Quasi-Hopf twist and elliptic Nekrasov factor

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\textbf{Abstract:} We investigate the quasi-Hopf twist of the quantum toroidal algebra of $\mathfrak{gl}_1$ as an elliptic deformation. Under the quasi-Hopf twist the underlying algebra remains the same, but the coproduct is deformed, where the twist parameter $p$ is identified as the elliptic modulus. Computing the quasi-Hopf twist of the $R$ matrix, we uncover the relation to the elliptic lift of the Nekrasov factor for instanton counting of the quiver gauge theories on $\mathbb{R}^4 \times T^2$. The same $R$ matrix also appears in the commutation relation of the intertwiners, which implies an elliptic quantum KZ equation for the trace of intertwiners. We also show that it allows a solution which is factorized into the elliptic Nekrasov factors and the triple elliptic gamma function.

\textbf{Keywords:} Conformal and W Symmetry, Conformal Field Theory, Quantum Groups, Supersymmetric Gauge Theory

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1 Introduction

The quantum toroidal algebra of type \( \mathfrak{gl}_1 \) (a.k.a. Ding-Iohara-Miki algebra [15, 46], the elliptic Hall algebra [11, 65, 66] and others) is the fundamental symmetry which controls five dimensional (or \( K \) theoretic lift of) Nekrasov partition function [51, 52, 54, 56]. Its manifestation is the celebrated AGT relation [1, 10, 48, 70] to the conformal block of deformed Virasoro and \( W \) algebras. The fact that the quantum toroidal algebra of type \( \mathfrak{gl}_1 \) has several names shows its ubiquity and broad applications to many areas in mathematics and physics. From the viewpoint of representation theory one of the advantages of the quantum toroidal algebra is that it has a coproduct which allows us to take the tensor product of the representations. In fact representations of the deformed Virasoro and \( W \) algebras by free bosons are derived from the tensor product of the Fock representations of the quantum toroidal algebra [20].
On the top of the hierarchy of supersymmetric gauge theories without coupling to the gravity is the six dimensional theory. Recall that the BPS state counting of five dimensional theories on $\mathbb{R}^4 \times S^1$ is naturally related to the supersymmetric quantum mechanics on the instanton moduli space, where $S^1$ is identified with the (periodic) time direction. In the same manner the partition function of BPS state counting of the six dimensional theories on $\mathbb{R}^4 \times T^2$ can be identified with the elliptic genus of the instanton moduli space [32]. On the algebraic side we thus expect the appearance of the elliptic algebra and elliptic integrable systems [57]. In fact an elliptic lift of Ding-Iohara-Miki (DIM) algebra together with its connection to the six dimensional Nekrasov function and the elliptic Virasoro algebra has been already discussed in [34, 58].

Most of the existing literatures employ a version of elliptic DIM algebra that was first introduced by Y. Saito [64] for the purpose of describing an elliptic version of Macdonald polynomials. This kind of elliptic algebra introduces a second set of deformed bosons, which is the same as the Clavelli-Shapiro method in old string theory [14]. The method allows us to rewrite the trace of the product of vertex operators (intertwiners) that appears in the one-loop diagrams in string theory by the vacuum expectation value with respect to the Fock vacuum of doubled boson system [29, 39]. This also reminds us of the method of thermo field dynamics in statistical mechanics [35, 61, 67]. However, there is another construction of an elliptic lift of DIM algebra, which is relatively unexplored [19]. This construction does not employ an additional boson, but makes use of the quasi-Hopf twist [36, 37]. In this paper we investigate the quasi-Hopf twist of DIM algebra. The original DIM algebra has two parameters $(q, t)$ and the quasi-Hopf twist introduces a third parameter $p$. As we have emphasized the quantum toroidal algebra has a coproduct. The quasi-Hopf twist deforms the coproduct by what is called twistor $\mathcal{F}(p)$. The generating currents of the algebra are also twisted accordingly. It turns out that the deformation parameter $p$ is identified with an elliptic parameter. In the commutation relations of the twisted currents there appears the theta function whose elliptic norm is $p$.

As a quantum group the quantum toroidal algebra has an universal $R$ matrix [21]. Since the quasi-Hopf twist deforms the coproduct, it also changes the universal $R$ matrix. As has been shown in [6–8], the Cartan part of the universal $R$ matrix is closely related to the Nekrasov factor through the generalized Knizhnik-Zamolodchikov (KZ) equation for the correlation function of intertwiners of DIM algebra. Just like the Wick theorem for the free fields, the solutions to the generalized KZ equation are factorized into a product of two point functions. In accord with the AGT correspondence the Nekrasov factor plays the role of the two point function [7, 13]. The elliptic Nekrasov factor is given by [5, 58, 71]:

$$
\begin{align*}
N_{\lambda \mu} (u|q, t, p) &= \prod_{i,j=1}^{\infty} \frac{\Gamma \left( uq^{\lambda_j - \mu_i} t^{i-j}; q, p \right)}{\Gamma \left( ut^{i-j}; q, p \right)} \cdot \frac{\Gamma \left( ut^{i-j+1}; q, p \right)}{\Gamma \left( uq^{\lambda_j - \mu_i} t^{i-j+1}; q, p \right)} \\
&= \prod_{\Box \in \lambda} \theta_p \left( uq^{a_{\lambda}(\Box)} t^{(\Box)} a_{\mu}(\Box) + 1 \right) \prod_{\Box \in \mu} \theta_p \left( uq^{-a_{\lambda}(\Box)} t^{-a_{\mu}(\Box)} \right),
\end{align*}
$$

(1.1)

However, see [41] and [42].
where \((\lambda, \mu)\) is a pair of partitions and \((q, t) = (e^{i\epsilon_1}, e^{-i\epsilon_2})\) is the \(\Omega\) background of Nekrasov \([54]\). \(\Gamma(u; q, p)\) is the elliptic gamma function. In this paper we will show that the universal \(R\) matrix after the quasi-Hopf twist is related to the elliptic Nekrasov factor by the relation;
\[
q^{\mid\lambda\mid + \mid\mu\mid}N_{\lambda\mu}(q^{-2}z|q, t, p) = \overline{R}_{\lambda\mu}(z; p)N_{\lambda\mu}(z|q, t, p),
\]
(1.2)
where \(q = \sqrt{t/q}\) and \(\overline{R}_{\lambda\mu}\) is the normalized \(R\) matrix of the vertical Fock representation. With appropriate specialization of the spectral parameter \(u\), the elliptic Nekrasov factor gives the contribution of the bifundamental matter hypermultiplet to the instanton partition function of the lift of \(N = 2\) quiver gauge theory to \(\mathbb{R}^4 \times T^2\), where the elliptic modulus of the two dimensional torus \(T^2\) is identified with \(p\). Hence, contrary to the case of \([34, 58]\), we do not have to introduce an additional boson (Heisenberg algebra) to obtain the elliptic Nekrasov factor (1.1). Only the quasi-Hopf twist suffices. This is one of the main messages of the present paper. The same normalized \(R\) matrix also appears in the commutation relations of the intertwiners \(\Psi_{\lambda}(v; p)\) and the dual intertwiners \(\Psi_{\lambda}^*(v; p)\);
\[
\Psi_{\lambda}(v; p)\Psi_{\mu}(w; p) = \frac{G_2 \left(\frac{v}{w}; p, q, t^{-1}\right)}{G_2 \left(\frac{q^{-2}z}{w}; p, q, t^{-1}\right)} \cdot \overline{R}_{\lambda\mu} \left(\frac{v}{w}; p\right)\Psi_{\mu}(w; p)\Psi_{\lambda}(v; p),
\]
(1.3)
\[
\Psi_{\lambda}^*(v; p)\Psi_{\mu}^*(w; p) = \frac{G_2 \left(\frac{v}{w}; p, q, t^{-1}\right)}{G_2 \left(\frac{v}{w}; p, q, t^{-1}\right)} \cdot \overline{R}_{\lambda\mu} \left(\frac{v}{w}; p\right)^{-1}\Psi_{\mu}^*(w; p)\Psi_{\lambda}^*(v; p),
\]
(1.4)
where \(p_*=pq^{-2}\) and \(G_2(u; p, q, t^{-1})\) is the double elliptic gamma function. The ratio of the double elliptic gamma functions, which is independent of \(\lambda\) and \(\mu\), comes from the vacuum contribution. Note that the elliptic parameter in \(G_2\) and \(\overline{R}_{\lambda\mu}\) for the commutation relation of \(\Psi_{\lambda}(v; p)\) is not \(p\), but \(p_*\).

Based on the braiding relations (1.3), (1.4) and the cyclic property of the trace, we can derive a difference equation for the trace of the product of intertwiners,
\[
\text{Tr} \left[ Q^{d_1}Q^{d_2}\Psi_{\mu_1}(w_1; p)\cdots\Psi_{\mu_n}(w_n; p)\Psi_{\lambda_1}(z_1; p)\cdots\Psi_{\lambda_n}(z_n; p) \right],
\]
(1.5)
where \((d_1, d_2)\) is a pair of grading operators of DIM algebra. The shift parameter of the difference equation is \(Q\) and the \(Q\)-shift produces a product of the \(R\) matrices \(\overline{R}_{\lambda\mu}\) (see section 6 for explicit forms). Thus, we can regard the \(Q\)-difference equation as a generalization of q-KZB equation for a genus one conformal blocks \([22, 23, 25]\). As in the case of \([7, 13]\), there is a solution whose building blocks are the Nekrasov factors. A typical example of such building blocks looks like
\[
G_3 \left(z; p, Q, q, t^{-1}\right) \cdot G_3 \left(Qz^{-1}; p, Q, q, t^{-1}\right) \prod_{k=0}^{\infty} N_{\lambda\mu} \left(Q^k z|q, t^{-1}, p\right) N_{\mu\lambda} \left(Q^{k+1}z^{-1}|q, t^{-1}, p\right),
\]
(1.6)
where the triple elliptic gamma function \(G_3\) represents the “vacuum” contribution. In particular, when all the partitions are trivial in (1.5), the Nekrasov factors become also trivial and only \(G_3\) factors, which are completely symmetric in four parameters \((p, Q, q, t^{-1})\), survive.

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The fact that we keep the underlying algebra and only twist the coproduct seems to have the following advantage. The DIM algebra has an $\operatorname{SL}(2, \mathbb{Z})$ automorphism, called Miki automorphism [46]. This is an automorphism of the associative algebra, but not of the bialgebra. Hence, the Miki automorphism survives after the quasi-Hopf twist, though it is not clearly seen in terms of the twisted (elliptic) currents. On the other hand if we introduce an additional boson as in [64], the existence of the Miki automorphism is not clear at all. Incidentally, we are also led to the following question; since the Miki automorphism is not an automorphism of the bialgebra, it deforms the coproduct structure. Hence, one can ask if the change of the coproduct by the automorphism is described by a twisting of the coproduct by an appropriate twistor.

One of the interesting aspects of the quasi-Hopf twist of DIM algebra is the emergence of $\operatorname{SU}(4)$ equivariant parameters. We have seen there appear two kinds of parameters $p$ and $p^*$ in the commutation relations (1.3) and (1.4). More basically, as we will see in the next section, in addition to the theta functions with the elliptic modulus $p$, the exchange relations of the twisted currents involve those with the elliptic modulus $p^* = pC^{-2}$, where $C$ is one of the central charges of DIM algebra. This is in sharp contrast with the elliptic DIM algebra defined in [64], where only the theta function with parameter $p$ appears. Recall that the original DIM algebra has parameters $(q_1, q_2, q_3)$ with $q_1q_2q_3 = 1$. The standard Fock representation in terms of a free boson has the central charge $C = q^{3/2}$. After the quasi-Hopf twist with the twist parameter $p$, the Fock representation has $p^* = pq_3^{-1} = pq_1q_2$. Hence defining $q_3 = p$ and $q_4 = p^*$, we obtain $\operatorname{SU}(4)$ parameters with $q_1q_2q_3q_4 = 1$. We should not forget that these parameters are not associated with the algebra itself, but only arise in its Fock representation. However, the emergence of the $\operatorname{SU}(4)$ parameters is quite suggestive. It is tempting to regard them as the equivariant parameters (or the $\Omega$ background) of the torus action on $\mathbb{C}^4$, which plays the role of the ambient space of the spiked instanton (or the gauge origami) proposed by Nekrasov [55]. It was introduced to provide a physical definition of the $qq$-character of the $\mathcal{N} = 2$ quiver gauge theories in terms of the brane configuration in type IIB string theory.

**Organization of material.** The materials of this paper are structured as follows: in section 2 we provide the definition of the quasi-Hopf twist of DIM algebra and the formula of its coproduct in terms of the twisted currents. Since we keep the underlying associative algebra, representations of the original DIM algebra also work as representations after the quasi-Hopf twist. We give corresponding representations in terms of the twisted currents in section 3. On the other hand, since the coproduct is deformed, the intertwiners will change. In section 4 after the intertwiner and the dual intertwiner are defined using the coproduct, we express them explicitly as operators on the Fock space of free bosons. We also provide formulas of the zero mode factor, which plays an important role in deriving their commutation relations. The quasi-Hopf twist of the $R$-matrix is derived from the universal $R$-matrix of the DIM algebra in section 5. We double-check the computation by confirming that it agrees with the coefficient which results from interchanging the elliptic Fock intertwiners themselves, and dual elliptic Fock intertwiners themselves. We also check the unitarity of the $R$ matrix and show a remarkable relation to the elliptic Nekrasov
factor. In section 6 a difference equation for the trace of intertwiners and dual intertwiners is derived from the cyclic property of the trace and the commutation relations among the intertwiners. Some of technical details and auxiliary contents are relegated to appendices.

**Elliptic functions.** The complete odd theta function is defined by

\[
\vartheta_1(z; p) := \sqrt{-1} \sum_{n \in \mathbb{Z}} (-1)^n p^{\frac{1}{2}(n+\frac{1}{2})^2} z^{n-\frac{1}{2}}.
\]  

(1.7)

By the Jacobi triple product formula

\[
\sum_{m \in \mathbb{Z}} p^{mb^2} z^m = \prod_{n=1}^{\infty} (1 - p^n) \left( 1 + p^{n-\frac{1}{2}} z \right) \left( 1 + p^{n+\frac{1}{2}} z^{-1} \right),
\]  

(1.8)

we see

\[
\sqrt{-1} (p; p)_{\infty} \cdot \theta_p(z) = p^{-\frac{1}{2}} z^{\frac{1}{2}} \vartheta_1(z; p),
\]  

(1.9)

where we have defined a “short” theta function by

\[
\theta_p(z) := (z; p)_{\infty} (pz^{-1}; p)_{\infty} = \exp \left( - \sum_{n \neq 0} \frac{z^n}{n(1 - p^n)} \right).
\]  

(1.10)

The “short” theta function enjoys the quasi-periodicity;

\[
\theta_p(p^n z) = (-z)^{-n} p^{-\frac{1}{2} n(n-1)} \theta_p(z),
\]  

(1.11)

and the inversion formula;

\[
\theta_p(z) = (-z) \theta_p(z^{-1}).
\]  

(1.12)

We also use the elliptic gamma function;

\[
\Gamma(z; q, p) := \frac{(q z^{-1}; q, p)_{\infty}}{(z; q, p)_{\infty}} = \exp \left( \sum_{n \neq 0} \frac{z^n}{n(1 - q^n)(1 - p^n)} \right).
\]  

(1.13)

The elliptic gamma function is symmetric in \( q \) and \( p \). It satisfies the \( q \)-difference equation;

\[
\Gamma(qz; q, p) = \theta_p(z) \Gamma(z; q, p).
\]  

(1.14)

In general we can define an elliptic deformation of the multiple \( q \)-Pochhammer symbol by

\[
(u; q_1, q_2, \ldots, q_n)_{\infty} := \exp \left( - \sum_{k=1}^{u} \frac{1}{k(1 - q_1^k)(1 - q_2^k) \cdots (1 - q_n^k)} \right)
\]  

\[
\quad \longrightarrow (u; q_1, \ldots, q_n, p)_{\infty} \cdot (q_1 \cdots q_n p u^{-1}; q_1, \ldots, q_n, p)_{\infty}^{-1}.
\]  

(1.15)

In the literature [59] a multi-parameter generalization of the elliptic gamma function is defined by

\[
G_n(u; q_0, \ldots, q_n) := (u; q_0, \ldots, q_r)_{\infty}^{-1} \cdot (q_0 \cdots q_n u^{-1}; q_0, \ldots, q_r)_{\infty},
\]  

(1.16)
such that \( G_0(u; q) = \theta_q(u), G_1(u; q_0, q_1) = \Gamma(u; q_0, q_1) \). Thus, the multiple elliptic gamma function \( G_n(u; p, q_1, \cdots, q_n)^{-1} \) provides the elliptic lift of the multiple \( q \)-Pochhammer symbol (1.15). They satisfy the recursion relation

\[
G_n(q_k u; q_0, \cdots, q_n) = G_{n-1}(u; q_0, \cdots, q_k) \cdot G_n(u; q_0, \cdots, q_n).
\]

The function \( G_2(u; q_0, q_1, q_2) \) is also called double elliptic gamma function.

2 Elliptic algebra from quasi-Hopf twist

2.1 Quasi-Hopf twist of Ding-Iohara-Miki algebra

Let us begin with a review of the quantum toroidal algebra of type \( \mathfrak{gl}_1 \), which we call Ding-Iohara-Miki (DIM) algebra in the present paper. The DIM algebra has the parameters \( (q_1, q_2, q_3) \) with \( q_1 q_2 q_3 = 1 \) and enjoys the triality of the permutation of \( q_i \). We assume they are generic in the sense that for any \( a, b, c \in \mathbb{Z} \),

\[
q_1^a q_2^b q_3^c = 1 \implies a = b = c.
\] (2.1)

We use the notation

\[
\kappa_n := \prod_{i=1}^{3} \left( q_i^{\frac{a}{2}} - q_i^{-\frac{a}{2}} \right) = \prod_{i=1}^{3} (q_i^a - 1) = \prod_{i=1}^{3} \left( 1 - q_i^{-n} \right) = \sum_{i=1}^{3} \left( q_i^n - q_i^{-n} \right),
\] (2.2)

which satisfies \( \kappa_{-n} = -\kappa_n \). By convention we often take \( q = q_1 \) and \( t = q_2^{-1} \) as independent parameters. It is convenient to introduce the notation \( q := q_3^{\frac{1}{2}} = \sqrt{t/q} \) as the parameter of quantum deformation.

We define the DIM algebra \( \mathcal{U} := U_{q,t}(\hat{\mathfrak{gl}}_1) \) to be the associative algebra with the generators \( E_k, F_k, K_0^\pm, H_r \) \( (k \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\}) \) and \( C \). It is convenient to introduce the generating functions (currents);

\[
E(z) = \sum_{k \in \mathbb{Z}} E_k z^{-k}, \quad F(z) = \sum_{k \in \mathbb{Z}} F_k z^{-k}, \quad K^\pm(z) = K_0^\pm \exp \left( \pm \sum_{r=1}^{\infty} H_{z^r} z^{-r} \right).
\] (2.3)

There are several conventions of the Cartan currents \( K^\pm(z) \). The original convention is \( K^\pm(z) = K^{\pm, \text{here}}(C^\frac{1}{2} z) \). The advantage of our convention is that we can eliminate \( C^\frac{1}{2} \) from the defining relations of the algebra. Some literatures use the convention \( K^\pm(z) = K^{\pm, \text{here}}(C z) \).

The DIM algebra has two-dimensional center spanned by \( (C, K_0^\pm) \). Note that \( K_0^+ \) is the inverse of \( K_0^- \) by definition. We will not write down the defining relations among the currents, since they can be recovered from the relations after the quasi-Hopf twist by putting the deformation (elliptic) parameter \( p = 0 \). But we only quote the commutation relation

\[
[H_r, H_s] = \delta_{r+s,0} \frac{\kappa_r}{r} (C^r - C^{-r}),
\] (2.4)
since it determines the normalization of the Cartan generators. Actually there are also the Serre’s relations in the defining relations, which we do not write down explicitly, since they are not used in this paper.

