SPECTRAL–PARAMETER DEPENDENT YANG–BAXTER OPERATORS AND YANG–BAXTER SYSTEMS FROM ALGEBRA STRUCTURES

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Abstract. For any algebra two families of coloured Yang-Baxter operators are constructed, thus producing solutions to the two-parameter quantum Yang-Baxter equation. An open problem about a system of functional equations is stated. The matrix forms of these operators for two and three dimensional algebras are computed. A FRT bialgebra for one of these families is presented. Solutions for the one-parameter quantum Yang-Baxter equation are derived and a Yang-Baxter system constructed.

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1. Introduction

The quantum Yang-Baxter equation (QYBE) plays a crucial role in analysis of integrable systems, in quantum and statistical mechanics and also in the theory of quantum groups. In the quantum group theory, solutions of the constant QYBE lead to examples of bialgebras via the Faddeev–Reshetikhin–Takhtajan (FRT) construction [3, 9]. On the other hand, the theory of integrable Hamiltonian systems makes great use of the solutions of the one-parameter form of the QYBE, since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion. The purpose of this paper is two-fold. First we study non-additive solutions of the two-parameter form of the QYBE, also known as the coloured QYBE. Such a solution is referred to as a coloured Yang-Baxter operator. Secondly, we construct the one-parameter Yang-Baxter operators and associate to it a Yang-Baxter system. We also compare our results with other well-known solutions. It is imperative to note that the Yang-Baxter operators presented here are obtained from algebra structures, and are therefore distinct from the $R$–matrices that arise from quasitriangular Hopf algebras. See [2] [9] [8] [11] for ordinary Yang-Baxter operators, [8] for one-parameter form of the QYBE, [5] [10] for the coloured QYBE, [4] [16] for Yang-Baxter maps and [13] [15] for coloured quantum groups.

2. Construction of coloured Yang–Baxter operators

Formally, a coloured Yang-Baxter operator is defined as a function

$$R : X \times X \to \text{End}_k(V \otimes_k V),$$

where $X$ is a set and $V$ is a finite dimensional vector space over a field $k$. Thus for any $u, v \in X$, $R(u, v) : V \otimes_k V \to V \otimes_k V$ is a linear operator. Starting with this
operator one constructs three operators acting on a triple tensor product $V \otimes_k V \otimes_k V$, $R_{12}(u, v) = R(u, v) \otimes_k I$, $R_{23}(v, w) = I \otimes_k R(v, w)$, and similarly $R_{13}(u, w)$ as an operator that acts non-trivially on the first and third factor in $V \otimes_k V \otimes_k V$. $R$ is a coloured Yang-Baxter operator if it satisfies the two-parameter form of the QYBE,
\begin{equation}
R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v)
\end{equation}
for all $u, v, w \in X$. This is essentially the spectral parameter dependent QYBE and non-additivity implies $R(u, v) \neq R(u - v)$. One of the aims of this paper is to construct two classes of coloured Yang-Baxter operators based on the use of associative algebra structures built on $V$. From now on we assume that $X$ is equal to (a subset of) the ground field $k$. The method of constructing solutions to equation (2.1) is based on the ideas applied in [2] [11] to the case of the constant QYBE:
\begin{equation}
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
\end{equation}
The key point of the construction is to suppose that $V = A$ is an associative $k$-algebra, and then to derive a solution to equation (2.1) from the associativity of the product in $A$. Guided by the observation that the operator
\[ R : A \otimes A \rightarrow A \otimes A \quad \text{given by} \quad a \otimes b \mapsto 1 \otimes ab + ab \otimes 1 - b \otimes a \]
satisfies the constant QYBE, we seek a solution to equation (2.1) of the following form
\begin{equation}
R(u, v)(a \otimes b) = \alpha(u, v)1 \otimes ab + \beta(u, v)ab \otimes 1 - \gamma(u, v)b \otimes a,
\end{equation}
where $\alpha, \beta, \gamma$ are $k$-valued functions on $X \times X$.

Now, inserting this ansatz into equation (2.1) one finds that $R(u, v)$ is a solution of the two-parameter QYBE if and only if the coefficients of certain terms are equal to zero. We present some of these calculations below.

