Chiral dynamics from the hadronic string: general formalism

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Abstract
QCD at long distances can be described by the chiral Lagrangian. On the other hand there is
overwhelming evidence that QCD and all non-abelian theories admit an effective string description.
Here we review a derivation of the (intrinsic) parity-even chiral Lagrangian by requiring that the
propagation of the QCD string takes place on a background where chiral symmetry is spontaneously
broken. Requiring conformal invariance leads to the equation of motion of the chiral Lagrangian. We
then proceed to coupling the string degrees of freedom to external gauge fields and we recover in this
way the covariant equations of motion of the gauge-invariant chiral Lagrangian at \(O(p^2)\). We consider
next the parity-odd part (Wess-Zumino-Witten) action and argue that this require the introduction of
the spin degrees of freedom (absent in the usual effective action treatment). We manage to reproduce
the Wess-Zumino-Witten term in 2D in an unambiguous way. In 4D the situation is considerably
more involved. We outline the modification of boundary interaction that is necessary to induce the
parity-odd part of the chiral Lagrangian.

1 Introduction: string propagation in a chirally broken background

The history of attempts to describe the hadrons in the framework of a string theory derived from, or at
least inspired by, QCD encompasses already more than 30 years (see, \[1\]-\[7\] as well as the reviews \[8\]-\[10\]
and an incomplete list of references therein). The commonly cited arguments to justify a stringy descrip-
tion of QCD are the dominance of planar gluon diagrams in the large \(N\) limit \[11\] ‘filling in’ a surface
(interpreted as the world-sheet of a string), the expansion in terms of surfaces built out of plaquettes
in strong-coupling lattice QCD \[12\], and to some extent the incarnation of Regge phenomenology \[13\]
within QCD \[14\]. Recently, the developments based on the Maldacena conjecture \[15\] and holographic
duality \[16\] have added further strength to these arguments. The last but not the least is an advent of
Nambu-Goto string in lattice QCD of static heavy quark and antiquark \[17\].
Clearly the simplest string models (bosonic string, supersymmetric string, ...) do not lead to realistic
amplitudes. The paradigmatic example is the Veneziano amplitude \[1\]; expanding it in powers of the
Mandelstam variables \(s\) and \(t\) one does not find the right Adler zeroes and, of course, reveals a tachyon
in the spectrum. The supersymmetric version \[18\] partially solves one of the problems by projecting out the
tachyon, but the wrong chiral behavior persists. Both difficulties are absent in the phenomenologically
inspired Lovelace-Shapiro amplitude \[19\], but this amplitude does not seem to derive from any known
string theory and there are good reasons to believe that it is anyway incompatible with QCD asymptotics
(see below).

Thus so far it is not yet clear what is a phenomenologically acceptable QCD string action, even
though there is a motivated agreement based on universality considerations that in a certain kinematic
regime the Nambu-Goto (or the Polyakov \[2\]) string action should be basically correct or, at least, provide
the basic description. A general characteristic of all the above amplitudes (including, incidentally the
Lovelace-Shapiro one) is that they lead to linearly rising trajectories. General arguments and recent
work \[20\] indicate that while this behavior corresponds to a confining theory with an infinite number
of narrow resonances (in the large \(N_c\) limit) it does not reproduce the chiral properties of the QCD
correlators. In fact, it can be proven that any strictly linearly rising Regge trajectory leads to complete
degeneracy between the vector and the axial-vector channels — not the way chiral symmetry is realized
in QCD. Exponentially small (of the form \( \exp(-an) \), \( n \) being the principal quantum number) deviations are required and that means that none of the existing amplitudes can reproduce the chiral properties of QCD.

It is quite plausible that the main reason for the presence of a tachyon in the spectrum and the wrong chiral properties is a wrong choice of the vacuum \( [24] \). One can make a parallel with scalar field theory with the potential \( V(\phi) = -\mu^2 \phi^2 + \lambda \phi^4 \), that generates spontaneous symmetry breaking with a sensible ground state, but where perturbing around \( \phi = 0 \) gives negative \( m^2 \) values for all components. Thus we assume that the string amplitudes obtained through the use of the canonical vertex operators correspond to amplitudes for excitations perturbed around the wrong, unphysical vacuum.

A possible way to take into account the non-trivial nature of the QCD vacuum was suggested in \( [22] \) and developed in \( [23] \). Namely, one can assume that in QCD chiral symmetry breaking takes place and the light (massless in the chiral limit) pseudoscalar mesons form the background of the QCD vacuum, whereas other massive excitations are assembled into a string. The massless pseudoscalars can be collected in a unitary matrix \( U(x) \). This matrix transforms as \( U(x) \rightarrow U'(x) = LU(x)R^\dagger \) under chiral transformations belonging to \( SU(3)_L \times SU(3)_R \) and describe excitations around the non-perturbative vacuum. From the string point of view \( U(x) \) is nothing but a bunch of couplings involving the string variable \( x_\mu(\tau, \sigma) \). The unitary matrix \( U(x) \) has to be somehow coupled to the boundary of the string, which is where flavor ‘lives’. Our goal is to find eventually a consistent string propagation in this non-perturbative background.

An essential property of string theory is, certainly, conformal invariance. In the limit of large \( N_c \) at least, the hadronic string action should obey re-parametization and conformal invariance as describing zero-width, point-like resonance states\(^1\). Since conformal invariance must hold when perturbing the string around any vacuum we demand the new coupling to chiral fields, living on the boundary, to preserve conformal invariance too (compare with \( [24] \)). The requirement of conformal invariance will provide the equations of motion of the background fields and, indirectly, their Lagrangian.

In the present paper we begin by describing the basic characteristics of our approach. We start by elucidating of how to incorporate ‘quarks’ at the end of the bosonic string, in a manner respectful with conformal and the chiral symmetry properties of QCD and its vacuum, by adding a suitable set of Grassmann variables \( [22] \), and, further on, establish the general setting of the formalism \( [23] \). We briefly review the results obtained in this way, most notably the phenomenologically successful prediction of the \( O(p^4) \) equations of motion and the related low-energy constants \( L_1, L_2 \) and \( L_3 \) of the effective chiral Lagrangian \( [20] \). We derive next a covariant version of the results at order \( p^2 \) by coupling external gauge fields. Then we proceed to the issue of deriving the odd intrinsic parity part of the action from this approach and we immediately deduce the need of including the spin degrees of freedom of the quarks (absent in the usual effective string treatment). By doing so we obtain rather easily the anomalous part of the effective action in two dimensions \( [20] \). Finally we turn to the four dimensional case that happens to be more involved. We discuss the general formalism and introduce a set of operators (that eventually turn out to be embedded into an algebra) that implement the spin-flavor coupling. In the subsequent sections we derive the counterterms at the one and two loop level that are subjected to vanish in order to guarantee conformal invariance. We see at once that the ‘quarks’ represented by the Grassmann variables at the end of the string cannot be in a \( s = 1/2 \) angular momentum state if one requires those counterterms to vanish and that they can be interpreted as parts of equations of motion of local non-linear sigma model.

General considerations regarding the algebra satisfied by the spin-flavor coupling operators are presented.

### 1.1 Basic concepts

The hadronic string is described by the following conformal field theory action which has four dimensional Euclidean space-time as target space

\[
W_{str} = \frac{1}{4\pi\alpha'} \int d^{2+1} \sigma \left( \frac{x_\mu}{\mu} \right)^{-\epsilon} \sqrt{|g|} g_{ik} \partial_i x_\mu \partial_k x_\mu , \quad (1)
\]

where, for \( \epsilon = 0 \), in the conformal gauge \( \sqrt{|g|} g_{ik} = \delta_{ik} \) and one takes

\[
x_\mu = x_\mu(\tau, \sigma); \quad -\infty < \tau < \infty, 0 < \sigma < \infty; \quad i = \tau, \sigma; \quad \mu = 1, \ldots, 4.
\]

\(^1\)Perhaps only in a dual manner – after all there is a natural scale in QCD and as we get to shorter and shorter distances the partonic picture eventually sets in.
The conformal factor \( \varphi(\tau, \sigma) \) is introduced to restore the conformal invariance in \( 2 + \epsilon \) dimensions\(^2\). The Regge trajectory slope (related to the inverse string tension) is known to be universal \( \alpha' \simeq 0.9 \text{ GeV}^{-2} \) from the meson phenomenology\(^2\).

We would like to couple the matrix in flavor space \( U(x) \) containing the meson fields in a chiral invariant manner to the string degrees of freedom while preserving general covariance in the two-dimensional coordinates and conformal invariance under local scale transformations of the two-dimensional metric tensor. Since the string variable \( z \) does not contain any flavor dependence, we introduce two dimensionless Grassmann variables (‘quarks’) living on the boundary of the string sheet. The boundary quark fields \( \psi_L(\tau), \psi_R(\tau) \) transform in the fundamental representation of the light-flavor group \( SU(3) \). The subscripts \( L, R \) are related to the chiral spinors produced by the projectors (in what follows we use Euclidean space-time),

\[
P_L = \frac{1}{2}(1 + \gamma_5), \quad P_R = \frac{1}{2}(1 - \gamma_5), \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3.
\] (2)

A local hermitian action \( S_b = \int d\tau L^{(f)} \) is introduced on the boundary \( \sigma = 0, -\infty < \tau < \infty \) to realize the interaction with background chiral fields \( U(x(\tau)) = \exp(i\Pi(x)/f_\pi) \) where \( f_\pi \simeq 90 \text{ MeV} \), the weak pion decay constant, relates the matrix field \( \Pi(x) \) to a bunch of light pseudoscalar mesons.

The boundary Lagrangian is chosen to be reparameterization invariant and in its minimal form reads

\[
L^{(f)}_{\text{min}} = \frac{1}{2} i \left( \bar{\psi}_L U(1 - z) \dot{\psi}_R - \dot{\bar{\psi}}_L U(1 + z) \psi_R + \bar{\psi}_R U^+(1 + z^*) \dot{\bar{\psi}}_L - \dot{\psi}_R U^+(1 - z^*) \psi_L \right),
\] (3)

where a dot implies a \( \tau \) derivative. The CP symmetry under transformation,

\[
\psi \rightarrow \gamma_0\psi; \quad \psi_R \rightarrow \gamma_0\psi_L; \quad \psi_L \rightarrow \gamma_0\psi_R; \quad U \rightarrow U^*,
\] (4)

requires purely imaginary constants \( z = -z^* = \pm i|z| \).

It is easy to see that the string action \( (1) \) is classically invariant under general coordinate transformations of the two-dimensional world sheet. The fermion action is also automatically conformally invariant, because it does not contain the two-dimensional world sheet metric tensor since it can be written as a line integral.

Upon obeying conformal invariance at the quantum level, one obtains the requirement of a vanishing beta-function; in this case a beta-functional of chiral fields and their derivatives as being coupling constants of boundary string interaction. This beta-functional constraints the chiral field \( U(x) \) in order to have a consistent (i.e. conformally invariant) string propagation. They have to be interpreted as the equations of motion for the collective field \( U(x) \). Adding the requirement of locality, the corresponding effective Lagrangian is uniquely reconstructed. In what concerns the parity-even part, this procedure will be explained in some more details in section 2.

