**BAYESIAN PREDICTION UNDER A CLASS OF MULTIVARIATE DISTRIBUTIONS**

ESSAM K. AL-HUSSAINI¹ AND SAIEED F. ATEYA² ³ ∗

¹Faculty of Science, Alexandria University, Alexandria, Egypt
²Mathematics & Statistics Department, Taif University, Hawia, Taif, Saudi Arabia
³Mathematics Department, Faculty of Science, Assiut University, Egypt

Abstract. In this paper the prediction problem is studied under members of a class $\mathcal{Z}$* of multivariate distributions, constructed by AL-Hussaini and Ateya [7–8]. More attention is paid to bivariate compound Rayleigh (BVCR) distribution, which is a member of this class, as illustrative example.

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1. Introduction

Suppose that a class $\mathcal{Z}$ of distribution functions is of the form

$$\mathcal{Z} = \left\{ F : F \equiv F_{X|\Theta}(x|\theta) = 1 - exp[-\theta \delta \eta(x)], 
\begin{array}{l}
0 \leq a < x < b \leq \infty, 
(\theta, \delta > 0, (\theta, \delta, \eta) \in \Omega)
\end{array} \right\},$$

where $a$ and $b$ are non-negative real numbers such that $a$ may assume the value zero and $b$ the value infinity, $\lambda_\eta(x)$ is a continuous, monotone increasing and differentiable function

*Corresponding author

E-mail addresses: ekalh2011@yahoo.com (E. K. AL-Hussaini), said_f_atya@yahoo.com (S. F. Ateya)

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of $x$ such that $\lambda_\eta(x) \to 0$ as $x \to a^+$, $\lambda_\eta(x) \to \infty$ as $x \to b^-$ and $\eta$ is a parameter (could be a vector), $(\theta, \delta, \eta)$ belongs to a parameter space $\Omega$. This class covers some important distributions such as the Weibull, exponential, Rayleigh, compound Weibull, compound exponential (Lomax), compound Rayleigh, Pareto, power function, beta, Gompertz and compound Gompertz distributions, among others. The failure rate and survival functions corresponding to $F \in \mathcal{S}$ are, respectively, $\delta \theta \lambda_\eta'(x)$ and $e^{-\theta \delta \lambda_\eta(x)}$, so that the probability density function (pdf) is given, for $0 \leq a < x < b \leq \infty$, by

$$
(2) \quad f_{X|\theta}(x|\theta) = \delta \theta \lambda_\eta'(x) \exp[-\theta \delta \lambda_\eta(x)], \quad f \equiv \frac{d}{dx}.
$$

The class $\mathcal{S}$ was used by AL-Hussaini and Osman [9], AL-Hussaini [4], Ahmad [1–2], Ahmad and Fawzy [3], AL-Hussaini and Ahmad [5–6] and Jafar et al [12].

1.1. A Class of multivariate distributions

AL-Hussaini and Ateya [7–8] constructed a class of multivariate distributions by compounding members of the class $\mathcal{S}$ with the gamma distribution. The resulting multivariate distributions form a class $\mathcal{S}^*$, given by

$$
\mathcal{S}^* = \left\{ F^* : F^* \equiv F_X(x) = \int f_X(u)du \right\},
$$

where $\int \equiv \int_0^{x_1} \ldots \int_0^{x_k} u = (u_1, \ldots, u_k), du = du_k \ldots du_1$ and $f_X(x)$ is the pdf of the random vector $X = (X_1, \ldots, X_k)$, given by

$$
(3) \quad f_X(x) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \left[ \prod_{i=1}^k c_i \lambda_\eta'(x_i) \right] \left[ 1 + \sum_{i=1}^k c_i \lambda_\eta(x_i) \right]^{-(\alpha+k)},
$$

$$
c_i = \delta_i / \beta, 0 \leq a < x_i < b \leq \infty, i = 1, 2, \ldots, k.
$$
It was assumed that $\Theta$ is a positive random variable following the $\text{gamma}(\alpha, \beta)$ distribution with pdf $g_{\Theta}(\theta)$ given by
\begin{equation}
g_{\Theta}(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta}, \theta > 0, (\alpha > 0, \beta > 0).
\end{equation}

The pdf $f_X(x)$ in (1.3) was obtained by writing
\begin{equation}
f_X(x) = \int_0^\infty \left[ \prod_{i=1}^k f_{X_i|\Theta}(x_i|\theta) \right] g_{\Theta}(\theta) d\theta.
\end{equation}

Maximum likelihood and Bayes estimation of the parameters of members of the class $\mathcal{I}^*$ were obtained by AL-Hussaini and Ateya [7–8] and particularly when the underlying population distribution is bivariate compound Weibull or bivariate compound Gompertz.

