A Geometric Approach to Modulus Stabilization

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Modulus stabilization, a must for explaining the hierarchy problem in the context of the Randall-Sundrum (RS) scenario, is traditionally achieved through the introduction of an extra field with ad hoc couplings. We point out that the stabilization can, instead, be achieved in a purely geometrodynamical way, with plausible quantum corrections in the gravity sector playing the key role. The size of the corrections that lead to acceptable phenomenology is also delineated.

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Notwithstanding the recent discovery of the Higgs boson\cite{1, 2}, the lack, so far, of any definitive signature of physics beyond the Standard Model (SM) is perplexing. Although the mass of the Higgs boson is such that new physics at a nearby scale is not demanded by considerations of triviality or vacuum stability, turning this around to imply that none exists until the Planck scale ($M_{Pl}$) is, at the least, aesthetically repugnant. Indeed, the hierarchy problem of the SM continues to be a vexing issue, and, over the years, several mechanisms have been suggested to ameliorate this. While most of these scenarios also do promise explanations of some of the other puzzles that beset the SM, no direct evidence for any of the new states intrinsic to these theories have been seen so far. Furthermore, several of these have, associated with them, some form of a little hierarchy problem.

An interesting approach (RS) to the hierarchy problem essentially does away with a fundamental weak scale, ascribing the apparent hierarchy to a geometrical origin \cite{3}. Envisaging space-time to be a slice of $AdS_5$, the known world is confined to one of a pair of three-branes that sit atop the two fixed points of a $S^1/Z_2$ orbifold. The metric has the non-factorizable form

$$ds^2 = e^{-2kr_c |y|} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 dy^2,$$

with $\eta_{\mu\nu} = diag(-1,1,1,1)$ being the Minkowski metric and $y \in [0, \pi]$ with $(x^\mu, -y) \equiv (x^\mu, y)$. On the (visible) brane at $y = \pi$, the natural mass scale is suppressed by a factor of $e^{-kr_c \pi}$ with respect to the fundamental scale, e.g. that operative at the (hidden) brane located at $y = 0$. With $k \approx O(M)$ arising naturally, having $kr_c \approx 12$ would ‘solve’ the hierarchy problem. However, the modulus $r_c$ is not determined by the dynamics. This can be cured by promoting $r_c$ to a dynamical field $M$ and inventing a mechanism that forces it to settle to $M = r_c$.

To this end, ref.\cite{4} introduced a new scalar field $\phi$ in the bulk with a quadratic potential. Interacting, as it does, with $M$ through the metric, integrating out $\phi$ would result in an effective potential $V_{\text{eff}}(M)$. A suitable form for $V_{\text{eff}}(M)$ can be arranged if $m_\phi \ll M_{Pl}$ as well as if $\phi$ has brane localized potentials that ensures appropriate classical value on the branes, leading to

$$kr_c \approx \frac{k^2}{m_\phi^2} \ln \left( \frac{\phi(y = 0)}{\phi(y = \pi)} \right).$$

The apparent success of this (GW) mechanism \cite{4} hinges on the ad hoc introduction of a new fundamental scalar, with masses and couplings being just so. This criticism can also be levelled at variations of this mechanism that have been attempted \cite{5}. It would be nice if the stabilization process could have emerged more naturally. To this end, we start by postulating the five-dimensional pure gravity action, in the Jordan frame, to be

$$S_{EH} = \int d^4 x dy \sqrt{\bar{g}} \left( 2M^3 f(\bar{R}) - 2\lambda M^5 \right) - \int d^4 x dy \sqrt{\bar{g}} \left[ \bar{\Lambda} \delta(y - \pi) + \lambda_v \delta(y) \right],$$

where $M$ is the fundamental mass scale and $\bar{g}_{ab}$ the metric with $\bar{g} = -\det(\bar{g}_{ab})$. While it could have been included in $f(\bar{R})$ itself, we prefer to write the putative cosmological term explicitly, with $\lambda \lesssim O(1)$. Similarly, $\lambda_{v,h}$ are the tensions associated, respectively, with the visible and the hidden brane.

