

EISENSTEIN POLYNOMIALS OVER FUNCTION FIELDS

EDOARDO DOTTI AND GIACOMO MICHELI

Abstract. In this paper we compute the density of monic and non-monic Eisenstein polynomials of fixed degree having entries in an integrally closed subring of a function field over a finite field.

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1. Introduction

Let us start with the definition of Eisenstein polynomial and natural density

Definition 1. Let $R$ be an integral domain. A polynomial $f(X) = \sum_{i=0}^{n} a_i X^i \in R[X]$ is said to be Eisenstein if there exists a prime ideal $p \subseteq R$ for which

- $a_i \in p$ for all $i \in \{0, \ldots, n-1\}$,
- $a_0 \notin p^2$,
- $a_n \notin p$.

Definition 2. A subset $A$ of $\mathbb{Z}^n$ is said to have density $a$ if

$$a = \lim_{B \to \infty} \frac{|A \cap [-B, B]^n|}{(2B)^n}.$$ 

A classical result from the literature is that any Eisenstein polynomial is irreducible. In addition, observe that any polynomial of degree at most $d$ and coefficients over $\mathbb{Z}$ can be regarded as an element of $\mathbb{Z}^{d+1}$, while any monic polynomial of degree $d$ can be regarded as an element of $\mathbb{Z}^d$. Recently, it has been of interest the explicit computation of the natural density of both degree $d$ Eisenstein polynomials and monic Eisenstein polynomials over $\mathbb{Z}$, see for example [3, 1].

As was first proved by Dubickas in [1], the natural density of monic Eisenstein polynomials over $\mathbb{Z}$ of fixed degree $d$ is

$$\prod_{p \text{ prime}} \left( 1 - \frac{p-1}{p^{d+1}} \right).$$

Heyman and Shparlinski extended the results of Dubickas to general Eisenstein polynomials and computed the error term of the density [3, Theorem 1, Theorem 2].

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In this paper we would like to establish a function field analogue of these results that will include all the cases in which \( R \) is selected as an integrally closed subring of a function field of a curve over a finite field.

The general case that we will analyse needs an appropriate definition of density which makes use of Moore-Smith convergence for directed sets, as described in [5]. For the moment, let us fix the notation for the basic structures we are going to deal with, which is essentially the same as in [6].

Let \( q \) be a prime power and \( \mathbb{F}_q \) be the finite field of order \( q \). Let \( F \) be a function field having full constant field \( \mathbb{F}_q \). Let \( \mathcal{P}_F \) be the set of places of \( F \) and \( S \) a non empty proper subset of \( \mathcal{P}_F \). Let us denote by \( \mathcal{O}_P \) the valuation ring at a place \( P \) of \( F \). Let \( H = \bigcap_{P \in S} \mathcal{O}_P \) be the holomorphy ring associated to \( S \) [6, Definition 3.2.2]. As it is well known, \( H \) is a Dedekind Domain therefore any prime ideal is also maximal. In addition the maximal ideals of \( H \) correspond exactly to the places in \( S \) see [6, Proposition 3.2.9]. Therefore, if \( P \) is a place of \( F \) which lies in \( S \) there exists a unique maximal ideal \( P_H \subseteq H \) corresponding to \( P \) for which \( P \cap H = P_H \). In order not to heavier the notation, we will denote \( P_H \) again by \( P \).

Let \( \mathcal{D} \) be the set of positive divisors of \( \text{Div}(F) \) having support outside the holomorphy set \( S \). It is easy to observe

\[
H = \bigcup_{D \in \mathcal{D}} \mathcal{L}(D)
\]

and that \( \mathcal{D} \) is a directed set.

Let now \( A \subseteq H^m \), we define the upper and lower density of \( A \) as

\[
\overline{D}(A) = \limsup_{D \in \mathcal{D}} \frac{|A \cap \mathcal{L}(D)^m|}{q^{m \ell(D)}},
\]

\[
\underline{D}(A) = \liminf_{D \in \mathcal{D}} \frac{|A \cap \mathcal{L}(D)^m|}{q^{m \ell(D)}},
\]

where the limit is defined using Moore-Smith convergence over the directed set \( \mathcal{D} \) (see [4, Chapter 2]). The density of \( A \) is then defined if \( \overline{D}(A) = \underline{D}(A) =: D(A) \).

