Many-body localization in a random $x-y$ model with the long-range interaction

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The possibility of many-body localization is investigated in a random $x-y$ model with a long-range interaction $V(R) \propto R^{-\alpha}$ in the thermodynamic limit of an infinite number of spins. We argue that a localization is possible only at $\alpha \geq 3d/2$ while at lower exponents $\alpha$ the delocalization takes place due to the combined effect of the $x-y$ interaction and the induced “Ising” interaction $R^{-2\alpha}$. These predictions are consistent with the numerical finite size scaling. Thus many-body localization can be easier attained in the systems of interacting spins with the only $x-y$ interactions ($\alpha \geq 3d/2$) compared to the spins coupled by mixed interactions ($\alpha \geq 2d$), which makes the spin systems with the interactions like in an $x-y$ model attractive for quantum information applications.

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Many-body localization-delocalization transition serves as a natural generalization of a single particle Anderson localization concept \cite{1} to interacting quantum systems at finite temperature \cite{2, 3} and as a quantum mechanical extension of a classical transition between deterministic and chaotic behaviors \cite{4, 5}. Localized system shows non-ergodic behavior where each small subsystem of it remembers its initial state during an infinite time. For example in the localized state of the system of interacting spins each spin remembers its initial projection to the quantization axis. In the delocalized state the rest of the system serves as a thermal bath for each part of it (spin or particle) promoting its relaxation to its thermal equilibrium. Many-body localization has been considered in a variety of physical systems \cite{6, 11} and it attracts growing attention because of its significance in quantum informatics \cite{12} where the memory of initial state is crucially important, and in cold atomic systems \cite{13, 14}, which can emulate various models for localization-delocalization transitions.

Many-body interaction often leads to the localization breakdown because it opens the new channels for system evolution \cite{2, 10, 11, 14, 16–19} (see however \cite{6, 7, 20} where the localizing effect of quasi-static interaction has been exploited). The long-range interaction $R^{-\alpha}$ is especially important since it creates the channels for the long-distance particle or energy hops dramatically increasing the number of delocalization pathways. For instance if the excitation hopping amplitude depends on the distance as $R^{-\alpha}$ with $\alpha < d$ where $d$ is the space dimension the excitations are delocalized at arbitrarily strong disordering in the limit of the infinite system size \cite{11, 21} while at the finite temperature and in the presence of long range Ising interaction $R^{-\beta}$ ($\beta < \alpha$) the inevitable delocalization is expected at $\beta < 2d$ \cite{11, 14, 13}.

A disordered $x-y$ model of interacting spins $1/2$ occupies an intermediate position between single particle and many-body problems \cite{22}. For the nearest neighbor interaction in one dimensional systems it can be reduced to non-interacting Fermions \cite{23} and a similar behavior has been expected in a strongly disordered regime with the long-range interaction $V(R) \propto R^{-d}$ \cite{24}. Consequently one can expect the many-body localized phase to be more stable in this model compared to the systems with significant Ising $S^z - S^z$ interactions \cite{11, 14, 19} so it can be of interest for quantum computing where the chaotic behavior should be avoided. Indeed, this model has been considered for quantum information transport \cite{25}. Random $x-y$ model with the long range interaction is relevant for a variety of physical systems ranging from Josephson junction arrays \cite{20} to cold atomic systems \cite{13, 14} and the long range interaction there can exist there due to dipolar, magnetic or elastic interactions. The many-body localization in the problem with mixed long-range Heisenberg interaction has been considered in many works \cite{11, 14, 19}, while to our knowledge $x-y$ model has not been analyzed yet. Here we perform such analysis.

In this paper we investigate the possibility of a many-body localization in a system of spins $1/2$ in strong random fields with the long-range hopping interaction $V_{ij} \propto R_{ij}^{-\alpha}$. Although this model is somewhat similar to the previously considered Heisenberg type models of interacting spins \cite{11, 14, 19} it lacks the Ising $S^z - S^z$ spin-spin interaction that is essentially responsible for the many-body localization breakdown there. This interaction creates the direct energy exchange between resonant pairs of spins and such interaction does not present in the $x-y$ model.

