Positronium in crossed electric and magnetic fields: the existence of a long-lived ground state

J. Shertzer  
*College of the Holy Cross, Worcester MA 01610*

J. Ackermann  
*Institut für molekulare Biotechnologie (IMB), Beutenbergstrasse 11, D-07745 Jena, Germany*

P. Schmelcher  
*Theoretische Chemie, Physikalisch-Chemisches Institut, Universität Heidelberg, Im Neuenheimer Feld 253, D-69120 Heidelberg, Germany*  
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It was earlier reported [PRL 78 199, (1997)] that long-lived excited states of positronium can be formed in crossed electric and magnetic fields at laboratory field strengths. Unlike the lower-lying states that are localized in the magnetically distorted Coulomb well, these long-lived states which can possess a lifetime up to many years are localized in an outer potential well that is formed for certain values of the pseudomomentum and magnetic field. The present work extends the original analysis and studies the dependence of the spectrum as a function of field strength and pseudomomentum over a wide range of parameters. We predict that in the limit of large pseudomomentum, the ground state of positronium atom in a magnetic field will become delocalized; for strong fields, the binding energy of this state is quite large, resulting in a ground state that is both stable against direct annihilation and against ionization by low frequency background radiation.

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I. INTRODUCTION

The problem of treating the two-body system in a magnetic field has a long history [1–9]. Early on it was realized that the center of mass motion cannot be separated from the internal motion. The motional electric field due to the collective motion of the system in the magnetic field gives rise to a Stark term in the Hamiltonian which is very similar to that arising from an external electric field. Hence, both problems can be treated in a unified way.

When the center of mass effects are treated correctly, the total momentum of the system $P$ is not a conserved quantity and a separation of the center of mass and internal motion is impossible. The pseudomomentum $K = P + \hat{z}B \times r$ is however a conserved quantity, where $P$ is the total momentum and $r$ is the relative vector of the two oppositely charged particles with charges $\pm e$ [1]. For neutral systems, one can carry out a pseudoseparation of the center of mass and internal motion, where the effective Hamiltonian for the internal motion includes $K$-dependent terms, where $K$ is the eigenvalue of the pseudomomentum [10]. Recently, it was shown that one can account for the effect of the motion of the center of mass on the internal motion via an effective potential that is gauge independent [12]. This potential gives rise to an outer well for certain values of $K$ and $B$, which leads to delocalized states. There have been several studies of the effects of a significant Stark term on the strong field behavior of one-electron atoms [13, 14].

Recently, the gauge-invariant potential was extended to the case of positronium, where rigorous numerical results indicated that for field strengths attainable in the laboratory, one could form delocalized states of positronium [10]. The probability density within an interparticle distance radius of hundreds of Angstroms is less than $10^{-16}$, indicating that the rates for direct annihilation and radiative decay to lower lying states localized in the Coulomb well are near zero. The existence of such delocalized states for positronium is important because it provides a mechanism for creating a stable bound state of a particle-antiparticle pair.

Our previous work established the existence of such a state at values of $K$ and $B$ that are attainable in the laboratory. In this paper, we attempt a more thorough investigation of the problem. In section II, we describe the potential surface of positronium (Ps) as a function of $K$ for values of the magnetic field from laboratory field strengths ($10^{-5} - 10^{-4}$ a.u.) up to superstrong fields (10-100 a.u.). We discuss qualitatively the effect of this potential on the energy spectrum. We present numerical results for select values of the parameters $K$ and $B$ and show that for large values of the pseudomomentum the ground state of Ps becomes a delocalized state. This is an important result because it means that any Ps formed under the proper conditions will be stable against annihilation. In the conclusion, we speculate on how one may obtain evidence of long-lived Ps in a laboratory.
Throughout this paper we use atomic units:

Charge \( e \)

Mass \( m \)

Length \( a_0 = \hbar^2/m_e^2 \)

Velocity \( v_0 = \alpha_c = e^2/\hbar \)

Momentum \( p_0 = m \alpha_c = m e^2/\hbar^2 \)

Energy \( m e^4/\hbar^2 \)

Magnetic field \( m^2 e^3/\hbar^4 \)

Electric field \( m^2 e^5/\hbar^4 \)

The effective Hamiltonian for the positronium atom in crossed electric and magnetic fields which results from the gauge invariant pseudoseparation [12] is

\[
H = \frac{p^2}{2} + \frac{1}{4}(K + B \times r)^2 - \frac{1}{r} + \mathbf{E} \cdot \mathbf{r}
\]

where \( T = \frac{p^2}{2} \) is the kinetic energy and \( V = \frac{1}{2}(\mathbf{K} + B \times r)^2 - \frac{1}{r} + \mathbf{E} \cdot \mathbf{r} \) is the potential of the relative motion. \( \mathbf{K} \) is the eigenvalue of the conserved pseudomomentum.

