THE COMETRIZABILITY OF GENERALIZED METRIC SPACES

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Abstract. A topological space $X$ is cometrizable if it admits a weaker metrizable topology such that each point $x \in X$ has a (not necessarily open) neighborhood base consisting of metrically closed sets. We prove that the class of cometrizable spaces includes all stratifiable spaces and all $\alpha$-cosmic spaces. Moreover, for any $k$-separable space $X$ and any cometrizable space $Y$, the function space $C_k(X, Y)$ endowed with the compact-open topology is cometrizable. For a topological space $X$ the function space $C_p(X)$ is cometrizable if and only if it is metrizable. Also, we present an example of a regular countable space of weight $\omega_1$, which is not cometrizable. Under $\omega_1 = \mathfrak{c}$ this space contains no infinite compact subsets and hence is $\text{cs}$-cosmic. Under $\omega_1 < \mathfrak{c}$ this countable space is Fréchet-Urysohn and is not $\text{cs}$-cosmic.

1. Introduction

In this paper we study the interplay between the class of cometrizable spaces and other classes of generalized metric spaces.

A topological space $X$ is called \textit{cometrizable} if it admits a metrizable topology such that each point $x \in X$ has a (not necessarily open) neighborhood base consisting of metrically closed sets. Equivalently, cometrizable spaces can be defined as spaces $X$ for which there exists a bijective continuous map $f : X \to M$ to a metrizable space $M$ such that for any open set $U \subset X$ and point $x \in U$ there exists a neighborhood $V \subset X$ of $x$ such that $f^{-1}(f(V)) \subset U$.

It is clear that each cometrizable space $X$ is regular and \textit{submetrizable}, i.e. admits a continuous bijective map onto a metrizable space.

Cometrizable spaces were introduced by Gruenhage [21] who proved the following interesting implication of PFA, the Proper Forcing Axiom [7].

\textbf{Theorem 1.1} (Gruenhage). Under PFA, a cometrizable space $X$ is cosmic if and only if $X$ contains no uncountable discrete subspaces and no uncountable subspaces of the Sorgenfrey line.

In [26, 8.5] Todorčević proved that this PFA-characterization of cosmic cometrizable spaces remains true under OCA (the Open Coloring Axiom, which follows from PFA).

These results of Gruenhage and Todorčević motivate a deeper study of cometrizable spaces. In this paper we establish some inheritance properties of the class of cometrizable spaces and using the obtained information study the relation of cometrizable space to some known classes of generalized metric spaces.

We start with the following simple (but important) observation.

\textbf{Proposition 1.2}. The class of cometrizable spaces contains all metrizable spaces and is closed under taking subspaces and countable Tychonoff products.

Next we show that the class of cometrizable spaces is stable under forming function spaces $C_k(X, Y)$.
Here for topological spaces $X, Y$ by $C(X, Y)$ we denote the set of all continuous functions from $X$ to $Y$. Given a family $\mathcal{K}$ of compact subsets of the space $X$ by $C_{\mathcal{K}}(X, Y)$ we denote the space $C(X, Y)$ endowed with the topology generated by the subbase consisting of the sets
\[ [K, U] := \{ f \in C(X, Y) : f(K) \subset U \} \]
where $K \in \mathcal{K}$ and $U$ is an open set in $Y$.

If $\mathcal{K}$ is the family of all compact (finite) subsets of $X$, then the function space $C_{\mathcal{K}}(X, Y)$ will be denoted by $C_k(X, Y)$ (resp. $C_p(X, Y)$). If $\mathcal{K}$ is the family of convergent sequences in $X$ (i.e., countable compact sets with a unique non-isolated point), then the function space $C_{\mathcal{K}}(X, Y)$ will be denoted by $C_{cs}(X, Y)$. The function spaces $C_k(X, \mathbb{R})$, $C_{cs}(X, \mathbb{R})$ and $C_p(X, \mathbb{R})$ are denoted by $C_k(X)$, $C_{cs}(X)$ and $C_p(X)$, respectively.

A family $\mathcal{K}$ of compact subsets of a topological space $X$ is called separable if
- each compact subset of any compact set $K \in \mathcal{K}$ belongs to the family $\mathcal{K}$;
- the union $\bigcup \mathcal{K}$ is dense in $X$;
- $\mathcal{K}$ contains a countable subfamily $\mathcal{C}$ such that each compact set $K \in \mathcal{K}$ can be enlarged to a compact set $\tilde{K} \in \mathcal{K}$ such that $\tilde{K} \cap \bigcup \mathcal{C}$ is dense in $\tilde{K}$.

**Theorem 1.3.** Let $X$ be a topological space and $\mathcal{K}$ be a separable family of compact subsets of $X$. Then for any cometrizable space $Y$ the function space $C_{\mathcal{K}}(X, Y)$ is cometrizable.

This theorem will be proved in Section 2. Now we derive some of its corollaries.

A topological space $X$ is defined to be
- **cs-separable** if $X$ contains a countable set $D \subset X$, which is sequentially dense in $X$ in the sense that each point $x \in X$ is the limit of some convergent sequence $\{x_n\}_{n \in \omega} \subset D$;
- **k-separable** if $X$ contains a countable subset $D \subset X$, which is $k$-dense in the sense that each compact set $K \subset X$ is contained in a compact set $C \subset X$ such that $K \subset C \cap D$;
- **$\sigma k$-separable** if $X$ contains a $\sigma$-compact set $D \subset X$ such that each compact set $K \subset X$ is contained in a compact set $C \subset X$ such that $K \subset C \cap D$.

It is clear that each $k$-separable space $X$ is $\sigma k$-separable (and cs-separable if all compact subsets of $X$ are Fréchet-Urysohn).

**Theorem 1.3** has two corollaries:

**Corollary 1.4.** For any $\sigma k$-separable topological space $X$ and any cometrizable space $Y$ the function space $C_k(X, Y)$ is cometrizable.

**Corollary 1.5.** For any cs-separable topological space $X$ and any cometrizable space $Y$ the function space $C_{cs}(X, Y)$ is cometrizable.

On the other hand, the cometrizability of the function spaces $C_p(X)$ is equivalent to the metrizability. Let $X, Y$ be two topological spaces. A topological space $X$ is defined to be $Y$-Hausdorff if any function $f : F \to Y$ defined on a finite subset $F \subset X$ can be extended to a continuous function $\tilde{f} : X \to Y$.

**Theorem 1.6.** For a separable topological space $Y$ containing more than one point and a $Y$-Hausdorff space $X$, the function space $C_p(X, Y)$ is cometrizable if and only if $X$ is countable and $Y$ is cometrizable.

This theorem will be proved in Section 3.

