Prescribed-time regulation of nonlinear uncertain systems with unknown input gain and appended dynamics

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Abstract
The prescribed-time stabilization problem for a general class of uncertain nonlinear systems with unknown input gain and appended dynamics (with unmeasured state) is addressed. Unlike the asymptotic stabilization problem, the prescribed-time stabilization objective requires convergence of the state vector to the origin in a finite time interval that can be arbitrarily picked (i.e., “prescribed”) by the control system designer irrespective of the system’s initial condition. The class of systems considered is allowed to have general nonlinear uncertain terms throughout the system dynamics as well as uncertain appended dynamics (that effectively generate a time-varying non-vanishing disturbance signal input into the nominal system). The control design is based on a nonlinear transformation of the time scale, dynamic high-gain scaling, adaptation dynamics with temporal forcing terms, and a composite control law that includes two components. The first component in the composite control law is analogous to prior dynamic high-gain scaling-based control designs, but with a time-dependent function in place of the unknown input gain, while the second component has a non-smooth form with time-dependent terms that ensure prescribed-time convergence in spite of the unknown input gain and the disturbances. The efficacy of the proposed control design is illustrated through numerical simulation studies on two example systems (a “synthetic” fifth-order system and a “real-world” electromechanical system).

KEYWORDS
adaptive control, high-gain control, nonlinear control systems, prescribed-time stabilization, uncertain systems

1 | INTRODUCTION

While the stabilization/regulation objective typically considered in control designs is formulated in term of asymptotic convergence (as time $t \to \infty$) of the state to a desired state value (e.g., the origin), the control objective of “finite-time” stabilization addresses the possibility of achieving the desired state and input convergence properties over a finite time interval. The length of this finite time interval that is attained depends, in general, on the system dynamics and the initial conditions. Requiring this finite time interval to be a constant that is independent of the initial condition, that is, requiring that the convergence should be attained within a fixed terminal time that is independent of the initial condition, yields...
the stronger control objective of “fixed-time” stabilization. Further requiring that the fixed finite time should be a parameter that can be arbitrarily chosen (i.e., “prescribed”) by the control designer irrespective of the system’s initial condition yields the even stronger control objective of “prescribed-time” stabilization. The prescribed-time stabilization problem addresses a finite time interval [0, T] where T is the designer-prescribed convergence time and considers applications where the closed-loop system objective is defined over this finite time interval.

To force convergence within the arbitrarily specified finite prescribed time, two general prescribed-time stabilizing controller design approaches that have been addressed in the literature can be viewed as state scaling or time scaling:

1. **State scaling by a time-dependent function.** By, for example, scaling the state x to define \( \tilde{x} = \mu(t)x \) where \( \mu(t) \) is a “blow-up” function defined such that \( \mu(t) \to \infty \) as \( t \to T \), a control design that keeps \( \tilde{x} \) bounded will implicitly make \( x \) go to 0 as \( t \to T \).
2. **Time scaling using a nonlinear temporal transformation.** Define, for example, \( \tau = a(t) \) with a being a function defined such that \( a(0) = 0 \) and \( \lim_{t \to T} a(t) = \infty \). Since this time scale transformation maps \( t \in [0, T] \) to \( \tau \in [0, \infty) \), a control design that achieves asymptotic convergence in terms of the time variable \( \tau \) implicitly achieves prescribed-time convergence in terms of the time variable \( t \).

The state scaling approach has been applied in References 21,22,28 to design prescribed-time stabilizing controllers for classes of systems such as chains of integrators coupled with uncertainties matched with the control input (i.e., systems in normal form). Prescribed-time stabilizing controllers have been designed for nonlinear strict-feedback-like systems31-35 using the time scaling approach to convert the prescribed-time stabilization problem into an asymptotic stabilization problem in terms of the transformed time variable and applying the dual dynamic high gain scaling based observer-controller design techniques37-42 to achieve asymptotic stabilization in terms of the transformed time variable. While31 considered the prescribed-time stabilization problem under state feedback, output feedback was addressed in Reference 32. The adaptation of the control design techniques from References 37-42 that were originally developed in the context of asymptotic stabilization to the prescribed-time setting necessitated introduction of time-dependent forcing terms into the high-gain scaling parameter dynamics and a set of modifications in the controller design and the Lyapunov analysis to achieve prescribed-time convergence instead of asymptotic convergence. Uncertain nonlinear systems with uncertain functions appearing throughout the system dynamics including uncertain parameters (with no a priori known magnitude bounds on unknown parameters) coupled with unmeasured state variables were addressed in Reference 33 and a dynamic output-feedback prescribed-time stabilizing controller was developed. A partial state-feedback prescribed-time stabilizing controller was designed for systems with uncertainties in the input gain and non-vanishing input-matched disturbances in addition to uncertain terms throughout the system dynamics in Reference 34. An output-feedback prescribed-time stabilizing controller was designed for systems with time delays of unknown magnitude in Reference 35. Apart from the state scaling and time scaling approaches, control designs based on time-varying gains have been developed in the literature such as the approach in Reference 36 that redesigned autonomous stabilizing controllers for systems in the form of chains of integrators into non-autonomous state-feedback controllers with bounded time-varying gains.

Based on the prescribed-time stabilizing control design in our earlier conference paper, we consider in this paper a general class of nonlinear systems that include an unknown input gain and time-varying non-vanishing disturbances generated by an uncertain appended dynamics in addition to nonlinear time-varying uncertain terms throughout the system dynamics. The uncertain terms in the system dynamics are allowed to contain both parametric and functional uncertainties without requiring magnitude bounds on the uncertain parameters. The detailed structure of the class of systems considered is presented in Section 2 and comprises of a nominal system with state \( x \) and an appended dynamics with state \( z \). While the state \( x \) of the nominal system is assumed to be measured, the state \( z \) of the appended dynamics is not measured. An unknown input gain \( h(z, x, u, t) \) is allowed multiplied with the input \( u \) in the dynamics of the nominal system \( x \). Compared with our earlier conference paper, the problem formulation considered in this paper is more general in three aspects: (1) while a lower bound on \( h \) was assumed to be known in Reference 34, such a known lower bound is not required in the control design in this paper; (2) while an upper bound on the additive non-vanishing component in the uncertain function in the system dynamics was required to be known in Reference 34, such an upper bound is not required to be known in this paper and the additive term is instead modeled as an uncertain nonlinear function dependent on the state of the uncertain appended dynamics with unmeasured state; (3) while31 considered that the appearance of the state \( z \) of the appended dynamics in the uncertain functions in the dynamics of \( x \) is via uniformly bounded terms, the design in this paper considers nonlinear terms involving \( z \) in the bounds on the uncertain functions.
in the dynamics of $x$; additionally, while\textsuperscript{14} considered appended dynamics $z$ that are bounded-input-bounded-state stable (BIBS), the control design in this paper considers two components in the appended dynamics state: $z_d$ whose dynamics is input-to-state stable (ISS) and $z_o$ whose dynamics is BIBS. Under these several generalizations of the system class and weakened assumptions, it will be seen in the control design and stability and convergence analysis in Sections 3 and 4 that prescribed-time stabilization can be attained for the considered class of systems through several novel ingredients including non-smooth components in the designed control law, time-dependent forcing terms in the dynamics of the adaptation variable and the scaling parameter, interconnections between the adaptation dynamics and the scaling parameter dynamics taking into account the structure of the time scale transformation $t \to \tau$, and analysis of the stability and convergence properties in the closed-loop system. In particular, the main novel contribution of this paper is the development of a methodology to address the unknown input gain $h$ by designing the control law for $u$ as a combination of two components: a first component $u_1$ that is analogous to the dynamic high-gain scaling-based design (e.g., Reference 32), but with a time-dependent function in place of the unknown input gain $h$, the second term $u_2$ that is designed as a non-smooth form with time-dependent terms that ensure that in the limit as $t \to T$, ensure convergence of the system state to 0 in spite of the unknown input gain and the disturbances. The class of systems addressed by the proposed prescribed-time control design significantly extends classes of systems considered in prior work in directions that are highly relevant to real-world applications. In particular, the proposed control design addresses unknown input gains that could arise from uncertainties in actuator characteristics in physical systems as well as unmodeled dynamic and time-varying effects. Furthermore, the proposed control design addresses parametric and functional uncertainties as well as unknown appended dynamics that could arise from unmodeled interactions with external subsystems/loads in real-world applications.

This paper is organized as follows. The class of systems considered and the assumptions imposed on the system are provided in Section 2. The control design is presented in Section 3. The main result of the paper is presented in Section 4. The control design for two numerical examples is addressed in Section 5 along with simulation studies. Concluding remarks are contained in Section 6.

2 \hspace{1em} \textbf{NOTATIONS, CLASS OF SYSTEMS, CONTROL OBJECTIVE, AND ASSUMPTIONS}

2.1 \hspace{1em} \textbf{Notations}

1. The notation $|a|$ denote Euclidean norm of a vector $a$ or absolute value of a scalar $a$. The notation $||M||$ denotes Frobenius norm of a matrix $M$.
2. The notation diag$(T_1, \ldots, T_m)$ denotes a diagonal matrix of dimension $m \times m$ with diagonal elements $T_1, \ldots, T_m$. Also, lowerdiag$(T_1, \ldots, T_{m-1})$ and upperdiag$(T_1, \ldots, T_{m-1})$ denote the matrices of dimensions $m \times m$ with the lower diagonal entries (i.e., entries at locations $(i+1, i)$ for $i = 1, \ldots, m-1$) and upper diagonal entries (i.e., entries at locations $(i, i+1)$ for $i = 1, \ldots, m-1$), respectively, being $T_1, \ldots, T_{m-1}$ and zeros everywhere else.
3. $I_m$ denotes the identity matrix of dimension $m \times m$.
4. The maximum and minimum eigenvalues of a symmetric positive-definite matrix $P$ are denoted by $\lambda_{\text{max}}(P)$ and $\lambda_{\text{min}}(P)$, respectively.
5. The notations $\max(a_1, \ldots, a_n)$ and $\min(a_1, \ldots, a_n)$ indicate the largest and smallest values, respectively, among numbers $a_1, \ldots, a_n$.
6. Given a vector $a = [a_1, \ldots, a_m]^T$, the notation $|a|_e \in \mathbb{R}^m$ denotes the vector comprised of element-wise magnitudes of the elements of $a$, i.e., $|a|_e = [|a_1|, \ldots, |a_m|]^T$. Given two vectors $a = [a_1, \ldots, a_m]^T$ and $b = [b_1, \ldots, b_m]^T$, the notation $a \leq b$ is used to indicate $|a_i| \leq |b_i|, i = 1, \ldots, m$, that is, the element-wise inequalities between each of the corresponding elements of the vectors $a$ and $b$.
7. Given a scalar $\delta$, the notation $S(\delta)$ denotes the sign of $\delta$, that is, $S(\delta) = 1$ if $\delta \geq 0$ and $S(\delta) = -1$ otherwise.

2.2 \hspace{1em} \textbf{Dilation functions}

Since the proposed control design methodology depends crucially on time scale transformations that map the infinite time interval $[0, \infty)$ in terms of the original time variable $t$ to the finite time interval $[0, T)$ in terms of a transformed time
variable \( i \), we define below the concept of a dilation function that captures the required mathematical properties for the time scale transformations.

**Definition:** A function \( \delta(t) \) is said to be a dilation function over time interval \([0, T]\) if it satisfies the following properties:

1. \( \delta \) is a monotonically increasing function over \([0, T]\) that is twice continuously differentiable and satisfying \( \delta(0) = 0 \) and \( \lim_{t \to T} \delta(t) = \infty \).
2. Denoting \( \delta'(t) = \frac{d\delta}{dt} \), the inequality \( \delta'(t) \geq \delta_0 \) is satisfied for all \( t \in [0, T] \) with \( \delta_0 \) being a positive constant. The first condition and this condition imply that the function \( \delta \) is invertible. Denote the inverse function by \( \delta^{-1} \), that is, \( t = \delta^{-1}(\tilde{t}) \) where \( \tilde{t} = \delta(t) \).
3. Denoting the function \( \delta'(t) = \frac{d\delta}{dt} \) expressed in terms of the time variable \( \tilde{t} \) by \( \delta'(\tilde{t}) \), that is, \( \delta'(\tilde{t}) = \delta'(\delta^{-1}(\tilde{t})) \), the function \( \delta'(\tilde{t}) \) grows at most polynomially as \( \tilde{t} \to \infty \), that is, a polynomial \( \delta'(\tilde{t}) \exists \) satisfying \( \delta'(\tilde{t}) \leq \delta(\tilde{t}) \) for all \( \tilde{t} \in [0, \infty) \).

Also, \( \frac{d\delta}{dt} \) grows at most polynomially as \( \tilde{t} \to \infty \).

For brevity, we denote the set of all dilation functions \( \delta(t) \) over time interval \([0, T]\) as class \( D_{[0,T]} \) functions. Also, we denote the set of all functions \( \delta'(t) \) as class \( D_{[0,T]}' \) functions.

