On the renormalization property and entropy conservation laws for the relativistic Vlasov-Maxwell system

Minh-Phuon Tran\textsuperscript{1}, Thanh-Nhan Nguyen\textsuperscript{2}

\textsuperscript{1}Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Viet Nam
Email: tranminhphuong@tdtu.edu.vn

\textsuperscript{2}Department of Mathematics, Ho Chi Minh City University of Education, Ho Chi Minh City, Viet Nam
Email: nguyenthnhan@hcmup.edu.vn

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Abstract

The aim of this paper is to improve the previous work on the relativistic Vlasov-Maxwell system, one of the most important equations in plasma physics. Recently in \cite{3}, C. Bardos et al. presented a proof of an Onsager type conjecture on renormalization property and the entropy conservation laws for the relativistic Vlasov-Maxwell system. Particularly, authors proved that if the distribution function $u \in L^\infty(0,T; W^{\alpha,p}(\mathbb{R}^6))$ and the electromagnetic field $E,B \in L^\infty(0,T; W^{\beta,q}(\mathbb{R}^3))$, with $\alpha, \beta \in (0,1)$ such that $\alpha \beta + \beta + 3\alpha - 1 > 0$ and $1/p + 1/q \leq 1$, then the renormalization property and entropy conservation laws hold. To determine a complete proof of this work, in the present paper we improve their results under a weaker regularity assumptions for weak solution to the relativistic Vlasov-Maxwell equations. More precisely, we show that under the similar hypotheses, the renormalization property and entropy conservation laws for the weak solution to the relativistic Vlasov-Maxwell system even hold for the end point case $\alpha \beta + \beta + 3\alpha - 1 = 0$. Our proof is based on the better estimations on regularization operators.

Keywords: Relativistic Vlasov-Maxwell system, Onsager type conjecture, renormalization property, entropy conservation laws.

1 Introduction

In recent years, mathematicians have devoted much attention to the relativistic Vlasov-Maxwell system, the most important equation describes the distribution of particles in phase space of a monocharged plasma under relativistic effects. There has been an increasing activity that studied the Vlasov-Maxwell system in kinetic plasma physics. It is well-known that the Vlasov equation describes the time evolution of particles in
a plasma, how the plasma response to electromagnetic fields. This equation finds the
unknown distribution function of particles \( u = u(t, x, \xi) \) satisfies:
\[
\partial_t u + v \cdot \nabla_x u + \mathcal{F} \cdot \nabla_\xi u = 0,
\]
where \((t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3\) represent time, position and momentum of particles,
respectively. The relativistic velocity \( v \) of a particle with momentum \( \xi \in \mathbb{R}^3 \) is given by
\[
v = \frac{\xi}{\sqrt{1 + |\xi|^2}}.
\]

The consideration of problem may be under electromagnetic, in which the Lorentz force \( \mathcal{F} = E + v \times B \) corresponds to the self-consistent electric field \( E = E(t, x) \) and magnetic field \( B = B(t, x) \) generated by the charged particles in the plasma. They are coupled satisfying Maxwell’s equations
\[
\begin{align*}
\partial_t E - \text{curl} B &= -j, & \partial_t B + \text{curl} E &= 0, \\
\text{div} E &= \rho, & \text{div} B &= 0,
\end{align*}
\]
where the quantities \( \rho = \rho(t, x) \) and \( j = j(t, x) \) are the charge density and electric current density of the plasma, respectively, defined by
\[
\rho(t, x) = \int_{\mathbb{R}^3} u(t, x, \xi) d\xi; \quad j(t, x) = \int_{\mathbb{R}^3} v(\xi) u(t, x, \xi) d\xi.
\]

Maxwell’s equations must be solved together with the Vlasov equation (1.1), so-called the Vlasov-Maxwell system. Here, we are interested in the Cauchy problem for system (1.1)-(1.5), where the initial data given as
\[
\begin{align*}
u(0, x, \xi) &= u_0(x, \xi) \geq 0, \\
E(0, x) &= E_0(x), & B(0, x) &= B_0(x), \\
\text{div} E_0 &= \rho_0 = \int_{\mathbb{R}^3} u_0 d\xi, & \text{div} B_0 &= 0.
\end{align*}
\]

There are many interesting problems that related to the Vlasov-Maxwell system (1.1)-(1.5) that make the range of its application has been considerably extended. For instance, the existence and uniqueness of analytical solutions to this, especially for high dimensions; regularity results for the system in some spaces; the conduction of sharp estimates for solutions; some numerical methods and simulations on the solutions, etc, are at the core of many researching topics at the moment.

The global existence of solution to this earlier has been studied intensively by several authors, such as R.J. Diperna and P.-L. Lions in [8], Y. Guo in [9, 10] or G. Rein in [12] and several references therein. Later, different approaches to the results related to this system were recently achieved and reviewed by other authors. In our knowledge, there has been a few results on the regularity of this system. Recently in 2018, N. Besse et al. have showed in [5] that if the macroscopic kinetic energy is in \( L^2 \), then the electric and magnetic fields belong to the Sobolev space \( H^s_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^3) \) with \( s = \frac{6}{13 + \sqrt{142}} \). Moreover, in [4, 11], authors have established the critical regularity of weak solutions to
a general system of entropy conservation laws which are related to the famous Onsager exponent 1/3. In the nearest research paper [3], Bardos et. al. gave a proof of an Onsager type conjecture on renormalization property and entropy conservation laws for the Vlasov-Maxwell equations. More precisely, their work devoted to the results that if the distribution function $u \in L^\infty(0,T; W^{\alpha,p}(\mathbb{R}^6))$ and the electromagnetic fields $E, B \in L^\infty(0,T; W^{\beta,q}(\mathbb{R}^3))$, with $\alpha, \beta \in (0,1)$ satisfying $\alpha \beta + \beta + 3\alpha - 1 > 0$ and $1/p + 1/q \leq 1$, then the renormalization property holds. As there have been too few results concerning to regularity of this system, such extensions have been promising to discussed under various assumptions and conditions of problem formulation. In the present paper, based on the regularity assumptions of weak solution to the Vlasov-Maxwell equations, a small portion of that result is improved, where the conclusion of this property holds even for $\alpha \beta + \beta + 3\alpha - 1 \geq 0$, the renormalization property and entropy conservation laws hold under the same hypotheses. To our knowledge, from the mathematical point of view, the end point case $\alpha \beta + \beta + 3\alpha - 1 = 0$, the proof is more challenging than what obtained in [3]. Compare to the previous study for the case $\alpha \beta + \beta + 3\alpha - 1 > 0$, ours have the advantage that for $\alpha \beta + \beta + 3\alpha - 1 \geq 0$, we work on the weaker regularity assumptions, and the effective technique is applied to extend the proof. The key idea comes from the better estimations on regularization operators that will be described later.

The rest of the paper is organized as follows. Next section 2 is devoted to some notations and definitions about the renormalization property and entropy conservation laws, and our main result of this paper is also stated therein. We then introduce in Section 3 some regularization operators and properties are also presented for later use. Finally, the last section gives a brief proof of the renormalization property and entropy conservation laws for Diperna-Lions weak solution to the Vlasov-Maxwell equations.

## 2 Main result

At the beginning of this section, let us recall some notations and definitions concerning to the problem.

