Steady state entanglement of two atoms created by classical driving field

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The stabilization of steady state entanglement caused by action of a classical driving field in the system of two-level atoms with the dipole interaction accompanied by spontaneous emission is discussed. An exact solution shows that the maximum amount of concurrence that can be achieved in Lamb-Dicke limit is 0.43, which corresponds to the entanglement $E_{\text{max}} = 0.285$ ebit. Dependence of entanglement on interatomic distance and classical driving field is examined numerically.

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I. INTRODUCTION

It is well known that quantum entanglement plays fundamental role in the quantum information processing and quantum computing [1, 2]. The practical application of entanglement requires the robust entangled states. This notion includes long enough lifetime of the states and high amount of entanglement (as close to maximum entanglement as possible).

The simplest entangled states of two or more qubits can be modelled physically by the states of two-level atoms, which have been successfully used for decades as the main tool for testing fundamentals of quantum mechanics [3]. In particular, a number of important experiments on generation and manipulation with entangled states in atomic systems have been performed recently (e.g., see [4] and references therein).

The lifetime of entangled states in atomic systems is usually governed by the spontaneous emission rate and therefore is quite short [5]. The problem of generation of robust entangled states in atomic systems is of high importance. In fact, this problem belongs to a more general class of problems related to the so-called quantum decoherence. This branch of modern physics examines an irreversible evolution of quantum systems under influence of a dissipative environment (see [6] and references therein).

Thus, using the ideas of quantum decoherence theory, we have to choose a proper environment for an atomic system, providing an irreversible evolution of this system towards a stable (or at least metastable) entangled state.

For example, it has been proposed in Refs. [7, 8] to use the three-level atoms with Λ-type configuration of levels instead of the two-level atoms. In this case, an unstable maximum entangled state is created by absorption of a photon, resonant with the transition between the ground and highest excited level in the system of the two atoms. This state rapidly decays either to the ground state or to the intermediate excited state. The latter process is accompanied by generation of the so-called Stokes photon and leads to creation of a new maximum entangled state of two atoms. If the Stokes photon, generated by the transition between the highest and intermediate excited states, is then discarded through the use of an appropriate dissipative environment, the system remains in the maximum entangled state for a long time determined by the non-radiative decay of the intermediate atomic state.

In the above example, the maximum entangled atomic state is induced by the photon exchange between the atoms. Another physical proposal is based on the use of direct dipole interaction between the two-level atoms [9]. In this case, the energy loss in the system due to the spontaneous emission is supposed to be compensate by interaction with an environment. In Refs. [9] it was proposed to use the interaction of atoms with the squeezed vacuum state in addition to the dipole-dipole interaction. It was shown that a steady state with reasonable amount of entanglement can be achieved in this case.

This is not an unexpected result. The point is that it has been recognized recently that entanglement is peculiarly connected with the amount of quantum fluctuations in the system [10, 11, 12]. This fact can be used as an operational definition of entanglement [12].

By the total amount of quantum fluctuations we mean the sum of variances of observables $O_i$ which form an orthonormal basis of Lie algebra of the dynamic symmetry group $G$ of quantum system [11]. For example, in the case of two qubits (two two-level atoms, the dynamic symmetry group is $G = SU(2) \times SU(2)$, while the local observables are provided by the Pauli (spin projection) operators.

The environment of Ref. [9] provided by the squeezed vacuum always manifests a high level of quantum fluctuations, and by means of interaction with the atomic system may transmit the quantum fluctuations to the atomic system, which is supposed to be initially unentangled.

It seems to be interesting to consider the case when an entangled state is generated from initially unentangled state in the presence of non-fluctuating classical environment. In this note, we show that a reasonable amount of entanglement can be achieved in a system of two-level atoms with dipole interaction in presence of a classical driving field. Although the classical driving field is devoid of quantum fluctuations, it helps to realize the potential ability of entanglement hidden in the structure of the eigenstates of the system. Let us note that the stabilizing influence of the classical driving field on entanglement in a system of two-level atoms in a cavity with high quality factor has been discussed in Ref. [13].
The paper is arranged as follows. In Sec. II, we discuss the model Hamiltonian and its properties. We examine the structure of eigenstates and show the presence of entanglement. We show that basis of the corresponding Hilbert space can be divided into the antisymmetric singlet part, represented by a certain maximum entangled state, and the triplet part. In Sec. III, we show that under certain physical assumptions the master equation can be solved in the triplet domain of the Hilbert space. Under the assumption of short interatomic distances (Lamb-Dicke limit), we find an exact steady state solution which give high enough amount of entanglement. We also examine entanglement as a function of interatomic separation and classical driving field. Sec. IV contains discussion of the obtained results and conclusions.

