Some Combinatorial Identities some of which involving Harmonic Numbers

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Abstract

A product difference equation is proved and used for derivation by elementary methods of four combinatorial identities, eight combinatorial identities involving generalized harmonic numbers and three combinatorial identities involving traditional harmonic numbers. For the binomial coefficients the definition with gamma functions is used, thus also allowing non-integer arguments in the identities. The generalized harmonic numbers in this case are harmonic numbers with a complex offset, where the traditional harmonic numbers are a special case with offset zero.

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1 Introduction

Binomial coefficients and their combination with harmonic numbers occur frequently in applied mathematics [12, 13, 15, 22]. Many ways to find and prove these identities can be found in literature [5, 16, 17, 18, 21, 23, 25]. In this paper a product difference equation is used for derivation of a number of such, often new, identities.

2 A Product Difference Equation

Theorem 2.1. Let n be a nonnegative integer, and let x and y be complex variables and let the set \{z_k\} be n complex variables and the set \{w_k\} be n complex variables. Then there is the following product difference equation:

\[
\prod_{k=1}^{n} (x - z_k)^{w_k} - \prod_{k=1}^{n} (y - z_k)^{w_k}
= \sum_{k=1}^{n} [(x - z_k)^{w_k} - (y - z_k)^{w_k}] \prod_{l=1}^{k-1} (x - z_l)^{w_l} \cdot \prod_{l=k+1}^{n} (y - z_l)^{w_l}
\]

(2.1)

where a sum of zero terms is zero and a product of zero terms is one.
Proof. Let $\alpha_k = (x - z_k)^w$ and $\beta_k = (y - z_k)^w$. Then (2.1) becomes:

$$\prod_{k=1}^{n} \alpha_k - \prod_{k=1}^{n} \beta_k = \sum_{k=1}^{n} (\alpha_k - \beta_k) \prod_{l=1}^{k-1} \alpha_l \cdot \prod_{l=k+1}^{n} \beta_l$$

$$= \sum_{k=1}^{n} \prod_{l=1}^{k} \alpha_l \cdot \prod_{l=k+1}^{n} \beta_l - \prod_{l=k}^{n} \alpha_l \cdot \prod_{l=k}^{n} \beta_l$$

(2.2)

On the right side the first product for each $k$ cancels the second product for each $k + 1$, only leaving the first product for $k = n$ and the second product for $k = 1$. These two remaining products are the two products on the left side.

In the case that all $z_k = 0$ and all $w_k = 1$, (2.1) reduces to the following power difference equation [19]:

$$x^n - y^n = (x - y) \sum_{k=1}^{n} x^{k-1} y^{n-k}$$

(2.3)

3 Combinatorial Identities

The definition of the binomial coefficient in terms of gamma functions for complex $x$, $y$ is [1, 10, 12]:

$$\binom{x}{y} = \frac{\Gamma(x + 1)}{\Gamma(y + 1)\Gamma(x - y + 1)}$$

(3.1)

For nonnegative integer $n$ and integer $k$ this reduces to [1, 7, 12, 13]:

$$\binom{n}{k} = \begin{cases} 
n! \\
0 
\end{cases} \frac{k!(n-k)!}{n!} \text{ if } 0 \leq k \leq n$$

(3.2)

From the recurrence formula for the gamma function for complex $s$ [1, 2, 11, 19, 24]:

$$\Gamma(s+1) = s \Gamma(s)$$

(3.3)

follows for integer $a \leq b$ and complex $s$:

$$\prod_{k=a}^{b-1} (s-k) = \frac{\Gamma(s-a+1)}{\Gamma(s-b+1)} = (-1)^b \frac{\Gamma(b-s)}{\Gamma(a-s)}$$

(3.4)

Combination of (2.1) with the $\{z_k\}$ consecutive integers and all $w_k = w$ and (3.1) and one of the right side expressions of (3.4) for $x$ and for $y$ in (2.1) yields the following four combinatorial identities.

