ON INHOMOGENEOUS HEAT EQUATION WITH INVERSE SQUARE POTENTIAL

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Abstract. We study inhomogeneous heat equation with inverse square potential, namely,
\[ \partial_t u + L_a u = \pm | \cdot |^{-b} |u|^{\alpha} u, \]
where \( L_a = -\Delta + a |x|^{-2} \). We establish some fixed-time decay estimate for \( e^{-tL_a} \) associated with inhomogeneous nonlinearity \( | \cdot |^{-b} \) in Lebesgue spaces. We then develop local theory in \( L^q \)-scaling critical and super-critical regime and small data global well-posedness in critical Lebesgue spaces. We further study asymptotic behaviour of global solutions by using self-similar solutions, provided the initial data satisfies certain bounds. Our method of proof is inspired from the work of Slimene-Tayachi-Weissler (2017) where they considered the classical case, i.e. \( a = 0 \).

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1. Introduction

We study heat equation associated to inverse square potential:
\[
\begin{aligned}
&u_t(t,x) + L_a u(t,x) = \frac{\mu}{|x|^2} |u(t,x)|^\alpha u(t,x) \\
&u(0,x) = \varphi(x)
\end{aligned}
\quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d
\tag{1.1}
\]
where \( d \geq 2, \mu \in \{ \pm 1 \} \) and \( b, \alpha > 0 \). (1.1) is also known as Hardy parabolic equation. The Schrödinger operator
\[ L_a = -\Delta + \frac{a}{|x|^2}, \]
with \( a \geq -\left( \frac{d-2}{2} \right)^2 \), is initially defined with domain \( C_\infty(\mathbb{R}^d \setminus \{0\}) \). See [14, Section 1.1]. It is then extended as an unbounded operator in \( L^p(\mathbb{R}^d) \) that generates a positive semigroup \( \{e^{-tL_a}\}_{t \geq 0} \).

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in $L^p(\mathbb{R}^d)$ for $s_1 < d/p < s_2 + 2$. Here,

$$s_1 := s_1(a) = \frac{d - 2}{2} - \sqrt{\frac{(d - 2)^2}{4} + a} \quad \text{and} \quad s_2 := s_2(a) = \frac{d - 2}{2} + \sqrt{\frac{(d - 2)^2}{4} + a}$$

are the roots of $s^2 - (d - 2)s - a = 0$, see [15, Theorems 1.1, 1.3]. Moreover, the semigroup \( \{e^{-t\mathcal{L}_a}\}_{t \geq 0} \) has the following smoothing effect (see also Remark 1.7 below).

**Theorem A** (Decay estimate, Theorem 5.1 in [11]). Assume $d \geq 2$, $a \geq -\frac{(d-2)^2}{4}$ and

$$\tilde{s}_1 = \max(s_1, 0) \quad \text{and} \quad \tilde{s}_2 = \min(s_2, d - 2).$$

Then, for all $\tilde{s}_1 < \frac{d}{q} \leq \frac{d}{p} < \tilde{s}_2 + 2$ and $t > 0$, we have

$$\|e^{-t\mathcal{L}_a}f\|_{L^q} \leq c t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}\|f\|_{L^p}.\quad(1.3)$$

This smoothing effect for $e^{-t\mathcal{L}_a}$ will play a vital role in our analysis of short and long time behaviour of (mild) solutions of (1.1). We note that the inverse-square potential breaks space-translation symmetry for (1.1). However, it retains the scaling symmetry. Specifically, if $u(t, x)$ solves (1.1), then

$$u_\lambda(t, x) = \lambda^{\frac{2+b}{\alpha}} u(\lambda^2 t, \lambda x)\quad(1.4)$$

also solves (1.1) with data $u_\lambda(0)$. The Lebesgue space $L^q$ is invariant under the above scaling only when $q = q_c := \frac{d}{2+b}$. We shall see that this constant $q_c$ will play important role in studying (1.1). The problem (1.1) is $L^q$—

$$\begin{cases}
\text{sub-critical} & \text{if } 1 \leq q < q_c \\
\text{critical} & \text{if } q = q_c \\
\text{super-critical} & \text{if } q > q_c.
\end{cases}$$

We point out that with $a = 0$, the heat equation (1.1) is extensively studied, see [6] and the references therein. Corresponding results in the context of Schrödinger equation, are investigated in [7, 2] and the references therein. With $a \neq 0$, the operator $\mathcal{L}_a$ is a mathematically intriguing borderline situation that can be found in a number of physical contexts, including geometry, combustion theory to the Dirac equation with Coulomb potential, and to the study of perturbations of classic space-time metrics such as Schwarzschild and Reissner–Nordström. See [14, 21, 4, 12, 17] for detailed discussions. In fact, substantial progress has been made to understand well-posedness theory for nonlinear Schrödinger and wave equation associated to $\mathcal{L}_a$, see for e.g. [16, 8, 13, 21] and references therein. The mathematical interest in these equations with $a|x|^{-2}$ however comes mainly from the fact that the potential term is homogeneous of degree -2 and therefore scales exactly the same as the Laplacian. On the other hand, it appears that we know very little about the well-posedness results for heat equation associated to $\mathcal{L}_a$, even when $b = 0$ in (1.1). In this note, we aim to initiate a systematic study of well-posedness theory for (1.1).

In this article, by a solution to (1.1) we mean mild solution of (1.1), that is, a solution of the integral equation

$$u(t) = e^{-t\mathcal{L}_a}\varphi + \mu \int_0^t e^{-(t-s)\mathcal{L}_a}(-b|u(s)|^\alpha u(s))ds.$$ 

We now state our first theorem.

**Theorem 1.1** (Local theory). Let $d \geq 2$, $a \geq -\frac{(d-2)^2}{4}$, $0 \leq b < \min(2, d)$, $\tilde{s}_1, \tilde{s}_2$ be as in (1.2) and $0 < \alpha < \frac{2+b}{\tilde{s}_1}$ and $\varphi \in L^q(\mathbb{R}^d)$. 

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(1) Assume that
\[
\max \left\{ \frac{d(\alpha + 1)}{s_2 + 2 - b}, q_c \right\} < q < \frac{d}{s_1},
\]

Then there exists a maximal time \( T_{\max} > 0 \) and a unique solution \( u \) of (1.1) such that \( u \in C([0, T_{\max}); L^q(\mathbb{R}^d)) \).

Moreover, blowup alternative holds, i.e. if \( T_{\max} < \infty \), then \( \lim_{t \to T_{\max}} \| u(t) \|_q = \infty \).

(2) Assume that
\[
q_c \leq q < \frac{d}{s_1}, \quad q > \frac{d}{s_2 + 2}
\]

(cf. Figure 1). Then
(i) there exist \( T > 0 \) and a solution \( u \) of (1.1) such that \( u \in C([0, T], L^q(\mathbb{R}^d)) \).

(ii) uniqueness in part (i) is guaranteed only among functions in
\[
\begin{align*}
- \{ u \in C([0, T]; L^q(\mathbb{R}^d) : \sup_{t \in [0, T]} t^{\frac{d}{q} + \frac{1}{2} - \frac{1}{r}} \| u(t) \|_r < \infty \} & \quad \text{where } r \text{ satisfies (3.1)} \\
- \{ u \in C([0, T]; L^q(\mathbb{R}^d) : \sup_{t \in [0, T]} t^{\frac{d}{q} + \frac{1}{2} - \frac{1}{r}} \| u(t) \|_r \leq M \} & \quad \text{for some } M > 0 \text{ and } r \text{ satisfies (3.5)} \text{ and } q = q_c.
\end{align*}
\]

Moreover, for the super-critical exponents, continuous dependency on data and the blowup alternative holds.

(3) In all the above cases, except where \( q = q_c \), the maximal existence time of the solution, denoted by \( T_{\max} \), depends on \( \| \varphi \|_q \), not \( \varphi \) itself.

As far as we know, Theorem 1.1 is new for \( a \neq 0 \). In [1, Theorem 1.1], Slimene, Tayachi and Weissler proved Theorem 1.1 when \( a = 0 \). Our method of proof is based on standard contraction argument and inspired from the work in [1]. The main key ingredient to prove Theorem 1.1 is Proposition 2.1. In order to prove Proposition 2.1, Theorem A and scaling properties of \( e^{-t\mathcal{L}_a} \) play a vital role.

Remark 1.1. For \( a = 0 \), (1.3) is valid for the end points, see Remark 1.7 below. Thus, in this case, whenever there is a strict inequality \( < \) involving \( s_1 \) in Theorem 1.1 it can be relaxed to non strict one \( \leq \).

Remark 1.2. The case \( a = b = 0 \) corresponds to standard nonlinear heat equation. This has been extensively studied in the literature since the pioneering work done in early 80s, see [20, 3]. In this situation, for the sub-critical case, Weissler [20] proved local well-posedness for (1.1) in \( L^q(\mathbb{R}^d) \) for \( q > q_c \geq 1 \). For sub-critical case, there is no general theory of existence, see [20, 3]. In fact, Haraux-Weissler [9] proved that if \( 1 < q_c < \alpha + 2 \), then there is a global solution (with zero initial data) in \( L^q(\mathbb{R}^d) \) for \( 1 \leq q < q_c \). But no such solution exists when \( \alpha + 2 < q_c \). For critical case, i.e. \( q = q_c \) the solution exists globally in time for small initial data.

Remark 1.3. For the sub-critical exponents, i.e. for \( q < q_c \), classical inhomogeneous heat equation, i.e. (1.1) with \( a = 0 \), is known to be ill-posed in \( L^q(\mathbb{R}^d) \). We believe that similar result hold for (1.1). However, shall not pursue this issue in the present paper.

After achieving local well-posedness with a contraction mapping, one then finds the following lower estimate for blow-up in the super critical case \( q > q_c \) as in the classical case \( a = b = 0 \).

**Theorem 1.2 (Lower blow-up rate).** Assume that \( q > \max(1, q_c) \) and \( T_{\max} < \infty \), where \( T_{\max} \) is the existence time of the resulting maximal solution of (1.1). Then under the hypotheses of Theorem 1.1, we have
\[
\| u(t) \|_q \geq (T_{\max} - t)^{-\frac{b}{2\alpha}}, \quad \forall t \in [0, T_{\max}).
\]
\[ \frac{1}{q} = \frac{1}{q_c} \]

(a) The case \( d = 3, a = -\frac{1}{8}, b = 1 \)

(b) The case \( d = 3, a \geq 0, b = 1 \)

**Figure 1.** Local well-posedness in mere \( L^q(\mathbb{R}^d) \) occurs in the darkly shaded (open) region by part (1) of Theorem 1.1. Local existence in \( L^q(\mathbb{R}^d) \) is guaranteed by part (2) of Theorem 1.1 in the total shaded (dark & light) region along with the open segment on boundary which is part of the curve \( \frac{1}{q} = \frac{1}{q_c} \).

On the other hand in the critical case \( q = q_c \), we achieve ‘small’ data global well-posedness where the smallness is in the sense described in the theorem below:

**Theorem 1.3** (Small data global existence). Let \( d \geq 2, 0 \leq b < \min(2, d) \) and \( \frac{d}{s_2 + 2} < q_c < \frac{d}{s_1} \) i.e.

\[ \frac{2 - b}{s_2 + 2} < \alpha < \frac{2 - b}{s_1} \]

Then we have the following.

1. If \( \varphi \in L^q(\mathbb{R}^d) \) and \( \|\varphi\|_{q_c} \) is sufficiently small, then \( T_{\text{max}}(\varphi) = \infty \).
2. If \( \varphi \in S'(\mathbb{R}^d) \) such that \( |\varphi| \leq c(1 + |\cdot|^2)^{-\sigma/2}, c \) sufficiently small and \( \sigma > \frac{2-b}{\alpha} \), then \( T_{\text{max}}(\varphi) = \infty \).
3. Let \( \varphi \in S'(\mathbb{R}^d) \) be such that \( |\varphi| \leq c \cdot |\cdot|^{-\frac{2+b}{\sigma}} \), for \( c \) sufficiently small. Then there exists a global time solution of (1.5), \( u \in C([0, \infty); L^q(\mathbb{R}^d)) \) for all \( q \in (q_c, \frac{d}{s_1}) \). Moreover \( u(t) \to \varphi \) in \( S'(\mathbb{R}^d) \) as \( t \to 0 \).

Theorem 1.3 is proved using more general Theorem 3.1 below. This is inspired from [1, Theorem 4.1] (that deal with \( a = 0 \) case), in which they uses idea from the earlier work [5, Theorem 6.1] (that deals with \( a = b = 0 \) case).

A solution of (1.1) is self-similar if \( u_\lambda = u \) for all \( \lambda > 0 \) where \( u_\lambda \) is defined in (1.4). In [10, 18], authors have established the existence of radially symmetric self-similar solutions and later in [1, Theorem 1.4] authors have proved the self-similar solutions that are not necessarily symmetric for classical inhomogeneous heat equation. In the next theorem, we establish similar result in the presence of inverse square potential.
Theorem 1.4 (Self-similar solutions). Let $d \geq 2$, $0 < b < \min(2, d)$ and $\frac{d}{s_2 + 2} < q < \frac{d}{s_1}$ i.e. 

$$\frac{2 - b}{s_2 + 2} < \alpha < \frac{2 - b}{s_1}.$$ 

Let $\varphi(x) = \omega(x)|x|^{-\frac{2b}{\alpha}}$, where $\omega \in L^\infty(\mathbb{R}^d)$ is homogeneous of degree 0 and $\|\omega\|_\infty$ is sufficiently small. Then there exists a global mild self-similar solution $u_S$ of (1.1) and $u_S(t) \to \varphi$ in $S'(\mathbb{R}^d)$ as $t \to 0$.

Using this self-similar solution, the next theorem gives the information about the asymptotic behaviour of global solutions achieved by Theorem 1.3, provided the data satisfies certain bounds.

Theorem 1.5 (Asymptotic behaviour). Let $d \geq 2$, $0 < b < \min(2, d)$ and $\frac{d}{s_2 + 2} < q < \frac{d}{s_1}$ i.e. \(\frac{2 - b}{s_2 + 2} < \alpha < \frac{2 - b}{s_1}\) and \(\frac{2 - b}{\alpha} \leq \sigma < \tilde{s}_2 + 2\).

Let $\varphi \in S'(\mathbb{R}^d)$ be such that 

$$|\varphi(x)| \leq c(1 + |x|^2)^{-\sigma/2}, \quad \forall x \in \mathbb{R}^d$$ 

for $c > 0$ sufficiently small, and 

$$|\varphi(x)| = \omega(x)|x|^{-\sigma}, \quad \forall |x| \geq A$$

for some constant $A > 0$ and some $\omega \in L^\infty(\mathbb{R}^d)$ homogeneous of degree 0 with $\|\omega\|_\infty$ sufficiently small.

