Four-wave mixing analysis of Brillouin dynamic grating in a polarization-maintaining fiber: theory and experiment

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Abstract: We investigate the Brillouin dynamic grating generation and detection process in polarization-maintaining fibers for the case of continuous wave operation both theoretically and experimentally. The four interacting light waves couple together through the material density variation due to stimulated Brillouin scattering. The four coupled equations describing this process are derived and solved analytically for two cases: moving fiber Bragg grating approximation and undepleted pump and probe waves approximation. We show that the conventional grating model oversimplifies the Brillouin dynamic grating generation and detection process, since it neglects the coupling between all the four waves, while the four-wave mixing model clearly demonstrates this coupling process and it is verified experimentally by measuring the reflection of the Brillouin dynamic grating. The trends of the theoretical calculation and experimental results agree well with each other confirming that the Brillouin dynamic grating generation and detection process is indeed a four-wave mixing process.

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1. Introduction

Since stimulated Brillouin scattering (SBS) was first observed in 1964 [1], it has remained an intensive research topic. Although SBS limits the channel power in optical fiber transmission systems since it could occur at a low input power in optical fibers, many applications could benefit from this nonlinear phenomenon, such as light storage [2], slow and fast light [3], Brillouin lasers and amplifiers [4], as well as distributed fiber sensors [5]. More recently, dynamic gratings generated in polarization-maintaining fibers (PMFs) via SBS, called Brillouin dynamic gratings (BDGs) [6], have attracted much attention. For example, it could be used to realize high-spatial resolution distributed sensing [7–10], to completely discriminate temperature and strain for Brillouin based sensors [11–14], to achieve tunable optical delays [15], and to measure birefringence of PMFs in a distributed manner [16].

SBS occurs when two pump waves, whose frequency difference equals to Brillouin frequency, counter-propagate in an optical fiber. The two pump waves beat together giving rise to density variation associated with an acoustic wave through electrostriction effect [17]. The acoustic wave introduces a moving periodically modulated refractive index acting like a moving fiber Bragg grating (FBG) which diffracts more Stokes wave reinforcing the SBS process. The property of the moving grating could be measured or detected by monitoring the diffracted wave from a third probe wave which is used to illuminate the grating. In order to discriminate the probe and the pump waves, a direct way is to decouple the polarization state of the pump and probe beams, so that most of the experiments are carried out in PMFs. This moving FBG was given the name of BDG in Ref. [6] where separation of the grating generation and diffracting processes was achieved: two pump waves with frequency difference of a Brillouin frequency polarized along the same axis of the PMF generates the BDG, while a probe wave polarized along the orthogonal axis is diffracted strongly when phase matching condition is satisfied. The frequency difference between the diffracted and the probe waves also equals to the Brillouin frequency. So far, it is widely accepted that the fundamental nature of the BDG is similar to that of an FBG so that conventional FBG theory [18] could be directly applied. Recent experiments [19, 20] also tend to confirm this by rigorously measuring the intrinsic spectra of the
BDG showing that the linewidth of the BDG appears inversely proportional to the BDG length. Note that BDG spectrum observed at the beginning with a broad bandwidth of $\sim 80$ MHz [6] or $\sim 320$ MHz [11] were contributed mainly due to the birefringence non-uniformity of PMFs as well as the relative frequency fluctuations between the multiple sources [14].

However, as discussed above, the BDG generation and detection process actually involves four waves interacting with each other through the material density variations; moreover, the diffracted wave has a Brillouin frequency difference with respect to the probe wave. Obviously, this would introduce a second SBS process between diffracted and probe waves, since these two waves have the same polarization state. More specifically, in most of the experiments [6,11,20], the diffracted wave would experience an SBS amplification since its frequency is downshifted by a Brillouin frequency from that of the probe beam whose propagation direction is the same as that of the BDG. Since the power level of the probe wave is relatively high in practice [11,19,20] in order to have strong diffracted wave for detection, this second SBS process is expected to play an important role. Physically speaking, this is a four-wave mixing (FWM) process that the longitudinal acoustic wave couples the four optical waves with different polarization states together. In this work, we will theoretically study this FWM process which is equivalent to the BDG generation and detection process in the case of continuous wave (CW) operation by deriving the four coupled light wave equations. These equations could be solved analytically for two cases: moving FBG model approximation and undepleted pump 1 (see Fig. 1) and probe waves approximation; showing that the FBG model clearly deviates from some of the reported experimental results [6]. In Section 3, experimental results will be provided to compare with our theoretical calculation confirming that the BDG generation and detection process is indeed an FWM process, and the moving FBG model oversimplifies this process. It is worth mentioning that even though this work is intent on studying the CW case, we believe that in the transit case (probe pulse follows one of the pump pulses) [7, 16, 21], when the power of the probe is high, the second SBS process is also expected to have influences on the reflectivity of the BDG characteristics.