Here we show that the induced Ising like interaction exists in the $x-y$ model and it is generated in the third order of perturbation theory in $x-y$ interactions (see Fig. \ref{fig:fig1}). This induced interaction decreases with the distance between spins as squared flip-flop interaction $V_{ij}^2 \propto R^{-2\alpha}$. The delocalization under these conditions can be approximately described as the consequence of resonant coupling of “extended pairs” \cite{14} leading to the unavoidable delocalization within the thermodynamic limit for $\frac{\alpha \beta}{\alpha + \beta} < d$, where $\beta$ is the power law exponent for the Ising spin-spin interaction. Since in our case of $x-y$ model the induced
Ising interaction is characterized by the exponent $\beta = 2\alpha$ an unavoidable breakdown of many-body delocalization can be expected for $\alpha < 3d/2$ (see the full summary of results in Table 1).

Below we present analytical derivation of delocalization criteria in $x - y$ model, discuss a general problem of mixed weak Ising and strong $x - y$ interactions not covered in earlier work, and report the numerical study of many-body localization in $x - y$ model for long-range $x - y$ interactions mostly supporting our expectations. As in the majority of considerations \[14, 19\] we examine the infinite temperature limit, which can be well represented by the states with energies close to 0 and zero total spin \[19\].

![FIG. 1: Many body interaction of two resonant pairs of spins (1,2) and (3,4) assisted by neighboring spin 5. Wavelines shows involved significant spin-spin interactions.]

We consider $N$ interacting spins in random independent fields $\phi_i$ along the z-axis ($\phi_i S_i^z$) uniformly distributed within the domain ($-W/2, W/2$). These spins are placed with the density $n$ into a $d$-dimensional hypercube and coupled by the flip-flop interaction $V_{ij} (S_i^+ S_j^- + S_j^+ S_i^-)$ decreasing with the distance according to the power law $V_{ij} = v_{ij}/R_{ij}^d$ with random uncorrelated interaction constants $v_{ij} \sim V_0$ (in numerical analysis we set $v_{ij} = \pm V_0$) so that the interaction between spins at the average distance can be estimated as $\tilde{V} \approx V_0 n^{\frac{d}{d-2}}$. Similarly to Refs. \[11, 14, 19\] we consider a limit of strong disordering $\tilde{V} \ll W$ and interaction exponents not less than the system dimension $\alpha \geq d$ so a single particle delocalization can be approximately neglected \[1, 21\].

Under these assumptions the spin dynamics is primarily associated with rare resonant pairs formed by the pairs of oppositely oriented spins having close random energies $|\phi_i - \phi_j| \leq |V_{ij}|$ (spins (1,2) and (3,4) in Fig. 1) so the static energy change due to flip-flop transition is compensated by their hopping interaction. The probability that the given pair of spins is resonant can then be estimated as $P_{ij} \approx \frac{|V_{ij}|}{W}$. If we consider the pairs of close energies formed by spins located at the distance of order of $R$ (say, $R < r_{ij} < 2R$) then the probability to form resonant pair of size $R$ can be estimated as $P(R) \sim n R^{d-1} \frac{V_0}{W^{d-1}} \approx \frac{V}{W(n R^d)^{d-2}}$ and their density is given by $n(R) = n P(R)$. Consequently the average distance between them is given by

$$r(R) = n(R)^{-\frac{d}{d-2}} = \left( \frac{W}{n V} \right)^{\frac{d}{d-2}} (n R R^d)^{-\frac{d}{d-2}} \sim R^{\frac{d-2}{d-2}}.$$ \hspace{1cm} (1)

According to Eq. (1) for the sufficiently small interaction law exponent $\alpha < 2d$ the distance between pairs increases with their size $R$ slower than that size. In the presence of an Ising interaction $U(r) \propto r^{-\alpha}$ that interaction induces delocalization in the subsystem of resonant pairs when these two sizes approach each other $r(R) \approx R$ \[11, 14, 19\] creating irreversible resonant energy $(V_0 R^{-\alpha})$ transport between pairs. Here in $x - y$ model we don’t have such interaction.

