QUANTITATIVE CONTROL OF SOLUTIONS TO AXISYMMETRIC NAVIER-STOKES EQUATIONS IN TERMS OF THE WEAK $L^3$ NORM

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Abstract. We are concerned with strong axisymmetric solutions to the 3D incompressible Navier-Stokes equations. We show that if the weak $L^3$ norm of a strong solution $u$ on time interval $[0, T]$ is bounded by $A \gg 1$ then for each $k \geq 0$ there exists $C_k > 1$ such that $\|D^k u(t)\|_{L^\infty(\mathbb{R}^3)} \leq t^{-(1+k)/2} \exp \exp A^{C_k}$ for all $t \in (0, T]$.

1. Introduction

We are concerned with the 3D incompressible Navier-Stokes equations,

$$\begin{cases}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \\
\text{div } u = 0
\end{cases}$$

for $t \in [0, T)$. While the question of global well-posedness of the equations remains open, it is well-known that the unique strong solution on a time interval $[0, T)$ can be continued past $T$ provided a regularity criterion holds, such as $\int_0^T \| \text{curl } u \|_\infty < \infty$ (the Beale-Kato-Majda [3] criterion), Lipschitz continuity up to $t = T$ of the direction of vorticity (the Constantin-Fefferman [9] criterion), or if $\int_0^T \| u \|_p^q < \infty$ for any $p \in [3, \infty], q \in [2, \infty]$ such that $2/q + 3/p \leq 1$ (the Ladyzhenskaya-Prodi-Serrin condition), among many others. The non-endpoint case $q < \infty$ of the latter condition was settled in the 1960s [13, 36, 29], while the endpoint case $L^\infty_t L^3_x$ was only settled many years later by Escauriaza, Seregin, and Šverák [10]. The main difficulty of the endpoint case is related to the fact that $L^3$ is a critical space for 3D Navier–Stokes, and [10] proved it with an argument by contradiction using a blow-up procedure and new unique continuation results. This result implies that if $T_0 > 0$ is a putative blow-up time of (1), then $\|u(t)\|_3$ has to blow-up at least along a sequence of times $t_k \to T_0^-$. While Seregin [33] showed that the $L^3$ norm must blow-up along any sequence of times converging to $T_0^-$, the question of quantitative control of the strong solution $u$ in terms of the $L^3$ norm remained open until the recent breakthrough work by Tao [39] who showed that

$$|\nabla^j u(x, t)| \leq \exp \exp(A^{O(1)}) t^{-\frac{j+1}{2}}$$

for all $t \in [0, T], j = 0, 1, x \in \mathbb{R}^3$, whenever

$$\|u\|_{L^\infty([0, T]; L^3(\mathbb{R}^3))} \leq A$$

for some $A \gg 1$. This result implies in particular a lower bound

$$\limsup_{t \to T_0^-} \frac{\|u(t)\|_3}{(\log \log \log (T_0 - t)^{-1}))^c} = \infty,$$

where $c > 0$ and $T_0 > 0$ is the putative blowup time, and has subsequently been improved in some settings. For example, Barker and Prange [2] and Barker [1] provided remarkable local
quantitative estimates, and the second author [27] proved that, in the case of axisymmetric solutions,
\[ |\nabla^j u(x, t)| \leq \exp\exp(A^{O(1)})t^{-\frac{j+1}{2}} \]
for all \( t \in [0, T], j = 0, 1, x \in \mathbb{R}^3 \), whenever
\[ \left\| r^{1-\frac{2}{d}} u \right\|_{L^\infty([0,T];L^q(\mathbb{R}^3))} \leq A \]
for some \( A \gg 1, q \in (2, 3] \). In another work [28] he generalized (2) to higher dimensions \((d \geq 4)\), where, due to an issue related to the lack of Leray’s intervals of regularity, one obtains an analogue of (2) with four exponential functions. Recently Feng, He, and Wang [11] extended (2) to the non-endpoint Lorentz spaces \( L^{3,q} \) for \( q < \infty \). We emphasize that all these generalizations rely on the same stacking argument by Tao [39]. In particular, the argument breaks down for the endpoint case \( q = \infty \).

1.1. Tao’s stacking argument and Type I blowup. In order to illustrate the issue at the endpoint space \( L^{3,\infty} \), let us recall that the main strategy of Tao [39] is to show that, if \( u \) concentrates at a particular time, then there exists a widely separated sequence of length scales \( (R_k)_{k=1}^K \) and \( \alpha = \alpha(A) > 0 \) such that \( \|u\|_{L^3(|x| \sim R_k)} \geq \alpha \) for all \( k \), which implies that
\[ \|u\|_3^3 = \int_{\mathbb{R}^3} |u|^3 \geq \sum_k \int_{|x| \sim R_k} |u|^3 \geq \alpha^3 K. \]  
(3)
The more singularly \( u \) concentrates at the origin, the larger one can take \( K \); thus the \( L^3 \) norm controls the regularity of \( u \). More precisely, if \( \|u\|_3 \leq A \) and \( u \) concentrates at a large frequency \( N \) at time \( T \), then one can take \( \alpha = \exp(-\exp(A^{O(1)})) \) and \( K \sim \log(NT^{\frac{1}{2}}) \), which leads to the conclusion \( N \leq T^{-\frac{1}{3}} \exp \exp \exp(A^{O(1)}) \).

Let us contrast this \( L^3 \) situation with that of general Lorentz spaces with interpolation exponent \( q \geq 3 \). In that case, \( \|u\|_{L^{3,q}(|x| \sim R_k)} \geq \alpha \) implies
\[ \|u\|_{L^{3,q}(\mathbb{R}^3)} \gtrsim \left\| \|u\|_{L^{3,q}(|x| \sim R_k)} \right\|_{L^q_k} \geq \alpha K^{\frac{q}{3}}, \]
and so one should expect the bounds from the stacking argument used in the Lorentz space \( L^{3,q} \) extension [11] to degenerate as \( q \to \infty \). Indeed, if \( |u(x)| = |x|^{-1} \), we have \( \|u\|_{L^{3,\infty}(|x| \sim R_k)} = 1/O(1) \) for every \( R > 0 \), yet \( \|u\|_{L^{3,\infty}(\mathbb{R}^3)} \sim 1 \) which shows that the first inequality in (3) fails for the \( L^{3,\infty} \) norm. For this reason, the approach of Tao [39] (and, for related reasons, of Escauriaza-Seregin-Šverák) to the \( L^3 \) problem cannot be extended to \( L^{3,\infty} \).

This issue is in fact closely related to the study of Type I blowups and approximately self-similar solutions to (1). Leray famously conjectured the existence of backwards self-similar solutions that blow up in finite time, a possibility later ruled out by Nečas, Růžička, and Šverák [22] for finite-energy solutions and by Tsai [40] for locally-finite energy solutions. The latter reference identifies the following as a very natural ansatz for blowup:
\[ u(t, x) = \frac{1}{(T_0 - t)^{\frac{1}{2}}} U \left( \frac{x}{(T_0 - t)^{\frac{1}{2}}} \right), \quad U(y) = a \left( \frac{y}{|y|} \right) \frac{1}{|y|} + o \left( \frac{1}{|y|} \right) \text{ as } |y| \to \infty, \]
(4)
where \( a : S^2 \to \mathbb{R}^3 \) is smooth. While Tsai [40] shows that there are no solutions exactly of this form, solutions that approximate this profile or attain it in a discretely self-similar way are promising candidates for singularity formation, as demonstrated by the Scheffer
constructions [23, 24, 31, 32], for example. Unfortunately, criteria pertaining to $L^3$ such as those in [10, 39, 27] are not effective at controlling such solutions because $|x|^{-1} \notin L^3(\mathbb{R}^3)$, which shows the relevance of the weak norm $L^{3,\infty}$.

Specializing to the case of axial symmetry, it is known, for instance due to Seregin’s result [34], that finite-time blowup cannot be of Type I. Thus, roughly speaking, no axisymmetric solution can approximate the profile (4) all the way up to a putative blowup time $T_0$. However, this regularity is only qualitative (indeed, the proof uses an argument by contradiction based on a “zooming in” procedure), and so explicit bounds on the solution have not been available.

The main purpose of this work is to make this regularity quantitative, in the same sense that Tao [39] made quantitative the of Escauriaza-Seregin-Šverák theroem [10]. This allows us to not only to rule out Type I singularities, but also to control how singular they can possibly become. For example it lets us estimate the length scale up to which a solution can be approximated by a self-similar profile, see Corollary 1.3 for details.

1.2. The main regularity theorem. We suppose that a strong solution to (1) on time interval $[0, T]$ is axisymmetric, meaning that

$$\partial_\theta u_r = \partial_\theta u_3 = \partial_\theta u_\theta = 0,$$

where $u_r, u_\theta, u_3$ denote (respectively) the radial, angular, and vertical components of $u$, so that

$$u = u_r e_r + u_\theta e_\theta + u_3 e_3$$

in cylindrical coordinates, where $e_r, e_\theta, e_3$ denote the cylindrical basis vectors.

We assume further that $u$ remains bounded in $L^{3,\infty}$,

$$\|u\|_{L^\infty([0, T]; L^{3,\infty}(\mathbb{R}^3))} \leq A$$

for some $A \gg 1$. We prove the following.

**Theorem 1.1** (Main result). Suppose $u$ is a classical axisymmetric solution of (1) on $[0, T] \times \mathbb{R}^3$ obeying (6). Then

$$\|\nabla^j u(t)\|_{L^\infty(\mathbb{R}^3)} \leq t^{-\frac{j+1}{2}} \exp(A^{O_j(1)})$$

for all $j \geq 0$, $t \in [0, T]$.

We note that, although our proof of the above theorem does use some of the basic a priori estimates (see Section 4.2) pointed out by Tao [39], it follows a completely different scheme. Our main ingredients are parabolic methods applied to the swirl $\Theta := ru_\theta$ near the axis, as well as localized energy estimates on

$$\Phi := \frac{\omega_r}{r} \quad \text{and} \quad \Gamma := \frac{\omega_\theta}{r}.$$ 

In a sense, we use those estimates to replace the Carleman inequalities appearing in Tao’s [39] approach.

To be more precise, our proof builds on the work of Chen, Fang, and Zhang [6], who showed that the energy norm of $\Phi, \Gamma$,

$$\|\Phi\|_{L^\infty_t L^2_x} + \|\Gamma\|_{L_t^\infty L^2_x} + \|\nabla \Phi\|_{L^2_t L^2_x} + \|\nabla \Gamma\|_{L^2_t L^2_x},$$

(8)
controls $u$ via an estimate on $\|u^2/\theta\|_{L^2}$ (see [6, Lemma 3.1]). They also observed that one can indeed estimate this energy norm as long as the angular velocity $u_\theta$ remains small in any neighbourhood of the axis, namely if

$$\|\partial^d u_\theta\|_{L_t^\infty([0,T];L^{3/(1-d)}(\{r<\alpha\}))}$$

is sufficiently small for some $\alpha > 0$ and $d \in (0,1)$. \(9\)

In fact, this can be observed from the PDEs satisfied by $\Phi$, $\Gamma$, namely that

\begin{align*}
\left( \partial_t + u \cdot \nabla - \Delta - \frac{2}{r} \partial_r \right) \Phi - (\omega_r \partial_r + \omega_3 \partial_3) \frac{u_r}{r} &= 0, \\
\left( \partial_t + u \cdot \nabla - \Delta - \frac{2}{r} \partial_r \right) \Gamma + 2 \frac{r^2 u_\theta \omega_r}{r} &= 0,
\end{align*}

which shows that, in order to control the energy of $\Gamma$, $\Phi$ one needs to control $u_r/r$, $\omega_r$, $\omega_3$ and $u_\theta$. However, $u_r/r$ can be controlled by $\Gamma$, in the sense that

$$\frac{u_r}{r} = \Delta^{-1} \partial_3 \Gamma - \frac{2}{r} \Delta^{-2} \partial_3 \Gamma$$

(see [6, p. 1929] for details), which is one of the main properties of function $\Gamma$. In particular, (11) lets us use the Calderón-Zygmund inequality to obtain that

$$\left\| D^2 \frac{u_r}{r} \right\|_{L^q} \leq \|\nabla \Gamma\|_{L^q}$$

for $q \in (1, \infty)$ (see [6, Lemma 2.3] for details). Moreover $\omega_r = r \Phi$, and $\omega_3 = \partial_r (ru_\theta)/r$, which shows that the $L^2$ estimate of $\Phi$, $\Gamma$ relies only on control of $u_\theta$. In fact, away from the axis, one can easily control $u_\theta$, while near the axis the smallness condition (9) is required in an absorption argument by the dissipative part of the energy, see [6, (3.11)–(3.14)] for details.

In this work we obtain such control of $u_\theta$ thanks to the weak-$L^3$ bound (6), by utilizing parabolic theory developed by Nazarov and Ural’tseva [20] in the spirit of the Harnack inequality. Namely, noting that the swirl $\Theta := ru_\theta$ satisfies the autonomous PDE

$$\left( \partial_t + \left(u + \frac{2}{r} \epsilon_r\right) \cdot \nabla - \Delta \right) \Theta = 0$$

everywhere except for the axis, one can deduce (as observed in [20, Section 4]) Hölder continuity of $\Theta$ near the axis. A similar observation, but in a case of limited regularity of $u$ was used by Seregin [34] in his proof of no Type I blow-ups for axisymmetric solutions. We quantify this approach (see Proposition 5.1 below) to obtain an estimate on the Hölder exponent in terms of the weak-$L^3$ norm, and hence we obtain sufficient control of the swirl in a very small neighbourhood of the axis. As for the outside of the neighbourhood, we obtain pointwise estimates on $u$ and all its derivatives, which are quantified with respect to $A$, and improve the second author’s estimates [27, Proposition 8]. This would enable one to close the energy estimates for the quantities in (8) if there exist sufficiently many starting times where the energy norms are finite. Unfortunately, there are no times when we can explicitly control these energies in terms of $A$ due to lack of quantitative decay in the $x_3$ direction. The standard approach of propagating $L^2$ control of $\Phi$, $\Gamma$ from the initial data at $t = 0$ (for instance, as in [6]) would lead to additional exponentials in Theorem 1.1. To avoid this issue and prove efficient bounds, we replace (8) with $L^2$ norms that measure $\Phi$ and $\Gamma$ uniformly-locally in $x_3$: namely, we consider

$$\|\Phi\|_{L_t^\infty L_{x_3-\text{uloc}}^2} + \|\Gamma\|_{L_t^\infty L_{x_3-\text{uloc}}^2} + \|\nabla \Phi\|_{L_t^2 L_{x_3-\text{uloc}}^2} + \|\nabla \Gamma\|_{L_t^2 L_{x_3-\text{uloc}}^2}.$$

(14)
See Proposition 6.1 below for an estimate of such energy norm, as well as (22) for the definition of the $L^2_{3-\text{uloc}}$ space. This issue gives rise to further challenges, such as the $x_3-\text{uloc}$ control of the solution $u$ itself in terms of (14) (see Section 7), as well as an estimate on $u_r$. We show that the former difficulty can be resolved by an $x_3-\text{uloc}$ generalization of the $L^4$ estimate on $u_\theta/r^{1/2}$ introduced by [6, Lemma 3.1], together with a $x_3-\text{uloc}$ bootstrapping via $\|u\|_{L_t^\infty L^6_{3-\text{uloc}}}^3$ and an inductive argument for the norms $\|u\|_{L_t^\infty W^{k-1,6}_\text{uloc}}$ with respect to $k \geq 1$, where "uloc" refers to the uniformly locally integrable spaces (in all variables, not only $x_3$); see Steps 2–4 in Section 7. As for the latter difficulty, we derive new $x_3-\text{uloc}$ estimates of $u_r$ in terms of $\Gamma$. To be more precise, instead of the global estimate (12), we require $L^2_{3-\text{uloc}}$ control of $u_r/r$, which is much more challenging, particularly considering the bilaplacian term in (11) above. To this end we develop bilaplacian Poisson-type estimate in $L^2_{3-\text{uloc}}$ (see Lemma 8.2), which enables us to show that

$$\|\nabla \partial_r \frac{u_r}{r}\|_{L^2_{3-\text{uloc}}}^3 + \|\nabla \partial_3 \frac{u_r}{r}\|_{L^2_{3-\text{uloc}}}^3 \lesssim \|\Gamma\|_{L^2_{3-\text{uloc}}} + \|\nabla \Gamma\|_{L^2_{3-\text{uloc}}}^3,$$

see Lemma 5.3. This lets us close the estimate of (14), and thus control all subcritical norms of $u$ in terms of $\|u\|_{L^3,\infty}$ (see Section 7 for details).

### 1.3. A comparison of the blowup rate.

We note that Theorem 1.1, together with the well-known blow-up criterion $\|u(t)\|_\infty \geq c/(T_0-t)^{1/2}$ (see [26, Corollary 6.25], for example), where $T_0 > 0$ is a putative blow-up time, immediately implies the following lower bound on the blow-up rate of $\|u(t)\|_{L^3,\infty}$.

**Corollary 1.2** (Blow-up rate of the weak-$L^3$ norm). If $u$ is a classical axisymmetric solution of (1) that blows up at $T_0$, then

$$\limsup_{t \to T_0^-} \frac{\|u(t)\|_{L^3,\infty}(\mathbb{R}^3)}{(\log \log (T_0-t)^{-1})^c} = +\infty. \quad (16)$$

This corollary is also a consequence of a recent theorem of Chen, Tsai, and Zhang [7], who proved\(^1\)

$$\limsup_{t \to T_0^-} \frac{\|b(t)\|_{B^{-1,\infty}_\infty(\mathbb{R}^3)}}{(\log \log \frac{100}{T_0-t})^{\frac{1}{48}}} = +\infty,$$

where $b := u_r e_r + u_3 e_3$ denotes the swirl-less part of the velocity field $u$ (see [18, Section 3.3] for the relevant definition of $B^{-1,\infty}_\infty$). Thus, since $B^{-1,\infty}_\infty(\mathbb{R}^3) \supset L^{3,\infty}$, the above blow-up rate implies (16). We conjecture that a variant of Theorem 1.1 holds with the weak-$L^3$ norm replaced by such a critical Besov norm and can be proved using the ideas presented here.

