Self-avoiding walks and polygons on quasiperiodic tilings

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Abstract

We enumerate self-avoiding walks and polygons, counted by perimeter, on the quasiperiodic rhombic Penrose and Ammann-Beenker tilings, thereby considerably extending previous results. In contrast to similar problems on regular lattices, these numbers depend on the chosen initial vertex. We compare different ways of counting and demonstrate that suitable averaging improves convergence to the asymptotic regime. This leads to improved estimates for critical points and exponents, which support the conjecture that self-avoiding walks on quasiperiodic tilings belong to the same universality class as self-avoiding walks on the square lattice. For polygons, the obtained enumeration data does not allow us to draw decisive conclusions about the exponent.

1 Introduction

Quasiperiodic tilings are most widely used for the description of quasicrystals. With appropriate atomic decorations of the vertices, they serve as structure models which explain physical properties of quasicrystals [10]. From a theoretical point of view, they are idealisations of real substances on which the usual models of statistical physics like the Ising model may be studied [28 29 25]. Quasiperiodic tilings arose before the discovery of quasicrystals, however, more as an object of aesthetic interest in geometry [27 20].

From a combinatorial point of view, they provide an interesting example of a non-periodic yet structured graph where typical problems of combinatorics like the counting of
objects on a lattice become more complex. This is fundamentally different from counting problems on semiregular lattices, where the underlying translational invariance is still present \[17\], and also from self-similar graphs, where the self-similarity allows for the solvability of some counting problems.

Consider for example the counting problem of \(n\)-step walks on a quasiperiodic tiling. This number depends on the chosen starting point. On a lattice, this phenomenon does not occur due to translation invariance. Questions arise, such as if the universal properties of the walks like critical exponents \[26\] are changed for quasiperiodic tilings\(^1\), and how different ways of counting affect asymptotic properties. The first question has been investigated for self-avoiding walks (SAWs) and self-avoiding polygons (SAPs). These are walks or loops which do not visit the same vertex twice. Extrapolation of exact enumeration data for a number of quasiperiodic tilings \[5, 28\] indicates that the critical exponents \(\gamma\) for SAWs and \(\alpha\) for SAPs are consistent with the corresponding values on regular lattices \(\alpha = 1/2\) and \(\gamma = 43/32\). In \[22\], the related problem of SAWs on Penrose random tilings \[15\] has been studied by Monte Carlo simulations, the results suggesting the same mean square displacement exponent as in the lattice situation. The studies \[5, 28\] suffered however from strong finite-size effects due to the relatively short series data available. This is mainly due to the fact that the finite-lattice method \[9\], being the most successful method known to date for walk enumeration on regular lattices \[7, 8\], cannot easily be applied here, and so we used the slower method of direct counting. The problem in applying the FLM is discussed in Section 3 below, see also \[29\]. Another reason for the pronounced finite-size behaviour is that the number of walks or loops depends on the chosen starting point. One might suspect that suitable averaging over different starting points reduces these effects, leading to behaviour comparable to the square lattice case. In this paper, we analyse three different methods of counting in detail. Whereas the first one depends on a chosen vertex of the tiling, the last two are averages over the whole tiling.

- **Fixed origin walks.** We count the number of \(n\)-step self-avoiding walks emanating from a given vertex. This number depends on the chosen vertex.

- **Mean number of walks.** We count all translationally inequivalent \(n\)-step self-avoiding walks which may occur anywhere in the infinite tiling, weighted by their occurrence probability. For tilings with quasicrystallographic \(k\)-fold symmetries, these probabilities are numbers in the underlying module \(\mathbb{Z}[e^{2\pi i/k}]\). This leads to a generating function which has non-integer coefficients.

- **Total number of walks.** Here we count the number of translationally inequivalent \(n\)-step self-avoiding walks which may occur anywhere in the tiling. This number is bigger than the number of fixed origin walks, and by definition, takes into account vertices over the whole tiling.

Note that two self-avoiding walks (polygons) are translationally equivalent iff they have, up to a translation, the same vertex coordinates. For self-avoiding polygons, we will employ

\(^1\)For ferromagnetic spin systems on quasiperiodic graphs, a heuristic criterion determines whether its behaviour is different from the lattice situation \[25\].
the second and the third method of counting. We do not distinguish between different fillings of the interior of the polygon. For SAPs, the second method has been implemented previously \cite{28} to obtain the high temperature expansion of the Ising model. The mean number of SAPs up to length $2n = 18$ has been determined on the Ammann-Beenker tiling \cite{18} and on the rhombic Penrose tiling \cite{27,6}.

