On the connection problem for the second Painlevé equation with large initial data

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Abstract
We consider two special cases of the connection problem for the second Painlevé equation (PII) using the method of uniform asymptotics proposed by Bassom \textit{et al.}. We give a classification of the real solutions of PII on the negative (positive) real axis with respect to their initial data. By product, a rigorous proof of a property associate with the nonlinear eigenvalue problem of PII on the real axis, recently revealed by Bender and Komijani, is given by deriving the asymptotic behavior of the Stokes multipliers.

Keywords Connection problem · uniform asymptotics · Painlevé II equation · Airy function

Mathematics Subject Classification Primary 33E17; Secondary 34M55 · 41A60

1 Introduction
The Painlevé equations are, in general, irreducible in the sense that their solutions cannot be expressed in terms of elementary functions or the classical special functions. Therefore, the study on the asymptotic behavior of the Painlevé functions and their connection problem is an important topic in the Painlevé theory. In this paper, we focus on the connection problem of the homogenous second Painlevé (PII) equation, \textit{i.e.} the following equation:

\[ \frac{d^2q}{dt^2} = 2q^3 + tq, \]

whose solutions are well studied. By collecting the results in the literatures \cite{5, 8, 10, 17}, we know that, when \( t \to -\infty \), the asymptotic behaviors of all the real solutions of (1.1) are divided into three types:

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(N1) If $|s_1| < 1$, the solutions are oscillating around the negative real axis, and satisfying
\[
q(t) = d(-t)^{-\frac{1}{3}} \cos \left( \frac{2}{3}(-t)^{3} - \frac{3d^2}{4} \ln(-t) + \phi \right) + O \left( (-t)^{-\frac{5}{10}} \right)
\]
as $t \to -\infty$, where
\[
d^2 = -\frac{1}{\pi} \ln(1 - s_1 s_3) \quad \text{and} \quad \phi = -\frac{3}{2} d^2 \ln 2 + \arg \Gamma \left( \frac{1}{2} + i d^2 \right) - \arg s_1;
\]

(N2) If $|s_1| = 1$, the solutions behave like
\[
q(t) = -\epsilon \sqrt{-\frac{t}{2}} + h(-t)^{-\frac{1}{4}} e^{-\frac{2\sqrt{2}}{3} (-t)^{3/2}} \left( 1 + O \left( (-t)^{-\frac{3}{2}} \right) \right)
\]
as $t \to -\infty$, where
\[
h = -\frac{s_2}{2^{1/4} \sqrt{\pi}}, \quad s_1 = i \epsilon;
\]

(N3) If $|s_1| > 1$, the solutions have infinite number of poles on the negative real axis and
\[
q(t) = \frac{\sqrt{-t}}{\sin \left( \frac{2}{3} (-t)^{3/2} + \frac{3}{2} \beta \ln(-t) + \varphi \right)} + O \left( (-t)^{-1/5} \right)
\]
as $t \to -\infty$, where
\[
\beta = \frac{1}{2\pi} \ln(s_1 s_3 - 1) - 1, \quad \varphi = 3 \beta \ln 2 - \arg \Gamma \left( \frac{1}{2} + i \beta \right) - \arg s_1.
\]

When $t \to +\infty$, the asymptotic behaviors of all the real valued PII functions can also be separated into two types:

(P1) If $s_2 \neq 0$, the solutions (a two-parameter family) satisfy
\[
q(t) = \sigma \sqrt{\frac{t}{2}} \cot \left( \frac{\sqrt{2}}{3} t^{3/2} + \frac{3\gamma}{4} \ln t + \chi \right) + O(t^{-1})
\]
as $t \to +\infty$ with $\sigma = \text{sgn}(s_2) = \pm 1$ and
\[
\gamma = \frac{1}{\pi} \ln |s_2|, \quad \chi = \frac{7\gamma}{4} \ln 2 - \frac{1}{2} \arg \Gamma \left( \frac{1}{2} + i \gamma \right) - \frac{1}{2} \arg(1 + s_2 s_3) + \frac{\pi}{2};
\]

(P2) If $s_2 = 0$, the solutions (a one-parameter family) behave like the Airy functions, i.e.
\[
q(t) = \kappa \text{Ai}(t)(1 + O(t^{-3/2}))
\]
as $t \to +\infty$, where
\[
\kappa = -\text{Im} s_1.
\]
In the above formulas, $s_k(k = 1, 2, 3)$ are the Stokes multipliers corresponding to the given solutions. They are to be given later in (1.19). According to the isomonodromy theory, one could build the connection formulas between the parameters in the asymptotic behaviors as $t \to +\infty$ and the ones as $t \to -\infty$. These connection formulas are well studied through the isomonodromy method [11, 19], the Deift-Zhou nonlinear steepest descent method [5, 6] and the uniform asymptotics method [2]. See in [8] for the detailed information of the connection formulas of PII and their applications. Nevertheless, how to connect the asymptotic behaviors showing above in (1.2)–(1.11) with the initial data of PII equation still needs further investigation. Actually, as is well know, it is an open problem proposed by Clarkson in several occasions.

It seems difficult to completely solve Clarkson’s open problem [13]. Nevertheless, it is interesting to consider the following initial value problem of (1.1)

\[
\begin{align*}
\frac{d^2 q}{dt^2} &= 2q^3 + tq, \\
q(0) &= a, \quad q'(0) = b,
\end{align*}
\]

when the initial data $a, b$ are two real constants and at least one of them is large. Set $a = \xi^{1/3}A(\xi)$ and $b = \xi^{3/2}B(\xi)$, where $\xi$ is a large positive real parameter and $A(\xi), B(\xi)$ are both real and bounded. We would consider the connection problem of (1.12) in the following two cases:

\[
(i) \quad A(\xi)^4 - B(\xi)^2 = 1;
(ii) \quad A(\xi)^4 - B(\xi)^2 = -1.
\]

The motivation for these special cases in (1.13) comes from Bender et al. [3, 4], in which the unstable separatrix solutions of PII on the negative real axis are studied numerically and analytically. They conclude that for any fixed initial value, say $q(0) = 0$, there exists a sequence of initial slopes $q'(0) = b_n$ that give rise to separatrix solutions, where the $b_n$’s are called as the nonlinear eigenvalues with the following asymptotic behavior

\[
b_n \sim B_{II} \cdot n^{\frac{2}{3}} = \left[\frac{3\sqrt{2\pi\Gamma(\frac{3}{4})}n}{\Gamma(\frac{1}{4})}\right]^{\frac{2}{3}} \text{ as } n \to \infty
\]

It is derived by studying the eigenvalue problem associated with the $\mathcal{PT}$-symmetric Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{q}^4$. However, part of their argument is based on numerical evidences (see [3 p. 8]). To give a rigorous proof of their results and to build the connection formulas between the initial data and the large negative (positive) $t$ asymptotics, further analysis is needed.

Although more attention of the authors of [3, 4] are paid to the separatrix solutions, their figures indicate an interesting phenomenon of the PII solutions as the initial slope varies. It seems that, when $b_{2n} < b < b_{2n+1}$, the solution with $q(0) = 0, q'(0) = b$ passes
through $n$ simple poles and then oscillates stably around the negative real axis, while for $b_{2n-1} < b < b_{2n}$, the solution with $q(0) = 0, q'(0) = b$ has infinite number of poles; see Figure 1. It is similar for fixed $q'(0)$ and large $q(0)$; see [3, Figures 14 and 15]. In this paper, instead of fixing $q(0)$ or $q'(0)$, we would like to regard $q(0), q'(0)$ as a whole term and only assume that at least one of $q(0), q'(0)$ is large. As we shall see later, this provides more convenience in analyzing the nonlinear eigenvalue problem of PII.

The heuristic numerical results of PII equation by Fornberg and Weideman [9] should also be noted. For example, it can be seen from [9, Figure 5] that the real solutions of PII equation have $n(n = 1, 2, \cdots)$ poles on the positive real axis when their initial data locate on the curves marked $n^+$. Similarly, there exists a sequence of regions marked $n^-$ such that the solutions, whose initial data in these regions, possess $n$ poles on the negative real axis. Hence, a natural question is how to describe these curves and regions theoretically. This motivates us again to study the precise relationship between the initial data of PII solutions and their large negative (positive) $t$ asymptotics.

We consider the initial problem (1.12) in the two special cases stated in (1.13). The main technique is based on the method of uniform asymptotics introduced by Bassom et al. [1], and further applied by [13, 14, 16, 20, 21, 22, 23].