In order to describe the relation of Corollary 1.2 to [7], we note that the argument in [7] proceeds by proving a pointwise estimate of the form

$$|ru_\theta| \leq C \exp(-c |\log r|^\gamma), \quad (17)$$

\(^1\)Let us note the existence of a substantial misprint in the published version of [7]: in their Theorem 1.4, as in our Corollary 1.2, the blowup rate is double-logarithmic.
where \( c, C > 0, \tau \in (0,1) \), for axisymmetric solutions obeying the slightly supercritical bound

\[
\frac{1}{R^2} \|u\|_{L^\infty((-R^2,0);L^2(B_R))} \leq K \left( \log \log \frac{100}{R} \right)^{\beta} \text{ for all } R \in (0,1/4)
\]

for some \( \beta \in (0, \frac{1}{8}) \) and \( K > 0 \). This is yet another application of Harnack inequality methods to axisymmetric Navier-Stokes equations. Rather than proving Hölder continuity of \( \Theta \) under a global control of a critical norm as we do in Proposition 5.1, [7] obtains (17) by an “almost Hölder continuity,”

\[
\text{osc}_{Q_\rho} \Theta \leq \exp \left( -c \left( \left( \log \frac{100}{\rho} \right)^{\tau} - \left( \log \frac{100}{R} \right)^{\tau} \right) \right) \text{osc}_{Q_R} \Theta
\]

for \( 0 < \rho < R \leq 1/4, \tau \in (0,1) \); see [7, Proposition 1.2]. A similar result in the case of \( \tau = 1/4 \) has been obtained independently by Seregin [35, Proposition 1.3]. Note that the case of \( \tau = 1 \) corresponds to Hölder continuity.

Let us emphasize that the main point of our work is not to improve the blowup rate but to give an explicit bound on \( u \) and its derivatives in terms of only the critical norm—this is a strictly stronger result in the sense that it pertains to all axisymmetric classical solutions, even those not blowing up. A naïve attempt to prove a similar quantitative theorem (e.g., using ideas of estimating axisymmetric vector fields from [17]) would lead to a bound which, compared to Theorem 1.1, would contain more iterated exponentials as well as severe dependence on the time \( t \) and subcritical norms of the initial data. Instead, Theorem 1.1 parallels the results in [39] and improves on those in [27] in the sense that the final bound depends only on \( \|u\|_{L^\infty_t L^3_x} \) and a dimensional factor in \( t \). This also leads to additional interesting corollaries: for instance, an explicit rate of convergence for \( u(t) \to 0 \) as \( t \to +\infty \), and the non-existence of nontrivial ancient axisymmetric solutions in \( L^\infty_t L^3_x \).

A comparison of these results with the work of Chen, Tsai and Zhang [7] raises the following question: Is it possible to efficiently control (in the sense of Theorem 1.1) \( u \) and its derivatives in terms of only \( b \) measured in some critical norm? In fact, in our proof of Hölder continuity of \( \Theta \) near the axis (Proposition 5.1) one can easily replace (6) with boundedness of \( \|b(t)\|_{L^3} \) in time, since “\( u \)” in (13) can be replaced by “\( b \)” due to axisymmetry. However, we do require \( L^3 \) control of all components of \( u \) for other quantitative estimates leading to Theorem 1.1. These include the basic estimates (Lemmas 4.2–4.4), quantitative decay away from the axis (Proposition 5.2), as well as energy estimates on \( \Gamma \) and \( \Phi \) (Proposition 6.1) and their implementation in the main argument (Section 7).

A related open problem is to explicitly control \( u \) in terms of \( u_\theta \) only. In fact, despite a number of works [6, 14, 16, 21, 35, 41] on the properties of the swirl \( ru_\theta \), its role in the regularity problem of axisymmetric solutions remains unclear.

### 1.4. An estimate on the self-similar length scale.

One of the remarkable properties of the quantitative estimate provided by Theorem 1.1 above is that it provides an estimate on the length scale up to which an axisymmetric solution to the NSE (1) can be approximated by a self-similar profile as in (4).

In order to make this precise, we will say that a vector field \( b \in L^\infty(\mathbb{R}^3;\mathbb{R}^3) \) is nearly-spherical if there exists \( \delta \in (0,1/2) \) such that for every \( R > 0 \), there exists \( x_0 \in \mathbb{R}^3 \) with
\[ |x_0| = R \text{ such that} \]
\[ |b(x_0)| \geq \frac{\|b\|_{\infty}}{2} \quad \text{and} \quad |b(x) - b(x_0)| \leq \frac{\|b\|_{\infty}}{4} \quad \text{for all} \ x \in B(x_0, \delta|x_0|). \]  

(19)

Clearly any spherical profile \( b(x) = a(x/|x|) \) is nearly-spherical for every \( a \in C(\partial B(0, 1)) \) (in which case the choice of \( \delta \) for \( 19 \) to hold can be made by a simple continuity argument). Let \( \psi \in C_c^\infty(\mathbb{R}^3; [0, 1]) \) be such that \( \int \psi = 1 \), and let \( \psi_l(x) := l^{-3}\psi(x/l) \) denote a mollifier of a given length scale \( l > 0 \). We also set \( \tilde{\psi}_l := \psi_l * \psi_l \).

We note that, letting \( R := 2l/\delta \), we can find \( x_0 \in \mathbb{R}^3 \) with \( |x_0| = 2l/\delta \) and satisfying (19). In particular

\[
\left| \left( \tilde{\psi}_l * \frac{b(\cdot)}{|\cdot|} \right)(x_0) \right| = \left| \int_{B(x_0,2l)} \tilde{\psi}_l(x_0 - y) \frac{b(y)}{|y|} \, dy \right| \geq \frac{|b(x_0)| - \|b\|_{\infty}/4}{(1 + \delta)|x_0|} \geq \frac{\delta\|b\|_{\infty}}{16l},
\]

which shows that

\[
\left\| \tilde{\psi}_l * \frac{b(\cdot)}{|\cdot|} \right\|_{\infty} \geq \frac{\delta\|b\|_{\infty}}{16l}
\]

(20)

for every length scale \( l > 0 \). This simple fact lets us deduce from Theorem 1.1 that, if an axisymmetric solution approximates a self-similar profile \( b(t, x)/|x| \) up to length scale \( l(t) \), where \( b \) is nearly-spherical uniformly on \([0, t]\), then \( l(t) \) cannot be smaller than a particular quantitative threshold.

**Corollary 1.3.** If \( u \) is a strong axisymmetric solution \( u \) of (1) on \([0, T]\),

\[
\left\| u(t) - \psi_l(t) * \frac{b(t, x)}{|x|} \right\|_{L^3,\infty} \leq \sigma\|b(t)\|_{\infty}
\]

(21)

for \( t \in [0, T] \), and \( \sigma < \epsilon \delta \), where \( \epsilon > 0 \) is a sufficiently small constant and \( b(T) \) is nearly-spherical with constant \( \delta \), then

\[
l(t) \geq \delta T \frac{\|b(T)\|_{\infty}}{\exp \left( \frac{\|b\|_{L^1([0,T] \times \mathbb{R}^3)}^{O(1)} \right)}.
\]

**Proof.** We note that, at time \( T \),

\[
\|u\|_{\infty} \geq \|\tilde{\psi}_l * u\|_{\infty}
\]

\[
\geq \left\| \tilde{\psi}_l * \frac{b(\cdot)}{|\cdot|} \right\|_{\infty} - \left\| \psi_l * \left( u - \tilde{\psi}_l * \frac{b(\cdot)}{|\cdot|} \right) \right\|_{\infty}
\]

\[
\geq \frac{\delta\|b\|_{\infty}}{16l} - CT^{-1} \left\| u - \psi_l * \frac{b(\cdot)}{|\cdot|} \right\|_{L^3,\infty}
\]

\[
\geq \left( \frac{\delta}{16} - C \sigma \right) \frac{\|b\|_{\infty}}{l}.
\]

Thus \( \|u(T)\|_{\infty} \geq \delta\|b(T)\|_{\infty}/32l \) if \( \sigma \in (0, \delta/32C) \). Since also

\[
\|u(t)\|_{L^3,\infty} \leq \left\| \tilde{\psi}_l(t) * \frac{b(t, \cdot)}{|\cdot|} \right\|_{L^3,\infty} + \left\| u(t) - \psi_l(t) * \frac{b(t, \cdot)}{|\cdot|} \right\|_{L^3,\infty} \leq C\|b(t, \cdot)\|_{\infty},
\]

for all \( t \in [0, T] \), Theorem 1.1 implies that

\[
\frac{\delta\|b(T)\|_{\infty}}{32 l(T)} \leq \|u(T)\|_{\infty} \leq T^{-1/2} \exp \left( \frac{\|b\|_{L^1([0,T] \times \mathbb{R}^3)}^{O(1)}}{T} \right),
\]

from which the claim follows. \( \square \)
1.5. Organization of the paper. The structure of the paper is as follows. In the following Section 2 we discuss preliminary concepts related to the Lorentz spaces $L^{p,q}$, the Bogovskiî operator, a simple Poisson-type tail estimate that we will later (in Section 8) expand to obtain our Poisson-type estimate (15) above, as well as some properties of cylindrical coordinates. In Section 3 we discuss some properties of axisymmetric functions, including an axisymmetric Bernstein inequality (Section 3.1) and a quantified version of Hardy’s inequality (Section 3.2). In Section 4 we present some quantitative estimates of the 3D Navier–Stokes equations, including the Picard iterates (Section 4.1), times of regularity, bounded total speed, and second derivatives estimates (Section 4.2), all of which remain valid without the assumption of axisymmetry. The following section, Section 5, is dedicated to quantitative estimates that are specific to the axisymmetric setting (5) of the equations (1). These include the statement of the Hölder estimate of the swirl $\Theta$ mentioned above (Section 5.1), pointwise estimates away from the axis (Section 5.2), as well as the statement of the Poisson-type $x_3$-uloc estimate on $u_t/r$ (15) (Section 5.3). In Section 6 we prove the energy estimate (14) for $\Gamma$ and $\Phi$ mentioned above, and Section 7 combines the developed methods to prove the main theorem, Theorem 1.1. The following section, Section 8, includes a detailed proof of the Poisson-type estimate (15), and Appendix A includes a detailed verification of the Hölder estimate of $\Theta$.

2. Preliminaries

Given $f : \Omega \to \mathbb{R}$ we let

$$\text{osc } f := \sup_{\Omega} f - \inf_{\Omega} f$$

dozen the oscillation of $f$ over $\Omega$. We also denote by $f_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} f$ the average over $\Omega$.

We use standard definitions of Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $W^{k,p}(\Omega)$, spaces of continuous functions $C(\Omega)$, spaces $C_c(\Omega)$ of continuous functions with compact support. For brevity of notation we often omit “$\Omega$” in the notation if $\Omega = \mathbb{R}^3$; for example $W^{1,\infty} \equiv W^{1,\infty}(\mathbb{R}^3)$. We use the convention $\| \cdot \|_p := \| \cdot \|_{L^p(\mathbb{R}^3)}$, and we reserve the notation $\| \cdot \| := \| \cdot \|_2$ for the $L^2(\mathbb{R}^3)$ norm. We also write $f := f_{\mathbb{R}^3}$. Given $p \in [1, \infty]$, we also define the uniformly local $L^p$ norms,

$$\|u\|_{L^p_{uloc}} := \sup_{x \in \mathbb{R}^3} \|u\|_{L^p_{\text{loc}}(B(x,1))} \quad \text{and} \quad \|u\|_{L^p_{x=uloc}} := \|u\|_{L^p_{\text{uloc}}} \|L^p_t,$$

as well as the norms that are uniformly local in $x_3$ only,

$$\|f\|_{L^p_{3=uloc}(\mathbb{R}^3)} := \sup_{z \in \mathbb{R}} \|f\|_{L^p_{\text{loc}}(\mathbb{R}^2 \times [z-1,z+1])}. \quad (22)$$

We let $\Psi(x,t) := (4\pi t)^{-3/2} e^{-x^2/4t}$ denote the heat kernel, which satisfies

$$\|\nabla^k \Psi(t)\|_p = C_{k,p} t^{-\frac{3}{2}(1-\frac{1}{p})} \frac{1}{t^\frac{k}{2}}. \quad (23)$$

We often use the notation $e^{i\Delta} f := \Psi(t) \ast f$.

Given $N \in \{2^k : k \in \mathbb{N}\}$ we let $P_N$ denote the $N$-th Littlewood-Paley projection. We recall a localized version of the Bernstein inequality

$$\|P_N f\|_{L^p(\Omega)} \lesssim_k N^{\frac{3}{2}} \frac{3}{k} \|P_N f\|_{L^{p_1}(\Omega_R)} + (RN)^{-k} N^{\frac{3}{2}-\frac{3}{q}} \|P_N f\|_{l^{p_2}}, \quad (24)$$

where $\Omega \subset \mathbb{R}^3$ is an open set, $k \geq 1$, $\Omega_R := \{x \in \mathbb{R}^3 : \text{dist}(x,\Omega) < R\}$, $q \in [1, \infty]$ and $p_1, p_2 \in [1, q]$; see [39, Lemma 2.1] for a proof.
2.1. Lorentz spaces. We recall the Lorentz spaces, defined by
\[ \|f\|_{L^{p,q}} := \|f\|_{L^{p,q}(\mathbb{R}^n)} \]
for \( q < \infty \) and
\[ \|f\|_{L^{p,\infty}} := \|\lambda|\{f| \geq \lambda\}|^{1/p}\|\lambda|\{f| \geq \lambda\}|^{1/p}\].

We recall the Hölder inequality for Lorentz spaces,
\[ \|fg\|_{L^{p,q}} \leq C_{p_1,p_2,q_1,q_2} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \]
whenever \( 1/p = 1/p_1 + 1/p_2, \ 1/q = 1/q_1 + 1/q_2, \ p_1,p_2,p \in (0,\infty), \ q_1,q_2,q \in (0,\infty) \). We
the reader to [38, Theorem 6.9] for a proof of (26). The Hölder inequality can be very
useful when estimating some localized integrals in terms of the \( L^{p,\infty} \) norm. For example, if
\( \phi \in C_0^\infty(\Omega) \) is a smooth cutoff function then we have the simple estimate
\[ \|\phi\|_{L^{p,1}} = p \int_0^\infty |\{f| \geq \lambda\}|^{1/p}d\lambda \leq p \int_0^{\|\phi\|_\infty} |\{f| \geq \lambda\}|^{1/p}d\lambda \leq p |\Omega|^{1/p} \|\phi\|_\infty, \]
which shows that, for example
\[ \int_{\Omega} fg \leq \|f\|_{L^{3,\infty}} \|g\|_{L^{1,2}}^{1/6}. \]
This simple method allows us to use the weak \( L^3 \) space to estimate some integrals over a
region close to the axis of symmetry.

We also note two Young’s inequalities involving weak \( L^p \) spaces
\[ \|f * g\|_{L^{p,\infty}} \lesssim \|f\|_{L^{1}} \|g\|_{L^{p,\infty}} \quad \text{for} \quad p \in (1,\infty), \]
(27)
\[ \|f * g\|_p \lesssim \|f\|_r \|g\|_{L^{q,\infty}} \quad \text{for} \quad p,q,r \in (1,\infty) \quad \text{with} \quad \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}, \]
(28)
see [18, Proposition 2.4(a)] and [30, Theorem A.16] for details (respectively).

2.2. The Bogovskiǐ operator. We recall that, given \( p \in (1,\infty) \), an open ball \( B \subset \mathbb{R}^3 \),
\( b \in W^{1,p}(B) \) such that \( \text{div} \ b = 0 \), and \( \phi \in C_0^\infty(\Omega; [0,1]) \) such that \( \phi = 1 \) on \( B/2 \) there exists
\( \tilde{b} \in W^{1,p}(\mathbb{R}^3) \) such that \( \tilde{b} = 0 \) outside \( B \) and inside \( B/2 \),
\[ \text{div} \ \tilde{b} = \text{div}(\phi b) \quad \text{and} \quad \|\tilde{b}\|_{W^{1,p}} \lesssim \|b\|_{W^{1,p}(B)}, \]
(29)
due to the Bogovskiǐ lemma (see [4, 5] or [12, Lemma III.3.1], for example). We note that
the Bogovskiǐ lemma often assumes that the domain is star-shaped (which is not the case
for \( B \setminus B/2 \)), but it can be overcome in this particular setting by applying the partition of
identity to \( \phi \); see [25, Section 2.3] for example.

2.3. A Poisson-type tail estimate. Here we are concerned with a Poisson equation of the
form \( -\Delta f = D^2g \), and we show that any \( W^{k,\infty}(B(0,1)) \) norm of \( \nabla f \) can be bounded by the
\( L^{1}_{uloc} \) norm of \( g \), if \( g = 0 \) on \( B(0,2) \).

To be more precise, we let \( \psi \in C_c^\infty(B(0,1); [0,1]) \) be such that \( \psi = 1 \) on \( B(0,1/2) \). Given
\( y \in \mathbb{R}^3 \) we set
\[ \psi_y(x) := \psi(x - y). \]
(30)
and
\[ \tilde{\psi} := \sum_{j \in \mathbb{Z}^3} \psi_j. \]
Lemma 2.1. Suppose that \( f = D^2(-\Delta)^{-1}(g(1 - \tilde{\psi})) \) for some \( g \in L^2 \). Then
\[
\|\psi \nabla f\|_{W^{k,\infty}} \lesssim_k \|g\|_{L^1_{\text{uloc}}} \quad \text{for } k \geq 0.
\]

Proof. We note that
\[
\partial_i f(x) = \int \frac{(x_i - y_i)g(y)(1 - \tilde{\phi}(y))}{|x - y|^5} dy
\]
for \( x \in \text{supp } \phi \), and so
\[
|\nabla f(x)| \leq \int_{\{|x-y|\geq 5\}} \frac{|g(y)|}{|x - y|^4} dy
\]
\[
\leq \sum_{j \in \mathbb{Z}^3} \int_{x_1 + j_1}^{x_1 + j_1 + 1} \int_{x_2 + j_2}^{x_2 + j_2 + 1} \int_{x_3 + j_3}^{x_3 + j_3 + 1} \frac{|g(y)|}{|x - y|^4} dy_3 dy_2 dy_1
\]
\[
\lesssim \|g\|_{L^1_{\text{uloc}}} \sum_{j \in \mathbb{Z}^3} \frac{|j|^{-4}}{\lesssim \|g\|_{L^1_{\text{uloc}}}},
\]
as required. An analogous argument applies to higher derivatives of \( f \). \(\square\)

The above proof demonstrates a simple method of tail estimation which we will later use to obtain a \( \tilde{L}^2_{3-\text{uloc}} \) estimate of \( u_r/r \) in terms of \( \Gamma \), mentioned in the introduction (recall (15)). In fact, to this end, a similar strategy can be applied in the \( x_3 \) direction only, and can be extended to the more challenging biLaplacian Poisson equation (see Lemma 8.2 below).

