COARSE GEOMETRY AND TOPOLOGICAL PHASES

ESKE ELLEN EWERT AND RALF MEYER

Abstract. We propose the Roe $C^*$-algebra from coarse geometry as a model for topological phases of disordered materials. We explain the robustness of this $C^*$-algebra and formulate the bulk–edge correspondence in this framework. We describe the map from the $K$-theory of the group $C^*$-algebra of $\mathbb{Z}^d$ to the $K$-theory of the Roe $C^*$-algebra, both for real and complex $K$-theory.

1. Introduction

Topological insulators are materials that are insulating in the bulk but allow a current to flow on the boundary. These boundary currents are protected by topological invariants and thus, in ideal cases, flow without dissipation. The mathematical description of a topological insulator uses a $C^*$-algebra $\mathcal{A}$ that contains the resolvent of the Hamiltonian $H$ of the system; this amounts to $H \in \mathcal{A}$ if $H$ is bounded. To describe an insulator, the spectrum of $H$ should have a gap at the Fermi energy $E$. Depending on further symmetries of the system such as a time reversal, particle–hole or chiral symmetry, the topological phase of the material may be classified by a class in the $K$-theory of $\mathcal{A}$ associated to the spectral projection of $H$ at the Fermi energy (see, for instance, [17, 27]). So the observable algebra $\mathcal{A}$ or rather its $K$-theory predicts the possible topological phases of a material.

At first, a material is often modelled without disorder and in a tight binding approximation. This gives a translation-invariant Hamiltonian acting on $L^2(\mathbb{Z}^d, \mathbb{C}^N)$ (see, for instance, [5, 11, 22]). Bloch–Floquet theory describes the Fermi projection through a vector bundle over the $d$-torus, with extra structure that reflects the symmetries of the system (see, for instance, [19, 24, 25]). The $K$-theory of the $d$-torus is easily computed. Once $d \geq 2$, many of the topological phases that are predicted this way are obtained by stacking a lower-dimensional topological insulator in some direction. Such topological phases are called “weak” by Fu–Kane–Mele [11]. They claim that weak topological phases are not robust under disorder.

Other authors have claimed instead that weak topological insulators are also quite robust, see [28]. Their proof of robustness, however, is no longer topological. Roughly speaking, the idea is that, although disorder may destroy the topological phase, it must be rather special to do this. Random disorder will rarely be so special. So in a finite volume approximation, the topological phase will remain intact in most places, and the small area where the randomness destroys it will become negligible in the limit of infinite volume. Such an argument may also work for the Hamiltonian of an insulator that is homotopic to a trivial one. Our study is purely topological in nature and thus cannot see such phenomena.

We are going to explain the difference between strong and weak topological phases and the robustness of the former through a difference in the underlying observable algebras. Namely, we shall model a material with disorder by the Roe $C^*$-algebra of

1991 Mathematics Subject Classification. 82C44; 46L80; 82D25; 81R15.

Key words and phrases. topological insulator; Roe $C^*$-algebra; disordered material; $K$-theory; $K$-homology.


\( \mathbb{R}^d \) or \( \mathbb{Z}^d \), which is a central object of coarse geometry. Roe [29, 30] introduced them to get index theorems for elliptic operators on non-compact Riemannian manifolds.

Before choosing our observable algebra, we should ask: What is causing topological phases? At first sight, the answer seems to be the translation invariance of the Hamiltonian. Translation-invariance alone is not enough, however. And it is destroyed by disorder. The subalgebra of translation-invariant operators on the Hilbert space \( \ell^2(\mathbb{Z}^d, \mathbb{C}) \) is the algebra of \( N \times N \)-matrices over the group von Neumann algebra of \( \mathbb{Z}^d \), which is isomorphic to \( L^\infty(\mathbb{T}^d, \mathbb{M}_N) \). If topological phases were caused by translation invariance alone, they should be governed by the K-theory of \( L^\infty(\mathbb{T}^d, \mathbb{M}_N) \). This is clearly not the case. Instead, we need the group \( C^* \)-algebra, which is isomorphic to \( C(\mathbb{T}^d, \mathbb{M}_N) \). The reason why the spectral projections of the Hamiltonian belong to the group \( C^* \)-algebra instead of the group von Neumann algebra is that the matrix coefficients of the Hamiltonian for \( (x, y) \in \mathbb{Z}^d \) are supported in the region \( \|x - y\| \leq R \) for some \( R > 0 \); let us call such operators controlled. The controlled operators do not form a \( C^* \)-algebra, and it makes no difference for K-theory purposes to allow the Hamiltonian to be a limit of controlled operators in the norm topology. We shall see below that this is equivalent to continuity with respect to the action of \( \mathbb{R}^d \) on \( B(\ell^2(\mathbb{Z}^d, \mathbb{C}) \) generated by the position observables. This action restricts to the translation action of \( \mathbb{R}^d \) on the group von Neumann algebra \( L^\infty(\mathbb{T}^d, \mathbb{M}_N) \), so that its continuous elements are the functions in \( C(\mathbb{T}^d, \mathbb{M}_N) \). Hence \( C(\mathbb{T}^d, \mathbb{M}_N) \subseteq B(\ell^2(\mathbb{Z}^d, \mathbb{C}) \) consists of those operators that are both translation-invariant and norm limits of controlled operators.

Since disorder destroys translation invariance, we should drop this assumption to model systems with disorder. The \( C^* \)-algebra of all operators on \( \ell^2(\mathbb{Z}^d, \mathbb{C}) \) that are norm limits of controlled operators is the algebra of \( N \times N \)-matrices over the uniform Roe \( C^* \)-algebra of \( \mathbb{Z}^d \). To get the Roe \( C^* \)-algebra, we must work in \( \ell^2(\mathbb{Z}^d, \mathcal{H}) \) for a separable Hilbert space \( \mathcal{H} \) and add a local compactness property, namely, that the operators \( (x \mid (H - \lambda)^{-1} \mid y) \in B(\mathcal{H}) \) are compact for all \( x, y \in \mathbb{Z}^d \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). This property is automatic for operators on \( \ell^2(\mathbb{Z}^d, \mathbb{C}) \). Working on \( \ell^2(\mathbb{Z}^d, \mathcal{H}) \) and assuming local compactness means that we include infinitely many bands in our model and require only finitely many states with finite energy in each finite volume. Kubota [21] has already used the uniform Roe \( C^* \)-algebra and the Roe \( C^* \)-algebra in the context of topological insulators. He prefers the uniform Roe \( C^* \)-algebra. We explain why we consider this a mistake.

Working in the Hilbert space \( \ell^2(\mathbb{Z}^d, \mathbb{C}) \) already involves an approximation. We ought to work in a continuum model, that is, in the Hilbert space \( L^2(\mathbb{R}^d, \mathbb{C}^k) \), where \( k \) is the number of internal degrees of freedom. This Hilbert space is isomorphic to

\[
L^2(\mathbb{R}^d, \mathbb{C}^k) \cong L^2(\mathbb{Z}^d \times (0, 1]^d) \otimes \mathbb{C}^k \cong \ell^2(\mathbb{Z}^d, \mathcal{H} \otimes \mathbb{C}^k)
\]

when we cover \( \mathbb{R}^d \) by the disjoint translates of the fundamental domain \((0, 1]^d\). This identification preserves both controlled and locally compact operators. Thus the Roe \( C^* \)-algebras of \( \mathbb{Z}^d \) and \( \mathbb{R}^d \) are isomorphic. For the Roe \( C^* \)-algebra of \( \mathbb{R}^d \), it makes no difference to replace \( L^2(\mathbb{R}^d) \) by \( L^2(\mathbb{R}^d, \mathbb{C}^k) \) or \( L^2(\mathbb{R}^d, \mathcal{H}) \): all these Hilbert spaces give isomorphic \( C^* \)-algebras of locally compact, approximately controlled operators. So there is only one Roe \( C^* \)-algebra for \( \mathbb{R}^d \), and it is isomorphic to the non-uniform Roe \( C^* \)-algebra of \( \mathbb{Z}^d \). We view the appearance of the uniform Roe \( C^* \)-algebra for \( \mathbb{Z}^d \) as an artefact of simplifying assumptions in tight binding models.

We describe some interesting elements of the Roe \( C^* \)-algebra of \( \mathbb{R}^d \) in Example 2.4.

In particular, it contains all \( C_0 \)-functions of the impulse operator \( P \) on \( L^2(\mathbb{R}^d) \) or, equivalently,

\[
\int_{\mathbb{R}^d} f(x) \exp(i x P) \, dx
\]
for \( f \in C^*(\mathbb{R}^d) \); this operator is controlled if and only if \( f \) has compact support. If \( V \in L^\infty(\mathbb{R}^d) \), then the operator of multiplication by \( V \) on \( L^2(\mathbb{R}^d) \) is controlled, but not locally compact. Its product with an operator as in (1) belongs to the Roe \( C^* \)-algebra.

The real and complex \( K \)-theory of the Roe \( C^* \)-algebra of \( \mathbb{Z}^d \) is well known: up to a dimension shift of \( d \), it is the \( K \)-theory of \( \mathbb{R} \) or \( \mathbb{C} \), respectively. In particular, the Roe \( C^* \)-algebra as an observable algebra is small enough to predict some distinct topological phases. These coincide with Kitaev’s periodic table [20]. This corroborates the choice of the Roe \( C^* \)-algebra as the observable algebra for disordered materials.

When we disregard disorder, the Roe \( C^* \)-algebra may be replaced by its translation-invariant subalgebra, which is isomorphic to

\[
C^*(\mathbb{Z}^d) \otimes \mathbb{K}(\mathcal{H}) \cong C(\mathbb{T}^d, \mathbb{K}(\mathcal{H})),
\]

where \( \mathbb{K}(\mathcal{H}) \) denotes the \( C^* \)-algebra of compact operators on an infinite-dimensional separable Hilbert space \( \mathcal{H} \). In the real case, the \( d \)-torus must be given the real involution by the restriction of complex conjugation on \( \mathbb{C}^d \subset \mathbb{T}^d \). The real or complex \( K \)-theory groups of the “real” \( d \)-torus describe both weak and strong topological phases in the presence of different types of symmetries. We show that the map

(2)

\[
K_*(C^*(\mathbb{Z}^d)_{\mathbb{F}}) \to K_*(C^*_{\text{Roe}}(\mathbb{Z}^d)_{\mathbb{F}})
\]

for \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \) is split surjective and that its kernel is the subgroup generated by the images of \( K_*(C^*(\mathbb{Z}^{d-1})_{\mathbb{F}}) \) for all coordinate embeddings \( \mathbb{Z}^{d-1} \to \mathbb{Z}^d \). That is, the kernel of the map in (2) consists exactly of the \( K \)-theory classes of weak topological insulators as defined by Fu–Kane–Mele [11]. The strong topological insulators are those that remain topologically protected even if the observable algebra is enlarged to the Roe \( C^* \)-algebra, allowing rather general disorder.

Following Bellissard [2,5], disorder is usually modelled by crossed product \( C^* \)-algebras \( \mathcal{A} = C(\Omega) \rtimes \mathbb{Z}^d \), where \( \Omega \) is the space of disorder configurations. It is more precise to say, however, that the space \( \Omega \) describes restricted disorder. Uncountably many different choices are possible. Such models are only reasonable when the physically relevant objects do not depend on the choice. But the \( K \)-theory of the crossed product depends on the topology of the space \( \Omega \). To make it independent of \( \Omega \), the space \( \Omega \) is assumed to be contractible in [27]. This fits well with standard choices of \( \Omega \) such as a product space \( \prod_{n \in \mathbb{Z}^d} [-1, 1] \) to model a random potential. The resulting \( K \)-theory then becomes the same as in the system without disorder. So another argument must be used to explain the difference between weak and strong topological phases, compare [27, Remark 5.3.5]. If one allows non-metrizable \( \Omega \), then there is a maximal choice for \( \Omega \), namely, the Stone–Čech compactification of \( \mathbb{Z}^d \). The resulting crossed product \( \ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \) is isomorphic to the uniform Roe \( C^* \)-algebra of \( \mathbb{Z}^d \), see [21]. Nevertheless, even this maximal choice of \( \Omega \) still contains a hidden restriction on disorder: the number of bands for a tight binding model is fixed, and so the disorder is also limited to a fixed finite number of bands. The Roe \( C^* \)-algebra of \( \mathbb{Z}^d \) also removes this hidden restriction on the allowed disorder. It is also a crossed product, namely,

\[
C^*_{\text{Roe}}(\mathbb{Z}^d) \cong \ell^\infty(\mathbb{Z}^d, \mathbb{K}(\mathcal{H})) \rtimes \mathbb{Z}^d.
\]

The \( C^* \)-algebra \( \ell^\infty(\mathbb{Z}^d, \mathbb{K}(\mathcal{H})) \) is not isomorphic to \( \ell^\infty(\mathbb{Z}^d) \otimes \mathbb{K}(\mathcal{H}) \): it even has different \( K \)-theory.

Since the Roe \( C^* \)-algebra has not been used much in the context of topological insulators, we recall its main properties in Section 2. We highlight its robustness or even “universality.” Roughly speaking, there is only one Roe \( C^* \)-algebra in each dimension, which describes all kinds of disordered materials in that dimension.
We describe the subalgebras of smooth, real-analytic and holomorphic elements without symmetries. The various symmetries (time-reversal, particle-hole, chiral) may be added by tensoring the real or complex Roe $C^*$-algebra with Clifford algebras, which replaces $K_0$ by $K_i$ for some $i \in \mathbb{Z}$. We shall not say much about this here. The Roe $C^*$-algebra of a coarse space is a coarse invariant. In particular, all coarsely dense subsets in $\mathbb{R}^d$ give isomorphic Roe $C^*$-algebras. Furthermore, the Roe $C^*$-algebras of $\mathbb{Z}^d$ and other coarsely dense subsets of $\mathbb{R}^d$ are isomorphic to that of $\mathbb{R}^d$. Thus it makes no difference whether we work in a continuum or lattice model. We also consider the twists of the Roe $C^*$-algebra defined by magnetic fields. The resulting twisted Roe $C^*$-algebras are also isomorphic to the untwisted one. This robustness of the Roe $C^*$-algebra means that the same strong topological phases occur for all materials of a given dimension and symmetry type, even for quasi-crystals and aperiodic materials.

We compute the K-theory of the Roe $C^*$-algebra in Section 3 using the coarse Mayer–Vietoris principle introduced in [13]. We prove the Mayer–Vietoris exact sequence in the real and complex case by reducing it to the representation of $\mathbb{R}^d$-algebras of a proper metric space and let $\varrho: C^*_Roe(X,F) \to C^*_Roe(\mathbb{Z}^d,F)$ be the zero map on K-theory. Hence the map $\varrho$ is an isomorphism. It shows also that the inclusion $C^*_Roe(\mathbb{Z}^d,F) \to C^*_Roe(\mathbb{Z}^d,F)$ induces the zero map on K-theory. Hence the map $\varrho$ is an isomorphism. We shall not say much about the fundamental class of the “real” $d$-torus $\mathbb{R}^d$. Except for an adaptation to “real” manifolds, this fundamental class is introduced in [16]. We show that the fundamental class extends to a K-homology class on the Roe $C^*$-algebra and that the pairing with this K-homology class is an isomorphism $K_{∗+d}(C^*_Roe(X,F)) \cong K_∗(F)$.

2. ROE $C^*$-ALGEBRAS

In this section, we define the real and complex Roe $C^*$-algebras of a proper metric space and prove that they are invariant under passing to a coarsely dense subspace and, more generally, under coarse equivalence. We prove that the twists used to encode magnetic fields do not change them. And we describe elements of the Roe $C^*$-algebra of a subset of $\mathbb{R}^d$ as those locally compact operators that are continuous for the representation of $\mathbb{R}^d$ generated by the position operators. We describe the subalgebras of smooth, real-analytic and holomorphic elements of the Roe $C^*$-algebra for this action of $\mathbb{R}^d$. We show that Roe $C^*$-algebras have approximate units of projections, which simplifies the definition of their K-theory.