II. PROPERTIES OF THE MODEL HAMILTONIAN

The two-atom system under consideration can be specified by the Hamiltonian

\[ H = \sum_{i=1}^{2} \left[ \frac{\Delta^{(i)}}{2} \sigma_{z}^{(i)} + \epsilon^{(i)} \right] \sigma_{+}^{(i)} e^{i k_{i} \cdot \vec{r}} + \sigma_{-}^{(i)} e^{-i k_{i} \cdot \vec{r}} \]

\[ + \Omega (\sigma_{+}^{(1)} \sigma_{-}^{(2)} + \sigma_{-}^{(1)} \sigma_{+}^{(2)}). \]

Here \( \Delta^{(i)} = \omega^{(i)} - \omega_{C} \) is the detuning between the atomic frequency \( \omega^{(i)} \) and classical driving field frequency \( \omega_{C} \) assumed to be small (\( \omega^{(i)} \gg |\Delta^{(i)}| \)). \( E^{(i)} \) represents the classical driving field at the location of the \( i \)-th atom, and the dipole-dipole coupling constant has the form \[ \Omega = \frac{3}{4} \sqrt{\Gamma^{(1)} \Gamma^{(2)}} \left[ (1 - |\vec{\mu}^{(i)}|^{2}) \cos k_{0} r \over k_{0} r 
\[ + (1 - 3|\vec{\mu}^{(i)}|^{2}) \left( \sin k_{0} r \over k_{0} r + \cos k_{0} r \over k_{0} r \right)^{2} \right], \] \]

where \( \vec{\mu}^{(i)} \) is the atomic dipole moment, \( k_{0} = (\omega^{(1)} + \omega^{(2)}) / 2c \), and

\[ \Gamma^{(i)} = \frac{k_{0}^{3} |\vec{\mu}^{(i)}|^{2}}{3\pi \hbar \epsilon_{0}}. \]

This Hamiltonian (1) is defined in the four-dimensional Hilbert space of two qubits \( \mathcal{H}_{2,2} \) spanned by the vectors

\[ |gg\rangle, \quad |eg\rangle, \quad |ge\rangle, \quad |ee\rangle, \]

where \( |\alpha\beta\rangle = |\alpha\rangle \otimes |\beta\rangle \) and \( |\alpha\rangle, |\beta\rangle \) denotes the excited and ground atomic states, respectively. This space \( \mathcal{H}_{2,2} \) represents in fact a product of the two two-dimensional "spin-1/2" spaces \( \mathcal{H}_{1/2} \) spanned by the vectors \( |e\rangle \) and \( |g\rangle \) each. This space can also be expanded as follows

\[ \mathcal{H}_{2,2} = \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_{0} \oplus \mathcal{H}_{1}, \]

where \( \mathcal{H}_{0} \) is the singlet space, consisting of the antisymmetric maximum entangled state

\[ |A\rangle = \frac{1}{\sqrt{2}} (|eg\rangle - |ge\rangle) \] (5)

("spin-0" space). Using the terminology of the phase-transition theory \[14\], we can associate this state with the "aniferromagnetic" order in the system of two dipoles.

In turn, the "spin-1" space \( \mathcal{H}_{1} \) is spanned by the triplet of symmetric states

\[ \begin{align*}
| +1\rangle &= |ee\rangle \\
|0\rangle &= \frac{1}{\sqrt{2}} (|eg\rangle + |ge\rangle) \\
| -1\rangle &= |gg\rangle
\end{align*} \] (6)

Formally, this basis (6) can be associated with the states of a "spin-1" object. Namely, the states \( |+1\rangle \), \( |0\rangle \), and \( |-1\rangle \) correspond to the states with projections of "spin" equal to 1, 0, and −1, respectively. It should be noted that the state \( |0\rangle \) of a qutrit always manifests maximum entanglement \[11, 12\].

