The (2 + 1)-dimensional axial universes—solutions to the Einstein equations, dimensional reduction points and Klein–Fock–Gordon waves

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Abstract

The paper presents a generalization and further development of our recent publications, where solutions of the Klein–Fock–Gordon equation defined on a few particular $D = (2 + 1)$-dimensional static spacetime manifolds were considered. The latter involve toy models of two-dimensional spaces with axial symmetry, including dimensional reduction to the one-dimensional space as a singular limiting case. Here, the non-static models of space geometry with axial symmetry are under consideration. To make these models closer to physical reality, we define a set of ‘admissible’ shape functions $\rho(t, z)$ as the $(2 + 1)$-dimensional Einstein equation solutions in the vacuum spacetime, in the presence of the $\Lambda_1$-term and for the spacetime filled with the standard ‘dust’. It is curious that in the last case the Einstein equations reduce to the well-known Monge–Ampère equation, thus enabling one to obtain the general solution of the Cauchy problem, as well as a set of other specific solutions involving one arbitrary function. A few explicit solutions of the Klein–Fock–Gordon equation in this set are given. An interesting qualitative feature of these solutions relates to the dimensional reduction points, their classification and time behavior. In particular, these new entities could provide us with novel insight into the nature of P- and T-violations and of the Big Bang. A short comparison with other attempts to utilize the dimensional reduction of the spacetime is given.

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1. Introduction

The idea of reducing the number of topological dimensions of the physical space at small distances (proposed recently [1]) was implemented in our previous papers [2, 3] for the $(2 + 1)$-dimensional spacetimes which comprise the two-dimensional static axial spaces with an arbitrary shape function $\rho(z) \geq 0$. This was done to develop general methods and get
an insight into possible features of physics in such a specific variable geometry, including dimensional reduction (DR).

In fact, DR of the physical space in general relativity (GR) can be regarded as an unrealized and as yet untapped consequence of the Einstein equations (EEqs) themselves which takes place around singular points of their solutions. The oldest indication of this still not studied phenomenon could be found as early as in the well-known 1921 Kasner solution of EEqS \[4, 5\]. For a short history of the Kasner solution and its modern applications see \[6\]. It is well known that if we consider, for example, a three-dimensional spacecube, according to the Kasner solution there are possible two types of its evolution, approaching the singularity.

1. Pancake-type evolution: in this case, one of the dimensions tends to zero and the cube becomes a two-dimensional square.

2. Cigar-type evolution: in this case, two of the dimensions tend to zero and the cube becomes a one-dimensional line.

Thus, the Kasner solution demonstrates a clear trend toward DR, but the evolution after the singular point has never been considered. The situation resembles the one in the two-body collision problem in the Newtonian gravity before the invention of the Lévi–Cevită continuation of the solutions after the collision. In this paper, we consider the DR as a dynamical problem for the EEqS.

To begin with, a more general time-dependent axial geometry of the two-dimensional space is analyzed. The \(\{t, z\}\) dependence of the shape function \(\rho(t, z) \geq 0\) is obtained by solving the EEqS in \((2 + 1)\)-dimensional spacetimes with axial spaces. For brevity, we refer to these specific spacetimes as the \((2 + 1)\)-dimensional axial universes (AxUs). It turns out that despite the fact that the EEqS fix quite firmly the axial geometry under consideration, there still remains a variety of dynamically admissible spacetime manifolds, including some of the previously studied static ones.

It is well known that in any \((2 + 1)\)-dimensional GR universe, the local degrees of freedom, which may be related to gravitational waves, are freezeed and we have no freely moving excitations of the gravitational field in spacetimes with trivial topology. There exist quite a large literature on \((2 + 1)\)-dimensional GR; see, for example \[7, 8\], and several hundred references therein. Unfortunately, there one cannot find the consideration of our AxUs which are specific solutions, not related to the \((2 + 1)\)-dimensional models of quantum gravity studied in the literature or with \((2 + 1)\)-dimensional black holes—the main subject and motivation for the previous investigations of the \((2 + 1)\)-dimensional GR. As a result, we can extract from the existing literature only some results of general character.

If the fundamental group of the spacetime is nontrivial, a finite number of global gravitational degrees of freedom remain and provide the classical basis for quantum theory of gravity. While this feature makes the theory simple, it does not quite make it trivial. For example, it drastically simplifies the analysis of the dynamics described by the EEqS in the \((2 + 1)\)-dimensional universes. In the case of AxUs, the axial symmetry yields an additional simplification of the physical problem and ensures the existence of a nontrivial fundamental group. As we shall see, it becomes possible to find the general solution of the EEqS for different matter contents of the \((2 + 1)\)-dimensional AxUs \(2.3\) and to study the novel physical phenomena related to the variable topological dimension in them, which is the main subject of this paper.

Another general result in \((2 + 1)\)-dimensional GR is the Birkhoff-like theorems \[9, 10\] which may be viewed as a no-hair theorem for stars in \((2+1)\) dimensions. The exterior geometry of an axially symmetric star is completely specified by the mass, angular momentum
and cosmological constant. Those results are only partially related to our topic, but some of those used in the cited articles' general considerations may also be valuable for us.

The solutions of the Klein–Fock–Gordon equation (KFGEq) in the $(2 + 1)$-dimensional AxUs, which are consistent with the EEq$s$, are studied. One can consider the related field excitation as test particles, since we ignore the back-reaction of these excitations on the metric. Thus, we reach the usual natural separation of macro- and micro-physics. Indeed, in the real world the geometry of the observable Universe, governed by the EEq$s$, is determined by about $10^{86}$ protons (the Eddington number) and the same number of electrons [15]. From a physical point of view, the influence of particles and fields, which we use for Earth-laboratory and Space experiments on this geometry, is obviously negligible. Hence, to obtain useful information for our domestic experiments, it is natural to study the behavior of test particles and fields on some spacetime background defined by the solutions of the EEq$s$ with some nonzero energy–momentum tensor on the rhs which describes a bulk of matter filling the Universe.

