The (Signless Laplacian) Spectral Radius (Of Subgraphs) of Uniform Hypergraphs

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Abstract. Let \(\lambda_1(G)\) and \(q_1(G)\) be the spectral radius and the signless Laplacian spectral radius of a \(k\)-uniform hypergraph \(G\), respectively. In this paper, we give the lower bounds of \(d - \lambda_1(H)\) and \(2d - q_1(H)\), where \(H\) is a proper subgraph of a \(f\)-edge-connected \(d\)-regular (linear) \(k\)-uniform hypergraph. Meanwhile, we also give the lower bounds of \(2\Delta - q_1(G)\) and \(\Delta - \lambda_1(G)\), where \(G\) is a nonregular \(f\)-edge-connected (linear) \(k\)-uniform hypergraph with maximum degree \(\Delta\).

1. Introduction

A hypergraph \(G = (V, E)\) is a pair consisting of a vertex set \(V = \{1, 2, \ldots, n\}\) and an edge set \(E = \{e_1, e_2, \ldots, e_m\}\), where \(e_i (1 \leq i \leq m)\) is a subset of \(V\). A hypergraph is called \(k\)-uniform if every edge contains precisely \(k\) vertices. We will use the term \(k\)-graphs in place of \(k\)-uniform hypergraphs. A hypergraph \(G\) is called linear provided that each pair of the edges of \(G\) has at most one common vertex [1]. Given two \(k\)-graphs \(G = (V, E)\) and \(H = (V', E')\), if \(V' \subseteq V\) and \(E' \subseteq E\), then \(H\) is said to be a subgraph (sub-hypergraph) of \(G\). If \(H\) is a subgraph of a \(k\)-graph \(G\), and \(H \neq G\), then \(H\) is called a proper subgraph of \(G\) [11]. A tensor \(A\) with order \(k\) and dimension \(n\) over the complex field \(C\) is a multidimensional array \(A = (a_{i_1i_2\ldots i_k}), 1 \leq i_1, i_2, \ldots, i_k \leq n\).

The tensor \(A\) is called symmetric if its entries are invariant under any permutation of their indices. For a vector \(x = (x_1, x_2, \ldots, x_n)^T \in C^n\), \(Ax^{k-1}\) is a vector in \(C^n\) whose \(i\)-th component is the following

\[
(Ax^{k-1})_i = \sum_{i_2, \ldots, i_k = 1}^n a_{i_1i_2\ldots i_k}x_{i_1}x_{i_2} \cdots x_{i_k}, \quad \forall i \in [n].
\]

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Let $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})^T \in \mathbb{C}^n$. If $Ax^{[k-1]} = \lambda x^{[k-1]}$ has a solution $x \in \mathbb{C}^n \setminus \{0\}$, then $\lambda$ is called an eigenvalue of $A$ and $x$ is an eigenvector associated with $\lambda$. And the spectral radius of $A$ is defined as $\lambda_1(A) = \max \{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. Also, a tensor $A$ of order $k$ and dimension $n$ uniquely determines a $k$-th degree homogeneous polynomial function $A x^k$, which is

$$x^T (Ax^{[k-1]}) = \sum_{i_1, i_2, \ldots, i_k = 1}^n a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The adjacency tensor [6] of a $k$-graph $G$ with $n$ vertices, denoted by $A(G)$, is an order $k$ dimension $n$ symmetric tensor with entries $a_{i_1 i_2 \ldots i_k}$ such that

$$a_{i_1 i_2 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\lambda$ be an eigenvalue of a $k$-graph $G$ with eigenvector $x$. Since $A(G)x^{[k-1]} = \lambda x^{[k-1]}$, we know that $cx$ is also an eigenvector of $\lambda$ for any nonzero constant $c$. So we can choose $x$ such that $\sum_{i=1}^n x_i^k = 1$. In this case, we have [6, 9]

$$\lambda = x^T (A(G)x^{[k-1]}) = k \sum_{e \in E(G)} x^e,$$

where $x^e = x_{i_1} x_{i_2} \cdots x_{i_k}$, $e = \{i_1, i_2, \ldots, i_k\}$.

For a $k$-graph $G$, we denote $N_G(v)$ as the set of neighbours of $v$ in $G$, and $E_G(v)$ as the set of edges containing $v$ in $G$. The degree of a vertex $v$ in $G$, denoted by $d_v = d_G(v)$, is $|E_G(v)|$. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimum degree and the maximum degree of $G$, respectively. If all vertices of $G$ have the same degree, then $G$ is called regular. Let $D = D(G)$ be a $k$-th order $n$-dimensional diagonal tensor with its diagonal element $d_{i,i}$ being $d_i$, the degree of vertex $i$ of $G$, for all $i \in [n]$. Then $Q(G) = D(G) + A(G)$ is the signless Laplacian tensor of the hypergraph $G$ [16]. The signless Laplacian eigenvalues refer to the eigenvalues of the signless Laplacian tensor. Let $q_1(G)$ be the signless Laplacian spectral radius of $G$.

