]
for the decreasing set $\mu_1(\omega) \geq \mu_2(\omega) \geq \ldots$ of eigenvalues of $A(\omega)$ (here $\mu_j(\omega) = -\lambda_j(\omega)$). Since $b_\alpha(u) \geq 0$ for any $u$, each $\mu_\alpha(\omega)$ is monotone and only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$ by (4-10), hence each is $\bar{\mathcal{F}}_2$-measurable by Lemma 5.2. Thus, as in the proof of Theorem 2.3 $\operatorname{tr}[\chi_F(A(\omega))]$ is $\bar{\mathcal{F}}_2$-measurable and we may integrate to get

$$(4-11) \quad \mathbb{E}\{\operatorname{tr}[\chi_F(A(\omega))]\} = \sum_{n \leq K} \mathbb{P}\{\mu_n(\omega) \in I'\} = \sum_{n \leq K} \bar{\mathbb{P}}\{\mu_n(T_2(v)) \in I'\},$$

where, using the notations of Lemma 4.3 $T_2 : \tilde{\Omega} \rightarrow \Omega$ is given by $T_2 : (v_\alpha) \mapsto (q - e^{v_\alpha})$, and we applied (4-7) to $g(v) := \chi_F(\mu_2(T_2(v)))$, noting that $(T_2 \circ T)(\omega) = \omega$.

Suppose now that (2-1) holds with $\zeta > 0$ and fix $u \in \mathcal{D}$. Since $r_u \circ T_2(v) = -a(u) + \sum_{\alpha \in \mathcal{I}_F} e^{v_\alpha - t} b_\alpha(u)$, given $v \in \tilde{\Omega}$ and $t \geq 0$ such that $v + t \cdot 1_F \in \Omega$, we have

$$(r_u \circ T_2)(v + t \cdot 1_F) = -a(u) + \sum_{\alpha \in \mathcal{I}_F} e^{v_\alpha + t} b_\alpha(u) = -a(u) + e^{\xi t} \sum_{\alpha \in \mathcal{I}_F} e^{v_\alpha - t} b_\alpha(u) \geq e^{\xi t} (r_u \circ T_2)(v)$$

since $-a(u) \geq -e^{\xi t} a(u)$.

Thus, if $\nu_n(v) := \mu_n(T_2(v))$, we get by (4-10)

$$\nu_n(v + t \cdot 1_F) \geq e^{\xi t} \nu_n(v).$$

Now note that if $\nu_n(v) \in I'$, then $\nu_n(v) \geq -E_2 = |E_2| > 0$. Hence,

$$(\nu_n(v + t \cdot 1_F) - \nu_n(v))\chi_{I'}(\nu_n(v)) \geq (e^{\xi t} \nu_n(v) - \nu_n(v))\chi_{I'}(\nu_n(v)) \geq (e^{\xi t} - 1)|E_2|\chi_{I'}(\nu_n(v)).$$

As $\zeta > 0$, $\nu_n(v)$ is monotone increasing in $v$, so using Lemma 4.4 we get

$$\bar{\mathbb{P}}\{\nu_n(v) \in I'\} \leq 2 \cdot |\mathcal{I}_F| \cdot s_F \left( \frac{1}{\zeta} \ln (1 + \frac{|I'|}{|E_2|}) \right) \leq 2 \cdot |\mathcal{I}_F| \cdot s_F \left( \frac{1}{\zeta} \ln (1 + \frac{|I'|}{|E_2|}) \right),$$

where we applied Lemma 4.3 with $\epsilon := \frac{1}{\zeta} \ln (1 + \frac{|I'|}{|E_2|})$. Using (4-11), the proof is complete for $\zeta > 0$. Now suppose that $\zeta < 0$ and put $\theta := -\zeta > 0$. Then

$$(r_u \circ T_2)(v) = -a(u) + \sum_{\alpha \in \mathcal{I}_F} e^{-\theta v_\alpha} b_\alpha(u) = -a(u) + e^{-\theta t} \sum_{\alpha \in \mathcal{I}_F} e^{-\theta(v_\alpha - t)} b_\alpha(u) \leq e^{-\theta t} (r_u \circ T_2)(v - t \cdot 1_F)$$

