THE ROTATING DYONIC BLACK HOLES OF KALUZA-KLEIN THEORY

Dean Rasheed*
DAMTP
Silver Street
Cambridge
CB3 9EW

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Abstract
The most general electrically and magnetically charged rotating black hole solutions of 5 dimensional Kaluza-Klein theory are given in an explicit form. Various classical quantities associated with the black holes are derived. In particular, one finds the very surprising result that the gyromagnetic and gyroelectric ratios can become arbitrarily large. The thermodynamic quantities of the black holes are calculated and a Smarr-type formula is obtained leading to a generalized first law of black hole thermodynamics. The properties of the extreme solutions are investigated and it is shown how they naturally separate into two classes. The extreme solutions in one class are found to have two unusual properties: (i). Their event horizons have zero angular velocity and yet they have non-zero ADM angular momentum. (ii). In certain circumstances it is possible to add angular momentum to these extreme solutions without changing the mass or charges and yet still maintain an extreme solution. Regarding the extreme black holes as elementary particles, their stability is discussed and it is found that they are stable provided they have sufficient angular momentum.

*dar17@amtp.cam.ac.uk
1 Introduction

In this paper we will investigate the rotating dyonic black holes of 5 dimensional Kaluza-Klein theory. Although the original 5 dimensional theory as it stands is not a realistic theory of nature, it continues to give insight into more sophisticated theories such as string theory and supergravity. The most elegant feature of Kaluza-Klein theory is the way in which the process of dimensional reduction leads naturally to electromagnetism coupled to 4 dimensional gravity without the need for the introduction of a source term on the right hand side of Einstein’s equations. All that is needed is the assumption of an extra fifth dimension which is assumed to be curled up to form a circle whose radius $R_{kk}$ is too small to be observed. The existence of extra spacetime dimensions has become an integral part of many theories in modern theoretical physics such as string theory.

Kaluza-Klein theory arises naturally in string theory and some of the Kaluza-Klein monopoles have been shown to correspond to exact solutions in string theory $[1]$. The monopoles may also be regarded as solutions of the $N = 8$ supersymmetric theory in 5 dimensions and they fit the same supermultiplets as the original fields of the $N = 8$ theory $[2]$. Kaluza-Klein theory has also been of interest recently in connection with noncommutative differential geometry $[3]$ which may be viewed as Kaluza-Klein theory in which the extra fifth dimension is taken to be a discrete set of points rather than a continuum.

Pure gravity in 4 dimensions admits a 2 parameter family of stationary, axi-symmetric black hole solutions, the Kerr solutions. The 2 parameters may be chosen to be the mass $M$ and the angular momentum $J$ and the condition $M^2 \geq |J|$ ensures cosmic censorship. When coupled to a single $U(1)$ Maxwell field the solutions generalize to the Kerr-Newman family described by an additional 2 parameters, the electric and magnetic charges $Q$ and $P$. The condition for the singularity to be hidden behind an event horizon then becomes $M^2 \geq Q^2 + P^2 + J^2/M^2$. These are the most general axi-symmetric black hole solutions of Einstein-Maxwell theory. This theory, however, requires the addition of a source term on the right hand side of the Einstein equations. The theory developed by Kaluza $[4]$ and Klein $[5]$ provides a way of unifying 4 dimensional gravity with electromagnetism without the need for such a source term. In this theory spacetime is considered as 5 dimensional and dimensional reduction of the 5 dimensional vacuum Einstein equations
then leads to 4 dimensional gravity coupled to a U(1) Maxwell field and a scalar dilaton field.

The strength of the coupling of the dilaton is fixed by the dimensional reduction process. It is also of interest, particularly in string theory, to consider more general dilaton couplings. The simplest extension of Einstein-Maxwell theory coupled to a scalar dilaton field $\sigma$ with coupling constant $b$ is described by the action

$$S = \int d^4 x \sqrt{g} \left[ R - 2(\partial \sigma)^2 - e^{2b\sigma} F^2 \right]$$

which leads to the following equations of motion

$$R_{\mu\nu} = 2(\partial \sigma_{\mu})(\partial \sigma_{\nu}) + 2e^{2b\sigma} T_{\mu\nu}$$

$$\nabla_{\mu} (e^{2b\sigma} F_{\mu\nu}) = 0$$

$$\Box \sigma = \frac{b^2}{2} e^{2b\sigma} F^2$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the Maxwell field

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F^2.$$ (1.3)

When $b = 0$ this reduces to Einstein-Maxwell theory. For $b \neq 0$ it is possible to consistently set $\sigma \equiv 0$ in (1.2) only when $F^2 = 0$. This is the Einstein-Maxwell embedding and includes the $Q = P$ Reissner-Nordström solutions but not, in general, the $Q = P$ Kerr-Newman solutions.

The $b = 1$ case of (1.1) arises naturally in string theory and the $b = \sqrt{3}$ case is the one given by Kaluza-Klein theory which will be the main subject of this paper. Less is known about other values of the dilaton coupling. The black hole solutions of this theory will have an additional charge $\Sigma$, the scalar dilaton charge given by

$$\sigma \sim \frac{\Sigma}{r} \quad \text{as} \quad r \to \infty.$$ (1.4)

In general this leads to the event horizon becoming singular unless $\Sigma$ takes a specific value determined by the other charges. Thus the stationary black
hole solutions can still be labelled by the 4 parameters $M, P, Q$ and $J$. The electrically charged static solutions for general $b$ are known [3], [7]. These have been generalized to slowly rotating solutions by expanding (1.2) linearly in angular momentum [8].

The aim of this paper is to find the most general axi-symmetric black hole solutions of the $b = \sqrt{3}$ Kaluza-Klein theory so that they may be compared and contrasted with those of Einstein-Maxwell theory and string theory.

2 Kaluza-Klein theory

The vacuum Einstein equations in 5 dimensions can be derived from the action

$$S = \int d^5x \sqrt{g^{(5)}} R.$$  \hspace{1cm} (2.1)

The extra coordinate $x^5$ is assumed to be periodic with period $2\pi R_{KK}$ and in addition $\frac{\partial}{\partial x^5}$ is assumed to be killing so that the 5 dimensional metric components are functions of $x^\mu (\mu = 0 \ldots 3)$ only. The 5 dimensional metric can be written in the form

$$ds^{2(5)} = e^{\lambda \sigma / \sqrt{3}} (dx^5 + 2 A_\mu dx^\mu)^2 + e^{-\lambda \sigma / \sqrt{3}} g_{\mu \nu} dx^\mu dx^\nu$$  \hspace{1cm} (2.2)

and the action (2.1) then reduces to

$$S = 2\pi R_{KK} \int d^4x \sqrt{g} [R - 2(\partial \sigma)^2 - e^{2\lambda \sigma / \sqrt{3}} F^2]$$  \hspace{1cm} (2.3)

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is equivalent to the $b = \sqrt{3}$ case of (1.1).