As noted in References 31, 32, an infinite number of functions exist that satisfy the conditions required on the function \( \delta \) above. For example, one choice for the function \( \delta \) is \( \delta(t) = \frac{\delta_0 t}{1-t^2} \) with \( \delta_0 \) being any positive constant. With this choice of the function \( \delta \), we have \( \delta'(t) = \frac{\delta_0}{(1-t^2)^2} \) and \( \delta''(\tilde{t}) = \delta_0 \left( \frac{\tilde{t}}{\delta_0^2} + 1 \right)^2 \). Note that \( \delta'(\tilde{t}) \) and \( \frac{d\delta}{dt} \) do indeed grow at most polynomially with the time \( \tilde{t} \) as required in the conditions introduced above on \( \delta' \).

### 2.3 Class of systems and control objective

We consider a class of uncertain nonlinear systems that can be written in the following form (or transformed into this form after an appropriate change of coordinates):

\[
\begin{align*}
\dot{x}_i &= \phi_i(z, x, u, t) + \phi_{i(i+1)}(x, t)x_{i+1}, \quad i = 1, \ldots, n-1 \\
\dot{x}_n &= \phi_n(z, x, u, t) + h(z, x, u, t)u \\
\dot{z}_a &= q_a(z_a, z_b, x, u, t), \quad \dot{z}_b = q_b(z_a, z_b, x, u, t); \quad z = [z_a^T, z_b^T]^T
\end{align*}
\]

(1)

that includes the nominal form with state \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) and an “appended dynamics” with state \( z = [z_a^T, z_b^T]^T \in \mathbb{R}^{n_a} \). \( n_a = n_x + n_z \) where \( z_a \in \mathbb{R}^{n_a} \) and \( z_b \in \mathbb{R}^{n_a} \) that is coupled with the \( x \) subsystem. In system (1), \( u \in \mathbb{R} \) is the input. \( \phi_{i(i+1)}, i = 1, \ldots, n-1 \) are known scalar real-valued continuous functions. \( \phi_i, i = 1, \ldots, n \), and \( h \) are continuous time-varying scalar real-valued uncertain functions of their arguments. \( q_a \) and \( q_b \) are uncertain continuous functions from their arguments to values in \( \mathbb{R}^{n_a} \) and \( \mathbb{R}^{n_b} \), respectively. The state \( x \) of the nominal system is measured while the state \( z \) of the appended dynamics is assumed to be unmeasured. The uncertain function \( h \) in (1) represents the unknown control input gain, which is allowed to be time-varying and state-dependent. Furthermore, while \( h \) is assumed to have known sign (without loss of generality, assumed positive) and lower-bounded in magnitude by a non-zero constant (to ensure controllability), the lower bound is not required to be a priori known unlike.\(^{34}\) The bounds imposed on functions \( \phi_i(z, x, u, t) \) in the assumptions on the system structure in Section 2 allow these functions to depend nonlinearly on the entire system state \( x \) as well as appearance of an uncertain parameter \( \theta \) (without requirement for a known magnitude bound) and coupling with the state of the appended dynamics. Furthermore, the assumed upper bound on \( \phi_{i(i+1)}(x, t) \) is allowed to include an uncertain additive term that is dependent on the unmeasured state of the appended dynamics; this term represents a non-vanishing disturbance that is not required to go to 0 when the state approaches the origin. Furthermore, a known constant upper bound on this non-vanishing disturbance is not required unlike.\(^{34}\)

**Control objective:** With \( T > 0 \) being any prescribed constant, the control objective is to design a dynamic control law for \( u \) using partial state feedback (with the measured signal being \( x \)) so that \( x(t) \to 0 \) as \( t \to T \) while also ensuring that \( z(t) \) and \( u(t) \) remain uniformly bounded over the time interval \( t \in [0, T] \), that is, \( \sup_{t \in [0, T]} |z(t)| < \infty \) and \( \sup_{t \in [0, T]} u(t) < \infty \).
2.4 Assumptions

The control design in this paper uses the following assumptions imposed on the system (1).

**Assumption 1.** [positive lower bounds on “upper diagonal” terms \(\phi_{(i+1)}\)] The inequalities \(\phi_{(i+1)}(x, t) \geq \sigma, 1 \leq i \leq n - 1\) are satisfied for all \(x \in \mathbb{R}^n\) and \(t \geq 0\) with \(\sigma\) being a positive constant.

**Assumption 2.** (upper bounds on magnitudes of uncertain functions \(\phi_i\)) The functions \(\phi_i, i = 1, \ldots, n,\) can be bounded as

\[
|\phi_i(z, x, u, t)| \leq \Gamma(x) \theta \sum_{j=i}^{n} |\phi_{ij}(x, t)| x_j
\]

for \(i = 1, \ldots, n - 1\) and

\[
|\phi_n(z, x, u, t)| \leq \Gamma(x) \left\{ \theta \sum_{j=1}^{n} |\phi_{nj}(x, t)| x_j + \phi_c(z, x, u, t) \right\}
\]

for all \(x \in \mathbb{R}^n, z \in \mathbb{R}^{k}, u \in \mathbb{R},\) and \(t \geq 0\) where \(\Gamma(x)\) and \(\phi_{ij}(x, t)\) for \(i = 1, \ldots, n, j = 1, \ldots, i\) are known continuous non-negative functions, \(\theta\) is an unknown non-negative constant, and \(\phi_c\) is an uncertain nonlinear function. The conditions that \(\phi_i\), \(\bar{\phi}_i\), \(\bar{\phi}_n\), \(\varepsilon_i\), \(\varepsilon_n\), \(\epsilon_i\), \(\epsilon_n\), \(\varepsilon_{i,j}\), and \(\varepsilon_{i,n}\) are known such that \(\forall x \in \mathbb{R}^n\) and \(t \geq 0,

\[
\frac{\phi_{i,1}(x, t)}{\phi_{i,2}(x, t)} \leq \epsilon_{i,1}, \quad 1, \ldots, n; \quad \frac{\phi_{i,2}(x, t)}{\sqrt{\phi_{i,2}(x, t)\phi_{i,3}(x, t)}} \leq \varepsilon_{i,2}, \quad i = 2, \ldots, n
\]

for all \(x \in \mathbb{R}^n\) and \(t \geq 0\).

**Assumption 3.** (positive lower bound on uncertain input gain \(h\)): The uncertain function \(h\) is lower bounded in magnitude by a positive constant \(\bar{h}\) that is not required to be known. Since \(h\) is a continuous function, this assumption can, without loss of generality, be stated as \(h(x, z, x, u, t) \geq \bar{h} > 0\) for all \(x \in \mathbb{R}^n, z \in \mathbb{R}^{k}, u \in \mathbb{R},\) and \(t \geq 0\).

**Assumption 4.** (bounds on relative sizes of “upper diagonal” terms \(\phi_{(i+1)}\)), \(i = 2, \ldots, n,\) that is, cascading dominance conditions): Positive constants \(\bar{\phi}_i\) exist such that \(\phi_{(i+1)}(x, t) \geq \bar{\phi}_i \phi_{(i-1)}(x, t), \quad i = 3, \ldots, n - 1, \forall x \in \mathbb{R}^n\) and \(t \geq 0\).

**Assumption 5.** (bounds on relative sizes of \(\phi_{(1,2)}\) and \(\phi_{(2,3)}\), that is, cascading dominance conditions between “upper diagonal” terms \(\phi_{(1,2)}\) and \(\phi_{(2,3)}\)): Continuous non-negative functions \(\bar{\phi}_{(1,2)}(x_1)\) and \(\bar{\phi}_{(2,3)}(x_1)\) exist such that

\[
\bar{\phi}_{(1,2)}(x_1) \leq \frac{\phi_{(1,2)}(x_1)}{\phi_{(2,3)}(x_1)} \leq \bar{\phi}_{(2,3)}(x_1)
\]

for all \(x \in \mathbb{R}^n\) and \(t \geq 0\).

**Assumption 6.** (stability properties of appended dynamics \(z\)): The appended dynamics with state \(z_0\) and input \((z_0, x, u, t)\) is ISS and a Lyapunov function \(V_z(z_0)\) exists such that for all \(x \in \mathbb{R}^n, z \in \mathbb{R}^{k}, u \in \mathbb{R}, t \geq 0\), the following inequalities are known to be satisfied\(^1\) with \(\kappa_z, \tilde{\kappa}_z,\) and \(\phi_{(z,0)}\) being known positive constants, \(\phi_{(z,j)}, j = 1, \ldots, n\) being known continuous non-negative functions, \(\theta_{x}\) being an unknown non-negative constant, \(\tilde{\kappa}_{z}\) a class \(\mathcal{K}_\infty\) function, and \(\varphi\) being a function in class \(\mathcal{D}'\wedge\mathcal{T}\)

\[
V_z = \frac{\partial V_z(z_0)}{\partial z_0} q_u(z_0, z_0, x, u, t) \leq -\varphi(t) \left[ \kappa_z |z_0| + \tilde{\kappa}_z \Gamma(x) \sum_{j=1}^{n} \phi_{(z,j)}(x, t) x_j^2 \right]
\]

\[
|\phi_{z}^2(z, x, u, t) | \leq \kappa_z^2 |z_0| + h^2(z, x, u, t) \phi_{(z,0)}^2; \quad \tilde{\kappa}_z V_z(z_0) \geq \varphi(t) V_z(z_0).
\]

The positive lower bound on \(\varphi(t)\) that is guaranteed by the definition of class \(\mathcal{D}'\wedge\mathcal{T}\) functions is denoted as \(\varphi_0\), that is, \(\varphi(t) \geq \varphi_0 > 0\) for all \(t \in [0, T]\). Similar to (4) in Assumption 2, the functions \(\phi_{(z,j)}\) satisfy the following inequalities

\[
\frac{\phi_{(z,1)}(x, t)}{\phi_{(1,2)}(x, t)} \leq \epsilon_{(z,1)}; \quad \frac{\phi_{(z,2)}(x, t)}{\sqrt{\phi_{(1,2)}(x, t)\phi_{(2,3)}(x, t)}} \leq \tilde{\epsilon}_{(z,2)}; \quad \frac{\phi_{(z,j)}(x, t)}{\phi_{(2,3)}(x, t)} \leq \epsilon_{(z,j)}, j = 2, \ldots, i.
\]
with \( e_{i,j}, i = 1, \ldots, n \), and \( e_{i,2} \) being known non-negative constants. The appended dynamics with state \( z_b \) and input \((z_a, x, u, t)\) is a BIBS stable system \( ^6 \).

### 2.5 Discussion on class of systems and assumptions

The class of systems (1) is in a strict-feedback-like structure for the state \( x \) of the nominal system and includes an uncertain appended dynamics \( z \) that generates disturbance inputs into the nominal system via the terms \( \phi_i, i = 1, \ldots, n \), and \( h \). Assumption 1 ensures controllability and uniform relative degree (of \( x_1 \) with respect to \( u \)). Assumption 2 imposes bounds on the uncertain terms \( \phi_i \) in the system dynamics and essentially requires known bounds on the uncertain terms with a triangular-like state dependence structure in the known bounds (up to coefficients dominated in a nonlinear function sense by the upper diagonal terms \( \phi_{i,l(i+1)} \)). The functions \( \phi_{i,l(i+1)} \) are referred to as “upper diagonal” terms since if the dynamics of \( x \) from (1) were to be written in a “linear-like” form \( \dot{x} = A(x, t)x + B(x, t)u + \phi(x, u, t) \) with \( \phi = [\phi_1, \ldots, \phi_n]^T \), the functions \( \phi_{i,l(i+1)} \) would appear on the upper diagonal of the matrix \( A(x, t) \). It is to be noted that the uncertain terms \( \phi_i \) can involve the state components \( z_a \) and \( z_b \) of the appended dynamics. While the dependence on the BIBS part’s state \( z_b \) is required to be uniformly bounded and therefore does not appear on the right hand side of the inequalities (2) and (3), the dependence on the ISS part’s state \( z_a \) is allowed to be of a general structure via the function \( \phi_\eta(z, x, u, t) \) appearing on the right hand side of (3). While more generally, the dependence on \( z_a \) of the upper bounds on \( \phi_i \) can be of a triangular structure analogous to Reference 37 (upper bound on \( \phi_i \) involving \( z_{a,1}, \ldots, z_{a,i} ; z_{a,i} \) driven by \( x_1, \ldots, x_i \) as characterized by Lyapunov inequalities analogous to (6)), we consider in this paper the case where the explicit appearance of a \( z \)-dependent term \( \phi_\eta \) is only in the upper bound on \( \phi_\eta \) for algebraic simplicity and to focus on the main challenge addressed in this paper which is the unknown input gain and unknown input-coupled additive disturbance. Assumption 3 requires that the uncertain input gain \( h \) should be bounded away from zero while allowing the lower bound itself to be unknown. Assumption 4 imposes constraints on the relative “sizes” (in a nonlinear function sense) of the upper diagonal terms \( \phi_{i,l(i+1)} \) and as discussed in Section 3.4, the inequalities in Assumption 4 enforce a “cascading dominance” relationship among the upper diagonal terms enabling solvability of a pair of coupled Lyapunov inequalities. While Assumption 4 relates to relative sizes of \( \phi_{i-1,l(i)} \) and \( \phi_{i,l(i+1)} \) for \( i = 3, \ldots, n - 1 \), Assumption 5 similarly imposes a constraint on the relative sizes (in a nonlinear function sense) between functions \( \phi_{i,2} \) and \( \phi_{i,3} \). Assumption 6 imposes requirements on the uncertain appended dynamics \( z \); specifically, \( z \) is allowed to be comprised of two subsystems: the first with state \( z_a \) satisfying an ISS condition and the second with state \( z_b \) satisfying a BIBS condition. The Assumptions 1–6 are not particularly stringent conditions and are satisfied by a wide class of systems. In particular, Assumptions 1, 4, and 5 are trivially satisfied under the most common classes of systems considered in the literature in which the upper diagonal terms \( \phi_{i,l(i+1)} \) are typically constants (e.g., 1 in the “chain of integrators” structures). Assumption 2 is also typically satisfied as long as the dependence of the uncertain functions \( \phi_i \) on the state variables \( x_2, \ldots, x_n \) can be bounded in a form that is linear in these state variables (up to nonlinear terms that also appear in \( \phi_{i,3} \)) while \( x_1 \) can appear nonlinearly throughout in these functions. Assumption 3 is typically satisfied since the input gain while uncertain would typically be lower bounded away from zero (albeit with an unknown lower bound) since controllability could be lost if the input gain can go to zero. Assumption 6 is a well-behavedness requirement on the appended dynamics stated formally in terms of ISS and BIBS conditions.