In a $k$-graph $G$, a path of length $l$ is defined to be an alternating sequence of vertices and edges $u_1, e_1, u_2, \ldots, u_l, e_l, u_{l+1}$, where $u_1, \ldots, u_{l+1}$ are distinct vertices of $G$, $e_1, \ldots, e_l$ are distinct edges of $G$ and $u_i, u_{i+1} \in e_l$ for $i = 1, \ldots, l$. For any two vertices $u$ and $v$, if there exists a path connecting $u$ and $v$, then $G$ is called connected. A hypergraph $G$ is called $f$-edge-connected if $G - U$ is connected for any edge subset $U \subseteq E(G)$ satisfying $|U| < f$. A hypergraph $G$ is called $f$-connected if there exist $f$ paths connecting $u$ and $v$ in $G$, where no pair of them have any other elements in common except $u$ and $v$, for any $u, v \in V(G)$ [27].

Spectral graph theory has a long history behind its development [2, 7]. It is natural to generalize spectral theory of graphs to hypergraphs. Recently, there are many work about the spectral theory of hypergraphs [8, 12, 14, 16, 17, 21–23]. In [20], Stevanović proposed a question: How small $\Delta - \lambda_1(G)$ can be when $G$ is an irregular graph with maximum degree $\Delta$ and spectral radius $\lambda_1(G)$? Cioabă et al. [5] gave a lower bound on $\Delta - \lambda_1(G)$ for irregular graphs, which improved previous bounds of Stevanović [20] and of Zhang [26]. Cioabă [4] obtained a lower bound on $\Delta - \lambda_2(G)$ for an irregular graph $G$ with maximum degree $\Delta$ and diameter $D$. Nikiforov [13] presented a lower bound on $\lambda_2(G) - \lambda_1(H)$ for a proper subgraph $H$ of a connected regular graph $G$. Shi [18] obtained a lower bound on $\Delta - \lambda_1(G)$ for a connected irregular graph $G$ in terms of its diameter and average degree. Ning et al. [15] gave a lower bound on $2\Delta - q_1(G)$ for a connected irregular graph $G$ in terms of the diameter. Shui et al. [19] gave a lower bound on $2\Delta - q_1(G)$ and $2\Delta - q_1(H)$ for a $k$-connected irregular graph $G$ and a proper spanning subgraph $H$ of a $\Delta$-regular $k$-connected graph, respectively. Li et al. [10] obtained the lower bounds on $\Delta - \lambda_1(G)$ for irregular connected $k$-graphs in terms of vertex degrees, the diameter, and the number of vertices and edges. Yuan et al. [25] gave some bounds on $\lambda_1(G)$ and $q_1(G)$ for a $k$-graph $G$ in terms of its degrees of vertices. Chen et al. [3] presented several upper bounds on $\lambda_1(G)$ and $q_1(G)$ for a $k$-graph $G$ in terms of degree sequences. We are inspired by two articles of Shui et al. [19] and Li et al. [10]. In this paper, we give the bounds of (signless Laplacian) spectral radius of subgraphs of $f$-edge-connected $d$-regular (linear) $k$-graphs. We also give the bounds of (signless Laplacian) spectral radius of connected nonregular (linear) $k$-graphs.
2. Preliminaries

In this section, we give some useful lemmas.

Let $G$ be a connected $k$-graph. By Perron-Frobenius theorem of nonnegative tensors [24], $\lambda_1(G)$ (resp., $q_1(G)$) is an eigenvalue of $\mathcal{A}(G)$ (resp., $Q(G)$), and there exists a unique positive eigenvector $x = (x_1, \ldots, x_n)^T$ corresponding to $\lambda_1(G)$ (resp., $q_1(G)$) with $\sum_{i=1}^{n} x_i = 1$, and $x$ is called the principal eigenvector of $\mathcal{A}(G)$ (resp., $Q(G)$).

The following Lemma 2.1 is from the proof of Theorem 4.1 in [10].

**Lemma 2.1.** (10) Let $G$ be a connected $k$-graph with $n$ vertices and $\lambda_1(G)$ be the spectral radius of $G$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i \in V(G)} |x_i|$ and $x_v = \min_{i \in V(G)} |x_i|$. Let $P : u = u_0, e_1, u_1, \ldots, u_r = v$ be a path from $u$ to $v$, where $e_i$ is an edge containing vertices $u_{i-1}$ and $u_i$. Then

$$\sum_{\{w_i, w_j \in E(P)\}} (x_{w_j} - x_{w_i})^2 \geq \frac{k}{2r} (\lambda_{w_2} - \lambda_{w_1})^2.$$ 

**Lemma 2.2.** (18) Let $a_1, \ldots, a_n$ be nonnegative real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} - (a_1 \cdots a_n)^{\frac{1}{n}} \geq \frac{1}{n(n - 1)} \sum_{1 \leq i < j \leq n} (\sqrt[n]{a_i} - \sqrt[n]{a_j})^2,$$

equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

**Lemma 2.3.** (18) Let $a, b, y_1, y_2$ be positive numbers. Then

$$a(y_1 - y_2)^2 + b y_2^2 \geq \frac{ab}{a + b} y_1^2,$$

equality holds if and only if $y_2 = \frac{ay_1}{a+b}$.

