Convergence of the Kähler–Ricci iteration

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Abstract

The Ricci iteration is a discrete analogue of the Ricci flow. According to Perelman, the Ricci flow converges to a Kähler–Einstein metric whenever one exists, and it has been conjectured that the Ricci iteration should behave similarly. This article confirms this conjecture. As a special case, this gives a new method of uniformization of the Riemann sphere.

1 Introduction

Let \( (M,g_1) \) be a compact Riemannian manifold. A Ricci iteration is a sequence of metrics \( \{g_i\}_{i \in \mathbb{N}} \) on \( M \) satisfying

\[
\text{Ric} g_{i+1} = g_i, \quad i \in \mathbb{N},
\]

where \( \text{Ric} g_{i+1} \) denotes the Ricci curvature of \( g_{i+1} \). One may think of (1) as a dynamical system on the space of Riemannian metrics on \( M \). Part of the interest in the Ricci iteration is that, clearly, Einstein metrics are fixed points, and so (1) aims to provide a natural theoretical and numerical approach to uniformization in the challenging case of positive Ricci curvature (different Ricci iterations can be defined in the context of non-positive curvature, but these are typically easier to understand and will not be discussed here). In essence, the Ricci iteration aims to reduce the Einstein equation to a sequence of prescribed Ricci curvature equations and can be thought of as a discretization of the Ricci flow. Going back to [26, 27], it has been studied since by a number of authors [4, 6, 9, 10, 12, 19, 18, 22, 25], see also the survey [29, §6.5].

Of particular interest has been the study of the Ricci iteration on Kähler manifolds (for the non-Kähler case results are scarce, see [25]). When \( (M,J,g_1) \) is Kähler, the Calabi–Yau Theorem [31] guarantees the existence and uniqueness of the sequence \( \{g_i\}_{i \in \mathbb{N}} \) if and only if \( M \) is Fano (i.e., has positive first Chern class \( c_1(M,J) \)) and the Kähler class associated to \( g_1 \) is \( c_1(M,J) \). Under a rather restrictive technical assumption, one of us showed that \( g_i \) converges smoothly to a Kähler–Einstein metric [27, Theorem 3.3] and made the following general conjecture [27, Conjecture 3.2]:

**Conjecture 1.1.** Let \( (M,J,g_1) \) be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to \( g_1 \) is \( c_1(M,J) \). Then the Ricci iteration (1) converges in the sense of Cheeger-Gromov to a Kähler–Einstein metric.

The best result so far on this conjecture is due to Berman et al. [6] who replace the technical assumption of [27, Theorem 3.3] concerning Tian’s \( \alpha \)-invariant by the weaker assumption of the Mabuchi energy being proper (both of these assumptions imply a Kähler–Einstein metric exists). Therefore, by a classical result of Tian [30], Conjecture 1.1 holds if \( M \) admits no holomorphic vector fields. However, the properness assumption is still too restrictive and
fails in general. For example, Conjecture 1.1 is still open even for $M = S^2$, the two-sphere. Furthermore, as recent counterexamples show [15], it is not possible to modify the properness assumption to simply hold on $K$-invariant metrics, where $K$ is the maximal compact subgroup of the holomorphic automorphism group of $M$.

The main result of the present article is the resolution of Conjecture 1.1, and in fact with a stronger convergence.

**Theorem 1.2.** Let $(M, J, g_1)$ be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to $g_1$ is $c_1(M, J)$ and let $\{g_i\}_{i \in \mathbb{N}}$ be given by (1). Then there exists holomorphic diffeomorphisms $h_k$ such that $h_k^*g_k$ converges smoothly to a Kähler–Einstein metric.

### 1.1 Uniformization of the two-sphere

As a very special case we obtain the following new method of uniformization. Fix a conformal class of volume $V$ on $S^2$. As we know, in this class there is a constant curvature metric, the round one. More precisely, let $\omega_c$ denote the round form of the constant $c$ Ricci curvature metric on $M = (S^2, J)$, given locally by

$$\omega_c = \frac{\sqrt{-1}}{c\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$  

Here $V = \int_{S^2} \omega_c = c_1([M])/c = 2/c$. Consequently, $c = 1/2\pi$ in case we are restricting the Euclidean metric of $\mathbb{R}^3$ to the unit sphere.

Let $\omega$ be any metric on $S^2$ with $\int_{S^2} \omega = V = 2/c$. Introduce $u_0 = 0$, and we solve iteratively to find $u_i \in C^\infty(S^2)$ satisfying

$$\Delta_\omega u_i = R_\omega - 2e^{u_i-1}, \quad \text{and} \quad \int_{S^2} e^{u_i} \omega = 2/c,$$

so that the scalar curvature of $\omega_i := e^{u_i} \omega$ satisfies $R_{\omega_i} = 2e^{u_i-1} - u_i$, or equivalently, $\text{Ric}_{\omega_i} = \omega_{1-1}$. (In two dimensions, $\text{Ric}_{\omega} = \frac{1}{2}R_{\omega_0} \omega$, where $R_{\omega}$ is the scalar curvature. If $\omega_0 = e^\phi \omega_0$, then the scalar curvatures of these two metrics satisfy

$$\Delta_\omega_0 \phi - R_{\omega_0} + R_{\omega_1} e^\phi = 0.$$  

We note that the conformal factor is often written $e^{2\phi}$ elsewhere, but this is compensated for here by the fact that $R_{\omega} = 2K_\omega$, where $K_\omega$ is the Gauss curvature.)

