Multiply robust two-sample instrumental variable estimation

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Abstract

Although instrumental variable (IV) methods are widely used to estimate causal effects in the presence of unmeasured confounding, the IVs, exposure and outcome are often not measured in the same sample due to complex data harvesting procedures. Following the influential articles by Klevmarken (1982) and Angrist & Krueger (1992, 1995), numerous empirical researchers have applied two-sample IV methods to perform joint estimation based on an IV-exposure sample and an IV-outcome sample. This paper develops a general semi-parametric framework for two-sample data combination models from a missing data perspective, and characterizes the efficiency bound based on the full data model. In the context of the two-sample IV problem as a specific example, the framework offers insights on issues of efficiency and robustness of existing estimators. We propose new multiply robust locally efficient estimators of the causal effect of exposure on the outcome, and illustrate the methods through simulation and an econometric application on public housing projects.

KEY WORDS: statistical matching, data fusion, multiply robust, instrumental variable
1 Introduction

Unmeasured confounding is a common challenge for drawing causal inferences from observational data. Under the regression setting for an outcome $Y$, this usually entails endogeneity of the exposure variable $A$, leading to biased causal effect estimates. Although instrumental variable (IV) methods are widely used in the health and social sciences as identification strategies (Bowden & Turkington 1990, Robins 1994, Angrist et al. 1996, Greenland 2000, Wooldridge 2010, Hernán & Robins 2006, Didelez et al. 2010), in many empirical scenarios the IV $Z$ and outcome $Y$ are observed in one sample, while $Z$ and $A$ are observed in another. Klevmarken (1982) and Angrist & Krueger (1992, 1995) have proposed the two-sample instrumental variable (TSIV) and two-sample two-stage least squares (TS2SLS) estimators, which have since been widely applied in econometrics and social sciences (Björklund & Jäntti 1997, Inoue & Solon 2010, Dee & Evans 2003, Olivetti & Pasceman 2015, Van den Berg et al. 2016). Ridder & Moffitt (2007) and Kmenta (2010) provide further details and review of two-sample IV methods, and Graham et al. (2016) studied the two-sample IV estimation problem as one specific example of a larger, general class of data combination or fusion problems. Recently, there has also been growing interest in genetic epidemiology in the two-sample IV setting to estimate causal relationship between an exposure and an outcome by leveraging on genetic markers as IVs. This is achieved through a method known as Mendelian randomization (Davey Smith & Ebrahim 2003, Lawlor et al. 2008, Burgess et al. 2017), by taking advantage of existing data sets from Genome-Wide Association Studies (GWAS) in which the exposure and disease outcome are not measured in the same sample (Pierce & Burgess 2013, Gamazon et al. 2015, Lawlor 2016, Zhao et al. 2017).

Traditionally, TSIV methods have been used to identify and estimate parameters indexing a system of structural linear models (Angrist & Krueger 1992, 1995, Inoue & Solon 2010, Pacini & Windmeijer 2016). Consider the following linear regression models with one endogenous exposure $A$ and $k \geq 1$ IVs,

$$Y = \beta A + X \eta + \epsilon_y$$

$$A = Z \theta_z + X \theta_x + \epsilon_a$$

(1)
where $Z$ is a $1 \times k$ vector of IVs, and $X$ is a $1 \times p$ vector of observed covariates. The target parameter of interest in (1) is $\beta \in \mathbb{R}$ which encodes the causal effect on $Y$ of changing $A$ by one unit (Holland 1988), and becomes identified when $\theta_2 \neq 0$ so that $Z$ is associated with $A$, $Z$ is excluded from the outcome model and $E(\epsilon|X,Z) = 0$. TSIV and TS2SLS are popular estimators of $\beta$ in the two-sample IV setting, and typically assume that the two samples are simple random draws from the primary population of interest. In addition, model (1) imposes a single set of strong assumptions on the underlying data generating process which conflates the definition, identification and estimation of the causal effect of interest (Wang & Tchetgen Tchetgen 2018). For example, the target parameter $\beta$ may not even be well-defined if model (1) is misspecified.

The TSIV and TS2SLS estimators remain consistent under conditional heteroskedasticity of the error terms (Inoue & Solon 2010), and Pacini & Windmeijer (2016) derived robust variance estimators under this setting. Choi et al. (2018) developed weak instrument robust inference based on model (1) and also relaxed the assumption of equal moments of exogenous covariates across the two samples. Inoue & Solon (2010) have noted that, unlike TSIV, the TS2SLS estimator with a correctly specified linear model for the IV-exposure relationship remains consistent with a stratified sampling mechanism (or propensity score) that can vary with $V = (X,Z)$. Graham et al. (2016) introduce doubly robust (DR) estimators under restricted model specification of nuisance parameters, and derive semiparametric efficiency bounds under a general class of moment conditions which allow sample moments of the common variables $V$ to differ significantly across the two datasets being combined. Recent works by Shu & Tan (2018) introduce DR estimators of $\beta$ which remain consistent when a parametric model for either the propensity score or IV-exposure relationship, but not necessarily both, is correctly specified. Zhao et al. (2017) introduced TSIV estimators that are robust to heterogeneous samples in which the distributions of the IVs are different, and relaxed the linearity assumptions of the IV-exposure and exposure-outcome equations.

In this paper, we develop a general semi-parametric framework for two-sample missing data problems, and investigate the two-sample IV setting as a specific example. We introduce a suite of semi-parametric estimators with improved robustness properties under misspecifications of some
of the working parametric models, and discuss the properties of existing two-sample IV estimators. The proposed methods include triply robust estimators which remain consistent in the union of three different observed data models involving combinations of working models for (i) the propensity score (ii) IV-exposure relationship, (iii) the conditional distribution of IV given observed covariates and (iv) the main effects of covariates on the outcome, and additionally attain the semi-parametric efficiency bound when all the models are correctly specified. We also introduce multiply robust estimation under a stronger ignoribility assumption. In section 2 we lay out the notation and assumptions. In section 3 we develop semi-parametric theory for the general two-sample setting, followed by discussions on new semi-parametric robust methods for the two-sample IV problem, as well as their implementations, in section 4. We examine and evaluate the finite sample performance of the proposed estimators in an extensive simulation study summarized in section 5, and illustrate the methods with an econometric application to public housing projects in section 6. We conclude in section 7 with a discussion. Throughout, proofs and derivations can be found in the appendix.

2 Preliminaries

2.1 Full data setting

Suppose the full data consists of \( n \) i.i.d. realizations of a vector \( L = (Y, A, V) \), where \( V = (X, Z) \). Let \( U \) denote a possibly multivariate unmeasured confounder of the effect of a scalar exposure \( A \) on a scalar outcome \( Y \). We assume \( Z \) to satisfy the following IV assumptions (Didelez & Sheehan 2007)

\[
A1. \ Z \perp A | X \quad (IV \ relevance),
\]

\[
A2. \ Z \perp Y | A, U, X \quad (Exclusion \ restriction), \ and
\]

\[
A3. \ Z \perp U | X \quad (IV \ independence).
\]

The first condition ensures that \( Z \) is a correlate of the exposure even after conditioning on \( X \). Condition A2 states that \( Z \) is independent of all unmeasured confounders of the exposure-outcome
association, while A3 formalizes the assumption of no direct effect of $Z$ on $Y$, which is not mediated by $A$. In the causal inference literature using the potential outcomes approach (Neyman 1923, Rubin 1974), conditions A2 and A3 are often alternatively formalized as the assumption (Robins 1994) that

$$A2'. \ Y^{(a)} \perp Z|X \ \forall a,$$

where $Y^{(a)}$ denote the outcome that would be observed if the exposure $A$ is set to value $a$. It is well known that while a valid IV satisfying A1-A3 suffices to obtain a valid statistical test of the sharp null hypothesis of no individual causal effect, the population average causal effect is itself not point identified with a valid IV without additional assumptions. Consider the following additive structural model for the outcome:

$$Y = m(X; \beta)A + h(X, U) + \epsilon$$

$$= m(X; \beta)A + E[h(X, U)|V] + \left\{h(X, U) - E[h(X, U)|V] + \epsilon\right\}$$

$$\epsilon_y (2)$$

where $m(X; \beta)$ is a known function of $X$ indexed by a finite-dimensional unknown parameter $\beta$ such that $m(X; \beta) = 0$ when $\beta = 0$, and captures the linear effect of $A$ on $Y$ while allowing for potential effect modification by $X$. In addition, $h(X, U)$ is an unknown function of $(X, U)$ and $E(\epsilon_y|V) = 0$. Model (2) imposes the same restrictions on the observed data law as the linear or additive structural mean model (Robins 1994), which can be derived without explicit reference to the unobserved confounders $U$ (Vansteelandt & Didelez 2018). Together with the IV assumptions A1-3, model (2) imposes the same restrictions on the observed data law (Robins & Rotnitzky 2004, Vansteelandt & Didelez 2018) as the model defined by

$$E\{Y - m(X; \beta)A|V\} = E\{h(X, U)|V\} = \omega(X),$$

(3)

which is a special case of the structural nested mean models (Robins 1994), and $\omega(X)$ is an unknown function in $X$. We denote the full data model (3) as $\mathcal{A}_{IV}$. Model (1) follows from (3) with the additional restrictions $m(X; \beta) = \beta$, $\omega(X; \eta) = X\eta$ and $E[A|V] = V\theta$.
2.2 Two-sample missing data setting

Suppose the observed data consists of \( n \) i.i.d. realizations of a vector \( L_O = (R, Y, A, V) \). Under the general two-sample missing data setting, \( Y \) is observed if \( R = 1 \) and \( Y = Y^* \) otherwise, while \( A \) is observed if \( R = 0 \) and \( A = A^* \) otherwise, where \( Y^* \) and \( A^* \) denote missing outcome and exposure values respectively. We always observe \( V = (X, Z) \). An individual is observed in the sample for \( R = 1 \) with probability \( \pi(V) = P(R = 1|V) \). In addition, we make the following assumptions on the sampling mechanism:

A4. **Positivity:** \( \rho < \pi(V) < 1 - \rho \) almost surely, for a fixed positive constant \( \rho \).

A5. **Ignorability:** \( R \perp (Y, A)|V \).

We denote the models satisfying assumptions A4 and A5 as \( \mathcal{A}_{TS} \). The positivity assumption A4 states that the probability of observing an individual in either sample is bounded away from both 0 and 1. We note that A4 is strictly weaker that the usual positivity assumption typically assumed in missing data problems which requires a positive probability of observing complete data for each individual. Assumption A5 is used extensively in econometrics and statistics to achieve identification in missing data and causal inference problems, for example in Robins et al. (1994, 1995) and Imbens (2004). The two-stage sampling design (Breslow et al. 2000, 2003) is an example of a sampling mechanism that satisfies both A4 and A5. We note that under the current setting, both samples are random draws from the population of interest, although the marginal distributions of the fully observed variables \( V \) can vary between the two samples. This development from a missing data perspective, e.g. Robins et al. (1994), differs from the “verify-out-of-sample” case (Chen et al. 2008, Graham et al. 2016, Shu & Tan 2018) in which the auxiliary and study samples are drawn from two distinct populations.

