Device-Independent Tests of Quantum States

Michele Dall’Arno

Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543, Singapore
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We construct a “Quantum Rosetta Stone”, that is, a correspondence between quantum states and the observable input-output correlations they are compatible with. Formally, for any family of states, one needs to provide: i) to the experimenter, all the measurements that generate extremal input-output correlations, and ii) to the theoretician, the full characterization of such correlations. Comparing the correlations observed in i) with those predicted by ii) corresponds to device-independently testing the states. We solve the problem in closed-form for the case of qubit states and tests, and as applications we specify our results to the case of any pair of pure states and to the case of pure states uniformly distributed on the Bloch equatorial plane.

Quantum systems are most generally described by quantum states, abstract vectors in a mathematical space with the quirky property of not being perfectly distinguishable – a property called superposition of pure states. However, all an observer can ultimately observe are just correlations among perfectly distinguishable events in usual space and time. How can the language of the quantum realm be translated into something intelligible by an observer? Here, we construct a “Quantum Rosetta Stone”, that unlocks the quantum language by providing a correspondence between quantum states and observable correlations among space-time events.

The problem is most generally framed as a game involving an experimenter, claiming to be able to prepare m quantum states \{ρ_x\} and to measure them, and a skeptical theoretician who is willing to base their conclusion on observed correlations only. At each run of the experiment, first the experimenter prepares state \(ρ_x\) upon input of \(x\), and then measures measurement \{π_{y|w}\} upon input of \(w\). Finally, the theoretician collects outcome \(y\), thus reconstructing correlation \(\{p_{y|x,w}\}\). The setup is as follows:

\[
p_{y|x,w} := \text{Tr}[ρ_xπ_{y|w}] = xρ_xwπ_{y|w} = y.
\]

Let us denote with \(S_n(ρ_x)\) the set of correlations generated by states \(\{ρ_x\}\) for any \(n\)-outcomes measurement \(\{π_y\}\), that is

\[
S_n(ρ_x) := \left\{ p \mid p_{y|x} = \text{Tr}[ρ_xπ_y] \right\}
\]

(we take \(y \in [0,n−1]\) since for \(y = n−1\) one simply has \(p_{n−1|x} = 1−\sum_{y=0}^{n−2} p_{y|x}\)). On the theoretician’s side, the problem amounts to fully characterize \(S_n(ρ_x)\), for any \(\{ρ_x\}\), in order to check if \(\{p_{y|x,w}\} \in S_n(ρ)\), for any \(w\). On the experimenter’s side, the problem amounts to choosing measurements \(\{π_{y|w}\}\) generating all the extremal correlations of \(S_n(ρ_x)\) (of course, the validity of the conclusion itself will be independent of \(\{π_{y|w}\}\)). Therefore, \(w\) represents a direction to be probed in the space of correlations in order to reconstruct \(S_n(ρ_x)\). Since, as shown later, \(S_n(ρ_x)\) is strictly convex, \(w\) is a continuous parameter.

Here, we provide a full closed-form solution of this problem for qubit states \(\{ρ_x\}\) and tests, that is measurements with \(n = 2\) outcomes. In particular, for any \(\{ρ_x\}\), we explicitly derive: i) the measurements \(\{π_{y|w}\}\) generating a correlation at the boundary of \(S_2(ρ_x)\) for any arbitrarily given direction \(w\); and ii) the full closed-form characterization of \(S_2(ρ_x)\). It turns out that \(S_2(ρ_x)\) is given by the convex hull of the two isolated points 0 and \(u\) (vectors with null and unit entries, respectively) and the 4-dimensional hyper-ellipsoid given by the system:

\[
\begin{align*}
(1−Q−1_{Q−1})(p−\frac{1}{2}u) &= 0, \\
(p−\frac{1}{2}u)^T Q^{-1}(p−\frac{1}{2}u) &\leq 1,
\end{align*}
\]

where \(Q_{x_0,x_1} = \frac{1}{2} \text{Tr}[ρ_{x_0}ρ_{x_1}] − \frac{1}{4} \). This situation is represented in Fig. 1. As applications, we explicitly discuss the case where \(m = 2\) and \(\{ρ_x\}\) are pure states, and the case where \(\{ρ_x\}\) are distributed on the \(m\)-vertices of a regular polygon on the Bloch equatorial plane.
Our results share analogies with previous works on device-independent testing of quantum dimension \cite{1,2} and entropy \cite{3}. Notice however that therein the aim is to test a specific scalar property of states \{\rho_x\} rather than their most general operatorial form, and the set of correlations is probed along an arbitrarily chosen direction rather than being fully reconstructed. Moreover, the present author has recently addressed the very related problems of device-independent tests of quantum channels \cite{4} and measurements \cite{5}.

