Multi-oriented props and homotopy algebras with branes

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Abstract

We introduce a new category of differential graded multi-oriented props whose representations (called homotopy algebras with branes) in a graded vector space require a choice of a collection of \( k \) linear subspaces in that space, \( k \) being the number of extra orientations (if \( k = 0 \) this structure recovers an ordinary prop); symplectic vector spaces equipped with \( k \) Lagrangian subspaces play a distinguished role in this theory. Manin triples is a classical example of an algebraic structure (concretely, a Lie bialgebra structure) given in terms of a vector space and its subspace; in the context of this paper, Manin triples are precisely symplectic Lagrangian representations of the 2-oriented generalization of the classical operad of Lie algebras. In a sense, the theory of multi-oriented props provides us with a far reaching strong homotopy generalization of Manin triples type constructions. The homotopy theory of multi-oriented props can be quite non-trivial (and different from that of ordinary props). The famous Grothendieck–Teichmüller group acts faithfully as homotopy non-trivial automorphisms on infinitely many multi-oriented props, a fact which motivated much the present work as it gives us a hint to a non-trivial deformation quantization theory in every geometric dimension \( d \geq 4 \) generalizing to higher dimensions Drinfeld–Etingof–Kazhdan’s quantizations of Lie bialgebras (the case \( d = 3 \)) and Kontsevich’s quantizations of Poisson structures (the case \( d = 2 \)).

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1 Introduction

1.1 Why bother with multi-oriented props?

A short answer to this question: “Because of the Grothendieck–Teichmüller group \( GRT_1 \)”. It is the latter beautiful and mysterious structure which is the main motiva-
tion for introducing and studying a new category of multi-oriented props as well as their representations (“homotopy algebras with branes”). In geometric dimensions 2 and 3, the group $GRT_1$ acts on some ordinary props of odd/even strong homotopy Lie bialgebras [16] and plays thereby the classifying role in the associated transcendental deformation quantizations of Poisson and, respectively, ordinary Lie bialgebra structures. In geometric dimension $d \geq 4$, the group $GRT_1$ survives in the form of symmetries of some multi-oriented props of even/odd homotopy Lie bialgebras so that deformation quantizations in higher dimensions (in which $GRT_1$ retains its fundamental classifying role) should involve a really new class of algebro-geometric structures—the homotopy algebras with branes. It is an attempt to understand what could be a higher ($d \geq 4$) analogue of two famous formality theorems, one for Poisson structures [6] (the case $d = 2$), and another for ordinary Lie bialgebras [4,13] (the case $d = 3$), that lead the author to the category of multi-oriented props after reading the paper [23] by Marko Živković and its predecessor [21] by Thomas Willwacher (see Sect. 5 for a brief but self-contained description of their remarkable results).

It is not that hard to define multi-oriented props in general, and multi-oriented generalizations of some concrete classical operads and props in particular, at the purely combinatorial level: the rules of the game with multi-oriented decorated graphs are more or less standard (see Sect. 2)—for any given $k \geq 1$ one just adds $k$ extra directions (=orientations) to each edge/leg of a 1-oriented graph, and defines rules for multi-oriented prop compositions via contractions along admissible multi-oriented subgraphs. However, it is much less evident how to transform that more or less standard rules into non-trivial and interesting representations (i.e. examples)—the intuition from the theory of ordinary (wheeled) props does not help much. Adding new $k$ directions to each edge of a decorated graph of an ordinary prop can be naively understood as extending that ordinary prop into a $2^k$-coloured one, but then the requirement that the new directions on graph edges do not create “wheels” (that is, closed directed paths of edges with respect to each of the new orientations) kills that naive picture immediately—the elements of the set of $2^k$ new colours start interacting with each other in a non-trivial way. We know which structure distinguishes ordinary props (the ones with no wheels in the given single orientation, i.e. the ones which are 1-oriented in the terminology of this paper) from the ordinary wheeled props (that is, 0-oriented 1-directed props in the terminology of this paper) in terms of representations in, say, a graded vector space $V$—it is the dimension of $V$. In general, a wheeled prop can have well-defined representations in $V$ only in the case dim $V < \infty$ as graphs with wheels generate the trace operation $V \times V^* \rightarrow \mathbb{K}$ which explodes in the case dim $V = \infty$; this phenomenon explains the need for 1-oriented props.

How to explain the need for all (or some) of the extra $k$ directions to be oriented? Which structure on a graded vector space $V$ can be used to separate (in the sense of representations) 0-oriented $(k+1)$-directed props from $(k+1)$-oriented ones? Perhaps the main result of this paper is a rather surprising answer to that question: one has to work again with a certain class of infinite-dimensional vector spaces $V$, but now
equipped with \( k \) linear subspaces \( W_1, \ldots, W_k \subset V \) together with their complements, and interpret a single element of a \((k+1)\)-oriented prop \( \mathcal{P} \) as a collection of \( k \) linear maps from various intersections of subspaces \( W_1 \) and their complements and their duals to themselves; then indeed graphs with wheels in one or another extra “coloured direction” get exploded under generic representations and hence must be prohibited. Representations when \( V \) has a symplectic structure and the subspaces \( W_1, \ldots, W_k \) are Lagrangian play a special role in this story, a fact which becomes obvious once all the general definitions are given (see Sect. 4).

A well-known example of such a “brane” algebraic structure in finite dimensions is provided by Manin triples [2]. In the context of this paper, Manin triples construction emerges as a (reduced symplectic Lagrangian) representation of the 2-oriented operad of Lie algebras (note that one cannot describe Manin triples using ordinary operads—one needs an ordinary prop of Lie bialgebras for that purpose). In a sense, multi-oriented props provide us with a far reaching strong homotopy generalization of Manin triples type constructions; they are really a new kind of “Cheshire cat smiles” controlling (via representations) homotopy algebras with branes and admitting in some interesting cases a highly non-trivial action of the Grothendieck–Teichmüller group [1,16].

This is the first of a sequence of papers on multi-oriented props. In the following paper [14], we study several transcendental constructions with multi-oriented props (elucidating their role as the construction material for building new highly non-trivial representations of ordinary props) and use them to prove several concrete deformation quantization theorems. This paper attempts to be as simple as possible and aims for more general audience: we explain here the main notion, illustrate it with examples, prove some theorems on multi-oriented resolutions, and, most importantly, discuss in full details the most surprising part of the story—the representation theory of multi-oriented props in the category of dg vector spaces with branes.

### 1.2 Finite-dimensionality versus infinite one in the context of ordinary props

The theory of (wheeled) operads and props originated in 60s and 70s in algebraic topology and has seen since an explosive development (see, e.g. the books [7,10] or the articles [8,9,22] for details and references). Operads and props provide us with effective tools to discover surprisingly deep and unexpected links between different theories and even branches of mathematics.

A building block of a prop(erad) \( \mathcal{P} = \{ \mathcal{P}(m,n) \}_{m,n \geq 0} \) is a graph (often called corolla)

![Corolla Diagram]

consisting of one vertex which has \( n \) incoming legs and \( m \) outgoing legs and which is decorated with an element of some \( S_m^{op} \times S_n \) module \( \mathcal{P}(m,n) \). Upon a representation of \( \mathcal{P} \) in a graded vector space \( V \), this \((m,n)\)-corolla gets transformed into a linear map.
from $V^\otimes n \to V^\otimes m$, i.e. every leg corresponds, roughly speaking, to a copy of $V$,

\[ \implies V. \]

Such linear maps can be composed which leads us to the idea of considering all possible graphs, for example, these ones

composed from corollas by connecting some output legs of one corolla with input legs of another corolla and so on. The graphs shown above—when translated into linear maps upon some representation of $\mathcal{P}$ in $V$—give us two very different situations: if the left graph makes sense for representations in both finite- and infinite-dimensional vector spaces $V$, the right graph gives us a well-defined linear map only for finite-dimensional vector spaces $V$ as it contains a closed path of directed edges (“wheel”) and hence involves a trace map $V \otimes V^* \to \mathbb{K}$ which is not a well-defined operation in infinite dimensions in general. Hence in order to be able to work in infinite dimensions\(^1\) one has to prohibit certain graphs—the graphs with wheels—and work solely with oriented (from bottom to the top) graphs. Similarly, in order to be able to work with certain completions (defined in 4.6) of various intersections of linear subspaces $W_1 \subset V, \ldots, W_k \subset V$ (“branes”), one has to prohibit certain divergent multi-directed graphs (which already have no wheels with respect to the basic direction!)—and this leads us to the new notion of $(k + 1)$-oriented prop which takes care about more sophisticated divergences associated with branes (the case $k = 0$ recovers the ordinary props); this important “divergency handling” part of our story is discussed in detail in Sect. 4.

There is a nice generalization of the notion of prop which takes care about collections of vector spaces $W_1, \ldots, W_N$. The corresponding props are called coloured props and, say, $N$-coloured (wheeled) prop $\mathcal{P}$ is generated by corollas

whose input and output legs are “coloured” (say, the unique vertex has $a_1$ input legs in “straight colour”, $a_2$ input legs in “dotted colour”, etc) and correspond to linear maps of the form

\[ W_1^{\otimes a_1} \otimes W_2^{\otimes a_2} \otimes \cdots \otimes W_N^{\otimes a_N} \longrightarrow W_1^{\otimes b_1} \otimes W_2^{\otimes b_2} \otimes \cdots \otimes W_N^{\otimes b_N}, \]

\[ a_1 + \cdots + a_N = n, \quad b_1 + \cdots + b_N = m. \]

\(^1\) As the symmetric monoidal category of infinite-dimensional vector spaces is not closed, one must be careful about the definition of the endomorphism prop $\text{End}_V$ in this category, see Sect. 4.1 for details.
Again it makes sense to talk about wheeled (i.e. 0-oriented 1-directed) and ordinary (i.e. 1-oriented) coloured props. In this theory, an oriented leg in “colour” \( i \in \{1, \ldots, N\} \) corresponds to the \( i \)-th vector space

\[
\cdots \quad \Leftrightarrow \quad W_i
\]

in the collection \( \{W_1, \ldots, W_i, \ldots, W_N\} \).

### 1.3 From branes to multi-directed props

A \((k + 1)\)-directed prop \( P_{k+1} = \{P_{k+1}^{(m,n)}\} \) is generated (modulo, in general, some relations) by corollas whose vertex is decorated with an element of some module \( P_{k+1}^{(m,n)} \) (see Sect. 2 for details) and whose input and output legs are decorated with extra (labelled by integers from 1 to \( k \) or by some colours – blue, red, etc—as in the picture above) directions. The “original” (or basic) direction is always shown in pictures in black colour as in the case of ordinary props; it is this basic direction which permits us to call this creature an \((m,n)\)-corolla (it can have different numbers of input and output legs with respect to directions in other colours).

Comparing this picture to the definition of an \( N \)-coloured prop, one can immediately see that a \((k + 1)\)-directed prop is just a special case of a coloured prop when the number of colours is a power of 2,

\[
N = 2^k.
\]  

If we allow all possible compositions of such multi-oriented corollas along legs with identical extra directions, then we get nothing but a \( 2^k \)-coloured prop indeed (called 0-oriented \( k + 1 \)-directed prop). Keeping in mind the key distinction between ordinary and wheeled props, one might contemplate the possibility of prohibiting compositions along graphs which have closed wheels along any of the extra orientations (and no wheels along the basic one), i.e. prohibiting compositions of the form
and allowing only compositions

along the subgraphs with no wheels with respect to any of the directions; let us call a prop generated by such multi-oriented corollas and equipped with such compositions laws (see Sect. 2 for the full list of axioms) a \((k+1)-oriented\) one.

Which structure on graded vector spaces \(V\) can be used to separate (in the sense of representations) 0-oriented \((k+1)\)-directed props from \((k+1)\)-oriented ones (or, more generally, \((l+1)\) oriented \((k+1)\)-directed with \(l \geq 0\))? Note that the compositions prohibited in the \((k+1)\)-oriented prop are still nicely oriented with respect to the basic direction, so the answer cannot be the dimension of \(V\) only.

To make sense of these new restrictions (which have no analogue in the theory of coloured props), we suggest to define a representation of a \((k+1)\)-oriented prop in a graded vector space \(V\) as follows. Assume \(V\) contains a collection of linear subspaces (satisfying certain restrictions in the infinite-dimensional case, see Sect. 4)

\[
W_1^+ \subset V, \quad W_2^+ \subset V, \ldots, \quad W_k^+ \subset V
\]

with chosen complements

\[
V/W_1^+ \simeq W_1^- \subset V, \quad V/W_2^+ \simeq W_2^- \subset V, \quad \ldots, \quad V/W_k^+ \simeq W_k^- \subset V.
\]

Then to a \((k+1)\)-directed outgoing leg, we associate (roughly) an intersection\(^2\)

\[
1 \quad 2 \quad \ldots \quad k \quad \Leftrightarrow \quad W_1^+ \cap W_2^- \cap \ldots \cap W_k^+
\]

\(^2\) Strictly speaking, this is true only in finite dimensions. In infinite dimensions, the subspaces \(W_i^+\) are defined as direct limits of systems of finite-dimensional spaces while their complements \(W_i^-\) always come as projective limits, so their intersection makes sense only at the level of finite-dimensional systems first (it is here where the interpretation of \(W^+\) and \(W^-\) as subspaces of one and the same vector space plays its role), and then taking either the direct or projective limit in accordance with the rule explained in Sect. 4.
obtained by the intersection of “branes” according to the rule:

- to the basic direction we always associate the “full” vector space $V$;
- to the $i$-th direction we associate the vector subspace $W_i^+$ if that direction is in agreement with the basic one, or its complement $W_i^-$ if it is not.

Then any multi-oriented corolla gets interpreted as a collection of $k$ linear maps, one map for each coloured orientation. For example, a 3-directed corolla gets represented in a graded vector space $V$ equipped with two branes $W_1^+, W_2^+ \subset V$ as two linear maps, one corresponding to one blue input and three blue outputs of the corolla,

$$W_1^+ \cap W_2^+ \longrightarrow (W_1^+ \cap W_2^+) \otimes (W_1^+ \cap W_2^-) \otimes (W_1^- \cap W_2^+)^*,$$

and another to three red inputs and one red output of the same corolla,

$$(W_1^+ \cap W_2^+) \otimes (W_1^- \cap W_2^+) \otimes (W_1^+ \cap W_2^-)^* \longrightarrow W_1^+ \cap W_2^+.$$

In finite dimensions, both maps are just re-incarnations of one and the same linear map

$$(W_1^+ \cap W_2^+) \otimes (W_1^- \cap W_2^+) \longrightarrow (W_1^+ \cap W_2^+) \otimes (W_1^+ \cap W_2^-),$$

which is far from being the case in infinite dimensions. Most importantly, this approach to the representation theory of multi-directed props explains nicely why compositions along graphs with wheels in at least one extra orientation must be prohibited. (We show explicit examples of the associated divergences in Sect. 4.) This approach also explains formula (1) for the associated number of “colours” on legs.

### 1.4 Structure of the paper

In Sect. 2, we give a detailed (combinatorial type) definition of multi-oriented props. In Sect. 3, we consider concrete examples. In particular, we introduce and study multi-oriented analogues, $\mathcal{A}ss^{(k+1)}$ and $\mathcal{L}ie^{(k+1)}$, of the classical operads of associative algebras and Lie algebras, and explicitly describe their minimal resolutions $\mathcal{A}ss_{\infty}^{(k+1)}$ and $\mathcal{L}ie_{\infty}^{(k+1)}$; we also construct surprising “forgetting the basic direction” maps from $\mathcal{A}ss^{(2)}$ to the dioperad of infinitesimal bialgebras, and from $\mathcal{L}ie^{(2)}$ to the dioperad of Lie bialgebras (proving that among representations of multi-oriented props we can recover sometimes classical structures); we also introduce a family of $(k+1)$-oriented props of homotopy Lie bialgebras $\mathcal{H}olie^{(c+d-1)}_{c,d}$ on which the Grothendieck–Teichmüller group acts faithfully (see Sect. 5). In Sect. 4, the main section of this paper, we define the notion of a representation of a multi-oriented prop in the category of graded vector spaces with branes and show, as an illustration, that Manin triples give us a class of symplectic Lagrangian representations of $\mathcal{L}ie^{(2)}$.
2 Multi-oriented props

2.1 $S$-bimodules reinterpreted

For a finite set $I$, let $S_I^{(1)}$ be the set of maps

$$s : I \to \{\text{out}, \text{in}\}$$

from $I$ to the set consisting of two elements called $\text{out}$ and $\text{in}$. A finite set $I$ together with a fixed function $s \in S_I^{(1)}$ is called 1-oriented. The collection of 1-oriented sets forms a groupoid $S^{(1)}$ with isomorphisms

$$(I, s) \to (I', s')$$

being bijections $\sigma : I \to I'$ of finite sets such that $s' = s \circ \sigma^{-1}$. The latter condition says that the groupoid $S^{(1)}$ can be identified with the groupoid of Cartesian products, $\{I_{\text{in}} := s^{-1}(\text{in}) \times I_{\text{out}} := s^{-1}(\text{out})\}$, of finite sets.

