Dilaton-scalar models in context of generalized affine gravity theories: their properties and integrability.

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Abstract

Nowadays it is widely accepted that the evolution of the universe was driven by some scalar degrees of freedom both on its early stage and at present. The corresponding cosmological models often involve some scalar fields introduced ad hoc. In this paper we cultivate a different approach, which is based on a derivation of new scalar degrees of freedom from fundamental modifications of Einstein’s gravity. In elaboration of our previous work, we here investigate properties of the dilaton-scalar gravity obtained by dimensional reductions of a recently proposed affine generalized gravity theory. We show that these models possess the same symmetries as related models of GR with ordinary scalar fields.

As a result, for a rather general class of dilaton-scalar gravity models we construct additional first integrals and formulate an integral equation well suited for solving by iterations.

1 Introduction

The cosmological observations of recent decades revealed that the universe expanded with acceleration on two different stages of its evolution: in the very beginning and at present time. In addition, the presence of new sorts of matter, the amount of which significantly exceeds that of the known matter, is also established very well. The models within the current paradigm of the Friedmann-Lemaître-Robertson-Walker cosmology and the Standard model physics can hardly describe all of them. This probably means that partial improvements of the existing theory may prove to be insufficient and the entire paradigm should be changed [1].

For example, one can try to extend the particle physics by adding inflaton or quint-essence fields, or consider the modified gravity, or use the paradigm of string/brane world (see, e.g., [2, 3, 4, 5] as a drop in the ocean of papers and reviews on the topic).

Incorporating new scalar fields into the Standard model could be a technically simplest solution, but this approach lacks a fundamental basis. Moreover, it can hardly be falsified, since any disagreement with the observational data can be cured by adding more fields to better suit the data. Actually, now it is widely accepted that the string theory, being a fundamental one, suffers the same problem, with numerous compactifications which can provide any effective theory.

Among the modifications of gravity there is a very interesting theory, originally introduced by Weyl, Eddington and Einstein, which has a fundamental distinction from General Relativity. It establishes the priority of geodesics within non-Riemannian geometry, while the metric arises at the level of the effective theory. On the other hand, the constraints on the initial theory come from the requirement to reproduce the metric theory of GR gravity. The constraints can be

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resolved, and the use of a special two-step Lagrangian formalism allowed Einstein to formulate the effective theory with normal curvature term, cosmological constant and some ‘dark’ matter fields interacting only with gravity. These fields looked quite abnormal, and since the actual goal at that time (around 1923) was to unify the gravity and electromagnetism, the theory was not investigated in detail, soon abandoned and completely forgotten.

Nowadays, the presence of dark fields is the fact of our life, and such a natural, almost inevitable, presence of the dynamical cosmological constant is welcomed very much. Indeed, one of the intriguing questions of the dark energy problem is: why does it look so much like \( \Lambda \)-term \([6, 7, 8, 9]\), while the origin of a so small \( \Lambda \)-term is very unlikely in the framework of GR and Standard model. Thus it looks quite natural to re-establish investigations of the affine generalization of GR \([10, 11]\), which can be called the ‘affine’ gravity.

One of the basic features of the affine gravity is the presence of a massive (normal or tachyonic) vector field. In cosmology, the scalar fields are usually considered, but recently \([11]\) we revealed the duality between this vector field (vecton) and the corresponding scalar field (scalaron) in case of reductions to two-dimensional theory. This allows us to investigate the basic properties of practically interesting cosmological and static solutions using the same approach as for gravity theories with scalar fields. In fact the affine gravity produces a scalaron field which significantly differs from the standard scalar field in GR, thus compelling us to use a significantly generalized description of the dilaton-scalar gravity (DSG). And the obtained results can be applied even outside the context of affine gravity. For instance, the dimensional reductions of Yang–Mills fields \([12, 13, 14, 15]\) also provide rather special scalar degrees of freedom.

On the other hand, DGS theories constructed for the cosmological purposes, are usually rather simple. In case of affine gravity theory, the corresponding effective DGS model, that we call the dilaton-scalaron gravity (DSG), is defined rather strictly. Actually it looks quite cumbersome, being significantly non-linear in dimensions \( D > 3 \) and non-integrable even in the simplest linear case. Nevertheless, in this paper we will investigate the basic properties of the DGS model and try to answer the question: is the DSG model really complicated?

The paper is organized in the following way. First, we discuss models which possess some fruitful properties, such as the presence of linear symmetry, the availability of iterative integration procedure, the existence of explicit non-trivial solutions. So we specify the set of simplest models within the class of DGS, mostly related to dimensional reductions. Then we show that DSG belongs to this set, despite its specific non-linearity. We thus conclude that affine gravity theory, being quite awkward at first sight, shares with common scalar field models in GR some useful properties. In Appendix we present some extra results which were derived while investigating the DGS gravity but can be applied to a rather general class of dynamical systems appearing in theories with vanishing Hamiltonian constraint.

## 2 Dilaton-gravity theory with scalar fields

A fairly general higher-dimensional field theories can be reduced to two-dimensional effective DGS models by taking into account their space-time symmetries. We specify the following generic action

\[
L^{(2)}_{\text{eff}} = \sqrt{-g} \left[ \varphi R(g) + W(\varphi)(\nabla_{\mu}\varphi)^2 + Z_{ij}(\varphi; \psi)\nabla_{\mu}\psi_i\nabla^{\mu}\psi_j + X(\varphi; \psi) \right], \quad \mu = (t, r),
\]

where \( \varphi \) is the dilaton field, and \( \psi_i \) denote all scalar fields of any origin, including the geometrical one. Note that in dimension \( D = 2 \) all the fields are dynamically equivalent to scalar fields or, in case of the massless vector, to the corresponding field charges. A practical transformation of the vecton into the scalaron in the dimensionally reduced affine gravity model was given in \([11]\).