Throughout the paper, we denote by $\mathcal{D}(\mathbb{R}^n)$, with $n \geq 1$, the space of infinitely differentiable function with compact support and by $\mathcal{D}'(\mathbb{R}^n)$ the space of distribution. For $\alpha \in (0,1)$, $1 \leq p \leq \infty$, the generalized fractional order Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ is defined for any function $f$ belonging to $W^{\alpha,p}(\mathbb{R}^n)$ if and only if the following Gagliardo-type norm is finite:

$$
\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} + \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}} \, dx \, dy \right)^{1/p} < +\infty,
$$

in the case $1 \leq p < \infty$ and

$$
\|f\|_{W^{\alpha,\infty}(\mathbb{R}^n)} := \max \left\{ \|f\|_{L^\infty(\mathbb{R}^n)}, \sup_{x \neq y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \right\} < +\infty,
$$

for $p = \infty$. Here and subsequently, $L^1(\mathbb{R}^6)$ denotes the set of non-negative almost
everywhere function $f$ such that
\[ \|f\|_{L^1(\mathbb{R}^6)} := \int_{\mathbb{R}^6} f(x,\xi) \sqrt{1 + |\xi|^2} \, dx \, d\xi < +\infty. \] (2.1)

In addition, the notation $S$ stands for the set of non-decreasing function $G \in C^1(\mathbb{R}^+; \mathbb{R}^+)$ such that
\[ \lim_{t \to +\infty} \frac{G(t)}{t} = +\infty. \]

The weak solution of a coupled set of relativistic Vlasov-Maxwell equations involves the distribution function $u$ (describes plasma components), electric and magnetic fields $E, B$ (self-consistently modified by particles). Here, we say that $(u, E, B)$ is a weak solution to relativistic Vlasov-Maxwell equations \([1.1]\) if $(u, E, B)$ satisfies the following weak formulation
\[ \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} u(\partial_t \varphi + v \cdot \nabla_x \varphi + \mathcal{F} \cdot \nabla_\xi \varphi) \, d\xi = 0, \]
for all $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^6)$. The existence result of a global in time weak solution to the relativistic Vlasov-Maxwell equations proposed by DiPerna-Lions is stated in the following theorem, where we refer the reader to [8] for details.

**Theorem 2.1** Let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ and $E_0, B_0 \in L^2(\mathbb{R}^3)$ be initial conditions with satisfy the constraints
\[ \text{div } B_0 = 0, \quad \text{div } E_0 = \int_{\mathbb{R}^3} u_0 \, d\xi, \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \]

Then there exists a global in time weak solution of the relativistic Vlasov-Maxwell system, i.e., there exist functions
\[ u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^6)), \quad E, B \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3)), \]
\[ \text{and } \rho, j \in L^\infty(\mathbb{R}^+; L^{4/3}(\mathbb{R}^3)), \] (2.2)
such that $(u, E, B)$ satisfy \([1.1]-[1.4]\) in the sense of distributions, where $\rho$ and $j$ are defined in \([1.5]\).

Let $(u, E, B)$ be a weak solution to the relativistic Vlasov-Maxwell system \([1.1]-[1.5]\), as in Theorem 2.1 Then for any smooth function $G \in C^1(\mathbb{R}^+; \mathbb{R}^+)$, we say that $(u, E, B)$ satisfies the renormalization property if
\[ \partial_t (G(u)) + \nabla_x \cdot (vG(u)) + \nabla_\xi \cdot (\mathcal{F}G(u)) = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^6) \] (2.3)
in the sense of distribution, that means,
\[ \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} G(u)(\partial_t \varphi + v \cdot \nabla_x \varphi + \mathcal{F} \cdot \nabla_\xi \varphi) \, d\xi = 0, \]
for all $\varphi \in \mathcal{D}'((0,T) \times \mathbb{R}^6)$. Otherwise, solution $(u, E, B)$ is said to satisfy the local in space entropy conservation law, if
\[ \partial_t \left( \int_{\mathbb{R}^3} G(u) \, d\xi \right) + \nabla_x \cdot \left( \int_{\mathbb{R}^3} vG(u) \, d\xi \right) = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \] (2.4)
and the local in momentum entropy conservation law, if

$$
\partial_t \left( \int_{\mathbb{R}^3} \mathcal{G}(u) dx \right) + \nabla_\xi \cdot \left( \int_{\mathbb{R}^3} \mathcal{F}(u) dx \right) = 0, \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^3),
$$

in the sense of distribution.

In this way, we can state that \((u, E, B)\) satisfies the global entropy conservation law, if we have

$$
\int_{\mathbb{R}^6} \mathcal{G}(u(t, x, \xi)) d\xi dx = \int_{\mathbb{R}^6} \mathcal{G}(u(s, x, \xi)) d\xi dx, \quad \text{for } 0 < s \leq t < T.
$$

Let us state our main result about the renormalized property and entropy conservation laws for the global weak solution of relativistic Vlasov-Maxwell equations. Related to the present note, it emphasizes that the regularity assumptions on the weak solution in our work are weaker than in the paper of C. Bardos et. al. \[3\], our improved results thus are more general. In particular, we prove that the renormalization property and entropy conservation laws for the global weak in time solution to the relativistic Vlasov-Maxwell’s system \((1.1)-(1.5)\) even hold for the end point case \(\alpha\beta + \beta + 3\alpha - 1 = 0\), as described in the following theorem.

**Theorem 2.2** Let \((u, E, B)\) be a weak solution of the relativistic Vlasov-Maxwell system \((1.1)-(1.5)\) given by Theorem \(2.7\). Assume moreover that this weak solution satisfies the additional regularity assumptions

$$
u \in L^\infty(0, T; W^{\alpha,p}(\mathbb{R}^6)) \quad \text{and} \quad E, B \in L^\infty(0, T; W^{\beta,q}(\mathbb{R}^3)), \quad (2.7)
$$

where \(\alpha, \beta \in (0, 1)\) such that \(\alpha\beta + \beta + 3\alpha - 1 \geq 0\), and \(p, q \in \mathbb{N}^*\) such that

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1 \text{ if } 1 \leq p, q < \infty, \quad \text{and } 1 \leq r < \infty \text{ is arbitrary if } p = q = \infty. \quad (2.8)
$$

Then for any entropy function \(\mathcal{G} \in C^1(\mathbb{R}^+; \mathbb{R}^+)\), the global weak solution \((u, E, B)\) satisfies the renormalization property \((2.3)\). Moreover, if \(\mathcal{G} \in \mathcal{S}\) and the mapping \(t \mapsto u(t, \cdot, \cdot)\) is uniformly integrable in \(\mathbb{R}^6\), for almost everywhere \(t \in [0, T]\), then the local entropy conservation laws \((2.4)-(2.5)\) and the global entropy conservation law \((2.6)\) hold.

### 3 Regularization operators

In this section, let us mention the important consequences of this work, that leading to the proof of our main result. It is devoted to study the standard regularization operators and their properties, that gives us the idea to prove main result in this paper. We will now show their descriptions and prove some preparatory lemmas that are necessary for later use.

