ON A WEIGHTED SPIN OF THE LEBESGUE IDENTITY

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Abstract. Alladi studied partition theoretic implications of a two variable generalization of the Lebesgue identity. In this short note, we focus on a slight variation of the basic hypergeometric sum that Alladi studied. We present two new partition identities involving weights.

1. Introduction

One of the fundamental identities in the theory of partitions and $q$-series is the Lebesgue identity:

\begin{equation}
\sum_{n \geq 0} \frac{(-aq)_n}{(q)_n} q^\frac{n(n+1)}{2} = \frac{(-aq^2;q^2)_\infty}{(q;q^2)_\infty},
\end{equation}

where $a$ and $q$ are variables and the $q$-Pochhammer symbol is defined as follows

\((a)_n := (a; q)_n := \prod_{i=0}^{n-1} (1 - aq^i),\)

for any $n \in \mathbb{Z} \cup \{\infty\}$. Some combinatorial implications of this result were studied by Alladi [3]. In the same paper, he also did a partition theoretic study of a summation formula due to Ramanujan [5, (1.3.13), p 13]

\begin{equation}
\sum_{n \geq 0} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n (bq)_n} = \frac{(-aq)_\infty}{(bq)_\infty}.
\end{equation}

Alladi called this identity and its dilated forms Generalized Lebesgue identities.

We would like to study a similar function that is not directly related to (1.2) or that satisfies a summation formula, but that still manifest beautiful relations. Let $a$, $z$ and $q$ be variables and define

\begin{equation}
F(a, z, q) := \sum_{n \geq 0} \frac{(za)_n}{(q)_n (zq)_n} z^n q^\frac{n(n+1)}{2}.
\end{equation}

Looking at $F(b, a, q)$ it is clear that this sum is —so to speak—a sibling of the Generalized Lebesgue identity (1.2), and $F(-aq, 1, q)$ is a cousin of the original Lebesgue identity (1.1) with an extra $q$-factorial, $1/(q)_n$, in the summand. This extra factor will be the source of the weights in the combinatorial/partition theoretic study of the identities related to the (1.3). For other references related to weighted partition identities of this spirit one can refer to [2,6,12], and in a wider perspective some other recent weighted partition identities can be found in [1,7,9].

Before any combinatorial study, we would like to note the following theorem.

Theorem 1.1. For variables $a$, $z$ and $q$, we have

\begin{equation}
\sum_{n \geq 0} \frac{(za)_n (zq^{n+1})^\infty}{(q)_n} z^n q^\frac{n(n+1)}{2} = \sum_{n \geq 0} \frac{(-za)_n (-zq^{n+1})^\infty}{(q)_n} (-z)^n q^\frac{n(n+1)}{2}.
\end{equation}
Please note that the only difference between the left- and right-hand sides of (1.4) is $z \mapsto -z$. In other words, the object is even in the variable $z$. In author’s view, the observed symmetry makes this identity visually highly pleasing.

The following sections are arranged as follows. In Section 2, we give a proof of Theorem 2 and note some Corollaries of this result. In Section 3, we study the partition theoretic interpretations of the results in Section 2.

2. Proof of Theorem 1.1

We require two main ingredients for the proof of (1.4). First, it is a known fact that

$$\lim_{\rho \to \infty} \frac{(\rho)_n}{\rho^n} = (-1)^n q^{\frac{n(n+1)}{2}},$$

and, second, Heine Transformation [10, p. 241, III.2]

$$\sum_{n \geq 0} \frac{(a)_n(b)_n}{(c)_n(c)_n} z^n = \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty} \sum_{n \geq 0} \frac{(abz/c)_n(b)_n}{(q)_n(bz)_n} \left(\frac{c}{b}\right)^n.$$

Proof of Theorem 1.1. The function $F(a, z, q)$ can be written as the following due to (2.1):

$$F(a, z, q) = \lim_{\rho \to \infty} \sum_{n \geq 0} \frac{(za)_n(\rho)_n}{(q)_n(zq)_n} \left(-\frac{zq}{\rho}\right)^n.$$

Then we can directly apply the Heine transformation (2.2), and after tending $\rho \to \infty$, one gets

$$F(a, z, q) = \frac{(-zq)_\infty}{(2q)_\infty} F(a, -z, q).$$

Multiplying both sides of (2.3) with $(zq)_\infty$, carrying the infinite $q$-Pochhammers inside the sums, and doing elementary simplifications in the summand level finishes the proof.