The term in the potential that depends explicitly on the external electric field can be eliminated by defining an effective pseudomomentum \( \mathbf{K}' = \mathbf{K} - 2\mathbf{v}_d \), where \( \mathbf{v}_d = \mathbf{E} \times \mathbf{B}/B^2 \) is the classical drift velocity of a free charged particle in crossed fields. Using this definition in Eq. (1), we have

\[
H = \frac{p^2}{2} + \frac{1}{4}(\mathbf{K}' + B \times r)^2 - \frac{1}{r} + \mathbf{K}' \cdot \mathbf{v}_d + \mathbf{v}_d^2
\]

It is now clear that the energy eigenvalues of positronium in an external electric field can be obtained from the zero external electric field results by replacing \( \mathbf{K} \) with \( \mathbf{K}' \) and shifting the energy \( \mathcal{E} \) by a constant amount:

\[
\mathcal{E}(\mathbf{K}, \mathbf{B}, \mathbf{E}) = \mathcal{E}(\mathbf{K}', \mathbf{B}, 0) + \mathbf{K}' \cdot \mathbf{v}_d + \mathbf{v}_d^2
\]

It is important to note that the ionization threshold \( I \), which is the zero point energy of the free particles in the presence of the fields, is different in the two cases. In a pure magnetic field \( I = B \). In crossed fields, the ionization threshold includes the energy due to the drift motion \( I = B + \mathbf{K}' \cdot \mathbf{v}_d + \mathbf{v}_d^2 \). The ionization threshold is shifted by the same amount as the energy levels. The external electric field can be used to control the value of the effective pseudomomentum \( \mathbf{K}' \) experimentally. For any combination of \( \mathbf{E} \) and \( \mathbf{K} \) that leaves the effective pseudomomentum \( \mathbf{K}' \) unchanged, the ionization energy \( I - \mathcal{E} \) is a constant.

Without loss of generality, we choose \( \mathbf{B} = B\hat{z} \), \( \mathbf{E} = 0 \), and \( \mathbf{K} = \tilde{K}\hat{y} \). Components of \( \mathbf{K} \) parallel to \( \mathbf{B} \) shift the energy by a constant amount. The effective Hamiltonian for the internal motion of positronium is given in Cartesian coordinates by

\[
H = p^2 + \frac{B^2}{4}(x^2 + y^2) + \frac{BKx}{2} - \frac{1}{\sqrt{x^2 + y^2 + z^2}} + \frac{K^2}{4}
\]

The Hamiltonian for other particle-antiparticle pairs can be obtained through a simple scaling of the Ps Hamiltonian. From the eigenvalues of the Hamiltonian for Ps \( \mathcal{E}(K, B) \), we can obtain results for the particle-antiparticle pair with \( m_1 = m_2 = m \) (m is a dimensionless scale parameter) as follows:

\[
\tilde{\mathcal{E}}(m, \tilde{K}, \tilde{B}) = m\mathcal{E}(K = \tilde{K}/m, B = \tilde{B}/m^2).
\]

All terms in the potential increase with increasing \( \sqrt{x}, \sqrt{y}, \) and \( \sqrt{z} \) with the exception of \( \frac{BKx}{2} \) which decreases for negative \( x \). In addition to the minimum at the Coulomb singularity \( V(0, 0, 0) = -\infty \), for sufficiently large \( K \) the potential has another local minimum \( V_o = V(x_0, 0, 0) \) where \( x_0 < 0 \) (see Fig. 1). To determine the critical value \( K_c \) for which the outer well on the negative \( x \)-axis appears, we set \( \partial V(x, 0, 0)/\partial x = 0 \):

\[
x^3 + \frac{K}{B}x^2 - \frac{2}{B^2} = 0, \quad x < 0.
\]
If $K \geq K_c = \sqrt[3]{27B}$, the roots of the equation are real and given by:

\[
x_1 = \frac{K}{3B} \left[ 2\cos \left( \frac{\theta}{3} \right) - 1 \right]
\]
\[
x_2 = \frac{K}{3B} \left[ 2\cos \left( \frac{\theta + 2\pi}{3} \right) - 1 \right]
\]
\[
x_3 = \frac{K}{3B} \left[ 2\cos \left( \frac{\theta + 4\pi}{3} \right) - 1 \right]
\]