It is well known that the bosonic string is inconsistent at \( d = 4 \) and that a dependence on the conformal factor appears for non-critical dimensions. We regard this issue as collateral here, the reason being that in a covariant treatment inconsistencies appear only at the one-loop level in string perturbation theory. For an effective hadronic string of the type discussed here, this would involve going beyond the \( 1/N_c \) limit and then the exact correspondence of QCD with an string theory can be called into question anyway. In fact, nowhere in the calculation there appears any interference between the requirement of conformal invariance for the \( \Pi \)-onic background and the string trace anomaly. The matter deserves further study though.

### 1.2 Feynman rules and perturbation theory

In order to develop a perturbative expansion we expand the function \( U(x(\tau)) \) in powers of the string coordinate field \( x_{\mu}(\tau) = x_{0,\mu} + \vec{x}_{\mu}(\tau) \) around a constant \( x_0 \),

\[
U(x(\tau)) = U(x_0) + \vec{x}_{\mu}(\tau)\partial_{\mu}U(x_0) + \frac{1}{2} \vec{x}_{\mu}(\tau)\vec{x}_{\nu}(\tau)\partial_\mu\partial_\nu U(x_0) + \ldots \equiv U(x_0) + \mathcal{V}(\vec{x}).
\] (5)

\(^2\)Finally this factor becomes a dilaton degree of freedom extending the four-component hadronic string to a five-component one that however is beyond the scope of the present paper.
and look for the potentially divergent one particle irreducible diagrams. We classify them according to the number of loops. Each additional loop comes with a power of $\alpha'$. One can find a resemblance to the familiar derivative expansion of chiral perturbation theory \[25\].

The free fermion propagator is

$$
\langle \psi_R(\tau) \bar{\psi}_L(\tau') \rangle = U^\dagger(x_0) \theta(\tau - \tau') .
$$

(6)

If we impose $CP$ symmetry then

$$
\langle \psi_L(\tau) \bar{\psi}_R(\tau') \rangle = \langle \psi_R(\tau) \bar{\psi}_L(\tau') \rangle^\dagger = U(x_0) \theta(\tau - \tau') ,
$$

(7)

for unitary chiral fields $U(x)$.

The free boson propagator projected on the boundary is

$$
\langle \tilde{x}_\mu(\tau) \tilde{x}_\nu(\tau') \rangle = \delta_{\mu\nu} \Delta(\tau - \tau') , \quad \Delta(\tau \to \tau') = \Delta(0) \sim \frac{\alpha'}{\epsilon} , \quad \partial_\tau \Delta(\tau \to \tau') = 0 ,
$$

(8)

the latter results hold in dimensional regularization (see below).

The normalization of the string propagator can be inferred from the definition of the kernel of the $N$-point tachyon amplitude for the open string\[8\].

$\Delta(\tau_j - \tau_i) = -2\alpha' \ln(|\tau_j - \tau_i|\mu)$.

(9)

Keeping in mind this definition let us determine the string propagator in dimensional regularization, restricted on the boundary. First we calculate the momentum integral in $2 + \epsilon$ dimensions,

$$
\Delta_\epsilon(\tau) = \alpha' \Gamma\left(\frac{\epsilon}{2}\right) \left| \frac{\tau \mu \sqrt{\pi}}{\varphi} \right|^{-\epsilon} .
$$

(10)

This dimensionally regularized propagator should be properly normalized to reproduce \[9\]. It can be done by subtracting from \[10\] its value at $\tau \mu = 1$

$$
\Delta_\epsilon(\tau)|_{reg} = \alpha' \Gamma\left(\frac{\epsilon}{2}\right) \left\{ \left| \frac{\tau \mu \sqrt{\pi}}{\varphi} \right|^{-\epsilon} - \left| \frac{\sqrt{\pi}}{\varphi} \right|^{-\epsilon} \right\} .
$$

(11)

Therefrom one unambiguously finds the relation

$$
\Delta(0) = -\alpha' \Gamma\left(\frac{\epsilon}{2}\right) \left| \frac{\sqrt{\pi}}{\varphi} \right|^{-\epsilon} \equiv -2\alpha' \left[ \frac{1}{\epsilon} + C + \ln \varphi \right] + O(\epsilon) \equiv \Delta_\epsilon - 2\alpha' \ln \varphi ,
$$

where following the recipe of dimensional regularization we have taken $\epsilon < 0$ and hence the first term in \[11\] vanishes at $\tau = 0$.

The two-fermion, $N$-boson vertex operators are generated by the expansion \[5\], from the generating functional $Z_b = \langle \exp(i S_b) \rangle$ and Eq.\[3\]. In particular, for the $L \to R$ transition one has vertices containing $N$ derivatives of the chiral field and $N$ bosonic coordinates $\tilde{x}$

$$
V = -\frac{1}{2} \left( (1 - z) \mathcal{V}(\tilde{x}) \partial_\tau + (1 + z) \partial_\tau [\mathcal{V}(\tilde{x}) \ldots] \right) ,
$$

(13)

and for the $R \to L$ transition the Hermitian conjugated vertex $V^+$ appears.

To implement the renormalization process we perform a loop (equivalent to a derivative) expansion and proceed to determine the counterterms required to make the theory finite. This will provide a beta functional for the couplings $U(x)$ and their derivatives, which shall be required to vanish up to the two loop level in order to implement the condition of vanishing conformal anomaly. The fact that we are working with a boundary field theory makes the required calculation quite manageable. For a detailed derivation of the different Feynman diagrams we refer the reader to \[28\] and herein will report only the final expressions.
2 Summary of the parity-even sector results

In this section we summarize the main results of [23] which are derived by using the previous Feynman rules.

At one-loop, the coefficient of the single pole gives the appropriate counterterms. First we determine the fermion propagator counterterm at the one-loop order. Power counting indicates that this should be of $O(p^2)$ in the chiral expansion; i.e., two derivatives acting on the $U(x)$ field. This gives the following counterterm
\[
\delta^{(2)} U = \Delta(0) \left[ \frac{1}{2} \partial_\mu^2 U - \frac{3 + z^2}{4} \partial_\mu U U^\dagger \partial_\mu U \right],
\]
(14)
The coupling constant must be imaginary to provide the CP symmetry. Its absolute value is determined by local integrability, i.e., by the requirement that the equation $\delta U = 0$, derives from a local Weinberg action,
\[
S_W = \int d^4 x \frac{f^2}{4} \text{tr} [\partial_\mu U \partial_\nu U^\dagger]; \quad \delta^{(2)} U = -\frac{\Delta(0)}{f^2} \frac{\delta S_W}{\delta U^\dagger(x)}.
\]
(15)
The latter one constraints $z^2 = -1$. This condition also ensures the (perturbative) unitarity of the chiral field.

The next step is to consider the renormalization of the one-loop divergences in vertices with any number of “bosons” – string coordinates $x_\mu(\tau, \sigma) = 0$. Some of the divergences are removed by the $U$ redefinition we just discussed, as this automatically implies a counterterm for $\partial_\mu U$, the one-boson, two-fermion tree-level vertex. This however is not sufficient to make these vertices finite and an extension of the boundary action is needed.

The relevant counterterms can be parameterized with three bare constants $g_1$, $g_2$ and $g_3$ (which are real if CP invariance holds),
\[
\Delta L_{\text{bare}} = i \frac{\alpha'}{4} (1 - z^2) \bar{\psi}_L \left( (1 - z) g \partial_\nu U U^\dagger \partial_\nu U - (1 + z) g^* \partial_\nu U U^\dagger \partial_\nu U \\
+ zg_3 \partial_\nu U U^\dagger \partial_\nu U \right) \psi_R + \text{h.c.},
\]
(16)
where the complex constant $g$ is related to real constants from [23] as follows,
\[
g_1 = 2(\text{Reg} + \text{Im}); \quad g_2 = 2(\text{Reg} - \text{Im}).
\]
(17)
This definition will be justified after generalization of boundary action in Sec. 5. Renormalization is accomplished by redefining the couplings $g_i$ in the following way
\[
g_i = g_{i,r} - \frac{\Delta(0)}{\alpha'}.
\]
(18)
In spite of the fact that the new vertices are higher-dimensional, it turns out [23] that their contribution into the trace of the energy-momentum tensor vanishes once the requirements of CP invariance and unitarity of $U$ are taken into account and therefore conformal invariance is not broken (see [28]). One can also prove [23] that vertices with more boson legs are made automatically finite once the renormalization of $U$ and $g_i$ has been performed — this completes the renormalization program at the one-loop level.

At two loops calculations are certainly more involved, but still relatively simple since we are working in a boundary field theory. We need to consider here only the renormalization of the fermion propagator. At this order there are several contributions: genuine two-loop contributions, one-loop $U$-counterterms inserted in one-loop diagrams (both in the propagator and in the vertices), and also the counterterms we just discussed inserted in the vertices of one-loop diagrams. We do not provide detailed formulae here because in section 5 we analyze in detail a generalization of these results that take into account spin effects. The expressions that are relevant for this section can be obtained from those in section 5 by simplifying to the present case.

After interpreting the vanishing beta-function condition as an equation of motion of the (parity-even) chiral Lagrangian one unambiguously obtains the low-energy constants [25] appearing in the order $p^4$.

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3As compared to [23] here we introduce the dimensionless constants $g_i$ factorizing out the Regge slope scale $\alpha'$. 
chiral Lagrangian. They are expressed in terms of the product of the Regge trajectory slope $\alpha' \simeq 0.9$ GeV$^{-2}$, $f_s^2$ and certain rational numbers (equivalently they can be characterized by the ratio of $f_s^2$ to the hadron string tension $T = 1/2\pi\alpha'$). The unique solution is

$$g_{1, r} = -g_{2, r} = -g_{3, r} = 2; \quad g = i;$$

$$2L_1 = L_2 = -\frac{1}{2}L_3 = \frac{f_s^2\alpha'}{8} = \frac{f_s^2}{16\pi T} \simeq 10^{-3} \equiv \xi.$$  \hspace{1cm} (19)

This prediction fits quite well the phenomenological values \[29\]. The small dimensionless parameter $\xi$ is an expansion parameter of Chiral Perturbation Theory and represents a natural scale for dimension-4 structural constants. Respectively in Section 5 it will be used in the parameterization of a most general local chiral Lagrangian.

Meantime the Lagrangian \[3\] only contains intrinsic parity-even terms and does not contain any operators which can eventually entail the anomalous P-odd part of the chiral dynamics. We turn to this interesting question next in sections 4 and 5.