Using the analogy with the phase transition theory, we can say that the unentangled states \(|+1\rangle\) and \(|-1\rangle\) correspond to the "ferromagnetic" (parallel) order of the dipoles, while the maximum entangled state \(|0\rangle\) in (6) corresponds to the disordered "paramagnetic" state.

Assume now that the atoms are identical when we may put \( \Delta^{(1)} = \Delta^{(2)} \equiv \Delta \) and \( \Gamma^{(1)} = \Gamma^{(2)} \equiv \Gamma \). In addition, we assume that the polarization of classical driving field is parallel to the interatomic axis, so that \( E^{(1)} = E^{(2)} = E \). It can be easily seen that under these conditions the antisymmetric singlet state (5) is an eigenstate of the Hamiltonian (1) with the eigenvalue \( \epsilon_{A} = -\Omega^{2} \):

\[ H |A\rangle = -\Omega^{2} |A\rangle. \]

In the triplet sector, equation for eigenvalues takes the form

\[ \epsilon^{3} - \Omega^{2} \epsilon^{2} - (\Delta^{2} + 4E^{2}) \epsilon + \Delta^{2} \Omega = 0. \] (8)

In the absence of external field (\( E = 0 \)), this equation has the roots \( \epsilon_{\pm} = \Delta, \epsilon_{0} = \Omega \), and \( \epsilon_{-} = -\Delta \), and the set of the corresponding eigenstates is provided by the states (6). Thus, at \( \Omega > |\Delta| \), the "antiferromagnetic" maximum entangled state (5) is the ground state of the system in the absence of the external field. Otherwise, one of the states with "ferromagnetic" (parallel) order of atomic dipoles becomes the ground state. This is quite natural result, because the pseudo-spin Hamiltonian (1) has the same structure as the model of the so-called plane rotator in transverse field, examined in the theory of phase transitions \[14\]. It is clear that similar behavior takes place at small values of the classical driving field.

In the absence of detuning (\( \Delta = 0 \)) and an arbitrary value of the classical driving field, there is the following symmetric maximum entangled eigenstate

\[ |\psi_{0}\rangle = \frac{1}{\sqrt{2}} (|ee\rangle - |gg\rangle) \]
in the triplet sector, which corresponds to the eigenvalue $\varepsilon_0 = 0$, and the two eigenstates
\[ |\psi_{\pm}\rangle = \frac{1}{\sqrt{4E^2 + \varepsilon_\pm^2}} [E\sqrt{2}(|ee\rangle + |gg\rangle) + \varepsilon_{\pm}|0\rangle] \] (9)

with the eigenvalues
\[ \varepsilon_{\pm} = \frac{\Omega}{2} \pm \frac{1}{2} \sqrt{\Omega^2 + 16E^2}. \] (10)

The concurrence of the states (9) is then given by the expression \[12\]
\[ C = 2|\det[\psi_{\pm}]| = \frac{|4E^2 - \varepsilon_{\pm}^2|}{4E^2 + \varepsilon_{\pm}^2}, \] (11)

where $[\psi_{\pm}]$ denotes the matrix of coefficients in the states (9). For the lowest state $|\psi_+\rangle$ with the eigenvalue $\varepsilon_-$ in (10), the concurrence (11) achieves the maximum value $C = 1$ at $E = 0$ and then monotonically decreases with the increase of $E$. Similar behavior takes place in the case of state $|\psi_+\rangle$ with the eigenvalue $\varepsilon_+$. Thus, the eigenstates of the system with the Hamiltonian (1) can manifests entanglement and even maximum entanglement (at least at a certain choice of parameters). Therefore, all one can assume is that the system under consideration has a good potential ability to evolve towards an entangled state under influence of a proper dissipative environment.

