On accompanying measures and asymptotic expansions in limit theorems for maximum of random variables.

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Abstract

A sequence of accompanying laws is suggested in the limit theorem of B. V. Gnedenko for maximums of independent random variables belonging to maximum domain of attraction of the Gumbel distribution. It is shown that this sequence gives an exponential power rate of convergence whereas the Gumbel distribution gives only a logarithmic rate. As examples, classes of Weibull and log-Weibull type distributions are considered in details. A scale for the Gumbel maximum domain of attraction is suggested as a continuation of the considered two classes.

Keywords: Gnedenko-Fisher-Tippet theorem, rate of convergence, correction terms, accompanying law

1 Introduction

Let $X_1, ..., X_n, ...$ be independent identically distributed random variables having distribution function $F(x)$. Denote $M_n := \max(X_i, i = 1, ..., n)$. By Gnedenko-Fisher-Tippet theorem, see [15], and also [17], [11], if for some positive $a_n$ and real $b_n$ there exists a non-degenerate limit of $P(M_n \leq a_n x + b_n)$ as $n \to \infty$, then the limit distribution function belongs to one of three types, that is Fréchet, Weibull, and Gumbel types. We deal in this note with the third one, that is, with distribution functions $F$ for which for any $x$,

$$
\lim_{n \to \infty} P(M_n \leq a_n x + b_n) = \Lambda(x) := \exp(-e^{-x}).
$$

One says in this case that the distribution belongs to Gumbel max domain of attraction, $F \in MDA(\Lambda)$. Moreover, we restrict ourselves by the case of infinite right end point of $F$, that is, $F(x) < 1$ for any positive $x$. It will be seen that our approach can be applied to

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any distributions from the three above mentioned domains of maximum attraction. We
take this domain because of it is extremely wide comparing with other two. Remark that
many applications require a division of this domain into reasonable in some sense sub-
domains. For example, this domain contains distributions with tails \( 1 - F(x) \), roughly
(logarithmic) equivalent to Weibull tails, that is, \( \log(1 - F(x)) \sim -Cx^p, C, p > 0, \) as
\( x \to \infty \), or log-Weibull tails, that is, \( \log(1 - F(x)) \sim -C \log^p x, C > 0, p > 1, \) wherein
one can take regularly varying on infinity functions instead of degrees in the right-hands
in the above relations, and many other distributions with heavier and lighter tails.
for comparison, recall that the Frechet maximum domain of attraction contains only
distributions with regularly varying tails on infinity, and Weibull maximum domain of
attraction contains only distributions with regularly varying tails at the right (finite) end
of the distribution. We discuss here an idea of scaling the distributions from \( MDA(\Lambda) \)
with mentioned above first two "scale points", that is, generalized Weibull-like and
generalized log-Weibull-like distributions, see definitions below.

By [6], the distributions from \( MDA(\Lambda) \) can be described using von Mises function,
that is, \( F \in MDA(\Lambda) \) if and only if for some \( x_0 \geq 0, \)
\[
1 - F(x) = c(x) \exp \left\{ - \int_{x_0}^{x} \frac{1}{f(t)} dt \right\}, \quad x \geq x_0, \tag{2}
\]
with \( f(x) \), positive and absolutely continuous on \( [x_0, \infty) \) with density \( f'(x), f'(x) \to 0, \)
and \( c(x) \to c > 0, \) as \( x \to \infty. \)

Often more flexible form of this assertion is convenient: \( F \in MDA(\Lambda) \) if and only if for some \( x_0 \geq 0, \)
\[
1 - F(x) = c(x) \exp \left\{ - \int_{x_0}^{x} \frac{g(t)}{f(t)} dt \right\}, \tag{3}
\]
with the same properties of \( f(x) \), and \( c(x) \) and \( g(x) \to 1, \) as \( x \to \infty. \) For both the
representations one can take
\[
b_n = F^{-1}(1 - n^{-1}), \quad a_n = f(b_n). \tag{4}
\]
See [11], [31]. Moreover, if \( (1) \) is fulfilled with \( b_n \) as in \( (4) \) and some sequence of positive
\( a_n \) then for \( a_n \) as in \( (1) \), \( a_n/\bar{a}_n \to 1 \) as \( n \to \infty \) and there exists a representation \( (3) \) with
other \( \bar{g} \) and \( \bar{f} \) such that \( \bar{a}_n = \bar{f}(b_n). \)

There is a wide bibliography on the quality of convergence in \( (1) \). We note here two
main directions of studies. First one is related with restrictions on the tail behavior
of \( F \) at infinity. First of all it is the second order condition and higher orders regular
behaviors of the tails of \( F, \) see [32], [25], [2], [17], [37], and references therein. Definitions
and comments see also in section 3 below.

The second direction related to concrete expressions of the distributions or families of
distributions, such as Gaussian, Gaussian like, Weibull, Weibull-like, log-Weibull-like, so
on. Here is also a wide bibliography, beginning with [20], [22], [28], see also monographs
[17] and [31].

Our study belongs rather to the second direction, we use von Mises structure \( (2) \)
and \( (3) \), but we suggest another approach: do not investigate quality of Gumbel ap-
proximation but first look for better approximations. Notice that this is very common
approach in the study of quality approximation in Central Limit Theorem. This is
Chebyshev-Hermite polynomials approximation, other types of approximation, other
types of accompanying laws and charges (signed measures), see [12], [16], [23], [29], [30],
[35]. In connection with, in 2002, one of the authors discussed with Laurens de Haan the
following result on Gaussian smooth stationary processes, only recently it is published
in [30], we then have agreed that such approach is interesting and promising.

Theorem 1 Let $X(t)$, $t \in \mathbb{R}$, be a twice differentiable in square mean Gaussian station-
ary process with $EX(t) = 0$, $EX^2(t) = 1$, $EX'(t)^2 = 1$. Assume that

$$
\int_0^\infty |r(t)|^a dt < \infty
$$

holds for its covariance function $r$ and some $a > 0$. Denote $l_T = \sqrt{2 \log \frac{T}{2\pi}}$. Denote also

$$
A_T(x) = \begin{cases}
  e^{-e^{-x-x^2/2l_T^2}}, & x \geq -l_T^{3/2}, \\
  0, & x < -l_T^{3/2},
\end{cases}
$$

(5)

$T > 0$. Then

1. For some $\gamma > 0$,

$$
P \left( \max_{t \in [0,T]} X(t) \leq l_T + \frac{x}{l_T} \right) - A_T(x) = O(T^{-\gamma}), \quad T \rightarrow \infty
$$

uniformly in $x \in \mathbb{R}$.

