Note

The Linet–Tian metrics are a restricted set of Krasiński’s solutions of Einstein’s field equations for a rotating perfect fluid

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Received 10 July 2021, revised 21 September 2021
Accepted for publication 27 October 2021
Published 22 November 2021

Abstract

In this note we show that the Linet–Tian family of solutions of the vacuum Einstein equations with a cosmological constant are a restricted set of the solutions of the Einstein field equations for a rotating perfect fluid previously found by Krasiński.

Keywords: 04.20.Jb, Einstein equations, exact solutions, cosmological constant

1. Introduction

There is an interesting family of solutions of the vacuum Einstein field equations with a cosmological constant \( \Lambda \), that can be positive or negative, found independently by Linet [1] and Tian [2]. The solutions are static, and contain also two other orthogonal Killing vectors. They may be written in the form [3],

\[
ds^2 = -y^{1/3+p_1/2}(1 - \Lambda y)^{1/3-p_1/2} C_1 \, dt^2 + \frac{1}{3y(1 - \Lambda y)} \, dy^2 \\
+ \ y^{1/3+p_2/2}(1 - \Lambda y)^{1/3-p_2/2} C_2 \, dz^2 + \ y^{1/3+p_3/2}(1 - \Lambda y)^{1/3-p_3/2} C_3 \, d\phi^2, \tag{1}
\]

where the \( C_i \) are arbitrary constants, and, for \( \Lambda > 0 \), \( y \) is restricted to \( 0 < y < 1/\Lambda \), while for \( \Lambda < 0 \) the range of \( y \) is \( 0 \leq y \leq \infty \). They satisfy Einstein’s equations [4],

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2}
\]
provided the parameters $p_i$ satisfy the relations,

\[ p_1 + p_2 + p_3 = 0 \]
\[ p_1^2 + p_2^2 + p_3^2 = \frac{8}{3} \] (3)

and this, in turn implies that the allowed values of the parameters $p_i$ are restricted to the range,

\[ -\frac{4}{3} \leq p_i \leq \frac{4}{3} \] (4)

and all values in that range are allowed. Clearly, (3) is still satisfied if we change every $p_i$ to $-p_i$, so that, given a solution of (3), we get a new solution by simply changing the sign of every $p_i$. However, this new solution is diffeomorphic to the original one, as can be seen by changing coordinates in (1), in accordance with,

\[ y = (1 - \Lambda z)/\Lambda \] (5)

which changes (1) to,

\[
ds^2 = -z^{1/3-p_1/2}(1 - \Lambda y)^{1/3+p_1/2}\tilde{C}_1 \, dr^2 + \frac{1}{3y(1 - \Lambda y)} \, dy^2 \\
+ y^{1/3-p_2/2}(1 - \Lambda y)^{1/3+p_2/2}\tilde{C}_2 \, dz^2 + y^{1/3-p_3/2}(1 - \Lambda y)^{1/3+p_3/2}\tilde{C}_3 \, d\phi^2,
\] (6)

where $\tilde{C}_i = C_i/\Lambda p_i$, which is identical to (1), up to a rescaling of the remaining coordinates, and the change $p_i \to -p_i$. Similarly, we get the same solution, up to a rescaling of appropriate coordinates, by exchanging $p_2$ and $p_3$. Finally we notice that we may solve (3) for $p_2$ and $p_3$ to get,

\[
p_2 = -\frac{p_1}{2} + \epsilon \frac{\sqrt{48 - 27p_1^2}}{6}, \\
p_3 = -\frac{p_1}{2} - \epsilon \frac{\sqrt{48 - 27p_1^2}}{6}
\] (7)

where $\epsilon = \pm 1$. This can also be seen as a proof of (4). As just discussed, we will set $\epsilon = 1$, without loss of generality. These properties will be important in the discussions that follows.