Now we study the relation of the class of cometrizable spaces to other known classes of generalized metric spaces. We start with stratifiable spaces and their generalizations.
A regular topological space $X$ is called

- **stratifiable** if each point $x \in X$ has a countable system of open neighborhoods $(U_n(x))_{n \in \omega}$ such that each closed subset $F$ of $X$ is equal to $\bigcap_{n \in \omega} U_n[F]$ where $U_n[F] = \bigcup_{x \in F} U_n(x);$  
- **semi-stratifiable** if each point $x \in X$ has a countable system of open neighborhoods $(U_n(x))_{n \in \omega}$ such that each closed subset $F$ of $X$ is equal to $\bigcap_{n \in \omega} U_n[F];$
- **quarter-stratifiable** if there exists a function $U : X \times \mathbb{N} \to \tau$ such that $X = \bigcup_{x \in X} U(x, n)$ for all $n \in \omega$ and for any point $x \in X,$ any sequence $(x_n)_{n \in \omega} \subset X$ with $x \in \bigcap_{n \in \omega} U(x_n, n)$ converges to $x;$
- a space with $G_\delta$-**diagonal** if the diagonal $\{(x, x) : x \in X\}$ is a $G_\delta$-set in $X \times X.$

Stratifiable spaces were introduced by Borges [9] but were known earlier as $M_\delta$-spaces of Ceder [10]. Semi-stratifiable spaces were introduced by Greede [18] and quarter-stratifiable spaces by Banakh, who proved in [1] that every semi-stratifiable space is quarter-stratifiable and every quarter-stratifiable space has $G_\delta$-diagonal. More information on stratifiable and semi-stratifiable spaces can be found in [19] and [20]. One of the main results of this paper is the following theorem, proved in Section 4.

**Theorem 1.7.** Each stratifiable space is cometrizable.

Next, we recall some local properties of topological spaces.

A topological space $X$ is defined to be

- **Fréchet-Urysohn** if for any subset $A \subset X$ and point $x \in \bar{A}$ there exists a sequence $(x_n)_{n \in \omega} \subset A$ that converges to $x;$
- **sequential** if for any non-closed set $A \subset X$ there exists a sequence $(x_n)_{n \in \omega} \subset A$ that converges to a point $x \in X \setminus A;$
- a $k$-**space** if for any non-closed set $A \subset X$ there exists a compact set $K \subset X$ such that $K \cap A$ is not closed in $K;$
- a $k_\mathbb{R}$-**space** if the continuity of a function $f : X \to \mathbb{R}$ is equivalent to the continuity of its restrictions $f|K$ onto compact subsets $K$ on $X;$
- **Ascoli** if for any compact subset $K \subset C_k(X)$ the map $K \times X \to \mathbb{R}, (f, x) \mapsto f(x),$ is continuous;
- **cs-Ascoli** (or **sequentially Ascoli**) if for any convergent sequence $K \subset C_k(X)$ the map $K \times X \to \mathbb{R}, (f, x) \mapsto f(x),$ is continuous.

By a **convergent sequence** we understand a compact countable set with a unique non-isolated point. Ascoli spaces were introduced and studied in [4]. By [4, 5.4] (and [4, 5.8]), a (Tychonoff) space $X$ is Ascoli if and only if the canonical map $\delta : X \to C_k(C_k(X))$ assigning to each $x \in X$ the Dirac measure $\delta_x : f \mapsto f(x)$ is continuous (if and only if the map $\delta$ is a topological embedding). Sequentially Ascoli spaces were studied in [14], [15] and in [3] (as spaces containing no strict $\text{Cld}^\omega$-fans). For any Tychonoff space $X$ we have the implications:

Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow$ $k$-space $\Rightarrow$ $k_\mathbb{R}$-space $\Rightarrow$ Ascoli $\Rightarrow$ cs-Ascoli.

The unique non-trivial implication ($k_\mathbb{R}$-space $\Rightarrow$ Ascoli) in this diagram is due to Noble [24], see [4, §5]. By [13, 2.9], any non-discrete $P$-space is cs-Ascoli but not Ascoli.

There is a useful characterization of cs-Ascoli spaces in terms of (strict) $\text{Cld}^\omega$-fans, which are defined as follows.

A sequence $(F_n)_{n \in \omega}$ of closed subsets of a topological space $X$ is called
A subset $U$ of a topological space $X$ is called a functionally open neighborhood of a set $A \subset X$ if there exists a continuous function $f : X \to [0, 1]$ such that $f(F) \subset \{0\}$ and $f(O \setminus A) \subset \{1\}$.

The following characterization of sequentially Ascoli spaces was proved in [3, 3.3.1, 2.9.6] (for the equivalence $(1) \iff (2)$, see also [17, 2.1] and [15]).

**Theorem 1.8.** For a topological space $X$ the following conditions are equivalent:

1. is sequentially Ascoli;
2. contains no strict $\text{Cld}^ω$-fans.

If $X$ is a normal $\aleph$-space, then the conditions $(1), (2)$ are equivalent to

3. $X$ contains no $\text{Cld}^ω$-fans.

The class of $\aleph$-spaces, appearing in Theorem 1.8 was introduced by O’Meara [22] and is one of many known classes of generalized metric spaces, which are defined with the help of networks.

A family $\mathcal{N}$ of subsets of a topological space $X$ is called

- a network if for each open set $U \subset X$ and point $x \in U$ there exists a set $N \in \mathcal{N}$ such that $x \in N \subset U$;
- a $k$-network if for each open set $U \subset X$ and compact set $K \subset U$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ such that $K \subset \bigcup \mathcal{F} \subset U$;
- a Pytkeev network if $\mathcal{N}$ is a network and for any open set $U \subset X$, a subset $A \subset X$ and point $x \in U \cap A \setminus A$, there exists a set $N \in \mathcal{N}$ such that $N \subset U$ and $N \cap A$ is infinite;
- an ap-network if for each open set $U \subset X$ and a sequence $\{x_n\}_{n \in \omega} \subset X$ of points that accumulate at a point $x \in U$ there exists a set $N \in \mathcal{N}$ such that $N \subset U$ and the set $\{n \in \omega : x_n \in N\}$ is infinite;
- an as-network if for each open set $U \subset X$ and a sequence $(S_n)_{n \in \omega}$ of closed subsets of $X$ that accumulates at a point $x \in U$ there exists a set $N \in \mathcal{N}$ such that $N \subset U$ and the set $\{n \in \omega : N \cap S_n \neq \emptyset\}$ is infinite.

The prefixes cs, ap, as are abbreviations of “convergent sequence”, “accumulating sequence of points”, and “accumulating sequence of closed sets”.

A regular topological space $X$ is called

- an $\aleph_0$-space if $X$ has a countable $k$-network;
- an $\aleph$-space if $X$ has a $\sigma$-discrete $k$-network;
- cosmic if $X$ has a countable network;
- cs-cosmic if $X$ has a countable cs-network;
- ap-cosmic if $X$ has a countable ap-network;
are real numbers.

Each Fréchet-Urysohn \(\sigma\)-space is stratifiable. Consequently, every \(\sigma\)-Ascoli normal \(\aleph\)-space \(X\) is a \(\sigma\)-space.

The classes of cosmic spaces and \(\aleph_0\)-spaces are two well-studied classes of generalized metric spaces, introduced by Michael [23] and considered in the surveys of Gruenhage [19], [20], [25]; \(\aleph\)-spaces were introduced and studied by O'Meara [22] in his dissertation (see also [13] and [19]). It can be shown (see [13]) that a \(\sigma\)-discrete (more generally, compact-countable) family of sets is a \(k\)-network if and only if it is a \(\sigma\)-network. This implies that a topological space is an \(\aleph_0\)-space (resp. an \(\aleph\)-space) if and only if it is \(\sigma\)-cosmic (res. a \(\sigma\)\(\sigma\)-space).

