EXPECTATION OF THE LARGEST BETTING SIZE IN LABOUCHÈRE SYSTEM

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For Labouchère system with winning probability \( p \) at each coup, we prove that the expectation of the largest betting size under any initial list is finite if \( p > \frac{1}{2} \), and is infinite if \( p \leq \frac{1}{2} \), solving the open conjecture in [GS01]. The same result holds for a general family of betting systems, and the proof builds upon a recursive representation of the optimal betting system in the larger family.

1. Introduction. The Labouchère system, also called the cancellation system, is one of the most well-known betting systems used in roulette. It was popularized by Henry Du Pré Labouchère, an English politician, writer and journalist. Before the betting, the bettor chooses an initial list \( L_0 \) of positive real numbers (e.g., \( L_0 = (1, 2, 3, 4) \)). During each betting, the betting size equals the sum of the first and last numbers on the list (if only one number remains on the list, then the betting size equals that number). After a win, the first and last terms are canceled from the list; after a loss, the amount just lost is appended to the last term of the list. This system is continued until the list is empty.

We introduce the following notations. Let \( p \in [0, 1] \) be the winning probability at each coup, where we assume that the outcomes at different coups are independent. Let \( L_n \) be the list after the \( n \)-th coup, \( l_n \) be the corresponding list length, \( B_n \) be the betting size at \( n \)-th coup, \( T_n \) be the remaining target profit (i.e., the sum of the numbers in the list) after \( n \)-th coup, and \( N \) be the stopping time that the list first becomes empty, i.e., \( L_N = \emptyset \). In this paper, we investigate the behavior of the largest betting size \( B^* \triangleq \max_{1 \leq n \leq N} B_n \) in the Labouchère system, and in particular, whether or not \( B^* \) has a finite expectation.

There is very limited literature on analyzing the Labouchère system. By the standard theory of asymmetric random walk, it is straightforward to show that \( E[N] < \infty \) if and only if \( p > \frac{1}{3} \). Downton [Dow80] found a formula for the distribution of the stopping time \( N \) in the case that the initial list \( L_0 \) is \( (1, 2, 3, 4) \), and Ethier [E+08] generalized this result to arbitrary initial list using a generalized version of the ballot theorem [Ber87, Bar87]. Specifically, the stopping time \( N \) has finite \( k \)-th moment for any \( k \) if and only if \( p > \frac{1}{3} \). However, Grimmett and Stirzaker [GS01] showed that both the largest target
\[ T^* \triangleq \max_{1 \leq n \leq N} T_n \] and the total amount of betting sizes \( \sum_{n=1}^{N} B_n \) have an infinite expectation if \( p = \frac{1}{2} \). They also raised the conjecture if the largest betting size \( B^* \) also has an infinite expectation if \( \frac{1}{3} < p \leq \frac{1}{2} \), which is the main focus of this paper.

There is also another betting system which is similar to the Labouchère system, i.e., the Fibonacci system. Instead of considering the first and last numbers in the list at each coup, the last two numbers are added or canceled in the Fibonacci system. Ethier [Eth10] showed that \( \mathbb{E}B^* = +\infty \) in Fibonacci system if and only if \( p \leq \frac{1}{2} \). However, the proof heavily relies on the fact that any list in a Fibonacci system is uniquely determined by its length, which does not hold for the Labouchère system where the list evolves in a more complicated “history dependent” manner.

2. Main Results. To study the Labouchère system, we first introduce a larger family of betting systems called list systems:

**Definition 1 (List System).** A list system consists of a target sequence \( \{T_n\} \), a bet sequence \( \{B_n\} \) and a length sequence \( \{l_n\} \), which evolve as follows:

1. At the beginning, \( T_0 > 0 \) and \( l_0 \in \{1, 2, \cdots\} \);
2. At \( n \)-th coup, the system makes a betting size \( B_n \in [0, T_{n-1}] \) which may depend on the entire history. Then the target and length sequences evolve as
   \[ T_n = \begin{cases} T_{n-1} - B_n & \text{if wins} \\ T_{n-1} + B_n & \text{if loses} \end{cases}, \quad l_n = \begin{cases} (l_{n-1} - 2)_+ & \text{if wins} \\ l_{n-1} + 1 & \text{if loses} \end{cases}. \]
3. Termination condition: if \( l_{n-1} \in \{1, 2\} \), we must have \( B_n = T_{n-1} \). Consequently, when \( l_m = 0 \), then \( T_n = B_n = l_n = 0 \) for any \( n \geq m \).