However, the effective Ising interaction shows up in an $x - y$ model. It appears in the third order of perturbation theory in hopping interaction $V_i$. Indeed, if we consider the random field Hamiltonian $\tilde{H} = \sum_{i=1}^{N} \phi_i S_i^z$ as a zeroth order approach and the “flip-flop” interaction as a perturbation there is no first order correction to energy while the second order corrections from each pair of spins $i, j$ reads $V_{ij}^2 (1/2 S_i^z (1/2 - S_j^z) + V_j^2 (1/2 - S_i^z) (1/2 + S_j^z)) = -\frac{V_j^2 (S_i^z - S_j^z)}{\phi_i - \phi_j}$ (the product of factors $1/2 \pm S^2$ chooses the spin states having non-zero perturbation matrix elements). This correction modifies random fields for each spin but does not lead to a spin-spin interaction because the two spin $x - y$ problem can be reduced to independent Fermions \[23, 24\].

The first non-zero contribution to interaction appears in the third order in $V$ from three spin loops (see spins 1, 3 and 5 in Fig. 1 cf. Ref. \[24\]). The specific contribution to the energy from the sequence of flip-flop transitions $(i, j), (j, k)$ and $(k, i)$ can be expressed as $V_{ij} V_{jk} V_{ki} \times \left( \frac{1}{(1/2 - S_i^z)(1/2 - S_j^z)(1/2 - S_j^z)} + (1/2 + S_j^z)(1/2 + S_j^z)(1/2 - S_i^z) \right) \left( \frac{1}{(\phi_i - \phi_k)(\phi_j - \phi_k)} \right)$

Non-zero contributions are associated with the constant term and the term containing binary products of spin $z$-projection operators. The constant term disappears after the summation of permutation contributions, while the binary interaction term is finite and can be expressed as

$$H_{int} = \frac{1}{2} \sum_{i,j,k} V_{ij} V_{jk} V_{ki} S_i^z S_j^z S_k^z \left( \frac{1}{(\phi_i - \phi_k)(\phi_j - \phi_k)} \right).$$ \hspace{1cm} (2)

The main contribution to the Ising spin-spin interaction of spins $i$ and $j$ comes from the “assisting” spins $k$ located either in the direct neighborhood of spin $i$ or spin $j$. This is because the interaction is weak $V_{ik}, V_{jk} \ll W$ and decreases with the distance faster than $r^{-d}$, so the sign variable sums in Eq. (2) are determined by short distances. The typical random field difference for few involved spins is generally given by the energy disordering $\phi_i - \phi_j \sim \phi_i - \phi_k \sim \phi_j - \phi_k \sim W$. Consequently, the Ising spin-spin interaction $U_{i,j} S_i^z S_j^z$ generated in the
TABLE I: Constraints for power law interaction exponents $\alpha$ for $x-y$ hopping interactions and $\beta$ for $z-z$ Ising anisotropic interactions permitting many-body localization in the thermodynamic limit $N \to \infty$.

| Model | $\alpha > \beta$ | $\alpha < \beta < 2\alpha$ | $2\alpha < \beta$ |
|-------|------------------|-----------------------------|------------------|
| Spins | $\beta > 2d$ | $\frac{\alpha \beta}{\alpha d + \beta} > d$ | $\alpha > 3d/2$ |

This result describes the interaction of resonant pairs separated by the distance $R$ (see Fig. 1) which is capable to transfer energy between them due to collective flip-flop transitions in each pair associated with the off-diagonal matrix elements of both $S^z$ operators. These matrix elements are of order of unity for resonant pairs [11, 14, 19].

