A JORDAN CURVE THEOREM FOR 2-DIMENSIONAL TILINGS

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Abstract. The classical Jordan curve theorem for digital curves asserts that the Jordan curve theorem remains valid in the Khalimsky plane. Since the Khalimsky plane is a quotient space of \( \mathbb{R}^2 \) induced by a tiling of squares, it is natural to ask for which other tilings of the plane it is possible to obtain a similar result. In this paper we prove a Jordan curve theorem which is valid for every locally finite tiling of \( \mathbb{R}^2 \). As a corollary of our result, we generalize some classical Jordan curve theorems for grids of points, including Rosenfeld’s theorem.

1. Introduction

The Jordan curve theorem asserts that a simple closed curve divides the plane into two connected components, one of these components is bounded whereas the other one is not. This theorem was proved by Camille Jordan in 1887 in his book *Cours d’analyse* [7].

During the decade of the seventies, Azriel Rosenfeld published a series of articles [19, 20, 21, 22, 23, 24] wherein he studies connectedness properties of the grid of points with integer coordinates \( \mathbb{Z}^2 \). For any point \((n, m) \in \mathbb{Z}^2\) Rosenfeld defines its 4-\textit{neighbors} as the four points \((n, m \pm 1)\) and \((n \pm 1, m)\), whilst the four points \((n \pm 1, m \pm 1)\) as well as the 4-neighbors are its 8-\textit{neighbors}. For \(k = 4\) or \(k = 8\), a \(k\)-\textit{path} is a finite sequence of points \((x_0, \ldots, x_n)\) in \(\mathbb{Z}^2\) such that for every \(i \in \{1, \ldots, n\}\), \(x_{i-1}\) is a \(k\)-neighbor of \(x_i\). A subset \(S\) of \(\mathbb{Z}^2\) is \(k\)-\textit{connected} if there is a \(k\)-path between any two elements of \(S\). A \(k\)-\textit{component} of \(S\) is a maximal \(k\)-connected subset. Lastly, a \textit{simple closed \(k\)-path} is a \(k\)-connected set \(J\) which contains exactly two \(k\)-neighbors for each of its points. In this case we can describe \(J\) as a \(k\)-path \((x_1, \ldots, x_n)\), where

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$x_i$ is a $k$-neighbor of $x_j$ iff $i \equiv j \pm 1 \pmod{n}$. In [20], Rosenfeld proves the following discrete Jordan curve theorem.

**Theorem 1.1.** Let $J \subset \mathbb{Z}^2$ be a simple closed 4-path with at least five points. Then $\mathbb{Z}^2 \setminus J$ has exactly two 8-components.

In the same decade, Efim Khalimsky developed a different approach for studying topological properties of the sets $\mathbb{Z}$ and $\mathbb{Z}^2$ [8, 9], endowing them with topologies that captured the proximity of its elements. These sets, together with the topologies proposed by Khalimsky, are known as the digital line (or Khalimsky line) and the digital plane (or Khalimsky plane), respectively. Khalimsky also proved a Jordan curve theorem for the digital plane (see [8, 9]). We explain this theorem later, in Section 2 (Theorem 2.6).

There are other versions of Jordan curve theorems for grids of points that are not the usual squared grid. For example, the case where the points are configured in a hexagonal grid has been studied in [15, Theorem 26] and [14]. In [17], V. Neumann-Lara and R. Wilson presented an analogue to the Jordan curve theorem in the context of graph theory (see Theorem 4.3). This result will be a key tool for the proof of our main theorem.

One way to define the topology of the digital plane is by an equivalence relation in $\mathbb{R}^2$ resulting of identifying the points in the same edge or the same face of a tiling of the plane by squares of the same size (see
equation 2.1 in Section 2). This approach for describing the topology of the digital plane leads us to question if it is possible to generalize Khalimsky’s Jordan curve theorem for every quotient space of $\mathbb{R}^2$ obtained by doing a similar identification in any nice enough tiling. Our main result, Theorem 4.4, states that this generalization is possible if the tiling is locally finite.

Our paper is organized as follows. In Section 2 we recall all basic notions concerning Alexandrov spaces and tilings. In Section 3 we introduce what the digital version of a tiling is and we prove some basic properties of these topological spaces. In Section 4 we introduce the definition of a digital Jordan curve and we prove our version of the Jordan curve theorem for tilings (Theorem 4.4). To finish the paper, in Section 5 we explore some basic properties of closed and open digital Jordan curves, that will be used in Section 6 to obtain conditions in order to guarantee that a digital Jordan curve encloses a face of the tiling.

2. Preliminaries

2.1. Alexandrov Topologies. Let $(X, \tau)$ be a topological space. The topology $\tau$ is called an Alexandrov topology if the intersection of an arbitrary family of open sets is open. In this case we say that $(X, \tau)$ is an Alexandrov discrete space.

It is not difficult to see that $(X, \tau)$ is an Alexandrov discrete space iff each point $x \in X$ has a smallest neighborhood, which we denote by $N(x)$. Clearly, $N(x)$ is the intersection of all open neighborhoods of $x$ and therefore $N(x)$ is open. Also, every subspace of an Alexandrov space is an Alexandrov space.

The first important example of an Alexandrov space concerning the contents of this work is the digital line, also known as the Khalimsky line.

**Example 2.1.** For every $n \in \mathbb{Z}$, let

$$N(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n - 1, n, n + 1\} & \text{if } n \text{ is even.} \end{cases}$$

Then the set of integers $\mathbb{Z}$, together with the topology $\tau_K$ given by the base $B = \{N(n) \mid n \in \mathbb{Z}\}$ is known as the digital line.

Another important example is the digital plane, also known as the Khalimsky plane.
Example 2.2. The Khalimsky plane (or digital plane) is the product space of the Khalimsky line \((\mathbb{Z}, \tau_K) \times (\mathbb{Z}, \tau_K)\). This product topology coincides with the quotient topology generated by the surjection \(p : \mathbb{R}^2 \to \mathbb{Z}^2\) defined by:

\[
(2.1) \\
p(x, y) = \begin{cases} 
(2n + 1, 2m + 1) & \text{if there are } n, m \in \mathbb{N} \text{ such that } (x, y) \in (2n, 2n + 2) \times (2m, 2m + 2), \\
(2n, 2m + 1) & \text{if there are } n, m \in \mathbb{N} \text{ such that } x = 2n, \ y \in (2m, 2m + 2), \\
(2n + 1, 2m) & \text{if there are } n, m \in \mathbb{N} \text{ such that } x \in (2n, 2n + 2), \ y = 2m, \\
(2n, 2m) & \text{if there are } n, m \in \mathbb{N} \text{ such that } x = 2n, \ y = 2m.
\end{cases}
\]

It is worth highlighting that the quotient map \(p\) induces an equivalence relation that identifies the interior, the edges (without vertices) and the vertices of the rectangles \([2n, 2n + 2] \times [2m, 2m + 2]\) in the plane. Thus, another way to understand the topology of the digital plane is to think of this space as the set of equivalence classes of this equivalence relation together with the quotient topology induced by \(\mathbb{R}^2\). This description of the digital plane is depicted in Figure 2. As we already mentioned, the digital plane is an Alexandrov space which can be readily verified by noting that

\[
N(n, m) = \begin{cases} 
\{(n, m)\} & \text{if } 2 \nmid n, \ 2 \nmid m, \\
\{n - 1, n, n + 1\} \times \{m - 1, m, m + 1\} & \text{if } 2 \nmid n, \ 2 \mid m, \\
\{n\} \times \{m - 1, m, m + 1\} & \text{if } 2 \mid n, \ 2 \nmid m, \\
\{n - 1, n, n + 1\} \times \{m\} & \text{if } 2 \mid n, \ 2 \mid m.
\end{cases}
\]

An important tool to understand the connectedness of an Alexandrov space is its connectedness graph.
Figure 2. The digital plane as equivalence classes of \( \mathbb{R}^2 \).

