Abstract Interpretation of Binary Code with Memory Accesses using Polyhedra

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Abstract. In this paper\footnote{An earlier version of this paper has been submitted to TACAS 2018 (http://www.etaps.org/index.php/2018/tacas) for peer-review. Compared to the submitted paper, this version contains more up-to-date benchmarks in Section 6.} we propose a novel methodology for static analysis of binary code using abstract interpretation. We use an abstract domain based on polyhedra and two mapping functions that associate polyhedra variables with registers and memory.

We demonstrate our methodology to the problem of computing upper bounds to loop iterations in the code. This problem is particularly important in the domain of Worst-Case Execution Time (WCET) analysis of safety-critical real-time code. However, our approach is general and it can applied to other static analysis problems.

1 Introduction

In real-time systems it is important to compute upper bounds to the execution times of every function, and check that they complete before their deadlines under all possible conditions. Worst-Case Execution Time (WCET) analysis consists in computing (an upper bound to) the longest path in the code. WCET analysis is usually performed on the binary code, because it needs information on the low-level instructions executed by the hardware processor in order to compute the execution time.

In this paper, we propose a static analysis of binary code based on abstract interpretation using polyhedra. Our motivation is the need to enhance existing WCET analysis by improving the computation of upper bounds on the number of iterations in loops, and by detecting unfeasible paths.

Most analyses by abstract interpretation proposed in the literature are performed on source code. However, there are several important advantages in performing static analysis of binary code: 1) we analyze the code that actually runs on the machine, hence no need for additional assumptions on how the compiler works; 2) by gaining access to the memory layout, we can precisely identify problems with pointers (alias, buffer overflows, etc.) that are not easily identified when working only on source code; 3) We can perform the analysis even without access to the source code.
To the best of our knowledge, no other work has tackled the problem of static analysis of binary code using abstract interpretation with polyhedra. The main reason is probably the difficulty in representing the state of the program.

In fact, binary code lacks important structural information: the type of variables, their structure and relations, their scope and lifetime, etc. More specifically, in most of the existing abstract interpretation papers in the literature, the abstract state of a program is represented by constraints on the program variables. The underlying assumption is that variables are well identified and in a relatively small number. In the binary code, the notion of program variable is lost, so we can only analyse processor registers and memory locations. Unfortunately, a representation of the abstract state that includes all registers and all possible memory locations is too large to be managed easily in the analysis.

Contributions of this paper. A key observation is that the number of memory locations that are effectively used in the binary code approximately corresponds to the number of program variables in the source code (at least, it is of the same order of magnitude). Based on this observation, in this paper we propose to identify the subset of registers and memory locations to be represented in the abstract state as the analysis progresses.

To this end, we propose an abstract domain consisting of 1) a polyhedron, to represent the linear constraints on the polyhedra variables; 2) a register mapping that maps register names to polyhedra variables; 3) a memory mapping that maps addresses and their values to the respective polyhedra variables. The abstract state is constructed and modified as the analysis progresses: in particular, new variables and constraints may be added to the polyhedra as new memory locations are discovered in the code, and the two mappings are modified accordingly. We must check that our abstract representation remains consistent at all stages, for example when joining the states from two different paths in the code. To do this, we carefully consider aliasing (or equivalence) between variables. We formally prove that the operations that manipulate the abstract state are semantically consistent with the abstract interpretation framework.

We present the application of this method to the problem of computing upper bounds to loop iterations. Finally, we evaluate the performance of our method on benchmarks and we compare with existing static analysis tools to evaluate the precision of our approach.

2 Related works

Many papers on static analysis are based on abstract interpretation. Approaches vary widely depending on the abstract domain, the type of target programs (source or binary code, language, application type, etc.), and the analysis goal (i.e. the kind of information we want to discover). Here we attempt to summarise the papers that are the most closely related to our research.

Abstract interpretation using polyhedra has been first described in [1]. It has been used extensively in the context of compilers. For instance, the PAGAI [2]
analyzer processes LLVM Intermediate Representation (IR) using various abstract domains (including polyhedra) to detect several properties (such as loop invariants). Compared to our approach, LLVM IR is closer to the source code, as it contains information on variables and their types.

An important problem when dealing with binary code analysis is to figure out the set of interesting data locations used by the program. This is related to pointer analysis (the so-called aliasing problem), and has been extensively studied [3,4]. While the majority of pointer analyses have been proposed in the context of compiler optimizations, a certain number of ideas can be borrowed and applied to binary code analysis.

An example of binary code analysis by abstract interpretation is described in [5] and later improved in [6]. Compared to our approach, this method uses a different abstract domain (interval analysis with congruence, and affine relations [7]). Furthermore, the set of abstract memory locations are computed by a preprocessing analysis using IDA Pro [8]: in contrast, we determine them dynamically during the analysis.

In this paper, our approach is applied to static loop bound estimation, in the context of WCET analysis, so we compare our results with other loop bound estimation tools. The oRange tool [9] is based on an abstract interpretation method defined in [10]. It provides a very fast estimation of loop bounds, but it is restricted to C source code. SWEET [11] features a loop bound estimator, which works on an intermediary representation (ALF format). The approach is based on slicing and abstract interpretation and it generally provides very tight loop bounds even in complex cases, but the running time of the analysis appears to depend on the loop bounds, and in our experience for large loop bounds the analysis did not terminate.

Compared to these existing works, our approach combines the polyhedral domain with binary code analysis, taking into account memory accesses; our method is sound and always terminates.

