QUASI-PERIODIC TRAVELLING WAVES FOR BEAM EQUATIONS WITH DAMPING ON 3-DIMENSIONAL RECTANGULAR TORI

BOCHAO CHEN
School of Mathematics and Statistics
Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University
Changchun, Jilin 130024, China
School of College of Mathematics, Jilin University
Changchun, Jilin 130012, China

YIXIAN GAO∗
School of Mathematics and Statistics
Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University
Changchun, Jilin 130024, China

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ABSTRACT. This paper concerns the mathematical analysis of quasi-periodic travelling wave solutions for beam equations with damping on 3-dimensional rectangular tori. Provided that the generators of the rectangular torus satisfy certain relationships, by excluding some values of two model parameters, we establish the existence of small amplitude quasi-periodic travelling wave solutions with three frequencies. Moreover, it can be shown that such solutions are either continuations of rotating wave solutions, or continuations of quasi-periodic travelling wave solutions with two frequencies, and that the set of two model parameters is dense in the positive quadrant.

1. Introduction. The classical linear theory of deformation derives the following Euler–Bernoulli beam equation

$$\rho(x)u_{tt} + (EI(x)u_{xx})_{xx} = 0, \quad t \in \mathbb{R}, x \in \Omega \subset \mathbb{R},$$

where $\rho$ is the mass density, $E$ is Young’s modulus of elasticity and the product $EI$ is the flexural rigidity. This model describes the motion of thin elastic beams for at least two hundred years old, see [22]. “Perhaps the most notable disadvantage associated with conservative systems is the fact that they do not occur in nature” ([5], p.433). In other words, in a real process, damping effect always exists in each dynamic system. “Dissipation mechanisms are called direct if they give rise

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∗ Corresponding author: Yixian Gao.
to supplementary dissipation terms in the original conservative equations” ([12]).

There are possibly three kinds of direct dissipation mechanisms

\begin{align*}
(i) \quad \alpha u_t, \quad & \quad (ii) \quad -\beta u_{xt}, \quad & \quad (iii) \quad \gamma u_{xxxx}
\end{align*}

contained in the damped beam equation. We call the term (i) external or viscous damping introduced by external, linear dampers. The term (ii) treated in [5] reflects that the damping force is proportional with the bending rate. It is the so-called structural damping. The term (iii) describes a situation where higher frequencies are more strongly damped than low ones. We call it internal or Kelvin–Voigt damping.

In this paper we are devoted to studying damped beam equations with nonlinear terms dependence in velocity on 3-dimensional rectangular tori as follows

\begin{equation}
\partial_1 u + \mu \Delta^2 u + \alpha \partial_2 u - \beta \partial_2 \Delta u + \gamma \partial_2 \Delta^2 u + m u = \lambda (\partial_2 u)^{2p+1}, \quad t \in \mathbb{R}, x \in \mathbb{T}_1^3,
\end{equation}

where $\Delta^2$ is the biharmonic operator with $\Delta = \sum_{k=1}^3 \partial_{x_k}^2$, and $p > 0$ is an integer. The parameter $\mu = EI/\rho$ stands for the elasticity coefficient. The parameter $m > 0$ describes the linear stiffness of the foundation, and the real numbers $\alpha, \beta, \gamma, \lambda$ are coefficients of friction.

Concerning damped beam equations, Herrmann [12] presented the oscillatory character of solutions for the following model

\begin{equation}
\partial_{tt} u + \mu \Delta^2 u + \alpha \partial_t u - \beta \partial_t \Delta u + \gamma \partial_t \Delta^2 u + m \partial_t u^+ - m \partial_t u^- = 0, \quad t \in \mathbb{R}, x \in \Omega \subset \mathbb{R},
\end{equation}

where $u^+ = \max\{u, 0\}$, and $m_+, m_-$ are constants (in particular, if $m_+ = m$, then $m_+ u^+ - m_- u^- = m u$). Emmerich and Thalhammer [10] obtained a weak solution of the beam equation with damping

\begin{equation}
\partial_{tt} u + \mu \Delta^2 u - \left( \mu_0 + \mu_1 \int_{\Omega} |\nabla u|^2 dx + \mu_2 \int_{\Omega} \nabla u \cdot \partial_2 \nabla u dx \right) \Delta u + \alpha \partial_t u - \beta \partial_t \Delta u + \gamma \partial_t \Delta^2 u + m u = f, \quad t \in (0, T), x \in \Omega \subset \mathbb{R}^d, q \in [2, \infty).
\end{equation}

For $\epsilon$ small enough, Kogelbauer and Haller [15] established the existence of spectral submanifolds (SSMs) for the Rayleigh beam equation with damping

\begin{equation}
\partial_{tt} u + \mu u_{xxxx} - \chi u_{xxtt} + \alpha u_t - \beta u_{xxt} + m u = f(u) + \epsilon g, \quad t \in \mathbb{R}, x \in (0, \pi)
\end{equation}

with respect to the initial data. Some recent results on beam equations with damping can be found in [9, 18, 21]. To the best of our knowledge, there is few articles on the existence of quasi-periodic travelling wave solutions for damped beam equations on rectangular tori so far. Generally, we say that $u$ is a time quasi-periodic traveling wave with rationally independent frequency vector $\omega = (\omega_1, \omega_2, \cdots, \omega_d) \in \mathbb{R}^d, d \geq 1$, namely, $\omega \cdot q \neq 0, \forall q \in \mathbb{Q}^d \setminus \{0\}$ (or $\omega \cdot j \neq 0, \forall j \in \mathbb{Z}^d \setminus \{0\}$) if there exists a function $\varphi : T^d \rightarrow \mathbb{R}$ such that

\begin{equation}
u(t, x) = \varphi(\omega_1 t + x_1, \omega_2 t + x_2, \cdots, \omega_d t + x_d).
\end{equation}

If $d = 1$, such a function corresponds to a periodic travelling wave. Let us review recent achievements associated with quasi-periodic travelling waves. Feola and Giuliani [11] proved the existence of small amplitude quasi-periodic travelling waves in time of the gravity water waves system with a periodic 1-dimensional interface in infinite depth. Berti et al. [1] concluded the existence of small amplitude quasi-periodic travelling waves in time of the gravity-capillary water waves equations with constant vorticity. Wilkening and Zhao [23] presented a numerical study of spatially quasi-periodic traveling waves on the surface of an ideal fluid of infinite depth.

We now introduce the definition of a rectangular torus.
Definition 1.1. Denote by $T^d_k$, $d \geq 1$ a $d$-dimensional rectangular torus defined by

$$T^d_k := (\mathbb{R}/2\pi L_1^1 \mathbb{Z}) \times (\mathbb{R}/2\pi L_2^2 \mathbb{Z}) \times \cdots \times (\mathbb{R}/2\pi L_d^d \mathbb{Z})$$

with generators $L_k > 0$, $k = 1, \ldots, d$.

Remark 1. (i) If at least one $L_k$ is “rationally independent” of the remaining ones, then $L_k$ cannot be written as a linear combination of the other $L_l$, $l \neq k$, $l = 1, \ldots, d$, with rational coefficients. In this case the corresponding torus is called irrational. When all $L_k$, $k = 1, \ldots, d$ are rational, the rectangular torus $T^d_k$ can be reduced to the standard one $T^d$ by a simple geometric consideration. That is, when $L_k = a_k/b_k$ for some $a_k, b_k \in \mathbb{N}$, we obtain the scaled standard torus $\kappa T^d = \mathbb{R}^d/(2\pi \kappa \mathbb{Z})^d$, where $\kappa$ is the least common multiple of $a_k$, $k = 1, \ldots, d$.

(ii) In particular, for $L_1 = L_2 = \cdots = L_d = 1$, such a torus becomes the $d$-dimensional standard one $T^d = \mathbb{R}^d/(2\pi \mathbb{Z})^d$.

In recent years there has been extensive interest in 3-dimensional irrational tori. In [2], Bourgain investigated the Strichartz estimates for Schrödinger equations on 3-dimensional irrational tori. Deng et al. [8] established the Strichartz estimates over large time scales for Schrödinger equations on 3-dimensional irrational tori, which are generic. Moreover, one is concerned with the phenomenon of growth of Sobolev norms of PDEs. Deng and Germain [7] gave polynomial bounds on the $H^s$ growth for Schrödinger equations on 3-dimensional irrational tori. Deng [6] also considered energy-critical Schrödinger equations on generic irrational tori, in dimension 3. In addition, Imekraz [13] proved a long time existence result for beam equations with small and smooth initial data on 2-dimensional irrational tori.

The present paper focuses on the existence of quasi-periodic travelling waves for damped beam equations on 3-dimensional rectangular tori. More precisely, we aim at constructing quasi-periodic travelling wave solutions for equation (1) on 3-dimensional rectangular tori of the form

$$u(t, x) = \varphi(\omega_1 t + L_1^{-1} x_1, \omega_2 t + L_2^{-1} x_2, \omega_3 t + L_3^{-1} x_3),$$

where $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ with $\omega \cdot q \neq 0$, $\forall q \in \mathbb{Q}^3 \setminus \{0\}$, and $\varphi$ is $(2\pi)^3$-periodic.

Define the energy functional as follows

$$\mathcal{E}(t) = \int_{T^d_k} \frac{1}{2} (\partial_t u(t, x))^2 + \frac{1}{2} \mu (\Delta u(t, x))^2 + \frac{1}{2} m(u(t, x))^2 \, dx.$$  

By periodic boundary conditions and integration by parts, we have

$$\frac{d}{dt} \mathcal{E}(t) = \int_{T^d_k} \partial_t u(t, x) (-\mu \Delta^2 u(t, x) - \alpha \partial_t u(t, x) + \beta \partial_t \Delta u(t, x) - \gamma \partial_t \Delta^2 u(t, x))$$

$$- m \partial_t u(t, x) u(t, x) + \mu \Delta u(t, x) \partial_t \Delta u(t, x) + m u(t, x) \partial_t u(t, x) + \lambda (\partial_t u(t, x))^2 dx$$

$$= \int_{T^d_k} -\alpha (\partial_t u(t, x))^2 - \beta \partial_t u(t, x) (-\Delta) \partial_t u(t, x)$$

$$- \gamma \partial_t u(t, x) \Delta^2 \partial_t u(t, x) + \lambda (\partial_t u(t, x))^2 dx.$$  

Because the operators $-\Delta, \Delta^2$ are positive semi-definite, for $\lambda > 0$, if we take $\alpha, \beta, \gamma \leq 0$, then $\mathcal{E}$ is increasing on $t$. Therefore, there will be a nontrivial smooth periodic/quasi-periodic solution to equation (1) if either $\alpha > 0$, or $\beta > 0$, or $\gamma > 0$. In fact, our model without external forcing describes the motion of a 3-dimensional beam with self-oscillation, see [14]. The phenomenon of self-oscillation is prevalent.
such as the heartbeat, the pupil light reflex, clocks and so on, see [14, 19]. Usually, a self-oscillator can generate and maintain a regular mechanical periodicity or quasi-periodicity without requiring a similar external periodicity or quasi-periodicity to drive it. In the experiments of [20], periodic and quasi-periodic vibrations of vehicle wheels were caused by time delay. Campbell et al. [3, 4] presented that a damped harmonic oscillator with time delay undergoes the bifurcation of not only periodic solutions, but also 2-dimensional tori. Recently, Kosovalić [16] constructed quasi-periodic travelling wave solutions of beam equations with damping on 2-dimensional standard tori.

Let $\mathbb{R}^+ := \{ x \in \mathbb{R} : x > 0 \}$ and $\mathbb{Q}^+ := \{ y \in \mathbb{Q} : y > 0 \}$. We now state precisely the main results.

**Theorem 1.2.** Fix positive generators $L_1, L_2, L_3$, with $L_1^2/L_2^2 \in \mathbb{Q}^+, L_1^2/L_2^2 \in \mathbb{Q}^+$, and positive integers $L_k^{-1} j_k^r \notin L_k^{-1} j_k^r$, $1 \leq k < k' \leq 3$. Let $\lambda > 0$ and $(\mu, m) \in S_{\mu, m}$, where

$$S_{\mu, m} := \left\{ (\mu, m) \in \mathbb{R}^+ \times \mathbb{Q}^+ : \mu L_1^{-4} \in \mathbb{Q}^+, \frac{m}{\mu} \not\in \mathbb{N}, 1 \leq p \leq 3, \left(\frac{\mu L_1^{-4}(j_1^r)^4 + m}{\mu L_1^{-4}(j_1^r)^4 + m}\right)^{\frac{1}{2}} \in \mathbb{R}^+ \setminus \mathbb{Q}, 1 \leq k < k' \leq 3, \right.$$

for $L_1 \in \mathbb{R}^+$, $L_2 \in \mathbb{R}^+$, $L_3 \in \mathbb{R}^+$, with $L_1^2/L_2^2 \in \mathbb{Q}^+$, $L_1^2/L_2^2 \in \mathbb{Q}^+$.