3 Covariant equations of motion

Let us incorporate external abelian gauge fields into the boundary chiral action \[3\]. The tree-level Lagrangian has to be translation- and time-reparameterization invariant and invariant under the gauge transformation, generated by an electric charge $Q$,

$$\psi(x) \mapsto e^{iA(x)Q}\psi(x), \quad A_\mu(x) \mapsto A_\mu(x) + Q\partial_\mu A(x),$$

$$U(x) \mapsto e^{iA(x)Q}U(x)e^{-iA(x)Q}. \hspace{1cm} (21)$$

Thus, in principle, the boundary Lagrangian can be constructed with the help of the covariant derivative projected on the boundary,

$$\dot{x}_\mu (\partial_\mu - iA_\mu(x)) = \partial_\tau - i\dot{x}_\mu A_\mu \Rightarrow e^{iA(x)Q} (\partial_\tau - i\dot{x}_\mu A_\mu) e^{-iA(x)Q}. \hspace{1cm} (22)$$

$$\mathcal{L} = \frac{i}{2} \bar{\Psi}_L \left\{ (1 - z)U(x)(\partial_\tau - i\dot{x}_\mu A_\mu) + (1 + z)(\partial_\tau - i\dot{x}_\mu A_\mu)U(x) \right\} \Psi_R + h.c. \hspace{1cm} (23)$$

However it turns out that for such a Lagrangian the corresponding fermion propagator is not gauge invariant, rather being bilocally covariant. As a consequence, the divergences do not form a gauge covariant combination and one ends up with equations of motion that do not derive from a manifestly gauge invariant Lagrangian. One has to proceed to the fermion fields dressed with the Dirac string phase factor so that the Lagrangian is written as

$$\mathcal{L} = \frac{i}{2} \bar{\Psi}_L \left\{ (1 - z)\bar{U}(x)(\partial_\tau - i\dot{x}_\mu \Delta A_\mu^\perp) + (1 + z)(\partial_\tau - i\dot{x}_\mu \Delta A_\mu^\perp)\bar{U}(x) \right\} \Psi_R + h.c. \hspace{1cm} (24)$$

Herein we redistribute the e.m. interaction between dressed fermions ($\Psi = e^{-i\varphi(x)||Q}\psi$), chiral fields

$$U(x) \to \bar{U}(x) = e^{-i\varphi(x)||Q}U(x)e^{i\varphi(x)||Q}; \hspace{1cm} (25)$$

and the covariant derivative. $\varphi_\parallel$ is defined as

$$\varphi_\parallel = \bar{x}_\mu(\tau)A_\mu(x_0) + \sum_{n=1}^{\infty} \frac{1}{(n + 1)!} \bar{x}_\mu(\tau)\bar{x}_{\nu_1}(\tau)\bar{x}_{\nu_2}(\tau)\cdots\bar{x}_{\nu_n}(\tau)\partial_{\nu_1}\cdots\partial_{\nu_n}A_\mu(x_0); \hspace{1cm} (26)$$

whereas the remaining transversal part of the covariant derivative reads

$$\Delta A_\mu^\perp = \sum_{n=1}^{\infty} \frac{n}{(n + 1)!} \bar{x}_{\nu_1}(\tau')\cdots\bar{x}_{\nu_n}(\tau')\partial_{\nu_1}\cdots\partial_{\nu_{n-1}}F_{\nu_\mu}(x_0). \hspace{1cm} (27)$$
Now in order to control the conformal symmetry we expand \( \bar{U}(x) \) around a constant background \( x_0 \) and look for the potentially divergent, one particle irreducible diagram:

\[
\bar{U}(x) = (1 - i\varphi \frac{1}{2}\vec{\varphi}^2 + \cdots) U(x_0) + \bar{x}_\mu \partial_\mu U(x_0) + \frac{1}{2} \bar{x}_\mu \bar{x}_\nu \partial_\mu \partial_\nu U(x_0) + \cdots \\
\times (1 + i\varphi \frac{1}{2}\vec{\varphi}^2 + \cdots)
\]

\[
= U(x_0) + \bar{x}_\mu D_\mu U(x_0) + \frac{1}{2} \bar{x}_\mu \bar{x}_\nu D_\mu D_\nu U(x_0) + \cdots
\]

(28)

where the covariant derivative acts in the adjoint representation

\[
D_\mu U \equiv [D_\mu, U] = \partial_\mu U - i[A_\mu, U].
\]

Using the perturbation expansion and vertex operators from Eq. (28) one arrives to the following expression:

\[
\frac{1}{4} U(x_0)^\dagger D_\mu U(x_0) U(x_0)^\dagger D_\mu U(x_0) U(x_0)^\dagger \{ (3 + z^2)\Delta(0) + (1 - z^2)\Delta(\tau_A - \tau_B) \} \Theta(\tau_A - \tau_B)
\]

\[
-\frac{1}{2} \Delta(0) \Theta(\tau_A - \tau_B) U(x_0)^\dagger D_\mu D_\mu U(x_0) U(x_0)^\dagger.
\]

(29)

The divergence is eliminated by introducing an appropriate counterterm

\[
\delta U = \frac{1}{2} \Delta(0) \left\{ D_\mu^2 U - \left( \frac{3 + z^2}{2} \right) D_\mu U U^\dagger D_\mu U \right\} = 0,
\]

(30)

and conformal symmetry is restored if \( \delta U \) vanishes as before. The value \( z^2 = -1 \) provides the integrability of this chiral dynamics, i.e. its origin from the local gauged Weinberg Lagrangian. We see that the dressed field Lagrangian produces the gauge invariant chiral dynamics which is determined unambiguously at one-loop level. The extension of the covariantization to the \( p^4 \) terms is in progress.

## 4 Two-dimensional QCD and the WZW term

The chiral bosonization of hadronic string presented in previous sections is certainly incomplete as it does not include any quark spin degrees of freedom and therefore does not generate parity-odd chiral dynamics in the form of the chiral anomaly in the equations of motion and the Wess-Zumino-Witten chiral Lagrangian. To understand the way parity-odd terms could emerge from the hadronic string built over the chirally broken QCD vacuum we investigate the toy model of two-dimensional QCD.

While a parity-even chiral-field interaction on the line may be qualitatively associated with vector quark currents a parity-odd interaction must have relation to axial-vector currents. However in QCD, in fact, vector and axial-vector fields couple to quarks with the same matrix vertex. Indeed, in two (Euclidean) dimensions the structure of Dirac \( \gamma \) matrices (in terms of the Pauli matrices \( \sigma_a \)),

\[
\gamma_0 = \sigma_1; \gamma_1 = \sigma_2; \gamma_2 \text{ (analog of } ''\gamma_5'' \text{)} = \sigma_3 = -i\gamma_0\gamma_1,
\]

allows to relate axial-vector and vector vertices as follows,

\[
\gamma_\mu \gamma_2 = i\epsilon_{\mu\nu} \gamma_\nu,
\]

(31)

in terms of antisymmetric tensor \( \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \epsilon_{01} = 1 \). Meantime the \( O(2) \) algebra is generated by \( \sigma_{\mu\nu} \equiv -\frac{1}{2}i[\gamma_\mu, \gamma_\nu] = \epsilon_{\mu\nu} \gamma_2 \).

Accordingly the boundary Lagrangian may equally well include two types of couplings,

\[
L^{(f)} \equiv \frac{1}{2} \left\{ \bar{\psi}_L \left[ \left\{ \partial_\tau, U \right\} + \tilde{F}_{\mu\nu} \bar{x}_\mu \partial_\nu U \right] \psi_R + \bar{\psi}_R \left[ \left\{ \partial_\tau, U^\dagger \right\} - \tilde{F}_{\mu\nu}^\dagger \bar{x}_\mu \partial_\nu U^\dagger \right] \bar{\psi}_L \right\};
\]

\[
\tilde{F}_{\mu\nu} \equiv z\delta_{\mu\nu} + igA\epsilon_{\mu\nu}.
\]

(32)
The CP symmetry \( z = -z^* \); \( g_A = g_A^* \); \( \tilde{F}_{\mu\nu} = -\tilde{F}_{\mu\nu}^\dagger \).

(33)

Now we develop string perturbation theory expanding the function \( U(x) \) in powers of the string coordinate field \( x_\mu(\tau) = x_{0,\mu} + \tilde{x}_\mu(\tau) \), then expanding the boundary action in powers of \( \tilde{x}_\mu(\tau) \) and finally looking for divergences, i.e. violations of conformal symmetry. At one loop one obtains the following condition to preserve conformal symmetry,

\[
-\partial_\mu^2 U + \frac{1}{2}(3 + z^2 - g_A^2)\partial_\mu U U^\dagger \partial_\mu U - ig_A \varepsilon_{\mu\nu} \partial_\mu U U^\dagger \partial_\nu U = 0.
\]

(34)

Unitarity of chiral fields and local integrability of Eqs. of Motion constrains the coupling constants to fulfill the relation \( g_A^2 - z^2 = 1 \). The naive QCD value (if we trust the arguments presented in Appendix A) is \( g_A = 1 \). This choice \( (z = 0, g_A = 1) \) corresponds to the correct value \( \beta \) of the dim-2 anomaly (last term in \( (34) \)). Thus in QCD2 the hadron string induces the WZW action from the vanishing the boundary \( \beta \) function already at one-loop level.

In turn, in QCD3 the anomaly and the WZW action have dimension 4 and 5 respectively and therefore they are generated by cancellation of two-loop divergences. Therefore the antisymmetric tensor \( \varepsilon_{\mu\nu\rho\lambda} \) in anomalies must arise from the algebra of \( \tilde{F}_{\mu\nu} \) matrices.

On the other hand, the boundary quark fields \( \psi_L(\tau), \psi_R(\tau) \) are, in fact, one-dimensional and one should not expect that they realize the fundamental, spin-1/2 representation of the Poincare group. This is because the projection on a line is not uniquely defined (see Appendix A) and to correct this projection consistently with the conformal symmetry and integrability we eventually have to introduce a more complicated algebra than the conventional Clifford one. Certain arguments in favor of this extension will be given in the next Section.

5 General formalism: renormalization at the one- and two-loop order

In the next sections we are going to translate the ideas above to the four dimensional case, develop the equations up to two loops, and try to set a general framework for the search of solutions satisfying unitarity of the \( U \) matrices and CP invariance. The goal of this section is to ensure the renormalizability at one- and two-loops of our model.

5.1 One loop fermion propagator: A first guess.

The starting point in this section is the following Lagrangian on the boundary for the fermions \( \psi_L \) and \( \psi_R \), analogous to Eq. (32).

\[
L^{(f)} = \frac{1}{2} \left\{ \bar{\psi}_L \left[ \partial_\tau, U \right] + F_{\mu\nu} \hat{x}_\mu \partial_\nu U \right\} \psi_R + \bar{\psi}_R \left[ \left[ \partial_\tau, U^\dagger \right] - F_{\mu\nu}^\dagger \hat{x}_\mu \partial_\nu U^\dagger \right] \psi_L ;
\]

\[
F_{\mu\nu} = z\delta_{\mu\nu} + g_\sigma \sigma_{\mu\nu} .
\]

(35)

The interaction term proportional to the \( F_{\mu\nu} \) encodes the spin degrees of freedom of the fermion variables once projected to the line (boundary of the string) where the fermions live. This is why, as a first choice, we have included \( \sigma_{\mu\nu} \) in analogy to its two-dimensional partner \( \tilde{F}_{\mu\nu} \).

