Research Article

Some Results on Preconditioned Mixed-Type Splitting Iterative Method

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We present a preconditioned mixed-type splitting iterative method for solving the linear system

\[ A x = b, \]

where \( A \) is a Z-matrix. And we give some comparison theorems to show that the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method. Finally, we give one numerical example to illustrate our results.

1. Introduction

For solving linear system,

\[ A x = b, \]

where \( A \) is an \( n \times n \) square matrix and \( x \) and \( b \) are \( n \)-dimensional vectors, the basic iterative method is

\[ M x^{k+1} = N x^k + b, \quad k = 0, 1, \ldots, \]

where \( A = M - N \) and \( M \) is nonsingular. Thus, (2) can be written as

\[ x^{k+1} = T x^k + c, \quad k = 0, 1, \ldots, \]

where \( T = M^{-1} N \) and \( c = M^{-1} b \).

Assuming that \( A \) has unit diagonal entries, let \( A = I - L - U \), where \( I \) is the identity matrix and \( -L \) and \( -U \) are strictly lower and strictly upper triangular parts of \( A \), respectively.

Transform the original system (1) into the preconditioned form as follows:

\[ P A x = P b. \]

Then, we can define the basic iterative scheme as follows:

\[ M_p x^{k+1} = N_p x^k + P b, \quad k = 0, 1, \ldots, \]

where \( P A = M_p - N_p \) and \( M_p \) is nonsingular. Thus, the equation above can also be written as

\[ x^{k+1} = T x^k + c, \quad k = 0, 1, \ldots, \]

where \( T = M_p^{-1} N_p \) and \( c = M_p^{-1} P b \).

In paper [1], Cheng et al. presented the mixed-type splitting iterative method as follows:

\[ (D + D_1 + L_1 - L) x^{k+1} = (D_1 + L_1 + U) x^k + b, \quad k = 0, 1, 2, \ldots, \]

with the following iterative matrix:

\[ T = (D + D_1 + L_1 - L)^{-1} (D_1 + L_1 + U), \]

where \( D_1 \) is an auxiliary nonnegative diagonal matrix, \( L_1 \) is an auxiliary strictly lower triangular matrix, and \( 0 \leq L_1 \leq L \).

In this paper, we will establish the preconditioned mixed-type splitting iterative method with the preconditioners \( P_\alpha = I + \alpha L \), \( P_\beta = I + \beta U \), and \( P_{\alpha \beta} = I + \alpha L + \beta U \) for solving linear systems. And we obtain some comparison results which show that the rate of convergence of the preconditioned mixed-type splitting iterative method with \( P_{\alpha \beta} \) is faster than that of the preconditioned mixed-type splitting iterative method with \( P_\alpha \) or \( P_\beta \). Finally, we give one numerical example to illustrate our results.
2. Preconditioned Mixed-Type Splitting Iterative Method

For the linear system (1), we consider its preconditioned form as follows:

\[ P_{a\beta}Ax = P_{a\beta}b, \quad (9) \]

with the preconditioner \( P_{a\beta} = I + \alpha L + \beta U \); that is,

\[ A_{a\beta}x = b_{a\beta}. \quad (10) \]

We apply the mixed-type splitting iterative method to it and have the corresponding preconditioned mixed-type splitting iterative method as follows:

\[ (D_{a\beta} + D_1 + L_1 - L_{a\beta}) x^{k+1} = (D_1 + L_1 + U_{a\beta}) x^k + b_{a\beta}, \quad k = 0, 1, 2, \ldots, \quad (11) \]

that is,

\[ x^{k+1} = (D_{a\beta} + D_1 + L_1 - L_{a\beta})^{-1} (D_1 + L_1 + U_{a\beta}) x^k + b_{a\beta}, \quad k = 0, 1, 2, \ldots, \quad (12) \]

So, the iterative matrix is

\[ \bar{T} = (D_{a\beta} + D_1 + L_1 - L_{a\beta})^{-1} (D_1 + L_1 + U_{a\beta}), \quad (13) \]

where \( D_{a\beta}, -L_{a\beta}, \) and \( -U_{a\beta} \) are the diagonal, strictly lower, and strictly upper triangular matrices obtained from \( A_{a\beta} \). \( D_1 \) is an auxiliary nonnegative diagonal matrix, \( L_1 \) is an auxiliary strictly lower triangular matrix, and \( 0 \leq L_1 \leq L_{a\beta} \).