Concentrating on the bulk action, it can be rewritten as

$$S_{\text{bulk}} = \int d^4 x dy \sqrt{\bar{g}} \left( 2M^3 \bar{R} F - U - 2\lambda M^5 \right),$$

where,

$$U = 2M^3 \left[ \bar{R} F - f(\bar{R}) \right] \text{ and } F \equiv f'(\bar{R}).$$

The non-minimal coupling above can be rotated away by a conformal transformation\cite{6–8}, viz.,

$$\bar{g}_{ab} \rightarrow g_{ab} = \exp(2\omega(x^\mu, y)) \bar{g}_{ab}$$
with the actual form of $\omega(x^\mu, y)$ yet to be specified. The Ricci scalars in the two frames are related through

$$\tilde{R} = e^{2\omega} \left[ R + 8 \Box \omega - 12 g^{ab} \partial_a \omega \partial_b \omega \right],$$

with $\Box$ representing the Laplacian operator appropriate for the Einstein frame (defined in terms of $g_{ab}$). Choosing a specific form of $\omega(x^\mu, y)$, viz.,

$$\omega = \frac{1}{3} \ln F \equiv \frac{\gamma \phi}{3}, \quad \gamma \equiv \frac{5}{4 \sqrt{3} M^{3/2}},$$

we have,

$$S = \int d^4x d\tilde{y} \sqrt{g} \left[ 2 M^3 R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right]$$

$$- \int d^4x d\tilde{y} \sqrt{g} e^{-\gamma \phi} \left[ \lambda_a \delta(\tilde{y} - \ell) + \lambda_b \delta(\tilde{y}) \right]$$

where

$$V(\phi) = [U(\phi) + 2 \lambda M^3] \exp(-\gamma \phi).$$

We have, thus, successfully traded the complex form of $f(\tilde{R})$ for the usual Einstein-Hilbert action, supplemented by a scalar field that essentially encapsulates the extra degree of freedom encoded in the higher powers of derivatives in $f(\tilde{R})$. As long as the potential $V(\phi)$ is bounded from below, the system would be free from Ostrogradski instabilities.

The exact form of $V(\phi)$ would, of course, hinge on the form of $f(\tilde{R})$. Some features of the scenario, though, are ubiquitous. Unlike in the original RS scheme, the two brane tensions would, in general, be unequal in magnitude. This could have been anticipated in the Jordan frame as well, for the tensions were necessary to allow for the discontinuity in the logarithmic derivative of the metric at the orbifold fixed points; and for $f(\tilde{R}) \neq \tilde{R}$, the two junction conditions can not be expected to be equivalent. What may seem even more problematic is the existence of the bulk scalar field $\phi$, which should need be considered. If the mass of the “fluctuation field” is small, then so is the energy contained in it and neglecting the corresponding back reaction is justifiable and constitutes a very useful first approximation.

Consider the case where $V(\phi)$ has a minimum at $\phi = \phi_{\text{min}}$. Given sufficient time, one would expect that $\phi$ would settle at $\phi_{\text{min}}$ with $V(\phi_{\text{min}})$ acting as the effective cosmological constant (i.e., it would assume the role of $\Lambda$ in [3]). To the leading order, only small deviations about $\phi_{\text{min}}$ should need be considered. If the mass of the “fluctuation field” is small, then so is the energy contained in it and neglecting the corresponding back reaction is justifiable and constitutes a very useful first approximation.