2. The $(2 + 1)$-dimensional time-dependent axial universes

Consider auxiliary flat Minkowski $(3 + 1)$-dimensional spacetime $\mathbb{R}^{(1,3)}_{\phi^1, \phi^2, \phi^3}$ with the interval
\[
ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.
\]
We introduce AxU as a hypersurface in $\mathbb{R}^{(1,3)}_{\phi^1, \phi^2, \phi^3}$ defined by the equations
\[
M^{(1,2)}_{\phi^1, \phi^2, \phi^3} : \begin{cases}
x^0 = t, \\
x^1 = \rho(t, z) \cos \phi, \\
x^2 = \rho(t, z) \sin \phi,
\end{cases}
\]
assuming $t \in (-\infty, \infty)$, $z \in (-\infty, \infty)$ and $\phi \in [0, 2\pi]$. This pseudo–Riemannian $(2 + 1)$-dimensional manifold has a structure $M^{(1,2)}_{\phi^1, \phi^2, \phi^3} = \mathbb{R}^{(1)}_{\phi^1, \phi^2} \otimes \mathbb{S}^{(1)}_{\phi^3}$, with $\mathbb{S}^{(1)}_{\phi^3}$ being a circle.

Clearly, the space $M^{(2)}_{\phi^1, \phi^2} = \mathbb{R}^{(1)}_{\phi^1, \phi^2} \otimes \mathbb{S}^{(1)}_{\phi^3}$ is a two-dimensional rotational surface with a time-dependent variable shape function $\rho(t, z) \geq 0$. Thus, for the AxU,
\[
\frac{\dot{\rho}}{\rho^2} = \frac{1}{1 - \dot{\rho}^2} \frac{d\tau^2}{ds^2} - 2 \rho \dot{\rho} \frac{d\tau}{ds} - (1 + \rho^2) \frac{d\phi}{ds}^2 - \rho^2 \frac{d\phi}{ds}^2;
\]
\[
\dot{\rho} = \frac{\partial \rho}{\partial t}, \quad \rho' = \frac{\partial \rho}{\partial z}.
\]

Further on, we consider the pseudo–Riemannian spacetimes (2.3), with restriction on the lapse function $(1 - \dot{\rho}^2) > 0$. This condition is needed to preserve the relativistic causality and the physical meaning of the time variable $t$. It ensures that the time variations of the shape of the axial space are not able to spread faster than the light, or with the velocity of light.

Note that the most general metric with axial symmetry for $(2 + 1)$-dimensional GR is a starting point in the articles [9, 10]. Under a proper choice of coordinates, the authors perform a detailed analysis of rotating axially symmetric $(2 + 1)$-dimensional vacuum spacetimes and derive the Birkhoff-like theorems. At the end, they arrive at the stationary but not static rotating Banados–Teitelboim–Zanelli vacuum metric for the exterior domain of a $(2 + 1)$-dimensional rotating star.

The AxUs with metric (2.3) present completely different spacetimes. Being neither stationary nor static, they describe non-rotating universes, both vacuum ones (without or with $\Lambda$ term) and filled with matter. As we shall see in the following sections, the metric (2.3) produces quite a different form of the EEq$s$ and yields essentially different physical solutions which are interesting for the study of the DR.

At the points where $\rho(t, z) = 0$, the dimension of the two-dimensional axial space reduces. We call these points the dimensional reduction points (DRPs). In general, they move along the $z$-axis, i.e. for the DRP we have $z = z^{\text{DRP}}(t)$. There exist two possibilities.
which shows that the component $G^\phi_\phi$ is not an independent one and can be expressed in terms of the other nontrivial components of the energy–momentum tensor of the matter sources:

$$T_{t}^\phi = T_{z}^\phi = T_{\phi}^t = T_{\phi}^z = 0.$$  \hfill (3.2)

These relations restrict the motion of the matter which creates a specific type of universe with metric (2.3).

After some algebra, one can write down equations (3.1a) in a much simpler form

$$\dot{\rho} = -\rho \dot{\gamma} T^z, \quad \rho'' = -\rho \dot{\gamma} T^\gamma, \quad \dot{\rho'} = \rho \dot{\gamma} T^\gamma.$$

Besides, one obtains from equations (3.1a) the compatibility condition $T^z = T^\gamma$ which is fulfilled by construction. The last equation (3.1b) in the Einstein system (3.1) yields the constraint

$$T^\phi = \dot{g}^2 (T^t T^\gamma - T^\gamma T^\gamma),$$

which shows that the component $T^\phi$ is not an independent one and can be expressed in terms of the other nontrivial components of the energy–momentum tensor in AxU. This constraint can be represented in the form

$$\det T = (T^\phi / g)^2 \geq 0,$$

where $\det T = \det |T^\gamma| |T^\gamma|$ is the determinant of the contravariant energy–momentum tensors of matter in the $(2 + 1)$-dimensional AxUs, subject to the conditions, equation (3.2). The constraint (3.5) shows that the determinant $\det T \geq 0$ is a non-negative quantity. This is compatible with the properties of the energy–momentum tensor for a physical matter in the $(2 + 1)$-dimensional AxUs, say for perfect fluid with a standard eigenvalue $\epsilon \geq 0$ ($\epsilon$ being

