Comparison of two forms of the $S$-matrix element of strong-field photoionization: II. $(n, l, m) = (2, 1, \pm 1)$ states

Jarosław H Bauer

Katedra Fizyki Teoretycznej, Wydział Fizyki i Informatyki Stosowanej Uniwersytetu Łódzkiego, Ul. Pomorska 149/153, 90-236 Łódź, Poland

E-mail: bauer@uni.lodz.pl

Received 7 September 2012, in final form 29 December 2012
Published 4 February 2013
Online at stacks.iop.org/JPhysB/46/045601

Abstract

In our previously published work (Bauer 2008 J. Phys. B: At. Mol. Opt. Phys. 41 185003) we numerically compared ionization rates and photoelectron energy (or momentum) spectra resulting from the so-called length gauge (LG) and velocity gauge (VG) forms of the Keldysh–Faisal–Reiss theory for a strong circularly polarized laser field. Our investigations concerned initial (bound) states of a hydrogen atom with the principal quantum number equal to 1 or 2. However, some of these states were not taken into account in this work, but only their linear combinations (quantum-mechanical superpositions). The main purpose of the present work is to fill this gap. It is appropriate to add that excited bound states (of the hydrogen atom) considered here are easier to achieve from an experimental point of view. As in our previous work, we find substantial differences in the shape of photoelectron energy spectra for both (LG and VG) theories. Differently from the VG theory, the LG one shows a multipeak envelope in photoelectron energy spectra for most initial states of the hydrogen atom. We give examples showing that the multipeak effect may be observed also in the polarization plane of the laser field. The field parameters (frequency and intensity) applied by us in the calculations indicate that one should be able to observe experimentally the above-mentioned effect in the near future.

(Some figures may appear in colour only in the online journal)

1. Introduction

The Keldysh–Faisal–Reiss (KFR) theory [1–4] describes nonresonant multiphoton processes such as the above threshold detachment of ions and the above threshold ionization (ATI) of atoms in an intense electromagnetic (laser) field. The theory, sometimes also called the strong-field approximation (SFA), utilizes the $S$-matrix theory and is one of the main analytical approaches describing the above-mentioned processes. We refer the reader to some older pioneering works [1, 4] or to our recent ones [5, 6] for details about the SFA and its basic assumptions. However, a very concise account will now follow. One should stress that our method of calculations, notations etc are exactly the same as in [6]. Therefore, a reader familiar with the latter article (which can be treated as part I) can skip the present introduction.

Throughout this work we assume non-relativistic and dipole approximations for atoms in strong laser fields. As a result, one drops a magnetic-field component of the laser. We use atomic units here: $\hbar = e = m_e = 1$, substituting explicitly $-1$ for the electronic charge (we keep any nuclear charge $Z$ in all of the equations given below but finally, in the numerical calculations, we set $Z = 1$, which corresponds to a hydrogen atom). Following Reiss [4], we introduce the intensity parameter $z$ such that

$$U_P = z\omega = \frac{I}{4\omega^2},$$

(1)

where $U_P$ stands for the ponderomotive energy (sometimes also called the ponderomotive potential; this is the time-averaged kinetic energy of a classical free charge oscillating in an electromagnetic plane-wave field), $\omega$ stands for the laser frequency and $I$ stands for the laser intensity in atomic...
units (1 au $\equiv 3.51 \times 10^{16}$ W cm$^{-2}$; for linear polarization $I = F^2$, and for circular polarization $I = 2F^2$, with $F$—the electric field vector amplitude). The following two conditions determine the lower and the upper applicability limits (for a given $I$, if $\omega = $ const) of the SFA [4]:

$$z_l = \frac{2U_p}{E_B} \gg 1, \quad \text{and} \quad z_r \equiv \frac{2U_p}{c^2} \ll 1.$$  \hspace{1cm} (2)

