ζ-function regularization and the interacting Bose gas at low temperature

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I. INTRODUCTION

The non-interacting Bose gas, and its behaviour as a function of temperature, is a key part of any study of statistical mechanics. (See for example Ref. [1, 2, 3, 4, 5, 6, 7, 8, 9].) However the neglect of interactions among the particles, which may be justified as a first approximation, cannot be valid at any fundamental level. (As emphasised in Ref. [8], the condensation of a free Bose gas has unphysical properties, such as infinite compressibility.) Accordingly the study of interparticle interactions is of obvious physical interest.

The interacting Bose gas received a great deal of attention about forty years ago. Textbook treatments may now be found. (See for example Ref. [10, 11, 12].) The methods used to study the interacting Bose gas go beyond what would normally be described as elementary. The pioneering studies include [9, 11, 12] who obtained the ground state energy of the interacting Bose gas at zero temperature using a variety of methods. The generalization to \( T \neq 0 \) was performed by Lee and Yang [13]. Much more recently it was shown how modern functional integral methods of quantum field theory could be used to study the interacting Bose gas to great effect [14]. It was also shown [15] how the results for the non-relativistic interacting Bose gas could be obtained as a limiting case of the relativistic gas.

The main purpose of the present paper is to show how it is possible to study the interacting Bose gas at low temperatures and densities in a relatively straightforward way. The method makes use of the background field method [16] to obtain the excitation spectrum of the gas in a manner akin to that of Bernstein and Dodelson [17]. There is no need to choose a specific form for the interaction. The zero temperature result for the ground state energy can then be understood as a regularized zero-point energy and studied in a similar way to the Casimir effect [18]. By adopting ζ-function regularization [19, 20, 21] and specialising the form of the interaction, is easy to derive the classic result [22, 23] for the ground state energy. Such a technique renders the discussion of renormalization of the theory trivial. By an examination of the thermodynamic potential it is possible to obtain finite temperature corrections to the zero temperature results in a straightforward way.

To further exemplify the methods used we will also study a mixture of two interacting Bose gases. Although there has been some work on the subject [24, 25, 26, 27] so far as we are aware, the analogue of the ground state energy calculations of Ref. [22, 23, 24, 25] have not been given before. We do this calculation for \( T = 0 \) as well as \( T \neq 0 \) below. It is worth pointing out that mixtures of two Bose gases have been achieved experimentally for trapped atomic gases [28]. (A small sample of some of the theoretical work on trapped mixtures includes [29].)

II. SINGLE SPECIES GAS

A. General approach

In order to see how the method of ζ-function regularization works in a simple setting we will first study the interacting single species gas. In the case where the interaction is quartic in the fields it will be shown how the classic low temperature results of Refs. [1, 2, 3, 4, 5, 6, 7, 8] are found in a relatively easy manner. After describing the basic setup in this section, we will obtain the ground state energy and the equation of state in Secs. II B and II C.

We begin with the action functional for a single complex scalar field \( \Psi \)

\[
S[\Psi, \Psi^\dagger] = \int dt \int_{\Sigma} d^3x \left\{ \frac{i}{2} (\Psi^\dagger \dot{\Psi} - \dot{\Psi}^\dagger \Psi) - \frac{1}{2m} \nabla \Psi |^2 + \mu |\Psi|^2 - U(|\Psi|^2) \right\}.
\]

Here \( \Sigma \) denotes the space of interest. Our concern here will be with flat 3-dimensional space, so that \( \Sigma \) can be thought of as a finite dimensional box with the infinite volume limit taken. The chemical potential \( \mu \) enforces
the conserved particle number. \( U(|\Psi|^2) \) is the interaction potential which we allow to be an arbitrary function. As the notation suggests, we assume that \( U \) depends on \( \Psi \) and \( \Psi^\dagger \) only through \( |\Psi|^2 \), and we take \( U \) to be real.

The thermodynamic potential \( \Omega \) at lowest order in quantum field theory is comprised of three pieces:

\[
\Omega = \Omega_{\text{class}} + \Omega_{T=0}^{(1)} + \Omega_{T\neq 0}^{(1)}.
\]

Here

\[
\Omega_{\text{class}} = V \left( U(|\bar{\Psi}|^2) - \mu|\bar{\Psi}|^2 \right)
\]

is the contribution from the classical part of the theory with \( \bar{\Psi} \) representing the non-zero condensate. (\( \Psi \) can be thought of as the background field in the background field method.) We take \( \bar{\Psi} \) to be constant here. \( V \) is the volume of \( \Sigma \). The terms in \( \Omega_{T=0}^{(1)} \) and \( \Omega_{T\neq 0}^{(1)} \) are the first order quantum corrections to the classical theory, with \( \Omega_{T\neq 0}^{(1)} \) representing the finite temperature corrections to the zero temperature theory described by \( \Omega_{T=0}^{(1)} \). The last two terms of (3) can be obtained in a consistent development using the background-field method; however, we will use a more elementary approach here.