Let $C$ be a symmetric monoidal category. A functor

$$\mathcal{P} : S^{(1)} \to C$$

$$(I, s) \to \mathcal{P}(I, s)$$

is called an $S^{(1)}$-module. An element $a \in \mathcal{P}(I, s)$ can be represented pictorially as a corolla with $\#I$ legs labelled by elements of $I$ and oriented via the rule: if $s(i) = \text{out}$ (resp., $s(i) = \text{in}$) we orient the $i$-labelled leg by putting the direction “$>$” from (resp., towards) the vertex; the vertex itself is decorated with $a$. For example, an element $a \in \mathcal{P}([6], s)$ can have a pictorial representation of the form

The category of finite sets has a skeleton whose objects are sets $[N] = \{1, 2, \ldots, N\}$ for some $N \geq 0$ (with $[0] = \emptyset$). For $I = [N]$, we often abbreviate $\mathcal{P}_s(N, s) := \mathcal{P}_s([N], s)$. Note that the above corolla is not assumed to be planar so that it can be equivalently represented in a more standard way,

which respects the flow of orientations going from the bottom to the top.

Any $S$-bimodule $E = \{E(m, n)\}_{m,n \geq 0}$ (with each $E(m, n)$ being an $S_m^{op} \times S_n$-module), gives rise to a $S^{(1)}$-module in the obvious way (and vice versa).
2.2 Multi-oriented modules

For a natural number $k \geq 0$ let $\mathcal{O}_{rk^+}$ be the set of all maps $m : [k^+] := \{0, 1, \ldots, k\} \rightarrow \{\text{out}, \text{in}\}$; the value $m(\tau) \subset \{\text{out}, \text{in}\}$ on $\tau \in [k^+]$ is called $\tau$-th orientation; the zeroth orientation $m(0)$ is called the basic one; the map $m$ is called itself a multi-direction. The elements of $[k^+]$ are often called colours. One can represent pictorially a multi-direction $m \in \mathcal{O}_{rk^+}$ as an “outgoing leg” if $m(0) = \text{out}$, or “ingoing leg” if $m(0) = \text{in}$

using the obvious rule: for any $\tau \in [k]$ the value $m(\tau)$ is represented by the $\tau$-coloured symbol “$\tau$” oriented in the same direction as $m(0) = \text{out}$, or in the opposite direction, “$\tau$”, if $m(\tau) \neq m(0)$.

For a finite set $I$, consider the associated set $S_I^{(k+1)}$ of maps

$$s : I \longrightarrow \mathcal{O}_{rk^+}$$

$$i \longrightarrow s_i := s(i) : [k^+] \rightarrow \{\text{out}, \text{in}\}. $$

For $i \in I$, the value $s_i$ on $\tau \in [k^+]$ is called $\tau$-th orientation (or $\tau$-th direction) of the element $i$. For any such a function $s$, there is associated the opposite function $s^{opp} : I \rightarrow \mathcal{O}_{rk^+}$ which is uniquely defined by the following condition: for each $i \in I$ and each $\tau \in [k^+]$ the value of $s^{opp}_i$ on $\tau$ is different from that of $s_i$ on $\tau$. Thus, the set $S_I^{(k+1)}$ comes equipped with an involution. The restriction of the function $s_i$ to the subset $[k] \subset [k^+]$ is denoted by $\tilde{s}_i$; hence, we can write

$$\tilde{s}_i \in \mathcal{O}_rk := \{[k] \rightarrow \{\text{out}, \text{in}\}\}, \quad \forall i \in I.$$ 

This function takes care about extra (i.e. non-basic) orientation assigned to an element $i \in I$. In some pictorial representations of multi-oriented sets $(I, s)$, we show explicitly only the basic orientation while compressing all the extra ones into this symbol $\tilde{s}_i$ (see below).

Note that for any given multi-oriented set $(I, s)$ and any fixed colour $\tau \in [k^+]$, there is an associated map

$$\tilde{s}_\tau : I \longrightarrow \{\text{out}, \text{in}\}$$

$$i \longrightarrow \tilde{s}_\tau(i) := s_i(\tau)$$

which we use in several constructions below.
Using the above pictorial interpretation of elements of $\mathcal{O}r_{k+}$ as multi-oriented legs, one can uniquely represent any element $s \in S_{[k+1]}^{I}$ as a *multi-directed (or multi-oriented) corolla*, that is, as a (non-planar) graph with one vertex $\bullet$ and $\#I$ legs such that each leg is (i) distinguished by an element $i$ from $I$ and (ii) decorated with the multi-direction $s_i \in \mathcal{O}r_{k+}$ as explained just above. For example, a corolla

![Corolla Diagram](image)

represents non-ambiguously some element $s \in S_{[6]}^{(2)}$. In the theory of ordinary props, corollas are often depicted in such a way that the orientation flow runs from the bottom to the top. In the multi-directed case, such a respecting flow representation (now non-unique—one for each coloured direction) also makes sense in applications.

A finite set $I$ together with a fixed function $s \in S_{[k+1]}^{I}$ is called $(k+1)$-oriented. The collection of $(k+1)$-oriented sets forms a groupoid $S^{(k+1)}$ with isomorphisms

$$(I, s) \rightarrow (I', s')$$

being isomorphisms $\sigma : I \rightarrow I'$ of finite sets such that $s' = s \circ \sigma^{-1}$. For example, the automorphism group of the object $([6], s)$ given by corolla (4) is $S_2 \times S_2$ as we can permute only labels (1, 3) and independently (5, 6) using morphisms in the category $S^{(2)}$.

Let $\mathcal{C}$ be a symmetric monoidal category. A functor

$$\mathcal{P}^{(k+1)} : S^{(k+1)} \rightarrow \mathcal{C}$$

$$(I, s) \rightarrow \mathcal{P}^{(k+1)}(I, s)$$

is called an $S^{(k+1)}$-module. Thus, an element of $\mathcal{P}^{(k+1)}(I, s)$ is a pair of the form

$$c = \begin{pmatrix} 6 & 1 & 3 \\ 2 & 4 & 5 \end{pmatrix}, \quad \mathcal{P}^{(k+1)}(c) \in \text{Objects}(\mathcal{C})$$

Note that $\mathcal{P}^{(k+1)}(c)$ carries a representation of the group $\text{Aut}(c)$ (in this particular case, of $S_2 \times S_2$). We shall work in this paper in category of topological vector spaces so that it makes sense to talk about elements $v$ of $\mathcal{P}^{(k+1)}(c)$. The pairs $(c, v)$ are called *decorated corollas* and are often represented pictorially by the corolla $c$ with its vertex decorated (often tacitly) by the vector $v$. Such decorated corollas span $\mathcal{P}^{(k+1)}(I, s)$.

When $k$ is clear, we often abbreviate $\mathcal{P} = \mathcal{P}^{(k+1)}$. The case $k = 0$ corresponds to the ordinary $S$-bimodule.
Given $S^{(k+1)}$-modules $\mathcal{P}$ and $\mathcal{P}'$. A natural transformation of functors
\[
\begin{align*}
f : \mathcal{P} &\rightarrow \mathcal{P}' \iff \{ f_{I,s} : \mathcal{P}(I,s) \rightarrow \mathcal{P}'(I,s) \mid f_{I,s'} \circ \mathcal{P}(\alpha) = \mathcal{P}'(\alpha) \circ f_{I,s} \forall \alpha : (I,s) \rightarrow (I,s') \}\end{align*}
\]
is called a morphism of $S^{(k+1)}$-modules.

### 2.3 Directed and multi-directed graphs

A graph with legs $\Gamma$ is, by definition, a finite set $H(\Gamma)$ (whose elements are called half-edges) equipped with

(a) a partition into the disjoint union of subsets, $H(\Gamma) = \bigsqcup_{\omega \in V(\Gamma)} H(\omega)$, parameterized by a set $V(\Gamma)$ whose elements are called vertices of $\Gamma$; the subset $H(\omega) \subset H(\Gamma)$ is called the set of half-edges attached to the vertex $\omega$ and its cardinality $\# H(\omega)$ is called the valency of $\omega$,

(b) an involution $i : H(\Gamma) \rightarrow H(\Gamma)$ whose fixed points are called legs of $\Gamma$ and whose orbits $(h,i(h)) \subset H(\Gamma)$ of cardinality two are called internal edges or simply edges; the set of edges is denoted by $E(\Gamma)$ and the set of legs is denoted by $L(\Gamma)$.

Any graph with legs $\Gamma$ can be identified with its geometric realization which is a topological space constructed as follows: (i) for each vertex $\omega$ consider the disjoint union of $\# H(\omega)$ copies of the unit interval $[0,1] \subset \mathbb{R}$ and identify all the copies of the end point 0, the result is a topological space (equipped with the quotient topology) which is called the corolla of $\omega$; (ii) consider the union of all stars and, for each internal edge $(h,i(h))$, identify the end-points 1 of the intervals $[0,1]$ labelled by $h$ and $i(h)$.

A graph with legs is called directed if every leg or internal edge in its geometric realization comes equipped with a fixed (one of the two possible) orientations. It is convenient to identify this orientation with a flow on a geometric edge making the latter into an arrow. Here is an example of a directed graph

![Directed Graph](image)

with three vertices and three edges.

If the involution $i$ has no fixed points, i.e. $L(\Gamma) = \emptyset$, the graph with legs is called simply a graph. We shall use directed graphs with legs below when defining multi-oriented props, while in this subsection we continue on only with graphs without legs (i.e. simply with graphs).

By a multi-directed, more precisely, $(k+1)$-directed graph we understand a pair $(\Gamma, s \in S^{(k+1)}_{E(\Gamma)})$ consisting of a directed graph and a function $s : E(\Gamma) \rightarrow \mathcal{O}_{k^+}$ such that for each $e \in E(\Gamma)$ the value of the associated function $s_e : [k^+] \rightarrow \{out,in\}$ takes value “out” (or, equivalently, “in”) at the “zero-th colour” 0 and is identified pictorially with the original (basic) direction of $e$. 

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Thus, the data \( \left( \Gamma, s \in S_{E(\Gamma)}^{(k+1)} \right) \) can be represented pictorially as a graph whose edges are equipped with \((k + 1)\)-directions. Here is an example of a 3-directed graph.

Let \( G^{0\uparrow k+1} \) denote the set of all such \((k + 1)\)-directed graphs. The permutation group \( S_{k+1} \) acts on this set via its canonical action on the set of colours \([k^+]\).

Let \( A \) be a subset of \([k^+]\). A \((k + 1)\)-directed graph \( \Gamma \in G^{0\uparrow k+1} \) is called \( A\)-oriented if \( \Gamma \) contains no closed directed paths of edges ("wheels" or "loops") in every colour \( c \in A \). The subset of \( A\)-oriented graphs is denoted by \( G^{A\uparrow k+1} \). If \( A \) is non-empty, then applying a suitable element of the automorphism group \( S_{k+1} \) we can (and will) assume without loss of generality that \( A = \{0, 1, 2, \ldots, l\} \) for some \( l \geq 0 \) and re-denote \( G^{l\uparrow l\uparrow k+1} := G^{A\uparrow k+1} \). If \( l = k \), we further abbreviate \( G^{(k+1)}\text{-or} := G^{k\uparrow k+1} \) and call its elements \( \text{multi-oriented} \) graphs.

### 2.4 From multi-directed graphs to endofunctors on \( S^{(k+1)}\)-modules

Fix an integer \( k \geq 0 \) and an integer \( l \) in the range \(-1 \leq l \leq k \). For a finite set \( I \) define \( G^{l\uparrow l\uparrow k+1}(I) \) to be the set of \((k + 1)\)-directed \((l + 1)\)-oriented graphs \( \Gamma \) equipped with an injection \( i : I \rightarrow V_1(\Gamma) \), where \( V_1(\Gamma) \subset V(\Gamma) \) is the subset of univalent vertices. The univalent vertices lying in the image \( L(\Gamma) := i(I) \) of this map are called \((k + 1)\)-directed legs of \( \Gamma \); each such leg is labelled therefore by an element \( i \) of \( I \) and is called an \( i \)-leg (in pictures we show it as a leg indeed with the 1-valent vertex erased and the index \( i \) put on its place); vertices in \( V_{\text{int}}(\Gamma) := V(\Gamma) \setminus L(\Gamma) \) are called internal. Edges connecting internal vertices are called internal; there is a decomposition \( E(\Gamma) = L(\Gamma) \sqcup E_{\text{int}}(\Gamma) \). Here are examples of 2-directed graphs, one with 4 internal vertices and 4 legs, the other with 2 internal vertices and 3 legs,

Given an internal vertex \( v \in V_{\text{int}}(\Gamma) \), there is an associated set \( H_v \) of edges attached to \( v \) and an obvious function ("the multi-oriented corolla at \( v \")

\[
\sigma_v : H_v \longrightarrow \mathcal{O}r_{k^+}
\]
There is also an induced function

\[ s : I = L(\Gamma) \longrightarrow \mathcal{O}r_{k^+} \]

on the set of legs defined uniquely by the pictorial rule explained in the first paragraph of Sect. 2.2. Let \( G^{l+1\uparrow k+1}(I, s) \subset G^{l+1\uparrow k+1}(I) \) be the subset of multi-directed (partially oriented, in general) graphs which have one and the same orientation function \( s \) on the set of legs \( I \).

For an \( S^{(k+1)} \)-module \( \mathcal{E} = \{ \mathcal{E}(I, s) \} \) in a symmetric monoidal category \( \mathcal{C} \) with countable coproducts and a graph \( \Gamma \in G^{l+1\uparrow k+1}(I, s) \) consider the unordered tensor product\(^3\) [cf. [8,10]]

\[ \Gamma\langle \mathcal{E}(I, s) \rangle := \left( \bigotimes_{v \in V_{int}(G)} \mathcal{E}(H_v, s_v) \right)_{\text{Aut}(\Gamma)} \]

where \( \text{Aut}(\Gamma) \) stands for the automorphism group of the graph \( \Gamma \), and define an \( S^{(k+1)} \)-module in \( \mathcal{C} \)

\[ \mathcal{F}ree^{l+1\uparrow k+1}(\mathcal{E}) : S^{(k+1)} \longrightarrow \mathcal{C} \]

\[ (I, s) \longrightarrow \mathcal{F}ree^{l+1\uparrow k+1}(\mathcal{E})(s, I) := \bigoplus_{\Gamma \in G^{l+1\uparrow k+1}(I, s)} \Gamma\langle \mathcal{E}(I, s) \rangle \]

As we shall see below, the \( S^{(k+1)} \)-module \( \mathcal{F}ree^{l+1\uparrow k+1}(\mathcal{E}) \) gives us an example of a \((l + 1)\)-oriented \((k + 1)\)-directed prop (called the free prop generated by the \( S^{(k+1)} \)-module \( \mathcal{E} \). For \( l = k = 0 \), this is precisely the ordinary free prop generated by the \( S^{(1)} \)-module \( \mathcal{E} \). For \( l = -1, k = 0 \), this is the free wheeled prop generated by \( \mathcal{E} \) [9,12]. If \( l = k \), i.e. if all directions are oriented, we abbreviate \( \mathcal{F}ree^{k+1\uparrow k+1}(\mathcal{E}) =: \mathcal{F}ree^{(k+1)}(\mathcal{E}) \).