Let $(X,d)$ be a locally compact, second countable, metric space. We assume the metric $d$ to be proper, that is, bounded subsets of $X$ are compact. We shall be mainly interested in $\mathbb{R}^d$ or a discrete subset of $\mathbb{R}^d$ with the restriction of the Euclidean metric. (All our results on general proper metric spaces extend easily to the more general coarse spaces introduced in [31].) Let $\mathcal{H}$ be a real or complex separable Hilbert space and let $\varrho: C_0(X) \to \mathcal{B}(\mathcal{H})$ be a nondegenerate representation. We are going to define the Roe $C^*$-algebra of $X$ with respect to $\varrho$, see also [12, Section 6.3]. Depending on whether $\mathcal{H}$ is a real or complex Hilbert space, this gives a real or complex version of the Roe $C^*$-algebra. Both cases are completely analogous.

Let $T \in \mathcal{B}(\mathcal{H})$. We call $T$ locally compact (on $X$) if the operators $\varrho(f)T$ and $T\varrho(f)$ are compact for all $f \in C_0(X)$. The support of $T$ is a subset $\text{supp} \subseteq X \times X$. Its complement consists of all $(x,y) \in X \times X$ for which there are neighbourhoods $U_x$, $U_y$ in $X$ such that $\varrho(f)T\varrho(g) = 0$ for all $f \in C_0(U_x)$, $g \in C_0(U_y)$. The operator $T$ is controlled (or has finite propagation) if there is $R > 0$ such that $d(x,y) \leq R$ for...
all \((x, y) \in \text{supp}\, T\). We sometimes write “R-controlled” to highlight the control parameter \(R\). The locally compact, controlled operators on \(\mathcal{H}\) form a \(*\)-algebra. Its closure in \(\mathcal{B}(\mathcal{H})\) is the Roe \(C^*\)-algebra \(C^*_{\text{Roe}}(X, \varrho)\).

The representation \(\varrho\) is called ample if the operator \(\varrho(f)\) for \(f \in C_0(X)\) is only compact for \(f = 0\).

**Theorem 2.1.** Let \(\varrho_i : C_0(X) \to \mathcal{B}(\mathcal{H}_i)\) for \(i = 1, 2\) be ample representations, where \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are both complex or both real. Then \(C^*_{\text{Roe}}(X, \varrho_1) \cong C^*_{\text{Roe}}(X, \varrho_2)\). Even more, there is a unitary operator \(U : \mathcal{H}_1 \sim \to \mathcal{H}_2\) with

\[
U C^*_{\text{Roe}}(X, \varrho_1) U^* = C^*_{\text{Roe}}(X, \varrho_2).
\]

Many references only assert the weaker statement that the Roe \(C^*\)-algebras for all ample representations have canonically isomorphic K-theory, compare [12 Corollary 6.3.13]. The statement above is [13, Lemma 2], and our proof is the same.

**Proof.** If \(X\) is compact, then \(C^*_{\text{Roe}}(X, \varrho_1) = \mathcal{K}(\mathcal{H}_i)\). Since \(\mathcal{H}_i\) for \(i = 1, 2\) are assumed to be separable, there is a unitary \(U : \mathcal{H}_1 \sim \to \mathcal{H}_2\), and it will do the job. So we may assume \(X\) to be non-compact. Fix \(R > 0\). The open balls \(B(x, R)\) for \(x \in X\) cover \(X\). Since \(X\) is second countable, there is a subordinate countable, locally finite, open covering \(X = \bigcup_{n \in \mathbb{N}} U_n\), where each \(U_n\) is non-empty and has diameter at most \(R\). Then there is a countable covering of \(X\) by disjoint Borel sets, \(X = \bigcup_{n \in \mathbb{N}} B'_n\), where each \(B'_n\) has diameter at most \(R\), and such that any relatively compact subset is already covered by finitely many of the \(B'_n\): simply take \(B'_n := U_n \setminus \bigcup_{j < n} U_j\). Next we modify the subsets \(B'_n\) so that they all have non-empty interior. Let \(\mathcal{M} \subseteq \mathbb{N}\) be the set of all \(n \in \mathbb{N}\) for which \(B'_n\) has non-empty interior. Let \(m \in \mathcal{M}\). Let \(K_m\) be the set of all \(k \in \mathbb{N}\) for which \(B'_k\) has empty interior and \(U_m \cap U_k \neq \emptyset\). The set \(K_m\) is finite because \(U_m\) is bounded and hence relatively compact. Let \(K^\circ_m := K_m \setminus \bigcup_{i, m < n} K_i\). Define

\[
B_m := B'_m \cup \bigcup_{k \in K^\circ_m} B'_k.
\]

This is a Borel set. It has non-empty interior because \(B'_m\) has non-empty interior. Its diameter is at most \(3R\) because all the \(U_k\) for \(k \in K_m\) intersect \(U_m\). The definition of \(K^\circ_m\) ensures that the subsets \(B_m\) for \(m \in \mathcal{M}\) are disjoint. We claim that \(\bigcup_{m \in \mathcal{M}} B_m = X\). This is equivalent to \(\bigcup_{m \in \mathcal{M}} K^\circ_m = \mathbb{N} \setminus \mathcal{M}\) because \(B'_m \subseteq B_m\) for all \(m \in \mathcal{M}\). This is further equivalent to \(\bigcup_{m \in \mathcal{M}} K_m = \mathbb{N} \setminus \mathcal{M}\). Let \(k_0 \in \mathbb{N} \setminus \mathcal{M}\), that is, \(B'_m\) has empty interior. Then \(B'_{k_0}\) does not contain \(U_{k_0}\). So there is some \(k_1 < k_0\) with \(U_{k_0} \cap U_{k_1} \neq \emptyset\). If \(k_1 \in \mathbb{N} \setminus \mathcal{M}\), then \(B'_{k_1}\) does not contain \(U_{k_0} \cap U_{k_1}\). So there is \(k_2 < k_1\) with \(U_{k_0} \cap U_{k_1} \cap U_{k_2} \neq \emptyset\). We continue like this and build a decreasing chain \(k_0 > k_1 > \ldots > k_t\) such that \(k_0, \ldots, k_{t-1} \in \mathbb{N} \setminus \mathcal{M}\) and

\[
U_{k_0} \cap U_{k_1} \cap \cdots \cap U_{k_t} \neq \emptyset.
\]

We eventually reach \(k_t \in \mathcal{M}\) because \(B'_1 = U_1\) is open and so \(1 \in \mathcal{M}\). We have \(k_0 \in K_{k_t}\). So \(\bigcup_{m \in \mathcal{M}} K_m = \mathbb{N} \setminus \mathcal{M}\) as asserted.

We have built a covering of \(X\) by disjoint Borel sets \(X = \bigcup_{m \in \mathcal{M}} B_m\) of diameter at most \(3R\), with non-empty interiors, and such that any relatively compact subset is already covered by finitely many of the \(B_m\). The set \(\mathcal{M}\) is at most countable, and it cannot be finite because then \(X\) would be bounded and hence compact.

Using the Borel functional calculus for the representation \(\varrho_i\), we may decompose the Hilbert space \(\mathcal{H}_i\) as an orthogonal direct sum, \(\mathcal{H}_i = \bigoplus_{m \in \mathcal{M}} \mathcal{H}_{i, m}\), where \(\mathcal{H}_{i, m}\) is the image of the projection \(\varrho_i(1_{B_m})\). Since each \(B_m\) has non-empty interior and our representations are ample, there is a non-compact operator on each \(\mathcal{H}_{i, m}\). So no \(\mathcal{H}_{i, m}\) has finite dimension. Hence there is a unitary \(U_m : \mathcal{H}_{1, m} \sim \to \mathcal{H}_{2, m}\) for each \(m \in \mathcal{M}\). We combine these into a unitary operator \(U = \bigoplus_{m \in \mathcal{M}} U_m : \mathcal{H}_1 \sim \to \mathcal{H}_2\).
Let $T \in B(H_1)$. We claim that $UTU^*$ is locally compact or controlled if and only if $T$ is. This implies $UC_{\text{Roe}}^*(X, \varrho_1)U^* = C_{\text{Roe}}^*(X, \varrho_2)$ as asserted. First, we claim that $T$ is locally compact if and only if $T\varrho_1(1_B_m)$ and $\varrho_1(1_B_m)T$ are compact for all $m \in M$. In one direction, this uses that there is $g \in C_0(X)$ with $1_B_m \leq g$ because $B_m$ has finite diameter. In the other direction, it uses that any relatively compact subset of $X$ is already covered by finitely many $B_m$. Since $U(H_{1,m}) = H_{2,m}$, the criterion above shows that $T$ is locally compact if and only if $UTU^*$ is so. Since the diameter of $B_m$ is at most $3R$, the operator $U_m$, viewed as a partial isometry on $H_1 \oplus H_2$, is $3R$-controlled. Thus $U$ is also $3R$-controlled. So $UTU^*$ is controlled if and only if $T$ is. 

**Corollary 2.2.** Let $\varrho$ be an ample representation and let $m \in \mathbb{N}_{\geq 2}$. Then 

$$C_{\text{Roe}}^*(X, \varrho) \cong M_m(C_{\text{Roe}}^*(X, \varrho)).$$

**Proof.** The direct sum representation $m \cdot \varrho$ is still ample. So $C_{\text{Roe}}^*(X, m \cdot \varrho) \cong C_{\text{Roe}}^*(X, \varrho)$. An operator on $H^m$ is locally compact or controlled if and only if its block matrix entries in $B(H)$ are so. Thus $C_{\text{Roe}}^*(X, m \cdot \varrho) = M_m(C_{\text{Roe}}^*(X, \varrho))$. \hfill $\Box$

The stabilisation $C_{\text{Roe}}^*(X, \varrho) \otimes \mathbb{K}(\ell^2 \mathbb{N})$, however, is usually not isomorphic to $C_{\text{Roe}}^*(X, \varrho)$.

**Example 2.3.** Let $X$ be discrete, for instance, $X = \mathbb{Z}^d$. The representation $\varrho$ of $C_0(X)$ on $\ell^2(X)$ by multiplication operators is not ample. It defines the **uniform Roe $C^*$-algebra** of $X$. To get the Roe $C^*$-algebra, we may take the representation of $C_0(X)$ on $\ell^2(X) \otimes \ell^2(\mathbb{N})$.

An operator $T$ on $\ell^2(X) \otimes \ell^2(\mathbb{N})$ is determined by its matrix coefficients $T_{x,y} = (x | T | y) \in \mathbb{B}(\ell^2(\mathbb{N}))$ for $x, y \in X$. It is locally compact if and only if all $T_{x,y}$ are compact. Its support is the set of all $(x, y) \in X^2$ with $T_{x,y} \neq 0$. So it is controlled if and only if there is $R > 0$ so that $T_{x,y} = 0$ for $d(x, y) > R$. The Roe $C^*$-algebra is the norm closure of these operators.

If $X$ is a discrete group equipped with a translation-invariant metric, then $C_{\text{Roe}}^*(X)$ is isomorphic to the reduced crossed product for the translation action of $X$ on $\ell^2(X, \mathbb{K}(\ell^2 \mathbb{N}))$ (compare [31] Theorem 4.28 for the uniform Roe $C^*$-algebra).

**Example 2.4.** Let $X = \mathbb{R}^d$. The representation $\varrho$ of $C_0(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d, dx)$ (real or complex) by multiplication operators is ample. Actually, all faithful representations of $C_0(\mathbb{R}^d)$ are ample. So they all give isomorphic Roe $C^*$-algebras by Theorem 2.1.

Let $T \in C_0(\mathbb{R}^d)$ (with real or complex values) act on $L^2(\mathbb{R}^d)$ by convolution. Then $T \cdot \varrho(f)$ and $\varrho(f) \cdot T$ are compact because they have a compactly supported, continuous integral kernel. And $T$ is controlled by the supremum of $\|x\|$ with $T(x) \neq 0$. So $T \in C_{\text{Roe}}^*(\mathbb{R}^d)$. Hence $C^*(\mathbb{R}^d) \subseteq C_{\text{Roe}}^*(\mathbb{R}^d)$. In particular, the resolvent of the Laplace operator or another translation-invariant elliptic differential operator on $\mathbb{R}^d$ belongs to $C_{\text{Roe}}^*(\mathbb{R}^d)$. Any multiplication operator is controlled. Thus multiplication operators are multipliers of $C_{\text{Roe}}^*(\mathbb{R}^d)$. And $L^\infty(\mathbb{R}^d) \cdot C^*(\mathbb{R}^d) \cdot L^\infty(\mathbb{R}^d)$ is contained in $C_{\text{Roe}}^*(\mathbb{R}^d)$. (Since the translation action of $\mathbb{R}^d$ on $L^\infty(\mathbb{R}^d)$ is not continuous, there is no crossed product for this action and it is unclear whether the closed linear spans of $L^\infty(\mathbb{R}^d) \cdot C^*(\mathbb{R}^d)$ and $C^*(\mathbb{R}^d) \cdot L^\infty(\mathbb{R}^d)$ are equal and form a $C^*$-algebra.)

**Proposition 2.5.** Let $V \in L^\infty(\mathbb{R}^n)$ and let $\Delta$ be the Laplace operator on $\mathbb{R}^d$. Then the resolvent of $V + \Delta$ belongs to the Roe $C^*$-algebra of $\mathbb{R}^d$.

We are indebted to Detlev Buchholz for pointing out the following simple proof.

**Proof.** View $V = V(Q)$ as an operator on $L^2(\mathbb{R}^d)$. Then $\|(ic + \Delta)^{-1}V\|^2 < 1$ for sufficiently large $c \in \mathbb{R}_{>0}$. Hence the Neumann series $\sum (-(ic + \Delta)^{-1}V)^n$ converges,
and
\[ \sum_{n=0}^{\infty} (-i\varepsilon + \Delta)^{-1} V^n = (1 + (-i\varepsilon + \Delta)^{-1} V)^{-1}. \]

We have already seen that \((i\varepsilon + \Delta)^{-1}\) and \((i\varepsilon + \Delta)^{-1} V\) belong to the Roe \(C^*\)-algebra. Hence so does \((i\varepsilon + \Delta + V)^{-1}\).

If \(\varrho\) is ample, then we often leave out \(g\) and briefly write \(C^*_\text{Roe}(X)_R\) or \(C^*_\text{Roe}(X)_C\), depending on whether \(g\) acts on a real or complex Hilbert space. Theorem 2.7 justifies this.

**Definition 2.6.** A closed subset \(Y \subseteq X\) is coarsely dense if there is \(R > 0\) such that for any \(x \in X\) there is \(y \in Y\) with \(d(x, y) \leq R\).

**Theorem 2.7.** Let \(Y \subseteq X\) be coarsely dense. Then \(C^*_\text{Roe}(Y)_R \cong C^*_\text{Roe}(X)_R\) and \(C^*_\text{Roe}(Y)_C \cong C^*_\text{Roe}(X)_C\). Both isomorphisms are implemented by unitaries between the underlying Hilbert spaces.

**Proof.** The proofs in the complex and real case are identical. Let \(g : C_0(X) \to C_0(Y)\) be the restriction homomorphism. Let \(g_Y : C_0(Y) \to \mathcal{B}(\mathcal{H}_Y)\) and \(g_X : C_0(X) \to \mathcal{B}(\mathcal{H}_X)\) be ample representations. Then \(g' := g_X \oplus g_Y \circ \pi\) is an ample representation of \(C_0(X)\) on \(\mathcal{H}' := \mathcal{H}_X \oplus \mathcal{H}_Y\).

**Definition 2.8.** Let \(f : X \to Y\) be a coarse map. For instance, the inclusion of a coarsely dense subspace is a coarse equivalence: the proof of Theorem 2.7 suffices for our purposes, but we mention that it extends to arbitrary coarsely dense subsets of \(X\).

**Theorem 2.9.** Let \(X\) and \(Y\) be coarsely equivalent. Then \(C^*_\text{Roe}(X)_R \cong C^*_\text{Roe}(Y)_R\) and \(C^*_\text{Roe}(X)_C \cong C^*_\text{Roe}(Y)_C\).
Proof. Here it is more convenient to work with coarse spaces. Let \( f: X \to Y \) be the coarse equivalence. We claim that there is a coarse structure on the disjoint union \( X \sqcup Y \) such that both \( X \) and \( Y \) are coarsely dense in \( X \sqcup Y \). This reduces the result to Theorem \( \ref{thm:coarse_equivalence} \). We describe the desired coarse structure on \( X \sqcup Y \). A subset \( E \) of \( (X \sqcup Y)^2 \) is called controlled if its intersections with \( X^2 \) and \( Y^2 \) and the set of all \( (f(x), y) \in Y^2 \) for \( (x, y) \in E \) or \( (y, x) \in E \) are controlled. This is a coarse structure on \( X \sqcup Y \) because \( f \) is a coarse equivalence. And the subspaces \( X \) and \( Y \) are coarsely dense for the same reason.

