Hidden Single-Qubit Topological Phase Transition without Gap Closing in Anisotropic Light-Matter Interactions

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Conventionally the occurrence of topological phase transitions (TPTs) requires gap closing, whereas there are also unconventional cases without need of gap closing. Although traditionally TPTs lie in many-body systems in condensed matter, both cases of TPTs may find analogs in few-body systems. Indeed, the ground-state node number provides a topological classification for single-qubit systems. While the no-node theorem of spinless systems is shown to also restrict the fundamental quantum Rabi model in light-matter interactions, it is demonstrated that the limitation of the no-node theorem can be broken not only in a small counter-rotating term (CRT) but also in the large-CRT regime, which stratifies a rich phase diagram with different TPTs. While these transitions are mostly accompanied with gap closing and parity reversal, a hidden node-phase transition is revealed that has neither gap closing nor parity change, which turns out to be an analog of the unconventional TPT in condensed matter. A hysteresis sign for the unconventional TPT is unveiled via the transition from amplitude squeezing to phase squeezing in the gapped phase. The imprints in the Wigner function are also addressed. The clarified mechanisms provide some special insights for the subtle role of the CRT.

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I. INTRODUCTION

Light-matter interactions play a ubiquitous role in our physical world and the past decade has witnessed both extraordinary experimental progresses1,2 and tremendous theoretical efforts3–5 on the investigations of light-matter interactions in the frontiers of modern quantum physics and quantum technologies. Indeed, the continuing experimental enhancements1,6–17 of the interaction strength have brought a new era with ultra-strong1,6–9,12–17 and even deep-strong couplings.17,18 The milestone work3 of revealing the Braak integrability for the quantum Rabi model (QRM), which is a most fundamental model of light-interactions, has stimulated intense theoretical studies.4,5,19–53

As well-known, under the rotating-wave approximation the QRM54 reduces to the Jaynes-Cummings model (JCM),55 which applies for weak/strong couplings or around resonance but fails in ultra-strong and deep-strong regimes.2,5,24,27,42 Nevertheless, the invalidity of the approximation does not exclude possible ultra-strong or deep-strong coupling of the JCM itself.56,57 These two models are also fundamental building blocks for quantum information and quantum computation1,58–61 and closely connected to models in condensed matter2 and relativistic systems.62 A full understanding of their difference, namely the counter-rotating term (CRT), is a main concern both experimentally6,63,64 and theoretically.27,31,34 The convenient model to study the role of the CRT is the anisotropic QRM which connects the QRM and the JCM by continuously tuning the CRT via the anisotropy of the coupling.27,31,34

One of the most fascinating phenomena in enhancing the coupling is possible onset of few-body phase transitions.5,23–25,31,33,34,65 When phase transitions traditionally lie in thermodynamic limit in condensed matter, the QRM possesses a few-body quantum phase transition in the low frequency limit.23–25 On the other hand, it was also suggested that whether the transition should be termed quantum or not is a matter of taste by considering the negligible quantum fluctuations in the photon vacuum state.43 Moreover, a single-qubit system can even exhibit multicriticalities either with different patterns of symmetry breaking31 or with the parity symmetry preserved.33 Interestingly, the anisotropic QRM exhibits a universality of criticality in the scaling of critical exponents.31 However, such a universality holds only under the condition of low frequency limit, while at finite frequencies the universality breaks down and diversity arises.34

Surprisingly, among the dominant diversity a new universality classification can be extracted from the topological feature of the ground state with different node numbers and the series transitions emerging at finite frequencies were found34 to be analogs of symmetry-protected topological phase transitions (TPTs) in condensed matter66–70 which are essentially different from the Landau class of phase transitions with symmetry breaking71. These emerging topological transitions in bridging the QRM and the JCM occur with gap closing and re-opening just as the conventional TPTs in condensed matter. Although gap closing is a necessary condition for the conventional TPTs of non-degenerate states, there are also some unconventional TPTs in condensed matter that occur without gap closing in some particular situations, such as in the presence of a strong electron-electron interaction in the quantum spin Hall effect72 or in the presence of disorder with Berry curva-
ture separation in the quantum anomalous Hall effect.\textsuperscript{73}
Since the single-qubit system of the anisotropic QRM can exhibit TPTs with gap closing as in the many-body systems,\textsuperscript{34} one may wonder whether a single-qubit system can also have any analog of the unconventional TPTs without gap closing.

To seek such a possibility we study in the present work the ground state and the first excitation gap of the anisotropic QRM in the full parameter space of coupling and anisotropy. We find that the limitation of the no-node theorem is broken not only in the small-CRT regime between the QRM and the JCM but also in the large-CRT regime beyond the QRM. While the TPTs in the both regimes are conventional ones with gap closing, we find there is a hidden transition between the two regimes that turns out to be an unconventional TPT without gap closing. We clarify the underlying mechanisms and show that the node in the conventional TPTs appears around the origin thus changing the parity, while the node in the unconventional TPT comes from the infinity thus keeping the parity unaffected. In the gapped phase we also reveal a transition from amplitude squeezing to phase squeezing which can be regarded as a hysteresis sign for the unconventional TPT.

The paper is organized as follows. Section II introduces the anisotropic QRM and the transform to quadrature representation. In Section III the no-node theorem is established for the QRM. Section IV shows the conventional TPTs and Section V reveals the unconventional TPT. The AS/PS transition is addressed in Section VI. The imprints in the Wigner function are demonstrated in Section VII. We clarify the mechanisms in Section VIII. An overview of the phase transitions is given in Section IX and finally Section X is devoted to conclusions and discussions.

II. MODEL AND SYMMETRY

The QRM and the JCM differ in the CRT which can be continuously tuned via the anisotropic QRM with the following Hamiltonian

\[
H = \omega a^\dagger a + \Omega \sigma_x + g \left[ (\tilde{\sigma}_- a^\dagger + \tilde{\sigma}_+ a) + \lambda (\tilde{\sigma}_+ a^\dagger + \tilde{\sigma}_- a) \right]
\]

where \(a^\dagger(a)\) creates (annihilates) a bosonic mode with frequency \(\omega\) and \(\sigma_{x,y,z}\) is the Pauli matrix. The coupling with strength \(g\) includes the rotating-wave term and afore-mentioned counter-rotating term (CRT), the latter is controlled by the anisotropy \(\lambda\). One can retrieve the QRM and the JCM by setting \(\lambda = 1\) and \(\lambda = 0\) respectively. Here the unconventional definition of spin raising and lowering operators \(\tilde{\sigma}_\pm = (\sigma_z \mp i \sigma_y) / 2\) is due to the adoption of the spin notation as in ref.\textsuperscript{42}, in which \(\sigma_z = \pm\) conveniently represents the two flux states in the flux-qubit circuit system,\textsuperscript{74} which is a mostly used platform for realization of ultra-strong couplings,\textsuperscript{1-3,9,12-17} The conventional form of the QRM and the JCM can be recovered by replacement \(\{ \sigma_z, \sigma_y, \sigma_x \} \rightarrow \{ \sigma_z, -\sigma_y, \sigma_x \}\) via a spin rotation around the axis \(\vec{x} + \vec{z}\), with the \(\Omega\) term denoting the atomic level splitting in cavity systems. The model has the parity symmetry, \([\hat{P}, H] = 0\) with \(\hat{P} = \sigma_x (-1)^{a^\dagger a}\), at any anisotropy.