\begin{align*}
R_{12}^{uv} \circ R_{13}^{uw} \circ R_{23}^{vw} &\equiv \beta(u, v)\beta(u, w)\beta(v, w)abc \otimes 1 \otimes 1 + \\
&\alpha(u, v)\beta(u, w)\beta(v, w)1 \otimes abc \otimes 1 - \gamma(u, v)\beta(u, w)\beta(v, w)bc \otimes a \otimes 1 + \\
&\beta(u, v)\alpha(u, w)\beta(v, w)bc \otimes 1 \otimes a + \alpha(u, v)\beta(u, w)\beta(v, w)1 \otimes bc \otimes a - \\
&\gamma(u, v)\alpha(u, w)\beta(v, w)bc \otimes 1 \otimes a - \beta(u, v)\gamma(u, w)\beta(v, w)bc \otimes 1 \otimes a - \\
&\alpha(u, v)\gamma(u, w)\beta(v, w)bc \otimes a + \gamma(u, v)\gamma(u, w)\beta(v, w)bc \otimes 1 \otimes a + \\
&\beta(u, v)\beta(u, w)\alpha(u, w)abc \otimes 1 \otimes 1 + \alpha(u, v)\beta(u, w)\alpha(u, w)1 \otimes abc \otimes 1 - \\
&\gamma(u, v)\beta(u, w)\alpha(u, w)1 \otimes abc \otimes 1 + \beta(u, v)\alpha(u, w)1 \otimes abc \otimes 1 + \\
&\alpha(u, v)\alpha(u, w)\alpha(u, w)1 \otimes abc \otimes 1 + \gamma(u, v)\alpha(u, w)\alpha(u, w)1 \otimes 1 \otimes abc - \\
&\beta(u, v)\gamma(u, w)\alpha(u, w)bc \otimes 1 \otimes a - \alpha(u, v)\gamma(u, w)\alpha(u, w)1 \otimes bc \otimes a + \\
&\gamma(u, v)\gamma(u, w)\alpha(u, w)v, w)1 \otimes bc \otimes a - \beta(u, v)\gamma(u, w)v, w)bc \otimes 1 \otimes a - \\
&\alpha(u, v)\beta(u, w)\gamma(v, w)bc \otimes 1 \otimes a + \gamma(u, v)\gamma(u, w)\beta(u, w)v, w)c \otimes ab \otimes 1 - \\
&\beta(u, v)\alpha(u, w)\gamma(v, w)c \otimes 1 \otimes ab - \alpha(u, v)\alpha(u, w)\gamma(v, w)c \otimes 1 \otimes ab + \\
&\gamma(u, v)\alpha(u, w)\gamma(v, w)c \otimes 1 \otimes ab + \beta(u, v)\gamma(u, w)\gamma(v, w)bc \otimes 1 \otimes a + \\
&\alpha(u, v)\gamma(u, w)\gamma(v, w)c \otimes b \otimes a
\end{align*}
The equality $R_{12}^{ab} \circ R_{13}^{uw} \circ R_{23}^{bc} (a \otimes b \otimes c) = R_{13}^{uw} \circ R_{12}^{ab} \circ R_{23}^{bc} (a \otimes b \otimes c)$ implies that the coefficients of the following terms are equal to zero: $1 \otimes abc \otimes 1$, $bc \otimes 1 \otimes a$, $1 \otimes bc \otimes a$, $c \otimes ab \otimes 1$ and $c \otimes 1 \otimes ab$. Thus, we obtain the following system of equations:

\[
\gamma(v, w)\alpha(u, w)\beta(u, v)1 \otimes abc \otimes 1 - \beta(v, w)\gamma(u, w)\beta(u, v)c \otimes ab \otimes 1 - \\
\alpha(v, w)\gamma(u, w)\beta(u, v)c \otimes 1 \otimes ab + \gamma(v, w)\gamma(u, w)\beta(u, v)c \otimes ab \otimes 1
\]

\[
\beta(v, w)\beta(u, w)\alpha(u, v)c \otimes ab \otimes 1 + \alpha(v, w)\beta(u, v)c \otimes 1 \otimes ab - \\
\gamma(v, w)\beta(u, v)c \otimes 1 \otimes ab + \alpha(v, w)\alpha(u, v)c \otimes 1 \otimes ab + \\
\beta(v, w)\gamma(u, w)\alpha(u, v)c \otimes ab \otimes 1 + \\
\alpha(v, w)\beta(u, v)c \otimes 1 \otimes ab - \gamma(v, w)\gamma(u, w)c \otimes ab \otimes 1
\]

\[
\beta(v, w)\beta(u, v)c - \gamma(u, v)\alpha(u, v)c \otimes ab \otimes 1 - \\
\alpha(v, w)\alpha(u, w)\gamma(u, v)c \otimes ab \otimes 1 + \\
\gamma(v, w)\gamma(u, v)c \otimes ab \otimes 1
\]

\[+(\beta(u, v) - \gamma(v, w))(\alpha(u, v)\beta(u, w) - \alpha(u, w)\beta(u, v)) = 0\]

\[+(\alpha(u, v) - \gamma(u, v))(\alpha(v, w)\beta(u, w) - \alpha(u, w)\beta(v, w)) = 0\]

\[+(\alpha(u, v) - \gamma(u, v))(\alpha(u, v)\gamma(u, v) - \alpha(u, w)\gamma(u, v)) = 0\]

\[+(\alpha(u, v) - \gamma(v, w))(\alpha(v, w)\gamma(u, v) - \alpha(u, w)\gamma(u, v)) = 0\]

The system of equations (2.4, 2.8) is rather non-trivial. Nonetheless, it has some remarkable symmetry properties which can be used to find some solutions. For example, let us observe that the equations (2.5) and (2.8) are in some sense dual to each other. Likewise, (2.6) and (2.7) are in some sense dual to each other. In this paper we find some families of solutions for this system. It is an open problem to classify all its solutions.