Two paths $P_1, P_2$ are called edge-disjoint if the edges of $P_1$ have no common with the edges of $P_2$.

**Lemma 2.4.** (27) A hypergraph $G$ is $f$-edge-connected if and only if there are $f$ mutual edge-disjoint paths between each pair of vertices.

**Lemma 2.5.** (27) If a hypergraph $G$ is $f$-connected, then there are $f$ mutual vertex-disjoint paths between each pair of vertices.

**Lemma 2.6.** (10) Let $G$ be a connected $k$-graph with $n$ vertices, minimum degree $\delta$ and maximum degree $\Delta$, and let $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $\mathcal{A}(G)$. Then $x_{\max} \geq \left(\frac{k}{\Delta} \frac{1}{\max_{i \in V(G)} x_i} + n - 1\right)^{-\frac{1}{2}}$, where $x_{\max} = \max_{1 \leq i \leq n} |x_i|$.

In fact, we can prove similarly that Lemma 2.6 also holds for the principal eigenvector of $Q(G)$, where $G$ is a connected $k$-graph with $n$ vertices.

3. The (signless Laplacian) spectral radius of subgraphs of $f$-edge-connected $d$-regular $k$-graphs

In this section, we will give a bound of the spectral radius and the signless Laplacian spectral radius of a subgraph of a $f$-edge-connected $d$-regular $k$-graph $G$, respectively. And we will give a bound on the the spectral radius and the signless Laplacian spectral radius of a subgraph of a $f$-connected $d$-regular linear $k$-graph $G$, respectively.
Lemma 3.1. Let $H$ be a maximal proper subgraph of a $f$-edge-connected $d$-regular $k$-graph $G$ such that $f \geq 2$, and $\lambda_1(H)$ be the spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Then
$$d - \lambda_1(H) = \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e \in \{v_1, v_2, \ldots, v_n\} \in E(H)} (x_{v_1}^k + \cdots + x_{v_n}^k - dx^k),$$
where $d_i$ is the degree of the vertex $i$ of $H$.

Proof. Let $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that $H$ is connected since $f \geq 2$. Let $x_u = \max_{v \in V(H)} |x_v|$ and $x_v = \min_{v \in V(H)} |x_v|$. We claim $u \neq u_i$ for any $1 \leq i \leq k$. Indeed, if $u = u_i$ for some $1 \leq i \leq k$, then
$$\lambda_1(H)x_u^{k-1} = \sum_{e \in \{v_1, v_2, \ldots, v_n\} \in E(H)} a_{v_1, v_2, \ldots, v_n} x_{v_1}x_{v_2} \cdots x_{v_n} \leq (d - 1)x_u^{k-1},$$
and thus $\lambda_1(H) \leq d - 1$, contradicting the fact that $\lambda_1(H) > \frac{\Delta(H)}{n} = d - \frac{k}{n} > d - 1$. We also find that
$$d - \lambda_1(H) = d \sum_{i=1}^{n} x_i^k - k \sum_{e \in E(H)} x^k$$
$$= d \sum_{i=1}^{n} x_i^k - d \sum_{i=1}^{n} d_i x_i^k + \sum_{i=1}^{n} d_i x_i^k - k \sum_{e \in E(H)} x^k$$
$$= \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e \in \{v_1, v_2, \ldots, v_n\} \in E(H)} (x_{v_1}^k + \cdots + x_{v_n}^k - dx^k).$$

\[\square\]

Lemma 3.2. Let $H$ be a maximal proper subgraph of a $f$-edge-connected $d$-regular $k$-graph $G$ such that $f \geq 2$ and $q_1(H)$ be the signless Laplacian spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Then
$$2d - q_1(H) = 2 \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e \in \{v_1, v_2, \ldots, v_n\} \in E(H)} (x_{v_1}^k + \cdots + x_{v_n}^k - dx^k),$$
where $d_i$ is the degree of the vertex $i$ of $H$.

Proof. Similarly, let $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that $H$ is connected since $f \geq 2$. Let $x_u = \max_{v \in V(H)} |x_v|$ and $x_v = \min_{v \in V(H)} |x_v|$. We claim $u \neq u_i$ for any $1 \leq i \leq k$. Indeed, if $u = u_i$ for some $1 \leq i \leq k$, then
$$q_1(H)x_u^{k-1} = a_{u_1, u_2, \ldots, u_n} x_{u_1}x_{u_2} \cdots x_{u_n} \leq 2(d - 1)x_u^{k-1},$$
and thus $q_1(H) \leq 2d - 2$, contradicting the fact that $q_1(H) \geq 2\lambda_1(H) > 2\frac{\Delta(H)}{n} = 2d - \frac{2k}{n} > 2d - 2$. We also find that
$$2d - q_1(H) = 2 \sum_{i=1}^{n} x_i^k - \sum_{i=1}^{n} d_i x_i^k - k \sum_{e \in E(H)} x^k$$
$$= 2 \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{i=1}^{n} d_i x_i^k - k \sum_{e \in E(H)} x^k$$
$$= 2 \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e \in \{v_1, v_2, \ldots, v_n\} \in E(H)} (x_{v_1}^k + \cdots + x_{v_n}^k - dx^k).$$

\[\square\]
Thus, we have
\[ d - \lambda_1(H') > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2)((\frac{d - 1}{2})^{\frac{n}{|H|}} + n - 1). \]

