The importance of level statistics for the decoherence of a central spin due to a spin environment

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We study the decoherence of a central spin-1/2 due to a closed environment composed of spin-1/2 particles. It is known that a frustrated spin environment, such as a spin glass, is much more efficient for decoherence of the central spin than a similar size environment without frustration. We construct a Hamiltonian where the degree of frustration is parametrized by a single parameter \( \kappa \). By use of this model we find that the environment can be classified by two distinct regimes with respect to the strength of level repulsion. These regimes behave qualitatively different with respect to decoherence of the central spin and might explain the strong enhancement of decoherence observed for frustrated environments.

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I. INTRODUCTION

Quantum decoherence, where coherence in a quantum system is reduced due to interaction with its environment, is a fundamental concept of physics. Testing of theories that go beyond unitary quantum mechanics requires deep understanding and control of the decoherence process in order to distinguish the breakdown of unitarity predicted in these theories from decoherence. Decoherence is also a fundamental problem in the branch of nanoscience, where one seeks to use and manipulate quantum states for application. Coherent manipulation and storage of quantum information are required in order to construct a working quantum computer and rely on reducing the decohering interaction between its basic elements, the qubits, and their environment.

Recently there has been increased experimental interest in electronic spin systems, where the most prominent source of decoherence is thought to be electronic, or, in samples with high purity, nuclear spins. These systems are nitrogen-vacancy centers in diamond semiconductor quantum dots and large-spin magnetic molecules. In addition, fluctuating two level defects are thought to be the major source of decoherence in solid state Josephson junction qubits, see Ref. [12] for a review. The coherence of a single spin interacting with a spin bath has been studied extensively in the limit of a non-interacting bath. Decoherence due to interacting spins has also been studied recently in the weak coupling limit and it was found that the coherence of the central spin decays rapidly when the environment is close to a phase transition.

Decoherence, relaxation and thermalization of a central system coupled to a closed, finite-size spin bath environment has been investigated in Refs. [10, 11]. In Refs. [10, 11], decoherence of a two-spin system was studied, and a large enhancement of decoherence was found for frustrated spin environments, the main conclusion being: “For the models under consideration, the efficiency of the decoherence decreases drastically in the following order: Spin glass - frustrated antiferromagnet - Bipartite antiferromagnet - One dimensional ring with nearest neighbour antiferromagnetic interactions”. A similar study found that the same was true also with regards to relaxation towards the ground state of the central system. Namely, frustrated environments are more efficient in relaxing the central system compared to an ordered environment. Whether or not the bath is chaotic or integrable was shown to be of less importance. Frustrated spin systems have been suggested to exist as localized electron states on the surface of superconducting quantum interference devices, such as SQUIDs and flux qubits, where they are thought to be a major source of magnetic flux noise.

However, a detailed understanding of the physics behind the importance of a frustrated environment is still lacking. In this work we construct a model where we can continuously tune the degree of frustration in the environment by a single parameter \( \kappa \), confirming that frustrated environments reduce the coherence of the central spin much more efficient compared to an environment with low degree of frustration, as previously found in Refs. [10, 11]. Using this model we study the structure of the eigenvalues of Hamiltonian of the environment, \( H_E \), in the presence of a central spin.

We find that we can explain the mechanism behind the efficiency of the frustrated environment by the structure of the eigenvalues of \( H_E \). The frustrated environment can be characterized by a Wigner-like distribution of eigenvalues, and therefore has large repulsion between energy levels. The presence of an external object, like a central qubit, will therefore result in the mixing of a large fraction of the eigenvectors of the unperturbed system. In an ordered environment, however, the level repulsion is very weak, and coupling to the central spin will only alter the set of eigenvectors of the environment slightly, preserving the coherence of the central spin.

