Out of equilibrium $O(N)$ linear-sigma system — Construction of perturbation theory with gap- and Boltzmann-equations

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Abstract

We establish from first principles a perturbative framework that allows us to compute reaction rates for processes taking place in nonequilibrium $O(N)$ linear-sigma systems in broken phase. The system of our concern is quasiuniform system near equilibrium or nonequilibrium quasistationary system. We employ the closed-time-path formalism and use the so-called gradient approximation. No further approximation is introduced. In the course of construction of the framework, we obtain the gap equation that determines the effective masses of $\pi$ and of $\sigma$, and the generalized Boltzmann equation that describes the evolution of the number-density functions of $\pi$ and of $\sigma$.

11.10.Wx, 11.30.Qc, 11.30.Rd

I. INTRODUCTION

Lattice Quantum Chromodynamics (QCD) results indicate that chiral symmetry is spontaneously broken at $T \sim 150$ MeV. Such temperatures may be reached in relativistic heavy-ion collisions. Thus, the time evolution of the chiral phase transition may be traced through observing predicted consequences, e.g., in the production rates of dilepton and of photon, etc. Earlier work is quoted therein.) Since Rajagopal and Wilczek proposed that heavy-ion collisions can generate large DCC domains, analyses of dynamical evolution of the system from a symmetric phase to a broken phase have energized. The analysis in [5] has been refined in [6], and has been extended [7] by incorporating more realistic descriptions of heavy-ion dynamics. The effects of quasiparticle excitations are studied in [8]. Refinements employing realistic initial conditions are made, e.g., in [9]. Since then, much work has been devoted to the analysis by systematically taking quantum and medium effects into account: For example, analysis on the basis of density-matrix formalism is made in [10], large-$N$ limit of the $O(N)$ linear-sigma model has been studied in [11], closed-time-path (CTP) formalism of nonequilibrium dynamics is employed in [12], Calderira-Leggett theory is applied in [13], a self-consistent-variational approach has been taken in [14], the time evolution of a particle distribution is studied in [15], and relaxation rate for long-wavelength fluctuations are analyzed in [16]. (For other related works, see, e.g., [17].) Different assumptions and approximations are employed in these analyses.

In this paper, as a first step toward the goal, we lay down from first principles a perturbative framework on the basis of a loop-expansion scheme. Only approximation we use is the so-called gradient approximation (see below). We use the standard framework of nonequilibrium statistical quantum-field theory that is formulated by employing the closed-time path, $-\infty \to +\infty \to -\infty$, in a complex-time plane, which is referred to as the CTP formalism.

Throughout this paper, we are interested in quasiuniform systems near equilibrium or nonequilibrium quasistationary systems. Such systems are characterized by two different spacetime scales; microscopic or quantum-field-theoretical and macroscopic or statistical. The first scale, the microscopic correlation scale, characterizes the reaction taking place in the system, while the second scale measures the relaxation of the system. For a weak coupling theory, in which we

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are interested in this paper, the former scale is much smaller than the latter scale.\footnote{It should be noted, however, that, as the system approaches the critical point of the phase transition, the microscopic correlation scale diverges. Thus, the formalism developed in this paper applies to the systems away from the critical point.} In a derivative expansion with respect to macroscopic spacetime coordinates $X$, we use the gradient approximation throughout:

$$F(X, \ldots) \simeq F(Y, \ldots) + (X - Y) \mu \partial_\mu F(Y, \ldots).$$

(1.1)

Let $\Delta(x, y)$ be a generic propagator. For the system of our concern, $\Delta(x, y)$, with $x$ and $y$ fixed, does not change appreciably in $(x + y)/2$. We refer the first term on the R.H.S. to as the leading part and the second term to as the gradient part. The self-energy part $\Sigma(x, y)$ enjoys a similar property. Thus, we choose $x - y$ as the microscopic coordinates while $X \equiv (x + y)/2$ as the macroscopic coordinates.

The plan of the paper is as follows. The $O(N)$ linear-sigma model is introduced in Sec. II and the forms of retarded and advanced bare propagators are given in Sec. III. In Sec. IV, a perturbative framework is constructed from first principles. The framework thus constructed allows us to compute reaction rates by using the reaction-rate formula \cite{21}. In Sec. V, a quasiparticle representation of the propagator is given. In Sec. VI, after constructing the self-energy-part resummed propagator, the gap equation and the generalized Boltzmann equation are derived. Section VII is devoted to conclusion and outlook. Concrete derivation of various formula used in the text is made in Appendices.

**II. O(N) LINEAR-SIGMA MODEL**

The Lagrangian (density) of the $O(N)$ linear-sigma model reads

$$\mathcal{L} = \frac{1}{2} \left[ \left( \partial_\mu \phi_0 \right)^2 - m_B^2 \phi_0^2 \right] - \frac{\lambda_B}{4!} \phi_0^4 + \frac{1}{2} \vec{H}_B \cdot \vec{\phi} + \frac{1}{2} \left( Z - 1 \right) \left( \partial_\mu \phi \right)^2 - \frac{Z_m Z - 1}{4} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4,$$

(2.1)

where $\vec{\phi}_B = (\phi_1^B, \phi_2^B, \ldots, \phi_N^B)$. When $m_B^2 < 0$, $\mathcal{L}$ describes the system whose ground state is in a broken phase in the classical limit. $\vec{H}_B$ is an external field, which explicitly breaks $O(N)$ symmetry. Noticing the fact that a renormalization scheme for the symmetric phase ($m_B^2 > 0$) works \cite{22} as it is for the broken phase ($m_B^2 < 0$), we introduce renormalized quantities, $\bar{\phi}_B = \sqrt{Z} \phi$, $m_B^2 = Z_m m^2$, and $\lambda_B = Z \lambda$, in terms of which $\mathcal{L}$ reads

$$\mathcal{L}(\bar{\phi}) = \frac{1}{2} \left[ \left( \partial_\mu \bar{\phi} \right)^2 - m^2 \bar{\phi}^2 \right] - \frac{\lambda}{4!} \bar{\phi}^4 + \frac{1}{2} \left( Z - 1 \right) \left( \partial_\mu \bar{\phi} \right)^2 - \frac{Z_m Z - 1}{4} m^2 \bar{\phi}^2 - \frac{\lambda}{4!} \bar{\phi}^4,$$

(2.2)

where $\vec{H} = \sqrt{Z} \vec{H}_B$.

The system in the broken phase is governed by the “sifted Lagrangian”

$$\mathcal{L}(\bar{\phi}(x); \bar{\varphi}(x)) \equiv \mathcal{L}(\bar{\phi}(x) + \bar{\varphi}(x)) - \frac{\partial \mathcal{L}(\bar{\varphi}(x))}{\partial \bar{\varphi}(x)} \cdot \bar{\varphi}(x).$$

To avoid too many notations, for $\mathcal{L}(\bar{\phi}(x); \bar{\varphi}(x))$, we have used the same letter “$\mathcal{L}$” as in Eqs. \[(2.1)\] and \[(2.2)\]. $\bar{\varphi}(x)$ is the (classical) condensate or order-parameter fields and $\bar{\phi}(x)$ is the quantum fields, which describes the fluctuation around $\bar{\varphi}(x)$. Straightforward manipulation yields

$$\mathcal{L}(\bar{\phi}(x); \bar{\varphi}(x)) = \mathcal{L}' + \mathcal{L}_{rc} + \ldots,$$

(2.3)

$$\mathcal{L}' = \frac{1}{2} \left[ \left( \partial_\mu \bar{\phi} \right)^2 - m^2 \bar{\phi}^2 \right] - \frac{\lambda}{12} \left[ \bar{\varphi}^2 \bar{\phi}^2 + 2 (t \bar{\varphi} \cdot \bar{\phi})^2 \right] - \frac{\lambda}{31} \left( t \bar{\varphi} \cdot \bar{\phi} \right) \bar{\phi}^2 - \frac{\lambda}{4!} \bar{\varphi}^4,$$

$$\mathcal{L}_{rc} = \frac{1}{2} \left( Z - 1 \right) \left( \partial_\mu \bar{\phi} \right)^2 - \frac{1}{2} \left( Z_m Z - 1 \right) m^2 \bar{\phi}^2 - \frac{\lambda}{12} \left[ \bar{\varphi}^2 \bar{\phi}^2 + 2 (t \bar{\varphi} \cdot \bar{\phi})^2 \right] + \frac{\lambda}{31} \left( t \bar{\varphi} \cdot \bar{\phi} \right) \bar{\phi}^2 + \frac{\lambda}{4!} \bar{\phi}^4.$$

(2.4)
In Eq. (2.3), ‘...’ stands for the terms that includes only c-number field $\varphi$, which is not necessary for the present purpose, but plays a role when spacetime evolution of the system is studied. We ignore ‘...’ throughout in the sequel. It is to be noted that $L$ enjoys $O(N - 1)$ symmetry.

For obtaining an efficient or rather physically-sensible (perturbative) scheme, we introduce \textsuperscript{[4,23–25]} weakly $x$-dependent masses, $M_\pi(x)$ and $M_\sigma(x)$, where ‘$x$‘ is macroscopic coordinates. A consistent perturbative scheme is obtained by assuming that, when compared to $m^2$, $\chi_\xi(x) \equiv M_\pi^2(x) - m^2$ ($\xi = \pi, \sigma$) are one-order higher in the loop expansion. How to determine $M_\pi^2(x)$ and $M_\sigma^2(x)$ will be discussed in Sec. VIB. We then rewrite $L(\phi(x); \varphi(x))$ in the form,

$$L(\phi(x); \varphi(x)) = L_0(\phi(x); \varphi(x)) + L_{\text{int}} + L_{rc} + L_{mc}.$$  \hfill (2.5)

Here $L_{rc}$ is as in Eq. (2.4) and

$$L_0(\phi(x); \varphi(x)) = \frac{1}{2} \int d^4y \, \phi^\dagger(x) \Delta^{-1}(x, y) \phi(y),$$  \hfill (2.6)

$$L_{\text{int}} = -\frac{\lambda}{3!} (\phi^\dagger \phi)^2 - \frac{\lambda}{4!} (\phi^\dagger)^2 \equiv L_{\text{int}}^{(3)} + L_{\text{int}}^{(4)},$$  \hfill (2.7)

$$L_{mc} = \frac{1}{2} \phi(x) \left[ \chi(1) P_{\pi}(x) + \chi(2) P_{\sigma}(x) \right] \phi(x),$$  \hfill (2.8)

where

$$\Delta^{-1}(x, y) \equiv -\left[ (\partial^2 + M_\pi^2(x)) P_{\pi}(x) + (\partial^2 + M_\sigma^2(x)) P_{\sigma}(x) \right] \delta^4(x - y),$$  \hfill (2.9)

$$P_{\pi}(x) = \mathbf{I} - \left| \phi(x) \right\rangle \left\langle \phi(x) \right|, \quad P_{\sigma}(x) = \left| \phi(x) \right\rangle \left\langle \phi(x) \right|,$$  \hfill (2.10)

$$M_\pi^2(X) \equiv M_{\pi 0}^2(X) + \frac{\lambda}{2 \xi^2}(x), \quad M_\sigma^2(X) \equiv M_{\sigma 0}^2(X) + \frac{\lambda}{2 \xi^2}(x).$$  \hfill (2.11)

In these equations, the boldface letters denote $N \times N$ matrices act on the real vector space, $P_{\pi}$ ($P_{\sigma}$) is the projection operator onto the $\pi$- ($\sigma$)-subspace, and $\mathbf{I}$ is an unit matrix. $\left| \phi(x) \right\rangle$ is a unit vector along $\varphi(x)$, and $\left\langle \phi(x) \right|$ is an adjoint of $\left| \phi(x) \right\rangle$.

