WELL-POSEDNESS AND CONVERGENCE OF A NUMERICAL SCHEME FOR THE CORRECTED DERRIDA-LEBOWITZ-SPEER-SPOHN EQUATION USING THE HELLINGER DISTANCE

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(Communicated by José A. Carrillo)

Abstract. In this paper we construct a unique global in time weak nonnegative solution to the corrected Derrida-Lebowitz-Speer-Spohn equation, which statistically describes the interface fluctuations between two phases in a certain spin system. The construction of the weak solution is based on the dissipation of a Lyapunov functional which equals to the square of the Hellinger distance between the solution and the constant steady state. Furthermore, it is shown that the weak solution converges at an exponential rate to the constant steady state in the Hellinger distance and thus also in the $L^1$-norm. Numerical scheme which preserves the variational structure of the equation is devised and its convergence in terms of a discrete Hellinger distance is demonstrated.

1. Introduction. Nonlinear evolution equations with higher-order spatial derivatives appear as approximate models in various contexts of mathematical physics. Besides the Cahn-Hillard equation [11], the most prominent models are various thin-film equations describing dynamics of the thickness of a thin viscous fluid film [16, 39, 37, 5]. In the case of free boundary film, the dynamics is driven by the competition between the surface tension and another potential force like gravity, capillarity, heating, Van der Waals, etc., which leads to a fourth-order evolution equation. Similarly, in the case when the fluid is covered by a thin elastic plate, then the pressure in the fluid is balanced by the sum of the bending of the plate and a potential force, which eventually leads to a sixth-order evolution equation [25, 34]. Many other higher-order models related to modelling of isolation oxidation of silicon in classical semiconductors [33], approximation of quantum effects in quantum semiconductors [18], description of Bose-Einstein condensate [26], image analysis [10], etc. can be found in the literature.

2020 Mathematics Subject Classification. Primary: 35B45, 35K30, 65M06, 65M12; Secondary: 35Q99, 65M15.

Key words and phrases. Fourth-order evolution equation, entropy methods, Hellinger distance, structure preserving numerical scheme, convergence.

This work has been supported by the Croatian Science Foundation under Grant agreement No. UIP-05-2017-7249 (MANDphy) and in part by the bilateral project No. HR 04/2018 between OeAD and MZO.
In this paper we study particular fourth-order evolution equation

$$\partial_t u = -\frac{1}{2} (u(\log u)_{xx})_{xx} + 2\delta \left( u^{3/4}(u^{1/4})_{xx} \right)_x,$$  \hspace{1cm} (1)

which has been derived in [7] as a corrected version of the well known Derrida-Lebowitz-Speer-Spohn (DLSS for short) equation. The latter first appeared in [19] in the form of (1) with $\delta = 0$. Unknown $u$ in (1) denotes the density function of a probability distribution which asymptotically describes the statistics of interface fluctuations between two phases of spins in the anchored Toom model. For the later reference we call (1) the corrected DLSS equation. To complete the problem for equation (1) we assume periodic boundary conditions, i.e. $x \in \mathbb{T} = [0, 1)$, where endpoints of the interval 0 and 1 are identified, and we prescribe a nonnegative initial datum $u(0) = u_0 \geq 0$ a.e. on $\mathbb{T}$.

Since the seminal paper by Bernis and Friedmann [4], analysis of higher-order nonlinear evolution equations, especially thin-film equations, has become an attractive field of interest in mathematics community. Existence of solutions and their qualitative properties like positivity, compact support, blow up, the long time asymptotics are among most important questions. Even the thin-film equation alone has very rich mathematical structure, which can be retrieved from [3, 17, 23, 14], and references therein. Adding lower (second) order unstable terms results in more complex dynamics [41, 38].

The original DLSS equation has been first analyzed by Bleher et. al. in [6]. Employing the semigroup approach they proved the local in time existence of positive classical solutions. Moreover, they proved equivalence between strict positivity and smoothness of the solution. The very same conclusions apply for the equation at hand. Namely, the third-order term from (1) enters into the “perturbation term” in [6], and all results apply analogously. The first construction of global in time weak nonnegative solutions to the DLSS equation, accompanied with the long time behavior analysis has been performed in [30]. Later on many results related to the DLSS equation have been achieved, we emphasize on [28], which generalizes the result from [30] to the multidimensional case, and [24], where the gradient flow structure has been rigorously justified. Namely, it has been shown that the DLSS equation constitutes the gradient flow of the Fisher information functional with respect to the $L^2$-Wasserstein metric. Let us point out at this place that the third-order term in equation (1) also possesses a geometric structure, it can be formally seen as a Hamiltonian flow of the Fisher information. A detailed discussion on this is postponed to Section 2. To conclude on the well-posedness for the DLSS equation, in [22] Fischer proved the uniqueness of weak solutions constructed by Jüngel and Matthes in [28].

Well-posedness and long time behaviour of solutions to equation (1) has been very recently discussed in [2] using Fourier analysis techniques and assuming smallness of initial data in a certain Wiener algebra. Our approach to the construction of global in time weak nonnegative solutions to equation (1) closely follows the ideas developed in [30] and [28]. Unlike there, where the key source of a priori estimates is the dissipation of the Boltzmann entropy, our construction is based on the dissipation of the functional

$$\mathcal{E}(u) = 2 \int_\mathbb{T} (u - 2\sqrt{u} + 1)dx.$$
\( \mathcal{E} \) is the only nontrivial “zero-order” functional, called entropy in further, for which we can formally prove the dissipation along solutions to (1), see Section 2 for more details. Observe that \( \mathcal{E}(u) \) is in fact proportional to the square of the Hellinger distance \( H(u, u_\infty) \) between \( u \) and \( u_\infty = 1 \) which is the constant steady state of equation (1). More precisely, \( \frac{1}{4} \mathcal{E}(u) = \frac{1}{2} \int_\Omega (\sqrt{u} - \sqrt{u_\infty})^2 \, dx \equiv H^2(u, u_\infty) \). The key source of our a priori estimates is the following entropy production inequality

\[
-\frac{d}{dt} \mathcal{E}(u(t)) \geq 4 \int_\Omega (u^{1/4})_x^2 \, dx , \quad t > 0 ,
\]

valid along smooth positive solutions of (1). The above inequality then motivates to rewrite the original equation (1) in a novel form in terms of \( u^{1/4} \) (cf. [32])

\[
\partial_t u = -2 \left( u^{1/2} \left( u^{1/4} (u^{1/4})_{xx} - (u^{1/4})_x^2 \right) \right)_x + 2 \delta (u^{3/4} (u^{1/4})_{xx})_x . \tag{2}
\]

Equation (2) is equivalent to (1) for smooth and strictly positive solutions. Now we can state the first result, which we prove in Section 3.

**Theorem 1.1.** Let \( u_0 \in L^1(\Omega) \) be given nonnegative function of unit mass and of finite entropy \( \mathcal{E}(u_0) < \infty \). Let \( T > 0 \) be given arbitrary time horizon. Then there exists a unique nonnegative unit mass function \( u \in W^{1,1}(0, T; H^{-2}(\Omega)) \) satisfying \( u^{1/4} \in L^2(0, T; H^2(\Omega)) \) and

\[
\int_0^T (\partial_t u, \phi)_{H^{-2}, H^2} \, dt \\
+ 2 \int_0^T \int_\Omega \left( u^{1/2} \left( u^{1/4} (u^{1/4})_{xx} - (u^{1/4})_x^2 \right) \phi_{xx} + \delta u^{3/4} (u^{1/4})_{xx} \phi_x \right) \, dx \, dt = 0
\]

for all test functions \( \phi \in L^\infty(0, T; H^2(\Omega)) \).

Concerning the question of the long time behaviour of weak solutions the approach is somewhat different than usual. This is due to the lack of a “global” Beckner type inequality for the parameter required by the entropy functional \( \mathcal{E} \), i.e. a lack of a “global” entropy-entropy production inequality. As a consequence, we cannot obtain time decay of the entropy functional \( \mathcal{E} \) at a universal exponential rate. Therefore, we employ appropriate “asymptotic” Beckner type inequality proved in [12], which will eventually provide an exponential time decay of the entropy functional \( \mathcal{E} \), but at a rate depending on the chosen initial datum \( u_0 \), or more precisely on \( \mathcal{E}(u_0) \).

Relation between the \( L^1 \) distance and the Hellinger distance, \( \|u - v\|_{L^1(\Omega)} \leq 2H(u, v) \), then implies the following result.

**Theorem 1.2.** Let \( u_0 \in L^1(\Omega) \) be a nonnegative function of unit mass such that \( \mathcal{E}(u_0) < +\infty \). Then the weak solution constructed in Theorem 1.1 converges at an exponential rate to the constant steady state \( u_\infty = 1 \) in the norm

\[
\|u(t) - u_\infty\|_{L^1(\Omega)} \leq \sqrt{\mathcal{E}(u_0)} e^{-\lambda t} , \quad t > 0 ,
\]

where \( \lambda = 4\pi^4 / (1 + C \sqrt{\mathcal{E}(u_0)}) \) and \( C > 0 \) depends only on \( \mathcal{E}(u_0) \).

It is very important that numerical schemes preserve some important features of equations of mathematical physics, for example positivity, conservation of mass, dissipation of certain functionals etc. Such schemes are then expected to be more reliable and robust to capture true behaviour of solutions, especially in the long run simulations. There are many such schemes in the literature devised for the original
DLSS equation [31, 13, 20, 8, 35, 36]. Here we discuss a discrete variational derivative (DVD) scheme [21], which is a slight modification of the scheme proposed in [8] for the original DLSS equation. DVD schemes are finite difference type schemes which respect the variational structure of equations. For smooth positive solutions, equation (1) can be rewritten in an equivalent (variational) form

$$\partial_t u = - \left( u \left( \frac{\sqrt{u}}{\sqrt{u}} \right)_x \right)_x + \delta \sqrt{u} (\sqrt{u})_{xxx},$$

which can be further written as

$$\partial_t u = (u \left( F'(u) \right)_x)_x - \delta \sqrt{u} (\sqrt{u} F'(u))_x,$$

where $F'(u) = - (\sqrt{u})_{xx}/\sqrt{u}$ denotes the variational derivative of the Fisher information. Form (4) of the equation obviously gives the dissipation of the Fisher information, and this is precisely the form of the $L^2$-Wasserstein gradient flow being justified in [24]. The main idea of DVD schemes is to construct a discrete analogue of (4), which will ensure the dissipation of the discrete version of the Fisher information on the discrete level.

However, we will not approximate directly (4), instead we approximate

$$\partial_t \sqrt{u} = \frac{1}{2\sqrt{u}} (u \left( F'(u) \right)_x)_x - \delta \frac{1}{2} (\sqrt{u} F'(u))_x.$$

Advantage of using this form has been already addressed in [6] and [22] for $\delta = 0$. The main cause lies in the monotonicity of the operator

$$K(v) = \frac{1}{v} \left( v^2 \left( \frac{\sqrt{v}}{v} \right)_x \right)_x,$$

which in our case will be the key ingredient for establishing the error estimates for the numerical scheme.

