A generating function approach to Markov chains undergoing binomial catastrophes

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Abstract. In a Markov chain population model subject to catastrophes, random immigration events (birth), promoting growth, are in balance with the effect of binomial catastrophes that cause recurrent mass removal (death). Using a generating function approach, we study two versions of such population models when the binomial catastrophic events are of a slightly different random nature. In both cases, we describe the subtle balance between the two birth and death conflicting effects.

Keywords: exact results, growth processes, population dynamics, stochastic processes

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1. Introduction, motivations and background

In a simple growth process, some organisms change the amount of their constitutive cells at random as follows. In the course of its lifetime, the organism alternates between random busy and idle periods. In a busy growth period, it produces a random or fixed number of new cells, which are being added to its current stock of cells (e.g. the incremental or batch cells). In an idle catastrophic period, the organism is at risk of being subject to, e.g. virus/bacteria attacks or radiation, resulting in each of its constitutive cells being either lysed or lost after mutations with some fixed mortality probability, independently of each other. Under such a catastrophic event, the population size is thus reduced according to a binomial distribution with survival probability, e.g. $c$; hence the name binomial catastrophes. It will happen so that the current size of the organism is reduced to zero at some random time. In a worst disaster scenario for instance, all cells can be lysed in a single idle period leading instantaneously to a first disastrous extinction event. From a first extinction event, the organism can then either recover, taking advantage of a subsequent busy epoch and starting afresh from zero, or not, being stuck at zero for ever. In the latter case, we shall speak of a local extinction, state zero being reflecting, while in the latter case eventual or global extinction is at stake, state zero being absorbing.

As just described in words, the process under concern turns out to be a Markov chain (MC) on the non-negative integers, displaying a subtle balance between generalized birth and death events. Whenever it is ergodic, there are infinitely many local extinctions and the pieces of sample paths separating consecutive passages to zero are called excursions, the height and length of which are of some relevance in the understanding of the population size evolution. Excursions indeed form independent and identically distributed (i.i.d.) blocks of this MC.
Stochastic models subject to catastrophes have a wide application in different fields, namely bioscience, marketing, ecology, computer and natural sciences, etc. For instance,

- In recent years, the study of catastrophe models has attracted much attention due to their wide application in computer-communication networks and digital telecommunications systems, where interruptions due to various types of virus attacks are referred to as catastrophes. Here, unfriendly events, like virus attacks, result in abrupt changes in the state of the system caused by the removal of some of its elements (packets), posing a major threat to these queueing systems. Continuous-time queueing models subject to total disasters have also been analyzed by a few researchers, such as [5, 6]. A detailed survey of catastrophic events occurring in communication networks has been carried out in [11].

- In oversimplistic physics models of earthquakes, the state of the process may be viewed as the total energy embodied in the Earth’s crust system, obtained as the sum of the energies of its constitutive blocks, each carrying, e.g. a fixed unit quantum of energy. A busy period corresponds to an accumulation of energy epoch in the system, while an idle period yields a stress release event. More realistic growth/collapse or decay/surge models in the same spirit, although in the continuum, were considered in [15–17], with physical applications in mind, including stress release mechanisms in the physics of earthquakes.

- In forestry, good management of the biomass depends on the prediction of how steady growth periods alternate with tougher periods, due to the occurrence of cataclysms, such as hurricanes, droughts or floods, hitting each tree in a similar way.

- In an economy, pomp periods often alternate with periods of scarcity when a crisis equally strikes all the economic agents (the Joseph and Noah effects).

Some aspects of related catastrophe models were recently addressed in [1, 4, 8, 13, 14], chiefly in continuous times. For instance, Economou [13] analyzed a continuous-time binomial catastrophe model wherein individuals arrive according to a compound Poisson process, while catastrophes occur according to a renewal process. He obtained the distribution of the population size at post-catastrophe, arbitrary and pre-catastrophe epochs. In [1], a population model with the immigration process subject to binomial and geometric catastrophes has been investigated; the authors obtained the removed population size and maximum population size between two extinctions (height of excursions). They also discussed the first extinction time and the survival time of a tagged individual. In [14] also, several approaches for the transient analysis of a total disaster model were introduced, with an extension of the methodologies to the binomial catastrophe model. The authors of [23] carried out the transient analysis of a binomial catastrophe model with birth/immigration-death processes and obtained explicit expressions for factorial moments, which are further used to develop the population size distribution. Results for the equilibrium moments and population size distributions are also given. Finally, recently, Yajima et al. [38] obtained the stationary queue length of the $M^X/M/\infty$ queue with binomial catastrophes in the heavy traffic system, using a central limit theorem.

Let us finally mention that not all aspects of binomial catastrophes are included in the above formulation. In [35], a density-dependent catastrophe model of a threatened population is investigated. Density-dependence is shown here to have a significant effect.
on population persistence (mean time to extinction), with a decreasing mean persistence
time at large initial population sizes and causing a relative increase at intermediate sizes.
A density-dependent model involving total disasters was designed in [21].

Discrete-time random population dynamics with catastrophes balanced by random
growth has a long history in the literature, starting with [25]. Mathematically, binomial
catastrophe models are MCs, which are random walks on the non-negative integers (as
a semigroup), so differ from standard random walks on the integers (as a group) in that
a one-step move down from some positive integer cannot take one to a negative state,
resulting in transition probabilities being state-dependent [12]. Such MCs may thus be
viewed as generalized birth and death chains. The transient and equilibrium behavior of
such stochastic population processes with either disastrous or mild binomial catastrophes
is one of the purposes of this work. We aim to study the equilibrium distribution of
this process and derive procedures for its approximate computation. Another issue of
importance concerns the measures of the risk of extinction, first extinction time, time
elapsed between two consecutive extinction times and the maximum population size
reached in between.

The detailed structure of the manuscript, attempting to realize this program, can be
summarized as follows:

• Section 2 is designed to introduce the model in probabilistic terms. We develop three
  particular important cases:

  * Survival probability \( c = 1 \). In this case, at a catastrophic event, the chain remains
    in its current state with no depletion of cells at all.
  * Survival probability \( c = 0 \). This is a case of total disasters for which, at a
    catastrophic event, the chain is instantaneously propelled to state 0.
  * The semi-stochastic growth/collapse scenario when the adjunction of incremental
    cells, at a growth event, is deterministic, being reduced to a single element, and
    \( c \neq \{0,1\} \) so that binomial mortality is not degenerate.

• In section 3, we discuss the conditions under which the chain is recurrent (positive
  or null) or transient. We emphasize that positive recurrence is generic, unless the
  batch random variables (rvs) have unrealistic very heavy tails. We use a generating
  function approach, which is well-suited to this analysis. When, as in the positive
  recurrent case, it is non-trivial, we discuss the shape of the invariant probability
  mass function (pmf) of the chain. Specifically,

  * When \( c \in [0,1) \), we show that the invariant pmf is the one of a compound Poisson
    (infinitely divisible (ID)) rv, which is moreover discrete self-decomposable (SD).
    As a result, it is unimodal, and we give conditions under which the mode is located
    at 0. We refer to [31] for the notions of discrete infinite divisibility (compounding
    Poisson), self-decomposability and unimodality.
  * When \( c = 0 \), (total disasters), the invariant pmf is the one of a geometric sum of
    the incremental (batch) rvs, so in the compound Poisson class, but not necessarily
    SD. We give some sufficient conditions under which it is SD, so unimodal and
    then with the mode necessarily at the origin.
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- In section 4, we analyze some important characteristics of the chain in the positive recurrent regime, making use of its Green (potential) kernel, including:
  - From the Green (potential) kernel at $(0, 0)$: the contact probability at state 0 and first return time to 0 (length of an excursion).
  - From the Green (potential) kernel at $(x_0, 0)$: the first extinction time at 0 when starting from $x_0 > 0$. We show that this rv has geometric tails. Some representation results on the mean (persistence) time to extinction are supplied.
  - From the Green (potential) kernel at $(x_0, y_0)$: first hitting time of state $y_0$ when starting from $x_0$.
  - Average position of the chain started at $x_0 > 0$ (transient analysis).
  - Height of an excursion making use of the idea of the scale or harmonic sequence of the chain.

- In section 5, we sketch some brief insights into the way the parameters of the chain could be estimated from an $N-$sample of observations.

- In section 6, we study an absorbed version of the binomial catastrophe chain in view of further analyzing the quasi-distribution of the process conditioned on being currently not extinct.

- Finally, a variant of the binomial catastrophe model is introduced and studied in section 7. It was motivated by the following observation: suppose at each step of its evolution, the size of an organism either grows by a random number of batch cells (not reduced to a fixed number) or shrinks deterministically by only one unit (a semi-stochastic decay/surge scenario ‘dual’ to the semi-stochastic growth/collapse one). The above standard binomial catastrophe model is not able to represent a scenario where at least one individual is removed from the population at catastrophic events. To remedy this, we therefore define and analyze a variant of the standard binomial model where, at a catastrophic event, the binomial shrinking mechanism applies to all currently alive cells but one, which is systematically removed. We also take control of the future of the population once it hits state zero. Despite the apparent small changes in the definition of this modified binomial catastrophe MC, the impact on its asymptotic behavior is shown to be drastic.

2. The binomial catastrophe model

We first describe the model as a toy model one.

2.1. The model

Consider a discrete-time MC $X_n$ taking values in $\mathbb{N}_0 := \{0, 1, 2, \ldots \}$. With $b_n (c)$, $n = 1, 2, \ldots$ an independent i.i.d. sequence of Bernoulli rvs with success parameter $c \in (0, 1)$, let $c \circ X_n = \sum_{m=1}^{X_n} b_m (c)$ denote the Bernoulli thinning of $X_n$.

In words, the thinning operation acting on a discrete rv is the natural discrete analog of scaling a continuous variable, i.e. multiplying it by a constant in $[0, 1]$.

\[ \text{https://doi.org/10.1088/1742-5468/abdfcb} \]
concern here are a balance between birth and death events according to \((p + q = 1)\):

\[
X_{n+1} = 1 \circ X_n + \beta_{n+1} = X_n + \beta_{n+1} \text{ with probability } p
\]

\[
X_{n+1} = c \circ X_n \text{ with probability } q = 1 - p.
\]

This model was considered by \([2, 25]\). We have \(c \circ X_n \sim \text{bin}(X_n, c)\) hence the name binomial catastrophe. The binomial effect is appropriate when, at a catastrophic event, the individuals of the current population each die (with probability \((wp) 1 - c\)) or survive (wp \(c\)) in an independent and even way, resulting in a drastic depletion of individuals at each step. Owing to: \(c \circ X_n = X_n - (1 - c) \circ X_n\), the number of stepwise removed individuals is \((1 - c) \circ X_n \text{ wp } q\). This way of depleting the population size (at shrinkage times) by a fixed random fraction \(c\) of its current size is very drastic, especially if \(X_n\) happens to be large. Unless \(c\) is very close to 1, in which case depletion is modest (the case \(c = 1\) is discussed below), it is very unlikely that the size of the upward moves will be large enough to compensate depletion while producing a transient chain drifting at \(\infty\). We will make this very precise below.