Following the procedure explained in Sec. 2 for the parity even part, and following the rules of Sec. 1.1 and 1.2 we will expand the \( U(x(\tau)) \) in powers of the coordinate fields \( x_\mu(\tau) = x_{0,\mu} + \tilde{x}_\mu(\tau) \). This will bring us a variety of operators; some vertices will contain \( F_{\mu\nu} \) (those coming from the expansion of the second term in the Lagrangian) and some will not (those coming from the first term in the Lagrangian). We will perform all computations with this expanded Lagrangian.
After computation of the diagrams we see that the divergence in the $R \rightarrow L$ part of the fermion propagator\(^4\) at one loop takes the form

$$O_1 = \frac{1}{4} \theta(A - B) \Delta(0) \left[-2 \partial^2 U + (3 \delta_{\alpha \beta} + F_{\beta \alpha} - F_{\alpha \beta} + F_{\alpha \mu} F_{\beta \mu}) \partial_\alpha U U^\dagger \partial_\beta U\right]. \quad (36)$$

We can compare this equation with Eq. (14). Borrowing the ideas from the two-dimensional case and motivated by the discussion in Appendix A we write (35).

Applying this definition to the one loop propagator we see that there appear two different channels. One channel related with the trace of the $O_1$, and another channel defined by its traceless part. Let us compute them separately.

On one side we can perform a trace in spinor space and find

$$\frac{1}{2} \text{tr} [O_1] = \frac{1}{4} \theta(A - B) \left[-2 \partial^2 U + (3 + z^2 + 3 g_\sigma^2) \partial_\mu U U^\dagger \partial_\mu U\right] \quad (37)$$

where the identity

$$\sigma_{\mu \nu} \sigma_{\nu \rho} = 3 \delta_{\mu \rho} - 2 i \sigma_{\mu \rho} \quad (38)$$

has been used. We can recover the already known result of [23] by making $g_\sigma = 0$.

The divergence in Eq. (37) is eliminated by introducing an appropriate counterterm $U \rightarrow U + \delta U$

$$\delta U = \Delta(0) \left[\frac{1}{2} \partial_\mu^2 U - \frac{3 + z^2 + 3 g_\sigma^2}{4} \partial_\mu U U^\dagger \partial_\mu U\right]. \quad (39)$$

Conformal symmetry is restored (the $\beta$-function is zero) if the above contribution vanishes. The unitarity of $U$ is compatible with the conformal symmetry saturation condition only if we demand

$$z^2 + 3 g_\sigma^2 = -1. \quad (40)$$

When taking into account the CP symmetry condition $z = -z^*$, $g_\sigma = -g_\sigma^*$ one find the following bounds on these couplings constants

$$0 \leq |z| \leq 1,$$

$$\frac{1}{\sqrt{3}} \geq |g_\sigma| \geq 0, \quad (41)$$

and if the ein-bein projector gives a correct hint (see Appendix A) then $|z| = |g_\sigma| = 1/2$.

Let’s explore now the other channel, i.e. the traceless part of $\delta U$. The latter is, in this case, the part proportional to $\sigma_{\mu \nu}$, thus

$$\tilde{O}_1 = \theta(A - B) i \frac{1}{2} \Delta(0) \left\{ -i g_\sigma - g_\sigma^2 \right\} \sigma^{\mu \nu} \partial_\mu U U^\dagger \partial_\nu U \quad (42)$$

This is a completely new term not observed before which comes directly from the inner space in $F_{\mu \nu}$.

We remark that for the $L \rightarrow R$ propagation the divergence is just complex conjugated, i.e. has the coefficient $\left\{ i g_\sigma^* + (g_\sigma^*)^2 \right\}$. This fixes $g_\sigma = 0, -i$ in order to make zero the non-scalar part. Both choices seem to be unacceptable because for $g_\sigma = 0$ one does not reproduce the Wess-Zumino action and for $g_\sigma = -i$ one cannot provide the vanishing $\beta$-function in the scalar channel for the CP invariant choice [33]. After this negative result we must accept that this, most intuitive, choice of $F_{\mu \nu}$ is not convenient for our purposes.

### 5.2 One loop fermion propagator: General form.

At one loop we have been already able to see the incompatibility of the guess \(^{35}\) with the unitarity and CP conditions for the model. At this point we must generalize our strategy allowing for more general

\(^{4}\)From now on we focus our analysis on the divergences in the $R \rightarrow L$ part of the fermion propagator having in mind that the $L \rightarrow R$ part is reproduced by means of hermitian conjugation.
forms of $F_{\mu\nu}$. This will make the spin content not well defined since general $F_{\mu\nu}$ can follow a more complicated algebra than the Clifford one. Exactly as in the previous guess, here we must consider that the $F_{\mu\nu}$ acts on an internal spinor space. This requires some care in the computations in order to keep the right ordering of the $F_{\mu\nu}$ ’s. This will be crucial in the renormalization process.

In this framework we recover the original Lagrangian leaving $F_{\mu\nu}$ unspecified for the time being. The complete one loop contribution to the fermion propagator in its general form reads.

\[
\frac{1}{2} \theta(A-B) \Delta(0) \{-\bar{U} \rho \partial^2 U U^\dagger + \frac{1}{2} U^\dagger \partial_\sigma U U^\dagger \partial_\lambda U U^\dagger \} [3 \delta_{\sigma\lambda} - (F_{\sigma\lambda} - F_{\lambda\sigma}) + F_{\sigma\gamma} F_{\lambda\gamma}] + \frac{1}{4} \theta(A-B) \Delta(A-B) U^\dagger \partial_\sigma U U^\dagger \partial_\lambda U U^\dagger \} \]

From this we can identify the general condition that unitarity of the \(U\) imposes,

\[
\delta_{\sigma\lambda} - F_{\sigma\lambda} + F_{\lambda\sigma} + F_{\sigma\gamma} F_{\lambda\gamma} = 0. \tag{44}
\]

This relation is a first constraint for the algebra we have alluded to.

The next step in the program is to compute the divergent part of one loop vertex with one boson and two fermions legs, compute the counterterms needed and try to see whether they are sufficient to renormalize all $n$-boson two fermion vertices.

### 5.3 One loop contribution to two-fermion one-boson vertex

In what follows we are going to compute the one loop contribution to the one boson two fermions vertex following the same lines as in Sec.2. The one-loop diagrams considered are the same considered in [23], with the additional $F_{\mu\nu}$ structure.

All contributions to this vertex have been summarized in a table contained in Appendix B. In this table we separate the different structures in derivatives of the $U$ matrices, and we focus on their $F$ structure.

The next step will be to use the relations

\[
\bar{x}_\rho(A) \theta(A-B) = - \int d \tau \partial_\tau \theta(A - \tau) \bar{x}_\rho(\tau) \theta(\tau - B),
\]

\[
x_\rho(B) \theta(A-B) = \int d \tau \theta(A - \tau) \bar{x}_\rho(\tau) \partial_\tau \theta(\tau - B),
\]

to convert the divergence in the one-boson vertex into a tree level contribution in the Lagrangian. The vertex operators extracted from this tree level expression are, of course, directly related to the counterterms we are looking for and they read,

\[
\int d \tau \frac{i}{2} \{ \bar{\psi}_L \bar{x}_\rho(\tau) \Phi^{(1)}_\rho \psi_R - \bar{\psi}_L \bar{x}_\rho(\tau) \Phi^{(2)}_\rho \psi_R \}, \tag{45}
\]

where,

\[
\Phi^{(1)}_\rho = -(\delta_{\rho\eta} - F_{\rho\eta}) \partial_\eta \delta U
\]

\[
- \Delta(0) \{ \frac{1}{4} \partial_\eta \partial_\sigma U U^\dagger \partial_\lambda U ((\delta_{\sigma\gamma} - F_{\sigma\gamma})(\delta_{\eta\rho} + F_{\eta \rho}) (\delta_{\lambda\gamma} - F_{\lambda\gamma}) - [F_{\eta \rho}, F_{\sigma\lambda}](\delta_{\lambda\gamma} - F_{\lambda\gamma})
\]

\[
- \frac{1}{4} \partial_\sigma U U^\dagger \partial_\lambda \partial_\eta U ((\delta_{\sigma\gamma} + F_{\sigma\gamma})(\delta_{\eta\rho} - F_{\eta \rho}) (\delta_{\lambda\gamma} + F_{\lambda\gamma}) - (\delta_{\sigma\gamma} + F_{\sigma\gamma})[F_{\lambda\gamma} , F_{\eta \rho}])
\]

\[
+ \frac{1}{2} \partial_\sigma U U^\dagger \partial_\delta U U^\dagger \partial_\lambda U (\delta_{\rho\sigma} F_{\sigma\lambda} + F_{\sigma\lambda}) - \delta_{\sigma\lambda} F_{\eta \rho} - F_{\sigma\gamma} F_{\eta \rho} F_{\mu \gamma} + \frac{1}{2} [F_{\eta \rho}, F_{\sigma\lambda}](\delta_{\lambda\gamma} - F_{\lambda\gamma})\}
\]

\[
\equiv -(\delta_{\rho\eta} - F_{\rho\eta}) \partial_\eta \delta U - \phi_\rho,
\]
\[
\Phi^{(2)}_{\rho} = -(\delta_{\eta\rho} + F_{\eta\rho}) \partial_\eta \delta U + \Delta(0)(\frac{1}{4} \partial_\eta \partial_\sigma U U^\dagger \partial_\eta U ((\delta_{\sigma\gamma} - F_{\sigma\gamma})(\delta_{\eta\rho} + F_{\eta\rho}))(\delta_{\lambda\gamma} - F_{\lambda\gamma}) - [F_{\eta\rho}, F_{\sigma\gamma}](\delta_{\lambda\gamma} - F_{\lambda\gamma}))
\]

\[
+ \frac{1}{4} \partial_\eta U U^\dagger \partial_\lambda \partial_\eta U ((\delta_{\sigma\gamma} + F_{\sigma\gamma})(\delta_{\eta\rho} - F_{\eta\rho}))(\delta_{\lambda\gamma} + F_{\lambda\gamma}) - (\delta_{\sigma\gamma} + F_{\sigma\gamma})[F_{\lambda\gamma}, F_{\eta\rho}])
\]

\[
+ \frac{1}{2} \partial_\eta U U^\dagger \partial_\eta U U^\dagger \partial_\lambda U (\delta_{\eta\rho} (F_{\sigma\lambda} + F_{\lambda\sigma}) - \delta_{\sigma\lambda} F_{\eta\rho} - F_{\sigma\gamma} F_{\eta\rho} F_{\lambda\gamma} + \frac{1}{2} [F_{\eta\rho}, F_{\sigma\gamma}](\delta_{\lambda\gamma} - F_{\lambda\gamma}))
\]

\[
= -(\delta_{\eta\rho} + F_{\eta\rho}) \partial_\eta \delta U + \phi_\rho .
\]

Herein the following relation (induced from Eq. (44)),

\[
[F_{\eta\rho}, F_{\sigma\gamma}](\delta_{\lambda\gamma} - F_{\lambda\gamma}) = (\delta_{\sigma\gamma} + F_{\sigma\gamma})[F_{\eta\rho}, F_{\lambda\gamma}] ,
\]

has been used in order to make CP invariance manifest.

