Explicit solutions for a class of nonlinear backward stochastic differential equations and their nodal sets

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Abstract

In this paper, we investigate a class of nonlinear backward stochastic differential equations (BSDEs) arising from financial economics, and give specific information about the nodal sets of the related solutions. As applications, we are able to obtain the explicit solutions to an interesting class of nonlinear BSDEs including the $k$-ignorance BSDE arising from the modeling of ambiguity of asset pricing.

Keywords: Cameron-Martin formula, Feynman-Kac formula, nodal set, nonlinear BSDE, parabolic equation

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1 Introduction

In a seminal paper [18], Pardoux and Peng (1990) studied a non-linear backward stochastic differential equation (BSDE)

$$dY_t = -g(t, Y_t, Z_t)dt + Z_t dB_t, \quad Y_T = \xi,$$

where $B$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, $T > 0$, and $\xi$ is measurable with respect to Brownian motion trajectories up to $T$. These authors proved, under some assumptions on the non-linear driver $g$ and the terminal value $\xi$, that BSDE (1) possesses a unique solution, a pair of adapted processes $Y$ and $Z$ satisfying stochastic integral equation

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$$

for $0 \leq t \leq T$. In past two decades, many researchers have worked on the theory of BSDEs and have obtained many excellent results about the solution pair $(Y_t, Z_t)$. Since the publication of [18], the theory of BSDEs has been applied to mathematical finance, stochastic control, partial differential equations, stochastic game and so on, see for example [5, 7, 14, 17, 9, 20, 21] and the literature therein. Explicit solutions to (1) are known only in few cases, mainly for the case where $g(t, y, z)$ is linear in $y$ and $z$. It is easy to see that the solution to a linear BSDE is given by Feynman-Kac’s formula (see for example Peng [18]). For a non-linear driver $g(t, y, z)$, little is known about $(Y_t, Z_t)$ due to lack of an explicit formula, but see [6], in which Chen et al. have obtained an interesting co-monotonic theorem of $(Z_t)$ for a non-linear but special driver $g(t, y, z)$. It remains a challenging problem in general to derive useful information about solutions of BSDEs.

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In applications to some problems in financial economics, it is useful to have an explicit expression for the solution of BSDE (1). For models appearing in mathematical finance, one needs to determine the signs of solutions \((Z_t)\), which allow to identify the monotone ranges of active hedging. Therefore researchers are very interested in determining the zeros of \((Z_t)\) for such BSDE models, i.e. the nodal set of the process \((Z_t)\).

The goal of this paper is to identify the nodal sets of solutions \((Z_t)\) to a class of non-linear BSDEs which arise from financial economics, and to identify the monotone ranges of \((Z_t)\) accordingly. Our results will cover the so-called \(k\)-ignorance model in continuous recursive utilities, studied by Chen and Epstein [5]. The model is a simple BSDE:

\[
Y_t = \xi + \int_t^T k|Z_s|ds - \int_t^T Z_s dB_s
\]

for \(0 \leq t \leq T\), where \(k > 0\) is a model parameter. (2) is perhaps the simplest non-linear BSDE. It has significant applications in discussing non-linear risk measures. Chen et al. [4, 6] have shown that if \(\xi = \varphi(B_T)\) and \(\varphi\) is monotonic, then the solution \((Y, Z)\) of (2) can be computed explicitly. In this case, Chen et al. [4, 6] observed that (2) can be reduced to an equivalent linear BSDE, so that an explicit formula may be obtained accordingly. If \(\varphi\) is not monotonic, it remains open to solve BSDE (2) explicitly. By exploring the information on the nodal set of \((Z_t)\), we are able to work out explicit solutions for a class of non-linear BSDEs, including the \(k\)-ignorance models, where \(\varphi\) is not necessary monotone. As an application, we therefore are able to give an explicit representation of the solution \((Y_t, Z_t)\) for the \(k\)-ignorance model (2), where the terminal value is Markovian and \(\varphi(x) = x^2\) or \(\varphi(x) = I_{[a,b]}(x)\). We should point out that the \(k\)-ignorance model (2) with these terminal values plays an important role in modeling ambiguity of asset pricing, and we will discuss this point in the last part of the article. For this aspect, the reader should also refer to Chen and Epstein [5] and the literature therein too.

The paper proceeds as follows. In Section 2 we first introduce some notions, notations and a few basic facts about BSDEs, which will be used through the paper. We then prove the main results of the paper in section 5 by discussing an application of our results in robust pricing in an incomplete market.

2 The main results

Let us begin with the notion of backward stochastic differential equations, recall the basic result on BSDEs and establish notations we will use in what follows. Let \((B_t)_{t \geq 0}\) be a standard one dimensional Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\). Let \((\mathcal{F}_t)\) be the \(\sigma\)-filtration generated by the Brownian motion, that is, \(\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\}\) for \(t \geq 0\).

The driver in formulating the BSDE to be studied in this paper is a deterministic real function \(g(t, y, z)\) for \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}\), which satisfies the following conditions:

(A.1) Lipschitz condition. There exists a constant \(k \geq 0\), such that

\[
|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq k(|y_1 - y_2| + |z_1 - z_2|)
\]

for all \(t \geq 0\), \(y_1, y_2 \in \mathbb{R}\) and \(z_1, z_2 \in \mathbb{R}\); and

(A.2) Normalization condition. \(g(t, y, 0) = 0\) for any \((t, y) \in [0, T] \times \mathbb{R}\).

We will use the standard notation that \(L^2(\Omega, \mathcal{F}, P)\) denote the space of \(\mathcal{F}_t\)-measurable and square integrable random variables on \((\Omega, \mathcal{F}, P)\) for each \(t \geq 0\). Let

\[
\mathcal{M}(0, T, \mathbb{R}) = \left\{(\nu_t)_{t \in [0, T]} : \text{real valued(} \mathcal{F}_t\text{)-adapted process with } E \left[ \int_0^T |\nu_t|^2 dt \right] < \infty \right\}.
\]
The fundamental result obtained in Pardoux-Peng [18] is the following. If $g$ satisfies (A.1), (A.2), and $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, BSDE (1) has a unique solution, i.e., there is a pair of adapted processes $Y, Z \in \mathcal{M}(0,T, \mathbb{R})$, which solve (1) in the sense that

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_sdB_s$$

for all $t \in [0,T]$.

We are interested in Markovian case, that is, the terminal value $\xi$ in (4) depends only on $B_T$, that is, $\xi = \varphi(B_T)$, so that

$$Y_t = \varphi(B_T) + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_sdB_s.$$  (5)

Let us isolate the following assumptions on $\varphi$, which will be used in our main results.

(H.1) There is $c \in \mathbb{R}$, $\varphi$ is symmetric about $c$, that is, $\varphi(c-x) = \varphi(c+x)$ for all $x \in \mathbb{R}$.

(H.2) $\varphi$ is monotone on $[c, \infty)$.