If we choose \( \beta = 0 \), we have the following corresponding iterative matrix:

\[ \bar{T} = (D_{a} + D_1 + L_1 - L_{a})^{-1} (D_1 + L_1 + U_{a}). \quad (14) \]

And if we choose \( \alpha = 0 \), we have the following corresponding iterative matrix:

\[ \bar{T} = (D_{\beta} + D_1 + L_1 - L_{\beta})^{-1} (D_1 + L_1 + U_{\beta}). \quad (15) \]

If we choose certain auxiliary matrices, we can get the classical iterative methods as follows.

(1) The PSOR method is

\[ D_1 = \frac{1}{r} (1 - r) D_{a\beta}, \quad L_1 = 0, \quad (16) \]

\[ \bar{T}_r = (D_{a\beta} - r L_{a\beta})^{-1} [(1 - r) D_{a\beta} + r U_{a\beta}] \cdot \quad (17) \]

(2) The PAOR method is

\[ D_1 = \frac{1}{w} (1 - w) D_{a\beta}, \quad L_1 = \frac{1}{w} (w - r) L_{a\beta}, \quad (18) \]

\[ \bar{T}_{rw} = (D_{a\beta} - r L_{a\beta})^{-1} [(1 - w) D_{a\beta} + (w - r) L_{a\beta} + w U_{a\beta}] \cdot \quad (19) \]

We need the following definitions and results.

Definition 1 (see [2]). A matrix \( A \) is a \( Z \)-matrix if \( a_{ij} \leq 0 \), for all \( i, j = 1, 2, \ldots, n \), such that \( i \neq j \). A matrix \( A \) is an \( L \)-matrix if \( a_{ij} > 0 \), \( i = 1, 2, \ldots, n \), and \( a_{ij} \leq 0 \), for all \( i, j = 1, 2, \ldots, n \), such that \( i \neq j \).

Definition 2 (see [2]). A matrix \( A \) is an \( M \)-matrix if \( A \) is a nonsingular \( Z \)-matrix, and \( A^{-1} \geq 0 \).

Definition 3 (see [2, 3]). Let \( A, N \in \mathbb{R}^{n \times n} \). Then, \( A = M - N \) is called a regular splitting if \( M^{-1} \geq 0 \) and \( N \geq 0 \); \( A = M - N \) is called an \( M \)-splitting if \( M \) is an \( M \)-matrix, \( N \geq 0 \).

Lemma 4 (see [2]). Let \( A \geq 0 \) be an irreducible matrix. Then,

(1) \( A \) has a positive real eigenvalue equal to its spectral radius;

(2) to \( \rho(A) \), there corresponds an eigenvector \( x > 0 \);

(3) \( \rho(A) \) is a simple eigenvalue of \( A \).

Lemma 5 (see [4]). Let \( A \) be a nonnegative matrix. Then,

(1) if \( ax \leq Ax \) for some nonnegative vector \( x, x \neq 0 \), then \( \alpha \leq \rho(A) \);

(2) if \( Ax \leq \beta x \) for some positive vector \( x \), then \( \rho(A) \leq \beta \). Moreover, if \( A \) is irreducible and if \( 0 \neq ax \leq Ax \leq \beta x \) for some nonnegative vectors \( x \), then

\[ \alpha \leq \rho(A) \leq \beta. \quad (18) \]

Lemma 6 (see [5]). Let \( A = M - N \) be an \( M \)-splitting of \( A \). Then, \( \rho(M^{-1}N) < 1 \) if and only if \( A \) is a nonsingular \( M \)-matrix.

Lemma 7 (see [6, 7]). Let \( A \) be a \( Z \)-matrix. Then, \( A \) is a nonsingular \( M \)-matrix if and only if there is a positive vector \( x \) such that \( Ax \geq 0 \).

Lemma 8 (see [8]). Let \( A = M - N \) be a regular splitting of \( A \). Then, the splitting is convergent if and only if \( A^{-1} \geq 0 \).