The most concrete results for these models can be obtained after the further reduction to the one-dimensional dynamical systems, describing static and cosmological states. For this, the system
is parameterized by a single coordinate, \( \tau \), which can be time-like or space-like. The signs of the one-dimensional Lagrangian components depend on the parametrization, but they usually may be consumed by the free parameters of the model.

So, the corresponding one-dimensional Lagrangian can be written in the following rather general form [11, 16, 17, 18, 19]:

\[
\mathcal{L} = -s \left[ h^{-1} \dot{h} \dot{\varphi} + W(\varphi) \dot{\varphi}^2 + Z_{ij}(\varphi; \psi) \dot{\psi}_i \dot{\psi}_j \right] + s^{-1} h X(\varphi; \psi). \tag{2}
\]

Here \( h \) is a metric function, chosen so that \( h > 0 \) is a cosmological state, and \( h < 0 \) is a static state. The presence of the remaining metric gauge degree of freedom \( s \), which is in fact the Lagrangian multiplier, leads to the vanishing Hamiltonian constraint:

\[
H = s \left[ h^{-1} \dot{h} \dot{\varphi} + W(\varphi) \dot{\varphi}^2 + Z_{ij}(\varphi; \psi) \dot{\psi}_i \dot{\psi}_j \right] + s^{-1} h X(\varphi; \psi) = 0. \tag{3}
\]

After imposing this constraint we may fix \( s \). For example, \( s = 1 \) is the lightcone gauge. In what follows we often replace the metric \( h \) with the new variable \( F \) defined by \( \dot{F} = h^{-1} \dot{h} + W(\varphi) \dot{\varphi} \).

This reduces the dilaton-metric kinetic term to \( F \dot{\varphi} \) and incorporates the Weyl transformation; the potential \( X(\varphi; \psi) \) accordingly acquires the factor \( \Omega(\varphi) = \exp(- \int W(\varphi) d\varphi) \).

### 2.1 Linear symmetries and bilinear integrals in DGS

Consider now the system [22] with just one scalar field \( \psi \) and the corresponding function, \( Z(\varphi) \), depending only on the dilaton variable. In this case, we may use a convenient gauge \( s = 1/Z(\varphi) \) and the new dilatonic variable \( \xi \) defined by \( \xi \equiv \varphi/Z(\varphi) \). Then our DGS dynamical system with the Hamiltonian constraint looks rather compact:

\[
\mathcal{L} = \dot{F} \dot{\xi} + \dot{\psi}^2 - e^F U(\xi, \psi), \quad H = \dot{F} \dot{\xi} + \dot{\psi}^2 + e^F U(\xi, \psi) = 0, \quad \text{where} \quad U = \Omega Z X. \tag{4}
\]

The corresponding Lagrangian system,

\[
\dot{\xi} = -e^F U, \quad \dot{F} = -e^F U_{\xi}, \quad \dot{\psi} = -e^F U_{\psi}/2, \tag{5}
\]

actually contains only two independent equations, while the remaining one can be derived from the above Hamiltonian constraint \( H = 0 \). Yet it is more convenient to use all the three equations when searching for linear symmetries of the system.

With this aim, let us introduce the coordinate vector \( q^i = (F, \xi, \psi) \) and represent our dynamics in a more general form,

\[
\mathcal{L} = A_{ij} \dot{q}^i \dot{q}^j - U(q), \quad H = A_{ij} \dot{q}^i \dot{q}^j + U(q) = 0, \tag{6}
\]

where in model [11]: \( A_{F \xi} = A_{\xi F} = 1/2 \), \( A_{\psi \psi} = 1 \), and \( U(q) = e^F U(\xi, \psi) \). The equations are

\[
2A_{ij} \ddot{q}^i = -\partial_i U(q) \quad \Rightarrow \quad \ddot{q}^k = -\frac{1}{2} (A^{-1})^{ki} \partial_i U(q). \tag{7}
\]

The last system can be contracted with a linear combination of the coordinates and some constants, \((\lambda A_{kj} + B_{kj}) q^j + c_k\), where \( B_{kj} \) is an arbitrary antisymmetric constant matrix, \( c_k \) is a constant vector, and \( \lambda \) is a constant. Then we have:

\[
[(\lambda A_{kj} + B_{kj}) q^j + c_k] q^k = [(\lambda A_{kj} + B_{kj}) q^j + c_k] q^k - \lambda A_{kj} q^k \dot{q}^j = -\frac{1}{2} [(\lambda A_{kj} + B_{kj}) q^j + c_k] (A^{-1})^{ki} \partial_i U(q). \tag{8}
\]
Using here constraint (9), $H = 0$, we replace the kinetic term, $A_{kj}q^i\dot{q}^j$, by the potential, $-U(q)$, and conclude that the bilinear form,

$$P = [(\lambda A_{kj} + B_{kj})q^i + c_k]q^k,$$

is the integral of motion if the potential satisfies the linear partial differential equation:

$$\lambda U(q) + \frac{1}{2} \left[ \lambda q^i + (B_{kj}q^j + c_k)(A^{-1})^{ki} \right] \partial_i U(q) = 0. \quad (10)$$