**Proof.** Let \( H \) be a maximal proper subgraph of \( G \), i.e., \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge \( \{u_1, u_2, \ldots, u_k\} \). Let \( \lambda_1(H) \) be the spectral radius of \( H \) with the principal eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \). Let \( x_u = \max_{v \in V(H)} \{x_i\} \) and \( x_v = \min_{v \in V(H)} \{x_i\} \). By Lemmas 2.2 and 3.1, we have
\[
    d - \lambda_1(H) > x_u^k + x_v^k + \cdots + x_v^k + \frac{1}{k - 1} \sum_{w, w' \in E(H)} (x_{w'w} - x_{ww})^2 \geq kx_v^k + \frac{1}{k - 1} \sum_{w, w' \in E(H)} (x_{w'w} - x_{ww})^2.
\]

By (3.1) and (3.2), we have
\[
    d - \lambda_1(H) > \sum_{w, w' \in E(H)} (x_{w'w} - x_{ww})^2 \geq \frac{k}{2r_t} (x_u^k - x_v^k)^2.
\]

Thus, we have
\[
    d - \lambda_1(H) > x_u^k + x_v^k + \cdots + x_v^k + \frac{1}{k - 1} \sum_{w, w' \in E(H)} (x_{w'w} - x_{ww})^2 \geq kx_v^k + \frac{1}{k - 1} \sum_{w, w' \in E(H)} (x_{w'w} - x_{ww})^2 \geq k(f - 1)^2 \sum_{t=1}^{f-1} (x_u^k - x_v^k)^2 \geq \frac{k(f - 1)^2}{2(m - 1)} (x_u^k - x_v^k)^2.
\]

By (3.1) and (3.2), we have
\[
    d - \lambda_1(H) > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2}(x_u^k - x_v^k)^2.
\]

The right hand side of the above inequality is a quadratic function of \( x_u^k \). By Lemma 2.3, we have
\[
    d - \lambda_1(H) > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2} x_u^k.
\]

By Lemma 2.6, we have
\[
    d - \lambda_1(H) > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2} x_u^k = \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2} x_u^k.
\]
Therefore, we have
\[ d - \lambda_1(H') > \frac{k(f-1)^2}{2(k-1)(m-1) + (f-1)^2[(\frac{d-1}{k}) \frac{\Delta}{\Delta(D)} + n - 1]}. \]
\[ \square \]

**Theorem 3.4.** Let \( G \) be a \( f \)-edge-connected \( d \)-regular \( k \)-graph with \( n \) vertices and \( m(= \frac{dn}{k}) \) edges, and \( H' \) be a proper subgraph of \( G \). If \( f, k \geq 2 \), then
\[ 2d - q_1(H') > \frac{2k(f-1)^2}{4(k-1)(m-1) + (f-1)^2[(\frac{d-1}{k}) \frac{\Delta}{\Delta(D)} + n - 1]}. \]

**Proof.** Let \( H \) be a maximal proper subgraph of \( G \), i.e., \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge \( \{u_1, u_2, \ldots, u_k\} \). Let \( q_1(H) \) is the signless Laplacian spectral radius of \( H \) with a principal eigenvector \( x \). Let \( x_u = \max_{i \in V(H)} \{x_i\} \) and \( x_v = \min_{i \in V(H)} \{x_i\} \). By Lemmas 2.2 and 3.2, we have
\[ 2d - q_1(H) > 2(x_u^k + x_v^k + \cdots + x_w^k) + \frac{1}{k-1} \sum_{w, j \in E(H)} (x_u^k - x_w^k)^2 \]
\[ \geq 2kx_u^k + \frac{1}{k-1} \sum_{w, j \in E(H)} (\frac{1}{k} x_u^k - \frac{1}{k} x_w^k)^2. \]

(3.3)

Since \( G \) is a \( f \)-edge-connected \( d \)-regular \( k \)-graph, there are at least \( f - 1 \) edge disjoint paths connecting \( u \) and \( v \) in \( H \). By (3.2) and (3.3), then we have
\[ 2d - q_1(H) > 2kx_u^k + \frac{k(f-1)^2}{2(k-1)(m-1)} (x_u^k - x_w^k)^2. \]

The right hand side of the above inequality is a quadratic function of \( x_w^k \). By Lemma 2.3, we have
\[ 2d - q_1(H) > \frac{2k(f-1)^2}{4(k-1)(m-1) + (f-1)^2} x_u^k. \]

By Lemma 2.6, we have
\[ 2d - q_1(H) > \frac{2k(f-1)^2}{[4(k-1)(m-1) + (f-1)^2][(\frac{d-1}{k}) \frac{\Delta}{\Delta(D)} + n - 1]} = \frac{2k(f-1)^2}{2(k-1)(m-1) + (f-1)^2[(\frac{d-1}{k}) \frac{\Delta}{\Delta(D)} + n - 1]}. \]

