MOTIVIC INTEGRATION AND MILNOR FIBER

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In memory of Masahiro Shiota

Abstract. We put forward in this paper a uniform narrative that weaves together several variants of Hrushovski-Kazhdan style integral, and describe how it can facilitate the understanding of the Denef-Loeser motivic Milnor fiber and closely related objects. Our study focuses on the so-called “nonarchimedean Milnor fiber” that was introduced by Hrushovski and Loeser, and our thesis is that it is a richer embodiment of the underlying philosophy of the Milnor construction. The said narrative is first developed in the more natural complex environment, and is then extended to the real one via descent. In the process of doing so, we are able to provide more illuminating new proofs, free of resolution of singularities, of a few pivotal results in the literature, both complex and real. To begin with, the real motivic zeta function is shown to be rational, which yields the real motivic Milnor fiber; this is an analogue of the Hrushovski-Loeser construction. We also establish, in a much more intuitive manner, a new Thom-Sebastiani formula, which can be specialized to the one given by Guibert, Loeser, and Merle. Finally, applying $T$-convex integration after descent, matching the Euler Characteristics of the topological Milnor fiber and the motivic Milnor fiber becomes a matter of simple computation, which is not only free of resolution of singularities as in the Hrushovski-Loeser proof, but is also free of other sophisticated algebro-geometric machineries.

Contents

1. Introduction ................................................................. 2
2. Hrushovski-Kazhdan style integration .................................. 8
   2.1. Categories of definable sets ..................................... 10
   2.2. A main theorem from the original construction .................. 12
   2.3. Integrating doubly bounded sets .................................. 14
   2.4. Uniform retraction to RES ........................................ 16
3. Motivic Milnor fiber ..................................................... 19
   3.1. Specialization to henselian subfields ............................... 20
   3.2. Grothendieck rings in real and complex geometry ............... 21
   3.3. Piecewise retraction to RES ..................................... 23
   3.4. Zeta function and motivic Milnor fiber ............................ 27
   3.5. Concerning the virtual Poincaré polynomial ....................... 30
4. Thom-Sebastiani formula ................................................ 32
   4.1. Combinatorial data and Galois actions of the torus ................ 32
   4.2. Categories with angular components ............................... 35
   4.3. Commuting with the convolution operators ....................... 39

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1. Introduction

Recent years have seen significant development in applying Hrushovski-Kazhdan’s integration theory to the study of Denef-Loeser’s motivic Milnor fiber and related topics. The main goal of this paper is to articulate a uniform narrative on such interactions, and thereby not only recover several fundamental results regarding motivic Milnor fiber but also subjugate them to the same principles afforded by the new perspective, and hopefully open up new fronts of inquiry in the process. This narrative is summarized in the diagram (1.2) below.

More concretely, we shall reconstruct motivic Milnor fibers as motivic integrals, establish a general type of Thom-Sebastiani formula, and retrieve invariants of the corresponding topological Milnor fibers, all without using resolution of singularities. In fact, there are several variants of the Hrushovski-Kazhdan style integration at play here and their synergy is the driving force of our telling. Among these variants, the central one is of course the original construction as developed in [22]. It works for any algebraically closed valued fields of equal characteristic 0 and is flexible enough to allow arbitrary choice of parameter spaces that satisfy certain mild conditions. Varying the parameter space enables one to study different categories of definable sets that are equipped with suitable Galois actions, which is highly desirable in the applications we are interested in. Such a perspective is first put forward in [23] for the purpose of finding a resolution-free construction of the complex motivic Milnor fiber, among other things (see also [29, 25] for further developments).

To begin with, by an (algebraic) variety over a field $k$, we mean a reduced separated $k$-scheme of finite type. We denote by $\text{Var}_k$ the category of varieties over $k$.

The Grothendieck semiring $K^+\mathcal{C}$ of a category $\mathcal{C}$ is the free semiring generated by the isomorphism classes of $\mathcal{C}$, subject to the usual scissor relation $[A \setminus B] + [B] = [A]$, where $[A]$, $[B]$ denote the isomorphism classes of the objects $A$, $B$ and “$\setminus$” is certain binary operation, usually just set subtraction; additional relation may be imposed, to be determined in context. Sometimes $\mathcal{C}$ is also equipped with a binary operation — for example, cartesian product of sets or (reduced) fiber product of varieties — that induces multiplication in $K^+\mathcal{C}$, in which case $K^+\mathcal{C}$ becomes a commutative semiring. The formal groupification $K\mathcal{C}$ of $K^+\mathcal{C}$ is then a commutative ring. If a group $G$ acts on the objects of $\mathcal{C}$ and the morphisms of $\mathcal{C}$ are $G$-equivariant, that is, they commute with $G$-actions, then the corresponding $G$-equivariant Grothendieck ring is denoted by $K^G\mathcal{C}$. If $G = \lim_n G_n$ is profinite then we shall always impose the condition that a $G$-action factor through some $G_n$-action. The archetype of this kind of Grothendieck rings is $K^\hat{\mu}\text{Var}_k$, where $\hat{\mu}$ is the procylic group of roots of unity (the limit of the inverse system of groups $\mu_n$ of $n$th roots of unity).

In this introduction, for simplicity, we shall just consider a nonconstant polynomial function $f : (C^d, 0) \to (C, 0)$ such that 0 is a singular point, that is, $\nabla f(0) = 0$. For $0 < \eta \ll \delta \ll 1$, the topological type (or even the diffeomorphism type) of the set $F_a = \bar{B}(0, \delta) \cap f^{-1}(a)$, where $\bar{B}(0, \delta)$ is the closed ball of radius $\delta$ centered at 0, is independent of the choice of $\eta$, $\delta$, and $a \in (0, \eta]$. This topological type, referred to as the (closed) Milnor fiber of $f$, is denoted by $F_f$. The open Milnor
fiber, where the open ball $B(0, \delta)$ is used, is also of interest, but more so in the real environment than in the complex one. We will come back to this later.

Let $\mathcal{L}$ be the space of formal arcs on $\mathbb{C}^d$ at 0. So each element in $\mathcal{L}$ is of the form $\gamma(t) = (\gamma_1(t), \ldots, \gamma_d(t))$, where $\gamma_i(t)$ is a complex formal power series and $\gamma_i(0) = 0$. Let $\mathcal{L}_m$ be the space of such arcs modulo $t^{n+1}$ (also referred to as “truncated arcs”). Consider the following locally closed subset of $\mathcal{L}_m$:

$$\mathcal{X}_{f,m} = \{ \gamma(t) \in \mathcal{L}_m \mid f(\gamma(t)) = t^m \mod t^{m+1} \}.$$  

It may be viewed in a natural way as the set of closed points of an algebraic variety over $\mathbb{C}$ and carries a natural $\mu_m$-action. The motivic zeta function attached to $f$ is then the generating series whose coefficients are in effect the “$\hat{\mu}$-equivariant motivic volumes” of the sets of truncated arcs above:

$$Z_f(T) := \sum_{m \geq 1} [\mathcal{X}_{f,m}] [\mathbb{A}]^{-nd} T^m \in \mathbb{K}^a \text{Var}_C[[\mathbb{A}]^{-1}][T].$$

It is shown in [9, 10] that $Z_f(T)$ is rational and the motivic Milnor fiber $\mathcal{J}_f := -\lim_{T \to \infty} Z_f(T)$ is then extracted from this rational expression via a formal process of sending the variable $T$ to infinity (this process is also summarized in [23, § 8.4]). Of course, to justify calling $\mathcal{J}_f$ a “Milnor so-and-so” one needs to show, at the very least, that invariants of the topological Milnor fiber $F_f$ can be recovered from it. This is indeed the case for, say, the Euler characteristic and the Hodge characteristic.

Originally, both the proof that $Z_f(T)$ is rational and the proof that the Euler (or Hodge) characteristics coincide rely on resolution of singularities. More recently, in [23], these results are established by way of a more conceptual construction, namely the Hrushovski-Kazhdan integration. To briefly outline the methodology, we work in the field $\mathbb{C}(t^\infty) = \bigcup_{n \in \mathbb{Z}^+} \mathbb{C}(t^{1/m})$ of complex Puiseux series, also simply denoted by $\tilde{\mathbb{C}}$. This field is the algebraic closure of the field $\mathbb{C}(t)$ of complex Laurent series. A typical element takes the form $x = \sum_{n \in \mathbb{Z}} a_n t^{n/m}$ for some $m \in \mathbb{Z}^+$ such that its support $\text{supp}(x) = \{ n/m \in \mathbb{Q} \mid a_n \neq 0 \}$ is well-ordered, in other words, there is a $q \in \mathbb{Q}$ such that $a_n = 0$ for all $n/m < q$. We think of $\mathbb{k} := \mathbb{C}$ as a subfield of $\tilde{\mathbb{C}}$ via the embedding $\mathbb{a} \mapsto \mathbb{a}t^0$. The map $\mathbb{v} : \tilde{\mathbb{C}}^\times \to \mathbb{Q}$ given by $x \mapsto \text{min supp}(x)$ is indeed a valuation, and its valuation ring $\mathcal{O} := \mathbb{C}(t^\infty)$ consists of those series $x$ with min supp$(x) \geq 0$ and its maximal ideal $\mathcal{M}$ of those series $x$ with min supp$(x) > 0$. Its residue field $\mathbb{k}$ admits a section onto $\mathbb{k}$ and hence is isomorphic to $\mathbb{C}$. It is well-known that $(\tilde{\mathbb{C}}, \mathcal{O})$ is an algebraically closed valued field.

For a series $x = \sum_{n \in \mathbb{Z}} a_n t^{n/m} \in \tilde{\mathbb{C}}$ with val$(x) = p/m$, let $\text{rv}(x) = a_p t^{p/m}$, which is called the leading term of $x$. Then the motivic zeta function attached to $f$ may be expressed as

$$Z_f(T) = \sum_{n \geq 1} H_m(\mathcal{X}_f) T^n,$$

where the coefficients $H_m(\mathcal{X}_f)$ are Hrushovski-Kazhdan integrals of definable sets that take values in $\mathbb{K}^a \text{Var}_C[[\mathbb{A}]^{-1}]$ and the so-called nonarchimedean Milnor fiber of $f$

$$\mathcal{X}_f = \{ x \in \mathcal{M}^d \mid \text{rv}(f(x)) = \text{rv}(t) \}$$

is a definable set over the parameter space (the “ground field”) $\mathbb{S} = \mathbb{C}(t)$. Formulated in this way, the rationality of $Z_f(T)$ essentially follows from certain computation rules of (convergent) geometric series. That the Euler characteristics of $\mathcal{J}_f$ and $F_f$ coincide follows from the fact that we can express both the Euler characteristic of each coefficient of $Z_f(T)$ and the Euler characteristic of $F_f$ in terms of traces of the monodromy action on the cohomological groups of $F_f$, where the
first expression relies on the resolution-free proofs of the A’Campo-Denef-Loeser formula (this is the main point of [23]) and quasi-unipotence of local monodromy (see [23, Remark 8.5.5]).

It is this kind of more conceptual viewpoint — no arbitrary choice of a resolution for computational purposes — we aim to emulate and develop further in this paper. Our discussion will lean toward real geometry, because that is where some of our new results are more pronounced. Here “real geometry” is broadly construed and may mean the study of varieties over \( \mathbb{R} \) or, more significantly, real varieties in the sense of [2] (real points of varieties over \( \mathbb{R} \)), or even semialgebraic (more generally, \( o \)-minimal) geometry. Accordingly, there is the issue of choosing or formulating an appropriate variant of the Hrushovski-Kazhdan integration theory as applied to the categories of definable sets in the real geometry as well, the parameter space for definable sets should not be essentially consists of the definable subsets of \( \text{VF} \), but rather \( \text{RV} \) (so the principal ideal \(( (\text{RV})_0) \) in \( \text{VF} \) ).

The two sorts \( \text{VF}, \text{RV} \) of the first-order language \( \mathcal{L}_{\text{RV}} \) are interpreted, respectively, as \( \hat{\text{C}}, \hat{\text{C}}^\times / 1 + \mathcal{M} \) (or, equivalently, the set of leading terms) and the cross-sort function \( r_v : \text{VF}^\times \to \text{RV} \) as the quotient map (or the leading term map described above). The obvious epimorphism from \( \text{RV} \) onto the value group \( \Gamma \), also referred to as the \( \Gamma \)-sort, with the kernel \( k^\times \), is denoted by \( r_v \). All this is encapsulated in the diagram (2.1) below. There is a slight difference. Since we intend to study real geometry as well, the parameter space for definable sets should not be \( C((t)) \) as in [23] but rather \( R((t)) \), and the Galois group \( \text{Gal}(\hat{C} / R((t))) \) is then identified with the profinite group \( \hat{\delta} := \hat{\mu} \times \text{Gal}(\hat{C} / \hat{R}) \), where \( \hat{R} \) is the field \( R((t)^\infty)) \) of real Puiseux series, that is, the real closure of \( R((t)) \).

The category \( \text{VF}_* \) essentially consists of the definable subsets of \( \text{VF}^n, n \geq 0 \), as objects and the definable bijections between them as morphisms. The category \( \text{RV}[k] \) essentially consists of the definable subsets of \( \text{RV}^k \) as objects and the definable bijections between them as morphisms. The category \( \text{RV}[*] \) is the coproduct of \( \text{RV}[k], k \geq 0 \), and hence is equipped with a gradation by ambient dimensions. One of the main results of [22] is the canonical isomorphism \( f \) in (1.2) between the Grothendieck rings, where \( P \) stands for the element \([r_v(1 + \mathcal{M})] - [r_v(\mathcal{M} \setminus 0)] \) in \( \text{KRV}[1] \) (so the principal ideal \(( (P - 1) \) is not homogenous).

The structure of \( \text{KRV}[*] \) can be significantly elucidated. To wit, it is isomorphic to a tensor product of two other Grothendieck rings \( \text{KRES}[*] \) and \( \text{KGamma}[*] \), where \( \text{RES}[*] \) is the category of
twisted constructible sets in the residue field \(k\) and \(\Gamma[\ast]\) is the category of definable sets in the value group \(\Gamma\) (as an o-minimal group), both are graded by ambient dimensions. The objects of \(\text{RES}[\ast]\) are twisted because the short exact sequence at the bottom of (2.1) does not admit a natural splitting, and \(K \Gamma[\ast]\) is not the Grothendieck ring of o-minimal groups because not all definable bijections are admitted as morphisms. Anyway, we have two retractions from \(K \text{RV}[\ast]\) onto a quotient \(|K| \text{RES}\) of \(K \text{RES}\) (the gradation is forgotten), reflecting the fact that there are two Euler characteristics in the \(\Gamma\)-sort; these are labeled \(E_b, E_g\) in (1.2). The isomorphism \(\Theta\) is constructed as in [23, § 4.3].

The motivic zeta function \(Z_f(T)\) now resides in \(K^\delta \text{Var}_R([A]^{-1})[T]\). However, the coefficients of \(Z_f(T)\) requires a kind of crude volume forms and the integral \(\int\) (or other variants in [22]) is not adequate for the task. Significant modifications are in order. This work has been carried out in [18] in order to correct an oversight in [23], resulting in the canonical isomorphism \(\int^\circ\) in (1.2). The category \(\mu \text{VF}^\circ[\ast]\) consists of the proper invariant objects of \(\text{VF}_\ast\) and the category \(\mu \text{RV}^\Phi[\ast]\) the doubly bounded objects of \(\text{RV}[\ast]\), all equipped with \(\Gamma\)-volume forms; see § 2.3 for the precise definitions. The nonarchimedean Milnor fiber \(\mathcal{X}_f\) of \(f\) is an object of \(\mu \text{VF}^\circ[\ast]\) (with the trivial volume form). Note that \(\mu \text{VF}^\circ[\ast]\) is also graded since, as in classical measure theory, gradation by ambient dimensions is a necessity in the presence of volume forms (a curve has no volume if considered as a subset of a surface). Also, the ideal \((P_\Gamma)\) is homogenous but is no longer principal. We may again express \(K \mu \text{RV}^\Phi[\ast]\) as a tensor product of two other Grothendieck rings \(K \mu \text{RES}[\ast]\) and \(K \mu \Gamma^\Phi[\ast]\). Since the objects of \(\mu \Gamma^\Phi[\ast]\) are doubly bounded, the two Euler characteristics coincide and consequently there is only one retraction onto \(|K| \text{RES}\), which is labeled \(E^\circ\) in (1.2).

The henselian field \(C((t^{1/m}))\), \(m \in \mathbb{N}\), is considered as an \(\mathcal{L}_{RV}\)-substructure of \(\mathcal{C}\) and, as such, its value group \(\Gamma(C((t^{1/m}))\) is identified with \(m^{-1}\mathbb{Z}\). Corresponding to each \(C((t^{1/m}))\) there is a homomorphism \(h_m\) from a subring \(K^\delta \mu \text{RV}^\Phi[\ast]\) of \(K \mu \text{RV}^\Phi[\ast]\) into \(K^\delta \text{Var}_R([A]^{-1})\) that vanishes on \((P_\Gamma)\). The integral \(\int^\circ[\mathcal{X}_f]\) indeed lands in \(K^\delta \mu \text{RV}^\Phi[\ast]\) and the coefficients \(H_m(\mathcal{X}_f)\) in (1.1) are given by \(h_m(\int^\circ[\mathcal{X}_f])\). Then \(\mathcal{S}_f\), that is, \(-\lim_{T \to \infty} Z_f(T)\), is equal to \((\Theta \circ E_b \circ \int^\circ)([\mathcal{X}_f])\) or \((\Theta \circ E_b \circ \int)([\mathcal{X}_f])\) in \(K^\delta \text{Var}_R([A]^{-1})\). Of course the element \((\Theta \circ E_b \circ \int)([\mathcal{X}_f])\) may be attached to \(f\) directly, but to establish its significance, we need to compare it with the zeta function construction.

It is this reason that forces us to work with an integral whose target only involves doubly bounded sets in \(RV\), namely \(\int^\circ\), instead of \(\int\), so as to facilitate the computation of the coefficients of \(Z_f(T)\).

Without the top row, the diagram (1.2) commutes with the dotted arrows too. The element \((\Theta \circ E_b \circ \int)([\mathcal{X}_f])\) may be attached to \(f\) directly as well, but then its significance is unclear, except in the bottom row. We will say more about this below.

Let \(RV\) be the category of real varieties in the sense of [2]. Taking real points and forgetting the \(\hat{\delta}\)-actions, we can specialize \(H_m(\mathcal{X}_f)\) to \(K \text{RVVar}\) and thereby obtain the real motivic Milnor fiber of \(f\) in \(K \text{RVVar}[[A]^{-1}]\). However, we are more interested in a subtler construction that is indigenous to the real algebraic environment.

Since \(f\) is assumed to be defined over \(R\), it may be realized as a real function \((R^d, 0) \longrightarrow (R, 0)\). The open and the closed Milnor fibers are constructed as before, but denoted by \(F^+_f, \bar{F}^+_f\) since, in the absence of monodromy, replacing \((0, \eta)\) with \([-\eta, 0)\) will, in general, result in different topological types \(F^-_f, \bar{F}^-_f\). So the qualifiers “positive” and “negative” should be tagged on in the terminology if we are to look at the whole picture. The difference between \(F^+_f\) and \(\bar{F}^+_f\) is more significant in real geometry.

The sets of real truncated arcs are denoted by \(\mathcal{L}_m(R)\). Replacing \(\mathcal{L}_m\) with \(\mathcal{L}_m(R)\) in \(\mathcal{X}_{f,m}\), we get a real variety \(\mathcal{X}_{f,m}^1\). The complexification \(\mathcal{X}_{f,m}^1 \otimes C\) of \(\mathcal{X}_{f,m}^1\) is a variety over \(C\), which is isomorphic to \(\mathcal{X}_{f,m}\), and carries a natural \(\delta_m\)-action, where \(\delta_m = \mu_m \rtimes \text{Gal}(C/R)\). Consequently,
\(X_{f,m}^1\) inherits a natural \(\mu_2\)-action from \(X_{f,m}^1 \otimes \mathbb{C}\). This is indeed how the homomorphism \(\Xi\) in (1.2) is constructed.

As a subfield, \(\tilde{R}\) inherits from \(\tilde{C}\) a valuation map, a valuation ring, a leading term map, etc. The pair \((\tilde{R}, \mathcal{O}(\tilde{R}))\) forms a henselian valued field. There is a general procedure to specialize the integral \(\tilde{f}\) to sets in any henselian subfield of \(\tilde{C}\), in particular, for those in \(\tilde{R}\) over \(\mathbb{R}(\!(t)\!)\). It only works for constructible sets, that is, quantifier-free definable sets (all definable sets in \(\tilde{C}\) are constructible because \(\tilde{C}\) eliminates quantifiers), since, after all, \(\tilde{R}\) is not an elementary submodel of \(\tilde{C}\). The corresponding homomorphisms between the Grothendieck rings appear in the middle row of arrows in (1.2).

Applying \(\Xi\) termwise to \(Z_f(T)\) brings about a (positive) motivic zeta function \(Z_f^1(T)\), which belongs to \(K^{\mu_2} \text{ RVar}[[\mathbb{A}^{-1}]]\); there is of course a negative one too. The rationality of \(Z_f^1(T)\) and hence the existence of the real motivic Milnor fiber \(\mathcal{S}_f^1\) in \(K^{\mu_2} \text{ RVar}[[\mathbb{A}^{-1}]]\) follows. Let \(X_f^1\) be the \(\tilde{R}\)-trace of \(X_f\). The image of \([X_f]\) in \(K \text{ VF}_{\tilde{R}}\) is \([X_f^1]\) and hence \(\mathcal{S}_f^1\) may indeed be computed purely in the real algebraic environment as \((O_{\tilde{R}} \circ E_{\tilde{R}} \circ \tilde{f})([X_f^1])[\mathbb{R}]\).

The next step is to justify calling \(\mathcal{S}_f^1\) a Milnor fiber by recovering invariants of \(\tilde{F}_f^+\) from \(\mathcal{S}_f^1\). Actually the only known additive invariant of \(\tilde{F}_f^+\) is the topological (or semialgebraic) Euler characteristic \(\chi(\tilde{F}_f^+)^\). It is shown in [5, Theorem 4.4] that \(\chi(\tilde{F}_f^+)\) does agree with \(\chi^{BM}(\mathcal{S}_f^1)\), where \(\chi^{BM}\) is the Borel-Moore Euler characteristic, also labeled as such in (1.2); note that the real motivic Milnor fiber there is the forgetful image of \(\mathcal{S}_f^1\) in \(K \text{ RVar}[[\mathbb{A}^{-1}]]\). Their method relies on a real analogue of the A’Campo-Denef-Loeser formula, which needs resolution of singularities. Unfortunately, with the absence of monodromy in the real case, we cannot follow the method of [23] outlined above to get a resolution-free proof, at least, perhaps, not without further elucidating the effect of the monodromy action on the complexification of the real Milnor fibers as suggested by [27].

Going through a different route, we use the theory of motivic integration for \(T\)-convex valued fields as developed in [34]. This theory is rich in expressive power and hence can handle all the definable objects in the algebraic environment. On the other hand, its expressive power is also its limitation in yielding algebro-geometric information since, in the corresponding categories of definable sets, there are much more morphisms that can cause loss of algebro-geometric data when passing to the Grothendieck rings. Nevertheless, it should retain much of the numerical information.

We work in \(\tilde{R}\), which is now viewed as a real closed field equipped with both a total ordering and a valuation (or more generally a polynomially bounded \(T\)-convex valued field). This structure is expressed in a first-order language \(L_{\text{TRV}}\), which still has two sorts VF and RV. The categories TVF\(_s\), TRV\(_s\), TRES\(_s\), etc., are all defined similarly as before. Again, there are the canonical isomorphism \(\int^T\) in (1.2) between the Grothendieck rings (this is the so-called generalized Euler characteristic of definable sets in \(\tilde{R}\), the tensor expression \(K \text{ TRES}\_s \otimes K \text{ TT}\_s\) of \(K \text{ TRV}\_s\), and the two retractions \(E^T_b\), \(E^T_g\) in (1.2). The definable sets in the residue field are precisely the semialgebraic sets and hence \(K \text{ TRES}\) is canonically isomorphic to \(\mathbb{Z}\); this is labeled \(\chi\) in (1.2) since it is indeed the semialgebraic Euler characteristic. Specializing \(Z_f^1(T)\) downwards in (1.2), we obtain a power series in \(\mathbb{Z}[T]\), which is understood as a topological zeta function attached to \(f\). The definable set \(X_f^1\) may be approximated by a sequence of semialgebraic sets \(\tilde{F}_r\), \(r \in \mathbb{R}^+\), whose semialgebraic homology eventually stabilizes. The Euler characteristic of this stabilized semialgebraic homology is equal to, on the one hand, \(\chi(\tilde{F}_f^+)\) and, on the other hand, \((\chi \circ E^T_b \circ \int^T)([X_f^1])\) and hence \(\chi^{BM}(\mathcal{S}_f^1)\).
The same argument shows that \( \chi(F^+_f) = (\chi \circ \mathbb{E}^T \circ f^T)(([X^f])) \). As a corollary, we get \( \chi([\tilde{F}_{a,r}]) = (-1)^{d+1}\chi([F_{a,r}]) \). This simply comes from an equality at the motivic level (the second and the third rows in (1.2)). However, at that level, it is unclear if there is a geometric interpretation of the image \((\Theta \circ \mathbb{E} \circ f)([X^f])\) of \([X^f] \) in \(K^\delta \mathcal{V} \mathcal{A} \mathcal{R}\).

This approach also works in the complex setting, considering \( \tilde{C} \) as \( \tilde{R}^2 \) and hence \( X^f \) as an object of TVF\textunderscore \textsc{v}. It shows in particular that \( \chi(F^f) \) is equal to the Euler characteristic of \( \mathcal{S}_f \), as in [23, Remark 8.5.5], but without using even quasi-unipotency of local monodromy. Note that, over \( C \), the Euler characteristics of the open and the closed Milnor fibers coincide, so if \( \mathcal{S}_f \) encodes information on both of them, one cannot see it at this level.

We can extend \( \Theta \circ \mathbb{E} \circ f \) further by composing the Hodge-Deligne polynomial map. It is shown in [30, Proposition 3.23] that this actually gives the Hodge-Deligne polynomial of the limit mixed Hodge structure associated with a variety. Extending \( \Theta_{\mathbb{R}} \circ \mathbb{E}_{\mathbb{R}} \circ \int_{\mathbb{R}} \) by composing the virtual Poincaré polynomial map, we get a similar homomorphism into \( \mathbb{Z}[u] \). It would be interesting to investigate if it too encodes information on limit structures. But of course we are ahead of ourselves here because such limit structures are not yet available in the real setting.

Finally, in showcasing the potential of the framework underlying (1.2), we describe another main result of this paper, namely a new (local) Thom-Sebastiani formula in mixed variables, extending the classical results and most of the later generalizations can only handle the case of separate \( f \) and often \( h \) is just a linear form. Our formula, on the other hand, is much more sophisticated.

Let \( g : (C^d,0) \rightarrow (C,0) \) be another nonconstant polynomial function, singular at 0, and \( h(x,y) \) a polynomial of the form \( y^N + \sum_{2 \leq i \leq \ell} x^m_i \). For each \( 1 \leq i \leq \ell \), let \( f^{(i)} = \sum_{2 \leq i \leq \ell} f^{m_i}, \) \( f^{(i)} = \bigoplus_{2 \leq i \leq \ell} f^{m_i} \), and \( \partial^{(i)} \) be the sequence \( (m_2/m_i, m_i/m_3)_{2 \leq i \leq \ell} \); here \( f^{(i)}, f^{(1)} \) both interpreted as the zero function and \( \partial^{(1)} \) as 1. Let \( X^g_f \) denote the restriction of \( g \) to the set \( \{ x \in M^d \mid \text{val}(f(x)) = 1 \} \), similarly for other functions into the affine line. The fibers of \( X^g_f \) over the set \( t+t M \) form precisely the nonarchimedean Milnor fiber \( X^f \), so \( X^g_f \) may be called the nonarchimedean Milnor fiber of \( f \) over \( G_m \). Let \( X^g_{g^N \oplus f^{(i)}} \) denote the restriction of \( g^N \oplus f^{(i)} \) to the set \( \{ x \in M^d \mid \text{val}((g^N \oplus f^{(i)})(x)) = \partial^{(i)} \} \).

The category \( \text{Var}^{g^{(i)}}_C \) consists of \((\partial^{(i)}, n)\)-diagonal varieties over \( G^*_m \), for some \( n \in \mathbb{Z}^+ \), with good \( G_m \)-actions; see § 4.2 for the unexplained terms. Each object of \( \text{Var}^{\delta^{(i)}}_C \) may be thought of as equipped with a \( \tau \)-action that factors through, for some \( n \in \mathbb{Z}^+ \), the canonical epimorphism \( \tau_n : \tilde{\tau} \rightarrow (C^\times)^n \). For \( i = 1 \) we write the Grothendieck ring \( K^{\delta^{(i)}}_C \) Var\(_C \) as \( K^\delta \text{Var}_C \); actually \( \text{Var}_C \)}
is just the category $\text{Var}^G_m$ in [20] and hence is equivalent to the category of varieties over $\mathbb{C}$ with $\hat{\mu}$-actions. There is a $K^1$-$\text{Var}_C$-module homomorphism

$$\Psi_{g(i)} : K^g(i) \text{Var}_C \rightarrow K^1 \text{Var}_C,$$

which is referred to as a convolution operator.

Suppose that $m_2 \ll N \ll m_3 \ll \ldots \ll m_\ell$. Then, for each $1 \leq i \leq \ell$, there is an operator $\Theta_{g(i)} : \mathbb{E}_{b,g(i)}(\int_{g(i)})$ on functions of the form $1 \mathbb{E}_{gN+f(i)}$, which may be roughly understood as $\Theta \circ \mathbb{E}_b \circ \int$ applied fiberwise. Abbreviate $(\Theta_{g(i)} : \mathbb{E}_{b,g(i)}(\int_{g(i)}))$ as $\mathcal{J}_{gN+f(i)}$, we call it the motivic Milnor fiber of $gN \oplus f(i)$ over $G_m$. Then our Thom-Sebastiani formula states that, in $K^1$-$\text{Var}_C$, $\mathcal{J}_{gN+f(i)}$ is equal to

$$\mathcal{J}_{gN}([Z_{f(i)}]) + \mathcal{J}_{f_2} + \sum_{2 \leq i \leq \ell} \mathcal{J}_{f_{i+1}}([Z_{gN+f(i)}]) - \sum_{2 \leq i \leq \ell} \Psi_{g(i)}(\mathcal{J}_{gN+f(i)});$$

here the first and the third terms are the motivic Milnor fibers over $G_m$ but restricted to the indicated zero sets (in [20] a variant of this is called iterated motivic vanishing cycles).

As before, the whole construction can be specialized to the real setting if $f$, $g$ are defined over $\mathbb{R}$, which enables us to recover the Thom-Sebastiani formula obtained in [3], in a more general form.

A novel perspective behind (1.3) is that $\mathcal{J}_{gN+f(i)}$ may be decomposed into terms corresponding to combinatorial data that can be read off of the tropical curve of $h(x, y)$. This actually suggests that our method can handle polynomials more complicated than $h(x, y)$, for instance, those with more variables and even mixed terms. However, the complexity of the combinatorics involved will become quite heavy, perhaps disproportionately so, as it is unclear how the ground gained can shed new light on the geometry and topology of the singularities in question. Thus we have chosen to just present a simple case that is already beyond what is known in the literature.

### 2. Hrushovski-Kazhdan style integration

The first-order language $L_{RV}$ has two sorts VF, RV and a cross-sort map $\text{rv} : \text{VF} \rightarrow \text{RV}$. Let ACVF denote the $L_{RV}$-theory of algebraically closed valued field of equal characteristic 0. We will not repeat the formal definition of the language $L_{RV}$ or the theory ACVF here, and refer the reader to [32, §2] for details. Every valued field $(K, \text{val})$ may be naturally interpreted as an $L_{RV}$-structure and, as such, its structure may be summarized as follows. Let $\mathcal{O}$, $\mathcal{M}$, and $k$ be the corresponding valuation ring, its maximal ideal, and the residue field, respectively. Let $\text{VF} = K$, $\text{RV} = K^\times/(1 + \mathcal{M})$, and $\text{rv} : K^\times \rightarrow \text{RV}$ be the quotient map. For each $a \in K$, the valuation map $\text{val}$ is constant on the set $a + a \mathcal{M}$ and hence there is an induced map $\text{rvv}$ from $\text{RV}$ onto the value group $\Gamma = \text{val}(K)$. The diagram

$$\begin{array}{ccc}
\mathcal{O} & \xrightarrow{\text{quotient}} & \text{VF}^\times \\
\downarrow & & \downarrow \text{val} \\
\mathcal{M} & \xleftarrow{\text{rv}} & \text{RV} \\
\downarrow \text{rvv} & & \downarrow \Gamma \\
k^\times & \xleftarrow{\text{rv}} & \text{RV} \\
\end{array}$$

then commutes, where the bottom sequence is exact.

A cross-section of $\Gamma$ is a group homomorphism $\text{csn} : \Gamma \rightarrow \text{VF}^\times$ such that $\text{val} \circ \text{csn} = \text{id}$. The corresponding reduced cross-section of $\Gamma$ is the function $\overline{\text{csn}} = \text{rv} \circ \text{csn} : \Gamma \rightarrow \text{RV}$. These are usually augmented by $\text{csn}(\infty) = 0$ and $\overline{\text{csn}}(\infty) = \infty$. If such a reduced cross-section exists then it
induces an isomorphism $RV \cong \Gamma \oplus k^\times$. In general this is not guaranteed, in other words, the short exact sequence above may not split.

**Example 2.1.** Let $K$ be the field $\mathcal{C}$ of complex Puiseux series and $\text{val} : \mathcal{C}^\times \to \mathbb{Q}$ the standard valuation. Let $RV = \mathcal{C}^\times / (1 + \mathcal{M})$ and $rv : \mathbb{R}^\times \to RV$ be the quotient map. This turns $\mathcal{C}$ into an $\mathcal{L}_{RV}$-structure, which is indeed an $ACVF$-model.

The leading term of a series in $\mathcal{C}^\times$ is its first term with nonzero coefficient. It is clear that two series $x, y$ have the same leading term if and only if $rv(x) = rv(y)$ and hence $RV$ is isomorphic to the subgroup of $\mathcal{C}^\times$ consisting of all the leading terms. There indeed exists a natural isomorphism given by $a_q t^n \mapsto (q, a_q)$ from this latter group of leading terms to the group $\mathbb{Q} \oplus \mathbb{C}^\times$, through which we may identify $RV$ with $\mathbb{Q} \oplus \mathbb{C}^\times$ (not definably, though).

We may think of $\hat{\mu}$ as the Galois group $\text{Gal}(\mathcal{C}/\mathbb{C}(t))$, since they are canonically isomorphic. For each element $\xi = (\xi_n)_n \in \hat{\mu}$, the assignment $n \mapsto \xi_n t^{1/n}$ indeed induces a reduced cross-section $\overline{csn}_\xi : Q \to RV$, and the map given by $\xi \mapsto \overline{csn}_\xi$ is indeed a bijection between $\hat{\mu}$ and the set $\Omega$ of reduced cross-sections $\overline{csn} : Q \to RV$ with $\overline{csn}(1) = rv(t)$; in other words, $\hat{\mu}$ acts freely and transitively on $\Omega$ via multiplication in the obvious way.

