Small width, low distortions: quasi-isometric embeddings with quantized sub-Gaussian random projections

Laurent Jacques*

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Abstract

Under which dimensionality conditions a subset $K$ of $\mathbb{R}^N$ can be embedded in another one of $\delta \mathbb{Z}^M$ for some resolution $\delta > 0$? In which norms can we measure the quality of this embedding and what is the nature of its distortions? This work addresses these general questions through the specific use of a quantized random linear mapping $A : \mathbb{R}^N \rightarrow \delta \mathbb{Z}^M$ combining a linear projection of $\mathbb{R}^N$ in $\mathbb{R}^M$ associated to a random matrix $\Phi \in \mathbb{R}^{M \times N}$ with a uniform scalar (dithered) quantization $Q$ of $\mathbb{R}^M$ in $\delta \mathbb{Z}^M$. The targeted embedding property relates the $\ell_2$-distance of any pair of vectors in $K$ with the $\ell_1$-distance of their respective mappings in $\delta \mathbb{Z}^M$, allowing for both multiplicative and additive distortions between these two quantities, i.e., describing a $\ell_2/\ell_1$-quasi-isometric embedding.

Extending previous results on quantized embeddings of finite vector sets or on 1-bit quantized random mappings, we show that the sought conditions depends on the Gaussian mean width $w(K)$ of the subset $K$. In particular, given a symmetric sub-Gaussian distribution $\varphi$ and a precision $\epsilon > 0$, if $M \geq C\epsilon^{-5}w(K)^2$ and if the sensing matrix $\Phi$ has entries i.i.d. as $\varphi$, then, with high probability, the mapping $A$ provides a $\ell_2/\ell_1$-quasi-isometry between $K$ and its image in $\delta \mathbb{Z}^M$. Moreover, in this embedding, the additive distortion is of order $\delta \epsilon$ while the multiplicative one grows with $\epsilon$. Interestingly, for non-Gaussian random $\Phi$, the multiplicative error is also impacted by the sparsity of the vectors difference, i.e., being smaller for “not too sparse” difference. As a special case, when $K$ is the set of bounded $K$-sparse vectors in any orthonormal basis, then only $M \geq C\epsilon^{-2}\log(cN/K\epsilon^{3/2})$ measurements suffice for achieving the same result.

Finally, we demonstrate that if $M \geq C\epsilon^{-4}w(K)^2$, then, with high probability, any pair of vectors of $K$ mapped by $A$ on the same quantization point and such that, for non-Gaussian sensing, their difference is “not too sparse”, have a distance smaller than $\epsilon$. For bounded $K$-sparse vectors in some orthonormal basis, the same result is reached if $M \geq C\epsilon^{-1}\log(cN/K\epsilon^{3/2})$. Equivalently, the distance between such consistent vectors, or consistency width, decays with high probability as $M^{-1/4}w(K)^{1/2}$ for a general subset $K$ and, up to some log factors, as $M^{-1}$ for sparse vector sets. Remark: all general values $C,c > 0$ above only depend on $\delta$ and on the distribution $\varphi$.

1 Introduction

There exists an ever growing trend in high (or “big”) dimensional data processing to design new procedures (or to simplify existing ones) using linear dimensionality reduction (LDR) methods in order to get faster or memory efficient algorithms. Provided this reduction does not bring

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*Image and Signal Processing Group (ISPGroup), ELEN Department, ICTEAM institute, Université catholique de Louvain (UCL), Belgium. The author is funded by Belgian National Science Foundation (F.R.S.-FNRS)
too much distortion between the initial data space and the reduced domain, as often allowed by an intrinsic “low dimensionality” properties of the input data, many techniques, such as nearest neighbors search in big databases [1 3], classification [5], regression [35], filtering [17], manifold processing [7] or compressed sensing [11 20] can be developed in this smaller domain with controlled loss of accuracy and stability with respect to data corruption (e.g., noise).

Most often, those LDR tools rely on defining a random projection matrix (sometimes called sensing matrix), with few rows $M$ than columns $N$, whose multiplication with data represented as a set of vectors in $\mathbb{R}^N$ provides a reduced representation (or sketch) of these. This is the scheme implicitly promoted for instance by the celebrated Johnson-Lindenstrauss (JL) lemma for finite set of vectors $S \subset \mathbb{R}^N$ [29]. This cornerstone result, and its subsequent developments [1 15], shows that, given a resolution $\epsilon > 0$, if $M \geq C\epsilon^{-2}\log S$ where $S$ is the cardinality of $S$ and $C > 0$ is a general constant, then a random matrix $\Phi \in \mathbb{R}^{M \times N}$ whose entries are independently and identically distributed (i.i.d.) as a centered sub-Gaussian distribution with unit variance defines an isometric mapping that preserves pairwise distances up to a multiplicative distortion $\epsilon$. In other words, $\Phi$ defines an $\epsilon$-isometry between $(S, \ell_2)$ and $(\Phi S, \ell_2)$, i.e., with high probability, for all $x, y \in S$,

$$
(1 - \epsilon)\|x - y\| \leq \frac{1}{\sqrt{M}}\|\Phi x - \Phi y\| \leq (1 + \epsilon)\|x - y\|.
$$

Equivalently, one observes that keeping the probability of success constant with respect to the random generation of $\Phi$ and inverting the requirement linking $M$ and $\epsilon$, such an isometry has a distortion $\epsilon$ decaying as $1/\sqrt{M}$ when $M$ increases, i.e., the distortion vanishes for large $M$ compared to $\log S$. Notice that variants of this embedding result exist with different “input/output” norms, see e.g., [33] for a unified treatment over a family of interpolation norms including $\ell_2$ and $\ell_1$ as special cases.

The JL lemma has been later generalized to any subsets $K \subset \mathbb{R}^N$, not only finite, whose typical “dimension” can be considered as small with respect to $N$ (see e.g., [7 18 36]). In other words, as soon as $K$ displays some internal structure that makes it somehow parametrisable with much fewer parameters than $N$, as for the set of sparse or compressible signals, the set low-rank matrices, signal manifolds, or union of low-dimensional subspaces, an $\epsilon$-isometry like [1] can be defined for all pair of vectors in $K$. This is for instance the essence of the restricted isometry property (RIP) and its link with the JL lemma where [1] holds with high probability for all $K$-sparse vectors provided $M \geq CK\log N/K$ [9 11].

However, there is one limitation of those isometric embeddings. Except for discrete sub-Gaussian random matrix $\Phi$ (e.g., Bernoulli) and finite $K$, the set $\Phi K \subset \mathbb{R}^M$ is not finite. An infinite number of bits is thus required if one needs to store, process or transmit $\Phi x$ without information loss for any possible $x \in K$. Moreover, knowing how many bits are required to represent such projection is also important theoretically for assessing and quantifying the level of information contained in the reduced data space or for improving specific data retrieval algorithms. Additionally, if this measure of information can be achieved, nothing prevents us to take $M \geq N$ as the sought “dimensionality reduction” can be concerned by minimizing that number of bits rather than by the space dimension. For instance, $\mathbb{R}$ defines locally sensitive hashing (LSH) as a procedure to turn data vectors into quantized hashes that preserve locality, i.e., close vectors induces, with high probability, close hashes. However, this method is specifically designed for boosting nearest neighbor searches over a finite set of vectors and is far to define isometry similar to [1].

As a more practical solution, the embedding realized by a random projection $\Phi$ is sometimes combined with a quantization procedure, e.g., with a uniform quantizer $Q : \mathbb{R} \to \delta\mathbb{Z}$ with resolution $\delta > 0$ and applied componentwise on the range of $\Phi$. A direct impact of this
combination is to induce a new additive distortion in (1) related to $\delta$ \cite{10}. Indeed, assuming $\Phi$ respects \(\Phi\) for all $x, y \in K$ for a certain subset $K \subset \mathbb{R}^N$, given a (mid-rise) uniform quantizer $Q$ of resolution $\delta > 0$ applied componentwise on vectors of $\mathbb{R}^M$, we have $|Q(\lambda) - \lambda| \leq \delta/2$ for all $\lambda \in \mathbb{R}$ and this involves $\|Q(u) - u\| \leq \sqrt{M}\delta/2$ for any $u \in \mathbb{R}^M$. Therefore, a simple manipulation of (1) provides

$$
(1 - \epsilon)\|x - y\| - \delta \leq \frac{1}{\sqrt{M}}\|Q(\Phi x) - Q(\Phi y)\| \leq (1 + \epsilon)\|x - y\| + \delta. \quad (2)
$$

In other words, as described in Sec. 2, the quantized mapping $A(\cdot) := Q(\Phi \cdot)$ defines now a quasi-isometric embedding between $(K \subset \mathbb{R}^N, \ell_2)$ and $(A(K) \subset \delta\mathbb{Z}^M, \ell_2)$.

However, while \cite{2} displays a constant additive distortion, several works in a few similar contexts have observed that such an additive error can decay as $M$ increases. First, when distances in the reduced space are measured with the $\ell_1$-norm and when $Q$ is combined with a dithering \cite{23}, a quasi-isometry similar to (2) holds with high probability for all vectors of a finite set $K = S$ \cite{26}. The additive distortion reads then $c\delta\epsilon$ for some absolute constant $\epsilon > 0$ and this error also decays as $1/\sqrt{M}$ with $M$ as does the multiplicative error $\epsilon$. Second, when combined with universal quantization \cite{10}, i.e., with periodic quantizer $Q$, an exponential distortion decay as $M$ grows can be reached, but, for the moment, only for sparse signal sets. Finally, recent works related to 1-bit compressed sensing (CS) have shown that for a quantization $Q$ reduced to the sign operator, i.e., $Q(\Phi \cdot) = \text{sign} (\Phi \cdot)$, angular distances between any pair of vectors of low-dimensionality sets $K$ are close to the Hamming distance of their mappings, up to an additive error decay with (some roots of) $M$. This is true for random Gaussian matrices and for the set of sparse signals \cite{25, 37}, for any sets with low “dimensions” as measured by their Gaussian mean width \cite{37, 39} (see below) and even for sub-Gaussian random matrices provided the projected vectors are not “too sparse” \cite{2}.

Considering these last observations, the objective of this paper is to show that:

(i) quasi-isometric embeddings can be obtained from scalar (dithered) quantization of random matrices, i.e., where both multiplicative and additive distortions co-exist, when, as in \cite{26}, distances between mapped vectors are measured with the $\ell_1$-norm;

(ii) the random sensing matrices are allowed to be generated from symmetric sub-Gaussian distributions provided embedded vector differences is not “too sparse” (as in the 1-bit case \cite{2});

(iii) those results holds with high probability for any subset $K$ of $\mathbb{R}^N$ as soon as $M$ is large compared to its typical dimension, i.e., to its squared Gaussian mean width.

Moreover, whenever it is possible, we aim at optimizing the requirements on $M$ (e.g., with respect to $\epsilon$ and $\delta$) that guarantee those results.

As a last objective, we also characterize the biggest distance separating vectors having identical quantized mappings, i.e., what we call the consistency width. While for any consistent vectors $x, y \in K$ this distance can be upper bounded from a quasi-isometric relation such as \cite{2} by setting $Q(\Phi x) = Q(\Phi y)$, we show that a better bound can be derived without imposing a quasi-isometry with even weaker conditions on $M$, i.e., with faster consistency width decay when $M$ increases. Notice that this extends also previous analyses of the consistency width \cite{27, 40} to any subsets $K$ and to the class of sub-Gaussian random matrices.

As an important aspect of our developments, we study the conditions for obtaining quasi-isometric embeddings of any bounded subsets $K \subset \mathbb{R}^N$ into $\delta\mathbb{Z}^M$. Following key procedures established in other works \cite{37, 38}, the typical dimension of these sets is measured through the
Gaussian mean width, i.e., 

\[ w(K) := \mathbb{E} \sup_{u \in K} |g^T u|, \]

with \( g \sim \mathcal{N}^N(0,1) \). This quantity, also called Gaussian complexity, has been recognized as central for instance in characterizing random processes \[33\], shrinkage estimators in signal denoising and high-dimensional statistics \[12\], linear inverse problem solving with convex optimization \[13\] or classification efficiency in randomly projected signal sets \[5\]. More specifically, the minimal number of measurements \( M \) necessary to induce, with high probability, an \( \ell_2/\ell_2 \)-isometric embedding of any subset \( K \subset \mathbb{R}^{N-1} \) into \( \mathbb{R}^M \) from sub-Gaussian random projections is known to be proportional to \( w(K)^2 \) \[36\]. Therefore, since \( w(K)^2 \lesssim \log |K| \) for finite set \( K \), we recover the condition defining the Johnson-Lindenstrauss lemma by imposing \( M \gtrsim \log |K| \) \[29\], while for the set of \( K \)-sparse vectors in an orthonormal basis (ONB) \( \Psi \in \mathbb{R}^{N \times N} \), \( w(K)^2 \lesssim K \log N/K \), which characterizes the conditions of the restricted isometry property (RIP) for sub-Gaussian random matrices \[6\].

