A CONTINUOUS TIME TUG-OF-WAR GAME FOR PARABOLIC $p(x,t)$-LAPLACE TYPE EQUATIONS

JOONAS HEINO

Abstract. We formulate a stochastic differential game in continuous time that represents the unique viscosity solution to a terminal value problem for a parabolic partial differential equation involving the normalized $p(x,t)$-Laplace operator. Our game is formulated in a way that covers the full range $1 < p(x,t) < \infty$. Furthermore, we prove the uniqueness of viscosity solutions to our equation in the whole space under suitable assumptions.

1. Introduction

In this paper, we study a two-player zero-sum stochastic differential game (SDG) that is defined in terms of an $n$-dimensional state process, and is driven by a $2n$-dimensional Brownian motion for $n \geq 2$. The players’ impacts on the game enter in both a diffusion and a drift coefficient of the state process. The game is played in $\mathbb{R}^n$ until a fixed time $T > 0$, and at that time a player pays the other player the amount given by a pay-off function $g$ at a current point. We show that the game has a value, and characterize the value function of the game as a viscosity solution $u$ to a parabolic terminal value problem

$$\begin{cases}
\partial_t u(x,t) + \nabla_N^{p(x,t)} u(x,t) + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i} (x,t) = ru(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\
u(x,T) = g(x) & \text{on } \mathbb{R}^n
\end{cases}$$

for $\mu \in \mathbb{R}^n$ and $r \geq 0$. Moreover, we show that the viscosity solution $u$ is unique under suitable assumptions. Here, the normalized $p(x,t)$-Laplacian is defined as

$$\nabla_N^{p(x,t)} u(x,t) := \left( \frac{p(x,t) - 2}{|Du(x,t)|^2} \right) \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) \frac{\partial u}{\partial x_i}(x,t) \frac{\partial u}{\partial x_j}(x,t) + \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x,t)$$

for $x \in \mathbb{R}^n$ and $t \in (0,T)$, provided that $Du(x,t) \neq 0$. The vector $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)^T$ is the gradient with respect to $x$, and the function

\[ Date: July 10, 2018. \]
\[ 2010 Mathematics Subject Classification. 91A15, 49L25, 35K65. \]
\[ Key words and phrases. normalized $p(x,t)$-Laplacian, parabolic partial differential equation, stochastic differential game, viscosity solution. \]
$p : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is Lipschitz continuous with values on a compact set $[p_{\min}, p_{\max}]$ for constants $1 < p_{\min} \leq p_{\max} < \infty$.

This work is motivated by a connection between $p$-harmonic functions and a stochastic game called tug-of-war, see the seminal papers [PSSW09, PS08, MPR12] in the elliptic case and [MPR10] in the parabolic case. Furthermore, Atar and Budhiraja [AB10] formulated a game in continuous time representing the unique viscosity solution to a certain elliptic inhomogeneous problem with the normalized $\infty$-Laplacian. The contribution of our work is the identification of a game in continuous time that corresponds to the parabolic normalized $p(\cdot, \cdot)$-Laplace operator. Moreover, our game covers the full range $1 < p(\cdot, \cdot) < \infty$. In the game formulation, we increased the dimension of the Brownian motion that drives our state process to let $p$ also get values below two. This approach is new even for constant $p$.

In this work, main difficulties arise from the variable dependence in $p$ and from the unboundedness of the game domain. It is simpler to approximate viscosity solutions and to prove comparison principles to our equations without the variable dependence in $p$. Furthermore, we overcome the loss of translation invariance on the SDG by utilizing the Hölder continuity of solutions to Bellman-Isaacs type equations. Because the game domain is unbounded, we need to eliminate solutions growing too fast when $|x| \to \infty$. We show that under a linear growth bound a viscosity solution to our equation is unique.

1.1. SDG formulation. We fix a time $T > 0$, and model $X(t), t \in [0, T]$ by a stochastic differential equation

$$
\begin{cases}
    dX(s) = \rho(G(s))\, ds + \sigma(X(s), G(s))\, d\overline{W}(s) \\
    X(0) = x,
\end{cases}
$$

where $x \in \mathbb{R}^n$, and $\overline{W}$ is a $2n$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ satisfying the standard assumptions. In our model, there are two competing players. We let

$$G(s) = (a(s), b(s), c(s), d(s)),$$

where

$$a(s), b(s) \in \mathbb{S}^{n-1}, c(s), d(s) \in [0, \infty), s \in [0, T]$$

are progressively measurable stochastic processes with respect to the filtration $\{\mathcal{F}_s\}$. Throughout the paper, $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$. The pairs $(a(s), c(s))$ and $(b(s), d(s))$ are called controls of the players. Roughly speaking, $a(s)$ and $b(s)$ are the directions, and $c(s)$ and $d(s)$ are the lengths taken by the players at the time $s$. Furthermore, let $\mu \in \mathbb{R}^n$. Then, for $s \in [0, T]$, we define the function $\rho$ in (1.1) by

$$\rho(G(s)) = \mu + (c(s) + d(s))(a(s) + b(s)).$$
Recall that $p : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a Lipschitz continuous function taking values on the compact set $[p_{\min}, p_{\max}]$. We define the $n \times 2n$ matrix $\sigma$ in (1.1) to be

$$
\sigma(X(s), G(s)) = \left[ a(s)\sqrt{p(X(s), s) - 1}; P_{a(s)}^\perp; b(s)\sqrt{p(X(s), s) - 1}; P_{b(s)}^\perp \right],
$$

where the $n \times (n - 1)$ matrices $P_{a(s)}^\perp$ and $P_{b(s)}^\perp$ are defined such that the matrices $P_{a(s)}^\perp(P_{a(s)}^\perp)^T$ and $P_{b(s)}^\perp(P_{b(s)}^\perp)^T$ are projections to the $(n - 1)$-dimensional hyperspaces orthogonal to the vectors $a(s)$ and $b(s)$ at the time $s$, respectively. For more details on $\sigma$, see Section 2 below.

We only allow players to use admissible controls. Roughly speaking, a player initially declares a bound $C < \infty$, and then plays as to keep $c(s) \leq C$ for all $s$, where $(a(s), c(s))$ is the admissible control of the player.

**Definition 1.1.** Given a control $A := (a(s), c(s))$, that is, a progressively measurable process with respect to the Brownian filtration $\{\mathcal{F}_s\}$ with $a(s) \in \mathbb{S}^{n-1}$, $c(s) \in [0, \infty)$, and $s \in [0, T]$, we set

$$
\Lambda(A) = \text{ess sup}_{\omega \in \Omega} \sup_{s \in [0, T]} c(s) \in [0, \infty].
$$

(1.2)

Then, we define the set of admissible controls by

$$
\mathcal{A}C = \{A \text{ control} : \Lambda(A) < \infty\}.
$$

Given an admissible control $A$, we say that the compact set $\mathbb{S}^{n-1} \times [0, \Lambda(A)]$ is an action set. A strategy is a response to the control of the opponent.

**Definition 1.2.** A strategy is a function

$$
S : \mathcal{A}C \to \mathcal{A}C
$$

such that for all $t \in [0, T]$, if

$$
\mathbb{P}(A(s) = \tilde{A}(s) \text{ for a.e. } s \in [0, t]) = 1 \text{ and } \Lambda(A) = \Lambda(\tilde{A}),
$$

then

$$
\mathbb{P}(S(A)(s) = S(\tilde{A})(s) \text{ for a.e. } s \in [0, t]) = 1 \text{ and } \Lambda(S(A)) = \Lambda(S(\tilde{A})).
$$

Given a strategy $S$, we set

$$
\Lambda(S) := \sup_{A \in \mathcal{A}C} \Lambda(S(A)) \in [0, \infty].
$$

(1.3)

Then, we define the set of admissible strategies by

$$
\mathcal{S} = \{S \text{ strategy} : \Lambda(S) < \infty\}.
$$
We define the lower and upper values of the game with the dynamics (1.1) by

\[
U^-(x, t) = \inf_{S \in \mathcal{S}} \sup_{A \in \mathcal{A}} \mathbb{E} \left[ e^{-r(T-t)} g(X(T)) \right], \\
U^+(x, t) = \sup_{S \in \mathcal{S}} \inf_{A \in \mathcal{A}} \mathbb{E} \left[ e^{-r(T-t)} g(X(T)) \right]
\] (1.4)

for all \((x, t) \in \mathbb{R}^n \times [0, T]\), where \(r \geq 0\), and \(g\) is the pay-off function. The game starts at a position \(x\) at a time \(t\), and the expectation \(\mathbb{E}\) is taken with respect to the measure \(\mathbb{P}\). The game is said to have a value at \((x, t)\), if it holds \(U^-(x, t) = U^+(x, t)\).

1.2. Statement of the main results. Let us denote

\[
F((x, t), u(x, t), Du(x, t), D^2 u(x, t)) \\
:= \triangle^N_{p(x,t)} u(x, t) + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i}(x, t) - ru(x, t)
\]

for all \((x, t) \in \mathbb{R}^n \times (0, T)\), where \(D^2 u\) is the matrix consisting of the second order derivatives with respect to \(x\). We consider the terminal value problem

\[
\begin{cases}
\partial_t u + F((x, t), u, Du, D^2 u) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, T) = g(x) & \text{on } \mathbb{R}^n,
\end{cases}
\] (1.5)

where \(g\) is a positive, bounded and Lipschitz continuous function. A common notion of a weak solution to this equation is a viscosity solution. In this paper, we prove the following main result.

**Theorem 1.3.** Let \(g\) be positive, bounded and Lipschitz continuous. Furthermore, let \(U^-\) and \(U^+\) be the lower and upper values of the stochastic differential game defined in (1.4), respectively. Then, the functions \(U^-\) and \(U^+\) are viscosity solutions to (1.5).

For completeness, we show that a viscosity solution to (1.5) is unique under suitable assumptions.

**Theorem 1.4.** Let \(g\) be positive, bounded and Lipschitz continuous. Then, a viscosity solution \(u\) to the equation (1.5) is unique, if \(u\) satisfies a linear growth bound

\[
|u(x, t)| \leq c(1 + |x|)
\] (1.6)

for all \((x, t) \in \mathbb{R}^n \times [0, T]\) and for \(c < \infty\) independent of \(x, t\).

Because \(g\) is bounded, the functions \(U^-\) and \(U^+\) satisfy (1.6). Thus, Theorems 1.3 and 1.4 imply the following.

**Corollary 1.5.** The game has a value at every \((x, t) \in \mathbb{R}^n \times [0, T]\).
As an application, one could study our model in the context of the portfolio option pricing. This would be based on the idea that, in addition to a random noise, the prices of the underlying assets are influenced by the two competing players. Roughly speaking, one can see the players as the issuer and the holder of the corresponding option. The issuer and the holder try, respectively, to manipulate the drifts and the volatilities of the assets to minimize and maximize, respectively, the expected discounted reward at the time $T$. The time $T$ can be interpreted as a maturity; it is the time on which the corresponding financial instrument must either be renewed or it will cease to exist. To some extent, we generalize the model developed by Nyström and Parviainen in [NP17]. Indeed, our contribution is the introduction of a local volatility $p$. The volatility of an asset may vary over the space and the time.

1.3. An outline of the proofs of Theorems 1.3 and 1.4. Our approach is influenced by the papers [Swi96, AB10, NP17]. First, we examine games with uniformly bounded action sets, and in the end, let the uniform bound tend to the infinity. Here, the important step is to connect the value functions under uniformly bounded action sets to the terminal value problems of Bellman-Isaacs type equations

\[
\begin{aligned}
\frac{\partial}{\partial t}u - F_m^-(x, t, u, Du, D^2u) &= 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, T) &= g(x) & \text{on } \mathbb{R}^n,
\end{aligned}
\]  

(1.7)

and

\[
\begin{aligned}
\frac{\partial}{\partial t}u - F_m^+(x, t, u, Du, D^2u) &= 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, T) &= g(x) & \text{on } \mathbb{R}^n.
\end{aligned}
\]  

(1.8)

The exact definitions of $F_m^-$ and $F_m^+$ are given in Section 2 below. Here, $m$ denotes the uniform bound on the controls. The uniqueness of viscosity solutions to (1.7) and (1.8) follows, for example, from [GGIS91, BL08]. Furthermore, the existence of viscosity solutions to the equations (1.7) and (1.8) follows by the construction of suitable barriers (Lemma 2.2) and by the use of Perron’s method.

In Section 3, the main result is Lemma 3.3 in which we show that a lower value function with uniformly bounded controls equals to the unique solution $u_m$ to (1.7). In the proof, we first regularize the solution $u_m$ by sup- and inf-convolutions, and then deduce the equality by utilizing Ito’s formula and passing to limits.

In section 4, we examine the problem (1.5). First, we prove Theorem 1.4. To prove a comparison principle, we double the variables and apply the celebrated theorem of sums, see [CI90]. Because we only consider solutions satisfying a linear growth bound in the whole space, we utilize a quadratic barrier function for the space infinity. Furthermore, we use the Lipschitz
continuity of $p$ to estimate the error coming from a penalty function. To continue, in Lemma 4.5 we show that

$$F_m \rightarrow F$$

as $m \rightarrow \infty$. Furthermore, in Lemma 4.6 we utilize the results of [KS80, Wan92] to show that the family

$$\{u_m : m \geq 1\}$$

is equicontinuous. Finally by the reduction of test functions (Lemma 4.4) and the stability principle for viscosity solutions, we can utilize the Arzelà-Ascoli theorem to find a solution $u$ to (1.5) and a subsequence $(u_{m_j})$ converging uniformly to $u$ as $j \rightarrow \infty$. To complete the proof of Theorem 1.3, we also need the fact that the subsequence of the corresponding lower value functions converges to the lower value function for the game without the uniform bound on the controls. In addition, all the proofs in the context of the equation (1.8) are analogous.

Acknowledgement. The author would like to thank Mikko Parviainen for many discussions and insightful comments regarding this work.