**Definition 2.3.** Let \( X \) be a topological space. The **connectedness graph** of \( X \) is the graph \( G = (V_G, E_G) \) where \( V_G = X \) and \( \{ x, y \} \in E_G \) if and only if \( \{ x, y \} \) is connected.

We refer the reader to [2] and [3] for any unknown notion concerning graph theory.

Let \( X \) be a topological space. For every \( x \in X \) the **adjacency set** of \( x \) is the set

\[
\mathcal{A}(x) = \{ y \in X \mid x \neq y, \{ x, y \} \text{ is connected} \}.
\]

Observe that \( x \in \mathcal{A}(y) \) if and only if \( y \in \mathcal{A}(x) \). In this case we say that \( x \) and \( y \) are adjacent.

Given two points \( x, y \in X \), we define a **digital path** from \( x \) to \( y \) as a finite sequence of elements of \( X \), \((x_0, x_1, \ldots, x_n)\), such that \( x = x_0 \), \( y = x_n \) and for every \( i \in \{0, 1, \ldots, n-1\} \), \( x_i \) and \( x_{i+1} \) are adjacent. We say that \( X \) is **digitally pathwise connected** if for every \( x, y \in X \) there is a digital path from \( x \) to \( y \). Lastly, a digital path \((x_0, \ldots, x_n)\) is a **digital arc** if \( n = 1 \) or if there is a homeomorphism from a finite interval \( I \) of the digital line into \( \{x_0, \ldots, x_n\} \). We say that \( X \) is **digitally arcwise connected** if for every \( x, y \in X \) there is a digital arc from \( x \) to \( y \).

From the definition of the connectedness graph and the definition of a digital path it is clear that \((x_0, \ldots, x_n)\) is a digital path in \( X \) if and only if \((x_0, \ldots, x_n)\) is a path in the connectedness graph \( G \) of \( X \). The following remark specifies the type of subgraphs in the connectedness
graph of \( X \) corresponding to digital arcs. For a proof of the non-trivial implication, see e.g. \cite{16} Theorem 5.6.

**Remark 2.4.** Let \( X \) be a topological space. Then \((x_0, x_1, \ldots, x_n)\) is a digital arc if and only if \((x_0, x_1, \ldots, x_n)\) is an induced simple path in the connectedness graph of \( X \).

The following theorem summarizes important results about connectedness in Alexandrov discrete spaces. Its proof can be found in \cite{11} Theorem 3.2] and \cite{15} Lemma 20].

**Theorem 2.5.** Let \( X \) be an Alexandrov discrete space.

1. For every \( x \in X \), \( \mathcal{A}(x) = (N(x) \cup \{x\}) \setminus \{x\} \).
2. The following are equivalent:
   a) \( X \) is connected.
   b) \( X \) is digitally pathwise connected.
   c) \( X \) is digitally arcwise connected.
   d) The connectedness graph of \( X \) is connected.
3. The connected components of \( X \) are open and closed.

Finally, we enunciate Khalimsky’s Jordan curve theorem for the digital plane. First, we say that a subset \( J \) of the digital plane is a *Jordan curve in the digital plane* if for every \( j \in J \) the complement \( J \setminus \{j\} \) is a digital arc, and \(|J| \geq 4\). For a detailed exposition of this theorem, see \cite{11}.

**Theorem 2.6.** Let \( J \) be a Jordan curve in the digital plane. Then \( \mathbb{Z}^2 \setminus J \) has exactly two connected components.

2.2. **Tilings.** To finish this section, let us recall some basic notions about tilings.

**Definition 2.7.** Let \( \mathcal{T} = \{T_n\}_{n \in \mathbb{N}} \) be a countable family of closed, connected subsets of \( \mathbb{R}^2 \). We say that \( \mathcal{T} \) is a tiling of \( \mathbb{R}^2 \) if:

(a) \( \bigcup_{n \in \mathbb{N}} T_n = \mathbb{R}^2 \),
(b) if \( i \neq j \), then \( \text{Int} (T_i) \cap \text{Int} (T_j) = \emptyset \).

In this case every element \( T_n \in \mathcal{T} \) is called a tile.
Given a tiling $\mathcal{T} = \{T_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}^2$, the points in the plane that are elements of three or more tiles are called vertices. If $i, j \in \mathbb{N}$, $i \neq j$, satisfy that

$$E_{i,j} := \{u \in \mathbb{R}^2 | u \in T_i \cap T_j, \text{ u is not a vertex} \} \neq \emptyset,$$

then each connected component of the set $E_{i,j}$ is called an edge. Lastly, the interior of a tile is called a face. If a vertex, an edge or a face are subsets of a tile $T$, we say that it is a vertex, edge or face of the tile, respectively.

As we mentioned in the introduction, for the purposes of this work, we limit our attention to a particular class of tilings, namely tilings that, besides (a) and (b) of the preceding definition, satisfy the following conditions:

- (c) $\mathcal{T}$ is a locally finite collection (namely, each point of the plane has a neighborhood that only intersects a finite number of elements of $\mathcal{T}$),
- (d) for each $n \in \mathbb{N}$, $T_n$ is homeomorphic to the closed unit disk,
- (e) each edge of the tiling is homeomorphic to the interval $(0, 1)$,
- (f) if two different tiles intersect each other, then their intersection is the disjoint union of a finite set of vertices and edges.

Consequently, from this point onwards, when we talk about tilings, we will refer to tilings that satisfy these requirements.

**Example 2.8.** The family $\{[2n, 2n+2] \times [2m, 2m+2] | (n, m) \in \mathbb{Z}^2 \}$ is a tiling of the plane.

In the following proposition we encapsulate basic properties of tilings that can be easily proved from the definition.

**Proposition 2.9.** Let $\mathcal{T}$ be a tiling.

1. If $S$ and $T$ are two tiles such that $S \cap T \neq \emptyset$, then $S \cap T = \partial S \cap \partial T$.
2. Every tile only intersects a finite number of tiles.
3. Every tile has at least two vertices.
4. Every tile has the same number of vertices and edges. Moreover, the boundary of a tile is the disjoint union of alternating vertices.
and edges, and the boundary of every edge is the set of two vertices that surround it.

From Proposition 2.9-(4), we infer that each edge is determined by two vertices. From its definition, it is clear that each edge is also determined by exactly two faces. Similarly, each face is determined by its boundary, that is, by the vertices and edges of the tile to whom it serves as interior. Also, a vertex is determined both by the faces of the tiles that intersect in it and by the edges that converge in it. In this way we can talk about the faces and edges of a vertex, the vertices and edges of a face, and the faces and vertices of an edge. We denote the sets of faces, edges and vertices of a tiling by \( \mathcal{F}_T, \mathcal{E}_T \) and \( \mathcal{V}_T \), respectively. Lastly, if \( v \) is a vertex, we denote the sets of edges and faces of \( v \) by \( \mathcal{E}_v \) and \( \mathcal{F}_v \), respectively. We can establish analogous notation for the vertices and edges of a face and for the faces and vertices of an edge.

### 3. Digital Topology of a Tiling

Let \( \mathcal{T} \) be a tiling of \( \mathbb{R}^2 \). Consider the set \( \mathcal{D}_T := \mathcal{F}_T \cup \mathcal{E}_T \cup \mathcal{V}_T \) and the surjection \( q : \mathbb{R}^2 \to \mathcal{D}_T \) given by

\[
q(x) = \begin{cases} 
  x, & \text{if } x \in \mathcal{V}_T, \\
  E, & \text{if } x \in E \in \mathcal{E}_T, \\
  F, & \text{if } x \in F \in \mathcal{F}_T.
\end{cases}
\]

Namely, \( q \) identifies the points in the same vertex, edge or face.

**Definition 3.1.** The set \( \mathcal{D}_T \) equipped with the quotient topology induced by \( q \) is called the digital version of \( \mathcal{T} \).