It is evident that some special cases of (1.4) (such as $(a, z, q) = (q, 1, q)$) can be summed by utilizing simple summation formulas (such as [10, II.2, p 354] and shown to be equal to $(q^2; q^2)_\infty$). This is not our motivation. We would like to look at special cases of (1.4) to extract some combinatorial information. The $(a, z, q) = (q, 1, q)$ and $(-q, 1, q)$ cases are presented in Corollary 2.1.

Corollary 2.1. Let $q$ be a variable, we have

$$\sum_{n \geq 0} (q^{n+1})_\infty q^{\frac{n(n+1)}{2}} = \sum_{n \geq 0} (-q^{n+1})_\infty \frac{(-q)_n}{(q)_n} (-1)^n q^{\frac{n(n+1)}{2}},$$

$$\sum_{n \geq 0} (-q^{n+1})_\infty (-1)^n q^{\frac{n(n+1)}{2}} = \sum_{n \geq 0} (q^{n+1})_\infty \frac{(-q)_n}{(q)_n} q^{\frac{n(n+1)}{2}}.$$

Another interesting corollary can be seen by picking $a = z = 1$ in (2.3) and using Jacobi Triple Product identity [10, p. 239, II.2],

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty.$$

Corollary 2.2. We have

$$\sum_{n \geq 1} (-1)^n q^{n^2} = \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q)_n(1 + q^n)}.$$

Proof. It is clear that only the $n = 0$ term of the sum on the left-hand side of (2.3) is non-zero when $a = z = 1$, and the total sum on the left hand side is 1:

$$1 = \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 0} \frac{(-1)_n}{(q)_n(-q)_n} (-1)^n q^{\frac{n(n+1)}{2}}.$$
We multiply both sides of this equation by \((q)_\infty/(-q)_\infty\) and observe that

\[
\frac{(q)_\infty}{(-q)_\infty} = \frac{(q; q^2)_\infty}{(-q)_\infty} = (q; q^2)^\infty(q^{n}\infty)(-q)_\infty = (q; q^2)^\infty(q^2; q^2)_\infty.
\]

The right-hand side of the last line is the same as the right-hand side of (2.6) with \(z = -1\). This yields

\[
(2.7) \quad \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \sum_{n \geq 0} \frac{(-1)_n}{(q)_n(-q)_n} (-1)^n q^{\frac{n(n+1)}{2}},
\]

where the left-hand side is coming from (2.6) and the right-hand side is \(F(1, -1, q)\). Splitting the bilateral sum on the left-hand side and using simple cancellations on \((-1)_n/(-q)_n\), using the definition of the \(q\)-factorials, on the right-hand side, we get

\[
(2.8) \quad 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2} = 1 + 2 \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q)_n(1 + q^n)}
\]

This shows claim.

Another proof of this result appears in the author’s joint paper with Berkovich as Lemma 4.1 [7]. Combinatorial interpretation of this identity was done by Bessenrodt–Pak [8] and later by Alladi [1] studies the combinatorial implications of this identity.

3. Partition Theoretic Interpretations of Corollary 2.1

We would like to interpret the identities (2.4) and (2.5) as weighted partition identities. To that end, we need to define what a partition is and some related statistics. A partition (in frequency notation [4]) is a list of the form

\[
(1^{f_1}, 2^{f_2}, 3^{f_3}, \ldots)
\]

where \(f_i \in \mathbb{N} \cup \{0\}\) and all but finitely many \(f_i\) are non-zero. When writing example partitions down, one tends to drop the zero frequency parts to keep the notation clean.

If none of the frequencies \(f_i\) are greater than 1, we call these partitions distinct. One can define the size of a partition \(\pi\) as

\[
|\pi| = \sum_{i \geq 1} i \cdot f_i,
\]

and the sum of all \(f_i\) is the number of parts in a partition, we denote this by \(#(\pi)\). The partition with \(f_i \equiv 0\) for all \(i \in \mathbb{N}\) is the only partition of 0 with 0 parts.

Let \(t(\pi)\) be the number of non-zero frequencies of a partition \(\pi\) starting from \(f_1\). In other words, one can think of \(t(\pi)\) as the length of the initial frequency chain. The length of the initial frequency chain seems to be an underutilized statistic in interpretations of \(q\)-series identities, the only other closely related statistic that the author knows of is used in [7, Thm 3.1]. Let \(p_j(\pi)\) be the maximum index \(i\) such that \(f_i \geq j\) in \(\pi\) and for all \(k \geq i\) has the property \(f_k < j\), if no positive value satisfies this we define \(p_j(\pi) = 0\). Let \(r_j(\pi)\) be the number of different parts with frequencies \(\geq j\).