For integer $a \leq b$ and complex $w$, $x$, $y$:

$$\sum_{k=a}^{b-1} (x+k+1)^w - (x - y + k)^w \left( \begin{array}{c} x+k \\ y \end{array} \right)^w$$

$$= (y+1)^w \left[ \left( \begin{array}{c} x+b \\ y+1 \end{array} \right)^w - \left( \begin{array}{c} x+a \\ y+1 \end{array} \right)^w \right]$$

(3.5)
\[
\begin{align*}
\sum_{k=a}^{b-1} [(y-x+k)^w - (y+k)^w] (-1)^w x^w y^k = \sum_{k=a}^{b-1} x^w \left[ (-1)^w \left( \frac{x-1}{y+b-1} \right)^w - (-1)^w \left( \frac{x-1}{y+a-1} \right)^w \right] & \quad (3.6) \\
\sum_{k=a}^{b-1} [(y-x+k)^w - (y-x+k+1)^w] (-1)^w x^w y^k = \sum_{k=a}^{b-1} (x+1)^w \left[ (-1)^w \left( \frac{x+1}{y+b} \right)^w - (-1)^w \left( \frac{x+1}{y+a} \right)^w \right] & \quad (3.7) \\
\sum_{k=a}^{b-1} [(x-k)^w - (y-k+1)^w] x^w y^k = \sum_{k=a}^{b-1} (y+1)^w \left[ \left( \frac{x}{b} \right)^w \left( \frac{y+1}{b} \right)^w - \left( \frac{x}{a} \right)^w \left( \frac{y+1}{a} \right)^w \right] & \quad (3.8)
\end{align*}
\]

For \( w = 1 \) these identities reduce to:
\[
\sum_{k=a}^{b-1} \left( \frac{x+k}{y} \right) = \left( \frac{x+b}{y+1} \right) - \left( \frac{x+a}{y+1} \right) \quad (3.9)
\]
\[
\sum_{k=a}^{b-1} (-1)^k \left( \frac{x}{y+k} \right)^{-1} = (-1)^a \left( \frac{x-1}{y+a-1} \right) - (-1)^b \left( \frac{x-1}{y+b-1} \right) \quad (3.10)
\]
\[
\sum_{k=a}^{b-1} (-1)^k \left( \frac{x}{y+k} \right)^{-1} = \frac{x+1}{x+2} \left[ (-1)^a \left( \frac{x+1}{y+a} \right)^{-1} - (-1)^b \left( \frac{x+1}{y+b} \right)^{-1} \right] \quad (3.11)
\]
\[
\sum_{k=a}^{b-1} \left( \frac{x}{k} \right)^{-1} \left( \frac{y}{k} \right)^{-1} = \frac{y+1}{x-y-1} \left[ \left( \frac{x}{b} \right)^{-1} \left( \frac{y+1}{b} \right)^{-1} - \left( \frac{x}{a} \right)^{-1} \left( \frac{y+1}{a} \right)^{-1} \right] \quad (3.12)
\]

which are equivalent to known identities [7].

For an example of \( w = 2 \), let be given [7, 15]:
\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \quad (3.13)
\]

Then (3.6) with \( w = 2, a = 0, b = n+1, x = n, y = 0 \) and (3.13) yields:
\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{n}{2} \binom{2n}{n} \quad (3.14)
\]

For an example of \( w = 3 \), let a formula from A.C. Dixon be given [8, 9, 18]:
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n} \binom{3n}{n} \quad (3.15)
\]
Then (3.6) with \( w = 3 \), \( a = 0 \), \( b = 2n + 1 \), \( x = 2n \), \( y = 0 \) and (3.15) yields:

\[
\sum_{k=0}^{2n} (-1)^k k(2n - k) \binom{2n}{k}^3 = (-1)^n \frac{4}{3} n^2 \binom{2n}{n} \binom{3n}{n} \tag{3.16}
\]

### 4 Combinatorial Identities with Harmonic Numbers

The definition of the harmonic numbers for nonnegative integer \( n \) is [13, 15]:

\[
H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{4.1}
\]

from which follows that \( H_0 = 0 \). The definition of the generalized harmonic numbers for nonnegative integer \( n \), complex order \( m \) and complex offset \( c \) is [14, 15]:

\[
H_{c,n}^{(m)} = \sum_{k=1}^{n} \frac{1}{(c+k)^m} \tag{4.2}
\]

from which follows that \( H_{c,0}^{(m)} = 0 \). There is the following symmetry formula:

\[
H_{c,n}^{(m)} = (-1)^m H_{-(c+n+1),n}^{(m)} \tag{4.3}
\]