Let \( \mathcal{T} \) be a tiling of \( \mathbb{R}^2 \). Clearly we have that the closure (in \( \mathbb{R}^2 \)) of any vertex of the tiling is the vertex itself. The closure of an edge is the union of the edge and its two vertices. And, the closure of a face is the union of the the face and all its vertices and edges. This allow us to describe the closure of any element \( x \) in \( \mathcal{D}_T \) as follows:

\[
\overline{x} = \begin{cases} 
  \{x\} & \text{if } x \in \mathcal{V}_T, \\
  \{x\} \cup \mathcal{V}_x & \text{if } x \in \mathcal{E}_T, \\
  \{x\} \cup \mathcal{V}_x \cup \mathcal{E}_x & \text{if } x \in \mathcal{F}_T.
\end{cases}
\]

In the following theorem we prove that \( \mathcal{D}_T \) is an Alexandrov space, by describing the smallest neighborhood of every element of \( \mathcal{D}_T \).
Theorem 3.2. Let $\mathcal{T}$ be a tiling. Then $\mathcal{D}_T$ is an Alexandrov space. Moreover, for every element $x \in \mathcal{D}_T$ the smallest neighborhood of $x$ is given by

$$N(x) = \begin{cases} 
\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x & \text{if } x \in \mathcal{V}_T, \\
\{x\} \cup \mathcal{F}_x & \text{if } x \in \mathcal{E}_T, \\
\{x\} & \text{if } x \in \mathcal{F}_T.
\end{cases}$$

Proof. Let $\mathcal{T}$ be a tiling, $q : \mathbb{R}^2 \to \mathcal{D}_T$ the quotient map associated to $\mathcal{D}_T$ and $x \in \mathcal{D}_T$ an arbitrary element.

If $x \in \mathcal{V}_T$ and $U$ is an open neighborhood of $x$, then $q^{-1}(U)$ is an open, saturated neighborhood of $q^{-1}(x)$. Since $x$ is a vertex, it is contained in the closure of its edges and faces, therefore $q^{-1}(U)$ intersects each element of $\mathcal{E}_x \cup \mathcal{F}_x$. Moreover, since $q^{-1}(U)$ is a saturated neighborhood then $q^{-1}(U)$ contains all of the faces and edges of $x$, that is, $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x) \subset q^{-1}(U)$. Next we see that $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$ is an open set. Let $y \in q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$. If $y$ is an element of a face of $x$, then that face is an open set contained in $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$. If $y$ is an element of an edge $E \in \mathcal{E}_x$, then there are two tiles $S, T \in \mathcal{T}$ that have $x$ as a common vertex and such that $y \in E \subset S \cap T$. Furthermore, for any other tile $R \in \mathcal{T} \setminus \{S, T\}$, the point $y$ does not belong to $R$. Since $\mathcal{T} \setminus \{T, S\}$ is a locally finite collection of closed sets, its union is a closed subset of $\mathbb{R}^2$ (Theorem 1.1.11.), thus

$$\mathbb{R}^2 \setminus \left( \bigcup_{R \in \mathcal{T} \setminus \{S, T\}} R \right)$$

is an open neighborhood of $y$ contained in $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$. Finally, if $y = q^{-1}(x)$ we consider the set $\mathcal{S}$ of tiles that contain $y$, and we recall that $\mathcal{S}$ is a finite set. By the previous argument $\mathbb{R}^2 \setminus \left( \bigcup_{T \in \mathcal{T} \setminus \mathcal{S}} T \right)$ is an open neighborhood of $y$ that is contained in $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$. This proves that $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$ is an open set such that $q^{-1}(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x) \subset q^{-1}(U)$.

Since $U$ was an arbitrary open neighborhood of $x$ in $\mathcal{D}_T$ we conclude that $(\{x\} \cup \mathcal{E}_x \cup \mathcal{F}_x)$ is the smallest neighborhood of $x$ in $\mathcal{D}_T$.

If $x \in \mathcal{E}_T$ and $U$ is an open neighborhood of $x$, then $q^{-1}(U)$ is an open, saturated neighborhood of $q^{-1}(x)$. Since $x$ is an edge we know that it is contained in the closure of its faces, so $q^{-1}(U)$ intersects the two
elements of $\mathcal{F}_x = \{F_1, F_2\}$. Moreover, since $q^{-1}(U)$ is a saturated neighborhood then $q^{-1}(U)$ contains $F_1 \cup F_2$. Thus, $q^{-1}(\{x\} \cup \mathcal{F}_x) \subset q^{-1}(U)$. In the same vein as the previous case, we can see that $q^{-1}(\{x\} \cup \mathcal{F}_x)$ is open in $\mathbb{R}^2$. Since $U$ is an arbitrary open neighborhood of $x$ in $\mathcal{D}_T$, we have that $(\{x\} \cup \mathcal{F}_x)$ is the smallest neighborhood of $x$ in $\mathcal{D}_T$.

In order to complete the proof, we see that if $x \in \mathcal{F}_T$, then $\{x\}$ is open in $\mathcal{D}_T$ since $x$ is the interior of a tile in $\mathbb{R}^2$. □

We infer from theorems 2.5 and 3.2 in combination with equality 3.2 that the adjacency set of any element $x \in \mathcal{D}_T$ is given by

$$
\mathcal{A}(x) = \begin{cases} 
\mathcal{E}_x \cup \mathcal{F}_x & \text{if } x \in \mathcal{V}_T, \\
\mathcal{V}_x \cup \mathcal{F}_x & \text{if } x \in \mathcal{E}_T, \\
\mathcal{V}_x \cup \mathcal{E}_x & \text{if } x \in \mathcal{F}_T.
\end{cases}
$$

This information allows us to describe the connectedness graph of any digital version of a tiling. In Figure 3 we illustrate a portion of the tiling of the plane with regular hexagons of the same size as well as the connectedness graph of its digital version. In this graph we depict with white points the vertices of the graph corresponding to the faces of the tiling, with black points the vertices of the graph corresponding to vertices of the tiling, and with black rectangles the vertices of the graph corresponding to the edges of the tiling.

![Figure 3. Tiling of the plane with regular hexagons and the connectedness graph of its digital version.](image-url)
In Figure 4 we present another example, illustrating a portion of a tiling of the plane with squares and hexagons, and the connectedness graph of its digital version. The vertices of the graph correspond to the elements of the digital version in the same way as in Figure 3.

Figure 4. Tiling of the plane with hexagons and squares and the connectedness graph of its digital version.

For the subsequent material, we recall the definition of a planar graph:

**Definition 3.3.** Let $G = (V_G, E_G)$ be a graph. We say that $G$ is a planar graph if there is a function $\psi : V_G \cup E_G \to \mathcal{P}(\mathbb{R}^2)$ such that:

1. The image of each element of $V_G$ is a singleton.
2. The image of each element of $E_G$ is a subset of $\mathbb{R}^2$ homeomorphic to $(0, 1)$.
3. For any two elements $x, y \in V_G \cup E_G$, $x \neq y$,
   $$\psi(x) \cap \psi(y) = \emptyset.$$  
4. If $v$ and $w$ are the vertices of the edge $e$, then
   $$\overline{\psi(e)} = \psi(e) \cup \psi(v) \cup \psi(w).$$

In this situation, we say that $|G| := \psi(V_G \cup E_G)$ is a planar embedding of $G$. Sometimes the set $|G|$ is also called a geometric realization of $G$.

