DUAL GRAPHS OF EXCEPTIONAL DIVISORS

JÁNOS KOLLÁR

Let $X$ be a complex algebraic or analytic variety. Its local topology near a point $x \in X$ is completely described by the link $L(x \in X)$, which is obtained as the intersection of $X$ with a sphere of radius $0 < \epsilon \ll 1$ centered at $x$. A regular neighborhood of $x \in X$ is homeomorphic to the cone over $L(x \in X)$; cf. [GM88, p.41].

One can study the local topology of $X$ by choosing a resolution of singularities $\pi: Y \to X$ such that $E_x := \pi^{-1}(x) \subset Y$ is a simple normal crossing divisor and then relating the topology of $E_x$ to the topology of the link $L(x \in X)$. This approach, initiated in [Mum61], has been especially successful for surfaces.

The topology of a simple normal crossing divisor $E$ can in turn be understood in 2 steps. First, the $E_i$ are smooth projective varieties, and their topology is much studied. A second layer of complexity comes from how the components $E_i$ are glued together. This gluing process can be naturally encoded by a finite cell complex $D(E)$, called the dual graph or dual complex of $E$; see Definition 5. Given $x \in X$ and a resolution $\pi: Y \to X$, the dual complex $D(E_x)$ depends on the resolution chosen, but its homotopy type does not; we denote it by $DR(x \in X)$ (see [Thu07, Ste08, ABW11]).

Using this approach [KK11] proved that for every finitely presented group $\Gamma$ there is a complex algebraic singularity $(x \in X)$ (of dimension 3) such that $\pi_1(L(x \in X)) \cong \Gamma$. The proof starts with first constructing a simple normal crossing variety $E$ such that $\pi_1(E) \cong \pi_1(D(E)) \cong \Gamma$ and then realizing $E$ as the exceptional divisor on a resolution of a singularity $(x \in X)$, yielding a chain of isomorphisms

$$\pi_1(L(x \in X)) \cong \pi_1(DR(0 \in X)) \cong \pi_1(D(E)) \cong \Gamma.$$

The aim of this note is to go further and prove that not just $\pi_1(DR(0 \in X))$ but $DR(0 \in X)$ can be arbitrary.

**Theorem 1.** Let $T$ be a connected, finite cell complex. Then there is a normal singularity $(0 \in X_T)$ whose dual complex $DR(0 \in X_T)$ is homotopy equivalent to $T$.

It is interesting to connect properties of $DR(0 \in X)$ with algebraic or geometric properties of the singularity $(0 \in X)$. A quasi projective variety $X$ has rational singularities if for one (equivalently every) resolution of singularities $p: Y \to X$ and for every algebraic (or holomorphic) vector bundle $F$ on $X$, the natural maps $H^i(X, F) \to H^i(Y, p^*F)$ are isomorphisms. That is, for purposes of computing cohomology of vector bundles, $X$ behaves like a smooth variety. See [KM98, Sec.5.1] for details.

It is known that if $X$ has rational singularities then $DR(0 \in X)$ is $\mathbb{Q}$-acyclic, that is, $H^i(DR(0 \in X), \mathbb{Q}) = 0$ for $i > 0$, see for example [KK11, Lem.39]. The fundamental groups of the $DR(0 \in X)$ for rational singularities were determined in [KK11]: these are exactly those finitely presented groups $G$ for which
$H^1(G, \mathbb{Q}) = H^2(G, \mathbb{Q}) = 0$ (sometimes called $\mathbb{Q}$-superperfect groups). Our next result determines the possible homotopy types of $\mathcal{D}(0 \in X)$ for rational singularities.

**Theorem 2.** Let $T$ be a connected, finite, $\mathbb{Q}$-acyclic cell complex. Then there is a rational singularity $(0 \in X)$ whose dual complex $\mathcal{D}(0 \in X)$ is homotopy equivalent to $T$.

While these are much stronger results than the fundamental group versions, most of the work needed to prove these theorems was already done in [Kol11, KK11].

The main technical result of [KK11] proves that for every compact simplicial complex $T$ there is a projective simple normal crossing variety $Z$ such that $\mathcal{D}(Z)$ is homotopy equivalent to $T$, while the main technical result of [Kol11] shows that for every projective simple normal crossing variety $Z$ there is a normal singularity $(0 \in X)$ and a partial resolution $\pi : X' \to X$ such that $Z \cong \pi^{-1}(0) \subset X'$.

If $\dim Z \leq 4$ then $X'$ has only very simple singularities which are easy to resolve. This was sufficient to control $\pi_1(\mathcal{D}(0 \in X))$. However, as $\dim Z$ increases, $X'$ has more and more complicated singularities given locally by equations of the form

$$x_1 \cdots x_n = t \cdot \det \begin{pmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & \ddots & \vdots \\ y_{m1} & \cdots & y_{mm} \end{pmatrix}$$

where $n, m$ are arbitrary and $(x_i, y_{ij}, t)$ are local coordinates.

If $\dim Z = 2$, then the only singularity that appears is the ordinary 3-fold double point $(x_1x_2 = ty_{11})$. The first somewhat complicated singularity

$$(x_1x_2x_3 = t \cdot (y_{11}y_{22} - y_{12}y_{21})) \subset \mathbb{C}^8$$

appears when $\dim Z = 6$.

In this paper we start with the varieties constructed in [Kol11, KK11] and resolve these singularities. Surprisingly, the resolution process described in [Kol11, KK11] leaves the dual complex unchanged and we get the following.