This theorem is applicable to the general constrained dynamics (gauge system) described by Eqs. (10) with any constant matrix $A_{ij}$ and any function $U(q^1, ..., q^n)$. For our DGS system with one scalar field, the only nonzero components of the $3 \times 3$ inverse matrix $A^{-1}$, are $A^{-1}_{12} = (A^{-1})^{21} = 2$, $A^{-1}_{33} = 1$ and the potential has the simple dependence on the metric, $U(q) = e^F U(\xi, \psi)$. In addition, in the three-dimensional coordinate space, the antisymmetric matrix $B_{kj}$ can be parameterized by a vector, $B_{kj} = \epsilon_{kj}b^k$. It follows that Eq. (10) splits into two independent equations for $U(\xi, \psi)$ (the second one expresses the condition of $F$-independence):

$$[(\lambda/2 + b_3)\xi - b_2\psi + c_1]U_\xi + \frac{1}{2} \left[ \lambda \psi - b_1 \xi + c_3 \right] U_\psi + (b_1 \psi + c_2 + \lambda)U = 0,$$

$$b_2 \partial_3 U \equiv b_2 U_\psi = (2b_3 - \lambda)U. \quad (11)$$

This equation depends on seven parameters, but $U$ is defined up to an arbitrary multiplier. Integral (12), also defined up to a multiplier, depends on the same parameters and now reads:

$$P = [(\lambda/2 + b_3)\xi - b_2\psi + c_1] \dot{F} + [(\lambda/2 - b_3)F + b_1 \psi + c_2] \dot{\xi} + [\lambda \psi + b_2 F - b_1 \xi + c_3] \dot{\psi}. \quad (12)$$

Note that when $b_2 \neq 0$ and $(\lambda - 2b_3) \neq 0$ the potential defined by equations (11) must be exponential in $\psi$, i.e., $U = u(\xi) \exp[(2b_3 - \lambda)\psi/b_2]$. Inserting this into the first equation we find that $u(\xi)$ is also exponential (or constant). So the model reduces to the well studied class of multi-exponential DGS theories with the completely integrable subclass of Toda - Liouville theories (see [20] and references there) and we therefore expect to find in the exponential case some integrable systems. Below, we mostly suppose that $b_2 = (\lambda - 2b_3) = 0$ that does not mean ignoring exponential potentials. In fact, supposing that $\lambda = b_1 = 0$, it is not difficult to find the general solution of (11),

$$Y(\xi, \psi) \equiv \ln |U| = f(c_3 \xi - 2c_1 \psi) + c_4 \xi - 2c_3^{-1}(c_1 c_4 + c_2), \quad (13)$$

where $f$ is an arbitrary function and $c_4$ — a new arbitrary constant. Making the simplest choice, $f(x) = c_5 x$ we find that $Y = g_1 \xi + g_2 \psi$, with $g_1 \equiv c_3 c_5 + c_4$ and $g_2 c_3 \equiv c_1 g_1 + c_2$. The potential depends only on the parameters $g_1, g_2$ and thus we are free to vary all the parameters $c_i$ while not changing $Y$. For example, we can take $c_1 = 0, 2c_2 = -g_2 c_3$ or $c_2 = 0, 2g_1 c_1 = -g_2 c_3$ and obtain two independent integrals discussed below.

These examples show that relations between the potential and integrals are in general rather complex. We hope to demonstrate that the construction given by Eqs. (9) - (12) is a powerful device, but it is not easy to use. In this paper, we only briefly discuss mathematics behind it and concentrate on simple but nontrivial examples. A very close but simpler approach to searching for integrals in DGS models can be found in [21].

1 Note that throughout the paper we consider the simplest DGS with one scalar matter mode. The general system is briefly discussed in Appendix A and multi-scalar DGS models – in Appendix B.
2.2 Master Integral Equation in DGS

One may reduce the order of the system (5) by choosing ξ as the independent variable with \(d\tau \equiv d\xi/\chi(\xi)\). The system of equations can be therefore simplified if we define \(F \equiv F - \ln \chi\), \(G \equiv \psi'\), and use the combination of constraint (4) with the first equation of the system instead of the second one. Then the system will contain only the first order equations:

\[
\chi' = -e^{F}U, \quad F' = -G^2, \quad \psi' = G, \quad (\chi G)' = -e^{F}U/2.
\]

(14)

Obviously, the solution for \(\psi\) and \(F\) can be written as formal integrals for \(G\) and \(G^2\):

\[
\psi(\xi) = \psi_0 + \int_{\xi_0}^{\xi} G(\bar{\xi})d\bar{\xi} \equiv I\{G; \xi\}, \quad F(\xi) = F_0 - \int_{\xi_0}^{\xi} G^2(\bar{\xi})d\bar{\xi} \equiv I\{G^2; \xi\}.
\]

(15)

Now use these integrals in the remaining equations for \(\chi\) and \(\chi G\) and integrate them as well:

\[
\chi(\xi) - \chi_0 = -\int_{\xi_0}^{\xi} \exp(I\{G^2; \bar{\xi}\}) U(\bar{\xi}, I\{G; \bar{\xi}\})d\bar{\xi} \equiv I_1\{G; \xi\},
\]

(16)

\[
2[\chi(\xi)G(\xi) - \chi_0 G_0] = -\int_{\xi_0}^{\xi} \exp(I\{G^2; \bar{\xi}\}) U_\psi(\bar{\xi}, I\{G; \bar{\xi}\})d\bar{\xi} \equiv I_2\{G; \xi\}.
\]

(17)

In the last equation replace \(\chi\) with the functional \(I_1\) and find the relation between \(G\) and the functionals depending on \(\mathcal{G}\) and the potential \(U\):

\[
G(\xi) = \frac{\chi_0 G_0 + I_2\{G; \xi\}/2}{\chi_0 + I_1\{G; \xi\}}.
\]

(18)

We call this the ‘Master Integral Equation’, or MIE. It can be used to obtain the approximate solutions for the general potential. With a clever choice of the trial function for \(G\) we can obtain a good approximation even with a few iterations. Although the integrations are in general rather cumbersome, the computations with MIE may be very effective near singular points or in vicinity of a horizon, where the solution can be expressed with the aid of convergent power series expansions (after explicitly resolving possible singularities).

3 Partially integrable DGS models

The models of gravity with scalar fields often can not be solved explicitly. Yet in case of just one massive scalar mode one may expect that deriving an additional first integral and/or applying iterative approximation procedure will allow to investigate the most important properties of the solutions. For example, in cosmology the slow-roll approximation technique provides a good inflation model easily derived for many non-integrable configurations. Having this in mind, consider some types of DGS configurations possessing the inner symmetries, which simplify their analysis.