Let $\mathbb{T}_N = \{x_i : i = 0, \ldots, N, x_0 \equiv x_N\}$ denotes an equidistant grid of mesh size $h$ on the one dimensional torus $\mathbb{T} \cong [0, 1)$ and let the vector $U^k \in \mathbb{R}^N$ with components $U^k_i$, $i = 0, \ldots, N - 1$, $k \geq 0$, approximates solution $u(t_k, x_i)$ at point $x_i \in \mathbb{T}_N$ and time $t_k = k\tau$, where $\tau > 0$ denotes the time step. Given $U^0 \in \mathbb{R}^N$, the DVD scheme for equation (5) is defined by the following nonlinear system with unknowns $V^{k+1}_i = \sqrt{U^{k+1}_i}$:

$$\frac{1}{\tau} (V^{k+1}_i - V^k_i) = \frac{1}{2W^i} (\sqrt{W^{k+1/2}_i} W^{k+1/2}_i \delta_i^+(\delta F_d(W^{k+1/2}_i)))$$

$$- \frac{1}{2\sqrt{u}} (\sqrt{u})_{xx} + \delta \sqrt{u} (\sqrt{u} F'(u)),

for all $i = 0, \ldots, N - 1$, $k \geq 0$, where $W^{k+1/2}_i = (V^{k+1} + V^k)/2$ and $\delta F_d(W)_i = -\delta^{(2)} W/W$ denotes the discrete variational derivative of the discrete Fisher information $F_d$ defined by (56). Above $\delta^+_i$, $\delta_i^{(1)}$ and $\delta_i^{(2)}$ denote finite difference operators precisely introduced in section 4.1. Note that DVD scheme (6) imitates equation (5) on the discrete level, and particular combination of discrete operators is justified by the following result.

**Theorem 1.3.** Let $u_0 \in H^1(\mathbb{T})$ be a strictly positive initial datum of unit mass and of finite entropy $E(u_0) < \infty$. Let $u$ be the weak solution from Theorem 1.1 and assume that $u(t)$ is strictly positive for a.e. $t \in (0,T)$. Let $N \in \mathbb{N}$, $h = 1/N$, and $\tau > 0$ be space-time discretization parameters and let $U^0 \in \mathbb{R}^N_+$ be a pointwise
approximation of the initial datum $u_0$ satisfying $h \sum_{i=0}^{N-1} U_i^0 = 1$ and $H_d(u_0, U^0) \leq C h^2$, where $H_d$ is defined below in (8). Then solutions $U^k \in \mathbb{R}^N$, $k \geq 1$, of scheme (6) are nonnegative, satisfy $h \sum_{i=0}^{N-1} U_i^k = 1$, and the discrete Fisher information is nonincreasing, i.e. $F_d(U^{k+1}) \leq F_d(U^k)$ for all $k \geq 0$. Furthermore, there exists a constant $C > 0$, independent of $\tau$ and $h$, such that

$$h \sum_{i=0}^{N-1} \left( \sqrt{u_i^k} - \sqrt{U_i^k} \right)^2 \leq C \tau^2 + h^4 \frac{1}{1-\tau} \quad \text{for all } k \geq 1,$$

where $u_i^k = u(k\tau, ih)$ for $i = 0, 1, \ldots, N-1$ and $k \geq 1$.

**Remark 1.** Inequality (7) provides a quantitative error estimate and thus convergence of the DVD scheme. Defining a discrete analogue of the Hellinger distance as

$$H_d(U, V)^2 = \frac{h}{2} \sum_{i=0}^{N-1} \left( \sqrt{U_i} - \sqrt{V_i} \right)^2, \quad \text{for } U, V \in \mathbb{R}_+^N,$$

inequality (7) can be interpreted as $\sup_{k \in \mathbb{N}} H_d(u^k, U^k) \leq C(\tau + h^2)$ for some $C > 0$.

The paper is organized as follows. In Section 2 we discuss some formal dissipation properties and the geometric structure of the equation. Section 3 is devoted to proofs of Theorems 1.1 and 1.2, while in Section 4 we introduce the numerical scheme and prove its properties summarized in Theorem 1.3.

2. Formal dissipation properties and geometric structure.

2.1. Entropy production estimates.** Before we undertake a thorough analysis on the wellposedness, let us discuss some formal dissipation properties of equation (1), which will be in the heart of rigorous proofs. For this purpose we assume the existence of smooth and strictly positive solutions to equation (1) and consider a parametrized family of functionals of the form

$$E_\alpha(u) = \frac{1}{\alpha(\alpha - 1)} \int_T (u^\alpha - \alpha u + \alpha - 1)dx, \quad \alpha \neq 0, 1,$$

$$E_1(u) = \int_T (u \log u - u + 1)dx, \quad \alpha = 1,$$

$$E_0(u) = \int_T (u - \log u)dx, \quad \alpha = 0.$$  

In particular, we are looking for those functionals satisfying the so called Lyapunov property, i.e. $(d/dt) E_\alpha(u(t)) \leq 0$ along solutions to (1) for all $t > 0$. Although having the opposite sign, functionals (9) are often named entropies due to their connection to the Boltzmann-Shannon entropy $H(u) = -E_1(u)$ and Tsallis entropies $T_\alpha(u) = -\alpha E_\alpha(u)$. For smooth and positive solutions we can write equation (1) in an equivalent polynomial representation

$$\partial_t u = \left( u P_\delta \left( \frac{u}{u}, \frac{u_x}{u}, \frac{u_{xx}}{u} \right) \right)_x,$$

where the polynomial $P_\delta$ is given by

$$P_\delta(\xi_1, \xi_2, \xi_3) = -\frac{1}{2} \xi_3 + \xi_1 \xi_2 - \frac{1}{2} \xi_1^3 + \delta \left( \frac{1}{2} \xi_2 - \frac{3}{8} \xi_1^2 \right).$$  

(10)
Calculating the entropy production we find
\[
-\frac{d}{dt} \mathcal{E}_\alpha(u) = \int_T u^\alpha \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx =: \int_T u^\alpha S_\delta \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u} \right) \, dx,
\]
with the polynomial \( S_\delta \) given by
\[
S_\delta(\xi) = \xi_1 P_3(\xi) = -\frac{1}{2} \xi_1 \xi_3 + \xi_1^2 \xi_2 - \frac{1}{2} \xi_1^4 + \delta \left( \frac{1}{2} \xi_1 \xi_2 - \frac{3}{8} \xi_1^3 \right).
\]

We are now looking for all \( \alpha \in \mathbb{R} \) such that the integral inequality \(-d/dt \mathcal{E}_\alpha(u(t)) \geq 0\) holds. In order to assert the integral inequality, we systematically use integration by parts formulae and transform integrands using their polynomial representation. Observe that equation (1) itself, and thus polynomial \( S_\delta \) as well, does not possess a homogeneity properties like those in [27]. However, the method of algorithmic construction of entropies proposed in [27] can be adjusted to the equation at hand. First we identify elementary integration by parts formulae, which are represented by so called shift polynomials \( T_i(\xi) \):
\[
\int_T \left( u^\alpha \left( \frac{u_x}{u} \right)^3 \right) \, dx = \int_T u^\alpha T_1 \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx,
\]
where \( T_1(\xi) = 3\xi_1^2 \xi_2 + (\alpha - 3)\xi_1^4 \),
\[
\int_T \left( u^\alpha \frac{u_x u_{xx}}{u} \right) \, dx = \int_T u^\alpha T_2 \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx,
\]
where \( T_2(\xi) = \xi_1^2 + (\alpha - 2)\xi_1^3 \xi_2 + \xi_1 \xi_3 \),
\[
\int_T \left( u^\alpha \frac{u_{xxx}}{u} \right) \, dx = \int_T u^\alpha T_3 \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx,
\]
where \( T_3(\xi) = (\alpha - 1)\xi_1 \xi_3 + \xi_4 \),
\[
\int_T \left( u^\alpha \left( \frac{u_x}{u} \right)^2 \right) \, dx = \int_T u^\alpha T_4 \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx,
\]
where \( T_4(\xi) = (\alpha - 2)\xi_1^3 + 2\xi_1 \xi_2 \),
\[
\int_T \left( u^\alpha \frac{u_{xx}}{u} \right) \, dx = \int_T u^\alpha T_5 \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx,
\]
where \( T_5(\xi) = (\alpha - 1)\xi_1 \xi_2 + \xi_3 \).

All other integration by parts formulae can be obtained as linear combinations of these.

Observe that
\[
\int_T u^\alpha T_i \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx = 0, \quad i = 1, \ldots, 5,
\]
thus, adding an arbitrary linear combination of the above integrals to (11) does not change the value of the entropy production, but only changes the integrand, i.e. for any \( c_1, \ldots, c_5 \in \mathbb{R} \)
\[
-\frac{d}{dt} \mathcal{E}_\alpha(u) = \int_T u^\alpha \left( S_\delta + \sum_{i=1}^5 c_i T_i \right) \left( \frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{\partial^4 u}{u} \right) \, dx.
\]
into the polynomial decision problem:

\[
(\exists c_1, \ldots, c_5 \in \mathbb{R}), (\forall \xi \in \mathbb{R}^4), \left( S_0 + \sum_{i=1}^{5} c_i T_i \right)(\xi) \geq 0,
\]

which is according to the Tarski [40] always decidable (solvable). First we eliminate decision variables \(\xi_3\) and \(\xi_4\) due to indefiniteness of terms \(\xi_1 \xi_3\), \(\xi_3\) and \(\xi_4\). This is done by choosing constants \(c_2 = 1/2\) and \(c_3 = c_5 = 0\). After straightforward calculations the decision problem amounts to

\[
(\exists c_1, c_4 \in \mathbb{R}), (\forall \xi_1, \xi_2 \in \mathbb{R}), \left( \frac{1}{2} \xi_2^2 + \left( 3c_1 + \frac{\alpha}{2} \right) \xi_1^2 \xi_2 + \left( (\alpha - 3)c_1 - \frac{1}{2} \right) \xi_1^4 + \frac{1}{2} \xi_1 \xi_2 + \left( (\alpha - 2)c_4 - 3\frac{\delta}{8} \right) \xi_1^3 \geq 0, \right.
\]

which has only four decision variables. Solution of (13) can be resolved for instance with Wolfram Mathematica, which gives

\[
\delta = 0 \text{ and } 0 \leq \alpha \leq \frac{3}{2} \quad \text{or} \quad \delta > 0 \text{ and } \alpha = \frac{1}{2}.
\]

First part of the sentence (14), \(\delta = 0\) and \(0 \leq \alpha \leq 3/2\), is the well known result for the original DLSS equation [27], while the second part, \(\delta > 0\) and \(\alpha = 1/2\), concerns our equation (1), and provides only

\[
\mathcal{E}_{1/2}(u) = 2 \int_T (u - 2\sqrt{u} + 1) dx
\]

as an entropy (Lyapunov functional) for equation (1).

