Computing Exact Guarantees for Differential Privacy

Antti Koskela\textsuperscript{1}, Joonas Jälkö\textsuperscript{2} and Antti Honkela\textsuperscript{3}

\textsuperscript{1} Helsinki Institute for Information Technology HIIT, Department of Mathematics and Statistics, University of Helsinki, Finland
\textsuperscript{2} Helsinki Institute for Information Technology HIIT, Department of Computer Science, Aalto University, Finland
\textsuperscript{3} Helsinki Institute for Information Technology HIIT, Department of Computer Science, University of Helsinki

Abstract

Quantification of the privacy loss associated with a randomised algorithm has become an active area of research and $\left(\varepsilon, \delta\right)$-differential privacy has arisen as the standard measure of it. We propose a numerical method for evaluating the parameters of differential privacy for algorithms with continuous one dimensional output. In this way the parameters $\varepsilon$ and $\delta$ can be evaluated, for example, for the subsampled multidimensional Gaussian mechanism which is also the underlying mechanism of differentially private stochastic gradient descent. The proposed method is based on a numerical approximation of an integral formula which gives the exact $(\varepsilon, \delta)$-values. The approximation is carried out by discretising the integral and by evaluating discrete convolutions using a fast Fourier transform algorithm. We give theoretical error bounds which show the convergence of the approximation and guarantee its accuracy to an arbitrary degree. Experimental comparisons with state-of-the-art techniques illustrate the efficacy of the method. Python code for the proposed method can be found in Github.\textsuperscript{1}

1 Introduction

Differential privacy (DP) \textsuperscript{8} has clearly been established as the dominant paradigm for privacy-preserving machine learning. Early work on DP machine learning focused on single shot perturbations for convex problems (e.g. \textsuperscript{6}), while contemporary research has focused on iterative algorithms such as DP stochastic gradient descent (SGD).

Evaluating the privacy loss of an iterative algorithm is based on the composition theory of DP. The so-called advanced composition theorem of Dwork et al. \textsuperscript{10} showed how to trade decreased $\varepsilon$ with slightly increased $\delta$ in $(\varepsilon, \delta)$-DP. This was further improved e.g. by Kairouz\textsuperscript{1}.\textsuperscript{1}

\textsuperscript{1}https://github.com/DPBayes/PLD-Accountant/
The privacy amplification by subsampling is another component that has been studied to improve the privacy bounds.

A major breakthrough in obtaining tighter composition bounds came from using the entire privacy loss profile of DP algorithms instead of single \((\varepsilon, \delta)\) values, first introduced by the moments accountant. This is especially useful for the Gaussian mechanism, where one mechanism has a continuum of characterisations. The development of Rényi differential privacy (RDP) allowed tight bounds on the privacy cost of composition, and recently proposed amplification theorem for RDP showed how subsampling affects the privacy cost of RDP.

Our contribution

Using the recently introduced privacy loss distribution formalism, we compute exact \((\varepsilon, \delta)\)-DP bounds on the composition of subsampled Gaussian mechanism, using discrete Fourier transforms to evaluate the required convolutions. We show numerically that our exact \((\varepsilon, \delta)\)-DP bounds are tighter than those obtained by Rényi DP compositions and the moments accountant.

2 Differential Privacy

We first recall some basic definitions of differential privacy. We use the following notation. An input data set containing \(N\) data points is denoted as \(X = (x_1, \ldots, x_N) \in \mathcal{X}^N\), where \(x_i \in \mathcal{X} \}, 1 \leq i \leq N\).

Next we give definitions for different types of neighbouring relations for datasets.

**Definition 1.** We say two datasets \(X\) and \(Y\) are neighbours in remove/add relation if you get one from removing/adding an element from/to another and denote it with \(\simeq_R\). We say \(X\) and \(Y\) are neighbours in substitute relation if you get one by substituting one element in another. We denote this with \(\simeq_S\).

The following gives a definition for \((\varepsilon, \delta)\)-differential privacy.

**Definition 2.** Let \(\varepsilon > 0\), \(\delta \in [0, 1]\) and let \(\simeq\) define a neighbouring relation. Mechanism \(\mathcal{M} : \mathcal{X}^N \to \mathbb{R}\) is \((\varepsilon, \delta, \simeq)\)-DP if for every \(X \simeq Y\) and every measurable set \(E \subset \mathbb{R}\), we have
\[
P(\mathcal{M}(X) \in E) \leq e^{\varepsilon}P(\mathcal{M}(Y) \in E) + \delta.
\]

When the neighbouring relation is clear from context or irrelevant, we will abbreviate these as \((\varepsilon, \delta)\)-DP.

3 Privacy loss distribution

For completeness, we prove certain lemmas for continuous one dimensional distributions. These results can be found in for discrete valued distributions.

We consider mechanisms \(\mathcal{M} : \mathcal{X}^N \to \mathbb{R}\) which give as an output distributions with support equaling the whole real line. The proofs apply for both neighbouring relations \(\simeq_S, \simeq_R\), which in this Section we simply refer as adjacency.
Definition 3. Let $\mathcal{M} : \mathcal{X}^N \rightarrow \mathbb{R}$ be a randomised mechanism and let $X, Y$ be adjacent datasets. Let $f_X(t)$ denote the density function of $\mathcal{M}(X)$ and $f_Y(t)$ the density function of $\mathcal{M}(Y)$. Assume $f_X(t) > 0$ and $f_Y(t) > 0$ for all $t \in \mathbb{R}$. We define the privacy loss function of $f_X$ over $f_Y$ as

$$L_{X/Y}(t) = \log \frac{f_X(t)}{f_Y(t)}.$$ 

Definition 4. Let $\Omega$ denote all the measurable subsets of $\mathbb{R}$ and let $S \in \Omega$. Again, let $\mathcal{M} : \mathcal{X}^N \rightarrow \mathbb{R}$ be a randomised mechanism and let $X, Y$ be adjacent datasets and let $f_X(t)$ denote the density function of $\mathcal{M}(X)$ and $f_Y(t)$ the density function of $\mathcal{M}(Y)$. The privacy loss distribution of $\mathcal{M}(X)$ over $\mathcal{M}(Y)$ is defined to be a random variable $\omega : \Omega \rightarrow \mathbb{R}$, such that

$$\omega(S) = \int_{\{t \in \mathbb{R} : L_{X/Y}(t) \in S\}} f_X(t) \, dt.$$ 

3.1 The density function of PLD

The following lemma gives a representation for the density function of the privacy loss distribution (PLD) (see also [10, Lemma 8]). We note that the assumptions on differentiability and bijectivity of the privacy loss function hold for the subsampled Gaussian mechanism analysed in Section 5.

Lemma 5. Let the assumptions of Definition 3 hold, i.e., $f_X(t)$ is the density function of $\mathcal{M}(X)$ and $f_Y(t)$ the density function of $\mathcal{M}(Y)$, and $\omega$ is the privacy loss distribution of $\mathcal{M}(X)$ over $\mathcal{M}(Y)$. Suppose $L_{X/Y} : \mathbb{R} \rightarrow D, D \subset \mathbb{R}$, is a continuously differentiable bijective function. Let $S \subset \mathbb{R}$ be a measurable set. Then,

$$\omega(S) = \int_S \frac{d\omega}{dy} \, dy,$$

where

$$\frac{d\omega}{dy} = \begin{cases} f_X(L_{X/Y}^{-1}(y)) \frac{dL_{X/Y}^{-1}(y)}{dy}, & \text{if } y \in L_{X/Y}(\mathbb{R}), \\ 0, & \text{else.} \end{cases}$$

Proof. By change of variables $t = L_{X/Y}^{-1}(y)$, we see that

$$\omega(S) = \int_{\{t \in \mathbb{R} : L_{X/Y}(t) \in S\}} f_X(t) \, dt = \int_{S \cap L_{X/Y}(\mathbb{R})} f_X(L_{X/Y}^{-1}(y)) \frac{dL_{X/Y}^{-1}(y)}{dy} \, dy.$$ 


4 Integral representation for exact DP-guarantees

We give an alternative definition of differential privacy which is equivalent to Definition 2 in the case of continuous one dimensional distributions. Throughout this section we denote for neighbouring datasets $X$ and $Y$ the density function of $M(X)$ with $f_X(t)$ and the density function of $M(Y)$ with $f_Y(t)$.