We will use the method of excitation energies as described in Ref. [12]. Begin with the equation of motion for \( \Psi \) and \( \Psi^\dagger \) which follow from (1) as

\[
0 = i\dot{\Psi} + \frac{1}{2m} \nabla^2 \Psi + \mu \Psi - U'(|\Psi|^2) \Psi,
\]

\[
0 = -i\dot{\Psi}^\dagger + \frac{1}{2m} \nabla^2 \Psi^\dagger + \mu \Psi^\dagger - U'(|\Psi|^2) \Psi^\dagger.
\]

If we expand \( \Psi = \bar{\Psi} + \Phi \), where \( \Phi \) represents the fluctuation about the background field \( \bar{\Psi} \), then \( \Phi \) satisfies

\[
0 = i\dot{\Phi} + \frac{1}{2m} \nabla^2 \Phi + \mu \Phi - U'(|\bar{\Psi}|^2) \Phi - U''(|\bar{\Psi}|^2) \left( |\bar{\Psi}|^2 \Phi + \bar{\Psi}^2 \Phi^\dagger \right),
\]

\[
0 = -i\dot{\Phi}^\dagger + \frac{1}{2m} \nabla^2 \Phi^\dagger + \mu \Phi^\dagger - U'(|\bar{\Psi}|^2) \Phi^\dagger - U''(|\bar{\Psi}|^2) \left( |\bar{\Psi}|^2 \Phi^\dagger + \bar{\Psi}^2 \Phi \right).
\]

If we define \( \{ f_k(x) \} \) to be a complete orthonormal set of solutions to

\[
-\nabla^2 f_k(x) = \sigma_k f_k(x),
\]

we can expand

\[
\Phi(t, x) = \sum_k A_k e^{-\text{i}E_k t} f_k(x),
\]

\[
\Phi^\dagger(t, x) = \sum_k B_k e^{-\text{i}E_k t} f_k(x),
\]

for independent expansion coefficients \( A_k \) and \( B_k \). \( E_k \) are the excitation energies. Because of the assumption of completeness for the \( f_k(x) \) we find, after substitution of (8) and (9) into (3) and (10),

\[
0 = \left( E_k - \frac{\sigma_k}{2m} + \mu - U'(|\bar{\Psi}|^2) - |\bar{\Psi}|^2 U''(|\bar{\Psi}|^2) \right) A_k
\]

\[
-\bar{\Psi}^2 U''(|\bar{\Psi}|^2) B_k ,
\]

\[
0 = \left( -E_k + \frac{\sigma_k}{2m} + \mu - U'(|\bar{\Psi}|^2) - |\bar{\Psi}|^2 U''(|\bar{\Psi}|^2) \right) B_k
\]

\[
-(\Phi^\dagger)^2 U''(|\bar{\Psi}|^2) A_k.
\]

For a non-trivial solution for the expansion coefficients \( A_k \) and \( B_k \) we must have

\[
E_k = \left( \frac{\sigma_k}{2m} + \mu + U'(|\bar{\Psi}|^2) \right)^{1/2}
\]

\[
\times \left( \frac{\sigma_k}{2m} - \mu + U'(|\bar{\Psi}|^2) + 2|\bar{\Psi}|^2 U''(|\bar{\Psi}|^2) \right)^{1/2}.
\]

We can now write down the last two contributions to the thermodynamic potential (3). We have

\[
\Omega_{T=0}^{(1)} = \frac{\hbar}{2} \sum_k E_k ,
\]

since at zero temperature the only contribution is the sum over zero-point energies. Also

\[
\Omega_{T\neq 0}^{(1)} = \hbar T \sum_k \ln(1 - e^{-\beta E_k}) ,
\]

which is the expression usually written down for the thermodynamic potential at finite temperature. We leave the factor of \( \hbar \) in front of both expressions (14) and (15) to denote that they are of first order in a systematic loop expansion. We will only work to order \( \hbar \).

\( \Omega_{T=0}^{(1)} \) is divergent, and to make sense of this we must regularize it in some way. We will use a version of \( \zeta \)-function regularization here. (See Ref. [15] for a pedagogical introduction to the method we will use.) This consists of defining an energy \( \zeta \)-function by

\[
E(s) = \sum_k E_k (\ell E_k)^{-s} ,
\]

where \( s \) is a complex variable suitably chosen so that the sum in (16) converges, and \( \ell \) is a constant with dimensions of inverse energy introduced so that \( E(s) \) has units of energy for any value of \( s \). The regularization consists of defining \( \Omega_{T=0}^{(1)} \) by

\[
\Omega_{T=0}^{(1)} = \frac{\hbar}{2} E(0) ,
\]

where \( E(0) \) represents the analytic continuation of \( E(s) \) from the region of the complex \( s \)-plane where (16) converges to \( s = 0 \), assuming that \( E(s) \) is regular at \( s = 0 \). It may be that \( E(s) \) has a simple pole at \( s = 0 \) in which case \( E(0) \) in (17) represents the analytic continuation of \( E(s) \) to a neighbourhood of \( s = 0 \). The pole can be dealt with by the usual machinery of renormalization. In the
theory considered here renormalization is not necessary as we will show that $E(s)$ is analytic at $s=0$.