For convenient to state the main results, we first briefly outline some important information in the isomonodromy theory for the PII equation. The reader is referred to [8, 12] for
more details. One of the Lax pairs for \((1.1)\) is the system of linear ordinary equations
\[
\begin{align*}
\frac{\partial \Psi}{\partial \lambda} &= \{-i(4\lambda^2 + t + 2q^2)\sigma_3 + 4\lambda q \sigma_1 - 2q' \sigma_2\} \Psi \\
\frac{\partial \Psi}{\partial t} &= (-i\lambda \sigma_3 + q \sigma_1) \Psi.
\end{align*}
\] (1.15)

Here and below the prime denotes the derivative with respect to \(t\), and
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
are the standard Pauli spin matrices. A direct calculation shows that the compatibility condition \(\Psi_{\lambda t} = \Psi_{t \lambda}\) reduces to (1.1). There exist canonical solutions \(\Psi_k(\lambda, t)\), defined in a neighborhood of the irregular singular point \(\lambda = \infty\) of the first equation of the Lax pair, with the following asymptotics behavior
\[
\Psi_k(\lambda, t) = \left( I + O \left( \frac{1}{\lambda} \right) \right) \exp \left\{ - \left( \frac{4}{3} \lambda^3 + it \lambda \right) \sigma_3 \right\}, \quad \lambda \to \infty, \quad \lambda \in \Omega_k,
\] (1.16)
where the canonical sectors are
\[
\Omega_k = \left\{ \lambda \in \mathbb{C} \left| \frac{(k - 2)\pi}{3} < \arg \lambda < \frac{k\pi}{3} \right. \right\}, \quad k \in \mathbb{Z}.
\] (1.17)

These canonical solutions are related by
\[
\Psi_{k+1}(\lambda, t) = \Psi_k(\lambda, x) S_k, \quad \lambda \in \Omega_k \cap \Omega_{k+1}, \quad k \in \mathbb{Z},
\] (1.18)
where \(S_k\) are Stokes matrix defined by
\[
S_{2j} = \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix}, \quad S_{2j-1} = \begin{pmatrix} 1 & 0 \\ s_{2j-1} & 1 \end{pmatrix}, \quad j \in \mathbb{Z},
\] (1.19)
where \(s_k\) are called Stokes multipliers, and independent of \(\lambda\) and \(t\) according to the isomonodromy condition. The Stokes multipliers are subject to the constraints
\[
s_{k+3} = -s_k, \quad k \in \mathbb{Z}, \quad \text{and} \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, s_1 = \overline{s_3}, s_2 = \overline{s_2}.
\] (1.20)

\section{Main results}

When \(A(\xi)^4 - B(\xi)^2 = 1\) \(i.e.\) \(a^4 - b^2 = \xi^2 \to +\infty\), we have the following result.
Theorem 1. Let \( q(t; a, b) \) be any solution of (1.12). There exists a sequence of curves 
\[
\Gamma_n : \quad a^4 - b^2 = f_n(a, b), \quad n = 1, 2, \cdots
\]
on the \((a, b)\) plane such that \( q(t; a, b) \) belongs to type (P2) solutions of PII when \((a, b)\) lies on these curves, where
\[
\kappa := \kappa_n = \frac{(-1)^n e^{2\xi E_1 - 2E_2 + o(1)}}{A(\xi)}
\]
as \( \xi \to +\infty \). As \( n \to \infty \), we have
\[
f_n(a, b) = \left[ \frac{n\pi - \frac{\pi}{2} + 2E_3 + o(1)}{2E_1} \right]^{\frac{4}{3}}
\]
Moreover if \( f_n(a, b) < a^4 - b^2 < f_{n+1}(a, b) \), then \( q(t; a, b) \) belongs to type (P1) with \( \sigma = \text{sgn}((-1)^{n+1}a) \) and
\[
\begin{align*}
\gamma &= \frac{1}{\pi} \left[-2\xi E_1 + 2E_2 + \ln |2A(\xi) \cos [2\xi E_1 - 2E_3 + o(1)]| + o(1)\right], \\
\chi &= \frac{7\gamma}{4} \ln 2 - \frac{1}{2} \arg \Gamma \left( \frac{1}{2} + i\gamma \right) - 2\xi E_1 + 2E_3 + o(1)
\end{align*}
\]
as \( \xi \to +\infty \). In the above formulas (2.2) and (2.3), \( E_1, E_2 \) and \( E_3 \) are defined by
\[
\begin{align*}
E_1 &= \frac{1}{6} B \left( \frac{1}{2}, \frac{1}{4} \right), \\
E_2 &= \frac{1}{16} \frac{B(\xi)}{A(\xi)} B \left( \frac{1}{4}, \frac{1}{2} \right) + \Re F(\xi), \\
E_3 &= \frac{1}{16} \frac{B(\xi)}{A(\xi)} B \left( \frac{1}{4}, \frac{1}{2} \right) + \frac{\pi}{8} + \Im F(\xi),
\end{align*}
\]
where \( B(\cdot, \cdot) \) is the beta function and
\[
F(\xi) := \frac{2A(\xi)^2 - (B(\xi)/A(\xi))^2}{8} \int_{\frac{1}{2\xi}}^{\infty} \frac{1}{s + \frac{B(\xi)}{2A(\xi)} \sqrt{s^4 + \frac{1}{4}}} ds.
\]
Specially, in the case that the initial slope \( y'(0) = b \) is fixed, noting that \( \xi \) is a large parameter and \( A(\xi)^4 - B(\xi)^2 = 1 \), we find that \( B(\xi) \to 0 \) and \( A(\xi) \to \pm 1 \). Moreover, a simple calculation yields \( F(\xi) = \frac{\pi}{8} + o(1) \) when \( B(\xi) \to 0 \). Hence, the following corollary is a direct consequence of Theorem 1 and is consistent with the corresponding result mentioned in [11, p11].

Corollary 1. For any fixed \( b \), there exists \( 0 < a_1 < a_2 < \cdots < a_n < \cdots \) such that \( y(t; a_n, b) \) belong to type (P2) solutions of PII, and
\[
a_n^4 - b^2 = \left[ \frac{n\pi + o(1)}{2E_1} \right]^{\frac{4}{3}} \quad \text{as} \quad n \to \infty.
\]
When $A(\xi)^4 - B(\xi)^2 = -1$ i.e. $a^4 - b^2 = -\xi^4 \to -\infty$, similar result are obtained.

**Theorem 2.** Let $q(t; a, b)$ be any solution of (1.12). There exists a sequence of curves

$$\Sigma_n : a^4 - b^2 = -g_n(a, b), \quad n = 1, 2, \ldots$$

on the $(a, b)$ plane such that $q(t; a, b)$ belongs to type (N2) solutions of PII equation when $(a, b)$ lies on these curves, where $\epsilon = \text{sgn}((-1)^n b)$ and

$$h := h_n = -\frac{1}{2^{7/4}\sqrt{\pi}} B(\xi) e^{2\sqrt{2}E_1 \xi - 2F_1} + o(1)$$

as $\xi \to +\infty$. As $n \to \infty$, we have

$$g_n(a, b) = \left[ \frac{n\pi - \pi + \text{sgn}(b) \cdot \frac{\pi}{2} - 2F_2 + o(1)}{2\sqrt{2}E_1} \right]^{\frac{2}{3}}.$$

Moreover,
(i) if \( g_{2n-1}(a,b) < -a^4 + b^2 < g_{2n}(a,b) \), then \( q(t;a,b) \) belongs to type (N1) with
\[
d^2 = -\frac{1}{\pi} \ln \left( \frac{2\cos(\sqrt{2}\xi E_1 + 2F_2)}{B(\xi)} \right) + \frac{2\sqrt{2}\xi E_1 + 2F_1}{\pi} + o(1),
\]
\[
\phi = -\frac{3}{2}d^2 \ln 2 + \arg \Gamma \left( \frac{1}{2}id^2 \right) + 2\sqrt{2}\xi E_1 + 2F_2 + o(1)
\]
as \( \xi \to +\infty \).

(ii) if \( g_{2n}(a,b) < -a^4 + b^2 < g_{2n+1}(a,b) \), then \( q(t;a,b) \) belongs to type (N3), where
\[
\beta = \frac{1}{2\pi} \ln \left( -\frac{2\cos(\sqrt{2}\xi E_1 + 2F_2)}{B(\xi)} \right) - \frac{2\sqrt{2}\xi E_1 + 2F_1}{2\pi} - 1,
\]
\[
\varphi = 3\beta \ln 2 - \arg \Gamma \left( \frac{1}{2} + i\beta \right) + 2\sqrt{2}\xi E_1 + 2F_2 + o(1)
\]
as \( \xi \to +\infty \).

In the above formulas, \( E_1 \) is the same as in Theorem 1 and
\[
F_1 = \frac{1}{2} \int_1^{\infty} \frac{A(\xi)}{B(\xi)} \cdot \frac{1 - A(\xi)^2}{2A(\xi)} \cdot \frac{1}{s^{1/2}} ds - \frac{1}{2} \log \left( \frac{2^{A(\xi)} + 1}{B(\xi)} \right)
\]
\[
F_2 = \frac{i}{2} \int_1^{\infty} \frac{A(\xi)}{B(\xi)} \cdot \frac{1 + A(\xi)^2}{2A(\xi)} \cdot \frac{1}{s^{1/2}} ds - \frac{i}{2} \log \left( \frac{2^{A(\xi)} + 1}{B(\xi)} \right)
\]

\[(2.10)\]

**Corollary 2.** For any fixed \( a \), there exist a sequence \( 0 < b_1 < b_2 < \cdots < b_n < \cdots \) such that \( y(t;a,b_n) \) belong to type (N2) solutions of PII, and
\[
a^4 - b_n^2 = -\left[ \frac{n\pi - \frac{\pi}{2} + o(1)}{2\sqrt{2}E_1} \right]^{1/2} \quad \text{as} \quad n \to \infty.
\]

\[(2.11)\]

**Remark 1.** It should be noted that \( E_2, E_3 \) and \( F_1, F_2 \) are all functions of \( a, b \). In fact, a combination of \( |a^4 - b^2| = \xi^{1/2} \) and \( a = \xi^{1/4} A(\xi) \) yields \( A(\xi) = a |a^4 - b^2|^{-1/4} \). Similarly, one may find that \( B(\xi) = b |a^4 - b^2|^{-1/2} \). Moreover, note from \[(2.10)\] that \( F_2(a,b) = F_2(-a,-b) + \pi \) for all \( b > 0 \). Hence from \[(2.7)\], we have \( g_n(a,b) = g_n(-a,-b) \).