**Experimental observations.** — We will make use of standard definitions and results in Quantum Information Theory \cite{6}. Any quantum state is represented by a density matrix \(\rho\), that is a unit-trace positive semi-definite operator. Any quantum measurement is represented by a positive-operator valued measure (POVM), that is a collection \(\{\pi_y\}\) of positive semi-definite operators such that \(\sum_y \pi_y = \mathbb{1}\). The conditional probability \(p_{y|x}\) of outcome \(y\) given input state \(\rho_x\) is given by the Born rule, that is \(p_{y|x} = \text{Tr}[\rho_x \pi_y]\).

The experimenter claims to be able to prepare states \(\{\rho_x\}\) and to measure them. Their task is to support such claims by generating all the correlations at the boundary of \(S_n(\rho_x)\). To this aim, for any direction \(w\) in the space of correlations, the experimenter must measure the POVM \(\{\pi_{y|w}\}\) that generates the correlation \(p_{y|x} := \text{Tr}[\rho_x \pi_{y|w}]\) that maximizes \(p^T w\). In this section, we derive any such a POVM for any given \(\{\rho_x\}\) and \(w\).

Formally, \(\{\pi_{y|w}\}\) is given by the solution of the following optimization problem:

\[
W(\rho_x, w) := \max_{\{\pi_y \geq 0\}} \sum_{x=0}^{m-1} \sum_{y=0}^{n-2} \sum_{x,k=0}^{m-1} w_{x,y} \text{Tr}[\rho_x \pi_y].
\]

In the following, we make the restriction \(n = 2\), hence \(p\) and \(w\) are column vectors with \(m\) entries. Therefore, the maximum in Eq. 2 is attained when \(\pi_0\) is the projector on \(\text{Pos}(\sum_x w_x \rho_x)\), where \(\text{Pos}(\cdot)\) denotes the positive part of operator (-), and, in this case one has

\[
W(\rho_x, w) = \text{Tr}\left[\text{Pos}\left(\sum_x w_x \rho_x\right)\right].
\]

Hence, our first result provides a closed-form characterization of the POVM \(\{\pi_{y|w}\}\) achieving the correlation \(p\) at the boundary of \(S_2(\rho_x)\) that maximizes \(p^T w\), for any given family \(\{\rho_x\}\) of states and direction \(w\).

**Proposition 1.** For any family \(\{\rho_x\}\) of states and direction \(w\) in the space of correlations, the POVM \(\{\pi_{y|w}\}\) generating the correlation \(p_{y|x} := \text{Tr}[\rho_x \pi_{y|w}]\) on the boundary of \(S_2(\rho_x)\) that maximizes \(p^T w\) is such that \(\pi_{0|w}\) is the projector on \(\text{Pos}(\sum_x w_x \rho_x)\) and \(\pi_{1|w} = \mathbb{1} - \pi_{0|w}\).

Proposition 1 restricts the set of POVMs \(\{\pi_{y|w}\}\) that need to be measured. Indeed, whenever \(\{\pi_{y|w}\}\) is such that \(\text{rank}\ \pi_0 = 0\) or \(\text{rank}\ \pi_1 = 0\), correlation \(p\) is trivial (i.e. \(p = 0\) or \(p = u\), respectively), thus direction \(w\) does not need to be probed.

**Theoretical predictions.** — The theoretician does not believe any of the claims made by the experimenter about the experimental setup. Their task is to test such claims by comparing the observed correlations with \(S_2(\rho_x)\). To this aim, in this section we provide a full closed-form characterization of \(S_2(\rho_x)\) under the restriction that \(\{\rho_x\}\) are qubit states.