When $\xi = \frac{\partial}{\partial t}$ is also killing the 5 dimensional metric can be further decomposed as

$$ds^{2(5)} = \lambda_{ab} (dx^a + \omega^a_i dx^i) (dx^b + \omega^b_j dx^j) + \frac{1}{\tau} h_{ij} dx^i dx^j$$  \hspace{1cm} (2.4)

where $\tau = -\det \lambda_{ab}$. $a, b, \ldots$ take the values 0 and 5, and $i, j, \ldots$ run from 1 to 3. $\lambda_{ab}, \omega^a_i$ and $h_{ij}$ are functions of the 3 spatial coordinates $x^i$. The $R_{ai}$ components of the vacuum Einstein equations imply that the $\omega^a_i$ can be expressed in terms of twist potentials $V_a$ satisfying

$$V_{a,i} = \tau \lambda_{ab} \varepsilon_{j}^{ik} \omega_{j,k}^b$$  \hspace{1cm} (2.5)
where $\varepsilon_{ijk}$ is the completely antisymmetric tensor of the 3 dimensional metric $h_{ij}$. Then, defining the symmetric, unimodular matrix

$$\chi = \begin{pmatrix} \lambda_{ab} - \frac{1}{\tau} V_a V_b & \frac{1}{\tau} V_a \\ \frac{1}{\tau} V_b & -\frac{1}{\tau} \end{pmatrix}$$

(2.6)

the remaining Einstein equations can be written as [9]

$$(\chi^{-1} \chi^{-i})_{;i} = 0$$

(2.7)

$$R_{ij} = \frac{1}{4} \text{Tr} \left( \chi^{-1} \chi^{-i} \chi^{-1} \chi^{-j} \right)$$

where $; \; \text{denotes the covariant derivative with respect to } h_{ij}$. These equations can be derived from the 3 dimensional $\sigma$-model action

$$S = \int d^3 x \sqrt{h} \left[ (3) R - \frac{1}{4} \text{Tr} \left( \chi^{-1} \chi^{-i} \chi^{-1} \chi^{-j} \right) \right].$$

(2.8)

Clearly equations (2.7) are invariant under $\text{SL}(3,\mathbb{R})$ transformations. Since $\chi$ is a symmetric matrix the most natural group action to consider is

$$\chi \mapsto N \chi N^T \quad N \in \text{SL}(3,\mathbb{R}).$$

(2.9)

In order for $\chi$ to represent an asymptotically Minkowskian spacetime it is necessary that

$$\chi \rightarrow \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{as} \; r \rightarrow \infty$$

(2.10)

and only the subgroup $\text{SO}(1,2)$ of $\text{SL}(3,\mathbb{R})$ transformations preserves this property, i.e. those $N$ satisfying $N^{-1} = \eta N^T \eta$. It is thus possible to transform continuously through the space of solutions obtaining new black hole solutions from old ones by applying $\text{SO}(1,2)$ transformations to $\chi$.

The only complication is that matrices $\chi$ of the form (2.10) may not represent asymptotically Minkowskian spacetimes (in either 4 or 5 dimensions). Indeed the magnetically charged solutions we seek will not be asymptotically Minkowskian from the 5 dimensional point of view, however some
SO(1,2) transformations will lead to solutions that are not even asymptotically Minkowskian in 4 dimensions. These solutions are Taub-NUT like solutions in 4 dimensions. In order to generate 4 dimensional black hole solutions it will be necessary to further restrict the transformations that can be applied to $\chi$. The restricted transformations, which no longer form a group, will be labelled by the 2 parameters $\alpha, \beta \in \mathbb{R}$.

3 Static Solutions

The simplest solutions of (2.7) take the form

$$\chi = \eta e^{A f(x^i)}$$

(3.1)

where $A$ is a constant matrix and $f$ is required to be a harmonic function of the spatial coordinates $x^i$ with respect to the 3 dimensional metric $h_{ij}$. These are geodesics in the symmetric space $SL(3,\mathbb{R})/SO(3)$ with metric $dS^2 = \text{Tr} (\chi^{-1} d\chi \chi^{-1} d\chi)$ [10]. The requirement that $\chi$ be symmetric and unimodular imposes 2 constraints on the matrix $A$:

$$\text{Tr} A = 0$$

(3.2)

$$A^T = \eta A \eta.$$ 

The most general matrix satisfying these constraints may be written in the suggestive form

$$A = \begin{pmatrix} -2M - 2\Sigma/\sqrt{3} & -2Q & 2N \\ 2Q & 4\Sigma/\sqrt{3} & 2P \\ 2N & -2P & 2M - 2\Sigma/\sqrt{3} \end{pmatrix}.$$  

(3.3)

If $f \sim \frac{1}{r}$, $N$ will be related to the NUT charge of the 4 dimensional spacetime. We will take $N$ to be zero from now on. In that case $M$ is the ADM mass of the 4 dimensional spacetime and $P$, $Q$ and $\Sigma$ are the magnetic, electric and scalar charges respectively.
Null geodesics in the symmetric space are given by the further constraint

\[ \text{Tr} \left( A^2 \right) = 0 \]  \hspace{1cm} (3.4)

which is equivalent to Scherk’s antigravity condition \cite{11}, \cite{12}

\[ M^2 + \Sigma^2 = P^2 + Q^2. \]  \hspace{1cm} (3.5)

which ensures a force balance between monopoles allowing for the possible existence of static multi-centre solutions. These are extreme black holes whose 3 dimensional metric \( h_{ij} \) is Ricci flat but which are only flat in the 3 cases described below – the extreme electric (plane wave) solution, the Gross-Perry-Sorkin magnetic monopole and the extreme \( P = Q \) Reissner-Nordström embedding.

The SO(1,2) transformation of \( \chi \)

\[ \chi \rightarrow N \chi N^T, \quad N^{-1} = \eta N^T \eta \]  \hspace{1cm} (3.6)

corresponds to a similarity transformation of the matrix \( A \)

\[ A \rightarrow M A M^{-1}, \quad M = \eta N \eta \in \text{SO}(1,2). \]  \hspace{1cm} (3.7)

There are thus 2 natural classes of solutions of the form (3.1) to be considered, depending on whether \( A \) is singular or non-singular. Consider first the case of non-singular \( A \). All matrices in this class are similar to the traceless diagonal matrix

\[
A = \begin{pmatrix}
-2M - 2\Sigma/\sqrt{3} \\
4\Sigma/\sqrt{3} \\
2M - 2\Sigma/\sqrt{3}
\end{pmatrix}.
\]  \hspace{1cm} (3.8)

Exponentiating this matrix and solving for the metric components using (2.2), (2.6), (2.7) and (3.1) gives the 5 dimensional metric. The 4 dimensional metric, extracted using (2.2), is

\[ ds^2(4) = - \left( 1 - \frac{2\tilde{M}}{r} \right)^{M/\tilde{M}} dt^2 + \left( 1 - \frac{2\tilde{M}}{r} \right)^{-M/\tilde{M}} \left[ dr^2 + r^2 \left( 1 - \frac{2\tilde{M}}{r} \right) d\Omega^2 \right], \]  \hspace{1cm} (3.9)

\[ \tilde{M} = \sqrt{M^2 + \Sigma^2}. \]
When $\Sigma = 0$ this is just the Schwarzschild solution and for $\Sigma \neq 0$ it is asymptotically like the Schwarzschild solution. However, when $\Sigma \neq 0$, the horizon at $r = 2\sqrt{M^2 + \Sigma^2}$ becomes singular and it is expected that all solutions in this class with $\det A \neq 0$ will be singular in this way.

