DETERMINANTAL REPRESENTATIONS OF SEMI-HYPERBOLIC
POLYNOMIALS

GREG KNESSE

Abstract. We prove a generalization of the Hermitian version of the Helton-Vinnikov determinantal representation for hyperbolic polynomials to the class of semi-hyperbolic polynomials, a strictly larger class, as shown by an example. We also prove that certain hyperbolic polynomials affine in two out of four variables divide a determinantal polynomial. The proofs are based on work related to polynomials with no zeros on the bidisk and tridisk.

1. Introduction

A homogeneous polynomial $P \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ is hyperbolic of degree $d$ with respect to $e \in \mathbb{R}^{n+1}$ if $P(e) \neq 0$ and if for all $x \in \mathbb{R}^{n+1}$ the one variable polynomial $t \mapsto P(x - te)$ has only real zeros. This concept was originally studied by Gårding for its relation to PDE (see [7], [15]) but it—and the related concept of stable polynomials—has since become important to convex optimization, combinatorics, probability, combinatorics, and analysis. See the papers and surveys [31], [13], [34], [29], [15], [26].

A deep result in the area is a determinantal representation for trivariate hyperbolic polynomials due to Helton-Vinnikov [16], [32] which solved a 1958 conjecture of Lax [17] (see [18]) and, as is mentioned in [15], can be used to develop the full Gårding theory of hyperbolicity.

Theorem A. Let $p \in \mathbb{R}[x_0, x_1, x_2]$ be hyperbolic of degree $d$ with respect to $e_2$ and monic in $x_2$. Then, there exist $d \times d$ real symmetric matrices $A_0, A_1$ such that

$$p(x_0, x_1, x_2) = \det(x_0 A_0 + x_1 A_1 + x_2 I).$$

If we relax the problem to finding self-adjoint matrices instead of real symmetric matrices, proofs more amenable to computations are possible (see [10], [30], [33]). The resulting theorem is just as useful for most purposes.

Theorem A*. Let $p \in \mathbb{R}[x_0, x_1, x_2]$ be hyperbolic of degree $d$ with respect to $e_2$ and monic in $x_2$. Then, there exist $d \times d$ self-adjoint matrices $A_0, A_1$ such that

$$p(x_0, x_1, x_2) = \det(x_0 A_0 + x_1 A_1 + x_2 I).$$

Our immediate goal is to prove a generalization of this result based on a result in Geronimo et al [8] and an extension to four variables based on a result in Bickel and Knese [2], while our larger goal is to advertise the close connection between determinantal representations of
hyperbolic polynomials and sums of squares decompositions for multivariable Schur stable polynomials. See [11], [12], [20], [21] for background on the latter topic.

Our main result establishes a determinantal representation with the assumption of hyperbolicity weakened. We shall call a homogeneous polynomial $P \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ a semi-hyperbolic polynomial with respect to the direction $e \in \mathbb{R}^{n+1} \setminus \{0\}$ if for every $x \in \mathbb{R}^{n+1}$ the univariate polynomial $t \mapsto P(x - te)$ is either identically zero or only has real roots. The key distinction between hyperbolic and semi-hyperbolic polynomials is that we do not assume $P(e) \neq 0$. Some references actually confuse the two, while Renegar [31] is the only reference we have found that emphasizes the distinction. We elaborate on our motivations in Section 6. We do need to allow for $t \mapsto P(x - te)$ to be identically zero, because for instance if $P(e) = 0$ and $x = 0$, then $P(-te) \equiv 0$. We give an example of a semi-hyperbolic polynomial that is not hyperbolic in any direction in Section 3.

Here is our main theorem.

**Theorem 1.** Let $p \in \mathbb{R}[x_0, x_1, x_2]$ of degree $d$ be semi-hyperbolic with respect to $e_2 = (0, 0, 1)$. Then, there exist $d \times d$ self-adjoint matrices $A_0, A_1, A_2$ with $A_2$ positive semi-definite and a constant $c \in \mathbb{R}$ such that

$$p(x) = c \det(x_0 A_0 + x_1 A_1 + x_2 A_2).$$

Assuming $p$ has no factors depending on $x_0, x_1$ alone, the above data can be chosen to additionally satisfy:

- $\text{rank } A_1 = \deg_1 p$, $\text{rank } A_2 = \deg_2 p$,
- $A_1 = B_+ - B_-$ with $B_\pm$ both positive semi-definite where $\text{rank } B_- = \text{the number of roots of } p(1, t, i)$ in the upper half plane and $\text{rank } B_- + \text{rank } B_+ = \text{rank } A_1$,
- and $B_- + B_+ + A_2 = I$.

See Section 2 for the proof. We can recover Theorem A* when $p(e_2) \neq 0$ since $p$ will then have degree $d$ in $x_2$ and then $A_2$ will be positive definite. We can then factor $A_2^{1/2}$ from the right and left of $\sum_{j=0}^2 x_j A_j$ in order to get a determinantal representation of the form given in Theorem A*, namely with $A_2 = I$.

There is nothing special about the vector $e_2$; a linear change of variables could be used to establish a determinantal representation for other semi-hyperbolic polynomials. The assumption of no factors depending on only $x_0, x_1$ is there to avoid certain annoyances that such trivial factors introduce. For instance $p(x_0, x_1, x_2) = x_1$ is certainly semi-hyperbolic in the direction $e_2$ but then $A_2 = A_0 = 0$ and the signature of the $1 \times 1$ matrix $A_1$ does not really provide any useful information.

