Congruences involving alternating multiple harmonic sum

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Abstract

We show that for any prime prime \( p \neq 2 \)
\[
\sum_{k=1}^{p-1} \left( -\frac{1}{k} \right)^\frac{p-1}{2} \equiv - \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^3}
\]
by expressing the l.h.s. as a combination of alternating multiple harmonic sums.

1 Introduction

In [8] Van Hamme presented several results and conjectures concerning a curious analogy between the values of certain hypergeometric series and the congruences of some of their partial sums modulo power of prime. In this paper we would like to discuss a new example of this analogy. Let us consider

\[
\sum_{k=1}^{\infty} \left( -\frac{1}{k} \right)^k \left( -\frac{1}{k} \right) = \left( \frac{1}{2} \right) + \frac{1}{2} \left( \frac{1}{2} \cdot \frac{3}{4} \right) + \frac{1}{3} \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \right) + \frac{1}{4} \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \right) + \cdots
\]

\[
= \int_0^{-1} - \frac{1}{x} \left( \frac{1}{\sqrt{1+x}} - 1 \right) \, dx = -2 \left[ \log \left( \frac{1 + \sqrt{1+x}}{2} \right) \right]_0^{-1} = 2 \log 2.
\]

Let \( p \) be a prime number, what’s the \( p \)-adic analogue of the above result? The real case suggests to replace the logarithm with some \( p \)-adic function which behaves in a similar way. It turns out that the right choice is the Fermat quotient

\[
q_p(x) = \frac{x^{p-1} - 1}{p}
\]

(which is fine since \( q_p(x \cdot y) \equiv q_p(x) + q_p(y) \pmod{p} \)), and, as shown in [7], the following congruence holds for any prime \( p \neq 2 \)

\[
\sum_{k=1}^{p-1} \left( -\frac{1}{k} \right)^k \left( -\frac{1}{k} \right) \equiv 2 q_p(2) \pmod{p}.
\]

Here we improve this result to the following statement.
Theorem 1.1. For any prime $p > 3$

$$
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \left( \frac{-\frac{1}{2}}{k} \right) \equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3}
$$

$$
\equiv \frac{(p-1)^2}{k} \pmod{p^3}
$$

where $B_n$ is the $n$-th Bernoulli number.

In the proof we will employ some new congruences for alternating multiple harmonic sums which are interesting in themselves such as

$$
H(-1, -2; p - 1) = \sum_{0 < i < j < p} \frac{(-1)^{i+j}}{ij^2} \equiv -\frac{3}{4}B_{p-3} \pmod{p},
$$

$$
H(-1, -1, 1; p - 1) = \sum_{0 < i < j < k < p} \frac{(-1)^{i+j}}{ijk} \equiv q_p(2)^3 + \frac{7}{8}B_{p-3} \pmod{p}.
$$

2 Alternating multiple harmonic sums

Let $r > 0$ and let $(a_1, a_2, \ldots, a_r) \in (\mathbb{Z}^*)^r$. For any $n \geq r$, we define the alternating multiple harmonic sum as

$$
H(a_1, a_2, \ldots, a_r; n) = \sum_{1 \leq k_1 < k_2 < \cdots < k_r \leq n} \prod_{i=1}^r \frac{\text{sign}(a_i)^{k_i}}{k_i^{a_i}}.
$$

The integers $r$ and $\sum_{i=1}^r |a_i|$ are respectively the depth and the weight of the harmonic sum.

From the definition one derives easily the shuffle relations:

$$
H(a; n) \cdot H(b; n) = H(a, b; n) + H(b, a; n) + H(a \oplus b; n),
$$

$$
H(a, b; n) \cdot H(c; n) = H(c, a, b; n) + H(a, c, b; n) + H(a, b, c; n) + H(a \oplus b, c; n) + H(a, b \oplus c; n)
$$

where $a \oplus b = \text{sign}(ab)(|a| + |b|)$.