We counted SAWs and SAPs on the Ammann-Beenker tiling and the rhombic Penrose tiling and compared different counting schemes, thereby extending and generalising the previous approaches to counting SAWs \cite{5} and SAPs \cite{28}. Generally speaking, averaging reduces oscillation of data due to finite size effects, providing improved estimates for critical points and critical exponents. Within numerical accuracy, we cannot rule out the universality hypothesis that SAWs on the Ammann-Beenker and on the rhombic Penrose tilings have the same exponents as on the square lattice. The data for the total number of walks (polygons) gives a different exponent, reflecting the fact that the number of patches grows quadratically with the patch size, in contrast to the lattice case \cite{21,24}. The limited quantity of SAP data does not allow us to draw decisive conclusions about exponents.

This paper is organised as follows. The next chapters describe the algorithms used for the generation of the tilings and for the computation of the numbers of walks and their mean values. The following chapter is devoted to the asymptotic analysis of the series and to a comparison of the different approaches. This is concluded by a discussion of possible future work.

## 2 Graph generation

Quasiperiodic tilings in $\mathbb{R}^d$ may be obtained by projecting certain subsets of lattices from a higher-dimensional space $\mathbb{R}^n$ into $\mathbb{R}^d$. This is described by a cut-and-project scheme, summarised in the following diagram.

\[
\begin{array}{cccc}
E_{||} & \rightarrow & E & \rightarrow & E_{\perp} \\
\pi_{||} & \downarrow & \pi_{\perp} & \downarrow & \pi_{\perp} \\
L_{||} & \cup & L_{\perp} & \cup & W \\
L_{||} = \pi_{||}(L) & \leftarrow & L & \leftarrow & W_{\text{polytope}}
\end{array}
\]

It consists of a Euclidean vector space $E$, together with orthogonal projections $\pi_{||}$ and $\pi_{\perp}$. The vector spaces $E_{||} = \pi_{||}(E)$ and $E_{\perp} = \pi_{\perp}(E)$ are called direct and internal space, respectively. Let $L \subset E$ be a lattice. The projections are such that $\pi_{||}|_L$ is one-to-one and $\pi_{\perp}(L)$ is dense in $E_{\perp}$ (or dense in some subspace of $E_{\perp}$). Let $W \subset E_{\perp}$ be a polytope (or a finite union of polytopes). The set $W$ is also called the acceptance window. The set of tiling vertices $\Lambda(W)$ is defined by

\[
\Lambda(W) = \{ x_{||} \in L_{||} \mid x \in L \text{ and } x_{\perp} \in W \}.
\]

The edges of the tiling are defined by the following rule: The tiling vertices $\pi_{||}(x)$ and $\pi_{||}(y)$ are adjacent iff the lattice vectors $x$ and $y$ are adjacent.
For the Ammann-Beenker tiling \[ \text{(1)}, \text{(18)} \], we have \( n = 4 \) and \( d = m = 2 \). The lattice is \( L = \mathbb{Z}^4 \). The projections \( \pi_\parallel \) and \( \pi_\perp \) are defined as follows. For \( x \in \mathbb{R}^n \), we set

\[
\begin{align*}
x_\parallel &= \begin{pmatrix}
1 & \cos \frac{\pi}{4} & \cos \frac{2\pi}{4} & \cos \frac{3\pi}{4} \\
0 & \sin \frac{\pi}{4} & \sin \frac{2\pi}{4} & \sin \frac{3\pi}{4}
\end{pmatrix} x, \\
x_\perp &= \begin{pmatrix}
1 & \cos \frac{3\pi}{4} & \cos \frac{6\pi}{4} & \cos \frac{9\pi}{4} \\
0 & \sin \frac{3\pi}{4} & \sin \frac{6\pi}{4} & \sin \frac{9\pi}{4}
\end{pmatrix} x.
\end{align*}
\]

The acceptance window \( W \subset \mathbb{R}^m \) is a regular octagon with unit side length centred at the origin, having edges perpendicular to the axes. A typical patch is shown in Figure 1.