In this paper, the prediction problem is studied under members of class $\mathcal{I}^*$. More attention is paid to bivariate compound Rayleigh ($\text{BVCR}$) distribution as illustrative example.

1.2. Generation of a multivariate random sample of size $n$ from the class $\mathcal{I}^*$

Knowing that $F_{X_i|\Theta}(x_i|\theta) = 1 - \exp[-\theta \delta_i \lambda_{\eta_i}(x_i)]$ and $g_{\Theta}(\theta) = \beta^\alpha \theta^{\alpha-1} e^{-\beta \theta}/\Gamma(\alpha)$, an observation $x_{ij}$ is obtained by first generating $\theta_j$ from $\text{Gamma}(\alpha, \beta)$, $u_i$ from $\text{Uniform}(0,1)$ and then setting $x_{ij} = \lambda_{\eta_i}^{-1} \left( - (\ln u_i)/\theta_j \delta_i \right)$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, k$. This is repeated until we obtain the required multivariate random sample.

1.3. One-sample prediction

Suppose that $X_1 < X_2 < \ldots < X_r$ is the informative sample, representing the first $r$ ordered lifetimes of a random sample of size $n$ drawn from a population with probability density function (pdf) $f_X(x)$, cumulative distribution function (cdf) $F_X(x)$ and reliability function (rf) $R(x)$. In one-sample scheme the Bayesian prediction intervals (BPI) for the remaining unobserved future $(n-r)$ lifetimes are sought based on the first $r$ observed ordered lifetimes.
For the remaining \((n - r)\) components, let \(Y_s = X_{r+s}\) denote the future lifetime of the \(s^{th}\) component to fail, \(1 \leq s \leq (n - r)\). The conditional density function of \(Y_s\) given that the \(r\) components had already failed is

\[
g_1(y_s|\theta) \propto [R(x_r) - R(y_s)]^{(s-1)}[R(y_s)]^{n-r-s}[R(x_r)]^{-(n-r)}f_X(y_s|\theta), y_s > x_r,
\]

\(\theta\) is the vector of parameters.

The predictive density function is given by

\[
g_1^*(y_s|x) = \int_\Theta g_1(y_s|\theta)\pi^*(\theta|x)d\theta, y_s > x_r,
\]

where \(\pi^*(\theta|x)\) is the posterior density function of \(\theta\) given \(x\) and \(x = (x_1, ..., x_r)\).

A \((1 - \tau)\)% BPI for \(y_s\) is an interval \((L, U)\) such that

\[
P(Y_s > L|x) = \int_L^\infty g_1^*(y_s|x)dy_s = 1 - \frac{\tau}{2}, L > x_r,
\]

\[
P(Y_s > U|x) = \int_U^\infty g_1^*(y_s|x)dy_s = \frac{\tau}{2}, U > x_r.
\]

By solving equations (7) and (8), we get the interval \((L, U)\).

1.4. Two-sample prediction

Let \(X_1 < X_2 < ... < X_r\) and \(Z_1 < Z_2 < ... < Z_m\) represent informative (type II censored) sample from a random sample of size \(n\) and a future ordered sample of size \(m\), respectively. It is assumed that the two samples are independent and drawn from a population with \((pdf)f_X(x)\), \((cdf)F_X(x)\) and \((rf)R(x)\).

Our aim is to obtain the BPI for \(Z_s, s = 1, 2, ..., m\). The conditional density function of \(Z_s\) given the vector of parameters \(\theta\), is

\[
g_2(z_s|\theta) \propto [1 - R(z_s)]^{(s-1)}[R(z_s)]^{m-s}f_X(z_s|\theta), z_k > 0,
\]

\(\theta\) is the vector of parameters.

The predictive density function is given by

\[
g_2^*(z_s|x) = \int_\Theta g_2(z_s|\theta)\pi^*(\theta|x)d\theta, z_s > 0,
\]
\( \pi^*(\theta|x) \) is the posterior density function of \( \theta \) given \( x \) and \( x = (x_1, ..., x_r) \).

A \((1 - \tau) \% \) BPI for \( z_s \) is an interval \((L, U)\) such that

\[
P(Z_s > L|x) = \int_{L}^{\infty} g_2^*(z_s|x)dz_s = 1 - \frac{\tau}{2},
\]

\[
P(Z_s > U|x) = \int_{U}^{\infty} g_2^*(z_s|x)dz_s = \frac{\tau}{2}.
\]

By solving equations (11) and (12), we get the interval \((L, U)\).

