ON SUMS OF FOURIER COEFFICIENTS OF CUSP FORMS

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The aim of this note is to evaluate asymptotically the sum

\[ F(x) := \sum_{n \leq x} f(n^2) \]

in case \( f(n) \) is the Fourier coefficient of a holomorphic or non-holomorphic cusp form. We shall first deal with the latter case, which is more complicated. Let as usual \( \{ \lambda_j = \kappa_j^2 + \frac{1}{4} \} \cup \{ 0 \} \) be the discrete spectrum of the non-Euclidean Laplacian acting on \( SL(2, \mathbb{Z}) \)-automorphic forms. Further let \( \rho_j(n) \) denote the \( n \)-th Fourier coefficient of the Maass wave form \( \varphi_j(z) \) corresponding to the eigenvalue \( \lambda_j \) to which the Hecke series

\[ H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s} \quad (\Re s > 1) \]

is attached (see Y. Motohashi [6] for an extensive account). For every \( n \in \mathbb{N} \) we have \( \rho_j(n) = \rho_j(1) t_j(n) \), so that one may consider sums of \( t_j(n) \) instead of sums of \( \rho_j(n) \). In [4] T. Meurman and the author proved that, for \( \kappa_j \leq x^{1-\alpha} \), we have uniformly in \( \kappa_j \)

\[ \sum_{n \leq x} t_j^2(n) = \frac{12x}{\pi^2 \alpha_j} + O(x^\varepsilon R(x)) \]

with

\[ R(x) := \kappa_j^{-\frac{1}{2}} x^{\frac{1}{2}} + \min \left( \frac{\kappa_j + 10}{\kappa_j}, x^{\frac{1}{2}} + x^{\frac{3+6\alpha}{5}}, \kappa_j^{-1} x^{1+2\alpha} \right) . \]

Here as usual

\[ \alpha_j = \frac{|\rho_j(1)|^2}{\cosh(\pi \kappa_j)} , \]

and \( \alpha \geq 0 \) is the constant for which

\[ t_j(n) \ll_n n^{\alpha+\varepsilon} \]

holds uniformly in \( \kappa_j \). Moreover, \( \varepsilon \) denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence, while \( \ll_n \) means that the \( \ll \)-constant depends on \( \varepsilon \). The accent in (2)-(3) is on uniformity in \( \kappa_j \), since in many applications \( \kappa_j \) may vary with \( x \). It is known that (5) holds with \( \alpha \leq \frac{5}{28} \) (see D. Bump et al. [1]). In what concerns the order of \( \alpha_j \), we have

\[ \kappa_j^{-\varepsilon} \ll_\varepsilon \alpha_j \ll_\varepsilon \kappa_j^{\varepsilon} . \]

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The lower bound in (6) was proved by H. Iwaniec [5], and the upper bound by Hoffstein–Lockhart [2].

It seems that there are no upper bounds in the literature for the sum \( F(x) \) in (1) when \( f(n) = t_j(n) \).

We shall prove

**THEOREM 1.** For \( \kappa_j \leq x^\varepsilon \),

(7) \[ 0 < c < \min \left( \frac{3 + 6\alpha}{2 + 20\alpha}, 1 - \alpha \right) \]

and a suitable constant \( A > 0 \) we have, uniformly in \( \kappa_j \),

(8) \[ \sum_{n \leq x} t_j(n^2) \ll \alpha_j^{-1} x \exp \left( -A \log^{3/5} x (\log \log x)^{-1/5} \right) . \]

**Corollary 1.** With the value \( \alpha \leq \frac{3}{28} \) it follows that (8) holds uniformly in \( \kappa_j \) for \( \kappa_j \leq x^\varepsilon \), and any constant \( c \) satisfying \( 0 < c < \frac{57}{78} \).

**Corollary 2.** If the Ramanujan-Petersson conjecture that \( \alpha = 0 \) is true, then (8) holds uniformly in \( \kappa_j \) for \( \kappa_j \leq x^\varepsilon \), and any constant \( 0 < c < 1 \).

**Remark 1.** The negative exponential in (8) comes from the sharpest known error term in the prime number theorem in the form (see e.g., [3, Chapter 12])

(9) \[ \sum_{n \leq x} \mu(n) \ll x \exp \left( -C \log^{3/5} x (\log \log x)^{-1/5} \right) \quad (C > 0). \]

Sharper forms of the prime number theorem, which would follow from a better zero-free region for the Riemann zeta-function \( \zeta(s) \) than the one that is currently known (see [3, Chapter 6]), would therefore lead to a better estimate than (8).

**Remark 2.** By a result of T. Meurman and the author [4] one has

\[ \sum_{n \leq x} t_j(n) \ll \kappa_j^{1+\varepsilon} \]

uniformly for \( \sqrt{x} < \kappa_j \leq x \), which may be compared to the bound in (8).

**Remark 3.** The oscillatory nature of the function \( t_j(n) \) accounts for the lack of a main term in (8). However, if one looks at the problem of evaluating \( F(x) \) in (1) when \( f(n) = d(n) \), the number of divisors of \( n \), then there will be a main term in the corresponding formula for the summatory function. Namely the function \( d(n^2) \) is generated by \( \zeta^3(s)/\zeta(2s) \), which has a pole of order three at \( s = 1 \). Consequently we have (see [3, eq. (14.29)]), for suitable constants \( B_1 (> 0), B_2, B_3, C (> 0) \),

\[ \sum_{n \leq x} d(n^2) = x(B_1 \log^2 x + B_2 \log x + B_3) + O \left( x \exp \left( -C \log^{3/5} x (\log \log x)^{-1/5} \right) \right) . \]