**Theorem 3.** Let $Z$ be a projective simple normal crossing variety of dimension $n$. Then there is a normal singularity $(0 \in X)$ of dimension $(n+1)$ and a resolution $\pi : Y \to X$ such that $E := \pi^{-1}(0) \subset Y$ is a simple normal crossing divisor and its dual complex $\mathcal{D}(E)$ is naturally identified with $\mathcal{D}(Z)$.

4 (Open problems).

(4.1) It might be possible to describe all complexes that occur on resolutions of $(n+1)$-dimensional varieties. It is clear that $\dim \mathcal{D}(E) \leq n$ but I do not know any other restrictions.

Starting with an $n$-dimensional complex $T$, the constructions of [Kol11, KK11] give a $(2n+1)$-dimensional singularity $(0 \in X)$ such that $\mathcal{D}(0 \in X)$ is homotopy equivalent to $T$. This increase of the dimension may not be necessary.

(4.2) The singularities constructed in Theorems 1-2 are not isolated. It would be interesting to construct isolated examples.

(4.3) As shown by [KK11], links of isolated singularities are much more complicated topologically than smooth projective varieties. As a starting point of further investigations, it would be useful to understand the precise relationship between $\mathcal{D}(0 \in X)$ and the topology of the link of an isolated singularity.
(4) As we noted, given a singularity \((0 \in X)\) and a resolution \(\pi : Y \to X\) such that \(E := \pi^{-1}(0)\) is a simple normal crossing divisor, the homotopy type \(\mathcal{D}R(0 \in X)\) of the dual complex \(\mathcal{D}(E)\) does not depend on the choice of \(\pi : Y \to X\).

Note that if \(p : X' \to X\) is a proper birational morphism such that \(E' := \pi^{-1}(0)\) is a simple normal crossing divisor, then the dual complex \(\mathcal{D}(E')\) is defined even if \(X'\) is singular.

It is possible that \(\mathcal{D}(E')\) is in fact homotopy equivalent to \(\mathcal{D}R(0 \in X)\) as long as \(X'\) has rational singularities. (The latter condition is actually quite weak, for instance it holds if none of the strata of \(E\) are contained in \(\text{Sing } X\).)

(5) Assume that \((X, \Delta)\) is dlt \([\text{KM}98, 2.37]\). Since dlt implies rational, \(\mathcal{D}R(x \in X)\) a \(\mathbb{Q}\)-acyclic for every point \(x \in X\). Furthermore, \(\pi_1(\mathcal{D}R(x \in X)) = 1\) by \([\text{Kol93}]\) and \([\text{Tak03}]\). Thus \(\mathcal{D}R(x \in X)\) is contractible iff it is \(\mathbb{Z}\)-acyclic. It would be very interesting to decide whether \(\mathcal{D}R(x \in X)\) is contractible or not. For quotient singularities this is proved in \([\text{KST11}]\).

Related results are treated in \([\text{Kol07a}]\) and \([\text{HX09}]\).

**Definition 5** (Dual graphs). Let \(X\) be a variety with irreducible components \(\{X_i : i \in I\}\). We say that \(X\) is a simple normal crossing variety (abbreviated as snc) if the \(X_i\) are smooth and every point \(p \in X\) has an open (Euclidean) neighborhood \(p \in U_p \subset X\) and an embedding \(U_p \hookrightarrow \mathbb{C}^{n+1}_p\) such that \(U_p \subset (z_1 \cdots z_{n+1} = 0)\). A stratum of \(X\) is any irreducible component of an intersection \(\cap_{i \in J} X_i\) for some \(J \subset I\).

The combinatorics of \(X\) is encoded by a cell complex \(\mathcal{D}(X)\) whose vertices are labeled by the irreducible components of \(X\) and for every stratum \(Z \subset \cap_{i \in J} X_i\) we attach a \((|J| - 1)\)-dimensional cell. Note that for any \(j \in J\) there is a unique irreducible component of \(\cap_{i \in J \setminus \{j\}} X_i\) that contains \(Z\); this specifies the attaching map. \(\mathcal{D}(X)\) is called the dual complex or dual graph of \(X\). (Although \(\mathcal{D}(X)\) is not a simplicial complex in general, it is an unordered \(\Delta\)-complex in the terminology of \([\text{Hat02}]\) p.534).

**Definition 6** (Dual graphs associated to a singularity). Let \(X\) be a normal variety and \(x \in X\) a point. Choose a resolution of singularities \(\pi : Y \to X\) such that \(E_x := \pi^{-1}(x) \subset Y\) is a simple normal crossing divisor. Thus it has a dual complex \(\mathcal{D}(E_x)\).

The dual graph of a normal surface singularity has a long history. Higher dimensional versions appear in \([\text{Kul77}]\ [\text{Per77}]\ [\text{Gor80}]\ [\text{FM83}]\) but systematic investigations were started only recently; see \([\text{Thu07}]\ [\text{Ste08}]\ [\text{Pay09}]\ [\text{Pay11}]\).

It is proved in \([\text{Thu07}]\ [\text{Ste08}]\ [\text{ABW11}]\) that the homotopy type of \(\mathcal{D}(E_x)\) is independent of the resolution \(Y \to X\). We denote it by \(\mathcal{D}R(x \in X)\).

The proof of Theorem 4 starts with the following, which is a combination of Theorem 29 and Lemma 39 of \([\text{KK11}]\).

