Outage Probability of Uplink Cell-Free Massive MIMO Network with Imperfect CSI Using Dimension-Reduction Method

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Abstract

In this paper, approximate outage probability (OP) expressions are derived for uplink cell-free massive multiple-input-multiple-output (CF-mMIMO) system. The access-points (APs) of the system considered have imperfect channel state information (CSI). The approximate expressions are derived first using conditional expectations and then using a novel dimension reduction method that approximates higher order integration by several single order integrations. Similar expressions are also derived for conventional massive MIMO (mMIMO) systems. The OP approximations are then used to characterize the performance of cell-edge users of CF-mMIMO systems and compare the designs of CF-mMIMO and mMIMO systems. The derived expressions have a close match with simulated expression for OP.

I. INTRODUCTION

Cell-free massive multiple-input multiple-output (CF-mMIMO) system, a contemporary architecture proposed for beyond 5G communication systems, provides better throughput than the popular small-cell system [1], [2]. In a small-cell system, each access point (AP) serves its exclusive set of user equipment (UE). On the other hand, in a CF-mMIMO system, all the APs connected to a central processing unit (CPU) jointly serve the UEs by coherent joint transmission and reception. Such a joint coherent transmission and reception is also the core idea of a Network MIMO (nMIMO) system. However, unlike a traditional nMIMO system, the CF-mMIMO system operates with more number APs than UEs. Therefore, the CF-mMIMO system

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benefits significantly from interference suppression and channel hardening when compared to nMIMO systems \cite{3}.

The CF-mMIMO systems have been extensively studied under different scenarios, such as downlink pilot training, channel hardening, limited back-haul, hardware impairments, non-orthogonal multiple access (NOMA) \cite{4}–\cite{12}. Several works derive spectral efficiency (SE) approximations and utilize them to optimize the performance of CF-mMIMO systems. That is, the characterization and optimization of CF-mMIMO systems depend upon the SE approximations. For example, the authors of \cite{7}, \cite{9} devise max-min power control algorithms to maximize the minimum UE rate. Furthermore, the approximations are obtained using the popular use-and-forget bound or exploiting the channel hardening conditions that average-out the interference powers using the central-limit theorem.

For deriving the exact expressions for SE or for other metrics such as outage probability (OP), characterization of the probability density function (pdf), or the cumulative distribution function (CDF) of the signal-to-interference-noise ratio (SINR) at the APs are essential. However, in a CF-mMIMO system, the numerator and denominator of the SINR are sums of correlated gamma random variables (RV) for a Rayleigh fading channel. Determining the pdf/CDF of the ratio of correlated gamma random variables is mathematically intractable \cite{13}. Also, one can utilize the use-and-forget bound only for deriving approximations for ergodic metrics like SE and not for metrics such as OP that depend on the tail characteristics of the SINR. Therefore, there is a shortage of open literature characterizing the OP of CF-mMIMO systems.

In massive MIMO, there are a few contributions towards deriving the approximate expressions for OP. For example, the authors of \cite{14} consider the downlink of a massive MIMO system with matched-filter precoding. The numerator term of SINR is treated as a deterministic quantity that can be replaced by its mean; the interference term is treated as an RV. The pdf of SINR can therefore be obtained by a simple transformation of the interference term’s pdf. A similar method is used in \cite{15}, where the authors have considered an mMIMO system in which the base station (BS) is equipped with ADCs of different resolution levels. Here, it is shown that the squared coefficient of variation (SCV) of all the terms except the interference term approaches zero as the number of antennas approaches infinity. Therefore, one can determine the pdf of the SINR by transforming the pdf of the interference term. However, it may not always be possible to show that the SCV of all but one of the terms of the SINR becomes zero. Even if there exist two non-zero terms, the pdf becomes intractable to characterize and such is the case in CF-mMIMO.
In [16], the SINR is approximated to a gamma RV by moment matching. The OP is then obtained in terms of the CDF of the gamma RV. However, the efficacy of moment-matching depends on the distribution to which the metric is matched. Also, in many cases, it is algebraically complex to determine the expressions for the moments. In a few other works such as [17], [18], the exact expressions for OP are derived under perfect CSI and i.i.d. channel. However, for CF-mMIMO, assuming perfect CSI knowledge at the APs and i.i.d. channels is not practical [1]. Therefore, to the best of our knowledge, there does not exist, in the open literature, any work that derives outage probability approximations for CF-mMIMO under the condition of imperfect CSI. Also, deriving such an expression involves solving a $M$th order integration, where $M$ denotes the number of APs. For a 50-AP CF-mMIMO system, a 50th order integral has to be solved to obtain an expression. Therefore, in this treatise, we explore the use of a novel dimension-reduction method to approximate the $M$-th order integration by $M$ single order integration. We contrast our contributions to prior art in Table I for completeness.

