Lorentzian quantum cosmology in novel Gauss-Bonnet gravity from Picard-Lefschetz methods

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Abstract

In this paper we study some aspects of classical and quantum cosmology in the novel-Gauss-Bonnet (nGB) gravity in four space-time dimensions. Starting with a generalised Friedmann-Lemaître-Robertson-Walker (FLRW) metric respecting homogeneity and isotropy in arbitrary space-time dimension \( D \), we find the action of theory in four spacetime dimension where the limit \( D \rightarrow 4 \) is smoothly obtained after an integration by parts. The peculiar rescaling of Gauss-Bonnet coupling by factor of \( D - 4 \) results in a non-trivial contribution to the action. We study the system of equation of motion to first order nGB coupling. We then go on to compute the transition probability from one 3-geometry to another directly in Lorentzian signature. We make use of combination of WKB approximation and Picard-Lefschetz (PL) theory to achieve our aim. PL theory allows to analyse the path-integral directly in Lorentzian signature without doing Wick rotation. Due to complication caused by non-linear nature of action, we compute the transition amplitude to first order in nGB coupling. We find non-trivial correction coming from the nGB coupling to the transition amplitude, even if the analysis was done perturbatively. We use this result to investigate the case of classical boundary conditions.

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I. INTRODUCTION

General relativity although enjoys the merit of explaining a wide range of physical phenomena over a large range of distance, however its validity becomes questionable beyond these regimes where it is expected to get modified. For example at ultra high energies motivated by lack of renormalizability of GR it is noticed that addition of higher-derivative terms \[1–3\] results in a better ultraviolet behavior of resulting quantum theory. It however comes with their own bag of issues regarding lack of unitarity. Some efforts have been made in \[4–7\], in asymptotic safety approach \[8, 9\] and ‘Agravity’ \[10\]. Such unitarity problems arises as the theory has more than two time-derivatives. Lovelock gravity \[11–13\] are a special class of higher-derivative gravity where equation of motion remains second order in time.

In four spacetime dimension the Lovelock gravity also known as Gauss-Bonnet gravity is topological and doesn’t contribute in the dynamical evolution of metric. However, they play a key role in path-integral quantization of gravity where it is used to classify topologies. Motivated by works of \[14, 15\] it is observed that Gauss-Bonnet gravity can contribute non-trivially if its coupling is rescaled by factor of \(D - 4\) (where \(D\) is spacetime dimensionality) \[16\]. Such rescaling introduces non-trivial features coming from Gauss-Bonnet in four spacetime dimensions. This has generated tremendous interest in novel Gauss-Bonnet gravity.

The novel Gauss-Bonnet gravity \[16\] action is following

\[
S = \frac{1}{16\pi G} \int \dd^D x \sqrt{-g} \left[ -2\Lambda + R + \frac{\alpha}{D - 4} \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right],
\]

where \(G\) is the Newton’s gravitational constant, \(\Lambda\) is the cosmological constant term, \(\alpha\) is the Gauss-Bonnet coupling and \(D\) is spacetime dimensionality. The Gauss-Bonnet coefficient has been defined with a \((D - 4)\) factor in denominator. The mass dimensions of various couplings are: \([G] = M^{2-D}\), \([\Lambda] = M^2\) and \([\alpha] = M^{-2}\).

It is seen that an integration by parts gets rid of \((D - 4)\) factors leaving behind an action with a well-defined \(D \to 4\) limit \[17, 18\]. Here the authors do a Kaluza-Klein dimensional reduction where the manifold is cross-product of two spaces \(M_D = M_4 \otimes M_{D-4}\), thereby implying that the full metric can be written as a four-dimensional metric on \(M_4\) and extra-dimension piece on \(M_{D-4}\). They notice that taking limit \(D \to 4\) after an integration by parts leads a well-defined action which is Horndeski type gravity. A similar study was conducted in \[19\] using ADM decomposition realises that for a well-defined limit and a consistent theory in four dimensions one either break (a part of) the diffeomorphism invariance or have an extra degree of freedom \[19\]. In doing a KK reduction it is seen that the four dimensional action retains a memory of the higher-dimensional manifold, which shows up as an appearance of additional scalar field.

Inspired by these studies we decided to explore quantum aspects of novel Gauss-Bonnet gravity in a cosmological setting. We start by considering a generic metric respecting spatial homogeneity and isotropicity in \(D\)-spacetime dimensions. It is a generalisation of FLRW metric in \(D\)-dimensions consisting of two unknown time-dependent functions: lapse and scale-factor. This is mini-superspace approximation of the metric. On plugging this metric in novel Gauss-Bonnet gravity action and performing integration by parts, we are left with a mini-superspace action of theory where a well-defined \(D \to 4\) limit can be taken \[20\]. This action contains non-trivial contribution from the Gauss-Bonnet term. This process of obtaining well-defined 4-dimensional action doesn’t involve KK type dimensional reduction.
as in [17, 18]. As a result the 4-dimensional action doesn’t have an additional scalar-field which is like a memory of higher-dimensional manifold.

In this paper we study the quantum gravity path-integral to compute the transition amplitude from one 3-geometry to another, and investigate the non-trivial contributions coming from the novel-Gauss-Bonnet gravity. Usually to study such transitions one has to study the behavior of the following path-integral

\[ G[g_1, g_2] = \int_C Dg_{\mu\nu} \exp \left( -I[g_{\mu\nu}] \right). \]  

Here \( g_{\mu\nu} \) is the metric whose gravitational action appears in the corresponding exponential and is given by \( I[g_{\mu\nu}] \). This is Euclideanised version of the original Lorentzian path-integral where the temporal part of the metric has been Wick rotated in order to have a well-defined convergent path-integral along the contour \( C \). In flat spacetime there is a meaningful time co-ordinate and enjoys the properties of global symmetries to cast Lorentz group into a compact rotation group under a transformation of time co-ordinate. This is hard to replicate in generic curved spacetime. In a sense Wick-rotation (a process of defining a convergent path-integral by transforming the highly oscillatory path-integral in Lorentzian signature to euclidean) in quantum field theory (QFT) on flat spacetime is more natural to implement than in curved spacetime where ‘time’ is just a parameter. The \(+i\epsilon\)-prescription by Feynman in flat spacetime QFT is a systematic way to choose a contour in complexified spacetime, which is done in such a manner so that contour doesn’t cross the poles of the free theory propagator. This offers relevant convergence to an otherwise highly oscillatory integral and naturally implements causality in path-integral in a systematic manner by requiring that the euclideanised version of two-point function must satisfy Osterwalder-Schrader positivity. Such benefits of flat spacetime is hard to replicate in generic Lorentzian spacetime, and it gets even more involved when spacetime becomes dynamical due to gravity and/or gravitational field is also quantized. A possibility exists to do a Wick rotation sensibly and obtaining the Lorentzian case from Euclidean by properly implementing Wick rotation in curved spacetime [21–24]. However, this direction is still in its infant stages and more work needs to be done.

Picard-Lefschetz theory offers a way to handle such kind of oscillatory path-integrals. In a sense it is a generalization of standard Wick-rotation where the process is adapted accordingly to deal with generic curved spacetime. Here one study them by integrating them along the path of steepest descent in the complexified plane where the contour is uniquely obtained by using generalised flow equation in complex plane. Such steepest descent flow lines are termed Lefschetz thimbles. Early attempts making use of knowledge of steepest descent contours occurred in the context of Euclidean quantum gravity [25, 26].

Motivation to study euclideanised gravitational path-integral was an expectation that similar to flat spacetime QFT one will have relevant convergence. This is a mistake. Gravitational path-integral are non-trivial. Apart from dealing with usual issues of path-integral measure, gauge-invariance (gauge-fixing), regularization, renormalizability and boundary conditions; it is equally important to choose a contour of integration carefully for necessary convergence. This last bit is obscure in curved spacetime, where the standard Feynman \(+i\epsilon\)-prescription (which works in flat spacetime QFT) no longer offers reliable results.

Picard-Lefschetz theory offers a systematic way to find this integration contour in a generic spacetime where the gravitational path-integral becomes absolutely convergent. This has been made use of in the simple models of quantum cosmology [27–29], where the authors
studied path-integral in the mini-superspace approximation. Earlier attempts employing similar strategy but in euclidean quantum cosmology goes back to 1980s [30–33] when issues of initial conditions was being explored. Such ventures lead to tunnelling proposal [30–32] and no-boundary proposal [25, 26, 33]. Euclidean path-integral of gravity (which is unbounded from below [34] due to famous conformal factor problem [35]) needs not only a sensible initial condition choice but also a choice of contour of integration [36–38]. Picard-Lefschetz theory allows one to pick the contour uniquely directly in Lorentzian spacetime and allows one to study scenarios involving various initial conditions in a systematic manner [27–29].

In this paper we make use of Picard-Lefschetz theory to analyse the path-integral of novel-Gauss-Bonnet gravity in the mini-superspace approximation. We ask a straightforward question: what is the transition probability from one state to another, where the states are specified by the boundary conditions and correspond to a geometry. We seek to answer this by building on the footsteps of the formalism developed in [42]. Due to complicated form of the mini-superspace action our efforts are limited to address the problem perturbatively in nGB coupling. We do the computation of transition amplitude to first order in nGB coupling.

The paper has following outline: section II deals with constructing a mini-superspace action for novel Gauss-Bonnet gravity. Section III solves the system of equations to first order in nGB coupling. In section IV we compute transition amplitude from one 3-geometry to another perturbatively to first order in nGB coupling. Section V deals with Picard-Lefschetz (PL) methods where beside reviewing the PL-technology, we use it to do the integration over lapse. In section VI we study the case of classical boundary conditions and apply the results obtained in previous section to compute the transition amplitude in the case of classical Universe. We conclude by summarizing our findings with a discussion in section VII.