2. Moreover,

$$
l_T^2 \left( P \left( \max_{t \in [0,T]} X(t) \leq l_T + \frac{x}{l_T} \right) - e^{-e^{-x}} \right) \rightarrow \frac{1}{2} e^{-e^{-x}} e^{-x} x^2,
$$

as $T \rightarrow \infty$, uniformly in $x \in \mathbb{R}$.

It follows from the second statement that the rate of convergence of the distribution
of the maximum to the Gumbel distribution is logarithmic. It also gives the second term
of the asymptotic expansion for the probability. The first statement gives the sequence
of approximating functions that approaches the maximum distribution with the power
rate. There are several results related to accompanying laws, see [32], [4], [9], [10],
where the mentioned above second order condition is exploited. Many of the results are
presented in celebrated monographs [31] and [17].

In the following section, an asymptotic expansion is derived in the theorem on conver-
gence to Gumbel distribution and speed of convergence is studied. It turns out that the
quality of convergence is as a rule logarithmic. Further, by analogy with corresponding
results on Central Limit Theorem, a sequence of accompanying charges (signed mea-
sures) is introduced. This sequence gives a power rate of convergence. Comparing our
results with the known ones is considered in Section 3. Section 4 contains examples
for distributions with smooth tails. Some of the examples of particular distributions
were subjects of student works in faculty of mechanics and mathematics of Lomonosov
Moscow state university, including one of the authors of the present work. We thank
Viktoria Maier, Ignat Melnilov, Viktor Troshin, Kirill Lisakov for theirs help. Finally, we
discuss in Section 5 a scale for distributions from $MDA(\Lambda)$ and some related problems.
2 Asymptotic expansions and accompanying measures

In contrast to similar problems related to the central limit theorem, [29], [23], [35], [36], the construction of asymptotic expansions and accompanying measures in limit theorems for maximums is much simpler, in a certain sense even trivial. Similar evaluations based on Taylor expansions one can find in many works on extreme distributions, see, for example, [27] and references given in Section 3. Nevertheless, we present these evaluations here since they are some background for similar calculations for particular distributions and distribution classes.

Assume (2). By (4),

$$\int_{x_0}^{b_n} \frac{1}{f(t)} dt = \log(nc(b_n)).$$

(6)

Further,

$$G_n(x) := F^n(a_n x + b_n) = \left(1 - c(a_n x + b_n)e^{-\int_{x_0}^{a_n x + b_n} \frac{1}{f(t)} dt}\right)^n.$$  

(7)

Taking logarithm, and denoting

$$g_n(x) := \int_{x_0}^{a_n x + b_n} \frac{1}{f(t)} dt - \log c(a_n x + b_n),$$

we have after easy calculations using Taylor,

$$\log G_n(x) = -ne^{-g_n(x)} \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-kg_n(x)}.$$  

(8)

Using (6),

$$g_n(x) = \int_{x_0}^{b_n} \frac{1}{f(t)} dt + \int_{b_n}^{a_n x + b_n} \frac{1}{f(t)} dt - \log c(a_n x + b_n) = \log n + \int_{b_n}^{a_n x + b_n} \frac{1}{f(t)} dt - \log \frac{c(a_n x + b_n)}{c(b_n)} =: \log n + \gamma_n(x).$$

Now (8) can be written as

$$\log G_n(x) = -e^{-\gamma_n(x)} \sum_{k=0}^{\infty} \frac{1}{(k+1)n^k} e^{-k\gamma_n(x)}.$$  

(9)

Taking out the first summand in the sum and passing to exponents, we get, that

$$G_n(x) = \exp \left(-e^{-\gamma_n(x)}\right) \exp \left(-\frac{1}{n} \Sigma(x)\right),$$  

(9)
with
\[ \Sigma(x) = \sum_{k=0}^{\infty} \frac{\exp(-(k+2)\gamma_n(x))}{(k+2)n^k}. \] (10)

From here and (11), using Taylor, we get that,
\[ P(M_n \leq a_n x + b_n) = \exp(-e^{-\gamma_n(x)}) + \frac{1}{n} \exp(-e^{-\gamma_n(x)}) \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \Sigma^{k+1}(x)}{(k+1)! n^k}. \] (11)

Writing summands in (10) as \( \exp(-F, \text{uniformly in } x \text{ algebra, } a \text{ with } g \text{ and for general } g \), Changing in (13) variables \( t \), One can easily see that, assuming (3) instead of (2), the expression for \( \gamma_n(x) \) correspondingly to the representations of \( F \) for the accompanying sequence, introduce therefore the \( h \) with the same is valid for the sum the second term on the right hand part of (11). Introduce therefore the accompanying sequence,
\[ B_n(x) = \left\{ \begin{array}{l} e^{-e^{-\gamma_n(x)}}, \quad \gamma_n(x) \geq -\log n, \\ 0, \quad \gamma_n(x) < -\log n. \end{array} \right. \] (12)

Turn now to \( \gamma_n(x) \). Notice first that for any \( x \), by (1) and (11), \( \gamma_n(x) \to x \) as \( n \to \infty \). Further, since \( a_n = f(b_n) \), rewrite the expression for \( \gamma_n(x) \) as
\[ \gamma_n(x) = \int_{b_n}^{a_n x + b_n} \left( \frac{1}{f(t)} - \frac{1}{f(b_n)} \right) dt - \log c\left(\frac{a_n x + b_n}{c(b_n)}\right) + x. \] (13)

One can easily see that, assuming (5) instead of (2), the expression for \( a_n \) is changed on \( a_n = f(b_n)/g(b_n) \), and the expression for \( \gamma_n(x) \) is changed on
\[ \gamma_n(x) = \int_{b_n}^{a_n x + b_n} \left( \frac{g(t)}{f(t)} - \frac{g(b_n)}{f(b_n)} \right) dt - \log c\left(\frac{a_n x + b_n}{c(b_n)}\right) + x. \] (14)

Changing in (13) variables \( t = b_n + v/b_n \) and using \( a_n = f(b_n) \), we have after some algebra,
\[ \gamma_n(x) = \int_{0}^{x} \left( \frac{f(b_n)}{f(b_n + a_n v)} - 1 \right) dv - \log c\left(\frac{a_n x + b_n}{c(b_n)}\right) + x; \] (15)
and for general \( g \),
\[ \gamma_n(x) = \int_{0}^{x} \left( \frac{g(b_n + a_n v)}{f(b_n + a_n v)} - 1 \right) dv - \log c\left(\frac{a_n x + b_n}{c(b_n)}\right) + x. \] (16)

Finally we get the following.

