Exact velocity of dispersive flow in the asymmetric avalanche process

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(March 22, 2022)

Using the Bethe ansatz we obtain the exact solution for the one-dimensional asymmetric avalanche process. We evaluate the velocity of dispersive flow as a function of driving force and the density of particles. The obtained solution shows a dynamical transition from intermittent to continuous flow.

PACS numbers:64.60.Lx,05.40.+j,47.55.Mh

I. INTRODUCTION AND THE MODEL DEFINITION.

The avalanche dynamics is a basic scenario of relaxation of unstable states in extremal systems where each movable element is near a border of stability. A typical long-tailed distribution of avalanche sizes leads to the dispersive transport of particles [1]. As an illustrative example, granular systems exhibit intermittent avalanches which enables one to use granular piles (sand piles, rice piles) for explanation of self-organized criticality in generic dissipative systems [2]. In the past decade, it has become clear that the dispersive transport can be recast in terms of interface depinning [3,4] and various growth models [5]. Recently, a dynamical transition from intermittent to continuous flow in a random sandpile model has been revealed [6]. Nevertheless, an explicit theoretical description of stochastic avalanche processes and an exact evaluation of characteristics of dispersive flow remains an open problem.

Despite drastic simplifications which were introduced to mimic real avalanches, exact results are scarce even for the deterministic dynamics. As to stochastic dynamics, it is es-
especially difficult as it is beyond the class of abelian models\cite{7}, where asymmetric processes appear to be solvable\cite{8}. The situation is to be compared with the theory of exclusion processes where many properties, such as steady states, average current, diffusion constant etc. have been calculated for an asymmetric one-dimensional case \cite{9,12}. The usual presentation of the asymmetric exclusion process (ASEP) is given by a master equation for the probability \( P_t(x_1, ..., x_P) \) of finding \( P \) particles at time \( t \) on sites \( x_1, ..., x_P \) of a ring consisting of \( N \) sites. During any time interval \( dt \), each particle jumps with probability \( dt \) to its right if the target site is empty. This elementary restriction leads to a non-trivial problem of evaluation of the steady state properties, which can be solved by the Bethe ansatz.

In a similar way, the simplest asymmetric avalanche process (ASAP) can be formulated as follows. In a stable state, each of \( N \) sites on a ring is either occupied by one particle or empty. The total number of particles \( P \) is fixed. During time interval \( dt \), each particle jumps with probability \( dt \) to its right. In the course of time, some site \( x \) may get unstable with occupation number \( n > 1 \). Then it must relax immediately to the stable state by transferring to its right either \( n \) particles with probability \( \mu_n \) or \( n - 1 \) particles with the probability \( 1 - \mu_n \). The quantity \( \mu_n \) can be associated with a driving force acting on the unstable group of \( n \) particles.

The main difference between the ASEP and ASAP lies in the depth of reconstruction of a configuration \( C = \{x_1, ..., x_P\} \) during the time interval \( dt \). In the ASEP, the total distance \( Y_t \) covered by all particles between time 0 and \( t \) increases by 1 during \( dt \) if the configuration \( C \) differs from a new one \( C' \) or remains unchanged if the motion is forbidden. In the ASAP, the motion of a particle is always possible and increase of \( Y_t \) is not bounded. Thus, the configuration \( C \) may be completely different from \( C' \) depending on numbers of particles spilled to right from each unstable site.

The present formulation of the ASAP is inspired by works\cite{13} where a model of activated random walks is introduced and\cite{13} where the directed avalanche dynamics is formulated in terms of continuous variables. Under an assumption about independence of variations of the avalanche size at each time step, the probability distribution of avalanche sizes is
found exactly [15], [16]. However, the configurational space of the continuous model is too complicated to determine steady state features.

Here, extending the Bethe ansatz approach to exclusion processes, we obtain the expression for the generating function of $Y_t$ for the discrete ASAP in the thermodynamic limit of large $N$ for a fixed density of particles $\rho = P/N$. We find two phases corresponding to a dispersive flow and a continuous flow, and evaluate the exact average velocity in the whole range of parameters of the first phase. We determine the separation line between two phases where avalanches are critical.