To define the quasi-Hopf twist of the DIM algebra, let us introduce the operators $b_{\pm n}$ defined via

$$K^+(z) = K_0^+ \exp \left( \sum_{n=1}^{\infty} b_n C_n z^{-n} \right), \quad K^-(z) = K_0^- \exp \left( -\sum_{n=1}^{\infty} b_{-n} z^n \right).$$

(2.5)

The coproduct of $b_{\pm n}$ is given by

$$\Delta(b_{\pm n}) = b_{\pm n} \otimes C_2^{-n} + 1 \otimes b_{\pm n}.$$  

(2.6)

Then in term of the twistor

$$\mathcal{F}(p) = \exp \left( \sum_{n=1}^{\infty} \frac{np^n}{\kappa_n (1 - p^n C_2^{-2n})} b_n \otimes b_{-n} \right) \in \mathcal{U} \otimes \mathcal{U}, \quad C_2 := 1 \otimes C,$$

(2.7)

we define the twisted coproduct by [19]

$$\Delta_p(a) = \mathcal{F}(p) \Delta(a) \mathcal{F}(p)^{-1}, \quad a \in \mathcal{U}.$$ 

(2.8)

Note that the twistor $\mathcal{F}(p)$ is invertible: $\epsilon \otimes \text{id}) \mathcal{F}(p) = (\text{id} \otimes \epsilon) \mathcal{F}(p) = 1$. By (2.6) one can check that it satisfies the shifted cocycle condition [19];

$$\mathcal{F}^{(23)}(p) (\text{id} \otimes \Delta) \mathcal{F}(p) = \mathcal{F}^{(12)} \left( p C_3^{-2} \right) (\Delta \otimes \text{id}) \mathcal{F}(p)$$

(2.9)

on $\mathcal{U}^{\otimes 3}$. We would like to emphasize that by the quasi-Hopf twist the underlying algebra $\mathcal{U}$ itself remains the same. But the coproduct has been deformed and hence the definition of tensor product representations will change. Originally the DIM algebra $\mathcal{U}$ is a (quasi-triangular) Hopf algebra. But due to the deformation of the coproduct it is no longer true and it becomes a quasi-Hopf algebra (hence the name “quasi-Hopf twist”) with the Drinfeld associator;

$$\Phi^{(D)} = \mathcal{F}^{(23)}(p) (\text{id} \otimes \Delta) \mathcal{F}(p) \cdot \left( \mathcal{F}^{(12)}(p) (\Delta \otimes \text{id}) \mathcal{F}(p) \right)^{-1}.$$ 

(2.10)

For a quasi-Hopf algebra the coassociativity is modified by $\Phi^{(D)}$. When $\Phi^{(D)} = 1$ the coassociativity holds and it is a Hopf algebra. Note that if there was no shift in the cocycle condition (2.9), we had $\Phi^{(D)} = 1$. Hence the shift of parameters in (2.9) causes the violation of the coassociativity. On the other hand in the case of the elliptic DIM algebra introduced in [64], the algebra is extended by an extra Heisenberg algebra, keeping the coproduct intact.

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2 $\epsilon$ denotes the counit, which is non-vanishing only on the central elements.
3 For a quantum group the cocommutativity is broken, but the universal $R$ matrix compensates it.
2.2 Coproduct and exchange relations among elliptic currents

The new coproduct $\Delta_p$ is neatly expressed in terms of the elliptic currents to be defined shortly. Introducing the twisting currents by

$$U^+(z; p) = \exp \left( - \sum_{n=1}^{\infty} \frac{p^n C^{-n} b_{-n} z^n}{1 - p^n C^{-2n}} \right),$$

$$U^-(z; p) = \exp \left( \sum_{n=1}^{\infty} \frac{p^n b_n z^{-n}}{1 - p^n} \right),$$

we define the elliptic generating currents by

$$E(z; p) = U^+(z; p)E(z), \quad F(z; p) = F(z)U^-(z; p),$$

$$K^+(z; p) = U^+(z; p)K^+(z)U^-(C^{-1} z; p),$$

$$K^-(z; p) = U^+(C^{-1} z; p)K^-(z)U^-(z; p).$$

Explicitly the twisted Cartan currents are

$$K^+(z; p) = K_0^+ \exp \left( - \sum_{n=1}^{\infty} \frac{p^n C^{-n} b_{-n} z^n}{1 - p^n C^{-2n}} \right) \exp \left( \sum_{n=1}^{\infty} \frac{C^n b_n z^{-n}}{1 - p^n} \right),$$

$$K^-(z; p) = K_0^- \exp \left( - \sum_{n=1}^{\infty} \frac{1}{1 - p^n C^{-2n}} b_{-n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{p^n b_n z^{-n}}{1 - p^n} \right).$$

Hence, they are related by the scaling of the spectral parameters;

$$K^+ \left( p^{-1} C z; p \right) = K_0^+ (K_0^-)^{-1} \cdot K^-(z; p).$$

Note that this relation ceases to hold for $p = 0$, since it involves $p^{-1}$.

In terms of the elliptic currents the coproduct is given by

$$\Delta_p(E(z; p)) = E(z; p) \otimes 1 + K^-(C_1 z; p_*) \otimes E(C_1 z; p),$$

$$\Delta_p(F(z; p)) = F(C_2 z; p_*) \otimes K^+(C_2 z; p) + 1 \otimes F(z; p),$$

$$\Delta_p(K^+(z; p)) = K^+(z; p) \otimes K^+(C_1^{-1} z; p),$$

$$\Delta_p(K^-(z; p)) = K^-(C_2^{-1} z; p_*) \otimes K^-(z; p),$$

where $C_1 := C \otimes 1, C_2 := 1 \otimes C$ and $p_* = p C_2^{-2}$. The deformed coproduct $\Delta_p$ becomes complicated in terms of the original currents. But the twisted currents makes it quite similar to the original coproduct $\Delta$. In fact $\Delta_p$ takes the same form as $\Delta$ except the shift of the elliptic parameter $p \to p_*$ in the first factor of the tensor product. Since the coproduct is a homomorphism; $\Delta_p(ab) = \Delta_p(a)\Delta_p(b)$, we can define the tensor product representation of $\rho_1$ and $\rho_2$ by $(\rho_1 \otimes \rho_2)(a) = \rho_1 \otimes \rho_2(\Delta_p(a))$. We would like to give a remark that the above coproduct is not a coproduct in the strict sense. The reason is that for each order $k \in \mathbb{Z}$ of $z^k$, the expression of the coproduct contains an infinite summation of generators, which is not well-defined in general.
Let us define an elliptic lift of the structure function of the DIM algebra by

$$ \mathcal{G}(x; p) := \frac{\theta_p(q_1^{-1}x) \theta_p(q_2^{-1}x) \theta_p(q_3^{-1}x)}{\theta_p(q_1x) \theta_p(q_2x) \theta_p(q_3x)} = \frac{\vartheta_1(q_1^{-1}x; p) \vartheta_1(q_2^{-1}x; p) \vartheta_1(q_3^{-1}x; p)}{\vartheta_1(q_1x; p) \vartheta_1(q_2x; p) \vartheta_1(q_3x; p)}. $$

(2.23)

Using the relation (1.11) and the property $q_1 q_2 q_3 = 1$, we can check that $\mathcal{G}(x; p)$ is periodic; $\mathcal{G}(px; p) = \mathcal{G}(x; p)$. Similarly (1.12) implies $\mathcal{G}(x^{-1}; p) = \mathcal{G}(x; p)^{-1}$. Then the exchange relations of among elliptic currents can be stated as follows;

$$ \begin{align*}
K^+(z;p)K^-(w;p) &= \frac{\mathcal{G}(w/z;p_*)}{\mathcal{G}(w/z;p)} K^+(w;p)K^-(z;p), \\
K^+(z;p)E(w;p) &= \frac{\mathcal{G}(w/Cz;p_*)}{\mathcal{G}(w/Cz;p)} E(w;p)K^+(z;p), \\
K^-(Cz;p)E(w;p) &= \frac{\mathcal{G}(w/z;p_*)}{\mathcal{G}(w/z;p)} E(w;p)K^-(Cz;p), \\
K^+(Cz;p)F(w;p) &= \frac{\mathcal{G}(w/z;p_*)}{\mathcal{G}(w/z;p)} E(w;p)F(z;p), \\
K^-(z;p)F(w;p) &= \frac{\mathcal{G}(w/z;p_*)}{\mathcal{G}(w/z;p)} F(w;p)K^-(z;p), \\
E(z;p)E(w;p) &= \frac{\mathcal{G}(w/z;p_*)}{\mathcal{G}(w/z;p)} E(w;p)E(z;p), \\
F(z;p)F(w;p) &= \frac{\mathcal{G}(w/z;p_*)}{\mathcal{G}(w/z;p)} F(w;p)F(z;p), \\
[E(z;p), F(w;p)] &= \hat{g} \left( \delta \left( \frac{Cw}{z} \right) K^+(z;p) - \delta \left( \frac{Cz}{w} \right) K^-(w;p) \right),
\end{align*} $$

(2.24 - 2.32)

where the normalization factor $\hat{g}$ of the commutation relation of $E(z;p)$ and $F(z;p)$ does not change under the quasi-Hopf twist. Hence we can keep the same normalization $\hat{g} = \kappa_1^{-1}$ as [13]. On the other hand, the elliptic DIM algebra of [64] choose a different normalization; the factor in $\hat{g}$ is lifted to the theta functions. Since we can change $\hat{g}$ by the rescaling of $E(z;p)$ and $F(z;p)$ without affecting other exchange relations, the rescaling

$$ \begin{align*}
E(z;p) &\rightarrow \frac{(1-q_1) (q_1^{-1}p; p)_\infty (q_2^{-1}; p)_\infty}{(p; p)_\infty (q_3^{-1}p; p)_\infty} E(z; p), \\
F(z;p) &\rightarrow \frac{(1-q_1^{-1}) (q_1p; p)_\infty (q_2^{-1}; p)_\infty}{(p; p)_\infty (q_3^{-1}p; p)_\infty} F(z; p).
\end{align*} $$

(2.33 - 2.34)

is allowed for the matching of the normalization. Namely, by the rescaling (2.33) and (2.34), we have

$$ \frac{1}{\kappa_1} \frac{(q_1; p)_\infty (q_1^{-1}p; p)_\infty (q_2^{-1}; p)_\infty}{(p; p)^2 (q_3^{-1}p; p)_\infty (q_3^{-1}; p)_\infty} = \frac{\theta_p(q_1) \theta_p(q_2)}{(p; p)^2 \theta_p(q_1q_2)}, $$

(2.35)

which exactly matches with the coefficient of the commutation relation (2) in [69].

\footnote{The second equality is due to the condition $q_1q_2q_3 = 1$.}
\footnote{The convention of the theta function in [69] is different from ours.}
Since $\mathcal{G}(x;p)$ has infinitely many poles, it is mathematically precise to write the exchange relation (2.30) in the following way;

\[ -\left(\frac{w}{z}\right)^3 \theta_{p_*} \left( q_1^{-1} \frac{z}{w} \right) \theta_{p_*} \left( q_2^{-1} \frac{z}{w} \right) \theta_{p_*} \left( q_3^{-1} \frac{z}{w} \right) \cdot E(z;p) E(w;p) \]

\[ = \theta_{p_*} \left( q_1^{-1} \frac{w}{z} \right) \theta_{p_*} \left( q_2^{-1} \frac{w}{z} \right) \theta_{p_*} \left( q_3^{-1} \frac{w}{z} \right) \cdot E(w;p) E(z;p). \]  

(2.36)

The same remark applies to other exchange relations.

The elliptic parameter appearing in the exchange relations involving $F(z;p)$ is $p$, while it is the shifted parameter $p_*$ for $E(z;p)$. Note also that the relations (2.24) – (2.32) are consistent with the scaling relation (2.18) of $K^\pm(z;p)$. In other words the relations involving $K^-(z;p)$ follow from those of $K^+(z;p)$. When $C = 1$ and hence $p_* = p$, these exchange relations agree with those of the elliptic DIM algebra introduced [64] up to the normalization factor $\tilde{g}$ of the commutation relation $[E(z;p), F(w;p)]$. This in particular implies that the vertical representations with $C = 1$ of Saito’s elliptic algebra are also the vertical representations of the quasi-Hopf twist of the DIM algebra.

3 Representations of the elliptic currents

Since the underlying algebra does not change as an associative algebra, the representations of the original DIM algebra provide also those of the quasi-Hopf twisted algebra as representations of the associative algebra. In particular there are central elements $(C,K^-)$ which are constant, if the representation is irreducible. Under the quasi-Hopf twist these values do not change. Since only integer powers of $q$ appear as the values of the central elements in the present paper, we take the additive convention and define a representation has level $(n,m)$, if $(C,K^-) = (q^n,q^m)$. On the other hand, the tensor product representations will change, since the coproduct is twisted. As we will see this leads to an issue on the construction of the vertical Fock representation. In this section we will express known representations of the original DIM algebra in terms of the elliptic currents. The advantage of using the twisted currents is that the coproduct $\Delta_p$ takes a simple form.

3.1 Vector representation

To obtain the vector representation of the elliptic currents, we first start with the vector representation of the DIM algebra [18], and then perform the twisting procedure. For each $v \in \mathbb{C}$ called spectral parameter, let $V(v)$ be the vector space over $\mathbb{C}$ with a basis $\{[v]_i | i \in \mathbb{Z}\}$. Recall that in the vector representation of the DIM algebra, we have

\[ K^+(z) [v]_i = \tilde{\psi} \left( q_1^i v / z \right) [v]_i, \quad K^-(z) [v]_i = \tilde{\psi} \left( q_1^{-i} z / v \right) [v]_i, \]  

(3.1)

where

\[ \tilde{\psi}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{z^n}{1-q_1^{-n}} \right). \]  

(3.2)

\footnote{Since the power of $p_*$ appears frequently, we have changed the original notation $p^*$ to $p_*$.}
Recall also that we define the operators $b_{\pm n}$ by (2.5). Since the vector representation has level $(0, 0)$, we see

$$b_{\pm n} [v]_i = \frac{\kappa_n}{n} \frac{1}{1 - q_1^{\pm n}} (q_1^i v)^{\pm n} [v]_i. \quad (3.3)$$

Hence, the twisting currents are given by

$$U^+ (z; p) [v]_i = \prod_{k=1}^{\infty} \tilde{\psi} \left( p^k q_1^{-i-1} z/v \right) [v]_i, \quad U^- (z; p) [v]_i = \prod_{k=1}^{\infty} \tilde{\psi} \left( p^k q_1^i v/z \right) [v]_i. \quad (3.4)$$

It is straightforward to check that in terms of the elliptic currents the vector representation $\rho^V_n$ is described by

$$K^+ (z; p) [v]_i = \frac{\theta_p (q_2^{-1} q_1^i v/z) \theta_p (q_3^{-1} q_1^i v/z)}{\theta_p (q_1^i v/z) \theta_p (q_1^{i+1} v/z)} [v]_i, \quad (3.5)$$

$$K^- (z; p) [v]_i = \frac{\theta_p (q_3 q_1^{-i} z/v) \theta_p (q_2 q_1^{-i} z/v)}{\theta_p (q_1^{-i} z/v) \theta_p (q_1^{i+1} z/v)} [v]_i, \quad (3.6)$$

$$E (z; p) [v]_i = \frac{(pq_2; p)_\infty (pq_3; p)_\infty}{(1 - q_1) (p; p)_\infty (pq_1; p)_\infty} \delta \left( q_1^{i+1} v/z \right) [v]_{i+1}, \quad (3.7)$$

$$F (z; p) [v]_i = \frac{(pq_2^{-1}; p)_\infty (pq_3^{-1}; p)_\infty}{(1 - q_1^{-1}) (p; p)_\infty (pq_1; p)_\infty} \delta \left( q_1^{i} v/z \right) [v]_{i-1}. \quad (3.8)$$

### 3.2 Vertical Fock representation by tensor product

As in the case of DIM algebra, we can construct the so-called vertical Fock representation from the vector representation via the inductive limit [18]. The first step of this procedure is to perform the tensor product of vector representations with appropriate shift of spectral parameters;

$$V^n (v) \overset{\text{def}}{=} V (v) \otimes V (q_2 v) \otimes \cdots \otimes V \left( q_2^{n-1} v \right). \quad (3.9)$$

For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, we define $|\lambda\rangle \in V^n (v)$ to be

$$|\lambda\rangle \overset{\text{def}}{=} [v]_{\lambda_1-1} \otimes [q_2 v]_{\lambda_2-1} \otimes \cdots \otimes [q_2^{n-1} v]_{\lambda_n-1}. \quad (3.10)$$

It is clear that $\{|\lambda\rangle \mid \lambda \in \mathbb{Z}^n\}$ forms a basis of $V^n (v)$. We can endow the structure of $\mathcal{U}$-module $\rho^{(n)} : \mathcal{U} \to \text{End}(V^n (v))$ to the vector space $V^n (v)$ by\footnote{Since the coassociativity does not hold for $\Delta_p$, there is an ambiguity in the definition of $\Delta_p^n$ in general. However, since $C = 1$ for the vertical representations, such a problem can be evaded. We use the definition in [9] and [13].}

$$\rho^{(n)} (a) |\lambda\rangle = \left[ \rho^V_v \otimes \rho^V_{q_2 v} \otimes \cdots \otimes \rho^V_{q_2^{n-1} v} \right] \Delta_p^{-1} (a) |\lambda\rangle, \quad a \in \mathcal{U}. \quad (3.11)$$
Note that here we can set $C_1 = C_2 = 1$ in the formulas of the coproduct (2.19) – (2.22), since we are focusing on the vector representations. For the elliptic currents the $n$-fold tensor product of vector representations is given by

$$
\rho^{(n)}(K^+ (z; p)) |\lambda\rangle = \left[ \prod_{i=1}^{n} \theta_p \left( q_1^{\lambda_i} q_2^{i-1} v/z \right) \theta_p \left( q_1^{\lambda_i-1} q_2^{i-2} v/z \right) \right] |\lambda\rangle, \quad (3.12)
$$

$$
\rho^{(n)}(K^- (z; p)) |\lambda\rangle = \left[ \prod_{i=1}^{n} \theta_p \left( q_1^{\lambda_i} q_2^{-i} v/z \right) \theta_p \left( q_1^{\lambda_i-1} q_2^{-1} v/z \right) \right] |\lambda\rangle, \quad (3.13)
$$

and

$$
\rho^{(n)}(E (z; p)) |\lambda\rangle = \sum_{k=1}^{n} \frac{(pq_2;p)_\infty (pq_3;p)_\infty}{(1-q_1)(p;p)_\infty (pq_1;p)_\infty} \delta \left( q_1^{\lambda_k} q_2^{k-1} v/z \right) \left[ \prod_{i=1}^{k-1} \theta_p \left( q_1^{\lambda_i} q_2^{i-1} \right) \theta_p \left( q_1^{\lambda_i-1} q_2^{i} \right) \right] |\lambda + 1_k\rangle, \quad (3.14)
$$

$$
\rho^{(n)}(F (z; p)) |\lambda\rangle = \sum_{k=1}^{n} \frac{\left( pq_2^{-1};p \right)_\infty \left( pq_3^{-1};p \right)_\infty}{(1-q_1^{-1})(p;p)_\infty (pq_1;p)_\infty} \delta \left( q_1^{\lambda_k-1} q_2^{k-1} v/z \right) \left[ \prod_{i=k+1}^{n} \theta_p \left( q_1^{\lambda_i-k+1} q_2^{-i-k} \right) \theta_p \left( q_1^{\lambda_i-k} q_2^{-i} \right) \right] |\lambda - 1_k\rangle, \quad (3.15)
$$

where $\lambda \pm 1_k$ means the shift of the $k$-th component $\lambda_k \to \lambda_k \pm 1$. Let us denote the set of partitions with length at most $n$ by

$$
\mathcal{P}_n := \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \text{ s.t. } \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \}. \quad (3.16)
$$

For later convenience we also introduce the set of all partitions $\mathcal{P}$. By the judicious choice of the $q_2$-shift of spectral parameters for a sequence of the vector representations, there is an invariant subspace

$$
W^{n,+}(v) \overset{\text{def}}{=} \text{span} \left\{ |\lambda\rangle \in V^n(v) | \lambda \in \mathcal{P}_n \right\}. \quad (3.17)
$$

This can be confirmed by investigating the positions of zeros appearing in the action of the creation operator $E(z)$ and the annihilation operator $F(z)$ [18].