2.4. Cylindrical coordinates. Given \( x \in \mathbb{R}^3 \) we denote by \( x' := (x_1, x_2) \) the horizontal variables, and \( r := (x_1^2 + x_2^2)^{1/2} \) denotes the radius in the cylindrical coordinates. We often use the notation
\[
\{r < r_0\} := \{x \in \mathbb{R}^3 : r < r_0\}
\]
for a given \( r_0 > 0 \).

We recall a version of the Hardy inequality
\[
\|r^{-1}f\|_{L^q(\Omega)} \lesssim C(\Omega) \|f\|_{L^q(\Omega)} + \|\nabla f\|_{L^q(\Omega)}
\]
(31)
where \( \Omega \) is a bounded domain and \( q \in (1, 2] \); see [6, Lemma 2.4] for a proof.

We recall the divergence operator in cylindrical coordinates: if \( v = v_r e_r + v_\theta e_\theta + v_3 e_3 \) then
\[
\text{div } v = \frac{1}{r} \partial_r rv_r + \frac{1}{r} \partial_\theta v_\theta + \partial_3 v_3
\]
(32)

We say that a vector field \( v \) is axisymmetric if
\[
\partial_\theta v_r = \partial_\theta v_3 = \partial_\theta v_\theta = 0.
\]
(33)

In such case we have
\[
|\nabla' v|^2 = (\partial_r v_r)^2 + (\partial_r v_\theta)^2 + (\partial_r v_3)^2 + \frac{1}{r^2}(v_r^2 + v_\theta^2),
\]
(34)
which implies the pointwise bounds
\[
\frac{|v_r|}{r}, \frac{|v_\theta|}{r} \leq |\nabla' v|.
\]
Here \( \nabla' \) refers to the gradient with respect to the horizontal variables \( x' \) only.
Moreover,
\[ |\partial_{rr} f| \lesssim |D^2 f| \] (35)

Indeed, since
\[ \partial_r = \cos \theta \partial_1 + \sin \theta \partial_2 = \frac{x_1}{|x'|} \partial_1 + \frac{x_2}{|x'|} \partial_2, \]
where \( x' := (x_1, x_2) \) refers to the horizontal variables, we can compute that
\[ \partial_{rr} = \frac{x_2}{|x'|^2} \partial_1 + 2 \frac{x_1 x_2}{|x'|^2} \partial_1 \partial_2 + \frac{x_2^2}{|x'|^2} \partial_{22}, \]
from which (35) follows. More generally,
\[ \partial_{rrr} = \frac{x_3}{|x'|^3} \partial_{111} + \frac{x_1^2}{|x'|^3} \partial_{111} \partial_2 + \frac{3 x_1 x_3}{|x'|^3} \partial_1 \partial_{22} + \frac{x_3^2}{|x'|^3} \partial_{222}, \]
\[ \partial_{rrrr} = \frac{x_4}{|x'|^4} \partial_{1111} + \frac{x_1^3}{|x'|^4} \partial_{1111} \partial_2 + \frac{4 x_1 x_3^2}{|x'|^4} \partial_{111} \partial_{22} + \frac{6 x_1 x_2^2}{|x'|^4} \partial_{11} \partial_{22} + \frac{4 x_1 x_2 x_3}{|x'|^4} \partial_1 \partial_{222} + \frac{x_4^2}{|x'|^4} \partial_{2222}. \]

This shows that
\[ |D_{r,x_3}^3 f| \lesssim |D^3 f| \]
and
\[ |D_{r,x_3}^4 f| \lesssim |D^4 f| \] (36)
for any axisymmetric \( f \) (here, for example, \( D^4 \) refers to all fourth order derivatives with respect to \( x_1, x_2, x_3 \)).

### 3. Properties of axisymmetric functions

Here we discuss some properties of axisymmetric functions, including an axisymmetric Bernstein inequality and a quantified Hardy’s inequality.

#### 3.1. Bernstein inequalities

Here we discuss a version of the axisymmetric Bernstein inequality provided by [27, Proposition 1] that involves the weak \( L^3 \) space.

**Lemma 3.1.** Let \( T_m \) be a Fourier multiplier whose symbol \( m \) is supported on \( B(0, N) \) with \( |\nabla^2 m| \leq MN^{-1} \) and \( 1 < q < p \leq \infty \). If either \( -\frac{1}{p} < \alpha < \frac{1}{q} - \frac{1}{p} \) or \( p = \infty \) and \( \alpha = 0 \), we have
\[ \|r^\alpha T_m u\|_{L^p} \lesssim MN^{\frac{3}{q} - \frac{3}{p} - \alpha} \|u\|_{L^{q, \infty}} \]
for all axisymmetric scalar- or vector-valued functions \( u \).

**Proof.** We normalize \( M = N = 1 \). Under these assumptions on \( p, \alpha \), Proposition 1 in [27] implies
\[ \|r^\alpha T_m u\|_{L^p} \lesssim \|P_{\leq 10} u\|_{L^{p, \infty}} \]
since \( T_m P_{\leq 10} = T_m \), for an \( \epsilon > 0 \) sufficiently small depending on \( p, q, \alpha \). Let \( \psi \) be the kernel such that \( P_{\leq 10} = \psi \ast \). Then by the weak Young inequality (28),
\[ \|P_{\leq 10} u\|_{L^{q, \infty}} \lesssim \|\psi\|_{L^{1+\alpha(\epsilon)}} \|u\|_{L^{q, \infty}} \lesssim \|u\|_{L^{q, \infty}}. \]
\[ \square \]
A useful consequence of the above lemma is the following heat kernel estimate

\[ \| r^\alpha \Delta \nabla^j f \|_{L^p} \leq \| r^\alpha \Delta \nabla^j P \|_{L^p} + \sum_{N>1} \| r^\alpha \Delta \nabla^j P_N f \|_{L^p} \]

\[ \lesssim \| f \|_{L^q,\infty} (1 + \sum_{N>1} e^{-N^2/100 N^{\frac{2}{q} - \frac{3}{p}}} ) \]

under the same assumptions on the parameters as in Lemma 3.1.

3.2. A quantified version of the Hardy inequality. By the classical Hardy inequality

\[ \| r^{-\frac{3}{p} + \frac{1}{2}} f \|_p \lesssim_p (\| f \|_2 + \| \nabla f \|_2) \]

for any axisymmetric \( f \), and \( p \in (2, 6) \) (see [6, Lemma 2.6], for example). Here we prove a version of this inequality, which is localized in the horizontal variables, “uloc” in \( x_3 \), and which has a quantified divergence of the constant near \( p = 2 \). Namely we prove the following.

**Lemma 3.2** (Quantified Hardy inequality). For \( p \in (2, 6 - \epsilon) \),

\[ \| r^{-\frac{3}{p} + \frac{1}{2}} f \|_{L^p_{uloc, r \leq 1}} \lesssim \epsilon (p - 2)^{-O(1)} \left( \| f \|_{L^2_{uloc, r \leq 1}} + \| \nabla f \|_{L^2_{uloc, r \leq 1}} \right) . \]

**Proof.** From the Sobolev embedding

\[ \| u \|_{L^{2p/(2-p)}(\mathbb{R}^2)} \lesssim (2 - p)^{-O(1)} \| \nabla u \|_{L^p(\mathbb{R}^2)} \]

for \( p < 2 \), (see, e.g., [37] where the sharp constant is computed), one can prove the two-dimensional Gagliardo-Nirenberg inequality

\[ \| f \|_{L^q(B(1))} \lesssim q \left( \| f \|_{L^6(B(1))}^{\frac{6}{q}} \| \nabla f \|_{L^1(B(1))}^{\frac{1}{q}} + \| f \|_{L^p(B(1))} \right) \]

(38)

for \( q > 6 \). Fix \( \epsilon > 0 \) to be specified. Then

\[ \| f \|_{L^p_{uloc, r \geq \epsilon}} \lesssim \| r^{-\frac{3}{p} + \frac{1}{2}} f \|_{L^{6q/(6-q)}_{uloc, r \geq \epsilon}} \| f \|_{L^{6q/(6-q)}_{uloc}(\mathbb{R}^2)} \lesssim \epsilon^{-\frac{1}{q} + \frac{1}{2}} \| f \|_{L^{6q/(6-q)}_{uloc}(\mathbb{R}^2)} . \]

Inside, for any \( \frac{1}{q} \in (\frac{3}{2p} - \frac{1}{4}, \frac{1}{p}) \), by (38),

\[ \| f \|_{L^p_{uloc, r \leq \min(1, \epsilon)}} \lesssim \| r^{-\frac{3}{p} + \frac{1}{2}} f \|_{L^{6q/(6-q)}_{uloc, r \leq \min(1, \epsilon)}} \| f \|_{L^{6q/(6-q)}_{uloc, r \leq \min(1, \epsilon)}} \]

\[ \lesssim \left( \frac{1}{s} - \frac{3}{2p} + \frac{1}{4} \right)^{-\frac{1}{s}} \left( \frac{1}{p} - \frac{1}{s} \right)^{-1} \left( \epsilon^{-\frac{1}{q} + \frac{1}{2}} \| f \|_{L^6_{uloc}(B(1))} + \| f \|_{L^6_{uloc}(B(1))} \right) . \]

Upon taking \( \epsilon = \| f \|_{L^6_{uloc}(B(1))}^3 \) and \( \frac{1}{s} = \frac{4}{3p} = \frac{4}{6} \),

\[ \| f \|_{L^p_{uloc}(B(1))} \lesssim (p - 2)^{-O(1)} \left( \| f \|_{L^6_{uloc}(B(1))} + \| f \|_{L^6_{uloc}(B(1))} + \| f \|_{L^6_{uloc}(B(1))} \right) . \]

Finally by Hölder’s inequality, Sobolev embedding, and Gagliardo-Nirenberg interpolation, we find

\[ \| f \|_{L^p_{uloc}(B(1) \times B_k(z, 1))} \lesssim (p - 2)^{-O(1)} \| f \|_{H^1(B_k^2(1) \times B_k(z, 1))} . \]
as required.

4. Basic estimates for the Navier-Stokes solutions

Here we discuss some estimates for the Navier-Stokes equations without the assumption of axisymmetry.

4.1. The Picard estimates. We define the flat and sharp Picard iterates

\[ u_n^\flat(t) := e^{(t-t_n)\Delta} u(t_n) - \int_{t_n}^t e^{(t-t')\Delta} \text{div}(u_{n-1}^\flat \otimes \nabla u_n^\flat(t')) \, dt', \quad u_n^\sharp := u - u_n^\flat \quad (39) \]

for all \( n = 1, 2, \ldots \) and \( t \geq t_n \) where \( t_n \in [0, \frac{1}{2}] \) is an increasing sequence of times, and \( u_0^\flat := 0, u_0^\sharp := u \). We have the following.

**Lemma 4.1** (Basic Picard estimates). Assume \( u \) solves (1) on \([0, 1] \times \mathbb{R}^3\) with the bound (6). If \( p \in (3, \infty) \) and \( -\frac{2}{p} < \alpha < \frac{1}{3} - \frac{1}{p} \) or \( p = \infty \) and \( \alpha = 0 \), we have

\[ \| r^\alpha \nabla^j u_n^\flat \|_{L_t^\infty L_x^p([\frac{1}{2}, 1] \times \mathbb{R}^3)} \leq A^{O_{n,j,p}(1)}, \quad (40) \]

\[ \| u_n^\flat \|_{L_t^\infty L_x^p([\frac{1}{2}, 1] \times \mathbb{R}^3)} \leq A^{O_{n,q}(1)} \quad \text{for all } q \in (1, 3), \quad (41) \]

\[ \| \nabla^j P_N u_n^\flat \|_{L_t^\infty L_x^p([\frac{1}{2}, 1] \times \mathbb{R}^3)} \leq e^{-N^2/O_{n,j}(1)} A^{O_{n,j}(1)}, \quad (42) \]

as well as the energy estimate

\[ \| \nabla u_n^\flat \|_{L_t^2 L_x^2([\frac{1}{2}, 1] \times \mathbb{R}^3)} \leq A^{O_{n}(1)}. \quad (43) \]

In particular,

\[ \| \nabla u \|_{L_t^2 L_x^2-u_{loc}([\frac{1}{2}, 1] \times \mathbb{R}^3)} \leq A^{O(1)}. \quad (44) \]

The proof of (40)--(42) above rely only on the definition (39) as well as basic heat estimates (23), which, together with the weak Young’s inequality (28), can be used in the same way as [39, (3.11)--(3.13)] and [28, Proposition 2.5] to obtain the estimates with \( \| u \|_{L^\infty([0,1];L^3,\infty)} \leq A \) on the right-hand side.

4.2. Basic estimates. Here we assume that \( u \) satisfies (1) with the weak \( L^{3,\infty} \) bound (6) on time interval \([0, T] \).

**Lemma 4.2** (Choice of time of regularity). If \( u \) solves (1) on a time interval \( I \) and satisfies \( \| u \|_{L_t^\infty L_x^{3,\infty}(I \times \mathbb{R}^3)} \leq A \), then there exists \( t_* \in I \) such that

\[ \| \nabla^j u(t_*) \|_{L_x^\infty} \leq |I|^{-\frac{j+1}{2}} A^{O(1)} \]

for all \( j = 0, 1, 2, \ldots, 10 \).

**Lemma 4.3** (Bounded total speed). We have the bounded total speed estimate

\[ \| u \|_{L_t^1 L_x^\infty(I/2 \times \mathbb{R}^3)} \leq |I|^{\frac{1}{2}} A^{O(1)}. \]

The 2 lemmas above follow by the same arguments in [39, Lemma 3.1] and [11, Propositions 3.1--2] using the estimates in Lemma 4.1. In particular, it is straightforward to check that the proofs of Propositions 3.1 and 3.2 in [11] are still valid in Lorentz spaces \( L^{p,q} \) with \( q = \infty \). Furthermore, we estimate \( \nabla^2 u \) in terms of \( A \).
Lemma 4.4 (2nd order derivatives estimates). If \( u \) solves (1) on \([0, T]\) and obeys (6), then
\[
\left\| \nabla^2 u \right\|_{L^{p}_{x}-uloc((\tau,T) \times \mathbb{R}^3)} \lesssim_p A^{O(1)} T^{\frac{5}{2p}}
\]
for \( p \in [1, \frac{4}{3}] \), where the local norm is considered at spatial scale \( T^{\frac{1}{2}} \).

Proof. We use an approach due to Constantin [8]. First rescale to make \( T = 1 \). For every \( \epsilon \in (0, \frac{1}{2}) \), we define the approximation to the function \( \langle x \rangle := (1 + |x|^2)^{\frac{1}{2}} \),
\[
q(x) := \langle x \rangle - \frac{1}{2(1 - \epsilon)} \langle x \rangle^{1 - \epsilon}
\]
which satisfies the properties
\[
|\nabla q| \leq 1, \quad (45)
\]
\[
\xi^T \nabla^2 q(x) \xi > \frac{\epsilon}{2} \langle x \rangle^{-(1 + \epsilon)} |\xi|^2, \quad (46)
\]
\[
\frac{1 - 2\epsilon}{2 - 2\epsilon} \langle x \rangle \leq q(x) \leq \langle x \rangle. \quad (47)
\]

With \( \tau \) a time scale to be specified, we define \( w := q(\tau \omega) \) which obeys the equation
\[
(\partial_t + u \cdot \nabla - \Delta) w = \tau \nabla q(\tau \omega) \cdot (\omega \cdot \nabla u) - \tau^2 \text{tr}(\nabla \omega^T \nabla^2 q \nabla \omega).
\]

Multiplying by a spatial cutoff at length scale \( R \) and integrating over \( \mathbb{R}^d \),
\[
\frac{d}{dt} \int_{\mathbb{R}^3} w \psi \leq \int_{\mathbb{R}^3} (u \cdot \nabla \psi + \Delta \psi) w + O(\tau |\nabla u|^2) \psi - \frac{\epsilon}{2} \tau^2 (\tau \omega)^{-(1 + \epsilon)} |\nabla \omega|^2 \psi.
\]

Let \( \tilde{\psi} \) be an enlarged cutoff function so that \( R |\nabla \psi| + R^2 |\Delta \psi| \leq 10 \tilde{\psi} \). We define the \( L^p_{x}-uloc,R \) norm to be the supremum of the \( L^p \) norm restricted to balls of radius \( R \). Integrating in time starting from a \( t_0 \) to be specified and taking a supremum over the balls,
\[
\left\| w(t) \right\|_{L^1_{x}-uloc,R} \lesssim \left( \left\| w(t_0) \right\|_{L^1_{x}-uloc,R} + \tau RA^{O(1)} \right) \exp(R^{-1} |t - t_0|) \left( \left\| w(t) \right\|_{L^1_{x}-uloc,R} + \tau R^{-1} \|u\|_\infty \right) dt' + \tau \left\| \nabla u \right\|_{L^2_{t,x}-uloc,R}^2.
\]

Grönwall’s inequality and (44) imply
\[
\left\| w(t) \right\|_{L^1_{x}-uloc,R} \lesssim \left( \left\| w(t_0) \right\|_{L^1_{x}-uloc,R} + \tau RA^{O(1)} \right) \exp(R^{-2} |t - t_0|) + R^{-1} A^{O(1)} |t - t_0|^{\frac{1}{2}}.
\]

Setting \( R = A^{C_1} \) and \( \tau = A^{-2C_1} \) for a sufficiently large \( C_1 \), we find
\[
\left\| \langle \tau \omega(t) \rangle \right\|_{L^1_{x}-uloc,R} \lesssim \left\| \langle \tau \omega(t_0) \rangle \right\|_{L^1_{x}-uloc,R}.
\]

By (44) and Hölder’s inequality, we can find a \( t_0 \in [1/4, 1/2] \) where the right-hand side is bounded by \( A^{O(1)} \). Therefore
\[
\int_{t_0}^t \int_{\mathbb{R}^3} \langle \tau \omega \rangle^{-(1 + \epsilon)} |\nabla \omega|^2 \psi \, dx \, dt \leq \epsilon^{-1} A^{O(1)}.
\]

We use Hölder’s inequality with the decomposition
\[
|\nabla \omega|^\frac{4}{3 + \epsilon} = \langle |\nabla \omega|^\frac{4}{3 + \epsilon} \rangle \langle \tau \omega \rangle^{-\frac{2}{3 + \epsilon}} \langle \tau \omega \rangle^{\frac{2}{3 + \epsilon}}
\]
to conclude
\[ \| \nabla \omega \|_{L^{4/(3+\epsilon)}_t L^{3+\epsilon}_x([t_0,t] \times \mathbb{R}^3)} \leq \epsilon^{-O(1)} A^{O(1)}. \]

To convert this into a bound on $\nabla^2 u$, fix a unit ball $B \subset \mathbb{R}^3$ and a cutoff function $\varphi \in C_c^\infty(3B)$ with $\varphi \equiv 1$ in $2B$. We decompose $\nabla^2 u = a + b$ where $a = \nabla^2 \Delta^{-1} \text{curl}(\varphi \omega)$. Note that $b = \nabla f$ where $f = \nabla \Delta^{-1} \text{curl}((1 - \varphi) \omega)$ is harmonic in $2B$ so for any $p \in [1, \frac{3}{2})$,
\[ \|a\|_{L_p^p([t_0,t] \times B)} \lesssim \|\nabla \omega\|_{L_p^p([t_0,t] \times 3B)} + \|\nabla \varphi\|_{L_p^\infty} \|\omega\|_{L_p^2([t_0,t] \times \mathbb{R}^3)} \leq \epsilon^{-O(1)} A^{O(1)} \]
and
\[ \|b\|_{L_{t,x}^p([t_0,t] \times B)} \lesssim \|\nabla \Delta^{-1} \text{curl}((1 - \varphi) \omega)\|_{L_{t,x}^p([t_0,t] \times 2B)} \lesssim \|\omega^2\|_{L_{t,x}^3([t_0,t] \times \mathbb{R}^3)} + \|\omega^b\|_{L_{t,x}^\infty([t_0,t] \times \mathbb{R}^3)} \leq A^{O(1)} \]
where we have used (44), Hölder’s inequality, (43), and (40).