3 In more general geometries without axial symmetry, the one-dimensional space can be part of a curved line.

4 The Riemannian scalar curvature of the interval (2.3) is

$$K = -2\dot{\rho}^{-2} (\dot{\rho}^2 \rho'' + 2\rho \dot{\rho} \dot{\rho'} - (1 + \rho^2) \ddot{\rho} + \rho^2 (\dot{\rho} \dot{\rho'} - (\dot{\rho'})^2)).$$

Then the causality condition ensures the absence of curvature singularities in the AxUs since $\dot{\rho}^2 < 1 \Rightarrow \dot{\rho} > 0$.\hfill (4)
the density of energy) and two identical negative ones, \(-p \leq 0\) (\(p\) being the pressure). Thus, relation (3.5) supports the compatibility of our models of AxUs with the standard physics [5].

Note that equations (3.3) replace EEqs (3.1) and govern the dynamics of geometry of the AxUs. To obtain the whole dynamics of the universe, filled with some matter, one has to add the continuity equation

\[ \nabla_i T^i_j = 0, \quad i, j = 1, 2, 3. \] (3.6)

It is a well-known consequence of the EEqs yielded by the restriction of the Bianchi identity \(\xi^i \equiv 0\) on the Einstein tensor and presents the GR dynamical equations for matter in any \((2 + 1)\)-dimensional universe. The third part of equations (3.6) \(\nabla_i T^i_j = 0\) is identically fulfilled in the AxUs at hand. Hence, for AxUs we have a specific universally conserved vector quantity

\[ T^i = T^i_j \xi^j, \quad \nabla_i T^i = 0, \] (3.7)

due to the axial symmetry which yields the obvious Killing vector \(\xi^i = \{\xi^t, \xi^z, \xi^\phi\} = \{0, 0, 1\}\) and conservation of the \(z\)-component of the angular momentum.

4. The solutions to the Einstein equations for the time-dependent AxU

4.1. The vacuum solutions of the Einstein equations

The \((2 + 1)\)-dimensional vacuum dynamical equations (3.3) with zero on the right-hand sides obey three simple solutions (related by Lorentz transformations):

(i) \(\rho(t, z) = v_0(t - t_0) + \rho_0\), where \(\rho_0 \geq 0\) is an arbitrary constant, \(v_0\) is the constant velocity of the expansion of the one-dimensional string all points of which are a non-isolated DRP (described by the equation \(\rho \equiv 0\) at the time instant \(t = t_0 - \rho_0/v_0\), if \(0 < |v_0| \leq 1\) on the surface of the cylinder of the radius \(\rho(t)\) which is independent of the coordinate \(z\);

(ii) \(\rho(t, z) = (z - z_0) \tan \alpha\), where \(\alpha\) is the constant angle at the vertex of a static cone. Further on, the short notation \(\sigma = \tan \alpha\) is used. The static isolated DRP is the point \(z = z_0 = \text{const}\);

(iii) \(\rho(t, z) = v_0(t - t_0) + \sigma(z - z_0)\), \(v_0 \neq 0\), is the velocity of a moving two-dimensional cone with the vertex angle \(\alpha \in (0, \pi/2)\). Here, an isolated running DRP \(z^{\text{drp}}(t) = z_0 - v(t - t_0)/\sigma\) moves with a constant velocity \(v_0/\sigma\).

As seen, in any case, we have the DRP of the EEq solutions, which are related to a reduction of the topological spacetime dimension from \((2 + 1)\) to \((1 + 1)\), or even to \((1 + 0)\).

4.2. The solutions with a positive \(\Lambda\) term

In the case of the positive lambda term \(\Lambda = 1/R^2 > 0\), one has the only solution of the EEqs \(G^i_j = \Lambda \delta^i_j\) in the AxU:

\[ \rho(t, z) = \sqrt{R^2 - (z - z_0)^2}. \]

It describes a two-dimensional static spherical surface of a constant radius \(R\). This solution was briefly discussed in [2]. On this sphere, we have two isolated static DRPs: \(z = z_0 \pm R\) which are not singular points of the very surface.

4.3. The solutions for the \((2 + 1)\)-dimensional AxU filled with dust

It is clear that putting some matter content like ‘dust’, perfect fluid, or different matter fields in the \((2 + 1)\)-dimensional universe with variable axial geometry, one can obtain much more
sophisticated solutions of EEqqs. Consider, for example, the case of this sort of a universe filled with dust. Then
\[ T^{ij} = \mu(t, z)u^i(t, z)u^j(t, z), \] (4.1)
where the standard notation was used [5].

4.3.1. The solution to the gravitational field equations. As a result, from equation (3.4) one obtains \( T^{00} \equiv 0 \) and the variable shape function \( \rho(t, z) \geq 0 \) has to be found according to equation (3.1b) by solving the well-known homogeneous Monge–Ampère equation [16]
\[ \ddot{\rho} \rho'' - (\dot{\rho})^2 = 0. \] (4.2)
Its general solution has the following implicit form in terms of two arbitrary functions \( a(v) \) and \( b(v) \):
\[ \rho = tv + a(v)z + b(v), \quad t + a_v(v)z + b_v = 0, \] (4.3)
where the comma denotes the corresponding partial differentiation. From the second equation one has to obtain the function \( v(t, z) = \rho(t, z) \). This is possible if and only if the following condition is fulfilled:
\[ a_{vv}(v)z + b_{vv} \neq 0. \] (4.4)
After that one obtains the solution \( \rho(t, z) \) from the first part of equations (4.3). In addition, one obtains the relations
\[ \rho' = a(v), \quad v' = v a_v, \quad \ddot{\rho} = \dot{v}, \quad \rho'' = v' a_v, \quad \rho' = \dot{v} a_v, \quad b(v) = \rho - t \dot{\rho} - z \rho', \] (4.5)
which reveal the meaning of the arbitrary functions in (4.3). In particular, the function \( b(v) \) describes the deviation of the shape function \( \rho(t, z) \) from a homogeneous function of degree 1.

