COLOURED $\mathfrak{sl}_r$ INVARIANTS OF TORUS KNOTS AND CHARACTERS OF $W_r$ ALGEBRAS

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Abstract. Let $p < \ p'$ be a pair of coprime positive integers. In this note, generalizing Morton’s work in the case of $\mathfrak{sl}_2$, we give a formula for the $\mathfrak{sl}_r$ Jones invariants of torus knots $T(p, p')$ coloured with the finite-dimensional irreducible representations $L_r(n\Lambda_1)$. When $r \leq p$, we show that appropriate limits of the shifted (non-normalized, framing dependent) invariants calculated along $L_r(n\Lambda_1)$ are essentially the characters of certain minimal model principal $W$ algebras of type $A$, namely, $W_r(p, p')$, up to some factors independent of $p$ and $p'$ but depending on $r$. In particular, these limits are essentially modular. We expect these limits to be the 0-tails of corresponding sequences of invariants. At the end, we formulate a conjecture on limits for $p < r$.

1. Introduction

Let $g$ be a finite dimensional simple Lie algebra over $\mathbb{C}$, let $\lambda$ be a dominant integral weight, and let $L_g(\lambda)$ be the finite dimensional irreducible representation with highest weight $\lambda$.

In [27], Rosso and Jones provided a formula for calculating the $L_g(\lambda)$ coloured invariants of torus knots $T(p, p')$ where $1 \leq p < p'$ and $(p, p') = 1$. In [26], Morton used this formula and calculated coloured $\mathfrak{sl}_2$ Jones invariants for all torus knots $T(p, p')$.

We show that Morton’s calculations generalize straight-forwardly to the case when $g = \mathfrak{sl}_r$ and $\lambda = n\Lambda_1$ (or $\lambda = n\Lambda_{r-1}$). The main ingredient driving these calculations is that all weights of $L_r(n\Lambda_1)$ have multiplicity one.

In the case when $r \leq p$, we then show that as $n \to \infty$, the resulting coloured (non-normalized, framing dependent) invariants have periodic limits. This period is related to the group $P_r/Q_r$ where $P_r$ and $Q_r$ respectively denote the weight and root lattices of $\mathfrak{sl}_r$. In particular, we show that if $n \to \infty$ along multiples of $r$ then the limit is essentially the character of the vertex operator algebra $W_r(p, p')$, up to some factors independent of $p, p'$ but depending on $r$. We expect these limits to be the 0-tails (see Definition 6.3 below) of the corresponding sequence of invariants, but at the moment, we do not have a full proof of this. The case $p < r$ is much more involved – our formula for (non-normalized, framing dependent) Jones polynomial involves a lot of cancellation, and their limit is 0, however, their tails are not. This case seems to have a lot of nice structure, see Conjecture 6.6 below. We hope to study this in detail in future.

The algebras $W_r(p, p')$ are known as principal $W$ algebras of type $A$ and due to results of Arakawa, [3] and [4], they satisfy important properties, namely, $C_2$-cofiniteness and rationality. Results of [13] now tell us that these characters are modular invariant with respect to appropriate congruence subgroups up to some factor $q'$ depending on the central charge of $W_r(p, p')$.

By now, there is an extensive literature on heads, tails and the stability of coloured Jones polynomials. We review some of the major highlights relevant for the present paper. The 0-stability of coefficients of coloured $\mathfrak{sl}_2$ Jones polynomials seems to have been first conjectured by Dasbach and Lin in [12], and proved for alternating and for adequate links by Armond [5] using skein-theoretic methods. A general notion of $k$-stability was introduced in [17] where Garoufalidis
and Lê proved that coloured $\mathfrak{s}l_2$ Jones polynomials of alternating links are $k$-stable for all $k$ (see also [21] and [7] for higher order stability). For some related, but purely $q$-series-theoretic studies, see [1], [25], [8], etc. In [19] Garoufalidis and Vuong conjectured that for a knot $K$, simple Lie algebra $\mathfrak{g}$ and dominant integral weight $\lambda$, the sequence of coloured $\mathfrak{g}$-invariants $J_K(L_\mathfrak{g}(n\lambda))$ has a property called cyclotomic stability. Roughly speaking, this property captures the periodic nature of limits as mentioned above. In the same paper [19], authors proved this conjecture for all torus knots and all rank 2 Lie algebras. Recently, Yuasa [30] has proved 0-stability for $L(n\Lambda_1)$-coloured $\mathfrak{s}l_3$ Jones polynomials for minus-adequate links using skein theory of $A_2$; see also [31] and [32].

We note that several other relations between invariants of knots, manifolds and characters of various algebras are known in the literature. For instance, in [9] tails for torus links $(2,2p)$ were related to characters of (irrational) singlet vertex operator algebras. Much earlier, in [23], Kashaev invariants of torus knots $T(p,p')$ were shown to be related to the Eichler integrals of $\text{Vir}(p,p') = W_2(p,p')$ minimal model characters. There are even more relations between certain quantum invariants of 3-manifolds and characters of certain logarithmic VOAs found recently, [10]. Finally, we mention [14] where HOMFLYPT polynomials of torus knots were related to characters of certain rational Cherednik algebras.

In another direction, we mention that recently, there has been much interest (see for example [2], [11], [15], [29], [24] and [28]) in understanding the combinatorics of $\mathcal{W}_3(3,p)$ characters. In general, $\mathcal{W}_r(p,p')$ characters are related to cylindric partitions; see [20], [15], etc. It will be very interesting to see if knot theory could provide other combinatorial approaches to these characters similar to how walks along the knot $T(2,2n+1)$ give rise to fermionic characters of $\mathcal{W}_2(2,2n+1)$ [6]. This has been our primary motivation for the present study.

This paper is organized as follows. We shall review some basic facts about Lie algebras $\mathfrak{sl}_r$ in Section 2. We will then review characters of $\mathcal{W}_r(p,p')$ algebras in Section 3 where we study the cases $r = p$ and $p < r$ in detail. Then, in Section 4 we explain the Rosso–Jones formula which governs coloured $\mathfrak{s}l_r$ invariants of torus knots. Section 5 presents the main calculation of $J_{T(p,p')}^{(1)}(L_r(n\Lambda_1))$, generalizing Morton’s arguments from [26]. Lastly, we study the $n \to \infty$ limits in Section 6. As mentioned above, the limits are trivial for $p < r$, but the tails are not. We present the example of $r = 3, p = 2$ in detail and show that our results match those given in [19]. Finally, we formulate a general conjecture in the case $p < r$.

Finally, we now explain certain choices made in this paper. Firstly, some of our main results are presented with respect to non-normalized and framing dependent invariants $J_{T(p,p')}^{(1)}$ where $T(p,p')$ is assumed to have writhe $pp'$. This has the nice consequence that for $p < p'$ with $(p,p') = 1$, $J_{T(p,p')}^{(1)}(L_r(kr\Lambda_1)) \in \mathbb{Z}[q]$ for all $ks$, see Remark 6.2. Secondly, it is common in vertex operator algebra literature to denote the parameters of $\mathcal{W}_r$ algebras as $p, p'$. While this becomes cumbersome especially in complicated formulas, we keep this notation since the role of parameters $p, p'$ is largely interchangeable and thus they merit similar notation. Indeed, we have $\mathcal{W}_r(p,p') \cong \mathcal{W}_r(p',p)$ and we also have $T(p,p') = T(p',p)$. Thirdly, all of the analysis presented here also works with weights $n\Lambda_{r-1}$, however, for convenience, we adhere to $n\Lambda_1$.

