Routh-Hurwitz criterion of stability and robust stability for fractional-order systems with order $\alpha \in [1, 2)$

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**Abstract**

Based on the generalized Routh-Hurwitz criterion, we propose a sufficient and necessary criterion for testing the stability of fractional-order linear systems with order $\alpha \in [1, 2)$, called the fractional-order Routh-Hurwitz criterion. Compared with the existing criterion, our one involves fewer and simpler expressions, which is significant for analyzing robust stability of fractional-order uncertain systems. All these expressions are explicit ones about the coefficients of the characteristic polynomial of system matrix, so the stable parameter region of fractional-order uncertain systems can be described directly. Some examples show the effectiveness of our method.

**Keywords:** Fractional-order system, Generalized Routh-Hurwitz criterion, Robust stability

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**1. Introduction**

Routh-Hurwitz criterion is an effective method for analyzing the stability of integer-order linear time-invariant systems. In 2006, some sufficient or necessary
stability conditions of fractional-order systems with order \((0, 1]\) were given based on the integer-order Routh-Hurwitz criterion\([1]\). Recently, for dimensions \(n = 2, 3\), some sufficient and necessary stability conditions for the case of order \((0, 2)\) were established; for dimension \(n = 4\), some sufficient or necessary stability conditions for the case of order \((0, 2)\) were given\([2]\). Some researchers proposed a sufficient and necessary stability condition of fractional-order systems with order \(\alpha \in [1, 2]\)\([3]\) which transforms a \(n\)-dimension fractional-order system into a \(2n\)-dimension integer-order system to analyze the stability. This method has beautiful form, but it has high computational complexity, especially for multi-parameter systems.

For fractional-order uncertain systems, some sufficient and necessary methods have been proposed in recent years. For example, \(\mu\)-analysis method to test the robust stability of fractional-order uncertain systems with \(\alpha \in [1, 2]\) can solve the robustness bound of perturbation parameters\([4]\). Moreover, based on cylindrical algebraic decomposition technique, some parameter space algorithms for analyzing the robust stability of fractional-order uncertain systems with \(\alpha \in (0, 2)\) were proposed, with which the range of perturbation parameters can be obtained\([5, 6]\). When we consider the robust stability of multi-parameter systems, with these methods, all robust stability results can not be obtained directly but by very complicated calculations.

In this paper, the problems of stability and robust stability on \(n\)-dimension fractional-order linear systems with \(\alpha \in [1, 2]\) are considered. Based on the generalized Routh-Hurwitz criterion, a sufficient and necessary criterion for testing the stability of fractional-order linear systems with order \(\alpha \in [1, 2]\), called the fractional-order Routh-Hurwitz criterion, is proposed. We also list complete, explicit expressions for \(n = 2, 3, 4\), respectively. Our criterion involves fewer and simpler expressions than the exiting method\([3]\). Meanwhile, all expressions in our method are explicit ones about the coefficients of the characteristic polynomial of system matrix. When we consider the robust stability of fractional-order systems with uncertain multiple parameters, our method can describe stability analysis results more easily than existing methods\([4–8]\).
2. Preliminaries

Let \( f(z) \) be a complex coefficient polynomial and satisfy:

\[
f(iz) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n + i \left( a_0 z^n + a_1 z^{n-1} + \cdots + a_n \right) (a_0 \neq 0),
\]

where \( a_j(j = 1, 2, \cdots, n) \) and \( b_j(j = 1, 2, \cdots, n) \) are real numbers.

The \( 2n \times 2n \) generalized Hurwitz matrix \( H_f \) is constructed from \( f(z) \) as follows:

\[
H_f = \begin{bmatrix}
a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\
b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\
0 & a_0 & \cdots & a_{n-1} & a_n & \cdots & 0 \\
0 & b_0 & \cdots & b_{n-1} & b_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & a_0 & \cdots & a_{n-1} & a_n \\
0 & \cdots & 0 & b_0 & \cdots & b_{n-1} & b_n
\end{bmatrix}.
\]

Lemma 1. \([9]\) (The Generalized Routh-Hurwitz Criterion)

All roots of \( f(z) \) have negative real parts if and only if \( \delta_k > 0 (k = 1, 2, \cdots, n) \), where \( \delta_k (k = 1, 2, \cdots, n) \) is the \( 2k \)-th order leading principle minor of \( H_f \).