The properties and applications of the Linet–Tian metrics have been the subject of a number of studies. A recent review of these and similar types of metrics has been recently presented in [6]. It apparently has escaped this review, as well as most the research papers centering on these type of metrics, that the Linet–Tian solutions are contained, as a restricted set, in a large family of solutions of the vacuum Einstein equations with a cosmological constant, previously found by Krasinski [7, 8]. These metrics can be obtained from the following Ansatz [9]. If one writes the metrics in the form,

\[
ds^2 = \frac{1}{v^{2/3}} \, dx_0^2 + 2 \frac{x_2}{v^{2/3}} \, dx_0 \, dx_1 + \frac{x_2^2 - V}{v^{2/3}} \, dx_1^2 \\
- \frac{J^2}{s^3 v^2} \exp \left( - \int \frac{x_2}{V} \, dx_0 \right) \, dx_2^2 - \frac{V}{s^{1/3}} \exp \left( - \int \frac{x_2}{V} \, dx_2 \right) \, dx_3^2,
\] (8)
where \( V = V(x_2), v = v(x_2), \) and \( J \) and \( s \) are constants, then, a necessary condition for the Einstein equation (2) to be satisfied is that \( V \) is a solution of,

\[
\frac{d^2V}{dx_2^2} - 2 = 0
\]  

and \( v \) is a solution of,

\[
\frac{d^2v}{dx_2^2} = \frac{1}{V} \left( \frac{dV}{dx_2} - x_2 \right) \frac{dv}{dx_2} + 3 \frac{V}{4V^2} \left( \frac{d^2V}{dx_2^2} - V + x_2 \frac{dV}{dx_2} - \left( \frac{dV}{dx_2} \right)^2 \right) v. \]  

Notice that (9) and (10) are independent of \( \Lambda \), and, in fact, of the signature of (8). The general (real) solution of (9) can be written in form,

\[
V(x_2) = (x_2 - p_0)(x_2 - q_0),
\]  

where \( p_0, \) and \( q_0, \) are constants, and we have three possibilities, namely, \( p_0, \) and \( q_0, \) are real and distinct, \( p_0 = q_0, \) (both real), and \( p_0 = q_0^*, \) i.e. complex conjugate of each other.

The Linet–Tian metrics (1) have three orthogonal Killing vectors, while this, in principle, is not the case for the Krasiński metrics (8). The terms in question are,

\[
d\sigma^2 = \frac{1}{v^{2/3}} dx_0^2 + 2 x_2 \frac{x_2 - V}{v^{2/3}} dx_0 dx_1 + \frac{x_2 - V}{v^{2/3}} dx_1^2
\]  

implying that \( \partial_{x_0}, \) and \( \partial_{x_1}, \) are not orthogonal. We, therefore, consider a (linear) change of coordinate basis of the form,

\[
x_0 = a_1 y_0 + a_2 y_1
\]

\[
x_1 = b_1 y_0 + b_2 y_1,
\]  

and find that the conditions for the orthogonality of \( \partial_{x_0}, \) and \( \partial_{x_1}, \) are,

\[
b_1 b_2 - a_1 a_2 p_0 q_0 = 0,
\]  

and either,

\[
a_2 q_0 + b_2 = 0,
\]  

or,

\[
a_2 p_0 + b_2 = 0.
\]  

Solving (14) for \( a_1, \) and (15) for \( a_2, \) we get,

\[
d\sigma^2 = \frac{a_1^2(x_2 - p_0)(q_0 - p_0)}{p_0^{2/3} v^{2/3}} dy_0^2 + \frac{b_2^2(q_0 - x_2)(q_0 - p_0)}{v^{2/3}} dy_1^2,
\]  

while solving (16) for \( a_2, \) and replacing in (12), we find,

\[
d\sigma^2 = \frac{a_1^2(q_0 - x_2)(q_0 - p_0)}{q_0^{2/3} v^{2/3}} dy_0^2 + \frac{b_2^2(x_2 - p_0)(q_0 - p_0)}{v^{2/3}} dy_1^2.
\]
We first notice that since $a_1$ and $b_2$ are arbitrary, (17) and (18) are equivalent up to irrelevant changes of names. Next we see that for complex $q_0$ and $p_0$ the transformation (13) leads to complex coefficients in $d\sigma^2$, while for $p_0 = q_0$ the transformation is singular. The only acceptable case is then for $q_0$ and $p_0$ real and distinct. In what follows we assume, without loss of generality, $q_0 > p_0$. The Krasiński metric (8) then takes the form,

$$d\sigma^2 = \frac{(x_2 - p_0)}{q_0^{3/2}} dy_0^2 + \frac{(q_0 - x_2)}{q_0^{3/2}} dy_1^2 - \frac{j^2}{s u^2} \exp \left( - \int \frac{x_2}{V} dx_2 \right) dx_2^2 - \frac{V}{s u^{3/2}} \exp \left( - \int \frac{x_2}{V} dx_2 \right) dx_3^2,$$

