A characterization of $\ell^p$-spaces symmetrically finitely represented in symmetric sequence spaces

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Received: 21 September 2021 / Accepted: 15 February 2022 / Published online: 22 March 2022
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Abstract

For a separable symmetric sequence space $X$ of fundamental type we identify the set $\mathcal{F}(X)$ of all $p \in [1, \infty]$ such that $\ell^p$ is block finitely represented in the unit vector basis $\{e_k\}_{k=1}^\infty$ of $X$ in such a way that the unit basis vectors of $\ell^p$ ($c_0$ if $p = \infty$) correspond to pairwise disjoint blocks of $\{e_k\}$ with the same ordered distribution. It turns out that $\mathcal{F}(X)$ coincides with the set of approximate eigenvalues of the operator $(x_k) \mapsto \sum_{k=2}^{\infty} \lambda_{[k/2]} x_k e_k$ in $X$. In turn, we establish that the latter set is the interval $[\alpha_X, \beta_X]$, where $\alpha_X$ and $\beta_X$ are the Boyd indices of $X$. As an application, we find the set $\mathcal{F}(X)$ for arbitrary Lorentz and separable Orlicz sequence spaces.

Keywords $\ell^p$ · Finite representability · Banach lattice · Symmetric sequence space · Dilation operator · Shift operator · Approximate eigenvalue · Boyd indices · Orlicz space · Lorentz space

Mathematics Subject Classification 46B70 · 46B42

1 Introduction

While a Banach space $X$ need not contain a subspace isomorphic to $\ell^p$ for some $1 \leq p < \infty$ or $c_0$ (as was shown by Tsirel’son [27] in 1974), at least one of these spaces is present in such a space $X$ locally. More precisely, for every $\varepsilon > 0$ there exist finite-dimensional subspaces of $X$ of arbitrarily large dimension $n$ which...
are $(1 + \varepsilon)$-isomorphic to $\ell_p^n$ for some $1 \leq p < \infty$ or $\ell_0^n$. This fact is the content of the famous result proved by Krivine in 1976 (we should recall here also the celebrated Dvoretzky theorem, which appeared long before Tsirelson’s example, in 1961, showing that for every $\varepsilon > 0$ any Banach space contains subspaces $(1 + \varepsilon)$-isomorphic to $\ell_2^n$ for each $n \in \mathbb{N}$; see [6]). Let us introduce some necessary definitions.

Suppose $X$ is a Banach space, $1 \leq p \leq \infty$, and $\{z_i\}_{i=1}^\infty$ is a bounded sequence in $X$. The space $\ell^p$, $1 \leq p \leq \infty$, is said to be block finitely represented in $\{z_i\}_{i=1}^\infty$ if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $0 = m_0 < m_1 < \cdots < m_n$ and $\alpha_i \in \mathbb{R}$ such that the vectors $u_k = \sum_{i=m_{k-1}+1}^{m_k} \alpha_i z_i$, $k = 1, 2, \ldots, n$, satisfy the inequality

$$(1 + \varepsilon)^{-1} \|a\|_p \leq \left\| \sum_{k=1}^n a_k u_k \right\|_X \leq (1 + \varepsilon) \|a\|_p$$

for arbitrary $a = (a_k)_{k=1}^n \in \mathbb{R}^n$. In what follows, $p \in [1, \infty)$.

Moreover, the space $\ell^p$, $1 \leq p \leq \infty$, is said to be finitely represented in a Banach space $X$ if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $x_1, x_2, \ldots, x_n \in X$ such that for any $a = (a_k)_{k=1}^n \in \mathbb{R}^n$

$$(1 + \varepsilon)^{-1} \|a\|_p \leq \left\| \sum_{k=1}^n a_k x_k \right\|_X \leq (1 + \varepsilon) \|a\|_p.$$ 

We can state now the above-mentioned famous result proved by Krivine in [14] (see also [1, Theorem 11.3.9]).

**Theorem 1** Let $\{z_i\}_{i=1}^\infty$ be an arbitrary normalized sequence in a Banach space $X$ such that the vectors $z_i$, $i = 1, 2, \ldots$, do not form a relatively compact set. Then $\ell^p$ is block finitely represented in $\{z_i\}_{i=1}^\infty$ for some $p$, $1 \leq p \leq \infty$.

This statement plays a prominent rôle in the modern theory of Banach spaces and operators (see, e.g., [7, 21, 22]). Theorem 1 is directly connected, in particular, with the important notions of the Rademacher type and cotype of a Banach space (see, e.g., [1, 16]). We mention here only the profound result due to Maurey and Pisier (see [20]); it states that, for every infinite-dimensional Banach space $X$, the spaces $\ell^p_{p_X}$ and $\ell^q_{q_X}$, where $p_X = \sup\{p \in [1, 2] : X \text{ has type } p\}$ and $q_X = \inf\{q \in [2, \infty) : X \text{ is of cotype } q\}$, are finitely represented in $X$. Similar result for Banach lattices and the notions of the upper and lower estimates, when the finite representability of $\ell^p$ in a Banach lattice $X$ has the additional property that the unit basis vectors of $\ell^p$ correspond to pairwise disjoint elements of $X$, was proved by Shepp in [25]. In this connection, it is natural to look for a description of the set of $p$ such that $\ell^p$ is finitely represented in a given Banach space.

In 1978, Rosenthal [24] established the following version of Theorem 1.
Theorem 2 Let \( \{e_i\}_{i=1}^{\infty} \) be a subsymmetric unconditional basis for a Banach space \( X \) such that \( c_0 \) is not finitely represented in \( X \). If the numbers \( s_n \) are defined by \( \| \sum_{i=1}^{n} e_i \|_X = n^{1/s_n} \), then \( \ell^p \) is block finitely represented in \( \{e_i\} \) provided that \( p \) belongs to the interval \( [\liminf_{n \to \infty} s_n, \limsup_{n \to \infty} s_n] \).

From an inspection of the proof of Theorem 2 (cf. the remark after the proof of Theorem 3.3 in [24]) it follows that for every \( p \), satisfying the hypothesis, any \( n \) and \( \varepsilon > 0 \) there exist pairwise disjoint blocks \( u_k, k = 1, 2, \ldots, n \), of the basis \( \{e_i\} \), with the same ordered distribution, such that we have (1). We will say that blocks \( u = \sum_{i=1}^{m} \alpha_i e_{n_i} \) and \( v = \sum_{i=1}^{m} \beta_i e_{k_i} \) of the basis \( \{e_i\} \) have the same ordered distribution if \( \beta_i = \alpha_{\pi(i)}, i = 1, 2, \ldots, m \), for some permutation \( \pi \) of the set \( \{1, 2, \ldots, m\} \).

Moreover, Theorem 2 suggests that a condition, ensuring that \( \ell^p \) is block finitely represented in a subsymmetric unconditional (in particular, symmetric) basis \( \{e_i\} \) of a Banach space so that the unit basis vectors of \( \ell^p \) correspond to blocks of this basis with the same ordered distribution, can be in a natural way expressed using suitable estimates for the norms of an appropriate dilation operator when it is restricted to the set \( \{e_i\} \).

Here, we consider a special class of Banach lattices, the symmetric (in other terminology, rearrangement invariant) sequence spaces \( X \) modelled on \( \mathbb{N} \) (for all necessary definitions see the next section). We will be interested in finding the set of all \( p \) such that \( \ell^p \) is block finitely represented in the unit vector basic sequence \( \{e_i\}_{i=1}^{\infty} \) in \( X \) so that the unit basis vectors of \( \ell^p \) (if \( p = \infty \)) correspond to pairwise disjoint blocks \( u_k, k = 1, 2, \ldots, n \), of the sequence \( \{e_i\} \), with the same ordered distribution. We will say in this case that \( \ell^p \) is symmetrically block finitely represented in the unit vector basic sequence \( \{e_i\}_{i=1}^{\infty} \) in \( X \).

We will further adopt also the following terminology. Sequences \( x = (x_k)_{k=1}^{\infty} \) and \( y = (y_k)_{k=1}^{\infty} \) are said to have the same ordered distribution\(^1\) if there is a permutation \( \pi \) of the set of positive integers such that \( y_{\pi(k)} = x_k \) for all \( k = 1, 2, \ldots \).

Definition 1 Let \( X \) be a symmetric sequence space, \( 1 \leq p \leq \infty \). We say that \( \ell^p \) is symmetrically finitely represented in \( X \) if for every \( n \in \mathbb{N} \) and each \( \varepsilon > 0 \) there exist pairwise disjoint sequences \( x_k \in X, k = 1, 2, \ldots, n \), with the same ordered distribution such that for every \( a = (a_k)_{k=1}^{n} \in \mathbb{R}^n \) we have

\[
(1 + \varepsilon)^{-1} \|a\|_p \leq \left\| \sum_{k=1}^{n} a_k x_k \right\|_X \leq (1 + \varepsilon) \|a\|_p.
\]

Clearly, if \( \ell^p \) is symmetrically block finitely represented in the unit vector basic sequence \( \{e_i\}_{i=1}^{\infty} \) in \( X \), then \( \ell^p \) is symmetrically finitely represented in \( X \).

We will need also the following formally weaker property.

Definition 2 We say that \( \ell^p \) is crudely symmetrically finitely represented in a symmetric sequence space \( X \) if there exists a constant \( C > 0 \) such that for every \( n \in \mathbb{N} \) we

\(^1\) Sometimes such sequences are called equimeasurable.
can find pairwise disjoint sequences \( x_k \in X, \ k = 1, 2, \ldots, n \), with the same ordered distribution such that for every \( a = (a_k)_{k=1}^n \in \mathbb{R}^n \)

\[
C^{-1} \|a\|_p \leq \left\| \sum_{k=1}^n a_k x_k \right\|_X \leq C \|a\|_p.
\]

The set of all \( p \in [1, \infty] \) such that \( \ell^p \) is symmetrically finitely represented (resp. crudely symmetrically finitely represented) in \( X \) we will denote by \( \mathcal{F}(X) \) (resp. \( \mathcal{F}_c(X) \)).

A key role will be played next by the following notion related to spectral properties of linear operators bounded in Banach spaces.

**Definition 3** Let \( T \) be a bounded linear operator on a Banach space \( X \). A sequence \( \{u_n\}_{n=1}^\infty \subset X, \|u_n\| = 1, \ n = 1, 2, \ldots, \) is called an approximate eigenvector corresponding to an approximate eigenvalue \( \lambda \in \mathbb{R} \) for \( T \) if \( \|Tu_n - \lambda u_n\|_X \to 0^2 \).

Clearly, the fact that \( \lambda \) is an approximate eigenvalue of \( T : X \to X \) means exactly that the operator \( T - \lambda I \), where \( I \) is the identity, is not an isomorphism from \( X \) onto its image \( \text{Im} (T - \lambda I) \).

The first main result of this paper (Theorem 3) indicates a direct connection of the problem of a description of the set \( \mathcal{F}(X) \), for a separable symmetric sequence space \( X \), with the identification of the set of approximate eigenvalues of the operator \( D := \tau_1 \sigma_2 \) in \( X \), where \( \sigma_2 \) and \( \tau_1 \) are the dilation and shift operators respectively defined by

\[
\sigma_2 x := \left( x_{\left\lceil \frac{j}{2} \right\rceil} \right)_{n=1}^\infty \quad \text{and} \quad \tau_1 x := (x_{n-1})_{n=1}^\infty, \quad \text{where} \quad x = (x_n)_{n=1}^\infty
\]

(we set \( x_j = 0 \) if \( j \notin \mathbb{N} \)). Specifically, we show that, for every separable symmetric sequence space \( X, \ell^p \) is symmetrically block finitely represented in the unit vector basis \( \{e_k\} \) of \( X \) (equivalently, \( p \in \mathcal{F}(X) \)) if and only if \( 2^{1/p} \) is an approximate eigenvalue of the operator \( D \) in \( X \).

As a consequence of Theorem 3, we prove that, for any symmetric sequence space \( X, \max \mathcal{F}(X) = 1/\alpha_X \) and \( \min \mathcal{F}(X) = 1/\beta_X \) (see Theorem 4). Let us mention that this result is stated without a proof in [16, Theorem 2.b.6] for both sequence and function symmetric spaces (in the case of function symmetric spaces on \( (0, \infty) \) the proof can be found in the paper [2]). In particular, from Theorem 4 it follows that \( \mathcal{F}(X) \neq \emptyset \) for every symmetric sequence space \( X \).

Observe that many properties of the dilation operator \( \sigma_2 \) (and hence those of \( D \)) in a symmetric space \( X \) are determined to a great extent by the values of the so-called Boyd indices \( \alpha_X \) and \( \beta_X \) of the space \( X \). In particular, it can be rather easily shown (see Corollary 3) that the set of all approximate eigenvalues of the operator \( D \) is contained in the interval \( [2^{\alpha_X}, 2^{\beta_X}] \).

---

2 In what follows, we consider real Banach spaces. Note however that every bounded linear operator in a Banach spaces over the complex field has at least one approximate eigenvalue (see, e.g., [21, 12.1])
The second main result of the paper (Corollary 10) shows that for a rather wide class of separable symmetric sequence spaces \( X \) of fundamental type the set of all approximate eigenvalues of the operator \( D \) coincides with the interval \([2^\alpha_X, 2^\beta_X]\). It should be emphasized that the latter condition, imposed on \( X \), is not so restrictive; all the most known and important symmetric sequence spaces, e.g., Orlicz, Lorentz, Marcinkiewicz spaces, are of fundamental type.

As a result, we obtain Theorem 5, which reads that, for any separable symmetric sequence space \( X \) of fundamental type, we have \( \mathcal{F}(X) = \mathcal{F}_c(X) = [1/\beta_X, 1/\alpha_X] \). Observe that, in contrast to Theorem 2, we get rid of the extra condition that \( c_0 \) is not finitely represented in \( X \).

Our approach to the problem of finding the set of approximate eigenvalues of the operator \( D \) in a symmetric sequence space \( X \) is based on its reduction to the similar task for the shift operator \( \tau_1 \) in some Banach sequence lattice \( E_X \) modelled on \( \mathbb{N} \) such that the dyadic block basis \( \left\{ \sum_{i=2^k-1}^{2^k-1} e_i \right\}_{k=1}^{\infty} \) is equivalent in \( X \) to the unit vector basis in \( E_X \) (see Proposition 7). We develop an idea applied by Kalton [9] to the special case of Lorentz function spaces and corresponding weighted \( \ell^p \)-spaces when studying problems related to interpolation theory of operators, in particular, to a characterization of Calderón couples of rearrangement invariant spaces (see also [8, Lemma 2.2]).

In the concluding part of the paper, as an application of the results obtained, we establish that \( \mathcal{F}(X) = [1/\beta_X, 1/\alpha_X] \) if \( X \) is an arbitrary Lorentz space or a separable Orlicz space (see Theorems 6 and 7).

Similar problems for symmetric function spaces on \((0, \infty)\) have been studied in [2] and more recently in [3]. On the one hand, in this case the situation gets a little more complicated than for sequence spaces; in particular, depending on the behaviour the doubling operator \( x(t) \mapsto x(t/2) \) in such a space \( X \) of fundamental type, the set of \( p \), for which \( \ell^p \) is symmetrically finitely represented in \( X \), may be also a union of two intervals. On the other hand, from the technical point of view, the case of function spaces is somewhat simpler, because the operator \( x(t) \mapsto x(t/2) \) is invertible (in contrast to the operator \( D \), which is clearly not surjective).

I would like to thank the referee for his/her valuable comments and remarks.

2 Preliminaries

2.1 Banach sequence lattices

Here, we recall some definitions and results that relate to Banach sequence lattices; for a detailed exposition see, for example, the monographs [4, 10, 16].

A Banach space \( E \) of sequences of real numbers is said to be a Banach sequence lattice (or Banach sequence ideal space) if from \( x = (x_k)_{k=1}^{\infty} \in E \) and \( |y_k| \leq |x_k| \), \( k = 1, 2, \ldots \), it follows that \( y = (y_k)_{k=1}^{\infty} \in E \) and \( \|y\|_E \leq \|x\|_E \).

Note that from the convergence in norm in a Banach sequence lattice \( E \) it follows the coordinate-wise convergence (see, e.g., [10, Theorem 4.3.1]).