By means of the same method [27] as briefly presented above, the entropy production can be further estimated from below by a positive nondegenerate functional as follows:

\[
- \frac{d}{dt} \mathcal{E}_{1/2}(u(t)) \geq 4 \int_T (u^{1/4})^2_{xx} dx, \quad t > 0
\]

along smooth positive solutions to equation (1). In this way we also prove the following key estimate for the construction of weak solutions (cf. [28]).

**Proposition 1** (Entropy production estimate). Let \(u \in H^2(\mathbb{T})\) be strictly positive, then the following functional inequality holds

\[
- \int_T u(u^{-1/2})_{xx} (\log u)_{xx} \geq 4 \int_T (u^{1/4})^2_{xx} dx.
\]

**Proof.** For \(u \in H^2(\mathbb{T})\) strictly positive, the entropy production bound can be written as

\[
4 \int_T (u^{1/4})^2_{xx} dx = \int_T u^{1/2} P \left( \frac{u_x}{u}, \frac{u_{xx}}{u} \right) dx,
\]

where \(P(\xi_1, \xi_2) = \frac{1}{4} \left( \xi_2 - \frac{3}{4} \xi_1^2 \right)^2 \). In order to prove functional inequality (15), it suffices to verify the following decision problem

\[
(\exists c_1, c_4 \in \mathbb{R}), (\forall \xi_1, \xi_2 \in \mathbb{R}), \left( \frac{1}{4} \xi_2^2 + \left( 3c_1 - \frac{1}{8} \right) \xi_1^2 \xi_2 - \left( \frac{5}{2} c_1 + \frac{41}{64} \right) \xi_1^4 + \frac{1}{2} \frac{\delta}{2} \xi_1 \xi_2 - \left( \frac{3}{2} c_4 + \frac{3\delta}{8} \right) \xi_1^3 \right) \geq 0,
\]
which is obtained by inserting $\alpha = 1/2$ and subtracting $P(\xi_1, \xi_2)$ in (13). Employing again the quantifier elimination procedure implemented in Wolfram Mathematica, it comes out that (16) is TRUE, hence, (15) holds.

Although the above dissipation results for equation (1) seem to be poor in comparison with the dissipation properties of the original DLSS equation, it turns out that estimate (15) is sufficient for the construction of global weak solutions, which we perform in a subsequent section.

2.2. Geometric structure. Besides the entropy $E_{1/2}$, there is another distinguished Lyapunov functional for the dynamics of (1), the Fisher information, which is defined by

$$F(u) = \int_T (\sqrt{u})^2_x \, dx . \quad (17)$$

The Lyapunov property of the Fisher information is directly seen from the following equivalent (for smooth positive solutions) formulation of equation (1)

$$\partial_t u = - \left( u \left( \frac{(\sqrt{u})_{xx}}{\sqrt{u}} \right)_x \right) + \delta \sqrt{u} (\sqrt{u})_{xxx} , \quad (18)$$

which can be further written as

$$\partial_t u = (u \left( F'(u) \right)_x)_x - \delta \sqrt{u} \left( \sqrt{u} F'(u) \right)_x , \quad (19)$$

where $F'(u) = - (\sqrt{u})_{xx}/\sqrt{u}$ denotes the variational derivative of the Fisher information.

The first term on the right hand side in (19) has the well known structure of the gradient flow with respect to the $L^2$-Wasserstein metric. This structure has been rigorously justified and exploited for the original DLSS equation posed on the whole space [24]. It has been shown that the DLSS equation constitutes the gradient flow of the Fisher information with respect to the $L^2$-Wasserstein metric.

We find it a remarkable fact that the second term on the right-hand side in (19) formally possesses the structure of a Hamiltonian flow of the Fisher information, which we discuss more in detail bellow. Hence, equation (18) can be formally written as a mixture flow, i.e. the sum of the gradient and the Hamiltonian flow

$$\partial_t u = - \nabla_{W^2} F(u) + X_F(u) . \quad (20)$$

2.2.1. Symplectic structure of the third-order term. Let us briefly discuss the structure of the third-order term in (18), i.e. we only consider equation

$$\partial_t u = \sqrt{u} (\sqrt{u})_{xxx} . \quad (21)$$

First of all, direct formal calculations reveal that all functionals

$$F_n(u) = \int_T (\partial^n_x \sqrt{u})^2 \, dx , \quad n \in \mathbb{N}_0 ,$$

are constants of motion (first integrals) for equation (21). Namely,

$$\frac{d}{dt} F_n(u(t)) = (-1)^n \mod 2 \int_T \partial^n_x \sqrt{u} \partial^2_x \sqrt{u} \, dx = - \frac{1}{2} \int_T \partial_x (\partial^n_x + 1 \sqrt{u})^2 \, dx = 0 .$$

Note that for the full equation (18), only $F_0$ (mass) is conserved and $F_1$ (Fisher information) is dissipated, while for all other $n \geq 2$ the Lyapunov property of functionals $F_n$ is an open question. Using the method of systematic integration by
parts like above, production of the Fisher information can also be bounded from below as follows
\[
- \frac{d}{dt} \mathcal{F}(u(t)) \geq \kappa \int_T \left( (\sqrt{u})^2_{xxx} + (\sqrt{u})^6_x \right) dx,
\]  
for some \( \kappa > 0 \), which can be explicitly determined.

Following [1] let \( \mathcal{M} \) denotes the set of smooth positive densities on \( T \) (Radon derivatives w.r.t. the Lebesque measure). For every \( u \in \mathcal{M} \), let
\[
T_u \mathcal{M} := \text{Cl}_{L^2(T, u dx)} \{ \partial_x \phi \mid \phi \in C^\infty(T) \}
\]
denotes the tangent space at \( u \), and by \( T \mathcal{M} \) we denote the tangent bundle. There is a natural orthogonal decomposition of \( L^2(T, u dx) \) according to
\[
L^2(T, u dx) = T_u \mathcal{M} \oplus [T_u \mathcal{M}]^\perp,
\]
where \([T_u \mathcal{M}]^\perp = \{ v \in L^2(T, u dx) : \partial_x(uv) = 0 \}\), and let \( \pi_u : L^2(T, u dx) \to T_u \mathcal{M} \) denotes the orthogonal projection. For fixed \( u \in \mathcal{M} \) we define operator \( J_u : C^\infty(T) \to (C^\infty(T))^* \) by
\[
(J_u \phi, \psi) := - \int_T \sqrt{u} \partial_x(\sqrt{u} \phi) \psi dx, \quad \forall \psi \in C^\infty(T).
\]
If \( u \) is positive and smooth enough, \( J_u \phi \) is given by its \( L^2 \)-representative
\[
J_u \phi = - \sqrt{u} \partial_x(\sqrt{u} \phi) \in L^2(T, u dx).
\]
In such a case we define the subbundle \( \hat{T} \mathcal{M} \) according to
\[
\hat{T}_u \mathcal{M} := \{ \pi_u(J_u \phi) : \phi \in C^\infty(T) \}, \quad u \in \mathcal{M},
\]
and on that bundle we define differential 2-form \( \omega_u : \hat{T}_u \mathcal{M} \times \hat{T}_u \mathcal{M} \to \mathbb{R} \), by
\[
\omega_u(\xi_1, \xi_2) := (J_u \phi_1, \phi_2) = - \int_T \sqrt{u} \phi_2 \partial_x(\sqrt{u} \phi_1) dx,
\]
where \( \xi_i = \pi_u(J_u \phi_i) \) for \( i = 1, 2 \).

Observe that for every \( u \in \mathcal{M} \), 2-form \( \omega_u \) is bilinear and skew-symmetric. Also, for every \( u \in \mathcal{M} \) and \( 0 \neq \xi = \pi_u(J_u \phi) \in T_u \mathcal{M} \), choosing \( \eta = \pi_u(\phi) \neq 0 \), it follows
\[
\omega_u(\eta \xi, \xi) = ||\phi||^2_{L^2} \neq 0,
\]
which shows that \( \omega_u \) is nondegenerate. In order to prove that \( \omega_u \) is symplectic, it remains to check that it is exact, i.e., its external derivative equals zero. The following formula holds,
\[
\omega_u[\xi_0, \xi_1, \xi_2] = D_u \omega_u(\xi_1, \xi_2)[\xi_0] - D_u \omega_u(\xi_0, \xi_2)[\xi_1] + D_u \omega_u(\xi_0, \xi_1)[\xi_2] - \omega_u([\xi_0, \xi_1]_u, \xi_2) + \omega_u([\xi_0, \xi_2]_u, \xi_1) - \omega_u([\xi_1, \xi_2]_u, \xi_0),
\]
where \( D_u \omega_u(\xi_1, \xi_2)[\xi_0] \) denotes the differential of \( \omega \) with respect to \( u \) at point \((u; \xi_1, \xi_2)\) in the direction of \( \xi_0 \), and \([\cdot, \cdot]_u\) denotes the Poisson bracket of vector fields at point \( u \), defined by
\[
[\xi_0, \xi_1]_u := \phi_1 \xi_0 - \phi_0 \xi_1 = \phi_1 \pi_u(J_u \phi_0) - \phi_0 \pi_u(J_u \phi_1).
\]
Directly from the definition (25) we calculate
\[
D_u \omega_u(\xi_1, \xi_2)[\xi_0] := \frac{d}{ds} \omega_u(x_{\xi_0}(\xi_1, \xi_2))_{|s=0}
= -\frac{1}{2} \int_T \frac{1}{\sqrt{u}} \left( \phi_2 \partial_x(\sqrt{u} \phi_1) - \phi_1 \partial_x(\sqrt{u} \phi_2) \right) \xi_0 dx,
\]
and analogously other two expressions. Also, by the definition
\[
\omega_u([\xi_0, \xi_1]_u, \xi_2) = -\int_T \sqrt{u} (\phi_1 \partial_x (\sqrt{u} \phi_0) - \phi_0 \partial_x (\sqrt{u} \phi_1)) \phi_2 \, dx.
\]
Then straightforward calculations yield \( d\omega_u[\xi_0, \xi_1, \xi_2] = 0 \).

Let \( \mathcal{H} : \mathcal{M} \to \mathbb{R} \) be a Hamiltonian (for instance the Fisher information \( F \)), the corresponding Hamiltonian vector field \( X_{\mathcal{H}} \) is defined through the identity
\[
d\mathcal{H}_u(\xi) = \omega_u(\xi, X_{\mathcal{H}}(u)) = -\int_T \sqrt{u} \phi_H \partial_x (\sqrt{u} \phi) \, dx
\]
for all \( \xi = \pi_u([\mathbb{I}_u \phi]) \), \( \phi \in C^\infty(\mathbb{T}) \) and \( X_{\mathcal{H}} = [\mathbb{I}_u \phi_H] \). On the other hand
\[
d\mathcal{H}_u(\xi) = \int_T \mathcal{H}'(u) \xi \, dx = -\int_T \mathcal{H}'(u) \sqrt{u} \phi \partial_x (\sqrt{u} \phi) \, dx = \int_\mathbb{R} \sqrt{u} \phi \partial_x (\sqrt{u} \mathcal{H}'(u)) \phi \, dx,
\]
which shows that \( X_{\mathcal{H}}(u) = [\mathbb{I}_u \mathcal{H}'](u) \).