(19)

where, without loss of generality, we have chosen (17), and assigned values to $a_1$, and $b_2$ so as to simplify the resulting expressions. With this restriction we still have to consider three separate cases, namely, $x_2 \neq q_0$, $q_0 \geq x_2 \geq p_0$, and $p_0 \geq x_2$. In what follows we analyze in detail the case $x_2 \geq q_0$. The other two cases can be straightforwardly analyzed along very similar lines, and will not be given explicitly here for brevity.

2. The case $x_2 \geq q_0$

2.1. The form of the metric

To continue our analysis of (8) we notice that for $x_2 \geq q_0$,

$$\exp \left( - \int \frac{x_2}{V} dx_2 \right) = \exp \left( - \int \frac{x_2}{(x_2 - p_0)(x_2 - q_0)} \right) = C(x_2 - p_0)^{\frac{p_0}{q_0 - j_0}}(x_2 - q_0)^{\frac{q_0}{q_0 - j_0}}, \tag{20}$$

where $C$ is a constant. Since the left-hand side of (20) is real and positive, we may set $C = 1$.

Again for $x_2 \geq q_0$, we may write the solution of (10) in the form,

$$V(x_2) = \frac{(x_2 - p_0)^{\frac{q_0 - j_0}{q_0 - p_0}} \sqrt{m^2 - m_0 m + m_0^2}}{\sqrt{(x_2 - q_0)^{p_0 - m_0} \sqrt{m^2 - m_0 m + m_0^2}}} \frac{(x_2 - q_0)^{2p_0 - m_0} \sqrt{m^2 - m_0 m + m_0^2}}{\sqrt{(x_2 - q_0)^{p_0 - m_0} + P(x_2 - p_0)^{\frac{q_0}{q_0 - m_0}}}}, \tag{21}$$

where $P$ and $Q$ are real constants. In what follows we restrict to $P > 0$, and $Q > 0$, to avoid singularities in $V(x_2)$.

Replacing (11), (20) and (21) in (8) we find that the full set of Einstein’s equation (2) is satisfied if we impose,

$$\Lambda = \frac{s Q P (q_0^2 - q_0 p_0 + p_0^2)}{3 j^2}. \tag{22}$$

We remark again that equation (2) are satisfied independently of the signature assigned to (8), or the particular signs of $j^2$ or $s$, and, therefore, (8) provides a (possibly large) family of solutions of (2). At this point it is convenient to go back to the ‘diagonal’ form (19). This restricted set of Krasiński’s metrics provides solutions of (2) with three commuting Killing vectors $(\partial_{\beta_0}, \partial_{\beta_1}$, and $\partial_{\beta_2})$, but we still need to fix the signature of the metrics. In the case of
\( \Lambda > 0 \), if we assume \( J^2 > 0 \), from (22) we must take \( s < 0 \). Without loss of generality we may take \( s = -1 \), and this makes the metric (19) static and with signature \(-, +, +, +\), and, in accordance with (22), corresponding to \( \Lambda > 0 \). In more detail, the ‘diagonal’ metric is then given by,

\[
\mathrm{d}s^2 = -\frac{(x_2 - q_0)}{v^{1/2}} \mathrm{d}y_1^2 + \frac{J^2(x_2 - p_0)\frac{\omega}{\omega + \rho_0} (x_2 - q_0)}{v^2} \mathrm{d}x_2^2 + \frac{(x_2 - p_0)^{-\frac{\omega}{\omega + \rho_0}} (x_2 - q_0)^{-\frac{\rho_0}{\omega + \rho_0}}}{v^{2/3}} \mathrm{d}x_3^2 + \frac{(x_2 - p_0)^{\frac{\omega}{\omega + \rho_0}}}{v^{3/2}} \mathrm{d}y_2^2.
\]

(23)

But, the Linet and Tian analysis shows that the static solutions of (2) with three orthogonal commuting Killing vectors are unique, up to diffeomorphisms. Therefore, the Linet–Tian metrics and Kransinski’s metrics should be related by a coordinate transformation. That this is the case is shown explicitly in the next section.

