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Une inégalité de Korn non linéaire sur une surface avec une majoration explicite de la constante

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Abstract. A nonlinear Korn inequality on a surface estimates a distance between a surface $\theta(\omega)$ and another surface $\phi(\omega)$ in terms of distances between their fundamental forms in the space $L^p(\omega)$, $1 < p < \infty$.

We establish a new inequality of this type. The novelty is that the immersion $\theta$ belongs to a specific set of mappings of class $C^1$ from $\omega$ into $\mathbb{R}^3$ with a unit vector field also of class $C^1$ over $\overline{\omega}$.

Résumé. Une inégalité de Korn non linéaire sur une surface estime une distance entre une surface $\theta(\omega)$ et une autre surface $\phi(\omega)$ en fonction des distances entre leur formes fondamentales dans l’espace $L^p(\omega)$, $1 < p < \infty$.

Nous établissons une nouvelle inégalité de ce type. La nouveauté réside dans l’appartenance de l’immersion $\theta$ à un ensemble particulier d’applications de classe $C^1$ de $\overline{\omega}$ dans $\mathbb{R}^3$ avec un champ de vecteurs normaux unitaires aussi de classe $C^1$ dans $\overline{\omega}$.

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1. Notation and definitions

Vector and matrix fields are denoted by boldface letters.

Given any open set $\Omega \subset \mathbb{R}^n$, any subset $V \subset Y$ of a finite-dimensional vector space $Y$, and any integer $\ell \geq 0$, the notation $\mathcal{C}^\ell(\Omega; V)$ designates the set of all fields $v = (v_i): \Omega \to Y$ such that $v(x) \in V$ for all $x \in \Omega$ and $v_i \in \mathcal{C}^\ell(\Omega)$. Likewise, given any real number $p > 1$, the notation $L^p(\Omega; V)$, resp. $W^{\ell,p}(\Omega; V)$, designates the set of all fields $v = (v_i): \Omega \to Y$ such that $v(x) \in V$ for almost all $x \in \Omega$ and $v_i \in L^p(\Omega)$, resp. $v_i \in W^{\ell,p}(\Omega)$.

The space of all real matrices with $k$ rows and $\ell$ columns is denoted $\mathbb{M}^{k \times \ell}$. We also let

$$
\mathbb{M}^k := \mathbb{M}^{k \times k}, \quad \mathbb{S}^k := \left\{ A \in \mathbb{M}^k; A = A^T \right\},
$$

$$
\mathbb{S}_+^k := \left\{ A \in \mathbb{S}^k; A \text{ is positive-definite} \right\}, \quad \mathbb{O}_k := \left\{ A \in \mathbb{M}^k; AA^T = I \right\}.
$$

A $k \times \ell$ matrix whose column vectors are the vectors $v_1, \ldots, v_\ell \in \mathbb{R}^k$ is denoted $(v_1 \ldots v_\ell)$. If $A \in \mathbb{S}_+^k$, there exists a unique matrix $U \in \mathbb{S}^k$ such that $U^2 = A$; this being the case, we let $A^{1/2} := U$.

The Euclidean norm in $\mathbb{R}^n$ is denoted $|\cdot|$. Spaces of matrices are equipped with the Frobenius norm, also denoted $|\cdot|$. The spaces $L^p(\Omega)$, $L^p(\Omega; \mathbb{R}^k)$, and $L^p(\Omega; \mathbb{M}^{k \times \ell})$, are respectively equipped with the norms denoted and defined by

$$
\|u\|_{L^p(\Omega)} := \left( \int_\Omega |u(x)|^p \, dx \right)^{1/p}, \quad \|v\|_{L^p(\Omega)} := \left( \int_\Omega |v(x)|^p \, dx \right)^{1/p},
$$

and

$$
\|A\|_{L^p(\Omega)} := \left( \int_\Omega |A(x)|^p \, dx \right)^{1/p}.
$$

A domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$, is a connected and open subset of $\mathbb{R}^n$ that is bounded and has a Lipschitz-continuous boundary, the set $\Omega$ being locally on the same side of its boundary (cf. Adams [1], Maz'ya [10], or Nečas [11]).