3. Well-posedness and long time behavior of nonnegative weak solutions.

3.1. Existence of weak solutions — proof of Theorem 1.1. Construction of the weak solution is divided into three main steps: analysis of the time discrete problem, passage to the limit \( \tau \downarrow 0 \) with the time step \( \tau \) and discussion on uniqueness.

3.1.1. Time discrete equation. Let \( \tau > 0 \) be given time step. We discretize equation (1) in time by means of the implicit Euler scheme. The semi-discrete equation then reads
\[
\frac{1}{\tau} (u - u_0) = -\frac{1}{2} (\log u)_{xx} + 2\delta \left( u^{3/4}(u^{1/4})_{xx} \right)_x \quad \text{on} \; T,
\]
where \( u_0 \geq 0 \) a.e. is given. Our aim is to solve nonlinear equation (27) by means of the fixed point method. For this purpose we divide our procedure into several steps. First we linearize and regularize equation (27). Linearization is performed by the change of variables \( y = \log u \), while we regularize it by adding an elliptic operator \( \varepsilon (\partial_y^2 y - y) - \varepsilon ((\log u)_y y)_x \), where \( \varepsilon > 0 \) is a small parameter. For given strictly positive function \( u = e^z \) and \( \sigma \in [0, 1] \) we then relax the above equation (27) into a linear elliptic equation in terms of \( y \):
\[
\frac{\sigma}{\tau} (e^z - u_0) = -\frac{1}{2} (e^z y_{xx})_x + 2\delta \sigma \left( e^z \left( \frac{z_{xx}}{4} + \frac{z_4}{16} \right) \right)_x + \varepsilon \left( \partial_y^2 y + (e^z y_y)_x - y \right).
\]
More precisely, for fixed \( \varepsilon > 0 \) we have formulated the fixed point mapping \( S_z : H^2(\mathbb{T}) \times [0, 1] \to H^2(\mathbb{T}) \) defined by \( S_z(z, \sigma) = y \), where \( y \in H^3(\mathbb{T}) \) is the unique solution to the elliptic problem (28).

The existence and uniqueness of \( y \) follows directly from the Lax-Milgram lemma. By standard arguments we also assert continuity and compactness of the operator \( S_z \). Observe that \( S_z(0, 0) = 0 \) for every \( z \in H^2(\mathbb{T}) \), while fixed points of \( S_z(\cdot, 1) \) will be solutions to the regularization of equation (27). In order to apply the Leray-Schauder fixed point theorem and conclude the existence of solutions we need a uniform (in \( \sigma \)) estimate on the set of fixed points of \( S_z(\cdot, \sigma) \) for all \( \sigma \in [0, 1] \). Let
\[ y \in H^3(\mathbb{T}) \] be such a fixed point. Employing the test function \( \phi = 2 - 2e^{-\eta/2} \in H^3(\mathbb{T}) \) in the weak formulation of (28) and using the estimate (15) we find
\[
\frac{\sigma}{\tau} E(e^y) + 4 \int_\mathbb{T} \left( e^{y/4} \right)_x^2 \, dx + \varepsilon \kappa \int_\mathbb{T} e^{-\eta/2} \left( y_{xxx} + y_x^2 \right) \, dx \leq \frac{\sigma}{\tau} E(u_0),
\]
where \( \kappa > 0 \) is some positive constant. Here we used the pointwise inequality \( a(1 - e^{-a/2}) \geq 0 \) for all \( a \in \mathbb{R} \). At this point it also becomes apparent why the regularization part contains the term \( \varepsilon \left( z^2 \right)_{yy} \). Namely, the linear regularization solely, would destroy the dissipation structure of the original equation, while adding this nonlinear term ensures the above uniform estimate. Estimate (29) also implies
\[
\| e^{y/2} - 1 \|_{L^2(\mathbb{T})} \leq C,
\]
which provides \( \| e^{y/4} \|_{L^2(\mathbb{T})} \leq C \), where \( C \) is independent of \( \sigma \). The latter conclusion together with (29) implies in further \( \| e^{y/4} \|_{H^2(\mathbb{T})} \leq C \), while the continuity of the Sobolev embedding \( H^2(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T}) \) asserts \( \| e^{y/4} \|_{L^\infty(\mathbb{T})} \leq C \).

Combining this again with (29) gives \( \sqrt{\varepsilon} \| y_{xxx} \|_{L^2(\mathbb{T})} \leq C \), while using the test function \( \phi = 1 \) in (28) gives \( \varepsilon \int_\mathbb{T} y \, dx \leq C \). The last two statements with help of the Poincaré inequality eventually provides the uniform (\( \sigma \)-independent) estimate \( \| y \|_{H^2(\mathbb{T})} \leq C \), i.e. \( \| y \|_{H^2(\mathbb{T})} \leq C \). The last constant \( C \) depends on \( \varepsilon \), but this is not an issue here. With this uniform estimate we conclude the existence of a fixed point \( y_\varepsilon \) of the mapping \( S_\varepsilon (\cdot, 1) \), and thus, the existence of a weak solution of
\[
\frac{1}{\tau} (u_\varepsilon - u_0) = -\frac{1}{2} \left( u_\varepsilon y_{xx} \right)_{xx} + 2\delta \left( u_\varepsilon^{4/3}(u_\varepsilon^{1/4})_{xx} \right)_x + \varepsilon \left( \partial_x^6 y_\varepsilon + (y_{xx})_x - y_\varepsilon \right),
\]
where \( u_\varepsilon = e^{y_\varepsilon} \), and therefore \( u_\varepsilon \) is strictly positive.

Our next step is to regularize equation (30), i.e. we consider the limit of all terms in (30) as \( \varepsilon \downarrow 0 \). Again from estimate (29) (with \( \sigma = 1 \)) and above discussion we conclude
\[
\sqrt{\varepsilon} \| y_{dx} \|_{L^2(\mathbb{T})} + \varepsilon^{1/6} \| y_{dx} \|_{L^6(\mathbb{T})} + \sqrt{\varepsilon} \| y \|_{L^2(\mathbb{T})} \leq C,
\]
where \( C > 0 \) is independent of \( \varepsilon \). The latter \( L^2 \) estimate essentially follows from (30) utilizing the pointwise inequality \( 2a(1 - e^{-a/2}) \geq a^2 - e^{a/2} - 5 \) for all \( a \in \mathbb{R} \).

As we already discussed above, we have
\[
\| u_\varepsilon^{1/4} \|_{H^2(\mathbb{T})} \leq C,
\]
which implies (up to a subsequence)
\[
u_\varepsilon^{1/4} \rightharpoonup u^{1/4} \quad \text{weakly in } H^2(\mathbb{T}).
\]
Since \( u_\varepsilon \) is strictly positive and smooth enough, we can write
\[
u_\varepsilon y_{dx} = 4u_\varepsilon^{1/2} \left( u_\varepsilon^{1/4}(u_\varepsilon^{1/4})_{xx} - (u_\varepsilon^{1/4})^2_x \right),
\]
and invoking compactness of Sobolev embeddings \( H^2(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T}) \) and \( H^2(\mathbb{T}) \hookrightarrow W^{1,4}(\mathbb{T}) \), we conclude:
\[
u_\varepsilon y_{dx} \rightharpoonup 4u^{1/2} \left( u^{1/4}(u^{1/4})_{xx} - (u^{1/4})^2_x \right) \quad \text{weakly in } L^2(\mathbb{T}),
\]
\[
u_\varepsilon^{3/4}(u_\varepsilon^{1/4})_{xx} \rightharpoonup (u^{3/4}(u^{1/4})_{xx})_x \quad \text{weakly in } L^2(\mathbb{T}),
\]
\[
u_\varepsilon \rightharpoonup u \quad \text{strongly in } L^\infty(\mathbb{T}).
\]
Moreover, uniform estimate (31) implies
\[
\varepsilon \left( \partial_x^6 y_\varepsilon + (y_{xx})_x - y_\varepsilon \right) \to 0 \quad \text{strongly in } H^3(\mathbb{T}),
\]
and we finally conclude that $u$ is a weak solution to
\[ \frac{1}{\tau}(u - u_0) = -2 \left( u^{1/2} \left( u^{1/4}(u^{1/4})_{xx} - (u^{1/4})^2 \right) \right)_{xx} + 2\delta \left( u^{3/4}(u^{1/4})_{xx} \right)_x \text{ on } T. \]

Employing the test function $\phi = 1$, it readily follows that $\int_T u \, dx = \int_T u_0 \, dx = 1$. Standard arguments of weak lower semicontinuity of both entropy and the entropy production bound provide the discrete entropy production inequality
\[ \mathcal{E}(u) + 4\tau \int_T (u^{1/4})^2_{xx} \, dx \leq \mathcal{E}(u_0), \tag{35} \]
which is essential for the next step of the procedure.

3.1.2. Passage to the limit $\tau \downarrow 0$. Let the time horizon $T > 0$ and the time step $\tau > 0$ be such that $T/\tau = N \in \mathbb{N}$. Then using recursively procedure from the previous step we construct solutions $u^k_\tau$ satisfying
\[ \frac{1}{\tau}(u^k_\tau - u^{k-1}_\tau) = -2 \left( u^{1/2}_\tau \left( u^{1/4}_\tau \left( (u^{1/4}_\tau)_{xx} - (u^{1/4}_\tau)^2 \right) \right)_{xx} \right. \]
\[ + 2\delta \left( u^{3/4}_\tau \left( (u^{1/4}_\tau)_{xx} \right)_x \right) \]
\[ \text{for } k = 1, \ldots, N, \]

Defining the step function $u_\tau$ according to
\[ u_\tau(0) := u_0, \]
\[ u_\tau(t) := u^k_\tau, \quad (k-1)\tau < t \leq k\tau, \quad k = 1, \ldots, N, \]

the sequence of equations in (36) sums up to
\[ \frac{1}{\tau} \int_0^T (u_\tau - \sigma_\tau u_\tau) \phi \, dx \, dt = -2 \int_0^T \left( u^{1/2}_\tau \left( u^{1/4}_\tau \left( (u^{1/4}_\tau)_{xx} - (u^{1/4}_\tau)^2 \right) \right) \right)_{xx} \phi_{xx} \, dx \, dt \]
\[ - 2\delta \int_0^T \int_T u^{3/4}_\tau \left( (u^{1/4}_\tau)_{xx} \right)_x \phi_x \, dx \, dt, \tag{38} \]

for all test functions $\phi \in L^1(0, T; H^2(\Omega))$, while inequality (37) results in
\[ \mathcal{E}(u^N_\tau) + 4 \int_0^T (u^{1/4}_\tau)_{xx}^2 \, dx \, dt \leq \mathcal{E}(u_0). \tag{39} \]

The last inequality directly implies uniform (in $\tau$) estimates
\[ \|(u^{1/4}_\tau)_{xx}\|_{L^2(0, T; L^2(\Omega))} \leq C, \]
\[ \|\sqrt{u_\tau} - 1\|_{L^\infty(0, T; L^2(\Omega))} \leq C, \]

which jointly imply $\|u^{1/4}_\tau\|_{L^2(0, T; H^2(\Omega))} \leq C$ and therefore we have (up to a subsequence) the weak convergence of the sequence $(u^{1/4}_\tau)$ to some $v \in L^2(0, T; H^2(\Omega))$, i.e.
\[ u^{1/4}_\tau \rightharpoonup v \quad \text{weakly in } L^2(0, T; H^2(\Omega)). \tag{40} \]

Estimate $\|\sqrt{u_\tau}\|_{L^\infty(0, T; L^2(\Omega))} \leq C$ implies $\|u^{1/4}_\tau\|_{L^\infty(0, T; L^4(\Omega))} \leq C$. Furthermore, the Gagliardo-Nirenberg inequality provides
\[ \|u^{1/4}_\tau(t)\|_{L^\infty(\Omega)} \leq C \|u^{1/4}_\tau(t)\|_{H^2(\Omega)}^{1/7} \|u^{1/4}_\tau(t)\|_{L^4(\Omega)}^{6/7}, \]
for a.e. \( t \in (0, T) \). Therefore, the Young inequality gives the uniform bound
\[
\|u^{1/4}_\tau\|_{L^7(0, T; L^\infty(\mathbb{T}))} \leq C. \tag{41}
\]
The above obtained estimates are now sufficient to conclude the uniform a priori estimate on the sequence of finite differences
\[
\tau^{-1} \|u_\tau - \sigma_\tau u_\tau\|_{L^1(\tau, T; H^{-2}(\mathbb{T}))} \leq C, \tag{42}
\]
where \( \sigma_\tau u_\tau = u_\tau (\cdot - \tau) \) denotes the left shift operator in time.