In our developments, we sometimes complete the characterization of sets provided by the Gaussian mean width with another important set measure: the Kolmogorov \( \epsilon \)-entropy of a set \( K \subset \mathbb{R}^N \) that we denote \( \mathcal{H}(K, \epsilon) \) \[32\]. This one is defined as the logarithm of the size of the smallest \( \epsilon \)-net of \( K \), i.e., a set \( \mathcal{C}_\epsilon(K) \subset K \) such that any vector of \( K \) cannot be farther than \( \epsilon \) from its closest vector in \( \mathcal{C}_\epsilon(K) \). From Sudakov inequality this entropy is connected to the Gaussian mean width through \( \mathcal{H}(K, \epsilon) \lesssim w(K)^2/\epsilon^2 \). In few cases (e.g., for \( K \)-sparse signal sets), this last inequality is too loose with respect to \( \epsilon \) and our analysis will consider specifically some of these special cases thanks to known tighter bounds on \( \mathcal{H} \).

The interested reader can find a summary of the main properties of the Gaussian mean width in Table 1, most of them being collected from \[15, 37, 38\]. This table could be helpful also to keep trace of these properties while reading our proofs.

The rest of the paper is structured as follows. In Sec. 2 we define the construction of our quantized sub-Gaussian random mapping. Additionally, this section characterizes the sub-Gaussianity of its linear ingredient, i.e., its internal random projection matrix, and its interplay with the “anti-sparse” nature of the mapped vectors. We also formalize and motivate the main objectives of the paper, e.g., explaining the shape and the origins of the targeted quasi-isometric embedding with its two specific distortions. Sec. 3 provides the main results of this work, namely, the possibility to create, with high probability, a quantized sub-Gaussian quasi-isometric embedding from our quantized mapping (Prop. 1), and its consistency width behavior (Prop. 2). Sec. 4 discusses those two propositions, analyzing them in a few specific settings in comparison with related works in the fields of dimensionality reduction and 1-bit compressed sensing. Finally, Sec. 5 and Sec. 6 contains the proofs of Prop. 1 and Prop. 2 respectively, auxiliary Lemmas being demonstrated in appendix.

Conventions: We find useful to summarize here our mathematical notations. Domain dimensions are denoted by capital roman letters, e.g., \( M, N, \ldots \). Vectors and matrices are associated to bold symbols, e.g., \( \Phi \in \mathbb{R}^{M \times N} \) or \( u \in \mathbb{R}^M \), while lowercase light letters are associated to scalar values. The identity matrix in \( \mathbb{R}^D \) reads \( I_D \) while \( I[A] \in \{0,1\} \) is the indicator function of a set \( A \subset \mathbb{R}^D \). An “event” is a set whose definition depends on the realization of some random variables, e.g., if \( X \in \mathbb{R} \) is a random variable, the event \( A = \{ X \leq 0 \} \) has probability \( P(X \leq 0) = \mathbb{E} [I[A] \] . The \( i^{th} \) component of a vector (or of a vector function) \( u \) reads either \( u_i \) or \( (u)_i \), and the vector \( u_i \) may refer to the \( i^{th} \) element of a set of vectors. The set of indices in \( \mathbb{R}^D \) is \( [D] = \{ 1, \ldots, D \} \). The cardinality of a finite set \( J \) reads \( |J| \). For any \( p \geq 1 \), the \( \ell_p \)-norm of \( u \) is \( \|u\|_p = \sum_i |u_i|^p \) with \( \|\cdot\| = \|\cdot\| \). The “\( \ell_0 \)-norm” of a vector \( u \in \mathbb{R}^N \) is
For a bounded set $A$, we have the uniform (dithered) quantizer $Q(t) = \delta \lfloor \frac{t}{\delta} \rfloor \in \delta \mathbb{Z}$, where $\delta > 0$. The $(N-1)$-sphere in $\mathbb{R}^N$ is $S^{N-1} = \{ x \in \mathbb{R}^N : \| x \| = 1 \}$, while the unit ball is denoted $B^N = \{ x \in \mathbb{R}^N : \| x \| \leq 1 \}$. The diameter of a bounded set $A \subset \mathbb{R}^N$ is written $\| A \| = \sup \{ \| u \| : u \in A \}$. The set of $K$-sparse signals in $\mathbb{R}^N$ is defined as $\Sigma_K := \{ u \in \mathbb{R}^N : \| u \|_0 \leq K \}$, while the set of $K$-sparse signals in an orthonormal basis (ONB) $\Psi \in \mathbb{R}^{N \times N}$, i.e., with $\Psi \Psi^T = \Psi^T \Psi = I_N$, reads $\Sigma_{K,\Psi} = \Psi \Sigma_{K,\Psi}$. The positive thresholding function is defined by $(\lambda)_+ := \frac{1}{2}(\lambda + |\lambda|)$ for any $\lambda \in \mathbb{R}$. For $t \in \mathbb{R}$, $\lfloor t \rfloor$ (resp. $\lceil t \rceil$) is the largest (smallest) integer smaller (greater) than $t$. A random matrix $\Phi \sim \mathcal{P}^{M \times N}(\Theta)$ is a $M \times N$ matrix with entries distributed as $\Phi_{ij} \sim \text{i.i.d. } \mathcal{P}(\Theta)$ given the distribution parameters $\Theta$ of $\mathcal{P}$ (e.g., $\mathcal{N}^{M \times N}(0, 1)$ or $\mathcal{U}^{M \times N}([0, 1])$). A random vector in $\mathbb{R}^M$ following $\mathcal{P}(\Theta)$ is defined by $\nu \sim \mathcal{P}^M(\Theta)$. Given two random variables $X$ and $Y$, the notation $X \sim Y$ means that $X$ and $Y$ have the same distribution. Since our developments are not focus on sharp bounds, we denote by $C, c, c'$ or $c''$ (possibly large) constants whose value can change between different lines. In a few places, for simplifying our notations, we write $f \lesssim g$ if there exists a constant $c > 0$ such that $f \leq cg$, and correspondingly for $f \gtrsim g$. Moreover, $f \sim g$ means that $f \lesssim g$ and $g \lesssim f$. Finally, for asymptotic relations most often given in plain text, we use the common Landau family of notations, i.e., the symbols $O$, $\Omega$ and $\Theta$ [31].

## 2 Quantized Sub-Gaussian Random Mapping

In this work, given a certain quantization resolution $\delta > 0$, we focus on the interaction of a uniform (dithered) quantizer $Q(t) = \delta \lfloor \frac{t}{\delta} \rfloor \in \delta \mathbb{Z}$, applied componentwise on vectors, with random projections of $\mathbb{R}^N$. In other words, for some random matrix $\Phi \in \mathbb{R}^{M \times N}$ whose distribution is

| Names | Properties |
|-------|------------|
| (P1) Definition | $w(A) = \mathbb{E} \sup_{u \in A} |\langle g, x \rangle|$ for $g \sim \mathcal{N}^N(0, 1)$. |
| (P2) Homogeneity | $w(\lambda A) = \lambda w(A)$ for $\lambda > 0$. |
| (P3) Set inclusion | if $A \subset B$, $w(A) \leq w(B)$. |
| (P4) Set difference | $w(A - B) = 2w(A)$. |
| (P5) Modularity | $w(A \cup B) + w(A \cap B) = w(A) + w(B)$. |
| (P6) Convex hull | $w(\conv(A)) = w(A)$. |
| (P7) Subspace | if $A_K$ is a $K$-dimensional subspace of $\mathbb{R}^N$, then $w(A_K \cap S^{N-1}) = w(A_K \cap B^N) \leq \sqrt{K}$. |
| (P8) Subspace addition | $w((A_K \cap B) \cap S^{N-1}) \leq K + w(B \cap S^{N-1})$. |
| (P9) Link with diameter | for $\| A \| = \sup_{u \in A} \| u \|$. |
| (P10) Symmetrization | $w(A) - (\frac{1}{2})^{1/2} \inf_{u \in A} \| u \| \leq \mathbb{E} \sup_{u \in A} |\langle g, x \rangle| \leq 2w(A)$. |
| (P11) Translation | $w(A - \{ t \}) \leq w(A + \{ t \}) \leq w(A) + (\frac{1}{2})^{1/2} \| t \|$. |
| (P12) Invariance under $\mathcal{O}_N$ | For all $B \in \mathcal{O}_N := \{ \mathcal{C} \in \mathbb{R}^{N \times N} : \mathcal{C}^T = \mathcal{C} \},$ $w(B \cdot A) = w(A)$. |
| (P13) Translation on origin | $w(A - \{ 0 \}) \leq w(A - \{ 0 \}) \leq 2w(A)$. |
| (P14) Sudakov inequality | $\forall \varepsilon > 0$ $\forall \mathcal{G}_\varepsilon \subset \mathcal{A}$ $\log |\mathcal{G}_\varepsilon| \leq \varepsilon^{-2} w(A)^2$. |

Table 1: Useful properties of the Gaussian mean width. Most of them come from [13, 37, 38]. Without additional explanations, all sets are subsets of $\mathbb{R}^N$. For a $\varepsilon$-net $\mathcal{G} \subset \mathcal{A}$, $\log |\mathcal{G}| \leq \varepsilon^{-2} w(A)^2$.
defined below, we study the properties of the mapping $A : \mathbb{R}^N \rightarrow \delta \mathbb{Z}^M$ with

$$A(x) := Q(\Phi x + \xi),$$

where $\xi \in U^M([0, \delta])$ is a uniform dithering stabilizing the action of $Q$ [9, 23, 26].

We specialize the mapping (3) on projection (or sensing) matrix $\Phi$ with entries independently and identically generated from a symmetric sub-Gaussian distribution. We recall that a random variable (r.v.) $X$ is sub-Gaussian if its sub-Gaussian (or $\psi_2$) norm [44]

$$\|X\|_{\psi_2} := \sup_{p \geq 1} p^{-1/2}(\mathbb{E}|X|^p)^{1/p}. \quad (4)$$

is finite.[1] Examples of sub-Gaussian r.v.’s are Gaussian, Bernoulli, uniform or bounded r.v.’s, as

$$\|X\|_{\psi_2} \leq \|X\|_{\infty} := \inf\{t \geq 0 : \mathbb{P}(\|X\| \leq t) = 1\}.$$ Sub-Gaussian r.v.’s respect several interesting properties described, e.g., in [44]. Their tail is for instance bounded as the one of a Gaussian r.v., i.e., there exists a $c > 0$ such that for all $\epsilon \geq 0$ and for a sub-Gaussian r.v. $X$,

$$\mathbb{P}(\|X\| > \epsilon) \lesssim \epsilon^{-c\epsilon^2/\|X\|_{\psi_2}^2}, \quad (5)$$

Moreover, since $\|X - \mathbb{E}X\|_{\psi_2} \leq \|X\|_{\psi_2} + \|\mathbb{E}X\|_{\psi_2} = \|X\|_{\psi_2} + \|\mathbb{E}X\| \leq \|X\|_{\psi_2} + \|\mathbb{E}X\| \leq 2\|X\|_{\psi_2},$ centering $X$ has no effect on its sub-Gaussianity.

By a slight abuse of notations, we denote collectively the distributions of symmetric sub-Gaussian r.v. with zero expectation, unit variance and finite sub-Gaussian norm $\alpha$ by $\mathcal{N}_{sg,\alpha}(0,1)$, with $\alpha \geq 1/\sqrt{2}$ from [1]. This means that if $X \sim \mathcal{N}_{sg,\alpha}(0,1)$, we do not fully specify the pdf of $X$ but we know that $X$ is centered, has unit variance and sub-Gaussian norm $\alpha$.

In this context, for a sub-Gaussian random matrix $\Phi = (\varphi_1, \ldots, \varphi_M)^T \sim \mathcal{N}_{sg,\alpha}^M(0,1)$, each row $\varphi_i$ is also isotropic, i.e., for all $i \in [M]$ and all $u \in \mathbb{R}^N$,

$$E(|\langle \varphi_i, u \rangle|^2) = \|u\|^2.$$ However, conversely to the Gaussian case where $E(|\langle g, u \rangle| = (\frac{\alpha}{2})^t\|u\|$ for $g \sim \mathcal{N}(0,1)$ and $u \in \mathbb{R}^N$ (since $(g, u) \sim \mathcal{N}(0, \|u\|^2)$), we do not necessarily have $E(|\langle \varphi, u \rangle| = c\|u\|$ for $\varphi \sim \mathcal{N}_{sg,\alpha}(0,1)$ and some absolute constant $c > 0$.