The class of \(\sigma\)-cosmic spaces coincides with the class of \(\aleph_0\)-spaces of Banakh in [2] and the class of \(\sigma\)\(\sigma\)-spaces (properly) contains the class of \(\aleph\)-spaces of Gabriyelyan and Kąkol [16].

The classes of \(\sigma\)\(\sigma\)-cosmic spaces and \(\sigma\)\(\sigma\)\(\sigma\)-spaces are new and are introduced in this paper with purpose to find in the class of cometrizable spaces a subclass of spaces defined by suitable network properties. These two new classes are studied in more details in the paper [6]. Here we announce just one result from [6].

**Theorem 1.9.** Each \(\sigma\)\(\sigma\)-network \(N\) in a \(\sigma\)-Ascoli normal \(\aleph\)-space \(X\) is an \(\sigma\)\(\sigma\)-network. Consequently, every \(\sigma\)-Ascoli normal \(\aleph\)-space \(X\) is a \(\sigma\)\(\sigma\)-space.

**Proof.** For convenience of the reader, we provide a short proof of this theorem. To show that the \(\sigma\)\(\sigma\)-network \(N\) is an \(\sigma\)\(\sigma\)-network, fix an open set \(U \subset X\) and a sequence \((F_n)_{n \in \omega}\) of closed subsets of \(X\) that accumulates at some point \(x \in U\). By Theorem 1.8 the \(\sigma\)\(\sigma\)-Ascoli \(\aleph\)-space \(X\) contains no \(Cld^\omega\)-fans. Consequently, the sequence \((F_n)_{n \in \omega}\) is not a \(Cld^\omega\)-fan and hence it is not compact-finite. So, we can find a compact set \(K \subset X\) such that the set \(\Omega = \{n \in \omega : K \cap F_n \neq \emptyset\}\) is infinite. For each \(n \in \Omega\) choose a point \(x_n \in K \cap F_n\). Since compact sets in \(\aleph\)-spaces are metrizable (by [19], 2.4, 4.6]), the sequence \((x_n)_{n \in \Omega}\) has a convergent subsequence \((x_{n_k})_{k \in \omega}\). Since \(N\) is a \(\sigma\)\(\sigma\)-network, there exists a set \(N \in N\) such that \(N \subset U\) and the set \(\{k \in \omega : x_{n_k} \in N\} \subset \{k \in \omega : N \cap F_{n_k} \neq \emptyset\}\) is infinite. Then the set \(\{n \in \omega : N \cap F_n \neq \emptyset\}\) is infinite too, witnessing that \(N\) is an \(\sigma\)\(\sigma\)-network for \(X\). \(\square\)

Let us also formulate a corollary of Theorem 1.7 and a result of Foged [13] (see [19] 11.4]) (saying that each Fréchet-Urysohn \(\aleph\)-space is stratifiable).

**Proposition 1.10.** Each Fréchet-Urysohn \(\aleph\)-space is stratifiable and hence cometrizable.

The location of cometrizable spaces among other generalized metric spaces is shown in Diagram 4 holding for any regular topological space. By simple arrows we denote the implications that hold under some additional assumptions (written at the arrows). The non-trivial (or not discussed sofar) implications of this diagram are established in the following theorem that will be proved in Section 5.

**Theorem 1.11.**

(1) Each cosmic space is \(\sigma\)\(\sigma\)-separable.

(2) Each \(\sigma\)\(\sigma\)-cosmic space is \(k\)-separable.

(3) Each \(\sigma\)\(\sigma\)-cosmic space is cometrizable.

Now we present some examples showing which implications cannot be added to the above diagram. Let us recall that the Sorgenfrey line is the real line endowed with the (first-countable) topology, generated by the base consisting of the half-intervals \([a, b)\) where \(a < b\) are real numbers.
**Example 1.12.** The Sorgenfrey line is cometrizable, first-countable, \( k \)-separable and quarter-stratifiable, but not semi-stratifiable.

The cometrizability of the Sorgenfrey line is witnessed by the standard Euclidean topology of the real line. The \( k \)-separability of the Sorgenfrey line is established in [5]. By Example 3.2 in [1], the Sorgenfrey line is quarter-stratifiable but not semi-stratifiable.

**Problem 1.13.** Is each cometrizable space quarter-stratifiable?

Our next example is more difficult and will be constructed in Section 6.

**Example 1.14.** There exists a regular countable space \( X \) of weight \( \omega_1 \), which is not cometrizable and hence not stratifiable. If \( \omega_1 = \mathfrak{c} \), then the space \( X \) contains no infinite compact sets and is \( \text{cs-cosmic} \). If \( \omega_1 < \mathfrak{p} \), the space \( X \) is Fréchet-Urysohn and is not \( \text{cs-cosmic} \).

The cardinal \( \mathfrak{p} \) is defined as the smallest character of a countable space with a unique non-isolated point, which is not Fréchet-Urysohn. It is known that \( \omega_1 \leq \mathfrak{p} \leq \mathfrak{c} \) and \( \mathfrak{p} = \mathfrak{c} \) under Martin’s Axiom, see [11], [27], [8].

**Example 1.15.** The weak topology of a Banach space \( X \) is cometrizable if and only if \( X \) is finite-dimensional. If \( X \) is separable and reflexive, then \( (X, \text{weak}) \) is a hemicompact submetrizable space.

Such an \( \aleph_0 \)-space (if it exists) cannot be \( \text{cs-Ascoli} \) according to Theorem 1.11(6). Looking at Proposition 1.10 and Theorems 1.9 and 1.11(3), it is natural to ask
Problem 1.16. Is each (sequential) $\sigma_{as}$-space cometrizable?

2. Proof of Theorem 1.13

Given a separable family $K$ of compact subsets of a topological space $X$ and a cometrizable space $Y$, we shall prove that the function space $C_K(X,Y)$ is cometrizable.

The family $K$, being separable, contains a countable subfamily $D \subset K$ such that each set $K \in K$ is contained in a set $\tilde{K} \in K$ such that $K$ is contained in the closure of the set $\tilde{K} \cap \cup D$. Then the density of $\cup K$ in $X$ implies the density of the union $\cup D$ in $X$.

Since the space $Y$ is cometrizable, there exists a weaker metrizable topology $\tau$ on $Y$ such that for each open set $U \subset X$ and point $y \in U$ there exists an open neighborhood $V \subset Y$ of $y$ whose closure $\overline{V}$ in the topology $\tau$ is contained in $U$. Denote by $Y_\tau$ the metrizable topological space $(Y, \tau)$.

By [12] 4.2.17, for every (compact) set $D \in D$ the function space $C_k(D, Y_\tau)$ is metrizable. Since the union $\cup D$ is dense in $X$, the map

$$ r : C_K(X,Y) \to \prod_{D \in D} C_k(D, Y_\tau), \quad r : f \mapsto (f|D)_{D \in D} $$

is injective. Let $\sigma$ be the (metrizable) topology on $C_K(X,Y)$ such that the map

$$ r : (C_K(X,Y), \sigma) \to \prod_{D \in D} C_k(D, Y_\tau) $$

is a topological embedding. We claim that the topology $\sigma$ witnesses that the space $C_K(X,Y)$ is cometrizable.

