Super-diffusion in one-dimensional quantum lattice models

Enej Ilievski, 1 Jacopo De Nardis, 2 Marko Medenjak, 3 and Tomáš Prosen 3

1Institute for Theoretical Physics Amsterdam and Delta Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands
2Département de Physique, École Normale Supérieure, PSL Research University, CNRS, 24 rue Lhomond, 75005 Paris, France
3Faculty of Physics, University for Mathematics and Physics, Jadranska 19, 1000 Ljubljana, Slovenia
(Dated: June 19, 2018)

We identify a class of one-dimensional spin and fermionic lattice models which display diverging spin and charge diffusion constants. These include, among others, several paradigmatic models of exactly solvable strongly correlated many-body dynamics such as the isotropic Heisenberg spin chains, the Fermi-Hubbard model, and the t-J model at the integrable point. Using the hydrodynamic transport theory, we derive an analytic lower bound on the spin and charge diffusion constants by calculating the curvature of the corresponding Drude weights at half filling. We establish that for isotropic interactions all components of the SU(N) Nöether charges exhibit super-diffusive transport at finite temperature and half filling.

Introduction. Understanding the microscopic mechanisms for the emergent macroscopic laws in many-body systems is one of the fundamental questions in condensed matter physics. Despite very long tradition, the question has mostly been pursued by studying certain simple classical dynamical systems [1], such as elastically colliding rigid objects [2, 3], but much less is known about strongly-correlated quantum dynamics. From the theoretical viewpoint, holographic theories [4, 5] and one-dimensional systems have a special role in this context. While, on the one hand, there is a plethora of powerful techniques available, which enable explicit analytic calculations, on the other hand one often encounters anomalous behavior, such as singular conductivities [6–18]. In contrast, only little is known about its regular part which characterizes transport on the sub-ballistic time scales, save for a few numerical studies which are often plagued by strong finite-size or finite-time effects [19–24]. In this respect, exactly solvable interacting models offer a unique opportunity to tackle this problem in a rigorous fashion which, moreover, permit efficient numerical simulations [22, 25–30] and sometimes allow for experimental realizations [31–35]. Yet, even in a very simple interacting system, such as the integrable Heisenberg spin-1/2 chain, the status of the spin dynamics on the sub-ballistic scales remains largely uncertain, despite several recent numerical studies [36–41]. Even though we focus on exactly solvable models, the hope is that they can serve as representative models for non-equilibrium universality classes.

In this Letter, we report on a class of quantum spin and electron models which exhibits diverging spin and charge diffusion constants. We built on an earlier proposal of ref. [42] which provides a lower bound on diffusion constants based on the behavior of the corresponding Drude weights in the vicinity of half-filled thermal equilibrium states. Here we reinterpret and optimize the bound in the framework of the hydrodynamic linear transport theory developed in [43, 44]. Focusing on the high-temperature regime, which cannot be probed by other theoretical methods [45–47], we derive analytic closed-form expression of the lower bound for several important examples of quantum lattice models. We find that if a system is invariant under a continuous non-abelian (and possibly graded) Lie group G, the components of the conserved Nöether charge at half filling exhibits diverging DC conductivity, implying super-diffusive transport.

Summary. The central result of this work is an analytic lower bound for the spin/charge diffusion constants for a family of interacting many-body one dimensional lattice systems. Let \( Q = \sum_x q_x \) denote a conserved \( U(1) \) charge of the model, with density \( \dot{q} \) satisfying a local conservation law \( \partial_t q_x(t) + \partial_x \dot{j}_x(t) = 0 \). The corresponding diffusion constant is defined according to the Kubo formula

\[
D^{(q)}(\beta) = \lim_{T \to \infty} \frac{\beta}{\chi(\beta)} \sum_x \int_0^T dt \left\langle \dot{j}^{(q)}_x(t) \dot{j}^{(q)}_x(0) \right\rangle, \tag{1}
\]

where \( \langle \cdot \rangle \) denotes the thermodynamic expectation value with respect to the grand-canonical Gibbs ensemble \( \hat{\rho}_{GC}(\beta) \simeq \exp(-\beta \hat{H} + \sum_i 2h_i \hat{N}_i) \) at inverse temperature \( \beta \), with \( \hat{N}_i \) denoting the full set of conserved \( U(1) \) charges of the model including \( Q \), whereas \( \chi(\beta) \) is the static susceptibility [48].

We concentrate on a few most prominent interacting systems which often play a pivotal role in the studies of strongly correlated one-dimensional materials, including the anisotropic Heisenberg spin chain, the Fermi–Hubbard model and the t-J model. These systems are exactly solvable and feature stable interacting particle excitations which undergo a completely elastic scattering. As a corollary, the average of the current density \( \dot{j}^{(q)} \) in general involves a dissipation-free component, which yields a singular DC conductivity characterized by a finite Drude weight

\[
D^{(q)}(\beta) = \lim_{T \to \infty} \frac{\beta}{2T} \sum_x \int_0^T dt \left\langle \dot{j}^{(q)}_x(t) \dot{j}^{(q)}_x(0) \right\rangle, \tag{2}
\]
signalling *ballistic* transport. Drude weights can be conveniently and efficiently computed using the hydrodynamic approach developed in [49, 50], which exploits the fact that the net effect of inter-particle interactions (which are fully accounted for by a two-body scattering amplitude) in the thermodynamic limit manifests itself as renormalization of particles’ bare quantities in the presence of a finite-density many-body background (e.g. a Gibbs thermal state or Generalized Gibbs states [51–55]). This mechanism is commonly referred to as the *dressing* (see e.g. [56–60]). Spectra of soluble models are described in terms of particles which are labelled by a discrete index \(A\), counting over (typically infinitely many) particle types, and a continuous rapidity variable \(u\) which parametrizes their bare momenta \(k_A(u)\) and energies \(\epsilon_A(u)\). The dressing of bare quantities \(q_A\) can be expressed as a linear transformation \(q_A \mapsto q_A^\text{dr}\). The effective velocity of propagation is obtained from the dressed dispersion relations, \(v_A^\text{eff} = \partial \epsilon_A / \partial q_A = \epsilon'_A / p'_A\), where \(p'_A = (k'_A)^\text{dr}\) and \(\epsilon'_A = (\epsilon'_A)^\text{dr}\), with prime denoting the rapidity derivative. In this picture, the hydrodynamic mode decomposition of the Drude weight reads [44, 60]

\[
D^{(q)}(\beta) = \frac{\beta}{2} \sum_A \int du D_A(u) \left[ q_A^\text{dr}(u) \right]^2 ,
\]

where \(D_A(u) = \rho_A(u)(1 - \partial_A(u))|v_A^\text{eff}(u)|^2\) is the ‘Drude kernel’. Dependence on the reference equilibrium state is encoded in the rapidity distribution functions \(\rho_A(u)\), which can be conveniently expressed in terms of mode occupation (filling) functions \(\vartheta_A(u) = \rho_A(u)/(2\pi \sigma_A p'_A(u))\) \((\sigma_A = \text{sgn}(k'_A(u)))\), while \(q_A^\text{dr}(u)\) denotes the dressed charges of individual excitations with respect to an equilibrium state (defined explicitly in [61]).

Given the full set of occupation functions \(\vartheta_A\), the dressing equations can be expressed as coupled linear integral equations, cf. SM [61]. Notice that computing the functions \(\vartheta_A\), which are obtained by minimizing the free energy as a functional of the densities \(\rho_A\), requires solving a system of non-linear equations which can only be done numerically via iterative schemes. Two important exceptions are the ground states and the high-temperature limit. In the latter case the occupation function become momentum-independent and the dressing transformation takes the form of an algebraic system of equations.

For spin and fermion lattice solvable Hamiltonians with global \(SU(M|N)\) symmetry considered here, the dressing equations can be solved fully analytically in a uniform group-theoretic way (see [61]). We wish to stress that that our motivation to focus on the high-temperature regime is of purely technical nature and, as detailed below, our main conclusions regarding the super-diffusive dynamics remain rigorously valid even at finite temperatures. We note that in our calculations we account for exact dressed dispersion relations of interacting excitations. Our results are thus inaccessible with effective field-theoretical methods [45–47, 62] or semi-classical approximations [63] which fail to capture the essential contributions of the bound states.

\[\Delta\]

**FIG. 1.** XXZ chain at infinite temperature: black curve shows diffusion bound (3), \(D^{(m)}(\beta) \geq 2C^{(m)}(0)/(\beta v_{LR})\), and the blue points display numerical values of the spin diffusion constant obtained by tDMRG in refs. [21] and [27]. The logarithmic divergence close to \(\Delta = 1\) is indicated by the dashed line. Notice that the bound does not vanish even in Ising the limit \(\Delta \to \infty\), in contrast to the dissipative case [64]. Inside the gapless interval we display \(\Delta = \cos \pi/\ell\) with \(\ell = 3, \ldots, 10\).

Drude weight curvature. In order to characterize transport on sub-ballistic time-scales, we exploit a relation between the curvature of the Drude weight and the diffusion constant. In the linear response theory, a small gradient of charge density is created in the system, and subsequently the induced current is measured at the origin. For finite times, the current density (initially localized at the origin) spreads only over a finite portion of the system, due to the locality of interactions. This implies that the probability of measuring the current in the sector away from half-filling is non-zero. In sectors away from half-filling the current grows indefinitely with time, however the probability that one finds the system away from the half-filled sector decreases with the system size. The interplay of vanishing probability and diverging conductivity permits the derivation of the lower bound on the diffusion constant [42]. The latter reads

\[
D^{(q)}(\beta) \geq \frac{1}{8\beta^2 \lambda^2(\beta)} \beta v_{LR}^2 D^{(q)}(\beta, h) \bigg|_{h=0},
\]

where \(v_{LR} = \max_{A,A'}v_A^\text{eff}(u)\) is the Lieb-Robinson velocity. Supposing the local degrees of freedom of a system carry the charge \(q \in \{-\frac{1}{2}(d-1), \ldots, \frac{1}{2}(d-1)\}\), the lower bound at infinite temperature reads

\[
D^{(q)}(0) \geq \lim_{\beta \to 0} \frac{18}{\beta(d^2 - 1)^2 v_{LR}^2} D^{(q)}(\beta, h) \bigg|_{h=0}.
\]

Subsequently we shall focus on the linear transport of global conserved \(U(1)\) charges, such as the total mag-
netization $\hat{S}^z = \sum_j \hat{S}_j^z$, and/or total electron charge $\hat{N}_e = \frac{1}{2} \sum_{j,\sigma=\uparrow,\downarrow} \hat{c}_j^\dagger \hat{c}_{j,\sigma}$. Provided that in the vicinity of half-filled spin/charge sector the Drude weights vanishes as

$$D^{(q)}(\beta, h) = C^{(q)}(\beta) \frac{h^2}{2} + \ldots \text{ for } h \sim 0,$$  \hspace{1cm} (6)

the bound (5) can be compactly expressed in terms of the Drude weight curvature

$$C^{(q)}(\beta) = \frac{\beta}{2} \sum_A \int du \mathcal{D}_A(u) \left[ \partial_h \left[ g_{\eta_s}^A(u) \right]^2 \right]_{h=0}. \hspace{1cm} (7)$$

A distinguished property of exactly solvable quantum lattice models is that elementary interacting excitations which carry spin and charge can form bound states. Let therefore an integer $s$ denotes a ‘bare mass’ or ‘bare charge’, i.e. the number of constituent particles within a bound state; for instance, in a spin system, such as the Heisenberg spin-1/2 chain, $s$ pertains to the number of bound magnons in a multi-magnon excitations, while in an electron system (e.g. the Fermi–Hubbard model) $s$ can be the number of bound spin-full electrons which combine form a spin singlet state etc. Crucially, in integrable models invariant under a global rotational symmetry of a (graded) Lie algebra $G = SU(N|M)$, the number of distinct bound states is infinite, i.e. there is no upper bound on $s$. We are able to show, rigorously, that when the particles’ scattering amplitudes are rational functions of the scattering momenta, the Drude weight curvature per particle decreases as $1/s$ for large mass $s$, in turn implying (logarithmically) divergent lower diffusion bound after performing the summation over all particle types.

**Anisotropic Heisenberg spin-1/2 chain.** The simplest model which features various anomalous transport properties is the Heisenberg XXZ spin-1/2 chain,

$$\hat{H}_{XXZ} = \sum_{j=1}^{L} \left( \hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right). \hspace{1cm} (8)$$

The spectrum is gapless in the regime $|\Delta| \leq 1$ and gapped when $|\Delta| > 1$. The value of anisotropy profoundly influences the spin transport, as we shortly summarize below. The exact analytic solution of the dressing equations in the high-temperature limit are known from the seminal paper [65]. An analytic hydrodynamic formula for the spin Drude weight curvature reads

$$C^{(m)}(\beta) = \frac{\beta}{2} \sum_{s \geq 1} \int \frac{du}{2\pi} \partial_s \left( 1 - \vartheta_s \right) \rho_s \left[ \nu_{\eta_s} \right]^2 \partial_h \left[ m_s^{dr} \right]^2 \big|_{h=0},$$  \hspace{1cm} (9)

where $m_s^{dr} = \partial_{2h} \log(\vartheta_s^{-1} - 1)$. Our conclusions are:

- Exactly at the $SU(2)$ isotropic point $\Delta = 1$, the finite-temperature spin diffusion constant $D^{(m)}$ diverges in the limit of half filling $h \to 0$. This is implied by the large-$s$ scaling of the dressed spin (magnetization), mode occupation functions and dressed dispersion relations,

$$m_s^{dr}(h) \simeq \frac{1}{4} (s + \kappa(\beta))^2 h + \mathcal{O}(h^3), \hspace{1cm} (10)$$

$$\lim_{h \to 0} \partial_s(\vartheta_s) \simeq (s + \kappa(\beta))^{-2}, \hspace{1cm} (11)$$

$$\lim_{h \to 0} \int_{-\infty}^{\infty} du \rho_s(u) \left[ \nu_{\eta_s}^\text{eff}(u) \right]^2 \simeq \frac{1}{s^3}, \hspace{1cm} (12)$$

for some temperature-dependent function $\kappa(\beta)$. The scaling (12) holds for any finite temperature, see Fig.2. For $\beta = 0$, relations (10) and (11) indeed become exact for all $s$, and we have $\kappa(0) = 1$.

- In the gapped regime, the interaction anisotropy $\Delta = \cosh(\eta)$ ($\eta > 0$) breaks the global $SU(2)$ symmetry to $U(1)$. The dressed spin and mode occupations of bound magnons in the limit of infinite temperature and vanishing chemical potential remain the same as in the isotropic case, cf. Eqs. (10),(11), with $\vartheta_s$ and $m_s^{dr}$ becoming independent of $u$ for large $s$. The key difference is that the bare dispersion relations become $\eta$-dependent functions, and the rapidity-dependent part of Eq. (9) scales as

$$\int_{-\pi/2}^{\pi/2} \frac{du}{2\pi} \rho_s(u) \left[ \nu_{\eta_s}^\text{eff}(u) \right]^2 \simeq e^{-\nu_s}. \hspace{1cm} (13)$$

In contrast to the isotropic case, this behavior implies a finite spin diffusion lower bound (5) which converges exponentially as a function of $s$.

- The gapless regime $|\Delta| < 1$ is rather exceptional. The spin transport exhibits finite Drude weight at finite temperatures and arbitrary filling, including the half-filled sector [9, 15, 43], where it is a non-continuous function of $\Delta$. Still, it remains an interesting question whether the sub-ballistic corrections to spin transport are normal, diffusive or anomalous sub/super-diffusive. Complete classification of the thermodynamic particle content of the model in this regime becomes quite involved (see [66]) and, in distinction to the gapped regime, now depends explicitly on the value of parameter $\Delta$ [43, 67]. For simplicity we restrict ourselves to the simplest cases $\Delta = \cos \pi/\ell$ for integer $\ell \geq 3$ (the Drude curvature at $\Delta = 0$ is not positive, consistently with the vanishing diffusion constant at the free fermionic point [68]), where the spectrum consist of $\ell$ distinct particle types. For $s = 1, \ldots, \ell - 2$, the particles represent bound states of $s$ magnons whose high-temperature dressed spin is given by Eq. (10) which therefore vanishes as $h \to 0$. In addition, there exists an exceptional doublet of particles which carry finite dressed spins $m_A^{dr} = \ell/2 \pm \kappa_\ell h$ ($\kappa_\ell > 0$) for $A = \ell - 1, \ell$, and are charged under the non-unitary local conservation laws found in [12, 13] which are responsible for a finite spin...
Drude weight even at half filling [43]. A non-vanishing contribution to the curvature $C^{(m)}$ is obtained by subtracting a finite Drude weight $D^{(m)} = \sum_{A=\ell} \int du \rho_A(u)(1 - \hat{\nu}_A(u))(\nu_A(u)\ell/2)^2$ and then expanding the remainder to the first order in $\hbar$. We find a finite lower bound for all $\ell < \infty$ which diverges as $\ell \to \infty$, namely $\Delta \to 1^-$, see Fig. 1.