It follows that a trivariate semi-hyperbolic polynomial $p$ can be lifted to a four variable polynomial

$$P(x_0, x_1, y_1, x_2) = c \det(x_0 A_0 + x_1 B_+ + y_1 B_- + x_2 A_2)$$

which is hyperbolic in the direction $(0, 1, 1, 1)$ and $P(x_0, x_1, -x_1, x_2) = p(x_0, x_1, x_2)$. So, we are projecting a hyperbolic polynomial (possessing a definite determinantal representation) of four variables to a set where it is not necessarily hyperbolic. It also follows that a trivariate semi-hyperbolic polynomial is a limit of hyperbolic polynomials. Indeed, writing $p$ as in Theorem 1 define for $\epsilon > 0$

$$p_\epsilon(z) = c \det(x_0 A_0 + x_1 A_1 + x_2 (A_2 + \epsilon I)).$$

(1.1)
Then, \( p_{\epsilon} \to p \) as \( \epsilon \searrow 0 \). We do not know if semi-hyperbolic polynomials in more than three variables are the limit of hyperbolic polynomials.

The theorem above has an curious asymmetry in its treatment of \( x_0 \) and \( x_1 \). This is partly due to idiosyncrasies of our proof but we also think there are some subtleties to resolve. To be specific, one could certainly break up \( A_0 \) into a difference of positive semi-definite matrices according to its signature but we have been unable to connect the signature of the \( A_0 \) we construct with geometric properties of \( p \). We have no reason to believe this cannot be done, especially because this issue does not arise in the hyperbolic case. Indeed, we can take \( A_2 = I \) and the number of zeros of \( t \mapsto p(tx_0, tx_1, i) \) in \( \mathbb{C}_+ \) equals the number of negative eigenvalues of \( x_0A_0 + x_1A_1 \). Similarly, the number of zeros of \( t \mapsto p(tx_0, tx_1, -i) \) equals the number of positive eigenvalues of \( x_0A_0 + x_1A_1 \). Thus, the signature of \( x_0A_0 + x_1A_1 \) can be derived from properties of \( p \) in the hyperbolic case. Notice that we evaluate \( p \) on the complex line \( (0, t, i) \) to determine the signature of \( A_1 \) while in the theorem above we evaluate on the line \( (1, t, i) \), which actually seems less natural. The example in Section 3 shows this is actually necessary: using the line \( (1, t, i) \) we get a correct count of the negative eigenvalues of \( A_1 \) while using the line \( (0, t, i) \) we get an incorrect count. The details are recorded in Section 3.

As a nice corollary, we can quickly recover the following variant of Theorem A*. The original proof, while not difficult, requires transforming a real stable polynomial to a hyperbolic polynomial through a linear transformation. Our signature count of \( A_1 \) in Theorem 1 makes the proof go smoothly.

**Corollary 1** (See Theorem 6.6 of [8]). If \( p \in \mathbb{R}[x_0, x_1, x_2] \) is homogeneous of degree \( d \) and hyperbolic with respect to all vectors in the cone \( \{(0, v_1, v_2) : v_1, v_2 > 0\} \), then there exist \( d \times d \) self-adjoint matrices \( A_0, A_1, A_2 \) and a constant \( c \in \mathbb{R} \) such that \( A_1, A_2 \) are positive semi-definite, \( A_1 + A_2 = I \), and

\[
p(x) = c \det(x_0A_0 + x_1A_1 + x_2A_2)\text{.}
\]

Since [8] uses Theorem A to prove the above result, all of the matrices can be taken to be real but our proof does not yield this. For \( p \) as in the corollary, \( p(1, x_1, x_2) \) is known as a real stable polynomial. This formula was used in the recent paper regarding the Kadison-Singer problem [20]. See Section 3 for the very short proof of the corollary.

The key tool for the proof of Theorem 1 is a determinantal representation proven in Geronimo-Iliev-Knese [8] for certain polynomials on the bidisk \( \mathbb{D}^2 = \mathbb{D} \times \mathbb{D} \) (here \( \mathbb{D} \) is the open unit disk in the complex plane \( \mathbb{C} \)). Define \( D(z) = z_1D_1 + z_2D_2 \) where the \( D_1, D_2 \) are \((n + m) \times (n + m)\) matrices given by

\[
D_1 = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}.
\]

For \( n = n_1 + n_2 \), define

\[
P_+ = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{n_2} & 0 \end{pmatrix},
\]

where the blocks correspond to the orthogonal decomposition \( \mathbb{C}^{n+m} = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \oplus \mathbb{C}^m \). Let \( \mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}, \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \).
Theorem B. Suppose $p \in \mathbb{C}[z_1, z_2]$ has bidegree $(n, m)$, no zeros in $(\mathbb{T} \times \mathbb{D}) \cup (\mathbb{T} \times \mathbb{E})$, and no factors depending on $z_1$ alone. Let $n_2$ be the number of zeros of $p(z_1, 0)$ in $\mathbb{D}$. Then, there exists an $(n + m) \times (n + m)$ unitary $U$ and a constant $c \in \mathbb{C}$ such that