Given the thermodynamic potential $\Omega$, the number of particles is given by

$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{\beta,V,\bar{\Psi}}, \tag{18}$$

and the internal energy density $\rho$ is

$$\rho V = \left(\frac{\partial}{\partial \beta} (\beta \Omega)\right)_{\mu,V,\bar{\Psi}}. \tag{19}$$

Using (18) it is easily seen that

$$\rho V = \left(\frac{\partial}{\partial \beta} (\beta \Omega)\right)_{\mu,V,\bar{\Psi}} + \mu N. \tag{20}$$

In addition, the condensate $\bar{\Psi}$ satisfies

$$0 = \left(\frac{\partial \Omega}{\partial \bar{\Psi}}\right)_{\beta,V,\mu}. \tag{21}$$

The exact equation for $\bar{\Psi}$ is very complicated due to the dependence of the excitation energies $E_k$ on $\bar{\Psi}$ in (13). However because we are only working to order $\hbar$, from (2) and (3) we can conclude that

$$\mu = U'(\bar{\Psi})^2 + \mathcal{O}(\hbar). \tag{22}$$

This means that in any expression which is already of order $\hbar$ we can simplify it using only the first term of (22). Of course in any term which is of lower order in $\hbar$ we must include the $\mathcal{O}(\hbar)$ correction to $\mu$ for consistency. We will encounter this in Sec. II C.

**B. Ground state energy**

We will introduce the number density $n$ by

$$n = \frac{N}{V}. \tag{23}$$

From (2), (15), (18) we find the number density of particles to be

$$n = |\bar{\Psi}|^2 + n_{T=0}^{(1)} + n_{T\neq 0}^{(1)}, \tag{24}$$

with

$$n_{T=0}^{(1)} = -\left(\frac{\partial \Omega_{T=0}^{(1)}}{\partial \mu}\right)_{\beta,V,\bar{\Psi}}, \tag{25}$$

and a similar expression holding for $n_{T\neq 0}^{(1)}$. Since $n_{T=0}^{(1)}$ and $n_{T\neq 0}^{(1)}$ are of order $\hbar$, it is obvious from (24) that

$$|\bar{\Psi}|^2 = n + \mathcal{O}(\hbar). \tag{26}$$

Using the same argument as for $\mu$ below (22), we can set $|\bar{\Psi}|^2 = n$ in any expression which is already of order $\hbar$.

If we use (2,3) along with (23) and (24) in the expression (20) for the energy density, we have

$$\rho = U(|\bar{\Psi}|^2) + \frac{1}{V} \Omega_{T=0}^{(1)} + \frac{1}{V} \left(\frac{\partial}{\partial \beta} (\beta \Omega_{T\neq 0}^{(1)})\right)_{\mu,V,\bar{\Psi}}$$

$$+ \mu (n_{T=0}^{(1)} + n_{T\neq 0}^{(1)}). \tag{27}$$

We can use (24) to simplify the argument of the energy $U(|\bar{\Psi}|^2)$. To order $\hbar$ we have

$$U(|\bar{\Psi}|^2) = U(n) - U'(n)(n_{T=0}^{(1)} + n_{T\neq 0}^{(1)}) + \cdots. \tag{28}$$

The last term of (27) may be simplified by using (22) and (23) to give

$$\mu (n_{T=0}^{(1)} + n_{T\neq 0}^{(1)}) = U'(n)(n_{T=0}^{(1)} + n_{T\neq 0}^{(1)}) + \cdots, \tag{29}$$

again working only to order $\hbar$. We therefore find

$$\rho = U(n) + \frac{1}{V} \Omega_{T=0}^{(1)} + \frac{1}{V} \left(\frac{\partial}{\partial \beta} (\beta \Omega_{T\neq 0}^{(1)})\right)_{\mu,V,\bar{\Psi}}, \tag{30}$$

from (27) holding to order $\hbar$. If we use (15), then the last term of (30) becomes

$$\rho_{T\neq 0}^{(1)} = \frac{1}{V} \left(\frac{\partial}{\partial \beta} (\beta \Omega_{T\neq 0}^{(1)})\right)_{\mu,V,\bar{\Psi}}, \tag{31}$$

$$= \frac{\hbar}{V} \sum_k E_k (e^{\beta E_k} - 1)^{-1}. \tag{32}$$

This is the usual expression for the internal energy density from statistical mechanics.