If \( E \) is a Banach sequence lattice, then the Köthe dual (or associated) lattice \( E' \) consists of all \( y = (y_k)_{k=1}^{\infty} \in E \) such that
\[ \|y\|_{E'} := \sup \left\{ \sum_{k=1}^{\infty} x_k y_k : \|x\|_E \leq 1 \right\} < \infty. \]

Observe that \( E' \) is complete with respect to the norm \( y \mapsto \|y\|_{E'} \) and is embedded isometrically into (Banach) dual space \( E^* \); moreover, \( E' = E^* \) if and only if \( E \) is separable \[10\], Corollary 6.1.2].

Every Banach sequence lattice \( E \) is continuously embedded into its second Köthe dual \( E'' \) and \( \|x\|_{E''} \leq \|x\|_E \) for \( x \in E \). A Banach sequence lattice \( E \) has the Fatou property (or maximal) if from \( x^{(n)} = (x_k^{(n)})_{k=1}^{\infty} \in E, \ n = 1, 2, \ldots, \sup_{n=1,2,\ldots} \|x^{(n)}\|_E < \infty \) and \( x_k^{(n)} \to x_k \) as \( n \to \infty \) for each \( k = 1, 2, \ldots \) it follows that \( x = (x_k)_{k=1}^{\infty} \in E \) and \( \|x\|_E \leq \lim \inf_{n \to \infty} \|x^{(n)}\|_E \). A Banach sequence lattice \( E \) has the Fatou property if and only if the natural inclusion of \( E \) into \( E'' \) is a surjective isometry \[10\], Theorem 6.1.7].

### 2.2 Symmetric sequence spaces

An important class of Banach lattices is formed by the so-called symmetric spaces.

If \( (u_k)_{k=1}^{\infty} \) is a bounded sequence of real numbers then, in what follows, \( (u^*_k)_{k=1}^{\infty} \) denotes the nonincreasing permutation of the sequence \( (|u_k|)_{k=1}^{\infty} \) defined by

\[ u^*_k := \inf_{\text{card } A=k-1} \sup_{i \in \mathbb{N} \setminus A} |u_i|, \ \ k \in \mathbb{N}. \]

A Banach sequence lattice \( X \) is called **symmetric sequence space** if \( X \subset \ell^\infty \) and the conditions \( y^*_k = x^*_k, \ k = 1, 2, \ldots, \) \( x = (x_k)_{k=1}^{\infty} \in X \) imply that \( y = (y_k)_{k=1}^{\infty} \in X \) and \( \|y\|_X = \|x\|_X \). In what follows, we will assume that any symmetric space is separable or has the Fatou property.

Clearly, if \( X \) is a symmetric sequence space, \( a = (a_k)_{k=1}^{\infty} \in X \) and \( \pi \) is an arbitrary permutation of \( \mathbb{N} \), then the sequence \( a_{\pi} = (a_{\pi(k)})_{k=1}^{\infty} \) belongs to \( X \) and \( \|a_{\pi}\|_X = \|a\|_X \).

Let \( X \) be a symmetric sequence space. The function \( \varphi_X(n) := \|\chi_{\{1,2,\ldots,n\}}\|_X, \ n \in \mathbb{N}, \) where \( \chi_A \) is the characteristic function of a set \( A \), is called the **fundamental function** of \( X \). It is a nondecreasing and positive function such that \( \varphi_X(n)/n \) is nonincreasing.

The most important examples of symmetric sequence spaces are the \( \ell^p \)-spaces, \( 1 \leq p \leq \infty \), with the usual norms

\[ \|a\|_{\ell^p} := \begin{cases} \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}, & 1 \leq p < \infty \\ \sup_{k=1,2,\ldots} |a_k|, & p = \infty \end{cases}. \]

Their generalization, the \( \ell^{p,q} \)-spaces, \( 1 < p < \infty, 1 \leq q \leq \infty \), are equipped with the quasi-norms

\[ \|a\|_{\ell^{p,q}} := \left\{ \sum_{k=1}^{\infty} (a_k^*)^q k^{q/p-1} \right\}^{1/q} \quad \text{if} \quad 1 \leq q < \infty, \]

and
The functional $a \mapsto \|a\|_{\ell^p,q} := \sup_{k=1,2,...} a_k^q 1/p$. The functional $a \mapsto \|a\|_{\ell^p,q}$ does not satisfy the triangle inequality for $p < q \leq \infty$, but it is equivalent to a symmetric norm (see, e.g., [4, Theorem 4.4.3]).

In turn, $\ell^p,q$-spaces, if $1 \leq q < p < \infty$, belong to a wider class of the Lorentz spaces. Let $1 \leq q < \infty$, and let $\{w_k\}_{k=1}^{\infty}$ be a nonincreasing sequence of positive numbers. The Lorentz space $\lambda_q(w)$ (see, e.g., [17] or [15, Chapter 4e]) consists of all sequences $a = (a_k)_{k=1}^{\infty}$ satisfying

$$\|a\|_{\lambda_q(w)} := \left( \sum_{k=1}^{\infty} (a_k^q w_k^q)^{1/q} \right)^{1/q} < \infty.$$ 

Since the classical Hardy-Littlewood inequality (see, e.g., [4, Theorem 2.2.2]) yields

$$\sup_{\pi} \left( \left\| \sum_{k=1}^{\infty} |a_{\pi(k)}|^q w_k^q \right\|^{1/q} \right),$$

where the supremum is taken over all permutations $\pi$ of the set of positive integers, the functional $a \mapsto \|a\|_{\lambda_q(w)}$ defines on the space $\lambda_q(w)$ a symmetric norm. Every Lorentz sequence space is separable and has the Fatou property.

Clearly, the fundamental function of $\lambda_q(w)$ is defined by

$$\phi_{\lambda_q(w)}(n) = \left( \sum_{k=1}^{n} w_k^q \right)^{1/q}, \quad n \in \mathbb{N}. \quad (2)$$

Another natural generalization of the $\ell^p$-spaces are Orlicz spaces (see [15, Chapter 4], [12, 19, 23]). Let $N$ be an Orlicz function, that is, an increasing convex continuous function on $[0, \infty)$ such that $N(0) = 0$ and $\lim_{t \to \infty} N(t) = \infty$. The Orlicz sequence space $l_N$ consists of all sequences $a = (a_k)_{k=1}^{\infty}$ such that

$$\|a\|_N := \inf \left\{ u > 0 : \sum_{k=1}^{\infty} N\left( \frac{|a_k|}{u} \right) \leq 1 \right\} < \infty.$$ 

Without loss of generality, we will assume that $N(1) = 1$. In particular, if $N(s) = s^p$, $1 \leq p < \infty$, we obtain $\ell^p$ with the usual norm. Every Orlicz space $l_N$ has the Fatou property and it is separable if and only if the function $N$ satisfies the $\Delta_2$-condition at zero, i.e.,

$$\limsup_{u \to 0} \frac{N(2u)}{N(u)} < \infty.$$ 

The fundamental function of $l_N$ can be calculated by the formula $\phi_{l_N}(n) = 1/N^{-1}(1/n)$, $n \in \mathbb{N}$, where $N^{-1}$ is the inverse function for $N$.

For more detailed information related to symmetric spaces we refer to the books [4, 13, 16].
2.3 Indices of Banach sequence lattices and symmetric sequence spaces

Let $E$ be a Banach sequence lattice modelled on $\mathbb{N}$ such that the shift operator $\tau_n(x) := (x_{k-n})_{k=1}^{\infty}$, where $x = (x_k)_{k=1}^{\infty}$, is bounded in $E$ for every $n \in \mathbb{Z}$ (we set $x_j = 0$ if $j \notin \mathbb{N}$). We define the following shift exponents:

$$k_+(E) := \lim_{n \to \infty} \| \tau_n \|_E^{1/n} \quad \text{and} \quad k_-(E) := \lim_{n \to \infty} \| \tau_{-n} \|_E^{1/n}.$$  

For each $m \in \mathbb{N}$ by $\sigma_m$ and $\sigma_{1/m}$ we define the dilation operators on the set of all numerical sequences as follows: if $a = (a_n)_{n=1}^{\infty}$, then

$$\sigma_m a = (a_{\frac{m+n}{m}})_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots)$$

and

$$\sigma_{1/m} a = \left( \frac{1}{m} \sum_{k=(n-1)m+1}^{nm} a_k \right)_{n=1}^{\infty}.$$  

These operators are bounded in every symmetric sequence space $X$ and $\| \sigma_m \|_X \leq m$, $\| \sigma_{1/m} \|_X \leq 1$ for each $m \in \mathbb{N}$ (see [13, Theorem II.4.5] or [4, Chapter 3, Exercise 15, p. 178]). This implies, in particular, that $1 \leq \| D \|_X \leq 2$, where $D := \tau_1 \sigma_2$, or equivalently $D(x_k) = \sum_{k=2}^{\infty} \frac{x_{k+1}}{2} e_k$.

The numbers $\alpha_X$ and $\beta_X$ given by

$$\alpha_X := - \lim_{m \to +\infty} \frac{\log_2 \| \sigma_{1/m} \|_X}{\log_2 m} = - \inf_{m \geq 2} \frac{\log_2 \| \sigma_{1/m} \|_X}{\log_2 m} \quad \text{(3)}$$

and

$$\beta_X := \lim_{m \to +\infty} \frac{\log_2 \| \sigma_m \|_X}{\log_2 m} = \inf_{m \geq 2} \frac{\log_2 \| \sigma_m \|_X}{\log_2 m}, \quad \text{(4)}$$

are called the Boyd indices of $X$ (cf. [13, § II.1.1 and § II.4.3]). Clearly,

$$\alpha_X = - \lim_{n \to +\infty} \frac{1}{n} \log_2 \| \sigma_{2^n} \|_X \quad \text{and} \quad \beta_X = \lim_{n \to +\infty} \frac{1}{n} \log_2 \| \sigma_{2^n} \|_X. \quad \text{(5)}$$

Let $\phi_X = \phi_X(n)$, $n \in \mathbb{N}$, be the fundamental function of a symmetric sequence space $X$. Then, we can introduce the following dilation functions of $\phi_X$:

$$M_X^0(n) := \sup_{m \in \mathbb{N}} \frac{\phi_X(m)}{\phi_X(mn)} \quad \text{and} \quad M_X^\infty(n) := \sup_{m \in \mathbb{N}} \frac{\phi_X(mn)}{\phi_X(m)}, \quad n \in \mathbb{N},$$

and the so-called fundamental indices of $X$ by

$$\mu_X = - \lim_{n \to +\infty} \frac{\log_2 M_X^0(n)}{\log_2 n} = - \inf_{n \geq 2} \frac{\log_2 M_X^0(n)}{\log_2 n}.$$
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and

$$\nu_X = \lim_{n \to \infty} \frac{\log_2 M^\infty_X(n)}{\log_2 n} = \inf_{n \geq 2} \frac{\log_2 M^\infty_X(n)}{\log_2 n}$$

[4, Exercise 14 for Chapter 3, p. 178]. Hence, we can write

$$\mu_X = -\lim_{n \to \infty} \frac{1}{n} \log_2 M^0_X(2^n) \quad \text{and} \quad \nu_X = \lim_{n \to \infty} \frac{1}{n} \log_2 M^\infty_X(2^n).$$

From the above definitions it follows easily that $0 \leq \alpha_X \leq \mu_X \leq \nu_X \leq \beta_X \leq 1$ for every symmetric sequence space $X$ (see also [13, § II.4.4]).

We will say that a symmetric sequence space $X$ is of fundamental type whenever the corresponding Boyd and fundamental indices of $X$ coincide, i.e., $\alpha_X = \mu_X$ and $\beta_X = \nu_X$.

The most known and important r.i. spaces, in particular, all Lorentz and Orlicz spaces, are of fundamental type.$^3$

Let us prove the last assertion for a Lorentz space $\lambda_q(w)$. In other words, taking into account formula (2), we need to show that

$$\alpha_{\lambda_q(w)} = -\lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{j \in \mathbb{N}} \left( \frac{\sum_{k=1}^{j} w^q_k}{\sum_{k=1}^{2^j} w^q_k} \right)^{1/q}$$

and

$$\beta_{\lambda_q(w)} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{j \in \mathbb{N}} \left( \frac{\sum_{k=1}^{2^j} w^q_k}{\sum_{k=1}^{2^j} w^q_k} \right)^{1/q}.$$  

We check only (7); equality (6) can be proved similarly. Since $(\sigma_{2^n} x)^* = \sigma_{2^n} (x^*)$, we can (and will) assume that a sequence $x = (x_k)_{k=1}^{\infty}$ is finitely supported and $x = x^*$. Then, applying the Abel transformation, we obtain for some $m \in \mathbb{N}$

$$\|\sigma_{2^n} x\|_{\lambda_q(w)}^q = \sum_{k=1}^{m} \sum_{i=2^n(k-1)+1}^{2^n k} x^q_i \sum_{i=1}^{m-1} (x^q_k - x^q_{k+1}) \sum_{i=1}^{2^n k} w^q_i + x^q_m \sum_{i=1}^{2^n m} w^q_i.$$  

Hence,

$$\|\sigma_{2^n} x\|_{\lambda_q(w)}^q \leq \sup_{j \in \mathbb{N}} \left( \frac{\sum_{i=1}^{j} w^q_i}{\sum_{i=1}^{2^j} w^q_i} \right)^q \left( \sum_{k=1}^{m-1} (x^q_k - x^q_{k+1}) \sum_{i=1}^{k} w^q_i + x^q_m \sum_{i=1}^{m} w^q_i \right)^q \leq \sup_{j \in \mathbb{N}} \left( \frac{\sum_{i=1}^{j} w^q_i}{\sum_{i=1}^{2^j} w^q_i} \right)^q \|x\|_{\lambda_q(w)}^q,$$

$^3$ The first example of a symmetric function space on $[0, 1]$ of non-fundamental type was constructed by Shimogaki in [26].
and consequently
\[
\beta_{l,N}(w) \leq \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{j \in \mathbb{N}} \left( \frac{\sum_{i=1}^{2^j} w_i^q}{\sum_{i=1}^{2^j} w_i^q} \right)^{1/q}.
\]

Since the opposite inequality is immediate, formula (7) is proved.

For the proof of the fact that every Orlicz space is of fundamental type see [5] or [18, Theorem 4.2]. Therefore, since
\[
\alpha_{l,N} = -\lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k=0,1,\ldots} \frac{N^{-1}(2^{-k}-n)}{N^{-1}(2^{-k})} \quad \text{and} \quad \beta_{l,N} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \geq n} \frac{N^{-1}(2^{-k+n})}{N^{-1}(2^{-k})}.
\]

One can readily check also that
\[
\alpha_{l,N} = \inf \left\{ q : \inf_{0<s,t \leq 1} \frac{N(st)}{N(s)N(t)} > 0 \right\} \quad \text{and} \quad \beta_{l,N} = \sup \left\{ q : \sup_{0<s,t \leq 1} \frac{N(st)}{N(s)N(t)^{1/q}} < \infty \right\}.
\]

(8)

2.4 Spreading sequence spaces

A sequence \( \{u_n\}_{n=1}^{\infty} \) in a Banach space \( X \) is called spreading if for every \( n \in \mathbb{N} \) and any integers \( 0 < m_1 < m_2 < \cdots < m_n \) and \( a_j \in \mathbb{R} \) we have
\[
\left\| \sum_{j=1}^{n} a_j u_m \right\|_X = \left\| \sum_{j=1}^{n} a_j x_j \right\|_X.
\]

Throughout, we denote by \( e_n, n \in \mathbb{N} \), the standard unit vectors in sequence spaces and by \( c_{0,0} \) the set of all finitely supported sequences, i.e., such that \( x_n = 0 \) for all sufficiently large \( n \).

A separable Banach sequence lattice \( E \) is spreading if the unit vector basis \( \{e_n\}_{n=1}^{\infty} \) is spreading. Clearly, every separable symmetric sequence space \( X \) is spreading.

For arbitrary \( x = (x_n) \in c_{0,0}, \ y = (y_n) \in c_{0,0} \), we denote by \( x \oplus y \) their disjoint sum. This is an arbitrary sequence whose nonzero entries coincide with all nonzero entries of \( x \) and \( y \). For instance, if \( n_0 = \max\{n \in \mathbb{N} : x_n \neq 0\} \), then for the disjoint sum of \( x \) and \( y \) we can take the sequence:
\[
x \oplus y = \sum_{n=1}^{n_0} x_n e_n + \sum_{n=n_0+1}^{\infty} y_{n-n_0} e_n.
\]

Clearly, if \( E \) is a spreading Banach sequence lattice, the norm \( \|x \oplus y\|_E \) does not depend on a specific choice for \( x \oplus y \).
Let $\varepsilon > 0$. We say that a sequence $x$ is replaceable (resp. $\varepsilon$-replaceable) in a spreading Banach sequence lattice $E$ by a sequence $y$ if for arbitrary $u \in c_{0,0}$, $v \in c_{0,0}$ we have

$$\|u \oplus x \oplus v\|_E = \|u \oplus y \oplus v\|_E$$

(resp.

$$|\|u \oplus x \oplus v\|_E - \|u \oplus y \oplus v\|_E| < \varepsilon$$.)