for any $t \geq 0$ such that $v - t \cdot 1_F \in \tilde{\Omega}$, since $-a(u) \leq -e^{-\theta t} a(u)$. Hence,

$$\nu_n(v) \leq e^{-\theta t} \nu_n(v - t \cdot 1_F),$$

and thus, noting that $(1 - e^{\theta t}) \leq 0$ we get

$$(\nu_n(v) - \nu_n(v - t \cdot 1_F))\chi_{I'}(\nu_n(v)) \leq (\nu_n(v) - e^{\theta t} \nu_n(v))\chi_{I'}(\nu_n(v)) \leq (1 - e^{\theta t})|E_2|\chi_{I'}(\nu_n(v)).$$

Furthermore, $\nu_n(v)$ is monotone decreasing. The claim of Theorem 2.4 for $\zeta < 0$ now follows as before using Lemma 4.4. \hfill \square
5. Appendix

5.1. Spectra of some Schrödinger operators. Let $G \subset \mathbb{Z}^d$ be non-empty, $B \subseteq \mathbb{R}$ a Borel set and consider the probability space $(\Omega, \mathcal{P})$, where $\Omega = \mathcal{B}^G$ and $\mathcal{P} = \bigotimes_{\alpha \in G} \mu$, for some probability measure $\mu$ on $\mathbb{R}$ with $\mu \subseteq B$. Define

$$H^\omega = H^0 + V^\omega \text{ on } \ell^2(\mathbb{Z}^d), \text{ where}$$

$$H^0 = -\Delta + V^0, \quad V^\omega = \sum_{\alpha \in G} \omega_\alpha \delta_\alpha,$$

$$H_\omega = H_0 + V_\omega \text{ on } L^2(\mathbb{R}^d), \text{ where}$$

$$H_0 = -\Delta + V_0, \quad V_\omega = \sum_{\alpha \in G} \omega_\alpha \chi_\alpha.$$

Here $\delta_\alpha$ and $\chi_\alpha$ are the characteristic functions of $\{\alpha\}$ and $[\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]^d$ respectively and $V^0, V_0$ are $\mathbb{Z}^d$-periodic bounded real potentials. We denote points in $\mathbb{R}^d$ by $(x^1, \ldots, x^d)$.

We now suppose that $G$ contains a half-space of $\mathbb{Z}^d$, i.e., there exists $r \in \mathbb{Z}$ and $i \in \{1, \ldots, d\}$ such that $(x^1, \ldots, x^d) \in G$ whenever $x^i > r$. Examples are half-spaces of $\mathbb{Z}^d$, and sets with a finite number of holes, i.e., with $\mathbb{Z}^d \setminus G$ finite. We can actually consider more general sets like quarter-spaces or rotated half-spaces. The only thing we need is that $G$ should contain arbitrarily large cubes of $\mathbb{Z}^d$. This excludes $(\mathbb{Z}^d)^d$ and thus excludes Delone sets. On the other hand, half-spaces are not Delone sets either since we allow for arbitrarily large cubes with no points of $G$. So the sets we consider here are neither a special case nor a generalization of Delone sets.

**Lemma 5.1.** If $G$ contains a half-space of $\mathbb{Z}^d$, then $\sigma(H^\omega) \supseteq \sigma(H^0) + \text{supp } \mu$ and $\sigma(H_\omega) \supseteq \sigma(H_0) + \text{supp } \mu$ almost surely.

**Proof.** We only prove the claim for $H_\omega$; the proof is identical for $H^\omega$. All the arguments actually go back to [34], [22]: one simply needs to choose $\Omega_k^{\lambda,q}(n)$ carefully below.

Assume $(x^1, \ldots, x^d) \in G$ whenever $x^i > r$. Let $E = \lambda + q \in \sigma(H_0) + \text{supp } \mu$. By Weyl’s criterion [27] Theorem 7.22], we may find $f_k \in C^\infty_c(\mathbb{R}^d)$, $\|f_k\| = 1$ such that $\|(H_0 - \lambda)f_k\| \to 0$ as $k \to \infty$. Choose $l_k = l_k(\lambda) \in \mathbb{N}^*$ such that $\supp f_k \subset \Lambda_{l_k}(0)$, put $I_k^q := [q - \frac{1}{k}, q + \frac{1}{k}]$ and consider the event

