Irreducible linear subgroups generated by pairs of matrices with large irreducible submodules

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Abstract

We call an element of a finite general linear group $GL(d, q)$ fat if it leaves invariant, and acts irreducibly on, a subspace of dimension greater than $d/2$. Fatness of an element can be decided efficiently in practice by testing whether its characteristic polynomial has an irreducible factor of degree greater than $d/2$. We show that for groups $G$ with $SL(d, q) \leq G \leq GL(d, q)$ most pairs of fat elements from $G$ generate irreducible subgroups, namely we prove that the proportion of pairs of fat elements generating a reducible subgroup, in the set of all pairs in $G \times G$, is less than $q^{-d+1}$. We also prove that the conditional probability to obtain a pair $(g_1, g_2)$ in $G \times G$ which generates a reducible subgroup, given that $g_1, g_2$ are fat elements, is less than $2q^{-d+1}$. Further, we show that any reducible subgroup generated by a pair of fat elements acts irreducibly on a subspace of dimension greater than $d/2$, and in the induced action the generating pair corresponds to a pair of fat elements.

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1 Introduction

Consider the finite general linear group $GL(d, q)$ for $d \geq 3$, that is the group of invertible $(d \times d)$-matrices over the finite field $F_q$ of order $q$. For a subgroup $G$ of $GL(d, q)$ the underlying vector space of row vectors of length $d$ over $F_q$ becomes a right $F_qG$-module via the natural “vector times matrix” action. We call this module the natural $F_qG$-module. An element $g \in GL(d, q)$ is said to be fat, or more precisely a fat($d, q; e$)-element, if the natural $F_qGL(d, q)$-module has an irreducible $F_q\langle g \rangle$-submodule of dimension $e > d/2$, or equivalently, if the characteristic polynomial for $g$ has an irreducible factor over $F_q$ of degree $e$.

Fat pairs, that is pairs of fat elements, relative to the (not necessarily distinct) integers $e_1, e_2$ are called fat($d, q; e_1, e_2$)-pairs. Further, a pair $(g_1, g_2)$ in $GL(d, q) \times GL(d, q)$ is said to be reducible or irreducible according as the natural $F_q\langle g_1, g_2 \rangle$-module has this property.

Let $SL(d, q)$ denote the finite special linear group, the group of all matrices in $GL(d, q)$ with determinant 1. Motivated by the wish to upgrade the Classical Recognition Algorithm [5] (see discussion in Section 2), we study fat pairs in $G \times G$ for a matrix group $G$ satisfying $SL(d, q) \leq G \leq GL(d, q)$.

We first give an explicit upper bound for the proportion of reducible fat pairs in the set of all pairs in $G \times G$. We denote this proportion by $\text{red} \text{fat}(G)$.

**Theorem 1.1.** Let $d \geq 3$. If $G$ is a group with $SL(d, q) \leq G \leq GL(d, q)$, then

$$\text{red} \text{fat}(G) < q^{-d+1}.$$ 

Let $\text{red} \text{iffat}(G)$ be the proportion of reducible pairs in the set of fat pairs in $G \times G$. Equivalently, we may define $\text{red} \text{iffat}(G)$ to be the (conditional) probability that, on a single random selection from the set of fat pairs in $G \times G$, we obtain a reducible pair. An upper bound for $\text{red} \text{iffat}(G)$ is given in

**Theorem 1.2.** Let $d \geq 3$. If $G$ is a group with $SL(d, q) \leq G \leq GL(d, q)$, then

$$\text{red} \text{iffat}(G) < 2q^{-d+1}.$$ 

Our next theorem shows that each reducible fat pair leads to an irreducible fat pair on a quotient space of dimension greater than $d/2$.