2.10. **Twists.** We show that magnetic twists do not change the isomorphism class of the Roe C*-algebra. We let \( X \) be a discrete metric space. Let \( g: C_0(X) \to B(H) \) be a representation. This is equivalent to a direct sum decomposition \( H = \bigoplus_{x \in X} H_x \), such that \( f \in C_0(X) \) acts by multiplication with \( f(x) \) on the summand \( H_x \). We assume for simplicity that each \( H_x \) is non-zero. This is weaker than being amenable, which means that each \( H_x \) is infinite-dimensional. So the following discussion also covers the uniform Roe C*-algebra of \( X \).

We describe an operator on \( H \) by a block matrix \( (T_{x,y})_{x,y \in X} \) with \( T_{x,y} \in B(H_y, H_x) \). These are multiplied by the usual formula, \((ST)_{x,y} = \sum_{z \in X} S_{x,z} T_{z,y} \).

We twist this multiplication by a scalar-valued function \( w: X \times X \times X \to \mathbb{T} \):

\[
(S * w T)_{x,y} = \sum_{z \in X} w(x, z, y) S_{x,z} T_{z,y}.
\]

This defines a bounded bilinear map at least on the subalgebra \( A(X, g) \subseteq B(H) \) of locally compact, controlled operators.

**Lemma 2.11.** The multiplication \( *_w \) on \( A(X, g) \) is associative if and only if

\[
w(x, z, y)w(x, t, z) = w(x, t, y)w(t, z, y)
\]

for all \( x, t, z, y \in X \).

**Proof.** For \( S, T, U \in A(X, g) \), we compute

\[
((S *_w T) *_w U)_{x,y} = \sum_{z,t \in X} w(x, z, y)w(x, t, z) S_{x,t} T_{t,z} U_{z,y},
\]

\[
(S *_w (T *_w U))_{x,y} = \sum_{z,t \in X} w(x, t, y)w(t, z, y) S_{x,t} T_{t,z} U_{z,y}.
\]

The condition \( \ref{eq:associativity} \) holds if and only if these are equal for all \( S, T, U \in A(X, g) \) because all \( H_x \) are non-zero.

**Proposition 2.12.** If the function \( w \) satisfies the cocycle condition in the previous lemma, then there is a function \( v: X \times X \to \mathbb{T} \) with

\[
w(x, z, y) = v(x, z)v(z, y)v(x, y)^{-1}.
\]

The map \( \varphi: (A(X, g), *_w) \to (A(X, g), \cdot), \ (T_{x,y})_{x,y \in X} \mapsto (v(x, y) \cdot T_{x,y})_{x,y \in X} \), is an algebra isomorphism.

**Proof.** Fix a “base point” \( e \in X \) and let \( v(x, y) := w(x, y, e) \). The condition \( \ref{eq:associativity} \) for \( (x, z, y, e) \) says that

\[
w(x, y, e)w(x, z, y) = w(x, z, e)w(z, y, e)
\]

holds for all \( x, z, y \in X \). So

\[
v(x, z)v(z, y)v(x, y)^{-1} = w(x, z, e)w(z, y, e)w(x, y, e)^{-1} = w(x, z, y).
\]
The map \( \varphi \) is a vector space isomorphism because \( v(x, y) \neq 0 \) for all \( x, y \in X \). The computation
\[
\varphi(S \ast_w T)_{x,y} = v(x,y) \sum_{z \in X} w(x,z,y) S_{x,z} T_{z,y}
\]
\[
= \sum_{z \in X} w(x,z,y)v(x,y)v(x,z)^{-1}v(z,y)^{-1}\varphi(S)_{x,z}\varphi(T)_{z,y} = \sum_{z \in X} \varphi(S)_{x,z}\varphi(T)_{z,y}
\]
shows that it is an algebra isomorphism. \( \square \)

So the twisted and untwisted versions of \( A(X, \varrho) \) are isomorphic algebras. Thus a magnetic field does not change the isomorphism type of the Roe \( C^* \)-algebra.

Now let \( X \) be no longer discrete. Then controlled, locally compact operators are not given by matrices any more. To write down the twisted convolution as above, we use the smaller \( \ast \)-algebra of controlled, locally \textit{Hilbert–Schmidt} operators; it is still dense in the Roe \( C^* \)-algebra. Let us assume for simplicity that the representation \( \varrho \) for which we build the Roe \( C^* \)-algebra has constant multiplicity, that is, it is the pointwise multiplication representation on \( L^2(X, \mu) \otimes \mathcal{H} \) for some regular Borel measure \( \mu \) on \( X \) and some Hilbert space \( \mathcal{H} \). Controlled, locally Hilbert–Schmidt operators on \( L^2(X, \mu) \otimes \mathcal{H} \) are the convolution operators for measurable functions \( T: X \times X \to \ell^2(\mathcal{H}) \) with controlled support and such that
\[
\int_{K \times K} \|T(x,y)\|^2 \, d\mu(x) \, d\mu(y) < \infty
\]
for all compact subsets \( K \subseteq X \) and such that the resulting convolution operator is bounded. The multiplication of such operators is given by a convolution of their integral kernels. This may be twisted as above, using a Borel function \( w: X^3 \to T \) that satisfies the condition \( 3 \). The resulting function \( v \) in Proposition \( 2.12 \) still works. Thus the twist gives an isomorphic \( \ast \)-algebra also in the non-discrete case.

Following Bellissard \( 2 \), a \( d \)-dimensional material is often described through a crossed product \( C^* \)-algebra \( C(\Omega) \rtimes_{\sigma} \mathbb{Z}^d \) for a compact space \( \Omega \) with a \( \mathbb{Z}^d \)-action by homeomorphisms and with an ergodic invariant measure on \( \Omega \). To encode a magnetic field, the crossed product is replaced by the crossed product twisted by a 2-cocycle \( \sigma: \mathbb{Z}^d \times \mathbb{Z}^d \to C(\Omega, \mathbb{T}) \). The space \( \Omega \) may be built as the “hull” of a point set or a fixed Hamiltonian, see \( 4 \). In this case, there is a \textit{dense} orbit \( \mathbb{Z}^d \cdot \omega \) in \( \Omega \) by construction. So assuming the existence of a dense orbit is a rather mild assumption in the context of Bellissard’s theory.

We briefly explain why all twisted crossed products \( C(\Omega) \rtimes_{\sigma} \mathbb{Z}^d \) as above are “contained” in the uniform Roe \( C^* \)-algebra of \( \mathbb{Z}^d \) and hence also in the Roe \( C^* \)-algebra. This observation is due to Kubota \( 21 \).

The main point here is the description of the uniform Roe \( C^* \)-algebra as a crossed product \( \ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \), see \( 31 \) Theorem 4.28. Let \( \omega \in \Omega \). Then we define a \( \mathbb{Z}^d \)-equivariant \( \ast \)-homomorphism \( \epsilon_\omega: C(\Omega) \to \ell^\infty(\mathbb{Z}^d) \) by \( (\epsilon_\omega f)(n) := f(n \cdot \omega) \) for all \( n \in \mathbb{Z}^d, f \in C(\Omega) \). This induces a \( \ast \)-homomorphism \( C(\Omega) \rtimes_{\sigma} \mathbb{Z}^d \to \ell^\infty(\mathbb{Z}^d) \rtimes_{\epsilon_\omega \circ \sigma} \mathbb{Z}^d \), where \( \epsilon_\omega \) denotes the crossed product twisted by a 2-cocyle \( \sigma \). The same argument that identifies the crossed product \( \ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \) with the uniform Roe \( C^* \)-algebra of \( \mathbb{Z}^d \) identifies \( \ell^\infty(\mathbb{Z}^d) \rtimes_{\epsilon_\omega \circ \sigma} \mathbb{Z}^d \) with a twist of the Roe \( C^* \)-algebra as above. Since all these twists give isomorphic \( C^* \)-algebras by Proposition \( 2.12 \) we get a \( \ast \)-homomorphism \( C(\Omega) \rtimes_{\sigma} \mathbb{Z}^d \to \ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \). If the orbit of \( \omega \) is dense, then the \( \ast \)-homomorphism \( \epsilon_\omega \) above is injective. Then the induced \( \ast \)-homomorphism \( C(\Omega) \rtimes_{\sigma} \mathbb{Z}^d \to \ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \) is also injective. Hence the uniform Roe \( C^* \)-algebra really contains the twisted crossed product algebra.
Now we turn to the continuum version of the above theory. Let \(\Omega\) be a compact space with a continuous action of \(\mathbb{R}^d\). This leads to crossed products \(C(\Omega) \rtimes_{\sigma} \mathbb{R}^d\) twisted, say, by Borel measurable 2-cocycles \(\sigma: \mathbb{R}^d \times \mathbb{R}^d \to C(\Omega, T)\); once again, the twist encodes a magnetic field. Restricting to the \(\mathbb{R}^d\)-orbit of some \(\omega \in \Omega\) maps \(C(\Omega) \rtimes_{\sigma} \mathbb{R}^d\) to \(C_b^0(\mathbb{R}^d) \rtimes_{e^{i\omega}} \mathbb{R}^d\) for the \(C^*\)-algebra \(C_b^0(\mathbb{R}^d)\) of bounded, uniformly continuous functions on \(\mathbb{R}^d\). We have seen in Example 2.4 that \(C_b^0(\mathbb{R}^d) \rtimes \mathbb{R}^d \subseteq L^\infty(\mathbb{R}^d)\). \(C^*(\mathbb{R}^d)\) is contained in the Roe \(C^*\)-algebra of \(\mathbb{R}^d\). This remains the case also in the twisted case because a Borel measurable 2-cocycle \(\sigma: \mathbb{R}^d \times \mathbb{R}^d \to C(\Omega, T)\) defines a Borel function \((\mathbb{R}^d)^3 \to \mathcal{T}\), which is untwisted by Proposition 2.12. So all twisted crossed products \(C(\Omega) \rtimes_{\sigma} \mathbb{R}^d\) map to the Roe \(C^*\)-algebra of \(\mathbb{R}^d\). As above, this map is an embedding if the orbit of \(\omega\) is dense in \(\Omega\).

The Roe \(C^*\)-algebras for \(\mathbb{R}^d\) and \(\mathbb{Z}^d\) are isomorphic by Theorem 2.7. So there is a unique Roe \(C^*\)-algebra in each dimension that contains all the twisted crossed product algebras that are used as models for disordered materials, both in continuum models and tight binding models. This fits interpreting the twisted crossed products \(C(\Omega) \rtimes_{\sigma} \mathbb{Z}^d\) or \(C(\Omega) \rtimes_{\sigma} \mathbb{R}^d\) as models for disorder with built-in \(a\ priori\) restrictions, whereas the Roe \(C^*\)-algebra describes general disorder.

### 2.13. Approximation by controlled operators as a continuity property

In order to belong to the Roe \(C^*\)-algebra, an operator has to be a norm limit of locally compact, controlled operators. Any such norm limit is again locally compact. The property of being a norm limit of controlled operators may be hard to check. A tool for this is Property A, an approximation property for coarse spaces that ensures that elements of the Roe \(C^*\)-algebra may be approximated in a systematic way by controlled operators, see [32]. We also mention the related Operator Norm Localization Property for subspaces of \(\mathbb{R}^d\).

We now specialise to the case where \(X\) is a closed subset of \(\mathbb{R}^d\) with the restriction of the Euclidean metric. Such spaces have Property A. We use it to define complex Roe \(C^*\)-algebras through continuity for a certain representation of \(\mathbb{R}^d\). We fix a representation \(\varrho: C_b(X) \to \mathcal{B}(\mathcal{H})\) on a complex Hilbert space \(\mathcal{H}\). Let \(\hat{\varrho}: C_b(X) \to \mathcal{B}(\mathcal{H})\) be its unique strictly continuous extension to the multiplier algebra. For \(t \in \mathbb{R}^d\), define \(e_t \in C_b(X)\) by \(e_t(x) := e^{ix \cdot t}\). The map \(t \mapsto e_t\) is continuous for the strict topology on \(C_b(X)\). Hence the representation \(\sigma\) of \(\mathbb{R}^d\) on \(\mathcal{H}\) defined by \(\sigma_t(\xi) := \varrho(e_t)(\xi)\) is continuous. This representation is generated by the position operators. If \(X \subseteq \mathbb{Z}^d\), then \(e_t = 1\) for \(t \in 2\pi \mathbb{Z}^d\), so that the representation \(\sigma\) descends to the torus \((\mathbb{R}/2\pi \mathbb{Z})^d\).

By conjugation, \(\sigma\) induces an action \(\text{Ad} \sigma\) of \(\mathbb{R}^d\) by automorphisms of \(\mathcal{B}(\mathcal{H})\). We call \(S \in \mathcal{B}(\mathcal{H})\) \(\text{continuous}\) with respect to \(\text{Ad} \sigma\) if the map \(\mathbb{R}^d \to \mathcal{B}(\mathcal{H}), t \mapsto \text{Ad} \sigma_t(S)\), is continuous in the norm topology on \(\mathcal{B}(\mathcal{H})\). The following theorem describes the Roe \(C^*\)-algebra through this continuity property:

**Theorem 2.14.** An operator \(S \in \mathcal{B}(\mathcal{H})\) is a norm limit of controlled operators if and only if it is continuous with respect to \(\text{Ad} \sigma\). And \(C^*_{\text{Roe}}(X, \varrho)\) is the \(C^*\)-subalgebra of all operators on \(\mathcal{H}\) that are locally compact and continuous with respect to \(\text{Ad} \sigma\).

**Proof.** For each \(S \in \mathcal{B}(\mathcal{H})\), the map \(\mathbb{R}^d \to \mathcal{B}(\mathcal{H})\) is continuous for the strong topology on \(\mathcal{B}(\mathcal{H})\). Therefore, the \(\mathcal{B}(\mathcal{H})\)-valued integral

\[
\int_{\mathbb{R}^d} f(t) \text{Ad} \sigma_t(S) \, dt
\]

makes sense for any \(f \in L^1(\mathbb{R}^d)\). Let \((f_n)_{n \in \mathbb{N}}\) be a bounded approximate unit in the Banach algebra \(L^1(\mathbb{R}^d)\). We claim that \(S\) is continuous if and only if \((f_n * S)_{n \in \mathbb{N}}\) converges in the norm topology to \(S\). It is well known that any continuous representation of \(\mathbb{R}^d\) becomes a nondegenerate module over \(L^1(\mathbb{R}^d)\).
Thus \((f_n \ast S)_{n \in \mathbb{N}}\) converges in norm to \(S\) if \(S\) is continuous. Conversely, operators of the form \(f \ast S\) are continuous because the action of \(\mathbb{R}^d\) on \(L^1(\mathbb{R}^d)\) is continuous. Since the set of continuous operators is closed in the norm topology, \(S\) is continuous if \((f_n \ast S)_{n \in \mathbb{N}}\) converges in norm to \(S\).

There is an approximate unit \((f_n)_{n \in \mathbb{N}}\) for \(L^1(\mathbb{R}^d)\) such that the Fourier transform of each \(f_n\) has compact support. For instance, we may use the Fejér kernel

\[
\Lambda(x_1, \ldots, x_n) := \prod_{j=1}^d \frac{\sin^2(\pi x_j)}{\pi^2 x_j^2},
\]

which has Fourier transform \(\prod_{j=1}^d (1 - |x_j|)_+\), and rescale it to produce an approximate unit for \(L^1(\mathbb{R}^d)\). We claim that \(f_n \ast S\) is controlled for each \(n \in \mathbb{N}\). More precisely, assume that \(f_n\) is supported in the ball of radius \(R\) in \(\mathbb{R}^d\). We claim that \(f_n \ast S\) is \(R\)-controlled.

This is easy to prove if \(X\) is discrete. Then we may describe operators on \(H\) using matrix coefficients \(S_{x,y} \in \mathcal{B}(H_x, H_y)\) for \(x, y \in X\). A direct computation shows that the matrix coefficients of \(f_n \ast S\) are \(f_n(x - y) \cdot S_{x,y}\). This vanishes for \(|x - y| > R\). So \(f_n \ast S\) is \(R\)-controlled as asserted. The following argument extends this result to the case where \(X\) is not discrete, such as \(X = \mathbb{R}^d\).