\section{2. Lie algebras $\mathfrak{sl}_r$}

Throughout, we will be working with the Lie algebra $\mathfrak{sl}_r$ of trace 0 $r \times r$ complex matrices with $r \geq 2$. We will let the Cartan subalgebra $\mathfrak{h}$ to be the set of diagonal matrices. Let $(\cdot, \cdot)$ be the trace form on $\mathfrak{sl}_r$, which is symmetric, invariant and non-degenerate and stays non-degenerate on $\mathfrak{h}$. We will identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via this form. In the usual way, we will embed $\mathfrak{h}^*$ into the vector space spanned by symbols $\epsilon_1, \ldots, \epsilon_r$ such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$. For $1 \leq i \leq r - 1$ we will have the simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1} \in \mathfrak{h}^*$ and the fundamental weights $\Lambda_i \in \mathfrak{h}^*$ such that $(\Lambda_i, \alpha_j) = \delta_{ij}$. The sets of roots and fundamental weights both form bases of $\mathfrak{h}^*$. We have the following sets of positive,
negative and all roots:

\[ \Phi_r^+ = \{ \pm(\alpha_i + \cdots + \alpha_j) | 1 \leq i \leq j \leq r - 1 \}, \quad \Phi_r = \Phi_r^+ \cup \Phi_r^- \]

The Weyl vector will be denoted by \( \delta \in \mathfrak{h}^* \) and it is uniquely determined by:

\[ (\delta, \alpha_i) = 1 \quad \text{for all } 1 \leq i \leq r - 1. \]

We have, with \( 1 \leq i \leq r - 1 \):

\[ \delta = \frac{(r - 1)}{2} \epsilon_1 + \frac{(r - 3)}{2} \epsilon_2 + \cdots + \frac{(1 - r)}{2} \epsilon_r, \]

\[ \Lambda_i = \frac{r - i}{r} (\epsilon_1 + \cdots + \epsilon_i) - \frac{i}{r} (\epsilon_{i+1} + \cdots + \epsilon_r). \]

We will let \( Q_r \subseteq P_r \) be the root and the weight lattices, respectively. Recall that \( Q_r \) is an even lattice, i.e., \( \| \lambda \| ^2 \in 2\mathbb{Z} \) for all \( \lambda \in Q_r \). \( P_r^+ \) will denote the set of dominant integral weights, i.e., \( \lambda \in P_r^+ \) iff \( (\lambda, \alpha_i) \in \mathbb{Z}_{\geq 0} \) for all \( i \). We will let \( P_r^0 \) denote the dominant integral weights that belong to the interior of the fundamental Weyl chamber, i.e., \( \lambda \in P_r^0 \) iff \( (\lambda, \alpha_i) \in \mathbb{Z}_{>0} \) for all \( i \).

The Weyl group of \( \mathfrak{sl}_r \) acts on \( \mathfrak{h}^* \) and is isomorphic to the symmetric group \( \mathfrak{S}_r \). \( \mathfrak{S}_r \) acts on \( \epsilon_i \)'s by permuting the subscripts and this determines its action on \( \mathfrak{h}^* \). Recall that \( Q_r, P_r \) and the form \( (\cdot, \cdot) \) are all \( \mathfrak{S}_r \)-invariant. With respect to the simple transpositions \( (i, i + 1) \), the length of the shortest expression for \( w \) will be denoted as \( \ell(w) \). The sign representation of \( \mathfrak{S}_r \) equals \( (-1)^{\ell(w)} \).

For \( \lambda \in P_r^+ \), let \( L_r(\lambda) \) denote the irreducible \( \mathfrak{sl}_r \) module with highest weight \( \lambda \). The modules \( L_r(n\Lambda_1) \) for \( n \geq 0 \) will be important to us in this note. We denote the set of weights of \( L_r(n\Lambda_1) \) by \( \Pi_{r,n} \). The following fact about \( \Pi_{r,n} \) will be crucial for us, and so we provide a quick proof; see also [16].

**Fact 2.1.** We have \( \lambda \in \Pi_{r,n} \) iff \( \lambda = n\Lambda_1 - a_1\alpha_1 - \cdots - a_{r-1}\alpha_{r-1} \) with \( n \geq a_1 \geq \cdots \geq a_{r-1} \geq 0 \). Additionally, every weight of \( L_r(n\Lambda_1) \) occurs with multiplicity 1.

**Proof.** Successively using the unbrokenness of the \( \mathfrak{sl}_2 \) strings with respect to the roots \( \alpha_1, \alpha_2, \ldots, \alpha_{r-1} \), we see that each \( \lambda = n\Lambda_1 - a_1\alpha_1 - \cdots - a_{r-1}\alpha_{r-1} \) with \( n \geq a_1 \geq \cdots \geq a_{r-1} \geq 0 \) belongs to \( \Pi_{r,n} \). This implies that the cardinality of \( \Pi_{r,n} \) is at least the \( n \)-th \((r - 1)\)-dimensional triangular number, which equals \( \binom{n+r-1}{r-1} \). This is in fact the same as the dimension of \( L_r(n\Lambda_1) \) as can be seen quickly from the Weyl dimension formula [16, Eq. 15.17]. Consequently, \( \Pi_{r,n} \) is precisely the set of such weights and moreover, each of them appears with multiplicity exactly 1. \( \square \)

In fact, similar argument works with \( L_r(n\Lambda_{r-1}) \). The weights of this module are of the form \( n\Lambda_{r-1} - a_{r-1}\alpha_{r-1} - \cdots - a_1\alpha_1 \) for \( n \geq a_{r-1} \geq \cdots \geq a_1 \geq 0 \) with each weight having multiplicity exactly 1. We will stick with \( n\Lambda_1 \) throughout this note but all the analysis also works for \( n\Lambda_{r-1} \).

Noting that \( r\Lambda_1 = (r-1)\alpha_1 + (r-2)\alpha_2 + \cdots + \alpha_{r-1} \in Q_r \), following two further facts are clear.

**Fact 2.2.** We have that \( \Pi_{r,n} \subseteq \Pi_{r,n+r} \).

**Fact 2.3.** We have that \( \bigcup_{j \geq 0} \Pi_{r,n+jr} = n\Lambda_1 + Q_r = t\Lambda_1 + Q_r \) where \( t \) is the residue of \( n \) modulo \( r \).

### 3. Characters of \( \mathcal{V}_r(p,p') \) algebras

In this section, we give character formulas for irreducible modules of the vertex operator algebra \( \mathcal{V}_r(p,p') \) following [2] and [15]. We then study the cases \( r = p \) and \( p < r \) separately, as they have interesting ramifications when we consider \( \mathfrak{sl}_r \) invariants of torus knots later. Note that this character formula requires \( r \leq p, p' \) with \( p, p' \) coprime, but we may continue to substitute other values of the parameters in the characters and treat them purely as \( q \)-series.
3.1. The characters. Let $2 \leq r < p' < p$ be integers with $p, p'$ relatively prime.

Given a non-negative integer $k$, we will let $P_{r,k}^+$ be the set of dominant integral weights $\lambda$ such that $(\lambda, \theta) \leq k$ with $\theta$ being the highest root $\alpha_1 + \cdots + \alpha_{r-1}$. This set is in bijection with the highest weights of integrable, irreducible, highest-weight, level $k$ modules for the affine Lie algebra $\mathfrak{sl}_r$.

The principal $W$ algebra of $\mathfrak{sl}_r$ with parameters $(p, p')$, denoted as $W_r(p, p')$, has inequivalent simple representations parametrized by $(\xi, \zeta) \in P_{r,p-r}^+ \times P_{r,p'-r}^+$. With $\eta(q) = q^{1/24}(q; q)_\infty$ denoting the Dedekind eta function, the corresponding characters are given as follows, [15, Eq. (91)]:

$$
\chi_{\xi, \zeta}^{r,p,p'} = \frac{1}{\eta(q)^{r-1}} \sum_{\alpha \in Q_r, \sigma \in \mathbb{S}_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2} pp'|\alpha-(\xi-\zeta)/p+\sigma(\zeta+\delta)/p'|^2}
$$

$$
= q^{\frac{1}{2} pp'|\xi-\zeta|^2-\frac{r-1}{24}} \sum_{\alpha \in Q_r, \sigma \in \mathbb{S}_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2} pp'|\alpha|^2-p'(\alpha, \alpha)+p(\alpha, \sigma(\zeta+\delta))-(\xi+\delta, \sigma(\zeta+\delta)-(\zeta+\delta))}
$$

$$
= q^{\frac{1}{2} pp'|\xi-\zeta|^2-\frac{r-1}{24}(q; q)_\infty} C_{\xi, \zeta}^r,
$$

(3.1)

where $C_{\xi, \zeta}^r \in 1 + q\mathbb{Z}[[q]]$ is the generating function of certain kinds of cylindrical partitions, [15, Eqn. (27)]. We may now normalize the character so that it belongs to $1 + q\mathbb{Z}[[q]]$:

$$
\bar{\chi}_{\xi, \zeta}^{r,p,p'} = (q; q)_\infty C_{\xi, \zeta}^r \in 1 + q\mathbb{Z}[[q]].
$$