Moreover, if $p$ is a prime, by replacing $k_i$ with $p - k_i$ we get the reversal relations:

$$
H(a, b; p - 1) \equiv H(b, a; p - 1)(-1)^{a+b}\text{sign}(ab) \pmod{p},
$$

$$
H(a, b, c; p - 1) \equiv H(c, b, a; p - 1)(-1)^{a+b+c}\text{sign}(abc) \pmod{p}.
$$

The values of several non-alternating (i.e. when all the indices are positive) harmonic sums modulo a power of prime are well known:

(i). ([4], [11]) for $a, r > 0$ and for any prime $p > ar + 2$

$$
H(\{a\}^r; p - 1) \equiv \begin{cases} 
(-1)^r \frac{a^{(ar+1)}}{2^{(ar+2)}} p^2 B_{p-ar-2} & \text{if } ar \text{ is odd} \\
(-1)^{r-1} \frac{a}{ar+2} p B_{p-ar-1} & \text{if } ar \text{ is even}
\end{cases} \pmod{p^3};
$$
Theorem 2.1. Let $H$ of depth $\leq H$. Moreover, by decomposing the sum $p$ values of multiple harmonic sums of depth $\leq 2$ when the indices are all negative.

The following result will allow us to compute the mod $p$ values of multiple harmonic sums

\begin{align*}
\text{(ii). (II) for any prime } p > 3 & \quad 
H \left( 1; \frac{p-1}{2} \right) = -2q_p(2) + p q_p(2)^2 - \frac{2}{3} p^2 q_p(2)^3 - \frac{7}{12} p^2 B_{p-3} \quad \text{(mod } p^3) \end{align*}

and for $a > 1$ and for any prime $p > a+1$

\begin{align*}
H \left( a; \frac{p-1}{2} \right) = \begin{cases} 
\frac{-2a^2-2}{a} B_{p-a} & \text{if } a \text{ is odd} \\
\frac{a(2a+1)}{2(a+1)} p B_{p-a-1} & \text{if } a \text{ is even} 
\end{cases} \quad \text{(mod } p^2) 
\end{align*}

\begin{align*}
\text{(iii). (II, II) for } a, b > 0 \text{ and for any prime } p > a+b+1 & \quad 
H(a, b; p-1) = \frac{(-1)^b}{a+b} \left( a \frac{b}{a} \right) B_{p-a-b} \quad \text{(mod } p) 
\end{align*}

(note that $B_{2n+1} = 0$ for $n > 0$).

The following result will allow us to compute the mod $p$ values of multiple harmonic sums of depth $\leq 2$ when the indices are all negative.

**Theorem 2.1.** Let $a, b > 0$ then for any prime $p \neq 2$

\begin{align*}
H(-a; p-1) = -H(a; p-1) + \frac{1}{2a-1} H \left( a; \frac{p-1}{2} \right), \\
2H(-a, -a; p-1) = H(-a; p-1)^2 - H(2a; p-1), 
\end{align*}

and

\begin{align*}
H(-a, -b; p-1) = -H(a, b; p-1) + \frac{2}{2a+b} \left( H \left( a, b; \frac{p-1}{2} \right) + (-1)^{a+b} H \left( b, a; \frac{p-1}{2} \right) \right). 
\end{align*}

**Proof.** The shuffling relation given by $H(-a; p-1)^2$ yields the second equation. As regards the first equation we simply observe that $(-1)^i/i^a$ is positive if and only if $i$ is even. We use a similar argument for the congruence: since $(-1)^i/(i^a j^b)$ is positive if and only if $i$ and $j$ are both even or if $(p-i)$ and $(p-j)$ are both even then

\begin{align*}
H(-a, -b; p-1) = -H(a, b; p-1) + \frac{2}{2a+b} \left( H \left( a, b; \frac{p-1}{2} \right) + (-1)^{a+b} H \left( b, a; \frac{p-1}{2} \right) \right). 
\end{align*}

