Inverse Lyndon words and Inverse Lyndon factorizations of words

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Abstract

Motivated by applications to string processing, we introduce variants of the Lyndon factorization called inverse Lyndon factorizations. Their factors, named inverse Lyndon words, are in a class that strictly contains anti-Lyndon words, that is Lyndon words with respect to the inverse lexicographic order. The Lyndon factorization of a nonempty word $w$ is unique but $w$ may have several inverse Lyndon factorizations. We prove that any nonempty word $w$ admits a canonical inverse Lyndon factorization, named ICFL($w$), that maintains the main properties of the Lyndon factorization of $w$: it can be computed in linear time, it is uniquely determined, it preserves a compatibility property for sorting suffixes. In particular, the compatibility property of ICFL($w$) is a consequence of another result: any factor in ICFL($w$) is a concatenation of consecutive factors of the Lyndon factorization of $w$ with respect to the inverse lexicographic order. As for the applications, experimental results on biological datasets shown that ICFL($w$) combined with the Lyndon factorization is intermediate between the Lyndon factorization and the LZ factorization with respect to the size of the factors. Moreover ICFL($w$) allows us to handle too long or too short factors in the Lyndon factorization.

1 Introduction

Lyndon words were introduced in \cite{24}, as \textit{standard lexicographic sequences}, and then used in the context of the free groups in \cite{6}. A Lyndon word is a word which is strictly smaller than each of its proper cyclic shifts for the lexicographical ordering. A famous theorem concerning Lyndon words asserts that any nonempty word factorizes uniquely as a nonincreasing product of Lyndon words, called its Lyndon factorization. This theorem, that can be recovered from results in \cite{6}, provides an example of a factorization of a free monoid, as defined in \cite{31} (see also \cite{4, 22}). Moreover, there are several results which give relations between Lyndon words, codes and combinatorics of words \cite{3}.

The Lyndon factorization has recently revealed to be a useful tool also in string processing algorithms \cite{2, 28} with strong potentialities that have not been completely explored and understood. This is due also to the fact that it can be efficiently computed. Linear-time algorithms for computing this factorization can be found in \cite{11, 12} whereas an $\mathcal{O}(\lg n)$-time parallel algorithm has been proposed in \cite{1, 10}. A connection between the Lyndon factorization and the Lempel-Ziv (LZ) factorization has been given in \cite{18}, where it is shown that in general the size of the LZ factorization is larger than the size of the Lyndon factorization, and in any case the size of the Lyndon factorization cannot be larger than a factor of 2 with respect to the size of LZ.

Relations between Lyndon words and the Burrows-Wheeler Transform (BWT) have been discovered first in \cite{9, 25} and, more recently, in \cite{20}. Variants of BWT proposed in the previous
papers are based on combinatorial results proved in [14] (see [29] for further details and [13] for more recent related results).

Lyndon words are lexicographically smaller than all its proper nonempty suffixes. This explain why the Lyndon factorization has become of particular interest also in suffix sorting problems. The suffix array (SA) of a word \( w \) is the lexicographically ordered list of the starting positions of the suffixes of \( w \). The connection between Lyndon factorizations and suffix arrays (SA) has been pointed out in [17], where the authors show a method to construct the Lyndon factorization of a text from its SA. Conversely, the computation of the SA of a text from its Lyndon factorization has been proposed in [5] and then explored in [26, 27].

The algorithm proposed in [26, 27] is based on the following interesting combinatorial result, proved in the same papers: if \( u \) is a concatenation of consecutive Lyndon factors of \( w = xuy \), then the position of a suffix \( u_i \) in the ordered list of suffixes of \( u \) (called local suffixes) is the same position of the suffix \( u_iy \) in the ordered list of the suffixes of \( w \) (called global suffixes). In turn, this result suggests a divide and conquer strategy for the sorting of the suffixes of a word \( w = w_1w_2 \): we order the suffixes of \( w_1 \) and the suffixes of \( w_2 \) independently (or in parallel) and then we merge the resulting lists (see Section 2.3 for further details).

However, in order to have a practical interest, the divide and conquer approach proposed in [27] would require two main ingredients: an efficient algorithm to perform the merging of two sorted lists, which is still to be improved, and a strategy to manage the size of the factors in a Lyndon factorization. Indeed, we may have extreme cases of very short or very long factors (a word \( a^k \) is factorized into factors of size 1, while we may have Lyndon words of huge size). On the other hand LZ may produce factorizations of very large sizes which are not useful for a good compression (as in the case of genomic sequences).

In this paper we face the following main question, raised by the above discussion: can we define a factorization of \( w \), which maintains some useful properties of the Lyndon factorization but that allows us to manage the size of the factors? We first introduce the notion of an inverse Lyndon word, that is a word greater than any of its proper nonempty suffixes (Section 4). The set of the inverse Lyndon words strictly contains that of Lyndon words with respect to the inverse lexicographic order (or anti-Lyndon words [15]). Then we give the definition of an inverse Lyndon factorizations of a word, whose factors are inverse Lyndon words (Section 4). The Lyndon factorization of a nonempty word \( w \) is unique but \( w \) may have several inverse Lyndon factorizations. As a main result, we define a canonical inverse Lyndon factorization of a nonempty word \( w \), denoted by ICFL(\( w \)). We prove that ICFL(\( w \)) can be still computed in linear time and it is uniquely determined. Moreover, if \( w \) is a Lyndon word different from a letter, then ICFL(\( w \)) has at least two factors and a converse holds for an inverse Lyndon word and its Lyndon factorization. Finally, we prove that ICFL(\( w \)) belongs to a special class of inverse Lyndon factorizations, called groupings (see Section 7), and then each of its factors is a concatenation of consecutive factors of the Lyndon factorization of \( w \) with respect to the inverse lexicographic order. Hence the compatibility property proved in [26] applies also to ICFL(\( w \)), with respect to the inverse lexicographic order.

In order to answer the above question, we propose to combine the Lyndon factorization of a word \( w \) with ICFL(\( w \)). We test our proposal by running an experimental analysis over two biological datasets. Experiments confirm that we obtain a factorization of intermediate size between that of LZ and that of the Lyndon factorization.

The paper is organized as follows. In Section 2 we gathered the basic definitions and known results we need. Inverse Lyndon words are discussed in Section 3. Inverse Lyndon factorizations and ICFL(\( w \)) are presented in Section 4. More precisely, for the construction of ICFL(\( w \)) we need a special prefix of \( w \), defined in Section 4.2 whereas we give the recursive definition of
the factorization in Section 4.3. A linear-time algorithm for computing ICFL(w) is presented in Section 6. This algorithm uses two subroutines described in Section 5. We introduce groupings in Section 7 and we prove that ICFL(w) falls in this class of factorizations in Section 7.3

2 Preliminaries

For the material in this section see [4, 7, 22, 23, 30].

2.1 Words

Let $\Sigma^*$ be the free monoid generated by a finite alphabet $\Sigma$ and let $\Sigma^+ = \Sigma^* \setminus \{1\}$, where 1 is the empty word. For a set $X$, $\text{Card}(X)$ denotes the cardinality of $X$. For a word $w \in \Sigma^*$, we denote by $|w|$ its length. A word $x \in \Sigma^*$ is a factor of $w \in \Sigma^*$ if there are $u_1, u_2 \in \Sigma^*$ such that $w = u_1xu_2$. If $u_1 = 1$ (resp. $u_2 = 1$), then $x$ is a prefix (resp. suffix) of $w$. A factor (resp. prefix, suffix) $x$ of $w$ is proper if $x \neq w$. Two words $x, y$ are incomparable for the prefix order, and we write $x \not\preceq y$, if neither $x$ is a prefix of $y$ nor $y$ is a prefix of $x$. Otherwise, $x, y$ are comparable for the prefix order. We write $x \preceq y$ if $x$ is a prefix of $y$ and $x \succeq y$ if $y$ is a prefix of $x$. The following result, named overlapping-suffix lemma in [8], is a direct consequence of the definitions.

Lemma 2.1 Let $x, y, w \in \Sigma^+$ such that $x$ and $y$ are both prefixes (resp. suffixes) of $w$. If $|x| < |y|$, then $x$ is a proper prefix (resp. proper suffix) of $y$. If $|x| = |y|$, then $x = y$. Conversely, let $x$ be a prefix (resp. a suffix) of $y$ and let $y$ be a prefix (resp. a suffix) of $w$. If $|x| < |y|$, then $x$ is a proper prefix (resp. a proper suffix) of $w$.

We recall that two words $x, y$ are called conjugate if there exist words $u, v$ such that $x = uv, y = vu$. The conjugacy relation is an equivalence relation. A conjugacy class (or necklace) is a class of this equivalence relation. The following is Proposition 1.3.4 in [21].

Proposition 2.1 Two words $x, y \in \Sigma^+$ are conjugate if and only if there exists $z \in \Sigma^*$ such that

$$xz = zy$$

(2.1)

More precisely, equality (2.1) holds if and only if there exist $u, v \in \Sigma^*$ such that

$$x = uv, \quad y = vu, \quad z \in u(vu)^*.$$  

(2.2)

A nonempty word $w$ is unbordered if no proper nonempty prefix of $w$ is a suffix of $w$. Otherwise, $w$ is bordered. A word $w$ is primitive if $w = x^k$ implies $k = 1$. An unbordered word is primitive. Let $r, w$ nonempty words over $\Sigma$. We say that two occurrences of $r$ as a factor of $w$ overlap if $w = xrxz = x'r'x'$ with $|x'| < |x| < |x'r'|$. Therefore $r$ is bordered. The following lemma will be used in Section 5.

Lemma 2.2 Let $x, y, w, r \in \Sigma^+$ be such that

$$w = xrz = ry,$$

with $|x| < |r|$, i.e., $r$ occurs twice in $w$ and these two occurrences of $r$ in $w$ overlap. Then there exists $r' \in \Sigma^+$ such that

$$w = x'r' = r'y',$$

with $|r'| < |x'|$, and $y', y$ start with the same letter.
Proof:

Let $x, y, r \in \Sigma^+$ be as in the statement. By Proposition 2.1, there are $u, v \in \Sigma^*$ and $n \in \mathbb{N}$ such that

$$x = uv, \quad y = vu, \quad r = u(vu)^n.$$  \hspace{1cm} (2.3)

Set

$$r' = \begin{cases} u & \text{if } u \neq 1 \\ v & \text{if } u = 1 \end{cases}$$  \hspace{1cm} (2.4)

and

$$x' = \begin{cases} (uv)^n = (uv)^{n+1} & \text{if } u \neq 1 \\ v^n & \text{if } u = 1 \end{cases} \quad \text{and} \quad y' = \begin{cases} (vu)^n y = (vu)^{n+1} & \text{if } u \neq 1 \\ v^n & \text{if } u = 1 \end{cases}$$  \hspace{1cm} (2.5)

By using Eqs. (2.3)-(2.5), we can easily see that $x' r' = r' y' = r y = x r$. Moreover, since $|x| < |r|$, we have $n \geq 1$ if $u \neq 1$ and $n \geq 2$ if $u = 1$, hence, in both cases, $|r'| < |x'|$. Finally, $y$ is a prefix of $y'$, therefore they start with the same letter. 

\section*{2.2 Lexicographic order and Lyndon words}

Definition 2.1 Let $(\Sigma, <)$ be a totally ordered alphabet. The lexicographic (or alphabetic order) $\prec$ on $(\Sigma^*, <)$ is defined by setting $x \prec y$ if

- $x$ is a proper prefix of $y$, or
- $x = ras, y = rbt, a < b$, for $a, b \in \Sigma$ and $r, s, t \in \Sigma^*$.

For two nonempty words $x, y$, we write $x \ll y$ if $x < y$ and $x$ is not a proper prefix of $y$. We also write $y \succ x$ if $x < y$. Basic properties of the lexicographic order are recalled below.

Lemma 2.3 For $x, y \in \Sigma^+$, the following properties hold.

1. $x < y$ if and only if $zx < zy$, for every word $z$.
2. If $x \ll y$, then $zu \ll vy$ for all words $u, v$.
3. If $x < y < xz$ for a word $z$, then $y = xy'$ for some word $y'$ such that $y' < z$.

Definition 2.2 A Lyndon word $w \in \Sigma^+$ is a word which is primitive and the smallest one in its conjugacy class for the lexicographic order.

Example 2.1 Let $\Sigma = \{a, b\}$ with $a < b$. The words $a, b, aaab, abbb, aabab$ and $aababaab$ are Lyndon words. On the contrary, $aba$ and $abaab$ are not Lyndon words. Indeed, $aab \prec aba$ and $aabab < abaab$.

Interesting properties of Lyndon words are recalled below.

Proposition 2.2 Each Lyndon word $w$ is unbordered.

Proposition 2.3 A word is a Lyndon word if and only if it is nonempty and smaller than all its proper nonempty suffixes, i.e., $w \neq 1$ and $w < v$, for each $u, v \neq 1$ such that $w = uv$. 

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2.3 The Lyndon factorization

A family \((X_i)_{i \in I}\) of subsets of \(\Sigma^+\), indexed by a totally ordered set \(I\), is a factorization of the free monoid \(\Sigma^*\) if each word \(w \in \Sigma^*\) has a unique factorization \(w = x_1 \cdots x_n\), with \(n \geq 0\), \(x_i \in X_{j_i}\) and \(j_1 \geq j_2 \geq \ldots \geq j_n\) \([4]\). A factorization \((X_i)_{i \in I}\) is called complete if each \(X_i\) is reduced to a singleton \(x_i\) \([4]\). Let \(L = L(\Sigma^*, <)\) be the set of Lyndon words, totally ordered by the relation \(<\) on \((\Sigma^*, <)\). The following theorem, shows that the family \((\ell)_{\ell \in L}\) is a complete factorization of \(\Sigma^*\).

**Theorem 2.1** Any word \(w \in \Sigma^+\) can be written in a unique way as a nonincreasing product \(w = \ell_1 \ell_2 \cdots \ell_h\) of Lyndon words, i.e., in the form

\[
\ell_1 \ell_2 \cdots \ell_h, \quad \text{with } \ell_j \in L \text{ and } \ell_1 \geq \ell_2 \geq \ldots \geq \ell_h
\]

(2.6)

The sequence \(CFL(w) = (\ell_1, \ldots, \ell_h)\) in Eq.(2.6) is called the Lyndon decomposition (or Lyndon factorization) of \(w\). It is denoted by \(CFL(w)\) because Theorem 2.1 is usually credited to Chen, Fox and Lyndon \([6]\). Uniqueness of the above factorization is a consequence of the following result, proved in \([11]\).

**Lemma 2.4** Let \(w \in \Sigma^+\) and let \(\ell_i \in L\) be such that \(w = \ell_1 \ell_2 \cdots \ell_h\), with \(\ell_1 \geq \ell_2 \geq \ldots \geq \ell_h\). Then the following properties hold:

(i) \(\ell_h\) is the nonempty suffix of \(w\) which is the smallest with respect to the lexicographic order.

(ii) \(\ell_h\) is the longest suffix of \(w\) which is a Lyndon word.

(iii) \(\ell_1\) is the longest prefix of \(w\) which is a Lyndon word.

Therefore, given \(w \in \Sigma^+\), if \(\ell_1\) is its longest prefix which is a Lyndon word and \(w = \ell_1 w'\), then \(CFL(w) = (\ell_1, CFL(w'))\). As a consequence of Theorem 2.1 for any word \(w\) there is a factorization

\[
w = \ell_1^{n_1} \cdots \ell_r^{n_r}
\]

where \(r > 0\), \(n_1, \ldots, n_r \geq 1\), and \(\ell_1 \succ \ldots \succ \ell_r\) are Lyndon words, also named Lyndon factors of \(w\). There is a linear time algorithm to compute the pair \((\ell_1, n_1)\) and thus, by iteration, the Lyndon factorization of \(w\). It is due to Fredricksen and Maiorana \([12]\) and it is also reported in \([23]\). It can also be used to compute the Lyndon word in the conjugacy class of a primitive word in linear time \([23]\). Linear time algorithms may also be found in \([11]\) and in the more recent paper \([16]\).

2.4 Inverse lexicographic order and anti-Lyndon words

We also need the following well-known definition.

**Definition 2.3** Let \((\Sigma, <)\) be a totally ordered alphabet. Let \(<_{in}\) be the inverse of \(<\), defined by

\[
\forall a, b \in \Sigma \quad b <_{in} a \iff a < b
\]

The inverse lexicographic or inverse alphabetic order, denoted \(<_{in}\), on \((\Sigma^*, <)\) is the lexicographic order on \((\Sigma^*, <_{in})\).

From now on, \(L_{in} = L(\Sigma^*, <_{in})\) denotes the set of the Lyndon words on \(\Sigma^*\) with respect to the inverse lexicographic order. A word \(w \in L_{in}\) will be named an anti-Lyndon word.
2.5 Sorting suffixes of a text

In [26, 27], the authors found interesting relations between the sorting of the suffixes of a word $w$ and that of its factors. Let $w, x, u, y \in \Sigma^*$, and let $u$ be a factor of $w = xuy$. Let $\text{first}(u)$ and $\text{last}(u)$ denote the position of the first and the last symbol of $u$ in $w$, respectively. If $w = a_1 \cdots a_n$, $a_i \in \Sigma$, $1 \leq i \leq n$, then we also set $w[i, j] = a_i \cdots a_j$. A local suffix of $w$ is a suffix of a factor of $w$, specifically $\text{suffix}_w(i) = w[i, \text{last}(u)]$ denotes the local suffix of $w$ at the position $i$ with respect to $u$, $i \geq \text{first}(u)$. The corresponding global suffix $\text{suffix}_w(i)$ of $w$ at the position $i$ is denoted by $\text{suffix}_w(i) = w[i, \text{last}(w)]$ (or simply $\text{suffix}(i)$ when it is understood). We say that $\text{suffix}_w(i)$ is associated with $\text{suffix}_w(i)$.

