Thermodynamic limit of random partitions and dispersionless Toda hierarchy

Kanehisa Takasaki\(^1\) and Toshio Nakatsu\(^2\)

\(^1\) Graduate School of Human and Environmental Studies, Kyoto University, Yoshida, Sakyo, Kyoto 606-8501, Japan
\(^2\) Institute for Fundamental Sciences, Setsunan University, Ikeda-Nakamachi, Neyagawa, Osaka 572-8508, Japan

E-mail: takasaki@math.h.kyoto-u.ac.jp and nakatsu@mpg.setsunan.ac.jp

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Abstract
We study the thermodynamic limit of random partition models for the instanton sum of 4D and 5D supersymmetric \(U(1)\) gauge theories deformed by some physical observables. The physical observables correspond to external potentials in the statistical model. The partition function is reformulated in terms of the density function of Maya diagrams. The thermodynamic limit is governed by a limit shape of Young diagrams associated with dominant terms in the partition function. The limit shape is characterized by a variational problem, which is further converted to a scalar-valued Riemann–Hilbert problem. This Riemann–Hilbert problem is solved with the aid of a complex curve, which may be thought of as the Seiberg–Witten curve of the deformed \(U(1)\) gauge theory. This solution of the Riemann–Hilbert problem is identified with a special solution of the dispersionless Toda hierarchy that satisfies a pair of generalized string equations. The generalized string equations for the 5D gauge theory are shown to be related to hidden symmetries of the statistical model. The prepotential and the Seiberg–Witten differential are also considered.

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1. Introduction
This is a sequel to our previous paper [1] on the correlation functions of loop operators of the 5D supersymmetric \(U(1)\) gauge theory in the \(\Omega\) background on \(\mathbb{R}^4 \times S^1\) [2]. As shown therein, a generating function of those correlation functions coincides with the partition function of the so-called melting crystal model deformed by a set of external potentials. The melting crystal model is a statistical model of random plane partitions (3D Young diagrams). Actually, by the technique of diagonal slicing [3], the partition function can be converted to a sum...
over ordinary partitions (Young diagrams). The latter may be thought of as a $q$-analog of the instanton sum (Nekrasov function) of $4D \mathcal{N} = 2$ supersymmetric gauge theories [4]. The undeformed melting crystal model was first proposed as a statistical model of the amplitude of A-model topological strings [5]. A slightly modified model was studied as the instanton sum of 5D supersymmetric gauge theories [6, 7]. The fully deformed model was introduced from the point of view of integrable structures [8] (see also the review [9]). It was, however, not clear what physical meaning the external potentials have in the context of the 5D gauge theory. Our previous paper [1] presented an answer to this question.

Another issue addressed therein is the thermodynamic limit of the melting crystal model, from which we attempted to derive a 5D analog of the Seiberg–Witten theory [10]. In the case of $4D \mathcal{N} = 2$ supersymmetric gauge theories, such a ‘microscopic’ derivation was first achieved by Nekrasov and Okounkov [11], and later extended to deformed models by Nekrasov and Marshakov [12, 13]. A clue to this problem is the notion of ‘limit shape’. In the thermodynamic limit, the partition function is dominated by Young diagrams of macroscopic sizes and a particular (rescaled) shape. This limit shape can be determined by a variational problem. The variational problem, in turn, can be converted to a Riemann–Hilbert problem for a complex analytic function. Solving the Riemann–Hilbert problem, Nekrasov and Marshakov [12, 13] could derive the Seiberg–Witten theory in the presence of external potentials (which they identified with local observables $\text{Tr} \phi^{k+1}/(k+1)$, $k = 1, 2, \ldots$, in the $\mathcal{N} = 2$ chiral multiplet). Inspired by their work, we studied the thermodynamic limit of the deformed melting crystal model, using a slightly different method that was developed for the undeformed model [7, 14]. Unfortunately, our calculations for the deformed model were wrong; we considered an inappropriate Riemann–Hilbert problem, which led to wrong results.

Our primary problem in this paper is, therefore, to formulate a correct Riemann–Hilbert problem for the deformed melting crystal model. We address this problem in a wider context. Namely, we treat both the 4D and 5D theories by the same method, thereby hoping to compare these two cases on a common ground. Moreover, this enables us, simultaneously, to compare our method with the method of Nekrasov and Marshakov. As it turns out, one can formulate (and solve) an appropriate Riemann–Hilbert problem for a function $W(z)$ that amounts to the so-called loop amplitude or resolvent function in the theory of random matrices. The solution is accompanied with a complex curve (a double covering of the $z$-plane), which may be thought of as the Seiberg–Witten curve of the $U(1)$ gauge theory. In the case of the deformed 4D theory, this result fully agrees with the result of Nekrasov and Marshakov.

We further present an interpretation of these calculations in the language of the dispersionless Toda hierarchy [15, 16]. The dispersionless Toda hierarchy is a long-wave limit of the Toda hierarchy [17] (see the review [18] for more details). It is known [8, 11, 13] that both the 4D and 5D Nekrasov functions are related to the 1D Toda hierarchy (higher time evolutions of the Toda lattice), hence to the Toda hierarchy (higher time evolutions of the 2D Toda fields) as well. On the other hand, as already observed by Marshakov and Nekrasov (loc. cit.), the thermodynamic limit corresponds to the long-wave limit of the Toda lattice, and the solution of the aforementioned Riemann–Hilbert problem in the 4D case gives a solution of the dispersionless 1D Toda hierarchy. We consider this issue from a different point of view, namely, the notion of ‘generalized string equations’ in the Toda and dispersionless Toda hierarchies. This notion stems from string theories [19–23], and found applications in interface dynamics [24–26], which eventually led to a detailed description of a quite general class of solutions of the dispersionless Toda hierarchy [27]. Employing this notion and techniques developed in earlier studies, we show that the solutions of the Riemann–Hilbert problem for both 4D and 5D cases can be (almost uniquely) characterized by a pair of generalized string
equations. ‘Almost’ means that the generalized string equations for the 5D case are suffered from a kind of ambiguity. The meaning of this ambiguity is left for a future study.

Let us mention that the notion of generalized string equations is also useful for a statistical model of the double Hurwitz numbers [28]. Remarkably, this statistical model and the melting crystal model have a common algebraic structure in their hidden symmetries. Following this analogy, one can derive the generalized string equations of the 5D theory from those hidden symmetries of the partition functions. We discuss this issue at the end of the paper.

This paper is organized as follows. Sections 2, 3 and 4 show the formulation of the random partition models, the prescription of the thermodynamic limit and the derivation of Riemann–Hilbert problems. In section 2, the deformed random partition models are formulated. A 2D free fermion system is used to formulate the main statistical weights and the external potentials in a compact form. In section 3, the partition functions are reformulated in terms of the density function of Maya diagrams. The energy functionals for the density function are specified. In section 4, the thermodynamic limit is formulated and the asymptotic form of the energy functionals is determined. The variational equation for the limit shape is derived and converted to a Riemann–Hilbert problem. Sections 5, 6, 7 and 8 present the main results on solutions of the Riemann–Hilbert problem and their relation to the dispersionless Toda hierarchy. The solution for the 4D theory is constructed in section 5, and examined in section 6 from the point of view of generalized string equations. The solution for the 5D theory is considered in the same way in sections 7 and 8. Section 8 also presents a derivation of the generalized string equations from hidden symmetries of the melting crystal model. Section 9 is devoted to concluding remarks on the prepotential and the Seiberg–Witten differential.

2. Random partition models of $U(1)$ gauge theories

2.1. Fermions

We use the same formulation of fermions as our previous papers [1, 8, 9]. Let $\psi_n, \psi_n^*, n \in \mathbb{Z}$, denote the Fourier modes of 2D complex fermion fields

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}.$$  

They satisfy anti-commutation relations

$$\psi_n \psi_m^* + \psi_m \psi_n^* = \delta_{m+n,0}, \quad \psi_n^* \psi_m + \psi_m^* \psi_n = \psi_n^* \psi_m^* + \psi_m^* \psi_n^* = 0.$$  

The charge-$s$ sector of the Fock space has the ground states

$$|s\rangle = (-\infty) \cdots |s_{-1}\rangle \psi_s^*, \quad |s\rangle = |s_1\rangle \cdots |s_1\rangle \psi_{s+1} \cdots \psi_{s-1}$$

and excited states

$$|\mu, s\rangle = (-\infty) \cdots |\mu, s_{-1}\rangle \psi_1^* \cdots \psi_{1+s}^* \cdots |\mu, s\rangle = |\mu, s\rangle \psi_{-s} \psi_{-s+1} \cdots |\mu, s\rangle$$

labelled by the set $\mathcal{P}$ of all partitions $\mu = (\mu_i)_{i=1}^\infty$, $\mu_1 \geq \mu_2 \geq \cdots \geq 0$, of arbitrary lengths. Let us introduce the fermion bilinears

$$J_k = \sum_{n \in \mathbb{Z}} \psi_{-n} \psi_n^*: $$

$$L_0 = \sum_{n \in \mathbb{Z}} n : \psi_{-n} \psi_n^*: \quad W_0 = \sum_{n \in \mathbb{Z}} n^2 : \psi_{-n} \psi_n^*: $$

The $J_k$ span a $U(1)$ current algebra, and $L_0$ and $W_0$ are zero-modes of Virasoro and $W^{(3)}$ algebras. $J_0$, $L_0$ and $W_0$ are diagonal with respect to the orthonormal basis $\langle \mu, s |$, $\langle \mu, s |$. The
diagonal elements can be calculated as

$$\langle \mu, s | h | \mu, s \rangle = \kappa(\mu) = \sum_{i \geq 1} \mu_i - 2i + 1.$$  

where

$$|\mu| = \sum_{i \geq 1} \mu_i.$$  

We use this fermionic language to formulate the random partition models of instanton sums for the 4D and 5D $U(1)$ gauge theories. The models are deformed by an infinite number of external potentials $\Phi_k(\mu, s)$, $k = 1, 2, \ldots$. The partition function of each model becomes a function $Z_s(t)$ of the coupling constants $t = (t_1, t_2, \ldots)$. The coupling constants play the role of time variables in an underlying integrable hierarchy.

### 2.2. Partition function for 4D theory

We start from the fermionic formula

$$Z_s^{4D}(t) = \langle s | e^{h^{-1}J_1} \Lambda_0 e^{H(t)} e^{h^{-1}J_1} s \rangle,$$  

where $\Lambda_0$ and $h$ are positive constants, and $H(t)$ is a linear combination

$$H(t) = \sum_{k=1}^{\infty} t_k H_k$$  

of the zero modes

$$H_k = \frac{k^{k+1}}{k+1} \sum_{n \in \mathbb{Z}} (n^{k+1} - (n-1)^{k+1}) \psi_n \psi_n^*$$  

of (a slightly modified version of) the ordinary $W_\infty$ algebra.

Since the action of $e^{J_1}$, $\Lambda_0 e^{H(t)}$ and $e^{J_1}$ preserves the charge-$s$ sector, one can insert the partition of unity

$$1 = \sum_{\mu \in P} |\mu, s \rangle \langle \mu, s|$$  

between these operators, where $P$ denotes the set of all partitions of arbitrary length. This leads to an expansion of $Z_s^{4D}(t)$ into a sum over partitions. Moreover, since $\Lambda_0 e^{H(t)}$ is diagonal, this becomes a single (rather than double) sum:

$$Z_s^{4D}(t) = \sum_{\mu \in P} \langle s | e^{h^{-1}J_1} | \mu, s \rangle \langle \mu, s | \Lambda_0 e^{H(t)} | \mu, s \rangle \langle \mu, s| e^{h^{-1}J_1} s \rangle.$$  

Let us specify the building blocks of this sum. Firstly, $e^{J_1}$ are known to act on the ground states as

$$\langle s | e^{h^{-1}J_1} = \sum_{\mu \in P} |\mu, s \rangle \frac{\dim \mu}{h^{\mu} |\mu|!}, \quad e^{h^{-1}J_1} s \rangle = \sum_{\mu \in P} \frac{\dim \mu}{h^{\mu} |\mu|!} |\mu, s \rangle,$$  

where $\dim \mu$ denotes the degree $\dim S^\mu$ of the irreducible representation $S^\mu$ of the symmetric group $S_r$, $r = |\mu|$ [29]. $\dim \mu$ has the hook length formula

$$\dim \mu = |\mu|! \prod_{\square \in \mu} h(\square)^{-1}.$$  

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We define a 5D analog of (2.3). Partition function for 5D theory of the zero-modes of Okounkov and Reshetikhin [3].

\[
\Phi(t, \mu, s) = \sum_{k=1}^{\infty} \Phi_k(\mu, s), \quad \Phi_k(\mu, s) = (\mu, s[H_k][\mu, s]).
\]

\(\Phi_k(\mu, s)\) has the formal (apparently divergent) expression

\[
\Phi_k(\mu, s) = \frac{h^{k+1}}{k+1} \sum_{i=1}^{\infty} ((s + \mu_i - i + 1)^{k+1} - (s - i + 1)^{k+1})
\]

which can be reorganized to a finite sum as

\[
\Phi_k(\mu, s) = \frac{h^{k+1}}{k+1} \sum_{i=1}^{\infty} ((s + \mu_i - i + 1)^{k+1} - (s - i + 1)^{k+1}) - \frac{(h\hbar)^{k+1}}{k+1}.
\]

Thus, \(Z^{4D}_s(t)\) can be converted to the sum

\[
Z^{4D}_s(t) = \sum_{\mu \in \mathcal{P}} \left( \frac{\dim \mu}{[h\hbar^{|\mu|} \mu !]} \right)^2 \Lambda_0^{2|\mu|+i(s+1)} e^{\Phi(t, \mu, s)}
\]

over all partitions. This is exactly the Nekrasov function of the 4D \(\mathcal{N} = 2\) supersymmetric \(U(1)\) gauge theory deformed by external potentials [12, 13].

2.3. Partition function for 5D theory

We define a 5D analog of (2.2) as

\[
Z^{5D}_s(t) = \langle |G_+ Q^s e^{H(t)} G_-| \rangle,
\]

where \(q\) and \(Q\) are constant with \(0 < q_1 < 1\) and \(0 < Q < 1\), \(H(t)\) is a linear combination

\[
H(t) = \sum_{k=1}^{\infty} t_k H_k
\]

of the zero-modes

\[
H_k = \sum_{n \in \mathbb{Z}} q^{q_n^k} \psi_{-n} \psi_{n}^*;
\]

of the quantum-torus algebra [9], and \(G_\pm\) are the transfer operators

\[
G_\pm = \exp \left( \sum_{k=1}^{\infty} \frac{q^k}{k(1 - q^k)} J_{\pm k} \right)
\]

of Okounkov and Reshetikhin [3].