**Theorem 1.3.** For integers $d, e_1, e_2$ satisfying $1 < d/2 < e_1, e_2 \leq d$, let $(g_1, g_2) \in GL(d, q) \times GL(d, q)$ be a fat($d, q; e_1, e_2$)-pair, and let $V$ be the natural $F_qGL(d, q)$-module. Then there exists an $F_q\langle g_1, g_2 \rangle$-composition factor $N$ of $V$ with $n = \dim(N) \geq \max\{e_1, e_2\} > d/2$, such that writing $\overline{g_1}, \overline{g_2}$ for the element in $GL(n, q)$ induced by $g_i$ on $N$, $(\overline{g_1}, \overline{g_2})$ is an irreducible fat($n, q; e_1, e_2$)-pair.
The proofs of Theorems 1.1 and 1.2 (see Subsections 5.2 and 5.3) rely on the following observation. A fat pair \((g_1, g_2) \in G \times G\), where \(G\) satisfies \(\text{SL}(d, q) \leq G \leq \text{GL}(d, q)\), is reducible, if and only if there exists a non-trivial and proper \(\langle g_1, g_2 \rangle\)-invariant subspace \(W \leq V\). In this case \(g_1, g_2\) lie in the maximal parabolic subgroup \(G_W \leq G\). The key ingredient to prove Theorems 1.1 and 1.2 is to show in Lemma 4.4 that for \(e > d/2\) the proportion of \(\text{fat}(d, q; e)\)-elements in \(G_W\) equals the proportion of \(\text{fat}(d, q; e)\)-elements in \(\text{GL}(d, q)\). The results then follow by summing the number of fat pairs over all possible maximal parabolic subgroups of \(G\). The proof of Theorem 1.3 is presented in Subsection 5.1. In Section 2 we motivate the results of this paper. The linear algebra background required is presented in Section 3, while the group theoretic preliminaries are in Section 4.

2 Motivation

The principal motivation for the work reported in this paper is the Classical Recognition Algorithm [5]. This is a one-sided Monte Carlo algorithm that, given a set of generating matrices for a subgroup \(G\) of the finite general linear group \(\text{GL}(d, q)\), examines whether \(G\) contains a “classical group” in its natural representation, that is whether (in its natural representation) \(G\) contains \(\text{SL}(d, q)\), or a \(d\)-dimensional symplectic, unitary or orthogonal group defined over \(\mathbb{F}_q\). The performance of the algorithm has been described by Leedham-Green in [3] as “one of the most efficient algorithms in the business”. The algorithm seeks particular kinds of elements, called ppd-elements, in \(G\) by making independent uniformly distributed random selections of elements from \(G\). A ppd-element, or more precisely a ppd\((d, q; e)\)-element for some integer \(e\) with \(e \leq d\), is an element \(g \in \text{GL}(d, q)\) such that \(g\) has order divisible by a prime divisor of \(q^e - 1\) which does not divide \(q^j - 1\) for any \(j < e\). It is shown in [5] that ppd\((d, q; e)\)-elements with \(e\) greater than \(d/2\) are very likely to occur in classical groups. Under some additional hypotheses, finding a pair of ppd-elements from \(G\) allows us to conclude that \(G\) contains a classical group. The proof of this relies on good estimates of the proportions of ppd-elements along with deep group theoretic analysis (depending on the simple group classification). In the long run, we wish to upgrade the Classical Recognition Algorithm in a threefold manner as described below. This paper takes a first step in this direction.

First, note that by [5] Lemma 5.1, given a ppd\((d, q; e)\)-element \(g\) with \(e > d/2\), there exists a unique irreducible \(e\)-dimensional \(\mathbb{F}_q\langle g\rangle\)-submodule of the natural \(\mathbb{F}_q\text{GL}(d, q)\)-module. In particular, \(g\) is a fat\((d, q; e)\)-element. While every ppd-element is fat the converse implication is not true, as the presence
of an $e$-dimensional irreducible $\mathbb{F}_q\langle g \rangle$-submodule of the natural $\mathbb{F}_qGL(d, q)$-module is not sufficient to guarantee that $g$ is a ppd$(d, q; e)$-element. For example in $GL(3, 3)$, an element of order 8 is a fat$(3, 3; 2)$-element but not a ppd$(3, 3; 2)$-element since $3^2 - 1 = 8$ has no prime divisors which do not divide $3 - 1 = 2$. However, even though fat elements do not necessarily need to be ppd-elements, most of them turn out to be. Our goal is to remove the restriction of looking for ppd-elements in the Classical Recognition Algorithm and evolve the algorithm into one based solely on elements with large irreducible submodules. Dropping the ppd-property should result in an even better performance of the algorithm as in practice fatness can be tested more cheaply than the ppd-property by finding an irreducible factor of degree greater than $d/2$ of the characteristic polynomial. The wish to waive the ppd-property raises the following problem which we intend to address in further work.

**Problem 2.1.** Describe all subgroups of $GL(d, q)$ containing an irreducible fat$(d, q; e_1, e_2)$-pair for $1 < d/2 < e_1, e_2 \leq d$.