Now consider those matrices of the form (3.3) which are singular. The condition $\det A = 0$ gives a cubic equation for $\Sigma$ in terms of the mass $M$ and the other charges $P$ and $Q$:

$$\frac{Q^2}{\Sigma + M\sqrt{3}} + \frac{P^2}{\Sigma - M\sqrt{3}} = \frac{2\Sigma}{3}.$$  

(3.10)

The general spherically symmetric solution in this class depending on one harmonic function $f$ may be obtained from the Schwarzschild solution by applying a restricted 2 parameter family of SO(1,2) transformations that do not introduce a NUT charge. The Schwarzschild solution is represented by the matrix

$$A = \begin{pmatrix} -2M & 0 \\ 0 & 2M \end{pmatrix}. \quad (3.11)$$

The solutions obtained from this correspond to the static, spherically symmetric dyonic solutions obtained in [13]–[16] and which were thoroughly investigated in [17]. These solutions are a special case of the more general rotating solutions which will be given in the next section and so we will postpone any detailed discussion of them and their derivation until then. A few important special cases are worth mentioning:

1. The 5 dimensional plane wave solution

$$ds^2 = \left(1 + \frac{4M}{r}\right)(dx^5)^2 + 2dx^5 dt + dx^i dx^i\quad (3.12)$$

which represents the extreme electric Kaluza-Klein monopole with $Q = 2M, P = 0$ and $\Sigma = M\sqrt{3}$ has

$$A = \begin{pmatrix} -4M & -4M & 0 \\ 4M & 4M & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad h_{ij} = \delta_{ij}, \quad f = \frac{1}{r}. \quad (3.13)$$
The Gross-Perry-Sorkin magnetic monopole \cite{18}, \cite{19}

\[ ds^2 = -dt^2 + \frac{1}{1 + \frac{4M}{r}} \left( dx^5 - 4M \cos \theta d\phi \right)^2 + \left( 1 + \frac{4M}{r} \right) dx \cdot dx \]  \hspace{1cm} (3.14)

with \( P = 2M, \ Q = 0, \ \Sigma = -M\sqrt{3} \),

\[ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4M & 4M \\ 0 & -4M & 4M \end{pmatrix}, \ h_{ij} = \delta_{ij} \ \text{and} \ f = \frac{1}{r}. \]  \hspace{1cm} (3.15)

This is the electromagnetic dual of the previous solution where the discrete electromagnetic duality transformation

\[ e^{2\sigma \sqrt{3}} F_{\mu\nu} \rightarrow ^* F_{\mu\nu}, \ \sigma \rightarrow -\sigma, \ g_{\mu\nu} \rightarrow g_{\mu\nu} \]  \hspace{1cm} (3.16)

is a symmetry of (1.2) and, in the non-rotating case, is equivalent to exchanging \( P \) and \( Q \), and changing the sign of \( \Sigma \).

(3). The extreme Reissner-Nordström embedding

\[ ds^2 = \left( dx^5 - M\sqrt{2} \cos \theta d\phi + \frac{M\sqrt{2}}{r} dt \right)^2 - \left( 1 - \frac{m}{r} \right)^2 dt^2 \]  \hspace{1cm} \hspace{1cm} + \left( 1 - \frac{m}{r} \right)^{-2} dr^2 + r^2 d\Omega^2 \]  \hspace{1cm} (3.17)

with \( P = Q = M/\sqrt{2}, \ \Sigma = 0 \),

\[ A = \begin{pmatrix} -2M & -M\sqrt{2} & 0 \\ M\sqrt{2} & 0 & M\sqrt{2} \\ 0 & -M\sqrt{2} & 2M \end{pmatrix}, \ h_{ij} = \delta_{ij} \ \text{and} \ f = \frac{1}{r}. \]  \hspace{1cm} (3.18)

These 3 solutions all have a flat 3 dimensional metric \( h_{ij} \) and so are easy to generalize to multi-centre solutions, simply by replacing \( f \) by

\[ f = \sum_{i=1}^{n} \frac{\lambda_i}{|x - x_i|} \]  \hspace{1cm} (3.19)
which gives \( n \) monopoles of masses \( \lambda_i M \) at \( x = x_i \). By explicitly solving equations (2.5) and (2.7) it can be shown that these 3 solutions are the only spherically symmetric solutions with flat spatial metric \( h_{ij} \) and so these will be the only multi-centre solutions of this form.

An important special case of an SO(1,2) transformation that can be applied to the matrix \( A \) is

\[
M = \begin{pmatrix}
\cosh \alpha & \sinh \alpha & 0 \\
\sinh \alpha & \cosh \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(3.20)

which corresponds to applying the Lorentz boost

\[
\begin{align*}
x^5 & \rightarrow \gamma (x^5 + vt) \\
t & \rightarrow \gamma (t + vx^5)
\end{align*}
\]

where \( \gamma = \left( 1 - v^2 \right)^{-\frac{1}{2}} = \cosh \alpha \).  

(3.21)

This gives a particularly easy way of obtaining the purely electrically charged solutions from neutral ones and it generalizes easily to the rotating case \([17]\), \([8]\) and \([20]\). The extreme electric Kaluza-Klein solution (3.12) above corresponds to the limit \( v \rightarrow 1 \) and thus represents a Schwarzschild black hole moving at the speed of light in 5 dimensions.