Fix any function $f \in C_K(X,Y)$ and an open neighborhood $O_f \subset C_K(X,Y)$. Without loss of generality, $O_f$ is of basic form $O_f = \bigcap_{i=1}^n [K_i, U_i]$ for some non-empty compact sets $K_1, \ldots, K_n \subset K$ and some open sets $U_1, \ldots, U_n \subset Y$. For every $i \leq n$ and point $x \in K_i$, find a neighborhood $V_{f(x)} \subset Y$ of $f(x) \in U_i$ whose $\tau$-closure $\overline{V}_{f(x)}$ is contained in $U_i$. Using the regularity of the cometrizable space $Y$, find two open neighborhoods $N_{f(x)}$, $W_{f(x)}$ of $f(x)$ such that $\overline{N}_{f(x)} \subset W_{f(x)} \subset \overline{W}_{f(x)} \subset V_{f(x)}$.

By the compactness of $K_i$ the open cover $\{f^{-1}(N_{f(x)}) : x \in K_i\}$ of $K_i$ has a finite subcover $\{f^{-1}(N_{f(x)}) : x \in F_i\}$ (here $F_i \subset K_i$ is a suitable finite subset of $K_i$). By choice of the family $D$, for every $x \in F_i$, the compact set $K_{i,x} := K_i \cap f^{-1}(\overline{N}_{f(x)}) \subset K$ can be enlarged to a compact set $\tilde{K}_{i,x} \subset K$ such that $K_{i,x}$ is contained in the closure of the set $\tilde{K}_{i,x} \cap \cup D$.

Replacing the set $\tilde{K}_{i,x}$ by $\tilde{K}_{i,x} \cap f^{-1}(\overline{W}_{f(x)})$, we can assume that $f(\tilde{K}_{i,x}) \subset \overline{W}_{f(x)} \subset V_{f(x)}$.

Consider the open neighborhood

$$ V_f = \bigcap_{i=1}^n \bigcap_{x \in F_i} [\tilde{K}_{i,x}, V_{f(x)}] $$

of $f$ in the function space $C_K(X,Y)$. We claim that its $\sigma$-closure $\overline{V}_f$ is contained in $O_f$.

Given any function $g \notin O_f$, we should find a neighborhood $O_g \subset \sigma$ of $g$ that does not intersect $V_f$. Since $g \notin O_f$, there exists $i \leq n$ and a point $z \in K_i$ such that $g(z) \notin U_i$. Find a point $x \in F_i$ with $z \in K_{i,x}$. Taking into account that $\overline{V}_{f(x)} \subset U_i \subset Y \setminus \{g(z)\}$, we conclude that $g(z) \notin \overline{V}_{f(x)}$. Since $z \in K_{i,n} \subset \tilde{K}_{i,n} \cap \cup D$, the continuity of the function $g : X \to Y_\tau$ yields a point $d \in \tilde{K}_{i,n} \cap \cup D$ such that $g(d) \notin \overline{V}_{f(x)}$. Then $O_g := \{[d], Y \setminus \overline{V}_{f(x)}\} \subset \sigma$ is a required $\sigma$-open neighborhood of $g$ that is disjoint with the set $V_f$. 
3. Proof of Theorem ??

Given a separable topological space $Y$ of cardinality $|Y| \geq 2$ and a non-empty $Y$-Hausdorff space $X$, we should prove that the function space $C_p(X, Y)$ is cometrizable if and only if $X$ is countable and $Y$ is cometrizable.

Since $C_p(X, Y) \subset Y^X$, the “if” part follows from Proposition 12.

To prove the “only if” part, assume that the function space $C_p(X, Y)$ is cometrizable and fix a metrizable topology $\tau$ witnessing this fact. Denote by $C_{\tau}(X, Y)$ the function space $(X, Y)$, endowed with the topology $\tau$. Observe that the space $Y$ is cometrizable, being homeomorphic to the subspace $\{ f \in C_{\tau}(X, Y) : |f(X)| = 1 \}$ of the cometrizable space $C_{\tau}(X, Y)$.

By [12, 2.7.10(d)], the separability of $X, Y$ ensures that the Tychonoff power $Y^X$ has countable cardinality. By the $Y$-Hausdorff property of $X$, the subspace $C_p(X, Y)$ is dense in $Y^X$ and hence has countable cardinality. Then the continuous image $C_{\tau}(X, Y)$ of $C_p(X, Y)$ also has countable cardinality. By [12, 4.1.15], the metrizable countably cellular space $C_{\tau}(X, Y)$ is second countable.

Let $D$ be a countable dense subset in the separable space $Y$ and $\Phi$ be the family of all functions $f : \text{dom}(f) \to D$, defined on finite subsets $\text{dom}(f) \subset X$. It is easy to see that for any countable set $Z \subset X$ the subfamily $\Phi[Z] := \{ f \in \Phi : \text{dom}(f) \subset Z \}$ is countable. For every function $f \in \Phi$, consider the subspace $C_{\tau}^f(X, Y) := \{ g \in C_{\tau}(X, Y) : g|\text{dom}(f) = f \}$ of $C_{\tau}(X, Y)$, which is not empty by the $Y$-Hausdorff property of $X$. Since $C_{\tau}(X, Y)$ is second-countable, the subspace $C_{\tau}^f(X, Y)$ is separable and hence contains a countable dense subset $\Psi[f]$.

For every function $g \in C(X, Y)$ fix a countable neighborhood base $\{ U_m(g) \}_{m \in \omega} \subset \tau$ at $g$ in the topology $\tau$. Since the topology $\tau$ is weaker than the topology of pointwise convergence, for every $m \in \omega$ we can find a finite set $F_{g,m} \subset X$ such that $\{ f \in C(X, Y) : f|F_{g,m} = g|F_{g,m} \} \subset U_m(g)$. By induction we can construct an increasing sequence $(Z_n)_{n \in \omega}$ of non-empty countable subsets of $X$ such that for every $n \in \omega$

$$\bigcup_{n \in \omega} \bigcup_{f \in \Phi[Z_n]} \bigcup_{g \in \Psi[f]} \bigcup_{m \in \omega} F_{g,m} \subset Z_{n+1}.$$ 

We claim that $X$ is equal to the countable set $Z = \bigcup_{n \in \omega} Z_n$. To derive a contradiction, assume that $X$ contains some point $x_0 \notin \bigcup_{n \in \omega} Z_n$.