II. THE DYNAMICAL RULES AND THE AVERAGE VELOCITY.

Even before going into details of our calculations, we can get important restriction on the toppling rules of the ASAP. For the problem to be solvable by Bethe ansatz, one has to make sure that many-particle problem can be reduced to the problem of two interacting particles. In other words, the toppling probabilities $\mu_n$ of unstable configuration with $n > 1$ particles at the same site should be determined recursively in terms of two-particle toppling probability $\mu_2$ only. Namely, the probability for $n$ particle to leave the site is the sum of two processes. In the first process two particles leave the site with probability $\mu_2$ and then $n - 2$ particles leave the site with probability $\mu_{n-2}$. The second process corresponds to spilling two particles with probability $1 - \mu_2$ when only one particle leave the site and then remaining $n - 1$ particles leave the site with probability $\mu_{n-1}$. Thus, we obtain the recursion relations which express all the probabilities trough the only constant $\mu$

$$\mu_1 = 0$$
$$\mu_2 = \mu,$$
$$\mu_n = \mu_{n-2}\mu + \mu_{n-1}(1 - \mu), \quad (1)$$

or in the form of one step recursion,

$$\mu_n = \mu(1 - \mu_{n-1}) \quad (2)$$
Although the exact solution of the Bethe anzats requires a long technical analysis, the average velocity of the dispersive flow can be obtained with minimal assumptions from simple combinatorial arguments. The average velocity of the particle flow in the ASAP is determined by the average number of steps of all particles involved into an avalanche during the time interval $t$ and can be written as

$$v = \frac{\langle Y_t \rangle}{Pt}$$  \hspace{1cm} (3)

The only assumption we use is that the probability for any site of the infinitely large lattice to be occupied does not depend on state of the other sites and is equal $\rho$. Let us consider the avalanche starting at the site $i$. To calculate the velocity, we introduce the probability $P_{i,j}(n)$ for $n$ particles to be transferred from the site $j$ to its right provided the avalanche is started at site $i$. Defining also the total probability to transfer exactly $n$ particles from any site during the whole avalanche $P(n) = \sum_{j=i}^{\infty} P_{i,j}(n)$ we can express the average velocity as follows

$$v = \sum_{n=1}^{\infty} nP(n)$$ \hspace{1cm} (4)

Using the translation invariance of the stationary state one can rewrite $P(n)$ as the sum of topplings of one site $i$ in all avalanches started on the left of the site $i$.

$$P(n) = \sum_{j=0}^{\infty} P_{i-j,i}(n).$$ \hspace{1cm} (5)

The dynamic rules of the model relate the values of $P(n)$ for two neighboring sites. Due to the translation invariance, $P(n)$ does not depend on site, and obeys the recurrent relations

$$P(n) = P(n-1)\rho\mu_n + P(n)(\rho(1 - \mu_{n+1}) + (1 - \rho)\mu_n) + P(n+1)(1 - \rho(1 - \mu_{n+1})),$$ \hspace{1cm} (6)

where we express $P(n)$ at the site $i$ via $P(n)$ at the site $i - 1$. For $P(1)$ we have different recurrent relation. The term with $P(n - 1)$ should be replaced by 1 which corresponds to the case when avalanche is starting at given site,

$$P(1) = 1 + P(1)(\rho(1 - \mu_2)) + P(2)(1 - \rho)(1 - \mu_2).$$ \hspace{1cm} (7)
To find the solution of these recurrent relations one should also fix the value of $P(1)$. If we want the system to be stationary we should make sure that the number of starting avalanches is equal to the number of dying ones. There is $P(1)(1 - \rho)$ avalanches ending at any site for each avalanche starting at the same site, what gives $P(1)(1 - \rho) = 1$. Then the solution of the system of the relations (6, 7) is

$$P(1) = \frac{1}{1 - \rho}$$

$$P(n) = \frac{\rho}{1 - \rho} \frac{\mu_n}{1 - \mu_n} P(n - 1),$$

or applying these relations recursively

$$P(n) = \frac{1}{\rho} \left( \frac{\rho}{1 - \rho} \right)^n \prod_{i=2}^{n} \frac{\mu_n}{1 - \mu_n}. \quad (10)$$