To facilitate our analysis we map the Hamiltonian to the effective spatial space

\[
H = \frac{\omega}{2} \hat{p}^2 + v_x + \left[ \frac{\Omega}{2} - g_y \sqrt{2} \hat{p} \right] \sigma_x^+ + \left[ \frac{\Omega}{2} + g_y \sqrt{2} \hat{p} \right] \sigma_x^- \tag{2}
\]

by the quadrature representation, \(a^\dagger = (\hat{x} - i \hat{p}) / \sqrt{2}\), \(a = (\hat{x} + i \hat{p}) / \sqrt{2}\), with momentum \(\hat{p} = -i \partial / \partial \hat{x}\) as well as spin raising and lowering on the \(\sigma_z = \pm\) basis, \(\sigma_x = \sigma_x^+ + \sigma_x^-\). \(\sigma_y = -i (\sigma_x^+ - \sigma_x^-)\). Let us define the dimensionless anisotropic coupling strengths \(g_y z, z = \sqrt{2} g_y z / \omega\) for \(g_y = (1 - \lambda \omega) g\) and \(g_z = (1 - \lambda \omega) g\). In such a formalism, the coupling effectively drives a displacement by \(g_y^\prime\) in the harmonic potentials \(v_x = \omega (\hat{x} + g_y^\prime \sigma_x^2)^2 / 2 + \epsilon_0^2\) in opposite directions for the two spin components, up to a constant \(\epsilon_0^2 = -\frac{1}{2} [g_y^2 + 1] \omega\). Then, rather than the aforementioned atomic splitting, the \(\Omega\) term now plays the role of spin flipping in the spin \(\sigma_x\) space and the role of tunneling in the effective spatial space.\textsuperscript{24,42} The \(g_y^\prime\) term actually resembles the Rashba spin-orbit coupling.

The Hamiltonian can be re-arranged to be an explicitly \(x-p\) dual form

\[
H_x = \frac{\omega}{2} \left[ \left( -i \frac{\partial}{\partial x} + g_y^\prime \sigma_y \right)^2 + (x + g_y^\prime \sigma_x^2)^2 \right] + \frac{\Omega}{2} \sigma_x + \epsilon_0^2 \tag{3}
\]

\[
H_p = \frac{\omega}{2} \left[ \left( -i \frac{\partial}{\partial p} - g_y^\prime \sigma_x \right)^2 + (p + g_y^\prime \sigma_y^2)^2 \right] + \frac{\Omega}{2} \sigma_x + \epsilon_0^2 \tag{4}
\]

where \(\epsilon_0^2 = -\omega (1 + g_y^2 + g_z^2) / 2\) and we have used \(\hat{x} = i \frac{\partial}{\partial p}\) which fits \([\hat{x}, \hat{p}] = i\). From (3) and (4) one can see directly that the positive-\(\lambda\) and the negative-\(\lambda\) regime are symmetric under the spin rotation and transform to momentum space

\[
\{ \sigma_x, \sigma_y, \sigma_z \} \rightarrow \{ \sigma_x, -\sigma_y, \sigma_y \}, \quad x \rightarrow p, \quad \lambda \rightarrow -\lambda. \tag{5}
\]

Note that in polaron picture the quantum phase transition at \(\tilde{g}^*_c = \frac{\tilde{2}}{\pi |\lambda|^2} g_s\), with \(g_s = \sqrt{\omega \Omega} / 2\), essentially is a wave packet splitting from a Gaussian-like wave packet into two wave packets in the potential separation.\textsuperscript{24,34} In the positive-\(\lambda\) regime, comparing (2) with its dual form indicates that \(g_y^\prime\) with a larger amplitude can generate a larger potential separation in \(x\) space than \(g_y^\prime\) in \(p\) space to bring the quantum phase transition, while the \(g_y^\prime\) term in \(x\) space and the \(g_y^\prime\) term in \(p\) space play little role for the quantum phase transition due to self-cancellation as the derivative of the Gaussian-like wave packet before the transition is an odd function.\textsuperscript{34} Thus the positive-\(\lambda\) regime is \(x\)-type in the sense that \((\tilde{x}^2)\) is more dominant than \((\tilde{p}^2)\), and vice versa, the negative-\(\lambda\) regime is \(p\)-type.\textsuperscript{31,34} Hereafter we discuss in the \(x\) space for \(\lambda > 0\), while the result is symmetrically available in the \(p\) space for \(\lambda > 0\).
Note that the no-node theorem applies for spinless particles or systems without spin interactions, now we extend the theorem to the QRM ($\lambda = 1$) which involves communications of the spin components via both the tunneling $\Omega$ term and the coupling $g$ term. The formalism (2) facilitates our analysis since the coupling $g$ term effectively gives rise to the harmonic potentials $v_{\sigma}(x)$ displaced in opposite directions for the two spin components. In the absence of the tunneling both the two spin components are effectively 1D systems which obey the no-node theorem. In the presence of the tunneling we can apply the similar wave-function deformation but now for both the spin components, $|\Psi\rangle = (|\Psi_+\rangle + |\Psi_-\rangle)/\sqrt{2}$, as in Figure 1c-h. Note that the system has the parity symmetry $\Psi_-(x) = P\Psi_+(x)$, where $P = \pm 1$, which involves the space inversion $x \rightarrow -x$ besides the spin reversion, the tunneling energy is then

$$E_\Omega = \frac{P\Omega^2}{2} \int_{-\infty}^{\infty} \Psi_+(x)\Psi_-(x)dx,$$

where $\Psi_+$ is chosen to be a real function as the ground state is non-degenerate.

Unlike the spinless systems in the conventional no-node theorem, here in the space inversion the node position $x_0$ becomes relevant and there are three possible situations: (i) In the first situation, as in Figure 1c, the nodes in the two spin components are located between the origin and the main peaks of $\Psi_\pm(x)$, in such a case a positive parity would be favorable to gain more negative tunneling energy around the peaks. After the deformation, as compared in Figure 1c,d, $\Phi(x)$ has the same the tunneling energy contribution from $|x| > |x_0|$ as $\Psi(x)$ but gains more negative tunneling energy from $|x| < |x_0|$ due to the opposite signs of $\Phi_+$ and $\Phi_-$. While $\Psi_+$ and $\Psi_-$ have the same sign. Unlike the afore-mentioned small local reduction in the kinetic energy of order $\epsilon$, the decrease in the tunneling energy is more global and finite thus being dominant. (ii) In the second situation, as in Figure 1e,
FIG. 2. Topological transitions with and without gap closing. a) Ground-state (GS) phase diagram of parity in $\lambda$-$g$ plane b) GS phase diagram of the node number $n_Z$ in $\lambda$-$g$ plane, the numbers mark some of $n_Z$ which increases by one for each stripe from blue to red regimes. c) The first excitation gap $\Delta$ versus $\lambda$ at $g = 1.5g_s$. d) $n_Z$ (blue solid line) versus $\lambda$ at $g = 1.5g_s$, the orange dashed lines mark the transitions with gap closing corresponding to c) while the red dot-dashed lines remind the transitions without gap closing. Here, $\omega = 0.5\Omega$.

