Surfactant-induced migration of a spherical drop in Stokes flow

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(Dated: May 18, 2009)

In Stokes flows, symmetry considerations dictate that a neutrally-buoyant spherical particle will not migrate laterally with respect to the local flow direction. We show that a loss of symmetry due to flow-induced surfactant redistribution leads to cross-stream drift of a spherical drop in Poiseuille flow. We derive analytical expressions for the migration velocity in the limit of small non-uniformities in the surfactant distribution, corresponding to weak-flow conditions or a high-viscosity drop. The analysis predicts that the direction of migration is always towards the flow centerline.

PACS numbers: 47.15.G-, 47.55.Dk

A classic result in microhydrodynamics is that lateral migration of a neutrally-buoyant, non-deformable spherical particle is prohibited in the creeping-flow limit. The result arises from the linearity of the Stokes equation and boundary conditions, and the symmetry of the problem under flow-reversal\cite{1}. However, such cross-stream drift may occur if the symmetry is lost, e.g. by particle deformation in a shear gradient or in the presence of a wall\cite{2}. A small-deformation solution for a drop with a clean (surfactant-free) interface predicts lateral motion in unbounded Poiseuille flow, with the direction of motion depending on the ratio of drop and suspending fluid viscosities\cite{3}. In this note, we investigate the possibility of cross-stream migration of a non-deforming spherical drop, induced by asymmetric interfacial tension resulting from redistribution of surfactant.

Let us consider a drop with radius $a$ and viscosity $\lambda \eta$, embedded in an ambient fluid with viscosity $\eta$. A surfactant, insoluble in the bulk phases, is adsorbed on the drop interface; diffusion of surfactant is neglected. The average surfactant concentration is $\Gamma_{\text{eq}}$; the corresponding interfacial tension is $\sigma_{\text{eq}}$, and $\sigma_0$ is the interfacial tension in the absence of surfactant. The drop is placed in a plane Poiseuille flow $\mathbf{v}^\infty = U_0 \alpha y^2 \hat{x}$, at position $(x_0, y_0, z_0)$. In a coordinate system centered on, and translating with, the drop, the flow becomes

$$\mathbf{v}^\infty = U_0 (\dot{\gamma} y + \alpha y^2) \hat{x} - \mathbf{U}_m,$$  

where $\alpha$ is a measure of the curvature of the flow profile, $\dot{\gamma} = 2\alpha y_0$ is the local shear rate, and the migration velocity $\mathbf{U}_m$ is the difference between the velocities of the drop and the undisturbed flow at the drop center. In creeping flows, a drop remains spherical provided that the capillary number is small

$$Ca = \frac{\eta U_0}{\sigma_{\text{eq}}} \ll 1.$$  

Convection by the surrounding flow creates non-uniformities in the surfactant distribution. The magnitude of this redistribution is determined by the inverse Marangoni number

$$Ma^{-1} = \frac{\eta U_0}{\Delta \sigma} \Delta \sigma = \sigma_0 - \sigma_{\text{eq}},$$  

which is the ratio of viscous stresses to the characteristic Marangoni stresses (gradients in surface tension). For a dilute surfactant monolayer, a linear “perfect gas” interfacial equation of state relates surface tension and local surfactant concentration: $\sigma(\Gamma) = \sigma(1) - Ma(\Gamma - 1)$. Henceforward, all quantities are normalized using $a$, $\eta$, $U_0$, and $\Gamma_{\text{eq}}$.

By the linearity of the Stokes equations, the drop migration arises as a superposition of that due to the flow about a clean drop, and that due to the flow driven by the Marangoni stresses\cite{4}. The clean drop velocity field gives rise to slip, but no lateral migration\cite{5}:

$$\mathbf{U}_m = \alpha \frac{\lambda}{3\lambda + 2} \hat{x} + \mathbf{U}_m^s(\Gamma).$$  