As will be clear below, our developments need anyway to characterize the deviations to this last equality. As in [2], we thus assume the existence of a constant $\kappa_{sg} \geq 0$ depending only the distribution of $\varphi \sim \mathcal{N}_{sg,\alpha}(0,1)$ such that

$$\int_0^{+\infty} |\mathbb{P}(|\langle \varphi, u \rangle| \geq t) - \mathbb{P}(|\langle g, u \rangle| \geq t)| \, dt \leq \kappa_{sg}\|u\|_{\infty}, \quad \forall u \in \mathbb{R}^N. \quad (6)$$

From a simple change of variable $t \rightarrow t\|u\|$ in the last integral, such an assumption is sustained by the Berry-Essen central limit theorem described in a simplified form in [2, Theorem 4.2]. This result shows basically that, for $u \in \mathbb{S}^{N-1}$, the LHS of (6) is bounded by $9E|\varphi|^3\|u\|^3 \leq 9\sqrt{27}\alpha^3\|u\|$ for $\varphi_i \sim_{i.i.d.} \varphi \sim \mathcal{N}_{sg,\alpha}(0,1).$ This means that $\kappa_{sg} \leq 9\sqrt{27}\alpha^3$ for any $\varphi \sim \mathcal{N}_{sg,\alpha}(0,1).$ However, this upper bound is not tight as $\kappa_{sg} = 0$ in the Gaussian case where $\alpha \neq 0.$

Thanks to assumption (6), we can precise the behavior of the first absolute moment function

$$\mu_{sg}(u) := E|\langle \varphi, u \rangle|.$$

[1] Notice that other equivalent definitions for sub-Gaussian r.v. exist, see e.g., [36].
Since $E[X] = \int_0^\infty P(|X| \geq t) \, dt$ for any r.v. $X$ and using Jensen’s inequality, we indeed observe that

$$
\mu_{sg}(u) \leq (E[\langle \varphi, u \rangle^2])^{1/2} = \|u\|,
$$

(8)

$$
|\mu_{sg}(u) - (\tilde{\varphi})^T u| \leq \kappa_{sg}\|u\|, \quad \text{for all } u \in \mathbb{R}^N.
$$

(9)

for all $u \in \mathbb{R}^N$. The last property, which is also considered in 1-bit CS with non-Gaussian projections [2], is key for characterizing quantized sub-Gaussian projection.

Having now fully described the elements composing our random quantized mapping $A$, we formally address the objectives defined in the Introduction by observing “when”, i.e., under which conditions over $M$, there exist two “small” distortions $\Delta_{\oplus}, \Delta_{\otimes} \geq 0$ such that the pseudo-distance $D(x, y) := \frac{1}{\pi\|A(x) - A(y)\|_1}$ is involved in the quasi-isometric relation

$$
|D(x, y) - (\tilde{\varphi})^T y| \leq \Delta_{\otimes}\|x - y\| + \Delta_{\oplus},
$$

(10)

for all pair of vectors taken in a general subset $K \subset \mathbb{R}^N$.

In particular, we aim to control the distortions $\Delta_{\oplus}$ and $\Delta_{\otimes}$ with respect to $M, N$, the non-Gaussian nature of $\Phi$ (i.e., through $\alpha$ and $\kappa_{sg}$), the “typical dimension” of $K$ (i.e., its Gaussian mean width) and possible additional requirements on $x$ and $y$.

Let us justify and comment the specific form taken by (10). First, $D$ is associated to a $\ell_1$-distance in the image of $A$. As detailed in Sec. 5, this choice establishes an equivalence between the evaluation of $D$ and a specific counting procedure, i.e., one that counts the number of quantization thresholds $D$ and possible additional requirements on $x$ and $y$.

Second, as explained in the Introduction, a special case specifies the constant $(\tilde{\varphi})^T$ in (10) and the existence of $\Delta_{\oplus}$ and $\Delta_{\otimes}$. When $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$, [29] has proved a quantized version of the Johnson Lindenstrauss (JL) Lemma showing that for a finite set $S \subset \mathbb{R}^N$ of size $S$, provided $M \geq \epsilon^{-2}\log S$, one has

$$
|D(x, y) - (\tilde{\varphi})^T y| \lesssim \epsilon\|x - y\| + \epsilon\delta,
$$

for all pairs $x, y \in S$ with a probability at least $1 - e^{-\epsilon^2 M}$. This shows that $A$ realizes a quasi-isometric mapping between $(S \subset \mathbb{R}^N, \ell_2)$ and $(A(S) \subset \delta \mathbb{Z}^M, \ell_1)$ with $\Delta_{\oplus} = \epsilon$ and $\Delta_{\otimes} = \delta \epsilon$. In other words, the approximation (10) cannot rely on a bi-Lipschitz embedding, i.e., with $\Delta_{\otimes} = 0$, as in the common JL Lemma [29] or in (RIP) [6].

Finally, the anti-sparse nature of $x - y$ must have an impact on the distortions involved in (10). Indeed, as proved in App. A

$$
E|\|x + \xi\| - |y + \xi\|| = |x - y|, \quad \forall x, y \in \mathbb{R}, \quad \xi \sim \mathcal{U}([0, 1]).
$$

(11)

Therefore, by definition of $Q$, from the independence of each component of $A$ and using the law of total expectation over $\xi$ and $\Phi$, we have

$$
E D(x, y) = E_{\varphi} E_{\xi} Q(\varphi^T x + \xi) - Q(\varphi^T y + \xi) = E_{\varphi} |\varphi^T(x - y)| = \mu_{sg}(x - y),
$$

(12)

with $\varphi \sim \mathcal{N}_{sg, \alpha}(0, 1)$ and $\xi \sim \mathcal{U}([0, \delta])$. From the assumption (9) and given $K_0 \in \mathbb{R}$, we then observe that

$$
|E D(x, y) - (\tilde{\varphi})^T y| = |\mu_{sg}(x - y) - (\tilde{\varphi})^T y| \leq \frac{\kappa_{sg}}{\sqrt{K_0}} \|x - y\|.
$$

(13)
for all vectors \( \mathbf{x} \) and \( \mathbf{y} \) such that \( \mathbf{x} - \mathbf{y} \) belongs to

\[
\mathcal{Z}_{K_0} := \{ \mathbf{u} \in \mathbb{R}^N : K_0 \| \mathbf{u} \|_\infty^2 \leq \| \mathbf{u} \|^2 \}.
\]

This last set amounts to considering vectors that are not “too sparse”, i.e., if \( \mathbf{u} \in \mathcal{Z}_{K_0} \) then \( \| \mathbf{u} \|_0 \geq K_0 \), which determines our notation \( \mathcal{Z}_{K_0} \) as opposed to \( \Sigma_K \). However, the converse is not true and \( \mathcal{Z}_{K_0} \neq \Sigma_{K_0} \). Since belonging to \( \mathcal{Z}_{K_0} \) prevents sparsity, we way that a vector \( \mathbf{u} \in \mathcal{Z}_{K_0} \) is an anti-sparse vector of level \( K_0 \geq 0 \). Actually, (13) states that, for vectors \( \mathbf{x} - \mathbf{y} \in \mathcal{Z}_{K_0} \), the expectation of \( D(\mathbf{x}, \mathbf{y}) \) is close to the one obtained with Gaussian random projections, i.e., close to the expectation \( (\frac{\pi}{2})^\frac{1}{2} \| \mathbf{x} - \mathbf{y} \| \) associated to \( \kappa_{sg} = 0 \). Therefore, if we expect to show that, for all vectors \( \mathbf{x} \) and \( \mathbf{y} \) in \( K \), \( D(\mathbf{x}, \mathbf{y}) \) concentrates around \( (\frac{\pi}{2})^\frac{1}{2} \| \mathbf{x} - \mathbf{y} \| \), we must take into account the (non) sparse nature of the difference \( \mathbf{x} - \mathbf{y} \), i.e., enforce this vector to belong to \( \mathcal{Z}_{K_0} \) for a sufficiently large \( K_0 \).

Combining these three observations, and anticipating over the next section, we can now affine the meaning of (10). We are actually going to show that, if \( M \) is bigger than some \( M_0 \) growing with the dimension of \( K \) and decreasing with \( \epsilon \) (see Sec. 3), then, with high probability,

\[
((\frac{\pi}{2})^\frac{1}{2} - \epsilon - \frac{\kappa_{sg}}{\sqrt{K_0}}) \| \mathbf{x} - \mathbf{y} \| - c\epsilon \delta \leq D(\mathbf{x}, \mathbf{y}) \leq ((\frac{\pi}{2})^\frac{1}{2} + \epsilon + \frac{\kappa_{sg}}{\sqrt{K_0}}) \| \mathbf{x} - \mathbf{y} \| + c\epsilon \delta,
\]

for all \( \mathbf{x}, \mathbf{y} \in \mathcal{K} \) and \( \mathbf{x} - \mathbf{y} \in \mathcal{Z}_{K_0} \).

Remark: As will be clear later, our developments greatly benefit of the tools and techniques developed in [37] where it is shown that, for a 1-bit mapping \( A' : \mathbb{R}^N \rightarrow \{\pm 1\}^M \) such that \( A'(\mathbf{x}) = \text{sign}(\Phi \mathbf{x}) \) with a random Gaussian matrix \( \Phi \sim \mathcal{N}^{M \times N}(0,1) \), and for the normalized Hamming distance \( D'(\mathbf{x}, \mathbf{y}) = M^{-1} \sum_i \mathbb{I}[A'((\frac{\pi}{2})^\frac{1}{2})] \leq M \), as soon as \( M \gtrsim \epsilon^{-4} w(\mathcal{K})^2 \), one has, with a probability at least \( 1 - \epsilon^{-c_2M} \) and for all \( \mathbf{x}, \mathbf{y} \in \mathcal{K} \),

\[
|D'(\mathbf{x}, \mathbf{y}) - \arccos(\frac{x^\top y}{\|x\| \|y\|})| \lesssim \epsilon.
\]

Our extension to non-Gaussian sensing matrix is also inspired by the corresponding result developed in [2] for 1-bit mappings and other generalized linear model.

3 Main Results

3.1 Quasi-Isometric Quantized Embedding

In regards of the context explained in the previous section, our first main result can be stated as follows.

**Proposition 1** (Quantized sub-Gaussian quasi-isometric embedding). Given \( \delta > 0 \), \( \epsilon \in (0,1) \), \( K_0 > 0 \), a bounded subset \( \mathcal{K} \subset \mathbb{R}^N \) and a sub-Gaussian distribution \( \mathcal{N}_{sg,a} \) respecting [9] for \( 0 \leq \kappa_{sg} < \infty \), there exist some values \( C, c, c' > 0 \), only depending on \( \alpha \), such that, if

\[
M \geq C \frac{1}{\delta c_2} w(\mathcal{K})^2,
\]

then, for \( \Phi \sim \mathcal{N}^{M \times N}_{sg,a}(0,1) \), a dithering \( \xi \sim \mathcal{U}^M([0,\delta]) \) and the associated quantized mapping \( \mathbf{u} \in \mathbb{R}^N \rightarrow A(\mathbf{u}) = Q(\Phi \mathbf{u} + \xi) \), we have with probability at least \( 1 - \epsilon^{-c_2M} \) and for all pairs \( \mathbf{x}, \mathbf{y} \in \mathcal{K} \) with \( \mathbf{x} - \mathbf{y} \in \mathcal{Z}_{K_0} \),

\[
((\frac{\pi}{2})^\frac{1}{2} - \epsilon - \frac{\kappa_{sg}}{\sqrt{K_0}}) \| \mathbf{x} - \mathbf{y} \| - c'\epsilon \delta \leq D(\mathbf{x}, \mathbf{y}) \leq ((\frac{\pi}{2})^\frac{1}{2} + \epsilon + \frac{\kappa_{sg}}{\sqrt{K_0}}) \| \mathbf{x} - \mathbf{y} \| + c'\epsilon \delta.
\]
Moreover, for the bounded set of $K$-sparse signals in an ONB $\Psi \in \mathbb{R}^{N\times N}$, i.e., if $K = \Sigma K_\Psi \cap \mathbb{B}^N$, then the condition \[(15)\] reduces to

$$M \geq C' \frac{1}{\gamma} K \log(\frac{N}{K\sqrt{\epsilon}}) $$

for some $C' > 0$ depending only on $\alpha$. In the Gaussian case, i.e., for $\Phi \sim \mathcal{N}^{M\times N}(0, 1)$, the conditions remain the same and \[(16)\] is simplified with $\kappa_{sg} = 0$, $K_0 = 1$ and $\mathcal{Z}_{K_0} = \mathbb{R}^N$.