In the case \( a_v(v)z + b_{vv} \equiv 0 \), equation (4.2) has a special solution
\[ \rho(t, z) = v_0 t + \rho_0(z), \quad |v_0| < 1, \] (4.6)
with \( v_0 \) being a constant velocity, not greater than the light velocity, and \( \rho_0(z) \geq 0 \) being an arbitrary time-independent shape function.

It is also not difficult to obtain the general solution of the Cauchy problem. Let \( \rho_0(z) \geq 0 \) and \( \dot{\rho}_0(z) \) be the Cauchy data. Then using equations (4.3) and (4.5), one obtains
\[ \rho(t, z) = \rho_0(\xi) + (z - \xi) \rho_0(\xi) - t \dot{\rho}_0(\xi). \] (4.7a)
where \( \xi \) is defined by the equation
\[ t \dot{\rho}_0(\xi) = (z - \xi) \rho_0(\xi) \Rightarrow \xi = \xi(t, z). \] (4.7b)

4.3.2. The solution to the matter equations. Since the gravitational field dynamics is already known, the description of the dynamics of matter is a simple algebraic task. From equations (3.3), relations (3.2), (4.1) and (4.5), as well as taking into account the normalization condition \( g_{ij}u^iu^j = 1 \) and assuming \( \dot{v} = \ddot{\rho} \neq 0 \), one obtains
\[ u^i(t, z) = -\frac{a_v}{\eta(v)}, \quad \dot{u}^i(t, z) = \frac{1}{\eta(v)}, \quad \ddot{u}^i(t, z) = 0, \] (4.8a)
\[ \mu(t, z) = -\ddot{\rho} \left( \frac{\eta(v)}{1 - v^2 + \delta^2} \right)^2 \geq 0, \quad \ddot{\rho} \leq 0, \] (4.8b)
thus reaching a complete description of the motion of matter which builds the universes under consideration. Here,
\[ \eta(v) = \sqrt{(1 - v^2)(a_v)^2 + v(a_v^2 - a^2 - 1)} \]
must be real. Hence, the quantity under the square root should be non-negative. The corresponding differential inequality can be represented in the form
\[ a_v \geq \frac{\sqrt{1 - v^2 + a^2 - va}}{1 - v^2} \quad \text{for} \quad v \in (-1, 1). \] (4.9)
This gives an additional restriction on the admissible functions \( a(v) \):
\[ 2a(v) \geq (1 - v)a(-1) - (1 + v)/a(-1) \quad \text{for} \quad v \in (-1, 1). \] (4.10)
One can simplify the consideration of matter dynamics using the standard co-moving frame (where the matter is at rest, i.e. \( u' = \delta'_0 \)), but we shall skip here the details.

4.3.3. Dynamics of the dimensional reduction points in the \((2 + 1)\)-dimensional AxUs filled with dust. The zeros of the initial shape function \( \rho_0(z) \geq 0 \) are DRPs for the axial space geometry. According to equations (4.7), in general, these are moving DRPs.

It is interesting to know whether it is possible to create additional DRPs which are not zeros of the initial shape function, or to annihilate some of the existing ones during the time evolution of our models. The following simple example shows that this is possible.

Consider the Cauchy initial data \( \rho_0(z) = \frac{1}{2} \rho_0 z_v^2/R + r, \rho_0(z) = \frac{1}{2} v_0 z_v^2/R^2, r, R > 0 \). Then from equation (4.7), one has \( \rho_0(t, z) = \frac{1}{2} z_v^2/(R + v_0 t) + r \). Here, at the initial moment \( t = 0 \) there are no real DRPs. Depending on the sign of the velocity constant \( v_0 \) two such DRPs \( z_{\pm 0}^o(t) = \pm \sqrt{2}r(-R - v_0 t) \) appear or disappear at the instant \( t = -R/v_0 \). Hence, we have a typical bifurcation problem. Since under the change of the corresponding bifurcation parameter the simple real roots of analytic functions occur in pairs, or disappear in pairs, this is also true for the DRPs in our problem assuming an analytical character of the Cauchy data.

One can point out several quite general examples of the \((2 + 1)\)-dimensional AxUs in which the number of the DRPs (finite, or even infinite one) is constant during the time evolution, see appendix A. It is possible that there exists an infinite sequence of DRPs which has a finite limiting point. Hence, the structure and the dynamics of the DRPs of the universe may be quite complicated.

5. The solutions of the KFGEq on the \((2 + 1)\)-dimensional AxUs which are the EEq solutions

The test particles and fields of any spin in the \((2+1)\)-dimensional AxUs have a common property. Due to the axial symmetry, the \( z \)-component of their angular momentum is a constant of motion. For the KFGEq,
\[ \Box \varphi - M^2 \varphi = 0, \quad \Box = -\frac{1}{\sqrt{|g|}} \partial_{\mu}(\sqrt{|g|}g^{\mu\nu} \partial_{\nu}); \] (5.1)
this means that one can separate the angular part of the field by the ansatz \( \varphi(t, z, \phi) = f_m(t, z) e^{im \phi} \), where the azimuthal number \( m = 0, \pm 1, \pm 2, \ldots \) is an integer.

