Non-maximal $\theta_{23}$, large $\theta_{13}$ and tri-bimaximal $\theta_{12}$ via quark-lepton complementarity at next-to-leading order

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Abstract – We present analytical formulae for the neutrino mixing angles at the next-to-leading order in the quark-lepton complementarity, and show that higher-order corrections are important to explain the observed pattern of neutrino mixing. In particular, the next-to-leading-order corrections 1) lead to a deviation of $\theta_{23}$ from maximal mixing, 2) reduce the predicted value of $\sin^2 2\theta_{13}$ by 9.8%, and 3) provide the same value of $\sin^2 \theta_{12}$ as that of the tri-bimaximal mixing.

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Introduction and motivation. – The main recent developments on neutrino mixing [1–5] are related to the relatively large value of $\theta_{13}$, which was measured by recent experiments [6–10], and indications of significant deviation of $\theta_{23}$ from maximal mixing [11,12]. In particular, non-maximal $\theta_{23}$ is strongly indicated by recent data, but the sign of $\theta_{23} - \pi/4$ is not yet determined. Results of global analysis are given in refs. [13–15].

To explain the observed pattern of neutrino mixing, the quark-lepton complementarity (QLC) has been widely investigated in the literature [16–42]. In particular, much attention has been paid to the class of models, $U_{PMNS} = V_{CKM}^\dagger V_M$, which can be obtained in grand unified theories (GUTs). The correlation matrix $V_M$ is simply defined by the product of the CKM and PMNS mixing matrices. In general $V_M$ is not determined by theories, because there is no relation between the Dirac and the Majorana mass operators [26]. In this class of models, the observed two large mixing angles $\theta_{12}$ and $\theta_{23}$ indicate that $V_M$ has two large mixing angles because all the mixing angles in $V_{CKM}$ are small. As the simplest possibility, for example, we can take $V_M$ being the bimaximal mixing matrix $V_{bn}$, $U_{PMNS} = V_{CKM}^\dagger V_{bn}$. (1)

In this paper we only consider this minimal model. In ref. [26], expanding $V_{CKM}$ to $O(\lambda^4)$ (where $\lambda \equiv \sin \theta_C \approx 0.2253$), it is found that

\[
\sin^2 2\theta_{13} = 2\lambda^2 + O(\lambda^4) = 0.102 + O(\lambda^4),
\]

\[
\sin^2 2\theta_{23} = 1 + O(\lambda^4),
\]

\[
\sin^2 2\theta_{12} = 1 - 2\lambda^2 + O(\lambda^4) = 0.898 + O(\lambda^4).
\]

These correspond to $\theta_{13} \simeq \theta_C/\sqrt{2} \simeq 9^\circ$, $\theta_{23} \simeq 45^\circ$ and $\theta_{12} \simeq 36^\circ$, which are consistent with experimental results at leading-order approximation. In particular, it is interesting that the predicted value of 1-3 mixing, $\theta_{13} \simeq 9^\circ$, has been confirmed by recent experiments [6–10].

However, the next-to-leading-order corrections, the $O(\lambda^4)$ terms in eqs. (2)–(4), should be calculated, because:

– the $O(\lambda^4)$ corrections may not be enough small with respect to the recent experimental errors; particularly, the Daya Bay Collaboration reported the precise value at $1\sigma$ [9],

\[
\sin^2 2\theta_{13} = 0.092 \pm 0.016{\text{(stat)}} \pm 0.005{\text{(syst)}},
\]

where the magnitude of systematic errors is the same order of that of $\lambda^4$;

– the same value of 1-3 mixing, $\theta_{13} \simeq 9^\circ$, can be obtained in various models with different schemes: flavor symmetries, texture, ansätze etc. [12]; to distinguish models, we need a more precise prediction;

– a deviation of $\theta_{23}$ from maximal mixing is the $O(\lambda^4)$ correction in eq. (3); to determine the magnitude of deviation, the $O(\lambda^4)$ corrections should be calculated;

– in the earlier works, eq. (1) was analyzed numerically; analytical formulae of higher-order terms are usually neglected; therefore, it is very unclear that which parameter is relevant at each order.