**Definition 6.** A randomised algorithm $M$ with an output of continuous one dimensional distributions satisfies $(\varepsilon, \delta)$-DP if for every set $S \subset \mathbb{R}$ and every neighbouring datasets $X$ and $Y$

$$\int_S f_X(t) \, dt \leq e^\varepsilon \int_S f_Y(t) \, dt + \delta \quad \text{and} \quad \int_S f_Y(t) \, dt \leq e^\varepsilon \int_S f_X(t) \, dt + \delta.$$  

We call $M$ tightly $(\varepsilon, \delta)$-DP, if there does not exist $\delta' < \delta$ such that $M$ is $(\varepsilon, \delta')$-DP.

The following auxiliary lemma is needed to obtain the representation given by Lemma 8 (see also [12, Lemma 1]).

**Lemma 7.** $M$ is tightly $(\varepsilon, \delta)$-DP with

$$\delta = \max_{X \approx Y} \left\{ \int_{\mathbb{R}} \max\{f_X(t) - e^\varepsilon f_Y(t), 0\} \, dt, \int_{\mathbb{R}} \max\{f_Y(t) - e^\varepsilon f_X(t), 0\} \, dt \right\}.$$  

(4.1)

**Proof.** Assume $M$ is tightly $(\varepsilon, \delta)$-DP. Then, for every set $S \subset \mathbb{R}$ and every neighbouring datasets $X$ and $Y$,

$$\int_S f_X(t) - e^\varepsilon f_Y(t) \, dt \leq \int_S \max\{f_X(t) - e^\varepsilon f_Y(t), 0\} \, dt \leq \int_{\mathbb{R}} \max\{f_X(t) - e^\varepsilon f_Y(t), 0\} \, dt.$$  

We get an analogous bound for $\int_S f_Y(t) - e^\varepsilon f_X(t) \, dt$. By Definition 6, this shows that

$$\delta \leq \max \left\{ \int_{\mathbb{R}} \max\{f_X(t) - e^\varepsilon f_Y(t), 0\} \, dt, \int_{\mathbb{R}} \max\{f_Y(t) - e^\varepsilon f_X(t), 0\} \, dt \right\}.$$  

To show that the above inequality is tight, consider the set

$$S = \{ t \in \mathbb{R} : f_X(t) \geq e^\varepsilon f_Y(t) \}.$$  

Then,

$$\int_S f_X(t) - e^\varepsilon f_Y(t) \, dt = \int_S \max\{f_X(t) - e^\varepsilon f_Y(t), 0\} \, dt \leq \int_{\mathbb{R}} \max\{f_X(t) - e^\varepsilon f_Y(t), 0\} \, dt.$$  

(4.2)
Next, consider the set
\[ S = \{ t \in \mathbb{R} : f_Y(t) \geq e^\varepsilon f_X(t) \}. \]
Similarly,
\[ \int_S f_Y(t) - e^\varepsilon f_X(t) \, dt = \int_\mathbb{R} \max\{ f_Y(t) - e^\varepsilon f_X(t), 0 \} \, dt. \quad (4.3) \]
From (4.2) and (4.3) it follows that there exists a set \( S \subset \mathbb{R} \) such that either
\[ \int_S f_X(t) \, dt = e^\varepsilon \int_S f_Y(t) \, dt + \delta \quad \text{or} \quad \int_S f_Y(t) \, dt = e^\varepsilon \int_S f_X(t) \, dt + \delta \]
for \( \delta \) given by (4.1). This shows that \( \delta \) given by (4.1) is tight.

The following lemma gives an integral representation for the right hand side of (4.1) involving the distribution function of the PLD (see also Lemma 5 and 10 of [13]).

**Lemma 8.** Let \( \mathcal{M} \) be defined as above. \( \mathcal{M} \) is tightly \((\varepsilon, \delta)\)-DP for
\[ \delta = \max_{X \geq Y} \max\{ \delta_{X/Y}, \delta_{Y/X} \}, \]
where
\[ \delta_{X/Y} = \int_{\mathcal{L}_{X/Y}(\mathbb{R}) \cap [\varepsilon, \infty]} (1 - e^{\varepsilon - y}) f_X \left( \mathcal{L}_{X/Y}^{-1}(y) \right) \frac{d\mathcal{L}_{X/Y}^{-1}(y)}{dy} \, dy, \]
\[ \delta_{Y/X} = \int_{\mathcal{L}_{Y/X}(\mathbb{R}) \cap [\varepsilon, \infty]} (1 - e^{\varepsilon - y}) f_Y \left( \mathcal{L}_{Y/X}^{-1}(y) \right) \frac{d\mathcal{L}_{Y/X}^{-1}(y)}{dy} \, dy. \]

**Proof.** Consider the privacy loss function \( \mathcal{L}_{X/Y}(t) = \log \frac{f_X(t)}{f_Y(t)} \). Denote \( y(t) = \mathcal{L}_{X/Y}(t) \).
Then, it clearly holds \( f_Y(t) = e^{-y(t)} f_X(t) \) and
\[ \max\{ f_X(t) - e^\varepsilon f_Y(t), 0 \} = \begin{cases} (1 - e^{\varepsilon - y(t)}) f_X(t), & \text{if } y > \varepsilon, \\ 0, & \text{otherwise}. \end{cases} \quad (4.4) \]
Consider next the integral \( \int_\mathbb{R} \max\{ 0, f_X(t) - e^\varepsilon f_Y(t) \} \, dt \). By making the change of variables \( t = \mathcal{L}_{X/Y}^{-1}(y) \) and using (4.4), we see that
\[ \int_\mathbb{R} \max\{ 0, f_X(t) - e^\varepsilon f_Y(t) \} \, dt = \int_\mathbb{R} \max\{ 0, (1 - e^{\varepsilon - y(t)}) f_X(t) \} \, dt \]
\[ = \int_{\mathcal{L}_{X/Y}(\mathbb{R})} \max\left\{ 0, (1 - e^{\varepsilon - y}) f_X \left( \mathcal{L}_{X/Y}^{-1}(y) \right) \frac{d\mathcal{L}_{X/Y}^{-1}(y)}{dy} \right\} \, dy \]
\[ = \int_{\mathcal{L}_{X/Y}(\mathbb{R}) \cap [\varepsilon, \infty]} (1 - e^{\varepsilon - y}) f_X \left( \mathcal{L}_{X/Y}^{-1}(y) \right) \frac{d\mathcal{L}_{X/Y}^{-1}(y)}{dy} \, dy, \]
The claim follows then from Lemma \ref{lemma3} and the fact that by definition \(\omega(S) \geq 0\) for all measurable sets \(S\). Analogously, we see that
\[
\int_{\mathbb{R}} \max \{0, f_Y(t) - e^\varepsilon f_X(t)\} \, dt = \int_{\mathcal{L}_{Y/X}(\mathbb{R}) \cap [\varepsilon, \infty)} (1 - e^{\varepsilon-y}) f_Y(\mathcal{L}^{-1}_{Y/X}(y)) \frac{d\mathcal{L}^{-1}_{Y/X}(y)}{dy} \, dy.
\]
The claim follows then from Lemma \ref{lemma5}.

**Corollary 9.** A randomised algorithm \(M\) with an output of continuous one dimensional distributions is tightly \((\varepsilon, \delta)\)-DP for
\[
\delta = \max_{X \succeq Y} \max \{\delta_{X/Y}, \delta_{Y/X}\},
\]
where
\[
\delta_{X/Y} = \int_\varepsilon^\infty (1 - e^{\varepsilon-y}) \frac{d\omega_{X/Y}}{dy} \, dy, \quad \delta_{Y/X} = \int_\varepsilon^\infty (1 - e^{\varepsilon-y}) \frac{d\omega_{Y/X}}{dy} \, dy,
\]
and \(d\omega_{X/Y}/dy\) and \(d\omega_{Y/X}/dy\) are defined as in Lemma \ref{lemma5}.