In such a list system, target \( T_n \) represents the remaining amount of money one would like to earn at the end of \( n \)-th coup; consequently, \( T_n \) shrinks after a win, and increases after a loss. Length \( l_n \) represents the length of the “list” at round \( n \), where it may be some real/virtual list which governs the betting process. The termination condition ensures that, as long as the list length \( l_n \) hits zero, the target must be fulfilled as well (i.e., \( T_n = 0 \)), and the betting process terminates. Clearly, both Labouchère system and Fibonacci system fall into the category of general list systems, and the specific numbers \( \{-2, 1\} \) in the length update can also be changed into different numbers to accommodate other betting systems.

Our first result characterizes the behavior of the largest betting size \( B^* \) under general list systems:
Theorem 1. For any list system, the following holds:

1. If $p > \frac{1}{2}$, we have $\mathbb{E}[B^*] < \infty$;
2. If $\frac{1}{3} < p < \frac{1}{2}$, we have $\mathbb{E}[B^*] = \infty$;
3. If $p \leq \frac{1}{3}$ and $B_n \geq c_1 l_{n-1} + c_2$ for some constants $c_1 > 0, c_2 \in \mathbb{R}$ almost surely, we have $\mathbb{E}[B^*] = \infty$.

Theorem 1 shows that for any list systems, the expectation $\mathbb{E}[B^*]$ of the largest betting size $B^*$ has a phase transition at $p = \frac{1}{2}$: the expectation is finite if the player is favored, and is infinite if the house takes the advantage. Consequently, we have the following corollary:

Corollary 1. For the Labouchère system with any initial list, we have $\mathbb{E}[B^*] < \infty$ if $p > \frac{1}{2}$ and $\mathbb{E}[B^*] = \infty$ if $p < \frac{1}{2}$.

The fair-game case $p = \frac{1}{2}$ requires more delicate analysis, and is summarized in the following theorem:

Theorem 2. Let $(\overline{b}_l)_{l=1}^\infty, (\underline{b}_l)_{l=1}^\infty$ be two sequences taking value in $[0, 1]$. Suppose that a list system satisfies $T_{n-1} \overline{b}_l B_{n-1} \leq B_n \leq T_{n-1} \underline{b}_l B_{n-1}$ for any $n$, and one of the following conditions holds:

1. $\lim_{l \to \infty} \overline{b}_l = 0$;
2. $\inf_l \overline{b}_l > 0$,

we have $\mathbb{E}[B^*] = \infty$ under $p = \frac{1}{2}$.

Note that $B_n/T_{n-1}$ is the betting proportion at $n$-th coup, and general list systems correspond to the case where $\overline{b}_l = 1, \underline{b}_l = 0$ for any $l$. Theorem 2 shows that, if the betting proportion either vanishes or is lower bounded from below as the list length $l$ grows, the largest betting size still has an infinite expectation in a fair game. The following corollary follows from Theorem 2:

Corollary 2. For the Labouchère system with any initial list, $\mathbb{E}[B^*] = \infty$ if $p = \frac{1}{2}$.

Combining Corollaries 1 and 2, we conclude that for the Labouchère system, $\mathbb{E}[B^*] = \infty$ if and only if $p \leq \frac{1}{2}$, solving the open conjecture in [GS01]. We remark that it also follows directly from Theorems 1 and 2 that for the Fibonacci system, $\mathbb{E}[B^*] = \infty$ if and only if $p \leq \frac{1}{2}$, recovering the result in [Eth10].

Based on Theorem 2, a natural question would be that whether $\mathbb{E}[B^*] = \infty$ general holds in any list systems. We have the following partial result:
Theorem 3. For any list system and $\epsilon > 0$, under $p > \frac{1}{2}$ we have

$$E\left[B^*(\log B^*)^{-(1+\epsilon)}\right] < \infty, \quad E[B^* \log B^*] = \infty.$$ 

Theorem 3 shows that, the moment $E[(B^*)^{\alpha}]$ always has a phase transition at $\alpha = 1$ in a fair game. However, the exact answer for $\alpha = 1$ is still unknown, and we leave it as a conjecture:

Conjecture 1. For any list systems, under $p = \frac{1}{2}$ we have $E[B^*] = \infty$.