$$p(z_1, z_2) = c \det((z_1 P_+ + P_+ + D_2) - U(P_+ + z_1 P_+ + z_2 D_2)).$$

This is referred to as a determinantal representation for “generalized distinguished varieties” in [8] since it generalizes a determinantal representation for the “distinguished varieties” of Agler and McCarthy [1] which correspond to the case $n_2 = 0$. Polynomials defining distinguished varieties are essentially a Cayley transform of real stable polynomials and distinguished varieties have their own motivation in terms of operator theory as shown in [1]. Theorem B is based on first proving a sums of squares decomposition for polynomials $p \in \mathbb{C}[z_1, z_2]$ with no zeros in $\mathbb{T} \times \mathbb{D}$ (“a face of the bidisk”) and no factors in common with $\tilde{p}(z) = z_1^n z_2^m p(1/\bar{z}_1, 1/\bar{z}_2)$. Namely,

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2)(|E_1(z)|^2 - |E_2(z)|^2) + (1 - |z_2|^2)|F(z)|^2$$

where $E_1 \in \mathbb{C}^{n_1}[z], E_2 \in \mathbb{C}^{n_2}[z], F \in \mathbb{C}^{n}[z], n = n_1 + n_2$ where $n_2$ is the number of zeros of $p(z_1, 0)$ in $\mathbb{D}$. This formula generalizes a sums of squares formula of Cole and Wermter [5] related to Andô’s inequality from operator theory (see also [9] and [22]). It would be interesting to characterize which polynomials possess such a sums of squares formula where $|F(z)|^2$ is also given by a difference of squares $|F_1(z)|^2 - |F_2(z)|^2$, and—going further—it would be interesting to see what sort of determinantal representation for real homogeneous polynomials comes out of the corresponding development from Theorem B to Theorem [1] presented here.

Beyond trivariate polynomials, there are many results on the existence or non-existence of determinantal representations. See [33], [19], [27], [28], [4], [25] for recent results and convenient summaries of the state of the art. Vinnikov [33] conjectures that hyperbolic polynomials always divide a hyperbolic polynomial which has a determinantal representation but with additional requirements placed on the set where the determinantal polynomial is positive. Our next theorem offers a step in the right direction for this conjecture albeit in a special situation. A polynomial $p$ is affine with respect to a variable $x_j$ if it has degree one in that variable.

Theorem 2. Let $p \in \mathbb{R}[x_0, x_1, x_2, x_3]$ be hyperbolic of degree $d$ with respect to the cone $\{(0, v_1, v_2, v_3) : v_1, v_2, v_3 > 0\}$. Assume $p$ is affine in $x_2$ and $x_3$ and of degree $n$ in $x_1$. Then, there exists $k \leq 2n + 4$ and $k \times k$ self-adjoint matrices $A_0, A_1, A_2, A_3$ such that $p$ divides

$$\det(\sum_{j=0}^{3} x_j A_j),$$

$A_1, A_2, A_3$ are positive semi-definite and $A_1 + A_2 + A_3 = I$.

See Section [5]. Theorem 2 seems to be one of the few higher dimensional situations where one gets a determinantal representation from simple hypotheses. The recent article of Kum-mer [24] proves the interesting result that a hyperbolic polynomial in $n$ variables with no real singularities divides a determinantal polynomial. This article also obtains bounds on the sizes of the matrices involved under the assumption that some power of the polynomial has a determinantal representation. Theorem 2 requires no assumptions of smoothness and
obtains general bounds on the sizes of the matrices involved, but Kummer’s result has the advantage that it works in $n$ variables and does not assume degree restrictions.

The key tool for Theorem 2 is the following sums of squares decomposition from Bickel-Knese [2].

**Theorem C** (Theorem 1.12 of [2]). Let $p \in \mathbb{C}[z_1, z_2, z_3]$ have multi-degree $(n, 1, 1)$ and no zeros on $\mathbb{D}^3$. Then, there exist column-vector valued polynomials $E_1 \in \mathbb{C}^n[z_1, z_2, z_3]$, $E_2, E_3 \in \mathbb{C}^2[z_1, z_2, z_3]$ such that for $z = (z_1, z_2, z_3), w = (w_1, w_2, w_3)$

$$p(z)p(w) - \tilde{p}(z)\tilde{p}(w) = \sum_{j=1}^{3} (1 - z_j \bar{w}_j)E_j(w)^*E_j(z)$$

where $\tilde{p}(z) = z_1^n z_2 z_3 p(1/z_1, 1/z_2, 1/z_3)$.

2. Proof of Theorem 1 from Theorem B

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}, \mathbb{C}_- = \{z \in \mathbb{C} : \Re z < 0\}$. Assume $P \in \mathbb{R}[x_0, x_1, x_2]$ is homogeneous of degree $d$ and for every $x \in \mathbb{R}^3$

$$t \mapsto P(x - te_2)$$

is either identically zero or only has real zeros. We will assume $P$ has no factors depending only on $x_0, x_1$ which can easily be incorporated into our final determinantal representation by appending diagonal blocks to our matrices.

Consider

$$q(z_1, z_2) = P(1, z_1, z_2)$$

which has no zeros in $(\mathbb{R} \times \mathbb{C}_+) \cup (\mathbb{R} \times \mathbb{C}_-)$. To see this, take $z = (a_1, a_2 + ib_2) \in (\mathbb{R} \times \mathbb{C}_+) \cup (\mathbb{R} \times \mathbb{C}_-)$ with $q(z) = 0$. Then, $P((1, a_1, a_2) + te_2)$ has the imaginary root $t = ib_2$, which would imply $t \mapsto P(1, a_1, a_2 + t)$ is identically zero. This means $x_1 - a_1 x_0$ divides $P$ which we have ruled out.