In these equations we have already separated two important parts. The first part is a first variation of \( U \) in the Lagrangian’s interaction part, so it is already under control. The remainder of \( \Phi^{(i)}_\rho \) is in fact the same in both \( i = 1 \) and \( 2 \) (only a sign makes a difference). Denoting the remainder as \( \phi_\rho \) and putting all together, one finds that the divergence is generated by the operators

\[
\int d\tau \left[ \frac{i}{2} \bar{\psi}_L \{ \bar{x}_\rho (\partial_\tau, \partial_\sigma) \psi_R + \frac{i}{2} \bar{\psi}_L \dot{x}_\rho (\tau) F_{\eta\rho} \partial_\eta (-\delta U) \psi_R \right] + \int d\tau \left[ \frac{i}{2} \bar{\psi}_L \dot{x}_\rho (\tau) \phi_\rho \psi_R \right]
\]

The two first terms are already taken care of by the \( \delta U \) counterterm, while the last one is dictating us the counterterms to introduce to guarantee the finiteness of this vertex. Obviously, if we set \( F_{\mu\nu} = 0 \) we recover the results of the spinless case.

### 5.4 Counterterms

In order to compensate the divergence in \( \phi_\rho \) we have to employ the counterterm

\[
\alpha' \int d\tau \frac{i}{2} \bar{\psi}_L \dot{x}_\rho \bar{\phi}_\rho \psi_R ,
\]

where the rescaling on the dimensional constant \( \alpha' \) has been introduced to simplify some algebraic relations that follow. The structure of \( \bar{\phi}_\rho \) is essentially determined by \( \phi_\rho \) and can be codified as follows

\[
\bar{\phi}_\rho = \partial_\eta \partial_\rho U U^\dagger \partial_\lambda U A^{(1)}_{\sigma\eta\lambda\rho} + \partial_\eta U U^\dagger \partial_\lambda U A^{(2)}_{\sigma\eta\lambda\rho} + \partial_\eta U U^\dagger \partial_\eta U U^\dagger \partial_\lambda U A^{(3)}_{\sigma\eta\lambda\rho} .
\]

Evidently the operator coefficients \( A^{(1)} \) and \( A^{(2)} \) are symmetric in a pair of indices,

\[
A^{(1)}_{\sigma\eta\lambda\rho} = A^{(1)}_{\eta\sigma\lambda\rho} ; \ \ A^{(2)}_{\sigma\eta\lambda\rho} = A^{(2)}_{\sigma\eta\lambda\rho} ,
\]

being contracted with symmetric chiral field tensors. One can find the similarities of the counterterm with its counterpart of Sec. 2 when \( F_{\mu\nu} \) reduces to \( 2\delta_{\mu\nu} \). In the present case the terms attached to each chiral field \( U \) structure are considerably more complex. The operator nature of \( F_{\mu\nu} \) is a reason for a larger set of coupling constants in the operator coefficients \( A^{(i)}_{\sigma\eta\lambda\rho} \)’s. As before the finite, renormalized part of this larger set of constants is to be determined by the consistency equations of vanishing beta functions as well as of local integrability of dimension-4 components of Eqs. of motion. Let us present
where the actual form of \( A^{(i)}_{\sigma_\eta_\lambda_\rho} \) more explicitly

\[
A^{(1)}_{\sigma_\eta_\lambda_\rho} = A^{(1,r)}_{\sigma_\eta_\lambda_\rho} + \frac{\Delta(0)}{8\alpha'} ((\delta_\sigma - F_\sigma_\gamma)(\delta_\eta_\nu + F_\eta_\nu)(\delta_\lambda_\gamma - F_\lambda_\gamma) - [F_\eta_\nu, F_\sigma_\gamma](\delta_\lambda_\gamma - F_\lambda_\gamma) + \{\sigma \leftrightarrow \eta\}) \\
= \left(\frac{1}{8}\right) \left( g^{(r)} - \frac{\Delta(0)}{\alpha'} \right) ((\delta_\sigma + F_\sigma_\gamma)(\delta_\eta_\nu - F_\eta_\nu)(\delta_\lambda_\gamma + F_\lambda_\gamma) - (\delta_\sigma + F_\sigma_\gamma)[F_\lambda_\gamma, F_\eta_\nu] + \{\eta \leftrightarrow \lambda\}) \\
A^{(2)}_{\sigma_\eta_\lambda_\rho} = A^{(2,r)}_{\sigma_\eta_\lambda_\rho} + \frac{\Delta(0)}{8\alpha'} ((\delta_\sigma + F_\sigma_\gamma)(\delta_\eta_\nu - F_\eta_\nu)(\delta_\lambda_\gamma + F_\lambda_\gamma) - (\delta_\sigma + F_\sigma_\gamma)[F_\lambda_\gamma, F_\eta_\nu] + \{\eta \leftrightarrow \lambda\}) \\
A^{(3)}_{\sigma_\eta_\lambda_\rho} = A^{(3,r)}_{\sigma_\eta_\lambda_\rho} - \frac{\Delta(0)}{4\alpha'} (\delta_\eta_\nu(F_\sigma_\lambda + F_\lambda_\sigma) - \delta_\sigma_\lambda F_\eta_\nu - F_\sigma_\gamma F_\eta_\nu F_\lambda_\gamma + \frac{1}{2}[F_\eta_\nu, F_\sigma_\gamma](\delta_\lambda_\gamma - F_\lambda_\gamma)) \\
= \frac{1}{4} \left( g^{(r)} - \frac{\Delta(0)}{\alpha'} \right) ((\delta_\eta_\nu(F_\sigma_\lambda + F_\lambda_\sigma) - \delta_\sigma_\lambda F_\eta_\nu - F_\sigma_\gamma F_\eta_\nu F_\lambda_\gamma + \frac{1}{2}[F_\eta_\nu, F_\sigma_\gamma](\delta_\lambda_\gamma - F_\lambda_\gamma)) ,
\]

where \( A^{(i,r)}_{\sigma_\eta_\lambda_\rho} \) are renormalized operators which depend on all finite parameters we referred above. As compared to Sec.2 these expressions contain the three similar constants \( g^{(r)}, \bar{g}^{(r)}, g_3^{(r)} \) but a more complicated algebraic structure.

The actual composition of the \( A^{(i,r)}_{\sigma_\eta_\lambda_\rho} \) is just a sum of products of the algebra elements \( F_{\mu_\rho} \) with independent finite constants. We follow a minimal renormalization scheme and restrict the form of the \( A^{(i,r)}_{\sigma_\eta_\lambda_\rho} \) by adopting only the same \( F \) combinations which appear in the corresponding infinite part.

We notice also that CP invariance of the Lagrangian imposes the relations

\[
A^{(1)}_{\sigma_\eta_\lambda_\rho} = -A^{(2)}_{\lambda_\sigma_\eta_\rho} , \quad A^{(3)}_{\sigma_\eta_\lambda_\rho} = -A^{(3)}_{\lambda_\eta_\sigma_\rho} ,
\]

while the CP invariance of the divergent part holds manifestly due to the Eq.\((54)\), it is the CP invariance of the renormalized part that we are interested in. This condition applied to the parameterization \((52)\) dictates that,

\[
\bar{g}^{(r)} = -(g^{(r)})^* , \quad g_3^{(r)} = (g_3^{(r)})^* ,
\]

hence we end up with three real variables \( \text{Re} \ g^{(r)}, \text{Im} \ g^{(r)} \) and \( g_3^{(r)} \) as in the scalar Lagrangian \((10)\).

Now one must examine the two-boson two-fermion vertex in order to prepare the two-loop renormalization of the fermion propagator. We do not display this part of the computation since it does not bring new counterterms and the algebraic expressions are rather cumbersome. All divergences in the one-loop two-boson two-fermion vertex are proven to be renormalized with the one-boson two-fermion vertex counterterms. Thereby by translational invariance \((24)\) all one-loop divergences in all n-boson two-fermion vertex are also entirely renormalized. Thus the renormalization program at one loop is completed. The inclusion of one-boson two-fermion counterterms \((50), (52)\) is sufficient to ensure the complete renormalization at one loop.

### 5.5 Dimension-4 divergences from one-loop counterterms and from two-loop contributions

There are ten two-loop one-particle irreducible diagrams which are listed in \((28)\). The divergences in the propagator at two-loops can be separated into five separate pieces

\[
\theta(A - B)[dt + dl_1 + dl_3 + dl_4] + dv[B].
\]

The first and second piece contain the double pole divergence \( \Delta^2(0) \), the third, fourth and fifth pieces contain the single pole divergence \( \Delta(0) \).

The piece \( dt \) represents ‘the second variation’, or one-loop divergence in the one-loop divergence and it is removed by the one-loop renormalization, hence it vanishes together with the one-loop \( \beta \)-function, \textit{i.e.} when the equations of motion are imposed.
The second part represents the remaining terms of order $\Delta^2(0)$ in two loop diagrams after subtraction of $d_I$. This part is made of the contributions generated by the one-boson line, after its insertion in a one-loop diagram.

$d_{II}$ contains those single-pole divergences, proportional to $\Delta(0)$, which are removed once the one-loop renormalization of $U$ in the finite nonlocal part of the fermion propagator at one loop is taken into account.

The inclusion of the counterterms modifies in fact the fermion propagator adding terms of higher order in derivatives (of dimension 4 in the count of Chiral Perturbation Theory). Eventually the genuine divergences that can contribute to the beta function (single poles). It must therefore be added of the counterterms when introduced in one-loop diagrams.

$\alpha$ genuine divergences linear in $\Delta(0)$ which come from the double integral in irreducible two-loop diagrams with maximal $\beta\mu$. Relevant part of arising overlapping divergences.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
Coefficient & U structure & F structure \\
\hline
$+\frac{1}{2\alpha'}\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\rho UU^\dagger\partial_\mu\partial_\nu UU^\dagger$ & $0$ \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\rho UU^\dagger\partial_\mu\partial_\nu UU^\dagger$ & $-2A^{1,r}_{\sigma\alpha\lambda\mu}(\delta_{\beta\mu} - F_{\beta\mu})$ \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\rho UU^\dagger\partial_\mu\partial_\nu UU^\dagger$ & $+2(\delta_{\alpha\mu} + F_{\alpha\mu})A^{2,2}_{\beta\mu\lambda\nu}\delta_{\beta\mu} - \delta_{\beta\mu}$ \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\lambda UU^\dagger\partial_\rho\partial_\nu UU^\dagger$ & $2((\delta_{\alpha\mu} + F_{\alpha\mu})A^{2,2}_{\beta\rho\lambda\nu}\lambda_{\beta\mu} - \delta_{\beta\mu})$ \\
\hline
\end{tabular}
\caption{Divergent contribution to the propagator from the vertex counterterms.}
\end{table}

In $d_{IV}$ we include the divergences that are eliminated when the additional counterterms in the one-boson vertices (those proportional to $A^{(4)}$) are included in the finite part of the one-loop fermion propagator. One can check (in a way similar to [23]) that all terms in the two loop fermion propagator linear in $\Delta(0)$ and in $\Delta(A, B)$ belong either to $d_{II}$ or to $d_{IV}$.