Lemma 9 (see [9]). Let \( A \) and \( B \) be two \( n \times n \) nonsingular lower triangular \( L \)-matrices. If \( A \geq B \), then \( B^{-1} \geq A^{-1} \geq 0 \).

3. Convergence Analysis and Comparison Results

Theorem 10. Let \( A \) be a nonsingular \( Z \)-matrix. Assume that \( D_i \geq 0, 0 \leq L_1 \leq L_{\alpha}, \alpha \in [0, 1], \) and \( T \) and \( T \) are the iterative
matrices given by (14) and (8), respectively. Consider the following.

(i) If \( \rho(T) < 1 \), then \( \rho(T) < \rho(T) < 1 \).

(ii) Let \( A \) be irreducible. Assume that \( 1 - \alpha \sum_{j=1}^{n} a_{i_j} a_{j} > 0 \) and \( a_{i} + \sum_{k=i+1}^{n} a_{i} a_{k} \leq 0 \);

then, one has

1. \( \rho(T) \geq \rho(T) \), if \( \rho(T) \geq 1 \),
2. \( \rho(T) \leq \rho(T) \), if \( \rho(T) < 1 \).

Proof. Let

\[
\begin{align*}
M_\alpha &= D_\alpha + D_1 + L_1 - L_\alpha, \\
N_\alpha &= D_1 + L_1 + U_\alpha, \\
M &= I + D_1 + L_1 - L, \\
N &= D_1 + L_1 + U, \\
E_\alpha &= (I + \alpha L) (I + D_1 + L_1 - L), \\
F_\alpha &= (I + \alpha L) (D_1 + L_1 + U).
\end{align*}
\]

Then, we have

\[
A = M - N = (I + D_1 + L_1 - L) - (D_1 + L_1 + U)
\]

is an \( M \)-splitting. Since \( \rho(T) < 1 \), it follows from Lemma 6 that \( A \) is a nonsingular \( M \)-matrix. Then, by Lemma 7, there is a positive vector \( x \) such that \( Ax \geq 0 \), so \( A \alpha x = (I + \alpha L) Ax \geq 0 \).

By Lemma 7, \( A_\alpha \) is also a nonsingular \( M \)-matrix.

Obviously, we can get that \( D_\alpha \) is a positive diagonal matrix. And from \( L_\alpha \) is nonnegative, we know that \( M_\alpha \) being a \( Z \)-matrix. Since \( D_\alpha^{-1} L_\alpha \geq 0 \) is a strictly lower triangular matrix, so that \( \rho(D_\alpha^{-1} L_\alpha) = 0 < 1 \).

So, we have \( (I + D_\alpha^{-1} D_1 + D_\alpha^{-1} L_1 - D_\alpha^{-1} L_\alpha)^{-1} \geq 0 \).

Then, \( M_\alpha^{-1} = (I + D_\alpha^{-1} D_1 + D_\alpha^{-1} L_1 - D_\alpha^{-1} L_\alpha)^{-1} \geq 0 \); hence, \( M_\alpha \) is a nonsingular \( M \)-matrix.

For \( j \neq i+1 \), it is obvious that \( (U_\alpha)_{i,j} = -a_{i+1,j} + \alpha a_{i+1,i} a_{i+1,j} \geq 0 \).

And for \( j = i+1 \), we have \( (U_\alpha)_{i,j} = (\alpha - 1)a_{i+1,j} \geq 0 \). Thus, \( U_\alpha \geq 0 \) and \( N_\alpha \geq 0 \).

We have proven that \( A_\alpha = M_\alpha - N_\alpha \) and \( A = M - N \) are both \( M \)-splittings and \( E_\alpha^{-1} F_\alpha = M^{-1} N \), two splittings \( A_\alpha = M_\alpha - N_\alpha = E_\alpha - F_\alpha \) are nonnegative.