3 Abstract domain

Abstract interpretation [12] is a static program analysis that provides a sound approximation of the semantics of the analyzed program. Instead of computing exact concrete program states (e.g. data valuation), abstract interpretation computes approximate abstract program states (e.g. inequalities on data valuation). The set of abstract states is called the abstract domain. Our analysis is based on the polyhedral abstract domain, to which we add information to track relations between polyhedra variables and registers or memory locations.

3.1 Polyhedra

A polyhedron \( P \) denotes a set of points in a \( \mathbb{Z} \) vector space bounded by linear constraints (equalities or inequalities). More formally, let \( C_n \) be the set of linear constraints in \( \mathbb{Z}^n \). Then \( \langle \{c_1, c_2, ..., c_m\} \rangle \) (with \( c_i \in C_n \) for \( 1 \leq i \leq m \)) denotes
the polyhedron consisting of all the vectors in \( \mathbb{Z}^N \) that satisfy constraints \( c_1, c_2, \ldots, c_m \). In our work, we only consider non-strict inequalities and equalities.

Like any other abstract domain, the polyhedral domain is equipped with operations that form a lattice \([13]\). We present these operations below, along with some other classic polyhedra operations \([13]\) that we will use in the following. Let \( p, p' \) be two polyhedra:

- We denote \( p \subseteq p' \) iff \( \forall s \in p, s \in p' \). This is the partial order of the lattice;
- We denote \( p'' = p \cup p' \) the smallest polyhedron such that \( p \subseteq p'' \) and \( p' \subseteq p'' \).
  This operation is the convex hull. It is the least upper bound of the lattice;
- We denote \( p'' = p \cap p' \) the intersection of \( p \) and \( p' \) (i.e. the union of the constraints of \( p \) and \( p' \)). It is the greatest lower bound of the lattice;
- The bottom state \( \bot \) of the lattice is the empty polyhedron (the set of constraints has no solution\(^2\)) and the top state \( \top \) is the polyhedron containing all points (the set of constraints is empty);
- We let \( \text{vars}(p) \) denote the set of variables appearing in the constraints of \( p \);
- Let \( |S| \) denote the cardinality of set \( S \). We let \( \text{proj}(p, x_1, \ldots, x_k) \) denote the projection of polyhedron \( p \) on space \( x_1, \ldots, x_k \), with \( k < |\text{vars}(p)| \) (this effectively removes other variables from the polyhedron constraints);
- We denote \( \max(p, x) \) the greatest value of \( x \) satisfying the constraints of \( p \).

### 3.2 Registers and memory

In polyhedral analysis of source code, variables of the polyhedra are related to variables of the source code. In our case, polyhedra variables are related to registers and memory locations. An abstract state is defined as a triple \((p, m, \ast)\), where \( p \) is a polyhedron, \( m \) is a register mapping and \( \ast \) is an address mapping. In the rest of the paper, the term variable refers to polyhedron variables.

Let us first consider registers. Let \( r \) be a register, \( v \) be a variable of \( \text{vars}(p) \) and \( m \) be a register mapping. Then we have \( m(r) = v \) iff \( v \) represents the value of \( r \) in \( p \). We denote \( \text{vars}_R(p) \) the image of mapping \( m \). We denote \( m[r : v] \) the mapping \( m' \) such that \( m'(r) = v \) and for every register \( r' \neq r \), \( m'(r') = m(r') \).

In other words, \( m[r : v] \) denotes a single mapping substitution.

Let us now consider memory locations. We want to keep track in our abstract state of the fact that, say variable \( x_0 \) represents a memory address, and the value at this address is represented by variable \( x_1 \). To to this, we introduce two more subsets \( \text{vars}_A(p) \) and \( \text{vars}_C(p) \) of \( \text{vars}(p) \). Variables in \( \text{vars}_A(p) \) define memory addresses, while variables in \( \text{vars}_C(p) \) define memory values. The subsets \( \text{vars}_R(p), \text{vars}_A(p) \) and \( \text{vars}_C(p) \) are disjoint, so we can distinguish between variables corresponding to memory values, memory addresses, registers or other. \( \ast \) is a partial bijection (some addresses may have no corresponding values) from \( \text{vars}_A(p) \) to \( \text{vars}_C(p) \), such that \( \ast(x_1) = x_2 \) iff \( x_2 \) represents the value at the memory address represented by \( x_1 \). The substitution \( \ast[x_1 : x_2] \) is defined similarly to \( m[r : v] \).

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\(^2\) Checking emptiness of a polyhedron over \( \mathbb{Z} \) is a complex operation. Therefore, in practice we approximate it by checking emptiness over the rationals.
3.3 Aliasing

In a general sense, aliasing occurs in a program when a data location can be accessed through several symbolic names. In our context, we define an aliasing relation between two variables $x_1$ and $x_2$ of a polyhedron $p$ as follows:

- Must alias or equivalent: the constraint $x_1 = x_2$ is true for every point in $p$. This is also denoted $x_1 \equiv x_2$. This holds iff $\langle \{x_1 = x_2\}\rangle \subseteq p$;
- May alias or overlapping: the constraint $x_1 = x_2$ is true for at least one point in $p$. This holds iff $\langle \{x_1 = x_2\}\rangle \cap p \neq \emptyset$;
- Cannot alias or independent: the constraint $x_1 = x_2$ is false for every point in $p$. This holds iff $\langle \{x_1 = x_2\}\rangle \cap p = \emptyset$.