### 2.5 Multi-oriented prop(erad)s

A (possibly disconnected) subgraph \( \gamma \) of a (connected or disconnected) graph \( \Gamma \in G^{l+1\uparrow l+1}(I, s) \) is called complete if every edge of \( \Gamma \) connecting a pair of (not necessarily distinct) vertices of \( \gamma \) belongs to \( \gamma \). Let \( \Gamma/\gamma \) be the graph obtained from \( \Gamma \) by contracting all internal vertices and all internal edges of \( \gamma \) to a single new vertex; note that the legs of \( \Gamma/\gamma \) are the same as in \( \Gamma \) so that \( \Gamma/\gamma \) comes equipped with the same orientation function \( s : L(\Gamma/\gamma) \to \mathcal{O}r_{k^+} \). A complete subgraph \( \gamma \subset \Gamma \) is called admissible if \( \Gamma/\gamma \) belongs to \( G^{l+1\uparrow k+1}(I, s) \), i.e. the contraction procedure does not create wheels in the first \( l + 1 \) coloured directions. Note that by its very definition \( \gamma \) belongs to \( \in G^{l+1\uparrow k+1}(I', s') \), where \( I' \) is the subset of \( E(\Gamma) \) consisting of (non-loop) those edges which are attached to precisely one vertex \( v \) of \( \gamma \), and the function \( s' : I' \to \mathcal{O}r_{k^+} \) is determined by the corresponding functions \( s_v \) in the obvious way.

---

3 The (unordered) tensor product \( \bigotimes_{i \in I} X_i \) of vector spaces \( X_i \) labelled by elements \( i \) of a finite set \( I \) of cardinality, say, \( n \) is defined as the space of \( S_n \)-coinvariants \( \left( \bigoplus_{\sigma \in [n]} \cong_{\sigma} \bigotimes_{i \in I} X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)} \right)_{S_n} \).
An \((l + 1)\)-oriented \((k + 1)\)-directed prop in a symmetric monoidal category (with countable colimits) \(\mathcal{C}\) is, by definition, an \(S^{k+1}\)-module \(\mathcal{P} = \{\mathcal{P}(I, s)\}\) in \(\mathcal{C}\) together with a natural transformation of functors

\[
\mu : \text{Free}^{l+1+k+1}(\mathcal{P}) \to \mathcal{P} \\
\mu_G : \Gamma \langle \mathcal{P} \rangle (I, s) \to \mathcal{P}(I, s)
\]

such that for any graph \(\Gamma \in G^{l+1+k+1}(I, s)\) and any admissible subgraph \(\gamma \subset \Gamma\) one has

\[
\mu_G = \mu_{\Gamma / \gamma} \circ \mu'_{\gamma},
\]

where \(\mu'_{\gamma} : \Gamma \langle \mathcal{P} \rangle (I, s) \to (\Gamma / \gamma) \langle \mathcal{P} \rangle (I, s)\) stands for the obvious map which equals \(\mu_{\gamma}\) on the (decorated) subgraph \(\gamma\) and which is identity on all other vertices of \(\Gamma\).

The most interesting case for us is \(k = l\). The associated props are called multi-oriented (more precisely, \((k + 1)\)-oriented). Thus, a multi-oriented prop is an \(S^{k+1}\)-module \(\mathcal{P}\) equipped, in particular, with

(i) a horizontal composition \(\boxtimes : \mathcal{P}(I_1, s_1) \otimes \mathcal{P}(I_2, s_2) \to \mathcal{P}(I_1 \sqcup I_2, s_1 \sqcup s_2)\),

(ii) a reduced vertical composition: for any two injections of the same finite set \(f_1 : K \to I_1\) and \(f_2 : K \to I_2\) such that the compositions

\[
K \xrightarrow{s_1 \circ f_1} \mathcal{O}r_{k^+} \quad , \quad K \xrightarrow{s_2 \circ f_2} \mathcal{O}r_{k^+}
\]

satisfy the condition \(s_1 \circ f_1 = (s_2 \circ f_2)^\text{opp}\) there is a vertical composition

\[
\mathcal{P}(I_1, s_1) \circ_K \mathcal{P}(I_2, s_2) \to \mathcal{P}_{s_{12}}((I_1 \setminus f_1(K)) \sqcup (I_2 \setminus f_2(K)), s_{12})
\]

where

\[
s_{12} : (I_1 \setminus f_1(K)) \sqcup (I_2 \setminus f_2(K)) \to \mathcal{O}r_{k^+}
\]

is defined by \(s_{12}(i) = s_1(i)\) for \(i \in I_1 \setminus f_1(K)\) and \(s_{12}(i) = s_2(i)\) for \(i \in I_2 \setminus f_2(K)\).

These compositions are required to satisfy the “associativity” axioms which essentially say that upon iterating such compositions the order in which we do it does not matter. In terms of decorated corollas, these compositions correspond to contraction maps (for \(k = 1\))

\[
\boxtimes : \begin{array}{c}
\begin{array}{c}
\bullet \\
I_1 \\
\bullet \\
I_2 \\
\bullet \\
I_3 \\
\bullet \\
I_4 \\
\bullet \\
\bullet
\end{array}
\end{array} \times \begin{array}{c}
\begin{array}{c}
\bullet \\
J_1 \\
\bullet \\
J_2 \\
\bullet \\
J_3 \\
\bullet \\
J_4 \\
\bullet \\
\bullet
\end{array}
\end{array} \to \begin{array}{c}
\begin{array}{c}
\bullet \\
I_1 \sqcup I_2 \\
\bullet \\
I_3 \sqcup I_4 \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\]

\(\boxtimes\) Springer
Note that compositions of the form

\[ I \rightarrow \bigcup J \]

are prohibited in 2-oriented props (as they contain at least one wheel in blue colour), but are allowed in 1- or 0-oriented 2-directed props. In the case of 1-oriented (i.e. ordinary) prop, the horizontal and reduced vertical compositions generate upon iteration any prop composition \( \mu_{\Gamma : \Gamma P(I, s) \rightarrow P(I, s)} \). However, this is not true in the case of generic \((k+1)\)-oriented props with \(k \geq 2\). For example, a prop composition

\[ \mu_{\Gamma} : \]

cannot be represented as an iteration of more elementary (i.e. two vertex) compositions. It is important to notice, however, that such a representation is always possible.
with respect to each coloured orientation, and different orientations produce different (sometimes incompatible as in this particular case) decompositions of $\mu_\Gamma$ into a sequence of horizontal and reduced vertical operations; moreover, it is this latter property which is really important when we consider representations of multi-oriented props (see Sect. 4 below).

If in the above definition of the natural transformation $\mu$ we restrict only to the subset $G^l+1\uparrow^{k+1}(I, s) \subset G^l+1\uparrow^{k+1}(I, s)$ of connected graphs, we get the notion of an $(l + 1)$-oriented $(k + 1)$-directed properad $P$ (cf. [22]). In this case, we do not have horizontal compositions in $P$, only vertical ones. There is an obvious exact functor from $(k + 1)$-directed properads to $(k + 1)$-directed props.

For any $S^{(k+1)}$-module $E$, the associated $S^{(k+1)}$-module $F_{ree}^{l+1\uparrow^{k+1}}(E)$ is a $(l + 1)$-oriented $(k + 1)$-directed prop with contraction maps $\mu_\Gamma$ being tautological. It is called the free multi-directed prop generated by $E$. If $l = k$, it is called the free multi-oriented (more precisely, $(k + 1)$-oriented) prop generated by $E$.

2.6 Morphisms of multi-oriented props

Let $P$ and $P'$ be $(l + 1)$-oriented $(k + 1)$-directed props. A morphism

$$f : P \longrightarrow P'$$

of $S^{(k+1)}$-modules is called a morphism of $(l + 1)$ oriented $(k + 1)$-directed props if for any $\Gamma \in G^l+1\uparrow^{k+1}(I, s)$ the associated diagram

$$\begin{array}{ccc}
\Gamma\langle P\rangle(I, s) & \xrightarrow{\mu_\Gamma} & P(I, s) \\
\downarrow^{f \otimes \# V(\Gamma)} & & \downarrow^{f_{I, s}} \\
\Gamma\langle P'\rangle(I, s) & \xrightarrow{\mu_\Gamma} & P(I, s)'
\end{array}$$

commutes.

As usual, any morphism $f : F_{ree}^{l+1\uparrow^{k+1}}(E) \rightarrow P'$ from a free prop is uniquely determined by its values on the generators $E$.

2.7 Multi-oriented operads

If a multi-oriented properad $P = \{P(I, s)\}$ is such that $P(I, s)$ vanishes unless $\tilde{s}_0^{-1}(out) = 1$ (i.e. the functor $P$ is non-trivial only on multi-oriented corollas with precisely one outgoing leg with respect to the basic direction) it is called a multi-oriented operad. Note that there is no such a restriction on non-basic directions.

2.8 Multi-oriented dioperads

Restricting the functor $F_{ree}^{l+1\uparrow^{k+1}}$ (and denoting it by $F_{ree}^{l+1\uparrow^{k+1}}_0$) and the compositions $\mu_\Gamma$ to connected graphs of genus zero, only one gets the notion of a
(k + 1)-directed (l + 1)-oriented dioperad. Note that a multi-oriented operad is a special case of a multi-oriented dioperad, and the difference between the notions is rather small.

2.9 Multi-directed wheeled prop(erad)s

The above definitions make sense in the case l = −1 as well; the associated 0-oriented (k + 1)-directed prop(erad)s \( P \) are called \((k + 1)\)-directed wheeled prop(erad)s (cf. [9,12]). In this case, graphs \( \Gamma \) can have an internal edge connecting one and the same vertex, and any prop composition \( \mu/\Gamma \) can decomposed into an iteration of horizontal compositions and reduced vertical composition defined above, and also of a trace map

\[
Tr_K : P(I, s) \rightarrow P \left( I \backslash [f_1(K) \sqcup f_2(K), s'] \right)
\]

which is defined for any two injections \( f_1 : K \rightarrow I, f_2 : K \rightarrow I \) of a finite set \( K \) such that \( f_1(K) \cap f_2(K) = \emptyset \) and \( s \circ f_1 = (s \circ f_2)^{opp} \); the orientation function \( s' \) is obtained from \( s \) by its restriction to the subset \( I \backslash [f_1(K) \sqcup f_2(K)] \).

Thus in the family of \((k + 1)\)-directed \((l + 1)\) oriented props, the special case \((l = -1, k \geq 0)\) corresponds to the ordinary \(2^k\)-coloured wheeled prop while the case \((l = 0, k \geq 0)\) to the ordinary \(2^k\) coloured prop. Thus, the really new cases must have \( k \geq l \geq 1 \).

In the next section Sect. 3, we introduce multi-oriented versions of some classical operads and props at the combinatorial level (which is straightforward). In Sect. 4, we discuss representations of multi-oriented props, i.e. explain what these multi-oriented graphical combinatorics give us in practice (and this step is, perhaps, not that straightforward).

2.10 Ordinary props as multi-oriented ones

By the very definition, the category of ordinary props is precisely the category of 1-oriented props. It is worth mentioning a canonical but very naive functor for any \( k \geq 1 \)

\[
O^{(k+1)} : \text{Category of ordinary props} \rightarrow \text{Category of (k + 1)oriented props}
\]

\[
P = \{ P(I, s) \} \rightarrow O^{(k+1)}(P) = \{ P(I, s^{(k+1)}) \}
\]

which simply associates to an \( S^{(1)} \)-module \( \{ P(I, s) \} \) an \( S^{(k+1)} \)-module \( \{ P(I, s^{(k+1)}) = P(I, s) \} \) which is non-trivial (and coincides with \( \{ P(I, s) \} \) only for those multi-orientations \( s^{(k+1)} \) in which all extra directions are aligned coherently with the basic one. More precisely, given an ordinary (1-oriented) prop

\[
P(I, s : I \rightarrow O_{r0^+} \equiv \{out, in\})
\]
we define

$$P(I, s^{(k+1)}) := \left\{ \begin{array}{ll} P(I, s) & \text{if } s^{(k+1)} \text{ satisfies } s^{(k+1)}_i(\tau) := s(i) \forall i \in I, \forall \tau \in [k^+], \\ 0 \text{ or } \emptyset & \text{otherwise.} \end{array} \right.$$ 

We do not use this naive functor in this paper (as it gives nothing new), but it is worth keeping in mind that all classical props can be “embedded” into the category of \((k + 1)\)-oriented props; at least nothing is lost.

Similarly one can interpret a \((k + 1)\)-oriented prop as a \((k + l + 1)\)-oriented prop for any \(l \geq 1\). In the next section, we consider much less naive extensions of classical operads and props to the multi-oriented setting.

3 Multi-oriented versions of some classical operads and props

3.1 Multi-oriented operad of (strongly homotopy) associative algebras

Let us recall an explicit combinatorial description of the operad \(Ass\) of associative algebras in terms of planar 1-oriented corollas. By definition, \(Ass\) is the quotient,

$$Ass := \mathcal{F}ree^{1-or} \langle A \rangle / (R)$$

of the free operad \(\mathcal{F}ree^{1-or} \langle A \rangle\) generated by an \(S\)-module \(A = \{A(n)\}_{n \geq 0}\) with

$$A(n) := \left\{ \begin{array}{ll} \mathbb{K}[S_2] \equiv \text{span} \left( \begin{array}{c} 0 \\ 1 \\ 2 \\ 2 \\ 1 \end{array} \right) & \text{if } n = 2, \\ 0 & \text{otherwise} \end{array} \right.$$ 

by the ideal generated by the relation (together with its \(S_3\) permutations),

$$\begin{array}{ccc} 0 & - & 0 \\ 1 & 2 & 3 \end{array} = 0$$

Its minimal resolution is a dg free operad, \(Ass_{\infty} := (\mathcal{F}ree \langle E \rangle, \delta)\) generated by the \(S\)-module \(E = \{E(n)\}_{n \geq 2}\) (whose generators we represent pictorially as planar corollas of homological degree \(2 - n\))

$$E(n) := \mathbb{K}[S_n][n - 2] = \text{span} \left( \begin{array}{c} 0 \\ 1 \\ 2 \\ 2 \\ 1 \end{array} \right)_{\sigma \in S_n}$$
and equipped with the differential given on the generators by

\[
\delta = \sum_{r=0}^{n-2} \sum_{l=2}^{n-r} (-1)^{r+l+n-r-l+1} \sum_{\bar{s} \in \mathcal{O}_{rk}} (-1)^{rl} \bar{s} \cdot \bar{s}_{r+1} \bar{s}_{r+2} \ldots \bar{s}_{n-1} \bar{s}_n \text{ if } \#I \leq 2
\]

\[
\delta = 0 \text{ if } \bar{s}_0^{-1}(out) \neq 1
\]

\[
\delta = \text{span}(ord(I'))[\#I - 3] \text{ otherwise.}
\]

Let us first consider the most naive multi-oriented generalization of \(\text{Ass}_\infty\) in which we enlarge the set of generators by decorating each leg of each planar corolla with \(k\) extra orientations in all possible ways while preserving its homological degree,

In some pictures, we show explicitly only the basic direction, while extra directions are indicated only by extra orientation functions \(\bar{s}_i\). Denote \(\mathcal{A}_{\text{subig}}^{(k+1)} := \mathcal{F}ree^{(k+1)}-\text{or}\{A^{(k+1)}\}\) be the free ("very big") operad generated by these corollas, more precisely, by the associated \(S^{(k+1)}\)-module \(A^{(k+1)} = \{A^{(k+1)}(I, s)\}\) defined formally as follows:

