FROM DOUBLE AFFINE HECKE ALGEBRAS TO QUANTIZED AFFINE SCHUR ALGEBRAS

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Abstract. We prove an equivalence between some category of modules of the double affine Hecke algebra of type $A$ and of the quantized affine Schur algebra.

Introduction

Let $F$ be a local non Archimedian field of residual characteristic $p$, $q$ the order of the residual field. Let $k$ be an algebraically closed field of characteristic $\ell$. Assume that $\ell = 0$, or $\ell > 0$ and $\neq p$. Let $G$ be a reductive group. Let $H$ be the affine Hecke algebra of $G$ over $k$. Cherednik has introduced a double affine Hecke algebra $H$, which may be viewed as an affine counterpart to $H$. It is natural to guess that $H$ takes some role in the representation theory of $kG(F)$. More precisely, let $B$ be the unipotent block in the category of smooth representations of $kG(F)$, i.e. the block containing the trivial representation. We expect $B$ to be equivalent to some category of representations of $H$.

The main result of this paper is a step in this direction. Assume that $G = GL_n$. Fix an Iwahori subgroup $I \subset G(F)$. Let $I_{\tau, \zeta}$ be the annihilator of the natural representation of the global Hecke algebra of $G(F)$ in $kG(F)/I$. The full subcategory $B' \subset B$ consisting of representations annihilated by $I_{\tau, \zeta}$ is an Abelian category. Let $Sc$ be the quantized affine Schur algebra of $G$ over $k$. Recall that $H, Sc$ are algebras over the ring $k[\tau^{\pm 1}, \zeta^{\pm 1}]$, while $H$ is an algebra over $k[\tau^{\pm 1}, \zeta^{\pm 1}]$. It is proved in [Vi] that $B'$ is equivalent to $(Sc_{\zeta = q})-Mod$. Note that $q$ is a root of unity in $k^\times$ if $\ell > 0$. Given $h_0, u_0 \in k$, set $\tau_0 = e^{u_0}$, $\zeta_0 = e^{u_0 h_0}$ and let $O_{\tau_0, \zeta_0} \subset (H_{\tau = \tau_0, \zeta = \zeta_0})-mod$ be the ‘category $O'$. If $k = \mathbb{C}$ and $u_0$ is generic we prove an equivalence between blocks of $O_{\tau_0, \zeta_0}$ and blocks of $(Sc_{\zeta = e^{h_0}})-mod$. The whole categories are not equivalent because they have different centers. To get the same categories one must either replace $Sc$ by some elliptic analogue (which is not known so far), or replace $O_{\tau_0, \zeta_0}$ by the corresponding category $O'_{h_0}$ of modules of the double affine graded Hecke algebra $H'$. See Section 5 for precise statements. We conjecture that our equivalence is still true if $k$ is any algebraically closed field of characteristic $\ell > 0$.

Roughly, the proof is as follows. We split $O_{\tau_0, \zeta_0}$ as a direct sum of subcategories $O_{\tau_0, \zeta_0} = \bigoplus \{\ell\} O_{\tau_0}. Each summand is equivalent to a category of modules, say $\{\lambda\} O_{h_0}$, of $H'$. The category $\{\lambda\} O_{h_0}$ is the limit of an inductive system of subcategories $\lambda O_{n, h_0}$ with $n \in \mathbb{Z}_{\geq 0}$. Although $\{\lambda\} O_{h_0}$ do not have enough projective objects, the categories $\lambda O_{n, h_0}$ are generated by a family of projective modules which

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are easily described. We construct an exact functor $\mathcal{M} : \lambda \mathcal{O}_{n,h_0} \to \text{mof}$ which is faithful on projective objects, under a mild restriction, using the trigonometric Knizhnik-Zamolodchikov connection. This functor is inspired from [GGOR]. In general we do not know how to compute the image by $\mathcal{M}$ of any projective generator. However, in some particular cases including the type $A$ case, this can be done via some deformation argument.

We may have proved our equivalence of categories with the geometric technics used in [V2]. From this viewpoint, one is essentially reduced to prove the injectivity conjectured in [V2, Remark 4.9]. By loc. cit., in type $A$, the simple object in $\mathcal{O}_{\widetilde{C}}$ are labelled by representations of a cyclic quiver, and the Jordan-Holder multiplicities of induced modules are the value at one of certain Kazhdan-Lusztig polynomials of type $A^{(1)}$. Our equivalence of categories may be viewed as an extension of these results. However, the present approach is more powerful in the sense that the $K$-theoretic construction does not adapt easily to the case of double affine Hecke algebras with several parameters.

1. Notations

1.1. Reminder on modules and categories. Let $k$ be a principal domain of characteristic zero. We will mainly assume that $k = A, F$ or $\mathbb{C}$, where $A = \mathbb{C}[[\tau]]$ and $F = \mathbb{C}((\tau))$. Let $k^\times \subset k$ be the multiplicative group. Given a $k$-algebra $A$, let $A$-$\text{Mod}$ be the category of left $A$-modules which are free over $k$, $A$-$\text{mod}$ be the full subcategory consisting of finitely generated modules, $A$-$\text{mof}$ be the full subcategory consisting of the modules of finite type over $k$.

Given an Abelian category $\mathcal{A}$ and a full subcategory $\mathcal{N} \subset \mathcal{A}$ stable under subquotients and extensions, let $\mathcal{A}/\mathcal{N}$ be the Serre quotient, see [G]. The category $\mathcal{A}/\mathcal{N}$ is Abelian and the obvious functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ is exact. Given an Abelian category $\mathcal{B}$ and an exact functor $F : \mathcal{A} \to \mathcal{B}$ such that $FM \simeq 0$ for all $M \in \mathcal{N}$, there is a unique exact functor $G : \mathcal{A}/\mathcal{N} \to \mathcal{B}$ such that $F = G \circ Q$. Conversely, given an Abelian category $\mathcal{C}$, an exact functor $Q : \mathcal{A} \to \mathcal{C}$ is called a quotient functor if and only if it induces an equivalence $\mathcal{A}/\ker Q \to \mathcal{C}$. Clearly, $Q$ is a quotient functor if and only if for any exact functor $F : \mathcal{A} \to \mathcal{B}$ such that $FM \simeq 0$ whenever $QM \simeq 0$ there is a unique exact functor $G : \mathcal{C} \to \mathcal{B}$ such that $F = G \circ Q$. If $\mathcal{A}$ is Artinian and $P \in \mathcal{A}$ is projective, the functor $\text{Hom}_{\mathcal{A}}(P, -)$ is a quotient functor from $\mathcal{A}$ to the category of right $\text{End}_A P$-modules of finite length.

1.2. Reminder on root systems. Let $\Delta$ be an irreducible root system. Let $\Delta_+ \subset \Delta$ be a system of positive roots, and $\Pi = \{\alpha_i ; i \in I\} \subset \Delta_+$ be the simple roots. Let $\theta \in \Delta_+$ be the maximal root, and $\rho = \frac{1}{2} \sum_{\beta \in \Delta_+} \beta$. The set of simple affine roots is $\{\alpha_I ; i \in \hat{I}\}$, where $\hat{I} = I \cup \{\emptyset\}$. For any subset $J \subset \hat{I}$ set $\Pi_J = \{\alpha_i ; i \in J\}$, $\Delta_J = \Delta \cap \mathbb{Z}\Pi_J$, and $\Delta_{J,+} = \Delta_+ \cap \Delta_J$. Let $\Delta^\vee$, $\Delta_{\emptyset}^\vee$, etc., denote the corresponding sets of coroots. Let $\Delta_{re} = \Delta \times \mathbb{Z}$, $\Delta_{\emptyset}^\vee = \Delta^\vee \times \mathbb{Z}$ be the sets of affine real roots and coroots.

Denote by $Y, Y^\vee$ the root and the coroot lattices, by $X, X^\vee$ the weights and the coweights lattices. Let $Y_+ \subset Y$ be the semigroup generated by $\Delta_+$, and write $Y_{\mathbb{R}}$ for $\mathbb{R}_{\geq 0} \otimes Y_+$.

Let $W, \hat{W}$ be the Weyl group and the affine Weyl group. Let $s_{\beta} \in W$ (resp. $s_{\emptyset} \in \hat{W}$) be the reflection relatively to the root $\beta \in \Delta$ (resp. $\emptyset \in \Delta_{re}$). We write $s_i$ for $s_{\alpha_i}$. Recall that $\hat{W} = Y \rtimes W$. We write $x_{\beta}$ for $(\beta, 0)$, and $s_{\emptyset}$ for $x_{\emptyset} s_{\emptyset}$. Let
\( \ell : W \to \mathbb{Z}_{\geq 0} \) be the length. We write \( \geq \) for the Bruhat order on \( \hat{W} \).

Let \( \Omega \subset \text{Aut}(\hat{W}) \) be the group of diagram automorphisms, and \( \hat{W} = W \times \Omega \) be the extended affine Weyl group. Then \( \hat{W} \cong X \ltimes W \). As above we write \( x_{\mu} \) for \( (\mu, 0) \). For each \( \pi \in \Omega \setminus \{1\} \) let \( \alpha_\pi \in \Pi \) be such that \( \pi(s_{\alpha_\pi}) = s_{-\alpha_\pi} \). Let \( \omega_\pi^\vee \) be the fundamental coweight dual to \( \alpha_\pi \), and \( \omega_\pi \) be the corresponding fundamental weight. We write \( \pi \) for the element \( (0, \pi) \in \hat{W} \). Let \( w_\pi \in W \) be such that \( \pi = \pi(0, w_\pi) \). Hence \( w_\pi \theta = -\alpha_\pi \), and \( w_\pi \alpha_i = \alpha_j \) if \( i, j \neq \circ \) and \( \pi(s_i) = s_j \).

Set \( X_k = k \otimes X \), \( X_k^\vee = k \otimes X^\vee \). The group \( W \) acts on \( X_k \) and \( X_k^\vee \) by \( s_\beta \lambda = \lambda - (\lambda : \beta') \beta \) and \( s_{\beta} \lambda \lambda^\vee = \lambda^\vee - (\beta : \lambda^\vee) \beta^\vee \), where \( (\cdot : \cdot) \) is the unique \( k \)-linear pairing \( X_k \times X_k^\vee \to k \) such that \( (\alpha_i : \omega_j^\vee) = \delta_{ij} \). We write \( \geq \) for the order on \( X_k \) such that \( \mu \geq \nu \) if and only if \( \mu - \nu \in Y_+ \).

Let \( S' \) be the symmetric algebra of \( X_k^\vee \). Given \( \lambda^\vee \in X^\vee \) set \( \xi_{\lambda^\vee} = 1 \otimes \lambda^\vee \in S' \). Put \( \xi_i = \xi_{\omega_i} \), and \( \xi_{\alpha_\pi} = 1 - \xi_{\theta_\pi} \). The group \( \hat{W} \) acts on the \( k \)-algebra \( S' \) by \( x_{\mu} \xi_{\lambda^\vee} = \xi_{w_\lambda \lambda^\vee} - (\mu : w_\lambda \lambda^\vee) \). The dual action on \( X_k \) is \( x_{\mu} \lambda = \mu + w_\lambda \). Put \( \xi_{\beta^\vee} = \xi_{\beta_\alpha \alpha_\beta} \) for \( (\beta : \beta') = (\beta, r) \in \hat{W} \). There is a unique \( \hat{W} \)-action on \( \hat{W} \) such that \( w_{\beta^\vee} \xi_{\beta^\vee} = \xi_{w_{\beta} \beta^\vee} \).

Set \( T = k^\times \otimes X^\vee \), and \( T^\vee = k^\times \otimes Y \). In the following \( \otimes \) means \( \otimes \), and \( e^\alpha \) means \( \exp(2ipz) \). For any \( \lambda \in X_k \) (resp. \( \lambda^\vee \in X_k^\vee \)) we write \( \lambda \) for the element \( (\lambda : \omega_j^\vee) \) in \( k \) (resp. \( \omega_j^\vee \) for \( \lambda^\vee \)) and \( e^\lambda \) for the element \( \prod_j e^{\lambda_j} \otimes \alpha_j \) in \( T^\vee \) (resp. \( e^{\lambda^\vee} \) for \( \prod_j e^{\lambda_j} \otimes \alpha_j \) in \( T \)).

Fix \( \tau \in k^\times \). Fix a \( m \)-th root \( \tau^{1/m} \) of \( \tau \), where \( m \) is the least natural number such that \( (X : X^\vee) = (1/m) \).

Set \( S = kX^\vee \). Given \( \lambda^\vee \in X^\vee \) let \( y_{\lambda^\vee} \) be the corresponding element in \( S \). Put \( y_i = y_{\omega_i} \), and \( y_{\alpha_\pi} = \tau y_{-\alpha_\pi} \). There is a unique ring isomorphism \( S \cong k[T] \) taking \( y_{\lambda^\vee} \) to the function \( z \otimes \gamma \mapsto \zeta(\gamma : \lambda^\vee) \). The group \( \hat{W} \) acts on the \( k \)-algebra \( S \) by \( x_{\mu} y_{\lambda^\vee} = y_{w_\lambda \lambda^\vee} \tau^{-1}(\mu : w_\lambda \lambda^\vee) \). The dual action on \( T^\vee \) is \( x_{\mu} \tau(z \otimes \gamma) = (z \otimes \tau \gamma)(\tau \otimes \mu) \).

Put \( y_{\beta^\vee} = y_{\beta_\alpha \alpha_\beta} \tau \) for \( (\beta : \beta') = (\beta, r) \in \hat{W} \). Set \( R = kY \). Given \( \beta \in Y \) let \( x_{\beta} \) denote also the corresponding element in \( R \). We write \( x_i \) for \( x_{\alpha_i} \). There is a unique ring isomorphism \( R \cong k[T] \) taking \( x_{\beta} \) to the function \( z \otimes \lambda^\vee \mapsto \zeta(\beta : \lambda^\vee) \). The group \( \hat{W} \) acts on the \( k \)-algebra \( R \) by \( w_{\beta} x_{\beta} = x_{w_{\beta} \beta} \). Let \( X_k \times R \to R \), \( (\xi, f) \mapsto \partial_\xi f \) be the unique \( k \)-linear action such that \( \partial_{\xi_{\lambda^\vee}} x_{\beta} = (\beta : \lambda^\vee) x_{\beta} \).

Given a root \( \beta \) let \( \partial_{\beta^\vee} : R \to R \), \( \partial_{\beta^\vee} : S' \to S' \) be the \( k \)-linear operators such that

\[
\partial_{\beta^\vee}(p) = \frac{p - sp}{\xi_{\beta^\vee}}, \quad \partial_{\beta^\vee}(f) = \frac{f - sf}{1 - x_{-\beta}}.
\]

1.3. Reminder on affine Weyl groups. For each subset \( J \subseteq \hat{I} \) let \( W_J \subset \hat{W} \) be the subgroup generated by \( \{s_i \mid i \in J\} \). It is finite. Let \( W_{J} \subseteq W \) be the set of elements \( \tau \) such that \( \ell(vu) = \ell(v) + \ell(u) \) for each \( u \in W_J \).

If \( \ell \in T^\vee \) we put \( \hat{W}_\ell = \{w \in \hat{W} \mid w\ell = \ell \} \) and \( W_\ell = W \cap \hat{W}_\ell \). If \( \lambda \in X_k \) we put \( \hat{W}_\lambda = \{w \in \hat{W} \mid w\lambda = \lambda \} \) and \( W_\lambda = W \cap \hat{W}_\lambda \). The group \( \hat{W}_\lambda \) is finite. If \( \tau \) is not a root of unity then \( \hat{W}_\ell \) is also finite. Let \( n_\lambda, n_\ell \) be the number of elements in \( \hat{W}_\lambda \), \( W_\ell \) respectively.

If \( k \) is an algebraically closed field the group \( W_\ell \) is generated by reflections, see [SS, Chap. II, Theorem 4.2].
2.1. The category \( \mathcal{O}' \)

Fix \( h_\beta \in k \) for each \( \hat{\beta} \in \hat{\Delta}_{\text{re}} \), such that \( h_\beta = h_{\alpha_i} \) if \( \hat{\beta} \in \hat{W}_{\alpha_i} \) and \( i \in \hat{I} \). We write \( h_i \) for \( h_{\alpha_i} \). Let \( H' \) be the degenerate double affine Hecke algebra. Recall that \( H' \) is the \( k \)-algebra generated by \( kW \) and \( S' \) with the relations

\[
(2.1.1) \quad s_i p - s_i p s_i = h_i \theta_{\alpha_i}(p),
\]

for all \( i \in \hat{I}, p \in S' \). Then

\[
(2.1.2) \quad \xi f - f \xi = \partial_\xi(f) - \sum_{\beta \in \Delta_i} h_\beta(\beta : \xi) \theta_\beta(f)s_\beta, \quad \forall f \in R, \forall \xi \in X'_{\text{k}}.
\]

The product yields an isomorphism \( R \otimes_k kW \otimes_k S' \to H' \). There is a unique action of \( \Omega \) on \( H' \) by algebra automorphisms such that \( \pi \in \Omega \) acts on \( \hat{W} \) and \( S' \) as in 1.2.

Let \( \mathcal{O}' \subset H'_{\text{-mod}} \) be the full subcategory consisting of the modules which are locally finite with respect to \( S' \). To avoid some ambiguity we may write \( H'_k \) for \( H' \).

Set \( \langle \mu \rangle = \{ p - p(\mu) ; p \in S' \} \), and \( \langle E \rangle = \bigcap_{\mu \in E} \langle \mu \rangle \) for each finite subset \( E \subset \hat{W} \lambda \). Let \( (\lambda) \mathcal{O} \subset \mathcal{O}' \) be the full subcategory consisting of the modules \( M \) such that for each element \( m \in M \) there is a finite subset \( E \subset \hat{W} \lambda \) and an integer \( n > 0 \) with \( \langle E \rangle^n m = \{ 0 \} \).

**Proposition.** (i) \( \mathcal{O}' \subset H'_{\text{-Mod}} \) is a Serre subcategory.

(ii) If \( k \) is an algebraically closed field then \( \mathcal{O}' = \bigoplus_{\lambda} \{ \lambda \} \mathcal{O} \), where \( \lambda \) varies in a set of representatives of the \( \hat{W} \)-orbits in \( X'_{\text{k}} \).