Consider the following \( n \)-dimension fractional-order linear time-invariant system:

\[
D^\alpha x = Ax,
\]

where \( \alpha \in [1, 2) \) is fractional order, \( A \in \mathbb{R}^{n \times n} \), \( x = (x_1, x_2, \cdots, x_n)^T \) is state vector.

Lemma 2. \([10–12]\) System(3) is asymptotically stable if and only if \( |\arg(\lambda_j)| > \frac{\alpha \pi}{2} \), where \( \lambda_j(j = 1, 2, \cdots, n) \) are the eigenvalues of matrix \( A \), \( \arg(\cdot) \) denotes the argument of a complex number.

Let

\[
\Omega := \{ \gamma \in \mathbb{C} \mid |\arg(\gamma)| > \frac{\alpha \pi}{2} \}
\]

\[
\Sigma := \{ \gamma \in \mathbb{C} \mid |\arg(\gamma)| < \frac{\alpha \pi}{2} \},
\]

\[
\Gamma := \{ \gamma \in \mathbb{C} \mid |\arg(\gamma)| = \frac{\alpha \pi}{2} \}
\]
call them the stable region, the unstable region and the critical line of system(3), respectively (as shown in Figure 1).

Figure 1: Ω, Σ and Γ of system(3)

Suppose the characteristic polynomial of matrix $A$ is

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$  \hspace{1cm} (5)

So,

$$f \left( \lambda \cdot e^{i \frac{\alpha \pi}{2}} \right) = \sum_{j=0}^{n} a_j \cdot \cos \left( \frac{(n-j) \cdot \alpha \pi}{2} \right) \cdot \lambda^{n-j} + i \sum_{j=0}^{n} a_j \cdot \sin \left( \frac{(n-j) \cdot \alpha \pi}{2} \right) \cdot \lambda^{n-j},$$  \hspace{1cm} (6)

where $a_0 = 1$.

From Lemma 2, we know that system(3) is asymptotically stable if and only if all roots of $f(\lambda)$ are in stable region Ω.

In this paper, based on the generalized Routh-Hurwitz criterion, we propose a sufficient and necessary criterion, called the fractional-order Routh-Hurwitz criterion, to analyze the stability and robust stability of n-dimension fractional-order systems with order $1 \leq \alpha < 2$. Compared with the existing criterion, our one involves fewer and simpler expressions. All expressions in our results are explicit ones about the coefficients of the characteristic polynomial of system matrix, so the stable parameter region of fractional-order uncertain systems can
be described directly.

3. Main Results

In this section, all notations are the same as above.

**Definition 1. (The Fractional-Order Routh-Hurwitz Matrix)**

For system (3), the $2n \times 2n$ fractional-order Routh-Hurwitz matrix $H_\alpha$ of $f(\lambda)$ is defined as follows:

$$H_\alpha = \begin{bmatrix}
    a_0 \sin \left( \frac{n-\alpha}{2} \right) & a_1 \sin \left( \frac{\alpha-1}{2} \right) & \cdots & 0 & \cdots & \cdots & 0 \\
    a_0 \cos \left( \frac{n-\alpha}{2} \right) & a_1 \cos \left( \frac{\alpha-1}{2} \right) & \cdots & a_n & \cdots & \cdots & 0 \\
    0 & a_0 \sin \left( \frac{n-\alpha}{2} \right) & \cdots & a_{n-1} \sin \left( \frac{\alpha}{2} \right) & \cdots & \cdots & 0 \\
    0 & a_0 \cos \left( \frac{n-\alpha}{2} \right) & \cdots & a_{n-1} \cos \left( \frac{\alpha}{2} \right) & a_n & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & \cdots & \cdots & a_0 \sin \left( \frac{n-\alpha}{2} \right) & \cdots & a_{n-1} \sin \left( \frac{\alpha}{2} \right) & 0 \\
    0 & \cdots & \cdots & a_0 \cos \left( \frac{n-\alpha}{2} \right) & \cdots & a_{n-1} \cos \left( \frac{\alpha}{2} \right) & a_n
\end{bmatrix}.$$ (7)

**Theorem 1. (The Fractional-Order Routh-Hurwitz Criterion)**

System (3) is asymptotically stable if and only if $\nabla_p > 0 (p = 1, 2, \cdots, n)$, where $\nabla_p (p = 1, 2, \cdots, n)$ is the 2p-th order leading principle minor of $H_\alpha$.