The space $Y$, being cometrizable, is Hausdorff. By the density of $D$ in $Y$, we can find two distinct points $y_0, y_1 \in D$ and an open set $W \subset Y$ such that $y_0 \in W$ and $y_1 \notin W$. Consider the constant function $g_0 : X \to \{ y_0 \} \subset D$ and its neighborhood $U = \{ f \in C_p(X, Y) : f(x_0) \in W \}$. By the choice of the topology $\tau$, the neighborhood $U$ contains a neighborhood $V \subset C_p(X, Y)$ whose closure $\overline{V}$ in the topology $\tau$ is contained in $U$. We lose no generality assuming that $V$ is of basic form $V = \{ f \in C_p(X, Y) : f(F) \subset W \}$ for some finite set $F \subset X$ and some open neighborhood $W' \subset W$ of $y_0$. Since $X$ is $Y$-Hausdorff, there exists a continuous function $\tilde{g} : X \to Y$ such that $\tilde{g}(x_0) = y_1 \notin W$ and $\tilde{g}(F \cap Z) \subset D \cap W'$. It is clear that $\tilde{g} \notin U$ and hence $\tilde{g} \notin \overline{V}$. Consider the function $f = \tilde{g}|F \cap Z$ and find $n \in \omega$ such that $F \cap Z \subset Z_n$. Observe that $f \in \Phi[Z_n]$. By the density of the set $\Psi[f]$ in $C_{\tau}^f(X, Y)$, there exists a function $g \in \Psi[f]$ with a neighborhood base of the topology $\tau$ at $g$, that exists $m \in \omega$ such that $U_m(g) \cap V = \emptyset$. Taking into account that $f \in \Phi[Z_n]$ and $g \in \Psi[f]$, we conclude that $F_{g,m} \subset Z_{n+1} \subset Z$ and hence $F_{g,m} \cap F \subset Z \cap F$. By the $Y$-Hausdorff property of $X$, there exists a continuous function $\tilde{g} : X \to Y$ such that $\tilde{g}|F_{g,m} = g|F_{g,m}$ and $\tilde{g}(F \setminus F_{g,m}) \subset \{ y_0 \} \subset W'$. Taking into account that $\tilde{g}(F \cap F_{g,m}) = g(F \cap F_{g,m}) \subset g(F \cap Z) = f(F \cap Z) = \tilde{g}(F \cap Z) \subset W'$, we conclude that $\tilde{g}(F) \subset W'$ and hence $\tilde{g} \in V$. On the other hand, $\tilde{g}|F_{g,m} = g|F_{g,m}$ and the
choice of the set $F_{g,m}$ ensure that $g \in U_m(g) \subset C_\tau(X, Y) \setminus V$. This is a desired contradiction, proving that the space $X = Z$ is countable.

4. PROOF OF THEOREM 1.7

Given a stratifiable space $X$, we should prove that $X$ is cometrizable. By the stratifiability of $X$, each point $x \in X$ has a countable system of open neighborhoods $(W_n(x))_{n \in \omega}$ such that every closed set $F$ in $X$ is equal to the intersection $\bigcap_{n \in \omega} W_n[F]$ where $W_n[F] = \bigcup_{x \in F} W_n(x)$.

By Theorem 5.9 [19], the stratifiable space $X$ is a $\sigma$-space. Consequently, $X$ has a $\sigma$-discrete network $\mathcal{N}$. Since the space $X$ is regular, we can replace each set $N \in \mathcal{N}$ by its closure and assume that $\mathcal{N}$ consists of closed subsets of $X$.

Write the $\sigma$-discrete family $\mathcal{N}$ as the countable union $\mathcal{N} = \bigcup_{k \in \omega} \mathcal{N}_k$ of discrete families $\mathcal{N}_k$ in $X$. The discreteness of $\mathcal{N}_k$ means that each point $x \in X$ has a neighborhood $O_x$ that intersects at most one set $N_k$. By Theorem 5.7 of [19], the stratifiable space $X$ is paracompact. Then we can find an open cover $\mathcal{U}_k$ of $X$ such that for every $U \in \mathcal{U}_k$ the $\mathcal{U}_k$-star $\text{St}(U, \mathcal{U}_k)$ is contained in some set $O_x$, $x \in X$, and hence intersects at most one set of the family $\mathcal{N}_k$. For every $N \in \mathcal{N}_k$ consider the $\mathcal{U}_k$-star $\text{St}(N, \mathcal{U}_k) = \bigcup\{U \in \mathcal{U}_k : N \cap U \neq \emptyset\}$. It follows that each set $U \in \mathcal{U}_k$ intersects at most one star $\text{St}(N, \mathcal{U}_k)$, $N \in \mathcal{N}_k$, which means that the family $(\text{St}(N, \mathcal{U}_k))_{N \in \mathcal{N}_k}$ is discrete in $X$.

For every $k \in \omega$, $N \in \mathcal{N}_k$, $n \in \omega$, consider the open neighborhood $O_{k,N,n} := \text{St}(N, \mathcal{U}_k) \cap W_n[N]$ of the closed set $N$ in $X$. Since stratifiable spaces are perfectly normal [9], there exists a continuous function $f_{k,N,n} : X \to [0, 1]$ such that $f_{k,N,n}^{-1}(0) = N$ and $f_{k,N,n}(0) = X \setminus O_{k,N,n}$.

Let $\ell_1(\mathcal{N}_k)$ be the Banach space of functions $f : \mathcal{N}_k \to \mathbb{R}$ such that $\|f\| := \sum_{x \in \mathcal{N}_k} |f(x)| < +\infty$. For every $N \in \mathcal{N}_k$ let $e_N : \mathcal{N}_k \to \{0, 1\}$ be the function such that $e_N^{-1}(1) = \{N\}$ (i.e., $e_N$ is the characteristic function of the singleton $\{N\} \subset \mathcal{N}_k$). Then $(e_N)_{N \in \mathcal{N}_k}$ is the standard unit basis of the Banach space $\ell_1(\mathcal{N}_k)$.

Since the family $(O_{k,N,n})_{N \in \mathcal{N}_k}$ is discrete in $X$, the function $f_{k,n} : X \to \ell_1(\mathcal{N}_k)$, $f_{k,n} : x \mapsto \sum_{N \in \mathcal{N}_k} f_{k,N,n}(x) \cdot e_N$ is well-defined and continuous.

The functions $f_{k,n}$, $k, n \in \omega$, compose a continuous function $f : X \to M \subset \prod_{k \in \omega} \ell_1(\mathcal{N}_k)^\omega$, $f : x \mapsto ((f_{k,n}(x))_{n \in \omega})_{k \in \omega}$ to the metrizable space $M = f(X) \subset \prod_{k \in \omega} \ell_1(\mathcal{N}_k)^\omega$.

Consider the topology $\tau = \{f^{-1}(U) : U$ is an open set in $M\}$ on $X$. We claim that $\tau$ is a metrizable topology $\tau$ witnessing that the space $X$ is cometrizable.

Take any point $x \in X$ and neighborhood $O_x \subset X$. By the regularity of the space $X$, there exists an open neighborhood $U \subset X$ of $x$ such that $\overline{U} \subset O_x$. Then the closed set $F = X \setminus U$ does not contain the point $x$. Since $F = \bigcap_{n \in \omega} W_n[F]$, there exists $n \in \omega$ such that $x \notin W_n[F]$. Then $V_x := X \setminus W_n[F]$ is an open neighborhood of $x$.