The Assumptions 1, 4, and 5 above are similar to Reference 31. The bounds in Assumption 2 are more general in structure than the corresponding assumption in Reference 31. While 31 assumed bounds of the form \( |\phi_i| \leq \Gamma(x) \sum_{j=1}^{i} \phi_{i,j}(x)|x_j| \), Assumption 2 above allows an uncertain parameter \( \theta \) (for which no magnitude bounds are required to be \textit{a priori} known), an additional term of form \( \Gamma(x) \phi_\eta(z, x, u, t) \) appearing in the bound on \( |\phi_i| \), and time dependence of the functions \( \phi_{i,l(i)} \). The additional \( \Gamma(x) \phi_\eta(z, x, u, t) \) term in the upper bound on \( |\phi_i| \) allows for the presence of uncertain non-vanishing disturbance inputs that are generated via the appended time-varying dynamics with unmeasured state \( z \). The Assumptions 3 and 6 do not have corresponding analogous assumptions in Reference 31. Assumption 3 requires that the unknown input gain \( h \) (that appears multiplied with the control input \( u \) in the system dynamics [11]) be lower bounded away from zero. While 31 assumed that the input \( u \) appears with a known gain as \( \mu_0(x)u \) with a known function \( \mu_0 \), the class of systems considered here are allowed to contain an uncertain time-varying state-dependent input gain \( h(z, x, u, t) \). Assumption 3 on this unknown input gain requires only a known lower bound \( h \) on \( h \) and does not require an upper bound. Assumption 6 relates to the appended dynamics \( z \) that were not considered in the control design in Reference 31. The role of the appended dynamics here is as a forcing function coupled with various uncertain terms in the system dynamics including \( \phi_i \) and \( h \). Also, the Assumptions 2 and 3 are weaker compared to the earlier conference version 34 of this paper. While 34 required a known upper bound \( \phi_{i,0} \) on the additive non-vanishing part of the uncertain function \( \phi_n \) and a known lower
bound $h$ on the input gain $h$, these requirements are relaxed in this paper. Specifically, the additive term in the upper bound on the uncertain function $\phi_n$ which was a known constant in Reference 34 is replaced by an uncertain function $\phi_r$ dependent on the unmeasured state of the appended dynamics and the lower bound $\tilde{h}$ on the input gain $h$ is allowed to be unknown. Removing the requirements in Reference 34 discussed above entails several modifications in the control design; specifically, while the control design in Reference 34 utilized $h$ and the constant upper bound on the additive non-vanishing part, additional time-dependent functions are introduced in this paper in their place. By designing these time-dependent forcing functions appropriately in combination with various modifications in the stability analysis, the need for knowledge of the constants $\phi_{00}$ and $h$ is removed. The weaker assumptions as outlined above significantly expands the class of systems that can be handled and addresses practically realistic considerations such as parametric and functional uncertainties in the system, unknown appended dynamics coupled with the nominal system, and unknown input gain (e.g., uncertainties in actuator characteristics in a physical system).

3 | CONTROL DESIGN

3.1 | Design of control law $u$

The control input $u$ is designed as comprised of two components:

$$u = u_1 + u_2$$  \hspace{1cm} (9)

where $u_1$ defined below is picked based on a pair of coupled Lyapunov inequalities as discussed in Section 3.4 and $u_2$ is designed as part of Section 3.6 based on a Lyapunov analysis that takes into account the various uncertain terms in the system dynamics including the unknown input gain $h$. The first component $u_1$ in (9) is designed as

$$u_1 = -\frac{r^h}{\gamma_1(t)} K_c \eta,$$  \hspace{1cm} (10)

In (10)

1. $r$ is a dynamic high-gain scaling parameter; the dynamics of $r$ that will be designed in Section 3.6 will be such that $r$ is a monotonically non-decreasing signal in time. Also, $r$ will be initialized such that $r(0) \geq 1$; hence, $r(t) \geq 1$ for all time $t$.
2. $\gamma_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function that will be designed as part of Section 3.6.
3. $\eta = [\eta_2, \ldots, \eta_n]^T$ with $\eta_i$ being scaled state variables (scaled by powers of the dynamic scaling parameter $r$) defined as:

$$\eta_2 = \frac{x_2 + \zeta(x_1, \hat{\theta})}{r}; \quad \eta_i = \frac{x_i}{r^{i-1}}, i = 3, \ldots, n.$$  \hspace{1cm} (11)

4. $\zeta$ is a function that is defined to be of the form

$$\zeta(x_1, \hat{\theta}) = \hat{\theta}x_1\zeta_1(x_1)$$  \hspace{1cm} (12)

with $\zeta_1$ being a function that will be designed as part of Section 3.6 and $\hat{\theta}$ is a dynamic adaptation parameter. The dynamics of $\hat{\theta}$ will also be designed as part of Section 3.6 such that $\hat{\theta}$ is a monotonically non-decreasing signal as a function of time. Also, $\hat{\theta}$ will be initialized such that $\hat{\theta}(0) \geq 1$; hence, $\hat{\theta}(t) \geq 1$ for all time $t$.
5. $K_c = [k_2, \ldots, k_n]$ with $k_i, i = 2, \ldots, n$, being functions of $x$ and $t$ that will be designed below.

The dynamics of the scaled state vector $\eta$ defined above under the control law given by (9) and (10) can be written as

$$\dot{\eta} = r A_c \eta - \frac{r}{r} D_x \eta + \Phi + H \eta_2 + \Xi - \frac{h - \gamma_1(t)}{\gamma_1(t)} r K_c \eta + B h \frac{u_2}{r^n - 1},$$  \hspace{1cm} (13)

where $A_c$ is the matrix of dimension $(n - 1) \times (n - 1)$ in which the $(i,j)^{th}$ element is given by $A_{c(i,j)} = \phi_{i+1,j+2}$, $i = 1, \ldots, n - 2$, $A_{c(n-1,j)} = -k_{j+1}$, $j = 1, \ldots, n - 1$, and zeros everywhere else. $B$ is the column vector of length $(n - 1)$ of form $[0, \ldots, 0, 1]^T$. Also,
\[ D_c = \text{diag}(1, 2, \ldots, n-1); \quad \Phi = \left[ \frac{\phi_2}{r}, \ldots, \frac{\phi_0}{r^{n-1}} \right]^T \]

\[ H = [\hat{\theta}[\zeta'_1(x_1)x_1 + \zeta_1]\phi_1, 0, \ldots, 0]^T \]

\[ \Xi = \left[ \left(\frac{\phi_1 - \zeta r\phi_1}{\tau}\right)\hat{\theta}[\zeta'_1(x_1)x_1 + \zeta_1] + \frac{\dot{\theta}}{\tau}\zeta_1, 0, \ldots, 0 \right]^T. \]

where \( \zeta'_1(x_1) \) denotes the partial derivative of the function \( \zeta_1 \) evaluated at \( x_1 \).

If the input gain \( h \) were a known function, \( h \) could be used in place of \( y_1(t) \) in (10) to cancel out the function \( h \) in the resulting dynamics of \( \eta \), that is, to remove the term involving \( (h - y_1(t)) \) in dynamics (13). However, since \( h \) is an uncertain function and even the lower bound \( h \) is unknown, the function \( y_1(t) \) is introduced in (10) and will be designed in Section 3.6 to handle the “mismatch” term involving \( (h - y_1(t)) \).

### 3.2 Time scale transformation

Define a time scale transformation \( \tau = a(t) \) with \( a \) being a class \( D_{[0, T]} \) function (as defined in Section 2.2) and denote the function \( a'(t) = \frac{da}{dt} \) expressed in terms of the time variable \( \tau \) by \( a(\tau) \), i.e., \( a(\tau) = a'(a^{-1}(\tau)) \). Specifically, if the appended dynamics component \( z_a \) is non-empty, then the function \( \varphi(t) \) from Assumption 6 is used to define the function \( a(t) \) as \( \varphi_a = \int_0^t \varphi(x)dx \) with \( \varphi_a \) being any positive constant. This definition of \( a \) implies that \( a'(t) = \varphi_a \varphi(t) \) and \( a(\tau) = \varphi_a \varphi(a^{-1}(\tau)) \).

On the other hand, if the appended dynamics component \( z_a \) does not exist in the system and therefore the first part of Assumption 6 is trivially satisfied, then the function \( a \) can be picked to be any class \( D_{[0, T]} \) function. The time scale transformation \( \tau = a(t) \) maps the time interval \([0, T]\) in terms of \( t \) to the time interval \([0, \infty)\) in terms of the transformed time variable \( \tau \). Hence, the prescribed-time control objective formulated as convergence objectives as \( t \to T \) are equivalent to the analogous convergence objectives as \( \tau \to \infty \). From the definition of the time scale transformation, we have

\[ dt = \frac{1}{a(\tau)} d\tau. \]

As discussed in section 2.2, an infinite number of functions exist that satisfy the conditions listed in the definition of class \( D_{[0, T]} \) functions. By the definition of class \( D_{[0, T]} \) functions, \( a(\tau) \) and \( \frac{da}{d\tau} \) grow at most polynomially as \( \tau \to \infty \). Also, from the definition of class \( D_{[0, T]} \) functions, we know that \( a(\tau) \) over \( \tau \in [0, \infty) \) and equivalently \( a'(t) \) over \( [0, T] \) are bounded below by a positive constant; denote this positive constant as \( a_0 \). In the stability and convergence analysis in Section 4, it will be seen that these polynomial growth conditions required from the definition of class \( D_{[0, T]} \) functions are indeed crucial in showing that the high-gain scaling parameter \( r \) grows at most polynomially with the time \( \tau \); since \( x_1 \) can be written in terms of combinations of \( \eta_2 \) and powers of \( r \), it will be seen that the polynomial growth property of \( r \) is crucial in inferring that exponential convergence of \( \eta \) to 0 as \( \tau \to \infty \) implies exponential convergence of \( x_1 \) to 0. Similarly, since the control law for \( u \) involves terms comprising of combinations such as \( r^n \eta \) as seen in (10), we will see in the analysis in Section 4 that the at-most-polynomial growth property of the scaling parameter \( r \) is crucial in inferring convergence to 0 of these terms in the control law from the exponential convergence of \( \eta \) to 0.

### 3.3 Lyapunov function

Define

\[ V_\xi = \frac{1}{2} \left( 1 + \frac{1}{r} \right) x_1^2 + r\eta^T \Phi \eta \]

where \( \Phi \) is a constant symmetric positive-definite matrix that will be defined in Section 3.4 based on the solution of a pair of coupled Lyapunov inequalities. Differentiating (18) and using the property that \( dt = \frac{dr}{a(\tau)} \), we have

\[
\frac{dV_\xi}{d\tau} = \frac{1}{a(\tau)} \left\{ x_1 [\Phi_1 + (r\eta_2 - \zeta)\Phi_1] + r^2 \eta^T \Phi \Phi \eta + 2r\eta^T \Phi \Phi \eta + 2r\eta^T \Phi \eta + \frac{\dot{\theta}}{\tau}\zeta_1 \right\} - \frac{dr}{d\tau} \eta^T \Phi \dot{\Phi} - \frac{1}{2r^2} \frac{dr}{d\tau} x_1^2.
\]
where \( D_c = D_c - \frac{1}{2} I_{n-1} \). Note that the Lyapunov function \( V_x \) explicitly includes a multiplicative term involving the dynamic high-gain scaling parameter \( r \) analogous to prior high-gain scaling based designs (e.g., References 33,37,38,40-42). Similar scalings with a dynamic gain in the Lyapunov function have also used in, for example, control of switched linear time-delay systems \(^{44}\) and dynamic weightings for Lyapunov functions. \(^{45}\)

### 3.4 Coupled Lyapunov inequalities

Assumption 4 on the relative sizes (in a nonlinear function sense) of the upper diagonal terms \( \phi_{(2,3)} \), \( \ldots \), \( \phi_{(n-1,n)} \) is seen to be the cascading dominance condition introduced in Reference 37; under this condition, it was shown in References 37,46 that a constant symmetric positive-definite matrix \( P_c \) and a function \( K_\nu(x,t) = [k_\nu(x,t), \ldots, k_\nu(x,t)] \) (whose elements appear in the definition of the matrix \( A_c \)) can be constructed to satisfy the coupled Lyapunov inequalities given by

\[
P_c A_c + A_c^T P_c \leq -v_c \phi_{(2,3)} I
\]

\[
v_c I \leq P_c D_c + D_c P_c \leq \overline{v}_c I
\]

where \( v_c, \overline{v}_c \), and \( \overline{v}_c \) are positive constants.

