ON THE FIXED NUMBER OF GRAPHS

I. JAVAID*, M. MURTaza, M. ASIF, F. IFTIKHAR

Abstract. An automorphism on a graph $G$ is a bijective mapping on the vertex set $V(G)$, which preserves the relation of adjacency between any two vertices of $G$. An automorphism $g$ fixes a vertex $v$ if $g$ maps $v$ onto itself. The stabilizer of a set $S$ of vertices is the set of all automorphisms that fix vertices of $S$. A set $F$ is called fixing set of $G$, if its stabilizer is trivial. The fixing number of a graph is the cardinality of a smallest fixing set. The fixed number of a graph $G$ is the minimum $k$, such that every $k$-set of vertices of $G$ is a fixing set of $G$. A graph $G$ is called a $k$-fixed graph if its fixing number and fixed number are both $k$. In this paper, we study the fixed number of a graph and give construction of a graph of higher fixed number from graph with lower fixed number. We find bound on $k$ in terms of diameter $d$ of a distance-transitive $k$-fixed graph.

1. Introduction

Let $G = (V(G), E(G))$ be a connected graph with order $n$. The degree of a vertex $v$ in $G$, denoted by $\deg_G(v)$, is the number of edges that are incident to $v$ in $G$. We denote by $\Delta(G)$, the maximum degree and $\delta(G)$, the minimum degree of vertices of $G$. The distance between two vertices $x$ and $y$, denoted by $d(x, y)$, is the length of a shortest path between $x$ and $y$ in $G$. The eccentricity of a vertex $x \in V(G)$ is $e(x) = \max_{y \in V(G)} d(x, y)$ and the diameter of $G$ is $\max_{x \in V(G)} e(x)$. For a vertex $v \in V(G)$, the neighborhood of $v$, denoted by $N_G(v)$, is the set of all vertices adjacent to $v$ in $G$.

An automorphism of $G$, $g : V(G) \rightarrow V(G)$, is a permutation on $V(G)$ such that $g(u)g(v) \in E(G) \Leftrightarrow uv \in E(G)$, i.e., the relation of adjacency is preserved under automorphism $g$. The set of all such permutations for a graph $G$ forms a group under the operation of composition of permutations. It is called the automorphism group of $G$, denoted by $\text{Aut}(G)$ and it is a subgroup of symmetric group $S_n$, the group of all permutations on $n$ vertices. A graph $G$ with trivial automorphism group is called rigid or asymmetric graph and such a graph has no symmetries. In this paper, all the graphs (unless stated otherwise) are connected graphs.

Key words and phrases. Fixing set; Stabilizer; Fixing number; Fixed number
2010 Mathematics Subject Classification. 05C25, 05C60.

* Corresponding author: imran.javaid@bzu.edu.pk.
otherwise) have non-trivial automorphism group i.e., $\text{Aut}(G) \neq \{\text{id}\}$. Let $u, v \in V(G)$, we say $u$ is similar to $v$, denoted by $u \sim v$ (or more specifically $u \sim^g v$) if there is an automorphism $g \in \text{Aut}(G)$ such that $g(u) = v$. It can be seen that similarity is an equivalence relation on vertices of $G$ and hence it partitions the vertex set $V(G)$ into disjoint equivalence classes, called orbits of $G$. The orbit of a vertex $v$ is defined as $O(v) = \{u \in V(G) | u \sim v\}$. The idea of fixing sets was introduced by Erwin and Harary in [5]. They used following terminology: The stabilizer of a vertex $v \in V(G)$ is defined as, $\text{stab}(v) = \{f \in \text{Aut}(G) | f(v) = v\}$. The stabilizer of a set of vertices $F \subseteq V(G)$ is defined as, $\text{stab}(F) = \{f \in \text{Aut}(G) | f(v) = v \text{ for all } v \in F\} = \bigcap_{v \in F} \text{stab}(v)$. A vertex $v$ is fixed by an automorphism $g \in \text{Aut}(G)$ if $g \in \text{stab}(v)$. A set of vertices $F$ is a fixing set if $\text{stab}(F)$ is trivial, i.e., the only automorphism that fixes all vertices of $F$ is the trivial automorphism. The cardinality of a smallest fixing set is called the fixing number of $G$. We will refer a set of vertices $A \subset V(G)$ for which $\text{stab}(A) \setminus \{\text{id}\} \neq \emptyset$ as non-fixing set. A vertex $v \in V(G)$ is called a fixed vertex if $\text{stab}(v) = \text{Aut}(G)$. Every graph has a fixing set. Trivially the set of vertices itself is a fixing set. It is also clear that a set containing all but one vertex is a fixing set. Following theorem gives a relation between orbits and stabilizers.