2. Bayesian prediction intervals for future bivariate observations

The main goal in this section is to study the one-sample and two-sample prediction problems in case of bivariate informative observations.

While ordering a set of univariate random variables is a clear and straight-forward matter as it can be done by simply ordering the set of random variables, such ordering is not as clear if we are dealing with a set of random vectors.

Barnett [10] classified the principles used for ordering multivariate data into four principles: marginal, reduced (aggregate), partial and conditional (sequential) ordering. An interesting detailed discussion of such principles with illustrative examples are given in Barnett’s paper.

In our paper, we wish to predict bivariate random vectors. The first components of the predicted random vectors are based on the ordered first components of the informative sample, as is done in the univariate case. To predict the second components, we compute the norms of each vector of the informative sample, order the norms and then predict the future norms as is done in the univariate case. The relation between the components of vectors and norms enables us to obtain the second components of the predicted vectors. In other words, we obtain the second component of a predicted vector from the knowledge of the values of the first component and the norm of the vector.

2.1. One-sample prediction
Let \((X_1, Y_1), \ldots, (X_r, Y_r)\) be the first \(r\) bivariate informative observations from a random sample of size \(n\) of bivariate observations. Suppose that the first components of such informative vectors are ordered, that is \(X_1 < X_2 < \ldots < X_r\) and that their norms are given by \(Z_1, Z_2, \ldots, Z_r\).

To obtain \(BPI\)'s for the remaining future vectors, denoted by \((X_1^*, Y_1^*), \ldots, (X_{n-r}^*, Y_{n-r}^*)\), where \(X_1^* < X_2^* < \ldots < X_{n-r}^*\) and norms \(Z_1^* < Z_2^* < \ldots < Z_{n-r}^*\) we apply the following steps:

1. based on ordered \(Z_1, Z_2, \ldots, Z_r\), denoted by \(Z_{1:r}, Z_{2:r}, \ldots, Z_{r:r}\) compute the \(BPI\)'s for \(Z_s^*, s = 1, 2, \ldots, (n-r)\), say \((L_{1s}, U_{1s})\),
2. based on \(X_1 < X_2 < \ldots < X_r\) compute the \(BPI\)'s for \(X_s^*, s = 1, 2, \ldots, (n-r)\), say \((L_{2s}, U_{2s})\),
3. from (1) and (2), compute the \(BPI\)'s for \(Y_s^*, s = 1, 2, \ldots, (n-r)\) which are \(((L_{1s}^2 - L_{2s}^2)^{1/2}, [U_{1s}^2 - U_{2s}^2]^2)^{1/2})\). This is true, since \(z_s^* = (x_s^* + y_s^*)^{1/2}\),
4. from (2) and (3), the \(BPI\)'s for \((X_s^*, Y_s^*), s = 1, 2, \ldots, (n-r)\) is \((L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2}, U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})\).

### 2.2. Two-sample prediction

In this case the first \(r\) bivariate informative observations \((X_1, Y_1), \ldots, (X_r, Y_r)\) from a random sample of size \(n\) is such that \(X_1 < X_2 < \ldots < X_r\) with norms \(Z_1, Z_2, \ldots, Z_r\). An independent future sample of size \(m\) is \((X_1^*, Y_1^*), \ldots, (X_m^*, Y_m^*)\), where \(X_1^* < X_2^* < \ldots < X_m^*\) and norms \(Z_1^* < Z_2^* < \ldots < Z_m^*\). To obtain the \(BPI\)'s of the future sample, we apply the following steps:

1. based on ordered \(Z_1, Z_2, \ldots, Z_r\), denoted by \(Z_{1:r}, Z_{2:r}, \ldots, Z_{r:r}\) compute the \(BPI\)'s for \(Z_s^*, s = 1, 2, \ldots, m\), say \((L_{1s}, U_{1s})\),
2. based on \(X_1 < X_2 < \ldots < X_r\) compute the \(BPI\)'s for \(X_s^*, s = 1, 2, \ldots, m\), say \((L_{2s}, U_{2s})\),
3. from (1) and (2), compute the \(BPI\)'s for \(Y_s^*, s = 1, 2, \ldots, m\) which are \(((L_{1s}^2 - L_{2s}^2)^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})\).
(4) from (2) and (3), the BPI’s for $\left(X_{s}^{*},Y_{s}^{*}\right)$, $s = 1,2,\ldots,m$

is $\left(L_{2s},\left[L_{1s}^{2} - L_{2s}^{2}\right]^{1/2}\right),\left(U_{2s},\left[U_{1s}^{2} - U_{2s}^{2}\right]^{1/2}\right)$.

