Minimizing the Lifetime Shortfall or Shortfall at Death

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Abstract: We find the optimal investment strategy for an individual who seeks to minimize one of four objectives: (1) the probability that his wealth reaches a specified ruin level before death, (2) the probability that his wealth reaches that level at death, (3) the expectation of how low his wealth drops below a specified level before death, and (4) the expectation of how low his wealth drops below a specified level at death. Young (2004) showed that under criterion (1), the optimal investment strategy is a heavily leveraged position in the risky asset for low wealth.

In this paper, we introduce the other three criteria in order to reduce the leveraging observed by Young (2004). We discovered that surprisingly the optimal investment strategy for criterion (3) is identical to the one for (1) and that the strategies for (2) and (4) are more leveraged than the one for (1) at low wealth. Because these criteria do not reduce leveraging, we completely remove it by considering problems (1) and (3) under the restriction that the individual cannot borrow to invest in the risky asset.

Keywords: Self-annuitization, optimal investment, stochastic optimal control, probability of ruin, borrowing constraints.

Mathematics Subject Classification: 90A09 (primary), 90A40 (secondary).

1. Introduction

We study an individual investment problem by using optimal stochastic control. The study of the investment problems faced by individuals is justified because a significant financial crisis is looming: it is projected that retired Americans’ living expenses will exceed their financial resources by $400 billion over the ten-year period 2020-2030 (VendDerhei and Copeland, 2003). This shortfall is due to the increased longevity of our aging population, changes in Social Security, inadequate private retirement savings, and the continuing switch from defined benefit plans to defined contribution plans, such as 401(k)s, which transfers the investment and longevity risk from the employer to the individual.

We consider the problem of how an individual should invest her wealth in a risky financial market in order to minimize her (1) probability of lifetime ruin, that is, the probability of running out of money before dying, (2) probability of ruin at death, that is, the probability of running out of money at the time of death (this objective can be used by people with bequest motives), (3) expected lifetime shortfall, and (4) expected shortfall at death. The latter two criteria are the counterparts of (1) and (2) in which the individual is penalized by the amount of loss. In (1) and (2) the penalty she gets is constant regardless
how low the wealth becomes with respect to the ruin threshold, the value of wealth at which the individual considers herself ruined.

As employers shift from defined benefit plans to defined contribution plans, the problem of outliving one’s wealth becomes more and more relevant to retirees because the income coming from guaranteed sources is projected to drop significantly; see, for example, Parikh (2003). We determine the optimal investment strategy of an individual who targets a given rate of consumption under each criterion and show how the strategies compare to each other. We assume that the rate of consumption (either nominal or real) is net of any income the retiree receives from pension plans, such as Social Security or a defined benefit plan. In finding the optimal strategy, we take into account that the time of death is random. This assumption differs from the one usually assumed by financial planners and common retirement planning software in that they generally assume a specific age of death.

We focus on minimizing the expected maximum lifetime shortfall and the shortfall at death. One might argue that the former penalty criterion is too severe because if an individual were to have a negative wealth at some point in life and later get out of debt, then the person should be as “happy” as someone who was never in debt but possesses the same current wealth. This argues that the latter objective function is more appropriate than the former; that is, all that matters to the individual is whether she is in debt at the time of death. On the other hand, if someone’s wealth were to become negative, then that could affect that person’s ability to borrow money in the future. Therefore, under this view, the first objective of minimizing the expected maximum shortfall during life is more appropriate.

The most common optimization criterion used in the mathematical finance literature is to maximize one’s expected discounted utility of consumption and bequest; see, for example, Merton (1992) and Karatzas and Shreve (1998, Chapter 3). Also, see Zari-phopoulou (1999, 2001) for helpful summaries of the work to date in this area. The goal of maximizing expected discounted utility of consumption and bequest may be difficult to implement because it depends on a subjective utility function for consumption and bequest. Minimizing the criteria we suggest might prove easier for individuals to understand because these criteria are arguably more objective. However, one should note there is a correspondence between the utility maximization problem and the ruin minimization problems when the utility function in question is HARA. But HARA is the only utility function when one finds a correspondence (see Bayraktar and Young (2007b) for details).

Recently, a variety of papers in the risk and portfolio management literature revitalized the Roy (1952) “Safety-First rule” and applied the concept in the context of maximizing the
probability of achieving certain investment goals before ruin. For example, Browne (1995, 1997, 1999a,b,c) derived the optimal strategy for a portfolio manager who is interested in maximizing the probability of reaching a safe level before ruin or minimizing the time expected time it takes to reach a goal, the probability of beating a stochastic benchmark, and the probability of reaching a goal by a deadline. Also, researchers have begun to study the problem of optimal investment to minimize the probability that an individual runs out of money before dying. See, for example, Milevsky, Ho, and Robinson (1997), Milevsky and Robinson (2000), and Young (2004). In insurance mathematics, the criteria of minimizing the probability of ruin was used to find the optimal reinsurance; see, for example, Schmidli 2001, Taksar and Markussen (2003), and Promislow and Young (2005).

Young (2004) studied the problem of finding the investment strategy to minimize the probability of lifetime ruin. She discovered that the individual leveraged her wealth when wealth approaches zero; that is, when wealth was low, the individual borrowed money to invest in the risky asset. In order to reduce or eliminate this leveraging, Bayraktar and Young (2007a) imposed borrowing constraints on the investment strategy. They first considered the case for which the individual is not allowed to borrow money in order to invest in the risky asset, and this certainly eliminated the leveraging that Young (2004) observed. They next considered the case for which the individual was allowed to borrow money but at a rate higher than the one earned by the riskless asset. They discovered that the leveraging in this case could be even worse than that in the case when the individual is allowed to borrow at the rate earned by the riskless asset.

In this paper, we show that the leveraging effect observed in the probability of lifetime ruin problem is not reduced by considering alternative penalty functions. We obtain two rather surprising results from our models. (1) We learn that the leveraging is exacerbated by considering the probability of ruin at death and the shortfall at death. (2) The optimal investment strategy for the lifetime shortfall is identical to the optimal investment strategy for the probability of lifetime ruin. This result is surprising because we believed that by penalizing the individual for the magnitude of her bankruptcy – not for just whether or not bankruptcy occurred – we would be able to temper her leveraging.

Our contributions to the applied probability literature are as follows: (1) By applying the Fenchel-Legendre transform to our value functions, we are able to linearize the corresponding non-linear HJB equations. The resulting linear differential equation is one with a free boundary, and we solve this free-boundary problem for the case of minimizing the probability of ruin at death. The corresponding minimum ruin probability is not explicitly available, but via we characterize it as the inverse Fenchel-Legendre transform of the solution of the free-boundary problem. (2) In the problem of minimizing the expectation of any
function of lifetime minimum wealth, we also rely on the Fenchel-Legendre transform to show that corresponding optimal investment strategy is identical to the one for minimizing the probability of lifetime ruin. Therefore, the optimally-controlled wealth processes for both problems are identical. From this observation, we develop a representation of lifetime shortfall in terms of the probability of lifetime ruin; see equation (3.16) below. (3) We provide verification theorems that show that the value function is the unique solution of an associated Hamilton-Jacobi-Bellman (HJB) equation. Our verification theorems are for a diffusion with killing, as well as include an additional state variable for the minimum wealth process.

The rest of the paper is organized as follows: In Section 2.1, we introduce the financial market and summarize the results of Young (2004) concerning the optimal investment strategy to minimize the probability of lifetime ruin. In Section 2.2, we solve the problem of minimizing probability of ruin at death. We discover that the optimal investment strategy for this criterion is always greater than the one for the probability of lifetime ruin from Section 2.1. In Section 2.3, we remove leveraging entirely by prohibiting borrowing.

In Section 3.1, we consider the problem of minimizing lifetime expected shortfall. We provide a verification theorem for the problem of minimizing a bounded, twice continuously differentiable function of minimum wealth. We construct a solution to this auxiliary problem. The value function of the lifetime shortfall is shown to be the limit of a sequence of appropriately-defined auxiliary problems. The optimal investment strategy for the lifetime shortfall is identical to the one for the probability of ruin. In Section 3.2, we consider the problem of minimizing shortfall at death and show that the optimal investment strategy is greater than the one for the probability of lifetime ruin. In Section 3.3, we eliminate leveraging in the lifetime shortfall problem by prohibiting borrowing. Section 4 concludes the paper.

2. Probability of Lifetime Ruin and Ruin at Death

In Section 2.1, we present the financial market and briefly review the problem of minimizing the probability of lifetime ruin. We provide an explicit expression for the optimal investment in the risky asset and point out the leveraging at low wealth. In Section 2.2, we consider the problem of minimizing the probability that wealth at death lies below a certain level. We show that the optimal investment in the risky asset under the latter problem is greater than under the former; therefore, the leveraging has increased, not decreased. Therefore, because the change in value function did not reduce the leveraging, in Section 2.3, we remove it completely by prohibiting borrowing to invest in the risky asset.
2.1. Financial Market and Probability of Lifetime Ruin

In this section, we first present the financial ingredients that make up the individual’s wealth throughout this paper, namely, consumption, a riskless asset, and a risky asset. We then determine the minimum probability of lifetime ruin. The individual consumes at a constant continuous rate $c$. We assume that the individual invests in a riskless asset whose price at time $t$, $X_t$, follows the process $dX_t = rX_t dt, X_0 = x > 0$, for some fixed rate of interest $r > 0$. Also, the individual invests in a risky asset whose price at time $t$, $S_t$, follows geometric Brownian motion given by

\[
\begin{cases}
    dS_t = \mu S_t dt + \sigma S_t dB_t, \\
    S_0 = S > 0,
\end{cases}
\]

in which $\mu > r$, $\sigma > 0$, and $B$ is a standard Brownian motion with respect to a filtration of the probability space $(\Omega, \mathcal{F}, P)$. Let $W_t$ be the wealth at time $t$ of the individual, and let $\pi_t$ be the amount that the decision maker invests in the risky asset at that time. It follows that the amount invested in the riskless asset is $W_t - \pi_t$. Thus, wealth follows the process

\[
\begin{cases}
    dW_t = [rW_t + (\mu - r)\pi_t - c(W_t)] dt + \sigma \pi_t dB_t, \\
    W_0 = w.
\end{cases}
\]

A process associated with this wealth process is the minimum wealth process. Let $M_t$ denote the minimum wealth of the individual during $[0, t]$; that is,

\[
M_t = \min \left[ \inf_{0 \leq s \leq t} W_s, M_0 \right],
\]

in which we include $M_0$ (possibly different from $W_0$) to allow for the individual to have a financial past.