For order \( m = 1 \) and nonnegative integer offset \( c = k \) these numbers can be expressed in the traditional harmonic numbers of (4.1):

\[
H_{k,n}^{(1)} = H_{k+n} - H_k \tag{4.4}
\]

from which follows:

\[
H_{0,n}^{(1)} = H_n \tag{4.5}
\]

Identities involving these numbers can be obtained by differentiation of the product [3.4] and using (4.3):

\[
\frac{d}{dx} \prod_{k=a}^{b-1} (x-k)^w = wH_{x-b,b-a}^{(1)} \prod_{k=a}^{b-1} (x-k)^w = -w \prod_{k=a}^{b-1} (x-k)^w \tag{4.6}
\]

and similarly for \( y \). These identities therefore are in pairs: one from differentiation to \( x \) and one from differentiation to \( y \), in each case using one of the right side expressions in (4.6). This results in the following eight combinatorial identities involving generalized harmonic numbers.

For integer \( a \leq b \) and complex \( w, x, y \):

\[
\sum_{k=a}^{b} [(x + k + 1)^w H_{x+a,k-a+1}^{(1)} - (x - y + k)^w H_{x+a,k-a}^{(1)}] (x + k)^w = (y + 1)^w \binom{x + b}{y + 1}^w H_{x+a,b-a}^{(1)} \tag{4.7}
\]
\[
\sum_{k=a}^{b-1} [(x + k + 1)^w H^{(1)}_{y - x - b, b - k - 1} - (x - y + k)^w H^{(1)}_{y - x - b, b - k}](x + k)^w
\]
\[
= -(y + 1)^w \binom{x + a}{y + 1} H^{(1)}_{y - x - b, b - a} \tag{4.8}
\]
\[
\sum_{k=a}^{b-1} [(y - x + k)^w H^{(1)}_{y - x + a - 1, k - a + 1} - (y + k)^w H^{(1)}_{y - x + a - 1, k - a}] \\
\cdot (-1)^w k \binom{x}{y + k} = x^w (-1)^w b \binom{x - 1}{y + b - 1} H^{(1)}_{y - x + a - 1, b - a} \tag{4.9}
\]
\[
\sum_{k=a}^{b-1} [(y - x + k)^w H^{(1)}_{y - x - b - k - 1} - (y + k)^w H^{(1)}_{y - x - b - k}] \\
\cdot (-1)^w k \binom{x}{y + k} = -x^w (-1)^w a \binom{x - 1}{y + a - 1} H^{(1)}_{y - x - b - a} \tag{4.10}
\]
\[
\sum_{k=a}^{b-1} [(-y - k - 1)^w H^{(1)}_{y + a, k - a + 1} - (x - y - k + 1)^w H^{(1)}_{y + a, k - a}] \\
\cdot (-1)^w k \binom{x}{y + k} = (x + 1)^w (-1)^w b \binom{x + 1}{y + b} H^{(1)}_{y + a, b - a} \tag{4.11}
\]
\[
\sum_{k=a}^{b-1} [(-y - k - 1)^w H^{(1)}_{x - y - b + 1, b - k - 1} - (x - y - k + 1)^w H^{(1)}_{x - y - b + 1, b - k}] \\
\cdot (-1)^w k \binom{x}{y + k} = -(x + 1)^w (-1)^w a \binom{x + 1}{y + a} H^{(1)}_{x - y - b + 1, b - a} \tag{4.12}
\]
\[
\sum_{k=a}^{b-1} [(x - k)^w H^{(1)}_{a - x - 1, k - a + 1} - (y - k + 1)^w H^{(1)}_{a - x - 1, k - a}] \binom{x}{k}^w \binom{y}{k}^w
\]
\[
= (y + 1)^w \binom{x}{b} \binom{y + 1}{b} H^{(1)}_{a - x - 1, b - a} \tag{4.13}
\]
\[
\sum_{k=a}^{b-1} [(x - k)^w H^{(1)}_{y - b + 1, b - k - 1} - (y - k + 1)^w H^{(1)}_{y - b + 1, b - k}] \binom{x}{k}^w \binom{y}{k}^w
\]
\[
= -(y + 1)^w \binom{x}{a} \binom{y + 1}{a} H^{(1)}_{y - b + 1, b - a} \tag{4.14}
\]