(7)

where $\cos \theta = \frac{27B}{K^2} - 1$. The angle is bound by $0 \leq \theta \leq \pi$. $\theta = 0$ corresponds to the critical value $K = K_c$; for $K > K_c$, the potential surface has two local minima. $\theta = \pi$ corresponds to $B \to 0$ or $K \to \infty$. It is clear from the form of the solutions that $x_2 < x_3 < 0 < x_1$: $x_1$ is unphysical because Eq. (7) holds only for $x < 0$. The location of the saddle $x_s$ and outer well minimum $x_o$ are given by

\[
x_s = x_3 = -\frac{K}{3B} \left[ 1 + \cos \left( \frac{\theta}{3} \right) - \sqrt{3} \sin \left( \frac{\theta}{3} \right) \right]
\]
\[
x_o = x_2 = -\frac{K}{3B} \left[ 1 + \cos \left( \frac{\theta}{3} \right) + \sqrt{3} \sin \left( \frac{\theta}{3} \right) \right]
\]

(8)

We now discuss the potential surface, the ground state, and energy spectrum for a fixed $B$ as a function of $K$. We will hereafter refer to the potential well that includes the origin as the magnetically distorted Coulomb well (MDCW). The additional well that forms on the negative $x-$axis will be referred to as the outer well (OW). The ionization threshold for positronium is given by the field strength $I = B$. All numerical results were obtained by applying the adaptive 3D finite element method (FEM) [17] to the solution of the Schrödinger equation for the Hamiltonian given in Eq. (1).

A. $K = 0$

At $K = 0$, the Hamiltonian simplifies to

\[
H = p^2 + \frac{B^2}{4} (x^2 + y^2) - \frac{1}{\sqrt{x^2 + y^2 + z^2}}
\]

(9)

where $p^2 - \frac{1}{2}$ is the Hamiltonian for Ps in the field free case. The field dependent term $\frac{B^2}{4}(x^2+y^2)$ destroys the rotational symmetry of the field-free Hamiltonian and the total angular momentum $L$ is no longer a good quantum number. The azimuthal symmetry of the problem remains and the $z$-component of the angular momentum is still a good quantum number. The field dependent term provides additional confinement in the $x$ and $y$ directions.

The field dependent term in the Hamiltonian is positive definite and thus the energy eigenvalues of the field free Hamiltonian are increased by the non-zero $B$ field. For the ground state, this effect is offset by the increase in the ionization threshold: the binding energy of the ground state increases with increasing $B$ field (see Table 1). For high lying states, the effect of the additional confinement perpendicular to the field is more pronounced and Rydberg states of the field free problem may be pushed into the continuum by the field.

The energy levels in the MDCW can be obtained by applying well-known scaling relations to the hydrogenic results in the infinite mass approximation which are available in the literature [18–20]. From the results $\tilde{E}(\tilde{B})$ for hydrogen (with $S = 0$ and $L_z = 0$), we can obtain results for Ps with zero pseudomomentum:

\[
\tilde{E}(K = 0, B) = \frac{1}{2} \tilde{E}(\tilde{B} = 4B)
\]

(10)

B. $0 \leq K \leq K_c$

As $K$ is increased monotonically from 0 up to the critical value, we analyze the effect of the two $K$-dependent terms in the potential for a fixed $B$. The term $K^2/4$ shifts the entire potential curve upward by a constant amount. The asymmetric term $\frac{BKx}{2}$ destroys the azimuthal symmetry for any $K > 0$ and further distorts the shape of the well.
The combined effect of both terms leads to a decrease in the binding energy of the ground state for fixed $B$ with increasing $K$ (see Table I). Second order perturbation theory can be used to calculate the energy shift of the ground state when both $B$ and $K$ are small. The average interparticle distance $\langle x \rangle$ for the ground state decreases ($\langle \langle x \rangle \rangle$ increases) with increasing $K$, indicating a `decentering' of the probability density. This effect can be quite dramatic for large $B$, where $K < K_c$ does not imply small $K$ (see Table I).

For high lying states, the asymmetry of the well becomes even more important resulting in greater decentering of the probability density than what is observed for the ground state. Because of the competing effects of the two $K$-dependent terms, it is impossible to predict the net effect on the energy levels near the ionization threshold. A full 3D solution of the Schrödinger equation is required for each set of parameters.

C. $K = K_c$

The critical value $K_c$ is the largest value of $K$ for which only a single minimum exists in the potential surface. At the critical value $K_c$ the saddle point $x_s$ and outer well minimum $x_o$ coincide at

$$x_c = - \frac{9}{K_c^2} = - \left( \frac{2}{B} \right)^{\frac{2}{3}} \quad (11)$$

The potential at this point is

$$V_c = V(x_c, 0, 0) = - \frac{K_c^2}{12} = - \frac{3}{12} \left( \frac{B}{2} \right)^{\frac{2}{3}} \quad (12)$$

For laboratory field strengths ($B \approx 10^{-5}$), the magnitude of the critical point is on the order of several thousand Angstroms; for strong fields ($B \approx 1$), the critical point is one the order of 1 Angstrom.