At very large size $R$ of resonant pairs and small interaction law exponent $\alpha < 2d$ the distances between adjacent resonant pairs $(r(R), \text{Eq. (1)})$ gets much smaller than their sizes. However, this small distance will be between only two of spins belonging to each pair (spins 1 and 3 in Fig. 1) while two others (2 and 4) are located at distance $R \gg r(R)$. Therefore the most significant interaction is between spins 1 and 3, i.e. $U_{13}S_i^zS_j^z$. According to Eq. (3) this interaction ($U_p(R) \sim U_{ij}(r_{ij} = r(R))$) can be estimated as

$$U_p(R) \sim V \left(\frac{V}{W}\right)^{\frac{2d}{d-a}} (n^{1/2}R)^{-\frac{2\alpha - d}{a}}. \tag{4}$$

Interactions Eq. (4) describe excitation hopping between resonant pairs having characteristic energy $V(R) \approx V_0R^{-\alpha}$. Following Refs. [11, 14, 19] we expect that this interaction results in delocalization of energy in the subsystem of resonant pairs if it exceeds the typical pair energy $U_p(R) > V(R)$. Comparing two expressions one can conclude that this happens inevitably in the limit of an infinite system size $R$ if $\alpha < 3d/2$. The critical system size $R_*$ corresponding to the delocalization $V(R_*) \sim U_p(R_*)$ at given disordering $W \gg V$ and the critical disordering needed for the many-body localization at a given size $R$ (number of spins $N = nR^d$) can be estimated accordingly as

$$R_* = n^{-\frac{1}{2}} \left(\frac{W}{V}\right)^{\frac{d-a}{d-a}}, \quad W_c \approx \left(n^{\frac{1}{2}}R\right)^{\frac{d}{3d-a}}. \tag{5}$$

The constraint $\alpha > 3d/2$ is consistent with the “extended pairs” criterion $\frac{\alpha}{\alpha + d} > d$ suggested in Ref. [14] for the case of flip-flop and Ising interactions decreasing with the distance as $R^{-\alpha}$ and $R^{-\beta}$, respectively, if we set there $\beta = 2\alpha$. Eq. (4) This criterion is most important in the case of $\alpha < \beta$, not considered in Ref. [14]. In the general case of two interactions and $\beta < \alpha$ one should use a minimum Ising interaction exponent $\beta_* = \min(\beta, 2\alpha)$ and use it in extended pair criterion (see Table I) to find the general constraint for interaction power law exponents permitting many-body localization in a thermodynamic limit.

To verify the analytical predictions (Table I) we performed the numerical finite size scaling analysis of many-body localization in a random one dimensional $x-y$ model (see [13, 31] for detail). The localization has been characterized using the ergodicity parameter defined as the local spin-spin correlation function $Q = \sum_m <m|S^z_i(t)S^z_i(0)|m > \delta(E_m)/\sum \delta(E_m)$ taken in the infinite time limit [30] and averaged over the narrow band of system eigenstates near zero energy and with the zero total spin. The latter choices correspond to the infinite temperature thermodynamic limit often used to study many-body localization [27, 28]. In that limit the average ergodicity parameter should approach zero in the delocalized state where correlations are subject to decay, while in the localized state it should be finite (unity in an infinite disordering limit). Our method is not as efficient in the definition of the localization transition point as recently developed analysis of the fluctuations of entanglement entropy [32]; yet it permits the easy determination of the scaling of the localization transition with the system size which is our main target.

![FIG. 2: Original (inset) and rescaled dependencies of ergodicity parameter $Q$ on disordering $W$ for the interaction power law exponent $\alpha = 1$.](image)
disordering ranging between \( W = 1 \) and \( W = 150 \) (we set \( V = n = 1 \) [31]). In all systems the localization transition can be seen with increasing disordering; i.e. \( Q \) approaches 0 at small \( W \) and tends to 1 in the opposite limit of \( W \to \infty \) (see Figs. 2[3] and [31]). However these transitions behave differently for different exponents \( \alpha \).