Inserting the partition of unity between operators, one can expand \(Z^{5D}_s(t)\) into a sum over all partitions. The building blocks of this expansion can be calculated as follows. According to Okounkov and Reshetikhin [3], \(G_\pm\) act on the ground states \(|s\rangle\), \(|s\rangle\) as

\[
\langle s | G_+ = \sum_{\mu \in \mathcal{P}} (\mu, s) s_\mu (q^\mu), \quad G_- |s\rangle = \sum_{\mu \in \mathcal{P}} s_\mu (q^\mu) |\mu, s\rangle.
\]
where \( s_\mu(q^\rho) \) is the special value of the Schur function \( s_\mu(x_1, x_2, \ldots) \) of infinite variables [30] at
\[
\rho = (q^{1/2}, q^{3/2}, \ldots, q^{n-1/2}, \ldots).
\]
This special value has the hook length formula
\[
s_\mu(q^\rho) = q^{-\kappa(\mu)/2} \prod_{\square \in \mu} \left( q^{-\ell(\square)/2} - q^{\ell(\square)/2} \right)^{-1},
\]
which may be thought of as a \( q \)-deformation of the hook length formula (2.4) of \( \mu \). The operator \( Q^L e^{H(t)} \) in the middle is diagonal, and the diagonal elements can be factorized as
\[
\langle \mu, s | Q^L e^{H(t)} | \mu, s \rangle = Q^L | \mu \rangle + s(s+1)/2 e^{\Phi_1(t, \mu, s)},
\]
where
\[
\Phi_1(t, \mu, s) = \sum_{i=1}^\infty t^i \Phi_k(\mu, s), \quad \Phi_k(\mu, s) = \langle \mu, s | H_k | \mu, s \rangle.
\]
\( \Phi_k(\mu, s) \) has the formal expression (apparently valid for \( |q| > 1 \))
\[
\Phi_k(\mu, s) = \sum_{i=1}^\infty q^{k(s+\mu_i-1)} - \sum_{i=1}^\infty q^{k(1-i)}.
\]
which can be reorganized to a finite sum as
\[
\Phi_k(\mu, s) = \sum_{i=1}^\infty (q^{k(s+\mu_i-1)} - q^{k(1-i)}) + \frac{q^k(1-q^k)}{1-q^k}.
\]
Thus, \( Z_{5D}^s(t) \) can be expanded to a sum over all partitions as
\[
Z_{5D}^s(t) = \sum_{\mu \in \mathcal{P}} s_\mu(q^\rho)^2 Q^{|\mu|+s+1/2} e^{\Phi(t, \mu, s)}.
\]
This is a 5D analog of the deformed 4D \( U(1) \) Nekrasov function (2.8). Although one can further insert a Chern–Simons term [6], we consider this simplest model in this paper.

3. Energy functional for the density function

3.1. Density function of the Maya diagram

We now recall the notion of density function of Maya diagrams [7, 14] that play a central role in our approach to the thermodynamic limit.

The Maya diagram of \( \mu \) is a configuration of particles located at the points \( x = \mu_i - i, i = 1, 2, \ldots, \) on a line. Let \( \rho_\mu(x) \) denote the density function
\[
\rho_\mu(x) = \sum_{i=1}^\infty \delta(x - \mu_i + i)
\]
of this particle configuration. The backward difference
\[
\Delta \rho_\mu(x) = \rho_\mu(x) - \rho_\mu(x-1)
\]
becomes, up to a constant factor, the second derivative of the so-called profile function \( f_\mu(x) \) [11] of the (45° rotated) Young diagram:
\[
\Delta \rho_\mu(x) = -\frac{1}{2} f''_\mu(x).
\]
The first few moments of $\Delta \rho_\mu(x)$ can be explicitly calculated as
\[
\begin{align*}
\int_{-\infty}^{\infty} dx \Delta \rho_\mu(x) &= -1, \\
\int_{-\infty}^{\infty} dx x \Delta \rho_\mu(x) &= 0, \\
\int_{-\infty}^{\infty} dx x^2 \Delta \rho_\mu(x) &= -2|\mu|, \\
\int_{-\infty}^{\infty} dx x^3 \Delta \rho_\mu(x) &= -3|\mu|.
\end{align*}
\]
(3.2)

The density function $\rho_{\mu,s}(x)$ for the charged partition $(\mu, s)$ is a parallel transform of $\rho_\mu(x)$:
\[
\rho_{\mu,s}(x) = \rho_\mu(x - s) = \sum_{i=1}^{\infty} \delta(x - s - \mu_i + 1).
\]
The first few moments read
\[
\begin{align*}
\int_{-\infty}^{\infty} dx \Delta \rho_{\mu,s}(x) &= -1, \\
\int_{-\infty}^{\infty} dx x \Delta \rho_{\mu,s}(x) &= -s, \\
\int_{-\infty}^{\infty} dx x^2 \Delta \rho_{\mu,s}(x) &= -2|\mu| - s^2, \\
\int_{-\infty}^{\infty} dx x^3 \Delta \rho_{\mu,s}(x) &= -3|\mu| - 6s|\mu| - s^3.
\end{align*}
\]
(3.3)

### 3.2. External potentials in terms of the density function

Let $O^\text{4D}_k(\mu, s)$ denote the $k$th moment of $-\Delta \rho_{\mu,s}(x)$:
\[
O^\text{4D}_k(\mu, s) = -\int_{-\infty}^{\infty} dx x^k \Delta \rho_{\mu,s}(x).
\]
These moments have a formal expression
\[
O^\text{4D}_k(\mu, s) = \sum_{i=1}^{\infty} (s + \mu_i - i + 1)^k - \sum_{i=1}^{\infty} (s + \mu_i - i)^k,
\]
(4.4)
which can be converted into a finite sum as
\[
O^\text{4D}_k(\mu, s) = \sum_{i=1}^{\infty} ((s + \mu_i - i + 1)^k - (s - i + 1)^k) - \sum_{i=1}^{\infty} ((s + \mu_i - i)^k - (s - i)^k) + s^k.
\]
(4.5)

The external potentials
\[
\Phi^\text{4D}_k(\mu, s) = \frac{\hbar^{k+1}}{k+1} \sum_{i=1}^{\infty} ((s + \mu_i - i + 1)^{k+1} - (s + \mu_i - i)^{k+1})
\]
of the 4D gauge theory can be thereby expressed as
\[
\Phi^\text{4D}_k(\mu, s) = \frac{\hbar^{k+1} O^\text{4D}_k(\mu, s)}{k+1} = -\int_{-\infty}^{\infty} dx \frac{(\hbar x)^{k+1}}{k+1} \Delta \rho_{\mu,s}(x).
\]
(3.6)
One can find a similar expression of the external potentials for the 5D gauge theory. Consider the \( q \)-analog
\[
O_{5D}^k(\mu, s) = -\int_{-\infty}^{\infty} dx \, q^{kx} \Delta \rho_{\mu,s}(x)
\]
of \( O_{4D}^k(\mu, s) \). The formal expression
\[
O_{5D}^k(\mu, s) = \sum_{i=1}^{\infty} q^{k(s+i-1)} - \sum_{i=1}^{\infty} q^{k(s-i)} + q^{ks}.
\]
(3.7)
can be converted to a finite sum as
\[
O_{5D}^k(\mu, s) = \sum_{i=1}^{\infty} q^{k(s+i-1)} - \sum_{i=1}^{\infty} q^{k(s-i)} + q^{ks}.
\]
(3.8)
The external potentials
\[
\Phi_{5D}^k(\mu, s) = \sum_{i=1}^{\infty} q^{k(s+i-1)} - \sum_{i=1}^{\infty} q^{k(s-i)} + q^{ks}.
\]
(3.9)

3.3. Hook product in terms of the density function

By the hook length formulae (2.4) and (2.11), the main part of the Boltzmann weights in (2.8) and (2.15) can be rewritten as
\[
\left( \frac{\dim \mu}{h^{[\mu]}} \right)^2 A_0^{2|\mu|} = \prod_{\square \in \mu} \left( \frac{\Lambda_0}{h} \right)^2 h(\square)^{-2},
\]
(3.10)
\[
s_\mu(q^\rho)^2 Q^{[\mu]} = q^{-e(\mu)/2} \prod_{\square \in \mu} Q_q h(\square)(1 - q h(\square))^{-2}.
\]
(3.11)

Let us recall the general formula [1]
\[
\sum_{\square \in \mu} f(h(\square)) = \frac{1}{2} \int_{-\infty}^{\infty} dx \, dy g(|x-y|) \Delta \rho_{\mu}(x) \Delta \rho_{\mu}(y),
\]
(3.12)
where \( f(x) \) is an arbitrary function and \( g(x) \) (referred to as the kernel function) is a function that satisfies the conditions
\[
g(x+1) + g(x-1) - 2g(x) = f(x), \quad g(0) = 0.
\]
(3.13)

By this formula, the logarithm of (3.10) and (3.11) can be converted to quadratic functionals of \( \Delta \rho_{\mu,s} \) as follows.

- **4D theory** [11]: one first obtains the expression
\[
- \log \left( \frac{\dim \mu}{h^{[\mu]}} \right)^2 A_0^{2|\mu|} = \int_{-\infty}^{\infty} dx \, dy \, g_{4D}(|x-y|) \Delta \rho_{\mu}(x) \Delta \rho_{\mu}(y),
\]
Thus, the main part of the Boltzmann weight becomes the exponential of a quadratic functional
\[ E = \int_{\mathbb{R}^n} \mathcal{L}(\rho(x)) \, dx. \]

\section{5D theory} \[ [1]: \] also using the last formula of (3.2) to the term \((\kappa(\mu)/2) \log q\), one first obtains the expression
\[ -\log s_\mu(q^r)^2 Q^{\mu} = \int_{-\infty}^\infty dx \, dy g^{5D}(x-y) \Delta \rho_\mu(x) \Delta \rho_\mu(y) - \frac{\log q}{6} \int_{-\infty}^\infty dx \, x^3 \Delta \rho_\mu(x), \]
where
\[ g^{5D}(x) = -\frac{x(x-1)}{4} \log q - \frac{x^2}{12} \log q + \frac{x(x-1)}{2} \log(1-q) + \log G_2(x+1; q). \]

G_2(x; q) is the second member of the multiple q-Gamma functions \( G_n(x; q) \), \( n = 0, 1, \ldots \) [31]. Note that
\[ \mathcal{E}^{5D}[\rho] = \int_{\mathbb{R}^n} \mathcal{L}(\rho(x)) \, dx. \]

Thus, the main part of the Boltzmann weight becomes the exponential of a quadratic functional of \( \Delta \rho_\mu \). The external potentials are already shown to have such a functional expression by (3.6) and (3.9). Thus, up to a simple factor independent of \( \Delta \rho_{\mu,s} \), the Boltzmann weight can be cast into the standard form \( e^{-\mathcal{E}[\rho, \lambda]} \) with the following energy functionals \( \mathcal{E}[\rho] \).

\subsection{4D theory:}
\[ \mathcal{E}^{4D}_\lambda[\rho] = \int_{\mathbb{R}^n} \mathcal{L}_\lambda(\rho(x)) \, dx + \int_{\mathbb{R}^n} dx \left( \sum_{k=1}^\infty \frac{t_k (h \lambda)^k}{k} \right) \Delta \rho(x). \]

\subsection{5D theory:}
\[ \mathcal{E}^{5D}_\lambda[\rho] = \int_{\mathbb{R}^n} \mathcal{L}_\lambda(\rho(x)) \, dx + \int_{\mathbb{R}^n} dx \left( \sum_{k=1}^\infty \frac{t_k q^k}{1 - q^k} \right) \Delta \rho(x). \]
We now have the following expression of the partition functions as statistical sums over the set $D_s$ of all density functions of the form $\rho = \rho_{\mu, s}$, $\mu \in \mathcal{P}$:

\[
Z_{4D}^s(t) = \Lambda_{0}^{|t|+1} \sum_{\rho \in D_s} e^{-\mathcal{E}_{4D}^{\text{int}}[\rho]},
\]

\[
Z_{5D}^s(t) = \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^2} \right) Q^{(s+1)/2} \sum_{\rho \in D_s} e^{-\mathcal{E}_{5D}^{\text{int}}[\rho]}.
\]

4. Thermodynamic limit and the Riemann–Hilbert problem

4.1. Thermodynamic limit for 4D theory

To achieve the thermodynamic limit for the 4D theory, we rescale $s$ and $t_k$ as

\[
s \to \hbar^{-1} s, \quad t_k \to \hbar^{-2} t_k,
\]

and let $\hbar \to 0$. The partition function is then dominated by partitions $\mu$ with $|\mu| = O(\hbar^{-2})$. The rescaled profile functions

\[
h \rho_{\mu, \hbar^{-1}}(h^{-1} u) = h \rho_{\mu}(h^{-1}(u - s))
\]

of these dominant partitions then tend to a continuous ‘limit shape’ $f_s^{(0)}(u)$. To formulate this limit shape in the language of the density functions of Maya diagrams, we assume that the dominant contribution to the partition function is due to density functions of the form

\[
\rho(h^{-1} u) = \rho^{(0)}(u) + O(\hbar) \quad \text{as} \quad \hbar \to 0,
\]

where $\rho^{(0)}(u)$ is a continuous function, monotonously decreasing, and differentiable in a weak sense, and the derivative $\rho^{(0)\prime}(u)$ satisfies the constraints

\[
\int_{-\infty}^{\infty} du \rho^{(0)\prime}(u) = -1, \quad \int_{-\infty}^{\infty} du u \rho^{(0)\prime}(u) = -s.
\]

Note that these constraints stem from the moment formula (3.3).

Under these assumptions, we can find the leading part (of order $\hbar^{-2}$) of the energy functional $\mathcal{E}_{4D}^s[\rho]$ explicitly. Note that the kernel function $g_{4D}$ in the rescaled coordinate behaves as

\[
g_{4D}(h^{-1} u) = h^{-2} g_{4D}^{(0)}(u) + O(\hbar^{-1}),
\]

where

\[
g_{4D}^{(0)}(u) = \frac{u^2}{2} \left( \log \frac{u}{\Lambda_0} - \frac{3}{2} \right).
\]

Also note that $\Delta \rho(x)$ in the rescaled coordinate can be expressed as

\[
\Delta \rho(h^{-1} u) = h \rho^{(0)\prime}(u) + O(\hbar^2).
\]

Consequently, the quadratic part of $\mathcal{E}_{4D}^s[\rho]$ turns out to have the following asymptotic form:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{x} - y) \Delta \rho(x) \Delta (y) = h^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{u} - v) \rho^{(0)\prime}(u) \rho^{(0)\prime}(v) + O(h^{-1}).
\]

This means that the derivative $\rho^{(0)\prime}$ can be singular at some exceptional points, but that the various integrals containing $\rho^{(0)\prime}$ are still meaningful.