5.1. The solutions of KFGEq on the \((2 + 1)\)-dimensional AxUs which are the EEq vacuum solutions

(1) For simplicity, we write down the first vacuum solution of section 4.1 in the form of \( \rho(t, z) = vt \) \( (v = \text{const}) \), which shows that it describes a cylinder with a radius
independent of the variable $z$. This cylinder collapses to a thread with a zero radius at instant $t = 0$ (for $v < 0$), or vice versa, and the thread expands to a cylinder with increasing radius $\rho(t, z) = vt$ (if $v > 0$). The corresponding KFGEq reads

$$-\frac{1}{1 - v^2} \frac{1}{|t|} \partial_t(|t| \partial_t \varphi) + \partial_z^2 \varphi + \frac{1}{v^2 t^2} \partial^2_{\varphi} \varphi - M^2 \varphi = 0.$$  

After separation of variables, one obtains its solutions in the interval $t \in (+0, +\infty)$,

$$\varphi_1(t, z, \phi) = J_v(\omega_p t) e^{i p_z z} e^{i m_p}, \quad \varphi_2(t, z, \phi) = Y_v(\omega_p t) e^{i p_z z} e^{i m_p},$$  

(5.2)

where $p_z$ is the $z$-component of the momentum of the Klein–Fock–Gordon field. Since the index $v$ of the Bessel functions $J_v$ and $Y_v$ is a purely imaginary number, in the limit $t \to +0$, when the space becomes one dimensional, both solutions oscillate infinitely many times remaining limited in the amplitude. The solutions cannot be continued directly through the DRP $t = 0$, which is an infinite-branching point on the real axis $t$.

(2) In the case 2 of section 4.1, the solution describes a static cone $\rho(t, z) = \sigma (z - z_0)$. The only static DRP lies on the $Oz$-axis at the point $z = z_0$. The different solutions of the KFGEq on this type of cone were found in [3]. We give them for comparison with other solutions considered here:

$$\varphi_1(t, z, \phi) = e^{-i \omega t} J_v (k_z z) e^{i m_p}, \quad \varphi_2(t, z, \phi) = e^{-i \omega t} Y_v (k_z z) e^{i m_p},$$  

(5.3)

where $\omega$ is, in general, the complex frequency with a positive imaginary part. As a result, the first solution $\varphi_1(t, z, \phi)$ vanishes at the static DRP $z = 0$, while the second one $\varphi_2(t, z, \phi)$ diverges.

In [3], one can also find a detailed description of highly nontrivial excitations of the Klein–Fock–Gordon field on a continuous manifold built of the parts of two static axial surfaces of type 1, given in section 4.1, but now with $v = 0$ and different constant radii $\rho_{0, 1} = r$ and $\rho_{0, 2} = R > r$, connected by the corresponding part of the static cone.

(3) Turn now to the third case described in section 4.1. The solution $\rho(t, z) = \sigma z + vt \ (\sigma, v \neq 0)$ represents a moving two-dimensional cone. It has a moving DRP $z_{drp}(t) = -vt/\sigma$.

After separation of the variables in the KFGEq by specific ansatz $\varphi(t, z, \phi) = F(\sigma z + vt) G(vz + \sigma t) e^{i m_p}$, one obtains two independent solutions:

$$\varphi_1(t, z, \phi) = e^{-i \frac{\omega}{\sqrt{v^2 - \sigma^2}}} J_v(\nu_\sigma (\sigma z + vt)) e^{i m_p}, \quad \varphi_2(t, z, \phi) = e^{-i \frac{\omega}{\sqrt{v^2 - \sigma^2}}} Y_v(\nu_\sigma (\sigma z + vt)) e^{i m_p},$$  

(5.4)

Here, the separation constant $\alpha$ plays the role of a spectral parameter. To obtain equation (5.2) from equations (5.4), one has to put $\sigma = 0$ and $\alpha = i p_z$. For obtaining equations (5.3), one has to put $v = 0$ and $\alpha = \omega$.

Note that in the vicinity of the moving DRP $z_{drp}(t) = -vt/\sigma$, solutions (5.4) of the KFGEq have a different behavior depending on the values of the constants $v$ and $\sigma = \tan \alpha$.

(i) If $v \cot \alpha \geq 1$, then $v$ is imaginary; the DRP moves with superluminal velocity and both solutions (5.4) are bounded in its vicinity but make an infinite number of oscillations approaching this point. For comparison, see the etalon case 1, i.e. equations (5.2) when $v \cot \alpha = \infty$. 

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(ii) If \( \cot \alpha \leq 1 \), then \( v > 0 \) is real; the DRP moves with subluminal velocity and the solution \( \psi_1(t, z, \phi) \to 0 \) but \( \psi_2(t, z, \phi) \to \infty \) in the vicinity of the singular point. For comparison, see the etalon case 2, i.e. equations (5.3) when \( v \cot \alpha = 0 \).

5.2. The solutions of the KFGEq on the (2+1)-dimensional AxU with a positive \( \Lambda \) term

In this case (see section 4.2.), the standard separation of the variables in the KFGEq leads to the following two solutions in terms of the Legendre functions\(^5\):

\[
\psi_{1,2}(t, z, \phi) = e^{-im\phi} \text{LegendreP} (v, |m|, \pm|z|/R) e^{im\phi}.
\] (5.5)

Then, \( \psi_1(t, z, \phi) \) is regular at the point \( z = R \) and singular at the point \( z = -R \), while \( \psi_2(t, z, \phi) \) is regular at the point \( z = -R \) and singular at the point \( z = R \).