the nodes appear outside the main peak region so that $\Psi$ has a negative parity with opposite signs of the main peaks. In this case, the tunneling energy after the deformation as in Figure 1f remains the same for $|x| < |x_0|$ but has a finite decrease for $|x| > |x_0|$. (iii) In the third situation, the node is located at the origin, which needs more delicate consideration as the energy variation order now depends on the round-off. In this case $P = 1$ for $\Psi$ and $P = -1$ for $\Phi$. Without the round-off, the tunneling energy before ($E_{\Psi}^\phi$) and after ($E_{\Phi}^\phi$) the deformation are equal. Taking the round-off into account we have a decrease of tunneling energy by order $\epsilon^3$ (see derivation in Appendix A):

$$E_{\Phi}^\phi - E_{\Psi}^\phi = - \left( \frac{4}{3} k^2 \Delta \rho \Omega \right) \epsilon^3,$$

(7)

where $\Delta \rho = \left| \sum_{n=0}^{\infty} |C_n|^2 \right| - \left| \sum_{n=0}^{\infty} (-1)^n |C_n|^2 \right| > 0$ and $C_n$ is the expansion coefficient on the basis of quantum harmonic oscillator $\Psi_+(x) = \sum_n C_n \phi_n$.

Thus the deformation always leads to an energy decrease from the tunneling energy, no matter finite or small, in addition to the energy reduction in the no-node theorem before the tunneling is added. Therefore, we can conclude that the ground state of the QRM has no nodes either.

IV. STATES WITH NODES AND CONVENTIONAL TOPOLOGICAL TRANSITIONS WITH GAP CLOSING

Note that the QRM has an isotropic coupling with equal weights ($\lambda = 1$) of the rotating-wave term and counter-rotating term, which together just right play the role of displacing the potentials by $g'$. In the presence of the coupling anisotropy an extra Rashba spin-orbit coupling arises as the $g_y$ term in the Hamiltonian (2). This $g_y$ term couples the spin components with a strength depending on the gradient of the wave function, thus inducing an effect very different from the direct spin flipping by the $\Omega$ term. Such a coupling breaks the limitation of the no-node theorem and brings about a series of phase transitions and a rich phase diagram.

As extracted by exact diagonalization, Figure 2a shows the phase diagram of the ground-state parity, by the example at $\omega = 0.5\Omega$, where red and blue colors represent positive and negative parities respectively. A transition occurs each time when the parity is reversed. As illustrated by Figure 2c, the parity reversal is always accompanied with a gap closing as the lowest two energy levels with different parities are crossing each other. Such transitions occur in the regime $|\lambda| \leq 1$ which connects the QRM ($\lambda = 1$) and the JCM ($\lambda = 0$) and were found to be topological phase transitions. The transitions change the node number of the ground state, $n_Z$, which characterizes the topological structure of the ground state, since by keeping a fixed number of $n_Z$ one cannot go to another $n_Z$ state by continuous shape deformation of the wave function, just as one cannot change a doughnut...
(torus) into a sphere by a continuous deformation. Note that generally speaking there is no universal topological invariant for all systems even in condensed matter. The mostly used Chern number based on many-body Brillouin zone structure is not either applicable for few-body systems. However, the node number provides a topological quantum number for the anisotropic QRM that is invariant within a topological phase and capable of distinguishing different phases of the ground state. Here we see that such TPTs in the anisotropic QRM not only occur in $|\lambda| \leq 1$ regime but also in $\lambda \leq 1$ regime. Indeed, by increasing $\lambda$ or $g$ the ground state experiences an infinite series of TPTs, thus forming a series stripes of phases.

V. HIDDEN UNCONVENTIONAL TOPOLOGICAL TRANSITIONS WITHOUT GAP CLOSING

As mentioned in Introduction, in condensed matter the conventional TPTs occur at gap closing, while there are also some unconventional TPTs that happen without gap closing. We find such an unconventional case also can appear in the anisotropic QRM. A full phase diagram of the ground-state node number $n_Z$ is figured out in Figure 2b where the numbers mark $n_Z$ which is zero at QRM lines $|\lambda| = 1$ while it increases by steps when anisotropy is introduced. For the regime $|\lambda| \leq 1$ and the regime with a large $|\lambda|$ the transitions in the node number correspond one by one to those in the phase diagram of the parity in Figure 2b where the parity changes are accompanied with gap closing. However, as one can notice, another transition boundary of the node number in Figure 2b disappears around $\lambda = 1$, with its duality around $\lambda = -1$, but has no match of parity change in Figure 2a. A clearer illustration at a fixed coupling $g = 1.5g_0$ is presented in Figure 2d where the orange dashed lines mark the transitions of $n_Z$ in large $\lambda$ that correspond to the gap closings in Figure 2b, whereas the transitions labeled by the red dot-dashed lines around $|\lambda| = 1$ have no gap closing. We find that the conventional transitions add nodes around the origin, as from Figure 1d,h to Figure 1c,g which change the parity, while the unconventional transition introduces a node from the infinity as from Figure 1f to Figure 1e which does not change the parity. We shall clarify the underlying mechanisms later on in Section VIII.

Unlike the TPTs with parity change and gap closing which are also accompanied with first-order transitions in some physical properties, the unconventional TPT has neither parity change nor gap closing to induce any abrupt changes of physical properties. The unconventional transition would be of infinite order as the node is coming from the infinity. The transition seems to completely lie in the topological feature of the ground-state wave function with the node number being invariant within a topological phase but changing across the phase boundary. Thus, the unconventional transition is hidden in the sense that it would be not easy to find unless one looks closely into the topological structure of the ground state. Nevertheless, although right at the unconventional transition there is no obvious change of physical properties apart from the change of the topological invariant, lagging behind the unconventional transition we can find a squeezing-type transition which can be regarded as a hysteresis sign of the unconventional transition, as addressed in the following.