Then for amplitudes $\rho = (\rho_1, \rho_2, \rho_3) \approx 0$, for $(\alpha, \beta, \gamma) = (\alpha(\rho), \beta(\rho), \gamma(\rho)) \approx 0$, with frequencies $\omega = (\omega_1, \omega_2, \omega_3)$ near $\omega_0 = (\omega_1^*, \omega_2^*, \omega_3^*)$, where

$$\omega_{j_1^r} = \left(\frac{\mu L_1^{-4}(j_1^r)^4 + m}{(j_1^r)^2}\right)^{\frac{1}{2}}, \quad \omega_{j_2^r} = \left(\frac{\mu L_1^{-4}(j_2^r)^4 + m}{(j_2^r)^2}\right)^{\frac{1}{2}} \quad \omega_{j_3^r} = \left(\frac{\mu L_1^{-4}(j_3^r)^4 + m}{(j_3^r)^2}\right)^{\frac{1}{2}},$$

equation (1) admits a family of small amplitude quasi-periodic travelling wave solutions with three frequencies of the form

$$u(\rho)(t, x) = \varphi(\rho)(\omega_1 t + L_1^{-1} x_1, \omega_2 t + L_2^{-1} x_2, \omega_3 t + L_3^{-1} x_3),$$

where $\varphi(\rho)$ is a real-valued function with $(2\pi)^3$-period of the form

$$\varphi(\rho)(\omega_1 t + L_1^{-1} x_1, \omega_2 t + L_2^{-1} x_2, \omega_3 t + L_3^{-1} x_3) = \sum_{k=1}^{3} 2\rho_k \cos(\omega_k t + L_k^{-1} x_k) + w(\rho)(\omega_1 t + L_1^{-1} x_1, \omega_2 t + L_2^{-1} x_2, \omega_3 t + L_3^{-1} x_3),$$

and $w(\rho) \in C^\infty(\mathbb{T}^3; \mathbb{R})$. The mapping $\rho \mapsto \varphi(\rho) \in H^s$ is $C^\infty$ for all $s > 0$.

Moreover, these quasi-periodic travelling wave solutions of equation (1) branch off of rotating wave solutions, in the sense that setting two of the amplitudes to zero, gives a one-parameter family of rotating wave solutions of equation (1), that is

$$u(\rho_1, 0, 0)(t, x) = \varphi(\rho_1, 0, 0)(\omega_1 t + L_1^{-1} x_1),$$

$$u(0, \rho_2, 0)(t, x) = \varphi(0, \rho_2, 0)(\omega_2 t + L_2^{-1} x_2),$$

$$u(0, 0, \rho_3)(t, x) = \varphi(0, 0, \rho_3)(\omega_3 t + L_3^{-1} x_3).$$

And these quasi-periodic travelling wave solutions of equation (1) also branch off of quasi-periodic travelling wave solutions with two frequencies, in the sense that
setting one of the amplitudes to zero, gives a two-parameter family of quasi-periodic 
travelling wave solutions with two frequency of equation (1), namely,

\[ u(0, \rho_2, \rho_3)(t, x) = \varphi(0, \rho_2, \rho_3)(\omega_2 t + L_2^{-1} x_2, \omega_3 t + L_3^{-1} x_3), \]

\[ u(\rho_1, 0, \rho_3)(t, x) = \varphi(\rho_1, 0, \rho_3)(\omega_1 t + L_1^{-1} x_1, \omega_3 t + L_3^{-1} x_3), \]

\[ u(\rho_1, \rho_2, 0)(t, x) = \varphi(\rho_1, \rho_2, 0)(\omega_1 t + L_1^{-1} x_1, \omega_2 t + L_2^{-1} x_2). \]

Furthermore, the set \( S_{\mu, m} \) is dense in the positive quadrant.

Remark 2. In particular, Theorem 1.2 holds for \((L_1, L_2, L_3) = (1, \sqrt{2}, \sqrt{3})\).

The other theorem shows the “directions of bifurcation”.

Theorem 1.3. Under the assumptions in Theorem 1.2, if there exist the noncon-
stant quasi-periodic travelling wave solutions to equation (1), then either \( \alpha > 0 \), or \( \beta > 0 \), or \( \gamma > 0 \).

Proof. This theorem follows from an energy argument as above. \qed

Our proof is mainly based on common branching methods including both a 
Lyapunov–Schmidt reduction and the implicit function theorem. Due to the bihar-
monic operator, we are able to avoid a “small divisors problem”. We first linearize 
this model around zero when the friction coefficients \( \alpha, \beta, \gamma \) vanish. According to 
Fourier expansion, an equivalent equation associated with the Fourier coefficients is 
obtained. We regard \((\mu, m)\) as two model parameters. By excluding some values of 
\((\mu, m)\), we can fix three frequencies which are rationally independent. We further 
solve the equivalent equation with fixed frequencies. Because this problem is con-
sidered on a rectangular torus of dimension 3, three generators will appear in the 
equivalent equation with fixed frequencies. In order to guarantee its solvability, let 
generators \( L_1, L_2 \) and \( L_3 \) satisfy certain relationships. Then we reduce equivalently 
to a bifurcation equation and a range equation. Finally, the bifurcation equation 
and the range equation can be solved, respectively.

The paper is structured as follows: In Section 2, we apply the Lyapunov–Schmidt 
reduction to get a bifurcation equation together with a range equation. The project 
of Section 3 is to solve the range equation and the bifurcation equation via the 
classical implicit function theorem, respectively.

2. Lyapunov–Schmidt reduction. The present section is devoted to obtaining a 
bifurcation equation and a range equation by implementing the Lyapunov–Schmidt 
reduction. Define

\[ \theta := (\theta_1, \theta_2, \theta_3), \quad \nu := (\nu_1, \nu_2, \nu_3), \quad \Delta_\nu := \sum_{k=1}^{3} \nu_k^2 \partial^2_{\theta_k} \]

with

\[ \theta_k = \omega_k t + \nu_k x_k, \quad \nu_k = L_k^{-1}, \quad k = 1, 2, 3. \]

Putting the ansatz (2) into equation (1) yields that

\[ (\omega \cdot \nabla)^2 \varphi + \mu \Delta^2_\nu \varphi + \alpha (\omega \cdot \nabla) \varphi - \beta (\omega \cdot \nabla) \Delta_\omega \varphi + \gamma (\omega \cdot \nabla) \Delta^2_\nu \varphi + m \varphi \]

\[ = \lambda ((\omega \cdot \nabla) \varphi)^{2p+1}. \quad (3) \]
The eigenvalues of $-\Delta_\nu$ are

$$\lambda_j = \sum_{k=1}^{3} \nu_k^2 j_k^2, \quad j = (j_1, j_2, j_3) \in \mathbb{Z}^3.$$ 

For $s \geq 0$, define the Sobolev space $H^s$ as follows

$$H^s := H^s(\mathbb{T}^3; \mathbb{R}) = \{ \varphi(\theta) = \sum_{j \in \mathbb{Z}^3} \varphi_j e^{ij\cdot \theta} : \varphi_j = \varphi_{-j}, \| \varphi \|^2_s := \sum_{j \in \mathbb{Z}^3} (1 + |j|^{2s}) |\varphi_j|^2 < \infty \},$$

where $\overline{\varphi_j}$ is the complex conjugate of $\varphi_j$ and

$$|j|^2 := j \cdot j = \sum_{k=1}^{3} j_k^2, \quad \varphi_j := \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \varphi(\theta) e^{-ij\cdot \theta} d\theta.$$ 

If $s > 3/2$, then $H^s$ is a Banach algebra with respect to multiplication of functions, that is

$$\| \varphi \varphi \|_s \leq C(s) \| \varphi \|_s \| \varphi \|_s, \quad \forall \varphi, \varphi \in H^s.$$ 

Denote by $C^k(\mathbb{T}^3; \mathbb{R})$ the set made of $k$ times differentiable functions on $\mathbb{T}^3$, with values in $\mathbb{R}$. One has that for $s > k + 3/2$ with $k \in \mathbb{N}$,

$$H^s \subset C^k(\mathbb{T}^3; \mathbb{R}) \quad \text{and} \quad \| \varphi \|_{C^k} \leq C \| \varphi \|_s.$$ 

For the sake of simplify, we rewrite equation (3) as

$$L_{\omega, \alpha, \beta, \gamma} \varphi = F(\omega, \varphi),$$

where $L_{\omega, \alpha, \beta, \gamma}$ stands for the following linear differential operator

$$L_{\omega, \alpha, \beta, \gamma} : H^{s+5} \longrightarrow H^s,$$

$$\varphi \longmapsto (\omega \cdot \nabla)^2 \varphi + \mu \Delta_\nu^2 \varphi + \alpha (\omega \cdot \nabla) \varphi - \beta (\omega \cdot \nabla) \Delta_\nu \varphi + \gamma (\omega \cdot \nabla) \Delta_\nu^2 \varphi + m \varphi,$$

and $F$ denotes the composition operator as follows

$$F : \mathbb{R}^3 \times H^s \longrightarrow H^{s-1}, \quad (\omega, \varphi) \longmapsto \lambda ((\omega \cdot \nabla) \varphi)^{2p+1}.$$ 

We first introduce the smoothness of the above composition operator.

**Lemma 2.1.** Let $p > 0$ be an integer. Define

$$F : \mathbb{R}^3 \times H^s \longrightarrow H^{s-1}, \quad (\omega, \varphi) \longmapsto \lambda ((\omega \cdot \nabla) \varphi)^{2p+1},$$

where $\omega = (\omega_1, \omega_2, \omega_3)$. For $s \geq 3$, the mapping $F$ is $C^\infty$ in the Fréchet sense with respect to $(\omega_k, \varphi)$ satisfying

$$\partial_{\omega_k} F(\omega, \varphi) = \lambda (2p + 1) ((\omega \cdot \nabla) \varphi)^{2p} \partial_{\omega_k} \varphi, \quad k = 1, 2, 3,$$

$$DF(\omega, \varphi)[h] = \lambda (2p + 1) ((\omega \cdot \nabla) \varphi)^{2p} (\omega \cdot \nabla) h.$$ 

**Proof.** The detail of the proof can be as seen in [17]. 

Now we linearize equation (4) about $\varphi = 0$ at $\alpha = \beta = \gamma = 0$, which leads to

$$(\omega \cdot \nabla)^2 \varphi + \mu \Delta_\nu^2 \varphi + m \varphi = 0.$$ 

Thanks to the Fourier series, we obtain

$$-(\omega \cdot j)^2 + \mu |j|_\nu^4 + m = 0, \quad \omega \in \mathbb{R}^3, j \in \mathbb{Z}^3,$$
where
\[ \omega \cdot \nu = \omega_1 j_1 + \omega_2 j_2 + \omega_3 j_3, \quad |\nu|^2 = \nu_1^2 j_1^2 + \nu_2^2 j_2^2 + \nu_3^2 j_3^2. \] (6)

Due to the existence of $|\nu|$, there are only finitely many Fourier modes satisfying equation (5). Moreover, by fixing $j_k = j_k^* \in \mathbb{Z}$, $k = 1, 2, 3$, observe that
\[ (\omega_{j_1}^*, \omega_2, \omega_3, j_1^*, 0, 0), \quad (\omega_1, \omega_{j_2}^*, \omega_3, 0, j_2^*, 0), \quad (\omega_1, \omega_2, \omega_{j_3}^*, 0, 0, j_3^*), \]
with
\[ \omega_{j_1}^* = \frac{\mu \nu_1^2 (j_1^*)^4 + m}{\nu_1^2 (j_1^*)^2}, \quad \omega_{j_2}^* = \frac{\mu \nu_2^2 (j_2^*)^4 + m}{\nu_2^2 (j_2^*)^2}, \quad \omega_{j_3}^* = \frac{\mu \nu_3^2 (j_3^*)^4 + m}{\nu_3^2 (j_3^*)^2}, \]
are three solutions to equation (5).

From now on, we consider the “critical frequency”
\[ \omega_{j}^* = (\omega_{j_1}^*, \omega_{j_2}^*, \omega_{j_3}^*) \quad \text{with} \quad \omega_{j_k}^* = \left( \frac{\mu \nu_k^2 (j_k^*)^4 + m}{\nu_k^2 (j_k^*)^2} \right)^\frac{1}{2}, \]
for fixed $\nu_1 > 0, \nu_2 > 0, \nu_3 > 0$, with $\nu_2^2/\nu_1^2 \in \mathbb{Q}^+, \nu_3^2/\nu_1^2 \in \mathbb{Q}^+$, and fixed positive integers $\nu_k j_k^* \neq \nu_k' j_k'^*$, $1 \leq k < k' \leq 3$, define the set of parameters $(\mu, m)$ as
\[ \hat{S}_{\mu, m} := \left\{ (\mu, m) \in \mathbb{R}^+ \times \mathbb{Q}^+ : \mu \nu_1^2 \in \mathbb{Q}^+, \frac{m}{\nu_1^2} \notin \mathbb{N}, 1 \leq p \leq 3, \right. \]
\[ \left( \frac{\mu \nu_k^2 (j_k^*)^4 + m}{\nu_k^2 (j_k^*)^2} \right)^\frac{1}{2} \in \mathbb{R}^+ \setminus \mathbb{Q}, \quad 1 \leq k < k' \leq 3, \quad \text{for} \ \nu_1 \in \mathbb{R}^+, \right. \]
\[ \nu_2 \in \mathbb{R}^+, \nu_3 \in \mathbb{R}^+, \text{ with } \frac{\nu_2^2}{\nu_1^2} \in \mathbb{Q}^+, \frac{\nu_3^2}{\nu_1^2} \in \mathbb{Q}^+, \}

Remark that $\hat{S}_{\mu, m}$ and $S_{\mu, m}$ are two identical sets. In addition, for fixed $\nu_k j_k^* \neq \nu_k' j_k'^*$, it follows from
\[ \frac{\omega_{j_k}^*}{\omega_{j_k'^*}} \in \mathbb{R}^+ \setminus \mathbb{Q}, \quad 1 \leq k < k' \leq 3. \] (7)

**Lemma 2.2.** Let $\mu > 0$. For fixed positive numbers $\nu_1, \nu_2, \nu_3$, with $\nu_2^2/\nu_1^2 \in \mathbb{Q}^+, \nu_3^2/\nu_1^2 \in \mathbb{Q}^+$, and fixed positive integers $\nu_k j_k^* \neq \nu_k' j_k'^*$, $1 \leq k < k' \leq 3$, if
\[ \mu \nu_1^2 \in \mathbb{Q}^+, \quad \left( \frac{\mu \nu_k^2 (j_k^*)^4 + m}{\nu_k^2 (j_k^*)^2} \right)^\frac{1}{2} \in \mathbb{R}^+ \setminus \mathbb{Q}, \quad 1 \leq k < k' \leq 3, \]
then $\omega_{j_1}^*, \omega_{j_2}^*, \omega_{j_3}^*$ are rationally independent.