By “outliving her wealth,” or equivalently “lifetime ruin,” we mean that the individual’s wealth reaches some specific value $x < c/r$ before she dies. Note that $c/r$ is the “safe” level above which the individual cannot ruin if she places at least $c/r$ in the riskless asset. Let $\tau_x$ denote the first time that wealth equals $x < c/r$, and let $\tau_d$ denote the random time of death of our individual. We assume that $\tau_d$ is exponentially distributed with parameter $\lambda$ (that is, with expected time of death equal to $1/\lambda$); this parameter is also known as the hazard rate of the individual.

Denote the minimum probability that the individual outlives her wealth by $\psi(w, m; x)$, in which the argument $w$ indicates that one conditions on the individual possessing wealth $w$ and minimum wealth $m$ at the current time, and the parameter $x$ reminds us of the
hitting level. Thus, \( \psi \) is the minimum probability that \( \tau_x < \tau_d \), in which one minimizes with respect to admissible investment strategies \( \pi \). A strategy \( \pi \) is *admissible* if it is \( \mathcal{F}_t \)-progressively measurable (in which \( \mathcal{F}_t \) is the augmentation of \( \sigma(W_s : 0 \leq s \leq t) \)) and if it satisfies the integrability condition \( \int_0^t \pi_s^2 \, ds < \infty \) almost surely for all \( t \geq 0 \).

Note that the event that \( \tau_d < \tau_x \) is the same event that \( M_{\tau_d} \leq x \) with probability 1; thus, we can express \( \psi \) as

\[
\psi(w, m; x) = \inf_{\pi} \mathbf{P}[M_{\tau_d} \leq x | W_0 = w, M_0 = m]. \tag{2.4}
\]

Young (2004) showed that

\[
\psi(w, m; x) = \begin{cases} 
1, & \text{if } m \leq x; \\
\left(\frac{c-r w}{c-r x}\right)^p, & \text{if } x < m \leq w < c/r; \\
0, & \text{if } x < m \text{ and } w \geq c/r; 
\end{cases} \tag{2.5}
\]

with

\[
p = \frac{1}{2r} \left[ (r + \lambda + \delta) + \sqrt{(r + \lambda + \delta)^2 - 4r \lambda} \right] > 1, \tag{2.6}
\]

and

\[
\delta = \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2. \tag{2.7}
\]

Young (2004) also showed that the corresponding optimal investment in the risky asset \( \pi^\psi \) for \( x < m \leq w < c/r \) is given by

\[
\pi^\psi(w) = \frac{\mu - r}{\sigma^2} \frac{1}{p - 1} \left( \frac{c}{r} - w \right), \tag{2.8}
\]
a positive, decreasing, linear function of wealth, independent of both \( m \) and \( x \). Note that as wealth increases towards \( c/r \), the amount invested in the risky asset decreases to zero. This makes sense because as the individual becomes wealthier, she does not need to take on as much risk to achieve her fixed consumption rate of \( c \). On the other hand, for wealth small, the optimal amount invested in the risky asset is greater than wealth; that is, the individual borrows money to invest in the risky asset in order to avoid the greater risk of lifetime ruin.

Browne (1997) considered a problem similar, but not equivalent, to minimizing the probability of ruin. He sought to maximize the probability of hitting the safe level, \( c/r \), before reaching zero wealth; Pestien and Sudderth (1985) solved a related problem for a general diffusion and showed that the optimal investment strategy is determined by maximizing the ratio of the drift to the variance. (As an aside, their technique is not applicable if \( \lambda \neq 0 \).) However, Browne’s problem does not have an optimal investment
strategy because the safe level is not attainable. Our problem of minimizing the probability of lifetime ruin, as given in (2.4), is not equivalent to maximizing the probability of reaching a certain level. An optimal strategy exists for (2.4), as given by (2.8), in contrast to Browne’s case.

Browne (1997) obtained an \( \epsilon \)-optimal policy for his problem that exhibits behavior similar to that given in (2.8), that is, with leveraging at low wealth. We believe that most people with small wealth will not borrow to invest in a risky asset to avoid ruin and that no credible financial advisor will give such advice. Therefore, in the next section, we modify the objective function to try to reduce this leveraging at low wealth.

2.2. Probability of Ruin at Death

In this section, we modify the objective function in hopes that the leveraging effect observed in (2.8) will be reduced. One might argue that the penalty that lifetime minimum wealth reaches a certain level is too severe and that the individual is happy enough as long as her wealth at death lies above that level, regardless of the path that wealth follows between now and then. However, we cannot simply minimize \( P[W_{\tau_d} \leq x|W_0 = w] \) for all \( w < c/r \) because we expect the minimum probability to be convex, as is \( \psi \) in (2.5) on the interval \((x, c/r)\). Recall that there exists no convex function on \((-\infty, c/r)\) such that the function is bounded and decreasing.

Therefore, consider the following related problem:

\[
\phi(w, m; x, M) = \inf_{\pi} P \left\{ W_{\tau_d} \leq x \cup \left\{ M_{\tau_d} \leq M \right\}\mid W_0 = w, M_0 = m \right\},
\]

in which \( M < x < c/r \). The domain of \( \phi \) with respect to \( w \) is effectively \([M, c/r)\) over which we expect \( \phi \) to be convex. By setting \( M \) to be a large negative number, \( \phi \) approximates what one might mean by the minimum probability of ruin at death.

For \( m \leq M \), \( \phi \) is identically 1, and for \( m > M \) and \( w \geq c/r \), \( \phi \) is identically 0. For \( M < m \leq w < c/r \), \( \phi \) solves the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
\begin{align*}
\lambda(\phi(w) - 1_{(w \leq x)}) &= (rw - c)\phi'(w) + \min_{\pi} \left[ (\mu - r)\pi \phi'(w) + \frac{1}{2} \sigma^2 \pi^2 \phi''(w) \right], \\
\phi(M) &= 1, \quad \phi(c/r) = 0,
\end{align*}
\]

(2.10)
in which we drop the notational dependence of \( \phi \) on \( m \) because \( \phi \) is independent of \( m \) if \( m > M \).

The solution of (2.10) will be \( C^1(M, c/r) \cap C^2((M, c/r) - \{x\}) \); therefore, because the solution is not smooth at \( w = x \), a corresponding verification theorem relies on the
approximation technique used in Øksendal (1998, proof of Theorem 10.4.1). Via a verification theorem similar to the ones in Bayraktar and Young (2007a) modified by this approximation technique, one can show that the solution of the boundary-value problem in (2.10) is the minimum probability in (2.9). We proceed to solve (2.10).

**Theorem 2.1.** For $m > M$ and $x \leq w < c/r$, the function $\phi$ is given by

$$\phi(w) = \beta \left(1 - \frac{r}{c}w\right)^p,$$

with $p$ as in (2.6) and $\beta > 0$. Thus, on $[x, c/r)$, $\phi$ is a multiple of the probability of lifetime ruin $\psi$. It follows that on $[x, c/r)$, the optimal investment strategy $\pi\phi$ equals $\pi\psi$ as given in (2.8).

**Proof.** From (2.10), for $x \leq w < c/r$, $\phi$ solves

$$\lambda \phi(w) = (rw - c)\phi'(w) + \min_{\pi} \left[(\mu - r)\pi\phi'(w) + \frac{1}{2}\sigma^2 \pi^2 \phi''(w)\right],$$

with boundary condition $\phi(c/r) = 0$. We also have the boundary condition $\phi'(c/r) = 0$, which we demonstrate now. Consider the minimum probability of lifetime ruin $\psi(w, m; x)$ given in (2.5). Certainly, we have $0 \leq \phi \leq \psi$ because the probability of ruinning at level $M$ before dying or at level $x > M$ at death is no greater than the probability of ruinning at level $x$ before dying. Note that $\psi'(c/r) = 0$ from (2.5), in which we differentiate with respect to $w$; therefore, $\phi'(c/r) = 0$ because $\phi$ is wedged between 0 and $\psi$ as wealth approaches $c/r$.

We hypothesize that $\phi$ is convex on $[x, c/r)$, and we consider its Fenchel-Legendre transform $\tilde{\phi}$ defined by

$$\tilde{\phi}(y) = \min_{w} [\phi(w) + wy].$$

Note that we can recover $\phi$ from $\tilde{\phi}$ by

$$\phi(w) = \max_{y} [\tilde{\phi}(y) - wy].$$

The minimizing value of $w$ in (2.13) equals $I(-y) = \tilde{\phi}'(y)$, in which $I$ is the inverse function of $\phi'$. Therefore, the maximizing value of $y$ in (2.14) equals $-\phi'(w)$.

Substitute $w = I(-y)$ in equation (2.12) to obtain

$$\lambda \tilde{\phi}(y) + (r - \lambda)y\tilde{\phi}'(y) - \delta y^2 \tilde{\phi}''(y) = cy,$$

in which $\delta$ is given in (2.7). The general solution of (2.15) is
\[
\tilde{\phi}(y) = A_1 y^{B_1} + A_2 y^{B_2} + \frac{c}{r} y, \quad (2.16)
\]
in which \(A_1\) and \(A_2\) are constants to be determined, and \(B_1\) and \(B_2\) are the positive and negative roots, respectively, of
\[
-\lambda - (r - \lambda + \delta)B + \delta B^2 = 0. \quad (2.17)
\]
Thus,
\[
B_1 = \frac{1}{2\delta} \left[ (r - \lambda + \delta) - \sqrt{(r - \lambda + \delta)^2 + 4\lambda\delta} \right] < 0. \quad (2.18)
\]
and
\[
B_2 = \frac{1}{2\delta} \left[ (r - \lambda + \delta) + \sqrt{(r - \lambda + \delta)^2 + 4\lambda\delta} \right] > 1. \quad (2.19)
\]
Note that \(B_1 = p/(p - 1)\).