First, we try to find solutions to the system of equations (2.4, 2.8) of the following form: $\alpha(u, v) = pu - p'v$, $\beta(u, v) = qu - q'v$ and $\gamma(u, v) = ru - r'v$. We obtain the solutions: $\alpha(u, v) = p(u - v)$, $\beta(u, v) = q(u - v)$ and $\gamma(u, v) = pu - qv$. Thus, we are guided to the following results.
Theorem 2.1. \( i \) For any two parameters \( p, q \in k \), the function 
\[ R : X \times X \to \text{End}_k(A \otimes A) \] 
defined by 
\begin{equation}
R(u, v)(a \otimes b) = p(u - v)1 \otimes ab + q(u - v)ab \otimes 1 - (pu - qv)b \otimes a,
\end{equation}
is a coloured Yang-Baxter operator.

\( ii \) If \( pu \neq qv \) and \( qu \neq pv \) then the operator (2.9) is invertible. Moreover, the following formula holds:
\[ R^{-1}(u, v)(a \otimes b) = \frac{p(u - v)}{(qu - pv)(pu - qv)}ba \otimes 1 + \frac{q(u - v)}{(qu - pv)(pu - qv)}1 \otimes ba - \frac{1}{(pu - qv)}b \otimes a. \]

Proof. \( i \) The proof can be done by following the steps presented above. Another way to prove this theorem is by direct calculations. The computations are quite involved.

\( ii \) This can be verified by direct calculations. ◼

Remark 2.2. For any coalgebra \( (C, \Delta, \varepsilon) \) and two parameters \( p, q \in k \), the function 
\[ R_C : X \times X \to \text{End}_k(A \otimes A) \] 
defined by 
\begin{equation}
R_C(u, v)(c \otimes d) = p(u - v)\varepsilon(c)\Delta(d) + q(u - v)\varepsilon(d)\Delta(c) - (pu - qv)d \otimes c,
\end{equation}
is a coloured Yang-Baxter operator. This follows from section 2, The transfer of the theory to coalgebras, of [2].

The system of equations (2.4–2.8) has another class of solutions: \( \alpha(u, v) = puq^v \), \( \beta(u, v) = ps^vb \) and \( \gamma(u, v) = pq^v \). Thus, we obtain the following theorem.

Theorem 2.3. \( i \) For any three parameters \( p, q, s \in k \), the function 
\[ R' : X \times X \to \text{End}_k(A \otimes A) \] 
defined by 
\begin{equation}
R'(u, v)(a \otimes b) = p^u(q^v1 \otimes ab + s^vab \otimes 1 - s^v b \otimes a)
\end{equation}
is a coloured Yang-Baxter operator.

\( ii \) If \( p \neq 0 \), \( q \neq 0 \), \( s \neq 0 \), then the operator (2.11) is invertible. Moreover, the following formula holds:
\[ R'^{-1}(u, v)(a \otimes b) = \frac{1}{pu} \left( \frac{1}{s^v}ba \otimes 1 + \frac{1}{q^v}1 \otimes ba - \frac{1}{s^v}b \otimes a \right). \]

Proof. Similar to that of Theorem 2.1. ◼

Remark 2.4. The system of equations (2.4–2.8) has also the following class of solutions: \( \alpha(u, v) = puq^v \), \( \beta(u, v) = ps^vb \) and \( \gamma(u, v) = pq^v \).
3. SOLUTIONS IN DIMENSIONS 2 AND 3

We consider the algebra \( A = \frac{[X]}{[X^2-\sigma]} \), where \( \sigma \in \{0,1\} \) is a scalar. Then \( A \) has the basis \( \{1, x\} \), where \( x \) is the image of \( X \) in the factor ring. We consider the basis \( \{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\} \) of \( A \otimes A \) and represent the operator (2.9) in this basis:

\[
R(u, v)(1 \otimes 1) = (qu - pv)1 \otimes 1 \\
R(u, v)(1 \otimes x) = p(u - v)1 \otimes x + (q - p)ux \otimes 1 \\
R(u, v)(x \otimes 1) = (q - p)v1 \otimes x + q(u - v)x \otimes 1 \\
R(u, v)(x \otimes x) = \sigma(p + q)(u - v)1 \otimes 1 - (pu - qv)x \otimes x
\]