2. Theoretical model and discussion

The diagram of the BDG generation and detection process is shown in Fig. 1. Two pump waves $\tilde{E}_1$ and $\tilde{E}_2$, whose frequencies are locked near the Brillouin frequency, are propagating oppositely inside a PMF with the same polarization state along slow ($\hat{x}$) axis. The frequency of pump 1 wave is assumed to be higher than that of the pump 2 wave, so that the generated BDG through SBS is propagating along $+z$ direction. A probe wave $\tilde{E}_3$ polarized along fast ($\hat{y}$) axis has the same propagation direction as pump 1 wave; when phase-matching condition is satisfied, a diffracted wave $\tilde{E}_4$ would be generated whose frequency is downshifted from the frequency of $\tilde{E}_3$ by an amount of about a Brillouin frequency which is determined by the frequency difference between the two pump waves. This diffracted wave would experience a second SBS amplification by $\tilde{E}_3$ along fast axis, since it has the same polarization state with the probe wave. Therefore, inside the PMF, the material density variation (acoustic wave) is...
contributed by two SBS processes which in turn couple the four optical waves together which could be considered as an FWM process.

In order to obtain a clear view of four waves interacting with each other in a PMF, we need to solve the Maxwell’s equations with the following material density equation [17]:

\[
\frac{\partial^2 \rho}{\partial t^2} - \Gamma_A \nabla^2 \frac{\partial \rho}{\partial t} - \nu_A \nabla^2 \rho = - \frac{1}{2} \varepsilon_0 \gamma_c \nabla^2 E_{tot}^2,
\]

where \( \rho = \rho - \rho_0 \) is a change in the local density from its average value \( \rho_0 \); \( \Gamma_A \) is the damping coefficient; \( \gamma_c = \rho_0 (d\varepsilon / dp)_{\rho=\rho_0} \) is the electrostrictive constant; \( \nu_A \) is the sound velocity inside the fiber; \( \varepsilon_0 \) is the vacuum permittivity. The total electric field, \( \mathbf{E}_{tot} \), inside the PMF could be expressed as \( \mathbf{E}_{tot} = \mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z + \mathbf{E}_d \), where

\[
\mathbf{E}_x = \frac{1}{2} \tilde{x} F(x,y) \tilde{A}_1(z,t) e^{i(k_1 z - \omega_1 t)} + F(x,y) \tilde{A}_2(z,t) e^{i(-k_2 z - \omega_2 t)}] + \text{c.c.},
\]

\[
\mathbf{E}_y = \frac{1}{2} \tilde{y} [F(x,y) \tilde{A}_3(z,t) e^{i(k_3 z - \omega_3 t)} + F(x,y) \tilde{A}_4(z,t) e^{i(-k_4 z - \omega_4 t)}] + \text{c.c.},
\]

where \( \tilde{A}_j(z,t) \ (j = 1 - 4) \) and \( F(x,y) \) are the slowly varying fields and dimensionless fundamental mode profile for the four interacting waves, respectively. Note that the mode profiles are assumed to be the same for all the four interacting waves for simplicity. \( \omega_j \) and \( k_j \) are, respectively, frequencies and propagation constants of optical waves; c.c. represents complex conjugate.

As discussed above, the material density variation is contributed by two SBS processes, which could be expressed as \( \tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2 \). When the frequency difference of the two pump lasers is locked at the Brillouin frequency, \( \Omega_B \), under steady-state condition with slowly varying envelope approximation, the coupled equations for all the four optical waves could be expressed as (a more detailed derivation could be found in Appendix A),

\[
\frac{\partial A_1}{\partial z} = -\kappa (|A_1|^2 + A_2 A_3^* e^{i\Delta k z}),
\]

\[
\frac{\partial A_2}{\partial z} = -\kappa (|A_1|^2 A_2 + A_1 A_3^* A_4 e^{-i\Delta k z}),
\]

\[
\frac{\partial A_3}{\partial z} = -\kappa (|A_3|^2 A_4 + A_1 A_2^* A_4 e^{-i\Delta k z}),
\]

\[
\frac{\partial A_4}{\partial z} = -\kappa (|A_3|^2 A_4 + A_1 A_2^* A_3 e^{i\Delta k z}),
\]

where the coupling constant is

\[
\kappa = \frac{8\pi^2 \gamma_c}{c \rho_0 \lambda \Omega_B A_{eff}^{opt}},
\]

with

\[
A_{eff}^{opt} = \left[ \frac{\langle F^2(x,y) \rangle}{\langle F^2(x,y) F_A(x,y) \rangle} \right]^2 \langle F_A^2 (x,y) \rangle,
\]

being the acousto-optic effective area [22, 23]; \( F_A(x,y) \) is the dimensionless mode profile of the acoustic wave (only the fundamental acoustic mode is considered); angular brackets denote averaging over transverse cross section of the fiber; \( c \) is the velocity of light in vacuum; \( \lambda \) is the operation wavelength; \( \Gamma_B \) is the Brillouin linewidth. \( \Delta k = k_3 + k_4 - k_1 - k_2 \) is the phase mismatch which is directly related to the frequency difference, \( \Delta \nu \), of pump 1 and probe waves [6]. Note that Eq. (3) have been normalized as that \( |A_j|^2 \ (j = 1 - 4) \) represents optical power.
Generally speaking, Eq. (3) needs to be solved numerically. However, with the approximation that pump 1 and probe waves are undepleted, these equations could be solved analytically which would give a clear physical insight for the FWM process. The analysis will be given in Sec. 2.2. Before that, we would like to discuss briefly the moving FBG model which is broadly accepted in all the reports so far, and show that such model oversimplifies the BDG generation and detection process.