II. MINI-SUPERSPACE ACTION

To compute the mini-superspace action here we first consider a generalization of FLRW metric in arbitrary spacetime dimension whose dimensionality is $D$. In polar co-ordinates $\{t_p, r, \theta, \cdots\}$ the FLRW metric can be expressed as

$$ ds^2 = -N_p^2(t_p)dt_p^2 + a^2(t_p) \left[ \frac{dr^2}{1 - kr^2} + r^2d\Omega_D^{2} \right], $$

(3)

where $N_p(t_p)$ is lapse function, $a(t_p)$ is scale-factor, $k = (0, \pm 1)$ is the curvature, and $d\Omega_D^{2}$ is the metric corresponding to unit sphere in $D-2$ spatial dimensions. The FLRW metric is conformally related to flat metric and hence its Weyl-tensor $C_{\mu\nu\rho\sigma} = 0$. For Riemann tensor the nonzero entries are [39–41]

$$ R_{0i0j} = - \left( \frac{a''}{a} - \frac{a'N_p'}{aN_p} \right) g_{ij}, $$

$$ R_{ijkl} = \left( \frac{k}{a^2} + \frac{a'^2}{N_p^2a^2} \right) (g_{ik}g_{jl} - g_{ij}g_{jk}), $$

(4)
where \( g_{ij} \) is the spatial part of the FLRW metric and \( (') \) denotes derivative with respect to \( t_p \). For the Ricci-tensor the non-zero components are

\[
R_{00} = -(D-1) \left( \frac{a''}{a} - \frac{a' N_p'}{aN_p} \right),
\]

\[
R_{ij} = \frac{(D-2)(kN_p^2 + a'^2)}{N_p^2 a^2} + \frac{a'' N_p - a N_p'}{aN_p^3} g_{ij},
\]

while the Ricci-scalar for FLRW is given by

\[
R = 2(D-1) \left[ \frac{a'' N_p - a N_p'}{aN_p^3} + \frac{(D-2)(kN_p^2 + a'^2)}{2N_p^2 a^2} \right].
\]

Weyl-flatness offers simplicity and allows one to express Riemann tensor in terms of Ricci-tensor and Ricci scalar.

\[
R_{\mu\nu\rho\sigma} = \frac{R_{\mu\rho\sigma} g_{\nu\nu} - R_{\mu\sigma\nu \nu} + R_{\nu\nu \rho \sigma} - R_{\nu \nu \rho \sigma}}{D-2} - \frac{R(g_{\mu \rho} g_{\nu \sigma} - g_{\nu \sigma} g_{\mu \rho})}{(D-1)(D-2)}.
\]

This identity is valid for all conformally flat metrics and allows one to express

\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{4}{D-2} R_{\mu \nu} R^{\mu \nu} - \frac{2R^2}{(D-1)(D-2)}.
\]

By making use of this identity for conformally flat metrics in the Gauss-Bonnet action one can obtain a simplified action of the theory. In such cases we have

\[
\int d^D x \sqrt{-g} \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu \nu} R^{\mu \nu} + R^2 \right) = \frac{D-3}{D-2} \int d^D x \sqrt{-g} \left( -R_{\mu \nu} R^{\mu \nu} + \frac{DR^2}{D-1} \right).
\]

On plugging the FLRW metric of eq. (3) in the action in eq. (1) one can get an action for \( a(t_p) \) and \( N_p(t_p) \). This action is given by,

\[
S = \frac{V_{D-1}}{16\pi G} \int dt_p \left[ a^{D-3} \left\{ (D-1)(D-2)kN_p^3 - 2\lambda a^2 N_p^3 - 2(D-1)aa'N_p' \right. \right.
\]

\[
+ (D-1)(D-2)a'^2 N_p + 2(D-1)N_p aa'' \left. \right\} + \frac{(D-1)(D-2)(D-3)\alpha}{D-4} \left\{ \frac{a^{D-5}N_p^3}{N_p} \right. \}
\]

\[
\times (kN_p^2 + a'^2)^2 + \frac{4a^{D-4}(kN_p^2 + a'^2)}{N_p^2} \frac{d}{dt_p} \left( \frac{a'}{N_p} \right) \right\}.
\]

where \( V_{D-1} \) is the volume of \( D-1 \) dimensional space. One can perform an integration by parts in the underlined terms to arrive at action where \( D \to 4 \) limit can be smoothly taken. Under an integration by parts the \( (D-4) \) factors are seen to cancel off. This resulting action in \( D = 4 \) is given by,

\[
S = \frac{V_3}{8\pi G} \int dt_p \left[ (3k - \Lambda a)N_p a - 3aa' N_p + 3\alpha \left\{ \frac{(kN_p^2 + a'^2)^2}{N_p^3} + \frac{4ka'^2}{N_p} + \frac{4a'^4}{N_p} \right\} \right].
\]
The Gauss-Bonnet term gives a non-trivial contribution in $D = 4$ which is possible as its coefficient has been defined with a $(D - 4)$ factor in denominator, which cancels off any $(D - 4)$ in numerator. With this action one can do further analysis. This action can be recast in to a more appealing form by a rescaling of lapse and scale factor.

$$N_p(t)p = \frac{N(t)}{a(t)}dt, \quad q(t) = a^2(t).$$

This set of transformation changes our original metric in eq. (3) into following

$$ds^2 = -\frac{N^2}{q(t)}dt^2 + q(t)\left[\frac{dt^2}{1 - kr^2} + r^2d\Omega^2_{D-2}\right],$$

and our action in $D = 4$ given in eq. (11) changes to following simple form.

$$S = \frac{V_3}{16\pi G} \int dt \left[ (6k - 2\Lambda q)N - \frac{3\dot{q}^2}{2N} + \frac{3\alpha}{8N^3q}(4kN^2 + q^2)(4kN^2 + 5\dot{q}^2) \right],$$

where () here represent derivative with respect to time $t$. It should be noticed that the action doesn’t contain any derivative of $N$, which happens as we have performed integration by parts previously. This is an interesting higher-derivative action which only depends on $q, \dot{q}$ and $N$.

### III. EQUATION OF MOTION

The action in eq. (14) lacks any derivative term for $N$ indicating that variation of action with respect to $N$ will result in a constraint equation. Varying action with respect to $q(t)$ however leads to a dynamical equation for the evolution of $q(t)$. We choose the ADM gauge $\dot{N} = 0$, which implies that $N(t) = N_c$ (constant). The equation of motion for $q(t)$ then is given by

$$-2N_c\Lambda + \frac{3\dot{q}}{N_c} + \frac{3\alpha}{8N_c^3} \left[ \frac{15\dot{q}^4}{q^2} - \frac{60\dot{q}^2\ddot{q}}{q} + 24kN_c^2 \left( \frac{q^2 - 2\ddot{q}}{q^2} \right) - \frac{16k^2N_c^4}{q^2} \right] = 0.$$

This equation contains higher-derivative contribution which is proportional to $\alpha$. It is a second order non-linear ODE. The higher-derivative contribution is novel here which doesn’t arise if the Gauss-Bonnet coupling wasn’t rescaled by factor of $(D - 4)$ [16]. Compared to the equation presented in [20], this has additional terms coming from non-zero $k$ (non-flat Universe). In principle one has to solve for $q(t)$ from the above equation for the boundary conditions

$$q(t = 0) = b_0, \quad q(t = 1) = b_1.$$  

One can then plug the $q(t)$-solution back into the action in eq. (14), where we are in constant-$N$ gauge. On integrating this with respect to time, we arrive at the action for the constant lapse $N_c$. One then look for saddle points solution for $N_c$ which are obtained by varying this action with respect to $N_c$. This will be the full saddle point solution of theory.

In practice this is not always possible. In the present case the evolution equation for $q(t)$ is quite complicated: higher-derivates and non-linear. We therefore approach to solve the system perturbatively. We start by expanding $q(t)$ in powers of $\alpha$.

$$q(t) = q_0(t) + \alpha q_1(t) + \cdots,$$

where $q_0$ is zeroth-order solution while $q_1$ is the first order solution.
A. zeroth-order

At the lowest ($\alpha^0$) order we have

$$\ddot{q} = \frac{2N_c^2\Lambda}{3}. \quad (18)$$

This linear second order ODE can be solved analytically. Its solution obeying the boundary condition stated in eq. (16) is given by

$$q_0(t) = \frac{\Lambda N_c^2}{3}(t^2 - t) + b_0(1 - t) + b_1 t. \quad (19)$$

We plug this back into the action in eq. (14) and integrate with respect to $t$. One gets zeroth-order action for $N_c$. This is given by

$$S_0 = \frac{V_3}{16\pi G} \left[ -\frac{3(b_0 - b_1)^2}{2N_c} + 6kN_c - (b_0 + b_1)N_c\Lambda + \frac{N_c^3\Lambda^2}{18} \right]. \quad (20)$$

From the zeroth order action for $N_c$ one can compute the zeroth order saddle points by varying action with respect to $N_c$. Then we see that $\partial S_0 / \partial N_c = 0$ whose solution gives $N_0$.

$$\frac{3(b_0 - b_1)^2}{2N_0^2} - (b_0 + b_1)\Lambda + \frac{N_0^2\Lambda^2}{6} + 6k = 0. \quad (21)$$

This is quadratic in $N_0^2$ and consist of four solutions which are given by

$$(N_0)_{\pm,\pm} = \pm \sqrt{\frac{3}{\Lambda}} \left( \sqrt{b_1 - \frac{3k}{\Lambda}} \pm \sqrt{b_0 - \frac{3k}{\Lambda}} \right). \quad (22)$$