As presented in [5], the Classical Recognition Algorithm takes as input a basis for the non-degenerate sesquilinear forms preserved by the subgroup $G \leq GL(d, q)$, as well as the knowledge that $G$ is irreducible on the underlying vector space. This requirement is reasonable as efficient algorithms for testing irreducibility exist (namely the Meataxe algorithm due to Richard Parker [6] and the improved, general purpose version of it developed by Holt and Rees [2]). Yet, we wish to develop a new (fat element based) recognition algorithm without the necessity to test for irreducibility. In order to evaluate how this move modifies the situation, Theorem 1.2 gives a good upper bound for the (conditional) probability of obtaining, on a single random selection from the set of fat pairs in $G \times G$ (where $SL(d, q) \leq G \leq GL(d, q)$), a reducible pair. We expect that similar bounds will hold if $\Omega(d, q) \leq G \leq N_{GL(d,q)}(\Omega(d, q))$ for any classical group $\Omega(d, q)$.

Finally, by Theorem 1.3 if for a given matrix group $G \leq GL(d, q)$ with $d \geq 3$, $G \times G$ contains a fat pair, then $G$ has a quotient $H$ which is isomorphic to a matrix group of degree $n > d/2$, such that $H \times H$ contains an irreducible fat pair. This suggests that recognition of groups containing classical groups could be generalised to test if a (reducible) subgroup of $GL(d, q)$ has a large quotient containing $SL(n, q)$ or an $n$-dimensional symplectic, unitary or orthogonal group, with $n > d/2$. 

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3 Linear algebra preliminaries

Throughout this section let $q$ be a power of a prime, $d$ a non-negative integer, and $V$ a $d$-dimensional vector space defined over the finite field $\mathbb{F}_q$.

The proofs of Theorems 1.1 and 1.2 involve counting certain subspaces in $V$. As usual we denote the number of $w$-dimensional subspaces in $V$ (for $0 \leq w \leq d$) by so-called Gaussian coefficients (see for example [4, p. 124]).

**Definition 3.1.** For a non-negative integer $w \leq d$, the Gaussian coefficient \( \binom{d}{w}_q \) is defined to be the number of $w$-dimensional subspaces in $V$.

An explicit formula for \( \binom{d}{w}_q \) is given for example in [4, (9.2.2)].

**Lemma 3.2.** Let $w \leq d$ be a non-negative integer. Then

\[
\binom{d}{w}_q = \frac{\prod_{i=d-w+1}^{d}(q^i - 1)}{\prod_{u=1}^{w}(q^u - 1)},
\]

and in particular \( \binom{d}{d}_q = \binom{d}{d-w}_q \).

For a rational number $r$ let $\lceil r \rceil$ be the smallest integer which is at least $r$.

**Lemma 3.3.** If $d \geq 3$, then $\sum_{i=1}^{[d/2]-1} \binom{d}{i}_q^{-1} < q^{-d+1}$.

**Proof.** Since for $i \in [1,d]$, \( \binom{d}{i}_q \) is the number if $i$-dimensional subspaces in $V$, we have \( \binom{d}{2}_q < \binom{d}{i}_q \) for $2 < i \leq \lceil d/2 \rceil - 1$, and obtain

\[
\sum_{i=1}^{[d/2]-1} q^{d-1}/\binom{d}{i}_q \leq q^{d-1}/\binom{d}{1}_q + (\lfloor d/2 \rfloor - 2)q^{d-1}/\binom{d}{2}_q.
\]

Note that $q^{d-1}/\binom{d}{1}_q < 1 - q^{-1} + q^{-d}$ and $q^{d-1}/\binom{d}{2}_q < q^{-d+3}$, whence

\[
\sum_{i=1}^{[d/2]-1} q^{d-1}/\binom{d}{i}_q < 1 - q^{-1} + q^{-d} + (\lfloor d/2 \rfloor - 2)q^{-d+3} =: \mu(d,q).
\]

If $d \in \{3,4\}$, then $\mu(d,q) < 1$. For $d \geq 5$ we use induction on $d$ to show that $\mu(d,q) < 1$. Now, $\mu(5,q) = 1 - q^{-1} + q^{-5} + q^{-2} < 1$. Next, assuming $\mu(d,q) < 1$, we have

\[
\mu(d+1,q) = 1 - q^{-1} + q^{-d+1} + (\lceil (d+1)/2 \rceil - 2)q^{-d+2} < 1 - q^{-1} + q^{-d} + (\lceil d/2 \rceil - 1)q^{-d+2} + (\lfloor d/2 \rfloor - 2)q^{-d+3} - (\lceil d/2 \rceil - 2)q^{-d+3} =: \mu(d+1,q).
\]
By assumption, $\mu(d, q) = 1 - q^{-1} + q^{-d} + \left\lfloor d/2 \right\rfloor - 2)q^{-d+3} < 1$, and thus

$$
\mu(d + 1, q) < 1 + \left\lfloor d/2 \right\rfloor - 1)q^{-d+2} - \left\lfloor d/2 \right\rfloor - 2)q^{-d+3}.
$$

