Abstract

The quantum string emission by Black Holes is computed in the framework of the “string analogue model” (or thermodynamical approach), which is well suited to combine QFT and string theory in curved backgrounds (particularly here, as black holes and strings possess intrinsic thermal features and temperatures). The QFT-Hawking temperature $T_H$ is upper bounded by the string temperature $T_S$ in the black hole background. The black hole emission spectrum is an incomplete gamma function of $(T_H - T_S)$. For $T_H \ll T_S$, it yields the QFT-Hawking emission. For $T_H \rightarrow T_S$, it shows highly massive string states dominate the emission and undergo a typical string phase transition to a microscopic “minimal” black hole of mass $M_{\text{min}}$ or radius $r_{\text{min}}$ (inversely proportional to $T_S$) and string temperature $T_S$.

The semiclassical QFT black hole (of mass $M$ and temperature $T_H$) and the string black hole (of mass $M_{\text{min}}$ and temperature $T_S$) are mapped one into another by a “Dual” transform which links classical/QFT and quantum string regimes.

The string back reaction effect (selfconsistent black hole solution of the semiclassical Einstein equations with mass $M_+$ (radius $r_+$) and temperature $T_+$) is computed. Both, the QFT and string black hole regimes are well defined and bounded: $r_{\text{min}} \leq r_+ \leq r_S$, $M_{\text{min}} \leq M_+ \leq M$, $T_H \leq T_+ \leq T_S$.

The string “minimal” black hole has a life time $\tau_{\text{min}} \simeq \frac{k_B c}{G \hbar} T_S^{-3}$.
In the context of Quantum Field Theory in curved spacetime, Black Holes have an intrinsic Hawking temperature Ref. [1] given by

\[ T_H = \frac{\hbar c}{4\pi k_B} \frac{(D - 3)}{r_S} \]

where \( r_S \equiv L_{cl} \) being the Schwarzchild’s radius (classical length \( L_{cl} \)).

In the context of Quantum String Theory in curved spacetime, quantum strings in black hole spacetimes have an intrinsic temperature given by

\[ T_S = \frac{\hbar c}{4\pi k_B L_q} \frac{(D - 3)}{4\pi} \]

\[ L_q = bL_S(D - 3) \]

\[ L_S \equiv \sqrt{\frac{\hbar \alpha'}{c}} \]

which is the same as the string temperature in flat spacetime (See Ref. [2] and Section 3 in this paper).

The QFT-Hawking temperature \( T_H \) is a measure of the Compton length of the Black Hole, and thus, of its “quantum size”, or quantum property in the semiclassical-QFT regime. The Compton length of a quantum string is a direct measure of its size \( L_q \). The string temperature \( T_S \) is a measure of the string mass, and thus inversely proportional to \( L_q \).

The \( \mathcal{R} \) or “Dual” transform over a length introduced in Ref. [3] is given by:

\[ \tilde{L}_{cl} = \mathcal{R}L_{cl} = L_q \]

\[ \tilde{L}_q = \mathcal{R}L_q = L_{cl} \]

Under the \( \mathcal{R} \)-operation:

\[ \tilde{T}_H = T_S \]

and

\[ \tilde{T}_S = T_H \]

The QFT-Hawking temperature and the string temperature in the black hole background are \( \mathcal{R} \)-Dual of each other. This is valid in all spacetime dimensions \( D \), and is a generic feature of QFT and String theory in curved backgrounds, as we have shown this relation for the respective QFT-Hawking temperature and string temperature in de Sitter space Ref. [3]. In fact, the \( \mathcal{R} \)-transform
Maps QFT and string domains or regimes.

In this paper, we investigate the issue of Hawking radiation and the back reaction effect on the black hole in the context of String Theory. In principle, this question should be properly addressed in the context of String Field Theory. On the lack of a tractable framework for it, we work here in the framework of the string analogue model (or thermodynamical approach). This is a suitable approach for cosmology and black holes in order to combine QFT and string study and to go further in the understanding of quantum gravity effects. The thermodynamical approach is particularly appropriated and natural for black holes, as Hawking radiation and the string gas \([4,5]\) posses intrinsic thermal features and temperatures.

In this approach, the string is a collection of fields \(\Phi_n\) coupled to the curved background, and whose masses \(m_n\) are given by the degenerate string mass spectrum in the curved space considered. Each field \(\Phi_n\) appears as many times the degeneracy of the mass level \(\rho(m)\). (Although the fields \(\Phi_n\) do not interact among themselves, they do with the black hole background).

In black hole spacetimes, the mass spectrum of strings is the same as in flat spacetime Ref. [2], therefore the higher masses string spectrum satisfies

\[
\rho(m) = \left(\sqrt{\frac{\alpha' c}{\hbar}} m\right)^{-a} e^{b \sqrt{\frac{\alpha' c}{\hbar}} m}
\]

\((a\) and \(b\) being constants, depending on the model, and on the number of space dimensions).\)

We consider the canonical partition function (\(\ln Z\)) for the higher excited quantum string states of open strings (which may be or may be not supersymmetric) in the asymptotic (flat) black hole region. The gas of strings is at thermal equilibrium with the black hole at the Hawking temperature \(T_H\), it follows that the canonical partition function, [Eq. (9)] is well defined for Hawking temperatures satisfying the condition

\[
T_H < T_S
\]

\(T_S\) represents a maximal or critical value temperature. This limit implies a minimum horizon radius

\[
r_{\text{min}} = \frac{b(D-3)}{4\pi} L_S
\]

and a minimal mass for the black hole \((BH)\).
We compute the thermal quantum string emission of very massive particles by a $D$ dimensional Schwarzschild $BH$. This highly massive emission, corresponding to the higher states of the string mass spectrum, is naturally expected in the last stages of $BH$ evaporation.

In the context of QFT, $BH$ emit particles with a Planckian (thermal) spectrum at temperature $T_H$. The quantum $BH$ emission is related to the classical absorption cross section through the Hawking formula Ref. [1]:

$$\sigma_q(k, D) = \frac{\sigma_A(k, D)}{(e^{E(k)/k_B T_H} - 1)}$$

The classical total absorption spectrum $\sigma_A(k, D)$ Ref. [6] is entirely oscillatory as a function of the energy. This is exclusive to the black hole (other absorptive bodies do not show this property).

In the context of the string analogue model, the quantum emission by the $BH$ is given by

$$\sigma_{\text{string}}(D) = \sqrt{\frac{\alpha' c}{\hbar}} \int_{m_0}^{\infty} \sigma_q(m, D) \rho(m) dm$$

$\sigma_q(m, D)$ being the quantum emission for an individual quantum field with mass $m$ in the string mass spectrum. $m_0$ is the lowest mass from which the asymptotic expression for $\rho(m)$ is still valid.

We find $\sigma_{\text{string}}(D)$ as given by [Eq. (34)] (open strings). It consists of two terms: the first term is characteristic of a quantum thermal string regime, dominant for $T_H$ close to $T_S$; the second term, in terms of the exponential-integral function $E_i$ is dominant for $T_H \ll T_S$ from which the QFT Hawking radiation is recovered.

For $T_H \ll T_S$ (semiclassical QFT regime):

$$\sigma_{\text{string}}^{(\text{open})} \simeq B(D) \beta_H^{(D-3)/2} \beta_S^{(D-3)/2} e^{-\beta_H m_0 c^2}, \quad \beta_H \equiv (T_H k_B)^{-1}$$

For $T_H \to T_S$ (quantum string regime):
\[ \sigma_{\text{string}} \simeq B(D) \frac{1}{(\beta_H - \beta_S)}, \quad \beta_S \equiv (T_S k_B)^{-1} \]

\( B(D) \) is a precise computed coefficient [Eq. (34.a)].