In Prop. \[1\] the condition $x - y \in \mathcal{Z}_{K_0}$, which disappears for Gaussian random projection where $\kappa_{sg} = 0$, seems unavoidable for guaranteeing \[(16)\] for any sub-Gaussian random matrix distributions. Let us take a simple example to convince us of this fact. Let $\Phi_i$ be a Bernoulli distribution, i.e., $\Phi_{ij} \sim_{iid} \mathcal{B}(\frac{1}{2})$ with $\mathbb{P}(\Phi_{ij} = 1) = \mathbb{P}(\Phi_{ij} = -1) = 1/2$ for all $1 \leq i \leq M$ and $1 \leq j \leq N$. We choose $x \in \mathbb{R}^N$ such that $x_i = 1$ for $1 \leq i \leq K$ and $x_i = 0$ for $i > K$, while we simply set $y = 0$ so that $A(0) = 0$. If $K = 1$, then, $A(x) = [\Phi_1 + \xi] = \Phi_1$ and $\mathcal{D}(x, y) = M^{-1}\|A(x) - A(y)\|_1 = 1 = \|x - y\| > \frac{1}{2}\|x - y\| = \frac{1}{2}$ in this case $x - y \in \mathcal{Z}_1$, but $x - y \notin \mathcal{Z}_{K_0}$ for $K_0 > K_1$. However, for $K > 1$ and assuming $\mathcal{K}$ contains both $x$ and $y = 0$, we have $\mathbb{E}|Q(\varphi^T x + \xi)| = \mathbb{E}|\varphi^T x|$ for $\varphi \sim \mathcal{B}(\frac{1}{2})^N$. The last quantity is actually twice the mean absolute deviation (MAD) of a Binomial distribution with $N$ degrees of freedom since

$$\mathbb{E}|\varphi^T x| = \mathbb{E}|\sum_{j=1}^{K} \varphi_j| = 2\mathbb{E}|\left(\sum_{j=1}^{K} \frac{1}{2}(\varphi_j + 1)\right) - \frac{1}{2}K| = 2\mathbb{E}|\beta - \mathbb{E}|\beta|,$$

with $\beta \sim \text{Bin}(K, \frac{1}{2})$. However, it is also known that $\mathbb{E}|\beta - \mathbb{E}|\beta| = \left(\frac{3}{2}\right)^{1/2} \sqrt{\mathbb{E}(\beta - \mathbb{E}|\beta|)^2} = O(K^{-1/2}) = \left(\frac{3}{2}\right)^{1/2} \frac{xK}{2} + O(K^{-1/2})$.

Since $\|x\| = \sqrt{K}$, this means that $\mathbb{E}|\varphi^T x| = \left(\frac{3}{2}\right)^{1/2} \|x\| + O(K^{-1/2})$, showing that in our simple example, despite a different behavior at $K = 1$, $\mathbb{E}|\varphi^T x|$ tends to $\left(\frac{3}{2}\right)^{1/2} \|x\|$ when $K$ increases, while in the same time $K\|x\|_{\Sigma_\varphi}^2 = \|x\|^2$ and $x = x - y \in \mathcal{Z}_K$ for $K$ growing.

In our example, where $x$ and $y$ are fixed, $\mathcal{D}(x, y)$ “concentrate” on a value that clearly depends on $x$ and $y$: it is equal to $\|x - y\|$ for $K = 1$ and, by the law of large number, tends with $M$ increasing to $\left(\frac{3}{2}\right)^{1/2} \|x - y\|$ with distortion $O(K^{-1/2})$ while $x - y \in \mathcal{Z}_K$. This shows that \[(16)\] must integrate a distortion that depends on the “anti-sparse” degree of $x - y$, i.e., the factor $\kappa_{sg}/\sqrt{K_0}$.

To conclude this section, let us observe that Prop. \[1\] improves a proof of existence of a quantized embedding given in \[27\] Theorem 1.10. In this work, it is shown that, provided $M \geq \epsilon^{-12}w(K - K_0)^2$, there exist an arrangement of $M$ affine hyperplanes in $\mathbb{R}^N$ and a scaling factor $\lambda$ such that

$$|\lambda \mathcal{D}_c(x, y) - \|x - y\|| \leq \epsilon,$$

where $\mathcal{D}_c$ denotes the fraction of affine hyperplanes that separates two vectors.

For reasons explained in Sec. \[5\] each element $\delta^{-1}|A_i(x) - A_i(y)|_1$ appearing in $\delta^{-1}\mathcal{D}(x, y) = \frac{1}{\pi\epsilon} \sum_{i=1}^M |A_i(x) - A_i(y)|_1$ actually counts the number of parallel affine hyperplanes in $\mathbb{R}^N$ normal to $\varphi_i$ and $\delta$ far apart with a dithering that randomly displaces the origin. Therefore, Prop. \[1\] basically constructs (randomly) an arrangement of $M$ such parallel hyperplanes arrangements, in $M$ different directions $\{\varphi_i/|\varphi_i|, i \in [M]\}$. Considering a Gaussian matrix $\Phi$ ($\kappa_{sg} = 0$), we have therefore proved that a random uniform tessellation of $\mathbb{R}^N$ exist with a minimal $M$ that grows like $\epsilon^{-8}$ rather than $\epsilon^{-12}$ when $\epsilon$ decays, as expressed in \[(15)\]. This is even reduced to $\epsilon^{-2}$ for pair of vectors taken $K$-sparse signal set.
3.2 Consistency Width Decay

As a second important result, we optimize the decay law with respect to $M$ of the distance of any pair of vectors $\mathbf{x}, \mathbf{y} \in \mathcal{K}$ whose difference is “not too sparse” when those are mapped by $\mathbf{A}$ on the same quantization point in $\delta \mathbb{Z}^M$, i.e., when they are consistent. We refer to this distance as the consistency width of $\mathbf{A}$.

The reader could notice that such a result can be obtained as a special case of Prop. 1 when $\mathcal{D}(\mathbf{x}, \mathbf{y}) = 0$, which provides $\|\mathbf{x} - \mathbf{y}\| \leq \epsilon \simeq M^{-1/5}$ (or $M^{-1/2}$ for $\mathcal{K} = \Sigma_{\Psi} \cap \mathbb{B}^N$ given an ONB $\Psi \in \mathbb{R}^{N \times N}$) for large $M$ respecting (15) (resp. (17)), $\delta$ fixed and $\kappa_{sg}/\sqrt{K}_0$ small. However, focusing on the conditions guaranteeing the consistency of $\mathbf{x}$ and $\mathbf{y}$, considering all quantities fixed but $M$, our result below reaches $\epsilon = O(M^{-1/4})$ for a general set $\mathcal{K}$ and $\epsilon = O(1/M)$ for $\mathcal{K} = \Sigma_{\Psi} \cap \mathbb{B}^N$. We prove the following proposition in Sec. 6.

**Proposition 2** (Consistency width). Given a quantization resolution $\delta > 0$, $\epsilon \in (0, 1)$, a sub-Gaussian distribution $\mathcal{N}_{sg, \alpha}(0, 1)$ respecting (5) for $0 \leq \kappa_{sg} < \infty$, and $\mathcal{K} \subset \mathbb{B}^N$ a bounded subset of $\mathbb{R}^N$ and $\delta > 0$, there exist some values $C, \epsilon > 0$ depending only on $\alpha$ and such that, if

$$M \geq C \left( \frac{(2+\delta)^4}{\delta^2} \right) w(\mathcal{K})^2,$$

then, for $\Phi \sim \mathcal{N}_{sg, \alpha}^M(0, 1)$, $\xi \sim \mathcal{U}(\{0, \delta\})$, the mapping $\mathbf{u} \in \mathbb{R}^N \to \mathbf{A}(\mathbf{u}) := \mathcal{Q}(\Phi \mathbf{u} + \xi)$ and for a $K_0$ such that $\sqrt{K}_0 \geq 16 \kappa_{sg}$, we have with probability at least $1 - 2 \exp(-\epsilon c M/(1 + \delta))$ and for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$ with $\mathbf{x} - \mathbf{y} \in \mathbb{Z}_{K_0}$,

$$A(\mathbf{x}) = A(\mathbf{y}) \implies \|\mathbf{x} - \mathbf{y}\| \leq \epsilon. \quad (19)$$

Moreover, in the case where $\mathcal{K} = \Sigma_{\Psi} \cap \mathbb{B}^N$ (for an ONB $\Psi \in \mathbb{R}^{N \times N}$), (18) simplifies to

$$M \geq C' \frac{2+\delta}{\epsilon^2} K \log \left( \frac{N}{K_0} \left( \frac{2+\delta}{\epsilon^2} \right)^{3/2} \right), \quad (20)$$

for some $C' > 0$ only depending on $\alpha$.

Unfortunately, we were unable to produce a convincing example of a pair of vectors with difference not in $\mathbb{Z}_{K_0}$ and failing to meet (19) under the conditions of Prop. 2. Therefore, it is not clear if the condition $\mathbf{x} - \mathbf{y} \in \mathbb{Z}_{K_0}$ is an artifact of the proof or if removing it worsen then dependence in $\epsilon$ in (18).

4 Discussions and Perspectives

Before delving into the proofs of Prop. 1 and Prop. 2 (see Sec. 5 and Sec. 6, respectively), let us discuss their meaning and limitations, providing also some perspectives for future works.

First, for both propositions, we can be concerned by the restriction imposed to the vector difference to be “not too sparse”, i.e., for $\mathbf{x}, \mathbf{y} \in \mathcal{K}$ there must exists a $K_0$ sufficiently big, either for having $\mathbf{x} - \mathbf{y} \in \mathbb{Z}_{K_0}$ and minimizing the distortion $\kappa_{sg}/\sqrt{K}_0$ in (16) or for helping in guaranteeing $\sqrt{K}_0 \geq 16 \kappa_{sg}$ in Prop. 2. However, in certain cases, it is possible to adapt the sensing matrix for increasing this $K_0$.

Indeed, assuming without loss of generality that the vectors $\mathbf{x} - \mathbf{y} \in \mathcal{K} - \mathcal{K}$ are expected to be “too sparse” only in $\Psi = \mathbf{I}$ when the sensing matrix is non-Gaussian (i.e., $\kappa_{sg} \neq 0$), we can always “rotate” $\mathcal{K}$ with an ONB $\Psi_0$ of $\mathbb{R}^N$ so that elements of $\mathcal{K}' - \mathcal{K}'$ have a higher anti-sparse
degree than those of $\mathcal{K} - \mathcal{K}$, i.e.,
\[
\max\{K_0 : (\mathcal{K}' - \mathcal{K}') \cap \mathbb{Z}_{K_0} \neq \emptyset\} = \min_{u \in \mathcal{K} - \mathcal{K}} \frac{\|u\|^2}{\|\mathcal{V}_0 u\|_\infty} \geq \min_{u \in \mathcal{K} - \mathcal{K}} \frac{\|u\|^2}{\|u\|_\infty} = \max\{K_0 : (\mathcal{K} - \mathcal{K}) \cap \mathbb{Z}_{K_0} \neq \emptyset\}
\] (21)
possibly trying to maximize the left hand side in the selection of $\Psi_0$.

Therefore, while the requirements imposed on $M$ in Prop. 1 and Prop. 2 are unchanged between $\mathcal{K}$ and $\mathcal{K}'$ in Prop. 1 (by the invariance (P12) of $\mathcal{V}(\mathcal{K})$ in Table. 1) and since $\|x' - y'\| = \|x - y\|$ for $x' = \Psi_0 x$ and $y' = \Psi_0 y$, “rotating” $\mathcal{K}$ with $\Psi_0$ helps to lighten the condition imposed on $x - y$. Moreover, this rotation is of course equivalent to directly build a sensing matrix $\Phi' = \Phi \Psi_0$ to quasi-isometrically embed the set $\mathcal{K}$ with the mapping $A(\cdot) := \mathcal{Q}(\Phi')$. Actually, in the case where $\Psi = \mathbb{I}$ as above, a good choice for $\Psi_0$ is the DCT basis, i.e., using the incoherence of those two bases that prevents sparse signal to be sparse in the frequency domain and exploiting possible advantage of the fast FFT-based matrix-vector multiplication offered by the DCT. Notice, however, that the procedure above cannot work if $\mathcal{K}$ is expected to generate differences of vectors that are sparse in different bases, e.g., a union of incoherent bases such has $\mathbb{I}$ and the DCT basis. In such case, it could be hard to maximizes the RHS of (21) over $\Psi_0$.