3.1 Integrable DGS models

Let us first explore the completely integrable configurations. They may arise in case of the exponential potential linear in its arguments: \(U = \exp (g_1 \xi + g_2 \psi)\). It possesses two commuting linear first integrals (the conditions for existence of several commuting first integrals can be found in Appendix). Indeed, such potential suits the condition (11) with the non-vanishing parameters \(c_2 = -c_1 g_1\), or \(c_2 = -c_3 g_2/2\). The two independent integrals,

\[
P_1 = \dot{\psi} - g_1 \dot{\xi}, \quad P_2 = 2\dot{\psi} - g_2 \dot{\xi},
\]

(19)
are in involution (commute). Taking into account the constraint, we find that the number of the integrals is equal to the number of variables and thus the system is integrable. This configuration with linear exponential potential may be related to the Toda-Liouville systems [19], or it can be derived from a quite special cylindrical reduction, as we will show below.

Another example comes from the effective $D = 2$ massless scalar-vector configuration

$$\mathcal{L} = -g^{(2)} \left[ \phi R(g^{(2)}) + W(\phi)(\nabla_\mu \phi)^2 + Z(\phi)\nabla_\mu \psi \nabla^\mu \psi + Y(\phi)B_{\mu\nu}B^{\mu\nu} + X(\phi) \right],$$  

(20)

where $B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the one-dimensional cosmological or static case, the scalar and vector fields can be integrated out, leaving as a trace the effective charge of the vector field, $Q_0$, and the integration constant of the scalar field, $c_0 = Z(\xi)\xi$. So the dynamics may be described by the effective Lagrangian supplied with the Hamiltonian constraint:

$$\mathcal{L} = -\dot{F} \xi + e^F U(\xi) + V(\xi), \quad H = \dot{\xi} + e^F U(\xi) + V(\xi) = 0.$$  

(21)

Here the original kinetic scalar field term contribute into the effective potential $V$, while the kinetic vector term contribute into $U$.

To derive an explicit solution of such a model we now need just one additional integral. Let us use here a sort of an inverse approach: first choose the suitable model which does possess such an integral, and then find a corresponding physical theory described by the model. First check the general condition (10) for the (bi)linear integral to exist. For the system (21) it can be satisfied only when one of the effective potentials, either $U$ or $V$, vanishes, because the total potential term, $e^F U(\xi) + V(\xi)$, does not possess the linear symmetry. Therefore, the mixed system with scalar and vector fields cannot have such kind of integral. With both potentials non-vanishing, there remains a possibility for a non-linear integral to exist. There is no general algorithm for constructing non-linear integrals but, nevertheless, an integral quadratic in momenta was found for a dynamical system of this sort (in [10], [17]).

Reproducing the procedure of [16] in our notation, first rewrite the equation for the metric:

$$\dot{F} = -e^F U'(\xi) - V'(\xi).$$  

(22)

For linear potentials, $U(\xi) = g_1 \xi + g_3$, $V(\xi) = g_2 \xi + g_4$, it becomes trivial and can be integrated:

$$\dot{F} = -g_1 e^F - g_2, \quad \frac{1}{2} \dot{F}^2 = C - g_1 e^F - g_2 F \equiv R(F).$$  

(23)

Here $C$ is the new first integral, quadratic in momenta and non-linear in the coordinate variable $F$. Multiplying it by $\xi$ and expressing $\dot{F} \xi$ by using the Hamiltonian constraint we find:

$$[e^F (g_1 \xi + g_3) + g_2 \xi + g_4] \dot{F} + 2(C - g_1 e^F - g_2 F) \dot{\xi} = 0.$$  

(24)

This expression is linear in momenta and may be more convenient to deal with.

Supposing that $g_1 \neq 0$, let us rescale $\xi$, $\tilde{\xi} = g_1 \xi + g_3$, and rewrite the above equation as

$$[(g_1 e^F + g_2) \tilde{\xi} + g_4 g_1 - g_2 g_3] \dot{F} + 2R(F) \dot{\tilde{\xi}} = 0.$$  

(25)

Since $g_1 e^F + g_2 = -R'(F)$, we find the equation

$$2R(F) \ddot{\xi} - \ddot{\xi} \ddot{R} + (g_4 g_1 - g_2 g_3) \dot{F} = 0,$$  

(26)

which can be integrated after dividing it by $R^{3/2}$:

$$2R^{-1/2}(F) \ddot{\xi} - 2R^{-1/2}(F_0) \ddot{\xi}_0 = (g_2 g_3 - g_4 g_1) \int_{F_0}^{F} R^{-3/2}(\tilde{F})d\tilde{F}.$$  

(27)
It is instructive to rewrite the the final expression for $\tilde{\xi}(F)$ in the form:

$$\tilde{\xi} = \sqrt{\frac{R(F)}{R(F_0)}} \left[ \xi_0 + \frac{1}{2} (g_2 g_3 - g_4 g_1) \sqrt{\frac{R(F_0)}{R(F)}} \int_{F_0}^F R^{-3/2}(\tilde{F})d\tilde{F} \right].$$

(28)

With $g_2 = 0$, the integral in r.h.s. will contain arctan, arctanh or exponential functions depending on the parameters, otherwise it cannot be expressed in terms of elementary functions. Let us also note that the model with $g_2 = 0$ may describe the spherical dimensional reduction of the massless scalar and vector configuration \[16\] [17]. For example, the Reissner–Nordström solution in the absence of the scalar field (when $c_0 = 0$) can be derived from (28) when $g_2 = g_4 = 0$.

We would like to mention that the full integrable model with all $g_i$ non-vanishing provides a rather rich dynamics as compared to the Schwarzschild or RN solutions. The investigation of such fairly complex integrable models can significantly improve our understanding of complex dynamics in non-integrable theories with massive fields.