The link between the response of the eigenvectors of
where $H_E$ to an external perturbation, and the decoherence of a central spin is found as follows. The initial state of the complete system is
\[
|\Phi(t = 0)\rangle = \left(1/\sqrt{2}\right)\left(|\uparrow\rangle_S + |\downarrow\rangle_S\right)\otimes |\psi_0\rangle_E,
\]
where the subscripts $S$ and $E$ denote the central system and the environment, respectively, and we have for simplicity assumed the central system to be in an initial symmetric superposition. The state $|\uparrow\rangle$ means that the system is in the eigenstate of the operator $S^z$ with eigenvalue $+1/2$ and $|\psi_0\rangle_E$ is the initial state of the environment. If the system and environment are coupled, the state of the system influences the dynamic evolution of the environment, and we can write the linear time evolution of the composite system as
\[
\left(|\uparrow\rangle_S + |\downarrow\rangle_S\right) |\psi_0\rangle_E \rightarrow |\uparrow\rangle_S \left|\psi(t)\right\rangle_E + |\downarrow\rangle_S |\psi(t)\rangle_E
\]
where $|\psi(t)\rangle_E$ denotes the time evolution of the environment conditioned upon that the initial state of the central system is $|\uparrow\rangle_S$. For now we assume that the system-environment coupling, $H_{SE}$, commutes with $S^z$, so that transitions between the levels of the central system is prohibited. We will characterize the decoherence by the off-diagonal matrix element of the density matrix,
\[
\rho_{\uparrow\downarrow}^S = \langle \psi(t)\downarrow | \psi(t)\rangle.
\]
Let us expand the state of the environment in the set of eigenstates,
\[
|\psi(t)\rangle = \sum_n \langle n\uparrow | \psi_0\rangle |n\uparrow\rangle e^{iE_n^t t},
\]
where $|n\uparrow\rangle$, $E_n^t$ denote the eigenstates and eigenvalues of the environment conditioned upon that the central spin points up, and similarly in the case where the central spin points down (throughout the paper we put $\hbar = 1$). Then the time evolution of the off diagonal element of the density matrix is given by the expression
\[
\rho_{\uparrow\downarrow}^S(t) = \sum_{n,m} \langle n\uparrow | \psi_0\rangle |m\downarrow\rangle \langle m\downarrow | \psi_0\rangle e^{i(E_n^t - E_m^t) t}.
\]
Thus $\rho_{\uparrow\downarrow}^S$ is determined by the magnitude of the overlap elements, unless the levels are degenerate. For degenerate states the corresponding phase factors of the overlap with each eigenstate of the degenerate level oscillate with the same phase.

The analysis is simplified if we assume that only the upper state of the central system couples to the environment, and that the environment is prepared in its ground state. In this case Eq. (3) simplifies to
\[
\rho_{\uparrow\downarrow}^S(t) = \sum_n \langle n\uparrow | 0\rangle |0\rangle e^{i(E_n^t - E_0)t}
\]
and the picture is more transparent.

Evolution of $\rho_{\uparrow\downarrow}^S$ is then determined by quantum beatings between the overlap contributions oscillating at frequencies $(E_n^t - E_0)$, i.e., by the differences between eigenvalues of $H_E$ and the eigenvalues of the environment in presence of the central spin. In the following we will investigate this further by numerical study of an explicit model.

The paper is organized as follows. In Sec. III we describe our model of a central spin-1/2 interacting with a spin environment with tuneable degree of frustration. In Sec. III A we study the different regimes of decoherence of our model, while in Sec. III B we explain the physical mechanism behind the enhancement of decoherence by frustration in detail. Furthermore, in Sec. III C we describe the sensitivity to the initial state and in Sec. III D we suggest a method to reduce the negative impact from frustrated environments on coherence. Finally the results will be discussed in Sec. IV and we conclude in Sec. V.

II. MODEL

We model a central spin-1/2 interacting with a spin environment by the Hamiltonian
\[
H = H_S + H_{SE} + H_E,
\]
\[
H_{SE} = \frac{1}{2} \sum_i \Delta_i \left(S^z - \frac{1}{2}\right) s_i^z,
\]
\[
H_E = \sum_{i,j,\alpha} \Omega_{ij}^{\alpha} s_i^\alpha s_j^\alpha
\]
where $H_S$, $H_{SE}$ and $H_E$ are the Hamiltonians for the central spin, the spin-environment coupling and the environment, respectively, $S$ is the operator of the central spin, while $s_i$ are the operators of the environmental spins. We set both the energy splitting and the tunneling element of the central system to zero. The parameters $\Delta_i$ and $\Omega_{ij}^{\alpha}$ specify the coupling strength along the $\alpha$-axis between the central spin and the environment and the intraenvironment coupling, respectively. The parameters $\Delta_i$ are chosen randomly in the interval $[-\Delta, \Delta]$.

In order to study the importance of frustration we specify $H_E$ as
\[
H_E = -\Gamma \sum_{i,j,\alpha} \left[(1 - \kappa) s_i^\alpha s_j^\alpha + \kappa \Omega_{ij} s_i^\alpha s_j^\alpha\right]
\]
where $\Omega_{ij}$ is a random number in the interval $[-\Omega, \Omega]$. The degree of disorder is then parametrized by $\kappa \in [0, 1]$. In this model we can continuously tune our environment by the parameter $\kappa$ from a perfect ferromagnet ($\kappa = 0$) to a highly frustrated spin-glass ($\kappa = 1$). In the following all the energies will be measured in the units of $\Gamma$, therefore $\Gamma = 1$. Correspondingly, time is measured in units of $\Gamma^{-1}$.