Define \(y_c = -\phi'(c/r) = 0\); that is, \(\tilde{\phi}'(0) = c/r\). From the definition of \(\tilde{h}\) in (2.13) and from \(\phi(c/r) = 0\), at \(y = y_c = 0\), we have
\[
\tilde{\phi}(0) = 0. \quad (2.20)
\]
It follows that \(A_2 = 0\). We can, then, recover \(\phi\) from (2.16) and (2.14) and obtain the expression for \(\phi\) in (2.11).

Because \(\phi\) is convex on \([x, c/r]\), the optimal policy \(\pi^\phi\) is given by the first-order necessary condition in (2.12). Thus, \(\pi^\phi(w) = \pi^\psi(w)\) for \(x \leq w < c/r\).

Next, we consider the solution of (2.10) for \(M < m \leq w < x\). On this domain, \(\phi\) solves the system
\[
\begin{align*}
\lambda(\phi(w) - 1) &= (rw - c)\phi'(w) + \min_{\pi} \left[ (\mu - r)\pi \phi'(w) + \frac{1}{2}\sigma^2\pi^2 \phi''(w) \right], \\
\phi(M) &= 1, \quad \phi(x)/\phi'(x) = -(c/r - x)/p,
\end{align*}
\quad (2.21)
\]
in which we assume that \(\phi\) and \(\phi'\) are continuous at \(w = x\). Again, we consider the Fenchel-Legendre transform of \(\phi\) and show how to solve for \(\tilde{\phi}\) explicitly. From (2.14), we can then determine \(\phi\).

By substituting \(w = I(-y)\) in (2.21), we obtain an equation similar to (2.15) whose general solution equals
\[ \tilde{\phi}(y) = D_1 y^{B_1} + D_2 y^{B_2} + \frac{c}{r} y + 1, \]  
(2.22)

in which \( B_1 \) and \( B_2 \) are given in (2.18) and (2.19), respectively, and \( D_1 \) and \( D_2 \) are constants to be determined. To determine \( D_1 \) and \( D_2 \), we rely on the boundary conditions of \( \phi \) at \( w = M \) and \( w = x \).

If we define \( y_x = -\phi'(x) = \beta p(r/c)(1 - rx/c)^{p-1} \), then we have

\[ \tilde{\phi}(y_x) = \frac{1}{p} y_x \left[ \frac{c}{r} + (p - 1)x \right], \]  
(2.23)

and

\[ \tilde{\phi}'(y_x) = x. \]  
(2.24)

Similarly, if we define \( y_M = -\phi'(M) \), then we have

\[ \tilde{\phi}(y_M) = 1 + My_M, \]  
(2.25)

and

\[ \tilde{\phi}'(y_M) = M. \]  
(2.26)

If we write these four equations in terms of the expression for \( \tilde{\phi} \) in (2.22), we obtain, respectively,

\[ D_1 y_x^{B_1} + D_2 y_x^{B_2} + \frac{c}{r} y_x + 1 = \frac{1}{p} y_x \left[ \frac{c}{r} + (p - 1)x \right], \]  
(2.27)

\[ D_1 B_1 y_x^{B_1 - 1} + D_2 B_2 y_x^{B_2 - 1} + \frac{c}{r} = x, \]  
(2.28)

\[ D_1 y_M^{B_1} + D_2 y_M^{B_2} + \frac{c}{r} y_M + 1 = 1 + My_M, \]  
(2.29)

and

\[ D_1 B_1 y_M^{B_1 - 1} + D_2 B_2 y_M^{B_2 - 1} + \frac{c}{r} = M. \]  
(2.30)

Solve equations (2.29) and (2.30) for \( D_1 \) and \( D_2 \) to obtain

\[ D_1 = \frac{1 - B_2}{B_1 - B_2} \left( M - \frac{c}{r} \right) y_M^{1-B_1} < 0, \]  
(2.31)
and

\[ D_2 = \frac{B_1 - 1}{B_1 - B_2} \left( M - \frac{c}{r} \right) y_M^{1 - B_2} < 0. \] (2.32)

Substitute these expressions for \( D_1 \) and \( D_2 \) into (2.28) to get an equation for \( y_x/y_M \):

\[
\frac{B_1(1-B_2)}{B_1-B_2} \left( \frac{c}{r} - M \right) \left( \frac{y_x}{y_M} \right)^{B_1-1} + \frac{B_2(B_1-1)}{B_1-B_2} \left( \frac{c}{r} - M \right) \left( \frac{y_x}{y_M} \right)^{B_2-1} = \frac{c}{r} - x. \] (2.33)

Equation (2.33) has a unique solution \( y_x/y_M \in (0,1) \) because (i) the left-hand side equals \( c/r - M > c/r - x \) when \( y_x/y_M = 1 \); (ii) as \( y_x/y_M \) approaches 0, the left-hand side goes to \(-\infty\); and (iii) the left-hand side is increasing with respect to \( y_x/y_M \).

Once we have the solution to (2.33), then we can solve for \( y_x \) as follows. First, substitute the expressions for \( D_1 \) and \( D_2 \) into equation (2.27) to obtain

\[
\frac{1}{y_x} = -\frac{p-1}{p} \left( \frac{c}{r} - x \right) + \frac{1-B_2}{B_1-B_2} \left( \frac{c}{r} - M \right) \left( \frac{y_x}{y_M} \right)^{B_1-1} + \frac{B_1-1}{B_1-B_2} \left( \frac{c}{r} - M \right) \left( \frac{y_x}{y_M} \right)^{B_2-1}. \] (2.34)

Substitute for \( y_x/y_M \) in the right-hand side of (2.34) then solve for \( y_x \). One technical point is that the right-hand side is required to be positive; this is true, but we omit the proof.

Now, \( y_M = y_x/(y_x/y_M) \), and finally we get \( D_1 \) and \( D_2 \) from (2.31) and (2.32), respectively. Also, note that \( \beta \) in (2.11) is given by

\[
\beta = \frac{c}{r p} \left( 1 - \frac{r}{c} x \right)^{1-p} y_x. \] (2.35)

We summarize these results in the following theorem:

**Theorem 2.2.** For \( M < m \leq w < x \), the function \( \phi \) is given by the inverse Fenchel-Legendre transform (2.14) of \( \tilde{\phi} \) in (2.22), in which \( D_1 \) and \( D_2 \) are given by (2.31) and (2.32), respectively. \( y_x/y_M \) is the unique solution of (2.33) in \((0,1)\), and \( y_x > 0 \) is given by (2.34). \( \square \)

Next, for \( M < m \leq w < x \), we compare the optimal investment strategy \( \pi^\phi \) with \( \pi^\psi \), as given in (2.8). Assume that the ruin level for the \( \psi \)-problem is less than or equal to \( M \). Recall that for \( m > M \) and \( x \leq w < c/r \), the two strategies are identical. For \( M < m \leq w < x \), \( \pi^\phi(w) > \pi^\psi(w) \) if and only if
\[-\frac{\phi'(w)}{\phi''(w)} > \frac{1}{p-1} \left( \frac{c}{r} - w \right). \] (2.36)

By substituting \( w = I(-y) \), in which \( I \) is the inverse of \( \phi' \), we obtain that \( \pi^\phi > \pi^\psi \) if and only if

\[-y\tilde{\phi}''(y) > \frac{1}{p-1} \left( \frac{c}{r} - \tilde{\phi}'(y) \right), \quad y_x < y < y_M. \] (2.37)

By substituting into (2.37) for \( \tilde{\phi} \) as given in (2.22) and by simplifying, we can show that inequality (2.37) is equivalent to \( B_2 < B_1 \), which is certainly true because \( B_2 < 0 \) and \( B_1 > 1 \). Thus, we have proved the following theorem:

**Theorem 2.3.** For \( M < m \leq w < x \), the optimal investment strategy \( \pi^\phi(w) > \frac{\mu - r}{\sigma^2} \frac{1}{p-1} \left( \frac{c}{r} - w \right) \).

Therefore, not only have we not reduced the leveraging by considering the probability of ruin at death, we have made it strictly worse for \( M < m \leq w < x \). The following example shows the extent of this worsening for some reasonable model parameters.

**Example 2.4.** Assume the following parameter values for a model in which consumption is expressed in real terms, that is, after inflation:

- \( \lambda = 0.04 \); the hazard rate is constant such that the expected future lifetime is 25 years.
- \( r = 0.02 \); the riskless rate of return is 2\% over inflation.
- \( \mu = 0.06 \); the risky asset’s drift is 6\% over inflation.
- \( \sigma = 0.20 \); the risky asset’s volatility is 20\%.
- \( c = 1 \); the individual consumes one unit of real wealth per year.
- \( x = 0 \); the ruin level at death is 0.
- \( M = -200 \); the lifetime ruin level is -200.

See Figure 1 for a graph of the optimal investment strategy \( \pi^\phi \) compared with the optimal investment strategy for the problem of minimizing lifetime ruin, as in Section 2.1, namely \( \frac{\mu - r}{\sigma^2} \frac{1}{p-1} \left( \frac{c}{r} - w \right) \). Note the discontinuity of \( \pi^\phi(w) \) when \( w = x = 0 \), as expected from the discontinuity of (2.10). More importantly, note that \( \pi^\phi(w) > \frac{\mu - r}{\sigma^2} \frac{1}{p-1} \left( \frac{c}{r} - w \right) \) for \( w < 0 \), as shown in Theorem 2.3.