The set \(S_2(\rho_x)\) is recovered by further optimizing \(W(\rho_x, w)\), as given by Eq. 3, over any direction \(w\), that is:

\[
S_2(\rho_x) = \left\{ p \mid p = \max_w (p^T w - W(\rho_x, w)) \leq 0 \right\}. \quad (4)
\]

Upon fixing a computational basis, \(\{\rho_x\}\) can be decomposed in terms of Pauli matrices \(\{\sigma_k\}\) as follows

\[
\rho_x = \frac{1}{2} \mathbb{1} + \sum_{k=1}^{3} S_{x,k} \sigma_k,
\]

where \(S_{x,k} := \frac{1}{2} \text{Tr}[\rho_x \sigma_k]\). Of course, our result will be independent of the choice of computational basis.

It is then a simple computation to find that

\[
W(\rho_x, w) = \max \left(0, \|w\|_1, \frac{1}{2} \|w\|_1 + \|S^T w\|_2\right),
\]

where \(\|\cdot\|_p\) denotes the \(p\)-norm of vector (-). The maximum is achieved by \(0\) and \(\|w\|_1\) if \(\{\pi_{y|w}\}\) is trivial (\(\pi_{0|w} = 0\) and \(\pi_{0|w} = \mathbb{1}\), respectively), and by \(\frac{1}{2} \|w\|_1 + \|S^T w\|_2\) if \(\{\pi_{y|w}\}\) is rank-one projective. If \(\{\pi_{y|w}\}\) is trivial, the optimization problem in Eq. 4 becomes

\[
\begin{cases}
\max_w p^T w \leq 0, & \text{if } \pi_0 = 0, \\
\max_w (p - \frac{1}{2} u)^T w \leq 0, & \text{if } \pi_0 = \mathbb{1}
\end{cases}
\]

which, as expected, are verified if and only if \(p = 0\) and \(p = u\), respectively.

If however \(\{\pi_{y|w}\}\) is rank-one projective, the optimization problem in Eq. 4 becomes

\[
\max_w \left( (p - \frac{1}{2} u)^T w - \|S^T w\|_2 \right) \leq 0. \quad (5)
\]

This optimization problem is formally equal to that in Eq. (5) of Ref. \cite{7}, where the problem of device-independent tests of quantum measurements was addressed. Notice however that the operational interpretation and, accordingly, the mathematical representation of the symbols are different. For example, in Ref. \cite{7} \(p\) represents the probability distribution of the outcomes of a POVM, and thus \(\sum_y p_y = 1\), while here \(p\) represents the vector of probabilities of outcome \(\pi_0\) given states \(\rho_x\), and thus there is no linear constraint on the sum of its.
elements. Analogous differences hold for $u$ ($t$ in Ref. [7]) and $S$. The consequences of these differences on the solution of Eq. (6) will be discussed at the end of this section.

Since Eq. (5) is left invariant by the transformation $w \rightarrow w = [(p - \frac{1}{2}u)^T]^T w^{-1}$ (we recall that $w$ only represents a direction in the space of correlations), without loss of generality one can take $(p - \frac{1}{2}u)^T w = 0, \pm 1$. When $(p - \frac{1}{2}u)^T w = 0, -1$, the inequality in Eq. (5) is of course satisfied, thus let $(p - \frac{1}{2}u)^T w = 1$. Equation (5) becomes

$$\max_{w, (p - \frac{1}{2}u)} \|ST^T w\|_2 \geq 1,$$

that is, a linearly-constrained quadratic-programming problem. It is known [9] that such a problem is solved by

$$\begin{cases}
Q w = -\lambda(p - \frac{1}{2}u), \\
(p - \frac{1}{2}u)^T Q w = 1
\end{cases}$$

where $Q = SS^T$ and $\lambda$ is a Lagrange multiplier. Upon denoting with $(\cdot)^{-1}$ the Moore-Penrose pseudoinverse [10] of matrix $(\cdot)$, it is known [10] that the system in Eq. (7) is solved by

$$\begin{cases}
w = -\lambda Q^{-1}(p - \frac{1}{2}u) + (\mathbb{1} - Q^{-1}Q)v, \\
-\lambda(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) + (p - \frac{1}{2}u)^T (\mathbb{1} - Q^{-1}Q)v = 1.
\end{cases}$$

Notice that the facts that $(\mathbb{1} - Q^{-1}Q)$ is a projector and that $Q^{-1} \geq 0$ imply the following system

$$\begin{cases}
(p - \frac{1}{2}u)^T (\mathbb{1} - Q^{-1}Q)(p - \frac{1}{2}u) \geq 0, \\
(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) \geq 0.
\end{cases}$$

Let us distinguish four cases.