Therefore, we have
\[ 2d - q_1(H') > \frac{2k(f-1)^2}{2(k-1)(m-1) + (f-1)^2[(\frac{d-1}{k}) \frac{\Delta}{\Delta(D)} + n - 1]}. \]
\[ \square \]

**Theorem 3.5.** Let \( G \) be a \( f \)-connected \( d \)-regular linear \( k \)-graph with \( n \) vertices, and \( H' \) be a proper subgraph of \( G \). If \( f, k \geq 2 \), then
\[ d - \lambda_1(H') > \frac{2k(f-1)^2}{(2n + d(k^2 - k - 2) + 4)(f-1)^2 + h'} \]
where \( h = k(k-1)(n-k-d+2)((n+2(f-2)^2 - (f-1)). \)
Proof. Let $H$ be a maximal proper subgraph of $G$, i.e., $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $(u_1, u_2, \ldots, u_k)$. We know that $H$ is connected since $f \geq 2$. Let $\lambda_1(H)$ be the spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_n = \max_{i \in V(H)}|x_i|$ and $x_0 = \min_{i \in V(H)}|x_i|$. By the proof of Lemma 3.1, we claim $u \neq u_i$ for $1 \leq i \leq k$. By Lemmas 2.2 and 3.1, we have

$$d - \lambda_1(H') \geq d - \lambda_1(H)$$

$$\geq x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k + \frac{1}{k-1} \sum_{v, w_j \in E(H)} (x_{w_1}^i - x_{w_j}^i)^2$$

$$\geq kx_u^k + \frac{1}{k-1} \sum_{v, w_j \in E(H)} (x_{w_1}^i - x_{w_j}^i)^2.$$

(3.4)

Since $G$ is a $f$-connected $d$-regular $k$-graph, by Lemma 2.5, there are at least $f-1$ vertex disjoint paths $P_1, P_2, \ldots, P_{f-1}$ connecting $u$ and $v$ in $H$. Thus, we have

$$\sum_{i=1}^{f-1} |V(P_i)| \leq n + 2(f-2).$$

Since $G$ is a linear $k$-graph, we have $|V(P_i)| \geq |E(P_i)| + 1$. Hence, $\frac{2|E(P_i)}{k} \leq \frac{|V(P_i)||V(P_i)|-1}{2}$. By Lemma 2.1, we have

$$\sum_{v, w_j \in E(H)} (x_{w_1}^i - x_{w_j}^i)^2 \geq \sum_{i=1}^{f-1} \sum_{v, w_j \in E(P_i)} (x_{w_1}^i - x_{w_j}^i)^2$$

$$\geq \sum_{i=1}^{f-1} \frac{k}{2 |E(P_i)|} (x_u^i - x_v^i)^2$$

$$\geq \sum_{i=1}^{f-1} \frac{2}{|V(P_i)| - 1} (x_u^i - x_v^i)^2$$

$$\geq \frac{2(f-1)^2}{2(f-1)^2} (x_u^i - x_v^i)^2$$

(3.5)

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So by (3.4) and (3.5), we have

$$d - \lambda_1(H') > kx_u^k + \frac{2(f-1)^2}{(k-1)(n + 2(f-2)^2 - (f-1))} (x_u^i - x_v^i)^2.$$
Case 1. If \( \sum_{i=1}^{k} x_{i} > C \), then from (3.4), we have
\[
d - \lambda_1(H') > (x_{u_1}^2 + x_{u_2}^2 + \cdots + x_{u_k}^2) + \frac{1}{k-1} \sum_{w,v_i \in E(H)} (x_{w}^2 - x_{v_i}^2)^2
\]
\[
> C + \frac{1}{k-1} \sum_{w,v_i \in E(H)} (x_{w}^2 - x_{v_i}^2)^2
\]
\[
> C.
\]

Case 2. Let \( x_{u_i} = \min_{1 \leq i \leq k} |x_{u_i}|. \) Since \( d_H(u_i) = d - 1 \), it is possible to choose at least \( d - 2 \) distinct vertices \( v_1, v_2, \ldots, v_{d-2} \) from \( N_H(u_i) \) such that \( u \notin \{v_1, v_2, \ldots, v_{d-2}\} \). If \( \sum_{i=1}^{d-2} x_{v_i}^2 \geq \frac{d(k-1)}{2} C \), by (3.4) again and Lemma 2.3, then we have
\[
d - \lambda_1(H') > \frac{2}{k-1} x_{u_1}^2 + \frac{1}{k-1} \sum_{i=1}^{d-2} (x_{v_i}^2 - x_{u_i}^2)^2
\]
\[
= \frac{1}{k-1} \sum_{i=1}^{d-2} \left( \frac{2}{d-2} x_{v_i}^2 + (x_{v_i}^2 - x_{u_i}^2)^2 \right)
\]
\[
\geq \frac{1}{k-1} \sum_{i=1}^{d-2} \frac{2}{d-2} x_{v_i}^2
\]
\[
\geq \frac{1}{k-1} \frac{2}{d} (d(k-1) C
\]
\[
= C.
\]