Let $u, u_S$ be the unique solutions to (1.1) with data $\varphi$, $\omega(x)|x|^{-\frac{2b}{\alpha}}$ given by Theorem 1.3, Theorem 1.4 respectively. Then

(1) if $\sigma = \frac{2 - b}{\alpha}$ and $q \in [r, \frac{d}{s_1})$ there exists $\delta > 0$ such that

$$\|u(t) - u_S(t)\|_q \leq Ct^{-\left(\frac{\sigma}{2} - \frac{d}{s_1}\right) - \delta}, \quad \text{for all } t > 0 \quad (1.5)$$

where $C$ is a positive constant. In particular, if $\omega \neq 0$, there exist $c_1, c_2 > 0$ such that for $t$ large

$$c_1 t^{-\left(\frac{2 - b}{\alpha} - \frac{d}{s_1}\right)} \leq \|u(t)\|_q \leq c_2 t^{-\left(\frac{\sigma}{2} - \frac{d}{s_1}\right)}.$$

(2) if $\frac{2 - b}{\alpha} < \sigma < (2 - b)(\frac{2s_2 + 2b}{s_1} - 1)$ and $q \in [r_1, \frac{d}{s_1})$ where $r_1$ as in Lemma 4.1, there exists $\delta > 0$ such that

$$\|u(t) - e^{-t\Delta}(\omega(x)|x|^{-\sigma})\|_q \leq Ct^{-\left(\frac{\sigma}{2} - \frac{d}{s_1}\right) - \delta}$$

for all $t$ large enough. In particular, if $\omega \neq 0$, there exist $c_1, c_2 > 0$ such that for $t$ large

$$c_1 t^{-\left(\frac{\sigma}{2} - \frac{d}{s_1}\right)} \leq \|u(t)\|_q \leq c_2 t^{-\left(\frac{\sigma}{2} - \frac{d}{s_1}\right)}.$$

Remark 1.4. As remarked earlier, see Remark 1.1, when $a = 0$, one can actually gets the above results even for $q = \frac{d}{s_1} = \infty$, recovering the result in [1].

Remark 1.5. In the case (1), since $\frac{\sigma}{2} - \frac{d}{s_1} = \frac{2 - b}{2\alpha} - \frac{d}{2s_1} > 0$, it follows from (1.5), that for $t$ large, the solution is close to a nonlinear self-similar solution. Thus in this case, the solution has nonlinear behaviour near $t = \infty$. By similar reasoning, in the case (2), the solution has linear behaviour near $t = \infty$, which can be related to scattering phenomenon. In both the case the final conclusion says $\|u(t)\|_q \to 0$ as $t \to \infty$.

Remark 1.6. The method of proof of Theorem 1.5 (2) differs at certain stage from the one in [1] (that deals with $a = 0$ case) due the fact that in the case $a \neq 0$, one do not have the decay estimate (1.3) for $q = \infty$, see Remark 4.1.
Remark 1.7.

(1) Notice that
\[ s_1 = \tilde{s}_1(a) = \begin{cases} s_1(a) & \text{for } a \in [-\frac{(d-2)^2}{4}, 0), \\ 0 & \text{for } a \in [0, \infty) \end{cases}, \quad \tilde{s}_2 = \tilde{s}_2(a) = \begin{cases} s_2(a) & \text{for } a \in [-\frac{(d-2)^2}{4}, 0), \\ d-2 & \text{for } a \in [0, \infty) \end{cases}. \]

(2) For \( a = 0 \), Theorem A says (1.3) valid for \( 1 < p \leq q < \infty \). However, since \( e^{-tL_0^a} = e^{\Delta f} = f \) where \( k_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \), the inequality (1.3) holds for all \( 1 \leq p \leq q \leq \infty \): In fact by Young’s inequality, \( \|e^{t\Delta f}\|_q \leq \|k_t\|_r \|f\|_p \) with \( \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} \) i.e. \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{q} \) (assumption \( p \leq q \) is to make sure \( r \geq 1 \)).

(3) For \( -\frac{(d-2)^2}{4} \leq a < 0 \), from Theorem A, (1.3) is valid for \( 1 < \frac{d}{s_2 + 2} < p \leq q < \frac{d}{s_1} < \infty \) (and do not have restrictions in involving \( s_1, s_2 \)) except the end point restrictions.

(4) On the other hand, for \( a > 0 \), from Theorem A, (1.3) is valid for all \( 1 < p \leq q < \infty \). Thus in this case the results match with the results for case \( a = 0 \) (and do not have restrictions in involving \( s_1, s_2 \)) except the end point restrictions.

The paper is organized as follows. In Section 2, we prove key estimate for \( e^{-tL_0^a}|\cdot|^{-b}f \). In Section 3, we prove Theorems 1.1, 1.3 and 1.4. In Section 4, we prove Theorem 1.5.

Notations. The notation \( A \lesssim B \) means \( A \leq cB \) for some universal constant \( c > 0 \). The Schwartz space is denoted by \( S(\mathbb{R}^d) \), and the space of tempered distributions is denoted by \( S'(\mathbb{R}^d) \).

2. Key ingredient

In order to incorporate the inhomogeneous nonlinearity, we first establish some fixed-time estimate for \( e^{-tL_0^a}|\cdot|^{-b}f \) in the next proposition.

Proposition 2.1. Let \( d \geq 2, a \geq -\frac{(d-2)^2}{4}, 0 \leq b < d \) and
\[ \tilde{s}_1 < \frac{d}{q} \leq b + \frac{d}{p} < \tilde{s}_2 + 2 \]
then for \( t > 0 \)
\[ \|e^{-tL_0^a}|\cdot|^{-b}f\|_{L^q} \leq c t^{\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{b}{2}} \|f\|_{L^p}. \]

Proof. Assume first that \( t = 1 \). Put \( m = \frac{d}{p} \). Since \( \frac{1}{q} < \frac{1}{m} + \frac{1}{p} < \frac{\tilde{s}_2 + 2}{d} \), we can choose \( \epsilon, \delta > 0 \) so that
\[ \frac{1}{q} < \frac{1}{m + \delta} + \frac{1}{p} < \frac{1}{m - \epsilon} + \frac{1}{p} < \frac{\tilde{s}_2 + 2}{d}. \]

Split \( |\cdot|^{-b} \) as follows
\[ |\cdot|^{-b} = k_1 + k_2 \quad \text{with} \quad k_1 \in L^{m-\epsilon}(\mathbb{R}^d), \quad k_2 \in L^{m+\delta}(\mathbb{R}^d), \]
for example one may take
\[ k_1 = \chi_{\{|x| \leq 1\}} |\cdot|^{-b}, \quad k_2 = \chi_{\{|x| > 1\}} |\cdot|^{-b}. \]

Let
\[ \frac{1}{s_1} = \frac{1}{m - \epsilon} + \frac{1}{p}, \quad \frac{1}{s_2} = \frac{1}{m + \delta} + \frac{1}{p} \]
and note that \( \tilde{s}_1 < \frac{d}{q} < \frac{d}{s_1} < \tilde{s}_2 + 2 \). By Theorem A and Hölder’s inequality, we have
\[ \|e^{-L_0^a}|\cdot|^{-b}f\|_q \leq \|e^{-L_0^a}(k_1 f)\|_q + \|e^{-L_0^a}(k_2 f)\|_q \leq \|k_1 f\|_{s_1} + \|k_2 f\|_{s_2} \leq (\|k_1\|_{m-\epsilon} + \|k_2\|_{m+\delta}) \|f\|_p \lesssim \|f\|_p. \]
Thus the case \( t = 1 \) is proved.

For \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), set \( D_\lambda \varphi = \varphi(\lambda \cdot) \). Then we claim that
\[
e^{-t\mathcal{L}_a}(D_\lambda \varphi) = D_\lambda(e^{-\lambda^2t\mathcal{L}_a} \varphi).
\]
In fact if \( v(t) = e^{-t\mathcal{L}_a}(D_\lambda \varphi) \), then \( v \) solves
\[
\begin{cases}
  v_t = \mathcal{L}_a v \\
  v(0, x) = D_\lambda \varphi(x) = \varphi(\lambda x).
\end{cases}
\tag{2.3}
\]
Put \( w(t) = u(\lambda^2 t) \) where \( u(t) = e^{-t\mathcal{L}_a} \varphi \). Then we have
\[
(D_\lambda w(t))_t = D_\lambda w(t) = \lambda^2 D_\lambda u(t) = \lambda^2 D_\lambda \mathcal{L}_a u(\lambda^2 t) = \lambda^2 D_\lambda \mathcal{L}_a w(t) = \lambda^2 \mathcal{L}_a w(t) = \mathcal{L}_a(w(t, \lambda \cdot)) = \mathcal{L}_a(D_\lambda w(t)),
\]
and
\[
(D_\lambda w)(0) = (D_\lambda u)(0) = D_\lambda \varphi.
\]
Thus, \( D_\lambda w \) also satisfies (2.3). Therefore, by uniqueness of solution of Banach valued ODE (semigroup theory), one has
\[
v(t) = D_\lambda w(t).
\]
This establishes the claim.

Using the above claim we have \( e^{\lambda^2t\mathcal{L}_a} \varphi = D_\lambda^{-1} e^{-t\mathcal{L}_a}(D_\lambda \varphi) = D_\lambda^{-1} e^{-t\mathcal{L}_a}(D_\lambda \varphi) \), putting \( \lambda = 1/\sqrt{t} \) one has \( e^{\mathcal{L}_a} \varphi = D_{\sqrt{t}} e^{-t\mathcal{L}_a}(D_{1/\sqrt{t}} \varphi) \). Then from the case \( t = 1 \) it follows that
\[
\| D_{\sqrt{t}} e^{-t\mathcal{L}_a} D_{1/\sqrt{t}}(\cdot \cdot \cdot) \|_q \lesssim \| \varphi \|_p.
\]
Since \( D_\lambda | \cdot |^{-b} = \lambda^{-b} | \cdot |^{-b} \), we get,
\[
t^{\frac{b}{2}} \| D_{\sqrt{t}} e^{-t\mathcal{L}_a} D_{1/\sqrt{t}} \varphi \|_q \lesssim \| \varphi \|_p.
\]
Replacing \( \varphi \) by \( D_{\sqrt{t}} \varphi \) we have
\[
t^{\frac{b}{2}} \| D_{\sqrt{t}} e^{-t\mathcal{L}_a}(\cdot \cdot \cdot \cdot) \|_q \lesssim \| D_{\sqrt{t}} \varphi \|_p,
\]
which implies
\[
t^{\frac{b}{2}} \| e^{-t\mathcal{L}_a}(\cdot \cdot \cdot \cdot) \|_q \lesssim t^{\frac{b}{2}} \| \varphi \|_p
\]
using \( \| D_\lambda f \|_p = \lambda^{-d/p} \| f \|_p \). This completes the proof.

\[\square\]

Remark 2.1. In view of Remark 1.7 (2), first strict inequality namely \( \tilde{s}_1 < \frac{d}{q} \) in (2.1) can be relaxed to \( \tilde{s}_1 \leq \frac{d}{q} \) when \( a = 0 \). But same cannot be done for the last inequality \( b + \frac{d}{p} < \tilde{s}_2 + 2 \) as a continuity argument is used to achieve (2.2).

Remark 2.2. When \( a = 0 \), the above result holds even in dimension \( d = 1 \). Then restriction for \( a \neq 0 \) is due to the fact that the operator \( \mathcal{L}_a \) is not defined on full \( \mathbb{R} \).
3. Existence of Solution

3.1. Local Existence.

**Proof of Theorem 1.1** (1). Let $K_t(\varphi) = e^{-tL_a(|\cdot|^{-b}|\varphi|^\alpha\varphi)}$. Since $\frac{d(\alpha+1)}{s_2+2-b} < q < \frac{d}{s_1}$, we have

$$\bar{s}_1 < \frac{d}{q} < b + \frac{d}{q/(\alpha+1)} < \bar{s}_2 + 2.$$ 

By Proposition 2.1, we may obtain

$$\|K_t(\varphi) - K_t(\psi)\|_q \lesssim t^{-\frac{d}{2}}(\bar{s}_2)^{\frac{1}{s_2}} \lesssim (\|\varphi\|_q^{\alpha} - \|\psi\|_q^{\alpha}) \lesssim t^{-\frac{d}{2}}\|\varphi - \psi\|_q^{\alpha}.$$ 

provided $\|\varphi\|_q, \|\psi\|_q \leq M$. Note that

- $K_t : L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is locally Lipschitz with Lipschitz constant $C_M(t) = t^{-\frac{d}{2}}M^\alpha$ in $\{\varphi \in L^q(\mathbb{R}^d) : \|\varphi\| \leq M\}$
- $C_M \in L^1(0, t)$ as $\frac{d}{2} + \frac{b}{2} < 1$, i.e. $q > \frac{d}{2} = q_c$
- $e^{-sL_a}K_t = K_{s+t}$
- $t \mapsto K_t(0) \equiv 0 \in L^1(0, t)$

By [19, Theorem 1, p. 279], the result follows. In order to ensure room for $q$, in between $\max(\frac{d(\alpha+1)}{s_2+2-b}, q_c)$ and $\frac{d}{s_1}$, the hypothesis $0 < \alpha < \frac{d-b}{s_1}$ is imposed, while $0 < b < 2$ is to make $q_c > 0$.

**Proof of Theorem 1.1** (2). Part (i) with the case $q > q_c$. (the case $q = q_c$ will be treated along with the global existence proof.)