The space of our static universe with the \( \Lambda \) term is a closed two-dimensional spherical surface. One obtains an infinite series of everywhere regular solutions of KFGEq \( \psi(t, z, \phi; n, m) \) which have a discrete spectrum with real frequencies,

\[
\omega_{n,m}^\pm = \sqrt{M^2 + (n \pm |m|)(n \pm |m| + 1)/R^2}, \quad n = 0, 1, 2, \ldots,
\] (5.6)

imposing the requirement for the linear dependence of the solutions \( \psi_1(t, z, \phi) \) and \( \psi_2(t, z, \phi) \), defined by equation (5.5). This assigns integer values to the parameter \( v \) in equation (5.5) and brings us to the associated Legendre polynomials.

5.3. Some solutions of the KFGEq on the (2+1)-dimensional AxU filled with dust

Consider for example solution (A1.a) which presents a moving wave \( \rho(t, z) = f(\sigma z + tv) \). After some algebra one separates the variables in the corresponding KFGEq by using again the ansatz \( \psi(t, z, \phi) = F(\sigma z + vt)G(\nu z + \sigma t)e^{im\phi} \) and obtains \( G(\nu z + \sigma t) = \exp(-i(\nu z + \sigma t)/\sqrt{\sigma^2 - v^2}) \). Now the function \( F(x) \) has to be a solution of the following ODE defined by the function \( f(x) \):

\[
F'' + \frac{f'(x)}{1 + (\sigma^2 - v^2)(f'(x))^2} F' + \frac{1 + (\sigma^2 - v^2)(f'(x))^2}{\sigma^2 - v^2} \left( a^2 - M^2 - \frac{m^2}{f(x)^2} \right) F = 0.
\] (5.7)

Here, the prime denotes differentiation with respect to the variable \( x = \sigma z + vt \). For some specific functions \( f(x) \), this equation has two independent solutions \( F_{1,2}(x) \) and one arrives at the KFGEq solutions in the form

\[
\psi_{1,2}(t, z, \phi) = \exp \left\{ -i \frac{(\nu z + \sigma t)}{\sqrt{\sigma^2 - v^2}} \right\} F_{1,2}(\sigma z + vt)e^{im\phi}.
\] (5.8)

For the simple case \( f(x) = x \), this gives the already obtained vacuum result (5.4). For \( f(x) = \sqrt{x} \), one obtains exact solutions of equation (5.8) in terms of the confluent Heun function:

\[
F_{1,2}(x) = x^{\pm} \exp \left\{ i \frac{\sqrt{a^2 - M^2}}{\sigma^2 - v^2} x \right\} \text{HeunC} \left( \alpha, \pm m, -\frac{3}{2}, \delta, \frac{3}{4} - \delta, -\frac{4x}{\sigma^2 - v^2} \right).
\] (5.9)

\(^5\) We use here the MAPLE notation for the Legendre functions assuming branch cuts on the real semi-axes \((-\infty, -1)\) and \((1, +\infty)\). This choice of the branch cuts is most convenient for our problem defined on the real interval \([-1, 1]\) and differs from the standard definitions of the Legendre functions in [17] and [18]. For each of the two admissible values of \( v: v = \pm \sqrt{1/4 + R^2(\alpha^2 - M^2)} - 1/2 \), one obtains two solutions (5.5) which are well defined on the interval \( z \in (-R, R) \) and are in general linearly independent.
where $\alpha = i\sqrt{(a^2 - M^2)(\sigma^2 - v^2)}$ and $\delta = \frac{m^2}{2} - \frac{1}{16}(\sigma^2 - v^2)(a^2 - M^2)$. Since $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, 0) = 1$, around the moving DRP $z_{\text{DRP}} = -vt/\sigma$, the behavior of the KFGEq solutions is

$$
\varphi_{1,2} \sim (\sigma z + vt)^{\pm \frac{m^2}{2}} \exp \left( i \left( \frac{\sigma \sqrt{a^2 - M^2} - av}{\sqrt{\sigma^2 - v^2}} z + \frac{v \sqrt{a^2 - M^2} - a\sigma}{\sqrt{\sigma^2 - v^2}} t \right) \right). \quad (5.10)
$$

6. Summary and outlook

Our main results are as follows.

1. A special kind of the $(2 + 1)$-dimensional toy models of time-dependent universes with axial space (axial universes) is introduced and considered in detail. Their variable geometry is defined by a single time-dependent shape function $\rho(t, z) \geq 0$. It is shown that these models allow the study of topological DR phenomena since the zeros of the shape function define the DRPs.

2. The time-dependent shape function $\rho(t, z)$ of the AxUs is determined by the solution of the EEqS for various energy–momentum tensors of matter. The exact solutions for the vacuum AxUs, AxUs filled with the $\Lambda$ term, as well as the exact general solution of the EEqS for the AxUs filled with dust are found. In the last case, we reduce the EEq solutions to the solutions of the homogeneous Monge–Amp`ere equation and obtain the general solution of the Cauchy problem for time-dependent AxUs, as well as three independent classes of solution each of which involves an arbitrary function. It is shown that some of the previously considered static AxUs are solutions to the EEqS.

3. The dynamics of the DRPs in the time-dependent AxUs is studied. It is found that these points can emerge and disappear in a real domain. Their dynamics is described in three essentially different cases.

4. The spreading of test particles in variable geometry, including the reduction of topological dimension, is studied. For this purpose, the exact solutions for test particles described by the Klein–Fock–Gordon equation in different time-dependent AxUs are presented in terms of special functions. The behavior of test particles in the vicinity of the DRPs is described explicitly.