**Lemma 3.4.** Let $\mathcal{T}$ be a tiling. The connectedness graph of $V_{\mathcal{T}} \cup E_{\mathcal{T}}$, with the subspace topology induced by $\mathcal{D}_{\mathcal{T}}$, is a planar graph.
Proof. Let $H = (V_H, E_H)$ be the connectedness graph of $\mathcal{V}_T \cup \mathcal{E}_T$. The set of vertices of $H$ is precisely the set $V_H = \mathcal{V}_T \cup \mathcal{E}_T$. We know that the edges of $H$ always denote the adjacency between an edge of the tiling and its two vertices. Therefore, every edge in $E_H$ is a pair of the form $\{A, z\}$, where $A \in \mathcal{E}_T \subseteq \mathcal{V}_T \cup \mathcal{E}_T$ and $z \in V_A \subseteq \mathcal{V}_T$. For every $A \in \mathcal{E}_T$, pick a point $x_A \in A \subseteq \mathbb{R}^2$. If $x \in \mathcal{V}_T$ is a vertex of $A \in \mathcal{E}_T$, we define $I_{x_A, x}$ as the connected component of $A \{x_A\}$ delimited by $x_A$ and $x$. It is clear that $\{x_A, x\} \cap I_{x_A, x} = \emptyset$. Since $A$ is an edge of the tiling, then $I_{x_A, x}$ is homeomorphic to the interval $(0, 1)$ and $\overline{I_{x_A, x}} = I_{x_A, x} \cup \{x_A, x\}$.

Moreover, if $\{x_A, x\} \neq \{x_B, y\}$ then $I_{x_A, x} \cap I_{x_B, y} = \emptyset$, considering that any two different edges of the tiling have an empty intersection. Therefore the function $\psi : V_H \cup E_H \to \mathcal{P}(\mathbb{R}^2)$ given by

$$
\psi(x) = \begin{cases} 
\{x\} & \text{if } x \in \mathcal{V}_T, \\
\{x_A\} & \text{if } x = A \text{ for some } A \in \mathcal{E}_T, \\
I_{x_A, z} & \text{if } x = \{A, z\} \in E_H, \text{ where } A \in \mathcal{E}_T \text{ and } z \in \mathcal{V}_T,
\end{cases}
$$

proves the planarity of $H$. $\square$

**Proposition 3.5.** Let $\mathcal{T}$ be a tiling. The connectedness graph of $\mathcal{D}_T$ is a planar graph.

Proof. Let $G = (V_G, E_G)$ be the connectedness graph of $\mathcal{D}_T$ and $H = (V_H, E_H)$ the connectedness graph of $\mathcal{V}_T \cup \mathcal{E}_T$. Then $H$ is a subgraph of $G$ and by Lemma 3.4 there is a bijection $\psi : V_H \cup E_H \to |H| \subset \mathcal{P}(\mathbb{R}^2)$ from the set of vertices and edges of $H$ to a planar embedding $|H|$ of $H$. In order to prove the planarity of $G$, we want to extend the definition of $\psi$ to a function $\phi : V_G \cup E_G \to \mathbb{R}^2$ satisfying conditions (1)-(4) of Definition 3.3. For that purpose, it suffices to define $\phi$ for every element of $\mathcal{F}_T$ and for every edge of $G$ connecting a face of the tiling with one of its vertices or edges.

Let $C \in \mathcal{F}_T$, we know that there is a unique tile $T$ that has $C$ as a face, and we know that $\mathcal{A}(C) = \mathcal{V}_C \cup \mathcal{E}_C$. Let $\gamma : \mathbb{B}^2 \to T$ be a homeomorphism between $\mathbb{B}^2$, the unit closed disk of $\mathbb{R}^2$, and $T$. Consider the set $\gamma^{-1}(\mathcal{V}_C) = \{w_1, w_2, \ldots, w_n\}$, which is contained in $\mathbb{S}^1$ since $\gamma$ is a homeomorphism and $\mathcal{V}_C \subset \partial T$. Without lost of generality we can suppose that $w_1, \ldots, w_n$ are numbered clockwise. Then we can
number the set \( \gamma^{-1}(E_C) = \{B_1, \ldots, B_n\} \) denoting by \( B_1 \) the arc delimited by \( w_1 \) and \( w_2 \) and then proceeding in a clockwise manner. For every \( i \in \{1, \ldots, n\} \), define \( A_i := \gamma(B_i) \), considering that \( \gamma \) is a homeomorphism we can assure that \( E_C = \{A_1, \ldots, A_n\} \). Being consistent with the notation introduced in Lemma 3.4, we have a point \( x_{A_i} \in A_i \) such that \( \psi(A_i) = \{x_{A_i}\} \). Let \( b_i := \gamma^{-1}(x_{A_i}) \), which is an element of \( B_i \subset S_1 \setminus (\gamma^{-1}(V_C)) \).

For every \( w \in \{w_1, \ldots, w_n, b_1, \ldots, b_n\} \), let \( L_w := \{tw \mid t \in (0, 1)\} \). Now we consider the family
\[
L_C := \{L_w \mid w \in \{w_1, \ldots, w_n, b_1, \ldots, b_n\}\},
\]
whose elements are pairwise disjoint and they are contained in \( \text{Int } B^2 \).

Let \( x_C := \gamma(0, 0) \). Since \((0, 0) \in \text{Int } B^2\), it follows that \( x_C \in C \). Each element of the family \( \gamma(L_C) = \{\gamma(L) \mid L \in L_C\} \) is an arc delimited by \( x_C \) and one of the points in the set \( \{v_1, \ldots, v_n, x_{A_1}, \ldots, x_{A_n}\} \), where \( v_i = \gamma(w_i) \in V_C \). Thus we can denote each arc \( \gamma(L) \in \gamma(L_C) \) as \( M_{x_C,v} \), where \( v \) is the point in \( \{v_1, \ldots, v_n, x_{A_1}, \ldots, x_{A_n}\} \) that, alongside \( x_C \), delimits \( \gamma(L) \). From its definition, it is clear that \( M_{x_C,v} \cap \{x_C, v\} = \emptyset \).

These arcs are pairwise disjoint, since \( \gamma \) is a homeomorphism and the elements of \( L_C \) are pairwise disjoint. Also, each arc \( M_{x_C,v} \) is contained in \( C \), since the elements of \( L_C \) are contained in \( \text{Int } B^2 \). From here we can infer that if \( D \) is another face of the tiling, then the elements of \( \gamma(L_C) \) and \( \gamma(L_D) \) are pairwise disjoint. Furthermore, we can also conclude that the elements of \( \gamma(L_C) \) do not intersect \( \partial T \). Lastly, note that \( M_{x_C,v} \in \gamma(L_C) \) implies
\[
\overline{M_{x_C,v}} = M_{x_C,v} \cup \{x_C, v\}.
\]

Therefore the function \( \phi : V_G \cup E_G \to \mathcal{P}(\mathbb{R}^2) \) given by
\[
\phi(x) = \begin{cases} 
\psi(x) & \text{if } x \in V_H \cup E_H, \\
\{x_C\} & \text{if } x = C \text{ for some } C \in \mathcal{F}_T, \\
M_{x_C,v} & \text{if } x = \{C, v\} \in E_G \setminus E_H, \text{ where } C \in \mathcal{F}_T \text{ and } v \in V_C, \\
M_{x_C,A} & \text{if } x = \{C, A\} \in E_G \setminus E_H, \text{ where } C \in \mathcal{F}_T \text{ and } A \in \mathcal{E}_C,
\end{cases}
\]
proves the planarity of \( G \). \( \square \)
4. A Jordan Curve Theorem for Tilings

In this section we present our generalization of Khalimsky’s Jordan curve theorem. To achieve this we rely on the connectedness graph of the digital version of a tiling.

First we generalize the definition of a digital Jordan curve that we have previously introduced in Section 2.

**Definition 4.1.** Let $T$ be a tiling. A subset $J$ of $D_T$ is a digital Jordan curve if $|J| \geq 4$ and if for every $j \in J$, $J \setminus \{j\}$ is a digital arc.

Recall that $J \setminus \{j\}$ is a digital arc in $D_T$ if and only if $J \setminus \{j\}$ is an induced simple path in the connectedness graph of $D_T$ (Remark 2.4). This correspondence allows us to define a Jordan curve in the context of graph theory.

**Definition 4.2.** Let $G$ be a graph. A graph-theoretical Jordan curve is an induced cycle of length equal or greater than four.