**Proof.** Assume that we could find three rational numbers $a, b$ and $c$, not all of which are zero, such that
\[ a \omega_{j_1}^* + b \omega_{j_2}^* + c \omega_{j_3}^* = 0. \] (8)

Then
\[ 2ab \omega_{j_2}^* \frac{\omega_{j_1}^*}{\omega_{j_2}^*} + a^2 \omega_{j_1}^* + b^2 \omega_{j_2}^* - c^2 \omega_{j_3}^* = 0. \]

When $ab \neq 0$, by the expressions of $\omega_{j_k}^*, k = 1, 2, 3$, we derive
\[ \mathbb{R}^+ \setminus \mathbb{Q} \equiv \frac{\omega_{j_1}^*}{\omega_{j_2}^*} = \frac{c^2 \nu_2^4 (j_2^*)^2 - a^2 \nu_1^4 (j_1^*)^2 - b^2 \nu_2^4 (j_2^*)^2 + m(\nu_1^2 - \nu_2^2 - \nu_3^2)}{2ab(\nu_2^4 (j_2^*)^2 + \frac{m}{\nu_1^2})}. \]
Since $\nu_2/\nu_4^2 \in \mathbb{Q}^+, \nu_4/\nu_4^4 \in \mathbb{Q}^+$ and $\mu\nu_4^4 \in \mathbb{Q^+}$, we derive $\mu\nu_4^2 \in \mathbb{Q^+}$ and $\mu\nu_4^3 \in \mathbb{Q^+}$, which leads to

$$c^2\mu\nu_4^2(j_2^2) - a^2\mu\nu_4^4(j_1^2)^2 - b^2\mu\nu_4^3(j_2^2)^2 + \frac{c^2}{(j_1^2)^2} - \frac{a^2}{(j_1^2)^2} - \frac{b^2}{(j_2^2)^2} \in \mathbb{Q}.$$ 

This leads to a contradiction. Hence either $a = 0$, or $b = 0$. In the first case, equality (8) is reduced to $b\nu_4^2/j_{\omega}^2 + c = 0$. Using (7) yields that $b = c = 0$. And in the latter, equality (8) becomes $a\nu_4^2/j_{\omega}^2 + c = 0$, which then gives $a = c = 0$. As a result, $a = b = c = 0$. Therefore we complete the proof of the lemma.

Moreover, we have to introduce an important fact.

**Lemma 2.3.** Let $q_1 \in \mathbb{R}\setminus\mathbb{Q}, q_2 \in \mathbb{R}\setminus\mathbb{Q}$ satisfy $q_1^2 \in \mathbb{Q}^+, q_2^2 \in \mathbb{Q}^+$ and $q_1/q_2 \in \mathbb{R}\setminus\mathbb{Q}$. Then $q_1, q_1q_2, q_2, 1$ are rationally independent.

**Proof.** Suppose that we could find four rational numbers $a, b, c$ and $d$, not all of which are zero, such that

$$aq_1 + bq_1q_2 + cq_2 + d = 0. \tag{9}$$

We further assume $d \neq 0$. By squaring equation (9), we conclude

$$(aq_1 + bq_1q_2)^2 = (-c q_2 - d)^2,$$

which leads to

$$(2abq_1^2 - 2cd)q_2^2 = c^2 q_2^2 + d^2 - a^2q_1^2 - b^2q_1^2q_2.$$ 

If $2abq_1^2 - 2cd \neq 0$, then

$$q_2 = \frac{c^2 q_2^2 + d^2 - a^2q_1^2 - b^2q_1^2q_2}{2abq_1^2 - 2cd}.$$ 

Since

$$q_2 \in \mathbb{R}\setminus\mathbb{Q}, \quad \frac{c^2 q_2^2 + d^2 - a^2q_1^2 - b^2q_1^2q_2}{2abq_1^2 - 2cd} \in \mathbb{Q},$$

this leads to a contradiction. Hence $c = \frac{2abq_1^2}{d}$. If we plug this back into (9), then

$$\left(bq_2^2 q_1/q_2 + d\right)(aq_1 + d) = 0.$$ 

It follows that either $bq_2^2 q_1 + d = 0$, or $aq_1 + d = 0$. In the first case $b = d = 0$, and in the latter $a = d = 0$. Hence the value of $d$ must be zero. This is in contradiction with the fact $d \neq 0$. As a result, $d = 0$.

For $d = 0$, equation (9) turns into

$$aq_1 + bq_1q_2 + cq_2 = 0.$$ 

By squaring the above equation, one has

$$2abq_1^2 q_2 = c^2 q_2^2 - a^2q_1^2 - b^2q_1^2q_2^2.$$ 

We can get that for $ab \neq 0$,

$$q_2 = \frac{c^2 q_2^2 - a^2q_1^2 - b^2q_1^2q_2^2}{2abq_1^2}.$$
Then it follows from (7) and Lemma 2.3 that
\[a^2q_2^2 - a^2q_1^2 - b^2q_1^2q_2^2 \in \mathbb{Q},\]
it follows that either \(a = 0\), or \(b = 0\). In the first case \(bq_1 + c = 0\), which gives rise to \(b = c = 0\). In the latter \(a\frac{q_1}{q_2} + c = 0\), which arrives at \(a = c = 0\). The above discussion shows that \(a = b = c = d = 0\).

Hence we arrive at the conclusion of the lemma.

The present goal aims at solving equation (5) with \(\omega = \omega_j^\ast\), i.e.,
\[-(\omega_j^\ast j_1 + \omega_j^\ast j_2 + \omega_j^\ast j_3)^2 + \mu(\nu^2_1 j^2_1 + \nu^2_2 j^2_2 + \nu^2_3 j^2_3)^2 + m = 0.\] (10)

**Lemma 2.4.** For fixed positive numbers \(\nu_1, \nu_2, \nu_3\) satisfying \(\nu^2_2/\nu^2_1 \in \mathbb{Q}^+, \nu^2_3/\nu^2_1 \in \mathbb{Q}^+, \) and fixed positive integers \(\nu_k j^+_k \neq \nu_{k'} j^+_k, 1 \leq k < k' \leq 3,\) if \((\mu, m)\) belong to \(\mathcal{S}_{\mu, m}\), then there exist only six solutions
\[j = (\pm j^+_1, 0, 0), (0, \pm j^+_2, 0), (0, 0, \pm j^+_3)\]
satisfying equation (10).

**Proof.** By the expressions of \(\omega_j^\ast, k = 1, 2, 3\), equation (10) is equivalent to
\[
2j_1 j_3 \frac{\omega^2_{j_1} + \omega^2_{j_3}}{\omega^2_{j_3}} + 2j_2 j_3 \frac{\omega^2_{j_2} + \omega^2_{j_3}}{\omega^2_{j_3}} + 2j_3 j_3 \frac{\omega^2_{j_3}}{\omega^2_{j_3}} - \frac{\mu \nu_1^2 (j^2_1 - (j^2_1)^2 j^2_1)}{\mu \nu_1^2 (j^2_1)^2 + \frac{m}{G_1^2}} - \frac{2 \mu \nu_2^2 (j^2_2 - (j^2_2)^2 j^2_2)}{2 \mu \nu_2^2 (j^2_2)^2 + \frac{m}{G_2^2}} - \frac{2 \mu \nu_3^2 (j^2_3 - (j^2_3)^2 j^2_3)}{2 \mu \nu_3^2 (j^2_3)^2 + \frac{m}{G_3^2}} + m(1 - \frac{j^2_1}{G_1^2} - \frac{j^2_2}{G_2^2} - \frac{j^2_3}{G_3^2})
\]

\[= 0.\]

Since \(\nu^2_2/\nu^2_1 \in \mathbb{Q}^+, \nu^2_3/\nu^2_1 \in \mathbb{Q}^+\) and \(\mu \nu_1^2 \in \mathbb{Q}^+, \mu \nu_3^2 \in \mathbb{Q}^+\), we deduce
\[\mu \nu_2^2 = \mu \nu_1^2(\nu^2_2/\nu^2_1)^2 \in \mathbb{Q}^+, \mu \nu_2^2 = \mu \nu_1^2(\nu^2_3/\nu^2_1)^2 \in \mathbb{Q}^+\).

Observe that
\[2j_1 j_3 \in \mathbb{Q}, 2j_1 j_2 \in \mathbb{Q}, 2j_2 j_3 \in \mathbb{Q}, \]
\[\mu \nu_1^2 (j^2_1 - (j^2_1)^2 j^2_1) + \mu \nu_2^2 (j^2_2 - (j^2_2)^2 j^2_2) + \mu \nu_3^2 (j^2_3 - (j^2_3)^2 j^2_3)
\]
\[\frac{\mu \nu_1^2 (j^2_1)^2 + \frac{m}{G_1^2}}{\mu \nu_1^2 (j^2_1)^2 + \frac{m}{G_1^2}} + \frac{2 \mu \nu_2^2 (j^2_2)^2 + \frac{m}{G_2^2}}{\mu \nu_2^2 (j^2_2)^2 + \frac{m}{G_2^2}} + \frac{2 \mu \nu_3^2 (j^2_3)^2 + \frac{m}{G_3^2}}{\mu \nu_3^2 (j^2_3)^2 + \frac{m}{G_3^2}} + m(1 - \frac{j^2_1}{G_1^2} - \frac{j^2_2}{G_2^2} - \frac{j^2_3}{G_3^2})
\]
\[\in \mathbb{Q}.
\]

Then it follows from (7) and Lemma 2.3 that
\[2j_1 j_3 = 0, 2j_1 j_2 = 0, 2j_2 j_3 = 0,\] (11)
\[\mu \nu_1^2 (j^2_1 - (j^2_1)^2 j^2_1) + \mu \nu_2^2 (j^2_2 - (j^2_2)^2 j^2_2) + \mu \nu_3^2 (j^2_3 - (j^2_3)^2 j^2_3)
\]
\[\frac{\mu \nu_1^2 (j^2_1)^2 + \frac{m}{G_1^2}}{\mu \nu_1^2 (j^2_1)^2 + \frac{m}{G_1^2}} + \frac{2 \mu \nu_2^2 (j^2_2)^2 + \frac{m}{G_2^2}}{\mu \nu_2^2 (j^2_2)^2 + \frac{m}{G_2^2}} + \frac{2 \mu \nu_3^2 (j^2_3)^2 + \frac{m}{G_3^2}}{\mu \nu_3^2 (j^2_3)^2 + \frac{m}{G_3^2}} + m(1 - \frac{j^2_1}{G_1^2} - \frac{j^2_2}{G_2^2} - \frac{j^2_3}{G_3^2})
\]
\[= 0.\] (12)
The rest of the proof is divided into the following two cases.

**Case 1.** \( j_1 \neq 0 \). By equality (11), we have \( j_2 = j_3 = 0 \). Then equality (12) becomes

\[
(m - \mu \nu_1^4(j_1^*)^2 j_1^*) ((j_1^*)^2 - j_1^2) = 0.
\]

If \( j_1^2 - (j_1^*)^2 \neq 0 \), then \( (j_1^*)^2 j_1^2 = \frac{m}{\mu \nu_1^4} \). Because of \( \frac{m}{\mu \nu_1^4} \notin \mathbb{N} \), we obtain \( j_1^2(j_1^*)^2 \neq \frac{m}{\mu \nu_1^4} \).

This leads to a contradiction. Hence \( (j_1^*)^2 - j_1^2 = 0 \), namely, \( j_1 = \pm j_1^* \).

**Case 2.** \( j_1 = 0 \). We further write equation (10) as

\[
-(\omega_{j_2} j_2 + \omega_{j_3} j_3)^2 + \mu (v_2^2 j_2^2 + v_3^2 j_3^2)^2 + m = 0. \tag{13}
\]

If \( j_2 \neq 0, j_3 \neq 0 \), then

\[
\frac{\omega_{j_2}}{\omega_{j_3}} = \frac{\mu \nu_1^2 (j_2^2 - (j_3^*)^2 j_3^2) + \mu \nu_1^2 (j_3^2 - (j_3^*)^2 j_3^2) + 2 \mu \nu_1^3 v_2^2 j_2 j_3}{2j_2 j_3 (\mu \nu_1^2 (j_3^*)^2 + \frac{m}{(\mu \nu_1^4)})} + \frac{m(1 - j_2^2 - j_3^2)}{2j_2 j_3 (\mu \nu_1^2 (j_3^*)^2 + \frac{m}{(\mu \nu_1^4)})}.
\]

This leads to a contradiction. Hence either \( j_2 = 0 \), or \( j_3 = 0 \). In the first case, we simplify equation (13) to

\[
(m - \mu \nu_1^3 (j_3^*)^2 j_3^*) ((j_3^*)^2 - j_3^2) = 0.
\]

Due to the fact \( \frac{m}{\mu \nu_3^3} \notin \mathbb{N} \), proceeding the similar technique as above yields that \( j_3 = \pm j_3^* \). And in the latter, by a similar argument as the case \( j_2 = 0 \), we get \( j_2 = \pm j_2^* \).