Proof. The coordinate system $xy$ counterclockwise turns through angle $\theta = \frac{(\alpha-1)\pi}{2}$ as the coordinate system $x'y'$ (as shown in Figure 2).

For system (3), in the new coordinate system $x'y'$, $f(\lambda)$ can be expressed as

$$g(\lambda) = f \left( \lambda \cdot e^{i \frac{\alpha}{2}} \right),$$ (8)

thus

$$g(i\lambda) = f \left( \lambda \cdot e^{i \frac{\alpha}{2}} \right).$$ (9)

Since $f(\lambda)$ is a real polynomial whose roots are symmetrical about the real axis in the coordinate system $xy$, its roots are in the stable region $\Omega$ if and only if they are in the left half plane of the coordinate system $x'y'$. Based on the above analysis, according to Lemma 1 and Lemma 2, we have Theorem 1. \qed
Remark 1. Consider the general 2-dimension fractional-order system as follows:

\[ D^\alpha x = Ax. \]  \hspace{1cm} (10)

where \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \) \( x = (x_1, x_2)^T. \)

System (10) is asymptotically stable if and only if the following integer-order system is asymptotically stable [3],

\[ \dot{x} = \begin{bmatrix} A \sin \left( \frac{\alpha \pi}{2} \right) & A \cos \left( \frac{\alpha \pi}{2} \right) \\ -A \cos \left( \frac{\alpha \pi}{2} \right) & A \sin \left( \frac{\alpha \pi}{2} \right) \end{bmatrix} x. \]  \hspace{1cm} (11)

The characteristic polynomial of system (11) is

\[ P(\lambda) = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4, \]  \hspace{1cm} (12)

where

\[ a_1 = -2(a_{11} + a_{22}) \sin \left( \frac{\alpha \pi}{2} \right), \]
\[ a_2 = (a_{11}^2 + 4a_{11}a_{22} - 2a_{12}a_{21} + a_{22}^2) \sin^2 \left( \frac{\alpha \pi}{2} \right) + (a_{11}^2 + 2a_{12}a_{21} + a_{22}^2) \cos^2 \left( \frac{\alpha \pi}{2} \right), \]
\[ a_3 = -2(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21}) \sin \left( \frac{\alpha \pi}{2} \right), \]
\[ a_4 = (a_{11}a_{12} - a_{12}a_{21})^2. \]  \hspace{1cm} (13)
The integer-order Routh-Hurwitz matrix $H_r$ of $P(\lambda)$ as follows:

$$H_r = \begin{bmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{bmatrix}. \tag{14}$$

Based on the integer-order Routh-Hurwitz criterion, we need to check $\Delta_i > 0 (i = 1, 2, 3, 4)$ to determine the stability of system (11), where $\Delta_i (i = 1, 2, 3, 4)$ is the $i$-th order leading principle minor of $H_r$ and

$$\begin{align*}
\Delta_1 &= a_1, \\
\Delta_2 &= a_1a_2 - a_3, \\
\Delta_3 &= -a_1^2a_4 + a_1a_2a_3 - a_3^2, \\
\Delta_4 &= -a_4(a_1^2a_4 - a_1a_2a_3 + a_3^2). \tag{15}\end{align*}$$

The above integer-order Routh-Hurwitz criterion needs to calculate 4 leading principle minors that are complex.