Let $G$ be a graph and $x$ a vertex of $G$. Keeping in mind the correspondence between a topological space and its connectedness graph, we denote by $\mathcal{A}(x, G)$ the subgraph induced by the vertices of $G$ adjacent to $x$. We say that $G$ is locally Hamiltonian if for every vertex $x$ of $G$, $\mathcal{A}(x, G)$ has a Hamiltonian cycle. The following theorem is crucial for our generalization and it is due to V. Neumann-Lara and R. Wilson ([17]).

**Theorem 4.3.** If $G$ is a locally Hamiltonian, connected graph and $J \subset G$ is a graph-theoretical Jordan curve, then

a) the complement of $J$ has at most two connected components in $G$,

b) if $G$ is planar, then the complement of $J$ has exactly two connected components in $G$.

It can be seen from Neumann-Lara and Wilson’s proof that if $G$ is an infinite, planar, locally Hamiltonian and connected graph and $J \subset G$ is a graph-theoretical Jordan curve, then out of the two connected components that make up the complement of $J$ in $G$, one is bounded whereas the other one is not. We call the set of vertices of the bounded component the interior of $V_G \setminus V_J$, and denote it by $I(J)$. In a similar manner, the set of vertices of the unbounded component, $O(J)$, are the exterior of $V_G \setminus V_J$. 
Theorem 4.4. Let $\mathcal{T}$ be a tiling and $J \subset \mathcal{D}_{\mathcal{T}}$ a digital Jordan curve. Then $\mathcal{D}_{\mathcal{T}} \setminus J$ has exactly two connected components.

Proof. Let $G$ be the connectedness graph of $\mathcal{D}_{\mathcal{T}}$, it suffices to see that $G$ satisfies the hypotheses of Theorem 4.3. Let $q : \mathbb{R}^2 \to \mathcal{D}_{\mathcal{T}}$ be the quotient function associated to the topology of $\mathcal{D}_{\mathcal{T}}$ (see equation 3.1). We know that $q$ is continuous and therefore the connectedness of $\mathcal{D}_{\mathcal{T}}$ is guaranteed by the connectedness of $\mathbb{R}^2$. Moreover, we know that $\mathcal{D}_{\mathcal{T}}$ is an Alexandrov discrete space, so by Theorem 2.5 we have that $G$ is connected. In addition, $G$ is a planar graph by Proposition 3.5.

It only rests us to prove that $G$ is a locally Hamiltonian graph. By Theorem 3.2 if $x$ is a vertex of $G$ we have that:

$$\mathcal{A}(x, G) = \begin{cases} \mathcal{E}_x \cup \mathcal{F}_x & \text{if } x \in \mathcal{V}_{\mathcal{T}}, \\ \mathcal{V}_x \cup \mathcal{F}_x & \text{if } x \in \mathcal{E}_{\mathcal{T}}, \\ \mathcal{V}_x \cup \mathcal{E}_x & \text{if } x \in \mathcal{F}_{\mathcal{T}}. \end{cases}$$

If $x \in \mathcal{V}_{\mathcal{T}}$, we know that at least three tiles intersect in $x$. Each of these tiles has exactly two edges whose vertex is $x$. This fact allows us to infer that if we consider the set $\mathcal{E}_x$ (which is finite) and we number their elements clockwise, then every two consecutive edges determine a face of $x$ (and none of these faces is determined by two different pairs of edges). We also know that every edge of $x$ is determined by exactly two faces of $x$. Lastly, we know that in the connectedness graph there are no two adjacent edges nor two adjacent faces. Thus the subgraph induced by $\mathcal{E}_x \cup \mathcal{F}_x$ is a cycle wherein the vertices of the graph corresponding to edges of the tiling and the vertices of the graph corresponding to faces of the tiling alternate.

If $x \in \mathcal{E}_{\mathcal{T}}$, $x$ has exactly two vertices and two faces, furthermore, both vertices are vertices of both faces. For that reason the subgraph of $G$ induced by $\mathcal{V}_x \cup \mathcal{F}_x$ is a cycle with exactly four vertices.

Finally, if $x \in \mathcal{F}_{\mathcal{T}}$ we recall the analysis made in Proposition 3.5 to conclude that the vertices and edges of $x$ alternate in $\partial x$, allowing us to verify that $\mathcal{V}_x \cup \mathcal{E}_x$ is a cycle.

In any case, we have proved that the subgraph induced by $\mathcal{A}(x, G)$ is a cycle and thus $G$ is locally Hamiltonian. In addition, we know that since $J$ is a digital Jordan curve, then the subgraph induced by $J$ in $G$ is a graph-theoretical Jordan curve. Hence, by Theorem 4.3 we can conclude that $G \setminus J$ has exactly two connected components in $G$. 
To finish the proof, recall that $\mathcal{D}_T$ is an Alexandrov space. Since $G \setminus J$ has exactly two connected components in $G$, by Theorem 2.5 we have that $\mathcal{D}_T \setminus J$ has exactly two connected components. Moreover, we know that the connected components of $\mathcal{D}_T \setminus J$ are precisely the vertices of the connected components of $G \setminus J$; in other words, the connected components of $\mathcal{D}_T \setminus J$ are its interior $I(J)$, which is bounded, and its exterior $O(J)$, which is not. □

5. Open and Closed Digital Jordan Curves

In [10], Khalimsky, Kopperman and Meyer developed a theory on the processing of digital pictures by means of studying boundaries. Their work relies on properties of the digital plane and on Khalimsky’s Jordan curve theorem. The digital plane is not a homogeneous space, nonetheless for any two faces $C_1, C_2 \in \mathcal{F}_T$ (where $T$ is the tiling of Example 2.8) there is a homeomorphism from the digital plane into itself that maps $C_1$ to $C_2$. For this reason, it is convenient to think about the pixels on a screen as the faces of the tiling.

Motivated by this approach, we are now interested in studying digital Jordan curves $J$ that satisfy two conditions:

(W1) half of the elements of $J$ are faces of the tiling (and they alternate with the non-faces elements of the curve).

(W2) $J$ encloses a face of the tiling (namely, $I(J) \cap \mathcal{F}_T \neq \emptyset$).

Lemma 5.1. Let $\mathcal{T}$ be a tiling and $J \subset \mathcal{D}_T$ a digital Jordan curve. Then, for every $x \in J$, $A(x) \cap I(J) \neq \emptyset$.

Proof. Let $x \in J$ and let $G$ be the connectedness graph of $\mathcal{D}_T$. We proceed by contradiction. Suppose that $A(x) \cap I(J) = \emptyset$ and let $x_0$ and $x_1$ be the two adjacent points to $x$ in $J$. It is clear that $x_0$ and $x_1$ are not adjacent. We know that the subgraph induced by $A(x)$ is a cycle with at least four elements. For that reason there are two paths contained in $A(x, G)$ that only intersect at $x_0$ and $x_1$. Let $a$ and $b$ be two vertices, different from $x_0$ and $x_1$, in each of these paths. Since $A(x) \cap I(J) = \emptyset$, then, with the exception of $x_0$ and $x_1$, the vertices of the paths we mentioned earlier are contained in $O(J)$. In particular, $a, b \in O(J)$.

Since $\mathcal{T}$ is a tiling, $G$ is a planar graph and thus there is a map $\psi : V_G \cup E_G \to \mathcal{P}(\mathbb{R}^2)$ that attests the planarity of $G$. Let $H$ be the subgraph induced by the vertices $V_H := I(J) \cup J$ in $G$. Being a subgraph
of $G$, $H$ is a planar graph too. Since $\mathcal{A}(x) \cap I(J) = \emptyset$, we have that $\mathcal{A}(x) \cap (I(J) \cup J) = \{x_0, x_1\}$, therefore $x$ is a vertex of exactly two faces of $H$. Let $C$ be the cycle that determines the bounded face of $H$ having $x$ as a vertex (the other face contains the exterior face of the subgraph induced by the vertices of $J$). Then we know that $\bigcup \psi(C)$ is the boundary of a set $D$, homeomorphic to a closed disk, and that $\psi(x_0), \psi(x), \psi(x_1) \subset \bigcup \psi(C) = \partial D$.