Equations (2) show that the ponderomotive energy of the outgoing electron should be much greater than the electron binding energy $E_B$ (in the atom or in the ion), and much less than the electron rest energy. The latter condition is due to a non-relativistic description of the process. The well-known Keldysh parameter $\gamma$ [1] (called the length gauge (LG) as well) approximations) can be written either in the form of Keldysh

$$\gamma = \frac{1}{\gamma^2}$$

or in the form of Reiss [4] (called the velocity gauge (VG) as well)

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where $T_N(\vec{p}, \omega, z)$ is the $T$-matrix element. The Dirac $\delta$-function reflects the conservation of the total energy after ionization and leads to the summation over $N$, which corresponds to different multiphoton channels. The differential ionization rate $\gamma(\vec{p})$ (the ionization probability per unit time) is found from

$$\gamma(\vec{p}) = \lim_{t \to \infty} \frac{1}{(S - 1)_{fi}}.$$

To obtain the total ionization probability per unit time $\Gamma$, one has to integrate the differential ionization rate over all the possible final momenta of the outgoing electron ($\vec{p}$, $\varphi$ are polar and azimuthal angles of $\vec{p}$)

$$\Gamma = \sum_{N=0}^{\infty} \Gamma_N$$

Finally, $\Gamma = \sum_{N=0}^{\infty} \Gamma_N$ where the number $N$ is a multiphoton order and $N_0 = [z + E_B/\omega] + 1$ may be treated as a minimal number of photons absorbed. (Here the symbol $[x]$ denotes the integer part of the number $x$.) The final kinetic energy $E_N$ of the ionized outgoing electron is quantized according to the following formula:

$$E_N = p_0^2/2 = N\omega - z_0 - E_B.$$  \hspace{1cm} (8)

Consecutive terms $\Gamma_N$ form the ATI energy spectrum, if we plot a curve connecting the points $(E_N, \Gamma_N)$ in the $OXY$ plane. These points are in a sequence separated by the photon energy $\omega$ along the $AX$ axis.

2. Discussion and results

Final expressions for ionization rates $\Gamma$ in the LG SFA and in the VG SFA were derived in [6] (appendix A). The following initial states (all normalized to unity in an entire space) of the hydrogen atom were taken into account (in the spatial representation: $\theta'$, $\varphi'$ denote the polar and azimuthal angles of $\vec{r}$—the electron's position):

$$\Phi_1(r) = \sqrt{\frac{Z}{\pi}} \exp(-Zr),$$

$$\Phi_2(r) = \sqrt{\frac{Z}{3Z^2}} \exp(-Zr/2),$$

$$\Phi_2p(r) = \sqrt{\frac{Z^2}{3Z^2}} \exp(-Zr/2) \sin\theta' \cos\varphi',$$

$$\Phi_2p'(r) = \sqrt{\frac{Z^2}{3Z^2}} \exp(-Zr/2) \sin\theta' \sin\varphi',$$

$$\Phi_2p(r) = \sqrt{\frac{Z^2}{3Z^2}} \exp(-Zr/2) \cos\theta'.$$

$\Phi_1$ has the well-known quantum numbers (omitting spin) $(n, l, m) = (1, 0, 0)$, $\Phi_2$ has $(n, l, m) = (2, 0, 0)$ and $\Phi_2p$ has $(n, l, m) = (2, 1, 0)$. (Thus, if $\Phi_{n,l,m}$ corresponds to $(n, l, m)$, we have $\Phi_1 = \Phi_{1,0,0}$, $\Phi_2 = \Phi_{2,0,0}$, and $\Phi_2p = \Phi_{2,1,0}$.) On
the other hand, \( \Phi_{2n} \) and \( \Phi_{2p} \), have \( n = 2 \) and \( l = 1 \), but are only linear combinations of states with \( m = -1 \) and \( m = 1 \):

\[
\Phi_{2n} = \frac{i}{\sqrt{2}} (\Phi_{2,1,-1} - \Phi_{2,1,1}), \\
\Phi_{2p} = \frac{i}{\sqrt{2}} (\Phi_{2,1,-1} + \Phi_{2,1,1}).
\]