The aliasing relation between a register $r$ and a variable $x$ is defined by the aliasing relation between $m(r)$ and $x$. Similarly, the aliasing relation between two registers $r_1, r_2$ is defined by the aliasing relation between $m(r_1)$ and $m(r_2)$.

As we will see in the following sections, while progressing in the analysis, it may happen that our algorithm generates equivalent variables to represent the same memory location. The presence of aliasing complicates the analysis, so we make sure that each memory location is represented by one single variable.

Definition 1. Let $s = (p, m, *)$ be an abstract state. We say that $s$ is a consistent state iff:

$$\forall \text{distinct } x_1, x_2 \in \text{vars}_A(p), x_1 \neq x_2$$

In a consistent state, each data location is defined by a single variable. Indeed, each register is defined by a single variable because mapping $m$ is a function. The same is true for values contained in memory locations because $*$ is a function and because the state is consistent.

To preserve the consistency of abstract states, if at some point in the analysis we detect equivalent address variables, we merge the variables and their constraints. Let $x_1, x_2$ be two distinct variables of $\text{vars}_A(p)$ such that $x_1 \equiv x_2$. We merge $x_1$ and $x_2$ as follows:

$$\text{Merge}((p, m, *), x_1, x_2) = (p_1 \cup p_2, m, *)$$

- $p_1 = p$ where $*(x_2)$ is replaced by $v_2$
- $p_2 = p$ where $*(x_1)$ is replaced by $v_1, x_2$ by $x_1$, and $*(x_2)$ by $*(x_1)$
- $* = *[x_2 : v_3]$
- $v_1 = \text{newFV}(), v_2 = \text{newFV}(), v_3 = \text{newFV}()$

Let us describe the operation in detail. We first transform $p$ into two different polyhedra $p_1$ and $p_2$. Polyhedron $p_1$ is the same as $p$, except that we eliminate the constraints on $*(x_2)$; this is done by replacing $*(x_2)$ with a new variable $v_2$ in all constraints. Polyhedron $p_2$ is the same as $p$, except that: 1) we remove the constraints on $*(x_1)$; 2) we rename $x_2$ as $x_1$ and $*(x_2)$ as $*(x_1)$ in all constraints. The polyhedron resulting from the merge is the convex-hull of $p_1$ and $p_2$. Also, $x_2$ is not needed anymore, so we can change its $*$ mapping to a new variable $v_3$. 
Equivalent address variables are merged whenever adding a new constraint, using the following unification function:

$$Unify(p, m, \ast) = \begin{cases} 
  \text{Merge}((p, m, \ast), x_1, x_2) & \text{if } \exists x_1, x_2 \in \text{vars}_A(p), x_1 \equiv x_2 \\
  (p, m, \ast) & \text{otherwise.}
\end{cases}$$

### 4 Computing abstract states

A program is represented by a graph $G =< I, E >$. The set of nodes $I$ is the set of instructions of the program. A directed edge $(b_1, b_2) \in E$ (where $E \subseteq I \times I$) represents a valid succession of two basic instructions in the program execution.

We say that a node $b_i \in I$ is a predecessor of $b_j$, and denote $b_i \rightarrow b_j$, iff $(b_i, b_j) \in E$. We say that $b_i$ dominates $b_j$, and denote $b_i \gg b_j$, iff all paths from the entry node to $b_j$ go through $b_i$. We say that node $h$ is a loop header if it has at least one predecessor $b_i$ such that $h \gg b_i$. We denote $l_h$ the loop associated to header $h$. An edge $(b_i, h)$ such that $h \gg b_i$ is called a back-edge of loop $l_h$; any other edge entering the loop header $h$ is an entry-edge.

We now describe our model of the processor architecture. We assume that all data locations have the same size and that memory accesses are aligned to the word size. We also assume that function calls are inlined. We consider a simplified instruction set made up of the following instructions\(^3\). Let $r_1, r_2, r_3$ be registers and $c$ be an integer constant:

- **OP** $r_1$ $r_2$ $r_3$: stores the result of operation $OP(r_2, r_3)$ in register $r_1$, where $OP$ denotes an arbitrary binary arithmetic or logic operation;
- **BOP** $r_1$ $r_2$: branches to the address contained in $r_1$ if condition $OP(r_2)$ is true, where $OP$ denotes an arbitrary unary logic operation;
- **LOADI** $r_1$ $c$: loads constant $c$ in register $r_1$;
- **LOAD** $r_1$ $r_2$: loads the value contained in the address designated by $r_2$ in register $r_1$;
- **STORE** $r_1$ $r_2$: stores the value of register $r_2$ at the address designated by $r_1$.

The abstract interpretation of a program consists in computing an abstract state for each edge of the program. For each node of the program, the analyses computes the state of the output edge(s) based on the state of the input edge(s). The state for the entry edge of the program is $(\top, \emptyset, \emptyset)$. The operations involved in the computation of abstract states are:

- **Update**: this is a monotonic function. It takes an instruction, the abstract state before the instruction, and returns the abstract state after it;
- **Join**: when an instruction has several input edges (due to branching) and is not a loop header, its input state is computed by the Join operation;
- **∇**: when an instruction is a loop header, its input state is computed by the widening operation $\nabla$.

\(^3\) Though this instruction set is very small, the principles of our analysis can be easily extended to support a richer set of instructions.
These operations are applied repeatedly on $G$ until a fixpoint is reached. Since the polyhedral domain admits infinite ascending chains, specific mechanisms are used to enforce the convergence to a fixpoint: this is ensured by the widening operation. A widening operation for the polyhedral domain has first been proposed in [14], and further improved in [15,16]. In our work, we use the widening operator of [15].