**Proof of Theorem 1.** From the multiplicative property (see [6, eq. (3.2.8)])

\[ t_j(mn) = \sum_{d|(m,n)} \mu(d) t_j \left( \frac{m}{d} \right) t_j \left( \frac{n}{d} \right) , \]
where $\mu(n)$ is the Möbius function, one has

\begin{equation}
  t_j(n^2) = \sum_{d|n} \mu(d) t_j^2 \left( \frac{n}{d} \right). 
\end{equation}

Therefore (10) gives

\begin{align*}
\sum_{n \leq x} t_j(n^2) &= \sum_{mn \leq x} \mu(m) t_j^2(n) \\
&= \sum_{m \leq \sqrt{x}} \mu(m) \sum_{n \leq x/m} t_j^2(n) + \sum_{\sqrt{x} < m \leq x/n} t_j(n^2) \sum_{n \leq x} \mu(m) \\
&= \sum_1 + \sum_2, 
\end{align*}

say. We set for brevity

$$\eta(x) = (\log x)^{3/5} (\log \log x)^{-1/5},$$

and let $0 < \beta < 1$, $\beta = \beta(\alpha)$ be such a number for which (2)-(3) give

\begin{equation}
\sum_{n \leq x} t_j^2(n) = \frac{12x}{\pi^2 \alpha_j} + O(x^\beta)
\end{equation}

uniformly for $\kappa_j \leq x^c$. If $C$ denotes generic positive constants, then

\begin{align*}
\sum_1 &= \frac{12x}{\pi^2 \alpha_j} \sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m} \, \mu \left( \sum_{m \leq \sqrt{x}} \left( \frac{x}{m} \right)^\beta \right) \\
&= \frac{12x}{\pi^2 \alpha_j} \sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m} + O(x^\beta x^{1-\beta}) \\
&\ll \alpha_j^{-1} x e^{-C\eta(x)},
\end{align*}

since in view of $0 < \beta < 1$ and the upper bound in (6) we have

$$x^{1+\beta} \ll \alpha_j^{-1} x e^{-C\eta(x)}.$$

Here we also used (9), partial summation and the well-known fact that

$$\sum_{m=1}^\infty \frac{\mu(m)}{m} = 0$$

to deduce that

\begin{equation}
\sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m} \ll \exp(-C\eta(x)).
\end{equation}

We also have, on using (9) and (11),

\begin{align*}
\sum_2 &\ll \sum_{n \leq \sqrt{x}} t_j^2(n) \left( \frac{x}{n} e^{-C\eta(x/n)} + \sqrt{xe^{-C\eta(x)}} \right) \\
&\ll e^{-C\eta(x)} \left( x \sum_{n \leq \sqrt{x}} \frac{t_j^2(n)}{n} + \sqrt{x} \sum_{n \leq \sqrt{x}} t_j^2(n) \right) \\
&\ll \alpha_j^{-1} x e^{-C\eta(x)}. 
\end{align*}
Now note that for \( \kappa_j \leq x^c \), \( c < 1 - \alpha \) we have
\[
\kappa_j^{\frac{1}{2} - 2\alpha} x^j \leq x^\beta, \quad \beta = \frac{c}{2 - 2\alpha} + \frac{1}{2} \quad (< 1).
\]
Moreover
\[
1 + 10\alpha + \frac{3}{3 + 6\alpha} c < \frac{1}{2} \quad \text{for} \quad c < \frac{3 + 6\alpha}{2 + 20\alpha},
\]
and \( \frac{3 + 6\alpha}{5} < 1 \). Hence (11) is satisfied if \( \kappa_j \leq x^c \) and \( c \) is any constant satisfying (7). This finishes the proof of Theorem 1.

The foregoing analysis may be applied to the case of a holomorphic cusp form
\[
\varphi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}
\]
of weight \( \kappa \) with respect to \( SL(2, \mathbb{Z}) \). If \( \varphi(z) \) is a normalized eigenform, i.e., an eigenfunction with respect to all Hecke operators and satisfies \( a(1) = 1 \), then all \( a(n) \in \mathbb{R} \). We have (see e.g., R.A. Rankin [7] and [8])
\[
\sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p (1 - a(p)p^{-s} + p^{\kappa-1-2s})^{-1} \quad (\sigma > \frac{1}{2}(\kappa + 1)),
\]
and
\[
a(m)a(n) = \sum_{d|(m,n)} d^{\kappa-1} a \left( \frac{mn}{d^2} \right) \quad (m, n \in \mathbb{N}).
\]
If we introduce the “normalized” function \( \hat{a}(n) \), namely
\[
a(n) = \hat{a}(n)n^{\frac{\kappa-1}{2}},
\]
then we can write (15) as
\[
\hat{a}(m)\hat{a}(n) = \sum_{d|(m,n)} \hat{a} \left( \frac{mn}{d^2} \right) \quad (m, n \in \mathbb{N}).
\]
When \( m = n \), (17) gives by the Möbius inversion formula
\[
\hat{a}(n^2) = \sum_{d|n} \mu(d) \left( \hat{a} \left( \frac{n}{d} \right) \right)^2,
\]
which is analogous to (10). If we use (18), the Rankin-Selberg formula
\[
\sum_{n \leq x} \hat{a}^2(n) = Cx + O(x^{3/5}) \quad (C > 0),
\]
then by employing the method of proof of Theorem 1 we shall obtain

**Theorem 2.** We have
\[
\sum_{n \leq x} \hat{a}(n^2) \ll x \exp \left( -C(\log x)^{3/5}(\log \log x)^{-1/5} \right) \quad (C > 0).
\]

From Theorem 2 we obtain by partial summation, on using (16),

**Corollary 3.** We have
\[
\sum_{n \leq x} a(n^2) \ll x^c \exp \left( -C(\log x)^{3/5}(\log \log x)^{-1/5} \right) \quad (C > 0).
\]
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