The computed \( \sigma_{\text{string}}(D) \) shows the following: At the first stages, the \( BH \) emission is in the lighter particle masses at the Hawking temperature \( T_H \) as described by the semiclassical QFT regime (second term in [Eq. (34)]). As evaporation proceeds, the temperature increases, the \( BH \) radiates the higher massive particles in the string regime (as described by the first term of [Eq. (34)]). For \( T_H \to T_S \), the \( BH \) enters its quantum string regime \( r_S \to r_{\text{min}}, M \to M_{\text{min}} \).

That is, “the \( BH \) becomes a string”, in fact it is more than that, as [Eq. (34)] accounts for the back reaction effect too: The first term is characteristic of a Hagedorn’s type singularity Ref. [5], and the partition function here has the same behaviour as this term. Its meaning is the following: At the late stages, the emitted \( BH \) radiation (highly massive string gas) dominates and undergoes a Carlitz’s type phase transition Ref. [5] at the temperature \( T_S \) into a condensed finite energy state. Here such a state (almost all the energy concentrated in one object) is a microscopic (or “minimal”) \( BH \) of size \( r_{\text{min}}, (\text{mass } M_{\text{min}}) \) and temperature \( T_S \).

The last stage of the \( BH \) radiation, properly taken into account by string theory, makes such a phase transition possible. Here the \( T_S \) scale is in the Planck energy range and the transition is to a state of string size \( L_S \). The precise detailed description of such phase transition and such final state deserve investigation.

A phase transition of this kind has been considered in Ref. [7]. Our results here supports and give a precise picture to some issues of \( BH \) evaporation discussed there in terms of purely thermodynamical considerations.

We also describe the (perturbative) back reaction effect in the framework of the semiclassical Einstein equations (\( c \)-number gravity coupled to quantum string matter) with the v.e.v. of the energy momentum tensor of the quantum string emission as a source. In the context of the analogue model, such stress tensor v.e.v. is given by:

\[ \langle \tau^\nu_{\mu}(r) \rangle = \frac{\int_{m_0}^{\infty} \langle T^\nu_{\mu}(r, m) \rangle \sigma_q(m, D) \rho(m) dm}{\int_{m_0}^{\infty} \sigma_q(m, D) \rho(m) dm} \]

Where \( \langle T^\nu_{\mu}(r, m) \rangle \) is the v.e.v. of the QFT stress tensor of individual quantum fields of mass \( m \) in the higher excited string spectrum. The solution to the semiclassical Einstein equations is given by ([Eq. (53)], [Eq. (56)], [Eq. (60)]) \( (D = 4) \):
The string form factor $\mathcal{A}$ is given by [Eq. (62)], it is finite and positive. For $T_H \ll T_S$, the back reaction effect in the QFT-Hawking regime is consistently recovered. Algebraic terms in $(T_H - T_S)$ are entirely stringly. In both cases, the relevant ratio $\mathcal{A}/r_S^6$ entering in the solution $(r_+, M_+, T_+)$ is negligible. It is illustrative to show it in the two opposite regimes:

$$
\left(\frac{\mathcal{A}}{r_S^6}\right)_{\text{open/closed}} \simeq \frac{1}{80640\pi} \left(\frac{M_{PL}}{M}\right)^4 \left(\frac{M_{PL}}{m_0}\right)^2 \ll 1
$$

$$
T_H \ll T_S
$$

$$
\left(\frac{\mathcal{A}}{r_{\min}^6}\right)_{\text{closed}} \simeq \frac{16}{735} \left(\frac{\pi}{b}\right)^3 \left(\frac{M_S}{M_{PL}}\right)^2 \left(\frac{M_S}{m_0}\right)^2 \ll 1
$$

$$
T_H \to T_S
$$

$M_{PL}$ being the Planck mass and $M_S = \frac{\hbar}{cL_S}$.

The string back reaction solution shows that the BH radius and mass decrease, and the BH temperature increases, as it should be. But here the BH radius is bounded from below (by $r_{\min}$ and the temperature does not blow up (as it is bounded by $T_S$). The “mass loss” and “time life” are:

$$
-\left(\frac{dM}{dt}\right)_+ = -\left(\frac{dM}{dt}\right) \left(1 + \frac{20}{21} \frac{\mathcal{A}}{r_S^6}\right)
$$

$$
\tau_+ = \tau_H \left(1 - \frac{8}{7} \frac{\mathcal{A}}{r_S^6}\right)
$$

The life time of the string black hole is $\tau_{\min} = \left(\frac{\mathcal{K}c}{\hbar}\right) T_S^{-3}$.

The string back reaction effect is finite and consistently describes both, the QFT regime (BH of mass $M$ and temperature $T_H$) and the string regime (BH of mass $M_{\min}$ and temperature $T_S$). Both regimes are bounded as in string theory we have:
The \( \mathcal{R} \) “Dual” transform well summarizes the link between the two opposite well defined regimes: \( T_H \ll T_S \) (ie \( r_S \gg r_{\text{min}}, M \gg M_{\text{min}} \)) and \( T_H \rightarrow T_S \) (ie \( r_S \rightarrow R_{\text{min}}, M \rightarrow M_{\text{min}} \)).

This paper is organized as follows: In Section 2 we summarize the classical BH geometry and its semiclassical thermal properties in the QFT-Hawking regime. In Section 3 we derive the bonds imposed by string theory on this regime and show the Dual relation between the string and Hawking temperatures. In Section 4 we compute the quantum string emission by the BH. In Section 5 we compute its back reaction effect. Section 6 presents conclusions and remarks.

## 2 THE SCHWARZSCHILD BLACK HOLE SPACE TIME

The \( D-\) dimensional Schwarzschild Black Hole metric reads

\[
ds^2 = -a(r)c^2 dt^2 + a^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2
\]

where

\[
a(r) = 1 - \left( \frac{r_S}{r} \right)^{D-3}
\]

being \( r_S \) the horizon (or Schwarzschild radius)

\[
r_S = \left( \frac{16 \pi GM}{c^2(D-2)A_{D-2}} \right)^{\frac{1}{D-3}} \tag{3.a}
\]

and

\[
A_{D-2} = \frac{2\pi^{\frac{(D-1)}{2}}}{\Gamma\left(\frac{(D-1)}{2}\right)}
\]

\[\]

\[\]
(surface area per unit radius). $G$ is the Newton gravitational constant. For $D = 4$ one has

$$r_s = \frac{2GM}{c^2} \quad (3.b)$$

The Schwarzschild Black Hole ($B.H$) is characterized by its mass $M$ (angular momentum : $J = 0$ ; electric charge : $Q = 0$). The horizon [Eq. (3)] and the thermodynamical magnitudes associated to the $B.H$ – temperature ($T$), entropy ($S$), and specific heat ($C_V$) – are all expressed in terms of $M$ (Table 1).

A brief review of these quantities is the following: in the context of QFT, Black Holes do emit thermal radiation at the Hawking temperature given by

$$T_H = \frac{\hbar \kappa}{2\pi \kappa B c} \quad (5.a)$$

where

$$\kappa = \frac{c^2(D - 3)}{2r_S} \quad (5.b)$$

is the surface gravity. For $D = 4$,

$$T_H = \frac{\hbar c^3}{8\pi \kappa B GM} \quad (5.c)$$

The $B.H$ Entropy is proportional to the $B.H$ area $A = r_S^{D-2} A_{D-2}$ [Eq. (4)]

$$S = \frac{1}{4} \frac{k_B c^3}{G \hbar} A \quad (6)$$

and its specific heat $C_V = T \left( \frac{\partial S}{\partial T} \right)_{V}$ is negative

$$C_V = -\frac{(D - 2)}{4} \frac{k_B c^3}{G \hbar} A_{D-2} r_S^{D-2} \quad (7.a)$$
In 4– dimensions it reads

\[ C_V = -\frac{8\pi k_B G}{\hbar c} M^2 \quad (7. b) \]

As it is known, [Eq. (5)], [Eq. (6)] and [Eq. (7)] show that the B.H— according to its specific heat being negative – increases its temperature in its quantum emission process (\(M\) decreases). Also, it could seem that, if the B.H would evaporate completely (\(M = 0\)), the QFT-Hawking temperature \(T_H\) would become infinite. However, at this limit, and more precisely when \(M \sim M_{PL}\), the fixed classical background approximation for the B.H geometry breaks down, and the back reaction effect of the radiation matter on the B.H must be taken into account. In Section 5, we will take into account this back reaction effect in the framework of string theory.