**Corollary 1.3.** We fix $c > 0$ and let $\omega$ be any Kähler form on $S^2$ with $\int_X \omega = 2/c$. We introduce $\{u_i\} \subset C^\infty(S^2)$ by repeatedly solving the Poisson equation (2). Then, there exist Möbius transformations $h_i$ such that $h_i^*(e^{u_1} \omega)$ converges smoothly to the round metric $\omega_c$.

### 1.2 Discretization of the Ricci flow

One of the original motivations for introducing the Ricci iteration, going back to [26, 27], is its relation to the Ricci flow. Hamilton’s Ricci flow on a Kähler manifold of definite or zero first Chern class is defined as $\{\omega(t)\}_{t \in \mathbb{R}^+}$ satisfying the evolution equation

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_\omega(t) + \mu \omega(t), \quad t \in \mathbb{R}^+,$$

$$\omega(0) = \omega,$$
where Ω is a Kähler class satisfying μΩ = c_1(M, J) for μ ∈ {-1, 0, 1} and [ω] = Ω [21].

The following dynamical system is seen to be a discrete version of this flow [27, Definition 3.1], obtained by a backward Euler discretization with time step τ.

**Definition 1.4.** Let Ω be a Kähler class satisfying μΩ = c_1(M, J) for μ ∈ {-1, 0, 1}. Given a Kähler form ω with [ω] = Ω and a number τ > 0, define the time τ Ricci iteration to be the sequence of forms {ω_{kτ}}_{k≥0} satisfying the equations

\[
\frac{ω_{kτ} - ω_{(k-1)τ}}{τ} = -\text{Ric}_ω ω_{kτ} + μω_{kτ}, \quad k \in \mathbb{N},
\]

ω_0 = ω.

Let us assume that μ = 1 from now on (for the cases μ ∈ {-1, 0} see [27, Theorem 3.3]). Observe that in the case when τ = 1, the time τ Ricci iteration is precisely the Ricci iteration from (1). Indeed, Conjecture 1.1 is in fact a special case of the following conjecture concerning the time τ Ricci iteration for any τ > 0 [27, Conjecture 3.2].

**Conjecture 1.5.** Let (M, J) be a compact Kähler manifold admitting a Kähler–Einstein metric. Let Ω be a Kähler class such that Ω = c_1(M, J). Then for any ω with [ω] = Ω and for any τ > 0, the time τ Ricci iteration exists for all k ∈ N and converges in the sense of Cheeger-Gromov to a Kähler–Einstein metric.

The case when τ > 1 is treated in [27, Theorem 3.3]. However, it is the case τ ≤ 1 that is the most interesting and challenging. The case τ = 1 is perhaps the most interesting due to the simple geometrical interpretation (1) while the cases τ < 1 are interesting due to the connection to the Kähler–Ricci flow. In this regime one may expect the Ricci iteration to converge to the Ricci flow in a certain scaling limit as τ → 0. The cases τ ≤ 1 are challenging since the a priori estimates are considerably harder then. While in the regime τ > 1 one has a uniform positive Ricci lower bound along the iteration, this is no longer true when τ ≤ 1. Thus, there is no a priori control on the diameter or the Poincaré and Sobolev constants. We work around these difficulties, by analyzing the Ricci iteration in the metric geometry of the space of Kähler potentials [13].

In this article we in fact confirm the more general Conjecture 1.5 and treat the iteration for all time steps τ by proving the following result of which Theorem 1.2 is a special case.

**Theorem 1.6.** Let (M, J, g_1) be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to g_1 is c_1(M, J) and let {ω_{kτ}}_{k∈\mathbb{N}} be the time τ Ricci iteration given by Definition 1.4. Then there exists holomorphic diffeomorphisms h_k such that h_k^*ω_{kτ} converges smoothly to a Kähler–Einstein form.

## 2 Energy functionals

Let (M, ω) denote a connected compact closed Kähler manifold. The space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

\[ \mathcal{H}_ω := \{ ϕ ∈ C^∞(M) : ω_ϕ := ω + \sqrt{-1} \partial \bar{\partial} ϕ > 0 \}, \]

can be identified with \( \mathcal{H} \times \mathbb{R} \), where

\[ \mathcal{H} = \{ ω_ϕ : ϕ ∈ C^∞(M), ω_ϕ > 0 \} \]
is the space of all Kähler metrics (or forms) representing the fixed cohomology class $[\omega]$.