Given an open subset $\Omega$ of $\mathbb{R}^n$ and any integer $\ell \geq 0$, the notation $\mathcal{C}^\ell(\overline{\Omega})$ designates the space of all functions $u \in \mathcal{C}^\ell(\overline{\Omega})$ such that $u$ and all its partial derivatives up to order $\ell$ possess continuous extensions to the closure $\overline{\Omega}$ of $\Omega$. If $\Omega \subset \mathbb{R}^n$ is a domain, then $\mathcal{C}^\ell(\overline{\Omega}) = \{f|_{\overline{\Omega}}; f \in \mathcal{C}^\ell(\mathbb{R}^n)\}$, where $f|_{\overline{\Omega}}$ denotes the restriction of the function $f$ to the set $\overline{\Omega}$, thanks to Whitney’s extension theorem: cf. Whitney [12]; see also Ciarlet & Mardare [5].

Given a connected open subset $\Omega$ of $\mathbb{R}^n$, the geodesic distance between two points $x, y \in \Omega$ is defined by

$$
\text{dist}_\Omega(x, y) := \inf \left\{ \ell \in \mathbb{R}; \text{there exists a path } c \in \mathcal{C}^1([0, \ell]; \mathbb{R}^n) \text{ such that } c(0) = x, \ c(\ell) = y, \ c(s) \in \Omega \text{ and } |c'(s)| = 1 \text{ for all } s \in (0, \ell) \right\}.
$$

If $\Omega \subset \mathbb{R}^n$ is a domain, then there exists a constant $C_\Omega \geq 1$ such that

$$
\text{dist}_\Omega(x, y) \leq C_\Omega |x - y| \text{ for all } x, y \in \overline{\Omega};
$$

see e.g. Anićić, Le Dret & Raoult [2, Proposition 5.1].

A generic point in an open subset $\omega$ of $\mathbb{R}^2$ is denoted $y = (y_1, y_2)$ and partial derivative operators with respect to $y_1$ and $y_2$ are denoted $\partial_1$ and $\partial_2$.

2. Main result

An immersion of class $\mathcal{C}^1$ from a two-dimensional domain $\omega \subset \mathbb{R}^2$ into the three-dimensional Euclidean space $\mathbb{R}^3$ is a mapping $\phi: \omega \to \mathbb{R}^3$ of class $\mathcal{C}^1$ such that the two vector fields $\partial_1 \phi: \omega \to \mathbb{R}^3$ and $\partial_2 \phi: \omega \to \mathbb{R}^3$ are linearly independent at each point of $\omega$. This means that the image
of \( \omega \) by \( \phi \) is a surface in \( \mathbb{R}^3 \) whose tangent plane at its point \( \phi(y), y \in \omega \), is spanned by the two vectors \( \partial_1 \phi(y) \) and \( \partial_2 \phi(y) \). Consequently,

\[
\nu(\phi) := \frac{\partial_1 \phi \wedge \partial_2 \phi}{|\partial_1 \phi \wedge \partial_2 \phi|}
\]

is a continuous unit vector field from \( \omega \) into \( \mathbb{R}^3 \) that is normal to the surface \( \phi(\omega) \).

Given an immersion \( \phi : \omega \to \mathbb{R}^3 \) of class \( \mathcal{C}^1 \), we let

\[
\nabla \phi := (\partial_1 \phi \mid \partial_2 \phi) \quad \text{and} \quad A(\phi) := \nabla^T \nu(\phi) .
\]

Note that \( \nabla \phi \) is field of \( 3 \times 2 \) matrices whose column vectors are the partial derivatives of \( \phi \) and that \( A(\phi) \) is a field of \( 2 \times 2 \) positive-definite symmetric matrices whose components are the covariant components of the first fundamental form associated with the immersion \( \phi \).

Given an immersion \( \phi : \omega \to \mathbb{R}^3 \) of class \( \mathcal{C}^1 \) such that the unit vector field \( \nu(\phi) : \omega \to \mathbb{R}^3 \) is also of class \( \mathcal{C}^1 \), we let

\[
\nabla \nu(\phi) := (\partial_1 \nu(\phi) \mid \partial_2 \nu(\phi)) \quad \text{and} \quad B(\phi) := \nabla^T \nabla \nu(\phi) .
\]

Note that \( \nabla \nu(\phi) \) is field of \( 3 \times 2 \) matrices whose column vectors are the partial derivatives of the vector field \( \nu(\phi) \) and that \( B(\phi) \) is a field of \( 2 \times 2 \) symmetric matrices whose components are the covariant components of the second fundamental form associated with the immersion \( \phi \).