D. $K < K_c$

1. General features of the potential

If we now allow $K$ to further increase above the critical value, we observe the formation of the OW, which is separated from the MDCW by the saddle. The outer well minimum is denoted $V_o = V(x_o, 0, 0)$ and the height of the saddle is $V_s = V(x_s, 0, 0)$. For fixed $B$, we observe the following general features of the potential:

1. The value of $x_o$ decreases with increasing $K$; the outer well moves away from MDCW along the negative $x$-axis ($x_o > x_s \geq -\infty$).
2. The minimum of the outer well $V_o$ increases with increasing $K$ but is bounded from above by zero ($V_o = 0$).
3. The value of $x_s$ increases with increasing $K$; the barrier maximum moves towards the origin ($x_c \leq x_s \leq 0$).
4. The height of the saddle $V_s$ increases without limit with increasing $K$ ($V_c \leq V_s < \infty$).

In Fig. 2, we show $|x_o|$ and $|x_s|$ as a function of $K$ for $B = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$, 1 and 10 on a log-log plot; the point of coalescence is the critical value $x_c$. Note that the curves are identical in shape. In Fig. 3, we show $V_o$ and $V_s$ as a function of $K$ at the same field strengths as Fig. 2; the point of coalescence is the value of the potential at the critical point, $V_c = V(x_c, 0, 0)$. The shapes of the curves are similar, with $V_s$ rising dramatically for $K > K_c$; $V_o$ approaches zero in the limit $K \to \infty$.

It is also important to note that the shape of the MDCW changes above the critical value of $K$. For $K > K_c$ and $|x| < 0.75|x_o|$, $\frac{BK|x|}{2} < \frac{k^2}{4}$ and the potential $V(x,0,0)$ near the origin rises sharply, resulting in a narrow well (see Fig. 1). The outer well, in contrast, becomes broader and deeper with increasing $K$. Although the well minimum is increasing (approaching zero from below) the barrier height is increasing at a much faster rate than the minimum is increasing.

Despite the similarity of the potential for different values of $B$ (see Fig. 1,3) the energy spectrum of the system - including the ground state - is dramatically different depending on the value of $B$. The bound states of the system depend not just on the potential, but also on the ionization threshold $I$.

When the two wells are distinct, that is, the saddle point energy is well above the ionization threshold, the bound states can be classified as OW states or MDCW states: their probability density is concentrated solely in one well. The bound states of the OW are called delocalized states because the average interparticle separation is large. The
overlap between an OW state and a MDCW state is essentially zero. All of the states that are localized in the MDCW are pushed upward in energy with increasing $K$ and into the continuum; the number of MDCW bound states decreases. The number of bound states in the outer well increases with increasing $K$.

However, before we reach the regime where the problem reduces to two separate wells, there is a range of $K$ for which the saddle point energy is less than the ionization threshold. If the energy of a state is considerably less than $V_s$, the state will be concentrated in the OW or the MDCW with very little probability in the barrier region and the above classification scheme for states is still valid. However, if the energy of a state is close to $V_s$, the distinction between an OW state and an MDCW states becomes murky. (It is important to remember that $V_s$ is the potential maximum in the $x$–direction only; the ionization threshold $I = B$ as well as the energy of the state are quantities that depend on the 3D wavefunction.) Under these very special conditions, $E \approx V_s < B$, we have a third kind of state: the saddle state.

2. Saddle states

In the regime $K > K_c$, but $V_s < I$, an interesting phenomena can occur: a MDCW state can undergo a continuous transformation into an OW state with increasing $K$ as the energy of the MDCW state approaches $V_s$. The MDCW state (see Fig. 4a) will begin to leak into the classically forbidden barrier region (see Fig. 4b). The amount of tunneling through the barrier increases with increasing $K$, until a saddle state forms, which has probability density in the MDCW, the barrier region and the OW (see Fig. 4c). At even higher $K$, more and more probability density is transferred from the MDCW to the OW (see Fig. 4d). Since the barrier height is increasing, the energy of the state is eventually less than $V_s$ and the state is fully delocalized in the OW (see Fig. 4e). The range of $K$ over which this process occurs can vary orders of magnitude depending on the relative values of $I$, $V_s$ and $E$ as a function of $K$ (see Table 1).

For low magnetic fields, the gap between the ionization threshold and the saddle point energy is already small at the critical value; hence, the formation of saddle states is limited to a very small range of $K$. For $B = 10^{-4}$, the critical value is $K_c = 0.1105$ and saddle point reaches the ionization threshold ($V_s = I$) at $K = 0.1414$. Saddle states can only form over a range $\Delta K \approx .03$. Many bound states still exist in the MDCW when the saddle point reaches the ionization threshold. In contrast, for high magnetic fields, the separation between the ionization threshold and the saddle point energy is orders of magnitude greater than what it is at low fields. For $B = 1$, the critical value is $K_c = 2.381$ and the saddle point reaches the ionization threshold at $K = 3.800$. Saddle states can form over a range $\Delta K \approx 1.4$. The ground state energy of the system is itself a saddle state in this range.