Let \(\varrho \in D(\mathbb{R}^+; \mathbb{R}^+)\) be a smooth non negative function such that

$$
\text{supp}(\varrho) \subset [1, 2], \quad \int_{\mathbb{R}} \varrho(\tau) d\tau = 1. \quad (3.1)
$$
For every $\delta > 0$ and $n \in \mathbb{N}^*$, the radially-symmetric compactly-supported Friedrichs mollifier

$$\varrho_\delta : \mathbb{R}^n \to \mathbb{R}^+, \quad x \mapsto \varrho_\delta(x),$$

is given by

$$\varrho_\delta(x) = \delta^{-n} \varrho \left( \delta^{-1} |x| \right), \quad x \in \mathbb{R}^n. \quad (3.2)$$

Let $\eta, \varepsilon, \delta$ be positive numbers and for any distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n)$ and $h \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n)$, we define their $\mathcal{C}^\infty$-regularization by

$$f^\delta(x) = \varrho_\delta(x) * f(x), \quad (3.3)$$

$$g^{\varepsilon,\delta}(t,x) = \varrho_\varepsilon(t) *_t \varrho_\delta(x) *_x g(t,x), \quad (3.4)$$

and

$$h^{\eta,\varepsilon,\delta}(t,x,\xi) = \varrho_\eta(t) *_t \varrho_\varepsilon(x) *_x \varrho_\delta(\xi) *_{\xi} h(t,x,\xi), \quad (3.5)$$

where the operator $*$ denotes the standard convolution product. We first establish two basic estimations for the relativistic velocity $v$ in (1.2) by the following lemmas.

**Lemma 3.1** Let $\delta > 0$ and $v$ be the relativistic velocity given by (1.2). Then we have the following estimations

$$|v(\xi - w) - v(\xi)| \leq 2|w|, \quad (3.6)$$

and

$$|v(\xi) - v^\delta(\xi)| \leq 4\delta, \quad (3.7)$$

for all $\xi, w \in \mathbb{R}^3$.

**Proof.** By a simple computation, we firstly get that

$$|\nabla_\xi v| = \frac{I_3}{\sqrt{1 + |\xi|^2}} - \frac{\xi \otimes \xi}{\sqrt{(1 + |\xi|^2)^3}} \leq 2, \quad (3.8)$$

where $I_3$ denotes the identity matrix of size 3. Combining inequality (3.8) and the fundamental theorem of calculus, we obtain the first basic estimate (3.6),

$$|v(\xi - w) - v(\xi)| \leq |w| \int_0^1 |\nabla v(\xi - sw)| ds \leq 2|w|.$$ 

By using this estimation, we obtain the second basic estimate (3.7) as follows

$$|v(\xi) - v^\delta(\xi)| = \left| \int_{\mathbb{R}^3} \varrho_\delta(w)(v(\xi) - v(\xi - w)) dw \right| \leq 2 \left| \int_{\mathbb{R}^3} \varrho_\delta(w)|w| dw \right| \leq 4\delta.$$
Remark 3.2 In fact, one may obtain the better estimate for \( |v - v^\delta| \) than that of Lemma 3.1, i.e., there exists a constant \( C \) depending only on the smooth function \( \varrho \) given by (3.1), such that
\[
|v(\xi) - v^\delta(\xi)| \leq C\delta^2, \quad (3.9)
\]

Proof. Using the fundamental theorem of calculus twice, for any \( i \in \{1, 2, 3\} \), in this way we obtain componentwise
\[
v_i - v_i^\delta = \int_{\mathbb{R}^3} d\varrho_\delta(w)(v_i(\xi) - v_i(\xi - w))
= \sum_{j=1}^{3} \int_{\mathbb{R}^3} d\varrho_\delta(w) w_j \int_0^1 d\tau \partial_j v_i(\xi - \tau w)
= \sum_{j=1}^{3} \partial_j v_i(\xi) \int_{\mathbb{R}^3} d\varrho_\delta(w) w_j
+ \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} d\varrho_\delta(w) w_j w_k \int_0^1 d\tau \int_0^1 ds \partial^2_{jk} v_i(\xi - s\tau w). \quad (3.10)
\]
Since the smooth function \( \varrho_\delta \) is radially symmetric, we have
\[
\int_{\mathbb{R}^3} d\varrho_\delta(w) w_j = 0, \quad \forall j \in \{1, 2, 3\},
\]
which deduces that the first term of the right hand side of (3.10) vanishes. Therefore, (3.10) becomes
\[
v_i - v_i^\delta = \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} d\varrho_\delta(w) w_j w_k \int_0^1 d\tau \int_0^1 ds \partial^2_{jk} v_i(\xi - s\tau w). \quad (3.11)
\]
Moreover, by the definition of the smooth function \( \varrho \), there exists a constant \( C \) depending only on the function \( \varrho \) such that
\[
\int_{\mathbb{R}^3} \varrho_\delta(w)|w_j||w_k|dw \leq \int_{\mathbb{R}^3} \varrho_\delta(w)|w|^2 dw \leq C\delta^2, \quad \forall j, k \in \{1, 2, 3\}, \quad (3.12)
\]
On the other hand, by using \( \delta_{ij} \) as the Kronecker notation and directly computation on the relativistic velocity \( v \), for all \( j, k \in \{1, 2, 3\} \), one obtains
\[
|\nabla^2_{jk} v_i(\xi)| = \left| \frac{\delta_{ij} \xi_k}{\sqrt{1 + |\xi|^2}^3} + \frac{\delta_{jk} \xi_i}{\sqrt{1 + |\xi|^2}^3} + \frac{\delta_{ik} \xi_j}{\sqrt{1 + |\xi|^2}^3} - \frac{3\xi_i \xi_j \xi_k}{\sqrt{1 + |\xi|^2}^3} \right|,
\]
which implies that
\[
|\nabla^2_{jk} v_i(\xi)| \leq 6.
\]
Combining this estimation together with the inequality (3.12), and from (3.11), it completes the proof.

We next present some well-known properties for \( C^\infty \)-regularization in the next lemma. For the proof of (ii) and (iii), it refers the reader to some papers found in [6, Proposition 4.2] or [7, Proof of Theorem 2.4], or in [2].
Lemma 3.3

(i) For any distribution \( f \in \mathcal{D}'(\mathbb{R}^n) \) and \( \varepsilon > 0 \), we have
\[
\langle f^\varepsilon, g \rangle = \langle f, g^\varepsilon \rangle, \quad g \in \mathcal{D}(\mathbb{R}^n),
\]
where \( \langle \cdot, \cdot \rangle \) denotes the dual bracket between spaces \( \mathcal{D}' \) and \( \mathcal{D} \).

(ii) Let \( \varepsilon > 0 \), \( \alpha \in (0, 1) \) and \( 1 \leq p \leq \infty \). Then for any function \( f \in L^1 \cap L^\infty \cap W^{\alpha,p}(\mathbb{R}^n) \), we have
\[
\|f^\varepsilon\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)},
\]
and
\[
\|f^\varepsilon\|_{W^{\alpha,p}(\mathbb{R}^n)} \leq \|f\|_{W^{\alpha,p}(\mathbb{R}^n)}.
\]

(iii) Let \( \alpha \in (0, 1) \) and \( 1 \leq p \leq \infty \). Then for any function \( f \in W^{\alpha,p}(\mathbb{R}^n) \), there exists a constant \( C \) such that
\[
\|f(\cdot - w) - f(\cdot)\|_{L^p(\mathbb{R}^n)} \leq C|w|^{\alpha}\|f\|_{W^{\alpha,p}(\mathbb{R}^n)}, \quad (3.13)
\]
for all \( w \in \mathbb{R}^n \).

Proof. For any distribution functions \( f, g \in \mathcal{D}'(\mathbb{R}^n) \) and \( \varepsilon > 0 \), we write
\[
\langle f^\varepsilon, g \rangle = \int_{\mathbb{R}^n} f^\varepsilon(x) g(x)dx
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(x-y) f(y)dy g(x)dx
\]
\[
= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} \varphi(x-y) g(x)dx dy
\]
\[
= \int_{\mathbb{R}^n} f(y) g^\varepsilon(y)dy
\]
\[
= \langle f, g^\varepsilon \rangle,
\]
that yields the proof of (i).