Our contributions through this paper are as follows:

- We consider an uplink CF-mMIMO system with imperfect CSI at the APs and Rayleigh faded channel. We determine OP approximations using conditional expectations and dimension-reduction method [19]. The dimension-reduction method circumvents the necessity to solve higher order integrals. We also derive novel rate approximations from the OP approximations.
- We also briefly discuss how one can derive similar expressions for mMIMO systems and compare them too with appropriate simulations.
- We also compare the OP performance of CF-mMIMO with that of mMIMO system. We also show through our OP derivation and corresponding simulations that a CF-mMIMO system provides significantly lower OP than a mMIMO system for cell-edge users.

The rest of the paper is structured as follows. In Section II, the uplink training and data transmission phases of CF-mMIMO system are discussed in detail. In Section III, we discuss the proposed dimension-reduction approach and derive the approximate OP and rate expressions. In Section IV, our simulation results are discussed.

1We have tried to approximate the SINR RV by gamma RVs but the resultant expressions did not match with the simulated OP. This is elaborated in section IV.

2Note that this advantage of CF-mMIMO is significant and its impact is best seen when one uses OP as a metric of comparison instead of rate.
TABLE I

II. System Model

We consider a cell-free massive MIMO system with $M$ APs and $K$ users where $M \gg K$, i.e., the number of APs is more than that of users. Note that this model is similar to the one assumed in [1]. We assume that the APs and the users have only one antenna each. The channel coefficient, denoted by $g_{mk} \in \mathbb{C}$, between the $m$-th AP and the $k$-th user, is modeled as follows:

$$g_{mk} = \beta_{mk}^{1/2} h_{mk},$$

where $\beta_{mk}$ represents the large-scale fading coefficient and $h_{mk} \sim \mathcal{CN}(0, 1)$ represents the small-scale fading coefficient. All the small-scale fading coefficients $h_{mk}$, $\forall m = 1, \ldots, M$ and $\forall k = 1, \ldots, K$ are assumed to be independent and identically distributed (i.i.d.) RVs. Let $\tau_c$ be the length of coherence interval (in samples). Typically, in a cell-free massive MIMO system, the coherence interval is partitioned into three phases, namely uplink training phase, uplink data transmission phase and, downlink data transmission phase [1]. In this work, we do not focus on downlink data transmission. Let $\tau_p$ be the length of the uplink training duration (in samples). Therefore, $(\tau_c - \tau_p)$ is the duration of the uplink data transmission phase. The process of uplink training and uplink data transmission are described in the following subsections.

A. Uplink Training

Before the transmission of uplink data by users, APs will acquire the channel state information (CSI) through a training phase. This acquired CSI is then used to process the received data symbols during the uplink data transmission phase. During this phase, all the users simultaneously transmit their pilot sequence to the APs. Let $\sqrt{\tau_p} \phi_k \in \mathbb{C}^{\tau_p \times 1}$ be the pilot sequence transmitted by the $k$-th user, $\forall k = 1, \ldots, K$ where $\|\phi_k\|^2 = 1$. Also, we assume that the pilot sequences transmitted by different users are orthogonal i.e. $\phi_k^H \phi_i = 0$, $\forall k \neq i$. The signal received at $m$-th AP during the training phase is

$$y_{p,m} = \sqrt{\tau_p \rho_p} \sum_{k=1}^{K} g_{mk} \phi_k + w_{p,m},$$

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where $\rho_p$ is the normalized transmit SNR of each pilot symbol, and $w_{p,m} \in \mathbb{C}^{\tau_p \times 1}$ is the noise vector whose entries are i.i.d. zero-mean complex Gaussian. Now, to estimate the channel coefficient using the observation $y_{p,m}$, we first project the received vector on $\phi_k^H$ and then use the MMSE estimator. Let $\hat{y}_{p,m} \triangleq \phi_k^H y_{p,m}$, i.e.,

$$
\hat{y}_{p,m} = \sqrt{\tau_p \rho_p} g_{mk} + \tilde{w}_{p,m},
$$

where $\tilde{w}_{p,m} = \phi_k^H w_{p,m}$ is a $CN(0,1)$ RV. The MMSE estimator is hence given by,

$$
\hat{g}_{mk} = \frac{E\{\hat{y}_{p,mk}^* \hat{y}_{p,mk}\}}{E\{|\hat{y}_{p,mk}|^2\}^{-1}} \hat{y}_{p,mk},
$$

where $c_{mk} = \frac{\sqrt{\tau_p \rho_p \beta_{mk}}}{\tau_p \rho_p \beta_{mk} + 1}$. Also, $\hat{g}_{mk} \sim CN(0, \gamma_{mk})$, where $\gamma_{mk} = \frac{\tau_p \rho_p \beta_{mk}^2}{\tau_p \rho_p \beta_{mk} + 1}$.