2.1. Moving FBG model

So far, most reports assumed the fundamental nature of the BDG to be similar to that of a moving FBG with the experimental confirmation [19,20]. Under this approximation, the frequencies of the two pump waves are locked at Brillouin frequency, the depletion of pump 1 wave and the amplification of pump 2 wave are both ignored. SBS process generates an acoustic wave (density variation) which is considered as a moving FBG. The probe wave whose frequency satisfies the phase-matching condition is diffracted by this grating, and there is no SBS process between probe and diffracted waves. Physically speaking, the diffraction of the probe wave by this moving FBG is actually an acousto-optic process under these assumptions. Mathematically, it could be considered as that Eq. (3) reduce to

\[
\frac{\partial A_1}{\partial z} = -\kappa A_1 A_2^* A_4 e^{-i\Delta k z}, \tag{6a}
\]

\[
\frac{\partial A_4}{\partial z} = -\kappa A_1^* A_2 A_3 e^{i\Delta k z}, \tag{6b}
\]

with \(A_1\) and \(A_2\) being constants. Equations (6a) and (6b) could then be solved analytically for the diffracted wave along with the boundary conditions, \(A_3|_{z=0} = A_3(0)\) which is a known quantity and \(A_4|_{z=L} = A_4(L) = 0\), where \(L\) is the length of the PMF [18],

\[
A_4(z) = \frac{2\kappa \exp(i\Delta k z/2) \sinh[g(L-z)/2]|A_1 A_2 A_3(0)|}{g \cosh(g L/2) - i\Delta k \sinh(g L/2)}, \tag{7}
\]

where \(g = \sqrt{4\kappa^2 |A_1|^2 |A_2|^2 - (\Delta k)^2}\). Under phase-matching condition, \(\Delta k = 0\), the reflectivity of the BDG could be expressed as

\[
R_{\text{FBG}} = \left| \frac{A_4(0)}{A_3(0)} \right|^2 = \tanh^2(\kappa |A_1| |A_2| L). \tag{8}
\]

Note that \(|A_1|\) and \(|A_2|\) are the square roots of the pump 1 and pump 2 powers, respectively. Figure 2 shows the properties of the functions of \(\tanh^2(x)\) and \(\tanh^2(\sqrt{x})\). It is clearly seen that the trends for \(\tanh^2(x)\) and \(\tanh^2(\sqrt{x})\) are very different. More specifically, when pump powers \(|A_1|^2\) and \(|A_2|^2\) are fixed, the reflectivity of the BDG varies with the length of the PMF following the trend showing as the solid curve. This has been confirmed by a recent experiment observing the intrinsic BDG spectrum [20]. On the other hand, when the length \(L\) and pump 2 power \(|A_2|^2\) are kept at constant, the BDG reflectivity versus pump 1 power \(|A_1|^2\) should follow the trend showing as the dashed curve. However, in the early experiment [6], an exponential growth of the BDG reflectivity with respect to pump 1 power has been observed, which clearly deviates from the result expressed by Eq. (8). Therefore, this approximated model condition oversimplifies the problem and needs to be modified by considering all the four coupling waves.

2.2. FWM process with undepleted pump 1 and probe waves

In most of the BDG experiments, the pump 1 power is much higher than pump 2 power [6, 16, 20]. Moreover, in order to have strong diffracted signal from the BDG, probe power...
is also chosen to be on the same order of the pump 1 power. Therefore, a more reasonable approximation compared to moving FBG model is that pump 1 and probe power are undepleted, since the length of PMF used in the experiments is short, usually less than tens of meters. In addition, since the probe power is also high in most of experiments, the diffracted signal would also experience an SBS amplification. Under these assumptions, Eq. (3) could be reduced to

\[
\frac{\partial A_2}{\partial z} = -\kappa (|A_1|^2 A_2 + A_1 A_3 A_4 e^{-i\Delta k z}), \tag{9a}
\]

\[
\frac{\partial A_4}{\partial z} = -\kappa (|A_3|^2 A_4 + A_1^* A_2 A_3 e^{i\Delta k}), \tag{9b}
\]

with \(A_1\) and \(A_3\) being constants. Equations (9a) and (9b) could be solved analytically along with the boundary conditions, \(A_2|_{z=L} = A_2(L)\) which is a known quantity and \(A_4|_{z=L} = 0\), as

\[
A_4(z) = 2\kappa A_1^* A_2(L) A_3 \frac{\sinh[\sqrt{\Phi^2 + 4\Psi(L-z)/2}]}{\sqrt{\Phi^2 + 4\Psi}} e^{\Phi(L-z)/2 + i\Delta k L}, \tag{10}
\]

where \(\Phi = \kappa (|A_1|^2 + |A_3|^2) - i\Delta k\) and \(\Psi = i\kappa |A_3|^2 \Delta k\). Under phase-matching condition, \(\Delta k = 0\), the reflectivity of the BDG could be expressed as (a more detailed derivation could be found in Appendix B),

\[
R_{FWM} = \left| \frac{A_4(0)}{A_3} \right|^2 = \left| A_1^2 A_2(L) \right|^2 \frac{e^{2\kappa (|A_1|^2 + |A_3|^2) L - 1^2}}{|A_1|^2 + |A_3|^2} \tag{11}
\]