Next, we shall take the inductive limit of the tensor product of vector representations constructed above. The reason why we are interested in taking the inductive limit is that we would like to remove the restriction on the length of the partitions $\lambda$. Thus, we consider the vector space $\mathcal{F}_v$ which is defined by

$$
\mathcal{F}_v \overset{\text{def}}{=} \lim_{n \to \infty} W^{n,+}(v), \quad (3.18)
$$

where the inductive limit is taken in the category of vector spaces. We would like to endow the structure of left $\mathcal{U}$-module on $\mathcal{F}_v$. At first glance, it is natural to define this representation $\rho^\mathcal{F}: \mathcal{U} \to \text{End}(\mathcal{F}_v)$ as follows; for each $\lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots)$ where $\lambda_n \neq 0$,

$$
\rho^\mathcal{F}(X(z; p)) |\lambda\rangle \overset{\text{def}}{=} \rho^{(n+1)}(X(z; p)) |\lambda\rangle = \rho^{(n+2)}(X(z; p)) |\lambda\rangle = \cdots, \quad (3.19)
$$
The reason is that the defining relation (2.32) must be satisfied. From [9], we expect that the modification factors are identified in the inductive limit. However, this is not the case, since from (3.12), we see that for \( \lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots) \) with \( \lambda_n \neq 0 \),

\[
\frac{\langle \lambda | \rho^{(n+2)}(K^-(z;p)) | \lambda \rangle}{\langle \lambda | \rho^{(n+1)}(K^+(z;p)) | \lambda \rangle} = \frac{\theta_p \left( q_1^{-1} q_2^n v/z \right) \theta_p \left( q_2^n v/z \right)}{\theta_p \left( q_1^{-1} q_2^{n+1} v/z \right) \theta_p \left( q_2^{n+1} v/z \right)} \neq 1. \tag{3.20}
\]

This means that we need certain modification factors in (3.19) for \( X = K^+ \). The same situation also occurs in the case \( X = K^- \). Hence, let us define

\[
\rho^\gamma(K^+(z;p))|\lambda\rangle \overset{\text{def}}= \beta_{n+1} \rho^{(n+1)}(K^+(z;p))|\lambda\rangle = \beta_{n+2} \rho^{(n+2)}(K^+(z;p))|\lambda\rangle = \cdots. \tag{3.21}
\]

Then we obtain a consistency condition for the modification factors \( \beta_n \):

\[
\beta_{n+2} \left( \frac{\theta_p \left( q_1^{-1} q_2^n v/z \right) \theta_p \left( q_2^n v/z \right)}{\theta_p \left( q_1^{-1} q_2^{n+1} v/z \right) \theta_p \left( q_2^{n+1} v/z \right)} \right) = 1. \tag{3.22}
\]

Therefore, we conclude that \( \beta_n \) takes the following form;

\[
\beta_n = f(v/z) \frac{\theta_p \left( q_1^{-1} q_2^{n-1} v/z \right)}{\theta_p \left( q_2^n v/z \right)}, \tag{3.23}
\]

where \( f(v/z) \) is a proportional factor which is independent of \( n \).

By the same line of arguments, if we define for each \( \lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots) \) with \( \lambda_n \neq 0 \),

\[
\rho^\gamma(K^-(z;p))|\lambda\rangle \overset{\text{def}}= \gamma_{n+1} \rho^{(n+1)}(K^-(z;p))|\lambda\rangle = \gamma_{n+2} \rho^{(n+2)}(K^-(z;p))|\lambda\rangle = \cdots. \tag{3.24}
\]

We then find that

\[
\gamma_n = g(z/v) \cdot q_3 \left( z/q_1^{-1} q_2^{n-1} v \right) \frac{\theta_p \left( z/q_1^{-1} q_2^{n-1} v \right)}{\theta_p \left( z/q_2^n v \right)}. \tag{3.25}
\]

From [9], we expect that the modification factors \( \beta_n \) for \( K^+(z;p) \) and \( \gamma_n \) for \( K^-(z;p) \) are the same. From this and the inversion formula (1.12) we obtain that \( f(v/z) = g(z/v) \).

On the other hand, in the case \( X = E \), the problem does not arise. Namely, if \( \lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots) \) with \( \lambda_n \neq 0 \), then the action

\[
\rho^\gamma(E(z;p))|\lambda\rangle \overset{\text{def}}= \rho^{(n+1)}(E(z;p))|\lambda\rangle = \rho^{(n+2)}(E(z;p))|\lambda\rangle = \cdots \tag{3.26}
\]

is well-defined. From this result, we immediately see that we also have to introduce a modification factor to the action of \( F(z;p) \), and, moreover, it has to be equal to \( \gamma_n \). That is,

\[
\rho^\gamma(F(z;p))|\lambda\rangle \overset{\text{def}}= \gamma_{n+1} \rho^{(n+1)}(F(z;p))|\lambda\rangle = \gamma_{n+2} \rho^{(n+2)}(F(z;p))|\lambda\rangle = \cdots. \tag{3.27}
\]

The reason is that the defining relation (2.32) must be satisfied.
Finally, to accomplish the task, we have to determine an explicit expression of \( f(v/z) \).
We require that the original Fock representation is recovered in the limit \( p \rightarrow 0 \). For simplicity, we also assume that \( f(v/z) \) does not depend on \( p \). Thus, we conclude that \( f(v/z) = g(z/v) = q^{-1} \).

As already mentioned, we can find the invariant subspace \( \mathcal{F}_v \), which is spanned by the set of partitions \( \mathcal{P} \). This is an irreducible subrepresentation generated by the empty partition \( \emptyset \). We call it vertical Fock representation. It is a highest weight representation with the empty partition \( \emptyset \) being the highest weight state. In summary, we have constructed a representation \( \rho^\mathcal{F} : \mathcal{U} \rightarrow \text{End}(\mathcal{F}_v) \) with the spectral parameter \( v \);

\[
\rho^\mathcal{F}(K^+(z;p)) |\lambda\rangle = q^{-1} \prod_{i=1}^{\infty} \frac{\theta_p \left( q_1^{\lambda_1} q_2^{i-1} v/z \right) \theta_p \left( q_1^{\lambda_1} q_2^{i-2} v/z \right)}{\theta_p \left( q_1^{\lambda_1} q_2^{i-1} v/z \right) \theta_p \left( q_1^{\lambda_1} q_2^{i-1} v/z \right)} |\lambda\rangle, \tag{3.28}
\]

\[
\rho^\mathcal{F}(K^-(z;p)) |\lambda\rangle = q \prod_{i=1}^{\infty} \frac{\theta_p \left( q_1^{-\lambda_1} q_2^{-i+1} z/v \right) \theta_p \left( q_1^{-\lambda_1+1} q_2^{-i+2} z/v \right)}{\theta_p \left( q_1^{-\lambda_1} q_2^{-i+1} z/v \right) \theta_p \left( q_1^{-\lambda_1+1} q_2^{-i+1} z/v \right)} |\lambda\rangle, \tag{3.29}
\]

\[
\rho^\mathcal{F}(E(z;p)) |\lambda\rangle = \sum_{k=1}^{\infty} \frac{(pq_2;p)_\infty (pq_3;p)_\infty}{(1-q_1)(pq_1;p)_\infty (pq_1^{-1};p)_\infty} \delta \left( q_1^{\lambda_1} q_2^{k-1} \frac{v}{z} \right) \prod_{i=1}^{k-1} \frac{\theta_p \left( q_1^{\lambda_1-\lambda_k} q_2^{k-i} \right) \theta_p \left( q_1^{\lambda_1-\lambda_k+1} q_2^{k-i+1} \right)}{\theta_p \left( q_1^{\lambda_1-\lambda_k} q_2^{k-i} \right) \theta_p \left( q_1^{\lambda_1-\lambda_k+1} q_2^{k-i} \right)} |\lambda + 1_k\rangle, \tag{3.30}
\]

\[
\rho^\mathcal{F}(F(z;p)) |\lambda\rangle = q^{-1} \sum_{k=1}^{\infty} \frac{(pq_2^{-1};p)_\infty (pq_3^{-1};p)_\infty}{(1-q_1^{-1})(pq_1;p)_\infty (pq_1^{-1};p)_\infty} \delta \left( q_1^{\lambda_1-1} q_2^{k-1} \frac{v}{z} \right) \prod_{i=k+1}^{\infty} \frac{\theta_p \left( q_1^{\lambda_1-\lambda_k+1} q_2^{-i+k} \right) \theta_p \left( q_1^{\lambda_1-\lambda_k} q_2^{-i+k} \right)}{\theta_p \left( q_1^{\lambda_1-\lambda_k+1} q_2^{-i+k} \right) \theta_p \left( q_1^{\lambda_1-\lambda_k} q_2^{-i+k} \right)} |\lambda - 1_k\rangle. \tag{3.31}
\]

These are universal formula which do not depend on the length of partition \( \ell(\lambda) \). Our prescription for the infinite product appearing in \( K^\pm(z;p) \) and \( F(z;p) \) is as follows; we make a successive cancellation of the factors in the denominator and the numerator for \( \lambda_n = 0 (\ell(\lambda) < n) \), which reduces the infinite product to a finite product once the partition \( \lambda \) is fixed. The factor \( q^\pm 1 \) is regarded as a result of the regularization of the infinite product by this prescription. Note that it does not appear for \( E(z;p) \) which does not require the infinite product. Similarly the infinite sum for \( E(z;p) \) and \( F(z;p) \) reduces to a finite sum up to \( \ell(\lambda) + 1 \), because when the adjacent lengths of the partition agree; \( \lambda_j = \lambda_{j+1} \), it is possible to have a factor \( \theta_p(1) = 0 \). This also implies that when \( \lambda + 1_k \) or \( \lambda - 1_k \) is no longer a partition, the corresponding coefficient automatically vanishes. From (3.29) we see that the vertical Fock representation has level \((0,1)\). After the scaling (2.33) and (2.34) of \( E(z;p) \) and \( F(z;p) \), our result agrees with the vertical representation in [69].

3.3 Vertical Fock representation by twisting

In the last subsection, we have constructed the vertical Fock representation by using the inductive limit of the tensor product of vector representations. On the other hand, since the
original DIM algebra has the vertical Fock representation [18], we may construct a vertical representation directly from the quasi-Hopf twist. In the vertical Fock representation, the operator \( b_n \) acts as follows:

\[
\begin{align*}
    b_n|\lambda\rangle &= \frac{v^n}{n} \frac{\kappa_n}{1 - q_1^n} \left[ \sum_{s=1}^{\ell(\lambda)} x_s^n + \frac{1}{1 - q_2^n} x_{\ell(\lambda)+1}^n \right] |\lambda\rangle \\
    &= \frac{v^n}{n} \frac{\kappa_n}{1 - q_1^n} \left[ \sum_{s=1}^{\infty} x_s^n \right] |\lambda\rangle, \quad x_s := q_1^{\lambda_s-1} q_2^{s-1}. \quad (3.32)
\end{align*}
\]

We note that the eigenvalues of \( b_n \) are proportional to \( A_\lambda(q_1^n, q_2^n) \) to be defined below (see (4.12)). Thus from (2.13), (2.14) and (2.15), we can check that the action of the elliptic Cartan currents \( K^\pm(z;p) \) is the same as (3.28) and (3.29). On the other hand, we find some discrepancy in the action of the elliptic currents \( E(z;p) \) and \( F(z;p) \). Namely there appear the following remainder factors against the formulas (3.30) and (3.31):

\[
R^{(k)}_\lambda(q_1, q_2; p) := \prod_{s=1}^{k-1} \frac{\left( pq_1^{\lambda_s-\lambda_k} q_2^{s-k}; p \right)_\infty}{\left( pq_1^{\lambda_s-\lambda_k-1} q_2^{s-k-1}; p \right)_\infty} \frac{\left( pq_1^{\lambda_k-\lambda_s+1} q_2^{k-s+1}; p \right)_\infty}{\left( pq_1^{\lambda_s-\lambda_k} q_2^{k-s}; p \right)_\infty}
\]

for \( E(z;p) \) and

\[
\tilde{R}^{(k)}_\lambda(q_1, q_2; p) = R^{(k)}_{\lambda-1\ell q_1, q_2; p}^{-1}
\]

for \( F(z;p) \). The relation (3.34) allows us to understand the remainders \( R^{(k)}_\lambda \) and \( \tilde{R}^{(k)}_\lambda \) in the following manner; in the vertical representations the Cartan currents \( K^\pm(z;p) \) are commuting and we employ a basis consisting of simultaneous eigenstates \( |\lambda\rangle \) of \( K^\pm(z;p) \).

Since the eigenvalues are non-degenerate, they are orthogonal. But there is an ambiguity of the (relative) normalization of \( |\lambda\rangle \), in particular it may depend on \( \lambda \) and the elliptic parameter \( p \). The change of the normalization does not affect the matrix elements of \( K^\pm(z;p) \), but the matrix elements of \( E(z;p) \) and \( F(z;p) \) will change, since they are off-diagonal. In fact let us consider the change of the normalization; \( |\lambda\rangle \rightarrow c_\lambda(q_1, q_2; p) |\lambda\rangle \), where \( c_\lambda(q_1, q_2; p) \) is determined by the recursion relation

\[
\frac{c_{\lambda+1\ell q_1, q_2; p}}{c_{\lambda}(q_1, q_2; p)} = R^{(k)}_\lambda(q_1, q_2; p). \quad (3.35)
\]

Then one can see this change of the normalization eliminates both \( R^{(k)}_\lambda \) and \( \tilde{R}^{(k)}_\lambda \). Finally with the initial condition \( c_\emptyset(q_1, q_2; p) = 1 \), the recursion relation is solved by

\[
c_\lambda(q_1, q_2; p) = \prod_{s \in \lambda} \left( pq_1^{a(s)} q_2^{-\ell(s)-1}; p \right)_\infty. \quad (3.36)
\]

Here \( a(s) := \lambda_i - j \) and \( \ell(s) := \lambda_j^\vee - i \) are the arm-length and the leg-length of the box \( s = (i, j) \) in the partition \( \lambda \). Note that

\[
C_\lambda(q_1, q_2) := \prod_{s \in \lambda} \left( 1 - q_1^{a(s)} q_2^{-\ell(s)-1} \right) \quad (3.37)
\]
is the normalization factor which appears in the integral form of the Macdonald polynomials [45].

### 3.4 Horizontal Fock representation

In subsection 2.2, we have seen the relation between the elliptic currents and the original DIM algebra, which has a Fock representation in terms of free bosons $\tilde{a}_n$ [19]. The free boson operators obey the commutation relations of the deformed Heisenberg algebra:

$$[\tilde{a}_n, \tilde{a}_m] = \frac{n}{\kappa_n} (q^n - q^{-n}) \delta_{n+m,0} = \frac{nq^{-n}}{(1 - q_1^n)(1 - q_2^n)} \delta_{n+m,0}. \quad (3.38)$$

By using the relations (2.13)–(2.15), we can obtain the horizontal Fock representation of the elliptic currents. Let $\mathcal{H}_u$ be a vector space over $\mathbb{C}$, which has

$$\left\{ \tilde{a}_{-\lambda_1} \cdots \tilde{a}_{-\lambda_n} | 0; u \right\} \Big| n \in \mathbb{Z}^>0, \lambda_1, \cdots, \lambda_n \in \mathbb{Z}^>0 \text{ s.t. } \lambda_1 \geq \cdots \geq \lambda_n \right\} \quad (3.39)$$

as a basis. Here we have introduced the horizontal spectral parameter $u$ and the state $| 0; u \rangle$ is defined to be annihilated by the positive mode operators $\{ \tilde{a}_n \} n \in \mathbb{Z}^>0$.

If we define $\rho_H^{(0)} : \mathcal{U} \rightarrow \text{End}(\mathcal{H}_u)$ by

$$\rho_H^{(0)} (K^+ (z; p)) = \exp \left( -\sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{p^n}{1 - p^n} \tilde{a}_{-n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{n/2}}{1 - q^n} \tilde{a}_n z^{-n} \right), \quad (3.40)$$

$$\rho_H^{(0)} (K^- (z; p)) = \exp \left( -\sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{-n/2}}{1 - q^n} \tilde{a}_{-n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{p^n q^{-n/2}}{1 - p^n} \tilde{a}_n z^{-n} \right), \quad (3.41)$$

$$\rho_H^{(0)} (E (z; p)) = \frac{u}{(1 - q_1)(1 - q_2)} \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{-n/2}(1 - q^n)}{1 - q^n} \tilde{a}_{-n} z^n \right)$$

$$\cdot \exp \left( -\sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{-n/2}}{1 - q^n} \tilde{a}_n z^{-n} \right), \quad (3.42)$$

$$\rho_H^{(0)} (F (z; p)) = \frac{u^{-1}}{(1 - q_1^{-1})(1 - q_2^{-1})} \exp \left( -\sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{n/2}(1 - p^n)}{1 - p^n} \tilde{a}_{-n} z^n \right)$$

$$\cdot \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{n/2}}{1 - p^n} \tilde{a}_n z^{-n} \right), \quad (3.43)$$

where $p_* = pq^{-2}$. This is a level $(1, 0)$ representation of $\mathcal{U}$. We can also obtain a level $(1, N)$ representation for any integer $N$. By definition, the zero modes of $K^\pm (z; p)$ are $q^\pm N$.

The zero modes of $E(z; p)$ and $F(z; p)$ are fixed by consistency;

$$e(z) f (q^{-1} z) = q^{-N}, \quad e (q^{-1} z) f (z) = q^N. \quad (3.44)$$

Using a canonical solution

$$e(z) = \left( \frac{q}{z} \right)^N, \quad f(z) = \left( \frac{q}{z} \right)^{-N}, \quad (3.45)$$
we define the homomorphism $\rho_H^{(N)}$;
\[
\begin{align*}
\rho_H^{(N)}(K^+(z;p)) &= q^{-N}\rho_H^{(0)}(K^+(z;p)), \\
\rho_H^{(N)}(E(z;p)) &= e(z)\rho_H^{(0)}(E(z;p)), \\
\rho_H^{(N)}(K^-(z;p)) &= q^N\rho_H^{(0)}(K^-(z;p)), \\
\rho_H^{(N)}(F(z;p)) &= f(z)\rho_H^{(0)}(F(z;p)).
\end{align*}
\] (3.46)

In summary in the same way as the original DIM algebra, the horizontal Fock representations are characterized by the level $N$ and the spectral parameter $u$. We will denote the free boson Fock space for the representation $H_u^{(N)}$ by $\mathcal{F}_u^{(N)}$.

4 Intertwiner and dual intertwiner

In this section we construct the intertwining operator and the dual intertwining operator. Historically they appeared in the theory of solvable lattice models associated with the quantum affine algebra $U_q(\hat{g})$, where they were called vertex operators of type II and of type I, respectively [47]. The vertical Fock representation $\mathcal{F}_v$ corresponds to the evaluation module in the case of the solvable lattice models and the horizontal Fock space $\mathcal{H}_u^{(N)}$ is a generalization of the level one highest weight module of the quantum affine algebra.

The intertwining operator and the dual intertwining operator for the elliptic DIM algebra introduced by Y.Saito [64] are constructed in [24, 71].

The intertwining operator $\Psi(v; p) : \mathcal{F}_v \otimes \mathcal{H}_u^{(N)} \rightarrow \mathcal{H}_w^{(N+1)}$ is determined by the following intertwining condition [2]:
\[
a\Psi(v; p) = \Psi(v; p)\Delta_p(a), \quad a \in \mathbb{U}.
\] (4.1)

Here $\mathcal{F}_v$ denotes the vertical Fock representation that has level $(0,1)$, while $\mathcal{H}_u^{(N)}$ and $\mathcal{H}_w^{(N+1)}$ are horizontal representations of level $(1,N)$ and $(1,N+1)$, respectively. Let $\Psi(v)$ be the intertwiner defined by the original coproduct $\Delta$, which is given in [2]. Since the twisted coproduct $\Delta_p$ is defined by (2.8), we see that schematically
\[
\Psi(v; p) = \Psi(v) \cdot (\rho_H \otimes \rho_H)(\mathcal{F}(p)^{-1})
\] (4.2)
satisfies the condition (4.1). Recall that $\{ |\lambda\rangle \}_{\lambda \in \mathfrak{F}}$ forms a basis of $\mathcal{F}_v$. We define the $\lambda$-component of the elliptic intertwiner $\Psi_\lambda : \mathcal{H}_u^{(N)} \rightarrow \mathcal{H}_w^{(N+1)}$ by
\[
\Psi_\lambda(v; p)(\bullet) = \Psi(v; p)(|\lambda\rangle \otimes \bullet).
\] (4.3)

Similarly the dual intertwiner $\Psi^*(v; p) : \mathcal{H}_u^{(N)} \rightarrow \mathcal{H}_w^{(N-1)} \otimes \mathcal{F}_v$ is determined by the dual intertwining relation;
\[
\Psi^*(v; p)a = \Delta_p(a)\Psi^*(v; p), \quad a \in \mathbb{U}.
\] (4.4)

Again, from (2.8), if $\Psi^*(v)$ is the dual intertwiner before the quasi-Hopf twist, then
\[
\Psi^*(v; p) = (\rho_H \otimes \rho_H)(\mathcal{F}(p)) \cdot \Psi^*(v)
\] (4.5)
gives a formal solution to the condition (4.4). We define the $\lambda$-component of the elliptic dual intertwiner $\Psi_\lambda^*(v; p) : \mathcal{H}_u^{(N)} \rightarrow \mathcal{H}_w^{(N-1)}$ by
\[
\Psi_\lambda^*(v; p)(\bullet) = \sum_\lambda \Psi_\lambda^*(v; p)(\bullet) \otimes |\lambda\rangle.
\] (4.6)
determine is fixed by the eigenvalues where \( \lambda \).