In matrix form, this operator reads

\[
R(u, v) = \begin{pmatrix}
qu - pv & 0 & 0 & \sigma(q + p)(u - v) \\
0 & p(u - v) & (q - p)v & 0 \\
0 & (q - p)u & q(u - v) & 0 \\
0 & 0 & 0 & qu - pv
\end{pmatrix}
\]

and satisfies the coloured QYBE. This equips us to look at the FRT bialgebra structure associated to this \( R \)-matrix operator. Let us recall [13, 15] (and references therein) that the coloured extension of a FRT bialgebra involves the generators to be parametrized by some continuously varying \textit{colour} parameters, and redefining the algebra and the coalgebra such that all Hopf algebraic properties remain preserved. Using the coloured FRT approach, the coloured \( RTT \)-relations are

\[
R(u, v)T_{1u}T_{2v} = T_{2v}T_{1u}R(u, v)
\]

where \( T_{1u} = T_u \otimes 1, T_{2v} = 1 \otimes T_v \). The generators are arranged in the matrices

\[
T_u = \begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix}, \quad T_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}
\]

The \( RTT \)-equation (3.13) then gives the algebra commutation relations:

\[
R(u, v)[a_u, d_v] - (q - p)(uc_vb_u - vc_ub_v) = 0
\]

(3.14)

\[
p(u - v)a_u c_v - (qu - pv)c_v a_u + (q - p)vc_u a_v = 0
\]

(3.15)

\[
(qu - pv)a_u b_v - p(u - v)b_v a_u - (q - p)ua_v b_u + \sigma(q + p)(u - v)c_u d_v = 0
\]

(3.16)

\[
p(u - v)b_v c_u - q(u - v)c_v b_v - (q - p)u(a_v d_v - a_v d_u) = 0
\]

(3.17)

\[
q(u - v)c_u d_v - p(u - v)d_v c_u - (q - p)uc_v d_u = 0
\]

(3.18)

\[
p(u - v)b_v d_v - (qv - pu)d_v b_u + (q - p)v d_u b_v - \sigma(q + p)(u - v)c_v a_u = 0
\]

(3.19)

\[
(qu - pv)[a_u, a_v] + \sigma(q + p)(u - v)c_u c_v = 0
\]

(3.20)

\[
(qu - pv)b_v b_u - (qv - pu)b_v a_u - \sigma(q + p)(u - v)(a_v a_u - d_u d_v) = 0
\]

(3.21)

\[
(qv - pu)c_u c_v - (qu - pv)c_v c_u = 0
\]

(3.22)

\[
[a_u, d_v] = -[a_u, a_v]
\]

(3.23)

\[
[a_u, d_v] = [a_v, d_u]
\]

(3.24)

\[
q(vc_u b_v - uc_v b_u) - p(vb_v c_u - ub_v c_v) = 0
\]

(3.25)
All the above relations (3.14–3.25) are symmetric with respect to the \( u \leftrightarrow v \) exchange. Note that relation (3.22) can be obtained from (3.20) when \( \sigma \neq 0 \), but holds independently of (3.20) if \( \sigma = 0 \). This algebra does not have an uncoloured counterpart i.e. the limit \( u = v \) does not hold. However, an interesting algebra arises in the limiting case of \( p = q \). The coproduct is \( \Delta(T_u) = T_u \otimes T_u \) where \( \otimes \) is the usual symbol for tensor product and matrix multiplication at the same time. The counit is \( \varepsilon (a_1 b_2) = (1_0 1) \). Finding a quantum determinant remains a difficult task, and we conjecture that the above algebra does not admit one.

In the limit \( p = q \), this algebra reduces to

\[
\begin{align*}
[a_u, d_v] &= 0 & [a_u, b_v] + 2\sigma c_u d_v &= 0 \\
[a_u, c_v] &= 0 & \{b_u, d_v\} - 2\sigma c_v a_u &= 0 \\
[b_v, c_u] &= 0 & \{a_u, a_u\} + 2\sigma c_u c_v &= 0 \\
\{c_u, d_v\} &= 0 & \{b_u, b_v\} - 2\sigma (a_v a_u - d_u d_v) &= 0 \\
\{c_u, c_v\} &= 0 & &
\end{align*}
\]

while relations (3.23) and (3.24) remain unchanged. Note that the relation (3.25) arises as a compatibility condition for the \( u \leftrightarrow v \) exchange symmetry, which does not have an analogue when \( p = q \). This limit is interesting since the deformation parameter \( q \) can be factored out from the \( R \)-matrix (3.12) itself which then depends only on the difference \( u - v \). Consequently, the algebra in this limit is also independent of \( q \).