Since $\mathcal{N}$ is a network, the open set $X \setminus \overline{U}$ coincides with the union $\bigcup \mathcal{N}'$ of the subfamily $\mathcal{N}' = \{N \in \mathcal{N} : N \subset X \setminus \overline{U}\}$. For every $k \in \omega$ let $\mathcal{N}_k' = \mathcal{N}' \cap \mathcal{N}_k$. It follows that for every $k \in \omega$ and $N \in \mathcal{N}_k'$ the set $W_n[N] \cap \text{St}(N, \mathcal{U}_k) = O_{k,N,n} = f_{k,N,n}^{-1}((0, 1))$ is $\tau$-open. Then the
union
\[ W = \bigcup_{k \in \omega} \bigcup_{N \in \mathcal{N}_k^0} O_{k,N,n} \]
also belongs to the topology \( \tau \). Observe that
\[ X \setminus O_x \subset X \setminus \overline{U} = \bigcup_{k \in \omega} \bigcup_{N \in \mathcal{N}_k^0} N \subset \bigcup_{k \in \omega} \bigcup_{N \in \mathcal{N}_k^0} O_{k,N,n} = W \subset \]
\[ \bigcup_{k \in \omega} \bigcup_{N \in \mathcal{N}_k^0} W_n[N] \subset W_n[X \setminus \overline{U}] \subset W_n[F] \subset X \setminus V_x. \]

Then \( V_x \subset X \setminus W \subset O_x \) and the \( \tau \)-closure of \( V_x \) is contained in the \( \tau \)-closed set \( X \setminus W \subset O_x \). This implies that the topology \( \tau \) satisfies the separation axiom \( T_1 \) and hence is metrizable, witnessing that the space \( X \) is cometrizable.

5. Proof of Theorem 1.11

The three statements of Theorem 1.11 are proved in the following three lemmas.

**Lemma 5.1.** Each cosmic space \( X \) is \( cs \)-separable.

*Proof.* By [19, 4.9], the cosmic space \( X \) is the image of a separable metrizable space \( M \) under a continuous map \( f : M \to X \). Let \( D \) be any countable dense set in \( M \). We claim that its image \( f(D) \) is sequentially dense in \( X \). Indeed, given any point \( z \in M \) with \( f(z) = x \) and choose a sequence \( \{z_n\}_{n \in \omega} \subset D \) that converges to \( z \). Then the sequence \( \{f(z_n)\}_{n \in \omega} \subset f(D) \) converges to \( x \) in the space \( X \). \( \square \)

We recall that the class of \( cs \)-cosmic spaces coincides with the class of \( \aleph_0 \)-spaces.

**Lemma 5.2.** Each \( cs \)-cosmic space \( X \) is \( k \)-separable.

*Proof.* By [19, p.494], the \( \aleph_0 \)-space \( X \) is the image of a separable metric space \((M, d)\) under a compact-covering map \( f : M \to X \). The compact-covering property of \( f \) means that each compact set \( K \subset X \) coincides with the image \( f(C) \) of some compact set \( C \subset M \). Let \( D \) be any countable dense set in \( M \). We claim that its image \( f(D) \) is \( k \)-dense in \( X \). Indeed, given any compact set \( K \subset X \), use the compact-covering property of \( f \) to find a compact set \( C \subset M \) with \( f(C) = K \). Fix a countable dense set \( \{c_n\}_{n \in \omega} \subset C \) and for every \( n, k \in \omega \) choose a point \( c_{n,k} \in D \) such that \( d(c_n, c_{n,k}) < \frac{1}{n+k} \). It is easy to see that the set \( \tilde{C} = C \cup \{c_{n,k} : n, k \in \omega\} \) is compact and \( D \cap \tilde{C} \supset \{x_{n,k} : n, k \in \omega\} \) is dense in \( \tilde{C} \). Then \( \tilde{K} = f(\tilde{C}) \) is a compact set, containing \( K \) and the set \( \tilde{K} \cap f(D) \supset \{f(c_{n,k})\}_{n,k \in \omega} \) is dense in \( \tilde{K} \). This shows that the countable set \( f(D) \) is \( k \)-dense in \( X \) and the space \( X \) is \( k \)-separable. \( \square \)

Our last lemma proves the (most difficult) third statement of Theorem 1.11

**Lemma 5.3.** Each \( as \)-cosmic space \( X \) is cometrizable.

*Proof.* Fix a countable \( as \)-network \( \mathcal{N} \) for the \( as \)-cosmic space \( X \). Let \( \mathcal{B} \) be a countable base of the topology of the real line \( \mathbb{R} \) such that \( \mathcal{B} \) is closed under finite unions. Let \( \tau \) be the zero-dimensional Hausdorff topology on the function space \( C(X, \mathbb{R}) \), generated by the countable subbase consisting of the sets
\[ [N, B] := \{f \in C(X, \mathbb{R}) : f(N) \subset B\} \text{ and } C(X, \mathbb{R}) \setminus [N, B] \]
where \( N \in \mathcal{N} \) and \( B \in \mathcal{B} \). Denote by \( C_\tau(X) \) the space \( C(X, \mathbb{R}) \) endowed with the topology \( \tau \).
Let $D$ be any countable dense subset of the second-countable space $C_τ(X)$. Let $K$ be the family of compact subsets $K \subset C_τ(X)$ such that either $K$ is finite or $K$ has a unique non-isolated point and $K \cap D$ is dense in $K$.

It is easy to see that $K$ is a separable family of compact sets in $C_τ(X)$. By Theorem [1.3], the function space $C_K(C_τ(X))$ is cometrizable.

Consider the canonical map $δ : X \to C_K(C_τ(X))$ assigning to each point $x \in X$ the Dirac measure $δ_x : f \mapsto f(x)$. Let us show that the Dirac measure $δ_x$ is a continuous function on $C_τ(X)$. Given any function $f \in C_τ(X)$ and an open neighborhood $U \in B$ of $δ_x(f) = f(x)$, find a set $N \in N$ such that $x \in N \subset f^{-1}(U)$. Then $[N, U] \in τ$ is a neighborhood of $f$ such that $δ_x([N, U]) \subset U$. This means that the functional $δ_x$ is continuous and the map $δ : X \to C_K(C_τ(X))$ is well-defined.

Let us show that the inverse map $δ^{-1} : δ(X) \to X$ is continuous. Take any point $x \in X$ and fix any neighborhood $O_x \subset X$ of $x$. The space $X$, being regular and Lindelöf (because of cosmic), is normal. Then we can find a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus O_x) \subset \{0\}$. Since the family $K$ contains all singletons in the space $C_τ(X)$, the singleton $\{f\}$ belongs to the family $K$. Consider the open set $U = \{r \in \mathbb{R} : r > \frac{1}{2}\}$ and the open set $[\{f\}, U] \subset C_K(C_τ(X))$, which contains the functional $δ_x$ as $δ_x(f) = f(x) = 1 ∈ U$. Since

$$δ^{-1}([\{f\}; U]) = \{z ∈ X : δ_x ∈ \{f\}; U\} = \{z ∈ X : δ_x(f) ∈ U\} = \{z ∈ X : f(x) > \frac{1}{2}\} \subset O_x,$$

the function $δ^{-1} : δ(X) \to X$ is continuous at the point $δ_x ∈ C_K(C_τ(X))$.

It remains to prove that the map $δ : X \to C_K(C_τ(X))$ is continuous. Since the base $B$ is closed under finite unions, it suffices to prove that for every compact set $K ∈ K$ and every basic open set $U ∈ B$ the preimage $δ^{-1}([K, U])$ is open in $X$. By the definition of the family $K$, the compact set $K$ is either finite or has a unique non-isolated point and $K \cap D$ is dense in $D$.

First assume that $K$ is finite. Then

$$δ^{-1}([K, U]) = \{x ∈ X : δ_x ∈ [K, U]\} = \bigcap_{f ∈ K} \{x ∈ X : δ_x(f) ∈ U\} = \bigcap_{f ∈ K} f^{-1}(U)$$

is open in $U$ by the continuity of the functions $f ∈ K$.