Moreover, by decomposing the sum $H(a, b; p-1)$ we obtain

\begin{align*}
H(a, b; p-1) = H \left( a, b; \frac{p-1}{2} \right) + H \left( a; \frac{p-1}{2} \right) (-1)^b H \left( b, \frac{p-1}{2} \right) + (-1)^{a+b} H \left( b, a; \frac{p-1}{2} \right). 
\end{align*}

that is

\begin{align*}
H \left( a, b; \frac{p-1}{2} \right) + (-1)^{a+b} H \left( b, a; \frac{p-1}{2} \right) \equiv H(a, b; p-1) - H \left( a; \frac{p-1}{2} \right) (-1)^b H \left( b, \frac{p-1}{2} \right). 
\end{align*}

and the congruence follows immediately.  \qed
Corollary 2.2. For any prime \( p > 3 \)
\[
H(-1; p - 1) = -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{1}{4}p^2 B_{p-3} \quad (\text{mod } p^3),
\]
\[
H(-1, -1; p - 1) = 2q_p(2)^2 - 2pq_p(2)^3 - \frac{1}{3}p B_{p-3} \quad (\text{mod } p^2).
\]
Moreover for \( a > 1 \) and for any prime \( p > a + 1 \)
\[
H(-a; p - 1) \equiv -\frac{2^a - 2}{a^2 - 1} B_{p-a} \quad (\text{mod } p).
\]

Proof. The proof is straightforward: apply Theorem 2.1 (i), (ii), and (iii).

The following theorem is a variation of a result presented in [9].

Theorem 2.3. Let \( r > 0 \) then for any prime \( p > r + 1 \)
\[
H(\{1\}^{r-1}, -1; p - 1) \equiv (-1)^{r-1} \sum_{k=1}^{p-1} \frac{2^k}{k^r} \quad (\text{mod } p).
\]

Proof. For \( r \geq 1 \), let
\[
F_r(x) = \sum_{0<k_1<\cdots<k_r<p} \frac{x^{k_r}}{k_1 \cdots k_r} \in \mathbb{Z}_p[x] \quad \text{and} \quad f_r(x) = \sum_{0<k<p} \frac{x^k}{k^r} \in \mathbb{Z}_p[x].
\]

We show by induction that
\[
F_r(x) \equiv (-1)^{r-1} f_r(1 - x) \quad (\text{mod } p)
\]
then our congruence follows by taking \( x = -1 \).

For \( r = 1 \), since \( \binom{p}{k} = (-1)^{k-1} \frac{p^k}{k!} \) (mod \( p^2 \)) for \( 0 < k < p \) then
\[
f_1(x) = \frac{1}{p} \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} x^k = -\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} (-x)^k = 1 - (1 - x)^p - x^p \quad (\text{mod } p).
\]

Hence \( F_1(x) = f_1(x) \equiv f_1(1 - x) \quad (\text{mod } p) \).

Assume that \( r > 1 \), then the formal derivative yields
\[
\frac{d}{dx} F_r(x) = \sum_{0<k_1<\cdots<k_r<p} \frac{k_r x^{k_r-1}}{k_1 \cdots k_r} = \sum_{0<k_1<\cdots<k_{r-1}<p} \frac{1}{k_1 \cdots k_{r-1}} \sum_{k_r=k_{r-1}+1}^{p-1} x^{k_r-1}
\]
\[
= \sum_{0<k_1<\cdots<k_{r-1}<p} \frac{1}{k_1 \cdots k_{r-1}} \cdot \frac{x^{p-1} - x^{k_{r-1}}}{x - 1}
\]
\[
= \frac{x^{p-1}}{x - 1} H(\{1\}^{r-1}; p - 1) - \frac{1}{x - 1} F_{r-1}(x) \equiv \frac{F_{r-1}(x)}{1 - x} \quad (\text{mod } p).
\]

Moreover
\[
\frac{d}{dx} f_r(1 - x) = - \sum_{0<k<p} \frac{(1 - x)^{k-1}}{k^{r-1}} = - \frac{f_{r-1}(1 - x)}{1 - x}
\]