The simulation procedure is the following. We select a set of model parameters. Then we compute the eigenstates and eigenvalues of $H$ by numerical diagonalization.
density matrix element, $\rho_{ij}^S(t)$, for $N = 9$ spins in the environment and different values for the disorder parameter $\kappa$, ranging from: 1. the spin glass phase ($\kappa = 1.0$, dashed, blue) to 4. the ferromagnetic phase ($\kappa = 0.1$, dashed/dots, black). Other arrangements are frustrated ferromagnet $\kappa = 0.5$ (2, solid, red) and for comparison we plot the time evolution for the completely disconnected bath with Heisenberg like $H_{SE}$ and $\kappa = 1$, $\Omega = 0.0$ (3, ‘+’, green), in this configuration there is no correlations between the different systems in the environment. The strength of the system-environment coupling is $\Delta = 3.0$.

The composite system is prepared in the state $|\psi_0\rangle$ where $|\psi_0\rangle$ is the initial state of the environment and the central spin is prepared in a superposition of eigenstates of $S^z$. Unless otherwise stated, the initial state of the environment is always the ground state in the absence of coupling, $|\psi_0\rangle_E = |0\rangle_E$. In general, the initial state is therefore a complicated superposition of eigenstates of the composite system $H$.

Decoherence is in this model solely due to entanglement between the central system and the environment. The state evolves according to the Schrödinger equation into an, in general, entangled state as in Eq. (2).

The reduced density matrix of the system is obtained by tracing over the degrees of freedom of the environment: $\rho(t) = \text{Tr}_E\{\Phi(t)\}$.

III. RESULTS

Using the simulation procedure described above we can study the dynamics of the reduced density matrix of the central system. The time evolution of the off-diagonal element of the density matrix for different values of the environment parameters, is shown in Fig. 1. We find that in general, higher degree of frustration, controlled by the parameter $\kappa$, result in stronger and more robust decay of $\rho_{ij}^S$. The initial evolution is similar and Gaussian in time for all values of $\kappa$, however for smaller $\kappa$ we find rapid revivals of coherence in the central system.

From Fig. 1 we see that it is useful to distinguish between the initial decoherence and the efficiency of decoherence. We define the initial decoherence as the evolution of coherence in the central system in the characteristic time during which $\rho_{ij}^S$ decays by a factor $e$ and the efficiency of decoherence as the mean value of the off-diagonal elements of the density matrix over a period that is large compared with the dynamics of the environment.

From Fig. 1, we thus find that initially $|\rho_{ij}^S|$ decays following the Gaussian law, $\rho_{ij}^S \propto e^{-(t/t^*)^2}$, with practically $\kappa$-independent decay time $t^*$. The efficiency of the decoherence is, however, much higher for the frustrated environment $\kappa = 1.0$. If the efficiency of decoherence is low, as for the ferromagnetic environment, the error might be corrected by use of quantum error correction. In fact, we show in Fig. 1 that a completely disconnected bath, $\Gamma = 0$, gives stronger decoherence than the ferromagnetic bath.

The picture we obtain is the following. The decoherence of the central spin is dependent on the sensitivity of the environment to the state of the central system. The response of the environment to an external system is closely related to the sensitivity of the Hamiltonian of the environment to a small perturbation. The latter can be, in turn, related to the so-called Loschmidt echo defined as the overlap between the two states evolving from the same initial wave function under the influence of two distinct Hamiltonians, the unperturbed $H_0$ and a perturbed one $H_0 + \Lambda$, see Ref. 24 for details. Therefore, in most cases, the Loschmidt echo of the environment and the efficiency of the decay of the off-diagonal elements of $\rho_S$ will be strongly correlated, even though there are exceptions. Thus our analysis applies to the purity of the central system as well as to the sensitivity to perturbations of the environment.