Since $\alpha < \frac{2s_2}{s_1} < \frac{2s_2}{s_1} + (\frac{d}{s_1} - 1)$ and $\frac{d}{s_2+2} < q < \frac{d}{s_1}$, we have

$$\max\left(\bar{s}_1, \frac{d/q - b}{\alpha + 1}\right) < \min\left(\bar{s}_2 + 2 - b, \frac{d}{\alpha + 1}, \frac{\bar{s}_2 + 2 - b}{\alpha + 1}, \frac{d}{q}\right).$$

Therefore, one can choose $r$ such that

$$\max\left(\bar{s}_1, \frac{d/q - b}{\alpha + 1}\right) < \frac{d}{r} < \min\left(\bar{s}_2 + 2 - b, \frac{d}{\alpha + 1}, \frac{d}{\alpha + 1}, \frac{d}{q}\right),$$

which implies

$$\bar{s}_1 < \frac{d}{r} < \frac{d}{q} < b + \frac{d(\alpha+1)}{r} < \bar{s}_2 + 2.\quad (3.1)$$

Thus Proposition 2.1 can be used with exponent pairs $(\frac{r}{\alpha+1}, q)$ and $(\frac{r}{\alpha+1}, r)$. From $q > q_c$, it follows that

$$\frac{d(\alpha+1)}{q} - 2 < \frac{d}{q} - b.$$ 

This and (3.1) implies that $\frac{d(\alpha+1)}{q} < \frac{d}{q} - b < \frac{d(\alpha+1)}{r}$ which gives $d(\frac{1}{q} - \frac{1}{r}) = 2 < \beta(\alpha + 1) < 2$ i.e. $\beta(\alpha + 1) < 1$ where $\beta = \frac{d}{2}(\frac{1}{q} - \frac{1}{r})$. Introduce the space

$$B_M^{\beta} = \{u \in C([0, T]; L^q(\mathbb{R}^d)) \cap C((0, T]; L^r(\mathbb{R}^d)) : \max\sup_{t \in [0, T]} \|u(t)\|_q, \sup_{t \in [0, T]} t^\beta\|u(t)\|_r \leq M\}.$$ 

This space is endowed with the metric

$$d(u, v) =: \max\sup_{t \in [0, T]} \|u(t) - v(t)\|_q, \sup_{t \in [0, T]} t^\beta\|u(t) - v(t)\|_r.$$
Consider the mapping
\[ \mathcal{J}_\varphi(u)(t) = e^{-tL_a}\varphi + \mu \int_0^t e^{-(t-s)L_a}(|\cdot|^b|u(s)|^\alpha u(s))ds. \]

Let \( \varphi, \psi \in L^q(\mathbb{R}^d) \) and \( u, v \in B^T_{M} \). By Proposition 2.1 with exponent pairs \((\frac{r}{\alpha+1}, q)\) we get
\[
\|\mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t)\|_q \\
\lesssim \|\varphi - \psi\|_q + \int_0^t (t-s)^{-\frac{d}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} \|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{\frac{r}{\alpha+1}}ds \\
\lesssim \|\varphi - \psi\|_q + \int_0^t (t-s)^{-\frac{d}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} (|u(s)|^\alpha + \|v(s)\|_r^\alpha)\|u(s) - v(s)\|_r, ds \\
\lesssim \|\varphi - \psi\|_q + M^\alpha \int_0^t (t-s)^{-\frac{d}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} s^{-\beta(\alpha+1)}ds \\
\lesssim \|\varphi - \psi\|_q + M^\alpha d(u, v) \int_0^1 (1 - \sigma)^{-\frac{d}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} \sigma^{-\beta(\alpha+1)}d\sigma
\]
as \( \beta = \frac{d}{2}(\frac{1}{q} - \frac{1}{r}) \). It follows from \( q > q_c \) that \( 1 - \frac{d}{2} - \frac{b}{2} > 0 \) and from \( r > q \) that
\[
-\frac{d}{2}(\frac{1}{r} - \frac{1}{q}) + \frac{b}{2} < \frac{d}{2}(-\frac{1}{r}) + \frac{b}{2} = \frac{d\alpha}{2r} + \frac{b}{2} < \frac{d\alpha}{2q} + \frac{b}{2} < 1.
\]
This together with \( \beta(\alpha+1) < 1 \) imply that \( \int_0^1 (1 - \sigma)^{-\frac{d}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} \sigma^{-\beta(\alpha+1)}d\sigma < \infty \). Hence,
\[
\|\mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t)\|_q \lesssim \|\varphi - \psi\|_q + M^\alpha T^{1 - \frac{d\alpha}{2q} - \frac{b}{2}}d(u, v). \tag{3.2}
\]

Similarly, by Proposition 2.1 with exponent pairs \((\frac{r}{\alpha+1}, r)\) we get
\[
\|\mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t)\|_r \\
\lesssim t^{-\beta}\|\varphi - \psi\|_q + \int_0^t (t-s)^{-\frac{d\alpha}{2r} - \frac{b}{2}} (|u(s)|^\alpha + \|v(s)\|_r^\alpha)\|u(s) - v(s)\|_r, ds \\
\lesssim t^{-\beta}\|\varphi - \psi\|_q + M^\alpha d(u, v) \int_0^t (t-s)^{-\frac{d\alpha}{2r} - \frac{b}{2}} s^{-\beta(\alpha+1)}ds.
\]

Hence,
\[
t^\beta\|\mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t)\|_r \\
\lesssim \|\varphi - \psi\|_q + M^\alpha d(u, v) t^\beta \int_0^t (t-s)^{-\frac{d\alpha}{2r} - \frac{b}{2}} s^{-\beta(\alpha+1)}ds \\
\lesssim \|\varphi - \psi\|_q + M^\alpha d(u, v) t^{1 - \frac{d\alpha}{2r} - \frac{b}{2}} \int_0^1 (1 - \sigma)^{-\frac{d\alpha}{2r} - \frac{b}{2}} \sigma^{-\beta(\alpha+1)}d\sigma \\
\lesssim \|\varphi - \psi\|_q + M^\alpha T^{1 - \frac{d\alpha}{2r} - \frac{b}{2}}d(u, v). \tag{3.3}
\]

By (3.2) and (3.3), we obtain
\[
d(\mathcal{J}_\varphi(u), \mathcal{J}_\psi(v)) \leq c\|\varphi - \psi\|_q + cM^\alpha T^{1 - \frac{d\alpha}{2r} - \frac{b}{2}}d(u, v) \tag{3.4}
\]
for some \( c > 0 \).
For a given \( \varphi \in L^q(\mathbb{R}^d) \), choose \( \rho > 0 \) so that \( \| \varphi \|_q \leq \rho \). Take \( M = 2c\rho \). Then for \( u \in B^T_M \), from (3.4) it follows that

\[
d(\mathcal{J}_\varphi(u), 0) \leq cp + cM^\alpha T^{\frac{d\alpha}{2\alpha - \frac{\beta}{2}} - \frac{b}{2}} d(u, 0) \leq \frac{M}{2} + cM^{\alpha + 1} T^{\frac{d\alpha}{2\alpha - \frac{\beta}{2}} - \frac{b}{2}} M
\]

provided \( T > 0 \) small enough so that \( cM^\alpha T^{\frac{d\alpha}{2\alpha - \frac{\beta}{2}} - \frac{b}{2}} M \leq \frac{1}{2} \). This is possible as \( 1 - \frac{d\alpha}{2\alpha} - \frac{b}{2} > 0 \) as a consequence of \( q > q_c \) as mentioned earlier. Thus \( \mathcal{J}_\varphi(u) \in B^T_M \). Note that \( T \sim M^{-\alpha(\frac{2\alpha - \beta}{2\alpha} - \frac{b}{2})^{-1}} \sim \rho^{-\alpha(\frac{2\alpha - \beta}{2\alpha} - \frac{b}{2})^{-1}} \) and thus this time \( T \) only depends on \( \| \varphi \|_q \) rather than on the profile of \( \varphi \) itself.

Also for \( u, v \in B^T_M \), from (3.4) we have

\[
d(\mathcal{J}_\varphi(u), \mathcal{J}_\varphi(v)) \leq cM^\alpha T^{\frac{d\alpha}{2\alpha - \frac{\beta}{2}} - \frac{b}{2}} d(u, v) \leq \frac{1}{2} d(u, v)
\]

which says \( \mathcal{J}_\varphi \) is a contraction in \( B^T_M \). Therefore, it has a unique fixed point say \( u \) i.e. there is a unique \( u \in B^T_M \) such that \( \mathcal{J}_\varphi(u) = u \). This completes the proof of part (i).

**Part (ii) (i.e. uniqueness) with the case \( q > q_c \).** Let \( u_1, u_2 \) be two solution satisfying

\[
\sup_{t \in [0, T]} \| u_j(t) \|_q < \infty, \quad \sup_{t \in [0, T]} \| t^\beta |u_j(t)|_r \| < \infty.
\]

Choose \( \tilde{M} \) big enough so that

\[
\sup_{t \in [0, T]} \| u_j(t) \|_q < \tilde{M}, \quad \sup_{t \in [0, T]} t^\beta \| u_j(t) \|_r < \tilde{M}.
\]

Then \( u_1, u_2 \in B^T_{\tilde{M}} \). Let \( t_0 \) be the infimum of \( t \) in \([0, T]\) where \( u_1(t) \neq u_2(t) \), then as in the above, one can choose a \( 0 < \tilde{T} < T - t_0 \) depending on \( \| u_1(t_0) \|_q = \| u_2(t_0) \|_q \). So that \( \tilde{\mathcal{J}}_{u_1(t_0)} \) given by

\[
\tilde{\mathcal{J}}_{u_1(t_0)}(u)(t) = e^{-(t-t_0)\zeta_a} \varphi + \mu \int_{t_0}^t e^{-(t-s)\zeta_a} (| \cdot |^{-b} |u(s)|^\alpha u(s)) ds
\]

has a unique fixed point in \( B^T_{\tilde{M}} \). Since \( u_j \) are solutions to (1.1) they are also fixed point of \( \tilde{\mathcal{J}}_{u_1(t_0)} \) and hence \( u_1(t) = t_2(t) \) for \( t \in [t_0, t_0 + \tilde{T}] \) which is a contraction.

Let \( u, v \) be the unique solutions with data \( \varphi, \psi \) respectively, the from (3.4), it follows that

\[
d(u, v) \leq c\| \varphi - \psi \|_q + cM^\alpha T^{\frac{d\alpha}{2\alpha - \frac{\beta}{2}} - \frac{b}{2}} d(u, v) \leq c\| \varphi - \psi \|_q + \frac{1}{2} d(u, v) \implies d(u, v) \leq 2c\| \varphi - \psi \|_q
\]

implying continuous dependency of solution on data.

Blowup alternatives, part (3) regarding \( T_{\max} \) are standard.

\[\square\]

### 3.2. Global Existence

Using (3.1) for \( q = q_c \), if \( \frac{2 - b}{\alpha + 2} < \alpha < \frac{2 - b}{s_1} \) i.e. \( \frac{d}{s_2 + 2} < q_c < \frac{d}{s_1} \) there exists \( r > q_c \) such that

\[
\tilde{s}_1 < \frac{d}{r} < \frac{d}{q_c} < b + \frac{d(\alpha + 1)}{r} < \tilde{s}_2 + 2. \tag{3.5}
\]

Set

\[
\beta = \frac{d}{2}(\frac{1}{q_c} - \frac{1}{r}) = \frac{2 - b}{2\alpha} - \frac{d}{2r}, \tag{3.6}
\]

as before. Note that using \( q_c = \frac{d\alpha}{2\alpha - \frac{\beta}{2}} \) and (3.5),

\[
\frac{d(\alpha + 1)}{q_c} - 2 = \frac{d}{q_c} - b < \frac{d(\alpha + 1)}{r}
\]

which implies \( \beta(\alpha + 1) < 1 \). Also \( \frac{d\alpha}{2\alpha} + \frac{b}{2} < \frac{d\alpha}{2q_c} + \frac{b}{2} = 1 \).
**Theorem 3.1** (global existence). Let $0 < b < \min(2, d)$ and $\tilde{s}_1 < \frac{\tilde{s}_2 + 2 - b}{\alpha + 1} - \frac{d}{2\tilde{s}_2 + 2} < q_c < \frac{d}{\tilde{s}_1}$. Let $r$ verify (3.5) and $\beta$ be as defined in (3.6). Suppose that $\rho > 0$ and $M > 0$ satisfy the inequality $c\rho + cM^{\rho+1} \leq M$, where $c = c(\alpha, d, b, r) > 0$ is a constant and can explicitly be computed. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\sup_{t > 0} t^{\beta} \|e^{-tL_{\varphi}}\varphi\|_r \leq \rho.$$  

(3.7)

Then there exists a unique global solution $u$ of (1.1) such that

$$\sup_{t > 0} t^{\beta} \|u(t)\|_r \leq M.$$  

Furthermore,

1. $u(t) - e^{-tL_{\varphi}} \varphi \in C([0, \infty); L^s(\mathbb{R}^d))$, for $\tilde{s}_1 < \frac{d}{q_c} < \frac{d}{\tilde{s}_2} < b + \frac{\alpha+1}{r} < \tilde{s}_2 + 2$
2. $u(t) - e^{-tL_{\varphi}} \varphi \in L^\infty((0, \infty); L^{q_c}(\mathbb{R}^d))$, if $\tilde{s}_1 < \frac{d}{q_c} < b + \frac{\alpha+1}{r} < \tilde{s}_2 + 2$
3. $\lim_{t \to 0} u(t) = \varphi$ in $L^s(\mathbb{R}^d)$ if $\varphi \in L^s(\mathbb{R}^d)$ for $s$ satisfying $\frac{d}{q_c} < \frac{d}{s} < b + \frac{\alpha+1}{r} < \tilde{s}_2 + 2$
4. $\lim_{t \to 0} u(t) = \varphi$ in $\mathcal{S}(\mathbb{R}^d)$ for $\varphi \in \mathcal{S}(\mathbb{R}^d)$
5. $\sup_{t > 0} \frac{t^{\beta}}{\|t\|^r} \|u(t)\|_q \leq C_M < \infty$, for all $q \in [r, \frac{d}{\tilde{s}_1}]$ with $C_M \to 0$ as $M \to 0$.

Moreover, if $\varphi$ and $\psi$ satisfy (3.5) and if $u$ and $v$ be respectively the solutions of (1.1) with initial data $\varphi$ and $\psi$. Then

$$\sup_{t > 0} \frac{t^{\beta}}{\|t\|^r} \|u(t) - v(t)\|_q \leq C \sup_{t > 0} t^{\beta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r, \text{ for all } q \in [r, \frac{d}{\tilde{s}_1}).$$

If in addition, $e^{-tL_{\varphi}}(\varphi - \psi)$ has the stronger decay property

$$\sup_{t > 0} t^{\beta + \delta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r < \infty,$$

for some $\delta > 0$ such that $\beta(\alpha + 1) + \delta < 1$, and with $M$ perhaps smaller, then

$$\sup_{t > 0} t^{\beta + \delta} \|u(t) - v(t)\|_r \leq C \sup_{t > 0} t^{\beta + \delta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r$$  

(3.8)

where $C > 0$ is a constant.

**Proof.** Let

$$B_M = \{u : (0, \infty) \to L^\infty(\mathbb{R}^d) : \sup_{t > 0} t^{\beta} \|u(t)\|_r \leq M\}$$

and

$$d(u, v) =: \sup_{t > 0} t^{\beta} \|u(t) - v(t)\|_r.$$  

Let $\varphi, \psi \in L^q(\mathbb{R}^d)$ and $u, v \in B_M$. Using Proposition 2.1 with exponent pairs $(\frac{\alpha}{\alpha+1}, r)$ we get

$$t^{\beta} \|\mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t)\|_r \leq t^{\beta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r + t^{\beta} \int_0^t (t - s)^{-\frac{\alpha}{\alpha+1}} \left(\|u(s)\|_r^\alpha + \|v(s)\|_r^\alpha\right) \|u(s) - v(s)\|_r ds$$  

(3.9)

$$\lesssim t^{\beta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r + t^{\beta} M^\alpha d(u, v) \int_0^t (t - s)^{-\frac{\alpha}{\alpha+1}} s^{-\beta(\alpha+1)} ds$$

$$\lesssim t^{\beta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r + M^\alpha d(u, v) t^{\beta + 1 - \frac{\alpha}{\alpha+1}} \int_0^1 (1 - \sigma)^{-\frac{\alpha}{\alpha+1}} \sigma^{-\beta(\alpha+1)} d\sigma.$$  

(3.10)

Since $\beta + 1 - \frac{\alpha}{2\alpha} - \frac{\alpha}{2} - \beta(\alpha + 1) = 0$, we get

$$d(\mathcal{J}_\varphi(u), \mathcal{J}_\psi(v)) \lesssim \sup_{t > 0} t^{\beta} \|e^{-tL_{\varphi}}(\varphi - \psi)\|_r + M^\alpha d(u, v).$$  

(3.11)
Putting $\psi, v = 0$, and using the hypothesis we have

$$
\sup_{t > 0} t^\beta \| \mathcal{J}_\varphi(u)(t) \|_r = d(\mathcal{J}_\varphi(u), 0) \leq c \sup_{t > 0} t^\beta \| e^{-tE_\alpha \varphi} \|_r + cM^\alpha d(u, 0) \leq c \rho + cM^\alpha \leq M
$$

implying $\mathcal{J}_\varphi(u) \in B_M$. Putting $\psi = \varphi$ in (3.11)

$$
d(\mathcal{J}_\varphi(u), \mathcal{J}_\varphi(v)) \leq cM^\alpha d(u, v)
$$

implying $\mathcal{J}_\varphi(u)$ is a contraction in $B_M$ as $cM^\alpha < 1$. Hence it has a unique fixed point in $B_M$.