### 3.2 DGS and cylindrical reduction

The general cylindrical reduction is, in fact, much more complicated than a spherical one. Consider the decomposition of the line interval in $D = 4$:

$$ds^2_4 = (g_{ij} + \varphi \sigma_{mn} \varphi^m \varphi^n) dx^i dx^j + 2 \varphi_{im} dx^i dy^m + \varphi \sigma_{mn} dy^m dy^n,$$

(29)

where $i, j = 0, 1, m, n = 2, 3$, all the metric coefficients depend only on the $x$-coordinates $(t, r)$, and $y^m = (\phi, z)$ are coordinates on the two-dimensional cylinder (torus). Note that $\varphi$ plays the role of a dilaton and $\sigma_{mn}$ (det $\sigma_{mn} = 1$) is the so-called $\sigma$-field.

As was shown in \[15\], the reduction of the Einstein part of the four-dimensional Lagrangian, $\sqrt{-g^{(4)}} R(g^{(4)})$, can be written as:

$$L_c = \sqrt{-g^{(2)}} \left[ \varphi R(g^{(2)}) + \frac{1}{2\varphi}(\nabla \varphi)^2 - \frac{\varphi}{4} \text{tr}(\nabla \sigma^{-1} \nabla \sigma^{-1}) - \frac{\varphi^2}{4} \sigma_{mn} \varphi_{ij}^m \varphi_{ij}^n \right],$$

(30)

where $\varphi_{ij}^m = \partial_i \varphi^m_j - \partial_j \varphi^m_i$. This complicated action can be simplified for some interesting cases. With a special choice of the matrix $\sigma_{mn}$,

$$\sigma_{22} = e^\psi, \quad \sigma_{33} = e^{-\psi}, \quad \sigma_{23} = \sigma_{32} = 0,$$

(31)

and the vanishing $\varphi_{ij}^m$, one can derive the simpler effective action:

$$L_c = \sqrt{-g^{(2)}} \left[ \varphi R(g^{(2)}) + \frac{1}{2\varphi}(\nabla \varphi)^2 - \frac{\varphi}{2} (\nabla \psi)^2 - \frac{Q_1^2}{2\varphi^2} e^{-\psi} \right],$$

(32)

where $Q_1$ is the contribution of the integrated out Abelian gauge field $\varphi_{ij}^m$. This gives us a serious motivation to study the DGS models in which scalar potentials are exponential in $\psi$. Their one-dimensional configuration can be described by a dynamical system \[4\], with the effective potential $U = e^{g_2 \psi} \Phi(\xi)$. Actually, in pure geometrical case one has $\varphi = e^{-\xi/2}$, since $\varphi/Z = \xi$ and $Z(\varphi) = -\varphi/2$. The potential is power-like in $\varphi$ (even taking into account the Weil transformation), and thus exponential in $\xi$: $\Phi(\xi) = e^{g_1 \xi}$. This is the integrable configuration considered above, but let us first consider a bit more general potentials with arbitrary $\Phi(\xi)$.

It may be of interest to find how does the general function $\Phi(\xi)$ break the integrability. The first integral is obviously $P = 2\psi - g_2 \xi$ (or, equivalently, $P = (g_2 - 2G) \chi$ using $\xi$ as the independent variable). If $P$ vanishes, we have the exact expression $G = g_2/2$ and then applying the MIE procedure easily derive the following special solution:

$$\psi = \frac{g_2}{2}(\xi - \xi_0), \quad F(\xi) = F_0 - \frac{g_2^2}{4}(\xi - \xi_0), \quad \chi(\xi) = \chi_0 - e^{(F_0 - g_2^2 \xi_0/4)} \int_{\xi_0}^\xi \Phi(\xi)e^{g_2^2 \xi/4}d\xi.$$

(33)
With $P \neq 0$ we return to Eqs. (14) and show that the equation for $\chi$ can be transformed into a simple Hamiltonian system with ‘dissipation’ vanishing of which transforms it into completely integrable system. Differentiating $\chi' = -e^{F + g_2 \Phi(\xi)}$ and using the other equation and integral $P$,

$$\psi' = \mathcal{G} = (g_2 - P/\chi)/2, \quad F' = -\mathcal{G}^2 = -(g_2 - P/\chi)^2/4,$$

we derive a somewhat unusual equation for $\chi$ containing a ‘dissipative’ term:

$$(\ln \chi')' = F' + g_2 \psi' + (\ln \Phi)' = \frac{g_2}{4} - \left( \frac{P}{2\chi} \right)^2 + (\ln \Phi)' .$$

This equation can be rewritten as a Hamiltonian system. If we denote $\chi' \equiv \varepsilon e^{\rho} (\varepsilon = \pm 1)$, we find its canonical formulation,

$$\chi' = \varepsilon e^{\rho} = \frac{\partial H}{\partial \rho}, \quad \rho' = \frac{g_2}{4} - \left( \frac{P}{2\chi} \right)^2 + (\ln \Phi)' = -\frac{\partial H}{\partial \chi},$$

with the $\xi$-dependent (‘time’ dependent) Hamiltonian,

$$H = H_0 + \varepsilon e^{\rho} - \frac{P^2}{4\chi} - \chi g_2^2/4 - \chi (\ln \Phi)'$$

The total derivative of this Hamiltonian on the equations of motion is simply

$$\frac{dH}{d\xi} \equiv -\chi(\xi) (\ln \Phi(\xi))'' \neq 0 .$$

For the integrable potential $\Phi = e^{g_1 \xi}$, the Hamiltonian is conserved on the equations of motion, $\frac{dH}{d\xi} = 0$, and one can derive the explicit exact solutions. In view of its simplicity, both technical and conceptual, this system may serve as a good laboratory for qualitative and quantitative studies of non-integrable DGS systems.