The sensitivity of the state of the environment to a perturbation (in our case, a flip of the central spin) and, therefore, the efficiency of decoherence can be characterized by overlaps between the initial state of the environment, $|0\rangle_E$, and the set of eigenstates of the environment in the presence of the perturbation, $\{|n\rangle\}$. We find that the largest of the overlap elements serves as a very good indicator for the decoherence of the central spin. We measure the efficiency of the environment by the modulus of the off-diagonal element of the reduced density matrix, $|\rho_{12}|_{\text{avg}}$, averaged over the interval $t \in [200, 300]$ that is long compared to the typical oscillation periods in $|\rho_{ij}^S(t)|$, cf. with Fig. 1. The relationship between $|\rho_{12}|_{\text{avg}}$ and the largest overlap element is plotted in Fig. 2. The fact that the largest overlap element correlates so well with the decoherence suggests that the probability of finding degenerate eigenstates among the states with the largest overlap element is relatively small and that the detailed distribution of overlapping vectors $\{|n\rangle\}$ is less important.

In the rest of the article we will use numerical simulations to clarify the difference with respect to decoherence of a central system interacting with a ferromagnetic or a frustrated environment. In view of the strong correlation demonstrated in Fig. 2, we will use $\max_n |\langle n | 0 \rangle_E|^2$ as a
In the presence of the central spin, which will therefore disturb the ground state of the environment given the perturbation represented by the central spin is not able to significantly alter the ground state. Close to \( \kappa = 0.5 \) we find a “phase transition” to a more disordered state. In this regime the coupling to the central spin is sufficient to alter the ground state of the environment. For \( \kappa \approx 1.0 \) the set of eigenstates are completely altered in the presence of the central spin, and the overlap with the original set is typically very small. The number of environmental spins is \( N = 9 \), \( \Delta = 3.0 \) and the same seed is used in generating the distributions of \( \Omega_{ij} \) and \( \Delta_i \) for each value of \( \kappa \) (solid line), while the markers correspond to a random seed for each value of \( \kappa \).

A. Decoherence in terms of the overlap with the initial state

We decompose the initial state of the environment in the eigenstates of \( H_E \) and use the ground state \( |\psi_0\rangle_E = |0\rangle \) as the initial state. In Ref. 17 decoherence was studied both using the ground state as initial state and a random superposition of eigenstates corresponding to “infinite temperature”. We will focus first on the ground state and address a more complicated initial state in Sec. III C.

If we now increase the disorder parameter \( \kappa \), the ground state of the environment will be only slightly altered, until the frustration in \( H_E \) given by \( \kappa \), together with the frustrated Ising type system-environment coupling \( H_{SE} \), becomes large enough to break the ferromagnetic order. This “phase transition” is evident from Fig. 3 where in these particular simulations it takes place at about \( \kappa \approx 0.5 \), but the value is in general dependent on the size of the system, and the strength and nature of \( H_{SE} \).

In the ferromagnetic phase, if the strength of the system-environment coupling is weaker than the intra-environment coupling, \( \Delta \ll N\Omega \), the presence of the central spin will not alter the ground state significantly. Therefore, the overlap between the ground state of the isolated environment with the ground state of the perturbed environment, \( \langle 0|0\rangle \), will be very close to one (i.e., the magnitude of all the terms of Eq. (9) will be close to zero except for the term \( \langle 0|0\rangle \), where \( |0\rangle \) is the ground state of the environment given the perturbation). Thus the ground state will still be ferromagnetic in the presence of the central spin, which will therefore not entangle sufficiently with its environment, preserving the coherence. In Fig. 3 we show numerical simulations for different values of frustration in the environment. As long as the disorder parameter \( \kappa \) is small, the largest overlap element between the unperturbed ground state of the environment \( |0\rangle_E \) and the set of eigenstates \( \{ |n\rangle \} \) when the system-environment coupling \( H_{SE} \) is turned on, is very close to one. The ground state is ferromagnetic and the interaction with the central spin is insufficient to break the ferromagnetic order.

In the absence of disorder, \( \kappa = 0 \), the ground state of \( H_E \) will be the ferromagnetic state where all spins point in the same direction \( \uparrow \uparrow \ldots \uparrow \), or in general a linear combination of the two degenerate ground states. In order to avoid the exact degeneracy, we use a small static symmetry breaking field acting on a single spin in the environment.

FIG. 2. Correlation between \( |\rho_{12}^{\text{avg}}(t)| \), and the largest overlap element between the ground state of \( H_E \) and the set of eigenstates \( \{ |n\rangle \} \) of the environment in presence of \( H_{SE} \). We see that the size of the largest overlap element is strongly correlated with the decoherence, \( |\rho_{12}^{\text{avg}}(t)| \), which is defined by the average of \( \rho_{12}(t) \) over the interval \( t \in [200, 300] \), i.e. after the initial rapid decoherence has taken place. We call this the efficiency of decoherence. The details of averaging do not matter as long as \( t \) is much larger than the initial decay time, and the averaging interval is much larger than the correlation length of the oscillations. The statistics are obtained by sampling over the parameter range \( \Omega \in [0, 1] \), \( \Delta \in [0, 3] \). The number of spins in the environment is 7.