The construction of perturbation theory based on $L$, Eq. (2.3), starts with constructing the Fock space of the quantized system, described by $\phi$, which is defined “on $\varphi(x)$.” As stated in Sec. I, we are concerned about the systems which are not far from equilibrium states or from stationary states. Then, the theory to be developed may be applied to the case where $\varphi(x)$ changes slowly. In one word, ‘$x$‘ of $\varphi(x)$ is macroscopic coordinates.

III. RETARDED AND ADVANCED PROPAGATORS

For the purpose of later use, we construct retarded and advanced propagators. The retarded (advanced) propagator $\Delta_R$ ($\Delta_A$) is an inverse of $\Delta^{-1}(x, y)$, Eq. (2.9), under the retarded (advanced) boundary condition. Derivation of the form of $\Delta_R$ and $\Delta_A$ in the gradient approximation is straightforward and, here, we display the result:

$$\Delta_R(x, y) \equiv \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \Delta_R^{(\xi)}(X; P),$$  \hfill (3.2)

$$\Delta^{(\xi)}_R(x, y) = 2i \left( \left| \phi(X) \right\rangle \hat{\partial}_X \left\langle \phi(X) \right| \right) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} P_\xi \Delta^{(\xi)}_R(X; P) \Delta^{(\sigma)}_A(X; P).$$  \hfill (3.3)

In Eq. (3.3), summation over $\xi$ runs over $\pi$ and $\sigma$, and, in Eqs. (3.2) and (3.3), $X = (x + y)/2$, $\hat{\partial}_X \equiv \partial_x - \partial_y$, and

$$\Delta_R^{(\xi)}(X; P) \equiv \begin{cases} \frac{1}{P^2 - M_\xi^2(X)} & (\xi = \pi, \sigma). \end{cases}$$

That the gradient part $\Delta^{(\xi)}_R(x, y)$ appears is a reflection of the fact that the internal reference frame, which defines $\sigma$ mode and three-$\pi$ modes, at the spacetime point $x$ is different from that at the point $y$. Let us see the meaning of $\left| \phi \right\rangle P_\xi \hat{\partial} \left\langle \phi \right|$ in $\Delta^{(\xi)}_R$. From Eq. (2.10), we can easily see that
\[ P_\xi |\hat{\varphi}| P \cdot \partial |\hat{\varphi}| = |\hat{\varphi}| P \cdot \partial |\hat{\varphi}|, \]
\[ P_\zeta |\hat{\varphi}| P \cdot \partial |\hat{\varphi}| P_\zeta = 0 \quad ((\xi, \zeta) \neq (\sigma, \pi)), \]

where use has been made of \( \hat{\varphi} \cdot \partial \hat{\varphi} = 0 \) (\( \hat{\varphi} \equiv \hat{\varphi}/|\hat{\varphi}| \)). \( |\hat{\varphi}| P \cdot \partial |\hat{\varphi}| \) enjoys similar property. Then, \( |\hat{\varphi}| P \cdot \partial |\hat{\varphi}| \) and \( |\hat{\varphi}| P \cdot \partial |\hat{\varphi}| \) “induce” transition between \( \pi \) mode and \( \sigma \) mode.

### IV. CONSTRUCTION OF PERTURBATIVE FRAMEWORK

#### A. Preliminary

The CTP formalism is formulated by introducing an oriented closed-time path \( C (= C_1 + C_2) \) in a complex-time plane, that goes from \(-\infty\) to \(+\infty\) \((C_1)\) and then returns from \(+\infty\) to \(-\infty\) \((C_2)\). The real time formalism is achieved by doubling every degree of freedom, \( \phi \to (\phi_1, \phi_2) \) and \( \varphi \to (\varphi_1, \varphi_2) \), where \( \phi(x_0, x) = \phi(x_0, x) \) with \( x_0 \in C_1 \) and \( \phi_2(x_0, x) = \phi(x_0, x) \) with \( x_0 \in C_2 \), etc. A classical contour action is written in the form

\[
\int C \, dx_0 \int dx \mathcal{L}(\tilde{\phi}(x); \tilde{\varphi}(x)) = \int_{-\infty}^{+\infty} dx_0 \int dx \tilde{\mathcal{L}}(x),
\]

\[
\tilde{\mathcal{L}} \equiv \mathcal{L}(\tilde{\phi}_1; \tilde{\varphi}_1) - \mathcal{L}(\tilde{\phi}_2; \tilde{\varphi}_2). \tag{4.1}
\]

\( \tilde{\mathcal{L}} \) here is sometimes called a hat-Lagrangian.

Taking \( \tilde{\mathcal{L}}_0 \), which corresponds to \( \mathcal{L}_0 \) in Eq. (2.4), for a free hat-Lagrangian, we construct from first principles a perturbative framework. Throughout this paper, we do not deal with initial correlations (see, e.g., [9]). Following standard procedure, the four kind of propagators emerges:

\[
\Delta_{11}^{\alpha\beta}(x, y) = -i \text{Tr} \left[ T \left( \phi^\alpha(x) \phi^\beta(y) \right) \rho \right], \quad \Delta_{22}^{\alpha\beta}(x, y) = -i \text{Tr} \left[ T \left( \phi^\alpha_2(x) \phi^\beta_2(y) \right) \rho \right],
\]
\[
\Delta_{12}^{\alpha\beta}(x, y) = -i \text{Tr} \left[ \phi^\alpha_2(y) \phi^\beta(y) \rho \right], \quad \Delta_{21}^{\alpha\beta}(x, y) = -i \text{Tr} \left[ \phi^\alpha(x) \phi^\beta_2(y) \rho \right], \tag{4.2}
\]

where \( \rho \) is the density matrix, \( T \) is a time-ordering symbol, and \( \mathbf{T} \) is an anti-time-ordering symbol. At the end of calculation we set \( \tilde{\phi}_1 = \tilde{\phi}_2 \) and \( \tilde{\varphi}_1 = \tilde{\varphi}_2 \) [4]. Let us introduce a matrix propagator \( \Delta(x, y) \), where the bold-face denotes, as above, the \( N \times N \) matrix and the ‘caret’ denotes the \( 2 \times 2 \) matrix: \( \Delta_{ij}^{\alpha\beta} \) is the \( (\alpha, \beta) \)-component of the \( N \times N \) matrix \( \Delta_{ij} \) and, at the same time, \( (i, j) \)-component of the \( 2 \times 2 \) matrix \( \Delta^{\alpha\beta} \). The matrix self-energy part \( \Sigma(x, y) \) is defined similarly.

We stress again that the argument ‘\( x \)’ of \( P_\xi(x) \) \((\xi = \pi, \sigma)\) in Eq. (2.9) is macroscopic spacetime coordinates. Let \( \hat{A} \) be a propagator or a self-energy part. Due to \( O(N - 1) \) symmetry of \( \mathcal{L} \), Eq. (2.5), we may write \( \hat{A} \) as

\[
\hat{A}(x, y) \approx P_\xi(x) \hat{A}(\xi)(x, y) P_\xi(y) + T_{\xi\xi}^\mu(x) \hat{A}_\mu(\xi)(x, y), \tag{4.3}
\]

where, for \( \xi = \pi (\sigma), \zeta = \sigma (\pi) \) and

\[
T_{\sigma\pi}(x) \equiv |\hat{\varphi}(x)| \partial_{x_\mu} |\hat{\varphi}(x)|, \quad T_{\pi\sigma}(x) \equiv |\hat{\varphi}(x)| \partial_{x_\mu} |\hat{\varphi}(x)| \tag{4.4}
\]

(cf. Sec. III). One can replace \( T_{\xi\xi}^\mu(x) \) in Eq. (4.3) with \( T_{\xi\xi}^\mu(y) \), since the arising difference is of higher order (cf. Eq. (1.1)). Fourier transforming \( \hat{A}(\xi) \) and \( \hat{A}_\mu(\xi) \) with respect to \( x - y \), we have

\[
\hat{A}(x, y) \approx P_\xi(x) \int \frac{d^4 P}{(2\pi)^4} e^{-i P \cdot (x-y)} \hat{A}(\xi)(X; P) P_\xi(y) + T_{\xi\xi}^\mu(x) \int \frac{d^4 P}{(2\pi)^4} e^{-i P \cdot (x-y)} \hat{A}_\mu(\xi)(X; P), \tag{4.5}
\]

where \( X \equiv (x + y)/2 \). In general, \( \hat{A}(\xi) \) in Eq. (4.5) consists of two pieces, \( \hat{A}(\xi) = \hat{A}(\xi)^0 + \hat{A}(\xi)^1 \), where \( \hat{A}(\xi)^0 \) is free from \( X \)-derivative and \( \hat{A}(\xi)^1(= \hat{A}(\xi)^0 (X; P)) \) contains explicit (first order) \( X_\mu \)-derivative:

\[
\hat{A}(x, y) \approx P_\xi(x) \int \frac{d^4 P}{(2\pi)^4} e^{-i P \cdot (x-y)} \hat{A}(\xi)^0(X; P) P_\xi(y) + P_\xi(x) \int \frac{d^4 P}{(2\pi)^4} e^{-i P \cdot (x-y)} \hat{A}(\xi)^1(X; P) P_\xi(y)
\]
\[
+ T_{\xi\xi}^\mu(x) \int \frac{d^4 P}{(2\pi)^4} e^{-i P \cdot (x-y)} \hat{A}(\xi)^0(X; P) P_\xi(y), \tag{4.6}
\]
The first term on the R.H.S. is the leading part of $\hat{A}$ while the second and third terms are the gradient parts (cf. above after Eq. (1.1)).