### 3 Proof of the main results

According to the isomonodromy theory, a general idea to solve connection problems is to calculate the Stokes multipliers of a specific solution in the two specific situations to be
connected. In the initial problem of PII, this means to calculate all $s_k$ when $t \to \pm \infty$ and $t = 0$. When $t \to \pm \infty$, as stated above in (1.3), (1.5), (1.7), (1.9) and (1.11), the Stokes multipliers well known. When $t = 0$, it seems much difficult to find the exact values of $s_k$. However, inspired by the ideas in Sibuya [18], we are able to obtain their asymptotic approximations in the special cases considered here, as a step forward. These approximations, together with (1.3), (1.5), (1.7), (1.9) and (1.11), suffice to prove our theorems.

**Lemma 1.** When $A(\xi)^4 - B(\xi)^2 = 1$, the asymptotic behaviors of the Stokes multipliers corresponding to the PII solution $y(t; a, b)$ are

$$s_2 = -s_{-1} = -2A(\xi)e^{-2\xi E_1 + 2E_2 + o(1)} \cos [2\xi E_1 - 2E_3 + o(1)],$$
$$s_1 = \overline{s_3} = -\overline{s_0} = \frac{e^{2(1+i)E_1 - 2(E_2+iE_3) + o(1)}}{A(\xi)}$$

as $\xi \to +\infty$, where $E_1, E_2, E_3$ are given in Theorem 1.

**Corollary 3.** There exists a sequence of curves $\Gamma_n : a^4 - b^2 = f_n(a, b)$ on the $(a, b)$ plane such that the Stokes multipliers corresponding to $y(t; a, b)$ satisfy

$$s_2 = s_{-1} = 0, \quad s_1 + s_3 = 0.$$ 

Moreover, we have (2.2) as $n \to \infty$.

**Proof of Corollary 3 and Theorem 1** Regard $s_2$ as a function of $\xi$. Then, from (3.1), we can define $f_n(a, b)$ by the $n$th zero of $s_2$ on the positive real axis. Moreover, we have the following two facts: (1) $s_2 = 0$ when $a^4 - b^2 = f_n(a, b)$; (2) arg $s_1 = n\pi - \frac{\pi}{2} - \arg A(\xi) + o(1)$ as $\xi \to +\infty$. It implies (2.2) immediately by substituting these two facts into (3.1).

When $s_2 = 0$, using the constraints of the Stokes multipliers in (1.20), we know that $s_1 = \overline{s_3}$ and $s_1 + s_3 = 0$. Hence, we obtain $\Im s_1 = \text{sgn}((-1)^{n+1})|s_1|$, which implies (2.1) immediately. When $s_2 \neq 0$, it is readily seen from Lemma 1 that $\text{sgn}(s_2) = \text{sgn}((-1)^{n+1})$ if $f_n(a, b) < a^4 - b^2 < f_{n+1}(a, b)$. Hence, substituting (3.1) into (1.9), we get (2.3). This finishes the proof of Theorem 1.

Similarly, according to the following lemma and corollary, one can easily get Theorem 2.

**Lemma 2.** When $A(\xi)^4 - B(\xi)^2 = -1$, the asymptotic behaviors of the Stokes multipliers corresponding to the PII solution $y(t; a, b)$ are

$$s_2 = -s_{-1} = B(\xi)e^{2\xi E_1 - 2F_1 + o(1)},$$
$$s_1 = \overline{s_3} = -\overline{s_0} = e^{-2\xi E_1 - 2iF_2 + o(1)}$$

as $\xi \to +\infty$, where $E_1, F_1, F_2$ are given in Theorem 2.
As an immediate consequence, we have the following corollary.

**Corollary 4.** There exists a sequence of curves \( \Sigma_n : a^4 - b^2 = -g_n(a, b) \) on the \((a, b)\) plane such that the Stokes multipliers corresponding to \( y(t; a, b) \) satisfy

\[
s_1 = \frac{s_3}{e^{-\text{sgn}(b)\frac{\pi}{2} - (n-1)\pi i}}.
\]

Moreover, we have \( g_n(a, b) = \left[ \frac{(n-1)\pi + \text{sgn}(b) \frac{\pi}{2} - 2F_2 + o(1)}{2\sqrt{2E_1}} \right]^\frac{4}{3} \) as \( n \to \infty \).

We leave the proofs of Lemmas 1 and 2 to the next section.

4 Uniform asymptotics and proofs of the lemmas

Let \( \Psi_k = ((\Phi_k)_1, (\Phi_k)_2)^T, k \in \mathbb{Z} \) be independent solutions of (1.15), and set

\[
\phi = \left( 4\lambda q - 2i \eta' \right)^{-\frac{3}{2}} (\Phi_k)_2,
\]

we get from (1.15) the second-order linear differential equation for \( \phi(\lambda) \)

\[
\frac{d^2 \phi}{d\lambda^2} = \left\{ -\left( 16\lambda^4 + 4\eta^4 - 4b^2 \right) + 4i \lambda + \frac{2b}{a} + i \left( \frac{(b/a)^2 - 2a^2}{\lambda - \frac{i b}{2a}} \right) + \frac{3}{4 \left( \lambda - \frac{i b}{2a} \right)^2} \right\} \phi.
\]

When \( t = 0 \), equation (4.2) is simplified to

\[
\frac{d^2 \phi}{d\lambda^2} = \left\{ -\left( 16\lambda^4 + 4\eta^4 - 4b^2 \right) + 4i \lambda + \frac{2b}{a} + i \left( \frac{(b/a)^2 - 2a^2}{\lambda - \frac{i b}{2a}} \right) + \frac{3}{4 \left( \lambda - \frac{i b}{2a} \right)^2} \right\} \phi.
\]

Our task is to derive the uniform asymptotic approximation of the solutions of (4.3) as \( \xi \to +\infty \). For convenient, we assume that \( A(\xi) \cdot B(\xi) > 0 \), and the analysis is similar when \( A(\xi) \cdot B(\xi) < 0 \) and only justified in a remark; (see Remark 2).

4.1 Case one: \( A(\xi)^4 - B(\xi)^2 = 1 \)

Make the scaling

\[
\lambda = -i\xi^{\frac{1}{2}} \eta
\]

and set \( Y(\eta, \xi) = \phi(\lambda) \). A straightforward calculation from (4.3) gives

\[
\frac{d^2 Y}{d\eta^2} = \xi^2 \left\{ 16(\eta^4 + \frac{1}{4}) - \frac{4\eta}{\xi} - 2B(\xi) - \frac{1}{\xi} \cdot \frac{2A(\xi)^2 - \left( \frac{B(\xi)}{A(\xi)} \right)^2}{\eta + \frac{B(\xi)}{2A(\xi)}} + g(\eta, \xi) \right\} Y
\]

\[
:= \xi^2 G(\xi, \eta) Y(\eta, \xi),
\]

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where \( g(\eta, \xi) = O\left(\frac{1}{\xi^2}\right) \) as \( \xi \to +\infty \) uniformly for all \( \eta \) bounded away from \( -\frac{B(\xi)}{2A(\xi)} \). For large \( \xi \), it follows from equation (4.5) that there are four simple turning points, say \( \eta_j, j = 1, 2, 3, 4 \). These turning points are near \( \alpha_1 = \frac{1}{\sqrt{2}} e^{-\frac{7}{4}} \), \( \alpha_2 = \frac{1}{\sqrt{2}} e^{\frac{7}{4}} \), \( \alpha_3 = \frac{1}{\sqrt{2}} e^{\frac{3 \pi i}{4}} \) and \( \alpha_4 = \frac{1}{\sqrt{2}} e^{-\frac{3 \pi i}{4}} \) respectively with \( |\eta_j - \alpha_j| = O(\xi^{-1}) \), \( j = 1, 2, 3, 4 \). Moreover, it also follows from (4.5) that \( \frac{1}{\eta_j - \alpha_j} = O(\xi) \), \( j = 1, 2, 3, 4 \) as \( \xi \to +\infty \).

Figure 3: Stokes geometry of \( G(\eta, \xi)d\eta^2 \)

According to [7, Figure 5], the limiting state of the Stokes geometry of the quadratic differential \( G(\eta, \xi)d\eta^2 \) as \( \xi \to +\infty \) is described in Figure 3. Therefore, following the main theorem in [1], we can approximate the solutions of (4.5) in terms of the Airy functions. Actually, if we define two conformal mappings \( \vartheta(\eta) \) and \( \zeta(\eta) \) by

\[
\int_0^\vartheta s^{1/2} ds = \int_\eta^{\eta_1} G(s, \xi)^{1/2} ds, \quad (4.6)
\]

and

\[
\int_0^\zeta s^{1/2} ds = \int_\eta^{\eta_2} G(s, \xi)^{1/2} ds, \quad (4.7)
\]

respectively in the neighborhoods of \( \eta = \eta_1 \) and \( \eta = \eta_2 \), then the following lemma is a direct implication of [1, Theorem 2].