Now, define

$$p(z_1, z_2) = q \left( \frac{1 + z_1}{1 - z_1}, \frac{1 + z_2}{1 - z_2} \right) \left( \frac{1 - z_1}{2i} \right)^n \left( \frac{1 - z_2}{2i} \right)^m$$

where $q$ has degree $n$ in $x_1$ and degree $m$ in $x_2$. Setting $x_0 = 1$ in $P(x_0, x_1, x_2)$ cannot lower the degree in $x_1$ or $x_2$, so $n = \deg_1 P, m = \deg_2 P$. Recall that

$$z \mapsto i \frac{1 + z}{1 - z}$$

is a conformal map of the unit disk onto the upper half plane sending $T$ to $\mathbb{R} \cup \{\infty\}$ where $1 \mapsto \infty$. Thus, $p$ has no zeros in $(\mathbb{T} \setminus \{1\}) \times \mathbb{D}$ as well as $(\mathbb{T} \setminus \{1\}) \times \mathbb{E}$ where $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$. We cannot have $p(1, z_2) = 0$ unless $p(z_1, z_2)$ has $z_1 - 1$ as a factor. This follows by Hurwitz’s theorem since the polynomials $z_2 \mapsto p(z_1, z_2)$ will have no zeros in $\mathbb{C} \setminus \mathbb{T}$ for $z_1 \in \mathbb{T}$ with $z_1 \rightarrow 1$, and then $p(1, z_2)$ will either have the same property or will be identically zero. However such factors cannot exist since they imply $q$ has degree less than $n$ in $x_1$. In any case, we can safely divide out factors of $p$ that depend only on $z_1$ since these can easily be incorporated into our final determinantal representation. Having done this, $p$ satisfies the hypotheses of Theorem B and we may write

$$p(z_1, z_2) = c \det((z_1 P_+ + P_+ + D_2) - U(P_+ + z_1 P_+ + z_2 D_2))$$
for a unitary $U$. Notice $n_2$ is the number of roots of $z_1 \mapsto p(z_1, 0)$ in $\mathbb{D}$ which is the same as the number of roots of $z_1 \mapsto q(z_1, i) = P(1, z_1, i)$ in $\mathbb{C}_+$.

We convert back to $q$ via $z \mapsto \frac{z - i}{z + i}$. So,

$$q(z_1, z_2) = p \left( \frac{z_1 - i}{z_1 + i}, \frac{z_2 - i}{z_2 + i} \right) (z_1 + i)^n(z_2 + i)^m$$

$$= p \left( \frac{z_1 - i}{z_1 + i}, \frac{z_2 - i}{z_2 + i} \right) \det((z_1 + i)D_1 + (z_2 + i)D_2)$$

$$= \frac{c \det((z_1 - i)P_- + (z_1 + i)P_+ + (z_2 + i)D_2)}{(-U((z_1 + i)P_- + (z_1 - i)P_+ + (z_2 - i)D_2))}$$

$$= \frac{c \det((I - U)D(z) - i(I + U)(P_- - P_+ - D_2))}{\pm c \det((-z_1P_- + z_1P_+ + z_2D_2) + i(I + U))}.$$  \hfill (2.1)

The last line comes from multiplying on the right by $\det(-P_- + P_+ + D_2)$. Letting $M(z) = -z_1P_- + z_1P_+ + z_2D_2$, we now form the spectral decomposition $U = V \left( \begin{smallmatrix} u & 0 \\ 0 & I \end{smallmatrix} \right) V^*$; $V$ is a unitary, $u$ is a $k \times k$ diagonal unitary with no 1’s on the diagonal, and $k$ is the rank of $U - I$. Factoring $V$ and $V^*$ out from the left and right of (2.1) leaves

$$q(z) = \pm c \det \left( \left( \begin{array}{cc} I - u & 0 \\ 0 & 0 \end{array} \right) V^*M(z)V + i \left( \begin{array}{cc} I + u & 0 \\ 0 & 2I \end{array} \right) \right)$$

$$= \pm c \det(I - u) \det \left( \left( \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right) V^*M(z)V + \left( \begin{array}{cc} a & 0 \\ 0 & 2I \end{array} \right) \right)$$

$$= \pm c \det(I - u) \det \left( (V^*M(z)V)_{kk} + a \begin{array}{cc} {} & * \\ 0 & 2I \end{array} \right)$$

$$= C \det((V^*M(z)V)_{kk} + a)$$

where $a = i(I + u)(I - u)^{-1}$ is a diagonal matrix with real entries, $(V^*M(z)V)_{kk}$ is the upper $k \times k$ block of $V^*M(z)M$, and $C$ is a constant. Now, $V^*M(z)V = -z_1V^*P_-V + z_1V^*P_+V + z_2V^*D_2V$ and if we set $A_0 = a$, $A_1 = (-V^*P_-V + V^*P_+V)_{kk}$, and $A_2 = (V^*D_2V)_{kk}$ we have a determinantal representation for $q$:

$$q(z) = C \det(A_0 + z_1A_1 + z_2A_2).$$

Notice $A_0, A_1, A_2$ are evidently self-adjoint with $A_2$ positive semi-definite, and since $\deg q = d$ we have $d \leq k$. Once we show $k = d$, we can homogenize to get the determinantal representation for $P$. It helps to first establish some of the additional details listed in Theorem 4.