Substituting the expression for $P(n)$ into Eq.(4), and using the recursion (2), we get

$$v = \frac{1}{\rho} \sum_{n=1}^{\infty} \frac{n}{\mu_{n+1}} \left( \frac{\mu_0}{1 - \rho} \right)^n \quad (11)$$

As we shall see below, this formula coincides with that we obtain from the exact Bethe anzats solution. This means that our assumption about uncorrelated stationary state is valid in the thermodynamic limit.

### III. BETHE ANZATS SOLUTION.

Consider the ASAP consisting of $P$ particles on a ring of $N$ sites and denote by $P_t(C)$ the probability of finding at time $t$ the system in a configuration $C$. The probability $P_t(C)$ satisfies

$$\frac{d}{dt} P_t(C) = \sum_{C'} [M_0(C, C') + M_1(C, C') P_t(C')]$$

where $M_1(C, C')dt$ is the probability of going from $C'$ to $C$ during the time interval $dt$, and $M_0$ is a diagonal matrix

$$M_0(C, C) = - \sum_{C' \neq C} M_1(C', C) \quad (13)$$
Before using the Bethe ansatz, it is instructive to note that in the region where the distances between every two neighbouring particles exceed 1, the master equation (12) becomes “free”:

$$\frac{d}{dt}P_t(x_1, \ldots, x_P) = \sum_k e^{\gamma} P_t(x_1, \ldots, x_{k-1}, \ldots, x_P) - P_t(x_1, \ldots, x_k, \ldots, x_P)$$

(14)

where \(\exp(\gamma)\) is activity of a single step. To compensate the difference between (12) and (14) when \(x_k - x_{k-1} = 1\) for some \(k\), we introduce the boundary conditions

$$P_t(\ldots, x, x, \ldots) = e^{\gamma}(1 - \mu)P_t(\ldots, x - 1, x, \ldots) + e^{2\gamma}\mu P_t(\ldots, x - 1, x - 1, \ldots)$$

(15)

This condition can be viewed as the recurrent relation where the ”intermediate” probability of an unstable configuration \(P_t(\ldots, x, x, \ldots)\) is given in terms of another unstable configuration \(P_t(\ldots, x - 1, x - 1, \ldots)\) and so on. All boundary conditions for more then two particles can be reduced to the two-particle case. This implies a recurrent relation for the probability \(\mu_n\) which is nothing but the two particle reducibility (4) discussed above.

Now, we can define the ASAP by (14) and (15) instead of (12) without even knowing the exact form of the matrix \(M(C, C')\) which is very cumbersome for the ASAP model. Specifying a configuration \(C\) by positions \(1 \leq x_1 < x_2 < \ldots < x_P \leq N\) of the \(P\) particles, we use the Bethe ansatz for an eigenvector of the matrix \(M_0 + M_1\) in the form

$$e^{\lambda t} \sum_Q A_Q \prod_{j=1}^{P} z^{-x_j}_{Q(j)}$$

(16)

where the sum is over all of the permutations \(Q\) of \(1, 2, \ldots, P\). The condition (15) fixes the two particle S-matrix \(A_{ij}/A_{ji}\) as

$$\frac{A_{jk}}{A_{kj}} = \frac{1 - (1 - \mu)e^{\gamma}z_j - \mu e^{2\gamma}z_jz_k}{1 - (1 - \mu)e^{\gamma}z_k - \mu e^{2\gamma}z_jz_k}$$

(17)

Imposing the periodic boundary conditions gives the Bethe equations

$$z_k^{-N} = (-1)^{N-1} \prod_{j=1}^{P} \frac{1 - (1 - \mu)e^{\gamma}z_j - \mu e^{2\gamma}z_jz_k}{1 - (1 - \mu)e^{\gamma}z_k - \mu e^{2\gamma}z_jz_k}$$