### 4.1 Privacy loss distribution of compositions

In order to use the representation given by Corollary \ref{corollary9} for a composition of several mechanisms, we need to be able to evaluate the privacy loss distribution for compositions. The solution is given by Theorem \ref{theorem10} which is a continuous version of \cite[Thm. 1]{Theorem10}.

**Theorem 10.** Let \(M : \mathcal{X}^N \rightarrow \mathbb{R}\) and \(M' : \mathcal{X}^N \rightarrow \mathbb{R}'\) be independent random mechanisms with outputs of continuous one dimensional random variables with supports equaling \(\mathbb{R}\). Let \(X, Y\) be adjacent datasets and let \(f_X(t)\) denote the density function of \(M(X)\), \(f_Y(t)\) that of \(M(Y)\), \(f_{X'}(t)\) that of \(M'(X)\) and \(f_{Y'}(t)\) that of \(M'(Y)\). Consider the PLD \(\omega^c_{X/Y}\) of the composition of \(M\) and \(M'\) (either \(M \circ M'\) or \(M' \circ M\)). Denote by \(\omega_{X/Y}\) the PLD of \(M(X)\) over \(M(Y)\) and by \(\omega_{X'/Y'}\) the PLD of \(M'(X)\) over \(M'(Y)\). The density function of \(\omega^c_{X/Y}\) is given by
\[
\frac{d\omega^c_{X/Y}}{dy}(y) = \int_{-\infty}^\infty \frac{d\omega_{X/Y}}{dy}(t) \frac{d\omega_{X'/Y'}}{dy}(y-t) \, dt.
\]

**Proof.** We first show that the privacy loss function of a composition is a sum of privacy loss functions. Let \(\mathcal{L}^c_{X/Y}\) denote the privacy loss function of the composition mechanism. Then,
\[
\mathcal{L}^c_{X/Y}(t_1, t_2) = \log \left( \frac{f_X(t_1) f_{X'}(t_2) f_Y(t_1) f_{Y'}(t_2)}{f_{X,Y'}(t_1, t_2)} \right) = \log \left( \frac{f_X(t_1)}{f_{Y,Y'}(t_1)} \right) + \log \left( \frac{f_{X'}(t_2)}{f_{Y'}(t_2)} \right) = \mathcal{L}_{X/Y}(t_1) + \mathcal{L}_{X'/Y'}(t_2).
\]
Let $S \in \mathbb{R}$ be a measurable set. By using the property \[14\] and by change of variables we see that
\[
\omega_{X/Y}(S) = \int\int_{\{(t_1, t_2) \in \mathbb{R}^2 : \mathcal{L}_X(t_1, t_2) \in S\}} f_{X,X'}(t_1, t_2) \, dt_1 \, dt_2
\]
\[
= \int\int_{\{(t_1, t_2) \in \mathbb{R}^2 : \mathcal{L}_{X/Y}(t_1) + \mathcal{L}_{X'/Y'}(t_2) \in S\}} f_X(t_1) f_{X'}(t_2) \, dt_1 \, dt_2
\]
\[
= \int\int_{\{y_1 + y_2 \in S\} \cap \{(L_{X/Y}(R)) + (L_{X'/Y'}(R))\}} f_X(L^{-1}_{X/Y}(y_1)) \frac{d\mathcal{L}^{-1}_{X/Y}(y_1)}{dy} \, dy_1 \, dy_2
\]
\[
= \int\int_{\{y_1 + y_2 \in S\}} f_X'(L^{-1}_{X'/Y'}(y_2)) \frac{d\mathcal{L}^{-1}_{X'/Y'}(y_2)}{dy} \, dy_1 \, dy_2
\]
\[
= \int_S \left( \int_{-\infty}^\infty \frac{d\omega_{X/Y}}{dy}(y_1) \frac{d\omega_{X'/Y'}}{dy}(y_2) (t - y_1) \, dt \right) \, dy_1
\]
From Corollary 9 and Theorem 10 we get the following integral formula for the function $\delta(\varepsilon)$.

**Corollary 11.** Consider $k$ consecutive applications of a mechanism $M$. Let $\varepsilon > 0$. The composition is tightly $(\varepsilon, \delta)$-DP for $\delta$ given by
\[
\delta = \max_{X \succeq Y} \max \{\delta_{X/Y}, \delta_{Y/X}\},
\]
where
\[
\delta_{X/Y} = \int_\varepsilon^\infty (1 - e^{-y}) \left( \frac{d\omega_{X/Y}}{dy} * k \frac{d\omega_{X/Y}}{dy} \right)(y) \, dy,
\]
where $(\frac{d\omega_{X/Y}}{dy} * k \frac{d\omega_{X/Y}}{dy})(y)$ denotes the density function obtained by convolving $\frac{d\omega_{X/Y}}{dy}$ by itself $k$ times (an analogous formula holds for $\delta_{Y/X}$).

## 5 Subsampled Gaussian mechanism

The main motivation for this work comes from privacy accounting of differentially private stochastic gradient descent (DP-SGD) (see e.g. \[2\]). DP-SGD is a general purpose optimisation method for minimising loss functions of the form
\[
f(\theta) = \frac{1}{N} \sum_{i=1}^N f(\theta, x_i),
\]
where \( f(\theta, x) \) is differentiable with respect to \( \theta \) for all \( \theta \) and \( x \). The algorithm for DP-SGD is given in Algorithm 1.

**Algorithm 1** Differentially private SGD

Input: Data records \( \{x_1, \ldots, x_N\} \), learning rate \( \eta_\ell \), noise level \( \sigma \), lot size \( L \), clipping constant \( C \).

Initialise \( \theta_0 \) randomly.

for \( \ell = 0, 1, \ldots \)

- Draw a batch \( B \), with \( q = L/N \).
- Compute the gradients: For every \( i \in B \), compute: \( \hat{f}_\ell(x_i) = \nabla_\theta L(\theta_\ell, x_i) \)
- Clip the gradients: \( \tilde{f}_\ell(x_i) \leftarrow \hat{f}_\ell(x_i) / \max\{1, \|\hat{f}_\ell(x_i)\|_2\} \)
- Add noise: \( M_\ell \leftarrow \frac{1}{L} \left( \sum_{i \in B} \tilde{f}_\ell(x_i) + \mathcal{N}(0, \sigma^2 I_d) \right) \)
- Descent: \( \theta_{\ell+1} \leftarrow \theta_\ell - \eta_\ell M_\ell \)

end for

5.1 DP analysis via one dimensional mixture distributions

For completeness, we show that the worst case analysis of DP-SGD can be carried out by analysis of one dimensional probability distributions. This can be seen as follows (see also [2, Proof of Lemma 3]). We see that the basic mechanism \( M \) of DP-SGD is of the form

\[
M(X) = \sum_{i \in B} f(x_i) + \mathcal{N}(0, \sigma^2 I_d),
\]

where \( B \) is a randomly drawn subset of \( \{1, \ldots, N\} \) of size \( L \) and \( \|f(x_i)\|_2 \leq 1 \) for all \( i \in B \).

First, consider the case of remove/add relation \( \simeq_R \) and let \( X \) and \( Y \) be neighbouring datasets. Suppose \( X = Y \cup \{x'\} \) and assume \( \|f(x')\|_2 = 1 \). Consider first the case \( L = N \).