VI. TRANSITIONS FROM AMPLITUDE SQUEEZING TO PHASE SQUEEZING IN GAPPED PHASE

After the afore-mentioned unconventional TPT in increasing the anisotropy from the QRM line, we also find a transition from amplitude squeezing (AS) to phase squeezing (PS) in the gapped phase. Actually, rather than polarons with coherent-state expansion for each wave packet, one wave packet in decomposing the wave function can be more compactly represented by a frequency-renormalized polaron depicted by

$$\varphi(\xi, \zeta) = e^{-\xi(x-\zeta)^2/2} (\xi/\pi)^{1/4}$$

with the displacement renormalization $\zeta$ and the frequency renormalization $\xi$. The squeezing is reflected by the frequency renormalization $\xi$, with $\xi < 1$ indicating an AS with a wave-packet extension in the $x$ space while $\xi > 1$ characterizing a PS with a wave-packet extension in the $p$ space. We can extract $\xi$ for the main peak of the wave function by the ratio

$$\xi = (r_G/r_\psi)^2$$

of the Gaussian half-peak radius $r_G = \sqrt{2\ln 2}$ and the wave-function half-peak radius $r_\psi$. To reduce the influence of the possible secondary wave packet, $r_\psi$ is extracted from the distance between the peak position and the half-peak position on the side farther from the origin. We show the absolute deviation of $\xi$ from 1, i.e. $|\xi - 1|$ in Figure 3a where the bright lines indicate the AS/PS transition with $\xi = 1$. The transition boundaries can be more clearly seen by the sign of the $\xi$ deviation from $\xi = 1$ in Figure 3b where the blue and red regions have an AS ($\xi - 1 < 0$) and a PS ($\xi - 1 > 0$) respectively. Another quantity that can reflect the squeezing is the fluctuation of the momentum, $\Delta \rho = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$, with the AS and the PS indicated by $\Delta \rho < 1/2$ and $\Delta \rho > 1/2$. The phase diagram of $\Delta \rho = 1/2$ is similar to that of $\xi - 1$ in Figure 3b, except for some small boundary discrepancy due to that $\xi$ is extracted from the main peak of wave packet while $\Delta \rho$ is the expectation over the total wave function.

In $\lambda < 1$ regime, the AS/PS boundary coincides with the first boundary of the conventional TPTs. However,
in $\lambda > 1$ regime, the AS/PS boundary in Figure 3b appears between the first conventional TPT in $\lambda > 1$ regime (dotted line) and the unconventional topological transition (dashed line). The AS/PS transition occurs before the dotted line where the parity changes and the gap closes for the first time in entering the $\lambda > 1$ regime as in Figure 2a,c, thus also without gap closing as the unconventional TPT at the dashed line. In other words, this AS/PS transition is lagging behind the dashed line where the first node enters the ground state when one leaves the nodeless QRM line at $\lambda = 1$. In Section VIII we will see this AS/PS transition is coming retarded due to awaiting enough strengthening of the nodal state after the unconventional TPT. Apart from the main AS/PS transition boundary in large-$g$ regime, a peculiar AS/PS transition is also hidden in small-$g$ and large-$\lambda$ regime in Figure 3b and a zoom-in view is provided in Figure 3c,d. Examples of $\xi$ and $\Delta p$ at fixed couplings are illustrated in Figure 3e,f which show the AS/PS transitions both in large-$g$ regime (blue) and small-$g$ regime (orange), as indicated by the dotted lines.

VII. IMPRINTS IN THE WIGNER FUNCTION

Both the squeezing and the nodal status may leave imprints in the Wigner function which is defined in the phase space and enables the visualization of information of the momentum:

$$W(x,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipy} |\psi|^2(x + \frac{y}{2})\psi(x - \frac{y}{2}) dy,$$  \hspace{1cm} (10)$$

where $|\psi\rangle = (\psi_+ |+\rangle + \psi_- |-) / \sqrt{2}$ and we have set $\hbar = 1$. We show four typical examples of the ground-state wave function in Figure 4a-d and the corresponding Wigner function in Figure 4e-h: (i) Figure 4a represents a nodeless state at $\lambda = 1$, the Wigner function of which has negative interference fringes (blue) away from the central line at $p = 0$ as shown in Figure 4e. (ii) Figure 4b illustrates a state with one node. In contrast to Figure 4e the Wigner function in Figure 4f has a central piece of the negative interference fringes as indicated by the dashed lines. This central negative interference fringe (CNIGF) turns out to be a sign of the node. Note that the node in this case is located around the origin $x = 0$. (iii) We show another nodal case in Figure 4c where the node is not around the origin but appears beyond the secondary peak, thus we have two positive peaks ($\alpha$, $\beta$) and a negative peak ($\gamma$) with the node between $\beta$ and $\gamma$. The Wigner function in Figure 4g has two regimes of interference fringes: one around $x = -0.5$ has no CNIGF and the other around $x = 1.5$ has a CNIGF, later on we will see the former comes from the $\alpha$-$\beta$ interference while the latter originates from the $\alpha$-$\gamma$ interference. Although this CNIGF is small due to the $\alpha$-$\gamma$ distance and the small weight of $\gamma$, a large negative central spot emerges around $x = 4.5$ which stems from the $\beta$-$\gamma$ interference. (iv) A state with one node around the origin and another node beyond the secondary peak is presented in Figure 4d. The Wigner function in Figure 4h now has a fringe regime with a CNIGF around $x = -0.5$, and a fringe regime without CNIGF around $x = 1.5$, and a central negative interference spot around $x = 3.5$.

To see the origins of the interference fringes and spots we can decompose the wave function into $n_p$ number of
FIG. 4. Imprints of nodal status in the Wigner function. a-d) Spin-up component of GS Wave function $\psi_+$ at $\{\lambda, g/g_s\} = \{1.0, 2.5\}$ ($n_Z = 0, P = -1$), $\{0.5, 3.0\}$ ($n_Z = 1, P = 1$), $\{2.5, 1.8\}$ ($n_Z = 1, P = -1$), and $\{4.0, 1.2\}$ ($n_Z = 2, P = 1$). Here the red dots are results of exact diagonalization (ED) while the solid lines are plotted by polaron picture (see parameters in Appendix B). e-h) The Wigner function $W(x, p)$ corresponding to a-d) by numeric integrals based on the ED wave function. The dashed lines in e),f) remind central or not for the negative peak position (blue) of the fringe. The dot-dashed line in e) is a guide to the eye for the curved fringe alignments. i-l) $W(x, p)$ in h) decomposed by three polarons into i) inner-polaron terms $\sum_{i=1}^{n_p} W_{ii}(x, p)$ and inter-polaron terms j) $W_{\alpha\beta}$, k) $W_{\alpha\gamma}$, and l) $W_{\beta\gamma}$. The horizontal and vertical dashed lines remind the central-peak signs and the positions $x_{ij}$ of the decomposed fringes. Here, $\omega = 0.5\Omega$ and the wave-function amplitude is amplified by $|\psi_+|^{1/2}$ to show the node more clearly and the Winger-function amplitude is amplified by $|W|^{1/4}$ to enhance the color contrast.