Lemma 3.4 Let \( \varepsilon > 0 \), \( \alpha \in (0, 1) \) and \( 1 \leq p \leq \infty \). Then for any function \( f \) belongs to \( L^1 \cap L^\infty \cap W^{\alpha,p}(\mathbb{R}^n) \), there exists a constant \( C \) such that
\[
\|f^\varepsilon - f\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^\alpha \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x-y| \leq 2\varepsilon} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}dxdy \right)^{\frac{1}{p}}, \quad (3.14)
\]
and
\[
\|\nabla f^\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^{\alpha-1} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x-y| \leq 2\varepsilon} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}dxdy \right)^{\frac{1}{p}}. \quad (3.15)
\]
Proof. For any \( y \in \mathbb{R}^n \), by the definition of \( \varrho_\varepsilon \) in (3.2) and using Hölder inequality, we obtain the following estimation
\[
|f^\varepsilon(y) - f(y)| = \left| \int_{\mathbb{R}^n} \varrho_\varepsilon(y - x)(f(x) - f(y))dx \right|
\leq C\varepsilon^{-n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} |f(x) - f(y)|dx
\leq C\varepsilon^{-\frac{n}{p}} \left( \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} |f(x) - f(y)|^p dx \right)^{\frac{1}{p}},
\]
where the constant \( C \) depends only on the function \( \varrho \) given in (3.1). It follows that
\[
\|f^\varepsilon - f\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} |f(x) - f(y)|^p dx dy.
\]
Otherwise, by multiplying two sides of this inequality by \( \varepsilon^{-\alpha p} \), it gives
\[
\varepsilon^{-\alpha p} \|f^\varepsilon - f\|_{L^p(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} |f(x) - f(y)|^p dx dy
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} |f(x) - f(y)|^p dx dy,
\]
which deduces the first inequality (3.14).
In order to obtain the second estimation, it will be necessary to remark that
\[
|\nabla f^\varepsilon(y)| = \left| \int_{\mathbb{R}^n} \nabla \varrho_\varepsilon(y - x)(f(x) - f(y))dx \right|
\leq \frac{C}{\varepsilon^{n+1}} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} |f(x) - f(y)| dx.
\]
and the same proof of (3.14), we obtain (3.15) the desired result.

\[\Box\]

Remark 3.5 For all \( f \in W^{\alpha,p}(\mathbb{R}^n) \), one can see that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha p}} dx dy \leq \|f\|_{W^{\alpha,p}(\mathbb{R}^n)}, \tag{3.16}
\]
Therefore, as the consequences of Lemma 3.4 one also obtains
\[
\|f^\varepsilon - f\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^{\alpha} \|f\|_{W^{\alpha,p}(\mathbb{R}^n)},
\]
and
\[
\|\nabla f^\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^{\alpha - 1} \|f\|_{W^{\alpha,p}(\mathbb{R}^n)}.
\]
For every function \( f \in W^{\alpha,p}(\mathbb{R}^n \times \mathbb{R}^n) \), we define a function \( \Theta_f \) as
\[
\Theta_f(\varepsilon) := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\varepsilon \leq |x - y| \leq 2\varepsilon} \frac{|f(x, \xi) - f(y, \xi)|^p}{|x - y|^{n+\alpha p}} dx dy d\xi \right)^{\frac{1}{p}}, \tag{3.17}
\]
the following Lemma is then stated and proved to give us a very important property related to this function.
Corollary 3.6 Let $\alpha \in (0, 1)$, $1 \leq p \leq \infty$ and the function $f \in W^{\alpha,p}(\mathbb{R}^n \times \mathbb{R}^n)$. Then for any $\varepsilon, \delta > 0$, there exists a constant $C$ such that

$$\|\nabla_x f^\varepsilon(x, \xi - w) - \nabla_x f^\varepsilon(x, \xi)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\varepsilon^{\alpha-1}|w|^\alpha \Theta_f(\varepsilon),$$

(3.18)

with $w \in \mathbb{R}^n$, and

$$\|(\nabla_x f^\varepsilon)^\delta - \nabla_x f^\varepsilon\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\varepsilon^{\alpha-1}\delta^\alpha \Theta_f(\varepsilon),$$

(3.19)

where the function $\Theta_f$ is defined by (3.17).

Proof. From Lemma 3.3, there exists a constant $C$ such that

$$\|\nabla_x f^\varepsilon(x, \xi - w) - \nabla_x f^\varepsilon(x, \xi)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C|w|^\alpha \|\nabla_x f^\varepsilon\|_{L^p(\mathbb{R}^n; W^{\alpha,p}(\mathbb{R}^n))}.$$  

The inequality (3.15) in Lemma 3.4 is then applied to get

$$\|\nabla_x f^\varepsilon(x, \xi - w) - \nabla_x f^\varepsilon(x, \xi)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\varepsilon^{\alpha-1}|w|^\alpha \Theta_f(\varepsilon).$$

To deal with the second estimation (3.19), by what obtained in Lemma 3.4 and Remark 3.5, there exists a constant $C$ such that

$$\|(\nabla_x f^\varepsilon)^\delta - \nabla_x f^\varepsilon\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\varepsilon^{\alpha-1}\delta^\alpha \Theta_f(\varepsilon).$$

Repeated application of the inequality (3.15) in Lemma 3.4 enables us to write

$$\|(\nabla_x f^\varepsilon)^\delta - \nabla_x f^\varepsilon\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\varepsilon^{\alpha-1}\delta^\alpha \Theta_f(\varepsilon),$$

and the proof is complete.

Lemma 3.7 Let $\alpha \in (0, 1)$, $1 \leq p \leq \infty$ and the function $f \in L^1(0,T; W^{\alpha,p}(\mathbb{R}^n \times \mathbb{R}^n))$. Then $\omega_f(\varepsilon, \delta)$ defined by

$$\omega_f(\varepsilon, \delta) := \int_0^T (\Theta_{f(t)}(\varepsilon) + \Theta_{f(t)}(\delta))dt,$$

(3.20)

vanishes as $\varepsilon$ and $\delta$ tend to 0.

Proof. By the definition of $\Theta_f$ in (3.17) and Remark 3.5, we have

$$\Theta_{f(t)}(\varepsilon) \leq \|f(t, \cdot, \cdot)\|_{W^{\alpha,p}(\mathbb{R}^n \times \mathbb{R}^n)} < \infty, \quad \forall t \in [0, T].$$

Apply Lebesgue dominated convergence theorem, it is clear that $\Theta_{f(t)}(\varepsilon)$ tends to 0 as passing $\varepsilon$ goes to 0 for all $t \in [0, T]$. The same conclusion is obtained for $\Theta_{f(t)}(\delta)$, and this guarantees that $\omega_f(\varepsilon, \delta)$ given by (3.20) vanishes as $(\varepsilon, \delta)$ goes to 0.
4 Proof of Theorem 2.2