To exemplify the statistics defined, let \(\pi = (1^4, 2^2, 3^4, 5^1, 6^1)\) then \(|\pi| = 31, #(\pi) = 12, t(\pi) = 3, p_1(\pi) = 6, p_2(\pi) = 3, p_3(\pi) = 3, p_4(\pi) = 3, p_5(\pi) = 0, \ldots, r_1(\pi) = 5, r_2(\pi) = 3, r_3(\pi) = 2, r_4(\pi) = 2, r_5(\pi) = 0, \ldots\)

With the statistics defined above, one can interpret Corollary 2.1 as a weighted partition theorem, where \(i = 1\) corresponds to (2.4) and \(i = 2\) refers to (2.5), as follows.

**Theorem 3.1.** Let \(\mathcal{D}\) be the set of distinct partitions and let \(\mathcal{A}\) be the set of partitions where all the partitions \(\pi \in \mathcal{A}\) satisfy \(p_2(\pi) \leq t(\pi)\). Then for \(i = 1\) and \(i = 2\), we have

\[
(3.1) \quad \sum_{\pi \in \mathcal{D}} w_i(\pi)q^{|\pi|} = \sum_{\pi \in \mathcal{A}} \hat{w}_i(\pi)q^{|\pi|},
\]
where
\begin{align}
w_i(\pi) &= 1 - f_1 \left( \frac{1 - (-1)^{t(\pi)}}{2} \right) (-1)^i \#(\pi), \\
\hat{w}_i(\pi) &= 2r_2(\pi) \left( \frac{-1}{2} t(\pi) + (1)p_2(\pi) \right) (-1)^{i-1}(r_1(\pi)+t(\pi)+p_2(\pi)).
\end{align}

We would like to exemplify Theorem 3.1 with relevant partitions of 6 in Table 1.

| $\pi \in \mathcal{D}$ | $t(\pi)$ | $w_1(\pi)$ | $w_2(\pi)$ | $\pi \in \mathcal{A}$ | $t(\pi)$ | $p_2(\pi)$ | $r_2(\pi)$ | $\hat{w}_1$ | $r_1(\pi)$ | $\hat{w}_2$ |
|---------------------|----------|--------------|--------------|---------------------|----------|-------------|-------------|-----------|--------------|-----------|
| $(6^1)$             | 0        | -1           | 1            | $(6^1)$             | 0        | 0           | 1           | 1         | 1           | -1        |
| $(1^1, 5^1)$        | 1        | 0            | 0            | $(1^1, 5^1)$        | 1        | 0           | 0           | 0         | 2           | 0         |
| $(2^1, 4^1)$        | 0        | 1            | 1            | $(2^1, 4^1)$        | 0        | 0           | 0           | 1         | 2           | 1         |
| $(1^1, 2^1, 3^1)$   | 3        | 0            | 0            | $(1^2, 4^1)$        | 1        | 1           | 1           | -2        | 2           | -2        |
| $(1^1, 2^1, 3^1)$   | 3        | 0            | 0            | $(1^3, 3^1)$        | 1        | 1           | 1           | -2        | 2           | -2        |
| $(1^1, 2^1, 3^1)$   | 3        | 0            | 0            | $(1^4, 2^1)$        | 2        | 1           | 1           | 0         | 2           | 0         |
| $(1^2, 2^2)$        | 2        | 2            | 2            | $(1^6)$             | 1        | 1           | 1           | -2        | 1           | 2         |
| Total:              | 0        | 2            |              |                     | 0        | 2           |              |           |              |           |

One key observation is that $w_2(\pi) = |w_1(\pi)| \geq 0$ for all distinct partitions. This proves that the series in (2.5), which is the analytic version of (3.1) with $i = 2$, have non-negative coefficients. We write this as a theorem using an equivalent form of the left-hand side series of (2.5).

**Theorem 3.2.** We have
\[
(-q, q)\infty \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)}}{(-q, q)_n} \geq 0,
\]
where $\geq 0$ is used to indicate that the series coefficients are all greater or equal than 0.

The sum in Theorem 3.2 is a false theta function that Rogers studied [11]. Although this series has alternating signs, its product with the manifestly positive factor $(-q, q)\infty$ has non-negative coefficients and the above key observation is a combinatorial explanation of this fact.

