On the response of a system with bound states of particles to the perturbation by the external electromagnetic field

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The response of the system, consisting of two kinds of opposite-charged fermions and their bound states (hydrogen-like atoms), to the perturbation by the external electromagnetic field in low particle kinetic energies region is studied. Expressions for Green functions that describe the system response to the external electromagnetic field and take into account the presence of particle bound states (atoms) are found. Macroscopic parameters of the system, such as conductivity, permittivity and magnetic permeability in terms of these Green functions are introduced. As an example, the perturbation of the ideal hydrogen-like plasma by the external electromagnetic field in low temperature region is considered. Such approach also enables to study the propagation properties of the signal, tuned up to the transition between two hyperfine ground state levels of alkali atoms that are considered in Bose-Einstein condensation (BEC) state. It is shown that the signal can propagate in such system with rather small energy loss. Such fact allows to introduce the group velocity concept and to study the slowing down conditions for the microwave signal that propagates in BEC.

Keywords: Green functions, bound states, response of systems, low-temperature hydrogen-like plasma, conductivity, magnetic permeability, BEC, ultra-slow electromagnetic waves.

PACS: 05.30.d; 05.30.Jp; 03.75.Mn; 03.75.Hh.

1 Introduction

In the process of describing a behavior of many-particle systems a class of problems appear, that concerned with the system response to the perturbing action of the external, in particular, electromagnetic field. Widespread approach to solving such kind of problems is based on using the Green functions formalism (see in that case e.g. [1]).

As well known, the most convenient method for physical processes describing in quantum many-particle theory is the second quantization method. Because of that in framework of the second quantization it is the most simple to formulate an approach for describing the system response to the perturbation by the external field, that is based on using Green functions. But if we try to realize such an approach, we can meet an essential difficulty, connected with the possibility of the particle bound states existence.

Really, the key role of the second quantization method consists in the introduction of creation and annihilation operators of particles in a certain quantum state. The operators of physical quantities are constructed in terms of creation and annihilation operators. Such a description of quantum many-particle systems implies the particles to be elementary (not consisting of other particles). Moreover, it is absolutely accurate despite of the possible existence of compound particles. Since the interactions between particles may lead to the formation of bound states, the standard second quantization method becomes too cumbersome. For this reason the construction of an approximate quantum-mechanical theory for many-particle systems consisting of elementary particles and their bound states represents an actual problem. In this theory it is necessary to introduce the creation and annihilation operators of bound states as the operators of elementary objects (not compound). In addition it must preserve the required information concerning internal degrees of freedom for bound states.

Such an approach is realized in [2]. In this work the possibility of constructing such theory is demonstrated for a system, that consists of two kinds of fermions, assuming that bound states (atoms or molecules) are formed by particles of two different kinds. The choice of such model is not dictated by the principle difficulties but by the desire to simplify calculations and obtain the visual results. In framework of this model a method of constructing the creation and annihilation operators of the bound state as a compound object is given. The substantiation of the conversion from the description of atoms as compound objects to elementary objects with the ordinary creation and annihilation Bose-operators is given. Such substantiation is considered in low-energy approximation in which the binding energy of a compound particle is much greater than its kinetic energy. In terms of the creation and annihilation operators of fermions and bosons (as ele-
mentary objects) a scheme for physical quantities operators construction is formulated. Explicit expressions for the operators of principal physical quantities, such as density and charge density, momentum and current density, system Hamiltonian, are found. The Maxwell–Lorentz system of equations, describing the interaction between electromagnetic field and matter, that may consist also of neutral “atoms” (low-energy quantum electrodynamic equations), is found.

In the present work we use this system of equations to study in the framework of Green functions formalism the response of the system with bound states to the perturbation by the external electromagnetic field. An essentially new moment in these considerations is the next circumstance. When we describe the system response to the perturbing action of the external electromagnetic field the approximate formulation of the second quantization method proposed in [2] gives us the possibility to account for the neutral bound states in sufficiently simple way.

2 Quantum electrodynamic equations for the low-temperature hydrogen-like plasma

The studied quantum-electrodynamic system, consisting of fermions of two different kinds and their bound states, in low-energy region, in fact, can be considered as a low-temperature hydrogen-like plasma. Before we turn to the description of such plasma response to the external electromagnetic field, let us obtain the main equations that describe an evolution of such system. Taking into account the interaction between radiation and matter the system’s Hamiltonian $\hat{H}(t)$, according to [2], can be written as

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_{\text{int}} + \hat{V}(t), \quad \hat{H}_0 = \hat{H}_f + \hat{H}_p,$$

where

$$\hat{H}_f = \sum_{k,\lambda} \omega_k \hat{C}^+_{k\lambda} \hat{C}_{k\lambda}$$

is the Hamiltonian for free photons ($\omega_k$ is the frequency of photon with wave number $k$, $\hat{C}^+_{k\lambda}$, $\hat{C}_{k\lambda}$ are the creation and annihilation operators of photon with wave number $k$ and polarization $\lambda$).

The value $\hat{H}_p$ in the equation (1) is the Hamiltonian for free particles (free fermions and their bound states)

$$\hat{H}_p = \sum_{j=1}^{2} \frac{1}{2m_j} \int dx \frac{\partial \hat{x}^+_j(x)}{\partial x} \frac{\partial \hat{x}_j(x)}{\partial x} + \sum_{\alpha} \int dX \left\{ \frac{1}{2M} \frac{\partial \hat{\eta}^+_\alpha(X)}{\partial X} \frac{\partial \hat{\eta}_\alpha(X)}{\partial X} + \varepsilon_\alpha \hat{\eta}^+_\alpha(X) \hat{\eta}_\alpha(X) \right\}, \quad M = m_1 + m_2,$$

where $\hat{x}^+_j(x)$, $\hat{x}_j(x)$ ($j = 1, 2$) are the creation and annihilation operators of a free fermion of $j$ kind and mass $m_j$ at the point $x$; $\hat{\eta}^+_\alpha(X)$, $\hat{\eta}_\alpha(X)$ are the creation and annihilation operators of bound states of two different fermions (“hydrogen-like atoms”) with the quantum numbers $\alpha$ at the point $X$; $\varepsilon_\alpha$ is the energy of an atom at the level with the quantum numbers $\alpha$.

The Hamiltonian $\hat{H}_{\text{int}}$ in the equation (11) describes the interaction between particles

$$\hat{H}_{\text{int}} = \hat{H}^1_{\text{int}} + \hat{H}^2_{\text{int}} + \hat{H}^3_{\text{int}},$$

where

$$\hat{H}^1_{\text{int}} = \int dx_1 dx_2 dy \hat{\phi}^+(x_2, y) \hat{\phi}(x_2, y) \times \{ (\nu_{11}(x_1 - x_2) + \nu_{21}(x_1 - y)) \hat{\chi}^+_1(x_1) \hat{\chi}_1(x_1) \}
\quad + (\nu_{22}(x_1 - y) + \nu_{12}(x_1 - x_2)) \hat{\chi}^+_2(x_1) \hat{\chi}_2(x_1) \},$$

$$\hat{H}^2_{\text{int}} = \frac{1}{2} \int dx_1 dx_2 dy_1 dy_2 \hat{\phi}^+(x_1, y_1) \times \hat{\phi}^+(x_2, y_2) \hat{\phi}(x_1, y_1) \times \{ \nu_{11}(x_1 - x_2) + \nu_{22}(y_1 - y_2) + \nu_{12}(x_1 - y_2) + \nu_{21}(y_1 - x_2) \},$$