### 3.3 DGS and spherical reduction

The spherical dimensional reduction implies the following decomposition of the line element:

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij} \, dx^i \, dx^j + \varphi^{2\nu} \, d\Omega_{D-2}^2 ,$$

where $\nu \equiv (D - 2)^{-1}$. It contains only the dilaton, so we consider the additional ordinary scalar field $\psi$ in the standard spherically reduced Einstein gravity:

$$L_s = \sqrt{-g^{(2)}} \left[ \varphi R(g^{(2)}) + k_\nu \varphi^{1-2\nu} + \frac{1-\nu}{\varphi}(\nabla \varphi)^2 + \varphi(\nabla \psi)^2 + \varphi \Phi(\psi) \right].$$

The curvature term, $k_\nu$, rarely supports the symmetry of the scalar field potential $\Phi(\psi)$. Since we are investigating the contribution of the scalar mode, we omit the curvature term. Or, one may consider the vanishing $k_\nu$ as the result of the trivial cylindrical reduction. The resulting dynamical system is quite similar to the discussed above for the non-trivial cylindrical reduction, but it represents the other variant of the half-exponential potential: now $U = e^{g_1 \xi} \Phi(\psi)$. Indeed, $Z(\varphi) = \varphi$, so $\varphi/Z = \xi$ provides $\varphi = e^\xi$. Taking account of the Weyl transformation and of the gauge choice we have $g_1 = \nu + 1$, where normally $\nu = 1, 1/2, \ldots, 0$, but we suppose that in this DGS model he parameter $g_1$ may be arbitrary.
In this model, there also exists a linear integral, \( P = \dot{F} - g_{11} \dot{\xi} = \chi(F' - g_{11}) + \chi' \), but with the arbitrary potential \( \Phi(\psi) \) this is insufficient for integrability. To find a partial result, let us combine first and last equations (14):

\[
G' = -\frac{\chi'}{\chi} \left( G - \frac{d\psi}{d\psi} \right), \quad w(\psi) \equiv \ln \sqrt{\Phi}.
\] (41)

If \( P = 0 \), we can obtain a simple first-order differential equation for \( y(\psi) \equiv 1/G(\psi) \). Recalling that \( \chi'/\chi = -F' + g_{1} = G'^{2} + g_{1} \) and \( d\xi = d\psi/G \), we have:

\[
y'(\psi) = \left( 1 + g_{1} y^{2} \right) \left[ 1 - y w'(\psi) \right].
\] (42)

A nice explicit solution can be obtained if we take \( g_{1} = 0 \): \( y = e^{-w} \int e^{w} d\psi \).

With non-vanishing \( P \), simple equations with explicit Hamiltonian interpretation can hardly be derived. For instance, one can express \( G = G(\chi, \chi') \) from the expression for \( P \) and put it into Eq. (11). Then, again using the relation \( \psi' = \dot{\chi} \), one obtains a rather cumbersome second-order differential equation for \( \chi(\psi) \) with arbitrary \( \Phi(\psi) \). Thus the system with the dilaton half-exponential potential looks more complicated than that with the scalar half-exponential potential.

### 3.4 DSG and spherical reduction

Finally we focus on the DSG model – a specific model of the DGS class, which is met in affine generalizations of gravity. The simplest case of DSG implies just one scalar mode, a scalaron \( \psi \), with the Lagrangian of the form (11):

\[
\mathcal{L}_{\text{DSG}} = \sqrt{-g^{(2)}} \left[ \varphi R(g^{(2)}) + k_{\nu} \varphi^{1-2\nu} + \frac{1-\nu}{\varphi} (\nabla \varphi)^{2} + \frac{1}{\varphi} (\nabla \psi)^{2} + \Phi(\psi^{2}/\varphi^{2}) \right].
\] (43)

Compared to the ordinary scalar field, it has an abnormal coupling of the kinetic term to gravity. Now the dependence of \( \xi \)-variable on the dilaton is \( \varphi = \sqrt{2\xi} \), since \( Z = 1/\varphi \) instead of the standard \( Z \sim \varphi \). In the gauge \( s = 1/Z \), after the Weyl transformation with the factor \( \Omega = \varphi^{\nu-1} \), we then obtain the effective power-like potential \( U(\xi, \psi) = \Omega Z \Phi = \varphi^{\nu/2-1} \Phi(\psi^{2}/\xi) \). The function \( \Phi(\psi^{2}/\xi) \) is actually specified in the DSG model, but it depends on the space-time dimension, and for \( D > 5 \) it cannot be given explicitly [10]. So it is much more convenient to treat it as an arbitrary function. For the ordinary scalar field potential, the dilaton field usually enters just as a factor. For the DSG, in contrast, the mixing of the dilaton and scalaron is essential.

Again, we omit the curvature term in the potential since it does not correspond to the inner symmetry of the DSG model. The pure DSG potential, \( U \), satisfies the condition of the linear symmetry (11), even in case of arbitrary \( \Phi(\psi^{2}/\xi) \), if we take \( \lambda = 2b_{3}, \ c_{2} = -\nu b_{3} \) with vanishing all remaining parameters. Then the bilinear integral (12) reads as

\[
P = \xi \dot{F} - \frac{\nu \dot{} \dot{\xi}}{2} + \psi \dot{\psi}.
\] (44)

Unfortunately, to find the other integrals or explicit solutions we have to require too much: \( P \) and \( \nu \) must vanish and the function \( \Phi \) has to be specified. Instead, we may use another approach, the MIE, introduced in [13]. It suits the DSG system very well and allows to derive a power-like solution, as we will see now.