Finally, function $newFV()$ returns a new fresh variable that has never been used at any other point during the analysis. It is very important that the variable is fresh globally (for the whole analysis) and not only locally (for the current state), so as to avoid using the same variable in two different states for representing unrelated constraints.

Figure 1 reports an example that will be used in the rest of the section to illustrate the operations on the abstract states.

1: LOADI R1 4 6: EQ R5 R1 R2 # R5←R1==R2
2: LOADI R2 5 7: BNZ R4 R5 # Branch to 9 if R5≠0
3: LOADI R3 1000 8: STORE R3 R2
4: LOADI R4 9 9: LOAD R6 R3
5: STORE R3 R1

| Edge       | Polyhedron                                      | Registers                                      | Memory                        |
|------------|-------------------------------------------------|------------------------------------------------|-------------------------------|
| $(L6, L7)$ | $p_1 = \{(x_1 = 4, x_2 = 5, x_3 = 1000, \ldots, x_7 = (x_1 - x_2)\}$ | $m_1 = \{R_1 : x_1, R_2 : x_2, R_3 : x_3, R_4 : x_4, R_5 : x_7\}$ | $*_1 = \{x_5 : x_6\}$ |
| $(L8, L9)$ | $p_1 \cap \{x_7 = 0, x_6 = x_2\}$              | $m_1$                                          | $*_1 = \{x_5 : x_6\}$ |
| $(L7, L9)$ | $p_3 = (p_1 \cap \{x_7 = 0, x_6 = x_2\}) \cup p_1 = \{x_7 = 0, x_6 = x_2\}$ | $m_1$                                          | $*_2 = \{x_5 : x_6\}$ |
| $(L9, \text{exit})$ | $p_1 \cap \{x_{10} = x_6\}$          | $m_1[R_6 : x_{10}]$                                      | $*_2$ |

**Fig. 1.** Example of analysis

### 4.1 Binary operation

We distinguish two possible cases. In the first case, the relation $r_1 = OP(r_2, r_3)$ is linear and the $Update$ function can be defined as follows:

\[
Update(OP \ r_1 \ r_2 \ r_3, (p, m, *)) = (p', m', *)
\]

\[
p' = p \cap \{x_i = OP(m(r_2), m(r_3))\}
\]

\[
m' = m[r_1 : x_i] \quad x_i = newFV()
\]

For instance, in Figure 1 Line 6 introduces the constraint $x_7 = (x_1 - x_2)$ (the equality test expressed as a linear constraint) and $m_1(R_5) = x_7$. 
If no linear relation can be determined, we assume that \( r_1 \) can contain any value after the update. In that case, the \( \text{Update} \) function is defined as follows:

\[
\text{Update}^\prime(\text{OP } r_1 r_2 r_3, (p, m, *)) = (p, m', *)
\]

\[
m' = m[r_1 : \text{newFV}()]
\]

### 4.2 Branching

Branching instructions have two out-edges, to which different abstract states can be associated. We use \textit{filtering} to represent the impact of the condition on the abstract state. It is applied only if the branching condition is a linear constraint, otherwise the branching condition is ignored. Let \textit{taken} denote the edge corresponding to the case where the branch condition is true, and \textit{not\_taken} denote the other edge. The abstract states for these edges are computed as:

\[
\text{Update}_{\text{taken}}(\text{BOP } r_1 r_2, (p, m, *)) = (p', m, *)
\]

\[
p' = p \cap \{\{\text{BOP}(m(r_2))\}\}
\]

\[
\text{Update}_{\text{not\_taken}}(\text{BOP } r_1 r_2, (p, m, *)) = (p', m, *)
\]

\[
p' = p \cap \{\{\neg \text{BOP}(m(r_2))\}\}
\]

For instance, in Figure 1 we add the constraint \( x_7 = 0 \) on edge \((L7, L8)\) (appearing also on edge \((L8, L9)\)). The constraint \( x_7 \neq 0 \) cannot be expressed as a linear constraint, so it is not added on edge \((L7, L9)\).

### 4.3 Load

The impact of the immediate load instruction is straightforward:

\[
\text{Update}(\text{LOADI } r_1 c, (p, m, *)) = (p', m', *)
\]

\[
p' = p \cap \{\{x_i = c\}\} \quad m' = m[r_1 : x_i] \quad x_i = \text{newFV}()
\]

Let us now consider the non-immediate load instruction. If the input state contains a memory address variable that is equivalent to the load address, then in the output state the value of the destination register is the value of the memory value mapped to this address:

\[
\text{Update}(\text{LOAD } r_1 r_2, (p, m, *)) = (p', m', *)
\]

\[
p' = p \cap \{\{x_i = *(a)\}\} \quad m' = m[r_1 : x_i] \quad a \equiv r_2 \quad x_i = \text{newFV}()
\]

Otherwise, the destination register value is undefined:

\[
\text{Update}(\text{LOAD } r_1 r_2, (p, m)) = (p, m')
\]

\[
m' = m[r_1 : \text{newFV}()]
\]

For instance, in Figure 1 Line 9 we have \( x_5 \equiv r_3 \) and \( *(x_5) = x_6 \), so we introduce the constraint \( x_{10} = x_6 \) and \( m'[R6 = x_{10}] \).
4.4 Store

Again, we need to consider the impact of aliases. First, we will define two helper functions. The Create operation is used to create a new memory mapping. It takes as parameters the register containing the address ($r_d$), the register holding the value to store at this address ($r_a$), and the current abstract state.

$$Create(r_d, r_a, (p, m, *)) = (p', m, s')$$

$$p' = p \cap \{x_i = m(r_d), x_j = m(r_a)\}$$

$$s' = *[x_i : x_j] \quad x_i = newFV() \quad x_j = newFV()$$

For instance, in Figure 1 Line 5 creates a new memory mapping: it introduces the constraints $x_5 = x_3$, $x_6 = x_1$ and $s_1(x_5) = x_6$.