(3.2)

The choice $\xi = \zeta = 0$ corresponds to the vertex operator algebra $W_r(p, p')$, and we record the following normalized character:

$$
\bar{\chi}(W_r(p, p')) = \bar{\chi}_{0,0}^{r,p,p'} = \frac{1}{(q; q)_\infty} \sum_{\alpha \in Q_r, \sigma \in \mathbb{S}_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2} pp'|\alpha|^2-p'(\alpha, \alpha)+p(\alpha, \sigma(\zeta+\delta))-(\delta, \sigma(\zeta+\delta)-(\zeta+\delta))`.
$$

(3.3)

Given a $\mu \in P_r$, it will be beneficial for us to define the $\mu$-shifted character:

$$
\bar{\chi}_{\mu}^{r,p,p'} = \frac{1}{(q; q)_\infty} \sum_{\alpha \in Q_r+\mu, \sigma \in \mathbb{S}_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2} pp'|\alpha|^2-p'(\alpha, \alpha)+p(\alpha, \sigma(\zeta+\delta))-(\delta, \sigma(\zeta+\delta)-(\zeta+\delta))`.
$$

(3.4)

These shifted characters are unchanged if we vary $\mu$ in a coset of $Q_r$. For $\mu \in Q_r$, we have:

$$
\bar{\chi}_{\mu}^{r,p,p'} = \bar{\chi}_{\xi, \zeta}^{r,p,p'}.
$$

3.2. The case $r = p$. We now show that the case $r = p$ is somewhat special in that all the $\mu$-shifted characters coincide up to a sign. In particular, when $\mu = k\Lambda_1$, it will help us in showing that shifted Jones invariants of torus knots $\Gamma_r(p, p')$ with $r < p'$ and $(r, p') = 1$ have a well-defined (non-periodic) limit with respect to the representations $L_r(n\Lambda_1)$ of $\mathfrak{sl}_r$.

To this end, define the following for $p' > r$ with $(r, p') = 1$, $\mu \in P_r$, $\zeta \in P_{r,p'-r}^+$ and $\sigma \in \mathbb{S}_r$:

$$
\Gamma_{0, \zeta}^{r,p,p'}(\sigma) = \sum_{\alpha \in Q_r+\mu} (-1)^{\ell(\sigma)} q^{\frac{1}{2} pp'|\alpha|^2-p'(\alpha, \alpha)+r(\alpha, \sigma(\zeta+\delta))-(\delta, \sigma(\zeta+\delta)-(\zeta+\delta))
$$

$$
= q^{\frac{1}{2} pp'|\mu|^2+(\mu, -p'+r\sigma(\zeta+\delta))-(\delta, \sigma(\zeta+\delta)-(\zeta+\delta))} \sum_{\alpha \in Q_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2} pp'|\alpha|^2+(\alpha, r\mu-p'+r\sigma(\zeta+\delta))}
$$

(3.4)

Before we proceed to manipulate these sums, we record the following useful property.

**Lemma 3.1.** Given any $\mu \in P_r$, there exists a $w \in \mathbb{S}_r$ and $\lambda \in Q_r$ such that:

$$
r\mu - \delta + w\delta = r\lambda \in rQ_r.
$$

(3.5)
Proof. If (3.5) holds for \( \mu \in P_r \) with \( w \in \mathcal{S}_r \) and \( \mu' \in P_r \) with \( w' \in \mathcal{S}_r \), then it holds for \( \mu + w\mu' \) with \( w\mu' \in \mathcal{S}_r \), because:

\[
r(\mu + w\mu') - \delta + w\mu'\delta = r\mu - \delta + w\delta + rw\mu' - w\delta + w\mu'\delta \in rQ_r.
\] (3.6)

Let us now prove (3.5) for \( \mu = \Lambda \). We have:

\[
r\Lambda_1 - \delta = \frac{r-1}{2}\epsilon_1 + \left( \frac{1-r}{2}\epsilon_2 + \frac{3-r}{2}\epsilon_3 + \cdots + \frac{r-3}{2}\epsilon_r \right).
\]

We may thus take \( w \) to be the cycle \( \sigma = (r,1,2,3,\ldots,r-1) \), so that \( r\Lambda_1 - \delta + \sigma\delta = 0 \in rQ_r \).

Iterating (3.6) with \( \mu = \mu' = \Lambda_1 \) and \( w = w' = \sigma \), we see that (3.5) holds for \( \mu_k = \Lambda_1 + w\Lambda_1 + \cdots + w^{k-1}\Lambda_1 \) with \( \sigma^k \). It is easy to see that for \( 1 \leq k \leq r \),

\[
\mu_k = \Lambda_1 + \sigma\Lambda_1 + \cdots + \sigma^{k-1}\Lambda_1 = \frac{(r-k)}{r}(\epsilon_1 + \cdots + \epsilon_k) - \frac{k}{r}(\epsilon_{k+1} + \cdots + \epsilon_r)
\]

which equals \( \Lambda_k \) if \( 1 \leq k \leq r-1 \) and is 0 if \( k = r \).

Any \( \nu \in P_r \) can now be written as \( \nu = \mu_k + \phi \) for some \( 1 \leq k \leq r \) and \( \phi \in Q_r \). Required statement for \( \nu \) now follows by taking \( \mu = \mu_k, \mu' = \phi, w = \sigma^k \) and \( w' = \text{Id} \) in (3.6). \( \square \)

We now have the following result.

**Proposition 3.2.** Fix \( \mu \in P_r \), and let \( \lambda \in Q_r \) and \( w \in \mathcal{S}_r \) be as in Lemma 3.1. Then, for all \( \sigma \in \mathcal{S}_r \) and \( \zeta \in P^+_{r,p'-r} \) we have:

\[
(-1)^{\ell(w)} \cdot \Gamma_{0,\zeta}^{\sigma,p';\mu}(\sigma) = \Gamma_{0,\zeta}^{\sigma,p';0}(w^{-1}\sigma).
\]

Consequently, we have:

\[
(-1)^{\ell(w)} \cdot \nabla_{0,\zeta}^{\sigma,p';\mu} = \nabla_{0,\zeta}^{\sigma,p'}.
\] (3.7)

Note that the sign \((-1)^{\ell(w)}\) depends solely on \( \mu \).

**Proof.** Pick and fix \( \sigma \in \mathcal{S}_r \). For convenience, denote \( v = w^{-1}\sigma \). We then have:

\[
\Gamma_{0,\zeta}^{\sigma,p';0}(v)
= \sum_{\alpha \in Q_r} (-1)^{\ell(v)} q^{\frac{1}{2}rp'\|\alpha\|^2-p'(\alpha-v(\zeta+\delta))+r(\alpha,v(\zeta+\delta))-r\alpha(\zeta+\delta)}

\]

\[
= \sum_{\alpha \in w^{-1}Q_r+\lambda} (-1)^{\ell(v)} q^{\frac{1}{2}rp'\|\alpha\|^2-p'(\alpha-v(\zeta+\delta))+r(\alpha,v(\zeta+\delta))-r\alpha(\zeta+\delta)}

\]

\[
= q^{-r\alpha(\zeta+\delta)-\zeta-\delta} + q^{\frac{1}{2}rp'\|\alpha\|^2-p'(\alpha-v(\zeta+\delta))+r(\alpha,v(\zeta+\delta))-r\alpha(\zeta+\delta)}

\]

\[
\sum_{\alpha \in Q_r} (-1)^{\ell(v)} q^{\frac{1}{2}rp'\|\alpha\|^2+(\alpha,rp'\lambda-p'w\delta+r\sigma(\zeta+\delta))}

\]

\[
= (-1)^{\ell(w)} q^{-r(\alpha,\zeta+\delta)-\zeta-\delta} + q^{\frac{1}{2}rp'\|\alpha\|^2-p'(\alpha-v(\zeta+\delta))+r(\alpha,v(\zeta+\delta))-r\alpha(\zeta+\delta)}