Consider the same fractional-order system (10), the characteristic polynomial of matrix $A$ is

$$f(\lambda) = \lambda^2 - (a_{11} + a_{22})x + a_{11}a_{22} - a_{12}a_{21}. \tag{16}$$

The fractional-order Routh-Hurwitz matrix $H_\alpha$ of polynomial (16) is as follows:

$$H_\alpha = \begin{bmatrix}
\sin (\alpha\pi) & -(a_{11} + a_{22}) \sin \left(\frac{\alpha\pi}{2}\right) & 0 & 0 \\
\cos (\alpha\pi) & -(a_{11} + a_{22}) \cos \left(\frac{\alpha\pi}{2}\right) & a_{11}a_{22} - a_{12}a_{21} & 0 \\
0 & \sin (\alpha\pi) & -(a_{11} + a_{22}) \sin \left(\frac{\alpha\pi}{2}\right) & 0 \\
0 & \cos (\alpha\pi) & -(a_{11} + a_{22}) \cos \left(\frac{\alpha\pi}{2}\right) & a_{11}a_{22} - a_{12}a_{21} \\
\end{bmatrix}. \tag{17}$$

Based on Theorem 1, we only need to check two even-order leading principle minors $\nabla_p > 0 (p = 1, 2)$ of $H_\alpha$ to determine the stability of system (10), where

$$\begin{align*}
\nabla_1 &= -(a_{11} + a_{22}) \sin \left(\frac{\alpha\pi}{2}\right), \\
\nabla_2 &= (a_{22}a_{11} - a_{21}a_{12}) \sin^2 \left(\frac{\alpha\pi}{2}\right) \left((a_{11} + a_{22})^2 - 4(a_{22}a_{11} - a_{21}a_{12}) \cos^2 \left(\frac{\alpha\pi}{2}\right)\right). \tag{18}\end{align*}$$
In this paper, $1 \leq \alpha < 2$, so $\nabla_1 > 0, \nabla_2 > 0$ is equivalent to
\[
\tilde{\nabla}_1 = -(a_{11} + a_{22}) > 0,
\tilde{\nabla}_2 = (a_{22}a_{11} - a_{21}a_{12}) \left((a_{11} + a_{22})^2 - 4(a_{22}a_{11} - a_{21}a_{12})\cos^2 \left(\frac{\alpha\pi}{2}\right)\right) > 0.
\]
(19)

Compared with the existing method, our one involves smaller numbers of leading principle minors. Each $\nabla_p$ is simpler than $\Delta_i, i = 2p$. Our method has less computational complexity than the existing method[3]. Especially for fractional-order uncertain systems, the advantage of low computational complexity is significant.

For system (3), suppose the characteristic polynomial of $A$ is $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$. Since systems with dimensions $n = 2, 3, 4$ are often used, based on Theorem 1, we have the following corollaries. In the following, always set $\cos^2 \left(\frac{\alpha\pi}{2}\right) = s$.

**Corollary 1.** In the case of $n = 2$, system (3) is asymptotically stable if and only if
\[
a_1 > 0, \quad a_2 \left(a_1^2 - 4a_2s\right) > 0
\]
(20)

**Corollary 2.** In the case of $n = 3$, system (3) is asymptotically stable if and only if
\[
a_1 > 0, \quad (4a_1a_3 - 4a_2^2)s + a_1^2a_2 - a_1a_3 > 0,
\]
\[
a_3 \cdot (64a_2^2s^3 - (16a_2a_3 + 48a_5^2)s^2 + (4a_1^3a_3 - 4a_1a_2a_3 + 4a_2^3 + 12a_3^2)s - a_1^2a_2^2 + 2a_1a_2a_3 - a_3^2) > 0.
\]
(21)

**Corollary 3.** In the case of $n = 4$, system (3) is asymptotically stable if and
only if

\[ a_1 > 0, \quad (4a_1a_3 - 4a_2^3)s + a_1^2a_2 - a_1a_3 > 0, \]

\[
(64a_1a_4^2 - 128a_2a_3a_4 + 64a_3^3)s^3 - (16a_1^2a_3a_4 + 16a_1a_2^2a_4 + 16a_1a_2a_3^2 - 64a_1a_4^2 \\
+ 96a_2a_3a_4 - 48a_3^2)s^2 + (4a_1^3a_2a_4 - 4a_1^2a_3^2 + 8a_1^2a_3a_4 + 4a_1a_2^2a_4 + 4a_1a_2a_3^2 - 4a_3^2a_3 \\
- 16a_1a_4^2 + 16a_2a_3a_4 - 12a_3^2)s - a_1^3a_2a_4 + a_1^2a_2a_3^2 + a_1a_3a_4^2a_1a_2a_3^2 + a_3^3 > 0,
\]