In this section, following the tradition in the model-theoretic literature, we work in a sufficiently saturated model $\mathcal{U}$ of $ACVF$, together with a fixed parameter space $\mathcal{S}$, which is a substructure of $\mathcal{U}$. This is of course a matter of convenience, otherwise one needs to change the model one is working in whenever compactness is applied. We assume that the map $rv$ is surjective in $\mathcal{S}$ and $RV$ to mean the $RV$-sort without the element $\infty$ — that is, $RV = rv(VF)$ — although the difference rarely matters (when it does we will of course indicate the difference).

A pillar of the structure of definable sets in $\mathcal{U}$ is $C$-minimality, meaning that every definable subset of $VF$ is a boolean combination of (definable) valuative discs.

**Notation 2.2.** There is a special element $\infty = rv(0)$ in the $RV$-sort. For simplicity, we shall write $RV$ to mean the $RV$-sort without the element $\infty$, and $RV_\infty$ otherwise — that is, $RV = rv(VF^\times)$ and $RV_\infty = rv(VF)$ — although the difference rarely matters (when it does we will of course provide further clarification). Also write $RV^\infty_\infty = rv(\mathcal{M})$ and $RV^{\infty_\infty} = RV^\infty_\infty \setminus \{\infty\}$.

**Terminology 2.3 (Sets and subsets).** By a definable set in $VF$ we mean a definable subset in $VF$, by which we just mean a subset of $VF^n$ for some $n$; similarly for other (definable) sorts or even structures in place of $VF$ that have been clearly understood in the context, such as $RV_\infty$, $k$, $\mathcal{M}$, or any substructure $\mathcal{M}$ of $\mathcal{U}$. In particular, a definable set without further qualification means a definable set in $\mathcal{U}$, that is, a definable subset of $VF^n \times RV^m_\infty$ for some $n, m \in \mathbb{N}$.

**Notation 2.4 (Coordinate projections).** For each $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, \ldots, n\}$. Let $A$ be a definable set. For $E \subseteq [n]$, we write $pr_E(A)$, or even $A_E$, for the projection of $A$ into the coordinates contained in $E$. It is often more convenient to use simple standard descriptions as subscripts. For example, if $E$ is a singleton $\{i\}$ then we shall always write $E$ as $i$ and $\tilde{E} := [n] \setminus E$ as $\tilde{i}$; similarly, if $E = [i], \{k \mid i \leq k \leq j\}, \{k \mid i < k < j\}, \{\text{all the coordinates in the sort } S\}$, etc., then we may write $pr_{\leq i}, \ pr_{[i,j]}, \ pr_{[i,j]}, \ A_{(i,j)}, \ AS$, etc.; in particular, we shall frequently write $A_{VF}$ and $A_{RV}$ for the projections of $A$ into the $VF$-sort and the $RV$-sort coordinates.

Unless otherwise specified, by writing $a \in A$ we shall mean that $a$ is a finite tuple of elements (or “points”) of $A$, whose length is not always indicated.
We shall write \( \{t\} \times A, \{t\} \cup A, A \setminus \{t\} \), etc., simply as \( t \times A, t \cup A, A \setminus t \), etc., when it is clearly understood that \( t \) is an element and hence must be interpreted as a singleton in these expressions.

For \( a \in A_E \), the fiber \( \{ b : (b, a) \in A \} \subseteq A_E \) over \( a \) is often denoted by \( A_a \). Note that the distinction between the two sets \( A_a \) and \( A_a \times a \) is usually immaterial and hence they may and shall be tacitly identified. In particular, given a function \( f : A \rightarrow B \) and \( b \in B \), the pullback \( f^{-1}(b) \) is sometimes written as \( A_b \) as well. This is a special case since functions are identified with their graphs. This notational scheme is especially useful when the function \( f \) has been clearly understood in the context and hence there is no need to spell it out all the time.

Another pillar of the structure of definable sets in \( U \) is the so-called orthogonality between the \( k \)-sort and the \( \Gamma \)-sort, meaning that every definable subset \( A \) of \( \mathbb{U}^n \) with \( \text{pr}_{\leq k}(A) \) in \( k \) and \( \text{pr}_{> k}(A) \) in \( \Gamma \) is a finite union of products \( A' \times A'' \subseteq k^k \times \Gamma^{n-k} \); in particular, if \( A \) is the graph of a function on \( \text{pr}_{\leq k}(A) \) or \( \text{pr}_{> k}(A) \) then its image is finite.

**Notation 2.5.** Semantically, we shall treat the value group \( \Gamma \) as a definable sort (the \( \Gamma \)-sort) consisting of “imaginary” elements (that is, classes of definable equivalence relations). However, syntactically, any reference to \( \Gamma \) may be eliminated in the usual way and we can still work with (much more cumbersome) \( \mathcal{L}_{\mathbb{R}V} \)-formulas for the same purpose.

We shall write \( \gamma_i^\sharp, \gamma \in \Gamma \), when we want to emphasize that it is the set \( \text{rv}^{-1}(\gamma) \subseteq \mathbb{R}V \) that is being considered. More generally, if \( I \) is a set in \( \Gamma \) then we write \( I^\sharp = \bigcup \{ \gamma_i^\sharp | \gamma_i \in I \} \). Similarly, if \( U \) is a set in \( \mathbb{R}V \) then \( U^\sharp \) stands for \( \bigcup \{ \text{rv}^{-1}(t) | t \in U \} \).

2.1. Categories of definable sets.

**Definition 2.6.** The VF-dimension of a definable set \( A \), denoted by \( \dim_{VF}(A) \), is the largest natural number \( k \) such that, possibly after re-indexing of the VF-coordinates, the projection \( \text{pr}_{\leq k}(A) \) has nonempty interior in the valuation topology for some \( t \in V_{RV} \).

It is a fact that if \( A \subseteq V_{\mathbb{U}} \) is definable then \( \dim_{VF}(A) \) equals the Zariski dimension of the Zariski closure of \( A \).

**Definition 2.7 (VF-categories).** The objects of the category \( VF[k] \) are the definable sets \( A \) of VF-dimension no more than \( k \) such that \( P_{VF} \uparrow A \) is finite-to-one. Any definable bijection between two such objects is a morphism of \( VF[k] \). Set \( VF_* = \bigcup_k VF[k] \).

As soon as one considers adding volume forms to definable sets in VF, the question of ambient dimension arises and, consequently, one has to take “essential bijections” as morphisms.

We will not recall the definition of the Jacobian \( Jcb_{VF} F \) of a morphism \( F \) of \( VF[k] \) here since there will be no use of it except in the following definition; see [32, Definition 9.6] for reference.

**Definition 2.8 (VF-categories with \( \Gamma \)-volume forms).** An object of the category \( \mu VF[k] \) is a definable pair \((A, \omega)\), where \( A \in VF[k] \), \( A_{VF} \subseteq VF^k \), and \( \omega : A \rightarrow \Gamma \) is a function, which is understood as a definable \( \Gamma \)-volume form on \( A \). A morphism between two such objects \((A, \omega)\), \((B, \sigma)\) is a definable essential bijection \( F : A \rightarrow B \), that is, a bijection that is defined outside definable subsets of \( A, B \) of VF-dimension \( < k \), such that, for almost every \( x \in A \),

\[
\omega(x) = \sigma(F(x)) + \text{val}(Jcb_{VF} F(x)).
\]

We also say that such an \( F \) is \( \Gamma \)-measure-preserving.

For example, there are an essential bijection between the sets \( M \) and \( M \setminus 0 \) and hence a morphism between the objects \((M, 0)\) and \((M \setminus 0, 0)\) in \( \mu VF[1] \).


Notation 2.9. In [22], the category $\mu VF[k]$ is denoted by $\mu VF[k]$ to indicate that the volume forms take values in $\Gamma$ as opposed to $RV$. Here the subscript “$\Gamma$” is dropped since we will not consider $RV$-volume forms.

In the definition above and other similar ones below, for the cases $k = 0$, the reader should interpret things such as $VF^0$ and how they interact with other things in a natural way. For instance, $VF^0$ may be treated as the empty tuple, the only definable set of $VF$-dimension $< 0$ is the empty set, and $Jcb_{VF}$ is always 1 on sets that have no $VF$-coordinates. So $(A, \omega) \in \mu VF[0]$ if and only if $A$ is a finite definable subset of $RV_n^\infty$ for some $n$.

Set $\mu VF[\leq k] = \bigoplus_{i \leq k} \mu VF[i]$ and $\mu VF[*] = \bigoplus_k \mu VF[k]$; similarly for the other categories below (with or without volume forms).

Remark 2.10. Let $F : (A, \omega) \to (B, \sigma)$ be a $\mu VF[k]$-morphism. Our intention is that such an $F$ should identify the two objects. However, if $F$ is not defined everywhere in $A$ then evidently it does not admit an inverse. We remedy this by introducing the following obvious congruence relation $\sim$ on $\mu VF[k]$. Let $G : (A, \omega) \to (B, \sigma)$ be another $\mu VF[k]$-morphism. Then $F \sim G$ if $F(a) = G(a)$ for all $a \in A$ outside a definable subset of $VF$-dimension $< k$. The morphisms of the quotient category $\mu VF/k/\sim$ have the form $[F]$, where $F$ is a $\mu VF[k]$-morphism. Clearly every $(\mu VF[k]//\sim)$-morphism is an isomorphism and hence $\mu VF[k]/\sim$ is a groupoid. In fact, all the categories of definable sets we shall work with should be and are groupoids.

It is certainly more convenient to work with representatives than equivalence classes. In the discussion below, this quotient category $\mu VF[k]/\sim$ will almost never be needed except when it comes to forming the Grothendieck semigroup or, by abuse of terminology, when we speak of two objects of $\mu VF[k]$ being isomorphic.

Definition 2.11 (RV-categories). The objects of the category $RV[k]$ are the pairs $(U, f)$ with $U$ a set in $RV_{\infty}$ and $f : U \to RV^k$ a definable finite-to-one function. Given two such objects $(U, f)$, $(V, g)$, any definable bijection $F : U \to V$ is a morphism of $RV[k]$.

The objects of the category $RV_{*}$ is obtained from $RV[*]$ by forgetting the function $f$ in the pair $(U, f)$. Any definable bijection between two such objects is a morphism of $RV_{*}$.

The category $RV_{*}$ will not be needed until § 5.2.2.

Note that the two categories $VF[0]$, $RV[0]$ are equivalent; similarly for other such categories.

Notation 2.12. We emphasize that if $(U, f)$ is an object of $RV[k]$ then $f(U)$ is a subset of $RV^k$ instead of $RV^\infty$ while $\infty$ can occur in any coordinate of $U$. An object of $RV[*]$ of the form $(U, \text{id})$ is often just written as $U$.

More generally, if $f : U \to RV^k_{\infty}$ is a definable finite-to-one function then $(U, f)$ denote the obvious object of $RV[\leq k]$. For example, the inclusion $\{\infty\} \to RV_{\infty}$ gives rise to an object of $RV[0]$, the inclusion $\{(1, \infty)\} \to RV_{\infty}^2$ gives rise to an object of $RV[1]$, and so on. Often $f$ will be a coordinate projection (every object in $RV[*]$ is isomorphic to an object of this form). In that case, $(U, \text{pr}_{\leq k})$ is simply denoted by $U_{\leq k}$ and its class in $K_+ RV[k]$ by $[U]_{\leq k}$, etc.

Definition 2.13 (RES-categories). The category $RES[k]$ is the full subcategory of $RV[k]$ such that $(U, f) \in RES[k]$ if and only if $\text{vr}(U)$ is finite.

Definition 2.14. Let $U \subseteq RV^m \times \Gamma^m$, $V \subseteq RV'^n \times \Gamma'^n$, and $C \subseteq U \times V$. The $\Gamma$-Jacobian of $C$ at $((u, \alpha), (v, \beta)) \in C$, written as $Jcb_{\Gamma} C((u, \alpha), (v, \beta))$, is the element

$$Jcb_{\Gamma} C((u, \alpha), (v, \beta)) = -\Sigma(\text{vr}(u), \alpha) + \Sigma(\text{vr}(v), \beta),$$

where $\Sigma(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \ldots + \gamma_n$. If $C$ is the graph of a function then we just write $C(u, \alpha)$ instead of $C((u, \alpha), (v, \beta))$. 
Every RV[k]-morphism \( F : (U, f) \rightarrow (V, g) \) induces a definable finite-to-finite correspondence \( F^\dagger \subseteq f(U) \times g(V) \). For \( u \in U \), we abbreviate \( \text{Jcb}_F F^\dagger(f(u), g \circ F(u)) \) as \( \text{Jcb}_F F^\dagger(u) \).

**Definition 2.15** (RV- and RES-categories with \( \Gamma \)-volume forms). An object of the category \( \mu_{RV}[k] \) is a definable triple \( (U, f, \omega) \), where \( (U, f) \) is an object of RV[k] and \( \omega : U \rightarrow \Gamma \) is a function, which is understood as a definable \( \Gamma \)-volume form on \( (U, f) \). A morphism between two such objects \( (U, f, \omega) \), \( (V, g, \sigma) \) is an RV[k]-morphism \( F : (U, f) \rightarrow (V, g) \) such that, for every \( u \in U \),

\[
\omega(u) = (\sigma \circ F)(u) + \text{Jcb}_F F^\dagger(f(u)).
\]

The category \( \mu_{RES}[k] \) is the obvious full subcategory of \( \mu_{RV}[k] \).

**Notation 2.16.** Let \( [1] \in K_+ \text{RES}[1] \) be the class of the singleton \( \{1\} \). The class of the singleton \( \{1\} \) in \( K_+ \text{RES}[0] \) is the multiplicative identity of \( K_+ \text{RES}[\ast] \) and hence is simply denoted by \( 1 \). We have \( \text{RV}_{\ast\ast} = [\text{RV}_{\ast\ast}] + 1 \) in \( K_+ \text{RV}[\leq 1] \).

**Definition 2.17.** Let \( U, V \) be sets in RV and \( f : U \rightarrow V \) a function. We say that \( f \) is vrv-contractible if \( (\text{vrv} \circ f)(U_{\gamma}) \) is a singleton for every \( \gamma \in \text{vrv}(U) \), where \( U_{\gamma} = U \cap \gamma^\sharp \). In that case, the induced function \( f_\gamma : \text{vrv}(U) \rightarrow \text{vrv}(V) \) is called the vrv-contraction of \( f \).

**Remark 2.18.** If \( \gamma \in \Gamma \) is definable then it is in the divisible hull \( Q \otimes \Gamma(S) \) of \( \Gamma(S) \), and vice versa. This does not mean, though, that the definable set \( \gamma^\sharp \subseteq \text{RV} \) contains a definable point unless \( \gamma \in \Gamma(S) \).

**Definition 2.19** (\( \Gamma \)-categories). An object of the category \( \Gamma[k] \) is a finite disjoint union of definable subsets of \( \Gamma^k \). A definable bijection between two such objects is a morphism of \( \Gamma[k] \) if and only if it is the vrv-contraction of a definable bijection.

The category \( \Gamma^\text{fin}[k] \) is the full subcategory of \( \Gamma[k] \) such that \( I \in \Gamma^\text{fin}[k] \) if and only if \( I \) is finite.

**Remark 2.20.** By [33, Remark 2.28], if a definable function between two sets in \( \Gamma \) is a vrv-contraction then it is \( Z \)-linear (with constant terms of the form \( \text{vrv}(t) \), where \( t \in \text{RV} \) is definable, that is, \( t \in \text{RV}(S) \)). Moreover, a definable bijection between two objects of \( \Gamma[k] \) is a \( \Gamma[k] \)-morphism if and only if it is definably a piecewise \( \text{GL}_k(Z) \)-transformation. The “if” direction is clear. For the “only if” direction, see [22, Lemma 10.1] or [33, Lemma 2.29].

**Definition 2.21** (\( \Gamma \)-categories with volume forms). An object of the category \( \mu\Gamma[k] \) is a definable pair \( (I, \omega) \), where \( I \in \Gamma[k] \) and \( \omega : I \rightarrow \Gamma \) is a function. A \( \mu\Gamma[k] \)-morphism between two objects \( (I, \omega) \), \( (J, \sigma) \) is a \( \Gamma[k] \)-morphism \( F : I \rightarrow J \) such that, for all \( \alpha \in I \),

\[
\omega(\alpha) = \sigma(F(\alpha)) + \text{Jcb}_F F(\alpha).
\]

The category \( \mu\Gamma^\text{fin}[k] \) is the obvious full subcategory of \( \mu\Gamma[k] \).

**Notation 2.22.** For each object \( (U, f, \omega) \in \mu_{RV}[k] \), write \( \omega_f : U \rightarrow \Gamma \) for the function given by \( u \mapsto \Sigma(\text{vrv} \circ f)(u) + \omega(u) \). Similarly, for \( (I, \sigma) \in \mu\Gamma[k] \), write \( \sigma_I : I \rightarrow \Gamma \) for the function given by \( \gamma \mapsto \Sigma\gamma + \sigma(\gamma) \).

2.2. **A main theorem from the original construction.** There is a natural map \( \Gamma[\ast] \rightarrow \text{RV}[\ast] \) given by \( I \mapsto (I^\sharp, \text{id}) \) (see Notation 2.5). This map induces a commutative diagram in the category of graded semirings:

\[
\begin{array}{ccc}
K_+ \Gamma^\text{fin}[\ast] & \rightarrow & K_+ \text{RES}[\ast] \\
\downarrow & & \downarrow \\
K_+ \Gamma[\ast] & \rightarrow & K_+ \text{RV}[\ast]
\end{array}
\]
where all the arrows are monomorphisms. The map
\[ K_+ \text{RES}[*] \times K_+ \Gamma[*] \longrightarrow K_+ \text{RV}[*] \]
determined by the assignment
\[ (\{U, f\}, [I]) \longmapsto [(U \times I^\sharp, f \times \text{id})] \]
is well-defined and is clearly \( K_+ \Gamma^{\text{fin}}[*]-\text{bilinear.} \) Hence it induces a \( K_+ \Gamma^{\text{fin}}[*]-\text{linear map} \)
\[ \Psi : K_+ \text{RES}[*] \otimes_{K_+ \Gamma^{\text{fin}}[*]} K_+ \Gamma[*] \longrightarrow K_+ \text{RV}[*], \]
which is a homomorphism of graded semirings. By the universal mapping property, groupifying a tensor product in the category of \( K_+ \Gamma^{\text{fin}}[*]-\text{semimodules} \) is, up to isomorphism, the same as taking the corresponding tensor product in the category of \( K \Gamma^{\text{fin}}[*]-\text{modules} \); the groupification of \( \Psi \) is still denoted by \( \Psi \). Similarly, there is a \( K_+ \mu \Gamma^{\text{fin}}[*]-\text{linear map} \)
\[ \mu \Psi : K_+ \mu \text{RES}[*] \otimes_{K_+ \mu \Gamma^{\text{fin}}[*]} K_+ \mu \Gamma[*] \longrightarrow K_+ \mu \text{RV}[*]. \]
We shall abbreviate \( \otimes_{K_+ \Gamma^{\text{fin}}[*]} \), \( \otimes_{K_+ \mu \Gamma^{\text{fin}}[*]} \) as “\( \otimes \)” below when no confusion can arise.

**Proposition 2.23.** \( \Psi \) and \( \mu \Psi \) are both isomorphisms of graded semirings.

See [22, Corollary 10.3, Proposition 10.10(1)] for proof.

Note that [22, Proposition 10.10(2)] does not hold. This oversight has caused some issues for certain constructions in [23] that depend on it. These issues are now resolved in [18], and the modified constructions there are also crucial for this paper, which we shall recall below in due course.

**Notation 2.24.** For simplicity, we often drop the constant \( \Gamma \)-volume form \( 0 \) from the notation. For instance, if \( U \) is an object of \( \text{RV}[*] \) then it may also denote the object \((U, 0) \) of \( \mu \text{RV}[*]. \)

**Notation 2.25.** Recall that \( \text{RV}^{\infty} \) = \( \text{RV}^{\circ} \) + 1 \( \in K_+ \text{RV}[\leq 1] \). Let \( I_{\text{sp}} \) be the (nonhomogenous) semiring congruence relation on \( K_+ \text{RV}[*] \) generated by the pair \(([1], \text{RV}^{\circ}) \). Let
\[ P = [1] - \text{RV}^{\circ} \in K \text{RV}[1]. \]
The corresponding principal ideal of \( K \text{RV}[*] \) is thus generated by the element \( P - 1 \).

Similarly, let \( \mu I_{\text{sp}} \) be the semiring congruence relation on \( K_+ \mu \text{RV}[*] \) generated by the pair \(([1], \text{RV}^{\circ}) \), which is homogenous, and the corresponding principal ideal of \( K \mu \text{RV}[*] \) is generated by the element \( P \).

**Notation 2.26.** For each \( U = (U, f) \in \text{RV}[k] \), let \( U_f \) be the set \( \bigcup \{ f(u)^2 \times u \mid u \in U \} \). Let \( \mathbb{L}_{\leq k} : \text{RV}[\leq k] \longrightarrow \text{VF}[k] \) be the map given by \( U \longmapsto U_f \). Set \( \mathbb{L} = \bigcup_k \mathbb{L}_{\leq k} \).

Let \( \mu \mathbb{L}_k : \mu \text{RV}[k] \longrightarrow \mu \text{VF}[k] \) be the map given by \( (U, \omega) \longmapsto (\mathbb{L}U, \omega) \), where \( \mathbb{L} \omega \) is the obvious function on \( \mathbb{L}U \) induced by \( \omega \). Set \( \mu \mathbb{L} = \bigoplus_k \mu \mathbb{L}_k \).

The map \( \mathbb{L} \) induces a surjective semiring homomorphism \( K_+ \text{RV}[*] \longrightarrow K_+ \text{VF} \) and the map \( \mu \mathbb{L} \) induces a surjective graded semiring homomorphism \( K_+ \mu \text{RV}[*] \longrightarrow K_+ \mu \text{VF}[*] \), see [22, §4, §6] or [32, Corollaries 7.7, 10.6]; we use the same notation for these homomorphisms as well as their groupifications. We have
\[ \mathbb{L}([1]) = [1 + \mathcal{M}] = [\mathcal{M}] \quad \text{and} \quad \mathbb{L}([\text{RV}^{\circ}]) = [\mathcal{M} \smallsetminus 0], \]
and hence \( \mathbb{L}(P - 1) = 0 \). Similarly, \( \mu \mathbb{L}(P) = 0 \) since, as we have seen above, \( \mathcal{M} \) and \( \mathcal{M} \smallsetminus 0 \) are in essential bijection. It so happens that these relations are the only ones needed to describe the kernels of \( \mathbb{L} \) and \( \mu \mathbb{L} \).
Theorem 2.27. For each $k \geq 0$ there is a canonical isomorphism of semigroups

$$\int_{+}: K_+ VF[k] \longrightarrow K_+ RV[\leq k]/I_{\text{sp}}$$

such that $\int_{+}[A] = [U]/I_{\text{sp}}$ if and only if $[A] = [LU]$. Since these isomorphisms are obviously compatible with the inductive systems, passing to the colimit of the groupifications, we obtain a canonical isomorphism of rings

$$\int: KVF_+ \longrightarrow KRV[*]/(P - 1).$$

Similarly, for each $k \geq 0$ there is a canonical isomorphism of semigroups

$$\int_{+}: K_+ \mu VF[k] \longrightarrow K_+ \mu RV[k]/\mu I_{\text{sp}}$$

such that $\int_{+}[A] = [U]/\mu I_{\text{sp}}$ if and only if $[A] = [\mu LU]$. Taking the direct sum of the groupifications yields a canonical isomorphism of graded rings

$$\int: K \mu VF[*] \longrightarrow K \mu RV[*]/(P).$$

This is a combination of two main theorems, Theorems 8.8 and 8.29, of [22]. But it is not enough for our purpose. Another such isomorphism is needed.

2.3. Integrating doubly bounded sets. We say that a set, possibly with $\Gamma$-coordinates, is bounded if, after applying the maps val, rv, id in the VF-, RV-, $\Gamma$-coordinates, respectively, it is contained in a box of the form $[\gamma, \infty]^n$, and doubly bounded if the box is of the form $[-\gamma, \gamma]^n$. We say that an object $(A, \omega) \in \mu VF[k]$ is bounded or doubly bounded if the graph of $\omega$ is so; similarly in the other categories. In particular, an object $(U, f, \omega) \in \mu RV[k]$ is bounded if the graphs of $f$ and $\omega$ are both bounded; actually, by [18, Lemma 3.26], if $U$ is doubly bounded then the images of these functions are necessarily doubly bounded.

We shall only be concerned with doubly bounded sets.

Notation 2.28. The full subcategories of $\mu RV[*]$, $\mu \Gamma[*]$ of doubly bounded objects are denoted by $\mu RV^d[*]$, $\mu \Gamma^d[*]$.

The corresponding restriction of $\mu \Psi$ is indeed an isomorphism:

$$(2.4) \quad K_+ \mu \text{RES}[\ast] \otimes K_+ \mu \Gamma^d[*] \longrightarrow K_+ \mu RV^d[*].$$

Terminology 2.29. A VF-fiber of a set $A$ is a set of the form $A_t$, where $t \in A_{RV}$ (recall Notation 2.4); in particular, a VF-fiber of a function $f: A \rightarrow B$ is a set of the form $f_t$ for some $t \in f_{RV}$ (here $f$ also stands for its own graph), which is indeed (the graph of) a function. We say that $A$ is open if every one of its VF-fibers is, $f$ is continuous if every one of its VF-fibers is, and so on.

Notation 2.30. For each $\gamma \in \Gamma$, denote the set $\{a \in VF \mid \text{val}(a) > \gamma\}$ by $M_\gamma$ and $\text{rv}(M_\gamma \times (0, 1))$ by $RV_\gamma^{\circ}$. Let $\pi_\gamma: VF \rightarrow VF/M_\gamma$ be the natural map. If $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ then $\pi_\gamma$ denotes the product of the maps $\pi_{\gamma_i}$; each coset of $M_\gamma$ in $VF^n$ is called a polydisc of radius $\gamma$.

Definition 2.31. Let $\alpha \in \Gamma^n$ and $\beta \in \Gamma^m$. We say that a function $f: A \rightarrow B$ with $A \subseteq VF^n$ and $B \subseteq VF^m$ is $(\alpha, \beta)$-covariant if it $(\pi_\alpha, \pi_\beta)$-contracts to a function $f_1: \pi_\alpha(A) \rightarrow \pi_\beta(B)$, that is, $\pi_\beta \circ f = f_1 \circ \pi_\alpha$. For simplicity, we shall often suppress parameters and refer to $(\alpha, \beta)$-covariant functions as $(\alpha, -)$-covariant or $(-, \beta)$-covariant or just covariant functions. A set $A \subseteq VF^n$ is $\alpha$-invariant if its characteristic function is $(\alpha, 0)$-covariant, or equivalently, if $A$ is a union of polydiscs of radius $\alpha$. 


More generally, for two sets $A$, $B$ with RV-coordinates, we say that a function $f : A \rightarrow B$ is covariant if every one of its VF-fibers $f_t$ is $(\alpha_t, \beta_t)$-covariant for some $(\alpha_t, \beta_t) \in \Gamma$ that depends on $t$. This is in line with Terminology 2.29. Accordingly, a set is invariant if (every VF-fiber of) its characteristic function is $(-,0)$-covariant.

For example, finite sets $A \subseteq VF$ are not invariant (or rather they are $\infty$-invariant, but $\infty$ is not allowed in the above definition). The maximal ideal $M \setminus M_0$ is $\alpha$-invariant for every $\alpha \in \Gamma^+$, whereas $M_0$ is not $\alpha$-invariant for any $\alpha \in \Gamma^+$, because the radii of its maximal open subdiscs tend to $\infty$ as they approach 0.

**Terminology 2.32.** A definable function $f : A \rightarrow B$ is proper covariant if

- the sets $A_{VF}$, $f(A)_{VF}$ are bounded and the sets $A_{RV}$, $f(A)_{RV}$ are doubly bounded,
- for each VF-fiber $f_t$ of $f$ there is a $t$-definable tuple $(\alpha_t, \beta_t) \in \Gamma$ such that $f_t$ is $(\alpha_t, \beta_t)$-covariant, $\text{dom}(f_t)$ is $\alpha_t$-invariant, and $\text{ran}(f_t)$ is $\beta_t$-invariant.

Accordingly, an invariant set $A$ is proper if its projection into the RV-coordinates is proper covariant, in particular, $A_{VF}$ is bounded and $A_{RV}$ is doubly bounded.

Note that if $U \in RV^\Phi[k]$ then $\mathbb{L}U$ is a doubly bounded proper invariant set.

**Definition 2.33.** Let $A \subseteq VF^n \times RV^m$, $B \subseteq VF^n \times RV^{m'}$ be objects of $VF_*$. We say that a morphism $G : A \rightarrow B$ is relatively unary or more precisely, relatively unary in the $i$th VF-coordinate, where $i \in [n]$, if $(pr_i \circ G)(x) = pr_i(x)$ for all $x \in A$.

Let $(U, f)$, $(V, g)$ be objects of $RV[k]$. We say that a morphism $F : U \rightarrow V$ is relatively unary in the $i$th coordinate, where $i \in [k]$, if $(pr_i \circ g \circ F)(u) = (pr_i \circ f)(u)$ for all $u \in U$.

Since identity functions are relatively unary in any coordinate, if a morphism is piecewise a composition of relatively unary morphisms then it is indeed a composition of relatively unary morphisms.

**Definition 2.34.** The subcategory $\muVF^\circ[k]$ of $\muVF[k]$ consists of the proper invariant objects and the morphisms that are compositions of relatively unary proper covariant morphisms whose inverses are also proper covariant.

**Remark 2.35.** Obviously the composition law holds in $\muVF^\circ[k]$ and hence it is indeed a category. Moreover, every morphism in it is a bijection, as opposed to merely an essential bijection, and is in effect required to admit an inverse. So $\muVF^\circ[k]$ is already a groupoid and there is no need to pass to a quotient category as in Remark 2.10. On the other hand, it does have nontrivial morphisms (see [18, Proposition 6.12, Remark 6.7]).

**Notation 2.36.** For each $\gamma \in \Gamma^+(S)$, let

$$P_\gamma = [RV^{\infty} \setminus RV_\gamma^\circ] + \{(t_\gamma)\} - [1] \in K RV^\Phi[1],$$

where $t_\gamma \in \gamma^\ell$ is any definable point. It also stands for the corresponding element in $K \mu RV^\Phi[1]$ (with the constant volume form 0 on each component). Of course $\{(t_\gamma)\} = [1]$ in $K RV^\Phi[1]$, but $\{(t_\gamma)\} \neq [1]$ in $K \mu RV^\Phi[1]$ unless $\gamma = 0$.

Clearly $P_\gamma$ does not depend on the choice of $t_\gamma \in \gamma^\ell$. The ideal of $K \mu RV^\Phi[*]$ generated by the elements $P_\gamma$ is denoted by $(P_\Gamma)$. The images of $(P_\Gamma)$ are contained in $(P - 1)$, $(P)$ under, respectively, the obvious homomorphisms

$$K \mu RV^\Phi[*] \rightarrow K RV[*], \quad K \mu RV^\Phi[*] \rightarrow K \mu RV[*].$$
By [18, Corollary 6.24], the map $\mu \mathbb{L}$ induces a surjective homomorphism, which is still denoted by $\mu \mathbb{L}$, between the graded Grothendieck rings

$$K \mu RV^\bullet[k] \longrightarrow K \mu VF^\circ[k].$$

By [18, Proposition 7.25], the kernel of $\mu \mathbb{L}$ in $K \mu RV^\bullet[*]$ is indeed $P_I$.

**Theorem 2.37 ([18, Theorem 7.27])**. There is a canonical isomorphism of graded Grothendieck rings:

$$\int^\circ : K \mu VF^\circ[*] \longrightarrow K \mu RV^\bullet[*] / (P_I).$$

### 2.4. Uniform retraction to $RES$. The objects of the category $RES$ are obtained from those of $RES[*]$ by forgetting the function $f$ in the pair $(U, f)$. Any definable bijection between two such objects is a morphism of $RES$. This is a full subcategory of $RV_*$ (see Definition 2.11)

**Notation 2.38.** Denote the half interval $(0, \infty) \subseteq \Gamma$ simply by $H$. We can associate two Euler characteristics $\chi_g, \chi_b$ with the $\Gamma$-sort, which are distinguished by $\chi_g(H) = -1$ and $\chi_b(H) = 0$; they agree on doubly bounded definable sets, though. The two induced ring homomorphisms $K \Gamma[*] \longrightarrow \mathbb{Z}$ will also be denoted by $\chi_g, \chi_b$, or both by $\chi$ when no distinction is needed.

**Notation 2.39.** Let $[\Lambda]$ denote the class of the affine line over the residue field (in any relevant Grothendieck ring). The class of the multiplicative torus $[\Lambda] - 1$ over the residue field is written as $[G_m]$. Note that the multiplicative identity $1$ of $K RES$ is indeed the class $[1]$, but $[1]$ is not the multiplicative identity of the graded ring $K RES[*]$.

Let $!$ be the ideal of $K RES$ generated by the elements $[\gamma^4] - [G_m]$, where $\gamma \in \Gamma$ is definable. The quotient ring $K RES / !$ is written as $!K RES$. The ideal $![*]$ of $K RES[*]$ and the (graded) quotient ring $!K RES[*] = K RES[*] / ![*]$ are constructed in the same way.

The quotient maps from “$K$” to “$!K$” will all be denoted by $\iota$. For simplicity, we will use the same notation for elements when passing from the former to the latter.

**Proposition 2.40.** There are two ring homomorphisms

$$E_g : K RV[*] \longrightarrow !K RES[*][[\Lambda]^{-1}] \quad \text{and} \quad E_b : K RV[*] \longrightarrow !K RES[*][[1]^{-1}].$$

such that

- their ranges are precisely the zeroth graded pieces of their respective codomains,
- $P - 1 \in K RV[1]$, vanishes under both of them,
- for all $x \in K RES[k]$ and all $y \in K \Gamma[l]$,

$$(2.5) \quad E_g(x \otimes y) = \chi_g(y)x[G_m][[\Lambda]^{-1}(k+l)} \quad \text{and} \quad E_b(x \otimes y) = \chi_b(y)x[G_m][[1]^{-1}(k+l)},$$

where $x \otimes y$ stands in for the element $\Psi^{-1}(x \otimes y) \in K RV[*]$.

For the constructions of $E_g$ and $E_b$, see [22, Theorem 10.5] or [18, § 5.1] or [23, § 2.5]. By the second clause above, they induce two homomorphisms on $K RV[*] / (P - 1)$, which we shall just denote by $E_g, E_b$.