At the zeroth order we don’t receive any correction from the Gauss-Bonnet term and they agree with the known saddles in the context of Lorentzian quantum cosmology [27, 28]. Corresponding to each $(N_0)_{\pm,\pm}$ we have corresponding $(q_0)_{\pm,\pm}$. Each of them leads to a different FLRW metric. Corresponding to each of them we have an on-shell action, which is given by

$$S_0^{\text{on-shell}} = \mp \frac{V_3}{4\pi G} \sqrt{\frac{\Lambda}{3}} \left[ \left( b_1 - \frac{3k}{\Lambda} \right)^{3/2} \pm \left( b_0 - \frac{3k}{\Lambda} \right)^{3/2} \right]. \quad (23)$$

B. First order

At first order in $\alpha$ the equations becomes more involved as the novel-Gauss Bonnet gravity starts to contribute. The evolution of $q(t)$ at first order is dictated by following equation

$$\ddot{q}_1 = -\frac{1}{8N_c^2} \left[ 15\dot{q}_0^4 \frac{q_0^2}{q_0^2} - 60\dot{q}_0^2 \ddot{q}_0 \frac{q_0^2}{q_0} + 24kN_c^2 \left( \frac{q_0^2}{q_0^2} - \frac{2\dot{q}_0}{q_0} \right) - \frac{16k^2N_c^4}{q_0^2} \right], \quad (24)$$

where $q_0$ is the zeroth order solution to $q(t)$ obtained before. This need to be solved along with the boundary conditions for $q_1$. The boundary conditions for $q_1(t)$ can be obtained from eq. (16) and those of $q_0$. This implies that

$$q_1(t = 0) = q_1(t = 1) = 0. \quad (25)$$
The ODE for \( q_1(t) \) can now be solved with these boundary conditions, and its solution is given by

\[
q_1(t) = \frac{5N_c^2\Lambda^2 t(t-1)}{3} - \frac{1}{4N_c^2} \left[ 5U + \frac{72N_c^2 k}{U} - \frac{432N_c^4 k^2}{U^3} \right] \left[ \left( b_0 - b_1 + \frac{N_c^2 \Lambda}{3} \right) (t-1) \times \tan^{-1} \left( \frac{3(b_0 - b_1) + N_c^2 \Lambda}{U} \right) + \left( b_0 - b_1 - \frac{N_c^2 \Lambda}{3} \right) t \tan^{-1} \left( \frac{3(b_1 - b_0) + N_c^2 \Lambda}{U} \right) + \left( b_1 - b_0 + \frac{N_c^2 \Lambda(2t-1)}{3} \right) \tan^{-1} \left( \frac{3(b_1 - b_0) + N_c^2 \Lambda(1 - 2t)}{U} \right) \right],
\]

where

\[
U = \sqrt{6(b_0 + b_1)^2 N_c^2 \Lambda - 9(b_0 - b_1)^2 - N_c^4 \Lambda^2}.
\]

Having obtained the first order correction to \( q(t) \), we can plug back the corrected solution \( q(t) = q_0 + \alpha q_1 \) in action in eq. (21) and perform the \( t \)-integration. This results in a first order corrected action for \( N_c \).

\[
S_1 = S_0 + \frac{V_4 \alpha}{16\pi G} \left[ \frac{5(b_0 - b_1)^2 \Lambda}{N_c} - \frac{5(b_0 + b_1)N_c \Lambda^2}{3} + \frac{10N_c^3 \Lambda^3}{27} + 12N_c \Lambda k + \left( \frac{5U^3}{36N_c^2} \right. \right.
\]

\[
- \left. \frac{6Uk}{N_c} + \frac{36N_c^2 k^2}{U} \right] \left\{ \tan^{-1} \left( \frac{3(b_0 - b_1) + N_c^2 \Lambda}{U} \right) + \tan^{-1} \left( \frac{3(b_1 - b_0) + N_c^2 \Lambda}{U} \right) \right\} + \cdots
\]

This first order corrected action can be varied with respect to \( N_c \) to obtain the correction to the zeroth order saddle points. To obtain this we substitute

\[
N_1 = N_0 + \alpha \nu_1 + \cdots.
\]

Then \( N_1 \) is the first order corrected saddle and can be obtained from

\[
\frac{\partial S_0}{\partial N_c} \bigg|_{N_c \to N_1 = (N_0 + \alpha \nu_1)} = 0.
\]

On solving this equation for \( \nu_1 \) we get

\[
\nu_1 = -\frac{5N_0 \Lambda}{2} + \frac{k N_0^3 \Lambda}{6k N_0} - \frac{3(b_0 - b_1)^2 + 6k N_0^2 - (b_0 + b_1)N_0^2 \Lambda}{6\sqrt{k} N_0} + 4\sqrt{k} \left[ \tan^{-1} \left( \frac{3(b_0 - b_1) + N_0^2 \Lambda}{6\sqrt{k} N_0} \right) + \tan^{-1} \left( \frac{3(b_1 - b_0) + N_0^2 \Lambda}{6\sqrt{k} N_0} \right) \right],
\]

where \( N_0 \) is given by eq. (21). This when combined with the zeroth order solution gives us the first order corrected saddles \( N_1 \).

\[
N_1 = N_0 - \frac{5\alpha N_0 \Lambda}{2} + \frac{k \alpha N_0^3 \Lambda}{6\sqrt{k} N_0} - \frac{3(b_0 - b_1)^2 + 6k N_0^2 - (b_0 + b_1)N_0^2 \Lambda}{6\sqrt{k} N_0} + 4\alpha \sqrt{k} \left[ \tan^{-1} \left( \frac{3(b_0 - b_1) + N_0^2 \Lambda}{6\sqrt{k} N_0} \right) + \tan^{-1} \left( \frac{3(b_1 - b_0) + N_0^2 \Lambda}{6\sqrt{k} N_0} \right) \right].
\]

From this we can compute the first order corrected on-shell action. This is given by

\[
S_{1 \text{on-shell}} = S_{0 \text{on-shell}} + \frac{V_3 \Lambda \alpha}{144G\pi} \left[ (b_0 - b_1)^2 \frac{4k N_0^2}{5} + (b_0 + b_1)N_0^2 \Lambda \right],
\]

where \( S_{0 \text{on-shell}} \) is the zeroth order on-shell action given in eq. (23).
IV. TRANSITION AMPLITUDE

Once we have action of theory then the real important question to ask is the role the theory plays in quantum regimes. Such issues can only be addressed when one has full action of theory. In our present case it is worthy to ask the transition amplitude from one 3-geometry to another. We aim to study this directly in Lorentzian signature by making use of WKB and Picard-Lefschetz theory [27, 36].

The relevant quantity that we wish to compute is the transition probability from one 3-geometry to another, which is a generalization of probability computation in usual quantum mechanics (or field theory) to the case of gravity. In mini-superspace approximation this means

\[ G[b_0, b_1] = \int_C \mathcal{D}N(t) \int_{b_0}^{b_1} \mathcal{D}q(t) \exp \left( \frac{i}{\hbar} S \right) , \]  

(34)

where \( q(t) \) satisfies the boundary condition given in eq. (16). \( S \) is given in eq. (14), ‘\( C \)’ is the contour of integration for \( N \) which is chosen using Picard-Lefschetz theory. The computation of the path-integral is a complicated task even in the mini-superspace approximation. The usual complication of defining measure, convergence, uncontrollable oscillations still exist. Often in quantum mechanical path-integral the measure is defined by discretising and convergence is obtained via Wick rotation. Using Picard-Lefschetz one can generalise Feynman \(+\i\epsilon\)-prescription in a unique way thereby leading to an absolutely convergent path-integral along the paths of steepest descent. We will study this system in WKB approximation. We have already worked out perturbative solution to equation of motion following from action in eq. (14). This will be required in the WKB approximation, which is also gaussian approximation.

In the WKB approximation we consider fluctuation around the solution to equation of motion keeping the end points fixed.

\[ q(t) = q_b(t) + Q(t) , \]  

(35)

where \( q_b(t) \) satisfies the equation of motion while \( Q(t) \) is the fluctuation around the background \( q_b \). We plug this in action given in eq. (14) and expand to second order in \( Q(t) \). In the expansion the first order terms identically vanish as \( q_b(t) \) satisfies equation of motion. The second order terms in the gauge \( \dot{N} = 0 \) are given by,

\[ S^{(2)} = \frac{V_3}{2} \int_0^1 dt \left[ -\frac{3\dot{Q}^2}{2N^3_c} + \frac{45\alpha}{4N^3_c} \left\{ \frac{\dot{q}_b^2}{q_b} + \frac{4kN^2_c}{5q_b} \right\} \dot{Q}^2 \right. \]

\[ + \frac{45\alpha}{8N^3_c} \left\{ \frac{2q_b^2\dot{q}_b}{q_b^2} - \frac{\dot{q}_b^4}{q_b^3} + 8kN^2_c \left( \frac{\dot{Q}}{q_b} - \frac{\dot{q}_b}{q_b^3} \right) + \frac{16k^2N^4_c}{15q_b^3} \right\} Q^2 \right] , \]

(36)

where we have set \((8\pi G) = 1\) (which we will continue to follow from now onward) and we have

\[ Q(0) = Q(1) = 0 . \]  

(37)

We have already performed integration by parts to obtain the second variation on eq. (36) which allowed us to combine certain terms. The \( q_b(t) \) entering here in the second variation is the full solution of the equation of motion, however in this paper we will focus on dealing with system to first order in \( \alpha \). This implies that if we plug \( q_b(t) = q_0(t) + \alpha q_1(t) \) in the eq.
(36) (where \( q_0(t) \) and \( q_1(t) \) are given in eq. (19) and (26) respectively) and expand to first order in \( \alpha \), then we have

\[
S^{(2)} = \frac{V_3}{2} \int_0^1 dt \left\{ -\frac{3}{2N_c} + \frac{45\alpha}{4N_c^3} \left( \frac{\dot{q}_0^2}{q_0} + \frac{4kN_c^2}{5q_0} \right) \right\} Q^2 \\
+ \frac{45\alpha}{8N_c^3} \left\{ \frac{2q_0^2}{q_0} - \frac{\dot{q}_0^4}{q_0} + \frac{8kN_c^2}{5} \left( \frac{\ddot{q}_0}{q_0} - \frac{\dddot{q}_0^2}{q_0} \right) + \frac{16k^2N_c^4}{15q_0^4} \right\} Q^2 ,
\]

where the terms proportional to \( \alpha \) constitute \( S_{\text{hGB}}^{(2)} \). After the decomposition we get the following form of the transition amplitude.

\[
G[b_0, b_1] = \int_{0+}^\infty dN_c \exp \left( \frac{iS_1}{\hbar} \right) \int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q(t) \exp \left( \frac{iS^{(2)}}{\hbar} \right) ,
\]

where \( S_1 \) is given in eq. (28). Our task first then is to compute the path-integral over \( Q(t) \). In the case when the second variation contains only terms coming from Einstein-Hilbert gravity (\( \alpha \to 0 \)), then the \( Q \)-path-integral is easy to perform exactly. For nonzero \( \alpha \) this is complicated as the coefficient of \( \dot{Q}^2 \) and \( Q^2 \) are \( t \)-dependent functions. However, to first order in \( \alpha \) one can perform path-integral perturbatively, which is what we will do. We will closely follow the strategy outlined in [42].