**Lemma 3.** Given any solution \( Y(\eta, \xi) \) of (4.5), there exist constants \( C_1, C_2 \) and \( \tilde{C}_1, \tilde{C}_2 \) such that,

\[
Y(\eta, \xi) = \left(\frac{\vartheta}{G(\eta, \xi)}\right)^{\frac{i}{2}} \left\{ [C_1 + o(1)] \text{Ai}(\xi^{\frac{2}{3}} \vartheta) + [C_2 + o(1)] \text{Bi}(\xi^{\frac{2}{3}} \vartheta) \right\}, \quad \xi \to +\infty
\]

uniformly for \( \eta \) on any two adjacent Stokes lines emanating from \( \eta_1 \); and

\[
Y(\eta, \xi) = \left(\frac{\zeta}{G(\eta, \xi)}\right)^{\frac{i}{2}} \left\{ [\tilde{C}_1 + o(1)] \text{Ai}(\xi^{\frac{2}{3}} \zeta) + [\tilde{C}_2 + o(1)] \text{Bi}(\xi^{\frac{2}{3}} \zeta) \right\}, \quad \xi \to +\infty
\]
Lemma 4. Let \( \vartheta \) be the Euler’s beta function. For large \( \xi \) and \( \eta \) with \( \eta \gg \xi \), we have

\[
\frac{2}{3} \vartheta^2 = \frac{4}{3} \eta^3 + (1 + i)E_1 - \frac{1}{2\xi} \ln(2\eta) - \frac{E_2 + iE_3}{\xi} + o\left(\frac{1}{\xi}\right),
\]

and

\[
\frac{2}{3} \zeta^2 = \frac{4}{3} \eta^3 + (1 - i)E_1 - \frac{1}{2\xi} \ln(2\eta) - \frac{E_2 - iE_3}{\xi} + o\left(\frac{1}{\xi}\right),
\]

where \( E_1 = \frac{1}{6} B\left(\frac{1}{2}, \frac{1}{4}\right) \), \( E_2 = \frac{B(\xi)}{16A(\xi)} B\left(\frac{1}{4}, \frac{1}{2}\right) + \Re F(\xi), \) \( E_3 = \frac{B(\xi)}{16A(\xi)} B\left(\frac{1}{4}, \frac{1}{2}\right) + \frac{\pi}{8} + \Im F(\xi) \), and

\[
F(\xi) := \frac{2A(\xi)^2 - (B(\xi)/A(\xi))^2}{8} \int_{\frac{1}{2} + i}^{\infty} \frac{1}{s + \frac{B(\xi)}{2A(\xi)}} \sqrt{s^4 + \frac{1}{4}} ds.
\]

Specially, when \( B(\xi) = 0 \), we find that \( A(\xi)^2 = 1 \). Then \( E_2 = 0 \) and \( E_3 = \frac{\pi}{4} \).

The proof of Lemma 4 will be given in Appendix A. It turns to the proof of Lemma 1.

**Proof of Lemma 1.** According to [1], in order to calculate the Stokes multipliers, one need to know the uniform asymptotic behaviors of \( Y(\eta, \xi) \) on two adjacent Stokes lines. Therefore the uniform asymptotic behaviors of the Airy functions is necessary. In fact, according to [15] Eqs.(9.2.12), (9.7.5), (9.2.10) (see also [13] Eqs.(4.5), (4.7)), we know that

\[
\begin{align*}
\text{Ai}(z) &\sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{3/2}}, & \text{arg } z \in (-\pi, \pi); \\
\text{Ai}(z) &\sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{3/2}} + \frac{i}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{3/2}}, & \text{arg } z \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right); \\
\text{Ai}(z) &\sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{3/2}} - \frac{i}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{3/2}}, & \text{arg } z \in \left(-\frac{5\pi}{3}, -\frac{\pi}{3}\right),
\end{align*}
\]

and

\[
\begin{align*}
\text{Bi}(z) &\sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{3/2}} + \frac{i}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{3/2}}, & \text{arg } z \in \left(\frac{\pi}{3}, \pi\right); \\
\text{Bi}(z) &\sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{3/2}} - \frac{i}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{3/2}}, & \text{arg } z \in \left(-\pi, \frac{\pi}{3}\right); \\
\text{Bi}(z) &\sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{3/2}} + \frac{i}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{3/2}}, & \text{arg } z \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right),
\end{align*}
\]

as \( z \to \infty \).
To derive $s_{-1}$, we should know the uniform asymptotic behaviors of $Y(\eta, \xi)$ on the two Stokes lines tending to infinity with $\arg \lambda \sim -\frac{2\pi}{3}$ and $\arg \lambda \sim -\frac{\pi}{3}$. In view of the transformation $\lambda = -i\xi^{\frac{1}{3}}\eta$, it is readily seen that, on these two lines, $\arg \eta \sim \pm \frac{\pi}{6}$ as $\eta \to \infty$. However, one may find in Figure 3 that the two adjacent Stokes lines $\gamma_{-1}$ and $\gamma_0$ with $\arg \eta \sim \pm \frac{\pi}{6}$ emanate from two different turning points. Hence we should build the relations between $(\text{Ai}(\xi^{\frac{2}{3}}\vartheta), \text{Bi}(\xi^{\frac{2}{3}}\vartheta))$ and $(\text{Ai}(\xi^{\frac{2}{3}}\zeta), \text{Bi}(\xi^{\frac{2}{3}}\zeta))$. Actually, it can be done by matching them on the Stokes line $\ell_1$ joining $\eta_1$ and $\eta_2$ in a similar way as in [13, p518-p519]. Precisely, we have

$$\zeta^{\frac{1}{3}} \left( \text{Ai}(\xi^{\frac{2}{3}}\zeta), \text{Bi}(\xi^{\frac{2}{3}}\zeta) \right) \sim \sqrt{\frac{\pi}{2}} \left( \text{Ai}(\xi^{\frac{2}{3}}\vartheta), \text{Bi}(\xi^{\frac{2}{3}}\vartheta) \right) \begin{pmatrix} e^{Q(\xi)} & -i e^{Q(\xi)} + e^{-Q(\xi)} \\ 0 & e^{-Q(\xi)} \end{pmatrix}$$

(4.14)

as $\xi \to +\infty$, uniformly for $\eta \in \gamma_0 \cup \gamma_1 \cup \ell_1$, where

$$Q(\xi) = \xi \int_{\eta_1}^{\eta_2} G(s, \xi)^{\frac{1}{3}} ds$$

and the branches are chosen such that $\arg(s - \eta_{1,2}) \in (-\pi, \pi)$. Note that

$$\int_{\eta_1}^{\eta_2} G(s, \xi)^{\frac{1}{3}} ds = \int_{\eta_1}^{\eta} G(s, \xi)^{\frac{1}{3}} ds - \int_{\eta_2}^{\eta} G(s, \xi)^{\frac{1}{3}} ds.$$  

(4.16)

Then a combination of (4.6), (4.7), (4.10) and (4.11) yields

$$Q(\xi) = i(2\xi E_1 - 2E_3) + o(1) \quad \text{as} \quad \xi \to +\infty.$$  

(4.17)

Now we begin to calculate $s_{-1}$ via $\Psi_0(\lambda) = \Psi_{-1}(\lambda)S_{-1}$ and $s_0$ via $\Psi_1(\lambda) = \Psi_0(\lambda)S_0$. If $\lambda \to \infty$ with $\arg \lambda \sim -\frac{\pi}{3}$, then $\arg \eta \sim \frac{\pi}{6}$, which implies that $\arg \zeta \sim \frac{\pi}{6}$. Hence, substituting (4.11) into [13, Eq.(3.11) and Eq.(3.13)], noting the transform $\lambda = -i\xi^{\frac{2}{3}}\eta$ and observing the definition of $G(\eta, \xi)$ in (4.5), we get

$$\begin{align*}
\sqrt{2\lambda a - ib} & \left( \frac{\zeta}{G(\eta, \xi)} \right)^{\frac{1}{3}} \text{Ai}(\xi^{\frac{2}{3}}\zeta) \sim c_1 e^{4i\lambda^3}, \\
\sqrt{2\lambda a - ib} & \left( \frac{\zeta}{G(\eta, \xi)} \right)^{\frac{1}{3}} \text{Bi}(\xi^{\frac{2}{3}}\zeta) \sim ic_1 e^{4i\lambda^3} + 2c_2 a e^{-4i\lambda^3},
\end{align*}$$

(4.18)

where

$$c_1 = \frac{a^{\frac{3}{2}} e^{-\frac{2i}{3} - \xi(1-i)E_1 + (E_2 - iE_3) + o(1)}}{2\sqrt{\pi}}, \quad c_2 = -\frac{\xi^{\frac{1}{3}} e^{-\frac{2i}{3} + \xi(1-\xi)E_1 - (E_2 - iE_3) + o(1)}}{2\sqrt{\pi} a^{\frac{1}{3}}}. $$

(4.19)

as $\xi \to +\infty$. In view of the fact that $(\Phi_k)_2 = ((\Phi_k)_{21}, (\Phi_k)_{22})$ and according to [8, Eq.(8.1.7)], we know that

$$(\Phi_0)_{21} \sim \frac{a}{2\lambda} e^{-\frac{4i}{3}\lambda^3}, \quad (\Phi_0)_{22} \sim e^{\frac{4i}{3}\lambda^3}, \quad \lambda \in \Omega_k$$

(4.20)
as $\lambda \to \infty$ with $\arg \lambda = \arg(-i\eta) \sim -\pi/3$. Combining (4.18) and (4.20), we have
\[
((\Phi_0)_{21}, (\Phi_0)_{22}) = \sqrt{2\lambda a - ib} \left( \frac{\zeta}{G(\eta, \xi)} \right)^{\frac{1}{4}} \left( \text{Ai}(\xi^2 \zeta), \text{Bi}(\xi^2 \zeta) \right) \left( -\frac{i}{2c_2} \frac{1}{c_1} \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
(4.21)
as $\xi \to +\infty$. Here and after, the $c_j$'s in (4.21) are not equal but asymptotically equal to the corresponding ones in (4.19) as $\xi \to +\infty$. By a little abuse of notations, we use the same symbol for the $c_j$'s in these two formulas, since we only care about the asymptotic behavior of the Stokes multipliers.