$$\hat{H}^3_{\text{int}} = \frac{1}{2} \int dx_1 dx_2 \times \{ \nu_{11}(x_1 - x_2) \hat{\chi}^+_1(x_1) \hat{\chi}^+_1(x_1) \hat{\chi}_1(x_1) + \nu_{22}(x_1 - x_2) \hat{\chi}^+_2(x_1) \hat{\chi}^+_2(x_1) \hat{\chi}_2(x_1) + 2 \nu_{12}(x_1 - x_2) \hat{\chi}^+_1(x_1) \hat{\chi}^+_2(x_1) \hat{\chi}_2(x_1) \}.$$
And, finally, the operator $\hat{V}(t)$ in (1) represents the Hamiltonian that describes the interaction of particles with the electromagnetic field

$$\hat{V}(t) = -\frac{1}{c} \int d\mathbf{x} \hat{A}(\mathbf{x}, t) \hat{J}(\mathbf{x}, t)$$

$$- \frac{1}{2c^2} \int d\mathbf{x} \hat{A}^2(\mathbf{x}, t) \sum_{i=1}^{2} \frac{e_i}{m_i} \hat{\sigma}(\mathbf{x})$$

$$+ \int d\mathbf{x} \varphi^{(e)}(\mathbf{x}, t) \hat{\sigma}(\mathbf{x}), \quad \hat{\sigma}(\mathbf{x}) = \sum_{i=1}^{2} \hat{\sigma}_i(\mathbf{x}).$$

In this expression we have taken into account an interaction of particles with the external electromagnetic field $\hat{A}^{(e)}(\mathbf{x}, t)$, $\varphi^{(e)}(\mathbf{x}, t)$ ($\varphi^{(e)}(\mathbf{x}, t)$ is the scalar potential of the external electromagnetic field) and the quantum electromagnetic field, that is described by the potential $\hat{a}(\mathbf{x})$ (Coulomb’s gauge):

$$\hat{A}(\mathbf{x}, t) = \hat{a}(\mathbf{x}) + \hat{A}^{(e)}(\mathbf{x}, t),$$

where $\hat{A}^{(e)}(\mathbf{x}, t)$ is the vector potential of the external electromagnetic field and $\hat{a}(\mathbf{x})$ is the quantum electromagnetic field operator, that is defined by the expression

$$\hat{a}(\mathbf{x}) = \sum_{k}^{2} \sum_{\lambda=1}^{2} \left( \frac{2\pi}{V \omega_k} \right)^{1/2} \left( e_{k\lambda} C_{k\lambda} e^{i\mathbf{k} \cdot \mathbf{x}} + h.c. \right)$$

($V$ is the system volume, $e_{k\lambda}$ is the photon polarization vector).

The charge density operators $\hat{\sigma}_i(\mathbf{x})$ for particles of $i$ kind (see (10)) are connected with the density operators $\hat{\rho}_i(\mathbf{x})$ (see 2)

$$\hat{\sigma}_i(\mathbf{x}) = e_i \hat{\rho}_i(\mathbf{x}),$$

$$\hat{\rho}_1(\mathbf{x}) = \hat{\chi}_1^+(\mathbf{x}) \hat{\chi}_1(\mathbf{x}) + \int d\mathbf{y} \int d\mathbf{Y} \delta(\mathbf{x} - \mathbf{y} - \frac{m_2}{M} \mathbf{y}) \hat{\varphi}^+(\mathbf{y}, \mathbf{Y}) \hat{\varphi}(\mathbf{y}, \mathbf{Y}),$$

$$\hat{\rho}_2(\mathbf{x}) = \hat{\chi}_2^+(\mathbf{x}) \hat{\chi}_2(\mathbf{x}) + \int d\mathbf{y} \int d\mathbf{Y} \delta(\mathbf{x} - \mathbf{y} - \frac{m_1}{M} \mathbf{y}) \hat{\varphi}^+(\mathbf{y}, \mathbf{Y}) \hat{\varphi}(\mathbf{y}, \mathbf{Y}),$$

and, as easy to see, in the equation (12) we have also taken into account a contribution that had been made by charged particles, that are represented in the bound states (see 8). The current density operator $\hat{J}(\mathbf{x}, t)$ in the formula (10) can be also expressed in terms of the creation and annihilation operators

$$\hat{J}(\mathbf{x}, t) = -\hat{A}(\mathbf{x}, t) \sum_{i=1}^{2} \frac{e_i}{m_i} \hat{\sigma}_i(\mathbf{x}) + \hat{j}(\mathbf{x}),$$

$$\hat{j}(\mathbf{x}) = \sum_{i=1}^{2} \frac{e_i}{m_i} \hat{\pi}_i(\mathbf{x}),$$

where the momentum density operators $\hat{\pi}_i(\mathbf{x})$ are defined by expressions

$$\hat{\pi}_1(\mathbf{x}) = -\frac{i}{2} \left( \frac{\hat{\chi}_1^+(\mathbf{x}) \partial \hat{\chi}_1(\mathbf{x})}{\partial \mathbf{x}} - \frac{\partial \hat{\chi}_1^+(\mathbf{x})}{\partial \mathbf{x}} \hat{\chi}_1(\mathbf{x}) \right),$$

$$- \frac{i}{2} \int d\mathbf{y} \int d\mathbf{Y} \delta(\mathbf{x} - \mathbf{Y} - \frac{m_2}{M} \mathbf{y})$$

$$\times \left[ \frac{\partial \hat{\varphi}(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{y}} - \frac{\partial \hat{\varphi}^+(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{y}} \hat{\varphi}(\mathbf{y}, \mathbf{Y}) \right]$$

$$+ \frac{m_1}{M} \left( \frac{\hat{\varphi}^+(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{Y}} \hat{\varphi}(\mathbf{y}, \mathbf{Y}) \right) \right),$$

$$\hat{\pi}_2(\mathbf{x}) = -\frac{i}{2} \left( \frac{\hat{\chi}_2^+(\mathbf{x}) \partial \hat{\chi}_2(\mathbf{x})}{\partial \mathbf{x}} - \frac{\partial \hat{\chi}_2^+(\mathbf{x})}{\partial \mathbf{x}} \hat{\chi}_2(\mathbf{x}) \right),$$

$$- \frac{i}{2} \int d\mathbf{y} \int d\mathbf{Y} \delta(\mathbf{x} - \mathbf{Y} - \frac{m_1}{M} \mathbf{y})$$

$$\times \left[ -\frac{\partial \hat{\varphi}(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{y}} + \frac{\partial \hat{\varphi}^+(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{y}} \hat{\varphi}(\mathbf{y}, \mathbf{Y}) \right]$$

$$+ \frac{m_2}{M} \left( \frac{\hat{\varphi}^+(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{Y}} \hat{\varphi}(\mathbf{y}, \mathbf{Y}) \right) \right].$$