We first choose \( G = C_{0} \xi^{-1/2} \) as a trial function and adjust the parameters \( \xi_{0}, \psi_{0}, \chi_{0} \) to cancel the contribution of the constant terms. Then we find that

\[
\psi = \psi_{0} + \mathcal{I}\{G\} = \psi_{0} + \frac{C_{0}}{2} \left( \xi^{1/2} - \xi_{0}^{1/2} \right) = \frac{C_{0}}{2} \xi^{1/2}.
\] (45)
Actually, we can keep the constant term in the function $\mathcal{F}$ and find

$$
\mathcal{F} = \mathcal{F}_0 + \mathcal{I}\{\mathcal{G}^2\} = \mathcal{F}_0 - \frac{C_0^2}{4} \ln \xi / \xi_0 = \mathcal{F}_1 - \frac{C_0^2}{4} \ln \xi,
$$

(46)

with the new parameter $\mathcal{F}_1 = \mathcal{F}_0 + \frac{C_0^2}{4} \ln \xi_0$.

Next, we see that, fortunately, $\Phi(\psi^2/\xi) = \Phi(C_0^2/4)$ is constant. Therefore we may continue applying the MIE procedure using an arbitrary DSG potential $\Phi$:

$$
\chi(\xi) = \chi_0 - e^{\mathcal{F}_1} \Phi(C_0^2/4) \left( \xi^{-C_0^2/4+\nu/2} - \xi_0^{-C_0^2/4+\nu/2} \right) = - e^{\mathcal{F}_1} \Phi(C_0^2/4) \xi^{-C_0^2/4+\nu/2}.
$$

(47)

The final integration in the first MIE iteration provides the improved function $\mathcal{G}$ that can be used in the next iteration:

$$
\chi(\xi) \mathcal{G}(\xi) = \chi_0 C_0 \xi_0^{-1/2} - e^{\mathcal{F}_1} C_0 \Phi'(C_0^2/4) \left( \xi^{-C_0^2/4+\nu/2-1/2} - \xi_0^{-C_0^2/4+\nu/2-1/2} \right) = - e^{\mathcal{F}_1} C_0 \Phi'(C_0^2/4) \xi^{-C_0^2/4+\nu/2-1/2}.
$$

(49)

(50)

Of course, to avoid singularities, we suppose that $\nu \neq C_0^2/2 + 1$, $\nu \neq C_0^2/2$.

We see that the new function $\mathcal{G}$ has the same form $\mathcal{G}(\xi) = C_1 \xi^{-1/2}$, where $C_1$ is a new constant,

$$
C_1 = \frac{C_0 \Phi'(C_0^2/4) [2 \nu - C_0^2]}{2 \Phi(C_0^2/4) [2 (\nu - 1) - C_0^2]}.
$$

(51)

If we choose $C_0$ satisfying the equation $C_1 = C_0$, the above expressions will give an exact solution.

It can be useful for the investigation of the trajectories in vicinity of some singular points.

## 4 Conclusion and outlook

In a set of previous works on the topic of affine generalization of gravity we discussed the general properties of the theory and derived a very useful DSG interpretation. Here we tried to advance toward investigation of the dynamics provided by the theory. Thus the specific properties of the DSG dynamical system were revealed. Compared to the closely related but different DGS models, the DSG configuration appeared to be one of the most favorable for the analysis, possessing a linear inner symmetry and being well suited for the iterative MIE procedure.

We suppose that this is not an accident but comes from the fundamental geometric origin of the affine gravity, the same as of the DGS models obtained by various dimensional reductions. In fact, such relations are usually hidden and can hardly be explicitly demonstrated on the level of Lagrangians. So, for the next step we will try to choose a somewhat different approach that would allow to establish a closer relation between the global (topological) properties of the set of all solutions and the inner symmetries of the DGS and DSG models.

In conclusion, we wish to emphasize that the methods discussed above can be applied outside the context of the affine generalization of gravity and can provide, as we hope, useful tools for a wide class of models with scalar fields.
APPENDIX A: Dynamical systems with linear symmetries

Consider a vanishing Hamiltonian for \( n \) dynamical variables \( q^i \), which is bilinear in momenta:

\[
H = A^{ij}(q)p_ip_j + U(q) = 0.
\]  
\( (A-1) \)

Let us introduce in the coordinate space an affine transformation that defines the vector field

\[
v^i = B^i_j q^j + c^i, \quad \text{where} \quad B^i_j, c^i = \text{const}.
\]  
\( (A-2) \)

For simplicity we will denote the vectors like \( \{q^i\} \) by the bold font, \( \mathbf{q} \). The Euclidean scalar product, \( \mathbf{p} \cdot \mathbf{v} \equiv (\mathbf{p}, \mathbf{v}) = p_i v^i \), is a so-called Hamiltonian of the vector field \( \mathbf{v} \). Its Poisson bracket with the original Hamiltonian \( H \) obviously is

\[
\{H, \mathbf{pv}\} = 2(\mathbf{p}, B\mathbf{Ap}) - \partial_v H,
\]  
\( (A-3) \)

where \( \partial_v = v^i \partial / \partial q^i \) is a derivative along the vector field \( \mathbf{v} \).

Consider the linear vector fields the matrices of which satisfy the conditions:

\[
(\mathbf{p}, B\mathbf{Ap}) = \lambda (\mathbf{p}, \mathbf{Ap}) \quad \Rightarrow \quad \{B, A\} \equiv BA + AB = \lambda A + B,
\]  
\( (A-4) \)

It follows that the Poisson bracket of the two Hamiltonians, \( H \) and \( \mathbf{pv} \), vanishes when

\[
\{B, A(q)\} = \lambda A(q) + B(q), \quad \partial_v A^{ij}(q) = 0, \quad \partial_v U(q) = -\lambda U(q).
\]  
\( (A-5) \)

So, in case of the corresponding symmetries of the kinetic matrix and the potential term, dynamical systems \( (A-1) \) admit the first integrals, which are linear in momenta and coordinates and are the Hamiltonians of the linear vector fields \( \mathbf{v} = B\mathbf{q} + \mathbf{c} \). Their algebra reads

\[
\{\mathbf{pv}, \mathbf{pv}'\} = \mathbf{pv}''', \quad \text{where} \quad v''' = [B', B]|\mathbf{q}| + (B'\mathbf{c} - B\mathbf{c}'),
\]  
\( (A-7) \)

where the square brackets denote the matrix commutator.