### B. Uplink Data Transmission

In the uplink data transmission phase, each user transmits its data symbol to all the APs. Let $p_k$ be the symbol of the $k$-th user, such that $E\{|p_k|^2\} = 1$. Hence, the received signal at the $m$-th AP is

$$
y_{u,m} = \sqrt{\rho_u} \sum_{k=1}^{K} g_{mk} p_k + w_{u,m},
$$

where $\rho_u$ is normalized uplink SNR and $w_{u,m}$ is the additive zero-mean unit-variance complex Gaussian noise. Since all the APs employ MRC, they multiply their copies of the received signal with the estimated channel coefficients $\hat{g}_{mk}^*$. The APs then send their received signal to the CPU. Therefore, the received signal at the CPU is given by

$$
r_{u,k} = \sum_{m=1}^{M} \hat{g}_{mk}^* y_{u,m}
$$

where $c_{mk} = \frac{\sqrt{\tau_p \rho_p \beta_{mk}}}{\tau_p \rho_p \beta_{mk} + 1}$. Also, $\hat{g}_{mk} \sim CN(0, \gamma_{mk})$, where $\gamma_{mk} = \frac{\tau_p \rho_p \beta_{mk}^2}{\tau_p \rho_p \beta_{mk} + 1}$.

### 3. It is assumed that the APs are connected with the CPU through a flawless backhaul network.
is the covariance matrix of $\hat{g}_k$. Using (6), the effective SINR of the $k$-th user is as follows:

$$\lambda_{u,k} = \frac{X_{u,k}}{Y_{u,k}} = \frac{\rho_u (\hat{g}_k^H \hat{g}_k)^2}{\rho_u \sum_{i \neq k} |\hat{g}_k^H \hat{g}_i|^2 + \hat{g}_k^H (I + \rho_u \Lambda_k) \hat{g}_k}, \quad (8)$$

where $\Lambda_k = \text{diag}((\beta_{1k} - \gamma_{1k}), \ldots, (\beta_{Mk} - \gamma_{Mk}))$ is a $M \times M$ diagonal matrix. Note this SINR expression is similar to the SINR expression given in [3, Eq. 12].

Using the effective SINR in (8), one can calculate various performance metrics such as achievable rate, outage probability, etc. The ergodic achievable rate for the uplink of a cell-free massive MIMO system has already been discussed in [1]. In the following section, we derive novel OP approximations utilizing the dimension-reduction method.

III. Outage Probability Analysis

The exact expression for OP involves characterizing the CDF of the SINR at the APs. Note that the numerator and denominator of the SINR involve correlated gamma RV and determining the CDF of their ratio is mathematically intractable [13]. Exact expressions are tractable only for perfect CSI conditions and i.i.d. channels as in the case of mMIMO channels [17], [18]. However, to assume that all the channels from the UEs to APs are i.i.d. or that perfect CSI is known at APs is impractical. Therefore, to derive approximate OP, we first derive the conditional OP assuming that $\hat{g}_K$ is a constant. We then integrate the conditional OP over $\hat{g}_K$. However, the integration results in a $M$th order integration which cannot be solved in close form or evaluated in Mathematica/MATLAB for large values of $M$. Therefore, we explore the use of a dimension-reduction method that approximates $M$th order integration with $M$ single order integrals.

**Theorem 1.** The outage probability of $K$th user is approximated by

$$P_{out}^K(T) \approx 1 - \sum_{i=1}^{K-1} \sum_{m=1}^{M} \mathbb{E} [f_i(1, \ldots, x_m, 1, \ldots, 1)] + (M - 1) \sum_{i=1}^{K-1} f_i(1, \ldots, 1), \quad (9)$$

where $T$ is the SINR threshold, $x_m, \forall m = 1, \ldots, M$ are exponential RV with unit power and

$$f_i(x_1, \ldots, x_M) = \frac{\left(\sum_{m=1}^{M} x_m \gamma_{mK} \gamma_{mi}\right)^{K-2}}{\prod_{j=1}^{K-1} \left(\sum_{m=1}^{M} x_m \gamma_{mK} \gamma_{mj} - \sum_{m=1}^{M} x_m \gamma_{mK} \gamma_{mi}\right)} \left[1 - e^{-\frac{d_K^T}{T \rho_u}}\right] U\left(d_K^T\right), \quad (10)$$

with

$$d_K^T = \left(\frac{\rho_u \left(\sum_{m=1}^{M} x_m \gamma_{mK}\right)^2 - T \sum_{m=1}^{M} x_m \gamma_{mK} (1 + \rho_u (\beta_{mK} - \gamma_{mK}))}{T \rho_u}\right). \quad (11)$$
and \( U(\cdot) \) denoting the unit-step function.