It is a general fact that for matrices $A, B$, the degree of $\det(tA + B)$ is at most rank $A$ (we leave this as an exercise). So, $\deg_j q \leq \text{rank } A_j$ for $j = 1, 2$. On the other hand, by construction rank $A_1 \leq \text{rank } (P_- + P_+) = \deg_1 q$ and rank $A_2 \leq \text{rank } D_2 = \deg_2 q$, yielding $\deg_j q = \text{rank } A_j$ for $j = 1, 2$. Next, setting $B_{\pm} = (V^*P_{\pm}V)_{kk}$ we have $A_1 = B_+ - B_-$. Since rank $A_1 = n_1 + n_2$ and rank $B_+ \leq n_1$ and rank $B_- \leq n_2$, we must have equality in both inequalities. This also shows the ranges of $B_+, B_-$ have trivial intersection by considering dimensions. Since $P_+ + P_- + D_2 = I$, we must have $B_+ + B_- + A_2 = I$. 


We see that \(d\) for diagonal matrices is non-singular for some \(t\) and thus \(k = d\). Note \(Q(t) = I + (t - 1)B_+ - (t + 1)B_-.\) By the spectral theorem

\[
B_+ = \sum_{j=1}^{n_1} \nu_j v_j v_j^* \quad B_- = \sum_{j=1}^{n_2} \mu_j w_j w_j^*
\]

where \(V = \{v_1, \ldots, v_{n_1}\}, W = \{w_1, \ldots, w_{n_2}\}\) form orthonormal sets of eigenvectors corresponding to the positive eigenvalues \(\{\nu_1, \ldots, \nu_{n_1}\}, \{\mu_1, \ldots, \mu_{n_2}\}\) of \(B_+, B_-\) respectively. Then, \(B = V \cup W \cup Y\) is a basis for \(C^k\). Let \(C\) be a basis dual to \(B\). (Two bases \(b_1, \ldots, b_N\), \(c_1, \ldots, c_N\) are dual if \(b_j^* c_k = \delta_{jk}\).) The matrix for \(Q(t)\) obtained by using \(C\) as a basis for the domain and \(B\) for the range is of the form

\[
\begin{pmatrix}
I + (t - 1)d_+ & 0 & 0 \\
0 & I - (t + 1)d_- & 0 \\
0 & 0 & I
\end{pmatrix}
\]

for diagonal matrices \(d_+, d_-\) containing the eigenvalues \(\nu_1, \ldots, \nu_{n_1}, \mu_1, \ldots, \mu_{n_2}\) on the diagonal. The determinant of this vanishes for only finitely many \(t\) and so \(Q(t_0)\) is certainly non-singular for some \(t_0\). Thus, \(k = d\) and we homogenize \(q\) at degree \(d\) to see that

\[
P(x) = C \det(x_0 A_0 + x_1 A_1 + x_2 A_2).
\]

This concludes the proof of Theorem 1.

3. Example

Renegar [31] has an example of a polynomial that is semi-hyperbolic but not hyperbolic in any direction (see Section 2 of that paper); however we have constructed an example that is more illustrative for our purposes.

Let

\[
p(x_0, x_1, x_2) = 2x_0^2 x_1 - (x_0^2 + 3x_1^2)x_2.
\]

Then, \(t \mapsto p(x - tc_2)\) clearly has only real roots for \(x \in \mathbb{R}^3\) since this one variable polynomial has degree 1 and real coefficients. Let

\[
A_0 = \frac{i}{3} \begin{pmatrix} 0 & -3 & -\sqrt{3} \\ 3 & 0 & \sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

We see that

\[
p(x) = 3 \det(x_0 A_0 + x_1 A_1 + x_2 A_2).
\]

As remarked in the introduction we can lift to

\[
P(x_0, x_1, y_1, x_2) = 3x_1 y_1 x_2 - (x_2 + x_1 + 3y_1)x_0^2
\]

which is hyperbolic in the direction \((0, 1, 1, 1)\) and \(P(x_0, x_1, -x_1, x_2) = p(x_0, x_1, x_2)\). We now explain why \(p\) is not hyperbolic in any direction.
We first show that \( \{x : p(x) \neq 0\} \) consists of the two connected components \( P_+ = \{x : p(x) > 0\}, \ P_- = \{x : p(x) < 0\}. \) I thank the referee for the following simplified explanation. The hypersurface \( \{x : p(x) = 0\} \) is the graph of the continuous function \((x_0, x_1) \mapsto \frac{2x_0^2x_1}{x_0^2+3x_1^2}\). Thus, \( \{x : p(x) \neq 0\} \) is divided into exactly two components: the part above the graph and the part below.

Next, neither component \( P_+, P_- \) is convex. For instance, \((-1, 0, -1), (1, 0, -1) \in P_+\) but \((0, 0, -1) \notin P_+\). One can similarly show \( P_- \) is not convex. This implies that \( p \) is not hyperbolic in any direction since it is a fundamental result of Gårding that if \( p \) is hyperbolic in some direction \( e \), then the connected component of \( \{x : p(x) \neq 0\} \) containing \( e \) is convex.

