A LÉ-GREUEL TYPE FORMULA FOR THE IMAGE MILNOR NUMBER

J.J. NUÑO-BALLESTEROS, I. PALLARÉS-TORRES

Abstract. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \) be a corank 1 finitely determined map germ. For a generic linear form \( p : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) we denote by \( g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) the transverse slice of \( f \) with respect to \( p \). We prove that the sum of the image Milnor numbers \( \mu_I(f) + \mu_I(g) \) is equal to the number of critical points of the stratified Morse function \( p|_{X_s} : X_s \to \mathbb{C} \), where \( X_s \) is the disentanglement of \( f \) (i.e., the image of a stabilisation \( f_s \) of \( f \)).

1. Introduction

The Lé-Greuel formula [4, 6] provides a recursive method to compute the Milnor number of an isolated complete intersection singularity (ICIS). We recall that if \( (X, 0) \) is a \( d \)-dimensional ICIS defined as the zero locus of a map germ \( g : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0) \), then the Milnor fibre \( X_s = g^{-1}(s) \) (where \( s \) is a generic value in \( \mathbb{C}^{n-d} \)) has the homotopy type of a bouquet of \( d \)-spheres and the number of such spheres is called the Milnor number \( \mu(X, 0) \). If \( d > 0 \), we can take \( p : \mathbb{C}^n \to \mathbb{C} \) a generic linear projection with \( H = p^{-1}(0) \) and such that \( (X \cap H, 0) \) is a \((d-1)\)-dimensional ICIS. Then,

\[
\mu(X, 0) + \mu(X \cap H, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(g) + J(g, p)},
\]

where \( \mathcal{O}_n \) is the ring of function germs from \((\mathbb{C}^n, 0)\) to \( \mathbb{C} \), \( (g) \) is the ideal in \( \mathcal{O}_n \) generated by the components of \( g \) and \( J(g, p) \) is the Jacobian ideal of \( (g, p) \) (i.e., the ideal generated by the maximal minors of the Jacobian matrix). Note that \( X_s \) is smooth and if \( p \) is generic enough, then the restriction \( p|_{X_s} : X_s \to \mathbb{C} \) is a Morse function and the dimension appearing in the right hand side of (1) is equal to the number of critical points of \( p|_{X_s} \).

The aim of this paper is to obtain a Lé-Greuel type formula for the image Milnor number of a finitely determined map germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \). Mond showed in [10] that the disentanglement \( X_s \) (i.e., the image of a stabilisation \( f_s \) of \( f \)) has the homotopy type of a bouquet of \( n \)-spheres and the number of such spheres is called the image Milnor number \( \mu_I(f, 0) \). The celebrated Mond’s conjecture says that

\[ A_{e-codim}(f) \leq \mu_I(f), \]

with equality if \( f \) is weighted homogeneous. Mond’s conjecture is known to be true for \( n = 1, 2 \) but it remains still open for \( n \geq 3 \) (see [10, 11]). We feel that our Lé-Greuel type formula can be useful to find a proof of the

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conjecture in the general case. In fact, it would be enough to prove that the module which controls the number of critical points of a generic linear function is Cohen-Macaulay and then, use an induction argument on the dimension n (see [H] for details about Mond’s conjecture).

We assume that \( f \) has corank 1 and \( n > 1 \). Then given a generic linear form \( p : \mathbb{C}^{n+1} \to \mathbb{C} \) we can see \( f \) as a 1-parameter unfolding of another map germ \( g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0) \) which is the transverse slice of \( f \) with respect to \( p \). This means that \( g \) has image \((X \cap H, 0)\), where \((X, 0)\) is the image of \( f \) and \( H = p^{-1}(0) \). The disentanglement \( X_s \) is not smooth but it has a natural Whitney stratification given by the stable types. If \( p \) is generic enough, the restriction \( p|_{X_s} : X_s \to \mathbb{C} \) is a stratified Morse function. Our Lê-Greuel type formula is