To determine the surfactant contribution $\mathbf{U}_m^s$, we solve for the velocity fields corresponding to the surfactant-covered drop using the formalism developed by Bławzdziewicz et al.\cite{6}. In the original work, the method was applied to the dynamics of a stationary surfactant-covered drop in a linear flow. We have generalized the approach to treat a migrating drop in higher-order flows. Owing to the spherical symmetry of the problem, all quantities are represented in terms of spherical harmonics (Appendix A). Accordingly, the surfactant concentration $\Gamma$ is expanded in scalar harmonics

$$\Gamma = 1 + \sum_{j=1}^{\infty} \sum_{m=-j}^{j} g_{jm} Y_{jm}.$$  

The velocity field is expanded in a set of fundamental solutions of the Stokes equations\cite{7,8}:

$$\mathbf{v}_{\text{out}} = c_{jmq}^+ \left[ \mathbf{u}_{jmq}^+(r) - \mathbf{u}_{jmq}^-(r) \right] + c_{jmq}^- \mathbf{u}_{jmq}^-(r),$$

$$\mathbf{v}_{\text{in}} = c_{jmq}^+ \mathbf{u}_{jmq}^+(r).$$  

Summation over repeated indices is implied. The velocity coefficients are decomposed into clean-drop and
surflect, and rotational components of the flow. The functions \( \mathbf{u}^{\pm}_{jm} \) are vector solid harmonics related to the harmonics in the Lamb solution. The function \( \mathbf{u}^{\pm}_{jm} \) is radial, while \( \mathbf{y}^{\pm}_{jm} \) and \( \mathbf{u}^{\pm}_{jm} \) are tangential to a sphere: \( \mathbf{u}^{\pm}_{jm} \) is surface-solenoidal (\( \nabla \times \mathbf{u}^{\pm}_{jm} = 0 \)). The far-field (imposed) flow is specified by \( \mathbf{v}^\infty = c_{jm}^{\infty} \mathbf{u}^{\pm}_{jm} \); coefficients for the Poiseuille flow \( \mathbf{u}^{\infty} \) are listed in Appendix B. The velocity fields defined in \( \mathbf{u}^{\infty} \) are naturally continuous across the interface because the basis fields \( \mathbf{u}^{\pm}_{jm} \) reduce to the corresponding vector spherical harmonics \( \mathbf{y}^{\infty} \) at \( r = 1 \). The velocity coefficients \( c_{jm}^{\pm} \) are determined from the stress balance equations listed in Appendix C. A fixed spherical shape limits the surface velocity to rigid motions. Thus, for \( j > 1 \), \( c_{jm}^{\pm} = 0 \), and the other two coefficients \( c_{jm}^{\infty} \) are determined from the tangential stress balances alone, as use of the normal stress balance in conjunction with shape specification over-constrains the problem. Both the tangential and normal stress balances are used to determine the \( c_{jm}^{\infty} \), which correspond to translational and rotational components of the flow. We obtain for the surfactant-driven flow:

\[
\begin{align*}
\mathbf{c}_{jm0} &= -\frac{1}{(3\lambda + 2)} \frac{\delta_{ij}}{\lambda + 1} \mathbf{g}_{jm} - \frac{(1 - \delta_{ij})}{\lambda + 1} \frac{\lambda j + 1}{(2j + 1)} \mathbf{g}_{jm}^{\infty}, \\
\mathbf{c}_{jm1} &= 0, \\
\mathbf{c}_{jm2} &= \frac{\delta_{ij}}{(3\lambda + 2)} \frac{\lambda}{\lambda + 1} \mathbf{g}_{jm}^{\infty} \\
\end{align*}
\]

where \( \delta_{kl} \) is the Kronecker delta. The solution for \( j > 1 \) is identical to that of Bławzdziewicz et al.\(^2\). The clean-drop flow is:

\[
\begin{align*}
\mathbf{c}_{jm0}^{\infty} &= \frac{1}{3\lambda + 2} \left[ (2\lambda + 3)\mathbf{c}_{jm0}^{\infty} + \sqrt{2}(\lambda - 1)\mathbf{c}_{jm2}^{\infty} \right] \\
&\quad + \frac{1}{\lambda + 1} \left[ 2\mathbf{c}_{jm0}^{\infty} - \frac{3}{\lambda + 1} \mathbf{c}_{jm2}^{\infty} \right], \\
\mathbf{c}_{jm1}^{\infty} &= \frac{2j + 1}{2 + j + \lambda(j - 1)} \mathbf{c}_{jm1}, \\
\mathbf{c}_{jm2}^{\infty} &= \frac{\delta_{ij}}{3\lambda + 2} \left[ \sqrt{2}(\lambda - 1)\mathbf{c}_{jm0}^{\infty} + (\lambda + 4)\mathbf{c}_{jm2}^{\infty} \right].
\end{align*}
\]