Comment. At the values $\Delta = \cos \frac{\pi}{\ell+1/N}$, the number of magnonic bound states is $\ell + \nu$ and, therefore, after subtracting a finite Drude weight, the Drude curvature diverges as $\nu \to \infty$, similarly to the case $\Delta \to 1^-$. This suggests that the spin diffusion constant, similarly to the Drude weight, is not a smooth function of $\Delta$, and that it diverges almost everywhere for $\Delta \in [-1,1]$.

Higher spins and higher rank symmetries. We have additionally investigated the role of higher degrees of freedom by (i) solving the (integrable) spin-$S$ isotropic Heisenberg chains and (ii) high-rank $SU(N)$-symmetric models of $N-1$ interacting particle species. The picture, explained above for the isotropic Heisenberg model ($N = 2$) of spin $S = \frac{1}{2}$, remains qualitatively unchanged. Indeed, by virtue of $SU(N)$ invariance, the results generalize to all components of the Nöether charge. Explicit results can be found in the Supplemental Material [61], which also includes refs. [69–76].

Fermionic models. Another class of models of particular importance are lattice models of fermions. The most prominent example is the one-dimensional Fermi–Hubbard model, describing spin-full electrons interacting via Coulomb repulsion,

$$\hat{H}_H = -\sum_{j=1}^L \sum_{\sigma \in \uparrow, \downarrow} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j+1,\sigma} + \hat{c}_{j+1,\sigma}^\dagger \hat{c}_{j,\sigma} + 4u \sum_{j=1}^L \hat{V}_{j,j+1}^H,$$

with $\hat{V}_{j,j+1}^H = \sum_{\alpha \in \uparrow, \downarrow} (\hat{n}_{j,\uparrow} - \frac{1}{2})(\hat{n}_{j,\downarrow} - \frac{1}{2})$. The model comprises spin and charge excitations which participate in the formation of bound states. The particle content consists of individual spin-up electrons, spin-singlet compounds made of 2a electrons with ($a \in \mathbb{N}$) and chargeless bound states of $s$ spin excitations with bare spin ($s \in \mathbb{N}$). Although spin and charge degrees of freedom mutually interact and undergo a non-trivial dressing (even in the limit of infinite temperature), the properties of spin and charge transport are once again in qualitative agreement with the isotropic Heisenberg chain. Specifically, in the vicinity of the half-filled regime $\hbar \to 0$, the dressed spin and thermal occupation functions scale with $s$ as $m_s^{\text{def}}(h) \sim h s^2$ and $\lim_{h \to 0} \hat{\rho}_s(h) \sim s^{-2}$, respectively. There is no dependence on charge chemical potential $\mu$ associated to the conservation of the number of electrons. An analogous reasoning applies for the transport of electron charge, see [61] for further details. Numerical evaluation shows that the momentum-dependent part of $D_A$ for the spin-carrying bound states once again scales as in Eq. (11), implying (logarithmically in $s$) diverging spin diffusion bound (5), in analogy to the isotropic Heisenberg model.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig2.pdf}
\caption{Large-$s$ scaling of $\log \Gamma_s = \log \int_{-\infty}^\infty \frac{d\varepsilon}{\rho_s(u)} [\nu_s^{\text{eff}}(u)]^2$, as in equation (12), computed for the isotropic Heisenberg chain for various temperatures, showing that the large $s$ scaling is independent of $\beta$ and therefore the chain displays super-diffusive transport for any $\beta$.}
\end{figure}

We have also solved the $SU(2)$ symmetric (EKS model [70]) and $SU(2|1)$ symmetric (t-J model) fermionic lattice models of spin-carrying electrons. The conclusions remain (qualitatively) unchanged provided the conserved charge $\hat{Q}$ belongs to a bosonic (i.e. even) $SU(2)$ sector of a model. Particularly, in addition to the conserved total magnetization $\hat{S}^z$, the $SU(2|1)$-invariant integrable $t$-J model conserves the total electron charge $\hat{N}_e$. The latter is however, in distinction to the total spin and charge in the Fermi–Hubbard model, a $U(1)$ charge which does not belong to an $SU(2)$ sub-algebra of the full $SU(2|1)$ symmetry of the Hamiltonian. As a consequence, the charge Drude weight can never be turned off in an equilibrium state with a finite fraction of electrons (finite chemical potential for the electrons). Similarly, in the $SU(2|2)$-symmetric model there exists, besides two independent spin $SU(2)$ sectors as in the Hubbard model, the third global $U(1)$ conserved charge which coincides with the Hubbard interaction $\hat{V}_H$. The latter yields a finite Drude weight for all values for chemical potentials. Additional information can be found in [61].

Conclusion. We identified and discussed a class of exactly solvable quantum lattice models with isotropic interactions where Nöether charges exhibit divergent diffusion constants. Super-diffusive transport is attributed to the existence of infinitely many bound states of magnons or electrons which behave (at any finite temperature, cf. Fig. 2 and Eqs. (11), (10)) as effective paramagnetic compounds of spin (or electrons) whose dressed spin (or charge) grows as $\sim h s^2$ as a function of their bare mass $s$ for small values of chemical potential $h$.

There are several related aspects to be addressed in
future. At the moment it is difficult to estimate the importance of integrability for the observed anomalous behavior. Although we have excluded normal diffusion at half filling, notice that we only invoked a lower bound which precludes to determine the exact super-diffusive dynamical exponent. Indeed, numerical simulations on the isotropic quantum [37] and classical Heisenberg magnet [77] give a firm indications of dynamical exponent $z = 2/3$ – which is consistent with the Kardar-Parisi-Zhang universality [78] that was also observed in random unitary circuits in $(1+1)D$ [79], in contrast to the standard diffusive exponent $z = 1/2$ observed in anisotropic models and strongly dissipative XXZ chains [80].

Another interesting open question is whether one can extract transport properties on the sub-ballistic scales for the conserved quantities which always possess finite Drude weights at finite charge density, such as e.g. the Lieb-Liniger model or the charge sector of the (integrable) t-J model, when our curvature bound is not directly applicable. Finally, it would be valuable to find a theoretical explanation for the anomalous diffusion which follows a collapse of a magnetic domain wall scrutinized in several recent studies [37–39].

Acknowledgments. The authors thank C. Karrasch for providing tDMRG data. J.D.N. acknowledges support from LabEx ENS-ICFP-ANR-10-LABX-0010/ANR-10-IDEX-0001-02 PSL*. E.I. is supported by VENI grant number 680-47-454 by the Netherlands Organisation for Scientific Research (NWO). M.M. and T.P. acknowledge the support from the ERC Advanced grant 694544 OMNES and the grant P1-0044 of Slovenian Research Agency.

[1] H. Spohn, Large Scale Dynamics of Interacting Particles (Springer Berlin Heidelberg, 1991).
[2] J. L. Lebowitz, Physical Review 133, A895 (1964).
[3] J. L. Lebowitz and J. K. Percus, Physical Review 155, 122 (1967).
[4] S. A. Hartnoll, Nature Physics 11, 54 (2014).
[5] M. Blake, Phys. Rev. Lett. 117, 091601 (2016).
[6] X. Zotos, F. Naef, and P. Prelovšek, Phys. Rev. B 55, 11029 (1997).
[7] R. G. Pereira, V. Pasquier, J. Sirker, and I. Affleck, Journal of Statistical Mechanics: Theory and Experiment 2014, P09037 (2014).
[8] C. Karrasch, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. 108, 227206 (2012).
[9] X. Zotos, Phys. Rev. Lett. 82, 1764 (1999).
[10] J. Herbrych, P. Prelovšek, and X. Zotos, Phys. Rev. B 84, 155125 (2011).
[11] C. Karrasch, T. Prosen, and F. Heidrich-Meisner, Phys. Rev. B 95, 060406 (2017).
[12] T. Prosen, Physical Review Letters 106 (2011), 10.1103/physrevlett.106.217206.
[13] T. Prosen and E. Ilievski, Phys. Rev. Lett. 111, 057203 (2013).
[14] C. Karrasch, J. Hauschild, S. Langer, and F. Heidrich-Meisner, Phys. Rev. B 87, 245128 (2013).
[15] E. Ilievski and T. Prosen, Communications in Mathematical Physics 318, 809 (2012).
[16] C. Karrasch, D. M. Kennes, and F. Heidrich-Meisner, Phys. Rev. B 91, 115130 (2015).
[17] V. Mastropietro and M. Porta, Journal of Statistical Physics (2018), 10.1007/s10955-018-1994-0.
[18] L. Mazza, J. Viti, M. Carrega, D. Rossini, and A. D. Luca, (2018), arXiv:1804.04476.
[19] T. Barthel, New Journal of Physics 15, 073010 (2013).
[20] R. Steinigeweg, J. Gemmer, and W. Brenig, Phys. Rev. Lett. 112, 120601 (2014).
[21] C. Karrasch, J. E. Moore, and F. Heidrich-Meisner, Phys. Rev. B 89, 075139 (2014).
[22] C. Karrasch, D. M. Kennes, and J. E. Moore, Phys. Rev. B 90, 155104 (2014).
[23] C. Karrasch, R. Ilan, and J. E. Moore, Phys. Rev. B 88, 195129 (2013).
The central ingredient of the Bethe Ansatz formalism is the notion of dressing. Integrable systems are interacting theories which support stable particle-like excitations which undergo elastic scattering, with no particle decay or production. Particles refer soliton-like objects which preserve their nature upon collisions. Non-interacting particles subjected to periodic boundary conditions correspond to plane waves whose momentum $k_L = 1$, irrespectively of other particles in the system. On the other hand, in interacting particle systems the quantization rule acquires an extra multiplicative phase factor,

$$e^{ik_{l}L}e^{i\Phi_{l}} = 1 \iff e^{ip_{l}L} = 1,$$

(A1)
due to an accumulated two-body phase shift $\delta_2$,

$$\Phi_{l} \equiv \sum_{i \neq j = 1}^{M} \delta_{2}(k_{i}, k_{j}).$$

(A2)

Supplemental Material

Super-diffusion in one-dimensional quantum lattice models

This Supplemental Material includes:

1. An exposition of the (nested) Thermodynamic Bethe Ansatz dressing formalism for a family of integrable quantum lattice models of spin or electron degrees.

2. A short derivation of the lower bound on the charge diffusion constants based on the Drude weight curvature.

Appendix A: Dressing formalism: Nested Bethe Ansatz

The central ingredient of the Bethe Ansatz formalism is the notion of dressing. Integrable systems are interacting theories which support stable particle-like excitations which undergo elastic scattering, with no particle decay or production. Particles refer soliton-like objects which preserve their nature upon collisions. Non-interacting particles subjected to periodic boundary conditions correspond to plane waves whose momentum $k_L = 1$, irrespectively of other particles in the system. On the other hand, in interacting particle system the quantization rule acquires an extra multiplicative phase factor,
The absence of diffraction means that multi-particle processes are completely factorizable in terms two-body collisions described by the scattering phase $\delta_2$. The dressing refers to renormalization of the bare momenta $k_i$ by absorbing the scattering shift into a redefinition of the excitation momentum, $k_i \rightarrow k_i^{\text{fr}} \equiv p_i$. In this case, $p_i$ depend implicitly on the dressed momenta of all other $M-1$ excitations in a given eigenstate. Computing the dressing for a state with finitely many excitations in a finite volume amounts to solve the celebrated Bethe Ansatz equations. In the thermodynamic limit, defined by as the scaling limit $L,N \rightarrow \infty$ with $N/L$ kept fixed, the particle momenta take continuous values, and equations (A1) can be converted to a set of coupled integral equation for a distribution function of the dressed momenta.

### Integrable graded lattice models

We devote our analysis to the class of homogeneous one-dimensional quantum lattice models with $SU(N|M)$-symmetric integrable Hamiltonians. Unless stated otherwise, the local physical degrees of freedom belong to the irreducible representations of $su(M|N)$ graded Lie algebras. A convenient property of this family of models is that they can be treated in a uniform way, which can be attributed to the fact that they all share the same elementary scattering amplitudes

$$ S_j(u) = \frac{u - j^{-1/2}}{u + j^{-1/2}}, \quad (A3) $$

for $j \in \mathbb{N}$, describing interactions between various particle excitations in their spectra. Specifically, the scattering amplitude for the collision of two elementary excitations, denoted by $S_{1,1}(u, w) = \exp(\iota \delta_2(u, w))$, is a rational function of the particles’ rapidity variables $u$ and $w$. The bare momentum $k_j$ of the fundamental particle reads

$$ k_j = k(u_j) = \iota \log S_1(u_j). \quad (A4) $$

The entire set of scattering amplitudes describing interactions among the composite (bound) particles is obtain by fusing the elementary ones,

$$ S_{j,k}(u, w) = S_{[j-k]}(u, w)S_{j+k}(u, w) \prod_{m=1}^{\min(j,k)-1} S_{2j-2k+2m}(u, w). \quad (A5) $$

By virtue of integrability, the many-body scattering process is elastic (i.e. free of diffraction) and completely factorizes into a sequence of two-body scattering events, irrespective of the ordering of the two-particle collisions. As a corollary, the set of outgoing momenta are simply a permutation of the momenta of incoming particles.

We shall primarily be interested in interacting models described by $SU(N|M)$-symmetric Hamiltonians

$$ \hat{H}^{N|M} = \sum_{j} \hat{h}_{j,j+1}^{N|M}. \quad (A6) $$

The interaction density $\hat{h}_{j,j+1}^{N|M}$ acts on adjacent lattice sites $j$ and $j+1$. Each lattice site is associated with the fundamental degree of freedom, namely the fundamental representation $V_\Box \cong \mathbb{C}^{N+M}$ of $su(M|N)$. The latter is a graded vector space spanned by vectors $v_i$ ($i = 1 \sim N+M$) equipped with Grassmann $\mathbb{Z}_2$-parity $|i| \in \{0,1\}$. There are $N$ bosonic states with parity $|i| = 0$, and $M$ fermionic states with parity $|i| = 1$. The linear algebra of operators acting in the fundamental representation $V_\Box$ is spanned by $su(N|M)$ generators $\hat{E}^{ij}$ which obey the graded commutations relations

$$ [\hat{E}^{ij}, \hat{E}^{kl}] = \delta_{jk} \hat{E}^{il} - (-1)^{(|i|+|j|)(|k|+|l|)} \delta_{il} \hat{E}^{kj}. \quad (A7) $$

The grading can be assigned arbitrarily. Inequivalent gradings are in one-to-one correspondence with Kac–Dynkin diagrams, consisting of $N+M-1$ nodes which are either bosonic when states $v_i$ and $v_{i+1}$ are of the same parity (open circles) or fermionic when $v_i$ and $v_{i+1}$ have different parities (crossed circles).

We consider a family of lattice Hamiltonians $\hat{H}^{N|M}$ on $V_\Box^{\otimes L}$ with nearest-neighbor interaction densities $\hat{h}^{N|M}$ acting on $V_\Box^{\otimes 2}$ of the form

$$ \hat{h}^{N|M} = 1 - \hat{P}^{N|M}, \quad (A8) $$
with
\[ \hat{P}^{N|M} = (-1)^{|a||b|} \sum_{a,b=1}^{N+M} \hat{E}^{ab} \otimes \hat{E}^{ba}, \]  
(A9)

begin the \textit{graded} permutation. Hamiltonians \( \hat{H}^{N|M} \) can be diagonalized by means of the \textit{nested Bethe Ansatz}. Below we summarize the main ingredients of this procedure. We note that constructing exact eigenstates in finite volume is not our main concern here. Instead, we are only interested in the complete spectrum of particle-like excitations and their properties.