**Theorem 7.** Let \(T\) be a finite cell complex. Then there is a projective simple normal crossing variety \(Z_T\) such that

1. \(\mathcal{D}(Z_T)\) is homotopy equivalent to \(T\),
2. \(\pi_1(Z_T) \cong \pi_1(T)\) and
3. \(H^i(Z_T, \mathcal{O}_{Z_T}) \cong H^i(T, \mathbb{C})\) for every \(i \geq 0\). □

**8** (Summary of the construction of \([\text{Kol11}]\)). Let \(Z\) be a projective local complete intersection variety of dimension \(n\) and choose any embedding \(Z \subset P\) into a smooth
Let $H$ be the complete intersection of $(N - n - 1)$ general sections of $L(-Z)$. Set

$$Y := B_{(-Z)}Y_1 := \text{Proj}_1 \sum_{m=0}^{\infty} \mathcal{O}_{Y_1}(mZ).$$

(Note that this is not the blow-up of $Z$ but the blow-up of its inverse in the class group.)

It is proved in [Kol11] that the birational transform of $Z$ in $Y$ is a Cartier divisor isomorphic to $Z$ and there is a contraction morphism

$$Z \subset Y \quad \downarrow \quad \downarrow \pi \quad (\text{S}1)$$

such that $Y \setminus Z \cong X \setminus \{0\}$. If $Y$ is smooth then $\mathcal{D}(0 \in X) = \mathcal{D}(Z)$ and we are done with Theorem. However, the construction of [Kol11] yields a smooth variety $Y$ only if $\dim Z = 1$ or $Z$ is smooth. (It is easy to see that not every simple normal crossing variety $Z$ can be realized as a hypersurface on a smooth variety, so this limitation is not unexpected.)

Thus we need to understand the singularities of $Y$ and resolve them.

In order to do this, we need a very detailed description of the singularities of $Y$. This is a local question, so we may assume that $Z \subset \mathbb{C}^N$ is a complete intersection defined by $f_1 = \cdots = f_{N-n} = 0$. Let $Z \subset Y_1 \subset \mathbb{C}^N$ be a general complete intersection defined by equations

$$h_{i,1}f_1 + \cdots + h_{i,N-n}f_{N-n} = 0 \quad \text{for} \quad i = 1, \ldots, N-n-1.$$

Let $H = (h_{ij})$ be the $(N-n) \times (N-n)$ matrix of the system and $H_i$ the submatrix obtained by removing the $i$th column. By [Kol11] or [Kol12, Sec.3.2], an open neighborhood of $Z \subset Y$ is defined by the equations

$$(f_i = (-1)^j \cdot t \cdot \det H_i : i = 1, \ldots, N-n) \subset \mathbb{C}^N \times \mathbb{C}_t. \quad (\text{S}2)$$

Assume now that $Z$ has simple normal crossing singularities. Up-to permuting the $f_i$ and passing to a smaller open set, we may assume that $df_2, \ldots, df_{N-n}$ are linearly independent everywhere along $Z$. Then the singularities of $Y$ all come from the equation

$$f_1 = -t \cdot \det H_1. \quad (\text{S}3)$$

Our aim is to write down local normal forms for $Y$ along $Z$.

On $\mathbb{C}^N$ there is a stratification $\mathbb{C}^N = R_0 \supset R_1 \supset \cdots$ where $R_i$ is the set of points where $\text{rank } H_i \leq (N-n-1) - i$. Since the $h_{ij}$ are general, $\text{codim}_W R_i = i^2$ and we may assume that every stratum of $Z$ is transversal to each $R_i \setminus R_{i+1}$. Let $S \subset Z$ be any stratum and $p \in S$ a point such that $p \in R_m \setminus R_{m+1}$. We can choose local coordinates $\{x_1, \ldots, x_d\}$ and $\{y_{rs} : 1 \leq r, s \leq m\}$ such that, in a neighborhood of $p$,

$$f_1 = x_1 \cdots x_d \quad \text{and} \quad \det H_1 = \det (y_{rs} : 1 \leq r, s \leq m).$$

Note that $m^2 \leq \dim S = n - d$, thus we can add $n - d - m^2$ further coordinates $y_{ij}$ to get a complete local coordinate system on $S$.

Then the $n$ coordinates $\{x_k, y_{ij}\}$ determine a map

$$\sigma : \mathbb{C}^N \times \mathbb{C}_t \to \mathbb{C}^n \times \mathbb{C}_t$$
such that \( \sigma(Y) \) is defined by the equation
\[
x_1 \cdots x_d = t \cdot \det(y_{rs} : 1 \leq r, s \leq m).
\]
Since \( df_2, \ldots, df_{N-n} \) are linearly independent along \( Z \), we see that \( \sigma|_Y \) is étale along \( Z \subset Y \).

We can summarize these considerations as follows.

**Proposition 9.** Let \( Z \) be a normal crossing variety of dimension \( n \). Then there is a normal singularity \( (0 \in X) \) of dimension \( n+1 \) and a proper, birational morphism \( \pi : Y \to X \) such that \( \text{red} \pi^{-1}(0) \cong Z \) and for every point \( p \in \pi^{-1}(0) \) we can choose local étale or analytic coordinates called \( \{ x_i : i \in I_p \} \) and \( \{ y_{rs} : 1 \leq r, s \leq m_p \} \) (plus possibly other unnamed coordinates) such that one can write the local equations of \( Z \subset Y \) as
\[
(\prod_{i \in I_p} x_i = t = 0) \subset (\prod_{i \in I_p} x_i = t \cdot \det(y_{rs} : 1 \leq r, s \leq m_p)) \subset \mathbb{C}^{n+2}. \]

**10** (Determinantal varieties). We have used the following basic properties of determinantal varieties. These are quite easy to prove directly; see [Har95, 12.2 and 14.16] for a more general case.