First, consider the case $(p - \frac{1}{2}u)^T (\mathbb{1} - Q^{-1}Q)(p - \frac{1}{2}u) > 0$ and $(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) > 0$. Let $v$ be as before. Then $\lambda$ is undetermined, $w = v$ and thus $\|ST^T w\|_2^2 = 0$.

Second, consider the case $(p - \frac{1}{2}u)^T (\mathbb{1} - Q^{-1}Q)(p - \frac{1}{2}u) > 0$ and $(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) = 0$. Let $v$ be as before. Then $\lambda$ is undetermined, $w = v$ and thus $\|ST^T w\|_2^2 = 0$.

Third, consider the case $(p - \frac{1}{2}u)^T (\mathbb{1} - Q^{-1}Q)(p - \frac{1}{2}u) = 0$ and $(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) > 0$. Let $v$ be as before. Then

$$\lambda = \left[(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u)\right]^{-1},$$

and $w = \lambda Q^{-1}(p - \frac{1}{2}u)$, and thus $\|ST^T w\|_2^2 = \lambda$.

Fourth, consider the case $(p - \frac{1}{2}u)^T (\mathbb{1} - Q^{-1}Q)(p - \frac{1}{2}u) = 0$ and $(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) = 0$. The latter equality implies $Q^{-1}(p - \frac{1}{2}u) = 0$, which together with the former inequality implies $p = \frac{1}{2}u$.

Hence, the solution of the optimization problem in Eq. (5) is given by

$$\begin{cases}
(\mathbb{1} - Q^{-1}Q)(p - \frac{1}{2}u) = 0, \\
(p - \frac{1}{2}u)^T Q^{-1}(p - \frac{1}{2}u) \leq 1.
\end{cases}$$

Finally, by explicit computation it immediately follows that $Q$ is the real symmetric matrix given by $Q_{x_0, x_1} = \frac{1}{2} \text{Tr}[\rho_x \rho_{x_1}] - \frac{1}{4}$; thus as expected the system in Eq. (5) does not depend on the choice of computational basis.

Then, our second main result provides a full closed-form characterization of the set $S_2(\rho_x)$ of correlations compatible with any arbitrary given qubit family $\{\rho_x\}$ of states.

**Proposition 2.** The set $S_2(\rho_x)$ of correlations generated by a given family $\{\rho_x\}$ of qubit states and any test $\{\pi_y\}$ is given by

$$S_2(\rho_x) = \text{conv}\{0, u, \text{Eq. (5)}\},$$

where $Q_{x_0, x_1} = \frac{1}{2} \text{Tr}[\rho_x \rho_{x_1}] - \frac{1}{4}$.

Let us provide a geometrical interpretation of Proposition 2. The system of equalities in Eq. (5) represents rank$(\mathbb{1} - Q^{-1}Q)$ linear constraints, while the inequality represents an $m$-dimensional cylinder with rank$(Q)$-dimensional hyper-ellipsoidal section. Thus, Eq. (5) represents a rank$(Q)$-dimensional hyper-ellipsoid embedded in an $m$-dimensional space. By denoting with $l$ the maximum number of linearly independent states in $\{\rho_x\}$, one has that $\text{rank}(Q) = l - 1$ if $\mathbb{1} \notin \text{span}(\rho_x)$, and rank$(Q) = l$ otherwise.

Notice as a comparison that, while in this case rank$Q$ is between 1 and 4, in the case of the device-independent tests of quantum measurements [7] rank$Q$ is between 2 and 3, due to the constraint on the sum of POVM elements. Furthermore, while in this case $S_2(\rho_x)$ includes the two isolated correlations 0 and u, no isolated correlations are included in the case of device-independent tests of quantum measurements.