The above expression, in conjunction with \( \mathbf{U}^{(1)} = \mathbf{U} \) yields the surfactant-induced migration:

\[
\mathbf{U}^{s} = \frac{Ma}{\sqrt{6\pi(3\lambda + 2)}} \left[ -(g_{11} - g_{1-1})\hat{x} - i(g_{11} + g_{1-1})\hat{y} + \sqrt{2}g_{10}\hat{z} \right].
\]

The surfactant distribution coefficients \( g_{jm} \) are determined from the evolution equation\(8\):

\[
\frac{\partial g_{jm}}{\partial t} = C_{jm} + [\Omega_{jmj2m2} + \Lambda_{jmj2m2}]g_{jm2} + Ma[W_{jm} + \Theta_{jmj1m1j1m1j2m2}]g_{jm2}g_{jm2}.
\]

The terms in this equation are defined in Appendix D. The first three terms involve the clean-drop flow: \( C_{jm} \) describes convection of the uniform part of the surfactant distribution by the imposed flow, \( \Omega \) describes rotation by the linear shear component of the flow, and \( \Lambda \) describes convection of the non-uniform part of the surfactant distribution by the non-rotational components of the flow. The terms proportional to \( Ma \) pertain to flows driven by Marangoni stresses. The linear term describes relaxation towards the equilibrium uniform surfactant distribution, while the quadratic term describes convection of surfactant by the surfactant-induced flow.

For arbitrary distortions of the surfactant concentration, equation\(12\) must be integrated numerically to determine \( \mathbf{U} \). Here we consider steady-state solutions for the surfactant distribution, which implies that the surfactant evolution occurs on a faster time scale than the drop migration, i.e., \( \eta a / \Delta \sigma \ll a / U \). Furthermore, we limit ourselves to small disturbances in the surfactant concentration, admitting an analytical solution for the migration velocity in the form of a regular perturbation expansion:

\[
\mathbf{U} = \mathbf{U}^{(0)} + \mathbf{U}^{(1)} + \ldots
\]

The choice of small parameter depends on the flow regime of interest. In weak flows, surfactant relaxation towards the equilibrium distribution is fast, the gradients in surface tension are small, and \( Ma^{-1} \) is a natural choice. If the drop is far away from the centerline, \( y_0 \gg 0 \), the local flow is dominated by the shear \( \gamma = 2a y_0 \), and surfactant inhomogeneities are limited by drop rotation. In this case, the ratio of rotational and extensional time scales\(3\) is \( \sim \lambda^{-1} \), so this is an appropriate small parameter. Next, we examine these two regimes in more detail.

**Weak flow / Nearly-incompressible surfactant:**

\( Ma^{-1} \ll 1 \) and \( \lambda = o(Ma) \)

If \( Ma^{-1} \ll 1 \), following the discussion in Bławzdziewicz et al.\(^5\), we introduce a regular expansion for the surfactant concentration:

\[
g_{jm} = \sum_{k=0}^{\infty} Ma^{-k-1} g_{jm}^{(k)}.
\]

At
leading order, the evolution equation \([12]\) becomes
\[
0 = C_{jm} + W(j)g_{jm}^{(0)}.
\]
Solving for \(g_{jm}^{(0)}\) yields
\[
g_{1m}^{(0)} = \frac{5}{\sqrt{2}} \left( e_{1m0}^{\infty} - \sqrt{2} e_{1m2}^{\infty} \right) \quad \text{and} \quad (14)
g_{j+1}^{(0)} = \frac{2j+1}{j(j+1)} \left[ 2\sqrt{j(j+1)}e_{jm0}^{\infty} - 3e_{jm2}^{\infty} \right] \quad \text{for} \quad j > 1.
\]
Inserting the \(g_{1m}^{(0)}\) expression in \((11)\) and \((4)\) gives
\[
U_{m}^{(0)} = \frac{\alpha}{3} \hat{x}.
\] (15)
The Marangoni stresses immobilize the interface, the surface flow is incompressible, and the drop behaves like a rigid sphere.

At next order, the equation is
\[
0 = [\Omega_{jm}g_{jm2} + A_{jm}g_{jm2}]g_{jm}^{(0)} + W(j)g_{jm}^{(1)} + \Theta_{jm} g_{jm}^{(0)} + \Theta_{jm} g_{jm}^{(0)},
\] (16)
and the \(g_{1m}^{(1)}\) rise to a cross-stream migration velocity
\[
U_{m}^{(1)} = -Ma^{-1} \alpha^{2} \gamma_{0} \frac{3\lambda + 17}{9(\lambda + 4)} \hat{y}.
\] (17)
This migration is directed towards the centerline for all values of the viscosity ratio \(\lambda\). Both the quadratic \(\alpha\) and shear \(\gamma\) components of the imposed flow disturb the surfactant distribution, and interact with the distortions created by the other. In addition, the slip velocity interacts with the shear component. The drop migrates in the direction of decreasing shear magnitude.