This brings up a potential paradox. Since \( p \) is a limit of hyperbolic polynomials \( p_\varepsilon \) as in equation (1.1), how is it possible that the connected components of \( \{x : p(x) \neq 0\} \) are non-convex in the above example? An answer is that a convex component of \( \{x : p_\varepsilon(x) \neq 0\} \) could shrink to an isolated point (in projective space) as \( \varepsilon \searrow 0\). This is something we have seen graphically using the above example.

Finally, in connection with our discussion after Theorem 1 regarding the signatures of \( A_0, A_1 \), let us point out that \( p(1, t, i) = 2t - (1 + 3t^2)i \) has one zero in \( \mathbb{C}_+ \), which agrees with the number of negative eigenvalues of \( A_1 \). On the other hand, \( p(0, t, i) = -3t^2i \) has no zeros in \( \mathbb{C}_+ \). Notice also that \( p(t, 1, i) = (2 - i)t^2 - 3i \) has one zero in \( \mathbb{C}_+ \). This matches the number of negative eigenvalues of \( A_0 \), which is what one would like to have more generally in order for Theorem 1 to have a more symmetric statement.

4. PROOF OF COROLLARY 1

Notice that \( t \mapsto p(x - te_2) \) is either identically zero or only has real roots by Hurwitz’s theorem since this polynomial can be obtained as the limit of as \( a \searrow 0 \) of \( t \mapsto p(x - t(ae_1 + e_2)) \).

Any factors depending only on \( x_0, x_1 \) can easily be dealt with separately so we may assume there are no such factors. So, \( p \) satisfies the hypotheses of Theorem 2. Also, \( t \mapsto p(1, t, i) \) can have no zeros in the upper half plane for if it had such a zero \( z = x + iy \) where \( y > 0 \), then
\[
 t \mapsto p((1, x, 0) + t(0, 0, 1))
\]
would have the non-real zero \( t = i \) contradicting hyperbolicity in the direction \((0, y, 1)\). This shows that rank \( B_- = 0 \) in Theorem 1 and therefore \( A_1 \) is positive semi-definite as desired.

5. PROOF OF THEOREM 2 FROM THEOREM C

We largely follow the scheme of [23]. Let \( P \in \mathbb{R}[x_0, x_1, x_2, x_3] \) be homogeneous of degree \( d \) of degree 1 in \( x_2, x_3 \) and of degree \( n \) in \( x_1 \). Assume \( P \) is hyperbolic with respect to the cone \( \{(v_1, v_2, v_3) : v_1, v_2, v_3 > 0\} \). Then, for \( x = (x_1, x_2, x_3) \)
\[
 q(x) = P(1, x)
\]
has no zeros in $\mathbb{C}^3 \cup \mathbb{C}^3$ and $\overline{q(x)} = q(x)$. Switching to the tridisk, we see that

$$f(z) = q \left( \frac{1 + z_1}{1 - z_1}, \frac{1 + z_2}{1 - z_2}, \frac{1 + z_3}{1 - z_3} \right) \left( \frac{1 - z_1}{2i} \right) ^n \left( \frac{1 - z_2}{2i} \right) \left( \frac{1 - z_3}{2i} \right)$$

has no zeros in $\mathbb{D}^3 \cup \mathbb{E}^3$. Note that we may as well assume $f$ is irreducible since otherwise $f$ will have a factor depending on one or two variables alone, in which case there is no issue with having a determinantal representation.

Let $1/\bar{z} = (1/\bar{z}_1, 1/\bar{z}_2, 1/\bar{z}_3)$ for $z \in \mathbb{C}^3$ and define

$$\tilde{f}(z) = z^n z_2 z_3 f(1/\bar{z})$$

$$\frac{\partial \tilde{f}}{\partial z_j} = z^n z_2 z_3 \frac{\partial f}{\partial z_j} (1/\bar{z}) \quad \text{for} \quad j = 1, 2, 3.$$

Since $q$ has real coefficients one can show that $\tilde{f} = f$ and

$$nf = z_1 \frac{\partial f}{\partial z_1} + \frac{\partial \tilde{f}}{\partial z_1}$$

$$f = z_j \frac{\partial f}{\partial z_j} + \frac{\partial \tilde{f}}{\partial z_j} \quad \text{for} \quad j = 2, 3$$

after some simple computations. Thus, $(n+2)f = p + \bar{p}$ where

$$p(z) = \sum_{j=1}^{3} \frac{\partial \tilde{f}}{\partial z_j} \quad \bar{p}(z) = \sum_{j=1}^{3} z_j \frac{\partial f}{\partial z_j}.$$

Let $f_t(z) = f(tz)$ for $0 < t < 1$. Then, $f_t$ has no zeros in $\overline{\mathbb{D}}^3$ and if we set $\tilde{f_t}(z) = t^{n+2} f(z/t)$, then $|f_t| = |\tilde{f_t}|$ on $\mathbb{T}^3$ (since $\tilde{f} = f$) and so $\tilde{f_t}/f_t$ is analytic and bounded by 1 in modulus for $z \in \mathbb{D}^3$ by the maximum principle. Now, for $z \in \overline{\mathbb{D}}^3$

$$0 \leq \lim_{t \to 1} \frac{|f(tz)|^2 - |t^{n+2} f(z/t)|^2}{1 - t^2} (n+2)$$

$$= (n+2)^2 |f(z)|^2 - 2 \text{Re}(\bar{p}(z)(n+2) \tilde{f}(z))$$

$$= |p(z)|^2 - |\bar{p}(z)|^2$$

with some computations omitted (see [23] for more details). This shows that if $p$ vanishes in $\mathbb{D}^3$, then so does $\bar{p}$ and so does $f$ which by assumption does not happen. Hence, $p$ has no zeros in $\mathbb{D}^3$.