$$\Omega_k^{\lambda,q}(n) := \{\omega \in \Omega : \omega_\alpha \in I_k^q \quad \forall \alpha \in \Lambda_{l_k}(x_{n,k})\},$$

where $x_{n,k} := (3nl_k + r)e^i$ and $e^i \in \mathbb{Z}^d$ has 1 in the $i$th coordinate and 0 otherwise. First note that $\Lambda_{l_k}(x_{n,k}) \cap G = \Lambda_{l_k}(x_{n,k})$, so that the above event is well defined. Moreover, $\Lambda_{l_k}(x_{n,k}) \cap \Lambda_{l_k}(x_{m,k}) = \emptyset$ for $n \neq m$, so the events $\{\Omega_k^{\lambda,q}(n)\}_{n \in \mathbb{N}}$ are independent and $\mathbb{P}(\Omega_k^{\lambda,q}(n)) = \mu(I_k^q)^{\lambda \Lambda_{l_k}}$ is the same for all $n$ and strictly positive since $q \in \text{supp } \mu$. It follows by Borel-Cantelli lemma II that if $\Omega_k^{\lambda,q} := \cap_{m \geq 1} \cup_{n \geq m} \Omega_k^{\lambda,q}(n)$, then $\mathbb{P}(\Omega_k^{\lambda,q}) = 1$.

Let $\Omega_k^{\lambda,q} := \cap_{k \in \mathbb{N}} \Omega_k^{\lambda,q}$, then $\mathbb{P}(\Omega_k^{\lambda,q}) = 1$.

Now fix $\omega \in \Omega_k^{\lambda,q}$ and let $k \in \mathbb{N}^*$. Then $\omega \in \Omega_k^{\lambda,q}$, so we may find $n \in \mathbb{N}^*$ such that $\omega \in \Omega_k^{\lambda,q}(n)$. But

$$\|(H_\omega - E)f_k(\cdot - x_{n,k})\| \leq \|(H_0 - \lambda)f_k(\cdot - x_{n,k})\| + \|(V_\omega - q)f_k(\cdot - x_{n,k})\|.$$

Since $V_0$ is periodic, $\|(H_0 - \lambda)f_k(\cdot - x_{n,k})\| = \|(H_0 - \lambda)f_k\| \to 0$. Moreover $\omega \in \Omega_k^{\lambda,q}(n)$, so $\omega_\alpha \in I_k^q$ for all $\alpha \in \Lambda_{l_k}(x_{n,k})$. Recalling that $\Lambda_{l_k}(x_{n,k}) \cap G = \Lambda_{l_k}(x_{n,k})$, we get

$$\|(V_\omega - q)f_k(\cdot - x_{n,k})\|^2 = \sum_{\alpha \in \Lambda_{l_k}(x_{n,k})} (\omega_\alpha - q)^2 \|\chi_\alpha f_k\|^2 \leq \frac{1}{k^2}\|f_k\|^2 \to 0.$$

Hence $f_k$ is a Weyl sequence for $E$. We thus showed that for any $\omega \in \Omega_k^{\lambda,q}$ we have $\lambda + q \in \sigma(H_\omega)$. Let $\Omega_0 := \bigcap_{k \in \mathbb{N}} \cap_{\alpha \in \Lambda_{l_k}(0)} Q_\omega \mathbb{R}^{\lambda,l_k} \supseteq \Omega_k^{\lambda,q}$. Then $\mathbb{P}(\Omega_0) = 1$ and for any $\omega \in \Omega_0$ we have $\sigma(H_\omega) \supseteq \sigma(H_0) \cap Q + \text{supp } \mu \cap Q$. Since $\sigma(H_\omega)$ is closed, the proof is complete. □
5.2. Technical details. We give here the details of some claims we made in Sections 1 and 4. Let \( \mu \) be a probability measure on \( \mathbb{R} \). To prove (1.2), let \( E \in \mathbb{R} \) and \( E_k := \left\lceil \frac{10^k E}{10^k} \right\rceil \), where \( \lfloor x \rfloor \) is the integer part of \( x \). Then \( E_k \not\subset E \) and \( c < E + \varepsilon \) if \( c < E_k + \varepsilon \) for some \( k \). Hence
\[
\mu(E, E + \varepsilon) = \mu(\cup_k (E, E_k + \varepsilon)) = \lim_{k \to \infty} \mu(E, E_k + \varepsilon) \\
\leq \lim_{k \to \infty} \mu(E, E_k + \varepsilon) \leq \sup_{F \in \mathcal{Q}} \mu(F, F + \varepsilon).
\]
Thus, \( \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) \leq \sup_{F \in \mathcal{Q}} \mu(F, F + \varepsilon) \). This proves (1.2).