In this section, we consider a global in time weak solution \((u, E, B)\) of Vlasov-Maxwell equations. The weak formulation for the Vlasov equation (1.1) reads

\[
\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} u(\partial_t \varphi + v \cdot \nabla_x \varphi + \mathbf{F} \cdot \nabla \xi \varphi) \, d\xi = 0,
\]

for all \(\varphi \in \mathcal{D}((0, T) \times \mathbb{R}^6)\). Let us choose a test function \(\varphi\) as follows

\[
\varphi = (G'(u^{n, \varepsilon, \delta}) \psi)^{n, \varepsilon, \delta} \in \mathcal{D}((0, T) \times \mathbb{R}^6),
\]

where \(\psi \in \mathcal{D}((0, T) \times \mathbb{R}^6)\) and \(G \in C^1(\mathbb{R}^+; \mathbb{R}^+)\). Integrating by parts this weak formulation yields that for all \(\psi \in \mathcal{D}((0, T) \times \mathbb{R}^6)\), there holds

\[
\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi G(u^{n, \varepsilon, \delta}) \left( \partial_t \psi + v^\delta \cdot \nabla_x \psi + \mathbf{F}^{n, \varepsilon, \delta} \cdot \nabla \xi \psi \right) + \psi G'(u^{n, \varepsilon, \delta}) \left[ \nabla_x \cdot \left( (vu)_{n, \varepsilon, \delta} - v^\delta u^{n, \varepsilon, \delta} \right) + \nabla_x \cdot \left( (\mathbf{F} u)_{n, \varepsilon, \delta} - \mathbf{F}^{n, \varepsilon, \delta} u^{n, \varepsilon, \delta} \right) \right] = 0. \quad (4.1)
\]

Following the renormalization property of solution \((u, E, B)\), it is sufficient to show that the second term in the left hand side of (4.1) vanishes as \((\eta, \varepsilon, \delta)\) tends to 0, for all \(\psi \in \mathcal{D}((0, T) \times \mathbb{R}^6)\). To do so, we firstly establish some commutator estimations which are presented in the next lemma.

For simplicity, the problem is considered with \(\alpha \in (0, 1)\) and \(1 \leq p, r \leq \infty\), with \(n = 3\) or \(n = 6\) and \(s = 1\) or \(s = \infty\), we will use the following notations in the remain part of our paper,

\[
\begin{align*}
\mathcal{L}_n^{s,p} &:= L^s(0, T; L^p(\mathbb{R}^n)), \\
\mathcal{L}_n^{s,p,r} &:= L^s(0, T; L^p(\mathbb{R}^3; L^r(\mathbb{R}^3))), \\
\mathcal{L}_n^{s,W^\alpha,p} &:= L^s(0, T; W^\alpha,p(\mathbb{R}^n)), \\
\mathcal{L}_n^{s,p,W^\alpha,p} &:= L^s(0, T; L^p(\mathbb{R}^3; W^\alpha,p(\mathbb{R}^3))).
\end{align*}
\]

Lemma 4.1 Let \((u, E, B)\) be a weak solution of the relativistic Vlasov-Maxwell system (1.1)-(1.6) given by Theorem 2.1 satisfying the regularity assumptions (2.7) of Theorem 2.2 with \(\alpha, \beta \in (0, 1)\) and \(p, q, r\) satisfy relations (2.8). Then for any positive numbers \(\eta, \varepsilon, \delta > 0\), there exists a constant \(C > 0\) depending only on the smooth function \(\varrho\) given by (3.11) such that

\[
\left\| \nabla_x \cdot \left( (vu)_{n, \varepsilon, \delta} - v^\delta u^{n, \varepsilon, \delta} \right) \right\|_{\mathcal{L}_n^{1,p}} \leq C \varepsilon^{-1} \delta^{\alpha+1} \omega_u(\varepsilon, \delta). \quad (4.2)
\]

Moreover, there exists a constant \(C_F > 0\) depending on \(\varrho\), \(\|u\|_{\mathcal{L}_1^{1,W^{\alpha,p}}}, \|E\|_{\mathcal{L}_\infty W^{\beta,q}_{\beta,q}}\) and \(\|B\|_{\mathcal{L}_\infty W^{\beta,q}_{\beta,q}}\) such that

\[
\left\| \nabla_\xi \cdot \left( (u\mathbf{F})_{n, \varepsilon, \delta} - \mathbf{F}^{n, \varepsilon, \delta} u^{n, \varepsilon, \delta} \right) \right\|_{\mathcal{L}_1^{1,p,r}} \leq C_F \left( \varepsilon^{\alpha+\beta} \delta^{\alpha-1} \omega_u(\varepsilon, \delta) + \delta^{\alpha} \right), \quad (4.3)
\]

where \(\mathcal{F} := E + v \times B\) is the Lorentz force field and the function \(\omega_u\) is given by (3.20).
Proof. We first consider the commutator estimate (4.2) for the free streaming term. It is easy to check that
\[(v u)_{\eta,\varepsilon,\delta} - v^{\delta} u_{\eta,\varepsilon,\delta} = K_\delta(v, u_{\eta,\varepsilon}) - (u^{\eta,\varepsilon,\delta} - u^{\eta,\varepsilon})(v - v^{\delta}), \tag{4.4}\]
where \(K_\delta\) is defined by
\[K_\delta(v, g)(t, x, \xi) = \hat{R}_3 \hat{\rho}_\delta(w)(v(\xi - w) - v(\xi)) \cdot (g(t, x, \xi - w) - g(t, x, \xi)) \, dw. \tag{4.5}\]
Passing to the limit \(\eta \to 0\) on the right hand side of (4.4) which can be justified by the Lebesgue dominated convergence theorem and regularity assumptions (2.7), we thus get that
\[
\|\nabla_x \cdot (v u)_{\eta,\varepsilon,\delta} - v^{\delta} u_{\eta,\varepsilon,\delta}\|_{L^1_{t,x}, p}^p \leq \|\nabla_x \cdot K_\delta(v, u^\varepsilon)\|_{L^1_{t,x}, p} + \|\nabla_x \cdot ((u^{\varepsilon,\delta} - u^\varepsilon)(v - v^{\delta}))\|_{L^1_{t,x}, p}. \tag{4.6}\]
By the definition of \(K_\delta\) in (4.5), one has
\[
\|\nabla_x \cdot K_\delta(v, u^\varepsilon)\|_{L^1_{t,x}, p} \leq \int_0^T \int_{\mathbb{R}^3} d\omega q_\delta w \int_0^T \Theta_{u(t)}(\varepsilon) dt \leq C \varepsilon^{-1/\alpha} \delta^{\alpha+1} \Theta_{u(t)}(\varepsilon), \tag{4.7}\]
where \(\omega_u\) given in (3.20). Additionally, from (3.7) and (3.19), there holds
\[
\|\nabla_x \cdot ((u^{\varepsilon,\delta} - u^\varepsilon)(v - v^{\delta}))\|_{L^1_{t,x}, p} \leq \|v - v^{\delta}\|_{L^1_{t,x}, p} \leq C \varepsilon^{-1/\alpha} \delta^{\alpha+1} \Theta_{u(t)}(\varepsilon) dt \leq C \varepsilon^{-1/\alpha} \delta^{\alpha+1} \omega_u(\varepsilon, \delta). \tag{4.8}\]
From what have already been proved, we obtain commutator estimate (4.2).

