Orbits in a central force field: Bounded orbits

Subhankar Ray
Dept of Physics, Jadavpur University, Calcutta 700 032, India

J. Shamanna
Physics Department, Visva Bharati University, Santiniketan 731235, India
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The nature of boundedness of orbits of a particle moving in a central force field is investigated. General conditions for circular orbits and their stability are discussed. In a bounded central field orbit, a particle moves clockwise or anticlockwise, depending on its angular momentum, and at the same time oscillates between a minimum and a maximum radial distance, defining an inner and an outer annulus. There are generic orbits suggested in popular texts displaying the general features of a central orbit. In this work it is demonstrated that some of these orbits, seemingly possible at the first glance, are not compatible with a central force field. For power law forces, the general nature of boundedness and geometric shape of orbits are investigated.

I. INTRODUCTION

The central force motion is one of the oldest and widely studied problems in classical mechanics. Several familiar force-laws in nature, e.g., Newton’s law of gravitation, Coulomb’s law, van-der Waals force, Yukawa interaction, and Hooke’s law are all examples of central forces. The central force problem gives an opportunity to test one’s understanding of the Lagrange’s equation, Hamilton’s equation, Hamilton Jacobi method, and classical perturbation. It also serves as an introduction to the concept of integrals of motion and conservation laws. We need only to appeal to the principles of conservation of energy and angular momentum to describe the nature and geometry of the possible trajectories in central force motion. Most books in classical mechanics, treatise, and advanced texts discuss the central force problem. In this article we present some interesting features of bounded orbits in a central field.

A. Kepler’s Laws

One of the most remarkable discoveries in the history of physics is that of Keplerian orbits. A tremendous wealth of data on planetary positions was collected by Tycho Brahe and Johannes Kepler after detailed observation spread over several decades. After a thorough analysis of this data Johannes Kepler formulated three empirical laws that described and correlated the motion of the five planets then known:

1. Each planet moves in an elliptical orbit, with the sun at one of its foci.
2. The radius vector from the sun to each planet sweeps out equal areas in equal times.
3. The square of the periods \(T^2\) of the planets are proportional to the cube of the lengths of the corresponding semimajor axes \(a^3\).

B. Newtonian Synthesis

Almost 100 years later Newton realized that the planets go about in their nearly circular orbits around the sun under the influence of the same force that causes an apple to fall to the ground, i.e., gravitation. Newton’s law of gravitation gave a theoretical basis to Kepler’s laws. Kepler’s laws can be derived from Newton’s law of gravitation; this is often referred to as the Newtonian synthesis. Kepler’s first and third laws are valid only in the specific case of inverse square force. There are, however, certain general features which are observed in all central field problems. They include (i) certain conserved quantities (energy, and angular momentum), (ii) planer nature of orbits, and (iii) constancy of areal velocity (Kepler’s second law). A large class of central forces allows circular orbits (stable or unstable), bounded orbits, and even closed and periodic orbits. Certain common characteristics about the generic shapes of bounded orbits can also be ascertained.

II. EQUATIONS OF MOTION AND THEIR FIRST INTEGRALS

A. Central field orbits: confinement in a plane

The central force motion between two bodies about their center of mass can be reduced to an equivalent one body problem in terms of their reduced mass \(m\) and their relative radial distance \(r\). Hence in this reduced system, a body having the reduced mass moves about a fixed center of force.

Consider the motion of a body under a central force, \( \mathbf{F} = \mathbf{F}(r) = f(r)\hat{r} \) with the origin as its force center. The potential \(V(r)\) from which this force is derived is also a function of \(r\) alone, \( \mathbf{F} = -\nabla V, V \equiv V(r) \).

On account of the central nature of the force, the mechanical properties of the body do not vary under rotation in any manner around the center of force. Let
the body be rotated through an infinitesimal angle \( \delta \theta \), where the magnitude \( \delta \theta \) is the angle of rotation while the direction is that of the axis of rotation \( \hat{n} \). The change in the radius vector from the origin to the body is \( | \delta \mathbf{r} | = r \sin(\theta) \delta \theta \), with \( \delta \mathbf{r} \) being perpendicular to \( \mathbf{r} \) and \( \delta \theta \). Hence \( \delta \mathbf{r} = \delta \theta \times \mathbf{r} \). The change in velocity is similarly given by \( \delta \mathbf{v} = \delta \mathbf{\theta} \times \mathbf{v} \). The Lagrangian of the system is a function of \( \mathbf{r} \) and \( \mathbf{r} \). The motion is governed by the Lagrange’s equation,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \tag{1}
\]