Suppose \( \mathbb{P} = \otimes \mu_n \) for some probability measures \( \mu_n \) on \( \mathbb{R} \). Then given \( A \in \mathcal{F} \), we have \( \mathbb{P}(A) = \int_{\mathcal{Y}} \mu_n(A \omega_n) d \mathbb{P}_n(\omega_n) \), so by [16, Corollary 10.4.15], \( \mu_{\omega_n} = \mu_n \mathbb{P}_{\omega_n} \)-a.s., so \( s_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{E}_F} \sup_{E \in \mathbb{R}} \mu_{\omega_n}(E, E + \varepsilon) \) using (1.2). Next, note that
\[
\mu(E, E + \varepsilon) = \mu(\cup_k (E + \frac{1}{k}, E + \varepsilon)) = \lim_{k \to \infty} \mu(E + \frac{1}{k}, E + \varepsilon) \\
\leq \lim_{k \to \infty} \mu(E + \frac{1}{k}, E + \varepsilon + \frac{1}{k}) \leq \sup_{E \in \mathbb{R}} \mu(F, F + \varepsilon),
\]
so \( \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) \leq \sup_{F \in \mathcal{Q}} \mu(F, F + \varepsilon) \) and this proves equality. Similarly, one checks that \( \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) \leq \sup_{F \in \mathcal{R}} \mu(F, F + \varepsilon) \), which proves (4.2).

We finally prove the following. Here \( \Omega = \mathcal{B}^\mathbb{R} \) with \( \mathcal{B} \subset \mathcal{R} \) a Borel set and \( \mathcal{I} \) is countable.

**Lemma 5.2.** If \( \mathbb{P} \) has no atoms, then any monotone \( \varphi : \Omega \to \mathbb{R} \) which depends on finitely many \( \omega_n \) is \( \mathcal{F}_{\mathbb{P}} \)-measurable, where \( \mathcal{F}_{\mathbb{P}} \) is the \( \mathbb{P} \)-completion of \( \mathcal{F} \).

**Proof.** Suppose \( \varphi \) is monotone increasing and only depends on \( (\omega_n)_{\alpha \in \mathcal{I}_m} \). For notational simplicity, assume \( \mathcal{I}_m = \{1, \ldots, m\} \). Put \( \mathcal{I}_k := \{1, \ldots, k\} \) for \( 1 \leq k \leq m \) and let \( \mathcal{F}_k \) be the \( \sigma \)-algebra generated by \( \{\omega_\alpha : \alpha \in \mathcal{I}_k\} \cup \mathcal{N}_k(\mathbb{P}) \), where \( \mathcal{N}_k(\mathbb{P}) := \{M \subset \mathcal{B}^{\mathbb{R}} : \mathbb{P}^*(M \times \mathcal{B}^{\mathbb{R}}) = 0\} \). Here \( \mathbb{P}^* \) is the outer measure defined by \( \mathbb{P} \) and \( \mathcal{I}_k = \mathcal{I} \setminus \mathcal{I}_k \). Then \( A \notin \mathcal{F}_{\mathbb{P}} \) implies \( A \times \mathcal{B}^{\mathbb{R}} \notin \mathcal{F}_{\mathbb{P}} \).