Using $q \geq 2$ and $d \geq 5$,

$$
\mu(d + 1, q) < 1 + \left\lfloor d/2 \right\rfloor - 1)q^{-d+2} - 2 \left( \left\lfloor d/2 \right\rfloor - 3 \right) q^{-d+3} \leq 1.
$$

We therefore have $\sum_{i=1}^{\left\lfloor d/2 \right\rfloor - 1} q^{d-1} / \binom{d}{i} q < \mu(d, q) < 1$, as asserted.

\[\square\]

4 Group theory preliminaries

In this section we assume that $d \geq 2$ is an integer, and $q$ is a power of a prime. Let $V$ be the natural $\mathbb{F}_q\text{GL}(d, q)$-module, that is the vector space of $d$-dimensional row vectors over $\mathbb{F}_q$ on which $\text{GL}(d, q)$ acts naturally.

For $G \leq \text{GL}(d, q)$ and a subspace $W \leq V$, we denote by $G_W$ the subgroup of $G$ which leaves $W$ invariant, that is $G_W = \{ g \in G \mid \forall g = W \}$. Using an argument very similar to [7, proof of Theorem 4.1] we obtain

Lemma 4.1. Let $e, w \in [0, d]$ be integers, and $W \leq V$ of dimension $w$.

(a) If $e + w \leq d$, then $\text{SL}(d, q)_W$ acts transitively on the set of all $e$-dimensional subspaces $U \leq V$ such that $U \cap W = \{0\}$.

(b) If $e \leq w \leq d$, then $\text{SL}(d, q)_W$ acts transitively on the set of all $e$-dimensional subspaces $U \leq W$.

In particular, $\text{SL}(d, q)$ is transitive on the all $e$-dimensional subspaces in $V$.

As specified in the introduction, we call an element $g \in \text{GL}(d, q)$ a fat($d, q; e$)-element, if $V$ has an irreducible $\mathbb{F}_q\langle g \rangle$-submodule of dimension $e > d/2$. In the remainder of this section we shall be concerned with the proportions of fat($d, q; e$)-elements in (maximal parabolic subgroups of) $G$, where $G$ satisfies $\text{SL}(d, q) \leq G \leq \text{GL}(d, q)$.

Definition 4.2. For an integer $e \in (d/2, d]$ and $G \leq \text{GL}(d, q)$, define fat($G; e$) to be the proportion of fat($d, q; e$)-elements in $G$. Set fat($e$) := fat($\text{GL}(e, q); e$).

Lemma 4.3. For an integer $e \geq 2$ we have $1/(e + 1) \leq \text{fat}(e) < 1/e$. 

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Proof. For \( e \geq 3 \), the lower bound is given in [4, Lemma 2.3]. From the proof of the same lemma it follows that for all \( e \geq 2 \) we have \( \text{fat}(e) = |C_0|/(e|C|) \), where \( C_0 \) is a proper subset of \( C \leq \text{GL}(e, q) \) with

\[
|C| = q^e - 1, \quad |C| - \sum_{f|e \text{ proper}} (q^f - 1) \leq |C_0| < |C|.
\]

For \( e = 2 \) we thus get (using \( q \geq 2 \))

\[
\text{fat}(2) = \frac{|C_0|}{e|C|} \geq \frac{q^2 - 1 - (q - 1)}{2(q^2 - 1)} = \frac{q}{2(q + 1)} \geq \frac{2}{2 \cdot 3} = \frac{1}{3},
\]

as required. Since \( |C_0|/|C| < 1 \), the upper bound follows (for all \( e \geq 2 \)).

The proof of the following lemma is based on [5, proof of Lemma 5.4].

**Lemma 4.4.** Let \( G \) be a group satisfying \( \text{SL}(d, q) \leq G \leq \text{GL}(d, q) \), and let \( W \leq V \). Let \( e \) be an integer such that \( e \in (d/2, d) \). Then, \( G_W \) contains a fat\((d, q; e)\)-element if and only if \( \dim(W) \in [0, d - e] \cup [e, d] \), and in this case

\[
\text{fat}(G_W; e) = \text{fat}(e).
\]

In particular, \( \text{fat}(G; e) = \text{fat}(e) \).