First, we will consider quantum strings in the fixed B.H background. We will see that even in this approximation, quantum string theory not only can retard the catastrophic process but, furthermore, provides non-zero lower bounds for the B.H mass (\(M\)) or horizon (\(r_S\)), and a finite (maximal) value for the B.H temperature \(T_H\) as well.
| Dimension: $D$ | $D = 4$ |
|---|---|
| $r_S$ | $\left[ \frac{16\pi GM}{c^4 (D-2) A_{D-2}} \right]^{\frac{1}{D-3}}$ | $\frac{2GM}{c^2}$ |
| $\kappa$ | $\frac{(D-3) c^2}{2r_S}$ | $\frac{c^2}{2r_S}$ |
| $A$ | $A_{D-2} r_S^{D-2}$ | $4\pi r_S^2$ |
| $T_H$ | $\frac{\hbar \kappa}{2\pi k_B c}$ | $\frac{\hbar c}{4\pi k_B r_S}$ |
| $S$ | $\frac{1}{4} \frac{k_B c^3}{G \hbar} A$ | $\frac{k_B \pi c^3}{G \hbar} r_S^2$ |
| $C_V$ | $-\frac{(D-2)}{4} \frac{k_B c^3}{G \hbar} A_{D-2} r_S^{D-2}$ | $-\frac{2\pi k_B c^3}{G \hbar} r_S^2$ |

Table 1: Schwarzschild black hole thermodynamics. $M$ (BH mass); $r_S$ (Schwarzschild radius); $\kappa$ (surface gravity); $A$ (horizon area); $T_H$ (Hawking temperature); $S$ (entropy); $C_V$ (specific heat); $G$ and $k_B$ (Newton and Boltzmann constants); $A_{D-2} = 2\pi \frac{(D-1)}{2} / \Gamma \left( \frac{(D-1)}{2} \right)$.

3 QUANTUM STRINGS IN THE BLACK HOLE SPACE TIME

The Schwarzschild black hole spacetime is asymptotically flat. Black hole evaporation – and any “slow down” of this process – will be measured by an observer which is at this asymptotic region. In Ref. [2] it has been found that the mass spectrum of quantum string states coincides with the one in Minkowski space. Critical dimensions are the same as well Ref. [2] ($D = 26$, open and closed bosonic strings; $D = 10$ super and heterotic strings).

Therefore, the asymptotic string mass density of levels in black hole spacetime will read as in Minkowski space.
\[ \rho(m) \sim \left( \frac{\alpha' c}{\hbar} m \right)^{-a} e^{b \sqrt{\frac{\alpha' c}{\hbar} m}} \] (8)

where \( \alpha' \equiv \frac{e^2}{2\pi T} \) (\( T \) : string tension) has dimensions of \((\text{linear mass density})^{-1}\); constants \(a/b\) depend on the dimensions and on the type of string Ref. [8]. For a non-compactified space-time these coefficients are given in Table 2.

| Dimension | String Theory       | a       | b       | \( k_B T_S / c^2 \) |
|-----------|---------------------|---------|---------|---------------------|
| \( D \)   | open bosonic        | \((D - 1)/2\) |         |                     |
|           | closed              | \( D \) | \( 2\pi \sqrt{\frac{D-2}{6}} \) | \( \left[ 2\pi \sqrt{\frac{(D-2)}{6}} \left( \frac{\alpha' c}{\hbar} \right) \right]^{-1} \) |
|          | 26 (critical)       |         |         |                     |
|           | open bosonic        | \( 25/2 \) |         | \( 4\pi \) \( (4\pi \sqrt{\frac{\alpha' c}{\hbar}})^{-1} \) |
|           | closed              | \( 26 \) | \( 4\pi \) |                     |
|          | 10 (critical)       |         |         |                     |
|           | open superstring    | \( 9/2 \) | \( \pi 2\sqrt{2} \) | \( \left[ \pi 2 \sqrt{2 \left( \frac{\alpha' c}{\hbar} \right)} \right]^{-1} \) |
|           | closed superstring (type II) | \( 10 \) | \( \pi (2 + \sqrt{2}) \) | \( \left[ \pi (1 + \sqrt{2}) \sqrt{2 \left( \frac{\alpha' c}{\hbar} \right)} \right]^{-1} \) |
|           | Heterotic           | \( 10 \) | \( \pi (2 + \sqrt{2}) \) |                     |

Table 2: Density of mass levels \( \rho(m) \sim m^{-a} \exp\{b\sqrt{\frac{\alpha' c}{\hbar} m}\} \). For open strings \( \alpha' \left( \frac{\xi}{\hbar} \right) m^2 \simeq n \); for closed strings \( \alpha' \left( \frac{\xi}{\hbar} \right) m^2 \simeq 4n \).

In this paper, strings in a \( B.H \) spacetime are considered in the framework of the string analogue model. In this model, one considers the strings as a collection of quantum fields \( \phi_1, \ldots, \phi_n \), whose masses are given by the string mass spectrum (\( \alpha' \left( \frac{\xi}{\hbar} \right) m^2 \simeq n \), for open strings and large \( n \) in flat spacetime). Each field of mass \( m \) appears as many times as the degeneracy of the mass level; for higher excited modes this is described by \( \rho(m) \) [Eq. (8)]. Although quantum fields do not interact among themselves, they do with the \( B.H \) background.
In the asymptotic (flat) B.H region, the thermodynamical behavior of the higher excited quantum string states of open strings, for example, is deduced from the canonical partition function Ref. [5]

\[ \ln Z = \frac{V}{(2\pi)^d} \sqrt{\frac{\alpha' c}{\hbar}} \int_{m_0}^{\infty} dm \rho(m) \int d^d k \ln \left\{ \frac{1 + \exp \left[ -\beta_H \left( m^2 c^4 + k^2 \hbar^2 c^2 \right)^{\frac{1}{2}} \right]}{1 - \exp \left[ -\beta_H \left( m^2 c^4 + k^2 \hbar^2 c^2 \right)^{\frac{1}{2}} \right]} \right\} \quad (9) \]

(\(d\): number of spatial dimensions) where supersymmetry has been considered for the sake of generality; \(\rho(m)\) is the asymptotic mass density given by [Eq. (8)]; \(\beta_H = (k_B T_H)^{-1}\) where \(T_H\) is the B.H Hawking temperature; \(m_0\) is the lowest mass for which \(\rho(m)\) is valid.

For the higher excited string modes, i.e., the masses of the B.H and the higher string modes satisfy the condition

\[ \beta_H m c^2 = \frac{4\pi m c}{(D-3)\hbar} \left[ \frac{16\pi G M}{c^2 (D-2) A_{D-2}} \right]^{\frac{1}{D-3}} \gg 1 \quad (10.a) \]

which reads for \(D = 4\)

\[ \beta_H m c^2 = \frac{8\pi G M m}{\hbar c} \gg 1 \quad (10.b) \]

(condition [Eq. (10.b)] will be considered later in section 4) the leading contribution to the r.h.s. of [Eq. (9)] will give as a canonical partition function

\[ \ln Z \simeq \frac{2V_{D-1} \left( \alpha' \frac{c^2}{\hbar} \right)^{-\frac{(a-1)}{2}}}{(2\pi \beta_H \hbar^{a+\frac{D-1}{2}})} \cdot \int_{m_0}^{\infty} dm m^{-a+\frac{D-1}{2}} e^{-(\beta_H - \beta_S)mc^2} \quad (11) \]

where \(\beta_S = (k_B T_S)^{-1}\), being \(T_S\) [Eq. (8)]

\[ T_S = \frac{c^2}{k_B b \left( \frac{\alpha' c}{\hbar} \right)^{\frac{a}{2}}} \quad (12) \]
the string temperature (Table 2). For open bosonic strings one divides by 2 the r.h.s. of [Eq. (11)] (leading contributions are the same for bosonic and fermionic sector).