\section{5. Estimates for axisymmetric Navier-Stokes solutions}

Here we provide some estimates of classical solutions of (1) that are specific to the axisymmetric assumption on the solutions.

We first note that $u_\theta$ satisfies
\[ \left( \partial_t + u \cdot \nabla - \Delta + \frac{1}{r^2} \right) u_\theta + \frac{u_r}{r} u_\theta = 0, \]
which in particular gives that the swirl $\Theta := ru_\theta$ satisfies
\[ \left( \partial_t + \left( u + \frac{2}{r} \hat{e}_r \right) \cdot \nabla - \Delta \right) \Theta = 0 \]
in $(\mathbb{R}^3 \setminus \{r = 0\}) \times (0, T)$. It then follows that, at each time, $u_\theta(r, x_3)$ is a continuous function on $\mathbb{R}_+ \times \mathbb{R}$ with $u_\theta(0, x_3) = 0$ for all $x_3$ (see [19, Lemma 1] for details). In particular
\[ \Theta(0, 0, x_3) = 0 \]

Moreover, since $\omega = \omega_r e_r + \omega_\theta e_\theta + \omega_3 e_3$ is a smooth vector field we see (also by [19, Lemma 1]) that $\Phi = \frac{\omega}{r}$, $\Gamma := \frac{\omega}{r} (\text{recall (7)})$ satisfy
\[ |\Phi(r, x_3, t)|, |\Gamma(r, x_3, t)| \lesssim C(x_3, t) \]
for $r \in [0, 1]$.

\subsection{5.1. Hölder continuity near the axis}

Here we consider the parabolic equation
\[ \mathcal{M}V := \partial_t V - \Delta V + b \cdot \nabla V = 0 \]
in a space-time cylinder
\[ Q_R(x_0, t_0) := B(x_0, R) \times (t_0 - R^2, t_0). \]
We assume that at each point of $Q_R$
\[ \begin{align*}
\text{either } \text{div } b &= 0 \quad \text{or } \quad V &= 0. \\
\end{align*} \]
We also assume that
\[ \mathcal{N}(R) := 2 + \sup_{R' \leq R} (R')^{-\alpha} \|b\|_{L_1^1 L_2^2(Q_{R'})} < \infty \]
where \( \alpha := \frac{3}{q} + \frac{2}{r} - 1 \in [0, 1) \). In such setting [20, Corollary 3.6] observed that \( V \) must be Hölder continuous in the interior of \( Q_R \), and in the proposition below we state a version of their result in which we quantify the dependence of the Hölder exponent in terms of \( N \).

**Proposition 5.1.** If \( V \) is a Lipschitz solution of (52) then

\[
\text{osc}_{B(r)} V(0) \leq \left( \frac{r}{R} \right)^\gamma \text{osc}_{Q(R)} V
\]

for all \( r \leq R \), where \( \gamma = \exp(-N^{O(1)}) \).

**Proof.** See Appendix A. \( \square \)

We note that the swirl \( \Theta \) satisfies (52) with \( b := u + 2e_r/r \) (recall (49) above). Moreover \( \text{div} b = 0 \) everywhere except for the axis, since \( \text{div} u = 0 \), \( \text{div}(e_r/r) = 0 \) (recall (32)) there. Moreover, \( V = 0 \) on the axis (recall (50)), and so the assumption (53) holds. Thus Proposition 5.1 shows that \( \Theta \) is Hölder continuous in a neighbourhood of the axis. We explore this in more detail in the proof of Theorem 1.1 below, where we quantify \( N \) in terms of the weak-\( L^3 \) bound \( A \) (see Step 1 in Section 6 below).

### 5.2. Pointwise estimates away from the axis.

The following is a more precise version of Proposition 8 in [27].

**Proposition 5.2** (Pointwise bounds away from the axis). Let \( u \) solve (1) on \([0, 1]\) satisfying (6). Then for every \( \epsilon \in (0, 4/15) \), we have

\[
|\nabla^j u| \leq \left( r^{-1-j} + r^{-\frac{1}{3}+\epsilon} \right) A^{O_{c,j}(1)}
\]

for each \( t \in [1/2, 1] \). We also have

\[
\|u\|_{L^p(r \geq 1)} \leq A^{O_p(1)}
\]

for each such \( t \), and \( p \in (3, \infty] \).

**Proof.** We first pick any \( \alpha \in (1/3 - \epsilon/2, 1/3) \) and \( c = c(j) > 0 \) sufficiently small so that

\[
(1 - \alpha + j)c < \epsilon/2 \quad \text{and} \quad c < \alpha/(1 - \alpha).
\]

We also pick \( n = n(j) \in \mathbb{N} \) sufficiently large so that

\[
n \geq (2 + j) \left( 1 + \frac{1}{c} \right)
\]

We set \( t_k := 1/2 - (1/2)^k \) and we define a sequence of regions \( \{ x \in \mathbb{R}^3 : r \geq R/2 \} = \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n = \{ x \in \mathbb{R}^3 : r \geq R \} \) such that \( \text{dist}(\Omega_i, \Omega_{i+1}) \geq R/2n \).

Given such sequence of times we now consider the corresponding Picard iterates \( u_k^\circ, u_k^\sharp \), for \( k \in \{0, 1, \ldots, n\} \).

**Step 1.** We show that

\[
\|P_N u_k^\circ(t)\|_{L^\infty(r \geq R/2)}, \|P_N u_k^\sharp(t)\|_{L^\infty(r \geq R/2)} \lesssim R^{-\alpha} N^{1-\alpha} A^{O_{\alpha}(1)}
\]

for all \( \alpha \in [0, 1/3) \), \( R > 0 \) and \( t \in [t_k, 1], k \geq 0 \).
In fact, we first observe that Lemma 3.1 gives that
\[ \|r^\alpha P_N u(t)\|_{L^\infty} \lesssim N^{1-\alpha} \|u(t)\|_{L^{3,\infty}} \lesssim N^{1-\alpha} A^{O(1)}. \] (58)

Thus, since the first inequality above is valid for any axisymmetric function, it remains to note that the second inequality is also valid for each \( u_k^\ast, \ u_k^\flat \), on \([t_k, 1], k \geq 0\). Indeed, the case \( k = 0 \) follows trivially, while the inductive step follows by applying Young’s inequality (27) for weak \( L^p \) spaces, and Hölder’s inequality (26) for Lorentz spaces

\[ \|u_k^\ast(t)\|_{L^{3,\infty}} \lesssim \|\Psi(t-t_k)\|_1 \|u(t_k)\|_{L^{3,\infty}} + \int_{t_k}^t \|\nabla \Psi(t-t')\|_1 \|(u_{k-1}^\ast \otimes u_{k-1}^\ast)(t')\|_{L^{3/2,\infty}}dt' \]

\[ \leq C_k A + C_k \|u_{k-1}^\ast\|_{L^\infty([t_k,1];L^{3,\infty})} \int_{t_k}^t (t-t')^{-\frac{1}{2}}dt' \leq A^{O_k(1)} \]

for \( t \in [t_k, 1] \), as required, where we also used the heat kernel bounds (23).

**Step 2.** We show that the inequality from Step 1 can be improved for \( u_k^\ast \) for large \( k \), namely

\[ \|P_N u_k^\ast\|_{L^\infty((\frac{1}{10}, 1] \times \{r \geq R\})} \leq N A^{O_k(1)} ((RN)^{-(k-1)\alpha} + N^{-(k-1)}) \] (59)

for every \( k \geq 1 \) and \( N \in 2^N \cap [100^k \max(1, R^{-1}), \infty) \).

We will show that,

\[ X_{k,N} \leq N^{-\frac{3}{4}} A^{O_k(1)} ((RN)^{-(k-1)\alpha} + N^{-(k-1)}) \]

(60)

for \( k \geq 1 \) and \( N \geq 100^k \max(1, R^{-1}) \), using induction with respect to \( k \), where

\[ X_{k,N} := \|P_N u_k^\ast\|_{L^\infty([t_k+1,1];L^{5/3}(\Omega_k))}. \]

Then (59) follows by the local Bernstein inequality (24).

As for the base case \( k = 1 \) we note that (37) gives that

\[ \|P_N u_1^\ast(t)\|_{L^{5/3}} \lesssim \int_{t_1}^t \|P_N e^{(t-t')\Delta} \mathbb{P} \text{div}(u \otimes u)(t')\|_{L^{5/3}}dt' \]

\[ \lesssim \int_{t_1}^t e^{-\frac{t-t'}{N^{2/\gamma}}} N^\frac{6}{5} \|(u \otimes u)(t')\|_{L^{\frac{6}{5},\infty}}dt' \]

\[ \lesssim N^\frac{6}{5} \|e^{-tN^{2/\gamma}}\|_{L^{1}(t_1,1)} \|u\|_{L^{3,\infty}}^2 \]

for \( t \in [t_1, 1] \). Thus

\[ X_{1,N} \leq \|P_N u_1^\ast\|_{L^\infty([t_2,1];L^{5/3})} \leq N^{-\frac{3}{4}} A^{O(1)}, \]

(61)

due to Hölder’s inequality for Lorentz spaces (26).
Moreover, the frequency-localized bounds (42) for \(u^a\) where we use the notation (41), as well boundedness of \(P\), where we used the weak \(L^1\) bound (6) and Lemma 3.1 for the \(u \otimes u\) term and (40) for the \(u^b \otimes u^b\) term. Thus we can use the paraproduct decomposition in the first term on the right-hand side to obtain

\[
X_{k,N} \lesssim N^{-1} \sum_{\Omega} \left| Y_1 + \cdots + Y_5 \right| L^\infty([t_k,1];L^{5/3}((\Omega_k\pm)_{\delta})) + N^{-\frac{1}{2}} (NR)^{-(k-1)\alpha} A^{O_k(1)},
\]

where

\[
Y_1 \coloneqq \sum_{N' \sim N} P_{N'} u^a_{k-1} \otimes P_{\leq N/100} u^a_{k-1},
\]

\[
Y_2 \coloneqq \sum_{N_1 \sim N_2 \geq N} P_{N_1} u^b_{k-1} \otimes P_{N_2} u^b_{k-1},
\]

\[
Y_3 \coloneqq \sum_{N' \sim N} P_{N_1} u^b_{k-1} \otimes P_{N_2} u^b_{k-1},
\]

\[
Y_4 \coloneqq \sum_{N' \sim N} P_{N'} u^b_{k-1} \otimes P_{\leq N/100} u^b_{k-1},
\]

\[
Y_5 \coloneqq \sum_{N' \sim N} P_{\leq N/100} u^b_{k-1} \otimes P_{N'} u^b_{k-1},
\]

where we use the notation \(a \odot b := a \otimes b + b \otimes a\). Using (57),

\[
\|Y_1\|_{L^\infty([t_k,1];L^{5/3}((\Omega_k\pm)_{\delta}))} \lesssim \sum_{N' \sim N} \sum_{N' \lesssim N} R^{-\alpha} (N')^{1-\alpha} A^{O_k(1)}
\]

\[
\lesssim R^{-\alpha} N^{1-\alpha} A^{O_k(1)} \sum_{N' \sim N} \sum_{N' \lesssim N} X_{k-1,N'},
\]

and

\[
\|Y_2\|_{L^\infty([t_k,1];L^{5/3}((\Omega_k\pm)_{\delta}))} \lesssim R^{-\alpha} A^{O_k(1)} \sum_{N' \sim N} (N')^{1-\alpha} X_{k-1,N'}.
\]

Moreover, the frequency-localized bounds (42) for \(u^b\) give that

\[
\|Y_3\|_{L^\infty([t_k,1];L^{5/3}(\Omega_k\pm)_{\delta}))} \lesssim A^{O_k(1)} \sum_{N' \sim N} e^{-N'^{2}/O_k(1)} N' X_{k-1,N'},
\]

and (41), as well as boundedness of \(P_{\leq N/100}\) on \(L^{5/3}\) give that

\[
\|Y_4\|_{L^\infty([t_k,1];L^{5/3}(\Omega_k\pm)_{\delta}))} \lesssim A^{O_k(1)} \sum_{N' \sim N} e^{-N'^{2}/O_k(1)} N' \lesssim e^{-N^{2}/O_k(1)} A^{O_k(1)}.
\]
Finally, using boundedness of $P_{\leq N/100}$ on $L^\infty$ and (40) we obtain
\[ \|Y_5\|_{L^\infty((t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N' \sim N} X_{k-1,N'} \]

Combining these estimates into (62), we have shown
\[ X_{k,N} \leq A^{O_k(1)} \left( \left((RN)^{-\alpha} + N^{-1}\right) \sum_{N' \sim N} X_{k-1,N'} + N^{-1}R^{-\alpha} \sum_{N' \geq N} (N')^{1-\alpha} X_{k-1,N'} \right) + N^{-1} \sum_{N' \geq N} e^{-\alpha} e^{-N'2/O_k(1)} X_{k-1,N'} + N^{-\frac{1}{5}} (NR)^{-(k-1)\alpha} + N^{-1} e^{-N'2/O_k(1)} \right) \]

(63)

Since the upper bounds on $X_{k-1,N'}$ provided by the inductive assumption (60) are comparable for all $N' \sim N$, up to constants depending only on $k$, we thus obtain that
\[ \sum_{N' \sim N} X_{k-1,N'} \leq A^{O_k(1)} N^{-\frac{1}{5}} \left((RN)^{-\alpha(k-2)} + N^{-k-2}\right), \]

\[ R^{-\alpha} \sum_{N' \geq N} (N')^{1-\alpha} X_{k-1,N'} \leq A^{O_k(1)} R^{-\alpha} \sum_{N' \geq N} (N')^{1-\alpha-\frac{4}{5}} \left((RN)^{-\alpha(k-2)} + (N')^{-(k-2)}\right) \]

\[ \leq A^{O_k(1)} N^{-\frac{4}{5}} \left((RN)^{-\alpha(k-1)} + N^{-(k-1)}\right), \]

where, in the last line we used the fact that $(k-1)(1-\alpha) - 4/5 < 0$ for any $k \geq 2$. A similar estimate for $\sum_{N' \geq N} e^{-\alpha} e^{-N'2/O_k(1)} N' X_{k-1,N'}$ now allows us to deduce from (63) that
\[ X_{k,N} \leq N^{-\frac{1}{5}} A^{O_k(1)} ((RN)^{-2(k-1)\alpha} + N^{-(k-1)}), \]

as required.

Step 3. We prove the claim.

We first consider the case $R \geq 100^{n/\epsilon}$, and we note that, by (57)
\[ \|P_{N \leq R^\epsilon} \nabla^j u_n^\#\|_{L^\infty_{t,\text{loc}}([\frac{1}{2},1]\times\{t \geq R\})} \leq \sum_{N \leq R^\epsilon} A^{O_n(1)} N^{1-\alpha+j} R^{-\alpha} \leq A^{O_n(1)} R^{-\alpha + (1-\alpha+j)\epsilon} \leq A^{O_n(1)} R^{-\frac{1}{4} + \epsilon}, \]

where we used the choice of $\alpha > 1/3 - \epsilon/2$ and the first property of our choice (55) of $c$ in the last inequality. On the other hand for $N > R^\epsilon$ we can use (59) with $k = n$ to obtain arbitrarily fast decay in $N$. Comparing the terms on the right-hand side of (59) we see that $N^{-(n-2)}$ dominates $(RN)^{-(n-2)\alpha}$ if and only if $N \leq R^\alpha/(1-\alpha)$, which allows us to apply the decomposition
\[ \|P_{N > R^\epsilon} \nabla^j u_n^\#\|_{L^\infty_{t,\text{loc}}([\frac{1}{2},1]\times\{t \geq R\})} \leq \sum_{R^\epsilon < N \leq R^\alpha/(1-\alpha)} A^{O_n(1)} N^{-n+2+j} \]
\[ + \sum_{N > R^\alpha/(1-\alpha)} A^{O_n(1)} N^{1+j} (RN)^{-(n-1)\alpha} \]
\[ \leq A^{O_n(1)} R^{c(n+2+j)} \]
\[ \leq A^{O_n(1)} R^{-1-j}. \]
where we used the second property of our choice (55) of \( c \) in the second inequality, and the choice (56) of \( n \) in the last inequality.

We now suppose that \( R \leq 100^{n/c} \). The low frequencies can be estimated directly from the weak \( L^3 \) bound (6),

\[
\| P_{<100^{2n/c}} \nabla^j u \|_{L^\infty_t([1/4,1] \times \{ r \geq R \})} \lesssim_{n,c} A^{O(1)} R^{-1-j}.
\]

On the other hand, for \( N > 100^{2n/c} R^{-1} \) we have in particular \( N > \frac{R^{\alpha}}{1 - \alpha} \), which shows that the dominant term on the right-hand side of (59) is \( (RN)^{\alpha} \), and so

\[
\| P_{>100^{2n/c} R^{-1}} \nabla^j u_n^b(t) \|_{L^\infty_t(\{ r \geq R \})} \lesssim_{N} N^{1+j} A^{O_{n}(1)} (RN)^{(n-\alpha)} \leq A^{O_{n}(1)} R^{-1-j}
\]

for every \( t \in [1/2, 1] \), as desired. As for the estimate for \( u^b \) we use (40) to obtain

\[
\| \nabla^j u_n^b \|_{L^\infty_t(\{ r \geq R \})} \leq R^{-1/3+\epsilon} \| r^{1/3-\epsilon} \nabla^j u_n^b \|_{\infty} \lesssim_{\epsilon} R^{-1/3+\epsilon} A^{O_{n,j}(1)},
\]

as needed.