The extreme solutions satisfy both (3.3) and (3.10) which, on eliminating \( \Sigma \) is equivalent to the astroid

\[
\left( \frac{Q}{M} \right)^{\frac{2}{3}} + \left( \frac{P}{M} \right)^{\frac{2}{3}} = \frac{2}{3}.
\]  

(3.22)

This is to be compared with the curve of spherically symmetric extreme solutions in Einstein-Maxwell theory \((b = 0)\)

\[
\left( \frac{Q}{M} \right)^2 + \left( \frac{P}{M} \right)^2 = 1
\]  

(3.23)

and in string theory \((b = 1)\)

\[
\left| \frac{Q}{M} \right| + \left| \frac{P}{M} \right| = \sqrt{2}.
\]  

(3.24)

So these different values of dilaton coupling \( b \) fit a family of power law curves of the form

\[
\left| \frac{Q}{M} \right|^n + \left| \frac{P}{M} \right|^n = K^n.
\]  

(3.25)
In the pure electric case, the extreme solution saturates the Bogomol’nyi bound \[21\]
\[
|Q|/M = \sqrt{1 + b^2}
\]
and so \(K = \sqrt{1 + b^2}\). The \(P = Q = M/\sqrt{2}\) Reissner-Nordström solution is an extreme solution for all values of the dilaton coupling \(b\) and this fixes \(n\) as a function of \(b\). So the curves of extreme solutions for different values of \(b\) can be summarized as

\[
\begin{align*}
|Q_M| &+ |P_M|^n = (1 + b^2)^{\frac{2}{b}} \\
n &\equiv \frac{2}{1 + \log_2(1 + b^2)}
\end{align*}
\]

(3.27)

which is a family of power law curves touching at one point, see Fig. 1. The formulae (3.27) are certainly true for \(b = 0, 1, \sqrt{3}\) and they may also be true more generally.
4 Rotating Solutions

A class of rotating solutions may be represented by totally geodesic surfaces in the symmetric space depending on 2 harmonic functions \( f(x^i) \) and \( g(x^i) \)

\[
\chi = \eta e^{A f} e^{B g}
\]

(4.1)

where \( A \) and \( B \) are constant, commuting matrices, both satisfying (3.2). This turns out not to be a convenient representation of the solutions, however. It is possible to generate all the rotating solutions from the Kerr solution by acting on the matrix \( \chi \) corresponding to the Kerr solution with a restricted set of SO(1,2) transformations of the form (3.6). The general rotating solution is expected to have a \( \chi \)-matrix which is asymptotically of the form

\[
\chi \sim \begin{pmatrix}
-1 + \frac{2M + 2\Sigma/\sqrt{3}}{r} & \frac{2Q}{r} & -\frac{2J \cos \theta}{r^2} \\
\frac{2Q}{r^2} & 1 + \frac{4\Sigma}{r\sqrt{3}} & \frac{2P}{r} \\
-\frac{2J \cos \theta}{r} & \frac{2P}{r} & -1 - \frac{2M - 2\Sigma/\sqrt{3}}{r}
\end{pmatrix}.
\]

(4.2)

Note that the (1,3) component is required to be \( O(\frac{1}{r}) \) in order that the 4 dimensional metric has no NUT charge and is asymptotically flat. This condition is not preserved by general SO(1,2) transformations (3.6) thus close attention must be payed to precisely which transformations can be applied to the Kerr solution to obtain other black hole solutions.

The Kerr solution of mass \( M_K \) and angular momentum \( J_K = aM_K \) is

\[
ds^2_{(5)} = \left( dx^5 \right)^2 - (1 - Z) \left( dt + \frac{aZ \sin^2 \theta}{1 - Z} d\phi \right)^2 + \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 + \frac{\Delta}{1 - Z} \sin^2 \theta d\phi^2
\]

where

\[
\left\{ \begin{aligned}
\rho &= r^2 + a^2 \cos^2 \theta \\
\Delta &= r^2 - 2M_K r + a^2 \\
Z &= \frac{2M_K r}{\rho}.
\end{aligned} \right.
\]

(4.3)

Therefore
\[ \lambda_{ab} = \begin{pmatrix} (1 - Z) & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau = 1 - Z \] 

\[ \omega^0 dx = \frac{aZ \sin^2 \theta}{1 - Z} d\phi, \quad \omega^5 = 0. \]

The twist potentials are then

\[ V_0 = \frac{-2M_K a \cos \theta}{\rho}, \quad V_5 = 0 \]

so the \( \chi \)-matrix for the Kerr solution is

\[ \chi_K = \begin{pmatrix} -2(1 - Z) - \frac{4M_K^2 a^2 \cos^2 \theta}{\rho^2 (1 - Z)} & 0 & -\frac{2M_K a \cos \theta}{\rho (1 - Z)} \\ 0 & 1 & 0 \\ -\frac{2M_K a \cos \theta}{\rho (1 - Z)} & 0 & -\frac{1}{1 - Z} \end{pmatrix} \]

We now apply an SO(1,2) transformation to this:

\[ \chi = N \chi_K N^T \]

but restricted to those transformations \( N \) that preserve the asymptotic form \((1.2)\). First we decompose the general SO(1,2) transformation into 2 boosts and a rotation, \( N = N_1 N_2 N_3 \), where

\[ N_1 = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 \\ \sinh \alpha & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ N_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & \sinh \beta \\ 0 & \sinh \beta & \cosh \beta \end{pmatrix}, \]

\[ N_3 = \begin{pmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{pmatrix}. \]
The requirement that $\chi$ have the asymptotic form (4.2) gives the following constraint on the matrix $N$

$$N_{(1,1)}N_{(3,1)} = N_{(1,3)}N_{(3,3)} \quad (4.9)$$

which is equivalent to

$$\tan 2\gamma = \tanh \alpha \sinh \beta. \quad (4.10)$$

Thus the allowed SO(1,2) transformations can be parametrized by the 2 boost parameters $\alpha, \beta \in \mathbb{R}$. Note that these restrictions (4.9), (4.10) giving the allowed transformation matrices $N$ are specific to the Kerr solution. A matrix $N$ applied to a different solution would have a different restriction on it. This is equivalent to the statement that the restricted set of transformations that do not introduce a NUT charge in the 4 dimensional metric do not form a group.

The transformed matrix $\chi$ can now be calculated in terms of $\alpha$, $\beta$ and $\chi_K$, and the 5 dimensional metric can be reconstructed from this. Some lengthy algebra gives the result

$$ds^2_{(5)} = \frac{B}{A} \left(dx^5 + 2A_\mu dx^\mu\right)^2 + \sqrt{\frac{A}{B}} ds^2_{(4)} \quad (4.11)$$

where

$$ds^2_{(4)} = -\frac{f^2}{\sqrt{AB}} \left(dt + \omega_\phi^0 d\phi\right)^2 + \frac{\sqrt{AB}}{\Delta} dr^2 + \sqrt{AB} d\theta^2 + \frac{\Delta \sqrt{AB}}{f^2} \sin^2 \theta d\phi^2 \quad (4.12)$$

and

$$A = \left(r - \Sigma/\sqrt{3}\right)^2 - \frac{2P^2 \Sigma}{\Sigma - M \sqrt{3}} + a^2 \cos^2 \theta + \frac{2JPQ \cos \theta}{(M + \Sigma/\sqrt{3})^2 - Q^2},$$

$$B = \left(r + \Sigma/\sqrt{3}\right)^2 - \frac{2Q^2 \Sigma}{\Sigma + M \sqrt{3}} + a^2 \cos^2 \theta - \frac{2JPQ \cos \theta}{(M - \Sigma/\sqrt{3})^2 - P^2}, \quad (4.13)$$

$$\omega_\phi^0 = \frac{2J \sin^2 \theta}{f^2} \left[r - M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2)(M + \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2}\right].$$
Here the radial coordinate has been translated

\[ r \mapsto r + M_K - M \quad (4.14) \]

so that now

\[ \Delta = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2 \quad (4.15) \]

and

\[ f^2 = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2 \cos^2 \theta. \quad (4.16) \]

