SMOCKED METRIC SPACES AND THEIR TANGENT CONES

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ABSTRACT. We introduce the notion of a smocked metric spaces and explore the balls and geodesics in a collection of different smocked spaces. We find their rescaled Gromov-Hausdorff limits and prove these tangent cones at infinity exist, are unique, and are normed spaces. We close with a variety of open questions suitable for advanced undergraduates, masters students, and doctoral students.

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1. Introduction

The asymptotic behavior of a metric space at infinity has been well studied by many mathematicians by taking a sequence of rescalings of the metric spaces and finding their Gromov-Hausdorff limit. This limit is called the tangent cone at infinity. This idea has ancient roots. For example, a hyperboloid rescaled in upon itself converges to a cone, which is equivalent to saying it is asymptotic to a cone at infinity. However for more abstract metric spaces, it can be more difficult to understand what it means to take a limit and it can be more difficult to find that limit. Often the limit is not unique and there is no reason for it to be a cone.

Here we introduce a new class of metric spaces that we call \textit{smocked spaces} inspired by the craft of smocking fabric. Each smocked space is defined by taking a Euclidean space with a pattern on it (as in Figure 1) and then pulling each stitch in the pattern to a single point. The notion of pulling a thread to a point has already been explored by metric geometers as it provides interesting counter examples to questions involving areas and perimeters (cf. [2] by Burago-Ivanov). However, this is the first time anyone has explored more complex patterns in which many threads are drawn to points. We discover that indeed we obtain some rather surprising tangent cones at infinity when studying these spaces.

The rigorous definition of a smocked space appears in Definition 3.1 after first reviewing the rigorous definition of pulling a thread to a point in Definition 2.9. We prove some lemmas about balls in these spaces, which allow us to explore the balls of all six spaces on an intuitive level. We then choose three smocked spaces to analyze more deeply, ultimately determining their tangent cones at infinity in

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The smocked spaces: $X_+$, $X_T$, $X_O$, and $X_H$, $X_o$, $X_z$.}
\end{figure}
Theorem 9.2, Theorem 9.7, and Theorem 9.12. We’ve also proven a general theorem one can apply to find the Gromov-Hausdorff limit of a large class of smocking spaces in Theorem 9.1 proving these tangent cones are unique and are normed spaces. In upcoming work by some of the authors [8], we complete the analysis of the other three smocked spaces and examine three dimensional versions of them as well. In upcoming work by Kazaras and Sormani [6], we will apply the results in these two papers to a question of Gromov and Sormani [4].

This paper can be read by anyone who has studied basic metric geometry as we review every concept before we need to apply it. Through the reading of this paper, one can learn the definition of Gromov-Hausdorff convergence, a standard technique for proving this convergence using correspondences, and the definition of the tangent cone at infinity of a metric space. As each concept is introduced, it is immediately applied to each of these patterns by various teams of students. It may be fun for the reader to find a smocking pattern online or in a smocking guide like [7] and work with it while reading this paper to discover something new. The reader might also slightly adapt the patterns explored within to see the consequences of altering the sizes of stitches and the spacing between them.

The paper closes with a section of open problems some of which are labeled as possible projects for an undergraduate or masters thesis. Other open problems are significantly more advanced. Students who would like to study metric geometry on a more advanced level are encouraged to read Burago-Burago-Ivanov’s award winning textbook [1].

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2. Background

Here we quickly review metric spaces and pulled thread spaces. See also the award winning textbook by Burago-Burago-Ivanov [1]. If you already know these topics, just briefly glance through for notation.
2.1. Metric spaces.

Definition 2.1. A metric space \((X, d)\) is a set \(X\) with a function \(d : X \times X \to \mathbb{R}\) satisfying the following:

- Nonnegativity: \(d(x, y) \geq 0\) \(\forall x, y \in X\)
- Definiteness: \(d(x, y) = 0 \iff x = y\).
- Symmetry: \(d(x, y) = d(y, x)\) \(\forall x, y \in X\)
- The Triangle Inequality: \(d(x, y) \leq d(x, z) + d(z, y)\) \(\forall x, y, z \in X\)

In this paper we will use the following notation:

Lemma 2.2. Given a point \(p\) in a metric space \(X\) and \(r > 0\), an open ball of radius \(r\) about \(p\):

\[
B_r(p) = \{x : d(x, p) < r\}.
\]

A closed ball of radius \(r\) about \(p\):

\[
\overline{B}_r(p) = \{x : d(x, p) \leq r\}.
\]

A sphere of radius \(r\) about \(p\):

\[
\partial B_r(p) = \{x : d(x, p) = r\}.
\]

Definition 2.3. Given a set \(K\) in a metric space \(X\) and \(r > 0\), the tubular neighborhood of radius \(r\) about \(K\):

\[
T_r(K) = \{x : \exists y \in K \text{ s.t. } d(x, y) < r\}.
\]

Lemma 2.4. Given a point \(p\) in a metric space \(X\) and \(s > 0\) we have

\[
T_s(B_r(p)) = B_{r+s}(p)
\]

Proof. Let \(q \in T_s(B_r(p))\). By the definition of tubular neighborhood, \(\exists y \in B_r(p)\) such that \(d(q, y) < s\). By the triangle inequality, \(d(q, p) < r+s\), and therefore \(q \in B_{r+s}(p)\) Now take \(q \in B_{r+s}(p)\). So \(d(q, p) < r+s\). If \(B_s(q) \cap B_r(p) = \emptyset\) then \(r+s \leq d(q, p)\) which leads to a contradiction. Hence \(\exists y \in B_r(p)\) such that \(d(q, y) < s\), and so \(q \in T_s(B_r(p))\).

Lemma 2.5. If \(K_1 \subset K_2\) then \(T_s(K_1) \subset T_s(K_2)\).

Proof. Let \(p \in T_s(K_1)\). Then \(\exists y \in K_1\) such that \(d(p, y) < s\). Since \(K_1 \subset K_2\), we have \(\exists y \in K_2\) such that \(d(p, y) \in s\). Thus \(p \in T_s(K_2)\).

Lemma 2.6. \(T_s(K_1 \cup K_2) = T_s(K_1) \cup T_s(K_2)\).

Proof. This follows from the definition of union:

\[
T_s(K_1 \cup K_2) = \{z : \exists w \in K_1 \cup K_2 \text{ s.t. } d(z, w) < s\}
\]

\[
= \{z : \exists w \in K_1 \text{ s.t. } d(z, w) < s \text{ OR } \exists w \in K_2 \text{ s.t. } d(z, w) < s\}
\]

\[
= \{z : \exists w \in K_1 \text{ s.t. } d(z, w) < s\} \cup \{z : \exists w \in K_2 \text{ s.t. } d(z, w) < s\}
\]

\[
= T_s(K_1) \cup T_s(K_2)
\]

Lemma 2.7. \(T_s(K_1 \cap K_2) \subset T_s(K_1) \cap T_s(K_2)\).
Proof. This follows from the definition of intersection:
\[
T_s(K_1 \cup K_2) = \{z : \exists w \in K_1 \cap K_2 \text{ s.t. } d(z, w) < s\}
\]
\[
= \{z : \exists w \in K_1 \text{ AND } K_2 \text{ s.t. } d(z, w) < s\}
\]
\[
\subseteq \{z : \exists w \in K_1 \text{ s.t. } d(z, w) < s\} \cap \{z : \exists w \in K_2 \text{ s.t. } d(z, w) < s\}
\]
\[
= T_s(K_1) \cap T_s(K_2)
\]
\[
\square
\]

Lemma 2.8. In $E^2$ we have the following
\[
T_s([a, b] \times y) \subset [a - s, b + s] \times [y - s, y + s]
\]
\[
T_s([x] \times [a, b]) \subset [x - s, x + s] \times [a - s, b + s].
\]

Proof. Let $(p_1, p_2) \in T_s([a, b] \times [y])$. Then $\exists (q_1, q_2) \in [a, b] \times [y]$ such that
\[
d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} < s.
\]
Hence $|p_1 - q_1| < s$ and $|p_2 - q_2| < s$. Therefore $p \in [a - s, b + s] \times [y - s, y + s]$, as required. A similar argument shows $T_s([x] \times [a, b]) \subset [x - s, x + s] \times [a - s, b + s]$. □

2.2. Pulled Thread Spaces. Before introducing a smocked metric space we recall the notion of a pulled thread metric space, particularly in the setting where one starts with a Euclidean space, $E^N$. The idea is that, if one views a Euclidean plane as a cloth, marks an interval on that cloth, sews along that interval, and pulls the thread tight, then one obtains a new metric space (called a pulled string space) in which the interval is now a point. See Figure 2.

![Figure 2](image.png)

Figure 2. The distance between two points in a pulled thread space is the minimum of the length of the direct path and the sum of the lengths of a pair of segments touching the interval.

Definition 2.9. Given a Euclidean space, $E^N$, and an interval, $I$, one can define the pulled thread metric space, $(X, d)$, in which the interval is viewed as a single point.
\[
X = \{x : x \in E^N \setminus I \} \cup \{I\}
\]
We have a pulled thread map $\pi : E^N \rightarrow X$ such that $\pi(x) = x$ for $x \in E^N \setminus I$ and $\pi(x) = I$ for $x \in I$. The distance function, $d : X \times X \rightarrow [0, \infty)$, is defined by
\[
d(x, y) = \min\{|x - y|, \min\{|x - z| + |z' - y| : z, z' \in I\}\} \quad \text{for } x, y \in E^N
\]
\[
d(x, I) = \min\{|x - z| : z \in I\} \quad \text{for } x \in E^N
\]
We can then define the pulled thread pseudometric $\tilde{d} : E^N \times E^N \rightarrow [0, \infty)$ to be
\[
\tilde{d}(x, y) = d(\pi(x), \pi(y)) = \min\{|x - y|, \min\{|x - z| + |z' - y| : z, z' \in I\}\}.
\]
Finally we define the distance to the interval, \( D : \mathbb{B}^N \to [0, \infty) \):

\[
D(x) = \min\{|x - z| : z \in I\} = d(\pi(x), I).
\]

**Notation:** For any \( p \in X \), \( p \neq I \), write \( z_p \) to denote the unique point in \( I \) such that \( |p - z_p| = d(p, I) \).

**Lemma 2.10.** A pulled thread space is a metric space.

**Proof.** This has three parts:

**Positive definiteness:** \( d(p, q) \geq 0 \) and \( d(p, q) = 0 \iff p = q \):

The fact that \( d(p, q) \geq 0 \) for all \( p, q \in X \) is clear from Definition 2.9. Suppose \( d(p, q) = 0 \) then \( p = \pi(v) \) and \( q = \pi(w) \) where \( d(v, w) = 0 \). Then either

\[
|v - w| = 0 \iff v = w \iff p = q
\]

or there exists \( z, z' \in I \) such that

\[
|v - z| + |z' - w| = 0
\]

which implies \( v = z \) and \( z' = w \) so \( v, w \in I \) so

\[
p = \pi(v) = \pi(w) = q.
\]

**Symmetry** \( d(p, q) = d(q, p) \):

This follows by taking \( p = \pi(x) \) and \( q = \pi(y) \) and noting that

\[
d(p, q) = \min\{|x - y|, \min\{|x - z| + |z' - y| : z, z' \in I\}
\]

\[
= \min\{|y - x|, \min\{|y - z| + |z' - z' : z, z' \in I\} = d(q, p).
\]

**Triangle inequality** \( d(a, b) \leq d(a, q) + d(q, b) \):

We prove the triangle inequality in cases using the notation that for any \( p \in X \) there exists \( v_p \in \mathbb{B}^N \) such that \( p = \pi(v_p) \) and if \( p \neq I \) then there exists \( z_p \in I \) such that

\[
d(p, I) = |v_p - z_p|.
\]

**Case I:** We assume \( a \in \pi(X \setminus \{I\}) \) and \( b = I \) (which also includes \( a \) and \( b \) switched by symmetry). This breaks into two cases:

**Case I.a:** We assume \( q = I \). This implies \( d(b, q) = 0 \) and \( d(a, q) = d(b, q) \) so

\[
d(a, b) = d(a, q) + d(q, b)
\]

**Case I.b:** We assume \( q \neq I \). This breaks into two deeper cases

**Case I.b.i:** We assume \( d(a, q) = |a - q| \) which implies

\[
d(a, b) = |v_a - z_a| < |v_a - z_q| \leq |v_a - v_q| + |v_q - z_q| = d(a, q) + d(q, b)
\]

**Case I.b.ii:** We assume \( d(a, q) = |v_a - z_a| + |v_q - z_q| \) which implies

\[
d(a, b) = |v_a - z_a| < |v_a - z_q| + 2|v_q - z_q| = d(a, q) + d(q, b)
\]

**Case II:** We assume \( a, b \in \pi(X \setminus \{I\}) \). This breaks into two cases:

**Case II.a:** We assume \( q = I \) which implies

\[
d(a, b) \leq |v_a - z_a| + |v_b - z_b| = d(a, q) + d(q, b)
\]

**Case II.b:** We assume \( q \neq I \). This breaks into four deeper cases:
Case II.b.i: We assume \( d(a, q) = |v_a - z_a| + |v_q - z_q| \) and \( d(q, b) = |v_b - z_b| + |v_q - z_q| \).

(24) 
So \( d(a, q) \leq |v_a - z_a| + |v_b - z_b| \)
(25) 
\( \leq |v_a - z_a| + |v_b - z_b| + 2|v_q - z_q| \)
(26) 
\( = d(a, q) + d(q, b) \).

Case II.b.ii: We assume \( d(a, q) = |v_a - v_q| \) and \( d(q, b) = |v_b - z_b| + |v_q - z_q| \).

(27) 
So \( d(a, b) \leq |v_a - z_a| + |v_b - z_b| \)
(28) 
\( \leq |v_a - v_q| + |v_q - z_q| + |v_b - z_b| \)
(29) 
\( = d(a, q) + d(q, b) \).

Case II.b.iii: We assume \( d(a, q) = |v_a - z_a| + |v_q - z_q| \) and \( d(q, b) = |v_b - v_q| \).

(30) 
So \( d(a, b) \leq |v_a - z_a| + |v_b - z_b| \)
(31) 
\( \leq |v_a - z_a| + |v_b - q| + |v_q - z_b| \)
(32) 
\( = d(a, q) + d(q, b) \)

Case II.b.iv: We assume \( d(a, q) = |v_a - v_q| \) and \( d(q, b) = |v_b - v_q| \).

(33) 
So \( d(a, b) \leq |v_a - v_b| \leq |v_a - v_q| + |v_b - v_q| = d(a, q) + d(q, b) \).

Remark 2.11. More generally pulled thread spaces can be defined starting with any geodesic metric space or length space. See Burago-Burago-Ivanov’s textbook [1].

Remark 2.12. Note that there is no particular reason for \( I \) to be a closed interval. It might be any compact set. But the intuition of a pulled thread is that the \( I \) is an interval.

2.3. Balls in Pulled Thread Spaces. Here we prove in three lemmas that the balls in pulled thread spaces have the form depicted in Figure 3.

![Figure 3](image-url)
Lemma 2.13. In a pulled thread space as in Definition 2.9

\( \pi^{-1}(B_r(I)) = T_r(I) \).

*Proof.* Note that \( v \in T_r(I) \) if and only if

\[ \exists z \in I \text{ such that } |v - z| < r. \]

By Definition 2.9, this is true if and only if

\[ d(\pi(v), \pi(I)) = \min(|v - z'| : z' \in I) < r. \]

This is true if and only if \( \pi(v) \in B_r(\pi(I)) \) which is true if and only if \( v \in \pi^{-1}(B_r(I)) \). \( \Box \)

Lemma 2.14. In a pulled thread space as in Definition 2.9

\( \pi^{-1}(B_r(\pi(x))) = B_r(x) \forall r \leq D(x). \)

*Proof.* If \( v \in \pi^{-1}(B_r(\pi(x))) \) then \( \pi(v) \in B_r(\pi(x)) \). Thus by Definition 2.9

\[ \min(|v - x|, |v - w| + |w' - x| : w, w' \in I) = d(\pi(v), \pi(x)) < r < D(x). \]

Since \( |w' - x| \geq D(x) \), we see that

\[ |v - x| = d(\pi(v), \pi(x)) < r \]

which implies that \( v \in B_r(x) \).

On the other hand, if \( v \in B_r(x) \), then \( |v - x| < r \). So

\[ d(\pi(v), \pi(x)) = \min(|v - x|, |v - w| + |w' - x| : w, w' \in I) \leq |v - x| < r \]

which implies that \( \pi(v) \in \pi^{-1}(B_r(\pi(x))) \). \( \Box \)

Lemma 2.15. In a pulled thread space as in Definition 2.9 if \( D(x) = h \) then

\( \pi^{-1}(B_{h+s}(x)) = B_{h+s}(x) \cup T_s(I) \forall s > 0. \)

*Proof.* Observe,

\[ v \in \pi^{-1}(B_{h+s}(\pi(x))) \]

\[ \iff \pi(v) \in B_{h+s}(\pi(x)) \]

\[ \iff d(\pi(v), \pi(x)) < h + s \]

\[ \iff \min(|v - x|, \min(|v - z_1| + |z_2 - x| : z_1, z_2 \in I}) < h + s \]

\[ \iff |v - x| < h + s \text{ OR } \exists z_1, z_2 \in I \text{ s.t. } |v - z_1| + |z_2 - x| < h + s \]

Note: we can take \( z_2 \) closest to \( x \) so \( |z_2 - x| = D(x) = h \). Thus

\[ v \in \pi^{-1}(B_{h+s}(\pi(x))) \]

\[ \iff v \in B_{h+s}(x) \text{ OR } \exists z_1 \in I \text{ s.t. } |v_1 - s| < s \]

\[ \iff v \in B_{h+s}(x) \text{ OR } v \in T_s(I) \]

\[ \iff v \in B_{h+s}(x) \cup T_s(I). \]
Proposition 2.16. In a pulled thread space as in Definition 2.9, if the length of the interval is \( L > 0 \), then
\[
B_r(x) \subset \pi^{-1}(B_r(\pi(x))) \subset B_{r+L}(x).
\]

**Proof.** We see that \( B_r(x) \subset \pi^{-1}(B_r(\pi(x))) \) because
\[
w \in B_r(x) \implies r > |w - x| \implies r > d(\pi(w), \pi(x)) \implies \pi(w) \in (B_r(\pi(x))).
\]
We see that \( \pi^{-1}(B_r(\pi(x))) \subset B_{r+L}(x) \) because
\[
v \in \pi^{-1}(B_r(\pi(x))) \implies \pi(v) \in B_r(\pi(x)) \implies d(\pi(v), \pi(x)) < r.
\]

Case 1: \( d(\pi(v), \pi(x)) = |v - x| \).
\[
\implies |v - x| < r \implies v \in B_r(x) \subseteq B_{r+L}(x).
\]

Case 2: \( d(\pi(v), \pi(x)) = |v - z_1| + |z_2 - x| \) for some \( z_1, z_2 \in I \)
\[
\implies |v - x| \leq |v - z_1| + |z_1 - z_2| + |z_2 - x| \leq r + L \implies v \in B_{r+L}(x).
\]

\[
□
\]

3. **Introducing Smocked Spaces**

We now introduce a new notion called a smocked space. In sewing there is a technique called smocking which is used to add texture to a cloth. See for example [7] for a few patterns and search “Canadian Smocking” in Pinterest for many more. To create such a smocked cloth, the seamstress follows a pattern. In Figure 1 we presented some such patterns. Each interval (or stitch) marked in black is sewn by a thread and pulled to a point. Sometimes the stitches are squares or plus signs. There are many other standard smocking patterns which are also periodic and many more which are not periodic.

In this section we rigorously define the metric space created from a plane by pulling every stitch in a smocking pattern to a point. We then describe the six patterns and their smocked spaces each on their own subsection along with graphics. It is recommended that the reader glance into these subsections while reading the definition.