**Remark 2.41.** The zeroth graded piece $(!K RES[*][[\Lambda]^{-1}])_0$ of $!K RES[*][[\Lambda]^{-1}]$ is canonically isomorphic to a colimit of the groups $!K RES[n]$, which is actually what appears in the construction of $E_g$. Thus there is an epimorphism from $(!K RES[*][[\Lambda]^{-1}])_0$ to the similar-looking ring $!K RES[[\Lambda]^{-1}]$. It is then routine to check that this epimorphism is also injective, and hence $(!K RES[*][[\Lambda]^{-1}])_0$ is canonically isomorphic to $!K RES[[\Lambda]^{-1}]$. Similarly, $(!K RES[*][[1]^{-1}])_0$ is canonically isomorphic to $!K RES[[1]^{-1}] \cong !K RES$. 
If the codomain of $E_b$ is changed to $!K \text{RES}[[A]^{-1}]$ in the obvious way then it may be compared with $E_g$. They are different since
\[(2.6) \quad E_b([1]) = 1 \quad \text{and} \quad E_g([1]) = E_g([RV^\infty]) + 1 = -[G_m][A]^{-1} + 1 = [A]^{-1}.
\]
We can equalize them by forcing $[A] = 1$ (hence $E_g(x \otimes y) = E_b(x \otimes y) = 0$ if $y \notin K \Gamma[0]$). The (complex version of the) resulting homomorphism is the one constructed in [23, (2.5.7)].

**Proposition 2.42** ([18, Proposition 5.15]). There is a graded ring homomorphism
\[\mu E^\db : K \mu RV^\db[*] \longrightarrow !K \text{RES}[*]\]
under which $P_\Gamma$ vanishes. Moreover, for all $x \in K \mu RES[k]$ and all $y \in K \mu \Gamma^\db[l]$,
\[\mu E^\db(x) = \phi(x) \otimes (\psi^\db \circ \lambda)(y),\]
where $\phi$ is the forgetful homomorphism $K \mu RES[*] \longrightarrow !K \text{RES}[*]$ and $\psi^\db \circ \lambda$ is a canonical homomorphism $K \mu \Gamma^\db[*] \longrightarrow !K \text{RES}[*]$. The composition of $\mu E^\db$ and the forgetful homomorphism $!K \text{RES}[*] \longrightarrow !K \text{RES}$ is denoted by $E^\circ$. By [18, Remark 5.14], the diagram commutes:
\[
\begin{array}{ccc}
K \mu RV^\db[*] & \longrightarrow & K RV[*] \\
\downarrow E^\circ & & \downarrow E_b \\
!K \text{RES} & & !K \text{RES}
\end{array}
\]
which may serve as an alternative and more direct construction of $E^\circ$.

**Remark 2.43.** The homomorphism $E_b$ will be used in the construction of motivic Milnor fiber in § 3, but not $E_g$, because it does not commute with $E^\circ$ (by (2.6), $E_g([1]) = [A]^{-1}$ whereas $E^\circ([1]) = E_b([1]) = 1$.

For the Thom-Sebastiani formula in § 4 to hold, we must also use $E_b$ (otherwise certain terms in the computation would not vanish, see Remark 4.23).

**Remark 2.44.** We shall only be interested in proper invariant objects of $VF_*$. For those objects, the homomorphisms $E_g \circ \int$, $E_b \circ \int$ only differ by a factor in $!K \text{RES}[[A]^{-1}]$. To see this, we first note that the proof of Theorem 2.37 in [18] shows that every proper invariant object $A \in VF_*$ with $\dim_{VF}(A) = n$ is isomorphic to an object of the form $\mathbb{L}U$ with $(U, 0) \in \mu RV^\db[n]$. So, in light of the isomorphism $\mu \Psi$ in (2.4) and the defining conditions of $E_g$, $E_b$ in (2.5), we have
\[(2.8) \quad E_g\left(\int [A]\right) = E_b\left(\int [A]\right)[A]^{-n}.
\]

In § 5, the $T$-convex versions of $E_b \circ \int$, $E_g \circ \int$ yield the Euler characteristics of the closed and the open topological Milnor fibers. Then the equality just described may be specialized to one between these two numerical quantities; see Corollary 5.18.

2.4.1. **With a reduced cross-section.** We can add a reduced cross-section $\text{csn} : \Gamma \longrightarrow RV$ to the language $L_{RV}$, denote by $L_{RV}^\dagger$, the extension, and consider the corresponding integration theory; this has been worked out in [33]. We shall, however, only need a few facts about definable sets in $RV$ in this setting. For the next few paragraphs we assume that $\mathbb{U}$ carries a reduced cross-section and work in an $L_{RV}^\dagger$-expansion $\mathbb{U}^\dagger$ of $\mathbb{U}$. Definability, if unqualified, is interpreted accordingly.
Definition 2.45. Let $A$ be a definable set in $RV$. A $\Gamma$-partition of $A$ is a definable function $p : A \rightarrow \Gamma^i_\infty$ such that, for all $\gamma \in \Gamma^i_\infty$, the set $\text{vrv}(A_\gamma)$ is a singleton and is $\overline{\text{csn}}(\gamma)-\mathcal{L}_{RV}$-definable. If $p$ is a $\Gamma$-partition of $A$ then the $RV^\dagger$-dimension of $p$, denoted by $\dim_{RV^\dagger}(p)$, is the number $\max\{\dim_{RV^\dagger}(A_\gamma) \mid \gamma \in \Gamma^i_\infty\}$.

Lemma 2.46 ([33, Lemma 3.2]). If $p_1$, $p_2$ are $\Gamma$-partitions of $A$ then $\dim_{RV^\dagger}(p_1) = \dim_{RV^\dagger}(p_2)$.

So the $RV^\dagger$-dimension $\dim_{RV^\dagger}(A)$ of a definable set $A$ in $RV$ may be defined as the $RV^\dagger$-dimension of any $\Gamma$-partition of $A$. It may also be shown that there is a definable finite-to-one function $f : A \rightarrow RV^k \times \Gamma^i_\infty$ if and only if there is a definable function $f : A \rightarrow RV^k$ such that all fibers of $f$ are of $RV^\dagger$-dimension 0 if and only if $\dim_{RV^\dagger}(A) \leq k$.

Definition 2.47 (RV^\dagger-categories). The objects of the category $RV^\dagger[k]$ are the pairs $(U, f)$ with $U$ a set in $RV_\infty$ and $f : U \rightarrow RV^k$ a definable function such that $\dim_{RV^\dagger}(U_t) = 0$ for all $t \in RV^k$. For two such objects $(U, f)$ and $(V, g)$, any definable bijection $F : U \rightarrow V$ is a morphism of $RV^\dagger[k]$.

The categories $RES^\dagger[k], RES^\dagger$ are formulated similarly to RES[k], RES.

Definition 2.48. The twistback function $\text{tbk} : RV_\infty \rightarrow k$ is given by $u \mapsto u/\overline{\text{csn}}(\text{vrv}(u))$, where $\infty/\infty = 0$. For any set $U \subseteq RV^\dagger_\infty$ and $\gamma \in \Gamma^\dagger_\infty$, the set $\text{tbk}(U_\gamma) \subseteq k^n$ is called the $\gamma$-twistback of $U$. If $\text{tbk}(U_\gamma) = \text{tbk}(U_{\gamma'})$ for all $\gamma, \gamma' \in \text{vrv}(U)$ then $U$ is called a twistoid, in which case we simply write $\text{tbk}(U)$ for the unique twistback.

Lemma 2.49 ([33, Corollaries 2.23, 3.4]). Every definable set in $\Gamma$ is $\mathcal{L}_{RV}$-definable and every definable set in $RV$ with $\text{vrv}(U)$ finite is $\overline{\text{csn}}(\text{vrv}(U))-\mathcal{L}_{RV}$-definable.

Lemma 2.50 ([33, Lemma 3.3]). Let $U \subseteq RV^n$ be a definable set. Then there is a definable finite partition $(D_i)_i$ of $D = \text{vrv}(U)$ such that each $U_i = U \cap D_i^\sharp$ is a twistoid and the corresponding twistback is $\mathcal{L}_{RV}$-definable.

A definable finite partition $(U_i)_i$ of $U$ is called a twistoid decomposition of $U$ if every $U_i$ is a twistoid. For instance, the partition $(U_i)_i$ of $U$ induced by $(D_i)_i$ above is a $\Gamma$-cohesive twistoid decomposition, which, by Lemma 2.49, is $\mathcal{L}_{RV}$-definable if $U$ is.

Definition 2.51. Suppose that $(U_i)_i$ is an $\mathcal{L}_{RV}$-definable twistoid decomposition of $U$. If, for each $i$, there is an $\mathcal{L}_{RV}$-definable bijection $f_i : U_i \rightarrow V_i \times \Gamma^\sharp_i$, where $V_i \in RES[\ast]$ and $I_i \in \Gamma[\ast]$, such that its $\text{vrv}$-contraction is a bijection and its graph is a twistoid as well, then we say that $(U_i)_i$ is bipolar. Naturally $U$ is called a bipolar twistoid if it admits a trivial bipolar twistoid decomposition.

Obviously a $\Gamma$-cohesive twistoid decomposition of a bipolar twistoid is also bipolar.

Lemma 2.52. Every $\mathcal{L}_{RV}$-definable set $U \subseteq RV^n$ admits a twistoid decomposition that is both bipolar and $\Gamma$-cohesive.

Proof. This follows from [22, Lemmas 3.21, 3.25]. In more detail, the proof of [22, Lemmas 3.21] shows that the definable finite partition $(D_i)_i$ given by Lemma 2.50 can be refined so as to make the following condition hold: each $U_i = U \cap D_i^\sharp$ is of the form $\{t \in D_i^\sharp : N_i t \in W_i\}$, where $N_i$ is an $n \times k$ matrix over $Z$ and $W_i$ is a definable subset of $\alpha_i^\sharp$ for some $\alpha_i \in \Gamma^k$. Next, there exists a matrix $M_i \in \text{GL}_n(Z)$ such that $N_i M_i$ is in lower echelon form (in general $M_i$ is not a product of “standard” column operations since $Z$ is not a field, but it exists over any principal ideal domain). Observe that if $N_i M_i$ does not have zero columns then $D_i$ must be a singleton. At any rate, the set $M_i^{-1} U_i$ must be of the form $\text{pr}_{\leq m}(U_i) \times I_i^\sharp \subseteq RV^n$, where $n - m$ is the number of zero columns in $N_i M_i$ and $\text{vrv}(\text{pr}_{\leq m}(U_i))$ is a singleton. \end{proof}
For each $U \in \RV^\dagger[*]$ and each twistoid decomposition $(U_i)_i$ of $U$, set

$$
\mathcal{E}_b^\dagger(U) = \sum_i \chi_b(\text{vr}(U_i))[\text{tbk}(U_i)] \in K \text{ RES}^\dagger.
$$

**Proposition 2.53** ([33, Propositions 3.21, 3.30]). The assignment (2.9) does not depend on the choice of the twistoid decomposition and is invariant on isomorphism classes. The resulting map $\mathcal{E}_b^\dagger : K \RV^\dagger[*] \to K \text{ RES}^\dagger$ is a ring homomorphism that vanishes on the ideal $(P - 1)$.

We consider $RV[*]$ as a subcategory of $RV^\dagger[*]$ and denote the induced homomorphism between the Grothendieck rings by

$$
\Lambda : K \RV[*] \to K \RV^\dagger[*].
$$

In the current environment, the transition from “$K$” to “$!K$” is superfluous since we already have $[\gamma^\dagger] = [G_m]$ in $K \text{ RES}^\dagger$ for all definable $\gamma \in \Gamma$ (every element in $K_+ \text{ RES}^\dagger$ is represented by a definable set in $k$). So there is a natural homomorphism $!K \text{ RES} \to K \text{ RES}^\dagger$, which is also denoted by $\Lambda$.

If $U$ is a bipolar twistoid then $\mathcal{E}_b([U])$ may be written as $\chi_b(\text{vr}(U))[U_\gamma]$, where $\gamma$ is any definable element in vr$(U)$ and $[U_\gamma]$ is the indicated class in $!K \text{ RES}$ which actually does not depend on $\gamma$, and hence $\mathcal{E}_b([U]) = (\Lambda \circ \mathcal{E}_b)([U])$. It follows from this and Lemma 2.52 (actually Proposition 2.23 suffices) that we have a commutative diagram

$$
\begin{array}{ccc}
K \RV[*] & \xrightarrow{\mathcal{E}_b} & !K \text{ RES} \\
\Lambda \downarrow & & \downarrow \Lambda \\
K \RV^\dagger[*] & \xrightarrow{\mathcal{E}_b^\dagger} & K \text{ RES}^\dagger
\end{array}
$$

One advantage of $\mathcal{E}_b^\dagger$ over $\mathcal{E}_b$ is that it makes computation easier, essentially because there is no need to (explicitly) decompose $K \RV[*]$ into a tensor product as before.

**Remark 2.54.** We may replace $\chi_b$ with $\chi_g$ in (2.9) and thereby obtain a ring homomorphism $\mathcal{E}_g^\dagger$ that also vanishes on the ideal $(P - 1)$. The diagram (2.10) still commutes if $\mathcal{E}_b$, $\mathcal{E}_b^\dagger$ are replaced by $\mathcal{E}_g$, $\mathcal{E}_g^\dagger$.

## 3. Motivic Milnor fiber

We begin with a brief discussion on “descent” to henselian substructures $M$, which is based on [22, § 12]. The main cases of interest are $M = \hat{C}$ (or $M = \hat{R}$, the field of real Puiseux series) and its henselian subfields $M = C((t^{1/m})), m \in \mathbb{Z}^+$. The value groups $\Gamma(\hat{C})$, $\Gamma(\hat{R})$ are identified with $Q$. The value group $\Gamma(C((t^{1/m})))$ is identified with $m^{-1}Z$. But to avoid confusion, we shall not write the residue field $k$ as $C$ (or $R$) — the latter being regarded as a subfield of $\hat{C}$ — even though they are canonically isomorphic.

The parameter space $S$ is going to be $R((t))$. It may seem at first glance that restricting parameters to $R((t))$ is unnecessary since every element in $\hat{R}$ is, after all, definable over $R((t))$. However, generally speaking, elements in $\hat{R}$ are definable over $R((t))$ only in $\hat{R}$, not in $\hat{C}$, in other words, they are not quantifier-free definable over $R((t))$ in $\hat{R}$ (to define them one needs to use the ordering, which is not quantifier-free definable).
3.1. Specialization to henselian subfields. Let $\mathbb{M}$ be a substructure of $\mathbb{U}$ in which the map $\text{rv}$ is surjective. Recall that the substructure $\mathbb{S}$ is a part of the language and hence all other substructures contain it. If $X \subseteq \text{VF}_m \times \text{RV}_m$ is a definable (and hence quantifier-free definable) set then the trace of $X$ in $\mathbb{M}$, denoted by $X(\mathbb{M})$, is the set of $\mathbb{M}$-rational points of $X$, that is,

$$X(\mathbb{M}) = X \cap (\text{VF}(\mathbb{M})^n \times \text{RV}(\mathbb{M})^m).$$

Such a trace is also called a constructible set in $\mathbb{M}$ since it is indeed quantifier-free definable in $\mathbb{M}$. Note that, however, if $f : X \to \Gamma$ is a definable function then the image $f(X(\mathbb{M}))$ is not necessarily a set in $\Gamma(\mathbb{M})$, but rather a set in the divisible hull $Q \otimes \Gamma(\mathbb{M})$ of $\Gamma(\mathbb{M})$. For instance, if $\mathbb{M} = \mathbb{C}((t))$ then $\Gamma(\mathbb{M}) = \mathbb{Z}$ and hence $\gamma \in \Gamma$ is definable if and only if $\gamma \in Q \otimes \Gamma(\mathbb{M}) = Q$. On the other hand, if $X$ is a set in $\Gamma$ and $f$ is a piecewise $GL_k(\mathbb{Z})$-transformation on $X$ then $f(X(\mathbb{M}))$ is of course a set in $\Gamma(\mathbb{M})$; this is the situation in the $\Gamma$-categories.

Remark 3.1. Suppose that $\mathbb{M}$ is definably closed and $\Gamma(\mathbb{M})$ is nontrivial, or equivalently, the valued field $(\text{VF}(\mathbb{M}), \text{O}(\mathbb{M}))$ is henselian (see [22, Example 12.8] or [18, Lemma 3.1]), then $\mathbb{M}$ is functionally closed, that is, for any definable set $X$ and any definable function $f$ on $X$, the image $f(X(\mathbb{M}))$ is a set that is definable in $\mathbb{M}$ (which then, ex post facto, is constructible in $\mathbb{M}$). This is all we need to deduce the results in this section.

The rest of the material in this subsection will be needed in later sections.

Remark 3.2. If $g : X(\mathbb{M}) \to Y(\mathbb{M})$ is a constructible bijection in $\mathbb{M}$ then obviously there may or may not be a definable bijection between $X$ and $Y$, let alone one whose trace equals $g$. But any quantifier-free formula that defines $g$ in $\mathbb{M}$ also yields a definable bijection $f : X' \to Y'$ with $X' \subseteq X$ and $Y' \subseteq Y$ such that the trace $f'(\mathbb{M}) : X'(\mathbb{M}) \to Y'(\mathbb{M})$ indeed equals $g$.

Our goal here is to derive a motivic integral that is associated with $\mathbb{M}$.

Definition 3.3 ($\mathbb{M}$-constructible categories). An object of the category $\text{RV}_m[k]$ is a pair of the form $U(\mathbb{M}) = (U(\mathbb{M}), f \mid U(\mathbb{M}))$, where $U \in \text{RV}_k$. Any constructible function of the form $F(\mathbb{M}) : U(\mathbb{M}) \to V(\mathbb{M})$, where $F : U \to V$ is a $\text{RV}_k$-morphism, is a morphism of $\text{RV}_m[k]$.

The categories $\text{VF}_m = \bigcup_k \text{VF}_m[k]$, $\Gamma_m[k]$, $\text{RES}_m[k]$, etc., are formulated analogously.

We call $\text{K}_+ \text{VF}_m$, etc., $\mathbb{M}$-constructible Grothendieck semirings associated with $\mathbb{M}$.

Since $\mathbb{M}$ is functionally closed, it is routine to verify that the following binary relation is well-defined and is indeed a semiring congruence relation:

$$\mathfrak{S}_m = \{([A], [B]) \in (\text{K}_+ \text{VF}_m)^2 \mid [A(\mathbb{M})] = [B(\mathbb{M})] \text{ in } \text{K}_+ \text{VF}_m\}.$$ 

The semiring congruence relations $\mathfrak{R}_m \subseteq (\text{K}_+ \text{RV}_m)^2$, $\mathfrak{G}_m \subseteq (\text{K}_+ \Gamma_m)^2$ are defined analogously. The restriction of $\mathfrak{R}_m$ to $(\text{K}_+ \text{RES}_m)^2$ and the corresponding ideal of $\text{K}_+ \text{RES}_m$ are both still denoted by $\mathfrak{R}_m$. We have $\text{K}_+ \text{VF}_m \cong \text{K}_+ \text{VF}_m / \mathfrak{S}_m$, etc.

Suppose that $([A], [B]) \in \mathfrak{S}_m$. Then there is a $\text{VF}_m$-morphism $F : A' \to B'$ between subsets of $A$, $B$ that witnesses this (existence is given by Remark 3.2). Let $A'' = A \setminus A'$ and $B'' = B \setminus B'$. By Theorem 2.27, there are $U, V \in \text{RV}_m$ such that $[A''] = [\text{L}U]$ and $[B''] = [\text{L}V]$. By functional closedness, if $U(\mathbb{M}) \neq \emptyset$ then $\text{L}U(\mathbb{M}) \neq \emptyset$. So $U(\mathbb{M}) = \emptyset$ and similarly $V(\mathbb{M}) = \emptyset$. This means that $([U], [V]) \in \mathfrak{R}_m$, in other words, $\int_+ [A''] = \int_+ [B'']$ modulo $\mathfrak{R}_m$. Therefore,

$$\int_+ [A] = \int_+ [A'] + \int_+ [A''] = \mathfrak{R}_m \int_+ [B'] + \int_+ [B''] = \int_+ [B].$$

Conversely, by a similar reasoning, if $([U], [V]) \in \mathfrak{R}_m$ then $\text{L}U(\mathbb{M})$, $\text{L}V(\mathbb{M})$ are isomorphic in $\text{VF}_m$, in other words, $([\text{L}U], [\text{L}V]) \in \mathfrak{S}_m$. Thus $f$ induces an isomorphism $\int_\mathbb{M}$ between $\text{K}_+ \text{VF}_m$ and $\text{K}_+ \text{RV}_m[*] / (P - 1)$.
Let $\mathcal{R}_M \otimes \mathcal{E}_M$ be the semiring congruence relation on $K_+ \text{RES}[\ast] \otimes K_+ \Gamma[\ast]$ generated by $\mathcal{R}_M$ and $\mathcal{E}_M$. By the universal mapping property of tensor product, there is a canonical isomorphism

$$K_+ \text{RES}[*] \otimes K_+ \Gamma[*] / \mathcal{R}_M \otimes \mathcal{E}_M \cong K_+ \text{RES}_M[*] \otimes K_+ \Gamma_M[*].$$

So the assignment (2.2) induces a $K_+ \Gamma_M^\alpha[*]$-linear map

$$\Psi_M : K_+ \text{RES}[*] \otimes K_+ \Gamma[*] / \mathcal{R}_M \otimes \mathcal{E}_M \rightarrow K_+ \text{RV}[*] / \mathcal{R}_M.$$

In light of Remark 3.2 and the surjectivity of $\Psi$, the proof of Proposition 2.23 still goes through and shows that $\Psi_M$ is an isomorphism as well.

Suppose that $\Gamma(M)$ is divisible. By $o$-minimal cell decomposition and induction on dimension, for any $I, J \in \Gamma[*]$, if $I(M) = J(M)$ then $\chi(I) = \chi(J)$, and hence this is so if, more generally, $([I], [J]) \in \mathcal{E}_M$. So $K \Gamma_M[*]$ also admits two ring homomorphisms into $\mathbb{Z}$ and the assignment (2.5) yields two ring homomorphisms $E_{g,M}, E_{b,M}$ from $K\text{RV}_M[*]/(P - 1)$ into $!K\text{RES}_M$.

**Remark 3.4.** Alternatively, if $\Gamma(M)$ is a $\mathbb{Z}$-group then we can replace the Euler characteristics in (2.5) with “formal summation of geometric series” as done in [4], and this should recover the integration theory developed therein.

All this is encapsulated in the commutative diagram

$$
\begin{array}{ccc}
K \text{VF}_* & \xrightarrow{f} & K \text{RV}[*] / (P - 1) \\
\downarrow / & & \downarrow / \\
K \text{VF}_M & \xrightarrow{f_M} & K \text{RV}_M[*] / (P - 1)
\end{array}
$$

(3.1)

where of course $E_b, E_{b,M}$ can be replaced by $E_g, E_{g,M}$.

### 3.2. Grothendieck rings in real and complex geometry.

The complexification of a variety $X$ over $\mathbb{R}$ is denoted by $X \otimes_\mathbb{R} \mathbb{C}$, which is a variety over $\mathbb{C}$ endowed with an antiholomorphic involution coming from the complex conjugation $c$ over $\mathbb{C}$; the Grothendieck ring of the corresponding category is denoted by $K^c \text{Var}_\mathbb{C}$. Conversely, to every quasi-projective variety $Y$ over $\mathbb{C}$ endowed with an antiholomorphic involution there corresponds a unique variety $X$ over $\mathbb{R}$ such that $Y \cong X \otimes_\mathbb{R} \mathbb{C}$. So extension of scalars induces an isomorphism $K \text{Var}_\mathbb{R} \rightarrow K^c \text{Var}_\mathbb{C}$.

Taking the fixed points of the set $X(\mathbb{C})$ of the complex points of a variety $X$ over $\mathbb{R}$ under the complex conjugation gives a real variety in the sense of [2]; this is denoted by $X(\mathbb{R})$. Such sets of real points of varieties over $\mathbb{R}$, considered with their sheaves of regular functions over $\mathbb{R}$, form the category $\text{RVar}$ of real varieties, and taking real points induces a homomorphism $K \text{Var}_\mathbb{R} \rightarrow K \text{RV}_\mathbb{R}$.

We consider also an equivariant version of the Grothendieck ring of complexified varieties over $\mathbb{R}$, taking into account group actions by roots of unity that are compatible with the complex conjugation.

**Notation 3.5.** Recall that the procyclic group $\mu = \lim_n \mu_n$ is canonically isomorphic to the Galois group $\text{Gal}(\mathbb{C}/\mathbb{C}((t)))$. Denote the dihedral group $\text{Gal}(\mathbb{C}/\mathbb{R}) \ltimes \mu_n$ by $\delta_n$, where the $\text{Gal}(\mathbb{C}/\mathbb{R})$-action on $\mu_n$ corresponds to taking the inverse. Set $\delta = \lim_n \delta_n$, which is canonically isomorphic to $\text{Gal}(\mathbb{C}/\mathbb{R}) \ltimes \mu$, where the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\mu$ corresponds again to taking the inverse. It may also be identified with the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}(t))$. 
The conjugation automorphism of $\mathcal{C}$ is also denoted by $c$. A straightforward computation shows that $c\sigma c\sigma = 1$ for any topological generator $\sigma$ of $\mu$ and hence for every element of $\hat{\mu}$. Indeed, for any integer $m \geq 0$, 
\[ c\sigma^{m+1}c\sigma^{m+1} = c\sigma^m c\sigma c\sigma c\sigma = c\sigma^m c\sigma^m, \]
and hence, by a routine induction, $c\sigma^m c\sigma^m = 1$.

**Definition 3.6.** A $\delta$-action $\hat{h}$ on a complexified variety $X$ over $\mathbb{R}$ is *good* if it factors through some $\delta_n$-action and the induced $\text{Gal}(\mathbb{C}/\mathbb{R})$-action is the canonical antiholomorphic involution.

The category of complexified varieties over $\mathbb{R}$ with good $\delta$-actions consists of objects of the form $X = (X, \hat{h})$, where $X$ is a complexified quasi-projective variety over $\mathbb{R}$ and $\hat{h}$ is a good $\delta$-action on $X$, and $\hat{\delta}$-equivariant morphisms between such objects.

The Grothendieck ring of this category is denoted by $K^{b,\delta}\text{Var}_\mathbb{R}$. The ring $K^{\delta}\text{Var}_\mathbb{R}$ is the quotient of $K^{b,\delta}\text{Var}_\mathbb{R}$ by the ideal generated by the elements of the form
\[(3.2) \quad [X \times (A^n_\mathbb{C}, \hat{h})] - [X \times (A^n_\mathbb{C}, c)], \]
where $\hat{h}$ is a good linear $\delta$-action.

**Remark 3.7.** Let $X = (X, \hat{h})$ be a complexified variety over $\mathbb{R}$ with a good $\hat{h}$-action. If $\sigma$ is a topological generator of $\mu$ then $c\sigma$ is another antiholomorphic involution on $X$ and hence gives rise to another complexified variety $X' = (X', \hat{h})$ over $\mathbb{R}$ with a good $\delta$-action such that $[X] = [X']$. So $\sigma$ induces a ring involution on $K^{b,\delta}\text{Var}_\mathbb{R}$ and also $K^{\delta}\text{Var}_\mathbb{R}$.

An arc $\text{Spec} \mathbb{C}[t] \rightarrow X$ on a variety $X$ over $\mathbb{C}$ may have branches, which are represented by Puiseux series in $\mathcal{C}$. Galois actions over $\mathbb{C}[t]$ on these branches encode certain information on the singularity in question and hence are an integral part of the construction in [23]. These Galois actions are gone when we restrict to real branches of real arcs, corresponding to the pair $\hat{\mathbb{R}}$ and $\mathbb{R}[t]$, albeit a faint trace remains.

**Remark 3.8.** Recall the discussion on reduced cross-sections in §2.4.1. We have seen in Example 2.1 above that there is a natural bijection between $\hat{\mu}$ and the set of reduced cross-sections $\text{csn} : Q \rightarrow \text{RV}$ with $\text{csn}(1) = \text{rv}(t)$ in $\mathcal{C}$. Similarly, there is such a bijection between $\hat{\delta}$ and such a set but with $\text{csn}(1) = \text{rv}(\pm it)$.

In contrast, there is only one such reduced cross-section in $\hat{\mathbb{R}}$, which is but another way of saying the fact that $\text{Gal}(\hat{\mathbb{R}}/\mathbb{R}(t))$ is trivial. Nevertheless, if $n$ is even then $\text{Gal}(\mathbb{R}((t^{1/n}))/\mathbb{R}(t)) \cong \mu_2$, and there are two such reduced cross-sections in $\mathbb{R}((t^{1/n}))$, determined by the two choices $\pm t^{1/n}$, and if $n$ is odd then there is only one.

**Definition 3.9.** The category of real varieties with $\mu_2$-actions consists of objects of the form $X = (X, h)$, where $X$ is a real variety and $h$ is a $\mu_2$-action on $X$, and $\mu_2$-equivariant morphisms between such objects.

The Grothendieck ring of this category is denoted by $K^{b,\mu_2}\text{RVar}$. The ring $K^{\mu_2}\text{RVar}$ is the quotient of $K^{b,\mu_2}\text{RVar}$ by the ideal generated by the elements of the form
\[(3.3) \quad [X \times (A^n_\mathbb{R}, h)] - [X \times (A^n_\mathbb{R}, \text{id})], \]
where $h$ is a linear $\mu_2$-action.

Let $[X] = [(X, h)] \in K^{\delta}\text{Var}_\mathbb{R}$. The $h$-orbit $O_x$ of any (closed) point $x \in X$ is finite because $h$ factors through some $\delta_n$-action. Since $\mu_n$ is cyclic, the induced $\hat{\mu}$-action on $O_x$ factors through a faithful $\mu_{dx}$-action with $dx|n$. So $h$ factors through a $\delta_{dx}$-action $h_{dx}$. Let $x \in X(\mathbb{R})$. For all
\[\sigma \in \mu_2, \text{ since } \mathbf{c}_x \sigma \mathbf{c}_x = 1, \text{ we have } h_{d_x}(\sigma) c h_{d_x}(\sigma)(x) = x. \] In particular, if \(d_x\) is even and \(\sigma \in \mu_2 \subset \delta_{d_x}\) then \(c h_{d_x}(\sigma)(x) = h_{d_x}(\sigma)(x)\).

**Definition 3.10.** The \(\mu_2\)-action \(\hat{h}(R)\) on \(X(R)\) is given by \(x \mapsto h_{d_x}(\sigma)(x)\) for \(\sigma \in \mu_2\) if \(d_x\) is even and \(x \mapsto x\) otherwise.

Let \([Y] \in K^{\delta}_\var{R}_C^R\). It is clear that if \(X, Y\) are isomorphic objects then \([X(R)] = [Y(R)]\) in \(K^{\mu_2}_\var{R}_C^R\). Since (3.2), (3.3) are essentially the same condition and the assignment \([X] \mapsto [X(R)]\) does respect addition, we have indeed constructed a group homomorphism

\[
\Xi : K^{\delta}_\var{R}_C^R \longrightarrow K^{\mu_2}_\var{R}_C^R.
\]

However, \(\Xi\) fails to respect product and hence is not a ring homomorphism: if \(x, y\) are two points belonging to \(X, Y\) then \(d_{(x,y)} = \gcd(d_x, d_y)\) and hence the \(\mu_2\)-action on \((x, y)\) as a point belonging to \(X \times Y\) is not necessarily the product of the \(\mu_2\)-actions on \(x, y\). On the other hand, in light of (3.2) and (3.3), it can be upgraded to an \(A_C\)-module homomorphism via the obvious ring homomorphism \(A_C \longrightarrow A_R\), where \(A_C\) is the subring of \(K^{\delta}_\var{R}_C^R\) generated by \([((A_C, C)]\) and \(A_R\) is the subring of \(K^{\mu_2}_\var{R}_C^R\) generated by \([A_R]\).

A similar construction at the level of \(K^{\delta,\beta}_\var{R}_C^R\) instead of \(K^{\delta}_\var{R}_C^R\) yields a group homomorphism \(\Xi^\beta : K^{\delta,\beta}_\var{R}_C^R \longrightarrow K^{\beta,\mu_2}_\var{R}_C^R\).

\[
\begin{array}{cccc}
K^{\beta,\mu_2}_\var{R}_C^R & \longrightarrow & K^{\mu_2}_\var{R}_C^R & \longrightarrow & K\var{R}_C^R \\
\Xi & \uparrow & \Xi & \uparrow \\
K^{\beta,\delta}_\var{R}_C^R & \longrightarrow & K^{\delta}_\var{R}_C^R & \longrightarrow & K\var{R}_C^R \\
\downarrow & & \downarrow & & \downarrow \\
K^{\delta,\beta}_\var{R}_C^R & \longrightarrow & K^{\beta}_\var{R}_C^R & \longrightarrow & K\var{R}_C^R 
\end{array}
\]

In [23], similar Grothendieck rings \(K^{\delta,\beta}_\var{C}_C^R, K^{\beta}_\var{C}_C^R\) are defined for categories of varieties over \(C\). We summarize the situation in the diagram (3.5), where the horizontal arrows are the obvious quotient maps, the first row of vertical arrows are obtained by taking real points, and the second row of vertical arrows are obtained by forgetting the antiholomorphic (that is, the real) structure. This diagram does commute except for the upper left square: the two routes from \(K^{\delta,\beta}_\var{R}_C^R\) to \(K^{\mu_2}_\var{R}_C^R\) are actually not identical (see Remark 3.21), and the construction below uses the one that passes through \(\Xi\).

**Remark 3.11** (Polynomial realizations). There is a unique ring homomorphism \(\beta : K\var{R}_C^R \longrightarrow \mathbb{Z}[u]\) that coincides with the Poincaré polynomial \(\sum_{i \in \mathbb{N}} \dim H_i(X, F_2) u^i\) for compact nonsingular real varieties \(X\); see [28]. Similarly, there is a unique group homomorphism

\[
\beta^{\mu_2} : K^{\mu_2}_\var{R}_C^R \longrightarrow \mathbb{Z}[u^{-1}][u]
\]

that coincides with the equivariant Poincaré series \(\sum_{i \in \mathbb{Z}} \dim H_i(X, F_2) u^i\) for compact nonsingular real varieties \(X\) endowed with \(\mu_2\)-actions; see [15].

3.3. **Piecewise retraction to RES.** For the rest of this section, fix a reduced cross-section \(\overline{\text{RES}}\) with \(\overline{\text{RES}}(1) = \text{rv}(t)\) in \(\hat{C}\) and denote the corresponding \(L^t_{\text{RV}}\)-expansion of \(\hat{C}\) by \(\hat{C}^t\), etc. We shall work in \(\hat{C}\) with \(S = \mathbb{R}(t)\), or in \(\hat{C}^t\) if \(\overline{\text{RES}}\) is used explicitly in defining objects. In the latter case, the
parameter space is in effect the definable closure \(\mathcal{S}^\dagger\) of \(R(\langle t \rangle)\) in \(\tilde{C}^\dagger\), and we have \(\text{VF}(\mathcal{S}^\dagger) = R(\langle t \rangle)\) but \(\text{RV}(\mathcal{S}^\dagger) \cong \text{rv}(\tilde{R})\). If \(\tilde{S}\) is in \(\tilde{R}\) then indeed \(\text{RV}(\mathcal{S}^\dagger) = \text{rv}(\tilde{R})\). Thus, in \(\tilde{C}^\dagger\), RV has no symmetries left other than the involution given by the complex conjugation. This also follows from Remark 3.8.