Let us now prove that
\[
\|u^{1/4}_\tau\|_{L^\infty(0, T; L^\infty(\mathbb{T}))} \leq C. \tag{43}
\]
Employing the Poincaré-type inequality together with the Gagliardo-Nirenberg inequality: for a.e. \( t \in (0, T) \), we find
\[
\left\|u^{1/4}_\tau(t) - u^{1/4}_\tau(0)\right\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2} \left\|\left(u^{1/4}_\tau(t)\right)_t\right\|_{L^1(\mathbb{T})} \leq C \|u^{1/4}_\tau\|_{H^2(\mathbb{T})}\|u^{1/4}_\tau(t)\|^{1-\theta}_{L^\infty(\mathbb{T})},
\]
where \( u^{1/4}_\tau(t) = \int_0^t u^{1/4}_\tau(t) \, dx \) and \( \theta = 2/(3q + 2) \) for any \( q > 1 \). Since \( u^{1/4}_\tau(t) \leq 1 \), employing the triangle inequality, it follows that
\[
\left\|u^{1/4}_\tau(t)\right\|_{L^\infty(\mathbb{T})}^{3q+2} \leq C \left(\|u^{1/4}_\tau\|_{L^2(0, T; H^2(\mathbb{T}))}^2 + \|u^{1/4}_\tau\|_{L^\infty(0, T; L^\infty(\mathbb{T}))}^3\right) + 1.
\]
Due to the continuity of the embedding \( L^q(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T}) \) and the Cauchy-Schwarz inequality, integrating the last inequality over \( (0, T) \) we further estimate
\[
\left\|u^{1/4}_\tau\right\|_{L^{3q+2}(0, T; L^\infty(\mathbb{T}))}^{3q+2} \leq C \left(\|u^{1/4}_\tau\|_{L^2(0, T; H^2(\mathbb{T}))}^2 + \|u^{1/4}_\tau\|_{L^\infty(0, T; L^\infty(\mathbb{T}))}^3\right)^2 + 1.
\]
Now employing (40) and (41), by the bootstrapping argument we conclude that \( u^{1/4}_\tau \) is uniformly bounded in \( L^r(0, T; L^\infty(\mathbb{T})) \) for every \( r > 1 \). Hence, (43) follows.

Next we want to prove
\[
\|\sqrt{u_\tau}\|_{L^2(0, T; H^2(\mathbb{T}))} \leq C. \tag{44}
\]
Notice that for smooth positive \( u \) the following identities hold pointwise:
\[
(u^{1/2})_x = 2u^{1/4}(u^{1/4})_x, \quad (u^{1/2})_{xx} = 2\left(u^{1/4}(u^{1/4})_{xx} + (u^{1/4})^2_x\right), \tag{45}
\]
Using an approximation argument, the same identities hold true pointwise a.e. for nonnegative \( u_\tau \) satisfying \( u^{1/4}_\tau \in L^2(0, T; H^2(\mathbb{T})) \). Using the Gagliardo-Nirenberg inequality, for a.e. \( t \in (0, T) \) we have
\[
\left\|\left(u^{1/4}_\tau(t)\right)_x\right\|_{L^4(\mathbb{T})} \leq C \left\|u^{1/4}_\tau(t)\right\|_{H^2(\mathbb{T})}^{1/2} \left\|u^{1/4}_\tau(t)\right\|_{L^\infty(\mathbb{T})}^{1/2},
\]
which implies
\[
\left\|\left(u^{1/4}_\tau(t)\right)_x\right\|_{L^4(0, T; L^4(\mathbb{T}))}^4 \leq C \left\|u^{1/4}_\tau(t)\right\|_{L^2(0, T; H^2(\mathbb{T}))}^2 \left\|u^{1/4}_\tau(t)\right\|_{L^\infty(0, T; L^\infty(\mathbb{T}))}^2 \leq C.
\]
Combining the latter with (40) and (43) in (45), estimate (44) follows.

Finally, we want to prove the uniform estimate
\[
\|u_\tau\|_{L^2(0, T; W^{2,1}(\mathbb{T}))} \leq C. \tag{46}
\]
Again observe that for smooth positive \( u \) it holds:
\[
u_x = 4u^{3/4}(u^{1/4})_x, \quad u_{xx} = 4u^{3/4}(u^{1/4})_{xx} + 12u^{1/2}(u^{1/4})^2_x.
\]
Standard approximation arguments and application of uniform estimates (40), (43) and (44) imply the desired bound (46). Having at hand estimates (42) and (46) we can invoke the Aubin-Lions lemma [15] and conclude the strong convergence (on a subsequence as $\tau \downarrow 0$)

$$u_\tau \to u \quad \text{strongly in } L^2(0, T; W^{1,6}(\mathbb{T})).$$

(47)

In order to pass to the limit as $\tau \downarrow 0$ in (38), we need to explore some more convergence results. First, using (47) we identify in (40) $v = u^{1/4}$. Then, employing [29, Proposition 6.1] on (40) and (47) we conclude

$$u_\tau^{3/4} \to u^{3/4} \quad \text{strongly in } L^8(0, T; W^{1,8}(\mathbb{T})),
$$

(48)

and similarly

$$u_\tau^{1/2} \to u^{1/2} \quad \text{strongly in } L^{12}(0, T; W^{1,12}(\mathbb{T})).$$

Using a stronger version of the entropy production inequality, namely

$$\mathcal{E}(u^N_t) + \kappa \int_0^T \int_\mathbb{T} \left( (u_\tau^{1/4})^2_{xx} + (u_\tau^{1/8})^2_x \right) dx dt \leq \mathcal{E}(u_0)$$

for some $\kappa > 0$, which can be proved in the same fashion as the basic one, we immediately have the uniform bound

$$\|u_\tau^{1/8}\|_{L^4(0, T; W^{1,4}(\mathbb{T}))} \leq C.$$

Combining the latter with (48) we conclude (again using [29, Proposition 6.1])

$$u_\tau^{1/4} \to u^{1/4} \quad \text{strongly in } L^4(0, T; H^2(\mathbb{T})).$$

The above convergence results are now sufficient to pass to the limit in (38).

3.1.3. Uniqueness. Uniform estimates of the previous subsection provide that weak solutions satisfy $u^{1/4} \in L^2(0, T; H^2(\mathbb{T}))$ and $u^{1/2} \in L^2(0, T; H^2(\mathbb{T}))$. This is precisely the required regularity in [22], which ensures the proof of the uniqueness of the global weak solution constructed by Jüngel and Matthes in [28] for the DLSS equation ($\delta = 0$ in (1)). Following [22], one first has to prove that weak solutions constructed above satisfy the square root form of the corrected DLSS equation

$$2\partial_t \sqrt{u} = -\sqrt{u}_{xxxx} + \frac{(\sqrt{u})^2_{xx}}{\sqrt{u}} + \delta(\sqrt{u})_{xxx}$$

(49)

in a weak sense. Recall, precisely this square root form of the DLSS equation has been employed in [6] for the proof of the local existence of a unique smooth and positive solution. Utilizing the weak formulation of (49) (see (70)), one then has to prove that the Hellinger distance between two weak solutions $u_1$ and $u_2$ verifies: for a.e. $t_2 > t_1 \geq 0$

$$H(u_1(t_2), u_2(t_2)) \leq H(u_1(t_1), u_2(t_1)).$$

(50)

This will imply the uniqueness.

Let $u$ be a weak solution of (2) constructed above, which satisfies $u^{1/4} \in L^2(0, T; H^2(\mathbb{T}))$, $u^{1/2} \in L^2(0, T; H^2(\mathbb{T}))$ and $u \in L^\infty(0, T; L^\infty(\mathbb{T}))$. Notice that identity

$$2\sqrt{u} \left( (u^{1/4})_{xx} - (u^{1/4})^2_x \right) = \sqrt{u} (\sqrt{u})_{xx} - (\sqrt{u})^2_x$$

(51)

holds in the sense of $L^2(0, T; L^2(\mathbb{T}))$. Thus, for the derivation of the evolution of $\sqrt{u}$ for the fourth-order operator, we can literally plug in calculations and arguments from [22, Lemma 15]. Regarding the third-order operator from (2), we provide all
3.2. Large time behavior of weak solutions. The discrete entropy production inequality

$$\mathcal{E}(u_N^T) + 4 \int_0^T \int_T (u_T^{1/4})^2_{xx} \, dx \, dt \leq \mathcal{E}(u_0)$$

provides the Lyapunov stability. Namely, using the weak lower semicontinuity of the functional on the left hand side, for the weak solution $u$ of (1) we have

$$\sup_{t \in (0,T)} \mathcal{E}(u(t)) \leq \mathcal{E}(u_0).$$

In order to conclude a stronger result, one needs an entropy – entropy production inequality which stems from a Beckner type inequality. Unfortunately, the “global” Beckner inequality of type

$$\frac{p}{p-1} \left( \int_T f^2 \, dx - \left( \int_T f^{2/p} \, dx \right)^p \right) \leq C_B \int_T (f_{xx})^2 \, dx$$

is valid for $p \in (1,2)$. In order to apply such inequality in our case, we would need inequality (52) with $p = 1/2$, which is out of the scope here. Therefore, we rely on an “asymptotic” Beckner inequality proved in [12, Corollary 2]: for any $p > 0$, $q \in \mathbb{R}$ and $\varepsilon_0 > 0$, there exists a positive constant $C$ (depending on $p,q$ and $\varepsilon_0$) such that, for any $\varepsilon \in (0,\varepsilon_0]$

$$\Sigma_{p,q}(f) := \frac{1}{pq(pq - 1)} \left( \int_T f^q \, dx - \left( \int_T f^{1/p} \, dx \right)^{pq} \right) \leq \frac{1 + C \varepsilon}{32p^2 \pi^4} \int_T (f_{xx})^2 \, dx$$

for all $f \in L^{p,q}_x = \{ f \in H^2(T) : f \geq 0 \ \text{a.e.}, \ \Sigma_{p,q}(f) \leq \varepsilon \}$ and $\int_T f^{1/p} \, dx = 1$.