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\[\text{J. Phys. A: Math. Theor. 45 (2012) 025403} \quad \text{K Takasaki and T Nakatsu}\]
On the other hand, the linear part, after rescaling $t_k \rightarrow h^{-2} t_k$, behaves as

$$
\int_{-\infty}^{\infty} dx \left( \sum_{k=1}^{\infty} \frac{h^{-2} t_k (h x)^{k+1}}{k+1} \right) \Delta \rho(x) = h^{-2} \int_{-\infty}^{\infty} du \left( \sum_{k=1}^{\infty} \frac{t_k u^{k+1}}{k+1} \right) \rho^{(0)}(u) + O(h^{-1}).
$$

Thus, the energy functional itself can be expressed as

$$
\mathcal{E}_4^{4D} \rho(\rho^0) = h^{-2} \int_{-\infty}^{\infty} du \left( \sum_{k=1}^{\infty} \frac{t_k u^{k+1}}{k+1} \right) \rho^{(0)}(u) + O(h^{-1}),
$$

(4.6)

Thus, the energy functional itself can be expressed as

$$
\mathcal{E}_4^{4D} \rho(\rho^0) = \int_{-\infty}^{\infty} du \left( \sum_{k=1}^{\infty} \frac{t_k u^{k+1}}{k+1} \right) \rho^{(0)}(u).
$$

(4.7)

4.2. Riemann–Hilbert problem for 4D theory

The density function

$$
\rho_\ast^{(0)}(u) = \frac{1 - f_\ast^{(0)}(u)}{2}
$$

of the limit shape is obtained as a minimizer of $\mathcal{E}_4^{4D} \rho(\rho^0)$ under the constraints (4.3). Following the well-known ansatz, we assume that $\rho_\ast^{(0)}(u)$ is constant outside an interval $[u_0, u_1]$. This amounts to the ‘one-cut’ condition in the large-$N$ limit of random matrix models. Since the test functions $\rho(\rho^0)$ in the minimization problem are monotonically decreasing continuous functions with the obvious boundary conditions

$$
\lim_{u \to -\infty} \rho^{(0)}(u) = 1, \quad \lim_{u \to +\infty} \rho^{(0)}(u) = 0,
$$

(4.8)

this means that

$$
\rho_\ast^{(0)}(u) = \begin{cases} 
1 & \text{for } u < u_0, \\
0 & \text{for } u > u_1.
\end{cases}
$$

(4.9)

If we limit the test functions to those with the last property (4.9), the variational equation (with Lagrange multiplier $\nu$)

$$
\frac{\delta}{\delta \rho^{(0)}} \left( \mathcal{E}_4^{4D} \rho(\rho^0) + \nu \left( \int_{-\infty}^{\infty} du u \rho^{(0)}(u) + s \right) \right) = 0
$$

reduces to solving the integral equation

$$
\frac{d}{du} \left( \int_{u_0}^{u} dv g_{4D}^{(0)}(|u - v|) \rho^{(0)}(v) \right) + \frac{V(u)}{2} + \frac{v}{2} = 0 \quad (u_0 \leq u \leq u_1),
$$

(4.10)

where

$$
V(u) = \sum_{k=1}^{\infty} hu^k,
$$

under the constraints (4.3) and the support condition

$$
\rho^{(0)}(u) = 0 \quad \text{for} \quad u \not\in [u_0, u_1].
$$

(4.11)

Actually, the derivative $\rho^{(0)}(u)$ of the minimizer turns out to have singularities at the endpoints $u_0, u_1$ of the support. The integrals in (4.10) and (4.3), however, have finite values.
Let us stress that the endpoints \(u_0, u_1\) are ‘dynamical’, namely determined along with the minimizer \(\rho^{(0)}(u)\).

Actually, rather than solving (4.10) directly, we consider the equation

\[
\text{PP} \int_{u_0}^{u_1} \text{d}v g_{4D}^{(0)}(\{u-v\}) \rho^{(0)}(v) + \frac{V'(u)}{2} = 0 \quad (u_0 \leq u \leq u_1)
\]

(4.12)

obtained by differentiating (4.10). Note that the integral on the left-hand side is now interpreted to be the principal part (indicated by the notation ‘PP’), because the kernel function

\[
g_{4D}^{(0)}(\{u-v\}) = \log \frac{|u-v|}{\Lambda_0}
\]

has a logarithmic singularity at \(v = u\).

We now introduce the complexified kernel function

\[
g_{4D}^{(0)}(z-v) = \log \frac{z-v}{\Lambda_0}
\]

and construct the complex analytic function

\[
W(z) = \int_{u_0}^{u_1} \text{d}v g_{4D}^{(0)}(z-v) \rho^{(0)}(v)
\]

from the derivative \(\rho^{(0)}(u)\) of the minimizer. \(W(z)\) is a holomorphic function on the cut-plane \(\mathbb{C} \setminus [u_0, u_1]\) with the boundary values

\[
W(u \pm i0) = \text{PP} \int_{u_0}^{u_1} \text{d}v g_{4D}^{(0)}(\{u-v\}) \rho^{(0)}(v) \mp \pi i \rho^{(0)}(u)
\]

(4.13)

along the real axis. One can thereby rewrite (4.12) as

\[
W(u + i0) + W(u - i0) = -V'(u) \quad (u_0 \leq u \leq u_1)
\]

(4.14)

and supplement this equation with the equation

\[
W(u + i0) - W(u - i0) = \begin{cases} 
0 & (u > u_1), \\
-2\pi i & (u < u_0)
\end{cases}
\]

(4.15)

off the support of \(\rho^{(0)}(u)\). Moreover, from the asymptotic form

\[
g_{4D}^{(0)}(z-v) = \log \frac{z-v}{\Lambda_0} - \frac{v}{z} + O(z^{-2}) \quad \text{as} \quad z \to \infty
\]

of the kernel function and the constraints (4.3), one can see that \(W(z)\) should satisfy the boundary condition

\[
W(z) = -\log \frac{z}{\Lambda_0} + \frac{s}{z} + O(z^{-2}) \quad \text{as} \quad z \to \infty.
\]

(4.16)

Conversely, constraints (4.3) can be recovered from this boundary condition.

Equations (4.14), (4.15) and (4.16) together form a kind of Riemann–Hilbert problem. Once a solution of this Riemann–Hilbert problem is found, the minimizer \(\rho^{(0)}(u)\) can be obtained from the imaginary part of \(W(u \pm i0)\) as

\[
\rho^{(0)}(u) = \mp \frac{\text{Im} W(u \pm i0)}{\pi}.
\]

(4.17)

Let us mention that one can translate the variational equation (4.10) itself, too, to a Riemann–Hilbert problem. This Riemann–Hilbert problem is formulated for the primitive function \(^5\)

\[
G(z) = \int_{\gamma} \text{d}z W(z)
\]

\(^5\) This amounts to the \(G\)-function in the large-\(N\) limit of random matrices.
In the case of the 5D theory, we set the parameters

\[ S(z) = 2G(z) + V(z) \quad (4.18) \]

rather than \( G(z) \) itself. This function \( S(z) \) coincides with the \( S \)-function in the theory of dispersionless integrable hierarchies [18].

### 4.3. Thermodynamic limit for the 5D theory

In the case of the 5D theory, we set the parameters \( q \) and \( Q \) in an \( \hbar \)-dependent form as

\[ q = e^{-R\hbar}, \quad Q = (R\Lambda_0)^2, \quad (4.19) \]

where \( R \) and \( \Lambda_0 \) are positive constants with \( R\Lambda_0 < 1 \), and rescale \( s \) and \( t_k \) as

\[ s \to \hbar^{-1} s, \quad t_k \to \hbar^{-1} t_k. \quad (4.20) \]

Assuming the same properties for the rescaled density function \( \rho(\hbar^{-1} u) \), one can derive an explicit form of the thermodynamic energy functional \( E^{5D}_{ss}[\rho^{(0)}] \). Note that the kernel function \( g_{SD} \) in the rescaled coordinate behaves as

\[ g_{SD}(h^{-1} u) = h^{-2} g^{(0)}_{SD}(u) + O(h^{-1}). \quad (4.21) \]

\( g^{(0)}_{SD}(u) \) is a somewhat complicated function; for our purpose, it is enough to know that this function can be characterized by the following conditions:

\[ g^{(0)\nu}_{SD}(u) = \log \frac{e^{Ra/2} - e^{-Ra/2}}{R\Lambda_0}, \quad g^{(0)\nu}_{SD}(0) = 0, \quad g^{(0)}_{SD}(0) = 0. \quad (4.22) \]

Hence, the quadratic part of \( E^{5D}_{ss}[\rho] \) has the following asymptotic form:

\[ \int \int_{-\infty}^{\infty} dx dy g_{SD}(|x - y|) \Delta \rho(x) \Delta(y) \]

\[ = h^{-2} \int_{-\infty}^{\infty} du d\rho^{(0)}(u) \rho^{(0)\nu}(u) + O(h^{-1}). \]

On the other hand, the building blocks of the linear part behave as

\[ -\frac{\log q}{6} \int_{-\infty}^{\infty} dx (x - \hbar^{-1} s)^3 \Delta \rho(x) = h^{-2} \frac{R}{6} \int_{-\infty}^{\infty} du (u - s)^3 \rho^{(0)\nu}(u) + O(h^{-1}) \]

and

\[ -\int_{-\infty}^{\infty} dx \left( \sum_{k=1}^{\infty} \frac{\hbar^{-1} t_k q^k}{1 - q^k} \right) \Delta \rho(x) = -h^{-2} \int_{-\infty}^{\infty} du \left( \sum_{k=1}^{\infty} \frac{t_k}{Rk} e^{-Rk\hbar} \right) \rho^{(0)\nu}(u) + O(h^{-1}). \]

The energy functional thus turns out to be expressed as

\[ E^{5D}_{h^{-1}}[\rho] = h^{-2} E^{5D}_{ss}[\rho^{(0)}] + O(h^{-1}), \quad (4.23) \]

where

\[ E^{5D}_{ss}[\rho^{(0)}] = \int \int_{-\infty}^{\infty} du d\rho^{(0)}(u) \rho^{(0)\nu}(v) \]

\[ + \frac{R}{6} \int_{-\infty}^{\infty} du (u - s)^3 \rho^{(0)\nu}(u) + \int_{-\infty}^{\infty} du \left( \sum_{k=1}^{\infty} t_k e^{-Rk}\hbar \right) \rho^{(0)\nu}(u). \quad (4.24) \]
4.4. Riemann–Hilbert problem for 5D theory

Assuming the same ‘one-cut’ condition (4.9) as in the case of the 4D theory, one can reduce the minimizing problem for $\mathcal{E}_{\mathbf{A}}^{(2D)}[\rho^{(0)}]$ to solving the integral equation

\[
\frac{d}{du} \left( \int_{u_0}^{u} dv g_{SD}^{(0)\nu}(u-v) \rho^{(0)\nu}(v) \right) + \frac{R(u-s)^2}{4} + \frac{V(u)}{2} + \frac{\nu}{2} = 0
\]

where \( V(u) = \sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k u} \),

under the constraints (4.3) and the support condition

\[
\rho^{(0)\nu}(u) = 0 \quad \text{for} \quad u \notin [u_0, u_1].
\]

To convert this variational equation to a Riemann–Hilbert problem, we introduce the complex analytic function

\[
W(z) = \int_{u_0}^{u_1} dv g_{SD}^{(0)\nu}(z-v) \rho^{(0)\nu}(v)
\]

with the complexified kernel

\[
g_{SD}^{(0)\nu}(z-v) = \log \left( \frac{e^{R(z-v)/\nu} - e^{-R(z-v)/\nu}}{R\Lambda_0} \right).
\]

\( W(z) \) is a holomorphic function on the \( z \)-plane with discontinuity along the intervals \([u_0 + 2\pi i n/R, u_1 + 2\pi i n/R], n \in \mathbb{Z} \). Moreover, \( W(z) + Rz/2 \) is periodic with respect to the translation \( z \mapsto z + 2\pi i R \). The once-differentiated form

\[
PP \int_{u_0}^{u_1} dv g_{SD}^{(0)\nu}(u-v) \rho^{(0)\nu}(v) + \frac{R(u-s)}{2} + \frac{V'(u)}{2} = 0
\]

of the variational equation (4.25) can be converted to the equation

\[
W(u + i0) + W(u - i0) = -R(u-s) - V'(u) \quad (u_0 \leq u \leq u_1)
\]

for \( W(u \pm i0) \). This equation is supplemented with the equation

\[
W(u + i0) - W(u - i0) = \begin{cases} 0 & (u > u_1), \\ -2\pi i & (u < u_0) \end{cases}
\]

off the support of \( \rho^{(0)\nu}_* \). The minimizer \( \rho^{(0)\nu}_*(u) \) can be read out from \( W(z) \) by the same formula as (4.17).

The constraints (4.3) can be translated into boundary conditions of \( W(z) \) at infinity. Unlike the 4D theory, \( W(z) \) exhibits different boundary behaviors as \( \text{Re} \, z \to -\infty \) and \( \text{Re} \, z \to +\infty \).

- As \( \text{Re} \, z \to -\infty \) and \( \pm \text{Im} \, z > 0 \),
  \[
  W(z) = \frac{R(z-s)}{2} \mp \pi i + \log(R\Lambda_0) + O(e^{R|z|}).
  \]

- As \( \text{Re} \, z \to +\infty \),
  \[
  W(z) = -\frac{R(z-s)}{2} + \log(R\Lambda_0) + O(e^{-R|z|}).
  \]
These conditions are derived from (4.3) as follows. As \( \text{Re} \, z \to -\infty \), the kernel function behaves as
\[
g^{(0)i'}_{5D}(z - v) = -\frac{R(z - v)}{2} \pm \pi i - \log(R\Lambda_0) + O(e^{Rz}).
\]
Consequently, as \( \text{Re} \, z \to -\infty \) under the condition \( \pm \text{Im} \, z > 0 \), \( W(z) \) behaves as
\[
W(z) = \int_{u_0}^{u_1} \, dv \left( -\frac{Rz}{2} + \frac{Rv}{2} \pm \pi i - \log(R\Lambda_0) \right) \rho^{(0)i'}(v) + O(e^{Rz})
\]
\[
= \frac{Rz}{2} - \frac{Rs}{2} \pm \pi i + \log(R\Lambda_0) + O(e^{Rz}).
\]
In much the same way, as \( \text{Re} \, z \to +\infty \),
\[
g^{(0)i'}_{5D}(z - v) = \frac{R(z - v)}{2} - \log(R\Lambda_0) + O(e^{-Rz});
\]

thus,
\[
W(z) = \int_{u_0}^{u_1} \, dv \left( \frac{Rz}{2} - \frac{Rv}{2} - \log(R\Lambda_0) \right) \rho^{(0)i'}(v) + O(e^{-Rz})
\]
\[
= \frac{Rz}{2} + \frac{Rs}{2} + \log(R\Lambda_0) + O(e^{-Rz}).
\]

Thus, \( W(z) \) turns out to satisfy (4.28), (4.29), (4.30) and (4.31). We have to solve these equations under the condition that \( W(z) + R(z - s)/2 \) be a periodic function with respect to the translation \( z \mapsto z + 2\pi i/R \). In other words, this is a Riemann–Hilbert problem on the cylinder \( \mathbb{C}/(2\pi i/R)\mathbb{Z} \) with a single cut along \([u_0, u_1]\).

5. Solution of the Riemann–Hilbert problem for 4D theory

5.1. Construction of the solution when \( t = 0 \)

When \( t = 0 \), we can solve the Riemann–Hilbert problem deductively (namely, without any heuristic consideration) as follows. This solution is a prototype of the solution in the case of \( t \neq 0 \).