From [Eq. (11)] we see that the definition of \( \ln Z \) implies the following condition on the Hawking temperature

\[
T_H < T_S
\]

(13)

Furthermore, as \( T_H \) depends on the \( B.H \) mass \( M \), or on the horizon \( r_S \), [Eq. (5.a)], [Eq. (5.b)] and [Eq. (3)], the above condition will lead to further conditions on the horizon. Then \( T_S \) represents a critical value temperature: \( T_S \equiv T_{cr} \). In order to see this more clearly, we rewrite \( T_S \) in terms of the quantum string length scale

\[
L_S = \left( \frac{\hbar \alpha'}{c} \right)^{\frac{1}{2}}
\]

(14)

namely

\[
T_S = \frac{\hbar c}{b k_B L_S}
\]

(15)

From [Eq. (13)], and with the help of [Eq. (5.a)], [Eq. (5.b)] and [Eq. (15)], we deduce

\[
r_S > \frac{b (D - 3)}{4\pi} L_S
\]

(16)

which shows that (first quantized) string theory provides a lower bound, or minimum radius, for the \( B.H \) horizon.

Taking into account [Eq. (3)] and [Eq. (16)] we have the following condition on the \( B.H \) mass

\[
M > \frac{c^2 (D - 2) A_{D-2}}{16\pi G} \left[ \frac{b (D - 3)}{4\pi} L_S \right]^{D-3}
\]

(17)

therefore there is a minimal \( B.H \) mass given by string theory.
For $D = 4$ we have

\[ r_S > \frac{b}{4\pi} L_S \tag{18} \]

\[ M > \frac{c^2 b}{8\pi G} L_S \tag{19} \]

these lower bounds satisfy obviously [Eq. (3.b)]. [Eq. (19)] can be rewritten as

\[ M > b \frac{M_{PL}^2}{8\pi M_S} \]

where $M_S = \frac{b}{L_{sc}}$ is the string mass scale ($L_S$ : reduced Compton wavelength) and $M_{PL} \equiv \left( \frac{\hbar c}{G} \right)^{\frac{1}{2}}$ is the Planck mass. The minimal $B.H$ mass is then [Eq. (14)] and [Eq. (19)]

\[ M_{\text{min}} = \frac{b}{8\pi G} \sqrt{\hbar c^2 \alpha'} \]

It is appropriate, at this point, to make use of the $\mathcal{R}$ or Dual transformation over a length introduced in Ref. [3]. This operation is

\[ \tilde{L}_{cl} = \mathcal{R} L_{cl} = \mathcal{L}_R L_{cl}^{-1} = L_q \quad \text{and} \quad \tilde{L}_{q} = \mathcal{R} L_{q} = \mathcal{L}_R L_{q}^{-1} = L_{cl} \tag{20.a} \]

where $\mathcal{L}_R$ has dimensions of (length)$^2$; and it is given by $\mathcal{L}_R = L_{cl} L_q$.

In our case, $L_{cl}$ is the classical Schwarzschild radius, and $L_Q \equiv r_{\text{min}} = (b(D-3)L_S)/4\pi$ [Eq. (16)].

The $\mathcal{R}$ transformation links classical lengths to quantum string lengths, and more generally it links QFT and string theory domains Ref. [3].

and the string temperature is

\[ T_H = \frac{\hbar c (D - 3)}{4\pi k_B L_{cl}} \tag{20.b} \]

For the $BH$, the QFT-Hawking temperature is
\[ T_S = \frac{\hbar c (D - 3)}{4\pi k_B L_q} \]  

(20.c)

Under the R operation we have

\[ \tilde{T}_H = T_S \quad \text{and} \quad \tilde{T}_S = T_H \]  

(20.d)

which are valid for all \( D \).

From the above equations we can read as well

\[ \tilde{T}_H \tilde{T}_S = T_S T_H \]

We see that under the \( R \)-Dual operation, the QFT temperature and the string temperature in the BH background are mapped one into another. This appears to be a general feature for QFT and string theory in curved backgrounds, as we have already shown this relation in the de Sitter background Ref. [3].

It is interesting to express \( T_H \) and \( T_S \) in terms of their respective masses

\[ T_H = \frac{\hbar c (D-3)}{4\pi k_B} \left( \frac{16\pi GM}{c^2(D-2)A_{D-2}} \right)^{-\frac{1}{2(D-3)}} \]

\[ T_H = \frac{\hbar c^3}{8\pi k_B GM} \quad (D = 4) \]

and

\[ T_S = \frac{c^2 M_S}{b k_B} \]

4 THERMAL QUANTUM STRING EMISSION FOR A SCHWARZSCHILD BLACK HOLE

As it is known, thermal emission of massless particles by a black hole has been considered in the context of QFT Ref. [1], Ref. [9], Ref. [10]. Here, we are going to deal with thermal emission of high massive particles which correspond to the higher excited modes of a string. The study will be done in the framework of the string analogue model.
For a static $D-$ dimensional black hole, the quantum emission cross section $\sigma_q(k, D)$ is related to the total classical absorption cross section $\sigma_A(k, D)$ through the Hawking formula Ref. [1]

$$\sigma_q(k, D) = \frac{\sigma_A(k, D)}{e^{E(k)\beta_H} - 1}$$

(21)

where $E(k)$ is the energy of the particle (of momentum : $p = \hbar k$) and $\beta_H = (k_B T_H)^{-1}$, being $T_H$ Hawking temperature [Eq. (5.a)] and [Eq. (5.b)].

The total absorption cross section $\sigma_A(k, D)$ in [Eq. (21)] has two terms Ref. [6], one is an isotropic $k-$ independent part, and the other has an oscillatory behavior, as a function of $k$, around the optical geometric constant value with decreasing amplitude and constant period. Here we will consider only the isotropic term, which is the more relevant in our case.

For a $D-$ dimensional black hole space-time, this is given by (see for example Ref.[2])

$$\sigma_A(k, D) = a(D) r_S^{D-2}$$

(22)

where $r_S$ is the horizon [Eq. (3.a)] and [Eq. (4)] and

$$a(D) = \frac{\pi^{(D-2)/2}}{\Gamma\left(\frac{D-2}{2}\right)} \left(\frac{D - 1}{D - 3}\right) \left(\frac{D - 1}{2}\right) \frac{2}{\pi^D}$$

(23)

We notice that $\rho(m)$ [Eq. (8)] depends only on the mass, therefore we could consider, in our formalism, the emitted high mass spectrum as spinless. On the other hand, as we are dealing with a Schwarzschild black hole (angular momentum equal to zero), spin considerations can be overlooked. Emission is larger for spinless particles Ref. [11].

The number of scalar field particles of mass $m$ emitted per unit time is

$$\langle n(m) \rangle = \int_0^\infty \langle n(k) \rangle \, d\mu(k)$$

(24)

where $d\mu(k)$ is the number of states between $k$ and $k + dk$.

$$d\mu(k) = \frac{V_d}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} k^{d-1} \, dk$$

(25)
and \( \langle n(k) \rangle \) is now related to the quantum cross section \( \sigma_q \) ([Eq. (21)] and [Eq. (22)]) through the equation.

\[
\langle n(k) \rangle = \frac{\sigma_q(k, D)}{r_S^{D-2}} \tag{26}
\]

Considering the isotropic term for \( \sigma_q \) [Eq. (22)] and [Eq. (23)] we have

\[
\langle n(k) \rangle = \frac{a(D)}{e^{E(k)\beta_H} - 1} \tag{27}
\]

where \( \beta_H = \frac{1}{k_B T_H} \), being \( T_H \) the BH temperature [Eq. (5)].