Here, \(ord(I')\) is the set of total orderings on the finite set \(I' = I \setminus \bar{s}_0^{-1}(out)\). The differential in \(\text{Ass}_\infty\) can be extended to \(\mathcal{A}_{\text{subig}}^{(k+1)}\) by summing over all possible ways of attaching extra directions \(\bar{s} \in \mathcal{O}_{rk}\) to the internal edge,
Hence, $\mathcal{As}vbig_{\infty}^{(k+1)}$ is just the $2^k$-coloured extension of $\mathcal{Ass}_{\infty}$. Note that the generating corollas in $\mathcal{As}vbig_{\infty}^{(k+1)}$ have at least one ingoing leg and at least one outgoing leg with respect to the basic direction (this condition kills “curvature terms” in that direction). As we shall see in the next chapter (where we introduce representations of multi-oriented props), it is actually extra directions (if present) which play the genuine role of inputs and outputs. Hence to avoid “curvature terms” with respect to any direction, we have to consider an ideal $I_1$ in the free operad $\mathcal{As}vbig_{\infty}^{(k+1)}$ generated by those corollas which have no output or no input leg(s) with respect to at least one extra orientation. It is easy to see that the above differential $\delta$ respects this ideal so that the quotient

$$\mathcal{As}big_{\infty}^{(k+1)} := \mathcal{As}vbig_{\infty}^{(k+1)}/I_1$$

is a dg free operad again. It is generated by a “smaller” set of generators, but still that set can be further reduced. Note that once the basic direction is fixed, the set of extra orientations $\mathcal{O}_k$ can be identified with the set of words of length $k$ in two letters, $>$ and $<$, and hence can be equipped with a lexicographic order $\leq$. Let us call a generating corolla

special if $\tilde{s}_1 \leq \tilde{s}_2 \leq \ldots \leq \tilde{s}_n$ (i.e. if the planar order agrees with the lexicographic one), and let $I_2$ be the ideal in the free operad $\mathcal{As}vbig_{\infty}^{(k+1)}$ generated by non-special corollas. It is again easy to see that the differential $\delta$ respects that second ideal so that the quotient

$$\mathcal{Ass}_{\infty}^{(k+1)} := \mathcal{As}big_{\infty}^{(k+1)}/I_2$$

is a dg free operad generated by the special corollas (essentially, the main point of this discussion is to motivate the claim that the derivation of $\mathcal{Ass}_{\infty}^{(k+1)}$ given on the generating special corollas by formula (8) is a differential). It is called the multi-oriented operad of strongly homotopy associative algebras. Let $J$ be the differential closure of the ideal in the free (viewed as a non-differential) operad $\mathcal{Ass}_{\infty}^{(k+1)}$ generated by the above corollas with $n \geq 3$. The quotient

$$\mathcal{Ass}^{(k+1)} := \mathcal{Ass}_{\infty}^{(k+1)}/J$$

is called a multi-oriented operad of associative algebras. We shall see below that this multi-oriented operad controls structures which are governed, in some special case, by ordinary dioperads. For example, a representation of $\mathcal{Ass}^{(2)}$ in a symplectic vector space with one Lagrangian brane can be identified with an infinitesimal bialgebra structure on that brane.
3.1.1 The simplest non-trivial case $k = 1$ in more detail

The dg operad $A_{ss}^{(2)}$ is generated by planar corollas of homological degree $2 - \#I - \#J$

\[
\delta = \sum_{I = I_1 \sqcup I_2 \sqcup I_3} (-1)^{\#I_1 + \#I_2 + \#I_3 + \#J + 1}
\]

\[
+ \sum_{J = J_1 \sqcup J_2 \sqcup J_3} (-1)^{(\#I + \#J_1)(\#J_2 + \#J_3 + 1)}
\]

\[
+ \sum_{I = I_1 \sqcup I_2 \quad J = J_1 \sqcup J_2} (-1)^{\#I_1(\#I_2 + \#J_1) + \#J_2 + 1}
\]

and similarly for the second corolla. Here, the summations run over decompositions of the totally ordered sets into disjoint unions of connected (with respect to the order) subsets.

The operad $A_{ss}^{(2)}$ is generated by the following planar corollas (in homological degree zero)
while the relations are given by

\[
\begin{align*}
    i_0 i_1 i_2 i_3 &= i_0 i_3 i_2 i_1, \\
    i_0 i_1 i_2 i_3 &= i_0 i_3 i_2 i_1 + i_0 i_2 i_3 i_1, \\
    i_0 i_1 i_2 i_3 &= i_0 i_3 i_2 i_1 + i_0 i_1 i_3 i_2.
\end{align*}
\]

(9)

\[
\begin{align*}
    i_0 i_1 i_2 i_3 &= i_0 i_3 i_2 i_1, \\
    i_0 i_1 i_2 i_3 &= i_0 i_3 i_2 i_1 + i_0 i_2 i_3 i_1, \\
    i_0 i_1 i_2 i_3 &= i_0 i_3 i_2 i_1 + i_0 i_1 i_3 i_2.
\end{align*}
\]

(10)

One can describe similarly the operad $\text{Ass}^{(k+1)}$ in terms of generators and relations.

3.1.2 Theorem

The natural projection $\text{Ass}^{(k+1)}_{\infty} \rightarrow \text{Ass}^{(k+1)}$ is a quasi-isomorphism.

Proof We have to show that $H^\bullet(\text{Ass}^{(k+1)}_{\infty}(I, s)) = \text{Ass}^{(k+1)}(I, s)$ for any $(k + 1)$-oriented set $(I, s)$. In fact, it is enough to show that the cohomology of the operad $H^\bullet(\text{Ass}^{(k+1)}_{\infty}(I, s))$ is concentrated in degree zero because that would imply the required equality due to the fact that the complex $\text{Ass}^{(k+1)}_{\infty}(I, s)$ is non-positively graded.

We shall prove the claim by induction over $\#I = n + 1$, and abbreviate the notation $\text{Ass}^{(2)}_{\infty}(n) := \text{Ass}^{(2)}_{\infty}(I, s)$ and $\text{Ass}^{(2)}(n) := \text{Ass}^{(2)}(I, s)$. When $n = 2$, the equality $H^\bullet(\text{Ass}^{(2)}_{\infty}(n)) = \text{Ass}^{(2)}(n)$ is obvious. Assume it is true for all multi-oriented sets with $\#I \leq n + 1$, and consider the complex $\text{Ass}^{(k+1)}_{\infty}(n + 1)$; we can assume without loss of generality that the input (with respect to the basic colour) legs of any graph from $\text{Ass}^{(2)}_{\infty}(n + 1)$ are labelled from left to right (in accordance with the planar structure) by $1, 2, \ldots, n + 1$ (while the root vertex by $n + 2$).

Consider first a filtration of $\text{Ass}^{(2)}_{\infty}(n + 1)$ by the total number of vertices lying on the path from the root edge to the leg labelled by 1 (and call it a special path), and let $\text{Gr}(n + 1)$ denote the associated graded. Consider next a filtration of $\text{Gr}(n + 1)$ by the total valency of vertices lying on the special path (and denote the set of such
vertices by \( V_{sp} \), and let \((\mathcal{E}_r, \delta_r)\) be the associated spectral sequence (converging to \(H^\bullet(\mathbb{A}_{ss}^{(2)}(n + 1))\)). The initial page \((\mathcal{E}_0, \delta_0)\) is isomorphic to the direct sum of tensor products of complexes of the form \(\mathbb{A}_{ss}^{(k+1)}(n')\) with all \(n' \leq n\) so that by the induction hypothesis we can easily describe the next page of the spectral sequence:

\[
\mathcal{E}_1 = H^\bullet(\mathcal{E}_0) \cong \bigoplus_{\text{special paths } n = \sum_{v \in V_{sp} n_v} v \in V_{sp}} \bigotimes_{n_v \geq 1} C_v(n_v)
\]

where \(C_v(n_v)\) is a complex spanned by planar corollas of the form

\[
\begin{array}{c}
\ldots \\
\vdots \\
\ldots \\
\end{array}
\]

whose dashed legs belong to the given special path (and are equipped with the induced multi-orientations from that special path) while solid legs are decorated by arbitrary elements of the unital extension of the operad \(\mathbb{A}_{ss}^{(k+1)}\),

\[
a_i \in \mathbb{A}_{ss}^{(k+1)}(n_i) := \begin{cases} 
\mathbb{A}_{ss}^{(k+1)}(n_i) & \text{if } \# n_i \geq 2 \\
\mathbb{K} & \text{if } \# n_i = 1, \ i \in [p], \\
0 & \text{if } \# n_i = 0
\end{cases}
\]

subject to the condition that

\[
\sum_{i=1}^{p} n_i = n_v,
\]

The differential on \(C_v(n_v)\) is non-trivial only on the root corolla on which it acts as follows (we suppress some extra orientations in the picture),

\[
\delta_1 = \begin{cases} 
\sum_{i=0}^{p-2} \sum_{\bar{s} \in \mathbb{O}_k} (-1)^{p-i+1} & \text{for } p \geq 3 \\
-\sum_{\bar{s} \in \mathbb{O}_k} & \text{for } p = 2 \\
0 & \text{for } p = 1.
\end{cases}
\]

Claim. The cohomology of the complex \(C_v(n_v)\) is concentrated in cohomological degree zero.
Indeed, consider a one-step filtration of $C_v(n_v)$ by the number of three-valent vertices of the form $\bullet$ and the associated two pages spectral sequence. It is easy to see that the complex on the initial page is a direct sum of a trivial complex spanned by graphs of the form $\bullet$ with $a_1 \in \text{Ass}^{(k+1)}(n_v)$ and a non-trivial complex which is quasi-isomorphic to the degree shifted (direct summand) subcomplex of $(\text{Ass}^{(k+1)}(n_v))[1, \delta]$ spanned by graphs with the orientation of the unique root leg fixed by the multi-orientation of the corresponding dashed edge of the given special path (indeed, take a filtration of the latter subcomplex by the valency of the root vertex and use the induction assumption). As $n_v \leq n$, we conclude (again by the induction assumption) that its cohomology is equal to

$$A := \text{span} \left( \begin{array}{c} \begin{array}{c} \bullet \\ a_1 \\ a_2 \end{array} \end{array} \right) \text{mod } \text{Ass}^{(k+1)}\text{-relations},$$

$$a_1 \in \text{Ass}^{(k+1)}(n_1), \ a_2 \in \text{Ass}^{(k+1)}(n_2), \ n_1 + n_2 = n_v$$

The induced differential on the next (and final) page of the spectral sequence is an injection

$$d : A \longrightarrow \text{Ass}^{(k+1)}(n_v) = \text{span} \left( \begin{array}{c} \begin{array}{c} \bullet \\ a \end{array} \end{array} \right), \ a \in \text{Ass}^{(k+1)}(n_v)$$

$$\sum_{\bar{s} \in \text{Or}_k} \bar{s}$$

which proves the CLAIM.

We conclude that the cohomology $H^\bullet(\text{Ass}^{(k+1)}(n+1))$ is generated by multi-oriented graphs of the form (modulo some relations corresponding to the image of the injection $d$)

$$\text{where } l := \#V_{sp}, \ a_{v_i} \in \text{Ass}^{(k+1)}(n_{v_i}), \ \sum_{i=1}^l n_{v_i} = n$$

which all have cohomological degree zero. Hence, $H^\bullet(\text{Ass}^{(k+1)}(n+1))$ is concentrated in degree zero implying its identification with $\text{Ass}^{(k+1)}(n+1)$. The induction argument and hence the proof the Theorem are completed.

In the next subsection, we discuss representations of $\text{Ass}^{(k+1)}$, that is, associative algebras with kbranes. Rather surprisingly, we recover, in particular, a well-known
notion of infinitesimal bialgebra as an associative algebra with one (symplectic Lagrangian) brane. This interesting fact can be seen already now (i.e. in purely combinatorial way) as follows.

3.2 Infinitesimal bialgebras as 2-oriented associative algebras

Recall that an ordinary (i.e. 1-oriented) dioperad of infinitesimal associative bialgebras is, by definition [5], the quotient of the 1-oriented free dioperad

\[ \mathcal{IB} := \mathcal{F}ree_{1-or}^1 \langle B \rangle / R \]

generated by an \( S \)-bimodule \( B = \{ B(m, n) \} \)

\[
B(m, n) := \begin{cases} 
\mathbb{K}[S_2] \otimes \mathbb{I}_1 \equiv \text{span} \left\{ \begin{array}{c} 1 \quad 2 \quad 1 \\ 0 \quad 0 \end{array} \right\} & \text{if } m = 2, n = 1, \\
\mathbb{I}_1 \otimes \mathbb{K}[S_2] \equiv \text{span} \left\{ \begin{array}{c} 0 \\ 2 \quad 2 \quad 1 \end{array} \right\} & \text{if } m = 1, n = 2, \\
0 & \text{otherwise}
\end{cases}
\]

by the ideal \( R \) generated by the following relations

\[
\begin{align*}
\begin{array}{c c c c}
1 & 2 & 3 \\
\circ & \circ & \circ
\end{array} - \begin{array}{c c c c}
1 & 3 & 2 \\
\circ & \circ & \circ
\end{array} &= 0, \\
\begin{array}{c c c c}
1 & 2 & 3 \\
\circ & \circ & \circ
\end{array} - \begin{array}{c c c c}
1 & 2 & 3 \\
\circ & \circ & \circ
\end{array} &= 0, \\
\begin{array}{c c c c}
1 & 2 & 3 & 4 \\
\circ & \circ & \circ & \circ
\end{array} - \begin{array}{c c c c}
1 & 3 & 4 & 2 \\
\circ & \circ & \circ & \circ
\end{array} - \begin{array}{c c c c}
1 & 2 & 3 & 4 \\
\circ & \circ & \circ & \circ
\end{array} &= 0.
\end{align*}
\]

Here, all internal edges and legs are assumed to be oriented along the flow running from the bottom of a graph to its top.

3.2.1 Proposition (cf. [19])

There is a (forgetting the basic orientation) morphism of dioperads

\[ \alpha : \mathcal{Ass}^{(2)} \longrightarrow \mathcal{IB} \]

given on the generators as follows:

\[
\begin{align*}
\alpha \left( \begin{array}{c}
\begin{array}{c c c c}
1 & 2 & 3 \\
\circ & \circ & \circ
\end{array}
\end{array} \right) &:= \begin{array}{c c c c}
0 & 0 & 1 \\
1 & 2 & 3
\end{array}, \\
\alpha \left( \begin{array}{c}
\begin{array}{c c c c}
1 & 2 & 3 \\
\circ & \circ & \circ
\end{array}
\end{array} \right) &:= \begin{array}{c c c c}
2 & 0 & 1 \\
1 & 0 & 2
\end{array}, \\
\alpha \left( \begin{array}{c}
\begin{array}{c c c c}
1 & 2 & 3 \\
\circ & \circ & \circ
\end{array}
\end{array} \right) &:= \begin{array}{c c c c}
0 & 0 & 1 \\
1 & 2 & 3
\end{array}
\end{align*}
\]
\[ \alpha \left( \begin{array}{c} 0 \\ 1 \\ 2 \\ \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 2 \\ \end{array} \right) \]

\textbf{Proof} It is straightforward to check that each of the six relations in (9)--(10) is mapped under \( \alpha \) into one of the above three relations in \( TB \). Hence, the map is well defined indeed.

This proposition indicates that the notion of \textit{representation} of a multi-oriented prop(erad) cannot be an immediate generalization of that notion for ordinary coloured prop(erad)s— the extra orientations are not really "colours", and the distinction between operads and dioperads should become void.

3.3 Example: Multi-oriented operad of Lie and \( \mathcal{L}ie_\infty \) algebras

Recall that the ordinary operad of strongly homotopy Lie algebras is the free operad \( \mathcal{L}ie_\infty := (\text{Free}^{1\text{-or}}(L), \delta) \) generated by an \( \mathbb{S} \)-module \( L = \{L(n)\}_{n \geq 2} \) with

\[ L(n) := sgn_n[n - 2] = \text{span} \left( \begin{array}{c} \sigma_2 \ldots \sigma_n \end{array} \right), \forall \sigma \in \mathbb{S}_n \]

where \( sgn_n \) is the 1-dimensional sign representation of \( \mathbb{S}_n \). The differential is given on the generators by

\[ \delta = \sum_{I=I_1\sqcup I_2} (-1)^{|I_2|+1} \]

It is essentially a skew-symmetrized version of \( \mathcal{A}ss_\infty \) (there is a canonical morphism of operads \( \mathcal{L}ie_\infty \to \mathcal{A}ss_\infty \) sending a generator of \( \mathcal{L}ie_\infty \) into a skew-symmetrization of the corresponding generator of \( \mathcal{A}ss_\infty \)). If \( I \) is the differential closure of the ideal in \( \mathcal{L}ie_\infty \) generated by all corollas with negative cohomological degree, then the quotient

\[ \mathcal{L}ie := \mathcal{L}ie_\infty / I \]

is an operad controlling Lie algebras. It is generated by degree zero planar skew-symmetric corollas

\[ = - \]

\( \text{Springer} \)
modulo the Jacobi relation,

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{fig1.png}
\end{array} & + & \begin{array}{c}
\includegraphics[width=0.1\textwidth]{fig2.png}
\end{array} + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{fig3.png}
\end{array} = 0
\end{array}
\end{align*}
\]

The natural surjection \( \mathcal{L}ie_{\infty} \to \mathcal{L}ie \) is a quasi-isomorphism.