The results for two limiting cases of the smallest exponent \( \alpha = 1 \) and the largest exponent \( \alpha = 2 \) are shown in the inset of Fig. 2 and in Fig. 3. It is quite clear that in the first case the transition shifts towards large disordering \( W \) with increasing the number of spins, while in the second case the transition is almost insensitive to the system size. This agrees with the qualitative expectation of the threshold exponent \( \alpha_c = 3/2 \) separating the regime where the localization is impossible in the large \( N \) limit \((\alpha < 3/2)\) and the opposite case where it is possible. Similar conclusions can be drawn for other exponents (see Ref. [31]).

For quantitative characterization of the critical disordering \( W_c \) dependence on the number of spins \( N \) we use the scaling function [19]

\[
f_\alpha(N) = \frac{N}{2(N-1)} \left( 2 \sum_{n=1}^{N/2-1} n^{\alpha-1} + \left(\frac{2}{N}\right)^{\alpha-1}\right).
\]

This function replaces the continuous power law dependence \( N^\alpha \) with the discrete sum of resonance probabilities which accounts better for finite size effects. For \( \alpha > 0 \) this function has the asymptotic behavior \( f_\alpha(N) \propto N^\alpha \).

In Fig. 2 we show the modified plot of the data for \( \alpha = 1 \) with disordering rescaled using the function Eq. [3] with the theoretically predicted exponent \( a = (\alpha - 3d/2)/(1 + \alpha) = 1/4 \) Eq. [3]. This rescaling leads to an excellent match between the graphs for different numbers of spins \( N \) indicating the good agreement of theoretically predicted and numerically found scaling of the transition point with the system size. Similar qualitative agreement is attained for other interaction law exponents [31].

In addition to the visual analysis we performed quantitative estimates of rescaling parameters for the data sets \( Q(W) \) with the same exponent \( \alpha \) and different \( N \)'s using the optimization procedure [19]. For each exponent \( \alpha \) we determined the set of optimum rescaling parameters \( c_\alpha(N) \) corresponding to the minimum of the squared deviation \( \sum_i (Q_{14}(W_i) - Q_N(c_\alpha(N))W_i)^2 \) with \( W_i \) changing with the step of 2 from 2 to 100. The results for different rescaling parameters \( c(\alpha, N) \) are shown in Fig. 3 by symbols defined within the graph.

To characterize the change of the rescaling factors and consequently the critical disordering \( W_c \) with the number of spins we fitted each data set by the function \( f_\alpha(N) \) Eq. [3] choosing the parameter \( a \) to attain the best agreement of the model with the numerical results. The lines show these optimum fits and the exponents \( a \) are also shown in the legend. The scaling function \( f_\alpha(N) \) serve as the crossover between the regimes of finite \((\alpha < 0)\) and infinite \((\alpha > 0)\) localization thresholds in the thermodynamic limit \( N \to \infty \). In agreement with the theoretical expectations of the crossover at \( \alpha = 3/2 \) we found \( a > 0 \) for \( \alpha = 1, 1.25, \alpha < 0 \) for \( \alpha = 1.75, 2 \), while situation is not clear at the threshold \( \alpha = 1.5 \) as shown in the inset to Fig. 4. The estimated exponents are, however, somewhat larger than the theory predictions which can be due to finite size effects. Also logarithmic increase
of the number of resonant interactions with increasing the number of spins for $\alpha = 1$ [33] can lead to observed deviations.

Thus we investigated many-body localization in an $x - y$ model with the long-range hopping interaction decreasing with the distance according to the law $R^{-\alpha}$. Generalizing the previous considerations we suggested the scenario of localization breakdown due to the Ising interaction of extended resonant pairs [14, 19]. This Ising interaction is generated in the third order of perturbation theory in the hopping interaction and decreases with the distance as $R^{-2\alpha}$. Our analysis predicts an inevitable breakdown of many-body localization in the thermodynamic limit of infinite system for sufficiently small interaction exponents $\alpha < 3d/2$. This result has been verified using the numerical finite size scaling.

