Block sparse signal recovery via minimizing the block $q$-ratio sparsity

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Abstract

In this paper, we propose a method for block sparse signal recovery that minimizes the block $q$-ratio sparsity \((\|z\|_2,1/\|z\|_2,q)^{\frac{1}{q-1}}\) with $q \in [0, \infty]$. For the case of $1 < q \leq \infty$, we present the theoretical analyses and the computing algorithms for both cases of the $\ell_2$-bounded and $\ell_{2,\infty}$-bounded noises. The corresponding unconstrained model is also investigated. Its superior performance in block sparse signal reconstruction is demonstrated by numerical experiments.

Keywords: Compressive sensing; Block $q$-ratio sparsity; $q$-ratio block constrained minimal singular value; Nonlinear fractional programming; Convex-concave procedure.

1. Introduction

The last two decades have seen increasing rapid advances in the field of compressive sensing (CS) (e.g., the monographs $^8$, $^{13}$ and references therein). In the standard CS model $y = Ax + \varepsilon$, where $y \in \mathbb{R}^{m \times 1}$ is the vector of measurements, $A \in \mathbb{R}^{m \times N}$ is the pre-given measurement matrix, $x \in \mathbb{R}^N$ is the unknown signal, $\varepsilon$ is the measurement error, and the number of measurements is much less than the length of the signal (i.e., $m \ll N$), we aim to recover the unknown signal $x$ by using the under-determined measurements $y$ and the known matrix $A$. Research in this area has shown that, under the sparsity assumption of the signal, that is $x$ has only a few nonzero entries, and the measurement matrix $A$ is properly chosen (usually has some randomness), we can reliably recover $x$ from $y$ by certain algorithms, such as the following constrained $\ell_1$-minimization $^6$:

$$\min_{z \in \mathbb{R}^N} \|z\|_1 \quad \text{subject to} \quad \|y - Az\|_2 \leq \eta. \quad (1)$$

Meanwhile, to gain better recovery performances, various non-convex algorithms have been proposed, including $\ell_p$ ($0 < p < 1$) $^3$, $^{12}$, $\ell_1 - \ell_2$ $^{20}$, transformed $\ell_1$ (TL1) $^{33}$, smoothly clipped absolute deviation (SCAD) $^{11}$, minimax concave penalty (MCP) $^{32}$, and $\ell_1/\ell_2$ $^{20}$, $^{24}$, among others. As a direct extension of the $\ell_1/\ell_2$ method, very recently $^{37}$ proposed a more general scale invariant approach for sparse recovery via minimizing the $q$-ratio sparsity measure $s_q(z) = (\|z\|_1/\|z\|_2)^{\frac{1}{q}}$ with $q \in [0, \infty]$. However, previous published studies on this kind of scale invariant approaches such as $^{20}$, $^{24}$, $^{37}$ are limited to the case of $q \leq 1$. 

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$^1$This work is supported by the Zhejiang Provincial Natural Science Foundation of China under Grant No. LQ21A010003.
non-block sparse signal recovery. The present paper sets out to investigate the minimization of the block $q$-ratio sparsity measure given in [34] for block sparse signal recovery.

When the nonzero entries of a sparse signal occur in clusters, we use block sparsity to characterize this additional structure. There are a lot of studies on the block sparse model, both on its wide range of practical applications [17, 18, 19] and on its theoretical analysis results [2, 3, 6, 10]. Suppose $N = \sum_{j=1}^{M} d_j$, then the $j$-th block of a length-$N$ vector $x$ over $\mathcal{I} = \{d_1, \cdots, d_M\}$ is denoted by $x[j]$. That means the $j$-th block is of length $d_j$, and the blocks are formed sequentially as follows:

$$ x = \begin{pmatrix} x_1 \cdots x_{d_1} \cdots x_{d_1+d_2} \cdots \cdots x_{d_1+\cdots+d_{M-1}+d_M} \end{pmatrix}^T. $$

(2)

Without loss of generality, for simplicity we may take $d_1 = d_2 = \cdots = d_M = d$ so that $N = Md$. Based on this setting, a vector $x \in \mathbb{R}^N$ is called block $k$-sparse if it has at most $k$ non-zero blocks. In other words, we have $\|x\|_{2,0} = \sum_{j=1}^{M} I(\|x[j]\|_2 \neq 0) \leq k$ for any block $k$-sparse vector $x$.

The corresponding extended versions of sparse algorithms have been developed to reconstruct block sparse signal, such as the mixed $\ell_2/\ell_1$ norm recovery algorithm given in [4]:

$$ \min_{z \in \mathbb{R}^N} \|z\|_{2,1} \quad \text{subject to} \quad \|y - Az\|_2 \leq \eta, $$

(3)

where $\|z\|_{2,1} = \sum_{j=1}^{M} \|z[j]\|_2$. The mixed $\ell_2/\ell_1$ method is the block version of the $\ell_1$-minimization method, while the block version of the non-convex $\ell_p$ ($0 < p < 1$) method is the mixed $\ell_2/\ell_p$ method [27, 28] by solving

$$ \min_{z \in \mathbb{R}^N} \|z\|_{2,p}^{p} \quad \text{subject to} \quad \|y - Az\|_2 \leq \eta, $$

(4)

with $\|z\|_{2,p} = (\sum_{j=1}^{M} \|z[j]\|_2^p)^{1/p}$. Other typical algorithms for block sparse recovery include the mixed $\ell_q/\ell_1$ ($q \geq 1$) norm recovery algorithm [10], group lasso [30], iterative reweighted $\ell_2/\ell_1$ recovery algorithms [31], the $\ell_2/\ell_1-2$ method (the block version of $\ell_1 - \ell_2$ via the minimization of $\|\cdot\|_2 - \|\cdot\|_1$) [26], the block version of Orthogonal Matching Pursuit (OMP) algorithm [1] and the extensions of the Compressive Sampling Matching Pursuit (CoSaMP) algorithm and of the Iterative Hard Thresholding (IHT) to the model-based CS [4], which includes block sparse model as a special case.

This work is inspired by [37], in which a $q$-ratio sparsity minimization based method was proposed for non-block sparse signal recovery. The benefit of this novel method is that it enjoys a superior performance when highly coherent measurement matrices are confronted. In the present paper, we extend this method to the framework of block sparse signal recovery via minimizing the block version of $q$-ratio sparsity given in [34] (namely the block $q$-ratio sparsity). Our main contributions are three folds:

1. We propose the minimization of the block $q$-ratio sparsity for block sparse signal recovery, which extends our previous work [37] from non-block case to block case.
We consider both the $\ell_2$-bounded and $\ell_{2,\infty}$-bounded noise cases, and obtain the stable and robust recovery results in terms of $q$-ratio block constrained minimal singular value (BCMSV). What’s more, the theoretical analyses for the unconstrained-version model are also established.

(3) We present the block version of convex-concave procedure algorithm given in [37] and conduct numerical experiments to show its good performances.

1.1. Organization and Notations

The overall structure of this paper takes the form of six sections. In Section 2, we present the definition of block $q$-ratio sparsity and propose the block sparse signal recovery methodology via minimizing the block $q$-ratio sparsity. In Section 3, we provide a verifiable sufficient condition for the exact block sparse recovery and derive the reconstruction error bounds based on $q$-ratio BCMSV for the proposed method in the case of $1 < q \leq \infty$, involving both constrained and unconstrained models. In Section 4, we design algorithms to solve the problem. Section 5 contains the numerical experiments. Finally, conclusions are included in Section 6.

Throughout the paper, we introduce the notations $[M]$ for the block index set $\{1, 2, \cdots, M\}$ and $|S|$ for the cardinality of a block index subset $S \subseteq [M]$. We write $S^c$ for the complement $[M] \setminus S$ of a set $S$ in $[M]$. The block support of a vector $x \in \mathbb{R}^N$ is the index set of its nonzero blocks, i.e., $\text{bsupp}(x) := \{j \in [M] : ||x[j]||_2 \neq 0\}$. The mixed $\ell_2/\ell_q$-norm $||x||_{2,q} = (\sum_{j=1}^M ||x[j]||_2^q)^{1/q}$ for any $q \in (0, \infty)$, while $||x||_{2,\infty} = \max_{1 \leq j \leq M} ||x[j]||_2$. For a vector $x \in \mathbb{R}^N$ and a block index subset $S \subseteq [M]$, $x_S$ will denote the vector equal to $x$ on the block index set $S$ and zero elsewhere.