As already observed in [5], if we specialize our definition of density to the case of the univariate polynomial ring over a finite field we get the usual definition of density for \( \mathbb{F}_q[x] \), see for example in [2, 7].

In addition, the final formulas for the density we get are analogous to the ones over the rational integers obtained in [1, 3].

The paper is structured as follows: in the next subsection we specify the notation we are going to use for the rest of the paper, in section 2 we compute the density of monic Eisenstein polynomials, in section 3 we apply a similar strategy to compute the density of general Eisenstein polynomials.

1.1. Notation. Throughout this paper, when \( Y \) is a set and \( m \) is a positive integer, we will denote by \( Y^m \) the cartesian product of \( m \)-copies of \( Y \). To avoid confusion, the square of an ideal \( Q \) will then be denoted by \( \hat{Q} = Q \cdot Q \). Furthermore notice that in
the whole paper we consider polynomials of degree $d > 1$. To easier the notation, we fix an enumeration $\{Q_1, Q_2, \ldots, Q_i, \ldots\}$ of the places of $S$. Since we will deal with the density of both monic and non-monic Eisenstein polynomials, we have to distinguish the notation, which we clarify in the following two paragraphs.

Notation for monic Eisenstein polynomials: with a small abuse of notation we identify $H^d$ with the set of all monic polynomials of degree $d$ having entries over $H$. In particular, if $(h_0, \ldots, h_{d-1}) \in H^d$ then $h_i$ denotes the coefficient of the monomial of degree $i$. Furthermore, we denote by $\mathcal{E} \subset H^d$ the set of monic Eisenstein polynomials of degree $d$ and by $\mathcal{N}$ its complement in $H^d$. We denote by $\mathcal{E}_i$ the set of monic polynomials which are Eisenstein with respect to $Q_i$:

$$\mathcal{E}_i = \{(h_0, \ldots, h_{d-1}) \in H^d : h_i \in Q_i \forall i \in \{0, \ldots, d-1\} \text{ and } h_0 \notin \hat{Q}_i\}.$$  

We denote by $\mathcal{N}_i$ the complement of $\mathcal{E}_i$.

Notation for Eisenstein polynomials: Analogously, we identify the set of all polynomials of degree $d$ having entries over $H$ with $H^{d+1}$. Let $\mathcal{E}^+ \subset H^{d+1}$ be the set of Eisenstein polynomials of degree $d$ and $\mathcal{N}^+$ be its complement in $H^{d+1}$. We denote by $\mathcal{E}_i^+$ the set of polynomials which are Eisenstein with respect to $Q_i$:

$$\mathcal{E}_i^+ = \{(h_0, \ldots, h_d) \in H^{d+1} : h_i \in Q_i \forall i \in \{0, \ldots, d-1\}, h_0 \notin \hat{Q}_i \text{ and } h_d \notin Q_i\}.$$  

We denote by $\mathcal{N}_i^+$ the complement of $\mathcal{E}_i^+$.

### 2. Monic Eisenstein Polynomials

In this section we compute the density of monic Eisenstein polynomials via approximating the complement of $\mathcal{E}$ (i.e. $\mathcal{N}$) with $\overline{\mathcal{N}}_t = \bigcap_{i=1}^t \mathcal{N}_i$. First we show that we can explicitly compute the density of $\overline{\mathcal{N}}_t$ (Proposition 3). Then, we give a criterion to check whether the approximation is “sharp”: i.e. whether the limit of the densities of $\overline{\mathcal{N}}_t$ converges to the density of $\mathcal{N}$ (Lemma 4). Finally, we verify that the conditions under which the approximation is sharp are verified (Theorem 5).

