A squeezing property and its applications to a description of long time behaviour in the 3D viscous primitive equations

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Abstract

We consider the 3D viscous primitive equations with periodic boundary conditions. These equations arise in the study of ocean dynamics and generate a dynamical system in a Sobolev $H^1$ type space. Our main result establishes the so-called squeezing property in the Ladyzhenskaya form for this system. As a consequence of this property we prove (i) the finiteness of the fractal dimension of the corresponding global attractor, (ii) the existence of finite number of determining modes, and (iii) ergodicity of a related random kick model. All these results provide a new information concerning long time dynamics of oceanic motions.

Keywords: 3D viscous primitive equations, periodic boundary conditions, global attractor, finite dimension, random kicks, ergodicity.

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1 Introduction

We deal with the 3D viscous primitive equations which arise in geophysical fluid dynamics for modeling large scale phenomena in oceanic motions. These equations are based on the so-called hydrostatic approximation of the 3D Navier-Stokes equations for velocity field $u$ which also contain a rotational (Coriolis) force and coupled to thermo- and salinity diffusion-transport equations (see, e.g., the survey [26] and the references therein). In this model a small variation of the density $\rho$ of the fluid is taken into account via the buoyancy term only and has the form $\rho = \rho_0 - \alpha T + \beta s$, where $T$ is the temperature and $s$ denotes the salinity. If we take into account one

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of the factors \((T \text{ or } s)\) or the diffusivities for \(T\) and \(s\) are equal, then their effect in the dynamics can be represented simply by the density \(\rho\), or equivalently the buoyancy \(b = \rho_0 - \rho = \alpha T - \beta s\) (see [26]). This is why we prefer to deal with the variables \(u\) and \(b\). In order to simplify our mathematical presentation we impose on \(u\) and \(b\) the periodic boundary conditions of the same type as in [24] (see also [26] and the references therein). Thus we consider our problem in the domain
\[
\mathcal{O} = (0, L_1) \times (0, L_2) \times (-L_3/2, L_3/2) \subset \mathbb{R}^3
\]
and denote the spatial variable in \(\mathcal{O}\) by \(\bar{x} = (x, z) = (x_1, x_2, z) \in \mathcal{O}\). We use the notation \(\nabla, \text{div and } \Delta\) for the gradient, divergence and Laplace operators in two dimensional variable \(x = (x_1, x_2)\). Below we also denote by \(\Delta_{x,z}\) the 3D Laplace operator. The notations \(\nabla_{x,z}\) and \(\text{div}_{x,z}\) have a similar meaning.

We consider the following equations for the fluid velocity field
\[
u_t + \nabla \cdot \mathbf{u} = \nabla P = G_f \text{ in } \mathcal{O} \times (0, +\infty),
\]
(1)
\[\text{div } \mathbf{u} + \partial_z w = 0 \text{ in } \mathcal{O} \times (0, +\infty),
\]
(2)
\[\partial_z P = b \text{ in } \mathcal{O} \times (0, +\infty),
\]
(3)
where \(\nu > 0\) is the dynamical viscosity, \(G_f\) and \(G_b\) are volume sources, and \(f\) is the Coriolis parameter. We denote \(\mathbf{v}^\perp = (-v_2, v_1)\). We supplement (1)–(4) with the periodic boundary conditions imposed on \((\mathbf{v}; w), P\) and \(b\). As in [24, 26] we also assume that \(\mathbf{v}\) and \(P\) are even with respect to \(z\) and \(w\) and \(b\) are odd with respect to \(z\) (hence \(w\big|_{z=0} = 0\) and \(b\big|_{z=0} = 0\)). These requirements and also [2, 3] lead to the following relations
\[w(\bar{x}, t) = -\int_0^z \text{div } \mathbf{v}(x, \xi, t) d\xi, \text{ } P(\bar{x}, t) = p(x, t) + \int_0^z b(x, \xi, t) d\xi\]
(5)
for every \(\bar{x} = (x, z) \in \mathcal{O}\). It is important (see [3] for the basic discussion) that the pressure \(p\) is independent of the vertical variable \(z\) and depends on 2D (horizontal) variable \(x\) only. Exploring intensively this observation the authors of the paper [3] have implemented a new effective approach for proving of the global well-posedness for problems like (1)–(4), see the discussion below.

The system in (1)–(4) was studied by many authors for the different types of boundary conditions (see the survey [26] and the references therein). The existence of week solutions [24] and (global) well-posedness of strong solutions were established (see [3] and also [17, 18, 24]). The existence of
a global attractor for the viscous 3D primitive equations was proved in [16] (see also the paper [24] devoted to the periodic case). However, to the best of our knowledge, the question concerning dimension of this attractor is still open. We note that the author of [16] claimed the finiteness of the dimension in 2007 with the reference to a forthcoming paper which is not published yet.

It should be also noted that stochastic perturbations of (1)–(4) were studied in [10, 14, 15] (see also the references in these publications) and the paper [11] deals with a random kick forcing of the primitive equations.

Our goal is to show that the system in (1)–(4) possesses an additional regularity and satisfies a squeezing property in the Ladyzhenskaya form (see [21, 22]). This property implies several important facts about dynamics of the system and, in particular, the finiteness of the fractal dimension of the attractor (see Corollary 3.6) and the existence of finite number of determining modes or functionals (Corollary 3.7). The squeezing property also plays an important role in the study of turbulent behaviour in some classes of dynamical systems with random kicks arising in the fluid dynamics (see, e.g., [19, 20]). In this paper we apply the squeezing property established and also the theory developed in [20] to present a result on ergodicity of (1)–(4) forced by random bounded kicks with appropriately chosen frequency (see Corollary 3.9).

As we will see below the main ingredient of our argument is the uniform smoothing stated in Theorem 3.1. There are two ways to obtain this asymptotic regularity property. One method involves appropriate (spatial) multipliers which produce higher kinetic energy type norms in the corresponding energy balance relation. Another method consists of two steps. In the first step the standard phase space estimates for higher time derivatives of solutions are established. Then, on the second step, spatial smoothness of solutions is derived via appropriate elliptic regularity of the corresponding stationary problem. Both methods requires some compatibility conditions involving the initial data and the nonlinearity (see, e.g., [27] for a general presentation). In the periodic case these compatibility relations are satisfied automatically. However the first method is a one-step method and looks simpler. Moreover, it requires minimal hypotheses concerning time regularity of external forces. This why it was used in [24, 26] and also in this paper. In the case of other boundary conditions (free type as in [3] or mixed free-Dirichlet as in [18]) we need rather careful analysis of compatibility conditions and elliptic properties of the stationary problem. We plan to analyze this issue in our further publications.

It is commonly recognized (see [3, 17, 18, 24, 26]) that the main difficulty arising in the study of the primitive equations in (1)–(4) is related to the hydrodynamical part. The calculations which should be made to obtain appropriate bounds for a buoyancy type variable are either standard or repeat the corresponding argument for the fluid variables in the simplified form. This is why below we state our results for the full problem (1)–(4), but
in the proof we concentrate on the fluid equations only. Moreover, for the transparency of our argument we even assume that the buoyancy equation is absent. This corresponds to the case of the zero buoyancy \( b(\bar{x}, t) \equiv 0 \) which is possible when \( G_b \equiv 0 \) and \( b(\bar{x}, t = 0) = 0 \).

The paper is organized as follows.