2. Minimization of the block $q$-ratio sparsity

The traditional block sparsity measure $||\cdot||_{2,0}$ has a severe practical drawback of being not sensitive to blocks with small $\ell_2$ norm. As a soft version, the entropy based block sparsity measure named block $q$-ratio sparsity was proposed in [34], which possesses many nice properties including continuity, scale-invariance, non-increasing with respect to $q$ and range equal to $[1, M]$. For more detailed arguments about this block sparsity measure, readers can refer to [34]. To be self-contained, here we give the full definition of block $q$-ratio sparsity.

**Definition 1.** ([34]) For any non-zero $z \in \mathbb{R}^N$ and non-negative $q \notin \{0, 1, \infty\}$, the block $q$-ratio sparsity level of $z$ is defined as

$$k_q(z) = \left( \frac{||z||_{2,1}}{||z||_{2,q}} \right)^{\frac{q}{q-1}}. \quad (5)$$

The cases of $q \in \{0, 1, \infty\}$ are evaluated as limits: $k_0(z) = \lim_{q \to 0} k_q(z) = ||z||_{2,0}$, $k_1(z) = \lim_{q \to 1} k_q(z) = \exp\left( - \sum_{j=1}^M \frac{||x[j]||_2}{||x||_{2,1}} \ln \frac{||x[j]||_2}{||x||_{2,1}} \right)$, $k_\infty(z) = \lim_{q \to \infty} k_q(z) = \frac{||z||_{2,1}}{||z||_{2,\infty}}$. 3
Based on this soft block sparsity measure, in this paper we propose the following non-convex minimization problems for block sparse signal recovery:

\[
\min_{z \in \mathbb{R}^N} \kappa_q(z) \quad \text{subject to} \quad \begin{cases} 
\|y - Az\|_2 \leq \eta, \\
\|A^T(y - Az)\|_{2,\infty} \leq \mu,
\end{cases}
\]  

(6a)

where \( y = Ax + \varepsilon \) with \( \|\varepsilon\|_2 \leq \eta \) or \( \|\varepsilon\|_{2,\infty} \leq \mu \), and some \( q \in [0, \infty] \) is pre-given. Here we consider both cases of the \( \ell_2 \)-bounded and \( \ell_{2,\infty} \)-bounded noises.

In order to illustrate the block sparsity promoting ability of the block \( q \)-ratio sparsity minimization problem, we revisit a toy example previously discussed in [20, 37]. Specifically, we let the measurement matrix

\[
A = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & -1
\end{pmatrix} \quad \in \mathbb{R}^{5\times6},
\]

and the measurement vector \( y = (0, 0, 20, 40, 18)^T \in \mathbb{R}^5 \). Then, it is straightforward to show that any solution of \( Az = y \) has the form of \( z = (t, t, t, 20 - 2t, 40 - 4t, 2(t - 9))^T \) for some \( t \in \mathbb{R} \). In this case we assume the block sizes go as that \( d_1 = d_2 = d_3 = 1, d_4 = 2, d_5 = 1 \). It is easy to notice that the block sparsest solution occurs at \( t = 0 \), where its block sparsity is 2. Other local solutions include \( t = 10 \) and \( t = 9 \) with block sparsity being 4. As can be seen in Figure 4 among the methods mentioned (including mixed \( \ell_2/\ell_1 \), mixed \( \ell_2/\ell_{0.5} \) and \( \ell_2/\ell_{1-2} \)), only \( \ell_2/\ell_{1-2} \) model can find the global minimizer \( t = 0 \). Moreover, according to the result displayed in Figure 2 our proposed methods with varying choices of \( q \) are all able to find the global minimizer at \( t = 0 \). Looking at Figure 4 it is apparent that the objective functions have two local minimizers \((t = 0 \text{ and } t = 10)\) when \( q = 1.5, 2, \infty \), while it has three local minimizers \((t = 0, t = 9 \text{ and } t = 10)\) when \( q = 0.5 \). This provides evidence that it is much harder to solve the minimization of block \( q \)-ratio sparsity for the case of \( 0 < q \leq 1 \) than for the case of \( 1 < q \leq \infty \).

On the other hand, we present the isosurface plots for the block \( q \)-ratio sparsity \( k_q(x) \) of \( x \in \mathbb{R}^3 \) with different values of \( q \). As shown in Figure 5 similar non-convex patterns arise while varying \( q \) from 0.1, 1.5, 2 and \( \infty \). The fact that the isosurface of \( k_2(x) \) approaches the planes \( x_3 = 0 \) and \( x_1 = x_2 = 0 \) as its value gets small reflects its ability to promote block sparsity. Meanwhile, the sparsity-promoting analysis technique used in [15] can also be adopted here to show that minimizing the block \( q \)-ratio sparsity in an orthant of the Euclidean space \( \mathbb{R}^N \) leads to solutions on the boundary, i.e., block sparser solutions. And it can be shown that minimizing the block \( q \)-ratio sparsity has the energy-promoting property, namely it promotes high-energy blocks while suppressing the rest low-energy blocks, see [15] for detailed discussions.

As done in [37], in this present paper we merely focus on the minimization problems with pre-given \( q \in (1, \infty) \), in which case they are equivalent to solve the constrained \( \ell_{2,1}/\ell_{2,q} \) minimization problems:

\[
\min_{z \in \mathbb{R}^N} \frac{\|z\|_{2,1}}{\|z\|_{2,q}} \quad \text{subject to} \quad \begin{cases} 
\|y - Az\|_2 \leq \eta, \\
\|A^T(y - Az)\|_{2,\infty} \leq \mu.
\end{cases}
\]  

(7a)
Figure 1: The objective functions of a toy example for the mixed $\ell_2/\ell_1$, the mixed $\ell_2/\ell_{0.5}$ and the $\ell_2/\ell_{1-2}$.

Figure 2: The objective functions of a toy example used to illustrate that minimizing the block $q$-ratio sparsity $k_q(\cdot)$ can find $t = 0$ as the global minimizer.
Figure 3: The isosurface plots of the block $q$-ratio sparsity $k_q(x)$ for $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, where $x$ has two blocks with $x[1] = (x_1, x_2)^T$ and $x[2] = x_3$.

3. Recovery analysis

The section below studies the global optimality results for the $\ell_2/\ell_{2,q}$ minimization with $q \in (1, \infty]$. We firstly establish a sufficient condition for the exact block sparse recovery using the $\ell_2/\ell_{2,q}$ minimization with $q \in (1, \infty]$. For some pre-given $q \in (1, \infty]$, we discuss the noiseless $\ell_2/\ell_{2,q}$ minimization problem:

$$\min_{z \in \mathbb{R}^N} \frac{\|z\|_{2,1}}{\|z\|_{2,q}} \quad \text{subject to } Az = Ax.$$  \hfill (8)

It can be easily verified that the sufficient and necessary condition for exactly recovering the block $k$-sparse $x$ via (8) is given by the following null space property \cite{35}:

$$\frac{\|x\|_{2,1}}{\|x\|_{2,q}} < \frac{\|x + h\|_{2,1}}{\|x + h\|_{2,q}}, \quad \forall h \in \ker(A) \setminus \{0\}.$$  \hfill (9)

As a consequence, we are able to obtain the following verifiable sufficient condition that guarantees the uniform exact block sparse recovery using the noiseless $\ell_2/\ell_{2,q}$ problem (8). It acts as a direct extension of Proposition 3 in \cite{37}.

**Proposition 1.** For some pre-given $q \in (1, \infty]$, if $x$ is block $k$-sparse such that

$$k < \inf_{h \in \ker(A) \setminus \{0\}} \frac{3 - 1}{3} k_q(h),$$  \hfill (10)

then the unique solution to the problem (8) is the truth $x$.

**Proof.** The proof of this proposition is almost identical to the proof of Proposition 3 in \cite{37}, with the major change being the substitution of the non-block norms for block norms. The proof is reproduced here for the sake of completeness. To prove the result, it suffices to verify the null space property (9) mentioned above.