**Proposition 3.** The density of $\overline{\mathcal{N}}_t$ is

$$\mathbb{D}(\overline{\mathcal{N}}_t) = \prod_{i=1}^t \left(1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1)\deg(Q_i)}}\right).$$

**Proof.** Consider the map

$$\tilde{\phi} : H^d \to \left(H/(\hat{Q}_1 \cdots \hat{Q}_t)\right)^d,$$

which is defined componentwise by the reduction modulo the ideal $(\hat{Q}_1 \cdots \hat{Q}_t)$. Observe also that $(H/(\hat{Q}_1 \cdots \hat{Q}_t))^d \simeq \prod_{i=1}^t (H/\hat{Q}_i)^d$ by the Chinese Remainder Theorem. Consider now a divisor $D \in \mathcal{D}$. In order to compute the density of $\overline{\mathcal{N}}_t$ it is enough to count how many elements there are in $\overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d$, when the degree of $D$ is large.
We start by showing that $\mathcal{L}(D)^d$ maps surjectively onto $\left(H/(\hat{Q}_1 \cdots \hat{Q}_t)\right)^d$ when the degree of $D$ is large enough.

For this consider the $\mathbb{F}_q$ linear map $\phi : \mathcal{L}(D) \to \left(H/(\hat{Q}_1 \cdots \hat{Q}_t)\right)$. We have $\ker(\phi) = \mathcal{L}(D) \cap (\hat{Q}_1 \cdots \hat{Q}_t)$, which represents the elements of $\mathcal{L}(D)$ having at least a double root at each $Q_i$. Hence $\ker(\phi) = \mathcal{L}(D - 2 \sum_{i=1}^t Q_i)$.

By Riemann’s theorem [6, Theorem 1.4.17], if the degree of $D$ is large enough, the dimension of the kernel as an $\mathbb{F}_q$ vector space is

$$(2) \quad \ell \left( D - 2 \sum_{i=1}^t Q_i \right) = \deg \left( D - 2 \sum_{i=1}^t Q_i \right) + 1 - g = \deg(D) - 2 \sum_{i=1}^t \deg(Q_i) + 1 - g,$$

where $g$ denotes the genus of the function field.

By the same theorem $\ell(D) = \deg(D) + 1 - g$. Hence we obtain

$$\dim_{\mathbb{F}_q}(\mathcal{L}(D)/\ker(\phi)) = \ell(D) - \ell \left( D - 2 \sum_{i=1}^t Q_i \right) = 2 \sum_{i=1}^t \deg(Q_i).$$

On the other hand, by the Chinese Remainder Theorem

$$\dim_{\mathbb{F}_q}\left( H/(\hat{Q}_1 \cdots \hat{Q}_t) \right)^{\text{CRT}} = \dim_{\mathbb{F}_q}\left( H/\hat{Q}_1 \times \cdots \times H/\hat{Q}_t \right) = 2 \sum_{i=1}^t \deg(Q_i).$$

Therefore when the degree of $D$ is large enough $\phi$ is surjective, thus $\hat{\phi}$ is surjective.

Let $\psi_i : \left( H/(\hat{Q}_1 \cdots \hat{Q}_t) \right)^d \to \left( H/\hat{Q}_i \right)^d$. We have the following situation:

$$\mathcal{L}(D)^d \xrightarrow{\hat{\phi}} \left( H/(\hat{Q}_1 \cdots \hat{Q}_t) \right)^d \xrightarrow{\psi_i} \prod_{i=1}^t \left( H/\hat{Q}_i \right)^d,$$

where $\psi = (\psi_1, \ldots, \psi_t)$. Notice that the check for $f \in H^d$ not to be Eisenstein with respect to $Q_i$ can be performed by looking at the reduction modulo $\hat{Q}_i$. Therefore $f \in \mathcal{N} \cap \mathcal{L}(D)^d$ if and only if $\psi_i \circ \hat{\phi}(f) \notin ((Q_i/\hat{Q}_i) \setminus \{0\}) \times \left( Q_i/\hat{Q}_i \right)^{d-1} =: E_i$ for all $i \in \{1, \ldots, t\}$.