Far from the centerline / High viscosity drops: \(\lambda^{-1} \ll 1\) and \(Ma = O(\lambda)\)

If \(\lambda^{-1} \ll 1\), we expand \(g_{jm} = \sum_{k=0}^{\infty} \lambda^{-k} g_{jm}^{(k)}\) and introduce \(\tilde{M}a = \lambda^{-1} Ma\). The leading order equation for the surfactant distribution is the \(\lambda \to \infty\) limit of \([12]\). Only the first convection term, linear Marangoni term, and rigid rotation within the \(\Omega\) term are \(O(1)\). Thus,
\[
0 = \tilde{C}_{jm} + \Omega_{jm}g_{jm}^{(0)} + \tilde{M}a \bar{W}(j)g_{jm}^{(0)},
\] (18)
where \(C(jm) = \lambda^{-1} \tilde{C}(jm)\) and \(W(j) = \lambda^{-1} \tilde{W}(j)\) as \(\lambda \to \infty\). The surfactant dynamics reflect both rotation and Marangoni relaxation balancing convection by the imposed flow. Solving for \(g_{jm}^{(0)}\) yields
\[
g_{1m}^{(0)} = 0; \quad g_{1m}^{(0)} = \frac{2\alpha}{\pm 2\tilde{M}a + 3i\alpha \gamma_{0}} \sqrt{\frac{2\pi}{3}}.
\] (19)
Inserting in \((11)\) and \((4)\) and keeping terms up to order \(\lambda^{-1}\) leads to both slip
\[
U_{m,x} = \frac{\alpha}{3} - \frac{2\alpha}{9\lambda} + \frac{8}{9\lambda} \frac{\alpha \tilde{M}a^{2}}{9\alpha^{2} \gamma_{0}^{2} + 4\tilde{M}a^{2}}; \quad (20)
\]
and cross-stream drift
\[
U_{m,y} = -\frac{4}{3\lambda} \frac{\alpha^{2} \gamma_{0} \tilde{M}a}{9\alpha^{2} \gamma_{0}^{2} + 4\tilde{M}a^{2}}.
\] (21)
As \(\tilde{M}a \to \infty, \lambda \to \infty\), or \(\gamma_{0} \to 0\), rigid sphere behavior is recovered; cross-stream motion vanishes, and the slip is given by the Faxén result \([13]\). While the cross-stream migration velocity is still always directed towards the centerline, it is now a non-monotonic function of the distance therefrom, with maxima at \(\gamma_{0} = \pm \frac{\sqrt{3}}{3} \tilde{M}a\).

In conclusion, we have shown that the presence of small amounts of surfactant can significantly affect drop motions in quadrangular flows. In contrast with a clean, deformable droplet, for which the direction of migration depends on viscosity ratio, a surfactant-covered spherical drop always migrates towards the flow centerline. This result is in agreement with those found for other particles, such as capsules and vesicles, whose interfaces are governed by Marangoni-like stresses. The present analysis may be extended to treat general surfactant redistributions, and the drop trajectory \(dy_{0}/dt = U_{m,y} [y_{0}, \Gamma(y_{0})]\) determined by numerical solution of the surfactant evolution equation.

PV acknowledges partial research support from ACS-PRF grant 48415-G9.