If $\lambda \to \infty$ with $\arg \lambda \sim -2\pi/3$, then $\arg \eta \sim -\pi/6$, which implies that $\arg \vartheta \sim -\pi/3$. Using a similar argument in the derivation of (4.21), we get
\[
((\Phi_{-1})_{21}, (\Phi_{-1})_{22}) = \sqrt{2(\lambda - a)} \left( \frac{\theta}{G(\eta, \xi)} \right)^{\frac{1}{4}} \left( \text{Ai}(\xi^2 \vartheta), \text{Bi}(\xi^2 \omega) \right) \left( \frac{i}{2c_2} \frac{1}{c_1} \begin{array}{c} 1 \\ 0 \end{array} \right),
\]
(4.22)
where
\[
\tilde{c}_1 = a^{\frac{1}{2}} e^{-\frac{\pi i}{4} - (1+i)E_1 + (E_2+iE_3)} \frac{2\sqrt{\pi}}{2\sqrt{a^2}}, \quad \tilde{c}_2 = -\xi \frac{1}{2} e^{\frac{\pi i}{4} - (1+i)E_1 - (E_2+iE_3)} \frac{2\sqrt{\pi a^2}}{2\sqrt{\pi a^2}}.
\]
(4.23)
as $\xi \to +\infty$. A combination of (4.14), (4.21), (4.22) and the fact $(\Phi_0)_{21} = (\Phi_{-1})_{22} S_{-1}$ yields
\[
S_{-1} = \left( \begin{array}{cc} \frac{i}{2c_2} & \frac{1}{c_1} \\ \frac{1}{2c_2} & 0 \end{array} \right)^{-1} \left( \begin{array}{cc} e^{Q(\xi)} & -i \left[ e^{Q(\xi)} + e^{-Q(\xi)} \right] \\ 0 & e^{-Q(\xi)} \end{array} \right) \left( \begin{array}{cc} \frac{i}{2c_2} & \frac{1}{c_1} \\ \frac{1}{2c_2} & 0 \end{array} \right)
\]
(4.24)
To get the last equality, one may note the fact that $\tilde{c}_2 = e^{Q(\xi)}$ which can be directly derived by (4.19), (4.23) and (4.17). Finally, substituting the explicit expressions of $\tilde{c}_1$ and $c_2$ into (4.24), we obtain $s_{-1} = 2A(\xi)e^{-2\xi E_1 + 2E_3 + o(1)} \cos[2\xi E_1 - 2E_3 + o(1)]$ as $\xi \to +\infty$.

For the derivation of $s_0$, we also need the uniform asymptotic behavior of $Y(\eta, \xi)$ on the Stokes line $\gamma_1$. In this case, $\arg \lambda = \arg(-i\eta) \sim 0$ as $|\eta| \to +\infty$. In view of (4.11), we find that $\arg \zeta \sim \pi$. Using the asymptotic behaviors of the Airy functions $\text{Ai}(z), \text{Bi}(z)$ in (4.12) and (4.13), it is readily seen that
\[
\begin{cases}
\sqrt{2\lambda a - ib} \left( \frac{\zeta}{G(\eta, \xi)} \right)^{\frac{1}{4}} \text{Ai}(\xi^2 \zeta) \sim c_1 e^{\frac{3}{4}a^2 \lambda^3} + ic_2 \frac{e^{\frac{3}{4}a^2 \lambda^3}}{2\sqrt{2\lambda a - ib}}, \\
\sqrt{2\lambda a - ib} \left( \frac{\zeta}{G(\eta, \xi)} \right)^{\frac{1}{4}} \text{Bi}(\xi^2 \zeta) \sim ic_1 e^{\frac{3}{4}a^2 \lambda^3} + c_2 \frac{e^{\frac{3}{4}a^2 \lambda^3}}{2\sqrt{2\lambda a - ib}}.
\end{cases}
\]
(4.25)
In a similar way of the above argument for deriving (4.21) or (4.22), we obtain
\[
((\Phi_1)_{21}, (\Phi_1)_{22}) = \sqrt{2\lambda a - ib} \left( \frac{\zeta}{G(\eta, \xi)} \right)^{\frac{1}{4}} \left( \text{Ai}(\xi^2 \zeta), \text{Bi}(\xi^2 \zeta) \right) \left( -\frac{i}{2c_2} \frac{1}{c_1} \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \frac{i}{2c_2} \frac{1}{c_1} \begin{array}{c} 1 \\ -\frac{i}{2c_1} \end{array} \right).
\]
(4.26)
Matching this result to the asymptotic behaviors of \((\Phi_1)_{21}, (\Phi_1)_{22}\) in (4.20), one can immediately get
\[
((\Phi_1)_{21}, (\Phi_1)_{22}) = ((\Phi_0)_{21}, (\Phi_0)_{22}) \begin{pmatrix} 1 & \frac{-i\xi}{c_1} \\ 0 & 1 \end{pmatrix}.
\] (4.27)
where \(c_1\) and \(c_2\) are given in (4.19). This implies \(s_0 = \frac{-i\xi}{c_1} = -\frac{e^{2\xi(1+i)E_1 - 2(E_2 - iE_3) + o(1)}}{A(\xi)}\) as \(\xi \to +\infty\).

Finally, in view of the restrictions to the Stokes multipliers in (1.20), we obtain
\[
s_2 = -s_1 = -2A(\xi) e^{-2\xi E_1 + 2E_2 + o(1)} \cos [2\xi E_1 - 2E_3 + o(1)]
\] (4.28)
and
\[
s_1 = \overline{s_3} = -\overline{s_0} = \frac{e^{2\xi(1+i)E_1 - 2(E_2 + iE_3) + o(1)}}{A(\xi)}
\] (4.29)
as \(\xi \to +\infty\).

**Remark 2.** If \(A(\xi) \cdot B(\xi) < 0\), the analysis is similar to the above argument, and the results are the same as the ones in (4.28) and (4.29). To get the corresponding results, the scaling (4.4) should be replaced by \(\lambda = i\xi^\frac{1}{2}\eta\). Moreover, one should also note the following fact:
\[
K^* = \frac{2A(\xi)^2 - (B(\xi)/A(\xi))^2}{8} \int_{A_1}^{\infty} \left[ \frac{1}{s + \frac{B(\xi)}{2A(\xi)}} + \frac{1}{s - \frac{B(\xi)}{2A(\xi)}} \right] \frac{1}{\sqrt{s^4 + \frac{1}{4}}} ds
\] (4.30)
where \(\alpha_1 = \frac{1}{\sqrt{2}} e^{-\frac{\pi i}{4}}\) and \(A(\xi)^4 - B(\xi)^2 = 1\). The proof of (4.30) is left in Appendix C.

### 4.2 Case two: \(A(\xi)^4 - B(\xi)^2 = -1\)

Make the scaling \(\lambda = -i\xi^\frac{1}{2}\eta\) and set \(Y(\eta, \xi) = \phi(\lambda, \xi)\). In this case, the second order equation (4.3) turns to
\[
\frac{d^2 Y}{d\eta^2} = \xi^2 \left\{ 16(\eta^4 - \frac{1}{4}) - \frac{8\eta}{\xi} + \frac{18A(\xi)\eta^2 - 4A(\xi)^3}{2A(\xi)\eta + B(\xi)} + \hat{g}(\eta, \xi) \right\} Y(\eta, \xi)
\] (4.31)
where \(\hat{g}(\eta, \xi) = O\left(\xi^{-\frac{3}{2}}\right)\) as \(\xi \to +\infty\) uniformly for all \(\eta\) provided that \(2A(\xi)\eta + B(\xi)\) is bounded away from 0. Note that there are four simple turning points, say \(\tilde{\eta}_j (j = 1, 2, 3, 4)\), near \(\tilde{\alpha}_1 = \frac{1}{\sqrt{2}}, \tilde{\alpha}_2 = \frac{1}{\sqrt{2}} e^{\frac{i\pi}{4}}, \tilde{\alpha}_3 = -\frac{1}{\sqrt{2}}\) and \(\tilde{\alpha}_4 = \frac{1}{\sqrt{2}} e^{-\frac{i\pi}{4}}\) respectively. It follows from (4.31) that \(\frac{1}{\tilde{\eta}_j - \tilde{\alpha}_j} = O(\xi) (j = 1, 2, 3, 4)\) as \(\xi \to +\infty\). According to [7], the limiting state of the Stokes geometry of the quadratic form \(G(\eta, \xi)d\eta^2\) as \(\xi \to +\infty\) is described in Figure 4.
Therefore, following [1, Theorem 2], we can approximate the solutions of (4.31) via the Airy functions.

Define two conformal mapping \( \rho(\eta) \) and \( \tau(\eta) \) by

\[
\int_0^\rho s^{1/2} ds = \int_\eta^{\hat{\eta}_1} \hat{G}(s, \xi)^{1/2} ds,
\]
and

\[
\int_0^\tau s^{1/2} ds = \int_\eta^{\hat{\eta}_2} \hat{G}(s, \xi)^{1/2} ds,
\]
respectively near the neighborhoods of \( \eta = \hat{\eta}_1 \) and \( \eta = \hat{\eta}_2 \). Then the following lemma is a consequence of [1, Theorem 2].