Motivated by these points, in this paper we perform analytical calculations of the $O(\lambda^4)$ corrections. We show
that the next-to-leading-order corrections

- reduce the value of $\sin^2 2\theta_{13}$ from 0.102 to 0.092, which is completely consistent with the result of Daya Bay Collaboration of eq. (5);
- lead to a deviation of $\theta_{23}$ from maximal mixing; the predicted value is $\sin^2 \theta_{23} = 0.446 + \mathcal{O}(\lambda^6)$, which corresponds to $\theta_{23} \simeq 41.9^\circ$;
- provide the value of $\sin^2 \theta_{12}$ of 0.335, which is very close to the predicted value of tri-bimaximal (TBM) mixing of 0.333.

**Neutrino mixing angles.**

1-3 mixing $\theta_{13}$. The recent experimental results on $\theta_{13}$ from two accelerator experiments, T2K [6] and MINOS [7], and from three reactor experiments, Double-Chooz [8], Daya Bay [9] and RENO [10] were very important developments in neutrino physics. The global fit value of $\theta_{13}$ is rather large, $\theta_{13} \simeq 9^\circ$ for the normal and inverted mass hierarchy, and $\theta_{13} = 0$ is now excluded at more than 10$\sigma$.

The 1-3 mixing angle $\theta_{13}$ is given by

$$\sin^2 2\theta_{13} = 4|U_{e3}|^2 (1 - |U_{e3}|^2). \quad (6)$$

Equation (6) indicates that $\theta_{13}$ is determined only by $U_{e3}$. In eq. (6), $\theta_{13}$ is a parameter defined by the standard parametrization, and the $U_{e3}$ is the matrix element of any parametrization of $U_{PMNS}$ [43].

Using eq. (A.28) in the appendix, we find that

$$\sin^2 2\theta_{13} = 2\lambda^2 - \{1 + 4A(1 - \overline{\theta})\} \lambda^4 + \mathcal{O}(\lambda^6),$$

$$= 0.1016 - 0.0098 + \mathcal{O}(\lambda^6),$$

$$= 0.092 + \mathcal{O}(\lambda^6), \quad (7)$$

where we have used the best-fit values of $\lambda$, $A$ and $\overline{\theta}$ of eq. (A.14). It was confirmed that varying values in the error range in eq. (A.14) are almost negligible. Equation (7) corresponds to $\theta_{13} \simeq 8.8^\circ$ (an another solution of $\theta_{13} \simeq 81.2^\circ$ is excluded, because $\sin^2 \theta_{13} = |U_{e3}|^2 < 0.5$).

The key points of eq. (7) are:

- $\sin^2 2\theta_{13}$ is determined only by one parameter $\lambda$ at leading order, and three parameters $\lambda$, $A$ and $\overline{\theta}$ are relevant at next-to-leading order; $\overline{\theta}$ is irrelevant up to $\mathcal{O}(\lambda^6)$;
- the next-to-leading-order corrections reduce the value of $\sin^2 2\theta_{13}$ from 0.102 to 0.092, which is completely consistent with the result of Daya Bay Collaboration of eq. (5); thus, the value of $\sin^2 2\theta_{13}$ becomes smaller by 9.8% due to $\mathcal{O}(\lambda^4)$ correction terms;
- the magnitude of the $\mathcal{O}(\lambda^4)$ correction terms is approximately twice larger than that of the systematic errors of Daya Bay Collaboration, and therefore the $\mathcal{O}(\lambda^4)$ corrections cannot be negligible.