The condition of \((\varepsilon, \delta)\)-differential privacy states that for every measurable set \( S \subset \mathbb{R}^d \):

\[
P(M(X) \in S) \leq e^\varepsilon P(M(Y) \in S) + \delta,
\]

and we easily see that this is then equivalent to the condition that for every measurable set \( S \subset \mathbb{R}^d \):

\[
P(N(x', \sigma^2 I_d) \in S) \leq e^\varepsilon P(N(0, \sigma^2 I_d) \in S) + \delta.
\]

Let \( U \in \mathbb{R}^{d \times d} \) be a unitary matrix such that

\[
U x' = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} =: e_1.
\]
Due to the unitarity of $U$, the condition \((5.2)\) is equivalent to the condition that for every measurable set $S \subset \mathbb{R}^d$:
\[
P(U\mathcal{N}(x', \sigma^2 I_d) \in US) \leq e^\varepsilon P(U\mathcal{N}(0, \sigma^2 I_d) \in US) + \delta
\]
\[
\Leftrightarrow
\]
\[
P(\mathcal{N}(e_1, \sigma^2 I_d) \in US) \leq e^\varepsilon P(\mathcal{N}(0, \sigma^2 I_d) \in US) + \delta,
\]
where $US = \{UX : x \in S\}$. Furthermore, we see that the condition \((5.3)\) is equivalent to the condition that for every measurable set $S \subset \mathbb{R}$:
\[
P(\mathcal{N}(1, \sigma^2) \in S) \leq e^\varepsilon P(\mathcal{N}(0, \sigma^2) \in S) + \delta.
\] \((5.4)\)
Thus, finding the parameters $\varepsilon$ and $\delta$ that satisfy \((5.1)\) amounts to finding values of $\varepsilon$ and $\delta$ that satisfy \((5.4)\). When $L < N$, we see that $x'$ is in $B$ with a probability $q = L/N$. By analogous reasoning this leads to the condition that for every measurable set $S \subset \mathbb{R}^d$:
\[
P(q\mathcal{N}(x', \sigma^2 I_d) + (1-q)\mathcal{N}(0, \sigma^2 I_d) \in S) \leq e^\varepsilon P(\mathcal{N}(0, \sigma^2 I_d) \in S) + \delta,
\]
where $q\mathcal{N}(x', \sigma^2 I_d) + (1-q)\mathcal{N}(0, \sigma^2 I_d)$ denotes a mixture distribution. Similarly, this leads to consider the one dimensional random variables $q\mathcal{N}(1, \sigma^2) + (1-q)\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \sigma^2)$. The case $\|x'\|_2 < 1$ would lead to a condition involving the distribution $\mathcal{N}(\|x'\|_2 e^\varepsilon, \sigma^2 I_d)$ which would give tighter $(\varepsilon, \delta)$-values and thus $\|x'\|_2 = 1$ gives the worst case.
Similarly, in the case of substitution relation $\approx_S$ the worst case is obtained by considering the mixture distributions $q\mathcal{N}(x', \sigma^2 I_d) + (1-q)\mathcal{N}(0, \sigma^2 I_d)$ and $q\mathcal{N}(-x', \sigma^2 I_d) + (1-q)\mathcal{N}(0, \sigma^2 I_d)$, where $\|x'\|_2 = 1$.

### 5.2 Neighbouring relation with remove/add

As shown above and also in \([2]\), for the analysis of DP-SGD in case of neighbouring relation $\approx_H$ it is sufficient to consider the one dimensional distributions (see also \([17]\) and \([12]\))
\[
f_X(t) = q\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-1)^2}{2\sigma^2}} + (1-q)\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{t^2}{2\sigma^2}},
\]
\[
f_Y(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{t^2}{2\sigma^2}}.
\]
Here the privacy loss function $\mathcal{L}_{X/Y}(t)$ is given by
\[
\mathcal{L}_{X/Y}(t) = \log q\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-1)^2}{2\sigma^2}} + (1-q)\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{t^2}{2\sigma^2}} = \log \left( q e^{\frac{2t-1}{2\sigma^2}} + (1-q) \right).
\]
We see that $\mathcal{L}_{X/Y}(\mathbb{R}) = (\log(1-q), \infty)$ and that $\mathcal{L}_{X/Y}$ is a monotonously increasing continuously differentiable function in the whole $\mathbb{R}$. Straightforward calculation shows that
\[
\mathcal{L}_{X/Y}^{-1}(y) = \sigma^2 \log \frac{e^y - (1-q)}{q} + \frac{1}{2}.
\]
Moreover, 
\[ \frac{d}{dy} \mathcal{L}_{X/Y}^{-1}(y) = \frac{\sigma^2 e^y}{e^y - (1 - q)}. \]

The privacy loss distribution \( \omega_{X/Y} \) can now be described with the density function 
\[ \frac{d \omega_{X/Y}}{dy}(y) = \begin{cases} f_X \left( L^{-1}_{X/Y}(y) \right) \frac{d}{dy} L^{-1}_{X/Y}(y), & \text{if } y > \log(1 - q), \\ 0, & \text{else.} \end{cases} \]

Using the properties of the privacy loss function (see [14, Lemma 2]), we find that 
\[ \text{supp} \frac{d \omega_{Y/X}}{dy} = \{ -y : y \in \text{supp} \frac{d \omega_{X/Y}}{dy} \} \]
and that for all \( y \in \text{supp} \frac{d \omega_{Y/X}}{dy} \) it holds 
\[ \frac{d \omega_{Y/X}}{dy}(y) = e^y \frac{d \omega_{X/Y}}{dy}(-y). \]

From this we can infer using Lemma 8 that \( \delta = \delta_{X/Y} \).

### 5.3 Neighbouring relation with substitution

Now we show the PLD for subsampled Gaussian mechanism for \((\varepsilon, \delta, \simeq S)\)-DP. In this case, without loss of generality, we may consider the density functions 
\[ f_X(t) = q \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(t-1)^2}{2\sigma^2}} + (1 - q) \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2}}, \]
\[ f_Y(t) = q \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(t+1)^2}{2\sigma^2}} + (1 - q) \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2}}. \]

The privacy loss function is then given by 
\[ \mathcal{L}_{X/Y}(t) = \log \left( \frac{q \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(t-1)^2}{2\sigma^2}} + (1 - q) \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2}}}{q \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(t+1)^2}{2\sigma^2}} + (1 - q) \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2}}} \right) = \log \left( \frac{q e^{\frac{2t-1}{2\sigma^2}} + (1 - q)}{q e^{\frac{-2t-1}{2\sigma^2}} + (1 - q)} \right). \]

We see that \( \mathcal{L}_{X/Y}(\mathbb{R}) = \mathbb{R} \) and again that \( \mathcal{L}_{X/Y} \) is a monotonously increasing continuously differentiable function in the whole \( \mathbb{R} \). Denote 
\[ x = e^{\frac{t}{\sigma^2}} \quad \text{and} \quad c = qe^{-\frac{1}{2\sigma^2}}. \]

Then, solving \( \mathcal{L}_{X/Y}(t) = y \) leads to the equation 
\[ \frac{cx + (1 - q)}{cx^{-1} + (1 - q)} = e^y \]
\[ \iff cx^2 + (1 - q)(1 - e^y)x - ce^y = 0 \]
\[ x \geq 0 \quad x = \frac{-(1 - q)(1 - e^y) + \sqrt{(1 - q)^2(1 - e^y)^2 + 4c^2e^y}}{2c}. \]

We find that 
\[ \mathcal{L}_{X/Y}^{-1}(y) = \sigma^2 \log \left( \frac{-(1 - q)(1 - e^y) + \sqrt{(1 - q)^2(1 - e^y)^2 + 4c^2e^y}}{2c} \right). \]
and
\[ \frac{d}{dy} \mathcal{L}^{-1}_{X/Y}(y) = \sigma^2 \frac{4c^2e^y - 2(1-q)^2e^y(1-e^y)}{2\sqrt{4c^2e^y + (1-q)^2(1-e^y)^2} - (1-q)(1-e^y)}. \]

Using [14, Lemma 2], Lemma 8 and the fact that \( L_{X/Y}(t) = -L_{X/Y}(-t) \) and \( f_Y(-t) = f_X(t) \), we infer that \( \delta = \delta_{Y/X} = \delta_{X/Y} \).

6 Numerical approximation of convolutions via DFT

The discrete Fourier transform (DFT) \( \mathcal{F} \) and its inverse \( \mathcal{F}^{-1} \) are linear operators \( \mathbb{C}^n \rightarrow \mathbb{C}^n \) that decompose a complex vector into a Fourier series, or reconstruct it from its Fourier series. One standard definition is given as follows (see e.g. [16]). Suppose \( x = (x_0, \ldots, x_{n-1}) \), \( w = (w_0, \ldots, w_{n-1}) \in \mathbb{R}^n \). Then, \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are defined as

\[ (Fx)_k = \sum_{j=0}^{n-1} x_j e^{-i2\pi kj/n}, \quad (F^{-1}w)_k = \frac{1}{n} \sum_{j=0}^{n-1} w_j e^{i2\pi kj/n}. \]

Evaluating \( \mathcal{F}x \) and \( \mathcal{F}^{-1}w \) as written above takes \( O(N^2) \) operations, however evaluation via the Fast Fourier Transform (FFT) [7] reduces the computational burden to \( O(N \log N) \).