Second, we can notice that for non-Gaussian random measurements, the term $\kappa_{sg} / \sqrt{K_0}$ in (16) is actually lower bounded. This is simply due to the relation $\|u\|^2 \leq N\|u\|_\infty^2$, which implies $K_0 \leq N$ whatever the properties of the vector $u \in \mathcal{K} - \mathcal{K} \subset \mathbb{R}^N$. Consequently,
\[
\frac{\kappa_{sg}}{\sqrt{K_0}} \geq \frac{\kappa_{sg}}{\sqrt{N}},
\]
which limits our hope to tighten the multiplicative error of quantized non-Gaussian quasi-isometric embeddings, except if one considers asymptotic regimes where $N$ can be considered as much larger than $\kappa_{sg}^2$.

Third, as already noticed in [28], Prop. 1 allows us to distinguish different regimes of the quasi-isometric embedding. If $\delta \simeq 0$, the quantization operator tends to the identity function and (16) converges to an $\ell_2/\ell_1$ variant of the RIP property generalized to any sets $\mathcal{K}$ and to sub-Gaussian random matrices. For $\delta \gg 2\|\mathcal{K}\|$ the embedding becomes purely quasi-isometric and, keeping the context defined in Prop. 1 (16) involves
\[
(\frac{2}{c})^{\frac{1}{2}}\|x - y\| - c(e\delta + \frac{\kappa_{sg}}{\sqrt{K_0}}) \leq D(x, y) \leq (\frac{2}{c})^{\frac{1}{2}}\|x - y\| + c(e\delta + \frac{\kappa_{sg}}{\sqrt{K_0}}),
\] (22)
for some absolute constant $c > 0$. However, in this case, the quantization becomes also essentially binary. In fact, it is exactly binary for random matrices whose entries are generated from a bounded symmetric sub-Gaussian distribution, i.e., from $\varphi \sim \mathcal{N}_{sg,a}(0, 1)$ with $\|\varphi\|_\infty \leq F$ for some $F > 0$. In this case, since $\mathcal{K}$ is assumed bounded, for all $u \in \mathcal{K}$, $\|\Phi u\|_1 \leq F\|\mathcal{K}\|$ and the components of $A(u) = \mathcal{Q}(\Phi u + \xi)$ with $\xi \sim \mathcal{U}(\mathbb{I})$ can only take two values, e.g., $\{-1, 0\}$ if $0 \in \mathcal{K}$. Moreover, if $\varphi$ is unbounded and $0 \in \mathcal{K}$, its sub-Gaussian nature makes that the fraction of quantized measurements that do not belongs to $\{-1, 0\}$ can be made arbitrarily close to 0 when $\delta$ increases. In conclusion, similarly to [30], we have basically defined a one-bit quantized embedding that preserves the norm of the projected vectors, as opposed to the mapping $A'(\cdot) = \text{sign}(\Phi \cdot)$ that looses this information [28] [30]. Notice there that the role of our dithering can be compared to the one of the threshold inserted in the sign quantization in [30].

Conversely to this work, however, we do not provide any algorithm to reconstruct a signal from its quantized mapping by $A$.\footnote{Strictly speaking, while $\text{det} \Psi_0 = 1$, $\Psi_0 \in \mathcal{O}_N$ is a rotation only if its determinant is 1.}
Fourth, it is not clear if Prop. 1 could be turned into a quasi-isometric embedding between $(\mathcal{K} \subset \mathbb{R}^N, \ell_2)$ and $(A(\mathcal{K}) \subset \delta \mathbb{Z}^M, \ell_2)$. As said earlier, for Gaussian random matrix and for finite set $\mathcal{K}$, an approximate quasi-isometric embedding can be found by integrating a non-linear distortion of the $\ell_2$-distance, i.e., in (16) for $\kappa_{eq} = 0$, $\|x - y\|$ is replaced by $g_\delta(\|x - y\|)$ for some non-decreasing function $g_\delta: \mathbb{R}_+ \to \mathbb{R}_+$. Interestingly, $|g_\delta(\lambda) - \lambda| = O(\sqrt{\delta} \lambda)$ for $\lambda \gg \delta$ and $|g_\delta(\lambda) - (\sqrt{2} \lambda / \sqrt{\pi})^{1/2}| = O(\lambda)$ for $\lambda < \delta$, so that for small $\delta$ or large $\lambda$, $g_\delta(\lambda) \approx \lambda$. Therefore, as soon as $\|x - y\| \gg \delta$, we get approximately a $\ell_2/\ell_2$ quasi-isometric embedding. Knowing if this extends to any subset $\mathcal{K}$ and to sub-Gaussian random matrices is left for a future work.

Fifth, beyond the mere analysis of the quasi-isometric properties of our quantized mapping and closer to the context of quantized compressed sensing, this paper does not say anything on the reconstruction algorithms that could be developed for recovering a signal $x$ from its observations $z = Q(\Phi x)$. A few algorithms exist for realizing this operation, some when $\delta$ is small compared to the expected dynamic of $\|\Phi x\|$ [14, 21] [45], others in the 1-bit CS setting [12, 28, 38, 39]. However, for the first category, their stability (or convergence) does not rely on a quasi-isometric embedding property but rather on the restricted isometry property [11, 14, 31] or on variations involving other norms [24, 25]. In future research, it is appealing to find a proof of the instance optimality of those algorithms, e.g., for the basis pursuit dequantizer (BPDQ), using the quasi-isometric property promoted by Prop. 1 even if recent interesting results show that optimal “non-RIP” proof can be developed for BPDQ [19].

We conclude this section by mentioning that it would be useful to prove Prop. 1 for structured random matrices, e.g., for random Fourier/Hadamard ensembles [21]. This would lead to fast computation of the resulting quantized mapping, with potential application in nearest neighbors search for databases of high-dimensional signals. An open question is also the possibility to extend this work to universally quantized embeddings [9, 10, 11], i.e., with periodic quantizer when defining $Q$ in (3). This could potentially lead to quasi-isometric embeddings with (exponentially) decaying distortions on vectors set with small Gaussian width and using sub-Gaussian random matrices.

5 Proof of Proposition 1

The architecture of this proof is inspired by the one developed in [37] for characterizing 1-bit random mapping $A': \mathbb{R}^N \to \{\pm 1\}^M$, $u \in \mathbb{R}^N \mapsto A'(u) = \text{sign}(\Phi u)$. As will be clear below, some of the ingredients developed there had of course to be adapted to the specificities of $A$ and of our scalar quantization. Compared to [37] we have also paid attention to optimize the dependency of $M$ to the desired level of distortions induced by $A$ in (3).

Prop. 1 is proved as a special case of a more general proposition based on a “softer” variant of $D$. This new pseudo-distance is established as follows. Defining the random mapping $u \in \mathbb{R}^N \mapsto \Phi'(u) := \Phi u + \xi$, with $\Phi'_i$ its $i^{th}$ component, we observe that for any $x, y \in \mathbb{R}^N$,

$$D(x, y) = \frac{\delta}{M} \sum_{i=1}^{M} \sum_{k \in \mathbb{Z}} ||\mathcal{E}(\Phi'_i(x) - k\delta, \Phi'_i(y) - k\delta)||,$$

with the distinct sign event $\mathcal{E}(a, b) := \{\text{sign } a \neq \text{sign } b\}$. In words, for each $i \in [M]$, the sum over $k$ above simply counts the number of thresholds in $\delta \mathbb{Z}$ separating $\Phi'_i(x) = \varphi_i^T x + \xi$ and $\Phi'_i(y) = \varphi_i^T y + \xi$ on the real line, since $||\mathcal{E}(\Phi'_i(x) - k\delta, \Phi'_i(y) - k\delta)||$ is equal to 1 for those and 0 for any other thresholds.

Notice that the decomposition (22) also justifies the observation made at the end of Sec. 3.2 namely the existence of uniform random tessellations of $\mathbb{R}^N$. Indeed, from the definition of $A$, for each $i \in [M]$, $\sum_{k \in \mathbb{Z}} ||\mathcal{E}(\Phi'_i(x) - k\delta, \Phi'_i(y) - k\delta)||$ also counts the number of parallel affine
For any Lemma 1.
both hyperplanes $\Pi_d$ evaluation of the impact of the distance softening by observing that leads to Prop. 1.

$F_d$ Fig. 1 explains how far apart, separating $x$ are created when counting the number of thresholds $k\delta$.

Figure 1: Behavior of the distance $d'(a, b)$ for $a, b \in \mathbb{R}$. On the top, $t > 0$ and forbidden areas determined by $F^t$ are created when counting the number of thresholds $k\delta$ separating $a$ and $b$. For instance, for an additional point $c \in \mathbb{R}$ as on the figure, $d'(a, b) = d'(c, b) = 3\delta$ but $3\delta = d'(a, b) = d'(c, b) + \delta \geq d'(c, b)$ as $c$ lies in one forbidden area. On the bottom figure, $t \leq 0$ and threshold counting procedure operated by $d^t$ is relaxed. Now $d'(a, b)$ counts the number of limits (in dashed) of the green areas determined by $F^t$, recording only one per thresholds $k\delta$, that separate $a$ and $b$. Here, for $e \in \mathbb{R}$ as on the figure, $d'(a, b) = d'(c, b) = 3\delta$ but $4\delta = d'(c, b) = d'(c, b) + \delta \geq d'(a, b)$.

hyperplanes $\Pi_i := \{u \in \mathbb{R}^N : \exists k \in \mathbb{Z}, \varphi_i^T u + \xi_i - k\delta = 0\}$, all normal to $\varphi_i$ and $\delta/\|\varphi_i\|$ far apart, separating $x$ and $y \in \mathbb{R}^N$, $t \leq 0$ and forbidden areas determined by $F^t$ as on the figure, $d^t(a, b) = d^t(c, b) = 3\delta$ but $4\delta = d^t(c, b) = d^t(c, b) + \delta \geq d^t(a, b)$.

Based on this observation, we introduce for some $t \in \mathbb{R}$ the new pseudo-distance

$$D^t(x, y) := \frac{\delta}{M} \sum_{i=1}^M \sum_{k \in \mathbb{Z}} [F^t(\Phi_i^T(x) - k\delta, \Phi_i^T(y) - k\delta)],$$

by defining the set

$$F^t(a, b) = \{a > t, b \leq -t\} \cup \{a < -t, b \geq t\}.$$  (25)

The pseudo-distance $D^t$ is a non-increasing function of $t$, with $F^0(a, b) = E(a, b)$ and

$$D^{|t|}(x, y) \leq D(x, y) \leq D^{-|t|}(x, y).$$

The behavior of $D^t$ is best understood by introducing the one-dimensional distance

$$d^t(a, b) := \delta \sum_{k \in \mathbb{Z}} [F^t(a - k\delta, b - k\delta)] \in \delta \mathbb{N}, \quad \text{for } a, b \in \mathbb{R},$$

so that

$$D^t(x, y) = \frac{1}{M} \sum_{i=1}^M d^t(\Phi_i^T(x), \Phi_i^T(y)).$$  (27)

Fig. 1 explains how $d^t(a, b)$ evolves for positive and negative $t$, observing that, for each $k \in \mathbb{Z}$, $F^t(a - k\delta, b - k\delta)$ determines forbidden or relaxed areas around the thresholds $k\delta$ separating $a$ and $b$ and counted by $d^t(a, b)$. Moreover, the next Lemma, proved in App. 3, provides a first evaluation of the impact of the distance softening by observing that $d^t(a, b)$ is not very far from both $|a - b|$ and $d^t(a, b)$ for $s$ close to $t$.

Lemma 1. For any $a, b \in \mathbb{R}$ and $t, s \in \mathbb{R}$,

$$|d^t(a, b) - d^s(a, b)| \leq 4(\delta + |t - s|),$$  (28)

$$|d^t(a, b) - |a - b|| \leq 4(\delta + |t|).$$  (29)

As announced above, we aim now at proving the next proposition whose special case $t = 0$ leads to Prop. 1.
Proposition 3. Given \( \delta > 0, \epsilon \in (0, 1), t \in \mathbb{R}, k_0 > 0 \), a bounded subset \( K \subset \mathbb{R}^N \) and a sub-Gaussian distribution \( N_{sg,a} \) respecting (3), for \( \delta \leq k_{sg} < \infty \), there exist some values \( C, c, c' > 0 \), only depending on \( \alpha \), such that, if

\[
M \geq C \max(\epsilon^{-2}H(K, \sqrt{\delta^2c^3}), \frac{1}{\sqrt{2\pi}} \log(R^{(\sqrt{\delta^2c^3})})),
\]

with \( H(K, \eta) \) the Kolmogorov-\( \eta \)-entropy of \( K \) and \( R_{\eta} := (K - K) \cap \eta \mathbb{R}^N \) for \( \eta > 0 \), then for \( \Phi \sim N_{sg,a}(0, 1) \), \( G \) is a dithering \( \xi \sim U(0, \delta) \) and the associated mapping \( A \) defined in (3), we have with probability at least \( 1 - e^{-c\epsilon^2M} \) and for all pairs \( x, y \in K \) with \( x - y \in \mathcal{K}_0 \),

\[
| \mathcal{D}(x, y) - (\frac{\epsilon}{2})^{|\mathcal{S}|} \|x - y\| | \leq (\epsilon + \sqrt{\frac{k_{sg}}{\sqrt{c}M}}) \|x - y\| + c'(t + \delta c).
\]

Proof. The proof sketch of Prop. 3 is as follows: (i) given \( x, y \in \mathbb{R}^N \), we first show that the r.v. \( \mathcal{D}(x, y) \) concentrates with high probability around \( (\frac{\epsilon}{2})^{|\mathcal{S}|} \|x - y\| \) up to a systematic bias \( \sqrt{\frac{k_{sg}}{\sqrt{c}M}} \|x - y\| \) due to the sub-Gaussian nature of \( \Phi \) and controlled by the anti-sparse level of \( x - y \); (ii) we take a finite covering of \( K \) by a \( \eta \)-net \( \mathcal{G}_\eta \subset K \) (for \( \eta > 0 \)) and we extend the concentration of \( \mathcal{D}(x, y) \) to all vectors of \( \mathcal{G}_\eta \) by union bound; (iii) we show that the softened pseudo-distance \( \mathcal{D} \) is sufficiently continuous in a neighborhood of each pair of vectors in \( \mathcal{G}_\eta \), which allows us then to extend (31) to all pair of vectors in \( K \), as stated by Prop. 3.