If we now require that the Lagrangian of the system remain invariant under this rotation we obtain

\[
\delta L = \frac{\partial L}{\partial \dot{\theta}} \cdot \delta \dot{\theta} + \frac{\partial L}{\partial \theta} \cdot \delta \theta = 0 .
\]

One can define generalized momentum as,

\[
\mathbf{p} = \frac{\partial L}{\partial \dot{\theta}}
\]

From the Lagrange’s equation we get,

\[
\dot{\mathbf{p}} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}
\]

Replacing \( \partial L/\partial \dot{\theta} \) by \( \mathbf{p} \) and \( \partial L/\partial r \) by \( \dot{\mathbf{p}} \) in the equation for \( \delta L \) we get,

\[
\dot{\mathbf{p}} \cdot \delta \mathbf{\theta} + \mathbf{p} \cdot \delta \mathbf{\theta} \times \mathbf{r} = 0 .
\]

As \( \delta \theta \) is arbitrary we conclude \( \mathbf{l} = \mathbf{r} \times \mathbf{p} \) is a conserved quantity. \( \mathbf{l} \) is called the angular momentum of the system. Since \( \mathbf{l} \) is a constant and is perpendicular to \( \mathbf{r} \) it follows that the radius vector of the particle lies in a plane perpendicular to \( \mathbf{l} \). This implies that the motion of the particle in a central field is confined to a plane.

### B. Lagrangian and equations of motion

As the motion in a central force field is confined to a plane, it suffices to use plane polar coordinates. One may write the Lagrangian of the particle as,

\[
L = \frac{1}{2}m\dot{r}^2 - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) . \tag{1}
\]

We assume the center of force to be at the origin. The coordinates of the body of mass \( m \) undergoing the central field motion are given by \( (r, \theta) \).

The Lagrange’s equation for the \( \theta \) and \( r \) coordinates are given respectively by,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \tag{2}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \tag{3}
\]

\[
\begin{align*}
\text{\( \text{Area} = \frac{1}{2} (r) (r \, d\theta ) \)}
\end{align*}
\]

\[
\text{\( \text{FIG. 1: Area swept by radius vector} \)}
\]

### C. First integrals and conservation laws

The canonical momentum corresponding to \( \theta \) is called the angular momentum (or rather the magnitude of the angular momentum that we discussed before),

\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = l
\]

As \( \theta \) is a cyclic coordinate, i.e., the Lagrangian is independent of \( \theta \), this angular momentum is conserved. This can be shown from the Lagrange’s equation for \( \theta \).

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0
\]

\[
\dot{p}_\theta = \frac{d}{dt}(mr^2 \dot{\theta}) = 0 .
\]

The corresponding integral of motion is,

\[
mr^2 \dot{\theta} = l \tag{4}
\]

and this \( l \) can easily be shown to be the magnitude of the angular momentum vector \( \mathbf{l} = \mathbf{r} \times \mathbf{p} \).

This conservation law is essentially equivalent to Kepler’s 2nd Law:

The elementary triangular area swept by the radius vector in an infinitesimal time interval \( dt \) is,

\[
dA = \frac{1}{2}r(r \, d\theta)
\]

hence it follows from Eq. (4) that the rate of areal sweep is a constant.

\[
\frac{dA}{dt} = \frac{1}{2}r^2 \dot{\theta} = \frac{l}{2m}
\]

There is another first integral of motion associated with the Lagrange’s equation for the \( r \) coordinate.

\[
\frac{d}{dt}(mr^2) - mr \dot{\theta}^2 + \frac{\partial V}{\partial r} = 0
\]
Angular momentum

The force in terms of the potential (conservative force) is given by, \( f(r) = -\partial V / \partial r \), whence the above equation becomes

\[
m\ddot{r} - m \dot{\theta}^2 = f(r)
\]

Using the first integral of motion, one can convert this second equation into an equation for \( r \) alone.

\[
m\ddot{r} = -\frac{d}{dr} \left( V + \frac{l^2}{2mr^2} \right)
\]

Integrating we get

\[
\frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{l^2}{mr^2} + V(r) = E
\]

where \( E \) is a constant of integration, called the energy. This is the law of conservation of total mechanical energy.