Section 2 contains some preliminary material. Here we represent the primitive equations in some standard form, introduce Sobolev type spaces and state the well-known result on well-posedness of strong solutions. In Section 3 we state our main results. The key outcome of Theorem 3.1 is the existence of an absorbing ball lying in an \( H^2 \) type space. This allows us not only guarantee the existence of a global attractor but also provides an important step in the proof of the squeezing in Theorem 3.5. In this section we also apply our main results to prove the finiteness of the dimension of the global attractor, the existence of finite number of determining functionals, and ergodicity of a random kicks forced model generated by (1)-(4). In Section 4 we demonstrate the main steps of the proofs of Theorems 3.1 and 3.5 in the case when the buoyancy variable \( b \) is absent.

2 Preliminaries

In this section we rewrite system (1)-(4) in the canonical form, introduce appropriate Sobolev type spaces and quote the result on the strong well-posedness.

Using the representations in (5) we arrive at the system

\[
v_t + (v, \nabla)v - \left[ \int_0^z \text{div} \, v \, d\xi \right] \partial_z v - \nu [\Delta v + \partial_{zz} v] + f v^\perp = -\nabla \left[ p(x, t) + \int_0^z b d\xi \right] + G_f \quad \text{in} \quad \mathcal{O} \times (0, +\infty),
\]

(6)

\[
b_t + (v, \nabla)b - \left[ \int_0^z \text{div} \, v \, d\xi \right] \partial_z b - \nu [\Delta b + \partial_{zz} b] = G_b \quad \text{in} \quad \mathcal{O} \times (0, +\infty),
\]

(7)

supplied with the conditions:

\[
\text{div} \int_{-L_3/2}^{L_3/2} v \, dz = 0; \quad \text{v is periodic in } \bar{x} \text{ and even in } z, \quad \int_{\mathcal{O}} v \, d\bar{x} = 0;
\]

(8)

and

\[
b \text{ is periodic in } \bar{x} \text{ and odd in } z.
\]

(9)

We also need to add initial data for \( v \) and \( b \):

\[
v(0) = v_0, \quad b(0) = b_0.
\]

(10)

We denote by \( \dot{H}^s_{\text{per}}(\mathcal{O}) \) the Sobolev space of order \( s \) consisting of periodic functions such that \( \int_\mathcal{O} f \, dx = 0 \). To describe fluid velocity fields we introduce the following spaces with \( s = 0, 1, 2 \):
\[ V_s = \left\{ v = (v^1; v^2) \in [H^s_{per}(O)]^2 : v^i \text{ is even in } z, \text{ div } \int_{-L_3/2}^{L_3/2} v dz = 0 \right\}. \]

We also denote \( H = V_0 \) and equip \( H \) with \( L_2 \) type norm \( \| \cdot \| \) and denote by \((\cdot, \cdot)\) the corresponding inner product. The spaces \( V_1 \) and \( V_2 \) are endowed with the norms \( \| v_1 \|_{V_1} = \| \nabla_{x,z} \cdot \| \) and \( \| v_2 \|_{V_2} = \| \Delta_{x,z} \cdot \| \).

To describe the buoyancy we need the spaces

\[ E_s = \left\{ b \in \dot{H}^s_{per}(O) : b \text{ is odd in } z \right\}, \quad s = 0, 1, 2, \]

equipped with the standard norms. We also denote \( W_s = V_s \times E_s \) with the standard (Euclidean) product norms.

As it was already mentioned, starting with [3] the global well-posedness of the equations in (6) and (7) was studied by many authors [17, 18, 24, 26]. In this section we quote the result from [24] (see also [26]) on well-posedness of strong solutions under the periodic boundary conditions.

**Proposition 2.1** ([24, 26]) Let \( G_f \in V_0 \) and \( G_b \in E_0 \). Then for every \( U_0 = (v_0; b_0) \in W_1 \) problem (6)–(10) has a unique strong solution \((v(t); b(t))\):

\[ U(t; U_0) \equiv (v(t); b(t)) \in C(\mathbb{R}_+; W_1) \cap L_2(0, T; W_2), \quad \forall T > 0. \]

Moreover, this solution continuously depends on time \( t \) and initial data \( U_0 \) and generates a dynamical system \((S_t, W_1)\) with the phase space \( W_1 \) and the evolution operator \( S_t \) defined by the relation \( S_t U_0 = U(t; U_0) \). This evolution operator \( S_t \) possesses the following Lipschitz property

\[ \| S_t U - S_t U_* \|_{W_1} \leq C_{T,R} \| U - U_* \|_{W_1}, \quad t \in [0, T], \]

for every \( T > 0 \) and \( U, U_* \in B(R) \equiv \{ U : \| U \|_{W_1} \leq R \} \).

**Remark 2.2** Formally [24, 26] do not contain the proofs of continuity of \( S_t U \) with respect to \( t \) and the Lipschitz property in \( W_1 \). However almost the same calculations as in [16] allow us to show these properties. We also note that in the case considered unique global solvability can be also established in smoother classes. For instance, if \( G_f \in V_{m-1} \), \( G_b \in E_{m-1} \) and \( U_0 = (v_0; b_0) \in W_m \) for some \( m \geq 2 \) then (see [24, 26]) the solution \( U(t) \) belongs to the class \( C(\mathbb{R}_+; W_m) \cap L_2(0, T; W_{m+1}) \) for every \( T > 0 \). This observation is important in our further considerations due to possibility to use smooth approximations of solutions in our calculations.

We conclude this preliminary section with the statement of the following well-known uniform Gronwall lemma.
Lemma 2.3 ([28]) Let \( g, h, y \) be nonnegative locally integrable functions on \([t_0, +\infty)\) such that
\[
\frac{dy}{dt} \leq gy + h,
\]
and
\[
\int_t^{t+1} g(s)ds \leq a_1, \quad \int_t^{t+1} h(s)ds \leq a_2, \quad \int_t^{t+1} y(s)ds \leq a_3
\]
for any \( t \geq t_0 \), where \( a_i > 0 \) are constant. Then
\[
y(t + 1) \leq (a_3 + a_2)e^{a_1} \quad \text{for any } t \geq t_0.
\]

3 Results

We start with the following assertion which partially follows from Proposition 2.1 and is based on the calculations given in [24].

**Theorem 3.1 (Smooth Absorbing Ball)** Let \( G_f \in V_1, G_b \in E_1 \) and also \((\partial_z G_f; \partial_z G_b) \in [L_6(O)]^3\). Then the dynamical system \((S_t, W_1)\) generated by problem (7)–(10) is dissipative and compact. Moreover, there exists \( R_* > 0 \) such that the ball
\[
\mathcal{B} = \{U \in W_2 : \|U\|_{W_2} \leq R_*\}
\]
is absorbing, i.e., for any bounded set \( B \) in \( W_1 \) there exists \( t_B \) such that
\[
S_t B \subset \mathcal{B} \quad \text{for all } t \geq t_B.
\]

In this ball \( \mathcal{B} \) there is a forward invariant absorbing set \( \mathcal{D} \).

We note that the dissipativity and compactness of \((S_t, W_1)\) stated in Theorem 3.1 is known from [24] under slightly weaker conditions on \( G_f \) and \( G_b \). However the existence of an absorbing compact set which is bounded in \( W_2 \) requires additional hypotheses and additional calculations.