(18)

The eigenvalue \(\lambda(\gamma)\) corresponding to (16) is

$$\lambda(\gamma) = -P + e^{\gamma} \sum_{i=1}^{P} z_i$$

(19)
The dependence of the eigenvalue on $\gamma$ allows one to use $\lambda(\gamma)$ as the large deviation function of $Y_t$. Specifically, the average velocity of the particle in ASAP is expressed through the derivative of $\lambda(\gamma)$.

$$v = \frac{1}{P} \frac{d\lambda(\gamma)}{d\gamma} \bigg|_{\gamma=0}$$

(20)

IV. THE BETHE EQUATIONS IN THERMODYNAMIC LIMIT.

The Bethe ansatz equations (18) together with (19) give the exact solution of the problem for all $N$ and $P$. The rest of the paper is devoted to evaluation of $v$ in the thermodynamic limit $N \to \infty$ for a fixed density of particles $\rho = P/N$. For a finite $N$, the largest eigenvalue $\lambda$ corresponds to the solution $\{z_j\}$ which converges to $z_j = 1, j = 1, \ldots, P$ as $\gamma \to 0$. For small $\gamma > 0$, the distance $|z_j - 1|$ grows rapidly with $N$ for all $j$ and becomes of order of 1 in the limit $N \to \infty$. Introducing a variable $\alpha$ by

$$z_j = \frac{1 - e^{i\alpha_j}}{1 + \mu e^{i\alpha_j} e^{-\gamma}}$$

(21)

and assuming the solutions $\{\alpha_j\}$ are distributed along a smooth curve in the complex plane $\alpha = (u + ir)$ with endpoints $(-a + ib)$ and $(a + ib)$, we obtain the Bethe equation in the form

$$p(\alpha) = 2\pi F(\alpha) + \frac{1}{2\pi} \int_{a+ib}^{a-ib} \theta(\alpha - \beta) R(\beta) d\beta - i\gamma$$

(22)

where we defined as usual a function $F(\alpha)$ such that $dF/d\alpha = -R(\alpha)/2\pi$ and $F(-a + ib) = -F(a + ib) = \rho/2$. The functions $p(\alpha)$ and $\theta(\alpha)$ are

$$p(\alpha) = -i \ln \left( \frac{1 - e^{i\alpha}}{1 + e^{i\alpha - 2i\nu}} \right)$$

(23)

and

$$\theta(\alpha) = -i \ln \left( \frac{\cosh(\nu + i\alpha/2)}{\cosh(\nu - i\alpha/2)} \right)$$

(24)

where $\nu = -\ln(\mu)/2$. Taking the derivative in (23), we get the integral equation for $R(\alpha)$.
\[- R(u, b) + \frac{1}{2\pi} \int_{-\alpha}^{\alpha} K(u - v) R(v, b) dv = \xi(u, b) \quad (25)\]

with

\[
\xi(\alpha) = \frac{\cosh \nu}{\sinh \nu - \sinh(\nu - i\alpha)} \quad (26)
\]

and

\[
K(\alpha) = \frac{\sinh 2\nu}{\cosh 2\nu + \cos \alpha} \quad (27)
\]

All that is very similar to the equations for the asymmetric 6-vertex model [17,18] (see also [19]) with an essential exception: both terms containing \(z_j\) and \(z_i z_j\) in (18) are negative, which is the reason for a dynamical transition, as we shall show below.

If \(a = \pi\), equation (25) can be solved by the Fourier transformation. To evaluate \(\partial_\lambda \lambda\), we have to find the solution of (25) in a vicinity of the point \(a = \pi\) which corresponds to a “conical” point, considered in [18]. Following Bukman and Shore, we write the solution \(R(u)\) as an expansion in \(\epsilon = \pi - a\) up to order of \(O(\epsilon^4)\)

\[
R(u) = R_0(u) + \epsilon^1 \delta R_1(u) + \epsilon^2 \delta R_2(u) + \epsilon^3 \delta R_3(u) \epsilon^4 \delta R_4(u) \quad (28)
\]