Let \( V \) be a smooth, affine variety, and \( \mathcal{L} \subset \mathcal{O}_V \) a finite dimensional sub vector space without common zeros. Let \( H = (h_{ij}) \) be an \( n \times n \) matrix whose entries are general elements in \( \mathcal{L} \). For a point \( p \in V \) set \( m_p = \text{corank} H(p) \). Then there are local analytic coordinates \( \{ y_{rs} : 1 \leq r, s \leq m_p \} \) (plus possibly other unnamed coordinates) such that, in a neighborhood of \( p \),
\[
\det H = \det(y_{rs} : 1 \leq r, s \leq m_p).
\]
In particular, \( \text{mult}_p(\det H) = \text{corank} H(p) \), for every \( m \) the set of points \( R_m \subset V \) where \( \text{corank} H(p) \geq m \) is a subvariety of pure codimension \( m^2 \) and \( \text{Sing} R_m = R_{m+1} \).

**11** (Inductive set-up for resolution). The object we try to resolve is a triple
\[
(Y, E, F) := (Y, \sum_{i \in I} E_i, \sum_{j \in J} a_j F_j)
\]
where \( Y \) is a variety over \( \mathbb{C} \), \( E_i, F_j \) are codimension 1 subvarieties and \( a_j \in \mathbb{N} \). (The construction \( \boxplus \) produces a triple \( (Y, E := Z, F := \emptyset) \). The role of the \( F_j \) is to keep track of the exceptional divisors as we resolve the singularities of \( Y \).)

We assume that \( E \) is a simple normal crossing variety and for every point \( p \in E \) there is a (Euclidean) open neighborhood \( p \in Y_p \subset Y \), an embedding \( \sigma_p : Y_p \hookrightarrow \mathbb{C}^{\text{dim} Y+1} \), subsets \( I_p \subset I \) and \( J_p \subset J \), a natural number \( m_p \in \mathbb{N} \) and local coordinates in \( \mathbb{C}^{\text{dim} Y+1} \) called
\[
\{ x_i : i \in I_p \}, \{ y_{rs} : 1 \leq r, s \leq m_p \}, \{ z_j : j \in J_p \} \quad \text{and} \quad t
\]
(plus possibly other unnamed coordinates) such that one can write the local equation of \( \sigma_p(Y_p) \subset \mathbb{C}^{\text{dim} Y+1} \) as
\[
\prod_{i \in I_p} x_i = t \cdot \det(y_{rs} : 1 \leq r, s \leq m_p) \cdot \prod_{j \in J_p} z_j^{a_j}. \]
Furthermore, \( \sigma_p(E_i) = (t = x_i = 0) \cap \sigma_p(Y_p) \) for \( i \in I_p \) and \( \sigma_p(F_j) = (z_j = 0) \cap \sigma_p(Y_p) \) for \( j \in J_p \). (We do not impose any compatibility condition between the local equations on overlapping charts.)

We say that \( (Y, E, F) \) is resolved at \( p \) if \( Y \) is smooth at \( p \).

The key technical result of the paper is the following.
Proposition 12. Let $(Y, E, F)$ be a triple as above. Then there is a resolution of singularities $\pi : (Y', E', F') \to (Y, E, F)$ such that

1. $Y'$ is smooth and $E'$ is a simple normal crossing divisor,
2. $E' = \pi^{-1}(E)$,
3. every stratum of $E'$ is mapped birationally to a stratum of $E$ and
4. $\pi$ induces an identification $D(E') = D(E)$.

Proof. The resolution will be a composite of explicit blow-ups of smooth subvarieties (except at the last step). We use the local equations to describe the blow-up centers locally. Thus we need to know which local subvarieties can be defined globally. For example, choosing a divisor $F_{j_1}$ specifies the local divisor $(z_{j_1} = 0)$ at every point $p \in F_{j_1}$. Similarly, choosing two divisors $E_{i_1}, E_{i_2}$ gives the local subvarieties $(t = x_{i_1} = x_{i_2} = 0)$ at every point $p \in E_{i_1} \cap E_{i_2}$. (Here it is quite important that the divisors $E_i$ are themselves smooth. The algorithm does not seem to work if the $E_i$ have self-intersections.) Note that by contrast $(x_{i_1} = x_{i_2} = 0) \subset Y$ defines a local divisor which has no global meaning. Similarly, the vanishing of any of the coordinate functions $y_{rs}$ has no global meaning.

To a point $p \in \text{Sing } E$ we associate the local invariant

$$\text{Deg}(p) := (\text{deg}_x(p), \text{deg}_y(p), \text{deg}_z(p)) = (|I_p|, m_p, \sum_{j \in I_p} a_j).$$

It is clear that $\text{deg}_x(p)$ and $\text{deg}_z(p)$ do not depend on the local coordinates chosen. We see in (14) that $\text{deg}_y(p)$ is also well defined if $p \in \text{Sing } E$. The degrees $\text{deg}_x(p), \text{deg}_y(p), \text{deg}_z(p)$ are constructible and upper semi continuous functions on $\text{Sing } E$.

Note that $Y$ is smooth at $p$ iff either $\text{Deg}(p) = (1, *, *)$ or $\text{Deg}(p) = (*, 0, 0)$. If $\text{deg}_x(p) = 1$ then we can rewrite the equation (11)2 as

$$x' = t \cdot \prod_j z_j^{a_j} \quad \text{where} \quad x' := x_1 + t \cdot (1 - \det(y_{rs})) \cdot \prod_j z_j^{a_j},$$

so if $Y$ is smooth then $(Y, E + F)$ has only simple normal crossings along $E$. Thus the resolution constructed in Theorem 3 is a log resolution.

The usual method of Hironaka would start by blowing up the highest multiplicity points. This introduces new and rather complicated exceptional divisors and I have not been able to understand explicitly how the dual complex changes.