In this case, equations (4.14) and (4.15) for \( W(u \pm i0) \) read
\[
W(u + i0) + W(u - i0) = 0 \quad \text{for} \quad u_0 \leq u \leq u_1
\]
and
\[
W(u + i0) - W(u - i0) = \begin{cases} 0 & \text{for} \quad u > u_1, \\ -2\pi i & \text{for} \quad u < u_0. \\ \end{cases}
\]

One can readily see from these equations that the identity
\[
e^{W(u+i0)} + e^{-W(u+i0)} = e^{W(u-i0)} + e^{-W(u-i0)}
\]
holds along the whole real axis. This means that the complex function \( e^{W(z)} + e^{-W(z)} \) has no discontinuity, and hence becomes a holomorphic function on the whole plane \( \mathbb{C} \). Moreover, by (4.16), this function has the asymptotic form
\[
e^{W(z)} + e^{-W(z)} = \frac{z - s}{\Lambda_0} + O(z^{-1})
\]as \( z \to \infty \). By Liouville’s theorem in a complex analysis, the \( O(z^{-1}) \) term disappears and one obtains the identity
\[
e^{W(z)} + e^{-W(z)} = \frac{z - s}{\Lambda_0}. \quad (5.1)
\]
Setting
\[ y = e^{-W(z)}, \]
one can rewrite the last identity as
\[ y + y^{-1} = \frac{z - s}{\Lambda_0}. \]  
(5.2)

Equation (5.2) may be thought of as the equation of a complex algebraic curve, which is exactly the Seiberg–Witten curve of the undeformed 4D \( U(1) \) gauge theory. One can solve (5.2) for \( y = y(z) \) as
\[ y(z) = \frac{z - s + \sqrt{P(z)}}{2\Lambda_0}, \quad P(z) = (z - s)^2 - 4\Lambda_0^2. \]  
(5.3)

\( u_0 \) and \( u_1 \) are determined to be the endpoints of the interval of \( \mathbb{R} \) where \( P(u) < 0 \), namely,
\[ u_0 = s - 2\Lambda_0, \quad u_1 = s + 2\Lambda_0. \]  
(5.4)

\( \sqrt{P(z)} \) is understood to be the branch on the cut plane \( \mathbb{C} \setminus [u_0, u_1] \) such that
\[ \sqrt{P(z)} = z - s + O(z^{-1}) \quad \text{as} \quad z \to \infty, \]
\[ \pm \text{Im} \sqrt{P(u \pm i0)} > 0 \quad \text{for} \quad u_0 < u < u_1. \]  
(5.5)

Thus, a (unique) solution of the Riemann–Hilbert problem can be obtained explicitly as
\[ W(z) = -\log y(z). \]  
(5.6)

As a cross check, let us examine the behavior of \( y(z) \) and \( W(z) \) along the real axis. \( y(z) \) has discontinuity along \([u_0, u_1]\) and takes real values on \( \mathbb{R} \setminus [u_0, u_1] \). The boundary values \( y(u \pm i0) \) for \( u_0 < u < u_1 \) are unimodular (\(|y(u \pm i0)| = 1\)), because they are mutually conjugate imaginary solutions of (5.2). Bearing these facts in mind, one can trace the behavior of \( y(u \pm i0) \) as \( u \) decreases along the real axis. When \( u \) starts from the right side of \( u_1 \), \( y(u) \) decreases towards the value \( y(u_1) = 1 \). When \( u \) passes \( u_1 \), \( y(u) \) bifurcates into the two imaginary numbers \( y(u \pm i0) \), which move on the unit circle in opposite directions towards \(-1 \) until \( u \) reaches \( u_0 \). Accordingly, when \( u \) decreases from \( u = u_1 \) to \( u = u_0 \), \( \arg y(u \pm i0) \) varies from \( 0 \) to \( \pm \pi \). For \( u < u_0 \), \( y(u) \) is negative and \( \arg y(u \pm i0) \) takes the constant values \( \pm \pi \). Translated into the language of \( W(z) \), this behavior of \( \arg y(u \pm i0) \) implies exactly that (4.15) holds. By (4.17), one can read an explicit expression of \( \rho^0_\star(u) \):
\[ \rho^0_\star(u) = \begin{cases} 0 & (u > u_1), \\ \pm \arg y(u \pm i0)/\pi & (u_0 \leq u \leq u_1), \\ 1 & (u < u_0). \end{cases} \]  
(5.7)

Let us mention that one can further rewrite this result to the celebrated ‘arc-sine law’ [34, 35].

5.2. Construction of the solution when \( t \neq 0 \)

When \( t \neq 0 \), we resort to a heuristic method. Namely, we seek a solution of the Riemann–Hilbert problem as a deformation of the foregoing solution. A naïve way will be to add \(-V'(z)/2\) as
\[ W(z) = -\log y(z) - \frac{V'(z)}{2}, \]
which does satisfy equations (4.14) and (4.15) for \( W(u \pm i0) \), but not the boundary condition (4.16). To fulfill the boundary condition, we modify this naïve form as
\[ W(z) = -\log y(z) + N(z)\sqrt{P(z)} - \frac{V'(z)}{2}, \]  
(5.8)

where
\( y(z) \) is a solution
\[
y(z) = \frac{z - \beta + \sqrt{P(z)}}{2\Lambda}, \quad P(z) = (z - \beta)^2 - 4\Lambda^2, \tag{5.9}
\]
of the equation
\[
y + \frac{1}{y} = \frac{z - \beta}{\Lambda} \tag{5.10}
\]
of the deformed Seiberg–Witten curve.

- \( \beta \) and \( \Lambda \) are functions \( \beta(s, t) \) and \( \Lambda(s, t) \) of \( s \) and \( t \) that reduce to the previous values at \( t = 0 \):
\[
\beta(s, 0) = s, \quad \Lambda(s, 0) = \Lambda_0. \tag{5.11}
\]

- \( u_0 \) and \( u_1 \) are the zeros of \( P(z) \):
\[
u_0 = \beta - 2\Lambda, \quad u_1 = \beta + 2\Lambda. \tag{5.12}
\]

- \( \sqrt{P(z)} \) is the branch on the cut plane \( \mathbb{C} \setminus [u_0, u_1] \) such that
\[
\sqrt{P(z)} = z - \beta + O(z^{-1}) \quad \text{as} \quad z \to \infty,
\]
\[
\pm \text{Im} \sqrt{P(u \pm i0)} > 0 \quad \text{for} \quad u_0 < u < u_1. \tag{5.13}
\]

- \( N(z) \) is a linear combination
\[
N(z) = \sum_{k=1}^{\infty} t_k N_k(z)
\]
of polynomials \( N_k(z), k = 1, 2, \ldots, \) in \( z \).
Equations (4.14) and (4.15) are already satisfied by the ansatz (5.8). To fulfill the boundary condition (4.16), we choose \( N_k(z) \) to satisfy the condition
\[
N_k(z) - \frac{k^{k-1}}{2\sqrt{P(z)}} = O(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{5.14}
\]
\( N_k(z) \) is uniquely determined by this condition as
\[
N_1(z) = 0, \quad N_k(z) = \frac{k}{2}(c_{k-1} z^{-2} + c_k z^{-3} + \cdots + c_{k-2}) \quad \text{for} \quad k \geq 2, \tag{5.15}
\]
where \( c_1, c_2, \ldots \) are the coefficients of the expansion
\[
\frac{1}{\sqrt{P(z)}} = z^{-1} + c_1 z^{-2} + c_2 z^{-3} + \cdots,
\]
and we set \( c_0 = 1 \) for convenience. \( N_k(z) \) can also be expressed by using a contour integral as
\[
N_k(z) = \frac{1}{2\pi i} \oint_C \frac{dx}{x - z} \frac{k^{k-1}}{2\sqrt{P(x)}} + \frac{k^{k-1}}{2\sqrt{P(z)}}, \tag{5.16}
\]
where \( C \) is a simple closed curve that encircles the interval \([u_0, u_1]\) anti-clockwise and leaves \( z \) outside. Rewriting \( W(z) \) as
\[
W(z) = -\log y(z) = \sum_{k=1}^{\infty} t_k \left( \frac{k^{k-1}}{2\sqrt{P(z)}} - N_k(z) \right) \sqrt{P(z)},
\]
\[
W(z) = -\log y(z) = \sum_{k=1}^{\infty} t_k \left( \frac{k^{k-1}}{2\sqrt{P(z)}} - N_k(z) \right) \sqrt{P(z)}.
\]
\[
W(z) = -\log y(z) = \sum_{k=1}^{\infty} t_k \left( \frac{k^{k-1}}{2\sqrt{P(z)}} - N_k(z) \right) \sqrt{P(z)},
\]
one can see that
\[
W(z) = - \log \frac{z - \beta + \sqrt{\Lambda(z)}}{2\Lambda} - \sum_{k=1}^{\infty} \frac{k t_k}{2} (c_{k-1} z^{-1} + c_k z^{-2} + \cdots) \sqrt{\Lambda(z)}
\]
\[
= - \log \frac{z}{\Lambda} - \sum_{k=1}^{\infty} \frac{k t_k c_{k-1}}{2} + \left( \beta - \sum_{k=1}^{\infty} \frac{k t_k (c_k - \beta c_{k-1})}{2} \right) z^{-1} + O(z^{-2})
\]
as \(z \to \infty\). Matching this expression with (4.16), one obtains the equations
\[
\log \frac{\Lambda}{\Lambda_0} = \sum_{k=1}^{\infty} \frac{k t_k c_{k-1}}{2} = 0,
\]
(5.17)
\[
\beta - \sum_{k=1}^{\infty} \frac{k t_k (c_k - \beta c_{k-1})}{2} = s
\]
(5.18)
for \(\beta = \beta(s, t)\) and \(\Lambda = \Lambda(s, t)\).

By the implicit function theorem, these equations do have a solution in a neighborhood of \(t = 0\) that satisfies the initial condition (5.11). Though we omit the details, (5.17) and (5.18) are equivalent to the equations that are derived by Marshakov and Nekrasov by their method based on the S-function (4.18) [12, 13].

Once \(W(z)\) is thus determined, one can use (4.17) to obtain an explicit expression of \(\rho_s^{(0)}(u)\):
\[
\rho_s^{(0)}(u) = \begin{cases} 
0 & (u > u_0), \\
\pm \arg y(u \pm i0)/\pi + N(u)\sqrt{|P(u)|} & (u_0 \leq u \leq u_1), \\
& (u < u_0).
\end{cases}
\]
(5.19)

Note that the structure of (5.7) is retained except that a new term proportional to \(\sqrt{|P(u)|}\) is added.

5.3. Rewriting the solution in terms of the Lax function

By construction, \(y = y(z)\) is (a branch of) the inverse function of \(z = z(y) = \beta + \Lambda(y + y^{-1})\).

We now interpret \(z(y)\) to be a long-wave limit of the well-known Lax operator
\[
\mathcal{L} = a(s)e^{\beta} + b(s) + a(s - 1)e^{-\beta}, \quad \partial_s = \partial/\partial s,
\]
(5.20)
of the Toda lattice [32], where \(e^{\pm \beta}\) are the shift operators, namely \(e^{\pm \beta} f(s) = f(s \pm 1)\). Note that one can rewrite the last term of (5.20) as
\[
a(s - 1)e^{-\beta} = e^{-\beta - a(s)}.
\]
(5.21)

In the long-wave (or ‘dispersionless’) limit, \(e^{\beta}\) turns into a c-number variable \(p\) (see section 6 for details). We identify \(y\) with this variable \(p\).

To pursue this analogy further, we introduce the truncation notations
\[
\begin{align*}
&\left( \sum_n a_n y^n \right)_{>0} = \sum_{n>0} a_n y^n, \\
&\left( \sum_n a_n y^n \right)_{<0} = \sum_{n<0} a_n y^n, \\
&\left( \sum_n a_n y^n \right)_m = a_m \quad (m \in \mathbb{Z})
\end{align*}
\]
for the Laurent series of \(y\); the same notations are used for difference operators as well. The time evolutions of the dispersionless 1D Toda hierarchy are generated by the Laurent
polynomials \((z(y^k))_{>0} - (z(y^k))_{<0}\), \(k = 1, 2, \ldots\), which are dispersionless limits of the generators \((Q_k)_{>0} - (Q_k)_{<0}\) of time evolutions of the 1D Toda hierarchy \([33]\).

We now show that the foregoing polynomials \(N_k(z)\) are closely related to these Laurent polynomials of \(y\). Note that \((z(y^k))_{>0} - (z(y^k))_{<0}\) is a linear combination of \(y^l - y^{-l}\), \(l = 1, 2, \ldots\), e.g.

\[
(z(y))_{>0} - (z(y))_{<0} = \Lambda (y - y^{-1}),
\]

\[
(z(y^2))_{>0} - (z(y^2))_{<0} = \Lambda^2 (y^2 - y^{-2}) + 2\Lambda \beta (y - y^{-1}), \ldots
\]

Since \(y^k - y^{-k}\) can be factorized as

\[
y^k - y^{-k} = (y^{k-1} + y^{k-2} + \cdots + y^{-k+2} + y^{-k+1})(y - y^{-1})
\]

and \((y(z))_{>0}\) and its inverse satisfy the relations

\[
y(z) + y(z)^{-1} = \frac{z - \beta}{\Lambda}, \quad y(z) - y(z)^{-1} = \frac{\sqrt{P(z)}}{\Lambda},
\]

\((z(y^k))_{>0} - (z(y^k))_{<0}\) can be expressed as

\[
(z(y^k))_{>0} - (z(y^k))_{<0} = Q_k(z) \sqrt{P(z)}.
\]

where \(Q_k(z)\) is a polynomial in \(z\), e.g.

\[
(z(y))_{>0} - (z(y))_{<0} = \sqrt{P(z)},
\]

\[
(z(y^2))_{>0} - (z(y^2))_{<0} = (z + \beta) \sqrt{P(z)}, \ldots
\]

The polynomial \(Q_k(z)\) can be determined as follows. Rewrite the last identity as

\[
\frac{(z(y^k))_{>0} - (z(y^k))_{<0}}{\sqrt{P(z)}} = Q_k(z)
\]

and note that

\[
\text{LHS} = \frac{z^k}{\sqrt{P(z)}} + O(z^{-1}) \quad \text{as} \quad z \to \infty.
\]

Truncating the negative powers of \(z\) yields the identity

\[
Q_k(z) = z^{k-1} + c_1 z^{k-2} + \cdots + c_{k-1}.
\]

One can thus derive the fundamental formula

\[
(z(y^k))_{>0} - (z(y^k))_{<0} = (z^{k-1} + c_1 z^{k-2} + \cdots + c_{k-1}) \sqrt{P(z}). \quad (5.22)
\]

Comparing this formula with \((5.15)\), one can readily see that

\[
N_k(z) = \frac{k}{2} \frac{(z(y^{k-1}))_{>0} - (z(y^{k-1}))_{<0}}{\sqrt{P(z)}}. \quad (5.23)
\]

Consequently, the second term on the right-hand side of \((5.8)\) turns out to be expressed as

\[
N(z) \sqrt{P(z)} = \sum_{k=1}^{\infty} \frac{kt_k}{2} ((z(y^{k-1}))_{>0} - (z(y^{k-1}))_{<0}). \quad (5.24)
\]

Moreover, since

\[
(z(y^k))_{>0} - (z(y^k))_{<0} = z^k - (z(y^k))_{>0} - 2(z(y^k))_{<0},
\]

one can rewrite \((5.22)\) as

\[
(z(y^k))_{>0} + 2(z(y^k))_{<0} = z^k - (z(y^k))_{>0} + (z(y^k))_{<0}
\]

\[
= z^k - (z^{k-1} + c_1 z^{k-2} + \cdots + c_{k-1}) \sqrt{P(z)}
\]

\[
= (c_k z^{k-1} + c_{k+1} z^{k-2} + \cdots) \sqrt{P(z)}
\]

\[
= c_k + (c_{k+1} - \beta c_k) z^{-1} + O(z^{-2}).
\]
Since \( z(y)^{-1} = (\Lambda y)^{-1} + O(y^{-2}) \) as \( y \to \infty \), one can re-expand the right-hand side in powers of \( y^{-1} \) and pick out the coefficients of \( y^0 \) and \( y^{-1} \). This leads to the identities

\[
(z(y)^k)_0 = c_k, \quad 2(z(y)^k)_{-1} = \frac{c_{k+1} - \beta c_k}{\Lambda}.
\]

One can thereby rewrite (5.17) and (5.18) as

\[
\log \frac{\Lambda}{\Lambda_0} - \sum_{k=1}^{\infty} \frac{k t_k}{2} (z(y)^{k-1})_0 = 0,
\]

\[
\beta - \Lambda \sum_{k=1}^{\infty} k t_k (z(y)^{k-1})_{-1} = s.
\]

In the next section, we derive the same equations from generalized string equations.