**Particle content**

The \( SU(N|M) \)-symmetric spin chain possesses \( N+M-1 \) types of elementary excitations which are in one-to-one correspondence with the nodes of the corresponding Kac–Dynkin diagram. For a particular choice of grading, each highest-weight eigenstate is characterized by \( N+M-1 \) unique sets of rapidity variables
\[
\left\{ u_j^{(k)} | j = 1 \sim N_k; k = 1 \sim N+M-1 \right\},
\]
(A10)

which solve a coupled set of algebraic equations known as the \textit{nested Bethe Ansatz} equations
\[
\exp (ip(u_i^{(\ell)} )L) \prod_{k=1}^{N+M-1} \prod_{j=1}^{N_k} S_{tk}^{(u_i^{(\ell)}, u_j^{(k)})} = -1,
\]
(A11)

where \( N_k \) is the number of Bethe roots of type \( k \) associated to the \( k \)th Dynkin node. The rational scattering amplitudes,
\[
S_{tk}^{(u_i^{(\ell)}, u_j^{(k)})} = \frac{u_i^{(\ell)} - u_j^{(k)}}{u_i^{(\ell)} - u_j^{(k)}} - \frac{i}{2} K_{t,k},
\]
(A12)

are parametrized with aid of the \textit{graded Cartan matrix},
\[
K_{t,k} = \delta_{t,k} (-1)^{[\ell]} + (-1)^{[\ell+1]} - (-1)^{[\ell+1]} \delta_{t+1,k} - (-1)^{[\ell]} \delta_{t-1,k},
\]
(A13)

The latter specifies the choice of the simple positive roots which, in physical terms, define the reference (Bethe) vacuum state.

**Particles and rectangular partitions.** Elementary excitations are either of bosonic or fermionic type. To describe the complete spectrum of a graded spin chain one has to additionally account for the compound (i.e. bound) particle excitations composed of the elementary ones. Such compounds are most commonly known in the literature the ‘Bethe strings’. Presently, for the class of \( SU(N|M) \)-symmetric homogeneous lattice models (cf. Eq. (A6)), the complete spectrum of particle excitations is in the bijective correspondence with the finite-dimensional (unitary) irreducible representations associated to rectangular partitions, i.e. Young tableaux \([a, s]\) with \( a \) rows and \( s \) columns. A distinguished property of rectangular irreducible representations is that they form a closed set of fusion rules, as dictated by the underlying \textit{classical} Lie algebra \( \mathfrak{g} \). Therefore it is natural to accordingly label the particle content by a pair of positive integers \( A = (a, s) \). In addition, each particle is characterized by a continuous rapidity variable \( u \).

**Inter-particle interactions.** The entire class of graded lattice models with \( \mathfrak{g} = su(N|M) \) symmetry shares a common ‘tight-binding’ four-vertex incidence (adjacency) matrix
\[
I_{AB} \equiv I_{(a,s), (a', s')} = \delta_{a,a'} (\delta_{s,s-1} + \delta_{s,s+1}) + \delta_{s,s'} (\delta_{a,a-1} + \delta_{a,a+1}),
\]
(A14)

which compactly encodes the fusion rules of the scattering amplitudes and can be used to express effective interactions among the particles. Its form reflects the internal structure of the bound states of elementary excitations.

The boundary conditions for \( I_{AB} \) depend on the rank of the algebra \( \mathfrak{g} \) and the number of bosonic states \( N \). This is best understood by recalling the bijective correspondence between the particles and rectangular partitions, which tells that the particles nicely arrange on a two-dimensional integer sub-lattice called the ‘fat hook’ [72, 76]. In the
simpler non-graded case \((M = 0)\), the anti-symmetric fusion can be applied at most \(N - 1\) times and the fat hook is coincides with a sub-lattice in the form of a semi-infinite strip with boundaries \(s \geq 0\) and \(0 \leq a \leq N\). On the other hand, there is no restrictions on the anti-symmetric fusion in the graded case provided \(s \leq M - 1\). These boundary conditions define an L-shaped sub-lattice within the \((a, s)\)-lattice.

There is no unique assignment of the particles to the nodes of the fat hook lattice. The prescription can be made unique by selecting a particular highest-weight Bethe vacuum which requires an appropriate embedding of the chosen Kac–Dynkin diagram inside the fat hook. For a general and comprehensive discussion on this we refer the reader to [74]. Below we instead only focus on particular physically relevant examples.

**Equilibrium ensembles**

A general equilibrium state in the system is uniquely characterized by a complete set of distribution functions \(\rho_A(u)\) pertaining to the densities of Bethe roots of the elementary and bound state solutions to the (nested) Bethe equations \((A11)\). A complete set of densities uniquely specifies a microcanonical statistical ensemble which are sometimes referred to as macrostates. Macrostates describe finite-entropy ensembles where each mode contribute entropy density

\[
\sigma = \log \rho_A(u) + \bar{\rho}_A(u) \log \left(1 + \frac{\rho_A(u)}{\bar{\rho}_A(u)}\right).
\]

The set of functions \(\bar{\rho}_A\) are called the hole densities and corresponds to the densities of the unoccupied solutions to the Bethe equations. The latter are uniquely determined given the densities \(\rho_A\) via a coupled system of linear integral equations

\[
\rho_A + \bar{\rho}_A = \sigma A K_A - K_{AB} * \rho_B,
\]

where \(*\) denotes the convolution-type integration over the rapidity, domain defined as

\[
(f * g)(u) = \sum_A \int dw f_A(u, w) g_A(w), \quad (F * g)(u) = \sum_B \int dw F_{AB}(u, w) g_B(w),
\]

assuming summation convention over repeated indices. The kernels entering in Eqs. \((A16)\) are the logarithmic derivatives of scattering amplitudes,

\[
K_A(u) = \frac{1}{2\pi i} \partial_u \log S_A(u), \quad K_{AB}(u) = \frac{1}{2\pi i} \partial_u \log S_{AB}(u).
\]

The \(\sigma\)-parity is a \(Z_2\) label defined as the sign of the bare momentum derivative, \(\sigma_A = \text{sign}(k'_A)\). The total density of the available states for a particle of type \(A\) is (due to interactions) a rapidity (i.e. momentum) dependent quantity,

\[
\rho'^{\text{tot}}_A(u) = \rho_A(u) + \bar{\rho}_A(u) = \sigma_A \frac{\rho'_A(u)}{2\pi}.
\]

An alternative way to characterize equilibrium state is via the mode occupation functions

\[
\vartheta_A(u) = \frac{\rho_A(u)}{\rho'^{\text{tot}}_A(u)},
\]

which are a natural generalization the Fermi–Dirac occupation function of non-interacting fermions to interacting particles subjected to the ‘exclusion principle’. In the dressing formalism of the Thermodynamic Bethe Ansatz, another useful set of quantities to define are hole to particle ratios,

\[
Y_A(u) = \frac{\bar{\rho}_A(u)}{\rho_A(u)},
\]

which are commonly known as the \(Y\)-functions.

The thermodynamic free energy \(f = -\log Z\) of a general equilibrium state is conveniently expressed a functional integral

\[
Z = \int D[\{\rho_A(u)\}] \exp \left(-L \sum_A \int du (\mu_A(u)\rho_A(u) + s_A(u))\right).
\]
where \( s_A(u) \) denotes the entropy density per particle given by Eq. (A15), and \( \mu_A(u) \) is a complete set of analytic (dynamical) chemical potentials as defined in [55]. Importantly, \( \mu_A(u) \) uniquely define a ‘generalized’ equilibrium Gibbs ensemble in the sense that the correspond a particular distribution of Bethe root densities \( \rho_A(u) \) or, equivalently, values of local conservation laws [75]. In the \( L \to \infty \) limit, the saddle-point integration yields a set of coupled nonlinear integral equations known as the canonical TBA equations. In particular, in terms of the \( Y \)-functions defined in Eq. (A21), these take the form

\[
\log Y_A = \mu_A + K_{AB} \ast \log(1 + 1/Y_B). \tag{A23}
\]

Hence, given \( \mu_A(u) \) one finds \( \vartheta_A(u) \) via Eq. (A23), and vice-versa. For instance, in the case of the grand-canonical Gibbs ensemble the chemical potentials take the form

\[
\mu_A^{GC}(u) = \beta e_A(u) + \sum_i 2\hbar_i n_{A,i}, \tag{A24}
\]

where \( \beta \) is the inverse temperature, \( e_A(u) \) denoted the one-particle energies, \( \lim_{L \to \infty} \frac{E}{L} = e_A \ast \rho_A \), and \( n_{A,i} \) one-particle bare \( U(1) \) charges.

**Universal dressing transformation**

An infinite sum over particle species on the right-hand side of Eq. (A23) can be removed by exploiting the following properties of fusion identities: introducing the *left* inverse of kernel \( (K + 1) \),

\[
C_{AB}(u) = (K_{AB}(u) + \delta_{AB})^{-1} = \delta_{AB} - \mathfrak{s}(u) I_{AB}, \tag{A25}
\]

where

\[
\mathfrak{s}(u) = \frac{1}{2 \cosh \left( \frac{\pi u}{2} \right)} , \tag{A26}
\]

is the solution to equation \( K_1 - s \ast K_2 = \mathfrak{s} \). The bare energies of *fundamental* particles are simply proportional to the elementary scattering kernels \( e_A(u) \simeq K_A(u) \), implying the quasi-local TBA source terms of the form

\[
(K + 1)^{-1}_A \ast K_B = C_{AB} \ast K_B = \delta_{A,15}. \tag{A27}
\]

In the case of the graded SU(\( N|M \))-symmetric quantum chains, the incidence matrix \( I_{AB} \) entering in the definition of the Baxter–Cartan matrix \( C_{AB} \) (cf. Eq. (A25)) splits into the horizontal and vertical parts,

\[
I_{s,s'} = \delta_{s+1,s'} + \delta_{s-1,s'}, \quad I_{a,a'} = \delta_{a-1,a'} + \delta_{a+1,a'}, \tag{A28}
\]

respectively, while leaving the boundary conditions which depend on \( N \) and \( M \) implicit for the time being.

Taking advantage of the fact that the entire TBA framework originates from the fusion rules for the Yangian extensions of classical characters for the rectangular Young tableaux, the entire TBA dressing formalism be realized in a *universal group-theoretic form*

\[
C_{s,s'} \ast L_{a,s'} - C_{a,a'} \ast L_{a,s} = \nu_{a,s} . \tag{A29}
\]

where indices \( (a, s) \in \mathbb{Z}^2 \) belong to the interior of the fat hook lattice. Physical interpretation of these equations depends on interpretation of variables \( L_{a,s} \) and \( \bar{L}_{a,s} \):

- for \( L_{a,s} \equiv \bar{\rho}_{a,s} \) and \( \bar{L}_{a,s} = -\rho_{a,s} \) one finds the Bethe–Yang integral equations (A16), expressing the hole distribution functions \( \bar{\rho}_{a,s} \) in terms of Bethe root densities \( \rho_{a,s} \), with source terms \( \nu_{a,s} = k'_{a,s} \),

- for \( L_{a,s} \equiv \log(1 + Y_{a,s}) \) and \( \bar{L}_{a,s} \equiv \log(1 + Y_{a,s}^{-1}) \) one finds quasi-local form of TBA equations. Dependence on equilibrium chemical potentials \( \mu_A(u) \) is contained in the source terms

\[
\nu_A = C_{AB} \ast \mu_B . \tag{A30}
\]
In practice, it is useful to use the differential form of Eqs. (A29). This yields the following system of coupled linear integral equations,

\[ C_{AB}^{(\vartheta)} \ast (q_B')^{dr} = q_A', \tag{A31} \]

or explicitly,

\[ C_{s,s'}^{(\vartheta)} \ast (q_{a,s'}')^{dr} + C_{a,a'}^{(\vartheta)} \ast (q_{a,s'}')^{dr} = q_{a,s}, \tag{A32} \]

where we have introduced the dressed Baxter–Cartan matrices

\[ C_{s,s}(u) = \delta_{s,s'} - s(u) I_{s,s'} \bar{\vartheta}_{a,s}(u), \tag{A33} \]

\[ C_{a,a}(u) = \delta_{a,a'} - s(u) I_{a,a'} \vartheta_{a,s}(u), \tag{A34} \]

where $\vartheta_{a,s}$ enter as input variables parametrizing the reference many-body vacuum (equilibrium state).

In the non-graded chains ($M = 0$), the fat-hook lattice is a semi-infinite strip and the indices range in $s = 1, 2, \ldots$, and $a = 1 \sim N$. In the graded cases (i.e. for $M > 0$), the exterior (interior) boundaries are along $(0, s \geq 0)$ and $(a \geq 0, 0)$ $(N, s \geq M)$ and $(N \geq a, M)$. An extra subtlety of the graded models is that there is an additional exceptional relation associated to the boundary node $(a, s) = (N, M)$ where Eqs. (A29) to no apply. Nonetheless, there is no ambiguity as the functional relations for quantum characters enforce its uniqueness.

**High-temperature expansion**

The TBA dressing equations cannot in general do not permit closed-form solutions. Two most important exceptions to this however are the ground-state limit $\beta^{-1} \to 0$ limit and the high-temperature $\beta \to 0$ limit. Below we specialize our treatment to the high-temperature limit of the grand canonical Gibbs ensembles where the dressing integral equations becomes a set of coupled algebraic equations, see e.g. [66]. Here we make use of the group-theoretical formulation by invoking the character formulae for classical (graded) Lie algebras.

We begin by the leading-order $\beta$-expansions of the TBA functions,

\[ \log Y_A = \log Y_A^{(0)} + \beta F_A + \mathcal{O}(\beta^2), \tag{A35} \]

\[ \log(1 + Y_A) = \log(1 + Y_A^{(0)}) + \beta \bar{\vartheta}_A^{(0)} F_A + \mathcal{O}(\beta^2), \tag{A36} \]

\[ \log(1 + 1/Y_A) = \log(1 + 1/Y_A^{(0)}) - \beta \vartheta_A^{(0)} F_A + \mathcal{O}(\beta^2). \tag{A37} \]

The major simplification in the $\beta \to 0$ limit which we consider below is that the occupation functions become constant (i.e. rapidity-independent) functions, and thus all convolution integrals reduce to scalar multiplication.

**Mode occupation functions**

Using the property of the $s$-kernel, $1 \ast s = \frac{1}{2}$, equations (A37) the $\beta \to 0$ limit reduce to the following non-linear functional relations for the $Y$-functions

\[ \log \left[ Y_{a,s}^{(0)} \right]^2 = I_{s,s'} \log(1 + Y_{a,s}^{(0)}) - I_{a,a'} \log(1 + 1/Y_{a,s}^{(0)}). \tag{A38} \]

An equivalent exponential form of this results is nothing but the simplified, ‘classical’, version of the $Y$-system relations [69]

\[ \left[ Y_{a,s}^{(0)} \right]^2 = \frac{(1 + Y_{a,s+1}^{(0)})(1 + Y_{a,s-1}^{(0)})}{(1 + 1/Y_{s,a-1}^{(0)})(1 + 1/Y_{s,a+1}^{(0)})}, \tag{A39} \]

for constant (i.e. rapidity-independent) $Y$-functions. To obtain unique solutions to this system of coupled recurrence relations we need in addition to supply the following $N + M - 2$ asymptotic conditions,

\[ \lim_{s \to \infty} Y_{a,s}^{(0)} = \exp(2h_a s), \quad a = 1 \sim N - 1, \tag{A40} \]

\[ \lim_{a \to \infty} Y_{a,s}^{(0)} = \exp(2\mu_s a), \quad s = 1 \sim M - 1. \tag{A41} \]
Parameters $h_a$ and $\mu_s$ are the chemical potentials associated with global conserved $U(1)$ charges of an equilibrium equilibrium state. In the graded chains, i.e. for $M > 1$, there is an additional chemical potential which is not encoded in the asymptotics of the TBA $Y$-functions. As we shall see on explicit examples, the latter enters through the equation at the corner node $(a,s) = (N,M)$ of the fat hook.

Any solution to equations (A39) admits an equivalent gauge-covariant parametrization in terms of classical characters $\chi_{a,s}$ which are defined through a non-linear transformation

$$Y_{a,s}^{(0)} = \frac{\chi_{a-1,s, \Delta} a_{s+1}}{\chi_{a-1,s, \Delta} a_{s+1}}.$$ (A42)

The infinite set of functions $\chi_{a,s}$ satisfy the simplified version of the Hirota bilinear relations

$$\chi_{a,s}^2 = \chi_{a-1,s, \Delta} a_{s+1} + \chi_{a-1,s, \Delta} a_{s+1}.$$ (A43)

Indeed, the above formula is a reduction of the full ‘quantum’ Hirota equation with spectral parameter (see e.g. [76]) in the ‘classical limit’ when dependence on the spectral parameter drops out. In fact, Eq. (A43) is the well-known identity for characters $\chi_{a,s} = \chi_{a,s}(G)$ of rectangular irreducible representations $(a,s)$ of classical graded algebras $g(N|M)$, with $G$ denoting an element of a $(N+M-1)$-dimensional Cartan subalgebra. Characters $\chi_{a,s}$ are thus only functions of the $U(1)$ chemical potentials which parametrize the infinite-temperature grand canonical Gibbs ensemble.