The periodic convolution of two vectors \( v = (v_0, \ldots, v_{n-1}) \), \( w = (w_0, \ldots, w_{n-1}) \in \mathbb{R}^n \) is defined as the vector \( v \ast w \), where

\[ (v \ast w)_k = \sum_{j=0}^{n-1} v_j w_{k-j}, \]

where the indices are modulo \( n \). The convolution theorem [15] states that for periodic convolutions

\[ v \ast w = \mathcal{F}^{-1}(\mathcal{F}v \odot \mathcal{F}w). \]

7 Description of the method

We next describe step by step the numerical method for computing exact DP-guarantees for continuous one dimensional privacy loss distributions. From here on, we simply denote \( \omega \) (instead of \( \frac{d\omega}{dy} \)) the density function of a PLD.

7.1 Approximation of integrals involving convolutions

We first approximate the convolution integrals on a truncated interval \([-L, L] \) as

\[ (\omega \ast \omega)(x) = \int_{-\infty}^{\infty} \omega(t)\omega(x-t) \, dt \approx \int_{-L}^{L} \omega(t)\omega(x-t) \, dt =: (\omega \odot \omega)(x). \]
Let \( \tilde{\omega} \) be a \( 2\pi \)-periodic extension of \( \omega \) such that \( \tilde{\omega}(t + n2\pi) = \omega(t) \) for all \( t \in [-L, L] \) and \( n \in \mathbb{Z} \). Then, we further approximate the convolutions as

\[
\int_{-L}^{L} \omega(t)\omega(x-t) \, dt \approx \int_{-L}^{L} \tilde{\omega}(t)\tilde{\omega}(x-t) \, dt = \int_{0}^{2L} \tilde{\omega}(t)\tilde{\omega}(x-t) \, dt. 
\] (7.1)

### 7.2 Discretisation

Divide the interval \([-L, L]\) on \( n \) equidistant points \( x_0, \ldots, x_{n-1} \) such that

\[ x_\ell = -L + \ell \Delta x, \text{ where } \Delta x = 2L/n. \]

Denote

\[ \omega_\ell = \omega(x_\ell) \text{ and } \tilde{\omega}_\ell = \tilde{\omega}(\ell \Delta x) \]

and consider the vectors

\[ \omega = \begin{bmatrix} \omega_0 \\ \vdots \\ \omega_{n-1} \end{bmatrix} \text{ and } \tilde{\omega} = \begin{bmatrix} \tilde{\omega}_0 \\ \vdots \\ \tilde{\omega}_{n-1} \end{bmatrix}. \]

Notice that from the periodicity it follows that \( \tilde{\omega} = D\omega \), where

\[
D = \begin{bmatrix} 0 & I_{n/2} \\ I_{n/2} & 0 \end{bmatrix}.
\]

Then, we may approximate the right hand side of (7.1) using trapezoidal rule and the convolution theorem as

\[
\int_{0}^{2L} \tilde{\omega}(t)\tilde{\omega}(\ell \Delta x - t) \, dt \approx \Delta x \sum_{j=0}^{n-1} \tilde{\omega}_j \tilde{\omega}_{\ell-j} \text{ (indices modulo } n) \\
= \Delta x \left[ \mathcal{F}^{-1}(\mathcal{F}(\tilde{\omega}) \odot \mathcal{F}(\tilde{\omega})) \right]_\ell,
\]

where \( \mathcal{F} \) denotes the discrete Fourier transform and \( \odot \) an elementwise product of vectors. Using the periodicity, we obtain the approximation

\[
(\tilde{\omega} \odot \tilde{\omega})(-L + \ell \Delta x) \approx \Delta x \left[ D\mathcal{F}^{-1}(\mathcal{F}(\tilde{\omega}) \odot \mathcal{F}(\tilde{\omega})) \right]_\ell.
\]

Inductively, we see that carrying out \( k \) convolutions of \( \omega \) with itself leads to the approximation

\[
(\tilde{\omega} \odot^k \tilde{\omega})(-L + \ell \Delta x) \approx (\Delta x)^{-1} \left[ D\mathcal{F}^{-1}(\mathcal{F}(\tilde{\omega}) \odot^k \tilde{\omega}) \right]_\ell \\
= (\Delta x)^{-1} \left[ D\mathcal{F}^{-1}(\mathcal{F}(D\omega \Delta x)^{\odot k}) \right]_\ell.
\]

where \( \odot^k \) denotes \( k \)th elementwise power of vectors.
7.3 Approximation of the integral

Denote \( C = (\Delta x)^{-1} \left[ D\mathcal{F}^{-1}(\mathcal{F}(D\omega \Delta x) \circ^k) \right] \) and \( \ell_x = \min\{\ell \in \mathbb{Z} : -L + \ell \Delta x > \varepsilon\} \). Using the vector \( C = \begin{bmatrix} C_0 & \cdots & C_{n-1} \end{bmatrix}^T \), we further approximate

\[
\delta(\varepsilon) = \int_{\varepsilon}^{\infty} (1 - e^{-y})(\omega \ast_k \omega)(y) \, dy \approx \Delta x \sum_{\ell = \ell_x}^{n-1} (1 - e^{\varepsilon - (-L + \ell \Delta x)}) C_\ell. \tag{7.2}
\]

7.4 Approximation for adaptive mechanisms

For a composition \( M_1 \circ \ldots \circ M_k \) of varying mechanisms \( M_1, \ldots, M_k \), the function \( \delta(\varepsilon) \) is given by Lemma 8 and Theorem 10 by an integral formula of the form

\[
\delta(\varepsilon) = \int_{\varepsilon}^{\infty} (1 - e^{-y})(\omega_1 \ast \ldots \ast \omega_k)(y) \, dy,
\]

where \( \omega_i \)'s are PLD distributions determined by the mechanisms \( M_i \), \( 1 \leq i \leq k \).

Denoting \( C = (\Delta x)^{-1} \left[ D\mathcal{F}^{-1}(F_1 \circ \ldots \circ F_k) \right] \), where \( F_i = \mathcal{F}(D\omega_i \Delta x) \), \( 1 \leq i \leq k \), and \( \omega_i \)'s are obtained from discretisations of \( \omega_i \)'s (as in Subsection 7.2), then \( \delta(\varepsilon) \) can be approximated as in (7.2).

8 Computing \( \varepsilon(\delta) \) using Newton’s method

We get \( \delta \) as a function of \( \varepsilon \) from the integral representation

\[
\delta(\varepsilon) = \int_{\varepsilon}^{\infty} (1 - e^{-y})(\omega \ast_k \omega)(y) \, dy. \tag{8.1}
\]

In order to get the function \( \varepsilon(\delta) \), we may use Newton’s method. From (8.1) it follows that

\[
\frac{d\delta}{d\varepsilon}(\varepsilon) = -\int_{\varepsilon}^{\infty} e^{-y}(\omega \ast_k \omega)(y) \, dy. \tag{8.2}
\]

Thus, in order to find \( \varepsilon \) such that \( \delta(\varepsilon) = \bar{\delta} \), we apply Newton’s method (see e.g. [16]) for the function \( \delta(\varepsilon) - \bar{\delta} \) which gives the iteration

\[
\varepsilon_{\ell+1} = \varepsilon_\ell - \frac{\delta(\varepsilon_\ell) - \bar{\delta}}{d\delta/d\varepsilon(\varepsilon_\ell)}.
\]

Evaluating \( \delta(\varepsilon) \) for different values of \( \varepsilon \) is cheap using the formulas (8.1) and (8.2) when we have a precomputed approximation of \( \omega \circ_k \omega \). We use as a stopping criterion

\[
|\delta(\varepsilon_\ell) - \bar{\delta}| \leq tol
\]

for some prescribed parameter \( tol \). The iteration was found to converge in all experiments with an initial value \( \varepsilon_0 = 0 \). If one wants to compute the \( \varepsilon(\delta) \)-values for several values of \( k \), each \( \varepsilon \)-value found serves as a good initial value for the subsequent Newton iteration.
9 Experiments

We compare the proposed method to the privacy accountant method included in the Tensorflow library \cite{tensorflow} which is the moments accountant described in \cite{abadi2016deep} and to the subsampled Rényi accountant method described in \cite{wang2017privacy}.