We assume that the block support of the block $k$-sparse $x$ is $\text{bsupp}(x) = S$ such that $|S| \leq k$. For any $q \in (1, \infty]$ and $h \in \ker(A) \setminus \{0\}$, it holds that

$$\frac{\|x + h\|_{2,1}}{\|x + h\|_{2,q}} \geq \frac{\|x\|_{2,1} + \|h\|_{2,1} - 2\|h_S\|_{2,1}}{\|x\|_{2,q} + \|h\|_{2,q}} \geq \min \left\{ \frac{\|x\|_{2,1}}{\|x\|_{2,q}} + \frac{\|h\|_{2,1} - 2\|h_S\|_{2,1}}{\|h\|_{2,q}} \right\},$$

where we adopt the facts that $\|x + h\|_{2,1} = \|x + h_S + h_{S^C}\|_{2,1} \geq \|x\|_{2,1} + \|h_{S^C}\|_{2,1} - \|h_S\|_{2,1} = \|x\|_{2,1} + \|h\|_{2,1} - 2\|h_S\|_{2,1}$ and $\|x + h\|_{2,q} \leq \|x\|_{2,q} + \|h\|_{2,q}$. 

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If \( k < 3^{-q} k_{q}(h) \), then we obtain that \( \frac{\| h \|_{2,q}}{\| h \|_{2,q}} > 3k^{1-1/q} > k^{1-1/q} + 2 \frac{\| h \|_{2,q}}{\| h \|_{2,q}} \), which leads to \( \frac{\| h \|_{2,q}}{\| h \|_{2,q}} > k^{1-1/q} \geq \frac{\| x \|_{2,q}}{\| x \|_{2,q}} \). Therefore, the null space property (9) holds and the proof is completed.

What follows is the stable and robust recovery analysis results for the \( \ell_{2,1}/\ell_{2,q} \) minimization problems involving both constrained and unconstrained models. We start with the definition of \( q \)-ratio block constrained minimal singular values (BCMSV), which is a computable quality measure for the measurement matrix. It is a block version of the \( q \)-ratio constrained minimal singular values (CMSV) proposed and systematically studied in [34, 36]. As an efficient theoretical analysis tool for block sparse recovery, \( q \)-ratio BCMSV has been successfully used in establishing reconstruction error bounds for the block basis pursuit (BBP), the block Dantzig selector (BDS), and the group lasso, see [22] for detailed arguments.

**Definition 2.** For any real number \( s \in [1,M] \), \( q \in (1, \infty) \) and matrix \( A \in \mathbb{R}^{m \times N} \), the \( q \)-ratio block constrained minimal singular value (BCMSV) of \( A \) is defined as

\[
\beta_{q,s}(A) = \min_{z \neq 0, k_{q}(z) \leq s} \frac{\| Az \|_{2}}{\| z \|_{2,q}}.
\]

### 3.1. Constrained Models

Let us now turn to the recovery analysis results for the constrained models (7a) and (7b) based on the \( q \)-ratio BCMSV. In addition to cover the main results in [37] for the non-block sparse recovery with \( \ell_{2} \)-bounded noise, this subsection also considers the \( \ell_{2,\infty} \)-bounded noise case. The corresponding results for the case that the true signal \( x \) is exactly block sparse are list as follows.

**Theorem 1.** Suppose \( x \) is non-zero and block \( k \)-sparse. For any \( 1 < q \leq \infty \) and \( \beta_{q,3,\frac{q}{q-1}k}(A) > 0 \),

1. If the noise in (7a) satisfies \( \| \varepsilon \|_{2} \leq \eta \), then the solution \( \hat{x} \) to the problem (7a) obeys

\[
\| \hat{x} - x \|_{2,q} \leq \frac{2\eta}{\beta_{q,3,\frac{q}{q-1}k}(A)},
\]

\[
\| \hat{x} - x \|_{2,1} \leq \frac{6k^{1-1/q}\eta}{\beta_{q,3,\frac{q}{q-1}k}(A)}.
\]

2. If the noise in (7b) satisfies \( \| A^{T}\varepsilon \|_{2,\infty} \leq \mu \), then the solution \( \hat{x} \) to the problem (7b) obeys

\[
\| \hat{x} - x \|_{2,q} \leq \frac{6k^{1-1/q}\mu}{\beta_{q,3,\frac{q}{q-1}k}(A)},
\]

\[
\| \hat{x} - x \|_{2,1} \leq \frac{18k^{2-2/q}\mu}{\beta_{q,3,\frac{q}{q-1}k}(A)}.
\]

**Proof.** As \( x \) is block \( k \)-sparse, let us assume that bsupp(\( x \)) = \( S \) and \( |S| \leq k \). For both of the constrained models (7a) and (7b), we denote the residual by \( h := \hat{x} - x \). Due to \( \hat{x} = x + h \) is the minimum among all \( z \) satisfying the constraints of the models (7a) and (7b), it follows that

\[
\frac{\| x + h \|_{2,1}}{\| x + h \|_{2,q}} \leq \frac{\| x \|_{2,1}}{\| x \|_{2,q}}.
\]
which leads to

\[ \|x + h\|_{2,1} \cdot \|x\|_{2,q} \leq \|x\|_{2,1} \cdot \|x + h\|_{2,q}. \]  

(16)

Meanwhile, we have

\[ \|x + h\|_{2,1} = \|x_S + h_S\|_{2,1} + \|x_{S'} + h_{S'}\|_{2,1} \geq \|x_S\|_{2,1} - \|h_S\|_{2,1} + \|h_{S'}\|_{2,1} = \|x\|_{2,1} - \|h_S\|_{2,1} + \|h_{S'}\|_{2,1}, \]

and \(\|x + h\|_{2,q} \leq \|x\|_{2,q} + \|h\|_{2,q}\). Consequently, we infer that

\[ (\|x\|_{2,1} - \|h_S\|_{2,1} + \|h_{S'}\|_{2,1}) \cdot \|x\|_{2,q} \leq \|x\|_{2,1} \cdot (\|x\|_{2,q} + \|h\|_{2,q}). \]

Then it holds that

\[ \|h_{S'}\|_{2,1} \leq \|h_S\|_{2,1} + \frac{\|x\|_{2,1} \cdot \|h\|_{2,q}}{\|x\|_{2,q}} = \|h_S\|_{2,1} + k_q(x)^{1-1/q} \|h\|_{2,q}, \]  

which implies that

\[ \|h\|_{2,1} = \|h_S\|_{2,1} + \|h_{S'}\|_{2,1} \leq 2\|h_S\|_{2,1} + k_q(x)^{1-1/q} \|h\|_{2,q} \leq (2k^{1-1/q} + k_q(x)^{1-1/q}) \|h\|_{2,q}. \]  

(18)

Thus we arrive at the conclusion that for any \(1 < q \leq \infty\), \(k_q(h) = \left( \frac{\|h\|_{2,1}}{\|h\|_{2,q}} \right)^{\frac{1}{q - 1}} \leq \left( \frac{2}{k^{1-1/q} + k_q(x)^{1-1/q}} \right)^{\frac{1}{q - 1}} \leq 3^q k\), by adopting \(k_q(x) \leq k_0(x) = \|x\|_{2,0} \leq k\).

(1) As for the problem (7a), because \(\hat{x}\) satisfies the constraint \(\|y - A\hat{x}\|_2 \leq \eta\) and \(\|y - Ax\|_2 = \|\epsilon\|_2 \leq \eta\), it follows that

\[ \|Ah\|_2 = \|A(\hat{x} - x)\|_2 \leq \|A\hat{x} - y\|_2 + \|y - Ax\|_2 \leq 2\eta. \]  

(19)

Then, according to the definition of \(q\)-ratio BCMSV and \(k_q(h) \leq 3\frac{q}{q-1} k\), it holds that

\[ \beta_{q,3\frac{q}{q-1} k}^\frac{1}{q-1} (A) \|h\|_{2,q} \leq \|Ah\|_2 \leq 2\eta \Rightarrow \|h\|_{2,q} \leq \frac{2\eta}{\beta_{q,3\frac{q}{q-1} k}^\frac{1}{q-1} (A)}. \]

Meanwhile, \(\|h\|_{2,1} \leq 3k^{1-1/q} \|h\|_{2,q} \Rightarrow \|h\|_{2,1} \leq \frac{6k^{1-1/q} \eta}{\beta_{q,3\frac{q}{q-1} k}^\frac{1}{q-1}(A)}. \) This completes the proof of results for the problem (7a).