Using formulas (12)–(14) we can write the expressions for the current and charge density operators in more suitable way:

$$\hat{\sigma}(\mathbf{x}) = \sum_{a} \hat{\sigma}_a(\mathbf{x}),$$

$$\hat{j}(\mathbf{x}) = \sum_{a} \hat{j}_a(\mathbf{x}), \quad a = 0, 1, 2,$$

where

$$\hat{\sigma}_i(\mathbf{x}) = e_i \hat{\chi}_i^+(\mathbf{x}) \hat{\chi}_i(\mathbf{x}), \quad i = 1, 2,$$

$$\hat{j}_a(\mathbf{x}) = -\frac{i e_i}{2m_i} \left( \hat{\chi}_i^+(\mathbf{x}) \frac{\partial \hat{\chi}_i(\mathbf{x})}{\partial \mathbf{x}} - \frac{\partial \hat{\chi}_i^+(\mathbf{x})}{\partial \mathbf{x}} \hat{\chi}_i(\mathbf{x}) \right),$$

$$\hat{\sigma}_0(\mathbf{x}) = \int d\mathbf{y} \int d\mathbf{Y} \left[ e_1 \delta(\mathbf{x} - \mathbf{Y} - \frac{m_2}{M} \mathbf{y}) + e_2 \delta(\mathbf{x} - \mathbf{Y} - \frac{m_1}{M} \mathbf{y}) \right] \hat{\varphi}^+(\mathbf{y}, \mathbf{Y}) \hat{\varphi}(\mathbf{y}, \mathbf{Y}),$$

$$\hat{j}_0(\mathbf{x}) = -\frac{i}{2} \int d\mathbf{y} \int d\mathbf{Y}$$

$$\times \left[ \frac{e_1}{m_1} \delta(\mathbf{x} - \mathbf{Y} - \frac{m_2}{M} \mathbf{y}) - \frac{e_2}{m_2} \delta(\mathbf{x} - \mathbf{Y} - \frac{m_1}{M} \mathbf{y}) \right]$$

$$\times \left( \frac{\partial \hat{\varphi}(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{y}} - \frac{\partial \hat{\varphi}^+(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{y}} \hat{\varphi}(\mathbf{y}, \mathbf{Y}) \right)$$

$$- \frac{i}{2} \int d\mathbf{y} \int d\mathbf{Y}$$

$$\times \left[ e_1 \delta(\mathbf{x} - \mathbf{Y} - \frac{m_2}{M} \mathbf{y}) + e_2 \delta(\mathbf{x} - \mathbf{Y} - \frac{m_1}{M} \mathbf{y}) \right]$$

$$\times \left( \hat{\varphi}^+(\mathbf{y}, \mathbf{Y}) \frac{\partial \hat{\varphi}(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{Y}} - \frac{\partial \hat{\varphi}^+(\mathbf{y}, \mathbf{Y})}{\partial \mathbf{Y}} \hat{\varphi}(\mathbf{y}, \mathbf{Y}) \right).$$
As is easy to see, the operators $\hat{\sigma}_0(x)$, $\hat{\mathbf{j}}_0(x)$ in these expressions define the bound states contribution to the charge and current densities.

In the momentum representation

$$\hat{\chi}_i(x) = \frac{1}{\sqrt{V}} \sum_{p} e^{i p x} \hat{a}_{i p}, \quad i = 1, 2,$$

$$\hat{\eta}_0(x) = \frac{1}{\sqrt{V}} \sum_{p} e^{i p x} \hat{a}_{0 p},$$

expressions (16) according to (8) will have the next form:

$$\hat{\sigma}_0(x) = \frac{1}{V} \sum_{p, p'} \sum_{\alpha, \beta} e^{i (p' - p) \cdot x} \sigma_{\alpha \beta}(p - p') \hat{a}_\alpha^+ \hat{a}_{\beta p'},$$

$$\hat{\mathbf{j}}_0(x) = \frac{e_1}{2m_1 V} \sum_{p, p'} e^{i (p' - p) \cdot x} (p + p') \hat{a}_\alpha^+ \hat{a}_{\beta p'},$$

and

$$\hat{\mathbf{j}}_0(x) = \frac{1}{V} \sum_{p, p'} \sum_{\alpha, \beta} e^{i (p' - p) \cdot x} (i_{\alpha \beta}(p - p') + \frac{(p + p')}{2m} \sigma_{\alpha \beta}(p - p')) \hat{a}_\alpha^+ \hat{a}_{\beta p'},$$

where (see (5))

$$\sigma_{\alpha \beta}(k) = \int d\mathbf{y} \varphi_\alpha^*(\mathbf{y}) \varphi_\beta(\mathbf{y})$$

$$\times \left[ e_1 \exp i \frac{m_2}{M} k \mathbf{y} + e_2 \exp (-i \frac{m_1}{M} k \mathbf{y}) \right],$$

$$I_{\alpha \beta}(k) = -\frac{i}{2} \int d\mathbf{y}$$

$$\times \left( \varphi_\alpha^*(\mathbf{y}) \frac{\partial \varphi_\beta(\mathbf{y})}{\partial \mathbf{y}} - \varphi_\alpha(\mathbf{y}) \frac{\partial \varphi_\beta^*(\mathbf{y})}{\partial \mathbf{y}} \right)$$

$$\times \left[ e_1 \exp i \frac{m_2}{M} k \mathbf{y} - e_2 \exp (-i \frac{m_1}{M} k \mathbf{y}) \right].$$

It is significant to note, that the Hamiltonian for free particles $\mathcal{H}_p$ (see (11), (13)) in the momentum representation can be written as

$$\mathcal{H}_p = \mathcal{H}_1 p + \mathcal{H}_2 p + \mathcal{H}_0 p,$$

$$\mathcal{H}_1 p = \sum_{p} \varepsilon_i^0(p) \hat{a}_{i p}^+ \hat{a}_{i p}, \quad i = 1, 2,$$

$$\mathcal{H}_0 p = \sum_{\alpha} \varepsilon_\alpha(p) \hat{a}_{\alpha p}^+ \hat{a}_{\alpha p},$$

$$\varepsilon_i^0(p) = p^2/2m_i, \quad \varepsilon_\alpha(p) = \varepsilon_\alpha + p^2/2M,$$

where $\varepsilon_\alpha$ is the energy of the atomic level with quantum numbers $\alpha$, $M$ is the bound state mass, $M = (m_1 + m_2)$.

The Maxwell equations for our system according to [2] can be written in the following form

$$\frac{\partial \hat{\mathbf{E}}}{\partial t} = -\mathbf{crot}\mathbf{E}, \quad \mathbf{div}\mathbf{E} = 4\pi (\hat{\sigma} + \sigma^{(c)}),$$

$$\frac{\partial \hat{\mathbf{H}}}{\partial t} = \mathbf{crot}\hat{\mathbf{H}} - 4\pi (\hat{\mathbf{j}} + \mathbf{J}^{(c)}), \quad \mathbf{div}\hat{\mathbf{H}} = 0,$$

where operators $\hat{\sigma}$, $\hat{\mathbf{j}}$ are still defined by the expressions (10), (12) and values $\sigma^{(c)}$, $\mathbf{J}^{(c)}$ are the external current and charge densities. The electric $\mathbf{E}$ and magnetic $\mathbf{H}$ field intensity operators in terms of the scalar and vector potentials can be expressed as (see (11), (2)) and also equations (10), (11))

$$\hat{\mathbf{E}} = \mathbf{rot}\hat{\mathbf{A}},$$

$$\hat{\mathbf{E}} = -\frac{1}{c} \frac{\partial \hat{\mathbf{A}}}{\partial t} - \frac{\partial}{\partial \mathbf{x}} \left( \varphi^{(c)} + \int d\mathbf{x}' \frac{\hat{\sigma}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right).$$

Note that in deriving the electrodynamic equations we used the Coulomb’s gauge.

