PARAMETERIZATIONS OF $cg$-FRAMES VIA OPERATORS

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Abstract. In this paper we introduce and show some new notions and results on $cg$-frames of Hilbert spaces. We define $cg$-orthonormal bases for a Hilbert space $H$ and verify their traits and relations with $cg$-frames. Actually, we present that every $cg$-frame can be present as a composition of a $cg$-orthonormal basis and an operator under some conditions. Also, we give for any $cg$-frame an induced $c$-frame and study their properties and relations. Moreover, we show that every $cg$-frame can be written as aggregate of two Parseval $cg$-frames. In addition, each $cg$-frame can be shown as a summation of a $cg$-orthonormal basis and a $cg$-Riesz basis.

1. Introduction

Frames (discrete frames) in Hilbert spaces were introduced by Duffin and Schaeffer [7] in 1952 to study some deep problems in nonharmonic Fourier series. After the illustrious paper [6] by Daubechies, Grossmann and Meyer, frame theory popularized immensely.

A frame for a Hilbert space allows each vector in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. Intuitively, a frame can be thought as a basis to which one has added more elements.

Generally, frames have been used in signal processing, image processing, data compression and sampling theory. Later, motivated by the theory of coherent states, this concept was generalized to families indexed by some locally compact space endowed with a Radon measure. This approach leads to the notion of continuous frames [2, 3, 11, 13]. Some results about continuous frames and their generalizations can be found in [8, 9, 10, 15].

In this paper we generalize some results inspired by [16] and [12] to $cg$-frames. The paper is organized as follows. In Section 2, we introduce the concept of $cg$-orthonormal bases for Hilbert spaces and discuss about their characteristics and their relations with $cg$-frames and $c$-frames. Our aim in Section 3 is describing every continuous $g$-frame as a sum of two Parseval continuous $g$-frames. We also present that every continuous $g$-frame can be written as a linear combination of an $cg$-orthonormal basis and a $cg$-Riesz basis.

Throughout this paper, $H$ is a separable Hilbert space, $(\Omega, \mu)$ is a measure space with positive measure $\mu$ and $\{H_\omega\}_{\omega \in \Omega}$ is a family of separable Hilbert spaces.

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We first review the definition of continuous frames and continuous \(g\)-frames.

**Definition 1.1.** ([15]) Suppose that \((\Omega, \mu)\) is a measure space with positive measure \(\mu\). A mapping \(f : \Omega \rightarrow H\) is called a continuous frame, or simply a \(c\)-frame, with respect to \((\Omega, \mu)\) for \(H\), if:

(i) For each \(h \in H\), \(\omega \mapsto \langle h, f(\omega) \rangle\) is a measurable function,

(ii) there exist positive constants \(A\) and \(B\) such that

\[
A \|h\|^2 \leq \int_{\Omega} |\langle h, f(\omega) \rangle|^2 d\mu(\omega) \leq B \|h\|^2, \quad h \in H.
\]

The constants \(A, B\) are called \(c\)-frame bounds. If \(A, B\) can be chosen such that \(A = B\), then \(f\) is called a tight \(c\)-frame and if \(A = B = 1\), it is called a Parseval \(c\)-frame. A mapping \(f\) is called \(c\)-Bessel mapping if the second inequality in (1.1) holds. In this case, \(B\) is called the Bessel bound.

Some operators associated to \(c\)-Bessel mappings can be useful to characterize them.

**Proposition 1.2.** ([15]) Let \((\Omega, \mu)\) be a measure space and \(f : \Omega \rightarrow H\) be a \(c\)-Bessel mapping for \(H\). Then the operator \(T_f : L^2(\Omega, \mu) \rightarrow H\), weakly defined by

\[
\langle T_f \varphi, h \rangle = \int_{\Omega} \varphi(\omega) \langle f(\omega), h \rangle d\mu(\omega), \quad h \in H,
\]

is well defined, linear, bounded, and its adjoint is given by

\[
T_f^* : H \rightarrow L^2(\Omega, \mu), \quad T_f^* h(\omega) = \langle h, f(\omega) \rangle, \quad \omega \in \Omega.
\]

The operator \(T_f\) is called synthesis operator and \(T_f^*\) is called analysis operator of \(f\).

If \(f\) is a \(c\)-Bessel mapping with respect to \((\Omega, \mu)\) for \(H\), then the operator \(S_f : H \rightarrow H\) defined by \(S_f = T_f T_f^*\), is called frame operator of \(f\). Thus

\[
\langle S_f h, k \rangle = \int_{\Omega} \langle h, f(\omega) \rangle \langle f(\omega), k \rangle d\mu(\omega), \quad h, k \in H.
\]

If \(f\) is a \(c\)-frame for \(H\), then \(S\) is invertible.

The converse of above proposition holds when \(\mu\) is \(\sigma\)-finite in the measure space \((\Omega, \mu)\).

**Proposition 1.3.** ([15]) Let \((\Omega, \mu)\) be a measure space where \(\mu\) is \(\sigma\)-finite. Let \(f : \Omega \rightarrow H\) be a mapping such that for each \(h \in H\), \(\omega \mapsto \langle h, f(\omega) \rangle\) is measurable. If the mapping \(T_f : L^2(\Omega, \mu) \rightarrow H\) defined by (1.2), is a bounded operator, then \(f\) is a \(c\)-Bessel mapping.

**Theorem 1.4.** ([15]) Suppose that \((\Omega, \mu)\) is a measure space where \(\mu\) is \(\sigma\)-finite. Let \(f : \Omega \rightarrow H\) be a mapping such that for each \(h \in H\), \(\omega \mapsto \langle h, f(\omega) \rangle\) is measurable. The mapping \(f\) is a \(c\)-frame with respect to \((\Omega, \mu)\) for \(H\) if and only if the operator \(T_f : L^2(\Omega, \mu) \rightarrow H\) defined by (1.2), is a bounded and onto operator.
**Definition 1.5.** Let \( \varphi \in \Pi_{\omega \in \Omega} H_\omega \). We say that \( \varphi \) is strongly measurable if \( \varphi \) as a mapping of \( \Omega \) to \( \Theta_{\omega \in \Omega} H_\omega \) is measurable, where
\[
\Pi_{\omega \in \Omega} H_\omega = \{ f : \Omega \rightarrow \bigcup_{\omega \in \Omega} H_\omega ; f(\omega) \in H_\omega \}.
\]

Now, we review the definition of continuous \( g \)-frames.

**Definition 1.6.** We call \( \{ \Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega \} \) a continuous generalized frame, or simply a \( cg \)-frame, for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \), if:
(i) each \( f \in H \), \( \{ \Lambda_\omega f \}_{\omega \in \Omega} \) is strongly measurable,
(ii) there are two positive constants \( A \) and \( B \) such that
\[
A \| f \|^2 \leq \int_\Omega \| \Lambda_\omega f \|^2 d\mu(\omega) \leq B \| f \|^2, \quad f \in H.
\]

We call \( A \) and \( B \) the lower and upper \( cg \)-frame bounds, respectively. If \( A, B \) can be chosen such that \( A = B \), then \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is called a tight \( cg \)-frame and if \( A = B = 1 \), it is called a Parseval \( cg \)-frame. A family \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is called \( cg \)-Bessel family if the second inequality in (1.4) holds.