It follows that $\mathcal{N} \cap \mathcal{L}(D)^d = \hat{\phi}^{-1} \left( \psi^{-1} \left( \prod_{i=1}^t \left( (H/\hat{Q}_i)^d \setminus E_i \right) \right) \right) \cap \mathcal{L}(D)^d$. Hence

$$|\mathcal{N} \cap \mathcal{L}(D)^d| = |\ker(\hat{\phi})| \cdot \prod_{i=1}^t \left| \left( H/\hat{Q}_i \right)^d \setminus E_i \right|,$$

where the last equality follows from (2). Now it remains to compute

$$\left| \left( H/\hat{Q}_i \right)^d \setminus E_i \right| = q^{2d \deg(Q_i)} \cdot \left| ((Q_i/\hat{Q}_i) \setminus \{0\}) \times \left( Q_i/\hat{Q}_i \right)^{d-1} \right|$$

$$= q^{\deg(Q_i)} \cdot q^{\deg(Q_i) - 1} \cdot q^{(d-1)\deg(Q_i)}$$

$$= q^{2d \deg(Q_i)} \left( 1 - q^{-(d+1)\deg(Q_i)} \right).$$
Therefore for $D$ of degree large enough
\[
\frac{|N_t \cap L(D)^d|}{|L(D)^d|} = \frac{q^d \left( \deg(D) - 2 \sum_{i=1}^t \deg(Q_i) + 1 - g \right)}{q^d (\deg(D) + 1 - g)} \prod_{i=1}^t q^{2 \deg(Q_i)} \left( 1 - q^{-\deg(Q_i)} + q^{-(d+1) \deg(Q_i)} \right)
\]
\[= \prod_{i=1}^t \left( 1 - q^{-\deg(Q_i)} + q^{-(d+1) \deg(Q_i)} \right) = \prod_{i=1}^t \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1) \deg(Q_i)}} \right).
\]

Hence
\[D(N_t) = \lim_{D \in D} \frac{|N_t \cap L(D)^d|}{|L(D)^d|} = \prod_{i=1}^t \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1) \deg(Q_i)}} \right).
\]

\[\square
\]

**Lemma 4.** Let $n \in \mathbb{N}$, $A \subseteq H^n$. Let $\{A_t\}_{t \in \mathbb{N}}$ be a family of subsets of $H^n$ such that $A_{t+1} \subseteq A_t$ and \( \bigcap_{t \in \mathbb{N}} A_t = A \). Assume also that $D(A_t)$ exists for all $t$. If $\lim_{t \to \infty} D(A_t \setminus A) = 0$, then $D(A) = \lim_{t \to \infty} D(A_t)$.

**Proof.** We start from the equality $|A_t \cap L(D)^n| = |A \cap L(D)^n| + |(A_t \setminus A) \cap L(D)^n|$, from which it follows

\[\liminf_{D \in D} \frac{|A \cap L(D)^n|}{|L(D)^n|} = \liminf_{D \in D} \left( \frac{|A_t \cap L(D)^n|}{|L(D)^n|} - \frac{|(A_t \setminus A) \cap L(D)^n|}{|L(D)^n|} \right) \geq \liminf_{D \in D} \frac{|A_t \cap L(D)^n|}{|L(D)^n|} - \limsup_{D \in D} \frac{|(A_t \setminus A) \cap L(D)^n|}{|L(D)^n|}.
\]

It follows that $D(A_t) - \overline{D}(A_t \setminus A) \leq D(A)$. Since $D(A_t)$ exists for all $t$ we get

$D(A_t) - \overline{D}(A_t \setminus A) \leq D(A)$.

Now notice that $\lim_{t \to \infty} D(A_t)$ exists since $D(A_t)$ is decreasing and bounded from below. By taking the limit in $t$, the last expression then becomes

\[\lim_{t \to \infty} D(A_t) - \lim_{t \to \infty} \overline{D}(A_t \setminus A) \leq D(A).
\]

Since $\lim_{t \to \infty} \overline{D}(A_t \setminus A) = 0$ by assumption, it follows that $\lim_{t \to \infty} D(A_t) \leq D(A)$.