**Theorem 1.1. (Orbit-Stabilizer Theorem)** Let $G$ be a connected graph and $v \in V(G)$.

$$|\text{Aut}(G)| = |O(v)||\text{stab}_{\text{Aut}(G)}(v)|.$$ 

Boutin introduced determining set in [3]. A set $D \subseteq V(G)$ is said to be a determining set for a graph $G$ if whenever $g, h \in \text{Aut}(G)$ so that $g(x) = h(x)$ for all $x \in D$, then $g(v) = h(v)$ for all $v \in V(G)$. The minimum cardinality of a determining set of a graph $G$, denoted by $\text{Det}(G)$, is called the determining number of $G$. Following lemma given in [7] shows equivalence between definitions of fixing set and determining sets.

**Lemma 1.2.** [7] A set of vertices is a fixing set if and only if it is a determining set.

Thus notions of fixing number and determining number of a graph $G$ are same.

The notion of fixing set is closely related to another well-studied notion, resolving set, defined in the following way: A vertex $v \in G$ resolves vertices $x, y \in V(G)$ if $d(x, v) \neq d(y, v)$. A set $W \subseteq V(G)$ is called a resolving set for $G$ if for every pair $x, y \in V(G)$, there exists a vertex $w \in W$ such that $d(x, w) \neq d(y, w)$. The idea of resolving set was introduced by Slater [15] and
he referred this set as a locating set. The cardinality of a minimum resolving set in a graph $G$, denoted by $\beta(G)$, is called the metric dimension of $G$. The resolving number $\text{res}(G)$ of $G$ is the minimum $k$ such that every $k$-set of vertices is a resolving set of $G$. The following proposition was independently proved by Erwin and Harary [5] (using fixing sets instead of determining sets) and Boutin [3].

**Proposition 1.3.** [3, 5, 8] If $S \subseteq V(G)$ is a resolving set of $G$ then $S$ is a fixing set of $G$. In particular, $\text{fix}(G) \leq \beta(G)$.

Jannesari and Omoomi have discussed the properties of resolving graphs and randomly $k$-dimensional graphs in [10] and [11] which were based on resolving number and metric dimension of $G$. In this paper, we define the fixed number of a graph, fixing graph and $k$-fixed graphs. We discuss properties of these graphs in the context of fixing sets and the fixing number.

The fixed number of a graph $G$, $\text{fxd}(G)$, is the minimum $k$ such that every $k$-set of vertices is a fixing set of $G$. It may be noted that $0 \leq \text{fix}(G) \leq \text{fxd}(G) \leq n-1$. A graph is said to be a $k$-fixed graph if $\text{fix}(G) = \text{fxd}(G) = k$. In this paper, the fixed number $k$, remains in the focus of our attention. A path of even order is a 1-fixed graph. Similarly a cyclic graph of odd order is a 2-fixed graph. We give a construction of a graph with $\text{fxd}(G) = r + 1$ from a graph with $\text{fxd}(G) = r$ in Theorem 2.8. Also a characterization of $k$-fixed graphs is given in Theorem 3.7.

2. The Fixed Number

Consider the graph $G$ in Figure 1. It is clear that $\text{Aut}(G) = \{e, (12)(34)(56)\}$. Also $\text{stab}(v) = \{id\}$ for all $v \in V(G)$. Thus $\{v\}$ for each $v \in V(G)$ forms a fixing set for $G$. Hence $\text{fix}(G) = \text{fxd}(G) = 1$ and $G$ is 1-fixed graph. Thus we have following proposition immediately from definition of fixing set.

![Figure 1. Graph G](image-url)
Proposition 2.1. Let $G$ be a connected graph and $fxd(G) = 1$, then

(i) $|O(v)| = |Aut(G)|$ for all $v \in V(G)$.

(ii) $G$ does not have fixed vertices.

Proof. (i) Since $|stab(v)| = 1$ for all $v \in V(G)$ and result follows by Theorem 1.1. (ii) As $stab(v) = Aut(G)$ for a fixed vertex $v \in V(G)$ and hence $\{v\}$ does not form a fixing set for $G$. □

The problem of ‘finding the minimum $k$ such that every $k$-subset of vertices of $G$ is a fixing set of $G$’ is equivalent to the problem of ‘finding the maximum $r$ such that there exist an $r$-subset of vertices of $G$ which is not fixing set of $G$’. Thus, the cardinality of a largest non-fixing set in $G$ helps in finding the fixed number of $G$. We can see $r = 0$ for the graph $G$ in Figure 1. We have following remarks about non-fixing sets.

Remark 2.2. Let $G$ be graph of order $n$.

(i) If $r$ ($0 \leq r \leq n - 2$) be the cardinality of a largest non-fixing subset of $G$, then $fxd(G) = r + 1$.