3 General two-sample observed data efficient score

The two-sample data combination problem is a special case of general censored data models (Van der Laan et al. 2003). Let \( \mathcal{H}, \mathcal{H}^F \) denote the observed and full data Hilbert spaces of all
$q$-dimensional, mean-zero, finite variance, measurable functions of $L_O$ and $L$ respectively, equipped with the covariance inner product. Suppose under the full data model $A_F$, the ortho-complement to the full data nuisance tangent space is given by elements of the form

$$\Lambda_{F}^\perp = \{ \phi^F(L; \beta) = \phi_1(Y, V; \beta) - \phi_2(A, V; \beta) \} \subset \mathcal{H}^F,$$

where $\beta$ is a $q$-dimensional parameter of interest. For example, under the outcome linear model in (1), the ortho-complement full data nuisance tangent space is given by (Tsiatis 2007)

$$\Lambda_{F}^{\perp \text{linear}} = \{ (Y - \beta A - X \eta) g(V) : g(V) \in \mathcal{H} \},$$

so that $\phi_1(Y, V; \beta) = Y g(V)$ and $\phi_2(A, V; \beta) = (\beta A + X \eta) g(V)$. Based on results for conditional mean models (Chamberlain 1987), $\Lambda_{F}^{\perp \text{linear}}$ also corresponds to the class of estimating functions for all regular and asymptotically linear (RAL) estimators of $\beta$ up to asymptotic equivalence. Given $\Lambda_{F}^{\perp}$ under a full data model $A_F$, we have the following result for the general two-sample data model $A_F \cap A_{TS}$, in which $A$ and $Y$ are never jointly observed:

**Result 1** The observed data ortho-complement nuisance tangent space under $A_F \cap A_{TS}$ is given by

$$\Lambda = \left\{ \left[ R \pi(V) \left[ \phi_1(Y, V) + a_1(V) \right] - \frac{1 - R}{1 - \pi(V)} \left[ \phi_2(A, V) + a_2(V) \right] + [a_1(V) - a_2(V)] \right] - \Pi\{\cdot\} \Lambda_\psi \right\} : \phi_1(Y, V) - \phi_2(A, V) = \phi^F(L) \in \Lambda_{F}^{\perp}, a_1(V), a_2(V) \in \mathcal{H},$$

where

$$\Lambda_\psi = \left\{ \left[ \frac{R}{\pi(V)} - \frac{1 - R}{1 - \pi(V)} \right] \mu(V) : \mu(V) \in \mathcal{H} \right\}$$

is the nuisance tangent space associated with the propensity score $\pi(V)$ and $\Pi$ is the projection operator. Furthermore, the linear subspace of efficient elements in $\Lambda$ for any fixed $\phi^F(L) \in \Lambda_{F}^{\perp}$.
is given by

\[ \mathcal{F}^{DR} = \left\{ \frac{R}{\pi(V)} [\phi_1(Y, V) - E(\phi_1|V)] - \frac{1 - R}{1 - \pi(V)} [\phi_2(A, V) - E(\phi_2|V)] + E[\phi^F(L)|V] : \phi_1(Y, V) - \phi_2(A, V) = \phi^F(L) \in \Lambda^{\perp} \right\} . \]

The fundamental results in Bickel et al. (1993) show that given any RAL estimator of \( \beta \), we can find a candidate estimating function of the form given in \( \Lambda^{\perp} \) so that its empirical solution is asymptotically equivalent to the estimator under standard regularity conditions. Therefore Result 1 identifies all estimating functions for \( \beta \) of interest under a general two-sample observed data setting, based on the full data restrictions. We denote \( \mathcal{F}^{DR} \) in Result 1 as the DR linear space, since the RAL estimators of \( \beta \) with influence functions proportional to elements in \( \mathcal{F}^{DR} \) are consistent if either the propensity score model \( \pi(V) \) or the conditional distributions \( f(Y|V) \), \( f(A|V) \) are correctly specified, but not necessarily both. We will illustrate the construction of such DR estimators for the two-sample IV problem in the next section. The observed data efficient score, which is an element of \( \mathcal{F}^{DR} \), is given in the next result.

**Result 2** Let \( S^F_{eff}(L) \) denote the full data efficient score under model \( A_F \). Then the observed data efficient score under model \( A_F \cap A_{TS} \) is given by

\[ S_{eff}(L_O) = \frac{R}{\pi(V)} [\phi_{1, eff}(Y, V) - E(\phi_{1, eff}|V)] - \frac{1 - R}{1 - \pi(V)} [\phi_{2, eff}(A, V) - E(\phi_{2, eff}|V)] + E[\phi^F_{eff}(L)|V], \]

where \( \phi^F_{eff}(L) = \phi_{1, eff}(Y, V) - \phi_{2, eff}(A, V) \) is the unique element \( \phi^F(L) \in \Lambda^{\perp} \) that solves

\[ \Pi \{ W^{-1} [\phi^F(L)] | \Lambda^{\perp} \} = S^F_{eff}(L) \]

and \( W(\cdot) \) denotes the linear operator \( W(\cdot) = E \{ E[\cdot|L_O] | L_O \} \). The semiparametric efficiency bound is given by \( E \{ S_{eff}(L_O)S^F_{eff}(L_O) \}^{-1} \).

The semiparametric efficiency bound in Result 2 is derived under model \( A_F \cap A_{TS} \), in contrast to
other efficiency results derived under a specific full data moment condition, for example in Chen et al. (2008). We note that in general, $\phi_{F_{\text{eff}}}$ is not equal to $S_{F_{\text{eff}}}$, so that the observed data efficiency bound derived under even the moment condition corresponding to $S_{F_{\text{eff}}}$ may not necessarily be equivalent to the semiparametric efficiency bound under model $A_F \cap A_{TS}$. Furthermore, $\phi_{F_{\text{eff}}}$ is generally not available in closed form, in the sense that it cannot be expressed explicitly as functions of the true distribution (Robins et al. 1995). However, as we will illustrate in section 5 in the context of the two-sample IV problem, a closed form expression for $\phi_{F_{\text{eff}}}$ exists with binary $Z$, and the result generalizes readily to polytomous $Z$. We will also derive approximately efficient influence functions in the case of continuous $Z$.

Remark 1 The semiparametric efficiency bound given in Result 2, which is derived without prior restrictions on the parametric form of $\pi(V)$, equals to the efficiency bounds derived when $\pi(V)$ is either known or assumed to belong to a parametric family. This is consistent with other missing data problems such as Robins et al. (1994) and the “verify-in-sample” case of Chen et al. (2008), in which the propensity score is ancillary to estimation of the parameters of interest. In contrast, knowledge about $\pi(V)$ reduces the efficiency bound under “verify-out-of-sample” problems (Chen et al. 2008, Shu & Tan 2018), such as in estimating average treatment effects on the treated (Hahn 1998).

4 Semiparametric Two-sample IV model

In this section, we consider the full data IV model $A_{IV}$ as a specific example of $A_F$, and assume that the parametric model for the main effects of the covariates on the outcome is correctly specified,

\[ A6. \text{The model for the covariate main effect } \omega(X; \eta) \text{ in the outcome regression model is correctly specified so that } \omega(X; \eta^\dagger) = \omega(X) \text{ for some } \eta^\dagger, \text{ where } \omega(X; \eta) \text{ is a function of } X \text{ smooth in } \eta, \text{ and } \eta^\dagger \text{ is an unknown finite-dimensional parameter. Let } \mathcal{M}_\omega \text{ denote the models satisfying } A6. \]

In the model $A_{IV} \cap \mathcal{M}_\omega$, it follows from general results on conditional mean models (Chamberlain 1987) that any RAL estimator of $\beta$ up to asymptotic equivalence has an influence function
which is proportional to an element in the linear subspace

\[ \{ [Y - \omega(X; \eta) - m(X; \beta)A] g(V) : g(V) \in \mathcal{H} \} . \]

A direct application of Result 1 leads to the observed data ortho-complement nuisance tangent space in the two-sample setting under model \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \mathcal{M}_\omega \),

\[
\Lambda_{iv,\omega}^\perp = \left\{ \left[ \frac{R}{\pi(V)} [Y + a_1(V)] - \frac{1 - R}{1 - \pi(V)} [\omega(X; \eta) + m(X; \beta)A + a_2(V)] + [a_1(V) - a_2(V)] \right] g(V) 
- \Pi \left\{ [\cdot] g(V) | \Lambda_\psi \right\} : g(V), a_1(V), a_2(V) \in \mathcal{H} \right\},
\]

and for fixed choices of \( g(V) \), the efficient elements in \( \Lambda_{iv,\omega}^\perp \) are given by the linear subspace

\[
\mathcal{F}^{\text{doubly}} = \{ U_g^{\text{doubly}} = H(L_O; \beta) g(V) : g(V) \in \mathcal{H} \},
\]

where

\[
H(L_O; \beta) = \left\{ \frac{R}{\pi(V)} [Y - \omega(X; \eta) - m(X; \beta)E(A|V)] - \frac{1 - R}{1 - \pi(V)} m(X; \beta) [A - E(A|V)] \right\}.
\]

### 4.1 Doubly robust estimation

The linear space \( \mathcal{F}^{\text{doubly}} \) is a subspace of the orthocomplement to the nuisance tangent space and therefore its elements are influence functions (IF) for RAL estimators of \( \beta \). IF-based estimating functions entails estimating the distribution of the observed data under parametric or non-parametric working models and then evaluating the estimating function under these models \((\text{Tsiatis} \ 2007)\). Although in principle the propensity score \( \pi(V) \) and regression function \( E(A|V) \) in \((\ref{H})\) can be estimated nonparametrically using methods such as sieve estimation \((\text{Hahn} \ 1998, \text{Hirano et al.} \ 2003, \text{Chen et al.} \ 2008)\), the resulting estimators of \( \beta \) typically exhibit poor finite sample behavior when \( V \) is of moderate or high dimension relative to the sample size, due to the curse of dimensionality \((\text{Robins \ & \ Ritov} \ 1997)\). This motivates the doubly robust approach in which parametric models are specified for both the propensity score and the regression function.
Consider the following additional assumptions:

A7. The propensity score model $\pi(V; \psi)$ is correctly specified so that $\pi(V; \psi^\dagger) = \pi(V)$ for some value $\psi^\dagger$, where $\pi(X; \psi)$ is a function of $V$ smooth in $\psi$, and $\psi^\dagger$ is an unknown finite-dimensional parameter. Let $M_\pi$ denote the models satisfying A7.