### B. Propagator

From the definition of $\Delta$'s, Eq. (4.2), with $\phi_1 = \phi_2$ (cf. above after Eq. (4.2)), we see that

$$\sum_{i,j=1}^{2} (-)^{i+j} \Delta_{ij} \Bigg|_{\phi_1 = \phi_2} = 0 \quad (4.7)$$

holds. Then, out of four $\Delta_{ij}$ ($i, j = 1, 2$), three are independent, for which we choose [19]

$$\Delta_R = \Delta_{11} - \Delta_{12}, \quad \Delta_A = \Delta_{11} - \Delta_{21}, \quad \Delta_c = \Delta_{12} + \Delta_{21}.$$  

Setting $\phi_1 = \phi_2 (\equiv \phi)$, we have

$$i\Delta_R^{\alpha\beta}(x, y) = \theta(x_0 - y_0) \text{Tr} \left\{ [\phi^\alpha(x), \phi^\beta(y)] \rho \right\} = \theta(x_0 - y_0) \left[ \phi^\alpha(x), \phi^\beta(y) \right] \rho,$$

$$i\Delta_A^{\alpha\beta}(x, y) = -\theta(y_0 - x_0) \text{Tr} \left\{ [\phi^\alpha(x), \phi^\beta(y)] \rho \right\} = -\theta(y_0 - x_0) \left[ \phi^\alpha(x), \phi^\beta(y) \right] \rho,$$

$$i\Delta_c^{\alpha\beta}(x, y) = -i \text{Tr} \left\{ (\phi^\alpha(x) \phi^\beta(y) + \phi^\beta(y) \phi^\alpha(x)) \rho \right\}.$$  

Thus, $\Delta_R$ and $\Delta_A$ are the retarded- and advanced-propagators, respectively, which have already been constructed in Sec. III. $\Delta_c$ is the correlation function. Expressing $\Delta$ in terms of them, we have

$$\hat{\Delta} = \frac{1}{2} \left( \Delta_R + \Delta_A \right) - \Delta_c \hat{A}_+, + \frac{1}{2} \Delta_c \hat{A}_-, \quad (4.9)$$

where

$$\hat{A}_{\pm} = \begin{pmatrix} 1 & 1 \\ \pm 1 & 1 \end{pmatrix}. \quad (4.10)$$

From the definition of $\Delta_c$ and Eq. (4.2) with $\phi_1 = \phi_2$, it follows that

$$(i\Delta_c(x, y))^* = i\Delta_c(x, y), \quad \Delta_c^{\alpha\beta}(x, y) = \Delta_c^{\beta\alpha}(y, x).$$  

To leading order (of derivative expansion), it is known that $\Delta_c$ takes the form (cf., e.g., [21])

$$\Delta_c(x, y) \sim P_\xi(x) \int \frac{d^4 P}{(2\pi)^4} e^{-iP(x-y) (1 + 2f_\xi(X; P)) \left( \Delta_R^{(\xi)}(X; P) - \Delta_A^{(\xi)}(X; P) \right)} P_\xi(y),$$

where $X \equiv (x + y)/2$ and, as will be seen below, $f_\xi$ is the real function that is related to the particle-number density. Then, to the gradient approximation, we may write

$$\Delta_c(x, y) = P_\xi(x) \Delta_c^{(\xi)}(x, y) P_\xi(y) + T_{\xi\xi}^{\nu}(x) \Delta^{(\xi)}_{\mu}(x, y), \quad (4.12)$$

where, $T_{\xi\xi}^{\nu}(x)$ is as in Eq. (4.4) and

$$\Delta^{(\xi)}_c = \Delta_c^{(\xi)} \cdot (1 + 2f_\xi) - (1 + 2f_\xi) \cdot \Delta_R^{(\xi)} + \Delta_A^{(\xi)} + \Delta_{c1}^{(\xi)}.$$  

where $\xi$ stands for $\pi$ or $\sigma$, and $\Delta_{c1}^{(\xi)}$ is a gradient part. It is clear that, in Eq. (4.13), although $\xi$ is a 'repeated index' on the R.H.S., summation should not be taken over $\xi$. This type of equations appears frequently in the sequel. Here we have used the short-hand notation $F \cdot G$, which is a function whose "$(x, y)$-component" is

$$[F \cdot G](x, y) = \int d^4 z F(x, z) G(z, y), \quad (4.14)$$
As will be shown later (cf. Eq. (6.8)), the function whose \((x, y)\)-component is \(\delta^4(x - y)\). For a given \(\rho\), \(\Delta_c\) is computed through Eq. (4.8). Eq. (4.12) with Eq. (4.13) is understood to be the defining equation of \(f_\xi\) and \(\Delta^{(c)}_\xi\). Physical meaning of \(f_\xi\) is clarified later (see, Sec. VIC). Using Eq. (4.11) in Eq. (4.12) with Eq. (4.13), we obtain the relations:

\[
\left(\Delta^{(c)}_\xi(x, y)\right)^* = -\Delta^{(c)}_\xi(x, y), \quad \left(\Delta^{(c)}_\mu(x, y)\right)^* = -\Delta^{(c)}_\mu(x, y),
\]

\[
\Delta^{(c)}_\xi(x, y) = \Delta^{(c)}_\xi(y, x), \quad \Delta^{(c)}_\mu(x, y) = \Delta^{(c)}_\mu(y, x).
\]

Note that \(T^{\mu}_{\xi} \Delta^{(c)}_\mu\) in Eq. (4.12) and \(\Delta^{(c)}_\xi\) in Eq. (4.13) are the gradient parts (cf. above after Eq. (4.6)). Applying \(\hat{\tau}_3(\partial^2 + M_\xi^2)\) in Eq. (4.11) in Eq. (4.12) with Eq. (4.13), we obtain the relations:

\[
\hat{\tau}_3 (\partial^2 + M_\xi^2(x)) P_\xi(x) \Delta_c(x, y) \hat{A}_{+} + \hat{A}_{+} \Delta_c(x, y)(\partial^2 + M_\xi^2(y)) P_\xi(y) \hat{\tau}_3
\]

\[
\simeq P_\xi(X) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[ 2i \left\{ f_\xi(X; P), P^2 - M_\xi^2(X) \right\} \Delta^{(c)}_\xi (X; P) \right.
\]

\[
+ 2i (P \cdot \partial P_\xi(X)) \left( 1 + 2f_\xi(X; P) \right) \left( \Delta^{(c)}_R (X; P) - \Delta^{(c)}_A (X; P) \right)
\]

\[
- \left( P^2 - M_\xi^2(X) \right) \left( \Delta^{(c)}_R (X; P) - \Delta^{(c)}_A (X; P) \right)
\]

\[
\Delta^{(c)}_c(X; P) - \left( P^2 - M_\xi^2(X) \right) \left( \Delta^{(c)}_R (X; P) - \Delta^{(c)}_A (X; P) \right)
\]

\[
\hat{A}_{+} \hat{\tau}_3,
\]

where

\[
\left\{ f_\xi(X; P), P^2 - M_\xi^2(X) \right\} = \frac{\partial f_\xi(X; P)}{\partial X_\mu} \frac{\partial (P^2 - M_\xi^2(X))}{\partial P_\mu} - \frac{\partial f_\xi(X; P)}{\partial P_\mu} \frac{\partial (P^2 - M_\xi^2(X))}{\partial X_\mu}
\]

\[
= 2P \cdot \partial X f_\xi(X; P) + \frac{\partial f_\xi(X; P)}{\partial P_\mu} \frac{\partial M_\xi^2(X)}{\partial X_\mu}.
\]

**Bare-N scheme**

The propagator matrix \(\Delta\) is an inverse of \(-\hat{\tau}_3(\partial^2 + M_\xi^2)\) in Eq. (4.11) and Eq. (2.6) with Eq. (2.9). Then, Eqs. (4.15) and (4.16) should vanish. From this condition, we obtain the following relations:

\[
\{ f_\xi, P^2 - M_\xi^2 \} = 0, \quad (4.17)
\]

\[
(P^2 - M_\xi^2(X)) \Delta^{(c)}_\xi(X; P) = 0, \quad (4.18)
\]

\[
\Delta^{(c)}_\mu(X; P) = -\Delta^{(c)}_\mu(X; P)
\]

\[
= \frac{2P_\mu}{M_\xi^2(X) - M_\xi^2(X)} \left( \left( 1 + 2f_\xi(X; P) \right) \left( \Delta^{(c)}_R(X; P) - \Delta^{(c)}_A(X; P) \right) \right.
\]

\[
- \left( 1 + 2f_\xi(X; P) \right) \left( \Delta^{(c)}_R(X; P) - \Delta^{(c)}_A(X; P) \right)
\]

\[
(4.19)
\]

As will be shown later (cf. Eq. (6.8)), \(f_\xi (\xi = \pi, \sigma)\) is related to the number density \(N_\xi\) of \(\xi\): \(f_\xi(X; \tau E_\xi^{(c)}; \hat{p}) = -\theta(-\tau) + \epsilon(\tau)N_\xi(X; E_\xi^{(c)}; \epsilon(\tau)\hat{p}) (\tau = \pm)\). Then, Eq. (4.17) is a “free Boltzmann equation.” One can construct a perturbation theory in a similar manner as in [20], where a complex-scalar field system with symmetric phase is treated. We call the perturbation theory thus constructed the bare-N scheme, since \(N_\xi\) obeys the “free Boltzmann equation.” This theory is equivalent [20] to the one obtained in the physical-N scheme, to which we now turn.
Physical-\(N\) scheme

We abandon Eq. (4.17), while we keep Eqs. (4.18) and (4.19). This means that \(f_\xi\) in the present (physical-\(N\)) scheme differs from \(f_\xi\) in the bare-\(N\) scheme. Specification of \(f_\xi\) is postponed until Sec. VIC, where we require the number density \(N_\xi\) to be as close as possible to the physical number density. Now, \(\Delta\) is not an inverse of \(-\hat{\tau}_3 (\partial^2 + M^2_\xi)\). It is straightforward to show in the gradient approximation that \(\Delta\) is an inverse of \(-\hat{\tau}_3 (\partial^2 + M^2_\xi)\) \(L'\), \(x, y\) \(\hat{A}_-\), where \(\hat{A}_-\) is as in Eq. (4.18) and

\[
L'(x, y) = iP_\xi (X) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left\{ f_\xi (X; P), P^2 - M^2_\xi (X) \right\} .
\]

Then the free action is

\[
A_0 = -\frac{1}{2} \int d^4x d^4y \hat{\phi} (x) \hat{\tau}_3 \left( \partial^2 + M^2_\xi (x) \right) \hat{\phi} (x) + \frac{1}{2} \int d^4x d^4y \hat{\phi} (x) L' (x, y) \hat{A}_- \hat{\phi} (y) .
\]

Note that the Lagrangian density corresponding to the last term of Eq. (4.21) is nonlocal not only in ‘space’ but also in ‘time.’ Here it is worth mentioning the so-called \(|p_0|\)-prescription. With this prescription, at an intermediate stage, we have \(L'(x, y)\), which is local in time. (For completeness, we briefly discuss the \(|p_0|\)-prescription in Appendix A.)

