Quenching for a Diffusion System with Coupled Boundary Fluxes

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Abstract: In this paper, we investigate a diffusion system of two parabolic equations with more general singular coupled boundary fluxes. Within proper conditions, we prove that the finite quenching phenomenon happens to the system. And we also obtain that the quenching is non-simultaneous and the corresponding quenching rate of solutions. This extends the original work by previous authors for a heat system with coupled boundary fluxes subject to non-homogeneous Neumann boundary conditions.

Keywords: Quenching, Quenching Rate, Quenching Point, Singular Term, Parabolic System

1. Introduction

In the present work, we mainly deal with the following diffusion system with singular coupled boundary fluxes

\[ \begin{align*}
    u_t(x,t) &= u_{xx}, & v_t(x,t) &= v_{xx}, & (x,t) & \in (0,1) \times (0,T), \\
    u_x(0,t) &= f(v(0,t)), & u_x(1,t) &= 0, & t & \in (0,T), \\
    v_x(0,t) &= g(u(0,t)), & v_x(1,t) &= 0, & t & \in (0,T), \\
    u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & x & \in [0,1].
\end{align*} \]

For functions \( u_0(x) \) and \( v_0(x) \), we always assume that initial data satisfies \( u_0(x), v_0(x) \geq 0 \) and \( u_0(x), v_0(x) \leq 0 \) with \( 0 < u_0(x), v_0(x) < 1 \). To facilitate the following research, we also suppose that functions \( f(v) \) and \( g(u) \) verify the assumptions:

- \((H_1)\) \( f(v) \) and \( g(u) \) are locally Lipschitz on \( u, v \in (0,1] \);
- \((H_2)\) \( f'(v) < 0 \) and \( g'(u) < 0 \) for \( u, v \in (0,1] \);
- \((H_3)\) \( \lim_{v \to 0^+} f(v) = \infty \) and \( \lim_{u \to 0} g(u) = \infty \).

In the model (1), \( u \) and \( v \) can be thought as the temperatures of two mixed media during the heat propagation. This is a one-dimensional heat conduction rod of length 1 with positive initial temperatures \( u_0(x), v_0(x) \). At the left end \( \{x=0\} \), heat is taken away with a rate \( f(v(0,t)) \) for \( u \) and \( v \), respectively. The right end \( \{x=1\} \) is thermal isolation with \( u_x(1,t) = v_x(1,t) = 0 \). Since the assumption proposed for the system implies that the two components are coupled completely and enhanced each other in the model. It is known that the singular negative flux at the boundary \( \{x=0\} \) may result in the so called finite time quenching of solutions, which makes it so interesting to investigate the quenching phenomenon of the solutions, see [2-5, 7-8, 11-13] and some survey papers [1, 6, 10]. Right here, we say that the solution \( (u,v) \) of the problem (1) quenches, if \( (u,v) \) exists in the classical sense and is positive for all \( 0 \leq t < T \) and satisfies

\[ \lim_{t \to T^-} \inf_{0 \leq r \leq 1} \min_{\{u(x,t), v(x,t)\}} = 0. \]

If this happens, \( T \) will be called as quenching time. Since a singularity develops in the absorption term at quenching time \( T \), thus the classical solution doesn't exist anymore.

Due to the great work by many previous researchers, the blow-up problems of parabolic equation have been studied gradually matured, thus plenty of authors have begun to pay attention to the quenching phenomena and become a heated study field.

Ferreira, Pablo and Quirs. etc in [2] studied a system of heat equations coupled at the boundary.
They obtained that if \( p, q \geq 1 \), and then quenching is always simultaneous. While if \( p < 1 \) or \( q < 1 \), non-simultaneous quenching indeed occurs. If \( 0 < p, q < 1 \), then there exists initial data such that simultaneous quenching produces. Besides, if quenching is non-simultaneous and, for instance \( u \) is the quenching variable, then

\[
1 \sim (T-t)^{q+1} u(t,x) \sim (T-t)^{q+1} v(t,x), \quad x \in [0,1].
\]

Zheng and Song in [3] studied phenomena of non-simultaneous quenching to a coupled heat system

\[
u_x(t,x) = u_{xx}, \quad \nu_x(t,x) = v_{xx}, \quad (x,t) \in (0,1) \times (0,T),
\]

\[
u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T),
\]

\[
u_x(t,0) = u_x(t,0) = 0, \quad t \in (0,T),
\]

\[
u_x(t,x) = u_x(t,x) = v_x(t,x) = 0, \quad x \in [0,1].
\]