We plot Eq. (11) with fixed length of the PMF, \(L\), by varying pump or probe powers. The results are shown in Fig. 3. The parameters used to plot these curves are as follows: \(\gamma_e = 0.902\), \(c = 3 \times 10^8\) m/s; \(\rho_0 = 2210\) kg/m\(^3\), \(A_{\text{eff}} = 80\) \(\mu\)m\(^2\), \(\Omega_B/2\pi = 10875\) MHz, \(\Gamma_B/2\pi = 30\) MHz, \(\lambda = 1550\) nm, \(L = 13\) m. From Fig. 3(a), we could see that when the probe and pump 2 powers are fixed, when the pump 1 power is increased, the reflectivity increases in a way that is very different from the moving FBG model. The trend could be approximately considered as a cubic curve by expanding the exponential term in Eq. (11) and keeping to the second order within the power range used for calculation, since the gain is not too high. When the pump 1 power is further increased, from Eq. (11), one could expect that the exponential growth would dominate until strong pump depletion occurs where numerical solution is needed for completely solving Eq. (3). However, this similar trend has already been observed in Ref. [6], and we will also show this in our experiment in the next section.
Figure 3(b) shows that when pump 1 and probe powers are fixed, the reflectivity of the BDG should show a linear increasing trend according to Eq. (11). Moreover, when the probe power is higher, the linear slope becomes steeper as well. The BDG reflectivity as a function of probe power is plotted in Fig. 3(c). For higher probe power, the diffracted wave undergoes a stronger SBS amplification which should result in a higher reflectivity. In the next section, we will demonstrate the experimental results (the power levels of the pump and probe waves are chosen to be consistent with the undepleted pump 1 and probe waves approximation), and show that all these trends are observed confirming that the BDG generation and detection process is indeed consistent with an FWM process.

3. Experiment validation

Figure 4 shows the experiment setup. Two narrow linewidth (∼3 kHz) fiber lasers operating at 1550 nm are used to provide pumps 1 and 2, and their frequency difference is locked by a phase-locking loop in a frequency counter via a photodetector (PD1). The frequency of Laser 1 is higher than Laser 2. A tunable laser (Agilent 81940A) whose linewidth is ∼100 kHz with a continuous sweep mode is used as the probe wave. Three polarization controllers (PCs) are used to adjust the polarization states of the three waves. A variable attenuator is used to adjust the power level of the pump 2 wave, while two erbium-doped fiber amplifiers (EDFAs) are...
employed to adjust power levels of pump 1 and probe waves. Pump 1 and pump 2 waves are launched into the PMF along slow axis in opposite direction to each other through polarization beam splitter/combiner (PBS/PBC), and the probe wave is injected along the orthogonal axis. Power meter 2 (PM2) is used to monitor pump 2 after passing through the PMF in order to determine whether a strong pump depletion happens. In the detection end, a 125 MHz photodetector (PD2) and a high performance oscilloscope (Agilent Infinium DSO81204B) are used to detect the back diffracted wave via a narrow bandwidth (0.1 nm) tunable filter (TF), whose function is to eliminate the leakage of pump 1 and 2 waves along the slow axis. After the frequency of the two pump waves are locked at the Brillouin frequency of the PMF to generate the BDG, the probe wave is fast swept and the diffracted spectrum of the BDG is recorded by the oscilloscope (A trigger is given to the oscilloscope when the sweep starts for synchronization).

Two kinds of standard PMFs, 13-m Panda and 10-m Bow-Tie fibers whose Brillouin frequencies are 10875 MHz and 10814 MHz at room temperature respectively, are tested in the experiment. Both fibers are under loose condition. When the frequency difference of the tun-
able laser and pump 1 wave, $\Delta \nu$, is near 50 GHz for Panda fiber and 43 GHz for Bow-Tie fiber, strong diffracted signal could be observed. Typical spectra of the corresponding BDGs generated in these two kinds of fibers are shown in Figs. 5(a) and 5(b). We intentionally set the position of highest peak as 0 MHz for comparison. The sweep speed of the tunable laser is 40 nm/s corresponding to $\sim$ 5 MHz/µs. The noise is reduced in the way that high resolution sampling mode of the oscilloscope is chosen; therefore, each BDG spectrum could be obtained within 200 µs for a 1 GHz sweep range. The frequency resolution of the measured BDG spectrum is $\sim$ 0.1 MHz (1 GHz/8003 points). Both spectra show multi-peaks due to the non-uniformity of the birefringence inside the PMF. This similar spectrum has been reported recently in a 5-m long PMF (Ref. [14]). In the other previous reports, the BDG spectrum shows only one broad peak, which is attributed to the relative frequency fluctuation between different laser sources. However, in our experiment, the sweep speed is fast enough to avoid such fluctuation. Actually, the sweep speed could be changed from 50 nm/s down to 10 nm/s, the spectrum remains the same for each sweep. However, when the sweep speed is further reduced, the spectrum starts to be different for each sweep, because the frequency fluctuation of the pump lasers cannot be ignored when the sweeping speed of the tunable laser is low. It is worth mentioning that the birefringence non-uniformity of the Panda fiber used in our experiment is much smaller than that of the Bow-Tie fiber, which could be confirmed by Fig. 5, since the overall width of spectrum in Fig. 5(a) is much narrower than that in Fig. 5(b). In the following experiment, we use the Panda fiber to measure the reflectivity of the BDG. As an example, the BDG spectrum changing with respect to the variation of the pump 1 power is shown in Fig. 6, where the pump 2 and probe powers are kept at 10 and 100 mW, respectively. It is clearly seen that the shape of the spectra remain unchanged except that the reflectivity is increasing when the pump 1 power increases. Note that although the birefringence non-uniformity could be suppressed by applying longitudinal strain to the PMF with the coating stripped completely [20], the spectrum obtained in our experiment is good enough to confirm the FWM process analyzed theoretically in last section by finding the maximum reflectivity of the BDG which is obtained by locating the highest peak in the spectrum.