It remains to prove the estimate in (4.3). To establish this commutator estimate for the Lorentz force term, it is possible for us to make the following decomposition as follows
\[
(F u)_{\eta,\varepsilon,\delta} - F u_{\eta,\varepsilon,\delta} = K_{\eta,\varepsilon}(E, u^\delta) - (E - E^{\eta,\varepsilon})(u^\delta - (u^\delta)^{\eta,\varepsilon}) + (v \times B u)^{\varepsilon,\delta} - v^{\delta} \times B^{x} u^{x,\delta}, \tag{4.9}\]
\[12\]
where $\mathcal{K}_{\eta,\varepsilon}$ is given by

$$
\mathcal{K}_{\eta,\varepsilon}(E, g)(t, x, \xi) = \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^3} dy \, g_\eta(\tau) g_\varepsilon(y) \\
\cdot \{(E(t - \tau, x - y, \xi) - E(t, x, \xi)) (g(t - \tau, x - y, \xi) - g(t, x, \xi))\}.
$$

(4.10)

For the sake of simplicity, in this work we will denote

$$
T_E := \mathcal{K}_{\eta,\varepsilon}(E, u^\delta) - (E - E^\varepsilon)(u^\delta - (u^\delta)^\varepsilon),
$$

(4.11)

and make the effective use of the Lebesgue dominated convergence theorem together with regularity assumptions (2.7), passing to the limit $\eta \to 0$ in $T_E$, yields that

$$
\|\nabla_\xi \cdot T_E\|_{L^{1, p, r}} \leq \|\nabla_\xi \cdot \mathcal{K}_{\varepsilon}(E, u^\delta)\|_{L^{1, p, r}} + \|\nabla_\xi \cdot ((E - E^\varepsilon)(u^\delta - (u^\delta)^\varepsilon))\|_{L^{1, p, r}}.
$$

(4.13)

By Hölder inequality, there holds

$$
\|\nabla_\xi \cdot \mathcal{K}_{\varepsilon}(E, u^\delta)\|_{L^{1, p, r}} \leq \int_{\mathbb{R}^3} \varrho_\varepsilon(y) \|E(t, x - y) - E(t, x)\|
\cdot \|\nabla_\xi u^\delta(t, x - y, \xi) - \nabla_\xi u^\delta(t, x, \xi)\|_{L^{1, p, r}} dy
\leq \int_{\mathbb{R}^3} \varrho_\varepsilon(y) \|E(t, x - y) - E(t, x)\|_{L^{\infty, \eta}}
\cdot \|\nabla_\xi u^\delta(t, x - y, \xi) - \nabla_\xi u^\delta(t, x, \xi)\|_{L^{1, p, r}} dy.
$$

Applying the estimate (3.13) in Lemma 3.3 and the regularity assumptions (2.7), we obtain that

$$
\|\nabla_\xi \cdot \mathcal{K}_{\varepsilon}(E, u^\delta)\|_{L^{1, p, r}} \leq C \int_{\mathbb{R}^3} \varrho_\varepsilon(y) |y|^{\alpha + \beta} \|E\|_{L^{\infty, W^{\alpha, \eta}_3}} \|\nabla_\xi u^\delta(t, x, \xi)\|_{L^{1, p, W^{\alpha, r}}} dy
\leq C \varepsilon^{\alpha + \beta} \|E\|_{L^{\infty, W^{\alpha, \eta}_3}} \|\nabla_\xi u^\delta(t, x, \xi)\|_{L^{1, p, W^{\alpha, r}}}.
$$

Thanks to (3.13) from Lemma 3.4, we have

$$
\|\nabla_\xi \cdot \mathcal{K}_{\varepsilon}(E, u^\delta)\|_{L^{1, p, r}} \leq C \varepsilon^{\alpha + \beta} \delta^{\alpha - 1} \|E\|_{L^{\infty, W^{\alpha, \eta}_3}} \int_0^T \Theta_u(t)(\delta) dt
\leq C \varepsilon^{\alpha + \beta} \delta^{\alpha - 1} \|E\|_{L^{\infty, W^{\alpha, \eta}_3}} \omega_u(\varepsilon, \delta).
$$

(4.14)

And the second term on the right hand side of (4.13) is then proved thanks to Hölder inequality,

$$
\|\nabla_\xi \cdot ((E - E^\varepsilon)(u^\delta - (u^\delta)^\varepsilon))\|_{L^{1, p, r}} \leq \|E - E^\varepsilon\|_{L^{\infty, \eta}} \|\nabla_\xi u^\delta - (\nabla_\xi u^\delta)^\varepsilon\|_{L^{1, p}}.
$$

Then, from (3.14) and (3.19), it deduces that

$$
\|\nabla_\xi \cdot ((E - E^\varepsilon)(u^\delta - (u^\delta)^\varepsilon))\|_{L^{1, p, r}} \leq C \varepsilon^{\alpha + \beta} \delta^{\alpha - 1} \|E\|_{L^{\infty, W^{\alpha, \eta}_3}} \int_0^T \Theta_u(t)(\varepsilon) dt
\leq C \varepsilon^{\alpha + \beta} \delta^{\alpha - 1} \|E\|_{L^{\infty, W^{\alpha, \eta}_3}} \omega_u(\varepsilon, \delta).
$$

(4.15)
From what have already been proved in (4.13), (4.14) and (4.15), we get
\[
\|\nabla_\xi \cdot T_E\|_{L^{1,p,r}} \leq C \varepsilon^\alpha \delta^\alpha - 1 \|E\|_{L^{\infty \gamma}} \|B\|_{L^{\infty ,q}} \int_0^T (\Theta_u(t)(\delta) + \Theta_\alpha(t)(\varepsilon)) dt
\lesssim C \varepsilon^\alpha \delta^\alpha - 1 \|E\|_{L^{\infty \gamma}} \|B\|_{L^{\infty ,q}} \omega_u(\varepsilon, \delta).
\] (4.16)

We next consider the term \(T_B\) given by (4.12), which can be decomposed as
\[
T_B = T_{B1} + T_{B2} + T_{B3},
\] (4.17)
where
\[
T_{B1} := \int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw \ \varrho_\eta(\tau) \varrho_\varepsilon(y) \varrho_\delta(w)
\left[(v(\xi - w) - v(\xi)) \times B(t - \tau, x - y)\right]u(t - \tau, x - y, \xi, w),
\]
\[
T_{B2} := v \times (B^\delta \eta, \varepsilon - B^\eta, \varepsilon(u^\delta \eta, \varepsilon),
\]
\[
T_{B3} := (v - v^\delta) \times B^\eta, \varepsilon(u^\delta \eta, \varepsilon).
\]
Let us now denote by
\[
\int dX = \int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw
\]
for simplicity of notations, the first term of (4.17) can be decomposed and rewritten as follows
\[
\nabla_\xi \cdot T_{B1} = \int dX \ \varrho_\eta(\tau) \varrho_\varepsilon(y) \nabla_w \varrho_\delta(w)
\cdot \left[(v(\xi - w) - v(\xi)) \times B(t - \tau, x - y)\right]u(t - \tau, x - y, \xi)
\]
\[
+ \int dX \ \varrho_\eta(\tau) \varrho_\varepsilon(y) \nabla_w \varrho_\delta(w) \cdot \left[(v(\xi - w) - v(\xi)) \times B(t - \tau, x - y)\right]
\cdot \left[u(t - \tau, x - y, \xi - w) - u(t - \tau, x - y, \xi)\right] =: I_1 + I_2.
\] (4.18)