**Proof.** The OP of the \( K \)-th user is given by

\[
P^K_{\text{out}}(T) = P\left( \lambda_{u,K} < T \right) = P\left( X_{u,K} < TY_{u,K} \right),
\]

(12)

Substituting from (8), we have

\[
P^K_{\text{out}}(T) = P\left( \rho_u \left( \hat{g}_K^H \hat{g}_K \right)^2 < T \rho_u \sum_{i=1}^{K-1} |\hat{g}_K^H \hat{g}_i|^2 + T \hat{g}_K^H (I + \rho_u \Lambda_K) \hat{g}_K \right).
\]

(13)

To simplify the above expression, we first calculate the conditional probability \( P^K_{\text{out}} \) for a fixed \( \hat{g}_K \). Therefore, for \( \hat{g}_K = b \), the OP is

\[
P^K_{\text{out}}(T) | (\hat{g}_K = b) = P\left( \rho_u \left( b^H b \right)^2 < T \rho_u \sum_{i=1}^{K-1} |b^H \hat{g}_i|^2 + T b^H (I + \rho_u \Lambda_K) b \right).
\]

(14)

Rearranging the constants to one side of the equality, we have

\[
P^K_{\text{out}}(T) = P\left( \sum_{i=1}^{K-1} |b^H \hat{g}_i|^2 > d^T_K \right),
\]

(15)

where

\[
d^T_K = \frac{\left( \rho_u \left( b^H b \right)^2 - T b^H (I + \rho_u \Lambda_K) b \right)}{T \rho_u}.
\]

(16)

To further simplify, the CCDF of \( \sum_{i=1}^{K-1} |b^H \hat{g}_i|^2 \) is required. Hence, note that \( Z_i = b^H \hat{g}_i \), is a complex Gaussian RV with mean [20, Eq. 15.25]

\[
E[Z_i|\hat{g}_K = b] = b^H E[\hat{g}_i|\hat{g}_K = b] = 0,
\]

(17)

and variance

\[
Var(Z_i|\hat{g}_K = b) = b^H Cov(\hat{g}_i|\hat{g}_K) b = \sum_{m=1}^{M} |b_m|^2 \gamma_{mi} := \alpha_i.
\]

(18)

Eq. (17) and (18) follow from the fact that \( \hat{g}_K \) and \( \hat{g}_i \) are independent \( \forall i \neq K \). Further, \( |Z_i|^2 \) is an exponential RV with parameter \( \alpha_i \). Hence, \( W = \sum_{i=1}^{K-1} |b^H \hat{g}_i|^2 \) is a sum of exponential RVs with CCDF

\[
P^K_{\text{out}}(T) | (\hat{g}_K = b) = 1 - P \left( W \leq d^T_K \right)
\]

\[
= 1 - \left( \prod_{i=1}^{K-1} \alpha_i^{K-2} \prod_{j=1}^{K-1} \frac{1}{\alpha_i - \alpha_j} \left[ 1 - e^{-\frac{d^T_K}{\alpha_j}} \right] \right) U\left( d^T_K \right).
\]

(19)
Now that we have obtained the conditional OP, to obtain the final OP, we integrate over the multivariate Gaussian pdf of $\hat{g}_K$. Hence,

$$P_{\text{out}}^K(T) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( 1 - \left( \sum_{i=1}^{K-1} \frac{\alpha_i^{K-2} \prod_{j=1}^{K-1} (\alpha_i - \alpha_j)}{\prod_{j=1}^{K-1} (\alpha_i - \alpha_j)} \right) \left[ 1 - e^{-\frac{d_T^2}{\alpha_i}} \right] \right) U(d_T)^{K-1} \left( \frac{db}{db} \right)$$

$$= 1 - \frac{1}{\pi^M} \prod_{m=1}^{M} \gamma_{mK}^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{K-1} \frac{\alpha_i^{K-2} \prod_{j=1}^{K-1} (\alpha_i - \alpha_j)}{\prod_{j=1}^{K-1} (\alpha_i - \alpha_j)} \left[ 1 - e^{-\frac{d_T^2}{\alpha_i}} \right] U(d_K)^{K-1} \left( \frac{db}{db} \right)$$

where the pdf of $\hat{g}_K(b)$ is

$$f_{\hat{g}_K}(b) = \frac{1}{\pi^M} \prod_{m=1}^{M} \gamma_{mK}^{-1} e^{-\sum_{m=1}^{M} \frac{|b_m|^2}{\gamma_{mK}}}.$$  