Now we construct a graph $\Gamma$ with the set of vertices $V_\Gamma = V_G \cup \{p\}$, where $p \notin V_G$, and the set of edges $E_\Gamma = E_G \cup \{\{p, w\} \mid w \in V_C\}$. We sustain that $\Gamma$ is a planar graph. Indeed, since $D$ is homeomorphic to a closed disk and $\psi(x_0), \psi(x), \psi(x_1) \subset \partial D$, we can replicate the construction of the point $x_C$ and the sets $M_{x_C, v}$ in the proof of Proposition 3.5 to prove the existence of a point $x_p \in \text{Int } D$ and sets $M_{x_p, w} \subset \text{Int } D$ such that the function $\phi : V_\Gamma \cup E_\Gamma \to \mathcal{P}(\mathbb{R}^2)$ defined as

$$\phi(x) = \begin{cases} 
\psi(x) & \text{if } x \in V_G \cup E_G, \\
\{x_p\} & \text{if } x = p, \\
M_{x_p, w} & \text{if } x \in E_\Gamma \setminus E_G, \text{ where } p \text{ and } w \in V_C 
\end{cases}$$

are the vertices of $x$,

proves the planarity of $\Gamma$.

However, if we take a look at the sets of vertices $\{p, a, b\}$ and $\{x_0, x, x_1\}$, we see that $\Gamma$ contains a subdivision of a complete bipartite graph $K_{3,3}$, which contradicts Kuratowski’s theorem ([3, Theorem 4.4.6]). Therefore we have proved that $\mathcal{A}(x) \cap I(J) \neq \emptyset$. □

**Proposition 5.2.** Let $\mathcal{T}$ be a tiling and $J \subset \mathcal{D}_\mathcal{T}$ a digital Jordan curve. The following are equivalent:

a) $J$ is closed.

b) $I(J)$ and $O(J)$ are open.

c) $J$ is nowhere dense.

d) $I(J) \cup O(J)$ is dense.

e) $J \cap F_\mathcal{T} = \emptyset$.

f) $J = \partial(I(J))$.

**Proof.** a) $\implies$ b). $I(J) \cup O(J)$ is the complement of $J$ and thus it is open. In addition, since $\mathcal{D}_\mathcal{T}$ is an Alexandrov space, Theorem 2.5
guarantees that $I(J)$ and $O(J)$ are open and closed in $I(J) \cup O(J)$, because they are the connected components of $I(J) \cup O(J)$. Since the latter is open in $\mathcal{D}_T$, we conclude that $I(J)$ and $O(J)$ are also open in $\mathcal{D}_T$.

b) $\implies$ c). We prove the contrapositive. Suppose there is an element $x \in \text{Int}(\overline{J})$. This implies that $N(x) \subset \overline{J}$. In particular, we have that $x \in J$ and hence $\{x\} \subset J$. Thus $\mathcal{A}(x) \cup \{x\} \subset J$. In this situation, $\overline{J}$ can not be a digital Jordan curve since the connectedness graph of $\mathcal{A}(x) \cup \{x\}$ contains at least one vertex, one edge and one face adjacent to each other; in other words, the connectedness graph of $\mathcal{A}(x) \cup \{x\}$ contains cycles of length 3. For that reason, $J$ is not closed and therefore $I(J) \cup O(J)$ is not open, which in turn implies that b) is false.

c) $\implies$ d). Suppose that $I(J) \cup O(J)$ is not dense. Since $\mathcal{D}_T$ is the disjoint union of $I(J)$, $O(J)$ and $J$, there is a point $x \in J$ such that $N(x) \cap (I(J) \cup O(J)) = \emptyset$. Therefore $N(x) \subset J$, so $x \in \text{Int}(J)$ and thus $\text{Int}(\overline{J}) \neq \emptyset$.

d) $\implies$ e). Assume there is a point $x \in J \cap \mathcal{F}_T$. Thus $N(x) = \{x\} \subset J$. Since $J \cap (I(J) \cup O(J)) = \emptyset$, we have that $N(x) \cap (I(J) \cup O(J)) = \emptyset$ and therefore $I(J) \cup O(J)$ cannot be dense.

e) $\implies$ f). First let us prove that $J$ is closed. By e), the elements of $J$ are alternating vertices and edges. The closure of a vertex is the vertex itself, and the closure of an edge is the union of the edge and its two vertices. Since the vertices and edges alternate in $J$, the vertices of each edge in $J$ are contained in $J$. Therefore the closure of each element of $J$ is contained in $J$. Since $\mathcal{D}_T$ is an Alexandrov discrete space this implies that $J$ is closed.

Now by the implication a) $\implies$ b), we infer that $I(J)$ and $O(J)$ are open. This proves that $\partial(I(J)) = (I(J)) \setminus I(J)$. Furthermore, since $\mathcal{D}_T$ is the disjoint union of $J$, $I(J)$ and $O(J)$, we conclude that $I(J) \cup J$ is a closed set that contains $I(J)$. Thus

$$\partial(I(J)) = (I(J)) \setminus I(J) \subset (I(J) \cup J) \setminus I(J) = J.$$
$J$ and $I(J)$ are disjoint, and that the vertices of $x$ are contained in $J$. Thus $\mathcal{A}(x) \cap I(J)$ contains at least one of the faces of $x$, implying that $N(x) \cap I(J) \neq \emptyset$. Lastly, if $x$ is a vertex, then $\mathcal{A}(x) \subset N(x)$ and therefore $N(x) \cap I(J) \neq \emptyset$. In any case, $J \subset \partial(I(J))$. 

f) $\implies$ a). This follows from the fact that the boundary of any subset of a topological space is always closed. 

Analogously, we can obtain a dual version of Proposition 5.2. We let the proof to the reader, since it is similar to the previous one.

**Proposition 5.3.** Let $\mathcal{T}$ be a tiling and $J \subset \mathcal{D}_\mathcal{T}$ a digital Jordan curve. The following are equivalent:

(a) $J$ is open.

(b) $I(J)$ and $O(J)$ are closed.

(c) $J \cap \mathcal{V}_\mathcal{T} = \emptyset$.

After Proposition 5.3-(c), we conclude that open digital curves are made exclusively of faces and edges (hence, they satisfy condition W1). Furthermore, if $J$ is an open digital Jordan curve, then $I(J)$ is closed, so for every $x \in I(J)$, $\{x\} \subset I(J)$, and whether $x$ is a vertex, an edge or a face, $\{x\}$ always contains a vertex of the tiling. This reasoning yields the following remark.

**Remark 5.4.** If $J$ is an open digital curve, then $I(J) \cap \mathcal{V}_\mathcal{T} \neq \emptyset$.

However, we cannot guarantee that $I(J)$ contains a face of the tiling. This situation is pictured for the tiling of the plane with regular hexagons in Figure 5. In this figure, the dotted line intersects all the elements of a digital Jordan curve consisting only of edges and faces of the tiling, whilst the thick black line intersects all the elements of its interior, none of which is a face of the tiling.

On the other hand, by Proposition 5.2-(d), every closed digital Jordan curve satisfies condition W2. However, this kind of curves may not be very interesting, since they do not contain any face (and therefore they do not satisfy condition W1). This is why, if we are looking for a class of digital Jordan curves satisfying W1 and W2, we need to add an extra condition. We formalize this in the following section.
6. Well-behaved Digital Jordan Curves

Intuitively, the problem of the example in Figure 5 is that the elements of \( J \) that are faces of the tiling, are very close to each other. In order to prevent this from happening, we need the following.

**Definition 6.1.** Let \( \mathcal{T} \) be a tiling and \( J \subset \mathcal{D}_\mathcal{T} \) a digital Jordan curve. We say that \( J \) is well-behaved if for every \( C \in \mathcal{F}_\mathcal{T} \cap J \) and every \( x \in \mathcal{V}_C \), \( \mathcal{F}_x \not\subseteq J \).