\]

\[
\sum_{\alpha \in Q_r} (-1)^{\ell(v)} q^{\frac{1}{2}rp'\|\alpha\|^2+(\alpha,rp'\mu-p'\delta+r\sigma(\zeta+\delta))}

\]

\[
= (-1)^{\ell(w)} q^{\frac{1}{2}rp'\left(\|\alpha\|^2-\|\mu\|^2\right)-(\lambda,\mu)p'\delta-(\mu,\mu)p'\delta-(\mu,\mu)p'\delta-(\mu,\mu)p'\delta-(\mu,\mu)p'\delta-(\mu,\mu)p'\delta-(\mu,\mu)p'\delta}.
\]

Now it is easy to check using \( v = w^{-1}\sigma \) and \( r\delta - \delta + w\delta = r\lambda \) that:

\[
-(\delta,v(\zeta+\delta)) + (\lambda,r\sigma(\zeta+\delta)) + (\delta,\sigma(\zeta+\delta)) - (\mu,r\sigma(\zeta+\delta)) = 0,
\]

\[
\frac{1}{2}r \left(\|\alpha\|^2-\|\mu\|^2\right) - (\lambda,w\delta) + (\mu,\delta) = 0.
\]

This immediately implies the first relation. The second relation follows by summing the first relation over \( \sigma \in \mathcal{S}_r \) and finally dividing by \( (q; q)^{r-1} \). \( \square \)
3.3. The case $p < r$. The character formula is not valid unless \( r \leq p, p' \) and \((p, p') = 1\). We now show that if \( p < r \) then these characters evaluate to 0 as \( q \)-series. This is related to the fact that our formulas for \( \mathfrak{sl}_r \) invariants of corresponding torus knots involve a lot of cancellations and thus one needs to be careful in calculating their tails.

We have the following lemma which essentially drives all the relevant cancellations.

**Lemma 3.3.** Let \( 1 \leq p \) and let \( w \in \mathcal{S}_r \). We have the following cases.

1. If \( p < r \) and \( u = w(1, p + 1) \) we have \( u\delta - w\delta \in pQ_r \) and \( (-1)^{\ell(u)} \neq (-1)^{\ell(w)} \).
2. If \( p = r - 1 \) then \( u\delta - w\delta \in (r - 1)Q_r \) iff \( u = w \) or \( u = w(1, r) \). In the latter case we have \( u\delta - w\delta = -(r - 1)w\theta \) where \( \theta = \alpha_1 + \cdots + \alpha_{r-1} \in \Phi_r \) is the highest root.
3. If \( p \geq r \) then \( u\delta - w\delta \in pQ_r \) iff \( u = w \).

**Proof.** For \( 0 < p \leq r - 1 \), we have
\[
(1, p + 1)\delta - \delta = p(\epsilon_1 - \epsilon_j) = p(\alpha_1 + \cdots + \alpha_{j-1}) \in pQ_r.
\]
This immediately proves the first part. The only pair of different coordinates of \( \delta \) whose difference is an integer divisible by \( r - 1 \) is \((r - 1)\epsilon_1/2 \) and \( (1 - r)\epsilon_r/2 \). Thus, if \( p = r - 1 \), \( \sigma\delta - \delta \in (r - 1)Q_r \) iff \( \sigma = 1 \) or \( \sigma = (1, r) \). Consequently, \( u\delta - w\delta \in (r - 1)Q_r \) iff \( u = w \) or \( u = w(1, r) \). The second part now follows easily. For \( p \geq r \), there is simply no way to subtract two different coordinates of \( \delta \) to get an integer divisible by \( p \). Indeed, the biggest possible difference is \( r - 1 \) when we swap \((r - 1)/2 \) with \((1 - r)/2 \). This implies the third part.

We now have the following result.

**Proposition 3.4.** Let \( 1 \leq p \leq r - 1, \mu \in P_r \). Then,
\[
\overline{\chi}_{0,0}^{r, p, p'; \mu} = 0.
\]

**Proof.** We will show below that for any fixed \( w \in \mathcal{S}_r \),
\[
\Gamma_{0,0}^{r, p, p'; \mu}(w) = -\Gamma_{0,0}^{r, p, p'; (1, p + 1)\mu}((1, p + 1) \cdot w).
\]
This will imply, upon summing over all \( w \) that:
\[
\overline{\chi}_{0,0}^{r, p, p'; (1, p + 1)\mu} = -\overline{\chi}_{0,0}^{r, p, p'; \mu}.
\]
But, noting that for \( \mu \in P_r \) we have \((1, p + 1)\mu - \mu = Q_r \), we also have:
\[
\overline{\chi}_{0,0}^{r, p, p'; (1, p + 1)\mu} = \overline{\chi}_{0,0}^{r, p, p'; \mu},
\]
which now gives the required result.

Fix \( w \in \mathcal{S}_r \). We write \( v = w^{-1}(1, p + 1) \) and note that \((-1)^{\ell(v)} = -(1)^{\ell(w)} \). Further, from Lemma 3.3, we have \((1, p + 1)\delta - \delta \in pQ_r \). We may thus write
\[
(1, p + 1)\delta - \delta = p\lambda
\]
for some \( \lambda \in Q_r \). We have:
\[
\Gamma_{0,0}^{r, p, p'; \mu}(w) = \sum_{\alpha \in Q_r + \mu} (-1)^{\ell(w)}q^{\frac{1}{2}p'p'\|\alpha\|^2 - p'(\alpha, \delta) + p(\alpha, w\delta) - (\delta, w\delta - \delta)}
\]
\[
= \sum_{\alpha \in Q_r + \mu} (-1)^{\ell(w)}q^{\frac{1}{2}p'p'\|pw^{-1}\alpha - w^{-1}\delta\|^2 + (pw^{-1}\alpha - w^{-1}\delta, \delta) + \|\delta\|^2 - \frac{p'}{2p}||\delta||^2}
\]
\[
= - \sum_{\alpha \in Q_r + \mu} (-1)^{\ell(v)}q^{\frac{1}{2}p'p'\|pw^{-1}(\alpha + \lambda) - v\delta\|^2 + (pw^{-1}(\alpha + \lambda) - v\delta, \delta) + \|\delta\|^2 - \frac{p'}{2p}||\delta||^2}
\]
\[
= - \sum_{\alpha \in Q_r + w^{-1}\mu} (-1)^{\ell(v)}q^{\frac{1}{2}p'p'\|\alpha - v\delta\|^2 + (\alpha - v\delta, \delta) + \|\delta\|^2 - \frac{p'}{2p}||\delta||^2}
\]
\[ \sum_{\alpha \in Q_r + v^{-1}w^{-1} \mu} (-1)^{\ell(v)} q^{\lfloor h_{\alpha} - 2 \rfloor} \left( (p v^a - d_{\delta}) + \| \delta \|^2 - \frac{1}{2} \right) \]
\[ = -\Gamma_{0,0}^r p^r (1, p+1)^a ((1, p + 1)w). \]

\[ \square \]

4. Torus knots and the Rosso–Jones formula

Fix the Lie algebra \( \mathfrak{sl}_r \).

Let \( \mathcal{R} \) denote the Grothendieck ring over \( \mathbb{C} \) of finite dimensional representations of \( \mathfrak{sl}_r \). As a vector space over \( \mathbb{C} \) it has a basis of finite dimensional irreducible modules, but as an algebra over \( \mathbb{C} \), it is generated by those modules whose highest weights are the fundamental weights. This ring is isomorphic to the ring \( \text{ch}(\mathcal{R}) = (\mathbb{C}[x_1, \ldots, x_r]/(x_1 \cdots x_r - 1))^\mathfrak{S}_r \) where \( \mathfrak{S}_r \) acts by permuting the variables. Naturally, the isomorphism is provided by considering characters of the modules. Let us also consider the bigger ring \( \widehat{\text{ch}}(\mathcal{R}) = \mathbb{C}[x_1, \ldots, x_r]/(x_1 \cdots x_r - 1) \) which is isomorphic to the group algebra of \( P_r \). This isomorphism is as follows. Given a \( \lambda = a_1 \Lambda_1 + \cdots + a_r \Lambda_{r-1} \in P_r \), the corresponding monomial in \( \widehat{\text{ch}}(\mathcal{R}) \) is:
\[ x^\lambda = \prod_{i=1}^{r-1} x_i^{a_i + a_{i+1} + \cdots + a_{r-1}}. \tag{4.1} \]

The characters of (finite-dimensional) irreducible modules are given by Weyl’s character formula:
\[ \text{ch}(L_r(\lambda)) = \sum_{\alpha \in \mathfrak{S}_r} (-1)^{\ell(v)} x^{\alpha(v) + \delta} = \frac{N(\lambda)}{\Delta_r} \]

The numerator and denominator of this formula do not belong to \( \text{ch}(\mathcal{R}) \) as they are alternating functions under the action of \( \mathfrak{S}_r \). They do, however, naturally live in the the bigger ring \( \widehat{\text{ch}}(\mathcal{R}) \) since it is isomorphic to the group algebra of \( P_r \).