An operad of \((k+1)\)-oriented strongly homotopy Lie algebras is defined as an obvious skew-symmetrization of the operad \( \mathcal{A}ss_{\infty}^{(k+1)} \) introduced in the previous subsection (so that there is again a canonical morphism of dg operads \( \mathcal{L}ie_{\infty}^{(k+1)} \to \mathcal{A}ss_{\infty}^{(k+1)} \)). More precisely, the operad \( \mathcal{L}ie_{\infty}^{(k+1)} \) is a free \( (k+1) \)-oriented operad generated by corollas with the same symmetries and degrees as in the case of \( \mathcal{L}ie_{\infty} \), but now with each leg decorated with extra \( k \)-orientations,

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{fig4.png}
\end{array}
\]

with the condition that there is at least one ingoing edge and outgoing edge with respect to each of the new directions. The differential is given by the same formula as in the case of \( \mathcal{L}ie_{\infty} \) except that now we sum over all possible (and admissible) new orientations attached to the new edge

\[
\delta = \sum_{[1,...,n]=I_1\sqcup I_2 \atop |I_1|\geq 1,|I_2|\geq 1} \sum_{\sigma \in \text{Or}_k} (-1)^{1+\#I_2+\text{sgn}(I_1,I_2)}
\]

where the first sum runs over decompositions of the ordered set \([n]\) into the disjoint union of (not necessarily connected) ordered subsets, and \( \text{sgn}(I_1,I_2) \) stands for the parity of the permutation \([n] \to I_1 \sqcup I_2 \).

In more detail, the operad of 2-oriented strongly homotopy Lie algebras is, by definition, a free 2-oriented operad generated by the following skew-symmetric planar corollas of degree \( 2-n, n \geq 2 \),

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig5.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig6.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig7.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig8.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig9.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig10.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig11.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig12.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig13.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig14.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig15.png}
\end{array} = (-1)^{\sigma+\tau}
\end{array}
\]
for any $\sigma \in S_r$ and $\tau \in S_l$. As in the case of $As_{\infty}^{(k+1)}$, we require that each corolla has at least one ingoing leg and at least one outgoing leg in each extra orientation (in order to avoid curvature terms in representations).

The differential is given by (and it is easy to check that $\delta$ is a differential indeed)

$$
\delta = \sum_{I=I_1 \sqcup I_2, J=J_1 \sqcup J_2} (-1)^{\# I_1 (\# I_2 + \# J_1) + \# J_2 + 1 + \text{sign}(I_1, I_2) + \text{sign}(J_1, J_2)}
$$

and similarly for the second class of corollas. Here, the sums run over all admissible decompositions of the ordered sets $I$ and $J$ into the disjoint unions of (not necessarily connected) ordered subsets, and $\text{sign}(I_1, I_2)$ (resp., $\text{sign}(J_1, J_2)$) stands for the parity of the permutation $I \rightarrow I_1 \sqcup I_2$ (resp., $J \rightarrow J_1 \sqcup J_2$).

If $I$ is the differential closure of the ideal in $\mathcal{L}ie^{(k+1)}_{\infty}$ generated by all corollas with negative cohomological degree, then the quotient

$$
\mathcal{L}ie^{(k+1)} := \mathcal{L}ie_{\infty}^{(k+1)}/I
$$

is called an operad of multi-oriented Lie algebras.

### 3.3.1 Theorem

*The natural projection $\mathcal{L}ie^{(k+1)}_{\infty} \rightarrow \mathcal{L}ie^{(k+1)}$ is a quasi-isomorphism.*

**Proof** It is enough to show that the cohomology of $\mathcal{L}ie^{(k+1)}_{\infty}$ is concentrated in degree zero, and this can be done by the arity induction in a close analogy to the proof of Theorem 3.1.2. We omit the details. $\square$

### 3.4 The operad of 2-oriented Lie algebras versus the ordinary dioperad of Lie bialgebras

The operad $\mathcal{L}ie^{(2)}$ can be explicitly described as follows: it is generated by the following list of degree 0 corollas,
modulo the following relations

\begin{align}
0_{123} &+ 0_{123} + 0_{123} = 0, \\
0_{123} - 0_{123} - 0_{123} &+ 0_{123} + 0_{123} = 0 \\
0_{123} &+ 0_{123} - 0_{123} - 0_{123} + 0_{123} = 0 \\
0_{123} - 0_{123} + 0_{123} &+ 0_{123} = 0
\end{align}

Representations of the operad of 2-oriented Lie algebras in symplectic vector spaces with one Lagrangian brane are studied in the next section where it is shown that they can be identified with famous Manin’s triples which give us an alternative (and often very useful) characterization of \textit{Lie bialgebras}. Hence, the combinatorics of the latter structure must be hidden in the combinatorics of the former one, and our next our purpose to make this inter-relation explicit.

Recall that the ordinary (i.e. 1-oriented) dioperad of Lie bialgebras is the quotient

\[ \mathcal{L}ieb_{\text{diop}} := \mathcal{F}ree_{0}^{1\text{-or}}(M) / J \]
of the free 1-oriented free dioperad generated by an \( \mathfrak{S} \)-bimodule \( M = \{ M(m, n) \} \) with

\[
M(m, n) := \begin{cases}
\text{sgn}_2 \otimes \mathbb{1}_1 \equiv \text{span} \left\{ \begin{array}{c}
\begin{array}{c}
1 \\
0
\end{array} = - \begin{array}{c}
2 \\
0
\end{array}
\end{array} \right\} & \text{if } m = 2, n = 1, \\
\mathbb{1}_1 \otimes \text{sgn}_2 \equiv \text{span} \left\{ \begin{array}{c}
\begin{array}{c}
0 \\
1
\end{array} = - \begin{array}{c}
2 \\
1
\end{array}
\end{array} \right\} & \text{if } m = 1, n = 2, \\
0 & \text{otherwise}
\end{cases}
\]

modulo the ideal \( J \) generated by the following relations

\[
\begin{align*}
\begin{array}{c}
1 \\
2 \\
3
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} = 0, \\
\begin{array}{c}
3 \\
2 \\
1
\end{array} + \begin{array}{c}
3 \\
2 \\
1
\end{array} = 0,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
1 \\
2 \\
3
\end{array} - \begin{array}{c}
1 \\
2 \\
3
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} - \begin{array}{c}
1 \\
2 \\
3
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} = 0.
\end{align*}
\]

3.4.1 Proposition (cf. [2])

There is a (forgetting the basic orientation) morphism of dioperads

\[
\beta : \mathcal{L}ie^{(2)} \longrightarrow \mathcal{L}ieb_{\text{diop}}
\]

given on the generators as follows:

\[
\beta \left( \begin{array}{c}
\begin{array}{c}
0 \\
1
\end{array}
\end{array} \right) := \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
0
\end{array}
\end{array}, \quad \beta \left( \begin{array}{c}
\begin{array}{c}
0 \\
1
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0
\end{array}
\end{array},
\]

\[
\beta \left( \begin{array}{c}
\begin{array}{c}
0 \\
1
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0
\end{array}
\end{array}, \quad \beta \left( \begin{array}{c}
\begin{array}{c}
0 \\
1
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0
\end{array}
\end{array},
\]

Proof It is straightforward to check that each of the eight relations in (12)–(14) is mapped under \( \beta \) into one of the above three relations for \( \mathcal{L}ieb_{\text{diop}} \). Hence, the map is well defined indeed. \( \square \)

This result gives us a purely combinatorial interpretation of the famous Manin triple construction [2].
3.5 Multi-oriented prop of homotopy Lie bialgebras

Let us recall a graded generalization of the classical prop of Lie bialgebras depending on two integer parameters $c, d \in \mathbb{Z}$. By definition [3] (one can also see [16] for more details), $\text{Lieb}_{c, d}$ is a quadratic properad given as the quotient,

$$\text{Lieb}_{c, d} := \mathcal{F}ree^{1-\text{or}}(Q)/(R),$$

of the free properad generated by an $S$-bimodule $Q = \{Q(m, n)\}_{m, n \geq 1}$ with all $Q(m, n) = 0$ except

$$Q(2, 1) := 1 \otimes \text{sgn}^c_2[c - 1] = \text{span}\left\{ \begin{array}{c} 2 \\ 1 \\ 3 \end{array} \right\} = (-1)^c \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$Q(1, 2) := \text{sgn}^d_2 \otimes 1[d - 1] = \text{span}\left\{ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = (-1)^d \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right\}$$

by the ideal generated by the following relations

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array} + \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array} + \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array}, \quad \begin{array}{c} 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \end{array} + \begin{array}{c} 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \end{array} + \begin{array}{c} 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \end{array}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array} - \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array} - (-1)^d \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array} - (-1)^{d+c} \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array} - (-1)^c \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array}$$

Its minimal resolution $\mathcal{H}_{\text{Lieb}}_{c, d}$ is a 1-oriented dg free properad generated by the following (skew)symmetric corollas of degree $1 + c(1 - m) + d(1 - n)$

$$\begin{array}{c} \sigma(1) \\ \tau(1) \end{array} \cdots \begin{array}{c} \sigma(m) \\ \tau(m) \end{array} = (-1)^{c|\sigma| + d|\tau|} \forall \sigma \in S_m, \forall \tau \in S_n$$

(15)

and has the differential given on the generators by

$$\delta = \sum_{[1, \ldots, m] = I_1 \sqcup I_2, \ |I_1| \geq 0, |I_2| \geq 1} \sum_{[1, \ldots, n] = J_1 \sqcup J_2, \ |J_1| \geq 1, |J_2| \geq 1} (-1)^{\#I_1 \#J_2 + \#I_1 + \#J_2 + c \text{sgn}(I_1, I_2) + d \text{sgn}(J_1, J_2)}$$



(16)
where sign($I_1$, $I_2$) (resp., sign($J_1$, $J_2$)) stands for the parity of the permutation $I \mapsto I_1 \sqcup I_2$ (resp., $J \mapsto J_1 \sqcup J_2$). The case $c = d = 0$ corresponds to ordinary strong homotopy Lie bialgebras \cite{11,22}, while the case $c = 1$, $d = 0$ to formal Poisson structures on graded vector spaces $V$ viewed as linear manifolds \cite{12}.

A $(k + 1)$-oriented generalization of $\mathcal{H}olieb_{c,d}$ is quite straightforward: the prop $\mathcal{H}olieb_{c,d}^{(k+1)}$ is a free $(k + 1)$-oriented prop generated by corollas with the same symmetries and degrees as in the case of $\mathcal{H}olieb_{c,d}$, but now with each leg decorated with $k$ extra orientations,

subject to the condition that there is at least one ingoing edge and one outgoing edge in each of the new directions. The differential is given by the same formula as in the case of $\mathcal{H}olieb_{c,d}$ except that now we sum over all possible (and admissible) new orientations attached to the new edge

$$
\delta = \sum_{[1,...,m]=I_1 \sqcup I_2} \sum_{[m+1,...,m+n]=J_1 \sqcup J_2} \sum_{|I_1|\geq 0,|I_2|\geq 1} \sum_{|J_1|\geq 1,|J_2|\geq 1} \sum_{\tilde{s} \in \mathcal{O}_{r_k}} (-1)^{|I_1| |J_2|+|I_2| |J_1|+c \cdot \text{sgn}(I_1,J_2)+d \cdot \text{sgn}(J_1,J_2)}
$$

The homotopy theory of such props can be highly non-trivial. As we discuss in more detail below in Sect. 5, the automorphism group of $\mathcal{H}olieb_{c,d}^{(c+d-1)}$ is equal to the Grothendieck–Teichmüller group $GRT = GRT_1 \times \mathbb{K}^*$ (for any $c, d \in \mathbb{Z}$ with $c+d \geq 2$), and hence, this prop can be a foundation for a rich deformation quantization theory in every geometric dimension $c+d \geq 2$. This fact was one of our main motivations to introduce and study the multi-oriented props.

Let $K$ be the differential closure of the ideal in $\mathcal{H}olieb_{c,d}^{(k+1)}$ generated by all corollas with total arity $\geq 4$ and denote the quotient by

$$
\mathcal{L}ieb_{c,d}^{(k+1)} = \mathcal{H}olieb_{c,d}^{(k+1)} / K.
$$

This prop gives us a $(k + 1)$ oriented version of $\mathcal{L}ieb_{c,d}$. It is quite easy to see that the automorphism group of the prop $\mathcal{L}ieb_{c,d}^{(c+d-1)}$ with $c, d \geq 1$ and $c+d \geq 3$ is almost trivial—it is equal to $\mathbb{K}^*$ acting by rescalings of the generators. In particular, the induced action of $GRT_1$ on $\mathcal{L}ieb_{c,d}^{(k+1)}$ with $c, d \geq 1$ and $c+d \geq 3$ is trivial; this is in sharp contrast to the 1-oriented case $c = d = 1$ where that action remains highly non-trivial.
Our main purpose in this paper is to introduce the notion of a representation of a multi-oriented prop in the category of dg vector spaces (with branes) which is done in the next section.

4 Multi-directed endomorphism prop and homotopy algebras with branes

4.1 Tensor algebra of infinite-dimensional vector spaces

By a countably infinite-dimensional graded vector space $V$ we understand in this paper any direct limit $V := \lim_{\rightarrow} V_p$ of a direct system of finite-dimensional vector spaces $V_p$, $p \geq 1$,

\[
V_0 \longrightarrow V_1 \overset{i_1}{\longrightarrow} V_2 \overset{i_2}{\longrightarrow} \cdots \overset{i_{p-1}}{\longrightarrow} V_p \overset{i_p}{\longrightarrow} V_{p+1} \overset{i_{p+1}}{\longrightarrow} \cdots \tag{18}
\]

where all arrows $i_p$ are proper injections. For example, $K^\infty = \lim_{\rightarrow} K^n$ with

\[
0 \longrightarrow K \overset{i_1}{\longrightarrow} K^2 \overset{i_2}{\longrightarrow} \cdots \overset{i_p}{\longrightarrow} K^p \overset{i_{p+1}}{\longrightarrow} K^{p+1} \overset{i_{p+2}}{\longrightarrow} \cdots ,
\]

\[
i_p(a_1, \ldots, a_p) := (a_1, \ldots, a_p, 0),
\]

is an example of a countably infinite-dimensional vector space.

Next, we define (non-countably) infinite-dimensional vector spaces

\[
\text{Hom}(\otimes^r V, \otimes^l V) = \lim_{\leftarrow} \left( V^*_{p_1} \otimes \cdots \otimes V^*_{p_r} \otimes V^{\otimes l} \right).
\]

which are equipped with the standard projective limit topology. An element $f \in \text{Hom}(\otimes^r V, \otimes^l V)$ is called a linear map

\[
f : \otimes^r V \longrightarrow \otimes^l V
\]

from $\otimes^r V$ to $\otimes^l V$. Such maps can be composed (no divergences) so that one has a well-defined endomorphism prop

\[
\mathcal{E}nd_V = \left\{ \mathcal{E}nd_V(l, r) := \text{Hom}(\otimes^r V, \otimes^l V) \right\}
\]

associated to $V$ and hence talk about representations of ordinary props in $V$.

Note that $\text{Hom}(V, V)$ can contain infinite sums of the form

\[
\sum_{n,m=1}^\infty a^n_m e^m \otimes e_n, \quad e^m \in V^*_m, e_n \in V_n, a^n_m \in K, \quad \text{all } a^n_m \neq 0,
\]
so that the trace operation on \( \text{Hom}(V, V) \) (and hence on \( \text{Hom}(\otimes^k V, \otimes^l V) \)) is not well defined in general so that \( V \) cannot be used for representations of wheeled props.