In our case, it turns out to be much better to look at a locus where $\text{deg}_y(p)$ is maximal but instead of maximizing $\text{deg}_x(p)$ or $\text{deg}_z(p)$ we maximize the dimension. Thus we blow up subvarieties along which $Y$ is not equimultiple. Usually this leads to a morass, but our equations separate the variables into distinct groups which makes these blow-ups easy to compute.

One can think of this as mixing the main step of the Hironaka method with the order reduction for monomial ideals (see, for instance, [Kol07b] Step 3 of 3.111).

After some preliminary remarks about blow-ups of simple normal crossing varieties the proof of (12) is carried out in a series of steps (111 110).

We start with the locus where $\text{deg}_y(p)$ is maximal and by a sequence of blow-ups we eventually achieve that $\text{deg}_y(p) \leq 1$ for every singular point $p$. This, however, increases $\text{deg}_z$. Then in 3 similar steps we lower the maximum of $\text{deg}_z$ until we achieve that $\text{deg}_z(p) \leq 1$ for every singular point $p$. Finally we take care of the singular points where $\text{deg}_y(p) + \text{deg}_z(p) \geq 1$. \hfill $\Box$

13 (Blowing up simple normal crossing varieties). Let $Z$ be a simple normal crossing variety and $W \subset Z$ a subvariety. We say that $W$ has simple normal crossing with $Z$
if for each point \( p \in Z \) there is an open neighborhood \( Z_p \), an embedding \( Z_p \hookrightarrow \mathbb{C}^{n+1} \) and subsets \( I_p, J_p \subset \{0, \ldots, n\} \) such that

\[
Z_p = (\prod_{i \in I_p} x_i = 0) \quad \text{and} \quad W \cap Z_p = (x_j = 0 : j \in J_p).
\]

This implies that for every stratum \( Z_f \subset Z \) the intersection \( W \cap Z_f \) is smooth (even scheme theoretically).

If \( W \) has simple normal crossing with \( Z \) then the blow-up \( B_W Z \) is again a simple normal crossing variety. If \( W \) is one of the strata of \( Z \), then \( \mathcal{D}(B_W Z) \) is obtained from \( \mathcal{D}(Z) \) by removing the cell corresponding to \( W \) and every other cell whose closure contains it. Otherwise \( \mathcal{D}(B_W Z) = \mathcal{D}(Z) \). (In the terminology of [Kol12 Sec.2.4], \( B_W Z \rightarrow Z \) is a thrifty modification.)

As an example, let \( Z = (x_1 x_2 x_3 = 0) \subset \mathbb{C}^3 \). There are 7 strata and \( \mathcal{D}(Z) \) is the 2-simplex whose vertices correspond to the planes \( (x_i = 0) \).

Let us blow up a point \( W = \{ p \} \subset Z \) to get \( B_p Z \subset B_p \mathbb{C}^3 \). Note that the exceptional divisor \( E \subset B_p \mathbb{C}^3 \) is not a part of \( B_p Z \) and \( B_p Z \) still has 3 irreducible components.

If \( p \) is the origin, then the triple intersection is removed and \( \mathcal{D}(B_p Z) \) is the boundary of the 2-simplex.

If \( p \) is not the origin, then \( B_p Z \) still has 7 strata naturally corresponding to the strata of \( Z \) and \( \mathcal{D}(B_p Z) \) is the 2-simplex.

We will be interested in situations where \( Y \) is a hypersurface in \( \mathbb{C}^{n+2} \) and \( Z \subset Y \) is a Cartier divisor that is a simple normal crossing variety. Let \( W \subset Y \) be a smooth, irreducible subvariety, not contained in \( Z \) such that

1. the scheme theoretic intersection \( W \cap Z \) has simple normal crossing with \( Z \)
2. \( \text{mult}_{Z \cap W} Z = \text{mult}_W Y \). (Note that this holds if \( W \subset \text{Sing} Y \) and \( \text{mult}_{Z \cap W} Z = 2 \).

Choose local coordinates \((x_0, \ldots, x_n, t)\) such that \( W = (x_0 = \cdots x_i = 0) \) and \( Z = (t = 0) \subset Y \). Let \( f(x_0, \ldots, x_n, t) = 0 \) be the local equation of \( Y \).

Blow up \( W \) to get \( \pi : B_W Y \to Y \). Up to permuting the indices \( 0, \ldots, i \), the blow-up \( B_W Y \) is covered by coordinate charts described by the coordinate change

\[
(x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n, t) = (x_0', x_1', x_0', \ldots, x_i', x_0', x_{i+1}, \ldots, x_n, t).
\]

If \( \text{mult}_W Y = d \) then the local equation of \( B_W Y \) in the above chart becomes

\[
(x_0')^{-d} f(x_0', x_1', x_0', \ldots, x_i', x_0', x_{i+1}, \ldots, x_n, t) = 0.
\]

By assumption (2), \( (x_0')^d \) is also the largest power that divides

\[
f(x_0', x_1', x_0', \ldots, x_i', x_0', x_{i+1}, \ldots, x_n, 0),
\]

hence \( \pi^{-1}(Z) = B_W \cap Z \).

Observe finally that the conditions (1–2) can not be fulfilled in any interesting way if \( Y \) is smooth. Since we want \( Z \cap W \) to be scheme theoretically smooth, if \( Y \) is smooth then condition (1) implies that \( Z \cap W \) is disjoint from \( \text{Sing} Z \).

(As an example, let \( Y = \mathbb{C}^3 \) and \( Z = (xyz = 0) \). Take \( W := (x = y = z) \). Note that \( W \) is transversal to every irreducible component of \( Z \) but \( W \cap Z \) is a non-reduced point. The preimage of \( Z \) in \( B_W Y \) does not have simple normal crossings.)