### 3.5 Inequality bounds on terms appearing in Lyapunov inequality (19)

Using the bounds on uncertain terms \( \phi_i \) in Assumption 2, the definition of \( \Phi \) in (14), the definitions of the scaled state variables \( \eta \) in (11), and the property of the scaling parameter \( r \) that \( r \geq 1 \), we have

\[
\Phi \leq \theta \Gamma \Phi_1 \frac{|x_1|}{r} + \theta \Gamma \Phi_M |\eta|_r + \theta \Gamma \Phi_2 \frac{\theta |x_1|}{r} + \Gamma \frac{\phi_c}{r^{n-1}}
\]

where

1. \( \Phi_1 = [\phi_{(1,2)}, \ldots, \phi_{(n,1)}]^T \); \( \Phi_2 = [\phi_{(2,2)}, \ldots, \phi_{(n,2)}]^T \)
2. \( \Phi_M \) is the matrix of dimension \((n-1) \times (n-1)\) with \((i,j)^{th} \) element \( \phi_{(i+1,j+1)} \) for \( i = 1, \ldots, n-1, j \leq i \), and zeros everywhere else

Hence (with some conservative overbounding for algebraic simplicity),

\[
2r \eta^T P_c \Phi \leq 2 \lambda_{max}(P_c) |\eta| |\eta| \{ |\Phi_1| |\eta| + |\Phi_2| |\eta| + |\Phi_M| |\eta| \} + 2 \Gamma \frac{\phi_c}{r^{n-2}} |\eta|^T P_c \eta \Gamma B \phi_c
\]

Hence,

\[
2r \eta^T P_c \Phi \leq \frac{2 \theta^2 \Gamma^2}{\zeta_0 \phi_{(1,2)}} \lambda_{max}(P_c) |\eta|^2 \{ |\Phi_1|^2 + |\Phi_2|^2 \frac{\phi_{(1,2)}}{r^2} \} + \zeta_0 |\phi_{(1,2)}|^2 x_1^2 + 2r \theta \Gamma \lambda_{max}(P_c) |\Phi_M| |\eta|^2 + 2 \Gamma \frac{\phi_c}{r^{n-2}} |\eta|^T P_c |B \phi_c
\]

with \( \zeta_0 \) being any positive constant and with \( |\eta|^T P_c |\eta| \) denoting the vector comprised of the element-wise magnitudes of the elements of the vector \( \eta^T P_c \eta \) as per the notation defined in Section 2.

Using Assumption 2, the other terms in the Lyapunov inequality (19) can also be upper bounded as:

\[
x_1 \phi_1 \leq \theta \Gamma x_1^2 \phi_{(1,1)}
\]

\[
x_1 r \phi_2 \phi_{(2,3)} \leq \frac{v_c}{4} r^2 \phi_{(2,3)} |\eta|^2 + \frac{1}{v_c} x_1^2 \phi_{(2,3)}^2
\]

\[
2r \eta^T P_c H \eta_2 \leq 2 \theta r \phi_{max}(P_c) |\phi_{(1,2)}|^2 |\phi_{(1,2)}|^2 + 2 \theta \Gamma \lambda_{max}(P_c) |\Phi_M| |\eta|^2 + 2 \Gamma \frac{\phi_c}{r^{n-2}} |\eta|^T P_c |B \phi_c
\]

\[
2r \eta^T P_c \Xi \leq \zeta_0 |\phi_{(1,2)}|^2 x_1^2 + \frac{2}{\zeta_0 \phi_{(1,2)}} \lambda_{max}(P_c) |\eta|^2 \{ \phi_{(1,2)}^2 + \theta \phi_{(1,2)}^2 \phi_{(1,2)}^2 (\phi_{(1,2)}^2 + \phi_{(1,2)}^2) \}
\]
Using Assumption 6, the term involving $\phi_c$ in (23) can be upper bounded as:

$$2 \frac{\Gamma}{r_{n-3}} |\eta^T P_c eB\phi_c| \leq 2 \frac{\Gamma}{r_{n-3}} |\eta^T P_c eB[\kappa_z \sqrt{Y_e} + h\phi(x,0)]|.$$  

(28)

Therefore,

$$2 \frac{\Gamma}{r_{n-3}} |\eta^T P_c eB\phi_c| \leq \frac{2\kappa_z}{\kappa_z} r_{\lambda_{max}(P_c)} \Gamma^2 |\eta|^2 + \frac{1}{2} r_{n-3} \kappa_z \kappa_z Y_e + 2 \frac{\Gamma}{r_{n-3}} |\eta^T P_c eBh\phi(x,0)|.$$  

(29)

Noting the appearance of the $Y_e$ term in (29) and the form of the Lyapunov inequality in Assumption 6, a composite Lyapunov function is defined as a combination of $V_x$ defined in (18) and $V_z$ from Assumption 6 as

$$V = V_x + \frac{\kappa_z}{r_{n-3}} V_z = \frac{1}{2} \left(1 + \frac{1}{r_0} \right) x_1^2 + r_0^T P_c \eta + \frac{\kappa_z \kappa_z}{\phi_0 r_{n-3}} V_z$$  

(30)

with $\kappa_z$ being any positive constant. Using (6) from Assumption 6 and the inequalities from Assumption 2, we obtain

$$\frac{d}{dt} \left( \frac{1}{r_{n-3}} V_z \right) \leq \frac{\rho(t)}{a(\tau)} \left\{ \frac{1}{r_{n-3}} Y(z, a) + \frac{1}{r_0} \theta_2 \Gamma(e_{(1)}, \phi(1,2) x_1^2 + \theta^* q_2(x_1) \phi(1,2) x_1^2 + \frac{\rho(t)}{r} \theta^* q_2(x_1) \phi(1,2) x_1^2 + \frac{\rho(t)}{r} \theta^* q_2(x_1) \phi(1,2) x_1^2 \\
+ r \theta \Gamma \phi(2,3) \sum_{j=1}^n e_{(j,2)} \eta_j^2 \right\} - \frac{1}{r_{n-3}} \frac{dr}{dt} V_z$$  

(31)

Using the inequalities in (20), (19), (23)–(27), and (31), it is seen that the Lyapunov function defined in (30) satisfies

$$\frac{dV}{dr} \leq \frac{1}{a(\tau)} \left\{ -x_1^2 \dot{\theta} \dot{\phi}_{(1,2)} + \frac{3}{4} \nu_c (\phi_{(2,3)})^2 |\eta|^2 + \frac{\rho(t) \kappa_z \kappa_z}{2 \phi_0 r_{n-3}} Y_e + q_1(x_1) \phi(1,2) x_1^2 + \theta^* q_2(x_1) \phi(1,2) x_1^2 + \frac{\rho(t)}{r} \theta^* q_2(x_1) \phi(1,2) x_1^2 \\
+ \frac{\rho(t) \theta^*}{r} q_3(x_1, \hat{\theta}) \phi(1,2) x_1^2 + r_1 w_1 \left( x_1, \hat{\theta}, \hat{\phi}_{(1,2)} \right) \phi(1,2) |\eta|^2 + \theta^* w_2(x_1, \hat{\theta}) \phi(2,3) |\eta|^2 + \frac{r \phi(t) \theta^* w_2(x_1, \hat{\theta}) \phi(2,3) |\eta|^2}{r_{n-3}} \\
- 2 r^3 h - \gamma_1(t) \frac{\eta^T P_c B K_e e + 2 \eta^T P_c B h \frac{u_2}{r_{n-3}} + 2 \frac{\Gamma}{r_{n-3}} |\eta^T P_c eBh\phi(x,0)|}{r_{n-3}} \right\} - \frac{dr}{dt} |\eta|^2 - \frac{1}{r_{n-3}} \frac{dr}{dt} x_1^2 - \frac{1}{r_{n-3}} \frac{dr}{dt} V_z$$  

(32)

where $\theta^* = (1 + \theta + \theta^2 + \theta_c)$ is an uncertain positive constant and

$$q_1(x_1) = \frac{1}{\nu_c} \phi(e_{(1,2)}(x_1)) + 2 \zeta_0; \quad q_2(x_1) = \Gamma(x_1) e_{(1,1)}$$  

(33)

$$q_{2c}(x_1) = \frac{\kappa_z \kappa_z}{\phi_0} \Gamma_e(x_1) e_{(1,1)}; \quad q_{3c}(x_1, \hat{\theta}) = 2 \frac{\kappa_z \kappa_z}{\phi_0} \Gamma_e(x_1) e_{(2,2)} \hat{\phi}_{(1,2)}(x_1)$$  

(34)

$$w_1 \left( x_1, \hat{\theta}, \hat{\phi}_{(1,2)} \right) = 2 \lambda_{\max}(P_c) \left| (\phi_{(1,2)}(x_1) x_1 + \phi_{(1,1)}(x_1)) \right| + \frac{2}{\zeta_0} \lambda_{\max}(P_c) \left( \phi_{(1,2)}(x_1) \right)^2 \zeta_1^2(x_1)$$  

(35)

$$w_2(x_1, \hat{\theta}) = 2 \lambda_{\max}(P_c) \Phi_1 \left( \phi_{(1,2)}(x_1) \right)^2 + \frac{4}{\zeta_0} \lambda_{\max}(P_c) \left| (\phi_{(1,2)}(x_1) x_1 + \phi_{(1,1)}(x_1)) \right|^2$$  

(36)

$$w_{2c}(x_1) = \frac{\kappa_z \kappa_z}{\phi_0} \Gamma_e(x_1) \phi_{(2,3)}(x_1) \left[ 2 \phi_{(1,2)}^2 + \sum_{j=3}^{n-2} e_{(j,2)}^2 \right].$$  

(37)
Note that the functions $q_1(x_1), q_2(x_1), q_{2z}(x_1), q_{3z}(x_1, \hat{\theta}), w_1 \left( x_1, \hat{\theta}, \frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}} \right), w_2(x_1, \hat{\theta}),$ and $w_{2z}(x_1)$ involve only known functions and quantities. The third argument of the definition of $w_1$ is written in terms of the combination given as $\frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}}$ rather than as simply $\dot{\theta}$ separately since it will be seen (in Lemma 2 in Section 4) that it can be shown that this combination $\frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}}$ grows at most polynomially as a function of the time $r$ and that this property can then be used to show (in Lemma 3 in Section 4) that the scaling parameter $r$ grows at most polynomially as a function of the time $r$. Keeping in mind the non-smoothness of the control component $u_2$ to be designed in Section 3.6, the solutions of the differential equations in this paper are understood in the sense of Filippov.\(^{47}\) In (32), it is to be noted that the term $\phi_{(1, 0)}$ in the right hand side is non-vanishing even when both $x$ and $z$ are zero.

### 3.6 Designs of functions $\zeta_1$ and $\gamma_1$, dynamics of $r$ and $\hat{\theta}$, and control law component $u_2$

The design freedoms (parameters/functions that can be picked by the control designer) appearing in the right hand side of (32) are $\zeta_1, \gamma_1, \frac{dr}{dt},$ and $u_2.$ In addition, the dynamics of $\hat{\theta}$, that is, $\frac{d\hat{\theta}}{dt}$ is also a design freedom. We will see in the design below that the dynamics of $r$ and $\hat{\theta}$ will be picked such that the signals $r$ and $\hat{\theta}$ are greater than or equal to $a(t)$ for all time in the maximal interval of existence of solutions. Since $a(t) = \varphi_u\varphi(t),$ this implies that the signals $r$ and $\hat{\theta}$ are greater than or equal to $\varphi_u\varphi(t)$ for all time in the maximal interval of existence of solutions. The function $\zeta_1$ is designed such that the negative $x_1^2 \hat{\theta} \zeta_1 \varphi_{(1, 2)}$ term in the right hand side of (32) dominates over the positive $q_1 \varphi_{(1, 2)x_1^2}^2$ and $\theta^* q_3 \varphi_{(1, 2)x_1^2}$ terms, but with the unknown constant $\theta^*$ replaced by $\hat{\theta}$, which is a dynamic adaptation state variable, and with $\frac{\varphi(t)}{\varphi_u}$ replaced by $\frac{1}{\varphi_u}$.