The estimate for \( \| u \|_{L^p_t(\{ r \geq 1 \})} \) follows by an \( L^p \) analogue of Step 1, as well as applying the \( X_{k,N} \) estimates (60) in the \( L^p \) variant of Step 3. \( \square \)

5.3. A Poisson-type estimate on \( u_r/r \). Here we discuss how derivatives of \( u_r/r \) can be controlled by \( \Gamma \) using the representation (11),

\[
\frac{u_r}{r} = \Delta^{-1} \partial_3 \Gamma - \frac{2}{r} \Delta^{-2} \partial_3 \Gamma,
\]

see [6, p. 1929], which will be an essential part of our \( 3 \)-uloc energy estimates for \( \Phi \) and \( \Gamma \) (see Proposition 6.1 below).

**Lemma 5.3** (The \( L^2_{3-\text{uloc}} \) estimate on \( u_r/r \)).

\[
\| \nabla \frac{u_r}{r} \|_{L^2_{3-\text{uloc}}} + \| \nabla \partial_3 \frac{u_r}{r} \|_{L^2_{3-\text{uloc}}} \lesssim \| \Gamma \|_{L^2_{3-\text{uloc}}} + \| \nabla \Gamma \|_{L^2_{3-\text{uloc}}}.
\]

**Proof.** See Section 8. \( \square \)

A version of the above estimate without the localization in \( x_3 \) has appeared in [6, Lemma 2.3]. As mentioned in the introduction, the localization makes the estimate much more challenging, particularly due to the bilaplacian term in (64).

6. Energy estimates for \( \omega/r \)

In this section, we assume the weak \( L^3 \) bound (6) on time interval \([0,1]\) and prove an energy bound for \( \Phi^2 + \Gamma^2 \) at time 1, that is we prove the following.

**Proposition 6.1** (An \( L^2_{3-\text{uloc}} \) energy estimate for \( \Phi \) and \( \Gamma \)). Let \( u \) be a classical solution of (1) satisfying the weak \( L^3 \) bound (6) on \([0,1]\). Then

\[
\| \Phi(1) \|_{L^2_{3-\text{uloc}}(\mathbb{R}^3)} + \| \Gamma(1) \|_{L^2_{3-\text{uloc}}(\mathbb{R}^3)} \leq \exp \exp A^{O(1)}.
\]

We note that we will only use (in (70) below) the bound on \( \Gamma \).
Proof. We fix a cutoff function $\phi \in C_0^\infty((-1,1);[0,1])$ such that $\phi \equiv 1$ in $[-1/2,1/2]$, and we define the translate
\[
\phi_z(y) := \phi(y - z).
\]
Clearly, we have the pointwise inequality
\[
\phi_z', \phi_z'' \lesssim \sum_{i=-2}^{2} \phi_{z+i}.
\]
We will consider the energies
\[
E(t) := \sup_{z \in \mathbb{R}} E_z(t), \quad E_z(t) := \frac{1}{2} \int_{\mathbb{R}^3} (\Phi(t, x)^2 + \Gamma(t, x)^2)\phi_z(x_3)dx,
\]
\[
F(t) := \sup_{z \in \mathbb{R}} F_z(t), \quad F_z(t) := \int_{t_0}^t \int_{\mathbb{R}^3} (\nabla \Phi(s, x)^2 + \nabla \Gamma(s, x)^2)\phi_z(x_3)dx ds
\]
for $t \in [t_0,1]$, where $t_0 \in [0,1]$ will be chosen in Step 3 below. Given $z \in \mathbb{R}$, we multiply the equations (10) by $\phi_z\Gamma$ and $\phi_z\Phi$, respectively, and integrate to obtain, at a given time $t$,
\[
E_z' \leq \int_{\mathbb{R}^3} \left(- (|\nabla \Phi|^2 + |\nabla \Gamma|^2)\phi_z + \frac{1}{2} (\Phi^2 + \Gamma^2 ) (u_z\phi_z' + \phi_z'')
\right.
\]
\[
+ (\omega_r \partial_r + \omega_3 \partial_3) \frac{u_r}{r} \Phi \phi_z - 2r^{-1} u_\theta \Phi \Gamma \phi_z \big)d x
\]
\[
=: - F_z'(t) + I_1 + I_2 + I_3.
\]
The second term on the right hand side can be bounded directly,
\[
I_1 \lesssim (1 + \|u_z\|_{L^\infty_x(\mathbb{R}^3)}) E(t).
\]
The remaining terms $I_2, I_3$ are more challenging. In order to estimate them, as well as choose $t_0$ and deduce the claim (65), we follow the steps below.

**Step 1.** We use the Hölder estimate (Proposition 5.1) to show that $|\Theta| \leq r^\gamma A^{O(1)}$ whenever $r \leq \frac{1}{2}$, where $\gamma = \exp(-A^{O(1)})$.

To this end we note that, due to incompressibility, $\text{div}(u + 2\epsilon_r) = 4\pi \delta_{x' = 0}$, which enables us to apply Proposition 5.1 to the equation for the swirl $\Theta$ (recall (13)).

Moreover, in the notation of Proposition 5.1, for every $R < \frac{1}{2}$, $t_0 \in [\frac{1}{2},1]$ and $x_0 \in (0,0) \times \mathbb{R}$ (i.e., on the $x_3$-axis),
\[
R^{-\frac{1}{2}} \|u + \epsilon_r\|_{L^\infty_t L^2_r(Q([t_0,x_0] \times \mathbb{R}))} \lesssim R^{-\frac{1}{2}} \|u\|_{L^\infty_t L^2_{uloc}([t_0-R^2,t_0] \times \mathbb{R}^2)} + 1 \leq A^{O(1)}
\]
by Hölder’s inequality and (44) applied on the timescale $R^2$. (In particular note that each scale $R$ leads to a different decomposition $u = u_\alpha + u_\beta$, but they all obey the same bounds up to being suitably rescaled.) Thus, for every $r \in (0,1/2)$, $\text{osc}_{B(x_0,r)} \Theta(t_0) \lesssim r^\gamma \text{osc}_{Q(1/2)} \Theta$ for $r \in (0,1/2)$, which implies the claim.

**Step 2.** We show that
\[
\int_{t_0}^t |I_2 + I_3| \lesssim \frac{1}{2} F(t) + r_0^{-10} + \int_{t_0}^t GE
\]
for each $t_0 \in [t/2, t]$, where

$$r_0 := e^{-\gamma^2},$$

(68)

$\gamma = \exp(-A^{O(1)})$ is given by Step 1, and

$$G := r_0^{-3} + \|u\|_\infty + \|D^2 u\|_{L^{5/4}} + \|\nabla u\|_{L^2_{uloc}}$$

at each $t' \in [t_0, t]$.

To this end, we proceed similarly to [6]. Using integration by parts, we compute

$$I_2 = 2\pi \int_0^\infty (-\partial_3 u_\theta \partial_r \frac{u_r}{r} \Phi + \frac{\partial_r (ru_\theta)}{r} \partial_3 \frac{u_r}{r} \Phi \phi_z (x_3)) r \, dr \, dx_3$$

$$= \int_0^\infty u_\theta (\partial_3 \frac{u_r}{r} \partial_3 \Phi \phi_z - \partial_3 \frac{u_r}{r} \partial_r \Phi \phi_z + \partial_r \frac{u_r}{r} \Phi \phi_z')$$

$$=: I_{2,1} + I_{2,2} + I_{2,3}.$$

Let us further decompose $I_{2,i} = I_{2,i,in} + I_{2,i,out} (i = 1, 2, 3)$ by writing

$$\int = \int_{\{r < r_0\}} + \int_{\{r \geq r_0\}}.$$

We decompose

$$I_{2,1,in} = I_{2,1,in,1} + I_{2,1,in,2}$$

where

$$I_{2,1,in,1} := \int_{\{r < r_0\}} u_\theta \left( \int_{\Omega} \partial_3 \frac{u_r}{r} \right) \partial_3 \Phi \phi_z$$

and $\Omega := \{x': r < 1\} \times \text{supp } \phi_z$. We compute using Hölder’s inequality and Sobolev embedding

$$\left| \int_{\Omega} \partial_3 \frac{u_r}{r} \right| \leq \|r^{-1} \partial_r u_r\|_{L^1(\Omega)} + \|r^{-2} u_r\|_{L^1(\Omega)}$$

$$\lesssim \|r^{-1}\|_{L^{15/8}(\Omega)} \|\nabla u\|_{L^{15/7}(\Omega)} \lesssim \|\nabla^2 u\|_{L^{5/4}(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \lesssim G.$$
where we have also applied the Poincaré’s inequality and our choice (68) of $r_0$. Thus

$$\int_{t_0}^t I_{2,1,in,2} \leq \frac{1}{20} F(t) + \int_{t_0}^t E.$$ 

An analogous argument, in which “$\partial_r$” and “$\partial_\theta$” are switch, gives us the same bound for $I_{2,2,in,2}$. As for $I_{2,2,in,1}$, we integrate by parts, and apply Hölder’s inequality for Lorentz spaces (26), and Young’s inequality, to obtain

$$|I_{2,2,in,1}| \leq \left| \int_{\Omega} \partial_\theta \frac{u_r}{r} \phi_z \right| \int_{\{r \leq r_0\} \cap \text{supp } \phi_z} |u_\theta \partial_r \Phi|$$

$$\lesssim \left| \int_{\Omega} \frac{u_r}{r} \phi_z \right| \|u_\theta\|_{L^1,\infty} \|\nabla \Phi\|_{L^2,\supp \phi_z} r_0^{-3}$$

$$\lesssim \sum_{i=-2}^{2} \|\nabla u\|_{L^1(\Omega)} A(F'_{z+i}) \frac{1}{2} r_0^{\frac{1}{2}}$$

$$\lesssim GA r_0^{1/3} \left( \sum_{i=-2}^{2} F'_{z+i} \right)^{\frac{1}{2}},$$

which, thanks to the smallness of $r_0 = \exp(-\exp(A^{O(1)}))$ (recall (68)), gives that

$$\int_{t_0}^t |I_{2,2,in,1}| \leq \frac{1}{20} F(t) + (t - t_0).$$

We similarly decompose $I_{2,3,in} = I_{2,3,in,1} + I_{2,3,in,2}$ to find

$$|I_{2,3,in,1}| = \left| \int_{\Omega} \partial_r \frac{u_r}{r} \right| \left| \int_{\{r \leq r_0\} \cap \text{supp } \phi_z} u_\theta \partial_r \phi_z' \right| \lesssim (\|\nabla u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^{5/4}(\Omega)}) AE \frac{1}{2} r_0^{\frac{1}{2}}$$

$$\lesssim G(E + 1)$$

where we have used Lemma 3.2 and change of variables, the pointwise estimate $|u_r/r| \leq |\nabla u|$, and Hölder’s inequality to bound

$$\left| \int_{\Omega} \partial_r \frac{u_r}{r} \right| \lesssim \int_{-\infty}^{\infty} \int_{0}^{1} \left( |\partial_r u_r| + \frac{|u_r|}{r} \right) \, dr \, dz$$

$$\lesssim \|r^{-1} \partial_r u_r\|_{L^1(\Omega)} + \|r^{-1} \nabla u\|_{L^1(\Omega)}$$

$$\lesssim \|\nabla u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^{5/4}(\Omega)},$$
where we used (34) in the third line, and the Hardy inequality (31) in the last line. Next

$$|I_{2,3,in,2}| = \left| \int_{\{r \leq r_0\}} u_\theta \left( \partial_r \frac{u_r}{r} - \int_{\Omega} \partial_r \frac{u_r}{r} \right) \Phi \phi_z' \right|$$

\[ \lesssim \|ru_\theta\|_{L^3(\{r \leq r_0\})} \left\| r^{-\frac{1}{2}} \left( \partial_r \frac{u_r}{r} - \int_{\Omega} \partial_r \frac{u_r}{r} \right) \right\|_{L^3(\mathbb{R}^2 \times \text{supp} \phi_z)} \left\| r^{-\frac{3}{2}} \Phi \right\|_{L^3(\mathbb{R}^2 \times \text{supp} \phi_z)} \]

\[ \leq A^{O(1)} r_0^{\frac{3}{2}} \left\| \nabla \partial_r \frac{u_r}{r} \right\|_{L^2_{3-\text{uloc}}} \left\| \nabla \Phi \right\|_{L^2_{3-\text{uloc}}}^{\gamma}, \]

where we have used the Hardy inequality (Lemma 3.2). Thus Lemma 5.3 and Young’s inequality imply that

$$\int_{t_0}^t |I_{2,3,in,2}| \leq \frac{1}{20} F(t) + \int_{t_0}^t E.$$

Next let us consider the contributions to $I_2$ from outside $B(r_0)$. Using Hölder’s inequality, we obtain that

$$|I_{2,1,out}| = \left| \int_{\{r > r_0\}} u_\theta \partial_r \frac{u_r}{r} \partial_3 \Phi \phi_z \, dx \right|$$

\[ \leq \|u_\theta\|_{L^6_{3-\text{uloc}}(\{r > r_0\})} \left\| r^{-1} \partial_r u_r - r^{-2} u_r \right\|_{L^1_{3-\text{uloc}}(\{r > r_0\})} \left\| \nabla \Phi \right\|_{L^2_{3-\text{uloc}}(\mathbb{R}^3)}. \]

Hence, since Proposition 5.2 shows that $|u| \leq A^{O(1)}(r^{-1} + r^{-1/4})$ and $|\partial_r u_r| \leq A^{O(1)}(r^{-2} + r^{-1/4})$, we see that the first two norms on the right hand side are finite and bounded by, say, $r_0^{-10}$. Thus, an application of Young’s inequality gives that

$$\int_{t_0}^t |I_{2,1,out}| \leq \frac{1}{20} F(t) + r_0^{-10}(t - t_0).$$

The remaining outer parts of $I_2$, i.e. $I_{2,2,out}$ and $I_{2,3,out}$ can be estimated in a similar way, with the latter bounded by, say, $E + r_0^{-10}$.

Finally let us consider $I_3$. Taking $p$ such that, for example, $\frac{1}{p} = \frac{1}{2} - \frac{\gamma}{4}$, we have $p - 2 = 2\gamma/(2 - \gamma) \geq \gamma$, and so our quantified Hardy’s inequality (Lemma 3.2) shows that

$$|I_{3,in}| \leq \left\| r^{-2+\frac{3}{p}} u_\theta \right\|_{L^{1+\frac{3}{p}}((\{r \leq r_0\}) \setminus \{r > r_0\})} \left\| r^{-\frac{3}{2}+\frac{1}{p}} \Phi \right\|_{L^p_{3-\text{uloc}}} \left\| r^{-\frac{3}{2}+\frac{1}{p}} \Gamma \right\|_{L^p_{3-\text{uloc}}}$$

\[ \lesssim \gamma^{-O(1)} r_0^{\gamma/2} \left( \|\Phi\|_{L^2_{3-\text{uloc}}} + \|\nabla \Phi\|_{L^2_{3-\text{uloc}}} \right) \left( \|\Gamma\|_{L^2_{3-\text{uloc}}} + \|\nabla \Gamma\|_{L^2_{3-\text{uloc}}} \right), \]

which gives that $\int_{t_0}^t |I_{3,in}| \leq \frac{1}{20} F(t) + \int_{t_0}^t E$. On the other hand, for $r \geq r_0$ we have the simple bound

$$|I_{3,out}| \leq 2 \|r^{-1} u_\theta\|_{L^p_{3-\text{uloc}}(\{r \geq r_0\})} \|\Phi\|_{L^2_{3-\text{uloc}}} \|\Gamma\|_{L^2_{3-\text{uloc}}} \leq r_0^{-5/4} E,$$

as required.

**Step 3.** Given $\tau > 0$ we use the choice of time of regularity (Lemma 4.2) to find $t_0 \in [1 - \tau, 1]$ such that $E(t_0) \lesssim A^{O(1)} \tau^{-3}$.

Indeed, Lemma 4.2 lets us choose $t_0 \in [1 - \tau, 1]$ such that

$$\|\nabla^2 u(t_0)\|_\infty \leq A^{O(1)} \tau^{-\frac{4}{3}}.$$
It follows from the axial symmetry and (34) that $|\Phi| + |\Gamma| \leq |\nabla \omega|$, and so
\[
\|\Phi(t_0)\phi_z^{1/2}\|_{L^2(\{r \leq 1\})} + \|\Gamma(t_0)\phi_z^{1/2}\|_{L^2(\{r \leq 1\})} \lesssim \|\nabla \omega(t_0)\|_{L^\infty(B(1) \times \mathbb{R})} \leq A^{O(1)} r^{-3/2}
\] (69)
for every $z \in \mathbb{R}$. Using the decomposition $\omega = \omega_3 + \omega_1$ on the interval $[0, 1]$, by (44), (40), and H"older’s inequality,
\[
\|\Phi(t_0)\phi_z^{1/2}\|_{L^2(\{r > 1\})} + \|\Gamma(t_0)\phi_z^{1/2}\|_{L^2(\{r > 1\})} \lesssim \|\omega_z^2\|_{L^2(\mathbb{R}^3)} + \|\omega_1\|_{L^2(\mathbb{R}^3)} + r^{-1}\|\omega_z\|_{L^2(\mathbb{R}^3)} + r^{-1}\|\omega_1\|_{L^2(\mathbb{R}^3)} \lesssim \sup_{\tau \in [0, 1]} E_{\omega}(\tau) \lesssim A^{O(1)}.
\]
This and (69) proves the claim of this step.