**Proof.** Any object in \( \mathcal{O}' \) is finitely generated over \( R \), because \( W \) is finite, \( H' = R \cdot kW \cdot S' \) and a module in \( \mathcal{O}' \) is finitely generated over \( H' \) and locally finite over \( S' \). Hence the category \( \mathcal{O}' \) is Abelian, because \( R \) is a Noetherian ring. The category \( \mathcal{O}' \) is obviously closed by subquotients and extensions.

For each module \( M \) in \( \mathcal{O}' \) let \( \{ \lambda \} M \subset M \) be the subspace consisting of the elements \( m \in M \) such that there is a finite subset \( E \subset \hat{W} \lambda \) and an integer \( n > 0 \) with \( \langle E \rangle^n m = \{ 0 \} \). Clearly \( \{ \lambda \} M \) lies in \( \{ \lambda \} \mathcal{O} \). We have \( M = \sum_{\lambda} \{ \lambda \} M \) because the \( S' \)-action on \( M \) is locally finite, and this sum is obviously direct. Claim (ii) follows. \( \square \)

For any group \( G \) acting linearly on \( X'_{\text{k}} \) and any \( \lambda \in X_{\text{k}} \), let \( [\lambda]_{G,k} \subset S' \) be the ideal generated by \( \langle \lambda \rangle^G \) (=the \( G \)-invariant elements in \( \langle \lambda \rangle \)). We write \( [\lambda] \) (or \( [\lambda]_k \) if necessary) for \( [\lambda]_{W_{\lambda,k}} \). Set \( S_{\lambda} = S' / [\lambda] \) (or \( S_{\lambda,k} \) if necessary). Note that \( S_{\lambda} \) is of finite type over \( k \) because \( \hat{W}_\lambda \) is finite, see [B, chap. V, §1, n° 9, Théorème 2]. If \( k \)
is a local ring then $S_λ$ is also a local ring. In this case let $m_λ \subset S_λ$ be the maximal ideal.

Let $E \subset \hat{W}λ$ be finite. Set $[E] = \cap_{μ \in E}[μ]$. The quotient $S_E = S'/[E]$ is of finite type over $k$. If $S_λ$ is free over $k$ then $S_E$ is also free because it embeds in $\bigoplus_{μ \in E} S_μ$ and $k$ is principal. If $k$ is a field then $S_E = \bigoplus_{μ \in E} S_μ$. If confusion is unlikely from the context we write $p$ again for the image in $S_E$ of an element $p \in S'$.

Let $λO \subset \{λ\}O$ be the full subcategory consisting of the modules such that for each element $m$ there is a finite subset $E \subset \hat{W}λ$ with $[E]m = \{0\}$. For a future use we prove the following technical lemma.

**Lemma.** If $S_λ$ is torsion free over $k$ there is a finite subset $F \subset \hat{W}λ$ containing $E$ such that $[F]s_i \subset [E] + [E]$ in $H'$.

**Proof.** Fix a finite subset $F \subset \hat{W}λ$ containing $E$ such that $s_i F = F$. We prove that $[F]s_i \subset [s_i E] + [F]$. For each $p \in [F]$ we have $p s_i = s_i p + \vartheta_λ(p)$ by (2.1.1). Hence we must prove that $\vartheta_λ(p) \in [F]$, i.e. that $\vartheta_λ([μ] ∩ [s_i μ]) \subset [μ] \cap [s_i μ]$ for each $μ \in F$

If $s_i μ = μ$ we are done because $[μ]$ is generated by $⟨μ⟩\hat{W}μ$, for all $p_1, p_2 \in S'$ we have $\vartheta_λ(p_1 p_2) = \vartheta_λ(p_1) p_2 + s_i p_1 \vartheta_λ(p_2)$, and $\vartheta_λ(⟨μ⟩\hat{W}μ) = \{0\}$.

Assume that $s_i μ \neq μ$. Fix $p \in [μ] \cap [s_i μ]$. It suffices to prove that $\vartheta_λ(p) \in [μ]$. We have $ξ_λ \vartheta_λ(p) = 0$ in $S_μ$. Let $K$ be the fraction field of $k$. Then $ξ_λ$ is invertible in $S_μ K$ because $S_μ K$ is a local ring and $ξ_λ \neq ⟨μ⟩$. Hence $\vartheta_λ(p) = 0$ in $S_μ$, because $S_μ$ is torsion free over $k$. □

**Remark.** Let $G$ be a finite group acting linearly on $X_C$. Fix $λ \in X_A$ whose image, $λ_0$, in $X_C$ is fixed by $G$. Then the algebra $R = S_A' / [λ]_{G,A}$ is free over $A$, and $R \otimes_A C = S'_{C}/[λ_0]_{G,C}$, because the graded ring associated to the decreasing filtration $(R \varpi^n)$ of $R$ is isomorphic to $(S_C'/[λ_0]_{G,C}) \otimes C A$. Indeed, for each $n$, the obvious map $S_A' \varpi^n \to S_{C}\varpi^n$ takes $S_A' \varpi^{n+1} + [λ]_{G,A} \varpi^n$ into $[λ_0]_{G,C} \varpi^n$, and the resulting ring homomorphism

$$S_A' \varpi^n / (S_A' \varpi^{n+1} + [λ]_{G,A} \varpi^n) \to (S_C' \varpi^n) / ([λ_0]_{G,C} \varpi^n)$$

is invertible (use averages over $G$).

### 2.2. Projective modules in $O'$. For each $μ \in \hat{W}λ$ we set $P(μ) = H'/H'[μ]$. To avoid some confusion we may write $P(μ)$ for $P(μ)_k$. Let $1_μ \in P(μ)$ be the image of the unity by the obvious projection $H' \to P(μ)$.

For a future use we set $M_μ = \{m \in M ; [μ]m = 0\}$ for each $H'$-module $M$. If $k$ is a field and $M$ lies in $λO$ then $M = \bigoplus_{μ \in \hat{W}λ} M_μ$, because for any $m \in M$ the map $S' \to M, p \mapsto pm$ factors through $S_E \to M$ for a finite set $E \subset \hat{W}λ$, and $S_E = \bigoplus_{μ \in E} S_μ$.

**Proposition.** Assume that $k$ is a field.

(i) $P(μ)$ is a projective object in $λO$.

(ii) The category $λO$ is generated by the modules $P(μ)$ with $μ \in \hat{W}λ$.

(iii) The category $O'$ is Artinian, and there are a finite number of simple objects in $λO$.

**Proof.** For each $w \in \hat{W}$ there is a finite subset $E \subset \hat{W}λ$ such that $[E]w \subset \sum_{w' \leq w} w' [μ]$ by Lemma 2.1. Then, $[E]w 1_μ = 0$ because $[μ]1_μ = 0$. Therefore $P(μ)$ belongs to $λO$. 
Given a map \( f : M \to N \) in \( \mathcal{O} \), we have \( f(M) = \bigoplus_{\mu} f(M_\mu) = \bigoplus_{\mu} f(M_\mu) \), and \( f(M_\mu) \subseteq f(M)_\mu \) for each \( \mu \). Therefore \( f(M_\mu) = f(M)_\mu \). Thus \( P(\mu) \) is projective because \( M_\mu = \text{Hom}_{\mathcal{H}'}(P(\mu), M) \) for each \( M \). Claim (i) is proved.

Each object \( M \) in \( \mathcal{O} \) is a quotient of a direct sum of modules isomorphic to some \( P(\mu) \), because \( M = \bigoplus_{\mu \in \widehat{W}} M_\mu \). Since \( M \) is finitely generated it is indeed the quotient of a finite direct sum of these modules. Claim (ii) is proved.

To prove that \( \mathcal{O}' \) is Artinian it is sufficient to check that \( P(\mu) \) has finite length over \( \mathcal{H}' \) for each \( \mu \). We have

\[
\upsilon \xi w - w \xi \in \sum_{w' < w} kw', \quad \forall \xi \in X_{k}', \, w \in \hat{W}.
\]

Let \( P(\mu) \subseteq P(\mu) \) be the right \( S_\mu \)-submodule spanned by \( \{w' 1_\mu ; w' \leq w\} \). Then \( P(\mu) \subseteq P(\mu) \) is a filtration of \( P(\mu) \) by \( S' \)-submodule, by (2.2.1). Let \( P(\mu) \) be the associated graded. The \( \mathcal{H}' \)-action on \( P(\mu) \) yields a \( k\hat{W} \times S' \)-action on \( P(\lambda)_\bullet \) by (2.2.1), where \( k\hat{W} \times S' \) is the semi-direct product relative to the \( k\hat{W} \)-action on \( S' \) in 1.2. It is sufficient to prove that \( P(\lambda)_\bullet \) has a finite length over \( k\hat{W} \times S' \). Note that \( P(\lambda)_\bullet \) is isomorphic to \( \bigoplus_{\mu \in \widehat{W}\lambda}(S_\mu)'^{\oplus \alpha_\mu} \) over \( S' \), and that a \( (k\hat{W} \times S') \)-submodule of \( P(\lambda)_\bullet \) is a sum of \( S' \)-submodules \( U_\mu \subseteq (S_\mu)'^{\oplus \alpha_\mu} \) such that \( w(U_\mu) = U_{w\mu} \) for all \( w \in \hat{W} \). Thus the length of \( P(\lambda)_\bullet \) is bounded by the length of \( (S_\lambda)'^{\oplus \alpha_\lambda} \) over \( S' \).

Hence it is finite because \( k \) is a field. By (ii), the last part of (iii) is a consequence of Proposition 2.3 below. \( \square \)

**Remarks.** (i) If \( k \) is a field, simple objects in \( \mathcal{O} \) have projective covers. However they do not have finite projective resolutions in general.

(ii) In general \( P(\mu) \) is not indecomposable over \( \mathcal{H}' \).

(iii) Assume that \( k \) is a field. Each object \( M \in \mathcal{O} \) has a filtration whose associated graded lies in \( \mathcal{O} \) (consider the submodule \( \{m \in M ; \exists E \text{s.t. } |E|m = 0\} \), which lies in \( \mathcal{O} \), and use the fact that \( M \) has a finite length). If \( k \) is algebraically closed and \( M \in \mathcal{O}' \) is simple then it lies in \( \mathcal{O} \) for some \( \lambda \in X_k \), because it lies in \( \mathcal{O} \) for some \( \lambda \in X_k \), hence it has a filtration whose associated graded lies in \( \mathcal{O} \).

2.3. **Intertwiners in \( \mathcal{O}' \).** Assume that \( k \) is a field. For any reduced decomposition \( w = s_{i_1}s_{i_2} \cdots s_{i_r} \in \hat{W} \) set \( \phi'_w = \phi'_1 \cdots \phi'_{r-1} \phi'_r \in \mathcal{H}' \), with \( \phi'_i = s_i \xi_{s_i} - h_i \) for all \( i \in \hat{I} \). Recall that \( wp\phi'_w = \phi'_w p \) for all \( p \in S' \). The intertwining operator \( \Phi'_i(\mu) : P(w\mu) \to P(\mu) \) is the unique \( \mathcal{H}' \)-homomorphism taking \( 1_{w\mu} \) to \( \phi'_w 1_{w\mu} \).

**Lemma.** The operator \( \Phi'_i(\mu) \) is invertible if and only if \( (\mu : \alpha'_i) \neq \pm h_i \). 

**Proof.** Set \( \psi'_i(\mu) = (\mu : \alpha'_i)s_i - h_i \). Let \( \Psi'_i(\mu) : P(s_i \mu) \to P(\mu) \) be the unique \( \mathcal{H}' \)-homomorphism taking \( wp \) to \( wp\psi'_i(\mu)p \) for each \( w \in \hat{W} \). The \( k \)-modules \( P(\mu)^{\geq k} = k\hat{W} \otimes_k (m_\mu)^k \), with \( k \geq 0 \), form a finite decreasing filtration of \( P(\mu) \).

We have \( \Psi'_i(\mu)(P(s_i \mu)^{\geq k}) \subseteq P(\mu)^{\geq k} \) for each \( k \), and \( \psi'_i(\mu) \), \( \phi'_i(\mu) \) coincide in the associated graded spaces. Hence \( \Phi'_i(\mu) \) is invertible if and only if \( \psi'_i(\mu) \) is invertible. The lemma follows. \( \square \)

For each \( \beta' \in \hat{\Delta}' \) we put \( H_{\beta'} = \{ \mu \in X_{\hat{R}} ; \xi_{\beta'}(\mu) = 0\} \). The connected components of \( X_{\hat{R}} \setminus \bigcup_{\beta' \in \hat{\Delta}'} H_{\beta'} \) are the alcoves. Let \( A_+ \) be the alcove containing \( \rho/k \) if \( k \gg 1 \), and \( A_w = \{w^{-1} \mu ; \mu \in A_+ \} \) for each \( w \in \hat{W} \).

The set \( \mathcal{H}_\lambda = \{ \beta' \in \hat{\Delta}' ; \xi_{\beta'}(\lambda) = \pm h_\beta \} \) is finite. Set \( U_\lambda = X_{\hat{R}} \setminus \bigcup_{\beta' \in \mathcal{H}_\lambda} H_{\beta'} \). The group \( \hat{W}_\lambda \) acts on \( U_\lambda \). An affine domain is a minimal subset in \( U_\lambda \) containing
a connected component and stable by $\hat{W}_\lambda$. Let $D_w$ be the unique affine domain containing $A_w$, and let $\mathcal{D}$ be the set of affine domains.

**Proposition.** (i) The $H'$-modules $P(w_1\lambda)$, $P(w_2\lambda)$ are isomorphic if $D_{w_1} = D_{w_2}$.

(ii) The modules $P(\lambda)$, $P(w\lambda)$ have the same composition factors for all $w \in \hat{W}$.

**Proof.** Fix $w \in \hat{W}$ and $i \in I$. The intertwining operator $\Phi'_{s_i}(w\lambda) : P(s_iw\lambda) \to P(w\lambda)$ is invertible if and only if $\xi_{w-1\alpha_i}(\lambda) \neq \pm h_i$. Thus $\Phi'_{w_1w_2}(w_2\lambda) : P(w_1\lambda) \to P(w_2\lambda)$ is invertible if $A_{w_1v_1}, A_{w_2v_1}$ are in the same affine domain for some $v_1, v_2 \in \hat{W}_\lambda$. This gives (i).

Fix $\lambda$, $w$. The modules $P(\lambda)$, $P(w\lambda)$ are isomorphic for generic parameters $h_i$ by (i). Hence (ii) follows by a standard argument, see [CG, Lemma 2.3.4] for instance. □

### 2.4. Induction

For any subset $J \subseteq \hat{I}$, the k-submodule $H'_J = kW_J \cdot S' \subset H'$ is a subring by (2.2.1). Set $\hat{H}' = H'_J$ and $\mathcal{O}' = \hat{H}'-mof$. For each $H'$-module $M$ let $\mathcal{I}(M) = H' \otimes_M \hat{H}'$. Put $P(\lambda) = \hat{H}'/H'[\lambda]$. Then $\mathcal{I}(P(\lambda)) = P(\lambda)$. If $M$ is finitely generated over $\hat{H}'$ then $\mathcal{I}(M)$ is finitely generated over $\hat{H}'$. If $M$ is locally finite over $S'$ then $\mathcal{I}(M)$ is also locally finite over $S'$ by (2.2.1), because $\mathcal{I}(M) \simeq kW \otimes_{kW} M$. Thus $\mathcal{I} \mathcal{O}$ factors through a functor $\mathcal{O}' \to \mathcal{O}'$.

### 2.5. The category $\mathcal{O}$

We do not assume anymore that $k$ is a field. Fix $\zeta_{\hat{\beta}} \in k^\times$ for each $\hat{\beta} \in \hat{\Delta}_{re}$, such that $\zeta_{\hat{\beta}} = \zeta_{\alpha_i}$ if $\hat{\beta} \in \hat{W}_\alpha$ and $i \in \hat{I}$. We write $\zeta_i$ for $\zeta_{\alpha_i}$. Let $H$ be the corresponding double affine Hecke algebra. It is the $k$-algebra generated by $S$ and the elements $t_w$ with $w \in \hat{W}$, modulo the following relations

$$(t_i - \zeta_i)(t_i + 1) = 0, \quad t_v t_w = t_{vw},$$

$$t_i y_{\lambda^\vee} - s_i y_{\lambda^\vee} t_i = (\zeta_i - 1)(y_{\lambda^\vee} - s_i y_{\lambda^\vee})(1 - y_{-\alpha_i^\vee})^{-1},$$

if $\ell(vw) = \ell(v) + \ell(w)$ and $t_i = t_{s_i}$. We may write $H_k$ for $H$ if necessary. There is a unique action of $\Omega$ on $H$ by algebra automorphisms such that $\pi(t_w) = t_{\pi(w)}$ for each $w \in \hat{W}$, and $\pi(p) = \pi p$ for each $p \in S$.

For any reduced decomposition $w = s_{i_1}s_{i_2} \cdots s_{i_r} \in \hat{W}$ set $\phi_w = \phi_{i_1}\phi_{i_2} \cdots \phi_{i_r} \in H$, with $\phi_i = t_i(y_{-\alpha_i^\vee} - 1) + \zeta_i - 1$ for all $i \in \hat{I}$. Recall that $w'\phi_w = \phi_w p$ for all $p \in S$.

Fix $\ell \in T^\vee$. Let $[\ell] = \{p - p(\ell); p \in S\}$. For any group $G$ acting on $T^\vee$, let $[\ell]_G \subset S$ be the ideal generated by $[\ell]^G$. We write $[\ell]$ (or $[\ell]_k$ if necessary) for $[\ell]_W, [\ell]_k$. Set $E = \{m \in E ; m \} = \{m \in E ; m \} = \{E \cap \ell \}$ if $E \subset \hat{W}$ is finite.

Let $\mathcal{O} \subset H-mod$ be the full subcategory consisting of the modules which are locally finite with respect to $S$. Let $G \subset \mathcal{O}$ (resp. $\hat{G} \subset \mathcal{O}$) be the full subcategory consisting of the modules which are locally finite with respect to $S$. For each element $m \in M$ there is a finite subset $E \subset \hat{W} \ell$ such that $E^n m = \{0\}$ if $n \gg 0$ (resp. such that $[E] m = \{0\}$).