We are now in a position to state our first result of the paper.

**Theorem 1.** Let $g \in C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$ satisfying (A.1) and (A.2), and $\varphi \in C^3(\mathbb{R})$. Assume that the derivatives $\varphi^{(i)}$ (where $i = 0, 1, 2, 3$) have at most polynomial growth.

1. Let $u(t,x)$ be the unique solution of Cauchy’s initial problem of the parabolic equation

$$
\begin{aligned}
\partial_t u &= \frac{1}{2} \partial^2_{xx} u + g(t,u,\partial_x u), \quad \text{in } (0,\infty) \times \mathbb{R}, \\
u(0,x) &= \varphi(x).
\end{aligned}
$$

2. If in addition $\varphi$ satisfies (H.1) and (H.2), and $g(t,y,z) = g(t,-y,-z)$ for any $t \in [0,T]$ and $y,z \in \mathbb{R}$, then

(i) $\partial_t u(t,c) = 0$ for every $t > 0$.

(ii) $w(t,x) = \partial_x u(t,x)$ is the unique solution to the initial value problem of the parabolic equation

$$
\partial_t w = \frac{1}{2} \partial^2_{xx} w + \partial_x g(t,u,w) \cdot \partial_x w + \partial_y g(t,u,w) \cdot w
$$

and

$$w(0,x) = \varphi'(x), \quad \text{for } x \in \mathbb{R}.$$  (8)

Moreover $w(t,c) = 0$ for $t \geq 0$.

(iii) Let $x \in \mathbb{R}$ and $0 \leq t \leq T$. Let

$$a_{s,t} = \partial_y g(t-s,u(t-s,X^x_s),w(t-s,X^x_s)), \quad b_{s,t} = \partial_z g(t-s,u(t-s,X^x_s),w(t-s,X^x_s))$$

for $0 \leq s \leq t$, where $X^x_t = x + B_t$. Define the stochastic exponential martingale

$$N_s = \exp \left\{ \int_0^s b_{r,t} dB_r - \frac{1}{2} \int_0^s b^2_{r,t} dr \right\}$$

for $0 \leq s \leq t$. Then

$$w(t,x) = E \left[ N_t \varphi'(X^x_t) \cdot e^{\int_0^t a_{s,t} ds} 1_{\{t < \tau\}} \right]$$

for every $t \geq 0$, where $\tau = \inf \{s \geq 0, X^x_s = c\}$.


Proof. Since \( \varphi \) is a \( C^3 \)-function with polynomial growth and \( g \in C^{1,3} \), so by the theory of parabolic equations of second order, (6) possesses a unique solution \( u(t,x) \) which belongs to \( C^{1,3}([0,T] \times \mathbb{R}) \), see for example [10].

By applying Itô's formula, \( Y_t = u(T-t,B_t), \ Z_t = \partial_t u(T-t,B_t) \) solve BSDE (5), and the conclusion follows from the uniqueness of the solution to BSDE (5), which proves the first claim.

Now we prove (2). Since \( g(t,y,z) \) is symmetric about 0, one can verify that \( u(t,c-x) \) and \( u(t,x+c) \) are solutions to the parabolic equation

\[
\partial_t v = \frac{1}{2} \partial_{xx}^2 v + g(t,v,\partial_x v), \text{ in } (0,\infty) \times \mathbb{R}
\]

and \( v(0,x) \) coincides with \( \varphi(c-x) \) and \( \varphi(c+x) \) respectively. Since \( \varphi \) is symmetric about \( c \), that is \( \varphi(c-x) = \varphi(c+x) \), by the uniqueness of the initial value problem for the parabolic equation (11) we may conclude that \( u(t,c+x) = u(t,c-x) \). It in turn yields that \( \partial_t u(t,c+x) = -\partial_t u(t,c-x) \) for \( (t,x) \in \mathbb{R}_+ \times \mathbb{R} \). In particular \( \partial_t u(t,0) = 0 \) for every \( t \geq 0 \). We have thus proven (i).

(ii) follows immediately by differentiating (6) in \( x \).

(iii) Under assumptions on \( g(t,y,z) \), \( a_{s,t} \) and \( b_{s,t} \) are bounded processes where \( 0 \leq s \leq t \leq T \). Since \( \varphi \in C^3 \) and \( \varphi^{(i)} \) (where \( i = 0,1,2,3 \)) possess at most polynomial growth, the unique strong solution \( u(t,x) \) to the problem (6) belongs to \( C^{1,3}([0,T] \times \mathbb{R}) \). In particular we have \( w \in C^{1,2}([0,T] \times \mathbb{R}) \).

Let us first consider the case where \( \varphi^{(i)} \) (where \( i = 0,1,2,3 \)) are bounded. For this case, the second order derivative of \( u(t,x) \), that is, \( \partial_{xx} u(t,x) \) are bounded in \([0,T] \times \mathbb{R}\). Let \( 0 \leq t \leq T \) be any but fixed.

Define \( q(s) \) by solving the ordinary differential equation: \( dq(s) = a_{s,t} q(s) ds + b_s \) with \( q(0) = 1 \). Then \( q(s) \) has finite variations, and is a bounded process. \( N_s \) is the solution to the exponential martingale equation: \( dN_s = N_s b_s dB_s \) and \( N_0 = 1 \). Of course \( N \) is just the stochastic exponential of \( \int_0^t b_s dB_s \), where \( (b_{s,t})_{s \leq t} \) is a bounded process (while its bound may depend on \( t \)). Let \( M_s = q(s) N_s w(t-s,X_s^t) \), where \( 0 \leq s \leq t \).

By Itô's formula we have

\[
dM_s = q(s) d[N_s, w(t-s,X_s^t)] + N_s w(t-s,X_s^t) a_{s,t} q(s) ds + q(s) N_s b_s w(t-s,X_s^t) dB_s + q(s) N_s a_{s,t} w(t-s,X_s^t) ds.
\]

Since \( w \) solves (7), so that

\[
dw(t-s,X_s^t) = \left( -\partial_{w} w(t-s,X_s^t) + \frac{1}{2} \partial_{x}^2 w(t-s,X_s^t) \right) ds + \partial_{t} w(t-s,X_s^t) dB_s
\]

Substituting this into the previous equality for \( M \), we obtain that

\[
dM_s = q(s) N_s [\partial_{w} w(t-s,X_s^t) + b_{s,t} w(t-s,X_s^t)] dB_s.
\]

We claim that \( M \) is a square integrable martingale. In fact, since

\[
|q(s) N_s [\partial_{w} w(t-s,X_s^t) + b_{s,t} w(t-s,X_s^t)]| \leq C_1 N_s
\]

for some positive constant \( C_1 \) depending on \( t \), but not on \( s \leq t \).