(ii) Let $A$ be a non-fixing set of $G$. For each non-trivial $g \in stab(A)$ there exist at least one set $B \subseteq V(G)$ such that $u \sim^g v$ for all $u, v \in B$.

Proposition 2.3. Let $G$ be a graph and $u, v \in V(G)$ such that $N(v) \{u\} = N(u) \{v\}$. Let $F$ be a fixing set of $G$, then $u$ or $v$ is in $F$.

Proof. Let $u, v \in V(G)$ such that $N(v) \{u\} = N(u) \{v\}$. Suppose on contrary both $u$ and $v$ are not in $F$. As $u$ and $v$ have common neighbors and $u, v \notin F$ so there exists an automorphism $g \in Aut(G)$ such that $g \in stab(F)$ and $g(u) = v$. Hence $stab(F)$ has a non-trivial automorphism, a contradiction. □

Theorem 2.4. Let $G$ be a connected graph of order $n$. Then, $fxd(G) = n - 1$ if and only if $N(v) \{u\} = N(u) \{v\}$ for some $u, v \in V(G)$.

Proof. Let $u, v \in V(G)$ such that $N(v) \{u\} = N(u) \{v\}$. Suppose on contrary that $fxd(G) \leq n - 2$, then $V(G) \{u, v\}$ is a fixing set for $G$. But, by Proposition 2.3, every fixing set contains either $u$ or $v$. This contradiction implies that, $fxd(G) = n - 1$.

Conversely, let $fxd(G) = n - 1$. Thus, there exists a non-fixing subset $T$ of $V(G)$ with $|T| = n - 2$. Assume $T = V(G) \{u, v\}$ for some $u, v \in V(G)$. Our claim is that $u, v$ are those vertices of $G$ for which $N(u) \{v\} = N(v) \{u\}$. Suppose on contrary $N(u) \{v\} \neq N(v) \{u\}$, then there exists a vertex $w \in T$ such that $w$ is adjacent to one of the vertices $u$ or $v$. Without loss of generality, let $w$ is adjacent to $u$ but not adjacent to $v$. Let a non-trivial automorphism $g \in stab(T)$ (such a non-trivial automorphism exists because $T$
is not a fixing set). Since \( g \) is non-trivial and \( V(G) \setminus T = \{u, v\} \), so \( g(u) = v \).
But \( u \) cannot map to \( v \) under \( g \), because \( g \in \text{stab}(w) \) and \( w \) is adjacent to \( u \) and not adjacent to \( v \). Hence \( g \) also fixes \( u \) and \( v \), i.e., \( g \in \text{stab}\{u, v\} \) and consequently \( g \) becomes trivial. Hence \( \text{stab}(T) \) is trivial, a contradiction. Thus \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \).

The following theorem given in [4] is useful for the proof of Corollary 2.6.

**Theorem 2.5.** [4] Let \( G \) be a connected graph of order \( n \). Then
\[ \text{fix}(G) = n - 1 \text{ if and only if } G = K_n. \]

**Corollary 2.6.** Let \( G \) be a graph of order \( n \) and \( G \neq K_n \). If \( G \) is \((n - 1)\)-fixed graph, then for each pair of distinct vertices \( u, v \in V(G) \), \( N(u) \setminus \{v\} \neq N(v) \setminus \{u\} \).

**Proof.** Let \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \) for some \( u, v \in V(G) \). Then by Theorem 2.4 \( \text{fxd}(G) = n - 1 \). Since \( G \neq K_n \), so by Theorem 2.5 \( \text{fix}(G) \neq n - 1 = \text{fxd}(G) \). Hence \( G \) is not \((n - 1)\)-fixed. \( \square \)

The fixing polynomial, \( F(G, x) = \sum_{i=\text{fix}(G)}^{n} \alpha_i x^i \), of a graph \( G \) of order \( n \) is a generating function of sequence \( \{\alpha_i\} \) \((\text{fix}(G) \leq i \leq n)\), where \( \alpha_i \) is the number of fixing subset of \( G \) of cardinality \( i \). For more detail about fixing polynomial see [12] where we discussed properties of fixing polynomial and found it for different families of graphs. For example \( F(C_3, x) = x^3 + 3x^2 \).

**Theorem 2.7.** Let \( G \) be a \( k \)-fixed graph of order \( n \).
\[ F(G, x) = \sum_{i=k}^{n} \binom{n}{i} x^i \]

**Proof.** Since \( \text{fix}(G) = \text{fxd}(G) = k \) and superset of a fixing set is also a fixing set, therefore each \( i \)-subset \((k \leq i \leq n)\) is a fixing set. Hence \( \alpha_i = \binom{n}{i} \) for each \( i \), \((k \leq i \leq n)\). \( \square \)

**Theorem 2.8.** Let \( G \) be a graph of order \( n \) and \( \text{fxd}(G) = r \). We can construct a graph \( G' \) of order \( n + 1 \), from \( G \) such that \( \text{fxd}(G') = r + 1 \).