**Lemma 5.** Given any solution \( \phi(\eta, \xi) \) of (4.31), there exist constants \( \tilde{d}_1, \tilde{d}_2 \) such that

\[
\phi(\eta, \xi) = \left( \frac{\rho}{\hat{G}(\eta, \xi)} \right)^{1/2} \left\{ [\tilde{d}_1 + o(1)] \text{Ai}(\xi^{2/3} \rho) + [\tilde{d}_2 + o(1)] \text{Bi}(\xi^{2/3} \rho) \right\}, \quad \xi \to +\infty
\]

uniformly for \( \eta \) on any two adjacent Stokes lines emanating from \( \hat{\eta}_0 \); and

\[
\phi(\eta, \xi) = \left( \frac{\tau}{\hat{G}(\eta, \xi)} \right)^{1/2} \left\{ [\tilde{d}_1 + o(1)] \text{Ai}(\xi^{2/3} \tau) + [\tilde{d}_2 + o(1)] \text{Bi}(\xi^{2/3} \tau) \right\}, \quad \xi \to +\infty
\]

uniformly for \( \eta \) on any two adjacent Stokes lines emanating from \( \hat{\eta}_2 \).

To calculate the connection matrix \( S_1 \) as \( \xi \to +\infty \), it is necessary to clarify the asymptotic behavior of \( \rho(\eta) \) and \( \tau(\eta) \) as \( \xi, \eta \to \infty \).
Lemma 6. For large $\xi$ and $\eta$ with $|\eta| \gg \xi$,
\[
\frac{2}{3} \rho^3 = \frac{4}{3} \eta^3 - \sqrt{2} E_1 - \frac{1}{\xi} \ln(2\eta) + \frac{1}{2\xi} \log \frac{2A(\xi)\eta + B(\xi)}{B(\xi)} + \frac{F_1}{\xi} + o(\xi^{-1}),
\]
and
\[
\frac{2}{3} \tau^3 = \frac{4}{3} \eta^3 + \sqrt{2i} E_1 - \frac{1}{\xi} \ln(2\eta) + \frac{1}{2\xi} \log(2A(\xi)\eta + B(\xi)) + \frac{\pi i}{2\xi} + iF_2 + o(\xi^{-1}),
\]
where $E_1$ is the same as in Lemma 4 and
\[
F_1 = \frac{1}{2} \int_{a_1}^{\infty} \frac{A(\xi)}{B(\xi)} \cdot \frac{1 - A(\xi)^2}{2 \frac{A(\xi)}{B(\xi)} s + 1} \frac{ds}{\sqrt{s^4 - \frac{1}{4}}} - \frac{1}{2} \int_{a_1}^{\infty} \frac{s \log(2 \frac{A(\xi)}{B(\xi)} s + 1)}{(s^2 + \frac{1}{2}) \sqrt{s^4 - \frac{1}{4}}} ds,
\]
and
\[
F_2 = \frac{i}{2} \int_{a_2}^{\infty} \frac{A(\xi)}{B(\xi)} \cdot \frac{1 + A(\xi)^2}{2 \frac{A(\xi)}{B(\xi)} s + 1} \frac{ds}{\sqrt{s^4 - \frac{1}{4}}} - \frac{i}{2} \int_{a_2}^{\infty} \frac{s \log(2 \frac{A(\xi)}{B(\xi)} s + 1)}{(s^2 - \frac{1}{2}) \sqrt{s^4 - \frac{1}{4}}} ds + \frac{i \log B(\xi)}{2}.
\]

Specially, if $A(\xi) = 0$, then $F_1 = 0$, and $F_2 = 0$ ($F_2 = \frac{-\pi}{2}$) when $B(\xi) = 1$ ($B(\xi) = -1$).

Remark 3. A more careful calculation to (4.39) yields $\text{Im} F_2 = 0$ when $A(\xi)^4 - B(\xi)^2 = -1$. Actually, we see from (4.39) that
\[
\text{Im} F_2 = -\int_1^{+\infty} \frac{(A(\xi)^2 + A(\xi)^4)}{(2A(\xi)^2 s + B(\xi)^2)\sqrt{s^4 - 1}} ds + \frac{1}{2} \int_1^{+\infty} \frac{\ln(2A(\xi)^2 s + B(\xi)^2)}{(s + 1) \sqrt{s^4 - 1}} ds
\]
\[
= -\frac{A(\xi)^2}{A(\xi)^2 - 1} \log A(\xi)^2 + \frac{A(\xi)^2}{A(\xi)^2 - 1} \log A(\xi)^2 = 0.
\]

The proof of Lemma 6 is left in Appendix B. Now we turn to prove Lemma 2, i.e. to calculate the Stokes $s_{-1}$ and $s_0$ as $\xi \to +\infty$ with $A(\xi)^4 - B(\xi)^2 = -1$.

Proof of Lemma 2. It can be seen from Fig. 4 that, in order to calculate $s_{-1}$, we should consider the uniform asymptotic behaviors of $\text{Ai}(\xi^2 \rho)$ and $\text{Bi}(\xi^2 \rho)$ on the two adjacent Stokes lines $\gamma_{-1}$ and $\gamma_0$. When $\arg \lambda \sim -\frac{2\pi}{3}$, we have $\arg \eta = \arg(i\lambda) \sim -\frac{\pi}{6}$, which implies $\arg \rho \sim -\frac{\pi}{3}$ as $\eta \to \infty$. Applying the asymptotic behavior of $\text{Ai}(z)$ and $\text{Bi}(z)$ in (4.12) and (4.13), we get
\[
\begin{align*}
\sqrt{2i} \lambda - id \left( \frac{\rho}{G(\eta, \xi)} \right)^{\frac{1}{4}} \text{Ai}(\xi^2 \rho) & \sim d_1 e^{\frac{4i}{3} \lambda^3}, \\
\sqrt{2i} \lambda - id \left( \frac{\rho}{G(\eta, \xi)} \right)^{\frac{1}{4}} \text{Bi}(\xi^2 \rho) & \sim -id_1 e^{\frac{4i}{3} \lambda^3} + 2d_2 a_2e^{-\frac{4i}{3} \lambda^3}
\end{align*}
\]
where
\[ d_1 = \frac{\xi^\frac{3}{2} B(\xi)^{\frac{1}{2}} e^{-\frac{\pi i}{4} + \xi \sqrt{2} E_1 - F_1 + o(1)}}{2 \sqrt{\pi}}, \quad d_2 = -\frac{\xi^\frac{1}{2} B(\xi)^{\frac{1}{2}} e^{\frac{\pi i}{4} - \xi \sqrt{2} E_1 + F_1 + o(1)}}{2 \sqrt{\pi}}. \]  
(4.42)
as \( \xi \to +\infty \). Comparing (4.41) to (4.20), we have
\[ ((\Phi_{-1})_{21}, (\Phi_{-1})_{22}) = \sqrt{2} \lambda - ib \left( \frac{\rho}{G(\eta, \xi)} \right)^{\frac{1}{4}} (\text{Ai}(\xi \frac{3}{2} \rho), \text{Bi}(\xi \frac{3}{2} \rho)) \left( \frac{1}{2d_2} \frac{1}{d_1} \right) \]  
(4.43)
as \( \xi \to +\infty \). Here and after, the \( d_j \)'s in (4.21) are also not equal but asymptotically equal to the corresponding ones in (4.42) as \( \xi \to +\infty \). By abuse of notations, we use the same symbol for the \( d_j \)'s in these two formulas, since we only care about the asymptotic behavior of the Stokes multipliers.

Similarly, when arg \( \lambda \sim -\frac{\pi}{3} \), we have arg \( \eta = \text{arg}(i \lambda) \sim \frac{\pi}{6} \), which implies arg \( \rho \sim \frac{\pi}{3} \). Making use of the uniform asymptotic behaviors of \( \text{Ai}(z) \) and \( \text{Bi}(z) \) again and noting (4.20), one can immediately obtain
\[ ((\Phi_0)_{21}, (\Phi_0)_{22}) = \sqrt{2} \lambda - ib \left( \frac{\rho}{G(\eta, \xi)} \right)^{\frac{1}{4}} (\text{Ai}(\xi \frac{3}{2} \rho), \text{Bi}(\xi \frac{3}{2} \rho)) \left( -\frac{i}{2d_2} \frac{1}{d_1} \right) \]  
(4.44)
as \( \xi \to +\infty \). Hence,
\[ ((\Phi_0)_{21}, (\Phi_0)_{22}) = ((\Phi_{-1})_{21}, (\Phi_{-1})_{22}) \left( \frac{1}{2d_2} \frac{1}{d_1} \right)^{-1} \left( -\frac{i}{2d_2} \frac{1}{d_1} \right) \]  
(4.45)
This implies \( s_{-1} = -i \frac{d_1}{d_2} = -B(\xi)e^{2\sqrt{2} E_1 \xi - 2F_1 + o(1)} \) as \( \xi \to +\infty \).