**Definition 2.4** [26, 27] Let $w \in \Sigma^*$ and let $u$ be a factor of $w$. We say that the sorting of the suffixes of $u$ is compatible with the sorting of suffixes of $w$ if for all $i, j$ with $\text{first}(u) \leq i < j \leq \text{last}(u)$,

$$\text{suffix}_u(i) < \text{suffix}_u(j) \iff \text{suffix}(i) < \text{suffix}(j).$$

Let $u = \ell_r \cdots \ell_s$ be a concatenation of consecutive Lyndon factors of $w$. Let $\mathcal{L}_{\text{loc}}(u, w) = (s_1, \ldots, s_l)$ be the ordered list of the suffixes of $u$ and let $\mathcal{L}_{\text{glob}}(u, w) = (s'_1, \ldots, s'_l)$ be the ordered list of the corresponding global suffixes of $w$. We name it the global list associated with $\mathcal{L}_{\text{loc}}(u, w)$. The following result proved in [26, 27] shows that each $s'_i$ in $\mathcal{L}_{\text{glob}}(u, w)$ is associated with $s_i$.

**Theorem 2.2** Let $w$ be a word and let $(\ell_1, \ldots, \ell_h)$ be its Lyndon factorization. Then, for any $r, s, 1 \leq r \leq s \leq h$, the sorting of the suffixes of $u = \ell_r \cdots \ell_s$ is compatible with the sorting of the suffixes of $w$.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two sorted lists of elements taken from any totally ordered set, then the result of the operation $\text{merge}(\mathcal{L}_1, \mathcal{L}_2)$ is a single sorted list containing the elements of $\mathcal{L}_1$ and $\mathcal{L}_2$. Theorem 2.2 could be considered in a merge sort algorithm for the sorting of the suffixes of $w$, as suggested in [26, 27]. Starting with the list $\text{CFL}(w) = (\ell_1, \ldots, \ell_h)$, it could operate as follows.

- Divide the sequence into two subsequences $(\ell_1, \ldots, \ell_r), (\ell_{r+1}, \ldots, \ell_h)$, where $r = \lceil k/2 \rceil$

- Let $\mathcal{L}_1$ be the list of the suffixes of $u = \ell_1 \cdots \ell_r$, let $\mathcal{L}_2$ be the list of the suffixes of $y = \ell_{r+1} \cdots \ell_h$. Sort the two subsequences $\mathcal{L}_1, \mathcal{L}_2$ recursively using merge sort, thus obtaining $\mathcal{L}_{\text{loc}}(u, w), \mathcal{L}_{\text{loc}}(y, w)$

- Merge the two subsequences $\mathcal{L}_{\text{glob}}(u, w), \mathcal{L}_{\text{glob}}(y, w)$ to produce $\mathcal{L}_{\text{glob}}(w, w)$.

Notice that in the third step we change $\mathcal{L}_{\text{loc}}(u, w)$ into $\mathcal{L}_{\text{glob}}(u, w)$. Alternatively, we can merge the two subsequences $\mathcal{L}_{\text{loc}}(u, w), \mathcal{L}_{\text{loc}}(y, w)$ and then produce $\mathcal{L}_{\text{glob}}(w, w)$. In this case, if an element $s$ occurs twice in $\mathcal{L}_{\text{loc}}(u, w), \mathcal{L}_{\text{loc}}(y, w)$, then change the second occurrence of it into $sy$.

We may not limit ourselves to the local lists of suffixes, as Example 2.2 shows.

**Example 2.2** Let $\Sigma = \{a, b, c, d\}$ with $a < b < c < d$. Let $w = bcbcacad$, thus $\text{CFL}(w) = (bcbc, bcacad)$. Then, $\mathcal{L}_{\text{loc}}(bcbc, w) = (bcbc, bc, c)$, $\mathcal{L}_{\text{loc}}(bcacad, w) = (bc, c)$, and $\mathcal{L}_{\text{loc}}(acad, w) = (acad, ad, cad, d)$. Notice that $c < cad$ but $bcacad > cad$. Let $u = bbbc, y = acad$, then

\[
\mathcal{L}_{\text{loc}}(bbbc, w) = \text{merge}(\mathcal{L}_{\text{glob}}(bcbc, u), \mathcal{L}_{\text{glob}}(bc, u)) = \text{merge}((bbbc, bbbc, cbc, (bc, c)) = (bcbc, bc, bcbc, c, cbc)
\]

\[
\mathcal{L}_{\text{glob}}(w, w) = \text{merge}((bcbcacad, bcacad, bcacad, cad, cbacad), (acad, ad, cad, d)) = (acad, ad, bbbcacad, bcacad, bcacad, acad, cad, cad, cbacad, d)
\]
Notice that, if in the third step we merged \( L_{loc}(bbc, w), L_{loc}(bc, w) \), we would obtain \((bbc, bc, bc, c, c)\). Then, according to the previous remark, we obtain \( L_{loc}(bbcbc, w) \).

### 3 Inverse Lyndon words

As mentioned in Section 1, the Lyndon factorization of a word may generate very long or too short factors, thus becoming unsatisfactory with respect to a parallel strategy. We face this problem in Section 4, where we introduce another factorization which maintains the main properties of the Lyndon factorization but that allows us to overcome the glitch. This factorization is based on the notion of inverse Lyndon words, given below.

**Definition 3.1** A word \( w \in \Sigma^+ \) is an inverse Lyndon word if \( s \prec w \), for each nonempty proper suffix \( s \) of \( w \).

**Example 3.1** The words \( a, b, bbba, baba, bbabab, bbababa \) are inverse Lyndon words on \( \{a, b\} \), with \( a < b \). On the contrary, \( aaba \) is not an inverse Lyndon word since \( aaba \prec ba \). Analogously, \( aabba \prec ba \) and thus \( aabba \) is not an inverse Lyndon word.

In Section 2.1 we will see that the set of inverse Lyndon words properly contains the set of Lyndon words with respect to the inverse lexicographic order. Some useful properties of the inverse Lyndon words are proved below. The following is a direct consequence of Definitions 2.1, 3.1.

**Lemma 3.1** If \( w \in \Sigma^+ \) is not an inverse Lyndon word, then there exists a nonempty proper suffix \( s \) of \( w \) such that \( w \ll s \).

Next lemma shows that the set of the inverse Lyndon words (with the empty word) is a prefix-closed set, that is, it contains the prefixes of its elements.

**Lemma 3.2** Any nonempty prefix of an inverse Lyndon word is an inverse Lyndon word.

**Proof**:
Let \( w \in \Sigma^+ \) be an inverse Lyndon word. By contradiction, assume that there is a proper nonempty prefix \( p \) of \( w = ps \) which is not an inverse Lyndon word. By Lemma 3.1 there is a proper nonempty suffix \( v \) of \( p \) such that \( p \ll v \). Hence, by item (2) in Lemma 2.3, \( w = ps \ll vs \). Thus \( w \) is smaller than its proper nonempty suffix \( vs \), in contradiction with the hypothesis that \( w \) is an inverse Lyndon word.

**Lemma 3.3** and 3.4 are needed for the definition of our new factorization of a word.

**Lemma 3.3** If \( w \in \Sigma^+ \) is not an inverse Lyndon word, then there exists a nonempty proper prefix \( p \) of \( w = ps \) such that \( p \ll s \).

**Proof**:
Let \( w \) be as in the statement. By Lemma 3.1 there exists a shortest nonempty proper suffix \( s \) of \( w = ps \) such that \( w \ll s \). Hence, \( p \) is nonempty. Moreover there are words \( r, t, q \in \Sigma^* \) and letters \( a, b \in \Sigma \), with \( a < b \), such that

\[
\begin{align*}
s &= rbq, \\
w &= rat = prbq
\end{align*}
\]
We show that \( p \ll s \). If \(|p| \geq |ra|\), then \( p \ll s \) and the proof is ended. Thus assume \( 0 < |p| \leq |r| \).
By Eq. (3.1), there exists a prefix \( t' \) of \( t \) such that \( pt = rat' \). Therefore, by Proposition 2.1 there are \( u, v \in \Sigma^* \) such that
\[
p = uv, \quad at' = vu, \quad r \in u(vu)^*.
\] (3.2)

If \( u = 1 \), then \( p \) starts with \( a \) and so does \( w \). Thus \( w \ll bq \) and, by the hypothesis on \( s \), we have \( s = bq \). Hence \( r = 1 \), a contradiction. The same argument applies if \( v = 1 \). Thus assume \( u \neq 1, v \neq 1 \). Set \( r = u(vu)^k \), \( k \geq 0 \). We also have \( k > 0 \), since, otherwise, by Eq. (3.2), \( p = rv \), with \( v \neq 1 \), and consequently \( |r| < |p| \leq |r| \), a contradiction. Finally, by Eqs. (3.1) and (3.2), \( w = ps = uvs = (uv)^{k+1}ubq \) and \( v \) starts with \( a \). Consequently, \( w \ll ubq \), with \( ubq \) shorter than \( s \), in contradiction with the hypothesis on the length of \( s \).

\[\blacksquare\]

**Corollary 3.1** A word \( w \in \Sigma^+ \) is not an inverse Lyndon word if and only if there are words \( r, u, t \in \Sigma^* \) and letters \( a, b \in \Sigma \), with \( a < b \) such that \( w = raurbt \).

**Proof:**
Let \( r, u, t \in \Sigma^* \) and \( a, b \in \Sigma \), with \( a < b \). By Definition 2.1 \( w = raurbt \ll rbt \), hence \( w \) is not an inverse Lyndon word. Conversely, assume that \( w \in \Sigma^+ \) is not an inverse Lyndon word. By Lemma 3.3 there exists a nonempty proper prefix \( p \) of \( w = ps \) such that \( p \ll s \). By Definition 2.1 there are \( r, u, t \in \Sigma^* \), \( a, b \in \Sigma \), with \( a < b \), such that \( p = rau \), \( s = rbt \), and thus \( w = ps = raurbt \).

The following lemma shows that there exists \( p \) satisfying Lemma 3.3 and which is in addition an inverse Lyndon word.

**Lemma 3.4** If \( w \in \Sigma^+ \) is not an inverse Lyndon word, then there exists a nonempty proper prefix \( p \) of \( w = ps \) such that \( p \ll s \).

**Proof:**
Let \( w \in \Sigma^+ \) a word which is not an inverse Lyndon word. The proof is by induction on \(|w|\). The shortest nonempty word which is not an inverse Lyndon word has the form \( w = ab \), with \( a, b \) letters such that \( a < b \). In this case the nonempty proper prefix \( a \) of \( w \) is an inverse Lyndon word and \( a \ll b = s \). Now assume \(|w| > 2 \). By Lemma 3.3 there exists a nonempty proper prefix \( p \) of \( w = ps \) such that \( p \ll s \). If \( p \) is an inverse Lyndon word, then the proof is ended. Otherwise, by induction hypothesis there exists a nonempty proper prefix \( p' \) of \( p = ps' \) such that \( p' \) is an inverse Lyndon word and \( p' \ll s' \). Hence, by item (2) in Lemma 2.3 \( p' \ll s' \) and \( s' \). Thus, \( p' \) is a nonempty proper prefix of \( w = p's's \) such that \( p' \) is an inverse Lyndon word and \( p' \ll s's \).

\[\blacksquare\]

## 4 Variants of the Lyndon factorization

### 4.1 Inverse Lyndon factorizations

We give below the notion of an inverse Lyndon factorization.

**Definition 4.1** Let \( w \in \Sigma^+ \). A sequence \((m_1, \ldots, m_k)\) of words over \( \Sigma \) is an inverse Lyndon factorization of \( w \) if it satisfies the following conditions.

1. \( w = m_1 \cdots m_k \),
(2) for any \( j \in \{1, \ldots, k\} \), the word \( m_j \) is an inverse Lyndon word,

(3) \( m_1 \ll m_2 \ll \ldots \ll m_k \).

Example 4.1 shows that a word may have different inverse Lyndon factorizations even with a different number of factors.

**Example 4.1** Let \( \Sigma = \{a, b, c, d\} \) with \( a < b < c < d \), let \( w = dabadabdadac \in \Sigma^+ \). The two sequences \((daba, dabdab, dadac)\), \((dabadab, dabda, dac)\) are both inverse Lyndon factorizations of \( w \). Indeed,

\[
w = (daba)(dabdab)(dadac) = (dabadab)(dabda)(dac).
\]

Moreover, \( daba, dabdab, dadac, dabadab, dabda, dac \) are all inverse Lyndon words. Furthermore,

\[
daba \ll dabdab \ll dadac, \quad dabadab \ll dabda \ll dac.
\]

As another example, consider the following two factorizations of \( dabdadacdddbdc \)

\[
(dab)(dadacd)(db)(dc) = (dabda)(dac)(ddbdc)
\]

It is easy to see that the two sequences \((dab, dadacd, db, dc)\), \((dabda, dac, ddbdc)\) are both inverse Lyndon factorizations of \( dabdadacdddbdc \). The first factorization has four factors whereas the second one has three factors.

In Section 2.3 we have given the definition of a complete factorization \((x_i)_{i \in I}\) of the free monoid \( \Sigma^* \). By a result of Schützenberger, if \((x_i)_{i \in I}\) is a complete factorization of \( \Sigma^* \), then the set \( X = \{x_i \mid i \in I\} \) is a set of representatives of the primitive conjugacy classes (see [4, Corollary 8.1.7]). In particular, any \( x_i \) is a primitive word. The fact that a word \( w \) may have several different inverse Lyndon factorizations is a consequence of this result since an inverse Lyndon word is not necessarily primitive (take \( baba \), with \( a < b \), for instance).

However, we focus on a special canonical inverse Lyndon factorization, denoted by \( ICFL(w) \), which maintains three important features of \( CFL(w) \): it is uniquely determined (Proposition 4.2), it can be computed in linear time (Section 6) and it maintains the compatibility property of the suffixes with respect to the inverse lexicographic order (Theorem 7.2). We give the definition of \( ICFL(w) \) in Section 4.3. It is based on the definition of the bounded right extension of a prefix of a word, defined in Section 4.2.

### 4.2 The bounded right extension

The bounded right extension, abbreviated \( bre \), of a prefix of a word \( w \), defined below, allows us to define the first factor in the inverse Lyndon factorization \( ICFL(w) \).

**Definition 4.2** Let \( w \in \Sigma^+ \), let \( p \) be an inverse Lyndon word which is a nonempty proper prefix of \( w = pv \). The bounded right extension \( \overline{p}_w \) of \( p \) (relatively to \( w \)), denoted by \( \overline{p} \) when it is understood, is a nonempty prefix of \( v \) such that:

1. \( \overline{p} \) is an inverse Lyndon word,
2. \( p'z' \) is an inverse Lyndon word, for each proper nonempty prefix \( z' \) of \( \overline{p} \),
3. \( p\overline{p} \) is not an inverse Lyndon word,
4. \( p \ll \overline{p} \).
Moreover, we set

\[ \text{Pref}_{\text{bre}}(w) = \{(p, \overline{p}) \mid p \text{ is an inverse Lyndon word which is a nonempty proper prefix of } w\} \]

Notice that, given a word \( w \) and a nonempty proper prefix \( p \) of \( w \), the bounded right extension \( \overline{p}_w \) of \( p \) may not exist. For instance, let \( \Sigma = \{a, b, c\} \) with \( a < b < c \). For the prefix \( ba \) of \( baababc \), \( ba \) does not exist since any nonempty prefix \( p' \) of \( ababc \) starts with \( a \), thus \( p' \prec ba \). On the contrary, for the prefix \( baa \) of \( baababc \), we have \( bab = baa \). As another example, for the prefix \( bab \) of \( babc \) we have \( c = bab \) but for the prefix \( ba \) of the same word \( babc \), \( ba \) does not exist. Moreover, it is clear that if \( w \) is a letter, then \( \text{Pref}_{\text{bre}}(w) \) is empty. More generally, if \( w \) is an inverse Lyndon words, then \( \text{Pref}_{\text{bre}}(w) \) is empty. A more precise result will be proved below. We will see that the set \( \text{Pref}_{\text{bre}}(w) \) is either empty or it is a singleton. In other words, given a word \( w \), either there is no prefix of \( w \) which has a bounded right extension or this prefix is unique. This result will be proved through Lemmas 4.3, 4.4 and Proposition 4.1. Lemmas 4.1, 4.2 below show interesting properties of \( \text{Pref}_{\text{bre}}(w) \).

**Lemma 4.1** Let \( w \in \Sigma^* \) and let \( (p, \overline{p}) \in \text{Pref}_{\text{bre}}(w) \). Then, there are \( r, s, t \in \Sigma^* \), \( a, b \in \Sigma \), with \( a < b \), such that \( p = ras \) and \( \overline{p} = rbt \).

**Proof:**
Let \( w \in \Sigma^+ \) and let \( (p, \overline{p}) \in \text{Pref}_{\text{bre}}(w) \). By Definition 4.2, \( p \prec \overline{p} \), hence \( p = ras \), \( \overline{p} = rbt \), with \( r, s, t \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \). Moreover \( t = 1 \), otherwise for any proper prefix \( t' \) of \( t \), the word \( z' = rbt' \) would be a proper prefix of \( \overline{p} \) such that \( p \ll z' \), thus \( pz' \ll z' \) and \( pz' \) would not be an inverse Lyndon word, in contradiction with Definition 4.2.

**Remark 4.1** Let \( w \in \Sigma^+ \), let \( p \) be a nonempty prefix of \( w = pv \) which is an inverse Lyndon word. By Lemma 4.1 if \( \overline{p} \) exists, then it is the shortest nonempty proper prefix of \( v \) such that the above conditions (1)-(4) holds. Indeed, \( p = ras \) and \( \overline{p} = rbt \). Hence, for any proper prefix \( r' \) of \( \overline{p} \), we have \( r' < p \).

Recall that, if \( w \in \Sigma^+ \) is not an inverse Lyndon word, then there exists a nonempty proper prefix \( p \) of \( w = ps \) such that \( p \ll s \) (Lemma 4.3).

**Lemma 4.2** Let \( w \) be a nonempty word which is not an inverse Lyndon word but all its proper nonempty prefixes are inverse Lyndon words. If \( p \) is the longest nonempty proper prefix of \( w = ps \) such that \( p \ll s \) for the corresponding suffix \( s \), then \( (p, s) \in \text{Pref}_{\text{bre}}(w) \). In other words, if \( (p, s) \in F = \{(p', s') \mid p's' = w, p' \ll s'\} \) and \( |p| \geq |p'| \), for any \( (p', s') \in F \), then \( (p, s) \in \text{Pref}_{\text{bre}}(w) \).