3.1. **The Definition of a Smocked Space.**

**Definition 3.1.** Given a Euclidean space, \( \mathbb{R}^N \), and a finite or countable collection of disjoint connected compact sets called *smocking intervals* or *smocking stitches*,
\[
\mathcal{I} = \{ I_j : j \in J \},
\]
with a positive **smocking separation factor**,
\[
\delta = \min\{|z - z'| : z \in I_j, z' \in I_{j'}, j \neq j' \in J\} > 0,
\]
one can define the **smocked metric space**, \( (X, d) \), in which each stitch is viewed as a single point.
\[
X = \{ x : x \in \mathbb{R}^N \setminus S \} \cup \mathcal{I}
\]
where $S$ is the smocking set or smocking pattern:

\begin{equation}
S = \bigcup_{j \in J} I_j.
\end{equation}

We have a smocking map $\pi : \mathbb{B}^N \to X$ defined by

\begin{equation}
\pi(x) = \begin{cases} 
  x & \text{for } x \in \mathbb{B}^N \setminus S \\
  I_j & \text{for } x \in I_j \text{ and } j \in J.
\end{cases}
\end{equation}

The smocked distance function, $d : X \times X \to [0, \infty)$, is defined for $y, x \notin \pi(S)$, and stitches $I_m$ and $I_k$ as follows:

\begin{align*}
d(x, y) &= \min \{d_0(x, y), d_1(x, y), d_2(x, y), d_3(x, y), \ldots \} \\
d(x, I_k) &= \min \{d_0(x, z), d_1(x, z), d_2(x, z), d_3(x, z), \ldots : z \in I_k \} \\
d(I_m, I_k) &= \min \{d_0(z', z), d_1(z', z), d_2(z', z), d_3(z', z), \ldots : z' \in I_m, z \in I_k \}
\end{align*}

where $d_k$ are the sums of lengths of segments that jump to and between $k$ stitches:

\begin{align*}
d_0(v, w) &= |v - w| \\
d_1(v, w) &= \min \{|v - z_1| + |z'_1 - w| : z_1, z'_1 \in I_{j_1}, j_1 \in J \} \\
d_2(v, w) &= \min \{|v - z_1| + |z'_1 - z_2| + |z'_2 - w| : z_i, z'_i \in I_{j_i}, j_1 \neq j_2 \in J \} \\
d_k(v, w) &= \min \{|v - z_1| + \sum_{i=1}^{k-1} |z'_i - z_{i+1}| + |z'_k - w| : z_i, z'_i \in I_{j_i}, j_1 \neq \cdots \neq j_k \in J \}
\end{align*}

We define the smocking pseudometric $\tilde{d} : \mathbb{B}^N \times \mathbb{B}^N \to [0, \infty)$ to be

\begin{equation}
\tilde{d}(v, w) = d(\pi(v), \pi(w)) = \min \{d_k(v', w') : \pi(v) = \pi(v'), \pi(w) = \pi(w'), k \in \mathbb{N} \}.
\end{equation}

We will say the smocked space is \textbf{parametrized by points in the stitches}, if

\begin{equation}
J \subset \mathbb{B}^N \quad \text{and} \quad \forall j \in J \quad j \in I_j.
\end{equation}

In Theorem 3.3 below, we will prove the minima are achieved and that the smocking spaces are metric spaces. In fact, we will see that the distances on these spaces can be determined by explicitly finding the shortest path of segments $z'_j$ to $z_j$ between a given pair of points. In Figure 4 we see such paths in one smocked space. If such a path has a single segment and jumps through no intervals, then $d(x, y) = d_0(x, y)$. If the path jumps through one interval, then $d(x, y) = d_1(x, y)$, and if the path jumps through $k$ intervals with $k+1$ segments, then $d(x, y) = d_k(x, y)$.

\textbf{Remark 3.2.} Note that often the $I_k$ are closed intervals. As one can see in the introduction, in some smocking patterns the stitches are replaced with squares or plus signs or wedges.

\subsection*{3.2. A Smocked Space is a Metric Space.}

\textbf{Theorem 3.3.} The smocked metric space is a well defined metric space and for any $v, w \in \mathbb{B}^N$

\begin{equation}
\exists N(v, w) \leq |v - w|/\lambda \quad \text{s.t.} \quad d_{N(v, w)}(v, w) \leq d_k(v, w) \quad \forall k \geq \mathbb{N},
\end{equation}
so the minimum in the definition of the smocking distance and pseudometric is achieved

\[ \bar{d}(v, w) = d_N(v, w) \text{ and } d(\pi(v), \pi(w)) = d_N(v, w). \]

**Proof.** Initially we consider all minima in the definition of a smocked space to be infima.

First, we fix \( v, w \in \mathbb{E}^N \), and prove \[ d_k(v, w) \geq k\delta \]
where \( \delta \) is the separation factor defined in (73) because each \( |z'_i - z_i| \geq \delta \) in the definition of \( d_k \). Thus

\[ d_0(v, w) = |v - w| \leq d_k(v, w) \geq N' = \lceil|v - w|/\lambda\rceil. \]

So we need only choose \( N_{v,w} \) such that

\[ d_{N_{v,w}}(v, w) = \min\{d_0(v, w), ..., d_{N'}(v, w)\}. \]

Now examine

\[ d_k(v, w) = \inf\{|v - z_1| + \sum_{i=1}^k |z'_i - z_i| + |z'_k - w| : z_i, z'_i \in I_{ji}, j_1 \neq \cdots \neq j_k \in J\}. \]

Since \( d_k(v, w) \leq |v - w| \), one need not consider smocking stitches such that \( I_j \cap B(v, |v - w|) = \emptyset \). Since the smocking stitches are a definite distance \( \lambda > 0 \) apart, there are only finitely many smocking stitches such that \( I_j \cap B(v, |v - w|) \neq \emptyset \). Since each smocking stitch is compact, the infima over choices of \( z_i \) and \( z'_i \) is also achieved. Thus this infimum is a minimum as well.

Now we prove \( d \) is symmetric. First observe that for any \( v, w \in \mathbb{E}^N \)

\[ d_k(v, w) = d_k(w, v) \]

because we can reverse the order of the segments:

\[ |v - z_1| + \sum_{i=1}^k |z'_i - z_i| + |z'_k - w| = |w - z'_1| + \sum_{i=1}^k |z'_i - z_i| + |z_1 - v|. \]

Taking the minimum of symmetric \( d_k \) we have \( \bar{d}(v, w) = \bar{d}(w, v) \) and thus

\[ d(\pi(v), \pi(w)) = d(\pi(w), \pi(v)). \]
This suffices to prove symmetry since the smocking map \( \pi \) is surjective.

To see that \( d \) is definite, we again take advantage of the fact that the smocking map is surjective, and refer to the points in \( X \) as \( \pi(v) \) and \( \pi(w) \). By (53),

\[
d(\pi(v), \pi(w)) = 0 \iff |v - w| = d_0(v, w) = d(v, w) = 0 \iff v = w.
\]

Note that \( \bar{d}(v, w) = 0 \) can happen when \( v, w \in I_j \), which is why \( \bar{d} \) is only a pseudometric.

To see that \( d \) and \( \bar{d} \) satisfy the triangle inequality, consider \( y, v, w \in \mathbb{B}^N \). By the achievement of the minima, we know there exists stitches, \( I_{j_1}, \ldots, I_{j_k} \) where \( k = N(y, v) \) and \( z_i, z'_i \in I_{j_i} \) such that

\[
\bar{d}(y, v) = |y - z_1| + \sum_{i=1}^{k} |z'_i - z_i| + |z'_k - v|
\]

and more stitches, \( I_{j_{k+1}}, \ldots, I_{j_{k+k'}} \) where \( k' = N(v, w) \) and \( z_i, z'_i \in I_{j_i} \) such that

\[
\bar{d}(v, w) = |v - z_{k+1}| + \sum_{i=k+1}^{k+k'} |z'_i - z_i| + |z_{k+k'} - w|.
\]

Adding these together and using the fact that

\[
|z'_k - v| + |v - z_{k+1}| \geq |z'_k - z_{k+1}|,
\]

we have

\[
\bar{d}(y, v) + \bar{d}(v, w) \geq |y - z_1| + \sum_{i=1}^{k+k'} |z'_i - z_i| + |z_{k+k'} - w| \geq \bar{d}(y, w).
\]

Applying the surjective smocking map we see \( d \) satisfies the triangle inequality. \( \square \)

### 3.3. The Diamond Smocked Space \( X_\diamond \)

In this section we study one of the most classic smocking patterns: diamond smocking (also called honeycomb smocking although there are no hexagons). The pattern used to create diamond smocking is depicted in Figure 5. It only appears to have diamonds after sewing the threads tight as seen in the same figure.

![Figure 5](image.png)

**Figure 5.** The classical diamond smocking pattern will be used to define the smocking stitches of \((X_\diamond, d_\diamond)\).

We need to define the smocking stitches to define this space rigorously:
Definition 3.4. Our metric space \((X_\diamond, d_\diamond)\) is a smocked plane defined as in Definition 3.1. We start with the Euclidean plane \(\mathbb{E}^2\). We define our index set (which will also be the center points of our stitches):

\[ J_\diamond = \{(j_1, j_2) : j_1 = 2n_1 - n_2, j_2 = n_2 \text{ where } n_1, n_2 \in \mathbb{Z}\} = \{(0, 0), (\pm 2, 0), (\pm 1, \pm 1), (0, \pm 2), (\pm 4, 0), (\pm 3, \pm 1), \ldots\}. \]  

\((65)\)

We define our stitches (which are all horizontal of unit length):
\[
I_{(j_1, j_2)} = [j_1 - 1/2, j_1 + 1/2] \times \{j_2\}.
\]

See Figure [5]

3.4. The Ribbed Smocked Space \(X_=\). In this section we study ribbed smocking, which is used to create fabric with a ribbed texture. The pattern used to create ribbed smocking is depicted in Figure [6]. It only appears to be ribbed after sewing the threads tight as seen in the same figure.

Let us begin by describing the stitches which will be used to create the smocked plane.

Definition 3.5. Our metric space \((X_=, d_=)\) is a smocked plane defined as in Definition 3.1. We start with the Euclidean plane \(\mathbb{E}^2\). We define our index set:

\[ J_= = 2\mathbb{Z} \times \mathbb{Z}. \]

We define our stitches: if \((j_1, j_2) \in J_=\), then
\[
I_{(j_1, j_2)} = [j_1 - 0.5, j_1 + 0.5] \times \{j_2\}.
\]

See Figure [6]

3.5. The Woven Smocked Space \(X_T\). In this section we study weaved smocking, which is used to create fabric with a basket weave texture. The pattern used to create weaved smocking is depicted in Figure [7]. In the same figure we have sewn the threads to a point in a cloth. In traditional smocking only the endpoints of these stitches are joined and then the pattern looks like like a garden lattice. However, here we have sewn each entire stitch to a point.

Let us begin by describing the stitches which will be used to create the smocked plane.
Definition 3.6. Our metric space \((X_T, d_T)\) is a smocked plane defined as in Definition 3.1. We start with the Euclidean plane \(\mathbb{E}^2\). We define our index set (which will also be the center points of our stitches):

\[
J_T = \{(j_1, j_2) : j_1 = 2n_1, j_2 = 2n_2, \ n_1, n_2 \in \mathbb{Z}\} = 2\mathbb{Z} \times 2\mathbb{Z}.
\]

We define our horizontal stitches (of length 2)

\[I_{(j_1, j_2)} = [j_1 - 1, j_1 + 1] \times \{j_2\} \text{ when } (j_1 + j_2)/2 \text{ is even}\]

and our vertical stitches (of length 2)

\[I_{(j_1, j_2)} = \{j_1\} \times [j_2 - 1, j_2 + 1] \text{ when } (j_1 + j_2)/2 \text{ is odd}.
\]

See Figure 7.

3.6. The Flower Smocked Space \(X_+\). In this section we study flower smocking, which is used to create fabric with flowers. The pattern used to create flower smocking is depicted in Figure 8. Each stitch appears to be a flower once it is sewn as seen in the same figure. Note that the smocking stitches here are compact sets formed by overlapping pairs of stitches.

Definition 3.7. Our metric space \((X_+, d_+)\) is a smocked plane defined as in Definition 3.1. We start with the Euclidean plane \(\mathbb{E}^2\). We define our index set (which will also be the center points of our stitches):

\[
J_+ = \{(j_1, j_2) : j_1 = 3n_1, j_2 = 3n_2, \ n_1, n_2 \in \mathbb{Z}\}.
\]
We define our stitches (which are + shapes)

\[ I_{(j_1, j_2)} = ([j_1 - 1, j_1 + 1] \times \{j_2\}) \cup ([j_1] \times [j_2 - 1, j_2 + 1]) \]

See Figure 8

3.7. The Checkered Smocked Space \( X_H \). In this section we study checkered smocking, which is used to create fabric with alternating diamonds puffed up and down. The pattern used to create checkered smocking is depicted in Figure 9. It only appears to be checkered after sewing the threads tight as seen in the same figure.

Let us begin by describing the stitches which will be used to create the smocked plane \( X_H \).

**Definition 3.8.** Our metric space \((X_H, d_H)\) is a smocked plane defined as in Definition 3.1. We start with the Euclidean plane \( \mathbb{E}^2 \). We define our index set in two parts:

\begin{align*}
J_H &= J_H^- \cup J_H^+ \\
\text{where} \\
J_H^- &= 3\mathbb{Z} \times 3\mathbb{Z} \\
\text{and} \\
J_H^+ &= (3\mathbb{Z} + 1.5) \times (3\mathbb{Z} + 1.5) .
\end{align*}

We define our stitches: if \((j_1, j_2) \in J_H^-\), \(I_{(j_1, j_2)}\) is the horizontal segment

\[ I_{(j_1, j_2)} = [j_1 - 0.5, j_1 + 0.5] \times \{j_2\} \]

and if \((j_1, j_2) \in J_H^+\), \(I_{(j_1, j_2)}\) is the vertical segment

\[ I_{(j_1, j_2)} = \{j_1\} \times [j_2 - 0.5, j_2 + 0.5] .
\]

See Figure 9
3.8. **The Bumpy Smocked Space** $X_{\square}$. In this section we study bumpy smocking, which is used to create fabric with a bumpy texture. The pattern used to create bumpy smocking is depicted in Figure 10. It only appears to be bumpy after sewing the threads tight as seen in the same figure.

Let us begin by describing the stitches which will be used to create the smocked plane.

**Definition 3.9.** Our metric space $(X_{\square}, d_{\square})$ is a smocked plane defined as in Definition 3.1. We start with the Euclidean plane $\mathbb{R}^2$. We define our index set (which will also be lower left corners of our squares):

$$J_{\square} = \{(j_1, j_2) : j_1 = 3n_1, j_2 = 3n_2, n_1, n_2 \in \mathbb{Z}\}.$$  

We define our stitches (which are squares of unit side lengths):

$$I_{(j_1, j_2)} = ([j_1, j_1 + 1] \times [j_2]) \cup ([j_1, j_1 + 1] \times [j_2 + 1]) \cup ([j_1 + 1] \times [j_2, j_2 + 1]) \cup ([j_1 + 1] \times [j_2, j_2 + 1]).$$

See Figure 10.

4. **The Smocking Constants: Depth, Lengths, and Separation Factor**

In this section we describe some essential parameters that we can later use to estimate the balls in smocked spaces and their distance functions. After providing the definitions we have subsections finding their value for the six smocked spaces that we have just defined above.

4.1. **Defining Depth and Lengths.** Given a smocked space, $(X, d)$, as in Definition 3.1 with smocking stitches $\{I_j : j \in J\}$ and smocking set $S = \bigcup_{j \in J} I_j$ we make the following definitions:

**Definition 4.1.** $D : \mathbb{R}^N \rightarrow [0, \infty)$, the distance of a point $x \in \mathbb{R}^N$ to the smocking set is defined to be

$$D(x) = \min\{|x - z| : z \in I_j, j \in J\}.$$  

**Lemma 4.2.** The minimum in Definition 4.1 is achieved.

**Proof.** For any $x \in S$, $D(x) = 0$ and hence is achieved. Now, fix $x \in \mathbb{R}^N \setminus S$. For any $r > 0$, the number of stitches inside $B(x, r)$ is finite. This is because the smocking separation factor $\delta$ is always positive, i.e.

$$\delta = \min\{|z - z'| : z \in I_j, z' \in I_{j'}, j \neq j' \in J\} > 0,$$
and only finitely many balls of radius $\delta$ cover $B(x, r)$, because the closure of $B(x, r)$ is compact.

Let $K \subset J$ be the index set of stitches which intersect $B(x, r)$. For each $k \in K$, let

$$D_k(x) = \inf \{|x - z| : z \in I_k\}.$$ \hfill (74)

There exists $z_{kn} \in I_k$ approaching this infimum, such that $|x - z_{kn+1}| < |x - z_{kn}|$ for all $n$. Since each stitch is compact, $(z_{kn})_n$ has a convergent subsequence, and hence there is a point $z_k \in I_k$ that achieves the infimum for $D_k(x)$. Then, $D(x) = \min\{D_k(x) : k \in K\}$ is achieved since $K$ is finite. \hfill $\square$

**Definition 4.3.** The smocking depth, $h$, is defined to be

$$h = \inf \{r : E^N \subset T_r(S)\} \in [0, \infty)$$ \hfill (75)

which, by definition of tubular neighborhood, is

$$h = \inf \{r : \forall x \in X \exists j \in J \exists z \in I_j \text{ s.t. } |x - z| < r\}.$$ \hfill (76)

**Lemma 4.4.** If the smocking depth, $h$, is finite, then

$$h = \sup\{D(x) : x \in E^N\}.$$ \hfill (77)

**Proof.** Since the smocking depth is finite, there exists $r \in (0, \infty)$ such that $E^N \subset T_r(S)$. So

$$D(x) \leq r \quad \forall x \in E^N,$$ \hfill (78)

since, if $D(x) > r$ for some $x \in E^N$, i.e. if the minimum distance of $x$ to any stitch is strictly more than $r$, then $x \notin T_r(S)$, by definition of tubular neighborhood.

Then, taking the supremum of the left side and infimum of the right side of inequality (78), we have

$$\sup\{D(x) : x \in E^N\} \leq \inf\{r : E^N \subset T_r(S)\}.$$ \hfill (79)

Let $r' = \sup\{D(x) : x \in E^N\}$. Then, $E^N \subset T_{r'}(S)$, since

$$\forall x \in E^N, \exists j \in J, \exists z \in I_j, \text{ such that } |x - z| = D(x) \leq r'.$$ \hfill (80)

Therefore it is not possible that $r' < r$, $r$ being the infimum. Thus $r' = h$, as defined in Definition 4.3. \hfill $\square$

**Definition 4.5.** The smocking lengths are defined either using the lengths of intervals

$$L_{\text{min}} = \inf\{L(I_j) : j \in J\} \in [0, \infty)$$ \hfill (81)

$$L_{\text{max}} = \sup\{L(I_j) : j \in J\} \in (0, \infty].$$ \hfill (82)

If $L_{\text{min}} = L_{\text{max}}$, we call this the smocking length. If the smocking stitches are not intervals we replace length with diameter in the above.

**Definition 4.6.** The smocking separation factor, $\delta = \delta_X$, is defined to be

$$\delta_X = \min\{|z - w| : z \in I_j, w \in I_k, j \neq k \in J\}.$$ \hfill (83)
Lemma 4.7. If a smocked space is parametrized by points in stitches as in (50), then

\[ \mathbb{E}^N \subset T_{h+L}(S) \]

where \( S = \bigcup_{j \in J} I_j \) is the smocking set, \( h \) is the smocking depth, and \( L = L_{\text{max}} \) is the maximum smocking length.

Proof. Given any \( x \in \mathbb{E}^N \), by the definition of smocking depth, we have a closest point in a closest stitch, \( z \in I_j \), such that

\[ d(x, I_j) = \bar{d}(x, z) = |x - z| \leq h. \]

Since our smocked space is parametrized by points in stitches, we have

\[ |z - j| \leq L. \]

So

\[ |x - j| \leq L + h. \]

Thus \( x \in T_{L+h}(S) \).

4.2. The Smocking Constants of \( X_\diamond \). Here we find the smocking constants for the Diamond Smocking Space, see Subsection 4.

Lemma 4.8. The smocking depth

\[ h_\diamond = \inf \{ r : \mathbb{E}^N \subset T_r(S) \} \in [0, \infty] = \frac{5}{8}. \]

Proof. Note that

\[ \mathbb{E}^2 = \{ [j_1 - \frac{1}{2}, j_1 + \frac{1}{2}] \times [j_2 - 2, j_2] : j \in J \}, \]

i.e. the rectangles

\[ R_j = [j_1 - \frac{1}{2}, j_1 + \frac{1}{2}] \times [j_2 - 2, j_2] \]

tile the Euclidean plane. Moreover, the rectangles \( R_j \) have a reflective symmetry along the line connecting \((j_1 - 1, j_2 - 1)\) and \((j_1 + 1, j_2 - 1)\). Therefore, it is enough to consider the square

\[ Q_j = [j_1 - \frac{1}{2}, j_1 + \frac{1}{2}] \times [j_2 - 1, j_2]. \]

The rectangle \( R_j \) and the square \( Q_j \) are shown in Figure 11. The tubular neighborhoods of radius \( r \) of the diamond smocked space for four of the stitches is shown in Figure 12.