\[
\mu_I(f) + \mu_I(g) = \#\Sigma(p|_{X_s}),
\]

where the right hand side is the number of critical points of \( p|_{X_s} \) as a stratified Morse function. The case \( n = 1 \) has to be considered separately, in this case we have

\[
\mu_I(f) + m_0(f) - 1 = \#\Sigma(p|_{X_s}),
\]

where \( m_0(f) \) is the multiplicity of the curve parametrized by \( f \). This makes sense, since \( \mu(X, 0) = m_0(X, 0) - 1 \) for a 0-dimensional ICIS \((X, 0)\).

2. Multiple point spaces and Marar’s formula

In this section we recall Marar’s formula for the Euler characteristic of the disentanglement of a corank 1 finitely determined map germ. We first recall the Marar-Mond [8] construction of the \( k \)th-multiple point spaces for corank 1 map germs, which is based on the iterated divided differences. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) be a corank 1 map germ. We can choose coordinates in the source and target such that \( f \) is written in the following form:

\[
f(x, z) = (x, f_n(x, z), \ldots, f_p(x, z)), \ x \in \mathbb{C}^{n-1}, \ z \in \mathbb{C}.
\]

This forces that if \( f(x_1, z_1) = f(x_2, z_2) \) then necessarily \( x_1 = x_2 \). Thus, it makes sense to embed the double point space of \( f \) in \( \mathbb{C}^{n+1} \) instead of \( \mathbb{C}^n \times \mathbb{C}^n \). Analogously, we will consider the \( k \)th-multiple point space embedded in \( \mathbb{C}^{n+k-1} \).

We construct an ideal \( I_k(f) \subset \mathcal{O}_{n+k-1} \) defined as follows: \( I_k(f) \) is generated by \((k-1)(p-n+1)\) functions \( \Delta_i^{(j)} \in \mathcal{O}_{n+k-1} \), \( 1 \leq i \leq k-1 \), \( n \leq j \leq p \). Each \( \Delta_i^{(j)} \) is a function only on the variables \( x, z_1, \ldots, z_{i+1} \) such that:

\[
\Delta_1^{(j)}(x, z_1, z_2) = \frac{f_j(x, z_1) - f_j(x, z_2)}{z_1 - z_2},
\]

and for \( 1 \leq i \leq k-2 \),

\[
\Delta_i^{(j)}(x, z_1, \ldots, z_{i+2}) = \frac{\Delta_i^{(j)}(x, z_1, \ldots, z_{i+1}) - \Delta_i^{(j)}(x, z_1, \ldots, z_i, z_{i+2})}{z_{i+1} - z_{i+2}}.
\]

**Definition 2.1.** The \( k \)th-multiple point space is \( D_k(f) = V(I_k(f)) \), the zero locus in \((\mathbb{C}^{n+k-1}, 0)\) of the ideal \( I_k(f) \).
We remark that the $k$th-multiple point space is denoted by $\tilde{D}^k(f)$ instead of $D^k(f)$ in \cite{8}.

If $f$ is stable, then, set-theoretically, $D^k(f)$ is the Zariski closure of the set of points $(x, z_1, \ldots, z_k) \in \mathbb{C}^{n+k-1}$ such that:

$$f(x, z_1) = \ldots = f(x, z_k), \quad z_i \neq z_j, \text{ for } i \neq j,$$

(see \cite{8,12}). But, in general, this may be not true if $f$ is not stable. For instance, consider the cusp $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $f(z) = (z^2, z^3)$. Since $f$ is one-to-one, the closure of the double point set is empty, but

$$D^2(f) = V(z_1 + z_2, z_1^2 + z_1z_2 + z_2^2).$$

This example also shows that the $k$th-multiple point space may be non-reduced in general.

The main result of Marar-Mond in \cite{8} is that the $k$th-multiple point spaces can be used to characterize the stability and the finite determinacy of $f$.