A8. The regression model $E(A|V; \theta) = h(V; \theta)$ is correctly specified so that $h(V; \theta^\dagger) = E(A|V)$ for some $\theta^\dagger$, where $h(V; \theta)$ is a function of $V$ smooth in $\theta$, and $\theta^\dagger$ is an unknown finite-dimensional parameter. Let $M_h$ denote the models satisfying A8.

Let $P_n$ denote the empirical mean operator $P_n f(L_O) = n^{-1}\sum_i f(L_{Oi})$. For a given parametric model $\pi(V; \psi)$ for the propensity score, let $\hat{\psi}$ denote the maximum likelihood estimator (MLE) of $\psi$ that solves $P_n \{S_\psi(R, V)\} = 0$, where

$$S_\psi(R, V) = \left\{ \frac{R}{\pi(V; \psi)} - \frac{1 - R}{1 - \pi(V; \psi)} \right\} \frac{\partial \pi(V; \psi)}{\partial \psi}$$

is the score function of $\psi$. For example, if we assume the logistic model $\pi(V; \psi) = \{1 + \exp[-b(V)\psi]\}^{-1}$, then $\hat{\psi}$ is the solution to the estimating equation $P_n \{[(R - \pi(V; \psi)b(V)]\} = 0$. By the ignorability assumption A3, $E(A|V) = E(A|R, V)$. Therefore for a given parametric model of the conditional expectation $E(A|V) = h(V; \theta)$, $\hat{\theta}$ can be obtained as the solution to the estimating equation $P_n \{U_{\theta,c}\} = 0$ where $U_{\theta,c} = (1 - R)c(V)[A - h(V; \hat{\theta})] \{m(X; \beta) h(V; \hat{\theta})\}$ for some vector function $c(V)$ that has the same dimension as $\theta$. For example, we can take $c'(V) = \frac{\partial h(V; \theta)}{\partial \theta}$.

Assume that the unknown true value $\beta^\dagger$ for the $p$-dimensional parameter of interest is in the interior of $\Theta_\beta$, where $\Theta_\beta \subset \mathbb{R}^p$ and is compact. Consider the estimator $(\hat{\beta}_g^{\text{doubly}}, \hat{\eta}^{\text{doubly}})$ which solves the estimating equation

$$P_n \left\{ U_g^{\text{doubly}} \left( \beta, \eta, \hat{\psi}, \hat{\theta} \right) \right\} = P_n \left\{ \left\{ \frac{R}{\pi(V; \hat{\psi})} \left[ Y - \omega(X; \eta) - m(X; \beta) h(V; \hat{\theta}) \right] - \frac{1 - R}{1 - \pi(V; \hat{\psi})} m(X; \beta) [A - h(V; \hat{\theta})] \right\} \times \left[ g^T(V), \lambda^T(X; \eta) \right]^T = 0, \right.$$  

(6)
where \( \lambda(X; \eta) = \frac{\partial \omega(X; \eta)}{\partial \eta} \) and \( g(V) \) are vector functions conformable to the dimensions of \( \eta \) and \( \beta \). For example, if \( X = (1, V) \) where \( V, Z \) are scalars and \( m(X; \beta) = \beta_1 + \beta_2 V \), \( h(V; \theta) = \theta_0 + \theta_1 Z \), we can choose \( g(V) = (Z, ZV) \), such that \( E[U_{g, doubly}^T U_{g, doubly}] < \infty \) and \( E[\frac{\partial}{\partial \tau} U_{g, doubly}(\tau)] \) is nonsingular, where \( \tau = (\beta, \eta) \).

**Result 3** Under standard regularity conditions, \( \hat{\beta}_{g, doubly} \) is a consistent and asymptotically normal (CAN) estimator of \( \beta \) in the union model

\[
\mathcal{M}_{doubly} = \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \mathcal{M}_\omega \cap (\mathcal{M}_\pi \cup \mathcal{M}_h).
\]

In addition, \( \hat{\beta}_{g, doubly} \) is efficient (for a fixed choice of \( g \)) at the intersection model \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \mathcal{M}_\omega \cap \mathcal{M}_\pi \cap \mathcal{M}_h \).

As corollaries to Result 3, inverse probability weighted (IPW) and exposure regression (ER) estimators of \( \beta \) can be constructed by specifying \( h(V; \theta) = 0 \) or \( \pi(V; \psi) = 0.5 \) a.s., respectively. An analytic expression for the asymptotic covariance matrix of \( \sqrt{n}(\hat{\beta}_{g, doubly} - \beta^T) \) is provided in the Appendix. Alternatively, nonparametric bootstrap can be performed to obtain estimates of the variance.

### 4.2 Relationship to existing estimators

Traditionally, the TSIV and TS2SLS estimators both assume that \( \pi(V) = P(R = 1|V) \) equals some constant \( 0 < \delta < 1 \) almost surely. The TSIV estimator solves the empirical version of the estimating function,

\[
U_{tsiv} = \frac{R}{\pi(V)}V^T Y - \frac{1 - R}{1 - \pi(V)}V^T (X\eta + A\beta),
\]  

(7)

where the propensity score model is assumed to be \( \pi(V) = \delta \), a constant almost surely. \( U_{tsiv} \) corresponds to the element in \( \Lambda_{iv, \omega}^T \) with \( \omega(X; \eta) = X\eta \), \( m(X; \beta) = \beta \), \( g(V) = Z \) and \( a_1 = a_2 = 0 \). The TS2SLS estimator proceeds by first estimating \( \theta = (\psi^T, \phi^T)^T \) in the population linear projection of \( A \) on \( V \) based on the second sample \( (R = 0) \), and then solving the empirical version of
the estimating function \( V^T [Y - V \hat{\theta} - X\eta] \) in the first sample \((R = 1)\), where \( \hat{\theta} \) is the first-stage least squares estimate for \( \theta \). Combining the two stages, we observe that the TS2SLS estimator solves an estimating function of the form

\[
U^{ts2sls} = \frac{R}{\pi(V)} V^T [Y - X\eta - V\theta\beta] - \frac{1 - R}{1 - \pi(V)} V^T (A - V\theta),
\]

where the propensity score model is assumed to be \( \pi(V) = \delta \). Since \( U^{ts2sls} \in \mathcal{F}^{doubly} \), TS2SLS is consistent even when the propensity score model is possibly misspecified, if the additional parametric model \( h(V; \theta) = V\theta \) is correctly specified. In addition, \( U^{ts2sls} \) is the efficient element of \( \Lambda_{iv,\omega}^+ \) for \( g(V) = V \), and therefore TS2SLS is more efficient than TSIV when both \( h(V; \theta) \) and \( \pi(V; \psi) \) are correctly specified. [Inoue & Solon (2010)] noted both of these two properties.

**Remark 2** Since TS2SLS is based on separately fitting the exposure and outcome models, its estimate may differ in finite samples compared to that obtained by solving the empirical version of \( U^{ts2sls} \) jointly, although asymptotically they are equivalent. However, when the true exposure effect depends on covariates \( X \), e.g. \( m(X; \beta) = X\beta \), TS2SLS is inefficient since it ignores the information on \( \beta \) from the second sample \((R = 0)\), which is apparent in the form of \( U^g \).

[Vansteelandt & Didelez (2018)] discuss these properties in more detail for general two-stage IV methods.

Since \( U^g \) is efficient for a fixed choice of \( g(V) \), we expect to find the efficient score among elements in the linear subspace \( \mathcal{F}^{doubly} \), which is given in the next result.

**Result 4** Suppose \( \hat{\beta}^g \) is a RAL estimator of \( \beta \) and

\[
\sqrt{n} \left( \hat{\beta}^g - \beta \right) \xrightarrow{D} \mathcal{N} \left( 0, E \left( \nabla_\beta U^g \right)^{-1} E \{ U^g U^g^T \} E \left( \nabla_\beta U^g \right)^{-1} \right)
\]

for some \( U^g \in \mathcal{F}^{doubly} \). Then \( \hat{\beta}^g \) achieves the semiparametric efficiency bound in the model \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \mathcal{M}_\omega \cap (\mathcal{M}_\pi \cup \mathcal{M}_h) \), at the intersection submodel \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \mathcal{M}_\omega \cap \mathcal{M}_\pi \cap \mathcal{M}_h \).
\[ g_{\text{opt}}(V) = -E[\nabla_\beta H(\beta)|V] E[H^2(\beta)|V]^{-1}. \]

### 4.3 Triply robust approach

The doubly robust approach above requires correct specification of a model for the covariate main effect in the outcome regression model. In this section we consider model (3) with unknown \( \omega(X) \), which is a special case of the structural nested mean models (Robins 1994, 2000). Based on results for semiparametric conditional mean independence models (Chamberlain 1987, Robins & Rotnitzky 2004), any RAL estimator of \( \beta \) in \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \) up to asymptotic equivalence has an influence function which is proportional to an element in the linear subspace

\[
\left\{ [Y - m(X; \beta)A] \{c(V) - E[c(V)|X]\} + \{d(V) - E[d(V)|X]\} : c, d \in \mathcal{H} \right\}.
\]

By Result 1, the observed data ortho-complement nuisance tangent space in the two-sample IV problem under model \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \) is given by

\[
\Lambda_{\psi}^+ = \left\{ \left[ \frac{R}{\pi(V)} [\phi_{1,c}(Y, V) + a_1(V)] - \frac{1 - R}{1 - \pi(V)} [\phi_{2,c}(A, V) + a_2(V)] + [a_1(V) - a_2(V)] \right] - \Pi\{[:,\Lambda_\psi]\} : c(V), a_1(V), a_2(V) \in \mathcal{H} \right\},
\]

where \( \phi_{1,c}(Y, V) = Y \{c(V) - E[c(V)|X]\} \) and \( \phi_{2,c}(A, V) = m(X; \beta)A \{c(V) - E[c(V)|X]\} \). For fixed choices of \( c(V) \), the efficient elements in \( \Lambda_{\psi}^+ \) are given by the linear subspace

\[
\mathcal{F}_{\text{triply}} = \{ U_{g_{\text{triply}}}^+ = J(L_O; \beta) \{c(V) - E[c(V)|X]\} : c(V) \in \mathcal{H} \},
\]

Where

\[
J(L_O; \beta) = \left\{ \frac{R}{\pi(V)} [Y - \omega(X) - m(X; \beta)E(A|V)] - \frac{1 - R}{1 - \pi(V)} m(X; \beta) [A - E(A|V)] \right\}.
\]
Constructing RAL estimators of \( \beta \) based on elements in \( F^{\text{triply}} \) involves the conditional distribution \( f(Z|X) \). Since \( X \) can be high-dimensional or contain continuous components, consider the following additional assumption:

**A9.** The conditional distribution \( f(Z|X; \zeta) \) is correctly specified so that \( f(Z|X; \zeta^\dagger) = f(Z|X) \) for some unknown finite-dimensional parameter \( \zeta^\dagger \). Let \( \mathcal{M}_f \) denote the models satisfying A9.

