THE CASIMIR ENERGY ANOMALY FOR A POINT INTERACTION

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Abstract

The Casimir energy for a massless, neutral scalar field in presence of a point interaction is
analyzed using a general zeta-regularization approach developed in earlier works. In addition
to a regular bulk contribution, there arises an anomalous boundary term which is infinite
despite renormalization. The intrinsic nature of this anomaly is briefly discussed.

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1 Introduction

This note deals with the Casimir effect for a massless, neutral scalar field in presence of an
external delta-type potential concentrated at a point. Similar models were previously considered
in the literature [2, 3, 7, 13, 14, 15, 16, 17], building on various mathematically sound descriptions
of the Schrödinger operator comprising the said singular potential (see, e.g., [4, 5, 6]).

Continuing the analysis begun in Ref. [13], here the total Casimir energy for the above model
is investigated within a general framework for zeta regularization developed in previous works
[8, 9, 10, 11, 12]. In addition to a regular bulk contribution which is finite after renormalization,
there also appears an anomalous boundary term which remains infinite even after implementing
the standard renormalization procedure. The arising of this anomaly is ascribed to an unnatural
interpretation of the model.

2 The Reference Model and Zeta Regularization

Consider \((1 + 3)\)-dimensional Minkowski spacetime, endowed with a set of inertial coordinates
\(x^\mu\)\(_{\mu=0,1,2,3} = (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3\), such that the metric has components \((\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)\)
(natural units are employed, so that \(c=1\) and \(\hbar=1\)).
The theory of a canonically quantized, massless, neutral scalar field \( \hat{\phi} \) living on Minkowski spacetime in presence of an external delta-type potential concentrated at the point \( \mathbf{x} = 0 \) can be described making reference to the space domain \( \Omega = \mathbb{R}^3 \setminus \{0\} \) and considering the Klein-Gordon equation \((-\partial_t^2 + A)\hat{\phi} = 0\), where \( A \) is a self-adjoint realization on \( L^2(\Omega) \) of the 3-dimensional Laplacian \(-\Delta\), accounting for suitable boundary conditions at \( \mathbf{x} = 0 \). More precisely, one has (see [4, 13])

\[
\text{Dom} \ A := \left\{ \psi = \varphi + \frac{4\pi \lambda}{1 - i\sqrt{2} \lambda} \varphi(0) G_z \mid z \in \mathbb{C} \setminus [0, \infty), \varphi \in H^2(\mathbb{R}^3) \right\},
\]

\[
A := (-\Delta) \mid \text{Dom} A \subset L^2(\Omega) \rightarrow L^2(\Omega),
\]

where \( \lambda \) is a real parameter related to the strength of the potential, \( G_z := \frac{e^{(\sqrt{\varepsilon} |\mathbf{x}|)/4\pi}}{4\pi |\mathbf{x}|} \) with \( \text{Im} \sqrt{\varepsilon} > 0 \) and \( H^2(\mathbb{R}^3) \) is the usual Sobolev space of order two. The non-negativity of \( A \) (necessary for a consistent formulation of the field theory; see [12]) is ensured assuming \( \lambda \geq 0 \); in this case, \( A \) has purely absolutely continuous spectrum \( \sigma(A) = [0, \infty) \). Besides, note that the choice \( \lambda = 0 \) corresponds to the free theory where no delta potential is present (correspondingly, \( \text{Dom} A \mid \lambda = 0 = H^2(\mathbb{R}^3) \)).

Next, consider the modified operator \( \mathcal{A}_\varepsilon := A + \varepsilon^2 \), where the fictitious mass \( \varepsilon > 0 \) plays the role of an infrared cut-off. For \( t > 0 \), the associated heat kernel reads (see [1, 13])

\[
e^{-\varepsilon^2 t} \left[ e^{-\frac{4\pi t}{\varepsilon^2} |\mathbf{x} - \mathbf{y}|^2} + \frac{2t}{|\mathbf{x}| |\mathbf{y}|} \left( e^{-\frac{(|\mathbf{x} - \mathbf{y}|^2 - 1)}{4t}} - \frac{1}{\lambda} \int_0^\infty dw \ e^{-\frac{(w + |\mathbf{x} - \mathbf{y}|^2)}{4t}} \right) \right].
\]

Note that the term multiplying \( \frac{2t}{|\mathbf{x}| |\mathbf{y}|} \) in the above expression vanishes for \( \lambda = 0 \).

Replacing \( \hat{\phi} \) with the zeta-regularized field \( \hat{\phi}^{\varepsilon} := (\kappa^{-2} \mathcal{A}_\varepsilon)^{-u/4} \hat{\phi} \) (\( u \in \mathbb{C} \) is the regulating parameter, \( \kappa > 0 \) is a mass scale parameter; see [12]), one obtains regularized observables whose vacuum expectation values (VEVs) can be expressed in terms of (derivatives of) the integral kernels \( \mathcal{A}_\varepsilon^{(1\pm u)/2} \) evaluated at \( \mathbf{y} = \mathbf{x} \) for Re \( u \) large enough. Renormalization of these VEVs is attained computing the regular part of their analytic continuation w.r.t. \( u \) at \( u = 0 \) (1) and then taking the limit \( \varepsilon \to 0^+ \) (see [12]).