Finally, some single-pole divergences remain and they are gathered in $d_I$. Namely, there are divergences linear in $\Delta(0)$ which come from the double integral in irreducible two-loop diagrams with maximal number of vertices (overlapping divergences),

$$J(A - B) = \int_B^A d\tau_1 \int_B^A d\tau_2 \partial_\tau_1 \Delta(\tau_1 - \tau_2) \partial_\tau_2 \Delta(\tau_1 - \tau_2) = 2\alpha'\Delta(0) + \text{finite part.} \quad (56)$$

These operators are described in the table.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
Coefficient & U structure & F structure \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\rho UU^\dagger\partial_\mu\partial_\nu UU^\dagger$ & $(F_{\alpha\beta}F_{\beta\alpha} - F_{\alpha\lambda}F_{\lambda\beta} + \{\alpha \leftrightarrow \sigma, \lambda \leftrightarrow \beta\})$ \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\rho UU^\dagger\partial_\mu\partial_\nu UU^\dagger$ & $(\delta_{\alpha\mu}F_{\alpha\beta}F_{\beta\mu} + (\alpha \leftrightarrow \sigma) \lambda \leftrightarrow \beta)\}$ \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\rho UU^\dagger\partial_\mu\partial_\nu UU^\dagger$ & $-\delta_{\alpha\mu}(\delta_{\alpha\mu} - F_{\alpha\mu})\delta_{\beta\mu} + (\alpha \leftrightarrow \sigma)$ \\
\hline
$+\frac{1}{2}\alpha'\theta(A-B)\Delta(0)$ & $U^\dagger\partial_\sigma\partial_\lambda UU^\dagger\partial_\rho\partial_\nu UU^\dagger$ & $-((\delta_{\alpha\beta} - F_{\alpha\beta})(\delta_{\sigma\gamma} + F_{\beta\gamma})\delta_{\lambda\mu} - (\delta_{\alpha\mu} - F_{\alpha\mu})\delta_{\beta\mu} + (\lambda \leftrightarrow \beta)$ \\
\hline
\end{tabular}
\caption{Relevant part of arising overlapping divergences.}
\end{table}

The terms in $d_I$ survive after adding all the counterterms and together with Table 1 are the only new genuine divergences that can contribute to the beta function (single poles). It must therefore be added to the equation of motion at the next order in the $\alpha'$ expansion. Thus two sets of operators listed in Tables 1 and 2 form the genuine contribution $U^\dagger\delta^{(4)}UU^\dagger$ of chiral dimension 4 into the beta-functional of the chiral field renormalization. To this order the condition of conformal invariance reads

$$\delta^{(2)}U + \delta^{(4)}U = 0, \quad (57)$$

and this equation must be identified with an equation of local chiral dynamics if we deal with the Goldstone boson physics of pseudoscalar mesons. However such an identification is not unique as one may have certain terms in $\delta^{(4)}U$ vanishing on the mass-shell $\delta^{(2)}U = 0$. This is a logic of Chiral Perturbation Theory. Therefore in the comparison of the beta-functional $\delta^{(4)}U$ and a relevant functional of local chiral dynamics one must include all possible operators vanishing on-shell.
6 Local integrability of dimension-4 part of Eqs. of motion

If the corresponding terms with four derivatives that we have found in the previous section originate from a dimension-four operators in a quasi-local effective Lagrangian then certain constraints are to be imposed on the constants $A^{\mu}_{\nu}$. On mass-shell such a Lagrangian has only three terms compatible with the chiral symmetry if we impose the dimension-two equations of motion (14), which are in principle acceptable are reduced to the set (58) with the help of integration by parts in the action and of the dimension-two equations of motion (14) (on-shell conditions).

Variation of the previous Lagrangian gives the following addition to the equations of motion,

$$
\frac{1}{f^2} \frac{\delta S^{(4)}}{\delta U} = -\frac{\alpha'}{8} U^\dagger \left[ 4K_1 \left[ \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma U + \partial_\mu \partial_\nu \partial_\rho U^\dagger U \partial_\sigma \right] - \partial_\mu U^\dagger \partial_\nu U \partial_\rho U \partial_\sigma \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma U \\
+ \partial_\mu U^\dagger \partial_\nu U \partial_\rho U \partial_\sigma \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma U \\
+ 2K_2 \left[ \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma U + \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma \partial_\mu \partial_\nu \partial_\rho U \partial_\sigma \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma U \\
+ \partial_\mu U^\dagger \partial_\nu \partial_\rho U \partial_\sigma \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma \partial_\mu \partial_\nu \partial_\rho U^\dagger \partial_\sigma U \\
- 2\partial_\mu U^\dagger \partial_\nu \partial_\rho U^\dagger \partial_\sigma U \partial_\mu \partial_\nu \partial_\rho U \partial_\sigma U \\
- 3\partial_\mu U^\dagger \partial_\nu \partial_\rho U^\dagger \partial_\sigma U \partial_\mu \partial_\nu \partial_\rho U \partial_\sigma U \partial_\mu \partial_\nu \partial_\rho U \partial_\sigma U \right] + K_3 \epsilon_{\alpha\beta\gamma\delta} \partial_\alpha U^\dagger \partial_\beta U \partial_\gamma U \partial_\delta U \right] U^\dagger. \tag{59}
$$

Now we proceed to the comparison of the beta-functional $\delta^{(4)}U$ and the four-derivative part of Eqs. of chiral dynamics. As it has been elucidated in the previous section one expects the entire identification on the dimension 2 mass-shell (i.e. after applying of the $O(p^2)$ Eqs. of motion). Off-shell one must extend to the following general set of operators and coefficients for the various chiral field structures,
In this Table the numeric, $C_i$ and operator, $B_i$ coefficients have been inserted so that they compensate each other in the total sum on the mass-shell. Evidently the two operators admit weaker algebraic restrictions on the operators than in the preceding table, with the same order as in the preceding table.

Let us identify this parameterization of local Lagrangian descendants with the coupling constants and operator coefficients arising from the vertices in Tables 1 and 2. In particular the coefficients $C_i$ and $B_i$ admit weaker algebraic restrictions on the operators $F_{\mu\nu}$. The pertinent consistency equations read, in the same order as in the preceding table,

\[(F_{\sigma\lambda}F_{\beta\alpha} - \delta_{\alpha\beta}F_{\sigma\gamma}F_{\lambda\gamma}) + (F_{\alpha\lambda}F_{\beta\sigma} - \delta_{\alpha\beta}F_{\alpha\gamma}F_{\lambda\gamma})
\]  
\[+ (F_{\sigma\beta}F_{\lambda\alpha} - \delta_{\alpha\lambda}F_{\sigma\gamma}F_{\beta\gamma}) + (F_{\alpha\beta}F_{\lambda\sigma} - \delta_{\alpha\sigma}F_{\alpha\gamma}F_{\lambda\gamma}) = C_0\delta_{\alpha\sigma}\delta_{\beta\lambda} + B_{aa\sigma}^{(0)}\delta_{\beta\lambda} + B_{aa\sigma}^{(1)}; \quad (61)
\]

\[(F_{\alpha\rho}(\delta_{\sigma\lambda} - F_{\lambda\sigma})\delta_{\beta\rho} + F_{\beta\rho} - F_{\sigma\gamma}(\delta_{\lambda\gamma} - F_{\lambda\gamma})(\delta_{\alpha\beta} + F_{\beta\alpha}))
\]  
\[+ (F_{\sigma\rho}(\delta_{\alpha\lambda} - F_{\lambda\alpha})\delta_{\beta\rho} + F_{\beta\rho} - F_{\alpha\gamma}(\delta_{\lambda\gamma} - F_{\lambda\gamma})(\delta_{\beta\gamma} + F_{\beta\gamma}))
\]  
\[-2A_{\sigma\alpha\lambda\mu}(\delta_{\beta\mu} - F_{\beta\mu}) = -(2K_1 + K_2)(\delta_{\alpha\beta}\delta_{\sigma\lambda} + \delta_{\alpha\lambda}\delta_{\sigma\beta})
\]  
\[+ B_{aa\sigma}^{(0)}\delta_{\beta\lambda} + \delta_{\alpha\sigma}(C_1\delta_{\beta\lambda} + B_{aa\beta}^{(2)}); \quad (62)
\]

\[((\delta_{\alpha\beta} - F_{\alpha\beta})(\delta_{\sigma\gamma} + F_{\sigma\gamma})F_{\lambda\gamma} - (\delta_{\alpha\rho} - F_{\alpha\rho})(\delta_{\sigma\lambda} + F_{\sigma\lambda})F_{\beta\rho})
\]  
\[+ ((\delta_{\alpha\lambda} - F_{\alpha\lambda})(\delta_{\sigma\gamma} + F_{\sigma\gamma})F_{\beta\gamma} - (\delta_{\alpha\rho} - F_{\alpha\rho})(\delta_{\sigma\beta} + F_{\sigma\beta})F_{\lambda\rho})
\]  
\[+ 2(\delta_{\alpha\mu} + F_{\alpha\mu})A_{\sigma\beta\lambda\mu}^{(2,r)} = -(2K_1 + K_2)(\delta_{\alpha\beta}\delta_{\sigma\lambda} + \delta_{\alpha\lambda}\delta_{\sigma\beta})
\]  
\[+ 2(\delta_{\alpha\mu} + F_{\alpha\mu})A_{\sigma\beta\lambda\mu}^{(1,r)} - A_{\alpha\sigma\beta\mu}^{(2,r)}(\delta_{\lambda\mu} - F_{\lambda\mu}) = -2K_2(\delta_{\alpha\sigma}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\sigma})
\]  
\[+ \delta_{\sigma\lambda}(C_3\delta_{\alpha\beta} + B_{aa\beta}^{(4)}); \quad (64)
\]
These equations dictate the consistency conditions for the algebra of the operators $F_{\mu\nu}$ and bound the values of the low-energy constants $K_1$, $K_2$, $K_3$ in the chiral Lagrangian. The simplest solution for $F_{\mu\nu} = z\delta_{\mu\nu}$ of this set of conditions was obtained in [23] and briefly described in Section 2. This solution, however, does not describe the spin degrees of freedom as implicitly assumes that quarks are scalar objects under rotations. The next simplest hypothesis is that $F_{\mu\nu}$ has an antisymmetric part proportional to $\sigma_{\mu\nu}$. Indeed, the coupling $\bar{\psi}_L\sigma_{\mu\nu}\psi_R \times \hat{x}_\mu\hat{x}_\nu U$ intuitively reflects the coupling between the string angular momentum (at the boundary) assuming that 'quarks' have $s = 1/2$ and the angular momentum of the $U$-field. As we have seen at the beginning of section 5 this not compatible with the one-loop renormalization properties of the model. We are then forced to conclude that the Grassmann variables are not in a state of well defined spin $s = 1/2$.

It is possible to use the previous set of equations (61)–(66) to further constrain the operators $F_{\mu\nu}$. This is a rather non-trivial task. In next Section we explore some identities for operators $F_{\alpha\beta\lambda\mu}$ and discuss possible realizations of this algebra. Even though we do not have a final answer and, in a sense, Eqs. (44), (61)–(66) are our final result, the problem is interesting enough, deserving more detailed considerations.

7 Algebra considerations

It is probably worth to recapitulate where we stand.

The elimination of all divergences at the one-loop order requires, in addition to redefining the unitary matrix $U$, additional counterterms that are given in Eqs. (50), (52) in terms of a certain number of constants $g^{(r)}$, $\bar{g}^{(r)}$, $g_3^{(r)}$. In spite of the rather large number of structures involving $F_{\mu\nu}$, only three independent combinations appear in the counterterms described in Eqs. (50), (52). This is somewhat reminiscent of the situation without the spin structures, where three additional constants $g_{s,r}$, each one accompanying a different chiral structure, are engaged. The complication here lies of course in the fact that the $F_{\mu\nu}$ operators are taking values in some algebra yet to be specified.