Let \( S_\zeta(Z,X) \) be the score function of \( \zeta \) based on a parametric model for the conditional density \( f(Z|X; \zeta) \), and let \( \hat{\zeta} \) be the solution to the estimating function \( \mathbb{P}_n \left\{ U_{\zeta,\hat{\psi}} = 0 \right\} \), where

\[
U_{\zeta,\hat{\psi}} = \frac{R}{\pi(V; \hat{\psi})} S_\zeta(Z,X) + \frac{1 - R}{1 - \pi(V; \hat{\psi})} S_\zeta(Z,X).
\]

Let \( (\hat{\beta}_{g}^{\text{triply}}, \hat{\eta}_{g}^{\text{triply}}) \) be the solution to the estimating equation

\[
\mathbb{P}_n \left\{ U_{g}^{\text{triply}}(\beta, \eta, \hat{\psi}, \hat{\theta}, \hat{\zeta}) = 0 \right\} = \mathbb{P}_n \left\{ \left\{ \frac{R}{\pi(V; \hat{\psi})} \left[ Y - \omega(X; \eta) - m(X; \beta)h(V; \hat{\theta}) \right] - \frac{1 - R}{1 - \pi(V; \hat{\psi})} m(X; \beta) \left[ A - h(V; \hat{\theta}) \right] \right\} \times \left\{ \left[ g(V) - E \left( g(V) \big| \hat{\zeta} \right) \right]^T, \lambda^T(X; \eta) \right\}^T = 0. \tag{11}
\]

The next result states that \( \hat{\beta}_{g}^{\text{triply}} \) is CAN if the analyst correctly specifies any one of three different observed data models.

**Result 5** Under standard regularity conditions, \( \hat{\beta}_{g}^{\text{triply}} \) is a CAN estimator of \( \beta \) in the union model

\[
\mathcal{M}_{\text{triply}} = \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \left\{ (\mathcal{M}_\pi \cap \mathcal{M}_\omega) \cup (\mathcal{M}_h \cap \mathcal{M}_\omega) \cup (\mathcal{M}_\pi \cap \mathcal{M}_f) \right\}.
\]

In addition, \( \hat{\beta}_{g}^{\text{triply}} \) is efficient (for a fixed choice of \( g \)) at the intersection model \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \mathcal{M}_\omega \cap \mathcal{M}_\pi \cap \mathcal{M}_h \cap \mathcal{M}_f \).

Although the efficient element in \( F^{\text{triply}} \) is not available in closed form, progress can be made when \( Z \) has discrete support. For binary \( Z \), any function \( c(\cdot) \) of \( V \) can be expressed as \( c(V) = \)
\[ Zc_1(X) + c_0(X), \text{ where } c_1(\cdot), c_0(\cdot) \text{ are arbitrary functions of } X. \] To find the efficient choice of \( c(\cdot) \), the triply robust linear subspace can be equivalently expressed as

\[
F_{\text{triply}} = \left\{ c_1(X) \tilde{J}(L_O; \beta) : c_1(X) \in \mathcal{H} \right\},
\]

where \( \tilde{J}(L_O; \beta) = J(L_O; \beta) \times [Z - P(Z = 1|X)]. \) Based on this equivalent expression, the efficient element in \( F_{\text{triply}} \), which is also the efficient score in the union model \( \mathcal{M}_{\text{triply}} \), is given by

\[
c_{1}^{opt}(X) \tilde{J}(L_O; \beta),
\]

where

\[
c_{1}^{opt}(X) = -E \left[ \nabla_{\beta} \tilde{J}(\beta)|X \right] E \left[ J^2(\beta)|X \right]^{-1}.
\]

The proof is similar to that of Result 4. A similar approach can be adopted for polytomous \( Z \) with \( s > 2 \) levels by noting that in this case

\[
c(V) = \sum_{k=1}^{s-1} I(Z = z_k)c_k(X) + c_0(X)
\]

and

\[
F_{\text{triply}} = \left\{ \sum_{k=1}^{s-1} c_k(X) \tilde{J}_k : c_k(X) \in \mathcal{H}, k = 1, 2, ..., s - 1 \right\},
\]

where \( \tilde{J}_k(L_O; \beta) = J(L_O; \beta) \times [I(Z = z_k) - P(Z = z_k|X)]. \) When \( Z \) contains continuous components, we adopt the general strategy proposed in Newey (1993) (see also Tchetgen Tchetgen & Robins (2010)) to construct an approximately locally efficient estimator by taking a basis system \( \psi_j(V), j = 1, 2, ... \) of functions dense in \( \mathcal{H} \), such as tensor products of trigonometric, wavelets or polynomial bases. In practice we let the \( p \)-dimensional \( c^K(V) = \tau \Psi_K \) where \( \tau \in \mathbb{R}^{p \times K} \) is a constant matrix and \( \Psi_K = \{\psi_1, \psi_2, ..., \psi_K\}^T \) for some finite \( K > p \).

To derive an approximately locally efficient estimating function for \( \beta \), let \( K \) denote the linear operator \( K(\cdot) = J(L_O; \beta) \times \{\cdot - E[\cdot|X]\} \) defined over the space of arbitrary functions of \( X \) and \( Z \) in \( \mathcal{H} \). Consider the linear space

\[
F_{\Psi_K} = \left\{ \tau K(\Psi_K) = \tau [K(\psi_1), K(\psi_2), ..., K(\psi_K)]^T : \tau \in \mathbb{R}^{p \times K} \right\}.
\]

Analogous to Result 4 it can be shown based on Theorem 5.3 in Newey & McFadden (1994) that
the efficient element in $F_{\Psi_K}$ is indexed by the constant matrix

$$\tau_{\text{opt}} = -E \left[ \nabla_\beta K(\Psi_K) \right] E \left[ K(\Psi) K^T(\Psi_K) \right]^{-1}.$$

In particular, the inverse of the asymptotic variance of the estimator indexed by $\tau_{\text{opt}}$ is

$$\Omega_K = E \left\{ \nabla_\beta K(\Psi_K) \right\}^T E \left\{ K(\Psi_K) K^T(\Psi_K) \right\}^{-1} E \left\{ \nabla_\beta K(\Psi_K) \right\} = E \left\{ S_\beta K^T(\Psi_K) \right\} E \left\{ K(\Psi_K) K^T(\Psi_K) \right\}^{-1} E \left\{ S_\beta K^T(\Psi_K) \right\}^T,$$

evaluated at the truth, and $S_\beta$ is the score vector with respect to $\beta$. Thus, $\Omega_K$ is the variance of the population least squares regression of $S_\beta$ on the linear span of $K(\Psi_K)$. Since $\Psi_K$ is dense in $H$, as the dimension $K \to \infty$ the linear span of $K(\Psi_K)$ recovers the subspace in the ortho-complement nuisance tangent space $\Lambda_{\text{iv}}^\perp$ containing the efficient score $S_{\text{eff}}(L_O)$ so that $\Omega_K \to ||\Pi (S_\beta | \Lambda_{\text{iv}}^\perp) ||^2 = \text{var} \{ S_{\text{eff}}(L_O) \}$, the inverse of the semiparametric information bound for estimating $\beta$ (Newey 1990).

4.4 Multiple robustness under strong ignorability

We note that the RAL estimator $\hat{\beta}_{\text{triplly}}$ is not consistent under the model $A_{TS} \cap A_{IV} \cap (M_h \cap M_f)$, in contrast to quadruply, or more generally, multiply robust estimators derived under factorized likelihood structures (Tchetgen Tchetgen 2009, Vansteelandt et al. 2007, Molina et al. 2017). This is due to variational dependence between $f(Z|X)$ and $\pi(V)$, if the former is to be identified from observed data. Specifically, $f(Z|X) = f(Z|X,R = 1)\kappa(X) + f(Z|X,R = 0)\{1 - \kappa(X)\}$, where $\kappa(X) = \int \pi(V)dF(Z|X)$. On the other hand, $f(Z|X) = f(Z|X,R = r), r = 1,2$ if we replace assumption A5 with the following:

A5'. Strong ignorability: $R \perp (Y, A, Z)|X$.

Assumption A5' is stronger than A5 and states that the sampling mechanism or the propensity score depends only on the observed covariates, i.e. $\pi(V) = \pi(X)$ almost surely. Let $A_{TS'}$ denote the model satisfying assumptions A4 and A5'. We have the following multiply robust result in the submodel $\{A_{TS'} \cap A_{IV}\} \subset \{A_{TS} \cap A_{IV}\}$:
Result 6 Under standard regularity conditions, \( \hat{\beta}_{g}^{\text{triply}} \) is a CAN estimator of \( \beta \) in the union model

\[
M_{\text{multiply}} = A_{TS} \cap A_{IV} \cap \{(M_{\pi} \cup M_{h}) \cap (M_{\omega} \cup M_{f})\}.
\]

In addition, \( \hat{\beta}_{g}^{\text{triply}} \) is efficient (for a fixed choice of \( g \)) at the intersection model \( A_{TS} \cap A_{IV} \cap M_{\pi} \cap M_{h} \cap M_{f} \).

It is straightforward to verify that the efficient score in \( M_{\text{multiply}} \) is of the same form as the efficient score in \( M_{\text{triply}} \), with \( \pi(V) \) replaced by \( \pi(X) \).

5 Simulation study

In this section, we report a simulation study evaluating the finite sample performance of the proposed estimators involving i.i.d. realizations of \((R,Y,A,V)\), where \( V = (Z,X_1,X_2) \). For each of the sample sizes \( n = 2500, 5000 \), we simulated 1000 datasets as followed. The random vector \( V \) is generated as

\[
\begin{pmatrix} Z \\ X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 & -0.5 \\ 0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix} \right\}.
\]

Similar to the simulation study in Shu & Tan (2018), \( A \) and \( Y \) are defined as

\[
A = Z + 0.6X_1 - 0.5X_2 + \epsilon_A
\]

\[
Y = 0.5A - 0.4X_1 + 0.5X_2 + \epsilon_Y,
\]

where \((\epsilon_A, \epsilon_Y)\) are distributed independently of \( V \) as

\[
\begin{pmatrix} \epsilon_A \\ \epsilon_Y \end{pmatrix} \sim \mathcal{N} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix} \right\}.
\]
so that $Z$ is a valid IV for the endogenous variable $A$. The sample indicator variable is generated as

$$R|V \sim \text{Bernoulli}\{\pi(V; \psi)\}, \quad \pi(V; \psi) = \{1 + \exp(-2.5 + Z + X_1 + X_2)\}^{-1},$$

such that $Y$ is only observed if $R = 1$, and $A$ is only observed if $R = 0$. Marginally, $\Pr(R = 1) \approx 0.58$. In addition to TSIV and TS2SLS, we implemented the IPW, ER, doubly (DR) and triply robust (TR) estimators of $\beta^\dagger = 0.5$ by solving the respective estimating functions with $g(V) = Z$ using the R package “BB” (Varadhan et al. 2009). We also evaluated the performance of the proposed estimators in situations where some models may be mis-specified, similarly as in Kang et al. (2007) and Shu & Tan (2018) by defining the transformed variables $W_0 = \exp(-0.5Z) + 5$, $W_1 = X_1/\{1 + 0.1 \exp(X_1)\} + 10$ and $W_2 = \exp(0.4X_2) + 3$. The misspecified models for $\pi$, $h$, $\omega$ and $f$ are

$$\pi^*(V; \psi) = \{1 + \exp(\psi_0 + \psi_1W_0)\}^{-1}, \quad h^*(V; \theta) = \theta_0 + \theta_1W_0 + \theta_2W_1 + \theta_3W_2,$$

$$\omega^*(X; \eta) = \eta_0 + \eta_1W_1 + \eta_2W_2, \quad \omega^*(Z|X; \zeta) \propto \exp\left\{- (Z - \zeta_0 - \zeta_1W_1 - \zeta_2W_2)^2 / 2\sigma^2\right\},$$

respectively. Figure 1 shows the boxplots of the differences between the estimates of $\beta$ and the true value 0.5; realizations of each estimator are censored within the range of the y-axis.