From [Eq. (24)] and [Eq. (27)], \( \langle n(m) \rangle \) will be given by

\[
\langle n(m) \rangle = F(D, \beta_H) m \frac{(D-3)}{2} (mc^2 \beta_H + 1)e^{-\beta_H mc^2} \tag{28}
\]

where

\[
F(D, \beta_H) \equiv \frac{V_{D-1}a(D)}{(2\pi)^{\frac{D-1}{2}}} \frac{(c^2)^{\frac{(D-3)}{2}}}{\beta_H^{\frac{(D-1)}{2}} (\hbar c)^{D-1}} \equiv A(D) \beta_H^{\frac{D+1}{2}} \tag{29}
\]

Large argument \( \beta_H m c^2 \gg 1 \), ie [Eq. (10.a)] and [Eq. (10.b)], and leading approximation have been considered in performing the \( k \) integral.

The quantum thermal emission cross section for particles of mass \( m \) is defined as

\[
\sigma_q(m, D) = \int \sigma_q(k, D)d\mu(k) \tag{30.a}
\]

and with the help of [Eq. (26)] we have

\[
\sigma_q(m, D) = r_S^{D-2} \langle n(m) \rangle \tag{30.b}
\]

where \( \langle n(m) \rangle \) is given by [Eq. (28)].
In the string analogue model, the string quantum thermal emission by a BH will be given by the cross section

\[ \sigma_{\text{string}}(D) = \sqrt{\frac{\alpha'}{\hbar}} \int_{m_0}^{\infty} \sigma_q(m, D) \rho(m) dm \]  

(31)

where \( \rho(m) \) is given by [Eq. (8)], and \( \sigma_q(m, D) \) by [Eq. (30)] and [Eq. (28)]; \( m_0 \) is the lowest string field mass for which the asymptotic value of the density of mass levels, \( \rho(m) \), is valid.

For arbitrary \( D \) and \( a \), we have from [Eq. (31)], [Eq. (30.b)], [Eq. (28)] and [Eq. (8)]

\[ \sigma_{\text{string}}(D) = F(D, \beta_H) r_S^{D-2} \left( \sqrt{\frac{\alpha'}{\hbar}} \right)^{-a+1} I_D(m, \beta_H - \beta_{cr}, a) \]  

(32)

where \( F(D, \beta_H) \) is given by [Eq. (29)] and

\[ \beta_{cr} \equiv \beta_S = \left( k_B T_S \right)^{-1} = \frac{b}{c^2} \sqrt{\frac{\alpha'}{\hbar}} \]  

(33.a)

and

\[ I_D(m, \beta_H - \beta_{cr}, a) \equiv \int_{m_0}^{\infty} m^{-a+\frac{D-3}{2}} (mc^2 \beta_H + 1) e^{-(\beta_H - \beta_{cr})mc^2} dm \]  

(33.b)

After a straightforward calculation we have

\[ I_D(m, \beta_H - \beta_S, a) = \frac{c^2 \beta_H}{[(\beta_H - \beta_S)c^2]^{-a+\frac{D+1}{2}}} \Gamma \left( -a + \frac{D+1}{2}, (\beta_H - \beta_S)c^2 m_0 \right) \]

\[ + \frac{1}{[(\beta_H - \beta_S)c^2]^{-a+\frac{D-1}{2}}} \Gamma \left( -a + \frac{D-1}{2}, (\beta_H - \beta_S)c^2 m_0 \right) \]  

(33.c)

where \( \Gamma(x, y) \) is the incomplete gamma function.

For open strings, \( a = \frac{(D-1)}{2} \) [\( D \) : non-compact dimensions], we have
\[ \sigma^{(\text{open})}_{\text{string}}(D) = A(D) \beta_H^{(D+1)/2} r_{\text{min}}^{D-2} \left( \frac{c^2 \beta_S}{b} \right)^{-(D-3)/2} \cdot \left( \frac{\beta_H}{\beta_H - \beta_S} \right) e^{-(\beta_H - \beta_S)c^2 m_0} - E_i \left( -((\beta_H - \beta_S)c^2 m_0) \right) \] (34)

where \( E_i \) is the exponential-integral function, and we have used [Eq. (29)] and [Eq. (33.A)].

When \( T_H \) approaches the limiting value \( T_S \), and as \( E_i(-x) \sim C + \ln x \) for small \( x \), we have from [Eq. (34)]

\[ \sigma^{(\text{open})}_{\text{string}}(D) = A(D) \beta_S^{(D+1)/2} r_{\text{min}}^{D-2} \left( \frac{c^2 \beta_S}{b} \right)^{-(D-3)/2} \cdot \left( \frac{\beta_S}{\beta_H - \beta_S} \right) - C - \ln ((\beta_H - \beta_S)c^2 m_0) \]

\[ = B(D) \beta_S^{-1} \left( \frac{\beta_S}{\beta_H - \beta_S} \right) - C - \ln ((\beta_H - \beta_S)c^2 m_0) \]

where

\[ r_{\text{min}} = \frac{\hbar c (D - 3) \beta_S}{4\pi} \]

and

\[ B(D) \equiv A(D) \left( \frac{\hbar c (D - 3)}{4\pi} \right)^{D-2} \left( \frac{c^2}{b} \right)^{-(D-3)/2} \] (34.a)

For \( \beta_H \to \beta_S \) the dominant term is

\[ \sigma^{(\text{open})}_{\text{string}}(D) \overset{T_H \to T_S}{\to} B(D) \frac{1}{(\beta_H - \beta_S)} \] (35.a)

for any dimension.

For \( \beta_H \gg \beta_S \), ie \( T_H \ll T_S \)

\[ \sigma^{(\text{open})}_{\text{string}}(D) \overset{T_H \ll T_S}{\to} A(D) \beta_H^{(D+1)/2} r_{\text{min}}^{D-2} \left( \frac{c^2}{b} \beta_S \right)^{-(D-3)/2} e^{-\beta_H c^2 m_0} \left( 1 + \frac{1}{\beta_H c^2 m_0} \right) \]
\[ \simeq B(D)\beta_H^{(D-5)}\beta_S^{(D-3)} e^{-\beta_H c^2 m_0} \]  
\hspace{1cm} (35.b)

as \( E_i(-x) \sim \frac{e^{-x}}{x} + \cdots \) for large \( x \). For \( D = 4 \),

\[ \sigma_{\text{string}}^{(\text{open})} (4) \simeq B(4) \left( \frac{1}{\beta_H \beta_S} \right)^{\frac{1}{2}} e^{-\beta_H c^2 m_0} \]

At this point, and in order to interpret the two different behaviours, we compare them with the corresponding behaviours for the partition function [Eq. (11)]. For open strings \( (a = (D - 1)/2) \), \( \ln Z \) is equal to

\[ \ln Z_{\text{open}} \simeq \frac{2V_{D-1}(\alpha')^\frac{(D-3)}{4}}{(2\pi \beta_H h^2)^\frac{(D-1)}{2}} \cdot \frac{1}{(\beta_H - \beta_S) c^2} \cdot e^{-(\beta_H - \beta_S) m_0 c^2} \]

For \( \beta_H \to \beta_S \):

\[ \ln Z_{\text{open}} \simeq \frac{2V_{D-1}(\alpha')^\frac{(D-3)}{4}}{(2\pi \beta_H h^2)^\frac{(D-1)}{2}} \cdot \frac{1}{(\beta_H - \beta_S) c^2} \]

and for \( \beta_H \gg \beta_S \):

\[ \ln Z_{\text{open}} \simeq \frac{2V_{D-1}(\alpha')^\frac{(D-3)}{4}}{(2\pi h^2)^\frac{(D-1)}{2}} \cdot \frac{1}{(\beta_H c^2)^{(D+1)/4}} \cdot e^{-\beta_H m_0 c^2} \]