Figure 1: A patch of the Ammann-Beenker tiling.

For the rhombic Penrose tiling \[ \text{(27)}, \text{(6)} \], we have \( n = 5 \), \( d = 2 \) and \( m = 4 \). The lattice is \( L = \mathbb{Z}^5 \). The projections \( \pi_\parallel \) and \( \pi_\perp \) are, for \( x \in \mathbb{R}^n \), defined by

\[
\begin{align*}
x_\parallel &= \begin{pmatrix}
1 & \cos \frac{2\pi}{5} & \cos \frac{4\pi}{5} & \cos \frac{6\pi}{5} & \cos \frac{8\pi}{5} \\
0 & \sin \frac{2\pi}{5} & \sin \frac{4\pi}{5} & \sin \frac{6\pi}{5} & \sin \frac{8\pi}{5}
\end{pmatrix} x, \\
x_\perp &= \begin{pmatrix}
1 & \cos \frac{4\pi}{5} & \cos \frac{8\pi}{5} & \cos \frac{12\pi}{5} & \cos \frac{16\pi}{5} \\
0 & \sin \frac{4\pi}{5} & \sin \frac{8\pi}{5} & \sin \frac{12\pi}{5} & \sin \frac{16\pi}{5}
\end{pmatrix} x.
\end{align*}
\]

The acceptance window \( W \subset \mathbb{R}^m \) is made up of four regular pentagons in the planes \( x_3 = 0 \), \( 1 \), \( 2 \), \( 3 \). The pentagons in the 0 and 3 \( x_3 \)-planes have unit side length and the others have side length \( 2 \cos \frac{\pi}{5} \). Each pentagon is centred at \( x_\perp = 0 \). Pentagons 0 and 2 have an edge crossing the positive \( x_1 \)-axis at right angles while pentagons 1 and 3 are rotated through \( \frac{2\pi}{5} \). A typical patch is shown in Figure 2.
We remark in passing that a more natural embedding of the rhombic Penrose tiling is the root lattice $A_4$, see also [3], but $Z^5$ is more convenient for computations. Moreover, the Ammann-Beenker tiling and the rhombic Penrose tiling may be alternatively defined by inflation rules for their prototiles [4, 11].

3 Enumeration

A self-avoiding walk on a graph is a path, beginning at an origin vertex, which never visits a vertex more than once. The SAW on the lattice $Z^d$ is a well-studied object (see, for example, [26, 23]). The number $c_n$ of translationally inequivalent $n$-step walks on a regular lattice is clearly independent of the choice of origin vertex. Hence this series is representative of the entire lattice. The enumeration of SAWs on non-periodic tilings introduces complications to the interpretation of the series $C(x) = \sum_{n \geq 0} c_n x^n$. This is because the possible origin vertices produce an infinite range of different series $C(x)$. Each origin produces a different series which is representative only of that vertex’s immediate neighbourhood in the tiling. The question then is, how do we obtain a SAW series which is representative of the whole tiling? In this paper we adopt three different approaches to enumerating SAWs on quasiperiodic tilings, as described in the introduction. They are:

- **Fixed origin walks.**
- **Mean number of walks.**
- **Total number of walks.**
3.1 Fixed origin walks

We take a random selection of origin vertices \( x \in L \) (if \( x_{\perp} \notin W \) the vertex is not a suitable choice and is ignored). For each suitable origin, we generate the neighbourhood of the vertex, including all vertices up to some Euclidean distance \( N \) away. Two such neighbourhoods are shown in Figure 1 and Figure 2. We enumerate all SAWs from the origin up to length \( n \) in the neighbourhood using backtracking \([31]\). This takes time proportional to the number of walks \( c_n \). Unfortunately, the transfer matrix approaches used to enumerate SAWs on regular two-dimensional lattices in less time cannot be used on this problem without major adaptation: Since a SAW of \( N \) steps may reach a vertex \( N \) steps from the origin, we would need to consider every possible tiling patch of radius \( N \). The finite-lattice method’s transfer matrix stage would then need to be adapted to each tiling patch, or generalised to handle them all.

\[
\begin{align*}
\mathbf{x}_{\perp} &= (0, 0) & \mathbf{x}_{\perp} &= (1, 0) & \mathbf{x}_{\perp} &= (1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & \mathbf{x}_{\perp} &= (1 - \sqrt{2}, 0) \\
\end{align*}
\]

Figure 3: The actual Ammann-Beenker neighbourhoods chosen for the enumerations.

\[
\begin{align*}
\mathbf{x}_{\perp} &= (0, 0, 1) & \mathbf{x}_{\perp} &= (0, 0, 0) & \mathbf{x}_{\perp} &= (0.5, 0.5, 1) & \mathbf{x}_{\perp} &= (0.25, 0.5, 0) \\
\end{align*}
\]

Figure 4: The actual rhombic Penrose neighbourhoods chosen for the enumerations.