While much of what follows below can be applied to a wide class of $f(\tilde{R})$, we choose to work with a series expansion in $\tilde{R}/M^2$, retaining only a few terms so as to facilitate an immediate examination of each step in the analysis, viz.

$$f(\tilde{R}) = \tilde{R} + a M^{-2} \tilde{R}^2 + b M^{-4} \tilde{R}^3.$$  \hspace{1cm} (11)

where $a, b$ are dimensionless free parameters with each, presumably, $\lesssim O(1)$. As would be expected, we need $b > 0$ for obtaining a sufficiently negative $V(\phi_{\text{min}})$ as also the desirability of a small second derivative at the minimum, whereas $a$ can assume either sign or even vanish. It is interesting to note that $f(\tilde{R}) = \tilde{R}^3$, typically, fails the twin test, unless $\beta$ is a certain very specific fraction. The corresponding potential has the form

$$V = 2 M^{-5} F^{-5/3} \left[ \lambda + a R^2(F) + 2 b R^3(F) \right],$$

where (for phenomenological reasons) we confine ourselves to a specific branch, namely

$$R(F) = \frac{-a - \sqrt{a^2 - 3b(1 - F)}}{3b}. \hspace{1cm} (13)$$

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1 This would also have been forced upon us in the GW case were back reaction to be taken into account.
We look for a situation whereby, in the Einstein frame, the only non-trivial dependence of the metric is on the coordinate \( \tilde{y} \equiv r_c y \), namely
\[
ds^2 = e^{-2\sigma(\tilde{y})} \eta_{\mu\nu} dx^\mu dx^\nu + d\tilde{y}^2
\] (14)

The Einstein’s equations reduce to
\[
6\sigma'' = \frac{1}{4M^3} \left[ \frac{1}{2} \phi'^2 - V \right],
\]
\[
3\sigma'' = \frac{1}{4M^3} \left[ \phi'^2 + e^{-\gamma\phi} (\lambda_h \delta(y) + \lambda_v \delta(y - l)) \right]
\] (15)

whereas the scalar field satisfies,
\[
\phi'' - 4\sigma' \phi' - \frac{dV}{d\phi} + \gamma e^{-\gamma \phi} [\lambda_h \delta(y) + \lambda_v \delta(y - l)] = 0.
\] (16)

Although the system can be solved numerically, it is instructive to consider an approximation so as to allow for closed form analytic expressions. Expanding around \( \phi = \phi_a \sim \phi_{\text{min}} \), we write
\[
\frac{V}{M^5} = V_0 + \left( \frac{V_1}{M^{7/2}} \right) \xi + \left( \frac{V_2}{M^2} \right) \xi^2,
\] (17)

where \( \xi(\tilde{y}) = M^{-3/2} (\phi - \phi_a) \) and \( V_i \) are constants. It might seem counterintuitive to consider \( \phi_a \neq \phi_{\text{min}} \); this, however, is useful to enhance the applicability of the approximation, which we require to be better than \( \sim 10\% \) over the range of interest (see Fig.1).

![Graph showing the potential for different values of \( a \) and \( b \)]

FIG. 1. The solid line denotes the potential for \( a = 0.01, b = 0.01, \lambda = -0.6 \), while the dashed line denotes a typical approximation for \( \phi_a = 5.06 M^{3/2} \).

Neglecting the back reaction altogether would reduce the system to the standard GW scenario with the corresponding warping \( \sigma_{(0)}(\tilde{y}) \) being linear in \( |\tilde{y}| \) with the coefficient determined by \( V_0 \). However, doing so is not really justified (even in the GW case) as it can be as much as 10\% or larger. Hence, we effect an inclusion by solving the bulk equations iteratively. Defining
\[
k^2 \equiv \frac{1}{24M^3} \left[ \frac{V_1^2}{4V_2} - V_0 \right],
\] (18)

we introduce the notation
\[
\kappa_{0,2} \equiv V_1/V_{0,2} \ , \quad N_{\pm} \equiv 2 \nu \pm 21
\] (19)