**Figure 1 about here**

In the next section, we remove the leveraging entirely by prohibiting borrowing of the riskless asset, as in Bayraktar and Young (2007a).

2.3. Probability of Lifetime Ruin with No Borrowing
Bayraktar and Young (2007a) consider the problem of minimizing the probability of lifetime ruin under the constraint that the individual cannot borrow; however, they consider only the case for which the ruin level \( x = 0 \). In Section 3.3, we require the corresponding minimum probability of ruin for an arbitrary level of ruin \( x < c/r \), so in this section, we consider that problem.

Let \( \psi_{nb}(w, m; x) \) denote the minimum probability that the individual’s wealth reaches \( x < c/r \) before she dies under the constraint that she cannot borrow money to invest in the risky asset. We will consider two cases when wealth reaches zero: (1) Welfare provides income at the rate of \( c \) to provide for the consumption needs of the individual. In that case, zero is an absorbing state for the wealth process, so if \( W_t = 0 \), then \( W_{t+s} = 0 \) for \( s \geq 0 \). (2) The individual is allowed to borrow but only to cover her consumption, that is, she cannot borrow to invest in the risky asset. In this case if \( W_t = 0 \), then \( W_{t+s} = -c(e^{rs} - 1)/r \) for \( s \geq 0 \).

When \( x = 0 \), Bayraktar and Young (2007a) show that \( \psi_{nb} \) is given by

\[
\psi_{nb}(w, m; 0) = \begin{cases} 
1, & \text{if } m \leq 0; \\
\h_0(w), & \text{if } 0 < m \leq w; \\
\beta_0 \left( 1 - \frac{r}{c} w \right)^p, & \text{if } 0 < m \text{ and } w < c/r; \\
0, & \text{if } 0 < m \text{ and } w \geq c/r; 
\end{cases}
\]  

(2.38)

in which \( w_l = \frac{\xi c}{1 + \xi r} \), where \( \xi = \frac{\mu - r}{\sigma^2} \frac{1}{p-1} \), and \( h_0 \) solves

\[
\begin{cases} 
\lambda h_0 = (\mu w - c) h_0' + \frac{1}{2} \sigma^2 w^2 h_0'', \\
h_0(0) = 1, \quad \frac{h_0(w_l)}{h_0'(w_l)} = -\frac{1}{p} \left( \frac{c}{r} - w_l \right).
\end{cases}
\]  

(2.39)

Once we have \( h_0 \), then we can compute \( \beta_0 = h_0(w_l)(1 - rw_l/c)^{-p} \). The corresponding optimal investment strategy for \( 0 < m \leq w < c/r \) is given by

\[
\pi_{nb}(w) = \begin{cases} 
w, & \text{if } 0 < w \leq w_l; \\
\frac{\mu - r}{\sigma^2} \frac{1}{p-1} \left( \frac{c}{r} - w \right), & \text{if } w_l < w < c/r.
\end{cases}
\]  

(2.40)

The value \( w_l \) is called lending level because when wealth is greater than \( w_l \), the individual “lends” money to the bank by buying the riskless asset.

We now extend Bayraktar and Young (2007a) to arbitrary \( x < c/r \) for two reasons: (1) The optimal investment strategy is independent of the ruin level \( x \), an interesting result in itself. (2) In Section 3.3, when we consider minimizing the expected lifetime shortfall under a no borrowing constraint, then we will use the \( \psi_{nb} \) to represent the value function. For the sake of brevity, we simply state the minimum probability of ruin \( \psi_{nb} \) and the
optimal investment strategy $\pi^{nb}$ because the proof of these results are similar to those of Bayraktar and Young (2007a).

**Case 1.** $w_l \leq x < c/r$. In this case, the constraint will not bind, and we have

$$
\psi^{nb}(w, m; x) = \begin{cases} 
1, & \text{if } m \leq x; \\
\left(\frac{c-rw}{c-rx}\right)^p, & \text{if } x < m \leq w < c/r; \\
0, & \text{if } x < m \text{ and } w \geq c/r;
\end{cases} \tag{2.41}
$$

and the corresponding optimal investment strategy for $x < m \leq w < c/r$ is given by

$$
\pi^{nb}(w) = \frac{\mu - r}{\sigma^2} \frac{1}{p-1} \left(\frac{c}{r} - w\right). \tag{2.42}
$$

Note that the investment strategy in (2.42) agrees with the one given in (2.40) on their common domain.

**Case 2.** $0 \leq x < w_l$. In this case, one can parallel the argument in Bayraktar and Young to show that the constraint does not bind for $w \in (w_l, c/r)$ and does bind for $w \in (x, w_l)$. Thus, we have

$$
\psi^{nb}(w, m; x) = \begin{cases} 
1, & \text{if } m \leq x; \\
h_x(w), & \text{if } x < m \leq w \leq w_l; \\
\beta_x \left(\frac{c-rw}{c-rx}\right)^p, & \text{if } x < m \text{ and } w_l < w < c/r; \\
0, & \text{if } x < m \text{ and } w \geq c/r;
\end{cases} \tag{2.43}
$$

in which $h_x$ solves (2.39) with the boundary condition $h_0(0) = 1$ replaced by $h_x(x) = 1$. Once we have $h_x$, then we can compute $\beta_x = h_x(w_l)(c - rw_l)/(c - rx)^{-p}$. The corresponding optimal investment strategy for $x < m \leq w < c/r$ is given by

$$
\pi^{nb}(w) = \begin{cases} 
w, & \text{if } x < w \leq w_l; \\
\frac{\mu - r}{\sigma^2} \frac{1}{p-1} \left(\frac{c}{r} - w\right), & \text{if } w_l < w < c/r.
\end{cases} \tag{2.44}
$$

As in Case 1, the investment strategy in (2.44) agrees with the one given in (2.40) on their common domain.

**Case 3.** $x < 0$. In this case, we make one of two assumptions as described in the paragraph preceding (2.38). Under the assumption that welfare will cover consumption if wealth reaches zero, we have that zero is an absorbing state. Also, if the process starts with $w < 0$, the process will remain there because of welfare. The corresponding trivial observation is that $\psi^{nb}(w, m; x) = 1$ if $m \leq x$, and $\psi^{nb}(w, m; x) = 0$ if $m > x$. There is no unique optimal investment strategy in this case.
Under the assumption that the individual is allowed to borrow to cover consumption, we have that \( \pi^{nb}(w) = 0 \) for \( w < 0 \), which immediately leads to the conclusion that

\[
\psi^{nb}(w, m; x) = \begin{cases} 
1, & \text{if } m \leq x; \\
\left( \frac{c - rw}{c - rx} \right)^{\lambda/r}, & \text{if } x < m \leq w \leq 0; \\
h_x(w), & \text{if } x < m \text{ and } 0 < w \leq w_l; \\
\beta_x \left( \frac{c - rw}{c - rx} \right)^{p}, & \text{if } x < m \text{ and } w_l < w < c/r; \\
0, & \text{if } x < m \text{ and } w \geq c/r; 
\end{cases}
\] (2.45)

in which \( h_x \) solves (2.39) with the boundary condition \( h_0(0) = 1 \) replaced by \( h_x(0) = (c/(c - rx))^{\lambda/r} \). The corresponding optimal investment strategy for \( x < m \leq w < c/r \) is given by

\[
\pi^{nb}(w) = \begin{cases} 
0, & \text{if } x < w \leq 0; \\
w, & \text{if } 0 < w \leq w_l; \\
\frac{\mu - r}{\sigma} \frac{1}{p - 1} \left( \frac{c}{r} - w \right), & \text{if } w_l < w < c/r. 
\end{cases}
\] (2.46)

In the next section, we parallel the work from this section with an objective function that penalizes for the amount that an individual’s wealth falls below a given level, not just whether or not wealth falls below this level.

3. Expected Lifetime Shortfall and Shortfall at Death

Initially, we anticipated that if the individual is penalized by the amount of loss, then she will take less chance with investing in the risky asset than if she were penalized simply for her wealth being low regardless of how low. However, we were surprised to learn that the investment strategy is the same when minimizing expected lifetime shortfall as when minimizing the probability of lifetime ruin, and we show in Section 3.1.

In Section 3.2, we minimize the expected shortfall at death and show that the optimal investment strategy is larger than if minimizing expected lifetime shortfall, in parallel to what we learned in Section 2.2. In other words, modifying the objective function to penalize the individual for the amount of loss does not ameliorate the leveraging effect, and considering wealth at death only makes it worse. Therefore, to reduce leveraging, in Section 3.3, we eliminate it completely by prohibiting borrowing in the problem in Section 3.1.

3.1. Expected Lifetime Shortfall

The individual’s objective is to minimize the maximum shortfall relative to \( x < c/r \) during life. Then, the relevant value function for this individual’s objective is given by
\[ V(w, m; x) = \inf_{\pi} \mathbb{E}[(x - M_{\tau_d})_+ | W_0 = w, M_0 = m]. \] (3.1)

Here \((x - m)_+ = \max(x - m, 0)\) denotes the negative part of the random variable \(M_{\tau_d} - x\), and \(\tau_d\) denotes the random time of death of our individual, as in Section 2. We refer to \((x - M_{\tau_d})_+\) as the lifetime shortfall relative to \(x\). The minimization in (3.1) is carried out over all admissible investment strategies, as defined in Section 2.1. In this section and in Section 3.2, we apply no further constraints on admissible investment strategies while in Section 3.3, we require that \(\pi_t \leq \max(0, W_t)\).

We first consider value functions of the form
\[ V^f(w, m) = \inf_{\pi} \mathbb{E}[f(M_{\tau_d}) | W_0 = w, M_0 = m], \] (3.2)
in which \(f\) is non-increasing, bounded, and continuously differentiable, and \(f(m) = 0\) for \(m \geq x\) for some \(x < c/r\). That is, \(f\) is a non-decreasing function of shortfall relative to \(x\). We provide a verification theorem (see Appendix A for its proof) and obtain (3.2) explicitly. As a consequence of this result, in Theorem 3.3, we find the optimal investment strategy for the problem of minimizing the maximum expected shortfall (3.1). We also evaluate the value function in (3.1) explicitly; see (3.19).