Under scenario (i) where all model specifications are correct, only the TSIV estimator show substantial bias due to stratified sampling which depends on the fully observed $V$. As discussed in section 4.2, TSIV is a type of IPW estimator in which the propensity score model is assumed to be a constant a.s., which is misspecified across all the scenarios considered in this simulation study. On the other hand, the TS2SLS estimator exhibits small bias as long as the models for $h$ and $\omega$ are correctly specified, due to its doubly robust property. In addition, the TS2SLS estimator has smaller variance compared to TSIV, in agreement with theory. Under scenario (ii) in which only models for $\pi$ and $\omega$ are correct, the TS2SLS and ER estimators are no longer unbiased, since their consistencies depend on a correct model for $h$. Under scenario (iii) in which only models for $h$ and $\omega$ are correct, the simple IPW estimator exhibit significant bias with very large variance, which is typical performance when the propensity score model $\pi$ is misspecified.
The DR estimator is unbiased across the scenarios (i)-(iii), due to its doubly robust property. The regression-based estimators TS2SLS and ER generally have smaller variances compared to estimators which leverage on inverse propensity score weighting, such as the IPW estimator; this difference in efficiency is characteristic of propensity score and outcome regression approaches, for example in causal inference problems (Tan 2007). Under scenario (iv) in which only models for \( \pi \) and \( f \) are correct, the TR estimator remains unbiased. Finally, when all models are incorrect under scenario (v), all the estimators are biased, except that the bias of the DR and TR estimators is substantially less than that of non-robust estimators, which is in agreement with previous results on multiply robust estimation (Molina et al. 2017).

Table 1 summarizes simulation results assessing the performance of our estimators of asymptotic variance and coverage of Wald confidence intervals using estimated standard errors for the six estimators under consideration. The results largely indicate that our standard error estimators are consistent in all scenarios where the point estimators are also consistent, including under partial model misspecification for the DR and TR estimators. However, our standard error estimators appear to break down whenever model misspecification induces bias in parameter estimates.
Figure 1: Boxplots of estimates of $\beta$ relative to 0.5.
Table 1: Bias, Monte Carlo standard error, empirical coverage rates based on 95% Wald confidence intervals of the proposed estimators, as well as accuracy of analytic standard error, under five scenarios: (i) all models correct, (ii) correct \((\pi, \omega)\) but incorrect \((h^*, f^*)\), (iii) correct \((h, \omega)\) but incorrect \((\pi^*, f^*)\), (iv) correct \((\pi, f)\) but incorrect \((h^*, \omega^*)\) and (v) all models incorrect. In each scenario, the first row presents results for \(n = 2500\) and the second row for \(n = 5000\).

|          | tsiv | ts2sls | er  | ipw | dr  | tr  |
|----------|------|--------|-----|-----|-----|-----|
| Bias     |      |        |     |     |     |     |
| (i)      | 0.80 | 0.00   | 0.00| 0.06| 0.00| 0.00|
|          | 0.79 | 0.00   | 0.00| 0.03| 0.00| 0.00|
| (ii)     | 0.80 | 0.33   | 0.22| 0.06| 0.02| 0.03|
|          | 0.79 | 0.33   | 0.21| 0.03| 0.01| 0.02|
| (iii)    | 0.80 | 0.00   | 0.00| 1.85| 0.00| 0.01|
|          | 0.79 | 0.00   | 0.00| 1.66| 0.00| 0.00|
| (iv)     | 0.81 | 0.33   | 0.21| 0.20| 0.22| 0.01|
|          | 1.32 | 0.83   | 0.72| 0.93| 0.54| 0.60|
| (v)      | 1.31 | 0.83   | 0.71| 0.96| 0.53| 0.59|

Monte Carlo SE

|          |      |        |     |     |     |     |
|----------|------|--------|-----|-----|-----|-----|
| (i)      | 0.13 | 0.06   | 0.06| 0.20| 0.09| 0.09|
|          | 0.09 | 0.04   | 0.04| 0.14| 0.07| 0.07|
| (ii)     | 0.13 | 0.11   | 0.09| 0.20| 0.11| 0.13|
|          | 0.09 | 0.08   | 0.06| 0.14| 0.09| 0.10|
| (iii)    | 0.13 | 0.06   | 0.06| 5.53| 0.11| 0.10|
|          | 0.09 | 0.04   | 0.04| 3.42| 0.08| 0.07|
| (iv)     | 0.22 | 0.14   | 0.12| 0.23| 0.20| 0.11|
|          | 0.22 | 0.12   | 0.09| 0.21| 0.19| 0.09|
| (v)      | 0.17 | 0.12   | 0.09| 0.79| 0.09| 0.38|

Coverage

|          |      |        |     |     |     |     |
|----------|------|--------|-----|-----|-----|-----|
| (i)      | 0.00 | 0.95   | 0.95| 0.94| 0.94| 0.94|
|          | 0.00 | 0.94   | 0.94| 0.92| 0.94| 0.94|
| (ii)     | 0.00 | 0.10   | 0.31| 0.94| 0.94| 0.91|
|          | 0.00 | 0.01   | 0.06| 0.92| 0.91| 0.91|
| (iii)    | 0.00 | 0.95   | 0.95| 0.93| 0.91| 0.92|
|          | 0.00 | 0.94   | 0.94| 0.54| 0.91| 0.93|
| (iv)     | 0.00 | 0.02   | 0.09| 0.41| 0.29| 0.96|
|          | 0.01 | 0.13   | 0.32| 0.96| 0.88| 0.70|
| (v)      | 0.00 | 0.02   | 0.09| 0.80| 0.86| 0.64|

SE ratio†

|          |      |        |     |     |     |     |
|----------|------|--------|-----|-----|-----|-----|
| (i)      | 0.94 | 1.02   | 1.02| 0.94| 1.00| 1.00|
|          | 0.90 | 1.04   | 1.04| 1.01| 1.05| 1.05|
| (ii)     | 0.94 | 1.00   | 1.01| 0.94| 1.07| 1.11|
|          | 0.90 | 1.03   | 1.02| 1.01| 1.11| 1.19|
| (iii)    | 0.94 | 1.02   | 1.02| 0.05| 1.35| 1.26|
|          | 1.50 | 1.33   | 1.34| 1.20| 2.10| 0.79|
| (iv)     | 1.71 | 1.54   | 1.46| 2.04| 2.75| 0.83|
|          | 1.50 | 1.34   | 1.34| 0.05| 1.33| 1.13|
| (v)      | 1.71 | 1.54   | 1.46| 0.12| 1.29| 2.69|

†: Monte Carlo SE/ Mean estimated SE
6 Application

Currie & Yelowitz (2000) study the effect of public housing participation on housing quality and educational attainment, and showed that project participation is associated with poorer outcomes based on data from the Survey of Income and Program Participation (SIPP). However, many unobserved factors such as social ties are likely to affect both project participation and outcomes, and the authors suspect that failure to control for this source of endogeneity would bias the estimated causal effects of living in projects downwards, since families in projects may be more likely to live in substandard housing in any case, and their children may be more likely to experience negative outcomes. Leveraging on the sex composition of children as an IV for project participation, Currie & Yelowitz (2000) use two-sample IV method to combine information from the 1990 Census data and 1990-1995 waves of the March Current Population Survey (CPS), and find that project households are less likely to suffer from over crowding or live in high-density complexes, and project children are less likely to have been held back. Their study is important as the results overturn the stereotype that project participation is harmful in terms of living conditions and children’s educational attainment.

In this analysis, we apply the proposed two-sample IV methods to estimate the causal effect of project participation (A) on the outcome “overcrowdedness” (Y), where \( Y = 1 \) if a family had three or less living/bedrooms and \( Y = 0 \) otherwise. Our substantive model of interest is \( Y = \beta A + \omega(X; \eta) + \epsilon_y \), where \( A = 1 \) for project participation and \( A = 0 \) otherwise. In line with the Currie & Yelowitz (2000) study, we specify \( \omega(X; \eta) = X\eta \) where \( X \) includes exogenous explanatory variables such as the household head’s gender, age, race, education, marital status and the number of boys in the family. We specify the exposure model \( h(V; \theta) = \theta_0 + \theta_1 Z + X\theta_x \), where \( Z \) is assumed to be a valid IV for \( A \) and \( Z = 1 \) if a family had a boy and a girl and \( Z = 0 \) if both are boys or girls. Families with two children of opposite genders will be eligible for three-bedroom apartments as opposed to two-bedroom apartments, and therefore will be more likely to participate in the housing project, although there is little reason to expect that the children’s sex composition will affect \( Y \). TSIV and TS2SLS estimation is based on \( n_1 = 279129 \) records for \((Y, V)\) from the 1990 Census \((R = 1)\) and \( n_0 = 21718 \) records for \((A, V)\) from CPS \((R = 0)\), where \( V = (Z, X) \),

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for a total sample size of \( n = 300847 \). We additionally implement IPW, DR and TR estimation by specifying the models \( \pi(V; \psi) = \{1 + \exp(\psi V)\}^{-1} \) and \( E_\zeta(Z|X) = X\zeta \). The analysis results are summarized in Table 2. The results from ER and TS2SLS estimation are similar and therefore only the latter estimate is presented.