**Comparison.** — Finally, we discuss the comparison of the set of correlations observed by the experimenter according to Proposition 1 and the set $S_2(\rho_x)$ predicted by the theoretician according to Proposition 2. Notice first that the inclusion relation $S_2(\rho_x) \supset S_2(\rho_x')$ induces a partial ordering among families of quantum states $\{\rho_x\}$ and $\{\rho_x'\}$, that is $\{\rho_x\} \supset \{\rho_x'\} \Leftrightarrow S_2(\rho_x) \supset S_2(\rho_x')$. Of course, if the experimenter produces some correlation not in $S_2(\rho_x)$, the theoretician must conclude that the prepared states $\{\rho_x'\}$ are such that

$$\{\rho_x'\} \not\supset \{\rho_x\}.$$

However, if the experimenter produces all the extremal correlations of $S_2(\rho_x)$ (as per Proposition 1), the theoretician must conclude that the prepared states $\{\rho_x'\}$ are
such that
\[ \{\rho'_x\} \succeq \{\rho_x\}. \]  
(10)
Since the ordering \(\succeq\) is partial, Eq. (10) is of course strictly stronger than Eq. (9), that is Eq. (10) implies Eq. (9) but the vice-versa is false. Informally, Eq. (10) allows the theorician to lower bound the “ability” to create input-output correlations of the states prepared by the experimenter.

An even stronger result can be achieved when \(m = 2\). In this case Proposition 2 provides for the first time the full closed-form quantum relative Lorenz curve for any pair \(\{\rho_0, \rho_1\}\) of qubit state, as illustrated by Fig. 1. Quantum relative Lorenz curves have been recently introduced by Buscemi and Gour [16] in the context of quantum relative majorization. As a consequence of a result therein, in turn based on a previous result by Alberti and Uhlmann [17], under the additional assumption that the prepared states \(\{\rho_0, \rho'_1\}\) are qubit states, Eq. (10) implies the existence of a quantum channel \(C\), that is a completely-positive trace-preserving linear map, such that
\[ C(\rho'_x) = \rho_x, \quad x = 1, 2. \]  
(11)
Therefore, Eq. (11) means that the states \(\{\rho'_x\}\) prepared by the experimenter are less noisy than the claimed states \(\{\rho_x\}\). However, it is known [18] that this implication fails if the assumption that the prepared states \(\{\rho'_x\}\) are qubit states is relaxed.

**Applications.** — As an application of the case \(m = 2\), we consider any pair of pure states \(\rho_x = |\psi_x\rangle\langle\psi_x|\), that can be written without loss of generality as
\[ |\psi_0\rangle = |0\rangle, \quad |\psi_1\rangle = \cos \frac{\alpha}{2} |0\rangle + \sin \frac{\alpha}{2} |1\rangle. \]
Since \( |\langle\psi_0|\psi_1\rangle|^2 = \cos^2 \frac{\alpha}{2} \), matrix \(Q_{x_0,x_1} := \frac{1}{2} (|\psi_x\rangle\langle\psi_x| - |\psi_0\rangle\langle\psi_0|) \) is given by \(Q = [(1+\cos\alpha) v_+ v_+^\dagger + (1-\cos\alpha) v_- v_-^\dagger]/4\), where \(v_\pm = 1/\sqrt{2} (1, \pm i)^T\). If \(\alpha \neq 0, \pi\), the system in Eq. (8) becomes
\[ \frac{1}{1 + \cos \alpha} (p_0 + p_1 - 1)^2 + \frac{1}{1 - \cos \alpha} (p_0 - p_1)^2 \leq \frac{1}{2}. \]
If \(\alpha = 0\) or \(\alpha = \pi\), that is \(|\psi_0\rangle = |\psi_1\rangle\) or \(|\psi_0\rangle = 0\) respectively, the system in Eq. (8) trivially becomes \(p_0 = p_1\) or \(p_0 = 1 - p_1\), respectively.