Broadly speaking, connections of false/partial theta functions and their implications in the theory of partitions have been studied in various places. Interested readers can refer to [1, 6].

**Proof of Theorem 3.1.** This theorem is a consequence of Corollary 2.1, the $i = 1$ and 2 cases correspond to the combinatorial interpretations of (2.4) and (2.5), respectively.

First we focus on the left-hand side summands. For a fixed $n$ and $\varepsilon_1 = \pm 1$, $(\varepsilon_1 q^{n+1})\infty$ is the generating function for the distinct partitions $\pi_d$ where every part is $\geq n+1$ counted with the weight $(-\varepsilon_1)^{\#(\pi_d)}$. We also interpret the $q$-factor, $\varepsilon_2^2 q^{n(n+1)/2}$ as the partition $\pi_i = (1^1, 2^1, \ldots, n^1)$ counted with the weight $\varepsilon_2^2$, where $\varepsilon_2 = \pm 1$. We can combine (add the frequencies of both partitions) $\pi_d$ and $\pi_i$ into a distinct partition $\pi$.

In the sum,
\[
\sum_{n \geq 0} (\varepsilon_1 q^{n+1})\infty \varepsilon_2^2 q^{n(n+1)/2},
\]
there are $t(\pi) + 1$ possible pairs $(\pi_d, \pi_i)$ that can yield $\pi$, and one needs to count the weights of these accordingly. Note that if $t(\pi) \geq 1$ since $\pi$ is a distinct partition $f_1 = 1$. For the total weight of $\pi$, one needs to sum from $k = 0$ to $t(\pi)$ of the alternating weights $(-\varepsilon_1)^{\#(\pi)-k} \varepsilon_2^k$:
\[
\sum_{k=0}^{t(\pi)} (-\varepsilon_1)^{\#(\pi)-k} \varepsilon_2^k.
\]
By reducing the summations of alternating weights, one finds that $w_i(\pi)$ can be represented as in (3.2) for $i = 1$ and 2, where $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_1 = \varepsilon_2 = -1$, respectively.

Similar to the left-hand side’s interpretation, one needs to look at the pieces of the right-hand side summand. For a fixed $n$, once again the parts $(\varepsilon_1 q^{n+1})_{\infty} \varepsilon_2 q^{n(n+1)/2}$ can be interpreted as the generating function for the partition pairs $(\pi_d, \pi_i)$ counted by some weights dependent of $\varepsilon_1$ and $\varepsilon_2$. The new factor $(−q)_n/(q)_n$ is the generating function for the number of overpartitions $\pi_o$, into parts $\leq n$. Overpartitions are the same as partitions counted with the weight $2^{\pi_1(\pi)}$. When we combine $\pi_d$, $\pi_i$, and $\pi_o$, we end up with a partition $\pi$ where some parts may repeat.

Any repetition of the parts in $\pi$ comes from the overpartition $\pi_o$ and these repetitions can only appear for part $\leq t(\pi)$. Note that $\pi_i$, has a single copy of every part size up to $t(\pi)$ and $\pi_o$ may add more occurrences of these parts. This modifies the overpartition related weight a little and we need to take the first occurrence of a part for granted. On the other hand, if a part appears more than once the repetition should be counted with the weight $2^{\pi_2(\pi)}$.

Here the summation bounds are slightly different than the previous case. One needs to sum all the possible $\varepsilon_1$ and $\varepsilon_2$ related weights from $k = p_2(\pi)$ to $t(\pi)$. Different than the previous one, $#(\pi)$ is replaced by the number of non-repeating parts above the initial chain $t(\pi)$, which is $r_1(\pi) - t(\pi)$. Moreover, one needs to replace $k$ by $k - p_2(\pi)$ to eliminate the effect of the parity of $p_2(\pi)$ on the alternating sum. Hence, the sum to reduce here is

$$\sum_{k=p_2(\pi)}^{t(\pi)} (-\varepsilon_1)^{r_1(\pi) - t(\pi) - p_2(\pi) - k} \frac{k}{\varepsilon_2}.$$ 

These sums, once reduced, can be seen to yield $\hat{w}_i(\pi)$ for $i = 1$ and 2, where $\varepsilon_1 = \varepsilon_2 = -1$ and $\varepsilon_1 = \varepsilon_2 = 1$, respectively. \hfill $\Box$

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