For \( w = 1 \) these identities reduce to:
\[
\sum_{k=a}^{b-1} (x + k)^w H^{(1)}_{x + a, k - a}
\]
\[
= (x + b)^w H^{(1)}_{x + a, b - a} - \frac{1}{y + 1} + \frac{1}{y + 1} (x + a) \tag{4.15}
\]

5
Changing all $k$ into $a+b-k-1$ in the summation term reverses the order of summation and yields valid identities. When the offset of the generalized harmonic numbers in these identities is zero, these identities yield one traditional harmonic number identity.

\[
\sum_{k=a}^{b-1} \binom{x+k}{y} H_{y-x-b,b-k-1}^{(1)} = - \binom{x+a}{y+1} H_{y-x-b,b-a}^{(1)} - \frac{1}{y+1} - \frac{1}{y+1} (x+b)
\]

\[
\sum_{k=a}^{b-1} (-1)^k \binom{x}{y+k} H_{y-x+a-1,k-a}^{(1)} = (-1)^{b+1} \binom{x-1}{y+b-1} H_{y-x+a-1,b-a}^{(1)} + \frac{1}{x} \binom{x-1}{y+a-1} - \frac{1}{x} \binom{x-1}{y+b-1}
\]

\[
\sum_{k=a}^{b-1} (-1)^k \binom{x}{y+k} H_{y-x-b,k-a}^{(1)} = \frac{x+1}{x+2} \left[ (-1)^{b+1} \binom{x+1}{y+b} H_{y+x+1,b-a}^{(1)} - \frac{1}{x+2} - \frac{1}{x+2} (x+1) \right]
\]

\[
\sum_{k=a}^{b-1} \binom{x}{k} \binom{y}{k} H_{a-x+1,k-a}^{(1)} = \frac{x+1}{x-y-1} \frac{y+1}{y-a} \binom{y+1}{a} H_{a-x+1,b-a}^{(1)} + \frac{1}{x-y-1} \frac{1}{y-x+1} (x+a) \binom{y+1}{a} \]

\[
\sum_{k=a}^{b-1} \binom{x}{k} \binom{y}{k} H_{y-b+1,b-k-1}^{(1)} = \frac{x+1}{y-x+1} \left[ \binom{y+1}{a} H_{y-b+1,b-a}^{(1)} - \frac{1}{y-x+1} \frac{1}{y-x+1} (x+a) \binom{y+1}{a} \right]
\]
and three combinatorial identities involving traditional harmonic numbers.

For nonnegative integer \( n \) and complex \( m, w \):

\[
\sum_{k=0}^{n} \left[ (k+1)^w - k^w \right] H_k = (n+1)^w H_{n+1} - H_{0,n+1}^{(1-w)} \tag{4.23}
\]

\[
\sum_{k=0}^{n} \binom{k}{m} H_k = \left( \frac{n+1}{m+1} \right) \left( H_{n+1} - \frac{1}{m+1} + \frac{1}{m+1} \binom{0}{m+1} \right) \tag{4.24}
\]

\[
\sum_{k=0}^{n} \binom{m}{n-k} \binom{n}{k}^{-1} H_k = \frac{n+1}{n-m+1} \left[ H_{n+1} + \frac{1}{n-m+1} \left( \binom{m}{n+1} - 1 \right) \right] \tag{4.25}
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n}{k}^{-1} H_k = \frac{m+1}{m+2} \left[ (-1)^n \binom{m+1}{n+1}^{-1} \left( H_{n+1} - \frac{1}{m+2} \right) - \frac{1}{m+2} \right] \tag{4.26}
\]

The identity (4.23) is an example of harmonic number identities with polynomials or rational functions in \( k \) \[20, 22\]. The identity (4.24) for nonnegative integer \( m \) was already listed and proved in literature \[12, 13\]. More complicated identities involving harmonic numbers with higher orders may be derived from the given identities by using (4.3) and the following relations \[3, 4, 6, 7\].

For nonnegative integer \( n \) and complex \( m, w, x, y \):

\[
\frac{d}{dx} H_{x+y}^{(m)} = -m H_{x+y,n}^{(m+1)} \tag{4.27}
\]

\[
\frac{d}{dx} \left( \frac{x+y}{n} \right)^w = wH_{x+y-n,n}^{(1)} \left( \frac{x+y}{n} \right)^w \tag{4.28}
\]

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