As the Tensorflow moments accountant method computes the privacy under $(\varepsilon,\delta,\simeq_R)$-DP, we compare it to the exact guarantee determined by the PLD given in Subsection 5.2 (see Fig. 1). The Rényi accountant method accounts the privacy under $(\varepsilon,\delta,\simeq_S)$-DP and we compare it with the exact guarantee using the PLD given in Subsection 5.3 (see Fig. 2).

We use here $q = 0.01$ and $\sigma \in \{1.0, 2.0, 3.0\}$, and compute for a number of compositions $k$ up to 4000. We set the parameters $L = 20$ and $n = 5 \cdot 10^6$ for the approximation of the exact integral.

Figure 3 shows the comparison of the exact guarantee and the Tensorflow moments accountant method in case of $(\varepsilon,\delta,\simeq_R)$-DP for values of $k$ up to $5 \cdot 10^4$. We set the parameters $L = 40$ and $n = 5 \cdot 10^6$.

Table 1 illustrates the convergence of the approximation with respect to the truncation parameters $n$ and $L$ in the case of the $(\varepsilon,\delta,\simeq_R)$-DP.

| $L$  | $\sigma = 1.0$ | $\sigma = 2.0$ | $n$     | $\sigma = 1.0$ | $\sigma = 2.0$ |
|------|----------------|----------------|--------|----------------|----------------|
| 2.5  | 2.49628225     | 2.34124153     | $5 \cdot 10^6$ | 7.05132200     | 2.44669858     |
| 5    | 4.97128256     | 2.44670580     | $1 \cdot 10^6$ | 6.98037918     | 2.44670204     |
| 10   | 6.90737515     | 2.44670554     | $2 \cdot 10^6$ | 6.90737388     | 2.44670423     |
| 20   | 6.90735948     | 2.44670515     | $4 \cdot 10^6$ | 6.90735948     | 2.44670515     |
| 40   | 6.90733782     | 2.44670417     | $8 \cdot 10^6$ | 6.90738305     | 2.44670559     |

Table 1: Convergence of the $\varepsilon(\delta)$-approximation with respect to the parameter $L$ (when $n = 4 \cdot 10^6$) and with respect to $n$ (when $L = 20$). We evaluate $\varepsilon(\delta)$ for $k = 10^4$, $q = 0.01$ and $\delta = 10^{-6}$.
Figure 1: Comparison of the Tensorflow moments accountant and the exact guarantee for $(\varepsilon, \delta, \approx_R)$-DP.

\[ (a) \ v(\delta) \text{ as a function of } k \text{ for } \delta = 10^{-6}. \]

\[ (b) \ \delta(v) \text{ as a function of } k \text{ for } v = 1.0. \]

Figure 2: Comparison of the Rényi accountant method and the exact guarantee for $(\varepsilon, \delta, \approx_S)$-DP.

\[ (a) \ v(\delta) \text{ as a function of } k \text{ for } \delta = 10^{-6}. \]

\[ (b) \ \delta(v) \text{ as a function of } k \text{ for } v = 1.0. \]
10 Error analysis for the subsampled Gaussian mechanism

We discuss next the four different approximations that affect the approximation error of the proposed method. We consider the PLD arising from a composition of subsampled Gaussian mechanisms with the neighbouring relation $\simeq_R$ (as described in Subsection 5.2). The purpose of the analysis is simply to show that the numerical approximation converges to the exact privacy guarantee as the truncation parameter $L \to \infty$ and the number of discretisation points $n \to \infty$. The questions of the convergence speed and a priori choice of parameters $L$ and $n$ are left open for future work. We do not consider here the error arising from the discrete Fourier transform, i.e., we assume that the discrete convolutions are evaluated exactly. From here on, we simply denote $\omega$ (instead of $\frac{d\omega}{dy}$) the density function of a PLD.

Let $\sigma > 0$ and $0 < q < 1$. Recall that in the case of the neighbouring relation $\simeq_R$ it is sufficient to analyse the distribution function

$$
\omega(y) = \begin{cases} 
    f(L^{-1}(y)) \frac{d}{dy} L^{-1}(y), & \text{if } y > \log(1-q), \\
    0, & \text{otherwise}. 
\end{cases}
$$

where

$$
f(t) = C \left[ q e^{-\frac{(t-1)^2}{2\sigma^2}} + (1-q) e^{-\frac{t^2}{2\sigma^2}} \right], \quad C = \frac{1}{\sqrt{2\pi\sigma^2}},
$$

(10.1)

$$
L^{-1}(y) = \sigma^2 \log \left( \frac{e^y - (1-q)}{q} \right) + \frac{1}{2}
$$

(10.2)

and

$$
\frac{d}{dy} L^{-1}(y) = \frac{\sigma^2 e^y}{e^y - (1-q)}.
$$

(10.3)
The total error can be decomposed as follows. By adding and subtracting terms and using the triangle inequality, the error can be bounded as

\[
\begin{align*}
&\left| \int_\varepsilon^\infty (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy - \Delta x \sum_{\ell=0}^{n-1} (1 - e^{\varepsilon-(t\Delta x)}) C_\ell \right| \leq \\
&\left| \int_\varepsilon^\infty (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy - \int_\varepsilon^L (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy \right| \\
+&\left| \int_\varepsilon^L (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy - \int_\varepsilon^L (1 - e^{\varepsilon-y}) (\omega \odot^k \omega)(y) \, dy \right| \\
+&\left| \int_\varepsilon^L (1 - e^{\varepsilon-y}) (\omega \odot^k \omega)(y) \, dy - \int_\varepsilon^L (1 - e^{\varepsilon-y}) (\tilde{\omega} \odot^k \tilde{\omega})(y) \, dy \right| \\
+&\left| \int_\varepsilon^L (1 - e^{\varepsilon-y}) (\tilde{\omega} \odot^k \tilde{\omega})(y) \, dy - \Delta x \sum_{\ell=0}^{n-1} (1 - e^{\varepsilon-(t\Delta x)}) C_\ell \right|. \\
\end{align*}
\]

(10.4)

We consider separately each of the four terms on the right hand side of (10.4) and thereby we show that the approximation converges to the exact privacy guarantee as \( L \to \infty \) and \( n \to \infty \).

10.1 First approximation: truncation of the integral representation

The first approximation we make is the truncation of the integral (for \( L > \varepsilon \))

\[
\int_\varepsilon^\infty (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy \approx \int_\varepsilon^L (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy.
\]

The error induced here is

\[
err_1(L) := \int_\varepsilon^L (1 - e^{\varepsilon-y})(\omega \ast^k \omega)(y) \, dy.
\]

Clearly,

\[
0 \leq err_1(L) < \int_\varepsilon^\infty (\omega \ast^k \omega)(y) \, dy. 
\]

(10.5)

Since \( \omega \) is a density function of a probability distribution, so is \( \omega \ast^k \omega \). Thus, we see from (10.5) that \( err_1(L) \to 0 \) as \( L \to \infty \). An analysis of the speed of convergence as \( L \to \infty \) is left here open.
10.2 Second approximation: truncation of the convolution integrals

The second approximation we carry out is

\[
\int_{\varepsilon}^{L} (1 - e^{\varepsilon-y})(\omega \ast^{k} \omega)(y) \, dy \approx \int_{\varepsilon}^{L} (1 - e^{\varepsilon-y})(\omega \ast^{k} \omega)(y) \, dy,
\]

i.e., we replace the limits in the convolution integrals from \( \int_{-\infty}^{\infty} \) to \( \int_{-L}^{L} \).