FIG. 3. The largest overlap element plotted versus the disorder strength \( \kappa \). For small values of \( \kappa \) the ferromagnetic ground state is energetically strongly favored and the perturbation represented by the central spin is not able to significantly alter the ground state. Close to \( \kappa = 0.5 \) we find a “phase transition” to a more disordered state. In this regime the coupling to the central spin is sufficient to alter the ground state of the environment. For \( \kappa \approx 1.0 \) the set of eigenstates are completely altered in the presence of the central spin, and the overlap with the original set is typically very small. The number of environmental spins is \( N = 9 \), \( \Delta = 3.0 \) and the same seed is used in generating the distributions of \( \Omega_{ij} \) and \( \Delta_i \) for each value of \( \kappa \) (solid line), while the markers correspond to a random seed for each value of \( \kappa \).
ical value $\kappa = 0.5$. Summarized, if the total frustration induced together by $\kappa$ and $H_{SE}$ is insufficient to break the ordered ground state, both $|0\rangle_E$ and $|0\rangle_E$ will have a large overlap with one of the states $|\uparrow \cdots \uparrow\rangle_E$ or $|\downarrow \cdots \downarrow\rangle_E$, according to Eq. (6), and in this regime the coherence of the central system will be preserved.

FIG. 4. (Color online) The largest overlap element computed according to Eq. (9), plotted versus the perturbation strength $\Delta$, for different values of the disorder parameter $\kappa$. The numbered curves correspond to the following values of $\kappa$: 1. spin glass $\kappa = 1.0$ (dashed, blue), 2. $\kappa = 0.8$ (solid, red), 3. $\kappa = 0.6$ (‘+’, green), 4. $\kappa = 0.4$ (‘*’, brown), 5. $\kappa = 0.2$ (dash-dotted, black). The environment is prepared in the ground state of $H_E$. The number of environmental spins are $N = 9$, and the same seed is used for each value of $\Delta$.

In Fig. 4 we follow the largest overlap element $\max_n |\langle n |0\rangle_E|^2$ as a function of the strength of the system-environment coupling $\Delta$, keeping $\kappa$ constant. In each of the simulations $H_{SE}$ is random and Ising-like. We find that for small values of $\kappa$, the strength of the random, frustrated system-environment coupling $H_{SE}$ needs to be sufficiently large in order to break the ferromagnetic interaction, in accordance with Eq. (9). Indeed, using Eq. (9) we predict the following values for the critical $\Delta$

| $\kappa$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|-------|-----|-----|-----|-----|---|
| $\Delta$ | 6.6 | 4.2 | 1.8 | -0.6 | -3.0 |

which agrees surprisingly well with Fig. 4. Until ferromagnetic order is destroyed by $H_{SE}$ the ground state of the perturbed system is very close to parallel with the unperturbed ground state, $\langle 0|0\rangle_E \approx 1$. For larger values of $\Delta$, we find strong oscillations in the size of the overlap element as a function of $\Delta$. This effect does not take place for $\kappa \geq 0.8$. In this regime the oscillations are less pronounced and the decay of the overlap element takes place for smaller values of $\Delta$. This regime is characterized by a highly frustrated ground state in the absence of $H_{SE}$. The presence of the central spin only alters microscopic details of the ground state, not its qualitative features.

In summary, an environment with frustrated interactions induces more effective decoherence than an unfraztured environment. This effect can be quantified by the strength of a perturbation (here $H_{SE}$) which alters the set of eigenstates $\{|n\rangle_E\}$. If the environment is dominated by frustrated interactions the set of eigenstates $\{|n\rangle_E\}$ in presence of the perturbation $H_{SE}$ will in general be very different from $\{|n\rangle_E\}$. We can think of this process as follows: In an environment with a large number of opposing interactions and a large set of almost degenerate low energy states, the presence of a central spin will in general cause a rotation of a subset of the eigenvectors $\{|n\rangle_E\}$. If there is a rotation and given that the subset contains the ground state $|0\rangle_E$, the maximal overlap element $\max_n |\langle n |0\rangle_E|^2$ and therefore the coherence of the central spin will decay. We will discuss the detailed physics behind this process in more detail in Sec. III B.

B. Decoherence in terms of avoided level crossings

In order to gain a deeper understanding of the differences between the ordered and the frustrated environment with respect to dephasing of the central spin, we study in detail the behavior of the eigenvalues. We use the same model as defined previously by Eq. (8) and an Ising-like random $H_{SE}$. Then we perform simulations where we gradually increase the coupling parameter $\Delta$ for different values of the disorder parameter $\kappa$.