Hence, by the induction hypothesis

\[(1 - x) \frac{d}{dx} (F_r(x) + (-1)^r f_r(1 - x)) \equiv F_{r-1}(x) + (-1)^{r-1} f_{r-1}(1 - x) \equiv 0 \pmod{p}.
\]

Thus \(F_r(x) + (-1)^r f_r(1 - x) \equiv c_1 \pmod{p}\) for some constant \(c_1\) since this polynomial has degree < \(p\). Substituting in \(x = 0\) we find that by (i)

\[F_r(x) + (-1)^r f_r(1 - x) \equiv c_1 \equiv F_r(0) + (-1)^r f_r(1) = (-1)^r H(r; p - 1) \equiv 0 \pmod{p}.
\]

With the next two corollaries we have a complete list of the mod \(p\) values of the alternating multiple harmonic sums of depth and weight \(\leq 3\).

**Corollary 2.4.** The following congruences mod \(p\) hold for any prime \(p > 3\)

\[
\begin{align*}
H(1, -1; p - 1) &\equiv -H(-1, 1; p - 1) \equiv q_p(2)^2, \\
H(-1, 2; p - 1) &\equiv H(1, -2; p - 1) \equiv H(2, -1; p - 1) \equiv H(-2, 1; p - 1) \equiv \frac{1}{4} B_{p-3}, \\
H(-1, -2; p - 1) &\equiv -H(-2, -1; p - 1) \equiv -\frac{3}{4} B_{p-3}.
\end{align*}
\]

**Proof.** By Theorem 2.6 and by 2

\[H(1, -1; p - 1) \equiv -\sum_{k=1}^{p-1} \frac{q_k^2}{k^2} \equiv q_p(2)^2 \pmod{p}.
\]

By (i) and by the shuffling relation given by the product \(H(-1; p - 1)H(2; p - 1)\) we get

\[H(-1, 2; p - 1) = \frac{1}{2} H(-1; p - 1)H(2; p - 1) - \frac{1}{2} H(-3; p - 1) \equiv \frac{1}{4} B_{p-3} \pmod{p}.
\]

By (ii) and by Theorem 2.1

\[H(-1, -2; p - 1) \equiv -\frac{3}{4} H(1, 2; p - 1) - \frac{1}{4} H \left(1; \frac{p - 1}{2}\right) H \left(2; \frac{p - 1}{2}\right) \equiv -\frac{3}{4} B_{p-3} \pmod{p}.
\]

The remaining congruences follow by applying the reversal relation of depth 2.

**Corollary 2.5.** The following congruences mod \(p\) hold for any prime \(p > 3\)

\[
\begin{align*}
H(-1, 1, -1; p - 1) &\equiv 0, \\
H(1, 1, -1; p - 1) &\equiv H(-1, 1, 1; p - 1) \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{24} B_{p-3}, \\
H(-1, -1, 1; p - 1) &\equiv -H(1, -1, 1; p - 1) \equiv q_p(2)^3 + \frac{7}{8} B_{p-3}, \\
H(1, -1, 1; p - 1) &\equiv \frac{2}{3} q_p(2)^3 + \frac{1}{12} B_{p-3}, \\
H(-1, -1, -1; p - 1) &\equiv -\frac{4}{3} q_p(2)^3 - \frac{1}{6} B_{p-3}.
\end{align*}
\]
Proof. By the reversal relation of depth 3, \( H(-1, 1, -1; p - 1) \equiv -H(-1, 1, -1; p - 1) \equiv 0 \).
By Theorem 3.1 and by \( B_d \)
\[
H(1, 1, -1; p - 1) \equiv \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p(2)^3 + \frac{7}{12} H(-3, p - 1) \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{24} B_{p-3} \pmod{p}.
\]
By the shuffling relations given by the products
\[
H(1, -1; p - 1) H(-1; p - 1), \ H(1, -1; p - 1) H(1; p - 1), \text{ and } H(-1, -1; p - 1) H(-1; p - 1)
\]
we respectively find that
\[
2H(1, -1; p - 1) \equiv H(1, -1; p - 1) H(-1; p - 1) - H(1, 2; p - 1) - H(-2, -1; p - 1),
\]
\[
H(1, -1; p - 1) \equiv -2H(1, 1, -1; p - 1) - 2H(2, -1; p - 1),
\]
\[
3H(-1, -1, -1; p - 1) \equiv H(-1, -1; p - 1) H(-1; p - 1) - 2H(2, -1; p - 1).
\]
The remaining congruences follow by applying the reversal relation of depth 3. \( \Box \)