**APPENDIX A: HARMONICS AND VELOCITY FIELDS**

Scalar and vector spherical harmonics are defined as
\[
Y_{jm} = \left[ \frac{2j+1}{4\pi} \frac{(j-m)!}{(j+m)!} \right]^{\frac{1}{2}} (-1)^{m} P_{jm}^{m}(\cos \theta) e^{im\varphi},
\]
\[
y_{jm0} = [j(j+1)]^{\frac{1}{2}} r \nabla_{\Omega} Y_{jm},
\] (A1)
\[
y_{jm1} = -i \hat{r} \times y_{jm0},
\]
\[
y_{jm2} = \hat{r} Y_{jm},
\]
where the \(P_{jm}^{m}\) are the Legendre polynomials, and \(\nabla_{\Omega}\) is the angular part of the gradient operator. The velocity basis functions are
\[
\mathbf{u}_{jm0} = \frac{1}{2} r^{j-2} (2 - j + jr^{-2}) y_{jm0}
\]
\[- \frac{1}{2} r^{j} (j(j+1))^{\frac{1}{2}} (1 - r^{-2}) y_{jm2} \]
\[- \frac{1}{2} r^{j} (j(j+1))^{\frac{1}{2}} (1 - r^{-2}) y_{jm2}, \quad (A2a)
\]
\[
\mathbf{u}_{jm1} = r^{-(j-1)} y_{jm1}, \quad (A2b)
\]
\[
\mathbf{u}_{jm2} = \frac{1}{2} r^{j-2} (2 - j + jr^{-2}) y_{jm0}
\]
\[+ \frac{1}{2} r^{j} (j + (2 - j)r^{-2}) y_{jm2}, \quad (A2c)
\]
\[ u_{jm0} = \frac{1}{2} r^{j-1} \left(- (j + 1) + (j + 3) r^2\right) y_{jm0} \]
\[ - \frac{1}{2} r^{j-1} \left[j (j + 1) + (j + 3) r^2\right] y_{jm2}, \]  
\[ u_{jm1} = r^j y_{jm1}, \]  
\[ u_{jm2} = \frac{1}{2} r^{j-1} \left(3 + j\right) \left(\frac{j+1}{j}\right)^\frac{3}{2} (1 - r^2) y_{jm0} + \frac{1}{2} r^{j-1} \left(j + 3 - (j + 1) r^2\right) y_{jm2}. \]  

**APPENDIX B: IMPOSED FLOW**

The unbounded plane Poiseuille flow (1) seen by a migrating particle is represented as
\[ \mathbf{v}^\infty = c_{jmq}^\infty \mathbf{u}_{jmq}, \]  
with coefficients
\[ \gamma_{3330} = \pm \alpha \sqrt{\frac{3 \pi}{4 \pi}}, \quad \gamma_{3332} = \pm \alpha \sqrt{\frac{3 \pi}{4 \pi}} \]
\[ \gamma_{3310} = \pm \frac{\alpha}{2} \sqrt{\frac{\pi}{4 \pi}}, \quad \gamma_{3312} = \pm \frac{\alpha}{2} \sqrt{\frac{3 \pi}{4 \pi}} \]
\[ \gamma_{2220} = \mp \frac{\alpha}{2} \sqrt{\frac{\pi}{4 \pi}}, \quad \gamma_{2222} = \pm \frac{\alpha}{2} \sqrt{\frac{3 \pi}{4 \pi}} \]
\[ \gamma_{2211} = - \frac{\alpha}{2} \sqrt{\frac{\pi}{4 \pi}}, \quad \gamma_{2212} = \mp \frac{\alpha}{2} \sqrt{\frac{3 \pi}{4 \pi}} \]
\[ \gamma_{1110} = \mp \frac{\alpha}{2} \sqrt{\frac{\pi}{4 \pi}}, \quad \gamma_{1112} = \pm \frac{\alpha}{2} \sqrt{\frac{3 \pi}{4 \pi}} \]
\[ \gamma_{1000} = -2 \alpha \sqrt{\frac{\pi}{4 \pi}}, \quad \gamma_{1002} = -2 \alpha \sqrt{\frac{3 \pi}{4 \pi}} \]

**APPENDIX C: HYDRODYNAMIC TRactions AND STRESS BALANCES**

Neglecting isotropic contributions, the hydrodynamic tractions on a sphere may be expanded in harmonics
\[ \mathbf{t}_{\text{hydr}} = \tau_{jmq} \mathbf{y}_{jmq}, \]  
with coefficients
\[ \tau_{jmq} = \sum_{q} \gamma_{jmq} \left(T_{q}^+ - T_{q}^\mp\right) + \gamma_{jmq} T_{q}^\pm, \]  
\[ \tau_{jmq} = \lambda \sum_{q} \gamma_{jmq} T_{q}^\pm, \]  
where \( T_{q}^+ = T_{q}^\pm \), \( T_{00} = \pm (2j + 1), \) \( T_{02} = \mp 3 \left(\frac{j+1}{2}\right)^\frac{1}{2} \), \( T_{12}^\pm = \pm \left(\frac{j+1}{2}\right)^\frac{1}{2} \), \( T_{22}^\pm = \pm (2j + 1) + \frac{3}{\pm (j+1)^2}, \) and \( T_{01}^0 = T_{12}^0 = 0. \)