The above definitions and notations apply as well to immersions \( \phi : \omega \to \mathbb{R}^3 \) and their associated unit vector fields \( \nu(\phi) : \omega \to \mathbb{R}^3 \) that are both of class \( \mathcal{C}^1 \) up to the boundary of \( \omega \), or of class \( W^{1,p} \) in \( \omega, 1 < p < \infty \). This being the case, we let

\[
\mathcal{C}^1(\overline{\omega}; \mathbb{R}^3) := \{ \phi \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3) ; |\partial_1 \phi \wedge \partial_2 \phi| > 0 \text{ in } \overline{\omega}, \nu(\phi) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3) \}
\]

and

\[
W^{1,p}(\omega; \mathbb{R}^3) := \{ \phi \in W^{1,p}(\omega; \mathbb{R}^3) ; |\partial_1 \phi \wedge \partial_2 \phi| > 0 \text{ a.e. in } \omega, \nu(\phi) \in W^{1,p}(\omega; \mathbb{R}^3) \}.
\]

The objective of this Note is to indicate how to establish a nonlinear Korn inequality with an explicit estimate of the constant that appears in it for mappings \( \phi \in W^{1,p}_+(\omega; \mathbb{R}^3) \) and \( \theta \in \mathcal{C}_+^1(\overline{\omega}; \mathbb{R}^3) \); see Theorem 1 below. We will show in particular that the estimate for the constant depends on \( \theta \) only via two scalar parameters, denoted \( \rho \) and \( \delta \) in what follows, which are related to the assumption that \( \theta \) is an immersion such that \( \theta \) and \( \nu(\theta) \) are continuously differentiable vector fields over \( \overline{\omega} \).

Note that the nonlinear Korn inequality of Theorem 1 constitutes an improvement, when \( n = 3 \), over two previous results by the authors about hypersurfaces in \( \mathbb{R}^n \), \( n \geq 3 \): see [7, Theorem 3.1 and Lemma 3.2], or [6, Lemma 2].

The definition of the constant \( C_\omega \) in the next statement is justified by relations (1)-(2) in Section 1.

**Theorem 1.** Given any domain \( \omega \subset \mathbb{R}^2 \) and any real numbers \( p > 1, 1 \geq \rho > 0 \) and \( \delta > 0 \), there exists a constant \( C = C(\omega, p, \rho, \delta) \) such that

\[
\inf_{R \in \mathcal{O}_r^+} \left( \| \nu(\phi) - R \nu(\theta) \|_{L^p(\omega)} + \| \nabla \phi - R \nabla \theta \|_{L^p(\omega)} + \| \nabla \nu(\phi) - R \nabla \nu(\theta) \|_{L^p(\omega)} \right)
\leq C \inf_{R \in \mathcal{O}_r^+} \left( \| \nu(\phi) - R \nu(\theta) \|_{L^p(\omega)} + \| \nabla \phi - R \nabla \theta \|_{L^p(\omega)} + \| \nabla \nu(\phi) - R \nabla \nu(\theta) \|_{L^p(\omega)} \right)
\leq C \sqrt{3} \left( \| A(\phi)^{1/2} - A(\theta)^{1/2} \|_{L^p(\omega)} + \| A(\phi)^{-1/2} B(\phi) - A(\theta)^{-1/2} B(\theta) \|_{L^p(\omega)} \right)
\]
for all mappings $\phi \in W^{1,p}_+(\omega;\mathbb{R}^3)$ and $\theta \in \mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$, where

$$
\mu > 0 \text{ is any real number such that } \mu \leq \frac{\rho^{11}}{468(1 + C_\omega)}.
$$

$C_\omega \geq 1$ is any constant such that $\text{dist}_\omega(y, \bar{y}) \leq C_\omega |y - \bar{y}|$ for all $y, \bar{y} \in \omega$.

and

$$
\mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3) := \left\{ \theta \in \mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3) : \inf_{y \in \overline{\omega}} |\partial_1 \theta(y) \land \partial_2 \theta(y)| \geq \rho, \sup_{y \in \overline{\omega}} |\nabla \theta(y)| \leq \frac{1}{\rho}, \sup_{y \in \overline{\omega}} |\nabla \nabla \theta(y)| \leq \frac{1}{\rho}, \right. \left. \begin{array}{c}
\sup_{y, \bar{y} \in \overline{\omega}, |y - \bar{y}| \leq \delta} |\nabla \theta(y) - \nabla \theta(\bar{y})| \leq \mu, \\
\sup_{y, \bar{y} \in \overline{\omega}, |y - \bar{y}| \leq \delta} |\nabla \nabla \theta(y) - \nabla \nabla \theta(\bar{y})| \leq \mu. 
\end{array} \right\}.
$$

The proof of the Theorem 1 is sketched in Section 3 below; the details are given in [9].