Case 3. Since \( G \) is a linear \( k \)-graph, we have \( v_t \neq u_i \) for \( 1 \leq t \leq d - 2 \), \( 2 \leq i \leq k \). If \( \sum_{i=1}^{k} x_{u_i}^2 \leq C \) and \( \sum_{i=1}^{d-2} x_{v_i}^2 < \frac{d(k-1)}{2} C \), then
\[
x_{u_i} \geq \frac{1}{n-k-d+2} (1 - \frac{d(k-1)}{2} C - \frac{2}{d} (d(k-1) C
\]
\[
\geq \frac{2(f-1)^2}{(n+2(f-2))^2 - (f-1)) + 2(f-1)^2 x_{v_i}^2 = C}.
\]
\( \square \)

Theorem 3.6. Let \( G \) be a \( f \)-connected \( d \)-regular linear \( k \)-graph with \( n \) vertices, and \( H' \) be a proper subgraph of \( G \). If \( f, k \geq 2 \), then
\[
2d - q_1(H') > \frac{2k(f-1)^2}{(n+d(k^2-k-1)+2)(f-1)^2 + h'}
\]
where \( h = k(k-1)(n-k-d+2)(n+2(f-2))^2 - (f-1)) \).

Proof. Let \( H \) be a maximal proper subgraph of \( G \), i.e., \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge \( \{u_1, u_2, \ldots, u_k\} \). We know that \( H \) is connected since \( f \geq 2 \). Let \( q_1(H) \) be the signless Laplacian spectral radius of \( H \) with a principal eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \). Let \( x_u = \max_{i \in V(H)} |x_i| \) and \( x_v = \min_{i \in V(H)} |x_i| \). By Lemmas 2.2 and 2.3, we have
\[
2d - q_1(H') \geq 2d - q_1(H) > 2(x_{u_1}^2 + x_{u_2}^2 + \cdots + x_{u_k}^2) + \frac{1}{k-1} \sum_{w,v_i \in E(H)} (x_{w}^2 - x_{v_i}^2)^2.
\]
By (3.5) and (3.7), similarly, we have

\[ 2d - q_1(H') > 2k x_i^k + \frac{2(f - 1)^2}{(k - 1)(n + 2(f - 2))^2 - (f - 1)} (x_i^1 - x_i^k)^2. \]  

(3.8)

Define

\[ C = \frac{2k(f - 1)^2}{(n + d(k^2 - k - 1) + 2)(f - 1)^2 + h'} \]

where \( h = k(k - 1)(n - k - d + 2)((n + 2(f - 2))^2 - (f - 1)). \)

**Case 1.** If \( \sum_{i=1}^{k} x_i^k > \frac{r}{2} \), then from (3.7), we have

\[
2d - q_1(H') > 2(x_i^1 + x_i^2 + \cdots + x_i^k) + \frac{1}{k - 1} \sum_{w, v, i \in E(H)} (x_i^1 - x_i^j)^2 \\
> 2 \frac{C}{2} + \frac{1}{k - 1} \sum_{w, v, i \in E(H)} (x_i^1 - x_i^j)^2 \\
= C.
\]

**Case 2.** Let \( x_i^k = \min_{1 \leq i \leq k} \{ x_i^k \} \). Since \( d_H(u_1) = d - 1 \), it is possible to choose at least \( d - 2 \) distance vertices \( \{ v_1, v_2, \ldots, v_{d-2} \} \) from \( N_H(u_1) \) such that \( u \notin \{ v_1, v_2, \ldots, v_{d-2} \} \). If \( \sum_{i=1}^{d-2} x_i^k \geq \frac{d(k-1)}{2} C \), by (3.7) again and Lemma 2.3, then we have

\[
2d - q_1(H') > \frac{2}{k - 1} x_i^1 + \frac{1}{k - 1} \sum_{i=1}^{d-2} (x_i^1 - x_i^k)^2 \\
= \frac{1}{k - 1} \sum_{i=1}^{d-2} \left( \frac{2}{d - 2} x_i^1 + (x_i^1 - x_i^k)^2 \right) \\
\geq \frac{1}{k - 1} \sum_{i=1}^{d-2} \frac{2}{d - 2} x_i^1 \\
\geq \frac{2}{k - 1} \frac{d(k - 1)}{2} C \\
= C.
\]

**Case 3.** Since \( G \) is a linear \( k \)-graph, we have \( v_i \neq u_t \) for \( 1 \leq t \leq d - 2, 2 \leq i \leq k \). If \( \sum_{i=1}^{k} x_i^k \leq \frac{r}{2} \) and \( \sum_{i=1}^{d-2} x_i^k \leq \frac{d(k-1)}{2} C \), then

\[
x_i^k \geq \frac{1 - \sum_{i=1}^{k} x_i^k - \sum_{i=1}^{d-2} x_i^k}{n - k - (d - 2)} > \frac{1}{n - k - d + 2} \left( 1 - \frac{d(k - 1)}{2} C \right) = \frac{1}{n - k - d + 2} (1 - \frac{d(k - d + 1)}{2} C),
\]

and from (3.8) and Lemma 2.3, we obtain

\[
2d - q_1(H') > \frac{2k(f - 1)^2}{k(k - 1)((n + 2(f - 2))^2 - (f - 1)) + (f - 1)^2 x_i^k} = C.
\]

\[ \square \]
4. The signless Laplacian spectral radius of connected nonregular (linear) $k$-graphs

In this section, we mainly study the upper bounds of the (signless Laplacian) spectral radius of a $f$-edge-connected nonregular $k$-graph $G$ with maximum degree $\Delta$, respectively.