3 The system response to the perturbation by the external electromagnetic field

In this section to study the system response to the perturbing action of the external electromagnetic field we will follow the principles that have been stated in (1). Let us consider a system that at some moment of time $t$ is characterized by statistical operator $\rho(t)$. Considering that the Hamiltonian of interaction $\hat{V}(t)$ is linear in respect to the external field and assuming that it is small in comparison with the Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}}$ (see (1)), we can develop the perturbation theory over the week interaction. In accordance with (1) the mean value of an arbitrary quasiloquid operator $\hat{a}(x)$ in linear approach for such system can be written as

$$\text{Sp}_\rho(t) \hat{a}(x) = \text{Sp}_\rho(0) + a^F(x, t),$$

$$a^F(x, t) = \int_{-\infty}^{\infty} dt' \int d\mathbf{x}'$$

$$\times G^{(s)}(\mathbf{x} - \mathbf{x}', t - t') F_1(\mathbf{x}', t'),$$

where $w$ is the Hibbs distribution operator

$$w = \exp \{ \Omega t - \beta (\hat{\mathcal{H}} - \mu_1 \hat{N}_1 - \mu_2 \hat{N}_2) \},$$

$\beta = 1/T$ is the reciprocal temperature, $\hat{N}_1, \hat{N}_2$ are the density operators of all fermions of the first and second kind (including fermions in bound states, see (12))

$$\hat{N}_1 = \int d\mathbf{x} \hat{\rho}_1(\mathbf{x}), \quad \hat{N}_2 = \int d\mathbf{x} \hat{\rho}_2(\mathbf{x})$$

and $\mu_1, \mu_2$ are the chemical potentials of fermions of the first and second kind. The thermodynamic parameters $\beta, \mu_1, \mu_2$ can be found from the relations

$$\text{Sp} \hat{\mathcal{H}} = \mathcal{H}, \quad \text{Sp} \hat{\mathcal{N}}_1 = N_1, \quad \text{Sp} \hat{\mathcal{N}}_2 = N_2.$$
and the thermodynamic potential $\Omega$ dependence on thermodynamic parameters is defined by the expression

$$\text{Sp}_w = 1.$$  

In the formula (22) $F_i(x, t)$ are the quantities, that define the external field and $\xi_i(x)$ are quasilocal operators, related to our system (see also (1)); the summation convention is assumed for the repeated index $i$.

And, finally, the quantity $G_{a\xi_i}^{(+)}(x - x', t - t')$ in the expression (22) is the two-time retarded Green function (note, that "tilde" over operators means that they are taken in the Heisenberg representation)

$$G_{a\xi_i}^{(+)}(x - x', t - t') = -i\theta(t - t')$$

$$\times \text{Sp}_w[\hat{a}(x, t), \xi_i(x', t')]$$

where $\theta(t)$ is Heaviside function

$$\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

Going over to Fourier transforms of values $a^F_i$, $F_i$

$$a^F_i(x, \omega) = \frac{1}{(2\pi)^4} \int dkd\omega x(t + k\omega) a^F_i(k, \omega),$$

$$F_i(x, \omega) = \frac{1}{(2\pi)^4} \int dkd\omega x(t + k\omega) F_i(k, \omega)$$

one obtains

$$a^F_i(k, \omega) = G_{a\xi_i}^{(+)}(k, \omega) F_i(k, \omega),$$

where

$$G_{a\xi_i}^{(+)}(k, \omega) = \int_{-\infty}^{\infty} dt \int d^3x e^{i(k\omega - kx)} G_{a\xi_i}^{(+)}(x, t).$$

It is significant, that in terms of the Fourier transforms of the introduced quantities we can express also the energy, transferred from field to matter. If to assume, that field is acting only for a limited period of time, the total energy $Q$, received by matter, is given by the expression (1)

$$Q = \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int d^3k \omega \int d^3k \omega \times F_i(-k, -\omega) G_{a\xi_i}^{(+)}(k, \omega) F_j(k, \omega).$$

Now we can apply these expressions for studying the response of the system, consisting of two kinds of oppositely charged fermions and their bound states. To make use of the Green function method, that was described above (see (22), (23), it is more convenient to represent the system Hamiltonian, that is defined by formulas (1), as

$$\hat{H}(t) = \hat{H} + \hat{V}^{(e)}(t), \quad \hat{H} = \hat{H}_0 + \hat{H}_{int} + \hat{V},$$

where $\hat{H}_0$ and $\hat{H}_{int}$ are given by the formulas (1), (7), $\hat{V}$ is defined by the expression (see also (10)–(13));

$$\hat{V} = -\frac{1}{c} \int dx \hat{a}^\dagger(x, t) \hat{J}^\dagger(x, t)$$

$$- \frac{1}{2c^2} \int dx \hat{a}^2(x, t) \sum_{i=1}^{m_i} e_i \hat{\sigma}_i(x),$$

and the Hamiltonian $\hat{V}^{(e)}(t)$ describes the system interaction with the external electromagnetic field

$$\hat{V}^{(e)}(t) = -\frac{1}{c} \int dx A^{(e)}(x, t) \hat{J}^\dagger(x)$$

$$+ \frac{1}{2c^2} \int dx A^{(e)}(x, t) \sum_{i=1}^{m_i} \frac{e_i}{m_i} \hat{\sigma}_i(x)$$

$$+ \int dx \varphi^{(e)}(x, t) \hat{\sigma}(x).$$

To get the Maxwell equations for the electromagnetic field in medium, it is necessary to average the equations (20) with the system statistical operator containing the information both about medium and electromagnetic field. To this end we will define the mean values of the electromagnetic fields $E(x, t)$, $H(x, t)$, acting in the matter

$$E(x, t) = \text{Sp}_P(t) E(x, t), \quad H(x, t) = \text{Sp}_P(t) H(x, t),$$

and also the induced charge and current averages (see (12)–(14))

$$J(x, t) = \text{Sp}_P(t) J(x, t), \quad \sigma(x, t) = \text{Sp}_P(t) \sigma(x).$$

The equations (20), averaged in accordance with the formulas (23)–(24), bring us to the Maxwell–Lorentz equations for the average fields in the matter

$$\frac{\partial E}{\partial t} = -\text{rot} E, \quad \text{div} E = 4\pi \sigma + \sigma^{(e)},$$

$$\frac{\partial H}{\partial t} = \text{rot} H - 4\pi (J + J^{(e)}), \quad \text{div} H = 0,$$

where quantities $\sigma^{(e)}$, $J^{(e)}$ still represent the extrinsic charge and current densities.

The next problem is to find the charge $\sigma(x, t)$ and current $J(x, t)$ densities induced by the external field. Calculating these quantities under assumption of week interaction between the system and the external field we will use the equations (22)–(25), considering the potentials $A^{(e)}(x, t)$, $\varphi^{(e)}(x, t)$ as $F_i(x, t)$, and $\hat{\sigma}(x)$ or $J(x, t)$ as quasilocal operator $a(x)$. As a result one
gets

\[ \bar{\sigma}(x, t) = \sum_a \sigma_a + \int_{-\infty}^{\infty} dt' \int d^3 x' \]
\[ \times \left[ -G_i^{(+)}(x - x', t - t') \frac{1}{c} A_i^{(+)}(x', t') + G^{(+)}(x - x', t - t') \varphi^{(+)}(x', t') \right], \]
\[ = \frac{1}{c} A_k^{(+)}(x, t) \sum_a \frac{e_a}{m_a} \sigma_a \]
\[ + \int_{-\infty}^{\infty} dt' \int d^3 x' \left[ -G_k^{(+)}(x - x', t - t') \frac{1}{c} A_k^{(+)}(x', t') + G^{(+)}(x - x', t - t') \varphi^{(+)}(x', t') \right], \]
\[ \bar{\sigma}(x, t) = \sum_a \sigma_a + \int dt' \int d^3 x' \]
\[ \times \left[ -G_i^{(+)}(x - x', t - t') \frac{1}{c} A_i^{(+)}(x', t') + G^{(+)}(x - x', t - t') \varphi^{(+)}(x', t') \right], \]
\[ + \int_{-\infty}^{\infty} dt' \int d^3 x' \left[ -G_k^{(+)}(x - x', t - t') \frac{1}{c} A_k^{(+)}(x', t') + G^{(+)}(x - x', t - t') \varphi^{(+)}(x', t') \right], \]
(36)

where \( \sigma_a = Spw \hat{\sigma}_a(0), a = 1, 2, 0 \) (see (17)), \( w \) is the Hibbs statistical operator (23). It is significant to note, that for quasineutral systems, where the number of fermions are equal \((N_1 = N_2, see (24))\) and their absolute charge values are also equal \(|e_1| = |e_2| \sum a \sigma_a = 0.\)