We now require the evaluation of $\Omega_{T=0}^{(1)}$. Eq. (16) can be used for this. A direct evaluation using the energy levels in (23) is difficult. Because we are only working to order $\hbar$, and $\Omega_{T=0}^{(1)}$ is already of order $\hbar$, we can simplify the excitation energies using $\mu = U'(n)$ and $|\bar{\Psi}|^2 = n$. This results in

$$E_k = \left(\frac{\sigma_k}{2m}\right)^{1/2} \left(\frac{\sigma_k}{2m} + 2nU''(n)\right)^{1/2}. \tag{33}$$

If we take the large volume limit we can replace $\sigma_k \to k^2$ and $\sum_k \to V \int \frac{d^3k}{(2\pi)^3}$; thus the energy $\zeta$-function to be evaluated is

$$E(s) = \frac{V}{\ell^8} \int \frac{d^3k}{(2\pi)^3} \left(\frac{k^2}{2m}\right)^{\frac{s-1}{2}} \left(\frac{k^2}{2m} + 2nU''(n)\right)^{\frac{s-1}{2}}. \tag{34}$$

(Note that the energy spectrum in Eq. (33) with $\sigma_k \to k^2$ was first given by Bogoliubov [29].) The integral in (24) is easily evaluated using the representation of the $\Gamma$-function to give

$$E(s) = (2m)^{3/2} V \frac{\Gamma(2 - \frac{s}{2})\Gamma(s - \frac{5}{2})}{4\pi^2\ell^8} \frac{\Gamma(s - \frac{3}{2})}{\Gamma(s - \frac{5}{2})} [2nU''(n)]^{5/2-s}. \tag{35}$$
This expression is analytic at \( s = 0 \) with the result
\[
E(0) = \frac{16}{15\pi^2} m^{3/2} V [nU''(n)]^{5/2} .
\] (36)

We then find from (31) that
\[
\frac{1}{V} \Omega^{(1)}_{T=0} = \frac{8\hbar}{15\pi^2} m^{3/2} [nU''(n)]^{5/2} .
\] (37)

If we use (32) we have
\[
\rho^{(1)}_{T\neq 0} = \hbar \int \frac{d^3k}{(2\pi)^3} \frac{E_k}{(e^{\beta E_k} - 1)} ,
\] (38)
where \( E_k \) is given by (33) after the large volume limit has been taken. Using spherical polar coordinates, performing the angular integration, and rescaling \( k \to (2mT)^{1/2}k \) results in
\[
\rho^{(1)}_{T\neq 0} = \frac{\hbar m^2}{2\pi^2 \beta} \left( \frac{2m}{\beta} \right)^{3/2} \int_0^\infty dk k^3 \sqrt{k^2 + 2\beta nU''(n)} \times \left[ e^{k \sqrt{k^2 + 2\beta nU''(n)}} - 1 \right]^{-1} .
\] (39)

Although it is not possible to evaluate the integral in (39) exactly, it is relatively straightforward to find the low (or high) temperature expansion. We concentrate on the low temperature expansion here and assume that \( 2\beta nU''(n) \gg 1 \). The first two terms in the asymptotic expansion of (39) give
\[
\rho^{(1)}_{T\neq 0} \simeq \frac{\hbar m^2}{30} \left( \frac{m}{nU''(n)} \right)^{3/2} T^4 \left\{ 1 - \frac{25\pi^4}{21} [\beta nU''(n)]^{-2} + \cdots \right\} .
\] (40)

We then have the low temperature expansion of the energy density in (30) being
\[
\rho(T) \simeq U(n) + \frac{8\hbar}{15\pi^2} m^{3/2} [nU''(n)]^{5/2}
\] (41)

\[
+ \frac{\hbar m^2 T^4}{30} \left[ \frac{m}{nU''(n)} \right]^{3/2} \left\{ 1 - \frac{25\pi^4}{21} [\beta nU''(n)]^{-2} \right\} .
\]

In the zero temperature limit only the first two terms survive and we have the exact result
\[
\rho(T = 0) = U(n) + \frac{8\hbar}{15\pi^2} m^{3/2} [nU''(n)]^{5/2} .
\] (42)

So far we have kept the potential \( U(|\bar{\Psi}|^2) \) arbitrary. In order to make contact with the standard results we will take
\[
U(|\bar{\Psi}|^2) = \frac{2\pi a}{m} |\bar{\Psi}|^4 ,
\] (43)

with \( a \) the s-wave scattering length. In this case it follows from (40) that
\[
\rho(T = 0) = \frac{2\pi a m^2}{m} \left[ 1 + \frac{128\hbar}{15\sqrt{\pi}} na^3 \right] ,
\] (44)

and from (40) that
\[
\rho^{(1)}_{T=0} \simeq \frac{\hbar m^3 T^4}{240\sqrt{\pi}} (na^3)^{-3/2} \left\{ 1 - \frac{25}{336} \left( \frac{mT}{na} \right)^2 + \cdots \right\} .
\] (45)

The result in (45) agrees with the classic calculations of Refs. [6, 7, 8] for the zero temperature gas. The first term of (44) reproduces the leading correction at low temperature found in Refs. [6]. (Note that Ref. [8] chooses \( m = 1/2 \).)