If one has a collection of \( k \) infinite-dimensional vector spaces, \( V_\tau = \lim_{\rightarrow} V_{\tau, p}, \tau \in [k], \) one can define similarly a \( k \)-coloured endomorphism prop \( \mathcal{E}nd_{V_1, \ldots, V_k} \) based on topological \( \mathbb{S} \)-modules

\[
\mathcal{E}nd_{V_1, \ldots, V_k} = \left\{ \text{Hom}(V_1^{\otimes r_1} \otimes \cdots \otimes V_k^{\otimes r_k}, V_1^{\otimes l_1} \otimes \cdots \otimes V_k^{\otimes l_k}) \right\}
\]

Its elements give us linear maps \( V_1^{\otimes r_1} \otimes \cdots \otimes V_k^{\otimes r_k} \rightarrow V_1^{\otimes l_1} \otimes \cdots \otimes V_k^{\otimes l_k} \) and hence can be used to define a representation of a \( k \)-coloured prop.

### 4.2 An infinite-dimensional graded vector space with \( k \) branes

Let \( V := \lim_{\rightarrow} V_p \) be a countably infinite-dimensional graded vector space. It is called a vector space with \( k \) branes (and denoted by \( (V, W_1, \ldots, W_k) \) or simply by \( V^{k\text{-br}} \)) if the following conditions hold:

(i) \( V \) comes equipped with a descending filtration

\[
V = F^0 V \supset F^1 V \supset F^2 V \supset \cdots \supset F^p V \supset F^{p+1} V \supset \cdots
\]

such that each quotient vector space \( V/F^p V \) is finite-dimensional and is isomorphic to \( V_p \) for any \( p \in \mathbb{N} \),

(ii) For any \( p \geq 0 \) we have \( k \) different non-trivial direct sum decompositions \( V_p = W^+_{\tau, p} \oplus W^-_{\tau, p}, \tau \in [k], \) which are compatible with the given injections \( i_p : V_p \rightarrow V_{p+1} \),

\[
i_p(W^\pm_{\tau, p}) \subset W^\pm_{\tau, p+1}
\]

Note that the inclusion \( F^{p+1} V \subset F^p V \) induces a projection

\[
\pi_{p+1} : V_{p+1} \equiv F^{p+1} V / F^{p+2} V \rightarrow V_p := F^p V / F^{p+1} V
\]

providing us with an inverse system of finite-dimensional vector spaces,

\[
\cdots \rightarrow W^\pm_{\tau, p} \rightarrow W^\pm_{\tau, p-1} \rightarrow \cdots \rightarrow 0
\]

Hence can consider two limits for branes (and their intersections, see below), the direct and projective ones,
\[
W_\tau^\pm := \lim_p W_{\tau,p}^\pm \subset \hat{W}_\tau^\pm := \lim_p \hat{W}_{\tau,p}^\pm,
\]
\[
(W_\tau^\pm)^\ast = \lim_p (W_{\tau,p}^\pm)^\ast \supset (\hat{W}_\tau^\pm)^\ast = \lim_p (W_{\tau,p}^\pm)^\ast, \quad \forall \tau \in [k].
\]

Note that the spaces \(W_\tau^\pm\) and \((\hat{W}_\tau^\pm)^\ast\) are always countably dimensional, while \((W_\tau^\pm)^\ast\) and \(\hat{W}_\tau^\pm\) are, in general, not (but as a compensation they come equipped with nice topologies). Note also that
\[
((\hat{W}_\tau^\pm)^\ast)^\ast = \hat{W}_\tau^\pm, \quad ((W_\tau^\pm)^\ast)^\ast = W_\tau^\pm.
\]

To define a suitable multi-oriented endomorphism prop out of an infinite-dimensional vector space with \(k\) branes, one has to work with both types of completions simultaneously. This fact motivates the extra filtration condition (i) in the definition of \(V^{k\text{-br}}\) above.

### 4.2.1 Basic example

Let \(\{x_1, x_2, \ldots, \}\) be a countably infinite set of formal variables of some homological degrees \(\lvert x_i \rvert \in \mathbb{Z}, i \in \mathbb{N}_{\geq 1}\). The graded vector space
\[
V = \text{span}(x_1, x_2, \ldots)
\]
is a typical example of an infinite-dimensional vector space satisfying conditions (i) and (ii) above with
\[
F^p V = \text{span}(x_i)_{i \geq p+1}, \quad V_p = \text{span}(x_1, x_2, \ldots, x_p).
\]

Let us choose \(k\) injections of countably infinite sets (i.e. \(k\) pairs of disjoint countably infinite subsets of \(\mathbb{N}_{\geq 1}\))
\[
f_\tau : \mathbb{N}_{\geq 1} \oplus \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}, \quad \tau \in [k],
\]
and define a family of finite-dimensional vector spaces (equipped with the \(\mathbb{Z}\)-grading induced in the obvious way from the homological grading of the formal variables \((x_1, x_2, \ldots, x_p)\))
\[
W_{\tau,p}^+ := \text{span}\left(\pi_+ \circ f_\tau^{-1}\{1, 2, \ldots, p\}\right), \quad W_{\tau,p}^- := \text{span}\left(\pi_- \circ f_\tau^{-1}\{1, 2, \ldots, p\}\right)
\]
where \(\pi_{\pm : \mathbb{N}_{\geq 1} \oplus \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}}\) is the projection to the first/second summand. The resulting data \((V, W_k^\pm, \ldots, W_k^\pm)\) is an example of an infinite-dimensional graded vector space with \(k\) branes.
4.3 Finite-dimensional case

A finite-dimensional vector space with k branes is simply a finite-dimensional vector space \( V \) equipped with \( k \) direct sum decompositions \( V = W^+_\tau \oplus W^-_\tau, \tau \in [k] \). We shall be most interested in the infinite-dimensional case and hence use all the time the direct/projective limit notation introduced in the previous section. The finite-dimensional version fits into that notation as a special case when \( V_p = V, W^\pm_{\tau, p} = W^\pm_\tau \) for all \( p \).

4.4 The (simplest non-trivial) case of a 2-directed endomorphism prop

Let \( V^{1-\text{br}} \) be an infinite-dimensional vector space with one brane, and consider

\[
W^+ = \lim_{\rightarrow} W^+_p, \quad \hat{W}^- = \lim_{\leftarrow} W^-_p, \quad (W^+)^* = \lim_{\rightarrow} (W^+_p)^*, \quad (\hat{W}^-)^* = \lim_{\rightarrow} (W^-_p)^*
\]

Note that out of these four spaces, only \( W^+ \) and \( (\hat{W}^-)^* \) are always countably dimensional. Introduce next an \( S^{(2)} \)-module, that is a functor

\[
\mathcal{E}nd_{V^{1-\text{br}}} : S^{(2)} \longrightarrow \text{Category of graded vector spaces}
\]

as follows. First notice that one can identify any 2-oriented set\(^4\)

\[
(I, s : I \to \mathcal{O}_{r^{1+}}) \equiv (I, s^\circ_0 : I \to \{\text{out}, \text{in}\}, s^\circ_1 : I \to \{\text{out}, \text{in}\} \text{ such that } s^\circ_0(i) := s_i(\hat{0}), s^\circ_1(i) := s_i(\hat{1}) \forall i \in I)
\]

with a 2-directed corolla (the subscript 0 indicates the basic orientation)

\[
\text{(20)}
\]

where

\[
I^{\text{out}, \text{out}^0} := s^{-1}_1(\text{out}) \sqcup s^{-1}_0(\text{out}), \quad I^{\text{in}, \text{out}^0} := s^{-1}_1(\text{in}) \sqcup s^{-1}_0(\text{out}),
\]

\[
I^{\text{out}, \text{in}^0} := s^{-1}_1(\text{out}) \sqcup s^{-1}_0(\text{in}), \quad I^{\text{in}, \text{in}^0} := s^{-1}_1(\text{in}) \sqcup s^{-1}_0(\text{in}),
\]

\(^4\) Here, we denote the elements of \([1^+]\) by \( \hat{0} \) and \( \hat{1} \) so that the value \( s_i \) of the map \( s \) on an element \( i \in I \) is itself a map of sets \( s_i : \{\hat{0}, \hat{1}\} \to \{\text{out}, \text{in}\} \).
Set
\[ \#I_{\text{out,out}_0} = m_1, \quad \#I_{\text{in,out}_0} = m_2, \quad \#I_{\text{in,in}_0} = n_1, \quad \#I_{\text{out,in}_0} = n_2. \]

Next, we define\(^5\) (cf. (19))
\[
\mathcal{E}nd_{V^1\text{-br}}(I, \mathfrak{s})
\]
\[
:= \lim_{p_a \to a} \lim_{p_b \to b} \left( \lim_{p_c \to c} \lim_{e \in I_{\text{out,in}_0}} \bigotimes_{e \in I_{\text{out,in}_0}} (W^+_p)^* \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \right)
\]
\[
= \lim_{p_a \to a} \lim_{p_b \to b} \lim_{p_c \to c} \lim_{e \in I_{\text{out,in}_0}} \Hom \left( \bigotimes_{a \in I_{\text{in,in}_0}} W^+_p \bigotimes_{e \in I_{\text{out,in}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \right)
\]
\[
= \Hom \left( \bigotimes_{a \in I_{\text{in,in}_0}} W^+_p \bigotimes_{e \in I_{\text{out,in}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \right)
\]

Thus, an element \( f \in \mathcal{E}nd_{V^1\text{-br}}(I, \mathfrak{s}) \) gives us a well-defined map
\[
f : \bigotimes_{a \in I_{\text{in,in}_0}} W^+_p \bigotimes_{e \in I_{\text{out,in}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \rightarrow \bigotimes_{a \in I_{\text{in,in}_0}} W^+_p \bigotimes_{e \in I_{\text{out,in}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c)
\]
between countably dimensional vector spaces, and hence, such elements can be composed along the “blue direction”. What about the basic direction? We can try rearranging tensor factors in \( \mathcal{E}nd_{V^1\text{-br}}(I, \mathfrak{s}) \) as follows:
\[
\mathcal{E}nd_{V^1\text{-br}}(I, \mathfrak{s})
\]
\[
:= \lim_{p_a \to a} \lim_{p_b \to b} \left( \lim_{p_c \to c} \lim_{e \in I_{\text{out,in}_0}} \bigotimes_{e \in I_{\text{out,in}_0}} (W^+_p)^* \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \right)
\]
\[
= \lim_{p_a \to a} \lim_{p_b \to b} \lim_{p_c \to c} \lim_{e \in I_{\text{out,in}_0}} \Hom \left( \bigotimes_{a \in I_{\text{in,in}_0}} W^+_p \bigotimes_{e \in I_{\text{out,in}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \right)
\]
\[
= \Hom \left( \lim_{p_a \to a} \lim_{p_c \to c} \bigotimes_{a \in I_{\text{in,in}_0}} W^+_p \bigotimes_{e \in I_{\text{out,in}_0}} (W^-_p)^* \bigotimes_{c \in I_{\text{out,out}_0}} (W^+_c) \bigotimes_{b \in I_{\text{out,out}_0}} (W^-_c) \right)
\]

However, in general,

---

\(^5\) Here we use the facts that for any vector space \( M \) and any inverse system of finite-dimensional vector spaces \( \{N_i\} \) one has \( \lim \Hom(N_i, M) \cong \Hom(\lim N_i, M) \) and \( \lim \Hom(M, N_i) \cong \Hom(M, \lim N_i) \), while \( \lim(N_i \otimes M) \cong (\lim N_i) \otimes M \) only if \( M \) is finite-dimensional. On the other hand, for any direct system \( \{N_i\} \) the equality \( \lim (M \otimes N_i) \cong M \otimes \lim N_i \) holds true for any \( M \), while the equality \( \lim \Hom(M, N_i) \cong \Hom(M, \lim N_i) \) is true if and only if \( M \) is finite-dimensional.
$$\otimes^{n_1} W^+ \otimes \otimes^{n_2} W^- := \lim_{\to} \lim_{\leftarrow} \otimes_{\substack{a \in I_{t_{in},t_{in_0}} \\text{out}_{t_{in},t_{in_0}}}}^{p_a} W^+ \otimes_{\substack{e \in I_{t_{out},t_{in_0}} \\text{out}_{t_{in},t_{in_0}}}}^{e} W^-$$

with the l.h.s. being a (proper, in general!) subspace of the r.h.s. Hence, elements of the Hom-spaces

$$\text{End}_{V_{1-br}}(I, s) \cong \text{Hom} \left( \otimes^{n_1} W^+ \otimes \otimes^{n_2} W^- , \otimes^{m_1} W^+ \otimes \otimes^{m_2} W^- \right)$$

cannot be composed, in general, along graphs of the type shown in (7). (Nevertheless, the latest formula shows that any element of $\text{End}_{V_{1-br}}(I, s)$ can be understood as some linear map along the basic direction.)

We conclude that the $S(2)$-module $\text{End}_{V_{1-br}}$ admits nice compositions $\mu_{\Gamma}$ along any graphs $\Gamma$ not containing closed paths of directed edges in blue colour as in (6) (with the “associativity” axioms are obviously satisfied) and hence gives us an example of 2-oriented prop. We call it the endomorphism prop of $V_{1-br}$.

Note that if $V_{1-br}$ is finite-dimensional (or at least if $W^-$ is finite-dimensional), then

$$\text{End}_{V_{1-br}}(I, s) \cong \text{Hom} \left( \otimes^{n_1} W^+ \otimes \otimes^{n_2} W^- , \otimes^{m_1} W^+ \otimes \otimes^{m_2} W^- \right)$$

$$\cong \text{Hom} \left( \otimes^{n_1} W^+ \otimes \otimes^{m_2} (W^-)^* , \otimes^{m_1} W^+ \otimes \otimes^{n_2} (W^-)^* \right)$$

and the compositions $\mu_{\Gamma}$ (in the definition of a multi-directed prop) make sense for any graphs $\Gamma \in G^{0 \uparrow k+1}$.

### 4.4.1 Definition

Let $P^{2-or}$ be a 2-oriented prop(erad). A morphism of 2-oriented prop(erad)s

$$\rho : P^{2-or} \longrightarrow \text{End}_{V_{1-br}}$$

is called a representation of $P^{2-or}$ in the vector space $V$ with one brane.

### 4.4.2 Example

A representation of a 2-oriented operad $\text{Ass}^{(2)}$ (resp., $\text{Lie}^{(2)}$) in $V_{1-br}$ is given by a collection of linear maps

$$W^+ \otimes W^+ \to W^+, \quad (W^-)^* \to (W^-)^* \otimes (W^-)^*,$$

$$W^+ \to W^+ \otimes (W^-)^*, \quad W^+ \otimes (W^-)^* \to (W^-)^*$$

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such that their compositions satisfy relations (9)–(10) [respectively, (12)–(14)].