2.2. A coordinate transformation

We consider again (1) and (23) and a coordinate change of the form \( y = y(x_2) \) that would take (1) into (23). Under this change we should have,

\[
\frac{1}{3y(1 - \Lambda y)} \left( \frac{\mathrm{d}y}{\mathrm{d}x_2} \right)^2 = \frac{J^2(x_2 - p_0)\frac{\omega}{\omega + \rho_0} (x_2 - q_0)^{-\frac{\rho_0}{\omega + \rho_0}}}{v(x_2)^2}
\]

or,

\[
\left[ \frac{1}{3y(1 - \Lambda y)} \right]^{1/2} \left( \frac{\mathrm{d}y}{\mathrm{d}x_2} \right) = \left[ \frac{J^2(x_2 - p_0)\frac{\omega}{\omega + \rho_0} (x_2 - q_0)^{-\frac{\rho_0}{\omega + \rho_0}}}{v(x_2)^2} \right]^{1/2}.
\]

(25)

This can be integrated to,

\[
\ln \left( 1 - 2\Lambda y - 2i\sqrt{\Lambda y} \sqrt{1 - \Lambda y} \right) = \ln \left[ \frac{\sqrt{Q(x_2 - q_0)^{\frac{\sqrt{\omega^2 + \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}} - iq_0(x_2 - p_0)^{\frac{\sqrt{\omega^2 + \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}}}{\sqrt{Q(x_2 - q_0)^{\frac{\sqrt{\omega^2 + \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}} + iq_0(x_2 - p_0)^{\frac{\sqrt{\omega^2 + \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}}}} \right],
\]

(26)

where we have used (21) to eliminate \( \Lambda \). Equation (26) can be solved for \( y \).

\[
y = \frac{P(x_2 - p_0)^{\frac{\sqrt{\omega^2 - \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}}}{Q(x_2 - q_0)^{\frac{\sqrt{\omega^2 - \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}} + P(x_2 - p_0)^{\frac{\sqrt{\omega^2 - \omega_0\rho_0 + \rho_0^2}}{\omega_0 + \rho_0}}} \Lambda.
\]

(27)

The range of \( x_2 \) is \((q_0 \leq x_2 \leq \infty)\), and, in accordance with (27), we have,

\[
\frac{1}{\Lambda} \geq y \geq \frac{P}{\Lambda(P + Q)}, \quad \text{for } q_0 \leq x_2 \leq \infty.
\]

(28)

We must remark that although there is a sign ambiguity in (25), we only need that (24) be satisfied, and one can check that (27) satisfies this requirement.
Next we consider the coefficient of $dy_1^2$ in (23). We have,
\[-x_2 - q_0 = \frac{(x_2 - q_0) - p_0 + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} (x_2 - p_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}}{(x_2 - q_0) - p_0 + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} + P(x_2 - p_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}} \]  
\[ (29) \]

On the other hand, going back to (1), if we consider the coefficient of $dt^2$, and change variables in accordance with (27), to get,
\[-C_1 y^4 + \frac{p_0}{2} (1 - \Lambda y)^{\frac{3}{2}} \]
\[ = - \frac{C_1 P^3 + \frac{p_0}{2}}{(x_2 - p_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} (x_2 - q_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} + \Lambda^3 + \frac{q_0^2}{2}} \frac{Q(x_2 - q_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} + P(x_2 - p_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}} \]  
\[ (30) \]

We then have that (30) will be equal to (29) if we set,
\[ C_1 = \Lambda^{\frac{3}{2}} + \frac{p_0}{2} P^{-\frac{1}{2}} + \frac{p_0}{2} Q^{-\frac{1}{2}} \]
\[ (31) \]
and,
\[ p_1 = \frac{2(2q_0 - p_0)}{3/\sqrt{\Lambda^2 - p_0 q_0 + q_0^2}} \]
\[ (32) \]