### C. Equation of state

The equation of state involves a determination of the pressure, which follows from the first law of thermodynamics as
\[
P = - \left( \frac{\partial \Omega}{\partial V} \right)_{T,\mu,\bar{\Psi}} .
\] (46)

Because \( \Omega \) is an extensive quantity, in the large volume limit we have simply
\[
P = - \frac{\Omega}{V} .
\] (47)

The evaluation of the equation of state involves the determination of \( \Omega \) to order \( h \). From (2 and 3) we have
\[
P = \mu |\bar{\Psi}|^2 - U(|\bar{\Psi}|^2) - \frac{\Omega^{(1)}_{T=0} + \Omega^{(1)}_{T\neq 0}}{V} .
\] (48)

Using (24) and (26) to eliminate \(|\bar{\Psi}|^2\) we find (to order \( h \))
\[
P = \mu n - U(n) - \frac{\Omega^{(1)}_{T=0} + \Omega^{(1)}_{T\neq 0}}{V} .
\] (49)

where we can set \(|\bar{\Psi}|^2 = n \) in \( \Omega^{(1)}_{T=0} \) and \( \Omega^{(1)}_{T\neq 0} \) since these two terms are already of order \( \hbar \). What makes the calculation of \( P \) more difficult than that of \( \rho \) is that we cannot simply use (22) for \( \mu \) in the first term; we need to know the \( \mathcal{O}(\hbar) \) contribution to \( \mu \) coming from the quantum corrections to the classical field equation for \( \bar{\Psi} \). Because we are interested only in the condensed phase, we can replace (21) with
\[
0 = \left( \frac{\partial \Omega}{\partial |\bar{\Psi}|^2} \right)_{\beta, V, \mu} .
\] (50)

This leads to
\[
\mu = U'(|\bar{\Psi}|^2) + \frac{1}{V} \frac{\partial (\Omega^{(1)}_{T=0} + \Omega^{(1)}_{T\neq 0})}{\partial |\bar{\Psi}|^2} |_{\mu, T, V} .
\] (51)

We now use (24) again in the first term to find
\[
\mu = U'(n) + \frac{1}{V} \left\{ \frac{\partial (\Omega^{(1)}_{T=0} + \Omega^{(1)}_{T\neq 0})}{\partial |\bar{\Psi}|^2} + U''(n) \frac{\partial (\Omega^{(1)}_{T=0} + \Omega^{(1)}_{T\neq 0})}{\partial \mu} \right\} .
\] (52)
Using this in (43) results in
\[ P = nU'(n) - U(n) + \frac{1}{V} \left\{ n \frac{\partial \Omega^{(1)}_{T=0}}{\partial \Psi} - \frac{\partial \Omega^{(1)}_{T=0}}{\partial \Psi} \right\} + \frac{1}{V} \left\{ n \frac{\partial \Omega^{(1)}_{T=0}}{\partial \Psi} - \frac{\partial \Omega^{(1)}_{T=0}}{\partial \Psi} \right\} \]
\[ + nU''(n) \left( \frac{\partial \Omega^{(1)}_{T=0}}{\partial \mu} - \frac{\partial \Omega^{(1)}_{T=0}}{\partial \mu} \right) \]
\[ + nU''(n) \left( \frac{\partial \Omega^{(1)}_{T=0}}{\partial \mu} - \frac{\partial \Omega^{(1)}_{T=0}}{\partial \mu} \right) \]
(53)

for the pressure to order \( \hbar \). The last term of (53) displays the finite temperature correction to the zero temperature result. In the last two terms of (72) we can set \( |\Psi|^2 = n \) and \( \mu = U'(n) \) to simplify the results after we have performed the indicated differentiated.

We regularize the terms in \( \Omega^{(1)}_{T=0} \) by the \( \zeta \)-function method as described in Sec. II A. From the excitation energy (53) it is easy to see that
\[ \frac{\partial E_k}{\partial \mu} = -\frac{1}{E_k} \left( \frac{\sigma_k}{2m} + nU''(n) \right) \]
(54)
\[ \frac{\partial E_k}{\partial |\Psi|^2} = \frac{1}{E_k} \left\{ \left( 2U''(n) + nU''(n) \right) \frac{\sigma_k}{2m} \right\} \]
(55)