**Theorem 2** Let (2) or (3) be fulfilled for a distribution function \( F(x) \). Let \( X_1, X_2, \ldots \), be i.i.d. random variables with distribution function \( F \), and \( M_n := \max(X_1, \ldots, X_n) \). Then for \( a_n, b_n \) defined by (4),
\[ P(M_n \leq a_n x + b_n) - B_n(x) = O(1/n), \text{ as } n \to \infty, \]
uniformly in \( x \in \mathbb{R} \), where \( B_n(x) \) is defined by (12) with \( \gamma_n(x) \) defined by (13) or (14) correspondingly to the representations of \( F \). Moreover, for all \( x \), \( B_n(x) \to \Lambda(x) \) as \( n \to \infty \).
That is, the sequence $B_n(x)$ is a natural sequence of accompanying charges (laws) in Gnedenko Limit Theorem. which gives the power rate of convergence to Gumbel distribution. Now one can see that the question on the rate of convergence to Gumbel distribution depends on how fast $\gamma_n(x)$ tends to $x$ as $n \to \infty$. Definitely, it depends on detailed tail behavior of $F(x)$. Generally, we may formulate an analogous of the second statement of Theorem 1 as follows.

**Corollary 1** In Theorem 2 conditions, for any $x$,

$$P(M_n \leq a_n x + b_n) - \exp \left(-e^{-x}\right) = \exp \left(-e^{-x}\right) e^{-x}(\gamma_n(x) - x)(1 + o(1)) + O(n^{-1})$$

as $n \to \infty$.

Indeed, it follows from Taylor expansion of $G_n(x)$. We point out that, in our opinion, a formulation of complete analogous of Theorem 1 statement 2 is possible only under sufficiently concrete tail behavior description. In Theorem 1 the behavior is given exactly. Remark that for exponential distribution, $f(x)$, $c(x)$ and $g(x)$ are constants, hence $\gamma_n(x) \equiv x$, that is the rate of convergence in (1) is $O(n^{-1})$, $n \to \infty$, the exact expression can be derived, using for example, (11). In Section 4 we prove several other refinements for distributions with different types of theirs tail behaviors, see Propositions 2, 3 and 4.

Thus in order to study the rate of convergence in (1) one has first to study the behavior of $\gamma_n(x) - x$ as $n \to \infty$ and in dependence of $x$. Then one should compare the asymptotic behavior of the difference with the behavior of the residual which is equal to $O(1/n)$. Notice that the two first summand in (14) both can play the main role in the rate of convergence.

Now give another obvious expression for $\gamma_n(x)$.

**Proposition 1** Let (1) be fulfilled, then for the corresponding representation (3), with $a_n = f(b_n)$,

$$\gamma_n(x) = -\log \frac{1 - F(b_n + a_n x)}{1 - F(b_n)} = \log \frac{1}{n(1 - F(b_n + a_n x)).}$$

(17)

Indeed, it follows from the above that

$$\gamma_n(x) = \int_{x_0}^{a_n x + b_n} \frac{g(t)dt}{f(t)} - \int_{x_0}^{b_n} \frac{g(t)dt}{f(t)} - \log c(a_n x + b_n) + \log c(b_n),$$

then use (3).

### 3 Relation to some known results.

As it was already mentioned, one of main approaches to studies the rate of convergence in limit theorems for maximums is introducing additional conditions on behavior of tails pf distribution functions. Here we consider how this conditions relate to the behavior of $\gamma_n(x)$. Notice that we consider $F \in MDA(\Lambda)$ with $F(x) < 1$ for all $x$. 


Definition 1 (Second order condition for functions from $\text{MDA}(\Lambda)$). There exists a function $A(n)$ of constant sign which tends to zero as $n \to \infty$ and such that there exists the limit
\[
\lim_{n \to \infty} \frac{e^{-\gamma_n(x)} - e^{-x}}{A(n)} = H(x), \tag{18}
\]
and $H(x)$ is not identically equals neither zero nor infinity.

This condition is introduced by L. de Haan, \[19\], in some other terms. We give equivalent formulation, based on Theorem 2.3.8, \[17\], and on representation (17) of $\gamma_n(x)$. From this definition immediately follows, see for example, \[17\], that $A(n)$ regularly varies on infinity with non positive index $\rho \leq 0$.

It is also known that for the considered here case of convergence to Gumbel distribution,
\[
H(x) = \frac{1}{\rho} \left( \frac{x^\rho - 1}{\rho} - \log x \right), \quad \text{â‰¥} \rho < 0, \tag{19}
\]
and
\[
H(x) = \frac{1}{2} \log^2 x, \quad \text{â‰¥} \rho = 0.
\]

Using above mentioned Theorem 2.3.8 and Proposition \[1\] we get the following.

Corollary 2 Let conditions of Theorem \[2\] and \[18\] be fulfilled. Then
\[
P(M_n \leq a_n x + b_n) = \exp \left\{ -e^{-x} - A(n)H(x)(1 + o(1)) \right\}
\times \exp \left( -\frac{1}{n} \sum_{k=0}^{\infty} \frac{1}{(k+2)n^k} \left( 1 - F(a_n x + b_n) \right)^{k+2} \right).
\]

Remark that if $\rho < -1$ the second exponent gives the main contribution in the rate of convergence; if $\rho = -1$, one should knows the behavior of $A(n) = n^{-1} \ell(n)$ more exactly, that is, the behavior of the corresponding slowly varied function $\ell(n)$; finally, if $\rho > -1$, then the second summand in the first exponent gives the main contribution.