frequency-renormalized polarons with weight $w_i^{24,36}$

$$\psi_+ = \sum_{i=1}^{n_p} w_i \varphi(\xi_i, \zeta_i),$$

which can reproduce well the wave function from the exact diagonalization (ED), as compared in Figure 4a-d where the blue solid lines are results in polaron picture which are in good agreements with the ED (red dots). With the polaron decomposition (11), we obtain an explicit expression of the Wigner function with different contributions from the inner-polaron part $W_{ii}$ and the inter-polaron part $W_{ij}$:

$$W(x, p) = \sum_{i=1}^{n_p} W_{ii}(x, p) + 2 \sum_{i<j}^{n_p} W_{ij}(x, p),$$

where

$$W_{ii} = \frac{w_i^2}{\pi N^2_p} e^{-\frac{\xi_i^2}{\xi_{ij}^2} - \frac{(x-x_i)^2}{\xi_{ij}^2}},$$

$$W_{ij} = \frac{w_i w_j}{\pi N^2_p} \left( \frac{\xi_i \xi_j}{\xi_{ij}} \right)^{1/4} e^{-\frac{\xi_j^2}{\xi_{ij}^2} - \frac{\xi_i^2}{\xi_{ij}^2} (x-x_i)^2} f_{ij},$$

$$f_{ij} = \cos[p \frac{(x-x_i)\xi_i - (x-x_j)\xi_j}{\xi_{ij}}].$$
and \( x_i = \zeta_i g' \) is the polaron position determined by the Gaussian center while \( x_{ij} = (x_i + x_j)/2 \) and \( \xi_{ij} = (\xi_i + \xi_j)/2 \) are averaged position and frequency renormalization for polaron \( i \) and \( j \). The normalization factor is decided by \( N_p = 1/\sqrt{|\sum_{i,j} S_{ij}}\) with \( S_{ij} = (\xi_i \xi_j/\xi_{ij})^{1/4} \exp[-\xi_{ij} / (2x_i - x_j)]^2 \). Figure 4i-l are the decomposed inner-polaron part \( W_{ij} \) and inter-polaron part \( W_{ij} \) which together well reproduce the total Wigner function by ED in Figure 4i-l. We see that inner-polaron part \( W_{ij} \) is round in the phase space if there is no squeezing \( \xi_i = 1 \), otherwise it becomes oval along \( x \) direction for amplitude squeezing with \( \xi_i < 1 \) or along \( p \) direction for phase squeezing with \( \xi_i > 1 \), as illustrated by Figure 4i which has a phase squeezing. The interference fringes are coming from \( f_{ij} \) in the inter-polaron part which oscillates with an average period

\[
T_p = \frac{2\pi}{|x_j - x_i|}
\]

as the momentum \( p \) is increasing, while the fringe position is decided by \( x_{ij} \). The expression (16) indicates that a larger polaron distance leads to denser interference fringes, thus accounting for the period difference in Figure 4j-l as the polarons \( \beta \) and \( \gamma \) for \( W_{ij} \) in Figure 4i are closer than those for \( W_{ij} \) and \( W_{ij} \) in Figure 4j,k.

It might be worthwhile to mention here that not only inner-polaron part \( W_{ij} \) and but also inter-polaron part \( W_{ij} \) can provide some squeezing information, with \( W_{ij} \) indicating the squeezing difference. In fact, when the two polarons have different frequencies, the fringe alignment will not be straight but become curved. The local curvature around \( p = 0 \) can be extracted to be

\[
K = \frac{2(\xi_i - \xi_j)}{(\xi_i + \xi_j)(x_j - x_i)},
\]

which is proportional to the frequency difference. The curving center is on the side of the polaron with a lower frequency, as illustrated in Figure 4e where the fringe alignment tends to curve around polaron \( \beta \) as \( \xi_i \approx 0.8 \) on the right side because \( \xi_i \approx 0.97 \).

Note a nodeless state has polaron weight \( w_i \) always positive thus \( W_{ij} \) is positive around \( p = 0 \), while the node introduces a negative polaron weight thus turns the positive peak of \( W_{ij} \) to be negative around \( p = 0 \), which accounts for the absence of the CNIGF in Figure 4e and the presence of the CNIGF in Figure 4f. When we have two nodes as in Figure 4d,h, the neighboring weight product \( \alpha \beta \) and \( \beta \gamma \) are negative while the weight of the next neighboring pair \( \alpha \gamma \) is positive, which produces a CNIGF in Figure 4j,l but not in Figure 4k as the horizontal lines remind. Thus, we see both the CNIGF and the negative interference spot at \( p = 0 \) are imprints of the nodes. The changes of patterns from e)→f) and g)→h) in Figure 4 are conventional TPTs in \( |\lambda| < 1 \) and \( |\lambda| > 1 \) regimes respectively, while e)→g) would indicate the unconventional TPT.

**VIII. MECHANISMS**

The topological transition without gap closing and the transitions of the different types of squeezing actually involve subtle energy competitions. Clarifying the mechanisms underlying these transitions would gain more insights for the role of the counter-rotating term in the coupling. In Section III we have seen that competitions of the kinetic energy, the potential energy and the tunneling energy are not favorable to introduce a node in the ground state, and the \( g_z \) term of the anisotropic coupling in (2) effectively just contributes to the potential. So the key to have a nodeful state is in the anisotropic \( g_y \) term of (2). We remind here we are focusing on the positive-\( \lambda \) regime while the analysis is the same for the negative-\( \lambda \) regime by exchange of \( \{g_y, x\} \) and \( \{-g_z, p\} \) as mentioned in (5). To facilitate the understanding we write the anisotropic \( g_y \) term together with the tunneling energy in the following form

\[
E_{\Omega} + E_{g_y} = \int_{-\infty}^{\infty} \psi_+(x) \left[ \frac{\Omega}{2} \psi_-(x) - \sqrt{2}g_y \partial_x \psi_-(x) \right] dx
\]

where the doubling of the contribution from spin exchange has canceled with the normalization factor \( 1/\sqrt{2} \). We remind here \( g_y = (1-\lambda)/2 \) \( \eta \), the sign of which will make an important difference.

**A. Energy Competitions in Nodeless State**

We first look at a nodeless state in the \(|\lambda| \leq 1 \) regime, as represented by an example in Figure 5a with \( \lambda = 0.7 \) and \( g = 2.8g_y \). As the \( \Omega \) term is more dominant over the small \( g_y \) with \( \lambda \) close to 1, the nodeless state has a negative parity to gain more negative energy in \( E_{\Omega} \). In this trend the peaks of \( \psi_+ \) and \( \psi_- \) tend to get located at the same positions to get a maximum wave-packet overlap as indicated by the vertical blue dashed lines in Figure 5a, while the two-peak structure is due to the interplay of the tunneling and the potential separation in the two spin components. Note that in this regime \( g_y \) is positive thus the anisotropic coupling is actually counter-acting against the \( \Omega \) term in a large region where \( \partial_x \psi_- (x) \) (orange solid line) and \( \psi_+ (x) \) (blue solid line) have opposite signs. The configuration with same peak positions of \( \psi_+ \) and \( \psi_- \) reduces the counteracting effect of \( g_y \) to a great degree as it is the zeros of \( \partial_x \psi_- \) that are meeting the peaks of \( \psi_+ \). On the other hand to gain more negative energy from \( E_{\Omega} \) the wave packets on the right and left sides tend to get more extended to get right-left overlap, which leads to an amplitude squeezing. Such a nodeless status will maintain until the two counteracting energies cancel each other and the parity reverses at a larger \( g_y \) at a critical anisotropy \( |\lambda|_{T1} = \sqrt{1 - 4g_y^2/g_z^2} \), while different parity represents different quantum states and parity change means level crossing and gap closing.
term, it is more favorable for the negative peaks of $\partial_x \psi$ as shown by two vertical dashed lines around $\psi$ rather than those of $g$ that the $g$ reverses the sign to be negative. In such a situation, the