**Theorem 4.1.** Let $G$ be a nonregular $f$-edge-connected $k$-graph with $n$ vertices, $m$ edges, minimum degree $\delta$ and maximum degree $\Delta$. Then

$$2\Delta - q_1(G) > \frac{2k(n\Delta - km)^2}{[4m(k - 1)(n\Delta - km) + kf^2][\frac{\Delta}{\delta} + n - 1]}.$$

**Proof.** Let $q_1(G)$ be the signless Laplacian spectral radius of $G$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{v \in V(G)} |x_v|$ and $x_v = \min_{v \in V(G)} |x_v|$. We also find that

$$2\Delta - q_1(G) = 2\Delta \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} d_i x_i^2 - k \sum_{e \in E(G)} x^e$$

$$= 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^2 + \sum_{i=1}^{n} d_i x_i^2 - k \sum_{e \in E(G)} x^e$$

$$= 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^2 + \sum_{e \in E(G)} (x_{w_1}^2 + \ldots + x_{w_k}^2 - kx^e),$$

where $d_i$ is the degree of the vertex $i$. By Lemma 2.2, we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x_u^k + \frac{1}{k - 1} \sum_{u,v \in E(G)} (x_u^k - x_v^k)^2. \quad (4.1)$$

Let $P_i : u = u_0, e_1, u_1, \ldots, u_n = v$ be a path from $u$ to $v$. By Lemma 2.1, we have

$$\sum_{w, v \in E(P_i)} (x_{w_1}^k - x_{v_2}^k)^2 \geq \frac{k}{2r_i} (x_u^k - x_v^k)^2.$$

Since $G$ is $f$-edge-connected, similar to (3.2), we have

$$\sum_{w, v \in E(G)} (x_{w_1}^k - x_{v_2}^k)^2 \geq \sum_{i=1}^{f} \frac{k}{2r_i} (x_u^k - x_v^k)^2 \geq \frac{kf^2}{2m}(x_u^k - x_v^k)^2. \quad (4.2)$$

By (4.1) and (4.2), we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x_u^k + \frac{kf^2}{2m(k - 1)}(x_u^k - x_v^k)^2.$$

The right hand side of the above inequality is a quadratic function of $x_u^k$. By Lemma 2.3, we have

$$2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{4m(k - 1)(n\Delta - km) + kf^2} x_u^k.$$

By Lemma 2.6, we have

$$2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{[4m(k - 1)(n\Delta - km) + kf^2][\frac{\Delta}{\delta} + n - 1]}.$$

$\square$
Theorem 4.2. Let $G$ be a nonregular $f$-connected linear $k$-graph with $n$ vertices, $m$ edges and maximum degree $\Delta$. Then

$$2\Delta - q_1(G) > \frac{2(n\Delta - km) f^2}{(n + 2(k - 1)(n\Delta - km) + (k - 2)(f - 1)) f^2 + h},$$

where $h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f)$.

Proof. Let $q_1(G)$ be the signless Laplacian spectral radius of $G$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{v \in V(G)} |x_v|$ and $x_v = \min_{v \in V(G)} |x_v|$. Consider the following two cases:

Case 1. Suppose $d_u \leq \Delta - 1$. Since $Qx^{k-1} = q_1 x^{k-1}$, we have

$$q_1(G)x^{k-1}_u = d_u x^{k-1}_u + \sum_{e = (u, w_1, \ldots, w_k) \in E(G)} x_{w_1} x_{w_2} \cdots x_{w_k} \leq 2(\Delta - 1)x^{k-1}_u.$$

Thus, we have $q_1(G) \leq 2\Delta - 2$. Consequently,

$$2\Delta - q_1(G) \geq 2 > \frac{2(n\Delta - km) f^2}{(n + 2(k - 1)(n\Delta - km) + (k - 2)(f - 1)) f^2 + h},$$

where $h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f)$.