The boundaries of the tubular neighborhoods which lie in \( Q_j \) are

1. a length one horizontal segment from the tubular neighborhood of \( I_{(j_1, j_2)} \), distance \( r \) below it,
2. a quarter of circle right of \( I_{(j_1 - 1, j_2 - 1)} \) of radius \( r \), and
3. a quarter of circle left of \( I_{(j_1 + 1, j_2 - 1)} \) of radius \( r \).
The smallest $r$ for which the tubular neighborhoods of the stitches cover $Q_j$ occurs when these boundary pieces intersect at a point $B$ as shown in Figure 13. This is because, otherwise, the point $B$ would not be covered.

Consider the triangle $OAB$ in figure 13, for which $AB = \frac{1}{2}L = \frac{1}{2}$, $OA = 1 - r$, and $OB = r$. By Pythagorean Theorem,

\begin{equation}
0 = \left(\frac{1}{2}\right)^2 + (1 - r)^2 - r^2 = \frac{1}{4} + 1 - 2r + r^2 - r^2 = \frac{5}{4} - 2r.
\end{equation}
Therefore, $r = \frac{5}{8}$ is the smallest $r$ such that
\begin{equation}
R_j \subset T_r(I_{(j_1, j_2)}) \cup T_r(I_{(j_1+1, j_2-1)}) \cup T_r(I_{(j_1-1, j_2+1)}).
\end{equation}

\[\square\]

**Lemma 4.9.** The smocking lengths are
\begin{align*}
L_{\min}^o &= \inf \{L(I_j) : j \in J_o \} = 1 \\
L_{\max}^o &= \sup \{L(I_j) : j \in J^o \} = 1,
\end{align*}
so the smocking length is 1.

**Proof.** All our stitches $I_j = \{j_1 - 1/2, j_1 + 1/2\} \times \{j_2\}$ have length
\begin{equation}
(j_1 + 1/2) - (j_1 - 1/2) = 1.
\end{equation}

\[\square\]

**Lemma 4.10.** The smocking separation factor is
\begin{equation}
\delta_o = 1
\end{equation}

**Proof.** Recall
\begin{equation}
\delta_o = \min \{|z - w| : z \in I_j, w \in I_k, j \neq k \in J\}.
\end{equation}
If we take
\begin{equation}
z_0 = (1/2, 0) \in I_{(0,0)} \in J_o
\end{equation}
and
\begin{equation}
w_0 = (1/2, 1) \in I_{(1,1)} \in J_o,
\end{equation}
then we see that
\begin{equation}
\delta_o \leq |z_0 - w_0| = 1.
\end{equation}
On the other hand, taking any $j \neq k$ we consider the following cases:

Case 1: $j_2 \neq k_2$. Then, by the definition of $J_o$, $|j_2 - k_2| \geq 1$. Now any $z \in I_j$, $w \in I_k$ has
\begin{equation}
|z_2 - w_2| = |j_2 - k_2| \geq 1,
\end{equation}
by the definitions of our stitches. Thus,
\begin{equation}
|z - w| \geq |z_2 - w_2| = |j_2 - k_2| \geq 1.
\end{equation}

Case 2: $j_2 = k_2$ and so $j_1 \neq k_1$. By the definition of $J_o$, $|j_2 - k_2| \geq 2$. Now any $z \in I_j$, $w \in I_k$ has
\begin{equation}
|z_1 - w_1| \geq 2 - (1/2) - (1/2) = 1.
\end{equation}
by the definitions of our stitches. Thus
\begin{equation}
|z - w| \geq |z_1 - w_1| \geq 2 - (1/2) - (1/2) = 1.
\end{equation}
Combining our cases, we see the minimum is 1.

\[\square\]
4.3. **The Smocking Constants of** $X_\infty$. Here we find the smocking constants for $X_\infty$, see Subsection 4.

**Lemma 4.11.** The smocking depth is

\[
h_\infty = \inf \{ r : \mathbb{E}^N \subset T_r(S) \} \in [0, \infty] = \frac{\sqrt{2}}{2}
\]

**Proof.** Let $A = [-\frac{1}{2}, \frac{5}{2}] \times [0, 1] \subset \mathbb{E}^2$. Note that the closest smocking interval to any point in $A$ must be contained in $S' = \{ I_0, I_{(0,1)}, I_{(2,0)}, I_{(2,1)} \} \subset S$. Since $X_\infty$ is invariant under translation by elements in the lattice $2\mathbb{Z} \times \mathbb{Z}$ and any point in $X_\infty$ is contained in some such translation of the set $A$, we have that

\[
h_\infty = \inf \{ r : A \subset T_r(S') \}.
\]

Consider the point $a = (1, \frac{1}{2}) \in A$. We have that $d(a, I_j) = \frac{\sqrt{5}}{2}$ for all $I_j$ in $S'$. Therefore, $h_\infty \geq \frac{\sqrt{5}}{2}$. On the other hand, it is clear that $A \subset T_{\sqrt{2}/2}(S')$. It follows that $h_\infty = \frac{\sqrt{5}}{2}$. \qed

The next two lemmas are very easy to see:

**Lemma 4.12.** The smocking lengths are

\[
L = L_{\min} = L_{\max} = 1.
\]

**Lemma 4.13.** The smocking separation factor is

\[
\delta_\infty = 1.
\]

4.4. **The Smocking Constants of** $X_T$. Here we find the smocking constants for $X_T$, see Subsection 4.

**Lemma 4.14.** The smocking depth of $X_T$ is

\[
h_T = 1.
\]

---

**Figure 14.** The four smocking stitches that intersect with $[0, 2] \times [0, 2]$. 

---
Proof. Our space is invariant under translation by elements in the lattice $2\mathbb{Z} \times 2\mathbb{Z}$. So we need only consider the point with the largest distance from any smocking stitches in $[0, 2] \times [0, 2]$. As one can see in Figure 14, there are four smocking stitches that intersect with this square:

(111) $I_{(0,2)} = \{0\} \times [1, 3]$ and $I_{(2,2)} = [1, 3] \times \{2\}$
(112) $I_{(0,0)} = [-1, 1] \times \{0\}$ and $I_{(2,0)} = \{2\} \times [-1, 1]$.

The point $(1, 1)$ in the center of this square has distance 1 from these stitches, while the other points in the square are each closer to one of these four smocking stitches. In other words, the farthest distance possible is $1 = h_T$. □

Lemma 4.15. The smocking lengths of $X_T$ are

(113) $L^{T}_{\text{min}} = \inf\{L(I_j) : j \in J_T\} = 2$
(114) $L^{T}_{\text{max}} = \sup\{L(I_j) : j \in J_T\} = 2$,

so the smocking length is $L_T = 2$.

Proof. This follows from the fact that each interval $I_{(j_1, j_2)}$ has length 2. □

Lemma 4.16. The smocking separation factor of $X_T$ is

(115) $\delta_T = 1$.

Proof. As above, we can restrict our study to the square in Figure 14. The shortest distance between distinct stitches is realized by neighboring horizontal and vertical intervals, which are all distance 1 apart, achieved by a segment running from the center of one smocking interval to the end point the other smocking interval. □

4.5. The Smocking Constants of $X_+$. Now we will compute the smocking constants for $(X_+, d_+)$, see Subsection 4.

Lemma 4.17. The smocking depth of $(X_+, d_+)$ is

(116) $h_+ = \inf\{r : B^N \subset T_r(S)\} = \sqrt{\frac{5}{2}}$.

Proof. The point $(0, 1) + \left(\frac{1}{2}, \frac{3}{2}\right)$ is distance

(117) $d = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{5}{2}}$

away from each of the four stitches surrounding it. Thus $h_+ \geq d$. Notice that, for any other point in the square

(118) $\square = [0, 3] \times [0, 3]$,

the distance to a stitch is less than $d$. This is because we can partition $\square$ into 8 right triangles, all sharing a vertex at $(0, 1) + \left(\frac{1}{2}, \frac{3}{2}\right)$ and having some leg intersecting one of the 4 surrounding stitches, so that the point $(0, 1) + \left(\frac{1}{2}, \frac{3}{2}\right)$ is the furthest away from the side of the triangle which intersects a stitch. See figure 15. □
Figure 15. The partition of the square $[0, 3] \times [0, 3]$ into 8 triangles.

**Lemma 4.18.** The smocking lengths are

\[
L_{\min}^+ = \inf\{L(I_j) : j \in J_+\} = 2
\]
\[
L_{\max}^+ = \sup\{L(I_j) : j \in J_+\} = 2.
\]

**Proof.** This follows from the fact that the diameter of a $+$ shaped stitch $I_j$ is 2. □

**Lemma 4.19.** The smocking separation factor of $(X_+, d_+)$ is

\[
\delta_+ = 1.
\]

**Proof.** Notice the stitches $I_{(0,0)}$ and $I_{(3,0)}$ are a distance 1 apart and the stitches $I_{(0,0)}$ and $I_{(3,3)}$ are a distance $\sqrt{2^2 + 2^2} = \sqrt{8}$ apart. By the symmetry of $X_+$, $\delta_+ = 1$. □

### 4.6. The Smocking Constants of $X_H$

Here we find the smocking constants for $X_H$, see Subsection 4.

**Lemma 4.20.** The smocking depth

\[
h_H = \inf\{r : \mathbb{B}^2 \subset T_r(S)\} \in [0, \infty] = 1.5.
\]

**Proof.** Note that the point $(0, 1.5)$ is exactly 1.5 far from the four stitches around it, i.e. $I_{(0,0)}$, $I_{(0,3)}$, $I_{(-1.5,1.5)}$, and $I_{(1.5,1.5)}$, so $h_H \geq 1.5$. So now we must show $\mathbb{B}^2 \subset T_{1.5}(J_H)$.

For each $j \in J_-$ let

\[
\Box_j = [j_1 - 1.5, j_1 + 1.5] \times [j_2, j_2 + 3],
\]

which is a 3 by 3 square above $I_j$ which touches $I_{(j_1,j_2+3)}$, $I_{(j_1-1.5,j_2+1.5)}$, or $I_{(j_1+1.5,j_2+1.5)}$.

Let

\[
\mathcal{W}_j = [j_1 - 1.5, j_1 - 0.5] \times [j_2, j_2 + 1]
\]
\[
\Box [j_1 + 0.5, j_1 + 1.5] \times [j_2, j_2 + 1]
\]
\[
\Box [j_1 - 1.5, j_1 - 0.5] \times [j_2 + 2, j_2 + 3]
\]
\[
\Box [j_1 + 0.5, j_1 + 1.5] \times [j_2 + 2, j_2 + 3]
\]

be four smaller unit squares in the corners of $\Box_j$. Notice

\[
\mathbb{B}^2 = \bigcup_{j \in J_H} \Box_j = \bigcup_{j \in J_H} \mathcal{W}_j \sqcup (\Box_j \setminus \Box_j).
\]

Let $(x, y) \in \mathbb{B}^2$. Then there exists $j \in J_H$ such that

\[
(x, y) \in \mathcal{W}_j \sqcup (\Box_j \setminus \Box_j).
\]
Suppose \((x, y) \in \mathbb{H}_j\). Since \(\mathbb{H}_j\) is a disjoint union of four sets, each with diameter \(\sqrt{2}\) and each intersecting \(S\), we know

\[
(126) \quad (x, y) \in T_{\sqrt{2}}(S) \subset T_{1.5}(S).
\]

Suppose \((x, y) \in \square_j - \mathbb{H}_j\). Note that

\[
\square_j - \mathbb{H}_j = [j_1 - 0.5, j_1 + 0.5] \times [j_2, j_2 + 1.5] \\
\quad \cup [j_1 - 0.5, j_1 + 0.5] \times [j_2 + 1.5, j_2 + 3] \\
\quad \cup [j_1, j_1 + 1.5] \times [j_2 + 1, j_2 + 2] \\
\quad \cup [j_1 - 1.5, j_1] \times [j_2 + 1, j_2 + 2].
\]

Therefore \((x, y)\) is in at most a 1.5 distance from one of the stitches \(I_{(j_1, j_2)}, I_{(j_1, j_2+3)}, I_{(j_1-1.5, j_2+3)}, I_{(j_1+1.5, j_2+1.5)}\). Thus

\[
(127) \quad (x, y) \in T_{1.5}(S).
\]

\[\square\]

**Lemma 4.21.** The smocking lengths are

\[
L = L^{H}_{\min} = L^{H}_{\max} = 1.
\]

**Proof.** For each \(j \in J\) we have \(L(I_j) = 1\). \(\square\)

**Lemma 4.22.** The smocking separation factor is

\[
(129) \quad \delta_{H} = \sqrt{2}.
\]

**Proof.** By the translational and reflective symmetry of the H-smocking pattern,

\[
(130) \quad \delta_{H} = \min\{|z - w| : z \in I_0, w \in I_j, j \in (J \setminus \{0\}) \cap \mathbb{R}^2_{\geq 0}\}.
\]

Suppose \(j \in J^0_H \cap \mathbb{R}^2_+\). Then

\[
\min\{|v - w| : v \in I_0, w \in I_j\} = |(0.5, 0) - (j_1 - 0.5, j_2)| \geq |(0.5, 0) - (3 - 0.5, 3)| = \sqrt{13}.
\]

Suppose \(j \in J^0_H \cap \mathbb{R}^2_+\). Then

\[
\min\{|v - w| : v \in I_0, w \in I_j\} = |(0.5, 0) - (j_1, j_2 - 0.5)| \geq |(0.5, 0) - (1.5, 1.5 - 0.5)| = \sqrt{2}.
\]

Suppose \(j \in J^0_H \cap (\{0\} \times \mathbb{R}_+\). Then

\[
\min\{|v - w| : v \in I_0, w \in I_j\} = |(0, 0) - (0, j_2)| \geq |(0, 0) - (0, 3)| = 3.
\]

Suppose \(j \in J^0_H \cap (\mathbb{R}_+ \times \{0\})\). Then

\[
\min\{|v - w| : v \in I_0, w \in I_j\} = |(0.5, 0) - (j_1 - 0.5, 0)| \geq |(0.5, 0) - (3 - 0.5, 0)| = 2.
\]

\[\square\]
4.7. The Smocking Constants of $X_{\boxplus}$. Here we find the smocking constants for $X_{\boxplus}$, see Subsection 4.

Lemma 4.23. The smocking depth of $X_{\boxplus}$ is

\[(131) \quad h_{\boxplus} = \sqrt{2}.\]

Proof. By the symmetry of the square lattice, we need only examine $I_{(0,0)}$, $I_{(0,3)}$, $I_{(3,0)}$, $I_{(3,3)}$. It is clear that the point simultaneously farthest from each square is $(2,2)$. It is $\sqrt{2}$ distance from every square. \hfill \Box

Lemma 4.24. The smocking lengths are

\[(132) \quad L_{\min}^{\boxplus} = \inf\{L(I_j) : j \in J_{\boxplus}\} = \sqrt{2},\]
\[(133) \quad L_{\max}^{\boxplus} = \sup\{L(I_j) : j \in J_{\boxplus}\} = \sqrt{2},\]

so the smocking length is $L_{\boxplus} = \sqrt{2}$.

Proof. For $X_{\boxplus}$, $L(I_j) = \text{Diam } I_j$. In this case of unit squares, it follows that $L(I_j) = \sqrt{2}$. \hfill \Box

Lemma 4.25. The smocking separation factor is

\[(134) \quad \delta_{\boxplus} = 2.\]

Proof. The minimum distance between any two squares is their Euclidean distance, which is 2. To prove this, we can use the symmetry of the pattern and compute any example. Consider the minimum distance between $I_{(0,0)}$ and $I_{(3,0)}$, the distance would be the distance between $(x_1, y_1) = (1,0)$ and $(x_2, y_2) = (3,0)$, which is 2. \hfill \Box

5. Balls in Smocked Spaces

In this section we examine how balls in smocked spaces look by describing their preimages under the smocking map. We begin with a few useful lemmas and propositions about balls and then draw some of the balls in our smocked spaces in subsequent subsections.

5.1. Useful Facts about Balls in Smocked Spaces. Recall the smocking map $\pi : \mathbb{B}^N \rightarrow X$ of Definition 3.1. In order to describe the balls in a smocked space precisely, we instead examine

\[(135) \quad \pi^{-1}(B_s(\pi(p))) = \{x \in \mathbb{B}^N : \pi(x) \in B_s(\pi(p))\} = \{x \in \mathbb{B}^N : d(\pi(x), \pi(p)) < r\} = \{x \in \mathbb{B}^N : \tilde{d}(x, p) < r\}.\]

Our first lemma will be applied repeatedly within this section:

Lemma 5.1. For all $p \in \mathbb{B}^N$ and all $r, s > 0$, we have

\[(138) \quad B_s(\pi(x)) \subset \pi^{-1}(B_s(\pi(x)))\]
and
\[(139) \quad T_s(\pi^{-1}(B_r(\pi(x)))) \subset \pi^{-1}(B_{r+s}(\pi(x))).\]
Proof. If \( v \in B_r(x) \), then \(|v - x| < s\) and so \( d(\pi(v), \pi(x)) < s\), which implies \( \pi(v) \in B_s(\pi(x)) \) and we get \( \textcolor{red}{[138]} \). If \( v \in T_s(\pi^{-1}(B_r(\pi(x)))) \), there exists \( z \in \pi^{-1}(B_r(\pi(x))) \) such that
\[
d(\pi(v), \pi(z)) < s \quad \text{and} \quad d(\pi(z), \pi(x)) < r.
\]
Then, by the triangle inequality,
\[
d(\pi(v), \pi(x)) \leq d(\pi(v), \pi(z)) + d(\pi(z), \pi(x)) < s + r.
\]
It follows that \( v \in \pi^{-1}(B_{r+s}(\pi(x))) \).

In our first proposition we examine a small ball about a stitch point whose radius is less than the separation factor of the smocked space:

**Proposition 5.2.** Suppose that \( I_j \) is a smocking stitch and \( r < \delta_X \) is defined as in Definition \( \textcolor{red}{[4.6]} \). Then
\[
\pi^{-1}(B_r(\pi(I_j))) = T_r(I_j).
\]

Proof. For each \( v \in T_r(I_j) \), there exists some \( z \in I_j \) such that \(|v - z| < r\). Therefore
\[
d(\pi(v), \pi(I_j)) = d_{0}(v, z) = |v - z| < r
\]
and so \( v \in \pi^{-1}(B_r(\pi(I_j))) \).