After a cartesian to polar transformation, $i.e., b_m = r_m e^{j\theta_m}$, and then using the transformation $\frac{r_m^2}{\gamma_{mK}} = x_m$, we finally obtain

$$P_{\text{out}}^K(T) = 1 - \sum_{i=1}^{K-1} \mathbb{E}[f_i(x_1, \ldots, x_M)]$$

$$= 1 - \sum_{i=1}^{K-1} \int_0^{\infty} \cdots \int_0^{\infty} f_i(x_1, \ldots, x_M) e^{-\sum_{m=1}^{M} x_mDx_1 \cdots Dx_M},$$

where $x_m$ is an exponential RV with unit power. To further evaluate (22), we need to solve $(K - 1)$ integrations of $M$-th order. For typical values of $M$ used in cell-free massive MIMO systems, say 32, it is impossible to solve a 32-th order integration even in popular softwares such as Matlab, Mathematica, etc. Thus, it is important to approximate (22) for evaluation and analysis. To circumvent the intractability, we propose to utilize the uni-variate dimension-reduction method from [19]. Using this method, one can be tightly approximate an $M$-th order integration by a sum of $M$ single-order integration. Therefore, (22) can be approximated as

$$P_{\text{out}}^K(T) \approx 1 - \sum_{i=1}^{K-1} \sum_{m=1}^{M} \mathbb{E}[f_i(\mu_1, \ldots, \mu_{m-1}, x_m, \mu_{m+1}, \ldots, \mu_M)] + (M - 1) \sum_{i=1}^{K-1} f_i(\mu_1, \ldots, \mu_M),$$

where $\mu_m = \mathbb{E}[x_m], \forall m = 1, \ldots, M$. Since, $\mathbb{E}[x_m] = 1, \forall m = 1, \ldots, M,$

$$P_{\text{out}}^K(T) \approx 1 - \sum_{i=1}^{K-1} \sum_{m=1}^{M} \mathbb{E}[f_i(1, \ldots, 1, x_m, 1, \ldots, 1)] + (M - 1) \sum_{i=1}^{K-1} f_i(1, \ldots, 1),$$

The simplification of $\mathbb{E}[f_i(1, \ldots, 1, x_m, 1, \ldots, 1)]$ is given in Appendix [A].
Corollary 1.1. For the mMIMO scenario, the outage probability of $K$th user is approximated by (9) with

$$f_i(x_1, \ldots, x_M) = \frac{\gamma_i^{K-2}}{\prod_{j=1 \atop j \neq i}^{K-1} (\gamma_i - \gamma_j)} \left[ 1 - e^{-\frac{d_K^T}{\gamma_i \gamma_i \sum_{m=1}^{M} x_m}} \right] U \left( d_K^T \right),$$

(25)

with

$$d_K^T = \frac{\gamma_K \sum_{m=1}^{M} x_m \left( \rho_u \gamma_K \sum_{m=1}^{M} x_m - T \left( 1 + \rho_u (\beta_K - \gamma_K) \right) \right)}{T \rho_u},$$

(26)

Corollary 1.1 follows from the fact that mMIMO system is a special case of the CF-mMIMO system where all the APs are collocated so we have $\beta_{mk} = \beta_k, \gamma_{mk} = \gamma_k, \forall m, k$.

Similarly, the approximate rate can also be determined from the OP approximations. Note that, the average rate is given by

$$R_K = \mathbb{E} \left[ \log_2 (1 + \lambda_{u,k}) \right].$$

(27)

Since $\log_2 (1 + \lambda_{u,k})$ is a positive RV,

$$\mathbb{E} \left[ \log_2 (1 + \lambda_{u,k}) \right] = \int_{t>0} \mathbb{P} \left[ \log_2 (1 + \lambda_{u,k}) > t \right] dt.$$  

(28)

Since logarithm is monotonically increasing in the SINR $\lambda_{u,k}$, we have

$$R_K = \int_{t>0} \mathbb{P} \left[ \lambda_{u,k} > 2^t - 1 \right] dt.$$  

(29)

Hence, the average rate of the $K$th user $R_{Kc}$ can be given in terms of OP evaluated at $2^t - 1$ as follows:

$$R_K = 1 - \int_{0}^{\infty} P_{out}^K (2^t - 1) dt.$$  

(30)

The above expression can be numerically evaluated using popular software such as MATLAB, Mathematica, etc. It provides an alternative to the popular use-and-forget bound for evaluating ergodic rate expressions.