Below we recall some basic facts about the character formulae for the non-graded $g(N)$ Lie algebras, where only the rectangular characters $\chi_{a,s}(G)$, with $G = \text{diag}(x_1, \ldots, x_N)$ denotes a general element of the Cartan subalgebra, will be of our interest. A character $\chi_{a,s}$ is expressible as a determinant including only the totally symmetric (anti-symmetric) characters $\chi_{1,s}$ ($\chi_{a,1}$) in accordance with the Giambelli-Jacobi-Trudi formula

$$\chi_{a,s}(G) = \text{Det} \left( \chi_{1,s+j-k} \right)_{1 \leq j,k \leq a}.$$ (A44)

An explicit parametrization in terms of the eigenvalues of the Cartan charges is given by the 1st Weyl character formula

$$\chi_{a,s}(G) = \text{Det} \left( x_k^{N-j+s \theta_{i,j}} \right)_{1 \leq i,k \leq N},$$ (A45)

with $\theta_{i,j} = 1$ if $i \geq j$ and zero otherwise. Functions $\chi_{a,s}(G)$ are related to Schur polynomials, i.e. completely symmetric polynomials of $N$ variables $x_1, \ldots, x_N$. The generating function for totally symmetric characters is

$$w(z) = \prod_{j=1}^N \frac{1}{1 - z x_j} = \sum_{s=0}^{\infty} z^s \chi_{1,s}(x_1, \ldots, x_N).$$ (A46)

Likewise, the totally anti-symmetric characters are generated from the inverse expansion $w^{-1}(z) = \sum_{s=1}^{\infty} (-1)^s \chi_{a,1} z^s$. For instance, in the simplest $N = 2$ case we have

$$\chi_{1,1}(x_1, x_2) = \begin{vmatrix} x_1^{s+1} & x_2^{s+1} \\ 1 & 1 \\ x_1 & x_2 \\ 1 & 1 \end{vmatrix} = \frac{x_1^{s+1} - x_2^{s+1}}{x_1 - x_2}.$$ (A47)

Cartan sectors of the $SU(2)$-symmetric fundamental spin chain is thus parametrized by a single parameter $h$, defined as $x_1/x_2 = \exp(2h)$, which pertains to the chemical potential for the conserved total spin $\hat{S}^z$. Therefore $\chi_{0,s} = \chi_{2,s} = 1$, implying

$$1 + Y_s^{(0)}(h) = [\chi_{1,s}(h)]^2,$$ (A48)

with symmetric characters

$$\chi_{1,s}(h) = \frac{e^{-s+1}h - e^{s+1}h}{e^{-h} - e^h}.$$ (A49)

At half filling limit $h \to 0$, we find

$$\lim_{h \to 0} \chi_{1,s}(h) = d_s = \dim V_s = s + 1,$$ (A50)

$$\lim_{h \to 0} Y_s^{(0)}(h) = s(s + 2).$$ (A51)
In the higher-rank $\mathfrak{su}(N)$-symmetric models we deal with $N - 1$ conserved number operators $\hat{N}_i$. We thus put

$$x_1 = 1, \quad x_j/x_{j+1} = \exp(2\hbar_j).$$

Finally, it is instructive to inspect the singular limit of vanishing chemical potentials $x_j \to 1$. For a general partition (Young tableaux) $\lambda = (\lambda_1, \ldots, \lambda_N)$, with $\lambda_j \geq \lambda_{j+1}$ defining a $\mathfrak{gl}(N)$ representation, the latter is the following specialization of Schur polynomials

$$s_\lambda(1,1,\ldots,1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$  

which is in fact the well-known hook-length formula yielding the multiplet dimension $\dim V_\lambda$. For the rectangular partitions $\lambda = (s^a) = (s, s, \ldots, s)$ this implies $\lim_{h_j \to 0} \chi_{a,s} = \dim V_{a,s}$.

**Dressing in the high-temperature limit**

In the infinite temperature limit $\beta \to 0$, the dressing transformation becomes an infinite system of coupled algebraic equations which admits closed-form solutions. The leading high-temperature contribution to the mode occupation functions and the dressed values for the $U(1)$ charges carried by the particles can be directly computed from the TBA $Y$-functions as outlined below. Specifically, the occupation functions are expressible as the following ratios of $\chi$-functions

$$\varrho^{(0)}_{a,s} = \frac{\chi_{a-1,s} \chi_{a+1,s}}{\chi_{a,s}^2}, \quad \bar{\varrho}^{(0)}_{a,s} = \frac{\chi_{a,s} \chi_{a+1,s}}{\chi_{a,s}^2}.$$  

The dressed values of conserved $U(1)$ charges are mostly computed from

$$m^{\text{dr}}_{a,s}(G) = \partial_{h_a} \log Y_{a,s}^{(0)}(G), \quad a = 1 \sim N - 1,$$

$$n^{\text{dr}}_{a,s}(G) = \partial_{\mu_s} \log Y_{a,s}^{(0)}(G), \quad s = 1 \sim M - 1,$$

where $G$ depends on a set of chemical potentials $h_i$, $\mu_i$. There is an extra chemical potential, denoted by $u$, which is not encoded in the asymptotic but explicitly enters in the equation for the corner node.

The high-temperature limit of the particles’ dressed dispersion relations in the leading order $\mathcal{O}(\beta)$ is found as a solution to the following system of coupled linear integral equations

$$F_{a,s} = C^{(0)}_{s,s'} \ast F_{a,s'} + C^{(0)}_{a,a'} \ast F_{a',s} - \nu_{a,s},$$

where $C^{(0)}_{a,a'}$ and $C^{(0)}_{s,s'}$ are the Baxter–Cartan matrices dressed by the infinite-temperature equilibrium state,

$$C^{(0)}_{a,a'} \ast F_{a,s'} = \delta_{a,a'} - I_{a,a'} \ast \bar{\varrho}^{(0)}_{a,s'} F_{a,s'},$$

$$C^{(0)}_{s,s'} \ast F_{a,s} = \delta_{s,s'} - I_{s,s'} \ast \varrho^{(0)}_{a,s} F_{a,s}.$$  

**Anisotropic Heisenberg spin-$1/2$ chain**

A prototype model of an integrable spin chain is the axially anisotropic Heisenberg spin-1/2 chain (XXZ model),

$$\hat{H}_{\text{XXZ}} = \sum_{j=1}^{L} \left( \hat{S}^x_j \hat{S}^x_{j+1} + \hat{S}^y_j \hat{S}^y_{j+1} + \Delta \hat{S}^z_j \hat{S}^z_{j+1} \right).$$

For $\Delta = 1$, the model exhibits a manifest global $SU(2)$ symmetry. Due to integrability, there is a non-manifest infinite dimensional quantum-group algebra known as the Yangian $\mathcal{Y}(\mathfrak{su}(2))$. For generic values $\Delta \neq 1$ the symmetry continuously deforms in the so-called ‘quantum deformed’ universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$. For the root-of-unity value $q = \pi(m/\ell)$, with co-prime integers $m < \ell$ and $\ell \geq 2$, the symmetry enlarges (see [66]).
Isotropic point

We first consider the isotropic point \(|\Delta| = 1\) regime, the particle content consists of magnons \((s = 1)\) and an infinite sequence of magnonic bound states \((s \geq 2)\). These particles are commonly referred to as the \(s\)-strings. The high-temperature limit of the dressing transformation is a three-point recurrence relation

\[
C_{s,s'}^{(1)}(h) \ast F_{s'} = \delta_{s,1} g, \quad \lim_{s \to \infty} F_s = 0. \tag{A61}
\]

The occupation functions are readily obtained from the character formulae as prescribed by Eqs. (A54),

\[
\bar{\varrho}_s^{(0)}(h) = \frac{1}{1 + Y_s(h)} = \frac{1}{\chi_s^2(h)}, \quad \bar{\varrho}_s^{(0)}(h) = 1 + 1/Y_s(h) = 1 - \varrho_s(h). \tag{A62}
\]

The solution to the recurrence relation (A61) reads

\[
F_s(h) = \frac{\chi_s(h)}{\chi_1(h)} \left( \frac{K_s}{\chi_{s-1}(h)} - \frac{K_{s+1}}{\chi_{s+1}(h)} \right). \tag{A63}
\]

In particular, at half filling we have

\[
\lim_{h \to 0} \bar{\varrho}_s^{(0)}(h) = \frac{s(s+2)}{(s+1)^2}, \quad \lim_{h \to 0} F_s(h) = \frac{s+1}{2} \left( \frac{K_s}{s} - \frac{K_{s+2}}{s+2} \right). \tag{A64}
\]

The dressed momentum in the \(\beta \to 0\) limit therefore reads

\[
p_s^{(0)}(u) = \frac{s+1}{2} \left( \frac{K_s(u)}{s} - \frac{K_{s+2}(u)}{s+2} \right). \tag{A65}
\]

Similarly, the dressed energy is of the form

\[
\varepsilon_s^{(0)}(u) = \lim_{\beta \to 0} \beta^{-1} \partial_u \log Y_s(u) = F_s'(h) = \frac{s+1}{2} \left( \frac{K_s'}{s} - \frac{K_{s+2}'}{s+2} \right). \tag{A66}
\]

The dressed spin is found as the solution of the following homogeneous recurrence

\[
m_s^{dr(0)}(h) - \frac{1}{2} I_{s,s'} \bar{\varrho}_s^{(0)}(h) m_{s'}^{dr(0)}(h) = 0, \quad \lim_{s \to \infty} m_s^{dr(0)}(h) = s. \tag{A67}
\]

For finite value of chemical potential \(h\), the solution reads

\[
m_s^{dr(0)}(h) = \partial_{2h} \log Y_s^{(0)}(h) = \frac{\sinh (h)}{\sinh ((s+1)h)} \left( \frac{s}{\sinh (sh)} - \frac{s+2}{\sinh ((s+2)h)} \right). \tag{A68}
\]

In the vicinity of half filling \(h \to 0\) we thus have

\[
m_s^{dr(0)}(h) \sim \frac{1}{3} (s+1)^2 h + \mathcal{O}(h^3). \tag{A69}
\]

Easy-axis (gapped) regime \(|\Delta| > 1\)

The easy-axis regime of the XXZ Hamiltonian (A60) is parametrized by anisotropy \(\Delta = \cosh (\eta), \) for \(\eta \in \mathbb{R}_+\). The elementary kernels undergo a trigonometric deformation

\[
K_s(u) = \frac{1}{2\pi i} \partial_u \log S(u) = \frac{1}{2\pi} \frac{2 \sinh(s \eta)}{\cosh(s \eta) - \cos(2u)}. \tag{A70}
\]

The fundamental zone of the rapidity integration is a compact interval \(u \in [-\eta/2, \eta/2]\). The kernels for the isotropic chain are recovered in the limit \(\eta \to 0^+\) after simultaneous rescaling the rapidity variables,

\[
K_s^{XXX}(u) = \lim_{\eta \to 0^+} K_s(\eta u) = \frac{1}{2\pi} \frac{s}{(s/2)^2 + u^2}. \tag{A71}
\]
The bare energies of the \( s \)-strings are
\[
e_s(u) = -\pi \sinh(\eta) K_s(u).
\]

In the large-\( s \) limit we the kernels behave as
\[
\lim_{s \to \infty} K_s(u) = \frac{1}{\pi}, \quad \lim_{s \to \infty} K_s'(u) = \frac{\sin(2u)}{\pi \cosh(s\eta)}.
\]

The rapidity derivatives of the dressed energies in the high-temperature limit at half filling read
\[
\varepsilon^{(0)}_s(u) = -\pi \sinh(\eta) \frac{s + 1}{2s(s + 2)} \left((s + 2)K_s' - sK_{s+2}'\right).
\]

The expressions for the rapidity-independent quantities \( m_s^{\text{dr}}(h) \) and \( \vartheta_s^{(0)}(h) \) are the same as those for the isotropic point.

**Easy-plane (gapless) regime** \(|\Delta| < 1\)

The easy-plane regime is parametrized by \( \Delta = \cos(\gamma) \), for \( \gamma \in \mathbb{R} \). We consider the root-of-unity value of the quantum deformation parameter \( q = e^{i\gamma} \), namely \( \gamma \) which are rational multiples of \( \pi \), \( \gamma/\pi = m/\ell \), in which case the centre of the quantum group symmetry algebra \( U_q(\mathfrak{sl}_2) \) enlarges. The structure of eigenstates and thermodynamic particle content becomes dependent on \( \Delta \) in a rather intricate way. The complete classification can be found in [66]. A key difference in compare to the \(|\Delta| \geq 1\) regime is that the number and types of excitations in the spectrum now explicitly depends on the value of \( \gamma \). To further simplify the analysis, we restrict our consideration to a discrete set of primitive roots of unity \( \gamma = \pi/\ell \), for \( \ell \in \mathbb{N} \) \( (\ell \geq 2) \), when there are \( \ell \) distinct species: for \( s = 1 \sim \ell - 1 \) we have the magnons and bound states thereof with bare spin \( n_s = s \), whereas the last particle labelled by \( s = \ell \) corresponds to an unbound magnon \( (n_\ell = 1) \) of negative \( \sigma \)-parity \( (\sigma_\ell = -1) \). The origin of such a truncation can be traced to the fact that at these particular values of \( q \) the \((\ell + 1)\)-dimensional irreducible representation of \( U_q(\mathfrak{sl}(2)) \) becomes reducible. The last two particles interact with other particles in a distinctive way. This special feature of the gapless regime is key to understand the nature of quantum spin transport [43].

The TBA \( Y \)-functions are denoted by \( Y_s = \tilde{\rho}_s/\rho_s \) for \( s = 1 \sim \ell - 2 \), whereas the special pair of particles is associated the following pair of \( Y \)-functions
\[
Y_{\ell-1} \equiv Y_0 = \tilde{\rho}_0/\rho_0, \quad Y_\ell \equiv Y_\bullet = \rho_\bullet/\tilde{\rho}_\bullet.
\]

The elementary scattering kernels depend on \( \gamma \) are read explicitly
\[
K_s(u) = \frac{2 \sin(\gamma q_j)}{\cosh(2u) + \cos(\gamma q_j)},
\]
with \( q_j = \ell - j \) for \( j = 1 \sim \ell - 1 \) and \( q_\ell = -1 \), and satisfying the following identities
\[
K_s - s \star (K_{s-1} + K_{s+1}) = 0, \quad s \leq \ell - 2,
\]
\[
K_\bullet = -K_\bullet \star K_{\ell-2},
\]

The \( s \)-kernel gets modified and now depends on the anisotropy,
\[
s_\ell(u) = \frac{\ell}{2\pi \cosh(2u)}.
\]

The quasi-local form of the Bethe–Yang equations for the string densities is compactly written as
\[
\rho_s^{\text{tot}} = I_{s,s'} \star \tilde{\rho}_{s'},
\]
where \( I^{(\ell)} \) stands for the \( \ell \)-dimensional incidence matrix of the \( D_\ell \) system,
\[
I^{(\ell)}_{jk} = \sum_{j=1}^{\ell-3} (\delta_{j,k-1} + \delta_{j,k+1}) + \delta_{\ell-2,k} + \delta_{\ell-1,\ell-2} + \delta_{\ell,\ell-2}.
\]
Similarly, the quasi-local TBA equation take the form
\[ \log Y_s = \nu_s + I_{s,s'} \sigma \log(1 + Y_{s'}) . \]  
(A82)

Notice that \( \nu_{\sigma \bullet} = \nu_{\sigma \bullet}(\ell, h) \).