Now assume that $K$ has a unique non-isolated point and the set $K \cap D$ is dense in $K$. Then $K = \{f_0\} \cup \{f_n\}_{n ∈ ω}$ for some sequence of functions $\{f_n\}_{n ∈ ω} ⊂ D$ that converge to a function $f_∞$ in the space $C_τ(X)$. Assuming that the set $δ^{-1}([K, U])$ is not open in $X$, we conclude that $δ^{-1}([K, U]) = \bigcap_{f ∈ K} f^{-1}(U)$ is not a neighborhood of some point $x ∈ δ^{-1}([K, U])$. Then each neighborhood $O_x$ of $x$ intersects infinitely many closed sets $X \setminus f_n^{-1}(U)$. Since $N$ is an as-network, there exists a set $N ∈ N$ such that $N ⊂ f_∞^{-1}(U)$ and $N$ intersects infinitely many closed sets $X \setminus f_n^{-1}(U)$. Since $[N, U]$ is a basic open neighborhood of the function $f_∞$ in the space $C_τ(X)$, there exists $m ∈ ω$ such that $f_m ∈ [N, U]$ for all $n ≥ m$. Then $N ⊂ f_n^{-1}(U)$ for all $n ≥ m$ and $N$ cannot intersect infinitely many sets $X \setminus f_n^{-1}(U)$. This contradiction completes the proof of the continuity of the map $δ : X \to C_K(C_τ(X))$.

Therefore, $δ$ is a topological embedding of the space $X$ into the cometrizable space $C_K(C_τ(X))$, which implies that the space $X$ is cometrizable.

6. The construction of the space from Example [1.14]

In this section we construct a regular topology $τ$ of weight $ω_1$ on the set of rational numbers $\mathbb{Q}$ such that the topological space $(\mathbb{Q}, τ)$ is not cometrizable (and contains no infinite compact subsets under CII).
Let \( \tau_0 \) be the standard metrizable topology on the set \( \mathbb{Q} \) of rational numbers. Let \( \mathcal{G} \) be the set of all injective functions \( s : \omega \to \mathbb{Q} \setminus \{0\} \) such that the sequence \((s(n))_{n \in \omega}\) converges in the topological space \((\mathbb{Q}, \tau_0)\).

Since the set \(\mathcal{G}\) has cardinality \(|\mathcal{G}| = \aleph_0\), it can be written as \( \mathcal{G} = \{s_{\alpha+1}\}_{\alpha \in \varepsilon} \). For each limit ordinal \( \alpha < \omega_1 \) let \( s_\alpha : \omega \to \mathbb{Q} \) be the map defined by \( s_\alpha(n) = n + 1 \) for \( n \in \varepsilon \).

By transfinite induction of length \( \omega_1 \), for every countable ordinal \( \alpha \) we select a regular second countable topology \( \tau_\alpha \) on \( \mathbb{Q} \) and a neighborhood \( U_\alpha \in \tau_\alpha \) of zero such that the following conditions are satisfied for every \( \alpha < \omega_1 \):

1. \( (1_\alpha) \bigcup_{\beta < \alpha} \tau_\beta \subset \tau_\alpha \);
2. \( (2_\alpha) \) the topological space \((\mathbb{Q}, \tau_\alpha)\) has no isolated points;
3. \( (3_\alpha) \) for every \( \beta < \alpha \) and every neighborhood \( V \in \tau_\alpha \) of zero the \( \tau_\beta \)-closure of \( V \) is not contained in \( U_{\beta+1} \);
4. \( (4_\alpha) \) the sequence \((s_\alpha(n))_{n \in \omega}\) is not convergent in the topological space \((\mathbb{Q}, \tau_\alpha)\).

Assume that for some non-zero ordinal \( \alpha \) and all ordinals \( \beta < \alpha \) we have constructed topologies \( \tau_\beta \) and open sets \( U_\beta \in \tau_\beta \) satisfying the conditions \((1_\beta)–(4_\beta)\).

If \( \alpha \) is a limit ordinal, let \( \tau_\alpha \) be the topology on \( \mathbb{Q} \), generated by the base \( \bigcup_{\beta < \alpha} \tau_\beta \), and observe that the conditions \((1_\alpha)–(4_\alpha)\) are satisfied. The condition \((4_\alpha)\) is satisfied since \( s_\alpha(n) = n + 1 \) for \( n \in \omega \).

Now assume that \( \alpha \) is a successor ordinal and hence \( \alpha = \gamma + 1 \) for some ordinal \( \gamma \). Let \( \xi : \omega \to \gamma = [0, \gamma) \) be a function such that for every \( \beta < \gamma \) the preimage \( \xi^{-1}(\beta) = \{ n \in \omega : \xi(n) = \beta \} \) is infinite.

Let \( \{V_n\}_{n \in \omega} \subset \tau_\gamma \) be a neighborhood base at zero such that \( V_0 = \mathbb{Q} \) and

\[
V_{n+1} \subset V_n \cap [2^{-n-1}, 2^{n+1}] \cap U_{\xi(n)+1}
\]

for all \( n \in \omega \). Since the metrizable countable space \((\mathbb{Q}, \tau_\gamma)\) is zero-dimensional, we can additionally assume that each \( V_n \) is closed-and-open in the topology \( \tau_\gamma \). If the limit point \( x_0 \) of the convergent sequence \((s_{\gamma+1}(n))_{n \in \omega}\) is not equal zero, then we shall additionally assume that the \( \tau_\gamma \)-closure of the set \( V_1 \) is disjoint with the compact set \( \{0\} \cup \{s_\alpha(n)\}_{n \in \omega} \) (which does not contain zero).

If \( x_0 \neq 0 \), then put \( x_{0,k} = s_\alpha(k) \) for all \( k \in \omega \) and observe that the sequence \((x_{0,k})_{k \in \omega}\) converges to the point \( x_0 \) in the topology \( \tau_0 \).

If \( x_0 = 0 \) and the sequence \((s_\alpha(n))_{n \in \omega}\) converges to zero in the topology \( \tau_\gamma \), then we can choose an increasing number sequence \((n_k)_{k \in \omega}\) such that the point \( x_{0,k} := s_\alpha(n_k) \) belongs to the neighborhood \( V_k \in \tau_\gamma \) of zero. In this case the sequence \((x_{0,k})_{k \in \omega}\) converges to zero in the topology \( \tau_\gamma \).

If \( x_0 = 0 \) but the sequence \((s_\alpha(n))_{n \in \omega}\) does not converge to zero in the topology \( \tau_\gamma \), then for every \( k \in \omega \) choose any point \( x_{0,k} \in V_k \) and observe that the sequence \((x_{0,k})_{k \in \omega}\) converges to zero in the topology \( \tau_\gamma \).