It is interesting to note that for motion in a general force field, \( l = mr^2 \dot{\theta} \) remains invariant even though \( r \) and \( \dot{\theta} \) vary with time (see Fig. 2). Similarly \( E = m \dot{r}^2/2 + l^2/(2mr^2) + V(r) \) remains a constant even though \( \dot{r} \) and \( r \) and hence \( m \dot{r}^2/2 \) and \( V(r) + l^2/(2mr^2) \) each varies with time (see Fig. 3).

Lagrange’s equations are two second order Ordinary Differential Equations (ODE) in \( r \) and \( \theta \). However they decouple, i.e., each equation is expressible in terms of either \( r \) or \( \theta \). On integrating each equation once we get the first integrals of motion namely, the total mechanical energy and the angular momentum. A further integration will yield the complete solution to the problem. This second integration introduces two more constants of integration namely, the initial radial \( (r_0) \) and angular \( (\theta_0) \) positions.

### D. Equation for the orbit

From the equation giving energy as a first integral of motion, we get the expression for radial velocity,

\[
\dot{r} = \sqrt{\frac{2}{m} \left( E - V - \frac{l^2}{2mr^2} \right)}
\]

On integration we get,

\[
t = \int_{r_0}^{r} \frac{dr}{\sqrt{2/m(E - V - l^2/(2mr^2))}}.
\]

This relation can be inverted to give \( r \) as a function of \( t \), \( r = r(t) \). The other first integral gives,

\[
\dot{\theta} = \frac{l}{mr^2}.
\]

Substituting the expression for \( r \) as a function of \( t \), and integrating,

\[
\theta - \theta_0 = \frac{l}{m} \int_{t_0}^{t} \frac{dt}{r(t)^2}.
\]

These two expressions for \( r(t) \) and \( \theta(t) \) express the equation of the orbit for the particle in a central field in terms of time \( t \) as a parameter.

Instead of expressing the orbit parametrically in terms of \( t \), one often wants to express the orbit directly as an equation connecting \( r \) and \( \theta \). Such an equation may be obtained by eliminating \( t \) from the above expressions for \( \dot{r} \) and \( \dot{\theta} \).

\[
\frac{d\theta}{dr} = \frac{l/(mr^2)}{\sqrt{2/m(E - V(r) - l^2/(2mr^2))}}
\]
On integration this yields,
\[ \theta - \theta_0 = \int_{r_0}^{r} \frac{l/(mr^2)}{\sqrt{2/m(E - V(r) - l^2/(2mr^2))}} \, dr \]

For understanding the qualitative nature of motion in a central field one looks at the equivalent one-dimensional problem.
\[ \dot{r} = \sqrt{\frac{2}{m}} \left( E - V(r) - \frac{l^2}{2mr^2} \right) = \sqrt{\frac{2}{m}} (E - \tilde{V}(r)). \]

We call \( \tilde{V} \) the effective potential, introduced to make the problem similar to that of a particle moving in a one-dimensional potential field. The effective radial force \( f(r) \) is connected to the effective radial potential \( \tilde{V}(r) \) by the expected relation,
\[ \dot{r} = -\frac{\partial \tilde{V}(r)}{\partial r} \quad \text{(10)} \]

At a point where the effective potential \( \tilde{V}(r) \) equals the energy \( E \), the radial velocity vanishes (\( \dot{r} = 0 \)). In one-dimensional motion this corresponds to a particle coming momentarily to rest, and having zero kinetic energy. However, in the case of central field, the motion is not really one-dimensional, and even for \( \dot{r} = 0 \), the particle is not at rest (\( \mathbf{v} = r\dot{\theta}\mathbf{\hat{r}} \)), and it has a non-zero kinetic energy \((1/2m)r^2\dot{\theta}^2\).

For the inverse square force, as in the case of gravitation, we have \( f(r) = -k/r^2 \) and \( \tilde{V}(r) = -k/r \). Effective potential (see Fig. 4) is given by,
\[ \tilde{V}(r) = -\frac{k}{r} + \frac{l^2}{2mr^2} \quad \text{(11)} \]

