Revisit on “Ruling out chaos in compact binary systems”

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Full general relativity requires that chaos indicators should be invariant in various spacetime coordinate systems for a given relativistic dynamical problem. On the basis of this point, we calculate the invariant Lyapunov exponents (LEs) for one of spinning compact binaries in the conservative second post-Newtonian (2PN) Lagrangian formulation without the dissipative effects of gravitational radiation, using the two-nearby-orbits method with projection operations and with coordinate time as an independent variable. It is found that the actual source leading to zero LEs in one paper but to positive LEs in the other does not mainly depend on rescaling, but is due to two slightly different treatments of the LEs. It takes much more CPU time to obtain the stabilizing limit values as reliable values of LEs for the former than to get the slopes (equal to LEs) of the fit lines for the latter. Due to coalescence of some of black holes, the LEs from the former are not an adaptive indicator of chaos for comparable mass compact binaries. In this case, the invariant fast Lyapunov indicator (FLI) of two nearby orbits, as a very sensitive tool to distinguish chaos from order, is worth recommending. As a result, we do again find chaos in the 2PN approximation through different ratios of FLIs varying with time. Chaos cannot indeed be ruled out in real binaries.

PACS numbers: 04.25.Nx, 05.45.Jn, 95.10.Fh, 95.30.Sf

The chaotic behavior of nonlinear dynamical systems has become a very interesting subject in relativistic astrophysics [1]. Especially the dynamics of binary systems of spinning compact objects in the frame of general relativity does deservedly receive a great deal of attention. Merging binaries are regarded as the most promising candidates for future ground- and space-based gravitational wave detectors, such as LIGO [2]. The successful detection is necessary to rely on the matched filtering technique with the theoretical gravitational wave templates matched to experimental data containing a lot of instrumental noise. However, chaos in the gravitational wave sources would affect the treatment of observational data, for example, signals not to be drawn out of the noise. For this reason, there have been a series of articles [3-9] for discussing whether spinning compact binaries can exhibit chaos. In the light of the method of fractal basin boundaries, an earlier paper [3] emphasized that the contribution of spin-orbit (SO) and spin-spin (SS) coupling is in favor of chaos for the case of comparable mass compact binaries in the Lagrangian formulation to 2PN order with the dissipative effects of gravitational radiation turned off. While another work [4] suggested ruling out chaos by finding no positive LEs of trajectories along the fractal basin boundaries of Ref. [3]. At once, as an answer to this claim, it was reported in Refs. [5,6] that the wrong results of Ref. [4] should be owing to the less rigorous calculation of the LEs of two nearby orbits, with unapt renormalization time steps adopted. Further some orbits with positive LEs were given.

Obviously, it is very surprise that Ref. [4] and Refs. [5,6] employed the same chaos index—the largest LE, but gave completely different dynamics to the same 2PN equations of motion for spinning compact binaries. Although the latter pointed out the problem of the former, such interpretation seems still to be ambiguous and puzzling in retrospect. Here are more thorough comments on these works. LEs, as a common chaos indicator, measure the rate of exponential divergence between neighboring trajectories in the phase space. There are two different methods to compute them. Historically, the tangent vectors $\xi(0)$ and $\xi(t)$ about a given trajectory at times 0 and $t$ are used to define the maximum LE: $\lambda = \lim_{t\to\infty} \chi(t)$, with $\chi(t) = (1/t) \ln \|\xi(t)\|$ about a given trajectory at times 0 and $t$. The technique for getting the LE is called as method 1 (M1). Usually it is a cumbersome task to derive the variational equations associated to the tangent vector for complicated dynamical systems. For an alternative procedure to M1, a simpler way, M2, is to adopt the distance $d(t)$ in the phase space between two nearby trajectories as an approximation to the norm of the tangent vector $\xi(t)$ such that $\chi(t) = (1/t) \ln |d(t)/d(0)|$. It is for a suitable choice of the starting separation $d(0)$ and of the rescaling interval that M2 gives almost the same values of LEs as M1 does for details, see Ref. [10]). As a practical application of M2, traditionally one plots a curve of $\ln \chi(t)$ vs $\ln t$. A negative constant slope of the curve means the regularity of the system. If the slope tends gradually to zero and $\ln \chi(t)$ reaches nearly a stabilizing value, the bounded system becomes chaotic. The diagram method is marked as M2a. In addition, there is another diagram method (M2b) by plotting $\ln [d(t)/d(0)]$ vs $t$. It is vital to perform a least-squares fit on the simulation data to work out the slope $\chi$ of the fit line $\ln [d(t)/d(0)] = \chi t$, as the largest LE. There should have been no difference between M2b and M2a in principle, but the former superior to the latter lies in that it is much easier to identify the linear growth of $\ln [d(t)/d(0)]$ than to identify the convergence