Once the saddle point reaches the ionization threshold for a fixed $B$, saddle states can not be formed and all of the MDCW states are pushed into the continuum with increasing $K$; the OW spectrum is continuously changing with increasing $K$.

3. Fate of the ground state

It is clear that this process of pushing MDCW states out of the well and into the continuum or into the OW continues until the MDCW spectrum is depleted of bound states. The fate of the MDCW ground state is quite different for low $B$ fields than for high $B$ fields. At low fields, there is a sudden transition at which the ground state energy of the OW drops below the ground state of the MDCW. Both states are well below the saddle point energy, which is much greater than the ionization threshold. (see Table 1, $B=0.001$, $K=1.0$, $1.1$) This is indicative of a sharp level crossing between two distinct states. The wavefunctions for these two states have zero overlap. The MDCW ground state will continue to be pushed upward in energy with increasing $K$ until it reaches the continuum.

In contrast, at high fields there is a slow migration of the ground state wavefunction away from the origin and towards the outer well. The ground state energy is greater than $V_s$ (but less than $I$) over a very wide range of $K$, allowing for a slow transfer of probability with increasing $K$ as the MDCW state forms a saddle state which eventually delocalizes. In Fig. 5 we show the value of the ground state wavefunction at the origin as a function of $K$ for several values of $B$. Note for small $B$ the abrupt drop at the point of the level crossing; for large $B$, there is a smooth decay indicative of the slow transfer of probability from the MDCW to the OW.

The ground state of positronium will become a delocalized state at sufficiently high $K$. This is an important result, because it implies that any positronium formed under these conditions will have a long-lived ground state that is stable against direct annihilation. For low fields, the ground state is extended in space and weakly bound; at high field strengths, the size of the atom in the high $K$ limit is small and the binding energy is large.

The decay rate for positronium is
where \( \sigma \) is the plane-wave cross section for free pair annihilation, \( v \) is the relative velocity of the electron and positron, and \( \rho \) is the square of the wavefunction evaluated at contact. In the field-free case, the lowest order decay rate for the ground state of parapositronium is \( \Gamma = 8.03 \times 10^{10} \text{s}^{-1} \); for orthopositronium, \( \Gamma = 7.21 \times 10^8 \text{s}^{-1} \). The distinction between ortho- and para-positronium is meaningless in the present discussion. The spin-spin interaction is dominated by the spin-field interaction at field strengths \( B > 10^{-5} \). We assume the spins are aligned with the field and the total energy and ionization energy are shifted by \( B \). Nevertheless, the basic physics contained in the decay rate formula is unchanged: the decay rate depends on the probability density at the origin and if the probability density is zero, the Ps atom is stable against annihilation.

At low fields \( 10^{-5} < B < 10^{-2} \), there is a well-defined crossing between the MDCW ground state and OW ground state near \( K = 1 \). The probability density at the origin drops dramatically at this crossing - resulting in a lifetime on the order of years. At higher fields, \( 0.1 < B < 10 \), the probability density at the origin decreases exponentially with increasing \( K \) (Fig. 5); for \( B/K^2 < 10^{-2} \), the probability density is less than \( 10^{-16} \), again resulting in a lifetime greater than a year.

E. \( K_e << K \)

In order to extract more quantitative conclusions about the spectrum of the outer well in the large \( K \) limit, we expand Eq. (8) in powers of \( \epsilon \), where \( \epsilon = \frac{n_B}{K} << 1 \). (Hereafter, the high \( K \) limit implies \( \epsilon << 1 \).) We obtain analytic expressions for the saddle and outer well quantities in the large \( K \) limit, retaining terms up to first order in \( \epsilon \):

\[
\begin{align*}
x_o &\to -\frac{K}{B} \left[ 1 - 2\epsilon \right] = -\frac{K}{B} + \frac{2}{K^2} \\
V_o &\to -\frac{B}{K} \left[ 1 + \epsilon \right] = -\frac{B}{K} - \frac{B^2}{K^4} \\
x_s &\to -\frac{K}{B} \left[ 0 + \sqrt{2\epsilon} + \epsilon \right] = -\sqrt{\frac{2}{BK}} - \frac{1}{K^2} \\
V_s &\to \frac{K^2}{4} \left[ 1 - \sqrt{2\epsilon} + 2\epsilon \right] = \frac{K^2}{4} - \frac{\sqrt{32BK}}{2K} \frac{B}{K}
\end{align*}
\]  