In Fig. 4 (top), we plot the 20 lowest eigenvalues against the coupling strength $\Delta$. The disorder parameter is set to $\kappa = 0.1$ and the environment is therefore dominated by the ferromagnetic interaction. In the absence of perturbation we have two almost degenerate eigenvalues, the gap to the third lowest state is large. For small values of $\Delta$ the overlap between the ground state $|0\rangle_E$ of $H_E$ and the ground state of the perturbed environment $|0\rangle_E$ is very close to one $\langle 0 |0\rangle_E \approx 1$. At $\Delta \approx 0.2$ there is an avoided level crossing between the two lowest levels. Close to the avoided crossing, the eigenvectors of the two states evolve rapidly in Hilbert space and end up switching directions. Thus, after the level crossing the first exited state overlaps completely with what was the the ground state before the level crossing took place $\langle 1 |0\rangle \approx 1$. The overlap with the the ground state of $H_E$ is, however, still very close to one as long as only two states take part in the crossing. The eigenvector corresponding to a large overlap with the original ground state has simply been swapped with its neighbour and the coherence of the central system is conserved according to Eq. (6).

When the disorder of $H_E$ increases, the picture becomes more complex. In Fig. 5 (middle) we plot the 20 lowest eigenvalues against $\Delta$, but we use a higher degree of disorder in the environment ($\kappa = 0.5$). Since the environment has a larger contribution from frustrated couplings in $H_E$, the spacing between the energy levels is more uniform due to the level repulsion effect. In this particular case, the energy of the original ground state $|0\rangle_E$ is shifted upwards by the perturbation. The energy of this state can be tracked by the dark line, highlighting the eigenvalues corresponding to eigenvectors with
Having developed the sufficient understanding, we are now also able to explain the oscillatory behavior during the “phase transition” in Fig. 3. In the region of the transition (κ ≈ 0.5), more levels are present close to the ground state, and the repulsion width increases with κ. When the ground state gets close enough to the first excited state to feel repulsion, the corresponding eigenvectors begin rotating in the subspace they span. The overlap element is initially reduced and transferred to the first excited state. Eventually, the first excited state will be the closest in Hilbert space to the original ground state, and the repulsion width increases with κ. When κ is increased even more, the picture grows more complex as several levels are involved.

In Fig. 5 (bottom) we have reduced the ferromagnetic part of $H_E$ to zero (κ = 1.0). In this spin-glass phase the effect of level repulsion is strongly pronounced. The space between levels at which the eigenvectors start to repel each other is related to the size of the off-diagonal elements of the Hamiltonian in the basis of the perturbation (in this particular case – the coupling to the central spin in the $S_z$ eigenbasis). When κ is large, the off-diagonal elements in the Hamiltonian, Eq. 4, are larger than the average level spacing. This means that avoided crossings take place continuously as the parameter Δ is increased. In the parameter range where the distance between levels is smaller than the width of repulsion, the eigenvectors will in general evolve with Δ in the Hilbert space spanned by the eigenvectors of the repelling levels.

Thus, we find a crossover between two regimes. In the weak repulsion regime, the repulsion width is smaller than the typical distance between levels. In this regime we will have few and pronounced avoided crossings, the crossings will typically involve only two levels and the probability of multi-level crossings is strongly suppressed. Each two level avoided crossing will result in a swap between the eigenvectors involved, but does not reduce the largest overlap element after the crossing has taken place. The overlap element is reduced only during the crossing, still the coherence of the central system is only slightly altered, due to the levels approaching degeneracy. In the second regime, we have strong level repulsion. In this regime the repulsion width is of the same order or larger than the typical distance between levels such that each level is for a large range of Δ repelled by more than one level at the same time. When the repulsion width is much larger than the average level splitting, a large fraction of the levels become connected in the sense that if a certain level is repelled by the one or more levels above and these again is repelled by the next few levels that are again repelled by new levels. The corresponding eigenvectors will then evolve continuously in the Hilbert space spanned by this cluster of levels.