Unlike in the isotropic chain or the gapped phase, the dressing equations in gapless regime for root-of-unity value of \( \gamma \) enclose a finite set of equations
\[
(1 + 1/Y_s)f_s - \delta_s \bullet \phi = \frac{1}{2} \left( \delta_{s-1} \phi \right), \quad s = 1 \sim \ell - 3, 
\]
(A83)
\[
(1 + 1/Y_{\ell-2})f_{\ell-2} - \delta_{\ell} \bullet \phi = \frac{1}{2} \left( \delta_{\ell-3} \phi + 2f_0 \right), \quad \ell = 1 \sim \ell - 2, 
\]
(A84)
\[
(1 + 1/Y_0)f_0 - \delta_0 \bullet \phi = \frac{1}{2} \left( \delta_{-1} \phi \right), \quad \ell = 0. 
\]
(A85)

where we wrote \( f_s = \tilde{\phi}_s F_s \). The general solution of equations (A85) is already known \[66\]
\[
\hat{f}_s(\kappa) = \frac{1}{\chi_1 \chi_{y_1 - 1} \chi_{s+y_1-1}} \left( \chi_{s+2y_1-1} \frac{\sinh \left( \frac{q_s}{p_0} \kappa \right)}{\sinh \left( \frac{\pi}{2} \kappa \right)} - \chi_{s-1} \frac{\sinh \left( \frac{q_s-p_0}{p_0} \kappa \right)}{\sinh \left( \frac{\pi}{2} \kappa \right)} \right), \quad s = 1 \sim \ell - 2, 
\]
(A86)
\[
\hat{f}_0(\kappa) = \frac{\sinh \left( \frac{q_0}{p_0} \kappa \right)}{2 \sinh \left( \frac{\pi}{2} \kappa \right)}, \quad \ell = 0, 
\]
(A87)

where
\[
y_1 = 1, \quad p_0 = \ell, \quad p_1 = 1, \quad q_s = \ell - s, \quad s = 1 \sim \ell - 1, \quad q_\ell = -1, 
\]
(A88)
(A89)
are the Takahashi–Suzuki numbers for the simple roots of unity \( q = \cos(\pi/\ell) \). In Fourier space the kernel read
\[
\hat{K}_s(\kappa) = \frac{\sinh \left( \frac{q_s}{p_0} \kappa \right)}{\sinh \left( \frac{\pi}{2} \kappa \right)}, \quad \sigma(\kappa) = \frac{1}{2} \cosh \left( \frac{\pi}{2} \kappa \right). 
\]
(A90)

The regular particles (s-strings) with indices ranging in s = 1 \sim \ell - 2 behave similarly to the regular s-strings in the isotropic and gapped regimes \(|\Delta| \geq 1\), and obey the standard Y-system relations
\[
\left[ Y_s(0)(h) \right]_s^2 = \left( 1 + Y_{s-1}(0)(h) \right) \left( 1 + Y_{s+1}(0)(h) \right), \quad s = 1 \sim \ell - 3. 
\]
(A91)

On the other hand, the pair of exceptional excitations which are due to the truncated particle spectrum require a separate analysis. For their Y-functions we find
\[
\log \left[ \frac{Y_0(h)}{Y_s(h)} \right] = 2h\ell, 
\]
(A92)

which is valid for any value of inverse temperature \( \beta \). We thus parametrize
\[
Y_\bullet(h) = e^{-2h\ell} Y_0(h), 
\]
(A93)

which implies that in the \( h \to 0 \) limit \( \lim_{h \to 0} \left( Y_s/Y_\bullet \right) = 1 \). Similarly, their total densities also coincide \( \rho_\sigma^\text{tot} = \rho_\sigma^{\text{tot}} \) (for any \( \beta \)), whereas the dressed momenta differ by an overall sign, \( p'_\sigma = -p'_\bullet \), due to opposite \( \sigma \)-parities \( \sigma_\sigma = -\sigma_\bullet \). In the \( \beta \to 0 \) limit and finite \( h \), the Y-system functional relations (A91) for the regular particles together with
\[
\left[ Y_{\ell-2}(0)(h) \right]_s^2 = \left( 1 + Y_{\ell-3}(0)(h) \right) \left( 1 + Y_{\ell-1}(0)(h) \right) \left( 1 + Y_\bullet(0)(h) \right), 
\]
(A94)

\[
\left[ e^{-h\ell} Y_\bullet(0)(h) \right]_s^2 = \left[ e^{h\ell} Y_\bullet(0)(h) \right]_s^2 = 1 + Y_{\ell-2}(0)(h), 
\]
(A95)

\[
Y_s(0)(h) = \chi^2_{1,s}(h) - 1, \quad s = 1 \sim \ell - 2, 
\]
(A96)
At half filling $h \to 0$ the high-temperature regular $Y$-functions become
\begin{equation}
\lim_{h \to 0} Y^{(0)}_s(h) = s(s + 2), \quad s = 1 \sim \ell - 2,
\end{equation}
and the regular particle carry the dressed spin
\begin{equation}
m^{dr(0)}_s(h) \sim \frac{1}{3} (s + 1)^2 h + \mathcal{O}(h^3),
\end{equation}
which agrees with the results found earlier for the $s$-string excitations in $|\Delta| \geq 1$ regime. For the special pair of excitations, the mode occupation functions at half filling depend on $\ell$ are read
\begin{equation}
\lim_{h \to 0} Y^{(0)}_{\circ \bullet}(h) = \ell - 1, \quad \lim_{h \to 0} \vartheta^{(0)}_{\circ \bullet}(h) = \frac{1}{\ell}.
\end{equation}
Their dressed energies and dressed spin are computed as
\begin{equation}
\varepsilon^{(0)}_{\circ}(u) = \varepsilon^{(0)}_{\bullet}(u) = \lim_{\beta \to 0} \beta^{-1} \partial_u \log Y_{\circ}(u) = \lim_{\beta \to 0} \partial_u (s \log(1 + Y_{\ell-2})) = \partial_u (s \log f_{\ell-2}) = (1 + 1/Y^{(0)}_{\circ}) f_{\circ}',
\end{equation}
where we have used the expansion $\log(1 + Y_{\ell-2}) = \log(1 + Y^{(0)}_{\ell-2}) + \beta f_{\ell-2} + \mathcal{O}(\beta^2)$. In the $h \to 0$ limit, the dressed spin of the special particles is
\begin{equation}
m^{dr(0)}_{\circ}(h) = -m^{dr(0)}_{\bullet}(h) = \partial_{2h} \log Y^{(0)}_{\circ}(h) \sim \pm \frac{\ell}{2} + \mathcal{O}(h).
\end{equation}

**Free fermionic point (XX spin chain).** The non-interacting point corresponds to $\ell = 2$. In this case there is no regular strings in the spectrum which now only comprises the exceptional particles $\circ$ and $\bullet$. Due to the absence of interactions, the scattering kernel simplify to $K_{\circ \circ} = K_{\bullet \bullet} = K_{(2,+)} = 0$ and $K_{\circ \bullet} = K_{\bullet \circ} = K_{(2,-)} = 0$, and the non-interacting TBA equations read
\begin{equation}
\log Y_{\circ} = 2h - 2\pi \beta K_1, \quad \log Y_{\bullet} = -2h - 2\pi \beta K_1.
\end{equation}
implies that the dressed spin is just the bare spin, $\lim_{h \to 0} s^{dr}_{\circ \bullet}(h) = \partial_{2h} \log Y^{(0)}_{\circ \bullet}(h)|_{h=0} = 1$, as expected. Indeed, the two particles with $\sigma$-particles $\sigma_\circ = 1$ and $\sigma_\bullet = -1$ at the non-interacting point $\gamma = \pi/2$ are nothing but two branches of a single free electron dispersion with momentum ranges $k \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $k \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$.

**Fermionic chains**

Integrable Hamiltonians given by Eq. (A6) with non-trivial grading are most naturally expressed in terms of canonical fermions. Below we consider a few most important examples which play an important role in the condensed matter literature. Before proceeding we owe a few general technical remarks. Unlike the ordinary (non-graded) semi-simple Lie algebras, the $\mathbb{Z}_2$-graded Lie algebras are rather special. To begin with, there is no unique choice of simple root system which affects the algebraic Bethe ansatz diagonalization procedure in the sense that the bare (Bethe) vacuum is no longer unique. This is to be contrasted with the non-graded spin chain where, for instance in the $SU(N)$-symmetric models, all distinct bare vacua share the same form of Bethe equations modulo particle relabelling.

In the fermionic (graded) cases, the total number of different bare vacua available equals the rank of $\mathfrak{g} = \mathfrak{su}(N|M)$, i.e. $\text{rank}(\mathfrak{g}) = N + M - 1$. Different choices are represented by their corresponding Kac–Dynkin diagrams which consist of $N + M - 1$ nodes (pertaining to elementary excitations in the spectrum), with convention that open (bosonic) circles correspond to adjacent states of equal Grassmann parity and crossed (fermionic) circles to adjacent states with opposite parities. All distinct possibilities which are nevertheless interrelated by the so-called fermionic duality transformations (cf. [74]) and permit to construct the same complete spectrum of (highest-weight) eigenstates. In the fundamental chains there is only one type of elementary excitations which carries momentum and energy, while the remaining excitations pertain to internal degrees of freedom and are referred to as the auxiliary particles.

Finally, we wish to emphasize that the notion of the momentum-carrying elementary particles and the assignment of the auxiliary excitations and bound states thereof depend explicitly on the choice of grading, the high-temperature $Y$-functions $Y^{(0)}_{a,s}$ attached to the interior nodes of the fat hook lattice always stay the same.
Fermi–Hubbard model

The one-dimensional Fermi–Hubbard model comprises spin-full electrons which interact via Coulomb repulsion,

\[ \hat{H}_H = - \sum_{j=1}^{L} \sum_{\sigma \in \uparrow, \downarrow} \hat{c}_j^{\dagger, \sigma} \hat{c}_{j+1, \sigma} + 4u \sum_{j=1}^{L} \hat{V}^H_{j,j+1}, \]  

with Hubbard interaction

\[ \hat{V}^H_{j,j+1} = \sum_{j=1}^{L} (\hat{n}_{j,\uparrow} - \frac{1}{2})(\hat{n}_{j,\downarrow} - \frac{1}{2}). \]

Strictly speaking, the model is not a member of a parameter-less family of SU(\(N|M\))-symmetric Hamiltonians (A6). Indeed, the Fermi–Hubbard chain has quite a special place among Bethe Ansatz solvable models as the underlying quantum algebra which governs the structure of eigenstates is related to a certain the degenerate limit of an exceptional central extension of su(2|2) graded Lie algebra [73].

There are four states per lattice site, \{\(|\emptyset\rangle, |\uparrow\rangle, |\downarrow\rangle, |\bullet\rangle\}\}, were \(|\emptyset\rangle\) denotes an empty site and \(|\bullet\rangle\) a doubly-occupied site. There are two types of elementary excitations which constitute two bosonic su(2) subalgebras: \{\{|\uparrow\rangle, |\downarrow\rangle\}\} are the spin degrees of freedom, while \{\{|\emptyset\rangle, |\bullet\rangle\}\} constitute the charge (\(\eta\)-spin) su(2) degrees of freedom. The bosonic generators of the spin and charge su(2) subalgebras are

\[ \hat{S}^\alpha_j = \sum_{j=1}^{L} \hat{s}^\alpha_j, \quad \hat{\eta}^\alpha_j = \sum_{j=1}^{L} \hat{\eta}^\alpha_j, \]

for \(\alpha \in \{z, +, -\}\). The Cartan generators of the global U(1) spin and charge in terms of local electron number operators, \(\hat{n}_\uparrow = |\uparrow\rangle \langle \uparrow| + |\bullet\rangle \langle \bullet|\) and \(\hat{n}_\downarrow = |\downarrow\rangle \langle \downarrow| + |\bullet\rangle \langle \bullet|\), reading explicitly

\[ \hat{S}^z_j = \frac{1}{2}(\hat{n}_{j,\uparrow} - \hat{n}_{j,\downarrow}), \quad \hat{\eta}^+_j = \frac{1}{2}(\hat{n}_{j,\uparrow} + \hat{n}_{j,\downarrow} - 1). \]

The local electron number operator \(\hat{n}_e = \hat{n}_\uparrow + \hat{n}_\downarrow\) and the total number of electrons on an L-site lattice is \(\hat{N}_e = 2\hat{n}_\uparrow + L\).

Despite the exceptional status of the Hubbard model, its spectrum may still be embedded in the previously described universal description of the SU(\(N|M\))-symmetric models. A few modifications are necessary though: the elementary \(S\)-matrices becomes a function of the coupling strength parameter \(u\),

\[ S_u(u) = \frac{u - n_i u}{u + n_i u}, \]

and the corresponding \(s\)-kernel is the following \(u\)-dependent function

\[ s(u) = \frac{1}{4u \cosh \left( \frac{\pi u}{2} \right)}. \]

The fused amplitudes for the scattering of \(u\) and \(w\)-roots are

\[ S_{n|u,w,m|u,w}(u) = S_{n|u,w,m|u,w}^{-1}(u) = S_{n|m|u,w}^{-1}(u), \]

whereas for the two-branched \(y\)-particle we have \(K_{\pm,n}(u) = \frac{1}{2\pi i} \partial_u \log S_n(u)\), where \(n\) is the integer index of either a \(s/u\)-string or a \(a|u,w\)-stack.

Bethe eigenstates in a finite-volume are characterized in terms of the solutions to the Lieb–Wu equations

\[ e^{i k(u_j)} \prod_{j=1}^{N_u} S_1(u_k, w_j) = 1, \]

\[ \prod_{j=1}^{N_u} S_1^{-1}(w_k, u_j) \prod_{j=1}^{N_u} S_{1,1}(w_k, w_j) = -1, \]
with $2N_w \leq N_u \leq L$, with the bare electron dispersion reading $u_j = \sin (k_j)$. This means that each Bethe root $u_j$ yields two distinct values of momenta $k_j$. To this end it is therefore useful to introduce a double-branched $y$-roots by virtue of the Zhukovsky transformation $u_j = \frac{1}{2} (y_j + y_j^{-1})$; the two branches are given by [73]

$$y_{\pm}(u) = x(u)^{\pm 1} = x(u \pm \pm 0), \quad x(u) = u + u \sqrt{1 - 1/u^2}. \quad (A112)$$

Note that $x(u)$ has a square-root branch cut along the interval $[-1,1]$.

Although the global symmetry of the Fermi–Hubbard is $SO(4)$, the thermodynamic particle content complies with fat hook lattice of $g = su(2|2)$. The assignment of particles to its nodes goes as follows:

1. The momentum-carrying unbound electrons, pertaining to the two-branched $y$-particles, $y_{\pm}(u)$ ($u \in [-1,1]$), are attached to the master node at $(1,1)$ and the corner node at $(2,2)$,

2. the auxiliary bound states of spin excitations forming regular $s$-strings are attached to nodes $(1, s + 1)$,

3. and the momentum-carrying $a|uw$-stacks representing spin-singlet bound states composed of $2a$ electrons and $a$ spin-down excitations are arranged along the nodes $(a + 1,1)$.

The canonical TBA equations are of the form

$$\log Y_y = \mu_y + K_M * \log (1 + Y_{M|uw}) - K_M * \log (1 + Y_{M|w}), \quad (A113)$$

$$\log Y_{M|uw} = \mu_{M|uw} + K_{MN} * \log (1 + 1/Y_{N|uw}) - K_M * \log (1 + 1/Y_+) + K_M * \log (1 + 1/Y_-), \quad (A114)$$

$$\log Y_{M|w} = \mu_{M|w} + K_{MN} * \log (1 + 1/Y_{N|w}) - K_M * \log (1 + 1/Y_+) + K_M * \log (1 + 1/Y_-), \quad (A115)$$

The canonical source terms depend on the bare energies and $U(1)$ chemical potentials and read

$$\mu_y(u) = \beta e_y(u) - \mu - h, \quad \mu_{a|uw}(u) = \beta e_{a|uw}(u) - 2a \mu, \quad \mu_{s|w}(u) = 2s h. \quad (A116)$$

The bare energies of momentum-carrying excitations are

$$e_{\pm}(u) = -2 \cos p_{\pm}(u) + 2u = \pm 2 \sqrt{1 - u^2} + 2u, \quad (A117)$$

$$e_{a|uw}(u) = e_+(u + Mu) + e_-(u - au) = 2 \sqrt{1 - (u + au)^2} + 2 \sqrt{1 - (u - au)^2}, \quad (A118)$$

and $e_{s|w} = 0$. One can get rid off the infinite sums in the last two equations in (A115) by convolving with respect to the Baxter–Cartan matrix $C$. Using the property $C_{aa'} \mu_{a'|uw} = \delta_{a,1} \beta \hat{s} \hat{s} (e_+ - e_-)$, one finds the quasi-local TBA equations

$$\log Y_{\pm} = \hat{s} \hat{s} \log (1 + Y_{1|uw}) - \hat{s} \hat{s} \log (1 + Y_{1|w}), \quad (A119)$$

$$\log Y_{s|w} = I_{s, a'} \hat{s} \hat{s} \log (1 + Y_{a'|uw}) - \delta_{a,1} \hat{s} \hat{s} \log \left( \frac{1 + 1/Y_-}{1 + 1/Y_+} \right), \quad (A120)$$

$$\log Y_{a|uw} = I_{a, a'} \hat{s} \hat{s} \log (1 + Y_{a'|uw}), \quad (A121)$$

subjected to the asymptotic conditions

$$\lim_{a \to \infty} \log Y_{a|uw}(\mu) = -2 \mu a, \quad \lim_{s \to \infty} \log Y_{s|w}(h) = 2h s. \quad (A122)$$

By furthermore performing the particle-hole transformations for all the particles lying along the vertical wing of the fat hook, namely $Y_{-} \rightarrow Y_{-}^{-1}$ and $Y_{a|uw} \rightarrow Y_{a|uw}^{-1}$, and making the following identifications, $Y_{1,s+1} \equiv Y_{s|w}$ for $s \geq 1$, $Y_{a+1} \equiv Y_{a|uw}$ for $a \geq 1$ and $Y_{-} = Y_{1,1}$, $Y_{+} = Y_{2,2}$, we recover the standard universal form of the $Y$-system functional relations. This time however, unlike in the $su(N|M)$ chains, the corner node is just a different branch of the same electronic excitations and there is no additional $U(1)$ chemical potentials besides the charge and spin chemical potential $\mu$ and $h$, respectively.