Figure 7(a) shows the BDG reflectivity with respect to pump 1 power for different pump 2 and probe powers. All the trends clearly differ from a linear increasing (the curves are the cubic fits of the data points obtained experimentally) which is consistent with the theoretical
Fig. 7. Experimentally obtained BDG reflectivity with respect to (a) pump 1 power $|A_1|^2$ for different pump 2 and probe powers (b) pump 2 power $|A_2|^2$ for different probe powers and fixed pump 1 power (c) probe power $|A_3|^2$ for different pump 1 powers and fixed pump 2 power.

calculation. We believe that further increasing the pump 1 power would show a more obvious exponential trend as that in Ref. [6], but the power level is limited by the maximum handling power of our optical components in CW operation. The reflectivity obtained experimentally is much smaller than calculated results because the birefringence of the PMF is nonuniform which results in phase mismatch. In addition, we did not account for other power losses of the signal in the configuration, especially for the TF which causes at least a 3-dB loss. However, this trend already provides a clear evidence that the BDG generation and detection is actually an FWM process, especially for the case that the probe power is comparable to the pump 1 power.

Figure 7(b) shows the BDG reflectivity with respect to pump 2 power for different probe powers with pump 1 power fixed. Both curves which are linear fits of the data points obtained experimentally show a well linear increase of the BDG reflectivity versus pump 2 power, which is also observed in Ref. [6]. For higher probe power, the reflectivity is higher and the slope of the linear increase is steeper as well which is consistent with the theoretical calculation. The reflectivity of the BDG with respect to the probe power is shown in Fig. 7(c). When the probe power is increasing, the BDG reflectivity is higher according to the theoretical analysis which is due to the stronger FWM process. For the moving FBG model, the reflectivity would be constant with respect to the variation of probe power. However, since the BDG generation and detection process in a PMF is actually an FWM process, the reflection of BDG is dependent on
the probe power which was ignored in all the previous reports. Even though the present work is focusing on CW case, we believe that for transit case, e.g., time-domain measurement of BDG spectrum (probe pulse follows one of the pump pulses) [7, 16, 21], the reflectivity is also dependent on probe power, since a strong probe pulse would result in an SBS amplification of the diffracted wave.

4. Conclusion

We have theoretically investigated the BDG generation and detection process in a PMF for the case of CW operation. The moving FBG model oversimplifies this process since the results derived from this model deviates from some of the experimental reports. By deriving the four coupled light wave equations, we could obtain analytical solutions under the undepleted pump and probe waves approximation. The trends of both the theoretical calculation and the experiment agree with each other very well showing that the reflectivity of the BDG is dependent on the probe power. This is a clear evidence indicating that the BDG generation and detection process is indeed an FWM process.

Appendix A: derivation of the four coupled wave equations

The theoretical model diagram is shown in Fig. 1. We start the analysis with the material density equation, Eq. (1). The total electric field, $\tilde{E}_{\text{tot}}$, on the right-hand side of Eq. (1) is expressed as Eqs. (2) and (3). Since acoustic wave is predominantly longitudinal, we neglect the transversal components of the acoustic wave for SBS process. Then, the right-hand of Eq. (1) takes the form,

$$\nabla^2 \tilde{E}_{\text{tot}} = -\frac{1}{2} F^2 \left[ \tilde{A}_1 \tilde{A}_2^* \tilde{q}_1^2 e^{i(q_1 z - \Omega t)} + \tilde{A}_3 \tilde{A}_4^* \tilde{q}_2^2 e^{i(q_2 z - \Omega t)} \right] + \text{c.c.}, \quad (A.1)
$$

where $\Omega = \omega_1 - \omega_2 = \omega_3 - \omega_4$, $q_1 = k_1 + k_2$ and $q_2 = k_3 + k_4$. Then, we are looking for the solution of the Eq. (1) in the form,