(2) With regard to the problem (7b), since \(\|AT\epsilon\|_{2,\infty} \leq \mu\), we have

\[ \|ATAh\|_{2,\infty} \leq \|AT(y - \hat{A}x)\|_{2,\infty} + \|AT(y - Ax)\|_{2,\infty} \leq 2\mu. \]  

(20)

Therefore,

\[ \|Ah\|_2^2 = \langle Ah, Ah \rangle = \langle h, ATAh \rangle \leq \|h\|_{2,1} \|ATAh\|_{2,\infty} \leq 2\mu \|h\|_{2,1}. \]

Thus, together with \(k_q(h) \leq 3\frac{q}{q-1} k\), we obtain that

\[ \beta_{q,3\frac{q}{q-1} k}^\frac{1}{q-1} (A) \|h\|_{2,q} \leq \|Ah\|_2^2 \leq 2\mu \|h\|_{2,1} \leq 6k^{1-1/q} \mu \|h\|_{2,q}, \]
which leads to
\[
\|h\|_{2,q} \leq \frac{6k^{1-1/q}\mu}{\beta^2 \beta_{q,3}^{2/q} k(A)}.
\] (21)

Hence, \( \|h\|_{2,1} \leq 3k^{1-1/q}\|h\|_{2,q} \leq \frac{18k^{2-2/q}\mu}{\beta^2 \beta_{q,3}^{2/q} k(A)} \). We have thus proved the theorem.

**Remark.** As studied in the Theorem 3 of [25], this sort of \( q \)-ratio BCMSV based condition \( \beta_{q,3}^{2/q} k(A) > 0 \) is fulfilled with high probability for subgaussian random matrix when the number of its measurements is reasonably large compared to the block sparsity level \( k \). And for any pre-given measurement matrix \( A \), its \( q \)-ratio BCMSV can be computed approximately so that the concise error bounds established in this theorem can be well computed.

The following corollary follows immediately from Theorem 1 by letting \( \eta = 0 \) in (23) or \( \mu = 0 \) in (25).

The sufficient condition that \( \beta_{q,3}^{2/q} k(A) > 0 \) for a perfect block sparse recovery via the noiseless \( \ell_{2,1}/\ell_{2,q} \) minimization presented here is a bit stronger than the condition that \( \beta_{q,2}^{2/q} k(A) > 0 \) given for the mixed \( \ell_{2}/\ell_{1} \)-minimization in [25].

**Corollary 1.** For any non-zero block \( k \)-sparse signal \( x \) and any \( q \in (1, \infty] \), if the condition \( \beta_{q,3}^{2/q} k(A) > 0 \) holds, then the unique solution of (23) is exactly the truth \( x \).

Having analyzed the case that the true signal is exactly block sparse in detail, we now move on to consider the case that it is block compressible, i.e., it can be well approximated by an exactly block sparse signal. For any \( x \in \mathbb{R}^N \), throughout this paper we denote by \( x^k \) its best block \( k \)-sparse approximation with respective to \( \|\cdot\|_{2,1} \) (i.e., \( x^k = \arg \min_{\|u\|_{2,1} \leq k} \|x - u\|_{2,1} \)).

**Theorem 2.** Let non-zero \( x \in \mathbb{R}^N \) and denote \( B_q(k,x) = (4k^{1-1/q} + k_q(x)^{1-1/q})^{1/q} \). For any \( 1 < q \leq \infty \) and \( \beta_{q,B_q(k,x)}(A) > 0 \),

1. If the noise in (76) satisfies \( \|\varepsilon\|_2 \leq \eta \), then the solution \( \hat{x} \) to the problem (76) obeys
   \[
   \|\hat{x} - x\|_{2,q} \leq \frac{2\eta}{\beta_{q,B_q(k,x)}(A)} + k^{1/q-1}\|x - x^k\|_{2,1},
   \] (22)
   \[
   \|\hat{x} - x\|_{2,1} \leq \frac{(4k^{1-1/q} + 2k_q(x)^{1-1/q})\eta}{\beta_{q,B_q(k,x)}(A)} + (4 + (k_q(x)/k)^{1-1/q})\|x - x^k\|_{2,1}.
   \] (23)

2. If the noise in (76) satisfies \( \|A^T\varepsilon\|_{2,\infty} \leq \mu \), then the solution \( \hat{x} \) to the problem (76) obeys
   \[
   \|\hat{x} - x\|_{2,q} \leq \frac{2B_q(k,x)^{1-1/q}\mu}{\beta_{q,B_q(k,x)}(A)} + k^{1/q-1}\|x - x^k\|_{2,1},
   \] (24)
   \[
   \|\hat{x} - x\|_{2,1} \leq \frac{(4k^{1-1/q} + 2k_q(x)^{1-1/q}B_q(k,x)^{1-1/q})\mu}{\beta_{q,B_q(k,x)}(A)} + (4 + (k_q(x)/k)^{1-1/q})\|x - x^k\|_{2,1}.
   \] (25)

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Proof. We assume that $S$ is the block index set over the $k$ blocks with largest $\ell_2$-norms of $x$ such that $\|x_S^*\|_{2,1} = \|x - x^k\|_{2,1}$ and let $h = \hat{x} - x$. Recall that (10) also holds here. Observe that

$$\|x + h\|_{2,1} = \|x_S + h_S\|_{2,1} + \|x_{S^c} + h_{S^c}\|_{2,1} \geq \|x_S\|_{2,1} - \|h_S\|_{2,1} - \|x_{S^c}\|_{2,1} + \|h_{S^c}\|_{2,1},$$

and $\|x + h\|_{2,q} \leq \|x\|_{2,q} + \|h\|_{2,q}$, we can obtain that

$$(\|x_S\|_{2,1} - \|h_S\|_{2,1} - \|x_{S^c}\|_{2,1} + \|h_{S^c}\|_{2,1}) \cdot \|x\|_{2,q} \leq (\|x_S\|_{2,1} + \|x_{S^c}\|_{2,1}) \cdot \|x\|_{2,q} + \|x\|_{2,1} \cdot \|h\|_{2,q}.$$  

Some simple manipulation yields

$$\|h_{S^c}\|_{2,1} \leq \|h_S\|_{2,1} + 2\|x_{S^c}\|_{2,1} + \|x\|_{2,q} \|h\|_{2,q} = \|h_S\|_{2,1} + 2\|x_{S^c}\|_{2,1} + k_q(x)^{1-1/q}\|h\|_{2,q}, \tag{26}$$

which implies

$$\|h\|_{2,1} = \|h_S\|_{2,1} + \|h_{S^c}\|_{2,1} \leq 2\|h_S\|_{2,1} + 2\|x_{S^c}\|_{2,1} + k_q(x)^{1-1/q}\|h\|_{2,q} \leq (2k^{1-1/q} + k_q(x)^{1-1/q})\|h\|_{2,q} + 2\|x_{S^c}\|_{2,1}. \tag{27}$$

(1) As for the problem (7A), suppose that $h \neq 0$ and $\|h\|_{2,q} > \frac{2\eta}{\beta_q, B_q(k,x)}$, otherwise (22) holds trivially. Due to $\|Ah\|_2 \leq 2\eta$, see (19), it follows that $\|h\|_{2,q} > \frac{\|Ah\|_2}{\beta_q, B_q(k,x)}$. Then it yields

$$\frac{\|Ah\|_2}{\|h\|_{2,q}} < \beta_q, B_q(k,x)(A) = \min_{z \neq 0, B_q(k,x) \leq B_q(k,x)} \frac{\|Az\|_2}{\|z\|_{2,q}} \Rightarrow k_q(h) > B_q(k,x) \Rightarrow \|h\|_{2,1} > B_q(k,x)^{1-1/q}\|h\|_{2,q} = (4k^{1-1/q} + k_q(x)^{1-1/q})\|h\|_{2,q}. \tag{28}$$

Together with (27), we infer that $\|h\|_{2,q} < k^{1/q-1}\|x_{S^c}\|_{2,1}$, which completes the proof of (22). The error $\ell_{2,1}$ norm bound (23) follows immediately from (22) and (27).