**Remark 3.2** In the case when \( G_f \equiv 0 \) and \( G_b \equiv 0 \) one can prove (see some argument below in Section 4.1) that \( \forall \varepsilon > 0, \exists T(R, \varepsilon) : \|S_tU\|_{W_1} \leq \varepsilon \quad \forall t \geq T(R, \varepsilon), \|U\|_{W_1} \leq R. \)

This means that in the case of vanishing sources the zero equilibrium state is asymptotically stable.

Using the standard results (see, e.g., one of the monographs [1, 5, 28]) on the existence of global attractors we can derive from Theorem 3.1 the following assertion.

\[\text{For the definitions see [1], for instance.}\]
Corollary 3.3 (Global Attractor) Let the hypotheses of Theorem 3.1 be in force. Then the dynamical system \((S_t, W_1)\) generated by problem (6)–(10) possesses a compact global attractor \(A\) which is a bounded set in \(W_2\). Moreover, in the case when \(G_b = 0\) the subspace \(V_1 \times \{0\} \subset W_1\) attracts the trajectories with exponential speed and the attractor \(A\) lies in \(V_2 \times \{0\}\).

Remark 3.4 The result on the existence of a global attractor for 3D viscous primitive equations was known before, see [16], and also Remark 2.1 in [24] for the periodic case. Our improvement is that we state the boundedness of \(A\) in an \(H^2\) type Sobolev space. One can also show that in the case of sufficiently smooth sources \(G_f\) and \(G_b\) the attractor \(A\) possesses additional spatial smoothness. Using the method presented in [24] we can even prove Gevrey regularity of the elements from the attractor provided the sources possess appropriate smoothness properties. However we do not pursue these improvements because our main goal is a squeezing property with application to dimension and ergodicity.

In the space \(W_1\) we consider the bilinear form
\[
a(U, U_*) = \int_O \left[ \nabla_{x,z} v \cdot \nabla_{x,z} v_* + \nabla_{x,z} b \cdot \nabla_{x,z} b_* \right] dx dz,
\]
where \(U = (v; b)\) and \(U_* = (v_*; b_*)\) are elements from \(W_1\). This form generates a positive self-adjoint operator \(A\) with a discrete spectrum. Due to periodicity of the problem the corresponding eigenfunctions and eigenvalues can be easily described. We assume that they are reorganized in such way that
\[
Ae_k = \lambda_k e_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{k \to +\infty} \lambda_k = +\infty,
\]
and, moreover, \(\{e_k\}\) is the orthonormal basis in \(W_0\). We denote by \(P_N\) the orthoprojector onto \(\text{Span}\{e_1, \ldots, e_N\}\) and \(Q_N = I - P_N\).

Now we can state our main result.

Theorem 3.5 (Squeezing) Let the hypotheses of Theorem 3.1 be in force. Then the dynamical system \((S_t, W_1)\) generated by problem (6)–(10) on the forward invariant absorbing set \(\mathcal{D}\) possesses the properties:

- **Lipschitz property:** For any \(U, U_* \in \mathcal{D}\) we have the relation
  \[
  \|S_t U - S_t U_*\|_{W_1} \leq C_{\mathcal{D}} e^{\alpha_{\mathcal{D}} t} \|U - U_*\|_{W_1}, \quad \forall t \geq 0,
  \]  
  where \(C_{\mathcal{D}}\) and \(\alpha_{\mathcal{D}}\) are positive constants.

- **Squeezing property:** For every \(T > 0\) and \(0 < q < 1\) there exists \(N = N(T, q)\) such that
  \[
  \|Q_N[S_T U - S_T U_*]\|_{W_1} \leq q \|U - U_*\|_{W_1}
  \]  
  for any \(U\) and \(U_*\) from the absorbing forward invariant set \(\mathcal{D}\).
The squeezing property stated in Theorem 3.5 allows us to apply Ladyzhenskaya Theorem on the dimension of invariant sets (see [21, 22] and also Theorem 8.1 in [5, Chap.1]). This yields the following assertion.

**Corollary 3.6 (Finite Dimension)** Under the hypotheses of Theorem 3.1 the global attractor $\mathcal{A}$ of the dynamical system $(S_t, W_1)$ generated by problem (6)–(10) has a finite fractal dimension.

This means that the asymptotic dynamics in this model is finite-dimensional and topologically can be represented by a compact set in $\mathbb{R}^d$ with appropriate $d < \infty$.

The next outcome of the squeezing property is the existence of a finite number of (asymptotically) determining modes. We note that this notion was introduced in [13] for 2D Navier-Stokes system and was studied by many authors (see the discussion in the recent paper [12]). For a general theory of the determining functionals we refer to [4], see also [5] and [8] for a development of this theory based on the notion of the completeness defect.

**Corollary 3.7 (Determining Modes)** Let the hypotheses of Theorem 3.1 be in force and $N$ be such that (12) holds for some $T > 0$ and $0 < q < 1$. Then the dynamical system $(S_t, W_1)$ possesses $N$ determining modes. This means that the relation

$\lim_{t \to +\infty} \|P_N[S_t U - S_t U_*]\|_W = 0$ for some $U, U_* \in W_1$  \(13\)

implies that $\|S_t U - S_t U_*\|_{W_1} \to 0$ as $t \to +\infty$.

**Proof.** The relation in (12) implies that

$\|S_T U - S_T U_*\|_{W_1} \leq q \|U - U_*\|_{W_1} + \|P_N[S_T U - S_T U_*]\|_{W_1}, \quad 0 < q < 1, \quad (14)$

for any $U$ and $U_*$ from the absorbing forward invariant set $\mathcal{D}$. Iterating (14) we obtain that

$\|S^n_T U - S^n_T U_*\|_{W_1} \leq q^n \|U - U_*\|_{W_1} + \sum_{k=1}^n q^{n-k} \|P_N[S^k T U - S^k T U_*]\|_{W_1}$  \(15\)

for every $n = 1, 2, \ldots$, where $U$ and $U_*$ are from $\mathcal{D}$. This inequality and also the Lipschitz property (11) makes it possible to prove that relation (13) implies that $\|S_t U - S_t U_*\|_{W_1} \to 0$ as $t \to +\infty$. For some details in the abstract situation we refer to the proof of Theorem 1.3 in [5, Chap.5].

**Remark 3.8** Using the same idea as in [8] it is also possible to derive from (15) the existence of other finite families of determining functionals. We also point out the recent paper [25] which contains the proof of the existence of a finite number of determining modes (and also nodes and local volume averages) by another method.
Now we consider an example of an application of the results above to ergodicity of a random kicks forced model generated by problem (6)–(10). We deal with the simplest situation and suppose that \( G_f \equiv 0 \) and \( G_b \equiv 0 \). Moreover, we consider a model in some fixed ball from \( W_1 \) and assume that the frequency of kicks is smaller than a certain critical value (depending on the radius of the ball and the amplitude of the kicks). We plan to remove all these restrictions and discuss further developments in our joint studies with S.Kuksin and A.Shirikyan.

Let \( \{ \eta_k \} \) be i.i.d. random variables in \( W_1 \). Assume that the support of the distribution \( D(\eta_1) \) in \( W_1 \) contains the origin and is bounded, i.e., there exists \( R_{kick} > 0 \) such that

\[
\text{supp} D(\eta_1) \subset B(R_{kick}) \equiv \{ U \in W_1 : \| U \|_{W_1} \leq R_{kick} \}.
\]

Here and below we keep the notation \( B(R) \) for the ball with the center at 0 of the radius \( R \) in the space \( W_1 \).