2. Preliminaries

Let $\overline{W} = (W^1, W^2)^T$ be a 2$n$-dimensional Brownian motion such that $W^1 = (W^1_1, \ldots, W^1_n)$ and $W^2 = (W^2_1, \ldots, W^2_n)$ are $n$-dimensional Brownian motions. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ denote a complete filtered probability space with right-continuous filtration supporting the process $\overline{W}$. As mentioned above, we consider the following stochastic differential equation

$$\begin{cases}
    dX(s) = \rho(G(s)) \, ds + \sigma(X(s), G(s)) \, d\overline{W}(s) \\
    X(0) = x
\end{cases}$$

(2.9)

for $s \in [0, T]$, $T > 0$ and $x \in \mathbb{R}^n$ with $G : [0, T] \rightarrow C\mathcal{S}$, $\rho : C\mathcal{S} \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times C\mathcal{S} \rightarrow M^{n \times 2n}$. Here, we define $C\mathcal{S} := S^{n-1} \times S^{n-1} \times [0, \infty) \times [0, \infty)$, where $C\mathcal{S}$ refers to control space. Furthermore, $M^{n \times 2n}$ is the set of $n \times 2n$ matrices.

We are interested in the following form of the functions $G$, $\rho$ and $\sigma$. Let $A_1 := (a(s), c(s))$ and $A_2 := (b(s), d(s))$ be admissible controls of the players in the sense of Definition 1.1, respectively. Furthermore, let $\mu \in \mathbb{R}^n$. Then, for $s \in [0, T]$, we define

$$G(s) = (a(s), b(s), c(s), d(s)),$$

and

$$\rho(G(s)) = \mu + (c(s) + d(s))(a(s) + b(s)).$$

Let $\nu \in S^{n-1}$, and denote the orthogonal complement of $\nu$ by

$$\nu^\perp := \{z \in \mathbb{R}^n : \langle z, \nu \rangle = 0\}.$$
We set $P_{\nu}^\perp$ to be a $n \times (n - 1)$ matrix such that the columns are $p_1^{(\nu)}, \ldots, p_{n-1}^{(\nu)}$, where \{p_1^{(\nu)}, \ldots, p_{n-1}^{(\nu)}\} is a fixed orthonormal basis of $\nu^\perp$.

$P_{\nu}^\perp = [p_1^{(\nu)} \cdots p_{n-1}^{(\nu)}].$

We can define the basis of $\nu^\perp$ such that the function $\nu \mapsto P_{\nu}^\perp$ is continuous.

In addition, let $p : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ be a Lipschitz continuous function such that

\[
 p_{\min} = \inf_{y \in \mathbb{R}^n \times [0, T]} p(y) > 1 \quad \text{and} \quad p_{\max} = \sup_{y \in \mathbb{R}^n \times [0, T]} p(y) < \infty. \tag{2.10}
\]

With respect to the time variable $t$, we only need that $p$ is Hölder continuous for all fixed $x$, but we minimize additional technical difficulties. Now, we define the $n \times 2n$ matrix $\sigma$ to be

\[
 \sigma(X(s), G(s)) = \begin{bmatrix} a(s) \sqrt{p(X(s), s)} - 1; & P_{a(s)}^\perp; & b(s) \sqrt{p(X(s), s)} - 1; & P_{b(s)}^\perp \end{bmatrix}.
\]

By the game dynamics (2.9), we get

\[
 dX_i(s) = \left[ \mu_i + (c(s) + d(s)) \left( a_i(s) + b_i(s) \right) \right] ds
 + \sqrt{p(X(s), s)} - 1 \left( a_i(s) dW_1^1(s) + b_i(s) dW_1^2(s) \right)
 + \sum_{k=2}^n (\overrightarrow{p_i})_{k-1} dW_k^1(s) + \sum_{k=2}^n (\overrightarrow{p_i})_{k-1} dW_k^2(s) \tag{2.11}
\]

for all $i \in \{1, \ldots, n\}$. Here, $(\overrightarrow{p_i})$ denotes the $i$-th row vector of $P_{\nu}^\perp$.

By a strong solution to the stochastic differential equation (2.9), we mean a progressively measurable process $(X(t))$ with respect to the Brownian filtration $\{\mathcal{F}_t\}$ such that the stochastic integral in right-hand side of (2.9) is defined and furthermore, $X(t)$ coincides with the right-hand side of (2.9) for all $t \in [0, T]$ almost surely. In addition, a strong solution is pathwise unique, if any two given solutions $(X(t), Y(t))$ satisfy

\[
 \mathbb{P}\left( \sup_{t \in [0, T]} |X(t) - Y(t)| > 0 \right) = 0.
\]

Let us denote by $| \cdot |_F$ the Frobenius norm

\[
 ||\sigma||_F := \sqrt{\text{trace}(\sigma \sigma^T)}
\]

for all $\sigma \in M^{n \times 2n}$. Then by (2.10), it holds

\[
 \mathbb{E} \int_0^T ||\sigma(X(l), G(l))||_F^2 \, dl \leq 2T(p_{\max} - 2 + n) < \infty. \tag{2.12}
\]
Hence, the stochastic integral in the right-hand side of (2.9) is well defined. Furthermore, the functions $\rho$ and $\sigma$ are continuous with respect to the control parameters. Because the controls of the players are admissible, it holds

$$
\mathbb{E} \int_0^T \left| \rho(G(s)) \right|^2 ds \leq (|\mu| + 2(\Lambda(A_1) + \Lambda(A_2)))^2 T < \infty \quad (2.13)
$$

for $\Lambda(A_1), \Lambda(A_2) < \infty$, where $\Lambda(\cdot)$ is defined in (1.2). Moreover, we can estimate

$$
\|\sigma(x, G(t)) - \sigma(y, G(t))\|_F \leq \sqrt{2}\left| \sqrt{p(x, t)} - 1 - \sqrt{p(y, t)} - 1 \right| \leq \frac{|p(x, t) - p(y, t)|}{\sqrt{2p_{\min}}} \leq \frac{L_p}{\sqrt{2p_{\min}}} |x - y|
$$

for all $x, y \in \mathbb{R}^n$ and $t \in [0, T]$ with $L_p$ denoting the Lipschitz constant of $p$. Therefore by combining this, (2.12), (2.13) and [Kry09, Theorem 2.5.7], the SDE (2.9) admits a pathwise unique strong solution.

Throughout, we denote by $\|\cdot\|$ a matrix norm

$$
\|M\| := \sup_{|x| = 1} |\langle Mx, x \rangle|
$$

for all $n \times n$ matrices $M$. Furthermore, $S(n)$ denotes the set of all symmetric $n \times n$ matrices, $I$ is the $n \times n$ identity matrix, and for $\xi \in \mathbb{R}^n$, we denote by $\xi \otimes \xi$ the $n \times n$ matrix for which $(\xi \otimes \xi)_{ij} = \xi_i \xi_j$. A function $\zeta : [0, \infty) \to [0, \infty)$ is said to be a modulus, if it is continuous, nondecreasing, and satisfies $\zeta(0) = 0$.

2.1. Viscosity solutions to Bellman-Isaacs equations with uniformly bounded action sets. We define $\Phi : CS \times \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ through

$$
\Phi(a, b, c, d; (x, t), \nu, M) = -\text{trace}\left( A_{a, b}^{(x, t)} M \right) - (c + d) \langle a + b, \nu \rangle - \langle \mu, \nu \rangle,
$$

where

$$
A_{a, b}^{(x, t)} := \frac{1}{2}\left( p(x, t) - 2 \right) (a \otimes a + b \otimes b) + I. \quad (2.14)
$$

Observe that the matrix $A_{a, b}^{(x, t)}$ is symmetric with eigenvalues between the values

$$
\lambda := \min\{1, p_{\min} - 1\} \quad \text{and} \quad \Lambda := \max\{1, p_{\max} - 1\}. \quad (2.15)
$$

Given $m \in \{1, 2, \ldots \}$, we let

$$
\mathcal{H}_m := \mathbb{S}^{n-1} \times [0, m],
$$
and define $F_m^-, F_m^+ : \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ through

$$F_m^-((x, t), \xi, \nu, M) = \inf_{(a, c) \in \mathcal{H}_m} \sup_{(b, d) \in \mathcal{H}_m} \Phi(a, b, c, d; (x, t), \nu, M) + r \xi,$$

$$F_m^+((x, t), \xi, \nu, M) = \sup_{(b, d) \in \mathcal{H}_m} \inf_{(a, c) \in \mathcal{H}_m} \Phi(a, b, c, d; (x, t), \nu, M) + r \xi$$

for $r \geq 0$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive bounded Lipschitz function such that

$$\sup_{x \in \mathbb{R}^n} g(x) + \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|g(x) - g(y)|}{|x - y|} < L_g$$

(2.16)

for some $L_g < \infty$. We study terminal value problems

$$\begin{aligned}
\begin{cases}
\partial_t u - F_m^-((x, t), u, Du, D^2u) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, T) = g(x) & \text{on } \mathbb{R}^n
\end{cases}
\end{aligned}$$

(2.17)

and

$$\begin{aligned}
\begin{cases}
\partial_t u - F_m^+((x, t), u, Du, D^2u) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, T) = g(x) & \text{on } \mathbb{R}^n.
\end{cases}
\end{aligned}$$

(2.18)

A common notion of weak solutions to these equations is viscosity solutions. We only consider solutions $u$ which satisfy a linear growth condition

$$|u(x, t)| \leq c(1 + |x|)$$

(2.19)

for all $(x, t) \in \mathbb{R}^n \times [0, T]$ and for some $c < \infty$ independent of $x, t$. We prove that there exists a unique viscosity solution to the equation (2.17) satisfying the condition (2.19). We omit the proof for (2.18), because it is analogous. The proofs are based on the comparison principle and Perron’s method.

**Definition 2.1.** (i) A lower semicontinuous function $\overline{u}_m : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a viscosity supersolution to (2.17), if it satisfies (2.19),

$$\overline{u}_m(x, T) \geq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and for all $\phi \in C^{2,1}(\mathbb{R}^n \times (0, T))$ such that

- $\overline{u}_m(x_0, t_0) = \phi(x_0, t_0)$
- $\overline{u}_m(x, t) > \phi(x, t)$ for all $(x, t) \neq (x_0, t_0)$

it holds

$$\partial_t \phi(x_0, t_0) - F_m^-((x_0, t_0), \phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \leq 0.$$

(ii) An upper semicontinuous function $\underline{u}_m : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a viscosity subsolution to (2.17), if it satisfies (2.19),

$$\underline{u}_m(x, T) \leq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and for all $\phi \in C^{2,1}(\mathbb{R}^n \times (0, T))$ such that
\begin{itemize}
  \item $u_m(x_0, t_0) = \phi(x_0, t_0)$
  \item $u_m(x, t) < \phi(x, t)$ for all $(x, t) \neq (x_0, t_0)$
\end{itemize}

it holds
\[
\partial_t \phi(x_0, t_0) - F_m^-((x_0, t_0), \phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \geq 0.
\]

(iii) If a function $u_m : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a viscosity supersolution and a subsolution to (2.17), then $u_m$ is a viscosity solution to (2.17).

Observe that we require the growth condition (2.19) as a standing assumption for viscosity super- and subsolutions. We start with the following lemma.

\textbf{Lemma 2.2.} Let $g \in \mathbb{R}^n$, $0 < \varepsilon < 1$, and let $L_g$ be the constant in (2.16) for $g$. Then, the functions
\[
\overline{a}(x, t) = g(y) + \frac{A}{\varepsilon^{1/2}}(T - t) + 2L_g(|x - y|^2 + \varepsilon)^{1/2},
\]
\[
\underline{a}(x, t) = g(y) - \frac{A}{\varepsilon^{1/2}}(T - t) - 2L_g(|x - y|^2 + \varepsilon)^{1/2}
\]
are viscosity super- and subsolutions to (2.17), respectively, if we choose $A$, independent of $y, \varepsilon$ and $m$, large enough.

\textbf{Proof.} Because $g$ is Lipschitz continuous with (2.16), we get
\[
\underline{a}(x, T) \leq g(x) \leq \overline{a}(x, T)
\]
for all $x \in \mathbb{R}^n$. Furthermore, $\underline{a}$ and $\overline{a}$ satisfy (2.19). First, we prove that $\overline{a}$ is a supersolution. To establish this, since $\overline{a}$ is a smooth function, we need to show that
\[
\partial_t \overline{a}(x, t) - F_m^-((x, t), \overline{a}(x, t), D\overline{a}(x, t), D^2\overline{a}(x, t)) \leq 0
\]
for all $(x, t) \in \mathbb{R}^n \times (0, T)$. Let $(x, t) \in \mathbb{R}^n \times (0, T)$. By a direct calculation, it holds
\[
D\overline{a}(x, t) = 2L_g(|x - y|^2 + \varepsilon)^{-1/2}(x - y)
\]
and
\[
D^2\overline{a}(x, t) = 2L_g(|x - y|^2 + \varepsilon)^{-1/2}\left(I - \frac{(x - y) \otimes (x - y)}{|x - y|^2 + \varepsilon}\right).
\]

Thus, we can estimate
\[
- \text{trace}\left( A^{(x, t)}_{a,b} D^2 \overline{a}(x, t) \right) = 2L_g(|x - y|^2 + \varepsilon)^{-1/2}.
\]
\[
\left\{ \text{trace}\left( A^{(x, t)}_{a,b} \left(|x - y|^2 + \varepsilon\right)^{-1}(x - y) \otimes (x - y) - I \right) \right\}
\]
\[
\geq -2nA_L L_g(|x - y|^2 + \varepsilon)^{-1/2}
\]
for all $a, b \in S^{n-1}$. Furthermore, we have $\partial_t \overline{a}(x, t) = -A\varepsilon^{-1/2}$. 