If $k = \mathbb{C}$ we write $h_{0i}, \zeta_{0i}, \tau_0$ for $h_i, \zeta_i, \tau_i$. Given $u_0 \in \mathbb{C}^\times$ such that

$$(2.5.1) \quad u_0(k + (\lambda_0 : \beta^\vee)) \notin \mathbb{Z} \setminus \{0\}, \quad \forall \beta^\vee \in \Delta^\vee \cup \{0\}, \forall k \in Z + \sum Z \tau_0,$$

we set $\zeta_{0i} = e^{u_0 h_{0i}}$, $\tau_0 = e^{u_0}$. Let $\Gamma \subset \mathbb{C}^\times$ be the subgroup generated by $e^{u_0}$.

Fix $\ell_0 \in T^\vee$. The set $\Delta_{(\ell_0)}^\vee = \{\alpha^\vee \in \Delta^\vee ; y_{\alpha^\vee}(\ell_0) \in \Gamma\}$ is a root system. Let $\Delta_{(\ell_0)} \subseteq \Delta$ be the dual root system. Let $\hat{W}_{(\ell_0)}$ be the affine Weyl group associated to
Δ(ℓ0). Let $H'_{(ℓ_0)}$ be the degenerated double affine Hecke algebra generated by $\hat{W}_{(ℓ_0)}$ and $S'$, modulo the relation analogous to \((2.1.1)\), relatively to the set of parameters \{\(h_{i,0}, \beta_0, \tau_0\) \| \(h_{i,0}, \beta_0, \tau_0\) are as above. If \(\hat{W}_{(ℓ_0)}\) is generated by reflections there is an equivalence of categories \(\{ℓ_0\}O\simeq \{λ_0\}O_{(ℓ_0)}\). Moreover, if \(\Delta_{(ℓ_0)}\) is also the set of coroots \(α^\vee\) such that \((λ_0 : α^\vee)\in \mathbb{Z} + \sum_i \mathbb{Z} h_{i,0}\) then the categories \(\{λ_0\}O_{(ℓ_0)}\) and \(\{λ_0\}O\) are equivalent.

**Proof.** Claims \((i), (ii)\) are proved as in \(2.1.3\). Claim \((iii)\) is ‘well-known’, but there is no proof in the literature. It is proved as in \([L]\), to which we refer for details. The proof consists of two parts (corresponding to the two reductions in \([L]\)), the first of which being an isomorphism between some completion of \(H', H\) similar to Cherednik’s isomorphism.

\(A\) The rings \(S'/\langle E \rangle^n\), with \(E \subset \hat{W}_{(ℓ_0)} \cdot λ_0\) finite and \(n \geq 0\), form an inverse system. Let \(\{λ_0\}S_{(ℓ_0)}\) be the projective limit. Set also \(\{ℓ_0\}S_{(ℓ_0)} = \lim\limits_{\leftarrow} S/\langle E \rangle^n\), with \(E \subset \hat{W}_{(ℓ_0)} \cdot ℓ_0\) finite and \(n \geq 0\). We have \(\hat{W}_{(ℓ_0)} \cap \hat{W}_{(ℓ_0)} = \hat{W}_{(ℓ_0)} \cap \hat{W}_{(ℓ_0)}\), because for any element \(w \in \hat{W}_{(ℓ_0)}\) we have

\[wλ_0 = λ_0 \iff (wλ_0 : α^\vee) = (λ_0 : α^\vee), \forall α^\vee \in Δ_{(ℓ_0)}\]

\[\iff yα^\vee(wℓ_0) = yα^\vee(ℓ_0), \forall α^\vee \in Δ_{(ℓ_0)}\]

\[\iff wℓ_0 = ℓ_0\]

since \(u0 \notin \mathbb{Q}\). Hence there is a bijection \(\hat{W}_{(ℓ_0)} : ℓ_0 \simeq \hat{W}_{(ℓ_0)} : λ_0\) which is compatible with the \(\hat{W}_{(ℓ_0)}\)-actions. It yields a ring isomorphism \(\{λ_0\}S_{(ℓ_0)} \simeq \{ℓ_0\}S_{(ℓ_0)}\) which is compatible with the \(\hat{W}_{(ℓ_0)}\)-actions. Let \(K', \{λ_0\}K_{(ℓ_0)}, K, \{ℓ_0\}K_{(ℓ_0)}\) be the fraction fields of \(S', \{λ_0\}S_{(ℓ_0)}, S, \{ℓ_0\}S_{(ℓ_0)}\). Let \(H_{(ℓ_0)}\) be the double affine Hecke algebra corresponding to \(H'_{(ℓ_0)}\). Set

\[\{λ_0\}H_{(ℓ_0)} = \{λ_0\}S_{(ℓ_0)} \otimes SS' H'_{(ℓ_0)}, \quad \{ℓ_0\}H_{(ℓ_0)} = \{ℓ_0\}S_{(ℓ_0)} \otimes SS H_{(ℓ_0)}\]

Set also

\[\{λ_0\}H_{K(ℓ_0)} = \{λ_0\}K_{(ℓ_0)} \otimes SS' H'_{(ℓ_0)}, \quad \{ℓ_0\}H_{K(ℓ_0)} = \{ℓ_0\}K_{(ℓ_0)} \otimes SS H_{(ℓ_0)}\]

For each \(w \in \hat{W}\) let \(ϕ'_w \in K'ϕ_w\) (resp. \(ϕ_w \in Kϕ_w\)) be normalized so that the map \(w \mapsto ϕ'_w\) (resp. \(w \mapsto ϕ_w\)) is a group homomorphism. The intertwiner \(ϕ_w\) is denoted by \(G_w\) in \([C2]\). An element in \(\{λ_0\}H_{K(ℓ_0)}\) is a finite sum \(\sum_w p_wϕ'_w\) with \(w \in \hat{W}_{(ℓ_0)}\) and \(p_w \in \{λ_0\}K_{(ℓ_0)}\). By Lemma 2.1 there is a unique \(\mathbb{C}\)-algebra structure on \(\{λ_0\}H_{(ℓ_0)}\), \(\{λ_0\}H_{K(ℓ_0)}\) extending \(H'_{(ℓ_0)}\). Idem for \(\{ℓ_0\}H_{(ℓ_0)}\), \(\{ℓ_0\}H_{K(ℓ_0)}\). There is a unique ring isomorphism

\[\{λ_0\}H_{K(ℓ_0)} \rightarrow \{ℓ_0\}H_{K(ℓ_0)}\]
This isomorphism takes \( \{\lambda_0\} \mathbb{H}_{(\ell_0)} \) onto \( \{\ell_0\} \mathbb{H}_{(\ell_0)} \) (use (2.5.1) and see [L, Theorem 9.3] for details).

(B) Given \( \ell \in \hat{W}_{\ell_0} \), let \( \Delta_{(\ell)} \) be the root system dual to \( \{ \alpha^\vee \in \Delta^\vee \mid y_{\alpha^\vee}(\ell) \in \Gamma \} \).

Note that \( \Delta_{(\ell)} = \Delta_{(\ell')} \) if \( \ell, \ell' \) belong to the same \( \Gamma \otimes Y \)-coset. The elements \( \ell, \ell' \) are said to be equivalent if they belong to the same \( \Gamma \otimes Y \)-coset and to the same \( \hat{W}_{(\ell)} \)-orbit. Let \( \mathcal{P} \) be the set of equivalence classes in \( \hat{W}_{\ell_0} \). We write \( \ell \in \mathcal{P} \) for the class of \( \ell \), i.e. \( \ell = \hat{W}_{(\ell)} \ell \). The group \( \hat{W} \) acts transitively on \( \mathcal{P} \), because \( \Delta_{(w\ell)} = w(\Delta_{(\ell)}) \) for all \( w \in \hat{W} \). The stabilizer of \( (\ell_0) \in \hat{W} \) is \( \hat{W}_{(\ell_0)} \hat{W}_{\ell_0} \). If \( s_\beta \in \hat{W}_{\ell_0} \) then \( y_{\beta^\vee}(\ell_0) = 1 \), hence \( \beta \in \Delta_{(\ell_0)} \). Thus \( \hat{W}_{\ell_0} \subseteq \hat{W}_{(\ell_0)} \), because \( \hat{W}_{\ell_0} \) is generated by reflections. Therefore the stabilizer of \( (\ell_0) \) in \( \hat{W} \) is \( \hat{W}_{(\ell_0)} \). Set \( \{\ell_0\} \mathcal{S} \) (resp. \( \{\ell_0\} \mathcal{S}_{(\ell)} \)) equal to the projective limit \( \lim \mathcal{S}/\langle E \rangle^n \) with \( E \subset \hat{W}_{\ell_0} \) (resp. \( E \subset (\ell) \)) finite and \( n \geq 0 \). Hence \( \{\ell_0\} \mathcal{S} \simeq \prod_{\ell \in \mathcal{P}} \{\ell_0\} \mathcal{S}_{(\ell)} \). The tensor product \( \{\ell_0\} \mathcal{H} = \{\ell_0\} \mathcal{S} \otimes \mathcal{S} \mathcal{H} \) is a ring. The ring \( \{\ell_0\} \mathcal{S}_{(\ell)} \) is a direct summand in \( \{\ell_0\} \mathcal{S} \). The identity in \( \{\ell_0\} \mathcal{S}_{(\ell)} \) is identified with an idempotent in \( \{\ell_0\} \mathcal{S} \), denoted by \( e_{(\ell)} \). The same computations as in [L, 8.13-16] yield a \( \{\ell_0\} \mathcal{S}_{(\ell)} \)-linear ring isomorphism

\[
\{\ell_0\} \mathbb{H}_{(\ell_0)} \to e_{(\ell_0)} \cdot \{\ell_0\} \mathbb{H} \cdot e_{(\ell_0)} \text{ such that } t_w \mapsto e_{(\ell_0)} t_w e_{(\ell_0)}, \ \forall w \in \hat{W}_{(\ell_0)}.
\]

By Proposition 7.2 and (A) we have a chain of equivalences

\[
\text{M}_\mathcal{P} \{\ell_0\} \mathbb{H}_{(\ell_0)} \to \text{mod} \to \{\ell_0\} \mathbb{H}_{(\ell_0)} \to \text{mod} \to \{\lambda_0\} \mathbb{H}_{(\ell_0)} \to \text{mod}.
\]

The rings \( \{\ell_0\} \mathbb{H}, \{\ell_0\} \mathbb{H}_{(\ell_0)}, \{\lambda_0\} \mathbb{H}_{(\ell_0)} \) are endowed with the topologies induced by the corresponding inverse systems, and \( \text{mod} \) is the category of smooth finitely generated modules, see 7.2 for the other notations. The restriction \( \{\ell_0\} \mathbb{H} \) yields an equivalence \( \{\ell_0\} \mathbb{H} \to \text{mod} \to \{\ell_0\} \mathcal{O} \). Similarly, the restriction \( \{\lambda_0\} \mathbb{H} \to \text{mod} \to \{\lambda_0\} \mathcal{O} \) are equivalent.

For each class \( (\ell) \in \mathcal{P} \) fix an element \( w_\ell \in \hat{W}_{(\ell)} \) such that \( (\ell) = w_\ell (\ell_0) \) and \( e_{(\ell)} \mathcal{S}_{(w_\ell)} \in \{\ell_0\} \mathbb{H} \cdot e_{(\ell_0)} \), see [L, 8.16]. We write \( \varphi_{(\ell)} \) for \( \varphi_{w_\ell} \). Let \( E_{(\ell),(\ell')} \) be the matrix with \( (\ell),(\ell') \)-th entry equal to \( h \) and all other entries equal to zero. The linear map

\[
\{\ell_0\} \mathbb{H} \to \text{M}_\mathcal{P} \{\ell_0\} \mathbb{H}_{(\ell_0)}, \ \varphi_{(\ell)} h \varphi_{(\ell')}^{-1} \to E_{(\ell),(\ell')} h,
\]

is an embedding of topological rings with a dense image, see [L, 8.16]. The restriction yields the desired equivalence \( \text{M}_\mathcal{P} \{\ell_0\} \mathbb{H}_{(\ell_0)} \to \text{mod} \to \{\ell_0\} \mathbb{H} \to \text{mod} \).

The last claim in (iii) is proved as in (B), see also [L, Section 8].

\[\Box\]

2.6. Intertwiners in \( \mathcal{O} \). Assume that \( k \) is a field. For any \( J \subseteq \hat{I} \) let \( \mathbb{H}_J = \bigoplus_{w \in W_J} t_w \mathcal{S} \subseteq \mathbb{H} \). It is a subring. We write \( \mathbb{H} \) for \( \mathbb{H}_I \), \( \mathcal{O} \) for \( \mathbb{H} \) mod, and \( [\ell] \) (or \( [\ell]_J \) if necessary) for \( [\ell]_{W_{J,K}} \). Set \( S_\ell = \mathcal{S}/[\ell], \ P(\ell) = \mathbb{H} \otimes \mathcal{S} S_\ell \), and \( 1_\ell = 1 \otimes 1 \in P(\ell) \).

Let \( \ell \mathbb{H} \) be the specialization of \( \mathbb{H} \) at the central character \( W\ell \in \text{Spec}(SW) \). Set \( \ell \mathcal{O} = \ell \mathbb{H} \) mod. The module \( P(m) \) lies in \( \ell \mathcal{O} \) for all \( m \in W\ell \) because \( \langle \ell \rangle W \subseteq [m] \) and \( \langle \ell \rangle W \) lies in the center of \( \mathbb{H} \). It is projective (as in Proposition 2.2(i)).
For each \( w \in W \) the intertwining operator \( \Phi_w(\ell) : D(\wp) \to D(\ell) \) is the unique \( H \)-homomorphism taking \( 1_{\wp} \ell \) to \( \wp_\ell \). The same argument as for Lemma 2.3 implies that \( \Phi_\wp(\ell) \) is invertible if and only if \( y_{\alpha^\vee}(\ell) \neq \zeta_{i}^{\pm 1} \).

The connected components of the set \( X_R \setminus \bigcup_{\beta^\vee \in \Delta^\vee} H_{\beta^\vee} \) are the chambers. Let \( C_+ \) be the chamber containing \( \pm \rho \), and \( C_w = \{ w^{-1} \mu ; \mu \in C_+ \} \).

Let \( \mathcal{H}_\ell = \{ \beta^\vee \in \Delta^\vee ; y_{\beta^\vee}(\ell) = \zeta_{i}^{\pm 1} \} \), and \( U_\ell = X_R \setminus \bigcup_{\beta^\vee \in \mathcal{H}_\ell} H_{\beta^\vee} \). The group \( W_\ell \) acts on \( U_\ell \). A domain is a minimal subset in \( U_\ell \) containing a connected component and stable by \( W_\ell \). Let \( D_w \) be the unique domain containing \( C_w \), and \( \mathcal{D} \) be the set of domains.

**Proposition.** (i) \( P(w_1 \ell), P(w_2 \ell) \) are isomorphic whenever \( D_{w_1} = D_{w_2} \).

(ii) Assume that \( W_\lambda = W_\ell \). There is a unique injection \( \| : \mathcal{D} \to \mathcal{D} \) such that \( D_{w_1}^\| = D_w \) if \( w_2 v = x_k w_1 \) and \( w \in W \), \( v \in W_\lambda \), \( k \in Y \) far enough inside \( C_+ \).

**Proof.** Claim (i) is immediate using the condition for the invertibility of the intertwining operator given above. Claim (ii) is easy and is left to the reader. \( \square \)

### 3. Reminder on Knizhnik-Zamolodchikov trigonometric connection

This section contains standard results on Knizhnik-Zamolodchikov trigonometric connection. See [GGOR] for the analogue in the rational case. The sheaf of regular functions on a scheme \( S \) is denoted by \( \mathcal{O}_S \). If \( S \) is smooth, the sheaf of differential operators on \( S \) is denoted by \( D_S \).

**3.1.** Assume that \( k \) is a field. Set \( T_0 = \{ x_\beta \neq 1 ; \forall \beta \in \Delta \} \subset T_0 \). Let \( D_0 \) be the ring of algebraic differential operators on \( T_0 \). For each \( j \in I \) set

\[
D_j = \partial_{\xi_j} - \sum_{\beta \in \Delta_+} h_\beta \beta_j \theta_\beta + \hat{\rho}_j \in D_0 \text{ with } \hat{\rho} = \frac{1}{2} \sum_{\beta \in \Delta_+} h_\beta \otimes \beta.
\]

Put \( R_0 = k[T_0] \), and \( H'_0 = R_0 \otimes_R H' \). Set \( \theta_\beta = (1 - x_{-\beta})^{-1} \otimes (1 - s_\beta) \in H'_0 \).

**Lemma.** (i) There is a unique \( k \)-algebra structure on \( H'_0 \) extending \( H' \).

(ii) There is a unique ring isomorphism \( D_0 \times kW \to H'_0 \) such that \( \partial_{\xi_j} \mapsto \nabla_j := \xi_j + \sum_{\beta \in \Delta_+} h_\beta \beta_j \theta_\beta - \hat{\rho}_j \), \( f \mapsto f \), \( w \mapsto w \) for all \( j \in I \), \( w \in W \) and \( f \in R_0 \).

**Proof.** (i) is immediate because \( H' \) is a free left \( R \)-module and \( \{ f \in R ; f(T_0) = 0 \} \) is a left Ore set in \( H' \) by (2.1.2). Let us check (ii). Observe that \( D_j \) preserves the subspace \( R \subset R_0 \). Identifying \( R \) with the module \( H' \otimes_{H'} k \) induced from the trivial representation of \( H' \) on \( k \), we get a representation of \( H' \) on \( R \) such that \( w(g) = wg \), \( \xi_j(g) = D_j(g) \) and \( f(g) = fg \) for each \( f, g \in R \) and \( w \in W \). This action extends obviously to an action of \( H'_0 \) on \( R_0 \).

Hence there is a ring homomorphism \( H'_0 \to D_0 \times kW \) such that \( \xi_j \mapsto D_j \), \( f \mapsto f \), \( w \mapsto w \). It is obviously surjective. It is also injective because the representation of \( H' \) on \( R \) above is faithful, by a well-known lemma of Cherednik. \( \square \)

**3.2.** For each \( H' \)-module \( M \) we set \( M_0 = R_0 \otimes_R M \). Composing the localization \( O' \to H'_0 \otimes \mathbf{Mod}, M \to M_0 \), the isomorphism 3.1, and the sheafification \( D_0 \otimes kW \times kW \to D_{T_0} \otimes kW \times kW \), we get a functor \( \mathcal{L} : O' \to D_{T_0} \otimes kW \times kW \). For any \( M \) in \( O' \) the module \( D_{T_0} \otimes kW \)-module \( \mathcal{L}(M) \) is locally free of finite rank over \( O_{T_0} \); because \( \mathcal{L}(M) \) is a \( D_{T_0} \)-module which is coherent over \( O_{T_0} \) (since \( M \) is finitely generated over \( R \)).
3.3. Set \( k = \mathbb{C} \). Let \( z_i, i \in I \), be the obvious coordinates on \( \mathbb{C}' \). For any \( \beta \in \Delta \) we write \( z^\beta \) for \( \prod z_i^{\beta_i} \). Let \( D_\infty \subset \mathbb{C}' \) be the divisor \( \{ \prod_{l \in I} z_i = 0 \} \). The map \((x, \xi) : T \to \mathbb{C}' \) is an isomorphism onto \( \mathbb{C}' \setminus D_\infty \). Set \( D_\Delta = \bigcup_{\beta \in \Delta} \{ z^\beta = 1 \} \), and \( D = D_\infty \cup D_\Delta \). Then \( T_0 \) is identified with the open set \( \mathbb{C}' = \mathbb{C}' \setminus D \).