To prove the converse, consider \( v \in \pi^{-1}(B_r(\pi(I_j))) \). Setting \( p = \pi(v) \), we have \( d(p, \pi(I_j)) < r \). This implies that
\[
d(p, \pi(I_j)) = \min\{d_0(y, y') : y \in \pi^{-1}(p), \ y' \in I_j\}
\]
since
\[
d_n(y, y') \geq \delta_X > r \quad \text{for} \ n \geq 2.
\]
By definition,
\[
d_1(y, y') = \min\{|y - z_1| + |z_1' - y'| : z_1, z_1' \in I_k\}
\]
for some \( I_k \). If \( d_1(y, y') < r < \delta_X \), then \( I_k = I_j \). It follows that
\[
d_1(y, y') \geq \min\{|y - z_1| : z_1 \in I_j\} = d_0(y, y').
\]
But also
\[
d_1(y, y') \leq \min\{|y - z_1| + |y' - y'| : z_1 \in I_j\} = d_0(y, y').
\]
Therefore
\[
d(p, \pi(I_j)) = \min\{d_0(y, y') : y \in \pi^{-1}(p), \ y' \in I_j\}
\]
\[
= \min\{|y - y'| : y \in \pi^{-1}(p), \ y' \in I_j\}
\]
and so
\[
\exists y' \in I_j \exists y \in \pi^{-1}(p) \text{ such that } |y - y'| < r.
\]
Since \( r < \delta_X \), it follows that \( y \notin S \). Thus
\[
\pi(v) = p = \pi(y) \notin \pi(S).
\]
Therefore \( v = y \) and
\[
\exists y' \in I_j \text{ such that } |v - y'| < r.
\]
So \( v \in T_r(I_j) \). \( \square \)

**Proposition 5.3.** Suppose that \( x \in \mathbb{R}^N \setminus S \) and \( r < D(x) \) is defined as in Definition 4.6. Then
\[
\pi^{-1}(B_r(\pi(x))) = B_r(x) = \{ y : |x - y| < r \}.
\]

**Proof.** Note that \( v \in \pi^{-1}(B_r(\pi(x))) \) if and only if \( \pi(v) \in B_r(\pi(x)) \). By the definition of the smocking distance, this is true if and only if
\[
\min\{d_0(\pi(v), \pi(x)), d_1(\pi(v), \pi(x)), d_2(\pi(v), \pi(x)), \ldots\} = d(\pi(v), \pi(x)) < r < D(x).
\]
By the hypothesis \( |z - x| \geq D(x) \) \( \forall z \in S \), so, by the definition of smocking distance, for all \( j \geq 1 \)
\[
d_j(\pi(v), \pi(x)) \geq \min\{|z - x| : z \in S\} \geq D(x).
\]
Thus the minimum in (155) is achieved by
\[
d_0(\pi(v), \pi(x)) = d(\pi(v), \pi(x)) < r < D(x).
\]
So (155) holds if and only if \( |v - x| < r \) which is true if and only if \( x \in B_r(p) \). \( \square \)

In our next proposition we explore how a ball grows when there are no smocking stitches to close to the original ball:

**Proposition 5.4.** Suppose that the points in the smocking set are a definite distance,
\[
\delta_r = \min \{|z - w| : z \in S \setminus \pi^{-1}(B_r(\pi(p))), w \in \pi^{-1}(B_r(\pi(p)))\} > 0,
\]
away from the points within the ball. Then for all \( r > 0 \) and \( s \in (0, \delta_r] \),
\[
\pi^{-1}\left( B_{r+s}(\pi(x)) \right) = T_s\left( \pi^{-1}(B_r(\pi(x))) \right).
\]

**Proof.** Suppose \( v \in \pi^{-1}(B_{r+s}(\pi(x))) \). If \( x \in \pi^{-1}(B_r(\pi(x))) \). Then clearly
\[
v \in T_s\left( \pi^{-1}(B_r(\pi(x))) \right),
\]
and we are done. If not, note that the distance \( d(\pi(v), \pi(x)) < r + s \), is achieved by a collection of segments starting at \( y \in \pi^{-1}(\pi(v)) \) and ending at \( y' \in \pi^{-1}(\pi(x)) \):
\[
d(\pi(v), \pi(x)) = |y - z_1| + \sum_{i=1}^{n} |z'_i - z_{i+1}| + |z'_k - y'|
\]
for some \( n \geq 0 \), where \( \pi(z_i) = \pi(z'_i) \) for each \( i \). Notice that for any \( z'_i \) in the minimizing sum, we must have
\[
d(\pi(v), \pi(x)) = d(\pi(v), \pi(z'_i)) + d(\pi(z'_i), \pi(x)).
\]
Since \( y \notin \pi^{-1}(B_r(\pi(x))) \) and \( y' \in \pi^{-1}(B_r(\pi(x))) \), at least one term in the sum must be of the form \( |a - b| \) where \( a \notin \pi^{-1}(B_r(\pi(x))) \) and \( b \in \pi^{-1}(B_r(\pi(x))) \). Since \( d(a, x) > r \)

and \( \bar{d}(b, x) < r \), the line segment from \( a \) to \( b \) must hit a point \( c \) such that \( \bar{d}(c, p) = d(\pi(c), \pi(x)) = r \), by the continuity of the pseudometric \( \bar{d} \). It follows that

\[
(163) \quad r + s > d_0(y, y')
\]

\[
(164) \quad \geq \ d(\pi(a), \pi(x))
\]

\[
(165) \quad = |a - b| + d(\pi(b), \pi(x))
\]

\[
(166) \quad = |a - c| + |c - b| + d(\pi(b), \pi(x))
\]

\[
(167) \quad = |a - c| + d(\pi(c), \pi(x))
\]

\[
(168) \quad = |a - c| + r.
\]

This implies that \( |a - c| < s \). If \( a \in S \), then \( |a - c| \geq \delta_r \geq s \), which is a contradiction. Therefore \( a \notin S \), which implies that \( a = v \). Hence \( |v - c| < s \), which implies that

\[
(169) \quad v \in T_s\left(\pi^{-1}(B_r(\pi(x)))\right) = T_s\left(\pi^{-1}(B_r(\pi(x)))\right).
\]

The other direction holds by Lemma 5.1.

Lemma 5.5. In a smocked metric space as in Definition 3.1 if there is a stitch, \( I_j \), such that

\[
(170) \quad \pi^{-1}(\bar{B}_r(\pi(x))) \cap I_j \neq \emptyset,
\]

then for all \( s > 0 \)

\[
(171) \quad T_s(\pi^{-1}(B_r(\pi(x)))) \cup T_s(I) \subset \pi^{-1}\left(B_{r+s}(\pi(x))\right).
\]

Proof. For each \( v \in T_s(I_j) \), there exists some \( z \in I_j \) such that \( |v - z| < s \). Therefore,

\[
(172) \quad d(\pi(v), \pi(I_j)) \leq d_0(v, z) = |v - z| < s
\]

so

\[
(173) \quad T_s(I_j) \subset \pi^{-1}(B_r(\pi(I_j))).
\]

By assumption, we have

\[
(174) \quad \pi(I_j) \in \bar{B}_r(\pi(x))
\]

which gives us

\[
(175) \quad B_r(\pi(I_j)) \subset B_{r+s}(\pi(x))
\]

and then by looking at the preimages we get

\[
(176) \quad \pi^{-1}(B_r(\pi(I_j))) \subset \pi^{-1}(B_{r+s}(\pi(x))).
\]

Combining this with line 173, we get

\[
(177) \quad T_s(I_j) \subset \pi^{-1}(B_r(\pi(I_j))) \subset \pi^{-1}(B_{r+s}(\pi(x))).
\]

In order to show that

\[
(178) \quad T_s(\pi^{-1}(B_r(\pi(x)))) \subset \pi^{-1}(B_{r+s}(\pi(p)))
\]

we apply Lemma 5.1.
Proposition 5.6. If we consider all the smocking stitches that just touch a given ball:

\[ J_r = \{ j \in J : I_j \cap \pi^{-1}(\bar{B}_r(x)) \neq \emptyset \text{ and } I_j \cap \pi^{-1}(B_r(x)) = \emptyset \} \]

and the distance to the nearest smocking interval that doesn’t touch this ball

\[ \delta_r = \min \{|z - w| : z \in S \setminus \pi^{-1}(\bar{B}_r(\pi(v))), w \in \pi^{-1}(\bar{B}_r(\pi(v)))\} > 0, \]

then for all \( s \leq \delta_r \) we have

\[ \pi^{-1}(B_{r+s}(\pi(v))) = T_s(\pi^{-1}(B_r(\pi(v)))) \cup \bigcup_{j \in J_r} T_s(I_j). \]

Proof. By applying Lemma 5.5 to each \( j \in J_r \), we know that

\[ T_s(\pi^{-1}(B_r(\pi(v)))) \cup \bigcup_{j \in J_r} T_s(I_j) \subset \pi^{-1}(B_{r+s}(\pi(v))). \]

So we need only show that

\[ \pi^{-1}(B_{r+s}(\pi(v))) \subset T_s(\pi^{-1}(B_r(\pi(v)))) \cup \bigcup_{j \in J_r} T_s(I_j). \]

We want to show that for any \( q \in B_{r+s}(\pi(v)) \), one of the following holds:

(A) \( \exists x' \in \pi^{-1}(B_r(\pi(v))) \text{ such that } |x' - \pi^{-1}(q)| < s \)

(B) \( \exists j \in J_r, z \in I_j \text{ such that } |z - \pi^{-1}(q)| < s. \)

First consider \( q \in (B_{r+s}(\pi(v))) \cap \pi(S) \). Since \( s < \delta_r \), all of the smocking intervals intersecting \( \pi^{-1}(B_{r+s}(\pi(v))) \) must either be contained inside \( \pi^{-1}(\bar{B}_r(\pi(v))) \), in which case (A) holds or be one of the \( I_j \) for some \( j \in J_r \), in which case (B) holds.

Now consider \( q \in (B_{r+s}(\pi(v))) \setminus \pi(S) \), which implies there exists a unique point \( y \in \mathbb{R}^N \) such that

\[ \pi(y) = q \text{ and } d(p, \pi(v)) = d(\pi(y), \pi(v)) < r + s. \]

Since \( \delta_X > 0 \), by Theorem 3.3 we know that one of the following cases holds:

Case I: \[ d(p, \pi(v)) = d(\pi(y), \pi(v)) = |y - \pi(v)| \]

Case II: \[ \exists I_j \in S \exists w, w' \in I_j \text{ s.t. } d(q, \pi(v)) = d(\pi(y), \pi(v)) = |y - w| + |w' - v|. \]

In Case I, we see that

\[ y \in T_s(B_r(\pi(v))) \subset T_s(\pi^{-1}(B_r(\pi(v)))) \]

by Lemma 5.1. So we have (A) because

\[ \exists x' \in \pi^{-1}(B_r(\pi(v))) \text{ such that } |x' - \pi^{-1}(q)| = |x' - y| < s. \]

In Case II,

\[ \exists I_j \in S \text{ w, w' } \in I_j \text{ s.t. } d(q, \pi(v)) = d(\pi(y), \pi(v)) = |y - w| + |w' - v| < r + s \]

and the interval \( I_j \) intersects with \( B_r(\pi(v)) \), so \( I_j \in B_r(\pi(v)) \). Consider the point \( z \in \bar{B}_r(\pi(p)) \) which minimizes \( d(q, z) \). Since the closest smocking interval lies
Thus the ball of radius \( R \), we know that \( z \) is either inside \( B_r(\pi(v)) \), in which case \( y \) is also inside \( B_r(\pi(v)) \) or \( z \) is on the boundary. In the second case, \( d(p, z) = r \) so

\[
d(q, z) = |y - \pi^{-1}(z)| < s.
\]

If the interval \( I_j \) does not intersect with \( B_r(\pi(p)) \), then it must be in \( J_r \). Then, since

\[
d(q, \pi(v)) = d(q, I_j) + d(I_j, \pi(v)) < r + s
\]

and \( d(I_j, \pi(v)) = r \), we have \( d(q, I_j) < s \) which gives us that \( \pi^{-1}(q) \in T_s(I_j) \) and so we have (B).

We can now use the maximum smocking length to estimate the size of a ball:

**Lemma 5.7.** Whenever the smocking stitches are all horizontal we have

\[
d((x_1, y_1), (x_2, y_2)) \geq |y_1 - y_2|.
\]

**Proof.** If we examine the minimum in Definition 3.1, observe that the first term in the minimum has

\[
|(x_1, y_1) - (x_2, y_2)| \geq |y_1 - y_2|.
\]

The second term satisfies is also bounded below by \( |y_1 - y_2| \): since \( z, z' \in I_j \) implies \( z_2' = z_2 \) due to the fact that the smocking intervals are horizontal, it follows that

\[
|(x_1, y_1) - (z_1, z_2) + (z_1' - z_2') - (x_2, y_2)| \geq |y_1 - z_2| + |z_2 - y_2| \geq |y_1 - y_2|.
\]

Similarly, all terms in the minimum in Definition 3.1 are bounded below by \( |y_1 - y_2| \).

**Remark 5.8.** The lemmas above allow one to draw the balls in these spaces using an art program like Art Studio as we will do in the subsequent sections. First one sets a grid and sets snap to grid and draws the pattern onto a transparent layer. Then take a thick pen with a round nib whose radius matches a radius \( r < \delta \) and draw over the central stitch \( I_{(0,0)} \) on a transparent layer beneath the pattern layer. With the round nib, it looks like a tubular neighborhood. Then on another transparent layer beneath that, one takes a new color, outlines the edge of this previous stitch, and adds in new stitches. Each concentric ball is added in using thick pens in new colors added on transparent layers beneath the previous layer.

5.2. **Exploring the Balls in \( X_o \).** Here we consider balls in \( X_o \) centered on the point,

\[
p_0 = I_{(0,0)} = [-1/2, 1/2] \times \{0\},
\]

by drawing their preimages \( \pi^{-1}(B_R(p_0)) \subset \mathbb{R}^n \). See Figure 5.2.

Observe that, by Lemma 5.2, we know that

\[
\pi^{-1}(B_R(p_0)) = T_R(I_{(0,0)}) \quad \forall R \in (0, 1]
\]

because the smocking separation factor was proven in Lemma 4.10 to be \( \delta_o = 1 \). Thus the ball of radius \( R = 1/2 \) is depicted in red and of radius \( R = 1 \) is depicted in orange in Figure 5.2.
Figure 16. Lifts of balls, $B_R(p_0)$, where $p_0 = I_{(0,0)}$ in the smocked space $(X_\circ,d_\circ)$.

We next apply Proposition 5.6, keeping in mind that $\delta_\circ = 1$ and observing that when $r = 1$ there are six stitches touching the ball of radius $r = 1$. This gives us the ball of radius $3/2$ depicted in yellow in Figure 5.2:

$$\pi^{-1}(B_{3/2}(p)) = T_{1/2}\left(\pi^{-1}(B_1(p))\right) \cup \bigcup_{j \in J_1} T_{1/2}(I_j)$$

where

$$J_1 = \{(1,1),(2,0),(1,-1),(-1,-1),(-2,0),(-1,1)\}.$$  

We then apply Proposition 5.4 with $r = 3/2$ and $s = 1/2$ to find the ball of radius 2 depicted in green in Figure 5.2:

$$\pi^{-1}(B_{2}(p)) = T_{1/2}(\pi^{-1}(B_{3/2}(p))).$$

We next apply Proposition 5.6, observing that when $r = 2$ there are twelve stitches touching the ball of radius $r = 2$:

$$J_2 = \{(0,\pm 2), (\pm 2, \pm 2), (\pm 3, \pm 1), (\pm 4, \pm 0)\}.$$  

This gives us the ball of radius $5/2$ depicted in pale green in Figure 5.2:

$$\pi^{-1}(B_{5/2}(p)) = T_{1/2}\left(\pi^{-1}(B_2(p))\right) \cup \bigcup_{j \in J_2} T_{1/2}(I_j).$$

We continue in this manner to complete the drawing in Figure 5.2 by eye.

While it can be rather complicated to provide formulas for these sets, we can nevertheless approximately describe their shapes. For large radii, these balls appear to have a hexagonal shape to them. In fact, it appears approximately to be the intersection of the strip, $y^{-1}(-r,r)$, with the diamond

$$\{(x,y) : |x| + |y| < R \}$$

which has vertices at $(\pm R,0)$ and $(0,\pm R)$ where $R = 2r$. There is an error in this approximation of about the length of a smocking interval.

Easier perhaps is to describe $J_r$ here since it seems always to change at integer values. We guess here that $J_r = \hat{J}_r \cup \tilde{J}_r$ where

$$\tilde{J}_r = \{(j_1,j_2) \in J_\circ : |j_2| = r \text{ and } |j_1| \leq |j_2|\}$$
$$\hat{J}_r = \{(j_1,j_2) \in J_\circ : |j_1 + j_2| \in [2r-1,2r]\}. $$
Due to the length of the proof required to rigorously prove this guess is true, we postpone studying this particular smocked space further in this paper. See [8] for a rigorous study of this space.

5.3. Exploring the Balls in $X_T$. Here we consider balls in $X_T$ centered on the point,

$$p_0 = I_{(0,0)} = [-1, 1] \times \{0\},$$

by drawing their preimages $\pi^{-1}(B_R(p_0)) \subset \mathbb{N}$. See Figure 17.

![Figure 17. Lifts of balls, $B_R(p_0)$, where $p_0 = I_{(0,0)}$ in the smocked space ($X_T, d_T$).](image)

Observe that, by Lemma 5.2 we know that

$$\pi^{-1}(B_R(p_0)) = T_R(I_{(0,0)}) \quad \forall R \in (0, 1]$$

because the smocking separation factor was proven in Lemma 4.16 to be $\delta_\diamond = 1$. Thus the ball of radius $R = 1$ is depicted in red in Figure 17.

We next apply Proposition 5.6 keeping in mind that $\delta_T = 1$ and observing that when $r = 1$ there are four stitches touching the ball of radius $r = 1$. This gives us the ball of radius 2 depicted in orange in Figure 17

$$\pi^{-1}(B_2(p)) = T_1/2\left(\pi^{-1}(B_1(p))\right) \cup \bigcup_{j \in J_1} T_1(I_j)$$

where

$$J_1 = \{(2, 0), (0, 2), (-2, 0), (0, -2)\}.$$

We next apply Proposition 5.6 observing that when $r = 3$ there are eight stitches touching the ball of radius $r = 2$:

$$J_2 = \{(0, \pm 4), (\pm 2, \pm 2), (\pm 4, \pm 0)\}.$$

This gives us the ball of radius 3 depicted in yellow in Figure 17

$$\pi^{-1}(B_3(p)) = T_1\left(\pi^{-1}(B_2(p))\right) \cup \bigcup_{j \in J_2} T_1(I_j).$$

We continue in this matter to easily complete the drawing in Figure 17.
A rigorous analysis of this space will be continued within this paper.

5.4. Exploring the Balls in \( X_{\pm} \). Here we consider balls in \( X_{\pm} \) centered on the point,

\[
p_0 = I_{(0,0)} = [-.5, .5] \times \{0\},
\]

by drawing their preimages \( \pi^{-1}(B_R(p_0)) \subset \mathbb{H}^N \). See Figure 18.

**Figure 18.** Lifts of balls, \( B_R(p_0) \), where \( p_0 = I_{(0,0)} \) in the smocked space \( (X_{\pm}, d_{\pm}) \).

Observe that, by Lemma 5.2, we know that

\[
\pi^{-1}(B_R(p_0)) = T_R(I_{(0,0)}) \quad \forall R \in (0, 1]
\]

because the smocking separation factor was proven in Lemma 4.16 to be \( \delta_o = 1 \). Thus the ball of radius \( R = 1 \) is depicted in red in Figure 18.

To find the ball of radius 2 depicted in orange in Figure 18, observe that we need to do things very carefully. Proposition 5.6 implies that

\[
\pi^{-1}(B_{1+s}(\pi(p))) = T_s\left(\pi^{-1}(B_1(\pi(p)))\right) \cup \bigcup_{j \in J_1} T_s(I_j)
\]

where

\[
J_1 = \{(2, 0), (0, 2), (-2, 0), (0, -2)\}
\]

and

\[
s \leq \tilde{\delta}_r = \sqrt{2} - 1
\]

because at \( \sqrt{2} \) we hit the intervals in

\[
J_{\sqrt{2}} = \{\pm 2, \pm 2\}.
\]

So

\[
\pi^{-1}(B_2(p)) = T_1\left(\pi^{-1}(B_1(p))\right) \cup \bigcup_{j \in J_1} T_1(I_j) \cup \bigcup_{j \in J_{\sqrt{2}}} T_{2-\sqrt{2}}(I_j).
\]

To find the ball of radius 3 we need

\[
J_3 = \{(-3, 0), (0, \pm 3)\} \quad J_{\sqrt{3}} = \{\pm 1, \pm 2\} \quad J_{1+\sqrt{2}} = \{\pm 2, \pm 1\}
\]

so we see that things become very complicated rapidly. Nevertheless, we roughly draw the balls by eye using an art program to produce Figure 18.

Although these balls appear to be converging to an ellipse, this has been shown to be false. Due to the lengthy nature of the estimates involved, we will not explore \( X_{\pm} \) further within this paper. Further work on this space will appear in [8].
5.5. Exploring the Balls in $X_+$. Let us consider balls in $X_+$ around

\( p_0 = I_{(0,0)} = \{(-1, 1) \times \{(0, 0)\} \cup \{(0, 0) \times [-1, 1]\} \)

by drawing their preimages $\pi^{-1}(B_R(p_0)) \subset \mathbb{B}$. See Figure 19.

\[ \text{Figure 19. Lifts of balls, } B_R(p_0), \text{ where } p_0 = I_{(0,0)} \text{ in the smocked space } (X_+, d_+). \]

Observe that, by Lemma 5.2, we know that

\[ \pi^{-1}(B_R(p_0)) = T_R(I_{(0,0)}) \forall R \in (0, 1] \]

because the smocking separation factor was proven in Lemma ?? to be $\delta_+ = 1$. Thus the ball of radius $R = 1$ is depicted in red in Figure 19.