Note also \(X_n = 0 \Rightarrow X_{n+1} = \beta_{n+1} \text{ wp } p\), \(X_{n+1} = 0 \text{ wp } q\) (translating reflection of \(X_n\) at 0 if \(q > 0\)).

Let \(B_n(p)\), \(n = 1, 2, \ldots\) be an i.i.d. sequence of Bernoulli rvs with \(P(B_1(p) = 1) = p\), \(P(B_1(p) = 0) = q\). Let \(C_n(p,c) = B_n(p) + cB_n(p) = cP_n(p)\), where \(P_n(p) = 1 - B_n(p)\).

The above process’s dynamics (driven by \(B_n(p)\beta_n\)) are compactly equivalent to

\[
X_{n+1} = cP_{n+1}(p) \circ X_n + B_{n+1}(p) \beta_{n+1}, \quad X_0 = 0
\]

\[
= B_{n+1}(p) \cdot c \circ X_n + B_{n+1}(p) (X_n + \beta_{n+1}),
\]

\((X_n, \beta_{n+1})\) being mutually independent. The thinning coefficients are now \(cP_{n+1}(p)\), so random.

With \(b_x := P(\beta = x)\), \(x \geq 1\) and \(d_{x,y} := \binom{x}{y} c^y (1 - c)^{x-y}\) the binomial pmf, the one-step-transition matrix \(P\) of the MC \(X_n\) is given by:

\[
P(0, 0) = q, \quad P(0, y) = pP(\beta = y) = pb_y, \quad y \geq 1
\]

\[
P(x, y) = q \left( \begin{array}{c} x \\ y \end{array} \right) c^y (1 - c)^{x-y} = qd_{x,y}x \geq 1 \text{ and } 0 \leq y \leq x
\]

\[
P(x, y) = pP(\beta = y - x) = pb_{y-x}x \geq 1 \text{ and } y > x.
\]

If \(\beta\) has first and second moment finite, with \(\overline{c} = 1 - c\), \(c \circ x \sim \text{bin}(x, \overline{c})\), as \(x\) gets large

\[
m_1(x) = \mathbb{E}((X_{n+1} - X_n) | X_n = x) = p\mathbb{E}(\beta) - q\overline{c}x \sim -q\overline{c}x
\]

\[
m_2(x) = \mathbb{E}((X_{n+1} - X_n)^2 | X_n = x) = p\mathbb{E}(\beta^2) + q(\overline{c}x + \overline{c}^2x(x - 1)) \sim q\overline{c}^2x^2
\]

with

\[
\frac{m_1(x)}{m_2(x)} \sim -\frac{1}{\overline{c}x}, \quad x \text{ large}.
\]

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Note the variance of the increment is
\[ \sigma^2 (X_{n+1} - X_n | X_n = x) = p \sigma^2 (\beta) + q c \bar{e} x \sim q c \bar{e} x. \]

2.2. Special cases

(a) When \( c = 1 \), the lower triangular part of \( P \) vanishes leading to
\[
P (0, 0) = q, \ P (0, y) = p P (\beta = y) = pb_y, \ y \geq 1 \\
P (x, y) = 0, \ x \geq 1 \text{ and } 0 \leq y < x, \ P (x, x) = q, \ x \geq 1 \\
P (x, y) = p P (\beta = y - x) = pb_{y-x}, \ x \geq 1 \text{ and } y > x.
\]

The transition matrix \( P \) is upper-triangular with diagonal terms. The process \( X_n \) is non-decreasing, so it drifts to \( \infty \).

(b) When \( c = 0 \) (total disasters),
\[
P (0, 0) = q, \ P (0, y) = p P (\beta = y) = pb_y, \ y \geq 1 \\
P (x, y) = 0, \ x \geq 1 \text{ and } 0 \leq y < x, \ P (x, 0) = q, \ x \geq 1 \\
P (x, y) = p P (\beta = y - x) = pb_{y-x}, \ x \geq 1 \text{ and } y > x.
\]

When a downward move occurs, it instantaneously takes \( X_n \) to zero (a case of total disasters), independently of the value of \( X_n \). This means that, defining \( \tau_{x_0,0} = \inf (n \geq 1 : X_n = 0 | X_0 = x_0) \), the first extinction time of \( X_n \), \( P (\tau_{x_0,0} = x) = qp^{x-1} \), \( x \geq 1 \), a geometric distribution with success parameter \( q \), with mean \( E (\tau_{x_0,0}) = 1/q \), independently of \( x_0 \geq 1 \). Note that \( \tau_{0,0} \), as the length of any excursion between consecutive visits to 0, also has a geometric distribution with the success parameter \( q \) and finite mean \( 1/q \). In addition, the height \( H \) of an excursion is clearly distributed like \( \sum_{x=1}^{\tau_{x_0,0}-1} \beta_x \) (with the convention \( \sum_{x=1}^{0} \beta_x = 0 \)). Finally, this particular MC clearly is always positive recurrent, whatever the distribution of \( \beta \). Consecutive excursions are the i.i.d. pieces of this random walk on the non-negative integers.

Some Markov catastrophe models involving total disasters are described in [21, 33].

(c) If \( \beta \sim \delta_1 \) a move up results in the addition of only one individual, which is the simplest deterministic drift upwards. In this case, the transition matrix \( P \) is lower-Hessenberg. This model constitutes a simple discrete version of a semi-stochastic growth/collapse model in the continuum [15].

3. Recurrence versus transience

Using a generating function approach, we start with the transient analysis before switching to the question of equilibrium.

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3.1. The transient analysis

Let \( B(z) = \mathbb{E}(z^\beta) \) be the probability generating function (pgf) of \( \beta = \beta_1 \), as an absolutely monotone function on \([0,1]\)^2. Let \( \pi'_n = (\pi_n(0), \pi_n(1), \ldots) \), where \( \pi_n(x) = \mathbf{P}_0(X_n = x) \), and \( \pi' \) denotes the transposition. With \( z = (1, z, z^2, \ldots) \), a column vector obtained as the transpose \( \pi' \) of the row vector \((1, z, z^2, \ldots)\), define

\[
\Phi_n(z) = \mathbb{E}_0(z^{X_n}) = \pi'_n z,
\]

the pgf of \( X_n \). With \( D_z = \text{diag}(1, z, z^2, \ldots), z = D_z \mathbf{1} \), the time evolution \( \pi'_{n+1} = \pi'_n P \) yields

\[
\Phi_{n+1}(z) = \pi'_{n+1} z = \pi'_n P z = \pi'_n P D_z \mathbf{1},
\]

leading to the transient dynamics

\[
\Phi_{n+1}(z) = pB(z) \Phi_n(z) + q\Phi_n(1-c(1-z)), \Phi_0(z) = 1. \tag{2}
\]

The fixed point pgf of \( X_\infty \), if it exists, solves

\[
\Phi_\infty(z) = pB(z) \Phi_\infty(z) + q\Phi_\infty(1-c(1-z)). \tag{3}
\]

(a) When \( c = 1 \), there is no move down possible. The only solution to \( \Phi_\infty(z) = pB(z) \Phi_\infty(z) + q\Phi_\infty(z) \) is \( \Phi_\infty(z) = 0 \), corresponding to \( X_\infty \sim \delta_\infty \). Indeed, when \( c = 1 \), combined with \( \Phi_\infty(1) = 1 \),

\[
\Phi_{n+1}(z) = (q + pB(z)) \Phi_n(z), \Phi_0(z) = 1
\]

\[
\Phi_n(z)^{1/n} = q + pB(z),
\]

showing that, if \( 1 \leq \rho := B'(1) = \mathbb{E}(\beta) < \infty \), \( n^{-1}X_n \to q + \rho \rho \geq 1 \) almost surely as \( n \to \infty \). The process \( X_n \) is transient in that, after a finite number of passages in state 0, it drifts to \( \infty \).

(b) When \( c = 0 \) (total disasters), combined with \( \Phi_\infty(1) = 1, \) (3) yields

\[
\Phi_\infty(z) = \frac{q}{1-pB(z)} = : \phi_\Delta(z), \tag{4}
\]

as an admissible pgf solution. In (4), we introduced the integral-valued rv \( \Delta \) whose pgf is \( \phi_\Delta(z) = \mathbb{E}(z^\Delta) = q/(1-pB(z)) \) obtained while compounding a shifted-geometric pgf \( q/(1-pz) \) with the pgf \( B(z) \) of the \( \beta \)'s. We conclude that in the total disaster setup when \( c = 0 \), the law of \( X_\infty \) is obtained as a compound shifted-geometric sum \( \Delta \) of the \( \beta \)'s, whatever the distribution of \( \beta \). Note that the \( \Delta \) clearly is the height of any total disaster excursion, as the sample path between any two consecutive visits of \( X_n \) to 0. The length of such excursions clearly is geometric with success probability \( q \), in that case.

---

2 A function \( B \) is said to be absolutely monotone on \((0,1)\) if it has all its derivatives \( B^{(n)}(z) \geq 0 \) for all \( z \in (0,1) \). Pgfs are absolutely monotone and the composition of two pgfs is a pgf.

3 A geometric \( q \) rv with success probability \( q \) takes values in \( \mathbb{N} = \{1,2,\ldots\} \). A shifted geometric \( (q) \) rv with success probability \( q \) takes values in \( \mathbb{N}_0 = \{0,1,2,\ldots\} \). It is obtained while shifting the former one by one unit.

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3.2. Existence and shape of an invariant pmf \((c \in [0, 1])\)

We shall distinguish two cases.

• The case \(c \in (0, 1)\).

From (3) and (4), the limit law pgf \(\Phi_\infty(z)\), if it exists, solves the functional equation

\[
\Phi_\infty(z) = \phi_\Delta(z) \Phi_\infty(1 - c(1 - z)),
\]

so that, formally

\[
\Phi_\infty(z) = \prod_{n \geq 0} \phi_\Delta(1 - c^n(1 - z)),
\]

as an infinite product pgf.

**Proposition.** The invariant measure exists for all \(c \in (0, 1)\) if, and only if, \(E(\log^+\Delta) < \infty\).

**Proof (theorem 2 in [25]):** By a comparison argument, we need to check the conditions under which \(\pi(0) = \Phi_\infty(0)\) converges to a positive number. We get

\[
\Phi_\infty(0) = \prod_{n \geq 0} \phi_\Delta(1 - c^n) > 0 \iff \sum_{n \geq 0} (1 - \phi_\Delta(1 - c^n)) < \infty
\]

\[
\iff \int_0^1 \frac{1 - \phi_\Delta(z)}{1 - z} \, dz < \infty \iff \sum_{x \geq 1} \log x P(\Delta = x) = E(\log^+\Delta) < \infty,
\]

meaning that \(\Delta\) has a finite logarithmic first moment.

For most \(\beta\)s, therefore, the process \(X_n\) is positive recurrent, in particular if \(\beta\) has a finite mean.