The electromagnetic vector potential is given by

\[ 2A_\mu dx^\mu = \frac{C}{B} dt + \left( \omega^5_\phi + \frac{C}{B} \omega^0_\phi \right) d\phi \quad (4.17) \]

where

\[ C = 2Q \left( r - \Sigma/\sqrt{3} \right) - \frac{2PJ \cos \theta \left( M + \Sigma/\sqrt{3} \right)}{(M - \Sigma/\sqrt{3})^2 - P^2} . \quad (4.18) \]

\[ \omega^5_\phi = \frac{2P\Delta}{f^2} \cos \theta - \frac{2QJ \sin^2 \theta \left[ r \left( M - \Sigma/\sqrt{3} \right) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2 \right]}{f^2 \left[ (M + \Sigma/\sqrt{3})^2 - Q^2 \right]} . \]

The new mass \( M \), electric charge \( Q \), magnetic charge \( P \), new angular momentum \( J \) and dilaton charge \( \Sigma \) are related to the old Kerr parameters \( M_K, J_K \) and the boost parameters \( \alpha, \beta \) by

\[ M = \frac{M_K \left( 1 + \cosh^2 \alpha \cosh^2 \beta \right) \cosh \alpha}{2\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} , \]

\[ \Sigma = \frac{\sqrt{3}M_K \cosh \alpha \left( 1 - \cosh^2 \beta + \sinh^2 \alpha \cosh^2 \beta \right)}{2\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} , \quad (4.19) \]

\[ Q = M_K \sinh \alpha \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta} , \quad P = \frac{M_K \sinh \beta \cosh \beta}{\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} , \]

\[ J = aM_K \cosh \beta \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta} . \]
It is easy to check that $\Sigma$ does indeed satisfy the cubic equation
\[
\frac{Q^2}{\Sigma + M\sqrt{3}} + \frac{P^2}{\Sigma - M\sqrt{3}} = \frac{2\Sigma}{3}
\] (4.20)
and the Kerr mass $M_K$ is related to the parameters of the new solution by
\[
M^2_K = M^2 + \Sigma^2 - P^2 - Q^2.
\] (4.21)

$J$ and $a$ are not independent parameters but are related via
\[
J^2 = a^2 \left[ \left( M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2 \right] \left[ \left( M - \frac{\Sigma}{\sqrt{3}} \right)^2 - P^2 \right] \left( \frac{M^2 + \Sigma^2 - P^2 - Q^2}{M^2 + \Sigma^2 - P^2 - Q^2} \right)
\] (4.22)

This is the general, rotating, dyonic black hole solution. It depends on 4 parameters, $M$, $P$, $Q$ and $J$, although for some purposes it is more convenient to use the parameters $M_K$, $a$, $\alpha$ and $\beta$.

5 Extreme Solutions

It is interesting to study the conditions under which (4.12) satisfies cosmic censorship. It turns out that the surface of extreme solutions is no longer a smooth surface as it is in Einstein-Maxwell theory, where it is a sphere. Instead it is made up of 2 distinct smooth surfaces $S$ and $W$ which intersect at a curve, see Fig. 2.

The horizons of (4.12) are given by the zeros of $\Delta$ and so a necessary condition for them to be present is
\[
M^2 \geq P^2 + Q^2 + a^2 - \Sigma^2
\] (5.1)

which is precisely the condition for the original Kerr solution to have horizons, $M_K \geq |a|$. Thus boosting the extremely rotating Kerr solution will give (part of) the surface of extreme rotating dyons. This is surface $S$ in Fig. 2.

This is only part of the surface, however, because when $M^2 + \Sigma^2 = P^2 + Q^2$, $a$ is necessarily zero but (4.22) breaks down and the angular momentum $J$ may be non-zero. This second set of extreme solutions forms a vertical surface ($W$ in Fig. 2) above the astroid of non-rotating extreme solutions. To see
Figure 2: Surfaces of Extreme Solutions in Kaluza-Klein Theory

these solutions, consider the $\beta \to \infty$ limit of the boosted Kerr solutions. In this limit

$$\frac{J}{M^2} = \frac{4a \sinh^3 \alpha}{M_K \cosh^6 \alpha}, \quad \frac{P}{M} = \frac{2}{\cosh^3 \alpha}, \quad \frac{Q}{M} = \frac{2 \sinh^3 \alpha}{\cosh^3 \alpha}. \quad (5.2)$$

Both $a$ and $M_K$ vanish in this limit but their ratio remains less than 1 and so

$$\left( \frac{P}{M} \right)^{\frac{2}{3}} + \left( \frac{Q}{M} \right)^{\frac{2}{3}} = 2^{\frac{2}{3}}, \quad J \leq PQ \quad (5.3)$$

which is the vertical wall $W$ in Fig. 2.

So we have the result that any extreme non-rotating dyon lying on the asteroid (3.22) can be given angular momentum $J \leq PQ$ with $M,P$ and $Q$ remaining fixed and the solution will remain extreme. This is an unexpected result since in Einstein-Maxwell theory adding angular momentum to an extreme solution would make it ultra-extreme. This unusual property of
Kaluza-Klein black holes only occurs when dyonic black holes are considered since the maximum amount of angular momentum that can be added without a naked singularity resulting is \( J = PQ \). It is a subject for further research to discover if this is a general feature of all theories with non-zero dilaton coupling.