The proof of the lemma is now completed. \( \square \)

Now let us verify the density of the set \( \tilde{S}_{\mu, m} \) in the space \( \mathbb{R}^+ \times \mathbb{R}^+ \).

**Lemma 2.5.** For fixed positive numbers \( \nu_1, \nu_2, \nu_3 \) satisfying \( \nu_2^2/\nu_1^2 \in \mathbb{Q}^+, \nu_3^2/\nu_1^2 \in \mathbb{Q}^+ \), and fixed positive integers \( \nu_k j_k^* \neq \nu_k j_{k'}^*, 1 \leq k < k' \leq 3 \), the set \( \tilde{S}_{\mu, m} \) is dense in the space \( \mathbb{R}^+ \times \mathbb{R}^+ \).

**Proof.** By the density of the set of rational numbers composed by ratio of distinct primes, the following set

\[ Q := \{ q \in \mathbb{Q}^+ : \sqrt{q} \in \mathbb{R}^+\setminus\mathbb{Q} \} \]

is dense in \( \mathbb{R}^+ \). For \( \nu_1 \in \mathbb{R}^+, \nu_2 \in \mathbb{R}^+, \nu_3 \in \mathbb{R}^+, \) with \( \nu_2^2/\nu_1^2 \in \mathbb{Q}^+, \nu_3^2/\nu_1^2 \in \mathbb{Q}^+ \), denote

\[
\tilde{S}_{\mu, m}^1 := \{ (\mu, m) \in \mathbb{R}^+ \times \mathbb{Q}^+ : \mu \nu_1^4 \in \mathbb{Q}^+, \frac{m}{\mu \nu_1^4} \notin \mathbb{N}, 1 \leq k \leq 3, \frac{\mu \nu_1^4 (j_k^*)^2 + m}{\mu \nu_1^4 (j_k^*)^2 + m} \in \mathbb{R}^+\setminus\mathbb{Q} \},
\]

\[
\tilde{S}_{\mu, m}^2 := \{ (\mu, m) \in \mathbb{R}^+ \times \mathbb{Q}^+ : \mu \nu_1^4 \in \mathbb{Q}^+, \frac{m}{\mu \nu_1^4} \notin \mathbb{N}, 1 \leq k \leq 3, \frac{\mu \nu_1^4 (j_k^*)^2 + m}{\mu \nu_1^4 (j_k^*)^2 + m} \in \mathbb{R}^+\setminus\mathbb{Q} \},
\]

\[ S_{\mu, m} := \{ (\mu, m) \in \mathbb{R}^+ \times \mathbb{Q}^+ : \mu \nu_1^4 \in \mathbb{Q}^+, \frac{m}{\mu \nu_1^4} \notin \mathbb{N}, 1 \leq k \leq 3, \frac{\mu \nu_1^4 (j_k^*)^2 + m}{\mu \nu_1^4 (j_k^*)^2 + m} \in \mathbb{R}^+\setminus\mathbb{Q} \}.
\]
and
\[
\tilde{S}^3_{\mu,m} := \left\{ (\mu, m) \in \mathbb{R}^+ \times \mathbb{Q}^+ : \mu \nu_1^4 \in \mathbb{Q}^+, \frac{m}{\mu^4} \notin \mathbb{N}, 1 \leq k \leq 3, \right. \\
\left. \left( \frac{\mu^4 \nu_1^4}{\mu^4 \nu_1^4} + m \right)^{1/2} \in \mathbb{R}^+ \setminus \mathbb{Q} \right\}.
\]

We first consider the density of the set \( \tilde{S}^1_{\mu,m} \) in the space \( \mathbb{R}^+ \times \mathbb{R}^+ \). For \((\mu_0, m_0) \in \mathbb{R}^+ \times \mathbb{R}^+\), we wish to find \((\tilde{\mu}, \tilde{m}) \in \tilde{S}^3_{\mu,m} \cap B_{\epsilon}(\mu_0, m_0), \forall \epsilon > 0\), where
\[
B_{\epsilon}(\mu_0, m_0) := \{ (\mu, m) : |\mu - \mu_0| < \epsilon, |m - m_0| < \epsilon \}.
\]

If \( \nu_1^4 \in \mathbb{Q}^+ \), then there exists \( \tilde{\mu} \in (\mu_0 - \epsilon, \mu_0 + \epsilon) \cap \mathbb{Q}^+ \) such that \( \tilde{\mu} \nu_1^4 \in \mathbb{Q}^+ \). On the other hand, if \( \nu_1^4 \in \mathbb{R}^+ \setminus \mathbb{Q} \), then we can seek \( \tilde{\mu} \in (\mu_0 - \epsilon, \mu_0 + \epsilon) \cap (\mathbb{R}^+ \setminus \mathbb{Q}) \) satisfying \( \tilde{\mu} \nu_1^4 \in \mathbb{Q}^+ \). Moreover, it follows from the facts \( \nu_2^2 / \nu_1^2 \in \mathbb{Q}^+, \nu_3^2 / \nu_1^2 \in \mathbb{Q}^+ \) that \( \tilde{\mu} \nu_2^2 \in \mathbb{Q}^+, \tilde{\mu} \nu_3^2 \in \mathbb{Q}^+ \). Hence we can find \( m' \in (m_0 - \epsilon, m_0 + \epsilon) \) satisfying \( \frac{m'}{\mu^2} \notin \mathbb{N}, 1 \leq k \leq 3 \). As a result, there exists an open interval \( \mathcal{U} \subseteq (m_0 - \epsilon, m_0 + \epsilon) \) satisfying \( \frac{m}{\mu^2} \notin \mathbb{N}, 1 \leq k \leq 3, \forall m \in \mathcal{U} \). For fixed \( \nu_1, \nu_2, j^*_1, j^*_2 \), with \( \nu_1 j^*_1 \neq \nu_2 j^*_2 \), and fixed \( \tilde{\mu} \in (\mu_0 - \epsilon, \mu_0 + \epsilon) \), we further define the following mapping
\[
\Theta: \mathcal{U} \rightarrow \mathbb{R}^+, \quad m \mapsto \tilde{\mu} \nu_1^4 (j^*_1)^4 + m \left( \frac{\mu^4 \nu_1^4}{\mu^4 \nu_1^4} + m \right)^{1/2}.
\]

Clearly, \( \Theta \) is a strictly monotone function. By the intermediate value theorem together with the density of \( \mathcal{Q} \), there exists \( \tilde{m} \in \mathcal{U} \) with
\[
\Theta(\tilde{m}) \in \mathbb{Q}^+, \quad \sqrt{\Theta(\tilde{m})} \in \mathbb{R}^+ \setminus \mathbb{Q}.
\]

Moreover, we set \( \Theta(\tilde{m}) = \frac{\tilde{\mu} \nu_1^4 (j^*_1)^4 + \tilde{m}}{\mu^2 \nu_2^2 (j^*_2)^4 + m} = q \in \mathbb{Q}^+ \) with \( q \neq 1 \). Then
\[
\tilde{m} = \frac{\tilde{\mu} \nu_2^4 (j^*_2)^4 - \tilde{\mu} \nu_1^4 (j^*_1)^4}{1 - q} \in \mathbb{Q}^+.
\]

Therefore, \((\tilde{\mu}, \tilde{m}) \in \tilde{S}^3_{\mu,m} \cap B_{\epsilon}(\mu_0, m_0) \). By proceeding a similar procedure as above, the sets \( \tilde{S}^3_{\mu,m} \), \( \tilde{S}^3_{\mu,m} \) are dense in the space \( \mathbb{R}^+ \times \mathbb{R}^+ \) as well. Thus \( \tilde{S}^3_{\mu,m} \cap \tilde{S}^3_{\mu,m} \cap \tilde{S}^3_{\mu,m} \) is dense in the space \( \mathbb{R}^+ \times \mathbb{R}^+ \). Consequently, we obtain the density of \( \tilde{S}^3_{\mu,m} \) in the space \( \mathbb{R}^+ \times \mathbb{R}^+ \).

Hence we arrive at the conclusion of the lemma. \(\square\)

By the above discussion, the critical cases in which \((\omega, \alpha, \beta, \gamma) = (\omega_{jr^+}, 0, 0, 0)\) will be considered. Set
\[
J = \left\{ j \in \mathbb{Z}^3 : j \neq (\pm j^*_1, 0, 0), (0, \pm j^*_2, 0), (0, 0, \pm j^*_3) \right\}.
\]

Then its orthogonal complement in \( \mathbb{Z}^3 \) is
\[
J^\perp = \left\{ j \in \mathbb{Z}^3 : j = (\pm j^*_1, 0, 0), (0, \pm j^*_2, 0), (0, 0, \pm j^*_3) \right\}.
\]

Denote by \( V \) and \( W \) the kernel space of the operator \( L_{\omega_{jr^+}, 0, 0, 0} \) and its orthogonal complement in \( H^0 \), respectively. Thanks to Lemma 2.4, we summarize the following lemma.

**Lemma 2.6.** *The space \( V \) is six-dimensional with*
\[
V = \left\{ v = \sum_{j \in J^\perp} v_j e^{ij \theta} \in H^0 \right\}.
\]
By Lemma 2.6, we write the space $W$ as

$$ W = \left\{ w = \sum_{j \in J} w_j e^{ij \cdot \theta} \in H^0 \right\}. $$

Observe that

$$ H^* = (V \cap H^*) \oplus (W \cap H^*). $$

Therefore, for every $\varphi \in H^*$, we write $\varphi = v + w$, where $v \in V \cap H^*$ and $w \in W \cap H^*$.

By implementing the Lyapunov–Schmidt reduction with respect to the above decomposition, equation (4) is equivalent to the range equation

$$ L_{\omega,\alpha,\beta,\gamma} w = \Pi_W F(\omega, v + w) \quad (14) $$

and the bifurcation equation

$$ L_{\omega,\alpha,\beta,\gamma} v = \Pi_V F(\omega, v + w), \quad (15) $$

where $\Pi_V : H^* \rightarrow V$ and $\Pi_W : H^* \rightarrow W$ stand for the projection operators on $V$ and $W$, respectively.

In the space $V$, we derive that for some phase $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{R}^3$,

$$ v(\theta) = 2 \operatorname{Re}(v_{j_1,0,0}) \cos(j_1 \theta_1) - 2 \operatorname{Im}(v_{j_1,0,0}) \sin(j_1 \theta_1) + 2 \operatorname{Re}(v_{0,j_2,0}) \cos(j_2 \theta_2) $$

$$ - 2 \operatorname{Im}(v_{0,j_2,0}) \sin(j_2 \theta_2) + 2 \operatorname{Re}(v_{0,0,j_3}) \cos(j_3 \theta_3) - 2 \operatorname{Im}(v_{0,0,j_3}) \sin(j_3 \theta_3) $$

$$ = \hat{\rho}_1 \cos(j_1 \theta_1 + \phi_1) + \hat{\rho}_2 \cos(j_2 \theta_2 + \phi_2) + \hat{\rho}_3 \cos(j_3 \theta_3 + \phi_3), $$

where

$$ \hat{\rho}_1 = 2 \sqrt{(\operatorname{Re}(v_{j_1,0,0}))^2 + (\operatorname{Im}(v_{j_1,0,0}))^2}, \quad \hat{\rho}_2 = 2 \sqrt{(\operatorname{Re}(v_{0,j_2,0}))^2 + (\operatorname{Im}(v_{0,j_2,0}))^2}, $$

$$ \hat{\rho}_3 = 2 \sqrt{(\operatorname{Re}(v_{0,0,j_3}))^2 + (\operatorname{Im}(v_{0,0,j_3}))^2}. $$

Moreover, if $\varphi(\theta)$ is a solution of equation (4), then so does $\hat{\varphi}(\theta) = \varphi(\theta_1 + \phi_1, \theta_2 + \phi_2, \theta_3 + \phi_3)$. Hence we can take $\phi = 0$, which leads to

$$ v(\hat{\phi})(\theta) = \hat{\rho}_1 \cos(j_1 \theta_1) + \hat{\rho}_2 \cos(j_2 \theta_2) + \hat{\rho}_3 \cos(j_3 \theta_3), $$