Let us fix \( \hat{R} > R_{kick} \) and choose \( T_c = T(\hat{R}, R_{kick}, \mathcal{D}) \geq 1 \) such that

\[
S_{t_1}B(\hat{R}) \subset B(\hat{R} - R_{kick}) \quad \text{and} \quad S_{t-1}B(\hat{R}) \subset \mathcal{D} \quad \text{for all} \quad t \geq T_c,
\]

where \( \mathcal{D} \) is the forward invariant absorbing set given by Theorem 3.1. This choice of \( T_c \) is possible due to Remark 3.2 and Theorem 3.1.

Now we fix \( T \geq T_c \) and define a Markov chain in \( W_1 \) by the formula

\[
U_k = F_k(U_{k-1}) \equiv S_{T_1}U_{k-1} + \eta_k, \quad k = 1, 2, \ldots \quad (16)
\]

This chain is called a random kicks model generated by (6)–(10) (with \( G_f \equiv 0 \) and \( G_b \equiv 0 \)). The parameter \( T^{-1} \) has meaning of the frequency of the kicks. We refer to [20] for motivation and physical importance of different types of kick models in the turbulence theory.

It is obvious that if \( U_0 \in B(\hat{R}) \), then we also have that \( U_k \in B(\hat{R}) \) for all \( k \). Thus we also have a chain in \( B(\hat{R}) \).

It follows from Proposition 2.1 that

\[
\| S_{T-1}U - S_{T-1}U_* \|_{W_1} \leq C_{T_1, \hat{R}} \| U - U_* \|_{W_1}, \quad \text{for all} \quad U, U_* \in B(\hat{R}).
\]

Let us fix \( 0 < q < 1 \) such that \( \eta = qC_{T_1, \hat{R}} < 1 \). Since \( S_{T-1}U, S_{T-1}U_* \in \mathcal{D} \), we can apply Theorem 3.5 and choose \( N \) such that

\[
\| Q_N [S_{T_1}U - S_{T_1}U_*] \|_{W_1} \leq q \| S_{T-1}U - S_{T-1}U_* \|_{W_1} \leq \eta \| U - U_* \|_{W_1}
\]

for all \( U, U_* \in B(\hat{R}) \). Thus we have the squeezing property on the ball \( B(\hat{R}) \).

Applying now Theorem 3.2.5[20] and the properties of the mapping \( S_t \) established above we arrive to the following assertion.

\[\text{For the readers convenience we state this theorem in the Appendix.}\]
Corollary 3.9 (Ergodicity) Assume that $P_{N\eta_1}$ and $Q_{N\eta_1}$ are independent and the distribution of $P_{N\eta_1}$ in the (finite-dimensional) space $P_NW_1$ has a density $p(U)$ with respect to the Lebesgue measure $dU$ on $P_NW_1$ such that

$$
\int_{P_NW_1} |p(U + V) - p(U)|dU \leq C\|V\|_{W_1}, \quad \forall V \in P_NW_1,
$$

where $C$ is a constant. Then for the chain (16) there is a unique invariant probability Borel measure $\mu$ on $B(\bar{R})$. Moreover, there is $0 < \gamma < 1$ such that

$$
\|F_k^*\lambda - \mu\|^*_L \leq C(\bar{R}, \lambda) \cdot \gamma^k, \quad k = 1, 2, \ldots
$$

for any probability Borel measure $\lambda$ on $B(\bar{R})$, where the Borel measure $F_k^*\lambda$ and the corresponding dual Lipschitz norm $\|\cdot\|^*_L$ are defined in the Appendix.

To apply [20, Theorem 3.2.5] (see Theorem A.1 in the Appendix) we first consider the extension $\hat{S_T}$ of $S_T$ from $B(\bar{R})$ on the whole space $W_1$ by the formula

$$
\hat{S_T}U = \begin{cases}
S_TU, & \|U\|_{W_1} \leq \bar{R}; \\
S_T\left(\frac{R}{\|U\|_{W_1}}\right), & \|U\|_{W_1} > \bar{R}.
\end{cases}
$$

One can see that $\hat{S_T}$ maps $W_1$ into $B(\bar{R} - R_{kick})$. Thus after the first step in (16) we are in $B(\bar{R})$ and therefore we have $S_T = \hat{S_T}$ for $k \geq 2$ in (16). We also note that $\hat{S_T}$ is globally Lipschitz (the proof of the latter property can be found in [9, p.64], for instance) and Conditions (A1)--(A4) of Theorem A.1 follow directly.

For other consequences of the Ladyzhenskaya squeezing property for the model considered we refer to [6].

4 Proofs

Our arguments are more or less standard and use the methods developed in [3, 16, 24] (see also the survey [26]). In fact they are some refinement of the calculations known now from [3], see also [16, 24, 26]. This is why we consider only key steps in the argument. Moreover, we deal with the reduced system which appears in the following way.

If we assume $G_b \equiv 0$ and $b_0 = 0$, we can take $b \equiv 0$ as a solution to (7). Thus we arrive to the problem: to find a (horizontal) fluid velocity field

$$
v(x, t) = (v^1(\bar{x}, t); v^2(\bar{x}, t)), \quad \bar{x} = (x, z),
$$

and the pressure $p(x, t)$ satisfying the equation

$$
v_t + (v, \nabla)v - \left[\int_0^z \text{div} \, \nu d\xi\right] \partial_z v - v[\Delta v + \partial_z v] + f v^\perp = -\nabla p + G_f \quad (17)
$$
in \( \mathcal{O} \times (0, +\infty) \) supplied with conditions (8) and with the initial data
\[
v(x, z, 0) = v_0(x, z).
\] (18)

It follows from Proposition 2.1 that for every \( G_f \in V_1 \) and \( v \in V_1 \) problem (17), (18), (8) has a unique strong solution
\[
v(t) \in C(\mathbb{R}_+; V_1) \cap L_2(0, T; V_2), \quad \forall T > 0.
\]

These solutions generate a dynamical system \((\tilde{S}_t, V_1)\) in \( V_1 \). Below we prove our main result for this system. Thus we deal with calculations involving the velocity field \( v \), but not the buoyancy. Moreover using approximation procedure and the results from [24, 26] we can even assume that all our calculations deal with smooth solutions.

We also note that in the case of full system (6)–(10) with \( G_b = 0 \) the multiplier \( b \) in (7) leads to the estimate \( \|b(t)\| \leq e^{-c_0 t}\|b_0\| \) for \( t > 0 \). Thus using dissipativity stated in Theorem 3.1 we can conclude that
\[
\|b(t)\|_{V_1} \leq C_B e^{-c_0 t}, \quad t > t_B,
\]
for all initial data from a bounded set \( B \subset W_1 \). This means that the subspace \( V_1 \times \{0\} \) is exponentially attracting and implies that the global attractors for (6)–(10) and (17) are the same.

### 4.1 Preliminary a priori estimates

In order to prove Theorem 3.1 for the system \((\tilde{S}_t, V_1)\) generated by (17) we need only to check the existence of absorbing ball in the space \( V_2 \). For this we need to repeat and partially refine some calculations in [16, 24].