For a dominant integral weight \( \lambda \) and a knot \( K \) along with a framing \( f \), let \( J^f_K(L_r(\lambda)) \) denote the (framing dependent) Reshetikhin-Turaev invariant of \( K \) with strands corresponding to \( L_r(\lambda) \). Let \( J^\circ_K(L_r(\lambda)) \) denote the (framing independent) coloured Jones invariant, i.e., the framing of \( K \) is altered to zero by introducing appropriate number of twists/curls. Further let \( J^{\circ,1}_K(L_r(\lambda)) \) (resp. \( J^{a,1}_K(L_r(\lambda)) \)) denote the framing dependent (resp. framing independent) normalized invariant that satisfies \( J^{\circ,1}_K(L_r(\lambda)) = 1 \) (resp. \( J^{a,1}_K(L_r(\lambda)) = 1 \)) where \( \circ \) is the unknot.

For a knot \( K \), we will view \( J^f_K, J^a_K : J^{\circ,1}_K, J^{a,1}_K \) as functions
\[ J^f_K : \mathcal{R} \to \mathbb{C}[q^1, q^{-1}] \]
\[ L_r(\lambda) \mapsto J^f_K(L_r(\lambda)). \]

where \( t \) is certain appropriate fraction. We will now use this upgraded notation where the argument of \( J^f_K \) is any element of \( \mathcal{R} \). By an abuse of the notation, we also use \( J^a_K \) to denote corresponding maps on \( \text{ch}(\mathcal{R}) \).

Let \( p < p' \) be a pair of coprime positive integers.

Given a knot \( K \) with framing \( f \), \( K(p, p') \) denotes the \( (p, p') \) cabling of \( K \), with framing \( \bar{f} \) inherited from the framing \( f \) of \( K \). Rosso–Jones formula relates coloured \( \mathfrak{sl}_r \) invariants of \( K(p, p') \) with that of \( K \). Two functions on \( \mathcal{R} \) enter into the Rosso–Jones formula [27], as explained by [26]. The first is the twist which is the action of ribbon element on the modules. On (finite-dimensional) irreducible modules, it acts by a scalar, given by \( (\lambda \in P^+_r) \):
\[ \theta^a_{L_r(\lambda)} = q^{\frac{1}{2}(\lambda, \lambda + 2\delta)}. \]
Second map is the $p$th Adams operation. On $\text{ch}(\mathcal{R})$ it acts by:

$$\psi_p(x_i) = x_i^p$$

Since $\mathcal{R}$ and $\text{ch}(\mathcal{R})$ are isomorphic, we will use the same notation $\psi_p$ to denote the operation on $\mathcal{R}$. In fact, $\psi_p$ is also well-defined on the bigger ring $\widehat{\text{ch}}(\mathcal{R})$.

Now, the Rosso–Jones [27] formula, as explained in [26], states:

**Theorem 4.1.** Let $p, p'$ be a pair of positive coprime integers. As maps on $\mathcal{R}$ we have:

$$J^p_{K(p,p')} = J^p_K \circ \theta^{p'/p} \circ \psi_p. \quad (4.2)$$

Let us specialize to the case $K = \odot$, the unknot with zero framing $u$. In this case, $\odot(p, p')$ gives rise to a framed torus knot $T(p, p')$ with writhe $pp'$. Call this framing $\overline{u}$.

Now, $J^u_\odot$ acts on the ring $\mathcal{R}$ simply as the quantum dimension. That is,

$$J^u_\odot(L_r(\lambda)) = \text{qdim}(L_r(\lambda)) = \prod_{\alpha \in \Phi^+_r} \frac{q^{\frac{1}{2}(\lambda+\delta,\alpha)} - q^{-\frac{1}{2}(\lambda+\delta,\alpha)}}{q^{\frac{1}{2}(\delta,\alpha)} - q^{-\frac{1}{2}(\delta,\alpha)}}. \quad (4.3)$$

Actually, this is known as the principal specialization of the character, and on $\widehat{\text{ch}}(\mathcal{R})$ it can be alternatively defined by:

$$\text{qdim}(x^\lambda) = q^{(\lambda,\delta)}. \quad (4.4)$$

Let $\lambda$ be a dominant integral weight, and suppose that

$$\psi_p(L_r(\lambda)) = \sum_{\mu \in P^+_r} m_{\lambda,\mu}^p L_r(\mu). \quad (4.5)$$

Only finitely many $m_{\lambda,\mu}^p \in \mathbb{C}$ are non-zero and they are in fact all integers. Then, we have:

$$J^u_\odot(L_r(\lambda)) = \sum_{\mu \in P^+} m_{\lambda,\mu}^p \cdot \text{qdim}(L_r(\mu)) \cdot \theta^{p'/p}_{L_r(\mu)}. \quad (4.6)$$

If we now alter the framing to have writhe 0, and then normalize, we have:

$$J^u_{T(p,p')} (L_r(\lambda)) = \theta^{-pp'}_{L_r(\lambda)} \sum_{\mu \in P^+} m_{\lambda,\mu}^p \cdot \text{qdim}(L_r(\mu)) \cdot \theta^{p'/p}_{L_r(\mu)};$$

$$J^u_{T(p,p')} (L_r(\lambda)) = \frac{\theta^{-pp'}_{L_r(\lambda)}}{\text{qdim}(L_r(\lambda))} \sum_{\mu \in P^+} m_{\lambda,\mu}^p \cdot \text{qdim}(L_r(\mu)) \cdot \theta^{p'/p}_{L_r(\mu)}. \quad (5.1)$$

### 5. $L_r(n\Lambda_1)$-coloured Jones polynomials

We now calculate $J^\sigma_{T(p,p')} (L_r(n\Lambda_1))$. Recall that $u$ denotes the zero framing of the unknot and $\overline{u}$ denotes the inherited framing on the $(p, p')$ cabling $T(p, p')$. For simplicity, we shall omit the $\overline{u}$.

We begin by understanding and extending various components of the formula (4.6).

#### 5.1. Extending the twist to $P$

First is the twist. If we view $\frac{1}{2} (\lambda, \lambda + 2\delta)$ as a polynomial function of $\lambda \in \mathfrak{h}^*$, it is not invariant with respect to the Weyl group. However, for $\lambda \in P^+_r$, consider $\lambda' = \lambda + \delta$. Then, $\lambda' \in P^0_r$, and we have: $\frac{1}{2} (\lambda, \lambda + 2\delta) = \frac{1}{2} (\lambda' - \delta, \lambda' + \delta) = \frac{1}{2} (\lambda', \lambda') - \frac{1}{2} (\delta, \delta)$. Now, extended to all of $\lambda' \in \mathfrak{h}^*$, this is clearly $\Theta$ invariant. We thus define:

$$\Theta_{\lambda'} = q^{\frac{1}{2}(\lambda', \lambda') - \frac{1}{2}(\delta, \delta)} \quad (5.1)$$

and we have:

$$\Theta_{w\lambda'} = \Theta_{\lambda'}, \quad (5.2)$$

$$\theta_{L_r(\lambda)} = \Theta_{\lambda + \delta} \quad (5.3)$$
for all $\lambda' \in \mathfrak{h}^*$ (we really only require $\lambda' \in P$) and all $\lambda \in P^+_r$.