Let \(U, V \subseteq \mathbb{R}^d\) be two relatively compact, open subsets of distance at least \(R\) and let \(g, h \in C^\infty(\mathbb{R}^d)\) be smooth functions supported in \(U \times V\), respectively. Then

\[
\int_{\mathbb{R}^d} g(y)e^{iy \cdot t} \cdot h(x)e^{-ix \cdot t} f_n(t) \, dt = g(y)h(x) f_n(y - x) = 0
\]

for all \(x, y \in \mathbb{R}^d\). We restrict \(g, h\) to \(X\) and compute

\[
\varrho(g)(f_n \ast S) \varrho(h) := \int_{\mathbb{R}^d} \varrho(g) \bar{\varrho}(e_t) S \varrho(e_{-t}) \varrho(h) f_n(t) \, dt
= \int_{\mathbb{R}^d} \varrho(g \cdot e_t) S \varrho(h \cdot e_{-t}) f_n(t) \, dt.
\]

Let \(C^\infty_0(U \times V)\) denote the Fréchet space of smooth function on \(\mathbb{R}^{2d}\) supported in \(U \times V\). We may identify this with the complete projective tensor product of \(C^\infty_0(U)\) and \(C^\infty_0(V)\). Hence there is a continuous linear map

\[
C^\infty_0(U \times V) \to \mathcal{B}(H), \quad g \otimes h \mapsto \varrho(g)S \varrho(h).
\]

The integral in \((4)\) converges to 0 in the Fréchet topology of \(C^\infty_0(U \times V)\). Hence \((4)\) implies \(\varrho(g)(f_n \ast S) \varrho(h) = 0\).

The continuous map \((5)\) still exists if \(U\) and \(V\) are not of distance \(R\). If \(S\) is locally compact, then \(\varrho(g)S \varrho(h)\) is a compact operator on \(H\) for all \(g \in C^\infty_0(U)\), \(h \in C^\infty_0(V)\). This remains so for all operators in the image of \((5)\) by continuity. Therefore, \(\varrho(g)(f_n \ast S) \varrho(h)\) is compact for all \(g, h\) as above. Choosing \(U\) large enough, we may take \(g\) to be constant equal to 1 on the \(R\)-neighbourhood of \(V\). Then \(\varrho(g)(f_n \ast S) \varrho(h) = (f_n \ast S) \varrho(h)\) because \(f_n \ast S\) is \(R\)-controlled. So operators of the form \((f_n \ast S) \varrho(h)\) with smooth, compactly supported \(h\) are compact. Since any continuous, compactly supported function is dominated by a smooth, compactly supported function, we get the same for all \(h \in C_c(X)\). A similar argument shows that \(\varrho(g)(f_n \ast S)\) is compact for all \(g \in C_c(X)\). Hence the operators \(f_n \ast S\) are locally compact, controlled operators if \(S\) is locally compact.

Property \(A\) is equivalent to the “Operator Norm Localization Property” for metric spaces with bounded geometry, see [33]. Roughly speaking, this property says that the operator norm of a controlled operator may be computed using vectors in the Hilbert space with bounded support. The support of a vector \(\xi \in H\) is the set of
all \( x \in X \) such that \( f \cdot \xi \neq 0 \) for all \( f \in C_0(X) \) with \( f(x) \neq 0 \). We formulate this property for subspaces of \( \mathbb{R}^d \):

**Theorem 2.15.** Let \( X \subseteq \mathbb{R}^d \) and let \( g: C_0(X) \to \mathcal{B}(\mathcal{H}) \) be a representation. Pick scalars \( R > 0 \) and \( c \in (0,1) \). Then there is a scalar \( S > 0 \) such that for any \( R \)-controlled operator \( T \in \mathcal{B}(\mathcal{H}) \), there is \( \xi \in \mathcal{H} \) with \( \|\xi\| = 1 \) such that the support of \( \xi \) has diameter at most \( S \) and \( \|T(\xi)\| \geq S \|T\| \geq c \cdot \|T(\xi)\| \).

**Proof.** The statement of the theorem is that the space \( X \) has the “Operator Norm Localisation Property” defined in [9]. This property is invariant under coarse equivalence and passes to subspaces by [9] Propositions 2.5 and 2.6. [9] Theorem 3.11 and Proposition 4.1 show that solvable Lie groups such as \( \mathbb{R}^d \) have this property, and hence also all subspaces of \( \mathbb{R}^d \).

\[ \blacksquare \]

2.16. **Dense subalgebras with isomorphic K-theory.** Let \( A \) be a \( C^* \)-algebra with a continuous \( \mathbb{R}^d \)-action \( \alpha: \mathbb{R}^d \to \text{Aut}(A) \). The action defines several canonical \( * \)-subalgebras of \( A \) with the same K-theory. The \( * \)-subalgebra of smooth elements is

\[ A^\infty := \{ a \in A : t \mapsto \alpha_t(a) \text{ is a smooth function } \mathbb{R}^d \to A \}. \]

This Fréchet \( * \)-subalgebra is closed under holomorphic functional calculus and also under smooth functional calculus for normal elements, see [6].

Let \( F \subseteq \mathbb{R}^d \) be a compact convex subset with non-empty interior and containing 0. Let \( \mathcal{O}(A, \alpha, F) \subseteq A \) be the set of all \( a \in A \) for which the function \( \mathbb{R}^d \ni t \mapsto \alpha_t(a) \) extends to a continuous function on \( \mathbb{R}^d + iF \) that is holomorphic on the interior of \( \mathbb{R}^d + iF \). This is a dense Banach subalgebra in \( A \), and the inclusion \( \mathcal{O}(A, \alpha, F) \hookrightarrow A \) induces an isomorphism on topological K-theory by [7] Théorème 2.2.1. Let \( \mathcal{O}^\infty(A, \alpha, F) \subseteq A \) be the set of those \( a \in A \) for which the function \( \mathbb{R}^d \ni t \mapsto \alpha_t(a) \) extends to a smooth function on \( \mathbb{R}^d + iF \) that is holomorphic on the interior of \( \mathbb{R}^d + iF \). The inclusion \( \mathcal{O}^\infty(A, \alpha, F) \hookrightarrow A \) induces an isomorphism on topological K-theory as well. If \( F_1 \subseteq F_2 \), then \( \mathcal{O}(A, \alpha, F_1) \hookrightarrow \mathcal{O}(A, \alpha, F_2) \). There are two important limiting cases of the subalgebras \( \mathcal{O}(A, \alpha, F) \).

First, let \( F \) run through a neighbourhood basis of 0 in \( \mathbb{R}^d \). Then the dense Banach subalgebras \( \mathcal{O}(A, \alpha, F) \) form an inductive system, whose colimit is the dense \( * \)-subalgebra \( A^* \subseteq A \) of all real-analytic elements of \( A \), that is, those \( a \in A \) with the property that each \( t \in \mathbb{R}^d \) has a neighbourhood on which \( s \mapsto \alpha_s(a) \) is given by a convergent power series with coefficients in \( A \). The subalgebra \( A^\omega \) is still closed under holomorphic functional calculus by [23] Proposition 3.46. This gives an easier explanation than Bost’s Oka principle why \( A^\omega \) has the same topological K-theory as \( A \).

Secondly, let \( F \) run through an increasing sequence whose union is \( \mathbb{R}^d \). Then the dense Banach subalgebras \( \mathcal{O}(A, \alpha, F) \) form a projective system, whose limit is the dense \( * \)-subalgebra \( \mathcal{O}(A, \alpha) \) of all holomorphic elements of \( A \), that is, those elements for which the map \( \mathbb{R}^d \ni t \mapsto \alpha_t(a) \) extends to a holomorphic function on \( \mathbb{C}^d \). This is a locally multiplicatively convex Fréchet algebra. Phillips [26] has extended topological K-theory to such algebras. The Milnor \( \lim^1 \)-sequence in [26] Theorem 6.5] shows that the inclusion \( \mathcal{O}(A, \alpha) \hookrightarrow A \) induces an isomorphism in topological K-theory.

We apply all this to the Roe \( C^* \)-algebra of \( \mathbb{Z}^d \) and the continuous \( \mathbb{R}^d \)-action \( \sigma \) defined in Section 2.13. Here this action descends to the torus \( \mathbb{T}^d \), which simplifies the study of the dense subalgebras above. We describe the dense subalgebras of smooth, real-analytic and holomorphic elements in \( C^*_\text{Roe}(\mathbb{Z}^d) \). All these have the same topological K-theory. Let \( g: C_0(\mathbb{Z}^d) \to \mathcal{B}(\mathcal{H}) \) be a representation on a separable Hilbert space, not necessarily amply. Let \( \mathcal{H}_x \) for \( x \in X \) be the fibres of \( \mathcal{H} \) with respect to \( g \). Describe operators on \( \mathcal{H} \) by block matrices \( (T_{x,y})_{x,y\in \mathbb{Z}^d} \) with
Proposition 2.17. A block matrix \((T_{x,y})_{x,y \in \mathbb{Z}^d}\) as above gives a smooth element for the \(\mathbb{R}^d\)-action \(\sigma\) on \(C^*_\text{Roe}(\mathbb{Z}^d)\) if and only if the function
\[
\mathbb{Z}^d \ni k \mapsto \sup_{n \in \mathbb{Z}^d} ||T_{n,n+k}||
\]
has rapid decay, that is, for each \(a > 0\) there is a constant \(C_a > 0\) such that
\[
||T_{n,n+k}|| \leq C_a (1 + ||k||)^{-a}
\]
for all \(n, k \in \mathbb{Z}^d\). It gives a real-analytic element for \(\sigma\) if and only if there are \(a > 0\) and \(C_a > 0\) such that
\[
||T_{n,n+k}|| \leq C_a \cdot \exp(-a||k||)
\]
for all \(n, k \in \mathbb{Z}^d\). It gives a holomorphic element for \(\sigma\) if and only if for each \(a > 0\) there is \(C_a > 0\) such that
\[
||T_{n,n+k}|| \leq C_a \cdot \exp(-a||k||).
\]

Proof. The \(j\)th generator of the \(\mathbb{R}^d\)-action \(\sigma\) maps a block matrix \((T_{x,y})_{x,y \in \mathbb{Z}^d}\) to
\[
\lim_{t \to 0} \frac{1}{t} (\sigma_{te_j}(T_{x,y}) - (T_{x,y}))_{x,y \in \mathbb{Z}^d} = ((x_j - y_j)T_{x,y})_{x,y \in \mathbb{Z}^d}.
\]
Hence polynomials in these generators multiply the entries \(T_{x,y}\) with polynomials in \(x - y \in \mathbb{Z}^d\). So \((T_{x,y})_{x,y \in \mathbb{Z}^d}\) belongs to a smooth element of \(C^*_\text{Roe}(\mathbb{Z}^d)\) if and only if \((p(x - y) \cdot T_{x,y})_{x,y \in \mathbb{Z}^d}\) belongs to a bounded operator for each polynomial \(p\) in \(d\) variables. It suffices to consider the polynomials \(1 + ||x - y||_2^2\) for \(b \in \mathbb{N}\). Since the operator norm for diagonal block matrices is the supremum of the operator norms of the entries, we see that the boundedness of \((1 + ||x - y||_2^2) \cdot T_{x,y})_{x,y \in \mathbb{Z}^d}\) for all \(b \in \mathbb{N}\) is equivalent to the boundedness of \(\sup_{k, n \in \mathbb{Z}^d} ||T_{n,n+k}|| (1 + ||k||_2^2)\) for all \(b \in \mathbb{N}\). This proves the claim about the smooth elements. The analytic extension of \(\sigma\) to \(i z \in \mathbb{C}^d\) must map \((T_{x,y})_{x,y \in \mathbb{Z}^d}\) to \(((\exp(z \cdot (x - y))T_{x,y})_{x,y \in \mathbb{Z}^d}\). Thus \((T_{x,y})_{x,y \in \mathbb{Z}^d}\) describes an element of \(\mathcal{O}^\infty(C^*_\text{Roe}(\mathbb{Z}^d), \sigma, F)\) if and only if
\[
\sup_{k, n \in \mathbb{Z}^d} ||T_{n,n+k}|| (1 + ||k||_2^2) \exp(z \cdot k) < \infty
\]
for all \(z \in F, b \in \mathbb{N}\). When we let \(F \cap \{0\} \subset F \supset \mathbb{C}^d\), we may leave out the polynomial factors because they are dominated by \(\exp(z \cdot k)\). This proves the claims about the real-analytic elements and holomorphic elements. \(\square\)

Estimates of the form \(||T_{x,y}|| \leq C_a \cdot \exp(-a \cdot ||x - y||)\) for some \(a > 0\), \(C_a > 0\) play an important role in the study of Anderson localisation; see, for instance, [1, Equation (2.3)].

2.18. Approximate unit of projections. Unlike the uniform Roe \(C^*\)-algebra, the Roe \(C^*\)-algebra of a proper metric space is never unital. Instead, it has an approximate unit of projections:

Proposition 2.19. Let \(X\) be a proper metric space and let \(\varrho : C_0(X) \to \mathbb{B}(H)\) be a representation. The Roe \(C^*\)-algebra \(C^*_\text{Roe}(X, \varrho)\) has an approximate unit of projections.

Proof. Any proper metric space contains a coarsely dense, discrete subspace. By Theorem 2.7 we may assume that \(X\) itself is discrete. By Theorem 2.11 we may further assume that the Roe \(C^*\)-algebra is built using the obvious representation of \(C_0(X)\) on \(l^2(X, C_0(\mathbb{N}))\). Then the Roe \(C^*\)-algebra contains \(l^\infty(X, C_0(\mathbb{N}))\) as multiplication operators. Any function \(h : X \to \mathbb{N}\) defines a projection in \(l^\infty(X, C_0(\mathbb{N}))\), namely, the characteristic function of \(\{(x, n) \in X \times \mathbb{N} : n < h(x)\}\). These projections form an approximate unit in the Roe \(C^*\)-algebra. \(\square\)
Let \((p_\alpha)_{\alpha \in S}\) be an approximate unit of projections in \(C^*_Roe(X, \varrho)\). Then \(C^*_Roe(X, \varrho)\) is isomorphic to the inductive limit

\[
C^*_Roe(X, \varrho) = \lim_{\to} p_\alpha C^*_Roe(X, \varrho) p_\alpha.
\]

Since K-theory commutes with inductive limits, we get

\[
(7) \quad K_*\left(C^*_Roe(X, \varrho)\right) \cong \lim_{\to} K_*\left(p_\alpha C^*_Roe(X, \varrho) p_\alpha\right).
\]

Each of the corners \(p_\alpha C^*_Roe(X, \varrho) p_\alpha\) is unital. This simplifies the definition of the groups \(K_*\left(p_\alpha C^*_Roe(X, \varrho) p_\alpha\right)\) for fixed \(\alpha\).

**Corollary 2.20.** Any class in \(K_0(C^*_Roe(X))\) is represented by a formal differences of projections in \(C^*_Roe(X)\). Two such differences \([p_+] - [p_-] + [q_+] - [q_-]\) represent the same class in \(K_0(C^*_Roe(X))\) if and only if there is a projection \(r\) in \(C^*_Roe(X)\) such that the projections \(p_+ \oplus q_- \oplus r\) and \(p_- \oplus q_+ \oplus r\) in \(M_k(C^*_Roe(X))\) are Murray–von Neumann equivalent.

**Proof.** Any class in \(K_0(C^*_Roe(X))\) is the image of a class in \(K_0\left(p_\alpha C^*_Roe(X) p_\alpha\right)\) for some \(\alpha\) by \([7]\). Since \(p_\alpha C^*_Roe(X) p_\alpha\) is unital, it is represented by a formal difference of two projections in \(M_k\left(p_\alpha C^*_Roe(X) p_\alpha\right)\) for some \(k \in \mathbb{N}_{\geq 1}\). These projections also belong to \(M_k(C^*_Roe(X))\), which is isomorphic to \(C^*_Roe(X)\) by Corollary 2.2. Even more, the isomorphism \(M_k(C^*_Roe(X)) \cong C^*_Roe(X)\) is by conjugation with a unitary \(k \times 1\)-matrix over the multiplier algebra of \(C^*_Roe(X)\). Hence any projection in \(M_k(C^*_Roe(X))\) is Murray–von Neumann equivalent to one in \(C^*_Roe(X)\). Thus any class in \(K_0(C^*_Roe(X))\) is represented by a formal difference of projections in \(C^*_Roe(X)\). Using \((7)\) and the definition of \(K_0\) for unital algebras once again, we see that \([p_+] - [p_-] = [q_+] - [q_-]\) holds if and only if there is a projection \(r\) in \(M_k\left(p_\alpha C^*_Roe(X) p_\alpha\right)\) for some \(k, \alpha\) such that \(p_+ \oplus q_- \oplus r\) and \(p_- \oplus q_+ \oplus r\) are Murray–von Neumann equivalent in \(M_{k+2}\left(p_\alpha C^*_Roe(X) p_\alpha\right)\). Here we may replace \(r\) by an equivalent projection and so reduce to \(k = 1\). And since any projection in \(C^*_Roe(X)\) is equivalent to one in \(p_\alpha C^*_Roe(X) p_\alpha\) for some \(\alpha\), we may allow \(r\) to be any projection in \(C^*_Roe(X)\).