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of $\ln \chi(t)$. In other words, generally it costs a rather long time for $\ln \chi(t)$ to converge a limit value in the chaotic case. As to the fit slope, perhaps it is not very true but can always easily be seen even if time is short. In particular, the difference is rather explicit when the authors of Ref. [4] and those of Refs. [5,6] used M2a and M2b to treat the LEs of compact binaries, respectively. Thus we think the very slow convergence of $\ln \chi(t)$ leading to the “false” LEs in Ref. [4], but do not agree with the authors of Refs. [5,6], who claimed that the wrong LEs in Ref. [4] depend on the rescaling. This will be further checked in our next numerical experiments, where both M2a and M2b adopt the same rescaling interval.

Now we conclude several problems that appear in Ref. [4]. (1) The initial distance problem. An important point to note is that relatively large and small initial separations are not permitted when M2 is used. For a machine double-precision environment with an order of $10^{-16}$, the starting distance $d(0)$ with a magnitude of $10^{-8}$ is viewed as the best choice [10]. However, Ref. [4] used $d(0) = 10^{-10}$ that must give rise to the overestimation of LEs if integration time is long enough. The rescaling that brings roundoff errors can certainly have an effect on LEs, but the initial distance is more important to affect LEs than the rescaling. (2) The integration time problem. In general, it is not true for the authors of Ref. [4] to declare the absence of chaos in compact binaries by finding no stabilizing values of $\ln \chi(t)$ only within a time span of a lower limit on the Lyapunov time, $t_\chi = 1/\lambda$, with many times greater than the typical inspiral time. Perhaps the authors considered that chaos after the inspiral time scale will not affect the dynamics of coalescing compact binaries. It is correct. However, for M2a it usually takes many and many times greater than the Lyapunov time (rather than the inspiral time) for $\ln \chi(t)$ to approach to a certain stabilizing value. For instance, Ref. [11] found that $\chi(t)$ of an orbit in 3-dimensional systems seems to have been stabilized to a value near 0.0005 up to $t = 220000$ and then abruptly jumps to a value around 0.01 up to $t = 1600000$ (see Fig. 10a in Ref. [11]). That is to say, it is completely impossible to arrive at the reliable value about 0.01 of LE when the orbit is integrated to the Lyapunov time, 100. In sum, the orbits of compact binaries must be integrated numerically for sufficiently long times, otherwise there are unreliable LEs. Unfortunately, coalescence does no longer give a chance to numerical integration. As an illustration, for the conservative system in which gravitational radiation is turned off, the coalescence is not a consequence of energy loss but just that these chaotic orbits happen to veer too close at some stage and merge. It should be possible in principle to find pairs that execute enough orbits that they do not coalesce before a lengthy integration has been performed. (3) The coordinate gauge invariance problem. There is a long history of the problem using LEs reliably in a curved space. In the mixmaster cosmology there was a long standing debate that the LEs were zero therefore there was no chaos [12-15]. This was a wrong conclusion and was an artifact of the spacetime slicing. Many independent groups have been engaged to this field. For example, Imponente and Montain [16] gave an invariant treatment of LEs by projecting a geodesic deviation vector for the Jacobi metric on an orthogonal tetradic basis so that they could successfully gain an insight into the dynamics of the mixmaster cosmology. So did Motter [17], who addressed directly the issue of the invariance of LEs. The invariant LEs in these works are mainly focused on the time evolution of the gravitational field itself. However, relativistic compact binaries are attributed to the geodesic or nongeodesic motion of particles in a given gravitational field. Ref. [3] used fractal basin boundary methods to detect chaos in black hole pairs. It should be mentioned that the original conclusion that there is chaos in spinning binaries was made using a coordinate invariant approach. There is no ambiguity in that approach. But the fractals can’t tell one more details of the dynamical features, such as the timescale for chaos to set in. Thus, it is fair to try to find a good invariant version of the LE. There are several points regarding this. From the physical point of view, it is questionable that the LEs, based on coordinate time $t$ and the Cartesian distance $d(t)$ between two nearby trajectories in the phase space of 12 dimensions, are used to discuss the dynamics of this system in Ref. [4]. Compact binaries are such strong fields that general relativistic effects become very apparent. On the other hand, general relativity admits a free choice of space and time coordinates so that the spacetime coordinates usually play book-keeping only for events. Therefore, physical observable quantities, such as the distance and the time, should be defined as proper quantities instead of coordinate quantities. This is a basic point from the theory of observation in general relativity. Following this idea, Ref. [18] used proper time $\tau$ of an “observer” and a proper configura-