For $\rho = (\rho_1, \rho_2, \rho_3)$, we further express $v$ as

$$ v(\rho)(\theta) = \rho_1(e^{ij_1 \theta_1} + e^{-ij_1 \theta_1}) + \rho_2(e^{ij_2 \theta_2} + e^{-ij_2 \theta_2}) + \rho_3(e^{ij_3 \theta_3} + e^{-ij_3 \theta_3}), \quad (16) $$

where $\rho_k = \frac{1}{2} \hat{\rho}_k, k = 1, 2, 3$. Denote

$$ \Pi_{(j_1,0,0)} F(\omega, v(\rho) + w) := \frac{1}{8\pi^2} \int_{T^3} F(\omega, v(\rho) + w) e^{-ij_1 \theta_1} d\theta, $$

$$ \Pi_{(-j_1,0,0)} F(\omega, v(\rho) + w) := \frac{1}{8\pi^2} \int_{T^3} F(\omega, v(\rho) + w) e^{ij_1 \theta_1} d\theta, $$

$$ \Pi_{(0,j_2,0)} F(\omega, v(\rho) + w) := \frac{1}{8\pi^2} \int_{T^3} F(\omega, v(\rho) + w) e^{ij_2 \theta_2} d\theta, $$

$$ \Pi_{(-0,j_2,0)} F(\omega, v(\rho) + w) := \frac{1}{8\pi^2} \int_{T^3} F(\omega, v(\rho) + w) e^{-ij_2 \theta_2} d\theta, $$

$$ \Pi_{(0,0,j_3)} F(\omega, v(\rho) + w) := \frac{1}{8\pi^2} \int_{T^3} F(\omega, v(\rho) + w) e^{ij_3 \theta_3} d\theta, $$

$$ \Pi_{(0,0,-j_3)} F(\omega, v(\rho) + w) := \frac{1}{8\pi^2} \int_{T^3} F(\omega, v(\rho) + w) e^{-ij_3 \theta_3} d\theta. $$

If we plug (16) back into (14) and (15), then

$$ L_{\omega,\alpha,\beta,\gamma} w = \Pi_W F(\omega, v(\rho) + w), \quad (17) $$
and
\[
\begin{align*}
\rho_1 \left( -\omega_1^2 (j_1)^2 + \mu v_1^2 (j_1)^4 + m + i \alpha \omega_1 j_1^* + i \beta \omega_1 j_1^* + i \gamma \omega_1 j_1^* \right) \\
= \Pi_{(j_1^*, 0, 0)} F(\omega, v(\rho) + w),
\end{align*}
\]
\[
\begin{align*}
\rho_1 \left( -\omega_1^2 (j_1)^2 + \mu v_1^2 (j_1)^4 + m - i \alpha \omega_1 j_1^* - i \beta \omega_1 j_1^* - i \gamma \omega_1 j_1^* \right) \\
= \Pi_{(j_1^*, 0, 0)} F(\omega, v(\rho) + w),
\end{align*}
\]
\[
\begin{align*}
\rho_2 \left( -\omega_2^2 (j_2)^2 + \mu v_2^2 (j_2)^4 + m + i \alpha \omega_2 j_2^* + i \beta \omega_2 j_2^* + i \gamma \omega_2 j_2^* \right) \\
= \Pi_{(0, j_2^*, 0)} F(\omega, v(\rho) + w),
\end{align*}
\]
\[
\begin{align*}
\rho_2 \left( -\omega_2^2 (j_2)^2 + \mu v_2^2 (j_2)^4 + m - i \alpha \omega_2 j_2^* - i \beta \omega_2 j_2^* - i \gamma \omega_2 j_2^* \right) \\
= \Pi_{(0, j_2^*, 0)} F(\omega, v(\rho) + w),
\end{align*}
\]
\[
\begin{align*}
\rho_3 \left( -\omega_3^2 (j_3)^2 + \mu v_3^2 (j_3)^4 + m + i \alpha \omega_3 j_3^* + i \beta \omega_3 j_3^* + i \gamma \omega_3 j_3^* \right) \\
= \Pi_{(0, 0, j_3^*)} F(\omega, v(\rho) + w),
\end{align*}
\]
\[
\begin{align*}
\rho_3 \left( -\omega_3^2 (j_3)^2 + \mu v_3^2 (j_3)^4 + m - i \alpha \omega_3 j_3^* - i \beta \omega_3 j_3^* - i \gamma \omega_3 j_3^* \right) \\
= \Pi_{(0, 0, j_3^*)} F(\omega, v(\rho) + w).
\end{align*}
\]

For \((\rho, \omega, \alpha, \beta, \gamma) \approx (0, \omega_*, 0, 0, 0)\), the remainder of this article is to seek solutions of (17) and (18), respectively.

3. Proof of the main results. The object of this section is to look for solutions of the range equation (17) and the bifurcation system (18), respectively. Moreover, we show \(C^\infty\) smoothness of solutions with respect to the amplitude parameters.

3.1. Solutions of the range equation. We take \(K \geq 1\) large enough. Define
\[
J_1 := \left\{ j \in J : |j|^2 \geq K \right\}, \quad J_2 := J \setminus J_1.
\]
Then \(J = J_1 \oplus J_2\). We further decompose \(W = Y \oplus Z\), where
\[
Y = \left\{ y = \sum_{j \in J_1} y_j e^{ij \theta} \in H^0 \right\}, \quad Z = \left\{ z = \sum_{j \in J_2} z_j e^{ij \theta} \in H^0 \right\}.
\]
Corresponding to the above decomposition, we split up (17) into
\[
L_{\omega, \alpha, \beta, \gamma} y - \Pi_Y F(\omega, v(\rho) + y + z) = 0, \quad (19)
\]
\[
L_{\omega, \alpha, \beta, \gamma} z - \Pi_Z F(\omega, v(\rho) + y + z) = 0. \quad (20)
\]
Observe that for \(y \in Y \cap H^{s+5}\),
\[
(L_{\omega, \alpha, \beta, \gamma} y)(\theta) = \sum_{j \in J_1} \eta(j, \omega, \alpha, \beta, \gamma) y_j e^{ij \theta}
\]
with
\[
\eta(j, \omega, \alpha, \beta, \gamma) := - (\omega \cdot j)^2 + \mu |j|^4 + m + i \alpha (\omega \cdot j) + i \beta (\omega \cdot j) |j|^2 + i \gamma (\omega \cdot j) |j|^4,
\]
where \(| \cdot |^2\) is as seen in (6). Moreover, denote by \(B_r(\omega_*)\) a neighborhood of \(\omega_*\) in \(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\), that is
\[
B_r(\omega_*) := \left\{ \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ : |\omega_k - \omega_*^k| < r \right\}. \quad (22)
\]
We first check the invertibility of the operator \(L_{\omega, \alpha, \beta, \gamma}\) restricted to \(Y \cap H^{s+5}\).
Lemma 3.1. Let $s > 0$. Then for all $\omega \in \mathcal{B}_r(\omega_j)$ and $(\alpha, \beta, \gamma) \in \mathbb{R}^3$, the operator $L_{\omega, \alpha, \beta, \gamma} : Y \cap H^{s+5} \to Y \cap H^s$ is invertible with
\[ L_{\omega, \alpha, \beta, \gamma}^{-1} : Y \cap H^s \to Y \cap H^{s+2}. \] Moreover there exists some constant $K \geq 1$ large enough such that for all $j \in J_1$, $\omega \in \mathcal{B}_r(\omega_j)$ and $(\alpha, \beta, \gamma) \in \mathbb{R}^3$,
\[ |\eta(j, \omega, \alpha, \beta, \gamma)| \geq K. \] Proof. Clearly, $L_{\omega, \alpha, \beta, \gamma}$ is a linear operator from $Y \cap H^{s+5}$ to $Y \cap H^s$. For all $\omega \in \mathcal{B}_r(\omega_j)$, there is $C = C(\omega_j, r) > 0$ such that $|\omega| \leq C$. We obtain that for all $j \in J_1$, $\omega \in \mathcal{B}_r(\omega_j)$ and $(\alpha, \beta, \gamma) \in \mathbb{R}^3$,
\[ |\eta(j, \omega, \alpha, \beta, \gamma)| \geq |-(\omega \cdot j)^2 + \mu |j|^4 + m| \geq m + \mu |j|^4 - (\omega_1 j_1 + \omega_2 j_2 + \omega_3 j_3)^2 \]
\[ \geq \mu (\min \{v_1, v_2, v_3\})^4 |j|^4 - C^2 (j_1^2 + j_2^2 + j_3^2 + 2 |j_1 j_2| + 2 |j_1 j_3| + 2 |j_2 j_3|) \]
\[ \geq |j|^2 (\mu (\min \{v_1, v_2, v_3\})^4 |j|^4 - 3 C^2) \geq |j|^2 \]
\[ \geq K \]
if $|j|^2 \geq K \geq \frac{3 C^2 + 1}{\mu (\min \{v_1, v_2, v_3\})^4}$. Hence $L_{\omega, \alpha, \beta, \gamma}$ is invertible.
It remains to verify (23). For $y \in Y \cap H^s$, one has
\[ (L_{\omega, \alpha, \beta, \gamma}^{-1} y)(\theta) = \sum_{j \in J_1} \frac{y_j}{\eta(j, \omega, \alpha, \beta, \gamma)} e^{ij \theta}. \]
By means of (24) and (25), we conclude
\[ \|L_{\omega, \alpha, \beta, \gamma}^{-1} y\|_{s+2} \leq \sum_{j \in J_1} \frac{1 + |j|^{2s+4}}{|j|^4} |y_j|^2 \leq \|y\|_{s}^2 < \infty. \]
Thus the proof of the lemma is now completed. \hfill \square

From Lemma 3.1, we rewrite equation (19) as
\[ y - L_{\omega, \alpha, \beta, \gamma}^{-1} \Pi_F (\omega, v(\rho) + y + z) = 0. \] (26)
Furthermore, by proceeding the similar procedure as in the proof of Lemma 3.1, we can show the continuity of the operator $L_{\omega, \alpha, \beta, \gamma}^{-1}$ with respect to $\omega, \alpha, \beta, \gamma$.

Lemma 3.2. Let $s > 0$. The mapping $\mathcal{B}_r(\omega_j) \times \mathbb{R}^3 \ni (\omega, \alpha, \beta, \gamma) \mapsto L_{\omega, \alpha, \beta, \gamma}^{-1} \in \mathcal{L}(Y \cap H^s)$ is continuous with respect to the uniform operator topology, where $\mathcal{L}(Y \cap H^s)$ stands for the space consisted of continuous linear operators from $Y \cap H^s$ to $Y \cap H^s$.

The following proposition addresses the existence of solutions to equation (26).

Proposition 1. Let $s \geq 3$ and $\sigma_0 = (0, \omega_j, 0, 0, 0, 0)$. Then there exists a solution
\[ y = y(\rho, \omega, \alpha, \beta, \gamma, z) \in Y \cap H^s \]
of equation (26) in a neighborhood of $\sigma_0$. Moreover, $y, \partial_{\rho} y, k = 1, 2, 3$ and $D_{\omega_j} y$ vary continuously near $\sigma_0$ in $\rho, \omega, \alpha, \beta, \gamma, z$. In particular, if $(\rho, z) = (0, 0)$, then
\[ y(0, \omega, \alpha, \beta, \gamma, 0) = 0, \quad \partial_{\rho} y(0, \omega, \alpha, \beta, \gamma, 0) = 0, \quad k = 1, 2, 3. \] (27)
Furthermore, there exists an $s$-independent neighborhood $\mathcal{B}_k(\sigma_0)$ of $\sigma_0$ such that equation (26) has a unique solution in $C^\infty(T^4; \mathbb{R})$ which coincides with the solution in $Y \cap H^s$. 