In quantum mechanics textbooks either \( \Phi_{n,l,m} \) or the states from equations (9)–(13) (and so on for \( n > 2 \)) are given as bound states of the hydrogen atom. In the previous work [6], we chose the latter set of initial states to compute ionization rates. In our calculations (in [6] and in this work), one assumes that the laser field propagates along the \( z \)-axis in its positive direction with the right or left circular polarization. The states \( \Phi_{2n} \) and \( \Phi_{2p} \) differ only in their orientation in space with respect to the \( z \)-axis (one can turn one of these two states around the \( z \)-axis through an angle of \( \pi/2 \) and obtain the other one). It follows from the symmetry of the problem that their ionization rates have to be identical in the same circularly polarized laser field, and it was confirmed in [6] indeed. Contrary to this situation, the states \( \Phi_{2,1,-1} \) and \( \Phi_{2,1,1} \) could ionize very differently, because they describe electrons initially rotating clockwise or anti-clockwise in the polarization plane (which is, of course, the \( xy \)-plane). As a result, the initially bound electron could rotate in the same or in the opposite direction with respect to the rotation of the electric-field component of the laser field. In other words, angular momentum components of the electron and the field could have the same or opposite signs along the propagation axis. Therefore, one has a physical motivation to investigate ionization from the states \( \Phi_{2,1,-1} \) and \( \Phi_{2,1,1} \) in the SFA. Furthermore, the states \( \Phi_{2,1,-1} \) and \( \Phi_{2,1,1} \) are certainly easier to achieve (than the states \( \Phi_{2n} \) and \( \Phi_{2p} \)) from an experimental point of view.

It appears that new calculations presented here are even simpler than the previous ones from [6]. Applying the same method, and assuming that the laser field is right circularly polarized one obtains (in equations (16)–(18) below one should set \( q = \sqrt{2Z/\omega} p_N \sin \theta \); cf equations (A.11)–(A.18) from [6])

\[
\Gamma_{2,1,-1}^{\text{VG}} = \Gamma_{2,1,1}^{\text{VG}} = \sum_{N=N_0}^{\infty} \frac{Z^2 p_N^2}{2(2E_B + p_N^2)} \int_0^\pi \mathrm{d}\theta \sin^3 \theta J_2(q),
\]

(16)

\[
\Gamma_{2,1,-1}^{\text{LG}} = \sum_{N=N_0}^{\infty} 2Z^2 p_N \int_0^\pi \mathrm{d}\theta \sin \theta \left[ \sum_{k=-\infty}^{\infty} C_k(p_N, \theta) \right]^2,
\]

(17)

\[
\Gamma_{2,1,1}^{\text{LG}} = \sum_{N=N_0}^{\infty} 2Z^2 p_N \int_0^\pi \mathrm{d}\theta \sin \theta \left[ \sum_{k=-\infty}^{\infty} C_k(p_N, \theta) \right]^2,
\]

(18)

where \( J_n(q) \) are ordinary Bessel functions and \( C_k(p_N, \theta) \) are Fourier coefficients given in equation (A.21) from [6]. Equations (17) and (18) are the main analytical results of the present work. (A generalization of the VG SFA to the excited states of the hydrogen atom is quite trivial, thus equation (16) could have probably been used somehow by other authors before.) Let us note that \( \Gamma_{2,1,-1}^{\text{LG}} \neq \Gamma_{2,1,1}^{\text{LG}} \)

\[
\Gamma_{2,1,-1}^{\text{LG}} = \frac{1}{2} (\Gamma_{2,1,-1}^{\text{LG}} + \Gamma_{2,1,1}^{\text{LG}})
\]

(19)

(cf appendix A in [6]). In the VG SFA analogous relations are yet simpler

\[
\Gamma_{2,1,-1}^{\text{VG}} = \Gamma_{2,1,1}^{\text{VG}} = \Gamma_{2,1,-1}^{\text{VG}} = \Gamma_{2,1,1}^{\text{VG}}.
\]

(20)