The high-temperature $Y$-functions read explicitly

$$Y_{-}^{(0)}(\mu, h) = Y_{+}^{(0)}(\mu, h) = \frac{e^\mu + e^{-\mu}}{e^h + e^{-h}}, \quad Y_{s|w}^{(0)}(h) = \chi_s^2(h) - 1, \quad Y_{a|uw}^{(0)}(\mu) = \chi_a^2(\mu) - 1, \quad (A123)$$

where, similarly as in the Heisenberg XXX model, the characters associated to the spin and charge wings are

$$\chi_s(h) = \frac{e^{-s+1}h - e^{s+1}h}{e^{-h} - e^h}, \quad \chi_a(\mu) = \frac{e^{-(a+1)\mu} - e^{(a+1)\mu}}{e^{-\mu} - e^\mu}. \quad (A124)$$
Notice that functions $Y_{s|w}$ do not depend on $\mu$ and, likewise, $Y_{a|uw}$ do not depend on $h$. This means that particles carrying charge are spin-less and conversely the ones carrying spin are charge-less. It is only the unbound electrons which is charged under both degrees of freedom. At half filling, these simplify to

$$\lim_{\mu \to 0} Y^{(0)}_{a|uw}(\mu, h) = a + 1, \quad \lim_{h \to 0} Y^{(0)}_{s|w}(\mu, h) = s + 1, \quad \lim_{\mu = h \to 0} Y^{(0)}_{y}(\mu, h) = 1.$$  \hspace{1cm} (A125)

The dressing transformation is written as a coupled system linear integral equations

$$F_{s|w} - s \ast I_{s,a} \tilde{\phi}^{(0)}_{a|uw} F_{s|w} + \delta_{s,1}(\tilde{\phi}^{(0)}_{a} F_{s} - \tilde{\phi}^{(0)}_{a} F_{s}) = 0,$$

$$F_{a|uw} - s \ast I_{a,a} \tilde{\phi}^{(0)}_{a|uw} F_{a|uw} - \delta_{a,1}(\tilde{\phi}^{(0)}_{a} F_{a} - \tilde{\phi}^{(0)}_{a} F_{a}) = 0,$$

$$F_{\pm} + s \ast (\tilde{\phi}^{(0)}_{1|uw} F_{1|uw} - \tilde{\phi}^{(0)}_{1|uw} F_{1|uw}) = f_{\pm} - s \ast f'_{1|uw},$$  \hspace{1cm} (A126-128)

where $\tilde{\phi}^{(0)}_{\pm} = \tilde{\phi}^{(0)}_{\pm} = \frac{1}{2}$. For the dressing of momentum (energy) we choose $f = k$ ($f = e$), with

$$k'_{\pm} = \mp (1 - u^2)^{-1/2}, \quad u \in [-1, 1],$$

$$k'_{1|uw}(u) = k'_{1}(u + iu) + k'_{-}(u - iu), \quad u \in \mathbb{R}.$$  \hspace{1cm} (A129-130)

In the high-temperature limit, the dressed values of particles’ spins $m_{s|w}^{dr}(h, \mu)$ and charges $n_{a|uw}^{dr}(h, \mu)$ are computed from the logarithmic derivatives of the Y-functions,

$$m_{s|w}^{dr}(h) = \partial_{2h} \log Y_{s|w}^{(0)}(h) = \frac{\partial h \chi_{s}(h)}{\chi_{s}(h) - 1/\chi_{s}(h)}, \quad m_{s|w}^{dr}(h) = \partial_{2h} \log Y_{s|w}^{(0)}(h, \mu),$$

$$n_{a|uw}^{dr}(\mu) = \partial_{2\mu} \log Y_{a|uw}^{(0)}(\mu) = \frac{\partial \mu \chi_{a}(\mu)}{\chi_{a}(\mu) - 1/\chi_{a}(\mu)}, \quad n_{a|uw}^{dr}(\mu) = \partial_{2\mu} \log Y_{a|uw}^{(0)}(h, \mu).$$  \hspace{1cm} (A131-132)

Notice also $m_{a|uw}^{dr}(h) = 0$ and $n_{a|uw}^{dr}(h) = 0$. An alternative route to compute the non-vanishing dressed spin and charge is to solve the following homogeneous dressing transformation,

$$C_{s,s} \ast m_{s|w}^{dr}(0) = 0, \quad \lim_{s \to \infty} n_{s|w}^{dr}(0) = 2s,$$

$$m_{y}^{dr}(0) - \frac{1}{2}(\tilde{\phi}^{(0)}_{1|uw} m_{1|uw}^{dr}(0) - \tilde{\phi}^{(0)}_{1|uw} m_{1|uw}^{dr}(0)) = 0,$$  \hspace{1cm} (A133-134)

and similarly

$$C_{a,a} \ast n_{a|uw}^{dr}(0) = 0, \quad \lim_{a \to \infty} n_{a|uw}^{dr}(0) = 2a,$$

$$n_{y}^{dr}(0) - \frac{1}{2}(\tilde{\phi}^{(0)}_{1|uw} n_{1|uw}^{dr}(0) - \tilde{\phi}^{(0)}_{1|uw} n_{1|uw}^{dr}(0)) = 0,$$  \hspace{1cm} (A135-136)

whence we conclude

$$n_{\pm}^{dr}(0) = \frac{1}{2} \tilde{\phi}^{(0)}_{1|uw} n_{1|uw}^{dr}(0), \quad m_{dr}^{dr}(0) = -\frac{1}{2} \tilde{\phi}^{(0)}_{1|uw} m_{1|uw}^{dr}(0).$$  \hspace{1cm} (A137)

In particular, in the vicinity of the half-filled charge and spin sectors $h = 0$ and $\mu = 0$ we find

$$m_{s|w}^{dr}(0) \sim \frac{1}{3} (s + 1)^2 h + O(h^3), \quad m_{y}^{dr}(0) \sim -\frac{1}{2} h + O(h^3),$$

$$n_{s|w}^{dr}(0) \sim \frac{1}{3} (a + 1)^2 \mu + O(\mu^3), \quad n_{y}^{dr}(0) \sim \frac{1}{2} \mu + O(\mu^3).$$  \hspace{1cm} (A138-139)

The derivatives of the dressed energies and momenta are computed from Eqs. (A128). Indeed, the structure of the recurrence relations in the spin and charge wings of the fat hook take same form as in the previously studied isotropic Heisenberg model, from where we readily obtain the expressions for the $s|w$-strings and the $a|uw$-stacks

$$F_{a|uw}^{(0)} = \frac{a + 1}{4} \left( \frac{f_{a|uw}}{a} - \frac{f_{a+2|uw}}{a + 2} \right), \quad F_{s|w}^{(0)} = -\frac{1}{4} \left( \frac{f_{s|w}}{s} - \frac{f_{s+2|w}}{s + 2} \right).$$  \hspace{1cm} (A140)
Functions \( F^{(0)}_A \) are interpreted as the derivatives of the dressed dressed momenta \( p^{(0)}_A \) or the derivatives of the dressed energy \( \varepsilon^{(0)}_A \), depending whether the sources are chosen as \( f'_A \leftarrow k'_A \) or \( f'_A \leftarrow e'_A \), respectively. Taking into account that \( \tilde{y}^{(0)}_{1|uw} f^{(0)}_A - \tilde{y}^{(0)}_{1|uw} f^{(0)}_A = \frac{3}{2} f^{(0)}_{1|uw} \), the remaining equation for the \( y \)-particles simplifies to

\[
F^{(0)}_+ - s * \frac{3}{2} F^{(0)}_{1|uw} = f'_+ - s * f'_{1|uw}.
\] (A141)

Taking the sum and the difference and, using \( f'_+ = -f'_- \), we find

\[
F^{(0)}_+ - F^{(0)}_- = f'_+ - f'_-,
\] (A142)

\[
F^{(0)}_+ + F^{(0)}_- = f'_{2|uw}.
\]

![Diagram](image)

FIG. 3. Thermodynamic particle content for (a) the Fermi–Hubbard model and (b) the \( SU(2|2) \)-symmetric chain of fundamental particles with respect to the non-distinguished vacuum \( \otimes \odot \otimes \). For (a), \((1,1)\) and \((2,2)\) are associated with the two-branched \( y \)-particle, while in (b) these separate into two distinct excitations with associated with \( z_\pm \)-roots. Particles in the horizontal wing on nodes \((1,s+1)\), with \( s \in \mathbb{N} \) are \( s \)-strings of \( w \)-roots for both (a) and (b). Particle in the vertical wing on nodes \((a+1,1)\), with \( a \in \mathbb{N} \) are \((a)\) \( uw \)-stacks made of both the \( y \)-roots and \( w \)-roots or in (b) \( z_+ w z_- \)-stacks made of \( a + 1 \) \( z_\pm \)-roots, \( a \) \( w \)-roots and \( a - 1 \) \( z_- \)-roots.

**SU(2|2) integrable fermionic chain**

The \( su(2|2) \)-invariant model, also known as the EKS model introduced in [70], is the simplest interacting integrable model of spin-full fermions on a lattice. While the model exhibits certain structural similarities to the Fermi–Hubbard model, there are some important differences. Both models in fact arise as certain degenerate limits of a more general integrable model of spin-full lattice fermions with correlated hopping called the Hubbard–Shastry model [73]. It will be thus convenient to characterize the spectrum with respect to the non-distinguished vacuum corresponding to the following grading of local Hilbert space configurations, \( |1| = |4| = 0 \) and \( |2| = |3| = 0 \), corresponding to following the Kac–Dynkin diagram

\[
\otimes \odot \otimes : \quad \mathcal{K} = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 0
\end{pmatrix}.
\] (A143)

The Bethe roots assigned to the nodes of the Kac–Dynkin diagram are labelled by \( z_{+,-} \), \( w_j \) and \( z_{-,j} \) when moving from the left to the right.

A short remark on the notation is in order here. The EKS model can be obtained from the Fermi–Hubbard model in the scaling limit of large Coulomb repulsion \( u \rightarrow 0 \), causing the rapidity domain of the two-branched \( y \)-particle which lies along the branch cut \([-1,1]\) opening up to the whole real line and splits into two independent particles species whose roots we denote by \( z_+ \) and \( z_- \). Notice that such a limiting procedure requires simultaneous rescaling of the rapidity variable to \( u/u \) which recovers the standard parametrization of the elementary scattering kernels \( K_{AB}(u) \) from Eq. (A18). Likewise, the model can be understood as a weak-coupling limit of a more general Hubbard–Shastry models [73].

The bare energies and momenta of fermionic excitations become [73]

\[
e_+ (u) = -2 \cos p_+ (u) - 2 = -4 \pi K_1, \quad e_- (u) = -2 \cos p_- (u) - 2 = 0,
\] (A144)

\[
p_+ (z_+) = \frac{z_+ + 1}{z_+ - 1}, \quad p_- = \pi.
\] (A145)
This signifies that $z_+$ are the only momentum-carrying roots while $z_-$ cease to be dynamical.

As a consequence of the decoupling of the $y$-roots, the Bethe Ansatz equations now involve two nested levels (assuming $L$ to be even)

$$e^{ip_{+}L} = \prod_{j=1}^{N_{w}} S_{1}^{-1}(z_{+}, w_j), \quad (A146)$$

$$e^{ip_{-}L} = \prod_{j=1}^{N_{w}} S_{2}^{-1}(w_j), \quad (A147)$$

$$e^{ip_{-}L} = (-1)^L = \prod_{j=1}^{N_{w}} S_{1}^{-1}(z_{-}, w_j), \quad (A148)$$

which is compatible with the fact that the rank of $su(2)$ equals three, with the standard rational scattering amplitudes (A5). Eigenstates are uniquely parametrized in terms of $N_{w}$ roots of type $z_+$ and $N_{w}$ w-roots. The $z_+$-roots are rapidities parametrizing bare momenta of electrons which occupy empty lattice sites. A key difference with respect to the Fermi–Hubbard model is that instead of a single conservation of electrons $N_{y}$ we have two independent conserved $U(1)$ charges $\tilde{N}_+$ and $\tilde{N}_-$. The third conservation law is indeed the Hubbard interaction $\tilde{V}^{H}$ which corresponds to conservation of doubly-occupied sites. Note that to generate configurations with doubly-occupied sites, all three types of roots need to be combined: first one creates singly occupied sites by adding $z_+$ $z_-$-roots, then adding $w$-roots for a subset of them to lower their spin, and finally adding $z_+$-roots to add a spin-up electron on a subset of sites with spin-down electrons. More specifically, the thermodynamic particle content of the $su(2)$ chain with respect to the non-disturbed vacuum state $\bigotimes_{\alpha \in \pm} \bigotimes_{\beta \in \pm} \bigotimes_{\gamma \in \pm}$ consists of

- the $a|z_+ w_-$-stacks (with $a = 1, 2, \ldots$) representing bound state compounds made of $a + 1$ $z_+$ roots, $a$ $w$-roots and $a - 1$ $z_-$ roots,
- the auxiliary non-dynamical $s|w$-strings ($s = 1, 2, \ldots$) representing bound state of $s$ second-level $w$-roots, with densities $\rho_s|uw$,
- the $z_\pm$ roots, corresponding to two independent unbound fermionic excitations, with $z_+$ being corresponding to physical momentum-carrying electronic excitations and $z_-$ being the third-level auxiliary real Bethe roots.

The rapidity derivatives of the bare momenta and energies for the momentum-carrying particles are

$$p_{a}^\prime(u) = -2\pi K_{1}(u), \quad (A149)$$

$$p_{a|z_+ w_-(u)}^\prime = -2\pi K_{a+1}(u), \quad (A150)$$

For the non-dynamical particles we obviously have $p_{a|w}^\prime = 0$ and $p_{a|w}^\prime = 0$.