Now we will prove (1). Let's prove the continuity at $t = 0$ first. Take $s$ satisfying

$$
\bar{s}_1 < d \leq \frac{d}{q_c} \leq \frac{d}{s} < \frac{b + (\alpha + 1)}{r} < \bar{s}_2 + 2.
$$

Using Proposition 2.1 for the pair $(\frac{r}{\alpha+1}, s)$ we have

$$
\| u(t) - e^{-tE_\alpha \varphi} \|_s \leq \int_0^t (t - \tau)^{-\frac{2}{\alpha+1} - \frac{1}{\beta}} \| u(\tau) \|_{r+1} d\tau
$$

\[
\leq M^{\alpha+1} \int_0^t (t - \tau)^{-\frac{2}{\alpha+1} - \frac{1}{\beta}} \tau^{-\beta(\alpha+1)} d\tau
\]

\[
= M^{\alpha+1} t^{\frac{\alpha+1}{2} - \frac{2}{\alpha+1} - \frac{1}{\beta}} \beta(\alpha+1) \int_0^1 (1 - \sigma)^{-\frac{2}{\alpha+1} - \frac{1}{\beta} - \beta(\alpha+1)} d\sigma
\]

\[
= M^{\alpha+1} t^{\frac{\alpha+1}{2} - \frac{2}{\alpha+1} - \frac{1}{\beta}} \beta(\alpha+1) \int_0^1 (1 - \sigma)^{-\frac{2}{\alpha+1} - \frac{1}{\beta} - \beta(\alpha+1)} d\sigma.
\]

Note that

$$
\frac{d}{2} (\frac{\alpha + 1}{r} - \frac{1}{s}) + \frac{b}{2} < \frac{d}{q_c} (\frac{\alpha + 1}{r} - \frac{1}{q_c}) + \frac{b}{2} < \frac{d}{2} (\frac{\alpha + 1}{r} - \frac{1}{r}) + \frac{b}{2} = \frac{d}{2} \sigma + \frac{b}{2} = 1
$$

and $\beta(\alpha + 1) < 1$ implying $\int_0^1 (1 - \sigma)^{-\frac{2}{\alpha+1} - \frac{1}{\beta} - \beta(\alpha+1)} d\sigma < \infty$. Thus

$$
\| u(t) - e^{-tE_\alpha \varphi} \|_s \leq M^{\alpha+1} \frac{d}{2} \frac{b}{\alpha+1}.
$$

(3.12)

Since $\frac{d}{2} - \frac{b\alpha}{2(\alpha+1)} > 0$ (this follows from the condition $\frac{d}{q_c} < \frac{d}{s}$ in (1)), we get

$$
\| u(t) - e^{-tE_\alpha \varphi} \| \to 0
$$

(3.13)

in $L^1(\mathbb{R}^d)$ as $t \to 0^+$ and hence continuity at $t = 0$.

For other $t > 0$, first note that $u(t) - e^{-tE_\alpha \varphi} - u(0) + e^{-tE_\alpha \varphi} = \int_0^t e^{-(t-s)E_\alpha} (\| \cdot \| \alpha) d\sigma$, but with $0 < t < t_0$

$$
\| \int_0^t e^{-(t-r)E_\alpha} (\| \cdot \| \alpha) u(\tau) d\tau \|_s = \int_0^t e^{-(t-r)E_\alpha} (\| \cdot \| \alpha) u(\tau) d\tau \|_s
$$

$$
\leq \int_0^{t_0} \chi_{[0,t]}(\tau) \| e^{-(t-r)E_\alpha} (\| \cdot \| \alpha) u(\tau) \|_s \to 0
$$

(3.14)

as $t \uparrow t_0$, using dominated convergence (in fact note that for each $\tau$, the integrand say $I(\tau)$ goes to zero as $t \uparrow t_0$ and now

$$
I(\tau) \leq \chi_{[0,t]}(\tau) \left[ \| e^{-(t-r)E_\alpha} (\| \cdot \| \alpha) u(\tau) \|_s \| + \| e^{-(t-r)E_\alpha} (\| \cdot \| \alpha) u(\tau) \|_s \right]
$$

$$
\leq \chi_{[0,t]}(\tau) \left[ (t - \tau)^{-\frac{2}{\alpha+1} - \frac{1}{\beta}} + (t_0 - \tau)^{-\frac{2}{\alpha+1} - \frac{1}{\beta}} \right] \| u(\tau) \|_{r+1}
$$

\[
\leq M^{\alpha+1} \chi_{[0,t]} \left[ (t - \tau)^{-\frac{2}{\alpha+1} - \frac{1}{\beta}} + (t_0 - \tau)^{-\frac{2}{\alpha+1} - \frac{1}{\beta}} \right] \tau^{-\beta(\alpha+1)}
\]
which is integrable). On the other hand for $\frac{a}{2} \leq t < t_0$,
\[
\| \int_0^t e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(t)|^\alpha u(t))} d\tau - \int_0^{t_0} e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau \|_s
= \| \int_t^{t_0} e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau \|_s
\leq M^{\alpha+1} \int_t^{t_0} (t_0-\tau)^{-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2} - \beta(\alpha+1)} d\tau
= M^{\alpha+1} 2^{\beta(\alpha+1)} t_0^{1-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2} - \beta(\alpha+1)} \int_t^{t_0} (1-\sigma)^{-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2}} d\sigma
\leq M^{\alpha+1} 2^{\beta(\alpha+1)} t_0^{1-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2} - \beta(\alpha+1)} \int_0^{1-t/t_0} \sigma^{-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2}} d\sigma
= M^{\alpha+1} 2^{\beta(\alpha+1)} t_0^{1-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2} - \beta(\alpha+1)} \int_0^{1-t/t_0} \frac{1}{1 - \frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2}} \left( 1 - \frac{t}{t_0} \right)^{1-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2}} \to 0
\]
as $t \uparrow t_0$. Using this and (3.14) we get $u(t) - e^{-t_0 \mathcal{L}_a} \varphi \to u(t_0) - e^{-t_0 \mathcal{L}_a} \varphi$ as $t \uparrow t_0$ and so left continuity is established. Now for $0 < t_0 < t$,
\[
\| \int_0^t e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau - \int_0^{t_0} e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau \|_s
= \| \int_0^{t_0} e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau \|_s
\leq \int_0^{t_0} (t_0-\tau)^{-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2} - \beta(\alpha+1)} d\tau
= t_0^{1-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2} - \beta(\alpha+1)} \int_0^{1} (1-\sigma)^{-\frac{d}{2} \left( \frac{\alpha+1}{\alpha-1} \right) - \frac{b}{2}} d\sigma
\to 0 \quad (3.15)
\]
as $t \downarrow t_0$. And arguing as (3.14)
\[
\| \int_0^t e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau - \int_0^{t_0} e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} d\tau \|_s
\leq \int_0^{t_0} \| e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} - e^{-(t_0-\tau)\mathcal{L}_a(|\cdot|^{-b}|u(\tau)|^\alpha u(\tau))} \|_s d\tau \to 0
\]
as $t \downarrow t_0$. This together with (3.15) implies $u(t) - e^{-t_0 \mathcal{L}_a} \varphi \to u(t_0) - e^{-t_0 \mathcal{L}_a} \varphi$ as $t \downarrow t_0$ so the right continuity is established. This proves (1).

(2) follows from (3.12).

If $\varphi \in L^r(\mathbb{R}^d)$, then $e^{-t \mathcal{L}_a} \varphi \to \varphi$ in $L^r(\mathbb{R}^d)$ as $t \to 0+$ (semigroup property). Thus, using (3.13), we conclude $u(t) \to \varphi$ in $L^r(\mathbb{R}^d)$. This proves (3) and hence (4).

For part (5) we use iteration. Set $r_0 = r$, $M_0 = M$ we note that we have
\[
\sup_{t > 0} \frac{\| | \cdot |^{-b} \mathcal{L}_a u(t) \|_{L^\alpha}}{r_0} \leq M_0.
\]
Take
\[
\frac{1}{r_1} = \begin{cases} \frac{1}{r_0} - \frac{1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_0} \right) & \text{if } \frac{1}{r_0} - \frac{1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_0} \right) > \frac{2}{d} \\ \text{any number in } \left( \frac{2}{d}, \frac{1}{r_0} \right) & \text{otherwise.} \end{cases}
\]
Then it is easy to see $\tilde{s}_1 < \frac{d}{r_1} < \frac{d}{r_0}$ as $\frac{2-b}{d} - \frac{\alpha}{r_0} > 0 \iff r_0 > q_{c}$. Thus using (3.5) we have

$$\tilde{s}_1 < \frac{d}{r_1} < \frac{d}{r_0} < b + \frac{d(\alpha + 1)}{r_0} < \tilde{s}_2 + 2 \quad (3.17)$$

If the first case occurs in (3.16), then $\frac{d}{r_0} \left( \frac{1}{r_0} - \frac{1}{r_1} \right) = \frac{1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_0} \right) < \frac{2-b}{d} - \frac{\alpha}{2r_0} \iff \frac{d}{2} \left( \frac{r_0}{r_1} - \frac{1}{r_1} \right) < 1 - \frac{b}{2}$. If the second case occurs in (3.16), then $\frac{1}{r_0} \left( \frac{1}{r_0} - \frac{1}{r_1} \right) = \frac{1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_0} \right) < \frac{2-b}{d} - \frac{\alpha}{2r_0} \Rightarrow \frac{d}{2} \left( \frac{1}{r_0} - \frac{1}{r_1} \right) < 1 - \frac{b}{2}$. Thus in both case in (3.16)

$$\frac{d}{2} \left( \frac{\alpha+1}{r_0} - \frac{1}{r_1} \right) < 1 - \frac{b}{2}.$$ 

In view of this and (3.17), using Lemma 3.1 with $(s, q) = (r_0, r_1)$ we have

$$\sup_{t>0} t \frac{2b}{\alpha} \frac{d}{2} \frac{r_0 - d}{d} ||u(t)||_{r_1} \leq cM_0(1 + M_0') =: M_1$$

If the second case occurs in (3.16), we stop the iteration, otherwise will next choose $r_2$ from $r_1$ as we have chosen $r_1$ from $r_0$ above.

Note that this iteration must stop (the second case must occur) at finite steps. If not we would find $\tilde{s}_1 < \cdots < \frac{d}{r_{i+1}} < \frac{d}{r_i} < \cdots < \frac{d}{r_0}$ with

$$\sup_{t>0} t \frac{2b}{\alpha} \frac{d}{2} \frac{r_0 - d}{d} ||u(t)||_{r_{i+1}} \leq cM_i(1 + M_i') =: M_{i+1} \quad (3.18)$$

and $\frac{1}{r_i} - \frac{1}{r_{i+1}} = \frac{1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_1} \right) > \frac{1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_0} \right) > 0$. Then $\frac{1}{r_0} = \left( \frac{1}{r_0} - \frac{1}{r_1} \right) + \left( \frac{1}{r_1} - \frac{1}{r_2} \right) + \cdots + \left( \frac{1}{r_i} - \frac{1}{r_{i+1}} \right) + \frac{1}{r_{i+1}} \geq \frac{i+1}{2} \left( \frac{2-b}{d} - \frac{\alpha}{r_0} \right) \to \infty$ which is a contradiction as $\frac{1}{r_0} < \infty$.

For $q \in (r_i, r_{i+1})$ we use interpolation inequality with (3.18) $||u(t)||_q \leq ||u(t)||_{r_i}^\theta ||u(t)||_{r_{i+1}}^{1-\theta}$ where $\frac{1}{q} = \frac{1}{r_i} + \frac{1}{r_{i+1}}$.

Further part: Using $\mathcal{J}_\varphi(u) = u$, $\mathcal{J}_\psi(v) = v$ in (3.9) we have

$$||u(t) - v(t)||_r \leq ||e^{-t\mathcal{L}_\varphi(\varphi - \psi)}||_r + \int_0^t (t-s)^{-\frac{d-\alpha}{2r} - \frac{b}{2}} \|(u(s))_r^\alpha + \|v(s))_r^\alpha||u(s) - v(s)||_r \cdot ds$$

$$\leq ||e^{-t\mathcal{L}_\varphi(\varphi - \psi)}||_r + M_\alpha \int_0^t (t-s)^{-\frac{d-\alpha}{2r} - \frac{b}{2}} s^{-\beta - \delta} \|u(s) - v(s)||_r \cdot ds.$$ 

Therefore

$$t^{\beta+\delta} ||u(t) - v(t)||_r \leq t^{\beta+\delta} ||e^{-t\mathcal{L}_\varphi(\varphi - \psi)}||_r + M_\alpha t^{\beta+\delta}$$

$$\cdot \int_0^t (t-s)^{-\frac{d-\alpha}{2r} - \frac{b}{2}} s^{-\beta - \delta} \|u(s) - v(s)||_r \cdot ds \leq t^{\beta+\delta} ||e^{-t\mathcal{L}_\varphi(\varphi - \psi)}||_r + M_\alpha t^{\beta+\delta} \sup_{r>0} t^{\beta+\delta} ||u(\tau) - v(\tau)||_r$$

$$\cdot \int_0^t (t-s)^{-\frac{d-\alpha}{2r} - \frac{b}{2}} s^{-\beta(\alpha+1) - \delta} ds \leq t^{\beta+\delta} ||e^{-t\mathcal{L}_\varphi(\varphi - \psi)}||_r + M_\alpha t^{\beta+\delta} \sup_{r>0} t^{\beta+\delta} ||u(\tau) - v(\tau)||_r$$

$$\cdot \int_0^1 (1-s)^{-\frac{d-\alpha}{2r} - \frac{b}{2}} s^{-\beta(\alpha+1) - \delta} ds$$