We next apply Proposition 5.6, keeping in mind that $\delta_T = 1$ and observing that when $r = 1$ there are four stitches touching the ball of radius $r = 1$. This gives us the ball of radius 2 depicted in orange in Figure 19.

\[ \pi^{-1}(B_2(p)) = T_{1/2} \left( \pi^{-1}(B_1(p)) \right) \cup \bigcup_{j \in J_1} T_1(I_j) \]

where

\[ J_1 = \{(\pm 3, 0), (0, \pm 3)\}. \]

We next apply Proposition 5.6 again, observing that when $r = 3$ there are eight stitches touching the ball of radius $r = 2$:

\[ J_r = \{(0, \pm 6), (\pm 3, \pm 3), (\pm 6, \pm 0)\}. \]

This gives us the ball of radius 3 depicted in yellow in Figure 19.

\[ \pi^{-1}(B_3(p)) = T_1 \left( \pi^{-1}(B_2(p)) \right) \cup \bigcup_{j \in J_2} T_1(I_j) \]

We continue in this matter to complete the drawing in Figure 19.

**Remark 5.9.** It appears that the balls are bumpy diamond shapes. That is if $k = \lceil r \rceil$, with $r > 1$ then

\[ S'_{3(k-2)} \subset \pi^{-1}(B_r(p)) \subset S'_{3k} \]
where
(222) \[ S'_r = \{ x \in X_+ : d_T(x, p) < r \} \]
with \( d_T \) being the taxicab metric:
(223) \[ d_T(x, y) = |x_1 - y_1| + |x_2 - y_2| \]
where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \).

**Remark 5.10.** If \( k = \lceil r \rceil \), then we define \( J^+_r \) to be
(224) \[ J^+_r = J^+_1 \cup J^+_2 \cup \cdots \cup J^+_{k-1} \cup J^+_k \]
where
(225) \[ J^+_k = \{ \pm (0, 3(k-1)), \pm (3 \cdot 1, 3(k-2)), \ldots, \pm (3 \cdot (k - 1), 0) \}. \]

A rigorous proof of these intuitive estimates will be provided in later sections.

5.6. **Exploring the Balls in** \( X_H \). Here we consider balls in \( X_H \) centered on the point,
(226) \[ p_0 = I_{(0,0)} = [-.5, +.5] \times \{0\} \]
by drawing their preimages \( \pi^{-1}(B_R(p_0)) \subset \mathbb{E}^N \). We also consider balls about the deepest point \( q_0 = (1.5, 1.5) \) because they have more symmetry. For very large balls, the center point should not matter because
(227) \[ B_R(p_0) \subset B_{R+C}(q_0) \subset B_{R+2C}(p_0) \]
where \( C = d_H(p_0, q_0) \). See Figure 20.

![Figure 20](image)

**Figure 20.** Balls centered on \( p_0 = I_{(0,0)} \) in \((X_H, d_H)\) have vertical and horizontal symmetry but balls centered at the deepest point \( q_0 = (1.5, 1.5) \) display octagonal symmetry.

Observe that, by Lemma 5.2, we know that
(228) \[ \pi^{-1}(B_R(p_0)) = T_R(I_{(0,0)}) \quad \forall R \in (0, \sqrt{2}] \]
because the smocking separation factor was proven in Lemma 4.22 to be \( \delta_o = \sqrt{2} \).
Thus the ball of radius \( R = 1 \) is depicted in red in Figure 21.
To find the balls of radius 2 depicted in dark orange in Figure 20, Proposition 5.6 implies that
\[
\pi^{-1}(B_2(\pi(p))) = T_1\left(\pi^{-1}(B_1(\pi(p)))\right) \cup \bigcup_{j \in J_{\sqrt{2}}} T_{2-\sqrt{2}}(I_{\sqrt{2}})
\]
where
\[
J_{\sqrt{2}} = \{\pm1.5, \pm1.5\}.
\]
For radius 3, we need to include two more intervals
\[
J_2 = \{\pm2.5, 0\}.
\]
For radius 4 depicted in yellow, we need to include
\[
J_3 = \{(0, \pm3)\} \text{ and } J_{2+\sqrt{2}} = \{\pm3, 0\} \text{ and } J_{2+\sqrt{2}} = \{(\pm4.5, \pm1.5)\}.
\]
We continuing drawing larger and larger balls by eye on the left side of Figure 20 and we cannot see any shape developing.

However, on the right side of Figure 20 drawn by eye, we see a nice octagonal shape forming. Due to the complicated nature of the balls in this space, further analysis of this space is postponed to [8].

5.7. **Exploring the Balls in** \(X_{\Box}\). Here we consider balls in \(X_{\Box}\) centered on the point,
\[
p_0 = I_{(0,0)} = ([0,1] \times \{0\}) \cup ([0,1] \times \{1\}) \cup \{0\} \times [0,1] \cup \{1\} \times [0,1]
\]
by drawing their preimages \(\pi^{-1}(B_R(p_0)) \subset \mathbb{E}^N\). See Figure 21.

![Figure 21. Lifts of balls, \(B_R(p_0)\), where \(p_0 = I_{(0,0)}\) in the smocked space \((X_{\Box}, d_{\Box})\).](image)

Observe that, by Lemma 5.2 we know that
\[
\pi^{-1}(B_R(p_0)) = T_R(I_{(0,0)}) \quad \forall R \in (0, 2]
\]
because the smocking separation factor was proven in Lemma 4.25 to be \(\delta = 2\).

Thus the ball of radius \(R = 1\) is depicted in red and the ball of radius \(R = 2\) is depicted in orange in Figure 21.
To find the balls of radius 3 and 4 depicted in shades of yellow in Figure 21, observe that we need to do things very carefully. Proposition 5.6 implies that

\[
\pi^{-1}(B_{2,s}(\pi(p))) = T_s \left( \pi^{-1}(B_2(\pi(p))) \right) \cup \bigcup_{j \in J_2} T_s(I_j)
\]

where

\[
J_2 = \{ (\pm 3, 0), (0, \pm 3) \}
\]

and

\[
s \leq \delta_r = \sqrt{8} - 2
\]

because at \( \sqrt{8} \) we hit the four stitches of

\[
J_{\sqrt{8}} = \{ \pm 3, \pm 3 \}.
\]

So

\[
\pi^{-1}(B_3(p)) = T_1 \left( \pi^{-1}(B_2(\pi(p))) \right) \cup \bigcup_{j \in J_2} T_1(I_j) \cup \bigcup_{j \in J_{\sqrt{8}}} T_{3-\sqrt{8}}(I_j)
\]

and

\[
\pi^{-1}(B_4(p)) = T_2 \left( \pi^{-1}(B_2(\pi(p))) \right) \cup \bigcup_{j \in J_2} T_2(I_j) \cup \bigcup_{j \in J_{\sqrt{8}}} T_{4-\sqrt{8}}(I_j).
\]

We draw the rest of the balls by eye using an art program to produce Figure 21.

To really better understand the shape of these balls on a large scale we computed

\[
J_R^R = \bigcup_{r < R} J_r^R \cup \{(0,0)\}
\]

for increasingly large values of \( r \) using \textit{Processing} 3. See Figure 5.7. It appears that the balls are becoming octagonal in shape. This space will be studied further within this paper.

Figure 22. Computer generated rescaled images capturing \( J_R^R \) for increasingly large values of \( R \) using \textit{Processing} 3.
6. Smocked Metric Spaces Are Complete and Noncompact

Recall Definition 3.1 of a smocked metric space. In this section, we prove the following theorem:

**Theorem 6.1.** Any smocked space \((X, d)\) is complete and noncompact.

Before we prove this theorem, we prove a couple useful lemmas.

**Lemma 6.2.** Suppose \((X, d)\) is a smocked metric space with separation factor \(\delta_X > 0\) and smocked length \(L_{\max} < \infty\). Then

\[
\bar{d}(v, w) < \delta \implies \bar{d}(v, w) = \min\{d_0(v, w), d_1(v, w)\}.
\]

In other words, the sum defining the smocking distance between \(\pi(v)\) and \(\pi(w)\) is of only one or two segments. Thus

\[
|v - w| \leq \bar{d}(v, w) + L_{\max} < \delta + L.
\]

**Proof.** By Theorem 3.3, we know there exists \(N = N(v, w)\) such that

\[
\bar{d}(v, w) = d_N(v, w).
\]

If \(N \geq 2\) then there are more than three segments and so there exists \(z, z'\) in different smocking stitches such that

\[
\bar{d}(v, w) \geq |z - z'| \geq \delta.
\]

This is a contradiction. So we have (242). If \(N = 0\), then

\[
\bar{d}(v, w) = |v - w| \geq |v - w| - L_{\max}
\]

and if \(N = 1\), then there exists \(z, z'\) in the same stitch such that

\[
\bar{d}(v, w) = |v - z| + |z' - w| \geq |v - w| - |z - z'| \geq |v - w| - L_{\max}
\]

by the triangle inequality. Thus in either case we have (243).

**Lemma 6.3.** Let \(I_j\) and \(I_{j'}\) be distinct smocking stitches \(j \neq j'\) and let \(p_j = \pi(I_j)\) and \(p_{j'} = \pi(I_{j'})\). Then

\[
T_r(I_j) \cap T_r(I_{j'}) = \emptyset \quad \forall r < \delta/2
\]

and

\[
B_r(p_j) \cap B_r(p_{j'}) = \emptyset \quad \forall r < \delta/2.
\]

where \(\delta\) is the separation factor.

**Proof.** First recall that by Lemma 5.2

\[
B_r(p_j) = \pi(T_r(I_j)).
\]

So (249) implies (248).

Suppose \(q \in B_r(p_j) \cap B_r(p_{j'})\). Then

\[
d(p_j, p_{j'}) \leq d(p_j, q) + d(q, p_{j'}) < \delta/2 + \delta/2 = \delta.
\]

However, if \(z \in I_j\) and \(z' \in I_{j'}\) we have

\[
d(p_j, p_{j'}) = \bar{d}(z, z') \leq |z - z'| < \delta,
\]

by the definition of the separation factor. Thus we have a contradiction. \(\square\)
We can now prove Theorem 6.1.

**Proof.** To see that \((X, d)\) is noncompact, we show it contains an infinite collection of pairwise disjoint balls of radius \(\delta/2\). When the smocking set is infinite we can just center these balls on the images of the smocking stitches and apply Lemma 6.3. When the smocking set is finite, we center the balls on the images of a sequence of points diverging to infinity in Euclidean space that are greater than \(\delta\) apart and are far from all the smocking stitches.

To prove completeness we must show that any Cauchy sequence \(x_n \in X\) converges to a point in \(X\). Let \(v_n \in \pi^{-1}(x_n) \subset \mathbb{E}^N\) so

\[
\forall \epsilon > 0 \ \exists N_{\epsilon} \ s.t. \ \forall n, m \geq N_{\epsilon} \ d(x_n, x_m) = \bar{d}(v_n, v_m) < \epsilon.
\]

Take \(\epsilon = \delta_X\) and \(N' = N_{\delta_X}\). Then

\[
d(x_n, x_m) = \bar{d}(v_n, v_m) < \delta_X \ \forall n, m \geq N'.
\]

By Lemma 6.2,

\[
|v_n - v_m| < \delta_X + L_{max} \ \forall n, m \geq N'.
\]

So \(v_n\) is a bounded sequence in \(\mathbb{E}^N\).

By Bolzano-Weierstrass, any bounded sequence in \(\mathbb{E}^N\) has a convergent subsequence, so there is \(y \in \mathbb{E}^N\) and a subsequence \(v_{n_k} \to y \in \mathbb{E}^N\). Since

\[
d(\pi(v_{n_k}), \pi(y)) = \bar{d}(v_{n_k}, y) \leq |v_{n_k} - y| \to 0,
\]

we see that the subsequence \(x_{n_k} = \pi(v_{n_k}) \to \pi(y)\).

If a Cauchy sequence has a converging subsequence, then the sequence itself converges to the same limit. So \(x_n \to \pi(y) \in X\). Thus \((X, d)\) is complete. \(\square\)

7. GH Convergence and Tangent Cones at Infinity

7.1. Review of the Definitions by Prof. Sormani. Gromov-Hausdorff convergence was first defined by David Edwards in [3] and rediscovered by Gromov in [5]. See the text by Burago-Burago-Ivanov [1] for an excellent introduction to this topic.

**Definition 7.1.** We say a sequence of compact metric spaces

\[
(X_j, d_j) \overset{GH}{\longrightarrow} (X_\infty, d_\infty)
\]

if and only if

\[
d_{GH}((X_j, d_j), (X_\infty, d_\infty)) \to 0,
\]

where the Gromov-Hausdorff distance is defined by

\[
d_{GH}(X_j, X_\infty) = \inf\{d^Z_H(\varphi_j(X_j), \varphi_\infty(X_\infty)) : \ Z, \ \varphi_j : X_j \to Z\}
\]

where the infimum is over all compact metric spaces, \(Z\), and over all distance preserving maps \(\varphi_j : X_j \to Z\):

\[
d_Z(\varphi_j(a), \varphi_j(b)) = d_j(a, b) \ \forall a, b \in X_j.
\]
The Hausdorff distance is defined by
\begin{equation}
(261)
\quad d_H(A_1, A_2) = \inf\{r : A_1 \subset T_r(A_2) \text{ and } A_2 \subset T_r(A_1)\}.
\end{equation}

Notice in the above definition that if a compact metric space \(X_j\) were replaced by another compact metric space \(Y\) which is isometric to it, then
\begin{equation}
(262)
\quad d_{GH}(X_j, X_\infty) = d_{GH}(Y_j, X_\infty).
\end{equation}

So the Gromov-Hausdorff distance between metric spaces is really a distance between isometry classes of metric spaces. Furthermore, Gromov proved that given two compact metric spaces, \(X\) and \(Y\),
\begin{equation}
(263)
\quad d_{GH}(X, Y) = 0 \iff X \text{ is isometric to } Y.
\end{equation}

**Example 7.2.** The taxi space
\begin{equation}
(264)
\quad (X_1, d_1) = (\mathbb{R}^2, d_{\text{taxi}}) \text{ where } d_{\text{taxi}}((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|
\end{equation}
and the rescaled taxi space
\begin{equation}
(265)
\quad (X_2, d_2) = (\mathbb{R}^2, d_{\text{taxi}}/2) \text{ where } d_{\text{taxi}}/2((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|)/2
\end{equation}
are isometric via the isometry
\begin{equation}
(266)
\quad F : X_1 \to X_2 \text{ where } F(x_1, x_2) = (2x_1, 2x_2)
\end{equation}
because
\begin{equation}
(267)
\quad d_2(F(x), F(y)) = (|2x_1 - 2y_1| + |2x_2 - 2y_2|)/2
\quad = |x_1 - y_1| + |x_2 - y_2| = d_1(x, y) \quad \forall x, y \in \mathbb{R}^2.
\end{equation}
The Gromov-Hausdorff distance between these spaces is 0. This can easily be proven using the following theorem by defining the correspondence
\begin{equation}
(268)
\quad C = \{(x, F(x)) : x \in X_1\}
\end{equation}

**Theorem 7.3.** If there exists a correspondence \(C \subset X \times Y:\)
\begin{equation}
(269)
\quad \forall x \in X \ \exists y \in Y \text{ s.t. } (x, y) \in C \forall y \in Y \ \exists x \in X \text{ s.t. } (x, y) \in C
\end{equation}
which is \(\epsilon\) almost distance preserving
\begin{equation}
(270)
\quad |d_X(x_1, x_2) - d_Y(y_1, y_2)| < \epsilon \quad \forall (x_i, y_i) \in C,
\end{equation}
then
\begin{equation}
(271)
\quad d_{GH}((X, d_X), (Y, d_Y)) < 2\epsilon.
\end{equation}

In this paper we are considering unbounded metric spaces and so we must consider the following definition by Gromov:

**Definition 7.4.** If one has a sequence of complete noncompact metric spaces, \((X_j, d_j)\), and points \(x_j \in X_j\), one can define pointed \(GH\) convergence:
\begin{equation}
(272)
\quad (X_j, d_j, x_j) \overset{\text{ptGH}}{\longrightarrow} (X_\infty, d_\infty, x_\infty)
\end{equation}
if and only if for every radius \( r > 0 \), the closed balls of radius \( r \) in \( X_j \) converge in the GH sense as metric spaces with the restricted distance to closed balls in \( X_\infty \):

\[
d_{GH}(\overline{B}_r(x_j) \subset X_j, d_j), (\overline{B}_r(x_\infty) \subset X_\infty, d_\infty) \to 0.
\]

We will be rescaling our metric spaces to see how they look from a distance. Unlike the taxi space in Example 7.2, most metric spaces are not isometric to their rescalings. So when we rescale a metric space repeatedly, we obtain a sequence of metric spaces. If the sequence or a subsequence converges in the Gromov-Hausdorff sense, then we obtain a space that is called the tangent cone at infinity.

**Definition 7.5.** A complete noncompact metric space with infinite diameter, \((X, d_X)\), has a tangent cone at infinity, \((Y, d_Y)\), if there is a sequence of rescalings, \(R_j \to \infty\), and points, \(x_0 \in X\) and \(y_0 \in Y\), such that

\[
(X, d/R_j, x_0) \xrightarrow{ptGH} (Y, d_Y, y_0).
\]

There are a variety of theorems in the literature concerning the existence and uniqueness of such tangent spaces at infinity. We will not be applying those theorems in this paper. We can prove our theorems directly using only what we’ve stated in this section.

### 7.2. Distances in Pulled thread Spaces.

**Lemma 7.6.** In a pulled thread space with an interval \( I \) of length \( L \), we have

\[
|\hat{d}(a, b) - |a - b|| \leq L \quad \forall a, b \in \mathbb{R}^N.
\]

**Proof.** By Definition 2.9

\[
\hat{d}(a, b) = \min\{|a - b|, |a - z| + |z' - b| : z, z' \in I\}.
\]

If the minimum is \(|a - b|\) then the lemma follows trivially. If the minimum occurs at \(z, z' \in I\), then

\[
|a - b| \geq \hat{d}(a, b) = |a - z| + |z' - b|
\]

\[
= |a - z| + |z - z'| + |z' - b| - |z - z'|
\]

\[
\geq |a - b| - |z - z'| \geq |a - b| - L
\]

because \(z, z' \in I \implies |z - z'| \leq L\). This implies the lemma. \(\square\)

Because of this uniform comparison we have the following surprising fact, even though the pair of spaces we consider are unbounded:

**Lemma 7.7.** If \(X\) is an \(N\) dimensional pulled thread space with an interval of length \(L\), then

\[
d_{GH}((X, d_X), (\mathbb{R}^N, d_E)) \leq 2L
\]

where \(d_E(v, w) = |v - w|\).
Proof. We set up the correspondence
\[ C = \{ (\pi(w), w) \in X \times \mathbb{E}^N : w \in \mathbb{E}^N \} \]
which is a correspondence because \( \pi : \mathbb{E}^N \to X \) is surjective. It is \( L \) distance
preserving because
\[ |d(\pi(v), \pi(w)) - |v - w|| = |\bar{d}(v, w) - |v - w|| \leq L. \]
The lemma then follows from Theorem 7.3. \( \square \)

7.3. Rescaling Pulled Thread Spaces.

Lemma 7.8. In a pulled thread space with an interval \( I \) of length \( L \), we have
\[ \lim_{R \to \infty} \frac{\bar{d}(Rx, Rx)}{R} = |x - y| \]
where the convergence is uniform on \( \mathbb{E}^N \):
\[ |\bar{d}(Rx, Ry)/R - |x - y|| \leq L/R \quad \forall a, b \in \mathbb{E}^N. \]

Proof. By Lemma 7.6 applied with \( a = Rx \) and \( b = Ry \), we have
\[ |\bar{d}(Rx, Ry)/R - |x - y|| \leq L/R \]
so \( \lim_{R \to 0} |\bar{d}(Rx, Ry)/R - |x - y|| = 0 \) uniformly on \( \mathbb{E}^N \). \( \square \)

Theorem 7.9. If \( X \) is an \( N \) dimensional pulled thread space with an interval \( I \) of
length \( L \), then it has a unique tangent cone at infinity which is \( \mathbb{E}^N \) endowed with
the standard Euclidean metric \( d_E(v, w) = |v - w| \).

Proof. Take any \( x_0 \in X \). By shifting the location of the interval, \( I \), we may assume
that \( \pi(0) = x_0 \) where \( \pi : \mathbb{E}^N \to X \) is the pulled thread map.