**Proof.** Since \( \text{fxd}(G) = r \), so \( G \) has a largest non-fixing set \( A \) of cardinality \( |A| = r - 1 \). By Remark 2.2(ii) for each non-trivial \( g \in \text{stab}(A) \), there exist at least one set \( B \subset V(G) \) such that \( u \sim^g v \) for all \( u, v \in B \). Consider \( B = \{v_1, v_2, \ldots, v_l\} \). Take a \( K_1 = \{x\} \) and join \( x \) with \( v_1, v_2, \ldots, v_l \) by edges \( xv_1, xv_2, \ldots, xv_l \). We call new graph \( G' \). This completes construction of \( G' \). We will now find a largest non-fixing subset of \( G' \). Since \( v_i \sim^g v_j \) \((i \neq j, 1 \leq i, j \leq l)\), then \( \text{fxd}(G') = r + 1 \).
l) in G and x is adjacent to v1, v2, ..., vl in G′. So we can find a g′ ∈ Aut(G′) such that

\[ g′(u) = \begin{cases} 
  x & \text{if } u = x, \\
  g(u) & \text{if } u \neq x 
\end{cases} \]

in G′. Clearly, g′ ∈ stab(x) ∩ stab(A) = stab({x} ∪ A) and vi ∼ g′ vj (i ≠ j, 1 ≤ i, j ≤ l) in G′. Since g′ is non-trivial and A is a largest non-fixing set in G, so A ∪ {x} is a largest non-fixing set in G′. Hence by Remark 2.2(i),

\[ fxd(G′) = |A ∪ \{x\}| + 1 = r + 1 \]

The following lemma is useful for finding the fixing number of a tree.

**Lemma 2.9.** Let T be a tree and F ⊂ V(T), then F fixes T if and only if F fixes the end vertices of T.

**Theorem 2.10.** For every integers p and q with 2 ≤ p ≤ q, there exists a graph G with fix(G) = p and fxd(G) = q.

**Proof.** For p = q, G = Kp+1 will have the desired property. So we consider 2 ≤ p < q. Consider a graph G obtained from a path w1, w2, ..., wp−p. Add p + 1 vertices u1, u2, ..., up+1 and p + 1 edges w1u1, w1u2, ..., w1up+1 with w1. Thus |V(G)| = q + 1. Consider set F ⊂ V(G), F = {u1, u2, ..., up} , then F fixes the set of end vertices \(\{u_1, u_2, ..., u_p, u_{p+1}\}\) of G. As G is a tree and \(w_{p-q}\) is a fixed end vertex, hence F fixes G by Lemma 2.9. Since F is a minimum fixing set for G, so fix(G) = |F| = p. Also fxd(G) = q because U = \(\{w_1, w_2, ..., w_{q-p}, u_1, u_2, ..., u_{p-1}\}\) is the largest non-fixing set with cardinality q − 1.



3. The Fixing Graph

Let G be a connected graph. The set of fixed vertices of G has no contribution in constructing the fixing sets of G, therefore we define a vertex set

\[ S(G) = \{v \in V(G) : v \sim u \text{ for some } u(\neq v) \in V(G)\} \]

(set of all vertices of G which are more than one in their orbits). Also consider \(V_s(G) = \{(u, v) : u \sim v (u \neq v) \text{ and } u, v \in V(G)\}\). Also, if G is an asymmetric graph, then \(V_s(G) = \emptyset\). Let x ∈ V(G), an arbitrary automorphism \(g \in \text{stab}(x)\) is said to fix a pair \((u, v) \in V_s(G)\) if \(u \sim g^u v\). If \((u, v) \notin V_s(G)\), then \(u \not\sim v\) and hence question of fixing pair \((u, v)\) by a \(g \in \text{stab}(x)\) does not arise. In this section, we use r and s to denote \(|S(G)|\) and \(|V_s(G)|\) respectively. It is clear that \(r ≤ n\) and \(\frac{n}{2} ≤ s ≤ \binom{n}{2}\) where s attains its lower bound in later inequality in case when r is even and pair \((u, v)\) is only fixed by automorphisms in \(\text{stab}(u, v)\) for all \((u, v) \in V_s(G)\). Consider the graph \(G_1\) in Figure 2 where \(r = 6\) and \(s = 7\). \(G_1\) has a fixed vertex \(v_1\) and \(S(G_1) = \{v_2, v_3, v_4, v_5, v_6, v_7\}\)
and $V_s(G_1) = \{(v_2, v_3), (v_4, v_5), (v_4, v_6), (v_4, v_7), (v_5, v_6), (v_5, v_7), (v_6, v_7)\}$. Since superset of a fixing set is also a fixing set, so we are interested in fixing set of minimum cardinality. Following remarks tell us the relation between a fixing set $F$ and $S(G)$.