Table 2: Estimates of \( \beta \), the effect of public housing project participation on overcrowdedness.

|               | tsiv   | ts2sls | ipw   | dr    | tr    |
|---------------|--------|--------|-------|-------|-------|
| point estimate| 0.069  | -0.159 | -0.212| -0.196| -0.196|
| standard error| 0.128  | 0.073  | 0.113 | 0.098 | 0.098 |
| 95% Wald CI   | (-0.183,0.321) | (-0.303,-0.016) | (-0.434,0.009) | (-0.388,-0.004) | (-0.388,-0.004) |

The TSIV point estimate is 0.069 with 95% Wald confidence interval covering 0. This result is possibly biased as the empirical distributions of \( V \) in sample \( R = 1 \) are largely significantly different from those in sample \( R = 0 \) at the 5% \( \alpha \)-level \((Shu & Tan 2018)\). The TS2SLS estimate of \(-0.159\) agrees with the two-sample IV point estimate presented in Table 4 of \(Currie & Yelowitz (2000)\), although the analytic standard error of 0.073 is larger than the value of 0.0624 reported by the original study, as the former takes into account the variability associated with the first-stage estimation. The IPW and DR estimates of \(-0.212\) and \(-0.196\) respectively suggest a larger causal effect of housing project participation in alleviating overcrowdedness in household living conditions, although the 95% Wald confidence interval for IPW covers 0 due to larger standard error, which is in agreement with simulation results. The DR and TR estimates are similar, which suggests that the covariate effect model \( \omega(X; \eta) \) and conditional distribution \( f(Z|X) \) may be specified nearly correctly \((Robins & Rotnitzky 2001)\). In addition, the similarity between the DR and IPW estimates (as opposed to DR and TS2SLS) suggests that the exposure model \( h(V; \theta) \) in this illustrative analysis may be misspecified; \(Tchetgen Tchetgen & Robins (2010)\) describe a formal specification test to detect which of the two baseline models \( \pi(V) \) and \( h(V) \) is correct under the union model \( \mathcal{M}_\pi \cup \mathcal{M}_h \).
7 Discussion

This paper develops a general semi-parametric framework for two-sample data combination problems from a missing data perspective which is practically relevant. As a specific example, the framework provides insights on issues of efficiency and robustness of two-sample IV methods which have been widely applied in empirical research for the health and social sciences. We characterize the semi-parametric efficiency bound in the two-sample IV setting and introduce novel multiply robust locally efficient estimators that can be used when non-parametric estimation is not possible.

There are several improvements and extensions for future work. Multiple valid IVs can be incorporated by adopting a standard generalized method of moments approach, and the proposed estimators can be improved in terms of efficiency (Tan 2006, 2010) and bias (Vermeulen & Vansteelandt 2015). With the introduction of the data source model \( \pi \) as an additional nuisance parameter, it will also be of interest to investigate multiply robustness for other causal inference problems under the two-sample setting.

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9 Appendix

We closely follow the semiparametric theory of Newey (1990) and Bickel et al. (1993), particularly in the missing data context (Tsiatis 2007).

Full data nuisance tangent space: Consider a full data parametric submodel \( f(L; \beta, \eta) \) for the joint distribution of \( L = (Y, A, V) \), where \( \beta \) is a \( q \)-dimensional parameter of interest. The joint density for \( L \) can be expressed as

\[
 f(L; \beta, \eta) = f(Y|A, V; \beta, \eta_1)f(A|V; \beta, \eta_2)f(V; \beta, \eta_3).
\]

The full-data nuisance tangent space is therefore given by

\[
 \Lambda^F_\eta = \{ s_1(Y|A, V) + s_2(A|V) + s_3(V) \}
\]

where \( E(s_1|A, V) = E(s_2|V) = E(s_3) = 0. \)

Observed data nuisance tangent space: Consider the observed data parametric submodel \( f(L_O; \beta, \eta) \) for the joint distribution of \( L_O = (R, RY, (1 - R)A, V) \). The joint density of \( L_O \) can be expressed as

\[
 f(L_O; \beta, \eta, \psi) = \pi(V; \psi) \left\{ \int f(Y|A, V; \beta, \eta_1)dF(A|V; \beta, \eta_2) \right\}^R \cdot f(A|V; \beta, \eta_2)^{1-R} f(V; \beta, \eta_3),
\]

where \( \psi \) is variationally independent of \((\beta, \eta)\) and indexes the parametric propensity score submodel. The score w.r.t. \( \eta_1 \) is

\[
 R \left\{ \frac{\int s_1(Y|A, V)f(Y|A, V; \beta, \eta_1)dF(A|V; \beta, \eta_2)}{\int f(Y|A, V; \beta, \eta_1)dF(A|V; \beta, \eta_2)} \right\} = R \left\{ \frac{\int s_1(Y|A, V)f(Y|A, V; \beta, \eta_1)dF(A|V; \beta, \eta_2)}{f(Y|V; \beta, \eta_1, \eta_2)} \right\} = R \left\{ \int s_1(Y|A, V)dF(A|Y, V; \beta, \eta_1, \eta_2) \right\} = RE(s_1|Y, V) = RE(s_1|Y, V) + (1 - R)E(s_1|A, V),
\]
since $E(s_1|A,V) = 0$. The score w.r.t. $\eta_2$ is

$$R \left\{ \int s_2(A|V) f(Y|A,V; \beta, \eta_1) dF(A|V; \beta, \eta_2) \right\} + (1 - R)s_2(A|V)$$

$$= R \left\{ \int s_2(A|V) f(Y|A,V; \beta, \eta_1) dF(A|V; \beta, \eta_2) \right\} + (1 - R)s_2(A|V)$$

$$= R \int s_2(A|V) dF(A|V, \beta, \eta_1, \eta_2) + (1 - R)s_2(A|V)$$

$$= RE(s_2|Y,V) + (1 - R)E(s_2|A,V),$$

Therefore, the observed data nuisance tangent space is given by $\Lambda = \Lambda_\psi \bigoplus \Lambda_\eta$, where

$$\Lambda_\psi = \left\{ s_\psi(R|V) = \left[ \frac{R}{\pi(V)} - \frac{1 - R}{1 - \pi(V)} \right] \mu(V) : \mu(V) \in H \right\},$$

and

$$\Lambda_\eta = \{ E(\alpha^F|L_O) : \alpha^F \in \Lambda^F_\eta \}.$$ 

The following two lemmas are from Tsiatis (2007).

**Lemma A1** Let $\mathcal{H}$ denote the observed data Hilbert space of all $q$-dimensional, mean-zero, finite variance, measurable functions of $L_O$ equipped with the covariance inner product. The space $\Lambda^\perp_\eta$ consists of all elements $h(L_O) \in \mathcal{H}$ such that $E(h|L) \in \Lambda^F_\eta$.

**Proof:** The space $\Lambda^\perp_\eta$ consists of all elements $h(L_O) \in \mathcal{H}$ such that

$$E \left[ h^T(L_O) E(\alpha^F(L)|L_O) \right]$$

$$= E \left\{ E \left[ h^T(L_O) \alpha^F(L) \right] | L_O \right\}$$

$$= E \left\{ h^T(L_O) \alpha^F(L) \right\}$$

$$= E \left\{ E \left[ h^T(L_O) \alpha^F(L) \right] | L \right\}$$

$$= E \left\{ E \left[ h^T(L_O) | L \right] \alpha^F(L) \right\} = 0, \quad \forall \alpha^F \in \Lambda^F_\eta$$

**Lemma A2** For any $\phi^F(L) \in \Lambda^F_\eta$, let $\mathcal{K}^{-1} \{ \phi^F(L) \}$ denote the space of elements $h(L_O) \in \mathcal{H}$
such that $\mathcal{K}[h(L_O)] = E[h(L_O)]L = \phi^F(L)$. If we could identify any element $h^*(L_O)$ such that $\mathcal{K}(h^*) = \phi^F(L)$, then $\mathcal{K}^{-1}\{\phi^F(L)\} = h^*(L_O) + \Lambda_g$, where $\Lambda_g$ is the linear subspace in $\mathcal{H}$ consisting of elements $g(L_O)$ such that $E[g(L_O)]L = 0$, i.e. $\Lambda_g = \mathcal{K}^{-1}(0)$.

Proof: If $h(L_O) \in h^*(L_O) + \Lambda_g$, then $h(L_O) = h^*(L_O) + g(L_O)$ for some $g(L_O) \in \Lambda_g$, so that $E[h(L_O)]L = E[h^*(L_O)]L + E[g(L_O)]L = \phi^F(L)$. Conversely, if $E[h(L_O)]L = \phi^F(L)$, then $h(L_O) = h^*(L_O) + [h(L_O) - h^*(L_O)]$ where clearly $[h(L_O) - h^*(L_O)] \in \Lambda_g$.

Proof of result 1:

Based on lemmas A1 and A2, under the assumptions, the space $\Lambda_{\eta}^+ \subset \mathcal{H}$ consists of all elements

$$
\left\{ \left\{ \frac{R}{\pi(V)} \phi_1(Y, V) - \frac{1 - R}{1 - \pi(V)} \phi_2(A, V) + g(L_O) : \right. \\
\phi_1(Y, V) - \phi_2(A, V) = \phi^F(L) \in \Lambda_{\eta}^{F^\perp}, g(L_O) \in \Lambda_g \right\}.
$$

It is straight forward to verify that the space $\Lambda_g$ consists of the elements

$$
\left\{ \left\{ g(L_O) = \frac{R}{\pi(V)} a_1(V) - \frac{1 - R}{1 - \pi(V)} a_2(V) - [a_1(V) - a_2(V)] : a_1(V), a_2(V) \in \mathcal{H} \right\}. 
$$

Therefore, we have

$$
\Lambda_{\eta}^+ = \left\{ \left\{ \frac{R}{\pi(V)} [\phi_1(Y, V) + a_1(V)] - \frac{1 - R}{1 - \pi(V)} [\phi_2(A, V) + a_2(V)] + [a_1(V) - a_2(V)] : \right. \\
\phi_1(Y, V) - \phi_2(A, V) = \phi^F(L) \in \Lambda_{\eta}^{F^\perp}, a_1(V), a_2(V) \in \mathcal{H} \right\}.
$$

Since the nuisance tangent space for the observed data law is given by $\Lambda = \Lambda_\psi \oplus \Lambda_\eta$ and $\Lambda_\psi \perp \Lambda_\eta$, the orthocomplement to the nuisance tangent space is given by

$$
\Lambda^\perp = \left\{ \left\{ \frac{R}{\pi(V)} [\phi_1(Y, V) + a_1(V)] - \frac{1 - R}{1 - \pi(V)} [\phi_2(A, V) + a_2(V)] + [a_1(V) - a_2(V)] : \right. \\
\phi_1(Y, V) - \phi_2(A, V) = \phi^F(L) \in \Lambda_{\eta}^{F^\perp}, a_1(V), a_2(V) \in \mathcal{H} \right\},
$$

where $\Pi$ is the projection operator. For a fixed element in the full data orthocomplement nuisance tangent space $\phi^F(L) \in \Lambda_{\eta}^{F^\perp}$, the space of elements in $\Lambda^\perp$ is a linear variety $V = x_0 + M$, with
the element \( x_0 = \frac{R}{\pi V} \phi_1(Y, V) - \frac{1-R}{1-\pi V} \phi_2(A, V) - \Pi \left[ \frac{R}{\pi V} \phi_1(Y, V) - \frac{1-R}{1-\pi V} \phi_2(A, V) \right] \Lambda_g \) and the linear subspace \( M = \Pi \left[ \Lambda_g \Lambda^\perp_g \right] \). The element in this linear variety with the smallest variance is given as \( x_0 - \Pi [x_0 | M] \). Following the results of Theorem 10.1 in [Tsatis (2007)] ,

\[
x_0 - \Pi [x_0 | M] = \frac{R}{\pi V} \phi_1(Y, V) - \frac{1-R}{1-\pi V} \phi_2(A, V) - \Pi \left[ \frac{R}{\pi V} \phi_1(Y, V) - \frac{1-R}{1-\pi V} \phi_2(A, V) \right] \Lambda_g .
\]