Similar calculations can be performed also for the $n$th order condition, see \[37\].

Let us give another result on the rate of convergence and accompanying laws, it is interesting to compared it with Theorem \[1\].

Theorem 3 (Theorem 2.1, \[9\]) Let \[18\] be fulfilled ans let $\rho < 0$, see \[19\]. Take $b_n = F^{-1/n}(e^{-1/n})$ and correspondingly $a_n = f(b_n)$. Then for any $\varepsilon > 0$ the relation
\[
\sup_x e^{(1-\varepsilon)x} \left| \frac{F^n(a_n x + b_n) - \exp(-e^{-x})}{A(n)} + \frac{1}{\rho} e^{-x+\rho x} e^{-e^{-x}} \right| \to 0
\]
takes place as $n \to \infty$.

Remark 1 This theoren, as well as Theorem 5.3.3, \[17\], can be obtained immediately from Theorem \[2\]. It is follow from proofs of the mentioned theorems.

Remark 2 It is interesting to use Theorem \[2\] in studying large deviations probabilities in Gnedenko Limit Theorem. For example, Corollary 2.1, \[9\] and Theorem 5.3.12, \[17\], under appropriate restrictions, can be obtained from suggested here asymptotical expansions. Notice also, that in \[37\], similar asymptotic expansions are used for this purpose.
4 Distributions with absolutely continuous tails

Assuming that $F$ is eventually, for large $x$, absolutely continuous, hence $c(x)$, is eventually absolutely continuous, and since it is additionally eventually strictly positive, there exists $x_0$ such that for all $x \geq x_0$,

$$
\log c(x) - \log c(x_0) = \int_{x_0}^{x} \frac{c'(t)dt}{c(t)}.
$$

Hence the representation (2) can be easily transformed to

$$
1 - F(x) = c \exp \left\{ - \int_{x_0}^{x} \frac{c(x) - c'(x)f(x)}{c(x)f(t)} dt \right\}, \ x \geq x_0,
$$

where we have re-defined $c(x)$ on $cc(x)$, and change $x_0$ to have positive and absolutely continuous $c'(x)$ for all $x \geq x_0$, with

$$
f(x)c'(x) \rightarrow 0, \text{ and } f^2(x)c''(x) \rightarrow 0 \text{ as } x \rightarrow \infty. \quad (20)
$$

In this conditions, the function

$$
\tilde{f}(x) := \frac{f(x)}{1 - \frac{c'(x)}{c(x)}f(x)}
$$

satisfies the same conditions as $f$, so that we can write

$$
1 - F(x) = c \exp \left\{ - \int_{x_0}^{x} \frac{1}{f(t)} dt \right\}, \ x \geq x_0. \quad (21)
$$

That is, a distribution function from the Gumbel domain of maximal attraction which has eventually sufficiently smooth tail, with introduced above restrictions on $c(t)$ is a von Mises function. Having this strengthened von Mises representation, we also have a corresponding shortened representation for $\gamma_n(x)$,

$$
\gamma_n(x) = \int_{0}^{x} \left( \frac{\tilde{f}(b_n)}{\tilde{f}(b_n + v)} - 1 \right) dv + x. \quad (22)
$$

We saw above that the behavior of $c(x)$ may give main contribution to the rate of convergence. On the contrary, in the introduced in this Section conditions, the influences of all the functions $c(x)$, $g(x)$ and $f(x)$ are aggregated by the function $\tilde{f}(x)$. So, let us consider the behavior of function

$$
\gamma(t; x) := \int_{0}^{x} \left( \frac{f(t)}{f(t + v)} - 1 \right) dv \quad (23)
$$

as $t \rightarrow \infty$. By (21) and Proposition 2, for any $x$, $\gamma(t; x) \rightarrow 0$. Moreover, since now $f$ is also eventually differentiable,

$$
\frac{f(t + v)}{f(t)} - 1 = \int_{0}^{v} \frac{f'(t + s)}{f(t)} ds,
$$
hence, for almost all $s$, \[
\frac{f'(t + s)}{f(t)} \to 0
\] as $t \to \infty$. Therefore, for smooth tails, having (20), we may investigate the rate of convergence only in terms of $f$. Below we consider two important classes of distributions which are subsets of $MDA(\Lambda)$, namely, distributions with Weibull-like and log-Weibull-like tails. Then we suggest a corresponding scale for $MDA(\Lambda)$.

Notice finally that since only ultimately behavior of the tail is in the frame of our consideration, we may modify $f$ and $c$ on any bounded fixed interval, in dependence of questions under consideration. For example we may put $x_0 = 0$ or $x_0 = 1$, changing correspondingly the functions in the von Mises representations.

### 4.1 Generalized Weibull-like distributions

Below on we assume that $F$ is absolutely continuous. For flexible von Mises representation (3), one can choose appropriately $g(t)$ on a finite interval, say, $[0, a]$, to have $c(x) \equiv 1$, $x_0 = 1$. Now take $f(t) = Ct^{1-p}$ with $p, C > 0$. That is, for all $x \geq 1$,

\[
1 - F(x) = \exp \left( - \int_{1}^{x} \frac{g(t)dt}{Ct^{1-p}} \right).
\] (24)

Such the distributions we call generalized Weibull-like distributions.