![Figure 3e](image.png)

Figure 3e. $\lambda$ competing relation arises instead. When $\lambda > 1$ regime has a different scenario as now $g_y$ reverses the sign to be negative. In such a situation, the $g_y$ term is not counteracting against the $\Omega$ term but a competing relation arises instead. When $\lambda$ is large so that the $g_y$ term becomes more dominant over the $\Omega$ term, it is more favorable for the negative peaks of $\partial_x \psi$, rather than those of $\psi_-$, to meet the positive $\psi_+$ peaks, as shown by two vertical dashed lines around $x = -3$ and $x = 2$ for the two positive $\psi_+$ peaks in Figure 5b with $\lambda = 2.5$ and $g = 1.8g_s$. It should be noted here the adjusting of peak positions has to pay the price to reduce the contribution of $E_\Omega$, thus requiring a large $\lambda$ and a considerable strength of $g$ (as $g_y$ is proportional to $g$). Nevertheless, that is not the only way to enhance the contribution of $E_{g_y}$, another way with $E_\Omega$ less affected is to introduce a node from the infinity or more exactly the infinity sides of the secondary peaks. Such a node entering not only reverses the sign of $\psi_+$ at the regime where $\partial_x \psi$ originally has the the same sign as $\psi_+$ as in $x > 4$ regime in Figure 5b, but also makes a quick change of $\psi_-$ thus increasing the amplitude of $\partial_x \psi$ close to the main peak of $\psi_+$ around $x = -3$. Since the main parts of the wave function and $E_\Omega$ are little affected, such a node introduction does not change the parity, thus needing no level crossing. This is the origin of the unconventional topological transition without gap closing. More interestingly, this topological transition can occur both at a weak anisotropy ($\lambda \rightarrow 1$) and in a weak coupling ($g \rightarrow 0$), as indicated in Figure 2b and Figure 3b (dashed lines).

After this unconventional topological transition, with the strengthening of the anisotropy and the coupling the node that enters from the infinity will move closer to the peak position to form the afore-mentioned peak meeting of $\psi_+(x)$ and $\partial_x \psi_-(x)$ as in Figure 5b. The further transitions will keep this optimized near-peak node configuration and add new nodes around the origin $x = 0$, as c) to d) in Figure 4, rather than from the infinity. Creation of a new node around the origin will braid $\psi_+(x)$ and $\psi_-(x)$ as from d,h) to c,g) in Figure 1, thus accompanied with parity reversal. States with opposite parities are different quantum states and the parity reversal means level crossing of the lowest levels, thus we have gap closing for further transitions which are then conventional TPTs.

![Figure 5](image.png)

Figure 5. Mechanisms for the topological transitions without gap and the squeezing transitions. The GS wave function $\psi_+(x)$ (blue solid), $\psi_-(x)$ (orange dotted) and its derivative $\partial_x \psi_-(x)$ (orange solid) at a) $\lambda = 0.7, g = 2.8g_s$, b) $\lambda = 2.5, g = 1.8g_s$, c) $\lambda = 3.0, g = 0.8g_s$, d) $\lambda = 5.5, g = 0.1g_s$. The vertical blue dashed lines mark the positions of positive or negative peaks of $\psi_+(x)$. Here, $\omega = 0.5\Omega$ and to increase the visibility of the nodes we have plotted the amplitudes by $|\psi_\pm|^{1/2}$ and $|\partial_x \psi_\pm|^{1/2}$.

B. Mechanism for the Unconventional Topological Transition without Gap Closing

The $\lambda > 1$ regime has a different scenario as now $g_y$ reverses the sign to be negative. In such a situation, the $g_y$ term is not counteracting against the $\Omega$ term but a competing relation arises instead. When $\lambda$ is large so that the $g_y$ term becomes more dominant over the $\Omega$ term, it is more favorable for the negative peaks of $\partial_x \psi$, rather than those of $\psi_-$, to meet the positive $\psi_+$ peaks, as shown by two vertical dashed lines around $x = -3$ and $x = 2$ for the two positive $\psi_+$ peaks in Figure 5b with $\lambda = 2.5$ and $g = 1.8g_s$. It should be noted here the adjusting of peak positions has to pay the price to reduce the contribution of $E_\Omega$, thus requiring a large $\lambda$ and a considerable strength of $g$ (as $g_y$ is proportional to $g$). Nevertheless, that is not the only way to enhance the contribution of $E_{g_y}$, another way with $E_\Omega$ less affected is to introduce a node from the infinity or more exactly the infinity sides of the secondary peaks. Such a node entering not only reverses the sign of $\psi_+$ at the regime where $\partial_x \psi$ originally has the the same sign as $\psi_+$ as in $x > 4$ regime in Figure 5b, but also makes a quick change of $\psi_-$ thus increasing the amplitude of $\partial_x \psi$ close to the main peak of $\psi_+$ around $x = -3$. Since the main parts of the wave function and $E_\Omega$ are little affected, such a node introduction does not change the parity, thus needing no level crossing. This is the origin of the unconventional topological transition without gap closing. More interestingly, this topological transition can occur both at a weak anisotropy ($\lambda \rightarrow 1$) and in a weak coupling ($g \rightarrow 0$), as indicated in Figure 2b and Figure 3b (dashed lines).

After this unconventional topological transition, with the strengthening of the anisotropy and the coupling the node that enters from the infinity will move closer to the peak position to form the afore-mentioned peak meeting of $\psi_+(x)$ and $\partial_x \psi_-(x)$ as in Figure 5b. The further transitions will keep this optimized near-peak node configuration and add new nodes around the origin $x = 0$, as c) to d) in Figure 4, rather than from the infinity. Creation of a new node around the origin will braid $\psi_+(x)$ and $\psi_-(x)$ as from d,h) to c,g) in Figure 1, thus accompanied with parity reversal. States with opposite parities are different quantum states and the parity reversal means level crossing of the lowest levels, thus we have gap closing for further transitions which are then conventional TPTs.