Now, let the space \( (\bigoplus_{\omega \in \Omega} H_\omega, \mu)_{L^2} \subseteq \Pi_{\omega \in \Omega} H_\omega \) be defined as follows,
\[
(\bigoplus_{\omega \in \Omega} H_\omega, \mu)_{L^2} = \{ \varphi \mid \varphi \text{ is strongly measurable, } \int_\Omega \| \varphi(\omega) \|^2 d\mu(\omega) < \infty \}.
\]

The space \( (\bigoplus_{\omega \in \Omega} H_\omega, \mu)_{L^2} \) is a Hilbert space with inner product
\[
\langle \varphi, \psi \rangle = \int_\Omega \langle \varphi(\omega), \psi(\omega) \rangle d\mu(\omega).
\]

**Proposition 1.7.** (11) Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a \( cg \)-Bessel family for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) with Bessel bound \( B \). Then the mapping \( T \) of \( (\bigoplus_{\omega \in \Omega} H_\omega, \mu)_{L^2} \) to \( H \) defined by
\[
\langle T\varphi, h \rangle = \int_\Omega \langle \Lambda^*_\omega \varphi(\omega), h \rangle d\mu(\omega), \quad \varphi \in (\bigoplus_{\omega \in \Omega} H_\omega, \mu)_{L^2}, \quad h \in H \tag{1.5}
\]
is linear and bounded with \( \| T \| \leq \sqrt{B} \). Furthermore for each \( h \in H \) and \( \omega \in \Omega \)
\[
T^*(h)(\omega) = \Lambda_\omega h \tag{1.6}
\]

The operators \( T \) and \( T^* \) are called synthesis and analysis operators of \( cg \)-Bessel family \( \{ \Lambda_\omega \}_{\omega \in \Omega} \), respectively.

Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a \( cg \)-frame for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) with frame bounds \( A, B \). The operator \( S : H \rightarrow H \) defined by
\[
\langle Sf, g \rangle = \int_\Omega \langle f, \Lambda^*_\omega \Lambda_\omega g \rangle d\mu(\omega), \quad f, g \in H \tag{1.7}
\]
is called the frame operator of \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) which is a positive and invertible operator.

Now, we state a known result that is helpful in proving some results.
Proposition 1.8. \cite{[4]} Let $K : H \rightarrow H$ be a bounded linear operator. Then the following hold.

(i) $K = \alpha(U_1 + U_2 + U_3)$, where each $U_j$, $j = 1, 2, 3$, is a unitary operator and $\alpha$ is a constant.

(ii) If $K$ is onto, then it can be written as a linear combination of two unitary operators if and only if $K$ is invertible.

Every closed-ranged operator has a right-inverse operator in the following sense:

Lemma 1.9. \cite{[5]} Let $H$ and $K$ be Hilbert spaces, and suppose that $U : K \rightarrow H$ is a bounded operator with closed range $R(U)$. Then there exists a bounded operator $U^\dagger : H \rightarrow K$ for which

$$UU^\dagger h = h, \quad h \in R(U).$$

The operator $U^\dagger$ is called the pseudo-inverse of $U$.

2. $\text{cg}$-Orthonormal Basis

Similar to continuous frames, we want to generalize orthonormal bases like continuous frames. Indeed, our purpose here is to define a mapping $f : \Omega \rightarrow H$ that has similar properties to an orthonormal basis of $H$.

Definition 2.1. Suppose $(\Omega, \mu)$ is a measure space. A mapping $f : \Omega \rightarrow H$ is called a $\text{c}$-orthonormal basis with respect to $(\Omega, \mu)$ for $H$, if:

(i) For each $h \in H$, $\omega \mapsto \langle h, f(\omega) \rangle$ is measurable,

(ii) for almost all $\nu \in \Omega$,

$$\int_{\Omega} |\langle h, f(\omega) \rangle|^2 d\mu(\omega) = \|h\|^2.$$

(iii) for each $h \in H$,

$$\int_{\Omega} \|\Lambda^* \omega h\|^2 d\mu(\omega) = \|h\|^2.$$

Now we define generalization of orthonormal basis in case of operators.

Definition 2.2. Assume $(\Omega, \mu)$ is a measure space. A family of operators $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called a continuous $g$-orthonormal basis or simply a $\text{cg}$-orthonormal basis, for $H$ with respect to $\{H_\omega\}_{\omega \in \Omega}$, whenever:

(i) For each $h \in H$, $\{\Lambda_\omega h\}_{\omega \in \Omega}$ is strongly measurable,

(ii) for almost all $\nu \in \Omega$,

$$\int_{\Omega} \langle \Lambda^*_\omega f_\omega, \Lambda^*_\nu g_\nu \rangle d\mu(\omega) = \langle f_\nu, g_\nu \rangle, \quad \{f_\omega\}_{\omega \in \Omega} \in (\oplus_{\omega \in \Omega} H_\omega, \mu)_{L^2}, \quad g_\nu \in H_\nu,$$

(iii) for each $h \in H$,

$$\int_{\Omega} \|\Lambda_\omega h\|^2 d\mu(\omega) = \|h\|^2.$$

If only conditions (i) and (ii) hold, $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}_{\omega \in \Omega}$ is called a $\text{cg}$-orthonormal system for $H$ with respect to $\{H_\omega\}_{\omega \in \Omega}$.

Example 2.3. Suppose that $\Omega = \{a, b, c\}$, $\Sigma = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$ and $\mu : \Sigma \rightarrow [0, \infty]$ is a measure such that $\mu(\emptyset) = 0$, $\mu(\{a, b\}) = 1$, $\mu(\{c\}) = 1$ and
\( \mu(\Omega) = 2 \). Let \( H \) be a 2 dimensional Hilbert space with an orthonormal basis \( \{e_1, e_2\} \). We define

\[
f : \Omega \rightarrow H
\]

by \( f = e_1 \chi_{\{a,b\}} + e_2 \chi_{\{c\}} \). So for each \( h \in H \),

\[
\langle f(\omega), h \rangle = \langle e_1, h \rangle \chi_{\{a,b\}}(\omega) + \langle e_2, h \rangle \chi_{\{c\}}(\omega), \quad \omega \in \Omega,
\]

hence \( \omega \mapsto \langle h, f(\omega) \rangle \) is measurable. Now, for each \( \omega \in \Omega \), we define

\[
\Lambda_\omega : H \rightarrow \mathbb{C}
\]

\[
\Lambda_\omega(h) = \langle h, f(\omega) \rangle.
\]

Actually, we consider for each \( \omega \in \Omega \), \( H_\omega = \mathbb{C} \). By an easy calculation, we have

\[
\Lambda_\omega^*(z) = f(\omega)z, \quad z \in \mathbb{C}.
\]

For any \( \nu \in \Omega \), \( x_\nu \in \mathbb{C} \) and any \( \{z_\omega\}_{\omega \in \Omega} \in (\oplus_{\omega \in \Omega} H_\omega, \mu)_{L^2} = L^2(\Omega, \mu) \), due to the Example 4.2 in [16], we have

\[
\int_{\Omega} \langle \Lambda_\omega^* z_\omega, \Lambda_\omega^* x_\nu \rangle d\mu(\omega) = \int_{\Omega} z_\omega \overline{x_\nu} \langle f(\omega), f(\nu) \rangle d\mu(\omega) = z_\nu \overline{x_\nu}.
\]

Also for each \( h \in H \),

\[
\int_{\Omega} \|\Lambda_\omega h\|^2 d\mu(\omega) = \int_{\Omega} |\langle h, f(\omega) \rangle|^2 d\mu(\omega) = \|h\|^2.
\]

Therefore \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a \( cg \)-orthonormal basis for \( H \) with respect to \( \{H_\omega\}_{\omega \in \Omega} \), where for each \( \omega \in \Omega \), \( H_\omega = \mathbb{C} \).