Renormalizing the two-loop order propagator (i.e. $U$) needs taking all these one-loop counterterms. Adding all the contributions up leads to the conditions listed in Tables 1–3 and to the set of Eqs. (61)–(66).

As explained, these relations equate the single pole divergent part of the fermion propagator (a combination of chiral fields $U$, their derivatives, and operators $F_{\mu\nu}$) with the equivalent terms arising in equations of motion derived from the local Chiral Lagrangian [18]. These equations of motion involve chiral fields and their derivatives, but not $F_{\mu\nu}$. If we insist, as we should, in making the two set of expressions equivalent this naturally brings about new relations involving the $F_{\mu\nu}$.

Through these equations we can learn more about the form of the $F_{\mu\nu}$ operator matrix and thus the way the spinor interaction degrees of freedom are implemented into this $F_{\mu\nu}$ operator, and of course, when possible, fix as much as we can the value of $K_1$, $K_2$ and $K_3$. These relations stem from the requirements of chiral invariance and locality of the effective action and they should be understood as restrictions that these conditions place on the algebra that the $F_{\mu\nu}$ satisfy.

To be specific, from Subsec. 5.2, Eq. (44) we have

$$F_{\sigma\gamma}F_{\lambda\gamma} = -\delta_{\sigma\lambda} + F_{\sigma\lambda} - F_{\lambda\sigma}; \quad F_{\lambda\gamma}F_{\lambda\gamma} = -4.$$ (66)

Next the fulfillment of Eq. (61) turns out to be very crucial as it removes the chiral field structure which is a serious obstruction for local integrability of Eqs. of motion. Therefrom, after contracting two of the

\begin{equation}
(\delta_{\alpha\rho} + F_{\alpha\rho}(\delta_{\sigma\gamma} + F_{\sigma\gamma})(\delta_{\beta\gamma} - F_{\beta\gamma})(\delta_{\lambda\rho} - F_{\lambda\rho})
+ (\delta_{\alpha\rho} - F_{\alpha\rho})(\delta_{\sigma\gamma} + F_{\sigma\gamma})(\delta_{\beta\rho} - F_{\beta\rho})(\delta_{\lambda\gamma} + F_{\lambda\gamma})
+ 2((\delta_{\alpha\mu} + F_{\alpha\mu})A_{\sigma\beta\lambda\mu}^{(3,r)} - A_{\alpha\sigma\beta\mu}^{(3,r)}(\delta_{\lambda\mu} - F_{\lambda\mu}))
= (2(2K_1 + K_2) - C_3)\delta_{\alpha\beta}\delta_{\sigma\lambda}
+ (2K_2 - C_0 - C_1 - C_2)\delta_{\alpha\sigma}\delta_{\beta\lambda}
+ 4(K_1 + K_2)\delta_{\alpha\lambda}\delta_{\beta\beta} + K_3\epsilon_{\alpha\sigma\lambda}\beta
- \delta_{\alpha\sigma}B_{\beta\lambda}^{(2)} - B_{\alpha\sigma}\delta_{\beta\lambda} - \delta_{\sigma\lambda}F_{\alpha\beta}^{(4)}.
\end{equation} (65)
indices with $\delta_{\alpha\beta}$, we obtain

$$F_{\gamma\sigma}F_{\gamma\lambda} = -(9 + C_0)\delta_{\sigma\lambda} - 5(F_{\sigma\lambda} - F_{\lambda\sigma}) - \text{tr}[F]F_{\sigma\lambda} - F_{\lambda\sigma}\text{tr}[F] + B_{\sigma\lambda}^{(0)} + B_{\sigma\lambda}^{(1)}.$$  

(67)

As the components of the operator $F_{\sigma\lambda}$ are antihermitian it comes out from (67) that,

$$C_0 = (C_0)^*; \quad (B_{\sigma\lambda}^{(0)} + B_{\sigma\lambda}^{(1)})^\dagger = B_{\sigma\lambda}^{(0)} + B_{\sigma\lambda}^{(1)}.$$  

(68)

As well from the further contraction of indices $\sigma = \lambda$ one determines the trace of the operator $F_{\sigma\lambda}$,

$$(\text{tr}[F])^2 = 2C_0 - 16 + \text{tr}\left[B_{(0)}^{(0)} + B_{(1)}^{(1)}\right].$$  

(69)

We however stress that in general it represents an operator relation when one of the traces is not a $c$-number.

Finally, a non-equivalent contraction allows us to fix the symmetric part of twist-contracted products of $F_{\gamma\sigma}$,

$$F_{\gamma\sigma}F_{\lambda\gamma} + F_{\gamma\lambda}F_{\sigma\gamma} = 2(C_0 - 1 + \frac{1}{4}\text{tr}\left[B_{(0)}^{(0)}\right])\delta_{\sigma\lambda} + 2B_{\sigma\lambda}^{(1)},$$  

(70)

that allows for the determination of twisted normalization of the operator $F_{\sigma\lambda}$,

$$F_{\gamma\lambda}F_{\lambda\gamma} = 4(C_0 - 1) + \text{tr}\left[B_{(0)}^{(0)} + B_{(1)}^{(1)}\right] = 28 + 2(\text{tr}[F])^2.$$  

(71)

All these algebraic relations originate from the requirement of local integrability of the would-be equations of motion. Notice that the last one (71) does not give us an explicit algebraic expression for the antisymmetric part. An ansatz admitting lineal in $F$ right-hand parts of Eqs. (67), (70) would close the algebra. However it happens to lead to a definite contradiction when the associativity of the algebra of contracted and twist-contracted products of three $F_{\gamma\sigma}$ is examined. Hence the ansatz is not correct and the algebra does not close.

Unfortunately, at the end of the day, we shall not have an explicit realization of the $F_{\mu\nu}$ satisfying all the previous requirements. Some obvious possibilities are however ruled out. We have already mentioned that the attempt of identifying the antisymmetric part of $F_{\mu\nu}$ with $B_{\mu\nu}$ fails (see Subsection 5.1). It is somewhat more surprising that if Eqs. (44), (61) – (66) are to be imposed, the described algebra spanned by the $F_{\mu\nu}$ does not close, so it must necessarily be embedded in a larger algebra.

We are then forced to somewhat loosen the requirement of closure of the algebra. At this point, we discontinue the analysis of the implications of the algebraic relations (44), (67), (70). We regard these equations as constraints that the algebra of the $F$’s must satisfy in order to provide consistent propagation of the hadronic string in a chirally non-invariant vacuum when the spin degrees of freedom are taken into account. The remaining relations (42) – (66) are rather tools for the estimation of all coupling constants introduced on the boundary as well as the chiral constants $K_1, K_2, K_3$. This program nevertheless requires first to discover the algebra of $F_{\mu\nu}$ to be predictive.

8 Conclusions

In this work we have analyzed in detail the conditions that the effective string conceived to describe the interactions between quarks at long distances in QCD must meet. An essential ingredient for this string is the assumption that in the real QCD vacuum chiral symmetry is broken and the propagation takes place in a background of $\Pi$-on fields (not states on the Regge trajectories). The condition of locality, chiral symmetry and conformal invariance place strong constraints on this background, eventually leading to vanishing beta functionals to be interpreted as equations of motion of the non-linear sigma model describing $\Pi$-on interactions.

The work reported here dwells on a previous analysis where quarks (represented by Grassmann variables living on a line) were considered to be scalars. But spin is indeed an important variable in Regge analysis (let us recall here the existence of the so-called $S$ and $D$ Regge trajectories). More importantly, it is not difficult to see that without considering angular momentum, the odd (internal) parity of the
Π-on Lagrangian (i.e. the Wess-Zumino-Witten action) will never be obtained as one of the byproducts of requiring conformal invariance.

In the preceding pages a number of new results have been obtained. We have managed to couple an external gauge field and in this way to derive the covariant $O(p^2)$ equations of motion. The analysis of the Wess-Zumino-Witten action in dimension 2 turns out to be rather straightforward and it reproduces well the expected results.

Angular momentum in two dimensions is somewhat special and this is reflected in its realization in terms of gamma matrices. In fact the calculation can be fully reformulated using scalar variables. When proceeding to the four-dimensional case, things become rather more involved. We construct the general coupling that involves some operator coupling $F_{\mu\nu}$ (acting on the angular momentum degrees of freedom of the quarks). Consistency conditions of the string propagation indeed remarkably seem to suggest that the quarks are not in a definite state of angular momentum. A deeper reason may be in that hadron string realizes the Reggeization of meson states which, in the spirit of quark-hadron duality, presumably follows from a Reggeization of quarks and gluons as it happens in the semi-hard high-energy scattering in QCD [14]. If such a quark-hadron duality holds then one cannot expect the boundary quarks to carry a definite spin. Rather they may be thought of in terms of an infinite-dimensional reducible representation of the Poincare group with any half-integer spin incorporated. Of course when $F_{\mu\nu}$ reduces to the scalar case, the results of [23] are fully reproduced. These results are in excellent agreement with phenomenology.

We finally spelled out the restrictions that locality, chiral symmetry and conformal invariance place on the couplings $F_{\mu\nu}$ and formulated the way to search for the consistent realization of the $F_{\mu\nu}$ algebra.

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Appendix A. Ein-bein projection of the Dirac operator on the string boundary

The Lagrangian (3) does not contain any operators that could give rise to the anomalous P-odd part of the Chiral Dynamics. To approach the required modification let us guess on what might be the form of boundary Lagrangian if one derives it, say, from the essential part of the Chiral Quark Model projecting it on the string boundary. In what follows the Minkowski space-time is employed to keep the axial-vector vertex to be Hermitian.

Let us introduce the constituent quark fields to control properly the chiral symmetry during the "ein-bein" projection,

$$Q_L \equiv \xi^\dagger \psi_L, \quad Q_R \equiv \xi \psi_R, \quad \xi^2 \equiv U. \tag{72}$$

Under chiral rotations $U \rightarrow \Omega_R U \Omega_L^+$ the fields $\xi$ transform as follows

$$\xi \rightarrow h_\xi \Omega_R \xi \Omega_L^+ = \Omega_R \xi h_\xi^+, \tag{73}$$

with $h_\xi$ being a nonlinear functional of fields $\xi$. As a consequence the hidden vector symmetry of the constituent field action replaces the original chiral invariance.

In these variables the CQM Lagrangian density and the pertinent E.o.M. read

$$\mathcal{L}_{CQM} = i \bar{Q} (\partial \psi + g_A \gamma_5 \phi^A) Q + \text{mass terms}; \quad i (\partial \phi^A + g_A \dot{\phi}^A) Q + \text{mass terms} = 0, \tag{74}$$

where

$$Q \equiv Q_L + Q_R, \quad \mathcal{A} \equiv \gamma^\mu A_\mu, \quad v_\mu = \frac{1}{2} (\xi^\dagger (\partial_\mu \xi) - (\partial_\mu \xi) \xi^\dagger), \quad a_\mu = -\frac{1}{2} (\xi^\dagger (\partial_\mu \xi) + (\partial_\mu \xi) \xi^\dagger), \tag{75}$$
and \( g_A = 1 - \delta g_A \) is an axial coupling constant of quarks to Π-ons. We skip all mass effects of the CQM, thereby neglecting the current quark mass in the chiral limit whereas relegating the effects of constituent quark mass to the gluodynamics encoded in the string interaction. Then one can decouple the left and right components of boundary fields in the process of dim-1 projection.