\(\square\)

3. **Coarse Mayer–Vietoris sequence**

We now recall how to compute the K-theory of the Roe C*-algebra of \(\mathbb{R}^d\) or \(\mathbb{Z}^d\) using the coarse Mayer–Vietoris sequence. This method is due to Higson–Roe–Yu \([13]\) and was also explained by Kubota in \([21\hbox{ Section 2.3.3}]\). The Roe C*-algebra is defined also for the half-space \(\mathbb{Z}^{d-1} \times \mathbb{N} \subseteq \mathbb{Z}^d\), and its K-theory vanishes. We identify the Roe C*-algebra of a subspace with a corner in the larger Roe C*-algebra and describe the ideal generated by this corner as a relative Roe C*-algebra. Then we recall the Mayer–Vietoris sequence for two ideals in a C*-algebra. We give an elegant proof due to Wodzicki, which reduces its exactness to ordinary long exact sequences for C*-algebra extensions. This also describes the boundary map in the Mayer–Vietoris sequence through the boundary map for a C*-algebra extension. Hence the many tools and formulas for the boundary maps for extensions of C*-algebras also apply to the boundary map in the Mayer–Vietoris sequence.

3.1. **Subspaces and corners.** Theorem 2.7 shows that \(C^*_Roe(Y)\) and \(C^*_Roe(X)\) are isomorphic if \(Y \subseteq X\) is coarsely dense. Now let \(Y \subseteq X\) be an arbitrary closed subset, still with the restriction of the metric from \(X\). We are going to relate the Roe C*-algebras of \(Y\) and \(X\). First, we are going to show that \(C^*_Roe(Y)\) is isomorphic to a corner in \(C^*_Roe(X)\); this is an easy consequence of Theorem 2.7. For the coarse Mayer–Vietoris sequence, we also need to know that the ideal generated by this corner is the relative Roe C*-algebra, defined as follows:
**Definition 3.2.** Let $Y \subseteq X$ and let $\varrho : C_0(X) \to \mathcal{B}(\mathcal{H})$ be a representation. An operator $T \in \mathcal{B}(\mathcal{H})$ is supported near $Y$ if there is $R > 0$ with

$$\text{supp}(T) \subseteq \{(x,y) \in X \times X : d(x,Y) < R \text{ and } d(y,Y) < R\}.$$ 

The relative Roe algebra $C^*_\text{Roe}(Y \subseteq X, \varrho)$ is the C*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the controlled locally compact operators supported near $Y$.

If $T$ is supported near $Y$ and $S$ is controlled, then $ST$ and $TS$ are again supported near $Y$, and $T^*$ is also supported near $Y$. Thus the controlled operators supported near $Y$ form a (two-sided) *-ideal in the *-algebra of controlled operators. Hence $C^*_\text{Roe}(Y \subseteq X, \varrho)$ is the closure of this *-algebra in $C^*_\text{Roe}(X, \varrho)$, and it is a closed two-sided *-ideal in $C^*_\text{Roe}(X, \varrho)$.

A corner in a C*-algebra $A$ is a C*-subalgebra of the form $PAP$ for a projection $P$ in the multiplier algebra of $A$. Any corner is a hereditary subalgebra. It is canonically Morita equivalent to the ideal generated by $P$, which we denote by $APA$ because it is the closed linear span of $a_1Pa_2$ for $a_1, a_2 \in A$. The imprimitivity bimodule is $AP$ with the obvious full Hilbert $APA, PAP$-bimodule structure, obtained by restricting the usual Hilbert $A, A$-bimodule structure on $A$.

**Theorem 3.3.** Let $Y \subseteq X$ be a closed subspace with the subspace metric. Then $C^*_\text{Roe}(X)$ is isomorphic to a corner in $C^*_\text{Roe}(Y)$. The ideal generated by it is the relative Roe C*-algebra $C^*_\text{Roe}(Y \subseteq X)$. This holds both in the complex and real case.

**Proof.** We prove the real case. The complex case is analogous. We choose ample representations $\varrho_X$ and $\varrho_Y$ of $C_0(X)$ and $C_0(Y)$ on separable real Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$, respectively. Using the restriction map $p : C_0(X) \to C_0(Y)$, we build another ample representation $\varrho' := \varrho_Y \circ p \circ \varrho_X$ of $C_0(X)$ on the separable real Hilbert space $\mathcal{H}_Y \oplus \mathcal{H}_X$. We use $\varrho'$ to build $C^*_\text{Roe}(X)$, which is allowed by Theorem 2.1. The projection $P$ onto the summand $\mathcal{H}_Y$ is a multiplier of $C^*_\text{Roe}(X)$ because it is 0-controlled. (It is only a multiplier because it is not locally compact.) Since the map $x \mapsto PxP$ on $\mathcal{B}(\mathcal{H}_Y \oplus \mathcal{H}_X)$ is bounded, the corner $PC^*_\text{Roe}(X)P$ is the closure of the space of all controlled, locally compact operators on $\mathcal{H}_Y$. Here we should use the representation $\varrho_Y \circ p$ of $C_0(X)$ to define controlled operators and local compactness. But since $Y$ carries the subspace metric from $X$ and $p$ is surjective, the representations $\varrho_Y \circ p$ of $C_0(X)$ and $\varrho_Y$ of $C_0(Y)$ define the same controlled or locally compact operators. Hence $PC^*_\text{Roe}(X, \varrho')P = C^*_\text{Roe}(Y, \varrho_Y)$. So $C^*_\text{Roe}(Y)$ is isomorphic to a corner in $C^*_\text{Roe}(X)$.

The corner $PC^*_\text{Roe}(X)P$ is contained in $C^*_\text{Roe}(Y \subseteq X)$. Since the latter is an ideal, the ideal $PC^*_\text{Roe}(X)PC^*_\text{Roe}(X)$ is also contained in $C^*_\text{Roe}(Y \subseteq X)$. For the converse inclusion, we must show that any controlled, locally compact operator $T$ that is supported near $Y$ belongs to $C^*_\text{Roe}(X)PC^*_\text{Roe}(X)$. Let $R > 0$ and let $T$ be supported in the $R$-neighbourhood of $Y$. This $R$-neighbourhood is a closed subspace $Y_R$ of $X$, and $Y \subseteq Y_R$ is coarsely dense by construction. The restriction of $\varrho'$ to the Hilbert subspace $\mathcal{H}_Y.R := \varrho'(1_{Y_R})(\mathcal{H}_Y \oplus \mathcal{H}_X)$ is an ample representation of $C_0(Y_R)$. Hence it defines $C^*_\text{Roe}(Y_R)$ by Theorem 2.1. This C*-algebra is simply the corner in $C^*_\text{Roe}(X)$ generated by the projection onto $\mathcal{H}_Y.R$. Theorem 2.7 gives a unitary $U : \mathcal{H}_Y \sim \mathcal{H}_Y.R$ such that $UC^*_\text{Roe}(Y)U^* = C^*_\text{Roe}(Y_R)$. The unitary $U$ is built in the proof of Theorem 2.7, and the construction there shows that it is controlled as an operator on $\mathcal{H}_Y \oplus \mathcal{H}_X$. So it is a multiplier of $C^*_\text{Roe}(X)$, where it is no longer unitary but a partial isometry. The operator $U^*TU$ belongs to $C^*_\text{Roe}(Y)$ because $T \in C^*_\text{Roe}(Y_R)$. Since multipliers of $C^*_\text{Roe}(X)$ are also multipliers of any ideal in $C^*_\text{Roe}(X)$, the operator $T = U(U^*TU)U^*$ belongs to the ideal in $C^*_\text{Roe}(X)$ generated by $C^*_\text{Roe}(Y) = PC^*_\text{Roe}(X)P$. Thus $C^*_\text{Roe}(Y_R) \subseteq C^*_\text{Roe}(X)PC^*_\text{Roe}(X)$. \qed
3.4. The coarse Mayer–Vietoris sequence.

Proposition 3.5 \([13]\). Let \(X\) be a proper metric space and let \(Y_1, Y_2 \subseteq X\) be closed subspaces with \(Y_1 \cup Y_2 = X\). Then

\[
C^*_\text{Roe}(Y_1 \subseteq X) + \text{C}^*_\text{Roe}(Y_2 \subseteq X) = \text{C}^*_\text{Roe}(X).
\]

We have \(\text{C}^*_\text{Roe}(Y_1 \subseteq X) \cap \text{C}^*_\text{Roe}(Y_2 \subseteq X) = \text{C}^*_\text{Roe}(Z \subseteq X)\) if and only if the following coarse transversality condition holds: for any \(R > 0\) there is \(S(R) > 0\) such that if \(x \in X\) satisfies \(d(x, Y_1) < R\) and \(d(x, Y_2) < R\), then \(d(x, Z) < S(R)\). The statements above hold both for real and complex Roe \(C^*\)-algebras.

Proof. The proof of Theorem 3.3 identifies \(\text{C}^*_\text{Roe}(Y_j \subseteq X)\) for \(j = 1, 2\) with corners in \(\text{C}^*_\text{Roe}(X)\). The ideal generated by these corners is all of \(\text{C}^*_\text{Roe}(X)\) because \(Y_1 \cup Y_2 = X\). That is, \(\text{C}^*_\text{Roe}(Y_1 \subseteq X) + \text{C}^*_\text{Roe}(Y_2 \subseteq X) = \text{C}^*_\text{Roe}(X)\).

An operator that is supported near \(Z\) is also supported near \(Y_1\) and near \(Y_2\). So

\[
\text{C}^*_\text{Roe}(Z \subseteq X) \subseteq \text{C}^*_\text{Roe}(Y_1 \subseteq X) \cap \text{C}^*_\text{Roe}(Y_2 \subseteq X),
\]

as these are closed ideals. The other inclusion uses the coarse transversality assumption above, which is called “\(\omega\)-excisiveness” in \([13]\).

Let \(T\) and \(U\) be locally compact operators that are \(R_T\) and \(R_U\)-controlled, respectively, and such that \(T\) is supported within distance \(P_T > 0\) of \(Y_1\) and \(U\) within distance \(P_U > 0\) of \(Y_2\). Let \(R := R_T + R_U + P_T + P_U\). If \((x, y) \in \text{supp}(TU)\), then there is \(z \in X\) with \((x, z) \in \text{supp}(T)\) and \((z, y) \in \text{supp}(U)\). Then

\[
d(x, Y_1 \cap Y_2) < S(R).\]

Hence \(d(x, Y_1 \cap Y_2) < S(R)\). A similar argument shows that \(d(y, Y_1 \cap Y_2) < S(R)\). So \(TU\) is supported near \(Y_1 \cap Y_2\). Thus \(\text{C}^*_\text{Roe}(Y_1 \subseteq X) \cdot \text{C}^*_\text{Roe}(Y_2 \subseteq X) \subseteq \text{C}^*_\text{Roe}(Z \subseteq X)\). This implies the claim because \(I \cap J = I \cdot J\) if \(I, J\) are closed ideals in a (real) \(C^*\)-algebra; the latter follows from the existence of approximate units. \(\square\)

Proposition 3.6. Let \(A\) be a real or complex \(C^*\)-algebra and let \(I, J \subseteq A\) be closed ideals with \(I + J = A\). Let \(\alpha_I : I \cap J \hookrightarrow I\), \(\alpha_J : I \cap J \hookrightarrow J\), \(\beta_I : I \hookrightarrow A\), and \(\beta_J : J \hookrightarrow A\) denote the inclusion maps and also the maps that they induce on \(K\)-theory. Then there is a long exact sequence in (real or complex) \(K\)-theory

\[
\cdots \to K_j(I \cap J) \xrightarrow{(-\alpha_I)} K_j(I) \oplus K_j(J) \xrightarrow{(\beta_I, \beta_J)} K_j(A) \xrightarrow{\partial_{\text{MV}}} K_{j-1}(I \cap J) \to \cdots.
\]

The boundary map \(\partial_{\text{MV}}\) is computed in \([5]\) below.

Proof. There is a commuting diagram

\[
\begin{array}{ccc}
I \cap J & \xrightarrow{\alpha_I} & I \\
\downarrow{\alpha_J} & & \downarrow{\beta_I} \\
J & \xrightarrow{\beta_J} & A \\
\end{array}
\]

whose rows are \(C^*\)-algebra extensions. The map between the quotients induced by \(\beta_J\) is an isomorphism because \(I/(I \cap J) \cong (I + J)/J = A/J\). The rows in the above diagram generate \(K\)-theory long exact sequences, which we view as exact chain complexes. The vertical maps generate a chain map between them. Its mapping cone is again exact. So we get an exact sequence

\[
\cdots \to K_j(I \cap J) \oplus K_{j+1}(A/J) \xrightarrow{(-\alpha_I, 0)} K_j(I) \oplus K_j(J) \xrightarrow{(\beta_I, \beta_J)} K_j(A/J) \xrightarrow{\partial_{\text{MV}}} K_{j-1}(I \cap J) \oplus K_j(A/J) \to \cdots.
\]
Since $\beta_{1*}$ is invertible, the boundary map restricts to an injective map on the summands $K_*(I/(I\cap J))$. So these summands and their images $B$ under the boundary map form an exact subcomplex. Dividing it out gives another exact chain complex. The direct summand $K_{j-1}(I\cap J)$ in $K_j(I\cap J)\oplus K_j(A/J)$ is complementary to $B$, and the projection to $K_{j-1}(I\cap J)$ that kills $B$ maps $x \in K_j(A/J)$ to $\delta(\beta_{1*}^{-1}(x))$ because the boundary map sends $\beta_{1*}^{-1}(x) \in K_j(I/(I\cap J))$ to $(-\delta(\beta_{1*}^{-1}(x)), x)$. Hence the quotient of the above complex by the exact subcomplex $K_*(I/(I\cap J)) \oplus B$ becomes the exact chain complex

$$\ldots \to K_j(I\cap J) \xrightarrow{(-\alpha_j, \alpha_j)} K_j(I) \oplus K_j(J) \xrightarrow{(\beta_1, \beta_2)} K_j(A) \xrightarrow{\delta(\beta_{1*})^{-1}\pi} K_{j-1}(I\cap J) \to \ldots.$$  

This is the desired long exact sequence. We have also computed the boundary map:

$$\partial_{MV} = \delta \circ (\beta_{1*})^{-1} \circ \pi : K_j(A) \xrightarrow{\pi} K_j(A/J) \xrightarrow{\beta_{1*}^{-1}} K_j(I/(I\cap J)) \xrightarrow{\delta} K_{j-1}(I\cap J),$$

where $\delta$ is the boundary map for the $\mathbb{C}^*$-extension $I \cap J \to I \to I/(I\cap J)$. \hfill $\square$

**Corollary 3.7.** Let $X$ be a proper metric space. Let $X = Y_1 \cup Y_2$ be a coarsely transverse decomposition as in Proposition 3.5 and let $Z := Y_1 \cap Y_2$. Then there is a long exact sequence

$$\ldots \to K_j(C^*_\text{Roe}(Z)) \xrightarrow{(-\alpha_j, \alpha_j)} K_j(C^*_\text{Roe}(Y_1)) \oplus K_j(C^*_\text{Roe}(Y_2)) \xrightarrow{(\beta_1, \beta_2)} K_j(C^*_\text{Roe}(X)) \xrightarrow{\partial_{MV}} K_{j-1}(C^*_\text{Roe}(Z)) \to \ldots.$$  

Here $\alpha_1, \alpha_2, \beta_1, \beta_2$ are the maps on $K$-theory induced by the $\*$-homomorphisms on Roe $\mathbb{C}^*$-algebras induced by the inclusion maps $Z \to Y_1$, $Z \to Y_2$, $Y_1 \to X$ and $Y_2 \to X$, respectively. The above holds both for real and complex Roe $\mathbb{C}^*$-algebras.