**Step 4.** We prove the claim.

Integration in time of the energy inequality (66) from initial time $t_0$ chosen in Step 3 above, taking $\sup_{z \in \mathbb{R}}$, and applying the estimate (67) for $I_1$ and Step 2 for $I_2, I_3$ we find that
\[
E(t) + \frac{1}{2} F(t) \leq E(t_0) + \frac{3}{2} + \int_{t_0}^{t} O(r_0^{-3}) + \|u\|_{L^4} + \|\nabla^2 u\|_{L^5} + \|\nabla u\|_{L^4} \leq A^{O(1)} r^{-3} + r_0^{-10} \exp\left(O\left(r_0^{-3}(t - t_0) + A^{O(1)}(t - t_0)^{1/2}\right)\right).
\]
for $t \in [t_0, 1]$. Thus, by Grönwall’s inequality,
\[
E(1) \leq (A^{O(1)} r^{-3} + r_0^{-10}) \exp\left(O\left(\frac{r_0^{-3}(t - t_0) + A^{O(1)}(t - t_0)^{1/2}}{t - t_0}\right)\right).
\]
Setting $\tau := r_0^3$, we see that the last exponential function is $O(1)$, and the prefactor gives the required estimate (65). \[\Box\]

**7. Proof of Theorem 1.1**

In this section we prove Theorem 1.1. Namely, given the $L^{3,\infty}$ bound (6) on time interval $[0, 1]$, we show that $|\nabla^3 u| \leq \exp \exp A^{O(1)}$ at time 1.

**Step 1.** We show that $\|b\|_{L^p_{3,\text{uloc}}(\mathbb{R}^3)} \leq C_p \exp \exp A^{O(1)}$ for each $p \in [3, \infty)$, $t \in [1/2, 1]$, where $b := u_r e_r + u_z e_z$ denotes the swirl-free part of the velocity field.

To this end we apply Proposition 6.1 to find
\[
\|\Gamma\|_{L^\infty L^3_{3,\text{uloc}}(\mathbb{R}^3)} \leq \exp \exp A^{O(1)}.
\]
(70)
On the other hand Proposition 5.2 shows that
\[
\|r^2 \omega\|_{L^p_{3,\text{x}}(\{r \leq 10\})} \leq A^{O(1)}.
\]
Interpolating between this inequality and (70) we obtain
\[
\|\omega_\theta\|_{L^p_{3,\text{uloc}}(\{r \leq 10\})} = \|\Gamma^{3/2}(r^2 \omega_\theta)^{3/2}\|_{L^p_{3,\text{uloc}}(\{r \leq 10\})} \lesssim \|\Gamma\|_{L^3_{3,\text{uloc}}}^{3/2} \|r^2 \omega_\theta\|_{L^3_{3,\text{x}}(\{r \leq 10\})} \leq \exp \exp A^{O(1)}
\]
for all $p \leq 3$.

Noting that
\[
\text{curl } b = \omega_\theta e_\theta, \quad \text{div } b = 0
\]
almost everywhere, and that \( \text{div} \, b = 0 \) we now localize \( b \) to obtain an \( L^p \) estimate near the axis. Namely, for any unit ball \( B \subset \{ r \leq 1 \} \), let \( \phi \in C^\infty_c(B) \) such that \( \phi \equiv 1 \) on \( B/2 \). Observe that for all \( p \in [1, 3) \) we can use Hölder’s inequality for Lorentz spaces \((26)\) to obtain
\[
\| \text{div}(\phi b) \|_{L^p(\mathbb{R}^3)} = \| b \cdot \nabla \phi \|_p \lesssim \| b \|_{L^{3,\infty}} \| \nabla \phi \|_{L^{3p/(3-p),1}} \lesssim A.
\]

Applying the Bogovskii operator \((29)\) to \( \text{div}(\phi b) \) on the domain \( B \setminus (B/2) \), we find \( \tilde{b} \in W^{1,p}(B) \) such that \( \text{div} \, \tilde{b} = 0 \), \( \| \tilde{b} - b \|_{W^{1,p}(B)} \leq A^{O(1)} \), \( \tilde{b} \equiv b \) in \( B/2 \), and \( \tilde{b} \equiv 0 \) outside \( B \). Then for any \( p \in (1, 3) \),
\[
\| \tilde{b} \|_{L^{3p/(3-p)}(B/2)} \leq \| \tilde{b} \|_{3p/(3-p)} \lesssim \| \nabla \tilde{b} \|_p \lesssim \| \text{curl} \, \tilde{b} \|_{L^p(B)} \leq \| \omega_\theta \|_{L^p(B)} + \| b - \tilde{b} \|_{W^{1,p}(B)} \leq \exp \exp A^{O(1)},
\]
which is our desired localized estimate. Here we have used the boundedness of the operator \( \nabla f \mapsto \text{curl} \, f \) in \( L^p \) (which is a consequence of the identity \( \text{curl} \, \text{curl} \, f = \nabla (\text{div} \, f) - \Delta f \), which in turn implies that \( \nabla f = \nabla (-\Delta)^{-1} \text{curl} \, (\text{curl} \, f) \) for divergence-free \( f \)). Combining this with the pointwise estimates away from the axis \((Proposition 5.2)\) gives the claim of this step.

**Step 2.** We show that there exists \( C_0 > 1 \) such that
\[
\left\| \frac{u_\theta(t)}{r^{\frac{3}{2}}} \right\|_{L^{3/2}_{3-\text{uloc}}}^4 \leq \left\| \frac{u_\theta(t_0)}{r^{\frac{3}{2}}} \right\|_{L^{3/2}_{3-\text{uloc}}}^4 + 1 + \exp \exp A^{C_0} \int_{t_0}^t \left\| \frac{u_\theta}{r^{\frac{3}{2}}} \right\|_{L^{3/2}_{3-\text{uloc}}}^4
\]
for each \( t_0 \in [1/2, 1] \) and \( t \in [t_0, 1] \).

To this end we provide a localization of the estimate of \( u_\theta / r^{1/2} \) in the spirit of \([6, Lemma 3.1]\). Indeed, one can calculate from the equation \((48)\) for \( u_\theta \) that for a smooth cutoff \( \psi = \psi(x_3) \),
\[
\frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^3} \frac{u_\theta^4}{r^2} \psi + \frac{3}{4} \int_{\mathbb{R}^3} |\nabla \frac{u_\theta^2}{r^2}|^2 \psi + \frac{3}{4} \int_{\mathbb{R}^3} \frac{u_\theta^4}{r^2} \psi z = -\frac{3}{2} \int_{\mathbb{R}^3} r u_\theta \frac{u_\theta^2}{r^2} \psi + \frac{1}{8} \int_{\mathbb{R}^3} r^2 u_\theta^2 (2u_\theta^2 u_z - \partial_z (u_\theta^2)) \psi' =: I_1 + I_2 + I_3.
\]

As before, we choose \( \psi \in C^\infty_c((-2, 2)) \) with \( \psi \equiv 1 \) in \([-1, 1]\) and define the translates \( \psi_z(x) := \psi(x - z) \) for all \( z \in \mathbb{R} \). Consider the energies
\[
E_z(t) := \frac{1}{4} \int_{\mathbb{R}^3} \frac{u_\theta^4}{r^2} \psi_z, \quad F_z(t) := \frac{3}{4} \int_{\mathbb{R}^3} \int_{t_0}^t \frac{\nabla u_\theta^2}{r} \psi z,
\]
\[
E(t) := \sup_{z \in \mathbb{R}} E_z(t), \quad F(t) := \sup_{z \in \mathbb{R}} F_z(t).
\]

By Step 1 and Sobolev embedding,
\[
|I_1| \lesssim \| u_r \|_{L^{5/2}_{3-\text{uloc}}} \left\| r^{-\frac{1}{2}} \frac{u_\theta^2}{r} \right\|_{L^{12/5}(\Omega)}^2 \
\leq \exp \exp A^{O(1)} \left( \left\| \frac{u_\theta^2}{r} \right\|_{L^2(\Omega)} \left\| \frac{\nabla u_\theta^2}{r} \right\|_{L^2(\Omega)} \left\| \frac{u_\theta^2}{r} \right\|_{L^2(\Omega)} \right).
\]
where $\Omega := \mathbb{R}^2 \times \text{supp } \psi$. It follows that
\[
\int_{t_0}^t |I_1| \leq \frac{1}{20} F(t) + \exp \exp A^{O(1)} \int_{t_0}^t E + (t-t_0).
\]
Similarly,
\[
|I_2| \lesssim \|u_r\|_{L^6_{3-\text{uloc}}} \left( \frac{u_\theta^3}{r} \right)_{L^4_{3-\text{uloc}}} \left( \frac{u_\phi^2}{r} \right)_{L^3(\Omega)}
\leq \exp \exp A^{O(1)} E^\frac{1}{2} \left( \left\| \frac{u_\theta^2}{r} \right\|_{L^2(\Omega)} \left\| \nabla \frac{u_\theta^2}{r} \right\|_{L^2(\Omega)} + \left\| \frac{u_\phi^2}{r} \right\|_{L^2(\Omega)} \right)
\]
which yields the same bound as $I_1$. Finally,
\[
|I_3| = \frac{1}{8} \int_{\mathbb{R}^3} \frac{u_\theta^2}{r} \frac{u_\phi^2}{r} \psi' \lesssim \left\| \frac{u_\theta}{r} \right\|_{L^4_{3-\text{uloc}}} \left\| \nabla \frac{u_\theta}{r} \right\|_{L^2(\Omega)}
\]
so we have
\[
\int_{t_0}^t |I_3| \leq \frac{1}{20} F(t) + \int_{t_0}^t O(E).
\]
Summing and taking the supremum over $z \in \mathbb{R}$ gives the claim of this step.

**Step 3.** We deduce that
\[
\|u\|_{L^\infty_t L^6_{3-\text{uloc}}([t_0, 1] \times \mathbb{R}^3)} \leq \exp \exp A^{O(1)},
\] (71)
where
\[
t_0 := 1 - \exp(- \exp A^{O(1)}).
\]
Indeed, Lemma 4.2 and Proposition 5.2 give a $t_0 \in [1 - \exp(- \exp A^{C_0}), 1]$ such that $\|r^{-\frac{1}{2}} u_\theta(t_0)\|_{L^4_{3-\text{uloc}}(\mathbb{R}^3)} \leq \exp \exp A^{2C_0}$. Therefore, applying Grönwall’s inequality to the claim of the previous step,
\[
\left\| \frac{u_\theta}{r^{\frac{3}{2}}} \right\|_{L^\infty_t L^4_{3-\text{uloc}}([t_1, 1] \times \mathbb{R}^3)} \leq \exp \exp A^{O(1)}.
\]
Combining this with Proposition 5.2 and Hölder’s inequality,
\[
\| u_\theta \|_{L^\infty_t L^6_{3-\text{uloc}}([t_1, 1] \times \mathbb{R}^3)} \leq \| r u_\theta \|_{L_x^\infty([t \leq 1])} \| r^{-\frac{3}{2}} u_\theta \|_{L^4_{3-\text{uloc}}([t_1, 1] \times \mathbb{R}^3)} + \| u \|_{L^\infty_t L^6_{3-\text{uloc}}([t_1, 1] \times \mathbb{R}^3)} \leq \exp \exp A^{O(1)},
\]
which, together with Step 1, implies (71).

We note that Step 3 already provides a subcritical local regularity condition of the type of Ladyzhenskaya-Prodi-Serrin, which guarantees local boundedness of all spatial derivatives of $u$, and can be proved by employing the vorticity equation for example (see [30, Theorem 13.7]). In the last step below we use a robust tail estimate of the pressure function (recall Lemma 2.1) to provide a simpler justification of pointwise bounds by $\exp \exp A^{O(1)}$. 

Step 4. We prove that, if \( \|u\|_{L^\infty([t_1-t_1];W_{\text{uloc}}^{k-1,\delta})} \lesssim \exp(\exp A^{O(1)}) \) for some \( k \geq 1 \) and \( t_1 = \exp(-\exp A^{O(1)}) \), then the same is true for \( k \) (with some other \( t_1 \) of the same order).

Let \( I = [a, b] \subset [t_1, 1] \), and let \( \chi \in C^\infty(\mathbb{R}) \) be such that \( \chi(t) = 0 \) for \( t < a + (b-a)/8 \) and \( \chi(t) = 1 \) for \( t > (a+b)/2 \). We set \( \phi \in C^\infty(\mathbb{R}; [0, 1]) \) such that \( \phi = 1 \) on \( (0, 1/2) \) and \( \sum_{j \in \mathbb{Z}^3} \tilde{\phi}_j = 1 \), where \( \phi_j := \phi(\cdot - j) \) for each \( j \in \mathbb{R}^3 \).

Letting \( v := \chi \phi \nabla^k u \) we see that \( v(t_1) = 0 \), and

\[
v_t - \Delta v = -\nabla^k u \chi \phi \cdot \nabla(\nabla^k u) - \chi \Delta \phi (\nabla^k u) - \chi \phi \text{div} (1 + T) \nabla^k (u \otimes u)
\]

\[
= f_1 - \phi \text{div} (1 + T) ((\chi \nabla^k u \otimes u + u \otimes \chi \nabla^k u) \tilde{\phi}) - \chi \phi \text{div} (1 + T) \sum_{|\alpha| + |\beta| + |\gamma| = k} C_{\alpha, \beta, \gamma} (D^\alpha u \otimes D^\beta u D^\gamma \tilde{\phi}) - \chi \phi \text{div} T \nabla^k (u \otimes u (1 - \tilde{\phi}))
\]

\[
= : f_1 + f_2 + f_3 + f_4.
\]

We can now estimate \( \|v(t)\|_6 \), by extracting the same norm on the right-hand side and ensuring that the length of the interval is sufficiently small, so that the norm can be absorbed. Namely,

\[
\|v(t)\|_6 = \left\| \int_a^t e^{(t-t')\Delta} f_1(t') dt' + \int_a^t e^{(t-t')\Delta} f_2(t') dt' + \int_a^t e^{(t-t')\Delta} f_3(t') dt' + \int_a^t e^{(t-t')\Delta} f_4(t') dt' \right\|_6
\]

\[
\leq \left( \|\chi \nabla^k u \tilde{\phi}\|_{L^\infty([a,t];L^6)} + \|\chi \nabla^{k-1} u \tilde{\phi}\|_{L^\infty([a,1],L^6)} \right) \int_a^t \|\Psi(t-t')\|_{W^{1,1}} dt'
\]

\[
+ \|\chi \nabla^k u \tilde{\phi}^{1/2}\|_{L^\infty([a,t];L^6)} \|u \tilde{\phi}^{1/2}\|_{L^\infty([a,1];L^6)} \int_a^t \|\Psi(t-t')\|_{W^{1,6/5}} dt'
\]

\[
+ \|u\|_{L^\infty([a,1];W^{k-1,6}_{\text{uloc}})} \int_a^t \|\Psi(t-t')\|_{W^{1,6/5}} dt'
\]

\[
+ \|\text{div} T (u \otimes u (1 - \tilde{\phi}))\|_{L^\infty([a,1];W^{k,6}(B(0,2)))} \int_a^t \|\Psi(t-t')\|_1 dt'
\]

\[
\leq \|\chi \nabla^k u\|_{L^\infty([a,t];L^6_{\text{uloc}})} ((b-a)^{1/2} + \exp(\exp A^{O(1)}(b-a)^{1/4}) + \exp(\exp A^{O(1)})
\]

for each \( t \in (a, b) \), where we used Young’s inequality, heat estimates (23) and the Calderón-Zygmund inequality. By replacing \( \phi \) (in the definition of \( v \)) by \( \phi_z \) for any \( z \in \mathbb{R}^3 \), we obtain the same bound, and so

\[
\|\chi \nabla^k u\|_{L^\infty([a,b];L^6_{\text{uloc}})} \leq \|\chi \nabla^k u\|_{L^\infty([a,b];L^6_{\text{uloc}})} (b-a)^{1/4} + \exp(\exp A^{O(1)} + \exp(\exp A^{O(1)})
\]

Thus, for any \( b, a \) such that \( t_1 \leq a < b \leq 1 \) and \( (b-a)^{1/4} \leq \exp(\exp A^{O(1)}/2 \) we can absorb the first term on the right-hand side by the left-hand side to obtain

\[
\|\nabla^k u\|_{L^\infty([a+b]/2,b];L^6_{\text{uloc}})} \leq \exp(\exp A^{O(1)}).
\]

Since the upper bound is independent of the location of \([a,b] \subset [t_1, 1]\), we obtain the claim.
Here we prove Lemma 5.3, namely that
\[
\left\| \nabla \partial_r \frac{u_r}{r} \right\|_{L^2_{3-\text{uloc}}} + \left\| \nabla \partial_3 \frac{u_r}{r} \right\|_{L^2_{3-\text{uloc}}} \lesssim \|\Gamma\|_{L^2_{3-\text{uloc}}} + \|\nabla\Gamma\|_{L^2_{3-\text{uloc}}}.
\] (72)

To this end we recall (64) that
\[
\frac{u_r}{r} = \Delta^{-1} \partial_3 \Gamma - 2 \frac{\partial_r}{r} \Delta^{-2} \partial_3 \Gamma.
\]
Since
\[
\frac{\partial_r}{r} = \Delta' - \partial_{rr},
\]
we have that
\[
\frac{u_r}{r} = -\Delta^{-1} \partial_3 \Gamma + 2(\partial_{rr} - \Delta') \Delta^{-2} \partial_3 \Gamma.
\] (73)
Thus, since \(|\nabla \partial_3 \frac{u_r}{r}| = |(\partial_r, \partial_3 \frac{u_r}{r}, \partial_3 \partial_3 \frac{u_r}{r})|\) (and similarly for \(|\nabla \partial_r \frac{u_r}{r}|\)), we can use (35) and (36) to observe that
\[
\left| \nabla \partial_3 \frac{u_r}{r} \right| + \left| \nabla \partial_r \frac{u_r}{r} \right| \lesssim \left| D^2_{r,x_3} \Delta^{-1} \partial_3 \Gamma \right| + \left| D^2_{r,x_3} (\partial_{rr} - \Delta') \Delta^{-2} \partial_3 \Gamma \right|
\lesssim \left| \nabla \Gamma \right| + \left| D^2 \Delta^{-1} \nabla' \Gamma \right| + \left| D^4 \Delta^{-2} \nabla' \Gamma \right|
\]
where we used \(\partial_{33} = \Delta - \Delta'\) in the last line. In particular, each of the terms on the right-hand side involves at least one derivative in the horizontal variables. Thus, in order to estimate the left-hand side of (72) it suffices to find suitable bounds on the last two terms, which we achieve in Lemmas 8.1–8.2 below. Their claims give us (72), as required.