We need to show that for all \( r > 0 \)
\[ \lim_{R \to \infty} d_{GH}((B_{Rr}(x_0), d_X/R), (B_r(0), d_E)) = 0 \]
and we will do this by finding a correspondence for each \( R, r > 0 \). Let
\[ U_{Rr}(x_0) = \pi^{-1}(\bar{B}_{Rr}(x_0)) \subset \mathbb{E}^N. \]
Note that by the fact that
\[ d(x, x_0) = \bar{d}(u, 0) \leq |u - 0| \text{ when } \pi(u) = x \]
we have
\[ B_{Rr}(0) \subset U_{Rr}(x_0). \]
We set up a correspondence
\[ C_R = \{ (\pi(w), f(w)) : w \in U_{Rr}(x_0) \} \subset B_{Rr}(x_0) \times B_r(0). \]
where \( \pi \) is the pulled thread map and \( f : U_{Rr}(x_0) \to \bar{B}_r(0) \) is defined:
\[ f(w) = \begin{cases} \frac{w}{R} & \text{if } |w| < rR \\ \frac{r}{|w|} & \text{if } w \geq rR. \end{cases} \]
This is a correspondence, because \( \pi : U_{Rr}(x_0) \to B_R(x_0) \) and \( f : U_{Rr}(x_0) \to \bar{B}_r(0) \) are surjective.

We claim that \( C \) is \((3L)/R\) almost distance preserving:

\[
|d_X(\pi(v), \pi(w))/R - |f(v) - f(w)|/R| \leq (3L)/R.
\]

Observe that

\[
|f(w) - w/R| = \begin{cases} 0 & \text{if } |w| < rR \\ (|w|/R) - r & \text{if } w \geq rR. \end{cases}
\]

and since \(|w| \leq Rr + L| on \ U_{Rr}(x_0)\), we have

\[
|f(w) - w/R| \leq L/R \quad \forall w \in U_{Rr}(x_0).
\]

So

\[
||f(v) - f(w)| - |v/R - w/R|| \leq 2L/R.
\]

Thus we have our claim:

\[
|d_X(\pi(v), \pi(w))/R - |f(v) - f(w)| | = |\bar{d}(v, w)/R - |v - w|/R + 2L/R
\]

\[
= (1/R)(|\bar{d}(v, w) - |v - w|| + 2L)
\]

\[
\leq (1/R)(L + 2L) = (3L)/R.
\]

So by Theorem 7.3 we have

\[
d_{GH}(\{B_{Rr}(x_0), d_X/R\}, \{B_{r}(0), d_E\}) \leq 2(3L)/R \to 0 \text{ as } R \to \infty.
\]

Note that we do not need a subsequence, nor does this depend on the base point, \( x_0 \).

\[ \square \]

8. Approximating the Smocking Pseudometric

In this section, we analyze the smocking pseudometric. In the first subsection we prove a key lemma which will allow us to estimate the smocking distances between points when one can only approximate the smocking distances between intervals. The next few subsections find the smocking pseudometric for three of our smocked spaces: \( X_T, X_+, \) and \( X_\Box \). The proofs for the other smocked spaces, \( X_\diamond, X_=, \) and \( X_H, \) are significantly more difficult, so we postpone them to our next paper [8].

8.1. Key Lemma.

Lemma 8.1. Given an \( N \) dimensional smocked space parametrized by points in intervals as in [50], with smocking depth, \( h \in (0, \infty) \), and smocking length \( L = L_{\text{max}} \in (0, \infty) \), if one can find a Lipschitz function \( F : \mathbb{E}^N \to [0, \infty) \) such that

\[
|d(I_j, I_{j'}) - [F(j) - F(j')]| \leq C,
\]

then

\[
|\bar{d}(x, x') - [F(x) - F(x')]| \leq 2h + C + 2dil(F)(h + L)
\]
where \( \text{dil}(F) \) is the dilation factor or Lipschitz constant of \( F \):

\[
\text{dil}(F) = \sup \left\{ \frac{|F(a) - F(b)|}{|a - b|} : a \neq b \in \mathbb{R}^N \right\}.
\]

**Proof.** Given any \( x, x' \in \mathbb{R}^N \), by the definition of smocking depth, we have closest points in closest intervals, \( z \in I_j \) and \( z' \in I_{j'} \), such that

\[
\tag{302}
d(x, I_j) = \bar{d}(x, z) = |x - z| \leq h \quad \text{and} \quad d(x', I_{j'}) = \bar{d}(x', z') = |x' - z'| \leq h.
\]

Since our smocked space is parametrized by points in intervals we have

\[
\tag{303}
|z - j| \leq L \quad \text{and} \quad |z' - j'| \leq L.
\]

So

\[
\tag{304}
|x - j| \leq L + h \quad \text{and} \quad |z' - j'| \leq L + h.
\]

By the definition of the smocking pseudometric, we have

\[
\tag{305}
\bar{d}(z, z') = d(\pi(z), \pi(z')) = d(I_j, I_{j'}).
\]

Thus by the \( \bar{d} \) triangle inequality twice we have

\[
\tag{306}
\bar{d}(x, x') \leq \bar{d}(x, z) + \bar{d}(z, z') + \bar{d}(z', x')
\]

\[
\tag{307}
\leq h + d(I_j, I_{j'}) + h.
\]

\[
\tag{308}
d(I_j, I_{j'}) = \bar{d}(z, z') \leq \bar{d}(x, z) + \bar{d}(x, x') + \bar{d}(z', x')
\]

\[
\tag{309}
\leq h + d(x, x') + h.
\]

So

\[
\tag{310}
|\bar{d}(x, x') - d(I_j, I_{j'})| \leq 2h.
\]

We are given that

\[
\tag{311}
|d(I_j, I_{j'}) - [F(j) - F(j')]| \leq C,
\]

so by the triangle inequality we have

\[
\tag{312}
|\bar{d}(x, x') - [F(j) - F(j')]| \leq 2h + C.
\]

By the definition of dilation and (304), we know

\[
\tag{313}
|F(j) - F(x)| \leq \text{dil}(F) |j - x| \leq \text{dil}(F) (h + L)
\]

\[
\tag{314}
|F(j') - F(x')| \leq \text{dil}(F) |j' - x'| \leq \text{dil}(F) (h + L).
\]

Thus

\[
\tag{315}
|[F(j) - F(j')] - [F(x) - F(x')]| \leq |F(j) - F(x)| + |F(j') - F(x')| \leq 2\text{dil}(F) (h + L).
\]

Combining this with (312), we have

\[
\tag{317}
|\bar{d}(x, x') - [F(x) - F(x')]| \leq 2h + C + 2\text{dil}(F) (h + L).
\]

\( \square \)
8.2. Estimating the Distances $d_+$. In this subsection we estimate the distance between the plus stitches in $X_+$. First, by examination it appears that the optimal paths of segments joining one plus stitch to another is found by traveling first vertically and then horizontally (see Figure 23). The sum of the length of vertical segments between two plus stitches is $1/3$ of the vertical distance between the centers of the plus stitches and the sum of the length of horizontal segments between two plus stitches is $1/3$ of the horizontal distance between the centers of the plus stitches. This intuitively allows us to guess the formula for the distance in lemma below. To prove the lemma rigorously one must ensure that there are no other shorter paths.

![Figure 23](image-url) Optimal paths of segments from various $I_{(j_1, j_2)}$ to $I_{(0, 0)}$ are drawn in white.

**Lemma 8.2.** For any two pairs $(j_1, j_2), (k_1, k_2) \in J_+$, we have the distance between smocking stitches

$$d_+(I_{(j_1, j_2)}, I_{(k_1, k_2)}) = \frac{|k_1 - j_1| + |k_2 - j_2|}{3}.$$  

**Proof.** We proceed by inducting on the sum $N = (|j_1 - k_1| + |j_2 - k_2|)/3$, which is a whole number since $j, k \in 3\mathbb{Z} \times 3\mathbb{Z}$. By symmetry, at any stage we may assume $(j_1, j_2) = 0$ and $k_1 \geq k_2 \geq 0$. Define

$$J_N = \left\{ (j_1, j_2) : \frac{|j_1| + |j_2|}{3} \leq N \right\}.$$  

For $N = 1$, (318) can be demonstrated directly because the distance is achieved by a single horizontal segment:

$$d_+((3, 0), (0, 0)) = |(2, 0) - (1, 0)| = 1 = (3 + 0)/3.$$  


Assume that (318) holds for all \((k_1, k_2), (j_1, j_2) \in J_+\) satisfying

\[ (|j_1 - k_1| + |j_2 - k_2|)/3 = N \]

for some whole number \(N \geq 1\). Now we must prove (318) for all \((k_1, k_2), (j_1, j_2) \in J_+\) satisfying

\[ (|j_1 - k_1| + |j_2 - k_2|)/3 = N + 1. \]

By symmetry, we can consider only \(k_1 > k_2 > 0\). Then taking \(i_1 = k_1 - 3\) and \(i_2 = k_2\), we have \((i_1, i_2) \in S_N\). By the triangle inequality and induction,

\[
d_+(I(k_1, k_2), I(0,0)) \leq d_+(I(k_1, k_2), I(i_1, i_2)) + d_+(I(i_1, i_2), I(0,0)) = 1 + (i_1 + i_2)/3 = (3 + k_1 - 3 + k_2)/3 = (|k_1| + |k_2|)/3.
\]

The distance between \(I(k_1, k_2)\) and \(I(0,0)\) is achieved by a sum of lengths of segments between intervals. Note that every one of these segments will head towards the origin, otherwise they could be replaced with one of the same length that does head towards the origin and the overall sum of segments would be shorter.

We claim it cannot be achieved by a single segment from \(I(k_1, k_2)\) to \(I(0,0)\) unless \(k_1 = 3\) and \(k_2 = 0\). If it could then the segment would connect the right tip of \(I(0,0)\) which is \((1, 0)\) to the bottom tip of \(I(k_1, k_2)\) which is \((k_1, k_2 - 1)\) since \(k_1 \geq k_2 > 0\). So

\[
d_+(I(k_1, k_2), I(0,0)) = |(k_1, k_2 - 1) - (1, 0)| = \sqrt{(k_1 - 1)^2 + (k_2 - 1)^2} = \sqrt{k_1^2 + k_2^2 - 2(k_1 + k_2)}
\]

\[
= \frac{1}{3} \sqrt{9(k_1^2 + k_2^2) - 18(k_1 + k_2)}
\]

\[
= \frac{1}{3} \sqrt{8(k_1^2 + k_2^2) + (k_1 - k_2)^2 + 2k_1k_2 - 18(k_1 + k_2)}
\]

\[
\geq \frac{1}{3} \sqrt{k_1^2 + k_2^2 + 2k_1k_2 + 7(k_1^2 + k_2^2) - 18(k_1 + k_2)}
\]

\[
= \frac{1}{3} \sqrt{k_1^2 + k_2^2 + 2k_1k_2 + 63\left((\frac{k_1}{3})^2 + (\frac{k_2}{3})^2\right) - 54\left((\frac{k_1}{3}) + (\frac{k_2}{3})\right)}
\]

\[
\geq \frac{1}{3} \sqrt{k_1^2 + k_2^2 + 2k_1k_2}
\]

because \(k_i/3 \geq 1\)

\[
= \frac{|k_1| + |k_2|}{3}.
\]

So the distance between the plus sewing stitches is achieved by a collection of segments which passes through some \(I(i_1, i_2)\). Since the segments head towards the origin, we have

\[ (i_1, i_2) \in J_N \text{ and } (k_1 - i_1, k_2 - i_2) \in J_N. \]
Since this interval \(I_{(i_1, i_2)}\) lies along a shortest collection of segments, we have
\[
d_+(I_{(k_1, k_2)}, I_{(0,0)}) = d_+(I_{(k_1, k_2)}, I_{(i_1, i_2)}) + d_+(I_{(i_1, i_2)}, I_{(0,0)})
\]
\[
= (|k_1 - i_1| + |k_2 - i_2|)/3 + (|i_1 - 0| + |i_2 - 0|)/3 \text{ by the ind. hyp.}
\]
\[
= (k_1 - i_1 + k_2 - i_2)/3 + (i_1 - 0 + i_2 - 0)/3 \text{ by } k_1 > i_1 > 0
\]
\[
= (k_1 + k_2)/3 = (|k_1| + |k_2|)/3.
\]

8.3. **Estimating the Distances** \(d_{\Box}\). In this section we prove Proposition 8.3. The proof itself will motivate the formula for the distance.

**Proposition 8.3.** For any two pairs \((i_1, i_2), (j_1, j_2) \in J_{\Box}\), we have the distance between square smocking stitches
\[
d_{\Box}(I_{(i_1, i_2)}, I_{(j_1, j_2)}) = 2 \sqrt{2} \min \left( \left\lfloor \frac{|i_1 - j_1|}{3} \right\rfloor, \left\lfloor \frac{|i_2 - j_2|}{3} \right\rfloor \right) + 2 \min \left( \left\lfloor \frac{|i_1 - j_1|}{3} - \frac{|i_2 - j_2|}{3} \right\rfloor \right).
\]

**Proof.** Thanks to the symmetry of the placement of the squares, \(d_{X_{\square}} = d_{\Box} = d\) is translation invariant. Thus we will fix \(I_{(0,0)}\) and see how to compute \(d(I_{(0,0)}, I_{(j_1, j_2)})\) for \(j_2 \geq j_1 > 0\). This will allow us to compute the distance between any two squares in the lattice since
\[
d(I_{(0,0)}, I_{(j_1, j_2)}) = d(I_{(0,0)}, I_{(|j_1|, |j_2|)}) = d(I_{(0,0)}, I_{(|j_2|, |j_1|)}),
\]
once again because of the symmetry of the square lattice. So we need only prove that, for \(j_2 \geq j_1 \geq 0\),
\[
d_{\Box}(I_{(0,0)}, I_{(j_1, j_2)}) = \sqrt{8} \min \left( \frac{j_2}{3} \right) + 2 \left( \frac{j_2 - j_1}{3} \right).
\]
We will work up to this general formula slowly. First consult Figure 24 for adjacent square stitches.

**Figure 24.** Key segments needed to estimate \(d_{\Box}(I_j, I_k)\).
We claim
\[(327) \quad d((I_{i,j}), I_{i+3,j+3}) = d(I_{0,0}, I_{0,3}) = 2.\]

It is quite easy to see that this will be the Euclidean distance between these two squares. Since we can compute the distance by looking at the distance between the closest corners of the two squares. In the case of \(I_{0,0}\) and \(I_{3,0}\) we will simply have to find the shortest distance between the points \((1, 1)\) and \((3, 1)\). This gives us
\[(328) \quad ||(3 - 1, 1 - 1)|| = \sqrt{2^2 + 0} = 2.\]

We claim
\[(329) \quad d((I_{i,j}), I_{i+3,j+3}) = d(I_{0,0}, I_{0,3}) = \sqrt{8}.\]

We use the Pythagorean theorem here, as the diagonal is shorter than going across the horizontal and then vertical. Since we can compute the distance by looking at the distance between the closest corners we will simply have to find the shortest distance between the points \((1, 1)\) and \((3, 3)\). This gives us \(||(3 - 1, 3 - 1)|| = ||(2, 2)|| = \sqrt{2^2 + 2^2},\) thus we arrive at a distance of \(\sqrt{8}\).

We claim
\[(330) \quad d((I_{i,j}), I_{i+3,j+6}) = d(I_{0,0}, I_{3,6}) = \sqrt{8} + 2 = \sqrt{8} + \left(\frac{8}{\frac{3}{2}} - 1\right).\]

Clearly the only contenders for shortest paths are the straight line from \(I_{0,0}\) to \(I_{3,6}\) of length \(\sqrt{29}\) and the path that takes one diagonal jump connecting \(I_{0,0}\) and \(I_{3,3}\) plus the straight line \(I_{3,3}\) to \(I_{3,6}\) of length \(\sqrt{8} + 2 < \sqrt{29}\).

We claim that \(j_2 \in 3\mathbb{Z}, j_2 > 3,\)
\[(331) \quad d(I_{0,0}, I_{3,j_2}) = \sqrt{8} + 2(j_2 - 3)/3,\]
which is achieved by taking one diagonal segment of length \(\sqrt{8}\) followed by \((\frac{j_2}{3} - 1)\) vertical segments of length 2. We must verify that there are no shorter paths.

We will prove this by induction on \(n\) taking \(j_2 = 3n\). Note the base case has been proven for \(n = 1\) in \((329)\) and also \(n = 2\) in \((330)\). Suppose we have \((331)\) for all \(j_2 \leq 3(n - 1)\). We prove it for \(j_2 = 3n\). Note that since \((331)\) holds for all \(j_2 \leq 3(n - 1)\), this means that it is always shorter to take a single diagonal jump connecting \(I_{0,0}\) and \(I_{3,3}\) followed by vertical segments, than it is to take any diagonals of slope greater than 1, except possibly to go on a single straight segment from \(I_{0,0}\) to \(I_{3,3n}\). The length of that single segment is
\[(332) \quad d(I_{0,0}, I_{3,3n}) = ||(1, 1), (3, 3n)|| = ||(2, 3n - 1)|| = \sqrt{9n^2 - 6n + 4}.\]

Now we square the distance of taking two jumps \(\sqrt{8 + 2(n - 1)}\) to get \(4n^2 - 4n + 8\sqrt{2}\). So to prove our induction step, we just have to prove that, for \(n > 1,\)
\[(333) \quad 9n^2 - 6n + 4 > 4n^2 - 4n + 8\sqrt{2}.\]

We will prove \((333)\) by induction. Plugging in \(n = 2\) we have
\[(334) \quad 28 > 24 > 8 + 8\sqrt{2},\]
that proves the base case. Now assuming the induction hypotheses

\[(335) \quad 9n^2 - 6n + 4 > 4n^2 - 4n + 8 \sqrt{2}\]

holds we wish to prove

\[(336) \quad 9(n + 1)^2 - 6(n + 1) + 4 > 4(n + 1)^2 - 4(n + 1) + 8 \sqrt{2}.
\]

To see this note that

\[
9(n + 1)^2 - 6(n + 1) + 4 = 9n^2 - 6n + 4 + 18n + 3
\]

\[
> 4n^2 - 4n + 8 \sqrt{2} + 18n + 3 \quad \text{by the ind. hyp.}
\]

\[
> 4(n + 1)^2 - 4(n + 1) + 8 \sqrt{2} \quad \text{by } n > 1.
\]

Thus (335) holds for all \(n\), which implies that the single segment is not shorter so we have (331).

Before we continue to larger values of \(j_2\), we observe that any line \(\{(x, y) : y = m(x - 1) + 1\}\) starting at \((1, 1)\) that does not hit a square with \(j_1 = 3\) as seen in Figure 25 must have slope

\[
(337) \quad 3/2 \leq m \leq 5/3.
\]

It follows that such a line must pass above \(I_{(3,i_2)}\) and below \(I_{(3,i_2+3)}\) for some \(i_2 \geq 3\). Thus when this line crosses \(x = 3\) and \(x = 4\), we have

\[(338) \quad i_2 + 1 < m(3 - 1) + 1 \quad \text{and} \quad m(4 - 1) + 1 < i_2 + 3,
\]

which implies (339).