The above strategy was employed in Ref. [13] to determine the renormalized VEV of the stress-energy tensor. Notably, the renormalized energy density \( \langle 0 | \hat{T}_{00}^{\varepsilon}(\mathbf{x}) | 0 \rangle_{\text{ren}} \) was shown to diverge in a non-integrable way near \( \mathbf{x} = 0 \); so, integrating it over \( \Omega = \mathbb{R}^3 \setminus \{0\} \) yields an infinite total energy. An alternative approach to treat this quantity entails integrating first the regularized total density \( \int_\Omega d\mathbf{x} \langle 0 | \hat{T}_{00}^{\varepsilon}(\mathbf{x}) | 0 \rangle \) is given by the sum of bulk and boundary contributions, respectively defined as (see [12])

\[
E_{\varepsilon}^{u,v} := \frac{\kappa^{u}}{2} \int_\Omega d\mathbf{x} A_{\varepsilon}^{\frac{u-1}{2}}(\mathbf{x}, \mathbf{y}) \big|_{\mathbf{y} = \mathbf{x}},
\]

\[
B_{\varepsilon}^{u,v} := \left( \frac{1}{4} - \xi \right) \kappa^{u} \int_{\partial\Omega} d\sigma(\mathbf{x}) \partial_{n_y} A_{\varepsilon}^{\frac{u-1}{2}}(\mathbf{x}, \mathbf{y}) \big|_{\mathbf{y} = \mathbf{x}}
\]

(\( \xi \) is the conformal parameter, \( \partial\Omega \) is the boundary of \( \Omega \), \( d\sigma(\mathbf{x}) \) is the induced measure on \( \partial\Omega \) and \( n_y \) is the outer unit vector normal to \( \partial\Omega \) at \( \mathbf{y} \)). Finally, recall the Mellin-type identities holding for Re \( u \) large (\( \Gamma \) is the Euler’s gamma function) (see [12])

\[
A_{\varepsilon}^{\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) = \frac{1}{\Gamma(\frac{u+1}{2})} \int_0^\infty dt t^{\frac{u+1}{2} - 1} e^{-t A_{\varepsilon}}(\mathbf{x}, \mathbf{y}).
\]

\(^1\)By definition, the regular part at \( u = 0 \) of a given meromorphic function \( f \) with Laurent expansion \( f(u) = \sum_{n=-1}^\infty f_n u^n \) is \( Rf \mid_{u=0} f(u) := f_0. \)
3 The Relative Bulk Energy

The “bare” regularized bulk energy $E^{u,\varepsilon}$ diverges for all $u \in \mathbb{C}$ due to a potential-independent empty space contribution, present even for $\lambda = 0$. Subtracting the latter, one can consider the regularized “relative” bulk energy (cf. Eq. (2.3))

$$\Delta E^{u,\varepsilon} := \frac{\kappa^u}{2} \int_{\Omega} dx \left( A_\varepsilon^{-\frac{u}{2}}(x, x) - A_\varepsilon^{-\frac{u}{2}}(x, x) |_{\lambda=0} \right).$$

From Eqs. (2.2), (2.5) and (3.1), passing to spherical coordinates (with $r := |x|$) one gets

$$\Delta E^{u,\varepsilon} = \frac{\kappa^u}{\sqrt{4\pi} \Gamma\left(\frac{u+1}{2}\right)} \int_0^\infty dr \int_0^\infty dt \frac{t^\frac{u}{2}-2}{\varepsilon^{2t}} e^{-\varepsilon^2 t} \left( e^{-\frac{r^2}{\varepsilon^2}} - \frac{1}{\lambda} \int_0^\infty dw e^{-\left(\frac{\lambda t + 2 \varepsilon^2 t}{4}\right)} \right).$$

The above representation makes sense for Re $u > 1$: besides, the order of integration can be permuted arbitrarily by Fubini’s theorem. Evaluating explicitly the integrals in $r$ and $w$, posing $\tau := \sqrt{t}/\lambda$ and integrating by parts twice one infers

$$\Delta E^{u,\varepsilon} = \frac{(\lambda \kappa)^u}{4 \kappa u \Gamma\left(\frac{u+1}{2}\right)} \int_0^\infty d\tau \frac{\gamma}{8 \sqrt{\pi}} \frac{d^2}{d\tau^2} \left( e^{-\varepsilon^2 \tau^2} e^{\varepsilon^2 \text{erfc}(\tau)} \right).$$

In view of the regularity and asymptotic features of the complementary error function erfc($\tau$), the latter expression provides the analytic continuation of $\Delta E^{u,\varepsilon}$ to a meromorphic function of $u$ for Re $u > -1$, with a simple pole at $u = 0$. Taking the regular part and evaluating the limit $\varepsilon \to 0^+$ (following the general approach of Ref. [12]), by dominated convergence theorem one obtains the renormalized relative bulk energy

$$\Delta E^{\text{ren}} := \lim_{\varepsilon \to 0^+} R P|_{u=\varepsilon} \Delta E^{u,\varepsilon}$$

$$= \frac{1}{\lambda} \int_0^\infty d\tau \frac{\gamma + 2 \log(2\kappa \lambda \tau)}{8 \sqrt{\pi}} \left( e^{\varepsilon^2 \text{erfc}(\tau)} \right) = \frac{\log(\kappa \lambda)}{2\pi \lambda}$$

($\gamma$ is the Euler-Mascheroni constant and the integral was evaluated using Mathematica).