There are, however, plenty of examples where \( Y \) is singular along \( Z \cap W \) and these are exactly the singular points that we want to resolve.
The resolution usually cannot be chosen equivariant.

14 (Resolving the determinantal part). Let $m$ be the largest size of a determinant occurring at a non-resolved point. Assume that $m \geq 2$ and let $p \in Y$ be a non-resolved point with $m_p = m$.

Away from $E \cup F$ the local equation of $Y$ is

$$\prod_{i \in I_p} x_i = \det(y_{rs} : 1 \leq r, s \leq m).$$

Thus, the singular set of $Y_p \setminus (E \cup F)$ is

$$\bigcup_{(i, i')} (\text{rank}(y_{rs}) \leq m - 2) \cap (x_i = x_{i'} = 0)$$

where the union runs through all 2-element subsets $\{i, i'\} \subset I_p$. Thus the irreducible components of $\text{Sing} Y \setminus (E \cup F)$ are in natural one-to-one correspondence with the irreducible components of $\text{Sing} E$ and the value of $m = \deg_p(p)$ is determined by the multiplicity of any of these irreducible components at $p$.

Pick $i_1, i_2 \in I$ and we work locally with a subvariety

$$W_p'(i_1, i_2) := (\text{rank}(y_{rs}) \leq m - 2) \cap (x_{i_1} = x_{i_2} = 0).$$

Note that $W_p'(i_1, i_2)$ is singular if $m > 2$ and the subset of its highest multiplicity points is given by $\text{rank}(y_{rs}) = 0$. Therefore the locally defined subvarieties

$$W_p(i_1, i_2) := (y_{rs} = 0 : 1 \leq r, s \leq m) \cap (x_{i_1} = x_{i_2} = 0),$$

glue together to a well defined global smooth subvariety $W := W(i_1, i_2)$.

$E$ is defined by $(t = 0)$ thus $E \cap W$ has the same local equations as $W_p(i_1, i_2)$. In particular, $E \cap W$ has simple normal crossings with $E$ and $E \cap W$ is not a stratum of $E$; its codimension in the stratum $(x_{i_1} = x_{i_2} = 0)$ is $m^2$.

Furthermore, $E$ has multiplicity 2 along $E \cap W$, hence (13.2) also holds and so

$$\mathcal{D}(B_{E \cap W}) = \mathcal{D}(E).$$

We blow up $W \subset Y$. We will check that the new triple is again of the form (11). The local degree $\deg(p)$ is unchanged over $Y \setminus W$. The key assertion is that, over $W$, the maximum value of $\deg(p)$ (with respect to the lexicographic ordering) decreases. By repeating this procedure for every irreducible components of $\text{Sing} E$, we decrease the maximum value of $\deg(p)$. We can repeat this until we reach $\deg_p(p) \leq 1$ for every non-resolved point $p \in Y$.

(Note that this procedure requires an actual ordering of the irreducible components of $\text{Sing} E$, which is a very non-canonical choice. If a finite groups acts on $Y$, the resolution usually cannot be chosen equivariant.)

Now to the local computation of the blow-up. Fix a point $p \in W$ and set $I_p^* := I_p \setminus \{i_1, i_2\}$. We write the local equation of $Y$ as

$$x_{i_1, i_2} \cdot L = t \cdot \det(y_{rs}) \cdot R \quad \text{where} \quad L := \prod_{i \in I_p^*} x_i \quad \text{and} \quad R := \prod_{j \in I_p^*} z_j^{a_j}.$$

There are two types of local charts on the blow-up.

1. There are two charts of the first type. Up to interchanging the subscripts 1, 2, these are given by the coordinate change

$$(x_{i_1}, x_{i_2}, y_{rs} : 1 \leq r, s \leq m) = (x'_{i_1}, x'_{i_2}, x'_{i_1}, y'_{rs} x'_{i_1} : 1 \leq r, s \leq m).$$

After setting $z_w := x'_{i_1}$ the new local equation is

$$x'_{i_2} \cdot L = t \cdot \det(y'_{rs}) \cdot (z_w^{m^2 - 2} \cdot R).$$

The exceptional divisor is added to the $F$-divisors with coefficient $m^2 - 2$ and the new degree is $(\deg_x(p) - 1, \deg_y(p), \deg_z(p) + m^2 - 2)$. 


(2) There are \( m^2 \) charts of the second type. Up to re-indexing the \( m^2 \) pairs \((r, s)\) these are given by the coordinate change
\[
(x_{i_1}, x_{i_2}, y_{rs} : 1 \leq r, s \leq m) = (x'_{i_1}, y''_{mm}, x'_{i_2}y''_{mm}, y'_{rs}y''_{mm} : 1 \leq r, s \leq m)
\]
except when \( r = s = m \) where we set \( y_{mm} = y''_{mm} \). It is convenient to set \( y''_{mm} = 1 \) and \( z_w := y''_{mm} \). Then the new local equation is
\[
x'_{i_1}x'_{i_2} \cdot L = t \cdot \det(y'_{rs} : 1 \leq r, s \leq m) \cdot (z_w^{m^2-2} \cdot R).
\]
Note that the \((m, m)\) entry of \((y'_{rs})\) is 1. By row and column operations we see that
\[
det(y'_{rs} : 1 \leq r, s \leq m) = det(y'_{rs} - y'_{rm}y'_{ms} : 1 \leq r, s \leq m - 1).
\]
By setting \( y''_{rs} := y'_{rs} - y'_{rm}y'_{ms} \) we have new local equations
\[
x'_{i_1}x'_{i_2} \cdot L = t \cdot \det(y''_{rs} : 1 \leq r, s \leq m - 1) \cdot (z_w^{m^2-2} \cdot R)
\]
and the new degree is \((\deg_x(p), \deg_y(p) - 1, \deg_z(p) + m^2 - 2)\).