**Figure 25.** Lines with slope greater than 2 must intersect the second column of squares.
We claim that, for $j_1 = 6$ and $j_2 \in \mathbb{Z}$, $j_2 \geq 6$,
\begin{equation}
    d(I(0,0), I(6,j_2)) = 2\sqrt{8} + 2(j_2 - 6)/3,
\end{equation}
which is achieved by taking two diagonal segment of length $\sqrt{8}$ followed by $(j_2 - 6)/3$ vertical segments of length 2. We must verify that there are no shorter paths. By (331) shifted, we know
\begin{equation}
    d(I(3,j_2), I(6,j_2)) = d(I(0,0), I(3,j_2 - i_2)) = \sqrt{8} + 2(j_2 - i_2)/3.
\end{equation}
So any shortest path of segments between $I(0,0)$ and $I(6,j_2)$ which touches a square on the middle column, $I(3,j_2)$, must consist of only diagonals of length $\sqrt{8}$ and verticals of length 2. Such a path has the length in (339). We need only show that a direct path would have slope as in (339). Thus our only concern is when $j_2 = 9$. However we see that the direct segment has longer length too:
\begin{equation}
    \|(6, 9) - (1, 1)\| = \sqrt{5^2 + 8^2} = \sqrt{89} > 2\sqrt{8} + 2
\end{equation}
because
\[
    (2\sqrt{8} + 2)^2 = 32 + 4\sqrt{8} + 4 = 32 + 8\sqrt{2} + 4 < 32 + 16 + 4 = 52 < 89 = \left(\sqrt{89}\right)^2.
\]
We complete the proof of the proposition inductively, still assuming $j_2 \geq j_1 \geq 0$:
Statement$(n)$: $\forall j_2 - i_2 \geq j_1 - i_1 = 3n$
\begin{equation}
    d_0(I(i_1,j_2), I(j_1,j_3)) = 2\sqrt{2}((j_1 - i_1)/3 + 2((j_2 - i_2) - (j_1 - i_1))/3
\end{equation}
which is achieved by taking $n = (j_1 - i_1)/3$ diagonal segment of length $\sqrt{8}$ followed by $(j_2 - j_1)/3$ vertical segments of length 2.
We’ve already proven two base cases $n = 1$ and $n = 2$. Assuming we have the induction hypothesis, Statement$(n)$, we now show Statement$(n+1)$. We first observe that any shortest path between the squares which passes through a square on the way must consist of a pair of shortest paths satisfying the induction hypotheses, and so all the segments are diagonal of length $\sqrt{8}$ or vertical of length 2. The only possible way to have a shorter path is to go directly between the endpoints without hitting any squares in between. We claim that for $n \geq 3$ there are no such direct paths. To see this we shift $I(i,j_2)$ to $I(0,0)$ so the line segment can be written as $\{(x, y) : y = m(x - 1) + 1\}$. By (339), such a direct path would have to have slope $m \in [3/2, 2]$. We must also avoid hitting the square $I(6,9)$ as depicted in Figure 25. However any line segment which passes beneath square $I(6,9)$ has slope
\begin{equation}
    m_{\text{under}} \leq (9 - 1)/(7 - 1) = 8/6 = 4/3 < 3/2
\end{equation}
and any line segment which passes above square $I(6,9)$ has slope
\begin{equation}
    m_{\text{above}} \leq (10 - 1)/(6 - 1) = 9/5 > 2.
\end{equation}
So there are no direct paths when $n \geq 3$. Thus we have proven the induction step and the proposition follows. \qed
8.4. **Estimating the Distances** $d_T$. In this subsection we prove a formula for the distance between the stitched line segments in $X_T$ [Proposition 8.6]. Recall that the $d_T$-distance between two points is the infimum of lengths of paths composed of straight line paths. We will show that every path between $I_{0,0}$ and another stitch $I_{i_1,j_2}$ is at least as long as a path which is constructed via a composition of horizontal and vertical lines of length 1 between neighboring stitches. See Figure 26 for an exploration as to how one might jump from one interval to the next.

![Figure 26](image)

**Figure 26.** Optimal paths of segments from various $I_{(j_1,j_2)}$ to $I_{(0,0)}$ are drawn in white.

To begin our discussion, we characterize $d_T$-minimizing line segments leaving $I_{(0,0)}$.

**Lemma 8.4.** Let $L \subset \mathbb{E}^2$ be a line segment minimizing the $d_T$-distance from $I_{0,0}$ to $I_{(k_1,k_2)}$ with $k_1, k_2 \geq 0$. Then either $L$ has infinite slope and begins at $(0,0) \in I_{(0,0)}$ or $L$ has non-negative slope and begins at $(1,0) \in I_{(0,0)}$.

**Proof.** We only consider the case when the $L$ does not intersect any other stitch $I_{(i_1,i_2)}$ since we may apply our argument to the first such segment of a general line. First assume $L$ is vertical. If $L$ does not leave from $(0,0)$, then it must terminate at stitch $I_{(0,4)}$ and have length 4. However, $d_T(I_{(0,0)}, I_{(0,4)})$ is evidently 2, giving a contradiction.

Now consider the case where $L$ has finite slope. Since $k_1, k_2 \geq 0$, $L$ must have non-negative slope. If $L$ leaves from point $(a,0)$ for $a < 1$, then $L$ does not even locally minimize distance from $I_{(0,0)}$. \hfill \Box

**Lemma 8.5.** Let $\gamma \subset \mathbb{E}^2$ be a collection of lines minimizing the $d_T$-distance from $I_{(i_1, i_2)}$ to $I_{(k_1,k_2)}$ with $k_1 \geq i_1$ and $k_2 \geq i_2$. Then $\gamma$ will intersect $I_{(i_1+2,i_2)}$, $I_{(i_1,i_2+2)}$ or $I_{(i_1+2,i_2+2)}$.

**Proof.** We first consider the case where $I_{(i_1, i_2)} = I_{(0,0)}$, which is a horizontal line segment. Notice that the line with slope 1 from $(1,0)$ intersects both $I_{(0,2)}$ and $I_{(2,2)}$. As the slope tends towards $\infty$ the line the line intersects $I_{(2,2)}$ and as the slope tends towards 0 the line intersects neighboring the vertical line $I_{(0,2)}$. So if the first
segment of \( \gamma \) has positive slope, Lemma \[ \text{8.4} \] says it must leave \((1, 0)\) and hence intersect \( I_{(2, 0)} \) or \( I_{(2, 2)} \). If the first segment of \( \gamma \) has infinite slope, Lemma \[ \text{8.4} \] says it must leave from \((0, 0)\) and hence intersect \( I_{(0, 2)} \).

If \( \gamma \) begins from some other horizontal stitch \( I_{(i_1, j_2)} \), we can translate \( I_{(0, 0)} \) to the horizontal line segment \( I_{(i_1, j_2)} \) under a \( d_T \)-isometry, which yields that a line with positive slope from \( I_{(i_1, j_2)} \) must intersect \( I_{(i_1+2, j_2)}, I_{(i_1, j_2+2)} \) or \( I_{(i_1+2, j_2+2)} \).

Now we consider the case where \( I_{(i_1, j_2)} \) is a vertical stitch. Consider the transformation

\[
T : \mathbb{E}^2 \rightarrow \mathbb{E}^2, \quad T(x_1, x_2) = (x_2 + 2, x_1),
\]

which is the composition of reflection about the line \( x_1 = x_2 \) and a horizontal translation. \( T \) preserves the stitching set and hence yields an isometry of \( X_T \). Applying this transformation to \( \gamma \) reduces us to the previous case since \( T \) takes vertical stitches to horizontal stitches. This completes the proof of Lemma \[ \text{8.5} \]. \( \square \)

Now we are ready to state and proof the formula for \( d_T \)-distances between stitches in \( X_T \).

**Proposition 8.6.** For any two pairs \( (j_1, j_2), (k_1, k_2) \) \( \in J_T \), we have

\[
d_T(I_{(j_1, j_2)}, I_{(k_1, k_2)}) = \frac{|j_1 - k_1| + |j_2 - k_2|}{2}.
\]

**Proof.** By making use of the reflectional and translational symmetry of the stitches that define \( X_T \), it suffices to demonstrate formula \[ \text{346} \] holds for \( j_1 = j_2 = 0 \) and \( k_1, k_2 \geq 0 \). We will show that

\[
d_T(I_{(0, 0)}, I_{(k_1, k_2)}) = \frac{k_1 + k_2}{2}
\]

by inducting on the whole number \( N = \frac{k_1 + k_2}{2} \). For a whole number \( N > 0 \), consider the collection

\[
B_N = \{(k_1, k_2) \in J_+ : k_1, k_2 \geq 0, \frac{k_1 + k_2}{2} = N\}.
\]

Formula \[ \text{347} \] holds for all \( (k_1, k_2) \in B_1 \) since \( I_{(k_1, k_2)} \) can be reached by unit-length segments emanating from \( I_{(0, 0)} \) and the tubular neighborhood of radius 1 about \( I_{(0, 0)} \) contains no other stitches.

Now consider a whole number \( N > 1 \) and assume formula \[ \text{347} \] holds for all points in \( B_n \) where \( n = 1, 2, \ldots, N \). Let \( (k_1, k_2) \in B_{N+1} \) and let \( \gamma \) be a distance-minimizing path from \( I_{(k_1, k_2)} \) to \( I_{(0, 0)} \). Consider \( \gamma_0 \subset \mathbb{E}^2 \), the component of \( \gamma \) which emanates from \( I_{(k_1, k_2)} \). Notice that \( \gamma_0 \) is a single line segment which realizes the \( d_T \)-distance between \( I_{(k_1, k_2)} \) and the stitch it terminates in, which we denote by \( I_{(i_1, i_2)} \).

Applying Lemma \[ \text{8.5} \] to \( \gamma_0 \), we find that \( I_{(i_1, i_2)} \) must be one of three possibilities: \( I_{(k_1 - 2, k_2)}, I_{(k_1 + 1, k_2 - 2)} \), or \( I_{(k_1 - 2, k_2 - 2)} \). We consider these cases.

If \( I_{(i_1, i_2)} = I_{(k_1 - 2, k_2)} \) or \( I_{(k_1, k_2 - 2)} \), then \( (i_1, i_2) \in B_N \) and

\[
d_T(I_{(0, 0)}, I_{(k_1, k_2)}) = d_T(I_{(0, 0)}, I_{(i_1, i_2)}) + d_T(I_{(i_1, i_2)}, I_{(k_1, k_2)}) = N + 1 = \frac{k_1 + k_2}{2}
\]
where we have used the inductive hypothesis. This verifies the desired formula (347) in this case.

In the case that \( I(i_1, i_2) = I(k_1 - 2, k_2 - 2) \), then \( (i_1, i_2) \in B_{N-1} \) and we may use the inductive hypothesis to find

\[
d_T(I(0, 0), I_{i_1, i_2}) = d_T(I(0, 0), I_{i_1, i_2}) + d_T(I_{i_1, i_2}, I_{k_1, k_2}) = (N - 1) + 2 = \frac{k_1 + k_2}{2}.
\]

This finishes the proof of the lemma. \( \Box \)

9. Rescaling Smocked Spaces

Recall that in Theorem 7.9 we proved that the tangent cone at infinity of a pulled thread space is unique and is Euclidean space with the standard Euclidean metric. Later in this section we will show that we also obtain unique tangent cones for various smocked spaces, which are normed spaces but not Euclidean space. The first subsection has a theorem which will be a key ingredient used in those proofs. The next few subsections find the tangent cone at infinity for three of our smocked spaces: \( X_T, X_+, \) and \( X_\square \). The proofs for the other smocked spaces, \( X_\triangledown, X_\ast, X_\times, \) and \( X_H \), are significantly more difficult, so we postpone them to our next paper [8].

9.1. Main Theorem.

**Theorem 9.1.** Suppose we have an \( N \) dimensional smocked space, \((X, d)\), as in Definition 3.1 such that

\[
|\tilde{d}(x, x') - [F(x) - F(x')]| \leq K \quad \forall x, x' \in E^N
\]

where \( F : E^N \to [0, \infty) \) is a norm. Then \((X, d)\) has a unique tangent cone at infinity,

\[
\left(\mathbb{R}^N, d_F\right) \quad \text{where } d_F(x, x') = \|x - x'\|_F = F(x - x').
\]

**Proof.** Take any \( x_0 \in X \). By shifting the smocking set, \( S \), we may assume that \( \pi(0) = x_0 \) where \( \pi : E^N \to X \) is the smocking map.

We need to show that for all \( r > 0 \)

\[
\lim_{R \to \infty} d_GH((B_{Rr}(x_0), d_X/R), (B_r(0), d_F)) = 0
\]

and we will do this by finding a correspondence for each \( R, r > 0 \). Let

\[
U_s(x_0) = \pi^{-1}(B_s(x_0)) = \{u \in E^N : \tilde{d}(u, 0) \leq s\}.
\]

Note that by the fact that

\[
\tilde{d}(u, 0) \leq F(u) + K \quad \text{when } \pi(u) = x,
\]

We have

\[
U_{Rr}(x_0) \supset \{u : F(u) \leq Rr - K\} = F^{-1}[0, Rr - K].
\]

We set up a correspondence

\[
C_R = \{(\pi(w), f(w)) : w \in U_{Rr}(x_0) \subset B_{Rr}(x_0) \times B_r(0)\}.
\]
Thus for all
\[ f : U_{R}(x_0) \to \bar{B}_r(0) = F^{-1}[0, r] \]
is defined to be
\[
f(w) = \begin{cases} 
  rw/(Rr - K) & \text{if } F(w) < rR - K \\
  rw/F(w) & \text{if } F(w) \geq rR - K.
\end{cases}
\]
This is a correspondence, because \( \pi : U_{R}(x_0) \to B_{Rr}(x_0) \) and \( f : U_{R}(x_0) \to \bar{B}_r(0) \) are surjective.

We claim that \( C_{R} \) is \( \epsilon_{R} \) almost distance preserving:
\[
|d_{X}(\pi(v), \pi(w)))/R - ||f(v) - f(w)||_{F}| \leq \epsilon_{R} \text{ where } \epsilon_{R} \to 0 \text{ and } R \to \infty.
\]
Observe that
\[
\left\| f(w) - \left( \frac{r}{(Rr - K)} \right) w \right\|_{F} = \begin{cases} 
  0 & \text{if } F(w) < rR - K \\
  (||w||_{F} - (Rr - K)) \left( \frac{r}{(Rr - K)} \right) & \text{if } F(w) \geq rR - K.
\end{cases}
\]
For \( w \in U_{R}(x_{0}) \), we have \( ||w||_{F} \leq Rr + K \), so
\[
\left\| f(w) - \left( \frac{r}{(Rr - K)} \right) w \right\|_{F} \leq \frac{2rK}{(Rr - K)}
\]
and
\[
\left\| \left( \frac{r}{(Rr - K)} \right) w - \frac{1}{R} w \right\|_{F} \leq \left( \frac{r}{(Rr - K)} - \frac{1}{R} \right) ||w||_{F} \leq \frac{K}{R(Rr - K)}(Rr + K).
\]
Thus for all \( v, w \in U_{R}(x_{0}) \), we have
\[
||f(v) - f(w)||_{F} - ||w - \frac{r}{R} w||_{F} \leq \Psi(R; K, r)
\]
where
\[
\Psi(R; K, r) = \frac{2rK}{Rr - K} + \frac{K(Rr + K)}{R(Rr - K)} \to 0 \text{ as } R \to \infty.
\]
Taking \( \epsilon_{R} = (1/R)K + \Psi(R; K, r) \), we see that \( C_{R} \) is \( \epsilon_{R} \) almost distance preserving:
\[
|d_{X}(\pi(v), \pi(w))/R - ||f(v) - f(w)||_{F}| = |\tilde{d}(v, w)/R - ||v - w||_{F}/R| + \Psi(R; K, r) = \frac{1}{R}|\tilde{d}(v, w) - ||v - w||_{F}| + \Psi(R; K, r) \leq \epsilon_{R}
\]
with \( \text{lim}_{R \to \infty} \epsilon_{R} = \text{lim}_{R \to \infty}(1/R)K + \Psi(R; K, r) = 0 \). So by Theorem [7.3] we have
\[
d_{GH}((B_{R}(x_{0}), d_{X}(R)), (B_{r}(0), d_{E})) \leq 2(3K)/R \to 0 \text{ as } R \to \infty.
\]
We do not need a subsequence nor did this limit depend on the base point, \( x_0 \). \( \square \)

9.2. The Tangent Cone at Infinity of \( X_{+} \). In this section we prove the following theorem, which is depicted in Figure [27].

**Theorem 9.2.** The tangent cone at infinity of \( (X_{+}, d_{+}) \) is \( (\mathbb{R}^2, d_{F+}) \), where
\[
d_{F+}(\tilde{x}, \tilde{y}) = F_{+}(\tilde{x} - \tilde{y}) \text{ where } F_{+}(\tilde{x}) = ||\tilde{x}||_{F} = \frac{|x_1| + |x_2|}{3}.
\]
Note that in the limit space, the shape of a ball, \{ \tilde{x} \in \mathbb{R}^2 : F_{+}(\tilde{x}) = R \}, is a diamond with vertices at \((\pm 3, 0)\) and \((0, \pm 3)\).
We will prove this by applying Theorem 9.1. Recall that the function $F_+$ was first found in Lemma 8.2.

\[(368) \quad d_+(I(j_1,j_2), I(j_1',j_2')) = \frac{|j_1 - j_1'| + |j_2 - j_2'|}{3} = F_+(\vec{j} - \vec{j}').\]

We will prove a series of lemmas to show we have all the hypotheses needed to apply Theorem 9.1 taking $F = F_+$.

**Lemma 9.3.** $F_+$ as in (367) is a norm.

**Proof.** It is definite:

\[(369) \quad 0 = F_+(\vec{x}) = \frac{|x_1| + |x_2|}{3} \iff x_1 = x_2 = 0.\]

It scales:

\[(370) \quad F_+(R\vec{x}) = \frac{|Rx_1| + |Rx_2|}{3} = |R| \frac{|x_1| + |x_2|}{3} = |R|F_+(\vec{x}).\]

It satisfies the triangle inequality:

\[(371) \quad F_+(\vec{x} + \vec{y}) = \frac{|x_1 + y_1| + |x_2 + y_2|}{3} = \frac{|x_1 + y_1|}{3} + \frac{|x_2 + y_2|}{3} = F_+(\vec{x}) + F_+(\vec{y}).\]

Next, we will estimate

\[(372) \quad \text{dil}(F_+) = \max_{\vec{a}, \vec{b} \in \mathbb{R}^2} \frac{|F(\vec{a}) - F(\vec{b})|}{|\vec{a} - \vec{b}|}.\]

**Lemma 9.4.** The function $F_+$ satisfies $\text{dil}(F_+) \leq 1$.

**Proof.** Recall that

\[(373) \quad ||\vec{x}||_{\text{taxi}} = |x_1| + |x_2|\]

is the taxicab norm, and

\[(374) \quad ||\vec{x}||_{\mathbb{R}^2} = \sqrt{(x_1)^2 + (x_2)^2}\]

is the Euclidean norm. We have

\[
\text{dil}(F_+) \leq \max_{\vec{a} \neq \vec{b}} \frac{1}{3} \frac{|a_1 - b_1 + a_2 - b_2|}{\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}} \leq \max_{\vec{a} \neq \vec{b}} \frac{1}{3} \frac{|a_1 - b_1| + |a_2 - b_2|}{\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}} = \max_{\vec{a} \neq \vec{b}} \frac{1}{3} ||\vec{a} - \vec{b}||_{\text{taxi}}.
\]
It is well-known that $||x||_{\ell^2} \leq ||x||_{\text{taxi}} \leq \sqrt{2}||x||_{\ell^2}$. Thus $\frac{1}{||x||_{\ell^2}} \leq \frac{\sqrt{2}}{||x||_{\text{taxi}}}$, which gives us

$$\max_{a \neq b} \frac{1}{3} \frac{||a - b||_{\text{taxi}}}{||a - b||_{\ell^2}} \leq \max_{a \neq b} \frac{\sqrt{2}}{3} \frac{||a - b||_{\text{taxi}}}{||a - b||_{\ell^2}} = \frac{\sqrt{2}}{3} \leq 1,$$

finishing the proof of Lemma 9.4.

Now we are ready to estimate the $d_+$-distance between arbitrary points.

**Lemma 9.5.** For any two points $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in \mathbb{R}^2$, we have

$$F_+ (\bar{x} - \bar{y}) - (4h + 2L) \leq d_+ ((x_1, x_2), (y_1, y_2)) \leq F_+ (\bar{x} - \bar{y}) + (4h + 2L)$$

where $L$ and $h$ are the smocking length and depth, respectively.

**Proof.** By the definition of the smocking depth, there exists some $I_{\bar{x}} = I(j_1, j_2)$ and some $I_{\bar{y}} = I(j_1', j_2')$ such that $d_+ (\bar{x}, I_{\bar{x}}) \leq h$ and $d_+ (\bar{y}, I_{\bar{y}}) \leq h$.

We first claim that

$$|d_+ (\bar{x}, \bar{y}) - d_+ (j_1, j_1')| \leq 2h.$$  
(376)

Note that

$$d_+ (\bar{x}, \bar{y}) \leq d_+ (\bar{x}, j_1') + d_+ (j_1', \bar{y}) \leq d_+ (\bar{x}, j_1') + d_+ (j_1', \bar{y}) + d_+ (j_1', \bar{y}).$$  
(377)

Thus

$$|d_+ (\bar{x}, \bar{y}) - d_+ (j_1, j_1')| \leq |d_+ (\bar{x}, j_1')| + |d_+ (j_1', \bar{y})| \leq 2h,$$

establishing (376).

Next, we claim that

$$|F_+ (j_1 - j_1') - F_+ (\bar{x} - \bar{y})| \leq 2(h + L).$$  
(379)

Indeed, by definition of $\text{dil}(F_+)$, we have

$$\frac{|F(j_1 - j_1') - F(\bar{x} - \bar{y})|}{|(j_1 - j_1') - (\bar{x} - \bar{y})|} \leq \text{dil}(F).$$  
(380)

By our choice of $j_1, j_1'$ and using Lemma 9.4, it follows that

$$|F_+ (j_1 - j_1') - F_+ (\bar{x} - \bar{y})| \leq |(j_1 - j_1') - (\bar{x} - \bar{y})| \text{dil}(F_+)$$

$$\leq (|j_1 - \bar{x}| + |j_1' - \bar{y}|) \text{dil}(F_+) \leq 2(h + L),$$

establishing inequality (379).