We present some equal conditions for \( cg \)-orthonormal bases.

**Theorem 2.4.** Let \( \{\Lambda_\omega\}_{\omega \in \Omega} \) be a \( cg \)-orthonormal system for \( H \) with respect to \( \{H_\omega\}_{\omega \in \Omega} \). Also assume that for each \( h \in H \), \( \int_{\Omega} \|\Lambda_\omega h\|^2 d\mu(\omega) < \infty \). Then the following conditions are equivalent:

(i) \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a \( cg \)-orthonormal basis for \( H \) with respect to \( \{H_\omega\}_{\omega \in \Omega} \).

(ii) For each \( h, k \in H \),

\[
\langle h, k \rangle = \int_{\Omega} \langle \Lambda_\omega h, \Lambda_\omega k \rangle d\mu(\omega).
\]

(iii) If \( \Lambda_\omega h = 0 \), a.e. \( [\mu] \), then \( h = 0 \).

(iv) For each zero measure set \( \Omega_0 \subseteq \Omega \), \( H = \overline{\text{span}\{\Lambda_\omega^*(H_\omega)\}_{\omega \in \Omega \setminus \Omega_0}} \).

**Proof.** (i) \( \iff \) (ii) Since \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a Parseval \( cg \)-frame for \( H \), so its frame operator \( S_\Lambda = I \). Hence (ii) is obvious. The converse side clearly holds.

(ii) \( \Rightarrow \) (iii) If \( \Lambda_\omega h = 0 \), a.e. \( [\mu] \), then for every \( k \in H \),

\[
\langle h, k \rangle = \int_{\Omega} \langle \Lambda_\omega h, \Lambda_\omega k \rangle d\mu(\omega) = 0.
\]

Therefore \( h = 0 \).

(iii) \( \Rightarrow \) (iv) Suppose that \( \Omega_0 \subseteq \Omega \) and \( h \perp \overline{\text{span}\{\Lambda_\omega^*(H_\omega)\}_{\omega \in \Omega \setminus \Omega_0}} \), so for almost
Proof. Let $\Lambda$ that

Suppose that Proposition 2.5. Therefore $\Lambda$ is a subspace of $H$. Also, it is closed, since if $\lim_{n \to \infty} h_n = h$, where $h_n$'s belong to $\mathcal{H}_k$, then

$$\langle h, k \rangle = \lim_{n \to \infty} \langle h_n, k \rangle = \lim_{n \to \infty} \int_{\Omega} \langle \Lambda \omega h_n, \Lambda \omega k \rangle d\mu(\omega).$$

According to assumption,

$$\int_{\Omega} |\langle \Lambda \omega h, \Lambda \omega k \rangle| d\mu(\omega) \leq \left( \int_{\Omega} \|\Lambda \omega h\|^2 d\mu(\omega) \right)^{\frac{1}{2}} \left( \int_{\Omega} \|\Lambda \omega g\|^2 d\mu(\omega) \right)^{\frac{1}{2}} < \infty.$$

So by Lebesgue’s Dominated Convergence Theorem,

$$\lim_{n \to \infty} \int_{\Omega} \langle \Lambda \omega h_n, \Lambda \omega k \rangle d\mu(\omega) = \int_{\Omega} \langle \Lambda \omega h, \Lambda \omega k \rangle d\mu(\omega),$$

which means $h \in \mathcal{H}_k$. For almost all $\nu \in \Omega$ and each $f \in H$, we have

$$\int_{\Omega} \langle \Lambda \omega \Lambda^*_\nu f, \Lambda \omega k \rangle d\mu(\omega) = \int_{\Omega} \langle \Lambda^*_\nu f, \Lambda^*_\nu \Lambda \omega k \rangle d\mu(\omega) = \langle \Lambda^*_\nu f, \Lambda^*_\nu k \rangle = \langle \Lambda^*_\nu f, k \rangle.$$

Therefore $\Lambda^*_\nu f \in \mathcal{H}_k$. Assume $f \perp \mathcal{H}_k$, then for almost all $\nu \in \Omega$,

$$0 = \langle \Lambda^*_\nu f, f \rangle = \|\Lambda^*_\nu f\|^2,$$

which gives $\Lambda^*_\nu f = 0$. For almost all $\nu \in \Omega$ and any $g_\nu \in H_{\nu}$,

$$\langle f, \Lambda^*_\nu g_\nu \rangle = \langle \Lambda^*_\nu f, g_\nu \rangle = 0.$$

So $f \perp \text{span}\{\Lambda^*_\nu(H_{\omega})\}_{\omega \in \Omega \setminus \Omega_0}$, where $\Omega_0$ is a zero measure subset of $\Omega$. By assumption (v), $\text{span}\{\Lambda^*_\nu(H_{\omega})\}_{\omega \in \Omega \setminus \Omega_0} = H$. Thus $f = 0$ and $\mathcal{H}_k = H$. Therefore

$$\langle h, k \rangle = \int_{\Omega} \langle \Lambda \omega h, \Lambda \omega k \rangle d\mu(\omega), \quad h, k \in H.$$

Proposition 2.5. Suppose that $\{\Theta_\omega\}_{\omega \in \Omega}$ is a cg-orthonormal basis for $H$ with respect to $\{H_\omega\}_{\omega \in \Omega}$ and $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is a family such that for each $h \in H$, $\{\Lambda_\omega h\}_{\omega \in \Omega}$ is strongly measurable. Then $\{\Lambda_\omega\}_{\omega \in \Omega}$ is a Parseval cg-frame for $H$ if and only if there exists a unique isometry $V \in B(H)$ such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$.