Since \( \varphi : \mathcal{B}^\mathbb{R} \to \mathbb{R} \) only depends on \( (\omega_\alpha)_{\alpha \in \mathcal{I}_m} \), then given \( a \in \mathbb{R} \), \( \{\omega : \varphi(\omega) \geq a\} = A' \times \mathcal{B}^{\mathbb{R}} \) for some \( A' \subseteq \mathcal{B}^\mathbb{R} \). So to show that \( \varphi \) is measurable, it suffices to show that \( A' \notin \mathcal{F}_{\mathbb{P}} \). But if we define \( \varphi_0 : \mathcal{B}^{\mathbb{R}} \to \mathbb{R} \) by \( \varphi_0(\omega^m) := \varphi(\omega^m, 0) \) for \( \omega^m = (\omega_\alpha)_{\alpha \in \mathcal{I}_m} \), then \( \varphi_0 \) is increasing and \( \{\varphi^m : \varphi_0(\omega^m) \geq a\} = A' \). Thus, it suffices to show that any monotone increasing \( f : \mathcal{B}^{\mathbb{R}} \to \mathbb{R} \) is \( \mathcal{F}_{\mathbb{P}} \)-measurable. For this, we proceed by induction, adapting an argument of Nathaniel Eldredge showing that monotone functions on \( \mathbb{R}^m \) are Lebesgue-measurable, following [10, Theorem 4.4].

For \( k = 1 \) the assertion is clear: if \( f : \mathcal{B} \to \mathbb{R} \) is increasing and \( A = \{t : f(t) \geq a\} \), then \( f = 0 \) or \( A = \emptyset \). Thus, \( A \notin \mathcal{B} \notin \mathcal{F}_1 \).

Now suppose \( f : \mathcal{B}^{\mathbb{R}} \to \mathbb{R} \) is increasing, fix \( a \in \mathbb{R} \) and define \( g : \mathcal{B}^{\mathbb{R}} \to \mathbb{R} \) by \( g(\omega^k) = \inf\{t \in \mathcal{B} : f(\omega^k, t) \geq a\} \). Then \( g \) is monotone decreasing, hence \( \mathcal{F}_{\mathbb{P}} \)-measurable by the induction hypothesis. So by [11, Proposition 3.3.4], we have \( E := \{(\omega^k, \omega_{k+1}) : g(\omega^k) < \omega_{k+1} \} \in \mathcal{F}_k \otimes \mathcal{B} \) and \( G := \{\omega_{k+1} : g(\omega^k) = \omega_{k+1} \} \in \mathcal{F}_k \otimes \mathcal{B} \). Moreover, for any \( \omega^k \in \mathcal{B}^{\mathbb{R}} \) and \( y \in \mathcal{B}^{\mathbb{R}} \), we have \( G_{\omega^k, y} := \{\omega_{k+1} : (\omega^k, \omega_{k+1}, y) \in \mathcal{F} \times \mathcal{F}_{\mathbb{P}} \} \). We may find \( F \subseteq \mathcal{F} \times \mathcal{F}_{\mathbb{P}} \) such that \( F \in \mathcal{F} \) and \( \mathbb{P}(F) \leq \mathbb{P}(F) \). The section \( F_{\omega^k, y} \) of such \( F \) is either a singleton or empty. Thus,
\[
\mathbb{P}(F \times \mathcal{B}^{\mathbb{R}}) = \mathbb{P}(F) = \mathbb{E}_{Y_{k+1}}(\sup_{E \in \mathbb{R}} \mu_{\omega_{k+1}}(E, E + \varepsilon)) \leq \mathbb{E}_{Y_{k+1}}\left\{ \sup_{E \in \mathbb{R}} \mu_{\omega_{k+1}}(E, E + \varepsilon) \right\}
\]
for any \( \varepsilon > 0 \). Since \( s(\mathbb{P}, \varepsilon) = 0 \), it follows that \( \mathbb{P}(G \times \mathcal{B}^{\mathbb{R}}) = 0 \).

Finally, if \( M = M' \times \mathcal{B} \) with \( M' \in \mathcal{N}_k(\mathbb{P}) \) and \( B \subseteq \mathcal{B} \), then \( \mathbb{P}^*(M \times \mathcal{B}^{\mathbb{R}}) = 0 \), hence \( \mathcal{F}_k \otimes \mathcal{B} \subseteq \mathcal{F}_{k+1} \) and \( E, G \in \mathcal{F}_{k+1} \). But if \( A = \{(\omega^k, \omega_{k+1}) : f(\omega^k, \omega_{k+1}) \geq a\} \), then \( E \subseteq A \) and \( \{A \setminus E\} \subseteq G \). Since \( E \in \mathcal{F}_{k+1} \) and \( \mathbb{P}^*(A \setminus E) \leq \mathcal{F}_{k+1} \times \mathcal{B}^{\mathbb{R}} = \mathbb{P}(G \times \mathcal{B}^{\mathbb{R}}) = 0 \), the proof is complete. \( \square \)
It is worthwhile to note that the completeness of \((\Omega, \mathcal{F}, \mathbb{P})\) is not only sufficient for the above argument to work, but also necessary. Indeed, there exist monotone increasing maps \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) which are not Borel-measurable; see \([40]\).