#### 4.1.1 A priori estimate in \( H \)

If we multiply (17) by \( v \) in \( H \), then we obtain
\[
\frac{d\|v\|^2}{dt} + \nu \left[ \|\nabla v\|^2 + \|\partial_z v\|^2 \right] \leq c \|G_f\|^2,
\]
which implies that
\[
\|v(t)\|^2 + \nu \int_t^{t+1} \left[ \|\nabla v\|^2 + \|\partial_z v\|^2 \right] d\tau \leq 2e^{-c_0 t}\|v(0)\|^2 + c_1 \|G_f\|^2
\]
for all \( t \geq 0 \). Thus there exists \( R_0 > 0 \) such that for any \( R > 0 \) there is \( t_R \geq 0 \) such that
\[
\|v(t)\|^2 + \nu \int_t^{t+1} \left[ \|\nabla v\|^2 + \|\partial_z v\|^2 \right] d\tau \leq R_0^2 \quad \text{for all } t \geq t_R
\] (19)
with \( \|v(0)\| \leq R \). Moreover, we can choose \( R_0^2 = \varepsilon + c_1 \|G_f\|^2 \) with arbitrary \( \varepsilon > 0 \), in this case \( t_R \) also depends on \( \varepsilon \).
4.1.2 Splitting

Let
\[
\tilde{v} = \frac{1}{L_3} \int_{-L_3/2}^{L_3/2} v dz \equiv \langle v \rangle_z \quad \text{and} \quad \tilde{v} = v - \tilde{v}.
\]

As in [3] one can see that the fields \(\tilde{v}\) and \(\tilde{v}\) satisfy the equations
\[
\tilde{v}_t + (\tilde{v}, \nabla)\tilde{v} + (\overline{\nabla}, \overline{\nabla})\tilde{v} - \nu \Delta \tilde{v} + f \tilde{v}^\perp = \nabla p + \overline{G}_f \quad (20)
\]
with \(\text{div} \, \tilde{v} = 0\) in \(\mathbb{T}^2 \equiv (0, L_1) \times (0, L_2)\), and
\[
\tilde{v}_t + (\tilde{v}, \nabla)\tilde{v} - \left[ \int_0^z \text{div} \, \tilde{v} dz \right] \partial_z \tilde{v}
\]
\[
+ (\tilde{v}, \nabla)\tilde{v} + (\overline{\nabla}, \overline{\nabla})\tilde{v} - \left[ (\tilde{v}, \nabla)\tilde{v} + (\overline{\nabla}, \overline{\nabla})\tilde{v} \right]
\]
\[
- \nu [\Delta \tilde{v} + \partial_z \tilde{v}] + f \tilde{v}^\perp = \overline{G}_f \quad (21)
\]
in \(O = \mathbb{T}^2 \times (-L_3/2, L_3/2)\).

4.1.3 \(H^1\)-estimates

Now we multiply \( (21) \) by \(|\tilde{v}|^4 \tilde{v}\) and integrate over \(O\). In the same way as in [3] [16] we obtain
\[
\frac{d}{dt} \|\tilde{v}\|_{L_6}^6 + \nu \int_O [\|\nabla \tilde{v}\|^2 + \|\partial_z \tilde{v}\|^2] \|\tilde{v}\|^4 d\tilde{x}
\leq C_0 \left[ \|\tilde{v}\|_2 \|\nabla \tilde{v}\|_2^2 + \|\nabla \tilde{v}\|_2 \right] \|\tilde{v}\|^6_{L_6} + C_1 \|G_f\|_{L_6} \|\tilde{v}\|_{L_6}^5.
\]

This implies that
\[
\frac{d}{dt} \|\tilde{v}\|_{L_6}^2 \leq C_0 \left[ 1 + \|v\|^2 \right] \|\nabla v\|^2 \|\tilde{v}\|^2_{L_6} + C_1 \|G_f\|_{L_6} \|\tilde{v}\|_{L_6}
\leq C_0 \left[ 1 + (1 + \|v\|^2) \|\nabla v\|^2 \|\tilde{v}\|^2_{L_6} + C_1 \|G_f\|^2_{L_6}.
\]

Since \(\|\tilde{v}\|_{L_6} \leq c [\|\nabla v\|^2 + \|\partial_z v\|^2]\), we can apply Lemma 2.3 [3] and relation \( (19) \) to obtain that there exists \(R_1 \geq R_0 > 0\) such that for any \(R > 0\) there exists \(t^*_R \geq t_R\) such that
\[
\|\tilde{v}(t)\|_{L_6} \leq R_1 \quad \text{for all} \quad t \geq t^*_R \quad \text{with} \quad \|v(0)\| \leq R. \quad (22)
\]

Moreover using \( (19) \) with \(R_0^2 = \varepsilon + c_1 \|G_f\|^2\) we can conclude that
\[
\forall \varepsilon \in (0, 1] : \quad \|\tilde{v}(t)\|_{L_6}^2 \leq C \left[ \varepsilon + \|G_f\|^2 \right] \quad \text{for all} \quad t \geq t^*_R, \varepsilon
\]
with \(\|v(0)\| \leq R\). Using \( (22) \) we can also assume that
\[
\int_t^{t+1} \int_O [\|\nabla \tilde{v}\|^2 + \|\partial_z \tilde{v}\|^2] \|\tilde{v}\|^4 d\tilde{x} d\tau \leq C(R_1) \quad \text{for all} \quad t \geq t^*_R \quad (23)
\]

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with \( \|v(0)\| \leq R \). In the case \( G_f \equiv 0 \) we can change \( C(R_1) \) into \( C\varepsilon \) and \( t^*_R \) into \( t^*_{R,\varepsilon} \).

Next we multiply (20) by \(-\Delta \bar{v}\) in \( L^2(\mathbb{T}^2) \). As in [3] using the relations

\[
\int_{\mathbb{T}^2} (\bar{v}, \nabla) \bar{v} \Delta \bar{v} dx = 0, \quad \int_{\mathbb{T}^2} \nabla p \Delta \bar{v} dx = 0, \quad \int_{\mathbb{T}^2} \bar{v}^+ \Delta \bar{v} dx = 0,
\]

we have that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \bar{v}\|^2 + \nu \|\Delta \bar{v}\|^2 = \int_{\mathbb{T}^2} \left[ (\bar{v}, \nabla) \bar{v} + (\nabla, \bar{v}) \bar{v} \right] \Delta \bar{v} dx - \int_{\mathbb{T}^2} G_f \Delta \bar{v} dx.
\]

This implies (see [3] for some details) that

\[
\frac{d}{dt} \|\nabla \bar{v}\|^2 + \nu \|\Delta \bar{v}\|^2 \leq C \left[ \|\nabla \bar{v}\|^2 + \int_{\mathcal{O}} |\bar{v}|^4 |\nabla \bar{v}|^2 d\bar{x} + \|G_f\|^2 \right].
\]

Therefore (22) and (23) give us that there exists \( R_2 > 0 \) such that for any \( R > 0 \) there is \( t^*_R \geq t^*_{R,\varepsilon} \) such that

\[
\|\nabla \bar{v}(t)\|^2 + \nu \int_t^{t+1} \|\Delta \bar{v}\|^2 d\tau \leq R_2^2 \text{ for all } t \geq t^*_R \text{ with } \|v(0)\| \leq R. \tag{24}
\]

As above in the case \( G_f \equiv 0 \) we can take \( R_2^2 \) of the order \( \varepsilon \) (with \( t^*_R \) depending on \( \varepsilon \)).