From now on let $\omega$ be a Kähler form on $M$, cohomologous to $c_1(M,J)$. The Aubin–Mabuchi functional was introduced by Mabuchi [24, Theorem 2.3],

$$\text{AM}(\varphi) := \frac{V^{-1}}{n+1} \sum_{j=0}^{n} \int_{M} \varphi \omega^{j} \wedge \omega^{n-j},$$

(5)

where $V := \int_{M} \omega^{n}_{\varphi} = \int_{M} \omega^{n}_{\varphi}$ is the total volume of the Kähler class. Integration by parts gives the useful estimates

$$\frac{1}{V} \int_{M} (u - v) \omega^{n}_{u} \leq \text{AM}(u) - \text{AM}(v) \leq \frac{1}{V} \int_{M} (u - v) \omega^{n}_{v}.$$  

(6)

The subspace

$$\mathcal{H}_0 := \text{AM}^{-1}(0) \cap \mathcal{H}_{\omega}$$  

(7)

is isomorphic to $\mathcal{H}$ [4], the space of Kähler metrics.

Let $f_{\omega_{\varphi}} \in C^\infty(M)$ denote the unique function (called the Ricci potential of $\omega_{\varphi}$) satisfying

$$\sqrt{-1} \partial \bar{\partial} f_{\omega_{\varphi}} = \text{Ric} \omega_{\varphi} - \omega_{\varphi},$$  

$$\frac{1}{V} \int_{M} e^{f_{\omega_{\varphi}}} \omega^{n}_{\varphi} = 1.$$  

The Ding and Mabuchi functionals are given by [16, 24]

$$D(\varphi) := -\text{AM}(\varphi) - \log \frac{1}{V} \int_{M} e^{f_{\omega_{\varphi}}} - \varphi \omega^{n},$$  

$$E(\varphi) := \frac{1}{V} \int_{X} \log \frac{\omega^{n}_{\varphi}}{e^{f_{\omega_{\varphi}}} \omega^{n}_{\varphi}} - \text{AM}(\varphi) + \frac{1}{V} \int_{M} \varphi \omega^{n}_{\varphi} + \frac{1}{V} \int_{M} f_{\omega} \omega^{n}.$$  

(8)

Notice that these functionals are invariant under addition of constants to $\varphi$, hence they descend to $\mathcal{H}$. Additionally, the critical points of these functionals are exactly the Kähler–Einstein metrics.

For $\varphi \in \mathcal{H}_{\omega}$ with $\int_{M} e^{f_{\omega_{\varphi}}} \omega^{n} = V$, Jensen’s inequality for the convex weight $t \to t \log t$ yields,

$$\text{Ent}(e^{f_{\omega_{\varphi}}} \omega^{n}, \omega^{n}_{\varphi}) := \frac{1}{V} \int_{X} \log \frac{\omega^{n}_{\varphi}}{e^{f_{\omega_{\varphi}}} \omega^{n}_{\varphi}} \omega^{n}_{\varphi} = \frac{1}{V} \int_{X} \log \frac{\omega^{n}_{\varphi}}{e^{f_{\omega_{\varphi}}} \omega^{n}_{\varphi}} e^{f_{\omega_{\varphi}}} \omega^{n} \geq 0.$$  

(9)

Thus,

$$E(\omega_{\varphi}) - \frac{1}{V} \int_{M} f_{\omega} \omega^{n} = \text{Ent}(e^{f_{\omega_{\varphi}}} \omega^{n}, \omega^{n}_{\varphi}) - \text{AM}(\varphi) \geq -\text{AM}(\varphi) = D(\omega_{\varphi}).$$  

Moreover, if

$$D(\omega_{\varphi}) = E(\omega_{\varphi}) - \frac{1}{V} \int_{M} f_{\omega} \omega^{n}$$

then equality holds in (9). As a result, $\omega^{n}_{\varphi} = e^{f_{\omega_{\varphi}}} \omega^{n} = e^{f_{\omega_{\varphi}}} \omega^{n}_{\varphi}$, i.e., $\omega_{\varphi}$ is Kähler–Einstein. This together with the fact that Kähler–Einstein metrics minimize both $D$ and $E$ allows to conclude the following result (see also [28, (24)]):

**Proposition 2.1.** For $\varphi \in \mathcal{H}_{\omega}$,

$$D(\omega_{\varphi}) \leq E(\omega_{\varphi}) - \frac{1}{V} \int_{M} f_{\omega} \omega^{n},$$

with equality if and only if $\text{Ric} \omega_{\varphi} = \omega_{\varphi}$.
3 The metric completion

All of the functionals introduced in the previous section can be extended to the potential space \( \mathcal{E}_1 \) introduced by Guedj–Zeriahi [20], that can be identified with a natural metric completion of \( \mathcal{H} \) [13]. The resulting metric theory provides essential tools for proving our main result concerning convergence of the Ricci iteration. We briefly recall this machinery, referring to [15, §4–5] and references therein for more details.