IV. RESULTS & DISCUSSION

In this section, we quantitatively study the OP performance of cell-free systems using the derived approximations (9). The simulation setup is similar to that in [11] and is repeated here for completeness. A cell-free massive MIMO system with various $M$ and $K$ values have been considered. All $M$ AP and $K$ users are dispersed in a square of area $D \times D$ km$^2$. The large-scale fading coefficient $\beta_{mk}$ models the path loss and shadow fading, according to

$$\beta_{mk} = PL_{mk}10^{\frac{\sigma_{th,mk}^2}{10}},$$

(31)
| Parameter                  | value             |
|----------------------------|-------------------|
| Carrier frequency          | 1.9 GHz           |
| Bandwidth                  | 20 MHz            |
| Noise figure               | 9 dB              |
| AP antenna height          | 15 m              |
| User antenna height        | 1.65 m            |
| $\sigma_{sh}$              | 8 dB              |
| $\bar{\rho}_p$, $\bar{\rho}_u$ | 100 mW          |

**TABLE II: Simulation parameters**

where $PL_{mk}$ represents the path loss, $\sigma_{th}$ represents the standard deviation of the shadowing and $z_{mk} \sim \mathcal{N}(0, 1)$. The relation between the path loss $PL_{mk}$ and the distance between the distance $d_{mk}$ between the $m$th AP and $k$th user is obtained using the three slope model in [1, Eq. 52]. The other parameters are summarized in Table II. The normalized transmit SNRs $\rho_p$ and $\rho_u$ are obtained by dividing the actual transmit powers $\bar{\rho}_p$ and $\bar{\rho}_u$ by the noise power respectively.

![Simulation vs. Target SINR (dB) for M = 20 and K = 5](image)

**Fig. 1: Outage Probability Vs. Target SINR (dB) For $M = 20$ and $K = 5$**

We first compare our OP approximation in [9] with the moment-matching approximation of [16] in Fig. 1. Two different moment-matching results are compared with our approximation. In the first, the whole SINR is approximated by a gamma RV, whereas in the second, the numerators...
and denominators of the SINR are separately approximated by gamma RVs. Note that the moment matching approximations significantly under-performs when compared to the dimension reduction approach. This is because such a matching succeeds only when the approximated RV is close to the approximating distribution, (in this case the gamma distribution). Other treatises such as [15] approximate the OP by proving that SCV of all but one component of the SINR is zero and obtaining OP by transforming the CDF of the remaining term. For our case, through extensive simulations, we determined that the SCV of more than one component of SINR is non-zero and hence the method cannot be applied.

![Image of Figure 2: Outage Probability Vs. Target SINR (dB) For M = 50 and various K](image)

**Fig. 2: Outage Probability Vs. Target SINR (dB) For M = 50 and various K**

We then investigate the OP performance of CF-mMIMO for various values of $M$ and $K$ through Fig. 2 and 3. OP decreases with a decrease in the number of users $K$; however, the decrease is more pronounced for a decrease in the number of users from $K = 10$ to $K = 5$ than for that of a decrease from $K = 15$ to $K = 5$. Similarly, with a doubling of $M$, the number of APs, the OP reduces due to an increase in spatial diversity. However, the drop in OP is significant when $M$ is increased to 128.

We also compare the OP of CF-mMIMO and mMIMO system evaluated using theorem 1 and corollary 1.1 with their corresponding simulations in Fig. 3. To make a fair comparison, we consider the users spread over the same geographical area in both cases. Also, we assume that the number of antennas at the BS in the mMIMO system is equal to the number of APs in a CF-mMIMO system. An interesting observation from the figure is that the CF-mMIMO system has a significantly smaller OP because a distributed AP provides better SINR for all
users. In the mMIMO system, the users located far away from the BS suffer from poorer SINR due to path-loss. Furthermore, with an increase of $M$, the drop in OP is also tremendous in a CF-mMIMO system. In Fig. 4 we compare the OP of cell-edge users of both the systems. All users that have an SINR lesser than 5dB are assumed to be cell-edge UEs. The number of UEs classified as cell-edge in CF-mMIMO and mMIMO systems is shown in Table III for comparison. Primarily, the number of users classified as cell-edge is lesser in CF-mMIMO than in mMIMO system. Furthermore, it can be observed from Fig. 4 that the cell-edge users also suffer from a lower outage in a CF-mMIMO system.
| No. of AP | Scenario   | $K = 4$ | $K = 8$ | $K = 12$ | $K = 16$ |
|----------|------------|---------|---------|----------|----------|
| $M = 32$ | CF-mMIMO   | 2       | 5       | 8        | 12       |
|          | mMIMO      | 3       | 7       | 10       | 14       |
| $M = 64$ | CF-mMIMO   | 1       | 3       | 6        | 9        |
|          | mMIMO      | 3       | 6       | 10       | 13       |
| $M = 128$| CF-mMIMO   | 1       | 2       | 4        | 6        |
|          | mMIMO      | 3       | 6       | 9        | 13       |

TABLE III: Average No. of Cell-Edge UE for CF-mMIMO and mMIMO

![Graph showing Outage Probability Vs. No. of APs for $T = -5dB$.]