![Fig. 1: (Colour on-line) Determination of the 1-3 mixing. Shown are the results from T2K [6], MINOS [7], Double Chooz [8], Daya Bay [9], RENO [10] and global fits of Forero et al. [13], Fogli et al. [14] and Gonzalez-Garcia et al. [15] for the normal hierarchy case (in the inverse hierarchy case the values do not differ by much). The QLC predictions are shown at leading and next-to-leading order.](image)

In fig. 1, we summarize the theoretical and experimental values of $\sin^2 2\theta_{13}$. The results from T2K [6] and MINOS [7] are shown for $\sin^2 2\theta_{23} = 0.4$ and $\delta = \pi$. For the QLC predictions, eq. (2) is shown for $\sin^2 2\theta_{13} = 0.102 \pm 5\times 10^{-4}$ (leading order), and eq. (7) is shown for $\sin^2 2\theta_{13} = 0.092 \pm 5\times 10^{-4}$ (next-to-leading order), which correspond to 0.089 < $\sin^2 2\theta_{13} < 0.114$ and 0.091 < $\sin^2 2\theta_{13} < 0.093$, respectively. Figure 1 shows that the QLC prediction at next-to-leading order is consistent with the results of five experiments [6-10] and three global fits [13-15] at the 1$\sigma$ level.

2-3 mixing $\theta_{23}$. The recent global analysis indicates that there is a solid deviation of $\theta_{23}$ from maximal mixing [11,12]. At present, the sign of $\theta_{23} - \pi/4$ is not yet determined [13-15].

Equation (3) shows that the 2-3 mixing is maximal at leading order. Equation (3) also shows that a deviation from maximal mixing is not $\mathcal{O}(\lambda^2)$ but the $\mathcal{O}(\lambda^4)$ effect. Therefore, a deviation is not so large.

The 2-3 mixing angle $\theta_{23}$ is given by

$$\sin^2 2\theta_{23} = \frac{4|U_{\mu3}|^2 |U_{\tau3}|^2}{(1 - |U_{e3}|^2)^2}. \quad (8)$$

Equation (8) shows that $\theta_{23}$ is determined by three matrix elements, $U_{\mu3}$, $U_{\tau3}$ and $U_{e3}$. Of these elements, two are independent because of the unitarity condition $|U_{e3}|^2 + |U_{\mu3}|^2 + |U_{\tau3}|^2 = 1$.  

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From eqs. (A.31), (A.34) and (A.36), we find that
\[
\sin^2 2\theta_{23} = 1 - 4 \left( \frac{1}{4} + A \right)^2 \lambda^4 + \mathcal{O}(\lambda^6),
\]
\[
= 1 - 0.0116 + \mathcal{O}(\lambda^6),
\]
\[
\lambda^4 \text{ term } = 0.988 + \mathcal{O}(\lambda^6).
\] (9)

Equation (9) indicates that the value of \(\sin^2 2\theta_{23}\) is determined only by two parameters, \(\lambda\) and \(A\) at next-to-leading order. The \(\mathbf{P}\) and \(\mathbf{\bar{P}}\) are irrelevant up to \(\mathcal{O}(\lambda^6)\).

To determine the sign of \(\theta_{23} - \pi/4\), we calculate \(\sin^2 \theta_{23} = |U_{e3}|^2/(1 - |U_{e3}|^2)\). From (A.35), we find that
\[
\sin^2 \theta_{23} = \frac{1}{2} - \left( \frac{1}{4} + A \right) \lambda^2 - \frac{1}{2} \left( \frac{1}{4} + A\mathbf{\bar{P}} \right) \lambda^4 + \mathcal{O}(\lambda^6),
\]
\[
= \frac{1}{2} - 0.0539 - 0.0005 + \mathcal{O}(\lambda^6),
\]
\[
\lambda^6 \text{ term } = 0.446 + \mathcal{O}(\lambda^6).
\] (10)

Equation (10) indicates \(\theta_{23} < \pi/4\). It is easy to confirm that eqs. (9) and (10) are consistent. The predicted value of \(\theta_{23}\) is \(\theta_{23} \simeq 41.9^\circ\).

In fig. 2, we summarize the values of \(\sin^2 \theta_{23}\). The results of global analysis of Forero et al. [13] and Fogli et al. [14] are shown for the normal (NH) and inverted (IH) hierarchy cases. For the QLC predictions, eq. (3) is shown for \(1 - 5\lambda^2 < \sin^2 2\theta_{23} < 1\) (leading order), and eq. (9) is shown for \(\sin^2 2\theta_{23} = 0.988 \pm 5\lambda^6\) (next-to-leading order), which correspond to \(0.443 < \sin^2 \theta_{23} < 0.500\) and \(0.444 < \sin^2 \theta_{23} < 0.447\), respectively. Figure 2 shows that the QLC prediction at next-to-leading order is consistent with the global fits [13–15] for \(\theta_{23} < \pi/4\) at the 2\(\sigma\) level.