We can assume \( x \neq y \), because otherwise the next term below is zero. It holds
\[
\inf_{(a,c)\in H_m} \sup_{(b,d)\in H_m} -(c+d)\langle a+b, D\pi(x,t) \rangle \\
\geq 2L_g(|x-y|^2 + \varepsilon)^{-1/2} \inf_{(a,c)\in H_m} -c\langle a - (x-y)/|x-y|, x-y \rangle \\
\geq 0.
\]
In addition, we can estimate
\[
|\langle \mu, D\pi(x,t) \rangle| \leq 2L_g|\mu||x-y|(|x-y|^2 + \varepsilon)^{-1/2} \leq 2L_g|\mu|.
\]
By combining our estimates above, we have
\[
\partial_t \pi(x,t) - F_m^-((x,t), \pi(x,t), D\pi(x,t), D^2\pi(x,t)) \\
\leq -A\varepsilon^{-1/2} + 2n\Lambda L_g(|x-y|^2 + \varepsilon)^{-1/2} + 2L_g|\mu| - r\pi(x,t) \\
\leq \varepsilon^{-1/2}( -A + 2n\Lambda L_g) + 2L_g|\mu|.
\]
Hence, if we choose
\[
A = 4L_g(n\Lambda + |\mu|),
\]
we can conclude that \( \pi \) is a supersolution to (2.17).

The proof that \( a \) is a subsolution to (2.17) is very similar to the above. We need to show that
\[
\partial_t a(x,t) - F_m^+((x,t), a(x,t), Da(x,t), D^2a(x,t)) \geq 0.
\]
Observe that for \( x \neq y \), we have this time
\[
\inf_{(a,c)\in H_m} \sup_{(b,d)\in H_m} -(c+d)\langle a+b, D\pi(x,t) \rangle \\
\leq 2L_g(|x-y|^2 + \varepsilon)^{-1/2} \sup_{(b,d)\in H_m} -d\langle (x-y)/|x-y| + b, x-y \rangle \\
\leq 0
\]
by estimating the infimum instead of the supremum. Thus, by repeating the argument above, we have
\[
\partial_t a(x,t) - F_m^+((x,t), a(x,t), Da(x,t), D^2a(x,t)) \\
\geq \varepsilon^{-1/2}(A - 2n\Lambda L_g) - 2L_g|\mu| - ra(x,t).
\]
Recall the assumption (2.16) implying \( -ra(x,t) \geq -rL_g \). Therefore by adjusting the constant \( A \) large enough, we can conclude that \( a \) is a subsolution to \( (2.17) \).

A useful tool for us is the comparison principle.

**Lemma 2.3.** Let \( u_m \) and \( \pi_m \) be continuous viscosity sub- and supersolutions to \((2.17)\) in the sense of Definition 2.1, respectively. Then, it holds
\[
u_m(x,t) \leq \pi_m(x,t)
\]
for all \((x, t) \in \mathbb{R}^n \times [0, T]\).

The proof of the comparison principle can be found from [BL08], see also [GGIS91]. Now, Lemmas \(2.2\) and \(2.3\) applied to Perron’s method yield the following result.

**Proposition 2.4.** There exists a unique viscosity solution \(u_m\) to \((2.17)\) in the sense of Definition 2.1.

Observe that by comparison with a sufficiently large constant, the unique solution \(u_m\) to \((2.17)\) is not merely of linear growth \((2.19)\). It is even bounded.

### 3. The SDG with uniformly bounded action sets

In this section, we examine the game dynamics under uniform bounds on the action sets of the players. In particular, we prove that the unique solution to \((2.17)\) equals the lower value function of the game under the uniform bound. For the upper value function, the proof is similar.

**Definition 3.1.** Let \(\mathcal{AC}\) be the set of admissible controls, and let \(\mathcal{S}\) be the set of admissible strategies in the sense of Definitions 1.1 and 1.2, respectively. For \(m \in \{1, 2, \ldots\}\), we set

\[
\mathcal{AC}_m := \{A \in \mathcal{AC} : \Lambda(A) \leq m\},
\]

\[
\mathcal{S}_m := \{S \in \mathcal{S} : \Lambda(S) \leq m\},
\]

where \(\Lambda(\cdot)\) is defined in \((1.2)\) and \((1.3)\).

Let \(m \in \{1, 2, \ldots\}\), and assume that the players choose their controls and strategies from the sets \(\mathcal{AC}_m\) and \(\mathcal{S}_m\), respectively. As before, the SDE \((2.9)\) admits a pathwise unique strong solution. We define the lower and upper value functions of the game with controls in \(\mathcal{AC}_m\) and strategies in \(\mathcal{S}_m\) by setting

\[
U_m^-(x, t) = \inf_{S \in \mathcal{S}_m} \sup_{A \in \mathcal{AC}_m} \mathbb{E}\left[e^{-r(T-t)}g(X(T))\right],
\]

\[
U_m^+(x, t) = \sup_{S \in \mathcal{S}_m} \inf_{A \in \mathcal{AC}_m} \mathbb{E}\left[e^{-r(T-t)}g(X(T))\right]
\]

for all \((x, t) \in \mathbb{R}^n \times [0, T]\), where \(g\) is the pay-off \((2.16)\). The game starts at \(x\) at a time \(t\), and the expectation \(\mathbb{E}\) is taken with respect to the measure \(\mathbb{P}\).

In Lemma 3.5 below, we assume that the solution \(u_m\) to \((2.17)\) is twice differentiable and that the solution and its derivatives of first and second order are Lipschitz continuous. Hence, we first study the so called sup- and inf-convolutions of the function \(u_m\). In particular, for a large \(j \in \mathbb{N}\), let us
denote \( T_j := T - j^{-1} \) and \( R_j^n := \mathbb{R}^n \times [j^{-1}, T_j] \). Then for \( j \) fixed and \( \varepsilon > 0 \) small, we define

\[
    u_\varepsilon(x, t) = \sup_{(z, s) \in \mathbb{R}^n \times [0, T]} \left( u_m(z, s) - \frac{1}{2\varepsilon}((t - s)^2 + |x - z|^2) \right)
\]

whenever \((x, t) \in R_j^n\). The sup-convolution \( u_\varepsilon \) has well-known properties. Indeed, \( u_\varepsilon \) is locally Lipschitz continuous, semiconvex and \( u_\varepsilon \searrow u_m \) as \( \varepsilon \to 0 \), see for example [CIL92]. Moreover, \( u_\varepsilon \) yields a good approximation of \( u_m \) in the viscosity sense. The proof of the following lemma follows [Ish95], where they consider an elliptic case. For the benefit of the reader, we give the proof in our parabolic setting.

**Lemma 3.2.** Let \( u_m \) be a viscosity solution to (2.17), and let \( u_\varepsilon \) be the sup-convolution of \( u_m \). Then for \( \varepsilon \) small enough, it holds

\[
    F_m^-(x, t, u_\varepsilon(x, t), Du_\varepsilon(x, t), D^2u_\varepsilon(x, t)) \leq \partial_t u_\varepsilon(x, t) + \zeta(\varepsilon)
\]

for a.e. \((x, t) \in R_j^n\) with a bounded modulus of continuity \( \zeta(\varepsilon) \).

**Proof.** By the comparison principle and the assumption (2.16) on \( g \), it holds \( 0 \leq u_m \leq L_g \). Therefore for all \((x, t) \in R_j^n\) and \( \varepsilon > 0 \) small enough, there exists a point \((x^*, t^*) \in \mathbb{R}^n \times [0, T]\), where the supremum used in the definition of \( u_\varepsilon \) is obtained. In particular, it holds

\[
    0 \leq u_m(x, t) \leq u_\varepsilon(x, t) \leq L_g - \frac{1}{2\varepsilon}((t - t^*)^2 + |x - x^*|^2).
\]

Hence, this yields \(|t - t^*| < j^{-1}\), if \( \varepsilon < 1/(2L_gj^2) \).

By the Lipschitz continuity and the semiconvexity of \( u_\varepsilon \), it holds

\[
    u_\varepsilon(z, s) \leq u_\varepsilon(x, t) + \partial_t u_\varepsilon(x, t)(s - t) + \langle Du_\varepsilon(x, t), z - x \rangle
    + \frac{1}{2}\langle D^2u_\varepsilon(x, t)(z - x), z - x \rangle + o(|s - t| + |z - x|^2)
\]

for a.e. \((x, t) \in R_j^n\) as \((z, s) \to (x, t)\), see [Jen88, Lemmas 3.3 and 3.15]. Here, we also applied the fundamental Aleksandrov’s theorem for convex functions, see for example [EG92, Theorem 6.4.1]. Moreover, the estimate (3.21) implies that we can choose \((x^*, t^*)\) such that

\[
    x^* = x + \varepsilon Du_\varepsilon(x, t),
    t^* = t + \varepsilon \partial_t u_\varepsilon(x, t)
\]

for a.e. \((x, t) \in R_j^n\), see [CIL92, Lemma A.5] or [Kat15, Theorem 4.7]. Let \((x, t) \in R_j^n\) such that (3.21) holds. We define \( v : R_j^n \to \mathbb{R} \) through

\[
    v(z, s) = \partial_t u_\varepsilon(x, t)(s - t) + \langle Du_\varepsilon(x, t), z - x \rangle
    + \frac{1}{2}\langle D^2u_\varepsilon(x, t)(z - x), z - x \rangle
\]
for \((z, s) \in \mathbb{R}^n\). We want to find a local maximum of a function at \((x^*, t^*, x, t)\) up to an error in order to use the parabolic theorem of sums. Because it holds \(v(x, t) = 0\) and
\[
\begin{align*}
    u_m(y, l) - \frac{1}{2 \varepsilon}((l - s)^2 + |y - z|^2) &\leq u_\varepsilon(z, s)
\end{align*}
\]
for all \((z, s), (y, l) \in \mathbb{R}^n\); we can estimate by (3.21)
\[
\begin{align*}
    u_m(y, l) - v(z, s) - \frac{1}{2 \varepsilon}((l - s)^2 + |y - z|^2) &\leq u_m(x^*, t^*) - v(x, t) - \frac{1}{2 \varepsilon}((t - t^*)^2 + |x - x^*|^2) \\
    &\quad + o(|s - t| + |z - x|^2)
\end{align*}
\]
for any \((y, l) \in \mathbb{R}^n\) as \((z, s) \to (x, t)\). By using this inequality, we can deduce
\[
\begin{align*}
    u_m(y, l) - v(z, s) &\leq u_m(x^*, t^*) - v(x, t) + \frac{1}{\varepsilon}(x^* - x, y - x^*) + \frac{1}{\varepsilon}(t^* - t)(l - t^*) \\
    &\quad + \frac{1}{\varepsilon}(x^* - x, z - x) + \frac{1}{\varepsilon}(t^* - t)(s - t) + \frac{1}{2 \varepsilon}(|y - x^*|^2 + |z - x|^2) \\
    &\quad - \frac{1}{\varepsilon}(y - x^*, z - x) + o(|s - t| + |l - t^*| + |z - x|^2)
\end{align*}
\]
for all \(y \in \mathbb{R}^n\) as \((z, s, l) \to (x, t, t^*)\). This is true, because by direct calculations it holds
\[
\begin{align*}
    \frac{1}{2 \varepsilon}((l - s)^2 - (t - t^*)^2) &\leq \frac{1}{2 \varepsilon}\left((t - s + l - t^*)^2 - 2(t^* - t)^2 + 2(t^* - t)(l - s)\right) \\
    &\leq \frac{1}{\varepsilon}(t^* - t)(l - t^*) + \frac{1}{\varepsilon}(t - t^*)(s - t) + o(|s - t| + |l - t^*|)
\end{align*}
\]
as \((s, l) \to (t, t^*)\) and
\[
\begin{align*}
    \langle x^* - x, y - x^* \rangle + \langle x - x^*, z - x \rangle + \frac{1}{2}(|y - x^*|^2 + |z - x|^2) - \langle y - x^*, z - x \rangle &\leq \frac{1}{2}(|y - z|^2 + |x - x^*|^2)
\end{align*}
\]
for all \(y, z \in \mathbb{R}^n\).

For the following notation and use of the parabolic theorem of sums, we refer the reader to [CIL92], see also [Kat15]. By the estimate (3.23), it holds
\[
\begin{align*}
    \left(\frac{1}{\varepsilon}(x^* - x), \frac{1}{\varepsilon}(t^* - t), \frac{1}{\varepsilon}(x - x^*), \frac{1}{\varepsilon}(t - t^*), \frac{1}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}\right) \\
    \in \mathcal{P}^{2,+}(u_m(x^*, t^*) - v(x, t)).
\end{align*}
\]
Thus by [Kat15, Theorem 6.7], there exist symmetric matrices $Y := Y(\varepsilon)$ and $Z := Z(\varepsilon)$ such that
\[
\left(\frac{1}{\varepsilon}(t^* - t), \frac{1}{\varepsilon}(x^* - x), Y\right) \in \mathcal{P}^2_+ u_m(x^*, t^*)
\]
and
\[
\left(\frac{1}{\varepsilon}(t^* - t), \frac{1}{\varepsilon}(x^* - x), Z\right) \in \mathcal{P}^2_- v(x, t)
\]
and
\[
\begin{bmatrix} Y & 0 \\ 0 & -Z \end{bmatrix} \leq \frac{3}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.
\] (3.24)

Therefore, because $u_m$ is a subsolution, this and (3.22) yield
\[
F^-_m((x^*, t^*), u_m(x^*, t^*), Du_\varepsilon(x, t), Y) \leq \partial_t u_\varepsilon(x, t).
\] (3.25)

Furthermore, since $D^2 v(x, t) = D^2 u_\varepsilon(x, t)$, the degenerate ellipticity of $F^-_m$ implies
\[
F^-_m((x, t), u_\varepsilon(x, t), Du_\varepsilon(x, t), D^2 u_\varepsilon(x, t)) 
\leq F^-_m((x, t), u_\varepsilon(x, t), Du_\varepsilon(x, t), Z).
\]

By combining this and (3.25), the proof is complete, if we can show that there exists a modulus $\zeta$ such that
\[
F^-_m((x, t), u_\varepsilon(x, t), Du_\varepsilon(x, t), D^2 u_\varepsilon(x, t), Z) \leq F^-_m((x^*, t^*), u_m(x^*, t^*), Du_\varepsilon(x, t), Y) + \zeta(\varepsilon).
\] (3.26)

We prove this inequality by utilizing (3.24).