On the other hand, since \( L_\alpha = D_\alpha - I + \alpha \bar{D} + L - \alpha L + \alpha L^2 + \alpha \bar{E} \), we get

\[
E_\alpha - M_\alpha = (I + \alpha L)(I + D_1 + L_1 - L) - (D_\alpha + D_1 + L_1 - L_\alpha) \\
= (I + D_1 + L_1 - L + \alpha L)(I + D_1 + L_1 - L) - (D_\alpha + D_1 + L_1 - L_\alpha) \\
= (I - L + \alpha L + \alpha L D_1 + \alpha L L_1 - \alpha L^2 - D_\alpha + L_\alpha) \\
= (I - L + \alpha L + \alpha L D_1 + \alpha L L_1 - \alpha L^2 - D_\alpha + D_\alpha) \\
= (I - \alpha D + L - \alpha L + \alpha L^2 + \alpha \bar{E}) \\
= \alpha (L D_1 + L L_1 + \bar{D} + \bar{E}) \geq 0,
\]

which implies that

\[
A_\alpha^{-1} E_\alpha - A_\alpha^{-1} M_\alpha = A_\alpha^{-1} (E_\alpha - M_\alpha) \geq 0.
\]

Therefore, \( A_\alpha^{-1} E_\alpha \geq A_\alpha^{-1} M_\alpha \geq 0 \). So, we have \( \rho(M_\alpha^{-1} N_\alpha) \leq \rho(E_\alpha^{-1} F_\alpha) \); that is,

\[
\rho(T) \leq \rho(T) < 1.
\]

(ii) Let \( A = I - L - U \) be irreducible. Since \( L + U \) is a nonnegative and irreducible matrix, and according to the proof of Lemma 4 in paper [9], we can obtain that \( T \) and \( T \) are nonnegative and irreducible matrices. Thus, from Lemma 4, we know that there exists a positive vector \( x = (x_1, x_2, \cdots, x_n)^T \) such that \( Tx = \lambda x \), where we denote \( \lambda = \rho(T) \), which is equivalent to

\[
(D_1 + L_1 + U) x = \lambda (I + D_1 + L_1 - L) x, \\
(U - \alpha L + \alpha L) x = [\lambda (I - L + \alpha L) x, \\
(U - \alpha L + \alpha L) x = [\lambda (I - L + \alpha L) x]
\]

Let \( LL = \bar{D} + \bar{E} + \bar{F} \), where \( \bar{D}, \bar{E}, \) and \( \bar{F} \) are the diagonal, lower triangular, and upper triangular parts of \( LL \), respectively. So,

\[
A_\alpha = D_\alpha - L_\alpha - U_\alpha \\
= (I - \alpha \bar{D}) - (L - \alpha L + \alpha L^2 + \alpha \bar{E}) - (U + \alpha F),
\]

where \( D_\alpha = I - \alpha \bar{D}, \alpha = L - \alpha L + \alpha L^2 + \alpha \bar{E}, U_\alpha = U + \alpha F \).

Now, we consider

\[
\bar{T} x - T x = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\
\times (D_1 + L_1 + U_\alpha) x - \lambda x \\
= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\
\times [(D_1 + L_1 + U_\alpha) - \lambda (D_\alpha + D_1 + L_1 - L_\alpha)] x \\
= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}
\]
\[ \begin{aligned}
&\times \left[ (D_1 + L_1 + U + \alpha F) - \lambda (I - \alpha D + D_1 + L_1 - L) - L + \alpha L - \alpha L^2 - \alpha E) \right] x \\
&= (D_\alpha + D_1 + L_1 - L)\alpha
\times \left[ (D_1 + L_1 + U) - \lambda (I + D_1 + L_1 - L) \right] x \\
&= (D_\alpha + D_1 + L_1 - L)\alpha
\times \left[ aF + \lambda (aD - \alpha L^2 + \alpha E) \right] x \\
&= (D_\alpha + D_1 + L_1 - L)\alpha
\times \left[ aF + \lambda (aD - \alpha L^2 + \alpha E) \right] x \\
&= (D_\alpha + D_1 + L_1 - L)\alpha
\times \left[ (\lambda - 1) \alpha L (D_1 + L_1) + (\lambda - 1) \alpha (D + E) \right] x \\
&= (\lambda - 1) (D_\alpha + D_1 + L_1 - L)\alpha
\times \alpha (LD_1 + LL_1 + D + E) x.
\end{aligned} \] (27)

Since \( D_\alpha + D_1 + L_1 - L \alpha \) is an M-matrix, and \( LD_1 + LL_1 + D + E \geq 0 \), we have the following.