$$\tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2 = \frac{1}{2} F_A (x,y) [Q_1(z,t) e^{i(q_1 z - \Omega t)} + Q_2(z,t) e^{i(q_2 z - \Omega t)}] + \text{c.c.} \quad (A.2)
$$

where $Q_1(z,t)$ and $Q_2(z,t)$ are the slowly varying amplitudes of the acoustic waves corresponding to the two SBS processes. $F_A (x,y)$ is the dimensionless acoustic wave mode profile which is the solution of the unperturbed Eq. (1) and thus satisfies the eigenvalue equation [23],

$$\nabla_{x}^2 F_A (x,y) + \left( \frac{\Omega_{\text{BM}}^2}{\nu_A^2} \right) F_A (x,y) = 0, \quad (m = 1, 2) \quad (A.3)
$$

where $\nabla_{x}^2 = \nabla_{x}^2 + \nabla_{y}^2$ is the transverse Laplacian operator; $\Omega_{\text{BM}}$ is the eigenfrequency of the solution of the modal acoustic equation that for a given wave vector $q_m$ which satisfies Eq. (A.3). Assuming steady-state condition with slowly varying envelope approximation, substituting Eqs. (A.1) and (A.2) into Eq. (1) and grouping the results by the exponential factors $e^{i(q_1 z - \Omega t)}$ and $e^{i(q_2 z - \Omega t)}$, one could obtain two equations for $Q_1$ and $Q_2$, respectively,

$$-\frac{1}{2} \{i \Omega \Gamma_{B1} F_A Q_1 + v_A^2 Q_1 [\nabla_{x}^2 F_A + \left( \frac{\Omega^2}{\nu_A^2} - q_1^2 \right) F_A] \} = \frac{1}{4} \varepsilon_0 \gamma F^2 q_1^2 \tilde{A}_1 \tilde{A}_2^*, \quad (A.4a)
$$

$$-\frac{1}{2} \{i \Omega \Gamma_{B2} F_A Q_2 + v_A^2 Q_2 [\nabla_{x}^2 F_A + \left( \frac{\Omega^2}{\nu_A^2} - q_2^2 \right) F_A] \} = \frac{1}{4} \varepsilon_0 \gamma F^2 q_2^2 \tilde{A}_3 \tilde{A}_4^*, \quad (A.4b)
$$
where $\Gamma_{Bm} = q_m^2 \Gamma_A$. Two additional approximations have been used in order to obtain Eq. (A.4): the term containing $\partial Q_m / \partial z$ has been dropped since the hypersonic phonons are strongly damped and thus propagate only over very short distances before being absorbed [17]; since the ratio $\Omega \Gamma_A / v_A^2 \approx 10^{-3}$ is very small, $\Omega \Gamma_A + v_A^2$ could be approximated by its real part [23], $v_A^2$, in the term which contains the derivative $\nabla_z^2 F_A$. Inserting Eq. (A.3) into Eq. (A.4), multiplying both sides of each resulting equation with $F_A(x,y)$ and integrating over the transverse plane, one could obtain that,

$$Q_1(z,t) = \frac{\epsilon_0 \gamma q_0^2 \tilde{A}_1(z,t) \tilde{A}_2^*(z,t)}{2(\Omega_{B1}^2 - \Omega^2 - i \Omega \Gamma_{B1})} \langle F_z^2(x,y)F_A(x,y) \rangle,$$

$$Q_2(z,t) = \frac{\epsilon_0 \gamma q_0^2 \tilde{A}_3(z,t) \tilde{A}_4^*(z,t)}{2(\Omega_{B2}^2 - \Omega^2 - i \Omega \Gamma_{B2})} \langle F_z^2(x,y)F_A(x,y) \rangle,$$

where angular brackets denote averaging over transverse cross section of the fiber. Note that

$$\langle F_1(z,t) \rangle = \frac{\epsilon_0 \gamma q_0^2 \tilde{A}_1(z,t) \tilde{A}_2^*(z,t)}{2(\Omega_{B1}^2 - \Omega^2 - i \Omega \Gamma_{B1})} \langle F_z^2(x,y)F_A(x,y) \rangle,$$

$$\langle F_2(z,t) \rangle = \frac{\epsilon_0 \gamma q_0^2 \tilde{A}_3(z,t) \tilde{A}_4^*(z,t)}{2(\Omega_{B2}^2 - \Omega^2 - i \Omega \Gamma_{B2})} \langle F_z^2(x,y)F_A(x,y) \rangle,$$

for frequencies and for wave vectors. Strictly speaking, it is the solution of the set of equations

$$q_1 = \frac{\omega n_{eff,x}(\omega_1)}{c} + \frac{(\omega_1 - \Omega) n_{eff,x}(\omega_1)}{c},$$

$$q_2 = \frac{\omega n_{eff,y}(\omega_1)}{c} + \frac{(\omega_1 - \Omega) n_{eff,y}(\omega_1)}{c},$$

and

$$\nabla_z^2 F_A(x,y) + \left( \frac{\Omega^2}{v_A^2} - q_m^2 \right) F_A(x,y) = 0, (m = 1, 2)$$