The energy levels of a system where the repulsion width is larger than the level splitting is expected to

large overlap element with the original ground state. In Fig. 5 (middle) we can compare the eigenvalues with the maximal overlap element. We find that the reduction in the maximal overlap element correspond to values of Δ where avoided level crossings take place. For large values of Δ the levels are closer, and we find avoided crossings

FIG. 5. (Color online) The 20 lowest eigenvalues plotted against the perturbation strength ∆ for different values of frustration in the environment. κ = 0.1 (top), 0.5 (middle), 1.0 (bottom). The overlap with the ground state of the unperturbed Hamiltonian $H_E$ is indicated by the color tone. A large overlap element increase the darkness of the corresponding eigenvalue (color bar is shown in the upper figure). The bottom plot shows the largest overlap element between the ground state of $H_E$ and the eigenstates of $H_E$ in presence of the interaction $H_{SE}$, $\max_n |\langle \kappa |0 \rangle|^2$. The number of spins in the environment is $N = 8$. where three or more levels are involved. Thus the overlap element is split between several states. The maximal overlap element is therefore reduced at these values of ∆.
be characterized by a distribution of energy levels following Wigner-Dyson statistics. In Fig. 6 we plot the level-spacing distribution of $H_E$ for different values of the disorder parameter $\kappa$. For large values of $\kappa$ we find that the distribution is consistent with the Wigner-Dyson distribution implying that the repulsion width is larger than the average splitting. At the same time, for small values of $\kappa$, where we have a ferromagnet, we find a special distribution of eigenvalues with two (almost) degenerate ground states (i.e. $\langle \uparrow \uparrow \cdots \uparrow \rangle_E$ and $\langle \downarrow \downarrow \cdots \downarrow \rangle_E$) and the next levels having a high degree of degeneracy. Each of the two ground states correspond to the bottom of a potential well, excited states belonging to different wells cannot be connected by flipping of two spins. The statistics obtained in Fig. 6 is therefore sorted by magnetization, the level statistics for each potential well of $H_E$ is treated separately.

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**FIG. 6.** (Color online) Level spacing distribution different degrees of disorder. 1 (solid, blue) - spin glass, $\kappa = 1$; 2 (dashed, red) - intermediate frustration, $\kappa = 0.5$; 3 (dashed/dots, black) - ferromagnetic phase, $\kappa = 0.25$. For comparison we also plot the Wigner-Dyson distribution $P(s) = (s\pi/2)e^{-s^2/4}$ (4, thin solid, grey). The number of spins in the environment is $N = 10$.

In summary, we find a weak repulsion regime, when $H_E$ has a low degree of disorder. In this regime the overlap element, $\langle n^\dagger | 0 \rangle_E$, between the original ground state and the set of eigenstates of the Hamiltonian in presence of the central spin is conserved even if we make the coupling to the central spin strong. In the second regime, when $H_E$ has high degree of disorder, we have strong repulsion between large clusters of states. In this regime, the set of eigenvectors of $H_E$ is very sensitive to the presence of the central spin. The largest overlap element, $\langle n^\dagger | 0 \rangle_E$, is therefore rapidly reduced as the coupling to the central spin is increased.

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**C. The initial state of the environment**

In Ref. [17], the importance of the initial state of the environment was studied. More efficient and stable decoherence was found for an initial state corresponding to infinite temperature, however no detailed explanation of this observation was given. If the initial state of the environment is no longer the ground state, but a linear combination of eigenstates from the set $\{|n\rangle_E\}$ such that $\psi_0 = \sum_i c_i |i\rangle_E$, where $|i\rangle_E \in \{|n\rangle_E\}$, Eq. (6) has to be replaced by

$$\rho_{\text{ext}}^c = \sum_{n,i} |c_i\langle n^\dagger | i \rangle_E|^2 e^{i(E_n - E_i)t}.$$  \(10\)

For finite temperature the overlaps are distributed over a number of eigenstates according to their Boltzmann weight, $e^{-E/kT}$. The coherence of the central spin, however, is conserved, $\rho_{\text{ext}}^c = 1$, as long as the perturbation introduced by the central spin does not alter the eigenvalues of the environment. If there is a significant perturbation, the coherence is reduced by an additional factor given by the square of the largest amplitude of the expansion $\psi_0 = \sum_i c_i |i\rangle$. The effect is shown in Fig. 7.

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**D. Enhancement of coherence by an external magnetic field**

As a consequence of the preceding analysis we find that the presence of an external magnetic field, $H_{\text{ext}}$, might enhance the coherence of the central system. The magnetic field will polarize the spins in the environment, resulting in a larger overlap element between the ground state $|\psi_0\rangle_E$ of the unperturbed environment $H_E$ and the set of eigenstates $\{|n\rangle_E\}$ in presence of the central spin. When the magnetic field is sufficiently strong to break the frustration in the ground state $|0\rangle_E$, i.e., when the magnitude of the external field is of the same order as the coupling between the spins in the environment, $H_{\text{ext}} \gg \sqrt{N}$, the presence of the central spin will not significantly alter the magnetized ground state of the
environment unless the coupling to the environment is strong compared to the external field. Thus, if the spin environment of the central system is disordered, magnetization is beneficial to the coherence of the central system. This procedure has already been applied experimentally, see Refs. 5 and 10.