For a more detailed account of the spreading property and related notions see, for instance, [1, Chapter 11].

By $\|T\|_E$ we will denote the norm of an operator $T$ bounded in a Banach space $E$. We write $f \simeq g$ if $cf \leq g \leq Cg$ for some constants $c > 0$ and $C > 0$ that do not depend on the values of all (or some) arguments of the functions (quasi-norms) $f$ and $g$. From one appearance to another the value of the constant $C$ may change.

3 A connection between symmetric finite representability of $\ell^p$ in symmetric sequence spaces and spectral properties of the operator $D$

Here, we prove the first main result of the paper, which reduces the problem of identification of the set $\mathcal{F}(X)$, where $X$ is a separable symmetric sequence space, to finding the set of approximate eigenvalues of the operator $D$ in $X$.

**Theorem 3** Let $X$ be a separable symmetric sequence space, $1 \leq p \leq \infty$. The following conditions are equivalent:

(a) $\ell^p$ is symmetrically block finitely represented in the unit vector basis $\{e_k\}$ in $X$;
(b) $\ell^p$ is symmetrically finitely represented in $X$;
(c) $\ell^p$ is symmetrically crudely finitely represented in $X$;
(d) $2^{1/p}$ is an approximate eigenvalue of the operator $D : X \rightarrow X$.

**Proof** Since implications $(a) \implies (b) \implies (c)$ are obvious, it remains only to prove that $(c) \implies (d)$.

$(c) \implies (d)$. Let $\ell^p$ is symmetrically crudely finitely represented in $X$. Then, by definition, there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ we can find pairwise disjoint sequences $x_k \in X$, $k = 1, 2, \ldots, 2^n$, with the same ordered distribution, satisfying for all $a_k \in \mathbb{R}$ the inequality

$$C^{-1}\left(\sum_{k=1}^{2^n} |a_k|^p\right)^{1/p} \leq \left\|\sum_{k=1}^{2^n} a_k x_k\right\|_X \leq C\left(\sum_{k=1}^{2^n} |a_k|^p\right)^{1/p} \quad (10)$$

(we assume that $p < \infty$; the case of $p = \infty$ is treated similarly). Since $X$ is a separable symmetric space, we always can ensure by approximation that $x_k \in c_{00}$, $x_k \geq 0$ and, moreover, in view of the definition of the operator $D$ (see Sect. 2.3), a
suitable spreading of the vector $x_1$ allows to achieve that the sequences $y_k := D^{k-1}x_1$, $k = 1, 2, \ldots, n$, are pairwise disjoint. Then, for arbitrary $b_k \in \mathbb{R}$ we have
\[
\| \sum_{k=1}^n b_k y_k \|_X = \| \sum_{k=1}^n b_k 2^{k-1} x_i \|_X,
\]
and from (10) it follows
\[
C^{-1} 2^{(k-1)/p} \leq \| y_k \|_X \leq C \cdot 2^{(k-1)/p} \quad \text{and} \quad C^{-1} n^{1/p} \leq \left\| \sum_{i=1}^n 2^{(1-i)/p} y_i \right\|_X \leq C n^{1/p}.
\]
Therefore, setting $z_k := 2^{(1-k)/p} y_k$, $k = 1, 2, \ldots, n$, and
\[
v_n := n^{-1/p} \sum_{k=1}^n z_k, \quad n = 1, 2, \ldots,
\]
we get
\[
C^{-1} \leq \| z_k \|_X \leq C, \quad C^{-1} n^{1/p} \leq \left\| \sum_{k=1}^n z_k \right\|_X \leq C n^{1/p} \quad (11)
\]
and
\[
C^{-1} \leq \| v_n \|_X \leq C, \quad n = 1, 2, \ldots \quad (12)
\]
Moreover, since
\[
Dz_k = 2^{(1-k)/p} Dy_k = 2^{(1-k)/p} y_{k+1} = 2^{1/p} z_{k+1}, \quad k = 1, 2, \ldots, n - 1,
\]
putting $D_\lambda := D - \lambda I$, where $\lambda = 2^{1/p}$ and $I$ is the identity in $X$, we have
\[
D_\lambda v_n = n^{-1/p} \left( \sum_{k=1}^n Dz_k - \lambda \sum_{k=1}^n z_k \right)
\]
\[
= n^{-1/p} \left( \lambda \sum_{k=1}^{n-1} z_{k+1} + Dz_n - \lambda \sum_{k=1}^n z_k \right)
\]
\[
= n^{-1/p} (Dz_n - \lambda z_1).
\]
Therefore, taking into account the inequality $1 \leq \|D\|_X \leq 2$ (see Sect. 2.3) and the first inequality in (11), we obtain $\|D_\lambda v_n\|_X \leq 4 C n^{-1/p}$, whence $\|D_\lambda v_n\|_X \to 0$ as $n \to \infty$. If now $w_n := v_n/\|v_n\|_X$, by (12), we see that $\|D_\lambda w_n\|_X \to 0$ as well. So, $\lambda = 2^{1/p}$ is an approximate eigenvalue of the operator $D$, which completes the proof of (d).

(d) $\implies$ (a). Suppose $\lambda = 2^{1/p}$ is an approximate eigenvalue of the operator $D$ and $\{g_l\}_{l=1}^\infty \subset X, \|g_l\|_X = 1, l = 1, 2, \ldots$, is the corresponding approximate eigenvector. Since $X$ is separable, there is no loss of generality in assuming that $g_l \in c_00, l = 1, 2, \ldots$, and
\[ \|Dg_l - \lambda g_l\|_X \leq \frac{1}{l}, \quad l = 1, 2, \ldots \]  

(13)

For every \( l \in \mathbb{N} \), we introduce a new sequence space \( \mathcal{E}_l \), which is being the completion of \( c_{00} \) with respect to the norm defined by

\[
\left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_l} := \| a_1 g_1 \oplus a_2 g_1 \oplus \cdots \oplus a_m g_l \|_X, \quad m \in \mathbb{N},
\]

(14)

where \( x \oplus y \) is the disjoint sum of sequences \( x \) and \( y \) (see Sect. 2.4).

Clearly, the unit vector basis \( \{ e_j \}_{j=1}^{\infty} \) in \( \mathcal{E}_l \) is isometrically equivalent to a sequence of disjoint copies of \( g_l \) in \( X \). In particular, \( \| e_j \|_{\mathcal{E}_l} = \| g_l \|_X = 1 \) for all \( j, l \in \mathbb{N} \). Hence, arranging arbitrarily all elements of \( c_{0,0} \) with rational coordinates in a sequence of collections \( (a_1^{(k)})_{k=1}^{\infty}, \ldots, a_m^{(k)}_{k=1}^{\infty} \), we can construct a family of infinite sequences \( (l^m_{i})_{i=1}^{\infty} \subset \mathbb{N} \), \( m = 1, 2, \ldots \), such that \( (l^1_{i})_{i=1}^{\infty} \supset (l^2_{i})_{i=1}^{\infty} \supset \cdots \) and for every \( m = 1, 2, \ldots \) the limit

\[
\lim_{i \to \infty} \left\| \sum_{j=1}^{r_k} a_j^{(k)} e_j \right\|_{\mathcal{E}_m}
\]

exists for all \( 1 \leq k \leq m \). The standard diagonal procedure yields then a sequence \( (l_s)_{s=1}^{\infty} \) that is included in each sequence \( (l^m_{i})_{i=1}^{\infty} \) up to finitely many terms. Hence, routine arguments based on the fact of density of the rationals in \( \mathbb{R} \) show that the limit

\[
\lim_{s \to \infty} \left\| \sum_{j=1}^{\infty} a_j e_j \right\|_{\mathcal{E}_{l_s}}
\]

exists for arbitrary \( a = (a_j) \in c_{0,0} \). Therefore, we can introduce a new norm \( \| a \|_{\mathcal{E}} \) on \( c_{00} \), which is equal to the latter limit. One can easily see that the completion of \( c_{00} \) in this norm, denoting by \( \mathcal{E} \), is a symmetric sequence space.

Let us prove that for every \( \epsilon > 0 \) and \( m \in \mathbb{N} \) there exists \( s_0 \in \mathbb{N} \) such that for all \( s \geq s_0 \) and arbitrary \( a_j \in \mathbb{R} \) we have

\[
(1 + \epsilon)^{-1} \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{l_s}} \leq \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}} \leq (1 + \epsilon) \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{l_s}}
\]

(15)

(cf. [24, Proposition 1.2]).

First, for every \( s \in \mathbb{N} \) and all \( a_j \in \mathbb{R} \), \( j = 1, 2, \ldots, m \), it holds

\[
\left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}} \leq m \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{l_s}}.
\]

(16)

Let \( \delta > 0 \) be such that \( \delta(m + 2 + \delta) < \epsilon \). Suppose that sequences \((b_j^1), (b_j^2), \ldots, (b_j^r)\) form a \( \delta \)-net of the set
\[ B := \left\{ (a_j) \in \mathbb{R}^m : \sum_{j=1}^{m} |a_j| \leq m \right\} \]

with respect to the \( \ell_1 \)-norm. Since

\[
\left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_1}} \leq \sum_{j=1}^{m} |a_j| \quad \text{and} \quad \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_2}} \geq \max_{j=1,2,\ldots,m} |a_j| \geq \frac{1}{m} \sum_{j=1}^{m} |a_j|
\]

for all \( s \in \mathbb{N} \) and \( a_j \in \mathbb{R}, j = 1, 2, \ldots, m \), then

\[ B \supseteq \bigcup_{s=1}^{\infty} \left\{ (a_j) \in \mathbb{R}^m : \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_1}} = 1 \right\} \]

and \( \{ (b_j^1), (b_j^2), \ldots, (b_j^r) \} \) is a \( \delta \)-net of the surface of the unit ball

\[ \left\{ (a_j) \in \mathbb{R}^m : \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_2}} = 1 \right\} \]

in \( \mathcal{E}_{i_1} \) with respect to the \( \mathcal{E}_{i_2} \)-norm for every \( s \in \mathbb{N} \). Moreover, by definition of the \( \mathcal{E} \)-norm, there is \( s_0 \in \mathbb{N} \) such that for all \( s \geq s_0 \) and \( k = 1, 2, \ldots, r \) we have

\[
(1 + \delta)^{-1} \left\| \sum_{j=1}^{m} b_j^k e_j \right\|_{\mathcal{E}_{i_1}} \leq \left\| \sum_{j=1}^{m} b_j^k e_j \right\|_{\mathcal{E}} \leq (1 + \delta) \left\| \sum_{j=1}^{m} b_j^k e_j \right\|_{\mathcal{E}_{i_2}} \tag{17}
\]

Fix \( s \geq s_0 \). By homogeneity, it suffices to prove inequality (15) in the case when \( \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_1}} = 1 \). First, there is a positive integer \( k_0 \) such that \( 1 \leq k_0 \leq r \) and

\[
\left\| \sum_{j=1}^{m} (a_j - b_j^{k_0}) e_j \right\|_{\mathcal{E}_{i_1}} \leq \delta.
\]

Then, combining this together with inequalities (16), (17) and the choice of \( \delta \), we get

\[
\left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}} \leq \left\| \sum_{j=1}^{m} (a_j - b_j^{k_0}) e_j \right\|_{\mathcal{E}} + \left\| \sum_{j=1}^{m} b_j^{k_0} e_j \right\|_{\mathcal{E}} \\
\leq m \left\| \sum_{j=1}^{m} (a_j - b_j^{k_0}) e_j \right\|_{\mathcal{E}} + (1 + \delta) \left\| \sum_{j=1}^{m} b_j^{k_0} e_j \right\|_{\mathcal{E}_{i_2}} \\
\leq m \delta + (1 + \delta) \left( \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_1}} + \delta \right) \\
= m \delta + (1 + \delta)^2 < 1 + \epsilon.
\]

Similarly,
A characterization of $\ell^p$-spaces symmetrically finitely…

$$\left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}} \geq \left\| \sum_{j=1}^{m} b_j^{k_0} e_j \right\|_{\mathcal{E}} - \left\| \sum_{j=1}^{m} (a_j - b_j^{k_0}) e_j \right\|_{\mathcal{E}}$$

$$\geq (1 - \delta) \left\| \sum_{j=1}^{m} b_j^{k_0} e_j \right\|_{\mathcal{E}_{i_k}} - m \left\| \sum_{j=1}^{m} (a_j - b_j^{k_0}) e_j \right\|_{\mathcal{E}_{i_k}}$$

$$\geq (1 - \delta) \left( \left\| \sum_{j=1}^{m} a_j e_j \right\|_{\mathcal{E}_{i_k}} - \delta \right) - m\delta$$

$$= (1 - \delta)^2 - m\delta > 1 - \varepsilon.$$ 

Thus, inequality (15) is proved.

In view of (14) and (15), for every $\varepsilon > 0$, $m \in \mathbb{N}$ and any pairwise disjoint sequences $v_k \in c_{00}$, $k = 1, 2, \ldots, m$, with the same ordered distribution there exist pairwise disjoint sequences $w_k \in c_{00}$, $k = 1, 2, \ldots, m$, with the same ordered distribution such that for arbitrary $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, m$,

$$(1 + \varepsilon)^{-1} \left\| \sum_{k=1}^{m} a_k w_k \right\|_X \leq \left\| \sum_{k=1}^{m} a_k v_k \right\|_{\mathcal{E}} \leq (1 + \varepsilon) \sum_{k=1}^{m} a_k w_k \left\| \right|_X. \quad (18)$$

We show now that the sum $e_1 + e_2$ is replaceable in $\mathcal{E}$ by $\lambda e_1$ (see Sect. 2.4). To this end, we first observe that, for all $l \in \mathbb{N}$, the latter sum is $1/l$-replaceable in $\mathcal{E}_l$ by $\lambda e_1$. Indeed, if $c = \sum_i c_i e_i \in c_{00}$, $d = \sum_i d_i e_i \in c_{00}$ and $c' = \oplus \sum_i c_i g_i$, $d' = \oplus \sum_i d_i g_i$, then, taking into account that the space $X$ is symmetric and applying (13), we get

$$\left\| c \oplus (e_1 + e_2) \oplus d \right\|_{\mathcal{E}_l} - \left\| c \oplus \lambda e_1 \oplus d \right\|_{\mathcal{E}_l}$$

$$= \left\| c' \oplus g_l \oplus g_l \oplus d' \right\|_X - \left\| c' \oplus \lambda g_l \oplus d' \right\|_X$$

$$\leq \left\| Dg_l - \lambda g_l \right\|_X \leq \frac{1}{l}.$$ 

Thus, by (15), for all $s \in \mathbb{N}$ large enough

$$\left\| c \oplus (e_1 + e_2) \oplus d \right\|_{\mathcal{E}} - \left\| c \oplus \lambda e_1 \oplus d \right\|_{\mathcal{E}}$$

$$\leq \left\| c \oplus (e_1 + e_2) \oplus d \right\|_{\mathcal{E}_{i_k}} - \left\| c \oplus \lambda e_1 \oplus d \right\|_{\mathcal{E}_{i_k}}$$

$$+ \varepsilon (1 + \varepsilon) \left( \left\| c \oplus (e_1 + e_2) \oplus d \right\|_{\mathcal{E}} + \left\| c \oplus \lambda e_1 \oplus d \right\|_{\mathcal{E}} \right)$$

$$\leq \frac{1}{l_s} + \varepsilon (1 + \varepsilon) \left( \left\| c \oplus (e_1 + e_2) \oplus d \right\|_{\mathcal{E}} + \left\| c \oplus \lambda e_1 \oplus d \right\|_{\mathcal{E}} \right).$$

Since the right-hand side in the last inequality can be made arbitrarily small, we obtain

$$\left\| c \oplus (e_1 + e_2) \oplus d \right\|_{\mathcal{E}} = \left\| c \oplus \lambda e_1 \oplus d \right\|_{\mathcal{E}}.$$

Hence, if $\lambda = 1$, it can be easily deduced that $\| e_1 + e_2 + \cdots + e_n \|_{\mathcal{E}} = 1$ for all $n \in \mathbb{N}$, which implies that $\mathcal{E}$ is isometric to $c_0$. Thus, applying (18) with $v_k = e_k$, $k = 1, 2, \ldots, n$, for arbitrary $\varepsilon > 0$ and $n \in \mathbb{N}$ we can find pairwise disjoint sequences
Let us consider on \( E \) two linear operators \( D_2 \) and \( D_3 \) given by

\[
D_2 e_q = e_{q/2} + e_{(q+1)/2}, \quad q \in \mathbb{Q}_0,
\]

and

\[
D_3 e_q = e_{q/3} + e_{(q+1)/3} + e_{(q+2)/3}, \quad q \in \mathbb{Q}_0.
\]

Then, since \( E(\mathbb{Q}_0) \) is a separable space, the operators \( D_2 \) and \( D_3 \), being in a sense counterparts of the dilation operators \( \sigma_2 \) and \( \sigma_3 \), are bounded on \( E(\mathbb{Q}_0) \) and moreover \( 1 \leq \|D_2\|_{E(\mathbb{Q}_0)} \leq 2 \) and \( 1 \leq \|D_3\|_{E(\mathbb{Q}_0)} \leq 3 \). In contrast to \( \sigma_2 \) and \( \sigma_3 \), the operators \( D_2 \) and \( D_3 \) possess also the following disjointness property.