Integrating by parts and since \(\nabla_w \cdot [(v(\xi - w) - v(\xi)) \times B(t - \tau, x - y)] = 0\), one observes that the first term also vanishes:
\[
I_1 = \int dX \ \varrho_\eta(\tau) \varrho_\varepsilon(y) \varrho_\delta(w)
\nabla_w \cdot [(v(\xi - w) - v(\xi)) \times B(t - \tau, x - y)]u(t - \tau, x - y, \xi) = 0,
\] (4.19)
and by Hölder inequality and estimate (3.16) in Lemma 3.1 it is easy to obtain that
\[
\|I_2\|_{L^{1,p,r}} \leq 2 \int dX \ \varrho_\eta(\tau) \varrho_\varepsilon(y) \nabla_w \varrho_\delta(w) \|w\|B_{L^{\infty ,q}}
\|u(t - \tau, x - y, \xi - w) - u(t - \tau, x - y, \xi)\|_{L^{1,p}}.
\]
We then apply the estimate (3.13) in Lemma 5.3 the restriction property for Sobolev spaces \(W^{\alpha,p}(\mathbb{R}^n)\) and regularity assumptions (2.7), it deduces from the above inequality that
\[
\|I_2\|_{L^{1,p,r}} \leq C \int_{\mathbb{R}^3} \|\nabla_w \varrho_\delta(w)\|w|^\alpha + 1\|B\|_{L^{\infty ,q}}\|u\|_{L^{1,p}}\|W^{\alpha,p}
\leq C \delta^\alpha \|B\|_{L^{\infty \gamma}} \|u\|_{L^{1,p}}\|W^{\alpha,p}.
\] (4.20)
It follows easily that from (4.18), (4.19) and (4.20), one has
\[ \| \nabla \xi \cdot T_{B_1} \|_{L^{1,p,r}} \leq C \delta^\alpha \| B \|_{L^\infty W_3^{\beta,q}} \| u \|_{L^1 W_6^{\alpha,p}}. \] (4.21)
To estimate \( \nabla \xi \cdot T_{B_2} \), we can now proceed analogously to what we have obtained in (4.16) for \( \nabla \xi \cdot T_E \), giving
\[ \| \nabla \xi \cdot T_{B_2} \|_{L^{1,p,r}} \leq C \varepsilon^{\alpha+\beta} \delta^{\alpha-1} \| B \|_{L^\infty W_3^{\beta,q}} \int_0^T (\Theta u(t)(\delta) + \Theta u(t)(\varepsilon)) dt \]
\[ \leq C \varepsilon^{\alpha+\beta} \delta^{\alpha-1} \| B \|_{L^\infty W_3^{\beta,q}} \omega_u(\varepsilon,\delta). \] (4.22)
Hölder inequality is used repeatedly to obtain
\[ \| \nabla \xi \cdot T_{B_3} \|_{L^{1,p,r}} \leq \| v - v^\delta \| \| B^{\eta,\varepsilon} \|_{L^\infty W_3^{\beta,q}} \| \nabla \xi u^{\eta,\delta,\varepsilon} \|_{L^{1,p}}. \]
Applying estimate (3.7) in Lemma 3.1 and Lemma 3.3 to this inequality, we have
\[ \| \nabla \xi \cdot T_{B_3} \|_{L^{1,p,r}} \leq C \delta \| B^{\eta,\varepsilon} \|_{L^\infty W_3^{\beta,q}} \| \nabla \xi u^{\eta,\delta,\varepsilon} \|_{L^{1,p}} \leq C \delta^\alpha \| B \|_{L^\infty W_3^{\beta,q}} \| u \|_{L^1 W_6^{\alpha,p}}. \] (4.23)
Gathering estimates (4.21), (4.22), we obtain from (4.17) that
\[ \| \nabla \xi \cdot T_{B} \|_{L^{1,p,r}} \leq C \varepsilon^{\alpha+\beta} \delta^{\alpha-1} \| B \|_{L^\infty W_3^{\beta,q}} \int_0^T (\Theta u(t)(\delta) + \Theta u(t)(\varepsilon)) dt \]
\[ + C \delta^\alpha \| B \|_{L^\infty W_3^{\beta,q}} \| u \|_{L^1 W_6^{\alpha,p}} \leq C \| B \|_{L^\infty W_3^{\beta,q}} \left( \varepsilon^{\alpha+\beta} \delta^{\alpha-1} \omega_u(\varepsilon,\delta) + \delta^\alpha \| u \|_{L^1 W_6^{\alpha,p}} \right). \] (4.24)
Finally, by estimates (4.16) and (4.24), we obtain (13) from (14) and therefore, the proof of Lemma is then complete.

**Proof of Theorem 2.2** We firstly use the notation
\[ \int dX = \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \]
for simplicity, the weak formulation for the Vlasov equation (1.1) reads
\[ \int dX \ u(\partial_t \varphi + v \cdot \nabla_x \varphi + F \cdot \nabla \xi \varphi) = 0, \quad \forall \varphi \in \mathcal{D}((0,T) \times \mathbb{R}^6), \] (4.25)
where \( F = E + v \times B \) denotes the Lorentz force field. We remark that integrals in (1.24) are finite since for DiPena-Lions weak solutions in \( \mathcal{S} \), it is known that \( u \in L^{\infty,2}_6 \) and \( E, B \in L^{\infty,2}_3 \). For every positive numbers \( \eta, \varepsilon \) and \( \delta \), let us take the test function in (4.26) as
\[ \varphi = (G'(u^{\eta,\varepsilon,\delta})(t)^{\eta,\varepsilon,\delta} \in \mathcal{D}((0,T) \times \mathbb{R}^6), \] (4.26)
with $\psi \in \mathcal{D}((0, T) \times \mathbb{R}^6)$ and $G \in C^1(\mathbb{R}^+; \mathbb{R}^+)$. By using Lemma 3.3 and successive integrations by parts, we obtain from (4.25) and (4.26) that

$$
\mathcal{D} G(u^{\eta, \varepsilon, \delta}) \left( \partial_t \psi + v^\delta \cdot \nabla_x \psi + \mathcal{F}^{\eta, \varepsilon, \delta} \cdot \nabla \xi \psi \right) + \psi \mathcal{D} G(u^{\eta, \varepsilon, \delta}) \left[ \nabla_x \cdot \left( (vu)^{\eta, \varepsilon, \delta} - v^\delta u^{\eta, \varepsilon, \delta} \right) + \nabla \xi \cdot \left( (\mathcal{F}u)^{\eta, \varepsilon, \delta} - \mathcal{F}^{\eta, \varepsilon, \delta} u^{\eta, \varepsilon, \delta} \right) \right] = 0,
$$

(4.27)

for all $\psi \in \mathcal{D}((0, T) \times \mathbb{R}^6)$. We now establish the renormalized Vlasov equation (2.3). Using regularity assumptions (2.7), Lemma 3.3, 3.6 and 4.1, we obtain that

$$
\left| \mathcal{D} G(u^{\eta, \varepsilon, \delta}) \left( \partial_t \psi + v^\delta \cdot \nabla_x \psi + \mathcal{F}^{\eta, \varepsilon, \delta} \cdot \nabla \xi \psi \right) \right| \leq C \left( \varepsilon^{\alpha-1} \delta^{\alpha+1} + \varepsilon^{\alpha+\beta} \delta^{\alpha-1} \right) \omega_u(\varepsilon, \delta) + C \delta^\alpha,
$$

(4.28)

where the function $\omega$ given in (3.20) and the constant $C$ depends on $\|u\|_{L^1 W^{\alpha,p}_6}, \|B\|_{L^\infty W^{9, q}_3}, \|E\|_{L^\infty W^{9, q}_3}, G$ and $\psi$. We see that

$$
\varepsilon^{\alpha-1} \delta^{\alpha+1} + \varepsilon^{\alpha+\beta} \delta^{\alpha-1} = \varepsilon^{\alpha-1} \delta^{\alpha-1} \left( \delta^{2} + \varepsilon^{\beta+1} \right).
$$

Therefore, to balance contributions coming from the free streaming and Lorentz force terms in the right hand side of (4.28), we may choose

$$
\delta^2 = \varepsilon^{\beta+1},
$$

which guarantees that

$$
\left| \mathcal{D} G(u^{\eta, \varepsilon, \delta}) \left( \partial_t \psi + v^\delta \cdot \nabla_x \psi + \mathcal{F}^{\eta, \varepsilon, \delta} \cdot \nabla \xi \psi \right) \right| \leq C \left( \varepsilon^{\alpha+\beta+3\alpha-1} \omega_u(\varepsilon, \delta) + \delta^\alpha \right).
$$

Under our general assumption $\alpha \beta + \beta + 3\alpha - 1 \geq 0$, we deduce that

$$
\left| \mathcal{D} G(u^{\eta, \varepsilon, \delta}) \left( \partial_t \psi + v^\delta \cdot \nabla_x \psi + \mathcal{F}^{\eta, \varepsilon, \delta} \cdot \nabla \xi \psi \right) \right| \leq C \left( \omega_u(\varepsilon, \delta) + \delta^\alpha \right).
$$

(4.29)

Thanks to Lemma 3.7 and $\alpha \in (0, 1)$, the right hand side of (4.29) vanishes as $(\varepsilon, \delta)$ goes to 0. So we obtain the renormalization property (2.3) of the Vlasov equation.