(i) Concentration of \( \mathcal{D}(x, y) \): Given a fixed pair \( x, y \in \mathbb{R}^N \), we show that \( \mathcal{D}(x, y) \) concentrates around its mean by bounding its sub-Gaussian norm as defined in (1). From (27),

\[
\mathcal{D}(x, y) = M^{-1} \sum_i Z_i^t \text{ with the } M \text{ random variables } Z_i^t := d'(\varphi_i^t x + \xi, \varphi_i^t y + \xi_i) \text{ for } 1 \leq i \leq M.
\]

However, the sum of \( D \) independent sub-Gaussian random variables \( \{X_1, \cdots, X_D\} \) is approximately invariant under rotation \( \mathcal{H} \), which means

\[
\| \sum_i (X_i - EX_i) \|_{\psi_2}^2 \lesssim \sum_i \|X_i - EX_i\|_{\psi_2}^2.
\]

Therefore, from (32), we find

\[
\| \sum_i (Z_i^t - EZ_i^t) \|_{\psi_2}^2 \lesssim \sum_i \|Z_i^t - EZ_i^t\|_{\psi_2}^2 = \sum_i \|Z_i^t - EZ_i^t\|_{\psi_2}^2 \lesssim 4M \|Z_1^t\|_{\psi_2}^2.
\]

As shown in following lemma (proved in App. C by using Lemma 1) \( \|Z_1\|_{\psi_2} \) can be upper bounded, and with it, the sub-Gaussian norm of \( \mathcal{D}(x, y) \).

Lemma 2. Let us take \( \varphi \sim N_{sg,a}(0, 1) \) and \( \xi \sim U([0, \delta]) \). For a fixed \( t \in \mathbb{R} \), the random variable \( Z^t := d'(\varphi^t x + \xi, \varphi^t y + \xi) \) is sub-Gaussian with a \( \psi_2 \)-norm bounded by

\[
\|Z^t\|_{\psi_2} \lesssim \delta + |t| + \|x - y\|.
\]

Moreover,

\[
|EZ^t - \mu_{sg}(x - y)| \lesssim |t|,
\]

with \( \mu_{sg}(x - y) = E[\langle \varphi, x - y \rangle] = (\frac{\epsilon}{2})^{|\mathcal{S}|} \|x - y\| \) if \( \varphi \sim N(0, 1) \).

Consequently, from (33) and (34), \( X := \frac{1}{\sqrt{M}} \sum_i (Z_i^t - EZ_i^t) \) is itself sub-Gaussian with \( \|X\|_{\psi_2} \leq \delta + |t| + \|x - y\| \). Therefore, from the tail bound (5), there exists a \( c > 0 \) such that for any \( \epsilon > 0 \)

\[
\mathbb{P}[\|X\|_{\psi_2} > \epsilon (\delta + |t| + \|x - y\|)] \leq 2 \exp(-c\epsilon^2M).
\]
Since $\mathbb{E}Z_i^t = \mathbb{E}Z_i^1$ and $\mathbb{E}Z_i^0 = \mathbb{E}[\varphi, x - y] = \mu_{sg}(x - y)$ for all $i \in [M]$, (35) provides

$$|\frac{1}{M} \sum_i(Z_i^t - \mathbb{E}Z_i^t)| = |D^t(x, y) - \mathbb{E}Z_i^1|$$

$$\geq |D^t(x, y) - \mu_{sg}(x - y)| - |\mathbb{E}Z_i^1 - \mu_{sg}(x - y)|$$

for some constant $c' > 0$, and

$$\mathbb{P}[|D^t(x, y) - \mu_{sg}(x - y)| > c'|t| + \epsilon (\delta + |t| + \|x - y\|)] \leq 2 \exp(-c\epsilon^2 M).$$

(ii) Extension to a covering of $K$: Given a radius $\eta > 0$ to be specified later, let $G_\eta$ an $\eta$-net of $K$, i.e., a finite vector set such that for any $x \in K$ there exists an $x_0 \in G_\eta$ with $\|x - x_0\| \leq \eta$. In particular, any vectors $x, y \in K$ can then be written as

$$x = x_0 + x', \quad y = y_0 + y',$$

for some $x_0, y_0 \in G_\eta$ and $x', y' \in (K - K) \cap \eta \mathbb{R}^N$. We also assume that the size of $G_\eta$ is minimal so that, by definition, $\log |G_\eta| = \mathcal{H}(K, \eta)$, with $\mathcal{H}$ the Kolmogorov $\eta$-entropy of $K$.

Since there are no more than $|G_\eta|^2$ distinct pairs of vectors in $G_\eta$, given $t \in \mathbb{R}$, a standard union bound over (36) shows that there exist some constant $C, c', c'' > 0$ such that, if $M \geq C\epsilon^{-2}\mathcal{H}(K, \eta)$

$$\mathbb{P}[\forall x_0, y_0 \in G_\eta, |D^t(x_0, y_0) - \mu_{sg}(x_0 - y_0)| \leq c'|t| + \epsilon (\delta + |t| + \|x_0 - y_0\|)]$$

$$\geq 1 - |G_\eta|^2 \exp(-c\epsilon^2 M) \geq 1 - 2 \exp(-c\epsilon^2 M).$$

(iii) Extension to $\mathbb{R}^N$ by continuity of $D^t$: We can extend the event characterized in (38) to all pairs of vectors in $K$ by analyzing the continuity property of $D^t$ in a limited neighborhood around the considered vectors. We propose here to analyze this continuity with respect to $\ell_2$-perturbations of those vectors, as compared to $\ell_1$-perturbations in (37). As will be clearer later, this allows us to reach a better control over $M$ with respect to $\epsilon$.

Lemma 3 (Continuity with respect to $\ell_2$-perturbations). Let $x_0, y_0, x', y' \in \mathbb{R}^N$. We assume that $\|\Phi x'\| \leq \eta \sqrt{M}, \|\Phi y'\| \leq \eta \sqrt{M}$ for some $\eta > 0$. Then for every $t \in \mathbb{R}$ and $P \geq 1$ one has

$$D^{t+\eta \sqrt{P}}(x_0, y_0) - 4(\frac{\delta}{P} + \frac{\eta}{\sqrt{P}}) \leq D^t(x_0 + x', y_0 + y') \leq D^{t-\eta \sqrt{P}}(x_0, y_0) + 4(\frac{\delta}{P} + \frac{\eta}{\sqrt{P}}).$$

The proof is given in App. [D]. Interestingly, the following proposition proved in App. [E] shows that $\|\Phi x'\|$ and $\|\Phi y'\|$ can indeed be bounded uniformly for all $x', y' \in \mathcal{R}_\eta := (K - K) \cap \eta \mathbb{R}^N$.

Lemma 4 (Diameter stability under random projections). Let $\mathcal{R} \subset \mathbb{R}^N$ be bounded, i.e., $\|\mathcal{R}\| : = \sup_{u \in \mathcal{R}} \|u\| < \infty$ and assume $\mathcal{R} \ni 0$. Then, for some $c > 0$, if

$$M \geq \frac{\alpha^{4\alpha}(\mathcal{R})^2}{\|\mathcal{R}\|^2},$$

for $\Phi \sim \mathcal{N}_{sg, \alpha}(0, 1)$ and with probability at least $1 - \exp(-c\alpha^{-4}M)$, we have for all $x \in \mathcal{R}$

$$\frac{1}{M}\|\Phi x\|^2 \leq \|\mathcal{R}\|^2,$$

i.e., $\|\Phi \mathcal{R}\| \leq \sqrt{M} \|\mathcal{R}\|$.
For the sake of simplicity, we consider below the sub-Gaussian parameter $\alpha$ as fixed and integrate it in explicit or hidden constants, as in the notations "$\gtrsim$" or "$\lesssim$". Noting that $\|R_\eta\| \lesssim \eta$ and using union bound over (38) and (40), we get that for 

$$M \gtrsim \max(\epsilon^{-2}H(K, \eta), \eta^{-2}w(R_\eta)^2),$$

with probability higher than $1 - 4\exp(-c'\epsilon^2 M)$, for all $x_0, y_0 \in \mathcal{G}_\eta$, and all $x', y' \in \mathcal{R}_\eta$,

$$\left| \mathcal{D}^{\pm \eta \sqrt{P}}(x_0, y_0) - \mu_{sg}(x_0 - y_0) \right| \leq c\|t - \eta \sqrt{P}\| + \epsilon (\delta + |t - \eta \sqrt{P}| + \|x_0 - y_0\|),$$

$$\left(41\right)$$

$$\left| \mathcal{D}^{\pm \eta \sqrt{T}}(x_0, y_0) - \mu_{sg}(x_0 - y_0) \right| \leq c\|t + \eta \sqrt{P}\| + \epsilon (\delta + |t + \eta \sqrt{P}| + \|x_0 - y_0\|),$$

$$\left(42\right)$$

$$\|\Phi x'\|_2 \leq \eta \sqrt{M}, \quad \|\Phi y'\|_2 \leq \eta \sqrt{M},$$

$$\left(43\right)$$

for some $C, c, c' > 0$ depending only on $\alpha$.

Therefore, for any $x, y \in \mathcal{K}$, using sequentially (37), (43), the upper bound given in Lemma 4 and (41) provides

$$\mathcal{D}^t(x, y) \leq \mathcal{D}^{t - \eta \sqrt{P}}(x_0, y_0) + 4(\frac{\delta}{P} + \frac{\eta}{\sqrt{P}})$$

$$\leq (c + \epsilon)t - \eta \sqrt{P} + \mu_{sg}(x_0 - y_0) + \epsilon \|x_0 - y_0\| + c\|t\| + 4(\frac{\delta}{\sqrt{P}} + \frac{\eta}{\sqrt{P}}).$$

However, given $\varphi \sim \mathcal{N}_{sg, \alpha}(0, 1)$, using Jensen’s inequality, the reverse triangular inequality and (3), we find

$$\left| \mu_{sg}(x_0 - y_0) - \mu_{sg}(x - y) \right| = |E(\langle \varphi, x_0 - y_0 \rangle) - E(\langle \varphi, x - y \rangle)|$$

$$\leq E(\langle \varphi, x' \rangle) + E(\langle \varphi, y' \rangle) \leq 2\eta.$$

Moreover, $\|x_0 - y_0\| - \|x - y\| \leq 2\eta$, so that,

$$\mathcal{D}^t(x, y) - \mu_{sg}(x - y) \leq \epsilon \|x - y\| + (c + \epsilon)(|t| + \eta \sqrt{P}) + 2\eta + 2\epsilon\eta + c\delta + 4(\frac{\delta}{\sqrt{P}} + \frac{\eta}{\sqrt{P}}).$$

If $x - y \in \mathcal{Z}_{K_0}$, then (13) induces $|\mu_{sg}(x - y) - (\hat{x})^{1/2}||x - y||| \leq \kappa_{sg}||x - y||/\sqrt{K_0}$ and assuming $\epsilon < 1$, there exists a $c > 0$ such that

$$\mathcal{D}^t(x, y) - (\hat{x})^{1/2}||x - y|| \leq (c + \kappa_{sg}\sqrt{K_0})||x - y|| + c(|t| + \eta \sqrt{P} + \eta + c\delta + \frac{\delta}{\sqrt{P}} + \frac{\eta}{\sqrt{P}}).$$