3. One-sample prediction in case of (BVCR) distribution

If, in (3), $k = 2, \lambda_{\eta}(x) = x^{2}, \lambda_{\eta}(y) = y^{2}, \delta_{1} = \delta_{2} = 1$ so that $c_{1} = c_{2} = 1/\beta = c$, then $(X,Y)$ has a bivariate compound Rayleigh (BVCR) pdf, given by

\[ f_{X,Y}(x,y) = 4\alpha (\alpha + 1) c^{2} xy \left[1 + c(x^{2} + y^{2})\right]^{-\left(\alpha + 2\right)}, x > 0, y > 0. \]

The marginal pdf’s of the random variables $X$ and $Y$ are given, respectively, by

\[ f_{X}(x) = 2\alpha c x \left[1 + c x^{2}\right]^{-\left(\alpha + 1\right)}, x > 0, \]
\[ f_{Y}(y) = 2\alpha c y \left[1 + c y^{2}\right]^{-\left(\alpha + 1\right)}, y > 0. \]

In this section we apply the steps given in Subsection 2.1.

Step 1

The norm $Z$ of the vector $(X,Y)$ is given by $Z = (X^{2} + Y^{2})^{1/2}$. In APPENDIX A the pdf and hence cdf and rf are derived. Such functions are given by

\[ f_{Z}(z) = 2\alpha (\alpha + 1) c^{2} z^{3} \left[1 + cz^{2}\right]^{-\left(\alpha + 2\right)}, z > 0, \]
\[ F_{Z}(z) = 1 - \alpha c z^{2} \left[1 + cz^{2}\right]^{-\left(\alpha + 1\right)} - \left[1 + cz^{2}\right]^{-\alpha}, z > 0, \]
\[ R(z) = \alpha c z^{2} \left[1 + cz^{2}\right]^{-\left(\alpha + 1\right)} + \left[1 + cz^{2}\right]^{-\alpha}, z > 0. \]

From (16) and (18), the conditional density of $Z_{s}^{*}$ given $(c,\alpha)$ is obtained (see APPENDIX B), as

\[ g_{1}(z_{s}^{*} | c,\alpha) \propto \sum B_{i,j,l,s} c^{k_{3}} \alpha^{k_{4}} (\alpha + 1) z_{s}^{* 2(2k_{1} - j) + 3} \left(1 + c z_{s}^{* 2}\right)^{-\alpha k_{1} - k_{1} + j - \alpha - 2} z_{r,r}^{2(k_{2} - l)} \left(1 + c z_{r,r}^{2}\right)^{-\alpha k_{2} - k_{2} + l}, \]
where

\[ \sum_{i=0}^{s-1} \sum_{j=0}^{k_1} \sum_{l=0}^{k_2} B_{i,j,l,s} = (-1)^i \binom{s-1}{i} \binom{k_1}{j} \binom{k_2}{l}, \]

\[ k_1 = n - r + i - s, k_2 = s - i - (n - r) - 1, k_3 = 1 - j - l, k_4 = -j - l. \]

Suppose that the prior belief of the experimenter is given by the pdf

\[ \pi(c, \alpha) = \pi_1(c|\alpha) \pi_2(\alpha), c|\alpha \sim \text{Gamma}(c_1, \alpha) \text{ and } \alpha \sim \text{Gamma}(c_2, c_3). \]

So that

\[ \pi(c, \alpha) \propto \alpha^{c_1+c_2-1} c^{c_1-1} e^{-\alpha(c+c_3)}. \]

The likelihood function of \((c, \alpha)\) given \(Z_1: r, ..., Z_r: r\) is given by

\[ L(c, \alpha|Z_1: r, ..., Z_r: r) \propto [R(Z_r: r)]^{n-r} \prod_{i=1}^{n-r} f(z_i) \]

\[ = 2^r \alpha^r c^{2r}(\alpha + 1)^r \left( \prod_{i=1}^{r} z_i \right)^{3} \left( \prod_{i=1}^{r} (1 + cz_i^2) \right)^{-(\alpha+2)} \sum_{l_1}^{n-r-l_1} \binom{n-r}{l_1} \alpha^{n-r-l_1} c^{n-r-l_1} \]

\[ z_{r: r}^{2(n-r-l_1)} (1 + cz_{r: r}^2)^{-\alpha(n-r)-(n-r)+l_1}. \]