\[
a_4 \cdot (4096a_4^3s^6 + (-1024a_1a_3a_4^2 - 8192a_4^3)s^5 + (256a_1^2a_2a_4 + 1536a_1a_3a_4^2 - 512a_2a_4^2 \\
+ 256a_2a_3a_4 + 6144a_4^3)s^4 + (-64a_1^2a_4^2 - 64a_1^2a_2a_4^2 - 64a_1a_3a_4^2a_4 - 1024a_1a_3a_4^2 \\
+ 512a_2a_4^2 - 64a_2a_3a_4 - 64a_4^2 - 2048a_4^3)s^3 + (48a_1^4a_4^2 + 16a_1^3a_3a_4 - 64a_1^2a_3a_4^2 \\
- 16a_1^2a_3a_4^2 - 32a_1a_2a_3a_4 + 16a_1a_2^2a_3^2 + 16a_2a_4 + 384a_1a_3a_4^2 - 128a_2a_4^2 - 64a_2a_3a_4^2 \\
+ 48a_4^4 + 256a_4^3)s^2 + (-12a_1^2a_4^2 + 4a_1a_2a_3a_4 - 4a_1^2a_3a_4 - 4a_1a_3a_4^2 - 4a_1a_2a_3^2 + 16a_2a_2^2a_4 + 8a_2a_3a_4 \\
+ 16a_1a_2^2a_3a_4 + 4a_1a_2a_3^2 - 4a_3a_4^2 - 64a_1a_3a_4^2 + 16a_2a_3a_4 - 12a_4^2)s + a_1^4a_4^2 - 2a_1^3a_2a_3a_4 \\
a_1^2a_3a_4^2 + 2a_1a_2a_3^2 + 2a_1a_2a_3^2 + a_3^4 > 0.
\]

(22)

The fractional-order Routh-Hurwitz criterion can be used to analyze the robust stability of fractional-order uncertain systems.

Consider fractional-order uncertain system as follows:

\[ D^\alpha x = A(\beta)x, \quad (23) \]

where \((\alpha, \beta) = (\alpha, \beta_1, \beta_2, \cdots, \beta_i)\) are uncertain parameters, \(x = (x_1, x_2, \cdots, x_n)^T\) is the state vector.

The characteristic polynomial of matrix \(A(\beta)\) is

\[ f(\lambda; \alpha, \beta) = \lambda^n + a_1(\alpha, \beta)\lambda^{n-1} + \cdots + a_n(\alpha, \beta). \]

(24)

Since all expressions in our method are explicit ones about the coefficients of the characteristic polynomial of system matrix, so Theorem 1 is also effective for analyzing the robust stability of system(23).

System(23) is of certain parameters for given parameter \((\alpha, \beta)\). We call the parameter \((\alpha, \beta)\) a stable parameter if the corresponding system is asymptotically stable. The set of all stable parameters is called the stable parameter.
region, denoted by Σ(α, β). According to Theorem 1, the stable parameter region of system(23) is the set of the solutions of \{∇_p > 0, p = 1, 2, \cdots, n\}. All expressions in our results are explicit ones about the coefficients of the characteristic polynomial of system matrix, so the stable parameter region can be described directly.

4. Illustrative Examples

Example 1. Consider the following fractional-order uncertain system:

\[
\frac{d^{1.5}x(t)}{dt^{1.5}} = (A_0 + \beta_1 A_1 + \beta_2 A_2)x(t), \tag{25}
\]

where

\[
A_0 = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.
\]

The characteristic polynomial of \(A_0 + \beta_1 A_1 + \beta_2 A_2\) is

\[
f(\lambda) = \lambda^2 + (2\beta_1 + 2 + \beta_2)\lambda + \beta_1^2 + \beta_2 - \beta_1 + 1, \tag{26}
\]
Based on Corollary 1, we know that system\((25)\) is asymptotically stable if and only if
\[
2\beta_1 + 2 + \beta_2 > 0, \quad \text{(27)}
\]
\[
(\beta_1^2 + \beta_2 - \beta_1 + 1) \left( (2\beta_1 + 2 + \beta_2)^2 - 2(\beta_1^2 + \beta_2 - \beta_1 + 1) \right) > 0.
\]
Parameters that satisfy the set of inequalities\((27)\) are stable parameters of system\((25)\). The solutions of the set of inequalities\((27)\) are shown in Figure 3, in which the stable parameter region is marked.