Proof. Let $\{\Lambda_\omega\}_{\omega \in \Omega}$ is a Parseval cg-frame for $H$. We define the operator $V$ weakly by

$$\langle V f, h \rangle = \int_{\Omega} \langle \Theta^*_\omega \Lambda \omega f, h \rangle d\mu(\omega), \quad f, h \in H.$$
For each $f, h \in H$, we have
\[
\|Vf, h\| \leq \left( \int_{\Omega} \|\Lambda_\omega f\|^2 d\mu(\omega) \right)^{\frac{1}{2}} \left( \int_{\Omega} \|\Theta_\omega h\|^2 d\mu(\omega) \right)^{\frac{1}{2}} \leq \|f\| \|h\|.
\]
So $V$ is well-defined and bounded. For almost all $\nu \in \Omega$ and each $f \in H$, $h_\nu \in H_\nu$,
\[
\langle \Theta_\nu Vf, h \rangle = \langle Vf, \Theta_\nu^* h \rangle = \int_{\Omega} \langle \Theta_\nu^* \Lambda_\omega f, \Theta_\nu^* h \rangle d\mu(\omega) = \langle \Lambda_\nu f, h \rangle.
\]
Since $\{\Theta_\omega\}_{\omega \in \Omega}$ is a $cg$-orthonormal basis, thus $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$.
For each $f \in H$,
\[
\|Vf\|^2 = \langle Vf, Vf \rangle = \int_{\Omega} \langle \Theta_\nu^* \Lambda_\omega f, Vf \rangle d\mu(\omega) = \int_{\Omega} \langle \Lambda_\omega f, \Theta_\nu Vf \rangle d\mu(\omega)
\]
\[
= \int_{\Omega} \langle \Lambda_\omega f, \Lambda_\omega f \rangle d\mu(\omega) = \|f\|^2.
\]
Therefore $V$ is an isometry.

Now, let $V_1$ and $V_2$ be two isometries such that $\Lambda_\omega = \Theta_\omega V_1$, a.e. $[\mu]$ and $\Lambda_\omega = \Theta_\omega V_2$, a.e. $[\mu]$. Then for each $f \in H$, $\Theta_\omega((V_1 - V_2)f) = 0$, a.e. $[\mu]$, which implies
\[
0 = \int_{\Omega} \|\Theta_\omega(V_1 - V_2)f\|^2 d\mu(\omega) = \|(V_1 - V_2)f\|^2,
\]
so $V_1 f = V_2 f$.

Conversely, let $V \in B(H)$ be a unique isometry such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$.
For any $f \in H$,
\[
\int_{\Omega} \|\Lambda_\omega f\|^2 d\mu(\omega) = \int_{\Omega} \|\Theta_\omega Vf\|^2 d\mu(\omega) = \|Vf\|^2 = \|f\|^2.
\]
Hence $\{\Lambda_\omega\}_{\omega \in \Omega}$ is a Parseval $cg$-frame for $H$.

\[\Box\]

**Theorem 2.6.** Assume $\{\Theta_\omega\}_{\omega \in \Omega}$ is a $cg$-orthonormal basis for $H$ with respect to $\{H_\omega\}_{\omega \in \Omega}$ and $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is a family such that for each $h \in H$, $\{\Lambda_\omega h\}_{\omega \in \Omega}$ is strongly measurable. Then $\{\Lambda_\omega\}_{\omega \in \Omega}$ is a $cg$-frame for $H$ with bounds $A$ and $B$ if and only if there exists a unique $V \in B(H)$ such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$ and $AI \leq V^*V \leq BI$.

**Proof.** Let $\{\Lambda_\omega\}_{\omega \in \Omega}$ be a $cg$-frame for $H$. Similar to proof of Proposition 2.5, the operator $V$ weakly defined by
\[
\langle Vf, h \rangle = \int_{\Omega} \langle \Theta_\nu^* \Lambda_\omega f, h \rangle d\mu(\omega), \quad f, h \in H,
\]
is a one-to-one and bounded operator such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$. Also for each $f \in H$,
\[
\|Vf\|^2 = \langle Vf, Vf \rangle = \int_{\Omega} \langle \Theta_\nu^* \Lambda_\omega f, Vf \rangle d\mu(\omega) = \int_{\Omega} \langle \Lambda_\omega f, \Theta_\nu Vf \rangle d\mu(\omega)
\]
\[
= \int_{\Omega} \|\Lambda_\omega f\|^2 d\mu(\omega).
\]
Therefore
\[ A(f, f) \leq \langle V^*V f, f \rangle \leq B(f, f), \]
which implies \( AI \leq VV^* \leq BI \).
The converse side is similar to Proposition 2.5. \(
\square
\)

**Theorem 2.7.** Suppose that \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) and \( V \in B(H) \). Then \( \{ \Lambda_\omega V \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) if and only if \( V \) is unitary.

**Proof.** Assume that \( \{ \Lambda_\omega V \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \). Since \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is also a cg-orthonormal basis for \( H \), for each \( f \in H \), we have
\[
\|Vf\|^2 = \int_{\Omega} \|\Lambda_\omega Vf\|^2 d\mu(\omega) = \|f\|^2.
\]
Hence \( V \) is an isometry and \( V^*V = I \). Considering \( \Theta_\omega = \Lambda_\omega V, \omega \in \Omega \), in Theorem 2.5, there exists a unique isometry \( U \in B(H) \) such that \( \Lambda_\omega = \Lambda_\omega VU \), a.e. \( [\mu] \). Let \( T_\Lambda \) and \( T_{AVU} \) be the synthesis operators of Parseval cg-frames \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) and \( \{ \Lambda_\omega VU \}_{\omega \in \Omega} \), respectively. Then \( T_\Lambda = (VU)^*T_\Lambda = U^*V^*T_\Lambda \). We deduce \( T_\Lambda T_\Lambda^* = U^*V^*T_\Lambda T_\Lambda \) or \( S_\Lambda = U^*V^*S_\Lambda \), where \( S_\Lambda \) is the frame operator of \( \{ \Lambda_\omega \}_{\omega \in \Omega} \). Since \( S_\Lambda = I \), so \( I = U^*V^* \) or equivalently \( VU = I \). This implies that \( V \) is onto. Also \( V \) is one-to-one, so \( V \) is invertible and \( V^{-1} = V^*, \) which means \( V \) is a unitary.

Conversely, suppose that \( V \) is a unitary operator. Now, we show that \( \{ \Lambda_\omega V \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \). For almost all \( \nu \in \Omega \), each \( g_\nu \in H_\nu \) and each \( \{ f_\nu \}_{\omega \in \Omega} \in \left( \Theta_{\omega \in \Omega} H_\omega, \mu \right)_{L^2} \), we have
\[
\int_{\Omega} \langle (\Lambda_\omega V)^*f_\omega, (\Lambda_\omega V)^*g_\nu \rangle d\mu(\omega) = \int_{\Omega} \langle V^*\Lambda_\omega^*f_\omega, V^*\Lambda_\omega^*g_\nu \rangle d\mu(\omega)
\]
\[
= \int_{\Omega} \langle \Lambda_\omega^*f_\omega, \Lambda_\omega^*g_\nu \rangle d\mu(\omega) = \langle f_\nu, g_\nu \rangle.
\]
Also for each \( f \in H \),
\[
\int_{\Omega} \|\Lambda_\omega Vf\|^2 d\mu(\omega) = \|Vf\|^2 = \|f\|^2.
\]
Therefore \( \{ \Lambda_\omega V \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \). \(
\square
\)

Concerning to cg-Riesz bases which are defined in [14], we have next result.