**Theorem 2.2.** \cite{8,2.12} Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ ($n < p$) be a finitely determined map germ of corank 1. Then:

1. $f$ is stable if and only if $D^k(f)$ is smooth of dimension $p-k(p-n)$, or empty, for $k \geq 2$.
2. $f$ is finitely determined if and only if for each $k$ with $p-k(p-n) \geq 0$, $D^k(f)$ is either an ICIS of dimension $p-k(p-n)$ or empty, and if, for those $k$ such that $p-k(p-n) < 0$, $D^k(f)$ consists at most of the point $\{0\}$.

The following construction is also due to Marar-Mond \cite{8} and gives a refinement of the types of multiple points.

**Definition 2.3.** Let $\mathcal{P} = (r_1, \ldots, r_m)$ be a partition of $k$ (that is, $r_1 + \ldots + r_m = k$). Let $I(\mathcal{P})$ be the ideal in $\mathcal{O}_{n-1+k}$ generated by the $k-m$ elements $z_i - z_{i+1}$ for $r_1, \ldots, r_j+1 \leq i \leq r_1, \ldots, r_j-1$, $r \leq j \leq m$. Define the ideal $I_k(f, \mathcal{P}) = I_k(f) + I(\mathcal{P})$ and the $k$-multiple point space of $f$ with respect to the partition $\mathcal{P}$ as $D^k(f, \mathcal{P}) = V(I_k(f, \mathcal{P}))$.

**Definition 2.4.** We define a generic point of $D^k(f, \mathcal{P})$ as a point

$$(x, z_1, \ldots, z_1, \ldots, z_m, \ldots, z_m),$$

($z_i$ iterated $r_i$ times, and $z_i \neq z_j$ if $i \neq j$) such that the local algebra of $f$ at $(x, z_i)$ is isomorphic to $\mathbb{C}[t]/(t^{r_i})$, and such that

$$f(x, z_1) = \ldots = f(x, z_m).$$

If $f$ is stable, then $D^k(f, \mathcal{P})$ is equal to the Zariski closure of its generic points (see \cite{8}). Moreover, we have the following corollary, which extends Theorem 2.2 to the multiple point spaces with respect to the partitions.

**Corollary 2.5.** \cite{8,2.15} If $f$ is finitely determined (resp. stable), then for each partition $\mathcal{P} = (r_1, \ldots, r_m)$ of $k$ satisfying $p-k(p-n+1) + m \geq 0$, the germ of $D^k(f, \mathcal{P})$ at $\{0\}$ is either an ICIS (resp. smooth) of dimension $p-k(p-n+1)+m$, or empty. Moreover, those $D^k(f, \mathcal{P})$ for $\mathcal{P}$ not satisfying the inequality consist at most of the single point $\{0\}$.

The next step is to observe that the $k$th-multiple point space $D^k(f)$ is invariant under the action of the $k$th symmetric group $S_k$. 


Definition 2.6. Let $M$ be a $\mathbb{Q}$-vector space upon which $S_k$ acts. Then the alternating part of $M$, denoted by $\text{Alt}_k M$, is defined to be

$$\text{Alt}_k M := \{ m \in M : \sigma(m) = \text{sign}(\sigma)m, \text{ for all } \sigma \in S_k \}.$$

The following theorem of Goryunov-Mond in [3] allows us to compute the image Milnor number of $f$ by means of a spectral sequence associated to the multiple point spaces.