**Proof.** Let $A := C^*_\text{Roe}(X)$, $I := C^*_\text{Roe}(Y_1 \subseteq X)$ and $J := C^*_\text{Roe}(Y_2 \subseteq X)$. Then $I + J = A$ and $I \cap J = C^*_\text{Roe}(Z \subseteq X)$ by Proposition 3.5. And the relative Roe $\mathbb{C}^*$-algebras above are Morita equivalent to the absolute ones by Theorem 3.3. Thus $K_*(C^*_\text{Roe}(Y_j \subseteq X)) \cong K_*(C^*_\text{Roe}(Z \subseteq X))$ for $j = 1, 2$ and $K_*(C^*_\text{Roe}(Z \subseteq X)) \cong K_*(C^*_\text{Roe}(Z))$. Plugging this into the Mayer–Vietoris sequence in Proposition 3.6 gives the assertion. \hfill $\square$

3.8. **Application to $\mathbb{Z}^d$.** We apply the coarse Mayer–Vietoris sequence to the decomposition

$$\mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{N} \cup \mathbb{Z}^{d-1} \times \{-\mathbb{N}\}$$

into two half-spaces, which intersect in $\mathbb{Z}^{d-1} \times \{0\}$. It is clearly coarsely transverse, so that Corollary 3.7 applies to it.

**Proposition 3.9 ([13 Proposition 1]).** The $K$-theory of the real and complex Roe $\mathbb{C}^*$-algebras of $X \times \mathbb{N}$ vanishes for any proper metric space $X$.

**Proof.** We sketch the proof in [13] for $K_0$ and then explain briefly why this argument also works for all other $K$-groups. We realise the Roe $\mathbb{C}^*$-algebra on $\ell^2(X \times \mathbb{N}, \mathcal{H})$ for a separable Hilbert space $\mathcal{H}$, which may be real or complex. The unilateral shift on $\ell^2(\mathbb{N})$ is a 1-controlled isometry. It also defines a controlled isometry $S$ on $\ell^2(X \times \mathbb{N}, \mathcal{H})$. If $n \in \mathbb{N}$, then the map $T \mapsto S^n T(S^n)^*$ on $C^*_\text{Roe}(X \times \mathbb{N})$ is a
\[ \varphi : \mathbb{B}(\ell^2(X \times \mathbb{N}, \mathcal{H})) \to \mathbb{B}(\ell^2(X \times \mathbb{N}, \mathcal{H}^\infty)), \quad T \mapsto \bigoplus_{n=0}^{\infty} S^n T(S^*)^n, \]

maps $R$-controlled operators on $\mathcal{H}$ to $R$-controlled operators on $\mathcal{H}^\infty := \mathcal{H} \otimes \ell^2(\mathbb{N})$. The matrix coefficients $(S^n T(S^*)^n)_{x,y}$ vanish for $0 \leq x, y < n$. Therefore, $\varphi(T)$ is locally compact if $T$ is locally compact. So $\varphi$ restricts to a $^*$-homomorphism from the Roe $C^*$-algebra of $X \times \mathbb{N}$ realised on $\ell^2(X \times \mathbb{N}, \mathcal{H})$ to the isomorphic Roe $C^*$-algebra of $X \times \mathbb{N}$ realised on $\ell^2(X \times \mathbb{N}, \mathcal{H}^\infty)$. We may identify these using a unitary operator $\mathcal{H} \cong \mathcal{H}^\infty$. We have $S_\varphi(T)S^* = \bigoplus_{n=0}^{\infty} S^n T(S^*)^n$. So $\varphi$ is equal to the direct sum of the canonical inclusion $\iota$ induced by the embedding $\mathcal{H} \to \mathcal{H}^\infty$, $\xi \mapsto \xi \otimes \delta_0$, and the $^*$-homomorphism $T \mapsto S_\varphi(T)S^*$.

In particular, if $P \in C^*_\text{Roe}(X \times \mathbb{N})$ is a projection, then $\varphi(P)$ is another projection in $C^*_\text{Roe}(X \times \mathbb{N})$. And $\varphi(P)$ is Murray–von Neumann equivalent to $(P) \oplus \varphi(P)$. Thus $\iota(P)$ is stably equivalent to $0$. That is, the inclusion $\iota$ induces the zero map on $K_0$. We may also identify $\mathcal{H}^\infty \cong \mathcal{H}^2$ so that the inclusion $\iota$ becomes the corner embedding

\[ C^*_\text{Roe}(X \times \mathbb{N}) \to \mathbb{M}_2(C^*_\text{Roe}(X \times \mathbb{N})), \quad T \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}. \]

This induces an isomorphism on $K_0$. So the zero map is an isomorphism on $K_0(C^*_\text{Roe}(X \times \mathbb{N}))$, which forces this group to vanish. The argument above works for any functor on the category of $C^*$-algebras and $^*$-homomorphisms that is matrix-stable because inner endomorphisms induced by isometries act by the identity on all such functors (see [10, Proposition 3.16]). In particular, the proof above works for all real and complex $K$-groups.

The proof above breaks down for the uniform Roe $C^*$-algebra, and indeed the result is wrong in that case.

**Corollary 3.10.** Let $d \in \mathbb{N}$. Then the boundary map in the coarse Mayer–Vietoris sequence for the decomposition $\Box$ is an isomorphism. So $K_{i+d}(C^*_\text{Roe}(\mathbb{Z}^d)_\mathbb{R}) \cong K_i(\mathbb{R})$ for $\mathbb{R} \in \{\mathbb{R}, \mathbb{C}\}$.

**Proof.** Apply the Mayer–Vietoris long exact sequence of Corollary 3.7 to the coarsely transverse decomposition $\Box$. Plug in that the $K$-theory vanishes for the two half-spaces (Proposition 3.9). Hence the boundary map is an isomorphism. Now an induction argument identifies $K_{i+d}(C^*_\text{Roe}(\mathbb{Z}^d)_\mathbb{R})$ with $K_i(C^*_\text{Roe}(\{0\})_\mathbb{R})$. Finally, $C^*_\text{Roe}(\{0\})_\mathbb{R}$ is isomorphic to the $C^*$-algebra of compact operators on a real or complex Hilbert space. This gives the statement because $K$-theory is $C^*$-stable. □

The well known $K$-theory computations for $\mathbb{R}$ and $\mathbb{C}$ give

\[ K_i(C^*_\text{Roe}(\mathbb{Z}^d)_\mathbb{C}) \cong \begin{cases} \mathbb{Z} & \text{if } i-d \equiv 0 \mod 2, \\ 0 & \text{if } i-d \equiv 1 \mod 2. \end{cases} \]

\[ K_i(C^*_\text{Roe}(\mathbb{Z}^d)_\mathbb{R}) \cong \begin{cases} \mathbb{Z} & \text{if } i-d \equiv 0 \text{ or } 4 \mod 8, \\ \mathbb{Z}/2 & \text{if } i-d \equiv 1 \text{ or } 2 \mod 8, \\ 0 & \text{if } i-d \equiv 3, 5, 6 \text{ or } 7 \mod 8. \end{cases} \]

In contrast, the $K$-theory of the uniform $C^*$-Roe algebra is far more complicated. The $K_0$-group of the complex uniform Roe $C^*$-algebra of $\mathbb{Z}^d$ is an uncountable Abelian group for all $d > 1$, see [14, Example II.3.4].

When we consider Hamiltonians with symmetries, then we should tensor the real Roe $C^*$-algebra of $\mathbb{Z}^d$ with a real or complex Clifford algebra. This gives
a $\mathbb{Z}/2$-graded $C^*$-algebra. Up to Morita equivalence, there are ten different real or complex Clifford algebras. So we get ten different observable algebras in each dimension. The resulting real or complex $K$-groups agree with those in Kitaev’s periodic table. Hence the latter agrees with the $K$-theory of the Roe $C^*$-algebra. We interpret it as saying that Kitaev’s table gives only the strong topological phases.

The real and complex $K$-groups of the point form a graded commutative, graded ring in a natural way, and the $K$-theory of any real or complex $C^*$-algebra is a graded module over this ring. The boundary map for an extension of real or complex $C^*$-algebras automatically preserves this module structure. In the complex case, the relevant ring is the ring of Laurent polynomials $\mathbb{Z}[\beta, \beta^{-1}]$ in $\beta \in \mathbb{C}$. This describes Bott periodicity. That a map on $K$-theory is a $K_0(\mathbb{C})$-module homomorphism only says that it is obtained by the maps on $K_0$ and $K_1$ and Bott periodicity. In other words, it is a homomorphism of $\mathbb{Z}/2$-graded groups. In the real case, the relevant ring is more complicated, and so the module structure contains more useful information.

In this section, we recall how to compute the boundary map of a Toeplitz extension, which is used by many authors to describe the bulk–edge correspondence, and the coarse Mayer–Vietoris sequence are compatible. So there is no need to compute the boundary map for the Roe $C^*$-algebras because the $K$-theory groups in question are so small, even in the real case.

The boundary map $\partial_{MV}$ is the incarnation of the bulk–edge correspondence in our Roe $C^*$-algebra context. It is shown by Kubota that the boundary maps in the Toeplitz extension, which is used by many authors to describe the bulk–edge correspondence, and the coarse Mayer–Vietoris sequence are compatible.

**Proposition 3.11.** Let $\varphi: \mathbb{Z}^{d-1} \to \mathbb{Z}^d$ be an injective group homomorphism. Then the induced map $\varphi_*: C^*_{Roe}(\mathbb{Z}^{d-1}) \to C^*_{Roe}(\mathbb{Z}^d)$ induces the zero map in $K$-theory, both in the real and complex cases.

**Proof.** Since $\varphi$ is an injective group homomorphism, it is a coarse equivalence from $\mathbb{Z}^{d-1}$ onto a subspace of $\mathbb{Z}^d$. This explains the definition of $\varphi_*: C^*_{Roe}(\mathbb{Z}^{d-1}) \to C^*_{Roe}(\mathbb{Z}^d)$. There is $x \in \mathbb{Z}^d$ so that the map $\mathbb{Z}^{d-1} \times \mathbb{Z} \to \mathbb{Z}^d$, $(a,b) \mapsto \varphi(a) + b \cdot x$, is injective. So the map $\varphi_*: C^*_{Roe}(\mathbb{Z}^{d-1}) \to C^*_{Roe}(\mathbb{Z}^d)$ factors through $C^*_{Roe}(\mathbb{Z}^{d-1} \times \mathbb{N})$. Since the $K$-theory of $C^*_{Roe}(\mathbb{Z}^{d-1} \times \mathbb{N})$ vanishes by Proposition 3.9, the map $\varphi_*$ induces the zero map on $K$-theory.

4. Comparison with the periodic case

Let $F \in \{\mathbb{R}, \mathbb{C}\}$. The observable algebra $C^*(\mathbb{Z}^d)_F$ or a matrix algebra over it describes periodic observables in the limiting case of no disorder, in the tight-binding approximation. This is contained in the corresponding Roe $C^*$-algebra $C^*_{Roe}(\mathbb{Z}^d)_F$. In this section, we recall how to compute the $K$-theory of $C^*_{Roe}(\mathbb{Z}^d)_F$ and we describe the map in $K$-theory induced by the inclusion $C^*(\mathbb{Z}^d)_F \to C^*_{Roe}(\mathbb{Z}^d)_F$. In particular, we show that this map is split surjective and that its kernel is generated by those elements that come from the $K$-theory of $C^*(\mathbb{Z}^{d-1})_F$ for a coordinate embedding.
We shall also use the “real” manifolds $A $, a “real” locally compact space where the quotient map is evaluation at $x$.

We only discuss the real case. It is convenient to replace real $C^*$-algebras by “real” ones, that is, complex $C^*$-algebras equipped with a real involution. We first recall some basic facts and definitions about “real” and real $C^*$-algebras and then describe the relevant “real” $d$-torus.

A real $C^*$-algebra $A$ corresponds to the “real” $C^*$-algebra $A \otimes_{\mathbb{R}} \mathbb{C}$ with the real involution $a \otimes z := a \otimes \bar{z}$. A “real” $C^*$-algebra $A$ corresponds to the real $C^*$-algebra

$$A_R := \{a \in A : \pi = a\}.$$ 

A “real” locally compact space $X$ is a locally compact space with an involutive homeomorphism $X \to X$, $x \mapsto \bar{x}$. Then we turn $C_0(X)$ into a “real” $C^*$-algebra using the real involution $\overline{f(x)} := f(\bar{x})$ for all $x \in X$, $f \in C_0(X)$. So $C_0(X)_R = \{f \in C_0(X) : f(\bar{x}) = \overline{f(x)} \text{ for all } x \in X\}$.

For a “real” $C^*$-algebra $A$, we define

$$K_R^*(A) := K_*(A_R).$$

For a “real” locally compact space $X$, we let

$$K_R^*(X) := K_{R*}(C_0(X)) = K_{R*}(C_0(X)_R).$$

Note the grading convention here, which is analogous to the numbering convention when a chain complex is treated as a cochain complex.

From now on, $C^*(\mathbb{T}^d)$ denotes the “real” $C^*$-algebra that corresponds to the real $C^*$-algebra $C^*(\mathbb{Z}^d)_R$. That is, the real involution acts on $f : \mathbb{Z}^d \to \mathbb{C}$ by pointwise complex conjugation. We give the $d$-torus $\mathbb{T}^d \subseteq \mathbb{C}^d$ the real involution by complex conjugation. So $C(\mathbb{T}^d)_R$ is the closed $\mathbb{R}$-linear span of the functions $z^k := \bar{z}^{k_1} \cdots \bar{z}^{k_d}$ on $\mathbb{T}^d$ for $k_1, \ldots, k_d \in \mathbb{Z}$. This is the unique real structure on $\mathbb{T}^d$ for which the Fourier isomorphism $C^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$ is an isomorphism of “real” $C^*$-algebras. Thus

$$K_*(C^*(\mathbb{Z}^d)_R) \cong K_R^*(C(\mathbb{T}^d)) = K^{\text{re}}_*(\mathbb{T}^d).$$

We shall also use the “real” manifolds $\mathbb{R}^{p,q}$ for $p, q \in \mathbb{N}$; this is $\mathbb{R}^{p+q}$ with the real involution $(x, y) := (x, -y)$ for $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$. We may also realise this as $\mathbb{R}^q \times i\mathbb{R}^p \subseteq \mathbb{C}^{p+q}$ with complex conjugation as real involution.

**Proposition 4.1.** The “real” $C^*$-algebra $C(\mathbb{T})$ is KK-equivalent to $\mathbb{C} \oplus C_0(\mathbb{R}^{0,1})$. And $C(\mathbb{T}^d)$ is KK-equivalent to a direct sum of copies of $C_0(\mathbb{R}^{0,j})$ for $j = 0, \ldots, d$, where the summand $C_0(\mathbb{R}^{0,j})$ appears $\binom{d}{j}$ times. The $K$-theory of $C^*(\mathbb{Z}^d)_R$ is a free $K_*(\mathbb{F})$-module of rank $2^d$, with $\binom{d}{j}$ generators of degree $-j \mod 8$.

**Proof.** The points $\pm 1 \in \mathbb{T}$ are real, that is, fixed by the real involution. The complement $\mathbb{T} \setminus \{1\}$ is diffeomorphic as a “real” manifold to $\mathbb{R}^{d-1}$, say, by stereographic projection at 1. Hence we get an extension of “real” $C^*$-algebras

$$C_0(\mathbb{R}^{0,1}) \to C(\mathbb{T}) \to \mathbb{C},$$

where the quotient map is evaluation at 1. This extension splits by embedding $\mathbb{C}$ as constant functions in $C(\mathbb{T})$. Since Kasparov theory is split-exact, also for “real” $C^*$-algebras, $C(\mathbb{T})$ is KK-equivalent to $C_0(\mathbb{R}^{0,1}) \oplus \mathbb{C}$.