Moreover, for any initial state of the form \( \Phi_{2p} = \alpha \Phi_{2,1,-1} + \beta \Phi_{2,1,1} \) (with \( \alpha \) and \( \beta \) being arbitrary complex numbers such that \( |\alpha|^2 + |\beta|^2 = 1 \)), the VG SFA ionization rate is again the same. (It follows from equations (4)–(7), as a result of the integration upon \( \varphi \)—the azimuthal angle of \( \vec{p} \)—and the following equality: \( |\Phi_{2,1,-1}|^2 = |\Phi_{2,1,1}|^2 \).) Consequently, the VG SFA fails to find any difference (also in relation to electron probability distributions) in strong-field ionization from the states \( \Phi_{2,1,-1} \) and \( \Phi_{2,1,1} \) (or their quantum-mechanical superpositions) by circularly polarized laser fields.

In figures 1 and 2 we show the VG and LG ionization rates (computed from equations (16)–(18)) as a function of the laser intensity (a.u.).
Figure 3. The final kinetic energy distributions (calculated from the VG and LG SFA) of photoelectrons for the hydrogen atom in the initial state \((n, l, m) = (2, 1, -1)\) (panel (a)) and \((n, l, m) = (2, 1, 1)\) (panel (b)) for fixed field parameters: \(\omega\) and \(z_1\) (see the text frame above). The classically most probable energy \(U_p\) is also shown for comparison (dashed vertical line).

Figure 4. As in figure 3, but for ten times higher intensity (and \(z_1\)).

The field’s intensity for the initial states with \((n, l, m) = (2, 1, -1)\) and \((n, l, m) = (2, 1, 1)\) are very similar to figures 1–4 from [6]. Only for relatively low intensities, differences between LG curves for quantum numbers \((2, 1, -1)\) and \((2, 1, 1)\) appear. Our calculations have been performed for four different frequencies of the laser field: \(\omega = 1/32, 1/8, 1/2, \) and 2. Each curve in figures 1 and 2 begins when \(z_1 = 1\) and ends when \(z_1 = 0.1\). (In figures 1 and 2 it is difficult to notice directly where most of the LG curves end, but their endings are situated for the same intensities as the VG ones.) The VG curves show structures connected with the existence of photon thresholds (when the number \(N_0\) as a function of intensity increases), which are particularly visible for higher frequencies (or photon energies). The structures in analogous LG curves may appear for different laser intensities and are not necessarily connected with photon thresholds. (In the LG, the \(N\)-photon ionization rate depends on several Bessel functions \(J_{N+k}, J_{N+k+1}\) and the Fourier coefficient \(C_k\). The total ionization rate is an infinite sum, which does not have to be dominated by its first term.) In both gauges the ionization rate for very high intensity becomes a decreasing function of intensity. However, in the LG such a pitch is much weaker, and the ionization rates are usually many orders of magnitude greater than their VG counterparts. The most striking difference observed between the VG and the LG in figures 1–4 in [6] is also present here. In figures 1 and 2 only for the LG does the ionization rate become independent of the laser frequency for a sufficiently strong laser field. The consecutive LG curves (for \(\omega = 1/32, 1/8, 1/2\) and 2, respectively) join some kind of envelope with increasing intensity. In the VG the ionization rate depends strongly both on frequency and intensity for the strongest laser fields considered here.