Since the posterior density \(\pi^*(c, \alpha|Z_1: r, ..., Z_r: r) \propto \pi(c, \alpha) L(c, \alpha|Z_1: r, ..., Z_r: r)\), it follows, from (19) – (21) that

\[ g_1(z_s^*| c, \alpha)\pi^*(c, \alpha| z_1: r, ..., z_r: r) = A \sum_{i,j,l,s} B_{i,j,l,s} c^{n+r+c_1-j-l-l_1} \]

\[ \alpha^{n+c_1+c_2-j-l-l_1-1}(\alpha + 1)^{r+1} \left( \prod_{i=1}^{r} z_i \right)^{3} \left( \prod_{i=1}^{r} (1 + cz_i^2) \right)^{-(\alpha+2)} \]

\[ z_{r: r}^{2(k_1-j+3)} (1 + cz_{r: r}^2)^{-\alpha k_1+1+j-\alpha-2} z_{r: r}^{2(s-i-l_1-l_1-1)} (1 + cz_{r: r}^2)^{-\alpha s-i-l_1+l_1+1+l_1} \]

\[ \exp[-\alpha c - \alpha c_3], \]

where \(A\) is a normalizing constant and

\[ \sum_{l_1=0}^{n-r} B_{i,j,l,s} \binom{n-r}{l_1}. \]

It then follows, from (6) and (22) that the predictive density function of \(Z_s^*\) is given by

\[ g_1(z_s^*| z_1: r, ..., z_r: r) = \int_0^\infty \int_0^\infty g_1(z_s^*| c, \alpha)\pi^*(c, \alpha| z_1: r, ..., z_r: r) dc d\alpha. \]
To obtain \((1 - \tau)\%\) BPI for \(Z_s^*\), say \((L_{1s}, U_{1s})\), we solve the following two nonlinear equations, numerically,

\begin{align}
(24) \quad P(Z_s^* > L_{1s} \mid z_{1:r}, \ldots, z_{r:r}) &= \int_{L_{1s}}^{\infty} g_1^*(z_s^* \mid z_{1:r}, \ldots, z_{r:r}) dz_s^* = 1 - \frac{\tau}{2}, L_{1s} > z_{r:r}, \\
(25) \quad P(Z_s^* > U_{1s} \mid z_{1:r}, \ldots, z_{r:r}) &= \int_{U_{1s}}^{\infty} g_1^*(z_s^* \mid z_{1:r}, \ldots, z_{r:r}) dz_s^* = \frac{\tau}{2}, U_{1s} > z_{r:r}.
\end{align}

**Step 2**

By using the pdf (14) and its cdf, the predictive density function of \(X_s^*\) can be written as follows

\begin{align}
(26) \quad g_1^*(x_s^* \mid x_1, \ldots, x_r) &= \int_0^\infty \int_0^\infty g_1(x_s^* \mid c, \alpha) \pi^*(c, \alpha \mid x_1, \ldots, x_r) dc d\alpha,
\end{align}

where

\begin{align}
g_1(x_s^* \mid c, \alpha) \pi^*(c, \alpha \mid x_1, \ldots, x_r) &= A_1 \sum_{i=0}^{s-1} B_{i,s} c^{c_1+r} \alpha^{c_1+c_2+r} \left( \prod_{i_1} x_{i_1} \right) \\
(27) \quad \left( \prod_{i_1} (1 + c x_{i_1}^{x_s^*}) \right)^{-\alpha c_3} x_s^* (1 + c x_s^{2s})^{-\alpha (n-r+i-s+1)-1} (1 + c x_r^{2s})^{-\alpha (s-i-1)} \\
&\exp[-\alpha c - \alpha c_3],
\end{align}

where \(A_1\) is a normalizing constant and \(B_{i,s} = (-1)^i \binom{s-1}{i}\).

To obtain \((1 - \tau)\%\) BPI for \(X_s^*\), say \((L_{2s}, U_{2s})\), we solve the following two nonlinear equations, numerically,

\begin{align}
(28) \quad P(X_s^* > L_{2s} \mid x_1, \ldots, x_r) &= \int_{L_{2s}}^{\infty} g_1^*(x_s^* \mid x_1, \ldots, x_r) dx_s^* = 1 - \frac{\tau}{2}, L_{2s} > x_r, \\
(29) \quad P(X_s^* > U_{2s} \mid x_1, \ldots, x_r) &= \int_{U_{2s}}^{\infty} g_1^*(x_s^* \mid x_1, \ldots, x_r) dx_s^* = \frac{\tau}{2}, U_{2s} > x_r.
\end{align}

**Step 3**

From steps 2 and 3, a \((1 - \tau)\%\) BPI for \(Y_s^*\) is \([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2}\).