**Proof.** We set \( H := G_W \). If \( \dim(W) \in [0, d - e] \cup [e, d] \), then it is easy to verify that \( H \) contains a fat\((d, q; e)\)-element.

Conversely, suppose that \( H \) contains a fat\((d, q; e)\)-element \( g \), and let \( \mathcal{U} \) be the irreducible \( \mathbb{F}_q(g)\)-submodule of \( V \) with \( \dim(\mathcal{U}) = e \). Note that \( \mathcal{U} \) is uniquely determined, as it is irreducible and of dimension \( e > d/2 \). The intersection \( \mathcal{U} \cap W \) is an \( \mathbb{F}_q(g)\)-submodule of \( \mathcal{U} \). Hence \( \mathcal{U} \cap W \in \{\{0\}, \mathcal{U}\} \), and in particular \( \dim(W) \leq d - e \) or \( \dim(W) \geq e \). Recall from Lemma 4.1 that \( H \) acts transitively on the set \( U \), where

\[
U := \begin{cases} 
\{U' \leq V \mid \dim(U') = e, \mathcal{U}' \cap W = \{0\}\}, & \text{if } \dim(W) \leq d - e, \\
\{U' \leq V \mid \dim(U') = e, \mathcal{U}' \leq W\}, & \text{if } \dim(W) \geq e.
\end{cases}
\]

Since \( \mathcal{U} \in U \), by the orbit stabiliser theorem \( |U| = |H : H_{\mathcal{U}}| \). Thus, the number of fat\((d, q; e)\)-elements in \( H \) equals \( |H : H_{\mathcal{U}}| \) times the number of fat\((d, q; e)\)-elements in \( H_{\mathcal{U}} \), that is \( \text{fat}(H; e)|H| = |H : H_{\mathcal{U}}| \text{fat}(H_{\mathcal{U}}; e)|H_{\mathcal{U}}| \), whence \( \text{fat}(H; e) = \text{fat}(H_{\mathcal{U}}; e) \).

Let \( \mathcal{X} : H_{\mathcal{U}} \to \text{GL}(e, q) \) be the representation afforded by \( \mathcal{U} \) as an \( \mathbb{F}_qH_{\mathcal{U}}\)-submodule of \( V \). Let \( \ker(\mathcal{X}) \) be the kernel of \( \mathcal{X} \). If for \( g \in H_{\mathcal{U}} \) the coset \( \ker(\mathcal{X})g \) contains a fat\((d, q; e)\)-element, then every element of \( \ker(\mathcal{X})g \)
is a \( fat(d, q; e) \)-element. It follows that the number of \( fat(d, q; e) \)-elements in \( H_\mathcal{U} \) equals \( |\ker(\mathcal{X})| \) times the number of \( fat(e, q; e) \)-elements in \( \mathcal{X}(H_\mathcal{U}) \), that is \( fat(H_\mathcal{U}; e)|\mathcal{H}_\mathcal{U}| = |\ker(\mathcal{X})|fat(\mathcal{X}(H_\mathcal{U}); e)|\mathcal{X}(H_\mathcal{U})| \). Then, using \( |H_\mathcal{U}| = |\ker(\mathcal{X})|\mathcal{X}(H_\mathcal{U})| \), we get \( fat(H_\mathcal{U}; e) = fat(\mathcal{X}(H_\mathcal{U}); e) \).

Finally, since \( e < d \), we have \( \mathcal{X}(H_\mathcal{U}) \cong GL(e, q) \), and thus \( fat(H_\mathcal{U}; e) = fat(e) \). This proves the assertion, as \( fat(H; e) = fat(H_\mathcal{U}; e) = fat(e) \).

By setting \( \mathcal{W} := \{0\} \) we obtain that \( fat(G; e) = fat(e) \).  

\[\square\]

5 Proofs of main results

Throughout this section let \( d \geq 3 \) be a positive integer, \( \mathbb{F}_q \) a finite field of order \( q \) for some prime power \( q \), and \( \mathcal{V} \) the natural \( \mathbb{F}_q GL(d, q) \)-module.