**Theorem 3.1.** Let \(D = \{(w, m) \in \mathbb{R} \times \mathbb{R} : w \geq m\}\). Suppose \(h : D \to \mathbb{R}\) is a bounded, continuous function that satisfies the following conditions:

(i) \(h(\cdot, m) \in C^2([m, c/r))\) for all \(m \in \mathbb{R}\);

(ii) \(h(w, \cdot)\) is continuously differentiable;

(iii) \(h_m(m, m) = 0\) for all \(m \in \mathbb{R}\);

(iv) \(h(w, m) = f(m)\) for \(w \geq c/r\);

(v) \(h_w(c/r, m) = 0\);

(vi) \(h\) solves the following (HJB) equation
\[ \lambda(h(w, m) - f(m)) = (rw - c)h_w(w, m) + \min_{\pi} \left[ (\mu - r)\pi h_w(w, m) + \frac{1}{2} \sigma^2 \pi^2 h_{ww}(w, m) \right]. \] (3.3)

Then the value function in (3.2) is given by
\[ V^f(w, m) = h(w, m), \quad -\infty < m \leq w < \infty. \] (3.4)

We next use Theorem 3.1 to solve for \(V^f\). We hypothesize that \(V^f\) is convex (in its first variable), satisfies (i), and solves the HJB equation (3.3) together with the boundary
conditions (iv) and (v) of Theorem 3.1. Under these assumptions we obtain an explicit expression for $V^f$ and later check that it satisfies (ii) and (iii).

Since we assume that $V^f$ is convex, we can compute its concave dual $\tilde{V}^f$ by the Fenchel-Legendre transform, as in the proof of Theorem 2.1. For ease of reference, we repeat the analogs of (2.13) and (2.14) here.

$$\tilde{V}^f(y, m) = \min_w [V^f(w, m) + wy]. \quad (3.5)$$

Note that we can retrieve the function $V^f$ from $\tilde{V}^f$ by the relationship

$$V^f(w, m) = \max_y [\tilde{V}^f(y, m) - wy]. \quad (3.6)$$

The minimizer of the right-hand side of (3.5) equals $I(-y, m) = \tilde{V}^f_y(y, m)$, where $I$ is the inverse function of $V^f_w$ with respect to $w$. Therefore, the maximizer of the right-hand side of (3.6) is equal to $-\tilde{V}^f_w(w, m)$.

Substitute $w = I(-y, m)$ into (3.3) to get

$$\lambda \tilde{V}^f(y, m) + (r - \lambda)y\tilde{V}_y^f(y, m) - \delta y^2\tilde{V}_{yy}^f(y, m) = cy + f(m), \quad (3.7)$$

in which $\delta$ is given in (2.7). The general solution of (3.7) is given by

$$\tilde{V}^f(y, m) = D_1 y^{B_1} + D_2 y^{B_2} + \frac{c}{r}y + f(m), \quad (3.8)$$

in which $B_1$ and $B_2$ are given in (2.18) and (2.19), respectively. $D_1$ and $D_2$ are functions of $m$ to be determined.

Define $y_c = -V^f_w(c/r, m) = 0$; that is, $\tilde{V}^f_y(0, m) = c/r$. From the definition of $\tilde{V}^f$ and from $V^f(c/r, m) = f(m)$, we have

$$\tilde{V}^f(0, m) = V^f(c/r, m) + \frac{c}{r}y_c = f(m). \quad (3.9)$$

Expression (3.9) implies that $D_2 = 0$ in (3.8). Now, we can calculate $V^f$ from

$$V^f(w, m) = \max_y \left[ D_1 y^{B_1} + \frac{c}{r}y + f(m) - wy \right]. \quad (3.10)$$

From the first-order condition, the maximizer of the right-hand side of (3.10) is given by

$$y = \left( -\frac{c/r - w}{D_1 B_1} \right)^{1/(B_1 - 1)}. \quad (3.11)$$

By substituting (3.11) into (3.10), we obtain
\[ V_f^f(w, m) = k(m) \left( \frac{c}{r} - w \right)^p + f(m), \quad (3.12) \]

for some \( k(m) > 0 \) yet to be determined and for \( p \) given in (2.6).

Now, we obtain an explicit expression for \( \pi^V \) by using the convexity of \( V_f^f \) and the corresponding first-order condition from (3.3), namely

\[ \pi^V(w, m) = -\frac{\mu - r}{\sigma^2} \frac{h^w(w, m)}{h^{ww}(w, m)}, \quad (3.13) \]

and obtain that \( \pi^V \) is identical to the investment strategy in (2.8) for the problem of minimizing the problem of lifetime ruin.

We next show how the minimum probability of lifetime ruin can be used to compute \( k(m) \) in (3.12). Denote by \( M^* \) the minimum wealth process if the individual follows the optimal investment strategy \( \pi^V = \pi^\psi \). Then, by using the fact that the optimal investment strategies are equal for \( V_f^f \) and for \( \psi \), we can write

\[ P(M^*_{\tau_d} \leq y | W_0 = w, M_0 = m) = \begin{cases} \left( \frac{c - rw}{c - ry} \right)^p, & \text{for } y < m < c/r, \\ 1, & \text{for } m \leq x < c/r, \end{cases} \quad (3.14) \]

for \( w < c/r \). It follows that we can write (3.2) as

\[ V_f^f(w, m) = \mathbb{E}_{w, m}[f(M^*_{\tau_d})] = \int_{-\infty}^{\infty} f(y) dP(M^*_{\tau_d} \leq y | W_0 = w, M_0 = m) \]
\[ = f(m) (1 - \psi(w, m; m)) + \int_{-\infty}^{m} f(y) d\psi(w, m; y) \quad (3.15) \]
\[ = f(m) - \int_{-\infty}^{m} f'(y) \left( \frac{c - rw}{c - ry} \right)^p dy, \]

for \( w < c/r \). \( \mathbb{E}_{w, m} \) denotes the conditional expectation given \( W_0 = w \) and \( M_0 = m \).

Note that if \( w \geq c/r \), then \( \psi \equiv 0 \), and \( V_f^f(w, m) = f(m) \), which is consistent with the first and second lines of (3.15). From (3.15) it follows that \( V_f^f(w, \cdot) \) is continuously differentiable because \( f \) is continuously differentiable and that \( V^f_{\cdot m}(m, m) = 0 \). We summarize our findings in the next theorem.

**Theorem 3.2.** With no constraints on borrowing, \( V_f^f \) defined by (3.2) is equal to

\[ V_f^f(w, m) = f(m) - \int_{-\infty}^{m} f'(y) \left( \frac{c - rw}{c - ry} \right)^p dy, \quad (3.16) \]
for \( m \leq w < c/r \). The optimal investment strategy is given by

\[
\pi^V(w) = \frac{1}{p-1} \mu - r \sigma^2 \left( \frac{c}{r} - w \right),
\]  

(3.17)

for \( w < c/r \), which is equal to the optimal investment strategy that attains the minimum probability of lifetime ruin \( \psi \).

As a result of Theorem 3.2, the following theorem (whose proof is in Appendix B) shows that \( \pi^V \) in (3.17) is also an optimal investment strategy to minimize the maximum lifetime shortfall.

**Theorem 3.3.** Let \( \pi^V \) be as in Theorem 3.2; then, the value function of the shortfall problem (3.1) satisfies

\[
V(w, m; x) = E_{w, m} \left[ (x - M_{\tau_d}^V)^+ \right];
\]

(3.18)

that is, \( \pi^V \) minimizes the expected maximum lifetime shortfall.

As a corollary to Theorem 3.3, we can use the expression in (3.16) to write (3.1) as

\[
V(w, m; x) = (x - m)^+ \left( 1 - \left( \frac{c - rw}{c - rm} \right)^p \right) + \int_{-\infty}^{m} (x - y)^+ \frac{\partial}{\partial x} \left( \frac{c - rw}{c - ry} \right)^p dy
\]

\[
= (x - m)^+ + \left( \frac{c - rw}{c - r(m \wedge x)} \right)^p \frac{c - r(m \wedge x)}{r(p - 1)}.
\]

(3.19)

Besides the useful representations of \( V^f \) and \( V \) in (3.16) and (3.19), respectively, the main fact to glean from this section is that the optimal investment strategy for the problems in (3.1) and (3.2) are identical and are identical to the one for minimizing the probability of ruin, as given in (2.8). Thus, introducing a penalty for the amount that one’s wealth drops below \( x \) does not reduce the leveraging at small wealth in the optimal investment strategy. On the other hand, the fact that the optimal investment strategies are the same leads to the useful representation of \( V^f \) in (3.16).

In the next section, we modify the penalty in (3.2) by considering the shortfall at death.

### 3.2. Expected Shortfall at Death

As we were motivated to consider the problem in Section 2.2, we hope that by introducing a penalty for the shortfall at death, we will reduce the leveraging in the optimal investment strategy. In Section 2.2, we were careful in modifying the definition of \( \psi \) so that \( \phi \) was convex. We take the same care here.
Note that one can write $\phi$ from (2.9) as
\[
\phi(w, m; x, M) = \inf_{\pi} E \left[ 1_{W_{rd} \leq x} 1_{M_{rd} > M} + 1_{M_{rd} \leq M} \right] W_0 = w, M_0 = m.
\] (3.20)
By comparing (3.20) with $\psi$ in (2.4), specifically with
\[
\psi(w, m; x) = \inf_{\pi} E \left[ 1_{M_{rd} \leq x} \right] W_0 = w, M_0 = m,
\] (3.21)
we now modify the penalty function in (3.1) in a similar manner and define $U$ as follows:
\[
U(w, m; x, M) = \inf_{\pi} E \left[ (x - W_{rd})_+ 1_{M_{rd} > M} + (x - M) 1_{M_{rd} \leq M} \right] W_0 = w, M_0 = m,
\] (3.22)
in which $M < x < c/r$.