As an application of the general case we consider \(m\) pure states \(\rho_x = |\phi_x\rangle\langle\phi_x|\) uniformly distributed in the Bloch equatorial plane, that can be written without loss of generality as
\[ |\phi_x\rangle = \cos \frac{\pi x}{m} |0\rangle + \sin \frac{\pi x}{m} |1\rangle. \]
Since \( |\langle\phi_x|\phi_x\rangle|^2 = \cos^2 \frac{\pi (x_0-x_1)}{m} \), matrix \(Q_{x_0,x_1} := \frac{1}{2} (|\phi_x\rangle\langle\phi_x| - |\phi_0\rangle\langle\phi_0|) \) is circulant, that is \(Q_{x_0+k,x_1+k} = Q_{x_0,x_1}\) for any \(x_0, x_1, k\). Therefore, it is lengthy but not difficult to show that its eigenvalues are given by
\[ \lambda_j = \frac{1}{4} \sum_{k=0}^{m-1} \cos \frac{2\pi k}{m} \exp \frac{2\pi ij(m-k)}{m} = \left( e^{\frac{2\pi ij}{m}} - 1 \right) \left( e^{\frac{2\pi (i+j)}{m}} - 2 \right) \left( e^{\frac{2\pi (i+j)}{m}} - 1 \right). \]
Hence, one has that \(\lambda_1 = \lambda_{m-1} = m/8\) and \(\lambda_j = 0\) otherwise, and two eigenvectors \(v\) corresponding to non-null eigenvalues are given by \(v_\pm = \frac{1}{\sqrt{m}} \exp(\pm 2\pi i k/m)\). Accordingly, one has that \(Q = \frac{m}{8} (v_+ v_+^\dagger + v_- v_-^\dagger)\), and the system in Eq. (8) becomes
\[ \left\{ (1 - v_+ v_+^\dagger - v_- v_-^\dagger)p = 0, \right\} \|p\|^2 \leq \frac{16}{m}. \]
For instance, consider the case of two mutually unbiased bases [11] (MUBs), obtained for \(m = 4\). MUBs have applications e.g. in classical communications over quantum channels [12], quantum cryptography [13], and locking of classical information in quantum states [14]. One has that \(v_\pm = (1, \pm i, -1, \mp i)^T\), from which the system in Eq. (8) becomes
\[ \left\{ p_0 + p_2 = p_1 + p_3 = 1, \right\} \|p\|^2 \leq \frac{1}{2}. \]

**Conclusion.** — In this work we have constructed a “Quantum Rosetta Stone”, that is, a correspondence between any given family \(\{\rho_x\}\) of \(m\) quantum states and the set \(S_m(\rho_x)\) of observables correlations they can generate for any POVM \(\{\pi_y\}\). Formally, for any \(\{\rho_x\}\), one needs to provide: i) to the experimenter, the measurement \(\{\pi_y|w\}\) that generates a correlation on the boundary of \(S_m(\rho_x)\) for \(any\) given direction \(w\), and ii) to the theorician, the full characterization of \(S_m(\rho_x)\). Comparing the correlations observed in i) with those predicted by ii) corresponds to device-independently testing the states. We have solved the problem in closed-form for the case of qubit states and tests, that is measurements with \(n = 2\) outcomes, and discussed the geometrical interpretation of our results. As applications, we have specified our results to the case of any pair of pure states and to the case of pure states uniformly distributed on the Bloch equatorial plane.

Natural open problems include relaxing some of the restrictions we considered, e.g. considering POVMs with arbitrary number of outcomes and states in arbitrary dimension. Furthermore, the characterization of the set \(S_m(\rho_x)\) of correlations compatible with an arbitrary dimensional family \(\{\rho_x\}\) of \(m = 2\) states might prove to be the key to solve a well-known longstanding conjecture by Shor [15], based on numerical work by Fuchs and
Peres: whether the accessible information of any binary ensemble is attained by a Von Neumann POVM. Finally, the full closed-form characterization of the quantum relative Lorenz curve for qubit states provided by Proposition 2 naturally leads to applications in quantum resource theories [19], within the general framework provided by the quantum Blackwell theorem [20].

We conclude by noticing that our results are remarkably suitable for experimental implementation. For any family of qubit states that an experimenter claims to be able to prepare, our framework only requires Von Neumann measurements to be performed in order to experimentally reconstruct the entire boundary of the set of compatible correlations.

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* cqtmda@nus.edu.sg

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