5.1 Proof of Theorem 1.3

If \( (g_1, g_2) \in GL(d, q) \times GL(d, q) \) is a \( fat(d, q; e_1, e_2) \)-pair for some integers \( e_1, e_2 > d/2 \), then by definition \( g_i \) determines an (uniquely determined) \( e_i \)-dimensional irreducible \( \mathbb{F}_q(g_i) \)-submodule \( \mathcal{U}_i \) of \( \mathcal{V} \) \((i = 1, 2) \). In addition, there may or may not exist a proper and non-trivial \( \mathbb{F}_q(g_1, g_2) \)-submodule \( \mathcal{W} \) of \( \mathcal{V} \) according as \((g_1, g_2)\) is reducible or not. The following lemma presents a basic, yet critical property of \( \mathcal{W} \) in such a setting. Note that, if \( \max\{e_1, e_2\} = d \), then \((g_1, g_2)\) is irreducible. Hence, in order that \( \mathcal{W} \) exists, we assume that each \( e_i < d \). We write \( \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\mathbb{F}_q(g_1, g_2)} \) for the intersection of all \( \mathbb{F}_q(g_1, g_2) \)-submodules in \( \mathcal{V} \) which contain \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \).

Lemma 5.1. Let \( e_1, e_2 \in \mathbb{N} \) with \( 1 < d/2 < e_1, e_2 < d \), and let \( (g_1, g_2) \) be a reducible \( fat(d, q; e_1, e_2) \)-pair in \( GL(d, q) \times GL(d, q) \). For \( i = 1, 2 \) let \( \mathcal{U}_i \) denote the irreducible \( \mathbb{F}_q(g_i) \)-submodule of \( \mathcal{V} \) of dimension \( e_i \), and let \( \mathcal{W} \notin \{\{0\}, \mathcal{V}\} \) be a \( \mathbb{F}_q(g_1, g_2) \)-submodule of \( \mathcal{V} \). Then exactly one of the following holds:

(a) \( \mathcal{W} \cap \mathcal{U}_i = \{0\} \), and \( 1 \leq \dim(\mathcal{W}) \leq d - \max\{e_1, e_2\} \), or

(b) \( \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\mathbb{F}_q(g_1, g_2)} \leq \mathcal{W} \) and \( \max\{e_1, e_2\} \leq \dim(\mathcal{W}) \leq d - 1 \).

In particular, \( \dim(\mathcal{W}) \in [1, d - \max\{e_1, e_2\}] \cup [\max\{e_1, e_2\}, d - 1] \).

Proof. For \( i = 1, 2 \) the intersection \( \mathcal{W} \cap \mathcal{U}_i \) is an \( \mathbb{F}_q(g_i) \)-submodule of \( \mathcal{U}_i \). Since \( \mathcal{U}_i \) is irreducible it follows that \( \mathcal{W} \cap \mathcal{U}_i \) is trivial or non-proper. Suppose that for some \( i \in \{1, 2\} \), \( \mathcal{W} \cap \mathcal{U}_i = \{0\} \) and \( \mathcal{W} \cap \mathcal{U}_{3-i} = \mathcal{U}_{3-i} \). Then \( \mathcal{U}_i \cap \mathcal{U}_2 = \{0\} \) which contradicts \( \dim(\mathcal{U}_i) = e_i > d/2 \). Thus either \( \mathcal{W} \cap \mathcal{U}_i = \{0\} \) for \( i = 1, 2 \), or \( \mathcal{W} \cap \mathcal{U}_i = \mathcal{U}_i \) for \( i = 1, 2 \). In the first case, \( 1 \leq \dim(\mathcal{W}) \leq d - \max\{e_1, e_2\} \) and (a) holds. In the second case, \( \max\{e_1, e_2\} \leq \dim(\mathcal{W}) \leq d - 1 \), and as each \( \mathcal{U}_i \leq \mathcal{W} \), also \( \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\mathbb{F}_q(g_1, g_2)} \leq \mathcal{W} \), so (b) holds.  

\[\square\]
Proof of Theorem 1.1 For $i = 1, 2$, let $\mathcal{U}_i$ denote the $F_q(g_i)$-submodule of $V$ with $\dim(\mathcal{U}_i) = e_i$. Let $\mathcal{X} := \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{F_q(g_1, g_2)}$ and let $\mathcal{Y}$ be an $F_q(g_1, g_2)$-submodule of $\mathcal{X}$ maximal by inclusion with respect to the property $\mathcal{U}_i \cap \mathcal{Y} = \mathcal{U}_2 \cap \mathcal{Y} = \{0\}$. Define $\mathcal{N} = \mathcal{X}/\mathcal{Y}$. For $i = 1, 2$, $\mathcal{U}_i \cong (\mathcal{U}_i \oplus \mathcal{Y})/\mathcal{Y} \leq \mathcal{X}/\mathcal{Y}$ can be viewed as a submodule of $\mathcal{N}$. It follows that $\dim(\mathcal{N}) \geq \max\{e_1, e_2\}$, and that the pair $(\mathcal{g}_1, \mathcal{g}_2)$ induced by $(g_1, g_2)$ on $\mathcal{N} \times \mathcal{N}$ is a fat($n, q; e_1, e_2$)-pair.