The intertwining relations (4.7) and (4.8) mean space is indicated. Below, where the change of the level and the spectral parameter of the horizontal Fock source and the target Fock spaces have to be related by intertwining relations are explicitly;

\[
K^+(z;p)\Psi_{\lambda}(v;p) = (\lambda | K^+(z;p_*) | \lambda) \Psi_{\lambda}(v;p) K^+(z;p),
\]

(4.7)

\[
K^-(q;z)\Psi_{\lambda}(v;p) = (\lambda | K^-(z;p_*) | \lambda) \Psi_{\lambda}(v;p) K^-(q;z),
\]

(4.8)

\[
E(z;p)\Psi_{\lambda}(v;p) = \sum_{k=1}^{\ell(\lambda)+1} (\lambda - 1)[E(z;p_*) | \lambda) \Psi_{\lambda+1_k}(v;p) + (\lambda | K^-(z;p_*) | \lambda) [\lambda](v;p)E(z;p),
\]

(4.9)

\[
F(z;p)\Psi_{\lambda}(v;p) = \sum_{k=1}^{\ell(\lambda)} (\lambda - 1)\sum_{k=1}^{\ell(\lambda)} F(q;v_*) [\lambda) \Psi_{\lambda-1_k}(v;p)K^+(q;z;p) + \Psi_{\lambda}(v;p)F(z;p),
\]

(4.10)

where \( p_* = pq^{-2} \). Note that here the inner product is calculated in the vertical representation \( \mathcal{F}_v \). For the existence of the intertwiner the horizontal spectral parameters of the source and the target Fock spaces have to be related by \( w = -uv \) [2] (see also appendix B).

We can express the elliptic Fock intertwiner \( \Psi_{\lambda}(v;p) \) by the trivalent diagram in figure 1 below, where the change of the level and the spectral parameter of the horizontal Fock space is indicated.

The intertwining relations (4.7) and (4.8) mean \( \Psi_{\lambda}(v;p) \) is an eigenstate of the adjoint action of \( K^+(z;p) \) and \( K^-(q;z) \) with eigenvalues \( (\lambda | K^\pm(z;p_*) | \lambda) \). These two conditions fix \( \Psi_{\lambda}(v;p) \) up to the overall factor \( z_{\lambda}(v)S_{\lambda}^{-1}(p_*) \) as follows;

\[
\Psi_{\lambda}(v;p) = z_{\lambda}(v)S_{\lambda}^{-1}(p_*) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-p^n}{1-p^n} q_n^{n/2} (q_1^n - 1) (q_2^n - 1) \tilde{a}_n v^{-n} A_{\lambda}(q_1^n, q_2^n) \right)
\]

\[
\cdot \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} q_n^{n/2} (q_2^n - 1) (q_1^n - 1) \tilde{a}_n v^{-n} A_{\lambda}(q_1^n, q_2^n) \right),
\]

(4.11)

where

\[
A_{\lambda}(q_1, q_2) := \sum_{(i,j) \in \lambda} x_{ij} - \frac{1}{(q_1 - 1)(q_2 - 1)}, \quad x_{ij} := q_1^{-1} q_2^{-1},
\]

(4.12)

is fixed by the eigenvalues \( (\lambda | K^\pm(z;p_*) | \lambda) \). The remaining two conditions (4.9) and (4.10) determine \( z_{\lambda}(v) \) and \( S_{\lambda}^{-1}(p_*) \) as follows;

\[
z_{\lambda}(v) = \prod_{(i,j) \in \lambda} \left( -q_2^{-1} x_{ij} \right) u \cdot e(x_{ij} v) = q_2^{n(\lambda)} (-v)^{-N |\lambda|} u |\lambda| f_{\lambda}(q_1, q_2)^{-N-1}, \quad v
\]

Figure 1. Trivalent diagram of the elliptic Fock intertwiner \( \Psi_{\lambda}(v;p) \).

4.1 Elliptic Fock intertwiner

For the intertwining relation (4.1), we have \( C_1 = 1 \) and \( C_2 = q \) in (2.19) – (2.22) and the intertwining relations are explicitly;

\[
K^+(z;p)\Psi_{\lambda}(v;p) = (\lambda | K^+(z;p_*) | \lambda) \Psi_{\lambda}(v;p) K^+(z;p),
\]

(4.7)

\[
K^-(q;z)\Psi_{\lambda}(v;p) = (\lambda | K^-(z;p_*) | \lambda) \Psi_{\lambda}(v;p) K^-(q;z),
\]

(4.8)

\[
E(z;p)\Psi_{\lambda}(v;p) = \sum_{k=1}^{\ell(\lambda)+1} (\lambda + 1_k)|E(z;p_*)|\lambda) \Psi_{\lambda+1_k}(v;p) + (\lambda | K^-(z;p_*) | \lambda) \Psi_{\lambda}(v;p) E(z;p),
\]

(4.9)

\[
F(z;p)\Psi_{\lambda}(v;p) = \sum_{k=1}^{\ell(\lambda)} (\lambda - 1k)F(q;v_*)|\lambda) \Psi_{\lambda-1_k}(v;p)K^+(q;z;p) + \Psi_{\lambda}(v;p)F(z;p),
\]

(4.10)
and
\[ G_\lambda(p_\nu) = \prod_{s \in \lambda} \left( \frac{(q_1^{-a(s)} q_2^{(s)+1}; q_2^{\infty})}{pq_1^{-a(s)} q_2^{(s)+1}; q_2^{\infty}} \right). \] (4.14)

Here the framing factor \( f_\lambda(q_1, q_2) \) is defined by
\[ f_\lambda(q_1, q_2) = \prod_{(i,j) \in \lambda} (-1)^j q_1^{j-1} q_2^{i-1} q^{-1} = \prod_{s \in \lambda} (-1)^a(s) + \frac{1}{2} q^{(s)+\frac{1}{2}}. \] (4.15)

The intertwining relations of \( E(z; p) \) and \( F(z; p) \) are responsible for the formula of the zero mode factor \( z_\lambda(v) \) the normalization \( G_\lambda(p_\nu) \). More precisely, \( z_\lambda(v) \) comes from the choice of the zero modes of the horizontal Fock representation and \( G_\lambda(p_\nu) \) depends on the normalization of the basis of the vertical Fock representation (see appendices A and B for computations of \( G_\lambda(p_\nu) \) and \( z_\lambda(v) \)). The appearance of \( G_\lambda(p_\nu) \) and \( z_\lambda(v) \) can be described in the following manner. Since the vertical Fock representation is constructed as the semi-infinite product of the vector representations (see subsection 3.2), we can express the elliptic Fock intertwiner (4.11) as a composition of elliptic intertwiners for the vector representations [9, 13]. The factor \( G_\lambda(p_\nu) \) is related to the normal ordering of this composition. By the relation (4.2) the zero factor \( z_\lambda(v) \) is the same as the intertwiners for the original coproduct \( \Delta \), which are e.g. given in [9].

We note that \( \Psi_\lambda(v; p) \) is expressed as a normal ordered product of the oscillator part \( \eta(z; p) \) of the elliptic current \( E(z; p) \):
\[ \Psi_\lambda(v; p) = z_\lambda(v) G_\lambda^{-1}(p) : \Psi_\lambda(v; p) \prod_{(i,j) \in \lambda} \eta(q_1^{j-1} q_2^{i-1}; v; p) : \] (4.16)
\[ \Psi_\lambda(v; p) := \prod_{i,j=1}^{\infty} \eta(q_1^{j-1} q_2^{i-1}; v;p)^{-1} :. \] (4.17)

The zero mode factor \( z_\lambda(v) \), which depends on the level \( N \) and the spectral parameters takes care of the zero mode factor of \( E(z; p) \). This structure is exactly the same as the intertwiner of the original DIM algebra, which is recovered by \( p \to 0 \), and explains the appearance of \( A_\lambda(q_1, q_2) \) which geometrically is the evaluation of the equivariant character of the tautological sheaf on the universal bundle of instantons at the fixed point labeled by \( \lambda \).

It is remarkable that the shift of the spectral parameter \( q_1^{j-1} q_2^{i-1} v \) in (4.16) and (4.17) comes from the way of constructing the vertical Fock representation. In fact we have used the matrix elements \( \langle \lambda | K^\pm(z; p_\nu) | \lambda \rangle \) to fix this part. It is the vertical Fock representation that arises naturally from the geometry of the Hilbert scheme of points on \( \mathbb{C}^2 \) [50]. In this way the Fock intertwiner incorporates the geometry of \( U(1) \) instanton moduli space into the vertex operators on the Fock space of free bosons. To describe the moduli space of \( U(N) \) instantons we have to take the \( N \)-fold tensor product of the boson Fock spaces.
\[ \Psi^*(v;p)K^+(z;p) = \langle \lambda|K^+(q^{-1}z;p)|\lambda \rangle K^+(z;p)\Psi^*(v;p), \tag{4.18} \]
\[ \Psi^*(v;p)K^-(z;p) = \langle \lambda|K^-(z;p)|\lambda \rangle K^-(z;p)\Psi^*(v;p), \tag{4.19} \]
\[ \Psi^*(v;p)E(z;p) = E(z;p)\Psi^*(v;p) + \sum_{k=1}^{\ell(\lambda)} \langle \lambda|E(qz;p)\lambda - 1_k \rangle K^-(qz;p)\Psi^*_{\lambda-1_k}(v;p), \tag{4.20} \]
\[ \Psi^*(v;p)F(z;p) = \langle \lambda|K^+(z;p)|\lambda \rangle F(z;p)\Psi^*_\lambda(v;p) + \sum_{k=1}^{\ell(\lambda)+1} \langle \lambda|F(z;p)|\lambda + 1_k \rangle \Psi^*_{\lambda+1_k}(v;p). \tag{4.21} \]

We can show that the solution of the intertwining relations (4.18) – (4.21) is
\[ \Psi^*_\lambda(v;p) = z^*_\lambda(v) G^\lambda_\lambda^{-1}(p) \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} q^{3n/2} (q_1^n - 1)(q_2^n - 1) \tilde{a}_n v^n A^\lambda(q_1^n, q_2^n) \right) \]
\[ \cdot \exp \left( \sum_{n=1}^{\infty} \frac{1 - p^n}{n(1 - p^n)} (q_1^n - 1)(q_2^n - 1) q^{3n/2} \tilde{a}_n v^{-n} A^\lambda(q_1^{-n}, q_2^{-n}) \right), \tag{4.22} \]
where
\[ z^*_\lambda(v) = q^{|\lambda|} \prod_{(i,j) \in \lambda} \left(-q_2^{a_{ij}-1}x_{ij}^{-1}\right) u^{-1} \cdot f(x_{ij}v) = q^{|\lambda|}q^{n(\lambda)}(-v)^{N(\lambda)} u^{-|\lambda|} f\lambda(q_1, q_2)^{N-1}, \tag{4.23} \]
and
\[ G^\lambda_\lambda(p) = \prod_{s \in \lambda} \left( \frac{q_1^{\ell(s)-\ell(s)-1}p}{p_\lambda q_1^{(a(s))} q_2^{\ell(s)-1}p} \right)_\infty. \tag{4.24} \]

For the existence of the dual intertwiner the horizontal spectral parameters of the source and the target Fock spaces have to be related by \( w = -u/v \) [2]. The elliptic dual Fock intertwiner \( \Psi^*_\lambda(v;p) \) is expressed by the trivalent diagram in figure 2 below.
We can observe a similarity to the case of the intertwiner. In fact we have
\[
\Psi^\Lambda_\rho(v; p) = z^\Lambda_\rho(v) J^{\rho - 1}_\Lambda(p) \cdot \Psi^\Lambda_\rho(v; p) \prod_{(i,j) \in \lambda} \xi(q_i^{-1} q_j^{-1} v; p), \tag{4.25}
\]
\[
\Psi^{\prime \Lambda}_\rho(v; p) := \prod_{i,j=1}^{\infty} \xi(q_i^{-1} q_j^{-1} v; p)^{-1}, \tag{4.26}
\]
where \(\xi(z; p)\) is the oscillator part of \(F(z; p)\). Namely, \(E(z; p)\) for \(\Psi^\Lambda_\rho(v; p)\) is simply replaced by \(F(z; p)\) for the dual intertwiner \(\Psi^{\prime \Lambda}_\rho(v; p)\). Again, the factor \(J^{\rho - 1}_\Lambda(p)\) appears by removing the normal ordered product, when we express the dual intertwiner as a composition of those for the vector representation. Note also that the factor \(z^\Lambda_\rho(v)\) depends on the level \(N\) of the horizontal Fock representation.

5 Vertical \(R\)-matrix and elliptic Nekrasov factor

In this section, we determine the \(R\)-matrix corresponding to the vertical Fock representation. According to [36], the quasi-Hopf twist of the universal \(R\)-matrix \(\tilde{\mathcal{R}}\) is
\[
\tilde{\mathcal{R}} = \mathcal{F}(21)(p) \cdot \mathcal{R} \cdot \mathcal{F}^{-1}(p), \tag{5.1}
\]
where \(\mathcal{R}\) is the universal \(R\)-matrix of the original DIM algebra and \(\mathcal{F}(p)\) is the twistor given by (2.7). Here the notation \(\mathcal{F}(21)(p)\) means that we interchange the order of the elements in the tensor product of the expression of \(\mathcal{F}(p)\);
\[
\mathcal{F}(21)(p) = \exp \left( \sum_{n=1}^{\infty} \frac{np^n C_1^n}{\kappa_n (1 - p^n C_1^{-2n})} b_{-n} \otimes b_n \right). \tag{5.2}
\]

According to [21], the universal \(R\)-matrix \(\mathcal{R}\) of DIM algebra factorizes as follows:
\[
\mathcal{R} = q^{-(c^+ \otimes d^+ + d^+ \otimes c^+)} \mathcal{R}_+ \mathcal{R}_0 \mathcal{R}_-, \tag{5.3}
\]
where \(q^c = K_0\) and \(d^+ = d_1\) (the grading operator for the principal degree). See also [28] for computations of the \(R\) matrix for the horizontal Fock representation. What is most relevant in the present paper is the Cartan factor \(\mathcal{R}_0\) with the contribution of the centers:
\[
\mathcal{R}_0' = q^{-(c^+ \otimes d^+ + d^+ \otimes c^+)} \exp \left( - \sum_{n=1}^{\infty} n \kappa_h h_{-n} \otimes h_n \right), \tag{5.4}
\]
where \(h_{\pm n}\) is defined via \(\kappa_n h_{\pm n} = \pm H_{\pm n}\). Recall that any explicit formula of the universal \(R\) matrix depends on the choice of the Borel subalgebra from which the quantum group is reconstructed as the Drinfeld double. It is interesting to find that the \(R\)-matrix \([\rho_{\ell_1} \otimes \rho_{\ell_2}] (\mathcal{R}_0)\), which appears shortly below, coincides with the infinite slope \(R\)-matrix \(R_\infty\) which is ubiquitous in the Khoroshkin-Tolstoy factorization of the slope \(s\) \(R\)-matrix introduced in [60]. From the viewpoint of the elliptic Hall algebra, \(\mathcal{R}_0'\) is a universal \(R\) matrix of the

\footnote{The definition of \(\kappa_n\) in this paper is \(-\kappa_n\) in [21]. We have changed the convention of the \(R\) matrix from [13].}
vertical (or slope infinity) Heisenberg subalgebra [53]. As noticed in [60], $R_{\infty}$ corresponds to multiplication by a class of normal bundles in $K$-theory and is diagonal in the fixed point basis of the torus action.

Since $c^\perp$ and $d^\perp$ commute with $b_\pm$, we obtain

$$\tilde{\mathcal{R}}_0 = q^{-(c^\perp+d^\perp+d^\perp+c^\perp)} \exp \left( \sum_{n=1}^{\infty} \frac{np^n C^n_1}{n \kappa_n \left( 1 - p^n C^n_2 \right)} b_{-n} \otimes b_n \right) \exp \left( - \sum_{n=1}^{\infty} n \kappa_n h_{-n} \otimes h_n \right) \exp \left( - \sum_{n=1}^{\infty} \frac{np^n C^n_2}{n \kappa_n \left( 1 - p^n C^n_2 \right)} b_n \otimes b_{-n} \right).$$

(5.5)

The $R$-matrix of the vertical Fock representation is evaluated as follows;

$$\{ \left[ \rho_{v_1}^\perp \otimes \rho_{v_2}^\perp \right] \left( \mathcal{R}_0 \right) \} \left( \left| \lambda, v_1 \right> \otimes \left| \mu, v_2 \right> \right) = R_{\lambda\mu} \left( v_1 \vphantom{v_2}; v_2 \right) \left( \left| \lambda, v_1 \right> \otimes \left| \mu, v_2 \right> \right),$$

(5.6)

where $\left| \lambda, v_1 \right> \in \mathcal{F}_{v_1}$ and $\left| \mu, v_2 \right> \in \mathcal{F}_{v_2}$. Since $c^\perp = 1$ for the Fock representation and $d^\perp = d_1$ counts the degree of the horizontal spectral parameter, we obtain

$$R_{\lambda\mu} \left( v_1 \vphantom{v_2}; v_2 \right) = q^{-|\lambda|+|\mu|} \exp \left( \sum_{n=1}^{\infty} \frac{1}{1-p^n} \frac{v_1^{-n}}{1-q_1^n} \left[ \sum_{s=1}^{\infty} x_{r,s,\lambda}^n \right] \frac{v_2^n}{n} \frac{\kappa_n}{1-q_1^n} \left[ \sum_{r=1}^{\infty} x_{r,\mu}^n \right] \right) \exp \left( - \sum_{n=1}^{\infty} \frac{p^n}{1-p^n} \frac{v_1^n}{1-q_1^n} \left[ \sum_{s=1}^{\infty} x_{r,s,\lambda}^n \right] \frac{v_2^{-n}}{n} \frac{\kappa_n}{1-q_1^n} \left[ \sum_{r=1}^{\infty} x_{r,\mu}^n \right] \right).$$

(5.7)

To simplify the expression (5.7) the following formula can be used;

$$\frac{1}{1-q_1} \left( \sum_{i=1}^{\infty} q_1^{\lambda_i} q_2^{i-1} \right) = -A_\lambda \left( q_1, q_2 \right),$$

(5.8)

where $A_\lambda(q_1, q_2)$ is defined by (4.12). Then, we see that

$$R_{\lambda\mu} \left( v_1 \vphantom{v_2}; v_2 \right) = q^{-|\lambda|+|\mu|} \exp \left( \sum_{n=1}^{\infty} \frac{1}{1-p^n} \frac{v_2^n}{v_1^n} \frac{\kappa_n}{n} \frac{A_\lambda \left( q_1^n, q_2^n \right) A_\mu \left( q_1^n, q_2^n \right)}{A_\lambda \left( q_1^n, q_2^n \right) A_\lambda \left( q_1^n, q_2^n \right)} \right) \exp \left( - \sum_{n=1}^{\infty} \frac{p^n}{1-p^n} \frac{\kappa_n}{n} \frac{v_1^n}{v_2^n} \frac{A_\mu \left( q_1^n, q_2^n \right)}{A_\lambda \left( q_1^n, q_2^n \right) A_\lambda \left( q_1^n, q_2^n \right)} \right).$$

(5.9)

Let us define the normalized $R$-matrix by

$$\overline{R}_{\lambda\mu} \left( z; p \right) := \frac{R_{\lambda\mu} \left( z; p \right)}{R_{\Theta^{\perp}} \left( z; p \right)},$$

(5.10)

so that $\overline{R}_{\Theta^{\perp}} \left( z; p \right) = 1$. Then $\overline{R}_{\lambda\mu} \left( z; p \right)$ may also be expressed in terms of the theta function $\theta_\mu(x)$ by the following lemma:
Lemma 5.1. 