If we apply this method to counting walks on a regular lattice it is clear that we would always produce the usual SAW series for that lattice. If each of the series in Table 1 and Table 2 showed lattice consistent properties, it would be a good indication that these properties belong to the entire tiling. The actual neighbourhoods chosen for the enumerations are shown in Figure 3 and Figure 4 for the Ammann-Beenker and Penrose tilings respectively.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$n$ & $\mathbf{x}_\perp = (0, 0)$ & $\mathbf{x}_\perp = (1, 0)$ & $\mathbf{x}_\perp = (1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ & $\mathbf{x}_\perp = (1 - \sqrt{2}, 0)$ \\
\hline
0 & 1 & 1 & 1 & 1 \\
1 & 8 & 3 & 4 & 5 \\
2 & 16 & 13 & 12 & 16 \\
3 & 48 & 34 & 46 & 42 \\
4 & 144 & 108 & 108 & 152 \\
5 & 448 & 292 & 374 & 388 \\
6 & 1088 & 952 & 976 & 1194 \\
7 & 3680 & 2458 & 3042 & 3412 \\
8 & 9584 & 7746 & 8330 & 9678 \\
9 & 28336 & 21348 & 24556 & 27218 \\
10 & 82960 & 61478 & 68376 & 79150 \\
11 & 225408 & 177230 & 197820 & 217562 \\
12 & 657536 & 495808 & 554108 & 628996 \\
13 & 1834768 & 1412152 & 1576464 & 1741464 \\
14 & 5140752 & 3985706 & 4400920 & 4968606 \\
15 & 14584112 & 11125408 & 12531794 & 13724682 \\
16 & 40222672 & 31617786 & 34541864 & 39209054 \\
17 & 114683280 & 87149372 & 98846548 & 107503768 \\
18 & 313146848 & 248799302 & 270221012 & 306845714 \\
19 & 896810944 & 680172768 & 773046904 & 840463852 \\
20 & 2437468000 & 1943692238 & 2109562128 & 2386875508 \\
21 & 6958267152 & 5303535884 & 6011045200 & 6548653714 \\
22 & 18981078176 & 15086983820 & 16431248782 & 18500898140 \\
23 & 53728620912 & 41295324398 & 46538635588 & 50883461478 \\
24 & 147472084608 & 116624466842 & 127704810544 & 142927122532 \\
25 & 413887940176 & & & \\
\hline
\end{tabular}
\caption{The number of $n$-step fixed origin SAWs for various starting points $\mathbf{x}_\perp$ in the Ammann-Beenker tiling, with starting point coordinates $(x, y)$ given in the internal space.}
\end{table}
Table 2: The number of fixed origin $n$-step SAWs for various starting points $\mathbf{x}_\perp$ in the rhombic Penrose tiling, with starting point coordinates $(x, y, z)$ given in the internal space.
3.2 Mean number of walks

Given that a pair of vertices from $\Lambda(W)$ are adjacent if and only if they are adjacent in $L$, we see that the neighbours of a vertex with image $x_\perp$ can be found by sequentially adding the $E_\perp$ image of all possible edges in $L$ to $x_\perp$ and testing if the new points lie in $W$. If it does the adjacent vertex exists in $\Lambda(W)$. By recursively checking all possible neighbours of a vertex, all possible walks on the lattices will be found.

Given an origin vertex $x^0$ in $\Lambda(W)$, we know its $E_\perp$ image $x^0_\perp$ must lie somewhere in $W$, i.e. $x^0_\perp \in W^0 = W$ ($W^n$ is the region $x^0_\perp$ can lie in given our knowledge of the $n$ steps in the walk). If we take a step $s$ (with projection $s_\perp$ onto $E_\perp$) to a possible adjacent vertex $x^1$, then we know $x^1_\perp = x^0_\perp + s_\perp$. Furthermore if $x^1 \in \Lambda(W)$ is true, $x^1_\perp \in W$. Hence $W^1 = (W \cap (W^0 + s_\perp)) - s_\perp$. The probability that the step $s$ is possible from a random $x^0$ is given by the ratio between the areas of $W^1$ and $W$.