5.2. Extending the plethysm multiplicities to $P_r$. Let $\lambda \in P^+_r$ and let

$$\psi_p(\text{ch}(L_r(\lambda))) = \sum_{\mu \in P^+_r} m^\mu_{\lambda,p} \text{ch}(L_r(\mu)) = \frac{1}{\Delta_r} \sum_{\mu \in P^+_r} \sum_{w \in S_r} (-1)^{\ell(w)} m^\mu_{\lambda,p} x^{w(\mu + \delta)}. \tag{5.4}$$

Given any $\mu' \in P^+_r$ (so that $\mu = \mu' - \delta \in P^+_r$), we define a new function

$$M^\mu_{\lambda,p} = m^{\mu'-\delta}_{\lambda,p}, \tag{5.5}$$

and extend this to an alternating function on the union of Weyl translates of $P^+_r$ by

$$M^{\mu''}_{\lambda,p} = (-1)^{\ell(w)} m^{\mu'-\delta}_{\lambda,p}, \tag{5.6}$$

where $\mu'' \in S_rP^+_r$, $w \in S_r$, $\mu' \in P^+_r$ such that $\mu'' = w\mu'$. We further define

$$M^{\mu''}_{\lambda,p} = 0 \tag{5.7}$$

whenever $\mu'' \in P_r \setminus S_rP^+_r$, which is the unique way to extend this function on all of $P_r$ so that it remains alternating. We thus have:

$$\psi_p(\text{ch}(L_r(\lambda))) = \frac{1}{\Delta_r} \sum_{\mu'' \in P_r} M^{\mu''}_{\lambda,p} x^{\mu''}. \tag{5.8}$$

Now we specialize to $\lambda = n\Lambda_1$. The crucial fact driving all of our calculations is that each weight of $L_r(n\Lambda_1)$ appears with multiplicity 1. On the one hand we have:

$$\psi_p(\text{ch}(L_r(n\Lambda_1))) = \sum_{\lambda \in \Pi_{r,n}} x^{\lambda}. \tag{5.9}$$

On the other hand, we have (5.8). Combining these two, the numbers $M^{\mu}_{n\Lambda_1,p}$ and in turn the numbers $m^{\mu}_{n\Lambda_1,p}$ are determined by the equation:

$$\Delta_r \cdot \psi_p(\text{ch}(L_r(n\Lambda_1))) = \sum_{\lambda \in \Pi_{r,n}} \sum_{\mu \in P_r} M^\mu_{n\Lambda_1,p} x^{\lambda} = \sum_{\mu \in P_r} M^\mu_{n\Lambda_1,p} x^{\lambda}. \tag{5.10}$$

We may also write

$$\Delta_r \sum_{\lambda \in \Pi_{r,n}} x^{\lambda} = \sum_{\mu \in P_r} (-1)^{\ell(w)} x^{\lambda + w\delta}, \tag{5.11}$$

using the explicit expression for the Weyl denominator $\Delta_r$. This finally gives us:

$$\sum_{\mu \in P_r} M^\mu_{n\Lambda_1,p} x^{\lambda} = \sum_{\lambda \in \Pi_{r,n}, w \in S_r} (-1)^{\ell(w)} x^{\lambda + w\delta}. \tag{5.12}$$

5.3. Jones polynomials. We now combine various extensions above with the Rosso–Jones formula to finally deduce our formula for the $L_r(n\Lambda_1)$ coloured invariants of the torus knots. The derivation is a straight-forward generalization from the $\mathfrak{sl}_2$ case previously studied by Morton [26].

Theorem 5.1. Let $p,p'$ be a pair of positive coprime integers. We have the following formula for the framing dependent (with framing inherited from $p,p'$ cabling of $\circ$) un-normalized invariant.

$$J_{T(p,p')}(L_r(n\Lambda_1)) = \frac{q^{-\frac{1}{2}||\delta||^2}}{q\text{dim}(\Delta_r)} \sum_{\lambda \in \Pi_{r,n}, w \in S_r} (-1)^{\ell(w)} q^{-\frac{1}{2}||\rho\lambda + w\delta||^2 + (p\lambda + w\delta,\delta)}. \tag{5.13}$$
Proof. We have:

$$J_{T(p,p')}(L_r(n\Lambda_1)) = \sum_{\mu \in \Pi_r^+} m_{n\Lambda_1,p}^\mu \cdot q \dim(L_r(\mu)) \cdot \theta_{L_r(\mu)}^{p'/p}$$

$$= \dim \left( \sum_{\mu \in \Pi_r^+} m_{n\Lambda_1,p}^{\mu'-\delta} \cdot \chi(L_r(\mu' - \delta)) \cdot \theta_{L_r(\mu' - \delta)}^{p'/p} \right)$$

$$= \dim \left( \frac{1}{\Delta_r} \sum_{\mu' \in \Pi_r^+} \sum_{w \in \mathbb{S}_r} (-1)^{\ell(w)} m_{n\Lambda_1,p}^{\mu'-\delta} \cdot x^{w\mu'} \cdot \theta_{L_r(\mu' - \delta)}^{p'/p} \right)$$

$$= \dim \left( \frac{1}{\Delta_r} \sum_{\mu \in \Pi_r^+} M_{n\Lambda_1,p}^{\mu} \cdot x^{w\mu} \cdot \Theta_{\mu}^{p'/p} \right)$$

$$(5.3)(5.6) \Rightarrow \dim \left( \frac{1}{\Delta_r} \sum_{\mu \in \Pi_r^+} M_{n\Lambda_1,p}^{\mu} \cdot x^{w\mu} \cdot \Theta_{\mu}^{p'/p} \right)$$

$$(5.7) \Rightarrow \dim \left( \frac{1}{\Delta_r} \sum_{\lambda \in \Pi_r^+, w \in \mathbb{S}_r} (-1)^{\ell(w)} x^{p\lambda + w\delta} \cdot \Theta_{p\lambda + w\delta}^{p'/p} \right)$$

$$(4.4),(5.1) \Rightarrow \frac{q^{-\frac{\ell(\mu')}{2p}}}{\dim(\Delta_r)} \sum_{\lambda \in \Pi_r^+, w \in \mathbb{S}_r} (-1)^{\ell(w)} q^{-\frac{\ell(\mu')}{2p}} (p\lambda + w\delta) \cdot \chi^{p\lambda + w\delta}(\lambda, \delta)$$

$$\sqrt{\frac{q}{x}}.$$  

6. Limits

In this section, we calculate the limits of the Jones polynomials that we have found. We expect these to equal the tails when $r \leq p$. Recall that $J_K$ without any superscripts is the framing dependent and non-normalized invariant.

6.1. The case $r \leq p$. There exists a unique element $w_0 \in \mathbb{S}_r$ such that $w_0(\Phi^+_r) = \Phi^-_r$, and in particular $w_0\delta = -\delta$. With respect to the simple transpositions $(i, i+1)$ this is the longest element of $\mathbb{S}_r$ and its length is $\ell(w_0) = |\Phi^+_r|$. 

**Theorem 6.1.** Let $p, p'$ be coprime positive integers. Let $j$ be such that $0 \leq j \leq r - 1$. Then, we have:

$$\lim_{n \to \infty} J_{T(p,p')}(L_r((j + nr)\Lambda_1)) = \frac{q; q^{r-1}}{\prod_{\alpha \in \Phi^+_r} (1 - q^{(\alpha, \delta)})} \cdot \chi^{p,p'; j\Lambda_1}$$  

(6.1)

**Proof.** From (5.12) we have:

$$J_{T(p,p')}(L_r(n\Lambda_1)) = \frac{1}{\dim(\Delta_r)} \sum_{\lambda \in \Pi_r^+, w \in \mathbb{S}_r} (-1)^{\ell(w)} q^{-\frac{\ell(\mu')}{2p}} (-p'(\lambda, w\delta) + p(\lambda, \delta) + (w\delta))$$

$$= \frac{(-1)^{\ell(w_0)}}{\dim(\Delta_r)} \sum_{\lambda \in \Pi_r^+, w \in \mathbb{S}_r} (-1)^{\ell(w_0)} q^{-\frac{\ell(\mu')}{2p}} (-p'(\lambda, w_0\delta) + p(\lambda, \delta) - (w_0\delta))$$

$$= \frac{(-1)^{\ell(w_0)}}{\dim(\Delta_r)} \sum_{\lambda \in \Pi_r^+, w \in \mathbb{S}_r} (-1)^{\ell(w)} q^{-\frac{\ell(\mu')}{2p}} (-p'(\lambda, w\delta) + p(\lambda, \delta) - (w\delta))$$
Remark 6.2. Let \( n = kr \) in (6.2) for some positive integer \( k \). In this case, \( \Pi_{r,kr} \subset Q_r \) since \( r\Lambda_1 \in Q_r \). So, \( \frac{1}{2} ||\lambda||^2 \in \mathbb{Z} \) by evenness of \( Q_r \). Additionally, \( (\lambda, \delta), (\lambda, w\delta), (\delta, \delta - w\delta) \in \mathbb{Z} \). Combined with (6.3), we now see that \( J_{T(p,p')}(|L_r(\lambda_k \Lambda_1)|) \in \mathbb{Z}[[q]] \).