It turns out that many-body localization in $x - y$ model is much more stable with respect to the long-range interaction then the model of spins having both Ising and $x - y$ long-range interactions [19]. For instance a three dimensional $x - y$ model with quadrupole interaction ($\alpha = 5$) can have many-body localization in thermodynamic limit, while the system with quadrupole Ising interaction cannot. Thus spin systems with only $x - y$ interaction can be more attractive for quantum memory applications. It will be of a great interest to implement an $x - y$ model in cold atomic systems [14], where the predictions of this work can be verified.

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Here we describe in detail the numerical finite-size scaling analysis performed for $x - y$ model. In this model we study $N$ spins $1/2$ placed into equally separated chain sites $i = 1, \ldots, N$, which can be described by the Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \phi_i S_i^z + \sum_{i \neq j} V_{ij} S_i^+ S_j^-.$$  \hfill (7)

Random potentials $\phi_i$ are uncorrelated in different sites and uniformly distributed within the domain $(-W/2, W/2)$. The interaction is defined in a periodic manner as

$$V_{ij} = \pm \frac{1}{(\min(|i-j|, |N-|i-j||))^{\alpha}},$$  \hfill (8)

where the sign of interaction constant is chosen randomly for each pair of spins. This model is similar to the models studied in Refs. [14, 19] with the only difference that the Ising spin-spin interaction is set equal to zero here.

To characterize the localization we use the ergodicity parameter defined as a configuration averaged spin-spin correlation function at infinite time $[30]$

$$Q = 4 \langle \delta S^z(\infty) \delta S^z(0) \rangle' = \frac{1}{N_\alpha} \sum_{\alpha} |\langle \alpha| \delta S^z |\alpha \rangle|^2,$$

$$\delta S^z = S^z - < S^z >.$$  \hfill (9)

The averaging $< ... >'$ is performed over the narrow band of eigenstates $\alpha$ of the Hamiltonian Eq. 7 around zero energy ($-\delta < E_\alpha < \delta$), $\delta = 0.04W\sqrt{N}$ and $N_\alpha$ is the number of states in this energy domain. The many-body density of states $g(E) \approx \exp \left( -\frac{E^2}{24\pi NW^2} \right) / \sqrt{24\pi NW^2}$ changes at the scale $\delta$ by around 1% which is the reason for our choice of the bandwidth. The function $g(E)$ has been estimated using the random potential part of the system Hamiltonian $-\sum_{k=1}^{N} \phi_k S_k^z$ assuming the law of large numbers ($N \gg 1$) and strong disordering $\tilde{U} \ll W$. In addition, following Refs. [14, 19] we consider only states with a zero total spin which corresponds to the maximum number of states and the strongest effect of resonant interactions. Consequently $< S^z > = 0$ and $\delta S^z = S^z$ in Eq. (9).

Our consideration corresponds to an infinite temperature limit. We expected that the ergodicity parameter $Q$ approaches zero in delocalization regime ($N \to \infty$) because of the correlation decay and remains finite otherwise [19].

Calculations of ergodicity parameter have been performed using Matlab software on the Linux cluster available through the Tulane University Center for Computational Science [34]. Random Hamiltonians have been generated.
for interaction exponents $\alpha = 1, 1.25, 1.5, 1.75, 2$ and disordering $2 < W < 150$, and total even numbers of spins $8 \leq N \leq 16$. The results have been averaged over a sufficiently large number of realizations chosen to make the relative error of the estimate less than 1% (5% for $N = 16$).