When \(\beta\) has finite first and second order moments, so do \(\Delta\) and \(X_\infty\) which exist. Indeed:

If \(B'(1) = E\beta = \rho < \infty\), (with \(E\Delta = (pp)/q\))

\[
\Phi'_\infty(1) = q \left( \frac{pp}{q^2} + \frac{c}{q} \phi'_\infty(1) \right) \Rightarrow \Phi'_\infty(1) = E(X_\infty) =: \mu = \frac{pp}{q(1 - c)} < \infty.
\]

If \(B''(1) < \infty\), \(X_\infty\) has finite variance:

\[
\Phi''_\infty(1) - \Phi'_\infty(1)^2 = \frac{p}{q} \frac{1}{1 - c^2} \left( B''(1) + 2 \frac{p}{q} \frac{2c}{1 - c} \mu^2 \right).
\]

Counter-example: with \(\beta, C > 0\), suppose that \(P(\beta > x) \sim x^\beta \cdot C(\log x)^{-\beta}\) translating that \(\beta\) has very heavy logarithmic tails (any other than logarithmic slowly varying function would do the job as well). Then \(E\beta^q = \infty\) for all \(q > 0\) and \(\beta\) has no moments of arbitrary positive order. Equivalently, \(B(z) \sim x \frac{1 - C}{(-\log(1 - z))^\beta}\). Therefore, with \(C' = pC/q\),

\[
\phi_\Delta(z) = \frac{q}{1 - pB(z)} \sim z \frac{1 - C'}{(-\log(1 - z))^\beta}.
\]

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translating that $P(\Delta > y) \sim y^{-\beta}C'(\log y)^{-\beta}$ shares the same tail behavior as $\beta$. From this, $P(\log \Delta > x) \sim y^{-\beta}C'x^{-\beta}$ so that $\log \Delta$ has a first moment if, and only if, $\beta > 1$. For such a (logarithmic tail) model of $\beta$, we conclude that $X$ remains positive recurrent if $\beta > 1$ and starts being transient only if $\beta < 1$. The case $\beta = 1$ is a critical null-recurrent situation.

Being strongly attracted to 0, the binomial catastrophe model exhibits a recurrence/transience transition, but only for such very heavy-tailed choices of $\beta$. Recall that:

- When positive recurrent, the chain visits state 0 infinitely often and the expected return time to 0 has a finite mean.
- When null recurrent, the chain visits state 0 infinitely often but the expected return time to 0 has an infinite mean.
- When transient, the chain visits state 0 a finite number of times before drifting to $\infty$ for ever after an infinite number of steps (no finite time explosion is possible for discrete-time MCs).

**Corollary.** If the process is null recurrent or transient, no non-trivial ($\neq 0$) invariant measure exists.

**Proof.** This is because, $\Phi_\infty (z)$ being an absolutely monotone function on $[0,1]$ if it exists,

$$\pi (0) = \Phi_\infty (0) = 0 \Rightarrow \Phi_\infty (1 - c) = 0 \Rightarrow \pi (x) = 0 \text{ for all } x \geq 1.$$ 

Clustering (sampling at times when thinning occurs, time change): let $G = \inf (n \geq 1 : B_n (p) = 0)$, with $P(G = k) = p^{k-1}q$, $E(z^{G-1}) = \frac{q}{1 - pz}$. $G$ is the time elapsed between two consecutive catastrophic events. So long as there is no thinning of $X$ (a catastrophic event), the process grows from $\Delta = \sum_{k=1}^{G-1} \beta_k$ individuals. Consider a time-changed process of $X$, whereby one time unit is the time elapsed between consecutive catastrophic events. During these laps of time, the original process $X_n$ grew from $\Delta$ individuals, before shrinking to a random amount of its current size in catastrophe times. We are thus led to consider the time-changed integral-valued Ornstein–Uhlenbeck process (also known as an integer-valued autoregressive of Order 1 (in short INAR(1)) process, see [24]):

$$X_{k+1} = c \circ X_k + \Delta_{k+1}, \quad X_0 = 0,$$

with $\Delta_k, k = 1, 2, \ldots$ an i.i.d. sequence of compound shifted geometric rvs.

In this form, $X_k$ is also a pure-death subcritical branching process with immigration, $\Delta_{k+1}$ being the number of immigrants at generation $k + 1$, independent of $X_k$. With $\Phi_k (z) = E_z (X_k)$, we have

$$\Phi_{k+1} (z) = \frac{q}{1 - pz} \Phi_k (1 - c (1 - z)), \quad \Phi_0 (z) = 1.$$

The limit law (if it exists) $\Phi_\infty (z)$ also solves (5). Thus

$$\Phi_\infty (z) = \prod_{n \geq 0} \phi_{\Delta} (1 - c^n (1 - z)),$$
corresponding to
\[ X_\infty \equiv \sum_{n \geq 0} c^n \circ \Delta_{n+1}. \]

As conventional wisdom suggests, the time-changed process has the same limit law as the original binomial catastrophe model, so if, and only if, the condition \( \mathbb{E} \log + \Delta < \infty \) holds.

**Proposition.** When the law of \( X_\infty \) exists, it is discrete SD.

**Proof.** This follows, for example, from theorem 3.1 of [7] and the INAR(1) process representation of \( (X_k) \). See [31] for an account of discrete SD distributions, as a remarkable subclass of compound Poisson ones.

The rv \( X_\infty \) being SD, it is unimodal, with the mode at the origin if \( \pi (1) < \pi (0) \), or with two modes at \( \{0, 1\} \) if \( \pi (1) / \pi (0) = 1 \) (see [32], theorem 4.20).

With \( P (\Delta = 1) = \phi'_\Delta (0) = pqP (\beta = 1) \), we have
\[
\begin{align*}
\pi (0) &= \Phi_\infty (0) = q\Phi_\infty (1 - c) \\
\pi (1) &= \Phi'_\infty (0) = P (\Delta = 1) \Phi_\infty (1 - c) + q e \Phi'_\infty (1 - c) \\
&= \frac{P (\Delta = 1)}{q} \pi (0) + q e \Phi'_\infty (1 - c) > p P (\beta = 1) \pi (0).
\end{align*}
\]

A condition for unimodality at 0 is thus
\[
(\log \Phi_\infty)' (1 - c) < \frac{1 - p P (\beta = 1)}{c}.
\]

Note also
\[
\begin{align*}
\pi (1) &= \Phi'_\infty (0) = \sum_{m \geq 0} c^n \phi'_\Delta (1 - c^n) \prod_{n \neq m} \phi_\Delta (1 - c^n) \\
&= \pi (0) \sum_{m \geq 0} c^n (\log \phi_\Delta)' (1 - c^n) \\
&= \pi (0) \sum_{m \geq 0} c^n \frac{p B' (1 - c^n)}{1 - p B (1 - c^n)}
\end{align*}
\]
giving a closed-form condition for unimodality at 0. For instance, if \( B (z) = z \), \( \pi (1) < \pi (0) \), if, and only if,
\[
\sum_{m \geq 0} \frac{p c^m}{q + p c^m} < 1.
\]

**Tails of** \( X_\infty \). The probabilities \( \pi (x) = [z^x] \Phi_\infty (z) \), \( x \geq 1 \) (the \( z^x \)–coefficient in the power series expansion of \( \Phi_\infty (z) \)) are hard to evaluate. However, some information on the large \( x \) tails \( \sum_{y>x} \pi (y) \) can be estimated in some cases.

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Consider a positive recurrent case with $B(z) \sim C \cdot (1 - z/z_c)^{-1}$ as $z \to z_c > 1$ so that $P(\beta > x) \sim C \cdot z^{-x}$ has geometric tails. As detailed below,

$$\phi_{\Delta}(z) \sim C' \cdot (1 - z/z_c')^{-1} \quad \text{as } z \to z_c' > 1$$

so that, with $C' = pqC/(1 - pC) < 1$, $P(\Delta > x) \sim C' \cdot z_c'^{-x}$ also has geometric (heavier) tails but with modified rate $z_c' = z_c(1 - pC) < z_c$.

Then

$$\Phi_{\infty}(z) \sim C'' \cdot (1 - z/z_c')^{-1} \quad \text{as } z \to z_c' > 1$$

so that $P(X_\infty > x) \sim C' \cdot z_c'^{-x}$ also has geometric tails with rate $z_c'$. Indeed, with $z_c'' = (z_c' - (1 - c)) / c > z_c'$,

$$C'' \cdot (1 - z/z_c')^{-1} \sim C' C'' \cdot (1 - z/z_c')^{-1} \cdot (1 - (1 - c(1 - z)) / z_c')^{-1}$$

$$= C' C'' \left(\frac{z_c'}{c z_c''}\right) \cdot (1 - z/z_c')^{-1} \cdot (1 - z/z_c'')^{-1}$$

showing that

$$\Phi_{\infty}(z) \sim C' C'' \left(\frac{z_c'}{c z_c''}\right) \cdot (1 - z/z_c')^{-1} \cdot (1 - z/z_c'')^{-1}, \quad \text{as } z \to z_c' > 1.$$

Consider a positive recurrent case with $B'(1) = E\beta = \infty$. This is the case if, with $\alpha \in (0, 1)$, $B(z) \sim 1 - (1 - z)^\alpha$ as $z \to 1$, or if (unscaled Sibuya, see [30]) $B(z) = 1 - (1 - z)^\alpha$. In this case $\phi_{\Delta}(z) \sim 1 - \frac{z}{q}(1 - z)^\alpha$ as a scaled Sibuya rv (with the scale factor $\frac{q}{q} = E(G - 1)$). Indeed,

$$\frac{q}{1 - p(1 - (1 - z)^\alpha)} = \frac{1}{1 + \frac{p}{q}(1 - z)^\alpha} \sim \frac{1}{1 - \frac{p}{q}(1 - z)^\alpha}.$$

Then, $X_n$ being recurrent, in view of $\Phi_{\infty}(z) = \frac{q}{1 - pB(z)} \Phi_{\infty}(1 - c(1 - z))$,

$$\Phi_{\infty}(z) \sim 1 - \gamma(1 - z)^\alpha$$

where $\gamma = p/ [q(1 - c)^\alpha]$. Indeed, as $z \to 1$,

$$1 - \gamma(1 - z)^\alpha \sim \left[1 - \frac{p}{q}(1 - z)^\alpha\right] \left[1 - \gamma c^\alpha(1 - z)^\alpha\right]$$

allowing one to identify the scale parameter $\gamma$. The three rvs $\beta, \Delta$ and $X_\infty$ have power law tails with index $\alpha$.

The case $c = 0$ (total disasters).

In that case, combined with $\Phi_{\infty}(1) = 1$,

$$\Phi_{\infty}(z) = \frac{q}{1 - pB(z)} = \phi_{\Delta}(z)$$
is an admissible pgf solution. The law of \( X_\infty \) is a compound shifted geometric of the \( \beta \)'s, whatever the distribution of \( \beta \).

The probabilities \( \pi (x) = [z^x] \Phi_\infty (z), \ x \geq 1 \) are explicitly given by the Faa di Bruno formula for compositions of two pgfs \([10]\). Let us look at the tails of \( X_\infty \).