The restriction in Theorem 1 that $\theta$ belongs to the subset $\mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ of the set $\mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ itself, is essential (i.e., not merely an artefact of the proof).

However, this inconvenience is alleviated by the fact that, as $\rho \to 0^+$ and $\delta \to 0^+$, the subset $\mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ becomes as large in $\mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ as one wants. More specifically, for each $\mu > 0$,

$$
\mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3) = \lim_{\rho \to 0^+} \left( \lim_{\delta \to 0^+} \mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3) \right),
$$

where the limits above are defined as the union of an increasing sequence of sets.

### 3. Sketch of the proof of Theorem 1

The proof is broken for clarity into six steps, numbered (i) to (vi).

**Proof.** As in the statement of the Theorem 1, let there be given a domain $\omega \subset \mathbb{R}^2$, a constant $C_\omega$ such that

$$
\text{dist}_\omega(y, \bar{y}) \leq C_\omega |y - \bar{y}| \text{ for all } y, \bar{y} \in \omega,
$$

four real numbers $p > 1$, $1 \geq \rho > 0$, $\delta > 0$, $\mu > 0$, and two mappings

$$
\theta \in \mathcal{C}^{1}_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3) \text{ and } \phi \in W^{1,p}_+(\omega;\mathbb{R}^3).
$$

Then let $\lambda := \rho / 3$, $\eta := 13\mu \rho^{-8}/3$, $\varepsilon := \rho^6/3$, let $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, and let $\Theta : \Omega^\varepsilon \to \mathbb{R}^3$ and $\Phi : \Omega^\varepsilon \to \mathbb{R}^3$ be the mappings defined by

$$
\Theta(x) := \theta(y) + x_3 \nu(\theta)(y) \text{ for all } x = (y, x_3) \in \Omega^\varepsilon
$$

and

$$
\Phi(x) := \phi(y) + x_3 \nu(\phi)(y) \text{ for almost all } x = (y, x_3) \in \Omega^\varepsilon.
$$

Note that the above definition of the constants $\lambda$, $\eta$ and $\varepsilon$ is justified by the estimates established in Step (iv) below.

**Step (i).** There exists a constant $C_1(p, \rho) > 0$ such that

$$
\inf_{R \in \mathcal{O}_{\rho}^1} \| \nabla \Phi - R \nabla \Theta \|_{L^p(\Omega^\varepsilon)} \geq C_1(p, \rho) \inf_{R \in \mathcal{O}_{\rho}^1} \left( \| \nu(\phi) - R \nu(\theta) \|_{L^p(\omega)} + \| \nabla \phi - R \nabla \theta \|_{L^p(\omega)} + \| \nabla \nu(\phi) - R \nabla \nu(\theta) \|_{L^p(\omega)} \right).
$$
The proof of this inequality relies on Clarkson’s inequalities in the space $L^p(\Omega^E)$ (see, e.g., Adams [1, Theorem 2.28]) and follows an argument previously used in Ciarlet, Gratte & Mardare [4, Proof of Theorem 4.2].

**Step (ii).** There exists a constant $C_2(p, \rho) > 0$ such that

$$\inf_{R \in \Omega^E_1} \left\| \nabla \Phi - R \nabla \Theta \right\|_{L^p(\Omega^E)} \leq C_2(p, \rho) \inf_{R \in \Omega^E_1} \left( \left| \nabla(\Phi) - R \nabla(\Theta) \right| + \left| \nabla(\Theta) - R \nabla \Theta \right| + \left| \nabla(\Phi) - R \nabla \Theta \right| \right)_{L^p(\omega)}.$$

The proof of this inequality uses either Jensen’s inequality if $p > 2$, or the inequality $(a + b + c)^{p/2} \leq a^{p/2} + b^{p/2} + c^{p/2}$ if $p \leq 2$ for some appropriate nonnegative real numbers $a$, $b$ and $c$, followed by an appropriate application of Fubini’s theorem.