Let us find norming sequences $a_n$ and $b_n$, taking in mind that in virtue of convergence to types theorem, see [16], [29], [11], the limit in (1), after changing constants $(a_n, b_n)$ on $(\tilde{a}_n, \tilde{b}_n)$, belongs to the same (Gumbel) type, if and only if

\[
\frac{a_n}{\tilde{a}_n} \to 1, \quad \frac{b_n - \tilde{b}_n}{a_n} \to 0.
\] (25)

Denote $\alpha(x) := g(x) - 1$. Integrating, we have,

\[
1 - F(x) = \exp \left( \int_{1}^{x} \alpha(t)dt \right) \exp \left( - \frac{x^p}{Cp} \right).
\] (26)

Since $\alpha(t) \to 0$ as $t \to \infty$,

\[
\int_{1}^{x} \frac{\alpha(t)dt}{Ct^{1-p}} = o(x^p)
\]
as $x \to \infty$. An equation for $b_n$ is

\[
\int_{1}^{x} \frac{g(t)dt}{t^{1-p}} = C \log n.
\]

Passing to $\alpha(t)$, integrating in parts, and denoting $y = x^p$, we come to the following equation,

\[
y + \int_{1}^{y^{1/p}} \frac{p\alpha(t)dt}{t^{1-p}} = Cp \log n + p.
\] (27)
From here we have, \( y = Cp \log n (1 + o(1)) \) as \( n \to \infty \). Now we apply the asymptotical iteration as \( n \to \infty \), see for example [24]. Denote \( u := Cp \log n \), write for convenience \( y = u(1 + \varepsilon(y, u))^p \), and find \( \varepsilon(y, u) \). Equation (27) becomes as following,

\[
y = u - \int_1^{u^{1/p}(1+\varepsilon(y,u))} \frac{p\alpha(t)dt}{t^{1-p}} = u - \int_1^{u^{1/p}} \frac{p\alpha(t)dt}{t^{1-p}} - \int_{u^{1/p}(1+\varepsilon(y,u))}^{u} \frac{p\alpha(t)dt}{t^{1-p}}.
\]

Since we may assume that \( \alpha(1) > 0 \), the latter integral is of order \( \varepsilon(y, u) \). Now, in order to find \( \varepsilon(y, u) \), one can again input this expression for \( y \) into (27), so on, repeating this several times.

Finally we get from equation (28) that

\[
b_n^p = Cp \log n + p - \int_1^{(Cp \log n)^{1/p}} \frac{p\alpha(t)dt}{t^{1-p}} - \varepsilon'(n, b_n) = \frac{Cp \log n}{p}.
\]

with

\[
\varepsilon'(n, b_n) = \int_1^{1+\varepsilon(b_n^{1/p},Cp \log n)} \frac{p\alpha((Cp \log n)^{-1/p})}{s^{1-p}}ds.
\]

Notice that since \( \alpha(t) \) vanishes as \( t \) increases, the second term on the right in (29) is infinitely smaller the first one. For example, depending on value of \( p \) and on rate of tending \( \alpha(t) \) to zero, one can take

\[
b_n = (Cp \log n)^{1/p} - (Cp \log n)^{1/p-1} \int_1^{(Cp \log n)^{1/p}} \frac{\alpha(t)dt}{t^{1-p}}.
\]

For an important particular case [24], expressions for \( a_n \) and \( b_n \) will be evaluated below from (29) by the same iteration method, with using the second relation in (25).

Using representation (17), we get for all sufficiently large \( n \) that

\[
\gamma_n(x) - x = -\log \frac{1 - F(b_n + a_n x)}{1 - F(b_n)} - x
\]

\[
= \int_{b_n + f(b_n)x}^{b_n + f(b_n)x} \frac{g(t)dt}{f(t)} - x = \int_{b_n}^{b_n + f(b_n)x} \frac{1}{f(t)}dt + \int_{b_n}^{b_n + f(b_n)x} \frac{\alpha(t)dt}{f(t)} - x. \tag{30}
\]

Using the expression for \( f \) and substituting \( t = b_nv \), we get, that the first integral on the right is equal to

\[
C^{-1}b_n^p \int_1^{1+Cb_n^p} \left( y^{p-1} - 1 \right) dy + x.
\]

Integrating and using Taylor for the integral (taking in mind that \( b_n \to \infty \) and \( p > 0 \), we get that this is equal to

\[
\frac{1}{2}C(p-1)b_n^{-p}x^2(1 + O(b_n^{-p})) = \frac{(p-1)x^2(1 + o(1))}{2p \log n}, \quad n \to \infty.
\]
Notice that for $p = 1$ this integral vanishes. For the second integral on the right in (30), changing $t = b_n + f(b_n)v$, we get that

\[ \int_{b_n}^{b_n + f(b_n)x} \frac{\alpha(t)dt}{f(t)} = (1 + O(b_n^{-p})) \int_{0}^{x} \alpha(b_n + Cb_n^{1-p}v)dv, \quad n \to \infty. \quad (31) \]

Therefore

\[ \gamma_n(x) - x = \frac{(1 + O(b_n^{-p}))}{2} C(p - 1)b_n^{-p}x^2 + (1 + O(b_n^{-p})) \int_{0}^{x} \alpha(b_n + Cb_n^{1-p}v)dv \]

\[ = (1 + o(1)) \left( \frac{(p - 1)x^2}{2p \log n} + \int_{0}^{x} \alpha(b_n + Cb_n^{1-p}v)dv \right), \quad n \to \infty. \quad (32) \]

Thus we have proven the following refinement of Corollary 1 for generalized Weibull-like distributions.

**Proposition 2** Let $F$ satisfies (24). Then for the correction term in Corollary 1, relation (32) is valid.

Obviously only for exponential like tail, $p = 1$, the rate of convergence may be better than logarithmic, it depends on the behavior of $\alpha(t) = g(t) - 1$. As we have already seen, if $\alpha(t) \equiv 0$, then, by virtue of Theorem 2, the rate of convergence is proportional to $n^{-1}$.

### 4.1.1 Weibull like distributions

Now turn to classical Weibull like distributions. A distribution on $\mathbb{R}_+$ with distribution function $F$ such that

\[ 1 - F(x) = \mathbf{1}_{x \geq 0} \ell(x)x^\alpha e^{-cx^p}, \quad \text{with} \quad p, c > 0, \quad \alpha \in \mathbb{R}, \quad (33) \]

and $\ell(x)$, slowly varying at infinity function, is called a Weibull-like distribution. By Theorem 1.3.1, [7], for any slowly varying $\ell(x)$, some $x_0$ and all $x \geq x_0$,

\[ \ell(x) = c(x) \exp \left( \int_{x_0}^{x} \delta(t) / t \ dt \right), \]

where $c(x)$ tends to a positive limit as $x \to \infty$ and $\delta(t) \to 0$ as $t \to \infty$. Assume for simplicity that $\ell(x)$ is normalized, [7], this means that $c(x)$ is a constant, $c(x) = c > 0$. Hence, for some other positive $c$,

\[ 1 - F(x) = c \exp \left( \int_{x_0}^{x} \frac{\alpha + \delta(t) - pc/t^p}{t} dt \right). \quad (34) \]

Hence in the model (24, 26),

\[ C = \frac{1}{cp} \quad \text{and} \quad \alpha(t) = -\frac{\alpha + \delta(t)}{cpt^p}. \quad (35) \]
Inputting this in (32) and integrating, we get after pretty tedious but standard evaluations

\[ \gamma_n(x) - x = \left( \frac{(p - 1)x^2}{2} - \alpha x + o(1) \right) \frac{1}{p \log n} \]  

(36)
as \( n \to \infty \).