**Theorem 2.8.** Let \( (\Omega, \mu) \) be a measure space where \( \mu \) is \( \sigma \)-finite. Suppose that \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \) and \( V \in B(H) \). Then \( \{ \Lambda_\omega V \}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \) if and only if \( V \) is invertible.

**Proof.** Let \( \{ \Lambda_\omega V \}_{\omega \in \Omega} \) be a cg-Riesz basis for \( H \). By definition of a cg-Riesz basis, the operator \( T_{AV} \) weakly defined by
\[
\langle T_{AV} \varphi, h \rangle = \int_{\Omega} \langle (\Lambda_\omega V)^* \varphi(\omega), h \rangle d\mu(\omega), \quad \varphi \in \left( \Theta_{\omega \in \Omega} H_\omega, \mu \right)_{L^2}, h \in H,
\]
is well-defined and there exist positive constants $A$ and $B$ such that
\[ A\|\varphi\| \leq \|T_{AV}\varphi\| \leq B\|\varphi\|, \quad \varphi \in (\oplus_{\omega \in \Omega} H_{\omega}, \mu)_{L^2}. \]

An easy calculation shows $T_{AV} = V^{*}T_{\Lambda}$, where $T_{\Lambda}$ is defined similarly for $\{\Lambda_{\omega}\}_{\omega \in \Omega}$. By Lemma 3.2 (i) in [14], $T_{AV}$ and $T_{\Lambda}$ both are invertible. So $V^{*} = T_{AV}T_{\Lambda}^{-1}$ is invertible and $V$ is invertible.

Conversely, let $V \in B(H)$ be invertible. If $\Lambda_{\omega}Vf = 0$, a.e. $[\mu]$, then $Vf = 0$ and it implies $f = 0$. The operator $T_{AV}$ defined by
\[ \langle T_{AV}\varphi, h \rangle = \int_{\Omega} \langle (\Lambda_{\omega} V)^{*} \varphi(\omega), h \rangle d\mu(\omega), \quad \varphi \in (\oplus_{\omega \in \Omega} H_{\omega}, \mu)_{L^2}, \quad h \in H, \]
is well-defined, bounded and $T_{AV} = V^{*}T_{\Lambda}$, where $T_{\Lambda}$ is defined similar to $T_{AV}$. Also, there are positive constants $A$ and $B$ such that
\[ A\|\varphi\| \leq \|T_{AV}\varphi\| \leq B\|\varphi\|, \quad \varphi \in (\oplus_{\omega \in \Omega} H_{\omega}, \mu)_{L^2}. \]

For each $\varphi \in (\oplus_{\omega \in \Omega} H_{\omega}, \mu)_{L^2}$, $\|T_{AV}\varphi\| = \|V^{*}T_{\Lambda}\varphi\| \leq B\|V^{*}\|\|\varphi\|$, and
\[ \|T_{AV}\varphi\| = \|V^{*}T_{\Lambda}\varphi\| \geq \frac{1}{\|V^{-1}\|}\|T_{\Lambda}\varphi\| \geq \frac{A}{\|V^{-1}\|}\|\varphi\|. \]
This shows that $\{\Lambda_{\omega} V\}_{\omega \in \Omega}$ is a cg-Riesz basis for $H$. \qed

Now, we define a cg-complete family as follows:

**Definition 2.9.** A family $\{\Lambda_{\omega} \in B(H, H_{\omega}) : \omega \in \Omega\}$ is called a cg-complete family for $H$ with respect to $\{H_{\omega}\}_{\omega \in \Omega}$, if:
\[ \{h : \Lambda_{\omega} h = 0, \text{ a.e. } [\mu]\} = \{0\}. \]

**Lemma 2.10.** $\{\Lambda_{\omega}\}_{\omega \in \Omega}$ is a cg-complete family for $H$ and $V \in B(H)$. Then $\{\Lambda_{\omega} V\}_{\omega \in \Omega}$ is a cg-complete family for $H$ if and only if $V$ is one-to-one.

**Proof.** Let $\{\Lambda_{\omega} V\}_{\omega \in \Omega}$ be a cg-complete family. If $Vh = 0$, then
\[ \Lambda_{\omega} Vh = 0, \quad \omega \in \Omega, \]
so $h = 0$ and $V$ is one-to-one.

Now, suppose $V$ is one-to-one and $\Lambda_{\omega} Vh = 0$, a.e. $[\mu]$. Since $\{\Lambda_{\omega}\}_{\omega \in \Omega}$ is cg-complete, $Vh = 0$. Hence $h = 0$, which implies $\{\Lambda_{\omega} V\}_{\omega \in \Omega}$ is cg-complete. \qed

**Proposition 2.11.** Let $(\Omega, \mu)$ be a measure space where $\mu$ is $\sigma$-finite and $\{\Lambda_{\omega}\}_{\omega \in \Omega}$ be a cg-Bessel family for $H$. Then $\{\Lambda_{\omega}\}_{\omega \in \Omega}$ is cg-complete if and only if $R(T_{\Lambda}) = H$, where $T_{\Lambda}$ is the synthesis operator of $\{\Lambda_{\omega}\}_{\omega \in \Omega}$.

**Proof.** Assume that $\{\Lambda_{\omega}\}_{\omega \in \Omega}$ is cg-complete. To show that $R(T_{\Lambda}) = H$, it is enough to prove that if $f \in H$ and $f \perp R(T_{\Lambda})$, so $f = 0$. Let $f \in H$ and $f \perp R(T_{\Lambda})$, then for each $F \in (\oplus_{\omega \in \Omega} H_{\omega}, \mu)_{L^2}$,
\[ 0 = \langle T_{\Lambda} F, f \rangle = \int_{\Omega} \langle \Lambda_{\omega}^{*} F(\omega), f \rangle d\mu(\omega). \]
Since \((\Omega, \mu)\) is \(\sigma\)-finite, there exists a family \(\{\Omega_n\}_{n=1}^\infty\) of disjoint measurable subsets of \(\Omega\), such that \(\Omega = \bigcup_{n=1}^\infty \Omega_n\) and \(\mu(\Omega_n) < \infty\), \(n \geq 1\). For each \(n \geq 1\), set
\[
F_n(\omega) = \begin{cases} 
\Lambda_\omega f, & \omega \in \Omega_n \\
0, & \text{otherwise}
\end{cases}
\]
then
\[
\langle T_\Lambda F_n, f \rangle = \int_\Omega \langle F(\omega), \Lambda_\omega f \rangle d\mu(\omega) = \|\Lambda_\omega f\|^2 \mu(\Omega_n) = 0.
\]
Thus \(\Lambda_\omega f = 0\), a.e. \([\mu]\), which implies \(f = 0\). So \(R(T_\Lambda) = H\).

Conversely, suppose \(R(T_\Lambda) = H\) and there exists a \(f \neq 0\) such that \(\Lambda_\omega f = 0\), a.e. \([\mu]\).

There exists a sequence \(\{F_n\}_{n=1}^\infty \subseteq \bigoplus_{\omega \in \Omega} H_\omega, \mu\) such that \(\lim_{n \to \infty} T_\Lambda F_n = f\).