The singular behaviour for \( \beta_H \to \beta_S \), and all \( D \), is typical of a string system with intrinsic Hagedorn temperature, and indicates a string phase transition (at \( T = T_S \)) to a condensed finite energy state (Ref. [5]). This would be the minimal black hole, of mass \( M_{\text{min}} \) and temperature \( T_S \).
5 QUANTUM STRING BACK REACTION IN BLACK HOLE SPACE TIMES

When we consider quantised matter on a classical background, the dynamics can be described by the following Einstein equations

\[ R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu}R = \frac{8\pi G}{c^4} \langle \tau_{\mu}^{\nu} \rangle \] (35)

The space-time metric \( g_{\mu\nu} \) generates a non-zero vacuum expectation value of the energy momentum tensor \( \langle \tau_{\mu}^{\nu} \rangle \), which in turn, acting as a source, modifies the former background. This is the so-called back reaction problem, which is a semiclassical approach to the interaction between gravity and matter.

Our aim here is to study the back reaction effect of higher massive (open) string modes (described by \( \rho(m) \), [Eq. (8)]) in black holes space-times. This will give us an insight on the last stage of black hole evaporation. Back reaction effects of massless quantum fields in these equations were already investigated Ref. [12], Ref. [13], Ref. [14].

As we are also interested in establishing the differences, and partial analogies, between string theory and the usual quantum field theory for the back reaction effects in black holes space-times, we will consider a 4- dimensional physical black hole.

The question now is how to write the appropriate energy-momentum tensor \( \langle \tau_{\mu}^{\nu} \rangle \) for these higher excited string modes. For this purpose, we will consider the framework of the string analogue model. In the spirit of this model, the v.e.v. of the stress tensor \( \langle \tau_{\mu}^{\nu} \rangle \) for the string higher excited modes is defined by

\[ \langle \tau_{\mu}^{\nu}(r) \rangle = \frac{\int_{m_0}^{\infty} \langle T_{\mu}^{\nu}(r, m) \rangle \langle n(m) \rangle \rho(m) \, dm}{\int_{m_0}^{\infty} \langle n(m) \rangle \rho(m) \, dm} \] (36)

where \( \langle T_{\mu}^{\nu}(r, m) \rangle \) is the Hartle-Hawking vacuum expectation value of the stress tensor of an individual quantum field, and the \( r \) dependence of \( \langle T_{\mu}^{\nu} \rangle \) preserves the central gravitational character of the problem; \( \rho(m) \) is the string mass density of levels [Eq. (8)] and \( \langle n(m) \rangle \) is the number of field particles of mass \( m \) emitted per unit time, [Eq. (28)].

For a static spherically symmetric metric
\[ ds^2 = g_{00}(r)c^2 dt^2 + g_{rr}(r)dr^2 + r^2d\Omega_2^2 \]  

(37)

where \( g_{00}(r) < 0 \) for a Schwarzschild black hole solution, the semiclassical Einstein equations [Eq. (36)] read

\[
\frac{8\pi G}{c^4} \langle \tau_r^r \rangle = g_{rr}^{-1} \left( \frac{1}{r} \frac{d \ln g_{00}}{dr} + \frac{1}{r^2} \right) - \frac{1}{r^2} \quad (39.a)
\]

\[
\frac{8\pi G}{c^4} \langle \tau_0^0 \rangle = g_{rr}^{-1} \left( \frac{1}{r^2} - \frac{1}{r} \frac{d \ln g_{rr}}{dr} \right) - \frac{1}{r^2} \quad (39.b)
\]

For Schwarzschild boundary conditions

\[ g_{00}(r) \cdot g_{rr}(r) = -1 \quad (40.a) \]

where

\[ g_{rr}(r) = \left( 1 - \frac{r_S}{r} \right)^{-1} \quad (40.b) \]

the solution to [Eq. (39.a)] and [Eq. (39.b)] is given by

\[
g_{rr}^{-1}(r) = 1 - \frac{2GM}{c^2 r} + \frac{8\pi G}{c^4 r} \int_{r_\infty}^{r} \langle \tau_0^0(r') \rangle r'^2 dr' \quad (41.a)
\]

\[
g_{00}(r) = -g_{rr}^{-1}(r) \cdot \exp \left\{ \frac{8\pi G}{c^4} \int_{r_\infty}^{r} \left( \langle \tau_r^r(r') \rangle - \langle \tau_0^0(r') \rangle \right) r' g_{rr}(r') dr' \right\} \quad (41.b)
\]

where \( M \) is the black hole mass measured from \( r_\infty \) (\( r_\infty \) may be infinite or the radius of a cavity where the black hole is put inside to maintain the thermal equilibrium).

In order to write \( \langle T^\mu_\mu(m, r) \rangle \) for an individual quantum field in the framework of the analogue model, we notice that \( \rho(m) \) [Eq. (8)] depends only on \( m \); therefore, we will consider for simplicity the vacuum expectation value of the stress tensor for a massive scalar field.

For the Hartle-Hawking vacuum (black-body radiation at infinity in equilibrium with a black hole at the temperature \( T_H \)), and when the (reduced) Compton wave length of the massive particle
\( \lambda = \frac{\hbar}{mc} \) is much smaller than the Schwarzschild radius \( r_S \)

\[
\frac{\hbar c}{2GMm} \ll 1 \quad (42)
\]

(same condition as the one of [Eq. (10.b)]), \( \langle T^r_r \rangle \) and \( \langle T^0_0 \rangle \) for the background B.H metric ([Eq. (1)], \( D = 4 \)) read Ref. [13]

\[
\frac{8\pi G}{c^4} \langle T^r_r \rangle = \frac{A}{r^8} F_1 \left( \frac{r_S}{r} \right) \quad (43.a)
\]

\[
\frac{8\pi G}{c^4} \langle T^0_0 \rangle = \frac{A}{r^8} F_2 \left( \frac{r_S}{r} \right) \quad (43.b)
\]

where

\[
A = \frac{M^2 L^6_{PL}}{1260\pi m^2} \quad (43.c)
\]

\[
F_1 \left( \frac{r_S}{r} \right) = 441 - \zeta 2016 + \frac{r_S}{r} (-329 + \zeta 1512) + O(m^{-4}) \quad (43.d)
\]

\[
F_2 \left( \frac{r_S}{r} \right) = -1125 + \zeta 5040 + \frac{r_S}{r} (1237 - \zeta 5544) + O(m^{-4}) \quad (43.e)
\]

\( M \) and \( m \) are the black hole and the scalar field masses respectively, \( \zeta \) (a numerical factor) is the scalar coupling parameter \((-\frac{\zeta R \phi^2}{2} ; R : \text{scalar curvature}, \phi : \text{scalar field})\) and \( L_P \equiv \left( \frac{hG}{c^3} \right)^{\frac{1}{2}} \) is the Planck length.