Proof of Theorem 1.2. Let $u_0$ be given initial datum, $\tau > 0$ and let $u_1^\varepsilon, u_2^\varepsilon, \ldots$ be the sequence of solutions to the semi-discrete problem constructed in section (3.1.1). The discrete entropy production inequality (35) provides

$$\mathcal{E}(u_k^\varepsilon) + 4\tau \int_T ((u_k^\varepsilon)^{1/4})^2_{xx} \, dx \leq \mathcal{E}(u_{k-1}^\varepsilon), \quad \forall k \in \mathbb{N}. \quad (54)$$

Employing inequality (53) with $p = 1/4$, $q = 2$, $\varepsilon_0 = \mathcal{E}(u_0)$ and $f = (u_k^\varepsilon)^{1/4}$ it readily follows

$$\mathcal{E}(u_k^\varepsilon) \leq \frac{1 + C \sqrt{\varepsilon_0}}{2 \pi^4} \int_T ((u_k^\varepsilon)^{1/4})^2_{xx} \, dx, \quad \forall k \in \mathbb{N}. \quad (55)$$

Combining (54) and (55) yields

$$\mathcal{E}(u_k^\varepsilon) + \frac{8\pi^4 \tau}{1 + C \sqrt{\varepsilon_0}} \mathcal{E}(u_k^\varepsilon) \leq \mathcal{E}(u_{k-1}^\varepsilon), \quad \forall k \in \mathbb{N},$$

which passing to the limit $\tau \downarrow 0$ implies

$$\mathcal{E}(u(t)) \leq \mathcal{E}(u_0)e^{-\frac{8\pi^4 \tau}{1 + C \sqrt{\varepsilon_0}}}, \quad t > 0.$$

The well known relation between the $L^1$ and the Hellinger distance finally provides

$$\|u(t) - 1\|_{L^1(T)} \leq 2\mathcal{H}(u(t),1) = \sqrt{\mathcal{E}(u(t))} \leq \sqrt{\mathcal{E}(u_0)}e^{-\frac{8\pi^4 \tau}{1 + C \sqrt{\varepsilon_0}}}, \quad t > 0.$$

□
4. A structure preserving numerical scheme.

4.1. Introduction of the scheme. In this section we devise a numerical scheme for equation (1), which respects its basic properties: nonnegativity, mass conservation and the dissipation of the Fisher information on the discrete level. More precisely, the scheme is a discretization of (18) with the time discretization inspired by (70). It is a discrete variational derivative (DVD) type scheme, which is a slight modification of the scheme from [8] proposed for the original DLSS equation. The advantage of the method proposed here is the error estimate given in terms of the discrete Hellinger distance.

Let $\mathcal{T}_N = \{x_i : i = 0, \ldots, N, x_0 \equiv x_N\}$ denote an equidistant discrete grid of mesh size $h$ on the one dimensional torus $\mathbb{T} \equiv [0, 1)$ and let the vector $U^k \in \mathbb{R}^N$ with components $U^k_i$ approximates the solution $u(t_k, x_i)$ for $i = 0, \ldots, N - 1$ and $k \geq 0$. We will use the following standard finite difference operators. For $U \in \mathbb{R}^N$ define:

- **forward difference:** $\delta^+_i U = h^{-1}(U_{i+1} - U_i)$,
- **backward difference:** $\delta^-_i U = h^{-1}(U_i - U_{i-1})$,
- **central difference:** $\delta^{(1)}_i U = (2h)^{-1}(U_{i+1} - U_{i-1})$,
- **2nd order central difference:** $\delta^{(2)}_i U = \delta^+_i \delta^-_i U = h^{-2}(U_{i+1} - 2U_i + U_{i-1})$.

To approximate the integral of one-periodic function $w$, we use the first-order quadrature rule $\sum_{i=0}^{N-1} w(x_i)h$. This rule is in fact of the second order, since due to the periodic boundary conditions it coincides with the trapezoidal rule $(w(x_0) + w(x_N))h/2 + \sum_{i=1}^{N-1} w(x_i)h$.

The first step is to define a discrete analogue of the Fisher information $\mathcal{F}_d : \mathbb{R}^N \to \mathbb{R}$ as an approximation of the true Fisher information $\mathcal{F}$. The basic idea of DVD methods is to perform a discrete variation procedure and calculate the corresponding discrete variational derivative. We approximate the Fisher information $\mathcal{F}(u)$ by

$$
\mathcal{F}_d[U] = \frac{1}{2} \sum_{i=0}^{N-1} ((\delta^+_i V_i)^2 + (\delta^-_i V_i)^2)h,
$$

(56)

where $U \in \mathbb{R}^N$ and $V_i = \sqrt{U_i}$ for $i = 0, \ldots, N - 1$. Applying the discrete variation procedure and using summation by parts formula (see [21, Proposition 3.2]) for periodic boundary conditions, we calculate:

$$
\mathcal{F}_d[U^{k+1}] - \mathcal{F}_d[U^k] = \frac{1}{2} \sum_{i=0}^{N-1} ((\delta^+_i V_i^{k+1})^2 - (\delta^+_i V_i^k)^2 + (\delta^-_i V_i^{k+1})^2 - (\delta^-_i V_i^k)^2)h
$$

$$
= \frac{1}{2} \sum_{i=0}^{N-1} \Big( \delta^+_i (V_i^{k+1} + V_i^k) \delta^+_i (V_i^{k+1} - V_i^k) + \delta^-_i (V_i^{k+1} + V_i^k) \delta^-_i (V_i^{k+1} - V_i^k) \Big)h
$$

$$
= -\sum_{i=0}^{N-1} \delta^{(2)}_i (V_i^{k+1} + V_i^k) (V_i^{k+1} - V_i^k)h
$$

$$
= -\sum_{i=0}^{N-1} \frac{\delta^{(2)}_i (V_i^{k+1} + V_i^k)}{V_i^{k+1} + V_i^k} (U_i^{k+1} - U_i^k)h
$$

for $k \geq 0$.

The discrete variational derivative, denoted by $\delta \mathcal{F}_d(U^{k+1}, U^k) \in \mathbb{R}^N$, is then defined componentwise by
by the following nonlinear system with unknowns $V_i$ rules, which can be readily checked by the definition of finite difference operators:

$$\delta F_d(U^{k+1}, U^k)_i := -\frac{\delta_i^{(2)}(V_i^{k+1} + V_i^k)}{V_i^{k+1} + V_i^k}, \quad i = 0, \ldots, N - 1,$$

and the main point is that the discrete chain rule holds

$$F_d(U^{k+1}) - F_d(U^k) = \sum_{i=0}^{N-1} \delta F_d(U^{k+1}, U^k)_i(U_i^{k+1} - U_i^k)h.$$  

Having this at hand, the DVD scheme for the corrected DLSS equation is defined by the following nonlinear system with unknowns $V_i^{k+1} = \sqrt{U_i^{k+1}}$:

$$\frac{1}{\tau}(V_i^{k+1} - V_i^k) = \frac{1}{2W_i^{k+1/2}} \delta_i^+ \left(W_i^{k+1/2}W_{i-1}^{k+1/2}\delta_i^- \left(\delta F_d(W^{k+1/2})_i\right)\right)$$

$$- \frac{\delta}{2}\delta_i^{(1)} \left(W_i^{k+1/2}\delta F_d(W^{k+1/2})_i\right),$$

for all $i = 0, \ldots, N - 1, k \geq 0$, where $W^{k+1/2} = (V^{k+1} + V^k)/2$, and $\delta F_d(W)_i = -\delta_i^{(2)}(W_i)/W_i$.

4.2. Convergence analysis — proof of Theorem 1.3. Basic properties of the scheme: conservation of mass and dissipation of the discrete Fisher information follow directly from the construction of the scheme, summation by parts and the above discrete chain rule. Our main aim is to prove the convergence of the scheme. For this purpose we first prove the monotonicity of the following discrete operator $A_d : \mathbb{R}^N_+ \to \mathbb{R}^N$ defined by

$$A_d(W)_i = -\frac{1}{W_i} \delta_i^+ \left(W_iW_{i-1}\delta_i^- \delta F_d(W)_i\right), \quad i = 0, \ldots, N - 1.$$

Operator $A_d$ is a discrete analogue of the differential operator

$$A(w) = \frac{1}{w} \left(w^2 \left(\frac{w_x x}{w}\right)_x\right),$$

whose monotonicity has been shown in [31].

Proposition 2. Operator $A_d : \mathbb{R}^N_+ \to \mathbb{R}^N$ defined by (59) is monotone.

Proof. Let $w, W \in \mathbb{R}^N$ be arbitrary vectors from the cone $\mathbb{R}^N_+$. Using the definition of $A_d$ and applying the summation by parts formula twice we compute

$$(A_d(w) - A_d(W)) \cdot (w - W)$$

$$= -\sum_{i=0}^{N-1} (\delta F_d(w) - \delta F_d(W))_i \delta_i^+ \left(w_iw_{i-1}\delta_i^- \left(\frac{w_i - W_i}{w_i}\right)\right)$$

$$+ \sum_{i=0}^{N-1} \delta F_d(W)_i \delta_i^+ \left(W_iW_{i-1}\delta_i^- \left(\frac{w_i - W_i}{W_i}\right) - w_iw_{i-1}\delta_i^- \left(\frac{w_i - W_i}{w_i}\right)\right)$$

$$=: S_1 + S_2.$$  

In order to resolve sums $S_1$ and $S_2$ we employ the following discrete differentiation rules, which can be readily checked by the definition of finite difference operators:

$$w_iw_{i-1}\delta_i^- \left(\frac{w_i - W_i}{w_i}\right) = w_{i-1}\delta_i^- (w - W) - (\delta_i^- w)(w - W)_{i-1},$$

$$\delta_i^+ (wW) = w_{i+1}(\delta_i^+ W) + W_i(\delta_i^+ w).$$
Utilizing (60) in $S_2$ we calculate
\[
W_i W_{i-1} \delta_i^- \left( \frac{w_i - W_i}{w_i} \right) - w_i W_{i-1} \delta_i^- \left( \frac{w_i - W_i}{w_i} \right) = W_{i-1} \delta_i^- (w - W) - (\delta_i^- W)(w - W)_{i-1}
\]
\[
- w_i \delta_i^- (w - W) + (\delta_i^- w)(w - W)_{i-1}
\]
\[
= (w - w)_{i-1} \delta_i^- (w - W) + \delta_i^- (w - W)(w - W)_{i-1} = 0.
\]
Therefore, $S_2 = 0$. Employing (60) and (61), respectively, we calculate
\[
\delta_i^+ \left( w_i w_{i-1} \delta_i^- \left( \frac{w_i - W_i}{w_i} \right) \right) = \delta_i^+ \left( w_{i-1} \delta_i^- (w - W) - (\delta_i^- w)(w - W)_{i-1} \right)
\]
\[
= w_i \delta_i^{(2)} (w - W) + \delta_i^- (w - W)(\delta_i^+ w_{i-1})
\]
\[
- (w - W) \delta_i^{(2)} w - (\delta_i^- w)(\delta_i^+ w_{i-1})
\]
\[
= -w_i (\delta_i^{(2)} W) + W_i (\delta_i^{(2)} w).
\]
Thus, we have
\[
\delta_i^+ \left( w_i w_{i-1} \delta_i^- \left( \frac{w_i - W_i}{w_i} \right) \right) = -w_i W_i \left( \frac{\delta_i^{(2)} W - \delta_i^{(2)} w}{w_i} \right)
\]
\[
= -w_i W_i (\delta F_d(w) - \delta F_d(W))_i,
\]
which implies
\[
S_1 = h \sum_{i=0}^{N-1} w_i W_i (\delta F_d(w) - \delta F_d(W))_i^2 \geq 0
\]
and proves the monotonicity of $A_d$. $\square$