On the other hand $A \subseteq A_t$ which implies $\overline{D}(A) \leq D(A_t)$. In particular $\overline{D}(A) \leq \lim_{t \to \infty} D(A_t)$. Combining all together we get

\[\lim_{t \to \infty} D(A_t) \leq D(A) \leq \overline{D}(A) \leq \lim_{t \to \infty} D(A_t),
\]

therefore the claim follows. $\square$

**Theorem 5.** The density of the set of monic Eisenstein polynomials with coefficients in $H$ is

\[D(E) = 1 - \prod_{Q \in S} \left( 1 - \frac{q^{\deg(Q)} - 1}{q^{(d+1) \deg(Q)}} \right).
\]
Proof. We make use of Lemma 4 for the family \( \{A_t\}_{t \in \mathbb{N}} \). Hence we want to show that
\[
\lim_{t \to \infty} \mathbb{D}(A_t \setminus \mathcal{N}) = 0.
\]
First note that
\[
\begin{align*}
\cdot \quad & \mathcal{N}_t \setminus \mathcal{N} = \bigcup_{r > t} \mathcal{E}_r \subseteq \bigcup_{r > t} Q_r^d, \\
\cdot \quad & Q_r^d \cap \mathcal{L}(D)^d = \mathcal{L}(D - Q_r) = 0, \text{ if } \deg(D) - \deg(Q_r) < 0.
\end{align*}
\]
Now we get
\[
\begin{align*}
\mathbb{D}(\mathcal{N}_t \setminus \mathcal{N}) &= \limsup_{d \in \mathcal{D}} \left| \frac{\left(\mathcal{N}_t \setminus \mathcal{N}\right) \cap \mathcal{L}(D)^d}{|\mathcal{L}(D)^d|} \right| \\
&\leq \limsup_{d \in \mathcal{D}} \left| \bigcup_{\deg(Q_r) \leq \deg(D)} Q_r^d \cap \mathcal{L}(D)^d \right| q^{-d \ell(D)} \\
&(3) = \limsup_{d \in \mathcal{D}} \left| \bigcup_{\deg(Q_r) \leq \deg(D)} \mathcal{L}(D - Q_r)^d \right| q^{-d \ell(D)} \\
&\leq \limsup_{d \in \mathcal{D}} \sum_{r > t} \left( \mathcal{L}(D - Q_r)^d \right)_r q^{-d \ell(D)} \\
&\leq \limsup_{d \in \mathcal{D}} \sum_{r > t} \frac{q^{d(1 + \deg(D) - \deg(Q_r))}}{q^{d(\deg(D) + 1 - g)}} \\
&= q^{d(g - \deg(Q_r))} \sum_{r > t} q^{-d \deg(Q_r)}.
\end{align*}
\]
Observe now that if \( \deg(D - Q_r) \geq 0 \) we have that \( \ell(D - Q_r) \leq \deg(D - Q_r) + 1 \) [6, Eq. 1.21] and also that \( \ell(D) \geq \deg(D) + 1 - g \) by Riemann’s theorem.
Hence we have that (3) is less or equal than
\[
\limsup_{d \in \mathcal{D}} \sum_{r > t} \frac{q^{d(1 + \deg(D) - \deg(Q_r))}}{q^{d(\deg(D) + 1 - g)}} \leq \sum_{r > t} q^{d(g - \deg(Q_r))} \sum_{r > t} q^{-d \deg(Q_r)}.
\]
We now notice that \( \sum_{r > t} q^{-d \deg(Q_r)} \) is the tail of a subseries of the Zeta function, which is absolutely convergent for \( d > 1 \). Letting \( t \) going to infinity the tail converges to 0, thus
\[
\lim_{t \to \infty} \mathbb{D}(A_t \setminus \mathcal{N}) = 0.
\]
We are now able to apply Lemma 4 with \( n = d, A_t = \mathcal{N}_t \) and \( A = \mathcal{N} \),
\[
\mathbb{D}(\mathcal{N}) = \lim_{t \to \infty} \mathbb{D}(A_t) = \lim_{t \to \infty} \prod_{i=1}^{t} \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1)\deg(Q_i)}} \right) = \prod_{Q \in \mathcal{S}} \left( 1 - \frac{q^{\deg(Q)} - 1}{q^{(d+1)\deg(Q)}} \right).
\]
We conclude by taking the complement
\[
\mathbb{D}(\mathcal{E}) = 1 - \mathbb{D}(\mathcal{N}) = 1 - \prod_{Q \in \mathcal{S}} \left( 1 - \frac{q^{\deg(Q)} - 1}{q^{(d+1)\deg(Q)}} \right).
\]
\[\square\]