The next step is the estimate for \( v_z \). We first note that \( u \equiv v_z \) solves the problem

\[
u u + (v, \nabla) u - \left[ \int_0^z \text{div} \, v d\xi \right] \partial_z u - \nu [\Delta u + \partial_{zz} u] = -f u^+ - (u, \nabla) v + (\nabla, v) u + \partial_z G_f \tag{25}
\]

in the class of periodic (odd in \( z \)) functions. Since \( u^+ u = 0 \) and

\[
\int_{\mathcal{O}} \left\{ (v, \nabla) u - \left[ \int_0^z \text{div} \, v d\xi \right] \partial_z u \right\} u d\bar{x} = 0,
\]

using the multiplier \( u \) after integration by parts we obtain that

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu [\|\nabla u\|^2 + \|\partial_z u\|^2] \leq \|\partial_z G_f\| \|u\| + \int_{\mathcal{O}} |v||u||\nabla u| d\bar{x},
\]

which implies (for some details we refer to [3]) that

\[
\frac{d}{dt} \|u\|^2 + \nu [\|\nabla u\|^2 + \|\partial_z u\|^2] \leq C \left[ \|v\|_{L^6}^4 \|u\|^2 + \|\partial_z G_f\|^2 \right] \leq C \left[ \|\bar{v}\|_{L^6}^4 + \|\nabla \bar{v}\|^4 \|u\|^2 + C \|\partial_z G_f\|^2 \right].
\]
Therefore there exists $R_3 > 0$ such that for any $R > 0$ there is $t_R^{(3)} \geq t_R^*$ such that

$$
\|v_z(t)\|^2 + \nu \int_t^{t+1} \left[ \|\nabla v_z\|^2 + \|v_{zz}\|^2 \right] d\tau \leq R_3^2
$$

for all $t \geq t_R^{(3)}$ with $\|v(0)\| \leq R$. Moreover, $R_3^2 \sim \varepsilon$ in the case $G_f \equiv 0$.

Next we estimate $\|\nabla v\|$. For this we multiply (17) by $-\Delta v$. As in [3] we obtain

$$
d\|\nabla v\|^2 + \nu \left[ \|\Delta v\|^2 + \|\nabla v_z\|^2 + \|\nabla \partial_z v\|^2 \right] d\tau \leq C \left[ \|\nabla v\|^4 L_\alpha + \|\nabla v\|^2 \|v_z\|^2 \|\nabla v\|^2 + C \|G_f\|^2 \right]
$$

which implies again that there exists $R_4 > 0$ such that for any $R > 0$ there is $t_R^{(4)} \geq t_R^{(3)}$ such that

$$
\|\nabla v(t)\|^2 + \nu \int_t^{t+1} \left[ \|\Delta v\|^2 + \|\nabla v_z\|^2 + \|v_{zz}\|^2 \right] d\tau \leq R_4^2
$$

for all $t \geq t_R^{(4)}$ with $\|v(0)\| \leq R$. Moreover one can see from the analysis above that in the case $G_f \equiv 0$ for any $0 < \varepsilon < 1$ there exists $t_{R,\varepsilon} > 0$ such that

$$
\|\nabla v(t)\|^2 + \nu \int_t^{t+1} \left[ \|\Delta v\|^2 + \|\nabla v_z\|^2 + \|v_{zz}\|^2 \right] d\tau \leq \varepsilon
$$

for all $t \geq t_{R,\varepsilon}$ provided $\|v(0)\| \leq R$.

The relation in (26) yields that the system $(\tilde{S}_t, V_1)$ is dissipative in $V_1$. In the case $G_f \equiv 0$ by (27) an absorbing ball can be chosen with an arbitrary small radius which implies that the zero equilibrium is asymptotically stable in this case (as it is claimed in Remark 3.2 when the buoyancy is presented).

### 4.2 Completion of the proof of Theorem 3.1

To conclude the proof we need to show uniform smoothing of trajectories. We note that the result for individual trajectories is known from [24] (see also [26]). However in these references the size of the corresponding ball is not controlled and thus we need some refinement of that results. This is why below we mainly follow the line of argument in [24, 26].

We multiply (17) by $(-\Delta_{x,z})^2 v$ to get

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_{x,z} v\|^2 + \nu \|(-\Delta_{x,z})^3/2 v\|^2 = - \int_O \left[ (v, \nabla) v + w(v) \partial_z v \right] (-\Delta_{x,z})^2 v d\bar{x} + (G_f, (-\Delta_{x,z})^2 v),
$$

where $w(v) = w(\bar{x}, t)$ is defined by (5). We also use the facts that

$$
\int_O v^\perp (-\Delta_{x,z})^2 v d\bar{x} = \int_O [\Delta_{x,z} v]^\perp \Delta_{x,z} v d\bar{x} = 0
$$
and
\[ \int_{\mathcal{O}} \nabla p(-\Delta_{x,z})^2 v d\bar{x} = 0. \]

The general structure of the nonlinear terms in (28) is given by the formula

\[ N(v) \equiv \int_{\mathcal{O}} [(v, \nabla) v + w(v) \partial_z v] (-\Delta_{x,z})^2 v d\bar{x} = \int_{\mathcal{O}} \Delta_{x,z} [(v, \nabla) v + w(v) \partial_z v] \Delta_{x,z} v d\bar{x} = \int_{\mathcal{O}} [(v, \nabla) \Delta_{x,z} v + w(v) \partial_z \Delta_{x,z} v] \Delta_{x,z} v d\bar{x} + \Sigma(v) = \Sigma(v). \]

Here
\[ \Sigma(v) \sim \sum_{k,i,j} \int_{\mathcal{O}} \left[ D^2 v^i D^1 v^j D^2 v^j + D^k w(v) D^{2-k} \partial_z v^j \Delta_{x,z} v^j \right] d\bar{x}, \quad (29) \]

where \( k, i, j = 1, 2 \), \( D^k \) is a differential operator of order \( k \), and the sign \( \sim \) means that we deal with linear combination of the terms in the right hand side.

Using the embeddings \( H^1(\mathcal{O}) \subset L_6(\mathcal{O}) \) and \( H^{1/2}(\mathcal{O}) \subset L_3(\mathcal{O}) \) (see [29]) and also interpolation one can see that

\[ \Sigma_1 \equiv \left| \int_{\mathcal{O}} D^2 v^i D^1 v^j \Delta_{x,z} v^j d\bar{x} \right| \leq C \|D^2 v_i\|_{L_6} \|D^1 v_j\|_{L_3} \|D^2 v_j\|_{L_3} \leq C \|v\|_{H^2}^2 \|D^2 v^j\|_{H^{1/2}} \|D^2 v^j\|_{H^{1/2}} \leq C \|v\|_{H^2}^2 \|\partial_z v\|_{L_6} \|v\|_{H^2}^2 \leq \xi \|[-\Delta_{x,z}]^{3/2} v\|^2 + C_\xi \|v\|_{H^2}^{10/3}. \]

To estimate the second term in (29) we start with the case \( k = 2 \):

\[ \int_{\mathcal{O}} D^2 w(v) \partial_z v^j \Delta_{x,z} v^j d\bar{x} \leq C \|D^2 w(v)\|_{L_6} \|\partial_z v^j\|_{L_6} \|D^2 v^j\|_{L_3} \leq C \|v\|_{H^3}^2 \|\partial_z v\|_{L_6} \|v\|_{H^2}^{1/2} \leq \xi \|[-\Delta_{x,z}]^{3/2} v\|^2 + C_\xi \|\partial_z v\|_{L_6} \|v\|_{H^2}^2. \]

To estimate the term with \( k = 1 \) in (29) we use the following lemma (see Proposition 2.2 in [2]).