The operator (2.11) in the same basis reads

\[
(3.26) \quad R'(u, v) = p^u \begin{pmatrix} q^v & 0 & 0 & \sigma(q^v + s^v) \\
0 & q^v & q^v - s^v & 0 \\
0 & 0 & s^v & 0 \\
0 & 0 & 0 & -s^v \end{pmatrix}
\]

Next, we present solutions in dimension three. Consider the algebra \( B = \frac{k[X]}{(X^3 - \epsilon X - \rho)} \), where \( \epsilon \) and \( \rho \) are scalars. Then \( B \) has the basis \( \{1, x, x^2\} \), where \( x \) is the image of \( X \) in the factor ring. We consider the basis \( \{1 \otimes 1, 1 \otimes x, 1 \otimes x^2, x \otimes 1, x \otimes x, x \otimes x^2, x^2 \otimes 1, x^2 \otimes x, x^2 \otimes x^2\} \) of \( B \otimes B \) and represent the operator (2.11) in this basis, in the matrix form

\[
(3.27) \quad R(u, v) = \begin{pmatrix}
w & 0 & 0 & 0 & 0 & \rho t' \lambda & 0 & \rho t' \lambda & 0 \\
0 & p \lambda & 0 & tv & 0 & \epsilon p \lambda & 0 & \epsilon p \lambda & \rho p \lambda \\
0 & 0 & p \lambda & 0 & tv & 0 & \epsilon p \lambda & \rho p \lambda & 0 \\
0 & tu & 0 & q \lambda & 0 & eq \lambda & 0 & \epsilon q \lambda & \rho q \lambda \\
0 & 0 & 0 & w' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w' & 0 \\
0 & 0 & tu & 0 & q \lambda & 0 & eq \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & w' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w' \\
w & 0 & 0 & 0 & 0 & \rho t' \lambda & 0 & \rho t' \lambda & 0
\end{pmatrix}
\]

where \( \lambda = u - v \), \( t = q - p \), \( t' = q + p \), \( w = qu - pv \), and \( w' = qv - pu \).
Similarly, the operator \((2.11)\) can be represented in the same basis and has the matrix form

\[
R'(u, v) = p^u =
\begin{pmatrix}
q^v & 0 & 0 & 0 & 0 & \rho(q^v + s^v) & 0 & \rho(q^v + s^v) & 0 \\
0 & q^v & 0 & q^v - s^v & 0 & \epsilon q^v & 0 & \epsilon q^v & \rho q^v \\
0 & 0 & q^v & 0 & q^v & 0 & q^v - s^v & 0 & \epsilon q^v \\
0 & 0 & s^v & 0 & 0 & \epsilon s^v & 0 & \epsilon s^v & \rho s^v \\
0 & 0 & 0 & 0 & -s^v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -s^v & 0 \\
0 & 0 & 0 & s^v & 0 & s^v & 0 & \epsilon s^v & 0 \\
0 & 0 & 0 & 0 & -s^v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s^v \\
\end{pmatrix}
\]

4. One-parameter Yang–Baxter operators

The analysis presented above leads naturally to look at solutions of the one-parameter QYBE. Let \(X\) be a set, \(Z \subset X \times X\) and \(V\) is a finite dimensional vector space over a field \(k\). Formally, an one-parameter Yang-Baxter operator is defined as a function

\[R : X \rightarrow \text{End}_k(V \otimes_k V).\]

Thus for any \(x \in X\), \(R(x) : V \otimes_k V \rightarrow V \otimes_k V\) is a linear operator. \(R\) is a one-parameter Yang-Baxter operator if it satisfies the one-parameter form of the QYBE,

\[(4.29)\quad R_{12}(x)R_{13}(\varphi(x, z))R_{23}(z) = R_{23}(z)R_{13}(\varphi(x, z))R_{12}(x)\]

for all \((x, z) \in Z\), where \(\varphi : Z \rightarrow X\).

We seek a solution to equation \((4.29)\) of the form

\[(4.30)\quad R(x)(a \otimes b) = \alpha(x)1 \otimes ab + \beta(x)ab \otimes 1 - \gamma(x)b \otimes a,\]

where \(\alpha, \beta, \gamma\) are \(k\)-valued functions on \(X\).

Now, inserting this ansatz into equation \((4.29)\) one finds that \(R(x)\) is a solution of the one-parameter QYBE if and only if the coefficients of certain terms are equal to zero. Thus, we obtain the following system of equations:
is a one-parameter Yang-Baxter operator, where \( \phi \)

\[
R \alpha \quad (4.31)
\]

\[
= (\beta(z) - \gamma(z))(\alpha(x)\beta(\varphi(x, z)) - \alpha(\varphi(x, z))\beta(x))
+ (\alpha(x) - \gamma(x))(\alpha(z)\beta(\varphi(x, z)) - \alpha(\varphi(x, z))\beta(z)) = 0
\]

(4.31)

Remark 4.2

8 FLORIN F. NICHITA AND DEEPAK PARASHAR

\[\alpha(x)\beta(z)(\alpha(\varphi(x, z)) - \gamma(\varphi(x, z))) + \alpha(z)\gamma(\varphi(x, z))(\gamma(x) - \alpha(x))\]