1. If \( \lambda \geq 1 \), then \( T x \geq T x = \lambda x \). By Lemma 5, we get \( \rho (\bar{T}) \geq \rho (T) \).

2. If \( \lambda < 1 \), then \( T x \leq T x = \lambda x \). By Lemma 5, we get \( \rho (\bar{T}) \leq \rho (T) \). \( \square \)

**Theorem 11.** Let \( A \) be a nonsingular Z-matrix. Assume that \( D_1 \geq 0, 0 \leq L \leq L_{\alpha}, \alpha, \beta \in [0, 1] \), and \( \bar{T} \) and \( T \) are the iterative matrices given by (13) and (8), respectively. Consider the following.

1. If \( \rho (T) < 1 \), then
\[ \rho \left( \bar{T} \right) < \rho \left( T \right) < 1. \] (28)

2. Let \( A \) be irreducible. Assume that
\[ 1 - \alpha \sum_{i=1}^{n} a_{ii} a_{jj} - \beta \sum_{k=1}^{n} a_{ik} a_{kj} > 0, \] (29)
\[ a_{ij} + \sum_{k=1}^{n} a_{ik} a_{kj} \leq 0; \]
Then, one has

1. \( \rho (\bar{T}) \geq \rho (T) \) if \( \rho (T) \geq 1 \).

2. \( \rho (\bar{T}) \leq \rho (T) \) if \( \rho (T) < 1 \).

**Proof.** Let
\[ A_{a\beta} = D_{a\beta} + D_1 + L_1 - L_{a\beta}, \]
\[ N_{a\beta} = D_1 + L_1 + U_{a\beta}, \]
\[ M = I + D_1 + L_1 - L, \]
\[ N = D_1 + L_1 + U, \] (30)

(i) By a similar proof of Theorem 10, we can prove that \( A_{a\beta} = M_{a\beta} - N_{a\beta} \) and \( A = M - N \) are both \( M \)-splitting and \( E_{a\beta} - F_{a\beta} = M^{-1} N \), two splittings \( A_{a\beta} = M_{a\beta} - N_{a\beta} = E_{a\beta} - F_{a\beta} \), are nonnegative.

On the other hand, since \( L_{a\beta} = D_{a\beta} - I + \alpha D + \beta D + L - \alpha L + \alpha L^2 + \alpha E + \beta E \), we get
\[ E_{a\beta} - M_{a\beta} = \left( I + \alpha L + \beta U \right) \left( I + D_1 + L_1 - L \right) \]
\[ - \left( D_{a\beta} + D_1 + L_1 - L_{a\beta} \right) \]
\[ = \left( I + D_1 + L_1 - L \right) + \alpha L \left( I + D_1 + L_1 - L \right) \]
\[ + \beta U \left( I + D_1 + L_1 - L \right) \]
\[ - \left( D_{a\beta} + D_1 + L_1 - L_{a\beta} \right) \]
\[ = I - L + \alpha L \left( I + D_1 + L_1 - L \right) \]
\[ + \beta U \left( I + D_1 + L_1 - L \right) - D_{a\beta} + L_{a\beta} \]
\[ = I - L + \alpha L \left( I + D_1 + L_1 - L \right) \]
\[ + \beta U \left( I + D_1 + L_1 - L \right) - D_{a\beta} + L_{a\beta} \]
\[ - I + \alpha D + \beta D + L - \alpha L \]
\[ + \alpha L^2 + \alpha E + \beta E \]
\[ = \beta \left( U - \bar{F} \right) + \beta U \left( D_1 + L_1 \right) \]
\[ + \alpha \left( D + E \right) + \alpha L \left( D_1 + L_1 \right) \geq 0, \] (32)

which implies that
\[ A_{a\beta}^{-1} E_{a\beta} - A_{a\beta}^{-1} M_{a\beta} = A_{a\beta}^{-1} \left( E_{a\beta} - M_{a\beta} \right) \geq 0. \] (33)