For $m \leq M$, $U$ is identically $x - M$, and for $m > M$ and $w \geq c/r$, $U$ is identically 0. For $M < m \leq w < c/r$, $U$ solves the following HJB equation:
\[
\begin{ cases }
\lambda(U(w) - (x - w)_+) = (rw - c)U'(w) + \min_{\pi} \left[ (\mu - r)\pi U'(w) + \frac{1}{2}\sigma^2\pi^2 U''(w) \right], \\
U(M) = x - M, \quad U(c/r) = 0,
\end{ cases }
\] (3.23)
in which we drop the notational dependence of $U$ on $m$ because $U$ is independent of $m$ if $m > M$.

We have the following theorem whose proof we omit because it is similar to that of Theorem 2.1.

**Theorem 3.4.** For $m > M$ and $x \leq w < c/r$, the function $U$ is given by
\[
U(w, m; x, M) = \beta \left( 1 - \frac{r}{c}w \right)^p,
\] (3.24)
with $p$ as in (2.6) and $\beta > 0$. Thus, on $[x, c/r]$, $U$ is a multiple of the probability of lifetime ruin $\psi$, or equivalently, a multiple of $V - (x - m)_+$ from (3.19). It follows that the optimal investment strategy $\pi^U$ equals $\pi^\psi$, as given in (2.8). \(\square\)

Next, we consider the solution of (3.23) for $M < m \leq w < x$. On this domain, $U$ solves the system
\[
\begin{ cases }
\lambda(U(w) - (x - w)) = (rw - c)U'(w) + \min_{\pi} \left[ (\mu - r)\pi U'(w) + \frac{1}{2}\sigma^2\pi^2 U''(w) \right], \\
U(M) = x - M, \quad U(x)/U'(x) = -(c/r - x)/p,
\end{ cases }
\] (3.25)
in which we assume that $U$ and $U'$ are continuous at $w = x$. As in Sections 2.2 and 3.1, we consider the Fenchel-Legendre transform of $U$, denoted by $\tilde{U}$. We show how to solve for $\tilde{U}$ numerically, and from (2.14) or (3.6), we can then determine $U$. Even though we can only calculate $U$ numerically, we show that $\pi^U > \pi^\psi$ for $M < m \leq w < x$.

By substituting $w = I(-y, m)$ in (3.25), we obtain the following equation:

$$\lambda \tilde{U}(y) + [(r - \lambda)y + \lambda]\tilde{U}'(y) - \delta y^2 \tilde{U}''(y) = \lambda x + cy.$$  \hspace{1cm} (3.26)

Write $\tilde{U}$ as the sum of a solution to the homogeneous problem and of a particular solution:

$$\tilde{U}(y) = \tilde{U}_h(y) + \frac{c}{r}y - \left(\frac{c}{r} - x\right).$$  \hspace{1cm} (3.27)

If we define $y_x = -U'(x) = \beta p(r/c)(1 - rx/c)^{p-1}$, then we have

$$\tilde{U}_h(y_x) = \left(\frac{c}{r} - x\right) \left(1 - \frac{p - 1}{p} y_x\right),$$  \hspace{1cm} (3.28)

and

$$\tilde{U}'_h(y_x) = x - \frac{c}{r}.$$  \hspace{1cm} (3.29)

Similarly, if we define $y_M = -U'(M)$, then we have

$$\tilde{U}_h(y_M) = \left(\frac{c}{r} - M\right) (1 - y_M),$$  \hspace{1cm} (3.30)

and

$$\tilde{U}'_h(y_M) = M - \frac{c}{r}.$$  \hspace{1cm} (3.31)

Therefore, the second-order linear differential equation for $\tilde{U}_h$, together with the four equations (3.28)-(3.31) at the two boundaries, are enough to determine $\tilde{U}_h$, $y_x$, and $y_M$.

To end this section, we compare the optimal investment strategy for this problem $\pi^U$ with the one for the problems in Sections 2.1 and 3.1, namely $\pi^\psi$ given in (2.8). Recall that for $m > M$ and $x \leq w < c/r$, the two strategies are identical. For $M < m \leq w < x$, $\pi^U(w) > \pi^\psi(w)$ if and only if

$$-y\tilde{U}''(y) > \frac{1}{p - 1} \left(\frac{c}{r} - \tilde{U}'(y)\right), \quad y_x < y < y_M.$$  \hspace{1cm} (3.32)

Compare this inequality with (2.37). In terms of $\tilde{U}_h$, we can rewrite (3.32) as follows:
\[ y\tilde{U}''_h(y) < \frac{1}{p-1}\tilde{U}'_h(y), \quad y_x < y < y_M. \]  

(3.33)

Substitute for \( \tilde{U}''_h \) in (3.33) from the homogeneous equation derived from (3.27) and simplify to obtain the inequality

\[ \frac{\tilde{U}_h(y)}{\tilde{U}'_h(y)} > \frac{p-1}{p}y - 1, \quad y_x < y < y_M. \]  

(3.34)

Thus, if we can demonstrate that inequality (3.34) holds, then \( \pi^U(w) > \pi^v(w) \) for \( M < m \leq w < x \). Inequality (3.34) does hold, and we demonstrate this in the proof of the following theorem:

**Theorem 3.5.** For \( M < m \leq w < x \), the optimal investment strategy \( \pi^U(w) > \frac{\mu-r}{\sigma^2} \frac{1}{p-1} \left( \frac{c}{r} - w \right) \).

**Proof.** Inequality (3.34) holds if and only if \( g(y) > z(y) \) for \( y_x < y < y_M \), in which \( g \) and \( z \) are defined by \( g(y) = \tilde{U}_h(y)/\tilde{U}'_h(y) \) and \( z(y) = [(p-1)/p]y - 1 \). For \( 0 < y_x < y < y_M \), \( g \) solves the following first-order non-linear differential equation:

\[ g'(y) = 1 - \frac{\lambda}{\delta y^2}g^2(y) - \frac{(r-\lambda)y + \lambda}{\delta y^2}g(y), \]  

(3.35)

with boundary condition at \( y = y_x \) determinable from (3.28) and (3.29). Specifically, \( g(y_x) = z(y_x) = [(p-1)/p]y_x - 1 \).

Suppose \( g(y) = z(y) \) for some \( y_x < y < y_M \), then from (3.35), we deduce that \( g'(y) > z'(y) = (p-1)/p \) if and only if \( rp > \lambda \), independent of \( y \). Now, \( rp > \lambda \) for \( r < \mu \); therefore, if \( g \) and \( z \) intersect at some point in \([y_x, y_M]\), then the slope of \( g \) is larger than the slope of \( z \) at that point. In particular, we know that \( g(y_x) = z(y_x) \); therefore, \( g'(y_x) \geq z'(y_x) \). In order for \( g \) to intersect \( z \) at \( y > y_M \), we must have \( g'(y) \leq z'(y) \) at the smallest such value of \( y \), a contradiction to \( g'(y) > z'(y) \) whenever \( g(y) = z(y) \). Therefore, no such intersection point \( y > y_x \) exists, and \( g \) is strictly larger than \( z \) on \([y_x, y_M]\). \( \square \)

From Theorem 3.5, we learn that as in Section 2.2, not only did we not reduce the leveraging exhibited in the investment strategy given in (2.8), we actually increased the leverage by considering a penalty that depends on the shortfall at death. Therefore, as in Section 2, our only real hope of reducing leverage is to prevent it by modifying the admissible investment strategies. We do that in the next section, in parallel to Section 2.3.

**3.3. No Borrowing**
In this section, we consider the problem of minimizing the expected maximum lifetime shortfall when the individual is not allowed to borrow money to invest in the stock market. We will observe that the optimal investment strategy that minimizes the probability of ruin (independent of the ruin level $x$) and the optimal investment strategy that minimizes the expected maximum shortfall are the same, subject to the constraint that $\pi_t \leq \max(0, W_t)$. As in Section 2.3, we consider two assumptions concerning negative wealth: (1) Welfare provides for consumption, and (2) the individual can borrow only to cover consumption.

As in Section 3.1, we first consider a continuously differentiable, bounded, non-increasing function $f$ of the minimum wealth such that $f(m) = 0$ for $m > x$ for some $x < c/r$. Without abusing the notation too much, we denote the corresponding value function by $V^f$, as in (3.2). We provide a verification lemma whose proof we omit because of its similarity to the proof of Theorem 3.1. Recall that $D = \{(w, m) \in \mathbb{R} \times \mathbb{R} : w \geq m\}$.

**Theorem 3.6.** Suppose $a : D \to \mathbb{R}$ is a bounded continuous function that satisfies the following conditions:

(i) $a(\cdot, m) \in C^2([m, c/r))$ for all $m \in \mathbb{R}$;

(ii) $a(w, \cdot)$ is continuously differentiable;

(iii) $a_m(m, m) = 0$ for all $m \in \mathbb{R}$;

(iv) $a(w, m) = f(m)$ for $w \geq c/r$;

(v) $a_w(c/r, m) = 0$;

(vi) $a$ solves the following HJB equation for $0 < w < c/r$

$$
\lambda(a(w, m) - f(m)) = (rw - c)a_w(w, m) + \min_{\pi \leq w} \left[ (\mu - r)\pi a_w(w, m) + \frac{1}{2}\sigma^2 \pi^2 a_{ww}(w, m) \right].
$$

(3.36)

(vii) If welfare exists, then $a(w, m) = f(m)$ for $w \leq 0$. If the individual is allowed to borrow to cover consumption when wealth is negative, then for $w \leq 0$, the function $a$ solves (3.36) with $\pi = 0$.