1-2 mixing \(\theta_{12}\). It has been known that a deviation of \(\theta_{12}\) from maximal mixing is large [1–5,44]. This can be naturally obtained in the model of eq. (1): eq. (4) shows that a deviation of \(\theta_{12}\) from maximal mixing is not \(\mathcal{O}(\lambda^2)\) but the \(\mathcal{O}(\lambda^4)\) effect, and therefore a large deviation is obtained.

The 1-2 mixing angle is given by
\[
\sin^2 2\theta_{12} = \frac{4|U_{e1}|^2|U_{e2}|^2}{(1 - |U_{e3}|^2)^2}.
\] (11)

Using eqs. (A.26), (A.27) and (A.37), we find that
\[
\sin^2 2\theta_{12} = 1 - 2\lambda^2 - 4A(1 - \mathbf{\bar{P}}) \lambda^4 + \mathcal{O}(\lambda^6),
\]
\[
= 1 - 0.0106 - 0.0073 + \mathcal{O}(\lambda^6),
\]
\[
\lambda^4 \text{ term } = 0.891 + \mathcal{O}(\lambda^6).
\] (12)

Equation (12) shows that the value of \(\sin^2 2\theta_{12}\) is determined by three parameters, \(\lambda\), \(A\) and \(\mathbf{\bar{P}}\) to \(\mathcal{O}(\lambda^6)\). The \(\mathbf{\bar{P}}\) is irrelevant up to \(\mathcal{O}(\lambda^6)\).

Equation (12) corresponds to \(\theta_{12} \simeq 35.4^\circ\) (an another solution of \(\theta_{12} \simeq 54.6^\circ\) is excluded, because \(\sin^2 \theta_{12} = [U_{e2}|^2/(1 - |U_{e3}|^2)] < 0.5\). The \(\sin^2 \theta_{12}\) written in terms of \(\lambda, A\) and \(\mathbf{\bar{P}}\) is given in the appendix). It is interesting that the value of eq. (12) is very close to the tri-bimaximal (TBM) value of \(\sin^2 2\theta_{12} = 0.889\) [45,46].

In fig. 3, we summarize the values of \(\sin^2 \theta_{12}\). In addition to the results of global analysis [13–15], the predicted TBM value of \(1/3\) [45,46], two proposed golden ratio (GR) values of 0.276(GR1 [47–50]) and 0.345(GR2 [50–52]) are shown. For the QLC predictions, eq. (4) is shown for \(\sin^2 2\theta_{12} = 0.898 \pm 5A\lambda^4\) (leading order), and eq. (12) is shown for \(\sin^2 2\theta_{12} = 0.891 \pm 5\lambda^6\) (next-to-leading order), which correspond to \(0.331 < \sin^2 \theta_{12} < 0.351\) and \(0.334 < \sin^2 \theta_{12} < 0.336\), respectively. Figure 3 shows that the QLC prediction at next-to-leading order is consistent with the global fits at the 2\(\sigma\) level.

Summary and conclusions. – In this paper we have presented analytical formulae of neutrino mixing angles at the next-to-leading order in the framework of eq. (1). It has been shown that higher-order corrections were important to explain the observed pattern of neutrino mixing. Some conclusions obtained in this paper are given below.

\(\theta_{13}\): \(\sin^2 2\theta_{13}\) is determined by three parameters \(\lambda\), \(A\) and \(\mathbf{\bar{P}}\) up to \(\mathcal{O}(\lambda^6)\). The \(\mathbf{\bar{P}}\) is irrelevant up to \(\mathcal{O}(\lambda^6)\). The \(\mathcal{O}(\lambda^4)\) corrections reduce the value of \(\sin^2 2\theta_{13}\) from 0.102 to 0.092, which is consistent with five experiments and three global fits at the 1\(\sigma\) level. A summary is shown in fig. 1.
In this paper we have shown that eq. (1) is consistent with experimental results with high precision. If it will be confirmed by experiments that some other corrections from eq. (1) are very small or negligible, it may indicate that there exists an unknown theoretical mechanism behind eq. (1). Therefore, it is very interesting to test eq. (1) by near future experiments.