The necessity of such a long expansion will be seen in further calculations. The Fourier transformation is defined by

\[
X(u) = \sum_{n=-\infty}^{\infty} (X)_n e^{-iu} \quad (29)
\]

where \(X\) stands for \(R_0, \delta R_m, \xi, K\). The non-zero Fourier coefficients of \(K\) and \(\xi\), for \(b \geq -2\nu\), are

\[
(K)_n = (-1)^n e^{-2\nu|n|} \quad (30)
\]

\[
(\xi)_n = -e^{bn} (1 - (-1)^n e^{2n\nu}), n < 0 \quad (31)
\]

Then, (25) gives
\[ R_0(u) = (R_0)_0 + \frac{e^{iu-b}}{1-e^{iu-b}} \]  

(32)

In the next order in \( \epsilon \), we have

\[ (\delta R_1)_n (K_n - 1) = \frac{R_0(\pi)}{\pi} (-1)^n (K)_n \]  

(33)

so, that \( R_0(\pi) = 0, (\delta R_1)_n = 0 \) and \( (R_0)_0 = 1/(1 + \exp b) \). The next terms in (28) are evaluated in [18]

\[ \delta R_2(u) = -\frac{1}{6} R''_0(\pi) \]  

(34)

\[ \delta R_3(u) = -\sum_{n\neq 0} \frac{in(K)_n R'_0(\pi)}{3\pi (1-(K)_n^2)} e^{-inu} \]  

(35)

\[ \delta R_4 = -\frac{1}{120} R^{(4)}_0(\pi) \]  

(36)

Thus, in the expansion of \( R(u) \), \( \delta R_2(u) \equiv \delta R_2 \) and \( \delta R_4(u) \equiv \delta R_4 \) are real constants and \( \delta R_3(u) \) is imaginary.

Now, we are ready to start a direct evaluation of \( \partial_\gamma \lambda \) in (13) using \( \partial_\gamma \lambda = \partial_\alpha \lambda / \partial_\alpha \gamma \). First, we find \( \partial_\alpha \gamma \). To this end, we put \( \alpha = a + ib \) in (22), and take the derivative by \( a \) at \( a = \pi \). Recalling the conditions \( F(a + ib) = -\rho/2 \) and \( \theta(0) = 0 \), we obtain

\[ \pi \partial_\alpha \rho + i \partial_\alpha \gamma = R(a,b) + (2\pi)^{-1} \theta(2a) R(-a,b) + \\
(2\pi)^{-1} \int_a^a \theta(a - v) \partial_\alpha R(v,b) dv \]  

(37)

To evaluate r.h.s. of (37), we express the values \( R(a,b), R(-a,b) \) and \( \theta(2a) \) by their Tailor expansions at \( a = \pi \) up to the order of \( O(\epsilon^3) \) using (24) and (28). The integral in (37) is treated as

\[ \int_{-\pi+\epsilon}^{\pi-\epsilon} f(v) dv = \int_{-\pi}^{\pi} f(v) dv + B(\epsilon) \]  

(38)

with

\[ B(\epsilon) = \sum_{m=1}^{\infty} \frac{1}{M!} \{ (-\epsilon)^m f^{(m-1)}(\pi) - (\epsilon)^m f^{(m-1)}(-\pi) \} \]
and, therefore, can be evaluated by the Fourier transformation. The Fourier coefficients of \( \theta(\pi - v) \) are

\[
(\theta(\pi - v))_n = \frac{2\pi i}{n} (e^{-2\nu n} - (-1)^n), n \neq 0
\](39)

and \((\theta(\pi - v))_0 = 2\pi^2\). The only \(v\)-dependent part of \( \partial_a R(v, b) \) is \( \delta R_3(v) \). Using (30, 34-36), and (39), we obtain the explicit expression for r.h.s. of (37)

\[
\epsilon \pi R''_0(\pi) - \epsilon^2 \frac{R'_0(\pi^2)}{\pi} + \epsilon^3 \pi R''_0(\pi)
\]