Note that $\int_0^1 (1-s)^{-\frac{d-\alpha}{2r} - \frac{b}{2}} s^{-\beta(\alpha+1) - \delta} ds < \infty$ as $\frac{d-\alpha}{2r} + \frac{b}{2} < 1$, $\beta(\alpha+1) + \delta < 1$. Thus

$$\sup_{t>0} t^{\beta+\delta} ||u(t) - v(t)||_r \leq ct^{\beta+\delta} ||e^{-t\mathcal{L}_\varphi(\varphi - \psi)}||_r + cM^{\alpha} \sup_{r>0} t^{\beta+\delta} ||u(\tau) - v(\tau)||_r$$

choosing $M > 0$ so that $cM^{\alpha} < \frac{1}{2}$ we achieve (3.8).
Moreover part: Note that
\[
  u(t) - v(t) = e^{-t\mathcal{L}_a/2}(u(t/2) - v(t/2)) + \mu \int_t^t e^{-(t-s)\mathcal{L}_a} (|\cdot|^{-b}\|u(s)\|^a u(s) - |v(s)|^a v(s)) ds.
\]
Therefore for \( q \in [r, \frac{d}{\alpha+1}] \) using (3.5), one has
\[
  \tilde{s}_1 < \frac{d}{q} < \frac{d}{r} < \tilde{s}_2 + 2 \quad \text{and} \quad \tilde{s}_1 < \frac{d}{q} < b + \frac{d(\alpha + 1)}{q} < \tilde{s}_2 + 2.
\]
Then using Theorem A for \((r, q)\) and Proposition 2.1 for \((\frac{d}{\alpha+1}, q)\) we have
\[
  \|u(t) - v(t)\|_q \lesssim t^{-\frac{d}{2} + \frac{q}{2}} \|u(t/2) - v(t/2)\|_r + \int_t^t (t-s)^{-\frac{d}{2} + \frac{q}{2}} (\|u(s)\|^a + \|v(s)\|^a) \|u(s) - v(s)\|_q ds
\]
\[
  \lesssim t^{-\frac{d}{2} + \frac{q}{2}} \|u(t/2) - v(t/2)\|_r + C_M^{\alpha} \int_t^t (t-s)^{-\frac{d}{2} + \frac{q}{2} - \alpha(\alpha+1)\beta(q)} ds
\]
using (5), \( \beta(q) := \frac{2-b}{2\alpha} - \frac{d}{2q} \).
Let \( \beta(q) = \frac{2-b}{2\alpha} - \frac{d}{2q} \).
Then
\[
  t^{\beta(q)} \|u(t) - v(t)\|_q \lesssim t^{\beta} \|u(t/2) - v(t/2)\|_r + C_M^{\alpha} t^{\beta(q)} \sup_{\tau > 0} \int_t^t (t-s)^{-\frac{d}{2} + \frac{q}{2} - \alpha(\alpha+1)\beta(q)} ds
\]
\[
  \lesssim (t/2)^{\beta} \|u(t/2) - v(t/2)\|_r + C_M^{\alpha} t^{\beta(q)} \int_t^t (t-s)^{-\frac{d}{2} + \frac{q}{2} - \alpha(\alpha+1)\beta(q) + 1} (1 - \sigma)^{-\frac{2-b}{2\alpha} - \frac{d}{2q} - \alpha(\alpha+1)\beta(q)} d\sigma.
\]
Since \( \beta(q) - \frac{d}{2q} - \frac{b}{2} - (\alpha + 1)\beta(q) + 1 = 0 \) and \( \frac{d}{2q} + \frac{b}{2} < \frac{d}{2q} + \frac{b}{2} < 1 \), we have
\[
  t^{\beta(q)} \|u(t) - v(t)\|_q \leq c \sup_{\tau > 0} \int_t^t (t-s)^{-\frac{d}{2} + \frac{q}{2} - \alpha(\alpha+1)\beta(q)} ds
\]
Choosing \( M > 0 \) small enough so that \( cC_M < \frac{1}{2} \), we get
\[
  \sup_{\tau > 0} t^{\beta(q)} \|u(t) - v(t)\|_q \lesssim \sup_{\tau > 0} t^{\beta} \|u(t) - v(t)\|_r \lesssim \sup_{\tau > 0} t^{\beta} \|e^{-t\mathcal{L}_a(\varphi - \psi)}\|_r
\]
by using (3.8) with \( \delta = 0 \). This completes the proof. \( \square \)

**Lemma 3.1** (A priori estimate). Suppose \( s < q \),
\[
  \tilde{s}_1 < \frac{d}{q} < b + \frac{d(\alpha + 1)}{s} < \tilde{s}_2 + 2, \quad \frac{d}{2}(\frac{\alpha + 1}{s} - \frac{1}{q}) < 1 - \frac{b}{2}.
\]
Assume \( u \) be solution to (1.1) satisfying
\[
  \sup_{\tau > 0} t^{\frac{2b}{2s} - \frac{q}{r}} \|u(t)\|_s \leq A < \infty
\]
then
\[
  \sup_{\tau > 0} t^{\frac{2b}{2s} - \frac{q}{2}} \|u(t)\|_q \leq A(1 + A^\alpha) =: C_A < \infty
\]
with \( C_A \to 0 \) as \( A \to 0 \).
Proof. Note that
\[ u(t) = e^{-\frac{b}{2} t} L_a u(t/2) + \mu \int_{t/2}^t e^{-(t-\sigma)L_a} (| \cdot |^{-b} |u(\sigma)|^a u(\sigma)) d\sigma. \]
Using Theorem A for \((p, q) = (s, q)\) and Proposition 2.1 for \((p, q) = \left(\frac{s}{s+1}, q\right)\) we have
\[ \|u(t)\|_q \lesssim t^{-\frac{d}{2} \left(\frac{1}{s} + \frac{1}{q}\right)} \|u(t/2)\|_s + \int_{t/2}^t (t-\tau)^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} \|u(\tau)\|^a u(\tau) \|\|_{\frac{s}{s+1}} d\tau \]
\[ \lesssim t^{-\frac{d}{2} \left(\frac{1}{s} + \frac{1}{q}\right)} A + \int_{t/2}^t (t-\tau)^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} \|u(\tau)\|_{s+1}^a d\tau \]
\[ \lesssim t^{-\frac{d}{2} \left(\frac{1}{s} + \frac{1}{q}\right)} A + A^{\alpha+1} \int_{t/2}^t (t-\tau)^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} \|u(\tau)\|_{s+1}^a d\tau \]
\[ \lesssim t^{-\frac{d}{2} \left(\frac{1}{s} + \frac{1}{q}\right)} A + A^{\alpha+1} t^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} \int_{1/2}^1 (1-\sigma)^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} \sigma^{-\beta(s)(\alpha+1)} d\sigma \]
\[ \lesssim t^{-\frac{d}{2} \left(\frac{1}{s} + \frac{1}{q}\right)} A + A^{\alpha+1} t^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} = A(1 + A^\alpha) t^{-\frac{d}{2} \left(\frac{1}{s} + \frac{1}{q}\right)} \]
as \[ \int_{1/2}^1 (1-\sigma)^{-\frac{d}{2} \left(\frac{a+1}{s+1} + \frac{1}{q}\right)} \sigma^{-\beta(s)(\alpha+1)} d\sigma < \infty. \]
This completes the proof. \(\square\)

Proof of Theorem 1.1 (2). The case \(q = q_c\). First note that if the condition (3.7) of Theorem 3.1 is satisfied on \([0, T]\) (in place of \((0, \infty)\) i.e. if
\[ \sup_{t \in [0, T]} t^\beta \|e^{-t L_a} \varphi\|_r \leq \rho, \quad (3.19) \]
happens, then the conclusion holds with \((0, \infty)\) replaced by \((0, T)\) i.e. there exists a unique solution \(u\) on \([0, T]\) such that
\[ \sup_{t \in [0, T]} t^\beta \|u(t)\|_r \leq M. \]

In view of this, it is enough to prove (3.19) with \(T > 0\) for a given \(\varphi \in L^{q_c}(\mathbb{R}^d)\). For \(\epsilon > 0\), there exists \(\psi \in L^{q_c}(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)\) such that \(\|\varphi - \psi\|_{q_c} < \epsilon\) (by density). Then for \(0 \leq t < T\), by Theorem A
\[ t^\beta \|e^{-t L_a} \varphi\|_r \leq t^\beta \|e^{-t L_a} (\varphi - \psi)\|_r + t^\beta \|e^{-t L_a} \psi\|_r \]
\[ \leq c t^\beta t^{-\beta} \|\varphi - \psi\|_{q_c} + c t^\beta \|\psi\|_r \]
\[ \leq c t^\beta t^{-\beta} \|\psi\|_r \]
by choosing \(T > 0\) small enough depending on \(\epsilon, \|\psi\|_r\) (which again depends on \(\epsilon\)). Now take \(\epsilon = \frac{\rho}{2c}\), to achieve (3.19). This completes the proof. \(\square\)

Proof of Theorem 1.3. For part (1), using Theorem A note that
\[ t^\beta \|e^{-t L_a} \varphi\|_r \leq c t^\beta t^{-\frac{d(1-\frac{a}{n})}{a}} \|\varphi\|_{q_c} = c \|\varphi\|_{q_c} \]
for all \(t > 0\) therefore it satisfies the hypothesis (3.7) of Theorem 3.1 for \(\|\varphi\|_{q_c}\) small enough. Hence the result follows from Theorem 3.1.

For part (2), with the given condition on \(\sigma\), the data satisfies the condition in part (1).

For part (3), write \(\cdot \ | -\frac{2}{n} \frac{b}{a}\) it follows that \(\sup_{\sigma \geq 0} t^\beta \|e^{-t L_a} \cdot | -\frac{2}{n} \frac{b}{a}\|_r \leq \infty\). Using positivity of \(e^{-t L_a}\), \(\|\varphi\| \leq c \) it follows that, \(\sup_{\sigma \geq 0} t^\beta \|e^{-t L_a} \varphi\|_r \leq \infty\). Choosing \(c > 0\) small enough, \(\varphi\) satisfies condition (3.7) of Theorem 3.1. \(\square\)
3.3. Selfsimilar Solution.

Proof of Theorem 1.4. Let \( r \) as in (3.5). Since \( |\varphi| \leq \|\omega\|_\infty \cdot |x|^{-\frac{2+b}{\alpha}} \), proceeding as in the proof of Theorem 1.3 (3) above, we achieve \( \varphi \) satisfies (3.7). Then (1.1) has a unique global solution \( u \) by Theorem 3.1 satisfying (3.8).

Since \( \varphi \) is homogeneous of degree \(-\frac{2+b}{\alpha}\) it follows that \( \varphi_\lambda = \varphi \) for all \( \lambda \) where \( \varphi_\lambda(x) = \lambda^{-\frac{2+b}{\alpha}} \varphi(\lambda x) \). Therefore \( \varphi_\lambda \) also satisfies (3.7) and has a unique global solution \( \tilde{u}_\lambda \) by Theorem 3.1 satisfying \( \sup_{t>0} t^{\beta} \|\tilde{u}_\lambda(t)\| \leq M \). By computation one has \( \tilde{u}_\lambda = u_\lambda \), where \( u_\lambda \) as defined in (1.4). Since \( \varphi_\lambda = \varphi \), by uniqueness of solution, we have \( u_\lambda = u \). \( \square \)

4. Asymptotic Behaviour

4.1. Case I: Nonlinear Behaviour.

Proof of Theorem 1.5 (1). Set \( \beta(q) = \frac{2-b}{\alpha} - \frac{d}{2q} \) and \( \psi(x) = \omega(x)|x|^{-\frac{2+b}{\alpha}} \). Note that \( |\varphi(x) - \psi(x)| = 0 \) for \( |x| \geq A \) and \( |\varphi(x) - \psi(x)| \leq (c + \|\omega\|_\infty)|x|^{-\frac{2+b}{\alpha}} \). Hence,

\[
|\varphi - \psi| \leq (c + \|\omega\|_\infty)\varphi_1
\]

where \( \varphi_1 = |x|^{-\frac{2+b}{\alpha}} \chi_{\{|x| \leq A\}} \in L^1(\mathbb{R}^d) \) with any 1 \( \leq s < \frac{d\alpha}{2b} = q_\epsilon \). Using (3.5), for any choice of \( s \) with \( d/q_\epsilon < d/s < \tilde{s}_2 + 2 \), we have by Theorem A and (3.5) that

\[
t^\beta(\frac{1}{s} - \frac{1}{r}) \|e^{-t\mathcal{L}_s}(\varphi - \psi)\|_r \lesssim \|\varphi - \psi\|_s \leq (c + \|\omega\|_\infty)\varphi_1 < \infty.
\]

This implies (letting \( \delta := \frac{d}{2} - \frac{1}{r} - \beta(r) = \frac{d}{2s} - \frac{2+b}{2\alpha} \))

\[
\sup_{t>0} t^{\beta(r)+\delta} \|e^{-t\mathcal{L}_s}(\varphi - \psi)\|_r < \infty \quad \text{for all 0 < } \delta < \tilde{s}_2 + 2 - \frac{2 - b}{2\alpha}.
\] (4.1)

Note that \( \beta(r)(\alpha + 1) < 1 \) and therefore for 0 < \( \delta < \tilde{s}_2 + 2 - \frac{2+b}{2\alpha} \) small enough one has \( \beta(r)(\alpha + 1) + \delta < 1 \), applying Theorem 3.1 (specifically (3.8)) we have

\[
\sup_{t>0} t^{\beta(r)+\delta} \|u(t) - u_S(t)\| < \infty
\] (4.2)

for \( \delta > 0 \) small enough. This proves the result for \( q = r \).