The presence of \(\tilde{S}\) induces an intrinsic isomorphism \(\text{RV} \cong k^\times \oplus \mathbb{Q}\). Consequently, \(K \text{ RES}^\dagger \cong K \text{ Var}_{R}\). As in [23, § 4.3], using the twistback function, we construct a homomorphism

\[
(3.6) \quad \Theta : !K \text{ RES} \longrightarrow K^\delta \text{ Var}_R
\]
such that \(\Phi \circ \Theta = \Lambda\) (the forgetful homomorphism \(\Phi\) is marked in (3.5)).

**Remark 3.12.** Several variants of \(\Theta\) will appear below. To show that they are injective, we may simply follow the argument in the proof of [23, Proposition 4.3.1]. For surjectivity, however, some modification is needed, and how much of it is needed varies.

For \(\Theta\) it is quite simple. Let \([(X, \tilde{h})] \in K^\delta \text{ Var}_R\) with \(\tilde{h}\) factoring through a \(\delta_n\)-action. We may assume that \(X\) is quasi-projective and irreducible. Considering the induced \(\mu_n\)-action on \(X\), we see that the quotient variety \(X/\mu_n\), which is also quasi-projective, carries an antiholomorphic involution and hence is defined over \(\mathbb{R}\). Then the Kummer-theoretic construction in the proof of [23, Proposition 4.3.1] yields a \(U \in \text{ RES}\) with \(\Theta([U]) = [(X, \tilde{h})]\).

The situation in \(\tilde{R}\) is somewhat trickier. Let \(U \in \text{ RES}_{\tilde{R}}\). For each \(u \in U\), let \(d_u\) be the least positive integer such that \(u\) is a tuple in \(\text{RV}(R(\langle t^{1/d_u} \rangle))\), or equivalently, \(\text{rv}(u) \in d_u^{-1}\mathbb{Z}\). As implied by Remark 3.8, there is a nontrivial \(\mu_2\)-action on a two-element set \(\{u, u'\} \subseteq U\) if \(d_u\) is even. Thus, similar to Definition 3.10, we can construct a \(\mu_2\)-action on \(U\) by \(u \mapsto u'\) if \(d_u\) is even and \(u \mapsto u\) otherwise. If \(\tilde{\mathcal{S}}\) is in \(\tilde{R}\) then the twistback function yields an isomorphism

\[
(3.7) \quad \Theta_{\tilde{R}} : !K \text{ RES}_{\tilde{R}} \longrightarrow K^\mu_2 \text{ RVar};
\]
it is surjective because, for \(\mu_2\)-actions, we can apply Kummer theory directly over \(\mathbb{R}\). There is also the commutative diagram

\[
(3.8) \quad \begin{array}{ccc}
!K \text{ RES} & \xrightarrow{\Theta} & K^\delta \text{ Var}_R \\
\Xi_{\tilde{R}} & \bigg\downarrow & \bigg\downarrow \Xi \\
!K \text{ RES}_{\tilde{R}} & \xrightarrow{\Theta_{\tilde{R}}} & K^\mu_2 \text{ RVar}
\end{array}
\]

where \(\Xi_{\tilde{R}}\) is obtained by taking traces in \(\tilde{R}\), similar to the construction of \(\Xi\) in (3.4) (so it is just an \(\mathcal{A}_C\)-module homomorphism). Of course, if we forget the \(\mu_2\)-actions then \(\Xi_{\tilde{R}}\) is indeed the ring homomorphism \(-/\mathfrak{R}_{\tilde{R}}\) with \(\mathfrak{M} = \tilde{R}\) in (3.1), which we shall denote by \(\Xi_{\tilde{R}}\).

**Notation 3.13.** We have pointed out above that if \(U\) is a bipolar twistoid then \(\mathcal{E}_b([U])\) may be written as \(\chi_b(\text{rv}(U))[[U_\gamma]]\), where \(\gamma \in \text{rv}(U)\) is definable and \([U_\gamma] \in !K \text{ RES}\) does not depend on \(\gamma\). Thus, in that case, we may denote the element \(\Theta([U_\gamma]) \in K^\delta \text{ Var}_R\) by \([\text{tbk}(U)]^\delta\).

**Notation 3.14.** For any ring \(R\), let \(R[T^Q]\) denote the ring of Puiseux polynomials over \(R\), that is, the group ring of \(Q\) over \(R\).

Let \(K^\delta \text{ Var}_R[[A]^{-1}][T, T^{-1}]\) denote the obvious subring of \(K^\delta \text{ Var}_R[[A]^{-1}][T^Q]\). The canonical image of \(K^\delta \text{ Var}_R[T^Q]\) in \(K^\delta \text{ Var}_R[[A]^{-1}][T^Q]\) is still denoted as such. The assignment \(T \mapsto [A]\) determines a ring homomorphism

\[
(3.9) \quad \eta : K^\delta \text{ Var}_R[[A]^{-1}][T, T^{-1}] \longrightarrow K^\delta \text{ Var}_R[[A]^{-1}].
\]
Recall Notation 2.22. Let $U = (U, f, \omega) \in \mu RV^d[k]$. The image $\omega_f(U_\gamma)$ is finite for every $\gamma \in vrv(U)$, because the $k$-sort and the $\Gamma$-sort are orthogonal to each other. It follows that there is a bipolar twistoid decomposition $(U_i)_{i}$ of $U$ such that every restriction $\omega_f \upharpoonright U_i$ vrv-contracts to a function $\sigma_i : I_i = vrv(U_i) \rightarrow \Gamma$. Write $I_{i,m}$ for the set $I_i(m^{-1}Z)$ of $m^{-1}Z$-rational points of $I_i$. We assign to $U$ the expression
\begin{equation}
    h_m(U) = \sum_{i} [tbk(U_i)]^\delta \sum_{\gamma \in I_{i,m}} T^{-m\sigma_i(\gamma)},
\end{equation}
which is a finite sum and hence belongs to $K^\delta \Var_R[[A]^{-1}][T^Q]$.

**Lemma 3.15.** The assignment (3.10) does not depend on the choice of the bipolar twistoid decomposition and is invariant on isomorphism classes.

**Proof.** Let $V = (V, g, \pi) \in \mu RV^d[k]$ and $f : U \rightarrow V$ be a $\mu RV^d[k]$-morphism. Let $D = (U_i)_{i}$, $E = (V_i)_{i}$ be bipolar twistoid decompositions of $U$, $V$ satisfying the condition above. We need to show that $h_m(U)$, which depends on $D$, and $h_m(V)$, which depends on $E$, are equal. This is clear if $U = V$, $E$ is trivial (so $U$ is already a bipolar twistoid and $\omega_f$ already vrv-contracts to a function on $vrv(U)$), and $D$ is $\Gamma$-cohesive or $vrv(U_i) = vrv(U_j)$ for all $i, j$. The case that $U = V$ and $D$ is a refinement of $E$ follows easily from this since there is a refinement $(U_{ij})_{ij}$ of $D$ such that $(U_{ij})_{ij}$ is a $\Gamma$-cohesive twistoid decomposition of $U_i$ and $vrv(U_{ij}) = vrv(U_{ij})$ for all $i, j$. If $\omega_f$, $\pi_g$ both vrv-contract to a function and $f$ vrv-contracts to a bijection then, by Lemma 2.52, we may assume that $U$, $V$ are already bipolar twistoids. In that case the desired equality follows because $C((t^{1/m}))$ is functionally closed (it is henselian, see Remark 3.1).

For the general case, we first remark that the image $vrv(f(U_\gamma))$ is finite for every $\gamma \in vrv(U)$. It then follows from Lemmas 2.50 and 2.52 that there is an $L_{RV}$-definable twistoid decomposition $(f_i)_{i}$ of $f$ such that every $f_i$ is a $\mu RV^d[k]$-morphism as in the last special case considered above and, moreover, is compatible with $D$, $E$ in the obvious sense, in other words, the domains and ranges of these $f_i$ induce bipolar refinements of $D$, $E$. The result follows. \qed

This means that $h_m$ may be viewed as a map on $K^\delta \mu RV^d[k]$. It is routine to check that we have in effect constructed a ring homomorphism
\[ h_m : K^\delta \mu RV^d[*] \rightarrow K^\delta \Var_R[[A]^{-1}][T^Q]. \]

Let $K_m^\delta \mu RV^d[*]$ denote the subring $h_m^{-1}(K^\delta \Var_R[[A]^{-1}][T, T^{-1}])$ of $K^\delta \mu RV^d[*]$.

**Lemma 3.16.** The homomorphism
\[ \eta \circ h_m : K_m^\delta \mu RV^d[*] \rightarrow K^\delta \Var_R[[A]^{-1}] \]
vanishes on $(P_\Gamma)$.

Note that the ideal $(P_\Gamma)$ of $K^\delta \mu RV^d[*]$ in Notation 2.36 is now generated by the elements $P_\gamma$ with $\gamma \in Z$ (because the point $t_\gamma$ there needs to be definable, which is possible only if $\gamma \in Z$ in the current setting).

**Proof.** For $\gamma \in Z$, the image of $[RV_\infty \setminus RV_\infty]_\gamma + \{t_\gamma\}$ under $h_m$ in $K^\delta \Var_R[[A]^{-1}][T, T^{-1}]$ is
\[ ([A] - 1) \sum_{i=1}^{m\gamma} T^{-i} + T^{-m\gamma}, \]
which, after passing to $K^\delta \Var_R[[A]^{-1}]$ via $\eta$, becomes $1 = \eta(h_m([1]))$. \qed
Remark 3.17. If $U = (U, f, l) \in \mu \mathbb{R} V^\Phi[*]$ with $l \in m^{-1} \mathbb{Z}$ (constant volume form) then the exponents in (3.10) are all integers and hence $[U] \in K^z_m \mu \mathbb{R} V^\Phi[*]$. Actually we shall only need the case $l = 0$.

The ring $\bigcap_{m \in \mathbb{Z}^+} K^z_m \mu \mathbb{R} V^\Phi[*]$ is denoted by $K^z \mu \mathbb{R} V^\Phi[*]$.

If $A = (A, l) \in \mu \mathbb{V} F^\circ[*]$ with $l$ constant then $\int^o A$ may be expressed as $[(U, f, l)]/(P_\Gamma)$, and hence if $l \in \mathbb{Z}$ then $\int^o A$ belongs to $K^z \mu \mathbb{R} V^\Phi[*]/(P_\Gamma)$. In that case, by Lemma 3.16, for every $m \in \mathbb{Z}^+$, the expression $(\eta \circ h_m \circ \int^o)([A])$ designates a unique element in $K^z \mathbb{V} \mathcal{F}^{\mathbb{R}}[[A]]^{-1}$.

Denote by RES$_m$ the full subcategory of RES such that $U \in$ RES$_m$ if and only if every $\gamma \in \mathbb{V} v v(U)$ is a tuple in $m^{-1} \mathbb{Z}$, or equivalently, $U \in$ RES$_m$ if and only if the action on $U$ of the kernel of the canonical projection $\hat{\mu} \longrightarrow \mu_m$ is trivial.

Let $\beta = (\beta_1, \ldots, \beta_n) \in (m^{-1} \mathbb{Z})^n$ and $A \subseteq O^n \times \mathbb{R} V^l$ be a proper $\beta$-invariant definable set such that $pr_{\mathbb{V} F} \upharpoonright A$ is finite-to-one. Then there is a set $A[m; \beta] \subseteq \prod_t C[t^{1/m}]/t^{(\beta_i+1)/m} \times \mathbb{R} V^l$

such that, for every $t \in \mathbb{R} V^l$, the VF-fiber $A(\mathbb{C}(t^{1/m}))/t$ is the pullback of the VF-fiber $A[m; \beta]_t$. Note that $pr_{\mathbb{V} F} \upharpoonright A[m; \beta]$ is still finite-to-one, for otherwise, by the $\beta$-invariance of $A$, $pr_{\mathbb{V} F} \upharpoonright A$ would fail to be finite-to-one. We may view $A[m; \beta]$ as a finite disjoint union of objects of RES$_m$ (see [23, § 4.2] for detail).

Lemma 3.18 ([23, Lemma 4.2.1]). Let $\beta' \in (m^{-1} \mathbb{Z})^n$ with $\beta_i \leq \beta'_i$ for all $i$. Then

\[(A[m; \beta'] = A[m; \beta][A]^{m \Sigma(\beta' - \beta)} \in K \text{RES}_m).

Thus the element $A[m; \beta][A]^{-m \Sigma \beta}$ in $K \text{RES}_m[[A]]^{-1}$ does not depend on $\beta$; we denote it by $\tilde{A}[m]$.

Lemma 3.19. $(\eta \circ h_m \circ \int^o)([(A, 0)]) = (\Theta \circ \iota)(\tilde{A}[m])$ in $K^z \mathbb{V} \mathcal{F}^{\mathbb{R}}[[A]]^{-1}$.

To show this lemma, the statement of Theorem 2.37 itself is not quite enough. We need the fact that there exists a special $\mu \mathbb{V} F^\circ[*]$-morphism $F : (A, 0) \longrightarrow (\mathbb{L} U, 0)$, called a proper special covariant bijection, with $U \in \mathbb{R} V^\Phi[*]$. Now both the definition of a proper special covariant bijection and the proof that such a morphism exists are quite involved. It is better that we do not repeat them here and instead refer the reader to [18, Definition 6.6, Proposition 6.12, Lemma 6.14] for a complete discussion. We only note that $F$ may be chosen with high enough “aperture” (a technical notion defined in [18]) so that $\mathbb{L} U$ is also $\beta$-invariant and $[A[m; \beta]] = [\mathbb{L} U[m; \beta]]$ in $K \text{RES}_m$ (actually $F$ induces a bijection between the two sets such that a point in $A[m; \beta]$ and its image in $\mathbb{L} U[m; \beta]$ only differ in the RV-coordinates).

Proof. We may assume that $A$ is already of the form $\mathbb{L} U$ for some $U \in \mathbb{R} V^\Phi[*]$. Let $D$ be a $\Gamma$-cohesive twistoid decomposition of $U$. Since both sides respect finite disjoint union of proper $\beta$-invariant definable sets, we may further assume that $D$ is actually trivial or even $\mathbb{V} v v(U)$ is a singleton. Then the equality follows from a simple computation. \qed

Recall the diagrams (3.5) and (3.8). Applying the ring homomorphism $\Xi \circ \Phi$, localized at $[A]$, to both sides of the equality in Lemma 3.19, we see that it also holds in $K \mathcal{R} \mathcal{V} \mathcal{F}^{\mathbb{R}}[[A]]^{-1}$; similarly in $K^{\mu_2} \mathcal{R} \mathcal{V} \mathcal{F}^{\mathbb{R}}[[A]]^{-1}$ if the $A_\mathbb{C}$-module homomorphism $\Xi$, localized at $((A_\mathbb{C}, c))$, is used. It is also possible to use $\overline{\Theta} \circ \Xi \circ \iota$ on the right-hand side instead and thereby obtain the same result in $K \mathcal{R} \mathcal{V} \mathcal{F}^{\mathbb{R}}[[A]]^{-1}$, where $\overline{\Theta}$ is obtained from $\Theta$ by forgetting the $\mu_2$-actions.
We can now proceed to replicate the construction in [23, § 8] (see also [18]) so to recover the motivic zeta function with coefficients in $K\hat{\delta}\Var_{\mathbb{R}}[[\mathbb{A}]]^{-1}$ (this is the purpose of Lemma 3.19) and the corresponding motivic Milnor fiber. The subsequent specialization to $K^{\mu_2}R\Var[[\mathbb{A}]]^{-1}$ is new and points to deeper phenomena in the real algebraic environment.

### 3.4. Zeta function and motivic Milnor fiber

Let $X$ be a nonsingular connected variety of dimension $d$ over $\mathbb{R}$ and $f$ a nonconstant morphism, also over $\mathbb{R}$, from $X$ to the affine line. Let $z \in f^{-1}(0)$ be an $\mathbb{R}$-rational point. Since $X$, $f$, and $z$ are fixed, we shall not always carry them in notation and terminology.

**Notation 3.20.** Let $\pi$ be the reduction map $X(\mathcal{O}) \longrightarrow X(\mathbb{C})$. The (complex) nonarchimedean Milnor fiber of $f$ is the set
\[
\mathcal{X} = \{ x \in X(\mathcal{O}) | \text{rv}(f(x)) = \text{rv}(t) \text{ and } \pi(x) = z \}.
\]

Note that $\mathcal{X}$ may be constructed in an affine neighborhood of $z$ and hence is indeed a (quantifier-free) definable set. Moreover, it is $\beta$-invariant for every $\beta \geq 1$. Therefore, $\mathcal{X}$ is an object of $\mu VF^{\circ}[\ast]$ equipped with the constant volume form $0$. The positive motivic zeta function of $f$ is the power series
\[
Z^1(T) = \sum_{m \in \mathbb{Z}^+} \left( \Xi \circ \eta \circ h_m \circ \int^\circ \right)([\mathcal{X}])T^m = K^{\mu_2}R\Var[[\mathbb{A}]]^{-1}[T].
\]

Now, recall from § 1, for each $m \in \mathbb{Z}^+$, the set of positive truncated arcs at $z$:
\[
\mathcal{X}^1_m = \{ \varphi \in X(\mathbb{R}[t]/t^{m+1}) | f(\varphi) = t^m \mod t^{m+1} \text{ and } \varphi(0) = z \}.
\]

The corresponding set of complex truncated arcs may be written as
\[
\mathcal{X}^1_m \otimes_{\mathbb{R}} \mathbb{C} \cong \{ \varphi \in X(\mathbb{C}[t^{1/m}]/t^{(m+1)/m}) | \text{rv}(f(\varphi)) = \text{rv}(t) \text{ and } \varphi(0) = z \}.
\]

So it makes sense to denote $\Xi([\mathcal{X}^1_m \otimes_{\mathbb{R}} \mathbb{C}])$ by $[\mathcal{X}^1_m]$. By Remark 3.17 and Lemma 3.19, we have, in $K^{\delta} \Var_{\mathbb{R}}[[\mathbb{A}]]^{-1}$,
\[
\left( \eta \circ h_m \circ \int^\circ \right)([\mathcal{X}]) = (\Theta \circ \iota)(\mathcal{X}[m]) = [\mathcal{X}^1_m \otimes_{\mathbb{R}} \mathbb{C}][\mathbb{A}]^{-md}.
\]

This shows that the coefficients of $Z^1(T)$ may also be written as $[\mathcal{X}^1_m][\mathbb{A}]^{-md}$.

**Remark 3.21.** The $\mu_2$-action on $\mathcal{X}^1_m$ considered here (also see Remark 3.12) is in general different from the one in [15], where it is simply induced by $t \mapsto -t$ for $m$ even. For instance, suppose that $X$ is the affine line, $f$ is the square function, and $z = 0$, then the $\mu_2$-action given by $\Xi$ on $\mathcal{X}^1_4 = \{ \pm t^2 + bt^3 + ct^4 \in \mathbb{R}[t]/t^5 | (b, c) \in \mathbb{R}^2 \} \simeq \{ x^2 = 1 \} \times \mathbb{R}^2$

swaps any two elements of the form $\pm t^2 + ct^4$ and hence induces the obvious nontrivial $\mu_2$-action on the first factor of $\{ x^2 = 1 \} \times 0 \times \mathbb{R}$, whereas the action induced by $t \mapsto -t$ is entirely trivial. Actually, the $\mu_2$-action induced by $t \mapsto -t$ corresponds to the dotted route from $K^{\delta,\hat{\delta}} \Var_{\mathbb{R}}$ to $K^{\mu_2}R\Var$ in (3.5).

The motivic zeta function $Z^G(T)$ studied in [15] is shown to be a rational series (the rational formula given therein needs to be revised, though) and hence one can take the limit as $T$ goes to infinity, as we shall do to $Z^1(T)$ too below. Unfortunately, this process of “taking the limit” kills off the $\mu_2$-actions on the coefficients of $Z^G(T)$ and, consequently, the limit of $Z^G(T)$ does not actually carry any $\mu_2$-action. In contrast, the limit of $Z^1(T)$ often retains a $\mu_2$-action and lives in $K^{\mu_2}R\Var[[\mathbb{A}]]^{-1}$. 
There is the negative counterpart $Z^{-1}(T)$ of $Z^1(T)$. Since the situation is the same, we shall concentrate on the positive one and drop the qualifier “positive” from the terminology. We do remark that, although complexification has seen success to some extent, for instance, the result on the Euler characteristics in [27] or the fact that the involution defined in Remark 3.7 exchanges $Z^1(T)$ and $Z^{-1}(T)$ (as observed in [16, Lemma 3.2] for truncated arcs), it is unclear how the duality of “the positive” and “the negative” here works.

**Notation 3.22.** Let $K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T]_\dag$ be the localization of $K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T]$ with respect to the multiplicative family generated by the elements $1 - [A]^a T^b$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$; it may be regarded as a subring of $K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T]$.

Applying the homomorphism $\Phi$ in (3.5) termwise to the coefficients of $Z^1(T)$, we obtain a zeta function $\bar{Z}^1(T)$. It is known from [14], using resolution of singularities, that $\bar{Z}^1(T)$ belongs to $K\mathbb{R}\text{Var}[[A]^{-1}][T]_\dag$ and, letting “$T$ go to infinity” as described in [23, § 8.4], we get a limit

$$\bar{Z}^1 := - \lim_{T \to \infty} \bar{Z}^1(T) \in K\mathbb{R}\text{Var}[[A]^{-1}],$$

which is understood as the real motivic Milnor fiber of $f$. The following finer result is in the same spirit.

**Remark 3.23.** Recall the homomorphism $E^\circ$ from (2.7). By Lemma 2.52 and Notation 3.13, we can construct the homomorphism

$$\Theta \circ E^\circ : K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][*] \longrightarrow K^{\delta}\text{Var}_R$$

simply using the expression in (2.9) with respect to bipolar twistoid decompositions.

The composition of $\Theta$ and the localization of $K^{\delta}\text{Var}_R$ at $[A]$ is still denoted by $\Theta$.

**Theorem 3.24.** The zeta function $Z^1(T)$ belongs to $K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T]_\dag$ and

$$Z([U])(T) = \sum_{m \in \mathbb{Z}^+} (\Xi \circ \delta \circ h_m)([U])T^m \in K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T].$$

Proof. Let $[U] = [(U, f, l)] \in K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][*]$ with $l \in \mathbb{Z}$ and consider the zeta function

(3.16)

$$Z([U])(T) = \sum_{m \in \mathbb{Z}^+} (\Xi \circ \delta \circ h_m)([U])T^m \in K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T].$$

If $[U]/(P_\dag) = \int^\circ([X])$ then this is $Z^1(T)$. Thus it is enough to show that $Z([U])(T)$ belongs to $K^{\mu_2}\mathbb{R}\text{Var}([[A]^{-1}][T]_\dag$ and $\lim_{T \to \infty} Z([U])(T)$ exists and equals $-(\Xi \circ \delta \circ E^\circ)([U])$.

Without loss of generality, we may assume that $U$ is already a bipolar twistoid and the function $l_f : U \longrightarrow \Gamma$ (recall Notation 2.22) vrv-contracts to a function $\sigma : I = vrv(U) \longrightarrow \Gamma$. Write $v = \Xi([tbk(U)])$. Then

$$(\Xi \circ \delta \circ E^\circ)([U]) = \chi(I)v.$$ 

Let $Z(v)(T) = \sum_{m \geq 1} vT^m$, which is an element in $K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T]_\dag$ with $\lim_{T \to \infty} Z(v)(T) = -v$. Write $I_m = I(m^{-1}\mathbb{Z})$ and let

$$Z(\sigma)(T) = \sum_{m \in \mathbb{Z}^+} \sum_{\gamma \in I_m} [A]^{-m\sigma(\gamma)}T^m.$$ 

By Remark 2.20, $\sigma$ is $\mathbb{Z}$-linear. So, by [23, Proposition 8.5.2], $Z(\sigma)(T) \in K^{\mu_2}\mathbb{R}\text{Var}[[A]^{-1}][T]_\dag$ and $\lim_{T \to \infty} Z(\sigma)(T) = -\chi(I)$. Since $Z([U])(T)$ is the Hadamard product of $Z(v)(T)$ and $Z(\sigma)(T)$, by [23, Lemma 8.4.1], $\lim_{T \to \infty} Z([U])(T) = -\chi(I)v$. □
Remark 3.25. The construction above no longer needs to go through this additional localization process “loc” employed throughout [23, § 8].

Remark 3.26. The expression $(\Xi \circ \Theta \circ \mathbb{E}^\circ)([U])$, as an element in $\mathbb{K}^\mu_{\varphi} \text{RVar}[\mathbb{A}]^{-1}$, does not actually involve $[\mathbb{A}]^{-1}$. Still, since the coefficients of $Z([U])(T)$ does involve $[\mathbb{A}]^{-1}$ and it is a fact that the natural homomorphism $\mathbb{K}^\delta_{\text{Var}_R} \rightarrow \mathbb{K}^\delta_{\text{Var}_R}[\mathbb{A}]^{-1}$ is not injective, we cannot really take the motivic Milnor fiber $\mathcal{S}^1$ of $f$ in $\mathbb{K}^\mu_{\varphi} \text{RVar}$, at least not if $\mathcal{S}^1$ is viewed as something obtained through $Z^1(T)$. It is this point of view that forces us to work with an integral whose target only involves doubly bounded sets in RV, namely $f^\circ$, instead of $f$, so as to facilitate the computation of the coefficients of $Z^1(T)$, and consequently with the nonarchimedean Milnor fiber $\mathcal{X}$, which is proper invariant (Definition 2.34), instead of, perhaps, the more obvious set

$$\{x \in X(\mathcal{O}) \mid f(x) = t \text{ and } \pi(x) = z\},$$

which is not proper invariant. This set is closely related to the analytic Milnor fiber introduced in [31] and does play a role in [23] (but not in this paper).

On the other hand, in light of Theorem 3.24, we can forego the zeta function point of view and recover $\mathcal{S}^1$ directly as $(\Xi \circ \Theta \circ \mathbb{E}^\circ \circ f^\circ)([\mathcal{X}])$. In that case there is truly no need to invert $[\mathbb{A}]$. In fact, we can also recover $\mathcal{S}^1$ directly as $\text{Vol}^\mu_{\varphi}([\mathcal{X}])$, where $\text{Vol}^\mu_{\varphi}$ is short for $\text{Vol}_{\mathbb{K}^\mu_{\varphi} \text{Var}_R}$,

$$\mathbb{K} \text{VF}_* \xrightarrow{\text{Vol}^\mu_{\varphi} = \Theta \circ \mathbb{E}^\circ \circ f} \mathbb{K}^\delta_{\text{Var}_R} \xrightarrow{\Xi} \mathbb{K}^\mu_{\varphi} \text{RVar},$$

and the result is the same because the diagram

$$\begin{array}{ccc}
\mathbb{K} \mu_{\text{VF}}^\circ[\ast] & \xrightarrow{f^\circ} & \mathbb{K} \mu_{\text{RV}}^\circ[\ast]/(\mathbb{P}_1) \\
\downarrow & & \downarrow id \\
\mathbb{K} \text{VF}_* & \xrightarrow{f} & \mathbb{K} \text{RV}[\ast]/(\mathbb{P} - 1)
\end{array}$$

(3.17)

commutes, where the first vertical arrow is induced by the obvious forgetful functor (it exists because the morphisms in $\mathbb{K} \mu_{\text{VF}}^\circ[\ast]$ are bijections instead of essential bijections, see Remark 2.35).

Remark 3.27. The real nonarchimedean Milnor fiber $\mathcal{X}^1$ of $f$ is the set $\mathcal{X}(\mathcal{R})$ of $\mathcal{R}$-rational points of $\mathcal{X}$. Appending (3.1) and then (3.8) to (3.17) with $\mathbb{M} = \mathcal{R}$ (this fulfills the requirement that $\Gamma(\mathbb{M})$ be divisible) and writing $\text{Vol}^\mu_{\varphi} = \Theta \circ \mathbb{E}_b \circ \int_{\mathcal{R}}$, we can calculate $\mathcal{S}^1$ as $\text{Vol}^\mu_{\varphi}([\mathcal{X}^1])$. This is sometimes much simpler than working with the complex nonarchimedean Milnor fiber $\mathcal{X}$; see Example 3.29 below. The reason is that $\mathcal{R}$ is real closed (and indeed o-minimal). This additional structure does give rise to a variant of the Hrushovski-Kazhdan construction, which we shall discuss in § 5.

Remark 3.28. We have seen in Remarks 2.43 and 2.44 that (3.17) almost commutes if $\mathbb{E}_b$ is replaced by $\mathbb{E}_g$. At any rate, one can still define a homomorphism $\text{Vol}^\mu_{\varphi}$ using $\mathbb{E}_g$ instead of $\mathbb{E}_b$. It would be interesting to give a geometric interpretation of the class $\text{Vol}^\mu_{\varphi}([\mathcal{X}])$ and relate it to the motivic Milnor fiber $\text{Vol}^\mu([\mathcal{X}])$. As has been mentioned earlier, we can indeed establish such a relation for the Euler characteristics, see Remark 5.17.

Example 3.29. Consider the polynomial function $f(x, y) = x^6 + x^2y^2 + y^6$ on the affine plane and take $z$ to be the origin. We decompose the real nonarchimedean Milnor fiber $\mathcal{X}^1$ into the following sets in RV($\mathcal{R}$):

$$A = \{y = 0\} \cap \{\text{rv}(x^6) = \text{rv}(t)\}, \quad A' = \{x = 0\} \cap \{\text{rv}(y^6) = \text{rv}(t)\}.$$
Clearly, in $\text{val}(\int)$ we shall perform a similar decomposition in $\tilde{\mathcal{X}}$ of $x,y$ and hence where the $\mu$-term. Then, applying the realization map $\mu$, if we forget the $\mathbb{C}$ of compact nonsingular real algebraic varieties with $\mathbb{R}$ for its existence there relies on the weak factorization theorem of [1] and Poincaré duality; the case $\mu$-action is given by $(3.18)$ $(x,y) \mapsto - (x,y)$ for the first term and $x \mapsto -x$ for the second term. Then, applying the realization map $\beta^\mu$ in Remark 3.11, we get
\[(3.18) \quad \langle \beta^\mu \circ \text{Vol}^\mu \rangle ([\mathcal{X}^1]) = 2u - (u - 1) = u + 1.\]
If we forget the $\mu$-action on $\text{Vol}^\mu ([\mathcal{X}^1])$, it becomes $2\{x^6 + x^2y^2 = 1\} - 2[G_m]$ in $\mathbb{K}$ RVar. If we take further the virtual Poincaré polynomial then it becomes 0, since $\{x^6 + x^2y^2 = 1\}$ has the same virtual Poincaré polynomial as the unit circle minus two points.

3.5. Concerning the virtual Poincaré polynomial. Let $R$ be a real closed field. An $R$-variety is defined in the same way as a real variety, but with $\mathbb{R}$ replaced by $R$. The corresponding category of $R$-varieties is denoted by $\text{RVar}$ and its Grothendieck ring by $\mathbb{K}$ RVar; we have seen the special case $R = \mathbb{R}$ in § 3.2.

The virtual Poincaré polynomial is an invariant of $\text{RVar}$, which is defined in [28]. The proof for its existence there relies on the weak factorization theorem of [1] and Poincaré duality; the former is valid over any field of characteristic 0 and the latter is available for singular homology of compact nonsingular real algebraic varieties with $\mathbb{F}_2$-coefficients. Replacing singular homology with semialgebraic homology $H^{sa}$ with $\mathbb{F}_2$-coefficients (see [7] or [2, § 11.7]), Poincaré duality still
holds (in the semialgebraic setting “compact” means “closed and bounded”). Thus the proof goes through almost verbatim for $R\text{Var}$:

**Theorem 3.30.** There exists a unique homomorphism $\beta^R : K R\text{Var} \to \mathbb{Z}[u]$ that assigns to each compact nonsingular $R$-variety $X$ its Poincaré polynomial $\sum_{i\in \mathbb{R}} \dim H^{sa}_i(X, \mathbb{F}_2) u^i$.

If $R = \mathbb{R}$ then we denote $\beta^\mathbb{R}$ simply by $\beta$ as in Remark 3.11.

**Remark 3.31.** Let $R \to R'$ be a real closed field extension. Let $X$ be an $R$-variety. Then the virtual Poincaré polynomial of the extension $X(R')$ of $X$ to $R'$ is equal to the virtual Poincaré polynomial of $X$. Actually, for $X$ compact and nonsingular, this follows immediately from the invariance of semialgebraic homology under real closed field extension. The general case follows from additivity, expressing the class of $X$ in terms of classes of compact nonsingular $R$-varieties via resolution of singularities.

Write $\text{Vol}_\mathbb{R}$ for the composition of $\text{Vol}^\mu_\mathbb{R}$ with the forgetful homomorphism $K^\mu_\mathbb{R} R\text{Var} \to K R\text{Var}$. Since $\check{R}\text{Var}$ is a subcategory of $VF_\mathbb{R}$, there is a natural homomorphism $K \check{R}\text{Var} \to K VF_\mathbb{R}$. Composing this with $\text{Vol}_\mathbb{R}$ and then the virtual Poincaré polynomial map $\beta$, we obtain a homomorphism $\beta^{\lim} : K R\text{Var} \to \mathbb{Z}[u]$. Thus we have found two homomorphisms $\beta^{\lim}, \beta^R$ from $K R\text{Var}$ to $\mathbb{Z}[u]$.

**Remark 3.32.** Over the algebraic closure of a henselian discretely valued field, it is shown in [30, Proposition 3.23] that the analogue of $\beta^{\lim}$, defined with the Hodge-Deligne polynomial instead of the virtual Poincaré polynomial, gives the Hodge-Deligne polynomial of the limit mixed Hodge structure associated with a variety. It would seem interesting to also compare $\beta^{\lim}$ with a similar map on limit structures, but such structures have not yet been constructed in the real framework.

Also, the duality of $\mathbb{E}_g$ and $\mathbb{E}_g$ described in Remarks 2.43 and 3.28 of course yields another homomorphism $\beta^{\lim}_g : K R\text{Var} \to \mathbb{Z}[u]$.

**Lemma 3.33.** For all compact nonsingular real variety $X$, $\text{Vol}^\delta([X(\check{C})]) = [X(C)]$ in $K^\delta \text{Var}_\mathbb{R}$ and hence $\text{Vol}^\mu_\mathbb{R}([X(\check{R})]) = [X]$ in $K^\mu_\mathbb{R} R\text{Var}$.

**Proof.** Let $n$ be the dimension of $X$ and choose a quasi-finite morphism $f : X \to \mathbb{A}^n$ over $\mathbb{R}$. Set $X = (X(C), f)$, which is treated as an object of $\text{RES}[n]$. Obviously $[X(C)] = \Theta(\mathbb{E}_b([X]))$ in $K^\delta \text{Var}_\mathbb{R}$. Thus, for the first clause, it is enough to show $\int [X(\check{C})] = [X]/(P - 1)$. This is essentially the content of [22, Lemma 13.3(2)] and the same proof works almost verbatim (the function $f$ needs to be adjusted so to become piecewise étale). The second clause is immediate from (3.1) and (3.8).