To derive \( s_0 \), we need to analyze the uniform asymptotic behaviors of \( \text{Ai}(\xi \frac{3}{2} \tau) \) and \( \text{Bi}(\xi \frac{3}{2} \tau) \) on the two adjacent Stokes lines \( \gamma_0 \) and \( \gamma_1 \) as \( \xi, |\eta| \to +\infty \). When \( \eta \to \infty \) on \( \gamma_0 \), we find that arg \( \eta = \frac{\pi}{6} \), which implies arg \( \tau \sim \frac{\pi}{3} \). Then according to (4.12) and (4.13), we obtain
\[ \left\{ \begin{array}{l} \sqrt{2} \lambda - ib \left( \frac{\tau}{G(\eta, \xi)} \right)^{\frac{1}{4}} \text{Ai}(\xi \frac{3}{2} \tau) \sim \tilde{d}_1 e^{\frac{3i}{2} \lambda^3}, \\
\sqrt{2} \lambda - ib \left( \frac{\tau}{G(\eta, \xi)} \right)^{\frac{1}{4}} \text{Bi}(\xi \frac{3}{2} \tau) \sim i \tilde{d}_1 e^{\frac{3i}{2} \lambda^3} + 2 \tilde{d}_2 \frac{2}{27} e^{-\frac{3i}{2} \lambda^3} \end{array} \right. \]  
(4.46)where
\[ \tilde{d}_1 = \frac{\xi^\frac{1}{4} e^{-\frac{3i}{2} \sqrt{2} \xi E_1 - \frac{3i}{2} F_2 + o(1)}}{2 \sqrt{\pi}}, \quad \tilde{d}_2 = \frac{\xi^\frac{1}{4} e^{\frac{3i}{2} \sqrt{2} \xi E_1 + \frac{3i}{2} F_2 + o(1)}}{2 \sqrt{\pi}}. \]  
(4.47)
as $\xi \to +\infty$. It can be further derived by matching (4.46) to (4.20) that
\[
((\Phi_0)_{21}, (\Phi_0)_{22}) = \sqrt{2\lambda a - ib} \left( \frac{\tau}{G(\eta, \xi)} \right) \left( \text{Ai}(\xi^{\frac{2}{3}} \tau), \text{Bi}(\xi^{\frac{2}{3}} \tau) \right) \left( \begin{array}{c} \frac{1}{2d_2} \\ 0 \end{array} \right)
\] (4.48)
as $\xi \to +\infty$. When $\eta \to \infty$ on $\gamma_1$, we know that $\arg \eta \sim \frac{\pi}{2}$, then $\arg \tau \sim \pi$. In a similar way to the above argument, we have
\[
((\Phi_1)_{21}, (\Phi_1)_{22}) = \sqrt{2\lambda a - ib} \left( \frac{\tau}{G(\eta, \xi)} \right) \left( \text{Ai}(\xi^{\frac{2}{3}} \tau), \text{Bi}(\xi^{\frac{2}{3}} \tau) \right) \left( \begin{array}{c} -\frac{i}{2d_2} \\ \frac{1}{2d_2} \end{array} \right)
\] (4.49)
A combination of (4.48) and (4.49) yields
\[
((\Phi_1)_{21}, (\Phi_1)_{22}) = ((\Phi_0)_{21}, (\Phi_0)_{22}) \left( \begin{array}{cc} -\frac{i}{2d_2} & \frac{1}{d_1} \\ \frac{1}{2d_2} & 0 \end{array} \right)^{-1} \left( \begin{array}{c} -\frac{i}{2d_2} \\ -\frac{1}{2d_1} \end{array} \right)
\] (4.50)
This means that $s_0 = -\frac{i}{d_1} = -e^{2\sqrt{2}iE_1 + 2iF_2 + o(1)}$ as $\xi \to +\infty$. Finally, in view of the restriction of the Stokes multipliers in (1.20), we obtain
\[
s_2 = -s_{-1} = B(\xi) e^{2\sqrt{2}iE_1 + 2iF_2 + o(1)}
\] (4.51)
and
\[
s_1 = \overline{s_3} = -\overline{s_0} = e^{-2\sqrt{2}iE_1 - 2iF_2 + o(1)}
\] (4.52)
as $\xi \to +\infty$.

Remark 4. When $A(\xi) \cdot B(\xi) < 0$, the analysis is similar to the above argument, and the corresponding results are agreement with (4.51) and (4.52).

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Appendix

A Proof of Lemma 4

The idea to prove Lemma 4 is to compute the asymptotic behavior of the integral on the right hand side of (4.7) as ξ, η → ∞. Here we only show the validity of (4.10). The proof of (4.11) is similar and hence omitted here.

Choose $T^*$ to be a point such that $|T^* - \eta_1| \sim \xi^{-\frac{3}{4}}$, then

$$
\int_{\eta_1}^{\eta} G(s, \xi) \frac{1}{4} ds = \int_{\eta_1}^{T^*} G(s, \xi) \frac{1}{4} ds + \int_{T^*}^{\eta} G(s, \xi) \frac{1}{4} ds := I_1 + I_2. \tag{A.1}
$$

Since $|T^* - \eta_1| \sim \xi^{-\frac{3}{4}}$ and $\frac{1}{\eta_1 - \alpha_1} = O(\xi)$ as $\xi \to +\infty$, then $|T^* - \alpha_1| \sim \xi^{-\frac{3}{4}}$ uniformly for all $s$ on the integral contour of $I_2$. It implies that $I_1 = o(\xi^{-1})$ as $\xi \to \infty$. By the Taylor expansion of $G(s, \xi)$ with respect to $\xi$, we have

$$
G(s, \xi) = G_0(s, \xi) - \frac{1}{\xi} G_1(s, \xi) + O\left(\frac{1}{\xi^\frac{3}{2}} \sqrt{s^4 + \frac{1}{4}} \right) , \tag{A.2}
$$

as $\xi \to +\infty$, where $G_0(s, \xi) = 4\sqrt{s^4 + \frac{1}{4}}$ and

$$
G_1(s, \xi) = \frac{s}{2\sqrt{s^4 + \frac{1}{4}}} + \frac{B(\xi)}{4A(\xi)} \frac{1}{\sqrt{s^4 + \frac{1}{4}}} + \frac{1}{8} \frac{2A(\xi)^2 - (B(\xi)/A(\xi))^2}{(s + \frac{B(\xi)}{2A(\xi)})\sqrt{s^4 + \frac{1}{4}}}. \tag{A.3}
$$

Hence

$$
I_2 = \int_{T^*}^{\eta} \left[ G_0(s, \xi) - \frac{1}{\xi} G_1(s, \xi) \right] ds + o(\xi^{-1}) = \int_{\alpha_1}^{\eta} \left[ G_0(s, \xi) - \frac{1}{\xi} G_1(s, \xi) \right] ds + o(\xi^{-1}). \tag{A.4}
$$

as $\xi \to +\infty$. To get the final approximation, one may only need to note the facts $|T^* - \eta_1| \sim \xi^{-\frac{3}{4}}$ and $\frac{1}{\eta_1 - \alpha_1} = O(\xi)$ as $\xi \to +\infty$. Furthermore, by integration by parts, we have

$$
\int_{\alpha_1}^{\eta} G_0(s, \xi) ds = \int_{\alpha_1}^{\eta} 4\sqrt{s^4 + \frac{1}{4}} ds = 2 \int_{\alpha_1}^{\eta} \frac{1}{\sqrt{4s^4 + 1}} ds \tag{A.5}
$$

$$
= 2\eta \sqrt{4\eta^4 + 1} - 2 \int_{\alpha_1}^{\eta} 4\sqrt{s^4 + \frac{1}{4}} ds + 4 \int_{\alpha_1}^{\eta} \frac{1}{\sqrt{4s^4 + 1}} ds. \tag{A.6}
$$

Hence, we get

$$
\int_{\alpha_1}^{\eta} G_0(s, \xi) ds = \frac{2}{3} \eta \sqrt{4\eta^4 + 1} + \frac{4}{3} \int_{\alpha_1}^{\eta} \frac{1}{\sqrt{4s^4 + 1}} ds
$$

$$
= \frac{4}{3} \eta^3 + \frac{4}{3} \int_{\alpha_1}^{\infty} \frac{1}{\sqrt{4s^4 + 1}} ds + O\left(\frac{1}{\eta}\right) \tag{A.7}
$$

$$
= \frac{4}{3} \eta^3 + (1 + i)E_1 + o(\xi^{-1}).
$$
as $\xi, \eta \to \infty$ provided that $|\eta| \gg \xi$, where
\[
E_1 := \frac{4}{3(1+i)} \int_{\alpha_1}^\infty \frac{1}{\sqrt{4s^4+1}} ds = \frac{1}{6} B \left( \frac{1}{4}, \frac{1}{2} \right). \tag{A.8}
\]

To derive the asymptotic behavior of $\int_{\alpha_1}^\eta G_1(s, \xi) ds$ as $\xi \to +\infty$, we still need to calculate $\int_{\alpha_1}^\eta \frac{s}{s^4+1/4} ds$. As a matter of fact,
\[
\int_{\alpha_1}^\eta \frac{s}{s^4+1/4} ds = \frac{1}{2} \ln(s^2 + \sqrt{s^4 + \frac{1}{4}}) \bigg|_{\alpha_1}^\eta = \ln(2\eta) + \frac{\pi i}{4} + o(1) \tag{A.9}
\]
as $\xi, \eta \to +\infty$ with $\eta \gg \xi$. Combining (A.3), (A.8) and (A.9), we have
\[
\int_{\alpha_1}^\eta G_1(s, \xi) ds = \frac{1}{2} \ln(2\eta) + E_2 + iE_3 + o(1) \tag{A.10}
\]
as $\xi, \eta \to +\infty$ with $\eta \gg \xi$ and $\arg \eta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where $E_2 = \frac{B(\xi)}{16A(\xi)} B \left( \frac{1}{4}, \frac{1}{2} \right) + \Re F(\xi)$, $E_3 = \frac{B(\xi)}{16A(\xi)} B \left( \frac{1}{4}, \frac{1}{2} \right) + \frac{\pi}{8} + \Im F(\xi)$, and
\[
F(\xi) := \frac{2A(\xi)^2 - (B(\xi)/A(\xi))^2}{8} \int_{\alpha_1}^\infty \frac{1}{(s + \frac{B(\xi)}{2A(\xi)}) \sqrt{s^4 + \frac{1}{4}}} ds.
\]

A combination of (A.4), (A.7) and (A.10) yields (4.10). Finally, noting the symmetry property $\vartheta(\eta) = \zeta(\eta) + o(\xi^{-1})$ as $\xi, \eta \to \infty$, we get (4.11).