It then follows that
\[ \frac{n}{V} \left\{ \frac{\partial \Omega^{(1)}_{T=0}}{\partial \Psi} + U''(n) \frac{\partial \Omega^{(1)}_{T=0}}{\partial \mu} \right\} = \frac{\hbar n}{2V} \left( 1 - s \right) e^{-s} \]
\[ \times \left[ U''(n) + nU''(n) \right] \sum_k E_k^{-1-s} \frac{\sigma_k}{2m} \]
(56)

with the right hand side analytically continued to \( s = 0 \). Taking the large volume limit it is easy to show that
\[ \frac{n}{V} \left\{ \frac{\partial \Omega^{(1)}_{T=0}}{\partial \Psi} + U''(n) \frac{\partial \Omega^{(1)}_{T=0}}{\partial \mu} \right\} = \frac{\hbar n}{8\pi^2} \left( 2m \right)^{3/2} \]
\[ \times \left[ U''(n) + nU''(n) \right] \left[ 1 - s \right] \]
\[ \times \frac{\Gamma(2-s/2) \Gamma(s-3/2)}{\Gamma \left( \frac{1+s}{2} \right)} \]
\[ \times \left[ 2mU''(n) \right]^{3/2-s} \]
\[ = \frac{4\hbar n}{3\pi^2} \left[ U''(n) + nU''(n) \right] \left[ mnU''(n) \right]^{3/2} \]
(57)
after analytic continuation to \( s = 0 \).

The pressure of the zero temperature gas follows from (53) as
\[ P(T = 0) = nU'(n) - U(n) + \frac{4\hbar}{5\pi^2} \left( \frac{n}{m} \right)^{5/2} \left[ nU''(n) \right]^{5/2} \]
\[ + \frac{4\hbar}{3\pi^2} \left( \frac{n}{m} \right)^2 \left[ nU''(n) \right]^{3/2} \]
(58)

if we use (37) and (33). Specialising to \( U(n) = 2\pi an^2/m \) as in (13) results in
\[ P_{T=0} = \frac{2\pi an^2}{m} \left[ 1 + \frac{64\hbar}{5\sqrt{\pi} na} \right] \]
(59)
in agreement with the standard results (4, 8).

To evaluate the finite temperature part of the pressure, we use (15) along with (24) and (53) to find
\[ \frac{n}{V} \left\{ \frac{\partial \Omega^{(1)}_{T=\neq 0}}{\partial \Psi} + U''(n) \frac{\partial \Omega^{(1)}_{T=\neq 0}}{\partial \mu} \right\} = \frac{\hbar n}{V} \left[ U''(n) + nU''(n) \right] \]
\[ \times \left[ \sum_k (e^{\beta E_k} - 1)^{-1} \left( \frac{\sigma_k}{\sigma_k + 4m\pi^2 U(n)} \right)^{1/2} \right] \]
(60)

In the large box limit we can evaluate the right hand side at low temperatures as we did for \( P^{(1)}_{T=\neq 0} \) in Sec. II B. A short calculation leads to
\[ \frac{n}{V} \left\{ \frac{\partial \Omega^{(1)}_{T=\neq 0}}{\partial \Psi} + U''(n) \frac{\partial \Omega^{(1)}_{T=\neq 0}}{\partial \mu} \right\} = \frac{\hbar n}{V} \left[ U''(n) + nU''(n) \right] \]
\[ \times \left[ 1 + \frac{nU''(n)}{U(n)} \left( 1 - 5\pi^2 \beta nU''(n) \right)^{-2} + \cdots \right] \]
(61)

where \( \beta nU''(n) \gg 1 \). In a similar way the low temperature expansion of (15) gives
\[ -\frac{\Omega^{(1)}_{T=\neq 0}}{V} = \frac{\pi^2}{90} T^4 \left[ \frac{m}{nU''(n)} \right]^{3/2} \left\{ 1 + \frac{3nU''(n)}{5U''(n)} \right\} \]
\[ -\pi^2 \left( \frac{9}{7} \right) \frac{nU''(n)}{U(n)} \left[ \beta nU''(n) \right]^{-2} + \cdots \] \( \mbox{(62)} \)

Combining (31) and (32) we have as the finite temperature correction to the pressure
\[ P_{T=\neq 0} \approx \frac{\pi^2}{36} T^4 \left[ \frac{m}{nU''(n)} \right]^{3/2} \left\{ 1 + \frac{3nU''(n)}{5U''(n)} \right\} \]
\[ -\pi^2 \left( \frac{9}{7} \right) \frac{nU''(n)}{U(n)} \left[ \beta nU''(n) \right]^{-2} + \cdots \] \( \mbox{(63)} \)
at low temperatures. When we specialise to \( U(n) = 2\pi an^2/m \) this result reduces to
\[ P_{T=\neq 0} \approx \frac{4\sqrt{\pi}}{288} m^3 T^4 (na)^{-3/2} \left\{ 1 - \frac{9}{118} \left( \frac{m}{\beta na} \right)^2 + \cdots \right\} . \]
(64)
The leading term agrees with the earlier calculation in Ref. [4].