The Replace operation is used to handle the replacement of the value of an already mapped memory address, overwriting the previous value. It takes as parameters the variable representing the memory address ($a$), the register holding the new value ($r_a$), and the current abstract state.

$$Replace(a, r_a, (p, m, *)) = (p', m, s')$$

$$p' = p \cap \{x_i = m(r_a)\}$$

$$s' = *[a : x_i] \quad x_i = newFV()$$

For instance, in Figure 1 Line 8 replaces a previous mapping: it introduces the constraint $x_8 = x_2$ and maps $x_5$ to $x_8$ (instead of $x_6$ previously).

Finally, the new abstract state is computed as follows (note that there is at most one address variable $a$ such that $a \equiv r_1$):

$$Update(\text{STORE } r_1, r_2, (p, m, *)) = \begin{cases} Replace(a, r_2, s') & \text{if } a \equiv r_1 \\ Create(r_1, r_2, s') & \text{otherwise} \end{cases}$$

$$s' = \{ a \in A | a \text{ overlaps } r_1 \} \quad (\text{Replace}(a, r_2, (p, m, *)), (p, m, *))$$

4.5 Join and widening

The procedure for joining two abstract states is detailed in Algorithm 1. Polyhedra are joined using the classic polyhedra convex hull (line 11). Concerning register and memory mappings, we first unify equivalent variables (lines 5 and 9), so that the same variable is used to represent the same register or memory location in both input states. Then, if a memory location or register is bound in one input state and unbound in the other, it is associated with a fresh variable in the output state (lines 13 and 16), meaning that we consider that there are no constraints concerning it. The widening operation is defined exactly in the same way, except that $\nabla$ is used in place of $\cup$.

This join operation is illustrated in in Figure 1 for $join(e, e')$. Here $s_1$ corresponds to the state of $e$ and $s_2$ to the state of $e'$. At the unification step, $x_5$ in $s_1$ is equivalent to $x_5$ in $s_2$, so we substitute $x_6$ for $x_8$ in $s_2$ and in the polyhedron
Algorithm 1 Computing \((p, m, *) = \mathit{Join}((p_1, m_1, *_1), (p_2, m_2, *_2))\)

1: \(p_{1,2} \leftarrow p_1 \cap p_2;\)
2: \((p'_1, m'_1, *'_1) \leftarrow (p_1, m_1, *_1)\)
3: \(\text{for each } (x_1, x_2) \in \mathit{vars}_A(p_1) \times \mathit{vars}_A(p_2) \text{ do}\)
4: \(\text{if } x_1 \text{ is equivalent to } x_2 \text{ then}\)
5: \(\text{Replace } x_1 \text{ by } x_2 \text{ and } *_1(x_1) \text{ by } *_2(x_2) \text{ in } p'_1, m'_1, \text{ and } *'_1\)
6: \(\text{end if}\)
7: \(\text{end for}\)
8: \(\text{for Each } r \in \mathit{Dom}(m'_1) \cap \mathit{Dom}(m_2) \text{ do}\)
9: \(\text{Replace } m'_1(r) \text{ by } m_2(r) \text{ in } p'_1, m'_1, \text{ and } *'_1\)
10: \(\text{end for}\)
11: \(p \leftarrow p'_1 \cup p_2\)
12: \(\text{for each } r \in \mathit{Dom}(m'_1) \text{ do}\)
13: \(m(r) \leftarrow (m'_1(r) = m_2(r)) ? m'_1(r) : \mathit{newFV}()\)
14: \(\text{end for}\)
15: \(\text{for each } a \in \mathit{Dom}(*_1) \text{ do}\)
16: \(*_1(a) \leftarrow (*'_1(a) = *_2(a)) ? *'_1(a) : \mathit{newFV}()\)
17: \(\text{end for}\)

constraints, so we obtain \(p'_1 = (p_1 \cap \{(x_7 = 0, x_6 = x_2)\})\). Then, the convex hull regroups the constraints of \(p'_1\) and \(p_2\) on \(x_6\) so we obtain \(x_1 \leq x_6 \leq x_2\). The convex hull also lifts the constraints on \(x_7\) and \(x_8\).

4.6 Soundness

The general principle of the proof of soundness of an abstract interpretation framework is based on a soundness relation \(\sigma\), which relates the concrete semantics \(c\) of a program \(p\) to its abstract semantics \(a\). The abstraction is sound (with respect to \(\sigma\)) iff \(\sigma(c, a)\) holds for any program \(p\) \([12]\).

In our case, the abstract semantics is defined by the abstract states assigned to the edges of a program. We define the concrete state of a program as the valuation of registers and memory locations. Note that, due to branching instructions, we may have several possible valuations for the same edge. Formally, a concrete state is a set of pairs \((m_c, *_c)\). A register valuation \(m_c\) maps registers to their value, while a memory valuation \(*_c\) maps memory addresses to their value. Valuations are partial because some registers or addresses may be mapped to no value (i.e. undefined). The soundness relation is stated in the following theorem.