The results of calculations of ergodicity parameter are shown for different exponents $\alpha$ and number of spins $N$ in Figs. 6 (a), 7 (a), and 8. All curves $Q(W)$ shows the transition from a delocalized regime $Q \ll 1$ to a localized regime $Q \approx 1$. Yet the graphs for different interaction exponents show different sensitivity to the number of spins. It is clear from straight visual inspection of all graphs that in the case of the smallest exponent $\alpha = 1$ there is a noticeable shift of the transition towards large disordering with increasing the number of spins $N$, while for the largest exponent $\alpha = 2$ there is almost no effect of increase in $N$ on the localization transition. This is qualitatively consistent with the theoretical definition of critical exponent $\alpha_c = 3/2$ suggested in the main body of the manuscript. At smaller exponent $\alpha < \alpha_c$ localization is impossible in the thermodynamic limit $N \to \infty$, while it is possible at larger exponents.
For more accurate characterization of size scaling of localization threshold we introduce the rescaling function \[19\]

\[
f_a(N) = \frac{N}{2(N-1)} \left( 2 \sum_{n=1}^{N/2-1} n^{a-1} + \left( \frac{2}{N} \right)^{a-1} \right). \tag{10}
\]

This function represents the size dependence for the number of resonant interactions per spin decreasing with the distance as \(n^{a-1}\), which is expected from the discrete version of the analytical derivation. The prefactor \(\frac{N/2}{N-1}\) accounts for the number of spins oriented in the opposite direction to the given spin so they can perform a joint flip-flop transition. For \(a > 1\) this function has the right asymptotic behavior \(f_a(N) \propto N^a\) and it accounts for finite size effects which takes place at small exponents \(a\) and finite numbers of particles.

**FIG. 8:** Dependencies of ergodicity parameter \(Q\) on disordering \(W\) for the interaction power law exponents \(\alpha = 1.75\) (a) and \(\alpha = 2\) (b).

To compare theory predictions for critical disordering size dependence \(W_c \propto f_a(N)\) with \(a = \frac{3d/2-\alpha}{d}\) (see the main body of the manuscript) with calculation results we plot rescaled dependencies \(Q_N(f_a(14)W/f_a(N))\) for \(\alpha = 1, 1.25, 1.5\) in Figs. 2 (b), 6 (b) and 7 (b). The visual inspection of graphs shows that the rescaled curves intersect nearly at the same point, which approximates a many-body localization transition.

In addition to the visual inspection we performed quantitative estimates of rescaling parameters using the optimum rescaling procedure \[19\]. For each exponent \(\alpha\) we determined the set of optimum rescaling parameters \(c_\alpha(N)\) corresponding to the minimum of the squared deviation \(\Delta = \sum_i (Q_{14}(W_i) - Q_N(c_\alpha(N)W_i))^2\) with \(W_i\) changing with the step of 2 from 2 to 100. The results for scaling parameters are shown in Fig. 9 (a) by symbols. Then these results have been fitted using the functions \(f_a(N)\) choosing optimum exponents \(a\) leading to the minimum deviation \(\Delta\). The function \(f_0(N)\) is shown in Fig. 9 (a) as the crossover line between the unlimitedly increasing critical disordering for \(N \to \infty\) at positive \(a\) and the critical disordering approaching the finite limit at \(a < 0\). As expected from the theory (see the main body of the manuscript) the crossover takes place at \(\alpha \approx 1.5\). Indeed, the estimated exponents \(a\) are positive for \(\alpha < 3/2\) and negative otherwise as shown in Fig. 9 (b). Consequently the numerical analysis is consistent with the theoretical expectations. The estimated exponents are, however, somewhat larger than the theory predictions which can be due to finite size effects. Particularly logarithmic increase of the number of resonant interactions with increasing the number of spins \[33\] for \(\alpha = 1\) can lead to the observed deviation.

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FIG. 9: Scaling parameters vs. the number of spins for different interaction exponent and the optimum fit of those dependencies using the function $f_\alpha(N)$ (a) and the dependence of estimated scaling exponents $a$ for critical disordering on the power law interaction exponents $\alpha$. Red line indicate the threshold value $a = 0$ (b).