Exact Solutions Related to Nonminimal Gravitational Coupling

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Abstract: We obtain exact analytic solutions for a typical autonomous dynamical system, related to the problem of a vector field nonminimally coupled to gravity.

Gravity is presently the only interaction that is not inserted in any of the unification schemes. Due to this fact, many toy models have appeared coupling gravity to fields that could have played significant role in the Early Universe, in such wise as to display desirable properties. Special interest has been put in gravity coupled nonminimally to other fields which could generate, among a number of new effects, a non-singular universe.

In this context, dynamical systems techniques have been applied to solve the problem of coupling of gravity to a vector field, whose Lagrangian can be written as

$$\mathcal{L} = \sqrt{-g} \left( \frac{R}{k} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta RW_\mu W^\mu \right),$$

where $g = \det g_{\mu\nu}$ ($g_{\mu\nu}$ is the metric tensor), $R$ is the scalar of curvature, $G = k/8\pi$ is the Newtonian gravitational constant in units that $\hbar = c = 1$, $|\beta| = 1$, $W_\mu$ is an arbitrary vector field and $F_{\mu\nu} = W_{[\mu,\nu]}$, where the comma represents ordinary differentiation and the square brackets represent the skew–symmetric part of $W_{\mu,\nu}$.

The Lagrangian (1) leads to a set of equations of motion that can be transformed into an autonomous dynamical system. By choosing the Robertson–Walker metric

$$ds^2 = dt^2 - S^2(t) \left[ d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \right],$$

with the ansatz $W^2 = W^2(t)$, the Lagrangian (1) leads to equations of motion which has a solution given by

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\[
W^2(t) = \frac{1}{k} \left( 1 - \frac{t}{S} \right),
\]
\[
S(t) = (t^2 + Q^2)^{1/2},
\]
where \( k \) is the Einstein constant and \( Q \) is also a constant. Afterwards one sets \( X = 3(S/\dot{S}) \) and \( Y = (\dot{\Omega}/\Omega) \), where \( \Omega = (1/k) + (\beta W^2) \), to write the equations of motion as
\[
\dot{X} = -\frac{1}{3} X^2 + XY, \tag{2}
\]
\[
\dot{Y} = -Y^2 - XY.
\]

Hence, the correct interpretation of functions \( X \) and \( Y \) is important to the knowledge of the evolution of the model.

Up to now there was no exact solution for this dynamical system and the analysis was carried out by using qualitative procedure. Here we exhibit the explicit solution in a very simple way.

One can rewrite eqs. (2) in polar coordinates, \( \rho = (X^2 + Y^2)^{1/2} \) and \( \theta = \arctan \frac{Y}{X} \), as
\[
\dot{\rho} = -\rho^2 \left( \frac{1}{3} \cos^3 \theta + \sin^3 \theta + \cos \theta \sin^2 \theta - \cos^2 \theta \sin \theta \right),
\tag{3}
\]
\[
\dot{\theta} = -2\rho \cos \theta \sin \theta \left( \frac{\cos \theta}{3} + \sin \theta \right).
\]

The phase diagram related to this autonomous dynamical system can be obtained through
\[
\frac{\dot{\rho}}{\dot{\theta}} = \frac{d\rho}{d\theta},
\]
which leads to
\[
\rho = C \exp \left( \frac{1}{2} I \right), \tag{4}
\]
where \( C \) is an arbitrary positive constant of integration and
\[
I = \int \frac{\frac{1}{3}(\cos^3 \theta + 3 \sin^3 \theta) + \cos \theta \sin^2 \theta - \cos^2 \theta \sin \theta}{\frac{1}{3}\cos^2 \theta \sin \theta + \cos \theta \sin^2 \theta} d\theta. \tag{5}
\]

After a little algebra, involving only basic trigonometric relations, one can show that the solution to integral \( I \) readily leads to
\[ \rho = C \left| \frac{\tan \theta}{{\cos \theta} + {3 \sin \theta}} \right|^{1/2} \] 

(6)

We point out that the precise knowledge of the function \( \rho(\theta) \) leads to the sketch of the desired phase diagram associate with the autonomous dynamical system without any qualitative analysis about the dynamical system in regions around the origin \(^1,4\).

On the other hand one can choose to study the dynamical system qualitatively by means of the projection on the Poincaré sphere \(^3,4\). This will provide useful informations concerning the asymptotic behaviour of the system. As defined in Ref. 3, the Poincaré sphere with unitary radius is placed over the \( xy \)-plane, this plane being tangent to the south pole of the sphere. Another frame is in order and it is placed in the centre of the sphere. Projections onto the sphere are taken by joining points of the diagram to the centre of the sphere. This process will give rise to a drawing on the sphere which is projected orthogonally on the \( xy \)-plane. The final portrait of the phase diagram is, then, in a circle with unitary radius where the behaviour at infinity is identified with the border of the circle.

Applying this method \(^3\) to (2) one can obtain the topologies around the singular points at infinity, as in Fig.1. The arrows show the evolution in time. Note that besides the equilibrium points \( \tilde{D}(0,0), \tilde{D}'(0,0), \tilde{C}(0,0) \) and \( \tilde{C}'(0,0) \), there are the points \( \tilde{B}(-1/3,0) \) and \( \tilde{B}'(1/3,0) \), the primes indicating antipodes points. The system refuses to give informations on the topology around the origin by the method of linearization. Dulac’s test shows there are no limiting cycles. Note that the above mentioned equilibrium points are consistent with eq. (6).

The complete phase diagram which compactifies infinities is shown in Fig.2. Its shape is very close to that in Ref.4; the difference is due to the unusual dimension used. The phase diagram in Ref.1 is obtained in a different way: the projection is taken on the plane which corresponds to \( X = 0 \) and crosses the poles of the sphere, the \( xy \)-plane being tangent to the north pole of the sphere.

Notice that any autonomous planar dynamical system

\[
\begin{align*}
\dot{X} &= P(X,Y), \\
\dot{Y} &= Q(X,Y).
\end{align*}
\]

allows a similar change of variables, and in polar form the phase diagram arises from

\[
\frac{d\rho}{d\theta} = \rho \left( P(\rho \cos \theta, \rho \sin \theta) + \tan \theta Q(\rho \cos \theta, \rho \sin \theta) \right) - \tan \theta P(\rho \cos \theta, \rho \sin \theta)
\]

(7)

For this reason, exact solutions may be achieved if it is possible to rewrite (7) as

\[
\frac{d\rho}{d\theta} = R(\rho) \Theta(\theta)
\]
Recently some cosmological models have been investigated in the so-called multidimensional scenario, and in a particular case a Maxwell field in higher dimensions is coupled to Einstein-Hilbert Lagrangian. This also leads to a qualitative dynamical system analysis, and, as in eq. (7), the search for exact solutions related to this problem is in progress.

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Figures Captions

Figure 1: Equilibrium points at infinity and the topology imposed by them. Empty balls means unstable points, and full balls stable ones.

Figure 2: Final aspect of the phase diagram showing the equilibrium points at infinity. Only one integral curve is chosen to each region in the diagram to avoid it to become entangled.