**Lemma A3** The projection of \( \frac{R}{\pi V} \phi_1(Y, V) - \frac{1-R}{1-\pi V} \phi_2(A, V) \) onto the linear space \( \Lambda_g \) is given by \( \frac{R}{\pi V} E(\phi_1 | V) - \frac{1-R}{1-\pi V} E(\phi_2 | V) - E(\phi_1 - \phi_2 | V) \)

Proof: Let \( \frac{R}{\pi V} a_1^*(V) - \frac{1-R}{1-\pi V} a_2^*(V) = [a_1(V) - a_2^*(V)] \in \Lambda_g \) be the unique projection of \( \frac{R}{\pi V} \phi_1(Y, V) - \frac{1-R}{1-\pi V} \phi_2(A, V) \) onto \( \Lambda_g \). Then,

\[
E \left\{ \left[ \frac{R}{\pi V} \left[ \phi_1(Y, V) - a_1^*(V) \right] - \frac{1-R}{1-\pi V} \phi_2(A, V) + a_2^*(V) \right] - [a_1(V) - a_2^*(V)] \right\} \times \\
E \left\{ \frac{1}{\pi V} E(\phi_1 | V) - a_1^*(V) \right\} a_1(V) \}
\]

\[
= E \left\{ \frac{1}{\pi V} [E(\phi_1 | V) - a_1^*(V)] a_1(V) \right\} + E \left\{ \frac{1}{1-\pi V} \left[ E(\phi_2 | V) - a_2^*(V) \right] a_2(V) \right\}
\]

\[
+ [a_1(V) - a_2(V)] \{ E(\phi_1 - \phi_2 | V) - [a_1^*(V) - a_2^*(V)] \} = 0, \forall a_1, a_2 \in \mathcal{H}.
\]

Since \( \rho < \pi(V) < 1 - \rho \) for a positive constant \( \rho \), the above equality holds if and only if \( a_1^*(V) = E(\phi_1(Y, V) | V) \) and \( a_2^*(V) = E(\phi_2(A, V) | V) \). Therefore, the linear subspace of efficient elements in the observed data orthocomplement nuisance tangent space for any fixed \( \phi^F(L) \in \Lambda^F_{\eta} \) is given by

\[
\mathcal{A}^{OR} = \left\{ \frac{R}{\pi V} \left[ \phi_1(Y, V) - E(\phi_1 | V) \right] - \frac{1-R}{1-\pi V} \left[ \phi_2(A, V) - E(\phi_2 | V) \right] + E[\phi^F(L) | V] : \\
\phi_1(Y, V) - \phi_2(A, V) = \phi^F(L) \in \Lambda^F_{\eta} \right\}.
\]

**Proof of result 2:**

The observed efficient score is unique and equal to \( S_{eff}(L_O) = s_\beta(L_O) - \Pi [s_\beta(L_O) | \Lambda] \), where \( s_\beta(L_O) = E \left[ s^F_\beta(L) | L_O \right] \) and \( s^F_\beta(L) \) is the full data score for \( \beta \). Since \( \Lambda = \Lambda_\psi \bigoplus \Lambda_\eta \) and \( \Lambda_\psi \bot \Lambda_\eta \),

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\( \Pi [s_\beta(L_o)|\Lambda] = \Pi [s_\beta(L_o)|\Lambda_\psi] + \Pi [s_\beta(L_o)|\Lambda_\eta]. \) For an arbitrary element \( g(L_o) \in \Lambda_g, \)

\[
E [g(L_o)s_\beta(L_o)] = E \left\{ E [g(L_o)s_\beta^F(L)|L_o] \right\} \\
= E \left\{ g(L_o)s_\beta^F(L) \right\} \\
= E \left\{ E [g(L_o)s_\beta^F(L)|L] \right\} \\
= E \left\{ s_\beta^F(L)E [g(L_o)|L] \right\} = 0.
\]

Since \( \Lambda_\psi \subset \Lambda_g, \) \( \Pi [s_\beta(L_o)|\Lambda_\psi] = 0. \) Let the unique projection \( \Pi [s_\beta(L_o)|\Lambda_\eta] \) be denoted by \( E (\alpha_{\text{eff}}^F|L_o) \) for some \( \alpha_{\text{eff}}^F \in \Lambda_\eta^F. \) Then we have

\[
S_{\text{eff}}(L_o) = E [s_\beta^F(L_o)] - E (\alpha_{\text{eff}}^F|L_o) \\
= E \left( s_\beta^F(L) - \alpha_{\text{eff}}^F|L_o \right) \quad (A1)
\]

In addition, since \( F^{DR} \) consists of efficient elements in \( \Lambda^\perp \) indexed by \( \phi^F \in \Lambda_\eta^F, \) the efficient score also has the representation

\[
S_{\text{eff}}(L_o) = \frac{R}{\pi(V)} [\phi_{\text{eff},1}(Y, V) - E(\phi_{\text{eff},1}|V)] - \frac{1 - R}{1 - \pi(V)} [\phi_{\text{eff},2}(A, V) - E(\phi_{\text{eff},2}|V)] + E[\phi^F(L)|V],
\]

\( (A2) \)

for some \( \phi_{\text{eff}}^F(L) \in \Lambda_\eta^{F^\perp}. \) Combining (A1) and (A2),

\[
E \left\{ E \left[ s_\beta^F(L) - \alpha_{\text{eff}}^F \right| L \right\} = \\
\frac{R}{\pi(V)} [\phi_{\text{eff},1}(Y, V) - E(\phi_{\text{eff},1}|V)] - \frac{1 - R}{1 - \pi(V)} [\phi_{\text{eff},2}(A, V) - E(\phi_{\text{eff},2}|V)] + E[\phi^F(L)|V] \left| L \right\} \\
\rightarrow \mathcal{W} \left\{ s_\beta^F(L) - \alpha_{\text{eff}}^F \right\} = \phi^F(L) \rightarrow \mathcal{W}^{-1} \left[ \phi^F(L) \right] = \left\{ s_\beta^F(L) - \alpha_{\text{eff}}^F \right\} \\
\rightarrow \Pi \left\{ \mathcal{W}^{-1} \left[ \phi_{\text{eff}}^F(L) \right] | \Lambda_{\eta^{F^\perp}} \right\} = \Pi \left\{ s_\beta^F(L) | \Lambda_{\eta^{F^\perp}} \right\} - \Pi \left\{ \alpha_{\text{eff}}^F | \Lambda_{\eta^{F^\perp}} \right\} = S_{\text{eff}}^F(L),
\]

since \( \alpha_{\text{eff}}^F \in \Lambda_\eta^F \subset \Lambda^F. \) That a unique \( \phi_{\text{eff}}^F(L) \in \Lambda_\eta^{F^\perp} \) exists which satisfies the above relationship is given in Lemma 11.1 of \cite{Tsiatis2007}.

**Proof of result 3:**
To show that \( \hat{\beta}_{g}^{\text{doubly}} \) is doubly robust, suppose \( \pi(V; \psi) \) is correctly specified so that \( \pi(V; \psi^\dagger) = \pi(V) \), but \( h^*(V; \theta) \) is possibly misspecified. Then under regularity conditions \( \hat{\psi} \overset{p}{\rightarrow} \psi^\dagger \), \( \hat{\theta} \overset{p}{\rightarrow} \theta^* \) and

\[
E_{\eta_1^\dagger, \psi_1^\dagger, \theta^*} \{ U_g^{\text{doubly}}(\beta^\dagger, \eta_1^\dagger, \psi^\dagger, \theta^*) \} = E_{\eta_1^\dagger, \psi_1^\dagger, \theta^*} \{ E \{ U_g^{\text{doubly}}(\beta^\dagger, \eta_1^\dagger, \psi^\dagger, \theta^*) \} | V \}
\]

\[
= E_{\eta_1^\dagger, \psi_1^\dagger, \theta^*} \left\{ \left\{ \frac{\pi(V; \psi^\dagger)}{\pi(V; \psi^\dagger)} m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h^*(V; \theta^*) \right] - \frac{1 - \pi(V; \psi^\dagger)}{1 - \pi^*(V; \psi^\dagger)} m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h^*(V; \theta^*) \right] \right\} \right\} = 0.
\]

Now suppose \( h(V; \theta) \) is correctly specified so that \( h(V; \theta^\dagger) = h(V) \), but \( \pi^*(V; \psi) \) is possibly misspecified. Then \( \hat{\theta} \overset{p}{\rightarrow} \theta^\dagger \), \( \hat{\psi} \overset{p}{\rightarrow} \psi^\ast \) and

\[
E_{\eta_1^\dagger, \psi_1^\dagger, \theta^*} \{ U_g^{\text{doubly}}(\beta^\dagger, \eta_1^\dagger, \psi^*, \theta^*) \} = E_{\eta_1^\dagger, \psi_1^\dagger, \theta^*} \{ E \{ U_g^{\text{doubly}}(\beta^\dagger, \eta_1^\dagger, \psi^*, \theta^*) \} | V \}
\]

\[
= E_{\eta_1^\dagger, \psi_1^\dagger, \theta^*} \left\{ \left\{ \frac{\pi(V; \psi^\dagger)}{\pi^*(V; \psi^\dagger)} m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h(V; \theta^\dagger) \right] - \frac{1 - \pi(V; \psi^\dagger)}{1 - \pi^*(V; \psi^\dagger)} m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h(V; \theta^\dagger) \right] \right\} \right\} = 0.
\]

Assuming the choice of \( g(V) \) is such that \( E \{ U_g^{\text{doubly}, T} U_g^{\text{doubly}} \} < \infty \) and \( E \{ \frac{\partial}{\partial \tau} U_g^{\text{doubly}}(\tau) \}_{\tau^\dagger} \) is nonsingular where \( \tau^\dagger = (\beta^\dagger, \eta^\dagger) \), under standard regularity conditions for method of moments estimation given in Theorem 3.1 of Newey & McFadden (1994), \( \hat{\beta}_g^{\text{doubly}} \overset{p}{\rightarrow} \beta^\dagger \) and \( \sqrt{n} \left( \hat{\beta}_g^{\text{doubly}} - \beta^\dagger \right) \overset{d}{\rightarrow} \mathcal{N}(0, \Sigma_{\beta^\dagger}) \).