Theorem 2.7. [3, 2.6] Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 map germ and $f_s$ a stabilisation of $f$, for $s \neq 0$ and $X_s$ the image of $f_s$. Then,

$$H_n(X_s, \mathbb{Q}) \cong \bigoplus_{k=2}^{n+1} \text{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})).$$

Note that since $X_s$ has the homotopy type of a wedge of $n$-spheres, the image Milnor number of $f$ is the rank of $H_n(X_s, \mathbb{Q})$. If we consider $H_n(X_s, \mathbb{Q})$ as a $\mathbb{Q}$-vector space,

$$\mu_I(f) = \dim_{\mathbb{Q}} H_n(X_s, \mathbb{Q}).$$

So, by Theorem 2.7, the image Milnor number is

$$\mu_I(f) = \sum_{k=2}^{n+1} \dim_{\mathbb{Q}} \text{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})).$$

By [3, Corollary 2.8], we can compute the alternating Euler characteristic of $D^k(f_s)$ as follows: for each partition $\mathcal{P}$, we set

$$\beta(\mathcal{P}) = \frac{\text{sign}(\mathcal{P})}{\prod_i \alpha_i!^{\alpha_i}},$$

where $\alpha_i := \# \{ j : r_j = i \}$ and $\text{sign}(\mathcal{P})$ is the number $(-1)^{k-\sum \alpha_i}$. Then,

$$\chi_{alt}(D^k(f_s)) = \sum_{|\mathcal{P}|=k} \beta(\mathcal{P}) \chi(D^k(f_s, \mathcal{P})).$$

Moreover, by Theorem 2.2 and Corollary 2.5, $D^k(f_s)$ (resp. $D^k(f_s, \mathcal{P})$) is a Milnor fibre of the ICIS $D^k(f)$ (resp. $D^k(f, \mathcal{P})$), and hence it has the homotopy type of a wedge of spheres of real dimension $\dim D^k(f) = n-k+1$ (resp. $\dim D^k(f, \mathcal{P})$). Thus,

$$\dim_{\mathbb{Q}} \text{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})) = (-1)^{n-k+1} \chi_{alt}(D^k(f_s)), $$

and

$$\chi(D^k(f_s, \mathcal{P})) = 1 + (-1)^{\dim D^k(f, \mathcal{P})} \mu(D^k(f, \mathcal{P})).$$

This gives the following strong version of Marar’s formula [7]:

$$\mu_I(f) = \sum_{k=2}^{n+1} (-1)^{n-k+1} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + (-1)^{\dim D^k(f, \mathcal{P})} \mu(D^k(f, \mathcal{P}))),$$

where the coefficients $\beta(\mathcal{P}) = 0$ when the sets $D^k(f, \mathcal{P})$ are empty, for $k = 2, \ldots, n+1$. 

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection such that $H = p^{-1}(0)$ is a generic hyperplane through the origin in $\mathbb{C}^{n+1}$. We can choose linear coordinates in $\mathbb{C}^{n+1}$ such that $p(y_1, \ldots, y_{n+1}) = y_1$. Then, we choose the coordinates in $\mathbb{C}^n$ in such a way that $f$ is written in the form

$$f(x_1, \ldots, x_{n-1}, z) = (x_1, \ldots, x_{n-1}, h_1(x_1, \ldots, x_{n-1}, z), h_2(x_1, \ldots, x_{n-1}, z)),$$

for some holomorphic functions $h_1, h_2$. We see $f$ as a 1-parameter unfolding of the map germ $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ given by

$$g(x_2, \ldots, x_{n-1}, z) = (x_2, \ldots, x_{n-1}, h_1(0, x_2, \ldots, x_{n-1}, z), h_2(0, x_2, \ldots, x_{n-1}, z)).$$

We say that $g$ is the transverse slice of $f$ with respect to the generic hyperplane $H$. If $f$ has image $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$, then the image of $g$ in $(\mathbb{C}^n, 0)$ is isomorphic to $(X \cap H, 0)$.