Let
\[
\text{PSH}(M, \omega) = \{ \varphi \in L^1(M, \omega^n) : \varphi \text{ is upper semicontinuous and } \omega_\varphi \geq 0 \}.
\]

Following Guedj–Zeriahi [20] Definition 1.1] we define the subset of full mass potentials:
\[
\mathcal{E}(M, \omega) := \{ \varphi \in \text{PSH}(M, \omega) : \lim_{j \to -\infty} \int_{\{ \varphi \leq j \}} (\omega + \sqrt{-1} \partial \overline{\partial} \max \{ \varphi, j \})^n = 0 \}.
\]

For each \( \varphi \in \mathcal{E}(M, \omega) \), define \( \omega_\varphi := \lim_{j \to -\infty} 1_{\{ \varphi > j \}} (\omega + \sqrt{-1} \partial \overline{\partial} \max \{ \varphi, j \})^n \). By definition, \( 1_{\{ \varphi > j \}}(x) \) is equal to 1 if \( \varphi(x) > j \) and zero otherwise, and the measure \( (\omega + \sqrt{-1} \partial \overline{\partial} \max \{ \varphi, j \})^n \) is defined by the work of Bedford–Taylor [3] since \( \max \{ \varphi, j \} \) is bounded. Consequently, \( \varphi \in \mathcal{E}(M, \omega) \) if and only if \( \int_M \omega_\varphi^n = \int_M \omega^n \), justifying the name of \( \mathcal{E}(M, \omega) \).

Next, define a further subset, the space of finite 1-energy potentials:
\[
\mathcal{E}_1 := \{ \varphi \in \mathcal{E}(M, \omega) : \int |\omega_\varphi^n| < \infty \}.
\]

Consider the following weak Finsler metric on \( \mathcal{H}_\omega \) [13]:
\[
\| \xi \|_\varphi := V^{-1} \int_M |\xi| \omega_\varphi^n, \quad \xi \in T_{\varphi} \mathcal{H}_\omega = C^\infty(M).
\]

We denote by \( d_1 \) the associated pseudo-metric and recall the result alluded to above, characterizing the \( d_1 \)-metric completion of \( \mathcal{H}_\omega \) [13] Theorem 2, Theorem 3.5]:

**Theorem 3.1.** (\( \mathcal{H}_\omega, d_1 \)) is a metric space whose completion can be identified with \( (\mathcal{E}_1, d_1) \), where
\[
d_1(u_0, u_1) := \lim_{k \to \infty} d_1(u_0(k), u_1(k)),
\]
for any smooth decreasing sequences \( \{ u_i(k) \}_{k \in \mathbb{N}} \subset \mathcal{H}_\omega \) converging pointwise to \( u_i \in \mathcal{E}_1, i = 0, 1 \).

Also, by [13] Theorem 3], we have the following qualitative estimates for the \( d_1 \) metric in terms of analytic quantities:
\[
\frac{1}{C} d_1(u, v) \leq \int_M |u - v| \omega_u^n + \int_M |u - v| \omega_v^n \leq C d_1(u, v), \quad u, v \in \mathcal{E}_1,
\]
where \( C > 1 \) only depends on \( \omega \).

A crucial fact is that the formulas defining the energy functionals discussed in [22] actually make sense on the metric completion \( \mathcal{E}_1 \), and then coincide with the greatest lower semi-continuous extension of the said functionals restricted to \( \mathcal{H}_\omega \) [15] Lemma 5.2, Proposition 5.19, Proposition 5.21];

**Lemma 3.2.** (i) \( AM, D : \mathcal{H}_\omega \to \mathbb{R} \) each admit a unique \( d_1 \)-continuous extension to \( \mathcal{E}_1 \) and these extensions still satisfy [5] and [8] respectively.
(ii) \( E : \mathcal{H}_\omega \to \mathbb{R} \) admits a \( d_1 \)-lower semi-continuous extension to \( \mathcal{E}_1 \) and the greatest such extension still satisfies [8].
Proposition 2.1 was generalized by Berman \cite[Theorem 1.1]{4} to the context of the metric completion (for a proof using the Ricci iteration see \cite[Proposition 4.42]{14}):\

**Theorem 3.3.** Proposition 2.1 holds more generally for all $\varphi \in E_1$.\

Let $G := \text{Aut}_0(M)$ denote the connected component of the complex Lie group of automorphisms (biholomorphisms) of $M$. The automorphism group acts on $\mathcal{H}$ by pullback:

$$f.\eta := f^*\eta, \quad f \in G, \quad \eta \in \mathcal{H}.\quad (12)$$

Given the one-to-one correspondence between $\mathcal{H}$ and $\mathcal{H}_0$ (recall (7)), the group $G$ also acts on $\mathcal{H}_0$. The precise action is described in the next lemma \cite[Lemma 5.8]{15}.