**Lemma 1** For every \( l, m \in \mathbb{N} \) the sequence \( D_2^l D_3^m e_{1/6} \) is a sum of \( 2^l 3^m \) different elements of the unit vector basis. Moreover, the sequences \( D_2^l D_3^m e_{1/6} \) and \( D_2^l D_3^m e_{1/6} \) are disjoint if at least one of the conditions \( l \neq l_1 \) or \( m \neq m_1 \) is satisfied.

We postpone for a moment the proof of Lemma 1 and proceed with that of the theorem.

Let
\[ u_n := n^{-2/p} \sum_{j=1}^{n} \sum_{k=1}^{n} 2^{-i/p} 3^{-k/p} D_j^k e_{1/6}, \quad n = 1, 2, \ldots. \]

Then, according to Lemma 1,
\[ D_j^2 D_3^k e_{1/6} = \sum_{q \in R_{jk}} e_q, \]
where the sets \( R_{jk} \subseteq Q_0 \) are mutually disjoint and \( \text{card} R_{jk} = 2^j 3^k, \quad j, k = 1, \ldots, n. \) Moreover, since \( \|D_j^2 D_3^k e_{1/6}\|_{\ell^p} = 2^j 3^k, \quad j, k = 1, \ldots, n, \) we have \( \|u_n\|_{\ell^p} = 1. \) Observe also that
\[ 2^{-1/p} D_2 u_n - u_n = n^{-2/p} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} 2^{-(j+1)/p} 3^{-k/p} D_j^{i+1} D_3^k e_{1/6} \right) - \sum_{j=1}^{n} \sum_{k=1}^{n} 2^{-i/p} 3^{-k/p} D_j^k D_3^k e_{1/6} \]
\[ = n^{-2/p} \left( - 2^{-1/p} \sum_{k=1}^{n} 3^{-k/p} D_2 D_3^k e_{1/6} + 2^{-(n+1)/p} \sum_{k=1}^{n} 3^{-k/p} D_2^{n+1} D_3^k e_{1/6} \right). \]

Hence, taking into account that the sequences \( D_2 D_3^k e_{1/6}, \quad D_2^{n+1} D_3^k e_{1/6}, \quad k = 1, 2, \ldots, n \) are pairwise disjoint, one can readily check that
\[ \|2^{-1/p} D_2 u_n - u_n\|_{\ell^p} = 2^{1/p} n^{-1/p}, \quad n \in \mathbb{N}. \]

Similarly,
\[ \|3^{-1/p} D_3 u_n - u_n\|_{\ell^p} = 3^{1/p} n^{-1/p}, \quad n \in \mathbb{N}. \]

Therefore, since the \( \mathcal{E} \)- and \( \ell^p \)-norms are equivalent on \( \mathcal{E} \), we get
\[ \lim_{n \to \infty} \|D_2 u_n - 2^{-1/p} u_n\|_\mathcal{E} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|D_3 u_n - 3^{-1/p} u_n\|_\mathcal{E} = 0. \]

Next, arguing in the same way as when going from the space \( X \) to the space \( \mathcal{E} \), we can construct, starting from \( \mathcal{E} \), the space \( \mathcal{E}' \) such that the sums \( e_1 + e_2 \) and \( e_1 + e_2 + e_3 \) will be replaceable in \( \mathcal{E}' \) by the elements \( 2^{1/p} e_1 \) and \( 3^{1/p} e_1 \), respectively. Then, by [1, Lemma 11.3.11(ii)], the space \( \mathcal{E}' \) is isometric to \( \ell^p \) and hence the formulae similar to (14) and (15), as above, imply that for every \( \epsilon > 0 \) and \( n \in \mathbb{N} \) there exist pairwise disjoint sequences \( u_1, u_2, \ldots, u_n \) from \( c_{0,0} \) with the same ordered distribution satisfying
\[ (1 + \epsilon)^{-1} \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} \leq \left\| \sum_{k=1}^{n} a_k u_k \right\|_\mathcal{E} \leq (1 + \epsilon) \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p}. \]
for arbitrary $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$. Furthermore, taking $u_k$ for $v_k$ in (18), for any $\varepsilon > 0$ and $n \in \mathbb{N}$ we can find pairwise disjoint sequences $w_1, w_2, \ldots, w_n$ from $e_{0,0}$ with the same ordered distribution such that

$$(1 + \varepsilon)^{-1} \left\| \sum_{k=1}^{n} a_k w_k \right\|_\varepsilon \leq \left\| \sum_{k=1}^{n} a_k w_k \right\|_X \leq (1 + \varepsilon) \left\| \sum_{k=1}^{n} a_k w_k \right\|_\varepsilon,$$

where again we can assume that $w_k, k = 1, 2, \ldots, n$, are subsequent blocks of the unit vector basis. Combining the last inequalities, we get

$$(1 + \varepsilon)^{-2} \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} \leq \left\| \sum_{k=1}^{n} a_k w_k \right\|_X \leq (1 + \varepsilon)^2 \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p}.$$

Since $\varepsilon > 0$ and $n \in \mathbb{N}$ are arbitrary, this concludes the proof of the theorem. \hfill \Box

**Proof of Lemma 1** First, for each $q \in \mathbb{Q}_0$ we have

$$D^l_{2^l} e_q = \sum_{i=0}^{2^{l-1}} e_{(q+i)2^{-l}} \quad \text{and} \quad D^m_{3^m} e_q = \sum_{j=0}^{3^{m-1}} e_{(q+j)3^{-m}}.$$

Therefore, to prove the first assertion of the lemma, it suffices to show that from the equation

$$\left( \frac{1}{6} + i \right) 2^{-l} + j = \left( \frac{1}{6} + i_1 \right) 2^{-l} + j_1,$$

for some $l \in \mathbb{N}$, $m \in \mathbb{N}$, $1 \leq i < 2^l$, $1 \leq i_1 < 2^l$, $1 \leq j < 3^m$ and $1 \leq j_1 < 3^m$, it follows that $i = i_1$ and $j = j_1$. Indeed, an easy calculation shows that $i - i_1 = 2^l(j_1 - j)$. Since $|i - i_1| < 2^l$, we deduce $j = j_1$ and hence $i = i_1$.

Now, we prove the second assertion. Clearly, supports of the sequences $D^l_{2^l} D^m_{3^m} e_q$ and $D^l_{2^l} D^m_{3^m} e_q$ are overlapped if and only if

$$\frac{(q + i)2^{-l} + j}{3^m} = \frac{(q + i_1)2^{-l} + j_1}{3^m},$$

or equivalently

$$\frac{q + i + 2^lj}{2^l 3^m} = \frac{q + i_1 + 2^l j_1}{2^l 3^m},$$

for some $l \in \mathbb{N}$, $m \in \mathbb{N}$, $l_1 \in \mathbb{N}$, $m_1 \in \mathbb{N}$, $1 \leq i < 2^l$, $1 \leq j < 3^m$, $1 \leq i_1 < 2^{l_1}$ and $1 \leq j_1 < 3^{m_1}$ such that at least one of the conditions $l \neq l_1$ or $m \neq m_1$ holds. Then, inserting into the latter equation $q = 1/6$, we obtain

$$\frac{1 + 6i + 2^l \cdot 6j}{2^l + 13^{m+1}} = \frac{1 + 6i_1 + 2^{l_1} \cdot 6j_1}{2^{l_1} + 13^{m_1+1}}.$$
Observe that both numbers $1 + 6i + 2l \cdot 6j$ and $1 + 6i + 2l' \cdot 6j_1$ are not divisible by 2 and 3. Hence, $l = l_1$ and $m = m_1$, and so the second assertion of the lemma is proved.

Next, we are going to prove that the set of approximate eigenvalues of the operator $D$ is contained in the interval $[2^{\alpha_X}, 2^{\beta_X}]$, where $\alpha_X$ and $\beta_X$ are the Boyd indices of $X$, while $2^{\alpha_X}$ and $2^{\beta_X}$ are always approximate eigenvalues of $D$.

**Proposition 2** Let $X$ be a symmetric sequence space. Then, the operator $D_\lambda = D - \lambda I$, $\lambda > 0$, is an isomorphic mapping in $X$ whenever $\lambda \notin [2^{\alpha_X}, 2^{\beta_X}]$. Moreover, $D_\lambda$ is not isomorphic if $\lambda$ is equal to $2^{\beta_X}$ or $2^{\alpha_X}$.

**Proof** First, let $\lambda > 2^{\beta_X}$. Then, by the definition of $\beta_X$ (see (5)), for all sufficiently large positive integers $n$ we have

$$\|D^n\|_X = \|\sigma_{2^n}\|_X < \lambda^n.$$ 

Therefore, the spectral radius $r(D)$ of the operator $D$ satisfies the inequality

$$r(D) = \lim_{n \to \infty} \|D^n\|_X^{1/n} < \lambda,$$ 

and hence the operator $D_\lambda$ is an isomorphism from $X$ onto $X$.

If $0 < \lambda < 2^{\alpha_X}$, similarly for the operator $D^{-1} := \sigma_{1/2} \tau_{-1}$ and for all $n \in \mathbb{N}$ large enough it holds

$$\|D^{-n}\|_X = \|\sigma_{2^{-n}}\|_X \leq \lambda^{-n},$$ 

whence the operator $D^{-1} - \lambda^{-1} I$ is an isomorphism from $X$ onto $X$. Then, in particular, for some $c > 0$ and all $x \in X$

$$\|(D^{-1} - \lambda^{-1} I)x\|_X \geq c\|x\|_X.$$ 

Consequently, since

$$D^{-1}D = \sigma_{1/2} \tau_{-1} \sigma_2 = \sigma_{1/2}^2 = I,$$ 

we have

$$\|D_\lambda x\|_X = \lambda \|(D^{-1} - \lambda^{-1} I)Dx\|_X \geq \lambda c\|Dx\|_X \geq \lambda c\|x\|_X,$$

and the first assertion of the proposition is obtained.

Let us prove the second assertion. Suppose first $\lambda = 2^{\beta_X}$. Then, by (4),

$$\|D^n\|_X = \|\sigma_{2^n}\|_X \geq 2^{n\beta_X}, \quad n \in \mathbb{N}.$$ 

Next, as in the proof of Lemma 11.3.12 in [1], we write

$$\lim_{n \to \infty} \|(n + 1)2^{-n\beta_X} D^n\|_X = \infty.$$
and then the Uniform Boundedness principle implies the existence of an element $a_0 \in X$, $\|a_0\|_X = 1$, such that

$$
\limsup_{n \to \infty} \|(n + 1)2^{-n\beta_x}D^n a_0\|_X = \infty.
$$

(19)

Clearly, $a_0$ may be assumed to be nonnegative.

Since $r(D) = 2^{\beta_x}$, the operator $D - rI$ is invertible in $X$ if $r > 2^{\beta_x}$. Consequently, $(D - rI)^{-2}$ can be represented for such a number $r$ as the following (converging) operator series:

$$(D - rI)^{-2} = \frac{1}{r^2} \sum_{n=0}^{\infty} (n + 1)r^{-n}D^n.$$

Since $a_0 \geq 0$ and $D \geq 0$, we conclude that

$$\|(D - rI)^{-2}a_0\|_X \geq r^{-2}\|(n + 1)r^{-n}D^n a_0\|_X$$

for every $r > 2^{\beta_x}$ and all $n \in \mathbb{N}$. Hence, by (19),

$$\lim_{r \to 2^{\beta_x}} \|(D - rI)^{-2}a_0\|_X = \infty.$$

Clearly, then there exists a sequence $\{r_n\}$, $r_n > 2^{\beta_x}$ and $r_n \to 2^{\beta_x}$ such that either

$$\lim_{n \to \infty} \|(D - r_n I)^{-1}a_0\|_X = \infty,$$

or

$$\lim_{n \to \infty} \frac{\|(D - r_n I)^{-2}a_0\|_X}{\|(D - r_n I)^{-1}a_0\|_X} = \infty.$$

It is easy to see that in either case, we can find a sequence $\{g_n\}_{n=1}^{\infty} \subset X$, $\|g_n\|_X = 1$, such that

$$\lim_{n \to \infty} \|(D - r_n I)g_n\|_X = 0.$$

Surely, since $r_n \to 2^{\beta_x}$, this implies that the operator $D_\lambda$ is not isomorphic in $X$ if $\lambda = 2^{\beta_x}$.

Let us proceed now with the case when $\lambda = 2^{\alpha_x}$. Then, in view of (3), we have

$$\|(D^{-1})^n\|_{X \to X} = \|\sigma_{2^{-n}}\|_{X \to X} \geq 2^{-n\alpha_x}, \quad n \in \mathbb{N},$$

and so, as in the preceding case,

$$\limsup_{n \to \infty} \|(n + 1)2^{n\alpha_x}D^{-n}b_0\|_X = \infty,$$

(20)

for some $b_0 \in X$ such that $\|b_0\|_X = 1$ and $b_0 \geq 0$. Since $r(D^{-1}) = 2^{-\alpha_x}$, the operator $D^{-1} - t^{-1}I$ is invertible in $X$ if $t < 2^{\alpha_x}$. Therefore, one can readily check that for $0 < t < 2^{\alpha_x}$
\[(D^{-1} - t^{-1}I)^{-2} = t^2 \sum_{n=0}^{\infty} (n+1)t^n D^{-n}. \quad (21)\]

Hence,

\[\|(D^{-1} - t^{-1}I)^{-2}b_0\|_X \geq t^2\|(n+1)t^n D^{-n}b_0\|_X\]

for every \(0 < t < 2^{\alpha_X}\) and all \(n \in \mathbb{N}\). Combining this together with (20), we get

\[\lim_{t \to 2^{\alpha_X}} \|(D^{-1} - t^{-1}I)^{-2}b_0\|_X = \infty,\]

which implies that there exists a sequence \(\{t_n\}, t_n < 2^{\alpha_X} \text{ and } t_n \to 2^{\alpha_X}\), such that either

\[\lim_{n \to \infty} \|(D^{-1} - t_n^{-1}I)^{-1}b_0\|_X = \infty, \quad (22)\]

or

\[\lim_{n \to \infty} \frac{\|(D^{-1} - t_n^{-1}I)^{-2}b_0\|_X}{\|(D^{-1} - t_n^{-1}I)^{-1}b_0\|_X} = \infty. \quad (23)\]

Assume first that we have (22). Denoting \(\gamma_k := \|(D^{-1} - t_k^{-1}I)^{-1}b_0\|_X^{-1}, k \in \mathbb{N}\), we define

\[h_k := \gamma_k(D^{-1} - t_k^{-1}I)^{-1}b_0.\]

Then, \(\gamma_k \to 0\) and hence

\[\lim_{k \to \infty} \|(D^{-1} - t_k^{-1}I)h_k\|_X = 0 \quad \text{and} \quad \|h_k\|_X = 1, k \in \mathbb{N}. \quad (24)\]

Moreover, since

\[h_k = \gamma_k(D^{-1} - t_k^{-1}I)^{-1}b_0 = \gamma_k \sum_{n=0}^{\infty} t_k^n D^{-n}b_0,\]

we have

\[Dh_k = \gamma_k \sum_{n=0}^{\infty} t_k^n D^{-n+1}b_0 = \gamma_k Db_0 + t_k h_k.\]

Therefore, it follows that

\[h_k = t_k^{-1}D(h_k - \gamma_k b_0) = Df_k, \quad \text{where} \quad f_k := t_k^{-1}(h_k - \gamma_k b_0), \quad k = 1, 2, \ldots,\]

and so

\[(D^{-1} - t_k^{-1}I)h_k = (D^{-1} - t_k^{-1}I)Df_k = -t_k^{-1}(D - t_kI)f_k.\]

Thus, since \(\gamma_k \to 0\) and \(t_k \to 2^{\alpha_X}\) as \(k \to \infty\), from (24) it follows that \(\|f_k\|_X \leq 1\) for \(k \in \mathbb{N}\) large enough and
\[
\lim_{k \to \infty} \| (D - 2^nx) f_k \|_X = 0.
\]

As a result, the operator \( D \) fails to be isomorphic for \( \lambda = 2^x \).