where $n_{eff,x}$ and $n_{eff,y}$ are the effective refractive indices of the optical mode along $\hat{x}$ and $\hat{y}$ axis, respectively. Moreover, approximations have been used as $n_{eff,x}(\omega_1) \approx n_{eff,x}(\omega_1)$ and $n_{eff,y}(\omega_1) \approx n_{eff,y}(\omega_3)$, since $\Omega \ll \omega_1, \omega_3$. Thus, for a fixed frequency $\omega_1$ or $\omega_3$, $\Omega_{Bm}$ could be calculated as the intersection of the two curves of $q_m(\Omega)$ obtained from Eqs. (A.6) and (A.7). However, since $q_m(\Omega)$ is a very weak function because of $\Omega \ll \omega_1, \omega_3$,

$$q_1 \approx \frac{n_{eff,x}(\omega_1)}{c} (2\omega_1 - \Omega) \approx \frac{2\omega_1 n_{eff,x}(\omega_1)}{c} \approx \frac{2\omega_1 n_x}{c},$$

$$q_2 \approx \frac{n_{eff,y}(\omega_3)}{c} (2\omega_3 - \Omega) \approx \frac{2\omega_3 n_{eff,y}(\omega_3)}{c} \approx \frac{2\omega_3 n_y}{c},$$

with $n_x$ and $n_y$ being the refractive index of the PMF along $\hat{x}$ and $\hat{y}$ axis, respectively. The obtained $q_m$ value could be substituted into Eq. (A.7) to find $\Omega_{Bm}$ directly. Moreover, as discussed before, for BDG generation and detection process, phase matching is required as that $\Delta k = 0$ which is equivalent to $q_1 = q_2 = q$; therefore, we could take $\Omega_{B1} = \Omega_{B2} = \Omega_B$, $\Gamma_{B1} = \Gamma_{B2} = \Gamma_B$ for further calculations. From Eqs. (A.2) and (A.5), the full material density variation could be expressed as,

$$\tilde{\rho}(x,y,z,t) = \frac{1}{4} F_A \langle F_{z}^2 F_A \rangle \epsilon_0 \gamma q_0^2 (\tilde{A}_1 \tilde{A}_2^* + \tilde{A}_3 \tilde{A}_4^*) e^{i(qz - \Omega t)} + c.c..$$

Next, we start to derive equations for the evolution of the four optical waves. Maxwell’s equations can be deduced to for the electric field $\tilde{E}_j (j = 1 - 4)$ inside the PMF as follows:

$$\nabla^2 \tilde{E}_j - \frac{c^2}{\epsilon_0} \frac{\partial^2 \tilde{E}_j}{\partial t^2} = \frac{1}{\epsilon_0 \gamma q_0^2} \frac{\partial^2 \tilde{P}_j}{\partial t^2},$$
where \( n = n_x \) for \( j = 1, 2 \) and \( n = n_y \) for \( j = 3, 4 \). The total nonlinear polarization induced by the acousto-optic interaction which gives rise to the source term in this equation is given by

\[
P_{\text{tot}} = e_0 \frac{\gamma}{\rho_0} \phi \mathbf{E}_{\text{tot}} = e_0 \mathbf{E}_{\text{NL}} \mathbf{E}_{\text{tot}},
\]

where

\[
e_{\text{NL}} = F_A U \left\{ \mathbf{A}_1 \mathbf{A}_2^* e^{i(k_1 + k_2)z - \Omega t} + \mathbf{A}_3 \mathbf{A}_4^* e^{i(k_3 + k_4)z - \Omega t} \right\} + \text{c.c.}
\]

with

\[
U = \frac{e_0 \gamma_0^2 q^2}{4\rho_0 (\Omega_B^2 - \Omega^2 - i\Omega B)} \langle F^2 F_A \rangle.
\]

By determining different parts of \( \mathbf{P}_{\text{tot}} \) that can act as phase-matched source terms for the four waves, one could obtain

\[
\begin{align*}
\tilde{P}_1 &= \frac{1}{2} e_0 U \mathbf{F}_\mathbf{A} (|\mathbf{A}_1|^2 |\mathbf{A}_2|^2 + 2 |\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4^* e^{i\Delta k z}) e^{i(k_1 z - \omega_1 t)} + \text{c.c.}, \quad (A.14a) \\
\tilde{P}_2 &= \frac{1}{2} e_0 U^* \mathbf{F}_\mathbf{A} (|\mathbf{A}_1|^2 |\mathbf{A}_2|^2 + 2 |\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4^* e^{-i\Delta k z}) e^{i(k_2 z - \omega_2 t)} + \text{c.c.}, \quad (A.14b) \\
\tilde{P}_3 &= \frac{1}{2} e_0 U \mathbf{F}_\mathbf{A} (|\mathbf{A}_3|^2 |\mathbf{A}_4|^2 + 2 |\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4^* e^{-i\Delta k z}) e^{i(k_3 z - \omega_3 t)} + \text{c.c.}, \quad (A.14c) \\
\tilde{P}_4 &= \frac{1}{2} e_0 U^* \mathbf{F}_\mathbf{A} (|\mathbf{A}_3|^2 |\mathbf{A}_4|^2 + 2 |\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4^* e^{i\Delta k z}) e^{i(k_4 z - \omega_4 t)} + \text{c.c.}, \quad (A.14d)
\end{align*}
\]