**Remark 3.1.** Let $G$ be a graph. A set $F \subseteq V(G)$ is a minimum fixing set of $G$, if $F \subseteq S(G)$ and an arbitrary $g \in \text{stab}(F)$ fixes $S(G)$.

The Fixing Graph, $D(G)$, of a graph $G$ is a bipartite graph with bipartition $(S(G), V_s(G))$. A vertex $x \in S(G)$ is adjacent to a pair $(u, v) \in V_s(G)$ if $u \not\sim^g v$ for $g \in \text{stab}(x)$. Let $F \subseteq S(G)$, then $N_{D(G)}(F) = \{(x, y) \in V_s(G) | x \not\sim^g y$ for $g \in \text{stab}(F)\}$. In the fixing graph, $D(G)$, the minimum cardinality of a subset $F$ of $S(G)$ such that $N_{D(G)}(F) = V_s(G)$ is the fixing number of $G$. Figure 2(b) shows the fixing graph of graph $G_1$ given in Figure 2(a). As $N_{D(G_1)}\{v_4, v_6\} = V_s(G_1)$, thus $\{v_4, v_6\}$ is a fixing set of $G_1$ and hence $\text{fix}(G_1) = 2$.

**Remark 3.2.** Let $G$ be graph and $F \subseteq S(G)$ be a fixing set of $G$, then $N_{D(G)}(F) = V_s(G)$.

Also $\{v_1, v_2, v_3, v_4, v_5\}$ is a largest non-fixing set of $G_1$. In fact every largest non-fixing set must have fixed vertex $v_1$. So we have following proposition

**Proposition 3.3.** Let $G$ be a graph and $A$ be a largest non-fixing subset of $G$. Then $A$ contains all fixed vertices of $G$.

![Figure 2.](image-url)
Proof. Let $x \in V(G)$ be an arbitrary fixed vertex of $G$. Suppose on contrary $x \notin A$. Then $\text{stab}(A \cup \{x\}) = \text{stab}(A) \cap \text{stab}(x) = \text{stab}(A) \cap \text{Aut}(G) = \text{stab}(A) \neq \{id\}$ ($A$ is non-fixing set). Consequently $A \cup \{x\}$ is non-fixing set, a contradiction that $A$ is largest non-fixing set. \hfill \Box

Let $t$ be the minimum number such that $1 \leq t \leq r$ and every $t$-subset $F$ of $S(G)$ has $N_{D(G)}(F) = V_s(G)$, then $t$ is helpful in finding the fixed number of a graph $G$. The following theorem gives a way of finding fixed number of a graph using its fixing graph.

**Theorem 3.4.** Let $G$ be a graph of order $n$ and $t$ ($1 \leq t \leq r$) be the minimum number such that every $t$-subset of $S(G)$ has neighborhood $V_s(G)$ in $D(G)$. Then

$$f_{xd}(G) = t + |V(G) \setminus S(G)|$$

**Proof.** By Remark [2.2(i)], we find a largest non-fixing subset $T$ of $V(G)$. By Proposition [3.3] $V(G) \setminus S(G)$ is a subset of largest non-fixing set $T$. Moreover, by hypothesis there is a $(t-1)$-subset $U$ of $S(G)$ such that $N_{D(G)}(U) \neq V_s(G)$. Then $U$ is non-fixing set for $G$ and hence $\{V(G) \setminus S(G)\} \cup U$ is a non-fixing set. Also $\{V(G) \setminus S(G)\} \cup U$ is a largest non-fixing set of $G$, because by hypothesis, a $t$-subset of $S(G)$ forms a fixing set of $G$. Further $\{V(G) \setminus S(G)\} \cap U = \emptyset$. Hence by Remark [2.2(i)],

$$f_{xd}(G) = |V(G) \setminus S(G)| + |U| + 1 = |V(G) \setminus S(G)| + t$$

\hfill \Box

In [13], we found an upper bound on the cardinality of edge set $|E(D(G))|$ of fixing graph $D(G)$ of a graph $G$.

**Proposition 3.5.** [13] Let $G$ be a $k$-fixed graph of order $n$, then

$$|E(D(G))| \leq n\left(\frac{n}{2} - k + 1\right).$$

Now we find lower bound on $|E(D(G))|$.