But

\[
E \left[ |N_s|^2 \right] = E \left[ \exp \left( 2 \int_0^s b_r dB_r - \int_0^s |b_r|^2 dr \right) \right] \\
\leq C_2 E \left[ \exp \left( 2 \int_0^s b_r dB_r - 2 \int_0^1 |b_r|^2 dr \right) \right] \\
= C_2 < \infty,
\]
where $C_2$ is a positive constant. Therefore,

$$E \left[ |M_t|^2 \right] = E \left( M_0 + \int_0^t [q(s)N_t \partial_t w(t-s,X^i_s) + b_{i,j}w(t-s,X^i_s)] dB_s \right)^2 \leq 2E(M_0^2) + 2E \left( \int_0^t [q(s)N_t \partial_t w(t-s,X^i_s) + b_{i,j}w(t-s,X^i_s)] dB_s \right)^2 \leq C_0 + 2C_1^2 \int_0^t E(N_s^2) ds < \infty$$

which implies that $(M_t)$ is a square integrable martingale up to time $t$.

Since

$$\tau = \inf \{ s \geq 0, X^i_s = c \} = \inf \{ s \geq 0, B_s = c - x \}$$

is a stopping time, finite almost surely, see (2.6) in [12], by stopping theorem for martingales, we have

$$E[w(t,x)] = E(w(t,x)) = E(w(t,x))$$

which allows to determine the sign of $w(t,x)$. Let $\phi$ be the solution pair of BSDE (5). Then the following conclusions hold.

1. If $\phi'(x) \geq 0$ and and $\phi'(x) \neq 0$ for all $x > c$, then $\sgn(Z_t) = \sgn(B_t - c)$ for all $t \geq 0$ almost surely.

2. Similarly, if $\phi'(x) \leq 0$ and $\phi'(x) \neq 0$ for all $x > c$, then $\sgn(-Z_t) = \sgn(B_t - c)$ for all $t \geq 0$ almost surely.

Proof. By Theorem 1, $Z_t = w(T-t,B_t)$, and

$$w(t,x) = E \left[ N_t \phi'(X^i_s) e^{\int_0^t \alpha_s ds} 1_{\{ \tau < t \}} \right]$$

which allows to determine the sign of $w(t,x)$ accordingly.

Note that if $x > c$, then $X^i_t > c$ on $t < \tau$, so, unless $\phi'(x)$ equals zero identically for $x > c$, we must have $P(N_t \phi'(X^i_s) 1_{\{ \tau < t \}} > 0) = 1$. Since $N_t > 0$, thus if $\phi' \geq 0$ and $\phi'$ does not vanish identically on $(c,\infty)$, we have $w(t,x) > 0$ for $x > c$ and $w(t,x) < 0$ for $x < c$, which implies $\sgn(Z_t) = \sgn(B_t - c)$. Similarly, if $\phi'(x) \leq 0$ and $\phi'(x) \neq 0$ for all $x > c$, we have $\sgn(Z_t) = -\sgn(B_t - c)$.

The proof of Theorem 2 is completed. \qed
Theorem 2 may be stated as the following “non-vanishing theorem”, which is the most useful form in our discussions below.

**Theorem 3.** Under the same assumptions on \(g\) and \(\varphi\) in Theorem 2, and suppose \((Y_t, Z_t)\) is the unique solution of BSDE (5). Then \(Z \neq 0\) with respect to the product measure \(dt \otimes dP\).

**Proof.** This is a direct consequence of Theorem 2, as \(\{Z \neq 0\} = \{B \neq c\}\) almost surely, \(B \neq c\) almost surely with respect to \(dt \otimes dP\). \(\square\)

While the conditions imposed on the initial data \(\varphi\) and the regularity imposed on the non-linear driver \(g(t, y, z)\) in Theorem 2 are too restrictive in applications, which are needed to achieve a general result, though these conditions are sufficient but very often not necessary. Here we do not seek for the best conditions in particular on \(\varphi\), and the approach put forward in Theorem 2 however also applies to situations where the regularity on the driver \(g(t, y, z)\) is not available. These instances however have to be treated case by case. In this article we deal with an important example, the \(k\)-ignorance model, with details.

The non-linear driver is only Lipschitz continuous in the \(k\)-ignorance model, which has a unique solution pair \((Y, Z)\) according to Pardoux-Peng [18]. One however can not apply the non-linear Feynman-Kac formula directly, as the solution \(u(t, x)\) to the corresponding parabolic equation is only \(C^{1+}\), but not \(C^2\) in the variable \(x\) in general. Thus the main effort is to derive a non-linear Feynman-Kac type formula for this case, and generalize the results in Theorem 1 to the current example.

**Theorem 4.** Let \(\varphi \in C^3(\mathbb{R})\) satisfying (H.1) and (H.2) with some constant \(c\), such that \(\varphi\) and \(\varphi'\) have at most polynomial growth, and let \(u\) be the unique weak solution to the non-linear parabolic equation

\[
\partial_t u = \frac{1}{2} \partial^2_{xx} u + k |\partial_x u|
\]  

with the initial condition that

\[
u(0, x) = \varphi(x).
\]  

Then \(\partial_x u(t, x)\) is Hölder continuous in any compact subset of \((0, \infty) \times \mathbb{R}\), and for every \(t > 0\) and \(x \in \mathbb{R}\)

\[
\partial x u(t, x) = E \left[ N_s \varphi'(B_s + x) \cdot 1_{\{t < \tau\}} \right],
\]

where

\[
N_s = \exp \left[ k \int_0^t \text{sgn}(w(t - r, B_r + x)) dB_r - \frac{k^2}{2} s \right]
\]

is a martingale for \(0 \leq s \leq t\), \(w(t, x) = \partial_x u(t, x)\) is the unique weak solution to the initial value problem of the parabolic equation

\[
\partial_t w = \frac{1}{2} \partial^2_{xx} w + k \text{sgn}(\partial_x u(t, x)) \cdot \partial_x w
\]

with the initial condition that

\[
w(0, x) = \varphi'(x),
\]

and \(\tau = \inf\{s \geq 0 : B_s + x = c\}\). Moreover \(Y_t = u(T - t, B_t)\) and \(Z_t = w(T - t, B_t)\) is the unique solution pair to the \(k\)-ignorance model.

**Proof.** According to the theory of parabolic equations [10, 13], there is a unique weak solution \(u(x, t)\) to the problem (12, 13), and \(\partial_x u(x, t)\) is Hölder continuous on any compact subset of \((0, T) \times \mathbb{R}^d\). According to Aronson’s estimate and Nash-Moser theory (see [1, 15] for details), it follows that the linear problem (15, 16) has a unique weak solution which is Hölder continuous in any compact set of \((0, T) \times \mathbb{R}\).