**B Proof of Lemma 6**

The idea to prove Lemma 6 is to compute the asymptotic behavior of the integral on the right hand side of (4.32) as $\xi, \eta \to \infty$. Here we only show the validity of (4.36). The proof of (4.37) is similar and hence omitted here.

Choose $T_1^*$ to be a point such that $|T_1^* - \hat{\eta}_0| \sim \xi^{-\frac{3}{4}}$, then
\[
\int_{\hat{\eta}_0}^\eta \hat{G}(s, \xi)^{\frac{1}{2}} ds = \int_{\hat{\eta}_0}^{T_1^*} \hat{G}(s, \xi)^{\frac{1}{2}} ds + \int_{T_1^*}^\eta \hat{G}(s, \xi)^{\frac{1}{2}} ds := J_1 + J_2. \tag{B.1}
\]

Note the fact that $\hat{G}(s, \xi) = O(\xi^{\frac{3}{4}})$ as $\xi \to +\infty$ uniformly on the integration contour of $J_1$. Then an easy approximation gives that $J_1 = o(\xi^{-1})$ as $\xi \to \infty$. Since $|T_1^* - \hat{\eta}_1| \sim \xi^{-\frac{3}{4}}$ and $\frac{1}{\hat{\eta}_1 - \hat{\alpha}_1} = O(\xi)$ as $\xi \to +\infty$, then $|T_1^* - \hat{\alpha}_1| > c \xi^{-\frac{3}{4}}$ for some constant $c > 0$. By the Taylor expansion of $\hat{G}(s, \xi)$ with respect to $\frac{1}{\xi}$, we have
\[
\hat{G}(s, \xi)^{\frac{1}{2}} = 4 \sqrt{s^4 - \frac{1}{4}} - \frac{1}{4} \frac{s}{\sqrt{s^4 - \frac{1}{4}}} + 1 \frac{2A(\xi) s^2 - A(\xi)^3}{2s^2 + 2A(\xi)s + B(\xi)} \frac{1}{\sqrt{s^4 - \frac{1}{4}}} + O \left( \frac{1}{\xi^{\frac{3}{4}} \sqrt{s^4 - \frac{1}{4}}} \right) \tag{B.2}
\]
as \( \xi \to +\infty \). Making use of the fact \( |T_1^* - \hat{\alpha}_1| > c\xi^{-\frac{3}{4}} \) again and noting that \( \frac{1}{\sqrt{s^4-1}} = O\left((s-\hat{\alpha}_1)^{-\frac{1}{2}}\right) \) as \( s \to \hat{\alpha}_1 \), we find that

\[
J_2 = \int_{\hat{\alpha}_1}^{\eta} 4\sqrt{s^4-\frac{1}{4}} ds - \frac{1}{\xi} \int_{\hat{\alpha}_1}^{\eta} \frac{s}{\sqrt{s^4-\frac{1}{4}}} ds + \frac{1}{2\xi} \int_{\hat{\alpha}_1}^{\eta} 2A(\xi)s^2 - A(\xi) \frac{1}{\sqrt{s^4-\frac{1}{4}}} ds
\]

\[
+ \frac{1}{2\xi} \int_{\hat{\alpha}_1}^{\infty} \frac{A(\xi) - A(\xi)^3}{2A(\xi)s + B(\xi)} \frac{1}{\sqrt{s^4-\frac{1}{4}}} ds + o(\xi^{-1})
\]

(B.3)

as \( \xi, \eta \to \infty \) with \( |\eta| \gg \xi \). The first integral of the right hand side in (B.3) can be calculated as follows:

\[
2 \int_{\hat{\alpha}_1}^{\eta} \sqrt{4s^4-1} ds = 2\eta\sqrt{4\eta^4-1} - 4 \int_{\hat{\alpha}_1}^{\eta} \sqrt{4s^4-1} ds - 4 \int_{\hat{\alpha}_1}^{\eta} \frac{1}{\sqrt{4s^4-1}} ds,
\]

which gives that

\[
2 \int_{\hat{\alpha}_1}^{\eta} \sqrt{4s^4-1} ds = \frac{2}{3} \eta\sqrt{4\eta^4-1} - \frac{4}{3} \int_{\hat{\alpha}_1}^{\eta} \frac{1}{\sqrt{4s^4-1}} ds
\]

\[
= \frac{4}{3} \eta^3 - \frac{4}{3} \int_{\hat{\alpha}_1}^{\infty} \frac{1}{\sqrt{4s^4-1}} ds
\]

(B.5)

as \( \xi, \eta \to \infty \) provided that \( |\eta| \gg \xi \). Next, we compute the integral \( \int_{\hat{\alpha}_1}^{\infty} \frac{1}{\sqrt{4s^4-1}} ds \) explicitly. Set \( s = \frac{t}{\sqrt{2}} \), then we have

\[
\int_{\hat{\alpha}_1}^{\infty} \frac{1}{\sqrt{4s^4-1}} ds = \frac{1}{\sqrt{2}} \int_{1}^{\infty} \frac{1}{\sqrt{t^4-1}} dt = \frac{1}{4\sqrt{2}} B\left(\frac{1}{2}, \frac{1}{4}\right).
\]

(B.6)

The last equality can be derived by setting \( u = t^{-4} \) and calculate as

\[
\int_{1}^{\infty} \frac{1}{\sqrt{t^4-1}} dt = \frac{1}{4} \int_{0}^{1} (1-u)^{-\frac{1}{2}} u^{-\frac{3}{4}} du = \frac{1}{4} B\left(\frac{1}{2}, \frac{1}{4}\right).
\]

(B.7)

Then, combining (B.5) and (B.6), we obtain

\[
\int_{\hat{\alpha}_1}^{\eta} 4\sqrt{s^4-\frac{1}{4}} ds = \frac{4}{3} \eta^3 - \frac{\sqrt{2}}{6} B\left(\frac{1}{2}, \frac{1}{4}\right) + o(\xi^{-1})
\]

as \( \xi, \eta \to \infty \) provided that \( |\eta| \gg \xi \). The second integral on the right hand side of (B.3) is

\[
\int_{\hat{\alpha}_1}^{\eta} \frac{s}{\sqrt{s^4-\frac{1}{4}}} ds = \frac{1}{2} \int_{\hat{\alpha}_1}^{\eta} \sqrt{s^4-\frac{1}{4}} ds = \frac{1}{2} \ln(s^2 + \sqrt{s^4-\frac{1}{4}})\big|_{\hat{\alpha}_1}^{\eta} = \ln(2\eta) + o(1).
\]

(B.9)
By integration by parts, we find that
\[
\int_{\hat{a}_1}^{\eta} \frac{2A(\xi) \left(s^2 - \frac{1}{2}\right)}{(2A(\xi)s + B(\xi))\sqrt{s^4 - \frac{1}{4}}} ds = \log(2A(\xi)\eta + B(\xi)) - \int_{\hat{a}_1}^{\infty} \frac{s\log(2A(\xi)s + B(\xi))}{(s^2 + \frac{1}{2}) \sqrt{s^4 - \frac{1}{4}}} ds + o(\xi^{-1})
\]
(B.10)
as \xi \to +\infty \text{ with } |\eta| \gg \xi.

Finally, combining (B.3), (B.8), (B.9) and (B.10), we obtain
\[
J_2 = \frac{4}{3} \eta^3 - \frac{\sqrt{2}}{6} B\left(\frac{1}{2}, \frac{1}{4}\right) - \frac{1}{\xi} \ln(2\eta) + \frac{1}{2\xi} \log(2A(\xi)\eta + B(\xi)) + \frac{1}{\xi} F_1 + o(\xi^{-1})
\]
as \xi \to +\infty \text{ with } |\eta| \gg \xi, \text{ where}
\[
F_1 = \frac{1}{2} \int_{\hat{a}_1}^{\infty} \frac{A(\xi) - A(\xi)^3}{2A(\xi)s + B(\xi)} \frac{1}{\sqrt{s^4 - \frac{1}{4}}} ds - \frac{1}{2} \int_{\hat{a}_1}^{\infty} \frac{s\log(2A(\xi)s + B(\xi))}{(s^2 + \frac{1}{2}) \sqrt{s^4 - \frac{1}{4}}} ds.
\]
(B.11)

This completes the proof of Lemma 6.

C Proof of (4.30)

In fact, with the transformation \( t = 2s^2 \), we have
\[
\int_{\hat{a}_1}^{\infty} \left[ \frac{1}{(s + B(\xi))} + \frac{1}{(s - B(\xi))} \right] \frac{1}{\sqrt{s^4 - \frac{1}{4}}} ds
= 2 \int_{e^{-\frac{\pi i}{4}}}^{\infty} dt \frac{dt}{t^2 + 1}
= \frac{4A(\xi)^2}{\sqrt{B(\xi)^4 + 4A(\xi)^4}} \left( \frac{\pi i}{2} - 2 \log |A(\xi)| \right).
\]
(C.1)

Making use of the fact \( A(\xi)^4 - B(\xi)^2 = 1 \), we further obtain
\[
K = \frac{2A(\xi)^2 - (B(\xi)/A(\xi))^2}{8} \frac{4A(\xi)^2}{\sqrt{B(\xi)^4 + 4A(\xi)^4}} \left( \frac{\pi i}{2} - 2 \log |A(\xi)| \right)
= \frac{\pi i}{4} - \log |A(\xi)|.
\]
(C.2)
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