\[
\exp \left( \sum_{n=1}^{\infty} \frac{1}{1-p^n} \frac{K_n}{n} z^n A_\lambda \left( q_1^{-n}, q_2^{-n} \right) A_\mu \left( q_1^{n}, q_2^{n} \right) \right) \\
= \prod_{(i,j) \in \lambda} \prod_{(k,l) \in \mu} \frac{(zq_3 x_{ij}^{-1}; q_1^\infty)}{(zq_3^{-1} x_{ij}; q_1^\infty)} \frac{(zq_2^{-1} x_{ij}; q_1^\infty)}{(zq_2 x_{ij}^{-1}; q_1^\infty)} \frac{(zq_3^{-1} x_{ij}; p)}{(z x_{ij}^{-1}; p)} .
\]

The result is

\[
\mathcal{R}_{\lambda \mu} \left( \frac{v_1}{v_2}; p \right) = q^{-|\lambda|+|\mu|} \prod_{(i,j) \in \lambda} \prod_{(k,l) \in \mu} \frac{\theta_p \left( q_1^{-1} v_2 x_{ij} \right)}{\theta_p \left( q_1^{-1} v_1 x_{ij} \right)} \frac{\theta_p \left( q_2^{-1} v_2 x_{ij} \right)}{\theta_p \left( q_2^{-1} v_1 x_{ij} \right)} \frac{\theta_p \left( q_3^{-1} v_2 x_{ij} \right)}{\theta_p \left( q_3^{-1} v_1 x_{ij} \right)} \\
\cdot \prod_{(i,j) \in \lambda} \frac{\theta_p \left( v_2 x_{ij} \right)}{\theta_p \left( v_1 x_{ij} \right)} \cdot \prod_{(i,j) \in \mu} \frac{\theta_p \left( q_1 q_2 x_{ij} \right)}{\theta_p \left( v_1^2 x_{ij} \right)} .
\]

By the inversion formula of the theta function (1.12), we can check the unitarity of the normalized \( R \) matrix;

\[
\mathcal{R}_{\mu \lambda} (z^{-1}; p) = \mathcal{R}_{\lambda \mu} (z; p)^{-1}.
\]

We can derive the relation of the \( R \) matrix for the vertical Fock representation and the elliptic Nekrasov factor. Recall that we have obtained

\[
R_{\lambda \mu} (z; p) = q^{-|\lambda|+|\mu|} \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - q_1^{-n} \right) \left( 1 - q_2^{-n} \right) \frac{1}{1-p^n} \sum_{i,j=1}^\infty \left( x_{i,j} x_{j,i}^{-1} z \right)^{-n} \right) \\
\cdot \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} q_1^n p^n \left( 1 - q_1^{-n} \right) \left( 1 - q_2^{-n} \right) \frac{1}{1-p^n} \sum_{i,j=1}^\infty \left( x_{i,j} x_{j,i}^{-1} z \right)^{-n} \right) \\
= q^{-|\lambda|+|\mu|} \prod_{i,j=1}^\infty \frac{\Gamma \left( q_1^{i-j} - \lambda_i^{i-j} z^{-1}; q_1, p \right) \Gamma \left( q_2^{i-j} - \lambda_j^{i-j} z^{-1}; q_2, p \right) \Gamma \left( q_3^{i-j} - \lambda_j^{i-j} z^{-1}; q_3, p \right)}{\Gamma \left( q_1^{i-j} - \lambda_i^{i-j} z^{-1}; q_1, p \right) \Gamma \left( q_2^{i-j} - \lambda_j^{i-j} z^{-1}; q_2, p \right) \Gamma \left( q_3^{i-j} - \lambda_j^{i-j} z^{-1}; q_3, p \right)} .
\]

On the other hand the elliptic Nekrasov factor is (cf. (1.1));

\[
N_{\lambda \mu} (u|q,t,p) = \prod_{i,j=1}^\infty \frac{\Gamma \left( u q_1^{i-j} - \lambda_i^{i-j} - q t^{i-j+1}; q, p \right)}{\Gamma \left( u q_1^{i-j} - \lambda_i^{i-j}; q, p \right)} \frac{\Gamma \left( u t^{i-j+1}; q, p \right)}{\Gamma \left( u t^{i-j}; q, p \right)} .
\]

Hence with \( t = q_2^{-1} \) we see

\[
N_{\lambda \mu} (z|q,t,p) = q^{\lambda|+|\mu|} R_{\mu \lambda} (z^{-1}; p) = q^{\lambda|+|\mu|} \mathcal{R}_{\mu \lambda} (z^{-1}; p) .
\]

By using the combinatorial identity (see appendix E of [3])

\[
\prod_{(i,j) \in \lambda} q^{\mu_i - j} \prod_{(i,j) \in \mu} q^{-\lambda_j + j-1} = \prod_{(i,j) \in \lambda} q^{\mu_i - j} \prod_{(i,j) \in \lambda} q^{-\lambda_j + j-1}
\]

(17.1)
and the inversion formula (1.12), we can prove
\[
\frac{N_{\lambda \mu}(z|q_1,q_2,p)}{N_{\lambda \mu}(q^{-2}z^{-1}|q_1,q_2,p)} = z^{\mu|+|\lambda} q^{\mu|+|\lambda} f_\lambda(q_1,q_2) \frac{f_\mu(q_1,q_2)}{f_\mu(q_1,q_2)},
\]
where the framing factor \( f_\lambda(q_1,q_2) \) is defined by (4.15). The formula (5.18) also confirms the unitarity of the (normalized) \( R \) matrix;
\[
\mathcal{R}_{\mu \lambda}(z^{-1};p) = q^{-\mu|-\lambda|} \frac{N_{\lambda \mu}(z|q_1,q_2,p)}{N_{\lambda \mu}(q^{-2}z^{-1}|q_1,q_2,p)} = q^{\mu|+|\lambda} \frac{N_{\mu \lambda}(q^{-2}z^{-1}|q_1,q_2,p)}{N_{\mu \lambda}(z^{-1}|q_1,q_2,p)} = \mathcal{R}_{\lambda \mu}(z;p)^{-1}.
\]
We can also show that the Fock \( R \)-matrix constructed above is obtained as the coefficient resulting from interchanging the intertwiners and the dual intertwiners themselves. The normal ordering of the vertex operators produces the elliptic Nekrasov factors. Taking the difference of the zero mode factors into account and using (5.18), we arrive at:
\[
\Psi_\mu^*(w;p) \Psi_\lambda(v;p) = \Upsilon \left( q^{-1} \left| \begin{array}{c} \frac{v}{w} \\ 0 \end{array} \right. \right) \Psi_\lambda(v;p) \Psi_\mu^*(w;p),
\]
\[
\Psi_\lambda(v;p) \Psi_\mu(w;p) = \mathcal{R}_{\mu \lambda}(v;w) \cdot \Upsilon \left( q^{-2} \left| \begin{array}{c} \frac{v}{w} \\ p_s \end{array} \right. \right) \Psi_\mu(w;p) \Psi_\lambda(v;p),
\]
\[
\Psi_\lambda^*(v;p) \Psi_\mu^*(w;p) = \mathcal{R}_{\lambda \mu}(w;v) \cdot \Upsilon \left( 1 \left| \begin{array}{c} \frac{v}{w} \\ p_s \end{array} \right. \right) \Psi_\mu^*(w;p) \Psi_\lambda^*(v;p),
\]
where
\[
\Upsilon(\alpha|z;p) \overset{\text{def}}{=} \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{\alpha^n}{(1-p^n)(1-q_1^n)(1-q_2^n)} (z^n - z^{-n}) \right) \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \frac{p_s^n \alpha^{-n}}{(1-p^n)(1-q_1^n)(1-q_2^n)} (z^n - z^{-n}) \right) = \frac{G_2(\alpha z^{-1};p,q_1,q_2)}{G_2(\alpha z;p,q_1,q_2)}. \tag{5.23}
\]
The equations (5.20) – (5.22) are elliptic generalization of eqs. (40) – (42) in [7]. It is remarkable that the exchange relation between \( \Psi_\lambda(v;p) \) and \( \Psi_\mu^*(w;p) \) is undeformed. The elliptic parameter for the exchange relation of the intertwiner is shifted \( p \to p_s \). Note also that
\[
\Upsilon(\alpha|z^{-1};p) = \Upsilon(\alpha|z;p)^{-1}. \tag{5.24}
\]
The emergence of the double elliptic gamma function \( G_2 \) is quite amusing, since it also appears some of computations in six dimensional supersymmetric gauge theory (note that it is symmetric in \( (p,q_1,q_2) \)) and topological strings [30, 31, 43, 44, 49].

6 Elliptic quantum Knizhnik-Zamolodchikov equation

In the elliptic case the quantum Knizhnik-Zamolodchikov (q-KZ) equation is a difference equation for the trace of intertwining operators [16, 17, 40], which is an analogue of the genus one conformal block of two dimensional conformal field theory;
\[
\text{Tr} \left[ q^{\ell_0} \phi_1(z_1) \cdots \phi_n(z_n) \right]. \tag{6.1}
\]
Here \( q = e^{2\pi i \tau} \) (not to be confused with the parameter of the DIM algebra), \( \tau \) is the modulus of the torus and \( L_0 \) is the zero mode of the Virasoro algebra. Contrary to the case of the vacuum expectation values (matrix elements) of the product of intertwining operators, which corresponds to the genus zero conformal block on \( \mathbb{P}^1 \), the shift parameter is not fixed for the trace.

Let us consider the trace of the intertwining operators;\(^9\)

\[
\mathfrak{S}_n \left( \vec{z}, \vec{w} | \vec{\lambda}, \vec{\mu} \right) := \text{Tr}_{(N)} \left[ \tilde{Q}^{d_1} Q^{d_2} \Psi^{*}_{\mu_1}(w_1) \cdots \Psi^{*}_{\mu_n}(w_n) \Psi_{\lambda_1}(z_1) \cdots \Psi_{\lambda_n}(z_n) \right],
\]

(6.2)

where \( d_1 \) and \( d_2 \) are the grading operators that count the degree of the horizontal and the vertical parameters (see section 2.3 of [13]). In particular

\[
\Psi_{\lambda}(Qz) = Q^{d_2} \Psi_{\lambda}(z) Q^{-d_2}, \quad \Psi^{*}_{\mu}(Qw) = Q^{d_2} \Psi^{*}_{\mu}(w) Q^{-d_2}.
\]

(6.3)

Note that we can express the trace of the intertwining operators (6.2) by the diagram in figure 3 below.

When we take the trace, the initial and the final Fock spaces must have the same level \( N \) and the same spectral parameter \( u \). Hence the number of the intetwiners and the dual intertwiners should agree. Since we can shift the horizontal spectral parameter by \( \tilde{Q}^{d_1} \), we make the spectral parameters coincide by tuning the parameter \( \tilde{Q} \) as

\[
\tilde{Q} = \prod_{i=1}^{n} \frac{w_i}{z_i}.
\]

(6.4)

Hence there remains a free parameter \( Q \). We can derive a difference equation with the shift parameter \( Q \) as follows; let us first consider the case \( z_k \to Qz_k \). By using (5.21) we can move \( \Psi_{\lambda_k}(Qz_k) \) to the rightmost position in the trace, then by the cyclic property of the trace it is moved to the leftmost position. Here it is important that this operation causes the change of the level and the spectral parameter of each intertwiner and dual intertwiner. We need the compensating factor \( C \) associated with it. Then the commutation with \( \tilde{Q}^{d_1} \) adjusts the zero mode factor of \( \Psi_{\lambda_k}(Qz_k) \) appropriately by the scaling of the horizontal spectral parameter \( u \) and the commutation with \( Q^{d_2} \) cancels the \( Q \)-shift of \( z_k \). Finally by using (5.20) and (5.21) again we can move \( \Psi_{\lambda_k}(z_k) \) to the original position. Note that (5.20)

\(^9\)For simplicity in this section we will suppress the \( p \)-dependence of the intertwiners.
and (5.21) already take care of the change of the level and the spectral parameter associated with the exchange of the intertwiners. Consequently no additional factors arise when we use them. After all these operations we arrive at

\[
Q^{z_k} q^{-\frac{a}{w_j}} \mathcal{G}_n(z, w) \bar{\lambda}, \bar{\mu} = \mathcal{C} \cdot \prod_{i=1}^{n} \mathcal{T}(q^{-1} \frac{z_i}{w_j}; 0) \prod_{i < k} \mathcal{R}_{\lambda_i \lambda_k} \left( \frac{z_i}{w_j}; p_s \right)^{-1} \mathcal{T}(q^{-2} \frac{z_k}{z_i}; p_s) \prod_{k < j} \mathcal{R}_{\lambda_k \lambda_j} \left( \frac{Q z_k}{z_j}; p_s \right) \mathcal{T}(q^{-2} \frac{Q z_k}{z_j}; p_s) \cdot \mathcal{G}_n(z, w) \bar{\lambda}, \bar{\mu}. \tag{6.5}
\]

Similarly in the case \( w_k \rightarrow Qw_k \), we can use (5.20) and (5.22) and obtain

\[
Q^{w_k} q^{-\frac{a}{w_i}} \mathcal{G}_n(z, w) \bar{\lambda}, \bar{\mu} = \mathcal{C}^* \cdot \prod_{i=1}^{n} \mathcal{T}(q^{-1} \frac{z_i}{Qw_k}; 0) \prod_{k < \ell} \mathcal{R}_{\mu_k \mu_\ell} \left( \frac{w_k}{w_\ell}; p \right)^{-1} \mathcal{T}(1 \frac{Qw_k}{w_\ell}; p) \prod_{\ell < k} \mathcal{R}_{\mu_k \mu_\ell} \left( \frac{w_k}{w_\ell}; p \right)^{-1} \mathcal{T}(1 \frac{w_k}{w_\ell}; p) \cdot \mathcal{G}_n(z, w) \bar{\lambda}, \bar{\mu}, \tag{6.6}
\]

with the factor \( \mathcal{C}^* \) for the move of \( \Psi^*_\mu(Qw_k) \) from the rightmost to the leftmost in the trace. We can regard these \( Q \)-difference equations as a generalization of elliptic \( q \)-KZB equation to DIM algebra.

### 6.1 Computation of \( \mathcal{C} \)

We define \( \mathcal{C} \) as follows:

\[
\mathcal{T} \left[ Q^{d_1} Q^{-d_1} Q^{d_2} \Psi^*_\mu_1(w_1) \cdots \Psi^*_\mu_n(w_n) \Psi_{\lambda_1}(z_1) \cdots \Psi_{\lambda_n}(z_n) \Psi_{\lambda_k}(Q z_k) \right] = \mathcal{C} \cdot \mathcal{T} \left[ \tilde{Q}^{d_1} Q^{d_2} \Psi_{\lambda_k}(z_k) \Psi^*_\mu_1(w_1) \cdots \Psi^*_\mu_n(w_n) \Psi_{\lambda_1}(z_1) \cdots \Psi_{\lambda_n}(z_n) \right]. \tag{6.7}
\]

Recall that when \( \Psi_\lambda(z) \) and \( \Psi_\mu(w) \) act on the Fock space of level \( N \) with the spectral parameter \( u \), their zero mode factors are

\[
\Psi_\lambda(z) \sim (-z)^{-N|\lambda|} u^{|\lambda|} f_A^{N-1}, \quad \Psi_\mu^*(w) \sim (-w)^{N|\mu|} u^{-|\mu|} f_\mu^{N-1}. \tag{6.8}
\]

Chasing the change of the level and the spectral parameter, we find the total zero mode factor on the left hand side is

\[
\mathcal{A} = \left[ (-Q z_k)^{-N|\lambda_k|} u^{|\lambda_k|} f_{\lambda_k}^{N-1} \right] \cdot \prod_{i=1}^{n} (-z_i)^{-(N+n-i+1)|\lambda_i|} \left( u \cdot (-Q z_k) \prod_{j=i+1}^{n} (-z_j) \right)^{|\lambda_i|} f_{\lambda_i}^{-N-2-n+i} \]

\[
\cdot \prod_{i=1}^{n} (-u_i)^{(N+i)|\mu_i|} \left( u \cdot (-Q z_k) \prod_{j=1}^{n} (-z_j) \prod_{l=i+1}^{k-1} (-z_l) \right)^{|\mu_i|} f_{\mu_i}^{-N-n+i-1} \]

\[
\cdot \prod_{i=1}^{n} (-w_i)^{(N+i)|\mu_i|} \left( u \cdot (-Q z_k) \prod_{j\neq k} (-z_j) \prod_{l=i+1}^{n} (-w_j) \right)^{-|\mu_i|} f_{\mu_i}^{N+i-1} \tag{6.9}
\]

\[= \mathcal{C} \cdot \mathcal{T} \left[ \tilde{Q}^{d_1} Q^{d_2} \Psi_{\lambda_k}(z_k) \Psi^*_\mu_1(w_1) \cdots \Psi^*_\mu_n(w_n) \Psi_{\lambda_1}(z_1) \cdots \Psi_{\lambda_n}(z_n) \right]. \]
where the initial level and spectral parameter of the horizontal Fock space are $N$ and $u$, respectively.

Next we are going to investigate the right hand side of (6.7). Similar consideration gives the total zero mode factor:

$$
\mathcal{B} = \left[ (-z_k)^{-(N-1)|\lambda_k|} \left( \frac{u \prod_{j \neq k} (-z_j)}{\prod_{j=1}^n (-w_j)} \right)^{|\lambda_k|} f_{\lambda_k}^{N-1} \right] \cdot \left[ \prod_{i=k+1}^n (-z_i)^{-(N+n-i)|\lambda_i|} \left( u \prod_{j=i+1}^n (-z_j) \right)^{|\lambda_i|} f_{\lambda_i}^{-(N-n+i-1)} \right] \cdot \left[ \prod_{i=1}^{k-1} (-z_i)^{-(N+n-i-1)|\lambda_i|} \left( u \prod_{j=i+1}^{k-1} (-z_j) \right)^{-|\mu_i|} f_{\mu_i}^{-N+n+i-2} \right] \cdot \left[ \prod_{i=1}^n (-w_i)^{(N+i-1)|\mu_i|} \left( u \prod_{j=i+1}^n (-z_j) \right)^{-|\mu_i|} \right]. \quad (6.10)
$$

After making the exchange with $Q^{d_2}$ which shifts all the vertical spectral parameters in $\mathcal{A}$ and $\mathcal{B}$; $Q^{d_2} \mathcal{A} = A' Q^{d_2}$ and $Q^{d_2} \mathcal{B} = B' Q^{d_2}$, we obtain

$$
\mathcal{C} = \frac{A'}{B'} = \frac{\prod_{i=1}^n u^{\mu_i} f_{\mu_i}^{N-1}}{\prod_{i=1}^n z_i^{|\lambda_i|} f_{\lambda_i}^{N-1}} \cdot \sum_{k=1}^n |\lambda_i| - \sum_{i=1}^n |\mu_i| \cdot Q \sum_{i=1}^n |\lambda_i| - \sum_{i=1}^n |\mu_i| Q^{-(N+1)|\lambda_k|}. \quad (6.11)
$$

### 6.2 Computation of $\mathcal{C}^*$

Now, we are going to compute the coefficient $\mathcal{C}^*$ defined by

$$
\text{Tr} \left[ \tilde{Q}^{d_1} Q^{d_1} Q^{d_2} \Psi_{\mu_k} (w_1) \cdots \Psi_{\mu_n} (w_n) \Psi_{\lambda_k} (z_1) \cdots \Psi_{\lambda_n} (z_n) \Psi^*_{\mu_k} (Q w_k) \right] = \mathcal{C}^* \cdot \text{Tr} \left[ \tilde{Q}^{d_1} Q^{d_2} \Psi_{\mu_k} (w_k) \Psi^*_{\mu_1} (w_1) \cdots \Psi_{\mu_n} (w_n) \Psi_{\lambda_k} (z_1) \cdots \Psi_{\lambda_n} (z_n) \right]. \quad (6.12)
$$