**Proof:**
Let \( w, p, s \) be words as in the statement. If \( s \) is an inverse Lyndon word, then conditions (1)-(4) in Definition 4.2 are satisfied with \( s = \overline{p} \) and we have done. Assume that \( s \) is not an inverse Lyndon word and set \( p = rax \), \( s = rbky \), where \( r, x, y \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \). We notice that \( y = 1 \), otherwise \( prb = raxrb \) would be a nonempty proper prefix of \( w = ps = prby \) which is not an inverse Lyndon word, since \( prb = raxrb \prec rb \), in contradiction with the hypotheses. Thus \( r \neq 1 \) (otherwise, \( s = b \) would be an inverse Lyndon word). Moreover, by Lemma 3.3 there exists a nonempty proper prefix \( q \) of \( s = rb = qt \) such that \( q \ll t \). Notice that \( q \) is a prefix of \( r = qq' \) and \( w = pq t \). Thus, by item (2) in Lemma 2.3 and \( q \ll t \), we get \( pq = qq'axq \ll t \), with \( pq \) longer than \( p \), a contradiction.
Lemma 4.3 Let $w \in \Sigma^+$. For any nonempty prefix $x$ of $w$, we have $\text{Pref}_{\text{bre}}(x) \subseteq \text{Pref}_{\text{bre}}(w)$.

Proof:
Let $w \in \Sigma^+$ and let $x$ be a proper nonempty prefix of $w = xv$. If $\text{Pref}_{\text{bre}}(x) = \emptyset$, then the proof is ended. Otherwise, let $(p, \overline{p}_x) \in \text{Pref}_{\text{bre}}(x)$. By Definition 4.2, $p$ is a nonempty prefix of $x = pv'$, thus of $w = pv'v$, which is an inverse Lyndon word. Moreover, $\overline{p}_x$ is a nonempty prefix of $v'$, thus of $v'v$, such that conditions (1)-(4) holds. Therefore, $(p, \overline{p}_x) = (p, \overline{p}_w) \in \text{Pref}_{\text{bre}}(w)$.

Lemma 4.4 A word $w \in \Sigma^+$ is not an inverse Lyndon word if and only if $\text{Pref}_{\text{bre}}(w)$ is nonempty.

Proof:
Let $w \in \Sigma^+$ be a word such that $\text{Pref}_{\text{bre}}(w) \neq \emptyset$ and let $(p, \overline{p}) \in \text{Pref}_{\text{bre}}(w)$. By Definition 4.2, the nonempty prefix $p \overline{p}$ of $w$ is not an inverse Lyndon word, thus $w$ is not an inverse Lyndon word by Lemma 3.2.

Conversely, let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. We prove that $\text{Pref}_{\text{bre}}(w)$ is nonempty by induction on $|w|$. The shortest nonempty word which is not an inverse Lyndon word has the form $w = ab$, with $a, b$ letters such that $a < b$ and $\text{Pref}_{\text{bre}}(ab) = \{(a, b)\}$.

Assume $|w| > 2$ and let $p'$ be the shortest nonempty prefix of $w$ which is not an inverse Lyndon word. Thus, $|p'| \geq 2$. If $|p'| < |w|$, then $\emptyset \neq \text{Pref}_{\text{bre}}(p') \subseteq \text{Pref}_{\text{bre}}(w)$ (induction hypothesis and Lemma 4.3). Otherwise, $p' = w$ and all nonempty proper prefixes of $w$ are inverse Lyndon words. Thus, by Lemma 3.3, there exists a nonempty proper prefix $p$ of $w = ps$ such that $p \ll s$. Choose it of maximal length. By Lemma 4.2, $(p, s) \in \text{Pref}_{\text{bre}}(w)$.

Proposition 4.1 If $w \in \Sigma^+$ is not an inverse Lyndon word, then $\text{Card}(\text{Pref}_{\text{bre}}(w)) = 1$.

Proof:
Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. By Lemma 4.4, the set $\text{Pref}_{\text{bre}}(w)$ is nonempty. By contradiction, let $(p, \overline{p}), (q, \overline{q}) \in \text{Pref}_{\text{bre}}(w)$, with $p \neq q$. Since $p \overline{p}$ and $q \overline{q}$ are both prefixes of $w$, they are comparable for the prefix order and one of the following three cases holds.

1. $p \overline{p}$ is a proper prefix of $q \overline{q}$,
2. $q \overline{q}$ is a proper prefix of $p \overline{p}$,
3. $p \overline{p} = q \overline{q}$.

In case (1), either $p \overline{p}$ is a prefix of $q$ or $p \overline{p} = qz'$, for a proper prefix $z'$ of $\overline{q}$. Since $p \overline{p}$ is not an inverse Lyndon word, both cases are impossible (the first contradicts Lemma 3.2, the second Definition 4.2). We may exclude case (2) by a similar reasoning, thus assume that $p \overline{p} = q \overline{q}$, with $q$ being a nonempty proper prefix of $p$ (the same argument applies if $p$ is a proper prefix of $q$).

By Lemma 4.1, there are $r, s, r', s' \in \Sigma^*$, $a, b, a', b' \in \Sigma$, with $a < b$, $a' < b'$, such that $p = ras$, $\overline{r} = rb$, $q = r'a's'$, $\overline{q} = r'b'$. Hence

$$b = b', \quad p \overline{p} = q \overline{q} = rasrb = r'a's'r'b, \quad a < b, \quad a' < b.$$  \hspace{1cm} (4.1)

Since $q$ is a nonempty proper prefix of $p$, we have $|q| < |p|$ which yields $|rb| = |\overline{p}| < |\overline{q}| = |r'b|$, i.e., $|r'| > |r|$. By Eq. (4.1), the word $ra$ is a prefix of $r'$, i.e., there is a word $x$ such that $r' = rax$. Furthermore, by Eq. (4.1) again, since $q$ is a nonempty proper prefix of $p$, $\overline{p} = rb$ is a nonempty proper suffix of $\overline{q}$, i.e., there is a nonempty word $y$ such that $\overline{q} = yrb$. In conclusion, $\overline{q} = r'b = raxb = yrb$, thus $\overline{q} \ll rb$, that is $\overline{q}$ is not an inverse Lyndon word, a contradiction.
4.3 Some technical lemmas

In this section we prove some technical results which allows us to compute the pair \((p, s)\) in \(\text{Pref}_{bre}(w)\), for a word \(w\) which is not an inverse Lyndon word. They will be used in Section 5.

In detail, Lemma 4.5 is a more precise reformulation of Lemma 3.3. Lemma 4.6 is preliminary to Lemma 4.7 which in turn characterizes \((p, s)\) through two simple conditions.

**Lemma 4.5** Let \(w \in \Sigma^+\). If \(w\) has no prefix with the form \(rurb\), where \(r, u \in \Sigma^*, a, b \in \Sigma\), with \(a < b\), then \(w\) is an inverse Lyndon word.

**Proof**: Let \(w \in \Sigma^+\) be as in the statement. If \(w\) were not an inverse Lyndon word, then, by Lemma 3.3, there would be \(r, u \in \Sigma^*, a, b \in \Sigma\), with \(a < b\) such that \(w = rurb\), a contradiction since \(rurb\) would be a prefix of \(w\).

**Lemma 4.6** Let \(w, u, v \in \Sigma^+, r \in \Sigma^*, \) and \(a, b \in \Sigma\) be such that

1. \(w = rvur\),
2. \(ru\) is an inverse Lyndon word,
3. \(u\) starts with \(a\), \(v\) starts with \(b\), with \(a < b\),
4. \(r\) is the shortest prefix of \(w\) such that conditions (1)-(3) holds.

Then \(w\) is not an inverse Lyndon word and \((ru, rb) \in \text{Pref}_{bre}(w)\).

**Proof**: Let \(w, u, v \in \Sigma^+, r \in \Sigma^*, \) and \(a, b \in \Sigma\) be as in the statement. By item (2) in Lemma 2.3, \(ru \ll rb\) implies \(rurb \ll rb\). Therefore, \(rurb\) is not an inverse Lyndon word, hence, by Lemma 3.2, \(w\) is not an inverse Lyndon word too. Moreover, by (2), each proper nonempty prefix of \(rurb\) is an inverse Lyndon word. Thus, by Lemmas 4.2, 4.3, if \(p\) is the longest nonempty proper prefix of \(rurb = ps\) such that \(p \ll s\) for the corresponding suffix \(s\), then \((p, s) \in \text{Pref}_{bre}(rurb) \subseteq \text{Pref}_{bre}(w)\). In particular, \(s\) is an inverse Lyndon word.

We claim that \(ru\) is the longest nonempty proper prefix of \(rurb\) such that \(ru \ll rb\) for the corresponding suffix \(rb\), hence \((ru, rb) = (p, s) \in \text{Pref}_{bre}(rurb) \subseteq \text{Pref}_{bre}(w)\). Indeed, otherwise there are \(p, s \in \Sigma^*\) such that \(rurb = ps\), \(p \ll s\) and \(|p| > |ru|\). Hence, there are \(r', x, y \in \Sigma^*, c, d \in \Sigma\), with \(c < d\), such that \(x\) starts with \(c\) and \(p = r'x\), \(s = r'dy\). Moreover, since any prefix of \(rurb\) is an inverse Lyndon word and \(r'x r'd\) is not an inverse Lyndon word, the word \(ru\) is a prefix of \(r'xr'd\), thus, by \(rurb = ps = r'xr'dy\), we have \(dy = b = d\) which implies

\[rurb = r'xr'b, \quad s = r'b\]  \hspace{1cm} (4.2)

Looking at Eq. (4.2), if \(|r'| = |r'|\), we would have \(r = r'\), hence \(p = ru\), a contradiction. Consequently, since \(r\) is the shortest prefix of \(w\) such that conditions (1)-(3) holds, we have \(|r'| > |r|\). Since \(u\) starts with \(a\), by Eq. (4.2) again, we get \(r' = ra' = u''r\). Thus, by Proposition 2.1 there are \(z, t \in \Sigma^*, n \geq 0\), such that

\[au' = zt, \quad u'' = tz, \quad r = t(zt)^n.\]

Of course \(zt \neq 1\). If \(z \neq 1\), \(z\) starts with \(a\) and \(s = r'b = t(zt)^nztb \ll tb\), a contradiction since \(s\) is a Lyndon inverse word. The same argument applies if \(z = 1\). In this case, \(t\) starts with \(a\) and \(s = r'b = t(zt)^nztb \ll b\), a contradiction since \(s\) is a Lyndon inverse word.
Lemma 4.7 Let \( w \in \Sigma^+ \) be a word which is not an inverse Lyndon word. Let \((p, \overline{p}) \in \text{Pref}_{\breve{\Sigma}}(w)\).

1. \( p\overline{p} \) is the shortest nonempty prefix of \( w \) which is not an inverse Lyndon word.

2. \( p = rau \) and \( \overline{p} = rb \), where \( r, u \in \Sigma^* \), \( a, b \in \Sigma \) and \( r \) is the shortest prefix of \( p\overline{p} \) such that \( p\overline{p} = rau rb \), with \( a < b \).

Proof:
Let \( w, p, \overline{p} \) be as in the statement.

1. Let \( x \) be the shortest nonempty prefix of \( w \) which is not an inverse Lyndon word. Since \( x \) and \( p\overline{p} \) are both prefixes of \( w \), they are comparable with respect to the prefix order. Since \( x \) is not an inverse Lyndon word whereas any proper nonempty prefix of \( p\overline{p} \) is an inverse Lyndon word (Definition 4.2), we have \(|x| \geq |p\overline{p}|\). Moreover, since \( x \) is of minimal length, we have \(|x| \leq |p\overline{p}|\), thus \(|x| = |p\overline{p}|\). Therefore, since \( x \) and \( p\overline{p} \) are comparable with respect to the prefix order, \( x = p\overline{p} \).

2. Let \( r \) be the shortest prefix of \( p\overline{p} \) such that \( p\overline{p} = rau rb \), with \( a < b \). The proper nonempty prefix \( raur \) of \( p\overline{p} \) is an inverse Lyndon word (Definition 4.2). Thus, by Lemma 4.6 \((raur, rb) \in \text{Pref}_{\breve{\Sigma}}(p\overline{p}) \subseteq \text{Pref}_{\breve{\Sigma}}(w)\).

Lemma 4.8 Let \( w \in \Sigma^+ \) be a word which is not an inverse Lyndon word. The shortest nonempty prefix of \( w \) which is not an inverse Lyndon word is the shortest nonempty prefix \( x \) of \( w \) such that \( x = rau \), where \( r, u \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \).

Proof:
Let \( w \) be a word which is not an inverse Lyndon word, let \( x' \) be the shortest nonempty prefix of \( w \) such that \( x \) is not an inverse Lyndon word, and let \( x \) be the shortest nonempty prefix of \( w \) such that \( x = rau \), where \( r, u \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \). By Corollary 3.1 there would be \( r', u', s \in \Sigma^* \), \( c, d \in \Sigma \), \( c < d \), such that \( x' = r'cu'r'ds \). By Corollary 3.1 again, \( r'cu'r'd \) is not an inverse Lyndon word and since \( x' \) is of minimal length with respect to this condition, we have \( x' = r'cu'r'd \). The words \( x, x' \) are both prefixes of \( w \), thus they are comparable with respect to the prefix order. If they were different, by the hypothesis on \(|x| \), \( x \) would be a proper nonempty prefix of \( x' \). This fact contradicts the hypothesis on \(|x'| \) because \( x \) is not an inverse Lyndon word (Corollary 3.1).

Example 4.2 Let \( \Sigma = \{a, b, c, d\} \) with \( a < b < c < d \), let \( w = cbabc \) \( \in \Sigma^+ \). The word \( w \) is not an inverse Lyndon word but any nonempty proper prefix of \( w \) is an inverse Lyndon word. By Lemma 4.7 item 1, we have \( w = p\overline{p} \), with \((p, \overline{p}) \in \text{Pref}_{\breve{\Sigma}}(w)\). We have \( cbab \ll cba \) but \( cba \) is not the shortest prefix \( r \) of \( w = p\overline{p} \) satisfying item 2 in lemma 4.7. Notice that \( cbad \) is not an inverse Lyndon word. Since \( cbabc \ll d, \) the shortest prefix \( r \) of \( w = p\overline{p} \) satisfying item 2 in lemma 4.7 is the empty word. Consequently, \((p, \overline{p}) = (cbabc, d)\).

Example 4.3 Let \( \Sigma = \{a, b, c, d\} \) with \( a < b < c < d \), let \( v = dagadabdag \). We can check that \( dagadabdag \) is the shortest nonempty prefix of \( v \) which is not an inverse Lyndon word, hence \( dagadabdag = p\overline{p} \), with \((p, \overline{p}) \in \text{Pref}_{\breve{\Sigma}}(v)\). The shortest prefix \( r \) of \( p\overline{p} \) satisfying item 2 in lemma 4.7 is \( da \), thus \((p, \overline{p}) = (dagadab, d) \subseteq \text{Pref}_{\breve{\Sigma}}(v)\). As another example, consider \( w = dagadabdagadac \). We can check that \( dagadabdagadac \) is the shortest nonempty prefix of \( w \) which is not an inverse Lyndon word, hence \( dagadabdagadac = q\overline{q} \), with \((q, \overline{q}) \in \text{Pref}_{\breve{\Sigma}}(w)\). The shortest prefix \( r \) of \( q\overline{q} \) satisfying item 2 in lemma 4.7 is \( dab \), thus \((q, \overline{q}) = (daba, dab) \subseteq \text{Pref}_{\breve{\Sigma}}(w)\).
where $v \in \mathcal{ICFL}(\Sigma)$ and let us compute $\mathcal{ICFL}(\Sigma)$. We can check that $cbabacbac$ is the shortest nonempty prefix of $v$ which is not an inverse Lyndon word, hence $v = cbabacbac = p\overline{p}$, with $(p, \overline{p}) \in \text{Pref}_{\text{bre}}(v)$. The shortest prefix $r$ of $p\overline{p}$ satisfying item (2) in lemma 4.7 is $cbab$, thus $(p, \overline{p}) = (cbaba, cbac) \in \text{Pref}_{\text{bre}}(v)$. As another example, consider $w = cbabacaacbababac$. We can check that $cbabacaacbabacb$ is the shortest nonempty prefix of $w$ which is not an inverse Lyndon word, hence $cbabacaacbabacb = q\overline{q}$, with $(q, \overline{q}) \in \text{Pref}_{\text{bre}}(w)$. The shortest prefix $r$ of $q\overline{q}$ satisfying item (2) in lemma 4.7 is $cbab$, thus $(q, \overline{q}) = (cbabacaac, cbabcb) \in \text{Pref}_{\text{bre}}(w)$.

4.4 A canonical inverse Lyndon factorization: $\mathcal{ICFL}(w)$

We give below the recursive definition of the canonical inverse Lyndon factorization $\mathcal{ICFL}(w)$.

Definition 4.3 Let $w \in \Sigma^+$.

(Basis Step) If $w$ is an inverse Lyndon word, then $\mathcal{ICFL}(w) = (w)$.

(Recursive Step) If $w$ is not an inverse Lyndon word, let $(p, \overline{p}) \in \text{Pref}_{\text{bre}}(w)$ and let $v \in \Sigma^*$ such that $w = pv$. Let $\mathcal{ICFL}(v) = (m'_1, \ldots, m'_k)$ and let $r \in \Sigma^*$ and $a, b \in \Sigma$ such that $p = rax$, $\overline{p} = rb$ with $a < b$.

$$\mathcal{ICFL}(w) = \begin{cases} (p, \mathcal{ICFL}(v)) & \text{if } \overline{p} = rb \leq_p m'_1 \\ (pm'_1, m'_2, \ldots, m'_k) & \text{if } m'_1 \leq_p r \end{cases}$$

Remark 4.2 With the same notations as in the recursive step of Definition 4.3 we notice that $rb$ and $m'_1$ are both prefixes of the same word $v$. Thus they are comparable for the prefix order, that is, either $rb \leq_p m'_1$ or $m'_1 \leq_p r$.