Since the last term of Eq. (4.21) is absent in the original action, we should introduce the counter action to compensate it,

\[
A_c = -\frac{1}{2} \int d^4x d^4y \hat{\phi} (x) L' (x, y) \hat{A}_- \hat{\phi} (y) ,
\]

which yields a vertex \(-iL' \hat{A}_-\) \((\equiv i\hat{V}_c)\). From Eqs. (4.9) and (4.22), \(\hat{V}_c \cdot \Delta \cdot \hat{V}_c = 0\) follows, and the \(\hat{V}_c\)-resummed propagator becomes

\[
\Delta_{c-\text{resum}} = \Delta \cdot \left[ 1 + \sum_{n=1}^\infty \left( -\hat{V}_c \cdot \hat{\Delta} \right)^n \right] = \Delta - \hat{\Delta} \cdot \hat{V}_c \cdot \hat{\Delta} \ (\equiv \Delta + \delta \hat{\Delta}) .
\]

Since \(\hat{V}_c\) is a gradient part, we obtain, to the gradient approximation,

\[
\delta \hat{\Delta} \approx iP_\xi (X) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left\{ f_\xi (P), P^2 - M^2_\xi (P) \right\} \Delta^{(c)}_R \Delta^{(c)}_A \hat{A}_+ .
\]

Note that Eq. (4.23) possesses pinch singularities in a \(p_0\)-plane, due to \(\Delta^{(c)}_R (X; P) \Delta^{(c)}_A (X; P)\). Since \(\delta \hat{\Delta}\) is proportional to \(\hat{A}_+\), it contributes to \(\Delta_c\) (cf. Eq. (4.9)). Then, including \(\delta \hat{\Delta}\) to \(\Delta_c \hat{A}_+ / 2\), we obtain for \(\Delta_c\) (within the gradient approximation),

\[
\Delta_c (x, y) = -iP_\xi (X) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left( p_0 (1 + 2f_\xi (X; P)) \delta (P^2 - M^2_\xi (X)) \right) P_\xi (y) \]

\[
+ \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[ -iP_\xi (X) \left\{ f_\xi (P), P^2 - M^2_\xi (P) \right\} \left( \Delta^{(c)}_R (X; P) - \Delta^{(c)}_A (X; P) \right)^2 \right.
\]

\[
+ \left. P_\xi (X) \Delta^{(c)}_c (X; P) + \left( |\hat{\varphi} (X)\rangle \hat{\partial}_X \langle \hat{\varphi} (X) | \right) \Delta^{(c)}_m (X; P) \right] ,
\]

where \(\Delta^{(c)}_m\) is as in Eq. (4.19). From Eq. (4.18), we see that \(\Delta^{(c)}_c (X; P) \propto \delta (P^2 - M^2_\xi (X))\), so that the term with \(\Delta^{(c)}_c\) in Eq. (4.24) may be absorbed into the first term of the R.H.S. by modifying the definition of \(f_\xi (X; P)\). Thus we shall drop the term with \(\Delta^{(c)}_c\) hereafter.

Substituting Eqs. (3.3), (4.3), and (4.24) into Eq. (4.8), we obtain

\[
\hat{\Delta} (x, y) = \hat{\Delta}^{(0)} (x, y) + \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[ \hat{\Delta}^{(p)} (X; P) + \hat{\Delta}^{(l)} (X; P) \right] ,
\]

\[
\hat{\Delta}^{(0)} (x, y) = P_\xi (x) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \Delta^{(c)} (X; P) P_\xi (y) ,
\]

\[
\hat{\Delta}^{(p)} (X; P) = -\frac{i}{2} P_\xi (X) \left\{ f_\xi (P), P^2 - M^2_\xi (P) \right\} \left( \Delta^{(c)}_R - \Delta^{(c)}_A \right)^2 \hat{A}_+ ,
\]

\[
\hat{\Delta}^{(l)} (X; P) = 2i|\hat{\varphi} (P)\rangle \hat{\partial}_\varphi \langle \hat{\varphi} | \hat{\Omega} .
\]

(4.26)
where $\Delta_R^{(\xi)} = \Delta_R^{(\xi)}(X; P)$, $f_\xi = f_\xi(X; P)$, etc., and

$$
\hat{\Delta}^{(\xi)}(X; P) = \left( \begin{array}{cc} \Delta_R^{(\xi)} + f_\xi \left( \Delta_R^{(\xi)} - \Delta_A^{(\xi)} \right) & f_\xi \left( \Delta_R^{(\xi)} - \Delta_A^{(\xi)} \right) \\ 1 + f_\xi & -\Delta_A^{(\xi)} + f_\xi \left( \Delta_R^{(\xi)} - \Delta_A^{(\xi)} \right) \end{array} \right),
$$

(4.27)

$$
\Omega = \left( \begin{array}{cc} \Delta_R^{(\sigma)} & 0 \\ \Delta_R^{(\sigma)} - \Delta_A^{(\sigma)} & -\Delta_A^{(\sigma)} \end{array} \right) + \omega \hat{A}_+,
$$

$$
\omega \equiv \frac{1}{M_2^2 - M_2^2} \left[ f_\sigma \left( \Delta_A^{(\sigma)} - \Delta_A^{(\sigma)} \right) - f_\pi \left( \Delta_A^{(\pi)} - \Delta_A^{(\pi)} \right) \right].
$$

As has been observed in Sec. III, $|\bar{\phi}| P \cdot \partial \langle \bar{\phi} |$ and $|\bar{\phi}| P \cdot \overleftarrow{\partial} \langle \bar{\phi} |$ in the gradient part $\Delta^{(t)}$ “induce” transition between $\pi$ mode and $\sigma$ mode. Thus, $\Delta^{(t)}$ contributes, e.g., to the one-loop two-point function with one-$\pi$ and one-$\sigma$ legs.

C. Feynman rules

Two fundamental elements of Feynman rules are the propagator and the vertices, which take the $(2 \times 2)$ matrix form. The propagator-matrix is given by Eq. (4.25). The vertex factors may be read off from Eq. (4.1) with Eqs. (2.5), (2.7), (2.8). As a matter of fact, from Eq. (4.1), it is obvious that there is no vertex that mixes $\phi_1$'s with $\phi_2$'s. The vertex factors for the fields $\phi_1$'s are the same as in vacuum theory. Each vertex factor for $\phi_2$'s is of opposite sign to the corresponding vertex factor for $\phi_1$'s. Thus, in matrix notation as for the propagator, every vertex-matrix $V$ is diagonal, $V = \text{diag}(\nu, -\nu)$, with $\nu$ the same as in vacuum theory. All other elements of Feynman rules, e.g., integration over every loop momentum, are the same as in vacuum theory.

Having thus constructed Feynman rules, one can compute reaction rates for processes taking place in the system, by using the reaction-rate formula [21].

V. QUASIPARTICLE REPRESENTATION OF THE PROPAGATOR

Here we obtain a quasiparticle representation [27] of $\Delta$, which tremendously simplifies the practical computation. Straightforward but lengthy calculation shows that $\Delta$ may be written in the form (cf. the definition (1.14)):

$$
\hat{\Delta} \simeq \hat{B}_L \cdot \left( \begin{array}{cc} \Delta_R & 0 \\ 0 & -\Delta_A \end{array} \right) \cdot \hat{B}_R,
$$

(5.1)

where

$$
\hat{B}_L \simeq \hat{B}_L^{(0)} + |\bar{\phi}(X)\rangle \overleftrightarrow{\partial} X^\mu \langle \bar{\phi}(X)|\alpha^\mu(X) \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right),
$$

$$
\hat{B}_R \simeq \hat{B}_R^{(0)} + |\bar{\phi}(X)\rangle \overleftrightarrow{\partial} X^\mu \langle \bar{\phi}(X)|\alpha^\mu(X) \left( \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right),
$$

with

$$
\hat{B}_L^{(0)} = P_\xi \cdot \left( \begin{array}{cc} 1 & f_\xi \\ 1 & 1 + f_\xi \end{array} \right) \cdot P_\xi, \quad \hat{B}_R^{(0)} = P_\xi \cdot \left( \begin{array}{cc} 1 + f_\xi & f_\xi \\ 1 & 1 \end{array} \right) \cdot P_\xi,
$$

(5.2)

$$
\alpha^\mu(X) = \frac{2i}{M_2^2(X) - M_2^2(X)} \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} P^\mu \left( f_\sigma(X; P) - f_\pi(X; P) \right).
$$

It can readily be shown that

$$
\hat{B}_L \cdot \hat{\tau}_3 \hat{B}_R \simeq \hat{B}_R \cdot \hat{\tau}_3 \hat{B}_L \simeq \hat{\tau}_3.
$$

(5.3)

It is obvious from the argument in Appendix A that, at an intermediate stage before taking the $|p_0|$-prescription, $\hat{B}_L$ and $\hat{B}_R$ are local in time and satisfy the relation [23]. Then, $\hat{B}_L$ and $\hat{B}_R$ at that stage are the generalized Bogoliubov-matrices [27]. Eq. (5.1) tells us that, through Bogoliubov-transforming $\phi_i$ ($i = 1, 2$), one can introduce
the stable quasiparticle modes, \( \phi_i \) (\( i = 1, 2 \)), whose matrix-propagator is \( \text{diag}(\Delta_R, -\Delta_A) \) in Eq. (5.1) (cf. [26] for more details).

It is worth mentioning that \( \mathbf{B}_L \) and \( \mathbf{B}_R \) may be written as

\[
\mathbf{B}_L(x, y) \simeq \mathbf{P}_\xi(x) \left( \begin{array}{cc} F_\xi(x, y) & \delta^4(x - y) + F_\xi(x, y) \\ \delta^4(x - y) & F_\xi(x, y) \end{array} \right) \mathbf{P}_\xi(y),
\]

\[
\mathbf{B}_R(x, y) \simeq \mathbf{P}_\xi(x) \left( \begin{array}{cc} \delta^4(x - y) + F_\xi(x, y) & F_\xi(x, y) \\ \delta^4(x - y) & F_\xi(x, y) \end{array} \right) \mathbf{P}_\xi(y),
\]

where

\[
F_\xi(x, y) = f_\xi(x, y) + 2 \frac{\tilde{\omega}_x \tilde{\omega}_y f_\xi(x, y) + f_\xi(x, y) \tilde{\omega}_y \tilde{\omega}_y}{\mathcal{M}_\xi^2(X) - \mathcal{M}_\xi^2(X)}
\]

with \( \tilde{\pi} = \sigma \) and \( \sigma = \pi \).