With the help of operator $A_d$ the discrete scheme (58) can be written as
\[
\frac{1}{\tau} (V_i^{k+1} - V_i^k) = -\frac{1}{2} A_d (W_i^{k+1/2}) + \frac{\delta}{2} \delta_i^{(1)} \left( \delta_i^{(2)} w_{k+1/2} \right) + f_i^{k+1/2}.
\]
(62)
Since the weak solution $u$ is assumed to remain strictly positive and also satisfies equation (49) in a weak sense (see eq. (70)), then according to [6] $u$ is smooth and we can write $u_i^k = u(t_k, x_i)$ pointwise for $i = 0, 1, \ldots, N - 1$ and $k \geq 1$. Let $v^k = \sqrt{u_i^k}$. Then we have
\[
\frac{1}{\tau} (v_i^{k+1} - v_i^k) = -\frac{1}{2} A_d (w_i^{k+1/2}) + \frac{\delta}{2} \delta_i^{(1)} \left( \delta_i^{(2)} w_{k+1/2} \right) + f_i^{k+1/2},
\]
(63)
where $w_i^{k+1/2} = (v_i^{k+1} + v_i^k)/2$ and values $f_i^{k+1/2}$ represent the local truncation error of the scheme. Subtracting (62) from (63) we get the discrete equation for the error vector $e_i^k := v^k - W^k$ which reads
\[
\frac{1}{\tau} (e_i^{k+1} - e_i^k) = -\frac{1}{2} \left( A_d (w_i^{k+1/2}) - A_d (W_i^{k+1/2}) \right) + \frac{\delta}{2} \delta_i^{(1)} \left( \delta_i^{(2)} e_i^{k+1/2} \right) + f_i^{k+1/2}.
\]
(64)
Multiplying (64) with $e_i^{k+1/2} = w_i^{k+1/2} - W_i^{k+1/2}$ and summing up over $i = 0, \ldots, N - 1$ we find
\[
\frac{h}{2\tau} \sum_{i=0}^{N-1} \left( (e_i^{k+1})^2 - (e_i^k)^2 \right)
\]
mula, we estimate $\delta A$.

Employing the monotonicity of the operator and the fact that $\sum_{i=0}^{N-1} \delta_i^{(2)} e^{k+1,k} = 0$ due to periodic boundary conditions and summation by parts formula, we estimate

$$h \sum_{i=0}^{N-1} \left( (e_i^{k+1})^2 - (e_i^k)^2 \right) \leq \tau h \sum_{i=0}^{N-1} \left( f_i^{k+1/2} \right)^2 .$$

Using the Cauchy-Schwarz, Young and Jensen’s inequalities we further estimate the right hand side and get

$$\frac{h}{2} \sum_{i=0}^{N-1} \left( (e_i^{k+1})^2 - (e_i^k)^2 \right) \leq \tau h \sum_{i=0}^{N-1} \left( f_i^{k+1/2} \right)^2 + \frac{\tau h}{4} \sum_{i=0}^{N-1} \left( (e_i^{k+1})^2 + (e_i^k)^2 \right).$$

The local truncation error of the scheme for smooth and positive solutions is proved to be of order $O(\tau) + O(h^2)$, see Lemma B.1. Therefore, summing up the last inequality for $k = 0, \ldots, M$ we have

$$(1 - \tau) \frac{h}{2} \sum_{i=0}^{N-1} \left( e_i^{M+1/2} \right)^2 \leq \frac{h}{2} \sum_{i=0}^{N-1} \left( e_i^0 \right)^2 + C(\tau^2 + h^4) + \frac{\tau h}{2} \sum_{k=0}^{M} \sum_{i=0}^{N-1} (e_i^k)^2 ,$$

where $C > 0$ is independent of $\tau$ and $h$. Utilizing the assumption on approximation of the initial datum, $H^2_d(u_0, U^0) = \frac{h}{2} \sum_{i=0}^{N-1} \left( e_i^0 \right)^2 \leq Ch^4$, the discrete Gronwall inequality implies (for $\tau < 1$)

$$\frac{h}{2} \sum_{i=0}^{N-1} \left( e_i^{M+1/2} \right)^2 \leq \frac{C(\tau^2 + h^4)}{1 - \tau} e^{\frac{\tau(M+1)}{1 - \tau}} \text{ for all } M \geq 0,$$

which concludes the proof of Theorem 1.3.

### 4.3. Implementation and illustrative examples

In this final subsection we illustrate numerical solutions to the corrected DLSS equation using the DVD method. Prior to that we expand terms of the scheme (58) and obtain a novel form in unknowns $W = W^{k+1/2} = (V^{k+1} + V^k)/2$:

$$W_i - V^k_i = -\frac{\tau}{4h^4} \left( W_{i+2} + 2W_i + W_{i-2} - \frac{(W_{i+1} + W_{i-1})^2}{W_i} \right)$$

$$+ \frac{\tau \delta}{8h^3} (W_{i+2} - 2W_{i+1} + W_{i-1} - W_{i-2}), \quad i = 0, \ldots, N - 1, \quad k \geq 0 .$$

Numerical solution $U^k$ of equation (18) is then resolved according to

$$U_i^{k+1} = (2W_i - V_i^k)^2, \quad i = 0, \ldots, N - 1, \quad k \geq 0 .$$

Note that system (66) is easier to treat numerically than the system (58). Moreover, (66) presents a Crank-Nicolson type discretization of equation (49). Numerical solutions are computed for two different initial conditions: (I) $u_0 = M_1^{-1}(\cos(\pi x))^6 + 0.1$ (first column of Figure 1) and (II) $u_0 = M_2^{-1}(\cos(2\pi x))^6 + 0.01$ (second column of Figure 1), where constants $M_1, M_2 > 0$ are taken such that $u_0$ have unit mass. Different rows in Figure 1 denote different dispersion parameter $\delta$, i.e. $\delta = 1,$
$\delta = 10$ and $\delta = 100$ in the first, second and third row of Figure 1, respectively. In each subfigure numerical evolution is sketched in five time instances starting from the initial datum $u_0$. Discretization parameters are taken to be $\tau = 10^{-6}$ and $h = 5 \cdot 10^{-3}$, and the nonlinear scheme (66) is solved by the Newton’s method using the solution from the previous time step as an initial guess for the solution on the current time step. Complete algorithm is implemented in Matlab. Figure 1 also illustrates convergence of numerical solutions to the constant steady state $u_\infty = 1$, as indicated by Theorem 1.2.

\begin{figure}[h]
\centering
\subfigure[ ]{
\includegraphics[width=0.4\textwidth]{figure1a.png}}
\subfigure[ ]{
\includegraphics[width=0.4\textwidth]{figure1b.png}}
\subfigure[ ]{
\includegraphics[width=0.4\textwidth]{figure1c.png}}
\subfigure[ ]{
\includegraphics[width=0.4\textwidth]{figure1d.png}}
\subfigure[ ]{
\includegraphics[width=0.4\textwidth]{figure1e.png}}
\subfigure[ ]{
\includegraphics[width=0.4\textwidth]{figure1f.png}}
\caption{Numerical evolution of the corrected DLSS equation for unit mass initial datum $u_0$ at different time moments: $t_1 = 5 \cdot 10^{-6}$, $t_2 = 4 \cdot 10^{-5}$, $t_3 = 2 \cdot 10^{-4}$, and $t_4 = 1.5 \cdot 10^{-3}$.}
\end{figure}

Numerical scheme (66) is additionally explored by testing its numerical convergence rates, both in space and time. For time convergence we set $\delta = 1$, $u_0 = M_1^{-1}(\cos(\pi x)^{16} + 0.1)$ and $h = 2 \cdot 10^{-3}$. The “exact solution” $\hat{u}$ is computed on the very fine time resolution $\tau = 10^{-9}$ and other numerical solutions $U_\tau$ are computed using the time step $\tau$. These solutions are compared “exact solution”
\( \hat{u} \) at time instance \( T = 5 \cdot 10^{-5} \) using the discrete Hellinger distance \( H_d \) defined by (8), i.e. we calculate the error \( \epsilon^M_T \) at time step \( M \) corresponding to time instance \( T \) as

\[
\epsilon^M_T := H_d(\hat{u}^M, U^M_T) = \left( \frac{h}{2} \sum_{i=0}^{N-1} \left( \sqrt{\hat{u}^M_i} - \sqrt{U^M_{T,i}} \right)^2 \right)^{1/2}.
\]

Numerical convergence rate is then estimated in a standard way. Assuming that \( \epsilon^M_T \approx C_{\tau^\kappa_{t}}, \) where \( \kappa_{t} \) is the convergence rate, simple algebra yields

\[
\kappa_{t} \approx \frac{\log \left( \frac{\epsilon^M_{T'}}{\epsilon^M_T} \right)}{\log \left( \frac{T}{T'} \right)},
\]

where \( \tau \) and \( \tau' \) are two consecutive time discretization parameters. Results of this numerical experiment are shown in Figure 2 (A) as well as in Table 1 (left). One can see that they are in agreement with the theoretical result of Theorem 1.3. Analogous numerical experiment has been carried out to demonstrate the space convergence of the numerical scheme and the results can be seen in Figure 2 (B) and in Table 1 (right).

![Figure 2](image_url)

**Figure 2.** Errors with respect to time and space discretization parameters. Dashed lines indicate theoretical convergence rates of the numerical scheme.

Appendix A. Calculations supplementing uniqueness of the weak solution.