Next we prove that \(Y_t = u(T - t, B_t)\) and \(Z_t = w(T - t, B_t)\) are the unique solution pair of BSDE (2). To this end, for \(\varepsilon \geq 0\), let \(g_\varepsilon(z) = k \sqrt{z^2 + \varepsilon}\). For \(\varepsilon > 0\), \(g_\varepsilon\) is smooth and \(|g_\varepsilon(z) - g_0(z)| \to 0\) as \(\varepsilon \to 0\) for every \(z \in \mathbb{R}\). Moreover \(g_\varepsilon'(z) = k \frac{z}{\sqrt{z^2 + \varepsilon}}\) so that \(|g_\varepsilon'(z)| \leq k\).
The condition that $\varphi$ possesses at most polynomial growth is sufficient to ensure the existence of uniqueness of a strong solution $u^\varepsilon(t, x)$ to the problem

$$\partial_t u^\varepsilon(x,t) = \frac{1}{2} \partial^2_{xx} u^\varepsilon(t, x) + g_\varepsilon(\partial_x u^\varepsilon(t, x))$$

(17)

together with the initial condition that

$$u^\varepsilon(x, 0) = \varphi(x),$$

(18)

for every $\varepsilon > 0$. According to the regularity theory (see [15]) of quasi-linear parabolic equations, $u^\varepsilon \in C^{1,\infty}((0,\infty) \times \mathbb{R})$, whose space derivative function $w^\varepsilon(t, x) = \partial_x u^\varepsilon(t, x)$ is the unique weak solution to the (linear) parabolic equation

$$\partial_t w^\varepsilon(t, x) = \frac{1}{2} \partial^2_{xx} w^\varepsilon(t, x) + g'_\varepsilon(\partial_x w^\varepsilon(t, x)) \cdot \partial_x w^\varepsilon(t, x)$$

(19)

subject to the initial value that

$$w^\varepsilon(0, x) = \varphi'(x).$$

(20)

By standard theory of parabolic equations, $u^\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$, where $u$ is the unique weak solution to the initial problem of the parabolic equation, that is the case where $\varepsilon = 0$ for the problem (17, 18).

We note that $g'_\varepsilon$ are uniformly bounded by $|k|$, which is crucial in our argument below, according to Nash’s continuity theory (see [15]), the solutions $\{w^\varepsilon(t, x)\}$ are uniformly Hölder continuous in any compact subset of $(0,\infty) \times \mathbb{R}$, and bounded in $L^2([0, T], H^1_{\text{loc}})$ (where $H^1$ is the usual Sobolev space), so that we may extract, if necessary, a sequence $\varepsilon_n \downarrow 0$, such that $w^\varepsilon_n(t, x)$ converges to $w(t, x)$ point-wise, uniform in any compact subset of $(0, T) \times \mathbb{R}$, and $w^{\varepsilon_n}$ converges weakly to $w$ in $L^2([0, T], H^1_{\text{loc}})$.

For every $\varepsilon > 0$, $w^\varepsilon$ is a strong solution to (19) so that, for every $\rho(x, t)$ with a compact support in $[0, T) \times \mathbb{R}$, by integration by parts

$$-\int_{\mathbb{R}} \rho(x,0) \varphi'(x) = -\frac{1}{2} \int_{\mathbb{R} \times (0,T)} \partial_x \rho(x,t) \partial_x w^\varepsilon(t,x) + \int_{\mathbb{R} \times [0,T]} \rho(x,t) g'_\varepsilon(\partial_x w^\varepsilon(t,x)) \cdot \partial_x w^\varepsilon(t,x).$$

Letting $\varepsilon \rightarrow 0$, we therefore obtain that

$$-\int_{\mathbb{R}} \rho(x,0) \varphi'(x) = -\frac{1}{2} \int_{\mathbb{R} \times (0,T)} \partial_x \rho(x,t) \partial_x w(t,x) + \int_{\mathbb{R} \times [0,T]} \rho(x,t) k \text{sgn}(w(t,x)) \cdot \partial_x w(t,x)$$

which implies that $w(t, x)$ is the unique weak solution to the problem (15, 16).

Since for every $n$, according to Itô’s formula

$$Y^n_t = \phi(B_T) + \int_t^T g^n_s(Z^n_s) ds - \int_t^T Z^n_s d{B}_s,$$

(21)

where $Z^n_s = w^{\varepsilon_n}(T - s, B_s)$, $Y^n_t = u^{\varepsilon_n}(T - t, B_t)$, and $Y^n_t = u^{\varepsilon_n}(T - t, B_t)$ as $n \rightarrow \infty$, and therefore $Y_t = u(T - t, B_t)$ and $Z_t = w(T - t, B_t)$ are the unique solution pair of BSDE

$$Y_t = \phi(B_T) + \int_t^T k |Z_s| ds - \int_t^T Z_s dB_s \quad \text{for } 0 \leq t \leq T.$$

Since $\varphi \in C^3(\mathbb{R})$ with polynomial growth, so that we may apply Theorem 1 to $u^\varepsilon(t, x)$. Thus for each $\varepsilon > 0$, $w^\varepsilon(t, x) = 0$ for all $t \geq 0$, and

$$w^\varepsilon(t, x) = E \left[ N^n_t \varphi'(B_t + x) \cdot I_{t < c} \right]$$

(22)

where $\tau = \inf \{ s \geq 0 : B_s + x = c \}$ and

$$N^n_t = \exp \left[ \int_t^0 g^n_s(w^\varepsilon(t-s, B_s + x)) dB_s - \frac{1}{2} \int_t^0 |g^n_s(w^\varepsilon(t-s, B_s + x))|^2 ds \right].$$
So by Lebesgue’s dominated convergence theorem,

\[ E \left[ N_{1}^{\varepsilon} \varphi'(B_{t} + x) \cdot I_{\{t < \tau\}} \right] \to E \left[ N_{1} \varphi'(B_{t} + x) \cdot I_{\{t < \tau\}} \right] \]

as \( n \to \infty \), and we may conclude that

\[ w(t, x) = E \left[ N_{1} \varphi'(B_{t} + x) \cdot I_{\{t < \tau\}} \right] \]

which completes the proof. \( \square \)

**Remark 5.** Let us supply some details we omitted in the proof in applying Nash’s theory and Aronson’s estimates to our situation, in proving that, we may extract a sequence \( \varepsilon_{n} \downarrow 0 \), so that \( w^{\varepsilon_{n}}(t, x) \to w(t, x) \), where \( w \) is the unique weak solution to the problem (15, 16). Let us use the same notations in the previous proof, but for simplicity \( b_{\varepsilon}(t, x) = g_{\varepsilon}(\partial_{t} u^{\varepsilon}(t, x)) \) for every \( \varepsilon > 0 \), and \( b_{0}(t, x) = k \text{sgn}(\partial_{t} u(t, x)) \). Then \( |b_{\varepsilon}(t, x)| \leq |k| \), a bound dependent of \( \varepsilon \). Then, according to Nash [15] and Aronson [1], the fundamental solution \( p_{\varepsilon}(s, x, t, y) \) to the parabolic equation

\[
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta - b_{\varepsilon}(t, x) \right) v = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}
\]

is jointly \( \alpha \)-Hölder continuous for some \( \alpha \) depending only on \(|k|\) (see page 328, Friedman [10] or Nash [15]), \( p_{\varepsilon}(s, x, t, y) \) satisfying a Gaussian lower and upper bounds uniformly in \( \varepsilon \geq 0 \), for \( 0 \leq s < t \leq T \), \( x, y \in \mathbb{R} \). This implies that \( p_{\varepsilon}(s, x, t, y) \) is \( \alpha \)-Hölder continuous in all its arguments, where \( \alpha \) and Hölder constant depend only on \(|k|\) but independent of \( \varepsilon > 0 \). According to Aronson [1], the unique weak solution \( w^{\varepsilon}(t, x) \) to (19, 20) (for all \( \varepsilon \geq 0 \)) has the representation

\[
w^{\varepsilon}(t, x) = \int_{\mathbb{R}} p_{0}(0, x, t, y) \varphi'(y)dy.
\]