These results allow us to express a hope for a possible further development.

- The time-dependent AxUs with variable spacetime geometry could give one a simple basis for commenting the real situation concerning the C, P and also T properties of the particles.
- The considered models of time-dependent AxUs inspire an intriguing idea: to treat the very Big Bang as a transition from the pre-Big–Bang–Universe with a lower topological dimension ($d = 1$, or $d = 2$): (or as a sequence of such transitions), to the present-day space with $d = 3$. For this purpose, a generalization of the model, described in [16], is needed.

At present, a quite intensive discussion of a different kind of reduction of space dimensions related to the theory of gravity and particle physics takes place in the literature (see appendix B and the references therein.). Here, we outline our answer to a very important question raised in [17]:

‘... there is no reason whatsoever for the theory ... to be close to 2+1 dimensional GR in the UV. Clearly, if this is to be a viable gravity theory it should resemble 4-dimensional GR at low energies. In fact, it is difficult to imagine how a theory that has this latter property can have the former property as well...’.
The same question also applies to our approach to DR, although it differs significantly from the others.

First of all, in our approach the DR is a dynamical consequence of EEqS in GR. As a result, one may expect that in the smaller dimensions, an analogous theory will be valid in the same manner as in the well-known Kaluza–Klein theories. This problem needs careful justification, especially for more realistic transitions between $(3 + 1)$-, $(2 + 1)$- and $(1 + 1)$-dimensional geometries, starting or ending with usual $(3 + 1)$-dimensional GR with a different kind of matter content of the Universe. This paper is to be considered only as a first step in this direction, based on the very simplified AxUs toy models.

Using the experience in the Kaluza–Klein-like theories (see, for example [15], and the references therein), we can shed more light on the above consideration.

Indeed, it is well known that after topological DR in the modern Kaluza–Klein theories, one remains with usual GR and some set of additional fields of various spins. This set is determined by the theory in higher dimensions. Hence, if we intend to reproduce our observable world, starting from a pre-Universe with lower dimension, the last has to be equipped with a definite set of primordial fields, precisely adjusted to reconstructing our four-dimensional world.

Therefore, it is reasonable to continue the development of this type of models by solving the inverse problem: to reconstruct the physics in the lower dimensional pre-Universe which allows one to reproduce the known physics in our real world. One of the important methods to relate the observable properties with the variable geometry is to use the ‘fingerprints’ of the shape of the junction domain between parts of the pre-Universe with different topological dimensions on the spectra of the observed particles [3].

Besides the use of lower spacetime dimensions in the place of higher ones, the main difference between the Kaluza–Klein theories with their further development (like superstring models) and our approach is that in the latter, instead of fixing very small compactification radii needed to make the introduced additional higher dimensions unobservable, we use the shape function\(^6\) $\rho(t, z) \geq 0$ (solution of the dynamic equations) which plays the role of a variable ‘compactification radius’. If $\rho(t, z) \to \infty$, we have a flat space. In the opposite case, when $\rho(t, z) \to 0$, the topological dimension of the spacetime reduces, as seen from the results of this paper\(^7\).

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\(^6\) Or a set of shape functions, see [20].

\(^7\) Thus, our model of topological DR is fundamentally different from models that use fractal, spectral or other possible concepts of dimensionality of spacetime, see appendix B.
Appendix A. Examples of time evolution of the set of dimensional reduction points in the (2 + 1)-dimensional AxU, filled with dust

One can shed some additional light on the evolution of DRPs considering the following three different types of solutions \( \rho(t, z) \) in which only one arbitrary function \( f(x) \) is involved [16]:

\[
\rho(t, z) = f(\sigma_0 z - v_0 t) + \sigma_1 z - v_1 t + \rho_1, \quad (A1.a)
\]

\[
\rho(t, z) = (\sigma_0 z - v_0 t + \rho_0) f \left( \frac{\sigma_1 z - v_1 t + \rho_1}{\sigma_0 z - v_0 t + \rho_0} \right) + \sigma_2 z - v_2 t + \rho_2, \quad (A1.b)
\]

\[
\rho(t, z) = (\sigma_0 z - v_0 t) f \left( \frac{vt}{z} \right) + \sigma_1 z - v_1 t + \rho_1, \quad (A1.c)
\]

with the arbitrary constants \( \sigma_0, \sigma_1, \sigma_2, v_0, v_1, v_2, \rho_0, \rho_1 \) and \( \rho_2 \).

Suppose \( x_0, x_1, x_2 \) to be the zeros of the corresponding function \( f(x) \) (which for this issue is assumed to be a bounded function) and consider the following three cases.

(i) Running waves of type (A1.a) with the equation \( \rho(t, z) = f(\sigma_0 z - v_0 t) \) (where \( \sigma_0, v_0 \neq 0 \)).

Then, the moving DRPs are

\[
z^\text{drp}_i(t) = (v_0 t + x_i)/\sigma_0, \quad i = 1, 2, \ldots
\]

The distance between the different DRPs remains constant during the time evolution.

(ii) Solutions of type (A1.b) with the equation \( \rho(t, z) = (\sigma_0 (z - z_0) - v_0 t) f \left( \frac{\sigma_1 z - v_1 t + \rho_1}{\sigma_0 (z - z_0) - v_0 t} \right) \) (where \( \sigma_{0,1}, v_{0,1} \neq 0 \)).