Extending this to a walk of length $n$ with steps $s^i, i = 1 \ldots n$, $W^k = (W \cap (W^{k-1} + \sum_{i=1}^k s^i_\perp)) - \sum_{i=1}^k s^i_\perp$, the probability of the walk existing is the ratio of the areas of $W^n$ and $W$. For example, consider the shaded $W^i$ in Figure 5 for the particular walk in the Ammann-Beenker tiling which steps west, south, south-west, north-west, west, north then north. For the Ammann-Beenker tiling, these probabilities are of the form $a + b\lambda$, where $\lambda = 1 + \sqrt{2}$ and $a, b \in \mathbb{Q}$.

Adding the self-avoiding constraint and summing the probabilities results in the expected number of SAWs beginning at a random origin. Again note that applying this method to a regular lattice would result in the usual SAW series. Due to the extra complexity of the rhombic Penrose tiling acceptance window, the area calculations are more involved, see also \[28\]. They lead to mean numbers of the form $a + b\tau$, where $\tau = (1 + \sqrt{5})/2$ is the golden number, and $a, b \in \mathbb{Z}$. The rhombic Penrose tiling also allows steps in ten directions, more than the Ammann-Beenker tiling’s eight. These facts combine to allow greater length series to be computed on the Ammann-Beenker tiling.
Figure 5: Examples of $W^0 \ldots W^7$ for a particular walk on the Ammann-Beenker tiling.

Table 3: The mean number of $n$-step SAWs for the Ammann-Beenker tiling and the rhombic Penrose tiling, where $\lambda = 1 + \sqrt{2}$ and $\tau = (1 + \sqrt{5})/2$. 

| $n$ | Ammann-Beenker | rhombic Penrose |
|-----|----------------|-----------------|
| 0   | 1              | 1               |
| 1   | 4              | 4               |
| 2   | 52-16$\lambda$ | 62 -30$\tau$   |
| 3   | 80-16$\lambda$ | -4 + 28$\tau$  |
| 4   | 444-134$\lambda$ | 914 -488$\tau$ |
| 5   | 1280-380$\lambda$ | -820 + 732$\tau$ |
| 6   | 4492-1430$\lambda$ | 13842 -7894$\tau$ |
| 7   | 10848-3248$\lambda$ | -17732 + 12860$\tau$ |
| 8   | 60988-21700$\lambda$ | 173876 -101988$\tau$ |
| 9   | 89800-27036$\lambda$ | -255784 + 173720$\tau$ |
| 10  | 643248-237732$\lambda$ | 1923078 -1143988$\tau$ |
| 11  | 979776-324200$\lambda$ | -3149856 + 2073192$\tau$ |
| 12  | 5486960-2043420$\lambda$ | 19566548 -11734706$\tau$ |
| 13  | 10785736-3819788$\lambda$ | -34951044 + 22612992$\tau$ |
| 14  | 45253532-16927618$\lambda$ | 192557132 -116151274$\tau$ |
| 15  | 110294592-40576780$\lambda$ | -366912524 + 234803904$\tau$ |
| 16  | 375796808-141368464$\lambda$ |
| 17  | 1058437232-398339560$\lambda$ |
| 18  | 3259350860-1238175678$\lambda$ |
| 19  | 9526156024-3632872284$\lambda$ |
| 20  | 29127575440-11192322668$\lambda$ |
| 21  | 81536068712-31337365980$\lambda$ |
| 22  | 259724099656-100797073134$\lambda$ |
3.3 Total number of walks

Investigating all possible walks as in the mean number of walks method, we count instead the number of non-zero contributions to the mean value. This counts the number of translationally inequivalent walks with $W^n$ having positive area or, equivalently, the number of translationally inequivalent walks which may occur anywhere in the tiling. Again we note that applying this method to a regular lattice gives the usual SAW series.