They gave an accurate non-simultaneous quenching classification and the corresponding quenching rates of (1.3) were determined as below:

\[
\frac{1}{q-1} u(1,t) \sim (T-t)^{2(pq-1)}, \quad v(1,t) \sim (T-t)^{2(pq-1)}, \quad \text{if } p, q > 1 \text{ or } p, q < 1;
\]

\[
\frac{1}{q-1} u(1,t) \sim (T-t)^{2q-1}, \quad v(1,t) \sim (T-t)^{2q-1}, \quad \text{if } p = q = 1;
\]

\[
\frac{1}{q-1} u(1,t) \sim (T-t)^{2(pq-1)}, \quad v(1,t) \sim (T-t)^{2(pq-1)}, \quad \text{if } q > p = 1;
\]

for simultaneous quenching, and

\[
\frac{1}{q-1} u(1,t) \sim (T-t)^{q+1}
\]

for non-simultaneous quenching.

Fila and Levine in [4] studied the following finite time quenching for the scalar equations

\[
u_x(t,x) = u_{xx}, \quad (x,t) \in (0,1) \times (0,T),
\]

\[
u_x(0,t) = 0, \quad u_x(1,t) = -v^{-q}(1,t), t \in (0,T),
\]

\[u(x,0) = u_0(x), \quad x \in [0,1].
\]

They obtained the quenching rate is

\[
u_x(t,x) = v_{xx}, \quad (x,t) \in (0,1) \times (0,T),
\]

\[
u_x(0,t) = 0, \quad u_x(1,t) = -v^{-q}(1,t), t \in (0,T),
\]

\[
u_x(t,0) = u_x(t,0) = 0, \quad t \in (0,T),
\]

\[
u_x(t,x) = u_x(t,x) = v_x(t,x) = 0, \quad x \in [0,1],
\]

\[
u_x(t,x) = u_{xx} - v^{-p}, \quad \nu_x(t,x) = v_{xx} - u^{-q}, \quad (x,t) \in (0,1) \times (0,T),
\]

\[
u_x(0,t) = u_x(1,t) = 0, \quad \nu_x(1,t) = 0, \quad (x,t) \in (0,1) \times (0,T),
\]

\[
u_x(t,0) = u_x(t,0) = 0, \quad t \in (0,T),
\]

\[
u_x(t,x) = u_x(t,x) = v_x(t,x) = 0, \quad x \in [0,1].
\]
Zhi and Mu in [7] studied the non-simultaneous quenching in a semilinear parabolic system

\[
\begin{align*}
 u_t &= u_{xx} + \log(\alpha v), \quad v_t = v_{xx} + \log(\beta u), \quad (x, t) \in (0,1) \times (0, T), \\
 u_x(0, t) &= u_x(1, t) = 0, \\
 v_x(0, t) &= v_x(1, t) = 0, \\
 u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x),
\end{align*}
\]

This implies the following mass estimates,

\[
0 < \int_0^1 u(x, t) \, dx \leq M - tf(N).
\]

Similarly, by integrating \((1)_1\) in the interval \([0,1]\), we can obtain

\[
\int_0^1 (v(x,t) - v_0(x)) \, dx \leq -tg(M),
\]

Consequently, there exists a finite time \(T\), such that quenching happens as \(t \to T\). Otherwise it will produce a contradiction if \(u, v\) are positive for all times.

**Lemma 2.2** There exists a positive constant \(\delta > 0\), such that

\[
\Phi(t) \leq -\delta g(\Phi(t)), \quad \Psi(t) \leq -\delta f(\Psi(t)), \quad t \in [0, T).
\]

**Proof:** Consider functions \(F = u_t + \delta v_x, \quad G = v_t + \delta u_x\), it's easy to check that \(F, G\) are solutions to the heat equation. If we choose \(\delta > 0\) small enough, for every \(x \in [0,1]\), we have \(F(x,0), G(x,0) < 0\). Notice that \(u\) and \(v\) are decreasing in time, so we can get \(F(1,t), G(1,t) < 0\). As to the flux at \(x = 0\), we have

\[
F_x = (f'(v) + \delta) v_t \geq -F(0,t), \quad G_x = (g'(v) + \delta) u_t \geq -G(0,t)
\]

with \(\delta\) small sufficiently. Thus by the maximum principle, we can obtain that \(F(x,t), G(x,t) \leq 0\) for every \(x \in [0,1]\) and \(t \in [0,T]\). The result in (2.5) is just the particular case for \(x = 0\).