Hence, noting that $\hat{\theta} \geq 1,$ we pick the function $\zeta_1$ such that

$$
\frac{1}{4} \zeta_1(x_1) = \max \left\{ \zeta, q_1(x_1) + q_2(x_1) + \frac{1}{\varphi_u} q_{2z}(x_1) \right\}
$$

(38)

with any constant $\zeta > 0.$

To design the dynamics of the high-gain scaling parameter $r,$ we use the basic motivation from the dynamic high-gain scaling control design approach for asymptotic stabilization (e.g., Reference 37) that the dynamics of the scaling parameter $r$ are to be designed in such a way that $r$ is “large enough” until $r$ itself becomes “large enough” where the “large enoughness” for both $r$ and $\hat{\theta}$ are in a nonlinear function sense based on a Lyapunov analysis. Specifically, the state-dependent form of these two “large enough” functions are to be designed based on Lyapunov analysis such that desired Lyapunov inequalities hold both under the case that the time derivative of $r$ is large enough and the case that $r$ is large enough. For this purpose, the dynamics of $r$ are designed to be of the form

$$
\frac{dr}{dt} = \lambda \left( R \left( x_1, \hat{\theta}, \frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}} \right) + a(r) - r \right) [\Omega(r, x, \hat{\theta}, \dot{\theta}, t) + \ddot{a}(r)]
$$

(39)

where $\lambda : \mathcal{R} \rightarrow \mathcal{R}^+$ is to be picked to be a continuous function and $\ddot{a}(r)$ denotes $\frac{da}{dt}.$ In particular, the continuous function $\lambda$ is to be picked such that $\lambda(s) = 1$ for all $s \geq 0$ and $\lambda(s) = 0$ for all $s \leq -c_r$ where $c_r$ can be picked to be any positive constant. With the continuous function $\lambda$ picked to satisfy this property, it can be seen that $\frac{dr}{dt}$ is “large” (i.e., $\frac{dr}{dt} = \Omega + \ddot{a}$) when $r$ is relatively small and on the other hand, when $r$ becomes “large” (i.e., $r \geq R + \epsilon_c$), $\frac{dr}{dt}$ goes to 0. The functions $R$ and $\Omega$ are picked as

$$
R \left( x_1, \dot{\theta}, \frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}} \right) = \max \left\{ \frac{r}{\varphi_u} \left[ w_1 \left( x_1, \hat{\theta}, \frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}} \right) \right] \right\}
$$

$$
\Omega(r, x, \hat{\theta}, \dot{\theta}, t) = \max \left\{ \frac{r}{\varphi_u} \left[ w_1 \left( x_1, \hat{\theta}, \frac{\dot{\phi}}{\phi_{(1, 2)(x, t)}} \right) \right] \right\}
$$

(40)

(41)
The function $R$ is chosen such that when $r \geq R$, two properties hold:

1. The negative term involving $\psi_3\phi_2(\beta_2,\beta_1)z^2$ in the right hand side of (32) dominates over the positive $rw_1\phi_1(\beta_2,\beta_1)\eta^2$, $r\phi_1(\beta_2,\beta_1)\eta^2$, and $r\psi_3\phi_2(\beta_2,\beta_1)\eta^2$ terms, but with the unknown constant $\theta^* > 0$ replaced by the dynamic adaptation parameter $\hat{\theta}$ and with $\psi(t)$ replaced by $\hat{\psi}_a$. The replacement of $\theta^*$ by $\hat{\theta}$ will be seen to be motivated by the design of the dynamics of $\hat{\theta}$ below that ensures that $\hat{\theta}$ will be greater than or equal to $\theta^*$ after some finite amount of time less than the prescribed time $T$. The replacement of $\psi(t)$ by $\hat{\psi}_a$ will also be seen to be motivated by the design of the dynamics of $\hat{\theta}$ that will ensure that $\hat{\theta} \geq a(\tau)$ for all time $\tau$, which implies that $\psi(t) \leq \hat{\psi}_a$ since $a(\tau) = \hat{\psi}_a\psi(t)$.

2. The negative term involving $\chi^2\phi_3(\beta_1,\beta_2)$ in (32) dominates over the positive term involving $\frac{\varphi_1(\beta_1,\beta_2)}{\psi_3}(\beta_2,\beta_1)x_1^2$, but again with the unknown constant $\theta^*$ replaced with the dynamic adaptation parameter $\hat{\theta}$ and with $\psi(t)$ replaced by $\hat{\psi}_a$.

The function $\Omega$ is chosen such that when $\frac{dr}{dt} \geq \Omega$, two properties hold:

1. The negative term involving $\psi_3 \frac{dr}{dt} \eta^2$ in the right hand side of (32) dominates over the positive $rw_1\phi_1(\beta_2,\beta_1)\eta^2$, $r\phi_1(\beta_2,\beta_1)\eta^2$, and $r\psi_3\phi_2(\beta_2,\beta_1)\eta^2$ terms, but with the unknown constant $\theta^*$ replaced by $\hat{\theta}$ and with $\psi(t)$ replaced by $\hat{\psi}_a$.

2. The negative term involving $\frac{1}{2r^2} \frac{dr}{dt} x^2$ dominates over the positive term involving $\frac{\varphi_1(\beta_1,\beta_2)}{\psi_3}(\beta_2,\beta_1)x_1^2$, but again with $\theta^*$ replaced with $\hat{\theta}$ and with $\psi(t)$ replaced by $\hat{\psi}_a$.

Hence, effectively, when $r$ is relatively small (i.e., when $r < R$), the derivative $\frac{dr}{dt}$ is relatively large (i.e., $\frac{dr}{dt} \geq \Omega$) by the form of the dynamics of $r$ in (39) and therefore the negative terms involving $\psi_3 \frac{dr}{dt} \eta^2$ and $\frac{1}{2r^2} \frac{dr}{dt} x^2$ in the right hand side of (32) dominate over the positive $rw_1\phi_1(\beta_2,\beta_1)\eta^2$, $r\phi_1(\beta_2,\beta_1)\eta^2$, and $r\psi_3\phi_2(\beta_2,\beta_1)\eta^2$ terms. On the other hand, when $r$ is sufficiently large (i.e., when $r \geq R$) the negative terms involving $\psi_3 \frac{dr}{dt} \eta^2$ and $\frac{1}{2r^2} \frac{dr}{dt} x^2$ in the right hand side of (32) dominate over the positive $rw_1\phi_1(\beta_2,\beta_1)\eta^2$, $r\phi_1(\beta_2,\beta_1)\eta^2$, and $r\psi_3\phi_2(\beta_2,\beta_1)\eta^2$ terms. Note that the factor $\frac{1}{a_0}$ has been introduced in the design of the function $\Omega$ in (41) as an upper bound for the term $\frac{dr}{dt}$ that appears multiplied with the terms in (32) that need to be dominated by the negative term involving $\psi_3 \frac{dr}{dt} \eta^2$. It is known by the construction of the function $\alpha$ in Section 3.2 that $\alpha(\tau) \geq a_0$ for all $\tau$.

As noted in Section 3.5, the appearance of $\hat{\theta}$ in the dynamics of $r$ is written in terms of the combination given as $\frac{dr}{dt}$ rather than simply $\hat{\theta}$ since this combination can be shown (Lemma 2 in Section 4) to grow at most polynomially as a function of the transformed time variable $\tau$; this property will be used in showing (Lemma 3 in Section 4) that $R$ and therefore $r$ grow at most polynomially as a function of $r$.

With the dynamics of the scaling parameter $r$ as designed in (39)–(41), it is seen that $\frac{dr}{dt} \geq 0$ for all time $\tau \geq 0$. It can be seen that the property $r \geq a(\tau)$ for all time $\tau$ in the maximal interval of existence of solutions follows from the following observations: (a) the initial condition for $r$ is picked such that $r(0) \geq a(0)$; (b) from (39), we have $\frac{dr}{dt} \geq \hat{a}(\tau) = \frac{dr}{dt}$ at any time instant at which $r \leq R + a(\tau)$ where $R \geq 0$ from (40).

The dynamics of $\hat{\theta}$ are designed as (the motivation for this form of the dynamics of $\hat{\theta}$ can be seen from the augmented Lyapunov function $V$ in (53) and its derivative (54)):

$$\frac{d\hat{\theta}}{dt} = \hat{a}(\tau) + a_0 \frac{\chi(r, x, \hat{\theta}, t)}{a(\tau)} \text{ with } \hat{\theta}(0) \geq \max\{1, a(0)\}$$

where

$$\chi(r, x, \hat{\theta}, t) = \phi_r(x_3, \hat{\theta})q_3(x_1)x_1^2 + \frac{\hat{\theta}}{r \psi_a} \phi_r(x_3, \hat{\theta})q_3(x_1) + q_3(x_1, \hat{\theta})x_1^2$$

$$+ rw_2(x_1, \hat{\theta})q_3(x_3, \hat{\theta})x_1^2 + r \frac{\hat{\theta}}{r \psi_a} \phi_r(x_3, \hat{\theta})q_3(x_1)|\eta|^2$$
where \(c_f\) is any positive constant. From (42), it is seen that \(\dot{\theta}(0) \geq a(0)\) and also \(\frac{d\theta}{dr} \geq \frac{d\alpha(t)}{dr}\) for all times \(r\). Hence, \(\dot{\theta} \geq a(r)\) at all times \(r\) in the maximal interval of existence of solutions, which confirms the motivation for the replacement of \(\varphi(t)\) by \(\frac{\theta}{\varphi}\) in several terms that were considered in the designs of the functions \(R\) and \(\Omega\) in (40) and (41) as discussed earlier in this section.

Now, noting the remaining non-negative/sign-indefinite terms in the right hand side of (32), that is, the terms in (32) involving \(2r^2\frac{h - \gamma(t)}{\gamma(t)}\eta^TP_cBK_c\eta \) and \(\frac{\varphi}{\gamma(t)}|\eta^TP_c|Bh\psi(\xi,\zeta,0)\), the component \(u_2\) of the overall control law (9) is designed such that the term \(2\eta^TP_cBh\frac{\varphi}{\gamma(t)}\) in the right hand side of (32) dominates over these two terms. Hence, the component \(u_2\) of the control input signal as defined in (9) is designed as:

\[
u_2 = -S(\eta^TP_cB)\left\{ |K_\eta|r^\alpha \left[ \frac{1}{\gamma_1(t)} + \dot{\theta} \right] + \Gamma(x_1)\phi(\xi,\zeta,0) \right\}. 
\] (44)

In (44), \(\dot{\theta}_1\) is an adaptation parameter, the dynamics of which will be designed below in (52). The dynamics of \(\dot{\theta}_1\) will be designed such that \(\dot{\theta}_1\) is a monotonically non-decreasing signal as a function of time and the initial condition of \(\dot{\theta}_1\) will be picked such that \(\dot{\theta}_1(0) \geq 0\). Hence, \(\dot{\theta}_1(t) \geq 0\) for all time \(t\). In (44), \(S(\cdot)\) is used to denote the sign of the scalar argument as per the notations summarized in Section 2. Analogous to (16), the time-dependent function \(\gamma_1(t)\) is used in place of \(h\) in the denominator of the term involving \(|K_\eta|r^\alpha\) in (44) since the function \(h\) is unknown.

Consider two cases given by: (a) \(r \geq R\); (b) \(r < R\). Under case (b), we have \(\frac{dr}{dr} \geq \Omega\) from the form of the dynamics of \(r\) in (39) corresponding to the property as discussed above that the dynamics (39) ensures that either \(r\) or its derivative \(\frac{dr}{dr}\) is “large”. Using (38)–(44), it is seen that in both cases (a) and (b), (32) reduces to

\[
\frac{dV}{dr} \leq \frac{1}{a(r)} \left\{ -\frac{1}{2} \eta^2 + \frac{\varphi(t)K_\eta}{\gamma(t)} \right\} + \frac{\varphi(t)K_\eta}{\gamma(t)} \sum_{i=1}^{n} \left[ \frac{h}{\gamma_1} + h\dot{\theta}_1 - \frac{|h - \gamma_1(t)|}{\gamma_1(t)} \right]. 
\] (45)

It was noted above that the dynamics (42) and (39) for \(\dot{\theta}\) and \(r\), respectively, imply that \(r \geq a(r)\) and \(\dot{\theta} \geq a(r)\) for all time \(r\). Hence, noting that \(a(r) = \varphi(t)\varphi_a\), (45) yields

\[
\frac{dV}{dr} \leq \frac{1}{a(r)} \left\{ -\frac{1}{2} \eta^2 + \frac{\varphi(t)K_\eta}{\gamma(t)} \right\} + \frac{\varphi(t)K_\eta}{\gamma(t)} \sum_{i=1}^{n} \left[ \frac{h}{\gamma_1} + h\dot{\theta}_1 - \frac{|h - \gamma_1(t)|}{\gamma_1(t)} \right]. 
\] (46)

Hence, noting the various terms on the right hand side of (46) and comparing with the definition of the Lyapunov function \(V\) given in (30), we have

\[
\frac{dV}{dr} \leq -\kappa V + (\theta^* - \dot{\theta}) \frac{\chi(r,x,\dot{\theta},t)}{a(r)} \left\{ -\frac{1}{a(r)} \left[ 2r^2|\eta^TP_cBK_c\eta| \left[ \frac{h}{\gamma_1} + h\dot{\theta}_1 - \frac{|h - \gamma_1(t)|}{\gamma_1(t)} \right] \right] \right\} 
\] (47)

where

\[
\kappa = \min \left\{ \frac{\zeta_0\sigma}{2}, \frac{\varphi_a\sigma}{2\lambda_{\max}(P_c)}, \frac{1}{2\varphi_a} \right\}. 
\] (48)

Noting that \(h\) and \(\gamma_1\) are non-negative, noting that \(h \geq h^*\), and defining \(\theta^*_1\) to be the unknown constant given as

\[
\theta^*_1 = \frac{1}{h^*}. 
\] (49)