Then, the minimum expected maximum shortfall is given by

$$
V^f(w, m) = a(w, m), \quad -\infty < m \leq w < \infty.
$$

(3.37)

Note that if welfare exists, then $V^f(w, m) = f(m)$ for $w \leq 0$. Alternatively, if the individual is allowed to borrow to cover consumption, then $\pi(w) = 0$ for $w \leq 0$, as in the case for minimizing the probability of ruin in Section 2.3. Thus, for $w \leq 0$,

$$
P(M^* \leq y|W_0 = w, M_0 = m) = \begin{cases} 
1, & \text{if } m \leq y; \\
\psi^{nb}(w, m; y) = \left( \frac{c-rw}{c-rx} \right)^{\lambda/r}, & \text{if } m > y.
\end{cases}
$$

(3.38)
It follows from (3.38) that we can write $V^f$ for $w \leq 0$ as follows:

$$V^f(w, m) = \mathbb{E}[f(M^*_r|W_0 = w, M_0 = m)] = \int_{-\infty}^{\infty} d\mathbb{P}(M^*_r \leq y|W_0 = w, M_0 = m)$$

$$= f(m) - \int_{-\infty}^{m} f'(x) \left( \frac{c- rw}{c- ry} \right)^{\lambda/r} dy. \quad (3.39)$$

Note that (3.39) gives us a boundary value for $V^f$ at $w = 0$ when borrowing is allowed for covering consumption. One can apply the arguments in Bayraktar and Young (2007a) to the more general problem outlined in Theorem 3.6 and, thereby, we prove the following theorem:

**Theorem 3.7.** If welfare exists, then $V^f(w, m) = f(m)$ for $m \leq w \leq 0$. Alternatively, if the individual is allowed to borrow to cover consumption when wealth is negative, then $V^f$ is given in (3.39) for $m \leq w \leq 0$.

For $0 < w \leq w_l$, $V^f$ solves

$$\lambda(V^f(w, m) - f(m)) = (\mu w - c)V^f_w(w, m) + \frac{1}{2} \sigma^2 w^2 V^f_{ww}(w, m), \quad (3.40)$$

with boundary condition at $w = 0$ given by $V^f(0, m) = f(m)$ if there is welfare or by $V^f(0, m)$ in (3.39) if borrowing is allowed to cover consumption and with boundary condition at $w = w_l$

$$\frac{V^f(w_l, m) - f(m)}{V^f_w(w_l, m)} = -\frac{1}{p} \left( \frac{c}{r} - w_l \right). \quad (3.41)$$

For $w_l < w < c/r$, $V^f(w, m) = k(m) \left( \frac{c}{r} - w \right)^p + f(m)$, in which $k(m) = V^f(w_l, m)(1- rw_l/c)^{-p}$. Moreover, $V^f(w, m) = f(m)$ for $w \geq c/r$.

If borrowing is allowed to cover consumption when wealth is negative, then for all $x < c/r$, the optimal investment strategy for the probability of ruin with ruin level $x$ and the optimal investment strategy for $V^f$ coincide on $[x, c/r)$. If welfare exists, then the optimal investment strategy for $V^f$ when wealth is positive is given by (2.40). \hfill \Box

We have a corollary that follows from the observation in Theorem 3.7 that the optimal investment strategy for the problem of minimizing lifetime shortfall is the same as the strategy for minimizing the probability of lifetime ruin. Recall that we assume that $f(m) = 0$ for $m \geq x$, in which $x$ is some number less than $c/r$.

**Corollary 3.8.** If borrowing is allowed to cover consumption when wealth is negative, then for $w < c/r$,

$$\mathbb{P}(M^*_r \leq y|W_0 = w, M_0 = m) = \begin{cases} 
\psi^{nb}(w, m; y), & \text{for } y < m < c/r, \\
1, & \text{for } m \leq y,
\end{cases} \quad (3.42)$$

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in which \( \psi^{nb} \) is given in Cases 1-3 in Section 2.3 and \( w < c/r \). Therefore, if borrowing is allowed to cover consumption,

\[
V^f(w, m) = f(m) - \int_{-\infty}^{m} f'(y) \psi^{nb}(w, m; y) dy. \tag{3.43}
\]

If welfare exists to cover consumption when wealth is negative, then

\[
P(M^*_\tau_d \leq y | W_0 = w, M_0 = m) = \begin{cases} 
0, & \text{for } y < m \leq w \leq 0, \\
0, & \text{for } y < \min(0, m) \text{ and } w > 0, \\
\psi^{nb}(w, m; y), & \text{for } 0 \leq y < m < c/r, \\
1, & \text{for } m \leq y,
\end{cases} \tag{3.44}
\]

in which \( \psi^{nb} \) is given in Cases 1 and 2 in Section 2.3 and \( w < c/r \). Therefore, if welfare exists,

\[
V^f(w, m) = \begin{cases} 
f(m), & \text{if } m \leq 0, \\
f(m) - \int_{0}^{m} f'(y) \psi^{nb}(w, m; y) dy, & \text{if } m > 0.
\end{cases} \tag{3.45}
\]

**Proof.** We prove (3.45) because (3.43) follows from (3.42) as (3.15) follows from (3.14). If \( m \leq 0 \), then \( P(M^*_\tau_d \leq y | W_0 = w, M_0 = m) = 0 \) if \( y < m \) and equals 1 if \( y \geq m \). Thus, if \( m \leq 0 \), \( V^f(w, m) = \int_{-\infty}^{\infty} f(y) dP(M^*_\tau_d \leq y | W_0 = w, M_0 = m) = f(m)(1 - 0) = f(m) \).

If \( m > 0 \), then

\[
V^f(w, m) = \int_{-\infty}^{\infty} f(y) dP(M^*_\tau_d \leq y | W_0 = w, M_0 = m) \\
= f(0) \psi^{nb}(w, m; 0) + \int_{0}^{m} f(y) d\psi^{nb}(w, m; y) + f(m)(1 - \psi^{nb}(w, m; m)) \tag{3.46}
\]

in which the last equality follows from integration by parts. \( \square \)

As in Theorem 3.3, we can show that the optimal investment strategy when \( f(m) = (x - m)_+ \) is identical to that in the problem of minimizing the probability of lifetime ruin. Denote the corresponding value function by \( V^{nb} \). Therefore, we have the following theorem:

**Theorem 3.9.** If borrowing is allowed to cover consumption when wealth is negative, then for \( f(m) = (x - m)_+ \), with \( x < c/r \), the value function of the constrained shortfall problem is given by
\[ V^{nb}(w, m; x) = (x - m)_+ + \int_{-\infty}^{m \wedge x} \psi^{nb}(w, m; y) \, dy. \]  
\hspace{1cm} (3.47)

If welfare exists to cover consumption when wealth is negative, then

\[ V^{nb}(w, m; x) = \begin{cases} 
(x - m)_+, & \text{if } m \leq 0, \\
(x - m)_+ + \int_0^{m \wedge x} \psi^{nb}(w, m; y) \, dy, & \text{if } m > 0.
\end{cases} \]  
\hspace{1cm} (3.48)

4. Summary and Future Research

In this paper, we examined the problems of minimizing (1) lifetime ruin probability, (2) ruin probability at death, (3) expected lifetime shortfall, and (4) expected shortfall at death. We showed that the leveraging effect, the fact that an individual borrows excessively at low wealth levels observed by Young (2004), is not reduced by the alternative penalty criteria. In fact, we learned that the optimal investment strategies of (1) and (3) are the same, and that the optimal amount of wealth traded in the risky asset for the cases of (2) and (4) exacerbate the leveraging observed in (1). We also introduced a no-borrowing to constraint to (1) and (3) to eliminate leveraging completely.

In work not shown here (for the sake of brevity), we also considered the penalty functions (3) in a setting where we allowed the individual to borrow but only at a rate greater than the one earned by the riskless asset and did not impose a constraint on the individual’s investment strategy. We found out that the leveraging effect is exacerbated in some cases, and that the optimal investment strategy is the same as in the corresponding problem for minimizing the probability of lifetime ruin, which is given in Bayraktar and Young (2007a). We also observed that changing the consumption function to a piece-wise linear function, \( c(w) = c + p(w - d)_+ \), does not alter the leveraging effect. Therefore, the leveraging effect is not a side effect of choosing a constant consumption.

The leverage effect might decrease if one were to introduce negative unbounded jumps, as found in Liu, Longstaff, and Pan (2003) in the context of maximizing expected utility of terminal wealth. In future research, we plan to consider the problem of minimizing the probability of lifetime ruin with stocks that are subject to such negative jumps.

Also, in the future, instead of prohibiting borrowing entirely, we will consider constraints of the from \( \pi_t \leq k(W_t + L) \). Here \( k, L > 0 \). Note that the constraint we considered in this paper corresponds to \( k = 1 \) and \( L = 0 \). In general, for \( k = 1 \), the quantity \( L \) might be the investor’s credit limit. The case for which \( L = 0 \) and \( 0 < k < 1 \) could represent the situation for which the individual is allowed to put only a fraction of his wealth into the risky asset.
Appendix A. Proof of Theorem 3.1

Assume that $h$ satisfies the conditions specified in the statement of Theorem 3.1. Let $N$ denote a Poisson process with rate $\lambda$ that is independent of the standard Brownian motion $B$ driving the wealth process. The occurrence of a jump in the Poisson process represents the death of the individual investor.

Let $\pi : \mathbb{D} \to \mathbb{R}$ be a function, and let $W^\pi$ and $M^\pi$ denote the wealth and the minimum wealth respectively, when the individual investor uses the investment policy $\pi_t = \pi(W_t, M_t)$. Assume that this investment policy is admissible. For a given $m$, define $\mathbb{D}^m = \{(w, m) : w \geq m\}$, and define $\bar{\mathbb{D}}^m = \mathbb{D}^m \cup \{\infty\}$ to be the one-point compactification of $\mathbb{D}^m$. The point $\infty$ is the “coffin state.” The wealth process is killed (and sent to the coffin) as soon as the Poisson process jumps (that is, when the individual dies), and we assign $W^\pi_{\tau_d} = \infty$. All functions $g$ on $\mathbb{D}^m$ are extended to $\bar{\mathbb{D}}^m$ by $g(\infty, m) = f(m)$. Observe that $h(c/r, m) = h(W^\pi_{\tau_d}, m) = f(m)$ for all $m \leq c/r$. Define the stopping time $\tau = \tau_d \land \tau_{c/r}$, where $\tau_{c/r} = \inf\{t > 0 : W_t = c/r\}$, with the convention that $\inf \emptyset = \infty$.