- If in particular, with \( C \in (0,1) \), \( B(z) \sim C \cdot (1 - z/z_c)^{-1} \) as \( z \to z_c > 1 \) so that \( \mathbf{P} (\beta > x) \sim C \cdot z_c^{-x} \) has geometric tails, then

\[
\Phi_\infty (z) \sim C' \cdot (1 - z/z'_c)^{-1} \quad \text{as} \ z \to z'_c > 1
\]

so that, with \( C' = pqC / (1 - pC) < 1 \), \( \mathbf{P} (X_\infty > x) \sim C' \cdot z_c^{-x} \) also has geometric (heavier) tails but with modified rate \( z'_c = z_c (1 - pC) < z_c \).

- If \( B(z) = \left( e^{\theta z} - 1 \right) / \left( e^\theta - 1 \right) \) (\( \beta \) is Poisson conditioned to be positive) is entire, \( \Phi_\infty (z) \) has a simple pole at \( z_c > 1 \) defined by

\[
\left( e^{\theta z} - 1 \right) / \left( e^\theta - 1 \right) = 1/p
\]

and \( X_\infty \) has geometric tails with rate \( z_c \).

- If, with \( \alpha \in (0,1) \), \( B(z) \sim 1 - (1 - z)^\alpha \) as \( z \to 1 \), or if (unscaled Sibuya) \( B(z) = 1 - (1 - z)^\alpha \), then \( \Phi_\infty (z) \sim 1 - \frac{q}{\alpha} (1 - z)^\alpha \) scaled Sibuya (with the scale factor \( \frac{q}{\alpha} = \mathbf{E} (G - 1) \):

\[
\frac{q}{1 - p (1 - (1 - z)^\alpha)} = \frac{1}{1 + \frac{q}{\alpha} (1 - z)^\alpha} \sim 1 - \frac{p}{q} (1 - z)^\alpha.
\]

- Suppose, with \( z_0 > 1 \),

\[
B(z) = \frac{1 - (1 - z/z_0)^\alpha}{1 - (1 - 1/z_0)^\alpha}
\]

If \( z_c > 1 : B(z_c) = 1/p \) exists, or if \( B(z_0) = 1 / (1 - (1 - 1/z_0)^\alpha) > 1/p \) (\( (1 - 1/z_0)^\alpha > q \) or \( z_0 > 1 / (1 - q^{1/\alpha}) \)), then \( \Phi_\infty (z) \) has a simple algebraic pole at \( z_c \). If no such \( z_c \) exists, \( \Phi_\infty (z) \) is entire. This is reminiscent of a condensation phenomenon. We finally observe that

\[
\frac{1 - \phi_\Delta (z)}{1 - z} = \frac{p (1 - B(z))}{(1 - z) (1 - pB(z))} = \frac{p (1 - B(z))}{1 - z} \phi_\Delta (z) ,
\]

\[
\mathbf{P} (\Delta > n) = \frac{p}{q} \sum_{m=0}^n \mathbf{P} (\beta > n - m) \mathbf{P} (\Delta = m) ,
\]

\[
\mathbf{P} (\Delta = n) = \frac{p}{q} \left( \sum_{m=0}^{n-1} \mathbf{P} (\beta = n - m) \mathbf{P} (\Delta = m) + \mathbf{P} (\Delta = n) \right).
\]

When is \( \Delta \) with \( \phi_\Delta (z) = \frac{q}{1 - pB(z)} - \phi_\Delta (z) \) itself SD? In any case, \( \Delta \) is at least ID or compound Poisson because \( \phi_\Delta (z) = \exp (-r (1 - \psi (z))) \), where \( r > 0 \) and \( \psi (z) \) is a pgf with \( \psi (0) = 0 \). Indeed, with \( q = e^{-r} \),

\[
\psi (z) = - \frac{\log (1 - pB(z))}{\log q}
\]

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is a pgf (the one of a Fisher-log-series rv).

**Proposition.** With \( b_x = [z^x] B(z) \), \( x \geq 1 \), the condition
\[
\frac{b_{x+1}}{b_x} \leq \frac{x - pb_1}{x+1} \quad \text{for any } x \geq 1,
\]
entails that \( \Delta \) is SD.

**Proof.** If \( \Delta \) is SD then (see [29], lemma 2.13)
\[
\phi_\Delta(z) = e^{-rf_1^{\frac{1-h(z)}{1-x}}dz},
\]
for some \( r > 0 \) and some pgf \( h(z) \) obeying \( h(0) = 0 \). We are led to check if
\[
\frac{pB'(z)}{1-pB(z)} = r \frac{1-h(z)}{1-z},
\]
for some pgf \( h \) and \( r = pb_1 \), where
\[
h(z) = 1 - \frac{1}{b_1} (1-z) \frac{B'(z)}{1-pB(z)} = \frac{1}{b_1} \frac{b_1 (1-pB(z)) - (1-z) B'(z)}{1-pB(z)}.
\]
Denoting the numerator \( N(z) \), a sufficient condition is that
\[
[z^x] N(z) \geq 0 \quad \text{for all } x \geq 1.
\]
But
\[
N(z) = \sum_{x \geq 1} z^x [(x-pb_1) b_x - (x+1) b_{x+1}].
\]

Let us show in four examples that these conditions can be met.

(a) Suppose \( B(z) = z \). Then \( 0 = \frac{b_{x+1}}{b_x} \leq \frac{x-p}{x+1} \) for all \( x \geq 1 \). The simple shifted-geometric rv \( \Delta \) is SD.

(b) Suppose \( B(z) = b_1 z + b_2 z^2 \) with \( b_2 = 1 - b_1 \). We need to check conditions under which \( \frac{b_x}{b_1} \leq \frac{1-pb_1}{2} \). This condition is met if, and only if, the polynomial \( pb_1^2 - 3b_1 + 2 \leq 0 \), which holds if, and only if, \( b_1 \geq b_1^* \), where \( b_1^* \in (0,1) \) is the zero of this polynomial in \( (0,1) \).

(c) Suppose \( B(z) = \alpha z / (1-\alpha z) \), \( \alpha \in (0,1) \), the pgf of a geometric(\( \alpha \)) rv, with \( b_x = \alpha x^{-1} \). The condition reads: \( \alpha \leq \frac{x-p}{x+q} \) for all \( x \geq 1 \), which is \( \alpha \leq q/(1+q) < 1 \) (or \( \alpha = b_1 \geq b_1^* = 1/(1+q) \)).

(d) Sibuya. Suppose \( B(z) = 1 - (1-z)^\alpha \), \( \alpha \in (0,1) \), with \( b_x = \alpha [\alpha]_{x-1}/x! \), \( x \geq 1 \) (where \( [\alpha]_x = \alpha (\alpha + 1) \ldots (\alpha + x - 1) \), \( x \geq 1 \) are the rising factorials of \( \alpha \) and \( [\alpha]_0 := 1 \)). The condition reads: \( \frac{\alpha x^{-1}}{x+1} \leq \frac{p}{x+1} \), which is always fulfilled. The shifted-geometric rv with the Sibuya distributed compounding rv is always SD.

**Proposition.** Under the condition that \( X_\infty \) is SD and so unimodal, \( X_\infty \) always has the mode at the origin.
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**Proof.** The condition is that, with \( \pi(0) = \Phi_\infty(0) = q \) and \( \pi(1) = \Phi'_\infty(0) = pqB'(0) = pqb_1 \), \( \pi(1)/\pi(0) = pb_1 < 1 \), which is always satisfied.

4. Green (potential) kernel analysis

In this section, we analyze some important characteristics of the chain, making use of its Green (potential) kernel.

4.1. Green (potential) kernel at \((0, 0)\): contact probability at 0 and first return time to 0

Suppose \( X_0 = 0 \). With then \( \Phi_0(z) = 1 \), define the double generating function \( \Phi(u, z) = \sum_{n\geq0} u^n \Phi_n(z) \), obeying \( \Phi(u, 1) = 1/(1-u) \), see [36]. (Note that \( \log z \) was the Laplace-conjugate to \( X_n \) and now \( \log u \) is the Laplace-conjugate to \( n \).) Then,

\[
\frac{1}{u} (\Phi(u, z) - 1) = pB(z) \Phi(u, z) + q\Phi(u, 1 - c(1 - z))
\]

\[
\Phi(u, z) = \frac{1 + qu\Phi(u, 1 - c(1 - z))}{1 - puB(z)}
\]

\[
\Phi(u, 0) = 1 + qu\Phi(u, 1 - c).
\]

With \( H(u, z) := 1/(1 - puB(z)) \), upon iterating, with \( \Phi(0, z) = \Phi_0(z) = 1 \)

\[
\Phi(u, z) = \sum_{n\geq0} (qu)^n \prod_{m=0}^{n} H(u, 1 - c^m(1 - z)).
\]

(10)

In particular

\[
\Phi(u, 0) = \sum_{n\geq0} (qu)^n \prod_{m=0}^{n} H(u, 1 - c^m) = 1 + \sum_{n\geq1} \prod_{m=1}^{n} \frac{qu}{1 - puB(1 - c^m)}.
\]

(11)

Note that

\[
G_{0,0}(u) := \Phi(u, 0) = \sum_{n\geq0} u^n P_0(X_n = 0) = (I - uP)^{-1}(0, 0)
\]

is the Green kernel of the chain at \((0, 0)\) (the matrix element \((0, 0)\) of the resolvent of \( P \)). Consequently,

**Proposition.** The Green kernel \( G_{0,0}(u) \) is given by (11).

With \( h_{m'} := [u^m] \prod_{m=0}^{n-1} (1 - puB(1 - c^m))^{-1} \), we have the following expression for the contact probability at 0:

\[
[u^n] \Phi(u, 0) = \Phi_n(0) = P_0(X_n = 0) = \sum_{m'=0}^{n-1} q^{n-m'} h_{m'}.
\]

(12)
Remark.

(a) Let us have a quick check of this formula. When $n = 1$, this leads to $P_0 (X_1 = 0) = q$ and, if $n = 2$, to $P_0 (X_2 = 0) = q^2 + q [u^1] (1 - puB (1 - c))^{-1} = q^2 + pqB (1 - c)$. The second part is not quite trivial because it accounts for any movement up in the first step (wp $p$) immediately followed by a movement down to 0. This is consistent however with the binomial formula $P (X_1 = 0 | X_0 = 0) = qP (c \circ x = 0) = q(1 - c (1 - z))^z|_{z=0} = q(1 - c)^x$ so that the second part is

\[
pq\sum_{x \geq 1} P (\beta = x) P (X_1 = 0 | X_0 = x) = pq\sum_{x \geq 1} b_x (1 - c)^x = pqB (1 - c).
\]

(b) With $B_m := pB (1 - c^m)$, $m \geq 1$, an increasing $[0, 1]$-valued sequence converging to $p$, decomposing the product $\prod_{m=1}^{n-1} (1 - uB_m)^{-1}$ into simple fraction elements, we get if $n > 1$

\[
h_{m'} = \sum_{m=1}^{n-1} A_{m,n} B_m^{m'},
\]

where $A_{m,n} = B_m^{n-2} / \prod_{m' \in \{1, \ldots, n-1\} \setminus \{m\}} (B_m - B_{m'})$. The term $A_{n-1,n-1} B_{n-1}^{n-1}$ contributes most to the sum. Hence, $P_0 (X_1 = 0) = q$ and if $n > 1$

\[
P_0 (X_n = 0) = q^n \sum_{m'=0}^{n-1} b^{m'-n} \sum_{m=1}^{n-1} A_{m,n} B_m^{m'} = q^n \sum_{m=1}^{n-1} A_{m,n} \sum_{m'=0}^{n-1} \left( \frac{B_m}{q} \right)^{m'}
\]

\[
= q^n \sum_{m=1}^{n-1} A_{m,n} \frac{q^n - B_m^n}{q - B_m}
\]

is an alternative representation of (12).