III. IRREVERSIBLE EVOLUTION TOWARDS ENTANGLEMENT

The irreversible evolution assumes dissipation of energy. In the system under consideration, it can be described by the losses due to spontaneous decay of the excited atomic states. We can specify the irreversible evolution in the system described by the Hamiltonian (1) with the assumptions made in the previous Section by the following master equation \[16\]
\[ \dot{\rho} = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{2} \Gamma^{(ij)} (2\sigma^{(i)}_+ \rho \sigma^{(j)}_- - \sigma^{(i)}_- \rho \sigma^{(j)}_+ - \sigma^{(i)}_+ \sigma^{(j)}_- \rho - \rho \sigma^{(i)}_- \sigma^{(j)}_+), \] (12)

where the coefficients $\Gamma^{(ij)}$ are responsible for the spontaneous decay and have the form $\Gamma^{(ii)} = \Gamma^{(i)}$ and
\[ \Gamma^{(12)} = \Gamma^{(21)} = -3\sqrt{\Gamma^{(1)}}\Gamma^{(2)} \left[ \cos k \delta r - \sin k \delta r \right]. \] (13)

We now show that under a certain choice of the initial state, the antiferromagnetic maximum entangled state (5) can be discarded from the evolution picture described by the master equation (12). Since (5) is the eigenstate of the Hamiltonian (1), it does not contribute into the unitary evolution, described by the first term in the right-hand side of (12). We also have
\[ \sum_{i=1}^{2} \sigma^{(i)}_- |A\rangle = 0. \]

Thus, it does not contribute into the Liouville term in the right-hand side of (12). This means that if the system is prepared initially in the maximum entangled antisymmetric state (5), it will stay in this state forever. If the initial state is chosen differently, the irreversible evolution of the system should be considered in the triplet sector (6).

This means that the density matrix in Eq. (12) takes the following block form
\[ \rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{21} & \rho_{22} & \rho_{23} & 0 \\ \rho_{31} & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix} \] (14)

Here the blocks correspond to the bases (6) and (5), respectively. The non-Hermitian effective Hamiltonian, corresponding to the evolution described by the master equation (12) with the density matrix (14), has the form
\[ H_{eff} = \begin{pmatrix} \Delta - i\frac{\Gamma}{2} & \sqrt{2}E & \sqrt{2}E & 0 \\ \sqrt{2}E & \Omega - \frac{i}{2}(\Gamma + \Gamma^{(12)}) & \sqrt{2}E & 0 \\ \sqrt{2}E & \Omega - \frac{i}{2}(\Gamma - \Gamma^{(12)}) & -\Delta & 0 \\ 0 & 0 & 0 & -\Omega - \frac{i}{2}(\Gamma - \Gamma^{(12)}) \end{pmatrix} \] (15)

Here again $\Gamma^{(1)} = \Gamma^{(2)} = \Gamma$. It is seen from the non-Hermitian Hamiltonian (15) that in the absence of the classical driving field, all states except $|gg\rangle$ are damped, so that the steady state entanglement at $E = 0$ is impos-
sible, and the system evolves towards the unentangled state \(|gg\rangle\).

In other words, only the presence of classical driving field can stabilize steady state entanglement in the model system with the Hamiltonian (1).

Because we are interested in the robust entanglement, let us consider the steady state solutions of the master equation (12) with the density matrix (14) in the triplet sector of the Hilbert space.

\[
\rho_S = \frac{1}{N} \begin{pmatrix}
64E^4 & -16iE^3\sqrt{2} & 8E^2(2\Omega - 1)
\\
16E^3\sqrt{2} & 8E^2(1 + 8E^2) & -2E\sqrt{2}(\Omega + i + 8iE^2)
\\
-8E^2(2\Omega + 1) & -2E\sqrt{2}(2\Omega - i - 8iE^2) & 4(\Omega^2 + 2E^2 + 16E^4) + 1
\end{pmatrix}
\]

Here \(N = Tr\rho\) is the normalization factor and \(\Omega\) and \(E\) are established for the dimensionless parameters \(\Omega/\Gamma\) and \(E/\Gamma\), respectively. It follows from the numerical analysis that the detuning \(\Delta\) has a weak influence upon the concurrence.

To determine the settings, leading to the maximum possible amount of entanglement in the system under consideration, we choose \(\Omega = \tau E^2\), where \(\tau\) is a dimensionless constant to be determined upon the maximization of concurrence. This factor in the Lamb-Dicke limit can be represented as follows

\[
\tau = \frac{3}{4\pi\alpha}[(k_0r)^3Q\bar{n}\bar{V}]^{-1},
\]

where \(\alpha = 1/137\) is the fine structure constant, \(Q\) denotes atomic quality factor \((Q = \omega_0T\), where \(T\) is the lifetime of the excited atomic state), \(\bar{n}\) is the mean number of photons per unit volume in classical driving field, and \(\bar{V}\) denotes the volume of interaction between atom and field, so that \(\bar{n}\bar{V}\) gives the mean number of photons interacting with atom during the time \(T\).