**Theorem 1.** For any program \(b\), for any edge \(e\) of \(b\), let \(C\) be the concrete state of \(e\) and let \((p, m, *)\) be the abstract state of \(e\) (at some point of the analysis). Then, all valuations \((m_c, *_c)\) of \(C\) satisfy the constraints of \(p\). This is denoted \(\sigma(C, (p, m, *))\).

**Proof.** We proceed by induction on the structure of the analysis (i.e. on the definition of the update, join and widening operations). The base of the induction is straightforward: the initial abstract state has no constraints so any concrete state is valid.
Now, we need to prove that, for each operation $Update$, $\nabla$, $Join$, assuming that $\sigma$ holds for the input state(s) of these operations, it also holds for their output state(s).

**Binary operation, Branching, LOADI:** Let us first consider binary operations. In the concrete semantics, we have:

$$Update_c(OP \ r_1 \ r_2 \ r_3, C) = \bigcup_{(m_c, *_c) \in C} (m_c[r_1 : OP(m_c(r_2), m_c(r_3))], *_c)$$

In the abstract semantics, if $OP$ does not define a linear relation, then no constraints are added, so $\sigma$ directly holds thanks to the induction hypothesis. Otherwise, the only new constraint is $x_i = OP(r_2, r_3)$, with $m(r_1) = x_i$. Considering the induction hypothesis and the fact that for all pairs $(m'_c, *'_c)$ of the output state, $m'_c(r_1) = OP(m'_c(r_2), m'_c(r_3))$, the new constraint holds. The $Branching$ and $LOADI$ cases are proved with similar reasoning.

**LOAD:** In the concrete semantics, we have:

$$Update_c((LOAD \ r_1 \ r_2, C) = \bigcup_{(m_c, *_c) \in C} (m_c[r_1 : *_c(m_c(r_2))], *_c)$$

The case where no address variable is equivalent to $r_2$ is trivial because it introduces no new constraints. Otherwise, we have $x_i = *a$, with $m[r_1 : x_i]$. Since $a$ is equivalent to $r_2$, the property holds.

**STORE:** In the concrete semantics, we have:

$$Update_c(STORE \ r_1 \ r_2, C) = \bigcup_{(m_c, *_c) \in C} (m_c, *_c[(m_c(r_1)) : m_c(r_2)])$$

Let us first assume that there are no polyhedron variables overlapping $r_1$. In the $create$ case, the new constraints are $x_i = m(r_1), x_j = m(r_2)$, with $*_c[x_i : x_j]$, so $\sigma$ holds. In the $replace$ case, the new constraint is $x_i = m(r_2)$, with $*_c[a : x_1]$ and $a$ equivalent to $r_1$, so $\sigma$ holds. Finally, if some variable $a$ overlaps $r_1$, we perform a $Join$ operation between $p$ and $p'$ plus the same constraints as for the $replace$ case. $a$ overlaps $r_1$ means that either $a \neq r_1$, in which case we add no constraints ($p' = p$), or $a = r_1$, in which case we add the same constraints as for the replace. The soundness of the $Join$ operation used here is proved below.

**Join and $\nabla$:** Concerning the $Join$ operation, in the concrete semantics we have:

$$Join_c(C_1, C_2) = C_1 \cup C_2$$

Let $(p_1, m_1, *_1)$ and $(p_2, m_2, *_2)$ be the abstract states corresponding to $C_1$ and $C_2$. In the abstract semantics, we join polyhedra by computing their convex hull.
By definition of the convex hull, and by the induction hypothesis, all the valuations of $C_1 \cup C_2$ satisfy the constraints of $p_1 \cup p_2$. Concerning register and memory mappings, the unification of equivalent variables (lines 8 and 9 of Algorithm 1) does not change constraints. The introduction of fresh variables, when a register or memory location is unbound in one of the input states (lines 13 and 16), effectively relaxes the constraints on it, so the soundness holds. Soundness of operation $\triangledown$ holds, from a similar reasoning.

4.7 Optimization

The complexity of our method depends on the number of variables and the number of constraints created as the analysis progresses. The number of variables introduced can be easily upper-bounded. For every instruction in the code, we introduce at most 5 variables: two new variables for STORE and 3 new variables for the Merge. This may seem a lot, however several optimizations are possible.

The most relevant is the elimination of unused variables from the polyhedra as the analysis progresses: any variable that is not in $m(.)$ or in $*(.)$ can be safely removed from the polyhedra by performing a projection on the remaining (used) variables. For example, the Merge operation unifies two existing variables, thus it is easy to see that after a Merge the number of useful variables in the polyhedron is actually reduced by one. The elimination of unused variables is implemented in the current version of our tool.

Several other optimizations are possible. An important one is to do the analysis at the level of functions: when the function ends, we can safely remove all variables that refer to the local context of the function. Also, by introducing techniques from data structure analysis, we can significantly reduce the number of variables that are necessary to investigate the properties of simple data structures like arrays. We plan to implement such optimizations as future work.

5 Loop bounds

In this section, we show how to apply our abstract interpretation to the problem of loop bounds estimation. We are interested in two types of bounds for each loop. The max bound specifies the maximum number of times the loop body will be executed, for each complete execution of the loop. The total bound specifies the total number of times the loop body will be executed in the whole program. The distinction makes sense only in the case of nested loops.