Proof. For $s \geq 3$, via Lemma 2.1 and Lemma 3.1, define
\[ G_1 : \mathbb{R}^3 \times B_{t_r}(\omega_j) \times \mathbb{R}^3 \times (Z \cap H^s) \times (Y \cap H^s) \rightarrow Y \cap H^s, \]
\[
(\rho, \omega, \alpha, \beta, \gamma, y) \mapsto y - L_{\omega, \alpha, \beta, \gamma}^{-1} \Pi_Y F(\omega, v(\rho) + y + z).
\]
Observe that $G(0, \omega, y) = 0$. It follows from Lemma 2.1 and Lemma 3.2 that $\partial_{\rho_k} G_1, D_y G_1$ exist, and $G_1, \partial_{\rho_k} G_1, D_y G_1$ are continuous with respect to $\rho, \omega, \alpha, \beta, \gamma, z$. Moreover, using Lemma 2.1 yields that for all $y \in Y \cap H^s$,
\[
D_y F(\omega, v(\rho) + y + z)[y] = \lambda(2p + 1)((\omega \cdot \nabla)(v(\rho) + y + z))^{2p}(\omega \cdot \nabla)y.
\]
It is evident that for $(\rho, z, y) = (0, 0, 0)$,
\[
D_y F(\omega, 0)[y] = 0.
\]
Therefore,
\[
D_y G_1(0, 0, 0, 0, 0)[y] = y.
\]
In view of the implicit function theorem, there is $y(\rho, \omega, \alpha, \beta, \gamma, z) \in Y \cap H^s$ satisfying equation (26) in a neighborhood of $\sigma_0$, and $y, \partial_{\rho_k} y, D_y y$ vary continuously in $\rho, \omega, \alpha, \beta, \gamma, z$. Notice that
\[
G(0, \omega, \alpha, \beta, \gamma, 0) = -L_{\omega, \alpha, \beta, \gamma}^{-1} \Pi_Y F(\omega, 0) = 0.
\]
The uniqueness coming from the implicit function theorem gives that
\[
y(0, \omega, \alpha, \beta, \gamma, 0) = 0.
\]
We further differentiate the implicit equation
\[
G_1(\rho, \omega, \alpha, \beta, \gamma, z, y(\rho, \omega, \alpha, \beta, \gamma, z)) = 0
\]
with respect to $\rho_k$. Then
\[
\partial_{\rho_k} G_1(\rho, \omega, \alpha, \beta, \gamma, z, y(\rho, \omega, \alpha, \beta, \gamma, z))
\]
\[
= \partial_{\rho_k} y(\rho, \omega, \alpha, \beta, \gamma, z)
\]
\[
- \lambda(2p + 1)L_{\omega, \alpha, \beta, \gamma}^{-1} \Pi_Y ((\omega \cdot \nabla)(v(\rho) + y(\rho, \omega, \alpha, \beta, \gamma, z) + z))^{2p}
\]
\[
\times (\omega \cdot \nabla)(\partial_{\rho_k} v(\rho) + \partial_{\rho_k} y(\rho, \omega, \alpha, \beta, \gamma, z)).
\]
For $(\rho, z, \partial_{\rho_k} y) = (0, 0, 0)$, one has $\partial_{\rho_k} G_1 = 0$. By uniqueness, we derive
\[
\partial_{\rho_k} y(0, 0, 0, 0, 0) = 0.
\]
In particular, fix $\delta \geq 3$. By the above discussion, equation (26) admits a solution $\tilde{y} : B_{t_r}(\sigma_0) \rightarrow Y \cap H^s$ satisfying
\[
\tilde{y}(\rho, \omega, \alpha, \beta, \gamma, z) = L_{\omega, \alpha, \beta, \gamma}^{-1} \Pi_Y F(\omega, v(\rho) + \tilde{y}(\rho, \omega, \alpha, \beta, \gamma, z) + z).
\]
From Lemma 2.1 and Lemma 3.1, we obtain
\[
L_{\omega, \alpha, \beta, \gamma}^{-1} \Pi_Y F(\omega, v(\rho) + \tilde{y}(\rho, \omega, \alpha, \beta, \gamma, z) + z) \in Y \cap H^{s+1},
\]
which then gives $\tilde{y}(\rho, \omega, \alpha, \beta, \gamma, z) \in Y \cap H^{s+1}$. Via a direct bootstrap argument, we have
\[
\tilde{y}(\rho, \omega, \alpha, \beta, \gamma, z) \in Y \cap H^{s+k}, \quad \forall k \geq 0.
\]
It follows from Sobolev embedding that $\tilde{y}(\rho, \omega, \alpha, \beta, \gamma, z) \in C^\infty(T^3; \mathbb{R})$. Hence we write $y = \tilde{y}$ by uniqueness. This ends the proof of the proposition.
Because of Proposition 1, there is an $s$-independent neighborhood of $\sigma_0$ such that equation (19) has a solution $y = y(\rho, \omega, \alpha, \beta, \gamma, z) \in C^\infty(T^3; \mathbb{R}) \cap (Y \cap H^s)$ with $s > 0$. Substituting it into equation (20) yields that

$$L_{\omega, \alpha, \beta, \gamma}z - \Pi_Z F(\omega, v(\rho)) + y(\rho, \omega, \alpha, \beta, \gamma, z) + z = 0. \quad (28)$$

Let us solve equation (28).

**Proposition 2.** Let $s > 0$ and $\sigma_1 = (0, \omega_j, 0, 0, 0)$. Then equation (28) admits a solution

$$z = z(\rho, \omega, \alpha, \beta, \gamma) \in C^\infty(T^3; \mathbb{R}) \cap (Z \cap H^s)$$

in an $s$-independent neighborhood $\mathcal{B}_{\delta_1}(\sigma_1)$ of $\sigma_1$ with $\delta_1 \leq \delta$. Furthermore, $z$ and $\partial_{\rho_k} z, k = 1, 2, 3$ vary continuously near $\sigma_1$ in $\rho, \omega, \alpha, \beta, \gamma$. In particular, we have that for $\rho = 0$,

$$z(0, \omega, \alpha, \beta, \gamma) = 0, \quad \partial_{\rho_k} z(0, \omega, \alpha, \beta, \gamma) = 0, \quad k = 1, 2, 3. \quad (29)$$

**Proof.** The definition of the set $J_2$ shows that the space $Z$ is finite dimensional. Define

$$G_2 : \mathbb{R}^3 \times \mathcal{B}_r(\omega_j^*) \times \mathbb{R}^3 \times (Z \cap H^s) \longrightarrow Z \cap H^s,$$

$$(\rho, \omega, \alpha, \beta, \gamma, z) \longmapsto L_{\omega, \alpha, \beta, \gamma}z - \Pi_Z F(\omega, v(\rho)) + y(\rho, \omega, \alpha, \beta, \gamma, z) + z.\)$$

Note that $G_2(0, \omega_{j^*}, 0, 0, 0, 0) = 0$. Thanks to Lemma 2.1, the mapping $G_2$ is continuous with respect to $\rho, \omega, \alpha, \beta, \gamma, z$, and $\partial_{\rho_k} G_2, D_z G_2$ exist varying continuously in $\rho, \omega, \alpha, \beta, \gamma, z$. In addition, we derive that for $(\rho, z, y) = (0, 0, 0)$,

$$D_z F(\omega, 0) = 0.$$

Since $Z$ is a subspace of the orthogonal complement of the kernel of the operator $L_{\omega_{j^*}, 0, 0, 0, 0}$, the following

$$D_z G_2(0, \omega_{j^*}, 0, 0, 0, 0) = L_{\omega_{j^*}, 0, 0, 0, 0}$$

is invertible from $Z \cap H^s$ to $Z \cap H^s$. Via the implicit function theorem, there exists a neighborhood of $\sigma_1$ such that $z = z(\rho, \omega, \alpha, \beta, \gamma)$, with values in $Z \cap H^s$, solves equation (28). Moreover, $\partial_{\rho_k} z$ is continuous in $\rho, \omega, \alpha, \beta, \gamma$.

In the rest, according to formula (27), proceeding as in the proof of Proposition 1 yields that these terms in (29) and $z(\rho, \omega, \alpha, \beta, \gamma) \in C^\infty(T^3; \mathbb{R})$. We have thus proved the proposition.

The following proposition addresses that if one of the amplitudes is taken as zero, then there is a two-parameter family of quasi-periodic travelling wave solutions with two frequencies of equation (17), and that if two of the amplitudes is set to zero, then there is a one-parameter family of rotating wave solutions of equation (17).

**Proposition 3.** Let $s > 0$. Then the range equation (17) has a solution $w = w(\rho, \omega, \alpha, \beta, \gamma)$, with values in $C^\infty(T^3; \mathbb{R}) \cap (W \cap H^s)$, in an $s$-independent neighborhood of $(0, \omega_{j^*}, 0, 0, 0)$ satisfying

(i) $w(\rho, \omega, \alpha, \beta, \gamma) = z(\rho, \omega, \alpha, \beta, \gamma) + y(\rho, \alpha, \beta, \gamma, z(\rho, \omega, \alpha, \beta, \gamma))$. 

where \( y \in C^\infty(T^3; \mathbb{R}) \cap (Y \cap H^s) \) and \( z \in C^\infty(T^3; \mathbb{R}) \cap (Z \cap H^s) \) are solutions of equation (19) and equation (20), respectively. Moreover,

(ii) \( w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma) \) is \( \theta_1 \)-independent, i.e.,
\[
\partial_{\theta_1} w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0,
\]

(iii) \( w(\rho_1, 0, \rho_3, \omega, \alpha, \beta, \gamma) \) is \( \theta_2 \)-independent, i.e.,
\[
\partial_{\theta_2} w(\rho_1, 0, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0,
\]

(iv) \( w(\rho_1, \rho_2, 0, \omega, \alpha, \beta, \gamma) \) is \( \theta_3 \)-independent, i.e.,
\[
\partial_{\theta_3} w(\rho_1, \rho_2, 0, \omega, \alpha, \beta, \gamma)(\theta) = 0,
\]

and

(v) \( w(0, 0, \rho_3, \omega, \alpha, \beta, \gamma) \) is \( \theta_1, \theta_2 \)-independent, i.e.,
\[
\partial_{\theta_1} w(0, 0, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0, \quad \partial_{\theta_2} w(0, 0, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0,
\]

Proof. Obviously, the property (i) follows from Proposition 1 and Proposition 2. Now let us verify the property (ii). For \( \rho \in \mathbb{R} \), we apply the Lyapunov–Schmidt reduction to get

\[
\partial_{\theta_1} w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0, \quad \partial_{\theta_2} w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0, \quad \partial_{\theta_3} w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)(\theta) = 0.
\]

where \( \tilde{\omega}(\theta) = \sum_{j \in \mathbb{Z}, j \neq 0} w_j \exp(j \omega \theta) \). Denote by \( \mathcal{Y} \) the space made of functions depending only on \( \theta_2, \theta_3 \). Let \( \mathcal{Z} \) be \( H^0 \)-orthogonal complement of \( \mathcal{Y} \). With respect to the following decomposition

\[
\mathbb{W} \cap H^* = (\mathcal{Y} \cap H^*) \oplus (\mathcal{Z} \cap H^*),
\]

we apply the Lyapunov–Schmidt reduction to get

\[
L_{\omega, \alpha, \beta, \gamma} y = \Pi_{\mathcal{Y}} F(\omega, \nu(\rho) + y + j),
\]

\[
L_{\omega, \alpha, \beta, \gamma} z = \Pi_{\mathcal{Z}} F(\omega, \nu(\rho) + y + j),
\]

where \( y = j + z \) with \( y \in \mathcal{Y} \), \( z \in \mathcal{Z} \), and \( \Pi_{\mathcal{Y}}, \Pi_{\mathcal{Z}} \) stand for the projectors onto \( \mathcal{Y}, \mathcal{Z} \), respectively. It follows from (16) that for \( \rho_1 = 0 \),

\[
v(0, \rho_2, \rho_3)(\theta) = 2\rho_2 \cos(j_2^* \theta_2) + 2\rho_3 \cos(j_3^* \theta_3).
\]

Suppose that \( 3|_{\rho_1 = 0} = 0 \). Then

\[
P_{3} F(\omega, 2\rho_2 \cos(j_2^* \theta_2) + 2\rho_3 \cos(j_3^* \theta_3) + \nu(\theta_2, \theta_3)|_{\rho_1 = 0})
\]

\[
= \lambda \Pi_{3}((\omega_1 \partial_{\theta_1} + \omega_2 \partial_{\theta_2} + \omega_3 \partial_{\theta_3})(2\rho_2 \cos(j_2^* \theta_2) + 2\rho_3 \cos(j_3^* \theta_3) + \nu(\theta_2, \theta_3)|_{\rho_1 = 0})^{2p+1}
\]

\[
= \lambda \Pi_{3}(\sum_{k=2}^{3} - 2j_k^* \omega_k \rho_k \sin(j_k^* \theta_k) + \omega_k \partial_{\theta_k} \nu(\theta_2, \theta_3)|_{\rho_1 = 0})^{2p+1} = 0.
\]

By the uniqueness coming from the implicit function theorem given by the proof of Proposition 1 and Proposition 2, \( 3|_{\rho_1 = 0} = 0 \) is a solution of equation (31) with \( \rho_1 = 0 \). Substituting it into equation (30) with \( \rho_1 = 0 \) yields that

\[
L_{\omega, \alpha, \beta, \gamma} y(\theta_2, \theta_3)|_{\rho_1 = 0} = \Pi_{\mathcal{Y}} F(\omega, 2\rho_2 \cos(j_2^* \theta_2) + 2\rho_3 \cos(j_3^* \theta_3) + \nu(\theta_2, \theta_3)|_{\rho_1 = 0}).
\]

Since the subspace \( \mathcal{Y} \cap H^* \) of \( \mathbb{W} \cap H^* \) is invariant for both \( L_{\omega, \alpha, \beta, \gamma} \) and \( F(\omega, \cdot) \), using the similar procedure as in the proof of Proposition 1 and Proposition 2, with
\( \rho_1 = 0 \), yields that \( w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma) = \eta(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma) \), which is a solution of equation (32). Therefore, \( H_3 w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma) = 0 \).

The remainder of the arguments on properties (iii)–(vii) can be stated by the analogous process as in the proof of the property (ii). Thus we obtain the conclusion of the lemma.

Now we are focused on the smoothness of solutions of equation (17) with respect to \( \rho, \omega, \alpha, \beta, \gamma \). We first introduce some notation. Define

\[
\zeta(j, \omega, \alpha, \beta, \gamma) := \frac{1}{\eta(j, \omega, \alpha, \beta, \gamma)} =: g(\eta(j, \omega, \alpha, \beta, \gamma)),
\]

where \( \eta \) is given by (21). Let \( p \in \mathbb{N} \). For all \( p \geq 1 \), denote by \( \tau := (\tau_1, \ldots, \tau_6) \) a multi-index with \( |\tau| = p + 1 \). Without loss of generality, we assume that \( \tau_4 \geq 1 \). Moreover, denote \( \zeta := (\tau_1, \tau_2, \tau_3, \tau_4 - 1, \tau_5, \tau_6) \). Observe that \( |\zeta| = p, p \geq 1 \).

Let us demonstrate \( C^\infty \) smoothness of \( L_{\omega, \alpha, \beta, \gamma}^{-1} \) with respect to \( \omega, \alpha, \beta, \gamma \).