Let \( \mathbb{C}' \to \mathbb{C}'_0 / W, u \mapsto [u] \) be the obvious projection. Fix \( \odot \in (0,1)^I \), and \( \lambda^\odot \in X^\odot \) such that \( e^{\lambda^\odot} = \odot \). The fundamental group \( \Pi_1(\mathbb{C}'_0 / W, \{ \odot \}) \) is generated by the homotopy classes of the paths \( \gamma_j, \tau_j : [0,1] \to \mathbb{C}'_0 / W \) such that

\[
\gamma_j(t) = [\odot \cdot e^{t \omega^\odot}], \quad \tau_j(t) = [\odot \cdot e^{-t(\alpha_j : \lambda^\odot)}].
\]

It is isomorphic to the affine braid group \( B_\hat{W} \) associated to \( \hat{W} \), see [H, §2] for more details and references.

From now on we assume that \( k = A, F \) or \( \mathbb{C} \). For any finite dimensional \( C \)-vector space \( V \) we call holomorphic function \( \mathbb{C}' \to V((\tau)) \) a formal series \( \sum_{n \gg -\infty} a_n \tau^n \) where each \( a_n \) is a holomorphic function \( \mathbb{C}' \to V \).

Given a \( W \)-equivariant \( k \)-vector bundle \( V \) over \( \mathbb{C}' \) with a \( W \)-invariant integrable connection \( \nabla \), let \( V^\nabla \) be the set of \( W \)-invariant holomorphic horizontal sections of \( V \) over the simply connected cover \( \mathbb{C}' \) of \( \mathbb{C}'_0 \). It is a free \( k \)-module of rank equal to the rank of \( V \).

The group \( B_\hat{W} \) acts on \( V^\nabla \) by monodromy. The functor \( V \mapsto V^\nabla \) is exact, from the category of \( W \)-equivariant vector bundles on \( \mathbb{C}'_0 \) with a \( W \)-invariant integrable connection to \( kB_\hat{W} \)-mof. It restricts to an equivalence from the category of \( W \)-equivariant vector bundles on \( \mathbb{C}'_0 \) with a regular integrable \( W \)-invariant connection to \( kB_\hat{W} \)-mof.

If \( k = A \) we have \( \mathbb{C} \otimes_A V^\nabla = (\mathbb{C} \otimes_A V)^\nabla \) and \( F \otimes_A V^\nabla = (F \otimes_A V)^\nabla \).

3.4. Let \( \mathcal{M} : \mathcal{O}' \to kB_\hat{W} \)-mof be the functor \( M \mapsto \mathcal{L}(M)^\nabla \).

Lemma. Fix \( M, N \in \mathcal{O}' \).

(i) The canonical map \( \text{Hom}_{\mathcal{O}'}(M, N) \to \text{Hom}_{kB_\hat{W}}(\mathcal{M}(M), \mathcal{M}(N)) \) is injective if \( N \) is torsion-free over \( R \).

(ii) The \( D_{\mathbb{C}'_0} \)-module \( \mathcal{L}(M) \) has regular singularities along \( D \).

Proof. The restriction \( \text{Hom}_H(M, N) \to \text{Hom}_{H}(M_0, N_0) \) is injective if \( N \) is torsion free over \( R \). Assigning to a horizontal section on \( \mathbb{C}'_0 / W \) its value in the fiber at a given point is an injective map. Thus the map

\[
\text{Hom}_{\mathcal{O}'}(M, N) \to \text{Hom}_{kB_\hat{W}}(\mathcal{M}(M), \mathcal{M}(N))
\]

is injective. Claim (i) is proved.

Fix a \( H \)-module \( M \) in \( \mathcal{O}' \). The horizontal sections of \( \mathcal{L} \mathcal{L}(M) \) are the elements in \( R_\phi \otimes_k M \) annihilated by the operator \( \nabla_j \) for all \( j \in I \), see Lemma 3.1. Using (2.1.2) we get

\[
\nabla_j = \partial \xi_j \otimes 1 + 1 \otimes \xi_j - \sum_{\beta \in A_+} h_{\beta} \beta \partial \beta - \tilde{\rho}_j.
\]

Hence the elements of \( \mathcal{M} \mathcal{L}(M) \) are the \( \mathcal{W} \)-invariant maps \( \mathbb{C}'_0 \to M \) which are annihilated by the connection \( d - \sum_{j} A_j dz_j/z_j \), with

\[
A_j = \tilde{\rho}_j - \xi_j - \sum_{\beta \in A_+} h_{\beta} \frac{\beta_j z^\beta}{1 - z^\beta} (1 - s_\beta).
\]
This connection is the trigonometric Knizhnik-Zamolodchikov connection on the vector bundle $\mathcal{C}_0 \times_W M_0$ over $\mathcal{C}_0/W$. It has regular singularities along $D$ and at infinity.

The category of $\mathcal{O}_{\mathcal{C}_0}$-coherent $D_{\mathcal{C}_0}$-modules with regular singularities is stable by subquotients. Therefore $\mathcal{L}(M)$ has regular singularities for each $M \in \mathcal{O}$ by Proposition 2.2.(ii).

The category of $\mathcal{O}_{\mathcal{C}_0}$-coherent $D_{\mathcal{C}_0}$-modules with regular singularities is stable by extensions. Therefore $\mathcal{L}(M)$ has regular singularities for each $M \in \mathcal{O}$. Then, (ii) follows from Proposition 2.1.(ii). □

**Notation.** If $M \in \mathcal{O}$ we write $M^\vee$ for $\mathcal{M}(M)$.

4. MONODROMY

We fix a branch of the logarithm. Put $e^a = \exp(a \log(z))$ for any $a$. Set $k = \mathbb{C}$. Fix $\lambda_0 \in X_\mathbb{C}$ such that $\hat{W}_{\lambda_0} \subseteq W$. Set $\ell_0 = e^{\lambda_0}$ and $\zeta^{1/2} = e^{h_{\lambda_0}/2}$. Note that $\hat{W}_{\lambda_0}$ by Lemma 1.3, hence $\hat{W}_{\lambda_0}$ is generated by reflections. We assume that $\zeta_0 \neq 1$, $-1$ for each $i$.

**Theorem.** (i) $\mathcal{P}(w\lambda_0)^\vee = \mathcal{P}(w\ell_0)$ for all $w \in \hat{W}$, $w \in W$ with $D_w = D_w^\dagger$.

(ii) $\mathcal{M}$ factors through a functor $\mathcal{O} \to \mathcal{O}$.

(iii) $\mathcal{M}$ is fully faithful on $\mathcal{I}(\mathcal{O})$.

**Proof of (i).** Fix $\mu_0 \in \hat{W}_{\lambda_0}$ and $m_0 = e^{\mu_0}$. The computation of $\mathcal{P}(\mu_0)^\vee$ uses a reduction to the rank one case as in [C3]. To do so, we first deform $\mathcal{P}(\mu_0)\mathcal{C}$ over $A$. Then we fix a fundamental matrix solution over $F$. From now on $k = A, F$ or $\mathbb{C}$.

(A) Set $X_0 = \{e \in X_\mathbb{C}; (we)_j \neq (w'e)_j, \forall w \neq w' \in W, \forall j \in I\}$.

Put $\mu = \mu_0 + \omega e$, with $e \in X_0$. Set $Q = \hat{W}_{\mu_0} \cdot \mu$. From now on let $\nu$ denote any element in $Q$. Set $\mathbf{S}_{Q,A}$ as in 2.1. The ring $\mathbf{S}_{Q,A}$ is local. Let $\mathbf{m}_{Q,A}$ be the maximal ideal. Set $\mathbf{S}_{Q,k} = k \otimes_A \mathbf{S}_{Q,A}$ for $k = F$ or $\mathbb{C}$. We claim that $\mathbf{S}_{Q,C} = \mathbf{S}_{\mu_0,C}$. We have $(\mathbf{S}'_A/[\mu]\mathbf{W}_{\mu_0,A}) \otimes_C \mathbb{C} = \mathbf{S}_{\mu_0,C}$ by Remark 2.1. Hence the obvious surjective map $\mathbf{S}'_A/[\mu]\mathbf{W}_{\mu_0,A} \rightarrow \mathbf{S}_{Q,A}$ specializes to a surjective map $\mathbf{S}_{\mu_0,C} \rightarrow \mathbf{S}_{Q,C}$. The claim follows, because $\dim(\mathbf{S}_{\mu_0,C}) = \hat{n}_{\mu_0}$ by Chevalley’s theorem, and $\dim(\mathbf{S}_{Q,C}) = \hat{n}_{\mu_0}$ because $\dim(\mathbf{S}_{Q,F}) = \hat{n}_{\mu_0}$, since $W_\nu = \{1\}$, and $\mathbf{S}_{Q,A}$ is free over $A$ because $\mathbf{S}_{Q,A} \subset \bigoplus \mathbf{S}_{\nu,A}$ and $\mathbf{m}_{\nu,A} = A$.

Put $\mathbf{P} = \mathbf{H}' \otimes_{\mathbf{S}} \mathbf{S}_{Q}$. The module $\mathbf{P}$ lies in $\mathcal{O}$ and $\mathbf{P}_{\mathcal{C}} = \mathcal{P}(\mu_0)\mathcal{C}$. Let $Y_j, T_j$ be the monodromy operators on $\mathbf{P}^\vee$ along $\gamma_j, \tau_j$ respectively. The assignment $y_j \mapsto e^{\beta_j}Y_j, t_j \mapsto \zeta_j^{1/2}T_j$ extends uniquely to a representation of $\mathbf{H}_F$ on $\mathbf{P}^\vee$ by [C1, Proposition 8]. The canonical maps $F \otimes_A \mathbf{P}^\vee \rightarrow \mathbf{P}^\vee$ and $\mathbb{C} \otimes_A \mathbf{P}^\vee \rightarrow \mathbf{P}^\vee$ commute to the $B_W$-action. Therefore the representation of $B_W$ on $\mathbf{P}^\vee$ factors also through $\mathbf{H}_k$ if $k = A, \mathbb{C}$.

(B) Assume that

\begin{equation}
(\mu_0 : \beta^{\vee}) \in \mathbb{R}_{<0} + i\mathbb{R}, \quad \forall \beta^{\vee} \in \Delta^{\vee}_+.
\end{equation}
We first prove that \( P^\nabla_S \) is cyclic over \( \mathcal{H}_\mathcal{C} \). Then we prove that \( P^\nabla_S \simeq P(m_0)_\mathcal{C} \).

Set \( \psi_w = \phi'_w \otimes 1 \in P \) for each \( w \in W \). Hence \( \psi_w \in w\psi_1 \pi_w + \sum_{w' < w} w' \psi_1 S_Q \), where \( \pi_w \) is the product of all \( \xi_{\alpha'} \) with \( \alpha' \in \Delta_+ \cap w^{-1} \Delta_- \). The image of \( \pi_w \) in \( S_{Q,A} \) is invertible; we have \( \pi_w \notin m_{Q,A} \) because the image of \( \pi_w \) in \( S_{\mu_0,\mathcal{C}} \) does not lie in \( (\mu_0)_\mathcal{C} \) since \( \mu_0 \) is regular by (4.1.1). Thus \( (\psi_w) \) is a \( S_{Q,A} \)-basis of \( P^\nabla_A \).

The obvious right \( S_Q \)-action on \( P \) commutes to the left \( \mathcal{H} \)-action, thus \( A_j \) (= the connection matrix in 3.4) is \( S_Q \)-linear, hence \( P^\nabla \) is a \((\mathcal{H}, S_Q)\)-bimodule. If \( k \in \mathbb{Z} \) is non-zero, the image of the element \( t = k + w^{-1} \xi_j - w^{-1} \xi_j \) in \( S_{Q,F} \) is invertible, because \( S_{Q,F} = \bigoplus_{\nu} S_{\nu,F} \) and the projection of \( t \) in \( S_{\nu,F} \) is invertible (since \( w^{-1} \xi_j \in (w\nu)_j + \langle \nu \rangle \) and \( \epsilon \in X_0 \)). Put \( A_{j \nu} = \tilde{\rho}_j - \xi_j \). The element \( A_{j\nu} \in \mathcal{S}' \) is identified with its projection in \( S_Q \) whenever needed. Set \( z^{A_{j\nu}} = \prod_j z_j^{A_{j\nu}} \). There is a unique \( S_{Q,F} \)-basis \( (\psi_w^\nabla) \) of \( P^\nabla_F \) such that the function

\[
(\zeta_j) \mapsto \psi_w^\nabla(\zeta_j) \cdot z^{-B},
\]

where \( B = w^{-1} \mu_0 \), is holomorphic on \( \mathcal{C}^I \setminus D_\Delta \) and equals \( \psi_w \) at 0. By Proposition 7.1 there is also a \( S_{Q,A} \)-basis \( (b_w^\nabla) \) of \( P^\nabla_A \) such that

\[
(4.1.2) \quad b^\nabla_w \in \psi^\nabla_w + \sum_{w' \mu_0 \prec w \mu_0} \psi^\nabla_{w'} \cdot S_{Q,F}.
\]

Let \( b_w \) be the image of \( b^\nabla_w \) by the unique \( S_{Q,F} \)-linear isomorphism \( P^\nabla_F \rightarrow P_F \) such that \( \psi^\nabla_w \mapsto \psi_w \). Let \( P^\nabla_A \) denote also the \( S_{Q,A} \)-span of \( (b_w) \). The monodromy \( \mathcal{H}_A \)-action on \( P^\nabla_F \) yields a representation of \( \mathcal{H}_A \) on \( P_F \) which preserves \( P^\nabla_F \). For each \( \eta_0, \eta'_0 \in X_\mathcal{C} \) we write \( \eta_0 > \eta'_0 \) if \( \eta_0 - \eta'_0 \in Y_\mathcal{R}^+ \setminus \{0\} + iX_\mathcal{R} \). Note that \( w\mu_0 > w' \mu_0 \) if \( w > w' \), by (4.1.1), or if \( w\mu_0 > w' \mu_0 \). We claim that

\[
(4.1.3) \quad t_w b_1 \in b_w \cdot S^\times_{Q,A} + \sum_{w' \mu_0 \prec w \mu_0} b_{w'} \cdot S_{Q,A}, \text{ and } S_A b_1 = b_1 \cdot S_{Q,A}.
\]

The proof is given in (D). Then an easy induction implies that \( P^\nabla_A = \mathcal{H}_A b_1 \), hence that \( P^\nabla = \mathcal{H}_c b_1 \).

The series \( e^{\xi_j} = \sum_{k \geq 0} (2i\pi \xi_j)^k / k! \) converges in \( S_{Q,A} \), because \( \xi_j \in (\mu_0) + m_{Q,A} \) and \( m_{Q,A} \) is pronilpotent. Let \( S_A \rightarrow S_{Q,A} \), \( p \mapsto p(e^{\xi_j}) \) be the ring homomorphism such that \( y_j \mapsto e^{\xi_j} \). It is surjective. The map \( p \mapsto p(e^{\xi_j}) \) factors through a ring isomorphism \( S_{m_0,\mathcal{C}} \rightarrow S_{\rho_0,\mathcal{C}} \) (fix \( \kappa \in Y \) such that \( \mu_0 + \kappa \in W\lambda_0 \); then \( \tilde{W}_{\mu_0} = x_\kappa^{-1} \tilde{W}_{\mu_0} x_\kappa \), because \( \tilde{W}_{\mu_0} = \tilde{W}_{\mu_0 + \kappa} x_\kappa \) and \( \tilde{W}_{\lambda_0} = W_{\lambda_0} \); the claim follows since \( \dim(S_{m_0,\mathcal{C}}) = \dim(S_{\rho_0,\mathcal{C}}) \). In particular \( [m_0] b_1 = 0 \) by (4.1.4), because \( [\mu_0] b_1 = 0 \).

Therefore there is a unique surjective \( \mathcal{H}_c \)-linear map \( P(m_0)_\mathcal{C} \rightarrow P^\nabla \) such that \( 1_{m_0} \mapsto b_1 \). It is invertible because both modules have the same dimension over \( \mathbb{C} \).

(C) Fix \( \tilde{w} \in \tilde{W}, w \in W \) as in (i). We may assume that \( wv = x_{\kappa} w' \tilde{w} \) with \( w' \in W, v \in \tilde{W}_{\lambda_0} \), and \( \kappa \in Y \) far inside \( C_{-1} \), because \( D_{\tilde{w}} = D_{\tilde{w}}^\dag \). In particular the alcove \( A_{\tilde{w}} \) is far inside \( D_{\tilde{w}} \). Put \( \mu_0 = w' \tilde{w} \lambda_0 \). Then (4.1.1) holds. Thus \( P^\nabla = P(m_0)_\mathcal{C} \) by (A). We have also \( m_0 = w_{\lambda_0} \) because \( \mu_0 + \kappa = w_{\lambda_0} \). Thus \( P(w_{\lambda_0})_\mathcal{C} = P((w_{\lambda_0})_\mathcal{C} \), because \( \Phi_{w'}(w_{\lambda_0}) : P(\mu_0)_\mathcal{C} \rightarrow P(w_{\lambda_0})_\mathcal{C} \) is invertible (since \( A_{\tilde{w}} \) is far inside \( D_{\tilde{w}} \)).
(D) Let \( G : \mathbb{C}_0 \to \text{End}(P_F) \) be the fundamental matrix solution such that \( G \psi_w = \psi_w^\nabla \). We have \( G = Hz^{\Delta_0} \) with \( H : \mathbb{C}^l \setminus D_\Delta \to \text{End}(P_F) \) holomorphic such that \( H(0) = \text{Id} \), and \( Y_j = G(ze^{\omega_j'})^{-1}G(z), T_j = G(s_jz)^{-1}s_jG(z) \). Thus

\[
(4.1.4) \quad w^p \psi = \psi \cdot p(e^t), \quad \forall p \in S_F, \forall w.
\]

Hence the second part of (4.1.3) is immediate.