Appendix: the matrix elements of the PMNS matrix. – In this appendix we present the PMNS matrix elements $U_{ai}(\alpha = e, \mu, \tau, i = 1, 2, 3)$. We use the Wolfenstein parametrization [53,54] of $V_{CKM}$.

The $V_{CKM}$ to $O(\lambda^4)$ has been widely used in the literature, however, $V_{CKM}$ to $O(\lambda^6)$ is necessary for calculations at next-to-leading order. We can write $V_{CKM}$ to $O(\lambda^6)$ in terms of the Wolfenstein parameters $\lambda$, $A$, $\overline{\rho}$ and $\overline{\eta}$,

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \quad (A.1)$$

$$V_{ud} = 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{8} + O(\lambda^6), \quad (A.2)$$
$$V_{us} = A + O(\lambda^6), \quad (A.3)$$
$$V_{ub} = A\lambda^3 \left(1 + \frac{\lambda^2}{2}\right) (\overline{\rho} - \overline{\eta}) + O(\lambda^6), \quad (A.4)$$
$$V_{cd} = -\lambda + A^2\lambda^5 \left(\frac{1}{2} - \overline{\rho} - \overline{\eta}\right) + O(\lambda^6), \quad (A.5)$$
$$V_{cs} = 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{8} (1 + 4A^2) + O(\lambda^6), \quad (A.6)$$
$$V_{cb} = A\lambda^2 + O(\lambda^6), \quad (A.7)$$
$$V_{td} = A\lambda^3 (1 - \overline{\rho} - \overline{\eta}) + O(\lambda^6), \quad (A.8)$$
$$V_{ts} = -A\lambda^2 + A\lambda^4 \left(\frac{1}{2} - \overline{\rho} - \overline{\eta}\right) + O(\lambda^6), \quad (A.9)$$
$$V_{tb} = 1 - \frac{A^2\lambda^4}{2} + O(\lambda^6), \quad (A.10)$$

where the Wolfenstein parameters $\lambda$, $A$, $\overline{\rho}$ and $\overline{\eta}$ are defined by

$$s_{12} = \lambda, \quad (A.11)$$
$$s_{23} = A\lambda^2, \quad (A.12)$$
$$s_{13} e^{i\delta} = \frac{A\lambda^3 (\overline{\rho} + \overline{\eta}) \sqrt{1 - A^2\lambda^4}}{\sqrt{1 - \lambda^2 [1 - A^2\lambda^4 (\overline{\rho} + \overline{\eta})]}}. \quad (A.13)$$

The quark mixing angles $s_{ij} = \sin \theta_{ij}$ and the Kobayashi-Maskawa $CP$ phase $\delta$ are defined by the standard parametrization. The CKM matrix written in terms of $\lambda$, $A$, $\overline{\rho}$ and $\overline{\eta}$ is unitary to all orders in $\lambda$. The values of the Wolfenstein parameters are given by [54]

$$\lambda = 0.22535 \pm 0.00065, \quad A = 0.811^{+0.022}_{-0.012},$$
$$\overline{\rho} = 0.131^{+0.026}_{-0.013}, \quad \overline{\eta} = 0.345^{+0.013}_{-0.014}. \quad (A.14)$$