(2) In the case of the problem (11), we assume that $h \neq 0$ and $\|h\|_{2,q} > \frac{2\beta_q(k,x)^{1-1/q}}{\beta_q, B_q(k,x)}$, otherwise (24) holds trivially. Since in this case $\|Ah\|_2^2 \leq 2\mu\|h\|_{2,1}$, it follows that $\|h\|_{2,q} > \frac{\beta_q(k,x)^{1-1/q} \|Ah\|_2^2}{\beta_q, B_q(k,x)}$. Then we get

$$\beta^2_q, B_q(k,x)(A) = \min_{z \neq 0, B_q(k,x) \leq B_q(k,x)} \frac{\|Az\|_2^2}{\|z\|_{2,q}^2} > \frac{\|Ah\|_2^2}{\|h\|_{2,q}^2} \cdot \left(\frac{B_q(k,x)}{k_q(h)}\right)^{1-1/q} \Rightarrow k_q(h) > B_q(k,x) \Rightarrow \|h\|_{2,1} > B_q(k,x)^{1-1/q}\|h\|_{2,q} = (4k^{1-1/q} + k_q(x)^{1-1/q})\|h\|_{2,q}. \tag{29}$$

Combining (27), we have $\|h\|_{2,q} < k^{1/q-1}\|x_{S^c}\|_{2,1}$, which completes the proof of (24). The error $\ell_{2,1}$ norm bound (26) follows immediately from (24) and (27). The proof of Theorem 2 is now completed.

3.2. Unconstrained Model

To date, there has been no research carried out on the rigorous stable and robust analysis of the unconstrained model. For the first time this subsection of this paper seeks to investigate the theoretical recovery analysis results for the unconstrained version of (8), that is when $1 < q \leq \infty$, we consider the problem

$$\min_{z \in \mathbb{R}^n} \frac{1}{2}\|y - Az\|_2^2 + \lambda\|z\|_{2,1}, \tag{30}$$
where $y = Ax + \varepsilon$ and $\lambda > 0$ is the regularization parameter.

As has already been done for the constrained models, this subsection provides the recovery analysis results for the problem \( (30) \) based on the $q$-ratio BCMSV. We start with the following main result for the case that the true signal $x$ is exactly block sparse.

**Theorem 3.** If $x$ is block $k$-sparse, $q \in (1, \infty]$ and $\beta > q, (\frac{3}{q-1})^{\frac{1}{q-1}} k(A) > 0$ with some $\kappa \in (0, 1)$, when

$$\lambda > \max\left( \frac{1 - \kappa}{2(k + 1)} \left\| \varepsilon \right\|_2^2 k^{1/q - 1}, \frac{\left\| A^T \varepsilon \right\|_{2, \infty}}{\kappa}, \frac{\left\| y \right\|_2}{\beta, (\frac{3}{q-1})^{\frac{1}{q-1}} k(A)} \right) ,$$

then the solution $\hat{x}$ to the problem \( (30) \) satisfies

$$\| \hat{x} - x \|_{2, q} \leq \frac{k^{1 - 1/q} \| A \|_2}{\beta^2, (\frac{3}{q-1})^{\frac{1}{q-1}} k(A)} \cdot \frac{3(\kappa + 1)}{1 - \kappa} \sqrt{\| \varepsilon \|_2^2 + 2k^{1 - 1/q} \lambda}, \quad (31)$$

$$\| \hat{x} - x \|_{2, 1} \leq \frac{k^{2 - 2/q} \| A \|_2}{\beta^2, (\frac{3}{q-1})^{\frac{1}{q-1}} k(A)} \cdot \frac{9(\kappa + 1)(1 - \kappa)^2}{1 - \kappa} \sqrt{\| \varepsilon \|_2^2 + 2k^{1 - 1/q} \lambda}. \quad (32)$$

**Proof.** Let $\alpha = \frac{z}{\| z \|_2, q}$ and $\beta = \| z \|_{2, q}$, then the problem \( (30) \) is equivalent to solve

$$\min_{\{\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^+\}} \frac{1}{2} \| y - \beta A \alpha \|_2^2 + \lambda \| \alpha \|_{2, 1}. \quad (33)$$

If the solution to \( (33) \) is $(\hat{\alpha}, \hat{\beta})$, then the solution to \( (30) \) is $\hat{x} = \hat{\beta} \hat{\alpha}$. By using the Karush-Kuhn-Tucker (KKT) condition of \( (33) \) with respect to $\beta$, we have

$$(A \hat{\alpha})^T (\hat{\beta} A \hat{\alpha} - y) = 0,$$

such that $\| \hat{x} \|_{2, q} = \hat{\beta} = \frac{(A \hat{\alpha})^T y}{\| A \hat{\alpha} \|_2}$. Therefore, we obtain that

$$\| \hat{x} \|_{2, q} = \frac{(A \hat{\alpha})^T y}{\| A \hat{\alpha} \|_2} \leq \| A \hat{\alpha} \|_2 \| y \|_2 \leq \| y \|_2,$$

Since $\hat{x}$ is the solution to \( (30) \), we have

$$\frac{1}{2} \| y - A \hat{x} \|_2^2 + \lambda \| \hat{x} \|_{2, 1} \leq \| \varepsilon \|_2^2 \| \hat{x} \|_{1, q} \leq \| \varepsilon \|_2^2 \| x \|_{1, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{2, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{1, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{1, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{1, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{1, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{1, q} \leq \frac{\varepsilon \| A \|_2}{2} + \lambda k \| \varepsilon \|_2^2 \| x \|_{1, q},$$

which implies that $\| \hat{x} \|_{1, q} \leq \| \varepsilon \|_2^2 / (2\lambda) + k \| \varepsilon \|_2^2$. Here we use the fact that $k_q(x) \leq k_0(x) = \| x \|_{2, 0} \leq k$. Hence, when $\lambda \geq \frac{1 - \kappa}{2(\kappa + 2)} \| \varepsilon \|_2^2 / (2\lambda) + k \| \varepsilon \|_2^2$ with some $\kappa \in (0, 1)$, we have $\frac{\| \hat{x} \|_{2, 1}}{\| x \|_{2, q}} \leq \frac{\kappa + 2}{1 - \kappa} k \| \varepsilon \|_2^2 + k \| \varepsilon \|_2^2 \leq \frac{3\kappa + 2}{1 - \kappa} k \| \varepsilon \|_2^2$.

As a consequence of $\hat{x} = \hat{\beta} \hat{\alpha}$, it follows that $k_q(\hat{\alpha}) \leq \left( \frac{3\kappa + 2}{1 - \kappa} \right)^{\frac{1}{q-1}} k$, which yields $\| A \hat{\alpha} \|_2 \geq \beta, (\frac{3}{q-1})^{\frac{1}{q-1}} k(A)$ since $\| \hat{\alpha} \|_{2, q} = 1$. To summarize what we have proved, we get

$$\| \hat{x} \|_{2, q} \leq \frac{\| y \|_2}{\beta, (\frac{3}{q-1})^{\frac{1}{q-1}} k(A)}, \quad (34)$$

when $\lambda \geq \frac{1 - \kappa}{2(\kappa + 2)} \| \varepsilon \|_2^2 / (2\lambda) + k \| \varepsilon \|_2^2$ with $\kappa \in (0, 1)$. 

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In addition, when \( \lambda \geq \frac{\|AT\|_2 \mathbb{E}}{\kappa} \), \( \frac{\|x\|_2}{\sqrt{q} (\frac{1}{\kappa \mathbb{E}}) \|\hat{x}\|_2} \), let \( h = \hat{x} - x \), then it holds that

\[
\lambda \frac{\|\hat{x}\|_2}{\|\hat{x}\|_2} \leq \frac{\|\hat{x}\|_2}{\|\hat{x}\|_2} - \frac{1}{2} \|A(\hat{x} - x)\|_2^2 + \lambda \frac{\|x\|_2}{\|x\|_2} \leq (A\hat{x} - x), \|A(\hat{x} - x)\|_2^2 + \lambda \frac{\|x\|_2}{\|x\|_2} \leq (Ah, \varepsilon) + \lambda \frac{\|x\|_2}{\|x\|_2} \leq \|h\|_2, A^T \varepsilon, \|h\|_2, \|x\|_2, \lambda \frac{\|x\|_2}{\|x\|_2}, \\
\leq \frac{\lambda \kappa \|h\|_2, \|x\|_2, \lambda \frac{\|x\|_2}{\|x\|_2}}{\|h\|_2, \|x\|_2}.
\]