Clearly the curve of Figure 5 is not well-behaved.

**Lemma 6.2.** Let \( \mathcal{T} \) be a tiling and \( J \subset \mathcal{D}_\mathcal{T} \) a well-behaved digital Jordan curve. If \( \mathcal{V}_\mathcal{T} \cap I(J) \neq \emptyset \) (in particular, if \( J \) is a well-behaved open digital Jordan curve), then \( \mathcal{F}_\mathcal{T} \cap I(J) \neq \emptyset \).

**Proof.** Since \( I(J) \) and \( O(J) \) are the components of \( Y := I(J) \cup O(J) \), we can find an open set \( U \subset \mathcal{D}_\mathcal{T} \) such that \( U \cap Y = I(J) \). Thus,

\[
I(J) \subset U \subset I(J) \cup J.
\]

Assume that \( \mathcal{F}_\mathcal{T} \cap I(J) = \emptyset \), and pick an element \( x \in \mathcal{V}_\mathcal{T} \cap I(J) \). Notice that

\[
\mathcal{F}_x \subset N(x) \subset U \subset I(J) \cup J.
\]
Now, we can use the assumption that $\mathcal{F}_T \cap I(J) = \emptyset$ to conclude that $\mathcal{F}_T \subset J$, which is impossible because $J$ is well-behaved. Therefore $\mathcal{F}_T \cap I(J) \neq \emptyset$, as desired. □

As an immediate consequence of Lemma 6.2, we have the following.

**Remark 6.3.** Let $\mathcal{T}$ be a tiling and $J \subset \mathcal{D}_\mathcal{T}$ a well-behaved open digital Jordan curve. Then $J$ satisfies condition $W1$ and $W2$.

To finish this paper we will use the theory we have developed in order to prove a generalization of Rosenfeld’s theorem that we presented on the introduction (Theorem 1.1).

Let $\mathcal{D}_\mathcal{T}$ be the digital plane. Recall that Rosenfeld works with a squared grid endowed with the $k$-adjacency relations $(k \in \{4, 8\})$ previously described. If we think of the faces of the digital plane as the points in Rosenfeld’s grid, two faces $C_1, C_2 \in \mathcal{F}_\mathcal{T}$ are $4$-adjacent if $\mathcal{E}_{C_1} \cap \mathcal{E}_{C_2} \neq \emptyset$. Similarly, we say that they are $8$-adjacent if $\mathcal{V}_{C_1} \cap \mathcal{V}_{C_2} \neq \emptyset$. We also give the corresponding definition of a simple closed 4-path with at least five points: $J \subset \mathcal{D}_\mathcal{T}$ is a 4-Jordan curve if $J$ is an open digital Jordan curve such that $|\mathcal{F}_\mathcal{T} \cap J| \geq 5$ and for every $C \in \mathcal{F}_\mathcal{T} \cap J$, $C$ has exactly two 4-adjacent faces in $J$. This reinterpretation of Rosenfeld’s work, motivates the following definition

**Definition 6.4.** Let $\mathcal{T}$ be a tiling and $\mathcal{D}_\mathcal{T}$ its digital version.

1. We say that two faces $C_1, C_2 \in \mathcal{F}_\mathcal{T}$ are edge-adjacent (or that $C_1$ and $C_2$ are edge-neighbors) if $\mathcal{E}_{C_1} \cap \mathcal{E}_{C_2} \neq \emptyset$.

2. We say that two faces $C_1, C_2 \in \mathcal{F}_\mathcal{T}$ are vertex-adjacent (or that $C_1$ and $C_2$ are vertex-neighbors) if $\mathcal{V}_{C_1} \cap \mathcal{V}_{C_2} \neq \emptyset$.

3. A finite sequence of faces $(C_0, \ldots, C_n)$ is an edge-path of faces (vertex-path of faces) if $C_i$ is edge-adjacent (vertex-adjacent) to $C_{i+1}$ and $C_{i-1}$, for every $i = 1, \ldots, n - 1$.

4. A set $Y \subset \mathcal{D}_\mathcal{T} \cap \mathcal{F}_\mathcal{T}$ is edge-connected (vertex-connected) if there is an edge-path (vertex-path) between any two elements $C_1, C_2 \in Y$, such that every element of the path belongs to $Y$.

5. A digital Jordan curve $J \subset \mathcal{D}_\mathcal{T}$ is an edge-Jordan curve, if $J$ is open and for every face $C \in J \cap \mathcal{F}_\mathcal{T}$ there are exactly two faces in $J$ that are edge-adjacent with $C$. 
Let $\mathcal{T}$ be a tiling and define
\[ \Delta(\mathcal{T}) = \sup\{|F_x| : x \in V_{\mathcal{T}} \}. \]
Observe that if $J$ is an edge-Jordan curve and $F_x \subset J$ for a certain $x \in V_x$, then every element of $F_x$ has exactly two edge-neighbors in $J$, and therefore $J$ cannot contain any other face of the tiling. This yields the following remark.

**Remark 6.5.** Let $\mathcal{T}$ be a tiling and $J \subset D_{\mathcal{T}}$ be an edge-Jordan curve. If $|J \cap \mathcal{F}_{\mathcal{T}}| \geq \Delta(\mathcal{T}) + 1$, then $J$ is well-behaved.

We can now generalize Rosenfeld’s theorem as follows.

**Theorem 6.6.** Let $\mathcal{T}$ be a tiling with $\Delta(\mathcal{T}) < \infty$. If $J \subset D_{\mathcal{T}}$ is an edge-Jordan curve with $|J \cap \mathcal{F}_{\mathcal{T}}| \geq \Delta(\mathcal{T}) + 1$, then $I(J) \cap \mathcal{F}_{\mathcal{T}} \neq \emptyset$, $O(J) \cap \mathcal{F}_{\mathcal{T}} \neq \emptyset$ and these two sets are vertex-connected.

**Proof.** Since $I(J) \cup J$ is a finite set and $\mathcal{F}_{\mathcal{T}}$ is infinite, we always have that $O(J) \cap \mathcal{F}_{\mathcal{T}} \neq \emptyset$. On the other hand, by Remark 6.5 and Lemma 6.2, we infer that $I(J) \cap \mathcal{F}_{\mathcal{T}} \neq \emptyset$.

Let us prove that $I(J) \cap \mathcal{F}_{\mathcal{T}}$ is vertex connected. We know that $I(J)$ is connected and closed. Thus, for every $C \in \mathcal{F}_{\mathcal{T}} \cap I(J)$ we have that $V_C \cup E_C \subset I(J)$.

**Claim:** For every $A \in E_{\mathcal{T}} \cap I(J)$, $\mathcal{F}_A \cap I(J) \neq \emptyset$. Indeed, if $A \in E_{\mathcal{T}} \cap I(J)$ is such that $\mathcal{F}_A \cap I(J) = \emptyset$, we can use the fact that $O(J)$ is closed to conclude that $\mathcal{F}_A \subset J$. Since $J$ is an open digital Jordan curve in $D_{\mathcal{T}}$, Proposition 5.3 guarantees that $J \cap V_{\mathcal{T}} = \emptyset$. Thus, each face of $A$ has a common edge with three faces of $J$, which contradicts the definition of $J$. This proves the claim.

Since $I(J)$ is a connected Alexandrov space, Theorem 2.5 ensures that $I(J)$ is digitally arc connected. Thus for any $C_1, C_2 \in I(J) \cap \mathcal{F}_{\mathcal{T}}$ there is a digital arc of minimum length $\alpha_0 = (x_0 = C_1, x_1, \ldots, x_n = C_2)$ in $I(J)$ from $C_1$ to $C_2$. We can also assume that $\alpha_0$ contains a maximum number of faces. Since $\alpha_0$ has minimum length, there are at most three consecutive elements of the same tile in $\alpha_0$. Let
\[ m := \max\{k \in \mathbb{N} : \forall i \leq k \colon x_{2i} \in \mathcal{F}_{\mathcal{T}} \}. \]
Suppose that $m \neq \frac{n}{2}$. First observe that if $2m = n - 1$, then $x_{2m}$ and $x_n$ would be two consecutive faces in $\alpha_0$, which is impossible. Thus, $0 \leq 2m < n - 1$. In this situation, $x_{2m}$ is a face of the tiling while
$x_{2m+2}$ is not. If $x_{2m+1} \in E_{x_{2m}}$, then $x_{2m+2} \in V_{x_{2m}}$, contradicting the fact that $\alpha_0$ is a digital arc. Therefore we infer that $x_{2m+1} \in V_{x_{2m}}$, $x_{2m+2} \in E_T \setminus E_{x_{2m}}$ and $x_{2m+1} \in V_{x_{2m+2}}$.