In order to evaluate normalizing sequences \( b_n \) and \( a_n \) on can apply again the iteration method which has been applied in proof of Proposition 2. After pretty complicated but standard calculations, see Appendix, we get the following.

For \( p = 1 \) one can take

\[ a_n = \frac{1}{c}, \quad b_n = \frac{1}{c} \log n + \frac{\alpha}{c} \log \left( \frac{1}{c} \log n \right) . \]  

(37)

For \( p \neq 1 \) one can take

\[ a_n = \frac{1}{cp} \left( \frac{1}{c} \log n \right)^{1/p - 1} ; \]  

\[ b_n = \left( c^{-1} \log n \right)^{1/p} \]  

\[ + \frac{1}{p} \left( c^{-1} \log n \right)^{1/p - 1} \left( \frac{a}{pc} \log(c^{-1} \log n) - c^{-1} \log \ell \left( \left( c^{-1} \log n \right)^{1/p} \right) \right) . \]  

(39)

Notice that in [13], the sequences \( a_n \) and \( b_n \) has been evaluates by another way, using so called \( W \) Lamperti functions, but only in case \( \ell(x) \) is constant. Thus we have the following refinement of Corollary 1 for Weibull-like distributions.

**Proposition 3** Let a distribution function \( F \) be ultimately absolutely continuous and (33) takes place for it. Then the correction term of Corollary 1 satisfies (36). Moreover, one can take \( a_n \) and \( b_n \) as in (38), (39), correspondingly.

4.1.2 Example. Weibull distribution.

Consider Weibull distribution, since it is extremely important in many fields, such as reliability theory, queuing theory, finances. That is, let \( \alpha = 0, \ell(x) \equiv 1 \) in (33). One can take

\[ b_n = \left( \frac{1}{c} \log n \right)^{1/p} \]  

\[ + \frac{1}{p} \left( \frac{1}{c} \log n \right)^{1/p - 1} \left( \frac{a}{pc} \log(c^{-1} \log n) - c^{-1} \log \ell \left( \left( c^{-1} \log n \right)^{1/p} \right) \right) . \]  

leaving the same \( a_n \). After integrating, we get that

\[ \gamma_n(x) - x = \log n \left( \left( 1 + \frac{x}{p \log n} \right)^p - 1 \right) - x \]  

\[ = \frac{1}{\log n} \sum_{k=0}^{\infty} \left( \frac{p}{k + 2} \frac{x^{k+2}}{p^{k+2} \log^k n} \right), \text{ for } p \neq 1; \]
and
\[ \gamma_n(x) = x, \quad \text{for } p = 1. \]
Hence, if \( p \neq 1 \), the decomposition in powers of \( \log n \) takes place,
\[ G_n(x) = \exp \left( -e^{-\gamma_n(x)} \right) + O(n^{-1}), \quad n \to \infty; \]
for \( p = 1 \), we have the decomposition in powers of \( n \),
\[ G_n(x) = \exp \left( -e^{-x} \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k e^{-kx}}{(k + 1)n^k} \right) \right). \]
Remark that for \( p = 2 \),
\[ \gamma_n(x) - x = \frac{x^2}{4 \log n}, \]
which corresponds to the member \( x^2/2t^2 \) in (5), with change \( n \) on \( T/(2\pi) \). Remind that for Gaussian process \( X(t) \) in Theorem 1, for any \( a \) and some \( \delta > 0 \) we have,
\[ P \left( \max_{t \in [0,a]} X(t) > u \right) = \frac{a}{2\pi} e^{-u^2/2} + P(X(0) > u) + O(e^{-(1+\delta)u^2/2}) \]
as \( u \to \infty \), see [30], where for proof of the Theorem we take \( a = a(T) \to \infty \), but \( a(T)/T \to 0, \quad T \to \infty \) in order to make negligible the probability in the right hand part and to use Weibull distribution with \( p = 2 \).

### 4.2 Generalized log-Weibull-like distributions

Assume again that \( 1 - F \) is ultimately absolutely continuous and take in (3), \( f(t) = Ct \log^{1-p} t, \quad p > 1 \). Also assume that conditions (20) are fulfilled. Then redefining \( g(t) \) on a finite interval, as earlier, consider distribution functions satisfying
\[ 1 - F(x) = \exp \left( - \int_1^x \frac{g(t)dt}{Ct \log^{1-p} t} \right) \]
\( x \geq 1 \), with an agreement that \( 1/\log^{1-p} 1 = 0 \). Remind that \( \alpha(t) = g(t)-1 \), and \( \alpha(t) \to 0 \) as \( t \to \infty \). Notice that for \( p \leq 1 \) such the functions do not belong to \( MDA(\Lambda) \).

Proof of the following refinement of Corollary [4] for generalized log-Weibull-like distributions (40) is given in Appendix.

**Proposition 4** For generalized log-Weibull-like distributions (40), assertion of Corollary 1 takes place with
\[ \gamma_n(x) - x = (1 + o(1)) \left( 2^{-1}C^{1/p}(1-p)/p x^2 \log^{1/p-1} n + \int_0^x \alpha(b_n + Cb_n^{1-p}v)dv \right) \]
as \( n \to \infty \).

That is, by Corollary 1 the rate of convergence in (1) for such distributions is proportional to the right hand part of this equality.