Note also that \( \{p_{\varepsilon}(s, x, t, y)\} \) is a family of equi-continuous functions on any compact set of \( 0 \leq s < t \), \( x, y \in \mathbb{R} \). Of course \( p_{\varepsilon}(s, x, t, y) \) depends on the solution \( w^{\varepsilon} \), which is a necessary feature for non-linear PDEs, but this does not cause any difficulty for us. Hence, by extracting a sequence, we can assume that \( p_{\varepsilon}(s, x, t, y) \) converges on \( \{0 \leq s < t\} \times \mathbb{R}^{2} \) to \( p(s, x, t, y) \), uniformly on any its compact subset, and therefore \( w^{\varepsilon}(t, x) \) converges to \( w(t, x) \) on \((0, \infty) \times \mathbb{R} \), and uniformly on any its compact subset, so that \( p(s, x, t, y) = p_{0}(s, x, t, y) \) and \( w(t, x) = w^{0}(t, x) \).

**Corollary 6.** Suppose that \( \varphi \in C^{3}(\mathbb{R}) \) satisfies (H.1) and (H.2) with some constant \( c \), such that \( \varphi \) and \( \varphi' \) have at most polynomial growth, and suppose that \( (Y_{t}, Z_{t}) \) is the unique solution of BSDE:

\[
Y_{t} = \varphi(B_{T}) + \int_{t}^{T} k|Z_{s}|ds - \int_{t}^{T} Z_{s}dB_{s},
\]

where \( k \) is a real constant.

(1) If \( \varphi' \geq 0 \) and \( \varphi' \not\equiv 0 \) on \((c, \infty)\), then

\[
\text{sgn}(Z_{t}) = \text{sgn}(B_{t} - c), \quad t \geq 0.
\]

(2) If \( \varphi' \leq 0 \) and \( \varphi' \not\equiv 0 \) on \((c, \infty)\), then

\[
\text{sgn}(-Z_{t}) = \text{sgn}(B_{t} - c), \quad t \geq 0.
\]

**Proof.** The conclusions follows from Theorem 4 follow now immediately. \( \square \)
3 Explicit solutions for some BSDEs

Firstly, Theorem 4 allows us to work out the explicit solution of BSDE(25). To this end, we recall the joint distribution \( P(B_t \in dx, L_t^\ell \in dy) \) of \( B_t \) and its local time \( L_t^\ell \) with respect to \( \ell \) given by

\[
P(B_t \in dx, L_t^\ell \in dy) = \frac{1}{\sqrt{2\pi t}} (y+|x-\ell|+|\ell|) \exp \left\{ \frac{-(y+|x-\ell|+|\ell|)^2}{2t} \right\} 1_{\{y>0\}}dxdy
+ \frac{1}{\sqrt{2\pi t}} \left[ \exp \left\{ -\frac{x^2}{2t} \right\} - \exp \left\{ -\frac{(|x|+|\ell|)^2}{2t} \right\} \right] 1_{\{y=0\}}dxdy,
\]

(26)

see [3] for example.

**Theorem 7.** Suppose \( \phi \in C^1(\mathbb{R}) \) satisfying (H.1) and (H.2) such that \( \phi(B_T) \) and \( \phi'(B_T) \) are square integrable. Then the unique solution of BSDE (25) is given by

\[
Y_t = H(B_t), \quad Z_t = \partial_t H(B_t),
\]

(27)

where \( H \) is defined in the following.

(i) If \( \phi' \geq 0 \) and \( \phi' \neq 0 \) on \( (c, \infty) \), then

\[
H(h) = e^{-\frac{1}{2}h^2(T-t)} \left\{ \int_{\mathbb{R}} \int_{y \geq 0} \phi(x+h)e^{k|x-c+h|-k|c-h|-ky} P(B_{T-t} \in dx, L_{T-t}^{c-h} \in dy) \right\}.
\]

(ii) If \( \phi' \leq 0 \) and \( \phi' \neq 0 \) on \( (c, \infty) \), then

\[
H(h) = e^{-\frac{1}{2}h^2(T-t)} \left\{ \int_{\mathbb{R}} \int_{y \geq 0} \phi(x+h)e^{-k|x-c+h|+k|c-h|+ky} P(B_{T-t} \in dx, L_{T-t}^{c-h} \in dy) \right\}.
\]

(28)

**Proof.** (i) Since \( \phi' \geq 0 \) and \( \phi' \neq 0 \) on \( (c, \infty) \), by Theorem 4, \( \text{sgn}(Z_t) = \text{sgn}(B_t - c) \), BSDE (25) can be rewritten as a linear BSDE in \( Z \)

\[
Y_t = \phi(B_T) + k \int_t^T \text{sgn}(B_t - c)Z_sds - \int_t^T Z_sdB_s
\]

whose solution is given by

\[
Y_t = E \left[ \phi(B_T) \cdot e^{-\frac{1}{2}k^2(T-t) + kT \text{sgn}(B_T-c)dB_T} \bigg| F_t \right].
\]

(29)

By Tanaka’s Formula,

\[
\int_0^T \text{sgn}(B_s - c)dB_s = |B_T - c| - |c| - L_T^c.
\]

Combine with (29), we have

\[
Y_0 = E_P \left[ \phi(B_T)e^{-\frac{1}{2}k^2T + kT \text{sgn}(B_T-c)dB_T} \right]
= E_P \left[ \phi(B_T) \cdot e^{-\frac{1}{2}k^2T + k(B_T-c) - |c| - L_T^c} \right]
= e^{-\frac{1}{2}k^2T} \int_{\mathbb{R}} \int_{y \geq 0} \phi(x) \exp \left\{ k|x-c| - k|c-h| - ky \right\} P(B_T \in dx, L_T^{c-h} \in dy).
\]

(30)

and

\[
Y_t = e^{-\frac{1}{2}k^2(T-t)} \int_{\mathbb{R}} \int_{y \geq 0} \phi(x+h) \exp \left\{ k|x-c+h| - k|c-h| - ky \right\} P(B_{T-t} \in dx, L_{T-t}^{c-h} \in dy)|_{h=B_t}
= H(B_t).
\]

According to Theorem 4, \( Z_t = \partial_t H(B_t) \).