Therefore, \( A_{a\beta}^{-1} E_{a\beta} \geq A_{a\beta}^{-1} M_{a\beta} \geq 0 \). So, we have \( \rho (M_{a\beta}^{-1} N_{a\beta}) \leq \rho (E_{a\beta}^{-1} F_{a\beta}) \); that is,
\[ \rho \left( \bar{T} \right) \leq \rho \left( T \right) < 1. \] (34)
(ii) Let $A_{αβ} = P_{αβ}A = (I + αL + βU)A = I - L - U + αL(I - L - U) + βU(I - L - U) = I - αD - βD - (L - αL + αL^2 + αE + βE)
- (U + αF - βU + βF + βU^2),

where $LU = D + E + F, UL = D + E + F$, and $D, E, F, D, E,$ and $F$ are the diagonal, strictly lower, and strictly upper triangular matrices of $LU$ and $UL$, respectively.

And denote $A_{αβ} = D_{αβ} - L_{αβ} - U_{αβ};$ then according to (35), we have

\[
D_{αβ} = I - αD + βD,
L_{α} = (L - αL + αL^2 + αE + βE),
U_{αβ} = U - βU + βU^2 + αF + βF.\]

By (25), we have

\[
̂T x - λ x = \left( (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \right. \times \left( D_1 + L_1 + U_{αβ} \right) \times (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} - λ \left( (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \times \left( [D_1 + L_1 + U + αF - βU + βF + βU^2] - λ \left( I - αD + βD + D_1 + L_1 - L + αL - αL^2 - αE - βE \right) \right) x
= (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \times \left( [D_1 + L_1 + U - αL - (I + D_1 + L_1 - L)] x + [αF - βU + βF + βU^2] - λ \left( -αD - βD + αL - αL^2 - αE - βE \right) \right) x
= (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \times \left( αF - βU + βF + βU^2 - λ \left( αF - βU + βF + βU^2 \right) - λ \left( αF - βU + βF + βU^2 \right) \right) x
= (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \times \left( αF - βU + βF + βU^2 - βU + βU^2 + λ (βD + βE) \right) x
= (λ - 1) \left( (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \times \left[ αL (D_1 + L_1) + α (D + E) \right. \right.
+ βU (D_1 + L_1) + β (U - F)] x
= (λ - 1) \left( (D_{αβ} + D_1 + L_1 - L_{αβ})^{-1} \times \left[ (αL + βU) (D_1 + L_1) \right. \right.
+ α (D + E) + β (U - F)] x.

(37)

If $a_{ij} + \sum_{k=i+1}^{n} a_{ik}a_{kj} ≤ 0$, then by the proof of Theorem 10, we have $U - F ≥ 0$.

Therefore, one has the following.

(1) If $λ ≥ 1$, then $̂T x - λ x ≥ 0$ but not equal to 0. Therefore, $̂T x ≥ λ x$. By Lemma 5, we get $ρ(̂T) ≥ λ = ρ(T)$.

(2) If $λ < 1$, then $̂T x - λ x ≤ 0$ but not equal to 0. Therefore, $̂T x ≤ λ x$. By Lemma 5, we get $ρ(̂T) ≤ λ = ρ(T)$.

**Remark.** If we choose $α = 0$ in Theorem 11, we have a similar result which is showed by the following corollary.

**Corollary 12.** Let $A$ be a nonsingular $Z$-matrix. Assume that $D_1 ≥ 0, 0 ≤ L_1 ≤ L_{αβ}, β ∈ [0, 1]$, and $T$ and $̂T$ are the iterative matrices given by (15) and (8), respectively. Consider the following.

(i) If $ρ(T) < 1$,

\[
ρ(̂T) < ρ(T) < 1. \tag{38}
\]

(ii) Let $A$ be irreducible. Assume that $1 - β \sum_{k=i+1}^{n} a_{ik}a_{kj} > 0, a_{ij} + \sum_{k=i+1}^{n} a_{ik}a_{kj} ≤ 0; \tag{39}

then, one has

(1) $ρ(̂T) ≥ ρ(T)$ if $ρ(T) ≥ 1$,
(2) $ρ(̂T) ≤ ρ(T)$ if $ρ(T) < 1$.