Combining this lemma with Remark 3.31, we get the following equality:

**Corollary 3.34.** For any real algebraic variety $X$,

$$\beta^{\lim}([X(\check{R})]) = \beta([X]) = \beta^{\check{R}}([X(\check{R})]).$$

However, the two homomorphisms do not coincide in general. Here is a counterexample:

**Example 3.35.** Consider the polynomial $f(x, y) = x^6 + x^2 y^2 + y^6$ again. Let $X \subseteq \check{R}^2$ be the $\check{R}$-variety given by the equation $f(x, y) = t$. Observe that we actually have $X \subseteq \mathcal{M}(\check{R})^2$ and hence $X$ is closed and bounded.

For any $t' \in VF$ with $\text{rv}(t') = \text{rv}(t)$, there is an immediate automorphism $\sigma$ of $\check{C}$ over $\mathbb{R}$ with $\sigma(t') = t$, where “immediate” means that $\sigma$ fixes RV pointwise. Therefore, changing $t$ to $t'$ in the
Polynomial is 0. So \( \beta(3.20) = 1 + u \) and hence, by (3.1), \( \int_{X} 1 = \int_{X} 1 \) where again \( X, X^1 \) are the complex and the real nonarchimedean Milnor fibers associated with \( f \).

Now, since \( X \) is nonsingular and has only one connected component, it follows that \( \beta^\text{R}([X]) = 1 + u \). On the other hand,

\[
(\beta \circ \text{Vol}_R)([X^1]) = (\beta \circ \text{Vol}_R)([X]) = \beta^\text{lim}([X]).
\]

The expression \( \text{Vol}_R([X^1]) \) may be understood as the motivic Milnor fiber \( \mathcal{F}^1 \) of \( f \) in (3.15), taken in \( K \text{RVar} \). The computation towards the end of Example 3.29 shows that its virtual Poincaré polynomial is 0. So \( \beta^\text{lim}([X]) \neq \beta^\text{R}([X]) \).

4. Thom-Sebastiani formula

Let \( X \) be a smooth connected variety and \( f, g \) nonconstant functions from \( X \) to the affine line, all defined over \( \mathbb{C} \). In this section we aim to establish a local motivic Thom-Sebastiani formula for composite morphisms on \( X \) of the form \( h(f, g) \), where \( h(x, y) \) is a polynomial of the form

\[
y^N + \sum_{2 \leq i \leq \ell} x^{m_i}, \quad m_2 \ll N \ll m_3 \ll \ldots \ll m_\ell;
\]

here we may take \( N = m_1 \), but it plays a special role and hence is denoted differently. The actual condition we shall assume is somewhat weaker than this, see Hypothesis 4.21.

4.1. Combinatorial data and Galois actions of the torus. The said formula expresses the motivic Milnor fiber of \( h(f, g) \) as a sum of (iterated) motivic Milnor fibers of morphisms derived from \( f, g \) and their convolution products. Before diving into technicalities, we first describe how the various terms in the sum are singled out based on certain combinatorial data that is read off from the tropical curve of \( h(x, y) \).

Consider the planes in \( \mathbb{Q}^+ \) defined by the following equations: \( z = 1, z = Ny, \) and \( z = m_i x \) for \( 2 \leq i \leq \ell \). The lowest points on these planes form the surface of a convex polyhedron whose edges are the pairwise intersections of the three planes \( z = 1, z = Ny, \) and \( z = m_2 x \). The tropical curve \( H \) of \( h(x, y) \) is the orthogonal projection of these edges in the \((x, y)\)-plane. Thus \( H \) consists of two rays \( H_1, H_2 \) and a line segment \( H_3 \), all emanating from the point \((1/m_2, 1/N)\), see the illustration on the left in Figure 1. Both \( H_1 \) and \( H_2 \) contribute a term of (iterated) motivic Milnor fiber to the formula.

For the other terms, we need to examine the vertical rectangular pane \( P \subseteq \mathbb{Q}^+ \) of height 1 standing on the line segment \( H_3 \), see the illustration on the right in Figure 1. For each \( 2 \leq i \leq \ell \), let \( \alpha_i = 1/m_i \) and \( \beta_i = m_2/Nm_i \). Each plane \( z = m_i x \) intersects \( P \) at the oblique line segment connecting \((0, 0, 0)\) and \((\alpha_i, \beta_i, 1)\) in \( \mathbb{Q}^+ \). Let \( L_i \subseteq \mathbb{Q}^+ \) be the open line segment between the two points \((\alpha_i, \beta_i)\) and \((\alpha_{i+1}, \beta_{i+1})\), where we set \( \alpha_{\ell+1} = \beta_{\ell+1} = 0 \). Then each point \((\alpha_i, \beta_i)\) with \( i > 2 \) also contributes a term of (iterated) motivic Milnor fiber. Finally, for each \( i \geq 2 \), the points above \((\alpha_i, \beta_i)\) that lie on the oblique line segments contribute another term, so does the corresponding open line segment \( L_i \), and the two of them are jointly referred to as a term of convolution product.

In this section, we choose to work with varieties over \( \mathbb{C} \) with \( \text{G}_m \)-actions instead of \( \hat{\mu} \)-actions. To that end, we shall mainly work in the ACVF-model \( \mathcal{C} \) with \( S = \mathbb{C} \cup \mathbb{Q} \). Even though \( \Gamma \cong N \) is only a definable sort of \( \mathcal{C} \), the Hrushovski-Kazhdan integration theory still goes through. This is
not explicitly stated in [22] but is included in the more general assumption of “effectiveness” there; in [32] and its sequels, \( \text{val}(\text{VF}(\mathbb{S})) \) is assumed to be nontrivial, but this is merely for convenience and is by no means an essential requirement.

Let \( z \in f^{-1}(0) \) be a \( C \)-rational point. As before, since the discussion below will be of a local nature, we may assume that \( X \) is actually affine (hence a definable subset of \( \text{VF}^n \) for some \( n \)) and, without loss of generality, \( z = 0 \). Write \( X \cap \mathcal{M}^n \) as \( X(\mathcal{M}) \). We shall consider definable sets of the form

\[
\mathcal{X}_\gamma = \{ x \in X(\mathcal{M}) \mid \text{val}(f(x)) = \gamma \}, \quad \gamma \in \Gamma^+;
\]

for simplicity, \( \mathcal{X}_\gamma^a \) shall just be written as \( \mathcal{X}_\gamma \), which is of primary interest, and the restriction \( f \upharpoonright \mathcal{X}_\gamma^a \) just as \( \mathcal{X}_\gamma^a \) (this will become a general notational scheme below). For each \( u \in \text{RV} \) and each \( a \in u^2 \subseteq \text{VF} \), let

\[
\mathcal{X}_a = \{ x \in X(\mathcal{M}) \mid f(x) = a \}
\]

and

\[
\mathcal{X}_u = \{ x \in X(\mathcal{M}) \mid \text{rv}(f(x)) = u \},
\]

which are \( a \)-definable sets; so \( \mathcal{X}_{\text{rv}(t)} \) is just the set called the nonarchimedean Milnor fiber of \( f \) above. The following equality relating \( \int [\mathcal{X}_a] \) and \( \int [\mathcal{X}_u] \) generalizes (3.20), and shall be used frequently (and often implicitly); the argument for it is the same as the one given thereabout.

**Lemma 4.1.** \( \int [\mathcal{X}_a] = [1] \int [\mathcal{X}_u] \).

For each \( a \in C^\times \), there is an automorphism \( C((t)) \rightarrow C((t)) \) sending \( t \) to \( at \). Thus there is a subgroup of \( \text{Gal}(C((t))/C) \) that may be identified with \( C^\times \); the preimage of \( C^\times \) along the canonical surjective homomorphism

\[
\text{Gal}(\hat{C}/C) \rightarrow \text{Gal}(C((t))/C)
\]

is denoted by \( \hat{\tau} \). A moment reflection shows that \( \hat{\tau} \cong \lim_n (C^\times)_n \), where each \( (C^\times)_n \) is just a copy of \( C^\times \) and the transition morphisms are the same as in the limit \( \hat{\mu} = \lim_n \mu_n \); so for each \( n \) there is a canonical epimorphism \( \tau_n : \hat{\tau} \rightarrow (C^\times)_n \), which is a part of the limit construction (in the category of groups, say). More concretely, the elements in \( \hat{\tau} \) may be identified as sequences \( \hat{a} = (a_n)_n \) of \( n \)th roots of \( a \), \( a \in C^\times \), satisfying \( a_{kn}^n = a_k \). Such an element acts on \( \hat{C} \) by \( \hat{a} \cdot t^{1/n} = a_n t^{1/n} \). We have a short exact sequence

\[
1 \rightarrow \hat{\mu} \rightarrow \hat{\tau} \rightarrow C^\times \rightarrow 1.
\]

This sequence does not split, though.

**Remark 4.2.** Here is a different perspective on \( \hat{\tau} \). By the structural theory of valued fields, an element \( \sigma \in \text{Gal}(\hat{C}/C((t))) \) is in the ramification subgroup if and only if it fixes \( \text{RV} \) pointwise (see [13, Lemma 5.3.2]). But it can be easily checked that every \( \sigma \in \text{Gal}(\hat{C}/C((t))) \) moves some element.
of RV unless $\sigma = \text{id}$. So $\text{Gal}(C/C((t))) \cong \hat{\mu}$ may be identified with $\text{Aut}(RV/\text{RV}(C((t))))$, where $\text{RV}(C((t)))$ is equal to the subgroup generated by $\text{rv}(t)$ over $k^\times$.

For each $u \in k^\times$, there is an automorphism $\text{Aut}(RV/k^\times)$ sending $\text{rv}(t)$ to $u \text{rv}(t)$; observe that an automorphism in $\text{Aut}(RV/k^\times)$ fixes $Q \cong RV/k^\times$ pointwise if and only if it is of this form. So $\hat{\tau}$ may also be identified with a subgroup of $\text{Aut}(RV/k^\times)$, namely $\text{Aut}(RV/k^\times \cup Q)$.

**Remark 4.3.** From yet a different perspective, recall from Remark 3.8 that there is a natural bijection between $\hat{\mu}$ and the set of reduced cross-sections $\overline{\text{csn}} : Q \longrightarrow RV$ with $\overline{\text{csn}}(1) = \text{rv}(t)$. Of course this is still the case if we change $\text{rv}(t)$ to any other element of the form $u \text{rv}(t)$, $u \in k^\times$. Consequently, we may identify $\hat{\tau}$ with the set of all such reduced cross-sections.

Since every reduced cross-section $\overline{\text{csn}}$ determines a reduced angular component $\overline{ac} : RV \longrightarrow k^\times$ via the assignment $u \mapsto \text{tbk}(u)$ and, conversely, every reduced angular component $\overline{ac}$ determines a reduced cross-section $\overline{\text{csn}}$ with $\overline{\text{csn}}(Q) = \overline{ac}^{-1}(1)$, we see that $\hat{\tau}$ may also be identified with the set of all such reduced angular components.

Intuitively, as we have seen above, all this is just saying that if any reduced angular component or reduced cross-section is chosen and added to the structure of RV then we have an intrinsic isomorphism $RV \cong k^\times \oplus Q$, and hence if both $k^\times$ and $Q$ are fixed pointwise then RV has no symmetries left other than the trivial one.

Therefore, similar to the case $S = C((t))$, elements in $K\text{RES}$ now carry *good* $\hat{\tau}$-actions, that is, those $\hat{\tau}$-actions that factor through some $\tau_n$ and hence may be considered as $G_m$-actions. To emphasize this and to distinguish it from the similar ring with good $\hat{\mu}$-actions, we shall denote $!K\text{RES}$ by $!K^\tau\text{RES}$ over $S = C \cup Q$ and by $!K^\mu\text{RES}$ over $S = C((t))$.

**Remark 4.4.** An action of an algebraic group $G$ on a variety $Y$, all defined over $C$, is *good* if every orbit is contained in an affine open subset of $Y$. If $G$ is finite and $Y$ is quasi-projective then this condition always holds, which is why we have not brought it up until now.

Let $Y$ be a variety over $C$ with a good $G_m$-action $h$. We say that $h$ is $n$*-weighted* for some $n \in Z^+$ if there is a morphism $\pi : Y \longrightarrow G_m$ such that $\pi(c \cdot y) = c^n\pi(y)$ for all $c \in G_m$ and all $y \in Y$. We also say that $h$ is $0$*-weighted* if it is trivial, and the only witness to this is the morphism $Y \longrightarrow 1$. Observe that if there is a $G_m$-equivariant isomorphism between $(Y, h)$ and $(Y', h')$ then $h$ is $n$-weighted if and only if $h'$ is $n$-weighted. Furthermore, if $h$ is $n$-weighted with a witness $\pi$ and $h'$ is $n'$-weighted with a witness $\pi'$ then $\pi(c \cdot y)\pi'(c \cdot y') = c^{n+n'}\pi(y)\pi'(y')$ for all $c \in G_m$, all $y \in Y$, and all $y' \in Y'$ and hence the good diagonal $G_m$-action $h \times h'$ on $Y \times Y'$ is $(n+n')$-weighted.

The category $\text{Var}_C^{\tau, n}$ consists of the varieties over $C$ with $n$-weighted $G_m$-actions and the $G_m$-equivariant morphisms between them. Let $\text{Var}_C^{\tau}$ denote the colimit of the inductive system of these categories $\text{Var}_C^{\tau, n}$, $n \in Z^+$, in which transition functors correspond to multiplication of integers (so there are no functors between $\text{Var}_C^{\tau, 0}$ and other $\text{Var}_C^{\tau, n}$). We may and do think of an object of $\text{Var}_C^{\tau}$ as equipped with a $\hat{\tau}$-action that factors through some $\tau_n$, hence the notation. The Grothendieck groups $K^{\tau, n}\text{Var}_C, K^\tau\text{Var}_C$ are constructed subject to the usual condition on trivializing $G_m$-actions on affine line bundles analogous to (3.2). Clearly $K^\tau\text{Var}_C$ is the colimit of $K^{\tau, n}\text{Var}_C, n \in Z^+$, and is indeed a commutative ring, with the product operation induced by that in $\text{Var}_C^{\tau}$.

**Remark 4.5.** Choose a reduced cross-section $\overline{\text{csn}} : Q \longrightarrow RV$; the point $\text{rv}(t) \in RV$ is not special in the present setting (the object of interest shall be $X^{\tau}$, not $X^{\tau(t)}$) and hence we no longer demand $\overline{\text{csn}}(1) = \text{rv}(t)$. Let $U \subseteq \gamma^\tau \subseteq RV^n$ be an object of $\text{RES}$, where $\gamma_1 \leq \ldots \leq \gamma_n$, and $d$ the least positive integer $d$ such that $U$ is a set in $\text{RV}(C((t^{1/d}))$; note that any other such integer is a multiple of $d$. Consider the function $\pi : U \longrightarrow RV$ given by $u \mapsto u^d_n$. Then $\pi(c \cdot u) = c^{\nu(d)}\pi(u)$ for all
$c \in G_m$ and all $u \in U$, where $c \in \mathbb{Z}^+$ with $\gamma_n = c/d$. So if $tbk(U)$ is a variety over $\mathbb{C}$ then it is $ed$-weighted, which is witnessed by $tbk(\pi)$.

Recall that the isomorphism in [23, § 4.3] is constructed via twistback; henceforth we denote it by $\Theta^\sharp : \!K^\sharp \ RES \longrightarrow K^\sharp \ Var_C$. Similarly, there is an isomorphism $\Theta^\flat : \!K^\flat \ RES \longrightarrow K^\flat \ Var_C$, and the diagram

$$
\begin{array}{ccc}
\!K^\flat \ RES & \longrightarrow & K^\flat \ Var_C \\
\downarrow & & \downarrow \\
\!K^\sharp \ RES & \longrightarrow & K^\sharp \ Var_C
\end{array}
$$

indeed commutes, where the first vertical arrow is induced by the subcategory relation and the second vertical arrow is induced by the obvious forgetful functor ($\hat{\mu}$ is a subgroup of $\hat{\tau}$).

**Remark 4.6.** The modified argument for the surjectivity of $\Theta^\flat$ is not as straightforward as that in Remark 3.12. Some model theory is needed.

Let $(Y, h) \in Var_C^{r/n}$ and $\pi : Y \longrightarrow G_m$ witness that $h$ is $n$-weighted. We may assume that $Y$ is irreducible and quasi-projective. By elimination of imaginaries in the first-order theory of algebraically closed fields, there is a definable surjection $\omega : Y \longrightarrow Z$, where $Z$ is a set in $G_m$, such that each fiber $\omega^{-1}(z)$ contains precisely one $h$-orbit. Then $\omega \times \pi$ is a definable finite-to-one surjection from $Y$ onto $Z \times G_m$ each of whose fibers inherits a $\mu_n$-action. By Kummer theory and compactness, there are a definable function $\eta : Z \longrightarrow G_m$ and a definable bijection $\zeta$ from the fiber $(\omega \times \pi)^{-1}(Z \times 1)$ to the set

$$Z = \{(z, 1, v) \in Z \times G_m \times G_m \mid \eta(z) = v^n\}.$$ 

If $\zeta_1(y) = (z, 1, v)$ and $c \in G_m$ then set $\zeta(c \cdot y) = (z, c^n, cv)$. It can be readily checked that if $c \cdot y = c' \cdot y'$ then $\zeta(c \cdot y) = \zeta(c' \cdot y')$ and hence $\zeta$ is a $G_m$-equivariant bijection from $Y$ onto a set $V \subseteq Z \times G_m \times G_m$, where the $G_m$-action on $V$ has just been given. Let $U \subseteq Z \times 1^i \times (1/\ell)^i$ such that $tbk(U) = V$. Then $\Theta^\flat([U]) = ([Y, h])$.

Following the notational scheme introduced in Remark 3.26, let us denote the composites $\Theta^\flat \circ E_b \circ f$, $\Theta^\flat \circ E_b \circ f$ by $Vol^\flat$, $Vol^\mu$. Relative to the chosen reduced cross-section $\tilde{\text{csm}}$, the fiber $X_{\tilde{\text{csm}}(1)}$ gives the motivic Milnor fiber $\mathcal{S}_f$ as constructed in [23, § 8.5], that is, $Vol^\mu([X_{\tilde{\text{csm}}(1)}]) = \mathcal{S}_f$, but for any $v \in k^\times$ other than $1$, $\mathcal{S}_f^v := Vol^\mu([X_{\tilde{\text{csm}}(1)}])$ is not equal to $\mathcal{S}_f$ in general. The $\hat{\mu}$-action on $\mathcal{S}_f^v$ corresponds to a cosect of $\hat{\mu}$ in $\hat{\tau}$, which in turn corresponds to the various reduced cross-sections $\tilde{\text{csm}} : Q \longrightarrow RV$ with $\tilde{\text{csm}}(1) = v \tilde{\text{csm}}(1)$.

All the relevant constructions above still go through if we replace $S = C(t)$ with $S = C(t^q)$ for any $q \in Q^+$.  

### 4.2. Categories with angular components.

To define convolution operators, we need to consider objects equipped with angular component maps and equivariant morphisms between them, as follows.

Ultimately we are only interested in the points $(\alpha_i, \beta_i) \in (Q^+)^2$ described above and the corresponding elements $m_i/m_s$, $2 \leq i \leq t$, in the interval $(0, 1] \subseteq Q^+$. But it is conceptually clearer to work in a more general setting. Thus let $\vartheta = (\vartheta_1, \ldots, \vartheta_t)$ be a sequence of elements in $(0, 1]$ such that if $\ell > 1$ then $\vartheta_1 = \vartheta_2 < \ldots < \vartheta_t$. Let $\lambda$ be the least positive integer such that $\lambda \vartheta_1 / \vartheta_t$ is an integer for every $2 \leq i \leq \ell$; so any other integer that has this property must be a multiple of $\lambda$.

Let $Y$ be a variety over $\mathbb{C}$ with a good $G_m$-action. Let $\pi : Y \longrightarrow G_m^\ell$ be a morphism and $\pi_i : Y \longrightarrow G_m$, $1 \leq i \leq \ell$, its coordinate projections. Suppose that there exists a morphism
For any $n \in \mathbb{Z}^+$ that is divisible by $\lambda$, we say that the morphism $\pi$ is $(\vartheta, n)$-diagonal if for all $c \in G_m$ and all $y \in Y$,

\[ \pi_1(c \cdot y) = c^{\vartheta_1/\vartheta} \pi_1(y) \quad \text{and} \quad \pi^*(c \cdot y) = c^{\vartheta/\vartheta} \pi^*(y). \]

This implies that $\pi(c \cdot y) = c^{\vartheta_1/\vartheta} \pi(y)$, where $c^{\vartheta_1/\vartheta} = (c^{\vartheta_i/\vartheta})_{1 \leq i \leq \ell}$; we also refer to $\pi$ as a $(\vartheta, n)$-diagonal variety over $G_m^\ell$ with a good $G_m$-action.

**Definition 4.7.** An object of the category $\text{Var}^\vartheta_{C,n}$ is a pair $(Y, \pi)$ such that $Y$ is a variety over $C$ with a good $G_m$-action and $\pi : Y \to G_m^\ell$ is a $(\vartheta, n)$-diagonal morphism. A morphism between two such objects is a morphism between the $(\vartheta, n)$-diagonal varieties over $G_m^\ell$ that is equivariant with respect to the $G_m$-actions.

The category $\text{Var}^\vartheta_{C,n}$ is the colimit of the inductive system of the categories $\text{Var}^\vartheta_{C,n}$, $n \in \mathbb{Z}^+$, in which transition functors correspond to multiplication of integers.

The Grothendieck groups $K^{\vartheta,n} \text{Var}_C$ are constructed as before. Fiber product (reduced) over $G_m^\ell$ with diagonal action induces a product operation on $K^{\vartheta,n} \text{Var}_C$, which turns the latter into a commutative ring. Set

\[ K^{\vartheta,n} \text{Var}_C = \lim_{\longrightarrow} K^{\vartheta,n} \text{Var}_C. \]

Observe that if $\ell = 1, 2$ then the entries in $\vartheta$ do not really have any bearing on the definitions of $(\vartheta, n)$-diagonality and $\text{Var}^\vartheta_{C,n}$, in which case we shall just say “$n$-diagonal” and write $\text{Var}^1_{C,n}$, etc., when $\ell = 1$ and $\text{Var}^2_{C,n}$, etc., when $\ell = 2$. Also note that $\text{Var}^1_{C,n}$ is just the category $\text{Var}^{G_m,n}$ as defined in [20, § 2.3] and hence, by [20, Lemma 2.5], it is equivalent to the category of varieties over $C$ with $\mu_n$-actions, in particular, $K^{1,n} \text{Var}_C \cong K^{\mu_n} \text{Var}_C$.

**Remark 4.8.** Denote by $\text{Var}_{G_m}^{\mu_n}$ the category of varieties $\xi : Z \to G_m$ over $G_m$ with $\mu_n$-actions such that its fibers are $\mu_n$-invariant, and by $\text{Var}_{G_m}^{G_m,n}$ the category of varieties $\xi : Z \to G_m^2$ over $G_m$ with good $G_m$-actions such that the fibers of the morphism $\xi_1$ are $G_m$-invariant and the morphism $\xi_2$ is $n$-diagonal. The morphisms in both categories are those that are equivariant with respect to the group in question. By (the proof of) [20, Lemma 2.5], these two categories are equivalent.

Assume $\ell > 1$. Let $(Y, \pi) \in \text{Var}^\vartheta_{C,n}$. For each $1 \leq i \leq \ell$, write $\pi_i = \text{pr}_i \circ \pi$, $\pi_{> i} = \text{pr}_{> i} \circ \pi$, etc. The composition

\[ \bar{\pi} : Y \xrightarrow{\pi_{\leq 2}} G_m^2 \xrightarrow{(x,y) \mapsto xy^{-1}} G_m \]

is a morphism in $\text{Var}_C$. It is clear from (4.1) that every fiber of $\bar{\pi}$ inherits a good $G_m$-action from $Y$ and hence $F_n(Y, \pi) := \bar{\pi} \times \pi^*$ is an object of $\text{Var}_{G_m}^{G_m,n/\lambda}$, where $\pi^*$ is the morphism that comes with the $(\vartheta, n)$-diagonality of $\pi$. Conversely, let $\xi : Z \to G_m^2$ be an object of $\text{Var}_{G_m}^{G_m,n/\lambda}$ and consider the morphism $\bar{\xi}$ on $Z$ given by

\[ z \longmapsto (\xi_1(z), 1, \ldots, 1)\xi_2(z)^{\lambda/\vartheta} \in G_m^\ell, \]

where $\xi(z)^{\lambda/\vartheta} = (\xi(z)^{\lambda_i/\vartheta})_{1 \leq i \leq \ell}$. Then $G_n(\xi) := (Z, \bar{\xi})$ is an object of $\text{Var}^\vartheta_{C,n}$.

It is straightforward to extend the two assignments $F_n, G_n$ to functors with $G_n(F_n(Y, \pi)) \cong (Y, \pi)$ and $F_n(G_n(\xi)) \cong \xi$, that is, they are quasi-inverse to each other. So

\[ K^{\vartheta,n} \text{Var}_C \cong K^{G_m,n/\lambda} \text{Var}_{G_m} \cong K^{\mu_n/\lambda} \text{Var}_{G_m}. \]
where the new Grothendieck groups are constructed as before. Consequently, all the operations on \( \mathbf{K}^{\vartheta, n} \text{Var}_C \) that will appear below may be considered as defined on \( \mathbf{K}^{m/\lambda} \text{Var}_{G_m} \) via this isomorphism. As is the case with [20, Proposition 2.6], the point here is that \( \mathbf{K}^{m/\lambda} \text{Var}_{G_m} \) is much closer to objects that have been studied extensively in the literature and hence for which we have a deeper understanding.

If \( Z \in \text{Var}_C^\varphi \) and \((Y, \pi) \in \text{Var}_C^{\vartheta,n}\) then \((Y \times Z, \pi \circ \text{pr}_Y) \in \text{Var}_C^{\vartheta,n}\), where the good \( G_m \)-action on \( Y \times Z \) is given by \( c \cdot (y, z) = (c \cdot y, c^n \cdot z) \) for all \( c \in G_m \). This is compatible with the inductive systems in question and hence, after passing to the colimits, we see that \( \mathbf{K}^\vartheta \text{Var}_C \) is indeed a \( \mathbf{K}^\varphi \text{Var}_C \)-module.

**Definition 4.9.** Assume \( \ell > 1 \). We construct a \( \mathbf{K}^\varphi \text{Var}_C \)-module homomorphism
\[
\Psi_\vartheta : \mathbf{K}^\vartheta \text{Var}_C \rightarrow \mathbf{K}^1 \text{Var}_C
\]
by induction on \( \ell \) as follows.

For \((Y, \pi) \in \text{Var}_C^{\vartheta,n}\), let \((\pi_1 + \pi_2)^{-1}(0)\) and \( Y \setminus (\pi_1 + \pi_2)^{-1}(0) \) denote, in \( \text{Var}_C \), the pullbacks of \( \pi_{\leq 2} : Y \rightarrow G^2_m \) along the antidiagonal of \( G^2_m \) and its complement, respectively. It is clear from (4.1) that both varieties inherit a good \( G_m \)-action from \( Y \).

For the base case \( \ell = 2 \), we consider the good \( G_m \)-action on \((\pi_1 + \pi_2)^{-1}(0) \times G_m \) whose second factor is given by \( c \cdot z = c^n z \). Then the expressions
\[
[(Y \setminus (\pi_1 + \pi_2)^{-1}(0), \pi_1 + \pi_2)], \quad [((\pi_1 + \pi_2)^{-1}(0) \times G_m, \text{pr}_{G_m})]
\]
designate two elements in \( \mathbf{K}^{1,n} \text{Var}_C \); they only depend on the class of \((Y, \pi)\) and hence may be denoted by \( \Psi_{2,n}([(Y, \pi)]) \), \( \Psi_{2,n}([(Y, \pi)]) \), respectively. These assignments respect the defining relations of \( \mathbf{K}^{2,n} \text{Var}_C \) and hence may be extended uniquely to two group homomorphisms \( \Psi_{2,n} \), \( \Psi_{2,n} \). These group homomorphisms in turn are compatible with the inductive systems in question and hence, after passing to the colimits, we obtain two group homomorphisms \( \Psi_{2}, \Psi_{2} \), which also respect the \( \mathbf{K}^\varphi \text{Var}_C \)-module structure. Set \( \Psi_2 = -(\Psi_2 - \Psi_2) \).

For the inductive step \( \ell > 2 \), let \( \vartheta' = (\vartheta_3, \vartheta_3, \ldots, \vartheta_\ell) \) and \( \lambda' \) be the least positive integer such that \( \lambda' \vartheta_i / \vartheta_i \) is an integer for every \( 3 \leq i \leq \ell \). Then \( \lambda' \) divides \( \lambda \). We consider the good \( G_m \)-action on \( G_m \times (\pi_1 + \pi_2)^{-1}(0) \) whose first factor is given by \( c \cdot z = c^{\vartheta_3 / \vartheta_i} z \). It follows that the expression
\[
(Y', \pi') := (G_m \times (\pi_1 + \pi_2)^{-1}(0), 1_{G_m} \times \pi_{22})
\]
designates an object of \( \text{Var}_C^{\vartheta', n} \) whose class only depends on that of \((Y, \pi)\) and hence may be denoted by \( \Psi_{2,n}([(Y, \pi)]) \). The assignments \( \Psi_{2,n}, n \in \mathbb{Z}^+ \), may be extended to group homomorphisms and their colimit \( \Psi_{2} : \mathbf{K}^{\varphi} \text{Var}_C \rightarrow \mathbf{K}^{\varphi} \text{Var}_C \) is a \( \mathbf{K}^\varphi \text{Var}_C \)-module homomorphism. Now we set
\[
\hat{\Psi}_{\vartheta} = \hat{\Psi}_{\varphi} \circ \Psi_{\vartheta}, \quad \tilde{\Psi}_{\vartheta} = \tilde{\Psi}_{\vartheta} \circ \Psi_{\vartheta}, \quad \Psi_{\vartheta} = \Psi_{\varphi} \circ \Psi_{\vartheta} = -(\hat{\Psi}_{\vartheta} - \tilde{\Psi}_{\vartheta}).
\]

We could have defined \( \Psi_{\vartheta} \) to be \( \hat{\Psi}_{\vartheta} - \tilde{\Psi}_{\vartheta} \) instead of \( -(\hat{\Psi}_{\vartheta} - \tilde{\Psi}_{\vartheta}) \). The negative sign at the front is inherited from the literature.

**Remark 4.10.** The case \( \ell = 2 \) is of course special. For \((X, \pi_X) \in \text{Var}_C^{1,m} \) and \((Y, \pi_Y) \in \text{Var}_C^{1,n} \), let \( \pi_X \oplus \pi_Y \) be the obvious morphism \( X \times Y \rightarrow G^2_m \). Then \((X \times Y, \pi_X \oplus \pi_Y) \) is an object of \( \text{Var}_C^{2,mn} \) whose class only depends on those of \((X, \pi_X)\) and \((Y, \pi_Y)\). We may then define a binary map on \( \mathbf{K}^1 \text{Var}_C \) by
\[
[(X, \pi_X)] \ast [(Y, \pi_Y)] = \Psi_2([(X \times Y, \pi_X \oplus \pi_Y)]) \in \mathbf{K}^1 \text{Var}_C.
\]
Although the category \( \text{Var}_C^2 \) is not the same one used in [20, § 5.1], the proof of [20, Proposition 5.2] still goes through verbatim, which justifies referring to (4.4) as a convolution product.
Definition 4.11. An object of the category $\text{RV}^{\varphi}_\vartheta[k]$ is a definable triple $(U, f, \overline{ac})$, where

- the pair $(U, f)$ consists of a set $U$ in $\text{RV}$ and a function $f : U \rightarrow \text{RV}^k$,
- $\overline{ac}$ is a function $U \rightarrow \vartheta^k$, which is referred to as an angular component map on $U$, such that, for each $r = (r_1, \ldots, r_\ell) \in \text{ran}(\overline{ac})$,
- the pair $(\overline{ac}^{-1}(r), f |_{\overline{ac}^{-1}(r)})$ is an object of $\text{RV}[k]$ (of course the category $\text{RV}[k]$ here is formulated relative to the additional parameters $r$),
- if $\ell > 2$ then there is an $r^* \in (\vartheta/\lambda)^\ell$ such that $r_i = (r^*)^{\lambda^i/\vartheta^i}$ for all $2 \leq i \leq \ell$.

A definable bijection $F : U \rightarrow V$ is a morphism between two such objects $(U, \overline{ac}_U), (V, \overline{ac}_V)$ if $\overline{ac}_U = \overline{ac}_V \circ F$. Set $\text{RV}^{\varphi}_\vartheta[*] = \bigoplus_k \text{RV}^{\varphi}_\vartheta[k]$.

The category $\text{RES}^{\varphi}_\vartheta$ is formulated in the same way, but with $\text{RV}[k]$ replaced by RES.

The ring structure of $K \text{RV}^{\varphi}_\vartheta[*]$ is induced by fiberwise disjoint union and fiberwise cartesian product in $\text{RV}^{\varphi}_\vartheta[*]$; similarly for other such categories. We may also think of $K \text{RV}^{\varphi}_\vartheta[*]$ as a $K \text{RV}[*]$-module and $!K \text{RES}^{\varphi}_\vartheta$ as a $!K^\varphi$ RES-module (the extra defining condition for "!K" in $!K \text{RES}^{\varphi}_\vartheta$ is in effect imposed fiberwise).

If $\ell = 1$ and $\vartheta = 1$ then the subscript $\vartheta$ shall be dropped from the notation.

Definition 4.12. Assume $\ell > 1$ and, for ease of notation, $\vartheta_\ell = 1$. We construct a $K \text{RV}[*]$-module homomorphism

$$\Pi_{\vartheta} : K \text{RV}^{\varphi}_\vartheta[*] \rightarrow K \text{RV}^{\varphi}_\vartheta[*]$$

by induction on $\ell$ as follows.

Let $(U, \overline{ac}) = (U, f, \overline{ac}) \in \text{RV}^{\varphi}_\vartheta[k]$. For each $1 \leq i \leq \ell$, let $\overline{ac}_i = \text{pr}_i \circ \overline{ac}$, $\overline{ac}_{>i} = \text{pr}_{>i} \circ \overline{ac}$, etc. Let $U'$ denote the subset of $U$ determined by the antidiagonal condition $\overline{ac}_1(u) = -\overline{ac}_2(u)$.