4. Two-sample prediction in case of (BVCR) distribution
In this case we apply the steps in Subsection 2.2 as follows

Step 1

Substituting from (16) and (18) in (9) and then using (20) and (21) we can write

\[ g_2(z^*_s \mid c, \alpha)\pi^*(c, \alpha \mid z_{1:r}, \ldots, z_{r:r}) = A \sum_{i,j,s,m} B_{i,j,s,m}^* e^{n+r+c_1-l_1+k-j+1} \]

\[ \alpha^{n+c_1+c_2+k-j-l_1} (\alpha + 1)^{r+1} \left( \prod_{i_1}^r z_{i_1} \right)^3 \left( \prod_{i_1}^r (1 + cz_{i_1}^2) \right)^{-\alpha - 2} \]

\[ (1 + c z_s^2)^{-\alpha k + k + \alpha - 2} x_{r:r}^{2(n-r-l_1)} (1 + c z_{r:r}^2)^{-\alpha (n-r)-(n-r)+l_1} \]

exp\[\left[-\alpha c - \alpha c_3, \right], \]

where

\[ \sum_{i=0}^{s-1} \sum_{j=0}^k \sum_{l_1=0}^{n-r} B_{i,j,s,m}^* = (-1)^i \binom{s-1}{i} \binom{k}{j} \binom{n-r}{l_1}, k = m - s + i, \]

and \( A \) is a normalizing constant.

It then follows that the predictive density function of \( Z^*_s \) is given by

\[ g_2^*(z^*_s \mid z_{1:r}, \ldots, z_{r:r}) = \int_0^\infty \int_0^\infty g_1(z^*_s \mid c, \alpha)\pi^*(c, \alpha \mid z_{1:r}, \ldots, z_{r:r}) dc d\alpha. \]

To obtain \((1 - \tau) \% BPI\) for \( Z^*_s \), say \((L_{1s}, U_{1s})\), we solve the following two nonlinear equations, numerically,

\[ P(Z^*_s > L_{1s} \mid z_{1:r}, \ldots, z_{r:r}) = \int_{L_{1s}}^\infty g_2^*(z^*_s \mid z_{1:r}, \ldots, z_{r:r}) dz^*_s = 1 - \frac{\tau}{2}, L_{1s} > 0, \]

\[ P(Z^*_s > U_{1s} \mid z_{1:r}, \ldots, z_{r:r}) = \int_{U_{1s}}^\infty g_2^*(z^*_s \mid z_{1:r}, \ldots, z_{r:r}) dz^*_s = \frac{\tau}{2}, U_{1s} > 0. \]

Step 2
Using the pdf (14), its cdf and the same prior as in (20) the predictive density function of 
$X^*_s$ is given by

$$g^*_2(x_s^*|x_1,...,x_r) = \int_0^\infty \int_0^\infty g_2(x_s^*|c,\alpha)\pi^*(c,\alpha|x_1,...,x_r)dc\,d\alpha,$$

where

$$g_2(x_s^*|c,\alpha)\pi^*(c,\alpha|x_1,...,x_r) = A_1 \sum_{i=0}^{s-1} B_{i,s} c^{r+c_1} \alpha^{c_1+c_2+r} \left( \prod_{i=1}^r x_i \right)$$

$$\left( \prod_{i=1}^r (1 + cx_i^2) \right)^{-(\alpha+1)} x_s^*(1 + cx_s^2)^{-(\alpha(m+i-s+1)-1)} (1 + cx_r^2)^{-\alpha(n-r)} \exp[-\alpha c - \alpha c_3],$$

where $A_1$ is a normalizing constant and

$$B_{i,s} = (-1)^i \binom{s-1}{i}.$$

To obtain $(1 - \tau)\%\ BPI$ for $X^*_s$, say $(L_{2s},U_{2s})$, we solve the following two nonlinear equations, numerically,

$$P(X^*_s > L_{2s}|x_1,...,x_r) = \int_{L_{2s}}^\infty g^*_2(x_s^*|x_1,...,x_r)dx_s^* = 1 - \frac{\tau}{2}, L_{2s} > 0,$$

$$P(X^*_s > U_{2s}|x_1,...,x_r) = \int_{U_{2s}}^\infty g^*_2(x_s^*|x_1,...,x_r)dx_s^* = \frac{\tau}{2}, U_{2s} > 0.$$

**Step 3**

From steps 2 and 3, a $(1 - \tau)\%\ BPI$ for $Y^*_s$ is $(\left[ L_{1s}^2 - L_{2s}^2 \right]^{1/2}, \left[ U_{1s}^2 - U_{2s}^2 \right]^{1/2})$.