**Proposition 3.6.** If $G$ is a $k$-fixed graph of order $n$, then

$$\left(\frac{r}{2}\right)(r - k + 1) \leq |E(D(G))|$$

**Proof.** Let $z \in V_s(G)$ and $A$ be a set of vertices of $S(G)$ which are not adjacent to $z$. Since $N_{D(G)}(A) \neq V_s(G)$, therefore $A$ is a non-fixing set of $G$. Our claim is $\text{deg}_{D(G)}(z) \geq r - k + 1$. Suppose $\text{deg}_{D(G)}(z) \leq r - k$, then $|A| \geq k$,
which contradicts that $fxd(G) = k$ ($A$ is non-fixing set with $|A| \geq k$). Thus, $\text{deg}_{D(G)}(z) \geq r - k + 1$ and consequently,

$$\left(\frac{r}{2}\right)(r - k + 1) \leq s(r - k + 1) \leq |E(D(G))|.$$  

Thus, on combining (1) and (2) we get

$$\left(\frac{r}{2}\right)(r - k + 1) \leq |E(D(G))| \leq n\left(\frac{r}{2}\right) - k + 1.$$  

**Theorem 3.7.** If $G$ is a $k$-fixed graph and $|S(G)| = r$, then $k \leq 3$ or $k \geq r - 1$.

**Proof.** For each $R \subseteq S(G)$, let $\overline{N}_{D(G)}(R) = V_s(G) \setminus N_{D(G)}(R)$. We claim that, if $R, T \subseteq S(G)$ with $|R| = |T| = k - 1$ and $R \neq T$, then $\overline{N}_{D(G)}(R) \cap \overline{N}_{D(G)}(T) = \emptyset$. Otherwise, there exists a pair $\{y, z\} \in \overline{N}_{D(G)}(R) \cap \overline{N}_{D(G)}(T)$. Therefore, $\{y, z\} \notin N_{D(G)}(R \cup T)$ and hence, $R \cup T$ is not a fixing set of $G$. Since, $R \neq T$, $|R \cup T| > |T| = k - 1$, which contradicts that $fxd(G) = k$. Thus, $\overline{N}_{D(G)}(R) \cap \overline{N}_{D(G)}(T) = \emptyset$.

Since, $\text{fix}(G) = k$, for each $R \subseteq S(G)$ with $|R| = k - 1$, $\overline{N}_{D(G)}(R) \neq \emptyset$. Now, let $\Omega = \{R \subseteq S(G) : |R| = k - 1\}$. Therefore,

$$|\bigcup_{R \in \Omega} \overline{N}_{D(G)}(R)| = \sum_{R \in \Omega} |N_{D(G)}(R)| \geq \sum_{R \in \Omega} 1 = \binom{r}{k-1}.$$  

On the other hand, $\bigcup_{R \in \Omega} \overline{N}_{D(G)}(R) \subseteq V_s(G)$. Hence, $|\bigcup_{R \in \Omega} \overline{N}_{D(G)}(R)| \leq s \leq \binom{r}{k-1}$. Consequently, $\binom{r}{k-1} \leq \binom{r}{k-1}$. If $r \leq 4$, then $k \leq 3$. Now, let $r \geq 5$. Thus, $2 \leq \frac{r+1}{2}$. We know that for each $a, b \leq \frac{n+1}{2}$, $\binom{r}{a} \leq \binom{r}{b}$ if and only if $a \leq b$. Therefore, if $k - 1 \leq \frac{r+1}{2}$, then $k - 1 \leq 2$, which implies $k \leq 3$. If $k - 1 \geq \frac{r+1}{2}$, then $r - k + 1 \leq \frac{r+1}{2}$. Since $\binom{r}{k-1} = \binom{r}{k-1}$, we have $\binom{r}{r-k+1} \leq \binom{r}{2}$ and consequently, $r - k + 1 \leq 2$, which yields $k \geq r - 1$. □

4. **The Distance-Transitive Graph**

We now study the fixed number in a class of graphs known as the distance-transitive graphs. A graph $G$ is called distance-transitive if $u, v, x, y \in V(G)$ satisfying $d(u, v) = d(x, y)$, then there exist an automorphism $g \in \text{Aut}(G)$ such that $u \sim^g x$ and $v \sim^g y$. For example, the complete graph $K_n$, the cyclic graph $C_n$, the Petersen graph, the Johnson graph etc, are distance-transitive. For more about distance transitive graphs see [2]. In this section, we use terminology as described in section 3 related to the fixing graph $D(G)$ of a graph $G$. Following proposition given in [2] tells that the distance transitive graph does not have fixed vertices.
Proposition 4.1. [2] A distance-transitive graph is vertex transitive.

Thus, if $G$ is a distance transitive graph, then $S(G) = V(G)$, $r = n$ and $V_s(G)$ consists of all $\binom{n}{2}$ pairs of vertices of $G$ (i.e., $s = \binom{n}{2}$).

Corollary 4.2. Let $G$ be a distance-transitive graph of order $n$. If $G$ is $k$-fixed, then $k \leq 3$ or $k \geq n - 1$.