These results have wider applicability than may be expected, because \( \epsilon \) is already small at the critical value: \( \epsilon(K = K_e) = 2 \). Below we compare the exact values of the saddle and OW parameters with the high \( K \) limit approximate results at \( B = 1 \); the critical value is \( K_e = 2.381 \).

| \( B \) | \( K \) | \( x_o \)- High \( K \) | \( V_o \)- High \( K \) | \( x_s \)- High \( K \) | \( V_s \)- High \( K \) |
|---|---|---|---|---|---|
|   |   | \( x_o \)- Exact | \( V_o \)- Exact | \( x_s \)- Exact | \( V_s \)- Exact |
| 1.0 | 5 | -4.920 | -.20160 | -.672 | 3.188 |
|   |   | -4.917 | -.20165 | -.680 | 3.195 |
| 1.0 | 10 | -9.9800 | -.100100 | -.4572 | 20.578 |
|   |   | -9.9799 | -.100100 | -.4578 | 20.579 |
| 1.0 | 20 | -19.99500 | -.0500625 | -.31873 | 93.7004 |
|   |   | -19.99500 | -.0500625 | -.31878 | 93.7006 |

For large \( K \), the energy of the bound states in the outer well are bounded from above and below by the well minimum and the ionization threshold respectively: \( V_o < E_{n}^{\text{OW}} < B \), where \( n \) refers to the set of quantum numbers used to label the OW states. In order to make estimates for the ground state, we use the anisotropic harmonic oscillator (AHO) approximation for the OW \[21,22\]. This approximation involves an expansion of the Coulomb potential around the minimum, retaining terms to order \( x_o^{-3} \); the AHO approximation is valid only when \( |x_o| \) is large and only for states such that \( (z^2 + y^2) << x_o^2 \). (For \( B > 1 \), the high \( K \) limit does not imply that the AHO approximation is valid: \( \frac{B}{K^2} << 1 \) is a stronger requirement than \( \frac{B}{K} << 1 \).)

The AHO energy spectrum is given by

\[
E_{n_x n_y n_z} = (n_x + \frac{1}{2})\omega_x + (n_y + \frac{1}{2})\omega_y + (n_z + \frac{1}{2})\omega_z + C.
\]  

where
In the high $K$-limit, $\omega_x \to B - 2\epsilon$, $\omega_y \to B + \epsilon$, and $\omega_z \to B\sqrt{2}\epsilon$ and the ground state energy is:

$$E_g \to B \left[ 1 + \sqrt{\frac{\epsilon}{2} - \frac{\epsilon^2}{2}} \right] - \frac{B}{K} \left[ 1 + 5\epsilon \right]$$

We now turn to the spectrum of OW excited states. For low-lying states, the AHO is still valid. Excitations in the z-direction dominate for large $K$. The level spacing between the harmonic states ($\omega_z = \sqrt{\frac{2B^2}{K^3}}$) is decreasing at a faster rate than the energy gap between the ionization threshold and the ground state ($I - E_g = \frac{B}{K}$): the density of low-lying bound states in the OW increases with increasing $K$. The AHO approximation will break down long before we reach the ionization threshold, although the potential remains approximately separable and harmonic in the $x$ and $y$. In the $z$ direction, the level spacing decreases as one approaches the ionization threshold (at constant $K$) due to the $\sqrt{x^2 + z^2}$ potential. We expect the number of OW bound states is infinite at high but finite $K$.

### III. CONCLUSION AND OUTLOOK

We have investigated the properties of the positronium atom in crossed electric and magnetic fields for a broad range of magnetic field strengths and values of the pseudomomentum. The occurrence of magnetically distorted Coulomb well states, saddle states and outer well states gives rise to a spectrum that is rich and unique. An intensive numerical study via the finite element approach allowed us to investigate the different regimes and study the transitions between different types of quantum states as a function of pseudomomentum.

We believe that there is rigorous theoretical and numerical evidence to support the prediction that long-lived states of positronium exist in crossed electric and magnetic fields. The near zero probability for particle overlap prevents direct annihilation. For large pseudomomentum ($B/K^3 << 1$), the ground state itself is a long-lived state.

Positronium in crossed fields is not an exotic system. Particle-antiparticle pairs occur in many different physical situations. Naturally occurring magnetic fields range from a few mGauss to $10^{12}$ in neutron stars. The answer to the question whether Ps has ever been produced under the appropriate conditions for stability is in our opinion yes. Detection of the system is a more difficult problem.