For a knot \( K \), let \( \hat{J}_K^r(L_r(n \Lambda_1)) \) denote \( J_K^r(L_r(n \Lambda_1)) \) divided by its trailing monomial (where \( \bullet \) stands for either framed or unframed, normalized or un-normalized invariant). Note that framing has no effect on \( \hat{J}_r^\bullet \).

Definition 6.3. Fix \( r \geq 2 \), \( a \geq 1 \) and \( 0 \leq b \leq a - 1 \). We say that the sequence \( \hat{J}_K^r(L_r((an + b)\Lambda_1)) \) is 0-stable if there exists a power series \( f \in \mathbb{Z}[[q]] \) such that for all \( n \),

\[
\hat{J}_K^r(L_r((an + b)\Lambda_1)) - f(q) \in q^{(an + b) + 1} \mathbb{Z}[[q]].
\]

In this case, we say that \( f \) is the 0-tail of the sequence \( \hat{J}_K^r(L_r((an + b)\Lambda_1)) \).

Several comments are now in order, encapsulated in the following remark.

Remark 6.4. Fix \( r \leq p < p' \) with \( p, p' \) coprime.

1. When \( j = 0 \), the limit (6.1) involves \( \chi_{p,p'}^{r,r} \) which is modular [13], [33] up to some factor \( q' \), since it is the character of \( \mathcal{W}_r(p, p') \) which is in turn a rational [4] and \( C_2 \) cofinite VOA [3]. Moreover, using (3.2) we see that the RHS of (6.1) is \( 1 + q \mathbb{Z}[[q]] \). This means that for all but finitely many \( n \), \( J_{T(p,p')}(L_r(n \Lambda_1)) = J_{T(p,p')}(L_r(n \Lambda_1)) \in 1 + q \mathbb{Z}[[q]] \).

2. Let us continue to take \( j = 0 \). It is clear that:

\[
\lim_{n \to \infty} J_{T(p,p')}(L_r(n \Lambda_1)) - J_{T(p,p')}(L_r(n \Lambda_1)) = \frac{q^{-\frac{1}{2p'}}||\delta||^2}{\operatorname{qdim}(r)} \sum_{\lambda \in \mathbb{Q}_r, \lambda \neq 0, \lambda \subset \mathbb{Z}} (-1)^{\ell(w)} q^\frac{1}{2p'}||w\lambda||^2 - p'(\lambda, \delta) + p(\lambda, \lambda) - (w, \delta) \cdot (w, \delta).
\]

Now, since \( ||\cdot||^2 \) is a positive definite quadratic form, the minimum \( q \) degree of the right-hand side grows as a quadratic in \( n \). This means that for all large enough \( n \), \( J_{T(p,p')}(L_r(n \Lambda_1)) \) matches the limit (6.1) at least up to first \( q^{g(n)} \) terms for some quadratic function \( g \). In effect, for all large enough \( n \), we must in particular have a match for the first \( nr \) terms.
However, at the moment, we do not know how to establish that this match works for all $n$ which would prove that the RHS of (6.1) is indeed the 0-tail.

(3) For $j \neq 0$, the modular properties of $\chi_{r,p,p';j\Lambda_1}$ are as yet unclear to us. Moreover, the minimum degree of $\chi_{r,p,p';j\Lambda_1}$ seems to depend on $r,p,p',j$.

(4) In the special case $r = p$, (3.7) gives that the limit is essentially independent of $j$, up to a sign. This means that in this case we have:

$$\lim_{n \to \infty} \hat{J}_{T(p,p')}(L_r(n\Lambda_1)) = \frac{(q;q)_{\infty}^{r-1}}{\prod_{\alpha \in \Phi^+_p} (1-q^{(\alpha,\delta)})} \chi_{r,p,p'}^{\infty,0}. \tag{6.5}$$

(5) Continuing to take $r = p$, similar to point 2 above, we expect the RHS of (6.5) to be the 0-tail of the sequence $J_{T(p,p')}(L_r(n\Lambda_1))$. In this case, $n$ is not confined to multiples of $r$.

(6) In the case of $sI_2$, there is another way in which $r = p = 2$ case is somewhat special – the coloured Jones polynomials for $T(p,p')$ satisfy a second order difference equation which reduces to first order for the torus knots $T(2,2m+1)$ [22].

### 6.2. A case study for $p < r$.

This case is quite a bit harder, since the limit (6.1) in this case is 0 due to Corollary 3.4. This means that there is cancellation in the formula for the Jones polynomial and it is quite tricky to pin point their minimum degrees. For general $p < r$, this appears to be a fairly involved problem, which we hope to address in the future. For now, we analyze the case $p = 2, r = 3$ to ensure that our results match with those given in [19]. We expect several considerations of this example to generalize to higher $r$. The following theorem gives the formula for $J_{T(2,p')}(L_3(n\Lambda_1))$ which agrees up to shifting, sign, framing and normalization with the one from [19]. For more calculations regarding $sI_3$ coloured invariants of $T(2, n)$, see [18].

**Theorem 6.5.** For any odd number $p' \geq 3$ we have:

$$J_{T(2,p')}(L_3(n\Lambda_1)) = (1)_{q} q^p \frac{q^{p^2}(q^{2n+2})^{-n}}{(1-q)(1-q^2)} \sum_{i=0}^{n} (-1)^{i} q^{p^2}(q^{i^2+i})^{-i} (1-q^{-n+i+1})(1-q^{2i+1})(1-q^{n+i+2}) \tag{6.6}$$

and so,

$$\lim_{n \to \infty} \hat{J}_{T(2,p')}(L_3(n\Lambda_1)) = \frac{1}{(1-q)(1-q^2)} \sum_{i=0}^{\infty} (-1)^{i} q^{p^2}(q^{i^2+i})^{-i} (1-q^{2i+1}) = \frac{(q;q)_{\infty}^{2,2,p'}}{(1-q)(1-q^2)} \tag{6.7}$$

**Proof.** Various features and steps of this proof are depicted in Figure 1 for the case $n = 4$. In the case of $sI_3$ we have $\delta = \epsilon_1 - \epsilon_3 = \theta$ where $\theta \in \Phi_3$ is the highest root and moreover, in this case $\mathcal{S}_3\delta = \Phi_3$. From Lemma 3.3, we have $w\delta - \delta \in 2Q_3$ with $1 \neq w \in \mathcal{S}_3$ iff $w = (1,3)$ and in this case, $(1,3)\delta - \delta = -2\theta$.

Now let $n \geq 1$. Equation (5.12) gives:

$$J_{T(2,p')}(L_3(n\Lambda_1)) = \frac{q}{q^{\dim(D_3)}} \sum_{(\lambda,w) \in \Pi_{3,n} \times \mathcal{S}_3} (-1)^{\ell(w)} q^{p_2} \chi_{\lambda+w\delta+2\lambda}(||2\lambda+w\delta||^2+||2\lambda+w\delta,\delta||). \tag{6.8}$$

This sum immediately involves cancellations which we now explain. Given $\lambda \in \Pi_{3,n}$, suppose there exist $\lambda \in \Pi_{3,n}$ and $\alpha \in \Phi_3$ such that $\lambda = \lambda + \alpha$. There is now a unique $w \in \mathcal{S}_3$ such that $\alpha = w\theta$ (among $sI_3\epsilon_r$, such a uniqueness only holds for $sI_2$ and $sI_3$). Let $\tilde{w} = w(1,3)$. We now have $2\lambda+w\delta = 2\tilde{\lambda} + \tilde{w}\delta$ and $(-1)^{\ell(w)} \neq (-1)^{\ell(\tilde{w})}$. This crucially implies that the contributions corresponding to $(\lambda, w)$ and $(\lambda, \tilde{w})$ cancel in (6.8). Consequently, we may omit these terms from the summation. The only terms $(\lambda, w)$ that now contribute to the summation are such that $\lambda + w\theta \notin \Pi_{3,n}$. 