**Lemma 4.1** For smooth functions \( \phi \) and \( \psi \) and vector field \( v = (v^1, v^2) \) we have
\[ \left| \int_{\mathcal{O}} \left[ \int_0^z \text{div} v(x, \xi) d\xi \right] \psi \phi d\bar{x} \right| \leq C \|v\|_{H^2}^{1/2} \|v\|_{H^1}^{1/2} \|\psi\|_{H^1}^{1/2} \|\psi\|_{H^1}^{1/2} \|\phi\|. \]

This lemma yields the following estimate
\[ \int_{\mathcal{O}} D^1 w(v) D^1 \partial_z v^j \Delta_{x,z} v^j d\bar{x} \leq C \|v\|_{H^2}^{1/2} \|v\|_{H^3}^{1/2} \|\partial_z v^j\|_{H^2}^{1/2} \|\partial_z v^j\|_{H^2}^{1/2} \|D^2 v\| \leq C \|v\|_{H^3} \|v\|_{H^2}^2 \leq \xi \|[-\Delta_{x,z}]^{3/2} v\|^2 + C_\xi \|v\|_{H^2}^4. \]
Thus we arrive at the following relation
\[ \frac{d}{dt} \| \Delta_{x,z} v \|^2 + \nu \| [-\Delta_{x,z}]^{3/2} v \|^2 \leq C_1 \left[ 1 + \| \Delta_{x,z} v \|^2 + \| \partial_z v \|_2^2 \| \Delta_{x,z} v \|^2 \right] + C_2 \| G_f \|_V^2. \] (30)

Our next step is an estimate for \( \| \partial_z v \|_{L_6} \). As above let \( u = \partial_z v \). We multiply (25) by \( |u|^4 u \). Using the relations \( u^4 u = 0 \) and
\[
\int_O \left\{ (v, \nabla) u - \left[ \int_0^z \text{div} \text{v} d\xi \right] \partial_z u \right\} |u|^4 u d\bar{x} = 0,
\]
we obtain that
\[
\frac{1}{6} \frac{d}{dt} \| u \|_{L_6}^6 + \nu \int_O \left[ |\nabla u|^2 + |\partial_z u|^2 \right] |u|^4 d\bar{x}
\leq \int_O \left\{ -(u, \nabla)v + (\nabla, v)u \right\} |u|^4 u d\bar{x} + (\partial_z G_f, |u|^4 u).
\]

In the same way as in [24, 26] after integration by parts we have that
\[
\int_O \left\{ -(u, \nabla)v + (\nabla, v)u \right\} |u|^4 u d\bar{x} \leq C \| v \|_{L_6} \int_O |\nabla u| |u|^5 d\bar{x}
\leq C \| v \|_{L_6} \left[ \int_O |\nabla u|^2 |u|^4 d\bar{x} \right]^{1/2} \| u \|_{L_6}^3
\leq \varepsilon \int_O |\nabla u|^2 |u|^4 d\bar{x} + C_\varepsilon \| v \|_{L_6}^2 \| u \|_{L_6}^6.
\]

This implies that
\[
\frac{d}{dt} \| u \|_{L_6}^6 + \nu \int_O \left[ |\nabla u|^2 + |\partial_z u|^2 \right] |u|^4 d\bar{x}
\leq C \left[ \| v \|_{H^2}^2 \| u \|_{L_6}^6 + \| \partial_z G_f \|_{L_6} \| u \|_{L_6}^6 \right].
\]

Thus we have that
\[
\frac{d}{dt} \| u \|_{L_6}^2 \leq C_1 \left[ 1 + \| \Delta_{x,z} v \|^2 \right] \| u \|_{L_6}^2 + C_2 \| \partial_z G_f \|_{L_6} \| u \|_{L_6}^2.
\]

Using Lemma 2.3 and the estimate in (26) we obtain that
\[
\| \partial_z v(t) \|_{L_6} \leq R_5 \quad \text{for all} \quad t \geq t_R^{(5)} \geq t_R^{(4)} \quad \text{with} \quad \| v(0) \| \leq R.
\]

Now returning to (30) and applying Lemma 2.3 allow us to obtain the following assertion.

**Proposition 4.2** Let \( G_f \in V_1 \) and \( \partial_z G_f \in L_6(\mathcal{O}) \). Then there exists \( R_5 > 0 \) such that the ball \( \mathcal{B} = \{ v \in V_2 : \| v \|_{V_2} \leq R_4 \} \) is absorbing for the dynamical system \( (\dot{S}_t, V_1) \) generated by problem (17). Moreover, there is a forward invariant absorbing set \( \mathcal{D} \) in this ball.

Thus the proof of Theorem 3.1 (in the case \( b_0 \equiv 0 \) and \( G_b \equiv 0 \)) is complete.
4.3 Squeezing estimate

Now we prove Theorem 3.5 (in the case $b_0 \equiv 0$ and $G_0 \equiv 0$). We start with the following assertion.

**Proposition 4.3** For any $v_*, v_{**}$ from the absorbing forward invariant set $D$ we have the relation

$$
\|\tilde{S}_t v_* - \tilde{S}_t v_{**}\|_{V_1} \leq C_D e^{\alpha_D t} \|v_* - v_{**}\|_{V_1}, \quad \forall t \geq 0,
$$

where $C_D$ and $\alpha_D$ are positive constants. Moreover, for $v_*(t) = \tilde{S}_t v_*$ and $v_{**}(t) = \tilde{S}_t v_{**}$ we also have that

$$
\|u(t)\|^2_{V_1} + \int_0^t \|u(\tau)\|^2_{V_2} d\tau \leq C_D e^{\alpha_D t} \|v_* - v_{**}\|^2_{V_1}, \quad \forall t \geq 0,
$$

where $u(t) = v_*(t) - v_{**}(t)$.

**Proof.** We use the same type of argument as in [16]. For this we write the equation

$$
u_t - \nu[\Delta u + \partial_{zz} u] + (u, \nabla)v_* + (v_*, \nabla)u - \left[ \int_0^z \text{div} u d\xi \right] \partial_z v_* - \left[ \int_0^z \text{div} v_{**} d\xi \right] \partial_z u + fu_\perp = -\nabla(p_* - p_{**}) \quad (31)
$$

for the difference $u(t) = v_*(t) - v_{**}(t) = S_t v_* - S_t v_{**}$. Then we multiply (31) by $(-\Delta_x, u)$ and apply the same argument as in [16]. At the final stage we use the fact that

$$
\|v_*(t)\|_{V_2} + \|v_{**}(t)\|_{V_2} \leq C_D, \quad t > 0,
$$

for initial data $v_*$ and $v_{**}$ from $D$. \hfill \Box

Let $\{\lambda_k\}$ be the eigenvalues of the corresponding Stokes type operator $A$ which arises in problem (17) and $\{e_k\}$ be the corresponding eigenvectors. We can assume that

$$
0 < \lambda_1 \leq \lambda_2 \leq \ldots; \quad \lim_{k \to \infty} \lambda_k = \infty.
$$

We denote by $P_N$ the orthoprojector in $H$ on $\text{Span}\{e_1, \ldots, e_N\}$. Let $Q_N = I - P_N$. We multiply (31) by $Q_N Au$ and obtain that

$$
\frac{1}{2} \frac{d}{dt} \|A^{1/2} Q_N u\|^2 + \nu \|AQ_N u\|^2 = -((u, \nabla)v_* + (v_{**}, \nabla)u, Q_N Au) + \int_{\partial O} \left[ \int_0^z \text{div} u d\xi \right] \partial_z v_* Q_N Au d\bar{x} - f(u^\perp, Q_N Au) + \int_{\partial O} \left[ \int_0^z \text{div} v_{**} d\xi \right] \partial_z u Q_N Au d\bar{x}.
$$

(32)
Now we need the following lemma which follows from the H"{o}lder inequality and from the standard Sobolev embedding (see, e.g., [29]):

\[ H^s(O) \subset L^p(O) \quad \text{when} \quad s - 3/2 = -3/p, \quad p \geq 2, \quad O \subset \mathbb{R}^3. \]

**Lemma 4.4** Let \( O \subset \mathbb{R}^3 \) and \( u, v \) be sufficiently smooth functions. Then

\[
\|uv\|_{L^2} \leq C\|u\|_{H^s}\|v\|_{H^r}
\]

for any \( 0 < s, r < 3/2 \) such that \( r + s = 3/2 \).