where \( \nu = \sqrt{4 + 2V_2/k^2} \). Then, the first order solution to the warping is
\[
\sigma_{(1)} = k|\tilde{y}| + \frac{1}{18M^3} \left[ c_1^2 e^{2\alpha_1|\tilde{y}|} + c_2^2 e^{2\alpha_2|\tilde{y}|} - \frac{c_1 c_2 V_2}{k^2} e^{4k|\tilde{y}|} \right]
\] (20)

where \( \alpha_{1,2} = (2 \pm \nu) k \). The dimensionless constants \( c_{1,2} \) can be determined by matching the discontinuities, leading to \( l \equiv r_c \pi \)
\[
0 = N_\pm^{-1} c_1 - N_\pm^{-1} c_2 + \left( 20\sqrt{3} + 25 \kappa_2 \right)
\]
\[
0 = N_\pm^{-1} c_1 e^{(2+\nu)kl} - N_\pm^{-1} c_2 e^{(2-\nu)kl} + \left( 20\sqrt{3} + 25 \kappa_2 \right).
\]
For large $kl$ (applicable since we need $kl \approx 36$ to explain the hierarchy), one obtains
\[
\begin{align*}
  c_1 &\simeq N_-^{-1} \left( \frac{20\sqrt{3} + 25\kappa_2}{20\sqrt{3} + 25\kappa_2} \right) \left[ e^{-2\nu kl} - e^{-(2+\nu)kl} \right], \\
  c_2 &\simeq N_+^{-1} \left( \frac{20\sqrt{3} + 25\kappa_2}{20\sqrt{3} + 25\kappa_2} \right) \left[ 1 + e^{-2\nu kl} - e^{-(2+\nu)kl} \right].
\end{align*}
\]

The nonlinear terms in eqn.(20) account for the leading backreaction due to the scalar field, which, to this order, is given by
\[
\xi(1)(\tilde{y}) = -\kappa_2/2 + M_3/2 \left[ c_1 e^{\alpha_1|\tilde{y}|} + c_2 e^{\alpha_2|\tilde{y}|} \right]
\]

One could extend this to even higher orders, with the additional corrections being given in terms of confluent hypergeometric functions. Substituting eq.(21) in the action and integrate over $\tilde{y}$, the effective potential for the modulus field is obtained to be
\[
\frac{V_{\text{eff}}}{M^3k} = \left[ d_0 + d_1(e^{-2\nu kl} - 2e^{-(2+\nu)kl}) + d_2e^{-4kl} \right]
\]
where
\[
\begin{align*}
  d_0 &= 24 - \frac{12\kappa_0\kappa_2}{\kappa_0\kappa_2 - 4} + \frac{(\nu - 2)(40\sqrt{3} + 25\kappa_2)^2}{4N_+^2} \\
  &\quad + \frac{25}{16N_-^2} \left( 5\kappa_2 + 8\sqrt{3} \right) \left( (67 - 4\nu)8\sqrt{3} + 125\kappa_2 \right), \\
  d_1 &= 250 N_-^2 N_-^{-2} \nu \left( 5\kappa_2 + 8\sqrt{3} \right) \left[ (\nu^2 + 21) \kappa_2 + 210\sqrt{3} \right], \\
  d_2 &= \frac{48 (21\nu - \nu^2 + 46)}{N_-^2} + \frac{250\sqrt{3}(\nu + 2)\kappa_2}{N_-^2} \\
  &\quad + \left( \frac{48}{\kappa_0\kappa_2 - 4} + \frac{625(4\nu - 17)\kappa_2^2}{16N_-^2} \right).
\end{align*}
\]

The consequent extrema are given by
\[
e^{(2-\nu)kl} = \frac{(2 + \nu) \pm \sqrt{(2 + \nu)^2 - 8\nu d_2/d_1}}{2\nu}
\]