**Lemma 3.3.** For \( y \in C^\infty(\mathbb{T}^3; \mathbb{R}) \cap (Y \cap H^s) \) with \( s > 0 \), the mapping \( \mathcal{B}_r(\omega, \rho) \times \mathbb{R}^3 \ni (\omega, \alpha, \beta, \gamma) \mapsto L_{\omega, \alpha, \beta, \gamma}^{-1} y \in Y \cap H^s \) is \( C^\infty \), where \( \mathcal{B}_r(\omega, \rho) \) is as seen in (22). Moreover,

\[
D^\kappa \zeta^{-1} \omega, \alpha, \beta, \gamma, y = \sum_{j \in J_1} D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma) y_j e^{ij\theta},
\]

where \( D^\kappa = \partial_{\omega_1}^{\kappa_1} \partial_{\omega_2}^{\kappa_2} \partial_{\omega_3}^{\kappa_3} \partial_{\alpha}^{\kappa_4} \partial_{\beta}^{\kappa_5} \partial_{\gamma}^{\kappa_6} \) with \( \kappa_i \in \mathbb{N}, i = 1, \cdots, 6 \).

**Proof.** Let us assert

\[
D^\kappa \zeta = \sum_{|\kappa|} g^{(k)}(\eta) P_k[D^{\kappa_1} \eta, \cdots, D^{\kappa_6} \eta], \quad |\kappa| = \sum_{i=1}^6 \kappa_i,
\]

where \( P_k \) is a polynomial in \( n(\kappa) \) variables of order at most \( |\kappa| \), that is

\[
P_k[D^{\kappa_1} \eta, \cdots, D^{\kappa_6} \eta] = \sum_{|\vartheta| \leq |\kappa|} C_0^\vartheta (D^{\kappa_1} \eta)^{\vartheta_1} (D^{\kappa_2} \eta)^{\vartheta_2} \cdots (D^{\kappa_6} \eta)^{\vartheta_6},
\]

where \( n(\kappa) \) is the number of partial derivatives of \( \eta \) with respect to \( \omega, \alpha, \beta, \gamma \) of order at most \( |\kappa| \), \( \vartheta_i, i = 1, \cdots, n(\kappa) \) are multi-indices of order at most \( |\kappa| \), and \( \vartheta = (\vartheta_1, \cdots, \vartheta_{n(\kappa)}) \) is an \( n(\kappa) \)-tuple of nonnegative integers. In fact, it is clear that \( D^\kappa \zeta = g^{(1)}(\eta) D^\kappa \eta \). Suppose that (34) could hold for \( |\kappa| = p, p \geq 2 \). Then for \( |\tau| = p + 1 \),

\[
D^\tau \zeta = \partial_\alpha (D^\kappa \zeta)
\]

\[
= \partial_\alpha \left( \sum_{k=1}^p \sum_{|\vartheta| \leq p} (\eta)^{\vartheta_1} \cdot (D^{\kappa_1} \eta)^{\vartheta_2} \cdots (D^{\kappa_6} \eta)^{\vartheta_6} \right)
\]

\[
= \sum_{k=1}^p g^{(k)}(\eta) \partial_\alpha \left( \sum_{|\vartheta| \leq p} C_0^\vartheta (D^{\kappa_1} \eta)^{\vartheta_1} (D^{\kappa_2} \eta)^{\vartheta_2} \cdots (D^{\kappa_6} \eta)^{\vartheta_6} \right)
\]

\[
= \sum_{k=1}^p \sum_{|\vartheta| \leq p} g^{(k)}(\eta) \partial_\alpha \left( \sum_{|\vartheta| \leq p} C_0^\vartheta (D^{\kappa_1} \eta)^{\vartheta_1} (D^{\kappa_2} \eta)^{\vartheta_2} \cdots (D^{\kappa_6} \eta)^{\vartheta_6} \right)
\]

\[
= \sum_{k=1}^{p+1} g^{(k)}(\eta) \sum_{|\vartheta| \leq p} C_0^\vartheta (D^{\kappa_1} \eta)^{\vartheta_1} (D^{\kappa_2} \eta)^{\vartheta_2} \cdots (D^{\kappa_6} \eta)^{\vartheta_6}.
\]

Denote by \( \Omega \) any bounded open set in \( \mathbb{R}^3 \). We claim that for all \( j \in J_1 \) and \( (\omega, \alpha, \beta, \gamma) \in \mathcal{B}_r(\omega, \rho) \times \Omega \),

\[
|D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma)| \leq C|j|^{|\kappa|}, \quad \forall |\kappa| \geq 1
\]

for some constant \( C = C(\tau, \Omega, \kappa, \nu_1, \nu_2, \nu_3) > 0 \). In fact, it follows from the definition of \( \eta \) (recall (21)) that

\[
D^\kappa \eta(j, \omega, \alpha, \beta, \gamma) = 0, \quad \forall |\kappa| \geq 3.
\]
Moreover, it follows from
\[ |D^\kappa \eta(j, \omega, \alpha, \beta, \gamma)| \leq C_1(1 + (\max\{\nu_1, \nu_2, \nu_3\})^4)|j|^6. \]
As a result,
\[ |P_k[|D^{\kappa_1}, \ldots, D^{\kappa_n}\eta]| \leq \sum_{|\kappa| \leq |\kappa|} |C_{\kappa}^k||(|D^{\kappa_1}\eta|)(\ell_1)|(|D^{\kappa_2}\eta|)(\ell_2)| \ldots |(|D^{\kappa_n}\eta|)(\ell_n)|
\leq \sum_{|\kappa| \leq |\kappa|} |C_{\kappa}^k|(1 + (\max\{\nu_1, \nu_2, \nu_3\})^4)^{|\kappa|}|j|^{|\kappa|}
\leq C_k'(1 + (\max\{\nu_1, \nu_2, \nu_3\})^4)|j|^{|\kappa|}.
\]
Moreover, it follows from \( g(\eta) = \frac{1}{\eta} \) that \( |g^{(k)}(\eta)| \leq \frac{C_k''}{|\kappa|^{K+1}} \). Combining this with (24) and (34) gives that
\[ |D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma)| \leq \sum_{|\kappa|} |C_{\kappa}^k|(1 + (\max\{\nu_1, \nu_2, \nu_3\})^4)|\kappa| |j|^{|\kappa|} \leq C|j|^{|\kappa|}.
\]
It remains to prove (33) and the continuity of the partial derivatives by an inductive argument. Suppose that (33) could hold for \( p \geq 1 \) (note that the case \( p = 1 \) can be handled in the same way). For all \( y \in C^\infty(T^3; \mathbb{R}) \cap (Y \cap H^s) \), we derive
\[
\frac{1}{v^2} \|D^sL^{-1}_{\omega, \alpha + v, \beta, \gamma}y - D^sL^{-1}_{\omega, \alpha, \beta, \gamma}y - vD_yD^sL^{-1}_{\omega, \alpha, \beta, \gamma}y\|_s^2
= \frac{1}{v^2} \sum_{j \in J_1} (1 + |j|^{2s})|y_j|^2 |R(j, \omega, \alpha, \beta, \gamma, v)|^2,
\]
where
\[ R(j, \omega, \alpha, \beta, \gamma, v) = D^\kappa \zeta(j, \omega, \alpha + v, \beta, \gamma) - D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma) - vD_\alpha D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma). \]
Note that
\[
|R(j, \omega, \alpha, \beta, \gamma, v)| = |\int_0^1 \partial_\alpha D^\kappa \zeta(j, \omega, \alpha + av, \beta, \gamma) - \partial_\alpha D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma)da||v|
\leq \max_{a \in [0, 1]} |\partial_\alpha D^\kappa \zeta(j, \omega, \alpha + av, \beta, \gamma) - \partial_\alpha D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma)||v|
= |\partial_\alpha D^\kappa \zeta(j, \omega, \alpha, \beta, \gamma)|\leq 2C|j|^{6(p+1)}.
\]
Thus we obtain that for \( v \) small enough,
\[
\frac{1}{v^2} \|D^sL^{-1}_{\omega, \alpha + v, \beta, \gamma}y - D^sL^{-1}_{\omega, \alpha, \beta, \gamma}y - vD_yD^sL^{-1}_{\omega, \alpha, \beta, \gamma}y\|_s^2
\leq 4C^2 \sum_{j \in J_1} (1 + |j|^{2s})|y_j|^2 |j|^{12(p+1)}
< \infty.
\]
This implies that \( D^sL^{-1}_{\omega, \alpha, \beta, \gamma}y = \partial_\alpha D^sL^{-1}_{\omega, \alpha, \beta, \gamma}y \). In addition, applying the similar process as above yields the continuity of the partial derivatives in \( \omega, \alpha, \beta, \gamma \). Hence the proof is now completed. \( \square \)

**Proposition 4.** Let \( s > 0 \) and \( \sigma_1 = (0, \omega, \alpha, 0, 0, 0) \). For \( w \in C^\infty(T^3; \mathbb{R}) \cap (W \cap H^s) \), there is an \( s \)-independent neighborhood \( B_{\sigma_1}(\sigma_1) \) of \( \sigma_1 \) such that the mapping \( \mathbb{R}^3 \times B_{\sigma_1}(\omega, \alpha) \times \mathbb{R}^3 \ni (\rho, \omega, \alpha, \beta, \gamma) \mapsto w(\rho, \omega, \alpha, \beta, \gamma) \in W \cap H^s \) is \( C^\infty \) smooth.
Proof. It follows from Proposition 3 that $w = y + z$, where $y \in C^\infty(T^4; \mathbb{R}) \cap (Y \cap H^*)$ and $z \in C^\infty(T^4; \mathbb{R}) \cap (Z \cap H^*)$. Thanks to Lemma 2.1, Lemma 3.3 and the definition of $G_1$, the mapping

$$(\rho, \omega, \alpha, \beta, \gamma, z, y) \mapsto G_1(\rho, \omega, \alpha, \beta, \gamma, z, y)$$

is $C^\infty$ smooth with respect to $\rho, \omega, \alpha, \beta, \gamma, z, y$. Hence, in view of the implicit function theorem, the mapping $(\rho, \omega, \alpha, \beta, \gamma, z) \mapsto y(\rho, \omega, \alpha, \beta, \gamma, z)$ is $C^\infty$. On the other hand, the space $Z$ is finite dimensional. We can further verify that the mapping

$$(\rho, \omega, \alpha, \beta, \gamma, z) \mapsto G_2(\rho, \omega, \alpha, \beta, \gamma, z)$$

varies in a $C^\infty$ way in $\rho, \omega, \alpha, \beta, \gamma, z$. Therefore, it follows from the implicit function theorem that the mapping $(\rho, \omega, \alpha, \beta, \gamma) \mapsto z(\rho, \omega, \alpha, \beta, \gamma)$ is $C^\infty$. This ends the proof of the lemma.

3.2. Solutions of the bifurcation equation. The goal of the present subsection is to look for solutions of the bifurcation system (18).

Via plugging the solution $w = w(\rho, \omega, \alpha, \beta, \gamma)$ of equation (17) back into the bifurcation system (18), one has

$$\begin{align*}
\rho_1 \left( -\omega_1^2(j_1)^4 + \mu \omega_1^4(j_1)^4 + m + \im \omega_1 j_1^4 + i \beta \omega_1 \nu_1^2(j_1)^3 + i \gamma \omega_1 \nu_4^1(j_1)^5 \right) &= \Pi_{(j_1, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)), \\
\rho_1 \left( -\omega_2^2(j_2)^4 + \mu \omega_2^4(j_2)^4 + m - \im \omega_2 j_2^4 - i \beta \omega_2 \nu_2^2(j_2)^3 - i \gamma \omega_2 \nu_4^2(j_2)^5 \right) &= \Pi_{(-j_2, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)), \\
\rho_2 \left( -\omega_2^2(j_2)^4 + \mu \omega_2^4(j_2)^4 + m + \im \omega_2 j_2^4 + i \beta \omega_2 \nu_2^2(j_2)^3 + i \gamma \omega_2 \nu_4^2(j_2)^5 \right) &= \Pi_{(0, j_2, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)), \\
\rho_2 \left( -\omega_2^2(j_2)^4 + \mu \omega_2^4(j_2)^4 + m - \im \omega_2 j_2^4 - i \beta \omega_2 \nu_2^2(j_2)^3 - i \gamma \omega_2 \nu_4^2(j_2)^5 \right) &= \Pi_{(0, -j_2, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)), \\
\rho_1 \left( -\omega_3^2(j_3)^4 + \mu \omega_3^4(j_3)^4 + m + \im \omega_3 j_3^4 + i \beta \omega_3 \nu_3^2(j_3)^3 + i \gamma \omega_3 \nu_4^3(j_3)^5 \right) &= \Pi_{(0, j_3, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)), \\
\rho_2 \left( -\omega_3^2(j_3)^4 + \mu \omega_3^4(j_3)^4 + m - \im \omega_3 j_3^4 - i \beta \omega_3 \nu_3^2(j_3)^3 - i \gamma \omega_3 \nu_4^3(j_3)^5 \right) &= \Pi_{(0, -j_3, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)).
\end{align*}$$