3 Proof of Theorem 1.1

The following useful identity appears in \( B_d \). Here we give an alternate proof by using
Riordan’s array method (see \( B_d \) for more examples of this technique).

Theorem 3.1. Let \( n \geq d > 0 \)
\[
d \sum_{k=1}^{n} \left( \frac{2k}{k+d} \right) \frac{x^{n-k}}{k} = \sum_{k=0}^{n-d} \left( \frac{2n}{n+d+k} \right) v_k - \left( \frac{2n}{n+d} \right)
\]
where \( v_0 = 2, v_1 = x - 2 \) and \( v_{k+1} = (x - 2)v_k - v_{k-1} \) for \( k \geq 1 \).

Proof. We first note that
\[
\left( \frac{2k}{k+d} \right) = \left( \frac{2k}{k-d} \right) = (-1)^{k-d} \frac{-k-d-1}{k-d} = \left[ z^{k-d} \right] \frac{1}{(1-z)^{k+d+1}} = \left[ z^{-1} \right] \frac{z^{d-1}}{(1-z)^{d+1}} \cdot \left( \frac{1}{z(1-z)} \right)^k.
\]
Since the residue of a derivative is zero then
\[
d \sum_{k=1}^{n} \left( \frac{2k}{k+d} \right) \frac{x^{n-k}}{k} = \left[ z^{-1} \right] x^n \frac{dz^{d-1}}{(1-z)^{d+1}} G \left( \frac{1}{xz(1-z)} \right)
\]
\[
= -\left[ z^{-1} \right] x^n \frac{dz^d}{(1-z)^d} G' \left( \frac{1}{xz(1-z)} \right) \cdot \left( \frac{1}{xz(1-z)} \right)'
\]
\[
= \left[ z^{-1} \right] \frac{z^{d-1}}{(1-z)^{n+d+1}} \frac{1 - x^n z^n (1-z)^n}{1 - xz + x^2 z^2} \cdot (1 - 2z)
\]
\[
= \left[ z^{-1} \right] \frac{z^{d-1}}{(1-z)^{n+d+1}} \frac{1 - 2z}{1 - xz + x^2 z^2}.
\]
Letting $F(z) = \sum_{k=0}^\infty v_k z^k$ and $G'(z) = \sum_{k=1}^n z^{k-1} = \frac{1-z^n}{1-z}$. Moreover
\[
\left( \frac{2n}{n+d+k} \right) = \left( \frac{2n}{n-d-k} \right) = (-1)^{n-d-k} \left( \frac{-n-d-k-1}{n-d-k} \right)
\]
\[
= [z^{n-d-k}] \frac{1}{(1-z)^{n+d+k+1}} = [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \cdot \left( \frac{z}{1-z} \right)^k
\]
Letting $F(z) = \sum_{k=0}^\infty v_k z^k = \frac{2-(x-2)z}{1-(x-2)z+z^2}$ then
\[
\sum_{k=0}^{n-d} \left( \frac{2n}{n+d+k} \right) v_k - \left( \frac{2n}{n+d} \right) = [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \cdot F \left( \frac{z}{1-z} \right) - [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}}
\]
\[
= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \left( \frac{(2 - xz)(1-z)}{1-xz + xz^2} - 1 \right)
\]
\[
= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \frac{1 - 2z}{1-xz + xz^2}.
\]

\[\square\]