In figures 3 and 4 we show the VG and the LG partial ionization rates, i.e. the \(\delta\)-function ATI peaks (computed as consecutive terms \(N = N_0, N_0 + 1, N_0 + 2, \ldots \) from equations (16)–(18)) as a function of the outgoing electron’s kinetic energy for a certain fixed frequency and intensity of the laser field. In figures 3–7 each curve (denoted as ‘VG’ or ‘LG’) is, in fact, a set of points, as equation (8) indicates. For some of the plots, particularly if the curve is not smooth everywhere, we have added solid or open circles. Each of the VG SFA spectra have been normalized to get the same area under both kinds of curves. (Usually the total VG SFA ionization rates are much smaller than the total LG SFA rates.) Figures 3(a) and (b) refer to the initial states with \((n, l, m) = (2, 1, -1)\) and \((n, l, m) = (2, 1, 1)\), respectively. In figures 3(a) and (b) \(\omega = E_B = 1/8\) and \(z_1 = 100\), which corresponds to \(I = 1.37 \times 10^{16} \text{W cm}^{-2}\). Figures 4(a) and (b) show analogous photoelectron energy spectra for \(\omega = E_B = 1/8\) and \(z_1 = 1000\) (thus \(I = 1.37 \times 10^{17} \text{W cm}^{-2}\)). Let us note that...
momentum distributions are also shown for the states that the double-peak shape of the ATI envelope (in the LG) for the initial state $(n, l, m) = (2, 1, 1)$ (blue solid line) and $(n, l, m) = (2, 1, 1)$ (red solid line) and $(n, l, m) = (2, 1, 1)$ (black dashed line) momentum distributions are also shown for comparison. Both latter theories predict the same distribution for the initial states $(2, 1, -1)$ and $(2, 1, 1)$. The classically most probable momentum $\sqrt{2U_P}$ is displayed as well (vertical dashed line).

**Figure 5.** The final momentum distributions (calculated from the LG SFA) of photoelectrons for the hydrogen atom in the initial state $(n, l, m) = (2, 1, 1)$ (red solid line) and $(n, l, m) = (2, 1, 1)$ (blue solid line) for fixed field parameters: $\omega$ and $z_1$ (see the text frame above). The LG SFA (black dotted line) and the Landau–Dykhne (black dashed line) momentum distributions are also shown for comparison. Both latter theories predict the same distribution for the initial states $(2, 1, -1)$ and $(2, 1, 1)$. The classically most probable momentum $\sqrt{2U_P}$ is displayed as well (vertical dashed line).

$$\frac{(2U_p)^{1/2}}{\Phi_{1,2}}$$

**Figure 6.** As in figure 5, but for $\omega = E_B/3$ ($z_1 = 100$) is the same.

\[
\frac{(2U_p)^{1/2}}{\Phi_1} = \frac{1}{z_1} (2U_p)^{1/2}
\]

for long laser pulses considered here, one does not observe finally any effect in the energy spectrum. For the initial state $(n, l, m) = (1, 0, 0)$ (the ground state) there is only one peak near $U_P$, and this value is the average energy absorbed by the ionized electron from the laser field as well. For the initial state $(n, l, m) = (2, 1, -1)$ the main peak exists below $U_P$, and for the initial state $(n, l, m) = (2, 1, 1)$ the main peak exists above $U_P$. These structures look like an interference pattern and some effort towards their explanation will be given below. For the initial state $(n, l, m) = (2, 1, 0)$, in which the electron does not rotate, the main peak exists near $U_P$, as expected classically (see figure 7(c) here or figures 5(d) and 6(d) in [6]). Although differences in the initial angular momentum component along the propagation axis of the laser field are small (the magnetic quantum number $m = -1, 0$ or 1), intense laser fields create large differences in the shape of the ATI envelopes for states with the same $n = 2$ and $l = 1$. (Note that initial momentum probability distributions are identical for $(n, l, m) = (2, 1, -1)$ and $(n, l, m) = (2, 1, 1)$, because $|\Phi_{2,1,1}|^2 = |\Phi_{2,1,-1}|^2$.) Comparing figures 3(a) with 4(a) and figures 3(b) with 4(b) we infer that the multipeak effect in the ATI envelope becomes more visible with increasing the laser intensity.

Figure 5 shows photoelectron momentum spectra for the same field parameters ($\omega = E_B/10$ and $z_1 = 100$, thus $I = 1.37 \times 10^{14} \text{W cm}^{-2}$) as figure 13(a) from [6]. It appears that a mysterious ‘transient shape in the LG’ for $H(2p_2)$ or $H(2p_1)$ atoms can be understood with the help of equation (19). Again, adding up the spectra for $(n, l, m) = (2, 1, -1)$ and $(n, l, m) = (2, 1, 1)$ one obtains the spectrum for $H(2p_2)$ or $H(2p_1)$. Figure 6 shows photoelectron momentum spectra for a higher laser frequency $\omega = E_B/3$ and the same $z_1 = 100$ (thus $I = 1.52 \times 10^{14} \text{W cm}^{-2}$; cf figure 13(c) from [6]). In figures 5 and 6 we also show the VG SFA and Landau–Dykhne [9, 10] results (as before in [6]). Both latter theories predict the same distribution for the initial states $(2, 1, -1)$, $(2, 1, 1)$, $H(2p_2)$, and $H(2p_1)$.