IV. DISCUSSION

In this article, we have considered the special case where the central spin is coupled diagonally to its environment. Then no transitions can take place between the eigenstates of the central system and the decoherence is entirely due to renormalization of its energy splitting (pure dephasing). This choice of coupling simplifies the treatment since the effect of the central spin upon the environment can be treated as a static perturbation. If we loosen this restriction and also include real transitions between the eigenstates (T1-processes) the central system will participate in the complex many-body dynamics of the total system. However, if the number of spins in the environment is large, the fine details of the coupling, $H_{SE}$, should not result in qualitatively different behavior of the environment with respect to level repulsion. The microscopic details of the dynamics will, of course, strongly depend on the exact nature of the coupling. We, therefore, believe that the central spin will preserve its coherence much longer in the ordered environment, compared to a frustrated environment also in the presence of non-diagonal system-environment coupling $H_{SE}$. The numerical analysis in Refs. 17 and 18 support this hypothesis.

We considered an arrangement where the central system coupled to each spin in the environment. In the presence of a very large environment, where the connectivity between the subsystems is limited, this approximation might fail. As an example the central spin might couple to only a few spins of the environment. However, even if the central spin couple only to few spins, in the presence of a ferromagnetic environment, this might be sufficient for coupling to collective modes of excitation (i.e., spin waves).

Since we treat a closed quantum system, we do not expect details of our analysis to carry on to realistic open systems. In the thermodynamic limit we expect that the environment will be damped, forgetting interactions with the central spin at times earlier than the correlation time. However, the analysis should be relevant to systems where the effective temperature is much less than the typical splitting between states in the environment.

We found it useful, in light of the correlations shown in Fig. 8 to discuss the decoherence of the central spin in terms of the overlap elements between the ground state of $H_E$ and the eigenstates $\{ | n \uparrow \rangle \}$ of the environment in presence of the central spin. However, the largest overlap element of Eq. (6) does not necessarily give the whole picture. The coherence of the central spin may differ from what was predicted by the overlap element due to the phase factor $e^{i (E_n^\uparrow - E_m^\uparrow) t}$. If the ground state of $H_E$ is degenerate due to symmetry, and the central system is unable to break this symmetry, then coherence will persist in the central system even if the overlap with the ground state of $H_E$ is split between several degenerate states. If the degeneracy is not exact, coherence might still decay extremely slowly if the difference $| E_n^\uparrow - E_m^\uparrow |$ of the states overlapping with $| \psi_0 \rangle_E$ is small.

V. CONCLUSION

In conclusion, we have analyzed the efficiency of decoherence using the overlap elements $\langle n \uparrow | 0 \rangle_E$, between the ground state of the isolated environment and the set of eigenstates of the environment in presence of the central spin. It was shown that the square of the largest overlap element, $\text{max}_{n} | \langle n \uparrow | \psi_0 \rangle_E \rangle |^2$, is a very good indicator for the efficiency of decoherence. The size of the largest overlap element tends to be much larger for an environment with no competing interactions, than in case of an environment with many frustrated coupling. The underlying mechanism behind this effect can be explained by the statistics of the eigenvalues of $H_E$. Coupling to an external object, e.g., a central spin, results in avoided level crossings between the levels of the environment. In the absence of frustration, the level repulsion is weak and the avoided crossings will take place in a short interval in the coupling parameter to the external object, $\Delta$. The eigenvectors corresponding to the involved levels will simply switch, and the overlap element remains unaltered. In this weak repulsion regime, multi-level crossings are strongly suppressed. In the opposite regime, characterized by strong level repulsion, eigenvalues within large fractions of Hilbert space are subject to mutual level repulsion. In this strong repulsion regime the corresponding eigenvectors will rapidly mix when increasing $\Delta$ resulting in very efficient decoherence of the central object.

As a side note – we have shown that a external mag-
ngetic field can transfer the environment from the strong to the weak repulsion regime provided it is stronger than the frustrated couplings present, thereby enhancing the coherence of the central spin. Thus, it should be possible to enhance the coherence time of a central spin in the presence of a spin-glass like environment, by applying an external magnetizing field that is of the same magnitude or larger than the internal coupling in the environment.