Suppose now that (22) does not hold. Since
\[
1 = \| b_0 \|_X \leq \| D^{-1} - t_n^{-1} b_0 \|_X \| (D^{-1} - t_n^{-1})^{-1} b_0 \|_X \leq C \| (D^{-1} - t_n^{-1})^{-1} b_0 \|_X
\]
for some \( C > 0 \) and all \( n \in \mathbb{N} \), passing to a subsequence if it is necessary, we can assume that
\[
0 < \liminf_{n \to \infty} \| (D^{-1} - t_n^{-1})^{-1} b_0 \|_X \leq \limsup_{n \to \infty} \| (D^{-1} - t_n^{-1})^{-1} b_0 \|_X < \infty.
\]

Moreover, we know that (23) holds. Consequently, putting
\[
h_k := \delta_k (D^{-1} - t_k^{-1} I)^{-2} b_0,
\]
where \( \delta_k := \| (D^{-1} - t_k^{-1} I)^{-2} b_0 \|_X^{-1} \), \( k \in \mathbb{N} \), as in the preceding case, we get (24).

Furthermore, by (21),
\[
h_k = \delta_k^2 \sum_{n=0}^{\infty} (n+1) t_k^n D^{-n} b_0,
\]
and hence
\[
Dh_k = \delta_k^2 \sum_{n=0}^{\infty} (n+1) t_k^n D^{-n+1} b_0 = \delta_k^2 \left( D b_0 + t_k \sum_{j=0}^{\infty} t_k^j D^{-j} b_0 \right)
\]
\[
= t_k h_k + \delta_k^2 \left( D b_0 + t_k \sum_{j=0}^{\infty} t_k^j D^{-j} b_0 \right) = t_k h_k + \delta_k^2 D b_0 + t_k v_k,
\]
where
\[
v_k := \delta_k^2 \sum_{j=0}^{\infty} t_k^j D^{-j} b_0 = \delta_k^2 (D^{-1} - t_k^{-1} I)^{-1} b_0.
\]

Therefore,
\[
h_k = t_k^{-1} D f_k - v_k, \quad \text{where} \quad f_k := h_k - \delta_k^2 t_k b_0, \quad k = 1, 2, \ldots,
\]
whence, in view of the definition of \( v_k \), we get
\[
(D^{-1} - t_k^{-1} I) h_k = t_k^{-1} (D^{-1} - t_k^{-1} I) D f_k - (D^{-1} - t_k^{-1} I) v_k
\]
\[
= - t_k^{-2} (D - t_k I) f_k - (D^{-1} - t_k^{-1} I) v_k
\]
\[
= - t_k^{-2} (D - t_k I) f_k - \delta_k^2 t_k^2 b_0.
\]
From (25) and (23) it follows that $\delta_k \to 0$. Combining this together with (24) and (26), we infer that $\|f_k\|_X \approx 1$ for all $k \in \mathbb{N}$ large enough and that $(D - t_k I)f_k \to 0$ in $X$. Therefore, since $t_k \to 2^{\alpha_x}$, we obtain

$$\lim_{k \to \infty} \|(D - 2^{\alpha_x} I)f_k\|_X = 0,$$

whence $D_\lambda$ is not isomorphic for $\lambda = 2^{\alpha_x}$. This completes the proof. $\square$

From Proposition 2 it follows

**Corollary 3** Suppose $X$ is a symmetric sequence space. Then, the set of approximate eigenvalues of the operator $D$ is contained in the interval $[2^{\alpha_x}, 2^{\beta_x}]$ and the numbers $2^{\alpha_x}$ and $2^{\beta_x}$ are approximate eigenvalues of $D$.

We get also the following result, which implies in particular that the set $\mathcal{F}(X)$ is non-empty for every symmetric sequence space $X$.

**Theorem 4** If $X$ is an arbitrary symmetric sequence space, then $\max \mathcal{F}(X) = 1/\alpha_X$ and $\min \mathcal{F}(X) = 1/\beta_X$.

**Proof** If $X$ is separable, the claim follows from Theorem 3 and Proposition 2. Otherwise, $X$ has the Fatou property. One can easily see that then $\|\sigma_m\|_X = \|\sigma_m\|_{X_0}$ and $\|\sigma_{1/m}\|_X = \|\sigma_{1/m}\|_{X_0}$ for each $m \in \mathbb{N}$, where $X_0$ is the separable part of $X$ (see Sect. 2.2). Thus, $\alpha_X = \alpha_{X_0}$ and $\beta_X = \beta_{X_0}$. We can assume that $E \neq L_\infty$. Therefore, $X_0$ is a separable symmetric space and so $\max \mathcal{F}(X_0) = 1/\alpha_X$ and $\min \mathcal{F}(X_0) = 1/\beta_X$. Since $X_0$ is a subspace of $X$, then we infer that $\max \mathcal{F}(X) \geq 1/\alpha_X$ and $\min \mathcal{F}(X) \leq 1/\beta_X$. On the other hand, if $\ell^p$ is symmetrically finitely represented in $X$, then from the definition of the Boyd indices it follows immediately that $1/\beta_X \leq p \leq 1/\alpha_X$, and hence the desired result follows. $\square$

4 On some correspondence between symmetric sequence spaces and Banach sequence lattices

As follows from Theorem 3, to find the set of all $p$ such that $\ell^p$ is symmetrically block finitely represented in the unit vector basis of a separable symmetric sequence space, it suffices to identify the set of all approximate eigenvalues of the dilation operator $D$. In this section, we show that the latter problem reduces to a similar task for the shift operator in a certain Banach sequence lattice.

We will associate to a symmetric sequence space $X$ the Banach sequence lattice $E_X$ equipped with the norm

$$\|a\|_{E_X} := \left\| \sum_{k=1}^{\infty} a_k \sum_{i=2^{k-1}}^{2^k-1} e_i \right\|_X.$$
We begin with a lemma which establishes simple connections between the Boyd indices $\alpha_X$ and $\beta_X$ of a symmetric sequence space $X$ and the shift exponents $k_+(E_X)$ and $k_-(E_X)$ of the lattice $E_X$.

**Lemma 4** For every symmetric sequence space $X$ we have

\[ \|\tau_n\|_{E_X} \leq 2\|\sigma_{2^n}\|_X, \quad n \in \mathbb{Z}, \]  

(27)

and

\[ \|\sigma_{2^n}\|_X \leq \|\tau_{n+1}\|_{E_X}, \quad n \in \mathbb{Z}. \]  

(28)

Therefore, $k_-(E_X) = 2^{-\alpha_X}$ and $k_+(E_X) = 2^{\beta_X}$.

**Proof** Let $n \in \mathbb{N}$ and let $a = (a_k)_{k=1}^\infty \in E_X$. It may be assumed that $a_k \geq 0$, $k = 1, 2, \ldots$. Then,

\[
\|\tau_n a\|_{E_X} = \left\| \sum_{k=n+1}^{\infty} a_k e_i \right\|_X = \left\| \sum_{k=1}^{\infty} a_k \sum_{i=2^k-1}^{2^k-1} e_i \right\|_X
\]

\[
= \left\| \sigma_{2^n} \left( \sum_{k=1}^{\infty} a_k \sum_{i=2^{k+1}-1}^{2^{k+1}-1} e_i \right) \right\|_X \leq \|\sigma_{2^n}\|_X \|a\|_{E_X}.
\]

Before proving the same estimate for negative integers, recall that for arbitrary $n \in \mathbb{N}$

\[
\sigma_{2^{-n}x} := 2^{-n} \sum_{k=1}^{2^n} \sum_{i=1}^{2^n} x_{(k-1)2^n+i} e_k, \quad \text{where} \quad x = (x_k)_{k=1}^\infty
\]

(see Sect. 2.3). Observe that for $k \geq n + 1$

\[
\sum_{i=2^{k+1}-1}^{2^k-1} e_i = \sum_{m=2^{k-n}-1}^{2^{k-n}-1} \sum_{j=0}^{2^n-1} e_{m2^n+j}
\]

and for every $m \in \mathbb{N}$

\[
\sigma_{2^{-n}} \left( \sum_{j=0}^{2^n-1} e_{m2^n+j} \right) \geq \sigma_{2^{-n}} \left( \sum_{j=1}^{2^n-1} e_{m2^n+j} \right) = 2^{-n}(2^n - 1)e_{m+1}.
\]

Therefore, we have

\[
\sigma_{2^{-n}} \left( \sum_{k=1}^{\infty} a_k \sum_{i=2^{k+1}-1}^{2^k-1} e_i \right) \geq \sum_{k=1}^{\infty} 2^{-n}(2^n - 1)a_{k+n} \sum_{i=2^{k+1}+1}^{2^k} e_i
\]

\[
\geq \frac{1}{2} \sum_{k=1}^{\infty} a_{k+n} \sum_{i=2^{k+1}+1}^{2^k} e_i.
\]

On the other hand,
\[ \tau_{-n}a = \sum_{k=1}^{\infty} a_{k+n} \sum_{i=2^{k-1}}^{2^k-1} e_i. \]

Thus, since \( X \) is a symmetric space, we get

\[ \|\tau_{-n}a\|_{E_X} \leq 2\|\sigma_{2^{-n}} \left( \sum_{k=1}^{\infty} a_k \sum_{i=2^{k-1}}^{2^k-1} e_i \right)\|_X \leq 2\|\sigma_{2^{-n}}\|_X \|a\|_{E_X}, \]

and inequality (27) is proved.

For the converse direction, we will need the natural averaging projection \( Q \) of \( X \) onto \( E_X \) defined by

\[ Qx := \sum_{k=1}^{\infty} 2^{-k+1} \sum_{j=2^{k-1}}^{2^k-1} x_j \sum_{i=2^{k-1}}^{2^k-1} e_i, \quad \text{where} \quad x = (x_k)_{k=1}^{\infty}. \quad (29) \]

It is well known that \( Q \) has norm 1 in each symmetric space \( X \) (see, e.g., [13, § II.3.2]).

Let \( n \in \mathbb{N} \) and \( a = (a_j)_{j=1}^{\infty} \in X \) be a nonincreasing and nonnegative sequence. Then, we have

\[ \|\tau_{n+1}Qa\|_{E_X} = \left\| \sum_{l=n+2}^{\infty} 2^{n+l+2} \sum_{j=2^{l-2}}^{2^{l+1}-1} a_j e_l \right\|_{E_X} \]
\[ = \left\| \sum_{l=n+2}^{\infty} 2^{n+l+2} \sum_{j=2^{l-2}}^{2^{l+1}-1} a_j \sum_{i=2^{j-1}}^{2^j-1} e_i \right\|_X \]
\[ = \left\| \sum_{l=1}^{\infty} 2^{-l+1} \sum_{j=2^{l+1}}^{2^{l+1}+1} a_j \sum_{i=2^{j-1}}^{2^j-1} e_i \right\|_X. \quad (30) \]

Let us show that

\[ \sum_{k=1}^{\infty} a_k \sum_{i=2^{n(k-1)+1}}^{2^k} e_i \leq \sum_{l=1}^{\infty} 2^{-l+1} \sum_{j=2^{l-1}}^{2^l-1} a_j \sum_{i=2^{j+1}}^{2^{j+1}-1} e_i. \quad (31) \]

Since both sequences are nonincreasing, it suffices to check that for each \( i \in \mathbb{N} \) the sequence from the right-hand side of (31) contains at least \( 2^i \) entries that are larger than or equal to \( a_i \).

Let \( i \in \mathbb{N} \) be arbitrary. Since \( a = (a_j)_{j=1}^{\infty} \in X \) is nonincreasing, the inequality \( 2^l - 1 \leq i \) ensures that \( a_i \leq 2^{-i+1} \sum_{j=2^l}^{2^l-1} a_j \). If \( l_0 = l_0(i) \) is the greatest positive integer satisfying the latter estimate, then \( 2^{l_0+1} > i + 1 \). Therefore, the number of entries in the sequence from the right-hand side of (31), which are not less than \( a_i \), satisfies the estimate
\[
\sum_{i=1}^{l_i} 2^{i+n} = 2^{i+n+1} - 1 \geq (i + 1)2^n - 1 \geq 2^ni.
\]

As was said above, this implies (31).

Combining (30) and (31) together with the fact that the left-hand side of (31) coincides with the sequence \(\sigma_{2^n}a\), we infer

\[
\|\sigma_{2^n}a\|_X \leq \|\tau_{n+1}Qa\|_{E_X} \leq \|\tau_{n+1}\|_{E_X} \|Qa\|_{E_X} = \|\tau_{n+1}\|_{E_X} \|a\|_X.
\]

Thus, taking into account that \((\sigma_{2^n}a)^* = \sigma_{2^n}(a^*)\), \(n \in \mathbb{N}\), we get (28) for positive \(n\).

Next, observe that for every \(n \in \mathbb{N}\) and hence

\[
\|\|\tau_{n+1}Qa\|_{E_X} \|
\]

and hence \(\sigma_{2^n}a = \sigma_{2^n}R_n a\), where \(R_n\) is the norm one averaging projection on \(X\) defined by

\[
R_n x := \sum_{k=1}^{\infty} 2^{-n} \left( \sum_{i=2^{n(k-1)+1}}^{2^n} x_i \right) \left( \sum_{i=2^{n(k-1)+1}}^{2^n} e_i \right), \text{ where } x = (x_k)_{k=1}^{\infty}
\]

(see, e.g., [13, § II.3.2]). On the other hand,

\[
\|\tau_{n+1}QR_n a\|_{E_X} = \left\| \sum_{k=1}^{\infty} (QR_n a)_{k+n-1} \sum_{l=2^{k-1}}^{2^k-1} e_l \right\|_{E_X} = \left\| \sum_{j=n}^{\infty} (QR_n a)_{j} \sum_{l=2^{j-n}}^{2^{j-n+1}-1} e_l \right\|_{E_X}.
\]

Assuming that \(a = a^*\), we prove that

\[
\sum_{k=1}^{\infty} c_k e_k \leq \sum_{j=n}^{\infty} (QR_n a)_{j} \sum_{l=1}^{2^{j-n+1}-1} e_l.
\]

To this end, again it suffices to check that for each \(k \in \mathbb{N}\) the sequence from the right-hand side contains at least \(k\) entries that are larger than or equal to \(c_k\).

Let \(k \in \mathbb{N}\) be fixed. From the definition of the operators \(Q\) and \(R_n\) it follows that

\[
(QR_n a)_{j} \geq c_k \text{ whenever } 2^j - 1 \leq k2^n.
\]

Now, if \(j_0 = j_0(k)\) is the greatest positive integer satisfying this inequality, then \(2^{j_0+1} > k2^n\) or \(2^{j_0-n+1} > k\). Thus, the number of entries in the sequence from the right-hand side of (34), which are not less than \(c_k\), is more than or equal to

\[
\sum_{j=n}^{j_0} 2^{j-n} = 2^{1+j_0-n} > k,
\]

and so we get (34).

As a result, from (32)–(34) it follows that
\[ \| \sigma_2 \cdot a \|_X \leq \| \tau_{-n+1} QR_n a \|_{E_X} \leq \| \tau_{-n+1} \|_{E_X} \| QR_n a \|_{E_X} \]
\[ = \| \tau_{-n+1} \|_{E_X} \| QR_n a \|_X \leq \| \tau_{-n+1} \|_{E_X} \| a \|_X. \]

Since \((\sigma_2 \cdot a)^* \leq \sigma_2 \cdot (a^*)\), \(n \in \mathbb{N}\), and \(X\) is a symmetric space, the latter inequality holds for every \(a \in X\). Hence, as a result, we obtain inequality (28) for negative \(n\).