Note that

\[
u(t) - u_S(t) = e^{-\frac{t}{\alpha+1}\mathcal{L}_\alpha}(u(t/2) - u_S(t/2)) + \mu \int_{t/2}^t e^{-(t-s)\mathcal{L}_s}(|\cdot|^{-b}|u(\sigma)|^\alpha u(\sigma) - |u_S(\sigma)|^\alpha u_S(\sigma))d\sigma
\]

Therefore for \( q \in [r, \frac{d}{\tilde{s}_1}] \) using (3.5), one has

\[
\tilde{s}_1 < \frac{d}{q} < \frac{d}{r} < \tilde{s}_2 + 2 \quad \text{and} \quad \tilde{s}_1 < \frac{d}{q} < b + \frac{d(\alpha + 1)}{q} < \tilde{s}_2 + 2.
\]

Thus applying Theorem A with the pair \( (r, q) \) and Proposition 2.1 with the pair \( (\frac{r}{\alpha+1}, q) \) we get

\[
t^{\beta(q)+\delta} \|u(t) - u_S(t)\|_q \lesssim t^{\beta(q)+\delta - \frac{d}{2} (\frac{1}{s} - \frac{1}{r})} \|u(t/2) - u_S(t/2)\|_r + t^{\beta(q)+\delta} \cdot \int_{t/2}^t (t - s)^{-\frac{d}{\alpha+1} - \frac{1}{2}} \||u(\sigma)|^\alpha u(\sigma) - |u_S(\sigma)|^\alpha u_S(\sigma)\|_q d\sigma.
\]

Using (4.2) for the first term in the LHS we have

\[
t^{\beta(q)+\delta} \|u(t) - u_S(t)\|_q \lesssim t^{\beta(q)+\delta - \frac{d}{2} (\frac{1}{s} - \frac{1}{r}) - \beta(r) - \delta} + t^{\beta(q)+\delta} \cdot \int_{t/2}^t (t - s)^{-\frac{d}{\alpha+1} - \frac{1}{2}} \||u(\sigma)|^\alpha u(\sigma) - |u_S(\sigma)|^\alpha u_S(\sigma)\|_q d\sigma.
\]
Using Theorem 3.1 part (5) we further have

\[
\begin{align*}
t^{β(q)+δ}\|u(t) - u_S(t)\|_q & \lesssim t^0 + C_M t^{β(q)+δ} \int_{1/2}^t (t - σ)^{-\frac{dα}{2q} - \frac{b}{2}\sigma - \frac{2b}{2} + \frac{dα}{2q}}\|u(σ) - u_S(σ)\|_q dσ \\
& \lesssim 1 + C_M t^{β(q)+δ} \int_{1/2}^t (t - σ)^{-\frac{dα}{2q} - \frac{b}{2}\sigma - \frac{2b}{2} + \frac{dα}{2q}} σ^{β(q)-δ} \\
& \quad \cdot \|u(σ) - u_S(σ)\|_q dσ \\
& \lesssim 1 + C_M t^{β(q)+δ} \int_{1/2}^t (t - σ)^{-\frac{dα}{2q} - \frac{b}{2}\sigma - \frac{2b}{2} + \frac{dα}{2q}} σ^{β(q)-δ} dσ \\
& \quad \cdot \sup_{τ > 0} τ^{β(q)+δ} \|u(τ) - u_S(τ)\|_q \\
& \lesssim 1 + C_M t^0 \int_{1/2}^1 (1 - σ)^{-\frac{dα}{2q} - \frac{b}{2}\sigma - \frac{2b}{2} + \frac{dα}{2q}} σ^{β(q)-δ} dσ \\
& \quad \cdot \sup_{τ > 0} τ^{β(q)+δ} \|u(τ) - u_S(τ)\|_q
\end{align*}
\]

where \(q\) is so that \(\frac{dα}{2q} + \frac{b}{2} < \frac{dα}{2q} + \frac{b}{2} < 1\) we obtain

\[
\sup_{t > 0} t^{β(q)+δ} \|u(t) - u_S(t)\|_q \lesssim c + cC_M \sup_{t > 0} t^{β(q)+δ} \|u(t) - u_S(t)\|_q
\]

and \(C_M \to 0\) as \(M \to 0\). Choosing \(M \to 0\) small enough so that \(C_M^α < 1\), we achieve

\[
\sup_{t > 0} t^{β(q)+δ} \|u(t) - u_S(t)\|_q \lesssim 1 \implies \|u(t) - u_S(t)\|_q \leq ct^{-β(q)-δ} \forall \ t > 0.
\]

Now \(u_S(t,x) = λ \frac{dα}{2q} u_S(λ^2 t, λx)\) for all \(λ\), and hence taking \(λ = \frac{1}{\sqrt{t}}\), we have \(u_S(t,x) = t^{-\frac{2b}{dα}} u_S(1, \frac{x}{\sqrt{t}})\) and hence

\[
\|u_S(t)\|_q = t^{-\frac{2b}{dα}} \left( \int_{R^d} |u_S(1, x/\sqrt{t})|^q dx \right)^{1/q} = t^{-\frac{2b}{dα}} \left( \int_{R^d} |u_S(1, y)|^q dy \right)^{1/q} = t^{-β(q)} \|u_S(1)\|_q.
\]

Therefore using (4.3), for large \(t \geq 1\) we have \(\|u(t) - u_S(t)\|_q \leq \frac{t^{-β(q)}}{2} \|u_S(t)\|_q\) and thus

\[
\|u(t)\|_q \geq \|u_S(t)\|_q - \|u(t) - u_S(t)\|_q \geq \frac{1}{2} \|u_S(t)\|_q = \frac{1}{2} t^{-β(q)} \|u_S(1)\|_q.
\]

Also

\[
\|u(t)\|_q \leq \|u_S(t)\|_q + \|u(t) - u_S(t)\|_q \leq \frac{3}{2} \|u_S(t)\|_q = \frac{3}{2} t^{-β(q)} \|u_S(1)\|_q.
\]

This completes the proof.

\(\square\)

### 4.2 Case II: Linear Behaviour.

For this case, we need the following technical result to be proved in the Appendix.

**Lemma 4.1**. Assume that \(0 < b < \min(2,d)\) and \(\frac{2b}{s_2 + 2} < α < \frac{2b}{s_1}\). Let \(α_1\) be real number such that

\[
\max \left( \frac{2 - b}{s_2 + 2}, \frac{s_1α}{s_2 + 2 - b - s_1α} \right) < α_1 < α < \frac{2 - b}{s_1}.
\]

Let \(r_1\) be a real number satisfying

\[
\max \left( \frac{(α_1 + 1)d}{s_2 + 2 - b}, \frac{dα_1}{2 - b} \right) < r_1 < \min \left( \frac{dα_1(α_1 + 1)}{(2 - b(α_1 + 1))}, \frac{dα_1}{s_1α} \right).
\]

Let

\[
r_2 = \frac{α}{α_1} r_1
\]
\[ \beta_1 = \frac{2 - b}{2\alpha_1} - \frac{d}{2r_1} \]
\[ \beta_2 = \frac{2 - b}{2\alpha} - \frac{d}{2r_2} \]
\[ r_{12} = \frac{\alpha + 1}{\alpha_1 + 1} \gamma_1 \]
\[ \beta_{12} = \frac{\alpha_1 + 1}{\alpha + 1} \beta_1. \]

Then one has the following

1. \( \beta_1, \beta_2, \beta_{12} > 0 \)
2. \( \tilde{s}_1 < \frac{d}{r_1} < b + \frac{(\alpha_1 + 1)d}{r_{12}} < \tilde{s}_2 + 2, \tilde{s}_1 < \frac{d}{r_2} < b + \frac{(\alpha + 1)d}{r_{12}} < \tilde{s}_2 + 2 \)
3. \( \frac{d}{2} \left( \frac{(\alpha_1 + 1)}{r_1} - \frac{1}{\gamma_1} \right) + \frac{b}{\gamma_1} = \frac{d\alpha}{2r_2} + \frac{b}{2} < 1 \)
4. \( \beta_2(\alpha + 1), \beta_{12}(\alpha + 1) < 1 \)
5. \( \beta_2 - \frac{d\alpha}{2r_2} - \frac{b}{2} - \beta_2(\alpha + 1) + 1 = 0 \)
6. \( \beta_1 - \frac{d}{2} \left( \frac{(\alpha_1 + 1)}{r_{12}} - \frac{1}{\gamma_1} \right) - \frac{b}{2} - \beta_{12}(\alpha + 1) + 1 = 0. \)

With the above, lemma in hand we now prove a variant of Theorem 3.1 which will be useful to prove Theorem 1.5 (2).

**Theorem 4.1.** Let \( 0 < b < \min(2, d) \) and \( \frac{2 - b}{s_2 + \gamma} < \alpha < \frac{2 - b}{s_1} \). Suppose that

\[ \max \left( \frac{2 - b}{\tilde{s}_2 + 2}, \frac{\tilde{s}_1 \alpha}{\tilde{s}_2 + 2} \right) < \alpha_1 < \alpha < \frac{2 - b}{\tilde{s}_1}. \]

Let \( r_1, r_2, r_{12}, \beta_1, \beta_2 \) are real numbers as in Lemma 4.1. Suppose further that \( M > 0 \) satisfies the inequality \( KM^\alpha < 1 \), where \( K \) is a positive constant. Choose \( R > 0 \) such that

\( cR + KM^{\alpha + 1} \leq M. \)

Let \( \varphi \) be a tempered distribution such that

\[ \sup_{t > 0} t^{\beta_1} \| e^{-tL_\alpha} \varphi \|_{r_1} \leq R, \quad \sup_{t > 0} t^{\beta_2} \| e^{-tL_\alpha} \varphi \|_{r_2} \leq R. \]

Then there exists a unique global solution \( u \) of (1.1) such that

\[ \sup_{t > 0} t^{\beta_1} \| u(t) \|_{r_1} \leq M, \quad \sup_{t > 0} t^{\beta_2} \| u(t) \|_{r_2} \leq M. \]

Furthermore,

1. \( \sup_{t \geq t_0} t^{\frac{2 - b}{s_1} - \frac{d}{r_1}} \| u(t) \|_q \leq C_M < \infty, \text{ for all } q \in [r_1, \frac{d}{s_1}] \) with \( C_M \to 0 \) as \( M \to 0 \)
2. \( \sup_{t > 0} t^{\frac{2 - b}{s_1} - \frac{d}{r_1}} \| u(t) \|_q \leq C_M < \infty, \text{ for all } q \in [r_2, \frac{d}{s_1}] \) with \( C_M \to 0 \) as \( M \to 0 \).

**Proof.** Let

\[ B_M := \{ u : (0, \infty) \to L^r(\mathbb{R}^d) : \sup_{t > 0} t^{\beta_1} \| u(t) \|_{r_1} \leq M, \sup_{t > 0} t^{\beta_2} \| u(t) \|_{r_2} \leq M \} \]

and

\[ d(u, v) = \max \left( \sup_{t > 0} t^{\beta_1} \| u(t) - v(t) \|_{r_1}, \sup_{t > 0} t^{\beta_2} \| u(t) - v(t) \|_{r_2} \right) \]

and

\[ \mathcal{J}_\varphi(u)(t) = e^{-tL_\alpha} \varphi + \mu \int_0^t e^{-(t-s)L_\alpha} (|\cdot|^{-b}u(s)|^\alpha u(s))ds. \]

for \( \varphi \) satisfies (4.6) and \( u \in B_M \).
Let \( \varphi, \psi \) satisfy (4.6) and \( u, v \in B_M \), then using Lemma 4.1, Proposition 2.1 for \((p, q) = (\frac{r_2}{\alpha+1}, r_1)\) we have
\[
\| \mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t) \|_{r_1} \\
\lesssim \| e^{-tL_c} \varphi - e^{-tL_c} \psi \|_{r_1} + \int_0^t \| e^{-(t-s)L_c} (| \cdot | - b)|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)) \|_{r_1} ds
\]
\[
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_1} + \int_0^t (t - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_1}\right) - \frac{\beta}{r_2}} \|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{r_1} ds
\]
\[
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_1} + \int_0^t (t - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_1}\right) - \frac{\beta}{r_2}} \|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{r_1} ds + \|u(s)\|_{r_1}^\alpha u(s) - |v(s)|^\alpha v(s)\|_{r_1} ds
\]
\[
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_1} + M^\alpha d(u, v) \int_0^t (t - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_1}\right) - \frac{\beta}{r_2}} s^{-(\alpha+1)\beta_2} ds
\]
as \( \frac{1}{r_{12}} = \frac{1}{r_1} + \frac{\alpha}{r_2} \) and hence for \( u \in B_M \),
\[
\|u(s)\|_{r_{12}} \leq \|u(s)\|_{r_1}^\alpha \|u(s)\|_{r_2}^\alpha \leq M s^{-\beta_1/(\alpha+1) - \alpha\beta_2/(\alpha+1)} = M s^{-\beta_1/(\alpha+1) + \beta_2 s^{-(\alpha+1)\beta_2}}.
\]
\[
\|u(s) - v(s)\|_{r_{12}} \leq \|u(s) - v(s)\|_{r_1}^\alpha \|u(s) - v(s)\|_{r_2}^\alpha \leq d(u, v) s^{-\beta_1/(\alpha+1) + \beta_2 s^{-(\alpha+1)\beta_2}}.
\]
Thus
\[
t^{\beta_1} \| \mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t) \|_{r_1} \lesssim t^{\beta_1} \| e^{-tL_c} (\varphi - \psi) \|_{r_1} + M^\alpha d(u, v) t^{\beta_1 - \frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_1}\right) - \frac{\beta}{r_2}} s^{-(\alpha+1)\beta_2} ds
\]
\[
\lesssim t^{\beta_1} \| e^{-tL_c} (\varphi - \psi) \|_{r_1} + M^\alpha d(u, v) \int_0^t (1 - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_1}\right) - \frac{\beta}{r_2}} s^{-(\alpha+1)\beta_2} ds
\]
using Lemma 4.1 (3), (4), (6). Now using Lemma 4.1 (2), Proposition 2.1 for \((p, q) = (\frac{r_2}{\alpha+1}, r_2)\) we have
\[
\| \mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t) \|_{r_2} \\
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_2} + \int_0^t \| e^{-(t-s)L_c} (| \cdot | - b)|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)) \|_{r_2} ds
\]
\[
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_2} + \int_0^t (t - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_2}\right) - \frac{\beta}{r_2}} \|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{r_2} ds
\]
\[
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_2} + \int_0^t (t - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_2}\right) - \frac{\beta}{r_2}} \|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{r_2} ds + \|u(s)\|_{r_2}^\alpha u(s) - |v(s)|^\alpha v(s)\|_{r_2} ds
\]
\[
\lesssim \| e^{-tL_c} (\varphi - \psi) \|_{r_2} + M^\alpha d(u, v) \int_0^t (t - s)^{-\frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_2}\right) - \frac{\beta}{r_2}} s^{-(\alpha+1)\beta_2} ds
\]
and hence using Lemma 4.1 (3), (4), (5) we get
\[
t^{\beta_2} \| \mathcal{J}_\varphi(u)(t) - \mathcal{J}_\psi(v)(t) \|_{r_2} \lesssim t^{\beta_2} \| e^{-tL_c} (\varphi - \psi) \|_{r_2} + M^\alpha d(u, v) t^{\beta_2 - \frac{\beta}{2}\left(\frac{r_2}{\alpha+1} - \frac{1}{r_2}\right) - \frac{\beta}{r_2}} s^{-(\alpha+1)\beta_2} ds
\]
\[
\lesssim t^{\beta_2} \| e^{-tL_c} (\varphi - \psi) \|_{r_2} + M^\alpha d(u, v). \tag{4.9}
\]
Using (4.8), (4.9) we have
\[
d(\mathcal{J}_\varphi(u), \mathcal{J}_\psi(v))
\]
\[
\leq c \max \left( \sup_{t > 0} t^{\beta_1} \| e^{-tL_c} (\varphi - \psi) \|_{r_1}, \sup_{t > 0} t^{\beta_2} \| e^{-tL_c} (\varphi - \psi) \|_{r_2} \right) + KM^\alpha d(u, v). \tag{4.10}
\]
Putting $\psi = 0, v = 0$ in (4.10) one has
\[ d(J_\varphi(u), 0) \leq c \max_{t > 0} \left( \sup_{t > 0} t^{\beta_1} \| e^{-tL_\alpha} \varphi \|_{r_1}, \sup_{t > 0} t^{\beta_2} \| e^{-tL_\alpha} \varphi \|_{r_2} \right) + KM^\alpha d(u, 0) \leq cR + KM^{\alpha + 1} \leq M \]
and hence $J_\varphi(u) \in B_M$. Thus $J_\varphi$ maps $B_M$ to itself. Putting $\psi = \varphi$ in (4.10)
\[ d(J_\varphi(u), J_\varphi(v)) \leq KM^\alpha d(u, v) < d(u, v) \]
and hence $J_\varphi$ is a contraction in $B_M$. Thus (1.1) has a unique solution in $B_M$ satisfying (4.7).