If $w \geq c/r$, then the individual can invest her wealth in the riskless asset and guarantee to finance the cost of her consumption, which is almost surely less than $\int_0^\infty ce^{-rt} dt = c/r$. Therefore, with this strategy, her wealth will almost surely be at least $c/r$ at the time of her death. This implies that $M_{\tau_d} = m$, and therefore $V^f(w, m) = f(m)$ for $w \geq c/r$. From this it follows that

$$V^f(w, m) = \inf_{\pi} \mathbb{E}[f(M_r)|W_0 = w, M_0 = m]. \quad (A.1)$$

Define

$$\tau_n = \inf\{t \geq 0 : W_t \geq n \text{ or } W_t \leq -n \text{ or } \int_0^t \pi_s^2 ds = n\}. \quad (A.2)$$

By applying Itô’s formula to $h$, we have

$$h(W^\pi_{t \land \tau \land \tau_n}, M^\pi_{t \land \tau \land \tau_n}) = h(w, m)$$
$$+ \int_0^{t \land \tau \land \tau_n} \left((\mu - rt)\pi_s - c)h_w(W^\pi_s, M^\pi_s) + \frac{1}{2}\sigma^2\pi_s^2 h_{ww}(W^\pi_s, M^\pi_s)\right) ds$$
$$+ \lambda \int_0^{t \land \tau \land \tau_n} (f(M^\pi_s) - h(W^\pi_s, M^\pi_s))ds + \int_0^{t \land \tau \land \tau_n} h_w(W^\pi_s, M^\pi_s)\sigma\pi_s dB_s$$
$$+ \int_0^{t \land \tau \land \tau_n} (f(M^\pi_s) - h(W^\pi_s, M^\pi_s))d(N_s - \lambda s) + \int_0^{t \land \tau \land \tau_n} h_m(W^\pi_s, M^\pi_s)dM^\pi_s. \quad (A.3)$$

The last integral in (A.3) is equal to zero almost surely because $dM_t$ is non-zero only when $M_t = W_t$, and $h_m(m, m) = 0$; $h_m$ denotes the left derivative of $h$ with respect to $m$. 

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Here we also used the fact that $M$ is non-decreasing, therefore the first variation process associated with it is finite almost surely, to conclude that the cross variation of $M$ and $W$ is zero almost surely. It follows from the definition of $\tau_n$ that

$$E_{w,m}\left[\int_0^{t \land \tau_n} h_w(W_s^\pi, M_s^\pi)\sigma \pi_s dB_s\right] = 0. \tag{A.4}$$

$E_{w,m}$ denotes the conditional expectation given $W_0 = w$ and $M_0 = m$. Moreover, the expectation of the fourth integral is zero since $f$ and $h$ are bounded; see, for example, Brémaud (1981).

Now, we have

$$E_{w,m}[h(W_{t \land \tau_n}^\pi, M_{t \land \tau_n}^\pi)] = h(w, m)$$

$$+ E_{w,m}\left[\int_0^{t \land \tau_n} \left( (rW_s^\pi + (\mu - r)\pi_s - c)h_w(W_s^\pi, M_s^\pi) + \frac{1}{2}\sigma^2\pi_s^2h_{ww}(W_s^\pi, M_s^\pi) \right) ds \right]$$

$$+ E_{w,m}\left[\int_0^{t \land \tau_n} \lambda(f(M_s^\pi) - h(W_s^\pi, M_s^\pi))ds \right]$$

$$\geq h(w, m), \tag{A.5}$$

where the inequality follows from assumption (vi) of the theorem. Because $h$ is bounded by assumption, it follows from the Dominated Convergence Theorem that

$$E_{w,m}[h(W_{t \land \tau_n}^\pi, M_{t \land \tau_n}^\pi)] \geq h(w, m). \tag{A.6}$$

Equation (A.6) shows that $(h(W_{t \land \tau_n}^\pi, M_{t \land \tau_n}^\pi))$ is a sub-martingale for any admissible strategy $\pi$.

Since $h(c/r, m) = h(W_{t \land \tau_n}^\pi, M_{t \land \tau_n}^\pi) = f(m)$ for all $m \leq c/r$, we have

$$h(W_\tau^\pi, M_\tau^\pi) = f(M_\tau). \tag{A.7}$$

If $\tau_d < \tau_{c/r}$, then obviously $M_\tau = M_{\tau_d}$. If $\tau_d \geq \tau_{c/r}$, then $M_\tau = M_{\tau_{c/r}}$. By taking the expectation of both sides of (A.7), we obtain

$$E_{w,m}[h(W_\tau^\pi, M_\tau^\pi)] = E_{w,m}[f(M_\tau)] \geq h(w, m). \tag{A.8}$$

The inequality in (A.8) follows from an application of the Optional Sampling Theorem because $(h(W_{t \land \tau_n}^\pi, M_{t \land \tau_n}^\pi))$ is a sub-martingale and $\sup_{t \geq 0} E_{w,m}[h(W_{t \land \tau_n}^\pi, M_{t \land \tau_n}^\pi)] < \infty$ because $h$ is bounded; see Theorem 3.15, page 17 and Theorem 3.22, page 19 of Karatzas and Shreve (1991). Together with (A.1) this implies that

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\[ V_f(w, m) = \inf_{\pi} \mathbb{E}[f(M_\pi^\tau)] \geq h(w, m). \quad \text{(A.9)} \]

If the individual investor follows a strategy \( \pi^V \) that minimizes the right-hand side of (3.3), then (A.5) is satisfied with equality, and an application of Dominated Convergence Theorem yields

\[ \mathbb{E}_{w,m}[h(W_{t\wedge \tau}^{\pi^V}, M_{t\wedge \tau}^{\pi^V})] = h(w, m), \quad \text{(A.10)} \]

which implies that \( (h(W_{t\wedge \tau}^{\pi^V}, M_{t\wedge \tau}^{\pi^V})) \) is a martingale. By following the same line of argument as above, we obtain

\[ V_f(w, m) = h(w, m), \quad \text{(A.11)} \]

which demonstrates that (3.4) holds and \( \pi^V \) is an optimal investment strategy.

**Appendix B. Proof of Theorem 3.3**

Let us introduce a sequence of increasing functions \( g_n \) that converges to \( g(y) = (x - y)_+ \), such that each element in this sequence is bounded and is a difference of two convex functions:

\[ g_n(y) = \begin{cases} 
    n, & \text{if } y \leq x - n, \\
    x - y, & \text{if } x - n < y \leq x, \\
    0, & \text{if } y > x.
\end{cases} \quad \text{(B.1)} \]

Note that \( g_n(y) = (x - y)_+ - (x - n - y)_+ \), and \( g_n(y) \leq n \). Observe that the sequence (B.1) indeed converges monotonically to \( g(y) = (x - y)_+ \) as \( n \to \infty \). Note that \( g_n \) is not continuously differentiable. We will approximate \( g_n \) with an increasing sequence \( \tilde{g}_{n,k} \) given by

\[ \tilde{g}_{n,k}(y) = \begin{cases} 
    n, & \text{if } y \leq x - n - \frac{1}{k}, \\
    n - \frac{k}{2} \left( y - x + n + \frac{1}{2} \right)^2, & \text{if } x - n - \frac{1}{k} < y \leq x - n, \\
    -\frac{1}{2k} - (y - x), & \text{if } x - n < y \leq x - \frac{1}{k}, \\
    \frac{1}{2k} - \frac{1-k(y-x)}{2} \left( y - x + \frac{1}{2} \right), & \text{if } x - \frac{1}{k} < y \leq x, \\
    0, & \text{if } y > x.
\end{cases} \quad \text{(B.2)} \]

The sequence \( (\tilde{g}_{n,k})_{k \in \mathbb{N}} \) increases to \( g_n \) for all \( n \). Each element of the sequence of \( (\tilde{g}_{n,k})_{k \in \mathbb{N}} \) is continuously differentiable.

We have
\begin{align*}
E_{w,m}[ (x - M^\pi_{\tau_d} )_+] & = E_{w,m} \left[ \lim_n g_n(M^\pi_{\tau_d} ) \right] = \lim_n E_{w,m} \left[ g_n(M^\pi_{\tau_d} ) \right] \\
& = \lim_n \lim_k E_{w,m} \left[ \tilde{g}_{n,k}(M^\pi_{\tau_d} ) \right] = \lim_n \lim_k \inf \pi E_{w,m} \left[ \tilde{g}_{n,k}(M^\pi_{\tau_d} ) \right] \\
& \leq \lim_n E_{w,m} \left[ g_n(M^\pi_{\tau_d} ) \right] \leq E_{w,m} \left[ (x - M^\pi_{\tau_d} )_+ \right].
\end{align*}

The second and the third equalities in (B.3) follow from the Monotone Convergence Theorem, and the fourth equality follows from Theorem 3.2. The inequalities follow from the fact that \( \tilde{g}_{n,k}(y) \leq g_n(y) \leq (x - y)_+ \). By taking an infimum over the admissible strategies, we obtain

\begin{align*}
E_{w,m}[ (x - M^\pi_{\tau_d} )_+] & \leq \inf \pi E_{w,m} \left[ (x - M^\pi_{\tau_d} )_+ \right].
\end{align*}

This proves that \( \pi^V \) in (3.17) is an optimal investment strategy. \( \Box \)

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Figure 1: Graph of $\pi^\phi$ and $\pi^\psi$. The solid line corresponds to $\pi^\phi$ and the dashed line to $\pi^\psi$. 