**Corollary 3.2.** For any $n > 0$
\[
4^n \sum_{k=1}^{n} \left( \frac{-1}{k} \right)^k \frac{(-1)^k}{k} = -4(-1)^n \sum_{d=0}^{n} \frac{(-1)^d}{n-d} \sum_{j=0}^{d-1} \frac{(2n)}{j} - 2(-1)^n \sum_{d=0}^{n} \frac{(-1)^d}{n-d} \frac{(2n)}{d}.
\]

**Proof.** Since
\[
0 = \sum_{d=-k}^{k} (-1)^d \left( \frac{2k}{k+d} \right) = \left( \frac{2k}{k} \right) + 2 \sum_{d=1}^{k} (-1)^d \left( \frac{2k}{k+d} \right)
\]
then for any $n \geq k$
\[
(-1)^k \left( \frac{-1}{k} \right)^k = 4^{-k} \left( \frac{2k}{k} \right) = -2 \cdot 4^{-k} \sum_{d=1}^{n} (-1)^d \left( \frac{2k}{k+d} \right).
\]
For $x = 4$ then $v_k = 2$ for all $k \geq 0$ and by Theorem 3.1
\[
4^n \sum_{k=1}^{n} \frac{(-1)^k}{k} \left( \frac{-1}{k} \right)^k = -2 \sum_{d=1}^{n} \frac{4^{n-k}}{k} \sum_{k=1}^{n} (-1)^d \left( \frac{2k}{k+d} \right) = -2 \sum_{d=1}^{n} (-1)^d \sum_{k=1}^{n} \frac{4^{n-k}}{k} \left( \frac{2k}{k+d} \right)
\]
\[
= -4 \sum_{d=1}^{n} (-1)^d \sum_{k=0}^{n-d} \left( \frac{2n}{n+d+k} \right) + 2 \sum_{d=1}^{n} (-1)^d \left( \frac{2n}{n+d} \right)
\]
\[
= -4 \sum_{d=1}^{n} (-1)^d \sum_{k=1}^{n-d} \left( \frac{2n}{n-d-k} \right) - 2 \sum_{d=1}^{n} (-1)^d \left( \frac{2n}{n-d} \right)
\]
\[
= -4(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \sum_{j=0}^{d-1} \left( \frac{2n}{j} \right) - 2(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \left( \frac{2n}{d} \right).
\]

\[\square\]

We will make use of the following lemma.
Lemma 3.3. For any prime \( p \neq 2 \) and for \( 0 < j < p \)
\[
\binom{2p}{j} \equiv -2p \frac{(-1)^j}{j} + 4p^2 \frac{(-1)^j}{j} H(1; j - 1) \pmod{p^3}
\]
and
\[
\binom{2p}{p} \equiv 2 - \frac{4}{3} p^3 B_{p-3} \pmod{p^4}.
\]

Proof. It suffices to expand the binomial coefficient in this way
\[
\binom{2p}{j} = -2p \frac{(-1)^j}{j} \prod_{k=1}^{j-1} \left( 1 - \frac{2p}{k} \right) = \frac{(-1)^j}{j} \sum_{k=1}^{j-1} (-2p)^k H(\{1\}^{k-1}; j - 1).
\]
and apply (i).