The next figures (unlike figures 1–6) have no exact counterparts in our previous work [6]. We hope that they contain some further pieces of information, which might be useful from the experimental point of view. It is (probably) easier to observe the ionized electrons, outgoing at a certain specific angle, than in a full solid angle. In figures 7(a)–(d) we show we the LG SFA and VG SFA differential ionization rates of the hydrogen atom in the following initial states: $(2, 1, -1)$, $(2, 1, 1)$, $(2, 1, 0)$ and $(2, 0, 0)$, respectively. The spectra are calculated for $\omega = E_B$, $z_1 = 100$ (thus $I = 1.37 \times 10^{14} \text{W cm}^{-2}$) and the fixed emission angle $\vartheta$, where most electrons are emitted. Usually this angle corresponds to the polarization plane of the laser field ($\vartheta = \pi/2$), but for $(n, l, m) = (2, 1, 0)$ there are two such angles: $\vartheta \cong 0.94\pi/2$ and $\vartheta \cong 0.94\pi - 0.94\pi/2$ in the LG SFA. In both VG and LG SFA for $(n, l, m) = (2, 1, 0)$, the differential ionization rate is equal to zero at $\vartheta = \pi/2$ (cf equations (A.14) and (A.18) in [6]). In the VG SFA one obtains the maximal differential rate for a slightly different angle $\vartheta \cong 0.88\pi/2$ and $\vartheta \cong \pi - 0.88\pi/2$. The multipeak effect is clearer in the polarization plane than in the full solid angle (cf figure 7(a) with figure 3(a), figure 7(b).
Figure 7. The LG SFA and VG SFA differential ionization rates of the $H(2p)$ or $H(2s)$ atom for the fixed emission angle $\vartheta$, where most electrons are emitted (usually this angle corresponds to the polarization plane of the laser field ($\vartheta = \pi/2$), but for $(n, l, m) = (2, 1, 0)$ there are two such angles: $\vartheta \approx 0.94 \pi/2$ and $\vartheta \approx \pi - 0.94 \pi/2$ in the LG SFA). Consecutive panels (a)–(d) correspond to different initial states $(n, l, m)$ and the same $\omega$ and $z_1$ (see the text frame above).

Moreover, like for the initial state $(2, 0, 0)$, these two kinds of the spectra may be very different.

3. Summary and conclusions

This work has supplemented our previous (although quite extensive) article [6] published in this journal. We have derived analytical expressions for ionization rates from initial states (of the hydrogen atom) described by the quantum numbers $(n, l, m) = (2, 1, -1)$ and $(n, l, m) = (2, 1, 1)$. Both the VG and LG SFA theories, which describe strong-field ionization in the circularly polarized laser field, have been studied. Total ionization rates calculated in the LG are usually orders of magnitude greater than their VG counterparts and show much weaker stabilization (defined as decreasing the ionization rate with increasing intensity). As in [6], an envelope has been found in the LG for curves showing the ionization rate as a function of the laser field’s intensity. It appears that for sufficiently strong fields the ionization rate is only a function of intensity, but not of frequency in the LG. Exactly the same statement was formulated in [6], but in relation to other initial states. In the LG the difference between total ionization rates for the states $(2, 1, -1)$ and $(2, 1, 1)$ is not large and nearly vanishes when increasing the laser intensity (for $z_1 \gg 1$). On the other hand, probability distributions of ionized electrons (for fixed field parameters) are much more sensitive to details of the initial-state wavefunction. This sensitivity increases when increasing the laser field’s intensity and frequency.