It is now easy to see that \( \lambda \) must belong to the boundary of \( \Pi_{3,n} \). There are three sides to this boundary:

\[
\lambda \in \{ n\Lambda_1 - i\alpha_1 \mid 0 \leq i \leq n \} \cup \{ n\Lambda_1 - n\alpha_1 - i\alpha_2 \mid 0 \leq i \leq n \} \cup \{ n\Lambda_1 - i\alpha_1 - i\alpha_2 \mid 0 \leq i \leq n \}.
\]

In the first case, we have \( w\theta = (1,2)\theta = \alpha_2 \) or \( w\theta = \theta = \alpha_1 + \alpha_2 \); in the second, we have \( w\theta = (3,1,2)\theta = -\alpha_1 \) or \( w\theta = (1,3)\theta = -\alpha_1 - \alpha_2 \); in the third, we have \( w\theta = (2,3)\theta = \alpha_1 \) or \( w\theta = (1,3,2)\theta = -\alpha_2 \).

Writing \( 2\lambda + w\delta = \mu \), we now see that each term in (6.8) is of the form

\[
(-1)^{\ell(w)} q^{(\mu, \frac{\mu}{\tau} \mu + \delta)},
\]

with \( \mu \) belonging to one of the following six sets, along with the appropriate sign \( (-1)^{\ell(w)} \):

\[
\begin{align*}
S_1 &= \{ 2n\Lambda_1 - 2i\alpha_1 + \alpha_1 + \alpha_2 \mid 0 \leq i \leq n \}, \quad (-1)^{\ell(w)} = 1, \\
S_2 &= \{ 2n\Lambda_1 - 2i\alpha_1 + \alpha_1 + \alpha_2 \mid 0 \leq i \leq n \}, \quad (-1)^{\ell(w)} = -1, \\
S_3 &= \{ 2n\Lambda_1 - 2n\alpha_1 - 2i\alpha_2 - \alpha_1 \mid 0 \leq i \leq n \}, \quad (-1)^{\ell(w)} = 1, \\
S_4 &= \{ 2n\Lambda_1 - 2n\alpha_1 - 2i\alpha_2 - \alpha_1 \mid 0 \leq i \leq n \}, \quad (-1)^{\ell(w)} = -1, \\
S_5 &= \{ 2n\Lambda_1 - 2i\alpha_1 - 2i\alpha_2 - \alpha_2 \mid 0 \leq i \leq n \}, \quad (-1)^{\ell(w)} = 1, \\
S_6 &= \{ 2n\Lambda_1 - 2i\alpha_1 - 2i\alpha_2 + \alpha_1 \mid 0 \leq i \leq n \}, \quad (-1)^{\ell(w)} = -1.
\end{align*}
\]

That is, we have:

\[
J_{T(2,\tau')} (L_\tau (n\Lambda_1)) = \frac{q^{-\frac{\ell(\mu)}{\tau} |\delta|^2}}{q \dim(\Delta_3)} \left( \sum_{\mu \in S_1} \sum_{\mu \in S_2} \sum_{\mu \in S_3} \sum_{\mu \in S_4} \sum_{\mu \in S_5} \sum_{\mu \in S_6} \right) q^{(\mu, \frac{\mu}{\tau} \mu + \delta)}
\]

It is now not too hard to see that this sum can be rearranged by grouping together the Weyl translates of those \( \mu \in S_1 \cup \cdots \cup S_6 \) that belong to the fundamental Weyl chamber. These are precisely the \( \mu \) in \( S_1 \) with \( 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \) or \( \mu \) in \( S_2 \) with \( 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \). In other words, we have:

\[
J_{T(2,\tau')} (L_\tau (n\Lambda_1)) = \frac{q^{-\frac{\ell(\mu)}{\tau} |\delta|^2}}{q \dim(\Delta_3)} \left( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{w \in \mathcal{O}_3} (-1)^{\ell(w)} q^{w(2n\Lambda_1 - 2i\alpha_1 + \alpha_1 + \alpha_2)} q^{\frac{\ell(\mu)}{\tau} w(2n\Lambda_1 - 2i\alpha_1 + \alpha_1 + \alpha_2)} \right)
\]

\[
\begin{align*}
&= \frac{q^{-\frac{\ell(\mu)}{\tau} |\delta|^2}}{q \dim(\Delta_3)} \sum_{i=0}^{\frac{n-1}{2}} \sum_{w \in \mathcal{O}_3} (-1)^{\ell(w)} q^{w(2n\Lambda_1 - 2i\alpha_1 + \alpha_1 + \alpha_2)} q^{\frac{\ell(\mu)}{\tau} w(2n\Lambda_1 - 2i\alpha_1 + \alpha_1 + \alpha_2)} \\
&= \frac{q^{-\frac{\ell(\mu)}{\tau} |\delta|^2}}{q \dim(\Delta_3)} \sum_{i=0}^{n} (-1)^{i+1} q^{\frac{\ell(\mu)}{\tau} (2n^2 + 2i^2 - 2i - 4n^2 + 4n)} \sum_{w \in \mathcal{O}_3} (-1)^{\ell(w)} q^{w(2n\Lambda_1 - i\alpha_1 + \alpha_1 + \alpha_2)} \\
&= \frac{q^{-\frac{\ell(\mu)}{\tau} |\delta|^2}}{q \dim(\Delta_3)} \sum_{i=0}^{n} (-1)^{i+1} q^{\frac{\ell(\mu)}{\tau} (2n^2 + 2i^2 - 2i - 4n^2 + 4n)} q^{-2n^2 + 4n} (1 - q^{i+1}) (1 - q^{2n^2 - 2i^2 + 1}) (1 - q^{2n^2 - i + 2}) \\
&= \frac{(-1)^{n+1} q^{\frac{\ell(\mu)}{\tau} (2n^2 + n) - 2n}}{q \ dim(\Delta_3)} \sum_{i=0}^{n} (-1)^i q^{\frac{\ell(\mu)}{\tau} (2n^2 + n) - 2n} (1 - q^{n-i+1}) (1 - q^{2i+1}) (1 - q^{n+2i+1}),
\end{align*}
\]

as required. Now the limit of shifted Jones polynomials follows easily. \( \square \)
Figure 1. Proof of theorem 6.5 with $n = 4$. The points $\bullet$ denote weights of $L_3(4\Lambda_1)$. The arrows $\rightarrow$ correspond to pairs $(\lambda, w)$ such that $\lambda \in \Pi_{3,4}$ but $\lambda + w \delta \notin \Pi_{3,4}$. These arrows originate at $\lambda$ and point in the direction of $w \delta$. Points $-$ and $+$ denote the actual contributions to the sum, with blue points giving a positive contribution and red giving negative. The gray region denotes fundamental Weyl chamber. Points $-$ and $+$ belonging to this region appear in the final summation (6.6).

Just like the $r = p$ case above, we see that the limit in this example is also independent of the congruence class of $n$ modulo $r$. Computer experiments suggest that this continues to happen for higher values of $r$ and with $p < r$. In fact, experiments with $r = 3, 4, 5$ suggest the following precise conjecture. Note that the case $p = r$ proved above also fits the pattern of this conjecture.

**Conjecture 6.6.** Let $2 \leq p < p'$ be a pair of coprime positive integers and let $2 \leq r$ be such that $p < r$. Letting $\delta_r$ denote the Weyl vector of $\mathfrak{sl}_r$ and $\Phi^+_r = \Phi^+_1 = \emptyset$ we have:

$$
\lim_{n \to \infty} \hat{J}_{T(p, p')} (L_r(n\Lambda_1)) = \frac{\prod_{\alpha \in \Phi^*_r - p} (1 - q^{(\alpha, \delta_r - p)})}{\prod_{\alpha \in \Phi^*_r} (1 - q^{(\alpha, \delta)})} (q; q)_\infty^{p-1} \cdot \chi_{p, p, 0}^{p, p'}.
$$

(6.9)

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