The concurrence is defined as follows

\[
C = \max(\lambda_1 - \lambda_2 - \lambda_3, 0),
\]

where \(\lambda\) denotes the spectrum of matrix \(R = (\sqrt{\bar{p}}\bar{\rho}\sqrt{\bar{p}})^{1/2}\) and \(\bar{p}\) denotes the complex conjugation of (16) in the so-called “magic basis” \([15]\). The maximum entangled state provides \(C = 1\), while the unentangled states give \(C = 0\).

One can see from Eq. (2) that at fixed \(\tau\) and in the Lamb-Dicke limit \(k_0\bar{r} \ll 1\), both dimensionless parameters \(\Omega/\Gamma, E/\Gamma \gg 1\). In this case, the density matrix (16) takes the form

\[
\rho_S \approx \frac{1}{\tau^2 + 48} \begin{pmatrix}
16 & 0 & 4i\tau
\\
0 & 16 & 0
\\
-4i\tau & 0 & 16 + \tau^2
\end{pmatrix}
\]

To our surprise, the concurrence (18) in this limit turns out to be rational function of \(\tau\)

\[
C(\tau) = \frac{8\tau - 16}{\tau^2 + 48}, \quad \tau \geq 2
\]

Consider first the Lamb-Dicke limit of short interatomic separation. Then, it follows from the definition of the decay rate (13) that

\[
\Gamma^{(12)} = \Gamma^{(21)} \approx \Gamma.
\]

In this case, making the further assumption that \(\Delta = 0\), for the steady state density matrix in triplet sector we get the following explicit expression

\[
\rho_S = \frac{1}{\tau^2 + 48} \begin{pmatrix}
16 & 0 & 4i\tau
\\
0 & 16 & 0
\\
-4i\tau & 0 & 16 + \tau^2
\end{pmatrix}
\]

extended by zero at \(\tau \leq 2\). Thus, entanglement is impossible if \(\tau \leq 2\). The maximum value of concurrence

\[
C_{max} = \frac{2}{\sqrt{13} + 1} \approx 0.43
\]

is attained at

\[
\tau_{max} = 2 + 2\sqrt{13} \approx 9.21.
\]

The corresponding amount of entanglement \([15]\) is

\[
\mathcal{E}_{max} = H \left(\frac{1 - \sqrt{1 - C_{max}^2}}{2}\right) \approx 0.285 \text{ ebit.}
\]

Taking into account the form of the dimensionless parameter \(\tau\) given by Eq. (17), we can examine the dimensionless interatomic distance \(k_0\bar{r}\), corresponding to the maximum entanglement provided by \(\tau_{max} = 9.21\), as a function of the number of photons \(\bar{n}\bar{V}\), which should obey the condition \(\bar{n}\bar{V} \gg 1\) in the case of classical driving field. The dependence is given by Fig. 1. It is seen that in the case of mean number of photons \(\bar{n}\bar{V} \sim 10\), the interatomic distance should be of the order of \(10^{-2}\lambda\) (where \(\lambda\) is the wavelength) to achieve the maximum possible amount of entanglement. Increase of the mean number of photons in the driving field, considered as a coherent state with \(|\alpha|^2 \gg 1\), leads to a decrease of interatomic distance, which is required to have maximum amount of entanglement.

The dependence of concurrence from the interatomic distance comes from the coupling constant \(\Omega\) (2). The results of numerical calculations for different interatomic separations and different values of the classical driving field are shown in Fig. 2. It is seen that deviation from the Lamb-Dicke limit decreases the concurrence.

Thus, the classical driving field stabilizes entanglement in the system under consideration.

We now turn to the discussion of the obtained results.
IV. SUMMARY AND CONCLUSION

We have examined the system of two identical two-level atoms, interacting with each other by means of dipole forces. The dissipation of energy in the system is provided by the spontaneous decay of the excited atomic states. The compensation of losses is provided by the presence of classical driving field.