6. Generalized string equations for 4D theory

6.1. 2D Toda hierarchy

Let us briefly recall the construction of the Toda hierarchy [17] (referred to as the ‘2D’ Toda hierarchy to distinguish it from the more classical ‘1D’ hierarchy [33]).

This integrable hierarchy is formulated in terms of two Lax operators \( L \) and \( \bar{L} \), which are difference (or, so to speak, ‘pseudo-difference’) operators with respect to the lattice coordinate \( s \). We now choose the ‘symmetric gauge’

\[
L = a e^h + \sum_{n=1}^{\infty} u_n e^{(1-n)h}, \quad \bar{L} = a^{-1} e^{-h} + \sum_{n=1}^{\infty} \bar{u}_n e^{(1+n)h},
\]

where \( a, u_n \) and \( \bar{u}_n \) are dynamical variables that depend on \( s \) and the two sets of time variables \( T = (T_1, T_2, \ldots) \) and \( \bar{T} = (\bar{T}_1, \bar{T}_2, \ldots) \). Let us introduce the truncation notations

\[
\left( \sum_n d_n e^{nh} \right)_{>0} = \sum_{n>0} d_n e^{nh}, \quad \left( \sum_n d_n e^{nh} \right)_{<0} = \sum_{n<0} d_n e^{nh},
\]

\[
\left( \sum_n d_n e^{nh} \right)_m = d_m \quad (m \in \mathbb{Z})
\]

for difference operators.

The time evolutions of the Lax operators are generated by Lax equations of the form

\[
\frac{\partial L}{\partial T_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{T}_k} = [\bar{B}_k, L],
\]

\[
\frac{\partial \bar{L}}{\partial T_k} = [B_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{T}_k} = [\bar{B}_k, \bar{L}].
\]

(6.1)

where

\[
B_k = (L^k)_{>0} + \frac{1}{2} (L^k)_0, \quad \bar{B}_k = (\bar{L}^k)_{<0} + \frac{1}{2} (\bar{L}^k)_0.
\]

These Lax equations are supplemented by another set of Lax equations

\[
\frac{\partial M}{\partial T_k} = [B_k, M], \quad \frac{\partial M}{\partial \bar{T}_k} = [\bar{B}_k, M],
\]

\[
\frac{\partial \bar{M}}{\partial T_k} = [B_k, \bar{M}], \quad \frac{\partial \bar{M}}{\partial \bar{T}_k} = [\bar{B}_k, \bar{M}].
\]

(6.2)

6 Since the first paper [17] was published, the notations for the Toda hierarchy have been considerably changed. Our notations are mostly based on the review [18].

7 The bar does not mean complex conjugate.
and the twisted canonical commutation relations
\[ [L, M] = L, \quad [\tilde{L}, \tilde{M}] = \tilde{L} \]
for the Orlov–Schulman operators
\[ M = \sum_{k=1}^{\infty} kT_kL^k + s + \sum_{n=1}^{\infty} \bar{v}_nL^{-n}, \quad \tilde{M} = -\sum_{k=1}^{\infty} k\bar{T}_k\tilde{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n\tilde{L}^{-n}. \]

6.2. Dispersionless 2D Toda hierarchy

The dispersionless Toda hierarchy \[15\] is a long-wave limit of the Toda hierarchy. In this limit, as briefly mentioned in section 5, the shift operator \( e^\hbar \) is replaced by a momentum-like variable \( p \), \( p \) and \( s \) then become a (twisted) canonical coordinate system of a 2D phase space of classical mechanics. This procedure is justified by applying the idea of semi-classical (or quasi-classical) approximation in quantum mechanics to the Toda hierarchy \[16\].

To derive the dispersionless Toda hierarchy, one starts from an \( \hbar \)-dependent formulation of the Toda hierarchy, in which \( e^\hbar, \partial/\partial T_k \) and \( \partial/\partial \bar{\alpha}T_k \) are replaced by \( e^{\hbar a}, \hbar \partial/\partial T_k \) and \( \hbar \partial/\partial \bar{\alpha}T_k \). The Lax and Orlov–Schulman operators are assumed to take an \( \hbar \)-dependent form as
\[ L = a e^{\hbar a} + \sum_{n=1}^{\infty} a_n e^{(1-n)\hbar a}, \quad \tilde{L} = a^{-1} e^{\hbar a} + \sum_{n=1}^{\infty} \bar{a}_n e^{(1+n)\hbar a}, \]
\[ M = \sum_{k=1}^{\infty} kT_kL^k + s + \sum_{n=1}^{\infty} \bar{v}_nL^{-n}, \quad \tilde{M} = -\sum_{k=1}^{\infty} k\bar{T}_k\tilde{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n\tilde{L}^{-n}, \]
where \( a, a_n, \bar{a}_n \) and \( v_n, \bar{v}_n \) depend on \( \hbar \) as well as \( (s, T, \bar{T}) \) and have semi-classical expansions
\[ a = a^{(0)} + \hbar a^{(1)} + \cdots, \]
\[ a_n = a_n^{(0)} + \hbar a_n^{(1)} + \cdots, \]
\[ v_n = v_n^{(0)} + \hbar v_n^{(1)} + \cdots, \]
\[ \bar{a}_n = \bar{a}_n^{(0)} + \hbar \bar{a}_n^{(1)} + \cdots, \]
\[ \bar{v}_n = \bar{v}_n^{(0)} + \hbar \bar{v}_n^{(1)} + \cdots, \]
as \( \hbar \to 0 \). They obey the Lax equations
\[ \hbar \frac{\partial L}{\partial T_k} = [B_k, L], \quad \hbar \frac{\partial L}{\partial \bar{T}_k} = [\bar{B}_k, L], \]
\[ \hbar \frac{\partial \tilde{L}}{\partial T_k} = [B_k, \tilde{L}], \quad \hbar \frac{\partial \tilde{L}}{\partial \bar{T}_k} = [\bar{B}_k, \tilde{L}], \]
\[ \cdots \quad \text{(equations of the same form for } M) \quad \cdots, \] (6.4)
and the twisted canonical commutation relations
\[ [L, M] = \hbar L, \quad [\tilde{L}, \tilde{M}] = \hbar \tilde{L}. \] (6.5)
One can derive, from these equations, the Lax equations
\[ \frac{\partial L}{\partial T_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{T}_k} = [\bar{B}_k, L], \]
\[ \frac{\partial \tilde{L}}{\partial T_k} = [B_k, \tilde{L}], \quad \frac{\partial \tilde{L}}{\partial \bar{T}_k} = [\bar{B}_k, \tilde{L}], \]
\[ \cdots \quad \text{(equations of the same form for } M) \quad \cdots, \] (6.6)
and the twisted canonical relations
\[ \{ L, M \} = L, \quad \{ \tilde{L}, \tilde{M} \} = \tilde{L} \] (6.7)
for the Laurent series
\[
\mathcal{L} = a^{(0)} + \sum_{n=1}^{\infty} u_n^{(0)} p^{-n}, \quad \tilde{\mathcal{L}} = a^{(0) - 1} + \sum_{n=1}^{\infty} \tilde{u}_n^{(0)} p^{-n},
\]
\[
\mathcal{M} = \sum_{k=1}^{\infty} k \tilde{T}_k \mathcal{L}^k + s + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}, \quad \tilde{\mathcal{M}} = -\sum_{k=1}^{\infty} k \tilde{T}_k \tilde{\mathcal{L}}^{-k} + s + \sum_{n=1}^{\infty} \tilde{v}_n^{(0)} \tilde{\mathcal{L}}^n
\]
of the new variable \(p\). \(\mathcal{B}_k\) and \(\tilde{\mathcal{B}}_k\) are determined by \(\mathcal{L}\) and \(\tilde{\mathcal{L}}\) as
\[
\mathcal{B}_k = (\mathcal{L}^k)_{>0} + \frac{1}{2} (\mathcal{L}^k)_0, \quad \tilde{\mathcal{B}}_k = (\tilde{\mathcal{L}}^{-k})_{<0} + \frac{1}{2} (\tilde{\mathcal{L}}^{-k})_0,
\]
where \((\quad)_{>0}, (\quad)_{<0}\) and \((\quad)_0\) are now understood to be truncation operations for the Laurent series of \(p\):
\[
\left(\sum_n a_n p^n\right)_{>0} = \sum_{n>0} a_n p^n, \quad \left(\sum_n a_n p^n\right)_{<0} = \sum_{n<0} a_n p^n,
\]
\[
\left(\sum_n a_n p^n\right)_m = a_m \quad (m \in \mathbb{Z}).
\]
\([, \,]\) denotes a Poisson bracket specified below. These equations are fundamental building blocks of the dispersionless Toda hierarchy.

This procedure may be thought of as a kind of ‘classical limit’ from quantum mechanics to classical mechanics. The Laurent series \(\mathcal{L}, \mathcal{M}, \tilde{\mathcal{L}}, \tilde{\mathcal{M}}\) are classical counterparts (referred to as the Lax and Orlov–Schulman functions) of \(L, M, \tilde{L}, \tilde{M}\), in which \(e^{\hbar h}\) turns into the c-number variable \(p\), and the coefficients are replaced by the leading terms of the \(h\)-expansion. The Poisson bracket \(\{, \,\}\) is defined as
\[
\{F, G\} = p \left( \frac{\partial F}{\partial p} \frac{\partial G}{\partial s} - \frac{\partial F}{\partial s} \frac{\partial G}{\partial p} \right)
\]
for functions on the 2D phase space \((p, s)\). In particular, \(p\) and \(s\) obey the twisted canonical relation
\[
[p, s] = p, \tag{6.8}
\]
which is exactly the classical limit of the quantum mechanical commutation relation
\[
[e^{\hbar h}, s] = \hbar e^{\hbar h}. \tag{6.9}
\]
As a technical remark, let us note that one can derive the \(h\)-dependent Toda hierarchy from the \(h\)-independent formulation by rescaling the variables \(s, T, \tilde{T}\) as
\[
s \to h^{-1} s, \quad T_k \to h^{-1} T_k, \quad \tilde{T}_k \to h^{-1} \tilde{T}_k \tag{6.10}
\]
and simultaneously rescaling \(M\) and \(\tilde{M}\) as
\[
M \to h^{-1} M, \quad \tilde{M} \to h^{-1} \tilde{M}. \tag{6.11}
\]
This remark is very important for our purpose. Note that (6.10) amounts to (4.1) and (4.20) in the prescription of the thermodynamic limit\(^9\). By the way, the energy functionals (3.18) and (3.19) depend on \(\hbar\) (through the relation \(q = e^{-\rho h}\) in the 5D case) before the rescaling.

\(^8\) Only after this rescaling, the twisted canonical commutation relations of the Lax and Orlov–Schulman operators take the correct form (6.5). Actually, this ad hoc reasoning can be justified more rigorously in the language of an auxiliary linear problem [18].

\(^9\) Equation (4.1) looks apparently different from (6.11), but actually, they are substantially the same. A careful inspection shows that \(H_0\) and \(\Phi_0(\mu, s)\) are defined to have an excess factor of \(\hbar\). Because of this factor, \(t_0\) have to be rescaled by \(h^{-2}\) rather than \(h^{-1}\).
This is not a contradiction, but necessary to obtain a non-trivial thermodynamics limit. In a similar sense, to obtain a non-trivial solution of the dispersionless Toda hierarchy, one has to start from an \( h \)-dependent solution of the \( h \)-independent Toda hierarchy, and let \( h \rightarrow 0 \) after rescaling the variables as (6.10) and (6.11). Moreover, for the rescaled solution to have a limit as \( h \rightarrow 0 \), one has to choose the \( h \)-dependent solution carefully.

### 6.3. Generalized string equations

Since we will not return to the Toda hierarchy in the rest of this paper, let us use the same notations \( L, M, \bar{L}, \bar{M} \), rather than \( \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}} \), for the Lax and Orlov–Schulman functions of the dispersionless Toda hierarchy. They are assumed to be the Laurent series of the form

\[
L = ap + \sum_{n=1}^{\infty} u_n p^{1-n}, \quad \bar{L} = a^{-1} p + \sum_{n=1}^{\infty} \bar{u}_n p^{1+n},
\]

\[
M = \sum_{k=1}^{\infty} k \bar{t}_k L^k + s + \sum_{n=1}^{\infty} v_n L^{-n}, \quad \bar{M} = -\sum_{k=1}^{\infty} k \bar{t}_k \bar{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n.
\]

More precisely, \( L \) and \( M \) are understood to be the Laurent series in a neighborhood of \( p = \infty \), and the other two in a neighborhood of \( p = 0 \). To consider generalized string equations, these four Laurent series are further assumed to have a common domain of convergence, say \( r_0 < |p| < r_1 \).

Solutions of the dispersionless Toda hierarchy in a general position [18] are characterized by functional equations of the form

\[
L = f(\bar{L}, \bar{M}), \quad M = g(\bar{L}, \bar{M}),
\]

(6.12)

where \( f = f(z, w) \) and \( g = g(z, w) \) are arbitrary functions of two variables that satisfy the symplectic condition

\[
\frac{df \wedge dg}{f} = dz \wedge dw.
\]

(6.13)

If the functional equations (6.12) have a solution, namely a quartet \((L, M, \bar{L}, \bar{M})\) of the Laurent series of the form shown above, they automatically give a solution of the dispersionless 2D Toda hierarchy [18]. Because of their origin in string theories [19–23], (6.12) are referred to as ‘generalized string equations’.

The goal of our consideration to the end of this section is to identify the solution (5.8) of the Riemann–Hilbert problem with a solution of the particular generalized string equations

\[
L = \bar{L}^{-1}, \quad L^{-1} M - \log \frac{L}{a_0} = -\bar{M} + \log \frac{\bar{L}^{-1}}{a_0},
\]

(6.14)

where \( a_0 \) is a positive constant that will be eventually identified with \( \Lambda_0 \). Let us mention that we have been unable to derive (6.14) from the first principle, namely from properties of the partition function. We reached (6.14) by guess work seeking analogy with the 5D counterparts. In the case of the 5D theory, we know a derivation from the first principle (see section 7).

It might be, however, possible to derive (6.14) from a suitable modification of the Eguchi–Yang model [36]. This random matrix model has a logarithmic term of the form \( \text{Tr}(X \log X - X) \) in the potential, which will lead to the logarithmic terms of (6.14). See also the work of Nekrasov and Marshakov [12, 13] on the origin of logarithmic terms in their approach to the thermodynamic limit.

A few technical remarks on (6.14) will be in order.

---

10 One can also consider a genuinely algebraic framework in which the coefficients of the Lax and Orlov–Schulman functions are meaningful as the formal Laurent series of \( p \). In such a formulation, the coefficients \( u_n, v_n, \bar{u}_n, \bar{v}_n \) are treated as the formal power series of \( T \) and \( \bar{T} \), whose coefficients are functions of \( s \) in a suitable class.
Equation (6.14) do not take the form shown in (6.12). This is not a serious problem. What is crucial is not this standard form itself but the symplectic condition (6.13). This condition ensures that the map \((z, w) \mapsto (f(z, w), g(z, w))\) connecting \((L, M)\) and \((\bar{L}, \bar{M})\) is symplectic with respect to the symplectic form \(dz \wedge dw\). In the case of (6.14), the relevant map \((z, w) \mapsto (\bar{z}, \bar{w})\) is defined implicitly by the equations
\[
z = \bar{z}^{-1}, \quad z^{-1}w - \log \frac{z}{a_0} = -\bar{z}\bar{w} + \log \frac{\bar{z}^{-1}}{a_0}. \tag{6.15}
\]
It is easy to show that this map is indeed symplectic:
\[
\frac{dz \wedge dw}{z} = \frac{d\bar{z} \wedge d\bar{w}}{\bar{z}}. \tag{6.16}
\]

The logarithmic terms in (6.14) have multi-valuedness in the vicinity of \(p = 0\) and \(p = \infty\) in which the Lax and Orlov–Schulman functions are assumed to have the Laurent expansion with respect to \(p\). This problem can be settled by a simple trick that we will present in the course of solving (6.14).