Let \( \phi \) denote the set of nuisance parameters, \( \phi = (\psi, \theta) \), and let \( \hat{\phi} \) denote the probability limit of \( \hat{\phi} = \left( \hat{\psi}, \hat{\theta} \right) \). In addition, let \( S_\phi = (S^T_\psi(R, V), U^T_{\theta, c}) \), \( U_{\tau, \phi} = (U_{g, \psi}^{\text{doubly}, T}(\tau, \phi), S_\phi^T) \) and

\[
G_{\tau} = E \left[ \frac{\partial}{\partial \tau} U_{\tau, \phi} \right]
\]

\[
G_{\phi} = E \left[ \frac{\partial}{\partial \phi} U_{\tau, \phi} \right]
\]

\[
M = E \left[ \frac{\partial}{\partial \phi} S_\phi \right]
\]

\[
\Psi = -M^{-1} S_\phi,
\]

where all the expectations are evaluated at the truth. Then under standard regularity conditions
for two-step method of moments estimation given in Theorem 6.1 of [Newey & McFadden (1994)],

\[
\Sigma_{\beta^*} = G_{\tau^*}^{-1} E \left\{ \left[ U_{\tau^*,\hat{\phi}} + G_{\hat{\phi}} \Psi \right] \left[ U_{\tau^*,\hat{\phi}} + G_{\hat{\phi}} \Psi \right]^T \right\} G_{\tau^*}^{-1}.\]

For inference, a consistent estimate \( \hat{\Sigma}_\beta \) of the asymptotic covariance matrix can be constructed by replacing all expected values with empirical averages evaluated at \( (\hat{\tau}^{\text{doubly}}, \hat{\phi}) \). Then a 95% Wald confidence interval for \( \beta_j \) is found by calculating \( \hat{\beta}_j \pm 1.96 \hat{\sigma}_j \), where \( \hat{\sigma}_j \) is the square root of the \( j \)th component of the diagonal of \( n^{-1} \hat{\Sigma}_\beta \). This concludes the proof for the first claim. The second claim in result 3 follows from result 1.

**Proof of result 4:**

The proof is based on the following lemma which is part of Theorem 5.3 in [Newey & McFadden (1994)].

**Lemma A4** If \( \exists \tilde{g}(V) \) satisfying

\[
-E \left[ g(V) \nabla_\beta H(\beta) \right] = E \left[ H^2(\beta)g(V)\tilde{g}(V)^T \right] \quad \forall g(V),
\]

then the estimator indexed by \( \tilde{g}(V) \) is most efficient.

Proof:

If \( g(V) \) and \( \tilde{g}(V) \) satisfy the equality in lemma A4 then the difference of the asymptotic variances of the respective estimators indexed by them is as follows:

\[
E \left[ H^2(\beta)g(V)\tilde{g}(V)^T \right]^{-1} E \left[ H^2(\beta)g(V)g(V)^T \right] E \left[ H^2(\beta)\tilde{g}(V)\tilde{g}(V)^T \right]^{-1} - E \left[ H^2(\beta)\tilde{g}(V)\tilde{g}(V)^T \right]^{-1}
\begin{equation}
= E \left[ H^2(\beta)g(V)\tilde{g}(V)^T \right]^{-1} E \left[ UU^T \right] E \left[ H^2(\beta)\tilde{g}(V)\tilde{g}(V)^T \right]^{-1},
\end{equation}
\]

where \( U = g(V) - E \left[ H^2(\beta)g(V)\tilde{g}(V)^T \right] E \left[ H^2(\beta)\tilde{g}(V)\tilde{g}(V)^T \right]^{-1} \tilde{g}(V) \) and \( E \left[ UU^T \right] \) is positive semi-definite.
We show that if \( \hat{g}(V) \) satisfies the equality in lemma A4 then \( \hat{g}(V) = g^{opt}(V) \).

\[
- E [g(V) \nabla_\beta H(\beta)] = E \left[ H^2(\beta)g(V)g^{opt}(V)^T \right] \quad \forall g(V),
\]

\[
\iff E \left\{ g(V) \left[ H^2(\beta)g^{opt}(V) + \nabla_\beta H(\beta) \right]^T \right\} = 0 \quad \forall g(V),
\]

\[
\iff E \left\{ g(V) E \left[ H^2(\beta)g^{opt}(V) + \nabla_\beta H(\beta) \bigg| V \right]^T \right\} = 0 \quad \forall g(V),
\]

\[
\iff E \left\{ E \left[ H^2(\beta)g^{opt}(V) + \nabla_\beta H(\beta) \bigg| V \right]^{\otimes 2} \right\} = 0,
\]

\[
\iff E \left[ H^2(\beta)g^{opt}(V) + \nabla_\beta H(\beta) \bigg| V \right] = 0,
\]

\[
\iff g^{opt}(V) = -E [\nabla_\beta H(\beta)|V] E \left[ H^2(\beta)|V \right]^{-1}.
\]

Due to Hájek’s representation theorem (Hájek 1970), the most efficient regular estimator is asymptotically linear and so the existence condition in lemma A4 holds when we consider only RAL estimators.

**Proof of result 5:**

The proof of result 3 still applies and therefore \( \hat{\beta}^{triply} \) is CAN in the model \( \mathcal{A}_{TS} \cap \mathcal{A}_{IV} \cap \{(\mathcal{M}_\pi \cap \mathcal{M}_\omega) \cup (\mathcal{M}_h \cap \mathcal{M}_\omega)\} \). In addition, suppose \( \pi(V; \psi) \) and \( f(Z|X; \zeta) \) are correctly specified, but \( h^*(V; \theta) \) and \( \omega^*(X; \eta) \) are possibly misspecified. Then \( \hat{\psi} \xrightarrow{p} \psi^\dagger, \hat{\zeta} \xrightarrow{p} \zeta^\dagger, \hat{\theta} \xrightarrow{p} \theta^*, \hat{\eta} \xrightarrow{p} \eta^* \) and

\[
E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^*, \zeta^\dagger} \left\{ U_g^{triply} \left( \beta^\dagger, \eta^*, \psi^\dagger, \theta^*, \zeta^\dagger \right) \right\} = E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^*, \zeta^\dagger} \left\{ E \left\{ U_g^{triply} \left( \beta^\dagger, \eta^*, \psi^\dagger, \theta^*, \zeta^\dagger \right) \right\} | V \right\}
\]

\[
= E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^*, \zeta^\dagger} \left\{ \left\{ \pi(V; \psi^\dagger) \left\{ \omega(X; \eta^\dagger) - \omega^*(X; \eta^*) + m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h^*(V; \theta^*) \right] \right\} - \frac{1 - \pi(V; \psi^\dagger)}{1 - \pi(V; \psi^\dagger)} m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h^*(V; \theta^*) \right] \right\} \left\{ g(V) - E[g(V)|X; \zeta^\dagger] \right\} \right\}^T \left\{ \lambda^T(X; \eta^*) \right\}^T
\]

\[
= E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^*, \zeta^\dagger} \left\{ \left\{ \omega(X; \eta^\dagger) - \omega^*(X; \eta^*) \right\} \left\{ g(V) - E[g(V)|X; \zeta^\dagger] \right\} \right\} \left\{ \lambda^T(X; \eta^*) \right\}^T
\]

\[
= E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^*, \zeta^\dagger} \left\{ \left\{ \omega(X; \eta^\dagger) - \omega^*(X; \eta^*) \right\} \left\{ g(V) - E[g(V)|X; \zeta^\dagger] \right\} \right\} \left\{ \lambda^T(X; \eta^*) \right\}^T \left| X \right\} = 0.
\]

Hence \( \hat{\beta}^{triply} \) is CAN in the model \( \mathcal{M}_{triply} \) under standard regularity conditions analogous to those presented in result 3. The second claim in result 5 follows from result 1.

**Proof of result 6:**
The proofs of results 3 and 5 still apply and therefore \( \hat{\beta}^{\text{triply}} \) is CAN in the model \( \mathcal{A}_{TS'} \cap \mathcal{A}_{IV} \cap \{ (\mathcal{M}_\pi \cap \mathcal{M}_\omega) \cup (\mathcal{M}_h \cap \mathcal{M}_\omega) \cup (\mathcal{M}_\pi \cap \mathcal{M}_f) \} \). In addition, suppose \( h(V; \theta) \) and \( f(Z|X; \zeta) \) are correctly specified, but \( \pi^*(V; \psi) \) and \( \omega^*(X; \eta) \) are possibly misspecified. Then \( \hat{\psi} \xrightarrow{p} \psi^* \), \( \hat{\zeta} \xrightarrow{p} \zeta^\dagger \), \( \hat{\theta} \xrightarrow{p} \theta^\dagger \), \( \hat{\eta} \xrightarrow{p} \eta^* \) and

\[
E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger} \left\{ U_g^{\text{triply}} \left( \beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger \right) \right\} = E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger} \left\{ E \left\{ U_g^{\text{triply}} \left( \beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger \right) \right\} | V \right\}
\]

\[
= E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger} \left\{ \frac{\pi(X; \psi^\dagger)}{\pi^*(X; \psi^*)} \left\{ \omega(X; \eta^\dagger) - \omega^*(X; \eta^\dagger) + m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h(V; \theta^\dagger) \right] \right\} \right\} - \frac{1 - \pi(X; \psi^\dagger)}{1 - \pi^*(X; \psi^*)} m(X; \beta^\dagger) \left[ h(V; \theta^\dagger) - h(V; \theta^\dagger) \right] \left\{ \{ g(V) - E[g(V)|X; \zeta^\dagger] \}, \lambda^T(X; \eta^*) \right\}^T \right\}
\]

\[
= E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger} \left\{ \frac{\pi(X; \psi^\dagger)}{\pi^*(X; \psi^*)} \left\{ \omega(X; \eta^\dagger) - \omega^*(X; \eta^\dagger) \right\} \left\{ \{ g(V) - E[g(V)|X; \zeta^\dagger] \}, \lambda^T(X; \eta^*) \right\}^T \right\}
\]

\[
= E_{\beta^\dagger, \eta^\dagger, \psi^\dagger, \theta^\dagger, \zeta^\dagger} \left\{ E \left\{ \frac{\pi(X; \psi^\dagger)}{\pi^*(X; \psi^*)} \left\{ \omega(X; \eta^\dagger) - \omega^*(X; \eta^\dagger) \right\} \left\{ \{ g(V) - E[g(V)|X; \zeta^\dagger] \}, \lambda^T(X; \eta^*) \right\}^T \right\} \right\}
\]

\[
= 0.
\]

Hence \( \hat{\beta}^{\text{triply}} \) is CAN in the model \( \mathcal{M}_{\text{multiply}} \) under standard regularity conditions analogous to those presented in result 3. The second claim in result 6 follows from result 1.