(4.33)

\[+ \gamma(z)(\alpha(x)\gamma(\varphi(x, z)) - \alpha(\varphi(x, z))\gamma(x)) = 0\]

(4.35)

\[+ (\beta(x) - \gamma(x))(\alpha(\varphi(x, z))\gamma(z) - \alpha(z)\gamma(\varphi(x, z))) = 0\]

Finding further solutions for this system of equations and their classification is also an open problem. From the above analysis it follows that the system of equations (4.31)−(4.35) has the following solution: \( \alpha(x) = x - 1, \quad \beta(x) = q(x - 1), \) and \( \gamma(x) = x - q, \) where \( \varphi(x, z) = xz. \) Thus, we are guided to the following results.

**Proposition 4.1.** i) For any parameter \( q \in k, \) the function \( R : X \to \text{End}_k(A \otimes A) \)

(4.36)

\[
R(x)(a \otimes b) = (x - 1)1 \otimes ab + q(x - 1)ab \otimes 1 - (x - q)b \otimes a,
\]

is a one-parameter Yang-Baxter operator, where \( \varphi(x, z) = xz. \)

ii) If \( x \neq q \) and \( xq \neq 1, \) then the operator (4.36) is invertible. Moreover, the following formula holds:

\[
R^{-1}(x)(a \otimes b) = \frac{x - 1}{(qx - 1)(x - q)}ba \otimes 1 + \frac{q(x - 1)}{(qx - 1)(x - q)}1 \otimes ba - \frac{1}{(x - q)}b \otimes a.
\]

iii) For any coalgebra \( (C, \Delta, \varepsilon) \) and a parameter \( q \in k, \) the function \( R_C : X \to \text{End}_k(C \otimes C) \)

(4.37)

\[
R_C(x)(c \otimes d) = (x - 1)\varepsilon(c)\Delta(d) + q(x - 1)\varepsilon(d)\Delta(c) - (x - q)d \otimes c,
\]

is a one-parameter Yang-Baxter operator, where \( \varphi(x, z) = xz. \)

**Remark 4.2.** We consider the algebra \( A = \frac{k[X]}{(X^2 - \sigma)} \) as before and the same basis of \( A. \) Then the operator (4.36) reads

\[
(4.38)
\]

\[
R(x) = \begin{pmatrix}
qx - 1 & 0 & 0 & \sigma(q + 1)(x - 1) \\
0 & (x - 1)(q - 1) & 0 & 0 \\
0 & (q - 1)x & q(x - 1) & 0 \\
0 & 0 & 0 & q - x
\end{pmatrix}
\]
The system of equations (4.31–4.35) has another class of solutions: $\alpha(x) = x$, $\beta(x) = 1$ and $\gamma(x) = 1$, where $\varphi(x, z) = z$. Thus, we obtain the following proposition.

**Proposition 4.3.** i) The function $R' : X \to \text{End}_k(A \otimes A)$ defined by

$$R'(x)(a \otimes b) = x1 \otimes ab + ab \otimes 1 - b \otimes a$$

is a one-parameter Yang-Baxter operator, where $\varphi(x, z) = z$.

ii) If $x \neq 0$, then the operator (4.39) is invertible. Moreover, the following formula holds:

$$R'^{-1}(x)(a \otimes b) = ba \otimes 1 + \frac{1}{x} \otimes ba - b \otimes a .$$

**Remark 4.4.** The system of equations (4.31–4.35) has also the following class of solutions: $\alpha(x) = 1$, $\beta(x) = x$, and $\gamma(x) = 1$, where $\varphi(x, z) = x$.

Let us now compare our solutions with other well-known spectral-parameter dependent solutions of the QYBE. Of particular interest is a solution appearing in the context of generalisation of some exactly solvable $q$–state vertex models [14]. This solution coincides with another solution associated to a multiparametric quantum deformation of the universal enveloping algebra of a symmetrisable Kac-Moody algebra [12]. Consider the formula (3.3) in [12] (which is the multiparameter $R$–matrix for $U_{q,Q}(\hat{\text{sl}}(n)))$ for $n = 2$. We obtain

$$\hat{R} = c_{11}(q^2x - 1)E_{11} \otimes E_{11} + c_{22}(q^2x - 1)E_{22} \otimes E_{22}$$

$$+ c_{22}\gamma_{12}(q^2 - 1)E_{11} \otimes E_{22} + c_{11}\gamma_{21}(q^2 - 1)xE_{22} \otimes E_{11}$$

$$+ c_{12}s_{12}q(x - 1)E_{12} \otimes E_{21} + c_{21}s_{21}q(x - 1)E_{21} \otimes E_{12}$$