C. Mechanism for the AS/PS transitions

Besides the adjustment of the peak positions and the introduction of the node from the infinity, there is a third way to enhance the contribution of $E_{g_y}$ which lies in reversion of the squeezing types. As afore-mentioned for Figure 5a a weak anisotropy has an amplitude squeezing when the $\Omega$ term is more important, however when the $g_y$ term comes to play a more dominant role at larger $|\lambda|$, a phase squeezing will increase the strength of $\partial_x \psi_-$
Thus leads to the AS/PS transition as described in Figure 3a,b. However such a AS/PS transition does not occur without the first onset of the unconventional TPT. Indeed, although the phase squeezing can amplify the amplitude of $\partial_x \psi_-$, the roughly antisymmetric profile of $\partial_x \psi_-$ around the $\psi_-$ peak position is however canceling itself if the peak of $\psi_+$ is located at the same position of $\psi_-$ in the absence of the node, as one can see in Figure 5a around the right dashed line at $x \approx 2.3$. So the enhancement of $\partial_x \psi_-$ by phase squeezing does not come to effect without a node. This situation is changed by the appearance of the node as in Figure 5b where the node around $x = 3.8$ deforms the symmetric profile of $\psi_+$ (blue solid line) which breaks the afore-mentioned cancelation effect of the antisymmetric $\partial_x \psi_-$. Of course the AS/PS transition does not occur immediately after the onset of the unconventional TPT, since the node needs to come from the infinity close enough to the peak to overcome the amplitude squeezing caused by the $\Omega$ term. Thus, the AS/PS transition is lagging behind the unconventional TPT. In this sense, the AS/PS transition can be regarded as the hysteresis transition of the unconventional TPT.

The AS/PS transition in the small-$g$ regime mentioned for Figure 3c,d is a bit different. Reversely, rather than starting with an amplitude-squeezing state, we first show a phase-squeezing case in Figure 5c with $\lambda = 5.5$ and $g = 0.1 g_0$ in the red small-$g$ region of Figure 3d. In this case $\psi_+$ and $\psi_-$ have a single-polaron profile and have just been separated a little bit due to small $g$. The profile of $\partial_x \psi_-(x)$ has two peaks with the negative one closer to the peak of $\psi_+$. Note with the large $\lambda$ here it is also favorable to get more contribution from the $g_y$ term which can be enhanced by driving the negative $\partial_x \psi$ peak closer to the origin. With little price to change the $\Omega$ term in the full overlap of $\psi_+$ and $\psi_-$, this can be realized by a larger $\xi$ as we can see from the Gaussian wave packet (8) that has a derivative-peak distance from its own peak position

$$d_{\psi} = \frac{1}{\sqrt{\xi}}$$  (19)

Thus a phase-squeezing effect is seen here. It should be mentioned this case has a node but too weak to have a considerable effect until the PS/AS transition. After the PS/AS transition we show a case in Figure 5d with $\lambda = 3.0$ and $g = 0.8 g_0$ with an amplitude squeezing in the blue region of Figure 3d. In this case the strength of $g$ is able to separate the $\psi_+$ and $\psi_-$ but still not enough to make the $g_y$ contribution overwhelming over the $\Omega$ term, consequently the negative peak of $\partial_x \psi_-(x)$ is located between the peaks of $\psi_+$ and $\psi_-$ as marked by the dashed line in Figure 5d. On the other hand, now the node is more visible but not yet fully reached the positive peak of $\partial_x \psi_+$. As indicated by (19) a smaller $\xi$ will push the negative $\partial_x \psi_-(x)$ peak farther from the peak of $\psi_-(x)$ but closer to the peak of $\psi_+(x)$, on the other hand it also drives the positive $\partial_x \psi_+(x)$ peak closer to the negative part of $\psi_+(x)$ beyond the node. This process enhances the contribution of the $g_y$ term, while it is also favorable for the wave-packet overlap extension in the $\Omega$ term. Thus, a PS/AS transition also occurs in the regime with large $\lambda$ and small $g$. Here we see the node enhancement is also a key driving factor for this PS/AS transition.

IX. OVERVIEW OF PHASE DIAGRAM: A MINI-WORLD OF PHASE TRANSITIONS

So far we have analyzed by standing in the positive-$\lambda$ regime, the analysis is similar for the negative-$\lambda$ regime by the mapping to momentum space in (5). A panorama over full parameter space can be obtained by the phase diagram of $\langle a^\dagger a \rangle = \langle \hat{\varphi}^2 \rangle / 2$ multiplied by the parity $P$, as shown in Figure 6. The parity is symmetric with respect to the sign reversal of $\lambda$, while $\langle a^\dagger a \rangle$ is antisymmetric as $\langle \hat{\varphi}^2 \rangle$ is dominant over $\langle \hat{p}^2 \rangle$ in the positive-$\lambda$ regime but it is reverse in the negative-$\lambda$ regime.

The dot-dashed line is the second-order boundary $g_0^\Lambda = 2/\sqrt{\lambda \xi}$ of the quantum phase transition in the low frequency limit, including the transition $g_0 = g_0$ in the QRM with $\lambda = 1$.

This transition is Landau class of phase transition with a hidden symmetry breaking despite that the parity symmetry is preserved.

Meeting with this Landau class of phase transition boundary at the hexacritical point $\{\lambda, g\} = \{0, 2g_0\}$ is a topological class of phase transition boundary $g_{T1} = 2g_0/\sqrt{1 - \lambda^2}$ without symmetry breaking. Following the hexacritical point are a series of quadruple points at larger $g$, formed by the series of symmetry-protected topological transitions crossing with the symmetry-breaking bound-
ary at the JCM line $\lambda = 0$. Bridging the QRM and the JCM are the criticality universality in the low frequency limit\textsuperscript{31} and the topological universality classification at finite frequencies reformed among the diversity and the breakdown of the criticality universality.\textsuperscript{34} These topological transitions in the $|\lambda| < 1$ regime are conventional type of TPTs with gap closing, here being of first order.

The stripes of phases in the $|\lambda| > 1$ regime are also adjoined by conventional TPTs with gap closing. The dashed lines are boundaries for the unconventional TPT without gap closing, which are plotted in different colors in large- and small-$g$ regimes to enhance the visibility. This unconventional TPT is of infinite order. Lagging behind the unconventional TPT and also in the gapped phase is the AS/PS transition with the white dashed boundaries.

At a final glance, it seems a bit surprising that such a single-qubit system actually opens a mini-world of phase transitions full of various ingredients: (i) different orders of transitions, (ii) multicritical point and multiple points, (iii) symmetry-breaking quantum phase transitions (Landau class) and symmetry-protected topological phase transitions (topological class), (iv) criticality universality and topological universality, (v) conventional TPTs with gap closing and unconventional TPTs without gap closing, (vi) transitions of amplitude squeezing and phase squeezing. As a Chinese saying goes: the sparrow may be small but it has all the vital organs. Through this abundant mini-world of phase transitions we might gain some deeper insights for the role of the counter-rotating term of light-matter interactions.

X. CONCLUSIONS AND DISCUSSIONS

We have extended the no-node theorem of the spinless-particle systems to the fundamental QRM which involves spin coupling via light-matter interaction. When the node number of the wave function characterizes the topological difference of the ground state of the single-qubit system and provides a topological classification, we have shown that the limitation of the no-node theorem can be broken not only in the regime of coupling anisotropy between the QRM and the JCM but also in the regime beyond the QRM. We have obtained a full phase diagram in the interplay of the coupling and the anisotropy, which compares the phase transitions identified by the changes of parity, the gap closing and the jumps of the node number. Most topological transitions in variations of the node number are accompanied with a parity reversal and a gap closing as in the conventional TPTs, whereas we have revealed a hidden transition in the node number without parity change or gap closing which is an analog of the unconventional TPT without gap closing. Our mechanism analysis shows that the conventional transitions occur with the nodes emerging at the origin while the unconventional transition happens with the node entering from the infinity.