Example 4.5 Let $\Sigma = \{a, b, c\}$ with $a < b < c$, let $v = cbabacbac$. Let us compute $\mathcal{ICFL}(v)$. As showed in Example 4.4 we have $(p, \overline{p}) = (cbaba, cbac) \in \text{Pref}_{\text{bre}}(v)$. Therefore, $cbaba, cbac$ are both inverse Lyndon words and we have $\mathcal{ICFL}(cbaba) = (cbaba)$, $\mathcal{ICFL}(cbac) = (cbac)$. Since $\overline{p} = cbac \leq_p m'$, we are in the first case of Definition 4.3 thus $\mathcal{ICFL}(v) = \mathcal{ICFL}(cbabacbac) = (p, \mathcal{ICFL}(\overline{p})) = (cbaba, cbac) = (m'_1, m'_2)$. Now, let $w = cbabacaacbabacbac$ and let us compute $\mathcal{ICFL}(w)$. In Example 4.4 we showed that $(q, \overline{q}) = (cbabacaac, cbabacb) \in \text{Pref}_{\text{bre}}(w)$. Thus, $w = qv$, where $v = cbabacbac$ is the above considered word and $\overline{q} = r'\overline{b}$, where $r' = cbabac$. Since $m'_1 = cbaba \leq_p cbabac$, we are in the second case of Definition 4.3 thus $\mathcal{ICFL}(w) = (qm'_1, m'_2) = (cbabacaacbababa, cbac)$.

Example 4.6 Let $\Sigma = \{a, b, c, d\}$ with $a < b < c < d$, let $v = dabdabdabadac$. Let us compute $\mathcal{ICFL}(v)$. As showed in Example 4.3 we have $(p, \overline{p}) = (dabda, dad) \in \text{Pref}_{\text{bre}}(v)$. Thus, $v = pv'$, where $v' = dadac = \overline{p}ac$. On the other hand, we can easily see that $dada$ is an inverse Lyndon word, hence $\mathcal{ICFL}(dada) = (dada)$. Since $\overline{p} = dad \leq_p dada$, by Definition 4.3 (first case), $\mathcal{ICFL}(v) = (dada, dadac)$. Now, let us compute $\mathcal{ICFL}(w)$, where $w = dabdabdabdadac$. In Example 4.3 we showed that $(q, \overline{q}) = (daba, dabd) \in \text{Pref}_{\text{bre}}(w)$. Thus, $w = qv$, where $v$ is the above considered word. Since $\overline{q} = dabd \leq_p dabdab$, by Definition 4.3 (first case), $\mathcal{ICFL}(w) = (daba, dabdab, dadac)$. In Example 4.4 we showed two different inverse Lyndon factorizations of $w$. $\mathcal{ICFL}(w)$ is that with the number of factors less than the other.

As proved below, $\mathcal{ICFL}(w)$ is uniquely determined.

Proposition 4.2 For any word $w \in \Sigma^+$, there is a unique sequence $(m_1, \ldots, m_k)$ of words over $\Sigma$ such that $\mathcal{ICFL}(w) = (m_1, \ldots, m_k)$. 

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The proof is by induction on $|w|$. If $w$ is a letter, then the statement is proved since $w$ is an inverse Lyndon word and $\text{ICFL}(w) = (w)$ (Definition 4.3). Thus, assume $|w| > 1$. If $w$ is an inverse Lyndon word we have done since, by Definition 4.3 $\text{ICFL}(w) = (w)$. Otherwise, $w$ is not an inverse Lyndon word and there is a unique pair $\langle p, \overline{p} \rangle$ in $\text{Pref}_{\text{pre}}(w)$ (Proposition 4.1). Let $\langle p, \overline{p} \rangle = (m_1', \ldots, m_k')$. Then $w = pm_1 \cdots m_k$ and $m_1 \preceq \cdots \preceq m_k$ and each $m_i$ is an inverse Lyndon word.

Proof:

The proof is by induction on $|w|$. If $w$ is a letter, then the statement is proved since $w$ is an inverse Lyndon word and $\text{ICFL}(w) = (w)$ (Definition 4.3). Thus, assume $|w| > 1$. If $h = 1$ we have done since, by Definition 4.3 $w$ is an inverse Lyndon word and $\text{ICFL}(w) = (w)$. Otherwise, $w$ is not an inverse Lyndon word. Let $\langle p, \overline{p} \rangle \in \text{Pref}_{\text{pre}}(w)$ and let $v \in \Sigma^+$ such that $w = pv$. Let $\text{ICFL}(v) = (m_1', \ldots, m_k')$ and let $r \in \Sigma^*$ and $a, b \in \Sigma$ such that $p = rax$, $\overline{p} = rb$ with $a < b$. By the recursive step of Definition 4.3 we have

$$\text{ICFL}(w) = \begin{cases} (p, \text{ICFL}(v)) & \text{if } rb \leq_p m_1' \\ (pm_1', m_2', \ldots, m_k') & \text{if } m_1' \leq_r r \end{cases}$$

In both cases, by the above arguments, the sequence $\text{ICFL}(w)$ is uniquely determined.

As a main result, we now prove that $\text{ICFL}(w)$ is an inverse Lyndon factorization of $w$.

**Lemma 4.9** For any $w \in \Sigma^+$, the sequence $\text{ICFL}(w) = (m_1, \ldots, m_k)$ is an inverse Lyndon factorization of $w$, that is, $w = m_1 \cdots m_k$, $m_1 \preceq \cdots \preceq m_k$ and each $m_i$ is an inverse Lyndon word.

**Proof:**

The proof is by induction on $|w|$. If $w$ is a letter, then the statement is proved since $w$ is an inverse Lyndon word and $\text{ICFL}(w) = (w)$ (Definition 4.3). Thus, assume $|w| > 1$. If $h = 1$ we have done since, by Definition 4.3 $w$ is an inverse Lyndon word and $\text{ICFL}(w) = (w)$. Otherwise, $w$ is not an inverse Lyndon word. Let $\langle p, \overline{p} \rangle \in \text{Pref}_{\text{pre}}(w)$ and let $v \in \Sigma^+$ such that $w = pv$. Let $\text{ICFL}(v) = (m_1', \ldots, m_k')$ and let $r \in \Sigma^*$ and $a, b \in \Sigma$ such that $p = rax$, $\overline{p} = rb$ with $a < b$. By the recursive step of Definition 4.3 we have

$$\text{ICFL}(w) = \begin{cases} (p, \text{ICFL}(v)) & \text{if } rb \leq_p m_1' \\ (pm_1', m_2', \ldots, m_k') & \text{if } m_1' \leq_r r \end{cases}$$

Moreover, since $|v| < |w|$, by induction hypothesis, $v = m_1' \cdots m_k'$, $m_1' \preceq \cdots \preceq m_k'$ and each $m_i'$ is an inverse Lyndon word. Thus, $w = pv = pm_1' \cdots m_k'$.

If $rb \leq_p m_1'$, then there is $z \in \Sigma^*$ such that $m_1' = rzb$ and $\text{ICFL}(w) = (p, m_1', \ldots, m_k')$. By Definition 4.2 $p$ is an inverse Lyndon word and so are all the words in $\text{ICFL}(w)$. Furthermore, $p \preceq m_1' \preceq \ldots \preceq m_k'$ and the proof is ended. Otherwise, there is $z \in \Sigma^*$ such that $r = m_1'z$, hence $p = rax = m_1'zax$ and $\text{ICFL}(w) = (pm_1', \ldots, m_k')$. The word $pm_1'$ is a proper prefix of $p\overline{p}$, thus, by Definition 4.2 $pm_1'$ is an inverse Lyndon word and so are all words in $\text{ICFL}(w)$. Finally, by $m_1' \preceq m_2' \cdots \preceq m_k'$ we have $pm_1' = m_1'zaxm_1' \preceq m_2' \cdots \preceq m_k'$ (Lemma 2.3).

We end this section with the following result showing that inverse Lyndon factorizations of Lyndon words are not trivial.

**Proposition 4.3** Any word $w \in \Sigma^+$ has an inverse Lyndon factorization $(m_1, \ldots, m_k)$. Moreover, if $w$ is a Lyndon word which is not a letter, then $k > 1$.

**Proof:**

The first part of the statement follows by Proposition 4.2 and Lemma 4.9. Assume that $w$ is a Lyndon word which is not a letter and let $(m_1, \ldots, m_k)$ be one of its inverse Lyndon factorizations. By Proposition 2.3 there is a proper nonempty suffix $v$ of $w$ such that $w \prec v$. Hence, $w$ is not an inverse Lyndon word, which yields $w \neq m_1$ and consequently $k > 1$.

Of course a converse of Proposition 4.3 can also be stated.
Proposition 4.4 Let \( w \in \Sigma^+ \) and let \( \text{CFL}(w) = (\ell_1, \ldots, \ell_h) \). If \( w \) is an inverse Lyndon word which is not a letter, then \( h > 1 \).

Proof:
Let \( w \) be an inverse Lyndon word which is not a letter and let \( \text{CFL}(w) = (\ell_1, \ldots, \ell_h) \). By Definition 3.1, there is a proper nonempty suffix \( v \) of \( w \) such that \( v < w \). Hence, by Proposition 2.3, \( w \) is not a Lyndon word, which yields \( w \neq \ell_1 \) and consequently \( h > 1 \).

5 An algorithm for finding the bounded right extension
In section 6 we will give a linear time recursive algorithm, called Compute-ICFL, that computes ICFL(\( w \)), for a given nonempty word \( w \). By Definition 4.3, we know that the computation of ICFL(\( w \)), when \( w \) is not an inverse Lyndon word, is based on that of the pair \((p, \overline{p}) \in \text{Pref}_{bre}(w)\). In this section we give algorithms to compute the above pair \((p, \overline{p})\). By Lemmas 4.7, 4.8, we are faced with the problems of

1. stating whether \( w \) is an inverse Lyndon word;
2. if not, finding the shortest prefix \( x \) of \( w \) such that \( x = raurb \), where \( r, u \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \). Therefore, \( x = pp \).
3. Finding the shortest \( r \) such that \( x = raurb \), with \( r, u, a, b \) as in (2). Thus \( p = rau \), \( \overline{p} = rb \).

The first two tasks are carried out by algorithm Find-prefix, the third is performed by algorithm Find-bre. Algorithm Find-prefix is very similar to Duval’s algorithm in [11] which computes the longest Lyndon prefix of a given string \( w \). This is not surprising because Find-prefix computes the longest prefix of \( w \) which is an inverse Lyndon word. Algorithm Compute-ICFL uses algorithm Find-bre as a subroutine. In turn, the latter uses the output of algorithm Find-prefix. Algorithm Find-bre also calls a procedure to compute the well known failure function (named the prefix function in [8]) of the Knuth-Morris-Pratt matching algorithm [19].

Firstly, we give a high-level description of algorithm Find-prefix followed by its pseudocode (Section 5.1) and we prove its correctness through some loop invariants (Section 5.2). Next, in Section 5.3, we recall the definition of the failure function and we prove some results concerning this function which will be useful later. Finally, we give a high-level description of algorithm Find-bre followed by its pseudocode (Section 5.4) and we end the section with the proof of its correctness through some loop invariants (Section 5.5).

5.1 Description of Find-prefix
Let us describe the high-level structure of algorithm Find-prefix. As already said, given \( w \), the algorithm looks for the shortest prefix \( raurb \) of \( w \), where \( r, u \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \) and \( raur \) is an inverse Lyndon word. This is equivalent to find, if it exists, a nonempty word \( r \) which is both a prefix of \( w \) and of a suffix of \( w \), and these two occurrences of \( r \) are followed by \( a, b \) as above.

As usual, the word is represented as an array \( w[1..n] \) containing the sequence of the letters in \( w \), with \( n = |w| \). The algorithm uses two indices \( i, j \) to scan the word \( w \). Initially, these indices denote the position of the first letter of a candidate common prefix \( r \).

A while-loop is used to compare the two letters \( w[i] \) and \( w[j] \). While \( j \) is incremented at each iteration, \( i \) is incremented only when \( w[i] = w[j] \). Notice that the algorithm does not test if eventually the two occurrences of \( r \) overlap. Lemma 2.2 allows us to avoid this test.
If \( w[i] > w[j] \), then the algorithm resets \( i \) to the first position of \( w \) and examines a new candidate common prefix \( r \), whose position of the first letter in the second occurrence of \( r \) is indicated by the value of \( j \) at this time.

The loop condition is false when \( w[i] < w[j] \) or when \( j \) denotes the last letter of \( w \). In the first case, our search has been successful and the algorithm returns \( raurb \). If \( j \) denotes the last letter of \( w \) (and \( w[i] \geq w[j] \)), then \( w \) has no prefix \( raurb \), with \( r, u, a, b \) as above, and the algorithm returns \( w\$ \), where \( \$ \) is a letter such that \( \$ \notin \Sigma \). In this case, by Lemma 4.5, we know that \( w \) is an inverse Lyndon word.

In conclusion, \textbf{Find-prefix} allows us to state whether \( w \) is an inverse Lyndon word or not and, in the latter case, it finds the prefix \( x \) of \( w \) such that \( x = p\bar{p} \), with \( (p, \bar{p}) \in \text{Pref}_{brw}(w) \). Algorithm 1 describes the procedure \textbf{Find-prefix}. It is understood that the empty array, namely \( w[j + 1..w] \) with \( j = n \), represents the empty word.

\begin{algorithm}
\begin{algorithmic}[1]
\Input A string \( w \)
\Output A pair of strings \((x, y)\), where \( xy = w \), and moreover \( x = w\$ \), \( y = 1 \) if \( w \) is an inverse Lyndon word, \( x = p\bar{p} \), with \((p, \bar{p}) \in \text{Pref}_{brw}(w) \), otherwise.
\end{algorithmic}
\begin{algorithmic}
\State 1 \textbf{if} \( |w| = 1 \) \textbf{then}
\State 2 \quad \textbf{return} \((w\$, 1)\);
\State 3 \quad \( i \leftarrow 1 \);
\State 4 \quad \( j \leftarrow 2 \);
\State 5 \quad \textbf{while} \( j < |w| \) \textbf{and} \( w[j] \leq w[i] \) \textbf{do}
\State 6 \quad \quad \textbf{if} \( w[j] < w[i] \) \textbf{then}
\State 7 \quad \quad \quad \( i \leftarrow 1 \);
\State 8 \quad \quad \textbf{else}
\State 9 \quad \quad \quad \( i \leftarrow i + 1 \);
\State 10 \quad \quad \( j \leftarrow j + 1 \);
\State 11 \quad \textbf{if} \( j = |w| \) \textbf{then}
\State 12 \quad \quad \textbf{if} \( w[j] \leq w[i] \) \textbf{then}
\State 13 \quad \quad \quad \textbf{return} \((w\$, 1)\);
\State 14 \quad \quad \textbf{return} \((w[1..j], w[j + 1..|w|])\);
\end{algorithmic}
\end{algorithm}

We notice that we can reach line (11) in two different cases: either when \( x = w \) or when \( w \) is an inverse Lyndon word. We distinguish these two cases: in the former case the output is \((w, 1)\) and in the latter case the output is \((w\$, 1)\) (see example below).

\textbf{Example 5.1} Let \( \Sigma = \{a, b, c\} \) with \( a < b < c \). Algorithm \textbf{Find-prefix}(bac) returns \((bac, 1)\). Indeed, \( |w| = 3 \) and \( w[2] < w[1] \) in the first iteration of the while-loop. Then, \( i = 1 \) (line (7)), \( j = 3 \) (line (10)), and we reach line (11) with \( j = 3 = |w| \). Since \( w[j] = w[3] = c > w[1] = w[i] \), the algorithm returns \((w[1..j], w[j + 1..|w|]) = (bac, 1)\).

Consider now the inverse Lyndon word \( w = bab \). Algorithm \textbf{Find-prefix}(bab) returns \((bab\$, 1)\). Indeed, \( |w| = 3 \) and, as before, \( w[2] < w[1] \) in the first iteration of the while-loop. Then, \( i = 1 \) (line (7)), \( j = 3 \) (line (10)), and we reach line (11) with \( j = 3 = |w| \). Since \( w[j] = w[3] = b = w[1] = w[i] \), the algorithm returns \((bab\$, 1)\). The same argument applies when \( w = baa \).

\textbf{Example 5.2} Let \( \Sigma = \{a, b\} \) with \( a < b \). Algorithm \textbf{Find-prefix}(bbababbabbb) outputs \((bbababbabbb, 1)\).
5.2 Correctness of Find-prefix

In this section we prove that Find-prefix does what it is claimed to do. To begin, notice that each time around the while-loop of lines (5) to (10), \( j \) is increased by 1 at line (10). Therefore, when \( j \) becomes equal to \(|w|\), if we do not break out of the while-loop earlier, the loop condition \( j < |w| \) will be false and the loop will terminate. In order to prove that Find-prefix does what it is intended to do, we define the following loop-invariant statement, where we use \( k \) to stand for one of the values that the variable \( j \) assumes and \( h_k \) for the corresponding variable \( i \), as we go around the loop. Finally \( h'_k \) depends on \( h_k \) and \( k \).

\[
S(k): \text{If we reach the loop test } |j| < |w| \text{ and } w[j] \leq w[i] \text{ with the variable } i \text{ having the value } h_k \text{ and the variable } j \text{ having the value } k, \text{ then}
\]

(a) \( h_k < k \). Moreover, set \( h'_k = k - h_k + 1 \), \( 1 < h'_k \leq |w| \).

(a1) \( w[1..h_k - 1] = w[h'_k..k - 1] \), that is, \( w[1..h_k - 1] \) is a proper prefix of \( w[1..k - 1] \) and \( w[1..h_k - 1] \) is also a suffix of \( w[1..k - 1] \).

(a2) For any \( t', 1 < t' < h'_k \), \( w[t'..k - 1] \) is not a prefix of \( w[1..k - 1] \).

(b) \( w[1..k - 1] \) has no prefix with the form \( raurb \), \( r, u \in \Sigma^*, a, b \in \Sigma, a < b \).

Therefore, by Lemma [4.5], \( w[1..k - 1] \) is an inverse Lyndon word.

Loosely speaking, at the beginning of each iteration of the loop of lines (5)-(10), indices \( i, j \) store the end of a candidate common prefix \( r \) (item (a1)) and we do not yet find a prefix with the required form in the examined part of the array (items (a2), (b)). Note that the examined part may be a single letter and \( r \) could be the empty word.

We need to show that the above loop invariant is true prior to the first iteration and that each iteration of the loop maintains the invariant. This will be done in Proposition 5.1 where we prove \( S(k) \) by (complete) induction on \( k \). Then, we show that the invariant provides a useful property to prove correctness when the loop terminates in Proposition 5.2. Even if \( S(k) \) clearly holds for \( k = 0, 1 \), we prove it for \( k \geq 2 \).