It remains to prove that $(\mathcal{g}_1, \mathcal{g}_2)$ is irreducible, that is $\mathcal{N}$ is an $F_q(g_1, g_2)$-composition factor of $V$. We do this by showing that $\mathcal{Y}$ is a maximal $F_q(g_1, g_2)$-submodule of $\mathcal{X}$. Suppose that there exists an $F_q(g_1, g_2)$-module $\mathcal{W}$ satisfying $\mathcal{Y} < \mathcal{W} < \mathcal{X}$. By Lemma 5.1, we either have $\mathcal{W} \cap \mathcal{U}_i = \{0\}$ for $i = 1, 2$, or $\langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{F_q(g_1, g_2)} \leq \mathcal{W}$. Since $\mathcal{X} = \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{F_q(g_1, g_2)}$ and $\mathcal{X} \nleq \mathcal{W}$, the latter case cannot occur. Hence, $\mathcal{W}$ is a proper $F_q(g_1, g_2)$-submodule of $\mathcal{X}$ that satisfies $\mathcal{W} \cap \mathcal{U}_i = \{0\}$ and properly contains $\mathcal{Y}$. This, however, is not true as we have chosen $\mathcal{Y}$ to be maximal with respect to this property. 

5.2 Proof of Theorem 1.1

Given a group $G$, which satisfies $\text{SL}(d, q) \leq G \leq \text{GL}(d, q)$, we wish to find a good upper bound for the proportion $\text{redandfat}(G)$ of reducible fat pairs in $G \times G$. As a first step, we consider the proportion of reducible fat pairs relative to some fixed parameters $e_1, e_2 > d/2$.

Definition 5.2. For a group $G$ such that $\text{SL}(d, q) \leq G \leq \text{GL}(d, q)$, and integers $e_1, e_2 \in (d/2, d]$ we define $\text{redandfat}(G; e_1, e_2)$ to be the proportion of reducible $\text{fat}(d, q; e_1, e_2)$-pairs in the set of all pairs in $G \times G$.

Lemma 5.3. Let $e_1, e_2 \in \mathbb{N}$ such that $d/2 < e_1, e_2 < d$, and let $G$ be a group satisfying $\text{SL}(d, q) \leq G \leq \text{GL}(d, q)$. Then

$$\text{redandfat}(G; e_1, e_2) < 2\text{fat}(e_1)\text{fat}(e_2)\frac{q^{-d+1}}{q^{-d+1}}.$$ 

Proof. The pair $(g_1, g_2) \in G \times G$ is a reducible $\text{fat}(d, q; e_1, e_2)$-pair if and only if there exists at least one non-trivial and proper subspace $\mathcal{W} \leq \mathcal{V}$ such that $g_i$ is a $\text{fat}(d, q; e_i)$-element in $G_{\mathcal{W}}$. By Lemma 5.1, $\dim(\mathcal{W}) \in [1, d - \max\{e_1, e_2\}] \cup [\max\{e_1, e_2\}, d - 1]$. We thus obtain the following upper bound for the number of reducible $\text{fat}(d, q; e_1, e_2)$-pairs in $G \times G$.

$$\text{redandfat}(G; e_1, e_2) \leq \sum_w \sum_{\mathcal{W}} \prod_{i=1,2} (\text{fat}(G_{\mathcal{W}}; e_i) | G_{\mathcal{W}}|),$$

where $w \in [1, d - \max\{e_1, e_2\}] \cup [\max\{e_1, e_2\}, d - 1]$, and $\mathcal{W} \leq \mathcal{V}$ with $\dim(\mathcal{W}) = w$. By Lemma 4.3, $\prod_{i=1,2} \text{fat}(G_{\mathcal{W}}; e_i) = \text{fat}(e_i)$, and hence

$$\text{redandfat}(G; e_1, e_2) \leq \text{fat}(e_1)\text{fat}(e_2) \sum_w \sum_{\mathcal{W}} |G : G_{\mathcal{W}}|^{-2},$$