Then, we can obtain that

\[
\|\hat{x}\|_2, 2, 1 + \|h S\|_2, 2, 1 \leq \|h + h\|_2, 2, 1 \leq \kappa \|h\|_2, 2, 1 \leq \kappa \|h\|_2, 2, 1 + \frac{\|\hat{x}\|_2, 2, 1}{\|x\|_2, 2, q} \cdot \|h + h\|_2, 2, q \leq \kappa \|h\|_2, 2, 1 + \frac{\|\hat{x}\|_2, 2, 1}{\|x\|_2, 2, q} \cdot \|h\|_2, 2, q \leq \kappa \|h\|_2, 2, 1 + \|x\|_2, 2, 1 + k^{1-1/2} \|h\|_2, 2, q,
\]

which implies that \( \|h S\|_2, 2, 1 \leq \|h S\|_2, 2, 1 + \kappa \|h\|_2, 2, 1 + k^{1-1/2} \|h\|_2, 2, q \). As a result,

\[
\|h\|_2, 2, 1 \leq \|h S\|_2, 2, 1 + \|h S\|_2, 2, 1 \leq 2 \|h S\|_2, 2, 1 + \kappa \|h\|_2, 2, 1 + k^{1-1/2} \|h\|_2, 2, q \leq 3k^{1-1/2} \|h\|_2, 2, q + \kappa \|h\|_2, 2, 1.
\]

Therefore, it follows that \((1 - \kappa) \|h\|_2, 2, 1 \leq 3k^{1-1/2} \|h\|_2, 2, q \), i.e., \( k_q(h) = (\|h\|_2, 2, 1) \|h\|_2, 2, q \leq \left( \frac{3}{1 - \kappa} \right)^{\frac{1}{2}} q \kappa \).

Moreover, \( \|Ah\|_2^2 = (Ah, Ah) \leq \|h\|_2, 2, 1 \|A^T Ah\|_2, 2, q \) and

\[
\|A^T Ah\|_2, 2, q \leq \|A^T (y - A\hat{x})\|_2, 2, q + \|A^T (y - A\hat{x})\|_2, 2, q \leq \|A^T \varepsilon\|_2, 2, q + \|A^T (y - A\hat{x})\|_2, 2, q.
\]

Meanwhile, the KKT condition of (33) with respective to \( \alpha \) implies that

\[
\hat{\beta} A^T (A\hat{x} - y) = -\lambda \partial \|\hat{\alpha}\|_2, 2, 1,
\]

where the sub-gradients in \( \partial \|\hat{\alpha}\|_2, 2, 1 \) for the \( i \)-th block are \( \hat{\alpha}[i] \|\hat{\alpha}[i]\|_2 \) when \( \hat{\alpha}[i] \neq 0 \) and is some vector \( g \) satisfying \( \|g\|_2 \leq 1 \) when \( \hat{\alpha}[i] = 0 \). Therefore, we get \( \hat{\beta} A^T (A\hat{x} - y) \|_2, 2, \leq \lambda \), i.e., \( \|A^T (A\hat{x} - y)\|_2, 2, \leq \frac{\lambda}{\|\hat{x}\|_2, 2, q} \).

As a consequence of \( \lambda \geq \frac{\|AT\|_2 \mathbb{E}}{\kappa} \|\hat{x}\|_2, 2, q \), it holds that

\[
\|A^T Ah\|_2, 2, q \leq \|A^T \varepsilon\|_2, 2, q + \|A^T (y - A\hat{x})\|_2, 2, q \leq (\kappa + 1) \frac{\lambda}{\|\hat{x}\|_2, 2, q},
\]

which leads to \( \|Ah\|_2^2 \leq \|h\|_2, 2, 1 \|A^T Ah\|_2, 2, q \leq (\kappa + 1) \frac{\lambda}{\|\hat{x}\|_2, 2, q} \|h\|_2, 2, 1 \).

Then, with \( k_q(h) = \left( \frac{3}{1 - \kappa} \right)^{\frac{1}{2}} q \kappa k \),

\[
\hat{\beta}^2 q, \left( \frac{3}{1 - \kappa} \right)^{\frac{1}{2}} q \kappa \frac{A}{A} \|h\|_2^2, q \leq \|Ah\|_2^2 \leq (\kappa + 1) \frac{\lambda}{\|\hat{x}\|_2, 2, q} \|h\|_2, 2, 1 \leq 3(\kappa + 1) \frac{k^{1-1/2} \|h\|_2, 2, q}{1 - \kappa} \|h\|_2, 2, q, \frac{\lambda}{\|\hat{x}\|_2, 2, q}.
\]
which implies that
\[
\|h\|_{2,q} \leq \frac{k^{1-1/q}}{\beta^2 q, (\frac{2}{\|x\|})^{\frac{q}{\|x\|}^q}} (A) \cdot \frac{3(\kappa + 1)}{1 - \kappa} \cdot \frac{\lambda}{\|\bar{x}\|_{2,q}}.
\]

Finally, we establish an upper bound for \( \frac{\lambda}{\|\bar{x}\|_{2,q}} \). By using the KKT condition again, we can obtain that
\[
\hat{\beta} \|A^T (A\bar{x} - y)\|_2 \geq \lambda \sqrt{\|\bar{\alpha}\|_{2,0}} \geq \lambda.
\]

Hence, it follows that
\[
\frac{\lambda}{\|\bar{x}\|_{2,q}} \leq \|A^T (A\bar{x} - y)\|_2 \leq \|A^T\|_2 \|A\bar{x} - y\|_2 \leq \|A\|_2 \sqrt{\|\varepsilon\|_2^2 + 2k^{1-1/q}\lambda}.
\]

Therefore, we get
\[
\|h\|_{2,q} \leq \frac{k^{1-1/q} \|A\|_2}{\beta^2 q, (\frac{2}{\|x\|})^{\frac{q}{\|x\|}^q}} (A) \cdot \frac{3(\kappa + 1)}{1 - \kappa} \sqrt{\|\varepsilon\|_2^2 + 2k^{1-1/q}\lambda},
\]
and
\[
\|h\|_{2,1} \leq \left(\frac{3}{1 - \kappa}\right) k^{1-1/q} \|h\|_{2,q} \leq \frac{k^{2-2/q} \|A\|_2}{\beta^2 q, (\frac{2}{\|x\|})^{\frac{q}{\|x\|}^q}} (A) \cdot \frac{9(\kappa + 1)}{(1 - \kappa)^2} \sqrt{\|\varepsilon\|_2^2 + 2k^{1-1/q}\lambda}.
\]

The proof is now completed.

Furthermore, the corresponding result for the case that the true signal is not exactly block sparse can be obtained as follows.

**Theorem 4.** For any \( x \in \mathbb{R}^N \), \( q \in (1, \infty) \) and some \( \kappa \in (0, 1) \), we denote \( B_{\kappa,q}(k, x) = \left(\frac{4k^{1-1/q}}{1 - \kappa} + \frac{k_q(x)^{1-1/q}}{1 - \kappa}\right) \). If \( \lambda \geq \max \left(\frac{1 - \kappa}{2(\kappa + 1)} \|\varepsilon\|_2^2 k_q(x)^{1/q-1}, \frac{\|y\|_2}{\beta^2 q, (\frac{2}{\|x\|})^{\frac{q}{\|x\|}^q}} (A)\right) \), then the solution \( \hat{x} \) to (37) obeys
\[
\|\hat{x} - x\|_{2,q} \leq \frac{(\kappa + 1) \|A\|_2}{\beta^2 q, (\frac{2}{\|x\|})^{\frac{q}{\|x\|}^q}} (A) \cdot B_{\kappa,q}(k, x)^{1-1/q} \cdot \frac{\|\varepsilon\|_2}{\|x\|_2} \sqrt{\|\varepsilon\|_2^2 + 2k_q(x)^{1-1/q}\lambda + k^{1/q-1}\|x - x^k\|_{2,1}}, \tag{36}
\]
\[
\|\hat{x} - x\|_{2,1} \leq \frac{(\kappa + 1) \|A\|_2}{\beta^2 q, (\frac{2}{\|x\|})^{\frac{q}{\|x\|}^q}} (A) \cdot B_{\kappa,q}(k, x)^{1-1/q} \cdot \frac{\|\varepsilon\|_2}{\|x\|_2} \sqrt{\|\varepsilon\|_2^2 + 2k_q(x)^{1-1/q}\lambda}
\]
\[
+ \left(\frac{4}{1 - \kappa} + \frac{(k_q(x) / k)^{1-1/q}}{1 - \kappa}\right) \|x - x^k\|_{2,1}, \tag{37}
\]