**Proposition 5.1** For any \( k \geq 2 \), \( S(k) \) is true.

**Proof:**

**(Basis)** Let us prove that \( S(2) \) is true. We reach the test with \( j \) having value 2 only when we enter the loop from the outside. Prior to the loop, lines (3) and (4) set \( i \) to 1 and \( j \) to 2. Therefore \( h_k = 1 < k = 2 \) and \( h'_k = k - h_k + 1 = 2 \), hence \( 1 < h'_k \leq |w| \). Part (a1) of \( S(2) \) is true. Indeed \( w[1..1 - 1] = w[2..2 - 1] \) holds since there are no elements in both the descriptions. Part (a2) of \( S(2) \) is also true since there are no integers \( t' \) such that \( 1 < t' < 2 \). Similarly, since \( w[1..2 - 1] = w[1] \), part (b) is true.

**(Induction)** We suppose that \( S(k) \) is true and prove that \( S(k + 1) \) is true. We may assume \( k < |w| \) and \( w[k] \leq w[h_k] \) (otherwise we break out of the while-loop when \( j \) has the value \( k \) or earlier and \( S(k + 1) \) is clearly true, since it is a conditional expression with a false antecedent). Moreover, by using the inductive hypothesis (part (a)), \( h_k < k \), \( w[1..h_k - 1] = w[h'_k..k - 1] \), where \( h'_k = k - h_k + 1 > 1 \) and \( w[t'..k - 1] \) is not a prefix of \( w[1..k - 1] \), \( 1 < t' < h'_k \).

To prove \( S(k + 1) \), we consider what happens when we execute the body of the while-loop with \( j \) having the value \( k \) and \( i \) having the value \( h_k \). Moreover, to prove part (a) of \( S(k + 1) \), we distinguish two cases:

(1) \( w[k] < w[h_k] \),
(2) \( w[k] = w[h_k] \).

(Case (1)) If \( w[k] < w[h_k] \), then line (7) set \( i \) to 1. Thus, when we reach the loop test with \( j \) having the value \( k + 1 \), the variable \( i \) has the value \( h_{k+1} = 1, h'_{k+1} = k + 1 - h_{k+1} + 1 = k + 1 \), hence \( 1 < h'_{k+1} = k + 1 \leq |w| \), and \( w[1..h_{k+1} - 1] = w[1..0] = w[k + 1..k] \). Hence part (a1) is proved in this case. Let us prove (a2). Let \( t' \) be any integer such that \( 1 < t' < h'_{k+1} = k + 1 \). If \( w[t'.k] \) were a prefix of \( w[1..k] \), then there would exist \( t \) such that \( w[1..t] = w[t'..k] \). Moreover, \( w[t'..k] = 1 \) would be a prefix of \( w[1..k] \), hence by inductive hypothesis (part (a2)), we would have \( t' \geq h'_{k} \). Hence \( |w[1..t]| = |w[t'..k]| \leq |w[h'_{k}..k]| = |w[h'_{k}..k] - 1| = 1 = |w[1..h_{k} - 1]| + 1 \) which implies \( t \leq h_{k} \). Notice that \( w[t] = w[k] < w[h_k] \), hence \( t \leq h_{k} - 1 \).

Since \( t \leq h_{k} - 1 \), the word \( w[1..t - 1] = w[t'..k - 1] \) is a proper prefix of \( w[1..h_{k} - 1] \) and, since \( t' \geq h'_{k} \), the same word \( w[1..t - 1] = w[t'..k - 1] \) is a suffix of \( w[h'_{k}..k - 1] = w[1..h_{k} - 1] \). Notice that each time around the while-loop, \( j \) is increased by 1 at line (10) and \( 2 \leq h_{k} + 1 \leq k \), by inductive hypothesis (part (a)). Therefore, when we reach the loop test with \( j \) having the value \( h_{k} + 1 \), the word \( w[1..h_{k}] \) has a prefix with the form \( raurb \), \( r, u \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \). Indeed, \( w[1..h_{k}] = raurb \), where \( a = w[t] < w[h_{k}] = b \) and \( r = w[1..t - 1] \) if the two occurrences of \( w[1..t - 1] \) (as a prefix and a suffix) in \( w[1..h_{k} - 1] \) do not overlap, otherwise it is its prefix given by Lemma 2.2 applied to \( w[1..t - 1] \). Hence part (b) of \( S(h_{k+1}) \) is not true, a contradiction.

(Case (2)) Assume that \( w[k] = w[h_k] \). Thus, by \( w[1..h_{k} - 1] = w[h'_{k}..k - 1] \), we also have \( w[1..h_{k}] = w[h'_{k}..k] \). Therefore, when we reach the loop test with \( j \) having the value \( k + 1 \), the variable \( i \) has value \( h_{k+1} = h_{k} + 1 \), hence \( h'_{k+1} = k + 1 - h_{k+1} + 1 = k + 1 \) and \( w[1..h_{k+1} - 1] = w[1..h_{k}] = w[h'_{k}..k] = w[h'_{k}..k] \), with \( 1 < h'_{k+1} = h'_{k} \leq |w| \). Thus part (a1) is proved in case (2). Furthermore, by inductive hypothesis (part (a2)), for any \( t' \), \( 1 < t' < h'_{k} = h'_{k+1} \), the word \( w[t'..k - 1] \) is not a prefix of \( w[1..k - 1] \), hence \( w[t'..k] \) is not a prefix of \( w[1..k] \) and part (a2) holds.

To prove part (b) of \( S(k+1) \), assume on the contrary that \( w[1..k] \) has a prefix with the form \( raurb \), \( r, u \in \Sigma^* \), \( a, b \in \Sigma \), \( a < b \). Thus, by using the inductive hypothesis (part (b)), we would have \( w[1..k] = raurb \), that is \( w[k] = b \). Moreover, there would exist \( t, t', 1 < t = |r| + 1 < k \), such that \( w[t] = a \) and \( r = w[1..t - 1] = w[t'..k - 1] \). Hence,

\[
 w[t] = a < b = w[k] \leq w[h_k].
\]

(5.1)

Since \( t < k \), the word \( w[1..t - 1] = w[t'..k - 1] \) is a prefix of \( w[1..k - 1] \). Thus, by using the inductive hypothesis, part (a2), we have \( t' \geq h'_{k} \). Now we apply the same argument as above. By \( t' \geq h'_{k} \), we have \( |w[1..t - 1]| = |w[t'..k - 1]| \leq |w[h'_{k}..k - 1]| = |w[1..h_{k} - 1]| \) which implies \( t - 1 \leq h_{k} - 1 \). Moreover \( t - 1 < h_{k} - 1 \), by Eq. (5.1). In turn, since \( t - 1 < h_{k} - 1 \), the word \( r = w[1..t - 1] = w[t'..k - 1] \) is a proper prefix of \( w[1..h_{k} - 1] \) and, since \( t' \geq h'_{k} \), the same word \( w[1..t - 1] = w[t'..k - 1] \) is a suffix of \( w[h'_{k}..k - 1] = w[1..h_{k} - 1] \). As already noticed, each time around the while-loop, \( j \) is increased by 1 at line (10) and \( 1 < t < h_{k} < k \) (see above and the inductive hypothesis, part (a)). Therefore, when we reach the loop test with \( j \) having the value \( h_{k} + 1 \), we have \( w[1..h_{k}] = r'au'r'b' \), where \( u \in \Sigma^* \), \( b' \in \Sigma \), \( a = w[t] < w[h_{k}] = b' \) (Eq. (5.1)) and \( r' = r \) if the two occurrences of \( r \) (as a prefix and a suffix) in \( w[1..h_{k} - 1] \) do not overlap, otherwise it is its prefix \( r' \) given by Lemma 2.2 applied to \( r \). Hence part (b) of \( S(h_{k} + 1) \) is not true, a contradiction.

Of course, when we reach the loop test with \( j \) having the value \( k + 1 \), the variable \( i \) has either value \( h_{k+1} = 1 \) or \( h_{k+1} = h_{k} + 1 \). We always have \( k + 1 > 1 \) and, if \( h_{k} < k \), then \( h_{k} + 1 < k + 1 \). Thus, we have proved the inductive step.
Definition 5.1: This result is given in Lemma 5.3.

Thus, by Lemma 2.1, the function $f$ guarantees, by Corollary 3.1, that $w$ and $b$, such that $a < b$.

Algorithm Find-prefix allows us to state whether $w$ is an inverse Lyndon word or not and, in the latter case, it finds the shortest prefix $z$ of $w$ such that $z = raurb$ where $r, u \in \Sigma^*$, $a, b \in \Sigma$, $a < b$.

**Proof:**

Let $n = |w|$. We have already shown that the while-loop will terminate. Evidently it terminates in two cases: when $w[i] < w[j]$, $j \leq n$, or when $j = n$ and $w[i] \geq w[n]$. Moreover, when we reach line (11), statement $S(j)$ must hold. In particular, $w[1..i-1] = w[h'_n..j-1]$, where $1 < h'_n = j - i + 1 \leq n$.

If $w[i] < w[j]$, the word $w[1..j]$ is a prefix of $w = w[1..n]$ which has the form $raurb$ where $r, u \in \Sigma^*$, $a = w[i] < w[j] = b$ and $r = w[1..i-1] = w[h'_n..j-1]$, if the two occurrences of $r$ (as a prefix and a suffix) in $w[1..j-1]$ do not overlap, otherwise it is its prefix $r'$ given by Lemma 2.2 applied to $r$. Moreover $raur = w[1..j-1]$ is an inverse Lyndon word, by part (b) of $S(j)$, which guarantees, by Corollary 3.1, that $raurb$ is the shortest prefix of $w$ with the above mentioned conditions.

If $j = n$ and $w[i] \geq w[n]$, then we prove that $w$ is an inverse Lyndon word by showing that $w$ has no prefix with the form $raurb$, $r, u \in \Sigma^*$, $a, b \in \Sigma$, $a < b$, thus by using Corollary 5.1. In turn, we prove the latter claim by arguments similar to that in the proof of Proposition 5.1. If $w$ had a prefix with the form $raurb$, $r, u \in \Sigma^*$, $a, b \in \Sigma$, $a < b$, then, by part (b) of $S(j) = S(n)$, we would have $w = raurb$, that is $w[n] = b$. Moreover, there would exist $t, t = |r| + 1$, such that $r = w[1..t-1] = w'[t..n-1]$ would be a prefix and a suffix of $w[1..n-1]$ and

$$w[t] = a < b = w[n] \leq w[i].$$  \hspace{1cm} (5.2)

By using part (a2) of $S(n)$, we would have $t' \geq h'_n$, hence, by $|r| = |w[1..t-1]| = |w'[t..n-1]| \leq |w[h'_n..n-1]| = |w[1..i-1]|$, we also have $t - 1 \leq i - 1$ and finally $t - 1 < i - 1$, by Eq. (5.2). In turn, since $t - 1 < i - 1$, the word $r = w[1..t-1] = w'[t..n-1]$ would be a proper prefix of $w[1..i-1]$ and, since $t' \geq h'_n$, the same word $r = w[1..t-1] = w'[t..n-1]$ would be a suffix of $w[h'_n..n-1] = w[1..i-1]$. In conclusion, $w[1..i] = r'aurb'$, where $b' \in \Sigma$, $a = w[t] < b = w[n] \leq w[i] = b'$, and $r' = r$ if the two occurrences of $r$ (as a prefix and a suffix) in $w[1..i-1]$ do not overlap, otherwise it is its prefix $r'$ given by Lemma 2.2 applied to $r$. Hence part (b) of $S(i+1)$ would be false, with $i + 1 \leq j = n$, a contradiction.

5.3 The failure function

We recall that, given a word, represented by the array $w[1..n]$, the failure function for $w$, is the function $f : \{1, \ldots , n\} \rightarrow \{0, \ldots , n-1\}$ such that

$$f(i) = \max\{k \mid k < i \text{ and } w[1..k] \text{ is a suffix of } w[1..i]\}.$$  

That is, $f(i)$ is the length of the longest prefix of $w[1..n]$ which is a proper suffix of $w[1..i]$. Thus, by Lemma 2.1, $f(i)$ is the length of the longest proper prefix of $w[1..i]$ which is a suffix of $w[1..i]$. It is known that by iterating the failure function $f$, we can enumerate all the prefixes $w[1..k]$ that are suffixes of a given prefix $w[1..i]$ (Lemma 5.2). A slightly more precise version of this result is given in Lemma 5.3.

**Definition 5.1:** For a positive integer $i$, we set

$$f^{(0)}(i) = i$$

$$f^{(\ell)}(i) = f(f^{(\ell-1)}(i)), \text{ for } \ell \geq 1$$

$$f^{*}(i) = \{i, f(i), f^{(2)}(i), \ldots , f^{(m)}(i)\}$$
where it is understood that the sequence in $f^*(i)$ stops when $f^{(m)}(i) = 0$ is reached.

**Lemma 5.1** For any positive integer $i$ and $\ell \geq 1$

$$f^{(\ell)}(i) = f(f^{(\ell-1)}(i)) = f^{(\ell-1)}(f(i)).$$

Consequently, $\text{Card}(f^*(f(i))) - 1 = \text{Card}(f^*(i)) - 2$.

**Proof:**

The proof is by induction on $\ell \geq 1$. Clearly $f^{(1)}(i) = f(f^{(0)}(i)) = f(i) = f^{(0)}(f(i))$. Let $\ell \geq 2$. Then, by using induction hypothesis,

$$f^{(\ell)}(i) = f(f^{(\ell-1)}(i)) = f(f(f^{(\ell-2)}(f(i)))) = f(f^{(\ell-2)}(f(i))) = f^{(\ell-1)}(f(i))$$

and the proof is complete. \(\blacksquare\)

The following result is proved in [8, Lemma 32.5] (Prefix-function iteration lemma).

**Lemma 5.2** Let $w$ be a word of length $n$ with failure function $f$. For any $i$, $1 \leq i \leq n$, we have $f^*(i) = \{k \mid k < i \text{ and } w[1..k] \text{ is a suffix of } w[1..i]\}$.

**Lemma 5.3** Let $w$ be a word of length $n$ with failure function $f$. For any $i$, $1 \leq i \leq n$, $f^*(i)$ lists the sequence of the lengths of all the words that are prefixes and suffixes of $w[1..i]$ in decreasing order. That is,

1. a word $x$ is a prefix and a suffix of $w[1..i]$ if and only if there is $k$, $0 \leq k \leq \text{Card}(f^*(i)) - 1$, such that $x = w[1..f^{(k)}(i)]$.
2. If $k < k'$, then $|w[1..f^{(k')}(i)]| < |w[1..f^{(k)}(i)]|$ and $w[1..f^{(k')}(i)]$ is a prefix and a suffix of $w[1..f^{(k)}(i)]$.

**Proof:**

(1) Let $x$ be a prefix and a suffix of $w[1..i]$. We prove that there is $k$, $0 \leq k \leq \text{Card}(f^*(i)) - 1$, such that $x = w[1..f^{(k)}(i)]$ by induction on $i$. If $i = 1$, then $x = w[1]$ or $x = 1$. Since $f^*(i) = \{1, 0\}$, the statement is proved for $i = 1$. Assume $i > 1$. If $x = w[1..i] = w[1..f^{(0)}(i)]$, then we have done. Otherwise, by the definition of the failure function, $x$ is a prefix and a suffix of $w[1..f(i)]$. Of course $f(i) < i$, hence, by induction hypothesis, there is $k$, $0 \leq k \leq \text{Card}(f^*(f(i))) - 1$, such that $x = w[1..f^{(k)}(f(i))]$. Thus, by Lemma 5.1 there is $k + 1$, $0 \leq k + 1 \leq \text{Card}(f^*(i)) - 1$, such that $x = w[1..f^{(k+1)}(i)]$. Conversely, by Lemma 5.2 if there is $k$, $0 \leq k \leq \text{Card}(f^*(i)) - 1$, such that $x = w[1..f^{(k)}(i)]$, then $x$ is a prefix and a suffix of $w[1..i]$.

(2) By the definition of the failure function, one has that the sequence $f^*(i)$ is strictly decreasing, that is, if $k < k'$, with $0 \leq k, k' \leq \text{Card}(f^*(i)) - 1$, then $f^{(k')}(i) < f^{(k)}(i)$ and consequently $|w[1..f^{(k')}(i)]| < |w[1..f^{(k)}(i)]|$. Therefore $w[1..f^{(k')}(i)]$ is a prefix and a suffix of $w[1..f^{(k)}(i)]$ by Lemma 2.1. \(\blacksquare\)

The longest proper prefix of $w[1..i]$ which is a suffix of $w[1..i]$ is also called a border of $w[1..i]$ [23]. It is known that there is an algorithm that outputs the array $f$ of $n = |w|$ integers such that $f(i)$ is the length of the border of $w[1..i]$ in time $O(n)$ (see [8] for a description of this procedure, called Compute-Prefix-Function or [23], where it is called Border).

**Example 5.3** Let us consider again $w = bbabbabbb$ of the Example 5.2. We have that $f = [0, 1, 0, 1, 2, 3, 4, 5, 2]$. 
5.4 Description of Find-bre

In this section we present **Find-bre**, which applies to the output \((x, y)\) of **Find-prefix**\((w)\) when \(w\) is not an inverse Lyndon word. In this case, \(x = \overline{p}p\), where \((p, \overline{p}) \in \text{Pref}_{bre}(w)\). As already said, the task of **Find-bre**\((x, y)\) is to find the shortest \(r\) such that \(x = raurb\), where \(r, u \in \Sigma^*\), \(a, b \in \Sigma\), \(a < b\). Hence, by Lemma 4.7, we have \(p = rau\) and \(\overline{p} = rb\). Therefore, **Find-bre**\((x, y)\) computes the prefix \(p\) and its bounded right extension \(\overline{p}\) and outputs the quadruple \((p, \overline{p}, y, |r|)\).

**Find-bre** uses the array \(f\) computed by **Border**\((raur)\).

**Algorithm 2: Find-bre**

Input : A pair of strings \((x, y)\), where \(w = xy\) is not an inverse Lyndon word, \(x = \overline{p}p = raurb\), with \((p, \overline{p}) \in \text{Pref}_{bre}(w)\), \(n = |raur| = |x| - 1\). The array \(f\) computed by **Border**\((raur)\).

Output: A quadruple \((x_1, x_2, y, k)\), where \((x_1, x_2, y, k) = (rau, rb, y, |r|)\).