Taking $P = \epsilon^{-1} \geq 1$ and $\delta \epsilon^{3/2} < \delta \epsilon$, which gives $\eta \sqrt{P} = \delta \epsilon$ and $\eta \sqrt{T} = \delta \epsilon^2 \leq \delta \epsilon$, we find for another $c > 0$

$$\mathcal{D}^t(x, y) - (\hat{x})^{1/2}||x - y|| \leq (c + \kappa_{sg}\sqrt{K_0})||x - y|| + c(|t| + \delta \epsilon).$$

Similarly, using (37), (43), the lower bound given in Lemma 3 and (42), we obtain

$$\mathcal{D}^t(x, y) - (\hat{x})^{1/2}||x - y|| \geq -(c + \kappa_{sg}\sqrt{K_0})||x - y|| - c(|t| + \delta \epsilon).$$

Finally, we have thus shown that there exist some $c, c' > 0$ such that for

$$M \gtrsim \max(\epsilon^{-2}H(K, \sqrt{\delta^2 \epsilon^3}), \frac{1}{\delta^2 \epsilon^3}, w(R_{\sqrt{\delta^2 \epsilon^3}})^2),$$

$$\left(44\right)$$

with probability at least $1 - 4\exp(-c'\epsilon^2 M)$ the bound

$$\left| \mathcal{D}^t(x, y) - (\hat{x})^{1/2}||x - y|| \right| \leq (c + \kappa_{sg}\sqrt{K_0})||x - y|| + c(|t| + \delta \epsilon)$$

holds for all $x, y \in \mathcal{K} \cap \mathcal{Z}_{K_0}$, which finishes the proof of Prop. 3.
Let us conclude this section by simplifying the condition on $M$ in Prop. 3, hence explaining the simpler requirements of Prop. 1. If $K = \Sigma^\Psi_K \cap \mathbb{B}^N$ with $\Psi$ an ONB of $\mathbb{R}^N$, then, since $w(\Sigma^\Psi_K \cap \mathbb{B}^N) = w(\Sigma_K \cap \mathbb{B}^N)$ by invariance under orthogonal group $O_N$ (see (12) in Table 1), we leverage the continuity of the pseudo-distance $D$ from the definition of $\Sigma^\Psi_K$ in (23). Moreover, it is known that $H(\Sigma_K \cap \mathbb{B}^N, \eta) \leq \log((\frac{\eta}{N})^k) \leq K \log(\frac{3eN}{K\eta^2}) \leq K \log(\frac{N}{K\sqrt{\delta t}e})$ (see, e.g., [16]). The Kolmogorov entropy is also invariant under $O_N$, i.e., $H(\Sigma_K \cap \mathbb{B}^N, \eta) = H(\Sigma^\Psi_K \cap \mathbb{B}^N, \eta)$. Additionally, since $\Sigma^\Psi_K$ is invariant under dilation, $\eta^{-1}K = \Sigma^\Psi_K \cap \eta^{-1}\mathbb{B}^N$ and

$$\eta^{-1}R_\eta = \eta^{-1}((K - K) \cap \eta\mathbb{B}^N) \leq ((\Sigma^\Psi_K \cap \eta^{-1}\mathbb{B}^N) - (\Sigma^\Psi_K \cap \eta^{-1}\mathbb{B}^N)) \cap \mathbb{B}^N \leq \Sigma^\Psi_{2K} \cap \min(2\eta^{-1}, 1)\mathbb{B}^N \subset \Sigma_{2K} \cap \mathbb{B}^N.$$ 

Therefore, from (F1) and (F3), $\eta^{-2}w(R_\eta)^2 = w(\eta^{-1}R_\eta)^2 \leq w(\Sigma^\Psi_{2K} \cap \mathbb{B}^N)^2 \leq K \log(N/K)$ and (44) is then satisfied if

$$M \geq \frac{1}{\epsilon^2} K \log(\frac{N}{K\sqrt{\delta t}e}),$$

as given in (17).

For a general bounded set $K$, since Sudakov inequality in (F14) provides $H(K, \eta) \leq w^2(K) \eta^2$, $R_\eta \subset (K - K)$, while (F2) and (F3) give $w(R_\eta) \leq w(K - K) \leq 2w(K)$, (44) holds if

$$M \geq \frac{1}{\epsilon^2} w(K)^2,$$

as imposed in (15).

### 6 Proof of Proposition 2

Using the context defined in Prop. 2 and for $M$ satisfying (18), we are going to show the contraposition of (19), i.e., that with probability at least $1 - 2e^{-\alpha M/(1+\delta)}$ for some $c > 0$ and for all $x, y \in K$ with $x - y \in \mathbb{Z}_{K_0}$, having $\|x - y\| > \epsilon$ involves $Q(\Phi x + \xi) \neq Q(\Phi y + \xi)$, or equivalently that

$$\|x - y\| > \epsilon \Rightarrow D(x, y) \geq \frac{\delta}{M},$$

from the definition of $D$ in (23).

The proof sketch is a follows. First, for some $\eta > 0$, we create a finite $\eta$-covering of the set $K \subset K \times K$ of vector pairs whose difference belongs to $\mathbb{Z}_{K_0}$. Second, in order to show (45), we leverage the continuity of the pseudo-distance $D$ under $\ell_2$-perturbations (Lemma 3), as it happens that all points of $K$ are obtained by $\ell_2$-perturbations of the $\eta$-covering and that, moreover, those perturbations are stable under projections by $\Phi$ (Lemma 4). Finally, we adjust $\eta$ and some additional parameters to show that, with high probability, the softened distance $D_t(x_0, y_0)$, for some $t$ depending on $\eta$, is large enough over all pairs $(x_0, y_0)$ of the covering compatible with $\|x - y\| \geq \epsilon$, hence inducing (15).

Let us define the set $\tilde{K} = \{(x, y) \in K \times K : x - y \in \mathbb{Z}_{K_0}\} \subset K \times K$. We introduce a minimal $\eta$-net $\tilde{G}_\eta \subset \tilde{K}$ of $\tilde{K}$ with $0 < \eta < \epsilon/2$ to be specified later, such that for all $(x, y) \in \tilde{K}$, there exists a $(x_0, y_0) \in \tilde{G}_\eta$ with

$$\|(x, y) - (x_0, y_0)\| \leq \eta,$$

which involves $\|x - x_0\| \leq \eta$ and $\|y - y_0\| \leq \eta$.

The size of this minimal $\eta$-net is bounded as $\log |\tilde{G}_\eta| \leq 2H(K, \eta/\sqrt{2})$. Indeed, by the semiadditivity of the Kolmogorov entropy [32, Theorem 2], $K \subset K \times K$ involves that $H(K, \rho) \leq$
\(\mathcal{H}(\mathcal{K} \times \mathcal{K}, \rho)\) for any \(\rho > 0\). Since a \(\rho\)-net of \(\mathcal{K} \times \mathcal{K}\) can be obtained by the product \(\mathcal{G}_{\rho'} \times \mathcal{G}_{\rho'}\), with \(\rho' = \rho/\sqrt{2}\) and \(\mathcal{G}_{\rho'}\) a \(\rho'\)-net covering of \(\mathcal{K}\), we obtain \(\mathcal{H}(\mathcal{K}, \rho) \leq 2\mathcal{H}(\mathcal{K}, \rho/\sqrt{2})\).

As for the proof of Prop. 1 in Sec. 5, by construction, all \((x, y) \in \bar{\mathcal{K})\) can also be written as

\[ (x, y) = (x_0, y_0) + (x', y'), \]

with \((x_0, y_0) \in \mathcal{G}_\eta\), \((x', y') \in (\mathcal{K} - \mathcal{K}) \cap \eta B^{2N}\). Notice that we have also \(x', y' \in \mathcal{R}_\eta := (\mathcal{K} - \mathcal{K}) \cap \eta B^{2N}\), since \(x, x_0, y, y_0 \in \mathcal{K}\) and \(\max(||x'||, ||y'||) \leq ||(x', y')|| \leq \eta\).

As stated by Lemma 4, the diameter of the residual set \(\mathcal{R}_\eta\) is stable with respect random projections. Since \(||\mathcal{R}_\eta|| \leq \eta\), there exist indeed two values \(C, c > 0\), only depending on the sub-Gaussian norm \(\alpha\), such that if

\[ M \geq C\eta^{-2}w(\mathcal{R}_\eta)^2 \]

and \(\Phi \sim N_{sg,\alpha}^M(0, 1)\), we have with probability at least \(1 - 2\exp(-cM)\),

\[ ||\Phi\mathcal{R}_\eta|| := \sup_{u \in \mathcal{R}_\eta} ||\Phi u|| \leq \sqrt{M||\mathcal{R}_\eta||} \leq \eta\sqrt{M}. \]

Therefore, \(||\Phi x'|| \leq \eta\sqrt{M}\) and \(||\Phi y'|| \leq \eta\sqrt{M}\) under the same conditions.

Moreover, if the previous event occurs, then, Lemma 3 for \(t = 0\) shows that for any \(P \geq 1\),

\[ \mathcal{D}(x, y) = \mathcal{D}^0(x_0 + x', y_0 + y') \geq \mathcal{D}^{\eta\sqrt{P}}(x_0, y_0) - 4(\frac{\eta}{P} + \frac{n}{\sqrt{P}}). \]

Consequently, for reaching \(\mathcal{D}(x, y) \geq \delta/M\) as expressed in \([45]\), since \(||x - y|| \geq \epsilon\) involves \(||x_0 - y_0|| \geq \epsilon - 2\eta\), the proof can be deduced if we can guarantee that, for all \((u, v) \in \mathcal{G}_\eta\) with \(||u - v|| \geq \epsilon - 2\eta\), the probability that \(\mathcal{D}^{\eta\sqrt{P}}(u, v) \geq 4(\frac{\eta}{P} + \frac{n}{\sqrt{P}}) + \frac{\delta}{M}\) tends (exponentially) to one with \(M\).

Let us upper bound the corresponding probability of failure. We can first observe the following result on a fixed pair of vectors. This one is proved in App. [P].

**Lemma 5.** Let \(u, v\) be in \(\mathbb{R}^N\) with \(u - v \in \mathcal{Z}_{K_0}\) for some \(K_0 > 0\) and \(||u - v|| \leq \epsilon_0\) for \(\epsilon_0 > 0\). For \(\delta > 0\), \(t \geq 0\), \(r \in [\eta, \eta]\), \(\Phi \sim N_{sg,\alpha}^N(0, 1)\), \(\xi \sim \mathcal{U}^N([0, \delta])\) and the pseudo-distance \(\mathcal{D}^t\) defined in \([24]\), we have

\[ \mathbb{P}[\mathcal{D}^t(u, v) \leq \frac{\delta}{M} r] \leq \exp(-\frac{(M - r)^2}{2Mr}) \]

with \(p := \mathbb{P}[d^t(\varphi^t u + \xi, \varphi^t v + \xi) \neq 0], \varphi \sim N_{sg,\alpha}^N(0, 1)\) and \(\xi \sim \mathcal{U}([0, \delta])\). Moreover, if \(\sqrt{K_0} \geq 16\kappa_{sg}\),

\[ p \geq \frac{1}{16(\delta + \epsilon_0)} ||u - v|| - 2\frac{\delta}{\delta + \epsilon_0}. \]

From the discrete nature of \(\mathcal{D}^t\), the previous lemma (with \(t\) sets to \(\eta\sqrt{P}\)) shows that for a fixed pair of vectors \(\mathcal{D}^{\eta\sqrt{P}}(u, v) \geq \frac{\delta}{M}(r + 1)\) holds with probability at least \(1 - \exp(-\frac{(M - r)^2}{2Mr})\). Moreover, if

\[ \frac{\delta}{M} r \geq 4(\frac{\eta}{P} + \frac{n}{\sqrt{P}}), \]

we have

\[ \mathcal{D}^{\eta\sqrt{P}}(u, v) \geq \frac{\delta}{M}(r + 1) \Rightarrow \mathcal{D}^{\eta\sqrt{P}}(u, v) \geq 4(\frac{\eta}{P} + \frac{n}{\sqrt{P}}) + \frac{\delta}{M}. \]