Now, one will provide some results to show the relations among $ρ(̂T), ρ(T),$ and $ρ(̂T)$.

**Theorem 13.** Let $A = (a_{ij}) \in R^{n×n}$ be a nonsingular $Z$-matrix.

Let $̂T$ and $T$ be iterative matrices given by (13) and (14), respectively. Assume that $α, β ∈ [0, 1], D_1 ≥ 0, 0 ≤ L_1 ≤ L_{αβ}$. If $1 - α \sum_{k=i+1}^{n} a_{ik}a_{kj} - β \sum_{k=i+1}^{n} a_{ik}a_{kj} > 0$ and $a_{ij} + \sum_{k=i+1}^{n} a_{ik}a_{kj} ≤ 0$, then

(1) $ρ(̂T) ≥ ρ(T)$ if $ρ(T) ≥ 1$;
(2) $ρ(̂T) ≤ ρ(T)$ if $ρ(T) < 1.$
Proof. Since \( D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta} \) and \( D_{\alpha} + D_1 + L_1 - L_{\alpha} \) are two lower triangular \( L \)-matrices with \( D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta} \leq D_{\alpha} + D_1 + L_1 - L_{\alpha} \), by Lemma 9, we have

\[
(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \geq (D_{\alpha} + D_1 + L_1 - L_{\alpha})^{-1} \geq 0.
\] (40)

By the proof of Theorems 10 and 11, we consider

\[
\bar{T}x - \bar{T}x = \bar{T}x - \lambda x - (\bar{T}x - \lambda x)
\]

\[
= (\lambda - 1) (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1}
\]

\[
\times \left[ (\alpha L + \beta U) (D_1 + L_1) + \alpha \bar{D} + \beta (U - \bar{F}) \right] x
\]

\[
- (D_{\alpha} + D_1 + L_1 - L_{\alpha})^{-1}
\]

\[
\times \left[ \alpha L (D_1 + L_1) + \alpha \bar{D} + \alpha \bar{E} \right] x
\]

\[
\geq (\lambda - 1) (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1}
\]

\[
\times \left[ \beta U (D_1 + L_1) + \beta (U - \bar{F}) \right] x.
\] (41)

In view of the proof of Theorem 11, we have \( \beta U (D_1 + L_1) + \beta (U - \bar{F}) \geq 0 \).

Therefore, one has the following.

(1) If \( \lambda \geq 1 \), the right-hand side of the above inequality is more than zero. By Lemma 8, \( \rho(\bar{T}) \geq \rho(\bar{T}) \).

(2) If \( \lambda < 1 \), the right-hand side of the above inequality is more than zero. By Lemma 8, \( \rho(\bar{T}) \leq \rho(\bar{T}) \). \( \square \)

**Theorem 14.** Let \( A = (a_{ij}) \in \mathbb{R}^{n\times n} \) be a nonsingular \( Z \)-matrix. Let \( \bar{T} \) and \( \bar{T} \) be iterative matrices given by (13) and (15), respectively. Assume that \( \alpha, \beta \in [0, 1] \), \( D_1 \geq 0 \), \( 0 \leq L \leq L_{\alpha\beta} \). If

\[
1 - \alpha \sum_{i=1}^{n-1} a_{ij} - \beta \sum_{k=i+1}^{n} a_{ik} a_{kj} > 0 \quad \text{and} \quad a_{i1} + \sum_{k=i+1}^{n} a_{ik} a_{kj} \leq 0,
\]

then

(1) \( \rho(\bar{T}) \geq \rho(\bar{T}) \) if \( \rho(T) \geq 1 \).

(2) \( \rho(\bar{T}) \leq \rho(\bar{T}) \) if \( \rho(T) < 1 \).