Let $a, b \in S^{n-1}$. We multiply from the left both sides in (3.24) by
\[
\begin{bmatrix} \mathcal{A}^{(x^*, t^*)}_{a,b} & \mathcal{A}^{(x, t), (x^*, t^*)}_{a,b} \\ \mathcal{A}^{(x, t), (x^*, t^*)}_{a,b} & \mathcal{A}^{(x, t)}_{a,b} \end{bmatrix},
\]
where
\[
\mathcal{A}^{(x, t), (x^*, t^*)}_{a,b} := \frac{1}{2} \left( \sqrt{p(x^*, t^*)} - 1 - \sqrt{p(x, t)} - 1 \right) (a \otimes a + b \otimes b) + I,
\]
and the matrices $\mathcal{A}^{(x, t)}_{a,b}$ and $\mathcal{A}^{(x^*, t^*)}_{a,b}$ are defined in (2.14). Then by taking traces and observing
\[
\text{trace}(a \otimes a + b \otimes b) = 2,
\]
we get
\[
- \text{trace} \left( \mathcal{A}^{(x, t)}_{a,b} Z \right) + \text{trace} \left( \mathcal{A}^{(x^*, t^*)}_{a,b} Y \right)
\leq \frac{3}{\varepsilon} \left( \text{trace} \left( \mathcal{A}^{(x^*, t^*)}_{a,b} + \mathcal{A}^{(x, t)}_{a,b} \right) - 2 \text{trace} \left( \mathcal{A}^{(x, t), (x^*, t^*)}_{a,b} \right) \right)
\leq \frac{3}{\varepsilon} \left( \sqrt{p(x, t)} - 1 - \sqrt{p(x^*, t^*)} - 1 \right)^2.
\] (3.27)
Because it holds $p_{\min} > 1$ and
\[
\sqrt{f} - \sqrt{h} = \frac{(\sqrt{f} + \sqrt{h})(\sqrt{f} - \sqrt{h})}{\sqrt{f} + \sqrt{h}} = \frac{f - h}{\sqrt{f} + \sqrt{h}}
\]
for any $f, h > 0$, we can estimate
\[
\frac{3}{\varepsilon} \left( \sqrt{p(x, t)} - 1 - \sqrt{p(x^*, t^*)} - 1 \right)^2 \leq \frac{3L_p^2}{2(p_{\min} - 1)} \cdot \frac{1}{2\varepsilon} \left( (t - t^*)^2 + |x - x^*|^2 \right)
\]
with $L_p$ denoting the Lipschitz constant of $p$. Therefore, because $H_m$ is compact, $\Phi$ is continuous with respect to the variables in $\mathcal{CS}$ and $a, b$ are arbitrary, this and $(3.27)$ imply
\[
F_m^-(\langle x, t \rangle, u_\varepsilon(x, t), Du_\varepsilon(x, t), Z) - F_m^-(\langle x^*, t^* \rangle, u_m(x^*, t^*), Du_\varepsilon(x, t), Y)
\]
\[
\leq \frac{3L_p^2}{2(p_{\min} - 1)} \cdot \frac{1}{2\varepsilon} \left( (t - t^*)^2 + |x - x^*|^2 \right).
\]
The solution $u_m$ is Hölder continuous, see Lemma 4.6 below. In particular, there exists a modulus $\zeta_u$, independent of $m$, such that
\[
\frac{1}{2\varepsilon} \left( (t - t^*)^2 + |x - x^*|^2 \right) \leq u_m(x^*, t^*) - u_m(x, t) \leq \zeta_u(\sqrt{2L_q\varepsilon}).
\]

Thus by denoting
\[
\zeta(\varepsilon) := \frac{3L_p^2}{2(p_{\min} - 1)} \zeta_u(\sqrt{2L_q\varepsilon})
\]
and recalling $(3.26)$, the proof is complete. \qed

We prove the following main lemma of this section.

**Lemma 3.3.** Let $u_m$ be the unique viscosity solution to the equation $(2.17)$. Furthermore, let $U_m^-$ be the lower value function of the game defined in $(3.20)$. Then, it holds
\[
u_m(x, t) = U_m^-(x, t)
\]
for all $(x, t) \in \mathbb{R}^n \times [0, T]$.

**Proof.** To establish the result, we regularize the solution $u_m$ first by the sup-convolution and then by the standard mollification. Then, we apply Lemma 3.5 below to the regularized function and finally pass to the limits.

Fix a large $j \in \mathbb{N}$ and a small $\varepsilon > 0$. By Lemma 3.2, it holds
\[
F_m^-\left(\langle x, t \rangle, u_\varepsilon(x, t), Du_\varepsilon(x, t), D^2u_\varepsilon(x, t)\right) \leq \partial_t u_\varepsilon(x, t) + \zeta(\varepsilon)
\]
for a.e. $(x, t) \in R^a_j$ with a bounded modulus of continuity $\zeta(\varepsilon)$. Let $\delta > 0$ be small, and denote by $\phi_\delta$ the standard mollifier in $\mathbb{R}^{n+1}$. Then for $\delta$ small enough, the function $u_\varepsilon^\delta := \phi_\delta \ast u_\varepsilon$ is well defined on $R^a_{j-1}$. Because $u_\varepsilon$ is bounded, the mollification ensures that $u_\varepsilon^\delta$ is bounded uniformly in $\delta$, and $u_\varepsilon^\delta$ is Lipschitz continuous. Moreover, $u_\varepsilon^\delta$ is smooth, and $Du_\varepsilon^\delta, \partial_t u_\varepsilon^\delta$ and $D^2u_\varepsilon^\delta$ are bounded and Lipschitz continuous on $R^a_{j-1}$. In addition, because $u_\varepsilon$ is
continuous on $R^n_j$, it holds that $u^\delta_\varepsilon \to u_\varepsilon$ uniformly as $\delta \to 0$ on $R^n_{j-1}$. We can also show that it holds
\[
Du^\delta_\varepsilon(x, t) \to Du_\varepsilon(x, t),
\]
\[
\partial_t u^\delta_\varepsilon(x, t) \to \partial_t u_\varepsilon(x, t),
\]
\[
D^2 u^\delta_\varepsilon(x, t) \to D^2 u_\varepsilon(x, t)
\]
as $\delta \to 0$ for a.e. $(x, t) \in R^n_j$, see for example [EG92]. Furthermore, we have\[
F_m^\varepsilon ((x, t), u^\delta_\varepsilon(x, t), Du^\delta_\varepsilon(x, t), D^2 u^\delta_\varepsilon(x, t)) \leq \partial_t u^\delta_\varepsilon(x, t) + \zeta(\varepsilon) + \gamma_\delta(x, t)
\]
for all $(x, t) \in R^n_j$, where it holds\[
\gamma_\delta(x, t) := \max \left\{ F_m^\varepsilon ((x, t), u^\delta_\varepsilon(x, t), Du^\delta_\varepsilon(x, t), D^2 u^\delta_\varepsilon(x, t)) - \partial_t u^\delta_\varepsilon(x, t), \zeta(\varepsilon) \right\}
\]
\[- \zeta(\varepsilon).
\]
By using the convergences above and (3.28), we see $\gamma_\delta \to 0$ as $\delta \to 0$ for a.e. on $R^n_j$. It also holds that $\gamma_\delta$ is uniformly continuous on $R^n_{j-1}$ and bounded from above uniformly with respect to $\delta$. This is true, because the operator $F^\varepsilon_m$ and the variables are uniformly continuous, and $u^\delta_\varepsilon$ is uniformly Lipschitz and semiconvex with respect to $\delta$. Now by doing minor adjustments to the proof of Lemma 3.5 below, we can argue that
\[
u^\delta_\varepsilon(x, t) \leq \inf_{S \in S_m} \sup_{A \in AC_m} \mathbb{E} \left[ \int_{t}^{T_{j-1}} e^{-r(l-t)} h^\delta_\varepsilon(X(l), l) \, dl \
+ e^{-r(T_{j-1}-t)} u^\delta_\varepsilon(X(T_{j-1}), T_{j-1}) \right]
\]
\tag{3.29}
for all $(x, t) \in R^n_j$ with $h^\delta_\varepsilon := \zeta(\varepsilon) + \gamma_\delta$ and $\varepsilon$ small enough. This is true, because $h^\delta_\varepsilon$ is uniformly continuous.

Next, for $j$ fixed, we let $\delta \to 0$ and $\varepsilon \to 0$. First, we make a rough estimate for the drift part and apply Doob’s martingale inequality for the diffusion part of the process $(X(l))$ to get the following. For all $\theta > 0$, we choose $R := R(\theta, m, \mu, n, p_{\text{max}}, T) > 0$, independent of controls and strategies, large enough such that
\[
P \left( \sup_{t \leq l \leq T} |X(l) - x| \geq R \right) \leq \theta,
\]
see for example [Eva13, Theorem 2.7.2.2]. Then by Egorov’s theorem, we find a set $\hat{U}_\theta \subset B_R(x) \times [0, T]$ such that $|\hat{U}_\theta| \leq \theta$ and
\[
\gamma_\delta \to 0 \text{ uniformly as } \delta \to 0 \text{ on } (B_R(x) \times [(j - 1)^{-1}, T_{j-1}]) \setminus U_\theta.
\tag{3.30}
\]
Now, we estimate
\[
\mathbb{E} \int_{t}^{T_{j-1}} e^{-r(l-t)} h^\delta_\varepsilon(X(l), l) \, dl \leq I_1^\varepsilon(\delta)(\theta) + I_2^\varepsilon(\delta)(\theta) + (C_\gamma + \zeta(\varepsilon))(T_{j-1} - t)\theta
\tag{3.31}
\]
where we denoted by $C_\gamma < \infty$ a constant such that $\sup_{j < 1} \gamma_\delta < C_\gamma$, and

\[
I_1^{\varepsilon, \delta}(\theta) := \mathbb{E} \int_t^{T_j-1} e^{-r(t-l)} R_\varepsilon(X(l), l) \chi_{U_{\theta}}(X(l), l) \, dl,
\]

\[
I_2^{\varepsilon, \delta}(\theta) := \mathbb{E} \int_t^{T_j-1} e^{-r(t-l)} R_\varepsilon(X(l), l) \chi_{(B_{R_\varepsilon}(x) \times [t, T_j-1]) \setminus U_{\theta}}(X(l), l) \, dl.
\]

By a fundamental estimate in \cite{Kry09, Theorem 3.4}, see also \cite{KS79}, it holds

\[
\mathbb{E} \int_t^{T_j-1} \left[ e^{-r(t-l)} \chi_{U_{\theta}}(X(l), l) \right] \, dl \leq C(T_j - t)|U_{\theta}|
\]

for a constant $C := C(n, p_{\text{min}}, \max, m, \mu, r) < \infty$. Hence, we have

\[
I_1^{\varepsilon, \delta}(\theta) \leq C(T_j - t)\theta(C_\gamma + \zeta(\varepsilon)).
\] (3.32)

Furthermore, because we have (3.30) and $\zeta(\varepsilon) \to 0$ as $\varepsilon \to 0$, it holds $I_2^{\varepsilon, \delta}(\theta) \to 0$ by first letting $\delta \to 0$ and then $\varepsilon \to 0$.

Combining this together with the estimates (3.29), (3.31) and (3.32), and letting $\delta, \theta, \varepsilon \to 0$, we have proven

\[
u_m(x, t) \leq \inf_{S \in S_m} \sup_{A \in A_{\varepsilon}} \mathbb{E} \left[ e^{-r(T_j - t)} u_m(X(T_{j-1}), T_{j-1}) \right]
\]

for all $(x, t) \in R_{j-1}^n$. Finally by recalling $T_{j-1} = T - (j - 1)^{-1}$ and letting $j \to \infty$, we see by utilizing the barrier constructed in Lemma 2.2 that

\[
\nu_m(x, t) \leq \inf_{S \in S_m} \sup_{A \in A_{\varepsilon}} \mathbb{E} \left[ e^{-r(T - t)} g(X(T)) \right].
\] (3.33)

Here, we also applied Jensen’s inequality, Ito’s isometry and (2.11) to get

\[
\mathbb{E} \left[ |X(T_{j-1}) - X(T)|^2 + j^{-1} \right]^{1/2} \leq \left( \mathbb{E} |X(T_{j-1}) - X(T)|^2 + j^{-1} \right)^{1/2}
\]

\[
\leq C(j - 1)^{-1} + j^{-1}
\]

with a constant $C := C(m, \mu, n, p_{\text{max}}) < \infty$ to estimate terms in the barrier.

The proof of the opposite inequality in (3.33) is analogous. In particular, we first apply the inf-convolution

\[
\tilde{\nu}_\varepsilon(x, t) = \inf_{(z,s) \in \mathbb{R}^n \times [0,T]} \left( u_m(z, s) + \frac{1}{2\varepsilon}((t-s)^2 + |x-z|^2) \right)
\]

whenever $(x, t) \in R_{j}^n$, and deduce an opposite type of inequality similar to (3.28) with the same modulus of continuity $\zeta$. Then, we make the standard mollification, and deduce the result by passing to the limits as before. Therefore, the proof is complete. \qed

In the result above, we utilized the following two lemmas.
**Lemma 3.4.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable, and let $a, b \in \mathbb{S}^{n-1}$ and $c, d \in \mathbb{R}^{n}$ with $m \in \mathbb{N}$. Furthermore, assume that $Du$ and $D^2u$ are Lipschitz continuous, and $D^2u$ is bounded. Then, the function

$$(x, t) \mapsto \Phi(a, b, c, d; (x, t), Du(x, t), D^2u(x, t))$$

is also Lipschitz continuous.