(ii) Similarly, if \( \phi' \leq 0 \) and \( \phi' \neq 0 \) on \( (c, \infty) \), by Theorem 4 we have \( \text{sgn}(Z_t) = -\text{sgn}(B_t - c) \). The rest can be proved in a similar manner as (i). The proof is completed. \( \square \)
Another application of Theorem 2 is to prove that the following different PDEs have the same solution under some assumptions on initial value $\phi$.

Let $v$ and $u$ be the solutions of the following initial value problems of parabolic equations

$$\begin{align*}
\partial_t v(t,x) &= \frac{1}{2} \partial^2_{xx} v(t,x) - k \cdot \text{sgn}(x-c) \partial_x v(t,x) \\
v(0,x) &= \phi(x),
\end{align*}$$

and

$$\begin{align*}
\partial_t u &= \frac{1}{2} \partial^2_{xx} u + \min_{|m| \leq k} (m \partial_x u) \\
u(0,x) &= \phi(x),
\end{align*}$$

respectively.

Shreve [25] show that for $\phi(x) = x^2$, then $v(t,x) = u(t,x)$ for $c = 0$. The following Corollary implies that both solutions may be same for all symmetric functions $\phi$ satisfying (H.1) and $\phi$ is increasing on $(c, \infty)$.

**Corollary 8.** Assume that $\phi$ satisfies (H.1), (H.2), moreover, $\phi' \geq 0$ on $(c, \infty)$ and $\phi' \not\equiv 0$ on $(c, \infty)$. Then PDE (31) and PDE (32) have the same solution, that is, $v(t,x) = u(t,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

**Proof.** It is obvious that PDE (32) is equivalent to the following PDE:

$$\begin{align*}
\partial_t u &= \frac{1}{2} \partial^2_{xx} u - k \cdot |\partial_x u| \\
u(0,x) &= \phi(x).
\end{align*}$$

As we have proven in Theorem 4, $Y_i = u(T-t, B_i)$, $Z_i = \partial_t u(T-t, B_i)$ is the unique solution pair of the following BSDE:

$$Y_i = \phi(B_T) - k \int_t^T |Z_s| ds - \int_t^T Z_s dB_s$$

By Theorem 4, when $\phi$ satisfies (H.1) and (H.2), $\phi' \not\equiv 0$ on $(c, \infty)$, $\text{sgn}(Z_i) = \text{sgn}(B_t - c)$, which means $\text{sgn}(\partial_t u(t,x)) = \text{sgn}(x-c)$. Thus the claim follows immediately. \qed

4 Applications

In this section, we give several examples to show how to get the explicit solution of BSDE for the cases where $\phi(x) = x^2$ and $\phi(x) = I_{|a \leq x \leq b|}$.

**Example 9.** The explicit solution $(Y_i, Z_i)$ of BSDE

$$Y_i = B_i^2 + k \int_t^T |Z_s| ds - \int_t^T Z_s dB_s,$$

is given by the following formulas:

$$\begin{align*}
Y_i &= \frac{1}{2k^2} \sqrt{T-t} \left| B_i \right| + k(T-t) + \frac{1}{k} \exp \left\{ - \frac{\left| B_i \right| + k(T-t)}{2(T-t)} \right\} \\
&\quad + \left\{ \left| B_i \right| + k(T-t) \right\} \left( \frac{\left| B_i \right| + k(T-t)}{\sqrt{T-t}} \right) \\
&\quad + e^{-k\left| B_i \right|} \left( \left| B_i \right| + T-t - \frac{1}{2k^2} \right) \Phi \left( \frac{-\left| B_i \right| - k(T-t)}{\sqrt{T-t}} \right)
\end{align*}$$

(34)
and

\[
Z_t = \sqrt{\frac{T-t}{2\pi}} \cdot \text{sgn}(B_t) \cdot \exp \left\{ -\frac{\|B_t\| + k(T-t)}{2(T-t)} \right\} \cdot \left\{ 1 + \frac{|B_t| + k(T-t) + \frac{1}{2}}{k} \cdot \left[ -\frac{|B_t| + k(T-t)}{(T-t)} \right] \right\}
\]

\[+ 2\text{sgn}(B_t) \cdot \left\{ |B_t| + k(T-t) \right\} \cdot \Phi \left( \frac{|B_t| + k(T-t)}{\sqrt{T-t}} \right)
\]

\[+ \sqrt{2\pi(T-t)} \cdot \exp \left\{ -\frac{|B_t| + k(T-t)}{2(T-t)} \right\}
\]

\[+ e^{-2k|B_t|} \Phi \left( -\frac{|B_t| - k(T-t)}{\sqrt{T-t}} \right) \cdot \left[ -2k \left( |B_t| + T-t - \frac{1}{2k^2} \right) + 1 \right]
\]

\[- e^{-2k|B_t|} \left( |B_t| + T-t - \frac{1}{2k^2} \right) \cdot \text{sgn}(B_t) \exp \left\{ -\frac{|B_t| - k(T-t)}{2(T-t)} \right\}.
\]

(35)

Here and in the sequel, \( \Phi \) is the standard normal cdf.

**Proof.** By Theorem 7,

\[
Y_t = e^{-\frac{1}{4}k^2(T-t)} \left\{ \int_{\mathbb{R}} \int_{y \geq 0} (x+h)^2 e^{k|x+h| - k|y|} \Phi(B_{t+h} \in dx, L_T^{h} \in dy) \right\} \bigg|_{h=B_t}
\]

By an elementary calculation, we have

\[
Y_t = \frac{1}{2k^2} + \sqrt{\frac{T-t}{2\pi}} \left\{ |B_t| + k(T-t) + \frac{1}{k} \right\} \exp \left\{ -\frac{|B_t| + k(T-t)}{2(T-t)} \right\}
\]

\[+ \left\{ \left( |B_t| + k(T-t) \right)^2 - (T-t) - \frac{1}{2k^2} \right\} \Phi \left( \frac{|B_t| + k(T-t)}{\sqrt{T-t}} \right)
\]

\[+ e^{-2k|B_t|} \left( |B_t| + T-t - \frac{1}{2k^2} \right) \Phi \left( -\frac{|B_t| - k(T-t)}{\sqrt{T-t}} \right)
\]

(36)

Therefore, by Theorem 4, we have \( Z_t = \partial_y H(B_t) \), and therefore we obtain \( Z_t \) as equation (35). The proof is completed. \( \square \)

It is interesting that our result may be also used to get the solution of the following PDE, its application can be found in [12] for \( k = -1 \).