Remark that in the case of log-Weibull-like distributions, since \( p > 1 \), the rate of convergence cannot be better than logarithmic and also depends through the second term above of how fast \( \alpha(t) \) tends to zero as \( t \to \infty \). Remark also that for \( p \leq 1 \) the distributions (40) do not belong to \( MDA(\Lambda) \).
4.2.1 Log-Weibull-like distributions

Now consider a particular case of distributions (40), a distribution on \([1, \infty)\) with distribution function \(F(x)\) such that

\[
1 - F(x) = \ell(x)x^\alpha e^{-c \log^p x}, \quad \text{with } p > 1, \ c > 0, \ \alpha \in \mathbb{R},
\]

and \(\ell(x)\), slowly varying on infinity function, is called a log-Weibull-like distribution. Similarly to representation (34), using (35) with corresponding modification \(\delta(t)\) on a finite interval, we get by simple calculus, that

\[
1 - F(x) = \exp \left( - \int_1^x \frac{cp}{t \log^{1-p} t} \left( 1 - \frac{\alpha + \delta(t)}{cp \log^{p-1} t} \right) dt \right),
\]

so that \(C = \frac{1}{cp}\), and \(g(t) = 1 - \frac{\alpha + \delta(t)}{cp \log^{p-1} t}\) (43).

Now apply Proposition 4. After simple calculations we get from (42), (43), that for \(\alpha \neq 0\),

\[
\alpha(b_nv) = - \frac{\alpha(1 + o(1))}{cp \log^{p-1}(b_nv)} = - \frac{\alpha(1 + o(1))}{(cp)^{(2p-1)/p} \ln^{(p-1)/p} n},
\]

that is, the second member on the right in (41) represents the rate of convergence. For \(\alpha = 0\) the rate can be better, in dependence of rate of tending \(\delta(t)\) to zero. In case \(\ell(t)\) is constant, the first member on the right in brackets of (41) is the rate of convergence.

Using again asymptotical iterations, one can evaluate expressions for \(a_n\) and \(b_n\), it is similar to corresponding evaluations for Weibull-like tails.

5 A scale in \(MDA(\Lambda)\) for distributions with smooth tails

Classes of generalized Weibull-like and log-Weibull-like distributions can be a beginning of a natural scale in \(MDA(\Lambda)\). As it is mentioned in Introduction, this domain is enormous wide, distributions with very different tail behaviors from this domain plays important role in financial, actuarial research, reliability theory, engineering applications. Therefore Gnedenko limit theorem (1) gives too far from complete information on the tail behavior of a distribution under estimation. For example, assignment of insurance premium is strongly depends on behavior of tail distribution of the time to insurance case. There are a plenty of studies in statistical discrimination of hypotheses about the distribution tails for Weibull and log-Weibull distributions, see [5], [33], [34] and references therein.

A continuation of a scale which begins with the two considered here distribution classes can be as following. In the two classes we have \(f(t) = C t^{1-p}, \ p > 0\) and \(f(t) = C t \log^{1-p} t, \ p > 1\), correspondingly. Distributions with heavier tails, say, with tails \(\exp(-C \log^p x \log \log^p x), \ p > 1\) can be described by von Mises function with \(f(t) = C t (\log \log t)^a, \ a < 0\), the constant \(C\) can be different. The next ”scale division” is.
\( f(t) = Ct (\log \log \log t)^{-a} \), so on. Obviously, for any natural \( k \), \( f(t) = C t (\log(\log) \log t)^{-a} \) satisfies conditions for representations (2) and 3). The repetitions number \( k \) of logarithms can be called Gumbel index, then the considered here classes of distributions have indexes \( k = 0 \) and \( k = 1 \), correspondingly.

It is interesting to consider behavior of \( \gamma_n(x) - x \) as \( n \to \infty \) for all the scale. As we see, it is sufficient to consider only representation (2), that is, \( g(t) = 1 \).

6 Appendix.

6.1 Derivation of normalizations \((37, 38, 39)\)

Taking logarithm of

\[
\ell(x) x^\alpha e^{-cx^p} = n^{-1}
\]

we have the equation for \( b_n \),

\[
\frac{1}{c} \log \ell(x) + \frac{\alpha}{c} \log x - x^p = -\frac{1}{c} \log n.
\]

Substituting \( u = u_n = c^{-1} \log n \), \( y = x^p \), we have,

\[
y - \frac{\alpha}{pc} \log y - \frac{1}{c} \log \ell_1(y) = u, \quad (44)
\]

where \( \ell_1(y) = \ell(y^{1/p}) \) is also slowly varying with \( \delta_1(t) = p\delta(t^p) \) in given above representation for smooth slowly varying functions. Now find asymptotically a root of \((44)\) as \( u \to \infty \) by asymptotic iteration. First, \( y = u(1 + o(1)) \). Substituting again this into the equation, we easily get, that

\[
y = u + L(u) + o(1), \quad u \to \infty,
\]

where

\[
L(u) := \frac{\alpha}{pc} \log u + \frac{1}{c} \log \ell_2(u),
\]

and \( \ell_2(y) = \ell_1(y(1 + o(1))) \), \( y \to \infty \), is slowly varying with corresponding \( \delta_2(t) \), obvious modification of \( \delta_1(t) \). Again inputting this in \((44)\), we get, that

\[
y = u + \frac{\alpha}{pc} \log (u + L(u) + o(1)) + \frac{1}{c} \log \ell_1(u + L(u) + o(1))
\]

\[
= u + \frac{\alpha}{pc} \log u + \frac{\alpha}{pc} \log \left(1 + \frac{L(u) + o(1)}{u}\right) + \frac{1}{c} \log \ell_1(u + L(u) + o(1))
\]

\[
= u + \frac{\alpha}{pc} \log u + \frac{\alpha}{pc} \frac{L(u) + o(1)}{u} + \frac{1}{c} \log \ell_1(u + L(u) + o(1)), \quad u \to \infty.
\]

Consider the last summand on the right. It is equal to

\[
\int_1^{u + L(u) + o(1)} \frac{\delta_1(t)}{t} \, dt = \log \ell_1(u) + \Delta(u),
\]
where

$$\Delta(u) = \int_u^{u+L(u)+o(1)} \frac{\delta_1(t)}{t} dt = \frac{\alpha L(u) \delta_1(\theta u)}{pc \ u \ \theta}$$

with

$$\theta \in \left[1, 1 + \frac{L(u) + o(1)}{u}\right].$$

Thus we have the following asymptotical equality,

$$b_n^p = u_n + L(u_n) + \frac{\alpha}{pc} \log u_n + \frac{1}{c} \log \ell_2(u_n) + \frac{1}{u_n} R(u_n), \ n \to \infty.$$  \hspace{1cm} (45)

It is easily follows from the above evaluations, that $R(u_n) = O(L(u_n))$ as $n \to \infty$.