3. Proof of Theorems 1 and 3. In this section, we first prove Theorem 3, and then applies Theorem 3 to proving Theorem 1.

3.1. Proof of Theorem 3. We make use of the asymptotic tail behavior of the stopping time $N$.

Lemma 1. [E+ 08] For $p > \frac{1}{3}$, we have

$$P_{l_0}(N \geq n + 1) \sim D_{l_0}(n)n^{-\frac{2}{3}} \rho^{\frac{n}{3}}$$

where $l_0$ is the length of the initial list, $D_{l_0}(n)$ is a constant only depending on $l_0$ and $n \pmod{3}$, and $\rho \triangleq \frac{27}{4}(1 - p)^2 < 1$.

Based on Lemma 1, we are about to prove Theorem 3. We first show that $E\left[B^*(\log B^*)^{-(1+\epsilon)}\right] < \infty$. Under $p = \frac{1}{2}$, the target sequence $\{T_n\}$ is a martingale, with $E[T_n] = T_0$. By Doob’s maximal inequality, for any $\lambda > 0$,

$$P\left(\max_{0 \leq m \leq n} T_m \geq \lambda\right) \leq \frac{E[T_n]}{\lambda} = \frac{T_0}{\lambda}.$$ 

Note that $B_n \leq T_{n-1}$, for $\lambda \geq 2$ we therefore have

$$P\left(\max_{1 \leq m \leq n} B_m (\log B_m)^{-(1+\epsilon)} \geq \lambda\right) = P\left(\max_{1 \leq m \leq n} B_m \geq C\lambda (\log \lambda)^{1+\epsilon}\right)$$

$$\leq P\left(\max_{0 \leq m \leq n} T_m \geq C\lambda (\log \lambda)^{1+\epsilon}\right)$$

$$\leq \frac{T_0}{C\lambda (\log \lambda)^{1+\epsilon}}$$

where $C > 0$ is some universal constant. As a result,

$$E\left[\max_{1 \leq m \leq n} B_m (\log B_m)^{-(1+\epsilon)}\right] = \int_0^\infty P\left(\max_{1 \leq m \leq n} B_m (\log B_m)^{-(1+\epsilon)} \geq \lambda\right) d\lambda$$

$$\leq \int_0^\infty \min\left\{1, \frac{T_0}{C\lambda (\log \lambda)^{1+\epsilon}}\right\} d\lambda < \infty$$
where in the last step we have used that
\[ \int_2^\infty \frac{dx}{x(\log x)^{1+\epsilon}} < \infty. \]
Choosing \( n \to \infty \), by monotone convergence we arrive at \( \mathbb{E} \left[ B^*(\log B^*)^{-(1+\epsilon)} \right] < \infty \).

Now we show that \( \mathbb{E}[B^* \log B^*] = \infty \). We recall the following Fenchel–Young inequality:
\[
xy \leq \psi(x) + \psi^*(y)
\]
where \( \psi(\cdot) \) is convex, and \( \psi^*(y) = \sup_x (xy - \psi(x)) \) is the Fenchel–Legendre dual of \( \psi \). For \( \psi(x) = e^{cx} \) with \( c > 0 \), we have
\[
\psi^*(y) = \sup_{x \in \mathbb{R}} (xy - e^{cx}) = \frac{y}{c} \left( \log \frac{y}{c} - 1 \right),
\]
and therefore
\[
\mathbb{E}[NB^*] \leq \mathbb{E}[\psi(N)] + \mathbb{E}[\psi^*(B^*)] = \mathbb{E}[e^{cN}] + \frac{1}{c} \mathbb{E} \left[ B^* \left( \log \frac{B^*}{c} - 1 \right) \right].
\]
By Lemma 1, for \( c > 0 \) sufficiently small we have \( \mathbb{E}[e^{cN}] < \infty \). Moreover, [GS01] shows that
\[
\mathbb{E}[NB^*] \geq \mathbb{E} \left[ \sum_{n=1}^{N} B_n \right] = \infty.
\]
A combination of the previous two inequalities yields \( \mathbb{E}[B^* \log B^*] = \infty \).