**Example 10.** The unique weak solution to the initial problem of the parabolic equation

\[
\left\{ \begin{array}{l}
\partial_t u(t,x) = \frac{1}{2} \partial^2_{xx} u(t,x) + k \cdot \text{sgn}(x) \partial_x u(t,x), \\
\lim_{t \to 0^+} u(t,x) = x^2,
\end{array} \right.
\]

is given as the following

\[
u(t,x) = \frac{1}{2k^2} + \sqrt{\frac{t}{2\pi}} \left( |x| + kt + \frac{1}{k} \right) \exp \left\{ -\frac{(|x| + kt)^2}{2t} \right\}
\]

\[+ \left\{ (|x| + kt)^2 + t - \frac{1}{2k^2} \right\} \Phi \left( \frac{|x| + kt}{\sqrt{t}} \right)
\]

\[+ e^{-2k|x|} (|x| + T-t - \frac{1}{2k^2}) \Phi \left( -\frac{|x| - kt}{\sqrt{t}} \right).
\]

The explicit solution agrees with the result of [12] when \( k = -1 \).
Proof. By Theorem 4, \( Y_t = u(T-t, B_t) \), \( Z_t = \partial_x u(T-t, B_t) \) are the unique solution pair to the BSDE
\[
Y_t = B_T^2 + k \int_t^T |Z_s| ds - \int_t^T Z_s dB_s.
\]
By Theorem 2, \( \text{sgn}(B_t) = \text{sgn}(Z_t) \) for \( \varphi(x) = x^2 \). Hence the expression for \( u(t,x) \) follows from (36) immediately.

Example 11. We now calculate the solution of the following BSDE:
\[
Y_t = I_{[a \leq B_T \leq b]} + \int_t^T k|Z_s| ds - \int_t^T Z_s dB_s. \tag{37}
\]

(1) For any \( a, b \in (-\infty, \infty) \), set \( c = \frac{a+b}{2} \),
\[
Y_t = \Phi \left( \frac{|B_t - c| - k(T-t) - \frac{b-a}{2}}{\sqrt{T-t}} \right) - e^{-k(b-a)} \Phi \left( \frac{|B_t - c| - k(T-t) + \frac{b-a}{2}}{\sqrt{T-t}} \right), \tag{38}
\]
and
\[
Z_t = -\text{sgn}(B_t - c) \left\{ e^{-\frac{|B_t - c| - k(T-t) - \frac{b-a}{2}|}{2(T-t)}} - e^{-k(b-a)} e^{-\frac{|B_t - c| - k(T-t) + \frac{b-a}{2}|}{2(T-t)}} \right\}. \tag{39}
\]

(2) If \( a = -\infty \), then for any \( b < \infty \),
\[
Y_t = \Phi \left( \frac{B_t - k(T-t) - b}{\sqrt{T-t}} \right), \quad Z_t = -\frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{|B_t - k(T-t) - b|^2}{2(T-t)}}.
\]

(3) If \( b = +\infty \), then for any \( a > -\infty \), we have
\[
Y_t = \Phi \left( \frac{B_t + k(T-t) - a}{\sqrt{T-t}} \right), \quad Z_t = \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{|B_t + k(T-t) - a|^2}{2(T-t)}}.
\]

Proof. (1) For any \( \varepsilon > 0 \), set
\[
\varphi_\varepsilon(x) = E[I_{[a,b]}(x + \sqrt{\varepsilon} \xi)] = \int_{-\infty}^{\infty} I_{[a,b]}(v) \frac{1}{\sqrt{2\pi\varepsilon}} \exp \left[ -\frac{(v-x)^2}{2\varepsilon} \right] dv
\]
where \( \xi \) is a standard normal distribution under probability measure \( P \). Then \( \varphi_\varepsilon \in C^\infty(\mathbb{R}) \) and \( \varphi_\varepsilon(x) \rightarrow I_{[a,b]}(x) \) as \( \varepsilon \rightarrow 0 \).

Consider the following BSDE
\[
Y_t^\varepsilon = \varphi_\varepsilon(B_T) + \int_t^T k|Z_s| ds - \int_t^T Z_s dB_s.
\]
By Theorem 7,
\[
Y_t^\varepsilon = e^{-\frac{1}{2}k^2(T-t)} \left\{ \int_{\mathbb{R}} \int_{y \geq 0} \varphi_\varepsilon(x+h) e^{\frac{|x-c+h| - k(c-h) - ky}{2}} P(B_{T-t} \in dx, L_{T-t}^h \in dy) \right\} \bigg|_{h=B_t}.
\]

where
\[
f(c,h,z) = \int_0^\infty \exp \{-k|x-c+h| + k|c-h| + ky\} f_1(c,h,x,y) dy + f_2(c,h,x),
\]

12
By solving the linear BSDE, we obtain (2).

In which the blue line is $B_t$.

For the study of the sign of $Z_t$,

Remark 3. From which we can see

where $f_1(c,h,x,y) = \frac{y + |x - (c - h)| + |c - h|}{\sqrt{2\pi(T-t)^3}} \exp \left[ -\frac{(y + |x - (c - h)| + |c - h|)^2}{2(T-t)} \right]$, and

\[ f_2(c,h,x) = \frac{e^{-k|x-c+h|+|c-h|}}{\sqrt{2\pi(T-t)}} \left\{ e^{\frac{-x^2}{2(T-t)}} - \exp \left[ -\frac{(|x - (c-h)| + |c-h|)^2}{2(T-t)} \right] \right\} \cdot \]

Now we prove that

\[ H_{\epsilon}(h) \rightarrow H(h) \quad \text{as} \quad \epsilon \rightarrow 0, \]

where

\[ H(h) = \Phi \left( -\frac{|h - c| - k(T - t) - \frac{b-a}{2}}{\sqrt{T-t}} \right) - e^{-k(b-a)} \Phi \left( -\frac{|h - c| - k(T - t) + \frac{b-a}{2}}{\sqrt{T-t}} \right). \]

Actually we have

\[ H(h) = e^{-\frac{x^2}{2(T-t)}} \left\{ \int_{\mathbb{R}} \int_{t \geq 0} I_{(a,b)}(x+h) e^{k|x-c+h| - k|c-h| - ky} P(B_{T-t} \in dx, L_{T-t}^{c-h} \in dy) \right\} \]

\[ = e^{-\frac{x^2}{2(T-t)}} \left[ \int_{-\infty}^{\infty} I_{(a,b)}(x+h) \cdot f(c,h,x)dx \right]. \]

By Lebesgue’s Dominated convergence theorem, we have $H_{\epsilon}(h)$ converges to $H(h)$ as $\epsilon \rightarrow 0$, which means $H_{\epsilon}(B_t)$ converges to $H(B_t)$ almost surely. Therefore, we have

\[ Y_t = H(B_t) = \Phi \left( -\frac{|B_t - c| - k(T - t) - \frac{b-a}{2}}{\sqrt{T-t}} \right) - e^{-k(b-a)} \Phi \left( -\frac{|B_t - c| - k(T - t) + \frac{b-a}{2}}{\sqrt{T-t}} \right). \]

By Corollary 4.1 in El Karoui, Peng and Quenez [18], we have $Z_t = \partial_0 H(B_t)$. So we get $Z_t$ given by (39). From which we can see

\[ \text{sgn}(Z_t) = -\text{sgn}(B_t - c), \]

which means Theorem 2 also holds for indicator function.