Let us prove the first part. We first claim that for each \( w \in W \) there is an invertible element \( p_w \in S_{Q,F} \) such that

\[
(4.1.5) \quad t_w \psi_1 \in \psi_w \cdot p_w + \sum_{w' < w} \psi_{w'} \cdot S_{Q,F}.
\]

To do so, observe that

\[
(4.1.6) \quad \psi_w \cdot S_{Q,F} = \{ \psi \in P_F ; w^p \psi = \psi \cdot p(e^t), \forall p \in S_F \}.
\]

Indeed, the direct inclusion is immediate, while the inverse one holds because \( P_F = \bigoplus_{w'} \psi_{w'} \cdot S_{Q,F} \) and, for each \( w \neq w' \), there is an element \( p \in S_F \) such that

\[
w^p(e^t) - w'(e^t) \in S_{Q,F}^\times
\]

(because \( S_{Q,F} = \bigoplus_{\nu} S_{\nu,F} \), and there is \( p \) such that \( p(we^\nu) \neq p(w'e^\nu) \) for each \( \nu \) since \( W_{we^\nu} = \{1\} \)). Then, (4.1.6) implies that \( \phi_w \psi_1 \in \psi_w \cdot S_{Q,F} \), and (4.1.5) follows.

Using (4.1.2) and (4.1.5) we get

\[
(4.1.7) \quad t_w b_1 \in b_w \cdot p_w + \sum_{w' \mu_0 < w \mu_0} b_{w'} \cdot S_{Q,A}
\]

for some \( p_w \in S_{Q,A} \cap S_{Q,F}^\times \). We must prove that \( p_w \in S_{Q,A}^\times \). We prove it by induction on \( \ell(w) \). Fix \( v \in W \) such that \( s_jv > v \). By (4.1.5) there is an element \( q \in S_{Q,F} \) such that

\[
(4.1.8) \quad t_j \psi_v \in \psi_{s_jv} \cdot q + \sum_{v' < s_jv} \psi_{v'} \cdot S_{Q,F}.
\]

By (4.1.2) and (4.1.7) we have \( q \in S_{Q,A} \). It is sufficient to prove that \( q \in S_{Q,A}^\times \). To simplify we write \( j \) for \( \{j\} \) and \( P_j \) for \( H'_j \otimes_{S_{vQ}} S_{vQ} \), where \( S_{vQ} \) is defined as \( S_Q \) in (A). From now on \( w \) is either \( v \) or \( s_jv \). Set \( \varphi_w = \phi^j_{w-1} \otimes 1 \in P_j \). Then \( (\varphi_w) \) is a \( S_{vQ} \)-basis of \( P_j \).

Let \( P^\nabla_j \) be the set of holomorphic functions \( f : \mathbb{C} \setminus \{0,1\} \to P_j \) such that

\[
z_j \partial_{z_j} f - A_{j0} f + h_{0j} z_j^\frac{1 - s_j}{1 - z_j} f = 0.
\]

It is a right \( S_{vQ} \)-module. Let \( Y'_j, T'_j \) the monodromy operators around 0 and 1. Since \( y_k \) lies in the center of \( H_j \) if \( k \neq j \), the assignement \( y_j \mapsto e^{\rho_j} Y'_j, t_j \mapsto \zeta^{1/2}_0 T'_j \) extends to a representation of \( H_j \) on \( P^\nabla_j \) such that \( y_k m = m \cdot e^{-1} \zeta_k \), for each \( k \neq j \) and each \( m \in P^\nabla_j \). Let \( G_j \) be the fundamental matrix solution such that
$G_j = H_j z_j^{A_j}$, with $H_j : \mathbb{C} \setminus \{1\} \to \text{End}(P_{j,F})$ holomorphic and $H_j(0) = \text{Id}$. Set $\varphi_w^\triangledown = G_j \varphi_w$. There is a unique $S_{Q,F}$-linear isomorphism $P_{j,F}^\triangledown \to P_{j,F}$ such that $\varphi_w^\triangledown \mapsto \varphi_w$. It yields a representation of $H_{j,F}$ on $P_{j,F}$.

Let $\theta_j : P_j \to \mathcal{P}$ be the $H_j'$-linear map such that $\varphi_w \mapsto \psi_w$. Note that $\theta_j(m \cdot v p) = \theta_j(m) \cdot p$ for each $m \in P_j$, $p \in S_Q$. We have

$$\theta_j(t_j \varphi_w) = \lim_{\varepsilon \to 0} \varepsilon D_j \circ t_j \circ \varepsilon^{-D_j}(\psi_w) \text{ with } D_j = \sum_{k \neq j} A_{k0},$$

because $\theta_j \circ G_j = \lim_{\varepsilon \to 0} (G \varepsilon^{-D_j})|_{C_\varepsilon} \circ \theta_j$ with $C_\varepsilon = \bigcap_{k \neq j} \{z_k = \varepsilon\} \subset \mathbb{C}$. Thus $t_j \varphi_v = \varphi_{s_j v} \cdot \psi_v$ modulo $S_{vQ}$, with $q$ as in (4.1.8), because $D_j(\psi_w) = \psi_w \cdot a$ for some element $a \in S_Q$ which is independent on $w \in \{v, \tau v\}$, and because $t_j \psi_v$ is a linear combination of the elements $\psi_{w'}$ with $w' \leq s_j v$ by (4.1.8). Therefore to prove (4.1.3) it suffices to check that

$$t_j \varphi_v \in \varphi_{s_j v} \cdot S_{vQ,A}^\times + \varphi_v \cdot S_{vQ,F}.$$

Since $S_{vQ,A} \subseteq \bigoplus_v S_{vQ,A}$, an element in $S_{vQ,A}$ is invertible if and only if its image in $S_{vQ,A}$ is invertible. There is a unique $H_j'$-linear map $P_j \to P_j(vv)$ taking $1 \otimes 1$ to $1 \otimes 1$. It commutes to the right actions of $S_{vQ}$ on $P_j$, and of $S_{uv}$ on $P_j(vv)$. Let $\tilde{\varphi}_w$ be the image of $\varphi_w$. Since $S_{vQ,A}$ is $A$ for each $v$, it is enough to prove that

$$t_j \tilde{\varphi}_v \in \varphi_{s_j v} \cdot A^\times + \varphi_v \cdot F.$$ (4.1.9)

Let $\Gamma$ be the gamma function. For each $z \in \mathbb{C} + \infty A^\times$ set $a(z) = (\zeta_{Qj}^{1/z} - \zeta_{Qj}^{-1/2})(e^z - 1)^{-1}$ and $b(z) = \Gamma(z) \Gamma(1 + z) \Gamma(h_{0j} + z)^{-1} \Gamma(1 - h_{0j} + z)^{-1}$. Then [C1, Theorem 10] yields

$$t_j \varphi_v = \varphi_{s_j v} \cdot b(-\gamma) + \varphi_v \cdot a(-\gamma),$$

with $\gamma = (vv : \alpha_j^\gamma)$. Note that $\gamma = (v\mu_0 : \alpha_j^\gamma) + \omega(v \varepsilon : \alpha_j^\gamma)$, where $(v\mu_0 : \alpha_j^\gamma) \notin \{0, \pm h_{0j}\} + \mathbb{Z}_{\geq 0}$ by (4.1.1) because $s_j v > v$. Thus $b(-\gamma) \in A^\times$, because $\Gamma$ does not vanish anywhere, and has a simple pole at each non positive integer. Hence (4.1.9) holds.

**Proof of (ii).** Set $k = F$. Fix $\lambda \in X_A$, $h_i \in A$ such that $(\lambda, h_i) = (\lambda_0, h_0i)$ modulo $\varpi$, and $(\lambda, h_i)$ is generic over $A$. Then $P(\mu)_F = P(\lambda)_F$ and $P(\lambda)_F^\Lambda = P(e^\lambda)_F$ for all $\mu \in \tilde{W}_\lambda$. Thus the $FB_{\lambda}$-action on $\mathcal{M}(M)$ factors through $\mathcal{H}$ for all $M$ in $^\lambda O_F$ by Proposition 2.2, yielding a functor $\mathcal{M} : {}^\lambda O_F \to {}^\Lambda O_F$.

Fix $k = A$, and $(\lambda, h)$ as above. For each $M$ in $^\lambda O_A$, $\mathcal{M}(M)$ is free over $A$, $\mathcal{M}(F \otimes_A M) = F \otimes_A \mathcal{M}(M)$, and $\mathcal{M}(F \otimes_A M) \in \mathcal{Q}_F$. Hence $\mathcal{M}(M) \in \mathcal{Q}_A$, thus $\mathcal{M}(C \otimes_A M) = C \otimes_A \mathcal{M}(M) \in \mathcal{Q}_C$.

Fix $k = \mathbb{C}$. Then $\mathcal{M}(^\lambda O) \subset \mathcal{Q}$. Therefore $\mathcal{M}(^\theta O) \subset \mathcal{Q}$, because an object in $^\theta O$ has a filtration whose associated graded lies in $^\lambda O$ and $\mathcal{M}$ is exact.

**Proof of (iii).** Fix $M, N \in \mathcal{Q}'. \text{ Since } \mathcal{I}(N) \text{ is torsion-free over } \mathcal{R} \text{ the natural map}$

$$\text{Hom}_{\mathcal{O}'}(\mathcal{I}(M), \mathcal{I}(N)) \to \text{Hom}_{B_{\mathcal{W}}}(M^{\triangledown}, N^{\triangledown})$$
is injective by Lemma 3.4.(i). The functor of horizontal sections yields an isomorphism
\[
\text{Hom}_{\mathbf{H}_c}(\mathcal{I}(M)_c, \mathcal{I}(N)_c) \to \text{Hom}_{\mathbf{B}_W}(M^\nabla, N^\nabla)
\]
by Lemma 3.1.(ii), Lemma 3.4.(ii). We must check that the restriction map
\[
\text{Hom}_{\mathbf{H}_c}(\mathcal{I}(M), \mathcal{I}(N)) \to \text{Hom}_{\mathbf{H}_c}(\mathcal{I}(M)_c, \mathcal{I}(N)_c)
\]
is surjective. An element \( f \in \text{Hom}_{\mathbf{H}_c}(\mathcal{I}(M)_c, \mathcal{I}(N)_c) \) is a horizontal \( W \)-invariant section of the bundle \( \text{Hom}_{\mathbf{R}_c}(\mathcal{I}(M)_c, \mathcal{I}(N)_c) \) over \( T_0 \). Given \( \beta \in \Delta_+ \), we expand \( f = \sum_{k \geq k_0} (1-z^\beta)^k f_k \) locally near a generic point of \( \{ z^\beta = 1 \} \), with \( f_k \) holomorphic on the divisor and \( f_{k_0} \) not identically zero. The residue of the connection on \( \{ z^\beta = 1 \} \) is constant and has eigenvalues \( 0, \pm 2h_{0\beta} \), see 3.4. Thus \( k_0 \geq 0 \) since \( 2h_{0\beta} \notin \mathbb{Z} \).

\[\square\]

**Remark.** Observe that \( P(\mu_0)^\nabla \neq P(e^{\mu_0}) \) in general. For instance, in type \( A_1 \), if \( \lambda_0 = \rho/2 \) and \( h_0 = 1/2 \) then \( e^{s_\rho \lambda_0} = \ell_0^{-1} \), and \( P(s_\rho \lambda_0)^\nabla = P(\ell_0) \neq P(\ell_0^{-1}) \). See 6.2 for more details.

### 4.2.

We do not know how to compute \( P(\mu_0)^\nabla \) for all \( \mu_0 \in \hat{W}\lambda_0 \). However we can prove a parabolic analogue to Theorem 4.1.(i) which is sufficient to recover the category \( \mathcal{O}' \) in type \( A \), see Section 5.

Fix a non-empty subset \( J \subseteq I \). The group \( W_J \) acts on \( \mathbf{H}' \) on the right by translations. The quotient is a left \( \mathbf{H}'_J \)-module which is naturally identified with \( S' \). Let \( O' \subseteq \hat{W}\lambda_0 \) be a finite subset such that \( W_J O' = O' \). The proof of Lemma 2.1 gives \( u[O'] \subseteq \sum_{i=0} \mathcal{O}'[u] \) for all \( u \in W_J \). Hence the ideal \( [O'] \subset S' \) is preserved by \( \mathbf{H}'_J \). Set \( P_J(O') = \mathbf{H}' \otimes \mathbf{H}'_J \mathcal{O}'_0 \), and \( 1_{O'} = 1 \otimes 1 \in P_J(O') \). The module \( P_J(O') \) lies in \( \mathcal{O}' \), and is generated by \( 1_{O'} \) over \( \mathbf{H}' \) with the defining relations \( [O']1_{O'} = 0 \) and \( W_J 1_{O'} = 1_{O'} \). If \( J \subseteq I \) then \( P_J(O') = P(J, P_J(O')) \), where \( P_J(O') = \mathbf{H}' \otimes \mathbf{H}'_J \mathcal{O}'_0 \).

From now on we assume that \( J \subseteq I \). Set \( C_{J,+} = \{ \mu_0 \in X_\mathbb{R} ; (\mu_0 : \alpha_j^\vee) = 0 \} \), \( (\mu_0 : \alpha_k^\vee) > 0 \), \( \forall j \in J, k \notin J \}. There is a unique representation of \( \mathbf{H}_J \) on \( S \) such that \( t_j 1 = \zeta_{0j} \) and \( S \) acts by multiplication. Set \( [E] \cap_{E \subseteq W_{\ell_0}} \) for any subset \( E \subseteq W_{\ell_0} \). If \( O \subseteq W_{\ell_0} \) is a \( W_J \)-orbit, the ideal \( [O] \subset S \) is preserved by \( \mathbf{H}_J \). Set \( P_J(O) = \mathbf{H}_J \otimes \mathbf{H}_J \mathcal{O}_0 \) and \( 1_O = 1 \otimes 1 \). The \( \mathbf{H}_J \)-module \( P_J(O) \) lies in \( \lambda_0 \mathcal{O}_0 \), and is generated by \( 1_O \) over \( \mathbf{H}_J \) with the defining relations \( [O]1_O = 0 \) and \( t_j 1_O = \zeta_{0j} 1_O \) for each \( j \in J \).

**Proposition.** (i) \( P_J(O') \) is projective in \( \lambda_0 \mathcal{O} \).

(ii) If \( m_0 \in W_{\ell_0}, \mu_0 \in \hat{W}\lambda_0 \) are such that \( e^{\mu_0} = m_0 \) and \( \mu_0 \in W\lambda_0 - \kappa \) with \( \kappa \in Y \) far enough inside \( C_{J,+} \), then \( P(W_J^\mu_0)^\nabla = P(W_J m_0) \).

**Proof.** Set \( M_O = \{ m \in M ; [O']m = 0 \} \) for each \( \mathbf{H}' \)-module \( M \). By Lemma 2.1 the subspace \( M_O \subset M \) is preserved by \( W_J \). Moreover \( \text{Hom}_{\mathbf{H}}(P_J(O'), M) = (M_O)^{W_J} \). Hence \( P_J(O') \) is projective in \( \lambda_0 \mathcal{O} \), because the functor \( M \mapsto (M_O)^{W_J} \) from \( \lambda_0 \mathcal{O} \) to vector spaces is exact. Claim (i) is proved.

The proof of (ii) is the same as the proof of Theorem 4.1.(i), to which we refer for notations and details. Set \( \mu_0 \in \hat{W}\lambda_0, O' = W_J : \mu_0, \) and \( O = W_J : m_0 \).

(A) We first prove that \( P_J(O')^\nabla \) is cyclic over \( \mathbf{H}_J \). To do so, we deform \( P_J(O')_C \). From now on \( k = A, F \) or \( \mathbb{C}, w, w' \in W, v, v' \in W_J, u \in W_J, \) and \( \nu_0, \nu'_0 \in O' \).
Put \( \mu = \mu_0 + \varepsilon \), with \( \varepsilon \in X_0 \). Set \( Q = W_j \hat{W}_{\mu_0} \cdot \mu \) and \( \nu_0 = Q \cap (\nu_0 + \varepsilon X_C) \). Let \( S_{Q,A}, S_{\nu_0,A} \) be as in 2.1. Set \( S_{Q,k} = k \otimes_A S_{Q,A} \) and \( S_{\nu_0,k} = k \otimes_A S_{\nu_0,A} \) if \( k = F, \mathbb{C} \). The ring \( S_{\nu_0,A} \) is local. Let \( m_{\nu_0,A} \) be the maximal ideal. We have \( S_{\nu_0,C} = S_{\nu_0,A}, \) see 4.1(A). Thus \( S_{Q,C} = S_{\nu_0,C} \), because \( S_{Q,A} = \bigoplus_{\nu_0} S_{\nu_0,A} \). Let \( \nu \) denote any element in \( Q \). The embedding \( S_{Q,A} \subseteq \bigoplus_{\nu} S_{\nu,A} \) is generically invertible.

The \( H'_j \)-action on \( S' \) descends to \( S_Q \) because \( W_j Q = Q \). Set \( \mathcal{P} = H'_j \otimes W_j S_Q \). The module \( \mathcal{P} \) lies in \( \mathcal{O}' \) and \( \mathcal{P}_C = \mathcal{P}_J(O' \cap C) \). Set \( \psi_v = \phi'_v \otimes 1 \in \mathcal{P} \). Assume that \( W_{\mu_0} \subseteq W_J \). We claim that \( \mathcal{P} = \bigoplus_v \psi_v \cdot S_Q \). It is enough to prove it for \( k = A \). Recall that \( \psi_v \in v^j \psi_1 \cdot \pi_v + \sum_{v' < v} v' \psi_1 \cdot S_{Q,A} \). The image of \( \pi_v \) in \( S_{Q,C} \) is invertible, because \( S_{Q,C} = \bigoplus_{\nu_0} S_{\nu_0,C} \) and \( \pi_v \notin m_{\nu_0,C} \) (indeed, \( \Delta_{s,+} \cap \Delta_{s,=} \neq \emptyset \), because \( v \in W_J \), hence \( \xi_{s,v} \neq 0 \) if \( \alpha \neq 0 \)). Therefore \( \psi_v \) is invertible in \( S_{Q,A} \). The claim follows.