Then
\[
\|f\|^2 = \langle f, f \rangle = \langle \lim_{n \to \infty} T_\Lambda F_n, f \rangle = \lim_{n \to \infty} \langle T_\Lambda F_n, f \rangle
\]
\[
= \lim_{n \to \infty} \int_\Omega \langle \Lambda_\omega^* F_n(\omega), f \rangle d\mu(\omega)
\]
\[
= \lim_{n \to \infty} \int_\Omega \langle F_n(\omega), \Lambda_\omega f \rangle d\mu(\omega) = 0,
\]
which is a contradiction. \(\square\)

**Remark 2.12.** Let \((\Omega, \mu)\) be a measure space and consider the family \(\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}\). Also, suppose that \(\{e_{\omega,k}\}_{\omega \in \Omega, k \in K_\omega}\) is an orthonormal basis for Hilbert space \(\bigoplus_{\omega \in \Omega} H_\omega\) such that for each \(\omega \in \Omega\), \(\{e_{\omega,k}\}_{k \in K_\omega}\) is an orthonormal basis of \(H_\omega\) and for each \(h \in H\), the mapping
\[
\Omega \times K \longrightarrow \mathbb{C}
\]
\[
(\omega, k) \longmapsto \langle h, e_{\omega,k} \rangle
\]
is measurable, where \(K = \bigcup_{\omega \in \Omega} K_\omega\).

The mapping \(h \longmapsto \langle \Lambda_\omega h, e_{\omega,k} \rangle, \omega \in \Omega, k \in K_\omega\), defines a bounded linear functional on \(H\). So there exist some \(u_{\omega,k} \in H\) such that
\[
\langle h, u_{\omega,k} \rangle = \langle \Lambda_\omega h, e_{\omega,k} \rangle, \quad h \in H, \ \omega \in \Omega, \ k \in K_\omega.
\]
Therefore \(\Lambda_\omega h = \sum_{k \in K_\omega} \langle h, u_{\omega,k} \rangle e_{\omega,k}, h \in H\). Since
\[
\sum_{k \in K_\omega} |\langle h, u_{\omega,k} \rangle e_{\omega,k}|^2 = \|\Lambda_\omega h\|^2 \leq \|\Lambda_\omega\|^2 \|h\|^2,
\]
so for each \(\omega \in \Omega\), \(\{u_{\omega,k}\}_{k \in K_\omega}\) is a \(c\)-Bessel family for \(H_\omega\). Also
\[
u_{\omega,k} = \Lambda_\omega^* e_{\omega,k}, \quad \omega \in \Omega, k \in K_\omega.
\]
(2.1)
The family \(\{u_{\omega,k}\}_{\omega \in \Omega, k \in K_\omega}\) is called the family induced by \(\{\Lambda_\omega\}_{\omega \in \Omega}\) with respect to \(\{e_{\omega,k}\}_{\omega \in \Omega, k \in K_\omega}\).
Consider the mapping \( u(\omega, k) : \Omega \times \mathbb{K} \rightarrow H \) defined by

\[
u(\omega, k) = \begin{cases} u_{\omega, k}, & k \in \mathbb{K}_\omega, \\ 0, & \text{otherwise} \end{cases}
\]

where \( \mathbb{K} = \bigcup_{\omega \in \Omega} \mathbb{K}_\omega \).

For each \( h \in H \), \((\omega, k) \mapsto \langle u_{\omega, k}, h \rangle \) is measurable and

\[
\int_{\Omega} \| \Lambda_\omega h \|^2 d\mu(\omega) = \int_{\Omega} \sum_{k \in \mathbb{K}_\omega} |\langle h, u_{\omega, k} \rangle|^2 d\mu(\omega)
= \int_{\Omega} \left( \int_{\mathbb{K}} |\langle h, u_{\omega, k} \rangle|^2 dl(k) \right) d\mu(\omega), \tag{2.2}
\]

where \( l : \mathbb{K} \rightarrow \mathbb{K} \) is the counting measure on \( \mathbb{K} \). If \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-frame for \( H \) with respect to \( \{H_\omega\}_{\omega \in \Omega} \), then \( u \) is a c-frame for \( H \) with respect to \( (\Omega \times \mathbb{K}, \mu \times l) \) and with the same bounds of \( \{\Lambda_\omega\}_{\omega \in \Omega} \).

The converse of that is true, too; if \( \{u_{\omega, k}\}_{\omega \in \Omega, k \in \mathbb{K}_\omega} \) is a c-frame for \( H \), then \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-frame for \( H \) with the same bounds of \( \{u_{\omega, k}\}_{\omega \in \Omega, k \in \mathbb{K}_\omega} \).

**Theorem 2.13.** Let \((\Omega, \mu)\) be a measure space where \( \mu \) is \( \sigma \)-finite. Consider the family \( \{\Lambda_\omega \in B(H, H_\omega) ; \omega \in \Omega \} \) and let \( \{u_{\omega, k}\}_{\omega \in \Omega, k \in \mathbb{K}_\omega} \) be defined as in (2.1). Then \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-frame (respectively, cg-Bessel family, tight cg-frame, cg-Riesz basis, cg-orthonormal basis) for \( H \) if and only if \( \{u_{\omega, k}\}_{\omega \in \Omega, k \in \mathbb{K}_\omega} \) is a c-frame (respectively, c-Bessel family, tight c-frame, c-Riesz basis, c-orthonormal basis) for \( H \).

**Proof.** We see from (2.2) that

\[
\int_{\Omega} \| \Lambda_\omega h \|^2 d\mu(\omega) = \int_{\Omega} \left( \int_{\mathbb{K}} |\langle h, u_{\omega, k} \rangle|^2 dl(k) \right) d\mu(\omega), \quad h \in H.
\]

Hence \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-frame (respectively, cg-Bessel family, tight cg-frame) for \( H \) if and only if \( \{u_{\omega, k}\}_{\omega \in \Omega, k \in \mathbb{K}_\omega} \) is a c-frame (respectively, c-Bessel family, tight c-frame).