By the condition \((3_\gamma)\), for every \( n \in \mathbb{N} \) the \( \tau_\xi(n) \)-closure of \( V_n \) is not contained in \( U_{\xi(n)+1} \). So, we can find a sequence \( \{x_{n,k}\}_{k \in \omega} \subset V_n \) of pairwise distinct points that converge to some point \( x_n \in X \setminus U_{\xi(n)+1} \) in the topology \( \tau_\xi(n) \). Since \( \tau_0 \subset \tau_\xi(n)+1 \), the sequence \((x_{n,k})_{k \in \omega}\) converges to \( x_n \) in the Euclidean topology \( \tau_0 \). Since \( V_n \subset [2^{-n}, 2^n] \), the point \( x_n \) belongs to the closed interval \([2^{-n}, 2^n]\) and hence the sequence \((x_n)_{n \in \omega}\) converges to zero in the topology \( \tau_0 \).

Replacing each sequence \((x_{n,k})_{k \in \omega}\) by a suitable subsequence, we can assume that the points \( x_{n,k} \), \((n,k) \in \mathbb{N} \times \omega \), are pairwise distinct and do not belong to the compact set \( \{x_0\} \cup \{x_{0,k}\}_{k \in \omega} \). Since \( \{x_n\} \cup \{x_{n,k}\}_{n,k \in \omega} \subset [2^{-n}, 2^n] \) for all \( n \in \mathbb{N} \), the subspace \( \{x_{n,k}\}_{n,k \in \omega} \) is closed and discrete in the (zero-dimensional) subspace \( \mathbb{Q} \setminus \{(0) \cup \{x_0\}_{n \in \omega}\} \) of \((\mathbb{Q}, \tau_0)\).
Consequently, for each \( n, k \in \omega \) we can find a closed-and-open neighborhood \( O_{n,k} \in \tau_0 \) of the point \( x_{n,k} \) such that the sets \( O_{n,k}, n, k \in \omega \), are pairwise disjoint and also are disjoint with the compact set \( \{0\} \cup \{x_n\}_{n \in \omega} \). Since the space \((\mathbb{Q}, \tau_0)\) has no isolated points, for every \( n \in \mathbb{N} \) and \( k \in \omega \) we can choose an open neighborhood \( V_{n,k} \in \tau_0 \) of the point \( x_{n,k} \) such that \( V_{n,k} \subseteq O_{n,k} \cap V_n \), \( V_{n,k} \) is not closed in the topology \( \tau_0 \) and the space \( O_{n,k} \setminus V_{n,k} \) has no isolated points in the topology \( \tau_0 \).

Let

\[
U_\alpha := \{0\} \cup \bigcup_{k \in \omega} O_{0,2k} \cup \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \omega} V_{n,k}
\]

and let \( \tau_\alpha \) be the topology generated by the subbase \( \tau_\gamma \cup \{U_\alpha, \mathbb{Q} \setminus U_\alpha\} \). It is easy to check that for the topology \( \tau_\alpha \) and the neighborhood \( U_\alpha \in \tau_\alpha \) of zero the conditions \((1_\alpha) - (3_\alpha)\) are satisfied.

Let us show that the condition \((4_\alpha)\) is satisfied, too. To derive a contradiction, assume that the sequence \((s_\alpha(k))_{k \in \omega}\) converges in the topological space \((X, \tau_\alpha)\). Since \( \tau_\gamma \subset \tau_\alpha \), it converges in the space \((X, \tau_\gamma)\) and hence \((x_{0,k})_{k \in \omega}\) is a subsequence of the sequence \((s_\alpha(n))_{n \in \omega}\) by the definition of the points \( x_{0,k}, k \in \omega \). Since the closed-and-open set \( U_\alpha \in \tau_\alpha \) contains the points \( x_{0,2k}, k \in \omega \), and does not contains the points \( x_{0,2k+1}, k \in \omega \), the sequence \((x_{0,k})_{k \in \omega}\) is not convergent in the topology \( \tau_\alpha \) and then the sequence \( \{s_\alpha(n)\}_{n \in \omega} \setminus \{x_{0,k}\}_{k \in \omega} \) also cannot be convergent in the topology \( \tau_\alpha \). Therefore, \((4_\alpha)\) is satisfied.

After completing the inductive construction, consider the topology \( \tau = \bigcup_{\alpha \in \omega_1} \tau_\alpha \) on \( \mathbb{Q} \) (this is a topology as \( \mathbb{Q} \) is countable). We claim that the space \((\mathbb{Q}, \tau)\) is not cometrizable. In the opposite case we could find a metrizable topology \( \sigma \subset \tau \) such that for every neighborhood \( U \in \tau \) of zero there exists a neighborhood \( V \in \tau \) of zero whose closure in the topology \( \sigma \) is contained in \( U \). Since the topology \( \sigma \subset \tau \) has a countable base, there exists a countable ordinal \( \beta \) such that \( \sigma \subset \tau_\beta \). By the choice of the topology \( \sigma \), for the neighborhood \( U_{\beta+1} \in \tau_\beta+1 \subset \tau \), there exists a neighborhood \( V \in \tau \) whose \( \sigma \)-closure in contained in \( U_{\beta+1} \). Since \( V \in \tau = \bigcup_{\alpha \in \omega_1} \tau_\alpha \), there exists a countable ordinal \( \alpha > \beta \) with \( V \in \tau_\alpha \). The inductive condition \((3_\alpha)\) ensures that the \( \tau_\beta \)-closure of \( V \) is not contained in \( U_{\beta+1} \). Since \( \sigma \subset \tau_\beta \), the \( \sigma \)-closure of \( V \) contains the \( \tau_\beta \)-closure of \( V \) and hence also is not contained in \( U_{\beta+1} \). This contradiction shows that the space \((\mathbb{Q}, \tau)\) is not cometrizable.

Finally, assuming that \( \omega_1 = \mathfrak{c} \), we shall prove that the space \((\mathbb{Q}, \tau)\) contains no infinite compact subsets. In the opposite case, we could find an injective function \( s : \omega \rightarrow \mathbb{Q} \setminus \{0\} \) such that the sequence \((s(n))_{n \in \omega}\) is convergent in the topological space \((\mathbb{Q}, \tau)\). Since \( \tau_0 \subset \tau \), this sequence remains convergent in the space \((\mathbb{Q}, \tau_0)\) and hence \( s \in \mathcal{G} = \{s_{\alpha+1}\}_{\alpha \in \omega_1} \). So, we can find a successor ordinal \( \alpha < \omega_1 \) such that \( s = s_\alpha \). In this case the inductive condition \((4_\alpha)\) ensures that the sequence \((s_\alpha(n))_{n \in \omega}\) is not convergent in the topology \( \tau_\alpha \) and then the sequence \((s(n))_{n \in \omega} = (s_\alpha(n))_{n \in \omega}\) cannot be convergent in the topology \( \tau \supset \tau_\alpha \), which contradicts the choice of \( s \). Since each compact subset of the space \( X \) is finite, the countable family \( \mathcal{N} = \{\{x\} : x \in X\} \) is a \( k \)-network for \( X \), which means that \( X \) is an \( \mathfrak{N}_0 \)-space and hence is \( \mathfrak{c} \)-space-

If \( \omega_1 < \mathfrak{p} \), then the space \( X \) is Fréchet-Urysohn by the definition of the cardinal \( \mathfrak{p} \). Assuming that \( X \) is an \( \mathfrak{N}_0 \)-space and applying [19, 11.4], we would conclude that that \( X \) is stratifiable and, by Theorem [17], \( X \) is cometrizable, which is not the case.
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