Case 2. Suppose $d_u = \Delta$. Since $G$ is a $f$-connected $k$-graph, there are at least $f$ vertex disjoint paths $P_1, P_2, \ldots, P_f$ connecting $u$ and $v$ in $G$. By Lemma 2.5, we have

$$\sum_{i=1}^{f} |V(P_i)| \leq n + 2(f - 1). \quad (4.5)$$

Thus, we have

$$2\Delta - q_1(G) = 2\Delta \sum_{i=1}^{n} x_i^f - \sum_{i=1}^{n} d_i x_i^f - k \sum_{e \in E(G)} x_e^f$$

$$= 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^f + \sum_{i=1}^{n} d_i x_i^f - k \sum_{e \in E(G)} x_e^f$$

$$= 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^f + \sum_{e = (w_1, w_2, \ldots, w_k) \in E(G)} (x_{w_1}^f + \cdots + x_{w_k}^f - kx^f),$$

where $d_i$ is the degree of the vertex $i$. By Lemma 2.2, we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x^f_u + \frac{1}{k - 1} \sum_{w, v \in V(G)} (x^f_{wv} - x^f_{uw})^2. \quad (4.6)$$

Similar to the proof of (3.5), we have

$$\sum_{w, v \in V(G)} (x^f_{wv} - x^f_{uw})^2 > \frac{2f^2}{(n + 2f - 2)^2 - f} (x^f_u - x^f_v)^2. \quad (4.7)$$

By (4.6), (4.7) and Lemma 2.3, we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x^f_u + \frac{2f^2}{(k - 1)((n + 2f - 2)^2 - f)} (x^f_u - x^f_v)^2$$

$$\geq \frac{2(n\Delta - km)f^2}{(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f) + f^2} x^f_u. \quad (4.8)$$
Thus, by (4.8), we have

\[ 2\Delta - q_1(G) > \frac{2(n\Delta - km)}{(k-1)(n\Delta - km)(n^2 - 1) + 1} x_v^k \]

and

\[ C = \frac{2(n\Delta - km)}{(n + 2(k-1)(n\Delta - km)) + h'} \]

where \( h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f) \).

**Case 2.1.** Suppose \( f = 1 \), we have

\[ 2\Delta - q_1(G) > \frac{2(n\Delta - km)}{(k-1)(n\Delta - km)(n^2 - 1) + 1} x_v^k \quad (4.9) \]

Case 2.1.1. If \( x_v^k \geq \frac{2n}{2(n\Delta - km)} \), then from (4.6) and (4.7), we obtain

\[ 2\Delta - q_1(G) > 2(n\Delta - km) \frac{C}{2(n\Delta - km)} + \frac{2f^2}{(k-1)((n + 2f - 2)^2 - f)} (x_v^k - x_v^k)^2 > C. \]

Case 2.1.2. If \( x_v^k < \frac{2n}{2(n\Delta - km)} \), then since \( \sum_{i=1}^{\frac{f-1}{2}} x_v^k = 1 \), we have

\[ x_u^k \geq 1 - x_v^k > \frac{1}{n-1}(1 - \frac{C}{2(n\Delta - km)}). \]

Thus, by (4.9), we have

\[ 2\Delta - q_1(G) > C. \]

Case 2.2. Suppose \( f \geq 2 \).

**Case 2.2.1.** If \( x_v^k \geq \frac{2n}{2(n\Delta - km)} \), then the result can be obtained using a similar argument of the case 2.1.1.

**Case 2.2.2.** Since \( G \) is a \( f \)-connected linear \( k \)-graph, we have \( d_v \geq f \). We can choose at least \( f - 1 \) vertices from \( N_G(v) \), denoted by \( \{v_1, v_2, \ldots, v_{f-1}\} \), such that \( u \not\in \{v_1, v_2, \ldots, v_{f-1}\} \). If \( \sum_{i=1}^{f-1} x_v^k > C(k - 1)(1 + \frac{f-1}{2(n\Delta - km)}) \), by (4.6), we have

\[ 2\Delta - q_1(G) > 2(n\Delta - km)x_v^k + \frac{1}{k-1} \sum_{i=1}^{f-1} (x_v^k - x_v^k)^2 \]

\[ \geq 2 \frac{(n\Delta - km)}{k-1} x_v^k + \frac{1}{k-1} \sum_{i=1}^{f-1} (x_v^k - x_v^k)^2. \]

Similar to the proof of the case 2 of Theorem 3.6, we have

\[ 2\Delta - q_1(G) > C. \]

**Case 2.2.3.** If \( x_v^k < \frac{2n}{2(n\Delta - km)} \) and \( \sum_{i=1}^{f-1} x_v^k \leq C(k - 1)(1 + \frac{f-1}{2(n\Delta - km)}) \), by \( \sum_{i=1}^{n} x_v^k = 1 \), then we have

\[ x_u^k \geq \frac{1}{n-f} (1 - x_v^k) \sum_{i=1}^{f-1} x_v^k > \frac{1}{n-f}(1 - \frac{2(k-1)(n\Delta - km) + (k-1)(f-1) + 1}{2(n\Delta - km)} C). \]

Thus, by (4.8), we have

\[ 2\Delta - q_1(G) > C. \]
Theorem 4.3. Let $G$ be a nonregular $f$-connected linear $k$-graph with $n$ vertices, $m$ edges and maximum degree $\Delta$. Then
\[ \Delta - \lambda_1(G) > \frac{2(n\Delta - km)}{2(n + (k - 1)(n\Delta - km) + (k - 2)(f - 1))^{f^2 + h'}} \]
where $h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f)$. 

Proof. The result can be obtained by using a similar argument of Theorem 4.2. \qed

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