Proof. Since $r = n$ for a distance-transitive graph, so result follows from Theorem 3.7.

Moreover an expression for bounds on $|E(D(G))|$ of a distance-transitive and $k$-fixed graph $G$ can be obtained by putting $r = n$ and $s = \binom{n}{2}$ in (2) and use the result in (3), we get

$$\binom{n}{2} (n - k + 1) \leq |E(D(G))| \leq n\binom{n}{2} - k + 1.$$ (4)

Also, the following two results given in [10] are useful in our later work.

Observation 4.3. [10] Let $n_1, ..., n_r$ and $n$ be positive integers, with $\sum_{i=1}^{r} n_i = n$. Then, $\sum_{i=1}^{r} \binom{n_i}{2}$ is minimum if and only if $|n_i - n_j| \leq 1$, for each $1 \leq i, j \leq r$.

Lemma 4.4. [10] Let $n, p_1, p_2, q_1, q_2, r_1$ and $r_2$ be positive integers, such that $n = p_i q_i + r_i$ and $r_i < p_i$, for $1 \leq i \leq 2$. If $p_1 < p_2$, then

$$(p_1 - r_1)\binom{q_1}{2} + r_1\binom{q_1+1}{2} \geq (p_2 - r_2)\binom{q_2}{2} + r_2\binom{q_2+1}{2}.$$ (5)

We define distance partition of $V(G)$ with respect to $v \in V(G)$, into distance classes $\Psi_i(v)$ ($1 \leq i \leq e(v)$) defined as: $\Psi_i(v) = \{x \in V(G)| d(v, x) = i \}$.

Proposition 4.5. Let $G$ be a distance transitive graph and $v, x, y \in V(G)$. Then $x, y \in \Psi_i(v)$ for some $i$ ($1 \leq i \leq e(v)$) if and only if $v$ is non-adjacent to pair $(x, y) \in V_s(G)$ in $D(G)$.

Proof. Let $x, y \in \Psi_i(v)$ for some $i$ ($1 \leq i \leq e(v)$), then $d(v, x) = d(v, y) = i$ and by definition of distance-transitive graph there exist an automorphism $g \in Aut(G)$ such that $v \sim^g v$ and $x \sim^g y$. Thus $x \sim^g y$ by an automorphism $g \in stab(v)$ and consequently the pair $(x, y)$ is not adjacent to $v$ in $D(G)$.

Conversely, suppose $v$ is non-adjacent to pair $(x, y) \in V_s(G)$, then $x \sim^g y$ by an arbitrary $g \in stab(v)$. Since $g$ is an isometry, therefore $d(v, x) = d(g(v), g(x)) = d(v, y) = i$ (say). Thus $x, y$ are in same distance class $\Psi_i(v)$.

Proposition 4.6. Let $G$ be a distance-transitive graph of order $n$. If $G$ is $k$-fixed, then for each $v \in V(G)$, $\deg_{D(G)}(v) = \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{\Psi_i(v)}{2}$.
Proof. By Proposition 4.5, the only pairs \((x, y) \in V_s(G)\) which are non-adjacent to \(v \in V(G)\) are those in which both \(x, y\) belong to same distance class \(\Psi_i(v)\) for each \(i \ (1 \leq i \leq e(v))\). So the number of such pairs in \(V_s(G)\) which are not adjacent to \(v\) is \(\sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}\). Therefore, \(\deg_{D(G)}(v) = \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}\) \(\square\)

Thus, an expression for \(|E(D(G))|\) can be obtained using Proposition 4.6 (5)

\[|E(D(G))| = \sum_{v \in V(G)} \left[ \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2} \right] = n \binom{n}{2} - \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}\]

From (4) and (5) we obtain

(6) \[n(k-1) \leq \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2} \leq \binom{n}{2}(k-1)\]

Theorem 4.7. Let \(G\) be a distance-transitive graph of order \(n\) and diameter \(d\). If \(G\) is \(k\)-fixed, then \(k \geq \frac{n-1}{d}\).