**Lemma 8.1.** Let \(f = \Delta^{-1} \nabla \Gamma\). Then
\[
\|D^2 f\|_{L^2_{3-\text{uloc}}} \leq \|\Gamma\|_{L^2_{3-\text{uloc}}} + \|\nabla \Gamma\|_{L^2_{3-\text{uloc}}}
\]

**Proof.** Let \(I(x)\) denote the kernel matrix of \(D^2(-\Delta)^{-1}\). We have that
\[
\left| \nabla^j I(x) \right| \leq \frac{C}{|x|^{3+j}} \quad \text{for } j = 0, 1,
\]
and
\[
D^2 f(x) = \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \nabla \Gamma(y) dy
\]
\[
= \text{p.v.} \int_{\mathbb{R}^3} \nabla \Gamma(y) \tilde{\phi}(y_3) I(x-y) dy + \text{p.v.} \int_{\mathbb{R}^3} \Gamma(y)(1 - \tilde{\phi}(y_3)) \nabla I(x-y) dy
\]
\[
=: f_1(x) + f_2(x).
\]
The Calderón-Zygmund inequality gives that
\[
\|f_1\|_{L^2_{3-\text{uloc}}} \leq \|\Gamma\|_{L^2_{3-\text{uloc}}} + \|\nabla \Gamma\|_{L^2_{3-\text{uloc}}}.
\]
Moreover, noting that \( \int_{\mathbb{R}^2} \frac{dx_1 dx_2}{(a^2 + x_1^2 + x_2^2)^2} = Ca^{-2} \), we can use Young’s inequality for convolutions to obtain

\[
\|f_2(\cdot, x_3)\|_{L^2} \leq \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, y_3)\|_{L^2}(1 - \tilde{\phi}(y_3))}{|x_3 - y_3|^2} dy_3
\]

\[
\leq \sum_{j \geq 1} \int_{\{|x_3 - y_3| \in (j, j+1)\}} \frac{\|\Gamma(\cdot, y_3)\|_{L^2}(1 - \tilde{\phi}(y_3))}{|x_3 - y_3|^2} dy_3
\]

\[
\leq \sum_{j \geq 1} j^{-2} \int_{\{|x_3 - y_3| \in (j, j+1)\}} \|\Gamma(\cdot, y_3)\|_{L^2} dy_3
\]

\[
\leq \|\Gamma\|_{L^2_{3-\text{uloc}}}.
\]

integration in \( x_3 \) over \( \text{supp} \phi \) finishes the proof. \( \square \)

For the double Laplacian term one needs to work harder:

**Lemma 8.2.** Let \( f = D^4 \Delta^{-2} \nabla T \). Then

\[
\|f\|_{L^2_{3-\text{uloc}}} \leq \|\Gamma\|_{L^2_{3-\text{uloc}}} + \|\nabla \Gamma\|_{L^2_{3-\text{uloc}}}.
\]

**Proof.** We have that

\[
f(x) = \text{p.v.} \int_{\mathbb{R}^3} \text{p.v.} \int_{\mathbb{R}^3} \partial_3 \Gamma(z) I(x - y) I(y - z) dz dy
\]

Recalling that \( \tilde{\phi} = \sum_{|j| \leq 10} \phi_j \), and \( \tilde{\phi} = \sum_{|j| \leq 20} \phi_j \) we use the partition of unity,

\[
1 = \tilde{\phi}(z_3) + (1 - \tilde{\phi}(z_3)) \tilde{\phi}(y_3) + \sum_{|j| > 10} \phi_j(y_3) \phi_k(z_3)
\]

\[
= \tilde{\phi}(z_3) + (1 - \tilde{\phi}(z_3)) \tilde{\phi}(y_3)
\]

\[
+ \sum_{|j| > 10} \phi_j(y_3) \begin{pmatrix}
\sum_{|k| > 20 \atop |k-j| \leq 10} \phi_k(z_3) + \sum_{|k| > 20 \atop k-j > 10} \phi_k(z_3) + \sum_{|k| > 20 \atop j/2 < k \leq 2j} \phi_k(z_3) + \sum_{|k| > 20 \atop k > 2j} \phi_k(z_3)
\end{pmatrix},
\]
to decompose \( f \) accordingly,

\[
f(x) = \text{p.v.} \int_{\mathbb{R}^3} \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \tilde{\phi}(z_3) I(x - y) I(y - z) dy \, dz \\
+ \text{p.v.} \int_{\mathbb{R}^3} I(x - y) \tilde{\phi}(y_3) \nabla' \Gamma(z)(1 - \tilde{\phi}(z_3)) I(y - z) dz \, dy \\
+ \text{p.v.} \int_{\mathbb{R}^3} I(x - y) \sum_{|j| > 10} \phi_j(y_3) \nabla' \Gamma(z) \sum_{|k| > 20 \atop |k - j| \leq 10} \phi_k(z_3) I(y - z) dz \, dy \\
+ \text{p.v.} \int_{\mathbb{R}^3} I(x - y) \sum_{|j| > 10} \phi_j(y_3) \nabla' \Gamma(z) \sum_{|k| > 20 \atop |k - j| > 10 \atop k \leq j/2} \phi_k(z_3) I(y - z) dz \, dy \\
+ \text{p.v.} \int_{\mathbb{R}^3} I(x - y) \sum_{|j| > 10} \phi_j(y_3) \nabla' \Gamma(z) \sum_{|k| > 20 \atop |k - j| > 10 \atop j/2 < k \leq 2j} \phi_k(z_3) I(y - z) dz \, dy \\
\]

\[
=: f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) + f_6(x). 
\]

Clearly \( f_1 \) involves localization of \( \nabla' \Gamma \) in \( z_3 \), and so we can use the Calderón-Zygmund inequality twice to obtain

\[
\|f_1\|_{L^2} \lesssim \|\nabla' \Gamma\|_{L^2_{-\text{uloc}}} 
\]

As for \( f_2 \) we integrate by parts in the \( z \)-integral (note that this does not conflict with the principal value, as the singularity has been cut off, and the far field has sufficient decay) and apply the Calderón-Zygmund estimate in \( x \) to obtain

\[
\|f_2\|_{L^2} \lesssim \left\| \tilde{\phi}(y_3) \int_{\mathbb{R}^3} \frac{\Gamma(z)(1 - \tilde{\phi}(z_3))}{|y - z|^4} dz \right\|_{L^2_{\tilde{\phi}}} \lesssim \sup_{y_3 \in \text{supp} \tilde{\phi}} \left\| \int_{\mathbb{R}^3} \frac{\Gamma(z)(1 - \tilde{\phi}(z_3))}{|y - z|^4} dz \right\|_{L^2_{\tilde{\phi}}} 
\]

\[
\lesssim \sup_{y_3 \in \text{supp} \tilde{\phi}} \int_{\mathbb{R}} \|\Gamma(\cdot, z_3)\|_{L^2} (1 - \tilde{\phi}(z_3)) d z_3 \\
\lesssim \sup_{y_3 \in \text{supp} \tilde{\phi}} \sum_{j \geq 1} j^{-2} \int_{|z_3 - y_3| \in (j, j + 1)} \|\Gamma(\cdot, z_3)\|_{L^2_{\tilde{\phi}}} d z_3 \lesssim \|\Gamma\|_{L^2_{-\text{uloc}}} 
\]

where we used Young’s inequality in the second line (as in the lemma above).

As for \( f_3 \), we integrate by parts in \( z \) and then in \( y \) to obtain

\[
|f_3(x)| \lesssim \sum_{|j| > 10} \int_{\mathbb{R}^3} \frac{\phi_j(y_3)}{|x - y|^4} \left| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{|k| > 20 \atop |k - j| \leq 10} \phi_k(z_3) I(y - z) dz \right| dy. 
\]
We note that the integration by parts is justified as
\[ f_3 = D^2(-\Delta)^{-1} \left( 1 - \sum_{|j| \leq 10} \phi_j(y_3) \right) D^2(-\Delta)^{-1} \left( \nabla \Gamma(1 - \sum_{k \in I} \phi_k(z_3)) \right), \]
where \( I := \{-20, \ldots, 20\} \cup \{j - 10, \ldots, j + 10\} \) is a finite index set. Thus, the operation of integration by parts above is equivalent to moving \( \nabla \) outside of the outer brackets, which in turn holds since the sums do not depend on \( x' \) and \( \nabla' \) commutes with other differential symbols.

Thus, using Young’s inequality in \( x' \)
\[ \| f_3(\cdot, x_3) \|_{L^2_y} \lesssim \sum_{|j| > 10} \left\| \frac{\phi_j(y_3)}{|x_3 - y_3|^2} \right\|_{L^2_y} \left\| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{|k| > 6} \frac{\phi_k(z_3) I(y - z)}{|k - j| \leq 2} \frac{dy_3}{L^2_y} \right\| \]
\[ \lesssim \sum_{|j| > 2} \left\| \frac{\phi_j(y_3)}{|x_3 - y_3|^2} \right\|_{L^2_y} \left\| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{|k| > 20} \frac{\phi_k(z_3) I(y - z)}{|k - j| \leq 10} \right\|_{L^2_y} \]
\[ \lesssim \sum_{|j| > 10} \left\| \frac{\phi_j(y_3)}{|x_3 - y_3|^2} \right\|_{L^2_y} \left\| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{|k| > 20} \frac{\phi_k(z_3) I(y - z)}{|k - j| \leq 10} \right\|_{L^2_y} \]
for each \( x_3 \in \text{supp} \phi \), where we applied the Cauchy-Schwarz inequality (in \( y_3 \)) in the second line.

As for \( f_4 \) we note that
\[ |y_3 - z_3| \geq |y_3| - |z_3| \geq (j - 1) - (k + 1) \geq \frac{j}{2} - 2 \geq (j + 2)/4 \geq (|y_3| + 1)/4 \geq |y_3 - x_3|/4 \]

Thus, we can integrate by parts in \( z \) to obtain
\[ |f_4(x)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{\Gamma(z)|1 - \tilde{\phi}(y_3)| (1 - \tilde{\phi}(y_3 - z_3))}{|x - y|^3 |y - z|} dz dy \]

Thus, applying Young’s inequality in \( x' \) and then in \( y' \) we obtain
\[ \| f_4(\cdot, x_3) \|_{L^2_y} \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{\Gamma(z)|1 - \tilde{\phi}(y_3)| (1 - \tilde{\phi}(y_3 - z_3))}{|y - z|^4} dz \left\| \int_{\mathbb{R}^2} \frac{dx_1 dx_2}{(|x_3 - y_3|^2 + x_1^2 + x_2^2)^{3/2}} \right\|_{L^2_{y'}} \]
\[ \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{\| \Gamma(\cdot, z_3) \|_{L^2(1 - \tilde{\phi}(y_3))(1 - \tilde{\phi}(y_3 - z_3))}}{|x_3 - y_3| |y_3 - z_3|^2} dz_3 dy_3 \]
(74)
Hence
\[ \|f_4(\cdot, x_3)\|_{L^2} \leq \int_{\mathbb{R}} \frac{1 - \tilde{\phi}(y_3)}{|x_3 - y_3|^{3/2}} \left( \sum_{j \geq 1} \int_{\{y_3 - z_3 \in (j, j+1)\}} \|\Gamma(\cdot, z_3)\|_{L^2}^2 \mathrm{d}z_3 \right) \mathrm{d}y_3 \]
\[ \lesssim \|\Gamma\|_{L^2_{3,\text{uloc}}} \int_{\mathbb{R}} \frac{1 - \tilde{\phi}(y_3)}{|x_3 - y_3|^{3/2}} \mathrm{d}y_3 \lesssim \|\Gamma\|_{L^2_{3,\text{uloc}}} . \]

As for \( f_5 \) we have
\[ \frac{1}{4} \leq \frac{|x_3 - y_3|}{|x_3 - z_3|} \leq 4, \]
since
\[ |x_3 - y_3| \leq |y_3| + |x_3| \leq j + 2 \leq 2j - 8 \leq 4k - 8 \leq 4(|z_3| - |x_3|) \leq 4|x_3 - z_3| \]
and
\[ |x_3 - z_3| \leq |z_3| + |x_3| \leq k + 2 \leq 2j + 2 \leq 4(j - 2) \leq 4(|y_3| - |x_3|) \leq 4|x_3 - y_3|. \]
In particular, the triangle inequality gives that
\[ |y_3 - z_3| \leq 5|x_3 - z_3| \]
Thus we can integrate by parts twice (in \( z \) and then in \( y \), so that the derivative falls on \( I(x - y) \)), and then use Young’s inequality twice (as in (74) above) and Tonelli’s Theorem to obtain
\[ \|f_5(\cdot, x_3)\|_{L^2} \leq \int_{\mathbb{R}} \int_{\{|y_3 - x_3|/4 \leq |x_3 - z_3| \leq 4|x_3 - y_3|\}} \|\Gamma(\cdot, z_3)\|_{L^2}^2 \frac{(1 - \tilde{\phi}(y_3 - z_3))(1 - \tilde{\phi}(z_3))}{|x_3 - y_3|^2 |y_3 - z_3|} \mathrm{d}z_3 \mathrm{d}y_3 \]
\[ \leq \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}^2 (1 - \tilde{\phi}(z_3))}{|x_3 - z_3|^2} \int_{\{|y_3 - z_3| \leq 4|x_3 - x_3|\}} \frac{1 - \tilde{\phi}(y_3 - z_3)}{|y_3 - z_3|} \mathrm{d}y_3 \mathrm{d}z_3 \]
\[ \lesssim \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}^2 (1 - \tilde{\phi}(z_3))}{|x_3 - z_3|^2} \log(5|x_3 - z_3|) \mathrm{d}z_3 \]
\[ \lesssim \sum_{j \geq 1} \int_{|y_3 - x_3| \in (j, j+1)} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}^2}{|x_3 - z_3|^2} \log(5|x_3 - z_3|) \mathrm{d}z_3 \]
\[ \lesssim \sum_{j \geq 1} j^{-2} \log(5j) \|\Gamma\|_{L^2_{3,\text{uloc}}}^2 \lesssim \|\Gamma\|_{L^2_{3,\text{uloc}}}^2 . \]

Finally, for \( f_6 \) we observe that
\[ \frac{1}{4} \leq \frac{|x_3 - z_3|}{|y_3 - z_3|} \leq 4, \]
since
\[ |y_3 - z_3| \geq |z_3| - |y_3| \geq k - j - 2 \geq k - \frac{8}{2} \geq \frac{k + 2}{4} \geq \frac{|x_3| + |z_3|}{4} \geq \frac{|x_3 - z_3|}{4} \]
and
\[ |y_3 - z_3| \leq |y_3| + |z_3| \leq j + k + 2 \leq \frac{3k + 4}{2} \leq 4(k - 2) \leq 4(|z_3| - |x_3|) \leq 4|x_3 - z_3|. \]
In particular, the triangle inequality gives that
\[ |x_3 - y_3| \leq 5|x_3 - z_3|. \]
Thus, similarly to the case of \( f_5 \) (although without integrating by parts in \( y \)), we apply Young’s inequality twice, and Tonelli’s Theorem to obtain
\[
\|f_6(\cdot, x_3)\|_{L^2} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2(1 - \bar{\phi}(y_3))(1 - \bar{\phi}(z_3))}}{|x_3 - y_3||y_3 - z_3|^2} \, dz_3 \, dy_3 \\
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2(1 - \bar{\phi}(z_3))}}{|x_3 - z_3|^2} \, dz_3 \\
\leq \lambda \sum_{j \geq 1} \int_{|z_3 - x_3| \in (j-1, j)} \frac{\|\Gamma(\cdot, z_3)\|_{L^2(1 - \bar{\phi}(z_3))}}{|x_3 - z_3|^2} \, dz_3 
\]
\[
\lesssim \sum_{j \geq 1} \log(5|j|)j^{-2} \|\Gamma\|_{L^2_{\text{loc}}} \lesssim \|\Gamma\|_{L^2_{\text{loc}}}
\]
for \( x_3 \in \text{supp } \phi \). Integration of the squares of the above estimates for \( f_3, f_4, f_5, f_6 \) gives the claim. \( \square \)

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**Appendix A. Quantitative parabolic theory**

Here we prove Proposition 5.1. Namely, we consider parabolic cylinders
\[
Q_R^{\lambda, \theta}(t_0, x_0) := [t_0 - \theta R^2, t_0] \times B(x_0, \lambda R), \quad Q_R^{\lambda, \theta} := Q_R^{\lambda, \theta}(0, 0), \quad Q_R := Q_R^{1, 1}
\]
and we consider Lipschitz solutions \( V \) of \( MV = 0 \) on \( Q_R^{\lambda, \theta} \), namely we suppose that
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} (\partial_t V \phi + \nabla V \cdot \nabla \phi + b \cdot \nabla V \phi) = 0 \tag{75}
\]
for all \( \phi \in C_c^\infty(Q_R^{\lambda, \theta}) \), where the (distributional) supports of \( \text{div } b \) and \( V \) are disjoint. Moreover we assume that (54) holds, namely
\[
\mathcal{N}(R) := 2 + \sup_{R' \leq 2R} (R')^{-\alpha} \|b\|_{L^q L^{\infty}(Q_{R'})} < \infty
\]
where \( \alpha := \frac{\alpha}{q} + \frac{\lambda}{2} - 1 \in [0, 1) \). We also say that \( V \) is a subsolution (or supersolution) of \( MV = 0 \), i.e. \( MV \leq 0 \) (or \( MV \geq 0 \)), if (75) holds with “=” replaced by “\( \leq \)” (or “\( \geq \)” for all nonnegative test functions.
We will show that
\[ \text{osc } V(0) \leq \left( \frac{r}{R} \right)^{\gamma} \text{osc } V \]
for all \( r \leq R \), where \( \gamma = \exp(-N^{O(1)}) \).

To this end we first prove the Harnack inequality for Lipschitz subsolutions of \( MV = 0 \).

**Lemma A.1** (based on Lemma 3.1 in [20]). Let \( V \) be a Lipschitz solution of \( MV \leq 0 \) in \( Q^\lambda_\theta \) where \( \lambda \in (1, 2] \) and \( \theta \in (0, 1] \). Then
\[ \sup_{Q^\lambda_\theta/2} V_+ \leq (\mathcal{N}/\theta)^C \left( \int_{Q^\lambda_\theta} V^2_+ \right)^{\frac{1}{2}}. \]

**Proof.** We first note that for any \( r, a \) such that
\[ \frac{3}{r} + \frac{2}{a} \in \left[ \frac{3}{2}, \frac{5}{2} \right] \]
we have the interpolation inequality
\[ \| \zeta U \|_{L^2_t L^4_x(Q^\lambda_\theta)} \lesssim_{\lambda, \theta} R^{\frac{4}{2} + \frac{2}{a} - \frac{2}{2}} \| \zeta U \|_{\mathcal{V}(Q^\lambda_\theta)} \]
by [15, (3.4) in Chapter II], where \( \mathcal{V} \) is the energy space \( L_t^\infty L^2_x \cap L^2_x \hat{H}^1_x \).