**Step (iii).** The following assertions hold:

$$A(\Theta) \in \mathcal{C}^0(\bar{\omega}; \mathbb{S}^2), \quad A(\Theta)^{-1} \in \mathcal{C}^0(\bar{\omega}; \mathbb{S}^2), \quad B(\Theta) \in \mathcal{C}^0(\bar{\omega}; \mathbb{S}^2), \quad \Theta \in \mathcal{C}^1(\bar{\Omega}^E; \mathbb{R}^3),$$

and

$$A(\Phi)^{1/2} \in L^p(\omega; \mathbb{S}^2), \quad A(\Phi)^{-1/2}B(\Phi) \in L^p(\omega; \mathbb{M}^2), \quad \Phi \in W^{1,p}(\bar{\Omega}^E; \mathbb{R}^3).$$

These assertions are straightforward generalisations of similar ones established in Ciarlet, Gratte & Mardare [3] for $p = 2$, and for this reason their proof is omitted.

**Step (iv).** The mapping $\Theta$ satisfies the following properties:

$$\det \nabla \Theta(x) \geq \lambda \quad \text{and} \quad \left| \nabla \Theta(x) \right| \leq \frac{1}{\lambda} \quad \text{for all} \quad x \in \bar{\Omega}^E,$$

and

$$\left| \nabla \Theta(x) - \nabla \Theta(\bar{x}) \right| \leq \eta \quad \text{for all} \quad x, \bar{x} \in \bar{\Omega}^E \quad \text{such that} \quad |x - \bar{x}| \leq \delta.$$

Using the estimates in terms of $\rho$ of the partial derivatives of $\theta$ and $\nabla(\Theta)$ appearing in the definition of the set $\mathcal{C}^1_{\rho, \delta, \mu}(\bar{\omega}; \mathbb{R}^3)$ (see the statement of Theorem 1), we first deduce from Weingarten’s equations that

$$\left| \nabla \Theta(x) \right| \leq \frac{7}{3\rho} \quad \text{and} \quad \det \nabla \Theta(x) \geq \frac{11\rho}{18} \quad \text{for all} \quad x = (y, x_3) \in \bar{\Omega}^E,$$

then we deduce from the definition of the vector field $\nabla(\Theta)$ in terms of the partial derivatives of $\Theta$ (see relation (3)) that

$$\left| \nabla(y) - \nabla(\bar{y}) \right| \leq \frac{3}{\rho^6} \left| \nabla \Theta(y) - \nabla \Theta(\bar{y}) \right| \quad \text{for all} \quad y, \bar{y} \in \bar{\omega}.$$

Combined with the definition of the mapping $\Theta$ in terms of $\Theta$ and the definition of the parameter $\varepsilon$ in terms of $\rho$, the last inequality implies that, for each $x = (y, x_3) \in \bar{\Omega}^E$ and each $\bar{x} = (\bar{y}, \bar{x}_3) \in \bar{\Omega}^E$,

$$\left| \nabla \Theta(x) - \nabla \Theta(\bar{x}) \right| \leq \left| \nabla \Theta(y) - \nabla \Theta(\bar{y}) \right| + \varepsilon \left| \nabla \Theta(y) - \nabla \Theta(\bar{y}) \right| + \left| \nabla(y) - \nabla(\bar{y}) \right|$$

$$\leq \left( 1 + \frac{3}{\rho^6} \right) \left| \nabla \Theta(y) - \nabla \Theta(\bar{y}) \right| + \frac{\rho^6}{3} \left| \nabla \Theta(y) - \nabla \Theta(\bar{y}) \right|.$$

Assume next that $x$ and $\bar{x}$ satisfy $|x - \bar{x}| \leq \delta$, so that, in particular, $|y - \bar{y}| \leq \delta$. Then we infer from the definition of the space $\mathcal{C}^1_{\rho, \delta, \mu}(\bar{\omega}; \mathbb{R}^3)$ and from the previous estimate that

$$\left| \nabla \Theta(x) - \nabla \Theta(\bar{x}) \right| \leq \left( 1 + \frac{3}{\rho^6} + \frac{\rho^6}{3} \right) \mu \leq \frac{13\mu}{3\rho^6}.$$
Step (v). Assume that the given constant \( \mu > 0 \) satisfies \( \mu \leq \frac{p^{11}}{488(1 + C_\omega)} \). Then there exists a constant \( C_3(\omega, p, \rho, \delta) \) depending only on \( \omega, p, \rho, \delta \) such that

\[
\inf_{R \in \mathcal{O}_3} \| \nabla \Phi - R \nabla \Theta \|_{L^P(\Omega^F)} \leq C_3(\omega, p, \rho, \delta) \inf_{R \in \mathcal{O}_3} \| \nabla \Phi - R \nabla \Theta \|_{L^P(\Omega^F)}.
\]