![FIG. 1: Invariant LEs of the considered three orbits, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, by use of the method M3a.](image-url)
tion space distance $\Delta L(\tau)$ between the observer and his “neighbor” particles to construct an invariant LE (M3): $\lambda = \lim_{\tau \to \infty} \chi(\tau)$, where $\chi(\tau) = (1/\tau) \ln[\Delta L(\tau)/\Delta L(0)]$. $\Delta L(\tau) = (h_{\alpha \beta} \Delta x^\alpha \Delta x^\beta)^{1/2}$, with the space projection operator of the observer $h_{\alpha \beta} = g^{\alpha \beta} + U^\alpha U^\beta$, and $\Delta x^\beta$ being the deviation vector from the observer to the neighbor. Here $g^{\alpha \beta}$ and $U^\alpha$ stand for metric tensor and 4-velocity of the observer, respectively. In practice, M3 is no other than a direct modified and refined version of M2. Naturally, M2a and M2b are corresponded to M3a and M3b. As an illustration, the coordinate time $t$ still remains a common time variable in the equations of motion for the two particles, while the proper time $\tau$ is from integration of the equation about $d\tau/dt$. For the special case of $\tau \sim \ln t$, M3 fails to work well. In spite of that, we do believe that M3 will be very useful and simple to investigate spinning compact binaries, because it is in the coordinate time $t$ that the equations of motion for the compact binaries have been given by Ref. [2], and $\tau$ and $t$ have no the approximately logarithmic relation at all. (4) The power spectra problem. The authors of Ref. [4] did not find chaotic behavior in terms of the power spectra. This is because the power spectra are difficult to distinguish among complicated periodic orbits, quasi-periodic orbits and weakly chaotic orbits. Usually the power spectra are not recommended to be a criterion for evaluating chaos.

One main aim of the present paper is to re-review the results of Ref. [4], as has been stated above. The other is more important to use the covariant chaos indicator M3 (M3a and M3b) to investigate spinning compact binaries so that we take the opportunity to examine the related results in some references [4-6]. Considering the slow convergence of LEs and the possible coalescence of two stars, we suggest adopting a sensitive tool for detecting chaos—the invariant FLI of two nearby trajectories in a curved spacetime [19]: $FLI(\tau) = \log_{10}[\Delta L(\tau)/\Delta L(0)]$. The related details and applications of FLIs can be seen in Refs. [19-23]. It stretches exponentially with (proper) time for the chaotic orbit, but grows linearly with time in the regular case. Throughout the work we use units $c = G = 1$ and the signature of a metric as $(-, +, +, +)$, and let Greek subscripts run from 0 to 3 and Latin indexes from 1 to 3.