We take $f_s$ a stabilisation of $f$ and denote by $X_s$ the image of $f_s$ (see [10] for the definition of stabilisation). Since $f$ has corank 1, $X_s$ has a natural Whitney stratification given by the stable types of $f_s$. In fact, the strata are the submanifolds $M^k(f_s, \mathcal{P}) := k(D^k(f_s, \mathcal{P})^0 \setminus k+1(D^k+1)(f_s))$, where $D^k(f_s, \mathcal{P})^0$ is the set of generic points of $D^k(f_s, \mathcal{P})$, $\mathcal{P} : \mathbb{C}^{n+k-1} \to \mathbb{C}^{n+1}$ is the map $(x, z_1, \ldots, z_k) \mapsto f_s(x, z_1)$ and $\mathcal{P}$ runs through all the partitions of $k$ with $k = 2, \ldots, n+1$. We can choose the generic linear projection $p : \mathbb{C}^{n+1} \to \mathbb{C}$ in such a way that the restriction to each stratum $M^k(f_s, \mathcal{P})$ is a Morse function. In other words, such that the restriction $p|_{X_s} : X_s \to \mathbb{C}$ is a stratified Morse function in the sense of [2]. We will denote by $\# \Sigma(p|_{X_s})$ the number of critical points as a stratified Morse function. Our first result in this section is for the case of a plane curve.

**Theorem 3.1.** Let $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be a corank 1 finitely determined map germ. Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection, then

$$\# \Sigma(p|_{X_s}) = \mu_1(f) + m_0(f) - 1,$$

where $m_0(f)$ is the multiplicity of $f$.

**Proof.** After a change of coordinates, we can assume that

$$f(t) = (t^k, c_m t^m + c_{m+1} t^{m+1} + \ldots),$$

where $k = m_0(f)$, $m > k$ and $c_m \neq 0$. The stabilisation $f_s$ is an immersion with only transverse double points. So, its image $X_s$ has only two strata: $M^2(f_s, (1, 1))$ is a 0-dimensional stratum composed by the transverse double points and $M^1(f_s, (1))$ is a 1-dimensional stratum given by the smooth points of $X_s$. Note that the number of double points of $f_s$ is the delta invariant of the plane curve, $\delta(X, 0)$, which is equal to $\mu_1(f)$ by [11] Theorem 2.3.

Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection such that $p|_{X_s}$ is a stratified Morse function. Then:

$$\# \Sigma(p|_{X_s}) = \# M^2(f_s, (1, 1)) + \# \Sigma(p|_{M^1(f_s, (1))}) = \mu_1(f) + \# \Sigma(p|_{M^1(f_s, (1))}).$$

Since $f_s$ is a diffeomorphism on the stratum $M^1(f_s, (1))$, the number of critical points of $p|_{M^1(f_s, (1))}$ is equal to the number of critical points of $p \circ f_s$. 


Assume that \( p(x, y) = Ax + By \) with \( A \neq 0 \). Then \( p \circ f_s \) is a Morsification of the function

\[
p \circ f(t) = At^k + B(c_m t^m + c_{m+1} t^{m+1} + \ldots)
\]

The number of critical points of \( p \circ f_s \) is equal to \( \mu(p \circ f) = k - 1 = m_0(f) - 1 \), which proves our formula.

Next, we state and prove the formula for the case \( n > 1 \).

**Theorem 3.2.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \) be a corank 1 finitely determined map germ with \( n > 1 \). Let \( p : \mathbb{C}^{n+1} \to \mathbb{C} \) be a generic linear projection which defines a transverse slice \( g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0) \). Then,

\[
\#\Sigma(p|_{X_s}) = \mu_1(f) + \mu_1(g).
\]

**Proof.** By Marar’s formula (4):

\[
\mu_1(f) + \mu_1(g) = \sum_{k=2}^{n+1} (-1)^{n-k+1} \sum_{|P|=k} \beta(P)(1 + (-1)^{\dim D^k(f, P)} \mu(D^k(f, P)))
+ \sum_{k=2}^{n} (-1)^{n-k} \sum_{|P|=k} \beta(P)(1 + (-1)^{\dim D^k(g, P)} \mu(D^k(g, P)))
\]