A. \( Q \)-integration

We first note that in the second variation the coefficient of \( \dot{Q}^2 \) and \( Q^2 \) has time dependence, which arise as the terms proportional to \( \alpha \) depend of \( q_0(t) \) and its derivatives. Although \( q_0(t) \) is a simple quadratic polynomial in \( t \), it still makes it tricky to evaluate the path-integral exactly. We note that as fluctuation \( Q(t) \) vanishes at the two boundary points, it implies that it has following decomposition

\[
Q(t) = \sum_{|k| \geq 1} c_k \exp \left( 2\pi kt \right) , \quad \text{with} \quad c_{-k} = c_k^* .
\]

The path-integral measure accordingly becomes the following

\[
\mathcal{D}Q(t) = \mathcal{N} \int_{-\infty}^{\infty} \prod_{|k| \geq 1} dc_k ,
\]

where the normalization \( \mathcal{N} \) needs to be fixed carefully. Usually the normalization is fixed in such a way so that it absorbs the infinities coming from the infinite-product or infinite summation. In the case of Einstein-Hilbert gravity (setting \( \alpha \to 0 \) above) we have the following \( Q \)-path-integral.

\[
\int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q(t) \exp \left( \frac{-3iV_3}{4N_c} \int_0^1 dt \dot{Q}^2 \right) = \left( \frac{3iV_3}{4\pi N_c} \right)^{1/2} .
\]

The expression on LHS is similar to path-integral of free particle. It can be evaluated exactly and it has a finite value on RHS. If we insert the decomposition of \( Q(t) \) from eq. (40) and
write the measure as in eq. (11), then by performing the path-integral one encounters infinities.

\[ \mathcal{N}_{\text{EH}} \int_{-\infty}^{\infty} \prod_{|k| \geq 1} \exp \left( \frac{3iV_3}{4\pi c \hbar} \sum_{k} |c_k|^2 \right) = \left( \frac{3iV_3}{4\pi c \hbar} \right)^{1/2}. \] (43)

The infinity arising from the infinite-product on the LHS can be absorbed by suitably defining \( \mathcal{N}_{\text{EH}} \). This will give

\[ \mathcal{N}_{\text{EH}} \prod_{k=1}^{\infty} \frac{8\pi c \hbar}{3(2\pi k)^2 V_3} = \left( \frac{3iV_3}{4\pi c \hbar} \right)^{1/2}. \] (44)

For the case of novel-GB gravity, the normalisation needs to be fixed accordingly. At this point it is best we also write \( c_n = a_n + i b_n \) and \( c_{-n} = c_n^* = a_n - i b_n \), where \( a_n \) and \( b_n \) are real numbers. Such a change of variables will lead to a Jacobian factor. The gravity action here consists of two parts: \( S^{(2)} = S^{(2)}_{\text{EH}} + S^{(2)}_{n\text{GB}} \). To first order in \( \alpha \) we then have

\[
\begin{align*}
\int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q(t) \exp \left( \frac{iS^{(2)}}{\hbar} \right) &= \mathcal{N}_{\text{EH}} (1 + \alpha N_1 + \cdots ) \\
\times &\int_{-\infty}^{\infty} \prod_{k=1}^{\infty} da_k db_k \left( \frac{2}{i} \right) \left( 1 + \frac{i}{\hbar} S^{(2)}_{n\text{GB}} + \cdots \right) \exp \left( \frac{iS^{(2)}_{\text{EH}}}{\hbar} \right), \\
= &\mathcal{N}_{\text{EH}} \int_{-\infty}^{\infty} \prod_{k=1}^{\infty} da_k db_k \left( \frac{2}{i} \right) \exp \left( \frac{iS^{(2)}_{\text{EH}}}{\hbar} \right) \\
+ &\mathcal{N}_{\text{EH}} \int_{-\infty}^{\infty} \prod_{k=1}^{\infty} da_k db_k \left( \frac{2}{i} \right) \left[ \alpha N_1 + \frac{i}{\hbar} S^{(2)}_{n\text{GB}} \right] \exp \left( \frac{iS^{(2)}_{\text{EH}}}{\hbar} \right) + \mathcal{O}(\alpha^2), \quad (45)
\end{align*}
\]

where \( \mathcal{N}_{\text{EH}} \) is given in eq. (41), \( N_1 \) is the infinite constant which will be adjusted to absorb the infinity coming from novel Gauss-Bonnet gravity part, while the factor \( 2/i \) arises due to Jacobian transformation. The EH action is quadratic in \( a_n \) and \( b_n \), which is easy to see once we plug the decomposition for \( Q(t) \) and integrate with respect to time. \( S^{(2)}_{n\text{GB}} \) on the other hand contains mixed terms. For example terms like \( a_m a_n, a_m b_n, \) and \( b_m b_n \) (where \( m \) and \( n \) need not be the same) occur. Such kind of terms don’t disappear even after the \( t \)-integration. The \( S^{(2)}_{n\text{GB}} \) is given by,

\[ S^{(2)}_{n\text{GB}} = \frac{45\alpha V_3}{8 N^3 c} \sum_{|k,k'| \geq 1} \int_0^1 dt \left[ \left( \frac{q_0^2}{q_0} + \frac{4k^2 N^2}{5q_0} \right) (4\pi^2 k k') \right. \\
+ \left. \left\{ \frac{q_0^2 q_0}{q_0^2} - \frac{q_0^4}{2q_0^2} + \frac{4k^2 N^2}{15q_0^2} \left( \frac{q_0^2}{q_0^2} - \frac{q_0^4}{3q_0^2} \right) + \frac{8k^2 N^4}{15q_0^2} \right\} c_k c_{k'} e^{2\pi i (k + k') t} \right], \quad (46)
\]

where \( q_0(t) \) is quadratic in \( t \) and is given in eq. (19). Here we need to perform \( t \)-integration. On plugging decomposition of \( c_k \)'s in terms of \( a_k \)'s and \( b_k \)'s, it is possible to write the above expression as a summation over only positive integer values of \( k \) and \( k' \). The resulting
expression will also contain mixed terms which are non-diagonal. We introduce a shorthand

\[ M(k, k') = \frac{45\alpha}{4N_c^2} \int_0^1 dt \left[ \left( \frac{\dot{q}_0^2}{q_0^2} + \frac{4kN_c^2}{5q_0} \right) (4\pi^2kk') \right. \]

\[ + \left\{ \frac{\ddot{q}_0^2/q_0}{q_0^2} - \frac{\dot{q}_0^4}{2q_0^4} + \frac{4kN_c^2}{5} \left( \frac{\dot{q}_0^2}{q_0^2} - \frac{\ddot{q}_0^2}{q_0^2} \right) + \frac{8k^2N_c^4}{15q_0^3} \right\} e^{2\pi i (k+k')t} \]

\[ = \int_0^1 dt \left[ 4\pi^2kk'A_1(t) + A_2(t) \right] e^{2\pi i (k+k')t}, \]  

(47)

where we have

\[ A_1(t) = \frac{45\alpha}{4N_c^2} \left( \frac{\dot{q}_0^2}{q_0^2} + \frac{4kN_c^2}{5q_0} \right), \]

(48)

\[ A_2(t) = \frac{45\alpha}{4N_c^2} \left\{ \frac{\ddot{q}_0^2/q_0}{q_0^2} - \frac{\dot{q}_0^4}{2q_0^4} + \frac{4kN_c^2}{5} \left( \frac{\dot{q}_0^2}{q_0^2} - \frac{\ddot{q}_0^2}{q_0^2} \right) + \frac{8k^2N_c^4}{15q_0^3} \right\}. \]

(49)

This shorthand is useful as it expresses the structure of the $S_{nGB}^{(2)}$ in a simple manner. This is given by,

\[ S_{nGB}^{(2)} = \frac{V_3}{2} \sum_{k, k' \geq 1} \left[ (M(k, k') + M(k, -k') + M(-k, k') + M(-k, -k')) a_ka_{k'} \right. \]

\[ + i (M(k, k') - M(k, -k') + M(-k, k') - M(-k, -k')) a_kb_{k'} \]

\[ + i (M(k, k') + M(k, -k') - M(-k, k') - M(-k, -k')) b_ka_{k'} \]

\[ + (-M(k, k') + M(k, -k') - M(-k, k') + M(-k, -k')) b_kb_{k'} \left. \right], \]

\[ = \frac{V_3}{2} \sum_{k, k' \geq 1} \left[ M_{11}(k, k')a_ka_{k'} + M_{12}(k, k')a_kb_{k'} + M_{21}(k, k')b_ka_{k'} + M_{22}(k, k')b_kb_{k'} \right]. \]  