The retarded charge and current Green functions, that are included in the expression (36), are determined in accordance with the formula (26) (see also [1]):

\[ G^{(+)}(x, t) = -i\theta(t)Spw[\hat{\sigma}(x, t), \hat{\sigma}(0)], \]
\[ G_k^{(+)}(x, t) = -i\theta(t)Spw[\hat{j}_k(x, t), \hat{\sigma}(0)] \]
\[ \bar{G}_k^{(+)}(x, t) = -i\theta(t)Spw[\hat{\sigma}(x, t), \hat{j}_k(0)], \]
\[ G_{kl}^{(+)}(x, t) = -i\theta(t)Spw[\hat{j}_k(x, t), \hat{j}_l(0)]. \]
(37)

As the charge and current density operators of particles of different kinds (see (17)) commute with each other

\[ [\hat{\sigma}_a, \hat{j}_b] = [\hat{\sigma}_a, \hat{j}_b] = [\hat{j}_b, \hat{j}_b] = 0, a \neq b, a, b = 1, 2, 0, \]
then, according to the equations (15), (17), (37), the contribution of different kinds of particles to Green functions will be additive

\[ G^{(+)}(x, t) = \sum_a G_a^{(+)}(x, t), \]
\[ G_k^{(+)}(x, t) = \sum_a G_{ak}^{(+)}(x, t), \]
\[ \bar{G}_k^{(+)}(x, t) = \sum_a G_{ak}^{(+)}(x, t), \]
\[ G_{kl}^{(+)}(x, t) = \sum_a G_{akl}^{(+)}(x, t), \]
\[ G_{ak}^{(+)}(x, t) = -i\theta(t)Spw[\hat{\sigma}_a(x, t), \hat{j}_k(0)], \]
\[ G_{akl}^{(+)}(x, t) = -i\theta(t)Spw[\hat{j}_k(x, t), \hat{j}_l(0)]. \]
(38)

With the help of direct calculations, following the method [1], we can see, that also in the presence of particle bound states the following correspondence between Green functions (37) takes place

\[ \bar{G}_k^{(+)}(x, t) = G_k^{(+)}(x, t), \]
\[ \frac{\partial G_k^{(+)}(x, t)}{\partial x_k} + \frac{\partial G^{(+)}(x, t)}{\partial t} = 0, \]
\[ \frac{\partial G_{kl}^{(+)}(x, t)}{\partial x_k} + \frac{\partial G_{kl}^{(+)}(x, t)}{\partial x_l} \]
\[ + \sum_a \frac{e_a}{m_a} \sigma_a \delta(t) \partial \delta(x) = 0. \]
(39)

For the Green functions Fourier transforms (see (28)) these relations can be written as

\[ \bar{G}_k^{(+)}(k, \omega) = G_k^{(+)}(k, \omega), \]
\[ G_{kl}^{(+)}(k, \omega) \delta_{kl} - \omega G_{kl}^{(+)}(k, \omega) = 0, \]
\[ G_{ij}^{(+)}(k, \omega) \delta_{ij} - \omega G_{ij}^{(+)}(k, \omega) + k_i \sum_a \frac{e_a}{m_a} \sigma_a = 0. \]
(40)

4 Green functions and macroscopic characteristics of the ideal low-temperature hydrogen-like plasma

If we neglect of all interactions between particles in the investigated system, it can be considered as an ideal hydrogen-like low-temperature plasma (we note, that kinetic energy of particles must be small in comparison with the binding energy of compound particles). For an ideal hydrogen-like plasma the Green functions, that was introduced earlier, can be calculated exactly. To do it we must take into consideration, that neglecting the quantum fields presence, the Hamiltonian \( \hat{\mathcal{H}} \) in the formula (30) must be interpreted as \( \hat{\mathcal{H}}_p \), see (11), (3), (19). Taking into account this fact the Heisenberg representation of charge and current density operators, that appear in the equations (37) for the Green functions, is defined by expressions:

\[ \hat{\sigma}_i(x, t) = \frac{e_i}{V} \sum_{p-p'} e^{-i\mathcal{C}(p-p')} e^{-it(\varepsilon_i(p) - \varepsilon_i(p'))} \hat{a}_{ip}^+ \hat{a}_{ip'}, \]
\[ \hat{j}_i(x, t) = \frac{1}{2m_i} \hat{\sigma}_i(x, t), \]
\[ \hat{\sigma}_0(x, t) = \frac{1}{V} \sum_{p-p'} \sum_{\alpha, \beta} e^{-i\mathcal{C}(p-p')} e^{-it(\varepsilon_0(p) - \varepsilon_\beta(p'))} \]
\[ \times \sigma_{\alpha\beta}(p - p') \hat{a}_{ip}^+ \hat{a}_\beta(p'), \]
\[ \hat{j}_0(x, t) = \frac{1}{V} \sum_{p-p'} \sum_{\alpha, \beta} e^{-i\mathcal{C}(p-p')} e^{-it(\varepsilon_0(p) - \varepsilon_\beta(p'))} \]
\[ \times \left( \frac{p + p'}{2M} \sigma_{\alpha\beta}(p - p') \]
where quantities $\sigma_{\alpha\beta}(k)$, $I_{\alpha\beta}(k)$ are given by the formulas [18]. If to substitute the operators (41) in (38) and to do some calculations, we will come to the following expressions for the Fourier transforms of scalar Green functions (see [25]):

$$G^{(+)}_1(k, \omega) = \frac{e^2}{V} \sum_p \frac{f_1(p-k) - f_1(p)}{\varepsilon_1(p) - \varepsilon_1(p-k) + \omega + i0}.$$  

$$G^{(+)}_2(k, \omega) = \frac{e^2}{V} \sum_p \frac{f_2(p-k) - f_2(p)}{\varepsilon_2(p) - \varepsilon_2(p-k) + \omega + i0}.$$  

$$G^{(+)}_0(k, \omega) = \frac{1}{V} \sum_p \sum_{\alpha, \beta} \frac{\sigma_{\alpha\beta}(k)\sigma_{\alpha\beta}(-k)}{\varepsilon_{\alpha}(p) - \varepsilon_{\beta}(p-k) + \omega + i0}.$$

(42)

Similar expressions for the vector Green functions have the form:

$$G^{(+)}_{1t}(k, \omega) = \frac{e^2}{2m_1V} \sum_p (2p - k)_l$$  

$$\times \frac{f_1(p-k) - f_1(p)}{\varepsilon_1(p) - \varepsilon_1(p-k) + \omega + i0},$$

$$G^{(+)}_{2t}(k, \omega) = \frac{e^2}{2m_2V} \sum_p (2p - k)_l$$  

$$\times \frac{f_2(p-k) - f_2(p)}{\varepsilon_2(p) - \varepsilon_2(p-k) + \omega + i0},$$

$$G^{(+)}_{0t}(k, \omega) = \frac{1}{V} \sum_p \sum_{\alpha, \beta} \frac{\sigma_{\alpha\beta}(k)[f_{\alpha}(p-k) - f_{\beta}(p)]}{\varepsilon_{\alpha}(p) - \varepsilon_{\beta}(p-k) + \omega + i0}.$$

(43)