To illustrate the divergency phenomenon, let us consider a generic representation of a prop in which compositions along graphs with non-trivial genus make sense. For example, a representation of, say, the prop of 2-oriented Lie bialgebras \( Lieb^{2-or} \) in \( V^{1-br} \) is given by maps as above plus the following ones

\[
\begin{align*}
\mu_1 : W^+ \to W^+ \otimes W^+, & \quad (\hat{W}^*)^* \to W^+ \otimes (\hat{W}^-)^* \quad \mu_3 : W^+ \otimes (\hat{W}^-)^* \to W^+,
\end{align*}
\]

satisfying certain quadratic relations. If \( \{x_{a^+}\}_{a^+ \in \mathbb{N}_{\geq 1}} \) is a countably infinite basis of \( W^+ \), \( \{x_{a^-}\}_{a^- \in \mathbb{N}_{\geq 1}} \) a countably infinite basis of \( W^- \), and \( \{y_{a^+}\}_{a^+ \in \mathbb{N}_{\geq 1}} \) and \( \{y_{a^-}\}_{a^- \in \mathbb{N}_{\geq 1}} \) the associated countably infinite dual bases of \( (\hat{W}^+)^* \) and, respectively, \( (\hat{W}^-)^* \) then the corresponding maps, say the following ones

\[
\begin{align*}
\mu_1 : W^+ \otimes W^+ \mu_1 W^+, & \quad W^+ \mu_2 \to W^+ \otimes (\hat{W}^-)^*, \quad W^+ \mu_3 \to W^+ \otimes W^+, \\
\mu_4 : W^+ \otimes (\hat{W}^-)^* \mu_4 W^+,
\end{align*}
\]

can be represented by the following infinite, in general, sums

\[
\begin{align*}
\mu_1 &= \sum_{a^+,b^+,c^+ \in \mathbb{N}_{\geq 1}} \Phi_{a^+b^+}^{c^+} y_{a^+}^{b^+} \otimes y_{b^+}^{c^+} x_{c^+}, \\
\mu_2 &= \sum_{a^+,b^-,c^+ \in \mathbb{N}_{\geq 1}} \Phi_{a^+b^-}^{c^+} x_{c^+}^{b^+} \otimes x_{a^+}^{b^-} \otimes y_{b^+}^{c^+}, \quad \Phi_{a^+b^+}^{c^+}, \Phi_{a^+b^-}^{c^+} \in \mathbb{K} \\
\mu_3 &= \sum_{a^+,b^+,c^+ \in \mathbb{N}_{\geq 1}} \Psi_{a^+b^+}^{c^+} x_{c^+}^{a^+} \otimes x_{a^+}^{b^+} \otimes x_{b^+}^{c^+}, \\
\mu_4 &= \sum_{a^+,b^-,c^+ \in \mathbb{N}_{\geq 1}} \Psi_{a^+b^-}^{c^+} x_{a^+}^{b^-} \otimes x_{b^-}^{c^+} \otimes x_{a^+}^{c^+}, \quad \Psi_{a^+b^+}^{c^+}, \Psi_{a^+b^-}^{c^+} \in \mathbb{K},
\end{align*}
\]

where the coefficients satisfy the conditions:

- for fixed \( a^+, b_- \) only finitely many \( \Phi_{a^+b^+}^{c^+} \neq 0 \); for fixed \( c^+ \) only finitely many \( \Phi_{c^+b_-}^{a^+} \neq 0 \);
- for fixed \( c^+ \) only finitely many \( \Psi_{c^+b^+}^{a^+b^-} \neq 0 \); for fixed \( a^+, b_- \) only finitely many \( \Psi_{a^+b^-}^{c^+} \neq 0 \).
Then, the element

$$\in \mathcal{L}ieb^{2-or}$$

gets represented in $V^{1-br}$ as a linear map

$$\sum_{c^+, d^+ \in \mathbb{N}_{\geq 1}} \left( \sum_{a^+, b^+ \in \mathbb{N}_{\geq 1}} \frac{\Phi_{a^+, b^+, \Psi_{a^+, b^+}}}{c^+ a^+ b^+} \right) y^{d^+} \otimes x_{c^+} : W^+ \to W^+$$

which is always well defined, while the element

$$\in \mathcal{L}ieb^{1\;\uparrow\;2}$$

gets represented in $V^{1-br}$ as a formal sum of linear maps

$$\sum_{c^+, d^+ \in \mathbb{N}_{\geq 1}} \left( \sum_{a^+, b^+ \in \mathbb{N}_{\geq 1}} \frac{\Phi_{a^+, b^+, \Psi_{a^+, b^+}}}{c^+ a^+ b^+} \right) y^{d^+} \otimes x_{c^+} : W^+ \to W^+$$

which in general diverges. Such a map makes sense in general in general only in the case $\dim W^+ < \infty$ or $\dim W^- < \infty$.

### 4.4.3 Symplectic vector space with Lagrangian branes

Let $(V, \omega : \wedge^2 V \to \mathbb{K})$ be a finite-dimensional vector space equipped with a symplectic form; in general $\dim V = 2n$ for some $n \in \mathbb{N}_{\geq 1}$. A subspace $W \subset V$ is called isotropic if

$$W \subset W^\perp := \{ v \in V \mid \omega(v, w) = 0 \forall w \in W \}.$$
Such a subspace is called Lagrangian if \( \dim W = n \). It is well known that a Lagrangian subspace \( W^+ \subset V \) always admits a complement \( W^- \subset V \) which is also Lagrangian. Moreover, in this case the symplectic form induces a canonical isomorphism

\[
\omega : (W^-)^* \longrightarrow W^+.
\]

The data \((V, W^+, W^-)\) is called a finite-dimensional symplectic vector space with one Lagrangian brane. Similarly, one defines finite-dimensional symplectic vector space with \( k \) Lagrangian branes, \((V, W^+_\tau, W^-_\tau)_{\tau \in [k]}\). We generalize this notion to infinite dimensions as follows.

Let \((V, W_1, \ldots, W_k)\) be a countably infinite-dimensional vector space with \( k \) branes such that for each \( p \) and each \( \tau \in [k] \) the vector space \( V_p = W^+_{\tau,p} \oplus W^-_{\tau,p} \) is a finite-dimensional symplectic vector space with \( k \) Lagrangian branes. Then, the symplectic form induces a linear map

\[
\omega : (\hat{W}_\tau^-)^* \longrightarrow W^+_{\tau}
\]

which is an isomorphism for each \( \tau \in [k] \). The resulting data \((V, W_1, \ldots, W_k, \omega)\) is called an infinite-dimensional symplectic vector space with \( k \) Lagrangian branes and denoted by \( V^{k\text{-br}}_{\text{symp}} \).

If we consider now a generic representation \( \rho \) of, say, \( \text{Ass}^{(2)} \) or \( \text{Lie}^{(2)} \) in \( V^{1\text{-br}}_{\text{symp}} \), then, due to the canonical isomorphism \((\hat{W}_\tau^-)^* = W^+_{\tau}\), we see that multi-oriented generators which differ only in the basic orientation stand for linear maps of the same type, for example

\[
\begin{align*}
\rho : W^+ \otimes W^+ &\longrightarrow W^+, \\
\rho : W^+ \otimes W^+ &\longrightarrow W^+,
\end{align*}
\]

and hence, it makes sense to identify them. We call a representation \( \rho \) in \( V^{1\text{-br}}_{\text{symp}} \) reduced symplectic Lagrangian if \( \rho \) takes identical values on all generating corollas of \( \mathcal{P}^{2\text{-or}} \) which become identical (as \( k \)-oriented graphs) once the basic directions on edges are forgotten. Then, we can reformulate observations Sects. 3.2.1 and 3.4.1 as follows.

4.4.4 Proposition

(i) There is a one-to-one correspondence between reduced symplectic Lagrangian representations of \( \text{Ass}^{(2)} \) in \( V^{1\text{-br}}_{\text{symp}} \) and infinitesimal bialgebra structures in the Lagrangian subspace \( W^+ \).

(ii) There is a one-to-one correspondence between reduced symplectic Lagrangian representations of \( \text{Lie}^{(2)} \) in \( V^{1\text{-br}}_{\text{symp}} \) and Lie bialgebra structures in the Lagrangian subspace \( W^+ \).
4.5 Remark

In principle, one can use symplectic structures on $V$ of homological degree $q \neq 0$ so that the induced isomorphism takes the form $\omega : (\hat{W}_\tau)^* \to W^+_\tau[q]$, but then the basic direction cannot be forgotten completely in representations as it stands now for a degree shift of linear maps.

4.6 Multi-directed endomorphism prop of a graded vector space with $k$ branes

Let $(I, s : I \to O_{r+k})$ be a multi-oriented set. Recall that for any fixed $i \in I$ there is an associated map

$$s_i : [k^+] \to \{\text{out}, \text{in}\}. $$

while for any fixed $\tau \in [k^+]$ there is a map

$$s_\tau : I \to \{\text{out}, \text{in}\}$$

$$i \to s_\tau(i) := s_i(\tau).$$

The latter map can be used to decompose $I$ into two disjoint subsets

$$I = s^{-1}_\tau(\text{out}) \sqcup s^{-1}_\tau(\text{in})$$

The basic direction $\tau = 0$ plays a special role. For any $\tau \neq 0$, i.e. for any $\tau \in [k]$ we can further decompose the set $I$ into four disjoint subsets

$$I = (s^{-1}_\tau(\text{out}) \sqcup s^{-1}_\tau(\text{in})) \cap (s^{-1}_0(\text{out}) \sqcup s^{-1}_0(\text{in})) := I^{\text{out, out}_0} \sqcup I^{\text{out, in}_0} \sqcup I^{\text{in, out}_0} \sqcup I^{\text{in, in}_0}$$

where

$$I^{\text{out, out}_0} := s^{-1}_\tau(\text{out}) \cap s^{-1}_0(\text{out}), \quad I^{\text{out, in}_0} := s^{-1}_\tau(\text{out}) \cap s^{-1}_0(\text{in}),$$

$$I^{\text{in, out}_0} := s^{-1}_\tau(\text{in}) \cap s^{-1}_0(\text{out}), \quad I^{\text{in, in}_0} := s^{-1}_\tau(\text{in}) \cap s^{-1}_0(\text{in}).$$

Given a graded vector space with $k$ branes, $V^{k-br} = (V = \varinjlim V_p, W_1, \ldots, W_k)$, consider a collection of linear subspaces for each $p \in \mathbb{N},$

$$W^m_p := W^{m(1)}_1, p \cap W^{m(2)}_2, p \cap \ldots \cap W^{m(k)}_{k, p},$$

one for each multi-direction $m : [k^+] \to \{\text{out}, \text{in}\}$ from $O_{r+k}$, where we set for each $\tau \in [k],$

$$W^{m(\tau)}_\tau, p := \begin{cases} W^+_\tau, p & \text{if } m(0) = m(\tau) = \text{out} \\ (W^+_\tau, p)^* & \text{if } m(0) = m(\tau) = \text{in} \\ W^-_\tau, p & \text{if } m(0) = \text{out}, \ m(\tau) = \text{in} \\ (W^-_\tau, p)^* & \text{if } m(0) = \text{in}, \ m(\tau) = \text{out} \end{cases}$$
Note that \((W^m_{\tau, p})^* = W^{m^{opp}}_{\tau, p}\). For example, for \(m = \bullet \rightarrow m(1) \rightarrow m(2) \rightarrow \ldots \rightarrow m(k) \rightarrow m(0)\), one has \(W^m_p = W^{-1}_{1, p} \cap W^+_{2, p} \cap \ldots \cap W^{-k}_{k, p}\).

while for \(m^{opp} = \bullet \rightarrow m(1) \rightarrow m(2) \rightarrow \ldots \rightarrow m(k) \rightarrow m(0)\), one has \(W^{m^{opp}}_p = (W^{-1}_{1, p})^* \cap (W^+_{2, p})^* \cap \ldots \cap (W^{-k}_{k, p})^*\).

We define a countably dimensional vector space

\[ W^m := \lim_{\longrightarrow} W^m_p, \]

Define an \(S^{(k+1)}\)-module \(\mathcal{E}nd_{V^{k-br}}\), that is, a functor

\[ \mathcal{E}nd_{V^{k-br}} : S^{(k+1)} \rightarrow \text{Category of dg vector spaces}, \]

by setting

\[ \mathcal{E}nd_{V^{k-br}}(I, s) := \bigcap_{\tau \in [k]} \text{Hom}_{\tau}(s, I), \]

where (cf. (19))

\[
\text{Hom}_{\tau}(s, I) := \lim_{\longrightarrow} \left( \lim_{\longrightarrow} \left( \bigotimes_{i \in I, p_i} W^{s_{i|p_i}} \bigotimes_{j \in I, p_j} W^{s_{j|p_j}} \right) \right) \\
= \lim_{\longrightarrow} \left( \bigotimes_{i \in I} W^{s_{i|p_i}} \bigotimes_{j \in I} W^{s_{j|p_j}} \right) \\
= \text{Hom} \left( \bigotimes_{i \in I} W^{s_{i|p_i}}, \bigotimes_{j \in I} W^{s_{j|p_j}} \right). \]
Thus a single element $f \in \text{End}_{V^{k-\text{br}}}(I, s)$ has $k$ incarnations as a linear map, one for each “coloured direction” $\tau \in [k]$. Note that all the $k$ spaces $\text{Hom}_\tau(s, I)$, $\tau \in [k]$, belong to one and the same vector space $\text{Hom}(s, I) := \lim_{\leftarrow} \bigotimes_{p_i} W^s_{p_i}$ (21) so that it makes sense to talk about their intersection. If $V$ is finite-dimensional, then, of course, $\text{Hom}_\tau(s, I) = \text{Hom}(s, I)$ for any $\tau \in [k]$.

Therefore, elements of $\mathcal{E}nd_{V^{k-\text{br}}}$ can be composed (when it makes sense) along each of the “coloured” direction, but in general they can not be composed along the basic direction (i.e., compositions of type (7) have no sense in general).

Let $\mathcal{P}^{(k+1)-\text{or}}$ be a $(k + 1)$-oriented prop(erad). A morphism of $(k + 1)$-oriented prop(erad)s

$$\rho : \mathcal{P}^{(k+1)-\text{or}} \longrightarrow \mathcal{E}nd_{V^{k-\text{br}}}$$

is called a representation of $\mathcal{P}^{(k+1)-\text{or}}$ in a vector space with one $k$ branes $V^{k-\text{br}}$. If $V^{k-\text{br}}$ happens to be a symplectic vector space with $k$ Lagrangian branes, then a representation $\rho$ is called reduced symplectic Lagrangian if $\rho$ takes identical values on all those generating corollas of $\mathcal{P}^{(k+1)-\text{or}}$ which become identical (as $k$-oriented decorated graphs) once the basic directions on edges are forgotten.

4.7 Example: 3-oriented endomorphism prop

Let $V^{2-\text{or}} = (V, W^+_1, W^+_2)$ be a countably dimensional graded vector space with 2 branes. We would like to describe in more details the structure of the associated endomorphism prop which is a functor

$$\mathcal{E}nd_{V^{2-\text{or}}} : S^{(3)} \longrightarrow \text{Category of dg vector spaces}$$

$$(I, s) = \longrightarrow \mathcal{E}nd_{V^{2-\text{or}}}(I, s)$$

$I_6$

The multi-orientation $s$ defines (and can be reconstructed from) the decomposition $I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_8$ as explained in the picture. If

$$\# I_i = m_i \text{ for } i \in \{1, 2, 3, 4\}, \quad \# I_i = n_{i-4} \text{ for } i \in \{5, 6, 7, 8\}.$$

then, by definition, $\mathcal{E}nd_{V^{2-\text{or}}}(I, s)$ is the intersection in (21) of two graded vector spaces $\mathcal{E}nd_{V^{2-\text{or}}}(I, s)$.
\[ \text{Hom}(I, s) := \text{Hom} \left( \otimes^{n_1} W^{++} \otimes^{n_2} W^{-+} \otimes^{m_2} (\hat{W}^{--})^* \right. \]
\[ \left. \otimes^{m_1} W^{++} \otimes^{n_3} W^{-+} \otimes^{n_2} (\hat{W}^{--})^* \otimes^{m_4} (\hat{W}^{--})^* \right) \]

and

\[ \text{Hom}(I, s) := \text{Hom} \left( \otimes^{n_1} W^{++} \otimes^{n_2} W^{+-} \otimes^{m_3} (\hat{W}^{+-})^* \otimes^{m_4} (\hat{W}^{--})^* \right. \]
\[ \left. \otimes^{m_1} W^{++} \otimes^{m_2} W^{+-} \otimes^{n_3} (\hat{W}^{+-})^* \otimes^{m_4} (\hat{W}^{--})^* \right) \]

where we set

\[ W^{++} := \lim_{p \to +} W_{1,p}^+ \cap W_{2,p}^+, \quad W^{-+} := \lim_{p \to +} W_{1,p}^- \cap W_{2,p}^+, \quad W^{+-} := \lim_{p \to +} W_{1,p}^+ \cap W_{2,p}^-, \]
\[ W^{--} := \lim_{p \to +} W_{1,p}^- \cap W_{2,p}^-, \quad \hat{W}^{++} := \lim_{p \to +} W_{1,p}^+ \cap W_{2,p}^+, \quad \hat{W}^{-+} := \lim_{p \to +} W_{1,p}^- \cap W_{2,p}^+, \]
\[ \hat{W}^{+-} := \lim_{p \to +} W_{1,p}^+ \cap W_{2,p}^-, \quad \hat{W}^{--} := \lim_{p \to +} W_{1,p}^- \cap W_{2,p}^- \]