In the positive recurrent case, as $n \to \infty$,

\[
P_0 (X_n = 0) \to \tau (0) = \prod_{n \geq 0} \phi_{\Delta} (1 - c^n) > 0.
\]

The Green kernel at $(0, 0)$ is thus $G_{0,0} (u) = \Phi (u, 0)$.

If $n \geq 1$, from the recurrence $P_0 (X_n = 0) = P^n (0, 0) = \sum_{m=0}^{n} P (\tau_{0,0} = m) P^{n-m} (0, 0)$, we see that the pgf $\phi_{0,0} (u) = E (u^{\tau_{0,0}})$ of the first return time to 0, $\tau_{0,0}$ and $G_{0,0} (u)$ are related by the Feller relation (see [3] pp 3–4 for example)

\[
G_{0,0} (u) = \frac{1}{1 - \phi_{0,0} (u)} \quad \text{and} \quad \phi_{0,0} (u) = \frac{G_{0,0} (u) - 1}{G_{0,0} (u)}.
\]

Hence,

Proposition. The pgf $\phi_{0,0} (u)$ of the first return time $\tau_{0,0}$ is

\[
\phi_{0,0} (u) = 1 - \frac{1}{G_{0,0} (u)}.
\]
where $G_{0,0}(u)$ is given by (11).

Note

$$G_{0,0}(1) = \sum_{n \geq 0} P_0(X_n = 0) = 1 + \sum_{n \geq 1} q^n \prod_{m=0}^n H(1, 1 - c^n) = \infty$$

if, and only if, $X$ is recurrent [26, 28]. And in that case, $\phi_{0,0}(1) = P(\tau_{0,0} = \infty) = 1 - \frac{1}{G_{0,0}(1)} = 1$. Positive (null) recurrence is when $\phi'_{0,0}(1) = E(\tau_{0,0}) = 1/\pi_0 < \infty (= \infty)$ [22].

Note finally $G_{0,0}(0) = 1$, so that $\phi_{0,0}(0) = P(\tau_{0,0} = 0) = 0$.

4.2. Starting from $x_0 > 0$: Green kernel at $(x_0, 0)$ and first extinction time $\tau_{x_0,0}$

Suppose now $X_0 = x_0 > 0$. After shifting $X_n$ of $x_0$, with $\Phi(u, z) = \sum_{n \geq 0} u^n E(z^{x_0 + X_n})$, we now get

$$\frac{1}{u}(\Phi(u, z) - z^{x_0}) = pB(z) \Phi(u, z) + q\Phi(u, 1 - c(1 - z)).$$

Then

$$\Phi(u, z) = \frac{z^{x_0} + qu\Phi(u, 1 - c(1 - z))}{1 - puB(z)}, \quad (14)$$

entailing

$$\Phi(u, z) = \sum_{n \geq 0} (qu)^n (1 - c^n(1 - z))^{x_0} \prod_{m=0}^n H(u, 1 - c^m(1 - z)),$$

$$\Phi(u, 0) = \sum_{n \geq 0} (qu)^n (1 - c^n)^{x_0} \prod_{m=0}^n H(u, 1 - c^m).$$

We obtained:

**Proposition.** The contact probability at 0 for the chain started at $x_0 > 0$ is given by

$$[u^n] \Phi(u, 0) = \Phi_n(0) = P_{x_0}(X_n = 0) = \sum_{m'=0}^{n-1} \left(1 - c^{n-m'} \right)^{x_0} q^{n-m'} h_{m'}. \quad (15)$$

Let us give the first two terms as compared to when $x_0 = 0$. As required, when $n = 1$, $P_{x_0}(X_1 = 0) = q(1 - c)^{x_0}$ and, when $n = 2$,

$$P_{x_0}(X_2 = 0) = q^2(1 - c)^{x_0} + pq(1 - c)^{x_0} B(1 - c)$$

a weighted sum of the two terms appearing in the above expression of $P_0(X_2 = 0)$.

**Corollary.**

(a) When $x_0$ is large and $n$ is fixed, the small but dominant term is when $m' = 0$, which is $q^n(1 - c^n)^{x_0}$. So $P_{x_0}(X_n = 0)$ decays geometrically with $x_0$. This expression quantifies the probability that the population is in an early state of extinction given

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The initial population size was large. Early is when \( c^n \gg 1/x_0 \) (so that \( (1 - c^n)x_0 \ll (1 - 1/x_0)x_0 \ll e^{-1} \)), so when \( n < -\log x_0 \).

(b) In the transient case, when \( n \) is large and \( x_0 \) is fixed, the dominant term is when \( m' = n - 1 \), which is \( q(1 - c)^{x_0}(pB(1 - c))^n \). So \( P_{x_0}(X_n = 0) \) decays geometrically with \( n \) at rate \( pB(1 - c) \). In the positive recurrent case, \( P_{x_0}(X_n = 0) \to \pi(0) > 0 \) as \( n \to \infty \), independently of \( x_0 \).

- The Green kernel at \((x_0,0)\) is thus \( G_{x_0,0}(u) = [z^0] \Phi(u,z) \) (the matrix element \((x_0,0)\) of the resolvent of \( P \)). It is related to the pgf of the first extinction time \( \tau_{x_0,0} \) by the Feller relation

\[
\phi_{x_0,0}(u) = E(u^{\tau_{x_0,0}}) = \frac{G_{x_0,0}(u)}{G_{0,0}(u)}.
\]

Therefore,

**Proposition.** With \( x_0 > 0 \), the pgf of the first extinction time \( \tau_{x_0,0} \) is

\[
\phi_{x_0,0}(u) = E(u^{\tau_{x_0,0}}) = \frac{\sum_{n \geq 1} (qu)^n (1 - c^n)^{x_0} \prod_{m=1}^n H(u,1-c^m)}{\sum_{n \geq 0} (qu)^n \prod_{m=0}^n H(u,1-c^m)}.
\]

In the recurrent case, state 0 is visited infinitely often, and so both \( G_{0,0}(1) \) and \( G_{x_0,0}(1) = \infty \), and

\[
P(\tau_{x_0,0} < \infty) = \phi_{x_0,0}(1) = \frac{G_{x_0,0}(1)}{G_{0,0}(1)} = 1.
\]

We finally note that, because state 0 is reflecting, \( \tau_{x_0,0} \) is only a local extinction time, followed by subsequent extinction times after \( \tau_{0,0} \). In the sequel, we shall let \( \overline{P} \) stand for the substochastic transition matrix obtained from \( P \) while deleting its first row and column.

There are two alternative representations of \( \phi_{x_0,0}(u) \).

- One alternative representation follows from the following classical first-step analysis:

Let \( X_1(x_0) \) be the position of \( X_n \) started at \( x_0 \).

Let \( X_+(x_0) \) be a positive rv with \( P(X_+(x_0) = y) = \overline{P}(x_0,y)/\sum_{y \geq 1} \overline{P}(x_0,y), y \geq 1 \).

With \( \tau'_{X_+(x_0),0} \) a statistical copy of \( \tau_{X_+(x_0),0} \), first-step analysis yields, (see [27, 36]):

\[
\tau_{x_0,0} = 1 \cdot 1_{\{X_1(x_0)=0\}} + 1_{\{X_1(x_0)>0\}} \cdot (1 + \tau'_{X_+(x_0),0}).
\]

Clearly, \( P(X_1(x_0) = 0) = P(x_0,0) = q(1-c)^{x_0} = x_0 \), \( P(X_1(x_0) > 0) = 1 - q_0 \). Therefore, \( \phi_{x_0,0}(u) := E(u^{\tau_{x_0,0}}) \) obeys the recurrence

\[
\phi_{x_0,0}(u) = q_0 u + up_{x_0} E\phi_{X_+(x_0),0}(u).
\]

With \( \phi(u) = (\phi_{1,0}(u),\phi_{2,0}(u),\ldots)^t \) the column vector of the \( \phi_{x_0,0}(u) = E(u^{\tau_{x_0,0}}) \), and \( q = (q_1,q_2,\ldots)^t \) the column vector of the matrix \( P \), \( \phi(u) \) then solves:

\[
\phi(u) = uq + uP\phi(u),
\]
whose formal solution is (compare with the explicit expression (17))

$$\phi(u) = u(I - u\overline{P})^{-1}q =: \overline{G}(u)\mathbf{q},$$

(19)

involving the resolvent matrix $\overline{G}(u)$ of $\overline{P}$. Note $\phi(1) = (I - \overline{P})^{-1}q$ gives the column vector of the probabilities of eventual extinction $\phi(1) := (\phi_{1,0}(1), \phi_{2,0}(1), \ldots)'$, so with $\phi_{x_0,0}(1) = \mathbf{P}(\tau_{x_0,0} < \infty)$ if $\overline{G}(1)q < \infty$. Clearly, $\phi(1) = 1$ (the all-one column vector) in the recurrent case. In that case, from (18), introducing the column vector $\mathbf{E}(\tau_{x,0}) := (\mathbf{E}(\tau_{1,0}), \mathbf{E}(\tau_{2,0}), \ldots)'$, where $\mathbf{E}(\tau_{x,0}) = \phi'_{x,0}(1)$ and observing $q + \overline{P}\phi(1) = 1$, we get

$$\phi'(1) := \mathbf{E}(\tau_{x,0}) = 1 + \overline{P}\mathbf{E}(\tau_{x,0}),$$
equivalently $\mathbf{E}(\tau_{x,0}) = (I - \overline{P})^{-1}\mathbf{1} = \overline{G}(1)\mathbf{1}$, or

$$\mathbf{E}(\tau_{x,0}) = \sum_{y \geq 1} \overline{G}_{x,y}(1),$$

where $\overline{G}_{x,y}$ is the matrix element $(x, y)$ of the resolvent of $\overline{P}$.