3.2. Proof of Theorem 1 and Corollary 1. Now we prove Theorem 1 using Theorem 3 and change of measure.

Fix any \( p > \frac{1}{2} \), let \( P \) be the probability measure over the betting process under winning probability \( p \), and \( Q \) be the counterpart under winning probability \( \frac{1}{2} \). Note that for any sample path \( \omega \) with stopping time \( N = n \), there must be \( \frac{n}{3} + c \) wins and \( \frac{2n}{3} - c \) losses, where \( c \) is a constant depending only on the initial length \( l_0 \). As a result, the likelihood ratio is
\[
\frac{dP}{dQ}(\omega) = \frac{p^{\frac{n}{3}+c}(1-p)^{\frac{2n}{3}-c}}{2^{-n}} = \left( \frac{p}{1-p} \right)^c \cdot \left( \frac{p(1-p)^2}{\frac{1}{2}(1-\frac{1}{2})^2} \right)^{\frac{n}{3}} \leq C \rho^n.
\]
where \( C > 0, \rho \in (0, 1) \) are numerical constants, and we have used that the function \( p \mapsto (1-p)^2 \) is strictly decreasing in \( p \in [\frac{1}{3}, 1] \). As a result,

\[
\mathbb{E}_P[B^*] = \mathbb{E}_Q \left[ B^* \cdot \frac{dP}{dQ} \right] \leq C \cdot \mathbb{E}_Q[\rho N B^*].
\]

Since \( T_n \leq T_{n-1} + B_n \leq 2T_{n-1} \) in any list system, \( B^* \leq \max_{0 \leq n \leq N} T_n \leq T_0 \cdot 2^N \), and therefore

\[
\mathbb{E}_Q[\rho^N B^*] \leq T_0^\epsilon \cdot \mathbb{E}_Q[(\rho 2^\epsilon)^N \cdot (B^*)^{1-\epsilon}],
\]

for any \( \epsilon > 0 \). Choosing \( \epsilon > 0 \) small enough such that \( \rho 2^\epsilon < 1 \), by Theorem 3 we conclude that \( \mathbb{E}_P[B^*] < \infty \).

For \( p \in (\frac{1}{3}, \frac{1}{2}) \), we use the same argument to obtain \( \frac{dP}{dQ} \geq C \rho^N \) for some \( \rho > 1 \). Then

\[
\mathbb{E}_P[B^*] \geq C \cdot \mathbb{E}_Q[\rho^N B^*] \geq C T_0^{-\epsilon} \cdot \mathbb{E}_Q[(\rho 2^{-\epsilon})^N (B^*)^{1+\epsilon}],
\]

and by choosing \( \epsilon > 0 \) small enough, Theorem 3 yields \( \mathbb{E}_P[B^*] = \infty \).

Finally, for \( p \leq \frac{1}{3} \), we have \( \mathbb{E}[\max_{0 \leq n \leq N} l_n] = \infty \) by the theory of asymmetric random walk. Hence, by assumption we have

\[
\mathbb{E}[B^*] \geq c_1 \mathbb{E}[\max_{0 \leq n \leq N} l_n] + c_2 = \infty
\]

as desired. The proof of Theorem 1 is completed.

As for Corollary 1, it suffices to verify that the condition \( B_n \geq c_1 l_{n-1} + c_2 \) holds for the Labouchère system. Let \( a > 0 \) be the minimum number in the initial list \( L_0 \), a simple induction on \( n \) yields that \( B_n \geq a(l_{n-1} - l_0)_+ \), which shows that the condition is fulfilled with \( c_1 = a > 0, c_2 = -a l_0 \).

4. Proof of Theorem 2 and Corollary 2. In this section, we first use a recursive representation of the optimal list system to prove Theorem 2. Then we investigate the specific properties of the Labouchère system and show that the condition in Theorem 2 holds, thereby proving Corollary 2.