**Lemma A.1.** Let \( u \) be a weak solution of equation (2), then the third-order operator

\[
2\delta \left( u^{1/4} (u^{1/4})_{xx} \right)_{x}
\]

in the weak square root form reads

\[
\delta \int_0^T \int_T \left( \sqrt{u} \right)_{xx} \phi_x \, dx \, dt
\]

for all test functions \( \phi \in L^\infty(0, T; W^{2,\infty}(\mathbb{T})) \cap W^{1,1}(0, T; L^\infty(\mathbb{T})) \) satisfying \( \phi(\cdot, T) \equiv 0. \)
First, using standard approximation arguments the following identity can be justified for a nonnegative function $w$ satisfying $w^{1/4} \in H^2(\mathbb{T})$ and $w^{1/2} \in H^2(\mathbb{T})$:

\[
2 \int_{T} w^{3/4}(u^{1/4})_{xx} \left( \varphi \ast \frac{\phi}{\sqrt{\varphi \ast w + \varepsilon}} \right)_{x} \, dx
= \int_{T} \left( \sqrt{w}(\sqrt{w})_{xx} - \frac{1}{2}(\sqrt{w})_{x}^2 \right) \varphi \ast \left( \frac{\phi_{x}}{\sqrt{\varphi \ast w + \varepsilon}} - \frac{\left(\varphi \ast w + \varepsilon\right)_{x} \phi}{\varphi \ast w + \varepsilon} \right) \, dx
\]

where $\phi \in C_{c}^{\infty}(\mathbb{T} \times [0, T])$, $\varphi \ast$ denotes a standard mollifier with respect to space and $\varphi \varepsilon > 0$ are positive parameters. Employing $\varphi \ast \frac{\phi}{\sqrt{\varphi \ast w + \varepsilon}}$ as a test function in the weak formulation of (2) and previous calculations, the third-order term reads

\[
2\delta \int_{0}^{T} \int_{T} w^{3/4}(u^{1/4})_{xx} \left( \varphi \ast \frac{\phi}{\sqrt{\varphi \ast w + \varepsilon}} \right)_{x} \, dx \, dt
= \delta \int_{0}^{T} \int_{T} \left( \sqrt{u}(\sqrt{u})_{xx} - \frac{1}{2}(\sqrt{u})_{x}^2 \right) \varphi \ast \frac{\phi_{x}}{\sqrt{\varphi \ast u + \varepsilon}} \, dx \, dt
- \delta \int_{0}^{T} \int_{T} \varphi \ast \left( \sqrt{u}(\sqrt{u})_{xx} - \frac{1}{2}(\sqrt{u})_{x}^2 \right) \left( \frac{\left(\varphi \ast u + \varepsilon\right)_{x} \phi}{\varphi \ast u + \varepsilon} \right) \, dx \, dt
= \delta \int_{0}^{T} \int_{T} \left( \sqrt{u}(\sqrt{u})_{xx} - \frac{1}{2}(\sqrt{u})_{x}^2 \right) \varphi \ast \frac{\phi_{x}}{\sqrt{\varphi \ast u + \varepsilon}} \, dx \, dt
- 2\delta \int_{0}^{T} \int_{T} \frac{1}{\sqrt{\varphi \ast u + \varepsilon}} \varphi \ast \left( \sqrt{u}(\sqrt{u})_{xx} - \frac{1}{2}(\sqrt{u})_{x}^2 \right) \left( \frac{\left(\varphi \ast u + \varepsilon\right)_{x} \phi}{\varphi \ast u + \varepsilon} \right) \, dx \, dt.
\]

With the help of convergence lemmas [22, Lemma 12, Lemma 14] we can pass to the limit as $\delta \downarrow 0$ and obtain

\[
d\int_{0}^{T} \int_{T} \left( \sqrt{u}(\sqrt{u})_{xx} - \frac{1}{2}(\sqrt{u})_{x}^2 \right) \left( \frac{\phi_{x}}{\sqrt{u + \varepsilon}} - \frac{(u + \varepsilon)_{x} \phi}{u + \varepsilon} \right) \, dx \, dt
= \delta \int_{0}^{T} \int_{T} \frac{\sqrt{u}}{\sqrt{u + \varepsilon}}(\sqrt{u})_{xx} \phi \, dx \, dt - \frac{\delta}{2} \int_{0}^{T} \int_{T} (\sqrt{u})_{x}^2 \frac{\phi_{x}}{\sqrt{u + \varepsilon}} \, dx \, dt - \frac{\delta}{2} \int_{0}^{T} \int_{T} (\sqrt{u})_{x}^2 \frac{(u + \varepsilon)_{x} \phi}{u + \varepsilon} \, dx \, dt.
\]
In order to pass to the limit $\varepsilon \downarrow 0$, we first integrate by parts in the second integral on the right hand side
\[
-\frac{\varepsilon}{2} \int_0^T \left( \frac{\phi_x}{\sqrt{u + \varepsilon}} \right) dx dt = \delta \int_0^T \int_T \frac{(\sqrt{u})_x(\sqrt{u})_{xx}}{\sqrt{u} + \varepsilon} \phi_x dx dt
- \frac{\delta}{2} \int_0^T \int_T (\sqrt{u})_x^2 \frac{(\sqrt{u} + \varepsilon)}{u + \varepsilon} \phi_x dx dt.
\]
The second term cancels out with the last term in (68) and remaining integrals can be reformulated according to
\[
\delta \int_0^T \int_T \frac{\sqrt{u}}{\sqrt{u} + \varepsilon} (\sqrt{u})_{xx} \phi_x dx dt + 2 \delta \int_0^T \int_T (\sqrt{u})_{xx} \frac{u^{1/4}}{u^{3/4}} \left( \frac{\sqrt{u}}{\sqrt{u} + \varepsilon} \right) \phi dx dt
- 2 \delta \int_0^T \int_T (\sqrt{u})_{xx} \frac{u^{3/4}}{u^{3/4}} (\sqrt{u}) \frac{(u + \varepsilon)^{1/4}}{x} \phi dx dt.
\]
Employing Lemma 12 from [22] we can pass to the limit as $\varepsilon \downarrow 0$ in (69), and since the last two terms cancel each other, we find the desired weak formulation of the third-order operator in the square root form to be
\[
\int_0^T \int_T (\sqrt{u})_x \phi_x dx dt.
\]
Therefore, according to [22, Lemma 15] and above calculations, weak solutions of equation (2) constructed in Section 3 also satisfy the square root form of the equation in the following weak sense:
\[
-2 \int_0^T \sqrt{u} \partial_t \phi dx dt - 2 \int_0^T \sqrt{u_0} \phi(\cdot, 0) dx = - \int_0^T \int_T (\sqrt{u})_{xx} \phi_{xx} dx dt
+ \int_0^T \int_T (\sqrt{u})_{xx} \phi_x dx dt - \delta \int_0^T \int_T (\sqrt{u})_{xx} \phi_x dx dt.
\]
for all test functions $\phi \in L^\infty(0, T; W^{2,\infty}(\mathbb{T})) \cap W^{1,1}(0, T; L^\infty(\mathbb{T}))$ satisfying $\phi(\cdot, T) \equiv 0$.

The following result is a direct extension of [22, Theorem 5].

Lemma A.2. Let $u_1$ and $u_2$ be two weak solutions of equation (2) with initial conditions $u_1(0)$ and $u_2(0)$, respectively. Then for a.e. $t_2 > t_1 \geq 0$
\[
\mathcal{H}(u_1(t_2), u_2(t_2)) \leq \mathcal{H}(u_1(t_1), u_2(t_1)).
\]

Proof. In the proof we only carry out calculations related to the third-order term in (70), while the rest of the proof can be completely taken from [22]. Take a smooth nonnegative function $\zeta \in C_c^\infty((0, T])$ and define $\phi_1 = \zeta \cdot \phi \ast (\phi \ast \sqrt{u_2})$ as a test function for $u_1$ satisfying (70), and vice versa, define $\phi_2 = \zeta \cdot \phi \ast (\phi \ast \sqrt{u_1})$ as a test function for $u_2$ satisfying (70). Summing up the two weak formulations, third-order terms read
\[
-\delta \int_0^T \int_T (\phi \ast (\sqrt{u_1})_{xx}) (\phi \ast \sqrt{u_2}) \zeta dx dt
- \delta \int_0^T \int_T (\phi \ast (\sqrt{u_2})_{xx}) (\phi \ast \sqrt{u_1}) \zeta dx dt.
\]
Performing the limit \( \varrho \downarrow 0 \) we find
\[
- \delta \int_0^T \int_T \left( \sqrt{u'_1} \right)_{xx} \left( \sqrt{u_2} \right)_x \zeta \, dt - \delta \int_0^T \int_T \left( \sqrt{u_1} \right)_{xx} \left( \sqrt{u_1} \right)_x \zeta \, dt.
\]
Integrating by parts in one of the integrals, these terms cancel each other, thus, showing in fact that the third-order term conserves the square of the Hellinger distance.

Appendix B. Calculations supplementing the error analysis.

**Lemma B.1.** Let \( u \) be a smooth and positive solution of equation (2). Then the local truncation error of the scheme (66) is of order \( O(h^2) + O(\tau) \), where \( h \) and \( \tau \) are discretization parameters with respect to space and time, respectively.

**Proof.** Let \( x_i = ih, i = 0, \ldots, N - 1 \) and \( t_k = k\tau, k \geq 0 \) denote space and time grid points, and let \( w_i = (v_i^{k+1} + v_i^k)/2 \). Employing Taylor expansions we can write:
\[
\begin{align*}
 w_i &= v_i^k + O(\tau), \\
 v_i^{k+1} &= v_i^k + \partial_t (v_i^k) \tau + O(\tau^2) \\
 w_{i\pm1} &= w_i \pm (w_i)_x h + \frac{1}{2} (w_i)_{xx} h^2 + O(h^3), \\
 w_{i\pm2} &= w_i \pm 2 (w_i)_x h + 2 (w_i)_{xx} h^2 \pm \frac{4}{3} (w_i)_{xxx} h^3 + \frac{2}{3} \partial_x^2 (w_i) h^4 \\
 &\quad \pm \frac{4}{15} \partial_x^3 (w_i) h^5 + O(h^6).
\end{align*}
\]
Straightforward calculations then provide
\[
\begin{align*}
 w_i - v_i^k &= \frac{1}{2} \partial_t (v_i^k) \tau + O(\tau^2), \quad (71) \\
 w_{i+2} + 2 w_i + w_{i-2} - \frac{(w_{i+1} + w_{i-1})^2}{w_i} &= \left( (w_i)_{xxxx} - \frac{(w_i)_{xx}^2}{w_i} \right) h^4 + O(h^6) \quad (72) \\
 &= \left( (v_i^k)_{xxxx} - \frac{(v_i^k)_{xx}^2}{v_i^k} + O(\tau) \right) h^4 + O(h^6), \\
 w_{i+2} - 2 w_{i+1} + 2 w_{i-1} - w_{i+2} &= 2 (w_i)_{xxx} h^3 + O(h^5) \quad (73) \\
 &= 2 \left( (v_i^k)_{xxx} + O(\tau) \right) h^3 + O(h^5).
\end{align*}
\]
Employing (71) – (73) in scheme (66) and utilizing the square root form of the equation (49) pointwise, yields the local truncation error of order \( O(h^2) + O(\tau) \).  

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Received January 2020; revised October 2020.

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