We may get $C^*(\mathbb{Z}^d)$ by tensoring $d$ copies of $C^*(\mathbb{Z})$. The tensor product of $C^*$-algebras descends to a bifunctor in KK-theory, also in the “real” case. So $C^*(\mathbb{Z}^d)$
is KK-equivalent to the $d$-fold tensor power of $\mathbb{C} \oplus C_0(\mathbb{R}^0)$. The tensor product of $C^*$-algebras is additive in each variable, and $C_0(\mathbb{R}^p) \otimes C_0(\mathbb{R}^q) \cong C_0(\mathbb{R}^{p+r}q)$. A variant of the binomial formula now gives:

$$C^*(\mathbb{Z}^d) \sim_{KK} (\mathbb{C} \oplus C_0(\mathbb{R}^0))^\otimes \cong \bigoplus_{j=0}^d \left(\begin{array}{c}d \\ j \end{array} \right) C_0(\mathbb{R}^j).$$

A Bott periodicity theorem by Kasparov shows that $C_0(\mathbb{R}^0)$ is KK-equivalent to $C_0(\mathbb{R}^0)$ with a dimension shift of $p - q$, see [15] Theorem 7. This implies the claim about K-theory. □

**Proposition 4.2.** Let $\varphi: \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be an injective group homomorphism. It induces an injective $^*$-homomorphism $\varphi_*: C^*(\mathbb{Z}^d)\varphi \rightarrow C^*(\mathbb{Z}^d)\varphi$ and a grading-preserving $K_\ast(\mathbb{F})$-module homomorphism $K_\ast(\varphi_*): K_\ast(C^*(\mathbb{Z}^d)\varphi) \rightarrow K_\ast(C^*(\mathbb{Z}^d)\varphi)$. The map $K_\ast(C^*(\mathbb{Z}^d)\varphi) \rightarrow K_\ast(C^*_{Roe}(\mathbb{Z}^d)\varphi)$ vanishes on the image of $K_\ast(\varphi_*).$

**Proof.** Proposition 3.11 shows that $\varphi$ induces the zero map on the K-theory of the Roe $C^*$-algebra. The canonical map $C^*_r(G) \hookrightarrow C^*_r(G)$ for a group $G$ is a natural transformation with respect to injective group homomorphisms. So there is a commuting square

$$
\begin{array}{ccc}
C^*(\mathbb{Z}^d)\varphi & \longrightarrow & C^*_r(\mathbb{Z}^d)\varphi \\
\varphi_* \downarrow & & \varphi_* \\
C^*(\mathbb{Z}^d) & \longrightarrow & C^*_r(\mathbb{Z}^d)
\end{array}
$$

This implies the statement. □

The coordinate embeddings

$$\iota_k: \mathbb{Z}^{d-1} \rightarrow \mathbb{Z}^d, \quad (x_1, \ldots, x_{d-1}) \mapsto (x_1, \ldots, x_{k-1}, 0, x_k, \ldots, x_{d-1}),$$

are injective group homomorphisms and induce injective $^*$-homomorphisms

$$\iota_k: C^*(\mathbb{Z}^{d-1}) \rightarrow C^*(\mathbb{Z}^d).$$

The Fourier transform maps $\iota_k(C^*(\mathbb{Z}^{d-1}))$ onto the $C^*$-subalgebra of $C(\mathbb{T}^d)$ consisting of all functions that are constant equal to 1 in the $k$th coordinate direction. We have seen that $K_\ast(C^*(\mathbb{Z}^d)\varphi)$ is a free $K_\ast(\mathbb{F})$-module of rank $2^d$ (with generators in different degrees). Now compute $K_\ast(C^*(\mathbb{Z}^d)\varphi)$ as in Proposition 4.1. The inclusion of functions that are constant in the $k$th direction corresponds in K-theory to the inclusion of those $2^{d-1}$ of the $2^d$ free $K_\ast(\mathbb{R})$-module summands in $K_\ast(C^*(\mathbb{Z}^d)\varphi)$ where we take the summand $\mathbb{R}$ in the $k$th factor. The summands in the image of $K_\ast(\iota_k)$ correspond to topological insulators that are built by stacking copies of a $d - 1$-dimensional insulator in the $k$th direction. Such topological insulators are considered weak by Fu–Kane–Mele [11]. So the map $K_\ast(C^*(\mathbb{Z}^d)\varphi) \rightarrow K_\ast(C^*_{Roe}(\mathbb{Z}^d)\varphi)$ kills the K-theory classes of weak topological insulators.

If $k$ varies, then all but one of the $2^d$ summands $\iota_k(C^*(\mathbb{Z}^d)\varphi)$ are in the image of $K_\ast(\iota_k)$ for some $k \in \{1, \ldots, d\}$. All these summands are mapped to 0 in $K_\ast(C^*_{Roe}(\mathbb{Z}^d)\varphi)$ by Proposition 4.2. The remaining summand is the K-theory of the ideal $C_0(\mathbb{R}^{d-1}) \subset C(\mathbb{T}^d)$. Here we identify $\mathbb{R}^{d-1}$ with an open subset of $\mathbb{T}^d$ using the stereographic projection in each variable, compare the proof of Proposition 4.1. Its “real” or complex K-theory is identified with $K_{\ast-d}(\mathbb{R})$ or $K_{\ast-d}(\mathbb{C})$ by Bott periodicity. Kasparov proves Bott periodicity isomorphisms $KR_\ast(C_0(\mathbb{R}^{p-q})) \cong K_{\ast+p-q}(\mathbb{R})$ using a canonical generator $\phi_{p,q}$ for the K-homology group

$$KR_{\ast-p}(C_0(\mathbb{R}^{p-q}), \mathbb{C}) \cong KK_0(C_0(\mathbb{R}^{p-q}, Cl_{p,q}), \mathbb{C});$$
here $\operatorname{Cl}_{p,q}$ is the Clifford algebra with $p + q$ anti-commuting, odd, self-adjoint generators $\gamma_1, \ldots, \gamma_{p+q}$ with $\gamma_i^2 = \gamma_i$ for $1 \leq i \leq p$ and $\gamma_i^2 = -\gamma_i$ for $p + 1 \leq i \leq p + q$. And we write $\otimes^k$ to highlight that the entries are treated as “real” $\mathbb{C}^*$-algebras.

The Bott generators $\alpha_{p,q}$ are generalised by Kasparov in [16, Definition and Lemma 4.2] to build a “fundamental class”

$$\alpha_X \in \text{KK}^0_0(\operatorname{Cl}_0(X, \operatorname{Cl} X), \mathbb{C})$$

for any complete Riemannian manifold $X$ (without boundary). Here $\operatorname{Cl} X$ is the bundle of “real” $\mathbb{C}^*$-algebras over $X$ whose fibre at $x \in X$ is the $\mathbb{Z}/2$-graded “real” Clifford algebra of the cotangent space $T^*_x X$ for the positive definite quadratic form induced by the Riemannian metric, and $\operatorname{Cl}_0(X, \operatorname{Cl} X)$ means the $\mathbb{Z}/2$-graded “real” $\mathbb{C}^*$-algebra of $\operatorname{Cl}_0$-sections of this Clifford algebra bundle. We now adapt Kasparov’s fundamental class to the case where $X$ is a “real” complete Riemannian manifold, in such a way that the fundamental class for $\mathbb{R}^{p,q}$ is the generator $\alpha_{p,q}$ of Bott periodicity from [15]. The only changes are in the real structure. In particular, all the analysis needed to produce cycles for Kasparov theory is already done in [16].

Recall that the real involution on $\operatorname{Cl}_0(X)$ is defined by $\overline{\mathcal{F}}(x) := \mathcal{F}(x)$ for $f \in \operatorname{Cl}_0(X)$. There is a unique conjugate-linear involution on the space of complex 1-forms on $X$ such that $d \overline{\mathcal{F}} = d \mathcal{F}$ for all smooth $f \in \operatorname{Cl}_0(X)$. There is a unique conjugate-linear involution on $\operatorname{Cl}_0(X, \operatorname{Cl} X)$ with

$$\overline{\omega_1 \cdots \omega_m} = \overline{\omega_1} \cdots \overline{\omega_m}$$

for all sections $\omega_1, \ldots, \omega_m$ of $T^* X \otimes \mathbb{C}$. This involution is also compatible with the multiplication and the $\mathbb{Z}/2$-grading. So it turns $\operatorname{Cl}_0(X, \operatorname{Cl} X)$ into a $\mathbb{Z}/2$-graded “real” $\mathbb{C}^*$-algebra.

Let $L^2(\Lambda^*(X))$ be the Hilbert space of square-integrable complex differential forms on $X$. This is the underlying Hilbert space of Kasparov’s fundamental class. It is $\mathbb{Z}/2$-graded so that sections of $\Lambda^{2q}(X)$ are even and sections of $\Lambda^{2q+1}(X)$ are odd. There is a unique conjugate-linear, isometric involution on $L^2(\Lambda^*(X))$ with

$$\overline{\omega_1 \wedge \cdots \wedge \omega_r} = \overline{\omega_1} \wedge \cdots \wedge \overline{\omega_r}$$

for all complex 1-forms $\omega_1, \ldots, \omega_r$. It commutes with the $\mathbb{Z}/2$-grading, so that $L^2(\Lambda^*(X))$ becomes a $\mathbb{Z}/2$-graded “real” Hilbert space.

Given a complex 1-form $\omega$ and a differential form $\eta$, let $\lambda_\omega(\eta) := \omega \wedge \eta$. These operators satisfy the relations

$$(\omega, \eta) = 0, \quad \lambda_\omega^* \lambda_\eta + \lambda_\eta \lambda_\omega^* = \langle \omega | \eta \rangle$$

for all complex 1-forms $\omega, \eta$, where $\langle \omega | \eta \rangle \in \operatorname{Cl}_0(X)$ denotes the pointwise inner product, which acts on $L^2(\Lambda^*(X))$ by pointwise multiplication. The representation of $\operatorname{Cl}_0(X, \operatorname{Cl} X)$ on $L^2(\Lambda^*(X))$ is defined by letting a complex 1-form $\omega$, viewed as an element of $\operatorname{Cl}_0(X, \operatorname{Cl} X)$, act by $\lambda_\omega + \lambda_\omega^*$. Here $\omega^*$ is the adjoint of $\omega$ in the $\mathbb{C}^*$-algebra $\operatorname{Cl}_0(X, \operatorname{Cl} X)$, that is, $\omega^*(x) = \omega(x)^*$ for all $x \in X$, where $\omega(x)^* \in T^*_x X \otimes \mathbb{C}$ is the pointwise complex conjugation in the second tensor factor $\mathbb{C}$. This defines a $^*$-representation of $\operatorname{Cl}_0(X, \operatorname{Cl} X)$ by [16]. It is grading-preserving and real as well.

Let $d$ be the de Rham differential, defined on smooth sections of $\Lambda^*(X)$ with compact support, and let $d^* := (d + d^*)$ be its adjoint. The unbounded operator $D := d + d^*$ is essentially self-adjoint because $X$ is complete. So

$$F := (1 + D^2)^{-1/2}D$$

is a well defined self-adjoint operator. The operator $d$ is odd and real. This is inherited by $D$ and $F$. Kasparov shows that $(1 - F^2) \cdot a$ and $[F, a]$ are compact for all $a \in \operatorname{Cl}_0(X, \operatorname{Cl} X)$. Thus $\alpha_X := (L^2(\Lambda^*(X)), F)$ is a cycle for the “real” Kasparov group $\text{KK}^0_0(\operatorname{Cl}_0(X, \operatorname{Cl} X), \mathbb{C})$. We call this the fundamental class of the “real” manifold $X$. (Kasparov calls it “Dirac element” instead.)
In particular, the fundamental class of the “real” manifold \( \mathbb{R}^n \) becomes the Bott periodicity generator \( \alpha_{R^d} \) from \([15]\) when we trivialise the Clifford algebra bundle on \( \mathbb{R}^n \) in the obvious way. So \( \alpha_{R^d} \in KK^R_0(C_0(\mathbb{R}^n)) \otimes \mathbb{C} \) is invertible.

We give \( T^d \) the \( \mathbb{Z}^d \)-invariant Riemannian metric to build its fundamental class. The torus \( T^d \) is parallelisable as a “real” manifold: its tangent bundle is isomorphic to \( T^d \times \mathbb{R}^{0,d} \). This induces an isomorphism \( C(T^d, \mathbb{C}) \otimes C(T^d) \cong C(T^d) \otimes C_{0,d} \). So the fundamental class \( \alpha_{T^d} \) also gives an element in \( KK^R_0(C(T^d), \mathbb{C}) \).

Let \( L \) be a separable “real” Hilbert space and build \( C_{Roe}(\mathbb{Z}^d) \) on the “real” Hilbert space \( L^2(\mathbb{Z}^d, \mathcal{L}) \). There is an obvious embedding \( C^*(\mathbb{Z}^d) \otimes K(\mathcal{L}) \subseteq C_{Roe}(\mathbb{Z}^d) \). Let

\[
\alpha_{T^d} \in KK^R_0(C(\mathbb{R}^d) \otimes C_{0,d} \otimes K(\mathcal{L}), \mathbb{C}) \cong KK^R_0(C^*(\mathbb{Z}^d) \otimes C_{0,d} \otimes K(\mathcal{L}), \mathbb{C})
\]

be the exterior product of the fundamental class \( \alpha_{T^d} \) and the Morita equivalence \( K(\mathcal{L}) \sim \mathbb{C} \). This is the Kasparov cycle with underlying \( \mathbb{Z}/2 \)-graded “real” Hilbert space \( L^2(\mathbb{Z}^d, \Lambda^*(\mathcal{C}^d)) \otimes \mathcal{L} \) with the operator \( \tilde{F} := F \otimes 1 \) with \( F \) as above for the manifold \( X = T^d \). So \( \tilde{F} \) is an odd, self-adjoint, real bounded operator with

\[
\begin{align*}
\left[ \tilde{F}, T \right], (1 - \tilde{F}^2) \cdot T \in \mathcal{K}(L^2(\mathbb{Z}) \otimes \Lambda^*(\mathcal{C}^d) \otimes \mathcal{L})
\end{align*}
\]

for all \( T \in C^*(\mathbb{Z}^d) \otimes C_{0,d} \otimes K(\mathcal{L}) \) (the commutator is the graded one).

**Theorem 4.3**. Equation \([11]\) still holds for \( T \in C_{Roe}(\mathbb{Z}^d) \otimes C_{0,d} \). This gives

\[
\alpha_{T^d} := [(L^2(\mathbb{Z}^d, \Lambda^*(\mathcal{C}^d)) \otimes \mathcal{L}, \tilde{F})] \in KK^R_0(C_{Roe}(\mathbb{Z}^d) \otimes C_{0,d}, \mathbb{C}).
\]

The following diagram in \( KK^R \) commutes:

\[
\begin{array}{ccc}
C_0(\mathbb{R}^0,d, C_{0,d}) & \xrightarrow{\text{inclusion}} & C(\mathbb{R}^d, C_{0,d}) \xrightarrow{\text{Fourier}} C^*(\mathbb{Z}^d) \otimes C_{0,d} \xrightarrow{\text{inclusion}} C_{Roe}(\mathbb{Z}^d) \otimes C_{0,d} \\
\cong & \downarrow & \alpha_{T^d} \downarrow & \alpha_{T^d} \downarrow \\
\mathbb{C} & \xleftarrow{\alpha_{T^d}} & \mathbb{C} & \xleftarrow{\alpha_{T^d}} & \mathbb{C}
\end{array}
\]

**Corollary 4.4**. The inclusion \( C_0(\mathbb{R}^0,d) \rightarrow C_{Roe}(\mathbb{Z}^d) \) induces a split injective map \( KR_* : C_0(\mathbb{R}^0,d) \rightarrow KR_* (C_{Roe}(\mathbb{Z}^d)) \). The map \( K_{*+d}(C_{Roe}(\mathbb{Z}^d)_F) \rightarrow K_* (\mathbb{F}) \) induced by \( \alpha_{T^d} \) is an isomorphism. Analogous statements hold in complex K-theory.