- Yet another alternative representation $\phi(u) := (\phi_{1,0}(u), \phi_{2,0}(u), \ldots)'$ is as follows. From the identity

$$\overline{P}^n(x, y) = \mathbf{P}_{x_0}(X_n = y, \tau_{x_0,0} > n),$$

we get $\mathbf{P}(\tau_{x,0} > n) = \overline{P}^n\mathbf{1}$, as the column vector of $(\mathbf{P}(\tau_{1,0} > n), \mathbf{P}(\tau_{2,0} > n), \ldots)'$, and so,

$$\sum_{n \geq 0} u^n\mathbf{P}(\tau_{x,0} > n) = \overline{G}(u)\mathbf{1}. $$

This leads in particular, as expected, to $\mathbf{E}(\tau_{x,0}) = \overline{G}(1)\mathbf{1}$ and (compare with (19) and the explicit expression (17)) to

$$\phi(u) = \sum_{n \geq 0} u^n\mathbf{P}(\tau_{x,0} = n) = 1 - (1 - u)\overline{G}(u)\mathbf{1}. $$

(20)

We obtained:

**Proposition.** We have $\phi(1) = \mathbf{P}(\tau_{x,0} < \infty)$ and so $\overline{G}(1)\mathbf{1} < \infty \Rightarrow \mathbf{P}(\tau_{x,0} < \infty) = 1$, meaning recurrence of $X_n$. In fact, positive recurrence is precisely when $\mathbf{E}(\tau_{x,0}) = \overline{G}(1)\mathbf{1} < \infty$. If $\overline{G}(1)\mathbf{1} = \infty$, the chain is null recurrent if $(1 - u)\overline{G}(u)\mathbf{1} \to 0$ as $u \to 1$, transient if $(1 - u)\overline{G}(u)\mathbf{1} \to \mathbf{P}(\tau_{x,0} = \infty)$ as $u \to 1$, a non-zero limit.

The matrix $\overline{P}$ is substochastic with the spectral radius $\rho \in (0, 1)$. With $\mathbf{r}$ and $\mathbf{l}'$ the corresponding right and left positive eigenvectors of $\overline{P}$, so with $P\mathbf{r} = \rho\mathbf{r}$ and $\mathbf{l}'\overline{P} = \rho\mathbf{l}'$, $\overline{P}^n \sim \rho^n \cdot \mathbf{r}\mathbf{l}'$ (as $n$ is large), where $\mathbf{r}\mathbf{l}'$ is the projector onto the first eigenspace. Using the Perron–Frobenius theorem [20, 34], we can normalize $\mathbf{1}$ to be of $l_1$-norm one to get

https://doi.org/10.1088/1742-5468/abdfcb
Proposition. In the positive recurrent case for \( X_n \) \( (\mathbb{E} \log_+ \Delta < \infty) \):

(a) With \( r(x_0) \) the \( x_0 \)-entry of \( r \),

\[
\rho^{-n} \mathbf{P} (\tau_{x_0,0} > n) \to r(x_0), \quad \text{as } n \to \infty,
\]

showing that \( \mathbf{P} (\tau_{x_0,0} > n) \) has geometric tails with rate \( \rho \) (extinction is fast).

(b) With \( l(y) \) the \( y \)-entry of \( l \), for all \( x_0 > 0 \),

\[
\mathbf{P}_{x_0} (X_n = y | \tau_{x_0,0} > n) \to l(y), \quad \text{as } n \to \infty,
\]

showing that the left eigenvector \( l \) is the quasi-stationary distribution of \( X_n \) (or the Yaglom limit [37]) [9].

Proof. In this case, with \( R = \rho^{-1} > 1 \), the convergence radius of \( \overline{G} \), \( \overline{G}(R) = \infty \) and \( \overline{P} \) is \( R \)-positive recurrent: (a) follows from \( \mathbf{P} (\tau_{x_0,0} > n) = \overline{P}^n \mathbf{1} \), and (b) from \( \mathbf{P}_{x_0} (X_n = y | \tau_{x_0,0} > n) = \overline{P}^n (x_0, y) / \overline{P}^n \mathbf{1} \).

Remark. The full Green kernel at \( (x_0, y_0) \) is \( G_{x_0,y_0}(u) = [z^{y_0}] \Phi(u, z) \). Hence,

\[
G_{x_0,y_0}(u) = \sum_{n \geq 0} (qu)^n [z^{y_0}] (1 - c^n (1 - z))^x_0 \prod_{m=0}^n H(u, 1 - c^m (1 - z))
\]

\[
= \sum_{n \geq 0} (qu)^n \sum_{y=0}^{y_0} h_{n,y}(u) g_{n,y_{0-y}}
\]

where

\[
g_{n,y} = [z^y] (1 - c^n (1 - z))^{x_0} = \left( \frac{x_0}{y} \right) c^n y (1 - c^n)^{x_0-y}
\]

and

\[
h_{n,y}(u) = [z^y] \prod_{m=0}^n H(u, 1 - c^m (1 - z)),
\]

which can be obtained from a decomposition into simple elements of the inner product.

Using \( P^n (x_0, y_0) = \sum_{m=1}^n \mathbb{P} (\tau_{x_0,y_0} = m) P^{n-m} (y_0, y_0) \), \( n \geq 1 \), we easily get the expression of the pgf of the first hitting times \( \tau_{x_0,y_0} = \inf (n \geq 1 : X_n = y_0 | X_0 = x_0) \), as

\[
\phi_{x_0,y_0}(z) = \frac{G_{x_0,y_0}(z)}{G_{y_0,y_0}(z)}.
\]

4.3. Average position of \( X_n \) started at \( x_0 \)

The double generating function \( \Phi(u, z) \) can be used to compute the evolution of \( \mathbb{E}_{x_0} (X_n) \). With \( \Phi'(u, z) = \partial_z \Phi(u, z) \),
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\[ \Phi'(u, 1) = x_0 \sum_{n \geq 0} (qc^u)^n H(u, 1)^{n+1} + \frac{pp}{1 - (p + qc)} \left( \frac{1}{1 - u} - \frac{1}{1 - u (p + qc)} \right) \]

\[ = x_0 \frac{H(u, 1)}{1 - qcuH(u, 1)} + \frac{pp}{1 - (p + qc)} \left( \frac{1}{1 - u} - \frac{1}{1 - u (p + qc)} \right) \]

\[ = \frac{x_0}{1 - (p + qc)} + \frac{pp}{1 - (p + qc)} \left( \frac{1}{1 - u} - \frac{1}{1 - u (p + qc)} \right). \]

This shows that, in the positive recurrent case, with \( \mu_n = E_{x_0}(X_n) \),

\[ \mu_n = [u^n] \Phi'(u, 1) = x_0(p + qc)^n + \frac{pp}{q(1 - c)} (1 - (1 - q(1 - c))^n) \]

\[ = x_0(p + qc)^n + E_0(X_n) \to \frac{pp}{q(1 - c)} \text{ whatever } x_0, \]

solving

\[ \mu_{n+1} = (p + qc) \mu_n + pp, \quad \mu_0 = x_0. \]

A similar analysis making use of the second derivative \( \Phi''(u, 1) \) would yield the transient evolution of the variance of \( X_n \), started at \( x_0 \). We skip the details.

4.4. The height \( H \) of an excursion

Assume \( X_0 = x_0 \) and consider a version of \( X_n \), which is absorbed at 0. Let \( X_{n \wedge \tau_{x_0}} \) stopping \( X_n \) when it first hits 0. Let us define the scale (or harmonic) function or sequence \( \varphi \) of \( X_n \) as the function which makes \( Y_n \equiv \varphi (X_{n \wedge \tau_{x_0}}) \) a martingale. The function \( \varphi \) is important because, for all \( 0 \leq x_0 < h \), with \( \tau_{x_0} = \tau_{x_0,0} \wedge \tau_{x_0,h} \) the first hitting time of \( \{0, h\} \) starting from \( x_0 \) (assuming \( \varphi (0) \equiv 0 \))

\[ P(X_{\tau_{x_0}} = h) = P(\tau_{x_0,h} < \tau_{x_0,0}) = \frac{\varphi (x_0)}{\varphi (h)}, \]

resulting from

\[ E\varphi (X_{n \wedge \tau_{x_0}}) = \varphi (x_0) = \varphi (h) P(\tau_{x_0,h} < \tau_{x_0,0}) + \varphi (0) P(\tau_{x_0,h} > \tau_{x_0,0}). \]

- The case \( \beta_1 \sim \delta_1 \). Let us consider the height \( H \) of an excursion of the original MC \( X_n \) first assuming \( \beta_1 \sim \delta_1 \) (a birth event adds only one individual); wp \( q, H = 0 \) and wp \( p \), starting from \( X_1 = 1 \), it is the height of a path from state 1 to 0 of the absorbed process \( X_n \). Using this remark, the event \( H = h \) is realized when \( \tau_{1,h} < \tau_{1,0} \) and \( \tau_{h,h+1} > \tau_{h,0} \), the latter two events being independent. Thus (with \( P(H = 0) = q \)):

\[ P(H = h) = p \frac{\varphi (h)}{\varphi (h + 1)} \left( 1 - \frac{\varphi (h)}{\varphi (h + 1)} \right), \quad h \geq 1. \]

We clearly have \( \sum_{h \geq 1} P(H = h) = p \) because partial sums form a telescoping series. But (27) is also

https://doi.org/10.1088/1742-5468/abdfcb
\[ P(H \geq h) = 1/\varphi(h), \ h \geq 1, \] (28)

with \( \varphi(1) = 1/p. \)

It remains to compute \( \varphi \) with \( \varphi(0) = 0 \) and \( P(H \geq 1) = 1/\varphi(1) = p. \) We wish to have: \( E_{x_0}(Y_{n+1}|Y_n = x) = x, \) leading to

\[ \varphi(x) = p\varphi(x+1) + q\sum_{y=1}^{x} \binom{x}{y} c^y (1-c)^{x-y} \varphi(y), \quad x_0 \geq 1. \]

The vector \( \varphi \) is the right eigenvector associated with the eigenvalue 1 of the modified version \( P^* \) of the stochastic matrix \( P \) having 0 as an absorbing state: \( P^*(0, 0) = 1, \) so with: \( \varphi = P^*\varphi, \ \varphi(0) = 0 \) [27]. The searched ‘harmonic’ function is increasing and given by recurrence, \( \varphi(1) = 1/p \) and

\[ \varphi(x+1) = \frac{1}{p} \left( \varphi(x) [1 - qc^x] - q\sum_{y=1}^{x-1} \binom{x}{y} c^y (1-c)^{x-y} \varphi(y) \right), \quad x \geq 1. \] (29)

The first two terms are

\[ \varphi(2) = \frac{1}{p} (1 - qc) \varphi(1) = \frac{1}{p^2} (1 - qc), \]

\[ \varphi(3) = \frac{1}{p} \left( \varphi(2) [1 - qc^2] - 2qc (1-c) \varphi(1) \right), \]

\[ = \frac{1}{p^3} (1 - qc) (1 - qc^2) - \frac{2q}{p} (1-c). \]

The sequence \( \varphi(x) \) is diverging when the chain \( X \) is recurrent.