Note that if \( \dim D^k(f, P) > 0 \), then \( \dim D^k(f, P) = 1 + \dim D^k(g, P) \). Otherwise, if \( \dim D^k(f, P) = 0 \), then \( D^k(g, P) = \emptyset \). So, we can separate the formula into two parts, the first one for partitions with \( \dim D^k(f, P) = 0 \), the second one for partitions with \( \dim D^k(f, P) > 0 \).

\[
\mu_1(f) + \mu_1(g) = \sum_{k=2}^{n+1} (-1)^{n+k-1} \sum_{\dim D^k(f, P)=0} \beta(P)(1 + \mu(D^k(f, P)))
+ \sum_{k=2}^{n} (-1)^{n+k-1} \sum_{\dim D^k(f, P)>0} \beta(P)(-1)^{\dim D^k(f, P)} (\mu(D^k(f, P)) + \mu(D^k(g, P)))
\]

If \( \dim D^k(f, P) = 0 \), the Milnor number of \( D^k(f, P) \) is

\[
\mu(D^k(f, P)) = \deg(D^k(f, P)) - 1,
\]

where \( \deg \) is the degree of the map germ that defines the 0-dimensional ICIS \( D^k(f, P) \).

We choose the coordinates such that \( p(y_1, \ldots, y_{n-1}) = y_1 \). We denote by \( \tilde{p} : \mathbb{C}^{n+k-1} \to \mathbb{C} \) the projection onto the first coordinate. Then:

\[
D^k(g, P) = D^k(f, P) \cap \tilde{p}^{-1}(0).
\]

By the Lê-Greuel formula for ICIS [4, 6],

\[
\mu(D^k(f, P)) + \mu(D^k(g, P)) = \#\Sigma(\tilde{p}|_{D^k(f, P)}).
\]
It is easy to check that $(-1)^{\dim D^k(f)} \text{sign}(\mathcal{P})(-1)^{\dim D^k(f, \mathcal{P})} = 1$ for any partition $\mathcal{P}$. Thus, we get:

$$\mu_I(f) + \mu_I(g) = \sum_{k=2}^{n+1} \sum_{|\mathcal{P}|=k} \frac{\deg(D^k(f, \mathcal{P}))}{\prod_i \alpha_i!^\alpha_i} \dim D^k(f, \mathcal{P}) = 0$$

$$+ \sum_{k=2}^{n} \sum_{|\mathcal{P}|=k} \frac{\#(\tilde{p}|D^k(f, \mathcal{P}))}{\prod_i \alpha_i!^\alpha_i} \dim D^k(f, \mathcal{P}) > 0$$

For any partition $\mathcal{P}$, the map $\epsilon^k : D^k(f_s, \mathcal{P})^0 \to \mathbb{C}^{n+1}$ is a covering map of degree $\prod_i \alpha_i!^\alpha_i$ onto its image. If $\dim D^k(f, \mathcal{P}) = 0$, then all the points of $D^k(f_s, \mathcal{P})$ are generic and moreover, $\epsilon^k(D^k(f_s, \mathcal{P}))$ does not contain points of $\epsilon^k+1(D^k(f_s, \mathcal{P}))$, so

$$\deg(D^k(f, \mathcal{P})) = \#D^k(f_s, \mathcal{P}) = (\prod_i \alpha_i!^\alpha_i) \#M^k(f_s, \mathcal{P}).$$

Otherwise, if $\dim D^k(f, \mathcal{P}) > 0$, then by the genericity of the projection we can assume that all the critical points of $\tilde{p}|D^k(f_s, \mathcal{P})$ are generic points of $D^k(f_s, \mathcal{P})$ and whose image is contained in $M^k(f_s, \mathcal{P})$, hence:

$$\#(\tilde{p}|D^k(f_s, \mathcal{P})) = (\prod_i \alpha_i!^\alpha_i) \#(p|M^k(f_s, \mathcal{P})).$$