**Lemma 3.4.** For $\varphi \in \mathcal{H}_0$ and $f \in G$ let $f.\varphi \in \mathcal{H}_0$ be the unique potential such that $f^*\omega_\varphi = \omega_{f.\varphi}$. Then,

$$f.\varphi = f.0 + \varphi \circ f.\quad (13)$$

Complementing the above, $G$ acts on $\mathcal{H}_0$ by $d_1$-isometries \cite[Lemma 5.9]{15}, which allows to introduce a natural (pseudo)metric on the space $\mathcal{H}_0/G$:

$$d_{1,G}(Gu,Gv) = \inf_{g \in G} d_1(u,g.v), \quad u,v \in \mathcal{H}_0.\quad (14)$$

### 4 Metric convergence of the iteration

We consider the $\tau$-step Ricci iteration equation:

$$\frac{\omega_{\psi(k+1)_{\tau}} - \omega_{\psi_{k\tau}}}{\tau} = \omega_{\psi_{(k+1)\tau}} - \text{Ric} \omega_{\psi_{(k+1)\tau}},$$

for $\tau \in (0,1]$. When $\tau = 1$, the iteration simply becomes $\text{Ric} \omega_{\psi_{k+1}} = \omega_{\psi_k}$. As explained in \cite[(33)]{27}, on the level of scalars the iteration can be written in the following manner:

$$\omega^n_{\psi(k+1)_{\tau}} = e^{f_\omega - \frac{1}{\tau}\psi_{k\tau} - (1 - \frac{1}{\tau})\psi_{(k+1)\tau}} \omega^n, \quad k \in \mathbb{N},\quad (15)$$

with the natural normalization

$$\frac{1}{V} \int_M e^{f_\omega - \frac{1}{\tau}\psi_{k\tau} - (1 - \frac{1}{\tau})\psi_{(k+1)\tau}} \omega^n = 1.\quad (16)$$

Other normalizations may be considered on the level of scalars. In our particular case, there will be special emphasis on working in the geodesically complete potential space $\mathcal{H}_0$, and we introduce accordingly:

$$\psi'_{k\tau} := \psi_{k\tau} - \text{AM}(\psi_{k\tau}) \in \mathcal{H}_0.\quad (17)$$

First we generalize an inequality of \cite{27} (in the case $\tau = 1$) that provides a comparison of the Ding and Mabuchi energies along the $\tau$-iteration:

**Proposition 4.1.** Suppose $\tau \in (0,1]$ and $(M,\omega_{\psi_{1}^{k}})$ is a compact Fano manifold. Then the following estimate holds:

$$E(\omega_{\psi_{(k+1)\tau}}) - \frac{1}{V} \int_M f_\omega \omega^n \leq \frac{1}{\tau} D(\psi_{k\tau}) + \left(1 - \frac{1}{\tau}\right) D(\omega_{\psi_{(k+1)\tau}}), \quad \forall \ k \in \mathbb{N}.\quad (18)$$


In the argument below (and thereafter) we will suppress the parameter \( \tau \) from superscripts whenever this will cause no confusion.

**Proof.** Using (8) and (15),

\[
E(\omega_{\psi_{k+1}}) - \frac{1}{V} \int_M f_\omega \omega^n = \frac{1}{V} \log \frac{\omega_n^{\psi_{k+1}} \omega_n^{\psi_{k+1}}}{e^{f_\omega} \omega_n^{\psi_{k+1}}} - \text{AM}(\psi_{k+1}) + \frac{1}{V} \int_M \psi_{k+1} \omega_n^{\psi_{k+1}} \\
= -\frac{1}{V} \int_M \left( \frac{1}{\tau} \psi_k + \left(1 - \frac{1}{\tau}\right) \psi_{k+1}\right) \omega_n^{\psi_{k+1}} - \text{AM}(\psi_{k+1}) + \frac{1}{V} \int_M \psi_{k+1} \omega_n^{\psi_{k+1}} \\
= \frac{1}{\tau V} \int_M (\psi_{k+1} - \psi_k) \omega_n^{\psi_{k+1}} - \text{AM}(\psi_{k+1}).
\]

Using this identity, to finish the proof, we notice that it is enough to prove the following two inequalities (and later add them up):

\[
\frac{1}{\tau V} \int_M (\psi_{k+1} - \psi_k) \omega_n^{\psi_{k+1}} - \text{AM}(\psi_{k+1}) \leq -\frac{1}{\tau} \text{AM}(\psi_{k+1}) - \left(1 - \frac{1}{\tau}\right) \text{AM}(\psi_{k+1}) \quad (19)
\]

\[
0 \leq -\frac{1}{\tau} \log \left(\frac{1}{V} \int_M e^{f_\omega - \psi_k \omega^n}\right) - \left(1 - \frac{1}{\tau}\right) \log \left(\frac{1}{V} \int_M e^{f_\omega - \psi_{k+1} \omega^n}\right) \quad (20)
\]

Notice that, after rearranging terms, (19) is seen to be equivalent to

\[
\frac{1}{V} \int_M (\psi_{k+1} - \psi_k) \omega_n^{\psi_{k+1}} \leq \text{AM}(\psi_{k+1}) - \text{AM}(\psi_k).
\]

Thus, (19) follows from (6). To address (20) we prove the following more general claim.