In compact binaries, the evolution equations about the relative position $\mathbf{x}$ and velocity $\mathbf{v}$ for body 1 relative to body 2 at 2PN order in harmonic coordinates are $\dot{\mathbf{x}} = \mathbf{a}_N + \mathbf{a}_{1PN} + \mathbf{a}_{1.5SO} + \mathbf{a}_{2PN} + \mathbf{a}_{2SS}$. The
numbers and the letters denote the order of the PN expansion and type of the contributions to the relative acceleration, respectively. The two spins precess by $\dot{\mathbf{S}}_i = \Omega_i \times \mathbf{S}_i$ ($i = 1, 2$). Their explicit forms can be seen in Ref. [2]. Now let $m_i$ be mass of body $i$, and the total mass $M = m_1 + m_2$. In addition, we specify $(\mathbf{y}_i, \mathbf{v}_i)$ as position and velocity of each body in the center-of-mass (CM) frame. The relations among three coordinates $y_1$, $y_2$ and $\tau$ at 2PN order are $y_1 = (m_2/M)x + Y_{1PN}(x, v) + Y_{1.5}(S_1, S_2) + Y_{2PN}(x, v)$ and $y_2 = -(m_1/M)x + Y_{1PN}(x, v) + Y_{1.5}(S_1, S_2) + Y_{2PN}(x, v)$ [24]. Here, the 1PN and 2PN terms can be found in Ref. [25], while the 1.5 order term is given by Ref. [26]. On the other hand, the proper time $\tau$ of body 1 in the CM frame satisfies the equation $d\tau/dt = -(g_{00} + 2g_{0i}v_i^1 + g_{ij}v_i^1v_j^1)^{1/2}$, $g_{0i}$, as a function of $(y_1, y_2; v_1, v_2)$, is the 2PN metric tensor at body 1. Each metric component is made of the related potentials at the location of body 1, and each potential is the sum of the non-spin piece and of the spin part. The non-spin part is presented by Ref. [27], and the spin piece is listed in Ref. [26]. See also Ref. [28] that contains the sum of the two parts. As an illustration, the 2.5 order terms in the references are dropped. This physically corresponds to dropping dissipative terms.

Clearly, the coordinate time $t$ plays an important role in connection with the motion of body 1 and of body 2, and the relative motion between the two bodies, but proper time does not since it differs for each of three motions. This gives us a good chance to apply M3 and the metric $g_{0i}$ to study the dynamics of orbits around body 1 in the CM frame [29]. The implementation is described briefly. We integrate the equations (7) of the relative motion, the spin equations (8) and the proper time equation (10) numerically together by using a fifth-order Runge-Kutta-Fehlberg algorithm with an adaptive coordinate time step. At once, we can get $S_1$, $S_2$, $x$, $v$, and $\tau$, at coordinate time $t$. Then $(y_1, y_2; v_1, v_2)$ are determined, and body 1 has its 4-velocity $\mathbf{U} = \left(\frac{d\mathbf{v}_1}{d\tau}, v_1^1, \frac{d\mathbf{v}_2}{d\tau}, v_2^1\right)$. Now body 1 is chosen as an observer, who can measure the proper distance $\Delta L$ to his neighboring orbit. Note, the neighboring orbit is not the orbit at which body 2 stays, and is from an orbit nearby body 1, caused by a slight separation of the relative position. In a word, numerical integration is to carry out in the relative coordinate system, but the relativistic dynamics is to investigate in the CM frame and whether chaos or not is measured by body 1. This is entirely different from the treatment of other references, where the Newtonian dynamical methods are used to consider the relative motion in spinning compact binaries.