**Proposition.** When \( \beta \sim \delta_1, \) equations (28) and (29) characterize the law of the excursion height \( H \) of the random walker in the recurrent case. In the transient case, \( \varphi(x) \) converges to a value \( \varphi^* \) and \( P(H = \infty) = 1/\varphi^* = P(\tau_{0,0} = \infty). \)

- General \( \beta_1. \) Whenever the law of \( \beta \) is general, the matrix \( P^* \) is no longer lower Hessenberg, and the harmonic vector \( \varphi = P^*\varphi, \) with \( \varphi(0) = 0, \) cannot be obtained by a recurrence. However, the event \( H \geq h \geq 1 \) is realized whenever a first birth event occurs with size \( \beta_1 \geq h \) or, if \( \beta_1 < h, \) whenever for all states \( h' \geq h \) being hit when the amplitude \( \beta \) of a last upper jump is larger than \( h' - h, \) then \( \tau_{\beta_1, h'} < \tau_{\beta_1, 0}. \) Hence,

**Proposition.** For a recurrent walker with general \( \beta, \) \( P(H = \infty) = 0 \) where, when \( h \geq 1, \)

\[ P(H \geq h) = pP(\beta_1 \geq h) + p\sum_{x=1}^{h-1} P(\beta_1 = x) \sum_{h' \geq h} \frac{\varphi(x)}{\varphi(h')} P(\beta > h' - h) \]

\[ = pP(\beta_1 \geq h) + p \sum_{x=1}^{h-1} P(\beta_1 = x) \varphi(x) \sum_{h'' \geq h} \frac{1}{\varphi(h + h'')} P(\beta > h'') \]

generalizing (28).
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5. Estimation from an \( N \)-sample of \( X \), e.g. \((x_0, x_1, \ldots, x_N)\)

We briefly sketch here how (in the presence of real data, which are suspected to be in the binomial catastrophe framework), to estimate its constitutive parameters.

From the transition matrix \( P(x, y) \), the log-likelihood function of the \( N \)-sample is

\[
L(x_0, x_1, \ldots, x_N) = \sum_{n=1}^{N} \left[ \log(qd_{x_{n-1}x_n}) \mathbf{1}(x_n \leq x_{n-1}) + \log(pb_{x_n-x_{n-1}}) \times \log \mathbf{1}(x_n > x_{n-1}) \right].
\]

If one knows that a population grows and decays according to the binomial catastrophe model with \( \text{E}\beta = \rho < \infty \), we propose the following estimators: the maximum-likelihood-estimator of \( p \) while setting \( \partial_p L = 0 \) is

\[
\hat{p} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}(x_n > x_{n-1}). \tag{30}
\]

With \( \rho = \text{E}\beta < \infty \), if the law of \( \beta \) is a known one-parameter \( \rho \)-family of pmfs,

\[
\hat{\rho} = \frac{1}{\sum_{n=1}^{N} \mathbf{1}(x_n > x_{n-1}) \sum_{n=1}^{N} (x_n - x_{n-1}) \mathbf{1}(x_n > x_{n-1})}.
\]

(31)

Note that if

\[
\hat{\rho} \hat{p} = \frac{1}{N} \sum_{n=1}^{N} (x_n - x_{n-1}) \mathbf{1}(x_n > x_{n-1}),
\]

then \( \hat{\rho} \hat{p} = \hat{\rho} \hat{p} \).

Also, in view of \( P(X_{n+1} = x | X_n = x) = qc^x \), \( x \geq 0 \),

\[
\hat{c} = \frac{1}{\sum_{n=1}^{N} \mathbf{1}(x_n = 1) \sum_{n=1}^{N} \mathbf{1}(x_n = 1, x_{n-1} = 1)}.
\]

(32)

6. Analysis of the absorbed version of \( X_n \) and quasi-stationarity

We consider again a version of \( X_n \) started at \( x_0 > 0 \), but which is now absorbed when first hitting 0. We aim to have additional insight into the quasi-stationary distribution, which is known to be a difficult issue. The absorbed process was considered in [19]. We work under the condition that the original (non-absorbed) process is not transient, because if it were, so would its absorbed version be, with a positive probability to drift to \( \infty \).

The only thing which changes in the transition matrix \( P \) from (1) is its first row, which becomes \( P(0, y) = \delta_{0,y} \), \( y \geq 0 \), leading to \( P' \). Letting \( \Phi_n(z) = \mathbf{E}(z^{X_n}) \) with \( \Phi_0(z) = z^{x_0} \),

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taking into account the behavior of $X_n$ at 0, with $\Phi_0 (z) = z^c \circ$, the recurrence (2) now becomes

$$
\Phi_{n+1} (z) = \Phi_n (1) = \Phi_0 (1) = 1 \text{ (no mass loss)},
$$

$$
\Phi_{n+1} (0) = p\Phi_n (0) + q\Phi_n (1 - c) \text{ else},
$$

$$
\Phi_{n+1} (0) - \Phi_n (0) = q (\Phi_n (1 - c) - \Phi_n (0)) > 0 \text{ if } n \geq 1,
$$

showing that $\Phi_n (0) = \mathbf{P} (X_n = 0 | X_0 = x_0)$ is increasing tending to 1 under the recurrence condition. Finally,

• $\Phi_n (0) = 1 \iff X_n = 0 \iff \Phi_n (z) = 1 \implies \Phi_n (1 - c) = 1 \implies \Phi_{n+1} (0) = 1 \text{ (0 is absorbing)}$
• $X_n = 0 \iff \tau_{x_0,0} \leq n \implies \Phi_n (0) = \mathbf{P} (\tau_{x_0,0} \leq n)$.
• $\mathbf{P} (\tau_{x_0,0} = n) = \mathbf{P} (\tau_{x_0,0} \leq n) - \mathbf{P} (\tau_{x_0,0} \leq n - 1) = \Phi_n (0) - \Phi_{n-1} (0) > 0$.

Here, $\tau_{x_0,0}$ are the first (and last) hitting time of 0 for $X_n$ given $X_0 = x_0 > 0$. It has the same distribution as the one obtained for the original non-absorbed chain. Considering the substochastic transition matrix $\overline{\mathbf{P}}$ of $X_n$, where the first row and column have been removed, the dynamics of $\Psi_n (z)$ are

$$
\Psi_n (z) = \mathbf{E} \left( z^{X_n} 1_{\tau_{x_0,0} > n} \right)
$$

are

$$
\Psi_{n+1} (z) = pB (z) \Psi_n (z) + q [\Psi_n (1 - c (1 - z)) - \Psi_n (1 - c)] , \Psi_0 (z) = z^c , \Psi_0 (0) = 0.
$$

(33)

Note $\Psi_n (0) = 0$ entails $\Psi_{n+1} (0) = 0$: as required, there is no probability mass at 0, because $\Psi_n (z)$ is the pgf of $X_n$, in the event $\tau_{x_0,0} > n$.

Furthermore, $\Psi_{n+1} (1) = p\Psi_n (1) + q [\Psi_n (1 - c (1 - z)) - \Psi_n (1 - c)] = \Psi_n (1) - q\Psi_n (1 - c)$, so that $\Psi_{n+1} (1) - \Psi_n (1) < 0$ translating a natural loss of mass.

We have $\Psi_n (1) = \mathbf{P} (\tau_{x_0,0} > n)$ (note $\Psi_n (1) = 1 - \Phi_n (0)$). Conditioning, with $\Phi_n^c (z) = \mathbf{E} \left( z^{X_n} | \tau_{x_0,0} > n \right)$, upon normalizing, we get

**Proposition.** Under the positive recurrence condition for $X_n$, as $n \to \infty$

$$
\Phi_n^c (z) := \frac{\Psi_n (z)}{\Psi_n (1)} \to \frac{\Psi_{\infty} (z)}{\Psi_{\infty} (1)}, \quad \text{as } n \to \infty.
$$

(34)
the pgf of the quasi-stationary distribution \( l \) (as a left eigenvector of \( \mathbf{P} \)). An explicit expression of \( \Psi_n (z) \) follows from (35) and (36) below.

The double generating function of \( \Psi_n (z) \) is

\[
\frac{1}{u} (\Psi (u, z) - z^{x_0}) = pB (z) \Psi (u, z) + q\Psi (u, 1 - c (1 - z)) - q\Psi (u, 1 - c).
\]

Its iterated version is

\[
\Psi (u, z) = \sum_{n \geq 0} (qu)^n (1 - c^m (1 - z)) \prod_{m=0}^{n} H (u, 1 - c^m (1 - z))
\]

(35)

\[
-\Psi (u, 1 - c) \sum_{n \geq 1} (qu)^n \prod_{m=0}^{n-1} H (u, 1 - c^m (1 - z)).
\]

If \( z = 1 - c \),

\[
\Psi (u, 1 - c) = \sum_{n \geq 0} (qu)^n (1 - c^{m+1}) \prod_{m=0}^{n} H (u, 1 - c^{m+1})
\]

(36)

\[
-\Psi (u, 1 - c) \sum_{n \geq 1} (qu)^n \prod_{m=0}^{n-1} H (u, 1 - c^m (1 - z)),
\]

so that

\[
\Psi (u, 1 - c) = \frac{\sum_{n \geq 0} (qu)^n (1 - c^{m+1}) \prod_{m=0}^{n} H (u, 1 - c^{m+1})}{1 + \sum_{n \geq 1} (qu)^n \prod_{m=0}^{n} H (u, 1 - c^m)}.
\]

(37)

Plugging (36) into (35) yields a closed-form expression of \( \Psi (u, z) \), and then of

\[
\frac{\Psi_n (z)}{\Psi_n (1)} = \frac{[u^n] \Psi (u, z)}{[u^n] \Psi (u, 1)}.
\]

The value of \( \Psi (u, z) \) at \( z = 1 \) is \( \Psi (u, 1) = \sum_{n \geq 0} u^n \mathbf{P} (\tau_{x_0,0} > n) \), so that \( \Psi (u, 1) = (1 - \mathbf{E} u^{\tau_{x_0,0}}) / (1 - u) \). With \( H (u, 1) = 1 / (1 - pu) \), from (35)

\[
\Psi (u, 1) = \sum_{n \geq 0} (qu)^n \prod_{m=0}^{n} H (u, 1) - \Psi (u, 1 - c) \sum_{n \geq 1} (qu)^n \prod_{m=0}^{n-1} H (u, 1)
\]

\[
= \frac{1}{1 - pu} \frac{1}{1 - qu / (1 - pu)} - \Psi (u, 1 - c) \frac{1}{1 - pu} \frac{1}{1 - qu / (1 - pu)}
\]

\[
= \frac{1}{1 - u} - \Psi (u, 1 - c) \frac{qu}{1 - u} = \frac{1}{1 - u} (1 - qu \Psi (u, 1 - c)).
\]

The pgf of \( \tau_{x_0,0} \) is thus also (a third representation of \( \phi_{x_0,0} (u) := \mathbf{E} u^{\tau_{x_0,0}} \), see (19) and (20))

\[
\phi_{x_0,0} (u) = qu \Psi (u, 1 - c),
\]

with \( \Psi (u, 1 - c) \) given by (36). This expression of \( \phi_{x_0,0} (u) \) is consistent with (17).

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7. A variant of the binomial catastrophe model

Suppose we are interested in the following simple semi-stochastic decay/surge model: at each step of its evolution, the size of a population either grows by a random number of individuals or shrinks by only one unit. The above binomial catastrophe model is not able to represent this scenario, where at least one individual is removed from the population at catastrophic events. To remedy this, we therefore define and study a variant of the above binomial model, whereby the transition probabilities in the bulk and at 0 are slightly modified to account for the latter decay/surge situation [17].