| $n$ | Ammann-Beenker | Penrose   |
|-----|----------------|-----------|
| 0   | 1              | 1         |
| 1   | 8              | 10        |
| 2   | 56             | 90        |
| 3   | 288            | 560       |
| 4   | 1280           | 2800      |
| 5   | 5344           | 12060     |
| 6   | 20288          | 48520     |
| 7   | 74192          | 182000    |
| 8   | 260336         | 658300    |
| 9   | 892800         | 2282400   |
| 10  | 2976512        | 7749440   |
| 11  | 9828256        | 25634920  |
| 12  | 31758112       | 83615140  |
| 13  | 101847216      | 268113660 |
| 14  | 322240144      | 850895040 |
| 15  | 1012048208     | 2668534600|
| 16  | 3147031584     |           |
| 17  | 9732815728     |           |
| 18  | 29852932384    |           |
| 19  | 91182029360    |           |
| 20  | 276695822928   |           |
| 21  | 836719766336   |           |
| 22  | 2516664888416  |           |

Table 4: The total number of $n$-step SAWs for the Ammann-Beenker tiling and the rhombic Penrose tiling.

3.4 Self-avoiding polygons

A self-avoiding polygon is equivalent to a self-avoiding walk in which the initial and final vertices are adjacent. In the enumeration of self-avoiding polygons, we do not distinguish between polygons having, up to a translation, the same boundary but different fillings of the interior.
SAPs may be enumerated in the same manner as we enumerate SAWs. The additional property of end point adjacency allows the backtracking algorithm to be pruned earlier. When enumerating SAPs up to size \( N \), a walk that visits a vertex after \( D \) steps that is further than \( N - D \) steps from the origin may be pruned from the search tree. Such a walk can never form part of a SAP of \( \leq N \) steps. This was used to extend the length of the rhombic Penrose SAP series. The Ammann-Beenker SAP series were calculated at the same time as the SAW series and hence are of the same length. The extensive run time requirements precluded further series extension.

For the computation of occurrence probabilities of self-avoiding polygons, the loop vertices are taken into account, as described in Section 3.2 for the mean number of walks. If the self-avoiding polygon has \( n \) loop vertices \( x_i \in \Lambda(W) \), where \( i = 1, \ldots, n \), the acceptance domain is \( W^n = \bigcap_i (W - x_i) \), and the occurrence probability is given by the ratio between the areas of \( W^n \) and \( W \), see also [28].

| \( n \) | mean number | total number | mean number | total number |
|-------|-------------|--------------|-------------|--------------|
| 2     | 4           | 8            | 4           | 10           |
| 4     | 8           | 48           | 8           | 80           |
| 6     | 12\( \lambda \) | 384         | 108-48\( \tau \) | 840         |
| 8     | 800-272\( \lambda \) | 2960    | 240-64\( \tau \) | 6480         |
| 10    | 2840-880\( \lambda \) | 21600   | 6192-3364\( \tau \) | 49760        |
| 12    | 28152-9984\( \lambda \) | 170256   | 25584-13248\( \tau \) | 394080       |
| 14    | 47712-9884\( \lambda \) | 1322048  | 179200-95340\( \tau \) | 3087140       |
| 16    | 869600-299392\( \lambda \) | 10194720 | 162976-5440\( \tau \) | 24020160      |
| 18    | 215712+294408\( \lambda \) | 79960896 | 2704140-1067580\( \tau \) | 183529440      |
| 20    | 14980920-3730840\( \lambda \) | 618248240 |  |
| 22    | 152588920-47048100\( \lambda \) | 4726263168 |  |

Table 5: The mean number of \( n \)-step SAPs and the total number of SAPs for the Ammann-Beenker tiling (first two columns) and for the rhombic Penrose tiling (last two columns), where \( \lambda = 1 + \sqrt{2} \) and \( \tau = (1 + \sqrt{5})/2 \).