The total zero mode factor on the left hand side is

$$
\mathcal{M} = \left[ (-Q w_k)^{N|\mu_k|} u^{-|\mu_k|} f_{\mu_k}^{N-1} \right] \cdot \left[ \prod_{i=1}^n (-z_i)^{-(N-1+n-i)|\lambda_i|} \left( \frac{u \prod_{j=1}^n (-z_j)}{Q w_k \prod_{j=i+1}^n (-z_j)} \right)^{|\lambda_i|} f_{\lambda_i}^{-(N-n+i-1)} \right] \cdot \left[ \prod_{i=k+1}^n (-z_i)^{-(N+n-i)|\lambda_i|} \left( \frac{u \prod_{j=i+1}^n (-z_j)}{Q w_k \prod_{j=i+1}^n (-w_j)} \right)^{-|\mu_i|} f_{\mu_i}^{-N+n+i-2} \right] \cdot \left[ \prod_{i=1}^{k-1} (-w_i)^{-(N+i)|\mu_i|} \left( \frac{u \prod_{j=1}^n (-z_j)}{Q \prod_{j=i+1}^n (-w_j)} \right)^{-|\mu_i|} \right]. \quad (6.13)
$$
On the other hand the zero mode factor on the right hand side of (6.12) is

\[
\mathcal{L} = \left[ (-w_k)^{(N+1)|\mu_k|} \left( \frac{u \prod_{j=1}^{n} (-z_j)}{ \prod_{\ell \neq k} (-w_{\ell})} \right)^{-|\mu_k|} f_{\mu_k}^N \right] \\
\cdot \left[ \prod_{i=1}^{n} (-z_i)^{-(N-n-i)|\lambda_i|} \left( \frac{u \prod_{j=i+1}^{n} (-z_j)}{ \prod_{\ell = i+1}^{n} (-w_{\ell})} \right)^{|\lambda_i|} f_{\lambda_i}^{-(N+n-i)-1} D_{\lambda_i} \right] \\
\cdot \left[ \prod_{i=k+1}^{n} (-w_i)^{(N+i)|\mu_i|} \left( \frac{u \prod_{j=i+1}^{n} (-z_j)}{ \prod_{\ell = i+1}^{n} (-w_{\ell})} \right)^{-|\mu_i|} f_{\mu_i}^{N+i-1} \right] \\
\cdot \left[ \prod_{i=1}^{k-1} (-w_i)^{(N+i+1)|\mu_i|} \left( \frac{u \prod_{j=i+1}^{n} (-z_j)}{ \prod_{\ell = i+1}^{n} (-w_{\ell}) \prod_{\ell = k+1}^{n} (-w_{\ell})} \right)^{-|\mu_i|} f_{\mu_i}^{N+i} \right].
\] (6.14)

As before defining \( M' = Q^{d_2} MQ^{-d_2} \) and \( \mathcal{L}' = Q^{d_2} \mathcal{L} Q^{-d_2} \), we obtain

\[
\mathcal{L}' = \frac{M'}{\mathcal{L}'} = \prod_{i=1}^{n} \frac{z^{|\lambda_i|}}{w_{\mu_i}^{|\mu_i|}} \frac{u \sum_{i=1}^{n} |\lambda_i| - \sum_{i=1}^{n} |\mu_i| - \sum_{i=1}^{n} |\lambda_i|}{Q^{-|\mu_k|} Q^{N-|\mu_k|} \prod_{i=1}^{n} |\mu_i| - \sum_{i=1}^{n} |\lambda_i|}. 
\] (6.15)

In [7] a pair of the q-KZ equations for the trace of intertwiners is derived and a solution is given rather explicitly (see eq. (84)). From the results we have derived in section 5, it is easy to guess a generalization of the solution in [7]. For example we can replace the Nekrasov factor (denoted as \( G_{\alpha \beta \gamma} \) in [7]) by the elliptic one. To define a generalization of the remaining factor related to \( \Upsilon (\alpha|z;p) \), let us introduce

\[
\tilde{\Upsilon} (\alpha|z;p,Q) := G_3(\alpha z;p,Q,q_1,q_2) \cdot G_3(\alpha Q z^{-1};p,Q,q_1,q_2) \\
= \exp \left( \sum_{n=1}^{\infty} \frac{\alpha^n (z^n + Q^n z^{-n})}{n (1-p^n) (1-Q^n) (1-q_1^n) (1-q_2^n)} \right) \\
\cdot \exp \left( - \sum_{n=1}^{\infty} \frac{\alpha^{-n} p_{\alpha}^n (z^n + Q^n z^{-n})}{n (1-p^n) (1-Q^n) (1-q_1^n) (1-q_2^n)} \right). 
\] (6.16)

Then the recursion relation (1.17) among multiple elliptic gamma functions implies

\[
\tilde{\Upsilon} (\alpha|Q z;p,Q) = \Upsilon (\alpha|z;p)^{-1} \cdot \tilde{\Upsilon} (\alpha|z;p,Q).
\] (6.17)

We define the building blocks

\[
\Theta_{\mu} (z|p,Q) := \left( \prod_{k=0}^{\infty} \mathcal{N}_{\lambda_k} \left( Q^k z | q_1, q_2, p \right) N_{\mu \lambda} \left( q^{-2} Q^{k+1} z^{-1} | q_1, q_2, p \right) \right) \tilde{\Upsilon} \left( q^{-1} | z ; p, Q \right),
\]
\[
\Phi_{\mu} (z|p,Q) := \left( \prod_{k=0}^{\infty} \mathcal{N}_{\lambda_k} \left( q^{-2} Q^k z | q_1, q_2, p \right) N_{\mu \lambda} \left( q^{-2} Q^{k+1} z^{-1} | q_1, q_2, p \right) \right) \tilde{\Upsilon} \left( q^{-2} | z ; p, Q \right),
\]
\[
\overline{\Phi}_{\mu} (z|p,Q) := \left( \prod_{k=0}^{\infty} \mathcal{N}_{\lambda_k} \left( Q^k z | q_1, q_2, p \right) N_{\mu \lambda} \left( Q^{k+1} z^{-1} | q_1, q_2, p \right) \right) \tilde{\Upsilon} \left( 1 | z ; p, Q \right).
\] (6.18)
It should be not an accident that, if we rewrite \( \tilde{\Upsilon}(\alpha|z;p, Q) \) in terms of the triple elliptic gamma function \( G_3(z;p, Q, q_1, q_2) \), the argument \( z \) agrees with those of \( N_{\lambda\mu} \) and \( N_{\mu\lambda} \) with \( k = 0 \). Then our claim is that

\[
\Theta_n \left( \bar{z}, \bar{w} | \lambda, \mu \right) = \mathcal{P} \cdot \prod_{i,j=1}^{n} \Theta_{\lambda\mu, j} \left( q^{-1} z_i / w_j | 0, Q \right)
\]

(6.19)

where the monomial prefactor

\[
\mathcal{P} := \prod_{i=1}^{n} z_i^{-(N+n-i)|\lambda_i| - \sum_{i=1}^{n} |\mu_i| + \sum_{j<i} |\lambda_j|} \cdot w_i^{(N+i)|\mu_i| + \sum_{j<i} |\mu_j|}
\]

(6.20)

is introduced for the matching of the \( Q \) dependence of the factor \( \mathcal{C} \) and \( \mathcal{C}^* \) evaluated above. The difference equation for (6.19) is derived from those for the building blocks \( \Theta_{\lambda\mu}, \Phi_{\lambda\mu} \) and \( \Phi_{\lambda\mu}^* \). When the shift parameter is \( Q \), by (6.17), a direct computation shows

\[
\frac{\Theta_{\lambda\mu}(Qz|p, Q)}{\Theta_{\lambda\mu}(z|p, Q)} = N_{\lambda\mu}(z|q_1, q_2, p)^{-1} N_{\mu\lambda}(q^{-2} z_1^{-1}|q_1, q_2, p) \Upsilon(q^{-1}|q_2|p)^{-1} = (qz)^{-|\lambda|-|\mu|} (f/\lambda) \Upsilon(q^{-1}|q_2|p)^{-1},
\]

(6.21)

\[
\frac{\Phi_{\lambda\mu}(Qz|p, Q)}{\Phi_{\lambda\mu}(z|p, Q)} = N_{\lambda\mu}(q^{-2} z_1^{-1}|q_1, q_2, p)^{-1} N_{\mu\lambda}(q^{-2} z_1^{-1}|q_1, q_2, p) \Upsilon(q^{-2}|z_2|p)^{-1} = z^{-|\lambda|-|\mu|} (f/\lambda) \bar{R}_{\lambda\mu}(z|p)^{-1} \Upsilon(1|z_2|p)^{-1},
\]

(6.22)

\[
\frac{\Phi_{\lambda\mu}^*(Qz|p, Q)}{\Phi_{\lambda\mu}^*(z|p, Q)} = N_{\lambda\mu}(z|q_1, q_2, p)^{-1} N_{\mu\lambda}(z^{-1}|q_1, q_2, p) \Upsilon(1|z_2|p)^{-1} = z^{-|\lambda|-|\mu|} (f/\lambda) \bar{R}_{\lambda\mu}(z|p) \Upsilon(1|z_2|p)^{-1},
\]

(6.23)

where we have also used (5.16) and (5.18). Hence, taking the power of \( Q \) coming from (6.20) into account, we can see (6.19) satisfies the difference equations (6.5) and (6.6) with exactly the same factor \( \mathcal{C} \) and \( \mathcal{C}^* \). Let us explain the origin of these factors. When we use the cyclic property of the trace and move \( \Psi_{\lambda\mu}(z_k) \) or \( \Psi_{\lambda\mu}^*(w_k) \) from right to left, it should be accompanied by the change of the level and the spectral parameter of the horizontal Fock space in the definition of the trace (6.2). When we move \( \Psi_{\lambda\mu}(z_k) \), the change is \( \text{Tr}_{\mathcal{H}_u^{(N)}} \rightarrow \text{Tr}_{\mathcal{H}_u^{(N+1)}} \). On the other hand, when we move \( \Psi_{\lambda\mu}^*(w_k) \) it is \( \text{Tr}_{\mathcal{H}_u^{(N)}} \rightarrow \text{Tr}_{\mathcal{H}_u^{(N-1)}} \).

We can check the total changes of the zero modes are exactly given by \( \mathcal{C} \) and \( \mathcal{C}^* \) up to the additional powers of \( Q \). Finally the power of \( \tilde{Q} \) comes from the exchange with \( \tilde{Q}^{d_1} \).

Since the coupled equations (6.5) and (6.6) are \( Q \)-difference equations, there is an ambiguity of “integration constant” or \( Q \)-periodic function in general. Let us illustrate this point in the simplest example of \( n = 1 \) with empty partitions;

\[
\Theta_1(z, w|\varnothing, \varnothing) = \exp \left( \sum_{n=1}^{\infty} - \frac{1}{n} \frac{(z/w)^n + Q^n q^{-2n}(w/z)^n}{n(1 - Q^n)(1 - q_1^n)(1 - q_2^n)} \right).
\]

(6.24)
Unfortunately the solution is \( p \) independent in this case. The parameter \( Q \) is identified with the gauge coupling (the parameter of instanton expansion), which is consistent with AGT dictionary \([1]\). Hence the substitution of \( Q = 0 \) gives the perturbative part;

\[
\mathcal{G}_1^{\text{pert}} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{(z/w)^n}{(1 - q_1^n)(1 - q_2^n)} \right). \tag{6.25}
\]

Then the instanton part is

\[
\mathcal{G}_1^{\text{inst}} = \mathcal{G}_1 / \mathcal{G}_1^{\text{pert}} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{Q^n((z/w)^n + q^{-2n}(w/z)^n)}{1 - Q^n(1 - q_1^n)(1 - q_2^n)} \right). \tag{6.26}
\]

On the other hand in this case we can compute the trace directly \([4, 27, 33, 62, 63]\). For example, we quote the formula from \([12]\):

\[
Z^{\text{inst}}(m, Q; q_1, q_2) = \exp \left( \sum_{n=1}^{\infty} \frac{Q^n(m^n - q_1^n)(m^n - q_2^n)}{nm^n(1 - Q^n)(1 - q_1^n)(1 - q_2^n)} \right), \tag{6.27}
\]

where \( m \) is the (exponentiated) mass of the adjoint matter hypermultiplet of \( N = 2^* \) \( U(1) \) gauge theory. Identifying \( m = z/w \), we find a complete agreement of (6.27) and (6.26) up to \( m \) independent factor.

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**A Normalization of the intertwiner**

In this appendix, we provide a detailed computation of the normalization factor \( \mathcal{S}_\lambda \) of the intertwiner. To set the stage, we first fix notations and provide some definitions. For a non-negative integer \( m \) we define

\[
B_m(v) := \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1 - p^n} q_2^{nm} q^{n/2} a_{-n} v^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{\tilde{a}_n v^{-n} q_2^{-nm} q^{-3n/2}}{1 - q_2^{n-1} v} \right). \tag{A.1}
\]

For a partition \( \lambda \), and for \( n \geq \ell(\lambda) \) we define

\[
\tilde{I}_\lambda^{[n]}(v) := \tilde{I}_{\lambda_1-1}(v; p) \tilde{I}_{\lambda_2-1}(q_2 v; p) \cdots \tilde{I}_{\lambda_{n-1}}(q_2^{n-1} v; p), \tag{A.2}
\]

where

\[
\tilde{I}_m(z; p) = \prod_{l=1}^{\infty} \eta \left( q_1^{m+l} z; p \right). \tag{A.3}
\]
and $\eta(z;p)$ is the oscillator part of the horizontal representation of $E(z;p)$ (see (3.42)).
Up to the zero mode factor, $\tilde{I}_n(z;p)$ gives a component of the intertwiner for the vector representation $[9, 13]$. From the definition it is straightforward to show that for $n \geq \ell(\lambda)$

$$
\tilde{\eta}^{[n]}_\lambda(v) B_n(v) = \tilde{\eta}^{[n+1]}_\lambda(v) B_{n+1}(v):
$$

(A.4)

**Remark A.1.** We can rewrite the intertwiner (4.11) as

$$
\Psi_\lambda(v;p) = z_\lambda S_\lambda^{-1} \cdot \tilde{I}^{[\ell(\lambda)]}_\lambda(v) B_{\ell(\lambda)}(v):
$$

$$
= z_\lambda S_\lambda^{-1} \cdot \tilde{I}_{\lambda-1}(v;p) \tilde{I}_{\lambda-1}(q_2v;p) \cdots \tilde{I}_{\lambda(\lambda)-1} (q_2^{\ell(\lambda)-1}v;p) B_{\ell(\lambda)}(v):
$$

(A.5)

Then, from (A.4) we get that

$$
\Psi_\lambda(v;p) = z_\lambda S_\lambda^{-1} \cdot \tilde{\eta}^{[n]}_\lambda(v) B_n(v): \quad (n \geq \ell(\lambda)).
$$

(A.6)

Now following [9], we define the coefficient appearing by removing the normal ordering of $\tilde{\eta}^{[n]}_\lambda(v) B_n(v):$. More precisely, for $n > \ell(\lambda)$,

$$
\tilde{\eta}^{[n]}_\lambda(v) B_n(v) \equiv G^{[n]}_\lambda \tilde{\eta}^{[n]}_\lambda(v) B_n(v),
$$

where $G^{[n]}_\lambda$ is defined by

$$
\tilde{\eta}^{[n]}_\lambda(v) B_n(v) : \equiv G^{[n]}_\lambda \tilde{\eta}^{[n]}_\lambda(v) B_n(v).
$$

(A.8)

Namely we factorize the coefficient into the $\lambda$ dependent part $S_\lambda$ and the $n$ dependent part $G^{[n]}_\lambda$. Hence, $S_\lambda$ should be independent of $n$ as long as $n > \ell(\lambda)$.

From (A.1) and (A.2), it is straightforward to show that for $m > \ell(\lambda)$,

$$
\tilde{I}^{[m]}_0(v;p) B_m(v) = \exp \left( - \sum_{n=1}^{\infty} \frac{1}{1-q_1^n} \frac{1}{n} - \frac{p^n}{1-p^n} \right) \tilde{I}^{[m]}_0(v;p) B_m(v):
$$

(A.9)

Thus, we obtain that

$$
G^{[m]} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{1-q_1^n} \frac{1}{n} - \frac{p^n}{1-p^n} \right).
$$

(A.10)

Similarly, replacing $\tilde{I}^{[m]}_0(v;p)$ by $\tilde{I}^{[m]}_\lambda(v;p)$ we can show

$$
G^{[m]} S_\lambda = \prod_{k=1}^{m-1} \exp \left( - \sum_{n=1}^{k-1} \frac{1}{1-q_1^n} \frac{1}{n} - \frac{p^n}{1-p^n} q_2^n \sum_{l=0}^{n-1} q_1^n q_2^{n(k-l)} q_1^{n(n\lambda_{l+1}-\lambda_{l+1})} \right)
$$

$$
\cdot \prod_{k=1}^{m} \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} - \frac{p^n}{1-p^n} q_2^n q_1^{n(n\lambda_{k-1}-\lambda_{k-1})} \left( \frac{q_1^{\lambda} q_2^{k-1}}{1-q_1^n} \right)^{-n} \right).
$$

(A.11)

According to (2) in page 338 of [45], we have the following identity:

$$
(1-q) \sum_{s \in \lambda} q^{a(s)} t^{s(n+1)+1} = \sum_{i=1}^{n} \left( t - q^\lambda t^{n+1-i} \right) - (1-t) \sum_{1 \leq i < j \leq n} q^{j-i},
$$

(A.12)
where \(a(s)\) and \(\ell(s)\) are the arm-length and the leg-length of the box \(s\) in the partition \(\lambda\), respectively. Equivalently,

\[
\sum_{k=1}^{m-1} \sum_{l=0}^{k-1} q_2^{n(k-l)} q_1^{n(\lambda_k+1-\lambda_{l+1})} = \frac{1}{1-q_2^n} \sum_{i=1}^{m} \left( q_2^n - q_1^{-n\lambda_i} q_2^{n(m+1-i)} \right) - \left( 1 - q_1^{-n} \right) \sum_{s \in \lambda} q_1^{-na(s)} q_2^{n(\ell(s)+1)} .
\]

Applying (A.13) to (A.11), we obtain that

\[
\mathcal{G}^{(m)}_{\lambda} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - p^n}{1 - q_1^n} \right) \prod_{k=1}^{m} \exp \left( \sum_{n=1}^{\infty} \frac{1 - p^n}{n} \frac{1}{1-q_1^n} q_2^{n(m-k+1)} q_1^{-n\lambda_k} \right) \cdot \exp \left( \sum_{n=1}^{\infty} \frac{1 - p^n}{n} \frac{1}{1-q_1^n} q_2^n q_1^{-na(s)} q_2^{n(\ell(s)+1)} \right) .
\]

So we get that

\[
\mathcal{G}_\lambda = \exp \left( - \sum_{n=1}^{\infty} \frac{1 - p^n}{n} \sum_{s \in \lambda} \frac{1}{1-q_1^n} q_2^{-na(s)} q_2^{n(\ell(s)+1)} \right) .
\]

From this we see that \(\mathcal{G}_\lambda\) does not depend on \(m\), as should be.

## B Intertwining relations for \(E\) and \(F\)

The free boson oscillator part of the horizontal representation is

\[
\eta(z; p) = \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{-n/2}(1-p^n)}{q^n - q^{-n}} \tilde{a}_n z^n \right) \cdot \exp \left( - \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{-n/2}(1-p^n)}{q^n - q^{-n}} \tilde{a}_n z^{-n} \right),
\]

\[
\xi(z; p) = \exp \left( - \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{n/2}(1-p^n)}{q^n - q^{n}} \tilde{a}_n z^n \right) \cdot \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \frac{q^{n/2}(1-p^n)}{q^n - q^{-n}} \tilde{a}_n z^{-n} \right).
\]

With the standard choice of the zero modes the elliptic currents are given by

\[
E(z; p) = \frac{(q/z)^{N} u}{(1-q_1)(1-q_2)} \eta(z; p), \quad F(z; p) = \frac{(q/z)^{-N} u^{-1}}{(1-q_1)(1-q_2)} \xi(z; p),
\]
where they act on the horizontal Fock space with level $N$ and the spectral parameter $u$. Note that the zero modes are independent of the elliptic parameter $p$. We have the following OPE relations with the intertwiner:

$$
\eta(z;p)\Psi_\lambda(v;p) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{v}{z} \right)^n \frac{(1-p^n)(1-q_2^n)}{1-p_s^n} \sum_{i=1}^{\infty} q_i^{n\lambda} q_2^{n(i-1)} \right) \eta(z;p) \Psi_\lambda(v;p)
$$

(B.4)

$$
\Psi_\lambda(v;p)\eta(z;p) = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{z}{v} \right)^n \frac{(1-p^n)(1-q_2^n)}{1-p_s^n} \sum_{i=1}^{\infty} q_i^{-n(\lambda-1)} q_2^{-n(i-1)} \right) \eta(z;p) \Psi_\lambda(v;p)
$$

(B.5)

$$
\xi(z;p)\Psi_\lambda(v;p) = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{v}{z} \right)^n q^n(1-q_2^n) \sum_{i=1}^{\infty} q_i^n q_2^{-n(i-1)} \right) \eta(z;p) \Psi_\lambda(v;p)
$$

(B.6)

Assume that $\Psi_\lambda(v;p)$ maps level $N$ and the spectral parameter $u$ to $N+1$ and $w$. The intertwining relation for the vacuum component $F(z;p)\Psi_\phi(v;p) = \Psi_\phi(v;p)F(z;p)$ implies

$$
(q/z)^{-N-1}w^{-1} \left( 1 - \frac{q^v}{z} \right) = (q/z)^{-N}u^{-1} \left( 1 - \frac{z}{q^v} \right).
$$

(B.8)

Hence we obtain the condition $w = -uv$ for the existence of $\Psi_\lambda(v;p)$.