Now, assume that \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \). So there are constants \( A, B > 0 \) such that the operator \( T_\Lambda : (\bigoplus_{\omega \in \Omega} H_\omega, \mu)_{L^2} \rightarrow H \) defined by

\[
\langle T_\Lambda F, h \rangle = \int_{\Omega} \langle \Lambda_\omega^* F(\omega), h \rangle d\mu(\omega), \quad F \in \left( \bigoplus_{\omega \in \Omega} H_\omega, \mu \right)_{L^2}, \quad h \in H,
\]

satisfies in

\[
A\|F\| \leq \|T_\Lambda F\| \leq B\|F\|, \quad F \in \left( \bigoplus_{\omega \in \Omega} H_\omega, \mu \right)_{L^2}.
\]

Consider the operator \( \Sigma : L^2(\Omega \times \mathbb{K}) \rightarrow H \) which is defined by

\[
\langle \Sigma \varphi, h \rangle = \int_{\Omega} \int_{\mathbb{K}} \varphi(\omega, k) \langle u_{\omega, k}, h \rangle dl(k) d\mu(\omega)
= \int_{\Omega} \sum_{k \in \mathbb{K}_\omega} \varphi(\omega, k) \langle u_{\omega, k}, h \rangle d\mu(\omega), \quad \varphi \in L^2(\Omega \times \mathbb{K}), \quad h \in H.
\]
To show that \( \{u_{\omega,k}\}_{k \in K} \) is a c-\( \text{Riesz} \) basis for \( H \), it is enough to show that \( \mathcal{F} \) is one-to-one (by Theorem 2.1 in [16]). If \( \mathcal{F}\varphi = 0 \), then for each \( h \in H \),

\[
0 = \langle \mathcal{F}\varphi, h \rangle = \int_{\Omega} \sum_{k \in K_{\omega}} \varphi(\omega, k) \langle \Lambda_{\omega}^{*}e_{\omega,k}, h \rangle d\mu(\omega)
\]

\[
= \int_{\Omega} \sum_{k \in K_{\omega}} \varphi(\omega, k) e_{\omega,k}, \Lambda_{\omega}h d\mu(\omega)
\]

\[
= \int_{\Omega} \langle \Lambda_{\omega}^{*}\psi, h \rangle d\mu(\omega) = \langle T_{\Lambda}^{*}\psi, h \rangle = 0,
\]

where \( \psi(\omega) = \sum_{k \in K_{\omega}} \varphi(\omega, k)e_{\omega,k}, \omega \in \Omega \). So \( T_{\Lambda}^{*}\psi = 0 \). Since \( T_{\Lambda} \) is bounded below, \( \psi = 0 \). But \( \|\psi\| = \|\varphi\| \), so \( \varphi = 0 \). Hence \( \mathcal{F} \) is one-to-one and it implies \( \{u_{\omega,k}\}_{k \in K} \) is a c-Riesz basis for \( H \).

Now, let \( \{u_{\omega,k}\}_{\omega \in \Omega, k \in K} \) be a c-Riesz basis for \( H \). By Theorem 3.3 in [14], we show that \( T_{\Lambda}^{*} \) is one-to-one. If \( T_{\Lambda}^{*}\phi = 0 \), then for each \( h \in H \),

\[
0 = \langle T_{\Lambda}^{*}\phi, h \rangle = \int_{\Omega} \sum_{k \in K_{\omega}} \varphi(\omega, k) e_{\omega,k}, \Lambda_{\omega}^{*}e_{\omega,k}, h d\mu(\omega)
\]

\[
= \int_{\Omega} \sum_{k \in K_{\omega}} \langle \phi(\omega), e_{\omega,k} \rangle \langle \Lambda_{\omega}^{*}e_{\omega,k}, h \rangle d\mu(\omega) = (*),
\]

set \( \varphi(\omega, k) = \langle \phi(\omega), e_{\omega,k} \rangle, \omega \in \Omega, k \in K_{\omega} \), then

\[
\int_{\Omega} \sum_{k \in K_{\omega}} |\langle \phi(\omega), e_{\omega,k} \rangle|^{2} d\mu(\omega) = \int_{\Omega} \|\phi(\omega)\|^{2} d\mu(\omega) = \|\phi\|^{2}.
\]

So \( \varphi \in L^{2}(\Omega, K) \) and \( \|\varphi\| = \|\phi\| \). Also

\[
(*) = \int_{\Omega} \int_{K} \varphi(\omega, k) \langle \Lambda_{\omega}^{*}e_{\omega,k}, h \rangle dk d\mu(\omega).
\]

So for each \( h \in H \), \( 0 = \langle \mathcal{F}\varphi, h \rangle = 0 \), hence \( \mathcal{F}\varphi = 0 \). Since \( \{u_{\omega,k}\}_{\omega \in \Omega, k \in K} \) is a c-Riesz basis for \( H \), so \( \mathcal{F} \) is invertible, which implies \( \varphi = 0 \) and \( \phi = 0 \). Thus \( T_{\Lambda}^{*} \) is one-to-one.

Now, suppose that \( \{\Lambda_{\omega}\}_{\omega \in \Omega} \) is a c-orthonormal basis for \( H \). For almost all \( \nu \in \Omega \) and all \( m \in K \),

\[
\int_{\Omega} \int_{K} \langle u_{\omega,k}, u_{\nu,m} \rangle dk d\mu(\omega) = \int_{\Omega} \int_{K} \langle \Lambda_{\omega}^{*}e_{\omega,k}, \Lambda_{\nu}^{*}e_{\nu,m} \rangle dk d\mu(\omega)
\]

\[
= \int_{K} \int_{\Omega} \langle \Lambda_{\omega}^{*}e_{\omega,k}, \Lambda_{\nu}^{*}e_{\nu,m} \rangle dk d\mu(\omega) = \int_{K} \langle e_{\nu,k}, e_{\nu,m} \rangle dk = 1.
\]

Also, for each \( h \in H \),

\[
\int_{\Omega} \|\Lambda_{\omega}h\|^{2} d\mu(\omega) = \int_{\Omega} \int_{K} |\langle h, u_{\omega,k} \rangle|^{2} dk d\mu(\omega) = \|h\|^{2}.
\]

So \( \{u_{\omega,k}\}_{\omega \in \Omega, k \in K} \) is a c-orthonormal basis. The converse side is similar. \( \square \)
3. \textit{cg-Orthonormal bases and cg-frames}

At first, we present some result on cg-frames which are constructed by composing with operators.

\textbf{Proposition 3.1.} Let \( \{\Lambda_\omega\}_{\omega \in \Omega} \) be a cg-frame for \( H \) with bounds \( A \) and \( B \) and \( V \in B(H) \). Then \( \{\Lambda_\omega V\}_{\omega \in \Omega} \) is a cg-frame for \( H \) if and only if there exists a positive constant \( \alpha \) such that
\[
\|Vf\|^2 \geq \alpha \|f\|^2, \quad f \in H.
\]

\textit{Proof.} Suppose \( \{\Lambda_\omega V\}_{\omega \in \Omega} \) is a cg-frame for \( H \) with bounds \( C \) and \( D \). For each \( f \in H \),
\[
C\|f\|^2 \leq \int_{\Omega} \|\Lambda_\omega Vf\|^2d\mu(\omega) \leq B\|Vf\|^2,
\]
so \( \|Vf\|^2 \geq \frac{C}{B}\|f\|^2 \). Set \( \alpha = \frac{C}{B} \), then the proof is done.

Conversely, let \( \alpha \) be such that
\[
\|Vf\|^2 \geq \alpha \|f\|^2, \quad f \in H.
\]
For each \( f \in H \),
\[
\int_{\Omega} \|\Lambda_\omega Vf\|^2d\mu(\omega) \leq B\|Vf\|^2 \leq B\|V\|^2\|f\|^2,
\]
and
\[
\int_{\Omega} \|\Lambda_\omega Vf\|^2d\mu(\omega) \geq A\|Vf\|^2 \geq A\alpha\|f\|^2.
\]
Hence \( \{\Lambda_\omega V\}_{\omega \in \Omega} \) is a cg-frame for \( H \). \( \square \)

\textbf{Corollary 3.2.} Let \( \{\Lambda_\omega\}_{\omega \in \Omega} \) be a cg-frame for \( H \) and \( V \in B(H) \). Then \( \{\Lambda_\omega V\}_{\omega \in \Omega} \) is a cg-frame for \( H \) if and only if \( V^* \) is onto.