4 Analysis of series

The various sequences were analysed using standard methods of asymptotic analysis of power series expansions as described in [12]. For self-avoiding walks and polygons, it is easy to prove that the limit \( \lim_{n \to \infty} \frac{c_n}{n^\gamma} \) exists by use of concatenation arguments [26]. We assume the usual asymptotic growth of the sequence coefficients \( c_n \), viz:

\[
c_n = A x_e^{-n} n^{\gamma - 1} \left[ 1 + O(n^{-\varepsilon}) \right] \quad (n \to \infty, 0 < \varepsilon \leq 1).
\] (4)

On the square lattice, there is overwhelming evidence [8] of the above asymptotic behaviour with \( \gamma = 43/32 \). There is however no proof of this assumption. For interesting new
developments see [23]. The above assumption results in an asymptotic growth of the ratios $r_n$

$$r_n = \frac{c_n}{c_{n-1}} = \frac{1}{x_c} \left( 1 + \frac{\gamma - 1}{n} + O(n^{-1-\epsilon}) \right) \quad (n \to \infty, 0 < \epsilon \leq 1),$$

which may be used to extrapolate numerical estimates of $x_c$ and $\gamma$. Whereas it has been proved for the square lattice that the limit $\lim_{n \to \infty} c_n/c_{n-2}$ exists and coincides with $x_c^{-2}$ [19], a similar statement for the ratios $r_n$ is not known. For some lattices, counterexamples are known [13].

Fig. 6 shows a plot of the ratios $r_n$ against $1/n$ for a typical fixed origin Ammann-Beenker walk (full circles) and for the ratios of the mean numbers of Ammann-Beenker walks (large empty circles). We notice that the fixed origin data suffers from dramatic fluctuations, which are smoothed out by averaging, but are still larger than the corresponding square lattice data [8], which is shown in small circles. The oscillating behaviour of the mean number of walks data is due to an additional singularity of the sequence generating function at $x = -x_c$, which, for the case of the square lattice, is well understood due to anti-ferromagnetic ordering [8]. To obtain estimates of $x_c$ and $\gamma$, we used the standard method described in [12] and first mapped away the singularity on the negative real axis by an Euler transform and then used Neville-Aitken series extrapolation.

We also used the method of differential approximants (DAs) [12]. The underlying idea is to fit a linear differential equation with polynomial coefficients to the generating function of the sequence, truncated at some order $n_0$. The critical points and critical exponents of the differential equation are expected to approximate the critical behaviour of the underlying sequence. Application of first order and second order DAs has proved useful for the analysis of square lattice SAWs [12]. A first order DA involves fitting the coefficients to the differential equation $P_1(x)xf'(x) + P_0(x)f(x) = R(x)$ where $P_1(x)$, $P_0(x)$
and \( R(x) \) are polynomials of degree \( N, M, \) and \( L \) respectively. We refer to this as a \([L/M; N]\) DA. We analysed the approximants \([L/N - 1; N], [L/N; N], [L/N + 1; N]\) for \( 1 \leq L \leq 8 \). We computed estimates for \( x_c \) and \( \gamma \) by averaging of the different DA results, given a fixed number of series coefficients \( n_0 \). Since this yields more accurate estimates of \( x_c \) and \( \gamma \) (typically of one digit better) than the ratio method and series extrapolation, we list in Table 6 only the results for the DA analysis. Note that the errors are no strict error bounds, but arise from averaging over approximants \([L/N_0; N_1]\) for different values of \( N_0, N_1 \) and \( L \), as explained in [12].

| \( x_c \)          | (0, 0)    | \( (1 - \frac{x_c}{\sqrt{2}}, \frac{x_c}{\sqrt{2}}) \) | (1, 0)    | \( (1 - \sqrt{2}, 0) \) |
|---------------------|-----------|-------------------------------------------------|-----------|-------------------------|
| 0.36414(18)         | 0.3659(14)| 0.3644(13)                                      | 0.3644(21)| 0.3657(16)              |
| \( \gamma \)       | 1.325(19) | 1.45(16)                                        | 1.30(14)  | 1.32(17)                |

Table 6: Estimates of \( x_c \) and \( \gamma \) for Ammann-Beenker SAWs. Numbers in brackets denote the uncertainty in the last two digits.

As suggested by the ratio plot, the estimate using the data for the mean number of walks yields the most precise estimates, which are, however, one order of magnitude in error worse than the corresponding estimates for square lattice SAWs.