The particles are charged under three Cartan charges, involving two independent bosonic $U(1)$ global charges $\hat{S}^z$, $\hat{h}^z$ pertaining to the spin and charge $su(2)$ sectors, (coupling to chemical potentials $2h$ and $2\mu$, respectively) and an additional fermionic $U(1)$ charge $\tilde{V}^{H}$ which couples to chemical potential $\mu$ (contributing a constant shift to particles’ bare dispersions). The values of bare charges are

$$m_{a|z_+ w_-} = 0, \quad m_{a|w} = -1, \quad m_{\pm} = \frac{1}{2}, \quad (A151)$$

$$n_{a|z_+ w_-} = 1, \quad n_{a|w} = 0, \quad n_{\pm} = \frac{1}{2}, \quad (A152)$$

$$n_{a|z_+ w_-} = 1, \quad n_{a|w} = 0, \quad n_{\pm} = \frac{1}{2}. \quad (A153)$$

The high-temperature limit of the free energy density, $\lim_{\beta \to 0} f_{2\beta}(\hat{h}; \mu, \mu, u)$, thus reads

$$f_{2\beta}(\hat{h}; \mu, u) = \sum_{\alpha \in \pm} (-\hat{h} - \mu \mp 2u) \ast \rho_{\pm} + (-2\mu - 4u) \ast \rho_{a|z_+ w_-} + 2h \ast s \ast \rho_{a|w}. \quad (A154)$$

The quasi-local TBA equations read

$$\log Y_{\pm} = \nu_{\pm} \ast s \ast \log \frac{1 + Y_{1|w}}{1 + Y_{1|w}}, \quad (A155)$$

$$\log Y_{a|z_+ w_-} = \nu_{a|w} \ast I_{a, a'} \ast s \ast \log (1 + Y_{a'|w}) + \delta_{a, 1} \ast s \ast \log \frac{1 + Y_{+}}{1 + \tilde{Y}_{-}}, \quad (A156)$$

$$\log Y_{s|w} = \nu_{s|w} \ast I_{s, s'} \ast s \ast \log (1 + Y_{s'|w}) + \delta_{s, 1} \ast s \ast \log \frac{1 + 1/\tilde{Y}_{+}}{1 + 1/\tilde{Y}_{-}}, \quad (A157)$$
with source terms

\[
\nu_+(u) = \beta \left( e_+(u) - s \ast e_1|_{w=} (u) \right), \quad \nu_-(u) = \beta \left( e_-(u) - s \ast e_1|_{w=} (u) \right) + 4u, \quad \nu_{a|_{w=}} = \nu_{s|w} = 0, \quad (A158)
\]

and asymptotic conditions

\[
\lim_{a \to \infty} \log Y_{a|_{w=}} = -2\mu a, \quad \lim_{s \to \infty} \log Y_{s|w} = 2h s. \quad (A159)
\]

Let us stress that \( u \) associated to the Hubbard charge \( \bar{Y}^H \) does not enter via the asymptotics, but instead explicitly appears in the equation for the distinguished corner role at \((2,2)\). Moreover, notice that Eqs. \((A158)\) become the standard \(Y\)-system relations upon particle-hole transformations along the vertical wing of the fat hook,

\[
Y_{1,1} = Y_{1}^{-1}, \quad Y_{2,2} = Y_-, \quad Y_{a+1,1} = Y_{a+1|w=}, \quad Y_{1,s+1} = Y_{s|w}. \quad (A160)
\]

In the high-temperature limit the quasi-local TBA equations take the form of coupled algebraic equations

\[
\left[ Y_{+}^{(0)} \right]^2 = \left[ e^{-4u} Y_{-}^{(0)} \right]^2 = \frac{1 + Y_{1|w=}^{(0)}}{1 + Y_{1|w}^{(0)}}, \quad (A161)
\]

\[
\left[ Y_{a|w=}^{(0)} \right]^2 = \left( 1 + Y_{a-1|w=}^{(0)} \right) \left( 1 + Y_{a+1|w=}^{(0)} \right) \left( \frac{1 + Y_{1|w}^{(0)}}{1 + Y_{1|w=}^{(0)}} \right)^{\delta_{a,1}}, \quad (A162)
\]

\[
\left[ Y_{s|w}^{(0)} \right]^2 = (1 + Y_{s-1|w=}^{(0)})(1 + Y_{s+1|w=}^{(0)}) \left( \frac{1 + Y_{1|w}^{(0)}}{1 + Y_{1|w=}^{(0)}} \right)^{\delta_{s,1}}. \quad (A163)
\]

The solution to these equations will once again be given in terms of \(\chi\)-functions. For the horizontal (spin) and vertical (charge) wings we find

\[
\chi_{1,s \geq 2}(h, \mu, u) = (e^\mu + e^{-\mu}) \sinh (s h) + e^{-2u} \sinh ((s-1) h) \sinh h + e^{2u} \sinh ((s+1) h) \sinh h, \quad (A164)
\]

\[
\chi_{a \geq 2,1}(h, \mu, u) = T_{1,a}(\mu, h, -u). \quad (A165)
\]

Here we adopted a convenient symmetric gauge-fixing condition \(\chi_{0,s} = \chi_{a,0} = 1\) for \(a \geq 0\) and \(s \in \mathbb{Z}\) for the exterior boundary, and

\[
\chi_{2,s \geq 3}(h, \mu, u) = \chi_{a \geq 3,2}(h, \mu, u) = 4 \left( 2 \cosh (h) \cosh (\mu) \cosh (2u) + \cosh^2 (h) + \cosh^2 (\mu) + \sinh^2 (2u) \right), \quad (A166)
\]

for the \(\chi\)-functions lying along the interior boundary of the fat hook. While the above gauge choice make the particle-hole symmetry between the spin and charge wings manifest, the price to pay is a mismatch at the corner node which forces us to define two independent corner \(\chi\)-functions

\[
\chi_{2,2}^{-\uparrow}(h, \mu, u) = \chi_{2,2}^{-\downarrow}(\mu, h, -u). \quad (A167)
\]

All the remaining \(\chi\)-functions are uniquely fixed by requiring the classical Hirota bilinear relation to hold in the respective wings. The is in principle no obstacle in demanding a unique corner \(\chi\)-functions, but this seems less natural as it induces asymmetry between the wings. We recall that it is the \(Y\)-functions which encode physical (i.e. gauge-invariant) information.

The character at the fundamental node is the logarithm of the free energy density \(f_{2|2}^{(0)} = -\log \chi_{1,1}(h, \mu, u)\), with

\[
\chi_{1,1}(h, \mu, u) = e^u (e^h + e^{-h}) + e^{-u} (e^\mu + e^{-\mu}). \quad (A168)
\]

The dressing transformation in high-temperature limit has the same structure as previously in the Hubbard model. In the present conventions it reads

\[
F_{\pm} - s \ast \left( \bar{\varphi}_{1|w=} F_{1|w=} - \bar{\varphi}_{1|w=} F_{1|w} \right) = f_{\pm} - s \ast f_{1|w=}, \quad (A169)
\]

\[
F_{s|w} - s \ast I_{s',s'} \bar{\varphi}_{s'|w} F_{s'|w} - \delta_{s,1} \left( \bar{\varphi}_{0} F_{-} - \bar{\varphi}_{0} F_{+} \right) = 0, \quad (A170)
\]

\[
F_{s|w=} - s \ast I_{s',s'} \bar{\varphi}_{s'|w=} F_{s'|w=} - \delta_{s,1} \left( \bar{\varphi}_{0} F_{-} - \bar{\varphi}_{0} F_{+} \right) = 0. \quad (A171)
\]
Recall that there is implicit dependence on all three $U(1)$ chemical potentials via the mode occupation functions.

As usual, we now inspect the properties of the dressed $U(1)$ charges in the high-temperature limit. The dressed spin and charge in the vicinity of the half-filled spin and charge sector respectively read,

$$m_{siw}^{dr(0)}(h, \mu, u) \sim \zeta_s(m)(\mu, u) \, h + \mathcal{O}(h^3), \quad m_{a|u}^{dr(0)}(h, \mu, u) \sim \zeta_a(m)(\mu, u) \, h + \mathcal{O}(h^3).$$ (A172)

and

$$n_{siw}^{dr(0)}(h, \mu, u) \sim \zeta_s(n)(h, \mu) \, \mu + \mathcal{O}(\mu^3), \quad n_{a|u}^{dr(0)}(h, \mu, u) \sim \zeta_a(n)(h, \mu) \, \mu + \mathcal{O}(\mu^3).$$ (A173)

A crucial difference in compare to the Hubbard model is that now the dressed spin (resp. charge) for any finite values of $u$ in the half-filled spin (resp. charge) sector explicitly depends on the other two chemical potentials, namely $\mu$ (resp. $h$) and $u$. This is a consequence of the third-level non-dynamical Bethe roots $\zeta_j^{(3)} \equiv z_{-j}$ which participate in the formation of doubly occupied lattice sites. In fact, since finite $u$ induces imbalance between $S^z$-spin and $\eta^z$-spin, the dressed spin and charges satisfy the following symmetry relations

$$m_{siw}^{dr(0)}(h, \mu, u) = n_{a|z+wz}^{dr(0)}(h, -\mu, -u), \quad m_{a|u}^{dr(0)}(h, \mu, u) = n_{siw}^{dr(0)}(\mu, h, -u),$$ (A174)

upon interchanging spin with charge $s \leftrightarrow a$ and flipping the sign of $u$. It is thus sufficient to examine the behaviour close the half-filled spin sector. We are not interested in the most general solution but mostly in the large-$s$ behavior. To this end it is useful to introduce the following ratios of the $\chi$-functions in the horizontal and vertical wings, namely

$$g_{\geq 1}^{-} = \frac{\chi_{s+1, s+1}}{\chi_{0, s+1} \chi_{2, s+1}}, \quad g_{\geq 1}^{+} = \frac{\chi_{s+1, s+1}}{\chi_{a+1, 0} \chi_{a+1, 2}}.$$ (A175)

The high-temperature limit of the dressed spin read

$$m_{s}^{dr(0)}(h, \mu, u) = \frac{\partial_h g_a(\mu, u)}{g_s(\mu, u) - 1/g_a(\mu, u)}.$$ (A176)

**$SU(2|1)$ spin chain (SUSY t–J model)**

The Hamiltonian of the t–J model expressed in terms of spin-full electrons takes the following form

$$\hat{H}_{t-J} = \hat{P} \left[ -t \sum_{j, \sigma} \hat{c}_{j, \sigma}^\dagger \hat{c}_{j+1, \sigma} + \hat{c}_{j+1, \sigma}^\dagger \hat{c}_{j, \sigma} \right] \hat{P} + J \sum_j \left( \hat{S}_j \cdot \hat{S}_{j+1} + \frac{1}{4} \hat{n}_j \hat{n}_{j+1} \right),$$ (A177)

Here $\hat{P} = \prod_{j=1}^{L} (1 - \hat{n}_j \hat{n}_j)$ is used project out configurations with doubly occupied sites. The model becomes integrable at the ‘supersymmetric point’ $J = 2t$, where $\hat{H}_{t-J}$ becomes proportional to the $SU(2|1)$-symmetric Hamiltonian $\tilde{H}^{21}$. The SUSY t–J model can also be retrieved from the large-repulsion $u \rightarrow \infty$ limit of the Hubbard model. The model can also be viewed as an extension of the $su(2)$ Heisenberg chain by allowing vacancies (i.e. empty lattice sites) which are treated as fermions. We find it most convenient to formulate the problem in the distinguished grading $|0| = |1| = 0$ and $|2| = 1$, corresponding to the diagram

$$\bigcirc \bigotimes : \quad \mathcal{K} = \left( \begin{array}{cc} 2 & -1 \\ -1 & 0 \end{array} \right).$$ (A178)

The vacuum state here is a completely polarized (ferromagnetic) state. The primary (physical) excitations are momentum-carrying spin-down magnonic excitations which form bound states, described by the primary Bethe roots $u^{(1)}_j$. The bare momentum of an elementary is

$$k(u^{(1)}_j) = i \log S_1 \left( u^{(1)}_j \right).$$ (A179)

In addition, we have an extra specie of fermionic excitations corresponding to vacancies (i.e. holes) of electrons, described by the second-level (auxiliary) rapidities $u^{(2)}_a$ which do not carry momenta and energy. The Bethe equations
with respect to the ferromagnetic background take the form

\[ e^{ik(u_j^{(1)})L} \prod_{k=1}^{N_1} S_2 \left( u_j^{(1)}, u_k^{(2)} \right) \prod_{l=1}^{N_2} S^{-1}_1 \left( u_j^{(1)}, u_l^{(2)} \right) = -1, \quad (A180) \]

\[ \prod_{j=1}^{N_1} S_1 \left( u_j^{(2)}, u_j^{(1)} \right) = 1. \quad (A181) \]

The number primary Bethe roots \( u_j^{(1)} \in \mathbb{C} \) is \( N_1 \), while the number of real charge rapidities \( u_n^{(2)} \) is \( N_2 \). The number of roots obey the following inequalities

\[ N_2 \leq N_1, \quad N_1 \leq \frac{1}{2}(L + N_2) \leq L. \quad (A182) \]

Here \( N_1 = N_\uparrow + N_\downarrow \) is the number of hole plus spin-down excitations and \( N_2 = N_\downarrow = N_1 - N_\uparrow \) is the total number of electron charge holes. The total spin and electron charge are \( S = \frac{1}{2}(N_\uparrow - N_\downarrow) = \frac{1}{2}(L - 2N_1 + N_\uparrow) \) \( (N_\uparrow = L - N_\downarrow - N_\uparrow) \) and \( N_e = N_\uparrow + N_\downarrow \) respectively. The \( U(1) \) chemical potentials which couple to total spin \( \hat{S}^z \) and number of holes \( \hat{N}_\downarrow = 1 - \hat{N}_e \) are denoted by \( 2h \) and \( \mu \), respectively. The electron filling fraction is \( \alpha_c = N_e/(N_\uparrow + N_\downarrow) = 1 - N_\downarrow/L \).

The total energy of a state is the sum of all spin-down excitations \( E \simeq \sum_{j=1}^{N_1} 2\pi K_1(u_j^{(1)}) \).

To reconcile the notation of the one used above, we relabel the Bethe roots as \( u_j^{(1)} \to u_j \in \mathbb{C} \) and \( u_l^{(2)} \to w_l \in \mathbb{R} \). The former represent charge-less bound spin excitations carrying bare spin

\[ m_s = -s, \quad n_s = 0, \quad (A183) \]

which form the standard \( s \)-strings with real centres

\[ u_{j,k}^{(s)} = u_j^{(s)} + \frac{i}{2}(s + 1 - 2k), \quad k = 1 \sim s, \quad (A184) \]

while the fermionic roots \( w_l \) correspond to electron holes which do not form bound states. The total number of primary Bethe roots is the number of spin-down excitations, \( N_1 = \sum_{s=1}^{\infty} s N_s \). Adding an electron hole amounts to remove a spin-up electron excitation and hence

\[ m_\otimes = -\frac{1}{2}, \quad n_\otimes = -1. \quad (A185) \]

Introducing rapidity distributions \( \rho_\otimes \) and \( \rho_s \), the canonical TBA equations take the form

\[ \log Y_s = \mu_s - K_s * \log(1 + Y_{\otimes}^{-1}) + K_{s,s'} * \log(1 + Y_{s'}^{-1}), \quad (A186) \]

\[ \log Y_{\otimes} = \mu_{\otimes} - K_s * \log(1 + Y_{\otimes}^{-1}), \quad (A187) \]

with chemical potentials

\[ \mu_{s|s} = -\beta e_{s|s} + 2h s, \quad \mu_\otimes = \mu + h, \quad (A188) \]

where \( e_s = 2\pi K_s \) are the bare energies of \( s \)-strings.

FIG. 4. Thermodynamic particle content for the \( SU(2|1) \) fundamental spin chain (SUSY integrable t–J model) with respect to the distinguished bare vacuum \( \otimes \rightarrow \bigotimes \). The momentum-carrying particles are bosonic \( s \)-strings attached to nodes \( (1, s) \), \( s \in \mathbb{N} \). The corner node \((2, 1)\) is assigned an auxiliary fermionic excitation representing electron vacancies.
It is instructive to remark that Eqs. (A187) are just a particular singular reduction of the $\mathfrak{su}(2|2)$ canonical TBA equations written for e.g. the distinguished (ferromagnetic) vacuum $\bigcirc \bigotimes \bigotimes \bigcirc$. Such a reduction is realized by freezing appropriate charge degrees of freedom in order to prohibit double occupancies while still allowing empty sites. Specifically, this amounts to remove the third-level Bethe roots responsible for the doubly-occupied configurations by decoupling the following $Y$-functions, $Y_{a,1} \to 0$, for $a \geq 3$, and $Y_{2,2} \to \infty$.

Below we outline how to transform the canonical TBA equations for the $\mathfrak{su}(2|1)$ chain to the quasi-local form. The starting point are the quasi-local TBA equations of the $\mathfrak{su}(2|2)$ chain. By taking the above limit, we find

\[ \log Y_{1,s} = \nu_{1,s} - \delta_{s,1} s \star \log(1 + 1/Y_{2,1}) + I_{s,s'} s \star \log(1 + Y_{1,s'}), \]
\[ \log Y_{2,1} = \nu_{2,1} - K_1 \star \log(1 + Y_{1,1}) - K_2 \star \log(1 + Y_{2,1}), \]

supplemented with large-$s$ asymptotics

\[ \lim_{s \to \infty} \log Y_{1,s}(h, \mu) = 2h s. \] (A191)

Let us remark that these equations are equivalent to those presented in ref. [71], which can be readily confirmed by first convolving with respect to $s$ and subsequently deconvolving with respect to $K_1$,

\[ \log Y_s = \nu_s + I_{s,s'} s \star \log(1 + Y_{s'}), \]
\[ \log Y_\otimes = \nu_\otimes - s \star \log(1 + Y_1) - K \star \log(1 + 1/Y_\otimes), \]

with $\tilde{K}(\kappa) = e^{-|k|}(1 + e^{-|k|})^{-1}$. The source terms are of the form

\[ \nu_s = -\beta \delta_{s,1} s, \quad \nu_\otimes = -\beta (s \star e_1) + \mu. \] (A194)

The high-temperature dressing transformation is presently of the form

\[ F_s - s \star I_{s,s'} \tilde{y}^{(0)}_{s'} F_{s'} - \delta_{s,1} s \star \tilde{y}^{(0)}_\otimes F_\otimes = -\delta_{s,1} s, \]
\[ F_\otimes + s \star \tilde{y}^{(0)}_1 F_{\otimes} - \tilde{K} \star \tilde{y}^{(0)}_\otimes F_\otimes = -s \star K_1. \] (A196)

We proceed by analysing the dressed spin and charge as functions of grand-canonical chemical potentials. In the high-temperature limit $\beta \to 0$ the $Y$-functions become constant and Eqs. (A193) turns into a set of algebraic relations

\[ \left[ Y_s^{(0)} \right]^2 = \frac{(1 + Y_{s-1}^{(0)})(1 + Y_{s+1}^{(0)})}{(1 + 1/Y_\otimes^{(0)})^\delta_{s,1}}, \] (A197)
\[ \left[ e^{-\mu} Y_\otimes^{(0)} \right]^2 = \frac{1}{(1 + 1/Y_\otimes^{(0)})(1 + Y_1^{(0)})}. \] (A198)

These take the standard form of the $Y$-system functional relations by identifying $Y_s \equiv Y_{1,s}$ and $Y_\otimes \equiv Y_{2,1}$. The self-coupling term in the last equation for the exceptional corner node is the remainder of the collapsed vertical wing of the $\mathfrak{su}(2|2)$ spectrum which leaves behind explicit dependence on the charge chemical potential $\mu$. This is reminiscent of the spin chemical potential entering explicitly in the truncated spectrum of the gapless regime of the XXZ Heisenberg model which was due to the root-of-unity restriction.