(50)

In the path-integral given in eq. (45) one has to take expectation value of $S_{nGB}^{(2)}$. We notice the occurrence of mixed terms in $S_{nGB}^{(2)}$ given in eq. (50). These non-diagonal terms don’t contribute as the action appearing in exponent is quadratic in $a_k$ and $b_k$. As a result only $M_{11}(k, k')$ and $M_{22}(k, k')$ contributes. Also, among them the non-vanishing contribution comes only when $k = k'$. These observations simplify our perturbative computations drastically. For $k = k'$ the expressions for $M_{11}(k, k)$ and $M_{22}(k, k)$ is given by,

\[ M_{11}(k, k) = \int_0^1 dt \left[ \left\{ A_1(2\pi k)^2 + A_2 \right\} 2\cos(4\pi kt) + 2 \left\{ -A_1(2\pi k)^2 + A_2 \right\} \right], \]

(51)

\[ M_{22}(k, k) = \int_0^1 dt \left[ - \left\{ A_1(2\pi k)^2 + A_2 \right\} 2\cos(4\pi kt) + 2 \left\{ -A_1(2\pi k)^2 + A_2 \right\} \right]. \]

(52)

Achieving great simplification we now only need to perform the integrations over $a_k$ and $b_k$ as dictated by the path-integral in eq. (45). This path-integral has two parts: the leading piece is the Einstein-Hilbert piece which has been computed before in eq. (43) while the second term is the correction term coming from the nGB. We will compute this piece now.
We now have the relevant ingredients necessary to write an expression for the transition probability. We plug them in eq. (39), which gives

\[ N_{\text{EH}} \int_{-\infty}^{\infty} \prod_{k=1}^{\infty} da_k db_k (\frac{2}{i}) \left[ \alpha N_1 + \frac{i}{\hbar} S_{\text{nGB}}^{(2)} \right] \exp \left( \frac{i S_{\text{EH}}^{(2)}}{\hbar} \right) \]

\[ = \left( \frac{3iV_3}{4\pi N_c \hbar} \right)^{1/2} \left[ \alpha N_1 - \frac{N_c}{3} \sum_{k=1}^{\infty} \{ M_{11}(k, k) + M_{22}(k, k) \} (2\pi k)^{-2} \right], \]

\[ = \left( \frac{3iV_3}{4\pi N_c \hbar} \right)^{1/2} \left[ \alpha N_1 - \frac{4N_c}{3} \sum_{k=1}^{\infty} \int_0^1 dt \{ A_1 + A_2(2\pi k)^{-2} \} \right], \]

\[ = -\frac{N_c}{18} \left( \frac{3iV_3}{4\pi N_c \hbar} \right)^{1/2} \int_0^1 dt A_2, \quad (53) \]

where we have absorbed the infinite piece by defining the infinite constant \( N_1 \) as

\[ N_1 = \frac{4N_c}{3\alpha} \sum_{k=1}^{\infty} \int_0^1 dt A_1, \quad (54) \]

and \( A_2(t) \) is given in eq. (49). Putting together all terms we find the value of the \( Q \)-integration to be

\[ \int_{Q[0]}^{Q[1]=0} DQ(t) \exp \left( \frac{i S_{\text{EH}}^{(2)}}{\hbar} \right) = \left( \frac{3iV_3}{4\pi N_c \hbar} \right)^{1/2} \left[ 1 - \frac{N_c}{18} \int_0^1 dt A_2 + O(\alpha^2) \right]. \quad (55) \]

The \( t \)-integration here \( A_2 \) can be performed using Mathematica. It carries crucial \( N_c \) dependence which is important in the \( N_c \)-integration of the path-integral for transition amplitude. We now have the relevant ingredients necessary to write an expression for the transition probability. We plug them in eq. (59), which gives

\[ G[b_0, b_1] = \int_{0^+}^{\infty} dN_c \exp \left( \frac{i S_1}{\hbar} \right) \left( \frac{3iV_3}{4\pi N_c \hbar} \right)^{1/2} \left[ 1 - \frac{N_c}{18} \int_0^1 dt A_2 + O(\alpha^2) \right], \quad (56) \]

where the form of \( A_2(t) \) is given in eq. (49) and \( S_1 \) is given in eq. (28). After performing the \( t \)-integration one obtains \( A_2 \). This is given by

\[ \int_0^1 dt A_2 = \frac{\alpha \{ 3(b_0 + b_1) - N_c^2 \} A}{48b_0^2 b_1^2 N_c^3} \left[ 5(b_0 - b_1)U^2 - 72(b_0^2 + b_1^2)kN_c^2 - 10N_c^4 \Lambda^2 b_0 b_1 \right] \]

\[ + \frac{9k^2 \alpha N_c}{b_0^2 b_1^2 U^4} \left[ (b_0^2 + b_1^2)kN_c^2 - 3(b_0 + b_1)(b_0^2 + 4b_0 b_1 + b_1^2) \right] + \frac{27\alpha k(U^2 + 18kN_c^2)}{b_0 b_1 N_c U^4} \]

\[ \times \left[ (b_0^2 + 6b_0 b_1 + b_1^2)N_c^2 \Lambda - 3(b_0 - b_1)^2(b_0 + b_1) \right] - \frac{3\alpha N_c^2 k}{2U} \left( 5 + \frac{36N_c^2 k}{U^2} + \frac{64N_c^4 k^2}{U^4} \right) \]

\[ \times \left\{ \tan^{-1} \left( \frac{3(b_0 - b_1) + N_c^2 \Lambda}{U} \right) + \tan^{-1} \left( \frac{3(b_1 - b_0) + N_c^2 \Lambda}{U} \right) \right\}, \quad (57) \]

where \( U \) is given in eq. (27). Our \( S_1 \) in eq. (28) consist of two parts: Einstein-Hilbert piece and a first order correction piece coming from nGB. The integration over \( N_c \) has to be performed carefully as the integrand has singularity at \( N_c = 0 \). In the complex \( N_c \) plane the integrand has a branch-cut along the positive real axis. We will use Picard-Lefschetz theory to study this integration in the complex plane.
V. PICARD-LEFSCHETZ AND $N_c$-INTEGRATION

Our task then reduces to the computation of the $N_c$-integration, which will be studied in complex plane. We make use of complex analysis and methods of Picard-Lefschetz theory [43–46]. In the complex plane we work out steepest descent/ascend paths which allow us to determined the relevant contours of integration. We then sum over the contribution of all such paths to find the transition amplitude. This powerful methodology offers a natural exponential damping along each thimble instead of an oscillatory integral.

To describe the process we start with the following generic path-integral

$$I = \int \mathcal{D}z(t) e^{iS(z)/\hbar},$$  \hspace{1cm} (58)

where the exponent is a functional of $z(t)$. In situations when the action $S(z)$ becomes large, then the integrand starts to oscillate violently. In flat spacetime field theory the usual strategy to tame such behavior is to Wick rotate the integration contour. This transforms the oscillatory integral into an exponentially damped integral. In PL-theory one lifts both $z$ and $S$ in complex plane where one interprets $S$ as an holomorphic functional of $z(t)$ satisfying a functional form of Cauchy-Riemann conditions

$$\frac{\delta S}{\delta \bar{z}} = 0 \Rightarrow \begin{cases} \frac{\delta \text{Re} S}{\delta x} = \frac{\delta \text{Im} S}{\delta y} \\ \frac{\delta \text{Re} S}{\delta y} = -\frac{\delta \text{Im} S}{\delta x} \end{cases}.$$  \hspace{1cm} (59)

A. Flow equations

Writing the complex exponential as $I = iS/\hbar = h + iH$ and $z(t) = x_1(t) + ix_2(t)$ then evolution downstream is defined as

$$\frac{dx_i}{d\lambda} = -g_{ij} \frac{\partial h}{\partial x_j},$$  \hspace{1cm} (60)

where $g_{ij}$ is a metric defined on the complex manifold, $\lambda$ is flow parameter and $(-)$ sign refers to downward flow. These are the steepest descent contours also known as thimbles and denoted by $J_\sigma$. Steepest ascent contours are defined by plus sign in front of $g_{ij}$ in the above equation, and are denoted as $K_\sigma$. Here $\sigma$ refers to the saddle point to which it is attached. This definition automatically implies that the real part $h$ (also called Morse function) decreases monotonically along the steepest descent contour as one moves away from the critical point along the flows. This can be seen by computing

$$\frac{dh}{d\lambda} = g_{ij} \frac{dx_i}{d\lambda} \frac{\partial h}{\partial x_j} = -\left( \frac{dx_i}{d\lambda} \frac{dx^i}{d\lambda} \right) \leq 0.$$  \hspace{1cm} (61)

It holds generically for any Riemannian metric. However, for simplicity we can consider $g_{z,\bar{z}} = g_{\bar{z},z} = 0$ and $g_{z,z} = g_{\bar{z},\bar{z}} = 1/2$. This leads to simplified version of flow equations

$$\frac{dz}{d\lambda} = \pm \frac{\partial \tilde{T}}{\partial \bar{z}}, \quad \frac{d\bar{z}}{d\lambda} = \pm \frac{\partial \tilde{T}}{\partial z}.$$  \hspace{1cm} (62)
An immediate outcome of these flow equations is that the imaginary part of $\text{Im} I = H$ is constant along all the flow lines.

$$\frac{dH}{d\lambda} = \frac{1}{2i} \frac{d(I - \bar{I})}{d\lambda} = \frac{1}{2i} \left( \frac{\partial I}{\partial z} \frac{dz}{d\lambda} - \frac{\partial \bar{I}}{\partial \bar{z}} \frac{d\bar{z}}{d\lambda} \right) = 0.$$  \hspace{1cm} (63)