VI. THE GAP EQUATION AND THE GENERALIZED BOLTZMANN EQUATION

A. Self-energy-part resummed propagator

We write the bare propagator \( \tilde{\Delta} \) in Eq. (4.23) as \( \tilde{\Delta} = \tilde{\Delta}^{(0)} + \tilde{\Delta}^{(1)} \) (\( \tilde{\Delta}^{(1)} \equiv \tilde{\Delta}^{(p)} + \tilde{\Delta}^{(c)} \)). \( \tilde{\Delta}^{(0)} \) is the leading part and the gradient part \( \tilde{\Delta}^{(1)} \) represents variation in the macroscopic spacetime coordinates \( X_\mu \), through first-order derivative \( \partial X_\mu \). Interactions among the fields give rise to reactions taking place in a system, which, in turn, causes a nontrivial change in the number density of quasiparticles. Thus, the self-energy part \( \Sigma \) ties with \( \tilde{\Delta}^{(1)} \). More precisely, \( \Sigma^{-1} \) is of the same order of magnitude as \( \tilde{\Delta}^{(1)} \). Hence, in computing \( \Sigma \) in the approximation under consideration, it is sufficient to keep the leading part (i.e., the part with no \( X_\mu \)-derivative):

\[
\hat{\Sigma}(x, y) \simeq \mathbf{P}_\xi(x) \hat{\Sigma}^{(c)}(x, y) \mathbf{P}_\xi(y) \simeq \mathbf{P}_\xi(x) \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x - y)} \hat{\Sigma}^{(c)}(X; P).
\]

We are adopting the loop-expansion. Then, to the gradient approximation, the relevant self-energy diagrams are the one-loop diagrams, together with relevant counter diagrams. There are two one-loop diagrams, the one is the tadpole diagram that includes one vertex coming from \( \tilde{L}^{(3)}_\mu \) (cf. the definition (2.7)), and the one is the diagram with two vertices coming from \( \tilde{L}^{(3)}_\mu \). The counter diagrams are the diagrams that include \( \tilde{L}_{rc} + \tilde{L}_{cr} \) (cf. Eq. (2.5) with Eqs. (2.4) and (2.8)). The contribution from the tadpole diagram includes no \( X_\mu \)-derivative. Yet higher-order contributions to \( \Sigma \) come in when one proceeds beyond the gradient approximation.

A \( \Sigma \)-resummed propagator \( \mathbf{G} \) obeys the Schwinger-Dyson equation:

\[
\hat{\mathbf{G}} = \hat{\Delta} + \tilde{\Delta} \cdot \hat{\Sigma} \cdot \hat{\mathbf{G}}.
\]

In Appendix B, we solve this within the gradient approximation. The result is

\[
\mathbf{G}(x, y) \simeq \mathbf{G}^{(0)}(x, y) + \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x - y)} \left[ \mathbf{G}^{(p)}(X; P) + \mathbf{G}^{(c)}(X; P) \right],
\]

\[
\mathbf{G}^{(0)}(x, y) = \mathbf{P}_\xi(x) \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x - y)} \hat{\Sigma}^{(c)}(X; P) \mathbf{P}_\xi(y),
\]

\[
\mathbf{G}^{(p)}(X; P) = \mathbf{G}^{(p)}_1(X; P) + \mathbf{G}^{(p)}_2(X; P),
\]

\[
\hat{\mathbf{G}}^{(c)}_1(X; P) = i \mathbf{P}_\xi(X) \left[ \left\{ f_\xi, \frac{P^2 - \mathcal{M}_\xi^2 - \Lambda_\xi^2}{\mathcal{M}_\xi^2} \right\} \left( \mathbf{G}_\xi^{(c)}(X; P) \right)^2 + \left\{ \tilde{\omega}_x, \frac{P^2 - \mathcal{M}_\xi^2 - \Lambda_\xi^2}{\mathcal{M}_\xi^2} \right\} \left( \mathbf{G}_\xi^{(c)}(X; P) \right)^2 \right] \hat{\Delta}_+,
\]

\[
\hat{\mathbf{G}}^{(c)}_2(X; P) = -i \mathbf{P}_\xi(X) \left[ \left\{ f_\xi, \frac{P^2 - \mathcal{M}_\xi^2 - \Lambda_\xi^2}{\mathcal{M}_\xi^2} \right\} \right] \mathbf{G}_R^{(c)} \mathbf{G}_A^{(c)} \hat{\Delta}_+,
\]

\[
\hat{\mathbf{G}}^{(c)}(X; P) = 2i \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \left( \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \left( \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \right) \right) \hat{\Delta}_+ + 2i \left[ \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \right] \hat{\Delta}_+ \hat{\Delta}_+ + \left[ \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \hat{\mathbf{G}}^{(c)}_1(X; P) \mathbf{P} \cdot \partial \right] \hat{\Delta}_+ \hat{\Delta}_+ \hat{\Delta}_+.
\]
where
\[ \hat{G}^{(\ell)}(X; P) = \left( G^{(\ell)}_R + f_\xi \left( G^{(\ell)}_R - G^{(\ell)}_A \right) \right) \left( 1 + f_\xi \left( G^{(\ell)}_R - G^{(\ell)}_A \right) \right), \]
\[ \hat{\Sigma}^{(\ell)}(X; P) = \left( G^{(\ell)}_R + f_\xi \left( G^{(\ell)}_R - G^{(\ell)}_A \right) \right) \left( 1 + f_\xi \left( G^{(\ell)}_R - G^{(\ell)}_A \right) \right), \]
\[ \omega_G = f_\pi G^{(\sigma)}_R \left( G^{(\pi)}_R - G^{(\pi)}_A \right) + f_\sigma G^{(\pi)}_A \left( G^{(\sigma)}_R - G^{(\sigma)}_A \right), \]
\[ \omega'_G = \omega_G [\pi \leftrightarrow \sigma]. \]

In the above equations, \( f_\xi = f_\xi(X; P), \) \( \Sigma's = \Sigma(X; P)'s, \) \( \mathcal{M}_\xi^2 = \mathcal{M}_\xi^2(X), \) and
\[ G^{(\ell)}_{R(A)}(X; P) = \frac{1}{P^2 - \mathcal{M}_\xi^2(X) - \Sigma^{(\ell)}_{R(A)}(X; P)}, \]
\[ \Sigma^{(\ell)}_{R(A)} = \Sigma^{(\ell)}_{11} + \Sigma^{(\ell)}_{12(21)}. \]

In Appendix C (cf. Eq. (C5)), we show that \( \Sigma^{(\ell)}_{\pi, \sigma}(X; P) = \left( \Sigma^{(\ell)}_{\pi}(X; P) \right)^{*}. \) The expression for \( \Sigma^{(\ell)}_{R} \) is given in Appendix D, and \( \Sigma^{(\ell)}_{12} \) and \( \Sigma^{(\ell)}_{21} \) in Eq. (6.4) on the mass-shell are computed in Appendix E, which plays a role in Sec. VIC.

**B. Gap equation**

We have introduced two “mass functions” \( \mathcal{M}_\xi^2(X) \) (\( \xi = \pi, \sigma \)), which is a generalization of \( \mathcal{M}_\xi^2, \) in which only one mass parameter is introduced for computing an effective potential for the equilibrium system. In this subsection, we determine so far arbitrary mass functions. Various methods are available to this end (see, e.g., [4]). Among those we employ the on-shell renormalization scheme,
\[ \text{Re} \left. \Sigma^{(\ell)}_{R}(X; P) \right|_{\text{s.p.}} = \text{Re} \left. \Sigma^{(\ell)}_{R}(X; p_0^2 = \mathcal{M}_\xi^2(X), p = 0) \right| = 0 \]
(\( \xi = \pi, \sigma \)). For a subtraction point, ‘s.p.,’ instead of \( (p_0^2 = \mathcal{M}_\xi^2(X), p = 0) \) adopted here, we can choose \( (p_0^2 = E_{p_s}^{(\xi)}, p = p_s) \), where \( E_{p_s}^{(\xi)} = \sqrt{p^2 + \mathcal{M}_\xi^2(X)} \) with \( p_s \) arbitrary. Eq. (6.7) is the gap equation, by which \( \mathcal{M}_\xi^2(X) \) (\( \xi = \pi, \sigma \)) are determined self consistently. The explicit form of the gap equation (6.7) to leading one-loop order is displayed in Appendix D. \( \mathcal{M}_\xi^2(X) \) thus determined depends on \( \varphi^2(X) \) and \( f_\xi. \) With \( \mathcal{M}_\xi^2(X) \) in hand, we can judge if \( \mathcal{M}_\xi^2(X) < 0, \) the log-wave-length modes, \( p^2 < |\mathcal{M}_\xi^2(X)|, \) have imaginary frequencies and thus unstable, which causes perturbative instability of the system [23]. Unfortunately, no consistent scheme for treating such a case is available. Then, when \( \mathcal{M}_\xi^2(X) < 0 \) happens, we abandon the condition [Sigma] and set \( \mathcal{M}_\xi^2(X) = 0. \) Perturbation theory in this case is less efficient.

**C. Boltzmann equation**

In order to find a physical meaning of \( f_\xi, \) we recall a momentum density of the system:
\[ \bar{\rho}(x) = \text{Tr} \left\{ \left( \frac{\partial \hat{\varphi}(x)}{\partial x} \cdot \nabla \hat{\varphi}(y) \right) \rho \right\}_{y=x} = -\frac{i}{2} \frac{\partial}{\partial x^0} \nabla_y G^{\alpha\alpha}(x, y)|_{y=x}. \]

We first analyze the contribution from \( G^{(0)}_c \) (= \( G^{(0)}_{11} + G^{(0)}_{22} \), Eq. (6.3)). When the interaction is switched off, \( G^{(0)}_c \) reduces to \( \Delta_c^{(0)} \) (= \( \Delta_c^{(0)} + \Delta^{(0)}_{22} \), Eq. (4.24)). Computation of the contribution from \( \Delta_c^{(0)} \) yields
\[ \bar{\rho}(x)|_{\Delta_c^{(0)}} = \frac{e_\xi}{2} \int \frac{d^3 p}{(2\pi)^3} \bar{p} \left\{ [f_\xi(x; E_p^{(\xi)}, \hat{p}) + 1/2] + [f_\xi(x; -E_p^{(\xi)}, \hat{p}) + 1/2] \right\}, \]
where \( c_\pi = N - 1, c_\sigma = 1, E_p^{(\xi)} = \sqrt{p^2 + M^2}\), and \( \hat{p} = p/|p| \). Similarly, the computation of the contribution to the free \( (\lambda = 0) \) energy density yields

\[
P_{\text{free}}^0(x)|_{\Delta^{(0)}} = \frac{c_\xi}{2} \int \frac{d^3p}{(2\pi)^3} E_p^{(\xi)} \left[ f_\xi(x; E_p^{(\xi)}, \hat{p}) - f_\xi(x; -E_p^{(\xi)}, \hat{p}) \right].
\]

Subtracting the contribution from the vacuum, we see that \( f_\xi (\xi = \pi, \sigma) \) is related to the number density \( N_\xi(x; E_p^{(\xi)}, \hat{p}) \) of \( \xi \) through

\[
N_\xi(x; E_p^{(\xi)}, \hat{p}) = f_\xi(x; E_p^{(\xi)}, \hat{p}) = -1 - f_\xi(x; -E_p^{(\xi)}, -\hat{p}).
\]  

(6.8)

It is to be noted that the argument \( x \) here is macroscopic coordinates. For the interacting system, the corresponding relation is obtained using \( G_c (= G_{11} + G_{22}) \), Eq. (6.3), and the total energy density \( P_0(x) \) in place of \( P_{\text{free}}^0(x) \) above.

The contribution to \( P^{(\mu)}(x) \) from the difference \( G_c^{(0)} - \Delta^{(0)} \) yields a correction to the relation (6.8). This is also the case for the contribution of \( (G_1^{(p)})_c \). \( (G^{(t)})_c \) yields vanishing contribution to the gradient approximation.