Another kind of envelope, namely the multipeak ATI envelope in photoelectron energy (or momentum) distributions, has been found in the LG, whereas no such effect has been found in the VG (it was also concluded before, in [6], but in relation to other initial states). The same concerns the electrons emitted not only in the full solid angle, but also at specific emission angles $\vartheta$, for example, in the polarization plane ($\vartheta = \pi/2$). For the states $(n, l, m) = (2, 1, 1)$ and $(n, l, m) = (2, 1, -1)$, the shift of the main peak (to the right or to the left of $U_p$) follows from the same or opposite initial rotation of the electronic cloud with respect to a rotation of the laser field. However, a presence of another, much smaller peak in the ATI envelope (on the other side of the vertical line showing $U_p$) is not so easy for an interpretation. This peak and additional peaks (except for the one near $U_p$) for the state $(n, l, m) = (2, 0, 0)$ are certainly connected with a non-trivial form of the initial-state wavefunction. The latter strongly affects the ionization probability amplitudes (cf equations (3) and (4)). Therefore, one should expect similar
and even stronger effects for higher lying bound states of the hydrogen atom (i.e. for \( n > 2 \)).

The authors of [11] used the saddle-point approximation to carry out the integration over time. According to them: ‘...it lends itself to an appealing physical interpretation of the results obtained.’ (p R227). The same method was used many times (see, for example, [11] and references therein) to obtain analytical results for the ATI energy spectra. The amplitude from equation (3) may be presented as

\[
(S-1)_{ji}^{LG} = i \int_{-\infty}^{\infty} dt \exp[i f(t)],
\]

where

\[
f(t) = -i \ln \left[ \Phi_1(\pi(t)) \left( \frac{1}{2} \pi(t)^2 + E_B \right) \right] + \frac{1}{2} \int_{-\infty}^{t} \pi(\tau)^2 d\tau + E_B t.
\]

(21)

Then one has to solve the saddle-point equation \( f'(t) = 0 \). The solutions to this equation \( t = t_s + i\nu \), with the condition: \( \text{Im} \, \nu > 0 \) [11], p 228) would describe a distribution in the saddle-point approximation (see equation (68) in [11], but their equation (9) is replaced by our equation (21), thus no problem with singularity in the matrix element exists in our Coulomb case). Let us note, that normally one uses a simpler version of the saddle-point equation \( f'(t) = 0 \), in the strong-field ionization context, namely such that the logarithmic term is omitted in equation (21). However, then such an equation does not describe the multipeak effect, because there is only one point \( t_s \) with \( \nu = \text{Im} \, t_s > 0 \). For the \( H(1s) \) atom, equation \( f'(t) = 0 \) takes the form \( (\pi(t)^2)^2 + 2E_B \pi(t)^2 + 2E_B^2 + 2\rho N(t)/c = 0 \). This expression describes a distribution with a single peak near \( U_p \). The distribution has a shape, which is similar to the one predicted by the LG SFA. There is a general formula, based on the saddle-point approximation, which describes the ATI spectrum

\[
(S-1)_{ji}^{LG} \sim \sum_{t_s} \left( \frac{2\pi}{i f'(t_s)} \right)^{1/2} \exp[i f(t_s)],
\]

(22)

where the sum is over all physically important saddle points. Then one has to calculate \( |(S-1)_{ji}^{LG}|^2 \), which is proportional to the ATI probability distribution. For the \( H(1s) \) atom (probably) only one point \( t_s \) contributes to the spectrum. For the \( H(2s) \) atom the analogous equation is

\[
\frac{1}{2} (\pi(t)^2)^2 + E_B (\pi(t)^2)^2 - 2E_B^2 \pi(t)^2 - 4E_B^3 + 2\rho N(t)(\pi(t)^2 - 6E_B) = 0,
\]

(23)

which gives three or four points \( t_s \) with \( \nu = \text{Im} \, t_s > 0 \). (We have checked this fact numerically for the field parameters from figure 7, taking \( \vartheta = \pi/2 \) and various values of \( E = p^2/2 \). There is always one point \( t_s \) with the same \( x = \text{Re} \, t_s \) for both \( H(1s) \) and \( H(2s) \). For \( H(2s) \) there are two other points \( t_s \) lying symmetrically on both sides (along the \( OX \) axis) of the first one. For some values of \( E = p^2/2 \) there is also a fourth point with the same \( x = \text{Re} \, t_s \) as the first one. Thus, equation (23) might reflect the multipeak effect in the ATI, but (probably) numerical calculations must be involved to obtain solutions to equation (23) (or its counterparts for \( n, l, m = (2, 1, \pm 1) \) states) and the respective ATI probability distribution. Since this was not the main goal of the present work, we delay such attempts to a later paper.