Let us re-calculate the LEs of three orbits that had been studied in Ref. [6]. The related initial conditions and parameters of the orbits are listed here. Orbit $\Gamma_1$: phase space variables $(x, v) = (5.5M, 0, 0, 0, 4, 0)$, mass ratio $\beta = m_2/m_1 = 1/3$, spin magnitudes $S_i = m_i^2$, and spin directions $\theta_1 = \pi/2$ and $\theta_2 = \pi/6$. $\Gamma_2$: $(x, v) = (5.0M, 0, 0, 0, 0.399, 0)$, $\beta = 1$, $S_i = m_i^2$, $\theta_1 = 38^\circ$, and $\theta_2 = 70^\circ$. $\Gamma_3$ is the same as $\Gamma_2$ but only 0.428 replaces 0.399. In addition, let only the first component $x$ of the initial relative position of each trajectory have a very small deviation, $\Delta x = 10^{-8}M$, then we get its corresponding neighboring orbit. Following M3a, we draw plots of $\log_{10} \chi(\tau)$ vs $\log_{10} \tau$ about the LEs of the three trajectories in Fig. 1. They all drop before proper time $\tau$ spans $10^6M \approx 3636T_0$ ($T_0 = 275M$, the average period of orbit $\Gamma_1$). For $\Gamma_1$, the LE looks to get a reliable value, $\tau_\lambda = 4675M = 17.07T_0$. It is larger than the value $\tau_\lambda = 3080M = 11.27T_0$ given in Ref. [6]. Perhaps one addresses a question whether $\chi$ can still remain the value of 1/$\tau_\lambda$ if numerical integration continues. We have no way to answer it since numerical integration has to end after $t = 1.37 \times 10^8M$ (or $\tau = 1335980M$), when the two objects coalesce. Above all, neither $\Gamma_2$ nor $\Gamma_3$ has any acceptable stabilizing value when integration time $t$ reaches $10^8M$. Additionally, we did not find any difference from these results by making several tests with different renormalization time steps. This seems to show that the results in Ref. [4] are reasonable. However, the case is completely different when M3b is used. Seen from the calculations including the rescaling, M3b is almost the same as M3a, but only a slight difference between them lies in plotting $\ln(\Delta L/\Delta L(0))$ vs $\tau$ instead of plotting $\log_{10} \chi(\tau)$ vs $\log_{10} \tau$. Another point to note is that M2b (rather than M3b) without rescaling was used in Ref. [7], but our M3b employs rescaling. In addition, there is a difference that the authors of Ref. [7] use the Hamiltonian formulation in ADM coordinates and not the Lagrangian formulation in harmonic coordinates. Although the two approaches are approximately related, they are not exactly equal. For instance, the constants of motion are exactly conserved in the Hamiltonian formulation, while they are approximately in the Lagrangian formulation. Ref. [7] also works to 3PN order. As shown in Fig. 2, it takes no long enough time to explicitly see the presence of positive slopes of the fit lines for $\Gamma_1$ and $\Gamma_2$, but to do that of about zero slope of the fit line for $\Gamma_3$. This means chaos of $\Gamma_1$ and $\Gamma_2$, while the regularity of $\Gamma_3$. It is what can be seen in Refs. [5, 6]. It is sufficiently argued that the LEs converge much faster for M3b than for M3a. As an illustration, $\tau_\lambda$ is more reliable than $\tau_\lambda$. This is because the longer numerical integration becomes, the more accurate the values of LEs are. In fact, the fit slope (its inverse being $4680/\tau_\lambda$) of $\Gamma_2$ vs log $T_0$ given in Ref. [6]. Perhaps one can argue that the invariant FLI given by Eq. (6). Its algorithm can be found in Ref. [19]. Fig. 3 displays that FLIs of $\Gamma_1$ and $\Gamma_2$
increase exponentially with $\log_{10} \tau$, but that of $\Gamma_3$ does algebraically. Thus $\Gamma_1$ and $\Gamma_2$ are chaotic, (chaos of $\Gamma_1$ is much stronger than that of $\Gamma_2$) but $\Gamma_3$ becomes ordered. It is worth emphasizing that the three orbits can be distinguished clearly in practice when proper time adds up to $10^5 M$. Consequently, the onset of chaos in the 2PN Lagrangian approximation is proved again through different ratios of FLIs varying with time.

The summary is included as follows. For conceptual clarity, it is necessary to apply chaos indicators independent of the choice of coordinate gauge to analyze the dynamics of relativistic gravitational systems. Since coordinate time is a good medium in connect with the mass centered motions of both body 1 and body 2, and the relative motion in spinning compact binaries, we think that M3, as an invariant indicator, is a good tool to study these systems. Using M3, we estimate the LEs on the mass centered motion of body 1 rather than on the relative motion considered by other references. We find that the orbits must be calculated for long enough times in order to get stabilizing limit values as reliable LEs for the case of comparable mass compact binaries. On the other hand, we track that the exact source of both the failure of Ref. [4] and the success of Refs. [5,6] in the computation of LEs does not stem from the rescaling, but is based on two slightly different treatments of LEs, M3a and M3b. At most cases, the LEs converge much faster for M3b than for M3a. However, coalescence of the black holes makes it impossible in some cases to have enough numerical integration. This shows that M3a is no longer a suitable indicator to quantify chaos in spinning compact binaries. Additionally, it should be noted that although it is rather easier to get the LEs for M3b than for M3a, long integration times are still needed to get reliable values of LEs when M3b is adopted. In this sense, the invariant FLI in a curved spacetime is a very fast and valid technique to detect chaos from order. Still, a reliable conclusion is that there is chaos in the conservative 2PN Lagrangian system. Of course, the 2PN approximation is so poor that there has been left an open question whether real binary systems with better approximations exhibit chaos [30]. Saying this another way, now one does not say that chaos can be ruled out in real binaries. In future, we will discuss a wider application of the FLI in detailedly investigating the dynamics of spinning compact binaries.