**Proof.** We let \( h = \hat{x} - x \) and suppose \( S \) is the block index set over the \( k \) blocks with largest \( \ell_2 \)-norms of \( x \) such that \( \|x_{S^c}\|_{2,1} = \|x - x^k\|_{2,1} \). Following similar arguments in the Proof of Theorem 3, when
\[ \lambda \geq \max \left( \frac{1-\kappa}{2(1+\kappa)} \|\varepsilon\|_2^2 k_q(x)^{1/q-1}, \frac{\|A^T\|_2 \|y\|_2}{\sigma \sqrt{(\frac{1}{\lambda^2} + \frac{\|y\|_2}{\lambda})}} \right), \]

we have

\[ \|\hat{x}\|_2 \leq \kappa \|h\|_2 + \frac{\|x\|_2}{\|x\|_2} \cdot \|x + h\|_2. \]

Hence,

\[ \|x\|_2 = \|x_S\|_2 + \|x_{S^c}\|_2 \geq \|\hat{x}\|_2 - \kappa \|h\|_2 - k_q(x)^{1-1/q} \|h\|_2, \]

which implies that

\[ \|h_{S^c}\|_2 \leq \frac{1+\kappa}{1-\kappa} \|h_S\|_2 + \frac{2}{1-\kappa} \|x_{S^c}\|_2 + \frac{k_q(x)^{1-1/q}}{1-\kappa} \|h\|_2. \]

As a result,

\[ \|h\|_2 = \|h_S\|_2 + \|h_{S^c}\|_2 \leq \frac{2}{1-\kappa} \|h_S\|_2 + \frac{2}{1-\kappa} \|x_{S^c}\|_2 + \frac{k_q(x)^{1-1/q}}{1-\kappa} \|h\|_2. \]

We assume that \( h \neq 0 \) and \( \|h\|_2 > \frac{\lambda}{\|x\|_2^2 + 2k_q(x)^{1-1/q} \|h\|_2} \) holds trivially. Because \( \|Ah\|_2^2 \leq (\kappa + 1) \|A\|_2 \|h_S\|_2 + (\kappa + 1) \|A\|_2 \|h_{S^c}\|_2 \|h_S\|_2 ^\frac{1}{2} \) holds, we have

\[ \|h\|_2 > \frac{B_{\kappa,q}(k,x)^{1-1/q}/(1-\kappa)}{\|Ah\|_2^2}. \]

Then we get

\[ \beta^2 > \frac{B_{\kappa,q}(k,x)}{(1-\kappa)^{q/2}} \implies k_q(h) > \frac{B_{\kappa,q}(k,x)}{(1-\kappa)^{q/2}} \implies \|h\|_2 > \frac{B_{\kappa,q}(k,x)^{1-1/q}}{1-\kappa} \|h\|_2 \geq \left( \frac{4}{1-\kappa} k_{1-1/q} + \frac{k_q(x)}{1-\kappa} \right) \|h\|_2. \]

Combining (38), we have \( \|h\|_2 < k^{1/q-1} \|x_{S^c}\|_2 \), which completes the proof of (38). The error \( \ell_{2,1} \) norm bound (37) follows immediately from (36) and (38).

At the end of this subsection, it should be pointed out the algorithms proposed in [7] for the generalized entropy function minimization problem can be used here in solving the unconstrained model (36) with some careful generalizations from non-block to block setting. Further work is required to evaluate the performances of the algorithms, which is out of the scope of this paper. Instead, in the following section we provide algorithms via a convex-concave procedure to solve the constrained models (7a) and (7b).
4. Algorithms

In fact, the minimization problems (7a) and (7b) belong to the nonlinear fractional programming, where both the numerator and denominator are convex functions. This specific nonlinear fractional programming was comprehensively discussed in Chapter 4 of [23], see also [21,22]. Basically there are two kinds of methods to solve it, namely parametric methods and a change of variable method.

- **Parametric methods.** To solve the fractional problems (7a) and (7b), a class of methods by iteratively solving the following difference of convex functions problems depending on a parameter $\lambda \in \mathbb{R}$:

$$\min_{z \in \mathbb{R}^N} \lambda \|z\|_{2,1} - \|z\|_{2,q}, \quad \text{subject to} \quad \left\{ \begin{array}{l} \|y - Az\|_2 \leq \eta, \\
\|A^T(y - Az)\|_{2,\infty} \leq \mu. \end{array} \right. \tag{40a}$$

(can be used, see [37] for detailed arguments within a non-block framework. However, how to solve these subproblems efficiently are left for future work.

- **Change of variable method.** As a more direct and faster solver compared to the parametric methods, in this section we mainly focus on the block version of the change of variable method proposed in [37] and provide detailed discussions in what follows.

With a change of variable by letting $v = \frac{z}{\|z\|_{2,1}} \in \mathbb{R}^N$ and $t = \frac{1}{\|z\|_{2,1}} \in \mathbb{R}^+$, the minimization problems (7a) and (7b) are equivalent to the following problems

$$\min_{v \in \mathbb{R}^N, t \in \mathbb{R}^+} \frac{1}{\|v\|_{2,q}} \quad \text{subject to} \quad \left\{ \begin{array}{l} t > 0, \|y - Av/t\|_2 \leq \eta \text{ and } \|v\|_{2,1} = 1, \\
\|A^T(y - Av/t)\|_{2,\infty} \leq \mu \text{ and } \|v\|_{2,1} = 1. \end{array} \right. \tag{41a}$$

Then we are able to replace the equality constraint $\|v\|_{2,1} = 1$ by $\|v\|_{2,1} \leq 1$, and change the minimization problem to a maximization problem, please see the arguments in Section 3 of [21] for details. Henceforth, it suffices to solve

$$\max_{v \in \mathbb{R}^N, t \in \mathbb{R}^+} \|v\|_{2,q} \quad \text{subject to} \quad \left\{ \begin{array}{l} t \geq t_0, \|y - Av/t\|_2 \leq \eta \text{ and } \|v\|_{2,1} \leq 1, \\
\|A^T(y - Av/t)\|_{2,\infty} \leq \mu \text{ and } \|v\|_{2,1} \leq 1, \end{array} \right. \tag{42a}$$

where $t_0 = 1/a$ for some $a \geq \max\{\|z\|_{2,1}\|y - Az\|_2 \leq \eta \text{ or } \|A^T(y - Az)\|_{2,\infty} \leq \mu\}$. If its solution is denoted as $\hat{v}$ and $\hat{t}$, then our final recovered signal goes to $\hat{x} = \hat{v}/\hat{t}$.

Hereafter, an algorithm via a convex-concave procedure (CCP) [16] is adopted to solve this convex-concave problem. The corresponding CCP algorithm goes as follows:

In this algorithm, we use the solution of the mixed $\ell_2/\ell_1$-minimization problem (3) as our $x^{(0)}$. In the case of $q = \infty$, the linearized term for $\|v\|_{2,\infty}$ at $v^{(k)}$ will be $\|v^{(k)}\|_{2,\infty} + \left(\frac{v^{(k)}[j]}{\|v^{(k)}\|_{2,1}}\right)^T(v[j] - v^{(k)}[j])$ if the block index to achieve the $\ell_{2,\infty}$ norm of $v^{(k)}$ is $j$, i.e., $\|v^{(k)}[j]\|_2 = \|v^{(k)}\|_{2,\infty}$. We solve all the convex sub-programs such as (43) using the CVX toolbox in Matlab [14].
Algorithm 1 CCP to solve \((42a)\) and \((42b)\).

Input: measurement matrix \(A\), measurement vector \(y\), error bounds \(\eta\) or \(\mu\), and \(t_0\).

Initialization: Given an initial point \(v^0 = \frac{x^{(0)}}{\|x^{(0)}\|_{2,1}}\) and \(k = 0\).

Iteration: Repeat until a stopping criterion is met at \(k = n\).