**Claim 4.2.** For \( \tau \in (0, 1] \) and \( g, h \in C^\infty(X) \) the following estimate holds:

\[
\left(\frac{1}{V} \int_M e^{f_\omega - g \omega^n}\right)^{\frac{1}{\tau}} \leq \frac{1}{V} \int_M e^{f_\omega - \frac{1}{\tau} (1 - \frac{1}{\tau}) h \omega^n}. \quad (21)
\]

By our choice of normalization (6), this inequality implies (20).

As (21) is seen to be invariant under adding constants to \( g \) and \( h \), we can assume that \( \frac{1}{V} \int_M e^{f_\omega - h \omega^n} = 1 \). In particular, we only have to argue that

\[
\left(\frac{1}{V} \int_M e^{-g + h} e^{f_\omega - h \omega^n}\right)^{\frac{1}{\tau}} \leq \frac{1}{V} \int_M (e^{-g + h})^\frac{1}{\tau} e^{f_\omega - h \omega^n}.
\]

This follows from Jensen’s inequality, as the function \( f(t) = t^\frac{1}{\tau} \) is convex for \( t > 0 \). \( \square \)

Next we show that in case a Kähler–Einstein metric exists, the iteration \( \{\psi'_k\}_k \) \( d_1 \)-converges up to pullbacks:

**Proposition 4.3.** Let \( \tau \in (0, 1] \). Suppose a Kähler–Einstein metric exists in \( \mathcal{H} \), and let \( \{\psi_{k+\tau}\}_{k \in \mathbb{N}} \) be the solutions of (15). Then there exist \( g_k \in G \) such that \( g_k \cdot \psi'_{k+\tau} \) \( d_1 \)-converges to a Kähler–Einstein potential.
Proof. Proposition 4.1 combined with Proposition 2.1 gives
\[ D(\omega_{\psi_{k+1}}) - E(\omega_{\psi_{k+1}}) - \frac{1}{V} \int_{M} f_{*} \omega^{n} \leq \frac{1}{\tau} D(\omega_{\psi_{k}}) + \left(1 - \frac{1}{\tau}\right) D(\omega_{\psi_{k+1}}), \quad k \in \mathbb{N}. \]
(22)
As a result, \( \{D(\omega_{\psi_{l}})\}_{l} \) is a decreasing sequence (this is proved in [27, Proposition 4.2(ii)] for \( \tau = 1 \)). We fix a Kähler–Einstein potential
\[ \psi_{\text{KE}} \in \mathcal{H}_{0}. \]
Existence of such a potential implies that both \( D \) and \( E \) are bounded below [2, 17]. Thus, the (monotone) sequence \( \{D(\omega_{\psi_{l}})\} \) converges. By (22), \( \{E(\omega_{\psi_{l}}) - \frac{1}{\tau} \int_{M} f_{*} \omega^{n}\}_{l} \) converges too and both of these sequences have the same limit \( l \in \mathbb{R} \).

Next we focus on the potentials \( \psi_{l}' \in \mathcal{H}_{0} \). By [15, Theorem 2.4], \( E \) is \( G \)-invariant and
\[ E(\psi_{l}') \geq C_{1}d_{1,G}(0, \psi_{l}') - C_{2}, \]
and so \( d_{1,G}(0, \psi_{l}') \leq C' \). By definition (see (14)), there exists \( g_{l} \in G \) such that
\[ d_{1}(\psi_{\text{KE}}, g_{l} \psi_{l}') \leq d_{1,G}(G \psi_{\text{KE}}, G \psi_{l}') + \frac{1}{l} \leq C' + 1. \]
(23)

Remark 4.4. In fact, there exists \( g_{l} \) which achieve the equality \( d_{1}(\psi_{\text{KE}}, g_{l} \psi_{l}') = d_{1,G}(G \psi_{\text{KE}}, G \psi_{l}') \) by [15, Proposition 6.8] but we do not have to know that for our proof here.

Denoting
\[ v_{l} := g_{l} \psi_{l}', \]
by \( G \)-invariance of \( E \), we obtain that \( E(v_{l}) \) is bounded. On the other hand, a combination of (11) and (23) gives that \( \text{AM}(v_{l}) = 0 \) and \( \int_{V} v_{l} \omega_{v_{l}}^{n} \) are bounded as well. Comparing with (4), we see that \( \text{Ent}(e^{f_{0} \omega^{n}}, \omega_{v_{l}}^{n}) \) is bounded too.

By (11), \( d_{1} \)-boundedness of potentials implies \( L^{1} \)-boundedness, which in turn implies boundedness of the supremum. As a result, we can apply the compactness result of [6] (see [15, Theorem 5.6] for a convenient formulation for our context) to conclude that \( \{v_{l}\}_{l} \) is \( d_{1} \)-precompact.