To see a specific example of this effect, consider the extreme \( P = Q \) solutions. These fall into 2 classes. First there are those solutions that which have \( P = Q = M/\sqrt{2} \) and \( J \leq PQ = M^2/2 \). These correspond to adding angular momentum to the \( P = Q \) extreme Reissner-Nordström solution (3.17) and they have a particularly simple 4 dimensional metric

\[
\begin{align*}
s^{2}(4) &= -\frac{(1-\frac{M}{r})^{2}}{F} \left( dt - \frac{2M_{a} \sin^{2} \theta}{r} d\phi \right)^{2} + \frac{F}{(1-\frac{M}{r})^{2}} dr^{2} + F r^{2} d\Omega^{2}, \\
F &= \sqrt{1 - \frac{4M^{2} a^{2} \cos^{2} \theta}{r^{4}}}. 
\end{align*}
\tag{5.4}
\]

Note that for \( J \neq 0 \) this is no longer the Einstein-Maxwell embedding since \( F_{\mu\nu} F^{\mu\nu} \) is no longer zero. The event horizon is at \( r = M \) and there is no ergoregion even when \( J \neq 0 \) (this is true of all the solutions on the surface \( W \) in Fig. 2.). The singularity can be found by looking at invariants formed from the Riemann tensor such as \( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) and this tells us that the 4 dimensional metric is everywhere non-singular except at

\[
r = \sqrt{2 \mid J \cos \theta \mid} \tag{5.5}
\]

and this singularity will be hidden safely behind the event horizon provided that \( J \leq M^{2}/2 \). The second class of extreme \( P = Q \) dyons can be obtained by boosting the \( a = M_{K} \ Kerr \) solution as above, subject to the constraint

\[
\sinh \alpha = \tanh \beta. \tag{5.6}
\]

The result is a curve of extreme solutions starting at the \( a = M_{K} \ Kerr \) solution \((\alpha = \beta = 0)\) and extending to the \( P = Q = M/\sqrt{2}, J = M^{2}/2 \) solution \((\sinh \alpha = 1, \beta = \infty)\). Plotting both these classes of \( P = Q \) extreme solutions on one graph gives a piecewise continuous curve, see Fig. 3., which is just the slice \( P = Q \) through the surface in Fig. 2.
6 Classical Properties

It will be useful here to calculate some of the classical quantities associated with the general rotating solution (4.12). First, unlike the non-rotating solutions, (4.12) has electric and magnetic dipole moments which are respectively

\[
D = \frac{P J (M + \Sigma/\sqrt{3})}{(M - \Sigma/\sqrt{3})^2 - P^2} \tag{6.1}
\]

and

\[
\mu = \frac{Q J (M - \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2}. \tag{6.2}
\]

Therefore the gyromagnetic and gyroelectric ratios are

\[
g_M = \frac{2M (M - \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2} \tag{6.3}
\]
Note that the discrete electromagnetic duality transformation (3.16) is equivalent to exchanging $P$ and $Q$ in (4.12) and also changing the signs of $J$ and $\Sigma$. This also exchanges $g_M$ and $g_E$. The purely electric solutions can be obtained by the Lorentz boost (3.21) applied to the Kerr solution which is equivalent to setting $\beta = 0$ in (4.7) and (4.8). This gives a gyromagnetic ratio of $2 - v^2$ in agreement with [17]. The gyroelectric ratio of the dual $Q = 0$ solutions is then also $2 - v^2$ where now $\gamma = (1 - v^2)^{-\frac{1}{2}} = \cosh \beta$. When $P = Q$, $\Sigma = 0$ and the gyromagnetic and gyroelectric ratios are equal:

$$g_M = g_E = \frac{2M^2}{M^2 - Q^2} = \frac{2M^2}{M^2 - P^2}. \tag{6.5}$$

Thus these 2 ratios range from 2 up to 4 in the extreme case. In general it turns out that $g_M$ and $g_E$ are bounded below by 1, with $g_M = 1$ only for the extreme electric (plane wave) solution and $g_E = 1$ only for the Gross-Perry-Sorkin magnetic monopole.

In the more general dyonic case, however, (6.3) and (6.4) show that $g_M$ and $g_E$ are not bounded above at all but can become arbitrarily large. For example, as $P \to 0$ and $Q \to 2M$, $\Sigma \to \sqrt{3}M$ and $g_E \to \infty$. This is to be contrasted with the Kerr-Newman dyonic black holes of Einstein-Maxwell theory which all have $g_M = g_E = 2$. The unusual behaviour of the gyromagnetic and gyroelectric ratios in Kaluza-Klein theory only becomes apparent when rotating dyons are considered. This may also be the case for other values of the dilaton coupling constant $b$.

(4.12) is stationary and axisymmetric, therefore $\xi = \frac{\partial}{\partial t}$ and $m = \frac{\partial}{\partial \phi}$ are Killing vector fields. The outer event horizon, given by the larger root of $\Delta$, $r = r_+$ where

$$r_\pm = M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} \tag{6.6}$$

is a Killing horizon of the Killing vector field

$$k = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi} \tag{6.7}$$
where $\Omega_H$ has the interpretation of the angular velocity of the event horizon. The requirement that $k$ be null on the horizon $H$ gives

$$
\Omega_H = - \frac{1}{\omega^0 \phi} \bigg|_{H} = \frac{a^2}{2J} \left[ r_+ - M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2) \left( M + \Sigma/\sqrt{3} \right)}{(M + \Sigma/\sqrt{3})^2 - Q^2} \right]^{-1}.
$$

(6.8)

In general the solution will possess an ergoregion which will be given by the region between $r = r_+$ and the larger zero of $f^2$:

$$
r = M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2 \cos^2 \theta}.
$$

(6.9)

The extreme solutions lying on the wall $W$ in Fig. 2., however, have $a = 0$ and therefore they will have no ergoregion and zero angular velocity even though they may have non-zero ADM angular momentum. In Einstein-Maxwell theory it is believed that a non-rotating black hole solution (in the sense that $k = \xi$, i.e. $\Omega_H = 0$) must be static and spherically symmetric [23]. The extreme dyons of Kaluza-Klein theory on $W$, however, may be non-rotating in this sense and still be neither static nor spherically symmetric. (5.5) is the simplest example of such a solution. Using (4.22) it can be seen that only those solutions with $(P/M)^4 + (Q/M)^4 = 2^4$ can have this unusual property, since only for these solutions is it possible for $a$ to be zero when $J$ is non-zero.

The surface gravity of the event horizon is

$$
\kappa = \frac{r_+ - M}{\sqrt{AB}} \bigg|_{r=r_+, \theta=0}
$$

(6.10)

and so this vanishes in the extreme limit (on both surfaces $S$ and $W$).