\textit{Proof.} By Lemma 2.4.1 (iii) in [\textit{5}], it is obvious. \( \square \)

\textbf{Corollary 3.3.} Let \( M \) be a close subspace of \( H \) and \( \{\Lambda_\omega\}_{\omega \in \Omega} \) be a cg-frame for \( H \) and \( V \in B(H,M) \). Then \( \{\Lambda_\omega V^*\}_{\omega \in \Omega} \) is a cg-frame for \( M \) if and only if there exists a positive constant \( \alpha \) such that
\[
\|V^*f\|^2 \geq \alpha \|f\|^2, \quad f \in M.
\]

\textbf{Corollary 3.4.} Let \( \{\Lambda_\omega\}_{\omega \in \Omega} \) be a tight cg-frame for \( H \) with frame bound \( A \) and \( V \in B(H) \). Then \( \{\Lambda_\omega V\}_{\omega \in \Omega} \) is a tight cg-frame for \( H \) with frame bound \( \alpha \) if and only if
\[
\|Vf\|^2 = \frac{\alpha}{A}\|f\|^2, \quad f \in H.
\]

\textbf{Proposition 3.5.} Suppose \( \{\Theta_\omega\}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \) with respect to \( \{H_\omega\}_{\omega \in \Omega} \) and \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-frame for \( H \) with respect to \( \{H_\omega\}_{\omega \in \Omega} \). Then there exists a bounded and one-to-one operator \( V \) on \( H \) such that \( \Lambda_\omega = \Theta_\omega V \), a.e. [\mu]. Furthermore, \( V \) is invertible if \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \) and \( V \) is unitary if \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \).
Proof. By the proof of Theorem 2.7, the first part is obvious. If \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \), then by definition of \( V \) in the proof of Theorem 2.7, \( V = T_0T_1^\star \). Theorem 3.3 in [14] implies \( T_1^\star \) is onto, So \( V \) is onto and consequently \( V \) is invertible. If \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-orthonormal for \( H \), then Theorem 2.7 implies the result.

**Proposition 3.6.** Suppose that \( \{ \Theta_\omega \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) and \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-frame for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \). Then there exist cg-orthonormal bases \( \{ \Psi_\omega \}_{\omega \in \Omega} \), \( \{ \Gamma_\omega \}_{\omega \in \Omega} \) and \( \{ \Phi_\omega \}_{\omega \in \Omega} \) for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) and a constant \( \alpha \) such that

\[
\Lambda_\omega = \alpha(\Psi_\omega + \Gamma_\omega + \Phi_\omega), \text{ a.e. } [\mu].
\]

Proof. Due to Proposition 3.5 and Proposition 1.8, we have an operator \( V \in B(H) \) so that \( V = \alpha(U_1 + U_2 + U_3) \), where each \( U_j \), \( j = 1, 2, 3 \), is a unitary operator and \( \alpha \) is a constant. Then \( \Lambda_\omega = \Theta_\omega V = \alpha(\Theta_\omega U_1 + \Theta_\omega U_2 + \Theta_\omega U_3) \), a.e. \([\mu]\). It is obvious that every \( \{ \Theta_\omega U_j \}_{\omega \in \Omega} \), \( j = 1, 2, 3 \), is a cg-orthonormal basis for \( H \). Assuming \( \Psi_\omega = \Theta_\omega U_1 \), a.e. \([\mu]\), \( \Gamma_\omega = \Theta_\omega U_2 \), a.e. \([\mu]\) and \( \Phi_\omega = \Theta_\omega U_3 \), a.e. \([\mu]\), the proof is completed.

**Proposition 3.7.** Consider \( \{ \Theta_\omega \}_{\omega \in \Omega} \) as a cg-orthonormal basis for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \). If \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \), then \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is sum of two cg-orthonormal bases for \( H \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \).

Proof. Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a cg-Riesz basis for \( H \). Proposition 3.5 implies that there exists an invertible \( V \in B(H) \) such that \( \Lambda_\omega = \Theta_\omega V \), a.e. \([\mu]\). Via Proposition 1.8 \( V \) can be written as \( V = aU_1 + bU_2 \), where \( U_1 \) and \( U_2 \) are unitary operators. The rest of proof is similar to the proof of Proposition 3.6.

Composing of a cg-orthonormal basis and an isometry, gives us a Parseval cg-frame.

**Proposition 3.8.** If \( V \in B(H) \) is an isometry and \( \{ \Theta_\omega \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \), then \( \{ \Theta_\omega \Lambda \}_{\omega \in \Omega} \) is a Parseval cg-frame for \( H \).

Proof. It is clear.

Every bounded operator \( V \) on \( H \) has a representation in the form \( V = U|V| \) (called the polar decomposition of \( V \)), where \( U \) is a partial isometry, \(|V|\) is a positive operator defined by \(|V| = \sqrt{V^*V}\) and \( \text{ker} U = \text{ker} V \).

Also, every positive operator \( P \) on \( H \) with \( \|P\| \leq 1 \) can be written in the form \( P = \frac{1}{2}(W + W^*) \), where \( W = P + i\sqrt{1 - P^2} \) is unitary.

Next theorem shows that we can state a cg-frame by some Parseval cg-frames.

**Theorem 3.9.** Suppose that \( \{ \Theta_\omega \}_{\omega \in \Omega} \) is a cg-orthonormal basis for \( H \). Every cg-frame for \( H \) can be written as a linear combination of two Parseval cg-frames.

Proof. By Proposition 3.5 there exists a bounded and one-to-one operator \( V \in B(H) \) such that \( \Lambda_\omega = \Theta_\omega V \), a.e. \([\mu]\). By above note, \( V \) can be written as \( V = \)
\( \frac{1}{2}(UW + UW^*) \), where \( U \) is an isometry and \( W \) is unitary. So \( UW \) and \( UW^* \) are isometries. Proposition \( \ref{prop:isometry} \) implies that \( \{ \Theta_{\omega}UW \}_{\omega \in \Omega} \) and \( \{ \Theta_{\omega}UW^* \}_{\omega \in \Omega} \) are Parseval \( cg \)-frames for \( H \). \( \square \)

Now, we can represent each \( cg \)-frame as a combination of a \( cg \)-orthonormal basis and a \( cg \)-Riesz basis of \( H \).

**Theorem 3.10.** Assuming \( \{ \Theta_{\omega} \}_{\omega \in \Omega} \) as a \( cg \)-orthonormal basis for \( H \), Every \( cg \)-frame for \( H \) is sum of a \( cg \)-orthonormal basis for \( H \) and a \( cg \)-Riesz basis for \( H \).

**Proof.** In a manner similar to the proof of Theorem 4.2 in \( \cite{16} \), we can prove this theorem. \( \square \)

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