We now turn to analyzing the remaining contribution that comes from \( (G_2^{(p)})_c \), in Eq. (6.3). Since \( (G_2^{(p)})_c \propto G_R^{(\xi)} G_A^{(\xi)} \), in the narrow-width approximation, \( Im\Sigma_R^{(\xi)}(X; P) \to -\epsilon(p_0)0^+ \), pinch singularity is developed. Then, the contribution of \( (G_2^{(p)})_c \) to \( P^{(\mu)}(x) \) diverges in this approximation. In practice, \( Im\Sigma_R^{(\xi)}(\propto \lambda^2) \) is a small quantity, so that the contribution, although not divergent, is large. This invalidates the perturbative scheme and a sort of “renormalization” is necessary for the number density \( \xi \). This observation leads us to introduce a condition \( (G_2^{(p)})_c = 0 \) or

\[
\left\{ f_\xi, P^2 - M^2 - Re\Sigma^{(\xi)} \right\} = \tilde{\Gamma}^{(p)}_{\xi} \quad (\xi = \pi, \sigma).
\]  

(6.9)

This serves as determining equation for so far arbitrary \( f_\xi \). Then \( \tilde{G} \) in Eq. (6.3) becomes \( \tilde{G} = G^{(0)} + G_1^{(p)} + G^{(t)} \), which is free from pinch singularity in the narrow-width approximation. It is obvious that, in the present scheme, above-mentioned large contributions do not appear.

In order to disclose the physical meaning of Eq. (6.9), we first define on the mass-shell, \( p_0 = \pm \omega_\xi (X; \pm p) (\equiv \pm \omega_\xi^{(\xi)}): \)

\[
Re\left(\Sigma_R^{(\xi)}(X; P)\right)^{-1} \bigg|_{p_0 = \pm \omega_\xi^{(\xi)}} = \left[ P^2 - M^2(X) - Re\Sigma_R^{(\xi)}(X; P) \right]_{p_0 = \pm \omega_\xi^{(\xi)}} = 0.
\]  

(6.10)

From Eq. (5.7), we see that, when \( M^2_{\xi}(X) > 0, \omega_\xi^{(\xi)}(X; 0) = M_{\xi}(X) \). We also introduce a wave-function renormalization factor,

\[
Z_{\xi}^{-1} = 1 - \frac{1}{2\omega_\xi(X; p)} \left. \frac{\partial Re\Sigma_R^{(\xi)}}{\partial p_0} \right|_{p_0 = \omega_\xi(X; p)}.
\]

It is now straightforward to show \( \xi \) that Eq. (6.9) becomes, on the mass-shell,

\[
\frac{\partial N_\xi}{\partial X_0} + v_\xi \cdot \nabla X N_\xi + \frac{\partial \omega_\xi(X; p) N_\xi}{\partial X_\mu} \bigg|_{p_0 = \omega_\xi(X; p)}
\]

\[
= \left. \frac{dN_\xi(X; \omega_\xi(X; p), \hat{p})}{dX_0} + \frac{\partial \omega_\xi(X; p)}{\partial \hat{p}} \cdot \frac{\partial N_\xi}{\partial X} - \frac{\partial \omega_\xi(X; p)}{\partial \hat{p}} \cdot \frac{dN_\xi}{d\hat{p}} \right.
\]

\[
\simeq Z_{\xi}^{-1} \tilde{\Gamma}_{\xi} \bigg|_{p_0 = \omega_\xi(X; p)}.
\]  

(6.11)

\[
\tilde{\Gamma}_{\xi} \bigg|_{p_0 = \omega_\xi(X; p)} = \frac{-i}{2\omega_\xi(X; p)} \left[(1 + N_\xi) \Sigma_1^{(\xi)} - N_\xi \Sigma_{21}^{(\xi)} \right] \bigg|_{p_0 = \omega_\xi(X; p)}.
\]  

(6.12)

Here \( N_\xi \) is as in Eq. (6.8) with \( E_p^{(\xi)} \to \omega_\xi(X; p) \) and \( v_\xi = \partial \omega_\xi(X; p)/\partial \hat{p} \) is the velocity of the quasiparticle mode with momentum \( p \). \( \tilde{\Gamma}_{\xi} \) in Eq. (6.12) is the net production rate of the quasiparticle of momentum \( p \). In fact, \( \tilde{\Gamma}_{\xi} \) is the difference between the production rate and the decay rate, so that \( \tilde{\Gamma}_{\xi} \) is the net production rate. In the case of an equilibrium system, \( \tilde{\Gamma}_{\xi} = 0 \) (detailed balance formula). \( N_\xi = N_\xi(X; \omega_\xi(X; p), \hat{p}) \) here is essentially (the main part of) the relativistic Wigner function, and Eq. (6.11) is the generalized relativistic Boltzmann equation (cf. \(^{29}\)).
Let us suppose the case $M_\xi^2(X) > \ Re \Sigma^{(\xi)}_R |_{p_0 = \pm \omega^{(\xi)}}$. The solution to Eq. (6.11) is $p_0 \simeq \pm E^{(\xi)}_p$. To one-loop order under consideration, only diagram that contributes to $\Sigma^{(\xi)}_{12(21)}$ in Eq. (6.12) is the one that includes two vertices coming from $\hat{\xi}^{(\xi)}$. Thus, $\Sigma^{(\xi)}_{12(21)}$ contains two $\Delta_{12(21)}$'s, each of which contains on-shell $\delta$-function, $\delta(\Omega^2 - M_\xi^2(X))$ (cf. Eq. (1.27)). One can easily see then that $\Sigma^{(\xi)}_{12(21)} |_{p_0 = E^{(\xi)}_p}$ vanishes unless $M_\sigma > 2M_\pi$, $\Sigma^{(\xi)}_{12(21)}(X; P)$ at $p_0 = E^{(\xi)}_p$ ($\xi = \pi, \sigma$) is computed in Appendix E. Let us turn to the case where $M_\xi^2(X) \leq O(\Sigma^{(\xi)}_R |_{p_0 = \pm \omega^{(\xi)}})$ with $\xi = \pi$ or $\xi = \sigma$ or $\xi = \pi$ and $\xi = \sigma$. For a hard momentum, $p > O(\Sigma^{(\xi)}_R |_{p_0 = \pm \omega^{(\xi)}})$, the same statement as above holds. For computing $\Sigma^{(\xi)}_{12(21)}(X; P)$ with soft $P$, i.e., $|P_0|$, $p \leq O(\Sigma^{(\xi)}_R |_{p_0 = \pm \omega^{(\xi)}})$, above two $\Delta_{12(21)}$'s contained in $\Sigma^{(\xi)}_{12(21)}$'s should be replaced by $G_{12(21)}$'s [20].

What we have shown here is that the requirement of the absence of $\hat{G}_2^{(p)} \propto G_R^{(\xi)} G_A^{(\xi)}$ from $\hat{G}$ leads to the Boltzmann equation for the quasiparticle-distribution functions. This means that the quasiparticles thus defined are the well-defined modes in the medium, in the sense that no large contribution appears in perturbation theory. Conversely, if we start with defining the quasiparticles such that their distribution functions subject to the Boltzmann equation, then, on the basis of them, well-defined perturbation theory may be constructed.

Comparison of our derivation of the generalized Boltzmann equation (GBE) with those in related works is made in [20]. It is worth recapitulating here the comparison with the derivation in nonequilibrium thermo field dynamics (NETFD) [27], which is a variant of nonequilibrium quantum field theory. We have imposed the condition (6.3) for determining $f_\xi$ or the number density $N_\xi$, so that pinch singularities (in narrow-width approximation) disappear. On the other hand, in NETFD, which employs the (space)time representation, the GBE is derived by imposing “an on-shell renormalization condition” for the propagator. Since the pinch singularity is a singularity in momentum space, it is not immediately obvious how to translate this condition into the (space)time representation, as adopted in NETFD. Nevertheless, closer inspection of the structure of both formalisms tells us that our condition is in accord with the on-shell renormalization condition in NETFD. Incidentally, reconciliation of the NETFD with the $|P_0|$-prescription (cf. Appendix A), a notion in momentum space, remains as an open problem.

VII. CONCLUSION AND OUTLOOK

In this paper we have constructed from first principles a perturbative framework for computing reaction rates of the processes taking place in the $O(N)$ linear-sigma system in a broken phase. Only approximation we have employed is the so-called gradient approximation, so that the framework applies to the quasiumiform systems near equilibrium or the nonequilibrium quasistationary systems.

The reactions taking place in the system causes a spacetime evolution of the system — development of phase transition. This is the next subject following to the present analysis. At the final stage of such an analysis, one should check whether or not the rate of the phase-change of the system is too large, so that the gradient approximation adopted in this paper is violated.

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APPENDIX A: $|P_0|$-PRESCRIPTION

One starts with $f_\xi(x, y)$ that is local in time, $f_\xi(x, y) = \delta(x_0 - y_0) \sum_{\tau = \pm} g^{(\tau)}(x, y; x_0)$. Here $\pm$ denotes positive/negative frequency part and the time coordinate ‘$x_0$’ is of macroscopic. Then, in place of Eq. (4.20), we have, with obvious notation,

$$L'(x, y) \simeq iP_\xi(X) \int \frac{d^3 p}{(2\pi)^3} e^{iP(x-y)} \sum_{\tau = \pm} \left[ 2i \delta'(x_0 - y_0) \partial_{X_0} g^{(\tau)}(X; p) \right]$$
\[ +\delta(x_0 - y_0) \left( 2p \cdot \partial_x - \frac{\partial M_\xi^2(X)}{\partial p} \right) g_\xi^{(r)}(X; p) \]

with \( X_0 = x_0 \). The form (A1) is local in time. It is well known through the analyses of equilibrium case that the following replacement \(|p_0|\)-prescription should be made:

\[ \theta(p_0) g_\xi^{(+)}(X; p) + \theta(-p_0) g_\xi^{(-)}(X; p) \rightarrow f_\xi(X; P) \]

with \( f_\xi(X; P) \) as in (4.20) in the text. Adopting this prescription, we attain Eq. (4.20). (See [26] for more details.)