Furthermore,

$$a_n = f(b_n) = \frac{1}{cp} b_n^{1-p}.$$  \hspace{1cm}

That is, for $p = 1$ one can take normalized sequences (37). Further, notice that from (45) it follows that any normalizing sequence $\tilde{b}_n$ is equal to $b_n + o(1)$. Hence for $p \neq 1$, applying in (45), we get that

$$b_n = u_n^{1/p} + \frac{1}{p} u_n^{1/p-1} \left( \frac{\alpha}{pc} \log u_n + \frac{1}{c} \log \ell_2(u_n) + \frac{1}{u_n} R(u_n) \right)$$

$$+ \frac{1}{2p} \left( \frac{1}{p} - 1 \right) u_n^{1/p-2} \left( \frac{\alpha}{pc} \log u_n + \frac{1}{c} \log \ell_2(u_n) + \frac{1}{u_n} R(u_n) \right)^2 + ...,$$  \hspace{1cm} (46)

If $p > 1$, then

$$a_n = \frac{1}{cp} u_n^{1/p-1}(1 + o(1)), \ n \to \infty,$$

and $a_n$ can be taken as in (38). Using that $\ell_2(y) = \ell(y^{1/p}) + O(\Delta(y))$, we see that $b_n$ can be taken as in (39). The same is valid also for $p < 1$. Indeed, in this case $a_n \to \infty$ as $n \to \infty$, and even if $p < 1/2$, the other members in (46) can tend to infinity with $n$, equalities (25) let us to remind the same members in (46) as in case $p > 1$.

### 6.2 Proof of Proposition 4

Similarly to evaluations rewrite equation (40) for $b_n$ as

$$\log^p x - \int_1^x \frac{pa(t)dt}{t \log^{1-p} t} = Cp \log n.$$  \hspace{1cm}

Denote $y = \log^p x$, $u = Cp \log n$, change $s = \log t$, the above equation takes the form

$$y - p \int_0^{y^{1/p}} \alpha(s) s^{p-1} ds = u.$$  \hspace{1cm} (47)
From here we get for the solution $y = u(1 + o(1))$, as $n \to \infty$. Write for convenience $y = u(1 + \varepsilon(y, u))^p$. Putting this to (47), we have,

$$y = u + p \int_{0}^{u^{1/p}(1+\varepsilon(y, u))} \alpha(e^s)s^{p-1}ds$$

$$= u + p \int_{0}^{u^{1/p}} \alpha(e^s)s^{p-1}ds + p \int_{u^{1/p}(1+\varepsilon(y, u))}^{u^{1/p}} \alpha(e^s)s^{p-1}ds$$

$$= u + p \int_{0}^{u^{1/p}} \alpha(e^s)s^{p-1}ds + pu^{-1} \int_{1+\varepsilon(y, u)}^{1} \alpha(\exp(u^{-1/p}v))v^{p-1}dv, \quad (48)$$

where we changed $v = u^{1/p}s$ in the last integral. Since $\alpha(1)$ can be chosen to be a positive constant, the last integral is of order $\varepsilon(y, u)$. Now, applying asymptotical iteration method, in order to find $\varepsilon(y, u)$, we input this again into the upper integration limit in (27), so on. Finally we get from (48) that

$$\log^p b_n = Cp \log n + p \int_{0}^{(Cp \log n)^{1/p}} \alpha(e^s)s^{p-1}ds + \frac{\varepsilon'(n, b_n)}{\log n}, \quad (49)$$

where

$$\varepsilon'(n, b_n) = \frac{1}{C} \int_{1}^{1+\varepsilon(\log^p b_n, Cp \log n)} \alpha(n^C v))v^{p-1}dv.$$

Now evaluate a rate of convergence in the limit theorem. Integrating in (30) with a new $\varepsilon(y, u)$ $f$, we get, that

$$\int_{b_n}^{b_n+f(b_n)x} \frac{dt}{f(t)}dt = \frac{1}{C} \int_{b_n}^{b_n+f(b_n)x} \log^{p-1}t \log t = \frac{1}{pC} \log^p t \bigg|_{b_n}^{b_n+f(b_n)x}$$

$$= \frac{1}{pC} \left( \log^p(b_n(1 + Cx \log^{1-p} b_n)) - \log^p b_n \right)$$

Applying several times Taylor expansion, after tedious but obvious evaluations we get, that the difference in latter brakets is equal to

$$Cpx + \frac{1}{2}C^2px^2 \log^{1-p} b_n + \frac{p(p-1)}{2}C^2x^2 \log^{-p} b_n + O(\log^{2-2p} b_n)$$

$$= Cpx + \frac{1}{2}C^2px^2 \log^{1-p} b_n + O(\log^{-p} b_n + \log^{2-2p} b_n).$$

Hence

$$\int_{b_n}^{b_n+f(b_n)x} \frac{dt}{f(t)}dt = x + \frac{1}{C}Cx^2 \log^{1-p} b_n(1 + O(\log^{-1\wedge(p-1)} b_n)).$$

Furthermore, similarly to (31), changing $t = b_n + f(b_n)v$, we get the following,

$$\int_{b_n}^{b_n+f(b_n)x} \frac{\alpha(t)dt}{f(t)} = (1 + O(\log^{1-p} b_n)) \int_{0}^{x} \alpha(b_n(1 + C \log^{1-p} v)dv, \quad n \to \infty.$$
\[ \gamma_n(x) - x = \frac{1 + O(\log^{-1(p-1)}b_n)}{2} C x^2 \log^{1-p} b_n + (1 + O(\log^{1-p} b_n)) \int_0^x \alpha(b_n + Cb_n^{-p} v) dv, \]

which follows (41).

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