Proof of Theorem 1.1. Letting \( n = p \) in the identity given by Corollary 3.2 we obtain
\[
4p \sum_{k=1}^{p} \frac{(-1)^k}{k} \left( \frac{-1}{k} \right) = 4 \sum_{0 \leq j < d < p} \frac{(-1)^d}{p - d} \left( \frac{2p}{j} \right) + 2 \sum_{0 \leq d < p} \frac{(-1)^d}{p - d} \left( \frac{2p}{d} \right).
\]
that is
\[
4p^{-1} \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \left( \frac{-1}{k} \right) = \frac{2 - \binom{2p}{p}}{4p} - \sum_{0 < d < p} \frac{(-1)^d}{d} + \sum_{0 < j < d < p} \frac{(-1)^d}{p - d} \left( \frac{2p}{j} \right) + \frac{1}{2} \sum_{0 < d < p} \frac{(-1)^d}{p - d} \left( \frac{2p}{d} \right).
\]
Now we consider each term of the r.h.s. separately. By Lemma 3.3
\[
\frac{2 - \binom{2p}{p}}{4p} \equiv \frac{1}{3} p^2 B_{p-3} \pmod{p^3}.
\]
By (ii)
\[
\sum_{0 < d < p} \frac{(-1)^d}{d} = H(-1; p - 1) = -2q_p(2) + p q_p(2)^2 - \frac{2}{3} p^2 q_p(2)^3 - \frac{1}{4} p^2 B_{p-3} \pmod{p^3}.
\]
Since for \( 0 < d < p \)
\[
\frac{1}{p - d} = -\frac{1}{d(1 - \frac{d}{p})} \equiv -\frac{1}{d} - \frac{p}{d^2} \pmod{p^2}
\]
then by Lemma 3.3 (i), and (iii) we have that
\[
\sum_{0 < d < p} \frac{(-1)^d}{d} \left( \frac{2p}{d} \right) \equiv \sum_{0 < d < p} \left( -\frac{(-1)^d}{d} - p \frac{(-1)^d}{d^2} \right) \left( -2p \frac{(-1)^d}{d} + 4p^2 \frac{(-1)^d}{d^3} H(1; d - 1) \right)
\]
\[
\equiv 2p H(2; p - 1) + 2p^2 H(3; p - 1) - 4p^2 H(1, 2; p - 1)
\]
\[
\equiv -\frac{8}{3} p^2 B_{p-3} \pmod{p^3}.
\]
In a similar way, by Lemma 3.3 and Corollaries 2.4 and 2.5 we get
\[
\sum_{0 < j < d < p} \frac{(-1)^d}{d} \left( \frac{2p}{j} \right) \equiv \sum_{0 < j < d < p} \left( -\frac{(-1)^d}{d} - p \frac{(-1)^d}{d^2} \right) \left( -2p \frac{(-1)^d}{j} + 4p^2 \frac{(-1)^d}{j^2} H(1; j - 1) \right)
\]
\[
\equiv 2p H(-1, -1; p - 1) + 2p^2 H(-1, -2; p - 1) - 4p^2 H(1, -1, -1; p - 1)
\]
\[
\equiv 4pq_p(2)^2 + \frac{4}{3} p^2 B_{p-3} \pmod{p^3}.
\]
Thus

\[ 4^{p-1} \sum_{k=1}^{p-1} \left( \frac{-1}{k} \right) \left( \frac{-1/2}{k} \right) = 2q_p(2) + 3pq_p(2)^2 + \frac{2}{3} p^2 q_p(2)^3 + \frac{7}{12} p^2 B_{p-3} \quad (\text{mod } p^3). \]

Since \( 4^{p-1} = (q_p(2)p + 1)^2 = 1 + 2q_p(2)p + q_p(2)^2 p^2 \) then

\[ 4^{-(p-1)} = (1 + 2q_p(2)p + q_p(2)^2 p^2)^{-1} \equiv 1 - 2q_p(2)p + 3q_p(2)^2 p^2 \quad (\text{mod } p^3). \]

Finally

\[ \sum_{k=1}^{p-1} \frac{\left( \frac{-1}{k} \right) \left( \frac{-1/2}{k} \right)}{k} \equiv (1 - 2q_p(2)p + 3q_p(2)^2 p^2) \left( 2q_p(2) + 3pq_p(2)^2 + \frac{2}{3} p^2 q_p(2)^3 + \frac{7}{12} p^2 B_{p-3} \right) \]

\[ \equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3} p^2 q_p(2)^3 + \frac{7}{12} p^2 B_{p-3} \quad (\text{mod } p^3). \]

Note that by (ii) the r.h.s. is just \(-H(1, (p - 1)/2) = - \sum_{k=1}^{(p-1)/2} \frac{1}{k} \).

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