We assume the quark fields be located on the dim-1 boundary with coordinates \( x_\mu \equiv x_\mu(\tau) \). The first step in projection of the E.o.M. (74) can be performed by their multiplication on \( \gamma^\mu \gamma_5 \) which leads to the following boundary equations,

\[
\{ i \{ \partial_\tau + \dot{x}_\mu v^\mu + g_A \gamma_5 \dot{x}_\mu a^\mu \} + \sigma^{\mu\nu} \dot{x}_\mu \{ \partial_\nu + v_\nu + g_A \gamma_5 a_\nu \} \} Q = 0; \quad \sigma^{\mu\nu} \equiv \frac{1}{2} [\gamma^\mu \gamma^\nu].
\]

Notice that this projected Dirac-type equation seems to be associated to the boundary action with a Lagrangian of type (63). But in order to provide the correct Hermitian properties of the Lorentz symmetry generators \( \sigma_{\mu\nu} \) one must involve the Dirac conjugated spinors, \( \bar{\psi} \equiv \psi^\dagger \gamma_0 \). As a consequence, the axial-vector part in the first, scalar contribution becomes anti-Hermitian as \( \gamma \) matrices anticommute. It can be cured by the prescription of analytic continuation \( g_A \rightarrow z = i g_A \) in the scalar part (only). At this place we must adopt an arbitrary constant \( z \) subject to the consistency conditions from the string with boundary.

Let us restore the current quark basis of fields \( \psi_L \) thereby going back to the original chiral fields. We use Eqs. (72) and multiply the left and right component of Eq. (76) by \( \xi \) and \( \xi^\dagger \) respectively. The result is that,

\[
\frac{1}{2} \left\{ i \left\{ \{ \partial_\tau, U^\dagger \} + z U^\dagger \right\} + \sigma^{\mu\nu} \dot{x}_\mu \left\{ \{ \partial_\nu, U^\dagger \} + g_A \partial_\nu U \right\} \right\} \psi_L = 0; \quad \frac{1}{2} \left\{ i \left\{ \{ \partial_\tau, U \} + z U \right\} + \sigma^{\mu\nu} \dot{x}_\mu \left\{ \{ \partial_\nu, U \} + g_A \partial_\nu U \right\} \right\} \psi_R = 0.
\]

Now the culminating point of the "ein-bein" projection consists of making the quark fields \( \psi \) truly one-dimensional. Namely we define their gradient in terms of the tangent vector \( \dot{x}_\mu \) and arbitrary matrix functions \( f(x_\mu), b(x_\mu) \) of \( x_\mu(\tau) \).

\[
\partial_\mu (f L \psi_L) + \partial_\mu (b L) \psi_L = \dot{\psi}_\mu \left[ \partial_\tau (f L \psi_L) + \partial_\tau (b L) \psi_L \right];
\]

\[
\partial_\mu (f R \psi_R) + \partial_\mu (b R) \psi_R = \dot{\psi}_\mu \left[ \partial_\tau (f R \psi_R) + \partial_\tau (b R) \psi_R \right].
\]

Keeping in mind our program we choose the functions \( f(x_\mu), b(x_\mu) \) to provide the correct chiral properties, translational and reparameterization invariance (in terms of chiral fields \( U \)). As well the operator appeared in projection must be anti-self-adjoint in respect to the dim-4 Dirac scalar product. All these requirements are satisfied by the choice,

\[
\{ \partial_\mu, U^\dagger \} \psi_L \rightarrow \frac{\dot{\psi}_\mu}{\dot{x}_\mu \dot{x}^\mu} \{ \partial_\tau, U^\dagger \} \psi_L; \quad \{ \partial_\mu, U \} \psi_R \rightarrow \frac{\dot{\psi}_\mu}{\dot{x}_\mu \dot{x}^\mu} \{ \partial_\tau, U \} \psi_R.
\]

Finally, the projected equations are originated from the boundary Lagrangian,

\[
L^{(f)} = \frac{1}{2} \left\{ \bar{\psi}_L \left[ \{ \partial_\tau, U \} + z U + g_\sigma \sigma^{\mu\nu} \dot{x}_\mu \partial_\nu U \right] \psi_R + \bar{\psi}_R \left[ \{ \partial_\tau, U^\dagger \} - z^* U^\dagger - g_\sigma \sigma^{\mu\nu} \dot{x}_\mu \partial_\nu U^\dagger \right] \psi_L \right\}; \quad \bar{\psi}_L \left[ \{ \partial_\tau, U \} + \tilde{F}^{\mu\nu} \dot{x}_\mu \partial_\nu U \right] \psi_R + \bar{\psi}_R \left[ \{ \partial_\tau, U^\dagger \} - \tilde{F}^{\mu\nu} \dot{x}_\mu \partial_\nu U^\dagger \right] \psi_L \right\};
\]

\[
\tilde{F}^{\mu\nu} = z g^{\mu\nu} + g_\sigma \sigma^{\mu\nu}; \quad \tilde{F}^{\mu\nu} \equiv \gamma_0 \left( \tilde{F}^{\mu\nu} \right)^\dagger \gamma_0.
\]

where we have obtained the indications that \( g_\sigma = -i g_A \). Still keeping in mind a certain ambiguity in the projection procedure we must consider both constants \( z \) and \( g_\sigma \) as arbitrary ones and search for their values from the consistency of the hadron string with chiral fields on its boundary.

The meaning of purely imaginary \( z \) and \( g_\sigma \) is clarified by the CP symmetry (4) of the Lagrangian (70). Indeed it is CP symmetric only if

\[
z = -z^*; \quad g_\sigma = -g_\sigma^*.
\]
Appendix B. One-loop two-fermion one-boson vertex

In this Appendix we present the calculation of 1-boson vertex for the boundary Lagrangian $F_{\mu\nu}$ including a more general spin structure $F_{\mu\nu}$.

| Coefficient | U structure | F structure |
|-------------|-------------|-------------|
| $-\frac{1}{2} \theta(A - B) \Delta(0)$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} \partial_{\tau} U U^{\dagger}$ | $\delta_{\sigma \lambda} [X_{\rho}(A)(\delta_{\eta \rho} + F_{\eta \rho}) + X_{\rho}(B)(\delta_{\eta \rho} - F_{\eta \rho})]$ |
| $\frac{1}{2} \theta(A - B) \Delta(0)$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\lambda} U U^{\dagger}$ | $[X_{\rho}(A)(\delta_{\sigma \lambda} + F_{\sigma \lambda})(\delta_{\eta \rho} + F_{\eta \rho}) + X_{\rho}(B)(2\delta_{\eta \rho} \delta_{\sigma \lambda} - F_{\eta \rho} \delta_{\sigma \lambda} - \delta_{\eta \rho} \delta_{F_{\lambda} \lambda} + \delta_{\eta \rho} F_{\lambda} \lambda - F_{\eta \rho} F_{\lambda} \lambda)]$ |
| $-\frac{1}{5} \theta(A - B) \Delta(0)$ | $U^{\dagger} \partial_{\sigma} U U^{\dagger} \partial_{\lambda} \partial_{\eta} U U^{\dagger}$ | $[X_{\rho}(B)(\delta_{\sigma \lambda} - F_{\sigma \lambda})(\delta_{\eta \rho} - F_{\eta \rho}) + X_{\rho}(A)(2\delta_{\eta \rho} \delta_{\sigma \lambda} - F_{\eta \rho} \delta_{\sigma \lambda} - \delta_{\eta \rho} \delta_{F_{\lambda} \lambda} + \delta_{\eta \rho} F_{\lambda} \lambda - F_{\eta \rho} F_{\lambda} \lambda)]$ |

Finite part $\propto \Delta(A - B)$

| $\frac{1}{2} \theta(A - B) \Delta(A - B)$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\tau} U U^{\dagger}$ | $X_{\rho}(A) \delta_{\eta \rho} (\frac{1}{2} \delta_{\sigma \lambda} + F_{\sigma \lambda} - \delta_{\eta \rho} - F_{\sigma \lambda} F_{\gamma \lambda} \gamma)$ |
| $\frac{1}{2} \theta(A - B) \Delta(A - B)$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\lambda} \partial_{\tau} U U^{\dagger}$ | $X_{\rho}(B) \delta_{\eta \rho} (\frac{1}{2} \delta_{\sigma \lambda} - F_{\sigma \lambda} - \delta_{\eta \rho} - F_{\sigma \lambda} F_{\gamma \lambda} \gamma)$ |
| $-\frac{1}{5} \theta(A - B) \Delta(A - B)$ | $U^{\dagger} \partial_{\sigma} U U^{\dagger} \partial_{\lambda} \partial_{\eta} U U^{\dagger}$ | $[X_{\rho}(A)(\delta_{\sigma \gamma} + F_{\sigma \gamma})(\delta_{\eta \rho} + F_{\eta \rho})(\delta_{\lambda \rho} - F_{\lambda \rho})(\delta_{\gamma \rho} - F_{\gamma \rho}) + X_{\rho}(B)(\delta_{\sigma \gamma} + F_{\sigma \gamma})(\delta_{\eta \rho} + F_{\eta \rho})(\delta_{\lambda \rho} - F_{\lambda \rho})(\delta_{\gamma \rho} - F_{\gamma \rho})]$ |

Finite part $\propto \int d\tau \dot{X}_{\rho}(\tau) \Delta(A - \tau) \equiv \text{Int}_{A}$

| $\frac{1}{2} \theta(A - B) \text{Int}_{A}$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\tau} U U^{\dagger}$ | $(\delta_{\sigma \gamma} + F_{\sigma \gamma})(\delta_{\lambda \rho} - F_{\lambda \rho})(\delta_{\eta \rho} - F_{\eta \rho})$ |
| $\frac{1}{2} \theta(A - B) \text{Int}_{A}$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\lambda} U U^{\dagger}$ | $(\delta_{\sigma \gamma} + F_{\sigma \gamma})(\delta_{\eta \rho} - F_{\eta \rho})(\delta_{\lambda \rho} - F_{\lambda \rho})$ |

Finite part $\propto \int d\tau \dot{X}_{\rho}(\tau) \Delta(\tau - B) \equiv \text{Int}_{B}$

| $\frac{1}{2} \theta(A - B) \text{Int}_{B}$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\tau} U U^{\dagger}$ | $(\delta_{\sigma \gamma} \eta \rho - F_{\sigma \gamma} \eta \rho)(\delta_{\lambda \rho} - F_{\lambda \rho})$ |
| $\frac{1}{2} \theta(A - B) \text{Int}_{B}$ | $U^{\dagger} \partial_{\eta} \partial_{\sigma} U U^{\dagger} \partial_{\lambda} U U^{\dagger}$ | $(\delta_{\sigma \gamma} \eta \rho - F_{\sigma \gamma} \eta \rho)(\delta_{\lambda \rho} - F_{\lambda \rho})$ |

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