This is a wonderful feature of flow-lines which can be exploited to determine them quickly. In the complex $N_c$-plane, in cartesian co-ordinates language, the flow equations corresponding to steepest descent (ascent) becomes the following

Descent $\Rightarrow$ \hspace{1cm} \frac{dx_1}{d\lambda} = -\frac{\partial \text{Re} I}{\partial x_1}, \quad \frac{dx_2}{d\lambda} = -\frac{\partial \text{Re} I}{\partial x_2}, \hspace{1cm} (64a)$

Ascent $\Rightarrow$ \hspace{1cm} \frac{dx_1}{d\lambda} = \frac{\partial \text{Re} I}{\partial x_1}, \quad \frac{dx_2}{d\lambda} = \frac{\partial \text{Re} I}{\partial x_2}, \hspace{1cm} (64b)$

as the $\delta (\text{Im} I) = 0$ along the flow lines. These equations can be used to determine the trajectories of the steepest descent and ascent in the complex $N_c$-plane emanating from the saddle point. Each saddle point has a steepest descent trajectory starting from it and a steepest ascent trajectory ending in it. Based on boundary condition and the values of various parameters, the location of saddles move accordingly. Similarly the behavior of trajectories and their shape also changes. Usually these equations are coupled ODEs and can be complicated to solve analytically in complicated system like in present case. These flow lines can also be determined by exploiting the knowledge that $H$ is constant along them, however to determine the ascent/descent one needs to compute the gradient of first derivative (second derivative at the saddle points).

**B. Choice of contour**

Once the set of steepest descent/ascent trajectories and saddle points are known, it can be used to determine the contour of integration in the complex $N_c$-plane. This is the deformed contour of integration to which the original contour is deformed. Along this new path of integration the $N_c$ integral becomes absolutely convergent as discussed in great detail in [27]. However, determining a suitable path of contour need some work. Part of the job is done once steepest descent $J_\sigma$ and ascent paths $K_\sigma$, and saddle points $N_\sigma$ are known.

In the complex $N_c$ plane one can study the behavior of $h$ and $H$, and determine the allowed region (region where integral is well-behaved) and forbidden region (region where integral diverges). The former is denoted by $J_\sigma$ while later is denoted by $K_\sigma$. It is seen that $h(J_\sigma) < h(N_\sigma)$, while $h(K_\sigma) > h(N_\sigma)$. Moreover, generically it is seen that $h$ goes to $-\infty$ along the steepest descent lines and ends in singularity, while steepest ascent contours end in singularity where $h \to +\infty$. These two lines intersect at only one point where they are both well-defined. Our task is to choose a contour of integration which lies in region $J_\sigma$ and follows along the steepest descent paths [27]. The relevance of saddle is decided when the steepest ascent path emanating from it intersects the original path of integration. The Lefschetz thimble passing through this saddle point becomes the relevant $J_\sigma$, as the intersection point of $J_\sigma$ and $K_\sigma$ smoothly moves over to the intersection of $K_\sigma$ with original contour. Thus cleanly deforming the original contour to path along Lefschetz thimbles.
Then the original integration \((0^+, \infty)\) is a summation over contribution from all the steepest descent contours passing through relevant saddles. Formally it can be expressed as

\[
(0^+, \infty) = \sum_{\sigma} n_{\sigma} \mathcal{J}_{\sigma},
\]

where \(n_{\sigma}\) takes values \(\pm 1, 0\) depending on the relevance of saddles, while \(\mathcal{J}_{\sigma}\) here refers to integration performed along the steepest descent path. Once we have deformed the contour from the original integration path to sum over various relevant thimbles we have

\[
I = \int_{\mathcal{C}} dz(t) e^{iS[z]/\hbar} = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} dz(t) e^{iS[z]/\hbar}.
\]

Usually more than one thimbles contribute leading to an occurrence of an interference. This is the Lorentzian path integral which is summation of contribution from various relevant thimbles. The integration over each thimble is absolutely convergent if

\[
\left| \int_{\mathcal{J}_{\sigma}} dz(t) e^{iS[z]/\hbar} \right| \leq \int_{\mathcal{J}_{\sigma}} |dz(t)| |e^{iS[z]/\hbar}| = \int_{\mathcal{J}_{\sigma}} |dz(t)| e^{\hbar(z)} < \infty.
\]

Defining the length along the curve as \(l = \int |dz(t)|\), then the above integral is convergent if \(e^{\hbar} \sim 1/l\) as \(l \to \infty\). Then the original integration becomes a sum of absolutely convergent steepest descent integrals. On doing an expansion in \(\hbar\) we get the following leading order piece

\[
I = \int_{\mathcal{C}} dz(t) e^{iS[z]/\hbar} = \sum_{\sigma} n_{\sigma} e^{iH(N_{\sigma})} \int_{\mathcal{J}_{\sigma}} dz(t) e^{\hbar} \approx \sum_{\sigma} n_{\sigma} e^{iS[N_{\sigma}]/\hbar [A_{\sigma} + \mathcal{O}(\hbar)]},
\]

where \(A_{\sigma}\) is the contribution coming after doing a gaussian integration around the saddle point \(N_{\sigma}\).

### C. Flow directions

The flow-directions can be determined by computing second derivative of action with respect to \(N_c\) at the saddle points. Writing \(N_{c} = N_{\sigma} + \delta N\) (where \(N_{\sigma}\) is any saddle point of action), the action has a power series expansion in \(\delta N\).

\[
S^{(0)} = S^{(0)}_{\sigma} + \frac{dS^{(0)}}{dN_c} \bigg|_{N_c-N_{\sigma}} \delta N + \frac{1}{2} \frac{d^2S^{(0)}}{dN_c^2} \bigg|_{N_c-N_{\sigma}} (\delta N)^2 + \cdots
\]

The first order terms vanish identically as \(N_{\sigma}\) are saddle points. The second order terms can be obtained directly from the action in eq. (28) by taking double derivative with respect to \(N_c\). From this the direction of flows can be determined. One should remember that the imaginary part of exponential \(H\) is constant along the flow lines. This immediately leads to \(\text{Im}[iS - iS(N_{\sigma})] = 0\). The second variation at the saddle point can be expressed as a complex number \(\frac{d^2S^{(0)}}{dN_c^2} \bigg|_{N_c-N_{\sigma}} (\delta N)^2 \sim n^2 e^{i(\pi/2 + 2\theta + \rho)}\), where \(r\) and \(\rho\) depends on boundary conditions. Near the saddle point the change in \(H\) will go like

\[
\Delta(H) \propto i \left( \frac{d^2S^{(0)}}{dN_c^2} \bigg|_{N_{\sigma}} \right) (\delta N_c)^2 \sim n^2 e^{i(\pi/2 + 2\theta + \rho)},
\]
where we have written $\delta N_c = ne^{i\theta}$ and $\theta$ is the direction of flow lines. As the imaginary part of $H$ remains constant along the flow lines, so this implies

$$\theta = \frac{(2k - 1)\pi}{4} - \frac{\rho}{2}, \quad (71)$$

where $k \in \mathbb{Z}$. The steepest descent/ascent flow lines have angles $\theta_{\text{des/aes}}$ respectively, where the phase for $\Delta H$ is such that it correspond to $e^{i(\pi/2 + 2\theta + \rho)} = \mp 1$. This implies

$$\theta_{\text{des}} = k\pi + \frac{\pi}{4} - \frac{\rho}{2}, \quad \theta_{\text{aes}} = k\pi - \frac{\pi}{4} - \frac{\rho}{2}. \quad (72)$$

These angles can be computed numerically for the given boundary conditions and for gravitational actions.

**D. Saddle-point approximation**

Once we have the information about the saddles, flow-lines, their directions, and steepest descent/ascent paths (denoted by $J_{\sigma}/K_{\sigma}$ respectively), it is then easy to figure out the relevant saddle points. When the steepest ascent path emanating from a saddle point coincides with the original contour of integration (which in this case is $(0^+, \infty)$), then it is a relevant saddle point. The original integration contour then becomes sum over the contribution coming from all the Lefschetz thimbles through relevant saddle points. The path-integral giving transition amplitude in eq. (56) then becomes following

$$G[b_0, b_1] = \sum_{\sigma} n_{\sigma} \left(\frac{3iV_3}{4\pi\hbar}\right)^{1/2} \int_{J_{\sigma}} \frac{dN_c}{\sqrt{N_c}} \exp \left(\frac{iS_1}{\hbar}\right) \left[1 - \frac{N_c}{18} \int_0^1 dtA_2 + O(\alpha^2)\right],$$

$$\approx \sum_{\sigma} n_{\sigma} \left(\frac{3iV_3}{4\pi\hbar}\right)^{1/2} \exp \left(\frac{iS_1^{\text{on-shell}}}{\hbar}\right) \frac{1}{\sqrt{N_{\sigma}}} \left[1 - \frac{N_{\sigma}}{18} \int_0^1 dtA_2(\sigma) + O(\alpha^2)\right]$$

$$\times \int_{J_{\sigma}} dN_c \exp \left(\frac{i(S_1)_{N_{\sigma}N_c}}{\hbar}(N_c - N_{\sigma})^2\right) \left(1 + O(\sqrt{N})\right), \quad (73)$$

where $N_{\sigma}$ are saddle points which to first order in $\alpha$ are given in eq. (32), $S_1$ is given in eq. (28), $S_1^{\text{on-shell}}$ is given in eq. (33) and $A_2(\sigma)$ can be computed from eq. (77). The second variation of action with respect to $N_c$ computed at the saddle point and to first order in $\alpha$ is given by following

$$(S_1)_{N_{\sigma}N_c} = \frac{V_3}{2} \left\{ -\frac{(2 + 3\alpha)\{(b_0 - b_1)^2 + 6kN_0^2 - (b_0 + b_1)N_0^2\Lambda\}}{2N_0^3} + \frac{k\alpha\Lambda\{(b_0 - b_1)^2 - 6kN_0^2 + (b_0 + b_1)N_0^2\Lambda\}}{N_0\{(b_0 - b_1)^2 + 6kN_0^2 - (b_0 + b_1)N_0^2\Lambda\}} - \frac{24\alpha\Lambda(3\Lambda - 3k)(\Lambda - 3)}{\sqrt{k}N_0^6\Lambda^2} \right\}^2$$

$$+ 12kN_0^2 - 2(b_0 + b_1)N_0^2\Lambda \left\{ \tan^{-1} \frac{3(b_0 - b_1) + N_0^2\Lambda}{6\sqrt{k}N_0} + \tan^{-1} \frac{3(b_1 - b_0) + N_0^2\Lambda}{6\sqrt{k}N_0} \right\}. \quad (74)$$