To complete the proof, it remains to observe that coincidence of the Boyd indices of \(X\) with the corresponding shift exponents of \(E_X\) follows immediately from their definition and inequalities (27) and (28).

Sometimes (see Sect. 6 below) it is useful to know that a lattice of the form \(E_X\) can be obtained also in a different way, when one starts from a Banach sequence lattice.

Let us note first that for every symmetric sequence space \(X\) and all \(x \in X\) it holds

\[ \| x \|_X \leq \sum_{k=0}^{\infty} x_{2^k} e_{k+1} \|_E_X \leq 5 \| x \|_X. \]  

(35)

Indeed, assuming (as we can) that \(x = x^*\), by the definition of \(E_X\), we get

\[ \| \sum_{k=0}^{\infty} x_{2^k} e_{k+1} \|_E_X = \| \sum_{k=0}^{\infty} x_{2^k} \sum_{i=2^k}^{2^{k+1}-1} e_i \|_X \geq \| \sum_{i=1}^{\infty} x_i e_i \|_X = \| x \|_X, \]

which gives the left-hand inequality.

Conversely, from (27) and the estimate \(\| \sigma_2 \|_{X \rightarrow X} \leq 2\) [13, Theorem II.4.5] it follows that

\[ \| \sum_{k=0}^{\infty} x_{2^k} e_{k+1} \|_E_X \leq \| x_1 e_1 \|_E_X + \| \sum_{k=1}^{\infty} x_{2^k} e_{k+1} \|_E_X \leq \| x \|_X + \| \tau_1 (\sum_{k=1}^{\infty} x_{2^k} e_k) \|_{E_X} \]
\[ \leq \| x \|_X + \| \tau_1 \|_{E_X} \| \sum_{k=1}^{\infty} x_{2^k} \sum_{i=2^k}^{2^k-1} e_i \|_X \]
\[ \leq \| x \|_X + 2 \| \sigma_2 \|_{X \rightarrow X} \| x \|_X \leq 5 \| x \|_X, \]

and the right-hand side inequality in (35) follows as well.

In the next result we show that, under some conditions, inequalities similar to (35) for a Banach lattice \(E\) imply that \(E = E_X\) with equivalence of norms. Although this result is in fact known (cf. [9, Proposition 5.1]), taking into account that in the case of sequence lattices it is stated in [9] without proofs, we provide here its complete proof for convenience of the reader.

**Proposition 5** Let \(E\) be a Banach sequence lattice, with \(k_(E) < 1\), and let \(X\) be a symmetric sequence space satisfying

\[ \| a \|_X \leq \sum_{k=0}^{\infty} a_{2^k} e_{k+1} \|_E. \]  

(36)
Then, $E_X = E$ (with equivalence of norms).

**Proof** At first, observe that the norms $\|a\|_{E_X}$ and $\|a\|_E$ are equivalent if $a = (a_k)_{k=1}^\infty$ is a decreasing nonnegative sequence (with the equivalence constant from (36)). Indeed, since $(\sum_{k=1}^\infty d_k \sum_{i=2^{k-1}}^{2^k-1} e_{i})_2 = a_{j+1}$, $j = 0, 1, \ldots$, by the definition of $E_X$ and (36), we have

$$\|a\|_{E_X} = \left\| \sum_{k=1}^\infty a_k \sum_{i=2^{k-1}}^{2^k-1} e_i \right\|_X \preceq \left\| \sum_{j=0}^\infty a_{j+1} e_{j+1} \right\|_E = \|a\|_E.$$

Assume now that $a = (a_k)_{k=1}^\infty \in E_X$ is arbitrary. The condition $k_-(E) < 1$ ensures that for some $\gamma > 0$ and $C > 0$

$$\|\tau_{-j}\|_{E \to E} \leq C 2^{-\gamma j}, \ j \in \mathbb{N}. \quad (37)$$

Consequently, since the sequence $(\max_{j \geq 0} |\tau_{-j} a|)_k$, $k = 1, 2, \ldots$, is decreasing and

$$|a_k| \leq (\max_{j \geq 0} |\tau_{-j} a|)_k, \ k = 1, 2, \ldots,$$

by the above observation, we have

$$\|a\|_{E_X} \leq \left\| \left( \max_{j \geq 0} |\tau_{-j} a| \right)_k \right\|_{E_X} \preceq \left\| \left( \max_{j \geq 0} |\tau_{-j} a| \right)_k \right\|_E \leq \sum_{j=0}^\infty \|\tau_{-j}\|_{E \to E} \|a\|_E = C \|a\|_E.$$

Conversely, for each $a = (a_k)_{k=1}^\infty \in E$ we have

$$\|a\|_E \leq \left\| \left( \max_{j \geq 0} |\tau_{-j} a| \right)_k \right\|_E \preceq \left\| \left( \max_{j \geq 0} |\tau_{-j} a| \right)_k \right\|_{E_X} \leq \sum_{k=1}^\infty \left( \max_{j \geq 0} |\tau_{-j} a| \right)_k \sum_{i=2^{k-1}}^{2^k-1} e_i \left\|_X \leq \sum_{k=1}^\infty \sum_{j=0}^\infty (|\tau_{-j} a|)_k \sum_{i=2^{k-1}}^{2^k-1} e_i \left\|_X \leq \sum_{j=0}^\infty \sum_{k=1}^\infty (|\tau_{-j} a|)_k \sum_{i=2^{k-1}}^{2^k-1} e_i \right\|_X. \quad (38)$$

Next, by the definition of the operator $\sigma_{2^{-n}}$, we have

$$\sum_{k=1}^\infty (|\tau_{-j} a|)_k \sum_{i=2^{k-1}}^{2^k-1} e_i = \sum_{k=1}^\infty a_{k+j} \sum_{i=2^{k-1}}^{2^k-1} e_i = \sum_{r=j+1}^\infty a_r \sum_{i=2^{r-j-1}}^{2^r-1} e_i = \sigma_{2^{-j}} a^{(j)},$$
where $a^{(j)} = \sum_{k=j+1}^\infty |a_k| \sum_{i=2^k-1}^{2^{k+1}-1} e_{i-2^k+1}$. Therefore, since for every $j = 0, 1, \ldots$

$$(a^{(j)})^* = \left( \sum_{k=1}^\infty a_k \sum_{i=2^{k-1}}^{2^{k-1}-1} e_i \right)^*$$

and $(\sigma_{2^{-n}}b)^* \leq \sigma_{2^{-n}}(b^*)$, $n \in \mathbb{N}$, from (38) it follows that

$$\|a\|_E \leq \sum_{j=0}^\infty \left\| \sigma_{2^{-j}} \left( \sum_{k=1}^\infty a_k \sum_{i=2^{k-1}}^{2^{k-1}-1} e_i \right)^* \right\|_X. \quad (39)$$

We claim that there exists a constant $C > 0$ such that for every nonincreasing and nonnegative sequence $b = (b_k)_{k=1}^\infty \in X$ and all integers $j \geq 0$

$$\|\sigma_{2^{-j}}b\|_X \leq C \|\tau_{j+1}\|_{E \rightarrow E} \|b\|_X. \quad (40)$$

Indeed, by (36) and the inequality $(2^k - 1)2^i + i \geq 2^{k+j-1}$ if $k, i \geq 1$, we have

$$\|\sigma_{2^{-j}}b\|_X \leq 2^{-j} \sum_{k=0}^\infty \sum_{i=1}^{2^j} b_{2^k-1}^{2^{k+j-1}} e_{k+1} \|E\|$$

$$\leq \left\| b_1 e_1 + \sum_{k=1}^\infty b_{2^k-1}^{2^k} e_{k+1} \right\|_E$$

$$= \left\| b_1 e_1 + \tau_{j+1} \left( \sum_{k=1}^\infty b_{2^k} e_{k+1} \right) \right\|_E$$

$$\leq \|b\|_X + \|\tau_{j+1}\|_{E \rightarrow E} \sum_{k=1}^\infty b_{2^k} e_{k+1} \|_E$$

$$\leq C \|\tau_{j+1}\|_{E \rightarrow E} \|b\|_X,$$

and (40) is proved.

Applying estimate (40) to the sequence $b = \sum_{k=1}^\infty a_k \sum_{i=2^{k-1}}^{2^{k-1}-1} e_i$, we get

$$\left\| \sigma_{2^{-j}} \left( \sum_{k=1}^\infty a_k \sum_{i=2^{k-1}}^{2^{k-1}-1} e_i \right)^* \right\|_X \leq C \|\tau_{j+1}\|_{E \rightarrow E} \sum_{k=1}^\infty a_k \sum_{i=2^{k-1}}^{2^{k-1}-1} e_i \|_X$$

$$= C \|\tau_{j+1}\|_{E \rightarrow E} \|a\|_{E_X}^*,$$

and so from (39) and (37) it follows that

$$\|a\|_E \leq C \sum_{j=0}^\infty \|\tau_{j+1}\|_{E \rightarrow E} \|a\|_{E_X} = C' \|a\|_{E_X}.$$

This completes the proof. \[
\]

From Proposition 5 and Lemma 4 it follows
Corollary 6 Let $X$ be a symmetric sequence space such that equivalence (36) holds for some Banach sequence lattice $E$ with $k_-(E) < 1$. Then, $\alpha_X > 0$.

In the concluding part of this section, we establish a direct connection between spectral properties of the dilation operator $D := \tau_1 \sigma_2$ in a symmetric sequence space $X$ and the shift operator $\tau_1$ in $E_X$.

Let $D_\lambda := D - \lambda I$ and $T_\lambda := \tau_1 - \lambda I$, $\lambda > 0$. Setting

$$Sx := \sum_{k=1}^{\infty} x_k \sum_{i=2^{k-1}}^{2^k-1} e_i,$$

where $x = (x_k)_{k=1}^{\infty}$, we get

$$D_\lambda Sa = \sum_{k=1}^{\infty} a_k \sum_{i=2^k}^{2^{k+1}-1} e_i - \lambda \sum_{k=1}^{\infty} a_k \sum_{i=2^{k-1}}^{2^k-1} e_i$$

$$= \sum_{k=2}^{\infty} a_{k-1} \sum_{i=2^{k-1}}^{2^k-1} e_i - \lambda \sum_{k=1}^{\infty} a_k \sum_{i=2^{k-1}}^{2^k-1} e_i$$

$$= \sum_{k=1}^{\infty} (T_\lambda a)_k \sum_{i=2^{k-1}}^{2^k-1} e_i = ST_\lambda a. \tag{41}$$

Proposition 7 Let $X$ be a symmetric sequence space and $\lambda > 0$. Then, we have:

(i) $D_\lambda$ is closed in $X$ if and only if $T_\lambda$ is closed in $E_X$;

(ii) $D_\lambda$ is an isomorphism in $X$ if and only if the operator $T_\lambda$ is an isomorphism in $E_X$.

Proof First, we observe that both operators $T_\lambda$ and $D_\lambda$ are injective for every $\lambda$. Indeed, for instance, the equation $T_\lambda a = 0$, with $a = (a_n)$, means that $a_1 = 0$ and $a_{n-1} = \lambda a_n$ if $n \geq 2$, whence $a_n = 0$ for all $n \in \mathbb{N}$. Therefore, it suffices to prove only (i).

Suppose first that the operator $D_\lambda$ is closed in the space $X$. Let $a^n = (a^n_k)_{k=1}^{\infty} \in E_X$, $n = 1, 2, \ldots$, and $T_\lambda a^n \to b = (b_k)$ in $E_X$. Since

$$T_\lambda a^n = -\lambda a_1^n e_1 + \sum_{k=2}^{\infty} (a^n_{k-1} - \lambda a_k^n) e_k$$

and $T_\lambda a^n \to b$ coordinate-wise as $n \to \infty$, we see that $b = T_\lambda a$, where $a = (a_k)_{k=1}^{\infty}$, $a_k := \lim_{n \to \infty} a^n_k$, $k \in \mathbb{N}$. It remains to show that $a \in E_X$.

By hypothesis, the operator $D_\lambda$ is closed in $X$ (and hence it is an isomorphic mapping). Denote by $D_\lambda^{-1}$ the left inverse operator to $D_\lambda$, defined on the subspace $\text{Im} D_\lambda$ of $X$. In view of (41), it holds
\[ D_\lambda S a^n = ST_\lambda a^n \quad \text{for all} \quad n = 1, 2, \ldots, \]

whence

\[ Sa^n = D^{-1}_\lambda ST_\lambda a^n, \quad n = 1, 2, \ldots \]

Then, since the convergence \( T_\lambda a^n \to b \) in \( E_X \) implies that \( ST_\lambda a^n \to Sb \) in \( X \), we get: \( Sa^n \to D^{-1}_\lambda Sb \) in \( X \). On the other hand, it is obvious that \( Sa^n \to Sa \) coordinate-wise as \( n \to \infty \). Hence, \( Sa = D^{-1}_\lambda Sb \in X \), which implies that \( a \in E_X \), as we wish.

Let us prove the converse. To get a contradiction, assume that the operator \( D_\lambda \) is not closed. Then, there exists a sequence \( \{x^{(n)}\} \subset X \) with the following properties:

\[ \|x^{(n)}\|_X = 1, \quad n = 1, 2, \ldots, \quad \text{and} \quad \|D_\lambda x^{(n)}\|_X \to 0. \quad (42) \]

Since \( X \) is separable or has the Fatou property, we have

\[ \|D_\lambda x^{(n)}\|_X \geq \|D(x^{(n)})^* - \lambda(x^{(n)})^*\|_X = \|D_\lambda (x^{(n)})^*\|_X \]

(see, e.g., [13, Lemma II.4.6 and Theorems II.4.9, II.4.10] or [4, § 3.7]). Consequently, we may assume that each of the sequences \( x^{(n)}, n \in \mathbb{N} \), is nonnegative and nonincreasing.

Observe that the operators \( Q \) (defined by (29)) and \( D_\lambda \) commute. Indeed, since \( D = \tau_1 \sigma_2 \), we have

\[
QD_\lambda x = \sum_{k=1}^{\infty} 2^{-k+1} \sum_{j=2^k-1}^{2^k-1} (D_\lambda x)_j \sum_{i=2^k-1}^{2^k-1} e_i \\
= \sum_{k=1}^{\infty} 2^{-k+1} \left( \sum_{j=2^k-1}^{2^k-1} x_j \sum_{i=2^k-1}^{2^k-1} e_i - \lambda \sum_{j=2^k-1}^{2^k-1} x_j \sum_{i=2^k-1}^{2^k-1} e_i \right) \\
= D_\lambda \left( \sum_{k=1}^{\infty} 2^{-k+1} \sum_{j=2^k-1}^{2^k-1} x_j \sum_{i=2^k-1}^{2^k-1} e_i \right) = D_\lambda Q x.
\]

Therefore, noting that in our notation \( Q x = Sa_x \), where

\[ a_x := \left( 2^{-k+1} \sum_{j=2^k-1}^{2^k-1} x_j \right)_{k=1}^{\infty}, \]

by (41), we get for all \( x \in X \)

\[ ST_\lambda a_x = D_\lambda S a_x = D_\lambda Q x = QD_\lambda x. \]

Since the projection \( Q \) is bounded on \( X \), then substituting \( x^{(n)}, n = 1, 2, \ldots \) for \( x \) into the latter formula and taking the limit as \( n \to \infty \), by (42), we get

\[ \lim_{n \to \infty} \|T_\lambda a_{x^{(n)}}\|_{E_X} = 0. \quad (43) \]
On the other hand, by the definition of $E_X$ and the monotonicity of each sequence $x^{(n)}$,
\[
\|a_{x^{(n)}}\|_{E_X} = \left\| \sum_{k=1}^{\infty} 2^{-k+1} \sum_{j=2^{k-1}}^{2^k-1} x^{(n)}_j \sum_{i=2^{k-1}}^{2^k-1} e_i \right\|_X \geq \left\| \sum_{k=1}^{\infty} x^{(n)}_j \sum_{i=2^{k-1}}^{2^k-1} e_i \right\|_X.
\]
Consequently, since it follows that
\[
(\sigma_{1/2}x^{(n)})_m = \frac{1}{2} \left( x^{(n)}_{2m-1} + x^{(n)}_{2m} \right), \quad m \in \mathbb{N}, \tag{44}
\]
it follows that
\[
\|a_{x^{(n)}}\|_{E_X} \geq \left\| \sum_{k=1}^{\infty} (\sigma_{1/2}x^{(n)})_{2^k-1} \sum_{i=2^{k-1}}^{2^k-1} e_i \right\|_X \geq \left\| \sum_{i=2^{k-1}}^{2^k-1} (\sigma_{1/2}x^{(n)})_i e_i \right\|_X = \|\sigma_{1/2}x^{(n)}\|_X.
\]
Moreover, taking into account that $X$ is a symmetric space and $x^{(n)} \geq 0$, in view of (44), we have
\[
\|x^{(n)}\|_X \leq \left\| \sum_{k=1}^{\infty} (x^{(n)})_{2k-1} e_{2k-1} \right\|_X + \left\| \sum_{k=1}^{\infty} (x^{(n)})_{2k} e_{2k} \right\|_X = \left\| \sum_{k=1}^{\infty} (x^{(n)})_{2k-1} e_k \right\|_X + \left\| \sum_{k=1}^{\infty} (x^{(n)})_{2k} e_k \right\|_X \leq 2 \max \left( \|x^{(n)}\|_{2k-1} \|x^{(n)}\|_{2k} \right) \leq 4 \|\sigma_{1/2}x^{(n)}\|_X.
\]
Combining the last inequalities together with (42) implies that
\[
\|a_{x^{(n)}}\|_{E_X} \geq \frac{1}{4}, \quad n \in \mathbb{N},
\]
and hence, by (43), the operator $T_\lambda$ fails to be an isomorphic embedding in the space $E_X$. Since it is injective, this means that $T_\lambda$ is not closed in $E_X$, which contradicts the assumption. \hfill \Box

The following result is an immediate consequence of Proposition 7.