1. \(i \leftarrow n;\)
2. \(\text{LAST} \leftarrow n + 1;\)
3. while \(i > 0\) do
   4. \[\text{if } w[f(i) + 1] < b\text{ then}\]
   5. \[\text{LAST} = f(i);\]
   6. \(i \leftarrow f(i);\)
7. \text{return} \((w[1..n - \text{LAST}], w[n - \text{LAST} + 1, n]b, y, \text{LAST})\); 

**Example 5.4** Let \(\Sigma = \{a, b\}\) with \(a < b\), let \(w = bbabbabbb\). In Example 5.2 we noticed that **Find-prefix**\((bbabbabbb)\) outputs \((bbabbabbb, 1)\). We can check that **Find-bre**\((bbabbabbb, 1)\) returns \((bbabba, bbb, 1, 2)\).

5.5 Correctness of Find-bre

In this section we prove that **Find-bre** does what it is claimed to do. To begin, notice that each time around the while-loop of lines (3) to (6), \(i\) decreases since \(f(i) < i\). Thus when \(i\) becomes zero, the loop condition \(i > 0\) will be false and the loop will terminate. Then, consider the following loop-invariant statement.

\[S(t):\] If we reach the loop test “\(i > 0\)” with the variable \(i\) having the value \(h\) and the variable \(\text{LAST}\) having the value \(k\), after \(t\) iterations of the while-loop, then

- \(h = f^t(n)\).
- \(k \geq h\). Precisely, if \(w[h + 1] < w[n + 1]\), then \(k = h\) else \(k > h\).

**Proposition 5.3** For any \(t \geq 0\), \(S(t)\) is true.

**Proof**: 
(Basis) Let us prove that \(S(t)\) is true for \(t = 0\). We reach the test after 0 iterations of the while-loop only when we enter the loop from the outside. Prior to the loop, lines (1) and (2) set \(i\) to \(n = f^0(n)\) and \(\text{LAST}\) to \(n + 1\). Therefore, we reach the test after 0 iterations of the while-loop with \(h = f^0(n)\) and \(k > h\). Thus, since \(w[h + 1] = w[n + 1]\), clearly \(S(0)\) is true.

(Induction) We suppose that \(S(t)\) is true and prove that \(S(t + 1)\) is true. Therefore, after \(t\) iterations of the while-loop, we reach the loop test “\(i > 0\)”, with the variable \(i\) having the value
\[ h = f^1(n) \]. We may assume \( h > 0 \) (otherwise we break out of the while-loop after \( t \) iterations or earlier and \( S(t+1) \) is clearly true, since it is a conditional expression with a false antecedent). Let us consider what happens when we run the \((t+1)\)th iteration of the while-loop and we execute the body of the while-loop with \( i \) having the value \( h \) and \( \text{LAST} \) having the value \( k \).

The variable \( i \) assumes value \( f(h) = f^{t+1}(n) \) on line (6). Moreover, if \( w[f(h)+1] = w[f^{t+1}(n)+1] < w[n+1] \), then \( \text{LAST} \) assumes value \( f^{t+1}(n) \) (lines (4)-(5)). Otherwise \( \text{LAST} \) remains unchanged, thus \( k \geq h = f^t(n) > f^{t+1}(n) \) (induction hypothesis). In both cases, \( S(t+1) \) is true.

**Proposition 5.4** Let \( w \in \Sigma^+ \) be a word which is not an inverse Lyndon word. Algorithm **Find-bre**, applied to the output \((x,y)\) of **Find-prefix**\( (w) \), outputs the quadruple \((p,\overline{p},y,|r|)\), where \( w = xy = pp\overline{y} \), \((p,\overline{p}) \in \text{Pref}_{bre}(w) \), \( p = rau \), \( \overline{p} = rb \).

**Proof:**
Let \( w \in \Sigma^+ \) be a word which is not an inverse Lyndon word. If \((x,y)\) is the output of **Find-prefix**\( (w) \), then \( x = pp\overline{y} \), where \((p,\overline{p}) \in \text{Pref}_{bre}(w) \), by Lemma 4.7 and Proposition 5.2. By Lemma 4.7 again, it suffices to prove that **Find-bre** outputs \((rau,rb,y,|r|)\), where \( r \) is the shortest prefix and suffix of \( z \) such that \( zb = x = pp\overline{y} = raurb \), with \( r,u \in \Sigma^* \), \( a,b \in \Sigma \), \( a < b \).

Of course after \( m = \text{Card}(f^*(n)) - 1 \) iterations, the while-loop of lines (3) to (6) terminates. Recall also that the sequence \( f^*(n) \) is strictly decreasing (Lemma 5.3). Let \( r \) be the shortest prefix and suffix of \( z \) such that \( w[r] < b = w[n+1] \). By Lemma 5.3 there is \( q \geq 0 \) such that \( |r| = f^q(n) \). By Proposition 5.3 \( S(q) \) is true. Therefore, \( i = f^q(n) = \text{LAST} \) since \( w[|r|] = w[i+1] = a < b = w[n+1] \). This value of \( \text{LAST} \) remains unchanged until we break out of the while-loop since, otherwise, for \( t > q \), we would have \( w[f^t(n)+1] = c < w[n+1] \) and a shorter word \( r' = w[1..f^t(n)] \) such that \( z = r'u'r' \), where \( u' \) starts with \( c \), a contradiction. Finally, **Find-bre** outputs \((w[1..n-\text{LAST}], w[n-\text{LAST}+1..n]b, y, \text{LAST}) \). Now \( w[n-\text{LAST}+1..n] \) is the suffix of \( z \) of length \( \text{LAST} = |r| \), hence \( w[n-\text{LAST}+1..n] = r \). Of course \( w[1..n-\text{LAST}] = rau \).

### 6 Computing ICFL in linear time

In this section we give a linear time algorithm, called **Compute-ICFL** to compute \( \text{ICFL}(w) \). For the sake of simplicity we present a recursive version of **Compute-ICFL**. In this case the correctness of the algorithm easily follows from the definition of ICFL. The output of **Compute-ICFL** is represented as a list denoted by \textit{list}.

Let us describe the high-level structure of algorithm **Compute-ICFL**\( (w) \). The algorithm firstly calls **Find-prefix**\( (w) \) (line (1)), that, in view of Proposition 5.2 allows us to state whether \( w \) is an inverse Lyndon word or not.

If \( w \) is an inverse Lyndon word, then **Find-prefix**\( (w) \) returns \((x,y)\) with \( x \) ending with \$ and **Compute-ICFL** stops and returns \((w)\) (lines (2)-(3)), according to Definition 4.3. If \( w \) is not an inverse Lyndon word, then **Find-prefix**\( (w) \) returns the pair \((x,y)\) such that \( w = pp\overline{y} \), \((p,\overline{p}) \in \text{Pref}_{bre}(w) \), and **Compute-ICFL** calls **Find-bre**\( (x,y) \) (line (4)). In turn, **Find-bre**\( (x,y) \) returns a quadruple \((x_1,x_2,y,\text{LAST}) \), where \( x_1 = p \), \( x_2 = \overline{p} \) and \( \text{LAST} = |r| \). Next, **Compute-ICFL** recursively calls itself on \( x_2y \) (line (5)) and returns \text{list} = \text{ICFL}(x_2y) = \text{ICFL}(\overline{p}y) \). Let \( z = m' \) be the first element of \textit{list} (line (6)). According to Definition 4.3 we have to test whether \( x_2 = \overline{p} = rb \leq_p m' = z \), that is if \( |z| > |r| = \text{LAST} \). This is done on line (7). If \( |z| > |r| = \text{LAST} \), then we add \( x_1 = p \) at the first position of \textit{list} (line (8)), otherwise
we replace \( z = m_1' \) in \( list \) with \( x_1z = pm_1' \) (line (10)). In both cases, \texttt{Compute-ICFL} returns \( list = \text{ICFL}(w) \).

It is worth of noting that there is no preprocessing of \( w \) for computing the failure function used by \texttt{Find-bre}(x, y). Each call to \texttt{Find-bre}(x, y) calls \texttt{Border}(x')}, where \((x, y)\) is the output of \texttt{Find-prefix}(w) and \( x' \) is the prefix of \( x \) of length \(|x| - 1\).

**Algorithm 3: Compute-ICFL**

| CALL                                      | RETURN                        |
|-------------------------------------------|-------------------------------|
| Compute-ICFL(\( cbabacaacbabaacb \))     | \( cbabacaacbaba, cbac \)    |
| Compute-ICFL(\( cbabacbac \))           | \( cbaba, cbac \)            |
| Compute-ICFL(\( cbac \))                 | \( cbac \)                   |

**Example 6.1** Let \( \Sigma = \{a, b, c\} \) with \( a < b < c \), let \( w = cbabacaacbabaacbac \), already considered in Example 4.5. Suppose we call \texttt{Compute-ICFL} on \( w \) and the empty list \( list \). The three tables below illustrate the sequence of calls made to \texttt{Compute-ICFL}, \texttt{Find-prefix} and \texttt{Find-bre} if we read the first column downward. For instance, since \texttt{Find-prefix}(\( cbac \)) returns \((\text{\( cbac \)}, 1)\), \texttt{Compute-ICFL}(\( cbac \)) returns \((\text{\( cbac \)})\) without invoking itself again and the recursion stops. \texttt{Compute-ICFL}(\( cbabacaacbabaacb \)) calls \texttt{Find-prefix} on \( cbabacaacbabaacb \), \texttt{Find-bre} on \((\text{\( cbabacaacbabaacb, ac \)})\) and then \texttt{Compute-ICFL}(\( cbabacbac \)) which returns \((\text{\( cbaba, cbac \)})\). Since \(|z| = |\text{\( cbac \)}| = 4 > \text{\( \text{LAST} = 3 \))\), \texttt{Compute-ICFL}(\( cbabacaacbabaacb \)) returns \((\text{\( cbaba, cbac \)})\). Finally, \texttt{Compute-ICFL}(\( w \)) calls \texttt{Find-prefix} on \( w \), \texttt{Find-bre} on \((\text{\( cbabacaacbabaacb, ac \)})\) and then \texttt{Compute-ICFL}(\( cbabacbac \)) which returns \((\text{\( cbaba, cbac \)})\). Since \(|z| = |\text{\( cbaba \)}| = 5 \leq \text{\( \text{LAST} = 6 \))\), \texttt{Compute-ICFL}(\( w \)) replaces \( cbaba \) with the concatenation of \( cbabaca \) and \( cbaba \) and returns \( list = (\text{\( cbabacaacbaba, cbac \)}) = \text{ICFL}(w) \).
6.1 Performance of Compute-ICFL

Let us compute the running time $T(n)$ of Compute-ICFL when $w$ has length $n$. We can check that the running time of $\text{Find-prefix}(w)$ is $O(|x|)$, when the output of the procedure is $(x,y)$. Then, Compute-ICFL calls the procedure $\text{Find-bre}(x,y)$, where $x = p \triangledown$ and $(p, \triangledown) \in \text{Pref-bre}(w)$. By Lemma 1.1 we know that $|\triangledown| \leq |p|$, and so the running time of $\text{Find-bre}$ is $O(|p| + |\triangledown|) = O(|p|)$. Let $\text{ICFL}(w) = (m_1, \ldots, m_k)$ and let $n_j$ be the length of $m_j$, $1 \leq j \leq k$. The recurrence for $T(n)$ is defined as $T(n) = T(n - n_1) + O(n_1)$, where $T(n_1)$ is $O(n_1)$, because there is no recursive call in this case. It is easy to see that the solution to this recurrence is $T(n) = \sum_{j=1}^{k} O(n_j) = O(n)$, since $w = m_1 \cdots m_k$.

7 Groupings

Let $(\Sigma, <)$ be a totally ordered alphabet. As we know, for any word $w \in \Sigma^*$, there are three sequences of words associated with $w$: the Lyndon factorization of $w$ with respect to the order $\prec$, denoted $\text{CFL}(w)$, the Lyndon factorization of $w$ with respect to the inverse lexicographic order $\prec_{in}$, denoted $\text{CFL}_{in}(w)$ and the inverse Lyndon factorization $\text{ICFL}(w)$ of $w$. In this section, we compare $\text{CFL}_{in}(w)$ and $\text{ICFL}(w)$.

We begin by proving some relations between inverse Lyndon words and anti-Lyndon words (Section 7.1). Then, we point out some relations between $\text{ICFL}(w)$ and $\text{CFL}_{in}(w)$. Precisely, starting with $\text{CFL}_{in}(w)$, we define a family of inverse Lyndon factorizations of $w$, called groupings of $\text{CFL}_{in}(w)$ (Section 7.2). We prove that $\text{ICFL}(w)$ is a grouping of $\text{CFL}_{in}(w)$ in Section 7.3.

Finally, we prove that Theorem 2.2 may be generalized to groupings when we refer to the sorting with respect to the inverse lexicographic order (Section 7.4).

7.1 Inverse Lyndon words and anti-Lyndon words

Let $(\Sigma, <)$ be a totally ordered alphabet, let $\prec_{in}$ be the inverse of $<$ and let $\prec_{in}$ be the inverse lexicographic order on $(\Sigma, <)$. The following proposition justifies the adopted terminology.

**Proposition 7.1** Let $(\Sigma, <)$ be a totally ordered alphabet. For all $x, y \in \Sigma^*$ such that $x \triangleright y$,

$$y \prec_{in} x \iff x \prec y.$$  

Moreover, in this case $x \preceq y$.

**Proof:**

Let $x, y \in \Sigma^*$ such that $x \triangleright y$. Assume $y \prec_{in} x$. Thus, by Definition 2.3 there are $a, b \in \Sigma$, with $b \prec a$, and $r, s, t \in \Sigma^*$ such that $y = rbs$, $x = rat$. Hence $a < b$ and $x < y$, by Definition 2.1. Conversely, if $x < y$ we have $y \prec_{in} x$ by a similar argument. The second part of the statement follows by Definition 2.1.

Lemma 7.1 is a dual version of item (2) in Lemma 2.3.

**Lemma 7.1** If $y \prec_{in} x$ and $x \triangleright y$, then $yu \prec_{in} xv$ for all words $u, v$.

**Proof:**

If $y \prec_{in} x$ and $x \triangleright y$, then $x \preceq y$, by Proposition 7.1. Then, by item (2) in Lemma 2.3 $xv \preceq yu$ for all words $u, v$. Since $xv \triangleright yu$, the conclusion follows by Proposition 7.1.

The following proposition characterizes the set $L_{in} = L(\Sigma^*, \prec_{in})$ of the anti-Lyndon words on $\Sigma^*$.
Proposition 7.2 A word $w \in \Sigma^*$ is in $L_{in}$ if and only if it is primitive and the largest one in its conjugacy class for the lexicographic order $<$ on $(\Sigma^*, <)$, i.e., if $w = uv$, with $u, v \neq 1$, then $w \succ vu$.

Proof: By Definition 2.2, if $w \in L_{in}$, then $w$ is nonempty and primitive. Moreover, if $w = uv$, with $u, v \neq 1$, then $w <_{in} vu$. Since $uv \not\succ vu$, by Proposition 7.1 one has $vu < w$, i.e., $w \succ vu$. A similar argument shows that if $w$ is a primitive nonempty word and $w$ is the largest one in its conjugacy class for the lexicographic order $<$ on $(\Sigma^*, <)$, then $w \in L_{in}$.

We state below a slightly modified dual version of Proposition 2.3. It shows that anti-Lyndon words are inverse Lyndon words.

Proposition 7.3 A word is in $L_{in}$ if and only if it is nonempty, unbordered and greater than all its proper nonempty suffixes, i.e., $w \neq 1$, $w$ is unbordered and $w \succ v$, for each $u, v \neq 1$ such that $w = uv$.

Proof: Let $w \subseteq L_{in} = L_{(\Sigma^*, <_{in})}$. Thus, $w$ is nonempty and unbordered (Definition 2.2, Proposition 2.2). Moreover, if $w = uv$, with $u, v \neq 1$, then $w <_{in} v$, by Proposition 2.3. In addition, $w \not\succ v$ since $w$ is unbordered and $|v| < |w|$. Hence, $w \succ v$ (Proposition 7.1).

Conversely, let $w = uv$ a nonempty unbordered word such that $w \succ v$, for each proper nonempty suffix $v$. Thus, $v \not\succ w$, hence $w <_{in} v$ (Proposition 7.1). By Proposition 2.3 the word $w$ is in $L_{in}$.

Of course there are inverse Lyndon words which are not anti-Lyndon words. For instance consider $\Sigma = \{a, b\}$, with $a < b$. The word $bab$ is an inverse Lyndon words but it is not unbordered, thus it is not an anti-Lyndon word.

The following result give more precise relations between words in $L_{in}$ and their proper nonempty suffixes.

Proposition 7.4 If $v$ is a proper nonempty suffix of $w \in L_{in}$, then $v \ll w$.

Proof: Let $w = uv \subseteq L_{in}$, with $u, v \neq 1$. By Proposition 7.3 we have $v \ll w$. Moreover, since $w$ is unbordered, we have $v \not\succ w$, hence $v \ll w$.

Some compositional properties of the inverse Lyndon words are proved below.

Proposition 7.5 For any $w \in L_{in}$ and $h \geq 1$, the word $w^h$ is an inverse Lyndon word on $(\Sigma^*, <)$.

Proof: Let $w \in L_{in}$ and let $vw^i$ be a proper nonempty suffix of $w^h$, $h \geq 1$. If $v = 1$, then $0 < i < h$, hence $w^i$ is a nonempty proper prefix of $w^h$. By Definition 2.4 $w^h \succ w^i$. Otherwise, $v$ is a proper nonempty suffix of $w$. Thus, by Proposition 7.4 $v \ll w$ and by item (2) in Lemma 2.3 $vw^i \ll w^h$, i.e., $w^h \succ vw^i$.

Proposition 7.6 Let $\ell_1, \ell_2 \in L_{in}$. If $\ell_2$ is a proper prefix of $\ell_1$, then $\ell_1 \ell_2$ is an inverse Lyndon word.
Proof:
Let \( \ell_1, \ell_2 \) be as in the statement. Set \( \ell_1 \ell_2 = \ell_2 x \ell_2 = w \). If \( s \) is a nonempty proper suffix of \( w \), then one of the following cases holds

1. \( s \) is a nonempty proper suffix of \( \ell_2 \)
2. \( s = \ell_2 \)
3. \( s = s' \ell_2 \), where \( s' \) is a nonempty proper suffix of \( \ell_1 \).