The “vacuum energy” $E_{\text{vacuum}} := -\lim_{\beta \to \infty} \partial_3 \log Z = 2\alpha(1 - \log(4\pi \alpha t))$ determined in accordance with Ref. [17] coincides with $\Delta E^{\text{ren}}$ if the renormalization length scale of [17] is fixed as $\ell = \epsilon^2/\kappa$ (recall also that $\alpha = 1/(4\pi \lambda)$, according to Ref. [13]).

4 The Anomalous Boundary Energy Term

The space domain $\Omega = \mathbb{R}^3 \setminus \{0\}$ has improper boundaries at $|x| \to \infty$ and $|x| \to 0$. Taking this and spherical symmetry into account, one can express the regularized boundary energy of Eq. (2.4) via appropriate limits of integrals over finite-size spheres:

$$B^{u,\varepsilon} = \left( \frac{1}{4} - \xi \right) \left[ \lim_{r \to \infty} B^{u,\varepsilon}_{\text{out}}(r) + \lim_{r \to 0} B^{u,\varepsilon}_{\text{in}}(r) \right];$$

$$B^{u,\varepsilon}_{\text{out/in}}(r) := \kappa^u \int_{|x|=r} d\sigma(x) \partial_{ny}^{\text{out/in}} A_{\varepsilon^{-\frac{u+1}{2}}}(x, y) |_{y=x}. \quad (r > 0).$$

In Eq. (4.2), $\partial_{ny}^{\text{out}}$ (resp. $\partial_{ny}^{\text{in}}$) denotes the derivative in the outer (resp. inner) radial direction normal to the sphere $|x|=r$. From Eqs. (2.2), (2.5) and (4.2) one infers

$$B^{u,\varepsilon}_{\text{out/in}}(r) = (-/+1) \frac{1}{r} \frac{\kappa^u}{\sqrt{\pi} \Gamma\left(\frac{u+1}{2}\right)} \int_0^\infty dt \frac{\frac{1}{t^{u-1}} - 2}{\varepsilon^2 t}$$

$$\times \left[ 1 + \frac{r^2}{t} \right] e^{-\frac{r^2}{\varepsilon^2} - \frac{1}{\lambda} \int_0^\infty dw e^{-\left(\frac{\lambda t + 2 w^2}{4}\right)} \left(1 + \frac{r(w+2r)}{2t}\right) \right].$$
Notably, $B_{\text{out/in}}^{u,\varepsilon}(r) \equiv 0$ for $\lambda = 0$; thus, there is no empty space contribution to $B^{u,\varepsilon}$.

On one hand, one easily infers by dominated convergence that $\lim_{r \to \infty} B_{\text{out}}^{u,\varepsilon}(r) = 0$ for any $u \in \mathbb{C}$ (and $\varepsilon > 0$). On the other hand, again by dominated convergence (though more careful estimates are demanded here), for any $\Re u > 0$ one gets

$$
\lim_{r \to 0^+} (r B_{\text{in}}^{u,\varepsilon}(r)) = \frac{k^u}{\sqrt{\pi} \Gamma\left(\frac{u+1}{2}\right)} \int_0^\infty dt \frac{t^{u-\frac{1}{2}} e^{-\varepsilon^2 t}}{1 \int_0^\infty dw e^{-(\frac{w}{2}+\frac{u^2}{4})}}. \tag{4.4}
$$

Since the r.h.s. is finite and not zero (in fact, it involves the integral of a positive function), Eq. (4.4) implies $\lim_{r \to 0^+} B_{\text{in}}^{u,\varepsilon}(r) = \infty$ for any $u \in \mathbb{C}$. This divergence entails an infinite contribution to the renormalized boundary energy; namely,

$$
B^{\text{ren}} := \lim_{\varepsilon \to 0^+} R P |_{u=0} B^{u,\varepsilon} = \lim_{\varepsilon \to 0^+} R P |_{u=0} \left( \lim_{r \to 0^+} B_{\text{in}}^{u,\varepsilon}(r) \right) = \infty. \tag{4.5}
$$

Thus, while no contribution arises from spatial infinity (as was to be expected), an infinite energy occurs where the potential is concentrated. Conceivably, the persistence after renormalization of this divergence is due to an unnatural use at small scales of an effective model meant to describe sensible physics only at large ones (cf. Ref. [16]).

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