Combining inequalities (376), (379), and the triangle inequality, we have

$$|d_+ (\bar{x}, \bar{y}) - F(\bar{x} - \bar{y})| \leq |d_+ (\bar{x}, \bar{y}) - d_+ (j_1, j_1')| + |F(j_1 - j_1') - F(\bar{x} - \bar{y})|4h + 2L,$$

finishing the proof of Lemma 9.5.

**Lemma 9.6.** For any two points $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2)$, we have

$$F_+ (\bar{x} - \bar{y}) = \lim_{R \to \infty} \frac{1}{R} \left[ d_+ (R\bar{x}_1, R\bar{x}_2), (R\bar{y}_1, R\bar{y}_2) \right].$$
Proof. By Lemma 9.5, we have
\[
\frac{1}{R} |\hat{d}_+(R\bar{x}_1, R\bar{x}_2, R\bar{y}_1, R\bar{y}_2) - F(R\bar{x} - R\bar{y})| \leq \frac{1}{R}(4h + 2L).
\]
Using the formula for \( F_+ \), we find
\[
\frac{1}{R} |\hat{d}_+(R\bar{x}_1, R\bar{x}_2, R\bar{y}_1, R\bar{y}_2) - F_+(\bar{x} - \bar{y})| \leq \frac{1}{R}(4h + 2L).
\]
Taking the limit of both sides,
\[
\hat{d}_{\infty}(\bar{x}, \bar{y}) = F_+(\bar{x}, \bar{y}) = \frac{1}{3} \left[ \frac{|x_1 - y_1|}{3} + \frac{|x_2 - y_2|}{3} \right],
\]
establishing the first part of Lemma 9.6.

Combining these lemmas with Theorem 9.1 we conclude Theorem 9.2.

9.3. The Tangent Cone at Infinity of \( X_\square \). In this section we prove the tangent cone at infinity of the \( \square \) smocked space is a normed space whose unit ball is an octagon. See Theorem 9.7, which is depicted in Figure 28.

**Figure 28.** Converging to the Tangent Cone at Infinity.

**Theorem 9.7.** The tangent cone at infinity of \((X_\square, d_\square)\) is \((\mathbb{R}^2, d_{F_\square})\), where
\[
d_{F_\square}(\bar{x}, \bar{y}) = F_\square(\bar{x} - \bar{y}) = ||\bar{x}||_\square = 2\sqrt{\frac{2}{3}} \min(|x_1|, |x_2|) + \frac{2}{3} ||x_1| - |x_2||.
\]
In the limit space, the shape of a ball, \( \{\bar{x} \in \mathbb{R}^2 : F_\square(\bar{x}) = R\} \), is an octagon.

Notice that, according to Lemma 8.3, for \( j, j' \in J_\square \)
\[
d_\square(I_j, I_{j'}) = F_\square(j - j').
\]
We will prove a series of lemmas to show we have all the hypotheses needed to apply Theorem 9.1 taking \( F = F_\square \).

**Lemma 9.8.** The function \( F_\square \) is a norm.

**Proof.** It is definite:
\[
0 = F_\square(\bar{x}) = 2\sqrt{2} \min\left(\frac{|x_1|}{3}, \frac{|x_2|}{3}\right) + 2\left|\frac{|x_1|}{3} - \frac{|x_2|}{3}\right| \iff x_1 = x_2 = 0.
\]
It scales:
\[
F_\square(R\bar{x}) = 2\sqrt{2} \min\left(\frac{|Rx_1|}{3}, \frac{|Rx_2|}{3}\right) + 2\left|\frac{|Rx_1|}{3} - \frac{|Rx_2|}{3}\right| = 2\sqrt{2} |R| \min\left(\frac{|x_1|}{3}, \frac{|x_2|}{3}\right) + 2|R| \left|\frac{|x_1|}{3} - \frac{|x_2|}{3}\right| = |R|F_\square(\bar{x}).
\]
It satisfies the triangle inequality:

\[ F_\square(\vec{x} + \vec{y}) = 2 \sqrt{2} \min \left( \frac{|x_1 + y_1| + |x_2 + y_2|}{3}, \frac{|x_1 + y_1| - |x_2 + y_2|}{3} \right) + 2 \left| \frac{|x_1 + y_1|}{3} - \frac{|x_2 + y_2|}{3} \right| \]

\[
= \frac{1}{3} \left( 2 \sqrt{2} \left( \frac{1}{2} (|x_1 + y_1| + |x_2 + y_2|) - \frac{1}{2} (|x_1 + y_1| - |x_2 + y_2|) \right) + 2 (|x_1 + y_1| - |x_2 + y_2|) \right) \\
= \frac{1}{3} \left( \sqrt{2} (|x_1 + y_1| + |x_2 + y_2|) + (2 - \sqrt{2}) |x_1 + y_1| - |x_2 + y_2| \right) \\
\leq \frac{1}{3} \left( \sqrt{2} (|x_1| + |y_1| + |x_2| + |y_2|) + (2 - \sqrt{2}) (|x_1| + |y_1| + |x_2| + |y_2|) \right) \\
= F_\square(\vec{x}) + F_\square(\vec{y}).
\]

\[ \square \]

Next, we will estimate

\[
\text{dil}(F_\square) = \max_{\vec{a}, \vec{b} \in \mathbb{R}^2} \frac{|F_\square(\vec{a}) - F_\square(\vec{b})|}{|\vec{a} - \vec{b}|}.
\]

**Lemma 9.9.** The function \( F_\square \) satisfies \( \text{dil}(F_\square) \leq 1 \).

**Proof.** Recall that

\[
|\vec{x}|_{\text{taxi}} = |x_1| + |x_2|
\]

is the taxicab norm, and

\[
|\vec{x}|_{\mathbb{E}^2} = \sqrt{(x_1)^2 + (x_2)^2}
\]

is the Euclidean norm. For two points \( \vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2) \), we want to estimate the quantity

\[
\frac{|F_\square(\vec{a}) - F_\square(\vec{b})|}{|\vec{a} - \vec{b}|_{\mathbb{E}^2}} = \frac{2}{3} \frac{\sqrt{2} (\min(|a_1|, |a_2|) - \min(|b_1|, |b_2|)) + |a_1 - a_2| - |b_1 - b_2|}{|\vec{a} - \vec{b}|_{\mathbb{E}^2}} \\
\leq \frac{2 \sqrt{2}}{3} \frac{\sqrt{2} (\min(|a_1|, |a_2|) - \min(|b_1|, |b_2|)) + |a_1 - a_2| - |b_1 - b_2|}{|a_1 - b_1| + |a_2 - b_2|}.
\]

By the symmetry of expression (388) in \( \vec{a} \) and \( \vec{b} \), it suffices to consider the following two cases: first when \( |a_1| \geq |a_2| \), \( |b_1| \geq |b_2| \) and second when \( |a_1| \geq |a_2| \), \( |b_1| \leq |b_2| \).

First, assume that \( |a_1| \geq |a_2| \) and \( |b_1| \geq |b_2| \). Then we may estimate (388) by

\[
\frac{|F_\square(\vec{a}) - F_\square(\vec{b})|}{|\vec{a} - \vec{b}|_{\mathbb{E}^2}} = \frac{2 \sqrt{2}}{3} \frac{\sqrt{2} (|a_2 - b_2| + |a_1 - a_2| - |b_1| + |b_2|)}{|a_1 - b_1| + |a_2 - b_2|} \\
\leq \frac{2 \sqrt{2}}{3} \frac{||a_1| - |b_1| + |a_2| - |b_2||}{|a_1| - b_1| + |a_2 - b_2|} \leq \frac{2 \sqrt{2}}{3} \frac{|a_1 - b_1| + |a_2 - b_2|}{|a_1 - b_1| + |a_2 - b_2|} = \frac{2 \sqrt{2}}{3},
\]

which gives the desired estimate.
Lemma 9.10. Establishing inequality (394).

By our choice of □ which gives the desired estimate and completes the proof of Lemma 9.9.

Proof. \( L \) and \( h \) are the smocking length and depth, respectively. Note that (391)

\[
(393)
\]

Thus establishing (391).

\[
\text{Indeed, by definition of } \text{dil}(F), \text{we have}
\]

\[
\frac{|F(\tilde{x} - \tilde{y})| - F(\tilde{x} - \tilde{y})|}{\text{dil}(F)} \leq |\tilde{e} - \tilde{e}| - (\tilde{x} - \tilde{y})| (\text{Lemma 9.9})
\]

\[
\leq (|\tilde{e} - \tilde{e}| + |\tilde{e} - \tilde{e}|) \text{dil}(F)
\]

\[
\leq 2(h + L),
\]

establishing inequality (394).
Combining inequalities (391), (394), and the triangle inequality, we have
\[
|\tilde{d}(\vec{x}, \vec{y}) - F(\vec{x} - \vec{y})| \leq |\tilde{d}(\vec{x}, \vec{y}) - \tilde{d}(\vec{j}, \vec{j}')| + |F(\vec{j} - \vec{j}') - F(\vec{x} - \vec{y})| \\
\leq 4h + 2L,
\]
finishing the proof of Lemma 9.10.

** Lemma 9.11.** For any two points \( \vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \), we have
\[
F(\vec{x} - \vec{y}) = \lim_{R \to \infty} \frac{1}{R} [\tilde{d}((Rx_1, Rx_2), (Ry_1, Ry_2))].
\]

**Proof.** By Lemma 9.10, we have
\[
\frac{1}{R}|\tilde{d}((Rx_1, Rx_2), (Ry_1, Ry_2)) - F(R\vec{x} - R\vec{y})| \leq \frac{1}{R}(4h + 2L).
\]
Since \( F \) is a norm, \( F(\lambda \vec{x}) = \lambda F(\vec{x}) \) for any \( \lambda \in \mathbb{R} \). It follows that
\[
\left| \frac{1}{R}\tilde{d}((Rx_1, Rx_2), (Ry_1, Ry_2)) - F(R\vec{x} - R\vec{y}) \right| \leq \frac{1}{R}(4h + 2L).
\]
Taking the limit of both sides,
\[
\tilde{d}_{\infty,T}(\vec{x}, \vec{y}) = F(\vec{x} - \vec{y}),
\]
as desired.

Combining these lemmas with Theorem 9.1 we conclude Theorem 9.7.

** 9.4. The Tangent Cone at Infinity of \( X_T \).** In this section, we prove the tangent cone at infinity of the T smocked space is a normed space whose unit ball, \( \{\vec{x} \in \mathbb{R}^2 : ||\vec{x}||_{\infty,T} = R\} \), is a square. See Theorem 9.12 which is depicted in Figure 29.

**Figure 29.** Converging to the Tangent Cone at Infinity.

**Theorem 9.12.** The tangent cone at infinity of \((X_T, d_T)\) is \((\mathbb{R}^2, d_{F_T})\), where
\[
d_{F_T}(\vec{x}, \vec{y}) = F_T(\vec{x} - \vec{y}) \text{ where } F_T(\vec{x}) = \|\vec{x}\|_T = \frac{|x_1| + |x_2|}{2}
\]
Note that in the limit space, the shape of a ball, \( \{\vec{x} \in \mathbb{R}^2 : ||\vec{x}||_{\infty,T} = R\} \), is a square with corners at \((\pm 2, 0)\) and \((0, \pm 2)\).

Recall that, according to Lemma 8.6, for \( \vec{j}, \vec{j}' \in J_T \)
\[
d_T(I_{\vec{j}}, I_{\vec{j}'}) = F_T(\vec{j} - \vec{j}').
\]
We will prove a series of lemmas to show we have all the hypotheses needed to apply Theorem 9.11 taking \( F = F_T \).

**Lemma 9.13.** The function \( F_T \) satisfies \( \text{dil}(F_T) \leq 1 \).
Proof. Recall $||\vec{x}||_{\text{taxi}} = |x_1| + |x_2|$ is the taxi norm, and $||\vec{x}||_{\mathbb{E}^2} = \sqrt{(x_1)^2 + (x_2)^2}$ is the Euclidean norm. It is well-known that $||\vec{x}||_{\text{taxi}} \leq \sqrt{2}||\vec{x}||_{\mathbb{E}^2}$. Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be points in the plane. We may estimate

$$\frac{|F_T(\vec{a}) - F_T(\vec{b})|}{||\vec{a} - \vec{b}||_{\mathbb{E}^2}} \leq \frac{|a_1| + |a_2| - |b_1| - |b_2|}{||\vec{a} - \vec{b}||_{\mathbb{E}^2}} \leq \sqrt{2} \left(\frac{|a_1 - b_1| + |a_2 - b_2|}{||\vec{a} - \vec{b}||_{\text{taxi}}}\right) = 1.$$

Now we are ready to estimate the $d_T$-distance between arbitrary points.

Lemma 9.14. For any two points $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \in \mathbb{E}^2$, we have

$$F_T(\vec{x} - \vec{y}) - (4h + 2L) \leq d_T((x_1, x_2), (y_1, y_2)) \leq F_T(\vec{x} - \vec{y}) + (4h + 2L)$$

where $L$ and $h$ are the smocking length and depth of $X_T$, respectively.

Proof. By the definition of the smocking depth,

$$\exists I_j = I_{(j_1, j_2)} \text{ and } I_{\vec{j}} = I_{(j_1', j_2')} \text{ s.t. } d_T(\vec{x}, I_j) \leq h \text{ and } d_T(\vec{y}, I_{\vec{j}}) \leq h.$$

We first claim that

$$|d_T(\vec{x}, \vec{y}) - d_T(\vec{j}, \vec{j}')| \leq 2h.$$  \hspace{1cm} (399)

To see this, note that

$$d_T(\vec{x}, \vec{y}) \leq d_T(\vec{x}, \vec{j}) + d_T(\vec{j}, \vec{j}') \leq d_T(\vec{x}, \vec{j}) + d_T(\vec{j}, \vec{j}') + d_T(\vec{j}', \vec{y}).$$

Thus

$$|d_T(\vec{x}, \vec{y}) - d_T(\vec{j}, \vec{j}')| \leq |d_T(\vec{x}, \vec{j})| + |d_T(\vec{j}, \vec{j}')| \leq 2h,$$

establishing (399).

Next, we claim that

$$|F_T(\vec{j} - \vec{j}') - F_T(\vec{x} - \vec{y})| \leq 2(h + L).$$

Indeed, by definition of $\text{dil}(F_T)$,

$$|F_T(\vec{j} - \vec{j}') - F_T(\vec{x} - \vec{y})| \leq \text{dil}(F_T).$$

By our choice of $\vec{j}, \vec{j}'$ and using Lemma 9.13 it follows that

$$|F_T(\vec{j} - \vec{j}') - F_T(\vec{x} - \vec{y})| \leq |(\vec{j} - \vec{j}') - (\vec{x} - \vec{y})| \text{dil}(F_T) \leq 2(h + L),$$

establishing inequality (402).

Combining inequalities (399), (402), and the triangle inequality,

$$|d_T(\vec{x}, \vec{y}) - F_T(\vec{x} - \vec{y})| \leq |d_T(\vec{x}, \vec{y})| - d_T(\vec{j}, \vec{j}') + |F_T(\vec{j} - \vec{j}') - F_T(\vec{x} - \vec{y})| \leq 4h + 2L,$$

finishing the proof of Lemma 9.14.  \hspace{1cm} □
Lemma 9.15. For any two points \( \vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \), we have

\[
F_T(\vec{x} - \vec{y}) = \lim_{R \to \infty} \frac{1}{R} \left[ \overline{d}_T((Rx_1, Rx_2), (Ry_1, Ry_2)) \right].
\]

Moreover, \( \|\vec{x}\|_{\infty,T} = F_T(\vec{x}) \) defines a norm on \( \mathbb{R}^2 \).

Proof. By Lemma 9.14, we have

\[
\frac{1}{R} |\overline{d}_T((Rx_1, Rx_2), (Ry_1, Ry_2)) - F_T(R\vec{x} - R\vec{y})| \leq \frac{1}{R} (4h + 2L).
\]

Inspecting the formula for \( F_T \), one can see that \( F_T(\lambda \vec{x}) = \lambda F_T(\vec{x}) \) for any \( \lambda \geq 0 \). It follows that

\[
|\frac{1}{R} \overline{d}_T((Rx_1, Rx_2), (Ry_1, Ry_2)) - F_T(\vec{x} - \vec{y})| \leq \frac{1}{R} (4h + 2L).
\]

Taking the limit of both sides,

\[
\overline{d}_{\infty,T}(\vec{x}, \vec{y}) = F_T(\vec{x} - \vec{y}),
\]

establishing the first part of Lemma 9.15.

Finally, since \( \|\cdot\|_{\infty,T} = d_{\infty,T}(\cdot, (0, 0)) \) is a positive multiple of the taxicab norm, it is also a norm. \( \square \)

Combining these lemmas with Theorem 9.1 we conclude Theorem 9.12.

10. Open Problems

In this section we propose a collection of open problems. Please be sure to contact Prof. Sormani if you are interested in working on some of these problems so that you are not in competition with other teams.

10.1. For Undergraduates and Masters students.

Problem 10.1. As suggested in the introduction: students may consider the balls, distances, and tangent cones at infinity for other periodic smocking patterns as in as in Figure 30. You might consider the various herringbone smocking patterns that are commonly sewn. You might also consider the patterns we have already studied but change the separation constants and lengths.

Problem 10.2. Three dimensional smocking patterns will be explored in [8]. You may wish to examine others.

Problem 10.3. Students might also explore the Gromov-Hausdorff limits of sequences of different periodic smocking patterns. What happens for example if you examine a sequence of patterns similar to one of our patterns as the length is taken to 0 with a fixed index set \( J \) as in Figure 31? Does it have a Gromov-Haudorff limit? Can you prove a general theorem about the Gromov-Hausdorff limits and what they are?

Problem 10.4. Students might also explore the Gromov-Hausdorff limit of a sequence of patterns obtained by rescaling one of our patterns horizontally but not vertically as in Figure 32? Does it have a Gromov-Haudorff limit?
Problem 10.5. Show that the smocking space $(X_Q, d_Q)$ defined by the smocking intervals
\[(408) \quad S_Q = \{ (j) \times [0, j] : j \in J_Q \} \text{ where } J_Q = \{ j_k : k \in \mathbb{N} \} \text{ with } \lim_{k \to \infty} j_k / j_{k+1} = 0.\]
Show that this smocked space does not have a unique tangent cone at infinity and the tangent cone is not always a normed space. First show that the GH limit when rescaling by $R_k = j_k \to \infty$ is a pulled thread space with a interval of unit length at $\{1\} \times [0, 1]$. Then rescale instead by $R_k = (j_k + j_{k+1})/2$ and show the GH limit is Euclidean space. What other possible tangent cones at infinity do you find?

10.2. More Advanced Questions.

**Problem 10.6.** Under what conditions is the rescaled limit of a smocked metric space a unique normed vector space? When does it exist? When is it unique? When is it a normed vector space? Here we are asking you to find hypotheses which allow you to apply Theorem 9.1 without having to carefully analyze the formula for the distance between intervals in the space.

**Problem 10.7.** Note that in this paper two of our tangent cones at infinity, that of $X_T$ and $X_{+}$ were isometric to one another as they are both just rescalings of taxispace (see Example 7.2). Is there a way to assess a pair of smocking metric spaces initially to determine if they have the same tangent cone at infinity without conducting a complete derivation of their distance functions?

**Problem 10.8.** If you have a sequence of smocked spaces defined using smocking sets, $S_k \subseteq \mathbb{R}^N$, and the sets $S_K$ converge in the Hausdorff sense to a smocking set $S_\infty$, then do the corresponding smocked metric spaces converge

$$d_H(S_k, S_\infty) \to 0,$$

then $d_GH(X_k, X_\infty) \to 0$?

Prove this or find a counter example.

**Problem 10.9.** You may read in [1] or [5] to learn Gromov’s Compactness Theorem, which describes when a sequence of metric spaces has a GH converging subsequence. This theorem requires a uniform upper bound on diameter and the number of balls of any given radius. Smocking spaces do not satisfy these hypotheses and yet we saw the rescaled sequences converged. When does a sequence of smocked metric space have a GH converging subsequence? Do uniform bounds on the smocking constants suffice? What properties do the limit spaces have?

**Problem 10.10.** Given a finite dimensional normed vector space, can one find a smocked space whose unique tangent cone at infinity is that given space?

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