(47) yields

\[
\frac{dV}{dr} \leq -\kappa V + (\theta^* - \dot{\theta}) \frac{\chi(r,x,\dot{\theta},t)}{a(r)} + h(\theta^*_1 - \dot{\theta}) \frac{\chi_1(r,x,\dot{\theta},t)}{a(r)} 
\] (50)
where
\[ \chi_1(r, x, \hat{\theta}, t) = 2r^2|\eta^TP_cBK_0\eta| \.
\] (51)

Based on the form of the dynamics in (50), the dynamics of \( \hat{\theta}_1 \) are designed as
\[ \frac{d\hat{\theta}_1}{dt} = \frac{c_{\theta_1}\chi_1(r, x, \hat{\theta}, t)}{a(r)}. \] (52)

The temporal forcing term \( \tilde{a}(r) \) is incorporated into the dynamics of \( \hat{\theta}_1 \) in (42) to ensure that \( \hat{\theta} \geq a(r), \) a property that is required to be able to infer (46) from (45). Noting that \( \frac{d}{dt}(\hat{\theta} - a(r)) = c_{\theta_1}\chi_1(r, x, \hat{\theta}, t)/a(r) \) from (42), it is seen from (50) that the signal \( (\hat{\theta} - a(r)) \) would suffice as the dynamic adaptation state variable to address the uncertain parameter \( \theta^* > 0. \)

Hence, considering an “augmented” Lyapunov function \( \overline{V} \) that is defined as the sum of \( V \) and an additional component involving \( (\hat{\theta} - a(r) - \theta^*)^2 \) as well as a quadratic component in terms of \( (\hat{\theta}_1 - \theta^*_1) \), that is,
\[ \overline{V} = V + \frac{1}{2c_{\theta_0}}(\hat{\theta} - a(r) - \theta^*)^2 + \frac{h}{2c_{\theta_1}}(\hat{\theta}_1 - \theta^*_1)^2, \] (53)

we have from (42) to (46) and noting that \( \chi(r, x, \hat{\theta}, t) \geq 0: \)
\[ \frac{d\overline{V}}{dt} \leq -\frac{1}{2}x^2_1\phi(1, 2) - \frac{1}{2}v_0\phi(2, 3)|\eta|^2 - \frac{K_2K_3}{2\phi_0r^{2\epsilon-3}}\phi \varepsilon. \] (54)

While, as we will see in Section 4, (54) can be used to show existence of solutions of the closed-loop dynamical system over the time interval \( r \in [0, \infty) \), it will not directly enable showing exponential convergence (since the quadratic terms involving the adaptation parameters do not appear on the right hand side of (54)). Showing exponential convergence of \( x_1 \) and \( \eta \) to 0 will be crucial in proving closed-loop stability since, for example, the boundedness of \( u_1 \) will be proved by showing that \( r \) grows at most polynomially as a function of the time \( r \) while \( \eta \) goes to 0 exponentially. Hence, to show exponential convergence, we will also want to ensure that an inequality of the form \( \frac{d\overline{V}}{dt} \leq -\kappa V \) is also satisfied at least after a sub-interval of the overall time interval \( r \in [0, \infty). \) From (47), we will for this purpose want to ensure that after some finite time, the following inequalities are satisfied:
\[ \hat{\theta} \geq \theta^* \] (55)
\[ \frac{h}{\gamma_1} + \hat{h}\hat{\theta}_1 \geq \frac{|h - \gamma_1|}{\gamma_1} \] (56)

are satisfied. From the dynamics of \( \hat{\theta}_1 \) in (42), it will be seen that the inequality (55) will be satisfied after some finite time. To ensure that (56) is satisfied after some finite time, we pick the function \( \gamma_1 \) such that \( \frac{1}{\gamma_1} \) goes to \( \infty \) as \( t \to T \), that is, as \( r \to \infty \), by defining
\[ \gamma_1(t) = \frac{1}{c_{\gamma_1}a(a(t)) + \tilde{c}_{\gamma_1}} \] (57)

with \( c_{\gamma_1} \) being any positive constant and \( \tilde{c}_{\gamma_1} \) being any non-negative constant. From the conditions imposed on the function \( a \) and the definition of the function \( \gamma_1 \) as (57), it is seen that \( \frac{1}{\gamma_1} \) grows at most polynomially as a function of the time \( r \). As a side observation, it is of interest to note that the above designs of the dynamics of \( \hat{\theta} \) and \( \hat{\theta}_1 \) and the time-varying function \( \gamma_1 \) can be interpreted from the viewpoint of adaptive control in terms of adaptations to account for uncertain terms in the system dynamics. Specifically, for example, the dynamics of \( \hat{\theta} \) are designed such that after some finite time, \( \hat{\theta} \) is bigger than the uncertain parameter \( \theta^* \). The function \( \gamma_1 \) is designed such that after some finite time, \( \gamma_1 \) is small enough relative to the unknown function \( h \) such that (57) is satisfied.
4 STABILITY ANALYSIS AND MAIN RESULT

In this section, a sequence of lemmas is established based on the adaptive controller design in Section 3. Let the maximal interval of existence of solutions of the closed-loop system formed by the system (1) in closed loop with the designed dynamic partial-state-feedback controller be \([0, \tau_f]\) in terms of the new time variable \(\tau\). Local existence and uniqueness of solutions of the closed-loop system follow from the assumptions on the functions appearing in the system dynamics and the continuity (by construction) of the functions appearing in the overall dynamic controller. From Lemmas 1–4, it is shown that \(\tau_f = \infty\), that is, solutions exist over the infinite time interval \(\tau \in [0, \infty)\). Thereafter, various convergence properties are shown in Lemmas 5–7. The main prescribed-time stabilization result of this paper (Theorem 1) is then stated and proved based on the Lemmas 1–7.

**Lemma 1.** The signals \(V, x_1, \sqrt{r|\eta|}, \frac{V}{\sqrt{r}}, (\hat{\theta} - a(\tau)), \) and \(\hat{\theta}_1\) are uniformly bounded over \([0, \tau_f]\).

**Proof.** From (54), it is seen that \(\frac{dV}{d\tau} \leq 0\) implying that the augmented Lyapunov function \(\overline{V}\) is uniformly bounded over the maximal interval of existence of solutions \([0, \tau_f]\) of the closed-loop system. Lemma 1 therefore follows by noting the definitions of \(V\) and \(\overline{V}\) in (30) and (53), respectively.

**Lemma 2.** The signals \(\hat{\theta}(a^{-1}(\tau))\) and \(\frac{\hat{\theta}(a^{-1}(\tau))}{\phi_{1,2}(x_1, a^{-1}(\tau))}\) grow at most polynomially in the time variable \(\tau = a(t)\) as \(\tau \to \infty\).

**Proof.** It was seen as part of Lemma 1 that \((\dot{\theta} - a(\tau))\) is uniformly bounded over \([0, \tau_f]\). Noting that \(a(\tau)\) grows at most polynomially in \(\tau\) due to the conditions imposed in Section 3.2 on the choice of the function \(a\), it follows that \(\hat{\theta}\) grows at most polynomially as a function of time \(\tau\). Noting the dynamics of the adaptation variable \(\hat{\theta}\) in (42) and using the Assumptions 1 and 5, it is seen that

\[
\frac{\dot{\hat{\theta}}}{\phi_{1,2}(x_1, t)} \leq a(\tau)\bar{a}(\tau) + c_\theta \left( q_2(x_1)x_1^2 + \frac{\dot{\theta}}{r_{\phi_a}}[q_{2z}(x_1) + q_{3z}(x_1, \hat{\theta})]x_1^2 + \frac{r}{\phi_{1,2}(x_1)}w_2(x_1, \hat{\theta})|\eta|^2 + r\frac{\hat{\theta}}{\phi_a\phi_{1,2}(x_1)}w_{2z}(x_1)|\eta|^2 \right).
\]

(58)

Note that \(a(\tau)\) and \(\bar{a}(\tau)\) grow at most polynomially in \(\tau\) by construction (Section 3.2). It was seen in Lemma 1 that \(x_1^2\) and \(r|\eta|^2\) are uniformly bounded over \([0, \tau_f]\). It was noted above that \(\hat{\theta}\) grows at most polynomially as a function of \(\tau\). From the definition of \(q_{3z}\) in (34) and \(w_2\) in (36), \(\hat{\theta}\) appears polynomially (as terms involving \(\hat{\theta}^2\)) in both \(q_{3z}\) and \(w_2\). Therefore, it follows from (58) that \(\frac{\dot{\hat{\theta}}}{\phi_{1,2}(x_1, t)}\) grows at most polynomially in the time \(\tau\).

**Lemma 3.** The signal \(r(a^{-1}(\tau))\) grows at most polynomially in time \(\tau\) as \(\tau \to \infty\).

**Proof.** Using Lemma 1 and Lemma 2 above, it is seen that \(q_{3z}(x_1, \hat{\theta}), w_1(x_1, \hat{\theta}, \frac{\dot{\theta}}{\phi_{1,2}(x_1, t)}), w_2(x_1, \hat{\theta}), \) and \(w_{2z}(x_1)\) defined in (34), (35), (36), and (37), respectively, grow at most polynomially in time \(\tau\). Hence, it is seen from (40) that

\[
R(x_1, \hat{\theta}, \frac{\dot{\theta}}{\phi_{1,2}(x, t)})\]

(40)

grows at most polynomially with time \(\tau\). From (39), it is seen that \(r = 0\) at any time instant \(\tau\) at which \(r \geq R(x_1, \hat{\theta}, \frac{\dot{\theta}}{\phi_{1,2}(x_1, t)}) + a(\tau) + \epsilon_r\). By the conditions imposed on the function \(a(\tau)\) in Section 3.2, \(a(\tau)\) is also polynomially upper bounded in \(\tau\). Hence, \(R(x_1, \hat{\theta}, \frac{\dot{\theta}}{\phi_{1,2}(x_1, t)}) + a(\tau) + \epsilon_r\) and therefore \(r\) as well grow at most polynomially as a function of time \(\tau\).

**Lemma 4.** The solution trajectories for the closed-loop dynamical system formed by the system (1) and the designed dynamic partial-state-feedback controller exist over the time interval \(\tau \in [0, \infty)\).

**Proof.** It is seen from Lemma 1 that \(x_1, \sqrt{r|\eta|}, \frac{V}{\sqrt{r}}, \) and \(\hat{\theta}_1\) are uniformly bounded over \([0, \tau_f]\) while it is seen from Lemmas 2 and 3 that \(\hat{\theta}\) and \(r\) grow at most polynomially in \(\tau\). Hence, it follows that all closed-loop signals are bounded over any finite time interval \(\tau \in [0, \tau_f]\) and therefore it is seen that solutions to the closed-loop dynamical system exist over the time interval \(\tau \in [0, \infty)\), that is, \(\tau_f = \infty\).

**Lemma 5.** A finite non-negative constant \(\tau_0\) exists such that for all time instants \(\tau \geq \tau_0\), the inequality \(\frac{dV}{d\tau} \leq -\kappa V\) is satisfied with the positive constant \(\kappa\) defined as in (48).
Proof. From the designed dynamics of the adaptation parameter \( \hat{\theta} \) in (42), it was noted in Section 3.6 that \( \dot{\hat{\theta}} \geq a(\tau) \) for all time \( \tau \), implying (due to the properties of the function \( a(\tau) \) defined in Section 3.2) that \( \dot{\hat{\theta}} \) goes to \( \infty \) as \( \tau \to \infty \). Hence, a finite positive constant \( \tau_1 \) exists such that (55) is satisfied for all time \( \tau \geq \tau_1 \). Similarly, from the construction of \( a(\tau) \) and the definition of \( \gamma_1 \), a finite constant \( \tau_2 \) exists such that (56) is satisfied for all times \( \tau \geq \tau_2 \). Hence, defining \( \tau_0 = \max(\tau_1, \tau_2) \), it is seen from (47) that for all times \( \tau \geq \tau_0 \), we have \( \frac{dV}{d\tau} \leq -\kappa V \) with \( \kappa \) given in (48).

Lemma 6. The signals \( V, x_1, \sqrt{r|\eta|}, \) and \( \frac{V}{\tau^2} \) go to 0 exponentially as \( \tau \to \infty \).

Proof. From Lemma 5, it is seen that a finite positive constant \( \tau_0 \) exists such that for all time instants \( \tau \geq \tau_0 \), the inequality \( \frac{dV}{d\tau} \leq -\kappa V \) is satisfied. Therefore, \( V \) goes to 0 exponentially as \( \tau \to \infty \). From the definition of the Lyapunov function \( V \) in (30), it is seen that \( x_1, \sqrt{r|\eta|}, \) and \( \frac{V}{\tau^2} \) go to 0 exponentially as \( \tau \to \infty \).

Lemma 7. The signal \( \eta \) goes to zero exponentially as \( \tau \to \infty \). Also, the signal \( u \) is uniformly bounded over the time interval \( \tau \in [0, \infty) \).