We first bound \( \omega \ast^{k} \omega - \omega \ast^{k} \omega \). By adding and subtracting, we may write

\[
\omega \ast^{k} \omega - \omega \ast^{k} \omega = \omega \ast (\omega \ast^{k-1} \omega - \omega \ast^{k-1} \omega) + (\omega \ast - \omega \ast)(\omega \ast^{k-1} \omega),
\]

where

\[
(\omega \ast - \omega \ast)(\omega \ast^{k-1} \omega)(x) = \int_{-L}^{L} \omega(t)(\omega \ast^{k-1} \omega)(x-t) \, dt - \int_{-\infty}^{\infty} \omega(t)(\omega \ast^{k-1} \omega)(x-t) \, dt = - \int_{L}^{\infty} \omega(t)(\omega \ast^{k-1} \omega)(x-t) \, dt,
\]

since \( \omega(y) = 0 \) for all \( y < \log(1 - q) \) and \( -L < \log(1 - q) \). Using Lemma 13 of Appendix, we see that for all \( x \),

\[
|((\omega \ast - \omega \ast)(\omega \ast^{k-1} \omega))(x)| \leq \max_{y \geq L} \omega(y) \int_{L}^{\infty} (\omega \ast^{k-1} \omega)(x-t) \, dt \leq \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(a^2 + C)^2}{2\sigma^2}},
\]

where \( C_2 = \sigma^2 \log\left(\frac{1}{2q}\right) - \frac{1}{2} \). Using again Lemma 13, we see that for all \( x \),

\[
|((\omega \ast - \omega \ast)(\omega \ast^{k-1} \omega))(x)| \leq \max_{y \geq L} \omega(y) \int_{L}^{\infty} \omega(x-t) \, dt \leq \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(a^2 + C)^2}{2\sigma^2}}.
\]

Using Lemma 12 of Appendix, the recursion (10.6) and the bounds (10.7) and (10.8), we see that for all \( x \),

\[
|((\omega \ast^{k} \omega - \omega \ast^{k} \omega)(x)| \leq 2\sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(a^2 L + C)^2}{2\sigma^2}} \left( (L + 1) - \frac{\sigma}{q \sqrt{2\pi e^{\sigma}}} \right)^{k-2}.
\]

From (10.2) it follows that for all fixed \( k \), the second approximation goes to zero as \( L \to \infty \).
10.3 Third approximation: periodisation

The third approximation we carry out is

\[
\int_{-\epsilon}^{L} (1 - e^{y}) (\omega \otimes \kappa \omega)(y) \, dy \approx \int_{-\epsilon}^{L} (1 - e^{y})(\bar{\omega} \otimes \kappa \bar{\omega})(y) \, dy,
\]

(10.10)

i.e., we replace the density function \(\omega\) by the periodic function \(\bar{\omega}\).

The operation \(\otimes\) is clearly linear w.r.t. both operands, and thus

\[
\omega \otimes \kappa \omega - \bar{\omega} \otimes \kappa \bar{\omega} = (\omega - \bar{\omega}) \otimes (\kappa \omega) + \bar{\omega} \otimes (\omega \otimes \kappa \omega - \bar{\omega} \otimes \kappa \bar{\omega}) = \bar{\omega} \otimes (\omega \otimes \kappa \omega - \bar{\omega} \otimes \kappa \bar{\omega}),
\]

(10.11)

since \(\omega = \bar{\omega}\) on the interval \([-L, L]\).

We see that

\[
\text{err}_2(x) := (\omega \otimes \omega - \bar{\omega} \otimes \bar{\omega})(x) = \int_{-L}^{L} \omega(t)[\omega(x - t) - \bar{\omega}(x - t)] \, dt
\]

\[
= \int_{\log(1 - q)}^{L} \omega(t)[\omega(x - t) - \bar{\omega}(x - t)] \, dt,
\]

since \(\omega(y) = 0\) for all \(y < \log(1 - q)\). By simple geometric reasoning, due to the periodicity of \(\bar{\omega}\), we see that when \(x < L + \log(1 - q)\), \(\text{err}_2(x) = 0\). When \(L + \log(1 - q) < x < L\), again due to the periodicity of \(\bar{\omega}\),

\[
\text{err}_2(x) = \int_{\log(1 - q)}^{L-x} \omega(t)[\omega(x - t) - \bar{\omega}(x - t)] \, dt
\]

\[
= \int_{\log(1 - q)}^{L-x} \omega(t)\omega(x - t) \, dt = \int_{\log(1 - q)}^{L-x} \omega(t)\omega(x - t) \, dt.
\]

Since inside the integration interval \(x - t \geq L\), by using Lemma 13 of Appendix, we see that in the integrand

\[
\omega(x - t) \leq \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(\sigma^2 L + C_2)^2}{2\sigma^2 x}}
\]

where \(C_2 = \sigma^2 \log\left(\frac{L}{2q}\right) - \frac{1}{2}\). Furthermore, using Lemma 12 of Appendix, we see that

\[
\text{err}_2(x) = \int_{\log(1 - q)}^{L-x} \omega(t)\omega(x - t) \, dt \leq \int_{\log(1 - q)}^{L-x} \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(\sigma^2 L + C_2)^2}{2\sigma^2 x}} \, dt
\]

\[
\leq -2 \log(1 - q) \frac{\sigma}{q\pi} e^{-\frac{(\sigma^2 L + C_2)^2}{2\sigma^2 x}} \left(\frac{1}{(1 - q)^2}\right) \frac{\sigma}{q\pi} e^{-\frac{\omega^2 L + C_2^2}{2\sigma^2 x}} + \frac{1}{x^2}
\]

(10.12)
Thus, we see from (10.12) and the recursion (10.11) that

\[ |(\omega \otimes^k \omega - \tilde{\omega} \otimes^k \tilde{\omega})(x)| \leq \left( \frac{\sigma}{q \sqrt{2\pi}} e^{\frac{x^2}{2}} (L + 1)^{k-2} \right) \log \left( \frac{1}{1-q} \right) \frac{\sigma}{q \pi} e^{-\frac{(x^2 L + C_2)^2}{2x^2 \sigma}} + \frac{1}{\sigma^2} \]  

(10.13)

We see from the bound (10.13) that the error converges uniformly to zero for all \( 0 < x \leq L \). Thus the error induced by the approximation (10.10) goes to zero for all \( k \in \mathbb{N}^+ \) as \( L \to \infty \).

### 10.4 Fourth approximation: discretisation

Divide the interval \([-L, L]\) on \( n \) equidistant points \( x_0, \ldots, x_{n-1} \) such that

\[ x_\ell = \ell \Delta x, \text{ where } \Delta x = 2L/n. \]

Denote

\[ C_\ell = (\tilde{\omega} \otimes^k \tilde{\omega})(\ell \Delta x). \]

The fourth approximation we carry out is

\[ \int_{-L}^{L} (1 - e^{\varepsilon - y})(\tilde{\omega} \otimes^k \tilde{\omega})(y) \, dy \approx \Delta x \sum_{\ell = k_x}^{n-1} (1 - e^{\varepsilon - (\ell \Delta x)}) C_\ell, \]  

(10.14)

where \( k_x = \min \{ k \in \mathbb{Z} : -L + k \Delta x > \varepsilon \} \).