- If \( X_n \geq 1 \), define
  \[
  X_{n+1} = 1 \circ X_n + \beta_{n+1} = X_n + \beta_{n+1} \, \wp \, p \\
  X_{n+1} = c \circ (X_n - 1) \, \wp \, q.
  \]

  Given a move down \( \wp \, q \): one individual out of \( X_n \) is systematically removed from the population (\( X_n \rightarrow X_n - 1 \)); each individual among the \( X_n - 1 \) remaining ones being independently subject to a survival/death issue (\( \wp \, c/1 - c \)) in the next generation.

- If \( X_n = 0 \) \( (p_0 + q_0 = 1) \) then,
  \[
  X_{n+1} = \beta_{n+1} \, \wp \, p_0, \\
  = 0 \, \wp \, q_0.
  \]

  Unless \( p_0 = p \), our model yields some additional control on the future of the population once it hits 0 (extinction event).

With \( b_x := P(\beta = x) \), \( x \geq 1 \), the one-step-transition matrix \( P \) of the modified MC \( X_n \) is now given by:

\[
P(0, 0) = q_0, \quad P(0, y) = p_0 \delta_1, \text{if } y \geq 1, \\
P(x, x) = 0 \quad \text{if } x \geq 1, \\
P(x, y) = q \left( \binom{x-1}{y} c^y (1 - c)^{x-1-y} \right), \text{if } x \geq 1 \text{ and } 0 \leq y < x, \\
P(x, y) = p \cdot P(\beta = y - x) = pb_{y-x}, \text{if } x \geq 1 \text{ and } y > x.
\]

Note that because at least one individual dies out in a shrinking event, the diagonal terms \( P(x, x) \) of \( P \) are now 0 for all \( x \geq 1 \).

Remark. One can introduce a holding probability \( r_x \) to stay in state \( x \) given \( X_n = x \), now filling up the diagonal of \( P \). This corresponds to a time change while considering a modified transition matrix \( \tilde{P} \) where: \( p \rightarrow p_x = p (1 - r_x) \) and \( q \rightarrow q_x = q (1 - r_x) \) \( (p_x + q_x + r_x = 1) \). So, with \( \rho := 1 - r \), a column vector with entries \( \rho_x = 1 - r_x \),

\[
P \rightarrow \tilde{P} = I + D_\rho (P - I).
\]
If the invariant measure $\pi$ obeying $\pi' = \pi' P$ (the fixed point of $\pi'_{n+1} = \pi'_{n} P$) exists, then that of $\tilde{P}$ also exists and obeys: $\pi' D\rho = \pi'$.

Suppose first $X_0 = 0$. Let $\mathbf{E} X^n := \Phi_n (z) = \overline{\Phi}_n (z) + \Phi_n (0)$ with $\Phi_0 (z) = 1$ translating $X_0 = 0$. Then, with $p + q = 1$,

$$
\Phi_{n+1} (z) = (q_0 + p_0 z) \Phi_n (0) + p B (z) \overline{\Phi}_n (z) \quad + \frac{q}{1 - c (1 - z)} \overline{\Phi}_n (1 - c) (1 - z) , \overline{\Phi}_0 (z) = 0,
$$

$$
\Phi_{n+1} (0) = q_0 \Phi_n (0) + \frac{q}{1 - c} \overline{\Phi}_n (1 - c) , \Phi_0 (0) = 1.
$$

Note $\Phi_n (0) = \mathbf{P} (X_n = 0)$ and $\overline{\Phi}_n (0) = 0$ for each $n \geq 0$.

Thus, if these fixed point quantities exist

$$
\overline{\Phi}_\infty (z) = p_0 (z - 1) \overline{\Phi}_\infty (0) + p B (z) \overline{\Phi}_\infty (z) + \frac{q}{1 - c (1 - z)} \overline{\Phi}_\infty (1 - c) (1 - z),
$$

$$
\Phi_\infty (0) = \frac{q}{p_0 (1 - c)} \Phi_\infty (1 - c).
$$

We shall iterate the first fixed point equation, which makes sense only when $c \neq 0$.

With $C (z) = \frac{p_0 (z - 1)}{1 - p B (z)}$ and $D (z) = \frac{q}{1 - c (1 - z) (1 - p B (z))}$, we get

$$
\overline{\Phi}_\infty (z) = \Phi_\infty (0) \sum_{n \geq 0} C (1 - c^n (1 - z)) \prod_{m=0}^{n-1} D (1 - c^m (1 - z))
$$

$$
= \Phi_\infty (0) \left[ C (z) + D (z) \sum_{n \geq 1} C (1 - c^n (1 - z)) \prod_{m=1}^{n-1} D (1 - c^m (1 - z))\right]
$$

$$
= \frac{p_0 \Phi_\infty (0) (z - 1)}{1 - p B (z)} \left[ 1 + \sum_{n \geq 1} (c q)^n \prod_{m=1}^{n} \frac{1}{1 - c^m (1 - z)} \frac{1}{1 - p B (1 - c^n (1 - z))}\right].
$$

Except when $c = 1$, the term inside the bracket has no pole at $z = 1$. Then $\overline{\Phi}_\infty (1) = 0$ and so, assuming $\Phi_\infty (0) = 0$, $\overline{\Phi}_\infty (z) = 1$ for all $z \in [0, 1]$ is the only possible solution to the first fixed point equation. Recalling from the second one that $\Phi_\infty (0) = \frac{q}{p_0 (1 - c)} \Phi_\infty (1 - c)$, we conclude that $\Phi_\infty (0) = 0$, and so $\Phi_\infty (z) = 0$ for all $z \in [0, 1]$. The only solution $\Phi_\infty (z)$ is the trivial null one.

It remains for us to study the cases $c = 1$ and $c = 0$.

• If $c = 1$. In this case, only a single individual can stepwise be removed from the population; the transition matrix $P$ is upper-Hessenberg. This constitutes a simple discrete version of a decay/surge model (some kind of time-reversed version of the simple growth/collapse model).

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\[ \Phi_\infty (z) = \frac{p_0 z (z - 1)}{z (1 - pB (z)) - q} \Phi_\infty (0) \]

\[ = \frac{p_0 z (z - 1)}{z - 1 + p(1 - zB (z))} \Phi_\infty (0). \]

with \( \Phi_\infty (0) = 0 \). Letting \( B(z) = (1 - B (z)) / (1 - z) \) the tail generating function of \( \beta \), this is also

\[ \Phi_\infty (z) = \frac{1 - p(1 + zB (z))}{1 - p(1 + zB (z))} \Phi_\infty (0) \]

\[ = \frac{p_0 z (z - 1)}{1 - p(1 + zB (z))} \Phi_\infty (0). \]

With \( B(1) = B' (1) = \mathbb{E} \beta = : \rho, \Phi_\infty (1) = \frac{p_0}{q - p} \Phi_\infty (0) = \frac{1}{1 - \rho} \Phi_\infty (1 - c) \).

We conclude:

**Proposition (Phase transition).**

- **Subcritical case:** \( \Phi_\infty (1) + \Phi_\infty (0) = 1 \Rightarrow \Phi_\infty (0) = (q - pp) / (q - pp + pq) \), well-defined as a probability only if \( pp < q \). In this case, the chain is positive recurrent. The term \( pp \) is the average size of a move up, which has to be smaller than the average size \( q \) of a move down.
- **Critical case:** if \( q = pp \), then \( \Phi_\infty (0) = 0 \Rightarrow \Phi_\infty (z) = 0 \) for each \( z \). The chain is null recurrent, and it has no non-trivial (\( \neq 0 \)) invariant measure \( \pi \).
- **Supercritical case:** if \( \infty \geq pp > q \), the chain is transient at \( \infty \).

**Examples:**

(a) In addition to \( c = 1 \), assume \( B(z) = \alpha z / (1 - \alpha z) \) (a geometric model for \( \beta \) with success probability \( \alpha \)). Then, if \( p < q \alpha, \Phi_\infty (0) = \pi (0) = (q \alpha - p) / (q \alpha + p_0 - p) \) is a probability and

\[ \Phi_\infty (z) = \frac{p_0 z}{1 - p(1 + z/(1 - \alpha z))} \Phi_\infty (0) = \frac{p_0 z (1 - \alpha z)}{q - z (\alpha + p \alpha)} \Phi_\infty (0). \]

Thus, with \( (\alpha < a := (\alpha + p \alpha) / q < 1) \), we obtain that

\[ \pi (x) = [z^x] \Phi_\infty (z) = \pi (0) \frac{p_0}{q} (a - \alpha) a^{-2}, \quad x \geq 1 \]

is the invariant probability measure of the chain, displaying geometric decay at rate \( a \).

(b) If in addition to \( c = 1 \), we assume \( B(z) = z, X \) is reduced to a simple birth and death chain (random walk) on the non-negative integers, reflected at the origin. In this case, we get \( \Phi_\infty (z) = \frac{p_0 z}{1 - p(1 + z)} \Phi_\infty (0) \) with \( (p = 1): \Phi_\infty (0) = (q - p) / (q - p + p) \). The corresponding chain is positive recurrent if \( p < 1/2 \), null recurrent if \( p = 1/2 \) and transient at \( \infty \) when \( p > 1/2 \). In the positive-recurrent
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In both examples, whenever the process is positive recurrent, the invariant measure has geometric decay at different rates $a$ and $p/q < 1$, respectively.

- If $c = 0$ (total disasters)
  
  $$
  \Phi_\infty (z) = p_0 (z - 1) \Phi_\infty (0) + pB (z) \Phi_\infty (z) + q\Phi_\infty (1)
  $$
  
  leading to
  
  $$
  \Phi_\infty (0) = \frac{q}{p_0} \Phi_\infty (1),
  $$

  which results in
  
  $$
  \Phi_\infty (z) = \frac{p_0 z}{1 - pB (z)} \Phi_\infty (0),
  $$

  or
  
  $$
  \Phi_\infty (z) = \left( 1 + \frac{p_0 z}{1 - pB (z)} \right) \Phi_\infty (0). \tag{38}
  $$

  We conclude:

  **Proposition.** In the total disaster case, the modified chain is always positive recurrent with the invariant measure having pgf (38), as a shifted compound geometric distribution.

  **Examples:**

  (a) In addition to $c = 0$, assume $B (z) = \alpha z / (1 - \alpha z)$ (a geometric model for $\beta$ with success probability $\bar{\pi}$). Then

  $$
  \Phi_\infty (z) = \left( 1 + \frac{p_0 z}{1 - pB (z)} \right) \Phi_\infty (0).
  $$

  (b) In addition to $c = 0$, assume $B (z) = z$. Then, with $\pi (0) = q / (p_0 + q)$,

  $$
  \Phi_\infty (z) = \left( 1 + \frac{p_0 z}{1 - pz} \right) \Phi_\infty (0),
  $$

  $$
  \pi (x) = [z^x] \Phi_\infty (z) = \pi (0) p_0 p^{x-1}, x \geq 1,
  $$

  a geometric distribution with decay rate $p$.

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