9
with \( w, \mathcal{W} \) as before. Since \( G \) acts transitively on the set of all \( w \)-dimensional subspaces in \( \mathcal{V} \) there is a total of \( |G : G_{\mathcal{W}}| \) such subspaces, whence

\[
\text{redandfat}(G; e_1, e_2) \leq \text{fat}(e_1)\text{fat}(e_2) \sum_{w} |G : G_{\mathcal{W}}|^{-1},
\]

where \( w \) as before. Using the notation from Definition 3.1 we write \( (d_w)_q = |G : G_{\mathcal{W}}| \). Then, since \( (d_w)_q = \binom{d}{d-w}_q \), and since \( d - \max\{e_1, e_2\} \leq [d/2] - 1, \)

\[
\text{redandfat}(G; e_1, e_2) \leq 2\text{fat}(e_1)\text{fat}(e_2) \sum_{w=1}^{[d/2]-1} \binom{d}{d-w}_q^{-1}.
\]

Then, by Lemmas 3.3 and 4.3 \( \text{redandfat}(G; e_1, e_2) < 2\text{fat}(e_1)\text{fat}(e_2)q^{-d+1} < 2/(e_1e_2)q^{-d+1}. \)

Note that for \( G \) with \( \text{SL}(d, q) \leq G \leq \text{GL}(d, q) \) we have \( \text{redandfat}(G) = \sum_{d/2 < e_1, e_2 \leq d} \text{redandfat}(G; e_1, e_2) \). This observation together with the upper bound given in Lemma 5.3 are the main ingredients of the

**Proof of Theorem 1.1** In the case \( d = \max\{e_1, e_2\} \) any fat \((d, q; e_1, e_2)\)-pair is irreducible, and thus \( \text{redandfat}(G; e_1, e_1) = 0. \) Hence, using Lemma 7.3

\[
\text{redandfat}(G) = \sum_{e_1, e_2} \text{redandfat}(G; e_1, e_2) < \sum_{e_1, e_2} 2/(e_1e_2)q^{-d+1},
\]

where \( \lceil(d+1)/2 \rceil \leq e_1, e_2 \leq d - 1. \) An easy argument estimating the sum by an integral shows that \( \sum_{i=0}^{d-1} (d+1)/2 i^{-1} < \ln(2). \) Hence, \( \text{redandfat}(G) < 2(\ln(2))^2q^{-d+1} < q^{-d+1}, \) as required.

### 5.3 Proof of Theorem 1.2

Recall that for a group \( G \) with \( \text{SL}(d, q) \leq G \leq \text{GL}(d, q) \) we write \( \text{redifat}(G) \) for the proportion of reducible fat pairs in the set of fat pairs from \( G \times G \). Our final task is to prove the upper bound for \( \text{redifat}(G) \) given in Theorem 1.2.

**Proof of Theorem 1.2** For integers \( e_1, e_2 \) with \( d/2 < e_1, e_2 \leq d \) we write \( \text{redandfat}(G; e_1, e_2) |G|^2 \) for the number of reducible fat \((d, q; e_1, e_2)\)-pairs in \( G \times G \), and \( \text{fat}(G; e_1)\text{fat}(G; e_2) |G|^2 \) for the number of fat \((d, q; e_1, e_2)\)-pairs in \( G \times G \). By Lemma 4.4 we have \( \text{fat}(G; e_i) = \text{fat}(e_i) \) for \( i = 1, 2 \), whence

\[
\text{redifat}(G) = \frac{\sum_{d/2 < e_1, e_2 \leq d} \text{redandfat}(G; e_1, e_2) |G|^2}{\sum_{d/2 < e_1, e_2 \leq d} \text{fat}(e_1)\text{fat}(e_2) |G|^2}.
\]
If \( \max\{e_1, e_2\} = d \), then every \( \text{fat}(d, q; e_1, e_2) \)-pair in \( G \) is irreducible, and hence \( \text{red and fat}(G; e_1, e_2) = 0 \) in that case. If \( e_1, e_2 \in (d/2, d) \), then

\[
\text{red and fat}(G; e_1, e_2) < 2 \text{fat}(e_1) \text{fat}(e_2) q^{-d+1}
\]

by Lemma 5.3. Note also that being a proportion \( \text{fat}(e_i) \geq 0 \), and thus

\[
\sum_{d/2 < e_1, e_2 < d} \text{fat}(e_1) \text{fat}(e_2) \geq \sum_{d/2 < e_1, e_2 < d} \text{fat}(e_1) \text{fat}(e_2).
\]

We obtain

\[
\text{red if fat}(G) < \frac{\sum_{d/2 < e_1, e_2 < d} 2 \text{fat}(e_1) \text{fat}(e_2) q^{-d+1}}{\sum_{d/2 < e_1, e_2 < d} \text{fat}(e_1) \text{fat}(e_2)} = 2q^{-d+1}.
\]

\[\Box\]

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