Approximating $\nu \simeq 2 + \epsilon$ with $\epsilon = V_2/2k^2$, and denoting
\[
n_0 = 1 - \frac{17}{8} \sqrt{\frac{25\kappa_0\kappa_3 - \kappa_2^3 - 708}{(\kappa_0\kappa_2 - 4)(1008 + 250\sqrt{3}\kappa_2 + 25\kappa_2^2)}}
\]
on one obtains the minimum (corresponding to the negative sign above) at
\[
kl \simeq -\epsilon^{-1} \ln n_0.
\]
giving us the interrelationship between the parameters that would lead to a desired hierarchy (see Fig.2). To understand the figure, note that for $V_1 \to 0$, $n_0 \to 0.0726$ leading to a direct correspondence between the hierarchy and $V_2/2k^2$. Since $V_2$ has only a very weak dependence on $b$, this implies $\lambda \propto 1/\sqrt{b}$ for a given $k\ell$, a relation exhibited to a very large degree by the curves in Fig.2. This clearly rules out the possibility of $b = 0$ and indicates the importance of $R^3$ term. On the other hand, $a = 0$ is clearly admissible. While the relationship between $a$ and $\lambda$ (Fig.3) is more complicated, it is interesting to note that the iso-hierarchy curves tend to a fixed point in this plane with the location depending on the value of $b$.

It is important to note that, unlike in the case of the original GW mechanism [4], we do not have the liberty of choosing an ‘appropriate’ ratio of the values of the scalar field on the two branes. Since our scalar field $\phi$ is of a geometrical origin, we are not even allowed to introduce brane-localized potentials unless accompanied by a corresponding change in the geometrodynamics. In other words, the values of $\phi$ at the two branes are fixed by the warp factor $\sigma(y)$. This, in turn, fixes the value of the brane tensions $\lambda_{c,b}$ which are no longer equal and opposite, but differ slightly in magnitude (to the same extent as $\sigma(y)$ differs from linearity) so that the entire solution is a self-consistent one. As mentioned in the beginning, this is but a consequence of incorporating the back-reaction (in the Einstein frame) or, equivalently, the non-linear form of $f(R)$ (in the Jordan frame).

As it turns out, the hierarchy contour is a fairly sensitive function of the $f(R)$ parameters. This, per se, is not a negative feature of the scenario, for the exact magnitude of the warping is unknown and depends on the exact value of the Higgs mass on the Planck brane. Indeed, there is a strong disagreement between the Kaluza-Klein (KK)-graviton masses in the theory derived under the assumption that the bulk curvature be sufficiently small compared to the fundamental mass scale $M$ so as to permit a classical treatment [10] and the experimental bounds on the same [11]. It has been argued [12] that this disagreement can be alleviated if there exists a cutoff in the theory approximately two orders of magnitude lower than $M$ with the (unknown) physics intervening between this cutoff and $M$ responsible for the remaining small hierarchy. While the prescription of ref.[12] is an ad hoc one, the scenario we discuss here provides a concrete threshold in the shape of $m$, the mass of the scalar. With the latter being of geometrodynamical origin, its value is determined the same quantum corrections that determine the hierarchy and, indeed, there is a nonzero correlation between the two. For example, if one were to start with a six-dimensional doubly-warped scenario [13] (which has been shown to evade this tension [14]), a non-trivial $f(R)$ would be generated, if one were to integrate out one of the warped directions.

To summarise, we have shown that the modulus field in the RS scenario can be stabilized in purely geometrical way. Appealing to plausible quantum corrections to the Einstein-Hilbert action, we trade the higher derivatives of the metric tensor for an equivalent scalar field with a complicated potential form and a nonminimal coupling to gravity. On going over to the Einstein frame (characterized by a nonminimal coupling), the corresponding potential is seen to have a local minimum leading to an negative effective bulk cosmological constant, and a fluctuation field with a naturally small mass. The resulting framework leads to the stabilization of the modulus without the need to appeal to boundary localized interactions or neglecting the backreaction. The correct hierarchy is obtained for a wide range of parameters. Moreover, the mechanism offers a natural way out of the tension between the theoretical expectations for the KK-graviton masses and the strong bounds obtained at the LHC.

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