**5. Numerical example**

In this section we follow the steps

1. given the set of prior parameters, generate the parameters $(c,\alpha)$,
2. using the generated population parameters, generate a bivariate random sample of size $n$, say $(X_1,Y_1),..., (X_n,Y_n)$ as shown in subsection 1.2
(3) follow steps in Subsections 2.1 and 2.2.

In Tables (1) and (2) 95\% BPI’s are computed in case of the one- and two-sample predictions, respectively, with the same parameters \( c, \alpha \), hyperparameters \( c_1, c_2, c_3 \) and using informative samples of different sizes, \( r \).

**Table (1): One-Sample prediction:** 95 \% BPI’s for \( Z_s^*, Y_s^* \) and \( X_s^*, s = 1, 2, 3 \).

| \( r \) | \( c_1 = 1.0, c_2 = 1.5, c_3 = 2.0 \) | \( x_1^* \) | \( x_2^* \) | \( x_3^* \) |
|---|---|---|---|---|
| Coverage Percentage | 97.43 | 98.45 | 98.97 |
| BPI | (3.9064,5.6565) | (4.4398,6.6373) | (4.8958,7.8809) |
| BPI Length | 1.7501 | 2.1975 | 2.9824 |
| Coverage Percentage | 96.33 | 97.42 | 97.99 |
| BPI | (3.8761,5.4953) | (4.4523,6.4451) | (4.8723,7.1942) |
| BPI Length | 1.6192 | 1.9928 | 2.3219 |
| Coverage Percentage | 95.80 | 96.12 | 96.87 |
| BPI | (3.7670,4.8779) | (4.3687,6.1819) | (4.7585,6.8615) |
| BPI Length | 1.1109 | 1.8132 | 2.1030 |

| \( r \) | \( y_1^* \) | \( y_2^* \) | \( y_3^* \) |
|---|---|---|---|
| Coverage Percentage | 96.11 | 98.41 | 98.84 |
| BPI | (2.4110,3.0393) | (2.7269,3.7051) | (3.1654,4.4564) |
| BPI Length | 0.6283 | 0.9782 | 1.2910 |
| Coverage Percentage | 95.88 | 96.23 | 97.16 |
| BPI | (2.3720,2.9688) | (2.5971,3.4690) | (3.0912,4.1933) |
| BPI Length | 0.5968 | 0.8719 | 1.1021 |
| Coverage Percentage | 95.41 | 95.92 | 96.10 |
| BPI | (2.2891,2.7694) | (2.4870,3.2379) | (2.9714,3.9531) |
| BPI Length | 0.4803 | 0.7509 | 0.9817 |

| \( r \) | \( z^*_1 \) | \( z^*_2 \) | \( z^*_3 \) |
|---|---|---|---|
| Coverage Percentage | 97.40 | 98.04 | 98.67 |
| BPI | (3.0736,4.7706) | (3.5036,5.5069) | (3.7389,6.4999) |
| BPI Length | 1.6970 | 2.0033 | 2.7610 |
| Coverage Percentage | 96.89 | 97.08 | 97.68 |
| BPI | (3.0665,4.6243) | (3.6164,5.4319) | (3.7661,5.8457) |
| BPI Length | 1.5588 | 1.8154 | 2.0796 |
| Coverage Percentage | 95.88 | 96.50 | 97.12 |
| BPI | (2.9917,4.0155) | (3.5917,5.2661) | (3.7167,5.6083) |
| BPI Length | 1.0238 | 1.6744 | 1.8916 |
Table (2): Two-Sample prediction: 95% $BPI'$s for $Z_s^*$, $Y_s^*$ and $X_s^*$, $s = 1, 2, 3$.

| $r$ | $c_1 = 1.0, c_2 = 1.5, c_3 = 2.0$ | $z_1^*$ | $z_2^*$ | $z_3^*$ |
|-----|---------------------------------|---------|---------|---------|
| 10  | Coverage Percentage              | 96.98   | 97.78   | 98.65   |
|     | BPI                             | (1.4319,2.1823) | (2.6267,3.4651) | (3.3804,5.2912) |
|     | BPI Length                      | 0.7501  | 1.2014  | 1.9108  |
| 20  | Coverage Percentage              | 95.79   | 96.45   | 97.03   |
|     | BPI                             | (1.4053,2.0401) | (2.2816,3.1608) | (3.2239,4.5159) |
|     | BPI Length                      | 0.6348  | 0.8792  | 1.2920  |
| 45  | Coverage Percentage              | 94.98   | 95.14   | 96.39   |
|     | BPI                             | (1.7919,1.9721) | (2.2502,3.0318) | (3.1705,4.1634) |
|     | BPI Length                      | 0.1801  | 0.7816  | 0.9925  |