3. Non-Monic Eisenstein Polynomials

In this section we compute the density of Eisenstein polynomials applying the same strategy of section 2. For this let \( \mathcal{N}_t^+ = \bigcap_{i=1}^{t} \mathcal{N}_i^+ \).

Proposition 6. The density of \( \mathcal{N}_t^+ \) is
\[
\mathbb{D}(\mathcal{N}_t^+) = \prod_{i=1}^{t} \left( 1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}} \right).
\]
Proof. Consider a divisor $D \in \mathcal{D}$. With the same reasoning of the monic case one can show that $\mathcal{L}(D)^{d+1}$ maps surjectively onto $\left( H/(\hat{Q}_1 \cdots \hat{Q}_t) \right)^{d+1}$ when the degree of $D$ is large enough.

Let $\psi_i : \left( H/(\hat{Q}_1 \cdots \hat{Q}_t) \right)^{d+1} \rightarrow \left( H/\hat{Q}_i \right)^{d+1}$ as before. The situation is now the following:

$$\mathcal{L}(D)^{d+1} \xrightarrow{\hat{\phi}} \left( H/(\hat{Q}_1 \cdots \hat{Q}_t) \right)^{d+1} \xrightarrow{\psi} \prod_{i=1}^{t} \left( H/\hat{Q}_i \right)^{d+1},$$

where $\psi = (\psi_1, \ldots, \psi_t)$.

Analogously to the case of monic polynomials we note that we can verify that $f \in H$ is not Eisenstein with respect to $Q_i$ by looking at the reduction modulo $\hat{Q}_i$. Hence $f \in \mathcal{N}_t^+ \cap \mathcal{L}(D)^{d+1}$ if and only if $\psi_i \circ \hat{\phi}(f) \notin \left( (Q_i/\hat{Q}_i) \setminus \{0\} \right) \times (Q_i/\hat{Q}_i)^{d-1} \times (H/\hat{Q}_i) \times (Q_i/\hat{Q}_i) =: E_i^+$ for all $i \in \{1, \ldots, t\}$.

Hence we get

$$\left| \mathcal{N}_t^+ \cap \mathcal{L}(D)^{d+1} \right| = \left| \ker(\hat{\phi}) \right| \prod_{i=1}^{t} \left| \left( H/\hat{Q}_i \right)^{d+1} \setminus E_i^+ \right|$$

where

$$\left| \left( H/\hat{Q}_i \right)^{d+1} \setminus E_i^+ \right| = q^{(d+1) \deg(Q_i)} - \left| \left( (Q_i/\hat{Q}_i) \setminus \{0\} \right) \times (Q_i/\hat{Q}_i)^{d-1} \times (H/\hat{Q}_i) \setminus (Q_i/\hat{Q}_i) \right|$$

$$= q^{2(d+1) \deg(Q_i)} - \left( q^{\deg(Q_i)} - 1 \right) q^{(d-1) \deg(Q_i)} \left( q^{2 \deg(Q_i)} - q^{\deg(Q_i)} \right)$$

$$= q^{2(d+1) \deg(Q_i)} \left( 1 - \frac{q^{2 \deg(Q_i)} - 2q^{\deg(Q_i)} + 1}{q^{(d+2) \deg(Q_i)}} \right)$$

Therefore for $D$ of degree large enough

$$\frac{\left| \mathcal{N}_t^+ \cap \mathcal{L}(D)^{d+1} \right|}{\left| \mathcal{L}(D)^{d+1} \right|} = \frac{q^{(d+1) \deg(D)} - \sum_{i=1}^{t} \deg(Q_i) + 1 - g}{q^{(d+1)(\deg(D) + 1) - g}} \prod_{i=1}^{t} q^{2(d+1) \deg(Q_i)} \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+2) \deg(Q_i)}} \right)^2$$

$$= \prod_{i=1}^{t} \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+2) \deg(Q_i)}} \right)^2.$$ 