And, finally, the tensor Green functions for the investigated system is given by expressions:

$$G^{(+)}_{1ls}(k, \omega) = \frac{e^2}{4m_1V} \sum_p \frac{(2p - k)_l(2p - k)_s}{\varepsilon_1(p) - \varepsilon_1(p-k) + \omega + i0}$$

$$\times \frac{f_1(p-k) - f_1(p)}{\varepsilon_1(p) - \varepsilon_1(p-k) + \omega + i0},$$

$$G^{(+)}_{2ls}(k, \omega) = \frac{e^2}{4m_2V} \sum_p \frac{(2p - k)_l(2p - k)_s}{\varepsilon_2(p) - \varepsilon_2(p-k) + \omega + i0}$$

$$\times \frac{f_2(p-k) - f_2(p)}{\varepsilon_2(p) - \varepsilon_2(p-k) + \omega + i0},$$

$$G^{(+)}_{0ls}(k, \omega) = \frac{1}{V} \sum_p \sum_{\alpha, \beta} \frac{[2p - k]_{\alpha\beta}(k)}{2M} \sigma_{\alpha\beta}(k)$$

$$\times \frac{f_{\alpha}(p-k) - f_{\beta}(p)}{\varepsilon_{\alpha}(p) - \varepsilon_{\beta}(p-k) + \omega + i0}.$$

(44)

In the expressions (43–44) we have introduced the distribution functions for free fermions of the first $f_1(p)$ and second $f_2(p)$ kind, and also the distribution functions $f_{\alpha}(p)$ for hydrogen-like atoms (bound states) with the set of quantum numbers $\alpha$

$$f_{\alpha}(p) = \{(\varepsilon_{\alpha}(p) - \mu_{\alpha})/T + 1\}^{-1},$$

$$f_0(p) = \{(\varepsilon_0(p) - \mu_0)/T + 1\}^{-1}$$

(45)

in accordance with the equations

$$\delta p a^+_{\alpha \rho} a_{\rho p} = \delta_{\alpha \rho} f_0(p), \ i = 1, 2,$$

$$\delta p \tilde{\eta}_{\alpha}(p) \tilde{\eta}_{\rho}(p) = \delta_{\alpha \rho} \delta_{p \rho} f_{\alpha}(p).$$

(46)

The particle energies $\varepsilon_{1, 2}(p)$, $\varepsilon_{0}(p)$ in formulas (42–45) are given by the expressions (19), and values $\delta_{\alpha \beta}, \delta_{p \rho}$ in (18) are the Kronecker symbols.

The particular feature of the obtained Green functions is that the contribution of bound states in the processes under consideration now is taken into account.

The expressions for Green functions that have been found allow us to get an expressions for the matter macroscopic parameters, such as conductivity, permittivity and magnetic permeability. To this end we will also use the method described in [11].

In accordance with the formulas (45), the next relation between Fourier transforms takes place

$$\tilde{J}_l(k, \omega) = \tilde{\sigma}^l(k, \omega) k_l \frac{k E^{(e)}(k, \omega)}{k^2}$$

$$+ \tilde{\sigma}^l(k, \omega) \frac{[k, E^{(e)}(k, \omega)]}{k^2},$$

$$\tilde{\sigma}(k, \omega) = \frac{1}{\omega} \tilde{\sigma}^l(k, \omega) k E^{(e)}(k, \omega),$$

(47)

where

$$\tilde{\sigma}^l(k, \omega) = \frac{i}{k^2} G^{(+)}_{ls}(k, \omega),$$

$$\tilde{\sigma}^t(k, \omega) = \frac{i}{\omega} \sum_a \frac{\varepsilon_a - \sigma_a}{m_a} \left( \frac{k}{\kappa^2} G^{(+)}_{ls}(k, \omega) \right)$$

$$+ \frac{1}{2} \left( \delta_{ij} - \frac{k i j}{k^2} \right) G^{(+)}_{ij}(k, \omega).$$

(48)

It is clear from equations (47) that the quantities $\tilde{\sigma}^l$ and $\tilde{\sigma}^t$, expressed in terms of Green functions according to formula (43), define the longitudinal and transversal current density components. They are usually interpreted as outer conductivity coefficients in contrast to inner (or true) longitudinal $\sigma^l$ or transversal $\sigma^t$ conductivity coefficients, that will be defined below. Note that according to (48) these coefficients are also additive quantities

$$\tilde{\sigma}^{l,t}(k, \omega) = \sum_a \tilde{\sigma}^{l,t}_{a}(k, \omega).$$

In terms of the introduced outer conductivity coefficients $\tilde{\sigma}^l$ and $\tilde{\sigma}^t$ we can express the energy, absorbed
from the external field sources (see (29)):  
\[ Q_{\omega k} = \frac{-2}{(2\pi)^3} \text{Im} \frac{1}{\omega} E^{(e)}(k, \omega) G^{(+)}_{ij}(k, \omega) E^{(e)}(k, \omega). \]

From this expression, in accordance with the formulas (47)–(48) one obtains
\[ Q_{\omega k} = \frac{2}{(2\pi)^3} \text{Re} \{ \hat{\delta}(k, \omega) \} | \frac{E^{(e)}(k, \omega)}{|E^{(e)}(k, \omega)|} |^2 \]
\[ + \hat{\delta}(k, \omega) | \frac{E^{(e)}(k, \omega)}{|E^{(e)}(k, \omega)|} |^2 \}. \]

In terms of these quantities (\( \hat{\delta}^t \) and \( \hat{\delta}^t \)) the expressions for permittivity and magnetic permeability can also be defined. The relation between the permittivity and outer conductivity (see (1)) is given by the formula:
\[ \epsilon = \left( 1 + \frac{4\pi \hat{\delta}^t}{i\omega} \right)^{-1}. \]

From this, according to the expression (48):
\[ \epsilon^{-1}(k, \omega) = 1 + \frac{4\pi}{k^2} G^{(+)}(k, \omega). \]

It is more convenient to express the magnetic permeability in terms of inner conductivity coefficients \( \sigma^t \) and \( \sigma^t \)
\[ \mu^{-1}(k, \omega) = 1 + \frac{4\pi \omega}{i\varepsilon^2 k^2} (\sigma^t - \sigma^t), \]
that are connected with the outer conductivity coefficients \( \hat{\delta}^t \) and \( \hat{\delta}^t \) by the relations:
\[ \sigma^t = \varepsilon \hat{\delta}^t, \quad \sigma^t = \frac{\hat{\delta}^t}{1 + \frac{4\pi \hat{\delta}^t}{i\omega} \left( 1 - \frac{k^2 \varepsilon^2}{\omega^2} \right)^{-1}}. \]

In accordance with the expression (49), their relations with the Green functions (12), (14) take the form:
\[ \sigma^t(k, \omega) = \frac{i\omega G^{(+)}(k, \omega)}{k^2 + 4\pi \omega G^{(+)}(k, \omega)}, \]
\[ \sigma^t(k, \omega) = \frac{k^2 \varepsilon^2 - \omega^2}{i\omega} \left( \omega^2 - k^2 \varepsilon^2 \right) + 4\pi A(k, \omega), \]
\[ A(k, \omega) = \frac{1}{n^2} \sum_{\alpha, \beta} \delta_{\beta} - \frac{k_{\beta}}{k^2} G^{(+)}(k, \omega). \]

So, we have defined the main macroscopic characteristics of the ideal hydrogen-like plasma in low temperature region. These characteristics allow us to solve a number of applied problems for our system. Let us demonstrate it on few examples.