The solution to Eqs. (A198) is

\[ Y_s(h, \mu) = \left( \frac{e^{-(s+1)h + \Phi} - e^{(s+1)h + \Phi}}{e^{h} - e^{h}} \right)^2 - 1, \quad Y_\otimes(h, \mu) = \frac{e^{-\mu}}{e^{h} + e^h + e^{\mu}}, \] (A199)

with

\[ \Phi(h, \mu) = \frac{1}{2} \log \left( \frac{1 + e^{-h + \mu}}{1 + e^{h + \mu}} \right). \] (A200)

The high-temperature limit of the grand canonical free energy density $f_{2|1}^{(0)} = f_{2|1}^{(0)}(h, \mu)$ is defined as

\[ f_{2|1}^{(0)}(h, \mu) = -\lim_{L \to \infty} \frac{1}{L} \log \text{Tr} \exp \left( 2h \hat{S}^z + \mu \hat{N}_h \right). \] (A201)
In terms of the $Y$-functions we have
\[ f_{21}^{(0)}(h, \mu) = -\frac{1}{2} \log[(1 + Y_{c21}^{(0)}(h, \mu))(1 + 1/Y_{s21}^{(0)}(h, \mu))] = -\log \chi_{1,1}(h, \mu), \] (A202)
where the fundamental $\chi$-function explicitly reads
\[ \chi_{1,1}^{(0)}(h, \mu) = e^h + e^{-h} + e^{\mu}. \] (A203)

Imposing boundary conditions $\chi_{0,s} = 1$ and $\chi_{a,0} = 1$, together with $\chi_{1,1}^{(0)}$ and $\chi_{2,1}^{(0)}$ determined from $1 + Y_{c21}^{(0)} = [\chi_{1,1}^{(0)}]^2/\chi_{2,1}^{(0)}$, uniquely fixes $\chi_{a,s}^{(0)}$ on the whole $(a, s)$-lattice. Specifically, the infinite tower of symmetric characters $\chi_{1,s}$ for $s \geq 1$ can be calculated recursively
\[ \chi_{1,s+1}^{(0)}(h, \mu) = (e^h + e^{-h})\chi_{1,s}^{(0)}(h, \mu) - \chi_{1,s-1}^{(0)}(h, \mu). \] (A204)

In the high-temperature limit, the dressed spin and charge are calculated as
\[ m_s^{dr(0)}(h, \mu) = \partial_{2h} \log Y_s^{(0)}(h, \mu), \quad n_s^{dr(0)}(h, \mu) = \partial_\mu \log Y_s^{(0)}(h, \mu). \] (A205)
The dressed values of particles’ spin can likewise be obtained from the following rapidity-independent recurrence relation
\[ m_s^{dr(0)} - \frac{1}{2} I_{s,s'} \gamma_s^{(0)} m_{s'}^{dr(0)} = 0, \quad \lim_{s \to \infty} n_s^{dr(0)} = s. \] (A206)

We subsequently specialize our attention to the half-filled spin sector. In the vicinity of the half-filled spin sector $h = 0$, $S^z(h, \mu) = \partial_{2h} f^c_0(h, \mu)$ (with $\lim_{h \to 0} S^z(h, \mu) = 0$ and $\lim_{h \to \pm \infty} S^z(h, \mu) = \pm \frac{1}{2}$, irrespective of $\mu$) we have
\[ m_s^{dr(0)}(h, \mu) = \frac{h}{6} \left( \frac{6}{e^{\mu} + 1} - 2 + 2s^2 + \frac{s(2s - 1)}{e^{\mu}(s + 1) - s} + \frac{(s + 2)(2s + 3)}{e^{\mu}(s + 1) + s + 2} \right) + O(h^3), \] (A207)
\[ n_s^{dr(0)}(h, \mu) = -\frac{h}{e^{\mu} + 2} + O(h^3). \] (A208)

For $\mu = 0$ these expressions further simplify to
\[ m_s^{dr(0)}(h, 0) \sim \frac{h}{12}(2s + 1)^2 + O(h^3), \quad m_{\theta}^{dr(0)}(h, 0) \sim -\frac{h}{3} + O(h^3). \] (A209)
The vanishing of the dressed spin at half filling (irrespectively of the chemical potential $\mu$) is actually implied by the bosonic symmetry, that is applying the spin-reversal transformation on the bosonic states. The electron charge transport behaves quite differently however. Let us examine the dressed charge in the high-temperature limit, given by
\[ n_s^{dr(0)}(h, \mu) = \partial_\mu \log Y_s(h, \mu), \quad n_{\theta}^{dr(0)}(h, \mu) = \partial_\mu \log Y_s(h, \mu) = -\frac{e^{-h} + e^h + 2e^{\mu}}{e^{-h} + e^h + e^{\mu}}. \] (A210)

For instance, hole excitations propagating in the half-filled spin background possess finite dressed charges
\[ \lim_{h \to 0} n_s^{dr(0)}(h, \mu) = -\frac{2e^{\mu}(e^{\mu} + 1)}{(e^{\mu} + 1)(e^{\mu}(s - 1) + (e^{\mu}(s + 1) + s + 2)), \quad \lim_{h \to 0} n_{\theta}^{dr(0)}(h, \mu) = -\frac{2(e^{\mu} + 1)}{e^{\mu} + 2}. \] (A211)

While spin degrees of freedom can be excited independently of hole excitations, the addition of a hole implies removing a spin-up electron from the state. Notice moreover that $N_h(h, \mu) = -\partial_\mu f_2^{(0)}(h, \mu)$ and hence the vanishing chemical potential corresponds to the third-filling $N_h(0, 0) = \frac{1}{4}$ (for $\mu \to -\infty$ we have $N_h = 0$). Imposing the filling fraction $\alpha_c = \lim_{L \to \infty} N_c/L$ (0 ≤ $\alpha_c$ ≤ 1) for arbitrary $h$ requires to adjust $\mu$ in accordance with
\[ e^{\mu} = \frac{e^{-h}\alpha_c + e^h\alpha_c}{1 - \alpha_c}. \] (A212)

In particular, for the half-filled charge sector $\alpha_c = \frac{1}{2}$ this means $e^{\mu} = e^{-h} + e^h$. 
The high-temperature mode occupation functions in the half-filled spin sector \((h = 0)\) and the third-filled charge sector \(\mu = 0\) are
\[
\begin{align*}
\lim_{\mu \to 0} \lim_{h \to 0} \vartheta_s^{(0)}(h, \mu) &= \frac{4}{(2s + 1)^2}, \\
\lim_{\mu \to 0} \lim_{h \to 0} \vartheta_s^{(0)}(h, \mu) &= \frac{3}{4}.
\end{align*}
\]
(A213)
(A214)

The \(s\)-string bound state corresponds to atypical (short) irreducible \(su(2|1)\) representations of dimension \(d_{1,s} = \lim_{G \to 1} \chi_{1,s}(G) = 2s + 1\). The solution to the dressing equation (A196) in the limit \(\mu \to 0\) and \(h \to 0\) reads
\[
F_s = \frac{2s + 1}{3} \left( \frac{K_s}{2s - 1} - \frac{K_{s+2}}{2s + 3} \right), \quad F_{\otimes} = \frac{4}{9} K_2.
\]
(A215)

The conclusion of above analysis we conclude that for all finite values of charge chemical potential \(\mu\) the dressed charges \(n_x^{\text{dr}(0)}\) and \(n_{\otimes}^{\text{dr}(0)}\) are always positive definite quantities. Near the half-filled spin sector \(h \to 0\) and the third-filled charge sector \(\mu \to 0\), the dressed electron charges read explicitly
\[
\begin{align*}
\lim_{h \to 0} n_x^{\text{dr}(0)}(h, \mu) &\sim \frac{2s + 1}{(2s - 1)(2s + 3)} - \frac{4s^2 + 4s + 5}{2(2s - 1)^2(2s + 3)^2} \mu + O(\mu^2), \\
\lim_{h \to 0} n_{\otimes}^{\text{dr}(0)}(h, \mu) &\sim \frac{4}{3} \frac{2}{9} \mu + O(\mu^2).
\end{align*}
\]
(A216)
(A217)

**Appendix B: Lower bound on diffusion constants**

In this section we re-derive a relation between the linear-response diffusion constants and the corresponding Drude weights, originally presented ref. [42]. We consider the linear transport the conserved \(U(1)\) charges
\[
\hat{Q} = \sum_{x = -L/2}^{L/2 - 1} \hat{q}_x,
\]
(B1)

Below we derive an explicit lower bound on the charge diffusion constant \(D\). For simplicity we first specialize to the infinite-temperature Gibbs equilibrium, and assume that the local charge density \(q\) has \(d\) distinct eigenvalues \(q \in \{-(d-1)/2, \ldots, (d-1)/2\}\) of the same multiplicity.

We consider a lattice of length \(L\), with an initial state described by the following density matrix
\[
\hat{\varrho}(\beta, \delta h) = Z^{-1}(\beta, \delta h) \exp \left( -\beta \hat{H} + \beta \delta h \sum_{x = -L/2}^{L/2 - 1} x \hat{q}_x \right),
\]
(B2)

with \(Z(\beta, \delta h) = \text{Tr} \hat{\varrho}(\beta, \delta h)\). Our aim is to compute the linear-response DC conductivity \(\sigma(\beta)\), which we define as the induced current density \(\hat{j}_0^{(q)}(t)\) in the limit of vanishing bias \(\delta h\),
\[
\sigma^{(q)}(\beta) = \lim_{t \to \infty} \lim_{L \to \infty} \lim_{\delta h \to 0} \frac{1}{\delta h} \left\langle \hat{j}_0^{(q)}(t) \right\rangle_{\beta, \delta h}.
\]
(B3)

Here the time propagation is governed by the Hamiltonian \(\hat{H} = \sum_x \hat{h}_x\), and the expectation value of the current is
\[
\left\langle \hat{j}_0^{(q)}(t) \right\rangle_{\beta, \delta h} = \text{Tr} \left( \hat{j}_0^{(q)}(t) \hat{\varrho}(\beta, \delta h) \right).
\]
(B4)

By resorting to the Lieb-Robinson theorem, any local perturbation on a lattice with bounded finite-range interactions \(\hat{h}\) propagates with a finite maximal velocity denoted by \(v_{LR}\). This permits to write Eq. (B3) as a single scaling limit \(t \to \infty\), provided the system size is scaled in accordance with the Lieb-Robinson velocity \(L = 2v_{LR} t\), yielding
\[
\sigma^{(q)}(\beta) = \lim_{t \to \infty} \lim_{\delta h \to 0} \frac{1}{\delta h} \left\langle \hat{j}_0^{(q)}(t) \right\rangle_{\beta, \delta h}.
\]
(B5)
The average value of the current density \( \langle j_0(t) \rangle_{\beta,\delta h,q} \) (see Eq. (B4)) can be written as a sum of averages \( \langle j_0(t) \rangle_{\beta,\delta h,q} \) over sectors with a fixed value of the charge density \( q \),

\[
q = \frac{2}{d - 1} \langle \hat{q} \rangle \in [-1, 1].
\]

(B6)

The DC conductivity \( \sigma^{(q)} \) is accordingly decomposed as a discrete sum over the charge sectors,

\[
\sigma^{(q)}(\beta) = \lim_{t \to \infty} \lim_{\delta h \to 0} \frac{1}{\delta h} \sum_{q = -1}^{1} P(q, 2v_{LR}t) \langle j_0(t) \rangle_{\beta,\delta h,q},
\]

with a step size

\[
\Delta q = \frac{2}{(d - 1)L}.
\]

(B8)

Here \( P(q, L) \) denotes the unbiased probability of finding a state with charge density \( q \) in a system of the length \( L \). For large times \( t \to \infty \) we first expand the current in \( q \)-sector as \[15\]

\[
\lim_{\delta h \to 0} \frac{1}{\delta h} \langle j_0(t) \rangle_{\beta,\delta h,q} = 2D^{(q)}(q)t + D_v^{(q)}(q) + O(t^{-1}),
\]

(B9)

where \( D^{(q)}(q) \) is the finite-temperature Drude weight, and \( D_v^{(q)}(q) \) denotes the leading \( O(t^0) \) sub-ballistic correction which is assumed to be positive, cf. refs. \([42, 68]\). This term will be subsequently disregarded. Furthermore, in the high-temperature limit \( \beta \to 0 \), the probability factor \( P(q, 2v_{LR}t) \) can be approximated by with the Gaussian distribution by neglecting the contributions from the sectors which become suppressed in the large-\( t \) limit \([42]\]

\[
P(q, L) \approx \sqrt{\frac{6}{(d^2 - 1)\pi L}} \exp \left( -\frac{3(d - 1)}{2(d + 1)} q^2 L \right).
\]

(B10)

Replacing the system size \( L \) with the Lieb-Robinson cone yields

\[
\sigma^{(q)} \geq \lim_{t \to \infty} \sum_q \sqrt{\frac{3}{(d^2 - 1)\pi v_{LR}t}} \exp \left( -\frac{3(d - 1)v_{LR}t}{(d + 1)} q^2 \right) D^{(q)}(q).\]

(B11)

Taking the \( t \to \infty \) limit and converting the sum in Eq. (B11) to an integral, we find

\[
\sigma^{(q)} \geq \frac{1}{6v_{LR}} \frac{d + 1}{d - 1} \partial_q^2 D^{(q)}(q).
\]

(B12)

Using Einstein relation \( D^{(q)} = \sigma^{(q)}/\chi \), where \( \chi \) denotes the static spin susceptibility, we obtain the following lower bound

\[
D^{(q)} \geq \frac{2}{\beta(d - 1)^2 v_{LR}} \partial_q^2 D^{(q)}(q).
\]

(B13)

In the infinite temperature limit, the scaled static susceptibility \( \tilde{\chi} = \lim_{\beta \to 0} (\chi/\beta) \) reads

\[
\tilde{\chi} = \frac{1}{12} (d^2 - 1).
\]

(B14)

The derivative with respect to the chemical potential \( h \) can be expressed as

\[
D^{(q)} \geq \left( \frac{\partial q}{\partial h} \right)^{-2} \frac{2}{\beta(d - 1)^2 v_{LR}} \partial_h^2 D^{(q)}(h) = \frac{18}{\beta(d^2 - 1)^2 v_{LR}} \partial_h^2 D^{(q)}(h),
\]

(B15)

where we have taken into account the relation

\[
q = \frac{1}{d - 1} (d \coth (d h) - \coth (h)).
\]

(B16)
The logic of the above derivation generalizes to finite temperatures by taking into account a temperature-dependent Gaussian approximation of the probability distribution $P(q, L)$ in the vicinity of the half filling,

$$\frac{\sum_x \exp(-\beta E_{x,h})}{\sum_{x,h'} \exp(-\beta E_{x,h'})} \approx \exp \left(-f(\beta) h^2 L \right). \quad (B17)$$

The finite temperature bound thus reads

$$D^{(q)}(\beta) \geq \left. \frac{\partial^2 D^{(q)}(\beta, h)}{4\chi(\beta)f(\beta)\nu_{LR}} \right|_{h=0}. \quad (B18)$$

Finally, we established the connection between the static susceptibility $\chi(\beta)$ and function $f(\beta)$. In order to achieve this, we need to related the average (B17) with the grand-canonical average with respect to $\hat{\varrho}(\beta, h) \simeq \exp(-\beta \hat{H} + 2h \hat{Q})$. In the Gaussian approximation we have

$$\frac{\text{Tr}(\hat{\varrho}(\beta, h))}{\text{Tr}(\hat{\varrho}(\beta, 0))} \approx \exp \left( -\frac{\beta}{2} \chi(\beta)(2h)^2 L \right), \quad (B19)$$

implying

$$f(\beta) = 2\beta \chi(\beta). \quad (B20)$$