Example 1 Consider the following code snippet:

```c
int k = 0;
for (int i = 0; i < 10; i++)
  for (int j = 0; j < i; j++)
    k++;
```

In this example, the max bound of the inner loop is 9, while its total bound is 45 (that is, $\sum_{n=1}^{9} n$). The max and total bound of the outer loop are equal to 10.
The general idea of our loop bound estimation is to count loop iterations using “virtual” registers. Before starting the analysis, for each loop \( l_h \) we create two virtual registers \( rm_{lh} \) (for the max bound) and \( rt_{lh} \) (for the total bound). The constraints on these registers are updated during the abstract interpretation of the program.

To analyse max bounds, special updates are applied on loop entries and on loop back-edges (in addition to the updates defined previously in Section 4). When interpreting an entry edge of loop \( l_h \), we assign value 0 to the corresponding max bound virtual register:

\[
UpdateEntryMax((p, m, d), l_h) = Update(LOADI rm_{lh}, 0, (p, m, d))
\]

When interpreting a back-edge of loop \( l_h \), we increment the max bound virtual register (to simplify the presentation, we abusively directly add a constant value instead of a register content):

\[
UpdateIterMax(p, m, d, l_h) = Update(ADD rm_{lh}, rm_{lh}, 1, (p, m, d))
\]

Let \((p_f, m_f, d_f)\) be the final abstract state, i.e. the state obtained as output of the exit node of the program. Once the analysis is complete, the max bound of a loop \( l_h \) is computed as \( \text{max}(p_f, m[rm_{lh}]) \).

The analysis of total bounds presented here only considers triangular loops. A triangular loop (see Example 1 for instance) consists of an inner loop \( l_i \) nested inside an outer loop \( l_o \), where the index of the inner loop depends on the index of the outer loop. The objective of the analysis is to try to establish a linear relation between them.

First, we define two auxiliary functions. The function \( \text{Relation}(p, x_1, x_2) \) tries to find a linear relation between variables \( x_1 \) and \( x_2 \) in \( p \):

\[
\text{Relation}(p, x_1, x_2) = \begin{cases} 
(A, B, C) & \text{if } A.x_1 + B.x_2 \leq C \text{ in } \text{proj}(p, x_1, x_2) \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

The function \( \text{Total}(A, B, C, M) \) computes a sum based on coefficients \( A \), \( B \), \( C \) (provided they are not undefined) and on a bound \( M \):

\[
\text{Total}(A, B, C, M) = \sum_{i=0}^{M-1} \max \left( 0, \left\lfloor \frac{C - Bi}{A} \right\rfloor \right)
\]

Let us now detail the analysis. At the program entry, each total bound virtual register is set to 0. The \( \text{UpdateIterTotal} \) function is applied on the back-edge of the inner loop. It updates the total bound virtual register:

\[
\text{UpdateIterTotal}((p, m, d), l_i) = Update(ADD rt_{li}, rt_{li}, 1, p, m, d)
\]
The $\text{UpdateExitTotal}$ function is applied on the exit-edge of the inner loop. It bounds the value of $rt_i$:

$$\text{UpdateExitTotal}((p, m, d), l, l_o) = (p', m, d)$$

$$p' = p \cap \{m[rt_i] \leq t\}$$

$$(A, B, C) = \text{Relation}(p, m[rm_l], m[rm_o])$$

$$M = \max(p, m[rm_o])$$

$$t = \text{Total}(A, B, C, M)$$

Let us consider Example 1. In the input state $(p, m, *)$ of the back-edge of $l_i$ the constraint $rm_o = rm_l$ holds, so $\text{Relation}(p, rm_l, rm_o) = (1, -1, 0)$. Since $\max(p, rm_o) = 10$, we get $\text{Total}(1, -1, 0, 10) = \sum_{n=1}^{10} n = 45$. Hence, we add constraint $rt_i \leq 45$ to $p$.

Note that the virtual register $rt_i$ is not actually used to compute the total loop bound. It can however be used to analyse code executed after the nested loops. For instance, in Example 1 we obtain the constraint $k \leq 45$, should $k$ appear somewhere later in the program.

### 6 Experimental results

Our methodology is implemented in a prototype called ABCPoly, as a plugin of OTAWA (version 2.0), an open source WCET computation tool [17]. ABCPoly relies on OTAWA for CFG construction and manipulation, and on PPL [13] for polyhedra operations. The analyses have been executed on a PC with an Intel Core i5 3470 at 3.2 Ghz, with 8 Gb of RAM. Every benchmark has been compiled with ARM crosstool-NG 1.20.0 (gcc version 4.9.1) with -O1 optimization level.

First, we report the results of our experiments on the Mälardalen benchmarks [18] in Table 1. We exclude benchmarks that are not supported by OTAWA, mainly due to floating point operations or indirect branching (e.g. switch). We compare ABCPoly with SWEET [19], Pagai [2] and oRange [9]. For each benchmark, we report: the number of lines of code (in the C source), the total number of loops, the number of loops that are correctly bounded by each tool, and the computation time. We do not report the computation time for SWEET because we only had access to it through an online applet. For oRange, computation time is below the measurement resolution (10ms), except for $edn$, where it reaches 50ms.

The execution time of ABCPoly is typically higher than that of Pagai because we introduce more variables and constraints. We believe we can reduce the gap with additional optimization of the method and of the code, however ABCPoly will probably remain more costly because it works at a lower level of abstraction.