Hence

$$\mathbb{D}(\mathcal{N}_t^+) = \lim_{D \in \mathcal{D}} \frac{\left| \mathcal{N}_t^+ \cap \mathcal{L}(D)^{d+1} \right|}{\left| \mathcal{L}(D)^{d+1} \right|} = \prod_{i=1}^{t} \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+2) \deg(Q_i)}} \right)^2.$$ 

$\square$
Theorem 7. The density of the set of Eisenstein polynomials with coefficients in $H$ is
\[
\mathbb{D}(\mathcal{E}^+) = 1 - \prod_{Q \in S} \left( 1 - \frac{(q^{\deg(Q)} - 1)^2}{q^{(d+2)\deg(Q)}} \right).
\]

Proof. Again by lemma 4 we have to show that $\lim_{t \to \infty} \mathbb{D}(\mathcal{N}_t^+ \setminus \mathcal{N}^+) = 0$.

Observe that $\mathcal{E}^+_r \cap \mathcal{L}(D)^{d+1} \subseteq Q_r^d \times \mathcal{L}(D)$. We get
\[
\mathbb{D}(\mathcal{N}_t^+ \setminus \mathcal{N}^+) = \limsup_{D \in \mathcal{D}} \left| \frac{(\mathcal{N}_t^+ \setminus \mathcal{N}^+) \cap \mathcal{L}(D)^{d+1}}{\mathcal{L}(D)^{d+1}} \right|
\]
\[
\leq \limsup_{D \in \mathcal{D}} \left| \bigcup_{r > t} \mathcal{E}^+_r \cap \mathcal{L}(D)^{d+1} \right| q^{-(d+1)\ell(D)}
\]
\[
\leq \limsup_{D \in \mathcal{D}} \left| \bigcup_{r > t} (Q_r^d \times \mathcal{L}(D)) \cap \mathcal{L}(D)^{d+1} \right| q^{-(d+1)\ell(D)}
\]
\[
\leq \limsup_{D \in \mathcal{D}} \sum_{r > t} \frac{|(Q_r^d \times \mathcal{L}(D)) \cap \mathcal{L}(D)^{d+1}|}{q^{(d+1)\ell(D)}}
\]
\[
= \limsup_{D \in \mathcal{D}} \sum_{r > t} \frac{|Q_r \cap \mathcal{L}(D)|}{q^{d\ell(D)}}
\]
\[
= \limsup_{D \in \mathcal{D}} \sum_{r > t} \frac{|Q_r \cap \mathcal{L}(D)|^d}{q^{d\ell(D)}}
\]
which is equation (3). Hence for $t$ going to infinity we obtain $\mathbb{D}(\mathcal{N}_t^+ \setminus \mathcal{N}^+) = 0$.

We now apply Lemma 4 with $n = d + 1$, $A_t = \mathcal{N}_t^+$ and $A = \mathcal{N}^+$ obtaining
\[
\mathbb{D}(\mathcal{N}^+) = \lim_{t \to \infty} \mathbb{D}(\mathcal{N}_t^+) = \lim_{t \to \infty} \prod_{i=1}^{t} \left( 1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}} \right) = \prod_{Q \in S} \left( 1 - \frac{(q^{\deg(Q)} - 1)^2}{q^{(d+2)\deg(Q)}} \right).
\]

We now take the complement
\[
\mathbb{D}(\mathcal{E}^+) = 1 - \mathbb{D}(\mathcal{N}^+) = 1 - \prod_{Q \in S} \left( 1 - \frac{(q^{\deg(Q)} - 1)^2}{q^{(d+2)\deg(Q)}} \right).
\]

\[
\square
\]

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Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland,

E-mail address: edoardo.dotti@uzh.ch

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland,

E-mail address: giacomo.micheli@math.uzh.ch