**APPENDIX B: DERIVATION OF EQ. (6.3)**

Here we solve the Schwinger-Dyson equation (6.2). Multiplying \( \tilde{B}_L^{(0)} \), Eq. (5.2), from left and \( \tilde{B}_R^{(0)} \) from right, we obtain

\[ \tilde{G} = \tilde{\Delta} + \tilde{\Delta} \cdot \tilde{\Sigma} \cdot \tilde{G} , \]

where

\[ \tilde{\Delta} = \begin{pmatrix} \Delta_R & 0 \\ 0 & -\Delta_A \end{pmatrix} , \]

\[ \Delta_{\text{off}}(x, y) \simeq \frac{2i}{\gamma} \gamma(X) \partial_\gamma \langle \gamma(\gamma) \rangle \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} P^\mu (f_\gamma(X; P) - f_\gamma(X; P)) \Delta_R^\gamma(X; P) \Delta_A^\gamma(X; P) \]

\[ + \frac{2i}{\gamma} \gamma(X) \partial_\gamma \langle \gamma(\gamma) \rangle \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} P^\mu (f_\gamma(X; P) - f_\gamma(X; P)) \Delta_R^\gamma(X; P) \Delta_A^\gamma(X; P) \]

\[ + iP_\xi(X) \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left\{ f_\xi + P^2 - M_\xi^2 \right\} \Delta_R^\xi(X; P) \Delta_A^\xi(X; P) \]

Using the identity (C1) in Appendix C, we obtain

\[ \tilde{\Sigma} = \begin{pmatrix} \Sigma_R & 0 \\ 0 & -\Sigma_A \end{pmatrix} , \]

\[ \Sigma_R = \Sigma_{11} + \Sigma_{12} , \quad \Sigma_A = \Sigma_{11} + \Sigma_{21} = -\Sigma_{22} - \Sigma_{12} , \]

\[ \Sigma_{\text{off}} = \Sigma_{12} \cdot (1 + P_\xi \cdot f_\xi \cdot P_\xi - P_\xi \cdot f_\xi \cdot P_\xi \cdot \Sigma_{21} + \Sigma_{11} \cdot P_\xi \cdot f_\xi \cdot P_\xi - P_\xi \cdot f_\xi \cdot P_\xi \cdot \Sigma_{11} . \]

Substituting the leading-order expression (6.1) for \( \Sigma \)'s and using Eq. (C1) in Appendix C, we obtain, to the gradient approximation,

\[ \Sigma_{\text{off}} \simeq P_\xi(x) \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left[ (1 + f_\xi(X; P)) \Sigma_{12}^\xi(X; P) - f_\xi(X; P) \Sigma_{21}^\xi(X; P) \right] P_\xi(y) \]

\[ + \frac{i}{2} P_\xi(x) \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left\{ f_\xi, \Sigma_{11}^\xi - \Sigma_{22}^\xi \right\} P_\xi(y) , \]

where use has been made of \( P_\xi(\partial P_\xi)P_\xi = 0 \) (no summation over \( \xi \)). To leading order under consideration, \( \Sigma_{12}^\xi(X; P) \) and \( \Sigma_{21}^\xi(X; P) \) are pure imaginary. Then, from Eq. (C2), \( Re \Sigma_{12}^\xi(X; P) = -Re \Sigma_{11}^\xi(X; P) \) and, from Eq. (C3) and Eq. (B4), we obtain \( Im \Sigma_{22}^\xi(X; P) = Im \Sigma_{11}^\xi(X; P) = i(\Sigma_{12}^\xi(X; P) + \Sigma_{21}^\xi(X; P))/2 \). Using these relations, we finally obtain

\[ \Sigma_{\text{off}}(x, y) \simeq P_\xi(x) \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \Sigma_{\text{off}}^\xi(X; P) P_\xi(y) , \]

\[ \Sigma_{\text{off}}^\xi(X; P) \simeq iP_\xi(X; P) + i \left\{ f_\xi, Re \Sigma_R^\xi \right\} . \]
Note that the first term of the R.H.S. of Eq. (B3), being proportional to $\lambda^2$, is proportional to the net-production rate (cf. above after Eq. (6.13)), which causes the change in the number density. The second term is proportional to $\lambda$ and includes derivatives with respect to $X_{\mu}$, and is of higher order. Then one can drop the second term. Nevertheless, we shall keep it in the following.

Substituting Eqs. (B2) and (B3) into Eq. (B1) and retaining up to the terms that are linear in $\Sigma_{\text{off}}$, we obtain

$$\hat{G} = \hat{\Delta} + \sum_{n=1}^{\infty} \hat{\Delta} \left[ \hat{\Sigma} \cdot \hat{\Delta} \right]^n$$

$$\approx \hat{\Delta} + \sum_{n=1}^{\infty} \hat{\Delta} \left[ \left( \begin{array}{cc} \Sigma_R & 0 \\ 0 & -\Sigma_A \end{array} \right) \cdot \hat{\Delta} \right]^n + \left( \begin{array}{cc} G_R & 0 \\ 0 & -G_A \end{array} \right) \left( \begin{array}{cc} 0 & \Sigma_{\text{off}} \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} G_R & 0 \\ 0 & -G_A \end{array} \right),$$

(Equation B6)

where higher-order terms have been dropped and

$$G_{R(A)} = \Delta_{R(A)} + \Delta_{R(A)} \sum_{n=1}^{\infty} \left( \Sigma_{R(A)} \cdot \Delta_{R(A)} \right)^n.$$

Keeping terms linear in $\Delta_{\text{off}}$ (Eq. (B2)), we can solve Eq. (B6):

$$\left( \mathbf{G}(x,y) \right)_{11} = \mathbf{P}_\xi(x) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} G^{(\xi)}_R(X;P) \mathbf{P}_\xi(y)$$

$$+ 2i |\hat{\phi}(X)\rangle \frac{\partial}{\partial \xi} \langle \hat{\phi}(X) | \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} P^\mu G^{(\pi)}_R(X;P) G^{(\sigma)}_R(X;P),$$

$$\left( \mathbf{G}(x,y) \right)_{12} = - \left( \mathbf{G}(x,y) \right)_{11} \bigg|_{R \to A},$$

$$\left( \mathbf{G}(x,y) \right)_{22} = 2i |\hat{\phi}(X)\rangle \frac{\partial}{\partial \xi} \langle \hat{\phi}(X) | \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} P^\mu (f_\sigma(X;P) - f_\pi(X;P)) G^{(\pi)}_R(X;P) G^{(\sigma)}_A(X;P),$$

$$- i \mathbf{P}_\xi(X) \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[ \hat{P}^\dagger \xi - \left\{ \hat{f}_\xi, \text{Re} \left( G^{(\xi)}_R \right)^{-1} \right\} \right] G^{(\xi)}_R(X;P) G^{(\xi)}_A(X;P),$$

where $G^{(\xi)}_{R(A)}(X;P)$ is as in Eq. (4.4) in the text. Computing $\mathbf{G} = \mathbf{B}_L^{(0)} \cdot \hat{\mathbf{G}} \cdot \mathbf{B}_R^{(0)}$ in the gradient approximation, we obtain Eqs. (5.3) - (5.5) in the text.

**APPENDIX C: PROPERTIES OF THE SELF-ENERGY PART**

It is obvious that the relation (4.7) holds for the full propagator $\hat{G}$ and also for the self-energy-part inserted propagator $\hat{\Delta} \cdot \hat{\Sigma} \cdot \hat{\Delta}$. From the latter, one can readily obtain the relation:

$$\sum_{i,j=1}^{2} \Sigma_{ij} = 0.$$

(Equation C1)

The expression for full $G_R$ and $G_A$ with $\bar{\phi}_1 = \bar{\phi}_2 \equiv \bar{\phi}$ are given by Eq. (4.8) with Heisenberg fields for $\phi$’s. From Eq. (4.8), we obtain

$$\left( \Delta^{\alpha\beta}_{R(A)}(x,y) \right)^* = \Delta^{\alpha\beta}_{R(A)}(x,y), \quad \Delta^{\alpha\beta}_{A}(y,x) = \Delta^{\alpha\beta}_{R}(x,y).$$

(Equation C2)

Eq. (C2) is also valid for self-energy-part-inserted propagator, $\hat{\Delta} \cdot \hat{\Sigma} \cdot \hat{\Delta}$:

$$\left( \Delta^{\alpha\beta}_{R(A)} \cdot \Sigma^{\alpha\beta}_{R(A)} \cdot \Delta^{\alpha\beta}_{R(A)} \right)^* = \Delta^{\alpha\beta}_{R(A)} \cdot \Sigma^{\alpha\beta}_{R(A)} \cdot \Delta^{\alpha\beta}_{R(A)},$$

$$\left[ \Delta^{\beta\alpha}_{A} \cdot \Sigma^{\beta\alpha}_{A} \cdot \Delta^{\beta\alpha}_{A} \right](y,x) = \left[ \Delta^{\beta\alpha}_{R} \cdot \Sigma^{\beta\alpha}_{R} \cdot \Delta^{\beta\alpha}_{R} \right](x,y).$$
which yields
\[
\left(\left[\Delta^\alpha\beta, \Sigma^\alpha\beta, \Delta^\beta\beta\right](x, y)\right)^* = \left[\Delta^\beta\beta, \Sigma^\beta\alpha, \Delta^\alpha\alpha\right](y, x).
\] (C3)

Applying \(\partial^2 + M^2_\xi P_\xi\) from both sides of Eq. (C3), we obtain
\[
\left(\Sigma^\alpha\beta(x, y)\right)^* = \Sigma^\beta\alpha(x, y).
\] (C4)

As discussed in Sec. VIA, it is sufficient to compute the leading part of \(\Sigma_{ij}\), Eq. (6.1). For a UV-renormalization scheme, as in [4], we use the \(\overline{MS}\) scheme. Computation is a straightforward generalization of [4] and the final result reads
\[
\Sigma_R(X; P) = P_\xi(X)\Sigma^{\xi}_R(X; P) - \chi_\xi(X),
\] (D1)
\[
\Sigma^{(\pi)}_R(X; P) = \frac{\lambda}{6} [(N + 1)I_\pi + I_\sigma] + \frac{\lambda^2}{9}\varphi^2(X)J_\pi\sigma(X; P),
\]
\[
\Sigma^{(\omega)}_R(X; P) = \frac{\lambda}{6} [(N - 1)I_\pi + 3I_\sigma] + \frac{\lambda^2}{18}\varphi^2(X) [(N - 1)J_\pi\pi(X; P) + 9J_\sigma\sigma(X; P)].
\]