(4.40)

where parameters $c$, $s$ and $\gamma$’s are defined in [12]. In line with Remark 2 of [12], we specialise these to be 1 to obtain a single-parameter version. Note that (4.40) is a solution of the braid equation, which in the matrix form reads

$$\hat{R} = \begin{pmatrix}
q^2x - 1 & 0 & 0 & 0 \\
0 & q^2 - 1 & q(x - 1) & 0 \\
0 & q(x - 1) & (q^2 - 1)x & 0 \\
0 & 0 & 0 & q^2x - 1
\end{pmatrix}$$

(4.41)

On the other hand, we also note that our $R$–matrix (3.12) reduces to (4.38) by setting $p = 1$ and adjusting an overall factor of $\frac{1}{v}$. In order to compare (4.38) with (4.41) we have to set $\sigma = 0$ and compose (4.38) with the twist map to obtain it as a solution of the braid equation, from that of the QYBE (2.1). It turns out that both solutions (4.41) and twisted (4.38), though distinct, are very similar to each other and if we further set $q = 1$, then both reduce to the same trivial $R$–matrix. It would be interesting to find further classes of solutions to the system of equations (4.31–4.35) that might relate to other known solutions.
5. Yang–Baxter systems

Yang-Baxter systems were introduced in [7] as a spectral-parameter independent generalisation of the quantum Yang-Baxter equation related to non-ultralocal integrable systems studied previously in [6]. Yang-Baxter systems are conveniently defined in terms of Yang-Baxter commutators. Consider three vector spaces $V, V', V''$ and three linear maps $R : V \otimes V' \to V \otimes V'$, $S : V \otimes V'' \to V \otimes V''$, and $T : V' \otimes V'' \to V' \otimes V''$. Then a Yang-Baxter commutator is a map $[R, S, T] : V \otimes V' \otimes V'' \to V \otimes V' \otimes V''$, defined by

\[ [R, S, T] = R_{12} \circ S_{13} \circ T_{23} - T_{23} \circ S_{13} \circ R_{12} \, . \]

In terms of a Yang-Baxter commutator, the quantum Yang-Baxter equation (2.2) is expressed simply as $[R, R, R] = 0$.

Definition 5.1. Let $V$ and $V'$ be vector spaces. A system of linear maps $W : V \otimes V \to V \otimes V$, $Z : V' \otimes V' \to V' \otimes V'$, and $X : V \otimes V' \to V \otimes V'$ is called a WXZ-system or a Yang-Baxter system, provided the following equations are satisfied:

\[ [W, W, W] = 0 \, , \]

\[ [Z, Z, Z] = 0 \, , \]

\[ [W, X, X] = 0 \, , \]

\[ [X, X, Z] = 0 \, . \]

There are several algebraic origins and applications of WXZ-systems. It has been observed in [17] that WXZ-systems with invertible $W$, $X$ and $Z$ can be used to construct dually-paired bialgebras of the FRT type, thus leading to quantum doubles. Given a WXZ-system as in Definition 5.1 one can construct an invertible solution to the constant Yang-Baxter equation provided $W$, $X$ and $Z$ are invertible (see [1] for details). In [1] it was also shown that a Yang-Baxter system can be constructed from any entwining structure and conversely, that Yang-Baxter systems of certain types lead to entwining structures.

The next result is a new construction of a Yang-Baxter system.

Theorem 5.2. Let $A$ be a $k$-algebra and $\lambda, \mu \in k$. The following is a Yang-Baxter system:

\[ W : A \otimes A \to A \otimes A, \quad W(a \otimes b) = \lambda 1 \otimes ab + ab \otimes 1 - b \otimes a \]
\[ Z : A \otimes A \to A \otimes A, \quad Z(a \otimes b) = 1 \otimes ab + \mu ab \otimes 1 - b \otimes a \]
\[ X : A \otimes A \to A \otimes A, \quad X(a \otimes b) = 1 \otimes ab + ab \otimes 1 - b \otimes a \]

Proof. This follows from Proposition 4.3, Remark 4.4, and [2]. □
We now represent the Yang-Baxter system from the theorem above in dimension two in the same basis as in the previous sections:

\[ W = \begin{pmatrix} \lambda & 0 & 0 & \sigma(\lambda + 1) \\ 0 & \lambda & \lambda - 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\[ Z = \begin{pmatrix} \mu & 0 & 0 & \sigma(\mu + 1) \\ 0 & 1 & 0 & 0 \\ 0 & \mu - 1 & \mu & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\[ X = \begin{pmatrix} 1 & 0 & 0 & 2\sigma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

6. Concluding remarks

One of our main results is Theorem 2.1 defining a class of coloured Yang-Baxter operators (2.9) and both (3.12) and (3.27) are the respective matrix forms, i.e. spin–\(\frac{1}{2}\) and spin–1 \(R\)–matrices, for most general algebras in dimensions 2 and 3, respectively. This is distinct from the solutions appeared in [8], (3.12) is likely to correspond to some seven-vertex solution of Baxter’s solvable models. Similarly, Theorem 2.3 produces another class of such operators. As remarked earlier, it is an open problem to classify all coloured Yang-Baxter operators solving the system of equations (2.4–2.8).