Thus, we conclude that

$$\mu_I(f) + \mu_I(g) = \sum_{k=2}^{n+1} \sum_{|\mathcal{P}|=k} \#M^k(f_s, \mathcal{P}) \dim D^k(f, \mathcal{P}) = 0$$

$$+ \sum_{k=2}^{n} \sum_{|\mathcal{P}|=k} \#(p|M^k(f_s, \mathcal{P})) \dim D^k(f, \mathcal{P}) > 0$$

which is nothing but the number of critical points of the stratified Morse function $p|X_s$. \hfill \Box

4. Examples

In this section, we give some examples to illustrate the formulas of theorems 3.1 and 3.2.

**Example 4.1.** (The singular plane curve $E_6$)

Let $f(z) = (z^3, z^4)$ be the singular plane curve $E_6$, let $f_s(z) = (z^3 + sz, z^4 + s^2sz^2)$ be a stabilisation of $f$, for $s \neq 0$.

Let $M^2(f_s, (1, 1))$ be the 0-dimensional stratum of $X_s$. It is composed by three points, they correspond to three double transversal points. Let $M^1(f_s, (1))$ be the 1-dimensional stratum. If we compose $f_s$ with $p(z, u) = z$ there are two critical points in a neighbourhood of the origin, so $\sum p|_{X_s} = 5.$
Now, since the multiplicity of \( f \), \( m_0(f) = 3 \) and the image Milnor number of \( f \) is \( \mu_1(f) = 3 \), \( \mu_1(f) + m_0(f) - 1 = 5 \). We conclude that the formula is true.

When \( n > 1 \), we proceed in the following way: Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \) be a corank 1 finitely determined map germ written as

\[
f(x, z) = (x, h_1(x, z), h_2(x, z)), \quad x \in \mathbb{C}^{n-1}, \quad z \in \mathbb{C}.
\]

Let \( f_s \) be a stabilisation of \( f \). The image of \( f_s \) is denoted by \( X_s \). First, we calculate the number of critical points of the restriction of \( p \) to \( X_s \), for the generic linear projection \( p(y_1, \ldots, y_{n+1}) = y_1 \). We separate the image set \( X_s \) in strata of different dimensions given by stable types, which correspond to the sets \( M^k(f_s, \mathcal{P}) \). The \( n \)-dimensional stratum, \( M^1(f_s, (1)) \), is composed by the regular part of \( f_s \). So, the restriction \( p|_{M^1(f_s)} \) has not critical points.

The \((n-1)\)-dimensional stratum is composed by \( M^2(f_s, (1, 1)) \). To calculate the critical points, we will work with the inverse image by \( e^2 \), that is, \( D^2(f_s, (1, 1)) = D^2(f_s) \). The double point space \( D^2(f_s) \) is a subset of \( \mathbb{C}^{n+1} \), but we take a projection of \( D^2(f_s) \) in the first \( n \) variables. So, we denote by \( D(f_s) \) the projection of double point space in \( \mathbb{C}^n \). The double point space \( D(f_s) \) is a hypersurface in \( \mathbb{C}^n \) given by the resultant of \( P_s \) and \( Q_s \) with respect to \( z_2 \), where \( P_s = \frac{h_1(x,z_2) - h_1(x,z_1)}{z_2 - z_1} \) and \( Q_s = \frac{h_2(x,z_2) - h_2(x,z_1)}{z_2 - z_1} \). This gives the defining equation of \( D(f_s) \), denoted by \( \lambda_s(x, z) = 0 \).

To calculate the critical points of the set \( D(f_s) \) we take the linear projection \( \tilde{p}(x_1, \ldots, x_{n-1}, z) = x_1 \). Note that the hypersurface \( D(f_s) \) also contains the critical points of the other \( k \)-dimensional strata, with \( k < n - 1 \). Then, it will be sufficient to compute critical points here, in order to have all the critical points. By definition, we say that \((x_1, \ldots, x_{n-1}, z) \) is a critical point of \( \tilde{p}|_{D(f_s)} \) if \( \lambda_s(x, z) = 0 \) and \( J(\lambda_s, \tilde{p})(x, z) = 0 \), where \( J(\lambda_s, \tilde{p}) \) is the Jacobian determinant of \( \lambda \) and \( \tilde{p} \).