For the base case $\ell = 2$, that is, $\vartheta = (1, 1)$, let $f_1 : U \rightarrow \text{RV}^{k+1}$ be the function given by $u \mapsto (f(u), \overline{ac}_1(u))$, similarly for $f_2$. The pairs $(U, f_1)$, $(U \times U', f_1)$, $(U', f_1)$ are more suggestively denoted, respectively, by

$$U_1, \ U_1 \times (\overline{ac}_1 + \overline{ac}_2)^{-1}(0), \ (\overline{ac}_1 + \overline{ac}_2)^{-1}(0).$$

The elements $\tilde{\Pi}_{(1,1)}([(U, \overline{ac})])$, $\tilde{\Pi}_{(1,1)}([(U, \overline{ac})])$ in $K \text{RV}^{\varphi}_\vartheta[k+1]$ are then given, respectively, by

$$(4.5) \quad [(U_1 \times (\overline{ac}_1 + \overline{ac}_2)^{-1}(0), \overline{ac}_1 + \overline{ac}_2)], \quad [((\overline{ac}_1 + \overline{ac}_2)^{-1}(0) \times 1_2, \text{pr}_{11})];$$

here the second term is such that each fiber of $\text{pr}_{11}$ is a copy of $(\overline{ac}_1 + \overline{ac}_2)^{-1}(0)$, and hence is indeed an element in $K \text{RV}^{\varphi}_\vartheta[k+1]$. These two assignments do not depend on the representative $(U, \overline{ac})$ or the choice between $f_1$ and $f_2$, and hence may be extended uniquely to two $K \text{RV}[*]$-module homomorphisms $\tilde{\Pi}_{(1,1)}$, $\tilde{\Pi}_{(1,1)}$ (the gradation has been shifted by 1). Then set $\Pi_{(1,1)} = -(\tilde{\Pi}_{(1,1)} - \tilde{\Pi}_{(1,1)})$.

For the inductive step $\ell > 2$, let $\vartheta' = (\vartheta_1, \vartheta_2, \ldots, \vartheta_\ell)$. Then the triple

$$(4.6) \quad (U', \overline{ac}') := (\vartheta_3^2 \times U', f \circ \text{pr}_{U'}, \text{id} \times \overline{ac}_{>2})$$

designates an object of $\text{RV}^{\varphi}_\vartheta[k]$ whose class only depends on that of $(U, \overline{ac})$ and the assignment $[(U, \overline{ac})] \mapsto [(U', \overline{ac}')]$ determines a $K \text{RV}[*]$-module homomorphism

$$\Pi_{\vartheta'} : K \text{RV}^{\varphi}_\vartheta[*] \rightarrow K \text{RV}^{\varphi}_\vartheta[*].$$

Thus we may set

$$\tilde{\Pi}_\vartheta = \tilde{\Pi}_{\vartheta'} \circ \Pi_{\vartheta'}, \quad \tilde{\Pi}_\vartheta = \tilde{\Pi}_{\vartheta'} \circ \Pi_{\vartheta'}, \quad \Pi_{\vartheta} = \Pi_{\vartheta'} \circ \Pi_{\vartheta'} = -(\tilde{\Pi}_\vartheta - \tilde{\Pi}_\vartheta).$$

There is a similar construction resulting in a $!K^\varphi$ RES-module homomorphism

$$!K \text{RES}^{\varphi}_\vartheta \rightarrow !K \text{RES}^{\varphi}_\vartheta,$$
which is denoted by $\Pi_\vartheta$ as well. Its construction is actually simpler since the categories $\text{RES}_0$, $\text{RES}^\vartheta$ are not graded and the function $f$ is irrelevant. Also, in light of the ring homomorphism $\mathbb{E}_0 : K \text{RV}[\ast] \to !K^2 \text{RES}$, this $\Pi_\vartheta$ may be viewed as a $K \text{RV}[\ast]$-module homomorphism.

For $(U, \overline{ac}_U) \in RV^\vartheta[k]$ and $(V, \overline{ac}_V) \in RV^\vartheta[l]$, let $\overline{ac}_U \oplus \overline{ac}_V$ be the obvious function from $U \times V$ into $(1, 1)^2$. Then the class

$$[(U \times V, \overline{ac}_U \oplus \overline{ac}_V)] \in K RV^\vartheta[(1, 1)]$$

only depends on the classes $[(U, \overline{ac}_U)], [(V, \overline{ac}_V)]$, not their representatives. Set

$$[(U, \overline{ac}_U)] \ast [(V, \overline{ac}_V)] = \Pi_{(1, 1)}([(U \times V, \overline{ac}_U \oplus \overline{ac}_V)]) \in K RV^\vartheta[k+l+1],$$

which may be thought of as a convolution product of the two classes.

The following lemma only serves to confirm the structural resemblance of the binary map $\ast$ here to the convolution product (4.4). It will not be of any use beyond this point.

**Lemma 4.13.** The binary map $\ast$ on $K RV^\vartheta[\ast]$ is commutative and associative. Moreover, for all $(U, \overline{ac}) = (U, f, \overline{ac}) \in RV^\vartheta[k]$,

$$[(U, \overline{ac})] \ast 1 = [(U, \overline{ac})][1] \in K RV^\vartheta[k+1].$$

Here $1 \in K RV^\vartheta[0]$ is the multiplicative identity in $K RV^\vartheta[\ast]$ and is represented by the triple $(1^2, \infty, \text{id})$, and $[1] \in K RV^\vartheta[1]$ is represented by the triple $(1^2, \text{id}, \text{id})$.

**Proof.** The formal computations involved are essentially the same as those in the proof of [20, Proposition 5.2]. We shall just write down some details for the second claim since the expected convolution identity 1 is actually off by a factor, namely $[1]$, in this setting.

The obvious function on $U \times 1^2$ induced by $\overline{ac}$ is still denoted by $\overline{ac}$, similarly for $f$ and $\text{id} \upharpoonright 1^2$. Write $(U \times 1^2, f \circ \overline{ac})$ as $U \times 1^2$. Then $[(U, \overline{ac})] \ast 1 \in K RV^\vartheta[k+1]$ is given by

$$[(U, \overline{ac})] \ast 1 = \Pi_{(1, 1)}([(U \times 1^2, \overline{ac} \oplus \overline{ac} + \text{id})^{-1}(0, \overline{ac} + \text{id}) \ast (1^2, \text{id}, \text{id})]).$$

In this expression, for each $r \in 1^2$, we have, in $K RV[k+1]$,

$$[(\overline{ac} + \text{id})^{-1}(r)] = [(U \setminus \overline{ac}^{-1}(r), f \oplus \overline{ac})] \quad \text{and} \quad [\text{pr}_{1^2}^{-1}(r)] = [(U, f \oplus \overline{ac})].$$

So (4.7) may also be written as

$$-([(U \times 1^2 \setminus (\overline{ac} - \text{id})^{-1}(0), \text{pr}_{1^2})] - [(U \times 1^2, \text{pr}_{1^2})]).$$

Of course this is just $([(\overline{ac} - \text{id})^{-1}(0), \text{pr}_{1^2})].$ Since we now have, for every $r \in 1^2$,

$$[\text{pr}_{1^2}^{-1}(r)] = [(\overline{ac}^{-1}(r) \times 1^2, f \circ \overline{ac} - \text{id})^{-1}(0), \text{pr}_{1^2})].$$

in $K RV[k+1]$, it follows that

$$[(\overline{ac} - \text{id})^{-1}(0), \text{pr}_{1^2})] = [(U, \overline{ac})][1]$$

in $K RV^\vartheta[k+1]$. \hfill \Box

4.3. **Commuting with the convolution operators.** We construct a composite homomorphism $K RV^\vartheta[\ast] \to K^\vartheta \text{Var}_C$, similar to $\Theta^\vartheta \circ \mathbb{E}_0 : K RV[\ast] \to K^\vartheta \text{Var}_C$, and show that it commutes with the various convolution operators.

**Remark 4.14.** Let $(U, \overline{ac}) \in RV^\vartheta[\ast]$. By Proposition 2.23 and compactness, there is a definable finite partition $(B_i)_i$ of $\vartheta^\vartheta$ such that every $\overline{ac}^{-1}(r)$ is $r$-definably bijective, uniformly over each $B_i$, to a disjoint union of products $U_{rij} \times D_{rij}^\vartheta$, where $U_{rij} \in \text{RES}[\ast]$ and $D_{rij} \in \Gamma[\ast]$; actually, we may write $D_{rij}$ as $D_{ij}$ since it must be the case that $D_{rij} = D_{rij}$ for any other $r' \in B_i$. Let $U_{ij} = \bigcup_{r \in B_i} U_{rij} \times r$. The obvious coordinate projection $U_{ij} \to \vartheta^\vartheta$ is denoted by $\overline{ac}_{ij}$. 

Remark 4.15. Keeping the notation of Remark 4.14, we see that, over each $B_i$, there are elements $[(V_i, \overline{ac}_i)], [(V_i', \overline{ac}'_i)]$ in $!K \text{ RES}^\varphi$, depending on the choice of $U_{ij}$ and $D_{ij}$, such that

$$E_0(\overline{ac}_i^{-1}(r)) = \overline{ac}_i^{-1}(r) - \overline{ac}'_i^{-1}(r).$$

This is an equality in $!K^\varphi \text{ RES}$ with $S = C(\{t^a t^b\})$, not with $S = C(t)$ unless $\varphi$ is a sequence of integers or the isomorphism $\Theta^\varphi$ is applied on both sides (in which case the equality happens in $K^\varphi \text{ Var}_C$). So the difference $[(V_i, \overline{ac}_i)] - [(V_i', \overline{ac}'_i)]$ does not depend on the choice of $U_{ij}$ and $D_{ij}$. Setting

$$[(U, \overline{ac})] \mapsto \sum_i \left([(V_i, \overline{ac}_i)] - [(V_i', \overline{ac}'_i)]\right)$$

yields a ring homomorphism

$$E_{b,\varphi}^\varphi : K \text{ RV}^\varphi[*] \rightarrow !K \text{ RES}^\varphi.$$

Note that $E_{b,\varphi}^\varphi(([(U, \overline{ac})]])$ may be understood as a function into $\varphi^\varphi$ whose fibers are of the form $E_0(\overline{ac}_i^{-1}(r)))$, which has nothing to do with the partition $(B_i)_i$. The point of the partition $(B_i)_i$ here is just to show the existence of such a finite sum as in (4.8).

Taking quotient by $(P - 1)$ fiberwise, we see that the corresponding ideal of $K \text{ RV}^\varphi[*]$, still denoted by $(P - 1)$, vanishes along $E_{b,\varphi}^\varphi$.

The map $\Pi_\varphi$ on $K \text{ RV}^\varphi[*]$ is indeed related to the map $\Pi_\varphi$ on $!K \text{ RES}^\varphi$ via $E_{b,\varphi}^\varphi$:

Lemma 4.16. As $K \text{ RV}[*]$-module homomorphisms, $E_{b,\varphi}^\varphi \circ \Pi_\varphi = \Pi_\varphi \circ E_{b,\varphi}^\varphi$, similarly for $\Pi_\varphi$ and hence for $\Pi_\varphi$.

Proof. Although the case of $\Pi_\varphi$ follows immediately from those of $\Pi_\varphi$ and $\Pi_\varphi$, we shall show this for $\Pi_\varphi$ directly using the same argument. It is enough to consider elements in $K \text{ RV}^\varphi[*]$ of the form $[(U, \overline{ac})]$, since the general case would follow from $K \text{ RV}[*]$-linearity. We proceed by induction on $\ell$ and, for simplicity, assume $\varphi = 1$.

For the base case $\ell = 2$, by Remark 4.14, using the notation there, we may write

$$[(U, \overline{ac})] = \sum_{ij} [(U_{ij}, \overline{ac}_{ij})][(D_{ij}^\varphi \times (1, 1)^\varphi, \text{ pr}_{(1,1)^\varphi})].$$

By the construction of $\Pi_{(1,1)}$, we have

$$\Pi_{(1,1)}([(U, \overline{ac})]) = \sum_{ij} [(D_{ij}^\varphi \times 1^\varphi, \text{ pr}_{1^\varphi})]\Pi_{(1,1)}([(U_{ij}, \overline{ac}_{ij})]).$$

Let $n_{ij} = \chi_b(D_{ij})$. Then, since the gradation is forgotten by $E_b^\varphi$, we have

$$(E_b^\varphi \circ \Pi_{(1,1)})([(U, \overline{ac})]) = \sum_{ij} n_{ij}\Pi_{(1,1)}([(U_{ij}, \overline{ac}_{ij})]),$$

where $(U_{ij}, \overline{ac}_{ij})$ stands for the obvious object of $\text{ RES}^\varphi_{(1,1)}$ in relation to $(U_{ij}, \overline{ac}_{ij})$. The right-hand side of this equality also equals $(\Pi_{(1,1)} \circ E_b^\varphi)([(U, \overline{ac})])$.

For the inductive step $\ell > 2$, let $\varphi' = \varphi$ as in Definition 4.12. Remark 4.15 and the construction of $\Pi_\varphi^\varphi$ together imply that

$$\Pi_\varphi^\varphi \circ E_{b,\varphi}^\varphi = E_{b,\varphi'}^\varphi \circ \Pi_\varphi^\varphi.$$

So the desired equality follows from the definition of $\Pi_\varphi$ and the inductive hypothesis. \qed
Remark 4.17. Let \( \eta = (\eta_1, \ldots, \eta_\ell) \) be another sequence of elements in the interval \((0,1] \subseteq \mathbb{Q}^+ \) such that \( \eta_i/\eta_1 = \delta_i/\delta_1 \) for every \( 1 \leq i \leq \ell \). Then there is a \( \sigma \in \text{Aut}(RV/\mathbb{k}^\times) \) with \( \sigma(\eta_1) = \vartheta_1 \) and hence \( \sigma(\eta_i) = \vartheta_i \) for every \( 2 \leq i \leq \ell \). If \( \sigma' \) is another such automorphism then there is a \( \tau \in \text{Aut}(RV/\mathbb{k}^\times \cup \mathbb{Q}) \) such that \( \sigma = \sigma' \circ \tau \), and hence \( \sigma, \sigma' \) induce the same endofunctors of \( RV^{[\star]} \), \( RES \). Therefore, there are canonical isomorphisms, both denoted by \( \Delta_\eta \) for simplicity, that fit in the following commutative diagram:

\[
\begin{array}{ccc}
\text{K} RV^{\bar{\varphi}}_{\eta}^{[\star]} & \xrightarrow{E_{\bar{\varphi},\eta}} & !\text{K} RES_{\eta}^{\bar{\varphi}} \\
\Delta_\eta & & \Delta_\eta \\
\text{K} RV^{\bar{\varphi}}_{\varphi}^{[\star]} & \xrightarrow{E_{\bar{\varphi},\varphi}} & !\text{K} RES_{\varphi}^{\bar{\varphi}}
\end{array}
\]

Recall that we may also identify \( \hat{\mu} \) with \( \text{Aut}(RV/\text{RV}(\mathbb{C}((t^{\varphi}))) \) and then interpret \( \hat{\tau} \), as well as Remarks 4.2 and 4.3, accordingly. It follows that the \( \hat{\tau} \)-action on an object \((U, \bar{\alpha}c) \in RES_{\varphi}^{\bar{\varphi}} \) must factor through some \( \tau_n \) such that every \( n\delta_i/\delta_1 \) is an integer and hence \( \lambda|n \) by the choice of \( \lambda \). Such a \( \hat{\tau} \)-action may also be interpreted as a \( G_m \)-action subject to the condition

\[ \bar{\alpha}c(c \cdot u) = rv(c^{n\delta_1/\delta_1}) \bar{\alpha}c(u), \] for all \( u \in U \) and all \( c \in G_m \).

Let \( RES_{\varphi,n}^{\bar{\varphi}} \) be the full subcategory of \( RES_{\varphi}^{\bar{\varphi}} \) of those objects for which this condition holds. Thus we have obtained an inductive system of categories \( RES_{\varphi,n}^{\bar{\varphi}} \) such that

\[ RES_{\varphi}^{\bar{\varphi}} = \operatorname{colim}_n RES_{\varphi,n}^{\bar{\varphi}} \quad \text{and} \quad !\text{K} RES_{\varphi}^{\bar{\varphi}} = \operatorname{colim}_n !\text{K} RES_{\varphi,n}^{\bar{\varphi}}; \]

Lemma 4.18. For each \( n \) there is a ring isomorphism

\[ \Theta_{\varphi,n}^{\bar{\varphi}} : !\text{K} RES_{\varphi,n}^{\bar{\varphi}} \rightarrow \text{K}^{\delta,n} \text{Var}_C, \]

determined by the assignment \([ (V, \bar{\alpha}C) ] \mapsto [ \text{tbk}(V, \bar{\alpha}C) ] \) for \( \text{vr}V \) a singleton, and hence a ring isomorphism

\[ \Theta_{\varphi}^{\bar{\varphi}} : !\text{K} RES_{\varphi}^{\bar{\varphi}} \rightarrow \text{K}^{\delta} \text{Var}_C. \]

Moreover, under the ring homomorphism \( \Theta^{\bar{\varphi}} \circ E_{\varphi} \), we have \( \Theta_{\varphi,n}^{\bar{\varphi}} \circ \hat{\Pi}_\varphi = \hat{\Psi}_\varphi \circ \Theta_{\varphi}^{\bar{\varphi}} \) as \( K RV^{[\star]} \)-module homomorphisms, similarly for \( \hat{\Pi}_\varphi, \hat{\Psi}_\varphi \) and hence for \( \Pi_\varphi, \Psi_\varphi \).

Note that the set \( \text{tbk}(V, \bar{\alpha}C) \) is definable without using the implicit reduced cross-section \( \bar{\alpha}C \); in other words, varying \( \bar{\alpha}C \) will not change \( \text{tbk}(V, \bar{\alpha}C) \), but does change the bijection in question, and that is why \( \text{tbk}(V, \bar{\alpha}C) \) inherits the \( \hat{\tau} \)-action on \((V, \bar{\alpha}C) \).

Proof. The situation here is very similar to that in [23, § 4.3] or in Remarks 4.5 and 4.6, so we shall be brief. If \( \text{vr}V \) is a singleton then the graph of \( \text{tbk}(\bar{\alpha}C) \) is just a constructible set, in fact uniformly so fiberwise. So the assignment induces a homomorphism \( \Theta_{\varphi,n}^{\bar{\varphi}} \) at the semiring level and hence at the ring level. If \( \text{tbk}(V, \bar{\alpha}C) \) and \( \text{tbk}(V', \bar{\alpha}'C) \) are isomorphic in \( \text{Var}_C^{\delta,n} \) then the isomorphism may be twisted to one between \((V, \bar{\alpha}C) \) and \((V', \bar{\alpha}'C) \) in \( RES_{\varphi,n}^{\bar{\varphi}} \). Thus \( \Theta_{\varphi,n}^{\bar{\varphi}} \) is injective. On the other hand, since objects in \( \text{Var}_C^{\delta,n} \) and \( RES_{\varphi,n}^{\bar{\varphi}} \) are all endowed fiberwise with \( \mu_n/\lambda \)-actions via restriction, the argument for surjectivity in Remark 4.6 can be easily modified to work for \( \Theta_{\varphi,n}^{\bar{\varphi}} \).

The second claim follows from an inductive argument, completely similar to the one in the proof of Lemma 4.16. \( \square \)
4.4. Decomposing the composite Milnor fiber. We shall make use of the homomorphism \( E_b^\dagger \circ \Lambda \) in (2.10). The composite homomorphism \( E_b^\dagger \circ \Lambda \circ f \) shall be abbreviated as \( \text{Vol}_b^\dagger \) below when we (tacitly) work in \( \tilde{C}^\dagger \).

Remark 4.19. Assume \( \ell = 1 \). Let \( A \) be a definable set in \( \mathcal{M} \) and \( \lambda : A \rightarrow \vartheta^\sharp \) a definable function. Note that \( \lambda \) must be surjective because no nonempty proper subset of \( \gamma^\sharp \) is definable for any \( \gamma \in \mathbb{Q}^+ \). For each \( r \in \vartheta^\sharp \), by Lemma 4.1, we have

\[
\int [\lambda^{-1}(r^\sharp)] = [U_r]/(P - 1),
\]

where \( \int [\lambda^{-1}(a)] = [U_r]/(P - 1) \) for any \( a \in r^\sharp \). By compactness and Theorem 2.27, the object \( U = \bigcup_{r \in \vartheta^\sharp} U_r \times r \) is such that \( \int [A] = [U]/(P - 1) \). Let \( \pi : U \rightarrow \vartheta^\sharp \) be the obvious coordinate projection. Then \( (U, \pi) \) is an object of \( \text{RV}_{\vartheta^\sharp}^\sharp \). We shall use the abbreviations

\[
[(U, \pi)]/(P - 1) = \int_\vartheta^\sharp [\lambda] \quad \text{and} \quad E_b^\pi \circ E_b^\vartheta \circ \int_\vartheta^\sharp = \text{Vol}_b^\vartheta. \]

By an obvious analogue of Lemma 3.15 and the construction of \( E_b^\pi \), computing \( \text{Vol}_b^\vartheta([\lambda]) \) boils down to computing \( \text{Vol}_b^\dagger([\lambda^{-1}(r^\sharp)]) \) for each \( r \in \vartheta^\sharp \).

There is usually no need to carry \( \vartheta \) in the notation, since it is implicit in the integrand \( \lambda \).

Notation 4.20. From here on, write the set \( \mathcal{X}^\vartheta_\gamma \) as \( \mathcal{X}^\vartheta_{f, \gamma} \), etc., to emphasize the dependency on the morphism \( f \) as given in § 4.1. Also keep in mind the convention that if \( \gamma = 1 \) then it is dropped from the notation.

For any definable set \( A \subseteq X(\mathcal{M}) \), the restriction \( f \upharpoonright (\mathcal{X}^\vartheta_\gamma \cap A) \) is just denoted by \( \mathcal{X}^\vartheta_\gamma \cap A \). We then write \( \text{Vol}_{\vartheta^\sharp}([\mathcal{X}^\vartheta_\gamma \cap A]) \) as \( \mathcal{S}_f^\vartheta([A]) \), or simply \( \mathcal{S}_f^\vartheta \) if \( A = X(\mathcal{M}) \).

Let \( m_2 < \ldots < m_\ell \) be positive integers. For each \( 1 \leq i \leq \ell \), let

\[
f^{(i)} = \sum_{2 \leq i \leq \ell} f^{m_i} \quad \text{and} \quad f^{(i)} = \bigoplus_{2 \leq i \leq \ell} f^{m_i},
\]

here \( f^{(i)}, f^{(1)} \) are both interpreted as the zero function. Naturally \( f^{(\ell)} - f^{(i)} \) denotes the function \( \sum_{i < i \leq \ell} f^{m_i} \).

Let \( g \) be another complex regular function from \( X \) to the affine line with \( g(0) = 0 \) and \( N \) another positive integer. We may think of \( N \) as \( m_1 \), but its role will be somewhat different and hence is denoted differently. For each \( 2 \leq i \leq \ell \), set

\[
\mathcal{X}^N_{g^N + f^{(i)}} = X(\mathcal{M}) \cap \{ \text{val} \circ (g^N + f^{(i-1)}) < 1 < \text{val} \circ f^{m_{i+1}} \} \cap \{ \text{val} \circ (g^N + f^{(i)}) = 1 \},
\]

\[
\mathcal{X}^N_{g^N + f^{(i)}} = X(\mathcal{M}) \cap \{ \text{val} \circ (g^N + f^{(i)}) = \text{val} \circ f^{m_i} = \text{val} \circ (g^N + f^{(i)}) = 1 \},
\]

\[
\mathcal{X}^N_{g^N + f^{(i)}} = X(\mathcal{M}) \cap \{ \text{val} \circ (g^N + f^{(i-1)}) > 1 \} \cap \{ \text{val} \circ (g^N + f^{(i)}) = 1 \}.
\]

Note that \( \text{val} \circ (g^N + f^{(i-1)}) = \text{val} \circ f^{m_i} < 1 \) is implied in the first line and \( \text{val} \circ f^{m_i} = 1 \) is implied in the third line. Also set

\[
\mathcal{Z}^N_{g^N + f^{(i)}} = X(\mathcal{M}) \cap \{ \text{val} \circ (g^N + f^{(i)}) > 1 \}, \quad 1 \leq i \leq \ell - 1,
\]

\[
\mathcal{Z}^f = X(\mathcal{M}) \cap \{ \text{val} \circ f^{(\ell)} > 1 \},
\]

\[
\mathcal{Z}^N_{g^N + f^{(i)}} = X(\mathcal{M}) \cap (g^N + f^{(i)})^{-1}(0), \quad 1 < i \leq \ell.
\]
If \( i = 1 \) then \( Z_{g^N + f(i)} \) is also written as \( Z_{g^N} \). The set \( Z_{g^N} \) will not play a role. So, for \( 2 \leq i \leq \ell \),

\[
X_{g^N + f(i)}^+ = X_{f(i)-f(i-1)}^+ \cap Z_{g^N + f(i-1)}^+.
\]

Thus we have

\[
X_{g^N + f(i)}^+ = (X_{g^N}^+ \cap Z_{f(i)}) \cup (X_{f(i)}^+ \cap Z_{g^N}) \cup \bigcup_{2 \leq j \leq \ell} X_{g^N + f(i)}^+ \cup \bigcup_{2 \leq j \leq \ell} (X_{g^N + f(i)}^- \cup X_{g^N + f(i)}^-).
\]

The restrictions of \( g^N + f(i) \) to the sets denoted by the union terms on the right-hand side are all definable functions onto \( 1^{\sharp}\), and hence Remark 4.19 may be applied to them; to curb excess of notation, these functions and other similar ones below shall just be denoted by their respective domains, as we have done for sets of the form \( X_{f,\gamma}^\sharp \).

**Hypothesis 4.21.** From here on we assume that, in the sequence \( (m_2, N, m_3, \ldots, m_\ell) \), each number is sufficiently large relative to the data in question that involve only the numbers before it. This condition will become clear and precise in the discussion below when it is needed, so we will not labor further here to explain it. We do note, however, that it is not necessarily the case that each number is greater than the numbers before it.

**Lemma 4.22.** Let \( \phi, \psi \) be complex regular functions from \( X \) to the affine line with \( \phi(0) = \psi(0) = 0 \). Let \( X_{\phi,\gamma}^\sharp, Z_\phi, Z\psi, X_{\phi,\gamma}^\sharp \cap Z_\psi \) (denoting both the set and the corresponding function), etc., be defined as above. If \( M \) is a sufficiently large positive integer then

\[
\text{Vol}^\mathbb{R}(X_{\phi,\gamma}^\sharp \cap Z_\psi) = \mathcal{J}_{\phi,\gamma}^\sharp([Z_\psi]) \quad \text{and} \quad \text{Vol}^\mathbb{R}(X_{\phi,\gamma}^\sharp \cap Z_{\phi,M}) = \mathcal{J}_{\phi,M}^\sharp.
\]

**Proof.** For the first equality, let \( X_{\phi,\gamma}^\sharp, Z_\phi, Z_\psi, X_{\phi,\gamma}^\sharp \cap Z_\psi \) denote the restriction of \( \phi^M \) to the set \( X_{\phi,\gamma}^\sharp \cap Z_\psi \), which is also a definable function onto \( 1^{\sharp} \). Clearly for every \( r \in 1^{\sharp} \),

\[
(X_{\phi,\gamma}^\sharp \cap Z_\psi)^{-1}(r^\sharp) = (X_{\phi,\gamma}^\sharp \cap Z_\psi)^{-1}(r^\sharp)
\]

and hence, by the construction of \( \text{Vol}^\mathbb{R} \), the two integrals in question are equal. So it is enough to show \( \text{Vol}^\mathbb{R}(X_{\phi,\gamma}^\sharp \cap Z_\psi) = \mathcal{J}_{\phi,\gamma}^\sharp([Z_\psi]) \).

We consider \( \phi \) instead of \( \phi^M \). For \( \gamma, \beta \in Q^+ \), denote by \( X_{\phi,\gamma,\psi,\beta}^\sharp \) the restriction of \( \phi \) to the set \( X_{\phi,\gamma}^\sharp \cap X_{\psi,\beta}^\sharp \), which is a definable function onto \( \gamma^\sharp \), and write

\[
\int [X_{\phi,\gamma,\psi,\beta}^\sharp] = [W_{\gamma,\beta}]/(P - 1).
\]

By Lemma 2.52, there is a \((\gamma, \beta)\)-definable finite partition \((D_{\gamma,\beta}; i)_i \) of \( \text{vr}(W_{\gamma,\beta}) \) such that, for each \( i \), the set \( W_{\gamma,\beta} \cap D_{\gamma,\beta,i}^\sharp \) is a bipolar twistoid. By compactness, there is a \( \gamma \)-definable finite partition \((E_{\gamma,j})_j \) of \( Q^+ \) such that, over each piece \( E_{\gamma,j} \), the partitions \((D_{\gamma,\beta}; i)_i \) may be achieved uniformly and, for each \( i \), the corresponding twistbacks are the same. So each class

\[
\int [X_{\phi,\gamma,\psi,\beta}^\sharp] = \text{vr}(W_{\gamma,\beta}) \quad \text{is indeed represented by a finite disjoint union of bipolar twistoids} \ W_{\gamma,i,j} \in \text{VR}^{[\ast]}\]
RV$^\infty [s]$ such that each $\operatorname{vrv}(W_{1/M,i})$ may be written in the form $\bigcup_{\beta \in (1, \infty)} D_{1/M, \beta, i} \times \beta$, where $\chi_b(D_{1/M, \beta, i}) \in \mathbb{Z}$ is constant over $\beta \in (1, \infty)$. A moment of reflection shows that the class $\int_{\mathbb{R}^\infty} \left[ \bigcup_{\beta \in (1, \infty)} X^\sharp_{\phi, \psi, \beta} \right]$ must admit a representative of this form as well, which then is annihilated by $\mathbb{E}_b^\infty$ because $\chi_b((1, \infty)) = 0$. This leaves only $Z_{\psi}$ in the computation. The first equality follows.

For the second equality, since the roles of $\phi^M$, $\psi$ are not exactly symmetric, a slightly different argument is needed. Let the restrictions $X^\sharp_{\phi, M, \psi, \beta}$ of $\psi$ and the partitions $(D_{\gamma, \beta, i})$, $(E_{\gamma, i})$ be defined as expected. Actually we will only need the case $\gamma = 1$ and hence will write $X^\sharp_{\phi, M, \beta, \psi}$, $D_{\beta, i}, E_{i}$ instead. Therefore, it is enough to show $\operatorname{Vol}^\infty([X^\sharp_{\phi, M, \psi}]) = \mathcal{S}_\psi$. Since $M$ is sufficiently large, $(0, 1/M] \subseteq E_{i}$ for some $i$. The class $\int_{\mathbb{R}^\infty} \left[ \bigcup_{\beta \in (0, 1/M]} X^\sharp_{\phi, M, \beta} \right]$ is represented by a finite disjoint union of bipolar twistoids satisfying the “regularity” condition in question, hence so is the class $\int_{\mathbb{R}^\infty} \left[ \bigcup_{\beta \in (0, 1]} X^\sharp_{\phi, M, \beta} \right]$. This is again annihilated by $\mathbb{E}_b^\infty$ because $\chi_b((0, 1]) = 0$. Since $X^\sharp_{\psi}$ is the union of $X^\sharp_{\psi, M, \beta}$ and $\bigcup_{\beta \in (0, 1]} X^\sharp_{\phi, M, \beta}$, the lemma follows.

Remark 4.23. It is not essential to use the bounded Euler characteristic $\chi_b$ for the second equality as the interval $(0, 1]$ vanishes under both, but $\chi_b$ is needed for the first equality.

Corollary 4.24. Substituting suitable functions for $\phi$, $\psi$ in Lemma 4.22, we obtain the following equalities:

- $\operatorname{Vol}^\infty([X^\sharp_{g, N} \cap Z_{f,i}]) = \mathcal{S}_{g, N}([Z_{f,i}])$,
- for $1 < i < \ell$, $\operatorname{Vol}^\infty([X^\sharp_{g, N + f_{i+1}}]) = \mathcal{S}_{f_{m+1}}([Z_{g, N + f_{i+1}}])$,
- $\operatorname{Vol}^\infty([X^\sharp_{f_{i+1}} \cap Z_{g, N}]) = \mathcal{S}_{f_{i+1}}$.

Proof. The second and the third equalities need additional explanation. For the second equality, recall (4.10). Similar to (4.12), for any $i$ (including $i = 1$) and any $r \in 1^\sharp$,

$$(\mathcal{S}_{f_{i+1}})^{-1}(r^\sharp) = (\mathcal{S}_{f_{i+1}})^{-1}(r^\sharp)$$

and hence $\mathcal{S}_{f_{i+1}}(r^\sharp) = \mathcal{S}_{f_{i+1}}(r^\sharp)$; more generally, $\mathcal{S}_{f_{i}}(A) = \mathcal{S}_{f_{m+1}}(A)$ for any definable set $A \subseteq X(\mathcal{M})$. For the same reason,

$$\operatorname{Vol}^\infty([X^\sharp_{f_{i}} \cap Z_{g, N + f_{i,1}}]) = \operatorname{Vol}^\infty([X^\sharp_{f_{m+1}} \cap Z_{g, N + f_{i,1}}]),$$

where as before the intersection on the right-hand side denotes the restriction of $g^N + f_{i+1}$ to the eponymous domain. Thus if $i > 1$ then the first equality of Lemma 4.22 may be applied with $\phi = f$, $M = m_{i+1}$, and $\psi = g^N + f_{i,1}$, and if $i = 1$ then the second equality of Lemma 4.22 may be applied with $\psi = f^m \phi = g$, and $M = N$.

Next we turn to the remaining terms in (4.11). To compute their values under $\operatorname{Vol}^\infty$, we need to make use of the convolution operators introduced in §4.2.

Remark 4.25. Here we extend the discussion in Remark 4.19. So assume $\ell > 1$. Let $A$ be a definable set in $\mathcal{M}$ and $\lambda: A \to \vartheta^\sharp$ a definable function of the form $\phi \oplus \bigoplus_{2 \leq i \leq \ell} \psi^m$.

Observe that $(pr, \circ \lambda)(A) = \vartheta^m_r$ for every $1 \leq r \leq \ell$, because no nonempty proper subset of $\gamma^\sharp$ is definable for any $\gamma \in Q^+$ (this fact has been used in Remark 4.19 and will be used implicitly several times below).

1. We assume that, for all $r \in \vartheta^\sharp$ and all $a \in r^\sharp \cap \lambda(A)$, $\int [\lambda^{-1}(a)] = [U_r]/(P - 1)$ only depends on $r$ and does not depend on the choice of $a$, and hence

$$(4.13) \int [\lambda^{-1}(r^\sharp)] = [U_r][V_r]/(P - 1),$$
where \( \int[r^2 \cap \lambda(A)] = [\mathbf{V}_r] \). Let \( U = \bigcup_{r \in \partial^2} U_r \times V_r \times r \) and \( \overline{ac} : U \to \partial^2 \) be the obvious coordinate projection. Then \( \int[A] = [U]/(P - 1) \) and \((U, \overline{ac})\) is an object of \( \text{RV}_{\partial^2}[\ast] \). Note that, unlike the situation in Remark 4.19, here the factor \( r \) in \( U \) is an object in \( \text{RV}[0] \), and we need it as a bookkeeping device.

The abbreviations \( \overline{\text{Vol}}[\lambda] \), \( \text{Vol}_{\partial^2}(\lambda) \) are defined accordingly as in Remark 4.19. Clearly if \( A' \subseteq A \) is definable then \( \overline{\text{Vol}}[\lambda \upharpoonright A'] \) is defined as well. As before, we shall simply write \( \overline{\text{Vol}} \) instead of \( \text{Vol}_{\partial^2} \), since the integrand \( \lambda \) already carries the information in question.