One possible means for creating stable positronium at laboratory field strengths is to prepare Landau states of $e^+$ and $e^-$ with low relative velocity in the $z$ direction. The initial positions and velocities of the two particles in the external fields define the conserved pseudomomentum. Once an OW bound state is formed, the large dipole moment $e_x\rho$ of the OW state should make detection easy. The difficulty is to detect the positronium atom before it collides with the container walls. It is not practical to use field gradients for trapping, as the external fields must be constant over the dimensions of the atom. Another possibility is to scatter positrons off hydrogen in crossed fields. A judicious choice of experimental parameters may enhance the cross section for positronium formation in an OW state.

The search for stable positronium need not be limited to the laboratory. The spectra from neutron stars may provide evidence of transitions between outer well states. These suggestions are not meant to confine the search for this system, but only to initiate a discussion that should challenge experimentalists in many subfields.

In conclusion, we believe that the existence of stable Ps will be relevant in many areas of physics. One can only speculate at this time on the possible applications of storing energy via a stable particle-antiparticle pair.

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Table I. The location of the saddle point \( x_s \), the value of the potential at the saddle \( V_s = V(x_s, 0, 0) \), the location of the outer well minimum \( x_o \), the value of the potential at the outer well minimum \( V_o = V(x_o, 0, 0) \), the ground state energy \( \varepsilon_g \), and the expectation value \( \langle x \rangle \) for the ground state as a function of \( K \) for several values of \( B \). All quantities are in atomic units.

| \( B \) | \( K \) | \( x_s \) | \( V_s \) | \( x_o \) | \( V_o \) | \( \varepsilon_g \) | \( \langle x \rangle \) |
|------|------|------|------|------|------|-------|------|
| 0.001 | 0.0 | -100.00 | 0.000 | -273.2 | -0.00348 | -0.2275 | -0.00540 |
| 0.1 | 0.1 | -78.92 | 0.0131 | -386.6 | -0.00254 | -0.2100 | -0.00720 |
| 0.2 | 0.2 | -68.04 | 0.0320 | -419.7 | -0.00202 | -0.1875 | -0.00900 |
| 0.3 | 0.3 | -60.91 | 0.0562 | -543.2 | -0.00167 | -0.1600 | -0.0108 |
| 0.4 | 0.4 | -55.72 | 0.0858 | -695.9 | -0.00143 | -0.1275 | -0.0126 |
| 0.5 | 0.5 | -51.70 | 0.1206 | -796.9 | -0.00125 | -0.0900 | -0.0144 |
| 0.6 | 0.6 | -48.46 | 0.1606 | -897.5 | -0.00111 | -0.04475 | -0.0162 |
| 0.7 | 0.7 | -45.78 | 0.2058 | -998.0 | -0.00100 | -0.0000 | -0.0018 |
| 0.8 | 0.8 | -43.97 | 0.2508 | -1499 | -0.00067 | -0.00034 | -0.00199 |
| 0.9 | 0.9 | -42.93 | 0.3081 | -1999 | -0.00050 | -0.00051 | -0.00199 |
| 1.0 | 1.0 | -41.92 | 0.3678 | -2498 | -0.00033 | -0.00067 | -0.00199 |
| 1.1 | 1.1 | -40.91 | 0.4305 | -2998 | -0.00020 | -0.00080 | -0.00199 |
|   |      |      |      |      |      |      |
|---|------|------|------|------|------|------|
| 1.2| -13.72 | .2095 | -118.6 | -.0084 | .00210 | -118.7 |
| 1.5| -12.04 | .3928 | -149.2 | -.0067 | .00367 | -149.1 |
| 2.0| -10.27 | .8026 | -199.4 | -.0050 | .00523 | -199.5 |
| 3.0| -8.280 | 2.007 | -299.7 | -.0033 | .00806 | -299.9 |
| 5.0| -6.365 | 5.935 | -499.9 | -.0020 | .00886 | -499.9 |
| 10.0| -4.482 | 24.55 | -1000 | -.0010 | .00902 | -1000 |
| 0.1|      |      |      |      |      |      |
| 0.0| -2.323 | 0.000 |      |      |      |      |
| 0.2| -2.234 | -0.2259 |      |      |      |      |
| 0.4| -1.969 | -0.4787 |      |      |      |      |
| 0.6| -1.532 | -0.8070 |      |      |      |      |
| 0.8| -0.938 | -1.363 |      |      |      |      |
| 1.0| -0.241 | -3.177 |      |      |      |      |
| 1.2| -5.583 | -0.0762 | -10.00 | -.00900 | .0219 | -9.780 |
| 1.5| -4.329 | 0.0537 | -13.98 | -.00507 | .0559 | -19.57 |
| 2.0| -3.479 | 0.3949 | -19.47 | -.00507 | .0559 | -19.57 |
| 3.0| -2.707 | 1.493 | -29.77 | -.00334 | .0702 | -29.81 |
| 5.0| -2.042 | 5.260 | -49.92 | -.00200 | .0818 | -49.93 |
| 1.0|      |      |      |      |      |      |
| 0.0|      |      |      |      |      |      |
| 0.2|      |      |      |      |      |      |
| 0.4|      |      |      |      |      |      |
| 0.6|      |      |      |      |      |      |
| 0.8|      |      |      |      |      |      |
| 1.0|      |      |      |      |      |      |
| 1.5|      |      |      |      |      |      |
| 2.0|      |      |      |      |      |      |
| 2.5| -1.281 | -0.4092 | -2.000 | -.04375 | .6970 | -2.255 |
| 3.0| -1.000 | 0.000 | -2.732 | -.03481 | .7481 | -2.831 |
| 4.0| -.7892 | 1.310 | -3.866 | -.02542 | .8095 | -3.910 |
| 5.0| -.6804 | 3.195 | -4.917 | -.02017 | .8451 | -4.939 |
| 6.0| -.6091 | 5.624 | -5.943 | -.01675 | .8688 | -5.960 |
| 8.0| -.5170 | 12.06 | -7.969 | -.01252 | .8991 | -7.977 |
| 10.0| -.4578 | 20.58 | -9.980 | -.01001 | .9178 | -9.985 |
| 20.0| -.3188 | 93.70 | -19.99 | -.05000 | .9567 | -19.996 |
| 10.0|      |      |      |      |      |      |
| 0.0|      |      |      |      |      |      |
| 0.4|      |      |      |      |      |      |
| 1.0|      |      |      |      |      |      |
| 1.5|      |      |      |      |      |      |
| 2.0|      |      |      |      |      |      |
| 3.0|      |      |      |      |      |      |
| 4.0|      |      |      |      |      |      |
| 5.0|      |      |      |      |      |      |
| 6.0| -.2337 | -.9246 | -.5284 | -.1764 | 9.109 | -.5800 |
| 7.0| -.2000 | 1.250 | -.6531 | -.1476 | 9.202 | -.6829 |
| 8.0| -.1795 | 4.054 | -.7659 | -.1277 | 9.281 | -.7857 |
| 9.0| -.1650 | 7.445 | -.8738 | -.1127 | 9.346 | -.8881 |
| 10.0| -.1537 | 11.40 | -.9791 | -.1010 | 9.400 | -.9901 |
| 15.0| -.1204 | 39.28 | -1.491 | -.06687 | 9.572 | -1.495 |
| 20.0| -.1027 | 80.26 | -1.995 | -.05006 | 9.663 | -1.997 |
| 50.0| -.0637 | 593.5 | -4.999 | -.2000 | 9.9474 | -5.000 |
| 100.0|      |      |      |      |      |      |
| 0.0|      |      |      |      |      |      |
| 1.0|      |      |      |      |      |      |
| 2.0|      |      |      |      |      |      |
| 4.0|      |      |      |      |      |      |
| 6.0|      |      |      |      |      |      |
| 8.0|      |      |      |      |      |      |
| 10.0|      |      |      |      |      |      |
| 15.0|      |      |      |      |      |      |
| 9.0|      |      |      |      |      |      |
FIG. 1. $V(x,0,0)$ at (a) $B = 0.001$ for $K = 0, K_c = 0.2381$, and $K = 0.4$ and (b) $B = 1.0$ for $K = 0, K_c = 2.381$, and $K = 4.0$. All quantities are in atomic units.