The following properties of the effective potential are easily noted:
1. \( \tilde{V}(r) = 1/r^2(l^2/2m - kr) \), as \( r \to 0 \) the term within bracket is essentially \( l^2/(2m) \) and hence \( \lim_{r\to0} \tilde{V}(r) \to +\infty \)
2. \( \lim_{r\to\infty} \tilde{V}(r) = 0 \)
3. \( \tilde{V}(r) = -1/r(k - l^2/2mr^2) \) and for large values of \( r \), \( l^2/2mr^2 \) is negligible compared to \( k \), hence \( \tilde{V}(r) \) has a negative value.
4. At \( r^* = l^2/2mk \) the function \( \tilde{V}(r) \) intersects the \( r \) axis, i.e., \( \tilde{V}(r^*) = 0 \).
5. \( \tilde{V}(r) \) reaches a minimum at \( r_0 = 2r^* = l^2/mk \). With \( \partial^2 \tilde{V}/\partial r^2|_{r_0} = 0 \) and \( \partial^2 \tilde{V}/\partial r^2|_{r_0} > 0 \).

For total energy \( E = \tilde{V}(r_0) \) the particle has zero radial velocity at \( r = r_0 \), and no other radial position is physically accessible, since \( \dot{r} \) becomes imaginary at \( r \neq r_0 \). This corresponds to circular motion.

For \( 0 < E < \tilde{V}(r_0) \) there is a range of radial positions \((r_{max} \geq r \geq r_{min})\) for which \( E \geq \tilde{V}(r) \). The particle can move in this range of \( r \) with varying \( \dot{r} \). The radial velocity \( \dot{r} \) vanishes at the end points \( r_{max} \) and \( r_{min} \) where the energy \( E \) equals the effective potential \( \tilde{V} \) and \( \dot{r} \) reaches a maximum at \( r = r_0 \) The points \( r_{max} \) and \( r_{min} \) are called the turning points. The central field particle cannot move beyond these points, as the energy becomes less than the effective potential, and the expression for radial velocity (\( \dot{r} \)) turns imaginary.

\( E < \tilde{V}(r_0) \) is a physically impossible situation, since no (radial) position is physically allowed for the particle.

For \( E \geq 0 \) we have an unbounded motion, where the particle can fly off to infinity. Thus in the case of inverse square force field we can have bounded \((E < 0)\) or unbounded \((E \geq 0)\) motion depending on the energy of the particle. In particular for \( E = 0 \) it is parabolic and for \( E > 0 \) it is hyperbolic.

III. EXISTENCE AND STABILITY OF CIRCULAR ORBITS FOR CENTRAL FORCES

At all positions other than where the effective potential is a minimum or a maximum, we have a net effective force \( f(r) = -\partial \tilde{V}/\partial r \neq 0 \). When the total energy \( E \) is not equal to the minimum of effective potential \( \tilde{V}(r) \), at the points of instantaneous zero radial velocity the particle is pushed away. If the effective potential has a minimum and energy is greater than that minimum then the radial distance has a lower and an upper bound and the particle moves between these radial limits. On the other hand if the effective potential has a maximum, and the energy is less than that maximum, the effective force pushes the
particle away from the positions of zero radial velocity in an inward or an outward spiral. If the total energy is equal to the maximum or minimum of the effective potential, the system can stay with zero radial velocity, and hence move in a circular orbit.

The condition for circular orbit is,

$\frac{\partial V}{\partial r} \bigg|_{r_0} = \frac{\partial V}{\partial r} \bigg|_{r_0} - \frac{l^2}{mr_0^2} = 0 \quad (12)$

whence we get,

$f(r_0) = -\frac{\partial V}{\partial r} \bigg|_{r_0} = -\frac{l^2}{mr_0^2} \quad (13)$

The negative sign on the right hand side clearly shows that the force must be attractive. The particle moves in a circular orbit, since the force of attraction due to the central field provides the necessary centripetal force.

For a given attractive central force ( $f(r)$ and $V(r)$ given ) it is possible to have a circular orbit of radius $r_0$ provided the angular momentum and energy of the particle are given by,

$l^2 = -mr_0^3 f(r_0) \quad (14)$

$E = V(r_0) + \frac{l^2}{2mr_0^2} \quad (15)$

However when the effective potential has a maximum the system is in an unstable circular orbit. A small deviation from this radial position causes the orbit to be unbounded. The effective force that comes into play makes the particle move away from the position of zero radial velocity in an inward or outward spiral (Fig. 5). When the effective potential has a minimum, the effective forces cause the particle to remain in a bound orbit confined in an annular space. For small deviations the annular radii are nearly the same, and hence the orbit is close to a circle. One can understand the stability question by studying the forces (restoring or unsettling) that come into play when the system is moved infinitesimally from the position of circular orbit.