Fig. 5: Outage Probability Vs. No. of APs For $T = -5dB$

More comparisons of OP of CF-mMIMO and mMIMO for different number of APs is provided in Fig. 5. An interesting inference is that a mMIMO system requires more APs than that of a CF-mMIMO system to obtain the same OP. For example, for $M = 32$ and $K = 8$, the OP of a CF-mMIMO system is nearly halved when compared to the OP of a mMIMO system. Furthermore, the OP of a mMIMO system saturates at $M = 64$ and any additional increase in the antennas does not result in a significant drop in the OP. On the contrary, the OP of a CF-mMIMO system nearly halves with every doubling in the number of APs. We also compared rate computed using (30) with the simulated rate and the rate obtained using the use and forget bounds. All three exhibit a similar performance.
V. CONCLUSIONS

In this work, we derived approximate OP expressions for a CF-mMIMO system and mMIMO system with imperfect CSI and Rayleigh faded channels. To derive the approximations, we used dimension reduction method that circumvents performing higher order integration and results in simpler expressions that are numerically solvable in popular mathematical software. The validity of the approximations, derived for both CF-mMIMO and mMIMO, was verified by Monte-Carlo simulations. Also, the expressions were used to explore the improvement in OP of cell-edge users in a CF-mMIMO system when compared to those of a mMIMO system. We showed that when \( M \) increases from 32 to 64 OP reduces by 70% in the case of CF-mMIMO system while it reduces by only 17% in case of mMIMO system. Further, the number of cell-edge users was 50% lesser in CF-mMIMO system than mMIMO system when \( M = 128 \) and \( K = 8 \). Thus, this OP based analysis further emphasizes the importance of CF-mMIMO even in the presence of imperfect CSI in tackling the cell edge problem when compared with conventional mMIMO.

APPENDIX

Note that,

\[
\mathbb{E} \left[ f_i (1, \ldots, 1, x_m, 1, \ldots, 1) \right] = \int_0^\infty \frac{\left( x_m c_{1,m}^i + c_{2,m}^i \right)^{K-2}}{\prod_{j=1, j \neq i}^{K-1} x_m c_{3,m}^i + c_{4,m}^i} \left[ 1 - e^{-\frac{x_m^2 c_5,m + x_m c_6,m + c_7,m}{x_m c_{1,m}^i + c_{2,m}^i}} \right] e^{-x_m} U \left( x_m^2 c_5,m + x_m c_6,m + c_7,m \right) \, dx_m,
\]

where,

\[
\begin{align*}
c_{1,m}^i &= \gamma_m K \gamma_{mi} \\
c_{2,m}^i &= \sum_{m' \neq m}^{M} \gamma_{m'} K \gamma_{m'i} \\
c_{3,m}^i &= c_{1,m}^i - c_{1,m}^i \\
c_{4,m}^i &= c_{2,m}^i - c_{2,m}^i
\end{align*}
\]
and
\[
c_{5,m} = \frac{1}{T} \gamma_{mK}
\]
\[
c_{6,m} = \frac{2}{T} \gamma_{mK} \left( \sum_{m' \neq m}^{M} \gamma_{m'K} \right) - \frac{1}{\rho_u} \gamma_{mK} \left( 1 + \rho_u (\beta_{mK} - \gamma_{mK}) \right)
\]
\[
c_{7,m} = \frac{1}{T} \left( \sum_{m' \neq m}^{M} \gamma_{m'K} \right)^2 - \frac{1}{\rho_u} \sum_{m' \neq m}^{M} \gamma_{m'K} \left( 1 + \rho_u (\beta_{m'K} - \gamma_{m'K}) \right).
\]

The presence of the unit-step function in Eq. (32), results in different domain of integration depending on the nature of the roots of quadratic equation \(x_m^2 c_{5,m} + x_m c_{6,m} + c_{7,m} = 0\).

A. When there are no real roots or both the roots are negative

In such a scenario, the region of integration will be the entire \(\mathbb{R}^+\). This is true for

- \(c_{6,m}^2 - 4c_{5,m}c_{7,m} < 0\)
- \(c_{6,m}^2 - 4c_{5,m}c_{7,m} \geq 0\) and \(c_{6,m} > 0\) and \(c_{7,m} > 0\).