where \( \Delta k = k_3 + k_4 - k_1 - k_2 \) is the phase mismatch. Neglecting the dependence of the optical modal profile on nonlinear effects, we can substitute the optical modal equation

\[
\nabla^2 f(x,y) + \left( \frac{\alpha^2 n^2}{c^2} - k_j^2 \right) f(x,y) = 0,
\]

along with Eqs. (A.14) into Eq. (A.10), using slowly varying envelope approximation, and multiplying \( f(x,y) \) on both sides of each resultant equation, integrating over the transverse cross section, to obtain four equations for four interacting optical waves under steady state,

\[
\begin{align*}
\frac{\partial A_1}{\partial z} &= i \eta_1 (|A_1|^2 A_2^2 + A_2 A_3 A_4^* e^{i\Delta k z}), \quad (A.16a) \\
\frac{\partial A_2}{\partial z} &= -i \eta_1^* (|A_1|^2 A_2 + A_1 A_3 A_4^* e^{-i\Delta k z}), \quad (A.16b) \\
\frac{\partial A_3}{\partial z} &= i \eta_2 (|A_3|^2 A_4^2 + A_1 A_2 A_4^* e^{-i\Delta k z}), \quad (A.16c) \\
\frac{\partial A_4}{\partial z} &= -i \eta_2^* (|A_3|^2 A_4 + A_1 A_2 A_3^* e^{i\Delta k z}), \quad (A.16d)
\end{align*}
\]

where

\[
\begin{align*}
\eta_1 &= \frac{\gamma_0^2 \omega_1^3}{\rho_0 e^4 (\Omega_B^2 - \Omega^2 - i\Omega B)|A_{\text{eff}}^{\text{ao}}|}, \quad (A.17a) \\
\eta_2 &= \frac{\gamma_0^2 \omega_2^3}{\rho_0 e^4 (\Omega_B^2 - \Omega^2 - i\Omega B)|A_{\text{eff}}^{\text{ao}}|}, \quad (A.17b)
\end{align*}
\]

with \( A_{\text{eff}}^{\text{ao}} \) being given by Eq. (5). Note that Eq. (A.16) has been normalized as that \( |A_j|^2 \) represents optical power by using \( |A_j|^2 = e_0 n |A_j|^2 \langle F^2 \rangle / 2 \). In addition, \( \omega_1 \approx \omega_2 \) and \( \omega_3 \approx \omega_4 \) are used to obtain Eqs. (A.16) and (A.17), and \( n_x \approx n_y \) is assumed for normalization. When the frequency difference of the two pump waves has been locked at Brillouin frequency, e.g., \( \Omega = \Omega_B \), Eq. (A.16) reduces to Eq. (3) where a further approximation of \( \lambda_1 \approx \lambda_3 \) (wavelengths of pump 1 and probe waves) is considered.
Appendix B: analytical solution of the four coupled wave equations under undepleted pump 1 and probe waves approximation

We calculate \( z \) derivative of Eq. (9b) with the help of (9a), considering that \( A_1 \) and \( A_3 \) are constants, and then the resultant equation for \( A_4 \) could be expressed as follows:

\[
\frac{\partial^2 A_4}{\partial z^2} + \Phi \frac{\partial A_4}{\partial z} - \Psi A_4 = 0, \tag{B.1}
\]

where \( \Phi = \kappa(|A_1|^2 + |A_3|^2) - i\Delta k \) and \( \Psi = i\kappa|A_3|^2\Delta k \). The general solution for it is

\[
A_4(z) = Ce^{p_+z} + De^{-p_-z}, \tag{B.2}
\]

with \( p_\pm = -\frac{1}{2}\Phi \pm \frac{1}{2}\sqrt{\Phi^2 + 4\Psi} \) and two constants \( C \) and \( D \) which could be determined by boundary conditions:

\[
A_4|_{z=L} = 0, \tag{B.3a}
\]

\[
\frac{\partial A_4}{\partial z}|_{z=L} = -\kappa A_1^* A_2(L)A_3 e^{i\Delta k L}. \tag{B.3b}
\]

Equation (B.3b) is obtained from Eq. (9b) after insertion of Eq. (B.3a). Substituting Eq. (B.2) into Eq. (B.3), the two constants could be determined as

\[
C = \frac{\kappa A_1^* A_2(L)A_3 e^{i\Delta k L} e^{-p_+L}}{p_+ - p_-}, \tag{B.4a}
\]

\[
D = -\frac{\kappa A_1^* A_2(L)A_3 e^{i\Delta k L} e^{-p_-L}}{p_- - p_+}. \tag{B.4b}
\]

Therefore, substituting Eq. (B.4) into Eq. (B.2), the diffracted wave \( A_4(z) \) could be obtained as expressed in Eq. (10). Under phase-matching condition, \( \Delta k = 0 \), so that \( \Phi = \kappa(|A_1|^2 + |A_3|^2) \) and \( \Psi = 0 \); the reflectivity could be expressed in Eq. (11). Note that the expression for pump 2 wave \( A_2 \) could also be obtained in the similar way.

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