**Proof.** Since \( D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta} \) and \( D_{\beta} + D_1 + L_1 - L_{\beta} \) are two lower triangular \( L \)-matrices with \( D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta} \leq D_{\beta} + D_1 + L_1 - L_{\beta} \), by Lemma 9, we have

\[
(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \geq (D_{\beta} + D_1 + L_1 - L_{\beta})^{-1} \geq 0.
\] (42)

By the proof of Corollary 12 and Theorem 11, we consider

\[
\bar{T}x - \bar{T}x = \bar{T}x - \lambda x - (\bar{T}x - \lambda x)
\]

\[
= (\lambda - 1) (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1}
\]

\[
\times \left[ (\alpha L + \beta U) (D_1 + L_1) + a \bar{D} + \beta (U - \bar{F}) \right] x
\]

\[
- (D_{\alpha} + D_1 + L_1 - L_{\alpha})^{-1}
\]

\[
\times \left[ \beta U (D_1 + L_1) + \beta (U - \bar{F}) \right] x.
\] (43)

Since \( \alpha L (D_1 + L_1) + a \bar{D} + a \bar{E} \geq 0 \), we get the following.

(1) If \( \lambda \geq 1 \), the right-hand side of the above inequality is more than zero. By Lemma 8, \( \rho(\bar{T}) \geq \rho(\bar{T}) \).

(2) If \( \lambda < 1 \), the right-hand side of the above inequality is more than zero. By Lemma 8, \( \rho(\bar{T}) \leq \rho(\bar{T}) \).

**Remark.** The results (theorems and corollaries) in Section 3 are in some sense the generalized Stein-Rosenberg-type theorems like those in the papers [10–13]. The results (theorems and corollaries) in Section 3 are the comparisons of spectral radius of iterative matrices between the mixed-type splitting method and the preconditioned mixed-type splitting method, while the results in the papers [10–13] are the comparisons of spectral radius of iterative matrices between the parallel decomposition-type relaxation method and its special case.

**4. Numerical Example**

Consider the following equation:

\[
-\Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f.
\] (44)

in the unit square \( \Omega \) with Dirichlet boundary conditions.

If we apply the central difference scheme on a uniform grid with \( N \times N \) interior nodes \( (N^2 = n) \) to the discretization of the above equation, we can get a system of linear equations with the coefficient matrix

\[
A = I \otimes P + Q \otimes I,
\] (45)

where \( \otimes \) denotes the Kronecker product,

\[
P = \text{tridiag} \left( \frac{2 + h}{8}, 1, \frac{2 - h}{8} \right),
\]

\[
Q = \text{tridiag} \left( \frac{1 + h}{4}, 1, \frac{1 - h}{4} \right)
\] (46)

are \( N \times N \) tridiagonal matrices, and the step size is \( h = 1/N \).
Table 1

| α(β) | 𝜌(𝑇) | 𝜌(̂𝑇) | 𝜌(̃𝑇) |
|------|------|------|------|
| 0    | 0.563691 | 0.563691 | 0.563691 |
| 0.05 | 0.563691 | 0.554915 | 0.554027 |
| 0.1  | 0.563691 | 0.545958 | 0.544027 |
| 0.15 | 0.563691 | 0.536827 | 0.533673 |
| 0.2  | 0.563691 | 0.527531 | 0.522949 |
| 0.25 | 0.563691 | 0.518081 | 0.511842 |
| 0.3  | 0.563691 | 0.508493 | 0.500331 |
| 0.35 | 0.563691 | 0.498782 | 0.488412 |
| 0.4  | 0.563691 | 0.488967 | 0.476073 |
| 0.45 | 0.563691 | 0.479065 | 0.463309 |
| 0.5  | 0.563691 | 0.469097 | 0.450121 |

We choose \( N = 5 \); then \( A \in \mathbb{R}^{25 \times 25} \).

If we choose

1. \( D_1 = 0.45D_\alpha, L_1 = 0.4L_\alpha, \alpha \in [0, 0.5] \),
2. \( D_1 = 0.45D_\beta, L_1 = 0.4L_\beta, \beta \in [0, 0.5] \),
3. \( D_1 = 0.45D_{\alpha\beta}, L_1 = 0.4L_{\alpha\beta}, \alpha = \beta \in [0, 0.5] \),

then we can obtain the following results by Theorems 10–14.

Table 1 shows that the rate of convergence of the preconditioned mixed-type splitting method is faster than that of the mixed-type splitting method. And it shows that the rate of convergence of the preconditioned mixed-type splitting method with \( P_{\alpha\beta} \) is faster than that of the preconditioned mixed-type splitting method with \( P_\alpha \) or \( P_\beta \).

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