The PMNS mixing matrix in the model of eq. (1) is

$$U_{PMNS} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} = V_{CKM}^\dagger V_{bm}, \quad (A.15)$$

where the bimaximal mixing matrix $V_{bm}$ is defined by

$$V_{bm} = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \quad (A.16)$$

We summarize the PMNS matrix elements $U_{ai}(\alpha = e, \mu, \tau, i = 1, 2, 3)$ and the squared $|U_{ai}|^2$ to $O(\lambda^6)$:

see eqs. (A.17)–(A.34) on the next page
\[ \begin{align*}
U_{e1} &= \frac{1}{\sqrt{2}} + \frac{\lambda}{2} - \frac{\lambda^2}{2\sqrt{2}} + \frac{A\lambda^3}{2}(1 - \bar{\nu} + \bar{\mu}) - \frac{\lambda^4}{8\sqrt{2}} - \frac{A^2\lambda^6}{2} \left( \frac{1}{2} - \bar{\nu} + \bar{\mu} \right) + O(\lambda^6), \\
U_{e2} &= \frac{1}{\sqrt{2}} - \frac{\lambda}{2} - \frac{\lambda^2}{2\sqrt{2}} - \frac{A\lambda^3}{2}(1 - \bar{\nu} + \bar{\mu}) - \frac{\lambda^4}{8\sqrt{2}} + \frac{A^2\lambda^6}{2} \left( \frac{1}{2} - \bar{\nu} + \bar{\mu} \right) + O(\lambda^6), \\
U_{e3} &= -\frac{\lambda}{\sqrt{2}} + \frac{A\lambda^3}{\sqrt{2}}(1 - \bar{\nu} + \bar{\mu}) + \frac{A^2\lambda^5}{\sqrt{2}} \left( \frac{1}{2} - \bar{\nu} + \bar{\mu} \right) + O(\lambda^6), \\
U_{\mu 1} &= \frac{1}{2} + \frac{\lambda}{\sqrt{2}} + \frac{\lambda^2}{2} \left( \frac{1}{2} - A \right) - \frac{\lambda^4}{8} \left( \frac{1}{8} + \frac{A^2}{2} + A \left( \frac{1}{2} - \bar{\nu} + \bar{\mu} \right) \right) + O(\lambda^6), \\
U_{\mu 2} &= \frac{1}{2} + \frac{\lambda}{\sqrt{2}} - \frac{\lambda^2}{2} \left( \frac{1}{2} - A \right) - \frac{\lambda^4}{8} \left( \frac{1}{8} + \frac{A^2}{2} + A \left( \frac{1}{2} - \bar{\nu} + \bar{\mu} \right) \right) + O(\lambda^6), \\
U_{\mu 3} &= \frac{1}{2} + \frac{\lambda^2}{2} + \frac{A\lambda^3}{\sqrt{2}}(\bar{\nu} + \bar{\mu}) - \frac{A^2\lambda^4}{4} + \frac{A\lambda^5}{2\sqrt{2}}(\bar{\nu} + \bar{\mu}) + O(\lambda^6), \\
U_{\tau 1} &= \frac{1}{2} + \frac{\lambda^2}{2} + \frac{A\lambda^3}{\sqrt{2}}(\bar{\nu} + \bar{\mu}) - \frac{A^2\lambda^4}{4} + \frac{A\lambda^5}{2\sqrt{2}}(\bar{\nu} + \bar{\mu}) + O(\lambda^6), \\
U_{\tau 2} &= \frac{1}{2} + \frac{\lambda^2}{2} + \frac{A\lambda^3}{\sqrt{2}}(\bar{\nu} + \bar{\mu}) - \frac{A^2\lambda^4}{4} + \frac{A\lambda^5}{2\sqrt{2}}(\bar{\nu} + \bar{\mu}) + O(\lambda^6), \\
U_{\tau 3} &= \frac{1}{\sqrt{2}} + \frac{A\lambda^2}{2} - \frac{A^2\lambda^4}{2\sqrt{2}} + O(\lambda^6); \\
\end{align*} \]

\[ |U_{e1}|^2 = \frac{1}{2} + \frac{\lambda}{\sqrt{2}} + \frac{\lambda^2}{4} - \frac{\lambda^3}{\sqrt{2}} \left( \frac{1}{2} - A(1 - \bar{\nu}) \right) + \frac{A\lambda^4}{2} \left( 1 - \bar{\nu} - \bar{\mu} \right) + \frac{\lambda^5}{8\sqrt{2}} \left( 1 + 2A \right) \left( 1 + 2A \right) + O(\lambda^6), \]