Proof. From Lemma 6, we see that \( \eta \) goes to 0 exponentially as \( \tau \to \infty \) since \( r \geq 1 \) for all time \( \tau \). Since, from Lemma 2, \( \dot{\hat{\theta}} \) grows at most polynomially while from Lemma 6, \( x_1 \) goes to 0 exponentially, it is seen that \( \zeta(x_1, \hat{\theta}) \) defined in (12) goes to 0 exponentially as \( \tau \to \infty \). Since \( r \) grows at most polynomially in time \( \tau \) from Lemma 3 while \( \eta \) goes to 0 exponentially, it follows from the definition of \( \eta_2, \ldots, \eta_n \) in (11) that \( x_2, \ldots, x_n \) go to 0 exponentially as \( \tau \to \infty \). Also, \( r^i \eta \) goes to 0 exponentially as \( \tau \to \infty \). Hence, noting that \( \frac{1}{r_1} \) grows at most polynomially from (57), it is seen that \( u_1 \) defined in (10) goes to 0 exponentially as \( \tau \to \infty \). Similarly, it is seen from the definition of the control law component \( u_2 \) in (44) that \( u_2 \) also remains uniformly bounded over the time interval \( \tau \in [0, \infty) \). Therefore, the overall control signal \( u = u_1 + u_2 \) defined in (9) is uniformly bounded over the time interval \( \tau \in [0, \infty) \).

Theorem 1. Under the Assumptions 1–6 given in Section 2, the closed-loop dynamical system formed by the given system (1) and the designed dynamic partial-state-feedback controller (of dynamic order 3 with state variables \( r, \hat{\theta}, \) and \( \hat{\theta}_1 \)) from Section 3 with \( T > 0 \) being arbitrarily picked by the designer satisfies the following property: starting from any initial conditions for the state variables \( x \) and \( z \), the signals \( x(t), z(t), \) and \( u(t) \) satisfy \( \lim_{\tau \to T} |x(t)| = 0, \sup_{t \in [0, T]} |u(t)| < \infty, \) \( \lim_{\tau \to T} |z_a(t)| = 0, \) and \( \sup_{t \in [0, T]} |z_b(t)| < \infty \).

Proof. Noting that \( x_2 = \eta_2 - \zeta \) and \( x_1 = \eta_1 r^{-1} \), \( i = 3, \ldots, n \), it follows from the Lemmas 3, 6, and 7 that \( x = [x_1, \ldots, x_n]^T \) goes to 0 exponentially as \( \tau \to \infty \). Also, it follows from Lemmas 3 and 6 and Assumption 6 that \( z_a \) goes to 0 as \( \tau \to \infty \). From Lemma 7 and Assumption 6, it is seen that \( u \) and \( z_b \) are uniformly bounded over time interval \( \tau \in [0, \infty) \). Since \( \tau \to \infty \) corresponds to \( t \to T \), these properties hold as \( t \to T \).

Remark 1. The designed prescribed-time stabilizing adaptive dynamic controller has dynamic order 3 with the state variables of the dynamic partial-state-feedback controller being the dynamic high-gain scaling parameter \( r \) and the dynamic adaptation parameters \( \dot{\hat{\theta}} \) and \( \dot{\hat{\theta}}_1 \). For the reader’s convenience, the overall control algorithm developed in this paper is summarized below:

1. Design \( P_\tau, k_2, \ldots, k_n \) based on the coupled Lyapunov inequalities (20) as in Section 3.4.
2. Design the time scale transformation \( \tau = a(\tau) \) as in Section 3.2. Define \( a(\tau) = \frac{d\tau}{d\tau} \). Define the function \( \gamma_1 \) as in (57).
3. Design the function \( \zeta \) as in (12) and (38), i.e., \( \zeta(x_1, \hat{\theta}) = \hat{\theta}_1 \zeta_1(x_1) \) and \( \zeta_1(x_1) = 4 \max \left\{ \zeta, q_1(x_1) + q_2(x_1) + \frac{1}{\eta_1} q_3(x_1) \right\} \)
where \( q_1, q_2, \) and \( q_3 \) are as in (33) and (34).
4. Design the dynamics of \( r \) as in (39) where the functions \( R \) and \( \Omega \) are shown in (40) and (41), respectively, where \( q_{3z}, w_1, w_2, \) and \( w_{3z} \) are as in (34), (35), (36), and (37), respectively.
5. Design the dynamics of \( \hat{\theta} \) and \( \hat{\theta}_1 \) as in (42) and (52), respectively, where the functions \( \chi \) and \( \chi_1 \) are as in (43) and (51).
6. Define the scaled state vector \( \eta \) as in (11), that is, \( \eta_2 = \frac{\zeta + \chi(x_1, \hat{\theta})}{r}; \eta_i = \frac{\eta_i}{r_1}, i = 3, \ldots, n \).
7. Define the control law \( u = u_1 + u_2 \) with \( u_1 \) and \( u_2 \) as in (10) and (44), that is, \( u_1 = \frac{\eta}{\tau_1(t)} K_c \eta \) and \( u_2 = -S(\eta^T P_c B) \left\{ |K_c| r^p \left[ \frac{1}{\tau_1(t)} + \hat{\theta}_1 \right] + \Gamma(x_1) \phi_i(x_2, 0) \right\} \).
Remark 2. It is to be noted that the fact the controller state variables \( r \) and \( \hat{\theta} \) go to \( \infty \) as \( t \to T \) is an intended property of the closed-loop system and is enforced by the presence of the time-dependent forcing terms in the dynamics of \( r \) and \( \hat{\theta} \). These time-dependent terms and the structure of the dynamics of \( r \) and \( \hat{\theta} \) are designed such that the growth rate of these state variables is slow enough (i.e., at most polynomial) such that the system state variables \( x \) and \( z \) and the control input \( u \) go to zero as \( t \to T \). Hence, although the controller state variables \( r \) and \( \hat{\theta} \) go to \( \infty \) as \( t \to T \), the rate at which these state variables go to \( \infty \) is regulated through the design of the dynamics of these state variables such that this growth rate is slow enough to preserve well-behavedness of the other state signals in the closed-loop system. The controller state variable \( \hat{\theta}_1 \) remains uniformly bounded as noted in Lemma 1. Also, the deviation between \( \hat{\theta} \) and \( \alpha(\tau) \) is uniformly bounded as also noted in Lemma 1.

Remark 3. As seen in the analysis above of the properties of the closed-loop system, multiple signals in the closed-loop system such as \( r, \alpha(\tau), \) and \( \hat{\theta} \) go to \( \infty \) as \( t \to T \) (but, with at-most-polynomial growth as a function of the time variable \( \tau \)). The polynomial growth of these signals implies that effective control gains go to \( \infty \) as \( t \to T \). This is essentially as expected since as has been discussed in References 21,22,28, it is to be noted that any approach for regulation in prescribed finite time (including optimal control designs with a terminal constraint and sliding mode based controllers with time-varying gains) will share the property that the effective control gains grow unbounded as \( t \) approaches the prescribed time \( T \). However, it is to be noted that, as proved above, the actual control signal \( u \) does indeed remain bounded over the prescribed time interval \([0, T]\). Also, \( x \) goes to zero as \( t \to T \). Nevertheless, it is possible that numerical challenges in the implementation of the controller can be posed by the fact that the effective control gains grow unbounded as \( t \to T \). As noted in References 31-33, numerical difficulties can be alleviated using several techniques such as adding a dead zone on the state \( x \), adding a saturation on the control gains, implementing the dynamics of the high-gain scaling parameter \( r \) via a temporally scaled version \( \tilde{r} = r(\tilde{t}) \), and setting the effective terminal time \( \tilde{T} \) in controller implementation to be a constant slightly larger than the desired prescribed time \( T \).

Remark 4. While the control design in this paper addressed the prescribed-time stabilization problem over the time interval \([0, T]\), the designed controller can be combined with a separate controller over the time interval \([T, \infty)\) if the behavior of the closed-loop system after time \( t = T \) is relevant (e.g., if it is desired in the particular application that the state be retained at/around the origin under disturbance inputs and other uncertainties). This separate controller can, for example, be designed using backstepping or dynamic scaling-based control design approaches, to retain \( x \) at/around zero under the uncertainties/disturbances.

5 | ILLUSTRATIVE EXAMPLES

5.1 | “Synthetic” fifth-order system

Consider the fifth-order system

\[
\begin{align*}
\dot{x}_1 &= (1 + x_1^2)x_2 \\
\dot{x}_2 &= (1 + x_1^4)x_3 + \theta_a \cos(x_2z_1)x_2 + \theta_b(1 + \cos(tu))e^{x_1^2}x_2^2 \sin(z_2) \\
\dot{x}_3 &= \left[1 + \frac{1}{2} \sin(t) \cos(z_2) + x_3^4(1 + e^{-|z_2|})\right]u + \theta_c x_1^2 \cos(x_2z_1)x_2 + \theta_d x_2^2 + \theta_e(1 + \cos(z_1)) \\
\dot{z}_1 &= -100z_1 + z_2 \\
\dot{z}_2 &= -100z_2 + x_3^4 + u
\end{align*}
\]

(59)

where \( \theta_a, \theta_b, \theta_c, \) and \( \theta_d \) are uncertain parameters with no magnitude bounds required to be known and \( \theta_e \) is an uncertain parameter with known bound \( \theta_e = 5 \). This system is of the form (1) with \( \phi_{1,3}(x_1) = 1 + x_1^2, \phi_{2,3}(x_1) = 1 + x_1^4, \) and \( h(z, x, u, t) = 1 + \frac{1}{2} \sin(t) \cos(z_2) + x_3^4(1 + e^{-|z_2|}) \). Assumption 1 is satisfied with the constant \( \sigma = 1 \). Assumption 2 is satisfied with \( \theta = \max\{\theta_a, 2\theta_b, \theta_c, \theta_d\}/c_\beta, \Gamma(x_1) = c_\beta \max(e^{x_1^2}|x_1|, 1 + x_1^2), \phi_e = \Theta_e(1 + \cos(z_1))/\Gamma(x_1), \phi_{(1,1)} = \phi_{(3,3)} = 0, \phi_{(3,1)} = |x_1|, \) and \( \phi_{(2,1)} = \phi_{(2,2)} = \phi_{(3,2)} = 1, \) with \( c_\beta \) being any positive constant. It is seen that inequalities (4) in Assumption 2 are trivially satisfied. Note that the forms of the various uncertain terms in the dynamics are not required to be known as long as bounds of the form in Assumption 2 are known to be satisfied. Assumption 3 is satisfied with \( h = 0.5 \). Assumption 4 is trivially satisfied since \( n = 3 \). Assumption 5 is satisfied with \( \bar{\phi}_{(1,2)} = \frac{3}{2} \) and \( \bar{\phi}_{(1,2)} = \frac{1 + x_2^4}{1 + x_2^4} \). Noting that the
z dynamics is a stable linear system with \( x_3 \) and \( u \) as inputs, it is seen that Assumption 6 is satisfied with \( z_a \) being empty, \( z_b = [z_1, z_2]^T \), \( \phi_0 = 2\theta_c \), and with the other terms in Assumption 6 such as \( \phi_{c(0)} \) and \( \phi(t) \) being not relevant with \( z_a \) being empty. Using the constructive procedure in References 37, 46 for solution of coupled Lyapunov inequalities, a symmetric positive-definite matrix \( P_c \) and functions \( k_2 \) and \( k_3 \) can be found to satisfy (20) as \( P_c = \tilde{\alpha}_c \begin{bmatrix} 3 & 0.75 \\ 0.75 & 1 \end{bmatrix} \), \( k_2 = 5\phi_{(2,3)} \), and \( k_3 = 4\phi_{(2,3)} \), and with \( v_c = 1.975\tilde{a}_c \), \( u_c = 1.5\tilde{a}_c \), and \( v_c = 4.5\tilde{a}_c \) with \( \tilde{a}_c \) being any positive constant. Since \( z_a \) is empty for this system, the time scale transformation \( a(t) \) can be picked to be any class \( D_{[0, T]} \) function as discussed in Section 3.2. In particular, we pick \( a(t) \) to be of the form \( a(t) = \frac{a_0 t^\gamma}{1 - t^\gamma} \) yielding the function \( a(\tau) = a'(a^{-1}(\tau)) = a_0 \left( \frac{1}{a_0 T} + 1 \right)^2 \). The function \( \gamma_1 \) is picked as in (57). Defining \( \eta_2 = \frac{x_2 + \theta_1 c_1}{r} \) and \( \eta_3 = \frac{x_3}{r^2} \), we have \( u_1 = -r^3[k_2\eta_2 + k_3\eta_3]/\gamma_1(t) \) from (10). Since \( \phi_{(1,1)} \) and therefore \( c_{(1,1)} \) are zero, we have \( q_2 = 0 \) from (33). Also, \( q_1 \) and therefore \( \zeta_1 \) are constants since \( \tilde{\theta}_{(1,2)} \) was found above to be a constant. Furthermore, \( q_{22}, q_{32}, \) and \( w_{22} \) are zero since \( z_a \) is empty. The control component \( u_c \) is defined as in (44) and the overall control input \( u \) is defined as \( u = u_1 + u_2 \) from (9). The dynamics of \( r \) are shown in (39) where the functions \( R \) and \( \Omega \) are computed following the procedure in Section 3 and using sharper bounds taking the specific system structure (59) into account and noting that several terms in the upper bounds vanish since \( \phi_{(1,1)} \), and so forth, are zero for this system and \( c_1 \) is a constant. The dynamics of the adaptation parameters \( \theta \) and \( \hat{\theta}_1 \) are as shown in (42) and (52).