If \( k \in \mathbb{Z} \) is non-zero the element \( t = k + v^{-\varepsilon} \xi_j - v^{-1} \xi_j \) is invertible in \( S_{Q,F} \), because \( S_{Q,F} = \bigoplus_v S_{v,F} \) and the projection of \( t \) in \( S_{v,F} \) is invertible (since \( \varepsilon \in X_0 \)). Thus there is a unique fundamental matrix solution \( G : C^I \rightarrow \text{End}(P_F) \) of the trigonometric Knizhnik-Zamolodchikov connection of the form \( G = H \sum \zeta^\alpha \), with \( H \) holomorphic on \( C^I \setminus D_\Delta \) and \( H(0) = Id \). It yields a \( F \)-linear isomorphism \( P_F^\nu \rightarrow P_F \). From now on we identify the \( F \)-vector spaces \( P_F, P_F^\nu \). The \( W \)-action on \( P_F^\nu \) factorizes through \( H \) by Theorem 4.1(ii). Thus \( P_F \) admits left actions of \( H' \) and \( H \), such that \( y_j = e^{\xi_j} \). Moreover \( P_F^\nu \subset P_F \) is a \( H_\alpha \)-submodule, and the canonical map \( C \otimes_A P_F^\nu \rightarrow P_F^\nu \) is an isomorphism of \( H_\alpha \)-modules.

We now fix \( \mu_0 \) as in (ii). Hence

\[
\nu_0 : \beta^v \in \mathbb{R}_{\leq 0} + i\mathbb{R}, \quad \forall \beta^v \in \Delta_+ \setminus \Delta_{s,+}, \forall \nu_0.
\]

In particular, \( W_{\mu_0} \subseteq W_J \). Assume that \( s_j v > v \) and \( s_j v \notin v W_J \). Hence \( s_j v \in W_J \). We claim that

\[
\forall p \in S_{Q,A}, \exists x \in H \text{ such that } x \psi_v \in \psi_{s_j v} \cdot p + \sum_{v' > s_j v} v' \psi_1 \cdot S_{Q,F}.
\]

We have \( S_A \psi_v = \psi_v \cdot S_{Q,A} \), because \( y_j \) acts as \( e^{\xi_j} \) in \( P_F \) and the ring homomorphism \( S_A \rightarrow S_{Q,A} \), \( p \mapsto p(e^{\xi}) \) is surjective. Therefore it is sufficient to prove that

\[
t_j \psi_v \in \psi_{s_j v} \cdot S_{Q,A}^\times + \sum_{v' < s_j v} v' \psi_1 \cdot S_{Q,F}.
\]

The same argument as for (4.1.5) implies that \( t_j \psi_v \in \sum_{v' < s_j v} \psi_{v'} \cdot S_{Q,F} \). Set \( P_j = H'_j \otimes W_j S_{Q,F} \). The ring \( H'_j \) acts on \( P_{j,F} \) by monodromy as in 4.1(D). Fix \( \varphi_v, \varphi_{s_j v} \in P_j \) as in 4.1(D). Let \( \theta_j : P_j \rightarrow P_j \) be the \( H'_j \)-linear embedding such that \( \varphi_v \mapsto \psi_w \) if \( w = v \) or \( s_j v \). Using \( \theta_j \) as in 4.1(D) we are reduced to prove that

\[
t_j \varphi_v \in \varphi_{s_j v} \cdot S_{Q,F}^\times + \varphi_v \cdot S_{Q,F}.
\]

An element in \( S_{Q,F} \) is invertible if and only if its image in \( S_{v,F} \) is invertible for each \( v \). The projection \( S_{Q,A} \rightarrow S_{v,A} \) yields a \( H'_j \)-linear map \( P_j \rightarrow P_j(v) \). Using this map we are reduced to prove (4.1.9) again. We have \( \nu^{-1} \alpha_j^v \in \Delta_+ \setminus \Delta_{s,+} \), because \( s_j v > v \) and \( s_j v \in W_J \). Hence \( \nu_0 : \alpha_j^v \notin \{0, \pm 1\} + \mathbb{Z}_{\geq 0} \) by (4.2.1). The claim (4.2.2) follows.
We now prove that (4.2.2) implies that $P^\gamma_C = H_C \psi_1$. If $\kappa$ is far enough inside $C_{I_+}$ there is an open convex cone $C \subset X_C \setminus \{0\}$ (i.e. $x+y, tx \in C$ for each $x, y \in C$ and $t \in \mathbb{R}_{>0}$) containing $Y_+ \setminus \{0\}$ such that $v \nu_0 - v' \nu'_0 \in C$ for each $v > v'$ and each $\nu_0, \nu'_0$. Given $\eta_0, \eta'_0 \in X_C$ we write $\eta_0 > \eta'_0$ if $\eta_0 - \eta'_0 \in C$. Then $v \nu_0 > v' \nu'_0$ if $v > v'$ or $v \nu_0 > v' \nu'_0$.

Fix a A-basis $(s_{\nu_0, t})$ of $S_{\nu_0, A}$ for each $\nu_0$. Write $\psi_{v, \nu_0, t}$ for $\psi_{v, s_{\nu_0, t}}$. By Proposition 7.1 there is a A-basis $(b_{v, \nu_0, t})$ of $P^\gamma_C$ such that

\[(4.2.4) \quad b_{v, \nu_0, t} \in \psi_{v, \nu_0, t} + \sum_{v' \nu'_0 < v \nu_0} \sum_{t'} \psi_{v', \nu'_0, t'} \cdot F.
\]

We first prove that $\psi_1 \in P^\gamma_C$. Since $\psi_1$ is a A-linear combination of the elements $\psi_{1, \nu_0, t}$, it suffices to check that $b_{1, \nu_0, t} = \psi_{1, \nu_0, t}$ for each $\nu_0, t$. By (4.2.4) it suffices to check that $v' \nu'_0 \not< \nu_0$ for each $\nu_0, v'$. If $v' \not= 1$ then $v' \nu'_0 - \nu_0 \in C$, hence $v' \nu'_0 \not< \nu_0$ because $(-C) \cap Y_+ = \emptyset$. If $v' = 1$ then $v'_0 - \nu_0 \not\in Y \setminus \{0\}$ because a direct computation, using $W_{\lambda_0} \subset W$ and $\nu_0 \in W \lambda_0 - \kappa$, yields $W_{\nu_0} \cap (Y \times W_J) \subset W_J$.

Given $\nu_0, t$ there is $x \in H_A$ such that $x \psi_1 \in \psi_{v, \nu_0, t} + \sum_{v' \nu'_0 < v} \psi_{v', \nu'_0} \cdot S_{Q, F}$ by (4.2.2) and an obvious induction on $\ell(v)$. Then

\[x \psi_1 \in b_{v, \nu_0, t} + \sum_{v' \nu'_0 < v \nu_0} \sum_{t'} b_{v', \nu'_0, t'} \cdot A.
\]

because $x \psi_1 \in P^\gamma_C$. Therefore $P^\gamma_C = H_A \psi_1$, hence $P^\gamma_C = H_C \psi_1$.

(B) Next, we prove that there is a unique surjective $H_C$-linear map $P_J(O)C \rightarrow P^\gamma_C$ such that $1_O \mapsto \psi_1$. To do so we must prove that $[O] \psi_1 = 0$, and $t_j \psi_1 = \zeta_{0j} \psi_1$ for all $j \in J$.

We have $W_{\nu_0} = x_\kappa^{-1} W_{\nu_0} x_\kappa$ for each $\nu_0$, because $\hat{W}_{\lambda_0} = W_{\ell_0}$ and $\nu_0 \in W \lambda_0 - \kappa$. Thus the map $p \mapsto p(e_\kappa)$ yields a ring isomorphism $S_{\nu, \nu_0, C} \rightarrow S_{\nu_0, C}$, see 4.1.(B). We have also a bijection of $W_J$-sets $O \simeq O'$, because $W_J \cap W_{m_0} = W_J \cap W_{\mu_0}$ (since $W_{m_0} = x_\kappa \hat{W}_{\mu_0} x_\kappa^{-1}$, and $W_J \cap x_\kappa \hat{W}_{\mu_0} x_\kappa^{-1} = W_J \cap W_{\mu_0}$ because $W_J$ centralizes $x_\kappa$). Hence the map $p \mapsto p(e_\kappa)$ yields a ring isomorphism $S_{O, C} \rightarrow S_{O', C}$. Hence $[O] \psi_1 = 0$, because $[O'] \psi_1 = 0$.

Assume that $j \in J$, $v = 1$. Then $t_j \psi_1 = \psi_1 \cdot p$ with $p \in S_{Q, F}$. We claim that $p = \zeta_{0j}$. For each $\nu$ the subspace $\psi_1 \cdot S_{(\nu, s_j, \nu)} \subset P$ is preserved by $H'_J$. Thus we are reduced to a computation in $S_{(\nu, s_j, \nu)}$ over $F$. The result follows from [C1].

There is a unique surjective $H_C$-linear map $P_J(O)C \rightarrow P^\gamma_C$ such that $1_O \mapsto \psi_1$. It is invertible because both modules have the same dimension (since $S_{O, C} \simeq S_{O', C}$).

\[\square\]

### 4.3.

Fix an integer $n > 0$, and a subset $J \subset I$. Given finite subsets $O \subset W_{\ell_0}$ and $O' \subset \hat{W} \lambda_0$ which are preserved by $W_J$, we put $S_{O', n} = S' / [O']^n$ and $S_{O, n} = S / [O]^n$. Set $P_J(O')_n = H'_J \otimes H'_J S_{O', n}$, $P_J(O)_n = P_J(O')_n$, and $P_J(O)_n = H \otimes H_J S_{O, n}$. Clearly $P_J(O')_n \in \{\lambda_0\} O$ and $P_J(O)_n \in \{\lambda_0\} O$. For each integer $n > 0$, let $\lambda^n O \subset \{\lambda_0\} O$ be the full subcategory consisting of the modules $M$ such that, for each $m \in M$, there is a finite subset $E \subset \hat{W} \lambda_0$ with $[E]^m = 0$. For a future use we need the following extension of 4.1-2.
2.2. Proof. For each $\mathbf{H}$-module $M$ we set $M_{O_n} = \{ m \in M ; [O'_n]m = 0 \}$. The functor $M \mapsto \{ m \in M ; [O'_n]m = 0 \}^W$ is exact on $\lambda^0O_n$ and is represented by $P_J(O'_n)$. Thus $P_J(O'_n)$ is projective in $\lambda^0O_n$. Claim (ii) is proved as in 4.2, replacing everywhere $S_Q$ by $S_{Q,n} = S'/[Q]^n$. The map in (iii) is injective by Lemma 3.4.(i) because $P_J(O'_n)\otimes \mathcal{O}$ is free over $\mathcal{O}$. Any projective and indecomposable module $N$ in $\lambda^0O_n$ is a direct summand of a module $P_u(\mu_0)$ with $\mu_0 \in W\lambda_0$, see Proposition 2.2.(ii). Since $P_u(\mu_0) \in \mathcal{I}(\mathcal{O})$, the functor $\mathcal{M}$ is fully faithful on the projective modules in $\lambda^0O_n$. Thus the map in (iii) is also surjective. □

5.1. Let $G^\vee$ be the simple simply connected and connected linear group whose weight lattice is $X$ and whose root system is $\Delta$. Thus $T^\vee$ is a maximal torus in $G^\vee$. Let $\mathfrak{g}^\vee$ be the Lie algebra of $G^\vee$ over $\mathbb{C}$.

Given $h_0 \in \mathbb{C}$, $\lambda_0 \in X_{\mathbb{C}}$ we set $\ell'_0 = e^{\lambda_0}$, $\zeta'_0 = e^{h_0}$, and

$$\ell'_0 N_{\zeta'_0} = \{ x \in \mathfrak{g}^\vee ; x \text{ is nilpotent and } \text{ad}(\ell'_0)(x) = \zeta'_0 x \}.$$ 

Let $\ell'_0 H \subseteq G^\vee(\mathbb{C})$ be the centralizer of $\ell'_0$. The group $\ell'_0 H$ acts on $\ell'_0 N_{\zeta'_0}$ by conjugation.

Given $u_0 \in \mathbb{C}^\times$ satisfying (2.5.1) we set $\zeta_0 = e^{u_0h_0}$, $\tau_0 = e^{u_0}$, $\ell_0 = e^{u_0\lambda_0}$, and

$$\ell_0 N_{\zeta_0,\tau_0} = \{ x(\varpi) \in \mathfrak{g}^\vee \otimes \mathbb{C} F ; x(\varpi) \text{ is nilpotent and } \text{ad}(\ell_0)(x(\tau_0\varpi)) = \zeta_0 x(\varpi) \}.$$ 

Let $\tilde{G}^\vee(F)$ be the Kac-Moody central extension of $G^\vee(F)$, and

$$\ell^0 H_{\tau_0} = \{ g(\varpi) \in \tilde{G}^\vee(F) ; \text{ad}(\ell_0)(g(\tau_0\varpi)) = g(\varpi) \}.$$ 

Note that $\ell^0 N_{\zeta_0,\tau_0} \subseteq \mathfrak{g}^\vee \otimes \mathbb{C} R$ and $\ell^0 H_{\tau_0} \subseteq \tilde{G}^\vee(R)$, where $R = \mathbb{C}[\varpi, \varpi^{-1}]$, because $\tau_0$ is not a root of unity. The group $\ell^0 H_{\tau_0} \times \mathbb{C}^\times \times \mathfrak{g}^\vee$ acts on $\ell^0 N_{\zeta_0,\tau_0}$: the first factor acts by conjugation, the second by ‘rotation of the loops’.

Lemma. The map $ev : \mathfrak{g}^\vee \otimes \mathbb{C} R \rightarrow \mathfrak{g}^\vee$, $x(\varpi) \mapsto x(1)$ factorizes through a bijection

$$\ell^0 N_{\zeta_0,\tau_0} / (\ell^0 H_{\tau_0} \times \mathbb{C}^\times) \rightarrow \ell^0 N_{\zeta'_0} / \ell^0 H.$$ 

Proof: We first claim that $ev$ restricts to an isomorphism $\ell^0 N_{\zeta_0,\tau_0} \rightarrow \ell^0 N_{\zeta'_0}$. Given $x(\varpi) \in \mathfrak{g}^\vee \otimes \mathbb{C} R$ we fix a decomposition $x(\varpi) = \sum_i x_i \otimes \varpi^k_i$, with $x_i \in \mathfrak{g}^\vee$, such that $x_i$ has the weight $\beta_i^\vee$ and the elements $x_i \otimes \varpi^{k_i}$ are linearly independent over $\mathbb{C}$. Using (2.5.1) we get

$$\text{ad}(\ell_0)(x(\tau_0\varpi)) = \zeta_0 x(\varpi) \Leftrightarrow (\lambda_0 + \beta_i^\vee) + k_i = h_0, \forall i.$$ 

In particular

$$\text{ad}(\ell_0)(x(\tau_0\varpi)) = \zeta_0 x(\varpi) \Rightarrow \text{ad}(\ell'_0)(x(1)) = \zeta'_0 x(1).$$
On the other hand, if $\text{ad}(\ell_0)(x) = \zeta_0 x$ and $x = \sum_i x_i$ with $x_i$ of weight $\beta_i^\vee$ and $\beta_i^\vee \neq \beta_j^\vee$ if $i \neq j$, then for each $i$ there is an integer $k_i$ such that $(\lambda_0 : \beta_i^\vee) + k_i = h_0$. Thus the element $x(\varpi) = \sum_i x_i \otimes \varpi^{k_i}$ satisfies

$$\text{ad}(\ell_0)(x(\tau_0 \varpi)) = \zeta_0 x(\varpi) \quad \text{and} \quad x(1) = x.$$ 

If $\text{ad}(\ell_0)(x(\tau_0 \varpi)) = \zeta_0 x(\varpi)$, $\text{ad}(\ell_0)(y(\tau_0 \varpi)) = \zeta_0 y(\varpi)$ and $x(1) = y(1)$, then, given decompositions $x(\varpi) = \sum_i x_i \otimes \varpi^{k_i}$, $y(\varpi) = \sum_j y_j \otimes \varpi^{k_j}$ as above, we get $\sum_i x_i = \sum_j y_j$, and $k_i = k_j$ whenever the weights of $x_i, y_j$ coincide. Thus $x(\varpi) = y(\varpi)$.

Obviously $x(1)$ is nilpotent if the element $x(\varpi)$ is nilpotent. Conversely assume that $x(1) \in \ell_0 \mathcal{N}_{\zeta_0}$ and $\text{ad}(\ell_0)(x(\tau_0 \varpi)) = \zeta_0 x(\varpi)$. Given $n > 0$ we set $y(\varpi) = \text{ad}(x(\varpi))^n \in \text{End}(g^\vee \otimes C \mathbb{R})$. Fix a decomposition $y(\varpi) = \sum_i y_i \otimes \varpi^{k_i}$, such that $y_i \in \text{End}(g^\vee)$ has the weight $\gamma_i^\vee$ and the elements $y_i \otimes \varpi^{k_i}$ are linearly independent over $C$. We have $(\lambda_0 : \gamma_i^\vee) + k_i = nh_0$ for all $i$. In particular $k_i = k_j$ whenever $\gamma_i^\vee = \gamma_j^\vee$. Thus the operators $y_i$ are also linearly independent. Hence, if $y(1) = 0$ then $y(\varpi) = 0$. Thus $x(\varpi) \in \ell_0 \mathcal{N}_{\zeta_0, \tau_0}$. The claim is proved.

Given $x(\varpi) \in \ell_0 \mathcal{N}_{\zeta_0, \tau_0}$ the orbit $\text{ad}(\ell_0 H_{\tau_0})(x(\varpi))$ is a cone because $x(\varpi)$ is nilpotent (use the Jacobson-Morozov theorem as in Claim 2 in the proof of [V2, Proposition 6.3] for instance). Fix $k_i, \beta_i^\vee$ as above. We have $(\lambda_0 : \beta_i^\vee) + k_i = h_0$ for all $i$. Hence for each $t \in C$ we get $x(e^{t \varpi}) = e^{h_0 t} \text{ad}(e^{-\lambda_0})(x(\varpi))$. Clearly, $e^{-\lambda_0} \in \ell_0 H_{\tau_0}$, thus $x(z \varpi) \in \text{ad}(\ell_0 H_{\tau_0})(x(\varpi))$ for each $z \in C^\times$, i.e. each $\ell_0 H_{\tau_0}$-orbit in $\ell_0 \mathcal{N}_{\zeta_0, \tau_0}$ is preserved by the action of $C^\times$ by rotation. Therefore $\ell_0 \mathcal{N}_{\zeta_0, \tau_0} / \ell_0 H_{\tau_0}$.