The above-mentioned findings confirm the physical relevance of our recent two-dimensional analogous SFA calculations [12], where similar regularities were found. It appears that probability distributions in the polarization plane for the real three-dimensional hydrogen atom are very similar to those obtained in its simplified two-dimensional model. Furthermore, testing the latter model by \( \text{ab} \, \text{initio} \) treatment may be more efficient than for the real atom [13]. High sensitivity to the initial quantum number \( n \) (for the ionization in nonperturbative circularly polarized laser fields) was theoretically predicted both recently [14] and long ago [15]. We are in hope that the LG SFA is able to describe correctly (at least qualitatively) what happens with the atom in an intense circularly polarized laser pulse. In our humble opinion, the advantage of the LG form of the SFA is connected with a matter of gauge invariance [16] (for a shorter version of the justification see appendix B in [6] or appendix C in [12]).

In the recent paper [17], the authors confirmed the \( m \)-dependence of the total ionization rates. However, in this experiment the Keldysh parameter \( \gamma \) was equal to 1.2 or 1.3, which corresponds to \( z_1 = 1.4 \) or \( z_1 = 1.2 \). The SFA is expected to be valid for \( z_1 \gg 1 \) (our curves in figures 1 and 2 begin at \( z_1 = 1 \)). The multipeak effect (visible enough) in the LG SFA appears for \( z_1 \geq 50 \) [12] or \( z_1 \geq 100 \) (this work).

In the recent experiment of the Keller group [18], the Keldysh parameter \( \gamma \) was equal to \( \gamma = 0.73 \), which corresponds to \( z_1 = 3.8 \). According to the LG SFA, this is still too weak a laser field to observe the multipeak effect and, indeed, no such structure has been observed experimentally so far. Our present work indicates that one should expect the multipeak effect for stronger laser fields. Detailed comparison with an experiment would require integration of the ionization rate over the pulse profile and a generalization of the LG SFA to any elliptical polarization. In the above-mentioned experiments the polarization was only ‘close-to-circular’; with the ellipticity equal to 0.78 [19]. Experiments of the Keller group were tunnelling ionization experiments. The SFA describes also over-the-barrier ionization. Maybe the multipeak effect can only be observed in the latter regime.

In conclusion, there are a few features, which tell two theories (the LG and VG SFA) apart from the hydrogen atom [20]. These features may be very important from experimenters’ points of view: (i) total ionization rates; (ii) presence or absence of the multipeak ATI envelope for certain bound states with \( n > 1 \); (iii) angles \( \vartheta \) of the maximal electronic emission, if there is no emission in the polarization plane (\( \vartheta = \pi/2 \); for example, for the initial state \((2, 1, 0)\)). There is also one issue, which is connected with item (i): the dependence of the total ionization rate on the laser frequency. If one keeps the laser intensity \( I = \text{const} \) and sets \( \omega \to 0 \) then, for any LG SFA expression, \( I_{\gamma}^{LG} \to \text{const} > 0 \). (We have verified this fact only numerically, but for many different intensities and initial states of the hydrogen atom with \( n = 1 \) or 2.) For \( n = 1 \) a reasonable agreement with the exact static-field result [21] was obtained a few years ago [5, 16]. In contrast, for any VG SFA expression \( I_{\gamma}^{VG} \to 0 \) in this limit [22, 23]. The latter result is unphysical.
Acknowledgments

The present paper has been supported by the University of Łódź. I thank the referees of this work for suggesting many valuable corrections.

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