Using this lemma to estimate the right hand side in (32) on the forward invariant set \( \mathcal{D} \) we can find that

\[
\frac{1}{2} \frac{d}{dt} \|A^{1/2}Q_Nu\|^2 + \nu\|AQ_Nu\|^2 \leq C_{R_*}\|u\|_{H^{3/2}}\|AQ_Nu\|,
\]

where \( R_* \) is the same as in the statement of Proposition 4.2. This yields

\[
\frac{d}{dt} \|A^{1/2}Q_Nu\|^2 + \nu\|AQ_Nu\|^2 \leq C_{R_*}\|u\|_{V_1}^2 + \varepsilon\|u\|_{V_2}^2
\]

for any \( \varepsilon > 0 \). This implies that

\[
\|A^{1/2}Q_Nu(t)\|^2 \leq e^{-\nu\lambda_{N+1}t}\|A^{1/2}Q_Nu(0)\|^2 \\
+ C_{R_*}\|u\|_{V_1}^2 \int_0^t e^{-\nu\lambda_{N+1}(t-\tau)}\|u\|_{V_1}^2 d\tau + \varepsilon \int_0^t e^{-\nu\lambda_{N+1}(t-\tau)}\|u\|_{V_2}^2 d\tau.
\]

Using Proposition 4.3 we obtain that

\[
\|A^{1/2}Q_Nu(t)\|^2 \leq e^{-\nu\lambda_{N+1}t} + \left( \frac{C_{R_*}\varepsilon}{\alpha_{R_*} + \nu\lambda_{N+1}} + \varepsilon C_{R_*} \right) e^{\alpha_{R_*}t} \|A^{1/2}Q_Nu(0)\|^2.
\]

Now for any fixed \( T > 0 \) and \( 0 < q < 1 \) we can choose \( \varepsilon \) and \( N \) such that

\[
\|A^{1/2}Q_Nu(T)\|^2 \leq q^2\|A^{1/2}Q_Nu(0)\|^2.
\]

Thus we arrive to the squeezing property: For every \( T > 0 \) and \( 0 < q < 1 \) there is \( N \) such that

\[
\|Q_N[\tilde{S}_Tv_1 - \tilde{S}_Tv_2]\|_{V_1} \leq q\|v_1 - v_2\|_{V_1}
\]

for any \( v_1, v_2 \) from the absorbing forward invariant set \( \mathcal{D} \).
A Appendix: Kuksin-Shirikyan ergodicity theorem

In this Appendix for the readers convenience we state Theorem 3.2.5 [20] on the ergodicity of a class of Markov chains.

Let $H$ be a separable Banach space. We denote by $B(R)$ the ball with the center at 0 of the radius $R$ in the space $H$. Let $S$ be a continuous (nonlinear) operator from $H$ into itself. Let $\{\eta_k\}$ be i.i.d. random variables in $H$ such that $\|\eta_k\| \leq r_*$ for some $r_* > 0$. We consider a Markov chain in $H$ defined by the relations

$$U_k = F_k(U_{k-1}) = SU_{k-1} + \eta_k, \quad k = 1, 2, \ldots$$  \hspace{1cm} (33)

We assume that the following hypotheses are satisfied.

(A1) $S$ is a Lipschitz mapping on every ball $B(\varrho)$ and for any $R > r > 0$ there exists $n_+ = n(R, r)$ such that $S^n B(R) \subset B(r)$ for all $n \geq n_+$.

(A2) There exists $\hat{R} > 0$ such that for every $R > 0$ we have that $A_k(R) \subset B(\hat{R})$ for all $k \geq k_*(R)$, where the sets $A_k(R)$ is defined by the relations

$$A_0(R) = B(R), \quad A_k(R) = S(A_{k-1}(R)) + B(r_*), \quad k = 1, 2, \ldots$$

(A3) There exist a finite dimensional projector $P$ in $H$ and a number $\eta < 1$ such that

$$\|(I - P)[SU - SU_*]\|_H \leq \eta\|U - U_*\|_H, \quad \forall U, U_* \in \text{Cl}[\bigcup_{k \geq 0} A_k(\hat{R})].$$

(A4) Assume that $P\eta_1$ and $(I - P)\eta_1$ are independent and the distribution of $P\eta_1$ in the (finite-dimensional) space $PH$ has a density $p(U)$ with respect to the Lebesgue measure $dU$ on $PH$ such that

$$\int_{PH} |p(U + V) - p(U)| dU \leq C\|V\|_H, \quad \forall V \in PH,$$

where $C$ is a constant.

We also recall (see, e.g., [20]) that the dual Lipschitz norm in the space of probability Borel measures on $H$ is defined by the relation

$$\|\mu_1 - \mu_2\|_L^* = \sup \left\{ \left| \int_H f(v) \mu_1(dv) - \int_H f(v) \mu_2(dv) \right| : f \in L_b(H), \|f\|_L \leq 1 \right\},$$

where $L_b(H)$ is the space of bounded Lipschitz functions on $H$ endowed with the norm

$$\|f\|_L = \sup_{v \in H} |f(v)| + \sup_{v_1, v_2 \in H} \|f(v_1) - f(v_2)\|_H.$$

Now we are in position to state Theorem 3.2.5 [20].
Theorem A.1 (Ergodicity; Kuksin-Shirikyan [20]) If we assume that hypotheses (A1)–(A4) are in force, then for the chain (33) there is a unique invariant probability Borel measure \( \mu \) on \( H \). Moreover, there is \( 0 < \gamma < 1 \) such that

\[
\| F^*_k \lambda - \mu \|_L \leq C e^{-\alpha k} \int \exp \{ \alpha k (\| U \|) \} \lambda(\mathrm{d}U), \quad k = 1, 2, \ldots
\]

for any probability Borel measure \( \lambda \) on \( H \) for which the integral in the right-hand side of the relation above is finite (the function \( k_\ast(R) \) is defined in (A2)). Here above the probability measure \( F^*_k \lambda \) is defined on a Borel set \( \Gamma \) by the formula

\[
(F^*_k \lambda)(\Gamma) = \int_H P_k(V, \Gamma) \lambda(\mathrm{d}V),
\]

where \( P_k(U_0, \Gamma) = \mathbb{P}(U_k \in \Gamma) \) is the transition probability for the chain.

We note that the result in Theorem A.1 formally deals with a Markov chain in a whole space. However one can see that the arguments from [20] remain true if we restrict the iteration procedure in (16) on an invariant subset. This observation was already used in [7] to prove ergodicity for random kick-forced 3D Navier-Stokes equations in a thin domain and can be also applied to obtain Corollary 3.9.

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