We further define

$$F_{1+}^1(\rho, \omega, \alpha, \beta, \gamma) := \begin{cases}
\frac{\partial \Pi_{(j_1, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))}{\partial \rho_1}, & \rho_1 \neq 0, \\
0, & \rho_1 = 0,
\end{cases}$$

$$F_{1-}^1(\rho, \omega, \alpha, \beta, \gamma) := \begin{cases}
\frac{\partial \Pi_{(j_1, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))}{\partial \rho_1}, & \rho_1 \neq 0, \\
0, & \rho_1 = 0,
\end{cases}$$

$$F_{2+}^1(\rho, \omega, \alpha, \beta, \gamma) := \begin{cases}
\frac{\partial \Pi_{(j_2, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))}{\partial \rho_2}, & \rho_2 \neq 0, \\
0, & \rho_2 = 0,
\end{cases}$$

$$F_{2-}^1(\rho, \omega, \alpha, \beta, \gamma) := \begin{cases}
\frac{\partial \Pi_{(j_2, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))}{\partial \rho_2}, & \rho_2 \neq 0, \\
0, & \rho_2 = 0,
\end{cases}$$

$$F_{3+}^1(\rho, \omega, \alpha, \beta, \gamma) := \begin{cases}
\frac{\partial \Pi_{(j_3, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))}{\partial \rho_3}, & \rho_3 \neq 0, \\
0, & \rho_3 = 0,
\end{cases}$$

$$F_{3-}^1(\rho, \omega, \alpha, \beta, \gamma) := \begin{cases}
\frac{\partial \Pi_{(j_3, 0, 0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))}{\partial \rho_3}, & \rho_3 \neq 0, \\
0, & \rho_3 = 0.
\end{cases}$$
\[ F_{\pm}^\mp (\rho, \omega, \alpha, \beta, \gamma) := \begin{cases} 
\text{Im}(\Pi(0,j_2,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)))/\rho_2, & \rho_2 \neq 0, \\
\partial_{\rho_2} \text{Im}(\Pi(0,j_2,0)F(\omega, v(\rho_1, 0, \rho_3) + w(\rho_1, 0, \rho_3, \omega, \alpha, \beta, \gamma))), & \rho_2 = 0, \\
\text{Re}(\Pi(0,0,j_1)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)))/\rho_3, & \rho_3 \neq 0, \\
\partial_{\rho_3} \text{Re}(\Pi(0,0,j_1)F(\omega, v(\rho_1, \rho_2, 0) + w(\rho_1, \rho_2, 0, \omega, \alpha, \beta, \gamma))), & \rho_3 = 0, 
\end{cases} \]

and
\[ F_{\pm}^\mp (\rho, \omega, \alpha, \beta, \gamma) := \begin{cases} 
\text{Im}(\Pi(0,0,j_1)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)))/\rho_3, & \rho_3 \neq 0, \\
\partial_{\rho_3} \text{Im}(\Pi(0,0,j_1)F(\omega, v(\rho_1, \rho_2, 0) + w(\rho_1, \rho_2, 0, \omega, \alpha, \beta, \gamma))), & \rho_3 = 0. 
\end{cases} \]

**Lemma 3.4.** In an s-independent neighborhood of \((0, \omega, \rho, 0, 0, 0)\), the functions Re(\(\Pi(0,j_1,0,0)F\)), Im(\(\Pi(0,j_1,0,0)F\)), Re(\(\Pi(0,j_2,0)F\)), Im(\(\Pi(0,j_2,0)F\)), Re(\(\Pi(0,0,j_1)F\)) and Im(\(\Pi(0,0,j_1)F\)) vary in a \(C^\infty\) way in \(\rho, \omega, \alpha, \beta, \gamma\) with
\[
\begin{align*}
\text{Re}(\Pi(0,j_1,0,0)F(\omega, v(\rho, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Re}(\Pi(0,j_1,0,0)F(\omega, v(\rho, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Re}(\Pi(0,j_2,0)F(\omega, v(\rho_1, 0, \rho_3) + w(\rho_1, 0, \rho_3, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Re}(\Pi(0,0,j_1)F(\omega, v(\rho_1, \rho_2, 0) + w(\rho_1, \rho_2, 0, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Im}(\Pi(0,j_1,0,0)F(\omega, v(\rho, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Im}(\Pi(0,j_1,0,0)F(\omega, v(\rho, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Im}(\Pi(0,j_2,0)F(\omega, v(\rho_1, 0, \rho_3) + w(\rho_1, 0, \rho_3, \omega, \alpha, \beta, \gamma))) &= 0, \\
\text{Im}(\Pi(0,0,j_1)F(\omega, v(\rho_1, \rho_2, 0) + w(\rho_1, \rho_2, 0, \omega, \alpha, \beta, \gamma))) &= 0.
\end{align*}

Moreover, the functions \(F_{\pm}^\pm, k = 1, 2, 3\) are \(C^\infty\) with respect to \(\rho, \omega, \alpha, \beta, \gamma\) satisfying
\[
\begin{align*}
F_{\pm}^\pm (0, \omega, \alpha, \beta, \gamma) &= 0, \\
\partial_\rho F_{\pm}^\pm (0, \omega, \alpha, \beta, \gamma) &= 0, \\
\partial_\omega F_{\pm}^\pm (0, \omega, \alpha, \beta, \gamma) &= 0, \\
\partial_\alpha F_{\pm}^\pm (0, \omega, \alpha, \beta, \gamma) &= 0, \\
\partial_\beta F_{\pm}^\pm (0, \omega, \alpha, \beta, \gamma) &= 0, \\
\partial_\gamma F_{\pm}^\pm (0, \omega, \alpha, \beta, \gamma) &= 0, 
\end{align*}
\]

\(k = 1, 2, 3\).

**Proof.** Observe that
\[
\begin{align*}
\text{Re}(\Pi(0,j_1,0,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))) &= \frac{1}{j_1} \int_{j_1} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)) \cos(j_1^* \theta_1) d\theta, \\
\text{Im}(\Pi(0,j_1,0,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))) &= -\frac{1}{j_1} \int_{j_1} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)) \sin(j_1^* \theta_1) d\theta, \\
\text{Re}(\Pi(0,j_1,0,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))) &= \frac{1}{j_2} \int_{j_2} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)) \cos(j_2^* \theta_2) d\theta, \\
\text{Im}(\Pi(0,j_1,0,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))) &= -\frac{1}{j_2} \int_{j_2} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)) \sin(j_2^* \theta_2) d\theta, \\
\text{Re}(\Pi(0,j_2,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))) &= \frac{1}{j_3} \int_{j_3} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)) \cos(j_3^* \theta_3) d\theta, \\
\text{Im}(\Pi(0,j_2,0)F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))) &= -\frac{1}{j_3} \int_{j_3} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)) \sin(j_3^* \theta_3) d\theta.
\end{align*}
\]

Thus the functions \(\text{Re}(\Pi(0,j_1,0,0)F), \text{Im}(\Pi(0,j_1,0,0)F), \text{Re}(\Pi(0,j_2,0)F), \text{Im}(\Pi(0,j_2,0)F), \text{Re}(\Pi(0,0,j_1)F)\) and \(\text{Im}(\Pi(0,0,j_1)F)\) are \(C^\infty\) in \(\rho, \omega, \alpha, \beta, \gamma\).
Moreover, thanks to Proposition 3 and formula (16), we conclude
\[
\text{Re}(\Pi_{(j^1,0,0)} F(\omega, v(0, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)))
= \frac{1}{8\pi^2} \int_{T_3} (w(2\rho_2 \cos(j_2^1 \theta_2) + 2\rho_3 \cos(j_3^1 \theta_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)(\theta_2, \theta_3))
\times \cos(j_1^1 \theta_1) d\theta
= \frac{\lambda}{8\pi} \int_{T_3} ((\omega_2 \partial_{\theta_2} + \omega_3 \partial_{\theta_3})(2\rho_2 \cos(j_2^1 \theta_2) + 2\rho_3 \cos(j_3^1 \theta_3))
+ w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)(\theta_2, \theta_3))^{2p+1} \cos(j_1^1 \theta_1) d\theta = 0.
\]
By the Taylor expansion of \(\text{Re}(\Pi_{(j^1,0,0)} F)\) at \(\rho_1 = 0\), we obtain
\[
\text{Re}(\Pi_{(j^1,0,0)} F(\omega, v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)))
= \partial_{\rho_1} \text{Re}(\Pi_{(j^1,0,0)} F(\omega, v(0, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)))\rho_1 + O(\rho_1^2).
\]
As a consequence,
\[
F_1^+(\rho_1, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma) = \partial_{\rho_1} \text{Re}(\Pi_{(j^1,0,0)} F(\omega, v(0, \rho_2, \rho_3) + w(0, \rho_2, \rho_3, \omega, \alpha, \beta, \gamma)))\rho_1 + O(\rho_1).
\]
Hence \(F_1^+\) is \(C^\infty\) with respect to \(\rho, \omega, \alpha, \beta, \gamma\).

Furthermore, it follows from (27) and (29) that \(w(0, \omega, \alpha, \beta, \gamma) = 0\). Therefore,
\[
\partial_{\rho_1} \text{Re}(\Pi_{(j^1,0,0)} F(\omega, v(0) + w(0, \omega, \alpha, \beta, \gamma)))
= \frac{1}{8\pi^2} \int_{T_3} D(F(\omega, v(0) + w(0, \omega, \alpha, \beta, \gamma))[\partial_{\rho_1} (v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma))]_{\rho = 0}]
\times \cos(j_1^1 \theta_1) d\theta
= \frac{\lambda(2p+1)}{8\pi} \int_{T_3} ((\omega \cdot \nabla)(v(0) + w(0, \omega, \alpha, \beta, \gamma)))^{2p}
\times (\omega \cdot \nabla)(\partial_{\rho_1} (v(\rho) + w(\rho, \omega, \alpha, \beta, \gamma)))_{\rho = 0}) \cos(j_1^1 \theta_1) d\theta
= 0,
\]
which leads to \(F_1^+(0, \omega, \alpha, \beta, \gamma) = 0\). In addition, one has that for \(k = 1, 2, 3\),
\[
\partial_{\omega_k} F_1^+(0, \omega, \alpha, \beta, \gamma) = \partial_{\omega_k} F_3^+(0, \omega, \alpha, \beta, \gamma) = 0,
\]
\[
\partial_{\beta} F_1^+(0, \omega, \alpha, \beta, \gamma) = \partial_{\beta} F_3^+(0, \omega, \alpha, \beta, \gamma) = 0.
\]
The remainder of the arguments on \(F_1^-, F_2^+, F_3^+\) can be given by the analogous process as above. Thus we get the conclusion of the lemma. \(\square\)

By Lemma 3.4, splitting up into real and imaginary parts of the above system (36) yields that
\[
\begin{cases}
- \omega_1^2 (j_1^1)^2 + \mu \omega_1^2 (j_1^1)^4 + m = F_1^+ (\rho, \omega, \alpha, \beta, \gamma),
\omega_1 j_1^1 + \beta_1 \omega_1^2 (j_1^1)^3 + \gamma \omega_1^2 (j_1^1)^5 = F_1^- (\rho, \omega, \alpha, \beta, \gamma),
- \omega_2^2 (j_2^1)^2 + \mu \omega_2^2 (j_2^1)^4 + m = F_2^+(\rho, \omega, \alpha, \beta, \gamma),
\omega_2 j_2^1 + \beta_2 \omega_2^2 (j_2^1)^3 + \gamma \omega_2^2 (j_2^1)^5 = F_2^- (\rho, \omega, \alpha, \beta, \gamma),
- \omega_3^2 (j_3^1)^2 + \mu \omega_3^2 (j_3^1)^4 + m = F_3^+(\rho, \omega, \alpha, \beta, \gamma),
\omega_3 j_3^1 + \beta_3 \omega_3^2 (j_3^1)^3 + \gamma \omega_3^2 (j_3^1)^5 = F_3^-(\rho, \omega, \alpha, \beta, \gamma),
\end{cases}
\]
(37)
System (37) are made of six equations in the nine unknowns
\[
(\rho_1, \rho_2, \rho_3, \omega_1, \omega_2, \omega_3, \alpha, \beta, \gamma).
\]
By linearizing system (37) with respect to $\omega_1, \omega_2, \omega_3, \alpha, \beta, \gamma$ at $(0, \omega_j^*, 0, 0, 0)$, we have the following matrix

$$A = \begin{pmatrix}
-2\omega_j^1 (j_1^*)^2 & 0 & 0 & 0 & 0 \\
0 & -2\omega_j^2 (j_2^*)^2 & 0 & 0 & 0 \\
0 & 0 & -2\omega_j^3 (j_3^*)^2 & 0 & 0 \\
0 & 0 & 0 & -2\omega_j^4 (j_4^*)^2 & 0 \\
0 & 0 & 0 & 0 & -2\omega_j^5 (j_5^*)^2 \\
\end{pmatrix}.$$

Note that

$$\det A = 8\omega_j^2 (j_1^*)^3 \omega_j^2 (j_2^*)^3 \omega_j^3 (j_3^*)^3 \nu_2^3 (j_2^*)^2 - \nu_1^3 (j_1^*)^2 - \nu_4^3 (j_3^*)^2 - \nu_3^3 (j_2^*)^2 \times (\nu_2^3 (j_2^*)^2 - \nu_3^2 (j_1^*)^2) \times \nu_1^2 (j_1^*)^2,$$

Since $\nu_k j_k^* \neq \nu_k' j_k'^*$, $1 \leq k < k' \leq 3$, we derive $\det A \neq 0$. Then it follows from the implicit function theorem that the mappings

$$\rho \mapsto \omega_k(\rho), \quad k = 1, 2, 3, \quad \rho \mapsto \alpha(\rho), \quad \rho \mapsto \beta(\rho), \quad \rho \mapsto \gamma(\rho)$$

are $C^\infty$, respectively.

Consequently, we complete the proof of Theorem 1.2.

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*E-mail address*: chenbc758@163.com

*E-mail address*: gaoyx643@nenu.edu.cn