The circular orbit is stable if,

$\frac{\partial^2 V}{\partial r^2} > 0. \quad (16)$

That is equivalent to,

$\frac{\partial^2 V}{\partial r^2} = \left( -\frac{\partial f}{\partial r} + \frac{3l^2}{mr_0^4} \right) \bigg|_{r_0} > 0$

$\frac{\partial f}{\partial r} \bigg|_{r_0} < \frac{3l^2}{mr_0^4}$

Using Eq. (13),

$\frac{\partial f}{\partial r} \bigg|_{r_0} < -\frac{3f(r_0)}{r_0} \quad (17)$

IV. NATURE OF BOUNDED ORBITS

General bounded motion has both lower and upper bounds. It means that the particle cannot approach nearer than some minimum or move farther than some maximum distance. One has to remember that the angular velocity has a constant sign, same as that of the constant angular momentum, throughout the motion. However its magnitude decreases with increase in the radial distance ($\sim 1/r^2$). Together with the angular motion, the radial distance changes from $r_{\text{max}}$ to $r_{\text{min}}$, then back to $r_{\text{max}}$ and so on. From this general nature of motion two families of generic orbits are suggested in popular texts (Fig. 6, Fig. 7). However it can be shown that the generic types shown in the later figure (Fig. 7) are not feasible for any attractive potential.

We consider below a general bounded orbit confined in an annular region. We would like to investigate the nature of the orbit close to the point where it touches the inner or the outer annulus. Let us choose the reference line (polar line) of the coordinate system such that the orbit touches the annular ring at $\theta = 0$, or more specifically at $r = R, \theta = 0$. Since at this point $P(r = R, \theta = 0)$ the orbit is at its closest (or farthest) approach from the pole, $(\partial r/\partial \theta)_{P} = 0$. As $r$ is a function of $\theta$, using Eq. (4) we get for motion along the trajectory,

$\frac{dr}{dt} = \frac{l}{mr^2} \frac{dr}{d\theta}$

hence

$r'(0) = \frac{dr}{d\theta} \bigg|_{\theta=0} = 0 \quad (18)$
The equation of motion for $r$ is given by,
$$m \ddot{r} - m r \dot{\theta}^2 = f(r)$$
From which we get $r''(\theta)$ along the trajectory,
$$m \frac{l}{mr^2} \frac{d}{d\theta} \left( \frac{l}{mr^2} \frac{dr}{d\theta} \right) - mr \frac{I^2}{mr^3} = f(r)$$
$$\frac{d^2 r}{d\theta^2} = \frac{mr^4}{I^2} \tilde{f}(r)$$
$$= r + \frac{mr^4}{I^2} f(r)$$

One can expand $r(\theta)$ in a Taylor series in $\theta$, and remember that $r'(0) = 0,$
$$r(\theta)_{\text{traj}} = r(0) + r'(0) \theta + r''(0) \frac{\theta^2}{2!} + h.o.$$
which can be expanded to give $r(\theta)_{\text{tan}}$ along the tangent,

$$r(\theta)_{\text{tan}} = r(0) + r(0) \frac{\theta^2}{2} + h.o.$$  

Notice that the constant or the $\theta$ independent terms in $r(\theta)_{\text{traj}}$ and $r(\theta)_{\text{tan}}$ are equal, and the first leading power of $\theta$ is $\theta^2$ in both the cases. The coefficient of $\theta^2$ (curvature) will determine the nature of the trajectory in reference to the tangent.

We first study the curvature of the orbit in reference to the tangent. The force being attractive $f(r(0)) < 0$, for small angular distance $(\Delta(\theta))$ away from the point $P$, we study the orbit near the point $P$ where it touches the outer annulus.

$$r(\Delta\theta)_{\text{traj}} < r(\Delta\theta)_{\text{tan}}$$  

which means that the orbit bends more sharply than the tangent and stays closer to the inner annulus for nearby points (Fig. 8). Hence the second type of orbits shown in some texts (Fig. 7) are not possible for central forces. For the outer annulus the analysis with respect to tangent is not very meaningful as one can set up an even stronger bound for its orbit, directly considering the potential $\tilde{V}(r)$. In this case the effective potential has a positive slope with respect to the radius, and hence a negative effective force ($f(r) < 0$). The orbit should not only remain nearer than the tangent, but even nearer than the outer annulus. This condition is confirmed if

$$r(\Delta\theta)_{\text{traj}} < r(0) = r_{\text{max}}$$  

This confirms that the outer annulus is indeed the outer bound of the trajectory. The nature of the orbit near the point of contact $P$ at the outer annulus is shown in figure 9.