Note that if $p$ and $\bar{p}$ had a common factor then this would be a factor of $f$ which we have already ruled out; we point out that $p$ and $\bar{p}$ cannot be multiples of one another since $\bar{p}$ vanishes at the origin. The conclusion of Theorem C holds for such a $p$ but since we have only stated it for polynomials with no zeros on $\overline{\mathbb{D}}^3$ (as opposed to $\mathbb{D}^3$) we must explain how to address the case at hand. The main point is that for $0 < t < 1$, $p_t(z) = p(tz)$ will satisfy the hypotheses of Theorem C and therefore there exist vector polynomials $E_1^t, E_2^t, E_3^t$ corresponding to $p_t$ as in Theorem C. Then,

$$\sup_{\mathbb{T}^3} |p(z)|^2 \geq (1 - |z_j|^2) |E_j^t(z)|^2$$

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shows the vector polynomials $E_j^l$ are locally bounded in $\mathbb{D}^3$ and hence we can choose subsequences of $t \nrightarrow 1$ such that that $E_1^l \in \mathbb{C}^{2n}[z], E_2^l, E_3^l \in \mathbb{C}^{2}[z]$ converge to vector polynomials $E_1 \in \mathbb{C}^{2n}[z], E_2, E_3 \in \mathbb{C}^{2}[z]$ and hence we will get a sums of squares decomposition as in Theorem C. Note the polynomials in $E_1, E_2, E_3$ necessarily have degree at most $(n - 1, 1, 1), (n, 0, 1), (n, 1, 0)$ (this is proven in [21] for instance) and they will be non-trivial since $p$ and $\tilde{p}$ have no factors in common. On the zero set $Z_f$ of $f, p = -\tilde{p}$ and therefore

\[(5.1)\quad 0 = \sum_{j=1}^{3} (1 - z_j \bar{w}_j)E_j(w)E_j(z)\]

for $z, w \in Z_f$. This equation ensures that the map

\[(5.2)\quad \begin{pmatrix} z_1E_1(z) \\ z_2E_2(z) \\ z_3E_3(z) \end{pmatrix} \mapsto \begin{pmatrix} E_1(z) \\ E_2(z) \\ E_3(z) \end{pmatrix}\]

defined initially for vectors of the above form with $z \in Z_f$, extends linearly to a well-defined $(2n + 4) \times (2n + 4)$ unitary $U$. (Some details: If a combination of vectors from the left side of (5.2) sums to zero, (5.1) shows the corresponding combination on the right sums to zero. So, we get a well-defined linear map from the span of the left side of (5.2) to the span of the right side. Now, (5.1) shows this map is an isometry. Since we are in finite dimensions it can be extended to a unitary.)

Note that $E_1, E_2, E_3$ cannot vanish identically in $Z_f$ without vanishing in all of $\mathbb{C}^3$ since the degrees are lower and $f$ is irreducible. Let $P_j$ for $j = 1, 2, 3$ be the projection onto the $j$-th component in the orthogonal decomposition of $\mathbb{C}^{2n+4} = \mathbb{C}^{2n} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ and let $M(z) = \sum_{j=1}^{3} z_j P_j$. By (5.2), for $z \in Z_f$

\[(I - UM(z)) \begin{pmatrix} E_1(z) \\ E_2(z) \\ E_3(z) \end{pmatrix} = 0\]

and therefore $\det(I - UM(z)) = 0$ for $z \in Z_f \setminus \{z : E_1, E_2, E_3 = 0\}$. Basic results in algebraic geometry (such as in Chapter 4, Section 4 of [6]) can be used to establish that this implies $\det(I - UM(z))$ vanishes for $z \in Z_f$ (i.e. $Z_f \setminus \{z : E_1, E_2, E_3 = 0\}$ is Zariski dense in $Z_f$) since $f$ is irreducible and none of $E_1, E_2, E_3$ vanish identically on $Z_f$.

Therefore $f$ divides $\det(I - UM(z))$. Write

\[f(z)g(z) = \det(I - UM(z))\]

for some polynomial $g$ of degree at most $(n, 1, 1)$. As with Section 2, we convert back to $g$. There is some repetition in what follows but since the situations are slightly different we include the details. Now,

\[(5.3)\quad q(z)r(z) = \det((\sum_{j=1}^{3} (z_j + i)P_j) - U(\sum_{j=1}^{3} (z_j - i)P_j)) = \det((I - U)M(z) + i(I + U))\]