**Proof.** By a direct computation, it holds

$$\langle (c + d)(a + b) + \mu, Du(x, t) - Du(z, s) \rangle$$

$$+ \text{trace} \left[ D^2u(x, t) - D^2u(z, s) \right]$$

$$\leq L(|x - z|^2 + (t - s)^2)^{1/2}$$

for all $(x, t), (z, s) \in \mathbb{R}^n \times [0, T]$ and for a constant $L := L(m, \mu, n, L_1, L_2)$ with $L_1$ denoting the Lipschitz constant of $Du$ and $L_2$ denoting the Lipschitz constant of $D^2u$, respectively. Furthermore, because $D^2u$ is bounded, we have

$$C_0 := \sup_{(z, l) \in \mathbb{R}^n \times [0, T]} \| D^2u(z, l) \| < \infty.$$  

Therefore, we can estimate

$$(p(x, t) - 2) \text{trace} \left( (a \otimes a + b \otimes b)D^2u(x, t) \right)$$

$$- (p(z, l) - 2) \text{trace} \left( (a \otimes a + b \otimes b)D^2u(z, l) \right)$$

$$= (p(x, t) - 2) \text{trace} \left( (a \otimes a + b \otimes b)(D^2u(x, t) - D^2u(z, l)) \right)$$

$$+ (p(x, t) - p(z, l)) \text{trace} \left( (a \otimes a + b \otimes b)D^2u(z, l) \right)$$

$$\leq \tilde{L}(|x - z|^2 + (t - s)^2)^{1/2}$$

for all $(x, t), (z, s) \in \mathbb{R}^n \times [0, T]$ and for a constant $\tilde{L} := (p_{\max}, n, L_2, L_p, C_0)$ with $L_p$ denoting the Lipschitz constant of $p$. Thus, this estimate, together with the estimate (3.34), completes the proof.

Let $u_m$ be the unique viscosity solution to the equation (2.17), and let $U_m^-$ be the lower value function of the game defined in (3.20). Furthermore, assume that $u_m$ is twice differentiable such that $u_m, \partial_t u_m, Du_m, D^2u_m$ are Lipschitz continuous, and $Du_m, D^2u_m$ are bounded in $\mathbb{R}^n \times [0, T]$. Then, it holds

$$u_m(x, t) = U_m^-(x, t)$$

for all $(x, t) \in \mathbb{R}^n \times [0, T]$.

**Proof.** The idea of the proof is to apply Ito’s formula to connect the solution $u_m$ and the lower value function $U_m^-$ with uniformly bounded controls. We
utilize discretized controls based on the solution $u_m$, and in the end, pass to
a limit with the discretization parameter.

Let $k \in \mathbb{N}$ be an integer, $(x, t) \in \mathbb{R}^n \times [0, T)$ and denote $\triangle t := (T - t)/k$
and $t_i := t + i\triangle t$ for all $i \in \{0, \ldots, k\}$. Note that $t_0 = t$ and $t_k = T$, and set
$E_i := [t_{i-1}, t_i)$ for all $i \in \{1, \ldots, k\}$. For the time interval $E_1$, we can choose
a constant control $(a^1, c_1) \in \mathcal{H}_m$ such that
\[
\sup_{(b, d) \in \mathcal{H}_m} \Phi(a^1, b, c_1, d, Du_m(x, t), D^2u_m(x, t)) + ru_m(x, t) \leq \partial_t u_m(x, t) + \frac{1}{k},
\]
(3.35)
since $u_m$ is a solution to (2.18). Let $s \in E_1$, and let $\{(b(l), d(l))\} \in \mathcal{AC}_m$ be
an arbitrary control. We define $X(s)$ as in (2.9) with $X(t) = x$ and controls
$(a^1, c_1)$ and $(b(l), d(l))$, $l \in [t, s]$. By the assumptions, $u_m$ is regular enough
utilize Ito’s formula. Thus, it holds
\[
u
\]
\[
\begin{align*}
\phi_m(X(s), s) &- \phi_m(x, t) \\
&= \int_t^s \partial_t \phi_m(X(l), l) \, dl + \sum_{i=1}^n \int_t^s \frac{\partial u_m}{\partial x_i}(X(l), l) \, dX_i(l) \\
&+ \frac{1}{2} \sum_{i,j=1}^n \int_t^s \frac{\partial^2 u_m}{\partial x_i \partial x_j}(X(l), l) \, d\langle X_i, X_j \rangle(l).
\end{align*}
\]
(3.36)
For brevity, we denote
\[
\phi_1(X)(s) := \phi(a^1, b(s), c_1, d(s); (X(s), s), Du_m(X(s), s), D^2u_m(X(s), s)),
\]
\[
\phi_2(s) := \phi(a^1, b(s), c_1, d(s); (x, s), Du_m(x, s), D^2u_m(x, s)).
\]
Therefore by utilizing (2.11) and (3.36), we get
\[
\begin{align*}
\phi_m(X(s), s) &= \phi_m(x, t) + \int_t^s (\partial_t \phi_m(X(l), l) - \phi_1(X)(l)) \, dl \\
&+ G(X(s), s).
\end{align*}
\]
(3.37)
Here, it holds
\[
G(X(s), s) = \sum_{i=2}^n \left( \int_t^s \langle Du_m(X(l), l), p^{i-1}_{\nu} \rangle \, dW_i(l) \\
+ \int_t^s \langle Du_m(X(l), l), p^{i-1}_{\nu} \rangle \, dW_i^2(l) \right) \\
+ \sqrt{p(X(l), l)} - 1 \left( \int_t^s \langle Du_m(X(l), l), c \rangle \, dW_i(l) \\
+ \int_t^s \langle Du_m(X(l), l), b \rangle \, dW_i^2(l) \right),
\]
where we recall that $p^{i}_{\nu}$ denotes the $i$-th column vector of the matrix $P_{\nu}^{\perp}$ for
all $\nu \in \mathbb{S}^{n-1}$.
We note that for any adapted one dimensional process \( \{ \theta(l) \}_{l \in [0,T]} \) with \( \mathbb{E} \int_0^T \theta^2(l) \, dl < \infty \), it holds
\[
\mathbb{E} \int_0^h \theta(l) \, dW(l) = 0
\]
for all \( h \in [0,T] \), where \( W \) is a one dimensional Brownian motion starting from the origin. Thus, because \( Du_m \) and \( p \) are assumed to be bounded, it holds
\[
\mathbb{E} G(X(s), s) = 0.
\]

Therefore by estimating the function \( (z, l) \mapsto e^{-rl}u_m(z, l) \) instead of \( (z, l) \mapsto u_m(z, l) \) in a similar way to (3.37), it holds
\[
\mathbb{E} \left[ e^{-rs}u_m(X(s), s) - e^{-rl}u_m(x, t) \right] = \mathbb{E} \int_t^s e^{-r(l-t)} \left( \partial_t u_m(X(l), l) - \Phi_1(l) - ru_m(X(l), l) \right) \, dl.
\]

This implies
\[
u_m(x, t) = \mathbb{E} \left[ e^{-r(s-t)}u_m(X(s), s) - \int_t^s e^{-r(l-t)} \left( \partial_t u_m(X(l), l) - \Phi_1(l) - ru_m(X(l), l) \right) \, dl \right].
\]

Next, we add and subtract terms so that we can utilize (3.35). In particular, it holds
\[
u_m(x, t) = \mathbb{E} \left[ e^{-r(s-t)}u_m(X(s), s) + K_1 + K_2 + K_3 \right.
\]
\[+ \left. \int_t^s e^{-r(l-t)} \left( - \partial_t u_m(x, t) + \Phi_1^x(t) + ru_m(x, t) \right) \, dl \right], \tag{3.38}
\]
where
\[
K_1 = \int_t^s e^{-r(l-t)} \left( \partial_t u_m(x, t) - \partial_t u_m(X(l), l) \right) \, dl,
\]
\[
K_2 = \int_t^s e^{-r(l-t)} \left( \Phi_1^x(l) - \Phi_1^x(t) \right) \, dl,
\]
\[
K_3 = \int_t^s e^{-r(l-t)} \left( ru_m(X(l), l) - ru_m(x, t) \right) \, dl.
\]

Hence by using (3.35) to estimate the last term in (3.38), we get
\[
u_m(x, t) \leq \mathbb{E} \left[ e^{-r(s-t)}u_m(X(s), s) + K_1 + K_2 + K_3 \right] + \frac{s-t}{k}. \tag{3.39}
\]

We recall that \( u_m, \partial_t u_m, Du_m, \) and \( D^2 u_m \) are Lipschitz continuous, and we denote the largest Lipschitz constant of these by \( L_m \). Then, we can estimate
\[
\mathbb{E} |K_1| + \mathbb{E} |K_3| \leq (1 + r) L_m \left[ (s-t)^2 + \mathbb{E} \int_t^s |X(l) - x| \, dl \right].
\]
Furthermore, let us denote
\[ C_{0,m} := \sup_{(z,l) \in \mathbb{R}^n \times [0,T]} \| D^2 u_m (z, l) \|, \]
which is assumed to be bounded. Then, Lemma 3.4 yields
\[ | \Phi^X_1 (l) - \Phi^X_1 (t) | \leq L (|X(l) - x|^2 + (s-t)^2)^{1/2} \]
for all \( l \in [t, s] \) and for a constant \( L := L(m, \mu, p, n, L_m, C_{0,m}, L_p) \). Here, recall that the constant \( L_p \) is the Lipschitz constant of \( p \). Therefore by applying these estimates with (3.39), we get
\[ u_m (x, t) \leq \mathbb{E} \left[ e^{-r(s-t)} u_m (X(s), s) \right] + C\mathbb{E} \int_t^s |X(l) - x| \, dl + C (s-t)^2 + \frac{s-t}{k} \]
for a constant \( C := C(m, \mu, p, n, L_m, C_{0,m}, L_p, r) \). By recalling (2.11) and utilizing Jensen’s inequality and Ito’s isometry, we see
\[ \int_t^s \mathbb{E} |X(l) - x| \, dl \leq \tilde{C} ((s-t)^2 + (s-t)^3/2) \]
for a constant \( \tilde{C} := \tilde{C}(m, \mu, p, n) \). Thus, combining this with (3.40) and letting \( s \to t_1 \), we have
\[ u_m (x, t) \leq \mathbb{E} \left[ e^{-r\Delta t} u_m (X(t_1), t_1) \right] + C(\Delta t)^2 + C(\Delta t)^{3/2} + \frac{\Delta t}{k} \]
for some generic constant \( C \).

Next, we replicate the same argument as above in the time interval \( E_2 \). By Lemma 3.4, it follows that there are a sequence \( C_2 := (a^{2,i}, c^{2,i})_{i=1}^{\infty} \) and a covering \( U_2 := (B(y^{2,i}, r_2,i))_{i=1}^{\infty} \) of \( \mathbb{R}^n \) such that
\[ \sup_{(b,d) \in \mathcal{H}_m} \left( \Phi \left( a^{2,i}, b, c^{2,i}, d, D u_m (y, t_1), D^2 u_m (y, t_1) \right) + r u_m (y, t_1) \right) \]
\[ \leq \partial_k u_m (y, t_1) + \frac{1}{k} \]
for all \( y \in B(y^{2,i}, r_2,i) \). For \( y \in \mathbb{R}^n \), let \( I_2(y) \) be the smallest index \( i \) for which \( y \in B(y^{2,i}, r_2,i) \) in the covering \( (B(y^{2,i}, r_2,i))_{i=1}^{\infty} \) of \( \mathbb{R}^n \). Then, we define a function \( z^2 : \mathbb{R}^n \to \mathcal{H}_m \) by
\[ z^2 (y) = (a^{2,i_2(y)}, c^{2,i_2(y)}) \]
for all \( y \in \mathbb{R}^n \). Observe that we can construct \( z^2 \) in such a way that it is Borel measurable. Furthermore, we define a control \( (a^2(l), c^2(l)) \) such that
\[ (a^2(l), c^2(l)) = \begin{cases} (a^1, c_1), & \text{if } l \in E_1, \\ z^2 (X(t_1)), & \text{if } l \in E_2. \end{cases} \]
By the inequality (3.42), we can now repeat the argument above to get
\[ u_m(X(t_1), t_1) \leq \mathbb{E}\left[ e^{-r\Delta t}u_m(X(t_2), t_2) \right] + C(\Delta t)^2 + C(\Delta t)^{3/2} + \frac{\Delta t}{k}. \]
Thus, combining this estimate with (3.41), it holds
\[ u_m(x, t) \leq \mathbb{E}\left[ e^{-r2\Delta t}u_m(X(t_2), t_2) \right] + 2C(\Delta t)^2 + 2C(\Delta t)^{3/2} + \frac{2\Delta t}{k}. \]

The idea is to repeat the argument in all time intervals \(E_1, \ldots, E_k\). Indeed, after the \(k\)-th iteration, we get a control \((a^k(l), c_k(l))\) such that
\[ (a^k(l), c_k(l)) = \begin{cases} (a^{k-1}(l), c_{k-1}(l)), & \text{if } l \in \bigcup_{i=1}^{k-1} E_i \\ z^k(X(t_{k-1})), & \text{if } l \in E_k. \end{cases} \]
Here, \(z^k\) corresponds to the triplet \((\tilde{C}_k, U_k, I_k(y))\) in the same way as above.

In particular, we have
\[ u_m(x, t) \leq \mathbb{E}\left[ e^{-r(T-t)}g(X(T)) \right] + (T-t)(C\Delta t + C(\Delta t)^{1/2}) + \Delta t, \quad (3.43) \]
because it holds \(k = (T-t)/\Delta t\) and \(u_m(z, T) = g(z)\) for all \(z \in \mathbb{R}^n\).

Let \(S \subset S_m\), and recall that the control \((b(l), d(l))\) is arbitrary. We set
\[ (b(l), d(l)) := S(a^k(l), c_k(l)) \]
for all \(l \in [0, T]\). Then by (3.43), it holds
\[ u_m(x, t) \leq \sup_{A \in Ac_m} \mathbb{E}\left[ e^{-r(T-t)}g(X(T)) \right] + (T-t)(C\Delta t + C(\Delta t)^{1/2}) + \Delta t. \]
Because \(S \subset S_m\) is arbitrary, by letting \(k \to \infty\), this yields
\[ u_m(x, t) \leq \inf_{S \in S_m} \sup_{A \in Ac_m} \mathbb{E}\left[ e^{-r(T-t)}g(X(T)) \right]. \]

The proof of the opposite inequality is analogous. Again, Lemma 3.4 implies that there is a sequence \(\tilde{C}_j := (b^{j,i}, d_{j,i})_{i=1}^\infty\) and a covering \(\tilde{U}_j := (B(\tilde{y}^{j,i}, \tilde{r}_{j,i}))_{i=1}^\infty\) of \(\mathbb{R}^n\) such that
\[ \inf_{(a,c) \in H_m} \left( \Phi(a, b^{j,i}, c, d_{j,i}, Du_m(y, t_{j-1}), D^2 u_m(y, t_{j-1})) + ru_m(y, t_{j-1}) \right) \geq \partial_t u_m(y, t_{j-1}) - \frac{1}{k} \]
for all \(y \in B(\tilde{y}^{j,i}, \tilde{r}_{j,i})\) and \(j \in \{2, \ldots, k\}\), because \(u_m\) is a solution to (2.18).