The prescribed terminal time is picked to be \( T = 0.2 \) s. To avoid numerical issues as discussed in Remark 3, the effective terminal time \( \tilde{T} \) in the controller implementation is defined as \( \tilde{T} = 0.205 \) s. The parameters in the definitions of the time-dependent functions \( a \) and \( \gamma_1 \) are picked as \( a_0 = 0.05, c_{r1} = 0.01, \) and \( c_{r2} = 0.5 \). Also, \( \zeta_0 = 1.0, \tilde{a}_c = 0.12, \) \( c_{r1} = 0.01, \) and \( c_r = 10^{-4} \). The values of the uncertain parameters \( \theta_a, \theta_h, \theta_c, \) and \( \theta_d \), and \( \theta_e \) are picked for simulations as \( \theta_a = \theta_h = \theta_c = \theta_d = \theta_e = 2.0 \). With the initial conditions for the system state \( [x_1, x_2, x_3, z_1, z_2]^T \) specified as \( [4, 1, 1, 1]^T \) and the initial conditions for the controller state \( [\hat{\theta}, \hat{\theta}_1, r]^T \) specified as \( [1, 0, 1]^T \), the closed-loop trajectories and control input signal are shown in Figure 1. For the numerical simulation, a variable-order ordinary differential equation (ODE) solver with adaptive step size was used and a step size of 0.001 s was used for plotting the signals computed by the solver. The plots shown in Figure 1 include the state variables \( x_1, x_2, \) and \( x_3 \) of the x subsystem, the state variables \( z_1 \) and \( z_2 \) of the appended dynamics, the adaptation state variables \( \hat{\theta} \) and \( \hat{\theta}_1 \) of the dynamic controller, the dynamic scaling parameter \( r \), and the control input signal \( u \). It is seen that the states \( x \) and \( z \) go to zero as \( t \to T \) as expected, while the adaptation state \( \hat{\theta} \) and the dynamic scaling parameter \( r \) go to \( \infty \) as \( t \to T \) also as expected. It is also seen that although the effective control gain grows unbounded as \( t \to T \), the control input \( u \) itself remains bounded.

To evaluate the performance improvements enabled by the proposed approach, the closed-loop signals utilizing the controller developed in our prior approach in the conference paper\(^{34} \) are shown in Figure 2. In Reference 34, a lower bound on \( h \) was required unlike the adaptive approach developed in this paper. Additionally, as discussed in Section 1, the proposed approach enables a more general structure of the dependence of the additive term in \( x_n \) on the unmeasured state of the appended dynamics as well as of the appended dynamics as a combination of ISS and BIIB components. To implement the approach in Reference 34, we consider an assumed lower bound on the uncertain function \( h \) as 20. Also, we freeze \( \hat{\theta}_1 \) at 0 since this adaptation term was not considered in Reference 34. It is seen in Figure 2 that the resulting closed-loop performance is significantly poorer than Figure 1 with pronounced oscillations (although the signals do eventually go to zero as \( t \to T \)), underscoring the utility of the adaptive approach in this paper.

### 5.2 “Real-world” electromechanical system

Consider a DC motor connected to a load that is affected by gravity. The dynamics of this system can be written as

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{B}{J}x_2 + \frac{K_i}{J}x_3 - \frac{1}{J}m_1gd_1 \sin(x_1) \\
\dot{x}_3 &= -\frac{R}{L}x_3 + \frac{1}{L}(u + u_b) - \frac{K_b}{L}x_2
\end{align*}
\]

(60)

where \( x_1 = \dot{q} \) and \( x_2 = \ddot{q} \) are the angular position and velocity, respectively, \( x_3 = i_a \) is the armature current, \( u \) is the input voltage and \( u_b \) is an uncertain input bias, \( J \) is the motor inertia, \( B \) is the viscous friction coefficient, \( m_1 \) is the mass of the load and \( d_1 \) is the distance of the load from the axis of rotation, \( g \) is the acceleration due to gravity, \( R \) and \( L \) are
the armature resistance and inductance, respectively, \(K_r\) is the torque coefficient, and \(K_b\) is the back-emf coefficient.

To model real-world constraints, we also consider a 10-bit quantization and saturation to \(\pm 48\) V, represented by \(u = \text{sat}([u_d], 2^{b-1} - 1)V_{\text{max}}/(2^{b-1} - 1)\) where \(b = 10\), \(V_{\text{max}} = 48\). \([\cdot]\) denotes rounding to the nearest integer, \(\text{sat}(a_1, a_2)\) denotes saturation of \(a_1\) to \(\pm a_2\), and \(u_d\) and \(u\) represent the digital control input command and effective control input to the system, respectively. For simulation, we use the following parameters: \(J = 5 \times 10^{-4}\) kg m \(^2\)/s, \(K_r = 0.05\) Nm/A, \(K_b = 0.05\) V/(rad/s), \(u_0 = 0.1\) V, \(m_1 = 0.1\) kg, \(d_1 = 0.1\) m, and \(g = 9.81\) m/s \(^2\).

For the design of the controller, we assume that all parameters of the system except \(K_f\) are uncertain. It is easily seen that this system satisfies the Assumptions 1 –6. This system is of the form (1) with \(\phi_{1,2}(x_1) = 1, \phi_{1,2,3}(x_1) = \frac{K_f}{L}\), and \(h(z, x, u, t) = \frac{1}{L}\). Assumption 1 is satisfied with the constant \(\sigma = 1\). Assumption 2 is satisfied with \(\theta = \max \left(\frac{B}{J}, \frac{m_1 g d_1}{L}, \frac{B}{L}\right)\), \(\Gamma(x_1) = c_\beta = \frac{u}{\Delta V(x_1)}, \phi_{1,1} = \phi_{1,3,1} = 0, \phi_{1,2} = \phi_{2,2} = \phi_{3,2} = \phi_{3,3} = 1\), with \(c_\beta\) being any positive constant. The inequalities (4) in Assumption 2 are trivially satisfied. Assumption 3 is trivially satisfied since \(\Delta\) is a positive constant (albeit uncertain) for this system. Assumption 4 is trivially satisfied since \(n = 3\). Assumption 5 is satisfied since \(\phi_{1,2}\) and \(\phi_{1,3}\) are constants for this system. Assumption 6 is also trivially satisfied for this system. Using the constructive procedure in References 37, 46 for solution of coupled Lyapunov inequalities, a symmetric positive-definite matrix \(P_c\) and functions \(k_2\) and \(k_3\) are found to satisfy (20) as \(P_c = \tilde{a}_c \begin{bmatrix} 2.5224 & 0.2477 & 0.2477 \\ 0.2477 & 0.3880 & 0.2477 \\ 0.2477 & 0.2477 & 0.3880 \end{bmatrix}, k_2 = 15\phi_{2,2}, k_3 = 7\phi_{2,3}\), and with \(v_c = 1.0002\tilde{a}_c, v_c = 1.0025\tilde{a}_c, \text{ and } \bar{v}_c = 2.6840\tilde{a}_c\) with \(\tilde{a}_c\) being any positive constant. The time scale transformation is picked as discussed in Section 3.2 to be of the form \(a(t) = \frac{a(t)}{\tilde{a}_c(t)}\) yielding the function \(a(t) = a'(r^{-1}(r)) = a_0\left(\frac{x}{a_0 T} + 1\right)^2\).

The function \(c_\nu\) is picked as in (57). With \(\eta_2 = \frac{x + \phi_{1,2}}{r}\) and \(\eta_1 = \frac{x}{r^2}\), the control input component \(u_1\) is given from (10) as \(u_1 = -r^2[k_3\eta_2 + k_3\eta_1]/\gamma_1(t)\). For this system, we noted above that \(\phi_{1,1} = 0\) implying that \(c_\nu = 0\) and therefore \(q_2 = 0\) from (33). Since \(\tilde{c}_\nu_{1,2}\) is a constant for this system, \(q_1\) and therefore \(\zeta_1\) are constants. The terms \(q_{2u}, q_{3u}, \text{ and } w_{2u}\) are zero since \(x_0\) is empty. The control input component \(u_2\) is given in (44) and the overall control input \(u\) is defined as \(u = u_1 + u_2\). The dynamics of \(r\) are shown in (39) with the functions \(R\) and \(\Omega\) computed as in Section 3 and using sharper bounds taking the specific system structure (59) into account and noting that several terms in the upper bounds vanish since \(\phi_{1,1}\), and so forth, are zero for this system and \(\zeta_1\) is a constant. The dynamics of the adaptation parameters \(\hat{\theta}\) and \(\hat{\theta}_1\) are as given in (42) and (52).
As in the example in Section 5.1, the prescribed terminal time is picked to be $T = 0.2$ s and the effective terminal time for controller implementation is defined as $\bar{T} = 0.205$ s to avoid numerical issues as discussed in Remark 3. The parameters in the definitions of the time-dependent functions $\alpha$ and $\gamma_1$ are picked as $a_0 = 0.05$, $c_{r1} = 0.01$, and $\bar{c}_{r1} = 0.5$. Also, $\zeta_0 = 4.0$, $\bar{\alpha}_c = 0.12$, $c_0 = 10^{-4}$, $c_{\theta_1} = 0.01$, and $c_\beta = 10^{-4}$. The initial conditions for the system state $[x_1, x_2, x_3]^T$ are specified for simulation as $[0.5, 0, 0]^T$ and the initial conditions for the controller state $[\hat{\theta}, \hat{\theta}_1, r]^T$ are specified as $[1, 0.1]^T$. The closed-loop trajectories of the state variables $x_1$, $x_2$, and $x_3$ and the control input signal $u$ are shown in Figure 3. The controller state variables $\hat{\theta}$, $\hat{\theta}_1$, and $r$ are shown in Figure 4. The numerical simulation uses a variable-order ODE
FIGURE 4 Simulations for closed-loop system formed by system (60) in closed loop with the prescribed-time stabilizing adaptive controller: Controller state variables ($\hat{\theta}$, $\hat{\theta}_1$, $r$)

A solver with adaptive step size. A step size of 0.0005 s was used for plotting the signals computed by the solver. It is seen in Figures 3 and 4 that the state $x$ goes to zero as $t \to T$. While the adaptation state variable $\hat{\theta}$ and the dynamic scaling parameter $r$ go to $\infty$ as $t \to T$, implying that the effective control gains go to $\infty$ as $t \to T$, it is seen that the control input itself remains bounded.

6 | CONCLUSION

It was shown that a prescribed-time stabilizing controller can be designed for the considered general class of nonlinear uncertain systems through a control design based on a combination of: a non-smooth control component ($u_2$ which includes a term based on the sign of $\eta^T P_c B$), time-dependent forcing functions in the definitions of both $u_1$ and $u_2$, a dynamic adaptation that incorporates time-dependent forcing terms, a time scale transformation from the time variable $t$ to a new time variable $\tau$, a dynamic high-gain scaling-based control design. The class of nonlinear systems considered allows several types of uncertainties including uncertain input gain and appended dynamics that effectively generate non-vanishing disturbances as well as a general structure of state-dependent uncertain terms throughout the system dynamics. While the adaptation parameter $\hat{\theta}$ and the dynamic scaling parameter $r$ grow at most polynomially as a function of the transformed time variable $\tau$, it was shown that the system state and input remain uniformly bounded and the system state $x$ converges to 0 in the prescribed time irrespective of the initial conditions of the system. Determining if similar control design approaches can be applied to other and more general classes of systems such as general cascade structures and non-triangular and feed forward systems as well as systems with unknown sign of the control gain remain topics for further research.

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CONFLICT OF INTEREST

The authors declare no conflicts of interest associated with this manuscript.

DATA AVAILABILITY STATEMENT

Data sharing not applicable since the article describes theoretical research.

ENDNOTES

*Throughout this paper, a dot above a symbol denoting a signal denotes the derivative of the signal with respect to time $t$, for example, $\dot{x}_i = \frac{dx_i}{dt}$. The derivative with respect to the transformed time variable $\tau$ that will be introduced as part of the control design will be written explicitly as, for example, $\frac{dx_i}{d\tau}$.
The notations $\mathcal{R}$, $\mathcal{R}^+$, and $\mathcal{R}^k$ are used to denote the set of all real numbers, the set of all non-negative real numbers, and the set of all real $k$-dimensional column vectors, respectively.

Note that requiring the inequality (6) with a function $\varphi(t)$ as defined above is a stronger requirement than simply ISS. This additional time-dependent function essentially imposes the prescribed-time ISS property\(^3\) that an ISS-like bound on the appended dynamics' state is satisfied within a prescribed time. Prescribed-time ISS properties have been studied in Reference \(^4\) where Lyapunov functions satisfying inequalities analogous to (6) were constructed with $\varphi(t)$ designed based on the "blow-up" function of form $\frac{1}{t^\alpha}$.

Since a prescribed-time problem is being addressed in this paper, only times $t$ in the interval $[0, T]$ are relevant and hence, the time dependence of the appended system is formulated in Assumption 6 with time $t$ considered as an input over bounded range in Assumption 6. However, if times $t$ after the prescribed time $T$ are to be considered (e.g., as in Remark 4), the condition on $z_p$ in Assumption 6 can be stated as requiring BIBS in terms of inputs $x$ and $u$ and uniform boundedness in terms of time $t$.

For convenience of notation, we drop the explicit arguments of functions whenever no confusion will result.

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