From [Eq. (36)], [Eq. (43.a)] and [Eq. (43.b)] the v.e.v. of the string stress tensor will read

\[
\frac{8\pi G}{c^4} \langle \tau^r_r \rangle = \frac{A}{r^8} F_1 \left( \frac{r_S}{r} \right) \quad (44.a)
\]

\[
\frac{8\pi G}{c^4} \langle \tau^0_0 \rangle = \frac{A}{r^8} F_2 \left( \frac{r_S}{r} \right) \quad (44.b)
\]

where
\[ A = \frac{M^2 L_{PL}^6}{1260\pi} \int_{m_0}^{\infty} m^{-2} \langle n(m) \rangle \rho(m) \, dm \]  

(45)

We return now to [Eq. (41.a)] and [Eq. (41.b)] which, with the help of [Eq. (44.a)], [Eq. (44.b)] and [Eq. (3.b)], can be rewritten as

\[ g_{rr}^{-1}(r) = 1 - \frac{r_S}{r} + \frac{A}{r} \int_{r_\infty}^{r} \frac{F_2 \left( \frac{r_S}{r} \right)}{r^6} \, dr' \]  

(46)

\[ g_{00}(r) = -g_{rr}^{-1}(r) \cdot \exp \left\{ A \int_{r_\infty}^{r} \left[ F_1 \left( \frac{r_S}{r} \right) - F_2 \left( \frac{r_S}{r} \right) \right] \frac{g_{rr}(r')}{r'^7} \, dr' \right\} \]  

(47)

A Schwarzschild black body configuration

\[ g_{00}(r) = -g_{rr}^{-1}(r) \]  

(48)

is obtained when [Eq. (47)]

\[ F_1 \left( \frac{r_S}{r} \right) - F_2 \left( \frac{r_S}{r} \right) \equiv (1566 - \zeta(7056)) \left( 1 - \frac{r_S}{r} \right) = 0 \]  

(49)

ie for \( \zeta = \frac{87}{392} \) for all \( r \).

Then from [Eq. (46)] and [Eq. (43.e)], we obtain

\[ g_{rr}^{-1} = 1 - \frac{r_S}{r} - \frac{A}{21r^6} \left[ 23 \left( \frac{r_S}{r} \right) - 27 \right] \]  

(50)

From the above equation it is clear that the quantum matter back reaction modifies the horizon, \( r_+ \), which will be no longer equal to the classical Schwarzschild radius \( r_S \). The new horizon will satisfy

\[ g_{rr}^{-1} = 0 \]  

(51.a)

ie

\[ r_+^7 - r_S r_+^6 + \frac{A}{21} r_+ - \frac{A}{21} r_S = 0 \]  

(51.b)
In the approximation we are dealing with \( O(m^{-4}) \) i.e. \( A^2 \ll A \), the solution will have the form

\[
 r_+ \simeq r_S (1 + \epsilon) , \quad \epsilon \ll 1
\]  

(52)

From [Eq. (51.b)] we obtain

\[
 r_+ \simeq r_S \left( 1 - \frac{4A}{21 r_S^6} \right)
\]  

(53)

which shows that the horizon decreases.

Let us consider now the surface gravity, which is defined as

\[
 k(r_+) = \frac{c^2}{2} \left. \frac{d g_{rr}^{-1}}{d r} \right|_{r=r_+}
\]  

(54)

(in the absence of back reaction, \( k(r_+) = k(r_S) \) given by [Eq. (5.b)] for \( D = 4 \)).

From [Eq. (50)], [Eq. (53)] and [Eq. (54)] we get

\[
 k(r_+) = \frac{c^2}{2 r_S} \left( 1 + \frac{1}{3} \frac{A}{r_S^6} \right)
\]  

(55)

The black hole temperature will then be given by

\[
 T_+ = \frac{\hbar \kappa (r_+)}{2 \pi k_B c} \simeq T_H \left( 1 + \frac{1}{3} \frac{A}{r_S^6} \right)
\]  

(56)

where \( T_H = \frac{\hbar c}{4 \pi k_B r_S} \)

([Eq. (5.a)] and [Eq. (5.b)] for \( D = 4 \)). The Black hole temperature increases due to the back reaction.

Due to the quantum emission the black hole suffers a loss of mass. The mass loss rate is given by a Stefan-Boltzman relation. Without back reaction, we have
\[- \frac{dM}{dt} = \sigma 4\pi r^2 T^4_H \quad (57)\]

where \( \sigma \) is a constant.

When back reaction is considered, we will have

\[- \frac{dM}{dt} + = \sigma 4\pi r^2 T^4_+ \quad (58)\]

where \( r_+ \) is given by [Eq. (53)] and \( T_+ \) by [Eq. (56)]. Inserting these values into the above equation we obtain

\[- \frac{dM}{dt} \simeq - \left( \frac{dM}{dt} \right) \left( 1 + 20 \frac{A}{21 r_6^6} \right) \quad (59)\]

On the other hand, the modified black hole mass is given by

\[ M_+ \equiv \frac{c^2}{2G} r_+ \simeq M \left( 1 - \frac{4A}{21 r_6^6} \right) \quad (60)\]

which shows that the mass decreases.

From [Eq. (59)] and [Eq. (60)], we calculate the modified life time of the black hole due to the back reaction

\[ \tau_+ \simeq \tau_H \left( 1 - \frac{8A}{7r_6^6} \right) \quad (61)\]

We see that \( \tau_+ < \tau_H \) since \( A > 0 \).

We come back to the string back reaction “form factor” \( A \) [Eq. (45)] which can be rewritten as

\[ A = \frac{M^2 L_{PL}^6}{1260\pi} \cdot \frac{N}{De} \quad (62)\]
where

\[ N = \int_{m_0}^{\infty} m^{-a+\frac{D-7}{2}} (mc^2\beta_H + 1) e^{-(\beta_H-\beta_S)mc^2} \, dm \]  

(63)

and [Eq. (33)]

\[ De = I_D(m, \beta_H - \beta_S, a) \]  

(64)

where use of [Eq. (28)] and [Eq. (8)] has been made (common factors for numerator and denominator cancelled out).

For arbitrary \( D \) and \( a \), \( N \) is given by

\[
N = \frac{c^2\beta_H}{[(\beta_H - \beta_S)c^2]^{-a+\frac{D-3}{2}}} \Gamma \left(-a + \frac{D - 3}{2}, (\beta_H - \beta_S)c^2m_0\right) 
+ \frac{1}{[(\beta_H - \beta_S)c^2]^{-a+\frac{D-5}{2}}} \Gamma \left(-a + \frac{D - 5}{2}, (\beta_H - \beta_S)c^2m_0\right)
\]

(65)

In particular, for open strings (\( a = \frac{(D-1)}{2} \)) we have for \( N \) and \( De \)

\[
N = c^4\beta_H(\beta_H - \beta_S)[E_i \left(-(\beta_H - \beta_S)m_0c^2\right) + \frac{e^{-(\beta_H-\beta_S)m_0c^2}}{(\beta_H - \beta_S)m_0c^2}] 
- \frac{(\beta_H - \beta_S)^2c^4}{2} [E_i \left(-(\beta_H - \beta_S)m_0c^2\right) + e^{-(\beta_H-\beta_S)m_0c^2} 
\cdot \left(\frac{1}{(\beta_H - \beta_S)m_0c^2} - \frac{1}{(\beta_H - \beta_S)^2m_0c^2}\right)\]
\]

(66)

and

\[ De = \frac{\beta_H}{\beta_H - \beta_S} e^{-(\beta_H-\beta_S)c^2m_0} - E_i \left(-(\beta_H - \beta_S)c^2m_0\right) \]  

(67)

For \( \beta_H \to \beta_S \ (M \to M_{\text{min}}, r_S \to r_{\text{min}}) \) we have for the open string form factor
\[ A_{\text{open}} \simeq \frac{M_{\text{min}}^2 L_P^6 (\beta_H - \beta_S)}{1260 \pi \beta_S} \left( \frac{1}{2m_0^2} + \frac{c^2 \beta_S}{m_0} \right) \]  

(68)

Although the string analogue model is in the spirit of the canonical ensemble—all (higher) massive string fields are treated equally – we will consider too, for the sake of completeness, the string “form factor” \( A \) for closed strings.