6.4. Solution of generalized string equations

Let us construct the solution of (6.14) that is to be identified with the solution of the Riemann–Hilbert problem. Apart from the trick for the logarithmic terms (which resembles a trick in the Eguchi–Yang model [36]), one can proceed in much the same way as the case of generalized string equations in the large-\(N\) Hermitian random matrix model [37].

The first equation of (6.14) implies that
\[
\mathcal{L} := L = \bar{L}^{-1}
\]
is a Laurent polynomial of the form
\[
\mathcal{L} = ap + b + ap^{-1}, \tag{6.17}
\]
where \(a\) and \(b\) are functions of \(s, T\) and \(\bar{T}\). This is nothing but the Lax function of the dispersionless 1D Toda hierarchy, namely the long-wave limit of (5.20).

As regards the second equation of (6.14), we add \(\log p\) to both sides and reorganize this equation as
\[
\mathfrak{M} := L^{-1}M - \log \frac{Lp^{-1}}{a_0} = -\bar{L}M + \log \frac{\bar{L}^{-1}p}{a_0}. \tag{6.18}
\]
The modified logarithmic terms, unlike \(\log(L/a_0)\) and \(\log(\bar{L}^{-1}/a_0)\), can be expanded to the Laurent series of \(p\) as
\[
\log \frac{Lp^{-1}}{a_0} = \log \frac{a}{a_0} + \log \left(1 + \frac{b}{a}p^{-1} + p^{-2}\right) = \log \frac{a}{a_0} + \frac{b}{a} p^{-1} + \left(1 - \frac{b^2}{2a^2}\right)p^{-2} + \cdots
\]
and
\[
\log \frac{\bar{L}^{-1}p}{a_0} = \log \frac{a}{a_0} + \log \left(1 + \frac{b}{a}p + p^2\right) = \log \frac{a}{a_0} + \frac{b}{a} p + \left(1 - \frac{b^2}{2a^2}\right)p^2 + \cdots.
\]
In particular, \(\mathfrak{M}\) itself is a Laurent series of \(p\).
Equation (6.18) splits into the following two equations:
\[
\mathcal{M} = L^{-1} M - \log \frac{L p^{-1}}{a_0}, \quad (6.19)
\]
\[
\mathcal{M} = -\tilde{L} M + \log \frac{\tilde{L}^{-1} p}{a_0}. \quad (6.20)
\]
Substituting
\[
M = \sum_{k=1}^{\infty} k T_k L^k + s + \sum_{n=1}^{\infty} v_n L^{-n},
\]
one can rewrite (6.19) as
\[
\mathcal{M} = \sum_{k=1}^{\infty} k T_k L^{k-1} + s L^{-1} + \sum_{n=1}^{\infty} v_n L^{-n-1} - \log \frac{a}{a_0} - \frac{b}{a^{-1}} + \cdots.
\]
The \((\ )_{>0}\) part and the \(p^0\) part give the equations
\[
(\mathcal{M})_{>0} = \sum_{k=2}^{\infty} k T_k (L^{k-1})_{>0} = \sum_{k=2}^{\infty} k T_k (\xi^{k-1})_{>0}, \quad (6.21)
\]
\[
(\mathcal{M})_0 = \sum_{k=1}^{\infty} k T_k (L^{k-1})_0 - \log \frac{a}{a_0} = \sum_{k=1}^{\infty} k T_k (\xi^{k-1})_0 - \log \frac{a}{a_0}. \quad (6.22)
\]
In the same way, (6.20) reads
\[
\mathcal{M} = \sum_{k=1}^{\infty} k \tilde{T}_k \tilde{L}^{k+1} - s \tilde{L} - \sum_{n=1}^{\infty} v_n \tilde{L}^{n+1} + \log \frac{a}{a_0} + \frac{b}{a} + \cdots,
\]
and the \((\ )_{<0}\) part and the \(p^0\) part give the equations
\[
(\mathcal{M})_{<0} = \sum_{k=2}^{\infty} k \tilde{T}_k (\tilde{L}^{1-k})_{<0} = \sum_{k=2}^{\infty} k \tilde{T}_k (\xi^{k-1})_{<0}, \quad (6.23)
\]
\[
(\mathcal{M})_0 = \sum_{k=1}^{\infty} k \tilde{T}_k (\tilde{L}^{1-k})_0 + \log \frac{a}{a_0} = \sum_{k=1}^{\infty} k \tilde{T}_k (\xi^{k-1})_0 + \log \frac{a}{a_0}. \quad (6.24)
\]
Equating the two expressions (6.22) and (6.24) of \((\mathcal{M})_0\), one obtains the explicit expression
\[
(\mathcal{M})_0 = \sum_{k=1}^{\infty} \frac{k (T_k + \tilde{T}_k)}{2} (\xi^{k-1})_0 \quad (6.25)
\]
of \((\mathcal{M})_0\) along with the equation
\[
\log \frac{a}{a_0} = \sum_{k=1}^{\infty} \frac{k (T_k - \tilde{T}_k)}{2} (\xi^{k-1})_0. \quad (6.26)
\]
From (6.21), (6.23) and (6.25), one finds that
\[
\mathcal{M} = \sum_{k=2}^{\infty} k T_k (\xi^{k-1})_{>0} + \sum_{k=2}^{\infty} k \tilde{T}_k (\xi^{k-1})_{<0} + \sum_{k=2}^{\infty} \frac{k (T_k + \tilde{T}_k)}{2} (\xi^{k-1})_0. \quad (6.27)
\]
Lastly, the \((\ )_{<0}\) part of (6.19) and the \((\ )_{>0}\) part of (6.20) yield some more equations. The equations from the \(p^{-1}\) terms of (6.19) and those from the \(p\) terms of (6.20) give the same equations
\[
\frac{s - b}{a} + \sum_{k=2}^{\infty} k (T_k - \tilde{T}_k) (\xi^{k-1})_{-1} = 0. \quad (6.28)
\]
This equation supplements (6.26) to determine $a$ and $b$ as functions of $(s, \mathbf{T}, \tilde{\mathbf{T}})$. The equations from higher orders of $p^{k+1}$ read

$$
\frac{v_n}{a^{n+1}} + \text{(linear combination of } v_1, \ldots, v_{n-1}) = \ldots (6.29)
$$

and

$$
\frac{\tilde{v}_n}{a^{n+1}} + \text{(linear combination of } \tilde{v}_1, \ldots, \tilde{v}_1) = \ldots (6.30)
$$

These equations determine $v_n$ and $\tilde{v}_n$ recursively once $a$ and $b$ are obtained. One can thus construct a (unique) solution of the generalized string equations (6.14).

6.5. Identification

As remarked above, the first equation of (6.14) is a reduction condition to the dispersionless 1D Toda hierarchy. If this equation holds, then

$$
B_k + \tilde{B}_k = L^k; \quad k = 1, 2, \ldots (6.31)
$$

hence,

$$
\frac{\partial \mathcal{L}}{\partial T_k} + \frac{\partial \mathcal{L}}{\partial \tilde{T}_k} = \{\mathcal{L}, \mathcal{L}\} = 0. (6.32)
$$

This implies that $a$ and $b$ (and, actually, all other dynamical variables) depend on $\mathbf{T}$ and $\tilde{\mathbf{T}}$ through the difference $\mathbf{T} - \tilde{\mathbf{T}}$ only:

$$
a = a(s, \mathbf{T} - \tilde{\mathbf{T}}), \quad b = b(s, \mathbf{T} - \tilde{\mathbf{T}}). (6.33)
$$

This explains why $T_k$ and $\tilde{T}_k$ show up in (6.26) and (6.28) in the form of the difference $T_k - \tilde{T}_k$.

Let us now restrict the time evolutions to the anti-diagonal subspace

$$
T_k = -\tilde{T}_k = \frac{t_k}{2}, \quad k = 1, 2, \ldots (6.34)
$$

The reduced Lax function $\mathcal{L}$ therein satisfies the standard Lax equations

$$
\frac{\partial \mathcal{L}}{\partial t_k} = \{A_k, \mathcal{L}\}, \quad A_k = \frac{1}{2}(\mathcal{L}^k)_{>0} - \frac{1}{2}(\mathcal{L}^k)_{<0} (6.35)
$$

of the dispersionless 1D Toda hierarchy. Moreover, (6.26), (6.28) and (6.27) take the reduced form

$$
\log \frac{a}{a_0} = \sum_{k=1}^{\infty} \frac{k t_k}{2} (\mathcal{L}^{k-1})_{>0}, (6.36)
$$

$$
\frac{s - b}{a} + \sum_{k=2}^{\infty} k t_k (\mathcal{L}^{k-1})_{-1} = 0, (6.37)
$$

$$
\mathcal{M} = \sum_{k=2}^{\infty} \frac{k t_k}{2} ((\mathcal{L}^{k-1})_{>0} - (\mathcal{L}^{k-1})_{<0}), (6.38)
$$
For (6.36) and (6.37) to coincide with (5.17) and (5.18), it is sufficient that the building blocks of the Riemann–Hilbert problem and the generalized string equations are identified as
\[ a_0 = \Lambda_0, \quad a = \Lambda, \quad b = \beta, \quad y = p, \quad z(y) = \mathcal{Z}. \quad (6.39) \]
Comparing (6.38) with (5.24), one finds that \( W(z) \) and \( \mathfrak{M} \) are related as
\[ W(z) + \log y(z) + \frac{V'(z)}{2} = N(z)\sqrt{P(z)} = \mathfrak{M}. \quad (6.40) \]
Thus, (5.8) can be identified with a solution of the dispersionless Toda hierarchy that satisfies the generalized string equations (6.14). Let us stress that the generalized string equations are a nonlinear analog of (so to speak, modern) matrix Riemann–Hilbert problems that are widely used in the theory of integrable systems [18]. We thus have a nonlinear (ultra-modern) Riemann–Hilbert problem for \( L, M, \bar{L}, \bar{M} \) on the \( p \)-plane. On the other hand, we have another (classic) Riemann–Hilbert problem for \( W(z) \) on the \( z \)-plane. This is a very interesting interplay of two Riemann–Hilbert problems of quite different natures.

7. Solution of the Riemann–Hilbert problem for 5D theory

7.1. Construction of the solution when \( t = 0 \)

As in the case of the 4D theory, we can deductively solve the Riemann–Hilbert problem in the case where \( t = 0 \).

When \( t = 0 \), one can rewrite (4.28) and (4.29) as
\[ \left( W(u + i0) + \frac{R(u - s)}{2} \right) + \left( W(u - i0) + \frac{R(u - s)}{2} \right) = 0 \quad \text{for} \quad u_0 \leq u \leq u_1 \]
and
\[ \left( W(u + i0) + \frac{R(u - s)}{2} \right) - \left( W(u - i0) + \frac{R(u - s)}{2} \right) = \begin{cases} 0 & (u > u_1), \\ -2\pi i & (u < u_0). \end{cases} \]

One can see from these equations that the complex function \( e^{W(z)+R(z-s)/2} + e^{-W(z)-R(z-s)/2} \) has no discontinuity along the real axis, and hence becomes a holomorphic function on the whole \( z \)-plane \( C \). Moreover, by the other conditions in the Riemann–Hilbert problem, this function is periodic with respect to the translation \( z \mapsto z + 2\pi i/R \) and behaves as
\[ e^{W(z)+R(z-s)/2} + e^{-W(z)-R(z-s)/2} = \frac{e^{-R(z-s)}}{R\Lambda_0} + O(1) \]
as \( \text{Re} \ z \to -\infty \) and
\[ e^{W(z)+R(z-s)/2} + e^{-W(z)-R(z-s)/2} = \frac{1}{R\Lambda_0} + R\Lambda_0 + O(e^{-R|z|}) \]
as \( \text{Re} \ z \to +\infty \). Therefore, by Liouville’s theorem, one can deduce that the identity
\[ e^{W(z)+R(z-s)/2} + e^{-W(z)-R(z-s)/2} = \frac{1}{R\Lambda_0} + R\Lambda_0 - \frac{e^{-R(z-s)}}{R\Lambda_0} \quad (7.1) \]
holds.

Let us introduce
\[ y = e^{-W(z)-R(z-s)/2}, \quad \beta = (1 + (R\Lambda_0)^2) e^{-Rs}, \quad \Lambda = \Lambda_0 e^{-Rs} \]
to rewrite the last identity as
\[ y + y^{-1} = \frac{\beta - e^{-Rs}}{R\Lambda}. \quad (7.2) \]
Viewed as an equation of a complex analytic curve, (7.2) defines substantially the same Seiberg–Witten curve as previously considered for the undeformed 5D $U(1)$ gauge theory [14].

One can solve this equation for $y$ as

$$y = y(z) = \frac{\beta - e^{-Rz} - \sqrt{P(z)}}{2R}, \quad P(z) = (e^{-Rz} - \beta)^2 - 4(R\Lambda)^2.$$  \hspace{1cm} (7.3)

$u_0$ and $u_1$ are determined to be the endpoint of the interval of $R$ where $P(u) < 0$, namely

$$u_0 = -\frac{\log(\beta + 2R\Lambda)}{R} = s - \frac{2\log(1 + R\Lambda)}{R},$$

$$u_1 = -\frac{\log(\beta - 2R\Lambda)}{R} = s - \frac{2\log(1 - R\Lambda)}{R}.$$  \hspace{1cm} (7.4)

$\sqrt{P(z)}$ is the branch on the $z$-plane cut along the intervals $[u_0 + 2\pi i n/R, u_1 + 2\pi i n/R, n \in \mathbb{Z}$, such that

$$\sqrt{P(z)} = e^{-Rz} - \beta + O(e^{Rz}) \quad \text{as} \quad \text{Re } z \to -\infty,$$

$$\sqrt{P(z)} = -\beta^2 - 4(R\Lambda)^2 + O(e^{-Rz}) \quad \text{as} \quad \text{Re } z \to +\infty,$$

$$\mp \text{Im} \sqrt{P(u \pm i0)} > 0 \quad \text{for} \quad u_0 < u < u_1.$$  \hspace{1cm} (7.5)

Solving

$$e^{-W(z) - R(z-s)/2} = y(z)$$

for $W(z)$ yields a (unique) solution of the Riemann–Hilbert problem explicitly as

$$W(z) = \frac{R(z-s)}{2} - \log y(z).$$  \hspace{1cm} (7.6)

As explained in the case of the 4D theory, it is easy to reconfirm directly that this function fulfills all requirements of the Riemann–Hilbert problems. The minimizer $\rho_4^{(0)}(u)$ has the same expression as the 4D case (5.7).

7.2. Construction of solution when $t \neq 0$

Let us now consider the case where $t \neq 0$. We seek a solution of the Riemann–Hilbert problem by modifying the solution for $t \neq 0$ as

$$W(z) = \frac{R}{2}(z-s) - \log y(z) + N(z)\sqrt{P(z)} - \frac{V'(z)}{2},$$  \hspace{1cm} (7.7)

where

- $y(z)$ is a solution

$$y(z) = \frac{\beta - e^{-Rz} - \sqrt{P(z)}}{2R}, \quad P(z) = (e^{-Rz} - \beta)^2 - 4(R\Lambda)^2.$$  \hspace{1cm} (7.8)

of the equation

$$y + y^{-1} = \frac{\beta - e^{-Rz}}{R\Lambda}$$  \hspace{1cm} (7.9)

of the deformed Seiberg–Witten curve.