We next establish the local in space entropy conservation law (2.4). For this purpose, we restrict entropy function $G \in \mathcal{S}$, this means $G$ is non decreasing function in $C^1(\mathbb{R}^+; \mathbb{R}^+)$ such that

$$
\lim_{t \to \infty} \frac{G(t)}{t} = \infty.
$$

Let us first take a function $\Gamma \in \mathcal{D}(\mathbb{R}^3)$ such that supp$(\Gamma) \subset B_2(0)$, $\Gamma \equiv 1$ on $B_1(0)$ and $0 \leq \Gamma \leq 1$ on $B_2(0) \setminus B_1(0)$, where $B_r(0)$ denotes the ball of radius $r$ and centered at 0 in $\mathbb{R}^3$. Then we introduce a function $\Gamma$ by

$$
\Gamma_R(\xi) = \Gamma \left( \frac{\xi}{R} \right), \quad \text{with } R > 0.
$$
It is obvious to check that $\Gamma_R \in D(\mathbb{R}^3)$ and
\[ \Gamma_R \to 1 \text{ and } \nabla \xi \Gamma_R \to 0, \text{ a.e. as } R \to \infty. \quad (4.30) \]

Now we choose a test function $\psi$ in (4.29) such that
\[ \psi(t, x, \xi) = \mu(t, x) \Gamma_R(\xi), \quad \text{with } \mu \in D((0, T) \times \mathbb{R}^3). \]

By assumption that the map $t \mapsto u(t, \cdot, \cdot)$ is uniformly integrable in $\mathbb{R}^6$, for almost everywhere $t \in [0, T]$, and the de La Vallée Poussin theorem, there exists a constant $C_G$ only depending on the entropy $G$ such that
\[ \int_{\mathbb{R}^3} dX \int_{\mathbb{R}^3} d\xi \ G(u^{\eta, \varepsilon, \delta}) \leq C_G < \infty. \quad (4.31) \]

Applying the Lebesgue dominated convergence theorem under the estimate (4.31), (4.30) and regularity assumptions (2.7), we obtain that
\[ \int_{\mathbb{R}^3} dX \ G(u^{\eta, \varepsilon, \delta}) \partial_t \mu \Gamma_R \to \int_{\mathbb{R}^3} dX \ G(u^{\eta, \varepsilon, \delta}) \partial_t \mu, \quad \text{as } R \to \infty, \quad (4.32) \]
\[ \int_{\mathbb{R}^3} dX \ G(u^{\eta, \varepsilon, \delta}) v^\delta \cdot \nabla x \mu \Gamma_R \to \int_{\mathbb{R}^3} dX \ G(u^{\eta, \varepsilon, \delta}) v^\delta \cdot \nabla x \mu, \quad \text{as } R \to \infty, \quad (4.33) \]
and
\[ \int_{\mathbb{R}^3} dX \ G(u^{\eta, \varepsilon, \delta}) F^{\eta, \varepsilon, \delta}_L u^{\eta, \varepsilon, \delta} \cdot \nabla \xi \Gamma_R \mu \to 0, \quad \text{as } R \to \infty. \quad (4.34) \]

Limits (4.32)-(4.34) are uniform in $(\eta, \varepsilon, \delta)$ and there exists a constant $c_1$ only depending on $\|u\|_{L^\infty_{t,x}}$, $\|B\|_{L^\infty_{t,x}}$, $\|E\|_{L^\infty_{t,x}}$, $\mu$ and $\Gamma$ such that
\[ \left| \int_{\mathbb{R}^3} dX \ G(u^{\eta, \varepsilon, \delta}) F^{\eta, \varepsilon, \delta}_L u^{\eta, \varepsilon, \delta} \cdot \nabla \xi \Gamma_R \mu \right| \leq c_1 R^{-1}. \quad (4.35) \]

Gathering estimates (4.32), (4.35), one obtains from (4.29) that
\[ \left| \int_{\mathbb{R}^3} dX \left( \partial_t \mu + v^\delta \cdot \nabla x \mu \right) G(u^{\eta, \varepsilon, \delta}) \right| \leq C(\omega_u(\varepsilon, \delta) + \delta^\alpha) + c_1 R^{-1}, \quad (4.36) \]
under the assumption $\alpha \beta + \beta + 3\alpha - 1 \geq 0$. Thanks to Lemma 3.7 again, the right hand side of (4.36) vanishes as $(\varepsilon, \delta) \to 0$ and $R \to \infty$. It deduces that the local in space conservation law (2.4) holds. The local momentum conservation law (2.5) can be obtained in a similar way.

The final task is now to establish the global entropy conservation law (2.6). Let us take a test function $\mu$ in (4.36) such that
\[ \mu(t, x) = \sigma(t) \Gamma_R(x), \quad \text{with } \sigma \in D((0, T)), \]
where $\Gamma_R$ is defined the the previous proof for local conservation laws. By (4.31)-(4.30) and regularity assumption (2.7), we may apply the Lebesgue dominated convergence theorem to obtain that

$$
\hat{d}X \mathcal{G}(u^{\eta,\varepsilon,\delta}) \partial_t \sigma \Gamma_R \to \hat{d}X \mathcal{G}(u^{\eta,\varepsilon,\delta}) \partial_t \sigma, \quad \text{as} \quad R \to \infty,
$$

(4.37)

$$
\int dX \mathcal{G}(u^{\eta,\varepsilon,\delta}) v^\delta \cdot \nabla_x \Gamma_R \sigma \to 0, \quad \text{as} \quad R \to \infty.
$$

(4.38)

Limits (4.37)-(4.38) are uniform in $(\eta, \varepsilon, \delta)$ and there exists a constant $c_2 > 0$ only depending on $C_G, \sigma$ and $\Gamma$ such that

$$
\left| \int dX \mathcal{G}(u^{\eta,\varepsilon,\delta}) v^\delta \cdot \nabla_x \Gamma_R \sigma \right| \leq c_2 R^{-1}.
$$

(4.39)

Combining between (4.36) to (4.37)-(4.39), we obtain that

$$
\left| \int dX \partial_t \sigma \mathcal{G}(u^{\eta,\varepsilon,\delta}) \right| \leq C(\omega_u(\varepsilon, \delta) + \delta^\alpha) + (c_1 + c_2) R^{-1},
$$

(4.40)

under the condition $\alpha \beta + \beta + 3\alpha - 1 \geq 0$. The global entropy conservation law (2.5) holds since the right hand side of (4.40) vanishes as $(\varepsilon, \delta) \to 0$ and $R \to \infty$ by Lemma 3.7. The proof of Theorem 2.2 is complete.

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