The area of the event horizon given by

$$
A = \int_{\theta=0}^{\pi} d\theta \int_{\phi=0}^{2\pi} d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \bigg|_{r=r_+}
$$

(6.11)

is

$$
A = \frac{8\pi J a}{a} \left[ r_+ - M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2) \left( M + \Sigma/\sqrt{3} \right)}{(M + \Sigma/\sqrt{3})^2 - Q^2} \right].
$$

(6.12)
We define the co-rotating electrostatic potential by

$$\Phi = k \cdot A = A_t + \Omega_H A_\phi.$$  \hspace{1cm} (6.13)

This is a gauge dependant quantity. In the gauge chosen in (4.17) and (4.18) \(\Phi \to 0\) as \(r \to \infty\) and on the horizon \(\Phi = \Phi_H\) where

$$\Phi_H = \left. -\frac{\omega^5_\phi}{2\omega^0_\phi} \right|_H.$$  \hspace{1cm} (6.14)

In the electrically charged, magnetically neutral case this leads to a Smarr-type formula \([22]\)

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q.$$  \hspace{1cm} (6.15)

In the magnetically charged, electrically neutral case one can use electromagnetic duality to define a co-rotating magnetostatic potential \(\Psi\) which may be obtained from \(\Phi\) by exchanging \(P\) and \(Q\) and changing the signs of \(J\) and \(\Sigma\). The Smarr formula will then simply be

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Psi_H P.$$  \hspace{1cm} (6.16)

In the dyonic case it turns out that the mass obeys the obvious generalization of (6.13) and (6.16)

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q + \Psi_H P.$$  \hspace{1cm} (6.17)

Therefore \(\Phi_H Q\) and \(\Psi_H P\) may be interpreted as the contributions to the total energy from the electric and magnetic charges respectively. Now \(M\) is a homogeneous function of degree 1 in the variables \(A^{\frac{1}{2}}, J^{\frac{1}{2}}, Q\) and \(P\) and so using Euler’s theorem (6.17) leads to the generalized first law of black hole thermodynamics

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ + \Psi_H dP.$$  \hspace{1cm} (6.18)

7 Stability

Extreme black holes often behave like elementary particles and so it is interesting to investigate their stability, i.e. whether it is possible for them to
split into smaller black holes. In Einstein-Maxwell theory this is forbidden by energy conservation. However, it has been pointed out \[24\] that in the case of extreme dilatonic black holes with \( b = 1 \) the relation between \( M, P \) and \( Q \) is such that energy conservation no longer forbids splitting of the black holes. It might also be argued that the second law of black hole thermodynamics would prevent such splitting. However, Kallosh et al. \[24\] showed that the entropy \( S \) vanishes for extreme dilaton black holes with \( b = 1 \) and later work \[25\], \[26\] has shown that \( S = 0 \) for all extreme black holes, so the second law does not forbid the splitting of extreme black holes \[27\]. Finally, in classical General Relativity the area law prevents extreme black holes from splitting but in a full quantum theory of gravity the emission of sufficiently small extreme black holes may be allowed. In this section we will only investigate the question of whether it is energetically favourable for extreme black holes to split.

Consider first the non-rotating extreme solutions of some general theory with conserved charges \( Q_1, Q_2, \ldots Q_n \) each having units of mass. The mass will be a homogeneous function of degree 1 of the charges. Hence

\[
\text{(i). } M(x) \geq 0 \quad \text{and} \quad M(x) = 0 \iff x = 0
\]

\[
\text{(ii). } M(\lambda x) = \lambda M(x) \quad \forall \lambda \in \mathbb{R}
\]

where \( x = (Q_1, Q_2, \ldots Q_n) \in \mathbb{R}^n \). A solution specified by \( x \in \mathbb{R}^n \) may be considered to be unstable if it is energetically favourable for it to decay into 2 new solutions labelled by \( x_1, x_2 \in \mathbb{R}^n \) with \( x_1 + x_2 = x \). The condition for all solutions to be stable is therefore

\[
\text{(iii). } M(x_1 + x_2) \leq M(x_1) + M(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n.
\]

The 3 conditions (i), (ii) and (iii) may be recognized as the conditions for \( M \) to define a norm on \( \mathbb{R}^n \), the stability condition being the triangle inequality. A convenient way of identifying stable mass functions is then given by the fact that the unit ball of a norm on \( \mathbb{R}^n \) is a convex subset of \( \mathbb{R}^n \). To see this consider \( x_1, x_2 \) in the unit ball of a norm \( M \) on \( \mathbb{R}^n \), \( M(x_1), M(x_2) \leq 1 \). Then \( M(\lambda x_1 + (1-\lambda)x_2) \leq \lambda M(x_1) + (1-\lambda)M(x_2) \leq 1 \). So \( \lambda x_1 + (1-\lambda)x_2 \) is also in the unit ball. Hence the unit ball is convex.

In the case of Einstein-Maxwell theory coupled to a dilaton field, the unit balls of the mass functions are given by the curves in Fig. 1. Hence
the extreme non-rotating dyons are stable for dilaton couplings $b \leq 1$ and unstable for $b > 1$. Another way of seeing this is to use Minkowski’s inequality which says that

$$M(Q, P) = \frac{1}{\sqrt{1+b^2}} (|Q|^n + |P|^n)^{\frac{1}{n}}$$

satisfies the triangle inequality for $n \geq 1$, i.e. for $b \leq 1$. Since Kaluza-Klein theory has $b = \sqrt{3} > 1$, its extreme non-rotating dyons are unstable.

The rotating dyons with $J \leq PQ$ on the surface $W$ of Fig. 2. have the same relation between $M$, $Q$ and $P$, independent of $J$, so they will be unstable. In the more general case, however, $M$ will be a function of $J$ as well and the requirement for stability becomes

$$M(Q_1 + Q_2, P_1 + P_2, J_1 + J_2) \leq M(Q_1, P_1, J_1) + M(Q_2, P_2, J_2). \quad (7.1)$$

Now $J$ has units of $(\text{mass})^2$ and so $M$ is no longer a homogeneous function and the argument used in the non-rotating case will not work. It is possible, however, to write $J$ uniquely as a function of $M$, $Q$ and $P$ (at least up to a sign) and $J$ will then be a homogeneous function of degree 2. Writing $\mathbf{x} = (M, Q, P)$, the stability condition becomes

$$J(\mathbf{x}_1 + \mathbf{x}_2) \geq J(\mathbf{x}_1) + J(\mathbf{x}_1). \quad (7.2)$$

Note that the inequality has become reversed. Hence if the solutions are \textit{unstable} the unit ball $J(\mathbf{x}) \leq 1$ will be a convex region of $(M, Q, P)$ space. For extreme rotating solutions on the surface $S$ of Fig. 2., the boundary of the unit ball in $(M, Q, P)$ space is given by setting $J = 1$ which gives

$$M = \frac{\cosh \alpha \left(1 + \cosh^2 \alpha \cosh^2 \beta \right)}{2 \sqrt{\cosh \beta \left(1 + \sinh^2 \alpha \cosh^2 \beta \right)}}^{\frac{1}{4}}$$

$$Q = \frac{\sinh \alpha \left(1 + \sinh^2 \alpha \cosh^2 \beta \right)}{\sqrt{\cosh \beta}}^{\frac{1}{4}} \quad (7.3)$$

$$P = \frac{\sinh \beta \sqrt{\cosh \beta}}{\left(1 + \sinh^2 \alpha \cosh^2 \beta \right)^{\frac{1}{4}}}.$$
A plot of this surface shows it to be concave and this is confirmed by the determinant of the matrix of second derivatives

\[
\begin{vmatrix}
\frac{\partial^2 M}{\partial Q^2} & \frac{\partial^2 M}{\partial Q \partial P} \\
\frac{\partial^2 M}{\partial P \partial Q} & \frac{\partial^2 M}{\partial P^2}
\end{vmatrix} = \frac{(1 + \sinh^2 \alpha \cosh^2 \beta)^{\frac{3}{2}}}{2 \cosh^4 \alpha \cosh \beta \left(3 \cosh^2 \alpha \cosh^2 \beta - 1\right)} \tag{7.4}
\]

which is everywhere positive. So the unit ball of \(J(M, Q, P)\) is concave, hence the rotating dyons on the surface \(S\) of Fig. 2. are stable.