(1) follows from Lemma 4.2 and iteration as in proof of Theorem 3.1. (2) follows from Lemma 3.1 and iteration as in proof of Theorem 3.1.

This is a variant of Lemma 3.1 used in the above result.

**Lemma 4.2 (A priori estimate).** Suppose $s < q$ and
\[ \tilde{s}_1 < \frac{d}{q} < b + \frac{d(\alpha + 1)}{s} < \tilde{s}_2 + 2, \quad \frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) < 1 - \frac{b}{2} \]
Assume $t_0 \geq 1$ and $u$ be solution to (1.1) satisfying
\[ \sup_{t > t_0} t^{\frac{\tilde{s}_1}{s} - \frac{d}{q}} \| u(t) \|_s \leq A < \infty \]
then
\[ \sup_{t \geq 2t_0} t^{\frac{\tilde{s}_1}{s} - \frac{d}{q}} \| u(t) \|_q \lesssim A(1 + A^\alpha) =: C_A < \infty \]
with $C_A \to 0$ as $A \to 0$.

**Proof.** Note that
\[ u(t) = e^{-\frac{t}{2}L_\alpha} u(t/2) + \mu \int_{t/2}^t e^{-(t-\tau)L_\alpha} (| \cdot | - b(|u(\tau)|^\alpha u(\tau))) d\tau \]
and therefore using Theorem A for $(p, q) = (s, q)$ and Proposition 2.1 for $(p, q) = \left( \frac{s}{\alpha + 1}, q \right)$ we have
\[ \| u(t) \|_q \lesssim \| e^{-\frac{t}{2}L_\alpha} u(t/2) \|_q + \int_{t/2}^t \| e^{-(t-\tau)L_\alpha} (| \cdot | - b(|u(\tau)|^\alpha u(\tau))) \|_q d\tau \]
\[ \lesssim t^{-\frac{d}{2} \left( \frac{1}{s} - \frac{1}{q} \right)} \| u(t/2) \|_s + \int_{t/2}^t (t - \tau)^{-\frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) - \frac{1}{2}} \| u(\tau) \|_s^\alpha u(\tau)) \|_s^\alpha d\tau \]
\[ \lesssim \frac{d}{2} - \frac{\tilde{s}_1}{s} t^{\frac{\tilde{s}_1}{s} - \frac{d}{q}} A + \int_{t/2}^t (t - \tau)^{-\frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) - \frac{1}{2}} \| u(\tau) \|_s^\alpha d\tau. \]
Now with $\beta_1(s) = \frac{2 - b}{2\tilde{s}_1} - \frac{d}{2s}$, and using $t/2 \geq t_0$
\[ \| u(t) \|_q \lesssim t^{\frac{d}{2} - \frac{\tilde{s}_1}{s}} A + A^{\alpha + 1} \int_{t/2}^t (t - \tau)^{-\frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) - \frac{1}{2}} - \beta_1(s)(\alpha + 1) d\tau \]
\[ \lesssim t^{\frac{d}{2} - \frac{\tilde{s}_1}{s}} A + A^{\alpha + 1} t^{1 - \frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) - \frac{1}{2}} - \beta_1(s)(\alpha + 1) \int_{1/2}^1 (1 - \sigma)^{-\frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) - \frac{1}{2}} - \beta_1(s)(\alpha + 1) d\sigma \]
\[ \lesssim t^{\frac{d}{2} - \frac{\tilde{s}_1}{s}} A + A^{\alpha + 1} t^{\frac{d}{2} - \frac{1}{2}} - \frac{2}{s} - \frac{2}{a_1} \int \frac{1}{(1 - \sigma)}^{-\frac{d}{2} \left( \frac{\alpha + 1}{s} - \frac{1}{q} \right) - \frac{1}{2}} - \beta_1(s)(\alpha + 1) d\sigma < \infty. \]
This completes the proof as $t \geq 1$ for $t/2 \geq t_0$. □

Following is again a technical result, to be used to prove Theorem 4.2 that further proves the final result.
Lemma 4.3. Let \( 0 < b < \min(2, d) \) and \( \frac{2-b}{s_2+2} < \alpha < \frac{2-b}{s_1} \). Let the real numbers \( \alpha_1 \) and \( \alpha \) be such that
\[
\max\left( \frac{2-b}{s_2+2}, \frac{s_1 \alpha}{s_2+2-b-s_1 \alpha} \right) < \alpha_1 < \alpha < \frac{2-b}{s_1}.
\]
Let \( r_1, r_2, \beta_1, \beta_2 \) are real numbers as in Lemma 4.1. Then there exists a real number \( \delta_0 > 0 \) such that, for all \( 0 < \delta < \delta_0 \), there exists a real number \( 0 < \theta_\delta < 1 \), with the properties that, the two real numbers \( \tilde{r} \) and \( \tilde{\beta} \) given by
\[
\frac{1}{\tilde{r}} = \frac{\theta_\delta}{r_1} + \frac{1-\theta_\delta}{r_2}, \quad \tilde{\beta} = \theta_\delta \beta_1 + (1-\theta_\delta) \beta_2
\]
satisfy the following conditions

(i) \( \tilde{s}_1 < \frac{d}{r_1} < b + \frac{d(\alpha+1)}{r} < \tilde{s}_2 + 2 \)

(ii) \( \beta_1 + \delta - \frac{d(\alpha+1)}{2r_1} - \frac{1}{r_2}\theta_\delta + \frac{\alpha+1}{r_2} < \frac{\alpha+1}{r_1}, \tilde{\beta} = \beta_1 - \beta_2 \)

(iii) \( \frac{d(\alpha+1)}{\alpha_1} - \frac{1}{r_1}\theta_\delta < 1, \tilde{\beta}(\alpha+1) < 1 \).

Moreover this \( \theta_\delta \) is given by
\[
\theta_\delta = \frac{1}{\alpha+1} + \frac{2\alpha_1 \alpha}{(2-b)(\alpha-\alpha_1)(\alpha+1)} \delta.
\]

Proof. Step I: If \((4.11)\) is true then \((ii)\) is equivalent to \((4.12)\):

First note that \( \frac{\alpha+1}{r} - \frac{1}{r_1} = (\alpha+1)(\frac{1}{r_1} - \frac{1}{r_2})\theta_\delta + \frac{\alpha+1}{r_2} - \frac{1}{r_1}, \tilde{\beta} = \theta_\delta \beta_1 - \beta_2 \). Thus
\[
\frac{1}{2} \left( \frac{\alpha+1}{r} - \frac{1}{r_1} \right) + \tilde{\beta}(\alpha+1) \\
= \frac{d(\alpha+1)}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \theta_\delta + \frac{d(\alpha+1)}{2r_1} - \frac{d}{2r_1} + \theta_\delta(\beta_1 - \beta_2)(\alpha+1) + \beta_2(\alpha+1).
\]

Then
\[
\beta_1 - \frac{d(\alpha+1)}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \theta_\delta - \frac{d(\alpha+1)}{2r_2} + \frac{d}{2r_1} - \theta_\delta(\beta_1 - \beta_2)(\alpha+1) - \beta_2(\alpha+1) \\
= (\beta_1 + \frac{d}{2r_1}) - (\alpha+1)(\beta_2 + \frac{d}{2r_2}) - \theta_\delta(\alpha+1)(\frac{d}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) + \beta_1 - \beta_2) \\
= \frac{2-b}{2\alpha_1} - (\alpha+1) \frac{2-b}{2\alpha} - \theta_\delta(\alpha+1)(\frac{2-b}{2\alpha_1} - \frac{2-b}{2\alpha}).
\]

Therefore \((ii)\) is equivalent to
\[
\frac{2-b}{2\alpha_1} - (\alpha+1) \frac{2-b}{2\alpha} + \delta - \frac{b}{2} + 1 = \theta_\delta(\alpha+1)(\frac{2-b}{2\alpha_1} - \frac{2-b}{2\alpha}) \\
\iff \frac{1}{\alpha_1} - \frac{\alpha+1}{\alpha} + \frac{2}{2-b} \delta + 1 = \theta_\delta(\alpha+1)(\frac{1}{\alpha_1} - \frac{1}{\alpha}) \\
\iff \frac{1}{\alpha_1} - \frac{1}{\alpha} + \frac{2}{2-b} \delta = \theta_\delta(\alpha+1)(\frac{1}{\alpha_1} - \frac{1}{\alpha})
\]
which is equivalent to \((4.12)\).

Step II: Validity of \((i), (iii)\):

Note that
\[
\theta_\delta = \frac{1}{\alpha+1} + \epsilon(\delta)
\]
Now $\theta_0 = \frac{4}{\alpha+1} + \epsilon(0) = \frac{1}{\alpha+1}$ and for this choice of $\theta_0$, one has $\tilde{r} = r_{12}$, $\tilde{\beta} = \beta_{12}$. So at $\theta = 0$, the inequalities (i), (iii) hold by Lemma 4.1. Then by continuity of $\epsilon$ with respect to $\delta$ one has (i), (iii) for $\delta > 0$ small enough. \hfill \Box

**Theorem 4.2.** Let $0 < b < \min(2, d)$ and $\frac{2-b}{2+b} < \alpha < \frac{2-b}{\alpha_1}$. Let the real numbers $\alpha_1$ and $\alpha$ be such that
\[
\max \left( \frac{2-b}{\alpha_2 + 2}, \frac{\tilde{s}_1 \alpha}{\tilde{s}_2 + 2 - b - \tilde{s}_1 \alpha} \right) < \alpha_1 < \alpha < \frac{2-b}{\tilde{s}_1}.
\]
Let $r_1, r_2$ be two real numbers as in Lemma 4.1. Let $\beta_1, \beta_2$ be given by Lemma 4.1 and define $\beta_1(q)$ by
\[
\beta_1(q) = \frac{2-b}{2\alpha_1} - \frac{d}{2q}, q > 1
\]
Let $\psi(x) = \omega(x)|x|^{-\frac{2+b}{\alpha_1}}$, where $\omega \in L^{\infty}(S^{d-1})$ is homogeneous of degree 0. Let $\varphi \in C_0(\mathbb{R}^d)$ be such that
\[
|\varphi(x)| \leq c(1 + |x|^2)^{-\frac{2+b}{\alpha_1}} \text{ for all } x \in \mathbb{R}^d, \quad |\varphi(x)| = \omega(x)|x|^{-\frac{2+b}{\alpha_1}} \text{ for all } |x| \geq A
\]
for some constant $A > 0$, where $c$ is a small positive constant and $\|\omega\|$ is sufficiently small.

Let $w$ be the solution of (1.1) with initial data $\varphi$, constructed by Theorem 4.1 and let $w$ be the self-similar solution of (1.1) constructed by Theorem 4.1 with $\mu = 0$, and with initial data $\psi$. Then there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, and with $M$ perhaps smaller, there exists $C_\delta > 0$ such that
\[
\|u(t) - w(t)\|_q \leq C_\delta t^{-\beta_1(q)} - \delta, \forall t \geq t_q, \quad (4.13)
\]
for all $q \in [r_1, \frac{d}{\alpha_1})$. In particular, if $\omega \neq 0$, for large time $t$
\[
c_1 t^{-\beta_1(q)} \leq \|u(t)\|_q \leq c_2 t^{-\beta_1(q)}.
\]

**Proof.** Let $\psi(x) = \omega(x)|x|^{-\frac{2+b}{\alpha_1}}$, $x \in \mathbb{R}^d$. Let $\varphi_1 = \varphi \chi_{\{|x| \leq 1\}}$ and $\varphi_2 = \varphi \chi_{\{|x| > 1\}}$ so that $\varphi = \varphi_1 + \varphi_2$. Note that
\[
|\varphi(x)| \leq c(1 + |x|^2)^{-\frac{2+b}{\alpha_1}} \leq c|x|^{-\frac{2+b}{\alpha_1}}, \forall x \in \mathbb{R}^d
\]
and thus $\varphi_1 \in L^s(\mathbb{R}^d)$ for $1 \leq s < \frac{d \alpha_1}{2+b}$ and $\varphi_2 \in L^\sigma(\mathbb{R}^d)$ for $\frac{d \alpha_1}{2+b} < \sigma \leq \infty$. Since $\alpha > \alpha_1$,
\[
|\varphi(x)| \leq c(1 + |x|^2)^{-\frac{2+b}{\alpha_1}} \leq c(1 + |x|^2)^{-\frac{2+b}{\alpha}} \leq c|x|^{-\frac{2+b}{\alpha}}, \forall x \in \mathbb{R}^d.
\]
Then applying Theorem A
\[
\|e^{-L_a \varphi}\|_{r_1} \leq \|e^{-L_a \varphi_1}\|_{r_1} + \|e^{-L_a \varphi_2}\|_{r_1} \leq \|\varphi_1\|_{r} + \|\varphi_2\|_{\sigma} < \infty
\]
by choosing $s, \sigma$ so that $\tilde{s}_1 < \frac{d}{r_1} < \frac{d}{\sigma} < \frac{2-b}{\alpha_1} < \frac{d}{s} < \tilde{s}_2 + 2$. Similarly using $r_2 > \frac{d \alpha_1}{2+b}$ one finds
\[
\|e^{-L_a \varphi}\|_{r_2} < \infty. \text{ Then using the homogeneity of } |\cdot|^{-\frac{2+b}{\alpha_1}}, |\cdot|^{-\frac{2+b}{\alpha}} \text{ (and positivity of } e^{-L_a}) \text{ we achieve}
\]
\[
\sup_{t>0} t^{\beta_1} \|e^{-tL_a \varphi}\|_{r_1} \leq R, \quad \sup_{t>0} t^{\beta_2} \|e^{-tL_a \varphi}\|_{r_2} \leq R
\]
after possibly choosing $c$ smaller.

Proceeding as (4.1), one finds
\[
\sup_{t>0} t^{\beta_1 + \delta} \|e^{-tL_a (\varphi - \psi)}\|_{r_1} < \infty \quad \text{for all } 0 < \delta \leq \frac{\tilde{s}_2 + 2}{2} - \frac{2-b}{2\alpha_1}. \quad (4.14)
\]
for $\delta > 0$ small enough.