Assume that case (1) holds. Thus \( s \ll \ell_2 \), by Proposition~7.4. Moreover \( \ell_2 \asymp s \), since \( \ell_2 \in L_{in} \). By item (2) in Lemma~2.3, \( w = \ell_2(x \ell_2) \succ s \). Of course, \( w = \ell_2 x \ell_2 \succ \ell_2 \) (case (2)), hence assume that case (3) holds, i.e., \( s = s' \ell_2 \), where \( s' \) is a nonempty proper suffix of \( \ell_1 \). Arguing as before, \( s' \ll \ell_1 \) (Proposition~7.4) and \( \ell_1 \asymp s' \), since \( \ell_1 \in L_{in} \). By item (2) in Lemma~2.3, \( w = \ell_1 \ell_2 \succ s' \ell_2 \).

Example 7.1 Let \( \ell_1, \ldots, \ell_h \) be words in \( L_{in} \) which form a non-increasing chain \( \ell_1 \geq_p \ell_2 \geq_p \cdots \geq_p \ell_h \) with respect to the prefix order, i.e., \( \ell_i \) is a prefix of \( \ell_{i+1} \), \( 1 < i < h \). The word \( \ell_1 \cdots \ell_h \) is not necessarily an inverse Lyndon word. As an example, let \( \Sigma = \{a, b\} \) with \( a < b \). The sequence \( baa, ba, b \) is such that \( baa \geq_p ba \geq_p b \). The word \( baabab \) is not an inverse Lyndon word since \( baabab \prec bab \).

7.2 A family of inverse Lyndon factorizations of words

Groupings of \( CFL_{in}(w) \) are special inverse Lyndon factorizations. They are constructed in a very natural way. We first give some needed definitions and results.

Definition 7.1 Let \( w \in \Sigma^+ \), let \( CFL_{in}(w) = (\ell_1, \ldots, \ell_h) \) and let \( 1 \leq r < s \leq h \). We say that \( \ell_r, \ell_{r+1}, \ldots, \ell_s \) is a non-increasing maximal chain with respect to the prefix order in \( CFL_{in}(w) \), abbreviated \( PMCI \), if \( \ell_r \geq_p \ell_{r+1} \cdots \geq_p \ell_s \). Moreover, if \( r > 1 \), then \( \ell_{r-1} \geq_p \ell_r \), if \( s < h \), then \( \ell_s \not\geq_p \ell_{s+1} \). Two \( PMCI \) \( C_1 = (\ell_r, \ell_{r+1}, \ldots, \ell_s) \), \( C_2 = (\ell_{r'}, \ell_{r'+1}, \ldots, \ell_{s'}) \) are consecutive if \( r' = s + 1 \) (or \( r = s' + 1 \)).

Lemma 7.2 Let \( x, y \) be nonempty words such that \( x \geq_{in} y \). Then either \( x \geq_{p} y \) or \( x \ll y \).

Proof:
Let \( x, y \) be nonempty words such that \( x \geq_{in} y \). Therefore \( x \) is not a proper prefix of \( y \). If \( y \) is not a prefix of \( x \), then \( x \asymp y \) and \( y \ll_{in} x \). Hence, by Proposition~7.1, we have \( x \ll y \).

The following is a direct consequence of Lemma~7.2.

Proposition 7.7 Let \( w \in \Sigma^+ \), let \( CFL_{in}(w) = (\ell_1, \ldots, \ell_h) \). Then

\[ (\ell_1, \ldots, \ell_h) = (C_1, \ldots, C_t), \]

where any \( C_j, 1 \leq j \leq t, \) is a \( PMCI \) in \( CFL_{in}(w) \). Moreover, \( C_j, C_{j+1} \) are consecutive and \( \ell \ll \ell' \), where \( \ell \) is the last word in \( C_j \) and \( \ell' \) is the first word in \( C_{j+1} \), for \( 1 \leq j \leq t - 1 \).

Proof:
Let \( w \in \Sigma^+ \) and let \( CFL_{in}(w) = (\ell_1, \ldots, \ell_h) \), i.e.,

\[ w = \ell_1 \cdots \ell_h, \text{ with } \ell_j \in L_{in} \text{ and } \ell_1 \geq_{in} \cdots \geq_{in} \ell_h \quad (7.1) \]
By Lemma 7.2, each symbol $\succeq_{in}$ in Eq. (7.1) may be replaced either by $\succeq_p$ or by $\ll$. Therefore, the conclusion follows.

The definition of a grouping of $\text{CFL}_{in}(w)$ is given below in two steps. We first define the grouping of a $\mathcal{PMCI}$. Then a grouping of $\text{CFL}_{in}(w)$ is obtained by changing each $\mathcal{PMCI}$ with one of its groupings.

**Definition 7.2** Let $\ell_1, \ldots, \ell_h$ be words in $L_{in}$ such that $\ell_i$ is a prefix of $\ell_{i-1}$, $1 < i \leq h$. We say that $(m_1, \ldots, m_k)$ is a grouping of $(\ell_1, \ldots, \ell_h)$ if the following conditions are satisfied.

1. $m_j$ is an inverse Lyndon word which is a product of consecutive $\ell_q$, $1 \leq j \leq k$.
2. $\ell_1 \cdots \ell_h = m_1 \cdots m_k$,
3. $m_1 \ll \ldots \ll m_k$.

We now extend Definition 7.2 to $\text{CFL}_{in}(w)$.

**Definition 7.3** Let $w \in \Sigma^+$ and let $\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$. We say that $(m_1, \ldots, m_k)$ is a grouping of $\text{CFL}_{in}(w)$ if it can be obtained by replacing any $\mathcal{PMCI}$ $C$ in $\text{CFL}_{in}(w)$ by a grouping of $C$.

Proposition 7.8 shows that groupings of $\text{CFL}_{in}(w)$ are inverse Lyndon factorizations of $w$. However, as Example 7.2 shows, there are inverse Lyndon factorizations which are not groupings.

**Example 7.2** Let $\Sigma = \{a, b, c, d\}$ with $a < b < c < d$, and let $w = dabadabdadac$. Thus, $d <_{in} c <_{in} b <_{in} a$ and $\text{CFL}_{in}(w) = (daba, dab, dab, dadac)$. The two sequences $(daba, dabdadabdadac)$, $(dabadab, dadab, dadac)$ are both inverse Lyndon factorizations of $w$ and moreover $\text{ICFL}(w) = (daba, dadab, dadac)$ (Examples 4.1, 4.6). However, $\text{ICFL}(w)$ is a grouping of $\text{CFL}_{in}(w)$, whereas $(dabadab, dadab, dadac)$ is not a grouping of $\text{CFL}_{in}(w)$. In Example 4.1 we also considered the following two inverse Lyndon factorizations of $z = dabdadacdbdbd$:

$$(dab)(dadac)(db)(dc) = (dabda)(dac)(dadbdc)$$

It is easy to see that $\text{CFL}_{in}(z) = (dabda, dac, dbdb)$ and it is a grouping. On the contrary, $(dab, dadacdb, db, dc)$ is not a grouping of $z$. Notice that $\text{CFL}_{in}(z) = \text{ICFL}(z)$ (see Corollary 7.1).

**Proposition 7.8** Let $w \in \Sigma^+$. If $(m_1, \ldots, m_k)$ is a grouping of $\text{CFL}_{in}(w)$, then $(m_1, \ldots, m_k)$ is an inverse Lyndon factorization of $w$.

**Proof:**

Let $w \in \Sigma^+$ and let $\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$. Therefore, by Proposition 7.7, we have

$$(\ell_1, \ldots, \ell_h) = (C_1, \ldots, C_t),$$

where any $C_j$, $1 \leq j \leq t$, is a $\mathcal{PMCI}$ in $\text{CFL}_{in}(w)$. Moreover, $C_j, C_{j+1}$ are consecutive and $\ell \ll \ell'$, where $\ell$ is the last word in $C_j$ and $\ell'$ is the first word in $C_{j+1}$, for $1 \leq j \leq t-1$.

Now let $(m_1, \ldots, m_k)$ be a grouping of $\text{CFL}_{in}(w)$. Any $m_j$ is an inverse Lyndon word since it is an element of a grouping of a $\mathcal{PMCI}$ in $\text{CFL}_{in}(w)$. Then, let $S_j$ be the product of the words in $C_j$, $1 \leq j \leq t$. Thus, it is clear that

$$w = \ell_1 \cdots \ell_h = S_1 \cdots S_t = m_1 \cdots m_k.$$
Finally, set $C_j = (\ell_i, \ldots, \ell_g)$ and $C_{j+1} = (\ell_{g+1}, \ldots, \ell_f)$, for $1 \leq j \leq t - 1$. Thus $\ell_g \ll \ell_{g+1}$.

It suffices to show that if $(m_1, \ldots, m_s)$, $(m_{s+1}, \ldots, m_v)$ are the groupings of $C_j$ and $C_{j+1}$ that replace them in $(m_1, \ldots, m_k)$, then $m_s \ll m_{s+1}$. But this is clear since $\ell_g$ is a suffix of $m_s$, thus also a prefix of $m_s$ ($C_j$ is a $\mathcal{PMCI}$) and $\ell_{g+1}$ is a prefix of $m_{s+1}$. Thus, by item (2) in Lemma 2.3, $\ell_g \ll \ell_{g+1}$ implies $m_s \ll m_{s+1}$.

\section*{7.3 ICFL(w) is a grouping of CFL\textsubscript{in}(w)}

Let us outline how we prove below that ICFL(w) is a grouping of CFL\textsubscript{in}(w). Let ICFL(w) = $(m_1, \ldots, m_k)$, let CFL\textsubscript{in}(w) = $(\ell_1, \ldots, \ell_h)$ and let $\ell_1, \ldots, \ell_q$ be a $\mathcal{PMCI}$ in CFL\textsubscript{in}(w), $1 \leq q \leq k$. The proof will be divided into four steps.

1. We prove that $m_1$ cannot be a proper prefix of $\ell_1$ (Proposition 7.9).
2. We prove that if $m_1$ is a prefix of $\ell_1 \cdots \ell_q$, then $m_1 = \ell_1 \cdots \ell_q'$, for some $q' \leq q$ (Proposition 7.11).
3. We prove that $\ell_1 \cdots \ell_q$ cannot be a proper prefix of $m_1$ (Proposition 7.12).
4. We complete the proof by induction on $|w|$ (Proposition 7.13).

We say that a sequence of nonempty words $(m_1, \ldots, m_k)$ is a factorization of $w$ if $w = m_1 \cdots m_k$.

It is worth of noting that steps (1) and (2) are proved under the more general hypothesis that $(m_1, \ldots, m_k)$ is a factorization of $w$ such that $m_1 \ll m_2$ and, for step (2), where $m_1$ is an inverse Lyndon word. Proposition 7.9 and Corollary 7.1 deals with two extremal cases where there is only one grouping of CFL\textsubscript{in}(w), namely ICFL(w).

\begin{proposition}
Let $(\Sigma, <)$ be a totally ordered alphabet. Let $w \in \Sigma^+$ and let CFL\textsubscript{in}(w) = $(\ell_1, \ldots, \ell_h)$. If $w$ is an inverse Lyndon word, then either $w$ is unbordered or $\ell_1, \ldots, \ell_h$ is a $\mathcal{PMCI}$ in CFL\textsubscript{in}(w). In both cases ICFL(w) = (w) is the unique grouping of CFL\textsubscript{in}(w).
\end{proposition}

\textbf{Proof}:

Let $w \in \Sigma^+$ be an inverse Lyndon word and let CFL\textsubscript{in}(w) = $(\ell_1, \ldots, \ell_h)$. We know that ICFL(w) = (w) (Definition 1.3). By Proposition 7.3 if $w$ is unbordered, then $w$ is an anti-Lyndon word. Hence, by item (iii) in Lemma 2.3 CFL\textsubscript{in}(w) = (w) and of course this is the unique grouping of CFL\textsubscript{in}(w).

Otherwise, $w$ is bordered and, again by Proposition 7.3 $h > 1$. By contradiction assume that $\ell_1, \ldots, \ell_h$ is not a $\mathcal{PMCI}$ in CFL\textsubscript{in}(w). By Lemma 7.2 there would be a smallest $q$, $1 \leq q \leq h - 1$ such that

$$\ell_1 \geq_p \cdots \geq_p \ell_q \ll \ell_{q+1}$$

Hence, by item (2) in Lemma 2.3 we would have $w \ll \ell_{q+1} \cdots \ell_h$, which is a contradiction since $w$ is an inverse Lyndon word and $\ell_{q+1} \cdots \ell_h$ is a proper nonempty suffix of $w$. Thus, $\ell_1, \ldots, \ell_h$ is a $\mathcal{PMCI}$ in CFL\textsubscript{in}(w) and ICFL(w) = (w) is a grouping of CFL\textsubscript{in}(w). By contradiction, assume that $(m_1, \ldots, m_k)$, $k \geq 2$, is another grouping of CFL\textsubscript{in}(w). Therefore, $m_1 \ll m_2$ and by item (2) in Lemma 2.3 we would have $w \ll m_2 \cdots m_k$, which is a contradiction since $w$ is an inverse Lyndon word and $m_2 \cdots m_k$ is a suffix of $w$.

\begin{proposition}
Let $(\Sigma, <)$ be a totally ordered alphabet. Let $w \in \Sigma^+$ and let CFL\textsubscript{in}(w) = $(\ell_1, \ldots, \ell_h)$. For any factorization $(m_1, \ldots, m_k)$ of $w$, with $k > 1$, if $m_1 \ll m_2$, then $m_1$ cannot be a proper prefix of $\ell_1$.
\end{proposition}

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Proof:
Let \((m_1, \ldots, m_k)\) a factorization of \(w\) such that \(m_1 \ll m_2\). By contradiction assume that \(m_1\) is a proper prefix of \(\ell_1\). Therefore, there are two nonempty words \(x, y\) such that \(m_1 = x, \ell_1 = xy\), with \(y\) being a nonempty prefix of \(m_2\). Thus, on one hand, by Proposition 7.4 we have \(y \ll \ell_1\). On the other hand, \(m_1 = x \ll m_2\), thus \(x = m_1 \ll m_2 \cdots m_k = yv\), for a word \(v\), by item (2) in Lemma 2.3. Consequently, \(x = ras, yv = rbt\), with \(a < b\). Hence either \(y\) is a prefix of \(r\), which is a contradiction since \(\ell_1 = xy = rasy\) is unbordered (Proposition 7.3), or \(rb\) is a prefix of \(y\), which is once again a contradiction since we would have \(x \ll y\) and consequently, by item (2) in Lemma 2.3, \(\ell_1 \ll y\).

Proposition 7.11 Let \((\Sigma, <)\) be a totally ordered alphabet. Let \(w \in \Sigma^+\) and let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\). Let \((m_1, \ldots, m_k)\) be a factorization of \(w\) such that \(k \geq 1\), \(m_1\) is an inverse Lyndon word and \(m_1 \ll m_2\). Let \(\ell_1, \ldots, \ell_q\) be a \(\mathcal{P}MCI\) in \(\text{CFL}_{in}(w)\), \(1 \leq q \leq h\). If \(m_1\) is a prefix of \(\ell_1 \cdots \ell_q\), then \(m_1 = \ell_1 \cdots \ell_{q'}\), for some \(q' \leq q\).

Proof:
Let \(w \in \Sigma^+\) and let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\). Let \((m_1, \ldots, m_k)\) be a factorization of \(w\) such that \(k \geq 1\), \(m_1\) is an inverse Lyndon word and \(m_1 \ll m_2\). Let \(\ell_1, \ldots, \ell_q\) be a \(\mathcal{P}MCI\) in \(\text{CFL}_{in}(w)\), \(1 \leq q \leq h\).

By contradiction, assume that there are two nonempty words \(x, y\) such that \(m_1 = \ell_1 \cdots \ell_{j-1}x, xy = \ell_j, 1 \leq j \leq q\) (where it is understood that \(m_1 = x\) when \(j = 1\)). By Proposition 7.10 we have \(j > 1\).

Since \(m_1 \ll m_2\), we have \(m_1 = \ell_1 \cdots \ell_{j-1}x \ll m_2 \cdots m_k = yv\), for a word \(v\). Hence, there are words \(r, s, t \in \Sigma^*\) and letters \(a, b \in \Sigma\), with \(a < b\) such that \(m_1 = ras, m_2 = yv = rbt\). If \(|r| < |y|\), then \(rb\) is a prefix of \(y\), hence it is a factor of \(\ell_j\) and so of \(\ell_{j-1}\). Hence \(rb\gamma\) is a suffix of \(m_1\), for a word \(\gamma, rb\gamma \neq rn_1 = ras\), and \(m_1 = ras \ll rb\gamma\), a contradiction, since \(m_1\) is an inverse Lyndon word. Thus \(|r| \geq |y|\), i.e., \(y\) is a prefix of \(r\) and thus it is a prefix of \(m_1\). The word \(\ell_j = xy\) is also a prefix of \(m_1\), hence \(y\) and \(\ell_j = xy\) are comparable for the prefix order. Therefore, \(y\) is both a nonempty proper prefix and a suffix of \(\ell_j\), which implies, by Lemma 2.2 that \(\ell_j = xy\) is not unbordered, a contradiction since \(\ell_j \in L_{in}\) (see Proposition 7.3).

Proposition 7.12 Let \((\Sigma, <)\) be a totally ordered alphabet. Let \(w \in \Sigma^+\), let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\) and let \(\text{ICFL}(w) = (m_1, \ldots, m_k)\). Let \(\ell_1, \ldots, \ell_q\) be a \(\mathcal{P}MCI\) in \(\text{CFL}_{in}(w)\), \(1 \leq q \leq h\). Then \(m_1 = \ell_1 \cdots \ell_{q'}\), for some \(q' \leq q\).

Proof:
Let \(w \in \Sigma^+\), let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\), and let \(\text{ICFL}(w) = (m_1, \ldots, m_k)\). We prove the statement by induction on \(|w|\). If \(|w| = 1\), then \(w\) is an inverse Lyndon word and we have done, by Proposition 7.9. Hence assume \(|w| > 1\). If \(w\) is an inverse Lyndon word, then the proof is ended, once again by Proposition 7.9. Therefore, assume that \(w\) is not an inverse Lyndon word.