Now let us turn to the intertwining relation for $E(z;p)$:

$$
E(z;p)\Psi_\lambda(v;p) = \sum_{k=1}^{\ell(\lambda)+1} \langle \lambda + 1_k | E(z;p_s) | \lambda \rangle \Psi_{\lambda+1_k}(v;p) + \langle \lambda | K^- (z;p_s) | \lambda \rangle \Psi_\lambda(v;p)E(z;p),
$$

(B.9)

Up to the zero mode factors we have

$$
\eta(z;p) \Psi_\lambda(v;p)
$$

$$
= \prod_{i=1}^{\infty} \left( p_s q_1^{i-1} q_2^{-2} (v/z) ; p_s \right) \left( q_1^{-i} q_2^{-i} (v/z) ; p_s \right) \left( q_1^{i-1} q_2^{i-1} (v/z) ; p_s \right) \eta(z;p) \Psi_\lambda(v;p).
$$

(B.10)
If a given rational function is
we assume that
the form
Fock space.

\[
p = 0
\]

Hence, incorporating the zero mode factor with \( w = -uv \), we find

\[
(1 - q_1)(1 - q_2) \left( E (z; p) \Psi_\lambda (v; p) - \langle \lambda | K^- (z; p) | \lambda \rangle \Psi_\lambda (v; p) E (z; p) \right)
\]

\[
= \left( \frac{q}{z} \right)^{N+1} (-uv) \prod_{i=1}^{\infty} \left( p \ast q_i^{\lambda} q_i^2 (v/z); p \ast \infty \right) \left( q_i^{\lambda} q_i^{-2} (v/z); p \ast \infty \right) \ast : \eta (z; p) \Psi_\lambda (v; p) : \right)
\]

\[
= \left( \frac{1}{1 - q_2^{(\lambda)} (v/z)} \prod_{i=1}^{\ell (\lambda)} 1 - q_i^{\lambda} q_i^2 (v/z) \right) \left( -z/v \left( 1 - q_2^{(\lambda)} (z/v) \right) \prod_{i=1}^{\ell (\lambda)} 1 - q_i^{\lambda} q_i^2 (z/v) \right).
\]

(B.12)

Note that the last factor in the big parentheses is independent of the elliptic parameter \( p \) and \( p \) dependence appears only through the factor \( (p \ast X; p \ast \infty) \), which becomes trivial when \( p = 0 \). The overall monomial factor is nothing but the zero mode \( \langle 3.45 \rangle \) for the target Fock space.

Now to evaluate the last factor we use the following lemma; for a rational function of the form

\[
f_+(z) = \prod_{i=1}^{n} \frac{1 - \alpha_i z}{1 - \beta_i z},
\]

we assume that \( \beta_j \) are all distinct and

\[
\prod_{i=1}^{n} \alpha_i = \prod_{i=1}^{n} \beta_i.
\]

Namely all the poles of \( f_+(z) \) are simple and

\[
\lim_{z \rightarrow \infty} f_+(z) = 1.
\]

(B.15)

If a given rational function is

\[
\tilde{f}_+(z) = \prod_{j=1}^{m} \frac{1 - \alpha_j z}{1 - \beta_j z}, \quad \prod_{i=1}^{n} \alpha_i = \prod_{j=1}^{m} \beta_j,
\]

(B.16)
with $m < n$ like (B.12). We can multiply $(1 - z)^{n-m}$ and consider
\[
f_+ (z) = (z - 1)^{n-m} \tilde{f}_+ (z). \tag{B.17}
\]
By using the condition (B.14), we can see
\[
f_- (z) = \prod_{i=1}^{n} \frac{1 - \alpha_i^{-1} z^{-1}}{1 - \beta_i^{-1} z^{-1}}, \tag{B.18}
\]
satisfies
\[
f_+ (z) = f_- (z) \tag{B.19}
\]
for $z \neq \beta_j^{-1}$. Then the following formula holds:
\[
f_+ (z) - f_- (z) = \sum_{k=1}^{n} \delta (\beta_k z) \prod_{i \neq k}^{n} \frac{1 - \beta_k z}{1 - \beta_i z}. \tag{B.20}
\]
Note that the coefficients of the delta function are the residues of $f_+ (z)$ at the corresponding poles.

We can prove (B.20) as follows; since $f_+ (z)$ is holomorphic on $\mathbb{P}^1$ with only simple poles at $z = \beta_j^{-1}$, the partial fraction decomposition of $f_+ (z)$ is\textsuperscript{10}
\[
f_+ (z) = 1 + \sum_{k=1}^{n} \frac{c_k}{1 - \beta_k z}, \quad \sum_{k=1}^{n} c_k = 0, \tag{B.21}
\]
where $c_k$ is the residue of $f_+ (z)$ at $z = \beta_k^{-1}$. Then by (B.19), we must have
\[
f_- (z) = 1 - \sum_{i=1}^{n} \frac{c_k \beta_k^{-1} z^{-1}}{1 - \beta_k z}, \tag{B.22}
\]
and hence
\[
f_+ (z) - f_- (z) = \sum_{k=1}^{n} c_k \left( \frac{1}{1 - \beta_k z} + \frac{\beta_k^{-1} z^{-1}}{1 - \beta_k z} \right) = \sum_{k=1}^{n} c_k \delta (\beta_k z). \tag{B.23}
\]
We can check that the last factor in (B.12) satisfies the above assumptions. Applying the lemma we obtain
\[
(1 - q_1) (1 - q_2) \left( E (z; p) \Psi_\lambda (v; p) - \langle \lambda | K^- (z; p_s) | \lambda \rangle \Psi_\lambda (v; p) E (z; p) \right)
= \left( \frac{q}{q_1 q_2 k_{1-1} v} \right)^{N+1} (-uv)^{\ell (\lambda)+1} \prod_{i=1}^{\infty} \left( (p_s q_i^{\lambda_i - \lambda_k} q_z^{i-k+1}; p_s) (p_s q_i^{\lambda_i - \lambda_k} q_z^{i-k-1}; p_s) \right) \times \sum_{k=1}^{\ell (\lambda)} \delta \left( q_i^{\lambda_k} q_2^{k-1} v \right) \prod_{i \neq k} \frac{1 - q_1^{\lambda_i - \lambda_k} q_2^{i-k+1}}{1 - q_1^{\lambda_i - \lambda_k} q_2^{i-k}} \cdot \eta \left( q_i^{\lambda_k} q_2^{k-1} v; p \right) \Psi_\lambda (v; p). \tag{B.24}
\]
\textsuperscript{10}Note (B.15).
Since
\[ \eta (q_1^{k} q_2^{k-1} ; p) \Psi_{\lambda} (v ; p) = z_{\lambda} \frac{G_{\lambda+1k}}{G_{\lambda} z_{\lambda+1k}} \Psi_{\lambda+1k} (v ; p), \] (B.25)

using (3.30) and assuming \((\lambda + 1) = 1\), we obtain recursion relations for the normalization factor \(G_{\lambda}\) and the zero mode factor \(z_{\lambda}\):

\[ \frac{z_{\lambda+1k}}{z_{\lambda}} = q_2^{k-1} (-uv) \left( \frac{q}{q_1} \right)^{N+1}, \] (B.26)

and

\[ \frac{G_{\lambda+1k}}{G_{\lambda}} \prod_{i \neq k} \frac{(q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty (p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}{(p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty} = q_2^{k-1} \prod_{i=1}^{\infty} \theta_{p_*} (q_1^{\lambda-k} q_2^{k-1}) \theta_{p_*} (q_1^{\lambda-k+1} q_2^{k-1}) \theta_{p_*} (q_1^{\lambda-k} q_2^{k-1}) \theta_{p_*} (q_1^{\lambda-k+1} q_2^{k-1}). \] (B.27)

Note that we have adjusted the monomial factor \(q_2^{k-1}\) between \(z_{\lambda}\) and \(G_{\lambda}\) to simplify the recursion relation for \(G_{\lambda}\) as follows:

\[ \frac{G_{\lambda+1k}}{G_{\lambda}} = \prod_{i=1}^{k-1} \frac{(p_*) \infty (p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}{(p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty} \prod_{i=k+1}^{\infty} \frac{(p_*) \infty (p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}{(p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}. \] (B.28)

We note the \(p\) dependent factor in the recursion relation is

\[ \prod_{i=1}^{k-1} \frac{(p_*) \infty (p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}{(p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty} \prod_{i=k+1}^{\infty} \frac{(p_*) \infty (p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}{(p_*) \infty (q_1^{\lambda-k} q_2^{k-1} ; p_*)_\infty}. \] (B.29)

It is striking that this is the same as the remainder factor \(R_{\lambda}^{(k)}\) appearing in section 3.3 by the change of variables \((q_1, q_2, p) \rightarrow (q_1^{-1}, q_2^{-1}, p_*)\). This means the base change discussed there eliminates the above \(p\) dependence by employing the “second” formula for the vertical Fock representation derived in section 3.3. When \(p \rightarrow 0\) or after the base change, the recursion reduces to

\[ \frac{G_{\lambda+1k}}{G_{\lambda}} = \prod_{i=1}^{k-1} \frac{1 - q_1^{\lambda-k} q_2^{k-i+1}}{1 - q_1^{\lambda-k+1} q_2^{k-i}} \prod_{i=k+1}^{\infty} \frac{1 - q_1^{\lambda-k} q_2^{i-k}}{1 - q_1^{\lambda-k+1} q_2^{i-k+1}}. \] (B.30)

With the initial condition \(G_{\lambda} = 1\) the recursion relation is solved by

\[ G_{\lambda} (q_1, q_2) = \prod_{s \in \lambda} \left( 1 - q_1^{-a(s)} q_2^{(s)+1} \right). \] (B.31)
After \((q_1, q_2) \rightarrow (q_1^{-1}, q_2^{-1})\), it agrees with the standard normalization factor for the integral form of the Macdonald function. In general for \(p \neq 0\) the solution of the recursion relation is

\[
G_\lambda(q_1, q_2; p) = \frac{\prod_{s \in \lambda} (q_1^{-a(s)}q_2^{\ell(s)+1}; q_2^{\ell(s)+1}; q_2; p)_\infty}{\prod_{s \in \lambda} (pq_1^{-a(s)}q_2^{\ell(s)+1}; q_2^{\ell(s)+1}; q_2; p)_\infty},
\]

which agrees with the result in appendix A. Finally the recursion relation for \(z_\lambda\) is solved by

\[
z_\lambda = q_2^{n(\lambda)}(-v)^{-N|\lambda|}u^{|\lambda|}f_\lambda(q_1, q_2)^{-N-1},
\]

where

\[
n(\lambda) := \sum_{k=1}^{\infty} \lambda_k(k - 1).
\]

C Free field representation and SU(4) Omega background

In this appendix we show an interesting connection of the free field representation employed in this paper and SU(4) Omega background \((q_1, q_2, q_3, q_4)\) or the equivariant parameters of the toric action on \(\mathbb{C}^4\), which opens a way of interpreting our computation from the viewpoint of eight dimensional gauge theory, or the spiked instantons of Nekrasov [55]. To motivate the SU(4) Omega background, let us look at the affine quiver \(\tilde{A}_0\) with a single node and a single loop (a.k.a. the Jordan quiver). Since it has a single node, the \(q\)-deformed Cartan matrix has a single component \(C = (1 - \mu^{-1})(1 - q^{-1}\mu)\), where \(\mu\) is a parameter associated with the loop of \(\tilde{A}_0\) quiver. The quiver gauge theory for \(\tilde{A}_0\) is the supersymmetric gauge theory with adjoint hypermultiplet, usually called \(N = 2^*\) theory and the parameter \(\mu\) is physically the exponentiated mass parameter. If we follow the prescription of [26] and [38], the commutation relation of the so-called “root boson” takes the form:

\[
[\lambda_n, \lambda_m] = -n(1 - q_1)(1 - q_2)C\delta_{n+m,0}
\]

\[
= -n(1 - q_1)(1 - q_2)(1 - q_3)(1 - q_4)\delta_{n+m,0},
\]

where we have defined \(q_3 = \mu^{-1}\) and \(q_4 = q^{-1}\mu\) with \(q = q_1q_2\). We are going to show that for each pair \((ij)\) with \(1 \leq i < j \leq 4\), there exists a Fock representation of the quasi-Hopf twist of DIM algebra with the central charge \(C = \sqrt{q_iq_j}\). Thus, we obtain a family of six Fock representations with various central charges, which seems to match with the six stacks of \(D3\) configuration for the spiked instanton, where the pair \((ij)\) specifies a choice of codimension four subspace of \(\mathbb{C}^4\) which are fixed by the toric action

\[
(z_1, z_2, z_3, z_4) \longrightarrow (q_1z_1, q_2z_2, q_3z_3, q_4z_4), \quad q_1q_2q_3q_4 = 1.
\]

One can define a completely \(S_4\) symmetric deformed Heisenberg algebra

\[
[a_n, a_m] = -n \left(1 - q_1^{\pm n}\right) \left(1 - q_2^{\pm n}\right) \left(1 - q_3^{\pm n}\right) \left(1 - q_4^{\pm n}\right)\delta_{n+m,0},
\]
and an \( S_4 \) symmetric vertex operator

\[
\Phi(z) = \exp \left( \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n(1-q_k^n)} z^{-n} \right). \tag{C.4}
\]

Then we define a quartet of the screening operators by

\[
S^{(k)}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{a_{-n}}{n(1-q_k^n)} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n(1-q_k^n)} z^{-n} \right), \quad (1 \leq k \leq 4) \tag{C.5}
\]

which satisfies

\[
\Phi(z) =: S^{(k)}(z) S^{(k)}(q_k z^{-1}) :. \tag{C.6}
\]

Writing the screening operator as

\[
S^{(k)}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} S^{(k)}_{-n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} S^{(k)}_n z^{-n} \right), \tag{C.7}
\]

we have the commutation relation

\[
[S^{(k)}_n, S^{(k)}_m] = -\frac{n}{n-m} \Pi_{k \neq k'} \frac{1-q_{k'}^n}{1-q_k^n} \delta_{n,m}. \tag{C.8}
\]

Computing OPE coefficients we obtain

\[
S^{(k)}(z) S^{(k)}(w) = \frac{\theta_{q_k} (w/z) \theta_{q_k} (q_i^{-1} q_j^{-1} w/z) \theta_{q_k} (q_i^{-1} z/w) \theta_{q_k} (q_j^{-1} z/w)}{\theta_{q_k} (z/w) \theta_{q_k} (q_i^{-1} q_j^{-1} z/w) \theta_{q_k} (q_i^{-1} w/z) \theta_{q_k} (q_j^{-1} w/z)} S^{(k)}(w) S^{(k)}(z), \tag{C.9}
\]

where \( \{i, j, k, \ell\} = \{1, 2, 3, 4\} \). Using the inversion formula (1.12), we can rewrite the relation (C.9) as

\[
S^{(k)}(z) S^{(k)}(w) = G^{(ij)}(w/z; q_k) S^{(k)}(w) S^{(k)}(z), \tag{C.10}
\]

where

\[
G^{(ij)}(u; q_k) := \frac{\theta_{q_k} (q_i u) \theta_{q_k} (q_j u) \theta_{q_k} (q_i^{-1} q_j^{-1} u)}{\theta_{q_k} (q_i^{-1} u) \theta_{q_k} (q_j^{-1} u) \theta_{q_k} (q_i q_j u)}. \tag{C.11}
\]

Note that since \( \theta_{p^{-1}}(u) = \theta_{p}(u^{-1})^{-1} \), we have

\[
G^{(ij)}(u; q_k^{-1}) = G^{(ij)}(u; q_k)^{-1}. \tag{C.12}
\]

\( G^{(ij)} \) is related to the structure function used in this paper by \( G^{(12)} = g^{-1} \). Later we will take \( q_3 = p_3^{-1} \) and (C.12) shows the consistency of this choice.

The commutation relation of the screening operators of different kind generates a new operator, which we identify with the Cartan current. For convenience let us denotes the OPE factors of two operators \( A(z) \) and \( B(w) \) by \( c(A(z), B(w)) \), namely

\[
A(z)B(w) = c(A(z), B(w)) : A(z)B(w) :, \quad |z| > |w|. \tag{C.13}
\]
We have
\[
c(S^{(k)}(z), S^{(\ell)}(w)) = \frac{(1 - q_k q_j \frac{w}{z}) (1 - q_k q_j \frac{w}{z})}{(1 - q_k \frac{w}{z}) (1 - q_\ell \frac{1}{w})},
\]
\[
c(S^{(\ell)}(w), S^{(k)}(z)) = \frac{(1 - q_\ell q_j \frac{z}{w}) (1 - q_\ell q_j \frac{z}{w})}{(1 - q_\ell \frac{1}{w}) (1 - q_k \frac{w}{z})}.
\]

(C.14)

Hence the support of the commutation relation \([S^{(k)}(z), S^{(\ell)}(w)]\) is only at the simple poles; \(q_i w/z = 1\) and \(q_j^{-1} w/z = 1\). Computing the residues there, we find
\[
\left[ S^{(k)}(z), S^{(\ell)}(w) \right] = \frac{(1 - q_i)(1 - q_j)}{1 - q_i q_j} \left( \frac{1}{1 - q_k \frac{w}{z}} - \frac{1}{1 - q_\ell \frac{1}{w}} + \frac{1}{1 - q_k \frac{1}{w}} - \frac{1}{1 - q_\ell \frac{w}{z}} \right) : S^{(k)}(z) S^{(\ell)}(w) :.
\]

(C.15)

Hence introducing \(\psi^{(k\ell)}(z) := :S^{(k)}(q_k z) S^{(\ell)}(z) := :S^{(k)}(z) S^{(\ell)}(q_\ell z) :\), we can express the commutation relation as
\[
\left[ S^{(k)}(z), S^{(\ell)}(w) \right] = \frac{(1 - q_i)(1 - q_j)}{1 - q_i q_j} \left( \delta \left( q_k \frac{w}{z} \right) \psi^{(k\ell)}(w) - \delta \left( q_\ell \frac{z}{w} \right) \psi^{(k\ell)}(z) \right).
\]

(C.17)

More explicitly the Cartan current is
\[
\psi^{(k\ell)}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1 - q_k^n q_\ell^n}{n(1 - q_k^n)(1 - q_\ell^n)} a_n z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{1 - q_k^n q_\ell^n}{n(1 - q_k^n)(1 - q_\ell^n)} a_n z^{-n} \right).
\]

(C.18)

To express the OPE factor of the Cartan current, it is convenient to temporarily use the notation
\[
q_1 = q_i, \quad q_2 = q_j, \quad q_3 = q_i^{-1} q_j^{-1}.
\]

(C.19)

Then we have
\[
c(\psi^{(k\ell)}(z), \psi^{(k\ell)}(w)) = \prod_{m=1}^{3} \frac{q_{m, q_k} w^{1/w} ; q_k, q_{\ell}^{-1}}{q_{m, q_{\ell^{-1}}} q_k, q_{\ell}^{-1}^{-1} q_{m, q_k}^{-1} q_k, q_{\ell}^{1}} \frac{q_{m, q_{\ell^{-1}}}^{-1} q_k, q_{\ell}^{-1}^{-1} q_{m, q_k}^{-1} q_k, q_{\ell}^{1}}{q_{m, q_{k^{-1}}}^{-1} q_k, q_{\ell}^{-1}^{-1} q_{m, q_k}^{-1} q_k, q_{\ell}^{1}}.
\]

(C.20)

Using the elliptic gamma function, we can simplify the exchange relation
\[
\psi^{(k\ell)}(z) \psi^{(k\ell)}(w) = \prod_{m=1}^{3} \frac{\Gamma_{q_k, q_{\ell^{-1}}^{-1}} q_{m, q_k} w^{1/w} ; q_k, q_{\ell^{-1}}} {\Gamma_{q_{m, q_k}^{-1}} q_k, q_{\ell^{-1}}^{-1} q_{m, q_k}^{-1}} \frac{\Gamma_{q_{m, q_{\ell^{-1}}}^{-1}} q_k, q_{\ell^{-1}}^{-1} q_{m, q_k}^{-1} q_k, q_{\ell}^{1}} {\Gamma_{q_{m, q_k}^{-1}} q_k, q_{\ell^{-1}}^{-1} q_{m, q_k}^{-1} q_k, q_{\ell}^{1}} \psi^{(k\ell)}(w) \psi^{(k\ell)}(z).
\]

(C.21)

\[\text{After the quasi-Hopf twist, we can treat } \psi^{\pm} \text{ on an equal footing.}\]
Finally by the difference relation,
\[ \Gamma_{p_1,p_2} (p_1 u) = \theta_{p_2} (u) \Gamma_{p_1,p_2} (u), \quad \Gamma_{p_1,p_2} (p_2 u) = \theta_{p_1} (u) \Gamma_{p_1,p_2} (u) \] (C.22)
we obtain
\[
\psi^{(k\ell)} (z) \psi^{(k\ell)} (w) = \prod_{m=1}^{3} \frac{G^{(ij)} \left( \frac{w}{z}, q_k \right)}{G^{(ij)} \left( \frac{w}{z}, q_{\ell}^{-1} \right)} \psi^{(k\ell)} (w) \psi^{(k\ell)} (z)
\]
\[
= G^{(ij)} \left( \frac{w}{z}, q_k \right) G^{(ij)} \left( \frac{w}{z}, q_{\ell} \right) \psi^{(k\ell)} (w) \psi^{(k\ell)} (z),
\] (C.23)
which is manifestly symmetric in \( i \leftrightarrow j \) and \( k \leftrightarrow \ell \). In the present \( S_4 \) symmetric formulation the exchange relations of \( S^{(k)}(z), S^{(\ell)}(z) \) and \( \psi^{(k\ell)}(w) \) are\(^{\text{12}}\)
\[
S^{(k)}(z) \psi^{(k\ell)}(w) = G^{(ij)}(q_k w/z; q_k) \psi^{(k\ell)}(w) S^{(k)}(z) = G^{(ij)}(w/z; q_k) \psi^{(k\ell)}(w) S^{(k)}(z),
\]
\[
S^{(\ell)}(z) \psi^{(k\ell)}(w) = G^{(ij)}(q_k w/z; q_\ell) \psi^{(k\ell)}(w) S^{(\ell)}(z) = G^{(ij)}(w/z; q_\ell) \psi^{(k\ell)}(w) S^{(\ell)}(z),
\] (C.24)
where we have used \( G^{(ij)}(p_x; p) = G^{(ij)}(u; p) \) which follows from \( \theta_p(p_x) = -x^{-1} \theta_p(x) \).