### 8 Thermodynamic Quantities

The thermodynamic quantities of (4.12) calculated earlier are more easily expressed in terms of the 4 independent variables \(M_K, a, \alpha, \beta\). The temperature defined by \(T = \frac{\kappa}{2\pi}\) and using (5.10) is

\[
T = \frac{1}{4\pi} \frac{\sqrt{M_K^2 - a^2}}{M_K \cosh \beta \left[M_K \cosh \alpha + \sqrt{M_K^2 - a^2} \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}\right]} \tag{8.1}
\]

which, as expected, vanishes for extreme solutions on \(S\) and \(W\) since \(|a| = M_K\) for these solutions. The entropy \(S = \frac{1}{4}A\) is

\[
S = 2\pi M_K \cosh \beta \left[M_K \cosh \alpha + \sqrt{M_K^2 - a^2} \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}\right]. \tag{8.2}
\]

The angular velocity of the event horizon (5.8) is

\[
\Omega_H = \frac{a}{2M_K \cosh \beta \left[M_K \cosh \alpha + \sqrt{M_K^2 - a^2} \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}\right]} \tag{8.3}
\]

The angular momentum is

\[
J = aM_K \cosh \beta \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}. \tag{8.4}
\]

The electrostatic potential on the horizon (6.14) is

\[
\Phi_H = \frac{\sinh \alpha \left[M_K + \sqrt{M_K^2 - a^2} \cosh \alpha \cosh^2 \beta \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}\right]}{2 \left[M_K \cosh \alpha + \sqrt{M_K^2 - a^2} \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}\right]} \tag{8.5}
\]
The electric charge is

\[ Q = M_K \sinh \alpha \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}. \]  

(8.6)

The magnetostatic potential on the horizon is

\[ \Psi_H = \frac{\sinh \beta \left[ M_K + \cosh \alpha \sqrt{M_K^2 - a^2 \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} \right]}{2 \cosh \beta \left[ M_K \cosh \alpha + \sqrt{M_K^2 - a^2 \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} \right]}, \]  

(8.7)

and the magnetic charge is

\[ P = \frac{M_K \sinh \beta \cosh \beta}{\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}}. \]  

(8.8)

Then the Smarr formula becomes

\[ M = 2TS + 2\Omega_H J + \Phi_H Q + \Psi_H P \]  

(8.9)

and the generalized first law is

\[ dM =TdS + \Omega_H dJ + \Phi_H dQ + \Psi_H dP. \]  

(8.10)

In the previous section it was shown that the only extreme solutions which could be unstable on energetic grounds are those on the surface \( \mathcal{W} \). It was argued in [25] and [26] that the entropy of extreme black holes is zero and so (8.2) does not apply to extreme solutions. Thus the second law does not forbid the splitting of extreme black holes on \( \mathcal{W} \). It turns out that (8.2) actually does give \( S = 0 \) for extreme solutions on \( \mathcal{W} \) (but not on \( \mathcal{S} \)) and so the event horizons of solutions on \( \mathcal{W} \) have zero surface area. Thus the area law does not forbid the splitting of extreme black holes on \( \mathcal{W} \) either and these solutions are expected to be genuinely unstable.

## 9 Conclusions

In this paper we have given an explicit form for the most general dyonic black hole solution of the 5 dimensional Kaluza-Klein theory. The formulae in terms of the physical parameters \( M, P, Q, J \) are fairly complicated and for
many purposes, such as thermodynamics, it is more convenient to use the parameters $M, a, \alpha, \beta$. The solution (4.12) reduces to the known non-rotating solutions [13]–[17] on setting $a = J = 0$. The magnetically neutral case $\beta = P = 0$ is simply the Lorentz boosted Kerr solution [17], [8], [20]. The electrically neutral, magnetically charged solutions $\alpha = Q = 0$ are the electromagnetic duals of the magnetically neutral, electrically charged solutions.

The gyromagnetic and gyroelectric ratios of the black holes in this theory have been calculated (6.3), (6.4) and they were found to be bounded below by 1 but not bounded above. This is a surprising result since the black holes of Einstein-Maxwell theory (the Kerr-Newman solutions) and those of heterotic string theory found by Sen [28] all have gyromagnetic and gyroelectric ratios bounded above by 2. ([28] only gives the gyromagnetic ratio for the electrically charged case. It would be interesting to see what happens to this ratio in the more general dyonic case.)

The surface of extreme solutions was found to be made up of 2 distinct surfaces $S$ and $W$ with $J > PQ$ and $J < PQ$ respectively. The 2 surfaces meet at a line given by $(P/M)^{\frac{2}{3}} + (Q/M)^{\frac{2}{3}} = 2^{\frac{2}{3}}, J = PQ$. Thus the full surface of extreme solutions is no longer a smooth surface, as it is in Einstein-Maxwell theory. The solutions on the 2 surfaces behave completely differently. Those on $S$ (which includes the extreme Kerr solution) show all the normal characteristics of rotating solutions, such as non-zero angular velocity of the event horizon and an ergoregion outside the event horizon. Those extreme solutions on $W$, however, behave more like non-rotating solutions since their event horizons have zero angular velocity and they have no ergoregion whilst they are not spherically symmetric and they have non-zero ADM angular momentum.

The stability of the extreme solutions viewed as elementary particles was investigated. It was found that extreme dyons on the surface $S$ of Fig. 2. with $J \geq PQ$ are stable, whereas those on the surface $W$ with $J < PQ$ are unstable. The solutions on $S$ with $J > PQ$ have $a \neq 0$, non-zero angular velocity and they have ergoregions and so they are expected to lose angular momentum by super-radiance. As the angular momentum decreases, however, and one moves down the surface $S$ towards $W$, $J \to PQ, a \to 0$, the ergoregion vanishes and the super-radiance is turned off. Therefore the extreme solutions never quite lose enough angular momentum to reach $W$ and become unstable, instead they asymptotically approach the $J = PQ$ curve.
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