If a critical point of \( \tilde{p}|_{D(f_s)} \) corresponds to a \( m \)-multiple point, then we will have \( m \) critical points in \( D(f_s) \) for one in the image of \( f_s \). Thus, once the critical points of each type are obtained, we have to divide by the multiplicity of the point. In this way, we obtain the number of critical points of \( p \) in the image of \( f_s \).
On the other hand, we compute separately the image Milnor numbers of $f$ and $g$ in order to check the formulas.

Example 4.2. (The germ $F_4$ in $\mathbb{C}^3$) Let $f(x, z) = (x, z^2, z^5 + x^3z)$ be the germ $F_4$. Let $f_s(x, z) = (x, z^2, z^5 + xsz^3 + (x^3 - 5xs - s)z)$ be a stabilisation of $f$, for $s \neq 0$. By [9], $f$ is a 1-parameter unfolding of the plane curve $A_4$, $g(z) = (z^2, z^5)$ and in fact, $g$ is the transverse slice of $f$.

![Figure 3. The germ $F_4$ and its stabilisation for $s > 0$](image)

Let $M^3(f_s, (1, 1, 1)) \cup M^2(f_s, (2))$ be the 0-dimensional strata of $X_s$. In our case, there are no triple points and they appear three cross caps in $M^2(f_s, (2))$.

Let $M^2(f_s, (1, 1))$ be the 1-dimensional stratum of $X_s$. As we said before, let $D^2(f_s)$ be the double point curve in $\mathbb{C}^3$, and by projecting in the first two coordinates, we have the double point curve in $\mathbb{C}^2$, denote by $D(f_s)$.

We compute the resultant of $P_s$ and $Q_s$ respect to $z_2$, where $P_s$ and $Q_s$ are the divided differences. The double point curve of $f_s$ in $\mathbb{C}^2$ is the plane curve

$$\lambda_s(x, z) = -s - 5sx + x^3 + sxz^2 + z^4.$$  

The critical points of the restriction $p|_{D(f_s)}$ are given by $\lambda_s(x_0, z_0) = 0$ and $J(\lambda_s, \tilde{p})(x_0, z_0) = 0$, where $\tilde{p}(x, z) = x$.

![Figure 4. Cusps and tacnodes in the double point curve](image)

Nine critical points are obtained. Three of these points are cusps in $g_{x,s}$, which correspond to the three cross caps of $f_s$. Then, the other six critical points in $\tilde{p}|_{\lambda_s(x_0, z_0)=0}$ correspond to three tacnodes in $g_{x,s}$, which are represented in the double point curve when a vertical line is tangent at two points of $D(f_s)$. So, each two of these critical points in $\lambda_s$ correspond to
one tacnode of $g_{x,s}$ in $M^2(f_s, (1, 1))$. Note that in the Fig. 4 there are only two tacnodes, that is because the other is a complex tacnode.

Finally, in the 2-dimensional stratum $M^1(f_s, (1))$ there are no critical points. So, the number of critical points in $X_s$ is $\#\Sigma p|X_s = 6$, three cusps, three tacnodes and zero triple points. Then, $\#\Sigma p|X_s = C + J + T$ where $C, J, T$ are the numbers of cusps, tacnodes and triple points respectively of $g_{x,s}$. By [9], $\mu_I(f) = C + J + T - \delta(g)$. Since $g$ is a plane curve, we have that $\mu_I(g) = \delta(g)$ (see [11]). So,

$$\#\Sigma p|X_s = C + J + T = \mu_I(f) + \mu_I(g).$$

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