An analysis of the total number of SAWs on the Ammann-Beenker tiling using first order DAs yields \( x_c = 0.3647(33) \) and \( \gamma = 3.14(37) \). Whereas the critical point estimate is consistent with the previous analysis, the exponent estimate deviates from the value of \( 43/32 = 1.34375 \) for fixed origin SAWs or the mean number of SAWs. This phenomenon reflects the fact that the number of Ammann-Beenker patches of radius \( r \) grows asymptotically as \( r^2 \). (More generally, for aperiodic Delone sets in \( \mathbb{R}^d \) described by a primitive substitution matrix, the number \( N(r) \) of patches of radius \( r \) grows like \( N(r) \approx r^d [24, 21] \).) Since the SAW has fractal dimension \( 4/3 \), we expect an asymptotic increase of the number of SAWs by \( n^{2\nu} \), where \( \nu = 3/4 \). Thus \( \gamma = 43/32 + 2\nu = 2.84375 \). Data extrapolation is consistent with this value.

The analysis of SAP data follows the same lines. However, the estimates suffer from large finite size errors due to the low number (11) of available coefficients. First order differential approximants for the mean number of SAPs yield \( x_c = 0.3688(41) \). We assume \( x_c(SAP) = x_c(SAW) \), which has been proven for the square lattice case [14].

For the critical exponent \( \alpha = 2 + \gamma \), we expect for the mean number of SAPs by universality that \( \alpha = 1/2 \), being the believed exact value for the square lattice (and numerically confirmed to very high precision [16]). Due to lack of data it is not possible to give estimates of critical exponents. An analysis of the total number of SAPs on the Ammann-Beenker tiling using first order DAs yields \( x_c = 0.3587(15) \).

The above analysis has also been applied to the rhombic Penrose tiling data. We observe qualitatively the same finite size behaviour as for the Ammann-Beenker tiling data, though the fluctuations are a bit less pronounced. In Table 7 we list estimates for \( x_c \) and \( \gamma \) obtained by analysing first order differential approximants.

An analysis of the total number of SAWs on the rhombic Penrose tiling using first
Table 7: Estimates of $x_c$ and $\gamma$ for Penrose SAWs. Numbers in brackets denote the uncertainty in the last two digits.

|   | mean no. | (0,0,0) | (0,0,1) | (0.25,0.5,0) | (0.5,0.5,1) |
|---|----------|---------|---------|--------------|-------------|
| $x_c$ | 0.36322(29) | 0.3621(12) | 0.3613(16) | 0.36248(83) | 0.36347(51) |
| $\gamma$ | 1.333(26) | 1.28(14) | 1.19(22) | 1.303(83) | 1.387(62) |

order DAs yields $x_c = 0.3638(31)$ and $\gamma = 2.77(21)$. The estimate of the critical point is consistent with the estimates from the other methods of counting. For the critical exponent, we again expect a value of $\gamma = 43/32 + 2\nu = 2.84375$, which agrees with the extrapolation within numerical accuracy.

For the analysis of SAP data, only 9 series coefficients are available. First order differential approximants for the mean number of SAPs yield $x_c = 0.372(11)$. An analysis of the total number of SAPs on the rhombic Penrose tiling using first order DAs yields $x_c = 0.3590(22)$. Again, due to lack of data, it is not possible to extrapolate reasonable estimates for the critical exponent $\alpha$.

**Conclusion**

We extended previous enumerations for self-avoiding walks and polygons on the Ammann-Beenker tiling and on the rhombic Penrose tiling and extracted estimates for the critical point and critical exponent, using different counting schemes. It turned out that averaging with respect to the occurrence probability in the whole tiling leads to the best estimates, whereas data produced by fixing an origin leads to strong finite size oscillations. The results support the universality hypothesis that the critical exponents appear to be the same as for the square lattice, within confidence limits. For the total number of walks (polygons) we obtain a new exponent reflecting the polynomial complexity of the number of patches of the underlying tiling.

Since the results were obtained using the enumeration method of backtracking, one might ask if more efficient enumeration methods can be applied in order to substantially increase the length of the series, and hence the accuracy of the estimates. Unfortunately, the successful finite-lattice method cannot be applied in this case, without substantial development.

On a mathematically rigorous level, it may be possible to show the equality of the critical points for SAPs and SAWs on quasiperiodic tilings by appropriately modifying the existing proofs for the hypercubic lattice [14]. Furthermore, it would be interesting to carry out an analysis to determine if random walk behaviour can be proved for dimensions greater than four [26].

Self-avoiding polygons may also be counted by area. Since this leads to a three-variable generating function due to the different areas of the prototiles, it is tempting to ask whether the scaling behaviour of these objects is different from that recently found [30] for self-avoiding polygons on two-dimensional lattices.
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