**Outcome.** After these blow ups we have a triple \((Y, E, F)\) such that at non-resolved points the local equations are
\[
\prod_{i \in I_p} x_i = t \cdot y \cdot \prod_{j \in J_p} z_{c_j}^{a_j} \quad \text{or} \quad \prod_{i \in I_p} x_i = t \cdot \prod_{j \in J_p} z_{c_j}^{a_j}.
\]
(Note that we can not just declare that \( y \) is also a \( z \)-variable. The \( z_j \) are local equations of the divisors \( F_j \) while \((y = 0)\) has no global meaning.)

15 (Resolving the monomial part). Following \((14.3)\), the local equations are
\[
\prod_{i \in I_p} x_i = t \cdot y^c \cdot \prod_{j \in J_p} z_{c_j}^{a_j} \quad \text{where} \quad c \in \{0, 1\}.
\]
We lower the degree of the \( z \)-monomial in 3 steps.

**Step 1.** Assume that there is a non-resolved point with \( a_{j_1} \geq 2 \).
The singular set of \( F_{j_1} \) is then
\[
\bigcup_{\{i, i'\}}(z_{j_1} = x_i = x_{i'} = 0)
\]
where the union runs through all 2-element subsets \( \{i, i'\} \subset I \). Pick an irreducible component of it, call it \( W(i_1, i_2, j_1) := (z_{j_1} = x_{i_1} = x_{i_2} = 0) \).
Set \( I^*_p := I_p \setminus \{i_1, i_2\} \) and write the local equations as
\[
x_{i_1}x_{i_2} \cdot L = t z_{c_j}^{a_j} \cdot R \quad \text{where} \quad L := \prod_{i \in I^*_p} x_i \quad \text{and} \quad R := y^c \cdot \prod_{j \in J_p} z_{c_j}^{a_j}.
\]
There are 3 local charts on the blow-up:

1. \((x_{i_1}, x_{i_2}, z_{j_1}) = (x'_{i_1}, x'_{i_2} x_{i_1}, z'_{j_1} x_{i_1})\) and, after setting \( z_w := x'_{i_1} \) the new local equation is
\[
x'_{i_2} \cdot L = t \cdot z_{w}^{a_j-2} z_{j_1}^{a_j} \cdot R.
\]
The new degree is \((\deg_x(p) - 1, \deg_y(p), \deg_z(p) + a_j - 2)\).

2. Same as above with the subscripts 1, 2 interchanged.

3. \((x_{i_1}, x_{i_2}, z_{j_1}) = (x'_{i_1}, x'_{i_2}, z'_{j_1}, z'_{j_1})\) with new local equation
\[
x'_{i_1}x'_{i_2} \cdot L = t \cdot z_{j_1}^{a_j-2} \cdot R.
\]
The new degree is \((\deg_x(p), \deg_y(p), \deg_z(p) - 2)\).
As before, the blow-up computation is the same as in Step 2. Assume that there is a non-resolved point with $a_{j_1} = a_{j_2} = 1$. The singular set of $F_{j_1} \cap F_{j_2}$ is then
$$
\bigcup_{(i, i')} \{ z_{j_1} = z_{j_2} = x_i = x_i' = 0 \}.
$$

where the union runs through all 2-element subsets $\{i, i'\} \subseteq I$. Pick an irreducible component of it, call it $W = \{ z_{j_1} = z_{j_2} = x_i = x_i = 0 \}$. Pick an irreducible component of it, call it $W$. Set $I_p' := I_p \setminus \{ i_1, i_2 \}$, $J' := J_p \setminus \{ j_1, j_2 \}$ and we write the local equations as
$$
x_i x_\iota : = 1.
$$

There are two types of local charts on the blow-up.

1. In the chart $(x_1, x_2, z_1, z_2) = (x_1', x_1', x_1', x_1', x_2', x_2', x_2', x_1')$ the new local equation is
$$
x_i' x_\iota : = 1.
$$

and the new degree is $(\deg_y(p) - 1, \deg_y(p), \deg_z(p))$. A similar chart is obtained by interchanging the subscripts $i_1, i_2$.

2. In the chart $(x_1, x_2, z_1, z_2) = (x_1', x_1', x_1', x_1', x_2', x_2', x_2', x_1')$ the new local equation is
$$
x_i' x_\iota : = 1.
$$

The new degree is $(\deg_y(p), \deg_y(p), \deg_z(p) - 1)$.

A similar chart is obtained by interchanging the subscripts $j_1, j_2$.

By repeated application of these two steps we are reduced to the case where $\deg_y(p) \leq 1$ at all non-resolved points.

Step 3. Assume that there is a non-resolved point with $\deg_y(p) = \deg_z(p) = 1$. The singular set of $Y$ is
$$
\bigcup_{(i, i')} \{ y = z = x_i = x_i' = 0 \}.
$$

Pick an irreducible component of it, call it $W(I_1, i_2) := \{ y = z = x_i = x_i = 0 \}$. The blow up computation is the same as in Step 2.

As before we see that at each step the conditions (12.1–2) hold, hence $D(E)$ is unchanged.

Outcome. After these blow-ups we have a triple $(Y, E, F)$ such that at non-resolved points the local equations are

$$
\prod_{i \in I_p} x_i = t \cdot y, \quad \prod_{i \in I_p} x_i = t \cdot z_1 \quad \text{or} \quad \prod_{i \in I_p} x_i = t. \quad (12.4)
$$

As before, the $y$ and $z$ variables have different meaning, but we can rename $z_1$ as $y$. Thus we have only one non-resolved local form left: $\prod x_i = ty$.