Then by a similar reasoning to the above, we construct a control \((b^k(l), d_k(l))\) to deduce
\[ u_m(x, t) \geq \mathbb{E}\left[ e^{-r(T-t)}g(X(T)) \right] - C(T-t)\Delta t \]
\[ - C(T-t)(\Delta t)^{1/2} - \Delta t. \quad (3.44) \]
Let \( A \in \mathcal{A}_m \). We construct \( S \in \mathcal{S}_m \) such that it holds
\[
S(A) = (b^k(l), d_k(l))
\]
for all \( l \in [0,T] \). Therefore, the inequality (3.44) implies
\[
\begin{align*}
    u_m(x,t) &\geq \mathbb{E} \left[ e^{-r(T-t)} g(X(T)) \right] - (T-t) \left( C \Delta t + C(\Delta t)^{1/2} \right) - \Delta t \\
    &\geq \inf_{S \in \mathcal{S}_m} \mathbb{E} \left[ e^{-r(T-t)} g(X(T)) \right] - (T-t) \left( C \Delta t + C(\Delta t)^{1/2} \right) - \Delta t.
\end{align*}
\]
Hence, by letting \( k \to \infty \), we get
\[
    u_m(x,t) \geq \inf_{S \in \mathcal{S}_m} \sup_{A \in \mathcal{A}_m} \mathbb{E} \left[ e^{-r(T-t)} g(X(T)) \right].
\]
Thus, the proof is complete. \( \square \)

4. Going to the limit: action sets without a uniform bound

In this section, we let bounds on the controls increase. To this end, we first show that viscosity solutions to the limiting equation are unique under suitable assumptions. Then by utilizing the stability principle and the equicontinuity of the families of viscosity solutions to the terminal value problems (2.17) and (2.18), we see that there exist subsequences of solutions to (2.17) and (2.18) converging uniformly to solutions of the limiting equation. The final part is to show that a subsequence of the corresponding value functions converges to a value function for the game without a uniform bound on the controls.

Let \( J_0 := \mathbb{R}^n \times [0,T] \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times S(n) \), and define \( F : J_0 \to \mathbb{R} \) through
\[
F((x,t),\xi,\nu,M) = (p(x,t) - 2) \frac{\langle M\nu,\nu \rangle}{|\nu|^2} + \text{trace}(M) + \langle \mu,\nu \rangle - r\xi.
\]
Then, the limiting terminal value problem for (2.17) and (2.18) as \( m \to \infty \) is
\[
\begin{align*}
    \partial_t u + F((x,t),u,Du,D^2u) &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\
    u(x,T) &= g(x) \quad \text{on } \mathbb{R}^n.
\end{align*}
\]
As before, this equation is understood in the viscosity sense. We take care of the points, where the gradient of the underlying function in the operator \( F \) vanishes, via semicontinuous envelopes. Let us denote
\[
F_*((x,t),\xi,\nu,M) := \liminf_{\bar{\nu} \to \nu} F((x,t),\xi,\bar{\nu},M)
\]
for all \( (x,t) \in \mathbb{R}^n \times [0,T], \xi \in \mathbb{R}, \nu \in \mathbb{R}^n \) and \( M \in S(n) \), and \( F^* := -(F)_* \).

The following definition parallels Definition 2.1.
Definition 4.1. (i) A lower semicontinuous function $\mathbf{u} : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a viscosity supersolution to (4.45), if it satisfies the growth bound (2.19),

$$\mathbf{u}(x, T) \geq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and for all $\phi \in C^{2,1}(\mathbb{R}^n \times (0, T))$ such that

- $\mathbf{u}(x_0, t_0) = \phi(x_0, t_0)$
- $\mathbf{u}(x, t) > \phi(x, t)$ for all $(x, t) \neq (x_0, t_0)$

it holds

$$\partial_t \phi(x_0, t_0) + F((x_0, t_0), \phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \leq 0$$

whenever $D\phi(x_0, t_0) \neq 0$, and

$$\partial_t \phi(x_0, t_0) + F^s((x_0, t_0), \phi(x_0, t_0), 0, D^2\phi(x_0, t_0)) \leq 0,$$

whenever $D\phi(x_0, t_0) = 0$.

(ii) An upper semicontinuous function $\mathbf{u} : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a viscosity subsolution to (4.45), if it satisfies the growth bound (2.19),

$$\mathbf{u}(x, T) \leq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and for all $\phi \in C^{2,1}(\mathbb{R}^n \times (0, T))$ such that

- $\mathbf{u}(x_0, t_0) = \phi(x_0, t_0)$
- $\mathbf{u}(x, t) < \phi(x, t)$ for all $(x, t) \neq (x_0, t_0)$

it holds

$$\partial_t \phi(x_0, t_0) + F((x_0, t_0), \phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \geq 0,$$

whenever $D\phi(x_0, t_0) \neq 0$, and

$$\partial_t \phi(x_0, t_0) + F^s((x_0, t_0), \phi(x_0, t_0), 0, D^2\phi(x_0, t_0)) \geq 0,$$

whenever $D\phi(x_0, t_0) = 0$.

(iii) If a function $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a viscosity supersolution and a subsolution to (4.45), then $u$ is a viscosity solution to (4.45).

Remark 4.2. Observe that for any test function $\phi \in C^{2,1}(\mathbb{R}^n \times (0, T))$ such that $D\phi(x_0, t_0) \neq 0$ or $D^2\phi(x_0, t_0) = 0$ in the Definition 4.1, it holds

$$F_s((x_0, t_0), \phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0))$$

$$= F^s((x_0, t_0), \phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0))$$

for all $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$.

To prove a comparison principle for the equation (4.45), we follow the path developed in [GGIS91], see also [CGG91, JLM01, KMP12]. Here, the main difficulties arise from the $(x, t)$ dependence in $F$ as well as from the unboundedness of the domain.
Theorem 4.3. Let $u$ and $\overline{u}$ be continuous viscosity sub- and supersolutions to (4.45) in the sense of Definition 4.1, respectively. Then, it holds

$$u(x, t) \leq \overline{u}(x, t)$$

for all $(x, t) \in \mathbb{R}^n \times [0, T]$.

Proof. The proof is by contradiction. We assume that

$$\alpha := \sup_{\mathbb{R}^n \times [0, T]} (u - \overline{u}) > 0. \quad (4.46)$$

Let $\varepsilon, \delta, \gamma > 0$, and define

$$w_{\varepsilon, \delta, \gamma}(x, y, t) = u(x, t) - \overline{u}(y, t) - \frac{1}{4\varepsilon}|x - y|^4 - B_{\delta, \gamma}(x, y, t)$$

for all $x, y \in \mathbb{R}^n$ and $t \in (0, T]$, where

$$B_{\delta, \gamma}(x, y, t) := \delta(|x|^2 + |y|^2) + \gamma t^{-1}. \quad (4.47)$$

The function $B_{\delta, \gamma}$ plays the role of a barrier for space infinity and $t = 0$.

We can show, see [GGIS91, Proposition 2.3], that there are constants $K, K' > 0$ independent of $x, y, t$ such that

$$u(x, t) - \overline{u}(y, t) \leq K|x - y| + K'(1 + t) \quad (4.48)$$

for all $x, y \in \mathbb{R}^n$ and $t \in [0, T]$. Indeed, because for $R' > 0$ it holds

$$|F((x, t), \xi, p, M)| \leq (p_{\max} - 2 + n + |\mu|)R' + r\xi < \infty$$

for all $(x, t, \xi, p, M) \in J_0$ such that $|p| \leq R'$ and $|M| \leq R'$, we can utilize the same arguments as in [GGIS91, Proposition 2.3]. Therefore by the estimate (4.48), it holds $\alpha < \infty$ in (4.46).

We denote by $(\hat{x}, \hat{y}, \hat{t})$ a maximum point of $w_{\varepsilon, \delta, \gamma}$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0, T]$. The growth condition (2.19) and the barrier (4.47) ensure that $w_{\varepsilon, \delta, \gamma}(x, y, t) < 0$, when $x, y$ are outside a compact set $E \subset \mathbb{R}^n \times \mathbb{R}^n$ depending on $\delta$, and $t \in (0, T]$. Therefore, because $w_{\varepsilon, \delta, \gamma}$ is continuous and (4.46) holds with $\alpha < \infty$, the maximum point exists for all $\delta, \gamma$ small enough and any $\varepsilon$. Furthermore by (4.46), we can find $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ such that

$$u(x_0, t_0) - \overline{u}(x_0, t_0) > \alpha - \varepsilon/3.$$

Because $u - \overline{u}$ is continuous, we may assume that $t_0 > 0$. Consequently, for $\varepsilon < \alpha$ there are $\delta_0 := \delta_0(\varepsilon) > 0$ and $\gamma_0 := \gamma_0(\varepsilon) > 0$ such that

$$w_{\varepsilon, \delta, \gamma}(\hat{x}, \hat{y}, \hat{t}) \geq u(x_0, t_0) - \overline{u}(x_0, t_0) - 2\delta|x_0| - \gamma t_0^{-1} > \alpha - \varepsilon \quad (4.49)$$

for all $\delta < \delta_0$ and $\gamma < \gamma_0$. Let $\varepsilon < \alpha/2, \delta < \delta_0$ and $\gamma < \gamma_0$. Then by (4.49) we can estimate

$$u(\hat{x}, \hat{t}) - \overline{u}(\hat{y}, \hat{t}) > \frac{1}{4\varepsilon}|\hat{x} - \hat{y}|^4 + B_{\delta, \gamma}(\hat{x}, \hat{y}, \hat{t}) \geq \frac{1}{4\varepsilon}|\hat{x} - \hat{y}|^4.$$

This and (4.48) imply

$$|\hat{x} - \hat{y}| \leq 4\varepsilon(K|\hat{x} - \hat{y}|^{-3} + K'(1 + T)|\hat{x} - \hat{y}|^{-4}).$$
Therefore, we have $|\hat{x} - \hat{y}| < C$ for some $C < \infty$ independent of $\varepsilon, \delta$ and $\gamma$. Moreover, it holds

$$|\hat{x} - \hat{y}| \leq \max \left\{ \varepsilon^{1/8}, 4K\varepsilon^{5/8} + 4K'(1 + T)\sqrt{\varepsilon} \right\} =: \zeta(\varepsilon). \quad (4.50)$$

By an analogous argument, we can deduce $\hat{t} > 0$. Because it holds $\underline{u}(z, T) \leq \overline{u}(z, T)$ for all $z \in \mathbb{R}^n$ by the assumptions, the inequality (4.49) yields $\hat{t} < T$.

In addition, because $|\hat{x} - \hat{y}|$ is bounded, the estimate (4.48) implies that $w_{\varepsilon, \delta, \gamma}(\hat{x}, \hat{y}, \hat{t})$ is uniformly bounded from above with respect to $\delta$. Hence, because $w_{\varepsilon, \delta, \gamma}(\hat{x}, \hat{y}, \hat{t})$ increases as $\delta \to 0$, the quantity $\lim_{\delta \to 0} w_{\varepsilon, \delta, \gamma}(\hat{x}, \hat{y}, \hat{t})$ exists. Therefore by denoting $(\hat{x}, \hat{y}, \hat{t})$ a global maximum point of $w_{\varepsilon, \delta, \gamma}$, we have

$$w_{\varepsilon, \delta, \gamma}(\hat{x}, \hat{y}, \hat{t}) \geq w_{\varepsilon, \delta, \gamma}(\hat{x}, \hat{y}, \hat{t}) + \delta/2(|\hat{x}|^2 + |\hat{y}|^2)$$

implying

$$\delta(|\hat{x}|^2 + |\hat{y}|^2) \to 0 \quad (4.51)$$

as $\delta \to 0$.