(2) We further assume that \( (r^2 \cap \lambda(A))_{a_2} = r_1^2 \) for all \( r = (r_1, r_2) \in \partial^2 \) and all \( a = (a_1, a_2) \in r^2 \cap \lambda(A) \). So in fact \( [\mathbf{V}_r \times r] \) is simply \([r, pr_{\leq 2}]\) for every \( r \in \partial^2 \).

**Notation 4.26.** For \( 1 \leq i < j \leq \ell \), let

\[
\lambda_{(i)} = \phi + \sum_{2 \leq i \leq \ell} \psi^{m_i}, \quad \lambda_{(j)} = \lambda_{(i)} \oplus \bigoplus_{i+1 \leq i \leq j} \psi^{m_i}, \quad \vartheta_{(i)} = (\vartheta_{i+1}, \vartheta_{i+1}, \ldots, \vartheta_j).
\]

If \( i = 1 \) then we shall write \( \lambda_{(i)}^{(j)}, \vartheta_{(i)}^{(j)} \) simply as \( \lambda^{(i)}, \vartheta^{(i)} \).

Observe that the two assumptions on \( \lambda \) in Remark 4.25 also holds for every \( \lambda_{(i)}^{(j)} \).

For the remainder of this subsection, fix an \( 1 \leq i < j \leq \ell \) and an \( \vartheta_{i+1} \leq \eta < \vartheta_{i+2} \). Then set \( A_\eta = (\text{val} \odot \lambda_{(i+1)})^{-1}(\eta) \). Since \( \text{val}(\lambda_{(i)}(a) - \lambda_{(i+1)}(a)) > \eta \) for all \( a \in A_\eta \), we have \( \lambda_{(i)}^{-1}(s^\ast) = \lambda_{(i+1)}^{-1}(s^\ast) \) for all \( s \in \pi^\ast \) and hence

\[
\overline{\text{Vol}}[\lambda_{(i)} \upharpoonright A_\eta] = \overline{\text{Vol}}[\lambda_{(i+1)} \upharpoonright A_\eta].
\]

Also, since \( \text{val}(\lambda_{(i)}(a)) = \text{val}(\psi(a)^{m_{i+1}}) = \vartheta_{i+1} \) for all \( a \in A_\eta \), we see that \( \lambda_{(i)}^{(j)} \) indeed restricts to a function \( A_\eta \to (\vartheta_{i+1}^{(j)})^{\pi^\ast} \).

**Remark 4.27.** Let \( r = (r_1, r_2) \in (\vartheta_i^{(j)})^{\pi^\ast} \) and \( a = (a_1, a_2) \in r^2 \cap \lambda_{(i)}^{(j)}(A_\eta) \). Then there is an \( a \)-definable finite set \( B_a \subseteq \lambda(A) \) such that \( (\lambda_{(i)}^{(j)})^{-1}(a) = \bigcup_{b \in B_a} \lambda^{-1}(b) \); indeed, for any \( b \in \lambda(A) \), \( b \in B_a \) if and only if \( \sum_{1 \leq i \leq j} b_i = a_1 \) and \( b_{>1} = a_{>1} \). Let \( a' \in r^2 \cap \lambda_{(i)}^{(j)}(A_\eta) \). It is easy to see that, by Remark 4.25(2), if \( a'_{>1} = a_{>1} \) then \( \text{rv}(B_a) = \text{rv}(B_{a'}) \). At any rate, there is an immediate automorphism of \( \tilde{C} \) over \( C \) sending \( a'_{>1} \) to \( a_{>1} \), which means that \( \text{rv}(B_a) = \text{rv}(B_{a'}) \) even if \( a'_{>1} \neq a_{>1} \). It follows from this that \( \overline{\text{Vol}}[\lambda_{(i)} \upharpoonright A_\eta] \) is defined according to Remark 4.25(1), and hence so is \( \overline{\text{Vol}}[\lambda_{(i)}^{(j)} \upharpoonright A_\eta] \) for every \( i < j \leq \ell \).

**Lemma 4.28.** We have the following equalities:

\[
\overline{\text{Vol}}[\lambda_{(i+1)} \upharpoonright A_\eta] = \left\{ \begin{array}{ll}
(\tilde{\Psi}_{(i+1)} \circ \overline{\text{Vol}})([\lambda_{(i+1)}]) & \text{if } \vartheta_{i+1} = \eta, \\
(\tilde{\Psi}_{(i+1)} \circ \overline{\text{Vol}})([\lambda_{(i+1)}]) & \text{if } \vartheta_{i+1} < \eta < \vartheta_{i+2}.
\end{array} \right.
\]

**Proof.** Let \( (\vartheta^{(j)})' \subseteq \partial^2 \) be the subset consisting those elements \( r \) with \( r_{\leq 2} \) on the antidiagonal of \( \vartheta^{(j)}_{\geq 2} \) and \( r_{>1} \in \text{rv}(\lambda(A))_{>1} \). Then \( (\vartheta^{(j)})' \subseteq \text{rv}(\lambda(A)) \) (the reason is stated in Remark 4.25). Let \( (\vartheta^{(j)})'' = \text{rv}(\lambda(A)) \setminus (\vartheta^{(j)})' \). For any coordinate projection \( pr_E \) on \( \partial^2 \), write \( \bigcup_{r \in (\vartheta^{(j)})'} U_r \times (r, pr_E) \) as \( U'_E \), similarly for \( U''_E \) with \( (\vartheta^{(j)})'' \) in place of \( (\vartheta^{(j)})' \).

We proceed by induction on \( i \). For the base case \( i = 1 \), note that \( [(\vartheta^{(j)})' \times pr_{\geq 2}] = [(\vartheta^{(j)}_{>1} \times pr_{1}] \), which is indeed equal to \( \int[\lambda(A)_{>1}] \) modulo \( (P - 1) \). Also, \( \vartheta_{\leq 2} = \vartheta^{(2)} \). By Remark 4.25(2), for all \( (a_1, a_{>1}) \in \lambda(A)_{>1} \) and all \( b \in (\vartheta^{(j)})'' \),

- if \( \vartheta_2 = \eta \) and \( (b - a_1, a_{>1}) \in \lambda(A) \) then \( \text{rv}(b - a_1)^{\ast} \times (a_1, a_{>1}) \subseteq \lambda(A) \),
- if \( \vartheta_2 < \eta < \vartheta_3 \) then \( (b - a_1, a_{>1}) \in \lambda(A) \);
thus, in the latter case, by Remarks 4.19 and 4.25(1), we have $\int [\lambda_{(2)}^{-1}(s^2)] = [U_2]'[s]/(P - 1)$ for every $s \in \eta$.

It follows that
\[
\int_{\eta}^\eta [\lambda_{(2)} \upharpoonright A_\eta] = \begin{cases}
[U''_2, \text{pr}_{\vartheta_1^2} + \text{pr}_{\vartheta_2^2}]/(P - 1) & \text{if } \vartheta_2 = \eta,
[U'_2 \times \eta^2, \text{pr}_{\vartheta_3^2}]/(P - 1) & \text{if } \vartheta_2 < \eta < \vartheta_3;
\end{cases}
\]

here $U''_2$ may also be used instead of $U_2''$ and the angular component map $\text{pr}_{\vartheta_1^2} + \text{pr}_{\vartheta_2^2}$ is interpreted in the obvious way. On the other hand, let $(U, \overline{ac}) \in \text{RV}_{\vartheta_{(2)}}[\#]$ be the representative of $\int [\lambda_{(2)}]$ that is constructed as in Remark 4.25(1); more precisely, $U = \bigcup_{r \in \vartheta^2} U_r \times (r, r^2)$ and $\overline{ac} : U \rightarrow (\vartheta^2)^\#$ is the obvious coordinate projection. Then
\[
\hat{\Pi}_{\vartheta_{(2)}}([(U, \overline{ac})]) = [(U''_2, \text{pr}_{\vartheta_1^2} + \text{pr}_{\vartheta_2^2})] \quad \text{and} \quad \hat{\Pi}_{\vartheta_{(2)}}([(U, \overline{ac})]) = [(U'_1 \times \vartheta_3^2, \text{pr}_{\vartheta_3^2})].
\]

Although the last term is not quite the same as $\int [\lambda_{(2)} \upharpoonright A_\eta]$ modulo $(P - 1)$ if $\vartheta_2 < \eta < \vartheta_3$, the difference disappears if we apply $\Theta^\# \circ \text{E}^\#$ to both them. So (4.14) follows from Lemmas 4.16 and 4.18.

For the inductive step $i > 1$, let $\tilde{A} = (\text{val} \circ \lambda_{(2)})^{-1}(\vartheta_3)$, which contains $A_\eta$. So $\lambda_{(2)}^{(i+1)}$ restricts to a function $\tilde{A} \rightarrow (\vartheta_{(2+i)}^2)^\#$, which shall be denoted by $\tilde{\lambda}$. Remark 4.25(2) also holds for $\tilde{\lambda}$. If we take $i = 2$ and $A_\eta = \tilde{A}$ in Remark 4.27 then it still goes through, and hence Remark 4.25(1) holds for $\tilde{\lambda}$ as well. In fact, the argument in Remark 4.27 shows that
\[
\int [\tilde{\lambda}] = [(U'_3 \times \vartheta_3^2, \text{pr}_{\vartheta_{(i+1)}^2})]/(P - 1);
\]

here the coordinate projection $\text{pr}_{\vartheta_{(i+1)}^2}$ is again interpreted in the obvious way. Let $(U, \overline{ac}) \in \text{RV}_{\vartheta_{(i+1)}}[\#]$ be as above, but of course with $\vartheta_{(2)}$ replaced by $\vartheta_{(i+1)}$. Then $\int [\lambda_{(i+1)}]$ is equal to $[(U, \overline{ac})]/(P - 1)$ and moreover
\[
\Pi_{\vartheta_{(i+1)}}([(U, \overline{ac})]) = [(U'_{\leq 2} \times \vartheta_3^2, \text{pr}_{\vartheta_{(i+1)}^2})] = [(U'_3 \times \vartheta_3^2, \text{pr}_{\vartheta_{(i+1)}^2})];
\]

recall from (4.6) that here the first $\vartheta_3^2$ in the middle is a bookkeeping device and is not really an object in $\text{RV}[1]$, whereas the second $\vartheta_3^2$ indeed stands for an object in $\text{RV}[1]$.

By the inductive hypothesis, (4.14) holds with $\tilde{\lambda}$ in place of $\lambda$. Since $\lambda_{(i+1)} \upharpoonright A_\eta = \tilde{\lambda}_{(i+1)} \upharpoonright A_\eta$ and $\tilde{\lambda}_{(i+1)}$ is just $\tilde{\lambda}$, we see that, by (4.15), (4.16), and Lemmas 4.16 and 4.18 (more precisely, the equality (4.9) and its implicit analogue in the proof of Lemma 4.18), if $\vartheta_{i+1} = \eta$ then
\[
\text{Vol}^\#([\lambda_{(i+1)} \upharpoonright A_\eta]) = \text{Vol}^\#([\tilde{\lambda}_{(i+1)} \upharpoonright A_\eta])
\]
\[
= (\hat{\psi}_{\vartheta_{(i+1)}} \circ \text{Vol}^\#)([\tilde{\lambda}_{(i+1)}])
\]
\[
= (\hat{\psi}_{\vartheta_{(i+1)}} \circ \Theta^\#_{\vartheta_{(i+1)}} \circ \text{E}^\#_{\vartheta_{(i+1)}} \circ \Pi_{\vartheta_{(i+1)}}([(U, \overline{ac})]/(P - 1))
\]
\[
= (\hat{\psi}_{\vartheta_{(i+1)}} \circ \hat{\psi}_{\vartheta_{(i+1)}} \circ \Theta^\#_{\vartheta_{(i+1)}} \circ \text{E}^\#_{\vartheta_{(i+1)}} \circ \Sigma^\#(U, \overline{ac})]/(P - 1))
\]
\[
= (\hat{\psi}_{\vartheta_{(i+1)}} \circ \text{Vol}^\#([\tilde{\lambda}_{(i+1)}])
\]

and similarly if $\vartheta_{i+1} < \eta < \vartheta_{i+2}$. \hfill $\square$

Next, we show how to manufacture such a function $\lambda$ as assumed in Remark 4.25. The trick is to replace $\phi$ with a large power of itself.
Remark 4.29. Assume $S = C$. Let $A$ be a definable set in $\mathcal{M}$ and $\phi, \psi : A \to \mathcal{M}$ definable functions. Let $\lambda : A \to \mathcal{M}^\ell$ be the function $\phi^N \oplus \bigoplus_{2 \leq i \leq \ell} \psi^m$, where $N \in \mathbb{Z}^+$ is assumed to be sufficiently large.

For each $a \in \mathcal{M}^2$, write $\int [(\phi + \psi)^{-1}(a)] = [U_a]/(P - 1)$. This equality makes sense over the larger substructure $S(a)$. However, unlike in Remark 4.19, $U_a$ does depend on the parameter $a$ now. Anyway, by compactness, there is a definable finite partition $(B_i)_i$ of $\mathcal{M}^2$ such that the objects $U_a$ are defined uniformly over each $B_i$. Since each $\text{val}(B_i) \subseteq (Q^+)^2$ is a cone based at the origin (because $S = C$), there are a $B_i$ and a $p \in Q^+$ such that $\alpha^{22} \times (p\alpha, \infty)^{22} \subseteq B_i$ for all $\alpha \in Q^+$. Let $\xi(x, y, \ldots)$ be a quantifier-free formula that defines the object $U_a$ over $a \in B_i$. Then this $p \in Q^+$ may be chosen so that for any $a = (a_1, a_2) \in B_i$ and every term in $\xi(x, y, \ldots)$ of the form $\text{rv}(F(x, y))$, where $F(x, y) \in C[x, y]$, if $p \text{val}(a_1) < \text{val}(a_2)$ then $\text{rv}(F(a)) = \text{rv}(F(a'))$ for any $a' \in \text{rv}(a)^\sharp$, and hence $U_a = U_{a'}$.

By the choice of $N$, we have $N\alpha > m_2\alpha$ for all $\alpha \in Q^+$. Thus, switching to the function $\lambda$, we see that for all $r = (r_1, r_2, \ldots) \in (RV^\infty)^r$ with $\text{rv}(r_1) \leq \text{rv}(r_2)$,

$$\int [\lambda^{-1}(r^\sharp)] = [U_r][(r, \text{pr} \leq 2)]/(P - 1),$$

where $\int [\lambda^{-1}(a)] = [U_r]/(P - 1)$ does not depend on any choice of $a \in r^\sharp \cap \lambda(A)$. It follows that the restriction of $\lambda$ to $\lambda^{-1}(\varrho^\sharp)$ satisfies the two conditions in Remark 4.25.

In particular, the foregoing discussion may be applied to the functions in (4.11), to which we shall return presently.

4.5. A local Thom-Sebastiani formula. Recall the points $(\alpha_i, \beta_i) \in (Q^+)^2$ and the corresponding line segments $L_i$ from § 4.1. Denote the open interval $\text{pr}_1(L_i)$ by $L'_i$. So if $a \in X^\alpha_{g^N + f(i)}$ then $\text{val}(f(a)) = \alpha_i$ and if $a \in X^-_{g^N + f(i)}$ then $\text{val}(f(a)) \in L'_i$; for each $\alpha \in L'_i$, let $X^-_{g^N + f(i)}$ be the subset of $X^-_{g^N + f(i)}$ determined by the condition $\text{val}(f(a)) = \alpha$.

Notation 4.30. For each $\alpha \in Q^+$, let $\alpha^{(i)} = (m_i \alpha)_{2 \leq i \leq \ell}$ and $\varrho^{(i)} = (m_2 \alpha, \alpha^{(i)})$. Then the set $X^\sharp_{g^N, m_2 \alpha} \cap X^-_{g^N, f(i)}$ may be viewed as the graph of a definable function $X^\sharp_{g^N, f(i), \alpha^{(i)}}$ induced by $g^N \oplus f(i)$, into $(\varrho^{(i)})_{22}$.

Write $\text{Vol}^\bigcirc(\lambda_{g^N + f(i)}, \alpha)$ as $\mathcal{X}^\sharp_{g^N \oplus f(i), \alpha}$. Also, since they are determined by $i$ alone, we may abbreviate $\varrho^{(i)}$, $X^\sharp_{g^N \oplus f(i), \alpha^i}$, $\mathcal{X}^\sharp_{g^N \oplus f(i), \alpha^i}$, etc., further as $\varrho^{(i)}$, $\mathcal{X}^\sharp_{g^N \oplus f(i), \alpha}$, $\mathcal{X}^\sharp_{g^N \oplus f(i), \alpha}$, etc.

Taking $\lambda = X^\sharp_{g^N + f(i)}$ and $\eta = 1$ in Lemma 4.28, we obtain

$$\text{Vol}^\bigcirc([X^\bullet_{g^N + f(i)}]) = \Psi^{(i)}(\mathcal{X}^\sharp_{g^N \oplus f(i)})$$

and, for each $\alpha \in L'_i$, 

$$\text{Vol}^\bigcirc([X^-_{g^N + f(i)}, \alpha]) = \Psi^{(i)}(\mathcal{X}^\sharp_{g^N \oplus f(i), \alpha}).$$

By the discussion in Remark 4.17, the right-hand side of the second equality is the same for any $\alpha \in Q^+$ and hence, in particular, may be written as $\Psi^{(i)}(\mathcal{X}^\sharp_{g^N \oplus f(i)})$. From another perspective, if we write $\int^\bigcirc[X^-_{g^N + f(i), \alpha}] = [U_{\alpha}]/(P - 1)$ then there is an $\alpha$-definable partition of $\text{val}(U_{\alpha})$ of the form $(D_{ka} \times \alpha)_k$, uniform over $Q^+$, such that each $U_{\alpha} \cap (D_{ka} \times \alpha)^\sharp$ is a bipolar twistoid. Thus, $\int^\bigcirc[X^-_{g^N + f(i)}]$ is represented by a finite disjoint union of bipolar twistoids $\mathcal{W}_k \in RV[\ast]$ with
The methodology of \cite{20} offers a geometric interpretation of “sufficiently large” \eqref{4.20} \eqref{4.21} $\Psi$

This implies that, for any sufficiently large $z$

Here

This latter formula may be written as

Corollary 5.16] as follows. In terms of motivic Milnor fibers instead of motivic vanishing cycles, note that the left-hand side of \eqref{4.21} is obviously commutative in the sense that $\Psi$ may now be interpreted as the Euler characteristic of a bounded open interval.

\textbf{Theorem 4.31.} In conclusion, we have derived a local Thom-Sebastiani formula in $K^1 \Var_C$:

$$
\mathcal{F}_{g^N+f(z)} = \mathcal{F}_{g^N}([Z_{f(z)}]) + \sum_{2 \leq i \leq \ell} \mathcal{F}_{f^i}([Z_{g^N+f(z)}]) - \Psi_g(\mathcal{F}_{g^N+f(z)}).
$$

The special case $\ell = 2$ and $m_2 = 1$ is related to the local Thom-Sebastiani formula in \cite[Corollary 5.16]{20} as follows. In terms of motivic Milnor fibers instead of motivic vanishing cycles, this latter formula may be written as

$$
\mathcal{F}_{f,z} - \mathcal{F}_{g^N+f,z} = \Psi_{\Sigma}(\mathcal{F}_{g^N,z}(\mathcal{F}_f)) - \mathcal{F}_{g^N,z}([f^{-1}(0)]).
$$

Here $z \in f^{-1}(0)$ is a $C$-rational point, which is implicit in Theorem 4.31 (recall the simplification made at the beginning of this section). The (local) motivic Milnor fibers $\mathcal{F}_{f,z}$ and $\mathcal{F}_{g^N+f,z}$ are constructed via motivic zeta functions with coefficients in $M^G_m$; see \cite[§3.6]{20} for details. The meaning of the term $\mathcal{F}_{g^N,z}([f^{-1}(0)])$ is established in \cite[Theorem 3.9]{20}, and it belongs to $M^G_m$. According to the nearby cycles formalism of \cite[§4.6]{20}, $\mathcal{F}_{g^N,z}(\mathcal{F}_f)$ belongs to $M^G_m$. But then, after applying the operator $\Psi_{\Sigma}$ as defined in \cite[§5.1]{20}, it comes down to $M^G_m$ as well. In a nutshell, the expression \eqref{4.18} is well-typed.

As we have mentioned above, $\Var^1_C$ is just the category $\Var^G_m$ as defined in \cite[§2.3]{20} and hence, by \cite[Proposition 2.6]{20}, there is a canonical ring isomorphism

$$
\Upsilon : K^1 \Var_C \longrightarrow K^\mu \Var_C;
$$

these two Grothendieck rings, if localized at $[A]$, are denoted by $M^G_m$, $M^\mu$ therein, respectively. It is routine to check that a similar construction via “taking the fiber at $\text{cm}(1)$” also yields an isomorphism $!K \RES \longrightarrow !K^\mu \RES$, which shall also be denoted by $\Upsilon$, and indeed $\Theta^\mu \circ \Upsilon = \Upsilon \circ \Theta^\mu$. Consequently, by \cite[Remak 3.13]{20} and the complex version of Theorem 3.24 (see \cite[Theorem 8.7]{18}), we have

$$
\mathcal{F}_{g^N+f,z} = \mathcal{F}_{g^N+f,z} = \mathcal{F}_{f,z}, \quad \mathcal{F}_{g^N}([Z_{f(z)}]) = \mathcal{F}_{g^N,z}([f^{-1}(0)]).
$$

This implies that, for any sufficiently large $N \in \mathbb{Z}^+$,

$$
\Psi_2(\mathcal{F}_{g^N+f}) = \Psi(\mathcal{F}_{g^N,z}(\mathcal{F}_f)).
$$

The methodology of \cite{20} offers a geometric interpretation of “sufficiently large $N \in \mathbb{Z}^+$” in terms of log-resolutions. Our interpretation lies in the proof of Lemma 4.22 and Remark 4.29, and is not as informative since it depends on compactness. It is not clear how to relate the two thresholds. Also note that the left-hand side of \eqref{4.21} is obviously commutative in the sense that $\mathcal{F}_{g^N+f} = \mathcal{F}_{f,g^N}$, and perhaps this can be translated into an expression on the right-hand side through a resolution-based analysis of the motivic zeta functions involved.

The setup for the motivic Thom-Sebastiani formula in \cite{8} involves a morphism $f'$ on another smooth variety $X'$ and the obvious morphism $f + f'$ on the product $Y = X \times X'$. This formula is a special case of \cite[Corollary 5.16]{20}, as demonstrated in \cite[Theorem 5.18]{20}, and hence can be
recovered from Theorem 4.31 as well, although we do need to check that it holds for $N = 1$ in that situation. Anyway, we can give a more direct proof. To begin with, write (4.11) as

$$
\mathcal{Y}^\sharp_{f + f'} = (\mathcal{X}^\sharp_f \times Z^\prime_f) \cup (\mathcal{X}^\sharp_{f'} \times Z^\prime_f) \cup \mathcal{Y}^+_{f + f'} \cup \mathcal{Y}^-_{f + f'}.
$$

Observe that the conclusion of Remark 4.29 already holds for the function $f \oplus f'$ on $Y(M) = X(M) \times X'(M)$ and indeed

$$
\text{Vol}^\text{df}(\mathcal{Y}^+_{f + f'} \cup \mathcal{Y}^-_{f + f'}) = -\Psi_2(\mathcal{I}^\sharp_{f \oplus f'}) = -\mathcal{I}^\sharp_f \ast \mathcal{I}^\sharp_{f'}.
$$

To compute the other two terms, now symmetric, the key is the following equality.

**Lemma 4.32.** $(E_b \circ f)([Z_f]) = 1$.

*Proof.* We actually show a more general claim: Over $\mathbb{S} = \mathbb{C}$, if $A$ is a definable set in $\mathcal{M}$ then $(E_b \circ f)([A]) = 0$ if $0 \not\in A$ and $(E_b \circ f)([A]) = 1$ otherwise. This is enough since enlarging the language (new parameters, new function symbols, etc.) will not change these equalities.

Since there is no definable point in $\Gamma \cong \mathbb{Q}$ except 0, we see that if $(U, f) \in \text{RES}[k]$ then $U, f(U)$ are just constructible sets in $\mathbb{C}$. Let $A$ be a definable set. Then $f(A)$ may be expressed as a finite sum $\sum U_i \otimes D_i$ modulo $(P - 1)$, where $[U_i] \in \text{K RES}[\ast]$ and $[D_i] \in \text{K} \Gamma[\ast]$. We may assume that either $[D_i] = 1$ (if $D_i \in \Gamma^\text{fin}[\ast]$ then it may be absorbed into $U_i$) or $D_i$ is infinite. In the latter case, for some coordinate projection, say $pr_1$, we may further assume that $pr_1(D_i)$ is $(-\infty, 0)$ or $(0, \infty)$ or $\mathbb{Q} \times 0$ and hence, by $\alpha$-minimality, $E_b(D_i) = 0$.

Thus, to compute $(E_b \circ f)([A])$, we write $f(A)$ as $\sum U_i$ modulo $(P - 1)$. By Theorem 2.27, there is a definable injection $g : \bigcup U_i \rightarrow A$. It is a basic model-theoretic fact that the $\Gamma$-sort is orthogonal to the $k$-sort, which implies that $\text{val}(g([U_i] \otimes D_i))$ is finite and hence only 0 and $\infty$ can occur in its coordinates; in the case we are interested in, that is, $A \subseteq \mathcal{M}_n$ for some $n$, only $\infty$ can occur, but then $A$ must contain the point 0. So $g([U_i] \otimes D_i)$ is either empty or is the singleton 0, which means that $\sum [U_i]$ is either 0 or 1, respectively. Since $(E_b \circ f)([A \times 0])$ also equals 1 or 0, we see that $(E_b \circ f)([A]) = 1$ if and only if $0 \in A$. \hfill \Box

By the same reasoning that leads to (4.17), the class $\int [Z^\prime_f \setminus Z_f]$ is represented by a finite disjoint union of bipolar twistoids $W_i \in \text{RV}[\ast]$ such that $\text{rvr}(W_i)$ is of the form $\bigcup_{\gamma \in (1, \infty)} D_r \gamma \times \gamma$ and $\chi_0(D_r)$ is constant over $(1, \infty)$ for every $i$. So $\text{Vol}^\text{df}([Z^\prime_f \setminus Z_f]) = 0$. So, by Lemmas 4.1 and 4.32, for every $r \in \mathbb{I}^2$,

$$
\text{Vol}^\text{df}((\mathcal{X}^\sharp_f \times Z^\prime_f)^{-1}(r^\ast)) = \text{Vol}^\text{df}([\mathcal{X}^\sharp_f, r][Z^\prime_f \setminus Z_f] + [\mathcal{X}^\sharp_{f, r}][Z_f^\prime]) = \text{Vol}^\text{df}([\mathcal{X}^\sharp_{f, r}]).
$$

In light of the last sentence of Remark 4.19, we deduce that $\text{Vol}^\text{df}([\mathcal{X}^\sharp_f \times Z^\prime_f]) = \mathcal{I}^\sharp_f$ and hence (4.22)

$$
\mathcal{I}^\sharp_{f + f'} = \mathcal{I}^\sharp_f + \mathcal{I}^\sharp_{f'} - \mathcal{I}^\sharp_f \ast \mathcal{I}^\sharp_{f'}.
$$

4.5.1. **The real case.** If we work in the ACVF-model $\tilde{\mathbb{C}}$ with $\mathbb{S} = \mathbb{R} \cup \mathbb{Q}$ and let the variety $X$, the morphism $f$, etc., be defined over $\mathbb{R}$ then the preceding discussion is still valid. In more detail, there is a subgroup of $\text{Gal}(\mathbb{C}(t)/\mathbb{R})$ that may be identified with $\text{Gal}(\mathbb{C}/\mathbb{R}) \times \mathbb{C}^\times$; its preimage along the canonical surjective homomorphism

$$
\text{Gal}(\tilde{\mathbb{C}}/\mathbb{R}) \rightarrow \text{Gal}(\mathbb{C}(t)/\mathbb{R})
$$

is denoted by $c\tau$, which may in turn be identified with $\lim_n(\text{Gal}(\mathbb{C}/\mathbb{R}) \times \mathbb{C}^\times)_n$. There is again an isomorphism $!K^{\text{et}} \text{ RES} \cong K^{\text{et}} \text{ Var}_R$ (for surjectivity, combine the arguments in Remarks 3.12 and 4.6).

The categories in Definition 4.7 and the corresponding Grothendieck groups are now written as $\text{Var}^{\text{et}}_R$ and $K^{\text{et}}_R \text{ Var}_R$. Note that, as in Definition 3.6, for an object $(Y, \pi)$ of $\text{Var}^{\text{et}}_R$, the good
Gal(\mathbb{C}/\mathbb{R}) \times \mathbb{C}^\times$-action on $Y$ and the morphism $\pi : Y \to \mathbb{G}_m^\ell$ are required to be compatible with the antiholomorphic involution in question; in particular, for the generator $c \in \text{Gal}(\mathbb{C}/\mathbb{R})$, the condition (4.1) should read

$$
(4.23) \quad \pi_1(c \cdot y) = c \cdot \pi_1(y) \quad \text{and} \quad \pi^*(c \cdot y) = c \cdot \pi^*(y),
$$

so if $y$ is a real point then $\pi_1(y)$, $\pi^*(y)$ must be real points as well. We can construct a $K^{c_\varphi}$-module homomorphism $\Psi_\varphi : K^0 \text{Var}_R \to K^1 \text{Var}_R$ as in Definition 4.9. Then Theorem 4.31 holds in $K^1 \text{Var}_R$ as well.

However, as in § 3.4, we are more interested in a statement that is indigenous to the real algebraic environment. In addition, we shall point out how to deduce the real Thom-Sebastiani formula in [3] from ours.

Let $K^{\delta}$-\text{Var} be the real analogue of $K^{\delta}$-\text{Var}, that is, the Grothendieck ring of the category of real varieties with weighted good $\mathbb{R}^\times$-actions. A morphism $\pi : Y(\mathbb{R}) \to (\mathbb{R}^\times)^\ell$ on a real variety $Y(\mathbb{R})$ with a good $\mathbb{R}^\times$-action is $(\vartheta, n)$-diagonal if the obvious analogue of (4.1) holds. The categories $\text{Var}^{\vartheta,n}$, $\text{Var}^{\vartheta}$, etc., are defined accordingly. The $K^{\delta}$-\text{Var}-module homomorphism $\Psi_\varphi$ in the bottom row of (4.24) is constructed as in Definition 4.9 again.

Given any $n$-weighted good $c_\varphi$-action $\hbar$ on $Y \otimes_R \mathbb{C}$, by considering the induced $\delta_n$-action in each fiber and the orbit size of each real point as in Definition 3.10, one sees that $\hbar$ gives rise to an $n$-weighted good $\mathbb{R}^\times$-action on $Y(\mathbb{R})$. Consequently, as in (3.5), taking real points yields $\mathcal{A}_C$-module homomorphisms $\Xi^\vartheta$, $\Xi^1$ in (4.24) (also one $K^{c_\varphi}$ $\text{Var}_R \to K^{\delta}$ $\text{Var}_R$).

\begin{equation}
\begin{array}{ccc}
K^0 \text{Var}_R & \xrightarrow{\Psi_\varphi} & K^1 \text{Var}_R \\
\Xi^\vartheta \downarrow \quad & \quad & \Xi^1 \\
K^0 \text{Var}_R & \xrightarrow{\Psi_\varphi} & K^1 \text{Var}_R \\
\end{array}
\end{equation}

By an inductive argument similar to the one in the proof of Lemma 4.16, noting also that, by (4.23), fibers of $\pi$ over genuinely complex points make no contributions to fibers over real points in (4.2) and (4.3), we deduce that the first square of (4.24) indeed commutes. It follows that Theorem 4.31 holds in $K^1 \text{Var}_R$ too, as a direct specialization of the same equality in $K^1 \text{Var}_R$ via $\Xi^\vartheta$ and $\Xi^1$.

This state of affairs may seem somewhat unsatisfactory as the supposedly real formula is in actuality computed from the complex objects in (4.11) and the volume operators $\Theta^{\varphi}_\vartheta \circ \Xi^{\varphi}_{b,\vartheta} \circ \int^\varphi$ over $\hat{\mathbb{C}}$. To remedy this, we can start the specialization procedure earlier, using the technique in § 3.1, as has been done in Remark 3.27, and obtain the same formula using the $\hat{\mathbb{R}}$-trace of (4.11) and the corresponding volume operators over $\hat{\mathbb{R}}$. No new perspective lies herein and hence we shall not labor further on it.

**Remark 4.33.** The second square of (4.24) also commutes, where the two horizontal arrows are constructed via taking the fiber at 1 as in (4.19). However, as another manifestation of the duality of the sign, taking the fiber at $-1$ yields a genuinely different ring homomorphism

$$
\Upsilon^{-1} : K^1 \text{Var}_R \to K^{\mu_2} \text{Var}_R
$$

Neither $\Upsilon^1$ nor $\Upsilon^{-1}$ is injective, not even taken as a pair (for instance any even power function on the torus gives the same class).

Now, the said formula in [3] is formulated in a specialization $\mathcal{M}_{\text{AS}}$ of $K^1 \text{Var}_R[[A]]^{-1}$, which is constructed using arc-symmetric (semialgebraic) sets and maps. In more detail, adapting the
method of [20], the (generalized) real motivic Milnor fiber $\mathcal{J}_f^T$ of $f$ is the limit of a motivic zeta function $Z^\times(T)$ whose coefficients are given by sets of truncated arcs of the form
\[ \{ \varphi \in X(\mathbb{R}[t]/t^{m+1}) \mid f(\varphi) = at^m \mod t^{m+1} \text{ with } a \in \mathbb{R}^\times \text{ and } \varphi(0) = z \} \]
together with the built-in angular component map sending $\varphi$ to $a$. Then an equality similar to the special case (4.22) may be established in $\mathcal{M}_\mathbb{AS}$; see [3, Corollary 6.20]. Here we point out that the process of “taking the limit” forces the $\mathbb{R}^+$-actions on the coefficients of $Z^\times(T)$ to factor through a $\mathbb{R}^+$-action and, consequently, the negative part of $\mathbb{R}^\times$ does not really figure in $\mathcal{J}_f^\times$; this is but another manifestation of what has been said in Remark 3.21 about the construction in [15].

Let us rather consider the same construction at the level of $K^1$ $\text{RVar}$ (hence finer, since full $\mathbb{R}^\times$-actions are retained). In order to show that [3, Corollary 6.20] can be obtained from the specialization of (4.22) to $K^1$ $\text{RVar}$, one needs to check that $\mathcal{J}_f^\times$ can indeed be recovered as $\text{Vol}^\mathbb{R}(\mathcal{A}^\sharp)$ over $\mathbb{R}$, similar to (4.20). We may attempt to reproduce the argument given there. To begin with, taking the fiber at 1 coefficientwise, we recover from $Z^\times(T)$ the motivic zeta function $Z^1(T)$ in (3.13) (taking the fiber at $-1$ gives its negative counterpart $Z^{-1}(T)$), and it is straightforward to check that this operation commutes with the operator $\text{"- limit}_{T \to \infty}$ in (3.15); actually this is just an analogue of [20, Remark 3.13], which we have also gone through in §3.4. However, this is as far as we can go since, unlike $\Upsilon$ in (4.19), $\Upsilon^1$ is not an isomorphism. In other words, although we know that the images of $\mathcal{J}_f^\times$, $\text{Vol}^\mathbb{R}(\mathcal{A}^\sharp)$ under $\Upsilon^1$ coincide in $K^\mu_2$ $\text{RVar}$, we cannot conclude that they themselves coincide in $K^1$ $\text{RVar}$.