- $\beta$ and $\Lambda$ are functions $\beta(s,t)$ and $\Lambda(s,t)$ of $s$ and $t$ that reduces to the previous values at $t = 0$:

$$\beta(s, 0) = (1 + (R\Lambda_0)^2)e^{-Rs}, \quad \Lambda(s, 0) = \Lambda_0 e^{-Rs}.$$  \hspace{1cm} (7.10)
\( u_0 \) and \( u_1 \) are the real zeros of \( P(z) \):

\[
\begin{align*}
u_0 &= -\log\left(\beta + \frac{2R\Lambda_1}{R}\right), & u_1 &= -\log\left(\frac{\beta - 2R\Lambda_1}{R}\right). \\
(7.11)
\end{align*}
\]

\( \sqrt{P(z)} \) is the branch on the \( z \)-plane cut along the intervals \([u_0 + 2\pi in/R, u_1 + 2\pi in/R], n \in \mathbb{Z}\), such that

\[
\begin{align*}
\sqrt{P(z)} &= e^{-Rz} - \beta + O(e^{Rz}) & \text{as } \text{Re } z \to -\infty, \\
\sqrt{P(z)} &= -\sqrt{\beta^2 - 4(R\Lambda_1)^2} + O(e^{Rz}) & \text{as } \text{Re } z \to +\infty, \\
\mp \text{Im } \sqrt{P(u \pm i0)} &> 0 \quad \text{for } u_0 < u < u_1.
\end{align*}
\]

(7.12)

\( N(z) \) is a linear combination

\[
N(z) = \sum_{k=1}^{\infty} t_k N_k(z)
\]

of polynomials \( N_k(z) \) in \( e^{-Rz} \).

Equations (4.28) and (4.29) are already satisfied by the ansatz (7.7). Moreover, since \( P(z) \), \( N(z) \) and \( V(z) \) are periodic with respect to the translation \( z \mapsto z + 2\pi i/R \), the periodicity condition of \( W(z) + R(z - s)/2 \) is also satisfied. Therefore, it is sufficient to fulfill the boundary conditions (4.30) and (4.31).

To this end, we choose \( N_k(z) \) to satisfy the condition

\[
N_k(z) + \frac{Rk}{2} e^{-Rkz} = O(e^{Rz}) \quad \text{as } \text{Re } z \to -\infty.
\]

(7.13)

\( N_k(z) \) is uniquely determined by this condition as

\[
N_k(z) = -\frac{Rk}{2} (e^{-R(k-1)z} + c_1 e^{-R(k-2)z} + \ldots + c_{k-1}),
\]

(7.14)

where \( c_1, c_2, \ldots \) are the coefficients in the expansion

\[
\frac{1}{\sqrt{P(z)}} = e^{Rz} + c_1 e^{2Rz} + c_2 e^{3Rz} + \ldots,
\]

and we set \( c_0 = 1 \) for convenience. Consequently, as \( \text{Re } z \to -\infty \) in the domain \( \pm \text{Im } z > 0 \), \( W(z) \) behaves as

\[
W(z) = -\frac{R(z - s)}{2} - \log\left(\frac{\beta - e^{-Rz} - \sqrt{P(z)}}{2R\Lambda_1}\right) + \sum_{k=1}^{\infty} \frac{Rk t_k c_k}{2} (c_k e^{Rz} + c_{k+1} e^{R(k+1)z} + \ldots) \sqrt{P(z)} = \frac{R(z + s)}{2} \mp \pi i + \log(R\Lambda_1) + \sum_{k=1}^{\infty} \frac{Rk t_k c_k}{2} + O(e^{Rz}).
\]

One thus finds the equation

\[
Rs + \log \frac{\Lambda}{\Lambda_1} + \sum_{k=1}^{\infty} \frac{Rk t_k c_k}{2} = 0
\]

(7.15)

for the first boundary condition (4.30) to be satisfied.
On the other hand, on the opposite end (Re \( z \to +\infty \)) of the \( z \)-plane, \( N(z) \), \( P(z) \) and \( y(z) \) converge to finite values as

\[
\lim_{\text{Re} z \to +\infty} N(z) = -\sum_{k=1}^{\infty} \frac{Rk_1c_{k-1}}{2},
\]

\[
\lim_{\text{Re} z \to +\infty} \sqrt{P(z)} = -\sqrt{\beta^2 - 4(R\Lambda)^2},
\]

\[
y_\infty : = \lim_{\text{Re} z \to +\infty} y(z) = \frac{\beta + \sqrt{\beta^2 - 4(R\Lambda)^2}}{2R\Lambda}
\]

and \( V(z) \) disappears. Therefore, \( W(z) \) behaves as

\[
W(z) = -\frac{R(z-s)}{2} - \log y_\infty + \sum_{k=1}^{\infty} \frac{Rk_1c_{k-1}}{2}\sqrt{\beta^2 - 4(R\Lambda)^2} + O(e^{-Rc}).
\]

One thus obtains the equation

\[
-\log y_\infty + \sum_{k=1}^{\infty} \frac{Rk_1c_{k-1}}{2}\sqrt{\beta^2 - 4(R\Lambda)^2} = \log(R\Lambda_0)
\]

for the second boundary condition (4.31) to be satisfied.

Thus, the problem reduces to solving (7.15) and (7.16) for \( \beta = \beta(s, t) \) and \( \Lambda = \Lambda(s, t) \).

The implicit function theorem ensures the existence of a solution in a neighborhood of \( t = 0 \) that satisfies the initial conditions (7.10). Note that (7.15) and (7.16) reduce to

\[
Rs + \log \frac{\Lambda}{\Lambda_0} = 0,
\]

\[
\frac{\beta + \sqrt{\beta^2 - 4(R\Lambda)^2}}{2R\Lambda} = \frac{1}{R\Lambda_0}
\]

when \( t = 0 \). It is easy to see that (7.10) gives a solution of these equations. Lastly, the minimizer \( \rho_{(0)}(u) \) turns out to have the same expression as the 4D case (5.19).

### 7.3. Rewriting solution in terms of the Lax function

Let us rewrite the foregoing solution of the Riemann–Hilbert problem in terms of the new variables

\[
Z = e^{-Rc}.
\]

Note that the cylinder \( C/((2\pi i/R)Z) \) is thereby mapped to the punctured plane \( C^* = C \setminus \{0\} \).

\( P(z) \) and \( N_k(z) \) thereby turn into polynomials in \( Z \) as

\[
P(z) = (Z - \beta)^2 - 4(R\Lambda)^2,
\]

\[
N_k(z) = -\frac{Rk}{2}(Z^{k-1} + c_1Z^{k-2} + \cdots + c_{k-1}).
\]

The \( c_n \) may be thought of as the coefficients of the Laurent expansion

\[
\frac{1}{\sqrt{(Z - \beta)^2 - 4(R\Lambda)^2}} = Z^{-1} + c_2Z^{-2} + c_3Z^{-3} + \cdots
\]

at \( Z = \infty \), and \( N_k(z) \) are uniquely determined by the condition

\[
N_k(z) + \frac{RkZ^k}{2\sqrt{(Z - \beta)^2 - 4(R\Lambda)^2}} = O(Z^{-1}) \quad \text{as} \quad Z \to \infty.
\]

Another expression of \( N_k(z) \) is the contour integral formula

\[
N_k(z) = -\frac{1}{2\pi i} \oint_{C} \frac{dX}{X - Z} \frac{RkX^k}{2\sqrt{(X - \beta)^2 - 4(R\Lambda)^2}} = \frac{RkZ^k}{2\sqrt{(Z - \beta)^2 - 4(R\Lambda)^2}},
\]

(7.18)
where $C$ is a simple closed curve that encircles the interval $[e^{-Rt_1}, e^{-Rt_0}]$ anti-clockwise and leaves $e^{-Rt}$ outside.

The function $y(z)$ can be redefined as a function of $Z$ as

$$y(Z) = \frac{\beta - Z - \sqrt{P(z)}}{2RA},$$

(7.19)

which is an inverse function of

$$Z(y) = \beta - R\Lambda(y + y^{-1}).$$

This function $Z(y)$ plays the role of a Lax function$^{11}$. As in the case of the 4D theory, one can derive the following identities:

$$(Z(y)^k)_{>0} - (Z(y)^k)_{<0} = (Z^{k-1} + c_1 Z^{k-2} + \cdots + c_{k-1}) \sqrt{P(z)},$$

(7.20)

$$(Z(y)^k)_{0} = c_k, \quad (Z(y)^k)_{-1} = -\frac{c_{k+1} - \beta c_k}{2RA}.$$  

(7.21)

One can thereby rewrite the third term of (7.7) as

$$N(z) \sqrt{P(z)} = -\sum_{k=1}^{\infty} \frac{Rk\ell_k}{2} ((Z(y)^k)_{>0} - (Z(y)^k)_{<0})$$

(7.22)

and (7.15) as

$$Rs + \log \frac{\Lambda}{\Lambda_0} + \sum_{k=1}^{\infty} \frac{Rk\ell_k}{2} (Z(y)^k)_0 = 0.$$  

(7.23)

Unlike (4.28), (4.29) and (4.30), it seems difficult to interpret (7.16) correctly within this framework. Equation (7.16) is derived to fulfill the boundary condition (4.31) as $\text{Re} z \to +\infty$; in other words, $Z \to 0 \ (\text{rather than } Z \to \infty)$. The situation of the 5D theory thus turns out to be considerably different from the 4D theory.

8. Solution of the dispersionless Toda hierarchy for 5D theory

8.1. Generalized string equations

The goal of this section is twofold. Firstly, following the method of section 6, we show that the solution (7.7) of the Riemann–Hilbert problem corresponds to a solution of the particular generalized string equations

$$L = \tilde{L}^{-1}, \quad e^{-RM} = Q^{-1} \tilde{L}^{-2} e^{R\tilde{M}}.$$  

(8.1)

Secondly, we derive these equations from hidden symmetries of a tau function $\tau_s(T, \tilde{T})$ of the Toda hierarchy [8, 9]. As we briefly review therein, this tau function reduces to the partition function $Z^{5D}_s(t)$ when the time variables $(T, \tilde{T})$ are restricted to the anti-diagonal subspace.

Let us remark that (8.1) can be converted to a form that resembles the generalized string equations (6.14) of the 4D theory. Rewrite the term $Q^{-1} \tilde{L}^{-2}$ on the right-hand side of the second equation as

$$Q^{-1} \tilde{L}^{-2} = \frac{L}{Q^{1/2}} \cdot \frac{\tilde{L}^{-1}}{Q^{1/2}},$$

move the factor $L/Q^{-1/2}$ to the left-hand side and take the logarithm of both hand sides. One thus obtains the equations

$^{11}$ Strictly speaking, it is $-Z(y)$ rather than $Z(y)$ that correspond to the Lax function $\Sigma$ (see section 8).
\[ L = \bar{L}^{-1}, \quad -RM - \log \frac{L}{Q^{1/2}} = R\bar{M} + \log \frac{\bar{L}^{-1}}{Q^{1/2}} \]  

(8.2)

with remarkable similarity with (6.14).

Unfortunately, (8.2) turn out to be not strong enough to characterize the solution of the Riemann–Hilbert problem uniquely. As one finds in the course of solving these string equations, (4.28), (4.29) and (4.30) are correctly encoded in the generalized string equations, but the second boundary condition (4.31) is missing therein. Because of this, our solution of the generalized string equations contains arbitrariness. This arbitrariness can be fixed by imposing (4.31) by hand, but its status in the framework of the dispersionless Toda hierarchy is obscure.

Thus, although looking very similar, the 5D generalized string equations (8.2) are drastically different from the 4D version (6.14). We can point out technical reasons leading to this difference, but a true reason is beyond our scope.

8.2. Solving generalized string equations

The method for solving (8.2) is parallel to the case of the 4D theory.

The first equation of (8.2) again leads to the Lax function

\[ \mathcal{L} := L = \bar{L}^{-1} = ap + b + ap^{-1} \]  

(8.3)

of the dispersionless 1D Toda hierarchy. \( a = a(s, T, \bar{T}) \) and \( b = b(s, T, \bar{T}) \) are arbitrary at this stage. We now derive equations for these functions from the second equation of (8.2).

Adding \( \log p \) to both hand sides, one can rewrite the second equation of (8.2) as

\[ \mathfrak{M} := -RM - \log \frac{L_{p^{-1}}}{Q^{1/2}} = R\bar{M} + \log \frac{\bar{L}^{-1}p}{Q^{1/2}}, \]  

(8.4)

and separate this equation to two equations:

\[ \mathfrak{M} = -RM - \log \frac{L_{p^{-1}}}{Q^{1/2}}, \]  

(8.5)

\[ \mathfrak{M} = R\bar{M} + \log \frac{\bar{L}^{-1}p}{Q^{1/2}}. \]  

(8.6)

The ( )\( >0 \) part of (8.5) and the ( )\( <0 \) part of (8.6) give

\[ (\mathfrak{M})_{>0} = -\sum_{k=1}^{\infty} RkT_k(\mathcal{L}^k)_{>0}, \quad (\mathfrak{M})_{<0} = -\sum_{k=1}^{\infty} Rk\bar{T}_k(\mathcal{L}^k)_{<0}. \]  

(8.7)

The ( )\( _0 \) parts of (8.5) and (8.6) read

\[ (\mathfrak{M})_0 = -Rs - \log \frac{a}{Q^{1/2}} - \sum_{k=1}^{\infty} RkT_k(\mathcal{L}^k)_0, \]  

(8.8)

\[ (\mathfrak{M})_0 = Rs + \log \frac{a}{Q^{1/2}} - \sum_{k=1}^{\infty} Rk\bar{T}_k(\mathcal{L}^k)_0. \]  

(8.9)

Equating these two expressions yields the explicit expression

\[ (\mathfrak{M})_0 = -\sum_{k=1}^{\infty} \frac{Rk(T_k + \bar{T}_k)}{2}(\mathcal{L}^k)_0 \]  

(8.10)
and the equation

\[ Rs + \log \frac{a}{Q^{1/2}} + \sum_{k=1}^{\infty} \frac{Rk(T_k - \tilde{T}_k)}{2} (\Omega^k)_0 = 0. \]  

Note that (8.11) amounts to (6.26) in the case of 4D theory. Thus, one finds that

\[ \mathcal{M} = - \sum_{k=1}^{\infty} RkT_k (\Omega^k)_{<0} - \sum_{k=1}^{\infty} Rk\tilde{T}_k (\Omega^k)_{>0} - \sum_{k=1}^{\infty} \frac{Rk(T_k + \tilde{T}_k)}{2} (\Omega^k)_0. \]  

On the other hand, from the \((\quad)_{<0}\) part of (8.5) and the \((\quad)_{>0}\) part of (8.6), one obtains the recursive formulae

\[ \frac{R\nu_n}{a^n} + \text{(linear combination of } \nu_1, \ldots, \nu_{n-1}) \]

\[ = - \sum_{k=1}^{\infty} Rk(T_k - \tilde{T}_k) (\Omega^k)_{<n} - (\log(Lp^{-1}/Q^{1/2}))_{<n} \]  

(8.13)

and

\[ \frac{R\tilde{\nu}_n}{a^n} + \text{(linear combination of } \tilde{\nu}_1, \ldots, \tilde{\nu}_1) \]

\[ = - \sum_{k=1}^{\infty} Rk(T_k - \tilde{T}_k) (\Omega^k)_{>n} - (\log(L^{-1}p/Q^{1/2}))_{>n} \]  

(8.14)

of \(\nu_n\) and \(\tilde{\nu}_n\) for \(n = 1, 2, \ldots\). Note that these equations are all that one can derive from the \((\quad)_{<0}\) part of (8.5) and the \((\quad)_{>0}\) part of (8.6). There is no equation that amounts to (6.28) in the case of the 4D theory.