Let $Z \subseteq \tilde{G}^\vee(F)$ be the kernel of the obvious projection $\tilde{G}^\vee(F) \to G^\vee(F)$. Thus $Z \simeq C^\times$. Obviously, we have $Z \subseteq \ell_0 H_{\tau_0}$. The map $ev : G^\vee(R) \to G^\vee(C)$, $g(\varpi) \mapsto g(1)$ restricts to an isomorphism $\ell_0 H_{\tau_0} / Z \to \ell_0 H$. Namely, both groups are connected by [V2, Lemma 2.13], $ev$ restricts to an injection $\ell_0 H_{\tau_0} / Z \to \ell_0 H$ (see [BEG, Proposition 5.13] for instance), and $ev$ yields an isomorphism of the Lie algebras of $\ell_0 H_{\tau_0} / Z$ and $\ell_0 H$. Therefore $ev$ yields a bijection $\ell_0 \mathcal{N}_{\zeta_0, \tau_0} \to \ell_0 \mathcal{N}_{\zeta_0} / \ell_0 H$. 

\[ \square \]

5.2. Set $k = C$. Put $\mathbf{T} = \bigoplus_{J \subseteq I} H \otimes_{\mathbb{H}} S$. The quantized affine Schur algebra is the ring $\mathbf{S}_C = \text{End}_H(\mathbf{T})$. The right $S^W$-action on $\mathbf{T}$ commutes to the left $H$-action. It yields a ring homomorphism $S^W \to \mathbf{S}_C$. Given $\ell'_0 \in T^\vee$, let $\{\ell'_0\}_C$ be the full subcategory of $\mathbf{S}_C$-mof consisting of the modules which are annihilated by some power of $\ell'_0$. Note that $\mathbf{S}_C$-mof $= \bigoplus_{\ell'_0} \{\ell'_0\}_C$ where $\ell'_0$ varies in a set of representatives of the $W$-orbits in $T^\vee$.

From now on let $\Delta$ be of type $A_{d-1}$. Since the parameters $\zeta_0i$ (resp. $h_0i$) are all equal we omit the subscript $i$. To keep track of the parameters, we will index the categories considered so far by $\zeta_0, \tau_0$, etc.

Lemma. The number of simple objects in $\{\ell'_0\}_C$ is not less than the number of $\ell_0 H$-orbits in $\ell_0 \mathcal{N}_{\zeta_0}$. 

Proof. We only sketch the proof because the arguments are standard. For any quasi-projective $\text{SL}_d(\mathbb{C}) \times \mathbb{C}^\times$-variety $X$, let $K(X)$ be the complexified Grothendieck group of $\text{SL}_d(\mathbb{C}) \times \mathbb{C}^\times$-equivariant coherent sheaves on $X$. Set $\mathbf{B} = K(\text{point})$. Recall
that $\text{Spec}(B) = (T'/W) \times \mathbb{C}^\times$. Let $\ell'_0 K(X)_{\zeta'_0}$ be the specialization of the $B$-module $K(X)$ at $(W_0, \zeta'_0) \in (T'/W) \times \mathbb{C}^\times$.

Let $X$ (resp. $\tilde{X}$) be the variety of $d$-steps flags in $\mathbb{C}^d$ (resp. the complete flag variety in $\mathbb{C}^d$). Let $Z \subseteq T^*X \times T^*X$ (resp. $\tilde{Z} \subseteq T^*\tilde{X} \times T^*\tilde{X}$, $Y \subseteq T^*X \times T^*X$) be the corresponding Steinberg varieties. We endow $K(Z)$, $K(\tilde{Z})$ with an associative unital $B$-linear product as in [CG]. We endow $K(Y)$ with the $(K(\tilde{Z}), K(Z))$-bimodule structure as in [GRV]. We have a ring isomorphism $\mathbb{H} \simeq K(\tilde{Z})$, and $\mathbb{T} \simeq K(Y)$ over $\mathbb{K}$. It gives rise to a ring homomorphism $K(Z) \to \mathbb{C}$ which is invertible generically over $\text{Spec}(B)$. The specialization map from the Grothendieck group of $K(Z) \otimes_{\mathbb{C}[z^\pm 1]} C(\zeta)$-mof to the Grothendieck group of $K(Z)_{\zeta'_0}$-mof is surjective. Thus the pull-back map from the Grothendieck group of $\ell'_0 \mathcal{S}_{\zeta'_0}$ to the Grothendieck group of $\ell'_0 K(Z)_{\zeta'_0}$-mof is also surjective. The simple $\ell'_0 K(Z)_{\zeta'_0}$-modules are labelled by $\ell'_0 N_{\zeta'_0}/\ell'_0 H$ following [V1] (see the remark below Theorem 4 if $\zeta'_0$ is a root of unity). We are done. \hfill \Box

5.3. Let $\zeta'_0$, $\zeta_0$, $\tau_0$ be as in 5.1.

**Theorem.** Assume that $h_0 \in \mathbb{C} \setminus (1/2)\mathbb{Z}$.

(i) $\mathcal{O}'_{h_0}$ and $\mathcal{S}'_{\zeta_0}$ are equivalent.

(ii) If $\ell_0$, $\ell'_0$ are as in 5.1 then $(\ell_0) \mathcal{O}_{\zeta_0, \tau_0}$ and $(\ell'_0) \mathcal{S}'_{\zeta'_0}$ are equivalent.

**Proof.** We first prove (i). Fix $\lambda_0 \in X_\mathbb{C}$. We can assume that $\check{W}_{\lambda_0} \subseteq W$. Namely for any $\pi \in \Omega$ the pull-back by the automorphism $\pi$ of $H'$ yields an equivalence of categories $(\pi_{\lambda_0}) \mathcal{O}_{h_0} \to (\lambda_0) \mathcal{O}_{h_0}$. Thus $(w_{\lambda_0}) \mathcal{O}_{h_0}$ and $(\lambda_0) \mathcal{O}_{h_0}$ are equivalent whenever $w \in \check{W}$. Observe also that, in type $A$, there is always an element $w \in \check{W}$ such that $\check{W}_{\lambda_0} \subseteq W$ by Lemma 1.3. In the other hand, for each $\pi$ the categories $(\pi_{\ell'_0}) \mathcal{S}'_{\zeta'_0}$ and $(\ell'_0) \mathcal{S}'_{\zeta'_0}$ are equivalent, because $\pi_{\ell'_0} = w_{\pi} (\ell'_0, e_{w_{\pi}^{-1} \omega \pi})$, the element $e_{w_{\pi}^{-1} \omega \pi} \in G^\vee$ is central, and for any $\ell \in T^\vee$ and $\omega$ in the center of $G^\vee$ we have $\ell' \mathbb{H} \simeq \ell' \mathbb{H}$ (use [CG, Proposition 8.1.5] for instance).

We have $\check{W}_{\ell'_0} = \check{W}_{\lambda_0}$ because

$$x_{\beta} w(\ell_0) = \ell_0 \iff (\tau_0 \otimes \beta)(\tau_0 \otimes w \lambda_0) = \tau_0 \otimes \lambda_0$$

$$\iff \beta + w \lambda_0 = \lambda_0$$

$$\iff x_{\beta} w(\lambda_0) = \lambda_0.$$

Thus $\check{W}_{\ell_0} = W_{\ell'_0}$, hence it is generated by reflections. Moreover,

$$\alpha^\vee \in \Delta^\vee(\ell'_0) \iff e_{w_0(\lambda_0 : \alpha^\vee)} \in \Gamma \iff (\lambda_0 : \alpha^\vee) \in \mathbb{Z} + h_0 \mathbb{Z}$$

by (2.5.1). Therefore, Proposition 2.5.(iii) yields an equivalence $(\ell_0) \mathcal{O}_{\zeta_0, \tau_0} \simeq (\lambda_0) \mathcal{O}_{h_0}$.

Since there is an involution of $H'$ taking $h_0$ to $-h_0$, we may assume that $h_0 \notin \mathbb{Q}_{>0}$. Thus the pair $(\tau_0, \zeta_0)$ is regular according to the terminology in [V2, Definition 2.14]. Hence, by [V2, Theorem 7.6 and Lemma 8.1] the simple objects in $(\ell_0) \mathcal{O}_{\zeta_0, \tau_0}$, hence in $(\lambda_0) \mathcal{O}_{h_0}$, are labelled by $\ell_0 N_{\zeta_0, \tau_0}/(\ell_0 H_{\tau_0} \times \mathbb{C}^\times)$. In the following we construct a quotient functor $(\lambda_0) \mathcal{O}_{h_0} \to (\ell'_0) \mathcal{S}'_{\zeta'_0}$. It is an equivalence by Lemmas 5.1 and 5.2.
For each integer $n > 0$ we set $\ell'_0 \mathbf{T}_n = \mathbf{T} \otimes S/[(\ell'_0)^n]_W$ and $\ell'_0 \mathbf{Sc}_n = \mathbf{Sc} \otimes S/[(\ell'_0)^n]_W$. Thus $\text{End}_H(\ell'_0 \mathbf{T}_n) = \ell'_0 \mathbf{Sc}_n$. Note that $[(\ell'_0)^n]_W = [W \ell'_0_n]$ by the Pittie-Steinberg theorem, because $W \ell'_0_n$ is generated by reflections. If $J \subseteq I$ then $S_{W \ell'_0, n} = \bigoplus O S_{O, n}$, where $O$ is any $W_J$-orbit in $W \ell'_0$. Hence $\ell'_0 \mathbf{T}_n = \bigoplus_{J \subseteq I} \bigoplus O P_J(O_i)_n$.

According to Proposition 4.3, for each $J, O$ we can fix a $W_J$-orbit $O'_n \subset \hat W \lambda_0$ such that $P_J(O'_n) = P_J(O_n)$. Set $\lambda_0 \mathbf{T}_n = \bigoplus O P_J(O'_n)$. Consider the functor $\mathcal{M} : \mathcal{O}'_{h_0} \to \mathcal{O}_{O'_n}$ introduced in Section 3. We have $\mathcal{M}(\lambda_0 \mathbf{T}_n) = \ell'_0 \mathbf{T}_n$, $\lambda_0 \mathbf{T}_n$ is projective in $\lambda_0 \mathcal{O}_{n, h_0}$, and $\text{End}_H(\lambda_0 \mathbf{T}_n) = \ell'_0 \mathbf{Sc}_n$ by Proposition 4.3.(i), (ii). Thus we have the quotient functor

$$F_n : \lambda_0 \mathcal{O}_{h_0} \to \ell'_0 \mathbf{Sc}_n - \text{mobj}, \quad M \mapsto \text{Hom}_H(\lambda_0 \mathbf{T}_n, M).$$

It is an equivalence because both categories have the same (finite) number of simple objects.

On the other hand

$$(\lambda_0) \mathcal{O}_{h_0} = \lim_{\longrightarrow n} \lambda_0 \mathcal{O}_{n, h_0}, \quad (\ell'_0) \mathbf{Sc}_n = \lim_{\longrightarrow n} (\ell'_0 \mathbf{Sc}_n - \text{mobj}),$$

where $\lambda_0$ stands for the inductive limit of categories. The functors $F_n$ are compatible with the inductive systems of categories. Consider the $H'$-module $\lambda_0 \mathbf{T}_\infty = \lim_{\leftarrow n} \lambda_0 \mathbf{T}_n$. Note that $\lambda_0 \mathbf{T}_\infty \notin \mathcal{O}_{h_0}$, because the $S'$-action is not locally finite. The natural map $F_\infty(M) \to \text{Hom}_H(\lambda_0 \mathbf{T}_\infty, M)$ is an isomorphism for each $M \in \lambda_0 \mathcal{O}_{n, h_0}$.

Hence the functor

$$F_\infty : (\lambda_0) \mathcal{O}_{h_0} \to (\ell'_0) \mathbf{Sc}_n, \quad M \mapsto \text{Hom}_H(\lambda_0 \mathbf{T}_\infty, M),$$

is an equivalence of categories. Claim (i) follows, because

$$\mathcal{O}_{h_0} = \bigoplus_{\lambda_0} (\lambda_0) \mathcal{O}_{h_0}, \quad \mathbf{Sc}_n = \bigoplus_{\ell'_0} (\ell'_0) \mathbf{Sc}_n,$$

where $\lambda_0$ (resp. $\ell'_0$) runs in a set of representatives of $\hat W$-orbits in $X_C$ (resp. of $W$-orbits in $X_C/Y$) and the map $\lambda_0 \mapsto \ell'_0$ is a bijection $X_C/\hat W \to (X_C/Y)/\hat W$.

The proof of (ii) follows immediately from the proof of (i). \hfill $\square$

6. ANOTHER EXAMPLE

6.1. For any $H'$-module $M$ in $\lambda_0 \mathcal{O}$, the character of $M$ is the element

$$\chi(M) = \sum_{\mu \in W \lambda_0} \text{dim}(M_\mu) \varepsilon^\mu \in \mathbb{Z} X_C,$$

where $M_\mu$ is as in 2.2. We do not assume that the root system is of type $A$ anymore, but we restrict our attention to one single block in $\mathcal{O}'$. Let $n$ be the Coxeter number. Fix a positive integer $k$ prime to $n$. Put $h_0 = h_0 = k/n \in \mathbb{Q}$, $\lambda_0 = \rho/n \in X_Q$, $\zeta_0 = \zeta_0 = e^{k/n}$, and $\ell_0 = \ell_0 = e^{\rho/n}$. Note that $\hat W \lambda_0 = \{1\}$. For any $j \in \mathbb{Z}$ we set $\Delta^\vee(j) = \{\beta^\vee \in \Delta^\vee \mid (\rho : \beta^\vee) = j\}$. Set $k = an + b$, with $0 < b < n$. We have

$$H_{\lambda_0} = \{\beta^\vee, -a, (\gamma^\vee, 1 - a) ; \beta^\vee \in \Delta^\vee(-b), \gamma^\vee \in \Delta^\vee(n - b)\}.$$

For each non-empty subset $J \subseteq J_k := \Delta^\vee(-b) \cup \Delta^\vee(n - b)$ we set

$$A_J = \{\mu \in X_R \mid (\mu : \beta^\vee), (\mu : \gamma^\vee) - 1 < a, \forall \beta^\vee \in J \cap \Delta^\vee(-b), \forall \gamma^\vee \in J \cap \Delta^\vee(n - b)\}.$$

The function $J \mapsto A_J$ is decreasing. Put $D_J = A_J \setminus \bigcup_{J' \supseteq J} A_{J'}$. The sets $D_J$ are the affine domains.
Lemma. The simple objects \( \{ V_J \} \) in \( \lambda_0 \mathcal{O} \) are uniquely labelled by non-empty subsets \( J \subseteq I_k \) in such a way that

\[
ch(V_J) = \sum_{A_w \subseteq D_J} \varepsilon^{w_\lambda_0}.
\]

Proof. Fix \( v_0 \in \mathbb{C}^\times \) not a root of unity, and set \( \zeta_0 = (v_0)^k, \tau_0 = (v_0)^n, \ell_0 = v_0^{\lambda_0} \).

By Proposition 2.5 the categories \( \{ \ell_0 \} \mathcal{O} \) and \( \{ \lambda_0 \} \mathcal{O} \) are equivalent. The simple modules in \( \{ \ell_0 \} \mathcal{O} \) are classified in [V2], and the Jordan-Hölder factors of induced modules are given there via intersection cohomology of some stratified variety. In our case, the corresponding variety is \( \mathbb{C}^I \), with the stratification induced by the coordinate hyperplanes. This yields

\[
\sum_{J' \supseteq J} ch(V_{J'}) = \sum_{A_w \subseteq A_J} \varepsilon^{w_\lambda_0}.
\]

\[\square\]

For all \( \mu_0 \in \hat{W}_\lambda_0 \) we have \( ch(P(\mu_0)) = \sum_{w \in W} \varepsilon^{w_\mu_0} \), because \( \hat{W}_\lambda_0 = \{ 1 \} \).

In particular \( P(\mu_0) \) is indecomposable, because it is generated by the one-dimensional subspace \( P(\mu_0)_{\mu_0} \). By the proposition above the modules \( P(\mu_0) \) and \( \bigoplus_J V_J \) are equal in the Grothendieck ring. There are \( 2r+1 - 1 \) affine domains in \( \mathbb{R}^r \), where \( r \) is the rank of \( \mathfrak{g}^\vee \). The corresponding projective objects in \( \lambda_0 \mathcal{O} \) are the projective covers \( P_J \) of the simple modules \( V_J \), for each non-empty subset \( J \subseteq I_k \). The set \( D_{I_k} \) is the unique bounded affine domain. We have \( \mathcal{M}(V_{I_k}) = 0 \) because \( V_{I_k} \) is finite-dimensional.

There are \( 2r+1 - 2 \) domains in \( \mathbb{R}^r \). The corresponding projective objects in \( \ell_0 \mathcal{O} \) are the modules \( \mathcal{M}(P_J) \) with \( J \subseteq I_k \) non-empty, by Theorem 4.1.(i). We claim that \( \mathcal{M}(P_{I_k}) = P_{I_k}(W_{\ell_0}) \). To prove the claim, observe that \( \text{Hom}_W(P_I(W_{\lambda_0}), V_{I_k}) = (\bigoplus_{\mu_0 \in W_\lambda_0}(V_{I_k})_{\mu_0})^W \). Hence \( P_I(W_{\lambda_0}) \) surjects to \( V_{I_k} \), because \( \bigoplus_{\mu_0 \in W_\lambda_0}(V_{I_k})_{\mu_0} \neq \{ 0 \} \) by the proposition above and \( V_{I_k} \) is simple. The module \( P_I(W_{\lambda_0}) \) is projective in \( \lambda_0 \mathcal{O} \). Hence it contains the projective cover of \( V_{I_k} \) as a direct summand. Thus \( P_I(W_{\lambda_0}) = P_{I_k} \), because \( ch(P_I(W_{\lambda_0})) = ch(P_{I_k}) \). On the other hand \( P_{I_k}(W_{\ell_0})^W = P_{I_k}(W_{\ell_0}) \) by Theorem 4.2 (with \( J = I \)). We are done.