Here
\[
I_\xi(X) = \mathcal{M}^2_\xi(X)/(4\pi^2) + \ln \frac{\mathcal{M}^2_\xi(X)}{\mu_d^2} + \int \frac{d^3q}{(2\pi)^3} \frac{N_\xi(X; E_\xi^\xi(X), q)}{E_\xi^\xi(X)} (\xi = \pi, \sigma),
\]
\[
J_\pi\sigma(X; P) = \frac{1}{(4\pi^2)} \left[ \ln \frac{\mathcal{M}^2_\xi(X)}{\mu_d^2} - \frac{P^2 - \mathcal{M}^2_\xi(X) + \mathcal{M}^2_\xi(X)}{2P^2} \ln \frac{\mathcal{M}^2_\xi(X)}{\mathcal{M}^2_\xi(X)} 
\right.
\]
\[
- \frac{S}{2P^2} \ln \left[ \frac{(S - P^2)^2 - (\mathcal{M}^2_\xi(X) - \mathcal{M}^2_\xi(X))^2}{(S + P^2)^2 - (\mathcal{M}^2_\xi(X) - \mathcal{M}^2_\xi(X))^2} \right]
\]
\[
+ \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{1}{E_\phi^\pi(X; q)} \frac{N_\sigma(X; E_\phi^\sigma(X), \hat{q})}{P^2 + \mathcal{M}^2_\phi(X) - 2p_0E_\phi^\sigma(X) + 2p \cdot q + i\epsilon(p_0 - E_\phi^\sigma(X))^0+}
\]
\[
+ \frac{N_\sigma(X; E_\phi^\sigma(X), \hat{q})}{P^2 + \mathcal{M}^2_\phi(X) - 2p_0E_\phi^\sigma(X) - 2p \cdot q + i\epsilon(p_0 + E_\phi^\sigma(X))^0+} \right\},
\] (D2)
\[
\mathcal{J}_{\xi\xi}(X; P) = \mathcal{J}_{\pi\sigma}(X; P) \bigg|_{\mathcal{M}^2_\pi = \mathcal{M}^2_\sigma \rightarrow \mathcal{M}^2_\xi} (\xi = \pi, \sigma),
\] (D3)

where \(E_\phi^\xi(X) = \sqrt{q^2 + \mathcal{M}^2_\phi(X)}, \)

\[
S = \sqrt{\left[ \mathcal{M}_\phi(X) - \mathcal{M}_\phi(X) \right]^2 - P^2} \left[ \mathcal{M}_\phi(X) + \mathcal{M}_\phi(X) \right]^2 - P^2
\] \(, \)

and \(\mu_d\) is an arbitrary parameter that appears in the dimensional-regularization scheme, as adopted here. The first term on the R.H.S. of Eq. (D2) is valid in the region \(P^2 < 0\). Its expressions in other regions of \(P^2\) are obtained through analytic continuation with \(\mathcal{M}^2_\pi(X) \rightarrow \mathcal{M}^2_\pi(X) - i\epsilon(p_0)^0+\).
B. Gap equation

The gap equation (6.7) with Eq. (D1) yields

\[ M_\xi^2(X) - m^2 = \delta M_\xi^2(X) \quad (\xi = \pi, \sigma) \]  

(D4)

with

\[
\delta M_\pi^2(X) = \frac{\lambda}{6} [(N + 1) I_\pi + I_\sigma] + \frac{\lambda^2}{9} \varphi^2(X) \left[ \frac{1}{(4\pi)^2} \left( \frac{M_\pi^2(X)}{2M_\pi^2(X)} \ln \frac{M_\pi^2(X)}{M_\pi^2(X)} + \ln \frac{M_\pi^2(X)}{e^2\mu_{\pi}^2} - \kappa^{(\pi)}_{\pi\pi} \right) 
+ \mathcal{H}^{(\pi)}_{\pi\sigma}(X; p_0 = M_\pi(X), 0) \right],
\]

(D5)

\[
\delta M_\sigma^2(X) = \frac{\lambda}{6} [(N - 1) I_\pi + 3I_\sigma] + \frac{\lambda^2}{18} \varphi^2(X) \left[ \frac{1}{(4\pi)^2} \left( (N - 1) \left( \frac{M_\pi^2(X)}{e^2\mu_{\pi}^2} - \kappa^{(\pi)}_{\sigma\pi} \right) + 3\sqrt{3} \pi + 9 \ln \frac{M_\pi^2(X)}{e^2\mu_{\pi}^2} \right) 
+ \left\{ (N - 1) \mathcal{H}^{(\pi)}_{\pi\sigma}(X; p_0 = M_\sigma(X), 0) + 9 \mathcal{H}^{(\pi)}_{\sigma\pi}(X; p_0 = M_\sigma(X), 0) \right\} \right].
\]

(D6)

\[
\kappa^{(\pi)}_{\pi\pi} = \theta \left( M_\pi^2(X) - 4M_\pi^2(X) \right) \frac{\sqrt{M_\pi^2(X)(M_\pi^2(X) - 4M_\pi^2(X))}}{2M_\pi^2(X)} \\
\times \left[ \ln \frac{M_\pi^2(X) - \sqrt{M_\pi^2(X)(M_\pi^2(X) - 4M_\pi^2(X))}}{M_\pi^2(X) + \sqrt{M_\pi^2(X)(M_\pi^2(X) - 4M_\pi^2(X))}} \\
- \ln \frac{M_\pi^2(X) - 2M_\pi^2(X) - \sqrt{M_\pi^2(X)(M_\pi^2(X) - 4M_\pi^2(X))}}{M_\pi^2(X) - 2M_\pi^2(X) + \sqrt{M_\pi^2(X)(M_\pi^2(X) - 4M_\pi^2(X))}} \right],
\]

and

\[
\kappa^{(\sigma)}_{\pi\pi} = \theta \left( M_\sigma^2(X) - 4M_\sigma^2(X) \right) \\
\times \frac{\sqrt{M_\sigma^2(X)(M_\sigma^2(X) - 4M_\sigma^2(X))}}{M_\sigma^2(X)} \ln \frac{M_\sigma^2(X) - \sqrt{M_\sigma^2(X)(M_\sigma^2(X) - 4M_\sigma^2(X))}}{M_\sigma^2(X) + \sqrt{M_\sigma^2(X)(M_\sigma^2(X) - 4M_\sigma^2(X))}} \\
+ \theta \left( 2M_\sigma^2(X) - M_\sigma^2(X) \right) \frac{\sqrt{M_\sigma^2(X)(M_\sigma^2(X) - 4M_\sigma^2(X))}}{M_\sigma^2(X)} \\
\times \left[ 2 \arctan \frac{\sqrt{M_\sigma^2(X)(M_\sigma^2(X) - 4M_\sigma^2(X))}}{M_\sigma^2(X)} - \pi \right],
\]

and

\[
\mathcal{H}^{(\pi)}_{\pi\sigma}(X; p_0, \mathbf{0}) = \int \frac{d^3q}{(2\pi)^3} \left[ \frac{M_\pi^2(X)}{E_q^{(\pi)}(X)} \left\{ M_\pi^2(X) - 4p_0^2(E_q^{(\pi)}(X))^2 \right\} \right. \\
+ \left. \frac{(2M_\pi^2(X) - M_\pi^2(X))}{E_q^{(\pi)}(X) \left\{ (2M_\pi^2(X) - M_\pi^2(X))^2 - 4p_0^2(E_q^{(\pi)}(X))^2 \right\}^2} \right],
\]

\[
\mathcal{H}^{(\pi)}_{\pi\pi}(X; p_0, \mathbf{0}) = 2 \int \frac{d^3q}{(2\pi)^3} \frac{M_\pi^2(X)}{E_q^{(\pi)}(X)} \left\{ M_\pi^2(X) - 4p_0^2(E_q^{(\pi)}(X))^2 \right\},
\]

(\xi = \pi, \sigma).
APPENDIX E: COMPUTATION OF $\Sigma_{12}$ AND $\Sigma_{21}$

Here, we compute the leading part of the one-loop contribution to $\Sigma_{12(21)}$: 

$$
\Sigma_{12(21)}(X; P) = \mathbf{P}_\pi(X)\Sigma_{12(21)}^{(\pi)}(X; P) + \mathbf{P}_\sigma(X)\Sigma_{12(21)}^{(\sigma)}(X; P),
$$

$$
\Sigma_{12(21)}^{(\pi)}(X; P) = \frac{-i\lambda^2\varphi^2(X)}{9} \int \frac{d^4Q}{(2\pi)^4} \Delta_{12(21)}^{(\pi)}(X; Q)\Delta_{12(21)}^{(\pi)}(X; P-Q),
$$

$$
\Sigma_{12(21)}^{(\sigma)}(X; P) = \frac{-i\lambda^2\varphi^2(X)}{18} \int \frac{d^4Q}{(2\pi)^4} \left[ (N-1)\Delta_{12(21)}^{(\sigma)}(X; Q)\Delta_{12(21)}^{(\pi)}(X; P-Q) + 9\Delta_{12(21)}^{(\sigma)}(X; Q)\Delta_{12(21)}^{(\pi)}(X; P-Q) \right].
$$

We compute $\Sigma_{12(21)}^{(\xi)}$ on the mass-shell $p_0 = E_p^{(\xi)}$. The contribution to $\Sigma_{12(21)}^{(\sigma)}$ from the term that accompanies two $\Delta_{12(21)}^{(\sigma)}$'s vanishes. Nonvanishing contributions emerge only when $2M_\pi < M_\sigma$. Computation is straightforward but lengthy. We only display the final forms:

$$
\Sigma_{12}^{(\pi)}(X; P) = \frac{i\lambda^2\varphi^2(X)}{72\pi} \frac{1}{p} \int_{\xi_{11}}^{\xi_{12}} d\xi N_\pi(X; \xi, \hat{q}) \left[ 1 + N_\pi(X; \xi - E_p^{(\pi)}, \hat{q} - p) \right]_{\hat{p} \cdot \hat{q} = z_1},
$$

$$
\Sigma_{21}^{(\pi)}(X; P) = \frac{i\lambda^2\varphi^2(X)}{72\pi} \frac{1}{p} \int_{\xi_{12}}^{\xi_{31}} d\xi N_\pi(X; \xi, \hat{q}) \left[ 1 + N_\pi(X; \xi + E_p^{(\pi)}, \hat{q} + p) \right]_{\hat{p} \cdot \hat{q} = z_2},
$$

$$
\Sigma_{12}^{(\sigma)}(X; P) = \frac{i(N-1)\lambda^2\varphi^2(X)}{144\pi} \frac{1}{M_\sigma} \int_{\xi_{13}}^{\xi_{31}} d\xi N_\pi(X; \xi, \hat{q}) N_\pi(X; E_p^{(\sigma)} - \xi, \hat{q} - p) \left[ 2 + N_\pi(X; E_p^{(\sigma)} - \xi, \hat{q} - p) \right]_{\hat{p} \cdot \hat{q} = z_3},
$$

$$
\Sigma_{21}^{(\sigma)}(X; P) = \frac{i(N-1)\lambda^2\varphi^2(X)}{144\pi} \frac{1}{M_\sigma} \int_{\xi_{13}}^{\xi_{31}} d\xi N_\pi(X; \xi, \hat{q}) N_\pi(X; E_p^{(\sigma)} - \xi, \hat{q} - p) \left[ 2 + N_\pi(X; E_p^{(\sigma)} - \xi, \hat{q} - p) \right]_{\hat{p} \cdot \hat{q} = z_3},
$$

for $\xi_{11} = \xi_{11}^{(1)} = \frac{M_\sigma E_p^{(\pi)} + \sqrt{M_\sigma^2 + 4M_\sigma^2}}{2M_\sigma}$, $z_1 = \frac{2E_p^{(\pi)} \xi - M_\sigma}{2p\sqrt{\xi^2 - M_\sigma^2}}$, 

$\xi_{12} = \xi_{11}^{(2)} - E_p^{(\pi)}$, $z_2 = \frac{2E_p^{(\pi)} \xi - M_\sigma^2 + 2M_\sigma^2}{2p\sqrt{\xi^2 - M_\sigma^2}}$, 

$\xi_{13} = \frac{M_\sigma E_p^{(\pi)} + \sqrt{M_\sigma^2 + 4M_\sigma^2}}{2M_\sigma}$, $z_3 = \frac{2E_p^{(\pi)} \xi - M_\sigma^2}{2p\sqrt{\xi^2 - M_\sigma^2}}$.

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