4.1. Proof of Theorem 2. If \( \inf \{l_i \} \geq c > 0 \), we have \( B^* \geq c \max_{0 \leq n \leq N} T_n \), which has an infinite expectation [GS01]. Now we assume that \( \lim_{l \to \infty} b_l = 0 \) and prove Theorem 2 by contradiction. We first introduce the following definition:

**Definition 2.** For any \( x > 0 \) and \( l \in \{1, 2, \cdots\} \), we define \( f(x, l) \) to be the infimum of \( \mathbb{E}[B^*] \) over all possible list systems with initial target \( x \) and initial length \( l \), such that \( B_n \leq T_{l_{n-1}} - T_{n-1} \) for any \( n \).
Definition 2 considers an optimal list system with initial target $x$ and initial length $l$, where optimality is measured in terms of a smallest expectation of the largest betting size $B^*$. The quantity $f(x, l) \in \mathbb{R}_+ \cup \{+\infty\}$ is the corresponding expectation, and it is well-defined even if the optimal list system does not exist. The next lemma presents recursive relations between $f(x, l)$ with different $l$.

**Lemma 2.** There exists some sequence $\{a_l\}$ taking value in $\mathbb{R}_+ \cup \{+\infty\}$ such that $f(x, l) = xa_l$ for any $x > 0$. Moreover, the sequence $\{a_l\}$ satisfies the following inequalities:

$$a_l \geq \min_{b \in [0, b_l]} \frac{\max\{b, (1-b)a_{l-2}\} + \max\{b, (1+b)a_{l+1}\}}{2}, \quad l \geq 3$$

$$a_1 \geq a_2 + \frac{1}{2} \geq a_3 + 1.$$

**Proof.** When the initial target $x$ is scaled by $\lambda > 0$, we may always scale all betting sizes by $\lambda$ to arrive at a new list system with the initial target $\lambda x$, and vice versa. Hence, $f(x, l)$ is proportional to $x$, and $f(x, l) = xa_l$.

For $l \geq 3$ and any list system, let $b \in [0, b_l]$ be any betting size at the first coup with initial target $T_0 = 1$ and initial length $l$. Let $B_1^*, B_2^*$ be the largest betting sizes (excluding the first bet) after winning/losing the first coup, respectively. Then by definition of $f(x, l)$, we have

$$\mathbb{E}B_1^* \geq f(1-b, l-2) = (1-b)a_{l-2},$$

$$\mathbb{E}B_2^* \geq f(1+b, l+1) = (1+b)a_{l+1}.$$ 

Note that $B^*$ is either $\max\{b, B_1^*\}$ or $\max\{b, B_2^*\}$, we have

$$\mathbb{E}[B^*] = \frac{\mathbb{E}\max\{b, B_1^*\} + \mathbb{E}\max\{b, B_2^*\}}{2} \geq \frac{\max\{b, \mathbb{E}B_1^*\} + \max\{b, \mathbb{E}B_2^*\}}{2} \geq \frac{\max\{b, (1-b)a_{l-2}\} + \max\{b, (1+b)a_{l+1}\}}{2}$$

where the first inequality is due to the convexity of $x \mapsto \max\{b, x\}$. Note that this inequality holds for any list systems, taking infimum over the LHS gives the desired inequality for $l \geq 3$. The other inequalities for $l \leq 2$ can be established analogously. 

Based on Lemma 2, we may investigate more properties of $a_l$. If $a_1 = \infty$, it is obvious that $a_l = \infty$ for any $l \in \mathbb{N}$ (since any initial list may evolve into
length one with non-zero probability), and Theorem 2 holds. Next we show that \( a_1 < \infty \) is impossible. Assume by contradiction that \( a_1 < \infty \), we will have the following lemma.

**Lemma 3.** If \( a_1 < \infty \), the sequence \( \{a_l\} \) will be strictly decreasing, i.e., \( a_1 > a_2 > a_3 > \cdots \).

**Proof.** For \( l \geq 3 \), by Lemma 2 we have

\[
a_l \geq \min_{b \in [0, \frac{5}{2}]} \frac{(1 - b)a_{l-2} + (1 + b)a_{l+1}}{2} \geq \min_{b \in [0, 1]} \frac{(1 - b)a_{l-2} + (1 + b)a_{l+1}}{2} = \min \left\{ \frac{a_{l-2} + a_{l+1}}{2}, a_{l+1} \right\},
\]

where in the last step we have used the fact that an affine function attains its minimum at the boundary. Consequently, if we already know that \( a_1 \geq a_2 \geq \cdots \geq a_l \), we must also have \( a_l \geq a_{l+1} \). Hence, by induction on \( l \), the sequence \( \{a_l\} \) is decreasing.