One can easily resolve the conditions \( (A-6) \) for the subgroup of transformations, \( \mathbf{v} = \lambda \mathbf{q} + \mathbf{c} \), which represent dilatations and translations:

\[
A^{ij}(\mathbf{q}) = \Phi_{ij}(\mathbf{q}_\perp),
\]  
\( (A-8) \)

\[
\lambda = 0 : \quad U(\mathbf{q}) = \Phi_1(\mathbf{q}_\perp), \quad \mathbf{q}_\perp = |\mathbf{c}|^2 \mathbf{q} - (\mathbf{q}, \mathbf{c})\mathbf{c};
\]  
\( (A-9) \)

\[
\lambda \neq 0 : \quad U(\mathbf{q}) = |\mathbf{v}|^{-2}\Phi_2(\mathbf{v}/|\mathbf{v}|).
\]  
\( (A-10) \)

Here \( \Phi_i, \Phi_{ij} \) are the arbitrary functions of their arguments, which are now not independent: \( \mathbf{q}_\perp \) is orthogonal to \( \mathbf{c} \), and \( \mathbf{v}/|\mathbf{v}| \) has unit length.

APPENDIX B: Linear integrals in DGS

Now let us return to the system \( (2) \) and try to construct integrals that are linear in momenta. For simplicity, we consider only the symmetries corresponding to the constant vector fields, \( \mathbf{v} = \mathbf{c} \). Then the first relation in the conditions \( (A-3) \) is automatically fulfilled for \( \lambda = 0 \), since \( B \) is now zero matrix.
The remaining conditions can be satisfied in the following way. Consider first the dependence of the kinetic matrix only on the dilaton, $A_{ij} = A_{ij}(\varphi)$, where the $h$-dependence was removed by using the new variable $\tilde{F} = h/h$. The coordinate vector is now $q = (F, \varphi, \psi_1, \ldots, \psi_N)$. Thus a possible ansatz has the form $c = (\alpha, 0, \beta^1, \ldots, \beta^N)$, where $\alpha, \beta^i$ are constants. To compactify the lengthy formulas let $\{\beta^i\}$ and $\{\psi^j\}$ be treated as vectors $\vec{\beta}, \vec{\psi}$ in $\mathbb{R}^N$.

All such constant fields, $c$, obviously satisfy $\partial_c A_{jk}(q) = e^F \partial_\beta A_{jk}(\varphi) / \partial q^i = 0$. The solution for the potential term for such vector field is already obtained in (A.9). Construct first the vector $q_{\perp}$ — the part of $q$ that is orthogonal to $c$:

$$q_{\perp}^F = |\vec{\beta}|^2 F - \alpha \vec{\beta} \vec{\psi}, \quad q_{\perp}^c = (\alpha^2 + |\vec{\beta}|^2) \varphi, \quad q_{\perp}^1 = (\alpha^2 + |\vec{\beta}|^2) \psi^i - (\alpha F + \vec{\beta} \vec{\psi}) \beta^i, \quad i = 1..N. \tag{B-1}$$

The potential can be an arbitrary function of $q_{\perp}$ components: $\Phi(F; \varphi; \psi_1, \ldots, \psi_N) = \Phi(q_{\perp})$. Yet, in the actual gravitational potential, the metric enters as $e^F$. So we can take the component $q_{\perp}^F$ as the argument of the exponent function. The metric can be excluded from the remaining components $q_{\perp}^c$ if we use the scalar products $q_{\perp}^c \beta_{\perp}^{(m)} = (\alpha^2 + |\vec{\beta}|^2) \beta_{\perp}^{(m)} \vec{\psi}$ with $N - 1$ linearly independent vectors orthogonal to $\vec{\beta}$: $\beta_{\perp}^{(m)} = 0, \ m = 1..N - 1$. Then we have the following expression for the potential:

$$e^F U(\varphi; \psi) = e^{\tilde{\psi} \beta a / |\vec{\beta}|^2} X(\varphi; \{\beta_{\perp}^{(m)} \psi^\alpha\}), \quad m = 1..N - 1, \tag{B-2}$$

where $X$ is an arbitrary function of $N$ arguments. The corresponding linear integral is

$$p c = \alpha \dot{\varphi} + 2Z_{ij}(\varphi) \beta^i \dot{\psi}^j = P. \tag{B-3}$$

In the same way we can add the dependence of $Z_{ij}$ on $\psi_{i_k}$, $k = 1, \ldots, K$. Then we should consider the constant vector $\vec{\beta}$ with vanishing those $i_k$-th coordinates, actually belonging to $\mathbb{R}^{N-K}$. And the potential satisfying our conditions will be

$$e^F U(\varphi; \psi) = e^{\tilde{\psi} \beta a / |\vec{\beta}|^2} X(\varphi; \{\psi_{i_k}\}, \{\beta_{\perp}^{(m)} \psi^\alpha\}), \quad k = 1..K, \ m = 1..N - K - 1, \tag{B-4}$$

where now $\beta_{\perp}^{(m)}$ are $N - K - 1$ arbitrary linearly independent constant vectors, belonging to the orthogonal complement to $\vec{\beta}$ in $\mathbb{R}^{N-K}$ (they also have vanishing $i_k$-th components). The first integral will be

$$p c = \alpha \dot{\varphi} + 2Z_{ij}(\varphi; \{\psi_{i_k}\}) \beta^i \dot{\psi}^j = P. \tag{B-5}$$

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