Let \(\ell_1, \ldots, \ell_q\) be a \(\mathcal{P}MCI\) in \(\text{CFL}_{in}(w)\), \(1 \leq q \leq h\). Since \(m_1\) and \(\ell_1 \cdots \ell_q\) are both prefixes of \(w, m_1\) and \(\ell_1 \cdots \ell_q\) are comparable for the prefix order. By contradiction, assume that \(m_1\) violates the statement. Therefore, by Propositions 7.10 and 7.11 \(m_1\) is not a prefix of \(\ell_1 \cdots \ell_q\). Hence, \(\ell_1 \cdots \ell_q\) is a proper prefix of \(m_1\), i.e., there are two words \(x, y\) and \(j\), with \(q < j \leq h\), such that \(m_1 = \ell_1 \cdots \ell_{j-1}x, xy = \ell_j\). Moreover, \(x \neq 1\) if \(j - 1 = q\) and \(\ell_q \ll \ell_{q+1}\). We must have \(j - 1 = q\) (thus \(x \neq 1\) also) and \(y \neq 1\). Indeed, otherwise \(j - 1 > q\) or \(x = \ell_{q+1}\). In both cases, \(\ell_{q+1} \cdots \ell_{j-1}x\) would be a proper nonempty suffix of \(m_1\) and \(\ell_q\) is a prefix of \(\ell_1\), thus \(\ell_q\) would be a prefix of \(m_1\). By item (2) in Lemma 2.3 applied to \(\ell_q \ll \ell_{q+1}\), we would have
$$m_1 \ll \ell_{q+1} \cdots \ell_{j-1} x, \text{ a contradiction since } m_1 \text{ is an inverse Lyndon word.} \text{ In conclusion, there are words } y', x \neq 1, \text{ and } y \neq 1 \text{ such that}$$

$$m_1 = \ell_1 \cdots \ell_q x, \quad \ell_{q+1} = xy, \quad m_2 \cdots m_k = yy', \quad \ell_1 \geq_p \cdots \geq_p \ell_q \ll \ell_{q+1} \quad (7.2)$$

Let $$(p, \bar{p}) \in \text{Pref}_{br}(w)$$. Let $$v \in \Sigma^+$$ be such that $$w = pv$$ and let $$\text{ICFL}(v) = (m_1', \ldots, m_k')$$. By Definition 4.3, one of the following two cases holds

1. $$m_1 = p$$
2. $$m_1 = pm_1'$$. 

In both cases, $$p$$ is a prefix of $$\ell_1 \cdots \ell_q x$$. Assume $$m_1 = pm_1'$$. If $$p = \ell_1 \cdots \ell_j$$, with $$j \leq q$$, then $$v = \ell_{j+1} \cdots \ell_h$$ and, by Theorem 2.1, $$\text{CFL}_{br}(v) = (\ell_{j+1}, \ldots, \ell_h)$$. Then, by Eq. (7.2), we would have $$m_1' = \ell_{j+1} \cdots \ell_q x, \quad \ell_{q+1} = xy, \quad \ell_q \ll \ell_{q+1}, \text{ and } x \neq 1, \text{ in contradiction with induction hypothesis applied to } v$$. On the other hand, by Definitions 4.2 and Definition 4.3, there are words $$p, v \in \Sigma, \text{ with } a < b$$ such that

$$p = ras, \quad \bar{p} = rb, \quad x_2 m_2 \cdots m_k = rbt \quad (7.4)$$

Since $$\ell_q \ll \ell_{q+1}$$, there are words $$z, f, g \in \Sigma^*$$ and letters $$c, d \in \Sigma$$, with $$c < d$$ such that

$$\ell_q = zcf, \quad \ell_{q+1} = xy = zdg \quad (7.5)$$

Observe that $$x$$ and $$zd$$ are both prefixes of $$\ell_{q+1}$$, hence they are prefix-comparable. If $$zd$$ would be a prefix of $$x$$, then for a word $$\gamma$$, $$zd \gamma$$ would be a proper nonempty suffix of $$m_1$$ (see Eq. (7.2)) and $$\ell_q$$ would be a prefix of $$\ell_1$$, thus of $$m_1$$ such that $$\ell_q \ll zd \gamma$$ (see Eq. (7.5)). By item (2) in Lemma 2.3 applied to $$\ell_q \ll zd \gamma$$, we would have $$m_1 \ll zd \gamma$$, a contradiction since $$m_1$$ is an inverse Lyndon word. Therefore, $$x$$ is a proper prefix of $$zd$$, i.e., there is $$z' \in \Sigma^*$$ such that

$$z = xz' = x_1 x_2 z'. \quad (7.6)$$

Eqs. (7.4) and (7.5) yield $$xy = zdg = xz'dg$$, therefore $$z'$$ is a prefix of $$y$$ and thus $$x_2 z'$$ is a prefix of $$x_2 m_2 \cdots m_k$$ (see Eq. (7.2)). On the other hand $$rb$$ is also a prefix of $$x_2 m_2 \cdots m_k$$ (see Eq. (7.4)). Thus, $$rb$$ and $$x_2 z'$$ are prefix-comparable and one of the following two cases is satisfied.

(i) $$rb$$ is a prefix of $$x_2 z'$$,

(ii) $$x_2 z'$$ is a prefix of $$r$$.

Assume that case (i) holds. In this case, since $$x_2 z'$$ is a suffix of $$z$$ (Eq. (7.6)) and $$z$$ is a prefix of $$\ell_q$$ (Eq. (7.5)), the word $$rb$$ would be a factor of $$\ell_q$$. Thus, by Eq. (7.2), there would be $$\gamma \in \Sigma^*$$ such that $$rb \gamma$$ would be a proper nonempty suffix of $$m_1$$. Since $$ras$$ is a prefix of $$p$$ (Eq. (7.4)), the word $$ras$$ would be a prefix of $$m_1$$ and we would have $$m_1 \ll rb \gamma$$, a contradiction since $$m_1$$ is an inverse Lyndon word. Case (ii) also leads to a contradiction. Indeed, set $$p' = \ell_1 \cdots \ell_q$$. Then, by Eqs. (7.3) and (7.6), $$px x_2 z' = p' x_1 x_2 z' = p' z$$, if $$x_2 z'$$ is a prefix of $$r$$, then $$px x_2 z' = p' z$$ is a
prefix of $pr$, thus $px_2z' = p'z$ is a proper prefix of $p\bar{p}$ (Eq. (7.4)), hence $px_2z' = p'z$ is an inverse Lyndon word, as all nonempty prefixes of $p'z$, by Definition 4.2. Moreover, by Eq. (7.5), $zc$ is a prefix of $\ell_q$, thus of $p'$. Hence we have $p' = zcf'$, for a word $f'$, and $p'zd$ is not an inverse Lyndon word since $p'zd \ll zd$. By Definition 4.2, the pair $(p', zd) \in \text{Pref}(w)$. Since $x_1 \neq 1$, we have $p' \neq p = p'x_1$, in contradiction with Proposition 4.1.

**Proposition 7.13** Let $(\Sigma, <)$ be a totally ordered alphabet. For any $w \in \Sigma^+$, ICFL($w$) is a grouping of CFL$_\text{in}(w)$.

**Proof:**
Let $w \in \Sigma^+$, let CFL$_\text{in}(w) = (\ell_1, \ldots, \ell_h)$, and let ICFL($w$) = $(m_1, \ldots, m_k)$. The proof is by induction on $|w|$. If $|w| = 1$, then $w$ is an inverse Lyndon word and we have done, by Proposition 7.9. Hence assume $|w| > 1$.

If $w$ is an inverse Lyndon word, once again by Proposition 7.9, ICFL($w$) = $(\ell_1 \cdots \ell_h)$ is a grouping of CFL$_\text{in}(w)$. Therefore, assume that $w$ is not an inverse Lyndon word. By Propositions 7.10, 7.11, there is $j$, $1 \leq j \leq h$ such that $m_1 = \ell_1 \cdots \ell_j$, where $\ell_1 \geq_p \cdots \geq_p \ell_j$.

Let $(p, \bar{p}) \in \text{Pref}(w)_{\text{bre}}$. Let ICFL($v$) = $(m'_1, \ldots, m'_{k'})$, where $v \in \Sigma^+$ is such that $w = pv$. By induction hypothesis, $(m'_1, \ldots, m'_{k'})$ is a grouping of CFL$_\text{in}(v)$. By Definition 4.3, one of the following two cases holds

1. $m_1 = p$, $(m_2, \ldots, m_k) = (m'_1, \ldots, m'_{k'})$, i.e., $k = k' + 1$, $m_j = m'_{j-1}$, $2 \leq j \leq k$.
2. $m_1 = pm'_1$, $(m_2, \ldots, m_k) = (m'_2, \ldots, m'_{k'})$, i.e., $k = k'$, $m_j = m'_{j}$, $2 \leq j \leq k$.

(Case 1). In this case, since $p = m_1 = \ell_1 \cdots \ell_j$, we have $v = \ell_{j+1} \cdots \ell_h$, where any $\ell_g$ is in $L_v$ and $\ell_{g+1} \succeq_{in} \cdots \succeq_{in} \ell_h$. By Theorem 2.1, CFL$_\text{in}(v) = (\ell_{j+1}, \ldots, \ell_h)$. Consequently, if $(\ell_{g+1}, \ldots, \ell_h) = (C_1, \ldots, C_t)$, where any $C_j$, $1 \leq j \leq t$, is a PMCI in CFL$_\text{in}(v)$, then $(\ell_1, \ldots, \ell_h) = (C, C_1, \ldots, C_t)$ where $C = \ell_1, \ldots, \ell_j$, any $C_j$, $2 \leq j \leq t$, is a PMCI in CFL$_\text{in}(w)$ and either $C, C_1$ represents two PMCI in CFL$_\text{in}(w)$ or it represents a single PMCI in CFL$_\text{in}(w)$. In both cases, since $m_1 = \ell_1 \cdots \ell_j$ and given that $(m_2, \ldots, m_k) = (m'_1, \ldots, m'_{k'})$ is a grouping of CFL$_\text{in}(v)$, we conclude that ICFL($w$) is a grouping of CFL$_\text{in}(w)$.

(Case 2). Set CFL$_\text{in}(v) = \ell'_{1}, \ldots, \ell'_{h'}$, and let $j'$ be such that $m'_1 = \ell'_{1} \cdots \ell'_{j'}$, $\ell'_{1} \geq_p \cdots \geq_p \ell'_{j'}$.

Hence

\[(pm'_1)(m_2 \cdots m_k) = (\ell_1 \cdots \ell_j)(\ell_{j+1} \cdots \ell_h) = (pl'_{1} \cdots l'_{j'})(l'_{j'+1} \cdots l'_{h'}) \quad (7.7)\]

Since $(pm'_1) = (\ell_1 \cdots \ell_j) = (pl'_{1} \cdots l'_{j'})$, Eq. (7.7) implies

\[(m_2 \cdots m_k) = (\ell_{j+1} \cdots \ell_h) = (l'_{j'+1} \cdots l'_{h'}) \quad (7.8)\]

Arguing as before, all the words $\ell_g, l'_g$ are in $L_v$ and $\ell_{g+1} \succeq_{in} \cdots \succeq_{in} \ell_h$, $l'_{g+1} \succeq_{in} \cdots \succeq_{in} l'_{h'}$, thus by Theorem 2.1 (applied to $v' = m_2 \cdots m_k = m'_2 \cdots m'_{k'})$, Eq. (7.8) implies $j = j'$, $h = h'$, and $\ell_g = l'_g$, $j \leq g \leq h$. Moreover, CFL$_\text{in}(v') = (\ell_{j+1}, \ldots, \ell_h)$ and $(m_2, \ldots, m_k) = (m'_2, \ldots, m'_{k'})$ is a grouping of CFL$_\text{in}(v')$. The rest of the proof runs as in Case (1). Namely, if $(\ell_{g+1}, \ldots, \ell_h) = (C_1, \ldots, C_t)$, where any $C_j$, $1 \leq j \leq t$, is a PMCI in CFL$_\text{in}(v')$, $(\ell_1, \ldots, \ell_h) = (C, C_1, \ldots, C_t)$ where $C = \ell_1, \ldots, \ell_j$, any $C_j$, $2 \leq j \leq t$, is a PMCI in CFL$_\text{in}(w)$ and either $C, C_1$ represents two PMCI in CFL$_\text{in}(w)$ or it represents a single PMCI in CFL$_\text{in}(w)$. In both cases, since $m_1 = \ell_1 \cdots \ell_j$ and given that $(m_2, \ldots, m_k) = (m'_2, \ldots, m'_{k'})$ is a grouping of CFL$_\text{in}(v')$, we conclude that ICFL($w$) is a grouping of CFL$_\text{in}(w)$.
Corollary 7.1 Let \((\Sigma, \prec)\) be a totally ordered alphabet. Let \(w \in \Sigma^+\) and let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\), with \(h > 1\). If \(\ell_1 \ll \cdots \ll \ell_h\), then \(\text{ICFL}(w) = \text{CFL}_{in}(w)\) and this is the unique grouping of \(\text{CFL}_{in}(w)\).

**Proof:** Let \(w \in \Sigma^+\) and let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\), with \(h > 1\). If \(\ell_1 \ll \cdots \ll \ell_h\), then \(\text{CFL}_{in}(w)\) is the unique grouping of \(\text{CFL}_{in}(w)\) (Definition 7.2). Then, by Proposition 7.13, \(\text{ICFL}(w) = \text{CFL}_{in}(w)\).

The following example shows that \(\text{ICFL}(w)\) is in general different from \(\text{CFL}_{in}(w)\).

**Example 7.3** Let \(\Sigma = \{a, b\}\) with \(a < b\), let \(w = bab \in \Sigma^+\). Therefore, \(b \prec a\) and \(\text{CFL}_{in}(w) = (ba, b)\). Since \(bab\) is an inverse Lyndon word, we have \(\text{ICFL}(w) = (bab)\).

The following example shows that there are words \(w\) such that \(\text{CFL}_{in}(w)\) has more than one grouping, thus there are groupings of \(\text{CFL}_{in}(w)\) different from \(\text{ICFL}(w)\).

**Example 7.4** Let \(\Sigma = \{a, b, c, d\}\) with \(a < b < c < d\), let \(w = dabadabdabldabdac \in \Sigma^+\). Therefore, \(d \prec c \prec b \prec a\). Moreover, \(\text{CFL}_{in}(w) = (daba, dab, dab, dab, dadac)\). The two sequences \((dabadab, (dab)^2, dadac), (daba, (dab)^3, dadac)\) are both groupings of \(\text{CFL}_{in}(w)\). Let us compute \(\text{ICFL}(w)\). The shortest prefix of \(w\) which is not an inverse Lyndon word is \(dabadab\) and \((daba, dabd) \in \text{Pref}_{rev}(w)\) by Lemma 7.7. Let \(w = dabay\), we have to compute \(\text{ICFL}(y)\). The shortest prefix of \(y = (dab)^3dadac\) which is not an inverse Lyndon word is \((dab)^3dad\) and \(((dab)^3, dad) \in \text{Pref}_{rev}(y)\) by Lemma 7.7. On the other hand, since \(dadac\) is an inverse Lyndon word, we have \(\text{ICFL}(dadac) = (dadac)\) and thus \(\text{ICFL}(y) = ((dab)^3, dadac)\) (see Definition 4.3). Finally, by Definition 4.3, \(\text{ICFL}(w) = (daba, (dab)^3, dadac)\).

### 7.4 Sorting suffixes in \(\text{ICFL}(w)\)

In this section we use the same notation and terminology as in Section 2.5. We prove that the same compatibility property proved in 7.1 holds between the sorting of the suffixes of a word \(w\) and that of its factors in \(\text{ICFL}(w)\) with respect to \(\prec_{in}\).

**Theorem 7.1** Let \(w\) be a word and let \((m_1, \ldots, m_k)\) be a grouping of \(\text{CFL}_{in}(w)\). Then, for any \(r, s, 1 \leq r \leq s \leq k\), the sorting of the suffixes of \(u = m_r \cdots m_s\) with respect to \(\prec_{in}\) is compatible with the sorting of the suffixes of \(w\) with respect to \(\prec_{in}\).

**Proof:** Let \(w\) and \((m_1, \ldots, m_k)\) be as in the statement. Let \(\text{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)\). Let \(u = m_r \cdots m_s\), with \(1 \leq r \leq s \leq k\). By Definitions 7.2, 7.3 any \(m_j\) is a concatenation of consecutive \(\ell_q\). Hence \(u\) is also a concatenation of consecutive \(\ell_q\). By Theorem 7.2, for all \(i, j\) with \(\text{first}(u) \leq i < j \leq \text{last}(u)\), we have

\[\text{suffix}(i) \prec_{in} \text{suffix}(j) \iff \text{suffix}(i) \prec \text{suffix}(j).\]  

(7.9)

The following corollary is a direct consequence of Proposition 7.13 and Theorem 7.1.

**Corollary 7.2** Let \(w\) be a word and let \(\text{ICFL}(w) = (m_1, \ldots, m_k)\). Then, for any \(r, s, 1 \leq r \leq s \leq k\), the sorting of the suffixes of \(u = m_r \cdots m_s\) with respect to \(\prec_{in}\) is compatible with the sorting of the suffixes of \(w\) with respect to \(\prec_{in}\).
On the contrary, we give below a counterexample showing that the compatibility property of local and global suffixes does not hold in general for inverse Lyndon factorizations with respect to $\prec_{in}$ (and with respect to $\prec$).

**Example 7.5** Let $(\Sigma, <)$ be as in Example 4.1 and let $w = daddbadc \in \Sigma^+$. Therefore, $d \prec_{in} c \prec_{in} b \prec_{in} a$. Consider the inverse Lyndon factorization $(dad,dba,dc)$ of $w$, with $dad \ll dba \ll dc$ and the factor $u = daddba$. Consider the local suffixes $a, addba$ of $u$ and the corresponding global suffixes $adc$ and $addbadc$. We have that $addbadc \prec_{in} adc$ while $a \prec_{in} addba$. Consequently, in general, the compatibility property does not hold for inverse Lyndon factorizations with respect to $\prec_{in}$. It does not hold also with respect to $<$ and even for ICFL. Indeed, let $w = dabadabdabdabdadac \in \Sigma^+$. We know that ICFL$(w) = (daba, (dab)^3, dadac)$ (see Example 7.4). For the local suffixes $dab, dabdab$ of $(dab)^3$ we have $dab \prec dabdab$ but for the corresponding global suffixes $dabadac, dabdabdac$ we have $dabadadac \prec dabdadac$.

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