For \( a = D \) (\( D \): non compact dimensions), from [Eq. (33.b)], [Eq. (33.c)] and [Eq. (64)], we have the following expressions

\[
D e = I_D(m, \beta_H - \beta_S, D) \\
= \frac{c^2 \beta_H}{[(\beta_H - \beta_S)c^2]^{\frac{D+1}{2}}} \Gamma \left( -\frac{D-1}{2}, (\beta_H - \beta_S)m_0 c^2 \right) \\
+ \frac{1}{[(\beta_H - \beta_S)c^2]^{\frac{D+1}{2}}} \Gamma \left( -\frac{D+1}{2}, (\beta_H - \beta_S)m_0 c^2 \right) 
\]  

(69)

and [Eq. (63)]

\[
N = \frac{c^2 \beta_H}{[(\beta_H - \beta_S)c^2]^{\frac{D+3}{2}}} \Gamma \left( -\frac{D+3}{2}, (\beta_H - \beta_S)c^2 m_0 \right) \\
+ \frac{1}{[(\beta_H - \beta_S)c^2]^{\frac{D+3}{2}}} \Gamma \left( -\frac{D+5}{2}, (\beta_H - \beta_S)c^2 m_0 \right) 
\]  

(70)

For \( \beta_H \to \beta_S \) and \( D \) even, we have then

\[
N = c^2 \beta_S [(\beta_H - \beta_S)c^2]^{\frac{(D+3)}{2}} \Gamma \left( -\frac{D+3}{2} \right) + \frac{c^2 \beta_S}{(m_0)\left(\frac{(D+3)}{2}\right)} \frac{\Gamma \left( -\frac{D+3}{2} \right)}{\left(\frac{D+3}{2}\right)} \\
+ [(\beta_H - \beta_S)c^2]^{\frac{(D+5)}{2}} \Gamma \left( -\frac{D+5}{2} \right) + \frac{1}{(\frac{D+5}{2})} \frac{\Gamma \left( -\frac{D+5}{2} \right)}{m_0\left(\frac{(D+5)}{2}\right)} 
\]  

(71)

and
\[ De = c^2 \beta_S [(\beta_H - \beta_S) c^2]^{(D-1)/2} \Gamma \left( -\frac{D - 1}{2} \right) + \frac{c^2 \beta_S}{(m_0)^{(D-1)/2} (D-1)} \]
\[ + \left[ c^2(\beta_H - \beta_S) \right]^{(D+1)/2} \Gamma \left( -\frac{D + 1}{2} \right) + \frac{1}{m_0^{(D+1)/2} (D+1)} \]  

(72)

Therefore, from [Eq. (62)], [Eq. (71)] and [Eq. (72)] \( A_{\text{closed}} \) is given by

\[ \left( \frac{N}{De} \right)_{\text{closed}} = \frac{c^2 \beta_S}{(m_0)^{(D+1)/2} (D+1)} + \frac{1}{(m_0)^{(D+1)/2}} \frac{(D+1)}{2} \Gamma \left( -\frac{D - 1}{2} \right) + \frac{c^2 \beta_S}{m_0^{(D+1)/2} (D+1)} \]  

(73)

and for \( D = 4 \), we have for \( \beta_H \to \beta_S \) \((M \to M_{\text{min}}, r_S \to r_{\text{min}})\)

\[ A_{\text{closed}} = \frac{M^2_{\text{min}} L^6_{PL}}{1260 \pi m_0^2} \left( \frac{c^2 \beta_S}{3} \frac{1}{b m_0} \right) \]  

(74)

From [Eq. (68)] and [Eq. (73)], we evaluate now the number \( A/r_{\text{S}}^6 \) appearing in the expressions for \( r_+ \) [Eq. (53)], \( T_+ \) [Eq. (56)], \( M_+ \) [Eq. (60)], and \( \tau_+ \) [Eq. (61)], for the two opposite limiting regimes \( \beta_H \to \beta_S \) and \( \beta_H \gg \beta_S \):

\[ \left( \frac{A_{\text{open}}}{r_{\text{min}}^6} \right)_{\beta_H \to \beta_S} \approx \frac{(\beta_H - \beta_S)}{\beta_S} \cdot \frac{16}{315} \left( \frac{\pi}{b} \right)^3 \left( \frac{M_S}{M_{PL}} \right)^2 \left( \frac{M}{m_0} \right) \]
\[ \approx (\beta_H - \beta_S) \cdot \frac{16}{315} \left( \frac{\pi}{b} \right)^3 \left( \frac{M_S}{M_{PL}} \right)^2 \left( \frac{M}{m_0} \right)^2 \frac{M_S^2 c^2}{b m_0} \ll 1 \]  

(75)

\[ \left( \frac{A_{\text{closed}}}{r_{\text{min}}^6} \right)_{\beta_H \to \beta_S} \approx \frac{16}{735b} \left( \frac{\pi}{b} \right)^3 \left( \frac{M_S}{M_{PL}} \right)^2 \left( \frac{M}{m_0} \right)^2 \ll 1 \]  

(76)

In the opposite (semiclassical) regime \( \beta_H \gg \beta_S \) i.e \( T_H \ll T_S \), we have from [Eq. (66)], [Eq. (67)], [Eq. (69)] and [Eq. (70)]

\[ \left( \frac{N}{De} \right)_{\beta_H \gg \beta_S} \approx \frac{1}{m_0^2} \approx \left( \frac{N}{De} \right)_{\beta_H \gg \beta_S} \]  

(77)
\[ \beta_H m_0 c^2 = 8\pi \left( \frac{M}{M_{PL}} \right) \left( \frac{m_0}{M_{PL}} \right) \gg 1 \tag{78} \]

\( (m_0, M \gg M_{PL}) \). Then, from [Eq. (62)]

\[ \left( \frac{A}{r^6_S} \right)_{\beta_H \gg \beta_S} \approx \frac{1}{80640\pi} \left( \frac{M_{PL}}{M} \right)^4 \left( \frac{M_{PL}}{m_0} \right)^2 \ll 1 \tag{79} \]

That is, in this regime, we consistently recover \( r_+ \approx r_S, T_+ \approx T_H, M_+ \approx M \) and \( \tau_+ \approx \tau_H = \frac{k_B c}{6\sigma G h T_H} \).

6 CONCLUSIONS

We have suitably combined QFT and quantum string theory in the black hole background in the framework of the string analogue model (or thermodynamical approach).

We have computed the quantum string emission by a black hole and the back reaction effect on the black hole in the framework of this model. A clear and precise picture of the black hole evaporation emerges.

The QFT semiclassical regime and the quantum string regime of black holes have been identified and described.

The Hawking temperature \( T_H \) is the intrinsic black hole temperature in the QFT semiclassical regime. The intrinsic string temperature \( T_S \) is the black hole temperature in the quantum string regime. The two regimes are mapped one into another by the \( \mathcal{R} \) - “Dual” transform.

String theory properly describes black hole evaporation: because of the emission, the semiclassical \( BH \) becomes a string state (the “minimal” \( BH \)), and the emitted string gas becomes a condensed microscopic state (the “minimal” \( BH \)) due to a phase transition. The last stage of the radiation in string theory, makes such a transition possible.

The phase transition undergone by the string gas at the critical temperature \( T_S \) represents (in the thermodynamical framework) the back reaction effect of the string emission on the \( BH \).

The \( \mathcal{R} \) - “Dual” relationship between QFT black holes and quantum strings revealed itself very interesting. It appears here this should be promoted to a Dynamical operation: evolution from classical to quantum (and conversely).
Cosmological evolution goes from a quantum string phase to a QFT and classical phase. Black hole evaporation goes from a QFT semiclassical phase to a string phase. The Hawking temperature, which we know as the black hole temperature, becomes the string temperature for the “string black hole”.

Acknowledgements

M.R.M. acknowledges the Spanish Ministry of Education and Culture (D.G.E.S.) for partial financial support (under project: “Plan de movilidad personal docente e investigador”) and the Observatoire de Paris - DEMIRM for the kind hospitality during this work.

Partial financial support from NATO Collaborative Grant CRG 974178 is also acknowledged.

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