| $r$ | $x_1^*$ | $x_2^*$ | $x_3^*$ |
|-----|---------|---------|---------|
| 10  | Coverage Percentage              | 97.53   | 97.99   | 98.36   |
|     | BPI                             | (0.8941,1.2541) | (1.3730,1.9512) | (2.1106,2.9016) |
|     | BPI Length                      | 0.3601  | 0.5782  | 0.7910  |
| 20  | Coverage Percentage              | 96.55   | 96.98   | 97.13   |
|     | BPI                             | (0.8714,1.2152) | (1.2537,1.6696) | (2.0943,2.7255) |
|     | BPI Length                      | 0.3438  | 0.4159  | 0.6311  |
| 45  | Coverage Percentage              | 95.81   | 96.30   | 97.03   |
|     | BPI                             | (0.8680,0.6083) | (1.2301,1.6013) | (2.0805,2.5665) |
|     | BPI Length                      | 0.2403  | 0.3709  | 0.5861  |

| $r$ | $y_1^*$ | $y_2^*$ | $y_3^*$ |
|-----|---------|---------|---------|
| 10  | Coverage Percentage              | 98.63   | 98.70   | 99.49   |
|     | BPI                             | (1.1184,1.7859) | (1.7985,3.2524) | (2.6405,4.4264) |
|     | BPI Length                      | 0.6676  | 1.4539  | 1.7840  |
| 20  | Coverage Percentage              | 97.97   | 98.13   | 99.01   |
|     | BPI                             | (1.1025,1.6387) | (1.9062,2.6839) | (2.4510,3.6011) |
|     | BPI Length                      | 0.5362  | 0.7776  | 1.1496  |
| 45  | Coverage Percentage              | 96.78   | 96.90   | 97.62   |
|     | BPI                             | (1.0681,1.5677) | (1.8842,2.5744) | (2.3924,3.2783) |
|     | BPI Length                      | 0.4816  | 0.6902  | 0.8859  |

6. Concluding remarks

In Tables (1) and (2) we take different sizes for the informative sample, 10, 20 and 45 and predict the first three future observations.

In these tables, we observe that
(1) The length of the BPI's and the number of samples which cover these intervals increase by increasing $s$ and decrease by increasing the informative sample size.
(2) The results become better as the informative sample size $r$ gets larger.
(3) In all cases, the simulated percentage coverages are at least 95%.
(4) There is no particular reason for choosing the hyperparameters $(c_1, c_2, c_3)$ as $(1, 1.5, 2)$.
(5) If the hyperparameters are unknown, they can be estimated by using the empirical Bayes method [see Maritz and Lwin[13]] or the hierarchical method [see Bernardo and Smith[11]].

APPENDIX A

Proof of equations (16)-(18)

From the joint density function of the random variables $X$ and $Y$ which is given by (13) and using the transforms $X = Z \cos \Theta$ and $Y = Z \sin \Theta$ we get the joint density function of the random variables $Z$ and $\Theta$ in the form

$$f_{Z,\Theta}(z, \theta) = 4 \alpha (\alpha + 1) c^2 z^3 \sin \theta \cos \theta [1 + c z^2]^{-(\alpha+2)}, z > 0, 0 \leq \theta \leq \pi/2. \quad (A.1)$$

Integrating (A.1) with respect to $\theta$, we get the density function of $Z$ as in (16).

The (cdf) of the random variable $Z$ is given by

$$F_Z(z) = 2 \alpha (\alpha + 1) c^2 \int_0^z u^3 [1 + cu^2]^{-(\alpha+2)}du. \quad (A.2)$$

The cdf(17) is obtained by integrating by parts the integral in (A.2). The rf is then obtained as in (18), since $R(z) = 1 - F_Z(z)$.

APPENDIX B

Proof of equation (19)

From (5), (16) and (18) we have

$$g_1(z^*_s|c, \alpha) \propto [R(z_{r,r}) - R(z^*_s)]^{(s-1)}[R(z^*_s)]^{n-r-s}[R(z^*_s)]^{-(n-r)}f_Z(z^*_s)$$

$$= \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} [R(z^*_s)]^{n-r-s+i} [R(z_{r,r})]^{s-i-(n-r)-1} f_Z(z^*_s), \quad (B.1)$$
where the reliability function $R(z)$, given by (18) yields

$$[R(z)]^k = \sum_{i=0}^{k} \binom{k}{i} c^{k-i} \alpha^{k-i} z^{2(k-i)} (1 + cz^2)^{-\alpha k + i}. \quad (B.2)$$

Using (B.2) and (16) in (B.1) we get (19)

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