**Corollary 8** Let $X$ be a symmetric sequence space. Then the operators $D$ in $X$ and $\tau_1$ in $E_X$ have the same set of positive approximate eigenvalues.
5 Approximative eigenvalues of the shift operator in Banach sequence lattices

Suppose $E$ is a Banach sequence lattice such that the shift operator $\tau_1(a_k) = (a_{k-1})$ and its inverse $\tau_{-1}(a_k) = (a_{k+1})$ are bounded in $E$. Let $s_k := \|e_k\|_E$, where $e_k, k = 1, 2, \ldots$, are elements of the unit vector basis.

Next, we suppose that the shift exponents $k_-(E)$ and $k_+(E)$ can be calculated when the shift operators $\tau_n(a_k) = (a_{k-n})$, $n \in \mathbb{Z}$, are restricted to the set $\{e_k\}_{k=1}^\infty$, or, more explicitly, it will be assumed that

$$k_+(E) = \lim_{n \to \infty} \left( \sup_{k>n} \frac{s_k}{s_{k-n}} \right)^{1/n} \quad \text{and} \quad k_-(E) = \lim_{n \to \infty} \left( \sup_{k \in \mathbb{N}} \frac{s_k}{s_{n+k}} \right)^{1/n}. \quad (45)$$

Hence, $1/k_-(E) \leq k_+(E)$.

As we show in this section, then we are able to identify, in terms of the exponents $k_-(E)$ and $k_+(E)$, the set of all parameters $\lambda > 0$, for which the operator $T_\lambda = \tau_1 - \lambda I$ (as usual, $I$ is the identity in $E$) is an isomorphic embedding in $E$.

**Proposition 9** Suppose a separable Banach sequence lattice $E$ satisfies assumption (45). Then, for every $\lambda > 0$ the following conditions are equivalent:

(i) the operator $T_\lambda$ is an isomorphic mapping in $E$;

(ii) the operator $T_\lambda$ is closed in $E$;

(iii) $\lambda \in (0, 1/k_-(E)) \cup (k_+(E), \infty)$.

Moreover, if $\lambda \in (k_+(E), \infty)$, then $\text{Im } T_\lambda = E$; if $\lambda \in (0, 1/k_-(E))$, then $\text{Im } T_\lambda$ is the closed subspace of $E$ of codimension 1 consisting of all $(a_k)_{k=1}^\infty \in E$ with

$$\sum_{k=1}^\infty \lambda^k a_k = 0. \quad (46)$$

**Proof** First of all, observe that the operator $T_\lambda$ is injective for every $\lambda$. Consequently, the equivalence of conditions (i) and (ii) is obvious.

Let us determine the possible form of the subspace $\text{Im } T_\lambda$. If a linear functional $f$ vanishes on $\text{Im } T_\lambda$, then $f(e_{n+1}) - \lambda f(e_n) = f(T_\lambda e_n) = 0$, and hence $f(e_{n+1}) = \lambda^n f(e_1)$, $n \in \mathbb{N}$. Let us define $f_\lambda$ as a unique linear functional on $c_{00}$ such that $f_\lambda(e_n) = \lambda^{n-1}$, $n \in \mathbb{N}$. Observe that if $a = (a_k)_{k=1}^\infty \in c_{00}$, then $f_\lambda(T_\lambda a) = 0$.

By assumption, $E$ is separable. Hence, the dual space $E^*$ coincides with the Köthe dual $E'$ (see Sect. 2.1), whence the condition $f_\lambda \in E^*$ is equivalent to the fact that

$$\sum_{k=1}^\infty \lambda^k a_k < \infty, \quad \text{for all } (a_k)_{k=1}^\infty \in E. \quad (47)$$

The restriction to $c_{00}$ of any linear functional $f$ on $E$ vanishing on $\text{Im } T_\lambda$ coincides with $f(e_1) \cdot f_\lambda$. Hence, by density of $c_{00}$ in $E$, all bounded linear functionals that vanish on $\text{Im } T_\lambda$ are proportional, and thus the annihilator $(\text{Im } T_\lambda)^\perp$ in $E^*$ has dimension
1 or 0, depending on the fact if \( f_\lambda \) is bounded on \( E \) or not, i.e., if condition (47) is verified or not. Consequently, the closure \( \overline{\text{Im} T_\lambda} \) has codimension 1 or 0. In the first case, \( f_\lambda \) has a (unique) bounded extension \( \tilde{f}_\lambda \) to \( E \) and

\[
\overline{\text{Im} T_\lambda} = \text{Ker} \tilde{f}_\lambda.
\] (48)

In the second case, \( \text{Im} T_\lambda \) is dense in \( E \), i.e.,

\[
\overline{\text{Im} T_\lambda} = E.
\] (49)

If \( \lambda \in (k_+(E), \infty) \), then the fact that the spectral radius of \( \tau_1 \) is equal to \( k_+(E) \) immediately implies that the operator \( T_\lambda \) is an isomorphism of \( E \) onto \( E \).

Let us consider now the case when \( \lambda \in (0, 1/k_-(E)) \). We show that the functional \( f_\lambda \) is bounded on \( E \) and the subspace \( \text{Im} T_\lambda \) is closed in \( E \). Once we prove this, by (48), \( \text{Im} T_\lambda = \text{Ker} \tilde{f}_\lambda \) and so being injective \( T_\lambda \) is an isomorphic mapping of \( E \) onto the closed subspace \( \text{Ker} \tilde{f}_\lambda \) consisting of all \( (a_k)_{k=1}^\infty \in E \) satisfying (46).

Choose \( \eta \in (\lambda, 1/k_-(E)) \). Then, \( 1/\eta > k_-(E) \) and hence, in view of (45), there is \( C > 0 \) such that

\[
\sup_{k \geq 0} \frac{s_k}{s_{k+n}} \leq C\eta^{-n}, \quad n = 1, 2, \ldots,
\]

whence

\[
s_n^{-1} \leq s_1^{-1} C\eta^{-n}, \quad n = 1, 2, \ldots
\]

On the other hand, since \( E \) is a Banach lattice, for every \( a = (a_k)_{k=1}^\infty \in E \) we have

\[
|a_k|\|\epsilon_k\|_E \leq \|a\|_E, \quad \text{i.e., } |a_k| \leq \|a\|_E s_k^{-1}.
\]

Combining the last inequalities, we get

\[
\sum_{n=1}^\infty \lambda^n a_n \leq \|a\|_E \sum_{n=1}^\infty \lambda^n s_n^{-1} \leq s_1^{-1} C\|a\|_E \sum_{n=1}^\infty (\lambda/\eta)^n < \infty.
\]

Thus, condition (47) is fulfilled for each \( a = (a_k)_{k=1}^\infty \in E \), yielding \( f_\lambda \in E^* \).

To prove that \( \text{Im} T_\lambda \) is closed, we represent \( E \) as follows:

\[
E = E_1 + E_\infty,
\]

where \( E_1 = \{ \{e_1\} \}_E \) and \( E_\infty = \{ \{e_n\}_{n \geq 2} \}_E \) (\( \{A\}_E \) is the closed linear span of a set \( A \) in the space \( E \)). Since \( \text{Im} T_\lambda = T_\lambda(E_1) + T_\lambda(E_\infty) \) and the space \( T_\lambda(E_1) \) is one-dimensional, it suffices to prove only that the set \( T_\lambda(E_\infty) \) is closed.

One can easily check that

\[
T_\lambda(E_\infty) = -\lambda \tau_1(\tau_{-1} - \lambda^{-1} I)(E_\infty),
\] (50)

Since the spectral radius of \( \tau_{-1} \) is equal to \( k_-(E) \), the operator \( \tau_{-1} - \lambda^{-1} I \) maps isomorphically \( E \) onto \( E \) whenever \( 0 < \lambda < 1/k_-(E) \). Consequently, since \( E_\infty \) is closed in \( E \), the set \((\tau_{-1} - \lambda^{-1} I)(E_\infty)\) is closed in \( E \) as well. Then, in view of the fact that \( \tau_1 \) is an isomorphic embedding in \( E \), from (50) it follows that the subspace \( T_\lambda(E_\infty) \) is closed.
Thus, in the case $0 < \lambda < 1/k_+(E)$ the operator $T_\lambda$ maps the space $E$ isomorphically onto the subspace of codimension 1 consisting of all $(a_k) \in E$ satisfying (46).

It remains to prove implication $(i) \implies (iii)$, i.e., that the operator $T_\lambda : E \to E$ fails to be an isomorphic embedding whenever $1/k_+(E) \leq \lambda \leq k_+(E)$.

Suppose first that $1/k_+(E) < \lambda < k_+(E)$. To the contrary, assume that there exists $c > 0$ such that for all $x \in E$

$$\|T_\lambda x\|_E \geq c\|x\|_E. \quad (51)$$

Let $n, k \in \mathbb{N}$, $k > n$, be arbitrary for a moment (they will be fixed later). We put $a := (I + \lambda^{-1} \tau_1 + \cdots + \lambda^{-n} \tau_1^n)^2 e_{k-n}$. A direct calculation shows that $a \geq n\lambda^{-n} e_k$, whence

$$\|a\|_E \geq n\lambda^{-n} s_k. \quad (52)$$

Now, we estimate the norm $\|T_\lambda^2 a\|_E$ from above. First,

$$T_\lambda^2 (I + \lambda^{-1} \tau_1 + \cdots + \lambda^{-n} \tau_1^n)^2 = \lambda (\tau_1 - \lambda I)(\lambda^{-n+1} \tau_1^{n+1} - I)(I + \lambda^{-1} \tau_1 + \cdots + \lambda^{-n} \tau_1^n) = \lambda^2 I - 2\lambda^{-(n-1)} \tau_1^{n+1} + \lambda^{-2n} \tau_1^{2n+2}.$$

Consequently,

$$T_\lambda^2 a = \lambda^2 e_{k-n} - 2\lambda^{-(n-1)} \tau_1 e_{n+k+2},$$

and from the triangle inequality it follows

$$\|T_\lambda^2 a\|_E = \lambda^2 s_{k-n} + 2\lambda^{-(n-1)} s_{k+1} + \lambda^{-2n} s_{n+k+2} \leq \lambda^2 s_{k-n} + 2\lambda^{-(n-1)} \|\tau_1\|_E s_k + \lambda^{-2n} \|\tau_1\|_E^2 s_{n+k}.$$

Hence,

$$\|T_\lambda^2 a\|_E - 2\lambda \|\tau_1\|_E \lambda^{-n} s_k \leq 2 \max(\lambda^2, \|\tau_1\|_E^2) \max(s_{k-n}, \lambda^{-2n} s_{n+k}).$$

Let us observe that (51) and (52) yield

$$\|T_\lambda^2 a\|_E \geq c^2 n\lambda^{-n} s_k.$$

Therefore, choosing $n \in \mathbb{N}$ so that

$$c^2 n > 2\lambda \|\tau_1\|_E + 2 \max(\lambda^2, \|\tau_1\|_E^2),$$

from the preceding inequality we get

$$\lambda^{-n} s_k < \max(s_{k-n}, \lambda^{-2n} s_{n+k}),$$

or, equivalently,

$$\nu_k < \max(\nu_{k-n}, \nu_{n+k}) \text{ for all } k > n, \quad (53)$$
where \( v_n := \lambda^{-n}s_n \).

By assumption, \( \lambda < k_+(E) \). Therefore, by (45), for the already chosen number \( n \in \mathbb{N} \) we can find \( k \in \mathbb{N} \) such that \( k > n \) and \( s_k > \lambda^n s_{k-n} \), i.e., \( v_k > v_{k-n} \). Hence, from (53) it follows that \( v_{k+n} > v_k \). Substituting \( k+n \) for \( k \) in (53) and taking into account that \( v_{k+n} > v_k \), we obtain \( v_{k+2n} > v_{k+n} \). Proceeding in the same way, we conclude that the sequence \( (v_{k+rn})_{r=0}^{\infty} \) with the above \( n, k \in \mathbb{N} \), is increasing.

Let \( j \geq k \) and \( m \geq n \). We find \( 1 \leq r_1 \leq r_2 \) such that
\[
k + (r_1 - 1)n \leq j \leq k + r_1 n \quad \text{and} \quad k + r_2 n \leq j + m \leq k + (r_2 + 1)n.
\]
Setting \( C_1 := \max_{i=1,2,\ldots,n} \| \tau_i \|_{E \to E} \), we have
\[
s_j \leq C_1 s_{k+r_1 n} \quad \text{and} \quad s_{k+r_2 n} \leq C_1 s_{j+m}.
\]

Therefore,
\[
\frac{s_{j+m}}{s_j} \geq C_1^{-1} \frac{s_{k+r_1 n}}{s_{k+r_1 n}} = C_1^{-1} \frac{\lambda^{rn} v_{k+r_1 n}}{\lambda^{rn} v_{k+r_1 n}} \geq \lambda^{(r_2-r_1)}.
\]

If \( \lambda \geq 1 \), then in view of the inequality \( m - 2n \leq (r_2 - r_1)n \), we deduce from (54) that
\[
\frac{s_{j+m}}{s_j} \geq C_1^{-1} \lambda^{m-2n} \quad \text{for all} \quad j \geq k \quad \text{and} \quad m \geq n,
\]
or
\[
\sup_{j \geq k} \frac{s_j}{s_{j+m}} \leq C_2 \lambda^{-m} \quad \text{for all} \quad m \in \mathbb{N}.
\]

Moreover, for all \( 1 \leq j < k \) we have
\[
\frac{s_j}{s_{j+m}} \leq \frac{s_j}{s_k} \cdot \frac{s_{k+m}}{s_{j+m}} \cdot \frac{s_k}{s_{k+m}} \leq C_3 \frac{s_k}{s_{k+m}},
\]
where \( C_3 := \max_{i=\pm 1,\pm 2,\ldots,\pm k} \| \tau_i \|_{E \to E} \). Combining the last estimates, we infer
\[
\sup_{j \in \mathbb{N}} \frac{s_j}{s_{j+m}} \leq C \lambda^{-m}
\]
for some constant \( C > 0 \) and all \( m \in \mathbb{N} \). Hence, in view of (45), we conclude that \( k_-(E) \leq 1/\lambda \), which contradicts the assumption.

If \( 0 < \lambda < 1 \), then from the inequality \( m \geq (r_2 - r_1)n \) and estimate (54) it follows that
\[
\frac{s_{j+m}}{s_j} \geq C_1^{-2} \lambda^m \quad \text{for all} \quad j \geq k \quad \text{and} \quad m \geq n.
\]
Then, reasoning precisely in the same way, we again obtain that \( k_-(E) \leq 1/\lambda \). As a result, the operator \( T_\lambda \) fails to be an isomorphic embedding in \( E \) for \( \lambda \in (1/k_-(E), k_+(E)) \).