Therefore,
\[
\mathbb{E} [f_i (1, \ldots, 1, x_m, 1, \ldots, 1)]
\]
\[
= \int_{0}^{\infty} \left[ \frac{1}{\prod_{j=1}^{K-1} x_m c_{3,m} + c_{4,m}} \right] e^{-x_m} - e^{-x_m} e^{-\left( \frac{x_m c_{5,m} + x_m c_{6,m} + c_{7,m}}{x_m c_{1,m} + c_{2,m}} \right)} dx_m.
\]

After decomposing into partial fractions, we obtain
\[
\mathbb{E} [f_i (1, \ldots, 1, x_m, 1, \ldots, 1)] = A_{1,m} + \sum_{j=1}^{K-1} \int_{0}^{\infty} A^{(j)}_{2,m} e^{-x_m} dx_m
\]
\[
- A_{1,m} \int_{0}^{\infty} e^{-\left( \frac{x_m c_{5,m} + x_m c_{6,m} + c_{7,m}}{x_m c_{1,m} + c_{2,m}} \right)} e^{-x_m} dx_m
\]
\[
- \sum_{j=1}^{K-1} \int_{0}^{\infty} A^{(j)}_{2,m} e^{-\left( \frac{x_m c_{5,m} + x_m c_{6,m} + c_{7,m}}{x_m c_{1,m} + c_{2,m}} \right)} e^{-x_m} dx_m.
\]
where
\[
A_{1,m} = \lim_{x_m \to \infty} \frac{(x_m c^i_{1,m} + c^j_{2,m})^{K-2}}{\prod_{j=1}^{K-1} x_m c^i_{j,m} + c^j_{4,m}^{}} = (c^i_{1,m})^{K-2} \prod_{j=1}^{K-1} c^j_{3,m}^{}
\]
\[
A_{2,m}^{(j)} = \frac{(x_m c^i_{1,m} + c^j_{2,m})^{K-2}}{\prod_{k=1}^{K-1} x_m c^k_{3,m} + c^j_{4,m}} = \frac{(c^i_{2,m} c^j_{3,m} - c^i_{1,m} c^j_{4,m})^{K-2}}{(c^j_{3,m}) \prod_{k=1}^{K-1} (c^i_{4,m} c^j_{3,m} - c^i_{3,m} c^j_{4,m})}
\]

(37)

1) Computation of 1st Term: By using the identity [21, Eq. (3.352.4)]

\[
1st \, \text{Term} = -\sum_{j=1}^{K-1} \frac{A_{2,m}^{(j)}}{c^i_{3,m}} e^{-c^i_{3,m} x_m} \text{Ei} \left( -\frac{c^j_{4,m}}{c^i_{3,m}} \right)
\]

(38)

where \( \text{Ei}(\cdot) \) is the exponential integral function.

2) Computation of \( \Pi^{rd} \) Term: After decomposing the argument of the exponential function into partial fractions,

\[
\Pi^{rd} \, \text{Term} = A_{1,m} e^{-c_{9,m}} \int_0^\infty e^{-c_{11,m} x_m} e^{-\frac{c_{10,m}}{x_m c^i_{1,m} + c^j_{2,m}}} dx_m,
\]

(39)

where
\[
c_{11,m} = c_{8,m} + 1
\]
\[
c_{8,m} = \frac{c_{5,m}}{c^i_{1,m}}
\]
\[
c_{9,m} = \frac{c_{6,m} c^i_{1,m} - c_{5,m} c^j_{2,m}}{(c^i_{1,m})^2}
\]
\[
c_{10,m} = \frac{c_{7,m} (c^i_{1,m})^2 - c_{6,m} c^i_{1,m} c^j_{2,m} + c_{5,m} (c^j_{2,m})^2}{(c^i_{1,m})^2}
\]

(40)

3) Computation of \( \Pi^{rd} \) Term: \( \Pi^{rd} \) Term can also be decomposed into

\[
\Pi^{rd} \, \text{Term} = \sum_{j=1}^{K-1} \int_0^\infty \frac{A_{2,m}^{(j)}}{x_m c^i_{3,m} + c^j_{4,m}} \exp \left( -c_{8,m} x_m - x_m - c_{9,m} - \frac{c_{10,m}}{x_m c^i_{1,m} + c^j_{2,m}} \right) dx_m
\]

(41)

\[
= \sum_{j=1}^{K-1} \int_0^\infty \frac{A_{2,m}^{(j)} e^{-c_{9,m}}}{x_m c^i_{3,m} + c^j_{4,m}} e^{-c_{11,m} x_m} e^{-\frac{c_{10,m}}{x_m c^i_{1,m} + c^j_{2,m}}} dx_m,
\]
B. One root is negative and other is positive

This is true when $c_{6,m}^2 - 4c_{5,m}c_{7,m} \geq 0$ and $c_{7,m} < 0$. In this case, the quadratic $x_m^2c_{5,m} + x_m c_{6,m} + c_{7,m} < 0$ for $0 \leq x_m \leq \kappa_m$, where $\kappa_m = \frac{-c_{6,m} + \sqrt{c_{6,m}^2 - 4c_{5,m}c_{7,m}}}{2c_{7,m}}$ is the positive root of the quadratic. Hence, (32) simplifies to

$$\int_{\kappa_m}^{\infty} \left( x_m^2c_{5,m} + x_m c_{6,m} + c_{7,m} \right) e^{-x_m} dx_m$$

Now (42) can be further simplified similar to subsection A

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