Thus, unlike the case of the 4D theory, we have only one equation (8.11) for the two unknown functions \(a\) and \(b\). Therefore, these functions remain to be fully determined. This is the aforementioned ‘arbitrarieness’. In this sense, these generalized string equations (8.2) are incomplete.

From a genuinely technical point of view, this phenomenon stems from a delicate difference of the structures of (6.14) and (8.2). In the second equation of (6.14), \(M\) and \(\mathcal{M}\) are multiplied by \(L^{-1}\) and \(\tilde{L}\). Because of the presence of these multipliers, when this equation is split into (6.19) and (6.20) and expanded in powers of \(p\), the \(p^{-1}\) terms in (6.19) and the \(p\) terms in (6.20) yield exceptional equations. These equations, which are mutually equivalent, are nothing but the second equation (6.28) for \(a\) and \(b\). In the case of the second equation of (8.2), there is no such exceptional terms, hence no counterpart of (6.28).

Let us now restrict the time evolutions to the anti-diagonal subspace. As the main result of our previous work [8] predicts, the reduced time variables \(t_k\) should be twisted by the signature factor \((-1)^k\) as

\[ T_k = -\tilde{T}_k = (-1)^k \frac{t_k}{2}, \quad k = 1, 2, \ldots \]  

(8.15)

Equations (8.11) and (8.12) then become the following equations:

\[ Rs + \log \frac{a}{Q^{1/2}} + \sum_{k=1}^{\infty} \frac{(-1)^k Rk t_k}{2} (\Omega^k)_0 = 0, \]  

(8.16)

\[ \mathcal{M} = - \sum_{k=1}^{\infty} \frac{(-1)^k Rk t_k}{2} ((\Omega^k)_{>0} - (\Omega^k)_{<0}). \]  

(8.17)

Equation (8.16) coincides with (7.15), if the building blocks of the Riemann–Hilbert problem and the generalized string equations are identified as

\[ Q = (R\Lambda_0)^2, \quad a = R\Lambda, \quad b = -\beta, \quad y = p, \quad Z = e^{-Rc} = -\Lambda. \]  

(8.18)
Note that the sign factor $(-1)^k$ in (8.15) is correlated with the negative sign in the last relation of (8.18) between $Z$ and $\mathcal{L}$. Comparing (8.17) with (7.22), one finds that $W(z)$ and $\mathcal{R}$ are related as

$$W(z) + \frac{R}{2}(z-s) + \log y(z) + \frac{V'(z)}{2} = N(z)\sqrt{P(z)} = -\mathcal{R}. \quad (8.19)$$

These are all that one can derive from the generalized string equations; equation (7.16) is missing here, and has to be added by hand for $a$ and $b$ to be determined. Thus, the aforementioned ‘arbitrariness’ is not resolved even in the diagonal subspace (8.15).

### 8.3. Deriving generalized string equations

We now turn to the second subject of this section, namely the origin of the generalized string equations in hidden symmetries of a tau function of the Toda hierarchy.

As observed in our previous work [8], the partition function of the 5D theory is connected with a tau function $\tau_s(T, \bar{T})$ of the Toda hierarchy as

$$Z^\text{5D}_s(t) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}\right) q^{-(s+1)(2r+1)/6} \times \tau_s(T, \bar{T})|_{T_k = -\bar{T}_k = (-1)^{s/2} q^{k/2}}. \quad (8.20)$$

This tau function $\tau_s(T, \bar{T})$ has the fermionic expression

$$\tau_s(T, \bar{T}) = \langle s | \exp\left(\sum_{k=1}^{\infty} T_k \gamma_k\right) g \exp\left(-\sum_{k=1}^{\infty} \bar{T}_k \gamma_{-k}\right) | s \rangle, \quad (8.21)$$

where $g$ is the $\text{GL}(\infty)$ element

$$g = q^{W_0/2} G_s Q^{k_s} G_s^* Q^{W_0/2}. \quad (8.22)$$

It is also pointed out in the same work (loc. cit.) that $g$ satisfies the algebraic relations

$$J_k g = g J_{-k}, \quad k = 1, 2, \ldots \quad (8.23)$$

These relations can be converted to the equations

$$\left(\frac{\partial}{\partial T_k} + \frac{\partial}{\partial \bar{T}_k}\right) \tau_s(T, \bar{T}) = 0, \quad k = 1, 2, \ldots \quad (8.24)$$

for the tau function, which thereby reduces to a function of $T - \bar{T}$:

$$\tau_s(T, \bar{T}) = \tau_s^{1D}(T - \bar{T}). \quad (8.25)$$

The function $\tau_s^{1D}(T)$ is a tau function of the 1D Toda hierarchy.

In the language of the Lax operators [21, 23], (8.23) amounts to the relations

$$L^k = \tilde{L}^{-k}, \quad k = 1, 2, \ldots, \quad (8.26)$$

which reduce to the single equation

$$\mathcal{L} := L = \tilde{L}^{-1}. \quad (8.27)$$

This implies that $\mathcal{L}$ becomes a difference operator of the form (5.20), namely the Lax operator of the 1D Toda lattice. One can thus reconfirm that the foregoing tau function is a special solution of the 1D Toda hierarchy.

Actually, (8.23) are merely a small part of a large set of relations satisfied by $g$. Those relations are derived from the following two types of ‘shift symmetries’ [8] among the elements:

$$V^{(k)}_m = q^{-km/2} \sum_{n \in \mathbb{Z}} q^{kn} \psi_{m-n} \psi^*_{n+},$$

$$\bar{V}^{(k)}_m = V^{(k)}_m - \frac{q^k}{1-q^k}\delta_{m,0}, \quad k, m \in \mathbb{Z},$$

of the quantum torus algebra.
First shift symmetry
\[ G_- G_+ \tilde{V}^{(k)}_m (G_- G_+)^{-1} = (-1)^k \tilde{V}^{(k)}_{m+k}. \]  

Second shift symmetry
\[ q_{W_0}^{\,2} \tilde{V}^{(k)}_m q_{W_0}^{\,2} = \tilde{V}^{(k-m)}_m \]
equivalently,
\[ q_{W_0}^{\,2} \tilde{V}^{(k)}_m q_{W_0}^{\,2} = \tilde{V}^{(k-m)}_m. \]

Using these symmetries repeatedly, one can rewrite \( V^{(k)}_m \) as
\[ \tilde{V}^{(k)}_m = q_{W_0}^{\,k} G_- G_+ L^m G_- G_+ q_{W_0}^{\,k} \]
\[ = (-1)^k q_{W_0}^{\,k} G_- G_+ L^{m+1} G_- G_+ q_{W_0}^{\,k} \]
\[ = (-1)^k q_{W_0}^{\,k} G_- G_+ L^{m+1} G_- G_+ q_{W_0}^{\,2k} \]
\[ = (-1)^k q_{W_0}^{\,k} G_- G_+ L^{m+1} G_- G_+ q_{W_0}^{\,2k} \]
\[ = q^{\,k} q_{W_0}^{\,k} G_- G_+ L^{m+1} G_- G_+ q_{W_0}^{\,2k} \]
\[ = q^{\,k} q_{W_0}^{\,k} G_- G_+ L^{m+1} G_- G_+ q_{W_0}^{\,2k} \]
\[ = q^{\,k} \tilde{V}^{(k-k)}_{m-2k}, \]

and thus obtains the relations
\[ \tilde{V}^{(k)}_m = g Q^{-k} \tilde{V}^{(k-k)}_{m-2k}. \]  
Equation (8.23) is contained therein as a special case where \( k = 0 \) and \( m \geq 1 \).

When \( m = 0 \) and \( k \geq 1 \), (8.31) becomes the relation
\[ \tilde{V}^{(k)}_0 = q^{\,k} \tilde{V}^{(k-k)}_{-2k}. \]

According to the general rule [21, 23, 28], \( \tilde{V}^{(k)}_m \) and \( \tilde{V}^{(k-k)}_{-2k} \) correspond to \( q^{-k} q^{\,k} L^m \) and \( q^{-k} q^{\,k} L^m \), and these relations imply the equations
\[ q^{\,k} = Q^{-k} q^{\,k} L^{-2k} q^{\,k}, \quad k = 1, 2, \ldots, \]  
for the Lax and Orlov–Schulman operators. Taking into account the twisted canonical relations (6.3), one can easily see that these equations reduce to the single equation
\[ q^{\,k} = (Q q)^{-1} L^{-2k} q^{\,k}. \]

Thus, the special solution of the Toda hierarchy determined by (8.22) turns out to satisfy (8.27) and (8.34). These equations may be thought of as the generalized string equations in the Toda hierarchy. Lastly, setting \( q = e^{\,R}, \) rescaling \( M \) and \( M' \) as (6.10) and letting \( h \to 0 \), these equations turn into the generalized string equations (8.1) for the dispersionless Toda hierarchy.

9. Concluding remarks

We have identified the Seiberg–Witten curves (5.10) and (7.9) for the deformed \( U(1) \) gauge theories. What about the Seiberg–Witten differential and prepotential? This issue is briefly discussed in our previous paper [1] (though not in a fully correct form, because of the incorrect formulation of the Riemann–Hilbert problem). We reconsider this issue in the present setup.

After the idea proposed in the previous paper, we consider the critical values
\[ \mathcal{E}_s^{\,4D(0)} = e^{\,4D(0)} |\rho_s^{(0)}|, \quad \mathcal{E}_s^{\,5D(0)} = e^{\,5D(0)} |\rho_s^{(0)}| \]
of the energy functionals as functions of \((s, t)\). As the following calculations demonstrate, they may be thought of as the prepotentials.

Let us show that the \(t\)-derivatives of these functions can be expressed as

\[
\frac{\partial \mathcal{E}^{(4D)}_s(s, t)}{\partial t_k} = \int_{u_0}^{u_1} \frac{d^{k+1}}{k+1} \rho_s^{(0)}(u), \quad (9.1)
\]

\[
\frac{\partial \mathcal{E}^{(5D)}_s(s, t)}{\partial t_k} = \int_{u_0}^{u_1} \frac{e^{-Ru}}{Rk} \rho_s^{(0)}(u) \quad (9.2)
\]

Since the treatment of the 4D and 5D cases are parallel, we illustrate the calculations for the 4D case only. Since \(\rho_s^{(0)}\) is a solution of \((4.12)\), the identity

\[
\mathcal{P}\int_{u_0}^{u_1} dv g^{(0)}_{4D}(u - v) \rho_s^{(0)}(v) + \frac{V(u)}{2} = 0 \quad (u_0 \leq u \leq u_1)
\]

holds. Integrated twice with respect to \(u\), it turns into the identity\(^\text{12}\)

\[
\int_{u_0}^{u_1} dv g^{(0)}_{4D}(u - v) \rho_s^{(0)}(v) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{t_k u^{k+1}}{k+1} + C_1 u + C_2 = 0 \quad (u_0 \leq u \leq u_1),
\]

where \(C_1\) and \(C_2\) are functions of \(t\) only. Differentiating the explicit form

\[
\mathcal{E}^{(4D)}_s = \int_{u_0}^{u_1} du dv g^{(0)}_{4D}(u - v) \rho_s^{(0)}(v) + \int_{u_0}^{u_1} du \left( \sum_{k=1}^{\infty} \frac{t_k u^{k+1}}{k+1} \right) \rho_s^{(0)}(u)
\]

of the critical value with respect to \(t_k\) yields

\[
\frac{\mathcal{E}^{(4D)}_s}{\partial t_k} = \int_{u_0}^{u_1} du \frac{\partial \rho_s^{(0)}(u)}{\partial t_k} \left( 2 \int_{u_0}^{u_1} dv g^{(0)}_{4D}(u - v) \rho_s^{(0)}(v) + \sum_{k=1}^{\infty} \frac{t_k u^{k+1}}{k+1} \right)
\]

\[
+ \int_{u_0}^{u_1} du \frac{u^{k+1}}{k+1} \rho_s^{(0)}(u).
\]

We can use \((9.3)\) to simplify the part inside the parenthesis as

\[
\frac{\mathcal{E}^{(4D)}_s}{\partial t_k} = -2 \int_{u_0}^{u_1} du \frac{\partial \rho_s^{(0)}(u)}{\partial t_k} (C_1 u + C_2) + \int_{u_0}^{u_1} du \frac{u^{k+1}}{k+1} \rho_s^{(0)}(u).
\]

On the other hand, since \(\rho_s^{(0)}\) satisfies \((4.3)\), the identities

\[
\int_{u_0}^{u_1} du \rho_s^{(0)}(u) = -1, \quad \int_{u_0}^{u_1} du u \rho_s^{(0)}(u) = 0
\]

hold. Differentiating them with respect to \(t_k\) gives

\[
\int_{u_0}^{u_1} du \frac{\partial \rho_s^{(0)}(u)}{\partial t_k} = 0, \quad \int_{u_0}^{u_1} du u \frac{\partial \rho_s^{(0)}(u)}{\partial t_k} = 0.
\]

Therefore, the integral containing \(C_1 u + C_2\) vanishes, and we obtain \((9.1)\).

By \((4.17)\), we can rewrite the right-hand side of \((9.1)\) and \((9.2)\) to contour integrals as

\[
\frac{\partial \mathcal{E}^{(4D)}_s(s, t)}{\partial t_k} = \frac{1}{2\pi i} \oint_{C} dz \frac{z^{k+1}}{k+1} dW(z), \quad (9.4)
\]

\[
\frac{\partial \mathcal{E}^{(5D)}_s(s, t)}{\partial t_k} = \frac{1}{2\pi i} \oint_{C} dz \frac{e^{-Rz}}{Rk} dW(z), \quad (9.5)
\]

\(^{12}\) The variational equations \((4.10)\) and \((4.25)\) can be recovered from \((4.12)\) and \((4.27)\) by differentiating once. Therefore, \(C_1\) can be identified with \(v/2\).
where $C$ is a simple closed curve encircling the interval $[u_0, u_1]$ anti-clockwise. Note that $dW(z)$ is a single-valued differential, so that the contour integrals are meaningful in the usual sense. Unfortunately, $W(z)$ is multi-valued on the cut plane, so that one cannot do integration by part to replace the differentials as

$$\frac{e^{-Rz}}{Rk} dW(z) \rightarrow -e^{-Rz} W(z) dz.$$

Apart from this problem, $W(z)dz$ may be thought of as a candidate of the Seiberg–Witten differential. Actually, one can add the exact form $dV(z)/2$ to replace it by

$$\frac{dS(z)}{2} = W(z) dz + \frac{dV(z)}{2},$$

where $S(z)$ is the $S$-function (4.18). Thus, we eventually reach the same conclusion as Marshakov and Nekrasov [12, 13]; that the total differential of the $S$ function gives the Seiberg–Witten differential. The numerical factor 2 is correlated with the denominator of (6.34) and (8.15); if $T_k$ (or $\bar{T}_k$) are used in place of $t_k$, this numerical factor disappears.

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