In summary, we have obtained a sextet of the Fock representations of the quasi-Hopf twist of DIM algebra; \( \mathcal{F}^{(k\ell)} = \mathcal{F}^{(k\ell)} \) \( (1 \leq k < \ell \leq 4) \) generated by \( S^{(k)}(z), S^{(\ell)}(z) \) and \( \psi^{(k\ell)}(z) \). Their commutation relations are:
\[
S^{(k)}(z) S^{(k)}(w) = G^{(ij)} \left( \frac{w}{z}, q_k \right) S^{(k)}(w) S^{(k)}(z),
\]
\[
\psi^{(k\ell)}(z) \psi^{(k\ell)}(w) = G^{(ij)} \left( \frac{w}{z}, q_k \right) G^{(ij)} \left( \frac{w}{z}, q_\ell \right) \psi^{(k\ell)}(w) \psi^{(k\ell)}(z),
\]
\[
S^{(k)}(z) \psi^{(k\ell)}(w) = G^{(ij)} \left( \frac{w}{z}, q_k \right) \psi^{(k\ell)}(w) S^{(k)}(z),
\]
\[
[S^{(k)}(z), S^{(\ell)}(w)] = \frac{(1 - q_k)(1 - q_\ell)}{1 - q_k q_\ell} \left( 1 - q_k \frac{w}{z} \right) \psi^{(k\ell)}(w) - \delta \left( q_\ell \frac{z}{w} \right) \psi^{(k\ell)}(z).
\] (C.28)
where \( \{ i, j, k, \ell \} = \{ 1, 2, 3, 4 \} \). In fact one can check that
\[
E(z) = S^{(k)}(\sqrt{q_k z}), \quad F(z) = S^{(\ell)}(\sqrt{q_\ell z}),
\]
\[
K^+(z) = \psi^{(k\ell)}(z/\sqrt{q_k}), \quad K^-(z) = \psi^{(k\ell)}(z/\sqrt{q_\ell}),
\] (C.29) (C.30)
gives a representation of the quasi-Hopf twist of DIM algebra with the central charge \( C = \sqrt{q_k q_\ell} \) and the following correspondence of the structure functions;
\[
G^{(ij)}(u; q_k) \leftrightarrow g(u; p_k), \quad G^{(ij)}(u; q_\ell) \leftrightarrow g(u; p)\] (C.31)
The normalization of the commutation relation \([E(z), F(z)]\) is
\[
\hat{g} = \frac{(1 - q_k)(1 - q_\ell)}{1 - q_k q_\ell}.
\] (C.32)

For example, take \( SU(4) \) parameters;
\[
q_1 = q, \quad q_2 = t^{-1}, \quad q_3 = p^{-1}, \quad q_4 = p,
\] (C.33)
\(^{\text{12}}\)Note that \( \psi^{(k\ell)} = \psi^{(k\ell)} \).
then $C = \sqrt{p/p^*} = q$ and for $p \neq 0$ we have

$$
E(z;p) = S^{(3)}(z/\sqrt{p^*}), \quad F(z;p) = S^{(4)}(\sqrt{p}z), \quad (C.34)
$$

$$
K^+(z;p) = \psi^{(34)}(\sqrt{p}z), \quad K^-(z;p) = \psi^{(34)}(z/\sqrt{p}). \quad (C.35)
$$

Note that after the quasi-Hopf twist $K^\pm(z)$ are related by the shift of the spectral parameter. Note also that (C.35) is valid only after the quasi-Hopf twist, since it involves $p \neq 0$.

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References

[1] L.F. Alday, D. Gaiotto and Y. Tachikawa, \textit{Liouville Correlation Functions from Four-dimensional Gauge Theories}, \textit{Lett. Math. Phys.} \textbf{91} (2010) 167 [\texttt{arXiv:0906.3219}] [\texttt{IN}SPIRE].

[2] H. Awata, B. Feigin and J. Shiraishi, \textit{Quantum Algebraic Approach to Refined Topological Vertex}, \textit{JHEP} \textbf{03} (2012) 041 [\texttt{arXiv:1112.6074}] [\texttt{IN}SPIRE].

[3] H. Awata and H. Kanno, \textit{Refined BPS state counting from Nekrasov’s formula and Macdonald functions}, \textit{Int. J. Mod. Phys. A} \textbf{24} (2009) 2253 [\texttt{arXiv:0805.0191}] [\texttt{IN}SPIRE].

[4] H. Awata and H. Kanno, \textit{Changing the preferred direction of the refined topological vertex}, \textit{J. Geom. Phys.} \textbf{64} (2013) 91 [\texttt{arXiv:0903.5383}] [\texttt{IN}SPIRE].

[5] H. Awata, H. Kanno, A. Mironov and A. Morozov, \textit{Elliptic lift of the Shiraishi function as a non-stationary double-elliptic function}, \textit{JHEP} \textbf{08} (2020) 150 [\texttt{arXiv:2005.10563}] [\texttt{IN}SPIRE].

[6] H. Awata, H. Kanno, A. Mironov, A. Morozov, Y. Ohkubo et al., \textit{Anomaly in RTT relation for DIM algebra and network matrix models}, \textit{Nucl. Phys. B} \textbf{918} (2017) 358 [\texttt{arXiv:1611.07304}] [\texttt{IN}SPIRE].

[7] H. Awata, H. Kanno, A. Mironov, A. Morozov, Y. Ohkubo et al., \textit{Generalized Knizhnik-Zamolodchikov equation for Ding-Iohara-Miki algebra}, \textit{Phys. Rev. D} \textbf{96} (2017) 026021 [\texttt{arXiv:1703.06084}] [\texttt{IN}SPIRE].

[8] H. Awata, H. Kanno, A. Mironov, A. Morozov, K. Suetake and Y. Zenkevich, \textit{(q,t)-KZ equations for quantum toroidal algebra and Nekrasov partition functions on ALE spaces}, \textit{JHEP} \textbf{03} (2018) 192 [\texttt{arXiv:1712.08016}] [\texttt{IN}SPIRE].

[9] H. Awata, H. Kanno, A. Mironov, A. Morozov, K. Suetake and Y. Zenkevich, \textit{The MacMahon R-matrix}, \textit{JHEP} \textbf{04} (2019) 097 [\texttt{arXiv:1810.07676}] [\texttt{IN}SPIRE].

[10] H. Awata and Y. Yamada, \textit{Five-dimensional AGT Conjecture and the Deformed Virasoro Algebra}, \textit{JHEP} \textbf{01} (2010) 125 [\texttt{arXiv:0910.4431}] [\texttt{IN}SPIRE].

[11] I. Burban and O. Schiffmann, \textit{On the Hall algebra of an elliptic curve, I}, \textit{Duke Math. J.} \textbf{161} (2012) 1171 [\texttt{math/0505148}].

[12] E. Carlsson, N. Nekrasov and A. Okounkov, \textit{Five dimensional gauge theories and vertex operators}, \textit{Moscow Math. J.} \textbf{14} (2014) 39 [\texttt{arXiv:1308.2465}] [\texttt{IN}SPIRE].

[13] P. Cheewaprunthiukarn and H. Kanno, \textit{MacMahon KZ equation for Ding-Iohara-Miki algebra}, \textit{JHEP} \textbf{04} (2021) 031 [\texttt{arXiv:2101.01420}] [\texttt{IN}SPIRE].
[14] L. Clavelli and J.A. Shapiro, *Pomeron factorization in general dual models*, *Nucl. Phys. B* **57** (1973) 490 [arXiv:1002.3100].

[15] J.-t. Ding and K. Iohara, *Generalization and deformation of Drinfeld quantum affine algebras*, *Lett. Math. Phys.* **41** (1997) 181 [arXiv:1002.3100].

[16] P. Etingof and A. Verchenko, *Traces of Intertwiners for Quantum Groups and Difference Equations, I*, *Duke Math. J.* **104** (2000) 391 [math/9907181].

[17] P. Etingof, O. Schiffmann and A. Verchenko, *Traces of Intertwiners for Quantum Groups and Difference Equations, II*, *Lett. Math. Phys.* **62** (2002) 143 [math/0207157].

[18] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Quantum continuous gl∞: Semi-infinite construction of representations*, *Kyoto J. Math.* **51** (2011) 337 [arXiv:1002.3100].

[19] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida, *A commutative algebra on degenerate CP1 and Macdonald polynomials* *J. Math. Phys.* **50** (2009) [arXiv:0904.2291].

[20] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi and S. Yanagida, *Kernel Function and Quantum Algebras*, *RIMS Kokyuroku* **1689** (2010) 133 [arXiv:1002.2485].

[21] B. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Finite Type Modules and Bethe Ansatz for Quantum Toroidal gl1*, *Commun. Math. Phys.* **356** (2017) 285 [arXiv:1603.02765].

[22] O. Foda, M. Jimbo, T. Miwa, K. Miki and A. Nakayashiki, *Vertex operators in solvable lattice models*, *J. Math. Phys.* **35** (1994) 13 [hep-th/9305100] [arXiv:1002.3100].

[23] E. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, *Commun. Math. Phys.* **146** (1992) 1 [arXiv:1002.3100].

[24] M. Fukuda, Y. Ohkubo and J. Shiraishi, *Non-Stationary Ruijsenaars Functions for κ = t−1/N and Intertwining Operators of Ding-Iohara-Miki Algebra*, *SIGMA* **16** (2020) 116 [arXiv:2002.00243] [arXiv:1002.3100].

[25] A. Garbali and J. de Gier, *The R-Matrix of the Quantum Toroidal Algebra Uq,t(ð1) in the Fock Module*, *Commun. Math. Phys.* **384** (2021) 1971 [arXiv:2004.09241] [arXiv:1002.3100].

[26] M. Ghoneim, C. Kozçaz, K. Kurşun and Y. Zenkevich, *4d higgsed network calculus and elliptic DIM algebra*, *arXiv:2012.15352* [arXiv:1002.3100].

[27] B. Haghighat, A. Iqbal, C. Kozçaz, G. Lockhart and C. Vafa, *M-Strings*, *Commun. Math. Phys.* **334** (2015) 779 [arXiv:1305.6322] [arXiv:1002.3100].

[28] B. Haghighat, C. Kozçaz, G. Lockhart and C. Vafa, *Orbifolds of M-strings*, *Phys. Rev. D* **89** (2014) 046003 [arXiv:1310.1185] [arXiv:1002.3100].

[29] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, *JHEP* **03** (2008) 069 [hep-th/0310272] [arXiv:1002.3100].
[33] A. Iqbal, C. Kozcaz and K. Shabbir, \textit{Refined Topological Vertex, Cylindric Partitions and the U(1) Adjoint Theory}, \textit{Nucl. Phys. B} \textbf{838} (2010) 422 [arXiv:0803.2260] [inSPIRE].

[34] A. Iqbal, C. Kozcaz and S.-T. Yau, \textit{Elliptic Virasoro Conformal Blocks}, arXiv:1511.00458 [inSPIRE].

[35] W. Israel, \textit{Thermo field dynamics of black holes}, \textit{Phys. Lett. A} \textbf{57} (1976) 107 [inSPIRE].

[36] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, \textit{Elliptic Virasoro Conformal Blocks}, arXiv:1511.00458 [inSPIRE].

[37] W. Israel, \textit{Thermo field dynamics of black holes}, \textit{Phys. Lett. A} \textbf{57} (1976) 107 [inSPIRE].

[38] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, \textit{Elliptic algebra }$U_{q,p}(SL_2)$: Drinfeld currents and vertex operators, \textit{Commun. Math. Phys.} \textbf{199} (1999) 605 [math/9802002] [inSPIRE].

[39] T. Kimura and V. Pestun, \textit{Quiver W-algebras}, \textit{Lett. Math. Phys.} \textbf{108} (2018) 1351 [arXiv:1512.08533] [inSPIRE].

[40] T. Kimura and V. Pestun, \textit{Quiver elliptic W-algebras}, \textit{Lett. Math. Phys.} \textbf{108} (2018) 1383 [arXiv:1608.04651] [inSPIRE].

[41] H. Konno, \textit{Elliptic Weight Functions and Elliptic q-KZ Equation}, arXiv:1706.07630 [inSPIRE].

[42] P. Koroteev and A. Sciarappa, \textit{Quantum Hydrodynamics from Large-N Supersymmetric Gauge Theories}, \textit{Lett. Math. Phys.} \textbf{108} (2018) 45 [arXiv:1510.00972] [inSPIRE].

[43] P. Koroteev and A. Sciarappa, \textit{On Elliptic Algebras and Large-N Supersymmetric Gauge Theories}, \textit{J. Math. Phys.} \textbf{57} (2016) 112302 [arXiv:1601.08238] [inSPIRE].

[44] G. Lockhart and C. Vafa, \textit{Superconformal Partition Functions and Non-perturbative Topological Strings}, \textit{JHEP} \textbf{10} (2018) 051 [arXiv:1210.5909] [inSPIRE].

[45] I.G. Macdonald, \textit{Symmetric functions and Hall polynomials}, 2nd ed., Oxford Mathematical Monographs, Oxford University Press (1995) [DOI].

[46] K. Miki, \textit{A $(q,\gamma)$ analog of the $W_{1+\infty}$ algebra}, \textit{J. Math. Phys.} \textbf{48} (2007) 123520.

[47] T. Miwa and M. Jimbo, \textit{Algebraic Analysis of Solvable Lattice Models}, Regional Conference Series in Mathematics \textbf{85} (1993) AMS.

[48] A. Mironov and A. Morozov, \textit{On AGT relation in the case of U(3)}, \textit{Nucl. Phys. B} \textbf{825} (2010) 1 [arXiv:0908.2569] [inSPIRE].

[49] A. Mironov, A. Morozov and Y. Zenkevich, \textit{Spectral duality in elliptic systems, six-dimensional gauge theories and topological strings}, \textit{JHEP} \textbf{05} (2016) 121 [arXiv:1603.00304] [inSPIRE].

[50] H. Nakajima, \textit{Lectures on Hilbert Schemes of Points on Surfaces}, University Lecture Series \textbf{18} (1999) AMS.

[51] H. Nakajima and K. Yoshioka, \textit{Instanton counting on blowup. I. 4-dimensional pure gauge theory}, \textit{Invent. Math.} \textbf{162} (2005) 313.

[52] H. Nakajima and K. Yoshioka, \textit{Instanton counting on blowup. II. K-theoretic partition function}, [math/0505553].

[53] A. Neguţ, \textit{The R-matrix of the quantum toroidal algebra}, arXiv:2005.14182 [inSPIRE].
[54] N.A. Nekrasov, Seiberg-Witten prepotential from instanton counting, *Adv. Theor. Math. Phys.* **7**(2003) 831 [hep-th/0206161] [inSPIRE].

[55] N. Nekrasov, BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and qq-characters, *JHEP* **03**(2016) 181 [arXiv:1512.05388] [inSPIRE].

[56] N. Nekrasov and A. Okounkov, Seiberg-Witten theory and random partitions, *Prog. Math.* **244**(2006) 525 [hep-th/0306238] [inSPIRE].

[57] N. Nekrasov, V. Pestun and S. Shatashvili, Quantum geometry and quiver gauge theories, *Commun. Math. Phys.* **357**(2018) 519 [arXiv:1512.05388] [inSPIRE].

[58] F. Nieri, An elliptic Virasoro symmetry in 6d, *Lett. Math. Phys.* **107**(2017) 2147 [arXiv:1511.00574] [inSPIRE].

[59] M. Nishizawa, An elliptic analogue of the multiple gamma function, *J. Phys. A* **34**(2001) 7411.

[60] A. Okounkov and A. Smirnov, Quantum difference equation for Nakajima varieties, [arXiv:1602.09007](https://arxiv.org/abs/1602.09007) [inSPIRE].

[61] I. Ojima, *Gauge Fields at Finite Temperatures: Thermo Field Dynamics, KMS Condition and their Extension to Gauge Theories,* *Annals Phys.* **137**(1981) 1 [inSPIRE].

[62] R. Poghossian and M. Samsonyan, Instantons and the 5D U(1) gauge theory with extra adjoint, *J. Phys. A* **42**(2009) 304024 [arXiv:0804.3564] [inSPIRE].

[63] E.M. Rains and S.O. Warnaar, A Nekrasov-Okounkov formula for Macdonald polynomials, *J. Algebr. Comb.* **48**(2018) 1 [arXiv:1606.04613] [inSPIRE].

[64] Y. Saito, Elliptic Ding-Iohara algebra and the free field realization of the elliptic Macdonald operator, *Publ. Res. Inst. Math. Sci.* 50 (2014): 411 [arXiv:1301.4912].

[65] O. Schiffmann, Drinfeld realization of the elliptic Hall algebra, *J. Alg. Comb.* **35**(2012): 237 [arXiv:1004.2575].

[66] R. Kaminski, G. Mennessier and S. Narison, Gluonium nature of the $\sigma/f_0(600)$ from its coupling to $K\bar{K}$, *Phys. Lett. B* **680**(2009) 148 [arXiv:0904.2558] [inSPIRE].

[67] Y. Takahasi and H. Umezawa, Thermo field dynamics, *Collect. Phenom.* **2**(1975) 55 [inSPIRE].

[68] M. Varagnolo, E. Vasserot, Schur duality in the toroidal setting, *Commun. Math. Phys.* **182**(1996) 469 [q-alg/9506026].

[69] L. Wang, K. Wu, J. Yang and Z. Yang, Representation of elliptic Ding-Iohara algebra, *Frontiers of Mathematics in China* **15**(2020) 155.

[70] N. Wyllard, $A(N-1)$ conformal Toda field theory correlation functions from conformal $N = 2$ SU($N$) quiver gauge theories, *JHEP* **11**(2009) 002 [arXiv:0907.2189] [inSPIRE].

[71] R.-D. Zhu, An Elliptic Vertex of Awata-Feigin-Shiraishi type for M-strings, *JHEP* **08**(2018) 050 [arXiv:1712.10255] [inSPIRE].