We have also unveiled a transition of amplitude squeezing (AS) and phase squeezing (PS) by tracking the frequency renormalization of the main wave-function peak and the variance/quantum fluctuation of the momentum. Such an AS/PS transition also occurs without gap closing but lagging behind the unconventional TPT. The mechanism analysis indicates that the AS/PS transition can be regarded as a hysteresis sign of the unconventional TPT.

Both the nodal status and the squeezing effect may leave imprints in the Wigner function. Apart from the shape deformation of the individual wave packets, the squeezing difference of different wave packets can lead to a local curvature of the interference fringe alignment. In particular, while the nodeless state has always positive central interference fringes, the node can induce a negative central interference fringe as well as an additional negative central interference spot around zero momentum in the phase space.

Note the anisotropic QRM is realistic and can be implemented in experimental setups of superconducting circuits.\textsuperscript{6,63,64} It may be worthwhile to stress that the hidden unconventional TPT can be triggered in all regimes of interaction strengths including small couplings, which would provide a great flexibility for experimental accessibility. The gapful situation of the unconventional TPT as well as the AS/PS transition might also be more favorable for the condition in quantum information processing. For an example, experimentally it is easier to cool the system down to a gapped ground state. On the other hand, the time to adiabatically prepare a quantum state in quantum metrology is inversely proportional to the gap,\textsuperscript{80} thus a transition without gap closing would have more advantages. Since the transformed Hamiltonian (2) with Rashba spin-orbit coupling has similarity with those in cold atoms\textsuperscript{82} as well as nanowires\textsuperscript{83,84} and note that nodal status could be observed in spatial density\textsuperscript{85} of Bose-Einstein condensates,\textsuperscript{82} we speculate our finding and analysis might also provide some insights for the cold-atom and nanowire systems, which we would like to address in some other works.

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Appendix A: Proof for $E_{\Omega}^\phi < E_{\Omega}^\psi$ for $x_0 = 0$

When the node of $\Psi$ is located away from the origin, after removing the node by the deformation to nodeless state $\Phi$ the energy reduction in the $\Omega$ term of the QRM is obviously finite as we have demonstrated in Figure 1c-f in the main text. However, when the node is right at the origin, i.e. $x_0 = 0$ as in Figure 1g,h, the energy reduction is more delicate and needs some more rigorous proof as presented here. The tunneling energy before and after
the deformation respectively are
\[ E_{\Omega}^{\Phi} = \Omega \int_{-\infty}^{\infty} \Psi_{\pm}(x)\Psi_{\pm}(-x)dx, \quad (A1) \]
\[ E_{\Omega}^{\Phi} = \Omega \int_{-\infty}^{\infty} \Phi_{\pm}(x)\Phi_{\pm}(-x)dx. \quad (A2) \]
Since the node is at the origin, we have \( \Psi_{\pm}(x) \approx \mp kx \)
around the node and \( \Phi_{\pm}(x) = \pm Nke \) for \( |x| \leq \epsilon \) while \( \Phi_{\pm}(x) = \pm \sqrt{N} |\Psi_{\pm}(x)| \) for \( |x| > \epsilon \), with the renormalization
factor \( N = 1/\sqrt{1+4k^2\epsilon^2}/3 \). Therefore,
\[ E_{\Omega}^{\Phi} = \Omega \left( \int_{-\epsilon}^{\epsilon} N^2 \Psi_{\pm}(x)\Psi_{\pm}(-x)dx - \Omega \int_{-\epsilon}^{\epsilon} N^2 (k\epsilon)^2 dx \right) \]
\[ = N^2 E_{\Omega}^{\Phi} - N^2 \Omega \frac{4}{3} k^2 \epsilon^3 \]
so that the difference is in an order of \( \epsilon^3 \):
\[ E_{\Omega}^{\Phi} = (N^2 - 1) E_{\Omega}^{\Phi} - N^2 \Omega \frac{4}{3} k^2 \epsilon^3 \quad (A4) \]
\[ = \Omega \frac{4}{3} k^2 \epsilon^3 \left( \int_{-\infty}^{\infty} \Psi_{\pm}(x)\Psi_{\pm}(-x)dx \right) - \Omega \frac{4}{3} k^2 \epsilon^3 + O(\epsilon^6) \]
\[ = -\Omega \frac{4}{3} k^2 \epsilon^3 \left( 1 - \int_{-\infty}^{\infty} \Psi_{\pm}(x)\Psi_{\pm}(-x)dx \right) + O(\epsilon^6) \]
\[ = -\Omega \frac{4}{3} k^2 \epsilon^3 \Delta_\rho + O(\epsilon^6) \quad (A5) \]
where \( \Delta_\rho \) is positive and finite
\[ \Delta_\rho = \sum_{n=0}^{\infty} |C_n|^2 - \sum_{n=0}^{\infty} (-1)^n |C_n|^2 \]
\[ = \sum_{k=0}^{\infty} \left( |C_{2k}|^2 + |C_{2k+1}|^2 \right) - \sum_{k=0}^{\infty} \left( |C_{2k}|^2 - |C_{2k+1}|^2 \right) > 0. \quad (A6) \]
Note here we have applied the expansion \( \Psi_{\pm}(x) = \sum_{n=0}^{\infty} C_n \phi_n(x) \) on the basis of quantum harmonic oscillator \( \phi_n(x) \) which gives
\[ \Psi_{\pm}(x) = \sum_{n=0}^{\infty} C_n (-1)^n \phi_n(x). \quad (A7) \]
Despite the small order, the deformation in removing the node always contributes an energy reduction in the \( \Omega \) term.

Appendix B: Polaron Parameters in Figure 4

The polaron parameters in Figure 4 are: a) \( \{ x_\alpha, x_\beta, w_\alpha/w_\alpha, \zeta_\alpha, \zeta_\beta \} = \{-2.45, 1.94, 0.094, 0.97, 0.80\} \), b) \( \{ x_\alpha, x_\beta, w_\beta/w_\alpha, \zeta_\alpha, \zeta_\beta \} = \{-2.16, 2.47, -0.098, 0.98, 1.21\} \), c) \( \{ x_\alpha, x_\beta, x_\gamma, w_\beta/w_\alpha, w_\gamma/w_\alpha, \zeta_\alpha, \zeta_\beta, \zeta_\gamma \} \) is \( \{-3.11, 2.2, 3.75, 0.105, -0.021, 1.07, 1.25, 1.3\} \), d) \( \{ x_\alpha, x_\beta, x_\gamma, w_\beta/w_\alpha, w_\gamma/w_\alpha, \zeta_\alpha, \zeta_\beta, \zeta_\gamma \} \) is \( \{-2.91, 2.0, 3.5, -0.155, 0.045, 1.2, 1.15, 1.3\} \).

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