for

\[r(z) = (z_1 + i)^n(z_2 + i)(z_3 + i)g\left(\frac{z_1 - i}{z_1 + i}, \frac{z_2 - i}{z_2 + i}, \frac{z_3 - i}{z_3 + i}\right).\]
Let $U = V \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} V^*$ be the spectral decomposition of $U$ where $u$ is $k \times k$ diagonal with unimodular entries, none of which equals 1. Here $k$ is the rank of $I - U$. As in Section 2 the determinant \((5.3)\) can be converted to
\[(5.4) \quad q(z)r(z) = (\text{const}) \det((V^*M(z)V)_{kk} + a)\]
where again $(V^*M(z)V)_{kk}$ refers to taking the upper $k \times k$ block of the given matrix, and $a = i(I + u)(I - u)^{-1}$. Finally, if we homogenize \((5.4)\) at degree $k$—note this is at most $2n + 4$—then
\[
P(x)R(x) = (\text{const}) \det(x_0a + \sum_{j=1}^3 x_jA_j)
\]
with $A_j = (V^*P_jV)_{kk}$ and $A_1 + A_2 + A_3 = (V^*IV)_{kk} = I$, and where $R(x) = x_0^{k-d}r((1/x_0)(x_1, x_2, x_3))$. This concludes the proof.

6. Concluding questions and remarks

We think it is worthwhile to discuss or rehash some of the motivations and lingering questions of this paper in more detail.

Semi-hyperbolic polynomials have perhaps been overlooked because they lack one of the key features of hyperbolic polynomials. Specifically, if $p$ is hyperbolic in the direction $e$, then the connected component of \(\{x : p(x) \neq 0\}\) containing $e$ is convex (see [31]). No such result holds for semi-hyperbolic polynomials (see Renegar [31] Section 2 or Section 3 above). This convexity property ties hyperbolic polynomials to optimization and is “the cornerstone of hyperbolic programming” [31]. This begs the question, why study semi-hyperbolic polynomials which may lack this property?

First, we think it is a good general principle in mathematics to understand the degenerate versions of objects of interest. Notice that the (local uniform) limit of a sequence of homogeneous polynomials of degree $d$ which are semi-hyperbolic with respect to a specific direction $e$ is either semi-hyperbolic or identically zero. This follows from Hurwitz’s theorem applied to each polynomial $t \mapsto P(x - te)$. Hyperbolic polynomials do not share this property. Somewhat related is the following question mentioned in the introduction.

**Question 1.** Every trivariate semi-hyperbolic polynomial is a limit of hyperbolic polynomials. Is this true more variables?

Second, our main theorem, Theorem 1, shows that trivariate semi-hyperbolic polynomials possess determinantal representations just as in the hyperbolic case. We think this in itself provides good justification for the study of semi-hyperbolic polynomials. This is a good point to formally state a question from the introduction.

**Question 2.** In Theorem 1, can the signature and rank of $A_0$ be determined directly from properties of $p$?

Our own personal motivations for studying semi-hyperbolic polynomials came from the natural connection we presented above between semi-hyperbolic polynomials and two variable polynomials with no zeros on $\mathbb{T} \times \mathbb{D}$. These latter polynomials appeared in [8] essentially because of the realization that some of the theory of polynomials with no zeros in $\mathbb{D}^2$ could be pushed further to the situation of no zeros in $\mathbb{T} \times \mathbb{D}$. It was realized later that this initially
unnatural condition is closely related to hyperbolicity and indeed is essentially equivalent to semi-hyperbolicity.

Finally, we wish to rehash our larger goal of the paper of connecting sums of squares formulas to determinantal representations. This description will be somewhat imprecise. The approach of this paper shows that if \( p(z_1, z_2, \ldots, z_n) \) has no zeros in \( \mathbb{D}^n \) and possesses a hermitian sums of squares formula

\[
(6.1) \quad |p|^2 - |\tilde{p}|^2 = \sum_{j=1}^{n} (1 - |z_j|^2)SOS_j
\]

(here each \( SOS_j \) term is a sum of squared moduli of polynomials) then \( p + \tilde{p} \) divides a unitary determinantal polynomial

\[
\det(I - UD(z)).
\]

Here \( U \) is a unitary matrix and \( D(z) \) is a diagonal matrix with coordinate functions on the diagonal. One can then convert \( p + \tilde{p} \) via Cayley transform and homogenization to a hyperbolic polynomial (hyperbolic with respect to all vectors with positive entries) and through some linear algebra get a self-adjoint determinantal polynomial. One can reverse engineer some of this: take a hyperbolic polynomial \( P \) (again hyperbolic with respect to vectors with positive entries) convert to a polynomial \( q \) satisfying \( q = \tilde{q} \). If \( q \) can be written as \( p + \tilde{p} \) where \( p \) satisfies (6.1) then \( P \) divides a determinantal representation. If \( n > 2 \), not every \( p \in \mathbb{C}[z_1, \ldots, n] \) with no zeros in \( \mathbb{D}^n \) satisfies an equation of the form (6.1); such polynomials are called Agler denominators. A polynomial is an Agler denominator if and only if \( \tilde{p}/p \) satisfies a multivariable von Neumann inequality (see [21]).

We have also presented a modification to hyperbolicity/semi-hyperbolicity with respect to a specific direction. In \( n \) variables this would entail, after various conversions, to understanding polynomials satisfying (6.1) where \( SOS_1, \ldots, SOS_{n-1} \) are replaced with differences of squares. With the exception of our work in [8], this is relatively uncharted territory.

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Washington University in St. Louis, Department of Mathematics, St. Louis, MO, 63130

E-mail address: geknese@wustl.edu