Now we have the DRP related to the first factor: \( z^\text{drp}_0(t) = z_0 - v_0 t/\sigma_0 \) and additional DRPs

\[
z^\text{drp}_i(t) = z_0 + \frac{v_1 - v_0 x_i}{\sigma_1 - \sigma_0 x_i} t - \frac{\rho_1 - z_0 \sigma_1}{\sigma_1 - \sigma_0 x_i}, \quad i = 1, 2, \ldots
\]

Since different DRPs move with different constant velocities, starting from a different initial position, their ordering may change, depending on the roots \( x_i \). During the time evolution some pairs of DRPs may coalesce. Indeed, the relative velocity between the points \( z^\text{drp}_i(t) \) and \( z^\text{drp}_j(t) \) is constant:

\[
v_{ij} = \frac{(x_i - x_j)(\sigma_1 v_1 - \sigma_1 v_0)}{(\sigma_1 - \sigma_0 x_i)(\sigma_1 - \sigma_0 x_j)}.
\]

Hence, if there are DRPs with \( v_{ij} < 0 \), they coalesce and after that go away from each other.

(iii) Solutions of type (A1.c) with the equation \( \rho(t, z) = (\sigma_0 z - v_0 t) f \left( \frac{vt}{z} \right) \) (with \( \sigma_0, v_0, v \neq 0 \)).

Related to the first factor is the DRP \( z^\text{drp}_0(t) = -v_0 t/\sigma_0 \). The other ones are

\[
z^\text{drp}_i(t) = vt/x_i, \quad i = 1, 2, \ldots
\]

In this case, all DRPs start from the common origin \( z = 0 \) at the time instant \( t = 0 \) and move with different velocities.

In all three cases, the number of DRPs is constant during the time evolution of the universe.
Appendix B. Remarks on some other approaches to dimensional reduction

In the literature (see, for example, [22–26, 29–32] and the references therein), one can find quite different attempts to consider spacetime with a variable dimension.

It seems that for the first time the idea of lowering the space dimension at short distances appeared in the models with causal dynamical triangulations (CDT) [22–25] in which the geometry emerges as the sum of all possible triangulations. There, a specific ‘spectral dimension’ was introduced as a probe of space dimension in CDT. The spectral dimension needs not to be an integer and furthermore is scale dependent. It can be thought of as the effective dimension as probed by a fictitious diffusion process.

In [26], a good agreement was found between the spectral dimensions of CDT and the completely different Horava–Lifshitz (HL) model of gravity based on a specific violation of Lorentz invariance at small distances.

Due to the articles [27, 28], now we know that the spectral dimension, as opposed to the topological dimension, is actually related to the kinematics of the fictitious diffusion process (as well as of other possible dynamical processes) in the region where spacetime curvature is small and the manifold is flat to a good approximation. Hence, the spectral dimension is not necessarily intrinsically geometric. It is ultimately equivalent to a dispersion relation for the differential operator which describes the dynamical process under consideration.

As we see, the above two approaches to DR are based on completely different physical properties and differ essentially from our approach, which is based on the standard topological dimension of the spacetime and considers physical processes on manifolds with variable topological dimension. Thus, the simultaneous use of the term ‘dimensional reduction’ both in CDT or HL and in our model is misleading.

Another class of models which make an attempt to use spacetime with variable dimension is described in [29–32] and references therein. Actually, in these articles, we have no well-defined model but only some general ideas, inspired by some yet not explained observational data from processes with cosmic ray particles detected in the Pamir mountains: ‘an alignment of the main energy fluxes along a straight line in a target plane’. This observation provokes the authors for a search of a layered structure of space.

At the beginning, the authors try to construct some crystal-like structure of the space (ordered string/brane lattice). The space consists of one-dimensional pieces at very small distances, joined in two-dimensional layers at larger distances. In its turn, the two-dimensional layers are joined in three-dimensional cells at usual scales and then the last are joined in a four-dimensional structure at distances bigger than the size of the observable Universe.

A somewhat different picture of the Universe, guided by the same motivation, is a folded string with folding given by the fundamental quantization scale \(L_1\) at very high energy. It then folds and interweaves forming a two-dimensional structure with the characteristic scale \(L_2 \gg L_1\) which in turn folds to make a three-dimensional structure at scale \(L_3 \gg L_2\), etc. It remains unclear which definition of dimension is used in the concept of ‘vanishing dimensions’. Nevertheless, we see that such concept is completely different from the DR scheme developed in this paper, based on the topological dimension of manifolds and EEq.

Using the above quite undefined general ideas, the authors of [31] propose a physical speculation according to which one can detect the vanishing dimensions via primordial gravitational wave astronomy. If the Universe was indeed \((2 + 1)\)-dimensional at some earlier epoch, according to the authors, we will not see gravitational waves from this epoch.

The last conclusion is strongly criticized in [17]. We can support this critic using another qualitative argument. According to the already discussed analogy with the Kaluza–Klein models, in the \((1 + 1)\)- and/or \((2 + 1)\)-dimensional phases of the Universe we have to
introduce primordial fields from which after the transition to \((3+1)\)-dimensional phase will be constructed the usual GR gravitational waves. In our model, the low-dimensional pre-Universe is not empty. As a result, in the \((3+1)\)-dimensional phase of the Universe, we can see the signals of the primordial fields in the form of standard \((3+1)\)-dimensional GR gravitational waves, since, according to the results of [3], the penetration of signals from the low-dimensional part of space into the higher dimensional one is possible.

It is important to stress that as well as in the Kaluza–Klein models, in our design of DR the numerical field degrees of freedom of the theory are preserved during the transition from one to another spacetime dimension. The only thing that changes is the grouping of these field degrees of freedom in different multiplets, because of the different symmetry groups in the tangent foliations of the parts with a different dimension of the spacetime manifold.

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