All the tensor factors shown in the above formulae for Hom(I, s) and Hom(I, s) are countably dimensional vector spaces. Let \( \{x_{A^{++}}\}, \{x_{A^{+-}}\}, \{x_{A^{-+}}\}, \{x_{A^{--}}\} \) be bases for the (direct limit) vector spaces \( W^{++}, W^{-+}, W^{+-}, W^{--} \), while \( y^{A^{++}}, y^{A^{+-}}, y^{A^{-+}} \) and \( y^{A^{--}} \) be the associated dual bases for (also direct limit) vector spaces \( (\hat{W}^{++})^*, (\hat{W}^{-+})^*, (\hat{W}^{+-})^* \) and \( (\hat{W}^{--})^* \). Then, the “big” vector space \( (21) \) consists of all formal power series of the form

\[ \sum_{A_{++}, A_{+-}, A_{-+}, A_{--}} F_{A^{++}A_{++}A_{+-}A_{--}} x_{A^{++}} \otimes x_{A_{++}} \otimes x_{A^{+-}} \otimes x_{A^{-+}} \otimes y^{B^{++}} \otimes y^{B^{+-}} \otimes y^{B^{++}} \otimes y^{B^{--}} \]
\[ F_{B^{++}B^{--}} \in \mathbb{K}, \]

its subspace Hom(I, s) is spanned by those formal series whose coefficients satisfy the condition

- for any fixed values of indices \( A^{++}, A_{+-}, B^{++} \) and \( B^{--} \) only finitely many \( F_{B^{++}B^{++}B^{--}B^{--}} \neq 0 \),

while the subspace Hom(I, s) is characterized by

- for any fixed values of indices \( A^{++}, A_{-+}, B^{+-} \) and \( B^{--} \) only finitely many \( F_{B^{++}B^{+-}B^{--}B^{--}} \neq 0 \),

This gives us a “down to earth” characterization of the endomorphism prop \( \mathcal{E}nd_{V2\text{-ve}} \cong \{ \text{Hom}(I, s) \cap \text{Hom}(I, s) \} \).
5 Action of the Grothendieck–Teichmüller group on some multi-oriented props

5.1 An operad of multi-oriented graphs

For any \( l \geq -1 \) and \( k \geq 0 \), let \( G_{n,p}^{l+1\uparrow k+1} \) be a set of \((l+1)\)-oriented \((k+1)\)-directed (see Sect. 2.3) graphs \( \Gamma \) with \( n \) vertices and \( p \) edges such that some bijections \( V(\Gamma) \to [n] \) and \( E(\Gamma) \to [p] \) are fixed, i.e. every vertex and every edge of \( \Gamma \) has a numerical label. There is a natural right action of the group \( S_n \times S_p \) on the set \( G_{n,p}^{l+1\uparrow k+1} \) with \( S_n \) acting by relabelling the vertices and \( S_p \) by relabelling the edges. For each fixed integer \( d \), consider a collection of \( S_n \)-modules \( Gr_d l+1\uparrow k+1 = [Gr_d l+1\uparrow k+1(n)]_{n \geq 1} \), where

\[
Gr_d l+1\uparrow k+1(n) := \prod_{p \geq 0} K(G_{n,p}^{l+1\uparrow k+1}) \otimes_{S_p} \text{sgn}_p^{\otimes |d-1|} [p(d-1)].
\]

where \( \text{sgn}_p \) is the 1-dimensional sign representation of \( S_p \). It has an (ordinary, i.e. 1-oriented!) operad structure with the composition rules

\[
o_i : Gr_d l+1\uparrow k+1(n) \times Gr_d l+1\uparrow k+1(m) \to Gr_d l+1\uparrow k+1(n+m-1), \quad \forall i \in [n]
\]

\[
(\Gamma_1, \Gamma_2) \to \Gamma_1 \circ_i \Gamma_2,
\]

given by substituting the graph \( \Gamma_2 \) into the \( i \)-labelled vertex \( v_i \) of \( \Gamma_1 \) and taking the sum over re-attachments of dangling edges (attached before to \( v_i \) ) to vertices of \( \Gamma_2 \) in all possible ways. If \( l = k \) we abbreviate \( Gr_d (l+1)\)-or \( = Gr_d l+1\uparrow k+1 \) and call it the operad of \((k+1)\)-oriented graphs.

Note also that for \( l > l' \) the operad \( Gr_d l+1\uparrow k+1 \) is a suboperad of \( Gr_d l'+1\uparrow k+1 \).

There is a canonical injection

\[
Gr_d l+1\uparrow k+1 \longrightarrow Gr_d l+1\uparrow k+2
\]
sending a \((k+1)\)-directed graph \( \Gamma \) into a sum of \((k+2)\)-directed graphs obtained from \( \Gamma \) by attaching the new \((k+2)\)nd direction to each edge in two possible ways.

Let \( \text{Lie}_d \) be a (degree shifted) ordinary operad of Lie algebras whose representations are graded Lie algebras equipped with the Lie bracket in degree \( 1-d \), and consider the standard (cf. [15]) deformation complex of the trivial morphism of operads,

\[
fGC_d l+1\uparrow k+1 := \text{Def} \left( \text{Lie}_d \longrightarrow Gr_d l+1\uparrow k+1 \right)
\]

\[
\simeq \prod_{n \geq 1} [Gr_d l+1\uparrow k+1(n) \otimes [d(1-n)]] \quad \forall k \geq 0, \ l \in \{-1, 0, 1, \ldots, k\}. \quad (22)
\]

This is a Lie algebra. Moreover, it admits a non-trivial Maurer–Cartan element \( \gamma_0 \) which corresponds to a morphism

\[
\gamma_0 : \text{Lie}_d \longrightarrow Gr_d l+1\uparrow k+1
\]
given explicitly on the generator (of homological degree $1 - d$)

\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array} \quad = (-1)^d \quad \begin{array}{c}
\text{2} \\
\text{1}
\end{array} \quad \in \text{Lie}_d(2)
\]

by the following explicit formula (cf. [20,21])

\[
\gamma_0 \left( \begin{array}{c}
\text{1} \\
\text{2}
\end{array} \right) = \sum_{a \in \text{Or}_k} \left( \begin{array}{c}
\text{1} \\
\text{2}
\end{array} \right) + (-1)^d \begin{array}{c}
\text{2} \\
\text{1}
\end{array} \right) =: \bullet \bullet
\]

where the summation runs over all possible ways to attach extra $k$ directions to the 1-oriented edge. Note that elements of $\text{fGC}_d^{l+1 \uparrow k+1}$ can be identified with graphs from $\text{Gr}_d^{l+1 \uparrow k+1}$ whose vertices’ labels are symmetrized (for $d$ even) or skew-symmetrized (for $d$ odd) so that in pictures we can forget about labels of vertices and denote them by unlabelled black bullets as in the formula above. Note also that graphs from $\text{Gr}_d^{l+1 \uparrow k+1}$ come equipped with a orientation which is a choice of ordering of edges (for $d$ even) or a choice of ordering of vertices (for $d$ odd) up to an even permutation in both cases. Thus, every graph $\Gamma \in \text{fGC}_d^{l+1 \uparrow k+1}$ has at most two different orientations, $\text{or}$ and $\text{or}^{opp}$, and one has the standard relation, $(\Gamma, \text{or}) = - (\Gamma, \text{or}^{opp})$; as usual, the data $(\Gamma, \text{or})$ is abbreviated to $\Gamma$ (with some choice of orientation implicitly assumed). Note that the homological degree of graph $\Gamma$ from $\text{fGC}_d^{l+1 \uparrow k+1}$ is given by

\[
|\Gamma| = d(\#V(\Gamma) - 1) + (1 - d)\#E(\Gamma).
\]

We show in [14] some other explicit Maurer–Cartan elements in the Lie algebra $\text{fGC}_d^{l+1 \uparrow k+1}$ given by transcendental formulae; in this paper, we need only $\gamma_0$.

The above Maurer–Cartan element (23) makes $(\text{fGC}_d^{l+1 \uparrow k+1}, [\ , \ ]$ into a differential Lie algebra with the differential

\[
\delta_0 := [\bullet \bullet , ].
\]

This dg Lie algebra contains a dg subalgebra $\text{fcGC}_d^{l+1 \uparrow k+1}$ spanned by connected graphs which in turn contains a dg Lie subalgebra $\text{GC}_d^{l+1 \uparrow k+1}$ spanned by connected graphs with at least bivalent vertices. It was proven in [20,21] (for the case $k = 0$ and $l = -1$, 0 but the arguments works in greater generality) that the latter two subalgebras are quasi-isomorphic,

\[
H^\bullet (\text{fcGC}_d^{l+1 \uparrow k+1}, \delta_0) = H^\bullet (\text{GC}_d^{l+1 \uparrow k+1}, \delta_0)
\]

It was also proven in [20,21] (in the cases $k = 0$ and $l \in \{-1, 0\}$ but the arguments work in greater generality) that

\[
H^\bullet (\text{fGC}_d^{l+1 \uparrow k+1}, \delta_0) = \bigcirc^\bullet \in (H^\bullet (\text{GC}_d^{l+1 \uparrow k+1}, \delta_0)[-d]) [d]
\]
so that there is no loss of information to working solely with $\mathcal{G}C_{d}^{l+1\uparrow k+1}$ instead of the full graph complex $f\mathcal{G}C_{d}^{l+1\uparrow k+1}$. There is a remarkable isomorphism of Lie algebras [20],

$$H^0(\mathcal{G}C_{2}^{0\uparrow 1}, \delta_0) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck–Teichmüller group $GRT_1$ introduced by Drinfeld in the context of the deformation quantization of Lie bialgebras. Nowadays, this group plays an important role in many areas of mathematics.

The multi-directed graph complexes have been introduced and studied in [23]; more precisely, Marko Živković studied fully oriented graph complexes which are dual to the complexes $(\mathcal{G}C_{d}^{k\uparrow 1\downarrow k}, \delta_0), k \geq 0$. We often abbreviate $\mathcal{G}C_{d}^{(k+1)-or} := \mathcal{G}C_{d}^{k+1\uparrow k+1}$ for $k \geq 0$ and $\mathcal{G}C_{d}^{0-or} := \mathcal{G}C_{d}^{0\uparrow 1}$.

Note that for $l' < l$ the Lie algebra $\mathcal{G}C_{d}^{l'+1\uparrow k+1}$ is a Lie subalgebra of $\mathcal{G}C_{d}^{l+1\uparrow k+1}$.

### 5.2 Cohomology of (partially) oriented multi-directed graph complexes

For any $k \geq 0$ and any $-1 \leq l \leq k$, there is an obvious map of graph complexes

$$i : \mathcal{G}C_{d}^{l+1\uparrow k+1} \rightarrow \mathcal{G}C_{d}^{l+1\uparrow k+2}$$

which sends an $(l+1)$-oriented graph with $k+1$ directions to an $(l+1)$-oriented graph with $k+2$ directions by taking a sum over all possible ways to attach a new $(k+2)$-nd direction to each $(k+1)$-directed edge.

#### 5.2.1 Theorem [20]

The injection $i : \mathcal{G}C_{d}^{l+1\uparrow k+1} \rightarrow \mathcal{G}C_{d}^{l+1\uparrow k+2}$ is a quasi-isomorphism of dg Lie algebras.

This theorem was proved by Thomas Willwacher in [20] in the case $k = 0, l \in \{-1, 0\}$, but the argument works in greater generality. This result implies

$$H^\bullet(\mathcal{G}C_{d}^{l+1\uparrow k+1}, \delta_0) = H^\bullet(\mathcal{G}C_{d}^{(l+1)-or}, \delta_0) \quad \forall k \geq 0, -1 \leq l \leq k.$$

Put another way, multidirections which are not oriented can be forgotten, they do not give us something really new.

Thomas Willwacher also proved the following

#### 5.2.2 Theorem [21]

$$H^\bullet(\mathcal{G}C_{d}^{l-or}, \delta_0) = H^\bullet(\mathcal{G}C_{d+1-or}^{l}, \delta_0) \text{ for any } d \in \mathbb{Z}.$$  

In particular, one has an isomorphism

$$H^0(\mathcal{G}C_{3}^{1-or}, \delta_0) = H^0(\mathcal{G}C_{2}^{0-or}, \delta_0) = \mathfrak{grt}_1$$
which plays an important role in the homotopy theory of (involutive) Lie bialgebras [16].

This theorem has been recently generalized to \((k + 1)\)-oriented graphs by Marko Živković.

5.2.3 Theorem [23]

\[
H^\bullet(GC_d^{(k+1)\text{-or}}, \delta_0) = H^\bullet(GC_{d+1}^{(k+2)\text{-or}}, \delta_0) \text{ for any } d \in \mathbb{Z} \text{ and any } k \geq 0.
\]

Theorems 5.2.1 and 5.2.3 imply the equalities

\[
H^\bullet(GC_d^{l+1\text{-or}k+1}, \delta_0) = H^\bullet(GC_{d+1}^{l+2\text{-or}k+2}, \delta_0) = \forall d \in \mathbb{Z}, \quad k \geq 0, \quad -1 \leq l \leq k.
\] (25)

In particular, we have isomorphisms of Lie algebras,

\[
H^0(GC_d^{d\text{-or}}, \delta_0) = H^0(GC_0^{d+2}, \delta_0) = \mathfrak{grt}_1,
\] (26)

for any \(d \geq 0\). For \(d = 2\) and \(d = 3\) the algebro-geometric meanings of the associated graph complex incarnations of the Grothendieck–Teichmüller group \(GRT_1\) are clear: the \(d = 2\) case corresponds to the action of \(GRT_1\) (through cocycle representatives in \(GC_2^{d+1}\)) on universal Kontsevich formality maps associated with the deformation quantization of Poisson structures (given explicitly with the help of suitable configuration spaces in the two dimensional upper half-plane [6]), while the case \(d = 3\) corresponds to the action of \(GRT_1\) (through cocycle representatives in \(GC_3^{d+1}\)) on universal formality maps associated with the deformation quantization of Lie bialgebras (see [18] where compactified configuration spaces in three dimensions have been used).

The above results tell us that the Grothendieck–Teichmüller group survives in any geometric dimension \(\geq 4\) but now in the multi-oriented graph complex incarnation. What can the associated to \(\mathfrak{grt}_1\) degree zero cocycles in \(GC_d^{d\text{-or}}\) act on? It is an attempt to answer this question which motivated much of the present work. In the first approximation, the answer is that it acts on the multi-oriented props \(\mathcal{H}_{\text{olieb}}^{(c+d-1)\text{-or}}\) (more precisely, on their genus completed versions \(\hat{\mathcal{H}}_{\text{olieb}}^{(c+d-1)\text{-or}}\), and it is not hard to see how. Recall the main result of [16] which says that there is a morphism of dg Lie algebras

\[
F: GC_1^{c+d+1} \to \text{Der}(\hat{\mathcal{H}}_{\text{olieb}}^{c+d+1})
\]

where \(\hat{\mathcal{H}}_{\text{olieb}}^{c+d+1}\) is the genus completion of \(\mathcal{H}_{\text{olieb}}^{c+d+1}\) and \(\text{Der}(\hat{\mathcal{H}}_{\text{olieb}}^{c+d+1})\) is the Lie algebra of continuous derivations of \(\hat{\mathcal{H}}_{\text{olieb}}^{c+d+1}\) (see [16] for some subtlety in its definition). This map is a quasi-isomorphism (up to one rescaling class), and it can be given by a simple formula: for any \(\Gamma \in GC_1^{c+d+1}\) one has

\[\]
\[ F(\Gamma) = \sum_{m,n \geq 1} \sum_{s,[n] \to V(\Gamma)} \hat{s} [m] \to V(\Gamma) \]

where the second sum in taken over all ways, \( s \) and \( \hat{s} \), of attaching the in- and outgoing legs to the graph \( \Gamma \), and then setting to zero every graph containing a vertex with valency \( \leq 2 \) or with no input legs or no output legs (there is an implicit rule of signs in-built into this formula). In a complete analogy, one can define an action of the dg Lie algebra \( GC^{(k+1)}_{c+d+1} \) as derivations on the multi-oriented dg prop \( \hat{H}olieb^{(k+1)}_{c,d} \), that is, a morphism of dg Lie algebras

\[ F : GC^{(k+1)}_{c+d+1} \to \text{Der}(\hat{H}olieb^{(k+1)}_{c,d}) \]

It was proven by Assar Andersson in [1] that this map is a quasi-isomorphism (up to one rescaling class). This result together with equality (26) implies a highly non-trivial action of \( GRT_1 \) on the infinite family of the multi-oriented props \( \hat{H}olieb^{(c+d-1)}_{c,d} \), \( c + d \geq 3 \).

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