The tangential stress boundary conditions are
\[ \tau_{jm0}^\text{out} - \tau_{jm0}^\text{in} = M \alpha \sqrt{j(j+1) g_{jm}}, \]
\[ \tau_{jm1}^\text{out} - \tau_{jm1}^\text{in} = 0, \]  
where a linear equation of state relating surface tension and local surfactant concentration is assumed: \( \sigma(\Gamma) = \sigma(1) - M \alpha (\Gamma - 1) \). The normal stress balance for a spherical drop requires
\[ \tau_{jm2}^\text{out} - \tau_{jm2}^\text{in} = -2 M \alpha g_{jm}. \]  
For \( j > 1 \), we set \( c_{jm2} = 0 \) and do not use the normal stress balance condition.

**APPENDIX D: SURFACTANT EVOLUTION EQUATION**

The surfactant conservation equation on a moving interface is
\[ \frac{\partial \Gamma}{\partial t} + \nabla \cdot (\mathbf{v}_s \Gamma) + \Gamma \mathbf{v} \cdot \nabla \mathbf{n} = 0, \]  
where \( \mathbf{v}_s \) is the velocity component tangential to the surface. In the case of a sphere, the mean curvature \( \nabla \cdot \mathbf{n} = 2 \). Representing all quantities as harmonics and decomposing the velocity field into clean-drop and surfactant contributions yields (12). The clean-drop terms are
\[ C_{jm} = \left[j(j+1)\right]^\frac{3}{2} c_{jm0} - 2 c_{jm2}, \]
\[ \Omega_{jmj_{2}m_{2}} = \left[j(j+1)\right]^\frac{3}{2} c_{jm1} \Gamma_{01}^{01} \]
\[ \Lambda_{jmj_{2}m_{2}} = \left[j(j+1)\right]^\frac{3}{2} c_{jm1} \Gamma_{00}^{00} - 2 c_{jm1} \Gamma_{22}^{22}. \]

Summation over \( j_1 \) and \( m_1 \) is implied, and the \( c_{jmq} \) coefficients refer to equations (8). Using equations (7) to define the vector \( \hat{W}(j) \)
\[ c_{jmq}^s \equiv \hat{W}(j) M \alpha g_{jm}, \]
the surfactant terms are
\[ W(j) = \left[j(j+1)\right]^\frac{3}{2} \hat{W}_0(j) - 2 \hat{W}_2(j), \]
\[ \Theta_{jmj_{1}m_{1}j_{2}m_{2}} = \left[j(j+1)\right]^\frac{3}{2} \hat{W}_0(j) g_{00} - 2 \hat{W}_2(j) C_{22}. \]

The Clebsch-Gordan coupling coefficients \( C \) are
\[ C_{00} = B \left( \begin{array}{ccc} j_1 & j_2 & j \end{array} \right) \]
\[ 0 0 0 \]  
\[ \times \left( \begin{array}{ccc} \hat{w}(j_1+1) + j(j+1) - j_2(j_2+1) \end{array} \right) \]
\[ \frac{(j(j+1)j_1(j_1+1))^{\frac{3}{2}}}{j_1(j_1+1)}, \]
\[ C_{01} = B \left( \begin{array}{ccc} j_1 & j_2 & j \end{array} \right) \]
\[ 0 0 0 \]  
\[ \times \left( \begin{array}{ccc} (s+1)(s-2j_2)(s-2j_1)(s-2j_1+1) \end{array} \right)^\frac{3}{2}, \]
\[ C_{22} = B \left( \begin{array}{ccc} j_1 & j_2 & j \end{array} \right) \]
\[ 0 0 0 \]  
\[ \left( \begin{array}{ccc} j_1 & j_2 & j \end{array} \right) \]
\[ m_1 m_2 -m \]  
where \( s = j + j_1 + j_2 \).

\[ B = \frac{(-1)^m}{2} \left[ \frac{(2j+1)(2j_1+1)(2j_2+1)}{4 \pi} \right] \]
\[ \times \left( \begin{array}{ccc} j_1 & j_2 & j \end{array} \right) \]
\[ m_1 m_2 -m \]  
and \( \left( \begin{array}{ccc} j_1 & j_2 & j \end{array} \right) \) denotes the Wigner 3j-symbol.
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