Since \( V \) is a subsolution, we have, for a non-negative test function \( \eta \),
\[ \int_{Q^\lambda_\theta} (\partial_t V \eta + \nabla V \cdot \nabla \eta + b \cdot \nabla V \eta) \leq 0. \]
We let \( \eta = \varphi'(V) \xi \) where \( \xi \) is a cutoff function vanishing on a neighborhood of the boundary of \( Q^\lambda_\theta \), and \( \varphi \) is a convex function vanishing on \( \mathbb{R}_- \). Taking \( U := \varphi(V) \) then gives
\[ \int_{Q^\lambda_\theta} (\partial_t U \xi + \nabla U \cdot \nabla \xi + \frac{\varphi''(V)}{\varphi'(V)^2} |\nabla U|^2 \xi + b \cdot \nabla U \xi) \leq 0. \]
We now take
\[ \varphi(\tau) := \tau^p \quad (p > 1) \quad \text{and} \quad \xi := \chi_{\{t < \overline{t}\}} U \zeta^2, \]
where \( \zeta \) is a smooth cutoff function in \( Q^\lambda_\theta \) and \( \overline{t} \in (-\theta R^2, 0) \),
\[ \int_{B_{\theta R}} (\zeta U)^2 dx + \int_{Q^\lambda_\theta \cap \{t < \overline{t}\}} (2 - p^{-1}) |\nabla U|^2 \zeta^2 + U \nabla U \cdot \nabla (\zeta^2) + \frac{1}{2} b \cdot \nabla (U^2) \zeta^2 - \partial_t (\zeta^2) U^2 \leq 0. \]
Using integration by parts and recalling the assumption \( \text{div } b \geq 0 \), we can apply Hölder’s inequality to obtain
\[ \int_{Q^\lambda_\theta \cap \{t < \overline{t}\}} b \cdot \nabla (\zeta^2) U^2 \geq - \int_{Q^\lambda_\theta \cap \{t < \overline{t}\}} b \cdot \nabla (\zeta^2) U^2 \]
\[ \geq - \|b\|_{L^p_t L^q_x(Q^\lambda_\theta)} \|U\|_{L^2_t L^4_x(Q)} \|\zeta\|_{L^2_t L^4_x(Q)} \|\nabla U\|_{L^p_t (1 + \frac{q}{p} - \frac{1}{2})^{-1} L^4_x (1 + \frac{q}{p} - \frac{1}{2})^{-1}} (Q) \]
\[ = - \|b\|_{L^p_t L^q_x(Q^\lambda_\theta)} \|U\|_{L^2_t L^4_x(Q^\lambda_\theta)} \|\zeta\|_{L^2_t L^4_x(Q)} \|\nabla U\|_{L^2_t L^4_x(Q^\lambda_\theta)}. \]
We now set
\[
\frac{1}{2s} + \frac{1}{q} + \frac{1}{r} \left( \frac{2}{q} - \frac{1}{s} \right) = 1, \quad \frac{1}{2s} + \frac{1}{\ell} + \frac{1}{a} \left( \frac{2}{q} - \frac{1}{s} \right) = 1.
\]
Applying Young’s inequality to separate the last term, and utilizing the interpolation inequality (77) (which is valid since
\[
\frac{3}{r} + \frac{2}{a} = \frac{3}{2} + 1 - 2 \left( 1 + \frac{2}{q} + \frac{2}{\ell} \right) \in (3/2, 11/6),
\]
as needed) we obtain, after plugging into the local energy inequality (78),
\[
\sup_{t \in [-\theta R^2,0]} \int_{B_{\lambda R}} (\zeta U)^2 dx + \int_{Q_{\lambda R}^M} (2 - p^{-1}) |\nabla U|^2 \zeta^2 + U \nabla U \cdot \nabla (\zeta^2) - \partial_t (\zeta^2) U^2
\]
\[
- O \left( R^2 \| U \|_{L^2(Q_{\lambda R}^M)}^2 \| \nabla \zeta \|_{L^2(Q_{\lambda R}^M)} \right) - \frac{1}{10} \| \zeta U \|_{V(Q_{\lambda R}^M)}^2 \leq 0.
\]
Absorbing $\nabla U$ from the term on the third term on the left-hand side by the second term we obtain
\[
\| \zeta U \|_{V(Q_{\lambda R}^M)}^2 \lesssim \int_{Q_{\lambda R}^M} \left( | \nabla \zeta |^2 + \zeta | \partial_t \zeta | + R^2 \| \zeta \|_{L^2(Q_{\lambda R}^M)}^2 \| \nabla \zeta |^{2s} \right) \zeta U^2.
\]
We now set
\[
\lambda_m := 1 + 2^{-m} (\lambda - 1) \quad \text{and} \quad \theta_m := \frac{1}{2}(1 + 4^{-m}),
\]
and we substitute $\zeta$ with $\zeta_m$ such that
\[
\zeta_m \equiv 1 \text{ in } Q_{\lambda_{m+1},\theta_{m+1}}^M, \quad \zeta_m \equiv 0 \text{ outside } Q_{\lambda_R}^{\lambda_m,\theta_{m}}, \quad \| \partial_t \zeta_m \| \leq \frac{4^m C}{\theta R^2}, \quad \frac{| \nabla \zeta_m |}{\zeta_m^{1-\frac{1}{q}}} \leq \frac{2^m C}{R},
\]
where $C$ may depend on $\lambda$. Then the energy estimate and (77), taken with $r = l = 10/3$, yield
\[
\| \zeta_m U \|_{L_{t,x}^{10/3}(Q_{\lambda R}^M)} \lesssim \| \zeta_m U \|_{V(Q_{\lambda R}^M)} \leq CR^{-1} (\theta^{-\frac{1}{2}} + 2^m + N^s) 2^m \| U \|_{L_{t,x}^2(Q_{\lambda R}^M)}.
\]
Recalling the definition of $U$ and replacing $p$ with $p_m := (5/3)^m$, Hölder’s inequality implies
\[
\left( \int_{Q_{\lambda_{m+1},\theta_{m+1}}^{M+1}} u_{2p_m}^{2p_m} \right)^{\frac{1}{2p_m}} \leq \left( \int_{Q_{\lambda_{m},\theta_{m}}^{M}} (\zeta_m U)^{10/3} \right)^{\frac{1}{2p_m}} \leq \left( C \theta_{m}^{-1} N^{2s} 4^{m(s+1)} \int_{Q_{\lambda_{m},\theta_{m}}^{M}} u_{2p_m}^{2p_m} \right)^{\frac{1}{2p_m}}.
\]
Iterating, we have
\[
\left( \int_{Q_{\lambda_{m},\theta_{m}}^{M}} u_{2p_m}^{2p_m} \right)^{\frac{1}{2p_m}} \leq \prod_{k=0}^{m-1} \left( \frac{C}{\theta} 4^{k(s+1)} N^s \right)^{\frac{1}{2p_{k}}} \left( \int_{Q_{\lambda_{M},\theta_{M}}^{M}} u_{2p_{M}}^{2p_{M}} \right)^{\frac{1}{2}}
\]
and we conclude by taking $m \to \infty$. \qed
In the next three lemmas we focus on nonnegative solutions to $\mathcal{M}V \leq 0$ and we find lower bounds on the mass distribution of such solutions. We first show that if $V \geq k$ in $Q_R$, except for a small (quantified) “portion of $Q_R$", then in fact $V \geq k/2$ everywhere in a smaller cylinder.

**Lemma A.2** (based on part 2 of Corollary 3.1 in [20]). If $V$ is a non-negative solution of $\mathcal{M}V \geq 0$ in $Q_R^{1,\theta}$ and

$$|\{V < k\} \cap Q_R^{1,\theta}| \leq (N/\theta)^{-5C}|Q_R^{1,\theta}|,$$

then

$$V \geq \frac{k}{2} \text{ in } Q_R^{1,\theta/2}.$$

**Proof.** We apply Lemma A.1 to $k - V$ to find

$$\sup_{Q_R^{1,\theta/2}} (k - V) \leq (N/\theta)^C \left( \int_{Q_R^{1,\theta}} (k - V)^2 \right)^{\frac{1}{2}} \leq N^{-1}k$$

which implies the result. \qed

We now show that, if the cylinder $Q_R^{1,\theta}$ is flat enough, then a lower bound on the bottom lid of $Q_R^{1,\theta}$ (i.e. at $t = -\theta R^2$) implies a similar lower bound at every $t$.

**Lemma A.3** (based on Lemma 3.2 in [20]). Suppose $V$ is non-negative with $\mathcal{M}V \geq 0$ in a neighbourhood of $Q_R^{1,\theta_0}$, and

$$|\{V(-\theta_0 R^2) \geq k\} \cap B_R| \geq \delta_0|B_R|$$

for some $\delta_0 > 0$ and $\theta_0 \leq C^{-1}\delta_0^6 N^{-1}$. Then

$$|\{V(t) \geq \frac{1}{3}\delta_0 k\} \cap B_R| \geq \frac{1}{3}\delta_0|B_R|$$

for all $t \in [-\theta_0 R^2, 0]$.

**Proof.** By the calculations in [20], with $\zeta$ a smooth cutoff function supported in $B_R$,

$$\int_{B_R} (V(t) - k)^2 \zeta^2 + \int_{Q_R^{1,\theta_0}} \chi_{(t<\bar{t})} |\nabla(V - k)|^2 \zeta^2 \leq \int_{B_R} (V(-\theta_0 R^2) - k)^2 \zeta^2$$

$$+ \int_{Q_R^{1,\theta_0}} \chi_{(t<\bar{t})} (V - k)^2 \left( O(|\nabla \zeta|^2) + b \cdot \nabla(\zeta^2) + (\text{div } b)\zeta^2 \right).$$

(79)

(80)

We choose $\zeta$ such that $\zeta \equiv 1$ in $B_{(1-\sigma)R}$ and $|\nabla \zeta| \leq \frac{2}{\sigma R}$ where $\sigma < 1$ is to be specified. Note that due to (53),

$$\int_{Q_R^{1,\theta_0}} \chi_{(t<\bar{t})} (V - k)^2 (\text{div } b)\zeta^2 \leq k^2 \int_{Q_R^{1,\theta_0}} \chi_{(t<\bar{t})} (\text{div } b)\zeta^2$$

$$= -k^2 \int_{Q_R^{1,\theta_0}} \chi_{(t<\bar{t})} b \cdot \nabla(\zeta^2).$$

Then the right-hand side of (79) is bounded by

$$k^2 \left( (1 - \delta_0)|B_R| + O(\theta_0 \sigma^{-2})|B_R| + \frac{4}{\sigma R} \|b\|_{L^q_L^q(Q_R)} \|1\|_{L^q_L^q(Q_R^{1,\theta_0})} \right).$$
From here one can proceed with the argument exactly as in [20] to arrive at
\[
\left| \left\{ V(t) < \frac{1}{3} \delta_0 k \right\} \cap B_R \right| \leq \left( 1 - \frac{1}{3} \delta_0 \right)^{-2} (1 - \delta_0 + O(\sigma + \sigma^{-2} \theta_0 + \sigma^{-1} \theta_0^{2/\ell'} \mathcal{N})).
\]
Setting \( \sigma = C^{-1/5} \delta_0^3 \) and \( \theta_0 \) as above proves the claimed bound.

We now show that for any given “portion of \( Q_R^{1,\theta} \)” (in the sense of a set with the measure arbitrarily close to \( |Q^{1,\theta}| \)) \( V \) is greater or equal a constant multiple of some lower bound, if, for each \( t \), the lower bound occurs at least on some “portion of \( B_R \).” Although this enables us to obtain a lower bound on almost the entire cylinder, we lose an exponential in the process.

**Lemma A.4** (based on Lemma 3.3 in [20]). Let \( V \geq 0 \) be a solution of \( \mathcal{M}V \geq 0 \) in \( Q_R^{\lambda,\theta} \) satisfying
\[
\left| \left\{ V(t) \geq k_0 \right\} \cap B_R \right| \geq \delta_1 |B_R| \text{ for all } t \in [-\theta R^2, 0]
\]
for some \( k_0 > 0, \delta_1 > 0 \). Then for any \( \mu > 0 \) and \( s > C(\mathcal{N} + \theta^{-1}) / (\delta_1 \mu)^2 \),
\[
\left| \left\{ V < 2^{-s} k_0 \right\} \cap Q_R^{1,\theta} \right| \leq \mu |Q_R^{1,\theta}|.
\]

**Proof.** With \( k_m = 2^{-m} k_0 \), we define
\[
\mathcal{E}_m(t) := \{ x \in B_R : k_{m+1} \leq V(x,t) < k_m \}; \quad \mathcal{E}_m := \{(t,x) \in Q^{1,\theta}_R : x \in \mathcal{E}_m(t) \}.
\]
Integrating the inequality \( \mathcal{M}V \geq 0 \) against the test function \( \eta = (V - k_m)_- \xi(x)^2 \) where \( \xi \) is a smooth cutoff vanishing in a neighborhood of \( \partial B_{\lambda R} \) and satisfying \( \xi \equiv 1 \) in \( B_R \),
\[
\int_{Q_R^{\lambda,\theta} \cap \{V < k_m\}} |\nabla V|^2 \xi^2 \leq \int_{Q_R^{1,\theta}} |\nabla (V - k_m)_- \xi|^2 \xi^2 \leq \int_{B_{\lambda R} \cap \{V < k_m\}} (V - k_m)_-^2 \xi^2 \bigg|_{t = -\theta R^2} + \int_{-\theta R^2}^{0} \int_{B_{\lambda R} \cap \{V < k_m\}} (V - k_m)_-^2 |\nabla \xi|^2 + 2(V - k_m)_-^2 \xi b \cdot \nabla \xi
\]
\[
\lesssim k_m^2 R^n (1 + \theta \mathcal{N}) \tag{81}
\]
by Hölder’s inequality and the trivial bound \( 0 \leq (V - k_m)_- \leq k_m \). From De Giorgi’s inequality [15, (5.6) in Chapter II],
\[
(k_m - k_{m+1}) \left| \left\{ V(t) < k_{m+1} \right\} \cap B_R \right| \lesssim \frac{R}{\delta_1} \int_{\mathcal{E}_m(t)} |\nabla V(t)|
\]
for all \( t \in [-\theta R^2, 0] \). Integrating in time, squaring, and applying Cauchy-Schwarz gives
\[
k_{m+1}^2 \left| \left\{ V < k_{m+1} \right\} \cap Q_R^{1,\theta} \right|^2 \lesssim \frac{R^2}{\delta_1^2} \int_{\mathcal{E}_m} |\nabla V|^2 dx dt |\mathcal{E}_m|.
\]
Combined with (81), this gives
\[
\left| \left\{ V < k_{m+1} \right\} \cap Q_R^{1,\theta} \right|^2 \lesssim \delta_1^{-2} R^{n+2}(1 + \theta \mathcal{N}) |\mathcal{E}_m|.
\]
We conclude

\[
s \left| \{ V < k_m \} \cap Q_R^{1,\theta} \right|^2 \leq \sum_{m=0}^{s-1} \left| \{ V < k_{m+1} \} \cap Q_R^{1,\theta} \right|^2 \]
\[
\lesssim \delta_1^{-2} R^{n+2}(1 + \theta N) \sum_{m=0}^{s-1} |E_m| \]
\[
\lesssim \delta_1^{-2}(\theta^{-1} + N)|Q_R^{1,\theta}|^2.
\]

\[\square\]

We can now combine Lemmas A.2–A.4 to obtain a pointwise lower bound for \( V \) in the interior of a cylinder, with an exponential dependence on \( N \).

**Lemma A.5** (based on part 1 of Corollary 3.2 in [20]). If \( V \) is a non-negative solution of \( \mathcal{M}V \geq 0 \) in \( Q_R^{2,1} \) and

\[
|\{ V(-\Theta R^2) \geq k \} \cap B_R| \geq \delta |B_R|
\]

for some \( k > 0 \) and \( \Theta \leq C^{-1}\delta^{\theta/2}N^{-1} \), then

\[
V \geq \exp(-\delta^{-2}(N/\Theta)^{20C})k \quad \text{in} \quad Q_R^{1,\Theta/2}.
\]

**Proof.** This is a straightforward application of Lemmas A.3, A.4, and A.2 in sequence, with the latter two applied with \( R \to \frac{3}{2}R \) to compensate for the shrinking domain in Lemma A.2.

\[\square\]

By considering \( V - \inf V \) and \( \sup V - V \) the above lemma now allows us to estimate oscillations of solutions to \( \mathcal{M}V = 0 \) with no sign restrictions.

**Lemma A.6** (based on Lemma 3.5 of [20]). If \( V \) solves \( \mathcal{M}V = 0 \) in \( Q_R^{2,1} \) then

\[
\text{osc}_{Q^{(1)}} V \leq (1 - \exp(-N^{50C})) \text{osc}_{Q^{(2)}} V
\]

where \( Q^{(1)} = Q_R^{1,\Theta/2} \), \( Q^{(2)} = Q_R^{2,1} \), and \( \Theta = C^{-2}N^{-1} \).

**Proof.** Consider the positive supersolutions \( V_1 = V - \inf_{Q^{(2)}} V \) and \( V_2 = \sup_{Q^{(2)}} V - V \). With \( k = \text{osc}_{Q^{(2)}} V \), clearly we must have \( |\{ V_i(-\Theta R^2) \geq k \} \cap B_{2R}| \geq |B_{2R}|/2 \) for either \( i = 1 \) or \( i = 2 \). Fix this \( i \), so \( V_i \) obeys the hypotheses of Lemma A.5. Let us assume for concreteness that \( i = 1 \); the other case is analogous. Then by the lemma,

\[
\inf_{Q^{(2)}} V + \exp(-N^{50C}) \text{osc}_{Q^{(2)}} V \leq V \leq \sup_{Q^{(2)}} V
\]

for all \( (t, x) \in Q^{(1)} \), which immediately implies the result.

\[\square\]

Finally, iterating Lemma A.6 we obtain the required Hölder continuity (76), i.e. we can prove Proposition 5.1.

**Proof of Proposition 5.1.** Iterating Lemma A.6, we have

\[
\text{osc}_{Q_R^{2,1}} V \leq (1 - \exp(-N^{50C}))^k \text{osc}_{Q_R^{2,1}} V.
\]

We conclude upon taking \( k = \lfloor \log \frac{R}{r}(\log \frac{2}{\delta})^{-1} \rfloor \).

\[\square\]
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