We would like to thank both the referee and F.A. Rasio for their honest comments and significant suggestions. We are also grateful to Professor Tian-Yi Huang of Nanjing University for his helpful discussion. This research is supported by the Natural Science Foundation of China under Contract No. 10563001. It is also supported by the Science Foundation of Jiangxi Province (0612034), the Science Foundation of Jiangxi Education Bureau (200655), and the Program for Innovative Research Team of Nanchang University.

1. X. Wu and H. Zhang, Astrophys. J. 652, 1466 (2006).
2. L.E. Kidder, Phys. Rev. D 52, 821 (1995).
3. J. Levin, Phys. Rev. Lett. 84, 3515 (2000).
4. J.D. Schnittman and F. A. Rasio, Phys. Rev. Lett. 87, 121101 (2001).
5. N.J. Cornish and J. Levin, Phys. Rev. Lett. 89, 179001 (2002).
6. N.J. Cornish and J. Levin, Phys. Rev. D 68, 024004 (2003).
7. M.D. Hartl and A. Buonanno, Phys. Rev. D 71, 024027 (2005).
8. S. Suzuki and K. Maeda, Phys. Rev. D 55, 4848 (1997).
9. Y. Xie and T.Y. Huang, Chinese J. Astron. Astrophys. 6, 705 (2006).
10. G. Tancredi, A. Sánchez and F. Roig, Astron. J. 121, 1171 (2001).
11. G. Contopoulos and B. Barbanis, Astron. Astrophys. 222, 329 (1989).
12. G. Contopoulos, N. Voglis and C. Efthymiopoulos, Celest. Mech. Dyn. Astron. 73, 1 (1999).
13. M. Szydlowski, Gen. Relativ. Gravit. 29, 185 (1997).
14. D. Hobill, D. Bernstein, D. Simpkins and M. Welge, Classical Quantum Gravity 8, 1155 (1991).
15. G. Cushman and J. Sniatycki, Rep. Math. Phys. 36, 75 (1995).
16. G. Imponente and G. Montani, Phys. Rev. D 63, 103501 (2001).
17. A.E. Motter and G. Montani, Phys. Rev. Lett. 91, 231101 (2003).
18. X. Wu and T.Y. Huang, Phys. Lett. A 313, 77 (2003).
19. X. Wu, T.Y. Huang and H. Zhang, Phys. Rev. D 74, 083001 (2006).
20. C. Froeschlé and E. Lega, Celest. Mech. Dyn. Astron. 78, 167 (2000).
21. C. Froeschlé, E. Lega and R. Gonczi, Celest. Mech. Dyn. Astron. 67, 41 (1997).
22. J.F. Zhu, X. Wu and D.Z. Ma, Chinese J. Astron. Astrophys. 7, 601 (2007).
23. X. Wu, H. Zhang and X.S. Wan, Chinese J. Astron. Astrophys. 6, 125 (2006).
24. The relation up to 2PN order is used to get the proper distance $\Delta L$ with the same accuracy. Of course, the terms higher than 2 order in $v_1$ and $v_2$ are turned off.
25. L. Blanchet and B.R. Iyer, Classical Quantum Gravity 20, 755 (2003).
26. H. Tagoshi, A. Ohashi and B.J. Owen, Phys. Rev. D 63, 044006 (2001).
27. L. Blanchet, G. Faye and B. Ponsot, Phys. Rev. D 58, 124002 (1998).
28. G. Faye, L. Blanchet and A. Buonanno, Phys. Rev. D 74, 104033 (2006).
29. As a reason to do this, the metric is easily given in the frame. In addition, the LE by M3 does not depend on a spacetime coordinate system. This implies that the LE of body 1 in the frame is equivalent to one of the relative motion orbit.
30. J. Levin, Phys. Rev. D 74, 124027 (2006).