FIG. 2. The location of the saddle point $x_s$ and the outer well minimum $x_o$ as a function of $K$ for various values of $B$ (in atomic units). The lowest point on each curve is the critical point $x_c$. All quantities are in atomic units.

FIG. 3. The value of the potential at the saddle point $V_s = V(x_s, 0, 0)$ and the OW minimum $V_o = V(x_o, 0, 0)$ as a function of $K$ for various values of $B$ (in atomic units). The saddle point point energy increases without limit from the critical value $V_c = V(x_c)$; the outer well minimum increases from the critical value and approaches zero in the limit $K \to \infty$.

FIG. 4. The evolution of a MDCW state into an OW state as a function of $K$ at $B = 0.1$; (a) MDCW state with minimal tunneling through the barrier; (b) significant tunneling of the MDCW state through the barrier; (c) saddle state; (d) decreased probability density in the MDCW; (e) a fully delocalized OW state.

FIG. 5. The value of the wavefunction at the origin as a function of $K$ for various values of $B$ (in atomic units).
\[ V(x,0,0) \]

- \[ B = 1.0 \]
- \[ K = 0 \]
- \[ K = 2.381 \]
- \[ K = 4 \]
transition for $B=0.1$

Wave functions $\psi(x,0,0)$

- $K=0.0$
- $K=1.1$
- $K=1.3$
- $K=2.0$