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**REFERENCES**

1. V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007.
2. A. Boutet de Monvel, V. Chulaevsky, P. Stollmann, and Y. Suhov, *Wegner-type bounds for a multi-particle continuous Anderson model with an alloy-type external potential*, J. Stat. Phys. **138** (2010), no. 4-5, 553–566.
3. A. Boutet de Monvel, D. Lenz, and P. Stollmann, *An uncertainty principle, Wegner estimates and localization near fluctuation boundaries*, Math. Z. **269** (2011), no. 3-4, 663–670.
4. A. Boutet de Monvel, S. Naboko, P. Stollmann, and G. Stolz, *Localization near fluctuation boundaries via fractional moments and applications*, J. Anal. Math. **100** (2006), 83–116.
5. V. Chulaevsky, *A Wegner-type estimate for correlated potentials*, Math. Phys. Anal. Geom. **11** (2008), no. 2, 117–129.
6. V. Chulaevsky and Y. Suhov, *Wegner Bounds for a Two-Particle Tight Binding Model*, Comm. Math. Phys. **283** (2008), 479–489.
7. J-M. Combes, P. Hislop, and F. Klopp, *Hölder continuity of the integrated density of states for some random operators at all energies*, Int. Math. Res. Not. (2003), no. 4, 179–209.
8. J-M. Combes, *An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators*, Duke Math. J. **140** (2007), no. 3, 469–498.
9. J. M. Combes, P. D. Hislop, F. Klopp, and S. Nakamura, *The Wegner estimate and the integrated density of states for some random operators*, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), no. 1, 31–53.
10. R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002.
11. A. Elgart and A. Klein, *Ground state energy of trimmed discrete Schrödinger operators and localization for trimmed anderson models*, preprint arXiv: 1301.5268v1 (2013).
12. A. Elgart, H. Krüger, M. Tautenhahn, and I. Veselić, *Discrete Schrödinger operators with random alloy-type potential*, Spectral analysis of quantum Hamiltonians, Oper. Theory Adv. Appl., vol. 224, Birkhäuser, 2012, pp. 107–131.
13. A. Elgart, M. Shamis, and S. Sodin, *Localisation for non-monotone Schrödinger operators*, preprint arXiv: 1201.2211 (2012).
14. A. Elgart, M. Tautenhahn, and I. Veselić, *Anderson localization for a class of models with a sign-indefinite single-site potential via fractional moment method*, Ann. Henri Poincaré **12** (2011), no. 8, 1571–1599.
15. P. Exner, M. Helm, and P. Stollmann, *Localization on a Quantum Graph with Random Potential on the Edges*, Rev. Math. Phys. **19** (2007), 923–939.
16. B. T. Graham and G. R. Grimmett, *Influence and sharp-threshold theorems for monotonic measures*, Ann. Probab. **34** (2006), no. 5, 1726–1745.
17. P. Hislop and F. Klopp, *The integrated density of states for some random operators with nonsign definite potentials*, J. Funct. Anal. **195** (2002), no. 1, 12–47.
18. D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, and I. Veselić, *Bounds on the spectral shift function and the density of states*, Comm. Math. Phys. **262** (2006), no. 2, 489–503.
19. W. Kirsch, *An invitation to random Schrödinger operators*, Random Schrödinger operators, Panor. Synthèses, vol. 25, Soc. Math. France, Paris, 2008, pp. 1–119.
20. W. Kirsch, *A Wegner estimate for multi-particle random Hamiltonians*, Zh. Mat. Fiz. Anal. Geom. **4** (2008), no. 1, 121–127, 203.
21. W. Kirsch and F. Martinelli, *On the ergodic properties of the spectrum of general random operators*, J. Reine Angew. Math. **334** (1982), 141–156.
22. W. Kirsch, P. Stollmann, and G. Stolz, *Localization for random perturbations of periodic Schrödinger operators*, Random Oper. Stoch. Eq. **6** (1998), no. 3, 241–268.