Let $v(t) = e^{-tL_a \varphi}$ then $u(t) = v(t) + \mu \int_0^t e^{-(t-\sigma)L_a} (|\cdot|^{-b}(u(\sigma))^a u(\sigma))d\sigma$, therefore using Proposition 2.1, Lemma 4.3
\[
\|u(t) - v(t)\|_{r_1} \leq \int_0^t \|e^{-(t-\sigma)L_a} (|\cdot|^{-b}(u(\sigma))^a u(\sigma))\|_{r_1} d\sigma
\]
\[ \begin{align*}
&\leq \int_0^t (t - \sigma)^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \frac{b}{2}} \|u(\sigma)|^\alpha u(\sigma)\|_{r_1} \, d\sigma \\
&= \int_0^t (t - \sigma)^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \frac{b}{2}} \|u(\sigma)\|_{r_1} \, d\sigma.
\end{align*} \]

Now note that
\[ \|u(\sigma)\|_{r_2} \leq \|u(\sigma)\|_{r_1} \] 

and hence
\[ t^{\beta_1 + \delta} \|u(t) - v(t)\|_{r_1} \leq M^{\alpha + 1} t^{\beta_1 + \delta} \int_0^t (t - \sigma)^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \frac{b}{2}} \sigma^{-(\alpha + 1)\beta} \, d\sigma \]
\[ \leq M^{\alpha + 1} t^{\beta_1 + \delta + 1} \int_0^t (1 - \sigma)^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \frac{b}{2}} \sigma^{-(\alpha + 1)\beta} \, d\sigma \]
\[ = M^{\alpha + 1} t^{\beta_1 + \delta} \int_0^t (1 - \sigma)^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \frac{b}{2}} \sigma^{-(\alpha + 1)\beta} \, d\sigma < \infty \]

using Lemma 4.3. This together with (4.14) imply
\[ \|u(t) - w(t)\|_{r_1} \leq \|u(t) - v(t)\|_{r_1} + \|e^{-t\mathcal{L}_a}\varphi - e^{-t\mathcal{L}_a}\psi\|_{r_1} \leq t^{-\beta_1 - \delta} \]
which proves (4.13) for \( q = r_1 \).

Now
\[ u(t) - w(t) = e^{-\frac{t}{2}}\mathcal{L}_a(u(t/2) - w(t/2)) + \mu \int_{t/2}^t e^{-(t - \sigma)\mathcal{L}_a} (|\cdot|^{-b}(|u(\sigma)|^\alpha u(\sigma))) \, d\sigma \]
using Lemma 4.3 for \( q \in [r_2, \frac{q}{s_1}] \),
\[ s_1 < \frac{d}{q} < \frac{d}{r_1} < s_2 + 2 \quad \text{and} \quad s_1 < \frac{d}{q} < b + \frac{d(\alpha + 1)}{q} < b + \frac{d(\alpha + 1)}{r_2} < s_2 + 2. \]
\[ \|u(t) - w(t)\|_q \]
\[ \leq t^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1})} \|u(t/2) - w(t/2)\|_q + \int_{t/2}^t (t - \sigma)^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \frac{b}{2}} \|u(\sigma)|^\alpha u(\sigma)\|_{r_1} \, d\sigma \]
\[ \leq t^{-\frac{q}{2}(\frac{\alpha + 1}{q} - \frac{1}{r_1}) - \beta_1 - \delta} + \int_{t/2}^t (t - \sigma)^{-\frac{d}{2r_2} - \frac{b}{2}} \|u(\sigma)\|_{r_1} \, d\sigma. \]

using (4.15). Now from (1), (2) in Theorem 4.1, for \( \sigma \geq t_q, \|u(\sigma)\|_q \leq C_M \sigma^{-\beta_1(q)} \) as well as \( \|u(\sigma)\|_q \leq C_M \sigma^{-\beta(q)} \) for all \( q \in [r_2, \frac{q}{s_1}] \). Therefore for \( \sigma \geq t_q, \)
\[ \|u(\sigma)\|_{r_1}^{\alpha + 1} \leq C_M \sigma^{-\alpha(1)[\beta_1(q)\theta_\delta + \beta(q)(1 - \theta_\delta)]} \]
and \( \beta_1(q)\theta_\delta + \beta(q)(1 - \theta_\delta) = \beta_1 \delta + \beta_2(1 - \theta_\delta) + \frac{d}{2} (\frac{1}{r_1} - \frac{1}{q}) \theta_\delta + \frac{d}{2} (\frac{1}{r_2} - \frac{1}{q}) (1 - \theta_\delta) = \tilde{\beta} + \frac{d}{2} (\frac{\theta_\delta}{r_1} + \frac{1 - \theta_\delta}{r_2} - \frac{d}{2q}) = \tilde{\beta} + \frac{d}{2} \left( \frac{\tilde{\beta}}{r_1} - \frac{d}{2q} \right). \)
Therefore for \( t \geq 2t_q, \)
\[ t^{\beta_1(q) + \delta} \|u(t) - w(t)\|_q \]
\[ \leq t^{\beta_1(q) - \beta_1 - \frac{d}{2} (\frac{\tilde{\beta}}{r_1} - \frac{d}{2q})} + t^{\beta_1(q) + \delta} \int_{t/2}^t (t - \sigma)^{-\frac{d}{2r_2} - \frac{b}{2}} \|u(\sigma)\|_{r_1} \, d\sigma \]
\[ \leq t^0 + C_M^{\alpha + 1} t^{\beta_1(q) + \delta} \int_{t/2}^t (t - \sigma)^{-\frac{d}{2r_2} - \frac{b}{2} - \sigma^{-(\alpha + 1)(\tilde{\beta} + \frac{d}{2} - \frac{d}{2q})} \, d\sigma \]
\[ \leq 1 + C_M^{\alpha + 1} t^{\beta_1(q) + \delta - \frac{d}{2r_2} - \frac{b}{2} - (\alpha + 1)(\tilde{\beta} + \frac{d}{2} - \frac{d}{2q}) + 1} \int_{t/2}^t (1 - \sigma)^{-\frac{d}{2r_2} - \frac{b}{2} - \sigma^{-(\alpha + 1)(\tilde{\beta} + \frac{d}{2} - \frac{d}{2q})} \, d\sigma. \]
But $\beta_1(q) + \delta - \frac{d\alpha}{2q} - \frac{b}{q} - (\alpha + 1)(\beta + \frac{d\alpha}{2q} - \frac{d}{q}) + 1 = \beta_1(q) + \delta - \frac{d\alpha - 1}{q} - \frac{b}{q} - (\alpha + 1)(\beta + \frac{d\alpha}{2q} - \frac{d}{q}) + 1 = \beta_1 + \delta - \frac{(\alpha + 1)}{r_1} - \frac{b}{2} - (\alpha + 1)\beta + 1 = 0$ and $\frac{d\alpha}{2q} + \frac{b}{q} < 1$. Thus $t^{\beta(q) + \delta} ||u(t) - w(t)||_q \leq 1$ for $t \geq 2t_q$. This proves (4.13) for $q \in [r_2, \frac{q}{2})$. To prove (4.13) for $(r_1, r_2)$, we use interpolation.

The final conclusion follows as in the proof in nonlinear case, see (4.4), (4.5) in Subsection 4.1. 

Remark 4.1. After proving (4.13) for $q = r_1$ (i.e. (4.15)) the authors in [1] proves it for $q = \infty$ and interpolates them to achieve (4.13) for $q \in (r_1, \infty)$. Since we do not have the decay estimate (1.3) for $q = \infty$, we could not achieve (4.13) for $q = \infty$ and hence we need to take a different path to achieve the result.

Proof of Theorem 1.5 (2). Since $\sigma > \frac{2 - b}{\alpha}$, we can find $\alpha_1 < \alpha$ so that $\sigma = \frac{2 - b}{\alpha_1}$.

To apply Theorem 4.2 we need

$$\max \left( \frac{2 - b}{s_2 + 2}, \frac{s_1 \alpha}{s_2 + 2 - b - s_1 \alpha} \right) < \alpha_1 < \alpha < \frac{2 - b}{s_1}.$$ 

which will follow if $\frac{2 - b}{s_2 + 2} < \frac{2 - b}{\sigma}$ i.e. $\sigma < s_2 + 2$ and $\frac{s_1 \alpha}{s_2 + 2 - b - s_1 \alpha} < \frac{2 - b}{\sigma}$ i.e. $\sigma < \frac{2 - b}{s_1 \alpha}(s_2 + b - s_1 \alpha)$.

APPENDIX

Lemma A1. Let $\max \left( \frac{2 - b}{s_2 + 2}, \frac{s_1 \alpha}{s_2 + 2 - b - s_1 \alpha} \right) < \alpha_1 < \alpha < \frac{2 - b}{s_1}$. Then one can choose $r_1$ so that

1. $s_1 < \frac{d\alpha_1}{r_1 \alpha} < \frac{d}{r_1} < b + \frac{(\alpha_1 + 1)d}{r_1} < s_2 + 2$
2. $\frac{d\alpha_1}{2r_1} + \frac{b}{2} < 1$
3. $\beta_1(\alpha_1 + 1) < 1$.

Proof. (i), (ii), (iii) are equivalent to $\frac{d\alpha_1 + 1}{s_2 + 2 - b} < r_1 < \frac{d\alpha_1}{s_1 \alpha} < \frac{d}{r_1} < b + \frac{(\alpha_1 + 1)d}{r_1} < s_2 + 2$. Then to make room for $r_1$, one thus needs

$$\left( \frac{d\alpha_1 + 1}{s_2 + 2 - b} \right) < r_1 < \left( \frac{d\alpha_1}{s_1 \alpha} \right) = \frac{d\alpha_1}{2 - b}$$

which is possible if $\max \left( \frac{2 - b}{s_2 + 2}, \frac{s_1 \alpha}{s_2 + 2 - b - s_1 \alpha} \right) < \alpha_1 < \alpha < \frac{2 - b}{s_1}$.

Lemma A2. Let $\max \left( \frac{2 - b}{s_2 + 2}, \frac{s_1 \alpha}{s_2 + 2 - b - s_1 \alpha} \right) < \alpha_1 < \alpha < \frac{2 - b}{s_1}$. Let $r_1, r_2$ be as in Lemma 4.1. Then

1. $r_1 < r_2$
2. $s_1 < \frac{d}{r_2} < b + \frac{(\alpha_1 + 1)d}{r_2} < s_2 + 2$, $s_1 < \frac{d}{r_2} < b + \frac{(\alpha_1 + 1)d}{r_2} < s_2 + 2$
3. $\frac{d\alpha_2}{2r_2} + \frac{b}{2} < 1$, $\frac{d\alpha_2}{2r_2} + \frac{b}{2} < 1$
4. $\beta_1(\alpha_1 + 1) < 1$, $\beta_2(\alpha_1 + 1) < 1$

Proof. (i) follows from $\alpha > \alpha_1$.

First part of (ii) is a consequence of Lemma A1. Note that $\frac{(\alpha_1 + 1)\alpha_1}{\alpha(\alpha_1 + 1)} < 1$ and hence $\frac{(\alpha_1 + 1)}{r_1} < (\alpha_1 + 1)\alpha_1$ imply the second part of (ii).

(iii) follows from (i) and Lemma A1 (ii).

First one in (iv) is exactly part (iii) in Lemma A1. Since $\beta_2(\alpha + 1) = \beta_1(\alpha + 1)\frac{(\alpha_1 + 1)\alpha_1}{\alpha(\alpha_1 + 1)}$ the last inequality in (iv) follows from the fact $\frac{(\alpha_1 + 1)\alpha_1}{\alpha(\alpha_1 + 1)} < 1$. 

□
Proof of Lemma 4.1. (1) $\beta_1 > 0$ follows from $r_1 > \frac{d\alpha_2}{2\beta_2}$. Then $\beta_2, \beta_{12} > 0$ follows from their definitions.

(2), (3), (4) are essentially consequences parts (ii), (iii), (iv) of Lemma A2 respectively.

(5), (6) follows by simple computation. □

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REFERENCES

[1] B. Ben Slimene, S. Tayachi, and F. B. Weissler, Well-posedness, global existence and large time behavior for Hardy-Hénon parabolic equations, Nonlinear Anal., 152 (2017), pp. 116–148.

[2] D. G. Bhimani, H. Hajaiej, S. Haque, and T. Luo, A sharp Gagliardo-Nirenberg inequality and its application to fractional problems with inhomogeneous nonlinearity, Evolution Equations and Control Theory, 12 (2023), pp. 362–390.

[3] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math., 68 (1996), pp. 277–304.

[4] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, J. Funct. Anal., 203 (2003), pp. 519–549.

[5] T. Cazenave and F. B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations, Math. Z., 228 (1998), pp. 83–120.

[6] N. Chikami, M. Ikeda, and K. Taniguchi, Well-posedness and global dynamics for the critical Hardy–Sobolev parabolic equation, Nonlinearity, 34 (2021), pp. 8094–8142.

[7] C. M. Guzmán and J. Murphy, Scattering for the non-radial energy-critical inhomogeneous NLS, J. Differential Equations, 295 (2021), pp. 187–210.

[8] S. Haque, Strichartz estimates for Schrödinger equation with singular and time dependent potentials and application to NLS equations, NoDEA Nonlinear Differential Equations Appl., 29 (2022), pp. Paper No. 2, 25.

[9] A. Haraux and F. B. Weissler, Nonuniqueness for a semilinear initial value problem, Indiana Univ. Math. J., 31 (1982), pp. 167–189.

[10] M. Hirose, Existence of global solutions for a semilinear parabolic Cauchy problem, Differential Integral Equations, 21 (2008), pp. 623–652.

[11] N. Ioku, G. Metafune, M. Sobajima, and C. Spina, $L^p - L^q$ estimates for homogeneous operators, Commun. Contemp. Math., 18 (2016), pp. 1550037, 14.

[12] H. Kalf, U.-W. Schmikelke, J. Walter, and R. Wüst, On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials, in Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens), Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975, pp. 114–115.

[13] R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng, The energy-critical NLS with inverse-square potential, Discrete Contin. Dyn. Syst., 37 (2017), pp. 3831–3866.

[14] R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng, Sobolev spaces adapted to the Schrödinger operator with inverse-square potential, Math. Z., 288 (2018), pp. 1273–1298.

[15] G. Metafune, N. Okazawa, M. Sobajima, and C. Spina, Scale invariant elliptic operators with singular coefficients, J. Evol. Equ., 16 (2016), pp. 391–439.

[16] C. Miao, J. Murphy, and J. Zheng, The energy-critical nonlinear wave equation with an inverse-square potential, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 37 (2020), pp. 417–456.

[17] J. L. Vázquez and E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal., 173 (2000), pp. 103–153.

[18] X. Wang, On the Cauchy problem for reaction-diffusion equations, Trans. Amer. Math. Soc., 337 (1993), pp. 549–590.

[19] F. B. Weissler, Semilinear evolution equations in Banach spaces, J. Functional Analysis, 32 (1979), pp. 277–296.

[20] ———, Local existence and nonexistence for semilinear parabolic equations in $L^p$, Indiana Univ. Math. J., 29 (1980), pp. 79–102.

[21] J. Zhang and J. Zheng, Scattering theory for nonlinear Schrödinger equations with inverse-square potential, J. Funct. Anal., 267 (2014), pp. 2907–2932.
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