Thus the apparent shortcut is blocked in the real environment, and we shall have to revert back to the zeta function point of view, that is, we need to show a version of Theorem 3.24 with respect to $Z^\times(T)$ and $\mathcal{J}_f^\times(\mathbb{R})$. Although some extra care is needed concerning the use of the integral $\int^\circ$, there is no new insight arising in this endeavor and, as above, we choose not to labor further on technicalities.

5. In $T$-convex valued fields

It is also shown in [23, §8] that one can recover, in a localization of $K^\mu \text{Var}_C[\mathbb{A}^{-1}]$, the motivic zeta function and then the motivic Milnor fiber $\mathcal{J}$ of $f$ from its nonarchimedean Milnor fiber $\mathcal{A}$. In [23, Remark 8.5.5], these results yield a proof, without using resolution of singularities but still using other sophisticated algebro-geometric machineries, that the Euler characteristic of $\mathcal{J}$ equals that of the topological Milnor fiber of $f$ (whether finer invariants such as the Hodge-Deligne polynomial can be recovered this way is still unknown). In this section, we aim to prove this equality and its real analogue using a geometric argument at the level of $T$-convex sets instead. Moreover, as is already mentioned in Remark 2.44, in the real environment, the difference between the bounded and the geometric Euler characteristics in the $\Gamma$-sort is manifested as an equality relating the Euler characteristics of the closed and the open topological Milnor fibers.

5.1. The universal additive invariant. We first summarize the main result of [34]. To begin with, let $T$ be a complete polynomially bounded $o$-minimal $L_T$-theory extending the theory RCF of real closed fields. It is not necessary in [34], but here we assume that $R$ is a $T$-model. Let $\mathcal{R} := (R, <, \ldots)$ be a nonarchimedean $T$-model containing $\mathbb{R}$ and $\mathcal{O} \subseteq R$ be the convex hull of $\mathbb{R}$. Then $\mathcal{O}$ is a proper $T$-convex subring of $\mathcal{R}$ in the sense of [12], that is, $\mathcal{O}$ is a convex subring of $\mathcal{R}$ such that, for every definable (no parameters allowed) continuous function $f : R \rightarrow R$, we have $f(\mathcal{O}) \subseteq \mathcal{O}$. According to [12], the theory $T_{\text{convex}}$ of the pair $(\mathcal{R}, \mathcal{O})$, suitably axiomatized in the language $L_{\text{convex}}$ that extends $L_T$ with a new unary relation symbol, is complete. We further assume that $T$ admits quantifier elimination and is universally axiomatizable, which can always
be arranged through definitional extension. Then $T_{\text{convex}}$ admits quantifier elimination too. It also follows that $\mathcal{R}$ is an elementary $\mathcal{L}_T$-substructure of $\mathcal{R}$.

We may also view $\mathcal{R}$ as an $\mathcal{L}_{\text{RV}}$-structure. To construct Hrushovski-Kazhdan style integrals in this environment, however, we need to work with a different language, which extends $\mathcal{L}_{\text{RV}}$. Since $1 + \mathcal{M}$ is a convex subset of $R^\times$, the total ordering on $R^\times$ induces a total ordering on RV. This turns RV into an ordered group and $k$ into an ordered field. By the general theory of $T$-convexity, there is a canonical way of turning $k$ further into a $T$-model, which is isomorphic to the $T$-model $\mathbb{R}$, with the isomorphism given by the residue map $\text{res}$. Let $k^+$ be the set of positive elements of $k$ (similarly for other totally ordered sets with a distinguished element), which forms a convex subgroup of RV.

**Notation 5.1.** Denote the quotient map $RV \to \Gamma := RV / k^+$ by $\text{rvv}$. The composition $\text{val} := \text{vrv} \circ \text{rv} : R^\times \to \Gamma$ is referred to as a *signed* valuation map. The corresponding value group is a “double cover” of the traditional value group. Consequently, the Euler characteristics, still denoted by $\chi_g$ and $\chi_b$, are slightly different from the ones in Notation 2.38.

All of this structure can be expressed in a two-sorted first-order language $\mathcal{L}_{\text{TRV}}$, in which $R$ is referred to as the VF-sort and $RV$ is taken as a new sort. The resulting theory $\text{TCVF}$ (see [34, Definition 2.7]) is complete and weakly o-minimal, and admits quantifier elimination. Informally and for all practical purposes, the language $\mathcal{L}_{\text{TRV}}$ may be viewed as an extension of the language $\mathcal{L}_{\text{convex}}$.

Henceforth we work in the unique (up to isomorphism, of course) $\text{TCVF}$-model $\mathcal{R}_\text{rv}$ that expands the $T_{\text{convex}}$-model $(\mathcal{R}, \mathcal{O})$, with all parameters allowed.

**Example 5.2.** If $T = \text{RCF}$ then we can turn $\mathbb{R}$ into a model of $\text{TCVF}$, with signed valuation, as follows. First note that rv is just the leading term map described in Example 2.1, and we may identify RV with $Q \oplus R^\times$. Then the ordering on RV is the same as the lexicographic ordering on $Q \oplus R^+$ or $Q \oplus R^-$ (but not both of them together due to the issue of sign). The quotient group $\Gamma = (Q \oplus R^\times)/R^+$ is naturally isomorphic to the subgroup $\pm e^Q := e^Q \cup -e^Q$ of $R^\times$, where $e = \exp(1)$, so that $Q$ is identified with $e^Q$ via the map $q \mapsto e^q$. Adding a new symbol $\infty$ to RV, now it is routine to interpret $\mathbb{R}$ as an $\mathcal{L}_{\text{TRV}}$-structure, with the signed valuation given by

$$x \mapsto \text{rv}(x) = (q, a_q) \mapsto \text{sgn}(a_q)e^{-aq},$$

where $\text{sgn}(a_q)$ is the sign of $a_q$. It is also a model of $\text{TCVF}$: all the axioms in [34, Definition 2.7] are more or less immediately derivable from the valued field structure, except (Ax. 7), which holds since RCF is polynomially bounded, and (Ax. 8), which follows from [12, Proposition 2.20].

The categories $\text{VF}[k]$, $\text{RV}[k]$, $\text{RES}[k]$, and $\text{RES}$ are defined as in § 2.2 and § 2.4. Of course all notions are now formulated relative to $\text{TCVF}$, in particular, “definable” means “$\mathcal{L}_{\text{TRV}}$-definable,” and so on. To distinguish them from the previous similar-looking categories, we shall write $\text{TVF}[k]$, $\text{TRV}[k]$, $\text{TRES}[k]$, and $\text{TRES}$ instead.

The $\Gamma$-categories contain subtle differences, though (recall Remark 2.20).

**Definition 5.3 (TT-categories).** The objects of the category $\text{TT}[k]$ are the finite disjoint unions of definable subsets of $\Gamma^k$. Any definable bijection between two such objects is a *morphism* of $\Gamma^k$. The category $\text{TT}^{\text{fin}}[k]$ is the full subcategory of $\text{TT}[k]$ such that $I \in \text{TT}^{\text{fin}}[k]$ if and only if $I$ is finite.

Clearly $\mathbf{K}\text{TT}^{\text{fin}}[k]$ is naturally isomorphic to $\mathbb{Z}$ for all $k$ and hence $\mathbf{K}\text{TT}^{\text{fin}}[\ast] \cong \mathbb{Z}[X]$.

**Lemma 5.4.** Every $\text{TT}[k]$-morphism $g$ is definably a piecewise GL$_k(Q)$-transformation. Consequently, $g$ is a vrv-contraction.
There is still a $\mathbf{K} \mathbb{T} \mathbb{T} \mathbb{F} \mathbb{I}^{\text{fin}}[*]$-linear map

$$\Psi^T : \mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}[*] \otimes_{\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}^{\text{fin}}[*]} \mathbf{K} \mathbb{T} \mathbb{F}[*] \longrightarrow \mathbf{K} \mathbb{F} \mathbb{V}[*],$$

which is an isomorphism of graded rings.

Remark 5.5 (Explicit description of $\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}$). The semiring $\mathbf{K}_+ \mathbb{T} \mathbb{F} \mathbb{I}$ is actually generated by isomorphism classes $[U]$ with $U$ a set in $\mathbb{k}^+$. We have the following explicit description of $\mathbf{K}_+ \mathbb{T} \mathbb{F} \mathbb{I}$. Its underlying set is $(0 \times \mathbb{N}) \cup (\mathbb{N}^+ \times \mathbb{Z})$, where the first coordinate indicates the dimension and the second the $o$-minimal Euler characteristic. For all $(a, b), (c, d) \in \mathbf{K}_+ \mathbb{T} \mathbb{F} \mathbb{I},$

$$(a, b) + (c, d) = (\max\{a, c\}, b + d), \quad (a, b) \times (c, d) = (a + c, b \times d).$$

The dimensional part is lost in the groupification $\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}$ of $\mathbf{K}_+ \mathbb{T} \mathbb{F} \mathbb{I}$, that is, $\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I} \cong \mathbb{Z}$, which is of course much simpler than $\mathbf{K}_+ \mathbb{T} \mathbb{F} \mathbb{I}$. The elements $[1]$, $P$, and $[A]$ in $\mathbf{K} \mathbb{F} \mathbb{V}[*]$, the lifting map $\mathbb{L}$, and the semiring congruence relation $I_{sp}$, are all defined as before.

Proposition 2.40 still holds in the current environment:

**Proposition 5.6 ([34, Proposition 4.24]).** There are two ring homomorphisms

$$E_g^T : \mathbf{K} \mathbb{F} \mathbb{V}[*] \longrightarrow \mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}[[A]]^{-1} \quad \text{and} \quad E_b^T : \mathbf{K} \mathbb{F} \mathbb{V}[*] \longrightarrow \mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}[[1]]^{-1}$$

such that

- their ranges are precisely the zeroth graded pieces $(\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}[[A]]^{-1})_0$, $(\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}[[1]]^{-1})_0$ of their respective codomains, and both of which are canonically isomorphic to $\mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I} \cong \mathbb{Z}$,
- $P - 1$ vanishes under both of them,
- for all $x \in \mathbf{K} \mathbb{T} \mathbb{F} \mathbb{I}[k]$ and all $y \in \mathbf{K} \mathbb{T} \mathbb{F} \mathbf{i}[l]$,

$$E_g^T (x \otimes y) = (-1)^k \chi_g(y)x \quad \text{and} \quad E_b^T (x \otimes y) = (-1)^l \chi_b(y)x,$$

where $x \otimes y$ stands in for the element $(\Psi^T)^{-1}(x \otimes y) \in \mathbf{K} \mathbb{F} \mathbb{V}[*].$

Here we can also write the last two equalities in a form that is not simplified so to make the similarity to Proposition 2.40 apparent (the classes are replaced by their Euler characteristics in the residue field):

$$E_g^T (x \otimes y) = \chi_g(y)x(-1)^l(-1)^{-(k+l)} \quad \text{and} \quad E_b^T (x \otimes y) = \chi_b(y)x(-1)^l1^{-(k+l)}.$$

Note that $-1$ in the expression $(-1)^l$ is the Euler characteristic of the half torus (think $\mathbb{R}^+$), not the torus (think $\mathbb{R}^x$); this is related to the use of signed valuation map, see Notation 5.1. Both $E_g^T$ and $E_b^T$ will be relevant to our construction below.

**Theorem 5.7 ([34, Theorem 5.40]).** For each $k \geq 0$ there exists a canonical isomorphism of semigroups

$$\int_+^T : \mathbf{K}_+ \mathbb{F} \mathbb{V}[k] \longrightarrow \mathbf{K}_+ \mathbb{F} \mathbb{V}[\leq k] / I_{sp},$$

such that $\int_+[A] = [U] / I_{sp}$ if and only if $[A] = [LU]$. Passing to the colimit yields a canonical isomorphism of semirings

$$\int_+^T : \mathbf{K}_+ \mathbb{F} \mathbb{V}[*] \longrightarrow \mathbf{K}_+ \mathbb{F} \mathbb{V}[*] / I_{sp}.$$
Theorem 5.8. There are a generalized Euler characteristic and two specializations to \( \mathbb{Z} \):

\[
\chi_T^g, \chi_b^T : K \text{TVF}_* \xrightarrow{f^T} K \text{TRV}[*]/(P - 1) \xrightarrow{\mathbb{E}_b^T} K \text{TRES} \cong \mathbb{Z}.
\]

Example 5.9. Let us compute the images of \([M]\) under these two generalized Euler characteristics. To begin with, \( \int^T[M] = [1]/(P - 1) \). Since \([1] + [A] = 0\) in \( K \text{TRES}[1] \), we have

\[
\chi_g^T([M]) = [1][A]^{-1} = -1 \in (K \text{TRES}[*][[A]^{-1}])_0.
\]

Similarly \( \chi_b^T([M]) = 1 \) in \( (K \text{TRES}[*][[A]^{-1}])_0 \). Thus \( \chi_g^T([M^+]) = -1 \) and \( \chi_b^T([M^+]) = 0 \). It follows that, for any interval \((0, a]\) with \( a \in \mathbb{R}^+ \), by additivity, we have

\[
\chi_g^T([0, a] \setminus M^+) = 1 \quad \text{and} \quad \chi_b^T([0, a] \setminus M^+) = 0.
\]

Remark 5.10. Denote by \( R \) the underlying henselian valued field of \( R_{rv} \), which is also considered as a substructure of \( U \). Then we can relate the isomorphism in Theorem 5.7 to the “purely algebraic” isomorphism in Theorem 2.27 via the following commutative diagram, extending (3.1) with \( M = R \):

\[
\begin{array}{ccc}
K \text{VF}_R & \xrightarrow{f_R} & K \text{RV}_R[*]/(P - 1) \xrightarrow{\mathbb{E}_b^R} \text{!K RES}_R \\
\downarrow & & \downarrow \\
K \text{TVF}_* & \xrightarrow{f^T} & K \text{TRV}[*]/(P - 1) \xrightarrow{\mathbb{E}_b^T} K \text{TRES}
\end{array}
\]

where the vertical arrows are all induced by the subcategory functors. Of course \( \mathbb{E}_b^R, \mathbb{E}_b^T \) may be replaced by \( \mathbb{E}_g^R, \mathbb{E}_g^T \) and the diagram still commutes; however, as we have pointed out in Remark 3.28, doing so would not extend (3.1) properly (off by a factor).

5.2. Link with the topological Milnor fiber. Denote by \( \text{Def}_T \) the category of \( \mathcal{L}_T \)-definable sets and \( \mathcal{L}_T \)-definable bijections. So \( \text{Def}_T \) is a subcategory of \( \text{TVF}_* \) and we have an induced homomorphism

\[
i : K \text{Def}_T \rightarrow K \text{TVF}._*
\]

Let \( \chi : K \text{Def}_T \rightarrow \mathbb{Z} \) be the \( \alpha \)-minimal Euler characteristic, which is of course an isomorphism; see [11]. On the other hand, \( K \text{Def}_T \) is also canonically isomorphic to \( K \text{TRES} \) (Remark 5.5). Since \( \chi, \chi_g^T \circ i, \) and \( \chi_b^T \circ i \) all agree on the class of the singleton \( \{1\} \), they must be equal.

5.2.1. The real case. In the case \( T = \text{RCF} \), that is, in semialgebraic geometry, the Borel-Moore homology is defined for locally compact semialgebraic sets and satisfies a long exact sequence, which gives rise to an additive (and multiplicative) Euler characteristic \( \chi^{BM} \). It is equal to the Euler characteristic of the singular cohomology with compact supports, also defined only for locally compact semialgebraic sets. One can compute \( \chi^{BM} \) on a cell decomposition, and the formula obtained can be used to extend the definition of \( \chi^{BM} \) to any semialgebraic set; see [6, § 1.8]. Consequently, \( \chi^{BM} \) coincides with \( \chi \). This holds in general for any \( \alpha \)-minimal theory, but we do not know a reference that contains a complete account of it.

Notation 5.11. Let \( X, f, \) and \( z \) be as in § 3.4. Recall that the (positive) closed topological Milnor fiber is instantiated by \( \mathcal{L}_T \)-definable sets (in \( \mathbb{R} \)) of the form

\[
\bar{F}_{a, r} = \{x \in X(\mathbb{R}) \mid \|x - z\| \leq r \text{ and } f(x) = a\}, \quad 0 < a < r \ll 1,
\]
where $\| \cdot \| : \text{VF}^d \to \text{VF}$ denotes the Euclidean norm restricted to $\mathbb{R}$. The (positive) open topological Milnor fiber is similarly instantiated by $L_T$-definable sets $F_{a,r}$, but with $\| x - z \| \leq r$ replaced by $\| x - z \| < r$.

Fix a $t \in \mathcal{M}^+$. For each $r \in \text{VF}^+$, the set $\bar{F}_r$ is defined as $F_{a,r}$, but with $X(\mathbb{R})$ replaced by $X(\text{VF})$ and $a$ by $t$ (since $t$ does not vary anymore, we drop it from the notation); similarly for $F_r$. So $\bar{F}_r$ is the topological closure of $F_r$. Let $\partial \bar{F}_r$ be the boundary of $\bar{F}_r$, that is,

$$\partial \bar{F}_r = \bar{F}_r \setminus F_r = \{ x \in X(\text{VF}) \mid \| x - z \| = r \text{ and } f(x) = t \}.$$ 

Set $F = \bigcap_{r \in \mathcal{U}^+} \bar{F}_r = \bigcap_{r \in \mathcal{U}^+} F_r$, where $\mathcal{U} = \mathcal{O} \setminus \mathcal{M}$, or equivalently,

$$F = \{ x \in X(\mathcal{O}) \mid \| x - z \| \in \mathcal{M} \text{ and } f(x) = t \}.$$ 

Since $\mathcal{O}$ is the convex hull of $\mathbb{R}$, we can also write $F = \bigcap_{r \in \mathcal{R}^+} \bar{F}_r = \bigcap_{r \in \mathcal{R}^+} F_r$. Note that $F$ is definable but is in general not $L_T$-definable.

**Proposition 5.12.** The $o$-minimal Euler characteristic $\chi([F_{a,r}])$ of the closed topological Milnor fiber is equal to $\chi^T([F])$. Similarly, for the open topological Milnor fiber, we have $\chi([F_{a,r}]) = \chi^T([F])$.

The proof essentially consists of the following two lemmas.

**Lemma 5.13.** If $r \in \mathcal{U}^+$ is sufficiently small then, in $K \text{TVF}_*$,

$$[F] = [F_r] - [(0, r) \setminus \mathcal{M}^+] \cup [F_r] = [F_r] - [(0, r) \setminus \mathcal{M}^+] \cup [\partial \bar{F}_r].$$

**Proof.** The second equality is clear. For the first equality, we shall think of the $L_T$-definable subset $A = \bigcup_{r \in \text{VF}^+} r \times \partial \bar{F}_r$ of $\text{VF} \times \text{VF}^+$ as a fibration over $\text{VF}^+$. By $o$-minimal trivialization (see [11, § 9.2.1]), there exists an interval $[a, b] \subseteq \text{VF}^+$ such that the sets $[a, b] \cap \mathcal{M}$, $[a, b] \setminus \mathcal{M}$ are both nonempty and the fibration $A$ is $L_T$-definably trivial over $[a, b]$, that is, there is an $L_T$-definable homeomorphism

$$h : [a, b] \times \partial \bar{F}_b \to \bigcup_{r \in [a, b]} r \times \partial \bar{F}_r,$$

compatible with the projections onto $[a, b]$ in the obvious sense. Now, by additivity, it suffices to compute $[F_b \setminus F]$ in $K \text{TVF}_*$. Since $h$ induces a definable bijection between $\bar{F}_b \setminus F$ and the product $((0, b) \setminus \mathcal{M}^+) \times \partial \bar{F}_b$, the desired equality follows. \hfill $\square$

**Lemma 5.14.** $\chi([F_{a,r}]) = \chi([F_r])$ and $\chi([F_{a,r}]) = \chi([F_r])$.

Note that all the occurrences of $\chi$ here stand for the $o$-minimal Euler characteristic, but on one side of the equality it is taken in $\mathbb{R}$, and in $\mathcal{R}$ on the other side.

**Proof.** Considering $F_{a,r}$ as a definable set in $\mathcal{R}$, it has the same Euler characteristic (since any cell decomposition in $\mathbb{R}$ is also a cell decomposition in $\mathcal{R}$) and, by $o$-minimal trivialization, there is a $t' \in \mathcal{M}^+$ such that $\chi([F_{a,r}]) = \chi([F_{t'}])$, where $F_{t'}$ is defined as $\bar{F}_r$ but with $t$ replaced by $t'$. Since $t, t'$ make the same cut in $\mathcal{R}$, there is an automorphism of $\mathcal{R}$ over $\mathbb{R}$ mapping $\bar{F}_r$ to $\bar{F}_{t'}$. The first equality follows. The second equality is similar. \hfill $\square$

**Proof of Proposition 5.12.** By Lemma 5.14, we may show $\chi^T_b([F]) = \chi([F])$ and $\chi^T_g([F]) = \chi([F])$ instead. This is immediate by Example 5.9 and Lemma 5.13. \hfill $\square$

Recall the (positive) nonarchimedean Milnor fiber $\mathcal{X}_1$ from Notation 3.20, which, in the presence of the Euclidean norm, may now be written as

$$\{ x \in X(\mathcal{O}) \mid \| x - z \| \in \mathcal{M} \text{ and } \text{rv}(f(x)) = \text{rv}(t) \}.$$
**Theorem 5.15.** We can recover the Euler characteristics of the closed and the open topological Milnor fibers by applying $\chi_b^T, \chi_g^T$ to $\mathcal{X}^1$. More precisely,

$$\chi_b^T(\mathcal{X}^1) = \chi([\bar{F}_{a,r}]) \quad \text{and} \quad \chi_g^T(\mathcal{X}^1) = -\chi([F_{a,r}]).$$

This follows from the equality:

**Lemma 5.16.** In $K\text{TRV}[*]/(P - 1)$, $\int^T[\mathcal{X}^1] = [1]\int^T[\mathcal{F}].$

**Proof.** The argument has already been given in Example 3.35. In the current setting, the immediate automorphisms in question are provided by [34, Lemma 2.22]. □

**Proof of Theorem 5.15.** Since $E_b^T([1]) = 1$ and $E_g^T([1]) = -1$, this is immediate by Lemma 5.16 and Proposition 5.12. □

**Remark 5.17.** If $T$ is RCF then we may simply take $\mathcal{R}_{ev} = \bar{\mathcal{R}}$. In that case, composing the three diagrams (3.17), (3.1), (5.1) together, we recover the real motivic Milnor fiber of $f$ as $\text{Vol}_{\bar{\mathcal{R}}}(\mathcal{X}^1)$ (this is not written $\text{Vol}_{\bar{\mathcal{R}}}^T(\mathcal{X}^1)$ as suggested in Remark 3.27 because all parameters are allowed in the current setting, which kills all the $\mu_2$-actions) and its Euler characteristic as $\chi_b^T(\mathcal{X}^1)$. In parallel with [23, Remark 8.5.5], the latter, by the preceding discussion, equals the (Borel-Moore) Euler characteristic of the closed topological Milnor fiber of $f$. This result has been previously obtained in [5, Theorem 4.12], whose method involves heavy dosage of resolution of singularities.

Of course we have also recovered the Euler characteristic of the open topological Milnor fiber of $f$ (up to sign) from $\mathcal{X}^1$ (for other method, see [5, Remark 4.10]), but this happens solely in the $T$-convex environment and, unlike the closed topological Milnor fiber, whether it comes from a motivic object, dual to $\text{Vol}_{\bar{\mathcal{R}}}(\mathcal{X}^1)$ in some sense, or not, is unclear; see Remark 3.28. The following equality might be a faint trace of this perceived duality.

**Corollary 5.18.** $\chi([\bar{F}_{a,r}]) = (-1)^{d+1}\chi([F_{a,r}])$

**Proof.** This is immediate from (2.8) and Theorem 5.15. □

This result has also been obtained in [5, Theorem 4.4]. It would be very interesting to categorify this equality, that is, lifting it to one between homology groups. One conceivable way to do this, as suggested by the work in [17], is to develop a sort of homology theory for definable sets in $\bar{\mathcal{R}}$, or in $\bar{\mathcal{C}}$, which might also shed light on the mystery alluded to in Remark 3.28. The existence of such a theory, however, is purely hypothetical.

**Example 5.19.** Consider the polynomial function $f(x, y) = x^p y^q$ on the affine plane, where $p, q \in \mathbb{Z}^+$, and take $z$ to be the origin. Let $m = \gcd(p, q)$. Without loss of generality, $p/m$ is odd. Then the assignment $(x, y) \mapsto (x^{p/m} y^{q/m}, y)$ gives a definable bijection between $\mathcal{X}^1$ and

$$\{(x, y) \in \mathcal{M}^2 \mid \text{rv}(x^m) = \text{rv}(t) \text{ and } 0 < \text{val}(y) < 1/q\},$$

which means that the integral $\int_{\bar{\mathcal{R}}}[\mathcal{X}^1]/(P - 1)$ works out at

$$[\{x^m = \text{rv}(t)\}] \otimes [0, 1/q]^2] \in K\text{RES}_{\bar{\mathcal{R}}}[1] \otimes K\Gamma_{\bar{\mathcal{R}}}[1].$$

So, in $K\text{RVar}[[A]^{-1}]$, we have

$$\text{Vol}_{\bar{\mathcal{R}},b}(\mathcal{X}^1) = -[G_m][\{x^m = 1\}] \quad \text{and} \quad \text{Vol}_{\bar{\mathcal{R}},g}(\mathcal{X}^1) = -[G_m][\{x^m = 1\}]A^{-2},$$

where the extra letters in the subscripts indicate which Euler characteristic is being used. Set $m' = 1$ if $m$ is odd and $m' = 2$ if $m$ is even. Then $\chi_b^T(\mathcal{X}^1) = 2m'$ is the Euler characteristic of the closed topological Milnor fiber and $-\chi_g^T(\mathcal{X}^1) = -2m'$ is the Euler characteristic of the open topological Milnor fiber.
Proposition 5.20. \(Z\) Proof. Since the obvious map \(X \rightarrow \mathbb{Z}\)
more detail, for each \(m \in \mathbb{Z}\) The complex case.
5.21 Example a by coefficients of the zeta function
which is just the semialgebraic Euler characteristic. Applying this homomorphism termwise to the
coefficients of the zeta function
\[\int_{\mathbb{R}}\chi[^T]\]
\(\text{var}\)
\[\mathbf{K} \mathbf{RVar}[\mathbb{A}^{-1}] \cong [!\mathbf{K} \mathbf{RES}_R[\mathbb{A}^{-1}]] \rightarrow \mathbf{K} \mathbf{TRES} \cong \mathbb{Z},\]
which is just the semialgebraic Euler characteristic. Applying this homomorphism termwise to the
coefficients of the zeta function \(Z_{\text{top}}(T)\) in \(\mathbb{Z}[T]\). This series is, up to sign, the positive topological zeta function considered in [24]. In more
detail, for each \(m \geq 1\), let \(X_m^+\) be the following set of truncated arcs at \(z\):
\[\{\varphi \in X(\mathbb{R}[t]/t^{m+1}) | f(\varphi) = at^m \mod t^{m+1} \text{ with } a \in \mathbb{R}^+ \text{ and } \varphi(0) = z\} \).

Proposition 5.20. \(Z_{\text{top}}(T) = -\sum_{m=1}^{\infty} (-1)^m \chi(X_m^+) \rightarrow \mathbb{Z}[T]\).

Proof. Since the obvious map \(X_m^+ \rightarrow \mathbb{R}^+\) is a trivial fibration by [24, Remark 1.1], we have
\(\chi(X_m^+) = \chi(\mathbb{R}^+) = \chi(X_m^+).\) So the equality follows from (3.14). \(\square\)

5.2.2. The complex case. We may consider the complex geometry of \(\tilde{C}\) over \(\mathbb{C}(t)\) in the TCVF-
model \(\tilde{R}\), since \(\tilde{R}\) may also be viewed as an \(\mathcal{L}_{\text{CVF}}\)-structure. Thus \(\text{VF}_\ast\), \(\text{RV}_\ast\), etc., refer to the
categories in § 2.1 with \(S = \mathbb{C}(t)\) and \(\text{VF}_\ast\), \(\text{RV}_\ast\), etc., refer to the categories in § 3.1 with \(\mathbb{M} = \mathbb{R}\). Also \(X, f\) are defined over \(\mathbb{C}\) and the point \(z\) is \(\mathbb{C}\)-rational.

The new perspective is that, as an \(\mathcal{L}_{\text{CVF}}\)-structure, there is an obvious interpretation, in the
model-theoretic sense, of \(\tilde{C}\) in \(\tilde{R}\); this is just a fancy way to say that, after fixing a square root
\(\sqrt{-1}\) of \(-1\), \(\tilde{C}\) may be identified with \(\mathbb{R}^2, \mathbb{C}(t)\) with \(\mathbb{R}(t)^2\), \(\mathbb{R}(C)\) with \(\mathbb{R}(\mathbb{R})^2\), and so on.

For convenience, we shall call \(\tilde{C}\) a complex field, \((\tilde{R}, 0) \subseteq \tilde{C}\) the real line in \(\tilde{C}\), and \((0, \tilde{R}) \subseteq \tilde{C}\) the
imaginary line in \(\tilde{C}\).

Example 5.21. We think of \(c \in \tilde{C}\) as \(a + \sqrt{-1}b\) but write it simply as a pair \((a, b) \in \tilde{R}^2\); also denote
\(a\) by \(\Re c\) and \(b\) by \(\Im c\). Let \(g, h \in \mathbb{C}[x_1, \ldots, x_n]\). Then the definable set
\(\{c \in \tilde{C}^n | \text{var } f(c) \leq \text{var } g(c)\} =: \{\text{var } f \leq \text{var } g\}\)
can also be described as the union of the following two subsets of \(\tilde{R}^{2n}\):
\(\{\text{var } \Re f \leq \text{var } \Re g\} \cap \{\text{var } \Re f \leq \text{var } \Im g\}, \{\text{var } \Im f \leq \text{var } \Re g\} \cap \{\text{var } \Im f \leq \text{var } \Im g\}\).

Since \(\tilde{C}\) is now interpreted in \(\tilde{R}\), there is an induced faithful functor \(\text{VF}_\ast \rightarrow \text{VF}_{\tilde{R}}\), which in
turn yields a homomorphism \(\mathcal{O} : \mathbf{K} \text{VF}_\ast \rightarrow \mathbf{K} \text{VF}_{\tilde{R}}\).

Remark 5.22. For the pair of RV-categories \(\text{RV}_\ast\) and \(\text{RV}_{\tilde{R}}\), although a similar functor
is available, we need to be more careful since these categories are graded.

To illustrate the concern, consider the object \(\text{RV}_{\infty}(\tilde{C}) = \text{RV}_{\infty}(\tilde{R})^2\). Since the real and the
imaginary lines have only one nonzero coordinate, this object has nonempty components in all of
the three categories \(\text{RV}_{\tilde{R}}[0], \text{RV}_{\tilde{R}}[1], \text{and RV}_{\tilde{R}}[2]\). This interpretation leads to an issue since,
for instance, the complex points \((1, 0)\) and \((1, 1)\) should certainly be isomorphic objects, but they
cannot be since they do not even belong to the same graded piece.

To resolve this issue, we can work with a dimension-free version of the Grothendieck ring \(\mathbf{K} \text{RV}_\ast\),
namely the zeroth graded piece \((\mathbf{K} \text{RV}_\ast[[[1]]^{-1}])^0 \subseteq \mathbf{K} \text{RV}_\ast[[[1]]^{-1}]\). This ring is indeed isomorphic to
\(\mathbf{K} \text{RV}_\ast\) (see Definition 2.11), for the same reason that \((\mathbf{K} \text{RES}_\ast[[1]]^{-1})^0 \subseteq \mathbf{K} \text{RES}_\ast\); see Remark 2.41. The obvious forgetful functor induces an epimorphism
\(\mathbf{K} \text{RV}_\ast \rightarrow \mathbf{K} \text{RV}_\ast\). The pushforward ideal of \((\mathbf{P} - 1)\) along this epimorphism is still denoted as such. It follows from the
construction of \(\mathbb{E}_0\) that there is a homomorphism
\(\mathbb{E}_0^\ast : \mathbf{K} \text{RV}_\ast/(\mathbf{P} - 1) \rightarrow !\mathbf{K} \text{RES}\).
whose composition with the epimorphism

\[ \mathbf{K} \mathbf{R} \mathbf{V}^*[\ast]/(P - 1) \longrightarrow \mathbf{K} \mathbf{R} \mathbf{V}^*/(P - 1) \]

is \( E_b \). All this also applies to the \( \mathbf{R} \)-constructible Grothendieck rings \( \mathbf{K} \mathbf{R} \mathbf{V}^*[\ast], \mathbf{K} \mathbf{R} \mathbf{V}^\mathbf{R}, \) and \( !\mathbf{K} \mathbf{R} \mathbf{E} \mathbf{S} \mathbf{R} \). The corresponding homomorphism is denoted by \( E_{b,R}^* \).

Since \( 1 - [\mathbf{R} \mathbf{V}^\infty(\mathbf{C})] = 1 - [\mathbf{R} \mathbf{V}^\infty(\mathbf{R})]^2 \) in \( \mathbf{K} \mathbf{R} \mathbf{V}^\mathbf{R} \), it is clear that the ideal \( (P - 1) \) of \( \mathbf{K} \mathbf{R} \mathbf{V}^* \) is included in the eponymous ideal of \( \mathbf{K} \mathbf{R} \mathbf{V}^\mathbf{R} \).

Since \( 1 \mid RV^\infty(\mathbf{C}) \) is defined as in Notation 5.11, but with \( RV^\ast(\mathbf{C}) \) and \( RV^\ast(\mathbf{R}) \), we still have, for all sufficiently small \( r \in U^+ \),

\[ [\mathcal{F}] = [\mathcal{F}_r] - [(0, r) \times \mathcal{M}^+](\mathbf{C})[\partial \mathcal{F}_r]. \]

Thus \( \chi^T_b([\mathcal{F}]) = \chi([\mathcal{F}_r]) \). On the other hand, since \( \chi(\partial \mathcal{F}_r) = \chi(\partial F_{a,r}) = 0 \) (the smooth compact complex manifold \( \partial F_{a,r} \) is of odd dimension), \( \chi^T_g([\mathcal{F}]) = \chi([\mathcal{F}_r]) \) as well. Now, observe that, in the present setting,

\[ \int^T [\mathcal{X}] = [1]^2 \int^T [\mathcal{F}] \in \mathbf{K} \mathbf{R} \mathbf{V}^*[\ast]/(P - 1), \]
and hence it makes no difference which one of the generalized Euler characteristics $\chi^T_a$, $\chi^T_b$ is used to relate $X$ and $F$. We recover thus the result in [23, Remark 8.5.5]:

**Theorem 5.24.** The Euler characteristic of the topological Milnor fiber of $f$ equals $\chi_C([X])$.

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