Using the developed theory it is not difficult to find the permittivity of an ideal gas of hydrogen-like (alkali) atoms at low temperatures. According to the expressions (48), (50) in neglect of free fermions contribution one gets
\[ \epsilon^{-1}(k, \omega) = 1 + \frac{4\pi}{k^2} \sum_{\alpha, \beta} \frac{1}{f_\alpha(p - k) - f_\beta(p)} \frac{\sigma_\alpha \beta(k) \sigma_\beta \alpha(-k)}{\varepsilon_\alpha(p) - \varepsilon_\beta(p - k) + \omega + i0}. \]

As good known, at extremely low temperatures the Bose-Einstein condensate (BEC) of alkali atoms can be formed. At the temperatures much lower the critical point temperature \( T < T_0 \), see e.g. (1), the bound states distribution functions \( f_\alpha(p) \) are proportional to the Dirac delta-function \( \delta(p) \). Therefore, according to the expressions (19), (53), after integration over momentum \( p \) the expression for permittivity of the studied gas in BEC state \( T < T_0 \) take the form:
\[ \epsilon^{-1}(k, \omega) \approx 1 + \frac{1}{2\pi^2 k^2} \sum_{\alpha, \beta} \sigma_\alpha \beta(k) \sigma_\beta \alpha(-k) \]
\[ \times \left[ \frac{\nu_\alpha}{\omega + \Delta \varepsilon_\alpha \beta - \varepsilon_k + i\gamma_\alpha \beta} + \frac{\nu_\beta}{\omega + \Delta \varepsilon_\alpha \beta + \varepsilon_k + i\gamma_\alpha \beta} \right], \]
where \( \nu_\alpha \) is the density of condensed atoms in the quantum state \( \alpha \), \( \varepsilon_k = k^2/2M \) and quantities \( \sigma_\alpha \beta(k) \) are still defined by the formula (18). Note that due to the damping processes in real systems we also introduced linewidth \( \gamma_\alpha \beta \), concerned with the transition probability from the state \( \alpha \) to the state \( \beta \).

As it easy to see, in the expression (54) at frequencies that are close to the energy interval \( \Delta \varepsilon_\alpha \beta (\Delta \varepsilon_\alpha \beta \equiv \varepsilon_\alpha - \varepsilon_\beta \) some peculiarities appear. In fact, such behavior can strongly reflect on the dispersion characteristics of the studied gas. It is a very interesting question, and we shall return to it in the section (5).

Basing on the developed theory we can also find the energy, that is dissipated by a charged particle when it passes through hydrogen-like plasma at low temperature (see in that case, e.g. (1)). In the case of a small dissipation the particle movement can be considered as uniform. Thus, the particle current density (the external current density in medium, see (20)) will be defined by the formula
\[ J^{(e)}(x, t) = ze \nu \delta(x - vt), \]
where \( ze \) is the particle charge and \( v \) is the particle velocity. It is easy to see, that the Fourier transform of the current density is given by the expression
\[ J^{(e)}(k, \omega) = 2\pi ze \nu \delta(\omega - kv). \]

Next we will use the expression (19) for the energy, absorbed in the matter from external field sources. In this expression, using (17), the Fourier transforms of the longitudinal and transversal components of the external field can be expressed in terms of longitudinal and transversal components of the current particle density (55). If we do the necessary calculations, we will come to the following expression for the energy \( dE_{\omega k} \), that was dissipated by the charged particle per unit time in the frequency \( d\omega \) and the wave vector \( dk \) intervals, when it passes through the hydrogen-like plasma:
\[ dE_{\omega k} = -q_k d\omega dk, \]
\[ q_{k\omega} = \frac{Q_{k\omega}}{T} = -\left( \frac{2\pi}{\varepsilon} \right)^2 \delta(\omega - kv)\omega \times \text{Im} \left[ \frac{v^2}{c^2} - \frac{1}{\varepsilon\mu} \right] \left( \frac{\omega^2}{c^2} - \frac{k^2}{\mu} \right)^{-1}, \]  

where \( T \) is the particle time of flight. To get the expression \( 56 \) it is necessary to use the formula

\[ \delta^2(\omega - kv) = \frac{T}{2\pi}(\delta(\omega - kv)). \]

The total dissipated particle energy \( E \) per unit length can be found by integrating the expression \( 56 \) over \( \omega \) and \( k \)

\[ \frac{dE}{dx} = -\frac{1}{\nu} \int d\omega d^3k q_{k\omega}. \]  

It is easy to see, that the main contribution in this integral comes from poles of the integrand (see \( 56 \))

\[ \varepsilon(k, \omega) = 0, \quad \frac{\omega^2}{c^2}\varepsilon(k, \omega)\mu(k, \omega) - k^2 = 0. \]  

The formulas \( 56 \)–\( 58 \) are similar to the expressions, that are given in [1], however, they take an account of particle bound states (atoms) to all processes that take place in our system (see \( 61 \)–\( 62 \), \( 122 \)–\( 140 \)). Note also that the expressions \( 58 \) represent the dispersion relations for free waves, that can spread in the studied system.

5 Light delay phenomenon for the two-level system in BEC state

In the previous section the expression for the permittivity of the system in BEC state (see Eq. \( 43 \)) was found. For the system with the frequency of the external laser field tuned up to the difference between two defined levels (marked below by indexes 1 and 2, \( \omega \approx \Delta\varepsilon_{21} \)) it can be written in more suitable form:

\[ \varepsilon^{-1}(k, \omega) \approx 1 + \frac{g_1 g_2 |\sigma_{12}(k)|^2}{2\pi^2 k^2} (\nu_1 - \nu_2) \delta(\omega - i\gamma). \]

Here \( \gamma \) is the line width related to the transition probability from the lower to the upper level. Note that we also neglected the term \( \varepsilon_k \), the substantiation of such operation will be discussed below.

To study the propagation properties it is more convenient to turn to the refractive index and damping factor quantities. To do it we set the magnetic permeability that enters the dispersion relation \( 65 \) close to unity \( \mu(k, \omega) = 1 \). In that case the refractive index and damping factor are concerned with the permittivity by well-known relations:

\[ n = \frac{\sqrt{\varepsilon'} + \sqrt{\varepsilon''^2 + \varepsilon''^2}}{\sqrt{2}}, \quad \kappa = \frac{\sqrt{-\varepsilon'} + \sqrt{\varepsilon''^2 + \varepsilon''^2}}{\sqrt{2}}. \]

Here \( \varepsilon' \) and \( \varepsilon'' \) are the real and imaginary part of the permittivity:

\[ \varepsilon(k, \omega) = \varepsilon'(k, \omega) + i\varepsilon''(k, \omega), \]

which can be found from the expression \( 59 \) by taking the real and imaginary part:

\[ \varepsilon' = \frac{\delta\omega(\delta\omega + a) + \gamma^2}{(\delta\omega + a)^2 + \gamma^2}, \quad \varepsilon'' = \frac{\gamma a}{(\delta\omega + a)^2 + \gamma^2}, \]  

(61)

where

\[ a(k) = \frac{(\nu_1 - \nu_2) g_1 g_2 |\sigma_{12}(k)|^2}{2\pi^2 k^2}. \]

Now one can find the dependence of the group velocity on the system parameters:

\[ v_g = \frac{c}{n + \omega(\partial n/\partial \omega)}. \]

(63)

As it easy to see, the ultraslow light phenomenon can be observed in case when matter has a rather strong dispersion, or, according to the formula (63),

\[ \omega(\partial n/\partial \omega) \gg 1. \]

(64)

It is known that to use the group velocity concept the energy dissipation must be rather small, or, in other words, the next condition must take place:

\[ |\varepsilon| \ll |\varepsilon''| \]  

(65)

If \( \varepsilon \sim 1 \), then according to the expression \( 60 \) and relation \( 65 \) \( (\partial n/\partial \omega) \approx 0.5(\partial \varepsilon'/\partial \omega) \). The partial derivative of the permittivity real part can be found from the formula \( 61 \):

\[ \frac{\partial \varepsilon'}{\partial \omega} = a \left[ (\delta\omega + a)^2 - \gamma^2 \right] \left[ (\delta\omega + a)^2 + \gamma^2 \right]^2 \]  

(66)

Using Eqs. \( 63 \)–\( 66 \) for the slowed signal group velocity finally we get:

\[ v_g \approx 2c \frac{(\delta\omega + a)^2 + \gamma^2}{\omega(\delta\omega + a)^2 - \gamma^2}. \]