We now prove (2). In fact, since $a = -\infty$, then for any $b \in \mathbb{R}$, $c = \frac{a+b}{2} = -\infty$. By Theorem 4, $\text{sgn}(Z_t) = -\text{sgn}(B_t - c) = -1$, which implies that $Z_t < 0$ for $t \in [0, T]$, and BSDE(37) is a linear BSDE:

\[ Y_t = I_{[B_t \leq h]} - \int_t^T kZ_s ds - \int_t^T Z_s dB_s. \]

By solving the linear BSDE, we obtain (2).

Similarly, we may deduce (3), and we omit the details. □

Remark 12. The BSDE (5) is associated with the parabolic equation

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + g(t,u,\nabla u), \quad u(0,x) = \varphi(x). \]  

(40)

For the study of the sign of $Z_t$, it is actually equivalent to the study the nodal set of $u$. It has a connection to the work of Qian and Xu (2018). For more details, see [24].

We plot one sample path of Brownian motion $B_t$ and the solution $Z_t$ of Example 11 in the Figure 1, in which the blue line is $B_t$ and the red is $Z_t$. We can see the relationship of the sign between $B_t - c$ and $Z_t$ intuitively in this figure.
5 Robust prices in incomplete markets

The Black-Scholes model studied by Black and Scholes (1973), Merton (1973,1991) is the most celebrated example of option pricing and hedging in a complete market using no-arbitrage theory and martingale methods. According to this theory, when a stock obeys the geometric Brownian motion

\[ dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = 1, \quad (41) \]

there exists a unique risk neutral martingale measure \( Q \) such that the price of the contingent claim \( \xi \) at time \( T \) was given by \( E_Q[\xi e^{-rT}] \), where

\[ \frac{dQ}{dP} \bigg|_{\mathcal{F}_T} = e^{\int_0^T (\frac{\mu - r}{\sigma^2}) dB_s - \frac{1}{2} \int_0^T \frac{\theta_s^2}{\sigma^2} ds}, \quad (42) \]

where \( r \) is the interest rate of a bond. Therefore, for \( \xi = I_{(a \leq S_T \leq b)} \), the price of the contingent claim \( \xi \) is given by

\[ E_Q(\xi e^{-rT}) = e^{-rT} \left[ \Phi \left( \frac{\ln b - (2\mu - r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \right) - \Phi \left( \frac{\ln a - (2\mu - r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \right) \right]. \quad (43) \]

In an incomplete market, the incompleteness of the market usually gives rise to infinitely many martingale measures, therefore upper and lower pricing was studied by El Karoui and Quenez (1995) [8], El Karoui and Peng (1997) [9]. They use the min-max pricing to show that, the pricing of an insurance or contingent claim equals the maximal (minimal) expectations with respect to a set of martingale measures. Chen and Epstein (2002) studied the ambiguity pricing under a set of special measures \( \mathcal{P} \), where

\[ \mathcal{P} = \left\{ Q : \frac{dQ}{dP} \bigg|_{\mathcal{F}_T} = \exp \left[ \int_0^T \theta_s dB_s - \frac{1}{2} \int_0^T \theta_s^2 ds \right], \quad \sup_{s \in [0,T]} |\theta_s| \leq k \right\}. \quad (44) \]

Set

\[ y_t = \text{ess inf}_{Q \in \mathcal{P}} E_Q[\xi | \mathcal{F}_t] \quad \text{and} \quad Y_t = \text{ess sup}_{Q \in \mathcal{P}} E_Q[\xi | \mathcal{F}_t]. \quad (45) \]

It is known that \( y_t \) and \( Y_t \) are respectively the minimum and maximum price of \( \xi \) in an incomplete market. Chen and Epstein (2002)[5] have shown that there exists an adapted \( z_t \) such that \( Y_t \) and \( y_t \) have the following representations:

\[ y_t = \xi - k \int_t^T |z_s| ds - \int_t^T z_s dB_s \quad (46) \]
and

\[ Y_t = \xi + k \int_t^T |z_s| ds - \int_t^T z_s dB_s. \]  \hspace{1cm} (47)

By using our results in the previous sections, we may give the explicit representation of the wealth \( Y_t \) when the stock price \( S_t \) obeys the geometric Brown motion

\[ S_t = \exp \left( (\mu - \frac{1}{2} \sigma^2) t + \sigma B_t \right) \]  \hspace{1cm} (48)

and \( \xi = I_{(a \leq S_T \leq b)} \).

In fact, when \( \xi = I_{(a \leq S_T \leq b)} \), that is,

\[ \xi = I_{\left\{ \ln a - \left( \mu - \frac{0.5 \sigma^2 T}{\sigma} \right) \leq B_T \leq \ln b - \left( \mu - \frac{0.5 \sigma^2 T}{\sigma} \right) \right\}}, \]  \hspace{1cm} (49)

According to the calculation in Example 11, with \( c = \frac{\ln(ab)}{2\sigma} - \frac{(\mu-0.5\sigma^2)T}{\sigma} \), we have the upper pricing which is given by

\[ Y_t = \Phi \left( -\frac{|B_t - c| - k(T-t) - \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}} \right) - e^{-k \frac{\ln(b/a)}{\sigma}} \Phi \left( -\frac{|B_t - c| - k(T-t) + \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}} \right) \]  \hspace{1cm} (50)

and the lower pricing is given as

\[ y_t = \Phi \left( -\frac{|B_t - c| + k(T-t) - \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}} \right) - e^{k \frac{\ln(b/a)}{\sigma}} \Phi \left( -\frac{|B_t - c| + k(T-t) + \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}} \right). \]  \hspace{1cm} (51)

In particular, let \( t = 0 \), then

\[ Y_0 = \Phi \left( -\frac{|c| - kT - \frac{\ln(b/a)}{2\sigma}}{\sqrt{T}} \right) - e^{-k \frac{\ln(b/a)}{\sigma}} \Phi \left( -\frac{|c| - kT + \frac{\ln(b/a)}{2\sigma}}{\sqrt{T}} \right) \]  \hspace{1cm} (52)

and

\[ y_0 = \Phi \left( -\frac{|c| + kT - \frac{\ln(b/a)}{2\sigma}}{\sqrt{T}} \right) - e^{k \frac{\ln(b/a)}{\sigma}} \Phi \left( -\frac{|c| + kT + \frac{\ln(b/a)}{2\sigma}}{\sqrt{T}} \right). \]  \hspace{1cm} (53)

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