First, the inequalities established in (iv) imply that the mapping \( \Theta \) belong to the set:

\[
\mathcal{C}^{-1}_{\lambda, \delta, \eta}(\Omega^F; \mathbb{R}^3)
\]

\[
:= \left\{ \Theta \in \mathcal{C}^{-1}(\Omega^F; \mathbb{R}^3) \mid \inf_{x \in \Omega^F} \det \Theta(x) \geq \lambda, \sup_{x \in \Omega^F} |\nabla \Theta(x)| \leq \frac{1}{\lambda}, \sup_{x, \tilde{x} \in \Omega^F, |x - \tilde{x}| \leq \delta} |\nabla \Theta(x) - \nabla \Theta(\tilde{x})| \leq \eta \right\}.
\]

Secondly, by (iii),

\[\Phi \in W^{1, p}(\Omega^F; \mathbb{R}^3).\]

Thirdly, the definition of the set \( \Omega^F \) in terms of \( \omega \), the definition of the geodesic distance in \( \Omega^F \) (see relation (1)), and the definition of the constant \( C_\omega \) (see the statement of Theorem 1), together show that, for each \( x = (y, x_3) \in \Omega^F \) and each \( \tilde{x} = (\tilde{y}, \tilde{x}_3) \in \Omega^F \),

\[\text{dist}_{\Omega^F}(x, \tilde{x}) \leq \text{dist}_{\omega}(y, \tilde{y}) + |x_3 - \tilde{x}_3| \leq C_\omega |y - \tilde{y}| + |x_3 - \tilde{x}_3| \leq (1 + C_\omega)|x - \tilde{x}|.\]

Fourthly, the assumption on \( \mu \) made in (v) implies that

\[\eta \leq \frac{\lambda^3}{4(1 + C_\omega)}.\]

The four observations above together imply that the assumptions of [8, Theorem 1 (a)] are satisfied by the domain \( \Omega^F \) and by the mappings \( \Theta \) and \( \Phi \) from \( \Omega^F \) into \( \mathbb{R}^3 \). Thus inequality (8) holds as a consequence of this theorem.

Step (vi). Combining the three inequalities established in steps (i), (ii) and (v) above yields the inequality

\[
\inf_{R \in \mathcal{O}_3} \left( \| v(\phi) - R v(\Theta) \|_{L^P(\omega)} + \| \nabla \phi - R \nabla \Theta \|_{L^P(\omega)} + \| \nabla v(\phi) - R \nabla v(\Theta) \|_{L^P(\omega)} \right)
\]

\[\leq C \inf_{R \in \mathcal{O}_3} \left( |v(\phi) - R v(\Theta)| + |\nabla \phi - R \nabla \Theta| + |\nabla v(\phi) - R \nabla v(\Theta)| \right),\]

where \( C := (C_1(p, \rho))^{-1}C_2(p, \rho)C_3(\omega, p, \rho, \delta) \). This establishes the first inequality of Theorem 1.

Using the polar decomposition of the \( 3 \times 3 \) matrix fields

\[(\nabla \Theta | v(\Theta)) \text{ and } (\nabla \Phi | v(\Phi))\]

and a method similar to one used in the proof of [7, Theorem 3.1], we next show that the following inequality holds almost everywhere in \( \omega \):

\[
\inf_{R \in \mathcal{O}_3} \left( |v(\phi) - R v(\Theta)|^2 + |\nabla \phi - R \nabla \Theta|^2 + |\nabla v(\phi) - R \nabla v(\Theta)|^2 \right)
\]

\[\leq |A(\phi)^{1/2} - A(\Theta)^{1/2}|^2 + |A(\phi)^{-1/2} B(\phi) - A(\Theta)^{-1/2} B(\Theta)|^2.\]
Consequently,
\[
\left\| \inf_{R \in O^+_3} \left( |v(\phi) - Rv(\theta)| + |\nabla v(\phi) - R\nabla v(\theta)| \right) \right\|_{L^p(\omega)} \\
\leq \sqrt{3} \left( \|A(\phi)^{1/2} - A(\theta)^{1/2}\|_{L^p(\omega)} + \|A(\phi)^{-1/2} B(\phi) - A(\theta)^{-1/2} B(\theta)\|_{L^p(\omega)} \right).
\]
This establishes the second inequality of Theorem 1. □

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