To show strict decreasing property, by Lemma 2 again we have

\[
a_l \geq \min_{b \in [0, \frac{5}{2}]} \frac{\max\{b, (1 - b)a_{l-2}\} + (1 + b)a_{l+1}}{2} \geq \min_{b \in [0, 1]} \frac{\max\{b, (1 - b)a_{l-2}\} + (1 + b)a_{l+1}}{2} = \frac{1}{2} \min_{b \in [0, 1]} \max\{b + (1 + b)a_{l+1}, (1 - b)a_{l-2} + (1 + b)a_{l+1}\}.
\]

For real numbers \( r_1, r_2, s_1, s_2 \) with \( r_1 > 0 \geq r_2, s_1 \leq s_2, r_1 + s_1 \geq r_2 + s_2 \), straightforward computation yields

\[
\min_{x \in [0, 1]} \max\{r_1 x + s_1, r_2 x + s_2\} = \frac{r_1 s_2 - r_2 s_1}{r_1 - r_2}.
\]

Hence,

\[
a_l \geq \frac{2a_{l-2}a_{l+1} + a_{l-2} + a_{l+1}}{2(a_{l-2} + 1)} = a_{l+1} + \frac{a_{l-2} - a_{l+1}}{2(a_{l-2} + 1)}.
\]

If we have \( a_l = a_{l+1} \), we will also have \( a_{l-2} = a_{l+1} \) based on the previous inequality. Due to the decreasing property of \( \{a_l\} \), \( a_{l-1} = a_l \) also holds, and repeating this process yields \( a_2 = a_3 \), a contradiction to Lemma 2. Hence \( a_l > a_{l+1} \) for any \( l \). \( \Box \)
Based on Lemmas 2 and 3, we are about to arrive at the desired contradiction. Fix any $\epsilon > 0$ such that $
exists \frac{1}{1+\epsilon} + \left(\frac{1}{1+\epsilon}\right)^2 > 1.$ Since $\lim_{l \to \infty} \frac{1}{b_l} = 0,$ we take $l_0 > 0$ large enough such that $\frac{1}{b_l} < \epsilon$ for any $l > l_0.$ Then for $l > l_0,$ Lemma 2 yields

$$a_l \geq \min_{b \in [a,b_1]} \frac{(1 - b)a_{l-2} + (1 + b)a_{l+1}}{2}$$

$$\geq \min_{b \in [a,\epsilon]} \frac{(1 - b)a_{l-2} + (1 + b)a_{l+1}}{2}$$

$$= \min_{b \in [0,\epsilon]} \frac{(a_{l+1} - a_{l-2})b + a_{l+1} + a_{l-2}}{2}$$

$$= \frac{(a_{l+1} - a_{l-2})\epsilon + a_{l+1} + a_{l-2}}{2}$$

where in the last step we have used $a_{l+1} \leq a_{l-2}$ by Lemma 3. A rearrangement of the previous inequality gives

$$a_l - a_{l+1} \geq \frac{1 - \epsilon}{1 + \epsilon} \cdot (a_{l-2} - a_l)$$

for any $l > l_0.$ Similarly,

$$a_{l+1} - a_{l+2} \geq \frac{1 - \epsilon}{1 + \epsilon} \cdot (a_{l-1} - a_{l+1})$$

$$\geq \frac{1 - \epsilon}{1 + \epsilon} \cdot (a_l - a_{l+1})$$

$$\geq \left(\frac{1 - \epsilon}{1 + \epsilon}\right)^2 \cdot (a_{l-2} - a_l).$$

Adding them together yields

$$a_l - a_{l+2} \geq \left[\frac{1 - \epsilon}{1 + \epsilon} + \left(\frac{1 - \epsilon}{1 + \epsilon}\right)^2\right] \cdot (a_{l-2} - a_l) = \rho(a_{l-2} - a_l).$$

Our choice of $\epsilon$ implies $\rho > 1,$ and therefore $a_{l+2k-2} - a_{l+2k} \geq \rho^k(a_{l-2} - a_l)$ for any $k \in \mathbb{N}$ and $l > l_0.$ Since $a_{l+2k-2} - a_{l+2k} \leq a_1,$ and $a_{l-2} > a_l$ by Lemma 3, this inequality implies that

$$a_1 \geq \rho^k(a_{l-2} - a_l)$$

for any $k = 1, 2, \ldots,$ a contradiction to the assumption $a_1 < \infty.$ The proof of Theorem 2 is complete.
4.2. Proof of Corollary 2. First we observe that it suffices to prove the case where the initial list consists of a single positive number. This observation is due to that there is a positive probability to reduce the list length to \( l_n = 1 \) after finitely many coups for any initial list \( L_0 \).