Finally, we need to check that neither \( T_{k_+(E)} \) nor \( T_{1/k_-(E)} \) is an isomorphic embedding in \( E \).

According to (48) and (49), if \( T_\lambda \) is an isomorphic embedding in \( E \), then it is both a Fredholm operator of index 0 or \(-1\). In particular, it was proved that \( T_\lambda \) is a Fredholm operator of index 0 (resp. \(-1\)) if \( \lambda \in (k_+(E), \infty) \) (resp. \( \lambda \in (0, 1/k_-(E)) \)). Moreover, we know that in the case when \( 1/k_-(E) < k_+(E) \) the operator \( T_\lambda \) fails to be an isomorphic embedding in \( E \) for \( \lambda \in (1/k_-(E), k_+(E)) \). Therefore, since the set of all Fredholm operators is open in the space of all bounded linear operators on \( E \) with respect to the topology generated by the operator norm (see, e.g., [11, Theorem III.21]), we conclude that each of the operators \( T_{k_+(E)} \) nor \( T_{1/k_-(E)} \) may not be isomorphic. The same result follows if \( 1/k_-(E) = k_+(E) \), because the set of Fredholm operators with a fixed index is also open (see, e.g., [11, Theorem III.22]).

\[ \square \]

**Corollary 10** Let \( X \) be a separable symmetric sequence space of fundamental type. Then, for every \( \lambda > 0 \) the following conditions are equivalent:

(i) the operator \( D_\lambda \) is an isomorphic mapping in \( X \);
(ii) the operator \( D_\lambda \) is closed in \( X \);
(iii) \( \lambda \in (0, 2^{\alpha_X}) \cup (2^{\beta_X}, \infty) \).

Hence, the set of all positive approximate eigenvalues of the operator \( D \) coincides with the interval \([2^{\alpha_X}, 2^{\beta_X}]\).

**Proof** Let us show that the Banach lattice \( E_X \) satisfies assumption (45). Recall that

\[
M_X^\infty(2^n) = \sup_{m \in \mathbb{N}} \frac{\phi_X(2^nm)}{\phi_X(m)}, \quad n \in \mathbb{N}
\]

(see Sect. 2.3). Choosing for each \( m \in \mathbb{N} \) a nonnegative integer \( k \) so that \( 2^k \leq m < 2^{k+1} \), we have \( \phi_X(2^{k}) \leq \phi_X(m) \leq 2\phi_X(2^{k}) \). Therefore, from the preceding formula it follows

\[
\sup_{k \in \mathbb{N}} \frac{\phi_X(2^{k+n})}{\phi_X(2^{k})} \leq M_X^\infty(2^n) \leq 2 \sup_{k \in \mathbb{N}} \frac{\phi_X(2^{k+n})}{\phi_X(2^{k})}, \quad n \in \mathbb{N},
\]

whence

\[
M_X^\infty(2^n) \leq \frac{\| \sum_{i=1}^{2^{k+n}} e_i \|_X}{\sum_{i=1}^{2^k} e_i \|_X}, \quad n \in \mathbb{N}.
\]

On the other hand, by the definition of \( E_X \), we have
which combined with the fact that $X$ is a symmetric space, implies

\[
\sup_{k \in \mathbb{N}} \frac{\| \sum_{i=1}^{2^k} e_i \|_X}{\sum_{i=1}^{2^k} e_i} = \sup_{k > n} \frac{\| \sum_{i=2^k}^{2^{k+n}} e_i \|_X}{\sum_{i=2^k}^{2^{k+n}} e_i} = \frac{s_k}{s_{k-n}},
\]

Thus,

\[
M_X^\infty (2^n) \leq \sup_{k > n} \frac{s_k}{s_{k-n}}, \quad n \in \mathbb{N},
\]

and hence, by the definition of $v_X$ and the assumption that $X$ is a space of fundamental type, we have

\[
\beta_X = v_X = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k > n} \frac{s_k}{s_{k-n}}.
\]

Applying now Lemma 4, we obtain the first equality in (45). The proof of the second is quite similar, and we skip it.

Since $E_X$ is separable together with $X$, all the assertions of the corollary follow now from Propositions 9 and 7 and Lemma 4.

From the last corollary and Theorem 3 it follows a complete description of the set $\mathcal{F}(X)$ for every separable symmetric sequence space $X$ of fundamental type.

**Theorem 5** Let $X$ be a separable symmetric sequence space of fundamental type. Then, the following conditions are equivalent:

(i) $\ell^p$ is symmetrically block finitely represented in the unit vector basis $\{e_k\}$;
(ii) $\ell^p$ is symmetrically finitely represented in $X$;
(iii) $\ell^p$ is crudely symmetrically finitely represented in $X$;
(iv) $p \in [1/\alpha_X, 1/\beta_X]$, where $\alpha_X$ and $\beta_X$ are the Boyd indices of $X$.

**Remark 1** By Theorem 4, if $p = 1/\alpha_X$ or $1/\beta_X$, the space $\ell^p$ is symmetrically finitely represented in every (not necessarily of fundamental type) symmetric sequence space $X$.

### 6 Symmetric finite representability of $\ell^p$ in Lorentz and Orlicz spaces

Results obtained allow us to find the set $\mathcal{F}(X)$ if $X$ is a Lorentz or a separable Orlicz sequence space.
6.1 Lorentz spaces

Let \( 1 \leq q < \infty \), and let \( \{w_k\}_{k=1}^{\infty} \) be a nonincreasing sequence of positive numbers. Recall that the Lorentz space \( \ell^q(w) \) consists of all sequences \( a = (a_k)_{k=1}^{\infty} \) such that

\[
\|a\|_{\ell^q(w)} = \left( \sum_{k=1}^{\infty} (a_k^q w_k^q) \right)^{1/q} < \infty.
\]

Since \( \ell^q(w) \) is a separable symmetric space of fundamental type (see Sect. 2.3), applying Theorem 5 implies

**Theorem 6** For every Lorentz sequence space \( \ell^q(w) \) the following conditions are equivalent:

(i) \( \ell^p \) is symmetrically block finitely represented in the unit vector basis \( \{e_k\} \) of \( \ell^q(w) \);
(ii) \( \ell^p \) is symmetrically finitely represented in \( \ell^q(w) \);
(iii) \( \ell^p \) is crudely symmetrically finitely represented in \( \ell^q(w) \);
(iv) \( p \in \left[ \frac{1}{\alpha_{\ell^q(w)}} \frac{1}{\beta_{\ell^q(w)}}, 1 \right] \), where the Boyd indices \( \alpha_{\ell^q(w)} \) and \( \beta_{\ell^q(w)} \) are defined by (6) and (7).

Suppose additionally that

\[
\lim_{n \to \infty} \left( \frac{\sup_{k=0,1,\ldots,2^n} w_{2^k}}{w_{2^{k+n}}} \right)^{1/n} < 2^{1/q}.
\]  (55)

By this assumption, we prove that the Banach lattice \( E_{\ell^q(w)} \) coincides with the weighted space \( l^q(\mu) \) equipped with the norm

\[
\|a\|_{\mu, q} := \left( \sum_{k=1}^{\infty} |a_k|^q \mu_k^q \right)^{1/q},
\]

where \( \mu_k = 2^{(k-1)/q} w_{2^{k-1}}, \ k = 1, 2, \ldots \). Moreover, in this case the Boyd indices of \( \ell^q(w) \) can be calculated by the simpler (than (6) and (7)) formulae:

\[
\alpha_{\ell^q(w)} = -\lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k=0,1,\ldots,2^n} \frac{w_{2^k}}{w_{2^{k+n}}} \text{ and } \beta_{\ell^q(w)} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k>n} \frac{w_{2^k}}{w_{2^{k-n}}}.
\]  (56)

Show first that

\[
\|x\|_{\ell^q(w)} \simeq \sum_{k=0}^{\infty} x^*_k e_{k+1} \|_{l^q(\mu)}.
\]  (57)

Indeed, assuming for simplicity of notation that \( x = x^* \), we get
On the other hand, one has

\[ \|x\|_{l^q(\mu)}^q = \sum_{k=1}^{\infty} \sum_{i=2^k-1}^{2^k-1} x_i^q w_i^q \leq \sum_{k=1}^{\infty} 2^{k-1} x_{2^k-1}^q w_{2^k-1}^q = \sum_{k=0}^{\infty} x_{2^k}^q 2^k w_{2^k}^q. \]

and so (57) follows.

Moreover, since \( \|e_k\|_{l^q(\mu)} = \mu_k, k = 1, 2, \ldots \), one can easily see that

\[ \|x\|_{l^q(\mu)}^q \geq x_1^q w_1^q + \sum_{k=1}^{\infty} x_{2^k}^q \sum_{i=2^k-1}^{2^k-1} w_i^q \geq x_1^q w_1^q + \sum_{k=1}^{\infty} x_{2^k}^q 2^{k-1} w_{2^k}^q \geq \frac{1}{2} \sum_{k=0}^{\infty} x_{2^k}^q 2^k w_{2^k}^q, \]

and so (57) follows.

From the first of these equalities and (55) it follows

\[ k_-(l^q(\mu)) = \lim_{n \to \infty} \|\tau_n\|_{l^q(\mu)}^{1/n} < 1. \]

This relation combined with equivalence (57) indicates that the Banach lattice \( l^q(\mu) \) satisfies all the conditions of Proposition 5, and hence we have \( l^q(\mu) = E_{l^q(\mu)} \) (with equivalence of norms). Moreover, by the above formulae for the norms of the shift operators in \( l^q(\mu) \) and Lemma 4, we get (56).

### 6.2 Orlicz spaces

Given an Orlicz function \( N \), the Orlicz sequence space \( l_N \) consists of all sequences \( a = (a_k)_{k=1}^\infty \), for which the norm

\[ \|a\|_{l_N} = \inf \left\{ u > 0 : \sum_{k=1}^{\infty} N\left( \frac{|a_k|}{u} \right) \leq 1 \right\} \]

is finite. Recall that \( l_N \) is separable if and only if the function \( N \) satisfies the \( \Delta_2 \)-condition at zero (see Sect. 2.2). It can be easily verified that the latter condition is equivalent to the fact that the lower Boyd index of \( l_N \) (see (8) or (9)) is positive. Since \( l_N \) is of fundamental type (cf. Sect. 2.3 or [5]), we arrive at the following characterization of the set of \( p \) such that \( \ell^p \) is symmetrically finitely represented in Orlicz spaces.
Theorem 7 Let $N$ be an Orlicz function such that for some $r > 0$

$$\inf_{0 < st \leq 1} \frac{N(st)}{N(s)r^r} > 0.$$  

Then, the following conditions are equivalent:

(i) $\ell^p$ is symmetrically block finitely represented in the unit vector basis $\{e_k\}$ of the Orlicz space $l_N$;

(ii) $\ell^p$ is symmetrically finitely represented in $l_N$;

(iii) $\ell^p$ is crudely symmetrically finitely represented in $l_N$;

(iv) $p \in \left[\frac{1}{\beta_N}, 1/\alpha_N\right]$, where $\alpha_N$ and $\beta_N$ are the Boyd indices of $l_N$ (cf. (8) or (9)).

In conclusion, we note that $E_{l_N} = U_N$ isometrically, where $U_N$ is the Banach sequence lattice equipped with the norm

$$\|a\|_{U_N} := \inf \left\{ u > 0 : \sum_{k=1}^{\infty} 2^{k-1} N\left(\frac{|a_k|}{u}\right) \leq 1 \right\}.$$  

Indeed, for every $a = (a_k)_{k=1}^{\infty}$, we have

$$\|a\|_{E_{l_N}} = \left\| \sum_{k=1}^{\infty} a_k \sum_{i=2^{k-1}}^{2^k-1} e_k \right\|_{l_N} = \inf \left\{ u > 0 : \sum_{k=1}^{\infty} \frac{1}{u} \sum_{i=2^{k-1}}^{2^k-1} N\left(\frac{|a_k|}{u}\right) \leq 1 \right\}$$

$$= \inf \left\{ u > 0 : \sum_{k=1}^{\infty} 2^{k-1} N\left(\frac{|a_k|}{u}\right) \leq 1 \right\} = \|a\|_{U_N}.$$  

References

1. Albiac, F., Kalton, N.J.: Topics in Banach Space Theory, Graduate Texts in Mathematics 233. Springer, New York (2006)

2. Astashkin, S.V.: On the finite representability of $\ell^p$-spaces in rearrangement invariant spaces. Algebra i Analiz 23(2), 77–101 (2011), (in Russian); English transl. in St. Petersburg Math. J. 23(2012), no. 2, 257–273

3. Astashkin, S.V.: Symmetric finite representability of $\ell^p$-spaces in rearrangement invariant spaces on $(0, \infty)$. Math. Ann. (2021). https://doi.org/10.1007/s00208-021-02277-5

4. Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, Boston (1988)

5. Boyd, D.W.: Indices for the Orlicz spaces. Pac. J. Math. 38, 315–323 (1971)

6. Dvoretzky, A.: Some results on convex bodies and Banach spaces, pp. 123–160. Jerusalem, Proc. Symp. on Linear Spaces (1961)

7. Johnson WB, Maurey B, Schechtman G, Tzafriri L (1979) Symmetric structures in Banach spaces, Mem. Am. Math. Soc. No. 217
8. Ivanov, S., Kalton, N.: Interpolation of subspaces and applications to exponential bases. Algebra i Analysis 13(2), 93–115 (2001), (in Russian); English transl. in St. Petersburg Math. J. 13(2002), no. 2, 221–239.

9. Kalton, N.J.: Calderón couples of rearrangement invariant spaces. Stud. Math. 106(3), 233–277 (1993).

10. Kantorovich, L.V., Akilov, G.P.: Functional analysis, 2nd edn. Pergamon Press, Oxford-Elmsford, New York (1982).

11. Kirillov, A.A., Gvishiani, A.: Theorems and Problems in Functional Analysis. Springer Verlag, New York-Heidelberg-Berlin (1982).

12. Krasnosel’skii, M.A., Rutickii, Ya. B.: Convex functions and Orlicz spaces. Noordhoff, Groningen (1961).

13. Krein, S.G., Petunin, Yu.I., Semenov, E.M.: Interpolation of linear operators, Nauka, Moscow, 1978 (in Russian); English transl. in Transl. Math. Monogr., vol. 54, Amer. Math. Soc., Providence, RI, (1982).

14. Krivine, J.L.: Sous-espaces de dimension finie des espaces de Banach réticulés. Ann. Math. (2) 104, 1–29 (1976).

15. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces. Springer-Verlag, Berlin-New York, I. Sequence Spaces (1977).

16. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces. Springer-Verlag, Berlin-New York, II. Function Spaces (1979).

17. Lorentz, G.G.: On the theory of spaces $\Lambda$. Pac. J. Math. 1, 411–429 (1951).

18. Maligranda, L.: Indices and interpolation. Dissertationes Math. 234, 1–54 (1985).

19. Maligranda, L.: Orlicz Spaces and Interpolation. Seminars in Mathematics 5, University of Campinas, Campinas (1989).

20. Maurey, B., Pisier, G.: Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. Stud. Math. 58, 45–90 (1976).

21. Milman, V.D., Schechtman, G.: Asymptotic Theory of Finite Dimensional Normed Spaces, Lect. Notes in Math., vol. 1200, Springer–Verlag, Berlin, (1986).

22. Pisier, G.: Factorization of Linear Operators and Geometry of Banach Spaces, Amer. Math. Soc., Providence, RI, CBMS 60 (1986).

23. Rao, M.M., Ren, Z.D.: Theory of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146 - Marcel Dekker Inc., New York, (1991).

24. Rosenthal, H.P.: On a theorem of J.L. Krivine concerning block finite representability of $\ell^p$ in general Banach spaces. J. Funct. Anal. 28, 197–225 (1978).

25. Schep, A.R.: Krivine’s theorem and the indices of a Banach lattice, Positive Operators and Semi-groups on Banach Lattices, (Curaçao, 1990). Acta Appl. Math. 27, 111–121 (1992).

26. Shimogaki, T.: A note on norms of compression operators on function spaces. Proc. Jpn. Acad. 46, 239–242 (1970).

27. Tsirel’son, B.S.: It is impossible to imbed $\ell^p$ or $c_0$ into an arbitrary Banach space. Funktsional. Anal. i Prilozhen. 8(2), 57–60 (1974), (in Russian); English transl. in Funct. Anal. Appl. 8 (1974), 138–141.