V. BOUNDED ORBITS FOR THE POWER LAW CENTRAL FORCE

A. Existence of stable circular orbit

For the case of a power law central potential

$$V(r) = \frac{k}{r^n}, \quad f(r) = -\frac{nk}{r^{n+1}}$$  

we find

$$\frac{(n+1)nk}{r_0^{n+2}} < -3 \left( -\frac{k}{r_0^{n+1}} \right) \frac{1}{r_0}$$

$$n < 2$$  

FIG. 9: Curvature of the trajectory in comparison to the outer annulus
Hence the circular orbit is stable for an attractive power law potential that varies slower than inverse square (or a force that varies slower than inverse cube).

**B. Study of boundedness of orbits**

1. \( V(r) = -a/r^n \) where \( n > 2 \)

Consider the case when \( n > 2 \). We have the effective potential,

\[
\tilde{V}(r) = -\frac{a}{r^n} + \frac{l^2}{2mr^2}
\]  

(23)

1. \( \tilde{V}(r) = -1/r^n(a - l^2 \cdot r^{n-2}/(2m)) \), and as \( r \to 0 \) we may neglect \( l^2 \cdot r^{n-2}/(2m) \) in comparison to \( a \), thus \( \lim_{r\to0} \tilde{V}(r) = -\infty \).

2. \( \lim_{r\to\infty} \tilde{V}(r) = 0 \).

3. For large but finite \( r \), \( \tilde{V}(r) = -1/r^n(a - l^2 - l^2/2m) \), and \( a/r^{n-2} \) is negligible compared to \( l^2/(2m) \), and hence \( \tilde{V}(r) \) is positive.

4. \( \tilde{V}(r) \) intersects the \( r \) axis at \( r^* \) where \( (r^*)^{n-2} = 2am/l^2 \).

5. \( \tilde{V}(r) \) has a maximum at \( r_0 \) where \( r_0^{n-2} = anm/l^2 \).

For a maximum point of \( \tilde{V} \) we should have \( \partial \tilde{V}/\partial r \big|_{r_0} = 0 \) and \( \partial^2 \tilde{V}/\partial r^2 \big|_{r_0} < 0 \).

\[
\frac{\partial \tilde{V}}{\partial r} = \frac{an}{r^{n+1}} - \frac{l^2}{mr^3} = 0.
\]

For the point of extremum

\[
r_0^{n-2} = \frac{anm}{l^2}.
\]

Finding the second derivative of \( \tilde{V} \) with respect to \( r \) at the point \( r_0 \),

\[
\frac{d^2 \tilde{V}}{dr^2} = -\frac{an(n+1)}{r^{n+2}} + \frac{3l^2}{mr^4}
\]

\[
= \frac{1}{r^4} \left( -\frac{an(n+1)}{r^{n-2}} + \frac{3l^2}{m} \right)
\]

\[
= \frac{1}{r_0^4} \left( -(n+1)\frac{l^2}{m} + \frac{3l^2}{m} \right)
\]

\[
= \frac{1}{r_0^4} \left( 2 - n \right) < 0
\]

Hence it is a point of maximum. For \( E > \tilde{V}(r_0) \) we always have an unbounded orbit. For \( E < \tilde{V}(r_0) \) the orbit is semibounded, bounded above or bounded below, according to its initial state. The particle either spirals in or spirals out. For \( E = \tilde{V}(r_0) \) we get an unstable circular orbit.

2. \( V(r) = -a/r^n \) where \( n = 2 \)

The effective radial potential is \( \tilde{V}(r) = -a/r^2 + l^2/(2mr) \). This is essentially an attractive or repulsive inverse square term.

\[
\tilde{V}(r) = -\frac{a}{r^2}, \quad \tilde{a} > 0 \text{ if } a > \frac{l^2}{2m}
\]

\[
\tilde{a} < 0 \text{ if } a < \frac{l^2}{2m}
\]

This potential cannot give circular orbit ever. If the effective potential is attractive and \( E < 0 \) it has an upper bound of radial distance. A typical orbit would therefore be an inward spiral. For a repulsive effective potential we will have outward spiral moving to infinite radial distance.