\[ |U_{e2}|^2 = \frac{1}{2} + \frac{\lambda}{\sqrt{2}} - \frac{\lambda^2}{4} + \frac{\lambda^3}{\sqrt{2}} \left( \frac{1}{2} - A(1 - \bar{\nu}) \right) + \frac{A\lambda^4}{2} \left( 1 - \bar{\nu} - \bar{\mu} \right) - \frac{\lambda^5}{8\sqrt{2}} \left( 1 + 2A \right) \left( 1 + 2A \right) + O(\lambda^6), \]

\[ |U_{e3}|^2 = \frac{\lambda^2}{2} - \frac{\lambda^4}{2} \left( 1 - \bar{\nu} \right) + O(\lambda^6), \]

\[ |U_{\mu 1}|^2 = \frac{1}{4} + \frac{\lambda}{\sqrt{2}} + \frac{\lambda^2}{2} \left( \frac{1}{2} + A \right) - \frac{\lambda^3}{\sqrt{2}} \left( \frac{1}{2} - A \right) - \frac{A\lambda^4}{2} \left( 1 - \bar{\nu} - \bar{\mu} \right) + \frac{\lambda^5}{8\sqrt{2}} \left( \frac{1}{8} + \frac{A^2}{2} + A \left( \frac{1}{2} - \bar{\nu} \right) \right) + O(\lambda^6), \]

\[ |U_{\mu 2}|^2 = \frac{1}{4} + \frac{\lambda}{\sqrt{2}} + \frac{\lambda^2}{2} \left( \frac{1}{2} + A \right) - \frac{\lambda^3}{\sqrt{2}} \left( \frac{1}{2} - A \right) - \frac{A\lambda^4}{2} \left( 1 - \bar{\nu} - \bar{\mu} \right) - \frac{\lambda^5}{8\sqrt{2}} \left( \frac{1}{8} + \frac{A^2}{2} + A \left( \frac{1}{2} - \bar{\nu} \right) \right) + O(\lambda^6), \]

\[ |U_{\mu 3}|^2 = \frac{1}{2} - \lambda^2 \left( \frac{1}{2} + A \right) + A\lambda^4 \left( 1 - \bar{\nu} \right) + O(\lambda^6), \]

\[ |U_{\tau 1}|^2 = \frac{1}{4} + \frac{A\lambda^2}{2} + \frac{A\lambda^2}{\sqrt{2}} \left( \frac{1}{2} - A \right) + O(\lambda^6), \]

\[ |U_{\tau 2}|^2 = \frac{1}{4} - \frac{A\lambda^2}{2} - \frac{A\lambda^2}{\sqrt{2}} \left( \frac{1}{2} - A \right) + O(\lambda^6), \]

\[ |U_{\tau 3}|^2 = \frac{1}{2} + A\lambda^2 + O(\lambda^6). \]

For convenience, we present $1/(1 - |U_{e3}|^2)$, $1/(1 - |U_{e3}|^2)^2$ and $\sin^2 \theta_{12}$ in terms of $\lambda$, $A$ and $\bar{\nu}$,

\[ \begin{align*}
1 - |U_{e3}|^2 &= 1 + \lambda^2 + \left( \frac{1}{4} - A(1 - \bar{\nu}) \right) \lambda^4 + O(\lambda^6), \\
1 - |U_{e3}|^2 &= 1 + \lambda^2 + \left( \frac{3}{4} - 2A(1 - \bar{\nu}) \right) \lambda^4 + O(\lambda^6), \\
\sin^2 \theta_{12} &= \frac{1}{2} - \frac{\lambda}{\sqrt{2}} - \frac{A^\lambda}{\sqrt{2}} \left( 1 - \bar{\nu} \right) \lambda^3 + \frac{\lambda^5}{\sqrt{2}} \left( \frac{1}{8} + A(1 - \bar{\nu}) + A^2 \left( \frac{1}{2} - \bar{\nu} \right) \right) + O(\lambda^6). \end{align*} \]

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