On writing $N_c - N_{\sigma} = ne^{i\theta}$, where $\theta$ is the angle Lefschetz thimble make with the real $N_c$-axis, then the above integration can be performed easily. It gives the following

$$G[b_0, b_1] \approx \sum_{\sigma} n_{\sigma} \sqrt{\frac{3iV_3}{4N_{\sigma}|(S_1)_{N_{\sigma}N_c}|}} \exp \left(i\theta_{\sigma} + \frac{iS_1^{\text{on-shell}}}{\hbar}\right) \left[1 - \frac{N_{\sigma}}{18} \int_0^1 dtA_2(\sigma) + O(\alpha^2)\right],$$

$$\approx \sum_{\sigma} n_{\sigma} \sqrt{\frac{3iV_3}{4N_{\sigma}|(S_1)_{N_{\sigma}N_c}|}} \exp \left(i\theta_{\sigma} + \frac{iS_1^{\text{on-shell}}}{\hbar}\right) \left[1 - \frac{N_{\sigma}}{18} \int_0^1 dtA_2(\sigma) + O(\alpha^2)\right], \quad (75)$$

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where $N_\sigma$ and to first order in $\alpha$ is given by eq. (32), $S_1^{\text{on-shell}}$ is given in eq. (33), $A_2(N_\sigma)$ is given in eq. (57) and $(S_1)_{N_c,N_c}$ is given in eq. (74). This is a general expression for the transitional amplitude and valid for various kind of boundary conditions in the saddle point approximation to first order in $\alpha$. The corrections coming from novel Gauss-Bonnet gravity are present in $S_1^{\text{on-shell}}, N_\sigma, A_2(N_\sigma)$ and $(S_1)_{N_c,N_c}$.

VI. CLASSICAL UNIVERSE

Here we study the transition probabilities in the classical Universe. This usually happens when $b_1 > b_0 > (3k/\Lambda)$. In this scenario the zeroth order saddle point solution for $N_c$ given in eq. (22) indicate the saddles are real. This furthermore leads to real on-shell zeroth order action as can be seen from eq. (23). For each of the saddle point one can compute the second variation of action with respect to $N_c$ to find the directions of the steepest ascent and descent flows. In the case of classical boundary conditions the second variation at saddle point is real. This is easy to see from eq. (74). From eq. (70) it immediately implies that $\rho = 0$. This then translates into expression for $\theta$.

$$\theta_{\text{des}} = k\pi + \frac{\pi}{4}, \quad \theta_{\text{aes}} = k\pi - \frac{\pi}{4}. \quad (76)$$

The steepest descent/ascent flow-lines have a nice property that the imaginary part of $I$ (which is $H$) remains constant along them. This feature can be exploited to find them. We use this property to plot these flow lines in the complex $N_c$-plane. For purpose of better understanding the things we considered the following values of parameters: $(8\pi G) = 1$, $k = 1$, $\Lambda = 3$. The novel-Gauss-Bonnet parameter has mass-dimensions $M^{-2}$. If in eq. (1) we take $G$ inside bracket then we see $\alpha / G$ is dimensionless. This allow us to write $\alpha = \tilde{\alpha} / M_P^2$, where $M_P$ is Planck mass, and $\tilde{\alpha}$ is dimensionless. As in our convention $(8\pi G) = 1$, this means $M_P^2 = 8\pi$. This then implies $\alpha = \tilde{\alpha} / (8\pi)$.

In the case of classical boundary conditions $b_1 > b_0 > (3k/\Lambda)$, all the four saddle points lie on real axis: two are positive while two are negative. The two positive ones lie on the original integration contour $(0, \infty^+)$ and become relevant saddle point. The steepest descent paths passing through them will be relevant thimbles which both will contribute in the Lorentzian path integral. In the case of novel-GB gravity we notice that in the first order perturbation the saddle points have shifted compared to their position in case of pure Einstein-Hilbert gravity [27]. We considered a simple example to study this situation where we depict the steepest descent/ascent flow lines (red/black lines), saddle points (blue cross-circle, blue-square, blue-dot), forbidden/allowed region (light-orange/light-green region) in figure 1. The light-green region has $h < h(N_\sigma)$ while light-orange region has $h > h(N_\sigma)$.

The first relevant saddle starts from origin, circles around a bit in first quadrant, passes through blue-square then asymptotes to negative imaginary axis. The second relevant thimbles runs up from negative imaginary axis, passes through blue-dot and asymptotes to infinity at an angle $\pi/6$. Both these thimbles contribute to the Lorentzian path-integral and their sum is deformable to the original contour of integration as explained in subsection VIB and in [27]. The integral being absolutely convergent along the steepest descent lines naturally leads to a generalization of Wick rotation and correct answer for the Lorentzian path-integral.

For the case of classical boundary conditions, the relevant saddles and their corresponding
FIG. 1. In this plot we consider the case of classical boundary conditions: \( b_0 = 2 \) and \( b_1 = 5 \). We take parameter values to be \( (8\pi G) = 1, k = 1, \Lambda = 3 \), while \( \alpha = 10^{-1}(8\pi)^{-1} \). Here the red lines are steepest descent lines (thimbles \( J_\sigma \)), while thin black lines are steepest ascent lines denoted by \( K_\sigma \). The saddle points \( N_\sigma \) are shown by blue. The two blue cross-circle are irrelevant saddle points, while the relevant saddle points are shown with blue-square and blue-dot respectively. Along the red and black lines \( H \) remains constant and is equal to the value of \( H(N_\sigma) \). The light-green region has \( h < h(N_\sigma) \), while the light-orange region has \( h > h(N_\sigma) \). The boundary of these region is depicted by light-blue lines and along them \( h = h(N_\sigma) \). The original contour of integration \((0, \infty^+)\) is shown by thick black line.

steepest descent flow lines will have

\[
\begin{align*}
n \bullet \mathbf{a} &= 1, \quad \theta \bullet \mathbf{a} = -\frac{\pi}{4}, \quad \theta \circ \mathbf{a} = \frac{\pi}{4}.
\end{align*}
\]

(77)

If we define shorthand variables (to avoid clutter)

\[
\begin{align*}
u &= \sqrt{\frac{b_0 \Lambda}{3k} - 1}, \quad v = \sqrt{\frac{b_1 \Lambda}{3k} - 1},
\end{align*}
\]

(78)

then in terms of them one can express saddle points, on-shell action and second variation in a compact form. They are given by following

\[
\begin{align*}
N_{\mathbf{a}} &= \frac{3\sqrt{k}(v-u)}{\Lambda} + \frac{\alpha \sqrt{k}}{2} \left[ 15(v-u) + \frac{v-u}{uv} + 8 \left( \tan^{-1} v - \tan^{-1} u \right) \right] + \mathcal{O}(\alpha^2), \\
N_{\bullet} &= \frac{3\sqrt{k}(v-u)}{\Lambda} + \frac{\alpha \sqrt{k}}{2} \left[ 15(v-u) + \frac{v-u}{uv} + 8 \left( \tan^{-1} v - \tan^{-1} u \right) \right] + \mathcal{O}(\alpha^2), \\
S_{1|\mathbf{a}} &= \frac{3\pi^2 (u^3 - v^3)}{2\Lambda} - \frac{1}{4} \pi^2 \alpha (5u^3 + 3u - 5v^3 - 3v) + \mathcal{O}(\alpha^2),
\end{align*}
\]

(79, 80, 81)
\[ S_1 \big|_ \bullet = -\frac{3\pi^2 (u^3 + v^3)}{2\Lambda} + \frac{1}{4}\pi^2 \alpha (5u^3 + 3u + 5v^3 + 3v) + O(\alpha^2). \]  

Using these one can write the leading order term for the transition amplitude in \(1/\hbar\) expansion. This is given by,

\[
G[b_0, b_1] = e^{i\pi/4} \sqrt{kuv} \exp \left[ \frac{i\pi^2}{4\Lambda} \left\{ -6v^3 + \alpha \Lambda v(3 + 5v^2) \right\} \right] \times \cos \left[ \frac{\pi^2}{4\Lambda \hbar} \left\{ 6u^3 - \alpha \Lambda (3u + 5u^3) \right\} - \frac{\pi}{4} \right] + O(\alpha),
\]

where we agree with the known results in \(\alpha \to 0\) results computed in \([27]\). This is the leading term in \(1/\hbar\) and first order in \(\alpha\). For the next order term, we will mention it in the Appendix \([A]\) due to its length.

**VII. SUMMARY AND CONCLUSION**

In this paper we study novel-Gauss-Bonnet (nGB) action by performing \(D \to 4\) limit carefully. We study this scenario in cosmology and consider a generalised FLRW Universe respecting homogeneity and isotropy in arbitrary spacetime dimensions. We compute an action for scale-factor \(a(t_p)\) and lapse \(N_p(t_p)\) in the nGB gravity, where we notice that an integration by parts allow us to take the \(D \to 4\) limit smoothly without encountering divergences. The residual finite action obtained is used to study the classical and quantum aspects of theory in empty Universe.