Next we claim that \( d_{1}(\psi_{\text{KE}}, v_{l}) \to 0 \). If this is not the case, then by possibly choosing a subsequence, we can assume that \( d_{1}(\psi_{\text{KE}}, v_{l}) > \varepsilon > 0 \). By possibly choosing another subsequence, we can assume that \( d_{1}(v_{l}, u) \to 0 \) for some \( u \in \mathcal{E}_{1} \). Lemma 3.2 gives that \( l = D(u) = E(u) - \frac{1}{V} \int_{M} f_{*} \omega^{n} \), in particular \( u \) is a Kähler–Einstein potential by Theorem 3.3.

By the Bando–Mabuchi uniqueness theorem \( u = h \psi_{\text{KE}} \) for some \( h \in G \) [2]. Combining this with (23), we conclude that
\[ d_{1}(v_{k}, \psi_{\text{KE}}) - \frac{1}{k_{l}} \leq d_{1,G}(Gv_{l}, G\psi_{\text{KE}}) \leq d_{1}(h^{-1} v_{l}, \psi_{\text{KE}}) = d_{1}(v_{l}, h \psi_{\text{KE}}) = d_{1}(v_{l}, u). \]
By choice, the right hand side converges to zero, and the lim inf of left hand side is bounded below by \( \varepsilon > 0 \), giving a contradiction. This implies that \( d_{1}(v_{k}, \psi_{\text{KE}}) \to 0 \), concluding the proof.

5 A priori estimates and smooth convergence

In this section we prove our main result by strengthening Proposition 4.3.
Theorem 5.1. Let $\tau \in (0, 1]$. Suppose a Kähler–Einstein metric exists in $\mathcal{H}$, and let $\{\psi_{k\tau}\}_{k \in \mathbb{N}}$ be the solutions of (15). Then there exist $g_k \in G$ such that $g_k \psi_{k\tau}^\prime$ converges smoothly to a Kähler–Einstein potential. In particular, $g_k^*\omega_{k\tau}$ converges smoothly to a Kähler–Einstein metric.

Proof. By Proposition 4.3 there exists $g_k \in G$ and a Kähler–Einstein potential $\psi_{KE} \in \mathcal{H}_0$ such that $d_1(g_k \psi_{k\tau}^\prime, \psi_{KE}) \to 0$. We show below that in fact $g_k \psi_{k\tau}^\prime \to C^\infty \psi_{KE}$.

Focusing on the $\tau$-step Ricci iteration recursion, we can write:

$$(24) \quad (g_{k+1}^{-1} \circ g_k)^* \text{Ric} \omega_{g_{k+1} \psi_{k+1}} = g_k^* \text{Ric} \omega_{\psi_{k+1}} = g_k^* \left( \frac{1}{\tau} \omega_{\psi_k} + \left( 1 - \frac{1}{\tau} \right) \omega_{\psi_{k+1}} \right)$$

$$= \frac{1}{\tau} \omega_{g_k \psi_k} + \left( 1 - \frac{1}{\tau} \right) \omega_{g_k \psi_{k+1}}$$

$$= \frac{1}{\tau} \omega_{g_k \psi_k} + \left( 1 - \frac{1}{\tau} \right) \omega_{(g_{k+1} \circ g_k) \cdot g_{k+1} \psi_{k+1}}.$$ (24)

Set

$$\varphi_k := g_k \psi_{k\tau}^\prime \in \mathcal{H}_0$$

and

$$f_k := g_k^{-1} \circ g_{k-1} \in G.$$ (24)

With this notation, (24) becomes:

$$\text{Ric} \omega_{f_{k+1} \cdot \varphi_{k+1}} = \frac{1}{\tau} \omega_{\varphi_k} + \left( 1 - \frac{1}{\tau} \right) \omega_{f_{k+1} \cdot \varphi_{k+1}}.$$ (25)

Without loss of generality we assume that $\omega$ (the reference form) is Kähler–Einstein. Using (25) we can write:

$$\sqrt{-1} \partial \bar{\partial} \left( \frac{1}{\tau} \varphi_{k-1} + \left( 1 - \frac{1}{\tau} \right) f_k \cdot \varphi_k \right) = \text{Ric} \omega_{f_k \cdot \varphi_k} - \text{Ric} \omega = \sqrt{-1} \partial \bar{\partial} \log \left( \omega^n / \omega^n_{f_k \cdot \varphi_k} \right).$$

This implies that

$$\frac{1}{\tau} \varphi_{k-1} + \left( 1 - \frac{1}{\tau} \right) f_k \cdot \varphi_k + \log(\omega^n_{f_k \cdot \varphi_k} / \omega^n) = B_j \in \mathbb{R}.$$ (25)