3. \( V(r) = -a/r^n \) where \( 2 > n > 0 \)

In this case the effective potential is,

\[
\tilde{V}(r) = -\frac{a}{r^n} + \frac{l^2}{2mr^2}
\]  

(24)

We have the following properties of \( \tilde{V}(r) \)

1. \( \tilde{V}(r) = 1/r^2(l^2/2m - a/r^{n-2}) \) and as \( r \to 0 \) we may neglect \( a/r^{n-2} \) in comparison to \( l^2/2m \), and thus \( \lim_{r\to0} \tilde{V}(r) \to +\infty \).

2. \( \lim_{r\to\infty} \tilde{V}(r) = 0 \).
3. \( \tilde{V}(r) = -1/r^n(a - l^2/2mr^2) \) and for large but finite values of \( r, l^2/2mr^2 \) is negligible compared to \( a \), hence \( \tilde{V}(r) \) is negative.

4. \( \tilde{V}(r) \) intersects the \( r \) axis at \( r^* \) where \( (r^*)^{2-n} = l^2/(2am) \).

5. \( \tilde{V}(r) \) reaches a minimum at \( r_0 \) where \( r_0^{2-n} = l^2/amn \).

For a minimum point of \( \tilde{V} \) we should have \( \partial \tilde{V}/\partial r \mid_{r_0} = 0 \) and \( \partial^2 \tilde{V}/\partial r^2 \mid_{r_0} > 0 \).

\[
\frac{\partial \tilde{V}}{\partial r} = \frac{an}{r^{n+1}} - \frac{l^2}{mr^3} = 0.
\]

For the point of extremum
\[
r_0^{2-n} = \frac{l^2}{amn}.
\]

Finding the second derivative of \( \tilde{V} \) with respect to \( r \) at the point \( r_0 \),
\[
\frac{d^2 \tilde{V}}{dr^2} = \frac{-an(n+1)}{r^{n+2}} + \frac{3l^2/mr^4}{m} = \frac{1}{r^4} \left( -\frac{an(n+1)}{r^{n-2}} + \frac{3l^2}{m} \right) = \frac{1}{r_0^4} \left( -(n+1)\frac{l^2}{m} + \frac{3l^2}{m} \right) = \frac{1}{r_0^4 m} \left( 2 - n \right) > 0
\]

Hence it is a point of minimum. For this case the orbit can be bounded or unbounded depending on the total energy \( E \) of the particle.

4. \( V(r) = a \ln r \)

This corresponds to one over \( r \) force \( f(r) \sim -1/r \). The effective potential is given by,
\[
\tilde{V}(r) = a \ln r + \frac{l^2}{2mr^2} \tag{25}
\]

We have the following properties of \( \tilde{V}(r) \)
1. \( \lim_{r \to 0} \tilde{V}(r) \to +\infty \).
2. \( \lim_{r \to \infty} \tilde{V}(r) \to +\infty \).
3. There is no point of intersection with the \( r \) axis, and \( \tilde{V}(r) \) is always positive.

4. \( \tilde{V}(r) \) reaches a minimum at \( r_0 = l/\sqrt{am} \).

For a minimum point of \( \tilde{V} \) we should have \( \partial \tilde{V}/\partial r \mid_{r_0} = 0 \) and \( \partial^2 \tilde{V}/\partial r^2 \mid_{r_0} > 0 \).

\[
\frac{\partial \tilde{V}}{\partial r} = \frac{a}{r} - \frac{l^2}{mr^3}.
\]

for the point of extremum
\[
r_0^2 = \frac{l^2}{am}.
\]

The second derivative of \( \tilde{V} \) at \( r = r_0 \),
\[
\frac{d^2 \tilde{V}}{dr^2} = \frac{a}{r^2} + \frac{3l^2/mr^4}{m} = \frac{1}{r^2} \left( -a + \frac{3l^2}{mr^2} \right) = \frac{1}{r^2} 2a > 0
\]
5. \( V(r) = ar^n \) where \( n > 0 \)

For this potential the effective potential is given by,

\[
\tilde{V}(r) = ar^n + \frac{l^2}{2mr^2} \quad n > 0 \tag{26}
\]

We have the following properties of \( \tilde{V}(r) \)

1. \( \lim_{r \to 0} \tilde{V}(r) = +\infty \).

2. \( \lim_{r \to \infty} \tilde{V}(r) = +\infty \).