16 (Resolving the multiplicity 2 part). Here we have a local equation $x_i \cdots x_i = ty$ where $d \geq 2$. We would like to blow up $(x_i = y = 0)$, but, as we noted, this subvariety is not globally defined. However, a rare occurrence helps us out. Usually the blow-up of a smooth subvariety determines its center uniquely. However, this is not the case for codimension 1 centers. Thus we could get a globally well defined blow-up even from centers that are not globally well defined.

Note that the inverse of $(x_i = y = 0)$ in the local Picard group of $Y$ is $E_{i_1} = (x_i = t = 0)$, which is globally defined. Thus

$$
\text{Proj}_Y \sum_{m \geq 0} \mathcal{O}_Y(mE_{i_1})
$$

where the union runs through all 2-element subsets $\{i, i'\} \subseteq I$. Pick an irreducible component of it, call it $W = \{ z_{j_1} = z_{j_2} = x_i = x_i = 0 \}$. Pick an irreducible component of it, call it $W$. Set $I_p' := I_p \setminus \{ i_1, i_2 \}$, $J' := J_p \setminus \{ j_1, j_2 \}$ and we write the local equations as
$$
x_i x_\iota : = 1.
$$

There are two types of local charts on the blow-up.

1. In the chart $(x_1, x_2, z_1, z_2) = (x_1', x_1', x_1', x_1', x_2', x_2', x_2', x_1')$ the new local equation is
$$
x_i' x_\iota : = 1.
$$

and the new degree is $(\deg_y(p) - 1, \deg_y(p), \deg_z(p))$. A similar chart is obtained by interchanging the subscripts $i_1, i_2$.

2. In the chart $(x_1, x_2, z_1, z_2) = (x_1', x_1', x_1', x_1', x_2', x_2', x_2', x_1')$ the new local equation is
$$
x_i' x_\iota : = 1.
$$

The new degree is $(\deg_y(p), \deg_y(p), \deg_z(p) - 1)$.

A similar chart is obtained by interchanging the subscripts $j_1, j_2$.

By repeated application of these two steps we are reduced to the case where $\deg_y(p) \leq 1$ at all non-resolved points.

Step 3. Assume that there is a non-resolved point with $\deg_y(p) = \deg_z(p) = 1$. The singular set of $Y$ is
$$
\bigcup_{(i, i')} \{ y = z = x_i = x_i' = 0 \}.
$$

Pick an irreducible component of it, call it $W(I_1, i_2) := \{ y = z = x_i = x_i = 0 \}$. The blow up computation is the same as in Step 2.

As before we see that at each step the conditions (12.1–2) hold, hence $D(E)$ is unchanged.

Outcome. After these blow-ups we have a triple $(Y, E, F)$ such that at non-resolved points the local equations are

$$
\prod_{i \in I_p} x_i = t \cdot y, \quad \prod_{i \in I_p} x_i = t \cdot z_1 \quad \text{or} \quad \prod_{i \in I_p} x_i = t. \quad (12.4)
$$

As before, the $y$ and $z$ variables have different meaning, but we can rename $z_1$ as $y$. Thus we have only one non-resolved local form left: $\prod x_i = ty$.

16 (Resolving the multiplicity 2 part). Here we have a local equation $x_i \cdots x_i = ty$ where $d \geq 2$. We would like to blow up $(x_i = y = 0)$, but, as we noted, this subvariety is not globally defined. However, a rare occurrence helps us out. Usually the blow-up of a smooth subvariety determines its center uniquely. However, this is not the case for codimension 1 centers. Thus we could get a globally well defined blow-up even from centers that are not globally well defined.

Note that the inverse of $(x_i = y = 0)$ in the local Picard group of $Y$ is $E_{i_1} = (x_i = t = 0)$, which is globally defined. Thus

$$
\text{Proj}_Y \sum_{m \geq 0} \mathcal{O}_Y(mE_{i_1})
$$
is well defined, and locally it is isomorphic to the blow-up $B_{(x_i = y = 0)} Y$. (A priori, we would need to take the normalization of $B_{(x_i = y = 0)} Y$, but it is actually normal.) Thus we have 2 local charts.

(1) \((x_i, y) = (x'_i, y'x'_i)\) and the new local equation is \((x_{i_2} \cdots x_{i_d} = ty')\). The new local degree is \((d-1,1,0)\).

(2) \((x_i, y) = (x'_i, y', y)\) and the new local equation is \((x'_i \cdot x_{i_2} \cdots x_{i_d} = t)\). The new local degree is \((d,0,0)\).

**Outcome.** After all these blow-ups we have a triple \((Y, \sum_{i \in I} E_i, \sum_{j \in J} a_j F_j)\) where \(\sum_{i \in I} E_i\) is a simple normal crossing divisor and \(Y\) is smooth along \(\sum_{i \in I} E_i\).

This completes the proof of Proposition 12. \(\Box\)

17 (Proof of Theorem 2). Assume that \(T\) is \(\mathbb{Q}\)-acyclic. Then, by 7 there is a simple normal crossing variety \(Z_T\) such that \(H^i(Z_T, \mathcal{O}_{Z_T}) = 0\) for \(i > 0\). Then [Kol11, Prop.9] shows that, for \(L\) sufficiently ample, the singularity \((0 \in X_T)\) constructed in 8 and 9 is rational. By 12 we conclude that \(DR(0 \in X_T) \cong D(Z_T)\) is homotopy equivalent to \(T\).

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Princeton University, Princeton NJ 08544-1000
kollar@math.princeton.edu