III. BINARY MIXTURE

A. General approach

We now generalize the results of Sec. II to a two-species Bose gas described by Schrödinger fields \( \Psi_1 \) and \( \Psi_2 \). We allow for self-interactions as well as mutual interactions between the two species in a general way. The action functional will be chosen to be
\[ S = \int dt \int d\Sigma \left\{ \sum_j \left[ \frac{i}{2} \left( \Psi_j^\dagger \partial_j \Psi_j - \Psi_j \partial_j \Psi_j^\dagger \right) - \frac{1}{2m_j} \left| \nabla \Psi_j \right|^2 \right] + \mu_j \left| \Psi_j \right|^2 \right\} - U \left( |\Psi_1|^2, |\Psi_2|^2 \right) . \]
(65)
Here $m_{1,2}$ denote the masses of the two species, and $\mu_{1,2}$ are the chemical potentials. We assume that the interaction, described by the potential $U$, only involves the magnitudes of the two fields. The theory has a rigid $U(1) \times U(1)$ symmetry in general, unless we further specify the potential. The field equations which follow from (28) are

$$0 = i\dot{\Psi}_j + \frac{1}{2m_j} \nabla^2 \Psi_j + \mu_j \Psi_j - U_j \left(|\Psi_1|^2, |\Psi_2|^2\right) \Psi_j \quad (66)$$

for $j = 1, 2$ along with the hermitian conjugate. Here $U_j = \frac{\partial}{\partial \Psi_j \bar{\Psi}_j} U$.

The first requirement is to identify the excitation energies by first setting $\Psi_j = \bar{\Psi}_j + \Phi_j$ and linearizing in $\Phi_j$. As in Sec. II A, $\bar{\Psi}_j$ denotes the background field which we assume to be constant. We find

$$0 = i\dot{\Phi}_j + \frac{1}{2m_j} \nabla^2 \Phi_j + \mu_j \Phi_j - \left(U_j + |\Phi_1|^2 U_{11}\right) \Phi_j - \bar{\Psi}_j \bar{\Psi}_j^2 U_{12} \Phi_2 - \bar{\Psi}_j \bar{\Psi}_j U_{12} \Phi_2^2, \quad (67)$$

along with its hermitian conjugate, and two other equations obtained by interchanging the labels 1 and 2. We assume $U_{jj'} = \frac{\partial^2}{\partial \Psi_j \partial \bar{\Psi}_{j'}} U = U_{jj'}$. All terms involving the potential are evaluated at a background fields. To get the excitation energies we set

$$\Phi_j(t, x) = \sum_k A_{kj} e^{-iE_k t} f_k(x), \quad (68)$$
$$\bar{\Phi}_j(t, x) = \sum_k B_{kj} e^{-iE_k t} f_k(x), \quad (69)$$

for independent coefficients $A_{kj}$ and $B_{kj}$. $(f_k(x)$ is still the solution to (6). After some calculation the excitation energies can be shown to be

$$E_{k \pm}^2 = \frac{1}{2} \left( a_j^2 + a_j^2 - |\bar{\Psi}_1|^4 U_{11} - |\bar{\Psi}_2|^4 U_{22} \right)$$
$$\pm \frac{1}{2} \left( a_j^2 - a_j^2 - |\bar{\Psi}_1|^4 U_{11} + |\bar{\Psi}_2|^4 U_{22} \right)^2$$
$$+ 16|\bar{\Psi}_1|^2 |\bar{\Psi}_2|^2 U_{12}^2$$
$$\times (a_1 - |\bar{\Psi}_1|^2 U_{11})(a_2 - |\bar{\Psi}_2|^2 U_{22})^{1/2} \quad (70)$$

where

$$a_j = \frac{\sigma_j}{2m_j} - \mu_j + U_j + |\Phi_j|^2 U_{jj} \quad (j = 1, 2). \quad (71)$$

The thermodynamic potential $\Omega$ has the form (31) where

$$\Omega_{\text{class}} = V \left[ U(|\bar{\Psi}_1|^2, |\bar{\Psi}_2|^2) - \mu_1 |\bar{\Psi}_1|^2 - \mu_2 |\bar{\Psi}_2|^2 \right] \quad (72)$$
$$\Omega_{T=0}^{(1)} = \frac{h}{2} \sum_k (E_k + E_{-k}), \quad (73)$$
$$\Omega_{T \neq 0}^{(1)} = hT \sum_k \left[ \ln \left( 1 - e^{-\beta E_k} \right) \right. \right.$$
$$\left. + \ln \left( 1 - e^{-\beta E_{-k}} \right) \right], \quad (74)$$

The number density for species $j$ is (see (18))

$$n_j = -\frac{1}{V} \left( \frac{\partial \Omega}{\partial \mu_j} \right)_{T, V, \Psi} \quad (j = 1, 2) \quad (75)$$

and the energy density is given by (13). It is easy to show that in place of (20) we have

$$\rho = \frac{1}{V} \left( \frac{\partial}{\partial \beta} \left( |\bar{\Psi}_j|^2 + |\bar{\Psi}_j|^2 \right) \right)_{\mu, V, \Psi} + \mu_1 n_1 + \mu_2 n_2. \quad (76)$$

The pressure is given by (47).