The consideration of the eigenstates of the Hamiltonian (1), describing the system under consideration, shows that the system carries a potential ability of creation of entanglement, hidden in the structure of the eigenstates.

Instead of conventional picture of two-qubit system, we prefer to specify the system by the triplet of states of an effective qutrit and by an “antiferromagnetic”, antisymmetric maximum entangled state, corresponding to a “spinless” quasiparticle.

It is shown that in the absence of the classical driving field, system evolves towards an unentangled state (both atomic dipoles are in the ground state). The presence of the classical driving field stabilizes the entanglement.

In the Lamb-Dicke limit of a point-like system, we obtained an exact solution for the steady state density matrix, that manifests high amount of entanglement (the concurrence $C_{\text{max}} = 0.43$ and the entanglement $\mathcal{E}_{\text{max}} = 0.285$ ebit). This amount is much higher than in a number of recent proposals. In particular, it is higher than that in the case when the squeezed vacuum is used for stabilization of entanglement instead of the classical driving field.

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Since the classical driving field is devoid of quantum fluctuations, the only source of the high-level quantum fluctuations peculiar for entanglement can be associated with the structure of the eigenstates of the Hamiltonian (1), describing the system. Thus, the classical driving field plays a double role. Namely, it compensates the losses of energy and stabilizes the entanglement carried by the system.

Such a picture suggests an analogy with the phase transitions in antiferromagnets, whose ground state can be considered as a collective entangled state. This ground state of such a system is realized at low temperatures. In other words, it is stabilized by the presence of a low-temperature environment (thermostat).

In our case, the system under consideration has very few degrees of freedom and therefore cannot manifest a thermodynamic phase transition. Nevertheless, the classical driving field together with the spontaneous decay plays a role similar to that of the thermostat.

In the investigation of steady state entanglement in the system, we always considered the triplet sector of the Hilbert space. From the physical point of view, this means that the initial states are symmetric. For example, the most natural choice is provided by the unentangled initial state $(gg)$ with both atoms in the ground state. In this case, a certain balance between the spontaneous decay and influence of the classical driving field lead to creation of the steady state entanglement in the system.

Let us now discuss the effect caused by the inclusion of the maximum entangled antisymmetric state (5) into the picture. This means that we choose the density matrix of the form

$$p|A\rangle\langle A| + q\rho_S,$$

in the whole four-dimensional Hilbert space. Here $\rho_S$ denotes the block, coming from the triplet sector and $p, q$ are the weights of the corresponding contributions ($p + q = 1$). Since the state $|A\rangle$ belongs to the “magic basis” (15), the concurrence of the steady state is given by formula similar to (17) applied to spectrum $q\lambda_1, q\lambda_2, q\lambda_3, p$. As far as $p < q\lambda_1$, that is $p < \lambda_1/(1 + \lambda_1)$, the presence of the antisymmetric component (5) decreases the concurrence.

Thus, to achieve a reasonable amount of robust entanglement, we have to prepare the system initially either in the antisymmetric maximum entangled state (5) or in a state from the symmetric sector (6) of the Hilbert space.

In the above consideration, we always assumed that atoms are identical. It seems to be interesting to extend our consideration on the case of non-identical atoms. In view of the result of Ref. [3], we can expect that this may lead to a significant increase of amount of entanglement.

We also restricted our consideration to the case of polarization of the classical driving field parallel to the interatomic axis. The alternative choice of the polarization perpendicular to the interatomic axis can lead to a strong changes of picture as well. First of all, the change of polarization changes the form of the coupling constant (2). Then, it causes the consideration of the different values of the classical driving field in the atomic locations. Finally, in this case, the antisymmetric state (5) cannot be discarded at all, so that the irreversible evolution should be examined in the whole four-dimensional Hilbert space.

The detailed analysis of the above mentioned two extensions of the model deserves special consideration.

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Fig. 1: Lamb-Dicke parameter $(\bar{k}_0 \cdot \bar{r})$ versus the number of photons in the case of dimensionless parameter (17), providing the maximal amount of entanglement for typical atomic quality factor $Q = 10^{-6}$.

Fig. 2: Numerical dependence of concurrence on the inter-atomic distance and classical driving field. The dimensionless quantities $\Omega/\Gamma$ and $E/\Gamma$ are used here.