Similarly, for the coefficient of $dy_0^2$ in (23) we have,
\[ x_2 - p_0 = \frac{(x_2 - q_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} (x_2 - p_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}}{(x_2 - q_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} + P(x_2 - p_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}} \]  
\[ (33) \]

while for the coefficient of $dz^2$ in (1) we have,
\[ C_2 y^4 + \frac{p_0}{2} (1 - \Lambda y)^{\frac{3}{2}} \]
\[ = C_2 P^3 + \frac{p_0}{2} \frac{(x_2 - p_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} (x_2 - q_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}}{(x_2 - q_0) - \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}} + P(x_2 - p_0) + \sqrt{\frac{q_0^2 - q_0 p_0 + p_0^2}{\Lambda^2 + q_0^2}}} \]  
\[ (34) \]

and we have equality of (34) and (33) imposing,
\[ C_2 = \Lambda^{\frac{3}{2}} + \frac{p_0}{2} P^{-\frac{1}{2}} + \frac{p_0}{2} Q^{-\frac{1}{2}} \]
\[ (35) \]
and,
\[ p_2 = \frac{2(2p_0 - q_0)}{3/\sqrt{\Lambda^2 - p_0 q_0 + q_0^2}} \]
\[ (36) \]
Finally, and using the same similar procedure, for the coefficient of \(d\phi^2\) we find,

\[C_3 = \Lambda \frac{h + \frac{p_i}{3}}{p - \frac{p_i}{3}} P + \frac{p_i}{3} Q \frac{h + \frac{p_i}{3}}{p - \frac{p_i}{3}}\]  

and,

\[p_3 = -\frac{2(p_0 + q_0)}{3\sqrt{p_0^2 - p_0q_0 + q_0^2}}\]  

We can immediately check that the \(p_i\) in (32), (36) and (38) satisfy (3). Moreover, considering again (32), (36) and (38) we have three different cases: \(p_0 > 0\), \(p_0 = 0\), and \(p_0 < 0\).

In the case \(p_0 > 0\) we may set \(p_0 = 1\), since the \(p_i\) depend only on the ratio \(q_0/p_0\), and then considering all \(q_0 > p_0\) we find for the \(p_i\) the ranges,

\[
\begin{align*}
\frac{2}{3} &\leq p_1 \leq \frac{4}{3} \\
\frac{2}{3} &\geq p_2 \geq -\frac{2}{3} \\
-\frac{4}{3} &\leq p_3 \leq -\frac{2}{3}
\end{align*}
\]  

Similarly, for \(p_0 < 0\) we have,

\[
\begin{align*}
-\frac{2}{3} &\leq p_1 \leq \frac{4}{3} \\
-\frac{2}{3} &\geq p_2 \geq -\frac{2}{3} \\
\frac{4}{3} &\leq p_3 \leq -\frac{2}{3}
\end{align*}
\]  

The case \(p_0 = 0\) corresponds just to \(p_1 = 4/3, p_2 = -2/3,\) and \(p_3 = -2/3\).

3. Summary of results

Equations (39) and (40), together with the properties already discussed of the \(p_i\) imply that the full relevant range of values of the \(p_i\) are covered with appropriate choices of \(p_0\) and \(q_0\). Considering the range of \(y\), we notice that once the \(p_i\) are fixed, the only relevant quantity is \(\Lambda\), since the \(C_i\) in (1) represent only rescalings of the coordinates. But according to (22), fixing \(\Lambda\) we still have a large freedom in the choices of \(P, Q,\) and \(J^2\). In accordance with (28), this freedom can be used to cover essentially the full range of \(y\) in (1), with the exception of the singular point \(y = 0\), so that we have shown that the Linet–Tian metrics are effectively a restricted set of the Krasinski metrics (8), in the case \(x_2 > q_0\). But essentially similar derivations show that this is also the case for \(q_0 > x_2 \geq p_0\), and \(p_0 \geq x_2\). In the first case the full range \(0 \leq y \leq 1/\Lambda\) is covered, while in the second the point \(y = 1/\Lambda\) is excluded. This, in turn, shows that the three possible ranges of \(x_2\) in (8) correspond, up to isometries, to the same solution, and also completes the proof of the equivalence of the metrics (1) and (8) in the cases where we take \(p_0\) and \(q_0\) in (11) as real and distinct.
Acknowledgments

I am grateful to A Krasiński for bringing to my attention the references to his work and its possible relation to the Linet–Tian metrics, and also for his comments on this note.

Data availability statement

No new data were created or analysed in this study.

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