**Proof of the corollary.** Both \( KR_* : C_0(\mathbb{R}^0,d) \rightarrow C_{Roe}(\mathbb{Z}^d) \) and \( KR_* (C_{Roe}(\mathbb{Z}^d)) \) are isomorphic to free \( K_* (\mathbb{R}) \)-modules with a generator in degree \(-d\). The Bott periodicity generator \( \alpha_{2G, d} \) maps the generator of \( KR_{-d} (C_0(\mathbb{R}^0,d)) \) onto a generator of \( K_0 (\mathbb{R}) \). The commuting diagram in Theorem 1.3 shows that its image in \( KR_{-d} (C_{Roe}(\mathbb{Z}^d)) \) must be a generator as well. So \( \alpha_{T^d} \) acts by multiplication with \( \pm 1 \) on a generator. Since \( \alpha_{T^d} \) is a K-homology class, the map on K-theory that it induces is a \( K_* (\mathbb{R}) \)-module homomorphism. Hence it is multiplication by \( \pm 1 \) everywhere once this happens on a generator. So the map on \( KR_* \) induced by \( \alpha_{T^d} \) is invertible. The same proof works for complex K-theory.

We have already shown that all but one of the free \( K_* (\mathbb{F}) \)-module summands in \( K_* (C_{Roe}(\mathbb{Z}^d)_F) \) are killed by the map to \( K_* (C_{Roe}(\mathbb{Z}^d)_F) \). When we combine this with the above corollary, it follows that the kernel of the map from \( K_* (C_{Roe}(\mathbb{Z}^d)_F) \) to \( K_* (C_{Roe}(\mathbb{Z}^d)_F) \) is exactly the sum of the images of \( K_* (\mathbb{F}) \) for \( k = 1, \ldots, d \), that is, the subgroup generated by the K-theory classes of weak topological insulators.

We still have to prove Theorem 1.3. The left triangle in the diagram in Theorem 1.3 commutes because of the following general fact:

**Proposition 4.5.** Let \( U \subseteq X \) be an open subset of a “real” manifold \( X \) that is invariant under the real involution. Give \( U \) and \( X \) some complete Riemannian metrics. The Kasparov product of the ideal inclusion \( j : C_0(U, \mathbb{C}) \rightarrow C_0(X, \mathbb{C}) \) and the fundamental class \( \alpha_X \in KK^R_0(C_0(X, \mathbb{C}), \mathbb{C}) \) is the fundamental class...
\( \alpha_U \in \text{KK}^\mathbb{R}_0(\mathcal{C}_0(U, \mathcal{C}U), \mathbb{C}) \). In particular, the fundamental class does not depend on the choice of the Riemannian metric.

**Proof.** Both Kasparov cycles \( j^*(\alpha_X) \) and \( \alpha_U \) live on Hilbert spaces of \( L^2 \)-differential forms on \( U \). Here square-integrability is with respect to different metrics. The resulting Hilbert spaces are isomorphic by pointwise application of a suitable strictly positive, smooth function \( X \to \mathcal{B}(\Lambda^*(X)) \). This isomorphism also respects the \( \mathbb{Z}/2 \)-grading and the real involution. The operator \( \mathcal{D} \) used to construct \( \alpha_X \) is a first-order differential operator. Hence \( F := (1 + \mathcal{D}^2)^{-1/2} \mathcal{D} \) is an order-zero pseudodifferential operator, and it has the same symbol as \( \mathcal{D} \). This is the function

\[
S^* X \to \mathcal{B}(\Lambda^*(X)), \quad (x, \xi) \mapsto \lambda_x + \lambda_x^*.
\]

The symbol of the operator \( F \) of \( \alpha_U \) is given by the same formula, except that the adjoint is for another Riemannian metric. So the unitary between the spaces of \( L^2 \)-forms will also identify these symbols. The class of the Kasparov cycle defined by an order-zero pseudodifferential operator \( F \) depends only on the symbol of \( F \). So \( j^*(\alpha_X) \) and \( \alpha_U \) have the same class in \( \text{KK}^\mathbb{R}_0(\mathcal{C}_0(U, \mathcal{C}U), \mathbb{C}) \). The last statement is the case \( U = X \) of the proposition. \( \square \)

Now we build the Kasparov cycle \( \alpha'_{U,j} \in \text{KK}^\mathbb{R}_0(\mathcal{C}_{\text{Ric}}(\mathbb{Z}^d) \otimes \mathcal{C}_{\mathbb{R}d}, \mathbb{C}) \). Let \( z_j : T^d \to \mathbb{C} \) be the \( j \)-th coordinate function and let

\[
z^k := z_1^{k_1} \cdots z_d^{k_d} \quad \text{for} \quad k = (k_1, \ldots, k_d) \in \mathbb{Z}^d.
\]

The real involution on \( T^d \) is defined so that these are real elements of \( \mathcal{C}(T^d) \). Hence the 1-forms \( z_j^{-1}dz_j \) for \( j = 1, \ldots, d \) are real. They form a basis of the space of 1-forms as a \( \mathcal{C}(T^d) \)-module. The differential forms

\[
z^k \cdot (z_{i_1} \cdots z_{i_k})^{-1}dz_{i_1} \wedge \cdots \wedge dz_{i_k}
\]

for \( k \in \mathbb{Z}^d \) and \( 1 \leq i_1 < i_2 < \cdots < i_k \leq d \) form a real, orthonormal basis of the Hilbert space \( L^2(\Lambda^*(T^d)) \). Hence there is a unitary operator

\[
U : \ell^2(\mathbb{Z}^d) \otimes \Lambda^*(\mathbb{C}^d) \simeq L^2(\Lambda^*(T^d)),
\]

\[
\delta_k \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto z^k \cdot (z_{i_1} \cdots z_{i_k})^{-1}dz_{i_1} \wedge \cdots \wedge dz_{i_k}.
\]

This unitary is grading-preserving and real for the \( \mathbb{Z}/2 \)-grading and real structure on \( \ell^2(\mathbb{Z}^d) \otimes \Lambda^*(\mathbb{C}^d) \) where the standard basis vector \( \delta_k \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} \) is real and is even or odd depending on the parity of \( \ell \).

The above trivialisation of the cotangent bundle of \( T^d \) gives the isomorphism

\[
\mathcal{C}(T^d) \otimes \mathcal{C}_{\mathbb{R}d} \simeq \mathcal{C}(T^d, \mathcal{C}T^d), \quad \gamma_j \mapsto iz_j^{-1}dz_j;
\]

recall that \( \gamma_1, \ldots, \gamma_d \) are the odd, self-adjoint, anti-commuting unitaries that generate \( \mathcal{C}_{\mathbb{R}d} \). The action of \( \mathcal{C}(T^d, \mathcal{C}T^d) \) on \( L^2(\Lambda^*(T^d)) \) now translates to an action of \( \mathcal{C}(T^d) \otimes \mathcal{C}_{\mathbb{R}d} \) on \( \ell^2(\mathbb{Z}^d) \otimes \Lambda^*(\mathbb{C}^d) \). Namely, the scalar-valued function \( z^k \in \mathcal{C}(T^d) \) acts by the shift \( (\tau_k f)(n) := f(n - k) \) for all \( k, n \in \mathbb{Z}^d \), \( f \in \ell^2(\mathbb{Z}^d, \Lambda^*(\mathbb{C}^d)) \). And the Clifford generator \( \gamma_j \in \mathcal{C}_{\mathbb{R}d} \) acts by

\[
(\gamma_j f)(n) = i\lambda_{\gamma_j} (f(n)) - i\lambda_{\gamma_j}^* (f(n)).
\]

The unitary \( U^* \) maps the domain of \( d \) to the space of rapidly decreasing functions \( \mathbb{Z}^d \to \Lambda^*(\mathbb{C}^d) \), where \( U^*dU \) acts by pointwise application of the function

\[
A : \mathbb{Z}^d \to \mathcal{B}(\Lambda^*(\mathbb{C}^d)), \quad n \mapsto \lambda_n = \sum_{j=1}^{d} n_j \cdot \lambda_{\gamma_j},
\]
Theorem 4.7

Assume that \( \xi_1, \xi_2 \) are sufficiently far below the Fermi level. Their inclusion only adds a trivial vector bundle. Several authors put in extra work to refine the classification of vector bundles in the most relevant cases. In particular, the operator \( F : U^*FU \otimes 1_C \) is real, odd, and self-adjoint. Let \( T \in C^*_Roe(\mathbb{Z}^d) \subseteq B(\ell^2(\mathbb{Z}^d, \mathbb{L})) \) and \( S \in Cl_{0,d} \). We must show that \( (1 - F^2) \cdot (T \otimes S) \) and \( [F^2, T \otimes S] \) are compact operators. The operator \( 1 - F^2 \) acts by pointwise multiplication with \( (1 + \|x\|^2)^{-1/2} \). Since \( T \) is locally compact and \( \Lambda^* C^d \) has finite dimension, the operator \( (1 - F^2) \cdot (T \otimes S) \) is compact. Describe \( T \) as a block matrix \((T_{x,y})_{x,y \in \mathbb{Z}^d}\) with \( T_{x,y} \in B(\mathbb{L}) \). The operator \( F \) anti-commutes with \( 1 \otimes S \). So the graded commutator \([A + A^*, T \otimes S] = [A + A^*, T \otimes 1] \cdot (1 \otimes S)\) corresponds to the block matrix with \((x, y)\)-entry

\[
T_{x,y} \otimes (\lambda_{x-y} + \lambda_{y-x}^*) S \in B(\mathbb{L} \otimes \Lambda^* C^d).
\]

Assume that \( T \) is \( R \)-controlled, that is, \( T_{x,y} = 0 \) if \( \|x - y\| > R \), and that \( \text{sup}_x \sum_y \|T_{x,y}\| \) and \( \text{sup}_y \sum_x \|T_{x,y}\| \) are bounded; block matrices with these two properties give bounded operators, and these are dense in the Roe \( C^* \)-algebra. For such \( T \), the commutator \([A + A^*, T \otimes S]\) satisfies analogous bounds because \( \|\lambda_{x-y} + \lambda_{y-x}^*\| \leq 2\|x - y\| \leq 2R \) whenever \( T_{x,y} \neq 0 \). So the set of \( T \in C^*_Roe(X) \) for which \([A + A^*, T \otimes S]\) is bounded is dense in \( C^*_Roe(X) \). Thus \( A + A^* \) defines a spectral triple over \( C^*_Roe(X) \otimes Cl_{0,d} \). As a consequence, \([F, T \otimes 1]\) is compact for all \( T \in C^*_Roe(X) \otimes Cl_{0,d} \). This finishes the proof of Theorem 4.3.

4.6. Another topological artefact of the tight binding approximation. We already argued in the introduction that the tight binding approximation may produce topological artefacts. Namely, it suggests to use the uniform Roe \( C^* \)-algebra instead of the Roe \( C^* \)-algebra, whose K-theory is much larger. We briefly mention another artefact caused by the tight binding approximation.

We work in Bloch–Floquet theory for greater clarity. The Fermi projection of a Hamiltonian is described by a vector bundle \( V \to \mathbb{T}^d \) over the \( d \)-torus, maybe with extra symmetries. Here \( d \) is the dimension of the material, which is 2 or 3 in the most relevant cases. In K-theory, two vector bundles \( \xi_1, \xi_2 \) are identified if they are stably isomorphic, that is, there is a trivial vector bundle \( \vartheta \) with \( \xi_1 \oplus \vartheta \cong \xi_2 \oplus \vartheta \). Several authors put in extra work to refine the classification of vector bundles (with symmetries) provided by K-theory to a classification up to isomorphism, see [18][19][24][25]. Here we argue that such a refinement of the classification is of little physical significance. The tight binding approximation leaves out energy bands that are sufficiently far below the Fermi level. Their inclusion only adds a trivial vector bundle – but this is the difference between stable isomorphism and isomorphism.

Theorem 4.7 ([14] Chapter 8, Theorem 1.5]). Let \( X \) be an \( n \)-dimensional CW-complex and let \( \xi_1 \) and \( \xi_2 \) be two \( k \)-dimensional vector bundles. Let \( c = 1, 2, 4 \)
depending on whether the vector bundles are real, complex or quaternionic. Assume $k \geq \lceil (n+2)/c \rceil - 1$. If $\xi_1$ and $\xi_2$ are stably isomorphic, then they are isomorphic.

So for a 3-dimensional space $X$, the isomorphism and stable isomorphism classification agree for real vector bundles of dimension at least 4, for complex vector bundles of dimension at least 2, and for all quaternionic vector bundles. For instance, consider the material Bi$_2$Se$_3$ studied in [22,35]. The model Hamiltonian in [22,35] focuses on four bands, of which half are below and half above the Fermi energy. But the dimension of the physically relevant vector bundle is $2 \cdot 83 + 3 \cdot 34 = 268$, the number of electrons per unit cell of the crystal; each atom of Bismuth has 83 electrons and each atom of Se has 34 electrons.

The theorem cited above does not take into account a real involution on the space $X$. The proof of Theorem [4,7] is elementary enough, however, to extend to “real” vector bundles over “real” manifolds. To see this, one first describes a “real” manifold as a $\mathbb{Z}/2$-CW-complex. The main step in the proof of Theorem [4,7] is to build nowhere vanishing sections of vector bundles, assuming that the fibre dimension is large enough. This allows to split off a trivial rank-1 vector bundle as a direct summand. Similarly, if two vector bundles with nowhere vanishing sections are homotopic, then there is a nowhere vanishing section for the homotopy if the dimension of the fibres is large enough. The only change in the “real” case is that we need a $\mathbb{Z}/2$-equivariant nowhere vanishing section of a “real” vector bundle to split off trivial summands. Such sections are built by induction over the cells of the $\mathbb{Z}/2$-CW-complex. The $\mathbb{Z}/2$-action on the interior of such a cell is either free or trivial. In the first case, a $\mathbb{Z}/2$-equivariant section is simply a section on one half of the cell. In the second case, the cell is contained in the fixed-point submanifold, and we need a nowhere vanishing section of a real vector bundle in the usual sense. So the argument in [14] allows to build nowhere vanishing real sections of “real” vector bundles under the same assumptions on the dimension as for real vector bundles.

References
[1] Michael Aizenman and Stanislav Molchanov, Localization at large disorder and at extreme energies: an elementary derivation, Comm. Math. Phys. 157 (1993), no. 2, 245–278, available at [http://projecteuclid.org/euclid.cmp/1104253939](http://projecteuclid.org/euclid.cmp/1104253939) MR 1244867
[2] Jean Bellissard, $K$-theory of $C^*$-algebras in solid state physics, Statistical mechanics and field theory; mathematical aspects (Groningen, 1985), Lecture Notes in Phys., vol. 257, Springer, Berlin, 1986, pp. 99–156, doi: 10.1007/3-540-16777-3_74 MR 862832
[3] Jean V. Bellissard, A. van Elst, and Hermann Schulz-Baldes, The noncommutative geometry of the quantum Hall effect, J. Math. Phys. 35 (1994), no. 10, 5373–5451, doi: 10.1063/1.530758 Topology and physics. [MR 1295473]
[4] J. Bellissard, D. J. L. Herrmann, and M. Zirnbauer, Hull of aperiodic solids and gap labeling theorems, Directions in mathematical quasicrystals, CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 207–258. MR 1798994
[5] B. Andrei Bernevig, Taylor L. Hughes, and Shou-Cheng Zhang, Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells, Science 314 (2006), no. 5806, 1757–1761, doi: 10.1126/science.113774
[6] Bruce Blackadar and Joachim Cuntz, Differential Banach algebras and smooth subalgebras of $C^*$-algebras, J. Operator Theory 26 (1991), no. 2, 255–282, available at [http://www.theta.ro/jot/archive/1991-026-002/1991-026-002-003.html](http://www.theta.ro/jot/archive/1991-026-002/1991-026-002-003.html) MR 1225517
[7] Jean-Benoît Bost, Principe d'Oka, K-théorie et systèmes dynamiques non commutatifs, Invent. Math. 101 (1990), no. 2, 261–333, doi: 10.1007/BF01231504 MR 1062964
[8] Jacek Brodzki, Chris Cave, and Kang Li, Exactness of locally compact groups, Adv. Math. 312 (2017), 209–233, doi: 10.1016/j.aim.2017.03.020 MR 3635811
[9] Xiaoman Chen, Romain Tessera, Xianjin Wang, and Guoliang Yu, Metric sparsification and operator norm localization, Adv. Math. 218 (2008), no. 5, 1496–1511, doi: 10.1016/j.aim.2008.03.016 MR 2419930
[10] Joachim Cuntz, Ralf Meyer, and Jonathan M. Rosenberg, Topological and bivariant K-theory, Oberwolfach Seminars, vol. 36, Birkhäuser Verlag, Basel, 2007. doi: 10.1007/978-3-7643-8399-2 MR 2340673