An elementary analysis shows that

\[ \left| \int_{-L}^{L} (1 - e^{\varepsilon - y})(\tilde{\omega} \otimes^k \tilde{\omega})(y) \, dy - \Delta x \sum_{\ell = k_x}^{n-1} (1 - e^{\varepsilon - (\ell \Delta x)}) C_\ell \right| \leq \max_{\varepsilon \leq y \leq L} \left| \frac{d}{dy} h(y) \right| \frac{(L - \varepsilon)^2}{2n}, \]  

(10.15)

where \( h(y) = (1 - e^{\varepsilon - y})(\tilde{\omega} \otimes^k \tilde{\omega})(y) \)

By using Lemma 12 of Appendix,

\[ |h'(y)| = \left| e^{\varepsilon - y}(\tilde{\omega} \otimes^k \tilde{\omega})(y) + (1 - e^{\varepsilon - y}) \frac{d}{dy} (\tilde{\omega} \otimes^k \tilde{\omega})(y) \right| \]

\[ \leq \frac{\sigma}{q \sqrt{2\pi}} e^{\frac{1}{2}} + \max_{y > \log(1-q)} \left| \frac{d}{dy} (\tilde{\omega} \otimes^k \tilde{\omega})(y) \right|. \]

Since

\[ \left| \frac{d}{dy} (\tilde{\omega} \otimes^k \tilde{\omega})(y) \right| = \int_{-L}^{L} (\tilde{\omega}(t_1) \int_{-L}^{L} (\tilde{\omega}(t_2) \ldots \int_{-L}^{L} (\tilde{\omega}(t_{k-2}) \frac{d}{dy} (\tilde{\omega}(y - t_1 - \ldots - t_{k-2}) dt_1 \ldots dt_{k-1} \right| \]

\[ \leq \max_{y > \log(1-q)} \left| \frac{d}{dy} \tilde{\omega}(y) \right|, \]

bounding \( \max_{\varepsilon \leq y \leq L} \left| \frac{d}{dy} h(y) \right| \frac{(L - \varepsilon)^2}{2n} \) amounts to finding a bound for \( \max_{y > \log(1-q)} \left| \frac{d}{dy} \tilde{\omega}(y) \right|. \)
We see that
\[
\frac{d}{dy} \omega(y) = f'(L^{-1}(y)) \left( \frac{dL^{-1}(y)}{dy} \right)^2 + f(L^{-1}(y)) \frac{d^2L^{-1}(y)}{dy^2}.
\]

A simple differentiation shows that
\[
\max_{\epsilon \leq y \leq L} f'(L^{-1}(y)) \leq \frac{1}{\sigma^2} \max_{\epsilon \leq y \leq L} L^{-1}(y) f(L^{-1}(y)) \left( \frac{dL^{-1}(y)}{dy} \right)^2 \tag{10.16}
\]
The two terms on the right hand side of (10.16) can be bounded using the technique of the proof of Lemma 12. Bounding the first term amounts to bounding the function \( f(x) = xe^{-\frac{(x^2+x^2+\frac{1}{2})^2}{2\sigma^2}-2x} \) for \( x < 0 \). As both are bounded function on the negative real axis, so is \( \max_{\epsilon \leq y \leq L} \left| \frac{d}{dy} h(y) \right| \) and thus
\[
\left| \int_{\epsilon}^{\infty} (1 - e^{\epsilon-y})(\tilde{\omega} \ast^k \tilde{\omega})(y) dy - \Delta x \sum_{\ell=0}^{n-1} (1 - e^{-\ell \Delta x}) C_\ell \right| \leq C(\sigma, q) \frac{(L - \epsilon)^2}{2n}
\]
for some constant \( C(\sigma, q) \) depending only on \( \sigma \) and \( q \).

10.5 Total error

Using the bounds given in subsections 10.1, 10.2, 10.3 and 10.4 to the four terms on the right hand side of (10.4), we see that the approximation converges to the exact privacy guarantee as the truncation parameter \( L \to \infty \) and the number of discretisation points \( n \to \infty \).

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A Appendix: Auxiliary results

These two Lemmas which are be needed in the analysis of the approximation error.

**Lemma 12.** For all $y \in \mathbb{R}$:

$$\omega(y) \leq \frac{\sigma}{q\sqrt{2\pi}} e^{\frac{1}{2\sigma^2}}.$$ 

**Proof.** Consider first the case $y \in (\log(1 - q), 0]$. We see from \([10.2]\) that for all $y \in (\log(1 - q), 0)$: $\mathcal{L}^{-1}(y) \in (-\infty, \frac{1}{2})$ and therefore

$$e^{-\frac{(\mathcal{L}^{-1}(y) - 1)^2}{2\sigma^2}} \leq e^{-\frac{\mathcal{L}^{-1}(y)^2}{2\sigma^2}}.$$ 

[10.2]
for all $y \in (\log(1-q), 0]$. Thus, we see from (10.1) that for all $y \in (\log(1-q), 0]$:

$$f(L^{-1}(y)) \leq Ce^{-\frac{L^{-1}(y)^2}{2\sigma^2}}. \quad (A.1)$$

Moreover, we see from (10.3) that for all $y \in (\log(1-q), 0]$:  

$$\frac{d}{dy}L^{-1}(y) \leq \frac{\sigma^2}{e^y - (1-q)}. \quad (A.2)$$

Using (A.1) and (A.2), we get the upper bound

$$\omega(y) \leq C_2 e^{-\frac{L^{-1}(y)^2}{2\sigma^2}} e^y - (1-q) \quad (A.3)$$

for all $y \in (\log(1-q), 0]$, where $C_2 = \frac{\sigma}{\sqrt{2\pi}}$. We make the change of variables

$$x = \log \left( \frac{e^y - (1-q)}{q} \right).$$

As $y \to 0$, $x \to 0$, and as $y \to \log(1-q)$, $x \to -\infty$. Also,

$$qe^x = e^y - (1-q). \quad (A.4)$$

Then, the right hand side of (A.3) is given by the function

$$f(x) = C_3 e^{-\frac{(\sigma^2 x + \frac{1}{2})^2}{2\sigma^2}} e^x,$$

where $C_3 = \frac{\sigma}{q\sqrt{2\pi}}$. By elementary calculus, we find that $f(x)$ obtains its maximum at $x = -\frac{3}{2\sigma^2}$. Therefore, when $y \in (\log(1-q), 0]$,

$$\omega(y) \leq \frac{\sigma}{q\sqrt{2\pi}} e^{-\frac{L^{-1}(\bar{y})^2}{2\sigma^2}} e^{\bar{y}} - (1-q) \quad (A.5)$$

for $\bar{y} = \log(q e^{-\frac{3}{2\sigma^2}} + (1-q))$. We see from (10.2) that

$$L^{-1}(\bar{y}) = \sigma^2 \left( -\frac{3}{2\sigma^2} \right) + \frac{1}{2} = -1,$$

and from (A.4) that

$$e^{\bar{y}} - (1-q) = q e^{-\frac{3}{2\sigma^2}}.$$  

Substituting these into (A.5), we find that for all $y \in (\log(1-q), 0]$:

$$\omega(y) \leq \frac{\sigma}{q\sqrt{2\pi}} e^{\frac{1}{2\sigma^2}}. \quad (A.6)$$
Assume next that $y > 0$. Then, we see from (10.3) that

$$
\frac{d}{dy} \mathcal{L}^{-1}(y) = \frac{\sigma^2 e^y}{e^y - (1 - q)} = \frac{\sigma^2}{1 - \frac{1-q}{e^y}} \geq \frac{\sigma^2}{1 - q}.
$$

Moreover, $f(\mathcal{L}^{-1}(y)) \leq \frac{1}{\sqrt{2\pi}\sigma}$ for all $y > 0$. Therefore, when $y > 0$,

$$
\omega(y) \leq \frac{\sigma}{q\sqrt{2\pi}}.
$$

The two cases (A.6) and (A.7) show the claim.\qed

**Lemma 13.** For all $y \geq 2$:

$$
\omega(y) \leq \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(\sigma^2 y + C_2)^2}{2\sigma^2}},
$$

where $C_2 = \sigma^2 \log(\frac{1}{2q}) - \frac{1}{2}$.

**Proof.** Suppose $y > 2$. Then,

$$
e^y - (1 - q) \geq \frac{1}{2} e^y
$$

and therefore

$$
\mathcal{L}^{-1}(y) = \sigma^2 \log \left( \frac{e^y - (1 - q)}{q} \right) + \frac{1}{2} \geq \sigma^2 y + C,
$$

where $C = \sigma^2 \log(\frac{1}{2q}) + \frac{1}{2}$. Also,

$$
f(\mathcal{L}^{-1}(y)) \leq \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(\sigma^2 y + C_2)^2}{2\sigma^2}},
$$

where $C_2 = \sigma^2 \log(\frac{1}{2q}) - \frac{1}{2}$. Furthermore, when $y > 2$,

$$
\frac{d}{dy} \mathcal{L}^{-1}(y) = \frac{\sigma^2 e^y}{e^y - (1 - q)} \leq 2\sigma^2.
$$

Thus, when $y > 2$,

$$
\omega(y) \leq \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{(\sigma^2 y + C_2)^2}{2\sigma^2}},
$$

where $C_2 = \sigma^2 \log(\frac{1}{2q}) - \frac{1}{2}$.\qed