Note that \( \mathcal{M}(P_{I_k}) = S_{W_{\ell_0}} \), and that \( \ell_0 \mathcal{H} = \bigoplus_{w \in W} P(w_{\ell_0}) \), hence \( \ell_0 \mathcal{H} \) is a sum (with positive multiplicities) of the modules \( \mathcal{M}(P_J) \) with \( J \subseteq I_k \). Thus there is a quotient functor \( \lambda_0 \mathcal{O} \to \text{End}_\mathcal{H}^{(\ell_0 \mathcal{H} \oplus S_{W_{\ell_0}})-mof} \). Therefore \( \lambda_0 \mathcal{O} \) is equivalent to \( \text{End}_\mathcal{H}^{(\ell_0 \mathcal{H} \oplus S_{W_{\ell_0}})-mof} \), because both categories have the same number of simple modules. More generally, let \( \{ \ell_0 \} \mathcal{C} \) be the full subcategory of \( \text{End}_\mathcal{H}(\mathcal{H} \oplus S)-mof \) consisting of the modules which are annihilated by some power of \( \langle \ell_0 \rangle^W \).

Proposition. The category \( \{ \lambda_0 \} \mathcal{O} \) is equivalent to \( \{ \ell_0 \} \mathcal{C} \).

6.2. We give more details in type \( A_1 \). Then \( \lambda_0 = \rho/2, h_0 = 1/2, \zeta_0 = -1 \), and \( \ell_0 = i \otimes \alpha_1 \). There are 3 simple objects \( V(s_0 \lambda_0), V(s_1 \lambda_0), V(\lambda_0) \) in \( \lambda_0 \mathcal{O} \), such that \( ch(V(\lambda_0)) = \varepsilon^{\lambda_0} \), and

\[
ch(V(s_0 \lambda_0)) = \sum_{j \in 1 + 4\mathbb{Z}_{<0}} (\varepsilon^{j\lambda_0} + \varepsilon^{-j\lambda_0}), \quad ch(V(s_1 \lambda_0)) = \varepsilon^{-\lambda_0} + \sum_{j \in 1 + 4\mathbb{Z}_{>0}} (\varepsilon^{j\lambda_0} + \varepsilon^{-j\lambda_0}).
\]
The representation of $H'$ on $V(\lambda_0)$ takes $\xi_1$ to 1/4, and $s_1, s_\varphi$ to 1. The module $V(s_j \lambda_0)$ is the quotient of $H'$ by the left ideal generated by $\{\xi_1 - (s_j \lambda_0), s_j + 1\}$ for each $j = \varphi, 1$. The modules $P(\lambda_0), P(s_\varphi \lambda_0), P(s_1 \lambda_0)$ are the projective covers of $V(\lambda_0), V(s_\varphi \lambda_0), V(s_1 \lambda_0)$ respectively in $N_0 O$.

There are 2 simple objects $V(\ell_0), V(\ell_0^\perp)$ in $\ell_0 O$. The module $V(\ell_0^\perp)$ is one-dimensional such that $t_1, y_1$ acts as $-1, \pm i$, and $\overline{P(\ell_0^\perp)}$ is the projective cover of $V(\ell_0^\perp)$ in $\ell_0 O$.

We have $\mathcal{M}(V(\lambda_0)) = 0$ because $\dim V(\lambda_0) < \infty$. We have $\mathcal{M}(V(s_1 \lambda_0)) = V(\ell_0)$ because $V(s_1 \lambda_0)$ is induced from the one-dimensional $H'$-module such that $W$ acts via the signature, and $V(\ell_0)$ is the one-dimensional $H$-module such that $t_j$ acts by -1. Similarly $\mathcal{M}(V(s_\varphi \lambda_0)) = V(\ell_0)$. Then, we get easily $\mathcal{M}(P(s_\varphi \lambda_0)) = \overline{P(\ell_0)}$ and $\mathcal{M}(P(s_1 \lambda_0)) = \overline{P(\ell_0)}$. There is an exact sequence

$$0 \to V(s_\varphi \lambda_0) \oplus V(s_1 \lambda_0) \to P(\lambda_0) \to V(\lambda_0) \to 0.$$  

It yields $\mathcal{M}(P(\lambda_0)) = V(\ell_0) \oplus V(\ell_0^-)$. Set $O' = \{-\lambda_0, \lambda_0\}$ and $O = \{\ell^- \setminus \ell_0\}$. Then $V(\ell_0) \oplus V(\ell_0^-) = P_0(O)$, and $P(\lambda_0) = P_0(O')$ because the map $P(\lambda_0) \to P_0(O')$, $1_{\lambda_0} \mapsto (\xi_1 + \frac{1}{4})1_{\pm \lambda_0}$ is surjective and $\text{ch} P(\lambda_0) = \text{ch} P_0(O')$. Thus $\mathcal{M}(P(\lambda_0)) = \overline{P(\ell_0)}$. To conclude, note that $SO = P_0(O)$ and $\ell_0 H = \overline{P(\ell_0)} \oplus \overline{P(\ell_0)}$.

7. Appendix

7.1. Recall that $A = \mathbb{C}[[\varpi]]$, $F = \mathbb{C}((\varpi))$. Fix a commutative $A$-algebra $S_A$ which is free of rank $e$ over $A$. Let $(s_u)$ be a $A$-basis of $S_A$. Set $S_k = k \otimes_A S_A$ if $k = \mathbb{C}$ or $F$. Assume that $S_C$ is a local Artinian ring with maximal ideal $m_C$. Then $S_A$ is also a local ring. Let $m_A \subset S_A$ be the maximal ideal. Let $V_A$ be a free right $S_A$-module of rank $d$, with basis $(e_r)$. From now on, $r, s$ belong to $\{1, 2, \ldots, d\}$, and $u, v$ to $\{1, 2, \ldots, e\}$. We write $e_{rv}$ for $e_r s_u$.

Let $\nabla = d - \sum_j A_j dz/j$ be a linear integrable meromorphic connection over $\mathbb{C}^t$, with $A_j = \sum_{\beta \geq 0} A_{j\beta} z^\beta$ and $A_{j\beta} \in \text{End}(V_A)$. The space of horizontal sections $V_A^\nabla$ is a free $A$-module of rank $de$. Set $V_k^\nabla = V_A^\nabla \otimes_A k$.

Assume that $A_{j0}(e_{ru}) = e_{ru} m_{rj}$ with $m_{rj} \in S_A$ such that $k + m_{rj} - m_{sj} \in S_F$ for each integer $k \neq 0$. Let $\mu_{rj}$ be the image of $m_{rj}$ in the residue field $S_A/m_A$. Set $m_r = \sum_j m_{rj} \otimes \alpha_j$ and $\mu_r = \sum_j \mu_{rj} \otimes \alpha_j$.

There is a unique fundamental matrix solution $G : \mathbb{C}^t \setminus D_\infty \to \text{End}(V_F)$ of the form $G = Hz^{n_0}$, with $H : \mathbb{C}^t \to \text{End}(V_F)$ holomorphic such that $H(0) = \text{Id}$. Set $f_{ru} = Ge_{ru}$. Then $(f_{ru})$ is a $F$-basis of $V_F^\nabla$.

There is an integer $k_0 \leq 0$ such that $f_{ru} \varpi^{-k_0} \in V_A^\nabla$ for each $u, r$. Put $\zeta_j = \log z_j$. Let $V_C[\zeta]$, be the set of $V_C$-valued polynomials in the $\zeta_j$'s, $W = V_C[\zeta][[\varpi]]\varpi^{-k_0}$, and $W[[\varpi]]$ be the set of $W$-valued formal series in the $\zeta_j$'s. Write $W[[\varpi]]' \subset W[[\varpi]]$ for the set of formal series without constant term. Then $f_{ru}$ has an expansion in $e_{ru} z^{m_r} + W[[\varpi]]' z^{\mu_r}$.

The following proposition is standard, but we have not found a convenient reference.

**Proposition.** There is an $A$-basis $(b_{ru})$ in $V_A^\nabla$ such that $b_{ru} \in f_{ru} + \sum_{\mu_r > \mu_r} \sum_v f_{sv} F$.

**Proof.** Note that $e_{ru} z^{m_r - \mu_r} \in W$ because $m_{rj} - \mu_{rj} \in m_A$. Consider a formal series $b_{ru} = \sum_{\beta \geq 0} b_{ru\beta} z^{\mu_r + \beta}$, with $b_{ru\beta} \in W$ and $b_{ru0} = e_{ru} z^{m_r - \mu_r}$. It is the
expansion of an horizontal section in \( V_A^\Sigma \) if and only if for all \( j \in I \) we have

\[
(7.1.1) \quad \partial_{\gamma} b_{\gamma \alpha} + b_{\gamma \alpha} (\beta_j + \mu_{rj}) - A_j (b_{\gamma \alpha}) = \sum_{\gamma < \beta} A_{\gamma, \beta - \gamma} (b_{\gamma \gamma}), \quad \forall \beta \geq 0.
\]

We have \( \partial_{\gamma} b_{\gamma 0} + b_{\gamma 0} (\mu_r) - A_j (b_{\gamma 0}) = 0 \) because \( A_j (b_{\gamma 0}) = b_{\gamma 0} m_{\gamma j} \). Assume that \( b_{\gamma \gamma} \) satisfies (7.1.1) for each \( \gamma < \beta \). Recall that for all \( j \in I, c \in W \) and \( B \in \text{End}(V_A) \), there is an element \( b \in W \) such that \( \partial_{\gamma} b - B(b) = c \) (solve the equation term by term using asymptotic expansions of \( b, c, B \) in series in \( \omega \). It is done inductively on the exponent of \( \omega \)). Hence, for each \( j \) there is a non empty set of solutions \( b_{\gamma \gamma} \in W \) to (7.1.1). There is a common solution for all \( j \) because \( \nabla \) is integrable. Therefore, for each \( (r, u) \) there is a horizontal section \( b_{ru} \in V_A^\Sigma \) with an expansion in \( e_{ru} z^{m_r} + W[[z]]' z^{\nu r} \). These sections form a A-basis of \( V_A^\Sigma \) because \( (e_{ru}) \) is a A-basis of \( V_A \). Fix elements \( x_{sv} \in F \) such that

\[
(7.1.2) \quad b_{ru} - \sum_{s, \nu} f_{sv} x_{sv} = 0.
\]

We must prove that \( \mu_s > \mu_r \) if \( x_{sv} \neq 0 \) and \( (s, \nu) \neq (r, u) \), and that \( x_{ru} = 1 \).

Consider expansions in \( \omega \) of the summands in (7.1.2). Given \( s \), let \( \beta_s z^{\mu_s} \) be the constant term in \(- \sum f_{sv} x_{sv} \), where the sum is over all \( v \) such that \( (s, v) \neq (r, u) \), and let \( \alpha_r z^{\nu r} \) the constant term in \( b_{ru} - f_{ru} x_{ru} \). Then \( \alpha_r, \beta_s \) are holomorphic with asymptotic expansions \( \alpha_r(z), \beta_s(z) \) in \( V_C[[z]] \). Moreover the constant term \( \beta_s(0) \in V_C[[z]] \) of the non zero series \( \beta_s \) are linearly independent. Fix \( \nu \geq 0 \) minimal such that \( \alpha_r z^{\nu r} = \gamma_r z^{\nu + \nu r} \) and \( \gamma_r \) has an asymptotic expansion in \( V_C[[z]] \) with non-zero constant term. Then (7.1.2) gives

\[
(7.1.3) \quad \gamma_r z^{\nu + \nu r} + \sum_s \beta_s z^{\mu_s} = 0.
\]

We claim that \( \nu > 0 \) and that there is an index \( s \) such that \( \beta_s \neq 0 \) and \( \nu + \mu_r = \mu_s \). Then, setting \( \gamma_r' = (\gamma_r + \sum_{s=0}^{\nu} \beta_s) z^{\nu - \nu} \) with \( \nu' \geq \nu \) minimal such that \( \gamma_r'(0) \neq 0 \), and \( \beta_s' = \beta_s \) if \( \mu_s \neq \nu + \mu_r \) and 0 else, (7.1.3) yields

\[
\gamma_r' z^{\nu' + \nu r} + \sum_s \beta_s' z^{\mu_s} = 0.
\]

Once again there is an index \( s \) such that \( \beta_s' \neq 0 \) and \( \nu' + \mu_r = \mu_s \). By induction we have proved that \( \mu_s > \mu_r \) for each pair \( (s, v) \neq (r, u) \) such that \( x_{sv} \neq 0 \). Moreover \( x_{ru} = 1 \) because \( \nu > 0 \). To prove the claim recall the following fact:

(7.1.4) given an equation \( \sum_{t=1}^{m} v_t z^{\nu_t} = 0 \) with \( v_t \in X_C \) and \( v_t \) holomorphic with an expansion \( v_t(z) \in V_C[\zeta][[z]] \), if the constant terms \( v_t(0) \) are non-zero then \( v_1, ... v_m \) are not all different.

(It is sufficient to prove this for \( I = \{ 1 \} \). If \( v_1, ... v_m \) are all different we can fix \( \zeta \in C \) such that \( |e^{\zeta}| < 1 \) and \( |e^{\zeta t} z^{\nu_t}| \), \( |e^{\zeta t} z^{\nu_t} z | \) are distincts. Assume that \( |e^{\zeta t + \zeta t} z^{t} z^{\nu_t}| > ... > |e^{\zeta m} z^{t} z^{\nu_t}| \). Setting \( \zeta = k \zeta \) with \( k \gg 0 \), the equation \( \sum_{t=1}^{m} v_t (e^{k \zeta}) e^{k \nu_t \zeta} = 0 \) yields \( v_1(0) = 0 \). If \( \nu = 0 \) then \( x_{ru} \neq 1 \). Hence the elements \( \gamma_r(0), \beta_s(0) \) with \( s \) such that \( \beta_s \neq 0 \) are linearly independent, and (7.1.3) yields a contradiction with (7.1.4). The rest of the claim is immediate from (7.1.4) again. \( \square \)
7.2. Let $A$ be a ring with a unity, and $S$ be an infinite (countable) set. Put $A^S = \bigoplus_{s \in S} A$, and $M_S(A) = \text{Hom}_A(A^S, A^S)$ (with respect to the right $A$-action on $A$). Elements in $M_S(A)$ may be viewed as infinite matrices whose columns have only finitely many entries. If $A$ is a topological ring we endow $M_S(A)$ with the finite topology: a system of neighborhoods of an element $f$ is formed by the subsets

$$\{f^i \in M_S(A) ; f(x) - f^i(x) \in U^S, \forall x \in A^E\},$$

where $E \subset S$ is finite and $U \subset A$ is an open neighborhood of zero. Recall that a $A$-module $M$ is smooth if the annihilator in $A$ of any element is open. Let $A\text{-mod}^\infty$ be the category of smooth finitely generated $A$-modules.

**Proposition.** The categories $A\text{-mod}^\infty$ and $M_S(A)\text{-mod}^\infty$ are equivalent.

**Proof:** Set $B = M_S(A)$. The ring $B$ acts on $A^S$ on the left, the ring $A$ acts on $A^S$ on the right. To simplify assume that the topology on $A$ is discrete. The general case is identical. We must prove that $A\text{-mod}$ and $B\text{-mod}^\infty$ are equivalent. Consider the functors

$$F : A\text{-Mod} \to B\text{-Mod}, \quad M \mapsto A^S \otimes_A M,$$

$$G : B\text{-Mod} \to A\text{-Mod}, \quad N \mapsto \text{Hom}_B(A^S, N).$$

The left $A$-action on $G(N)$ comes from the right $A$-action on $A^S$. The functor $G$ is exact because $A^S$ is projective in $B\text{-Mod}$. The functor $F$ is obviously exact.

(i) We have

$$GF(M) = \text{Hom}_B(A^S, A^S \otimes_A M) = \text{Hom}_B(A^S, A^S) \otimes_A M,$$

because $A^S$ is finitely generated over $B$. The right $A$-action on $\text{Hom}_B(A^S, A^S)$ comes from the right $A$-action on $A^S$. Using commutation with elementary matrices, we get

$$\text{Hom}_B(A^S, A^S) = \{f_a : A^S \to A^S, v \mapsto va ; a \in A\} \simeq A$$

(isomorphism of $(A, A)$-bimodules). Thus $GF(M) = M$.

(ii) The natural evaluation map

$$\phi_N : FG(N) = A^S \otimes_A \text{Hom}_B(A^S, N) \to N$$

is a morphism of $B$-modules. We claim that $\phi_N$ is bijective if $N \in B\text{-mod}^\infty$.

To prove the surjectivity it is sufficient to assume that $N$ is smooth and cyclic. For any finite set $E \subset S$, set $I_E = \{f \in B ; f(x) = 0, \forall x \in A^E\}$. Then it is enough to assume $N = B/I_E$, because the ideals $I_E$ form a basis of open neighborhoods of zero in $B$. Clearly $B/I_E \simeq (A^S)^E$ over $B$. Moreover $FG(A^S)^E = F(A)^E = (A^S)^E$; by (i), and $\phi_N$ is the identity if $N = (A^S)^E$.

We now prove the injectivity. The exact sequence

$$0 \to \text{Ker}(\phi_N) \to FG(N) \to N \to 0$$

yields an exact sequence

$$0 \to G(\text{Ker}(\phi_N)) \to G(N) \to G(N) \to 0$$
by (i), where the third map is $G(\phi_N) = \text{Id}_{G(N)}$. Thus $G(\text{Ker}(\phi_N)) = \{0\}$. The $B$-module $\text{Ker}(\phi_N)$ is smooth, because $FG(N)$ is smooth. Hence, for any finitely generated submodule $N' \subset \text{Ker}(\phi_N)$ we have $G(N') = \{0\}$ and the map $\phi_{N'}$ is surjective. Thus $N' = \{0\}$. Therefore $\text{Ker}(\phi_N) = \{0\}$.

(iii) It is sufficient to check $G(B\text{-mod}^\infty) \subset A\text{-mod}$ on smooth cyclic $B$-modules. Thus it is enough to prove that $G(B/I_E) \in A\text{-mod}$ for each finite set $E \subset S$, see (ii). This is obvious because $G(B/I_E) \simeq G(A^S)^E = A^E$ by (i).

(iv) The inclusion $F(A\text{-mod}) \subset B\text{-mod}^\infty$ is obvious because $A^S \subset B\text{-mod}^\infty$.

$\square$

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