**Remark 2.** System\((25)\) has been considered\([4, 5]\). By using existing methods, the robustness bound of a single parameter can be obtained, or the stable parameter region can be determined by taking points to test the stability of some corresponding systems. All expressions in our results are explicit ones about the coefficients of the characteristic polynomial of system matrix, the relationship among multiple parameters can be described and the stable parameter region can be solved directly.

**Example 2.** Consider a fractional-order system with \(n = 3\), where \(\alpha \in [1, 2)\) and \(\beta \in (0, 10)\) are uncertain parameters. Suppose the characteristic equation of the system matrix is:
\[
\lambda^3 + (\beta - \alpha)\lambda^2 + 2\beta\lambda + 4 = 0. \quad \text{(28)}
\]

Based on Corollary 2, we know that the system is asymptotically stable if and only if
\[
\beta - \alpha > 0,
\]
\[
(16(\beta - \alpha) - 16\beta^2) s + 2(\beta - \alpha)^2 \beta - 4\beta + 4\alpha > 0,
\]
\[
-4(1024s^3 - (128(\beta - \alpha)\beta + 768)s^2 + (16(\beta - \alpha)^3 - 32(\beta - \alpha)\beta + 32\beta^3 + 192)s - 4(\beta - \alpha)^2 \beta^2 + 16(\beta - \alpha)\beta - 16 > 0. \quad \text{(29)}
\]
The solutions of the set of inequalities\((29)\) are shown in Figure 4, in which the stable parameter region is marked.
Remark 3. Consider fractional-order systems with uncertain order, the existing methods can not describe the relationship between order parameter and stability directly\cite{6,8}. By using our method, systems with uncertain order and uncertain other parameters can be analyzed easily and all results are explicit expressions about the coefficients of the characteristic polynomial of system matrix.

Example 3. \cite{13} Consider the following fractional-order uncertain system with $n = 4$:

$$\frac{d^{1.5}x(t)}{dt^{1.5}} = (\beta_1 A_1 + \beta_2 A_2)x(t)$$

(30)

where $\beta_1 + \beta_2 = 1$, $\beta_1, \beta_2 \geq 0$, $A_1 = \Gamma - \varepsilon b * c$, $A_2 = \Gamma + \varepsilon b * c$,

$$\Gamma = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ -10 & -10 & -20 & -6 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad c' = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}.$$

Let $\beta_2 = 1 - \beta_1$, the characteristic polynomial of system matrix is:

$$f(\lambda) = \lambda^4 + 12\lambda^3 + 67\lambda^2 + (6\beta_1 \varepsilon - 3\varepsilon + 142) \lambda + 12\beta_1 \varepsilon - 6\varepsilon + 96,$$  

(31)
based on Corollary 3, the solutions of the set of inequalities are shown in Figure 5, the stable parameter region of system (30) is marked. For system (30), since \( \beta_1 \) satisfies \( 0 \leq \beta_1 < 1 \), which means the set of inequalities has solutions for all \( \beta_1 \in [0, 1) \), so the maximum value of \( \varepsilon \) is 7.274.

**Remark 4.** The above examples showed the effectiveness of our method for analyzing fractional-order systems with \( n = 2, 3, 4 \). For general \( n \), based on the existing method \([3]\), we need to test \( \Delta_i > 0, i = 1, 2, \cdots, 2n \) to determine the stability of system (23). And as a comparison in this case, using our method, we only need to test whether \( \nabla_p > 0, p = 1, 2, \cdots, n \) to analyze the stability. The number of leading principle minors of our method is smaller than existing methods.

5. Conclusions

Based on the generalized Routh-Hurwitz criterion, we give a fractional-order Routh-Hurwitz criterion for analyzing the stability and robust stability of fractional-order linear systems with \( 1 \leq \alpha < 2 \). Compared with existing
methods, our one involves fewer and simpler expressions, so it has low computational complexity. All expressions in our results are explicit ones about the coefficients of the characteristic polynomial of system matrix, so the stable parameter region of fractional-order uncertain systems can be described directly. Our method is suitable for some complex cases just like systems with uncertain order and uncertain other parameters.

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