By theorem of sums, see [CIL92, Theorem 8.3], there exist symmetric matrices $X := X(\varepsilon, \delta)$ and $Y := Y(\varepsilon, \delta)$, and real numbers $\tau_{\underline{u}}$ and $\tau_{\overline{u}}$, such that $\tau_{\underline{u}} - \tau_{\overline{u}} = \partial_t B_{\delta, \gamma}(\hat{x}, \hat{y}, \hat{t}) = -\gamma \hat{t}^{-2}$ and

$$\left( \tau_{\underline{u}}, \varepsilon^{-1} |\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) + 2 \delta \hat{x}, X \right) \in \mathcal{P}^{2,+}_{\underline{u}}(\hat{x}, \hat{t}),$$

$$\left( \tau_{\overline{u}}, \varepsilon^{-1} |\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) - 2 \delta \hat{y}, Y \right) \in \mathcal{P}^{2,-}_{\overline{u}}(\hat{y}, \hat{t}). \quad (4.52)$$

Furthermore by computing the second derivatives of the function $B_{\delta, \gamma}(x, y, t) + \frac{1}{4\varepsilon}|x - y|^4$, it holds

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq (1 + 4\varepsilon\delta) \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} + 2\varepsilon \begin{bmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{bmatrix}$$

$$+ 2\delta(1 + 2\delta) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (4.53)$$

with

$$M := \varepsilon^{-1}\left(2(\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y}) + |\hat{x} - \hat{y}|^2 I \right),$$

and

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \geq -(\varepsilon^{-1} + 3\varepsilon^{-1}|\hat{x} - \hat{y}|^2 + 2\delta) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (4.54)$$

Thus, because $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution, it holds by (4.52)

$$\tau_{\underline{u}} + F^*((\hat{x}, \hat{t}), \underline{u}(\hat{x}, \hat{t}), \varepsilon^{-1}|\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) + 2\delta \hat{x}, X) \geq 0,$$

$$\tau_{\overline{u}} + F^*((\hat{y}, \hat{t}), \overline{u}(\hat{y}, \hat{t}), \varepsilon^{-1}|\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) - 2\delta \hat{y}, Y) \leq 0, \quad (4.55)$$

see also Remark 4.2.
We consider two different cases depending on the behavior of \( \hat{x} - \hat{y} \) as \( \delta \to 0 \). First, assume that \( \hat{x} - \hat{y} \to 0 \) as \( \delta \to 0 \). Then by the estimate (4.53), it holds
\[
\limsup_{\delta \to 0} \langle X z, z \rangle \leq 0 \quad \text{and} \quad \liminf_{\delta \to 0} \langle Y z, z \rangle \geq 0
\]
for all \( z \in \mathbb{R}^n \). Thus by combining this with (4.55), and recalling (4.49), the degenerate ellipticity of \( F \) and \( \delta \hat{x}, \delta \hat{y} \to 0 \) as \( \delta \to 0 \) by (4.51), we can estimate
\[
\gamma T^{-2} \leq \limsup_{\delta \to 0} F^* \left( (\hat{x}, \hat{t}), \mu(\hat{x}, \hat{t}), 0, 0 \right) - \liminf_{\delta \to 0} F^* \left( (\hat{y}, \hat{t}), \pi(\hat{y}, \hat{t}), 0, 0 \right) \leq 0.
\]
Hence, because it holds \( \gamma > 0 \), we have found a contradiction.

Next, we assume \( \hat{x} - \hat{y} \to \eta \neq 0 \) for some subsequence still denoted by \( (\delta) \). For brevity, let us denote
\[
\bar{\xi}_x := \varepsilon^{-1} |\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) + 2 \delta \hat{x},
\]
\[
\bar{\xi}_y := \varepsilon^{-1} |\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) - 2 \delta \hat{y},
\]
\[
\xi_x := \bar{\xi}_x / |\bar{\xi}_x| \quad \text{and} \quad \xi_y := \bar{\xi}_y / |\bar{\xi}_y| \quad \text{assuming \( \bar{\xi}_x, \bar{\xi}_y \neq 0 \).}
\]
Then, because of (4.49) and (4.55), we can estimate
\[
0 < \langle p(\hat{x}, \hat{t}) - 2 \rangle \langle X \xi_x, \xi_x \rangle - \langle p(\hat{y}, \hat{t}) - 2 \rangle \langle Y \xi_y, \xi_y \rangle + \sum_{i=1}^n \lambda_i (X - Y) + 2 \langle \mu, \delta \hat{x} + \delta \hat{y} \rangle - r \alpha / 2,
\]
where \( \lambda_i \) denotes the \( i \)-th eigenvalue of the corresponding matrix. Because the first two matrices in the right-hand side of (4.53) annihilate, we have
\[
X - Y \leq 4 \delta (1 + 2 \delta) I.
\]
Thus to complete the proof, we need to estimate the first two terms in the right-hand side of (4.56).

Let us define \( \xi_\delta := (\hat{x} - \hat{y}) / |\hat{x} - \hat{y}| \in \mathbb{S}^{n-1} \) for all \( \delta \) small enough. Then, it holds
\[
\xi_\delta \to \eta / |\eta|
\]
as \( \delta \to 0 \). Observe that by the convergence (4.51), it also holds
\[
\xi_x, \xi_y \to \eta / |\eta|
\]
as \( \delta \to 0 \). Furthermore by (4.53) and (4.54), \( X \) and \( Y \) are uniformly bounded with respect to \( \delta \), see also [Ish89, Lemma 5.3]. Thus, because the function \( p \) is bounded, the convergences (4.58) and (4.59) imply
\[
\left( p(\hat{x}, \hat{t}) - 2 \right) \langle X \xi_x, \xi_x \rangle - \left( p(\hat{y}, \hat{t}) - 2 \right) \langle Y \xi_y, \xi_y \rangle = \left( p(\hat{x}, \hat{t}) - 1 \right) \langle X \xi, \xi_\delta \rangle - \left( p(\hat{y}, \hat{t}) - 1 \right) \langle Y \xi_\delta, \xi_\delta \rangle + E_\delta(\hat{x}, \hat{y}, \hat{t})
\]
(4.60)
for some error $E_\delta(\hat{x}, \hat{y}, \hat{t})$ such that

$$E_\delta(\hat{x}, \hat{y}, \hat{t}) \to 0$$

as $\delta \to 0$. For the vector

$$(\xi_\delta^T \sqrt{p(\hat{x}, \hat{t}) - 1}, \xi_\delta^T \sqrt{p(\hat{y}, \hat{t}) - 1}) \in \mathbb{R}^{2n}$$

in the estimate (4.53), it holds

$$(p(\hat{x}, \hat{t}) - 1)\langle X\xi_\delta, \xi_\delta \rangle - (p(\hat{y}, \hat{t}) - 1)\langle Y\xi_\delta, \xi_\delta \rangle$$

$$\leq \left( \sqrt{p(\hat{x}, \hat{t}) - 1} - \sqrt{p(\hat{y}, \hat{t}) - 1} \right)^2 \left( (1 + 4\varepsilon \delta)\langle M\xi_\delta, \xi_\delta \rangle \right.$$  \bigg)  

$$+ 2\varepsilon \langle M^2\xi_\delta, \xi_\delta \rangle \bigg) + 4(p_{\text{max}} - 1)\delta(1 + 2\delta)$$  \bigg)$$

$$\leq \frac{L_p^2}{4(p_{\text{min}} - 1)} |\hat{x} - \hat{y}|^4 \left( (1 + 4\varepsilon \delta)3\varepsilon^{-1}|\hat{x} - \hat{y}|^2 + 18\varepsilon^{-1}|\hat{x} - \hat{y}|^4 \right)$$

$$+ 4(p_{\text{max}} - 1)\delta(1 + 2\delta),$$

where $L_p$ is the Lipschitz constant of $p$. Moreover by the estimates (4.49) and (4.50), it holds

$$\frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} < u(\hat{x}, \hat{t}) - \bar{u}(\hat{y}, \hat{t}) - \alpha + \varepsilon$$

$$\leq \sup_{|x - y| < \zeta(\varepsilon), t \in [0, T]} (u(x, t) - \bar{u}(y, t)) - \alpha + \varepsilon.$$

This estimate, together with (4.46), implies

$$\lim_{\varepsilon \to 0} \limsup_{\delta, \gamma \to 0} \frac{|\hat{x} - \hat{y}|^4}{\varepsilon} = 0.$$

Therefore by combining this, (4.51), (4.57), (4.60) and (4.61) with the estimate (4.56), we have found a contradiction by first letting $\delta, \gamma \to 0$ and then $\varepsilon \to 0$. Hence, the proof is complete. \qed

A typical phenomenon for equations of $p$-Laplacian type is that the set of test functions used in their definition can be reduced.

**Lemma 4.4.** Let $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ be continuous. Then, to test whether or not $u$ is a viscosity super- or subsolution at $(x_0, t_0)$ in the sense of Definition 4.1, it is enough to consider test functions $\phi \in C^{2,1}(\mathbb{R}^n \times (0, T))$ such that either

- $D\phi(x_0, t_0) \neq 0$
- $D\phi(x_0, t_0) = 0$ and $D^2\phi(x_0, t_0) = 0$.

**Proof.** We only provide the proof in the context of supersolutions. Let $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$. Assume that there exist $\delta > 0$ and a test function
\( \phi \in C^{2,1}(\mathbb{R}^n \times (0, T)) \) such that \( u(x_0, t_0) = \phi(x_0, t_0) \), \( u(x, t) > \phi(x, t) \) for \((x, t) \neq (x_0, t_0)\), \( D\phi(x_0, t_0) = 0 \), \( D^2\phi(x_0, t_0) \neq 0 \) and
\[
0 < \partial_t \phi(x_0, t_0) + F_*(\phi(x_0, t_0), 0, D^2\phi(x_0, t_0) - \delta) \quad (4.62)
\]
Observe that \( u - \phi \) has a strict global minimum at \((x_0, t_0)\). We define a function
\[
w_j(x, t, y, s) := u(x, t) - \phi(y, s) + \frac{j}{4}|x - y|^4 + \frac{j}{2}(t - s)^2
\]
for \( x, y \in \mathbb{R}^n, t, s \in [0, T] \). Let \( R := \max\{2|x_0|, 1\} > 0 \), and denote by \((x_j, t_j, y_j, s_j)\) a minimum point of \( w_j \) on a compact set \( K := \overline{B}_R(0) \times [0, T] \times \overline{B}_R(0) \times [0, T] \). Because \( w_j(x_j, t_j, y_j, s_j) \) increases as \( j \) increases, and it is bounded from above by \( w_j(x_0, t_0, x_0, t_0) = 0 \) for all \( j \), the limit
\[
\lim_{j \to \infty} w_j(x_j, t_j, y_j, s_j) < \infty
\]
exists. Consequently, the estimate
\[
w_{j/2}(x_j/2, t_j/2, y_j/2, s_j/2) \leq w_j(x_j, t_j, y_j, s_j) - \frac{j}{8}|x_j - y_j|^4 - \frac{j}{4}(t_j - s_j)^2
\]
implies
\[
4|x_j - y_j|^4 + 4(t_j - s_j)^2 \to 0 \quad (4.63)
\]
as \( j \to \infty \). Furthermore, because the global minimum of \( u - \phi \) is strict, it holds
\[
(x_j, t_j, y_j, s_j) \to (x_0, t_0, x_0, t_0) \quad (4.64)
\]
as \( j \to \infty \). In particular, the point \((x_j, t_j, y_j, s_j)\) is not on the boundary of the set \( K \) for all \( j \) large enough, because it holds \((x_0, t_0) \in B_R(0) \times (0, T) \).

We prove the case \( x_j = y_j \) for an infinite sequence of \( j \):s, and consider only such indices \( j \). The proof in the case \( x_j \neq y_j \) for all \( j \) large enough is similar to the proof of Theorem 4.3, see also [CGG91, JLM01]. By denoting \( \varphi(x, y) := \frac{2}{7}|x - y|^4 \), it holds
\[
D_x \varphi(x_j, y_j) = -D_y \varphi(x_j, y_j) = 0 \quad \text{and} \quad D^2_{xx} \varphi(x_j, y_j) = D^2_{yy} \varphi(x_j, y_j) = 0.
\]
Furthermore, the function
\[
(y, s) \mapsto \phi(y, s) - \varphi(x_j, y) - \frac{j}{2}(t_j - s)^2
\]
has a local maximum at \((y_j, s_j)\). These imply \( D\phi(y_j, s_j) = -D_y\varphi(x_j, y_j) = 0 \), \( \partial_t \phi(y_j, s_j) = -j(t_j - s_j) \) and \( D^2\phi(y_j, s_j) \leq -D^2_{yy} \varphi(x_j, y_j) = 0 \). Thus, because \( p \) and \((y, s) \mapsto \lambda_i(D^2\phi(y, s)) \) for any \( i \) are continuous with \( \lambda_i \) denoting the \( i \)-th eigenvalue of the corresponding matrix, the assumption (4.62) and
the convergence (4.64) yield
\[ 0 < \partial_t \phi(y_j, s_j) + \lambda_{\max} \left( p(y_j, s_j) - 1 \right) D^2 \phi(y_j, s_j) \]
\[ + \sum_{i \neq \text{im}in} \lambda_i \left( D^2 \phi(y_j, s_j) \right) - r \phi(y_j, s_j) - \frac{\delta}{2} \]
\[ \leq -j(t_j - s_j) - r \phi(y_j, s_j) - \frac{\delta}{2} \]
for all \( j \) large enough. Furthermore, because the function
\[ (x, t) \mapsto \Psi(x, t) := -\varphi(x, y_j) - \frac{j}{2} (t - s_j)^2 + \varphi(x, y_j) + \frac{j}{2} (t_j - s_j)^2 \]
+ \( u(x_j, t_j) \)
tests \( u \) from below at \( (x_j, t_j) \), and it holds \( D_x \Psi(x_j, t_j) = 0 \), we have
\[ 0 \geq \Psi_t(x_j, t_j) + F_*((x_j, t_j), u(x_j, t_j), 0, D^2_{xx} \Psi(x_j, t_j)). \]
Thus, because it holds \( \Psi_t(x_j, t_j) = -j(t - s_j) \) and \( D^2_{xx} \Psi(x_j, t_j) = 0 \), by combining this and (4.65), we get
\[ 0 < r \left( u(x_j, t_j) - \phi(y_j, s_j) \right) - \delta/2. \]
Hence, because \( u \) is continuous and (4.64) holds, we find a contradiction for all \( j \) large enough. \( \square \)

The following lemma suggests that \( F \) is the correct limiting equation in our setting. The proof for the equation \( F_m^+ \) is analogous.

**Lemma 4.5.** Let \( (x_m, t_m), (x, t) \in \mathbb{R}^n \times [0, \infty) \), \( \xi_m, \xi \in \mathbb{R}, \nu_m, \nu \in \mathbb{R}^n \setminus \{0\} \) and \( M_m, M \in S(n) \) be such that
\[ (x_m, t_m) \to (x, t), \ \xi_m \to \xi, \ \nu_m \to \nu \text{ and } M_m \to M \]
as \( m \to \infty \). Then, it holds
\[ F_m^-((x_m, t_m), \xi_m, \nu_m, M_m) \to -F((x, t), \xi, \nu, M) \]
as \( m \to \infty \).

**Proof.** It is clear that \( \langle \mu, \nu_m \rangle \to \langle \mu, \nu \rangle \) and \( r \xi_m \to r \xi \) as \( m \to \infty \). To complete the proof, we utilize the key inequality
\[ \langle \nu_m/|\nu_m| + \xi, \nu_m \rangle \geq 0 \] (4.66)
whenever \( \xi \in \mathbb{S}^{n-1} \).