23. W. Kirsch and I. Veselić, Wegner estimate for sparse and other generalized alloy type potentials, Proc. Indian Acad. Sci. Math. Sci. 112 (2002), no. 1, 131–146.
24. Y. Kitagaki, Wegner estimates for some random operators with Anderson-type surface potentials, Math. Phys. Anal. Geom. 13 (2010), no. 1, 47–67.
25. Y. Kitagaki, Generalized eigenvalue-counting estimates for some random acoustic operators, Kyoto J. Math. 51 (2011), no. 2, 439–465.
26. A. Klein, Unique continuation principle for spectral projections of Schrödinger operators and optimal weigner estimates for non-ergodic random Schrödinger operators, preprint arXiv: 1209.4863, to appear in Commun. Math. Phys. (2013).
27. A. Klein and S.T. Nguyen, The bootstrap multiscale analysis for the multi-particle Anderson model, J. Stat. Phys. 151 (2013), no. 5, 938–973.
28. F. Klopp, Localization for some continuous random Schrödinger operators, Comm. Math. Phys. 167 (1995), no. 3, 553–569.
29. F. Klopp and K. Pankrashkin, Localization on Quantum Graphs with Random Vertex Couplings, J. Stat. Phys. 151 (2013), no. 5, 938–973.
30. F. Klopp, Localization for some continuous random Schrödinger operators, Comm. Math. Phys. 167 (1995), no. 3, 553–569.
31. F. Klopp and H. Zenk, The integrated density of states for an interacting multiparticle homogeneous model and applications to the Anderson model, Adv. Math. Phys. (2009), Art. ID 679827, 15.
32. H. Krüger, Localization for random operators with non-monotone potentials with exponentially decaying correlations, Ann. Henri Poincaré 13 (2012), no. 3, 543–598.
33. P. Kuchment, Quantum graphs I. Some basic structures, Waves Random Media 14 (2004), no. 1, S107–S128.
34. H. Kunz and B. Souillard, Sur le spectre des opérateurs aux différences finies aléatoires, Comm. Math. Phys. 78 (1980), no. 2, 201–246.
35. D. Lenz, N. Peyerimhoff, O. Post, and I. Veselić, Continuity of the integrated density of states on random length metric graphs, Math. Phys. Anal. Geom. 12 (2009), no. 3, 219–254.
36. N. Peyerimhoff, M. Tautenhahn, and I. Veselić, Wegner estimates for alloy-type models with sign-changing exponentially decaying single-site potentials, TU Chemnitz Preprint (2011).
37. M. Reed and B. Simon, Methods of modern mathematical physics. IV. Analysis of operators, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
38. C. Rojas-Molina, The Anderson model with missing sites, preprint arXiv: 1302.3640 (2013).
39. C. Rojas-Molina and I. Veselić, Scale-free unique continuation estimates and applications to random Schrödinger operators, Comm. Math. Phys. 320 (2013), no. 1, 245–274.
40. M. Sabri, Étude de la localisation pour des systèmes désordonnés sur un graphe quantique, PhD Thesis, in preparation.
41. B. Simon, Lifschitz tails for the Anderson model, J. Statist. Phys. 38 (1985), no. 1-2, 65–76.
42. P. Stollmann, Wegner estimates and localization for continuum Anderson models with some singular distributions, Arch. Math. (Basel) 75 (2000), no. 4, 307–311.
43. P. Stollmann, From uncertainty principles to Wegner estimates, Math. Phys. Anal. Geom. 13 (2010), no. 2, 145–157.
44. I. Veselić, Wegner estimate for discrete alloy-type models, Ann. Henri Poincaré 11 (2010), no. 5, 991–1005.
45. I. Veselić, Wegner estimates for sign-changing single site potentials, Math. Phys. Anal. Geom. 13 (2010), no. 4, 299–313.
46. F. Wegner, Bounds on the density of states in disordered systems, Z. Phys. B 44 (1981), no. 1-2, 9–15.
47. J. Weidmann, Linear operators in Hilbert spaces, Graduate Texts in Mathematics, vol. 68, Springer-Verlag, New York, 1980.