To study the combinatorial properties of the Labouchère system, we introduce the following definition:

**Definition 3.** A list of positive real numbers \((a_1, a_2, a_3, \ldots, a_n)\) is called good if it satisfies the following conditions:

- Every element in the list is positive, i.e., \( a_i > 0 \) for any \( i \);
- The list is non-decreasing, i.e., \( a_1 \leq a_2 \leq \cdots \leq a_n \);
- The difference of the list is non-decreasing with difference at most \( a_1 \), i.e., \( a_2 - a_1 \leq a_3 - a_2 \leq \cdots \leq a_n - a_{n-1} \leq a_1 \).

The key properties of a good list are summarized in the following lemmas.

**Lemma 4.** If the initial list \( L_0 \) is good, the list \( L_n \) after \( n \)-th coup is also good for any \( n \).

**Proof.** It suffices to prove that, if \( L_{n-1} = (a_1, \cdots, a_l) \) is a good list, so is \( L_n \). Based on the outcome at \( n \)-th coup, there are only two possibilities:

- \( L_n = (a_1, a_2, \cdots, a_l, a_1 + a_l) \), or
- \( L_n = (a_2, a_3, \cdots, a_l-1) \).

In either case, one can check from Definition 3 directly that \( L_n \) is a good list, as desired. \( \Box \)

**Lemma 5.** If the list \( L_{n-1} \) is good and has length \( l \geq 2 \), in Labouchère system we have

\[
\frac{B_n}{T_{n-1}} \leq \sqrt{\frac{2}{l} + \frac{2}{l^2}}.
\]

**Proof.** Let \( L_{n-1} = (a_1, \cdots, a_l) \). By definition, for any \( k \leq l \) we have

\[ a_l \leq a_{l-1} + a_1 \leq a_{l-2} + 2a_1 \leq \cdots \leq a_{l-k} + ka_1. \]

As a result,

\[ a_l \leq \frac{1}{k} \sum_{j=1}^{k} (a_{l-j} + ja_1) \leq \frac{1}{k} \sum_{j=1}^{l} a_j + \frac{k+1}{2} \cdot a_1. \]
Note that the current betting size is $B_n = a_1 + a_l$, and the current target is $T_{n-1} = \sum_{j=1}^{l} a_j$. Hence, for any $k \leq l$ we have

$$
\frac{B_n}{T_{n-1}} = \frac{a_1 + a_l}{\sum_{j=1}^{l} a_j} \leq \frac{(k + 3)a_1}{2 \sum_{j=1}^{l} a_j} + \frac{1}{k} \leq \frac{k + 3}{2l} + \frac{1}{k}.
$$

Setting $k = \lceil \sqrt{2l} \rceil \leq l$ arrives at

$$
\frac{B_n}{T_{n-1}} \leq \frac{\lceil \sqrt{2l} \rceil + 3}{2l} + \frac{1}{\lceil \sqrt{2l} \rceil} \leq \frac{\sqrt{2l} + 4}{2l} + \frac{1}{\sqrt{2l}} = \sqrt{\frac{2}{l} + \frac{2}{l}},
$$

as claimed. \hfill \Box

Note that the initial list $L_0$ consisting of a single positive number is good, by Lemma 4 we know that all future lists $L_n$ are also good. Moreover, by setting

$$
\bar{b}_l = \min \left\{ \sqrt{\frac{2}{l} + \frac{2}{l}}, 1 \right\},
$$

by Lemma 5 we know that $B_n \leq \bar{b}_{n-1} T_{n-1}$ always holds. Note that $\lim_{l \to \infty} \bar{b}_l = 0$, Theorem 2 yields $\mathbb{E}[B^*] = \infty$ in Labouchère system, as desired.

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