Due to (32), the first and third terms in r.h.s. of (40) are real, the second one is imaginary. Therefore, we have

\[
\partial_a \gamma = i \epsilon^2 \frac{R'_0(\pi^2)}{\pi}
\]

(41)

The expression for \( \partial_a \lambda \) can be found in a similar way. In this case, we take the derivative by \( a \) in

\[
\frac{\lambda}{N} = \frac{1}{2\pi} \int_{-a}^{a} R(u, b)(z(u, b) - 1)du
\]

(42)

where \( z(u, b) \), according to (21), is

\[
z(u, b) = \frac{1 - e^{iu-b}}{1 + e^{iu-2\nu-b}}
\]

(43)

The obtained derivative is similar to (37) but contains \( z(v, b) \) instead of \( \theta(a - v) \). So, we need the Fourier coefficients of \( z(v, b) \) which are

\[
(z)_n = (1 + e^{2\nu})(-1)^n e^{(2\nu+b)n}, n < 0
\]

and \((z)_0 = 1\). Continuing as in (37), we get

\[
\frac{1}{N} \partial \lambda \partial_a = \epsilon^2 \frac{R'_0(\pi^2)z'(\pi, b)}{\pi} - 3\epsilon^2 \sum_{n=1}^{\infty} (z)_{-n}(\delta R_3)_n
\]

(45)

We can see that the term \( \delta R_3(u) \) in (28) is relevant. As to \( \delta R_4(u) \), it is sufficiently that it is a constant and does not lead to a divergency by integration.
Substituting the explicit expressions for \((z)_n\) and \((\delta R_3)_n\) gives the second term in r.h.s. of (45) in the form

\[
\epsilon^2 (1 + e^{2\nu}) R_0'(\pi) \sum_{n=1}^{\infty} \frac{(-1)^n e^{-(4\nu+b)n}}{1 - (-1)^n e^{-2nu}}
\]

Due to (41), \(R_0'(\pi)\) is cancelling in \(\partial_a \lambda / \partial_\alpha \gamma\). Then, using (44) and the identity \(\rho = (R_0)_0 = 1/(1 + \exp b)\) we obtain the final result

\[
v = \frac{(1 - \rho)(1 + \mu)}{(1 - \rho(1 + \mu))^2} + \frac{1 + \mu}{\mu \rho} \sum_{n=1}^{\infty} \frac{(-1)^n n \mu^2}{1 - (-1)^n \mu} \left( \frac{\rho}{1 - \rho} \right)^n,
\]

which exactly coincides with the formula (11) after some algebra.

The velocity of flow \(v\) diverges at \(\rho_c = 1/(1+\mu)\) which implies a transition to the phase of continuous flow. The value of critical density \(\rho_c\) can be easily understood from the condition of a balance between gaining \((\rho_c)\) and losing \((1 - q_\infty)\) one particle at each step of a large avalanche.

The considered model is a directed version of the model of activated random walks introduced in [13] to see how a conservative dynamical system with the sandpile toppling rules approaches criticality. It has been shown in [14] that the relaxation time \(\tau\) and correlation length \(\xi\) diverge as \(\tau \sim |\rho_c - \rho|^{-\nu_1}\) and \(\xi \sim |\rho_c - \rho|^{-\nu_2}\). The exponents \(\nu_1\) and \(\nu_2\) have been determined numerically for several kinds of toppling rules. In the directed case, \(\xi\) coincides with \(\tau\) and is proportional to the average size of avalanches \(\langle s \rangle\). On the other hand, \(\langle s \rangle = v\) so we have from (47) \(\langle s \rangle \sim (\rho_c - \rho)^{-2}\) and \(\nu_1 = \nu_2 = 2\). An extension of this result to the symmetrical case [20] is a very interesting and difficult problem.

V. ACKNOWLEDGMENTS.

This work is supported by the grant of Russian Foundation for Basic Research number 99-01-00882 and by the grant of Swiss National Science Foundation. One of us (VBP) is grateful to D.Dhar for helpful comments. VBP acknowledges T.Dorlas for reading the manuscript and DIAS, Dublin for hospitality.
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