By the claim, we can pick a face $C \in F_{x_{2m}+2} \cap I(J)$. Once again, since $\alpha_0$ is a digital arc, we conclude that $C \notin \alpha_0$ and $x_{2m+3} \in V_C$ (if $2m + 2 < n$). Thus, $\{x_{2m_0+1}, x_{2m_0+2}, x_{2m_0+3}\} \subset \{C\}$, implying that $x_{2m_0+4} \notin \{C_{x_{2m_0}+2}\}$.

For $i \in \{0, \ldots, n\}$, let

$$y_i = \begin{cases} x_i & \text{if } i \neq 2m_0 + 2, \\ C & \text{if } i = 2m_0 + 2. \end{cases}$$

and define $\alpha_1 := (y_0, y_1, \ldots, y_n)$. Then $\alpha_1$ is a digital arc from $C_1$ to $C_2$ in $I(J)$ containing more faces than $\alpha_0$. This contradicts the fact that $\alpha_0$ contains a maximum number of faces. Thus, we can conclude that $m = \frac{n}{2}$ and therefore $(x_0, x_2, \ldots, x_{2m})$ is a simple vertex-path between $C_1$ and $C_2$. This proves that $I(J) \cap F_T$ is vertex-connected.

Similarly we can prove that $O(J) \cap F_T$ is vertex-connected. This completes the proof. \hfill \Box

It is clear that Theorem 6.6 directly implies Rosenfeld’s theorem (Theorem 1.1). Moreover, we can also deduce similar results for every grid of points induced by the faces of a regular tiling of the plane, such as the one given by Kopperman in [15, Theorem 26] for a hexagonal grid of points.

Finally, there is a dual theorem, also due to Rosenfeld (see [24, Theorem 3.3]), that is obtained by interchanging the roles of 4 and 8 in Theorem 1.1. To finish this paper, we present a generalization of this result in the context that we have been working on. In order to do this, consider a tiling $T$ and define a vertex-Jordan curve as a digital Jordan curve $J \subset D_T$ such that $J \subset F_T \cup V_T$ and with the property that every $C \in F_T \cap J$ has exactly two vertex-neighbors in $J$.

**Theorem 6.7.** Let $T$ be a tiling with $\Delta(T) < \infty$. Suppose that $J \subset D_T$ is a vertex-Jordan curve such that $|J \cap F_T| \geq \Delta(T) + 1$. Then

1. $J$ is well-behaved.
2. $I(J) \cap F_T \neq \emptyset$ and $O(J) \cap F_T \neq \emptyset$. 

(3) $I(J) \cap F_T$ and $O(J) \cap F_T$ are edge-connected.

Proof. Since $I(J)$ and $O(J)$ are the components of $Y := I(J) \cup O(J)$, we can find two sets $U, K \subset D_T$, with $U$ open and $K$ closed, such that

$$U \cap Y = I(J) = K \cap Y.$$ 

(1) If $J$ is not well-behaved, there exists a vertex $x \in V_T$, such that $F_x \subset J$. Now, for every $C_1, C_2 \in F_x$, $C_1$ and $C_2$ are vertex-adjacent. Since $J$ is a vertex-Jordan curve and $|F_x| \geq 3$, we infer that $J \cap F_T = F_x$. Thus

$$|F_x| = |J \cap F_T| \geq |\Delta(T)| + 1 \geq |F_x| + 1,$$

a contradiction.

(2) Since we always have that $O(J) \cap F_T \neq \emptyset$, we only need to prove the other inequality. By contradiction, assume that $I(J) \cap F_T = \emptyset$. We claim that $V_T \cap I(J) \neq \emptyset$. Indeed, if $V_T \cap I(J) = \emptyset$, then every element of $I(J)$ is an edge. Thus, for every $E \in I(J)$, we get the following inclusions

$$F_E \subset N(E) \subset U \subset I(J) \cup J \text{ and } V_E \subset \overline{E} \subset K \subset I(J) \cup J.$$ 

This implies that $\mathcal{A}(E) = V_E \cup F_E \subset J$. Since $J$ is a digital Jordan curve, $J$ cannot contain any other element of $D_T$. This implies that $J$ contains only two faces, which contradicts the hypothesis $|J \cap F_T| \geq |\Delta(T)| + 1$. Therefore $V_T \cap I(J) \neq \emptyset$ and since $J$ is well behaved, we infer from Lemma 6.2 that $I(J)$ contains a face of the tiling. This contradicts the original assumption that $I(J) \cap F_T = \emptyset$.

(3) Since $I(J)$ is connected, for any $C_1, C_2 \in I(J) \cap F_T$, there exists a digital path $\alpha_0 = (C_1 = x_0, x_1, \ldots, x_n = C_2)$ completely contained in $I(J)$. Furthermore, we can assume that $\alpha_0$ contains a minimum number of vertices (namely, every other path between $C_1$ and $C_2$ contains at least the same amount of vertices than $\alpha_0$).

If $\alpha_0$ contains a vertex, let

$$m := \min \{ k \in \{1, \ldots, n\} \mid x_k \text{ is a vertex} \}.$$ 

In this case $x_{m-1}$ and $x_{m+1}$ belong to $\mathcal{A}(x_m) = E_{x_m} \cup F_{x_m}$.

Observe that $\mathcal{A}(x_m)$ can only contain one element of $J$, at most. Indeed, if $D_1, D_2 \in J \cap \mathcal{A}(x_m)$, then $\{D_1, D_2\} \subset F_{x_m}$, because $J$ does not have any edge. This implies that $D_1$ and $D_2$ are vertex-adjacent, but since $x_m \notin J$, there must exist two other faces, $D, D' \in J$, different from $D_1$ and $D_2$, such that $D_1$ is vertex adjacent to $D$ and $D'$.

Hence $J$ cannot be a vertex-Jordan curve, a contradiction.
Since $\mathcal{A}(x_m)$ induces a cycle in the connectedness graph of $D_T$, we can find a digital path $\beta = (x_{m-1} = y_0, y_1, \ldots, y_k = x_{m+1})$, where each $y_i$ belongs to $\mathcal{A}(x_m) \setminus J$. In particular, there are no vertices in $\beta$. Furthermore, since $N(x_m) \subset U \subset I(J) \cup J$ and $\overline{\{x_m\}} \subset K \subset I(J) \cup J$, we get that
\[
\{y_1, \ldots, y_k\} \subset \mathcal{A}(x_m) \setminus J \subset (I(J) \cup J) \setminus J = I(J).
\]
Therefore the path
\[
\alpha_1 := (C_1 = x_0, \ldots, x_{m-1}, y_1, \ldots, y_{k-1}, x_{m+1}, \ldots, x_n = C_2)
\]
is a digital path between $C_1$ and $C_2$, which is completely contained in $I(J)$ and it contains one vertex less than $\alpha_0$, a contradiction. Thus $\alpha_0$ does not contain any vertex, and therefore $x_k \in F_T$ if $k$ is even and $x_k \in E_T$ if $k$ is odd. Then
\[
\alpha := (C_1 = x_0, x_2, \ldots, x_n = C_2)
\]
is an edge-path of faces, which proves that $I(J)$ is edge-connected, as desired. Analogously we can prove that $O(J)$ is edge-connected. \qed

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