Proof. Note that, for each \(v \in V(G)\), \(|\bigcup_{i=1}^{e(v)} \Psi_i(v)\| = n-1\). For \(v \in V(G)\), let \(n-1 = q(v)e(v) + r(v)\), where \(0 \leq r(v) < e(v)\). Then, by Observation 4.3 \(\sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}\) is minimum if and only if \(|\Psi_i(v)| - |\Psi_j(v)|\ | \leq 1\), where \(1 \leq i, j \leq e(v)\). This condition will be satisfied if there are \(r(v)\) distance classes having \(q(v) + 1\) vertices and \(e(v) - r(v)\) distance classes having \(q(v)\) vertices. Thus, the number of pair of vertices in \(\Psi_i(v)\) having \(q(v) + 1\) vertices is \(r(v)\binom{q(v)+1}{2}\) and the number of pair of vertices in \(\Psi_i(v)\) having \(q(v)\) vertices is \((e(v) - r(v))\binom{q(v)}{2}\). Thus,

(7) \[(e(v) - r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2} \leq \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}\]

Let \(w \in V(G)\) with \(e(w) = d\), \(r(w) = r\), and \(q(w) = q\), then \(n-1 = qd + r\). Since for each \(v \in V(G)\), \(e(v) \leq e(w)\), by Lemma 4.4 \(d-r)\binom{q}{2} + r\binom{q+1}{2} \leq (e(v) - r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2}\).

Therefore,

\[n[(d-r)\binom{q}{2} + r\binom{q+1}{2}] \leq \sum_{v \in V(G)} [(e(v) - r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2}]\]

Thus, by relation (6) and (7)

\[n[(d-r)\binom{q}{2} + r\binom{q+1}{2}] \leq \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2} \leq \binom{n}{2}(k-1)\].
Hence, \( q[(d-r)(q-1)+r(q+1)] \leq (n-1)(k-1) \), which implies, \( q[(r-d)+(d-r)q+r(q+1)] \leq (n-1)(k-1) \). Therefore, \( q(r-d)+q(n-1) \leq (n-1)(k-1) \).

Since \( q = \left\lceil \frac{n-1}{d} \right\rceil \), we have

\[
k - 1 \geq q + q \frac{r - d}{n - 1} = q + \frac{qr}{n - 1} - \frac{qd}{n - 1} = q + \frac{qr}{n - 1} - \frac{\left\lceil \frac{n-1}{d} \right\rceil d}{n - 1} \geq q + \frac{qr}{n - 1} - 1.
\]

Thus, \( k \geq \left\lceil \frac{n-1}{d} \right\rceil + \frac{qr}{n-1} \). Note that, \( \frac{qr}{n-1} \geq 0 \). If \( \frac{qr}{n-1} > 0 \), then \( k \geq \left\lceil \frac{n-1}{d} \right\rceil \), since \( k \) is an integer. If \( \frac{qr}{n-1} = 0 \), then \( r = 0 \) and consequently, \( d \) divides \( n - 1 \).

Thus, \( \left\lfloor \frac{n-1}{d} \right\rfloor = \left\lceil \frac{n-1}{d} \right\rceil \). Therefore, \( k \geq \left\lceil \frac{n-1}{d} \right\rceil \geq \frac{n-1}{d} \).

\[\Box\]

References

[1] M. O. Albertson and D. L. Boutin, Using determining sets to distinguish Kneser graphs, The Electronic Journal of Combinatorics 14 (2007), R#20.
[2] N. Biggs, Algebraic Graph Theory (2nd ed.), Cambridge University Press, pp. 155 - 163, chapter 20.
[3] D. L. Boutin, Identifying graph automorphisms using determining sets, Electronic Journal of Combinatorics 13 (2006), R#78.
[4] J. Caceres and D. Garijo, On the determining number and the metric dimension of graphs, The Electronic Journal of Combinatorics 17 (2010), R#63.
[5] D. Erwin and F. Harary, Destroying automorphisms by fixing nodes, Discrete Mathematics, 306 (2006) pp. 3244 - 3252.
[6] R. Frucht, Herstellung von graphen mit vorgegebener abstrakter gruppe., Compositio Mathematica 6 (1939), pp. 239 - 250.
[7] C. R. Gibbons and J. D. Laison, Fixing numbers of graphs and groups, The Electronic Journal of Combinatorics 16 (2009), #R39.
[8] F. Harary, Methods of destroying the symmetries of a graph, Bulletin of the Malaysian Mathematical Sciences Society 24(2) (2001), pp. 183 - 191.
[9] F. Harary and R. A. Melter, On the metric dimension of a graph, Ars Combinatoria, 2 (1976), pp. 191 - 195.
[10] M. Jannesari and B. Omoomi, Characterization of randomly k-dimensional graphs, http://arxiv.org/abs/1103.3169.
[11] M. Jannesari and B. Omoomi, On randomly k-dimensional graphs., http://arxiv.org/abs/1103.3169.
[12] I. Javaid and M. Fazil and U. Ali and M. Salman, On some parameters related to fixing sets in graphs, submitted for publication.
[13] I. Javed and H. Benish and U. Ali and M. Murtaza, On some automorphism related parameters in graphs, arXiv:1411.4922 [math.CO].
[14] Skiena S., Automorphism groups, 5.2.2 in Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica, Reading MA : Addison-Wesley, (1990) pp. 184 - 187.
[15] P. J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549 - 559.