We set
\[ \Phi_m := \inf_{(a, c) \in \mathcal{H}_m, (b, d) \in \mathcal{H}_m} \sup_{(a, c) \in \mathcal{H}_m, (b, d) \in \mathcal{H}_m} \left[ -\text{trace} \left( \mathcal{A}^{(x_m, t_m)}_{a, b, c} M_m \right) - (c + d) \langle a + b, \nu_m \rangle \right]. \]
Because \( (\nu_m/|\nu_m|, 0) \in \mathcal{H}_m \), it holds
\[ \Phi_m \leq \sup_{(b, d) \in \mathcal{H}_m} \left[ -\text{trace} \left( \mathcal{A}^{(x_m, t_m)}_{b, d} M_m \right) - d \langle \nu_m/|\nu_m| + b, \nu_m \rangle \right]. \]
Therefore, this estimate and (4.66) imply
\[ \tilde{\Phi}_m \leq n\Lambda \|M_m\|, \]
where \( \Lambda \) is defined in (2.15). Hence, \( \tilde{\Phi}_m \) is bounded from above as \( m \to \infty \).

Because \(( -\nu_m/|\nu_m|, m) \in H_m \), we can estimate
\[ \tilde{\Phi}_m \geq \inf_{(a,c) \in H_m} \left[ -\operatorname{trace} \left( A_{a,-\nu_m/|\nu_m|}^{(x_m,t_m)} M_m \right) - (c + m)\langle a - \nu_m/|\nu_m|, \nu_m \rangle \right]. \]

Now, (4.66) implies that the second term after the infimum is bounded from below as \( m \to \infty \). Hence by the definition of the infimum, there exists \((a_m, c_m) \in H_m \) such that
\[ \tilde{\Phi}_m \geq -\operatorname{trace} \left( A_{a_m,-\nu_m/|\nu_m|}^{(x_m,t_m)} M_m \right) - (c_m + m)\langle a_m - \nu_m/|\nu_m|, \nu_m \rangle - \frac{1}{m}. \] (4.67)

Next, we prove that
\[ a_m \to \frac{\nu}{|\nu|} \] (4.68)
as \( m \to \infty \). To establish this, it suffices to show that for given \( \eta > 0 \), there is \( m_0 := m_0(\eta) \) such that
\[ \langle a_m, \nu_m \rangle \geq |\nu_m| - \eta \]
for all \( m \geq m_0 \). We assume, on the contrary, that there is \( \eta > 0 \) such that
\[ \langle a_m, \nu_m \rangle < |\nu_m| - \eta. \]
Thus in this case, (4.67) implies
\[ \tilde{\Phi}_m \geq -n\Lambda \|M_m\| + \eta(c_m + m) - \frac{1}{m}. \]
This contradicts the boundedness of \( \tilde{\Phi}_m \) as \( m \to \infty \), and hence, (4.68) holds.

Recall that the function \( p \) is continuous which implies \( p(x_m, t_m) \to p(x, t) \) as \( m \to \infty \). Therefore by combining the assumptions, (4.66) and (4.68) with (4.67), we get
\[ \liminf_{m \to \infty} \tilde{\Phi}_m \geq -\operatorname{trace} \left( A_{\frac{\nu}{|\nu|}, \frac{\nu}{|\nu|}}^{(x,t)} M \right) - \langle M\nu, \nu \rangle \]
\[ = -(p(x, t) - 2)\frac{\langle M\nu, \nu \rangle}{|\nu|^2} - \operatorname{trace}(M). \]
Thus, we have proven
\[ \liminf_{m \to \infty} F_m^\alpha((x_m, t_m), \xi_m, \nu_m, M_m) \geq -F((x, t), \xi, \nu, M). \]

Next, we prove that
\[ \limsup_{m \to \infty} F_m^\alpha((x_m, t_m), \xi_m, \nu_m, M_m) \leq -F((x, t), \xi, \nu, M). \] (4.69)
Lemma 4.6. Let \( \Phi \) belong to \( \lambda, \mathcal{R} \) is equicontinuous on \( A \) below, because we can use (4.66), and estimate the supremum in \( \Phi \) with the choice \( (-\nu_m/|\nu_m|, 0) \in \mathcal{H}_m \). Moreover, \( \tilde{\Phi} \) bounded also from below, because we can use (4.66) and estimate the supremum in \( \tilde{\Phi} \) with the choice \( (-\nu_m/|\nu_m|, 0) \in \mathcal{H}_m \). This and the estimate (4.70) imply \( b_m \to -\nu/|\nu| \) as \( m \to \infty \) in a similar way to the above. Therefore, this, together with the estimate (4.66) in the inequality (4.70), by taking \( \limsup_{m \to \infty} \), completes the proof of (4.69).

For all \( M \in S(n) \), we utilize the Pucci operators

\[
P^+(M) := \sup_{A \in A_{\lambda, \lambda}} \text{trace}(AM)
\]

and

\[
P^-(M) := \inf_{A \in A_{\lambda, \lambda}} \text{trace}(AM),
\]

where \( A_{\lambda, \lambda} \subset S(n) \) is the set of symmetric \( n \times n \) matrices whose eigenvalues belong to \( [\lambda, \Lambda] \).

Lemma 4.6. Let \( u_m \) be the unique solution to (2.17) ensured by Proposition 2.4. Then, the function \( u_m \) is Hölder continuous on \( \mathbb{R}^n \times [0, T] \) with a Hölder constant independent of \( m \). In particular, the sequence

\[
\{u_m : m \geq 1\}
\]

is equicontinuous on \( \mathbb{R}^n \times [0, T] \).

Proof. Let \( m \geq 1 \) and \( (x, t) \in \mathbb{R}^n \times (0, T) \). Furthermore, let \( \varphi \in C^2(\mathbb{R}^n \times (0, T)) \) test \( u_m \) from below at \( (x, t) \). First, we assume \( D\varphi(x, t) \neq 0 \). Because \( u_m \) is a supersolution to (2.17), we can find a vector \( b_m \) on a compact set \( S^{n-1} \) such that

\[
0 \geq \partial_t \varphi(x, t) + \text{trace} \left( A^{(x, t)} \frac{\partial^2 \varphi(x, t)}{|\partial^2 \varphi(x, t)|} b_m \right) + \langle \mu, D\varphi(x, t) \rangle - r\varphi(x, t)
\]

\[
\geq \partial_t \varphi(x, t) + P^-(D^2 \varphi(x, t)) + \langle \mu, D\varphi(x, t) \rangle - r\varphi(x, t).
\]

Next, we assume \( D\varphi(x, t) = 0 \). Now, since there is no more gradient dependence in \( \Phi \), the term inside \( \inf \sup \) in \( \Phi \) is always bounded, and hence for any \( \nu \in S^{n-1} \), there is \( b_m \in S^{n-1} \) such that

\[
0 \geq \partial_t \varphi(x, t) + \text{trace} \left( A^{(x, t)} b_m D^2 \varphi(x, t) \right) + \langle \mu, D\varphi(x, t) \rangle - r\varphi(x, t)
\]

\[
\geq \partial_t \varphi(x, t) + P^-(D^2 \varphi(x, t)) + \langle \mu, D\varphi(x, t) \rangle - r\varphi(x, t).
\]
Let $\phi \in C^2(\mathbb{R}^n \times (0,T))$ test $u_m$ from above at $(x,t)$. In a similar way to the above, if $D\phi(x,t) \neq 0$, we can find $a_m \in \mathbb{S}^{n-1}$ such that

$$0 \leq \partial_t \phi(x,t) + \text{trace} \left( A^{(x,t)}_{a_m} D^2 \phi(x,t) \right) + \langle \mu, D\phi(x,t) \rangle - r\phi(x,t)$$

$$\leq \partial_t \phi(x,t) + P^+(D^2 \phi(x,t)) + \langle \mu, D\phi(x,t) \rangle - r\phi(x,t),$$

because $u_m$ is a subsolution to (2.17). Furthermore, if $D\phi(x,t) = 0$, for any $\nu \in \mathbb{S}^{n-1}$, there is $a_m \in \mathbb{S}^{n-1}$ such that

$$0 \leq \partial_t \phi(x,t) + \text{trace} \left( A^{(x,t)}_{a_m,\nu} D^2 \phi(x,t) \right) + \langle \mu, D\phi(x,t) \rangle - r\phi(x,t)$$

$$\leq \partial_t \phi(x,t) + P^+(D^2 \phi(x,t)) + \langle \mu, D\phi(x,t) \rangle - r\phi(x,t).$$

Thus, we have shown that $u_m$ is a super- and a subsolution to the equations

$$\begin{cases} 
\partial_t u_m(x,t) + P^-(D^2 u_m(x,t)) + \langle \mu, D u_m(x,t) \rangle - ru_m(x,t) = 0, \\
\partial_t u_m(x,t) + P^+(D^2 u_m(x,t)) + \langle \mu, D u_m(x,t) \rangle - ru_m(x,t) = 0,
\end{cases}$$

respectively. Therefore, the classical result of [Wan92, Theorem 4.19], see also [KS80], implies that the function $u_m$ is Hölder continuous with a Hölder constant independent of $m$. \hfill \square

We are now in a position to prove the main theorem of the paper.

\textbf{Proof of Theorem 1.3.} By the comparison principle Lemma 2.3 and (2.16), we see that the sequence $(u_m)$ of solutions to (2.17) is uniformly bounded in $m$. Hence, because Lemma 4.6 holds, by the Arzelà-Ascoli theorem, there exist $u$, continuous on $\mathbb{R}^n \times [0,T]$, and a subsequence $(m_j)$ such that it holds

$$u_{m_j} \to u$$

uniformly on $\mathbb{R}^n \times [0,T]$ as $j \to \infty$. By Lemmas 4.4 and 4.5, the stability principle for viscosity solutions yield that $u$ is a viscosity solution to (4.45). Therefore by Lemma 3.3, the final part is to show that the value function $U_m^+$ with uniformly bounded controls converges to the value function $U^-$ as $m \to \infty$. This follows from the properties of the infimum and the supremum, because the boundary values $g$ are bounded, and for the set of admissible strategies, it holds $\mathcal{S} = \bigcup_m S_m$. For more details, see for example [NP17, the proof of Theorem 1.2].

The corresponding proofs in the context of $U^+$, $U^+_m$ and the equation (2.18) are analogous to the above. In particular, let $u^+_m$ be the unique viscosity solution to (2.18). The proof of Lemma 3.2 for $u^+_m$ and $F^+_m$ is essentially the same as before. Then by minor adjustments to the proofs of Lemmas 3.3 and 3.5, we can show that $u^+_m = U^+_m$ on $\mathbb{R}^n \times [0,T]$. Finally, the uniform boundedness and the equicontinuity of the family $(u^+_m)$, together with the convergence of $U^+_m$ to $U^+$ as $m \to \infty$, follows as before. Therefore, the proof is complete. \hfill \square
References

[AB10] R. Atar and A. Budhiraja. A stochastic differential game for the inhomogeneous ∞-laplace equation. *Ann. Probab.*, 38(2):498–531, 2010.

[BL08] R. Buckdahn and J. Li. Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations. *SIAM J. Control Optim.*, 47(1):444–475, 2008.

[CGG91] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[CII90] M. G. Crandall and H. Ishii. The maximum principle for semicontinuous functions. *Differential Integral Equations*, 3(6):1001–1014, 1990.

[CI90] M. G. Crandall, H. Ishii, and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[EG92] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[Eva13] L. C. Evans. *An introduction to stochastic differential equations*. American Mathematical Society, Providence, RI, 2013.

[GGIS91] Y. Giga, S. Goto, H. Ishii, and M.-H. Sato. Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. *Indiana Univ. Math. J.*, 40(2):443–470, 1991.

[Ish89] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs. *Comm. Pure Appl. Math.*, 42(1):15–45, 1989.

[Ish95] H. Ishii. On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions. *Funkcial. Ekvac.*, 38(1):101–120, 1995.

[Jen88] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.*, 101(1):1–27, 1988.

[JLM01] P. Juutinen, P. Lindqvist, and J. J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.*, 33(3):699–717, 2001.

[Kat15] N. Katzourakis. *An introduction to viscosity solutions for fully nonlinear PDE with applications to calculus of variations in L∞*. SpringerBriefs in Mathematics. Springer, Cham, 2015.

[KMP12] B. Kawohl, J. J. Manfredi, and M. Parviainen. Solutions of nonlinear PDEs in the sense of averages. *J. Math. Pures Appl.*, 97(2):173–188, 2012.

[Kry09] N. V. Krylov. *Controlled diffusion processes*. Volume 14 of Stochastic Modeling and Applied Probability. Springer-Verlag, Berlin, 2009.

[KS79] N. V. Krylov and M. V. Safonov. An estimate for the probability of a diffusion process hitting a set of positive measure. *Dokl. Akad. Nauk SSSR*, 245(1):18–20, 1979.

[KS80] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980.

[MPR10] J.J. Manfredi, M. Parviainen, and J.D. Rossi. An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM J. Math. Anal.*, 42(5):2058–2081, 2010.

[MPR12] J.J. Manfredi, M. Parviainen, and J.D. Rossi. On the definition and properties of p-harmonious functions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 11(2):215–241, 2012.

[NP17] K. Nyström and M. Parviainen. Tug-of-war, market manipulation, and option pricing. *Mathematical Finance*, 27(2):279–312, 2017.
[PS08] Y. Peres and S. Sheffield. Tug-of-war with noise: a game-theoretic view of the $p$-Laplacian. *Duke Math. J.*, 145(1):91–120, 2008.

[PSSW09] Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson. Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.*, 22(1):167–210, 2009.

[Swi96] A. Swiech. Another approach to the existence of value functions of stochastic differential games. *J. Math. Anal. Appl.*, 204(3):884–897, 1996.

[Wan92] L. Wang. On the regularity theory of fully nonlinear parabolic equations. I. *Comm. Pure Appl. Math.*, 45(1):27–76, 1992.

Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland

E-mail address: joonas.heino@jyu.fi