3. There is no point of intersection with the \( r \) axis, and \( \tilde{V}(r) \) is always positive.

4. \( \tilde{V}(r) \) has a minimum at \( r_0 \), \( r_0^{n+2} = \frac{l^2}{amn} \).

For a minimum point of \( \tilde{V} \) we should have \( \frac{\partial \tilde{V}}{\partial r} |_{r_0} = 0 \) and \( \frac{\partial^2 \tilde{V}}{\partial r^2} |_{r_0} > 0 \).

\[
\frac{\partial \tilde{V}}{\partial r} = anr^{n-1} - \frac{l^2}{mr^3}
\]

for the point of extremum

\[
r_0^{n+2} = \frac{l^2}{amn}
\]

Hence it is a point of minimum. The orbit is always bounded.

The second derivative of \( \tilde{V} \) with respect to \( r \),

\[
\frac{d^2 \tilde{V}}{dr^2} = an(n-1)rn^{-2} + \frac{3l^2}{mr^4}
\]

\[
= \frac{1}{r^4} \left( an(n-1)r^{n+2} + \frac{3l^2}{m} \right)
\]

\[
= \frac{1}{r^4} \frac{l^2}{m} (2 + n) > 0
\]

Hence it is a point of minimum. The orbit is always bounded.

The findings for the general power law potential can be summarized as follows,

\[
V(r) = \text{sign}(n)ar^n \quad n \neq 0 \tag{27}
\]

\[
f(r) = -\text{abs}(n)ar^{n-1} \tag{28}
\]

\[
V(r) = b \ln r \tag{29}
\]

\[
f(r) = -\frac{b}{r} \tag{30}
\]

**TABLE I: Dependence of boundedness on power law.**

| \( n \) of \( V(r) \sim ar^n \) | Nature of boundedness |
|-------------------------------|-----------------------|
| \( ) - \infty, -2( \)        | always unbounded      |
| \(-2 \)                       | spiralling orbit       |
| \( ) - 2, 0( \)              | bounded or unbounded depending on \( E \) |
| \( 0 \ (\ln r) \)             | always bounded        |
| \( ) 0, +\infty( \)          | always bounded        |

**C. Stable bounded orbit, geometric shape**

The stable bounded orbits in a power law central field can have the following forms

1. \( V(r) = -a/r^n, \ 0 < n \leq 2 \)

2. \( V(r) = b \log r \)

3. \( V(r) = a \cdot r^n, \ n > 0 \)

In all the above cases the derivative of \( V(r) \) with respect to \( r \) is always positive, and hence the force is necessarily attractive \( (f(r) = -\partial V(r)/\partial r < 0) \).

In the first case the effective one dimensional potential \( \tilde{V}(r) \) goes to infinity as \( r \to 0 \). \( \tilde{V}(r) \) has a minimum at some \( r = r_0 \), and it has a negative slope for all \( r < r_0 \), and positive slope for all \( r > r_0 \). For \( E = \tilde{V}(r_0) \) we get stable circular orbit, and for \( \tilde{V}(r_0) < E < 0 \) the orbits are still bounded. For cases (2) and (3) the circular orbit is stable, and orbits are always bounded for any energy.
TABLE II: Dependence of stability of circular orbits on power law.

| n of $V(r) \sim ar^n$ | stability of circular orbits |
|------------------------|-----------------------------|
| $-\infty, -2$          | unstable                    |
| $-2, \infty$           | stable                      |

VI. CONCLUSION

Existence of bounded orbit for a large class of attractive central field has been discussed. The generic nature of central field bounded orbits is analytically derived. Certain class of these orbits (Fig. 7) presented in popular texts, are shown to be non-feasible.

Small deviation from circularity in the case of central field is often expressed in terms of inverse of radial distance ($u = 1/r$).

$$u = u_0 + a \cdot \cos(\beta \theta)$$

$$r = \frac{r_0}{1 + a \cdot r_0 \cdot \cos(\beta \theta)}$$

where $r_0 = 1/u_0$. It is interesting to note that these orbits are sometime mistakenly identified with diagrams of the form shown in Fig. 7. The above expression in fact corresponds to a class of orbits that look generically like those shown in Fig. 6.

The generic features of central force orbits discussed here have been verified by computer simulation for a large class of central force fields. The figures shown here (Fig. 2, 3, 5 and 6) were generated by these simulations. Students interested in studying and generating such orbits will find Ref.12 helpful.

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* Electronic address: [subho@juphys.ernet.in](mailto:subho@juphys.ernet.in)

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