(67)

Now let us consider the cases in which the light delay phenomenon for the two-level system in BEC state can be observed. To do it we shall go to the limit \( \delta\omega \to 0 \). In that case according to Eq. \( 61 \) the real and imaginary parts of permittivity will have the following limits:

\[ \lim_{\delta\omega \to 0} \varepsilon' = \frac{\gamma^2}{\gamma^2 + a^2}, \quad \lim_{\delta\omega \to 0} \varepsilon'' = \frac{\gamma a}{\gamma^2 + a^2}. \]

Now one can see that the condition for the dissipation smallness \( 65 \) is equivalent to the following condition:

\[ \frac{|a|}{\gamma} \ll 1, \]
Note, that it is necessary also to add the slowing down (strong dispersion) condition (64), which in this case can be written as:

$$\frac{c}{v_g} \sim \Delta \varepsilon_{21} \eta \gg 1,$$

As for the defined system with fixed energy spectrum structure the one parameter that can be changed is occupation difference \(\delta_\nu = (\nu_1 - \nu_2)\), which is included in the parameter \(a\) (see (62)), thus, basing on these relations we get the expression that characterizes the region where the mentioned phenomenon can be observed:

$$\frac{\gamma}{\Delta \varepsilon_{21}} \ll \frac{|a|}{\gamma} \ll 1,$$

Let us demonstrate that such region can exist on the example of cesium hyperfine structure ground state levels. The choice of hyperfine levels is stimulated by their stability and pumping capability (see more in that case in Ref. [3]). Note that for such levels the dipole transitions are forbidden, thus the transitions come from the higher order effects that reflects on the extremely little values of linewidths. It will be shown below that such fact gives the opportunity for signal to propagate with small loss of energy.

It should be mentioned that the description analogically can be spread to the other hydrogen-like atoms and other type of levels. For example, it can be used for description the experiments with ultraslow light in BEC of sodium atoms [4], where for the pulse slowing the three-level system had been used.

As good known, alkali metals (\(^{133}\)Cs in particular) in the ground state do not have the dipole moment, thus the charge density matrix element \(\sigma_{12}\) (see definition (18)) must be expanded to the second order over \((ky) \ll 1\). As a result, one gets:

$$\sigma_{12}(k) \approx \frac{e}{3}(kr_0)^2,$$

where \(r_0\) is the atomic radius (for the cesium ground state \(r_0 \approx 2.6 \times 10^{-8}\) cm [5]), \(e\) is the electron charge. Taking \(g_1 = 7\), \(g_2 = 9\), the linewidth \(\gamma \approx 3.8 \times 10^{-21}\) eV (or \(10^{-6}\) Hz in frequency units, as in "cold atomic clocks" experiments [8]), \(k \approx (\Delta \varepsilon_{21}/c)\), where \(\Delta \varepsilon_{21} \approx 3.8 \times 10^{-5}\) eV (microwaves with frequency \(9.2\) GHz) and basing on the expressions (62), (68) one can find the next region for the occupation difference when light delay phenomenon can be observed:

$$\approx 10^{-3}\ \text{cm}^{-3} \ll |\nu_1 - \nu_2| \ll 3 \times 10^{13}\ \text{cm}^{-3}.$$

In the experiments for the BEC regime reaching the cesium atoms with peak density of \(\nu = 7 \times 10^{10}\) cm\(^{-3}\) was kept in the trap [7], so, according to the inequality (70), one can conclude that also for such experiments the microwave pulse slowing down phenomenon can be observed. Note that the effect becomes greater with the density difference increasing until it reaches the upper limit of the expression (70), when damping effects in the system are great.

Also we must note the following. As it easy to see from Eq. (67) in the limit \(\delta \omega \to 0\), the sign of the group velocity \(v_g\) depends directly on the sign of the quantity \(a\), that in turn depends on the sign of the difference \((\nu_1 - \nu_2)\). In other words, it depends whether population is normal or inverse.

In case of normal population \((\nu_1 > \nu_2)\) the group velocity of signal is negative. It is traditionally considered that the group velocity for the transparent matter is positive. But here one can conclude that due to the relation (68), the signal can propagate in the system with rather small dissipation (i.e., in fact, the matter is transparent) and rather slow velocity. Let us note that an observing of electromagnetic pulses with negative group velocity is not so abnormal. The existence of such kind of phenomena for physical systems when the wave frequency is close to atomic (or molecular) resonances was pointed out and studied in many works (both theoretical, see e.g. [8], and experimental [3, 10]).

In case of inverse population \((\nu_1 < \nu_2)\) more "normal" situation takes place because the group velocity of the slowed pulse is positive.

We stress that such rather unusual phenomenon occurs due to the unique property of hyperfine splitted ground state levels (69). One can show that for the two-level system with allowed dipole transitions the signal will not propagate due to large absorption. The most obvious way out from this situation is the second (coupling) laser using that leads to the electromagnetically induced transparency (EIT, see more in that case in Refs. [4, 11]).

Now, let us say a few words about the quantity \(\varepsilon_k\), that had been neglected in deriving the equation (59). If to take \(k = (\Delta \varepsilon_{21}/c)\), one can find for cesium \(\varepsilon_k \approx 3.5 \times 10^{-30}\) eV. So, even at the point \(\delta \omega = 0\) it is small in comparison with linewidth \(\gamma\), thus, one can note that used approximation is correct.

## 6 Conclusion

Thus, by using the microscopic approach, we studied the linear response of the system with bound states of particles to disturbing effect of an external electromagnetic field. Our approach was based on a novel formulation of the second quantization method in the presence of bound states of particles [2]. Such approach allowed to obtain the expressions for the macroscopic characteristics of the ideal hydrogen-like plasma at low temperatures taking into account not only the contribution of free charged fermions but also their bound states (alkali atoms). The expression for the dielectric permittivity of the ideal gas of alkali atoms in the presence of Bose-Einstein condensation phase was also obtained. The dispersion equation for the waves propagating in the system was derived and the existence of
resonance frequencies was found.

Our approach gave the opportunity to study the propagation properties of the microwave signal, tuned up to the transition between two hyperfine ground state levels of alkali atoms that are considered in BEC state. In contrast to the dipole allowed transitions, it was shown that the signal could propagate in such system with rather small energy loss. Such fact allowed to introduce the group velocity concept. The slowing down conditions for the signal that propagates in BEC were studied. Moreover, we revealed the dependence of the group velocity sign on the population difference sign.

References

[1] Akhiezer A. and Peletminskii S. Methods of Statistical Physics. Pergamon, Oxford (1981).

[2] Peletminskii S. and Slyusarenko Yu. J. Math. Phys. 46 (2005), 022301; quant-ph/0605159.

[3] Slyusarenko Yu. and Sotnikov A. Low Temp. Phys. 33 (2007), 30.

[4] Z. Dutton, N.S. Ginsberg, C. Slowe, and L. V. Hau, Europhysics News 35 (2004).

[5] Clementi E., Raimondi D., and Reinhardt W. J. Chem. Phys. 38 (1963), 2686.

[6] Tiesinga E., Verhaar B., Stoof H., van Bragt D. Phys. Rev. A 4519922671

[7] Guéry-Odelin D., Söding J., Desbiolles P., and Dalibard J. Europhys. Lett. 44 (1998), 25.

[8] Kadomtsev B. Collective phenomena in plasmas. Pergamon, New York (1978).

[9] Chu S. and Wong S. Phys. Rev. Lett. 48 (1982), 738.

[10] Segard B. and Macke B. Phys. Lett. 109A (1985), 213.

[11] Harris S. Physics Today 50 (1997), 36.