In the first part of paper we reproduce the results obtained in the paper \([20]\) for classical cosmic evolution but for nonzero \(k\). As in \([20]\) we do a redefinition of scale factor \(a\) and lapse \(N_p\), thereby writing the theory in term of \(q(t)\) and \(N(t)\). The resulting action is a function of \(N, q\) and \(\dot{q}\) only, and doesn’t contain any \(t\)-derivative of lapse \(N\). Varying this action with respect to \(q\) gives equation of motion for \(q(t)\), while varying with respect to \(N\) gives a constraint. We solve the equation of motion for \(q(t)\) perturbatively to first order in \(\alpha\) for non-zero \(k\) for given boundary conditions. On plugging this back in to action of theory, gives us an action for lapse \(N\), which can be varied to obtain saddle points for \(N\). This has to be done order by order.

In the second part of paper we study the quantum aspects of the mini-superspace action of theory in the nGB gravity. We ask a straight-forward question what is the amplitude of transition from one 3-geometry to another in the case when gravity is getting modified due to novel-Gauss-Bonnet term? To answer this we study the path-integral of the mini-superspace theory by doing path-integration over \(q(t)\) and lapse \(N\). We study this directly in Lorentzian signature without doing a Wick-rotation of time co-ordinate to obtain Euclideanised theory. This is Lorentzian quantum cosmology of novel-GB gravity. We follow the strategy described in \([27, 36, 37]\) to analyse the path-integral in the mini-superspace approximation. We study this in gauge \(\dot{N} = 0\) (implying \(N(t) = N_c\), a \(t\)-independent parameter). This path-integral consist of two segments: path-integral over \(q(t)\) and an ordinary integral over \(N_c\). We study the former using WKB approximation while for the later we use combination of Picard-Lefschetz methods and WKB to compute the transition amplitude. We follow the footsteps of formalism developed in \([42]\) to compute this transition probability.

Then do the path-integral for \(q(t)\) using WKB we write \(q(t) = q_b(t) + Q(t)\), where \(Q(t)\) is fluctuation around the background solution \(q_b(t)\) which is computed perturbatively to first
order in $\alpha$. This gives us a path-integral over $Q(t)$ satisfying vanishing boundary conditions. In sub-section IV A we perform the $Q$-integral. Due to non-linear nature of the original mini-superspace action our abilities are limited and we compute this $Q$-integral perturbatively to first order in $\alpha$, following the strategy outlined in [42].

For the $N_c$ integral we make use of techniques of Picard-Lefschetz methods to analyse the integral in complex $N_c$ plane. For a generic set of boundary conditions we compute the saddle points in complex $N_c$ plane. We make use of flow equations the find the behavior of Morse function $h$ and $H$. It is noticed that $H$ remains constant along the steepest descent/ascent flow lines, a property which is used later to numerically draw a graph on the complex $N_c$ plane. Depending on boundary conditions it is seen that not all saddle points are relevant, as steepest ascent paths from only some will interest the original integration contour. The steepest descent paths from these relevant saddles will constitute the relevant thimbles contributing to the path-integral. The original contour can be deformed in to a contour passing through these relevant thimbles. The $N_c$-integral is then performed along these thimbles, taking contribution from all relevant thimbles. This is a generalization of Wick rotation. We obtain an expression for transition amplitude $G[b_0, b_1]$ to first order in $\alpha$ and in $1/\hbar$ expansion. This is given in eq. (75).

We use this to investigate the case of classical boundary conditions where $b_1 > b_0 > (3k/\Lambda)$. In this case the saddle points are all real and their corresponding on-shell action is also real. In this case we compute numerically the flow lines, and determine the angles they make with real axis. Out of the four real saddles (two positive and two negative), only the two positive ones are relevant, as they lie on the original integration contour. This implies that only two steepest descent curves are relevant, which will contribute in the $N_c$-integral. Combining all the ingredients we were finally able to write the leading order term in the transition amplitude for the case of classical boundary conditions, which is given in eq. (83). In the limit $\alpha \to 0$ this agrees with the result in [27], and we write the next order terms in the appendix. We notice that novel-GB gravity gives non-trivial correction to transition amplitude even though our analysis was done perturbatively.

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Appendix A: $S_{N_c N_c}$ at saddles, $A_2$ and $O(\alpha)$ terms

Here we write the expression for second variation of action at saddle points. For the classical boundary conditions the second variation at the two relevant saddle points is given by,

\begin{align*}
(S_1)_{N_c N_c}^\uparrow &= -\frac{2\sqrt{k}\Lambda uv}{v-u} + \frac{\sqrt{k}\Lambda^2 v^2}{3(v-u)} \left[ \frac{v^2 + uv + u^2}{uv} - 9uv + \frac{24u^2v^2}{(v-u)} \tan^{-1} v - \tan^{-1} u \right], \\
(S_1)_{N_c N_c}^\downarrow &= -\frac{2\sqrt{k}\Lambda uv}{v+u} + \frac{\sqrt{k}\Lambda^2 v^2}{3(v+u)} \left[ \frac{v^2 - uv + u^2}{uv} - 9uv - \frac{24u^2v^2}{(v+u)} \tan^{-1} v + \tan^{-1} u \right].
\end{align*}
In the computation of transition amplitude to first order in $\alpha$ we require to compute $A_2$ at the saddle points. Its expression at the two relevant saddle point it given by,

$$A_2 = -\frac{5\alpha \Lambda^3(u + v)(uv + 1)}{6k^{3/2} (u^2 + 1)^2 (v^2 + 1)^2} + \frac{\alpha \Lambda^4}{11664k^{5/2} (u^2 + 1)^2 (v^2 + 1)^2 (u - v)^3}$$

$$\times \left[ \sqrt{k}(u - v) \left( u^4 + 2u^2 + v^4 + 2v^2 + 2 \right) + 3 \left( u^2 + v^2 + 2 \right) \left( u^4 + u^2 \left( 4v^2 + 6 \right) + v^4 + 6v^2 + 6 \right) \right]$$

$$+ \frac{\alpha \Lambda^2(uv + 1)}{4\sqrt{k} (u^2 + 1)^2 (v^2 + 1)^2 (u - v)} \left[ u^4 (5v^2 + 9) - 10u^3 (v^3 + v) + u^2 (5v^2 + 2v^2 + 13) \right]$$

$$- 10u (v^3 + v) + 9v^4 + 13v^2 + 8 \right] + \frac{13\alpha \Lambda^2 (\tan^{-1} u - \tan^{-1} v)}{8\sqrt{k}}, \quad (A2)$$

$$A_2 = -\frac{5\alpha \Lambda^3(u - v)(uv - 1)}{6k^{3/2} (u^2 + 1)^2 (v^2 + 1)^2} + \frac{\alpha \Lambda^4}{11664k^{5/2} (u^2 + 1)^2 (v^2 + 1)^2 (u + v)^3}$$

$$\times \left[ \sqrt{k}(u + v) \left( u^4 + 2u^2 + v^4 + 2v^2 + 2 \right) - 3 \left( u^2 + v^2 + 2 \right) \left( u^4 + u^2 \left( 4v^2 + 6 \right) + v^4 + 6v^2 + 6 \right) \right]$$

$$+ \frac{\alpha \Lambda^2(uw - 1)}{4\sqrt{k} (u^2 + 1)^2 (v^2 + 1)^2 (u + v)} \left[ u^4 (5v^2 + 9) + 10u^3 (v^3 + v) + u^2 (5v^2 + 2v^2 + 13) \right]$$

$$+ 10u (v^3 + v) + 9v^4 + 13v^2 + 8 \right] - \frac{13\alpha \Lambda^2 (\tan^{-1} u + \tan^{-1} v)}{8\sqrt{k}}. \quad (A3)$$

The order $\alpha$ correction piece to the transition amplitude is given by,

$$G[b_0, b_1]|_{\text{correct}} = \frac{\alpha \sqrt{uv}}{139968u^3v^3 (u^2 + 1)^2 (v^2 + 1)^2} \left[ \frac{\Lambda}{(u - v)^2} \right]$$

$$\times \left\{ -9720\Lambda u^2v^2(u + v)(uv + 1)(u - v)^3 + \Lambda^2 u^2v^2 \left( (u - v) \left( u^4 + 2u^2 + v^4 + 2v^2 + 2 \right) \right. \right.$$
\[ \times \exp \left\{ \frac{i\pi^2}{4\hbar\Lambda} \left( -6v^3 - 6u^3 + \alpha\Lambda(3u + 3v + 5u^3 + 5v^3) \right) \right\} \]

\[ -i \frac{(13u^2 + 70uv + 13v^2 - 32)}{u - v} \tan^{-1}(u) \exp \left\{ -\frac{i\pi^2}{4\hbar\Lambda} \left( 6v^3 - 6u^3 + \alpha\Lambda(3u - 3v + 5u^3 + 5v^3) \right) \right\} \]

\[ + \frac{(13u^2 - 70uv + 13v^2 - 32)}{u + v} \tan^{-1}(v) \exp \left\{ \frac{i\pi^2}{4\hbar\Lambda} \left( -6u^3 - 6v^3 + \alpha\Lambda(3u + 3v + 5u^3 + 5v^3) \right) \right\} \]

\[ + i \frac{(13u^2 + 70uv + 13v^2 - 32)}{u - v} \tan^{-1}(v) \exp \left\{ -\frac{i\pi^2}{4\hbar\Lambda} \left( 6v^3 - 6u^3 + \alpha\Lambda(3u - 3v + 5u^3 - 5v^3) \right) \right\} \]

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