**B. Ground state energy**

Using (28) we can write

$$n_j = |\bar{\Psi}_j|^2 + n_{j=0}^{(1)} + n_{j \neq 0}^{(1)} \quad (j = 1, 2) \quad (77)$$

as in (24)(23). This allows us to conclude that $|\bar{\Psi}_j|^2 = n_j + O(\hbar)$. Since only $\Omega_{T=0}^{(1)}$ has an explicit $\beta$ dependence, we find

$$\rho = U(|\bar{\Psi}_1|^2, |\bar{\Psi}_2|^2) + \frac{1}{V} \left( \frac{\partial \Omega_{T=0}^{(1)}}{\partial \beta} \right)_{\mu, V, \Psi}$$
$$+ \mu_1 n_1 + \mu_2 n_2 - |\bar{\Psi}_2|^2. \quad (78)$$

To order $h$ we have $\mu_j = U_j$ as in (22). We can simplify (78) to order $h$ as

$$\rho = U(n_1, n_2) + \frac{1}{V} \left( \frac{\partial \Omega_{T=0}^{(1)}}{\partial \beta} \right)_{\mu, V, \Psi} + \mu_1 n_1 + \mu_2 n_2. \quad (79)$$

where

$$\rho_{T \neq 0} = \frac{1}{V} \left( \frac{\partial \Omega_{T \neq 0}^{(1)}}{\partial \beta} \right)_{\mu, V, \Psi}$$
$$= \frac{1}{V} \sum_k \left\{ E_k + (e^{\beta E_k} - 1)^{-1} \right. \right.$$
$$\left. + E_{-k} - (e^{\beta E_{-k}} - 1)^{-1} \right\} \quad (80)$$

is the finite temperature correction to the zero-temperature result.

We were not able to obtain analytical results for the ground state energy density except in two special cases. One case is $U_{12} = 0$, which means that the two species do not interact with each other. This is not very interesting since it just results in a sum of terms for the two species, with each term given as in Sec. II. The second case where we can obtain exact analytic results at $T = 0$ occurs if we take $m_1 = m_2$. A physical system where this is relevant concerns different hyperfine spin states of a given gas. This specialisation allows a sufficient simplification of (78) to enable analytic results.

We are free to set $|\bar{\Psi}_j|^2 = n_j$ in the second two terms of (28), as well as to take $\mu_j = U_j$. With these two simplifications made, and $m_1 = m_2 = m$ taken, the excitation energies (70, 71) become

$$E_{k \pm}^2 = \frac{\sigma_k}{2m} \left( \frac{\sigma_k}{2m} + a_\pm \right) \quad (81)$$
where
\[\alpha_\pm = n_1 U_{11} + n_2 U_{22} \pm [(n_1 U_{11} - n_2 U_{22})^2 + 4n_1 n_2 U_{12}^2]^{1/2}.\]

The result for \(\Omega_{T=0}^{(1)}\) may be defined using the energy \(\zeta\)-function method exactly as in Sec. II. In fact the energy \(\zeta\)-function for the energies \(E_{k\pm}\) may be obtained from [13] by replacing \(2nU''\) with \(\alpha_\pm\). The energy density of the zero temperature gas is therefore
\[\rho_{T=0} = U(n_1, n_2) + \frac{\hbar}{20\pi^2} (2m)^{3/2} \left(\alpha_+^{5/2} + \alpha_-^{5/2}\right).\]

The finite temperature correction, at low temperatures, can be obtained from [14] with the replacement \(2nU''\) with \(\alpha_\pm\). The leading term at low temperature is
\[\rho_{T \neq 0} \simeq \frac{\pi^2}{30} T^4 \left[\left(\frac{2m}{\alpha_+}\right)^{3/2} + \left(\frac{2m}{\alpha_-}\right)^{3/2}\right].\]

If we set \(U_{12} = 0\), then the energy density reduces to the sum of the single species gas contributions. The presence of mutual interactions of the two species complicates the result.

It is possible to generalize the method described for the single species gas in Sec. II to deal with the binary mixture. However as the details are rather messy, and the result not particularly simple we will not pursue this here.

IV. DISCUSSION AND CONCLUSIONS

We have shown how \(\zeta\)-function techniques may be used to obtain the thermodynamic properties of a single species as well as a two species Bose gas. The interaction in either case was arbitrary. For the binary mixture analytical results at zero temperature were obtained in the case where the masses of the two constituents were equal. Although this restriction might seem artificial at first sight, and perhaps not of great physical relevance, it has been realised experimentally for trapped gases [20]. Some progress can be made in the more general case of unequal masses, but analytical results do not appear possible.

The overwhelming interest at the present time is in the behaviour of trapped Bose gases. In the present formalism this entails the introduction of an additional term to \(\Omega\) to represent the trapping potential. This loses the feature of the calculation above which enabled analytical results to be obtained; namely, that \(\Psi\) was constant. Nevertheless the background field method works whether \(\Psi\) is constant or not, and it seems likely that the use of generalised \(\zeta\)-function methods like those presented above can be used to provide a different approach to the problem of trapped interacting gases.

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