It is pertinent to note that this work is in the spirit of references [2, 11], i.e. seeking a coloured generalisation of the (constant) Yang-Baxter operators that were derived from algebra structures which also produced new classes of solutions to the constant QYBE, again distinct from those that had appeared in the literature (see for instance [5]). This makes the framework of Yang-Baxter operators somewhat different and perhaps more general than the traditional approach of solving the QYBE.

We believe it could be of physical interest to look at Yang-Baxter operators and their coloured generalisations to make contact with integrability and vertex models in a way analogous to the additive solutions of the spectral parameter dependent QYBE. Furthermore, we have shown that one can associate a FRT bialgebra to a coloured Yang-Baxter operator exhibiting explicitly for the 2–dimensional case. It would be useful to investigate further such algebras (including the coloured quantum groups) which on the one hand seem quite constrained, but on the other, possess nice properties of the colour exchange symmetries.

Proposition 4.1 defines a class of one-parameter Yang-Baxter operators and Theorem 5.2 is another main result which associates a Yang-Baxter system to such operators. Further work along these lines will lead to a better understanding of spectral-parameter dependent Yang-Baxter operators.
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References

[1] Brzeziński, T. and Nichita, F. Yang-Baxter systems and entwining structures. Comm. Algebra 33 (2005), 1083–1093.
[2] Dăscălescu, S. and Nichita, F. Yang-Baxter operators arising from (co)algebra structures. Comm. Algebra 27 (1999), 5833–5845.
[3] Faddeev, L. D., Reshetikhin, N. Y. and Takhtajan, L. A. Quantisation of Lie groups and Lie algebras. Leningrad Math. J. 1 (1990) 193–225.
[4] Goucharenko, V. M. and Veselov, A. P. Yang-Baxter maps and matrix solitons. (in New trends in integrability and partial solvability) pp 191–197, NATO Sci. Ser. II Math. Phys. Chem. 132, Kluwer Academic Publishers, Dordrecht (2004).
[5] Hietarinta, J. Solving the two-dimensional constant Yang-Baxter equation J. Math. Phys. 34 (1993) 1725–1756.
[6] Hlavatý, L., Kundu, A., Quantum integrability of nonultralocal models through Baxterization of quantised braided algebra, Int. J. Mod. Phys. A11 (1996), 2143–2165.
[7] Hlavatý, L., Snobl, L., Solution of the Yang-Baxter system for quantum doubles, Int. J. Mod. Phys. A14 (1999), 3029–3058.
[8] Hlavatý, L. On solutions of the Yang-Baxter equations without additivity. J. Phys. A: Math. Gen. 25 (1992) 1395–1397 and references therein; Wang, S-K., Yang, H-T. and Wu, K. Classification of seven-vertex solutions of the coloured Yang-Baxter equation. arXiv:math-ph/9807003.
[9] Kassel, C. Quantum Groups. Graduate Texts in Mathematics 155. Springer Verlag (1995).
[10] Lambe, L. and Radford, D. Introduction to the quantum Yang-Baxter equation and quantum groups: an algebraic approach. Mathematics and its Applications 423. Kluwer Academic Publishers, Dordrecht (1997).
[11] Nichita, F. Self-inverse Yang-Baxter operators from (co)algebra structures. J. Algebra 218 (1999), 738–759.
[12] Okado, M. and Yamane, H. R–matrices with gauge parameters and multiparameter quantized enveloping algebras. (in Special functions, Okayama, 1990 eds. M. Kashiwara and T. Miwa) pp 289–293, IC–90 Satell. Conf. Proc., Springer, Tokyo (1991).
[13] Parashar D. Duality for coloured quantum groups. Lett. Math. Phys. 53 (2000) 29–40; The coloured quantum plane. J. Geom. Phys. 44 (2003) 481–488.
[14] Perk, J. H. H. and Schultz, C. L. Families of commuting transfer matrices in q–state vertex models. (in Nonlinear integrable systems – classical theory and quantum theory, Kyoto 1981 eds. M. Jimbo and T. Miwa) pp 135–152, World Scientific Publishing, Singapore (1983).
[15] Quesne, C. Coloured quantum universal enveloping algebras. J. Math. Phys. 38 (1997) 6018–6039; Duals of coloured quantum universal enveloping algebras and coloured universal T–matrices. J. Math. Phys. 39 (1998) 1199–1222.
[16] Veselov, A. P. Yang-Baxter maps and integrable dynamics. Phys. Lett. A 314 (2003) 214-221.

[17] Vladimirov, A.A., A method for obtaining quantum doubles from the Yang-Baxter $R$-matrices, Mod. Phys. Lett. A, 8 (1993), 1315–1321.

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