Since log is a concave function, by Jensen’s inequality,

$$\frac{1}{V} \int_M \log(\omega^n_{f_k \cdot \varphi_k} / \omega^n) \omega^n \leq \log \frac{1}{V} \int_M \omega^n_{f_k \cdot \varphi_k} = 0.$$ (25)

By the triangle inequality, for $k$ sufficiently large,

$$d_1(0, \varphi_{k-1}) \leq d_1(\psi_{KE}, 0) + 1.$$ (25)

Using (25) we conclude that $\int_M \varphi_{k-1} \omega^n \leq C$. These last two estimates combine to give

$$B_j - \left( 1 - \frac{1}{\tau} \right) \frac{1}{V} \int_M f_k \cdot \varphi_k \omega^n = \frac{1}{V} \int_M \varphi_{k-1} \omega^n + \frac{1}{V} \int_M \log(\omega^n_{f_k \cdot \varphi_k} / \omega^n) \omega^n \leq C.$$ (25)

Since $f_k \cdot \varphi_k \in \text{PSH}(M, \omega)$, it is well known that $\int_M f_k \cdot \varphi_k \omega^n$ and $\sup_M f_k \cdot \varphi_k$ are comparable. As a result,

$$B_j - \left( 1 - \frac{1}{\tau} \right) \sup_M f_k \cdot \varphi_k \leq C,$$ (25)
hence we can write:
\[
\omega_t^n = e^{B_j - (1 - \frac{1}{\tau})f_k \varphi_k} \varphi_k^{-\frac{1}{\tau}} \omega_t^{n-1} \leq e^{C - \frac{1}{\tau} \varphi_k} \omega_t^n. \tag{26}
\]

Moreover, by Zeriahi’s version of the Skoda integrability theorem \[82\] (see \[15\] Theorem 5.7) for a formulation that fits our context most), there exists \( C > 0 \) such that, say,
\[
\int_M e^{-\frac{3}{2} \varphi_k} \omega_t^n \leq C, \ k \in \mathbb{N}.
\]
Combining this estimate with \(26\), we get that
\[
||\omega_{f_k \varphi_k}/\omega_t^n||_{L^3(M, \omega_t^n)} \leq C.
\]
Now Kołodziej’s estimate \[71 22\] allows to conclude that the oscillation satisfies \( \text{osc} f_k \varphi_k \leq C \) for some uniform \( C \). Note that for any \( u \in H_0 \), it follows from \(6\) that
\[
\inf u \leq \frac{1}{V} \int u \omega_t^n \leq 0 \leq \frac{1}{V} \int u \omega_t^n \leq \sup u,
\]
so \( u \) changes signs on \( M \). Thus, since \( f_k \varphi_k \in H_0 \), the oscillation bounds implies a uniform bound
\[
||f_k \varphi_k||_{L^\infty(M)} \leq C. \tag{27}
\]
Consequently, \(11\) yields
\[
d_1(0, f_k \varphi_k) = d_1(f_k^{-1} \varphi_k, 0) \leq C.
\]
Thus,
\[
d_1(f_k^{-1} \varphi_k, 0) \leq d_1(f_k^{-1} \varphi_k, 0) + d_1(\varphi_k, 0) \leq C'.
\]
By the arguments in the proof of \([15\) Proposition 6.8] (see also \([5\] Lemma 2.7] and \([15\] Claim 7.11], \( \{f_k^{-1}\}_k \) is contained in a bounded set of \( G \). In particular, all derivatives up to order \( m \), say, of \( f_k^{-1} \) are bounded by some \( C_m \) independently of \( k \). So, to finish the proof, it suffices to estimate derivatives of
\[
h_k := f_k \varphi_k
\]
(since that will imply the same estimates on \( f_k^{-1} f_k \varphi_k = \varphi_k \)).

Note that \( |\Delta \omega h_k| < C \) by the Chern–Lu argument of \([27\) pp. 1539–1540] since by \(25\) we have
\[
\text{Ric } \omega_{h_{k+1}} = \text{Ric } \omega_{f_k \varphi_{k+1}} \geq \left(1 - \frac{1}{\tau}\right) \omega_{f_k^{-1} \varphi_k \omega_{h_{k+1}}} = \left(1 - \frac{1}{\tau}\right) \omega_{h_{k+1}}.
\]
(cf. \[29\ Corollary 7.8 (i)] with \( C_1 = 0 \) and \( C_2 = \left(\frac{1}{\tau} - 1\right) \)). The \( C^{2,\alpha} \) and higher order estimates then follow the same way as in \([27\] (or by applying \([8\) Theorem 5.1] directly to \(26\)).

As we already have that \( d_1(\varphi_k, \psi_{KE}) \to 0 \), an application of \(11\) and the Arzelà-Ascoli compactness theorem finishes the argument.

We note that in our arguments above the estimates depend on a positive lower bound to \( \tau > 0 \). If this could be avoided, then one could hope that these estimates also hold in a scaled limit, as the iteration should converge to the Kähler–Ricci flow.

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