Moreover, we have the following estimates via directly integrating for inequalities (2.5),

\[
\int_0^T \Phi(t) \, dt \geq C(T-t), \quad \int_0^T \Psi(t) \, dt \geq C(T-t).
\]

Within these estimates we can obtain the following corollary.

**Corollary 2.1**

\[
\int_0^T \Phi(t) \, dt \geq C(T-t), \quad \int_0^T \Psi(t) \, dt \geq C(T-t).
\]
Corollary 2.2 The quenching time is continuous with respect to the initial data.

Since the proof is similar to the Theorem 2.1 in [1], we omit here.

Lemma 2.3 There exists a constant $C > 0$ such that,

$$
\Phi'(t) \geq - C \frac{d}{dt} \left( \frac{d}{dt} \Phi(t) \right) g(\Phi), \quad \Psi'(t) \geq - C \frac{d}{dt} \left( \frac{d}{dt} \Psi(t) \right) f(\Psi).
$$

Proof: Let $H = u_x - \phi(x)f(u), \quad I = v_x - \phi(x)g(u)$, where $\phi : [0,1] \rightarrow [0,1]$ is a nonnegative, non-increasing, convex $C^2$ function such that $\phi(0) = 1, \phi(1) = 0$, and $\phi(u) \leq u_0^0(x)f(v_0(x)), \phi(u) \leq v_0^0(x)g(u_0(x))$ for $u \in [0,1]$. It’s easy to find that $H$ and $I$ are nonnegative at $t = 0$.

$$
u_t(0,t) = u_{xx}(0,t) \geq \phi'(0)f(\nu(0,t)) + f'(\nu(0,t))v_x(0,t) \geq -C\phi'(0)f(\nu(0,t))g(u(0,t)).$$

And the analogous estimate holds for $v$.

To this end, the proof of this lemma is complete.

Lemma 2.4 The quenching point is only the origin $x = 0$.

Proof: Since $H(x,t) \geq 0$, we have $u_x(x,t) \geq \phi(x)f(x) \geq \frac{f(N)}{3}$ for every such that $\phi(x_0) = \frac{1}{3}$. Therefore, we obtain $u(x,t) \geq u(0,t) + Cx$. The similar estimate also holds for $v$. Thus we can obtain that the quenching point is the origin.

Theorem 2.1 Let $(u_t, v_t)$ be time-derivatives of $(u, v)$, and then $(u_t, v_t)$ will blow up at quenching point $x = 0$ simultaneously.

Theorem 2.2 If quenching is non-simultaneous and let $u$ be the quenching variable, then

$$u(0,t) \sim g^{-1}(C(T-t)), \quad u_t(0,t) \sim C(T-t) \quad \text{and} \quad u(x,T) \sim x.$$

Proof: In Lemma 2.1, we have given the lower bound of the non-simultaneous rate, while the upper bound can be obtained easily by integrating the first estimate in (11). Using that $\Psi \geq C > 0: g(\Phi(t)) \leq C(T-t)$. As $x \rightarrow 0$, by lower estimate given in Corollary 2.1, then upper estimate follows directly from the fact that $u$ is concave; therefore

$$u_t(x,t) \leq u_x(0,t) = f(\nu(0,t)) \leq C.$$

To this end, the proof of Theorem 2.1 is complete.

3. Conclusion

Throughout this paper, we have studied the solutions of a parabolic system of heat equations coupled at the boundary through a singular flux. This system displays a singularity in finite time, which is called quenching in the literature. We obtained the quenching point is the origin, non-simultaneous quenching rates. To some degree, our work extends the
original work by previous authors for a heat system with coupled boundary fluxes for a more general boundary flux.

We have to admit that there are still many possible improvements and extensions of our results. One possibility is that we consider the diffusion process in a higher dimension. If we study the radial solutions in a ball, some similar results may hold as well. Besides, we can extend the local diffusion to nonlocal diffusion, which may be more effective to describe the real situation. Another aspect for us to improve is to find a method to identify the non-simultaneous quenching and simultaneous quenching, which once was determined by some parameters.

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