Concerning loop bounds there are two benchmarks for which ABCPoly did not find any loop bound: for bench $edn$, ABCPoly is unable to prove that there is no array out-of-bound accesses which could potentially overwrite the loop index. For $janne\_complex$, the difficulty is that it contains complex loop index updates inside a if-then-else. Furthermore, note that Pagai does not compute total loop bounds.
We further illustrate the differences between tool capabilities on the two examples of Figure 2. For both examples, ABCPoly provides the correct max and total loop bounds. For the example `foo1`, oRange fails to compute the max and total bounds of the inner loop, because it does not notice that `i-x<10`. For `foo2`, Pagai does not find the max loop bound (the loop is considered unbounded), because it does not infer that `*ptr=&bound` when executing instruction `*ptr=15`.

```c
foo1( int x) {
    int i = 0, j = 0, k = 0;
    for (i = x; i < (x + 10); i++)
        for (j = 0; j < (i - x); j++);
}

foo2() {
    int 1, bound = 10;
    int *ptr = &bound;
    ptr++; ptr--; *ptr = 15; k = 0;
    for (i = 0; i < bound; i++)
        for (i = 0; i < (x + 10); i++)
            for (j = 0; j < (i - x); j++);
}
```

Fig. 2. Loop examples

7 Conclusion

In this paper we propose a novel technique for performing abstract interpretation of binary code using polyhedra. Our method consists in adding new variables to the polyhedra as the analysis progresses, and maintaining a correspondence with registers and memory addresses. Thanks to the relational properties of polyhedra, our technique naturally provides information on pointer aliasing when compared to other techniques based on non-relational domains. While the complexity of our method is currently still relatively high, we believe that there is room for improvement: we are planning to apply some well-known techniques from static analysis to reduce the number of variables to analyse.
References

1. Cousot, P., Halbwachs, N.: Automatic discovery of linear restraints among variables of a program. In: Proceedings of the 5th ACM SIGACT-SIGPLAN symposium on Principles of programming languages, ACM (1978) 84–96
2. Henry, J., Monniaux, D., Moy, M.: Pagai: A path sensitive static analyser. Electronic Notes in Theoretical Computer Science 289 (2012) 15–25
3. Hind, M.: Pointer analysis: Haven’t we solved this problem yet? In: Proceedings of the 2001 ACM SIGPLAN-SIGSOFT Workshop on Program Analysis for Software Tools and Engineering, PASTE ’01, New York, NY, USA, ACM (2001) 54–61
4. Hardekopf, B., Lin, C.: The ant and the grasshopper: fast and accurate pointer analysis for millions of lines of code. ACM SIGPLAN Notices 42(6) (2007) 290–299
5. Balakrishnan, G., Reps, T.: Analyzing memory accesses in x86 executables. In: Compiler Construction, Springer (2004) 2732–2733
6. Reps, T., Balakrishnan, G.: Improved memory-access analysis for x86 executables. In: Compiler Construction, Springer (2008) 16–35
7. Müller-Olm, M., Seidl, H.: Precise interprocedural analysis through linear algebra. ACM SIGPLAN Notices 39(1) (2004) 330–341
8. Eagle, C.: The IDA pro book: the unofficial guide to the world’s most popular disassembler. No Starch Press (2011)
9. Bonenfant, A., de Michiel, M., Sainrat, P.: orange: A tool for static loop bound analysis. In: Workshop on Resource Analysis, University of Hertfordshire, Hatfield, UK. (2008)
10. Ammarguellat, Z., Harrison, III, W.L.: Automatic recognition of induction variables and recurrence relations by abstract interpretation. In: Proceedings of the ACM SIGPLAN 1990 Conference on Programming Language Design and Implementation. PLDI ’90, New York, NY, USA, ACM (1990) 283–295
11. Ermedahl, A., Sandberg, C., Gustafsson, J., Bygde, S., Lisper, B.: Loop Bound Analysis based on a Combination of Program Slicing, Abstract Interpretation, and Invariant Analysis. In Rochange, C., ed.: 7th International Workshop on Worst-Case Execution Time Analysis (WCET’07). Volume 6 of OpenAccess Series in Informatics (OASIcs)., Dagstuhl, Germany, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik (2007)
12. Cousot, P., Cousot, R.: Abstract interpretation frameworks. Journal of Logic and Computation 2(4) (1992) 511–547
13. Bagnara, R., Hill, P.M., Zaffanella, E.: The parma polyhedra library: Toward a complete set of numerical abstractions for the analysis and verification of hardware and software systems. Science of Computer Programming 72(1) (2008) 3–21
14. Halbwachs, N., Poy, Y., Roumanoff, P.: Verification of real-time systems using linear relation analysis. Formal Methods in System Design 11(2) (August 1997) 157–185
15. Bagnara, R., Hill, P.M., Ricci, E., Zaffanella, E.: Precise widening operators for convex polyhedra. In: Proceedings of the 10th International Conference on Static Analysis. SAS’03, Berlin, Heidelberg, Springer-Verlag (2003) 337–354
16. Simon, A., King, A. In: Widening Polyhedra with Landmarks. Springer Berlin Heidelberg, Berlin, Heidelberg (2006) 166–182
17. Ballabriga, C., Cassé, H., Rochange, C., Sainrat, P.: Otawa: An open toolbox for adaptive wcet analysis. In: Software Technologies for Embedded and Ubiquitous Systems. Volume 6399 of Lecture Notes in Computer Science. Springer Berlin Heidelberg (2010) 35–46
18. Gustafsson, J., Betts, A., Ermedahl, A., Lisper, B.: The målardalen wcet benchmarks: Past, present and future. In: OASICS-OpenAccess Series in Informatics. Volume 15., Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik (2010)

19. Lisper, B.: Sweet—a tool for wcet flow analysis. In: International Symposium On Leveraging Applications of Formal Methods, Verification and Validation, Springer (2014) 482-485