1. Convexify: Linearize \(\|v\|_{2,q}\) with the approximation

\[
\|v^{(k)}\|_{2,q} + \nabla(\|v\|_{2,q})_{v=v^{(k)}}(v - v^{(k)}) = \|v^{(k)}\|_{2,q} + \left[\|v^{(k)}\|_{2,q}^{1-q} v^{(k)}_* \odot v^{(k)}\right]^T (v - v^{(k)}),
\]

where \(v^{(k)}_* = \begin{bmatrix} \|v^{(k)}[1]\|_2^{-2}, \cdots, \|v^{(k)}[1]\|_2^{-2}, \|v^{(k)}[2]\|_2^{-2}, \cdots, \|v^{(k)}[2]\|_2^{-2}, \cdots, \|v^{(k)}[M]\|_2^{-2}, \cdots, \|v^{(k)}[M]\|_2^{-2} \end{bmatrix}^T\)

with \(\|v^{(k)}[i]\|_2\) denoting the \(\ell_2\) norm of the \(i\)-th block of \(v^{(k)}\) for \(i \in [M]\), and \(\odot\) denoting the Hadamard product.

2. Solve: Set the value of \(v^{(k+1)} \in \mathbb{R}^N, t^{(k+1)} \in \mathbb{R}^+\) to be a solution of

\[
\max_{v \in \mathbb{R}^N, t \in \mathbb{R}^+} \|v^{(k)}\|_{2,q} + \left[\|v^{(k)}\|_{2,q}^{1-q} v^{(k)}_* \odot v^{(k)}\right]^T (v - v^{(k)})
\]

\[
\text{s.t. } t \geq t_0, \|v\|_{2,1} \leq 1, \|y - Av/t\|_2 \leq \eta \text{ for } (42a), \left\{\|A^T(y - Av/t)\|_{2,\infty} \leq \mu \text{ for } (42b)\right\}. \tag{43}
\]

3. Update iteration: \(k = k + 1\).

Output: The recovered signal \(\hat{x} = v^{(\bar{n})}/t^{(\bar{n})}\).
5. Numerical experiments

In this section, we conduct numerical experiments to illustrate the performance of our proposed method in block sparse signal reconstruction from different perspectives. In all the following experiments, the block \( k \)-sparse signal is generated by choosing \( k \) blocks uniformly at random, and then choosing the non-zero values from the standard normal distribution for these \( k \) blocks.

5.1. A test

In this set of experiments, we tested the CCP recovery algorithms for a block sparse signal \( x \in \mathbb{R}^{400} \) reconstruction with a Gaussian random measurement matrix \( A \in \mathbb{R}^{120 \times 400} \). We fixed the block size \( d = 2 \). We considered two cases, one is that the true signal \( x \) has a block sparsity level of 20 and the measurements are noise free, the other case is that the true signal is block 10-sparse and the measurements are noisy with either \( \ell_2 \)-bounded noise or \( \ell_2,\infty \)-bounded noise. For the \( \ell_2 \)-bounded noise, we set \( \varepsilon = 0.1u/\|u\|_2 \) to ensure that \( \|\varepsilon\|_2 \leq 0.1 \) with \( u = \text{randn}(120, 1) \), while for the \( \ell_2,\infty \)-bounded noise, we set \( \varepsilon = 0.1u/\|A^T u\|_{2,\infty} \) such that \( \|A^T \varepsilon\|_{2,\infty} \leq 0.1 \).

As shown in Figure 4 when there is no noise in the measurements, a perfect recovery can be achieved via the \( \ell_{2,1}/\ell_{2,q} \) minimization problem with \( q = 2 \), while it would not be impaired almost at all with slightly noisy measurements regardless the noise types.

![Figure 4: A numerical test for the \( \ell_{2,1}/\ell_{2,q} \) minimization via CCP with \( q = 2 \), left panel (Noise Free) and right panel (Noisy with \( \eta = \mu = 0.1 \)).](image)

5.2. Different choices of \( q \)

In this subsection, we compared the performances of the proposed \( \ell_{2,1}/\ell_{2,q} \) minimization problem with different \( q \) varying from 1.1, 1.5, 2, 4, \( \infty \). In this study, \( A \) is an \( m \times 200 \) random matrix generated as Gaussian...
with \( m = \{20, 40, 60, 80, 100, 120, 140\} \). The true signal \( x \in \mathbb{R}^{200} \) is simulated as block 10-sparse with block size \( d = 2 \). For each \( q \), we replicated the noiseless experiments 100 times with different \( A \) and \( x \). It is recorded as one success if the relative error \( \frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq 10^{-3} \).

Figure 5 shows the success rate over the 100 replicates for various values of parameter \( q \) and number of measurements \( m \). It can be seen that \( q = 1.5 \) is the best among all tested values of \( q \), and the results for \( q = 1.1 \) and \( q = 2 \) are better than those for \( q = 4 \) and \( q = \infty \). The ability to choose suitable values of \( q \) enables us to fully exploit the block sparsity promoting power of the proposed models and to achieve better reconstruction performances.

![Figure 5: Reconstruction performance comparison for the \( \ell_{2.1}/\ell_{2,q} \) minimization with Gaussian random measurements and different \( q \) varying from 1.1, 1.5, 2, 4, \( \infty \).](image)

### 5.3. Comparison on different block sparse recovery methods

In the final part of this section, comparisons between the proposed \( \ell_{2.1}/\ell_{2.1.5} \) minimization and other state-of-the-art block sparse signal recovery methods including mixed \( \ell_2/\ell_p \) with \( p = 0.2, 0.5, 0.8 \), group lasso and \( \ell_2/\ell_{1.2} \) are performed. For each recovery method, we replicated the noiseless experiments 100 times with different \( A \) and \( x \) and evaluated its performance in terms of success rate.

#### 5.3.1. Gaussian random matrices

We start with the comparison of block sparse signal recovery by using the Gaussian random matrices as the measurement matrices. We set \( m = 60, N = 200 \) and choose the block size \( d = 2 \), and the block sparsity level \( k = \{7, 9, 11, 13, 15, 17, 19, 21, 23\} \).
Figure 6: Recovery performance comparison for different algorithms with Gaussian random matrices.

As Figure 6 shows, the mixed $\ell_2/\ell_{0.2}$ and mixed $\ell_2/\ell_{0.5}$ perform the best for the Gaussian case, while our proposed $\ell_{2,1}/\ell_{2,1.5}$ tends to perform better than other methods including the mixed $\ell_2/\ell_{0.8}$, group lasso and $\ell_2/\ell_{1-2}$.

5.3.2. Block-coherent random matrices

Lastly, we conducted experiments with the block-coherent random matrices to verify the advantageous performance of our block $q$-ratio sparsity minimization based method in block sparse signal reconstruction. We construct the highly block-coherent random matrices $A$ as done in [26] by using $A = P \otimes D$ with $D = H/\sqrt{d}$ and $P \in \mathbb{R}^{m/d \times N/d}$ is a randomly oversampled partial discrete cosine transform (DCT) matrix with its $i$-th column being $\sqrt{\frac{d}{m}} \cos(2\pi \omega (i - 1)/F), i = 1, 2, \cdots, N/d$ and $\omega$ is a random vector uniformly distributed in $[0, 1]^{m/d}$. As shown in [26], a larger $F$ yields a more block-coherent matrix. In this set of experiments, we set $m = 60$, $N = 500$, $d = 2$, $F = 5$ and $k = \{3, 5, 7, 9, 11, 13, 15, 17\}$.

In contrast to earlier findings for the Gaussian case, however, from Figure 7 we can see that the proposed $\ell_{2,1}/\ell_{2,1.5}$ minimization gives the best result for the block-coherent case, even better than the $\ell_2/\ell_{1-2}$ method. Taken together, these findings suggest that the $\ell_{2,1}/\ell_{2,1.5}$ can achieve satisfactory block sparse recovery results which is robust to the block coherence of the measurement matrix.
Figure 7: Recovery performance comparison for different algorithms with block-coherent random matrices.

6. Conclusion

In this paper, we studied the block sparse signal recovery approach via minimizing the block q-ratio sparsity. In the case $1 < q \leq \infty$, it reduces to a problem of minimizing the ratio of the mixed $\ell_2/\ell_1$ and the mixed $\ell_2/\ell_q$ norms. We gave a verifiable sufficient condition for the exact block sparse recovery and established the corresponding reconstruction error bounds in terms of q-ratio BCMSV. Both constrained and unconstrained models were considered. A computational algorithm was proposed to approximately solve this non-convex problem. In addition, varieties of numerical experiments were conducted to illustrate the good performance of our proposed approach.

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