Therefore, setting \(r = \lfloor M/2 \rfloor \geq M/2\), \([49]\) gives

\[ \mathbb{P}[\mathcal{D}^{\eta\sqrt{P}}(u, v) \geq 4(\frac{\eta}{P} + \frac{n}{\sqrt{P}}) + \frac{\delta}{M}] \geq 1 - \exp(-\frac{(M - r)^2}{2Mr}) > 1 - 2\exp(-\frac{M}{8}), \]
if, from (51),
\[
p \geq \frac{8}{3}(\frac{\delta}{\sqrt{\alpha}} + \frac{\eta}{\sqrt{\beta}}).
\]
We have thus to adjust \( P \) and \( \eta \) for satisfying (52). Noting that \( \epsilon - 2\eta \leq \|u - v\| \leq 2 \) if \( K \subset \mathbb{B}^N \), i.e., we can set \( \epsilon_0 = 2 \) in Lemma 5, this adjustment can be done from (50) by imposing \( B \geq C \) in
\[
p \geq \frac{1}{16(\delta + 2)}(\epsilon - 2\eta) - 2\eta\sqrt{P} \geq C := 8(\frac{1}{\beta} + \frac{\eta}{\sqrt{\beta}}).
\]
(53)
A solution is to set, for some \( c \geq 1 \) and \( d > 0 \) to be specified later, \( P = c^{2+\delta} \epsilon \geq 1 \) and
\[
\eta = d \frac{c^{3/2}}{\sqrt{2+\delta}} \leq d \epsilon.
\]
Then
\[
\epsilon - 2\eta \geq (1 - 2d)\epsilon, \quad \eta\sqrt{P} = cd \epsilon, \quad \frac{1}{\beta} = \frac{1}{c^{2+\delta}}\epsilon, \quad \eta = \frac{d}{c} \frac{2}{\delta} \epsilon,
\]
so that
\[
B \geq \frac{1 - 2d - 32cd}{16(\delta + 2)} \epsilon, \quad C \leq \frac{8}{c^{2+\delta}}(1 + cd^2) \epsilon.
\]
Fixing \( d = \frac{1}{2}(32)^{-2} \frac{\delta}{\delta + 2} < \frac{1}{2} (32)^{-2} \) and \( c = 32 \), a few estimations show finally that
\[
\epsilon^{-1} B \geq \frac{1 - (32)^{-2} - \frac{1}{2}}{16(\delta + 2)} \geq \frac{1}{33(\delta + 2)}, \quad \epsilon^{-1} C \leq \frac{8}{(32)^2(\delta + 2)}(1 + \frac{2}{64(\delta + 2)}) < \frac{1}{64(\delta + 2)},
\]
proving that for our choice of parameters, i.e., for \( P = (32)^2 \epsilon \geq 1 \) and \( \eta = \frac{1}{2}(32)^{-2} \delta(\frac{r}{2+\delta})^{3/2} \), (52) can be satisfied since \( B \geq C \). Moreover, for this choice of parameters, (52) provides
\[
p \geq \frac{\epsilon}{33(\delta + 2)}.
\]
We are now ready to complete the proof. Using the previous developments, defining \( \tilde{G}_\eta := \{(u, v) \in \tilde{G}_\eta : \|u - v\| \geq \epsilon - 2\eta\} \subset \tilde{G}_\eta \) with \( \eta \geq \delta \epsilon^{3/2} (2 + \delta)^{-3/2} \) fixed as above and \( \log |\tilde{G}_\eta| \leq \log |\tilde{G}_\eta| \leq 2 \mathcal{H}(K, \eta/\sqrt{2}) \) as explained before, by a simple union bound there exist some constants \( C, c, c' > 0 \) such that if
\[
M \geq C \frac{2+\delta}{\epsilon} \mathcal{H}(K, c \delta(\frac{r}{2+\delta})^{3/2}),
\]
then, the event
\[
\mathcal{D}^\eta \mathcal{T}(u, v) \geq 4(\frac{\delta}{\sqrt{\alpha}} + \frac{\eta}{\sqrt{\beta}}) + \frac{\delta}{3\sqrt{7}}, \quad \forall u, v \in \tilde{G}_\eta,
\]
(54)
holds with probability at least
\[
1 - 2 \exp(2 \mathcal{H}(K, \frac{\eta}{\sqrt{2}}) - M \frac{\eta}{\sqrt{\alpha}}) \geq 1 - 2 \exp(2 \mathcal{H}(K, \frac{\eta}{\sqrt{2}}) - M \frac{\eta}{33(\delta + 2)}) \geq 1 - 2 \exp(-c' \frac{M \eta}{2+\delta}).
\]
Remembering that for having (48) the diameter of \( \mathcal{R}_\eta \) must remains small under random projections by \( \Phi \) (as stated in (17) by imposing (46), we find again by union bound that for other constants \( C, c, c' > 0 \), if
\[
M \geq C \max(\frac{(2+\delta)^3}{\delta c^2} w(\mathcal{R}_\delta(\frac{r}{2+\delta})^{3/2})^2, 2+\delta \mathcal{H}(K, c \delta(\frac{r}{2+\delta})^{3/2}))
\]
(55)
then, with probability at least \( 1 - 4 \exp(-c' M \epsilon/(2 + \delta)) \), for all \( x, y \in K \) with \( x - y \in \mathcal{K}_0 \) and \( \|x - y\| \geq \epsilon \), (48) combined with (54) provides
\[
\mathcal{D}(x, y) \geq \frac{\delta}{37},
\]
as requested at the beginning.
We conclude the proof by simplifying the involved condition \([55]\) reusing to the simplification realized at the end of Sec. 5 for proving Prop. 4.

First, if \(K = \Sigma \Psi \cap B^N\) with \(\Psi\) an ONB of \(\mathbb{R}^N\), then we got in Sec. 5 that \(w^2(K) \lesssim K \log N/K\), \(\mathcal{H}(K, \eta) \lesssim K \log(\frac{3eN}{K\eta})\) and \(\eta^{-1}w(\mathcal{R}_\eta) \lesssim w(\Sigma \Psi \cap B^N)\). Therefore, \([55]\) is then satisfied if

\[
M \geq C \frac{2+\delta}{\epsilon} K \log \left( \left( \frac{N}{K^2} \right)^{3/2} \right),
\]

for some \(C > 0\).

For a general bounded set \(K\), Sudakov inequality (P. 14) and Sec. 5 provide \(\mathcal{H}(K, \eta) \lesssim \frac{w^2(K)}{\eta^2}\) and \(w(\mathcal{R}_\eta) \lesssim 2w(K)\), so that \([55]\) holds if

\[
M \geq C \frac{(2+\delta)\delta}{\epsilon^2} w(K)^2,
\]

for another constant \(C > 0\).

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A On the absolute expectation of dithered floorings difference

This short appendix proves the equality

\[
\mathbb{E}[|x + \xi| - |y + \xi|] = |x - y|, \quad \forall x, y \in \mathbb{R}, \quad \xi \sim \mathcal{U}([0, 1])
\]

Denoting \(a = \lfloor x \rfloor \in \mathbb{Z}\), \(b = \lfloor y \rfloor \in \mathbb{Z}\), \(x' = x - a \in [0, 1)\) and \(y' = y - b \in [0, 1)\), since \(\lfloor \lambda - n \rfloor = \lfloor \lambda \rfloor - n\) for any \(\lambda \in \mathbb{R}\) and \(n \in \mathbb{Z}\), we can always write

\[
\mathbb{E}[|x + \xi| - |y + \xi|] = \mathbb{E}[a - b + X],
\]

with \(X = |x' + \xi| - |y' + \xi|\). Without loss of generality, we can assume the r.v. \(X\) positive, i.e., \(x' \geq y'\) (just flip the role of \(x\) and \(y\) if this is not the case). Moreover, since \(x', y' \in [0, 1)\), \(X \in \{0, 1\}\) and

\[
\mathbb{P}(X = 0) = \mathbb{P}(x' + \xi < 1, y' + \xi < 1) + \mathbb{P}(x' + \xi \geq 1, y' + \xi \geq 1)
\]

\[
= \mathbb{P}(x' + \xi < 1) + \mathbb{P}(y' + \xi \geq 1) = 1 - x' + y'.
\]

Therefore,

\[
\mathbb{E}[a - b + X] = (|a - b| - |a - b + 1|) \mathbb{P}(X = 0) + |a - b + 1|
\]

\[
= |a - b| - (x' - y')(|a - b| - |a - b + 1|).
\]

If \(x' = y'\), then \(\mathbb{E}[a - b + X] = |a - b| = |x - y|\). Let us consider now the case \(x' > y'\). If \(x - y \geq 0\), then \(a - b \geq y' - x' > -1\) since \(x' < 1\), i.e., \(a - b \geq 0\) since \(a - b \in \mathbb{Z}\). Consequently, \([56]\) provides \(\mathbb{E}[a - b + X] = a - b + x' - y' = x - y\). When \(x - y < 0\), \(b - a > x' - y' > 0\), i.e., \(a - b \leq a - b + 1 \leq 0\), and we get \(\mathbb{E}[a - b + X] = b - a - (x' - y') = x - y\). In summary, \(\mathbb{E}[a - b + X] = |x - y|\) in all cases, which proves the result.
B Proof of Lemma 1

We start by observing that

\[ \frac{1}{\delta} |d^t(a, b) - d^t(a, b)| \leq \sum_{k \in \mathbb{Z}} |\mathbb{I}[\mathcal{F}^t(a - k\delta, b - k\delta)] - \mathbb{I}[^\mathcal{F}^t(a - k\delta, b - k\delta)]| \leq \sum_{k \in \mathbb{Z}} |\mathcal{H}^t,s(a - k\delta, b - k\delta)| \]

with

\[ \mathcal{H}^t,s(a, b) := \mathcal{F}^t(a, b) \Delta \mathcal{F}^s(a, b) := (\mathcal{F}^t(a, b) \cup \mathcal{F}^s(a, b)) \setminus (\mathcal{F}^t(a, b) \cap \mathcal{F}^s(a, b)). \]

For \( t \geq s \), \( \mathcal{F}^t(a, b) \subseteq \mathcal{F}^s(a, b) \) and \( \mathcal{H}^t,s(a, b) = \mathcal{F}^s(a, b) \setminus \mathcal{F}^t(a, b) \), while for \( t < s \), \( \mathcal{H}^t,s(a, b) = \mathcal{F}^t(a, b) \setminus \mathcal{F}^s(a, b) \). Moreover, a careful piecewise analysis made on the different sign combinations for \( s \) and \( t \) show that \( \mathcal{H}^t(a, b) \subseteq \{|a| \in [r_-, r_+]| \cup \{|b| \in [r_-, r_+]| \) with \( r_+ := \max(|s|, |t|) \) and \( r_- \) equals to \( \min(|s|, |t|) \) if \( ts \geq 0 \) and 0 otherwise. Consequently, writing \( r = r_+ - r_- \leq |t - s| \),

\[ |d^t(a, b) - d^s(a, b)| \leq \delta |\sum_{k \in \mathbb{Z}} \mathbb{I}[\{|a - k\delta| \leq |r_-|] \cup \{|b - k\delta| \leq |r_-|\}| \leq 2\delta \left( \frac{2|t|}{\delta} + 2 \right) = 4(|t - s| + \delta). \]

Moreover, if \( s = 0 \), since then \( r_- = 0 \) and \( r_+ = r = |t| \),

\[ \sum_{k \in \mathbb{Z}} \mathbb{I}[\{|a - k\delta| \leq |t|\} \cup \{|b - k\delta| \leq |t|\}] \leq 2\delta \left( \frac{2|t|}{\delta} + 1 \right) = 4|t| + 2\delta, \]

and we find

\[ |d^t(a, b) - a - b| \leq |d^t(a, b) - d(a, b)| + |d(a, b) - a - b| \]

\[ = |d^t(a, b) - d^s(a, b)| + |Q(a) - Q(b)| - |a - b| \]

\[ \leq (4|t| + 2\delta) + 2\delta = 4(|t| + \delta). \]

C Proof of Lemma 2

Let us define \( \tilde{Z} := |\varphi^T(x - y)| = |a - b| \) with the two r.v.’s \( a = \varphi^T x + \xi \) and \( b = \varphi^T y + \xi \). From \( \mathbb{E}Z = \mathbb{E}Z^0 \). Moreover, from the approximate rotational invariance property \( (32) \), \( \tilde{Z} \) is sub-Gaussian with \( \|\tilde{Z}\|_{\psi_2} = \|\varphi^T(x - y)\|_{\psi_2} \lesssim \|x - y\| \), and using Lemma 1 and the bound \( \|\cdot\|_{\psi_2} \leq \|\cdot\|_{\infty} \), we find

\[ \|Z^t\|_{\psi_2} \leq \|Z^t - \tilde{Z}\|_{\psi_2} + \|\tilde{Z}\|_{\psi_2} \]

\[ \lesssim \|d^t(a, b) - a - b\|_{\psi_2} + \|x - y\| \]

\[ \lesssim \delta + |t| + \|x - y\|, \]

which demonstrates the sub-Gaussianity of \( Z^t \).

For the expectation, writing \( a = a' + \xi \) and \( b = b' + \xi \) with \( a' = \varphi^T x \) and \( b' = \varphi^T y \), by Jensen’s inequality and the law of total expectation, we find

\[ \mathbb{E}[Z^t - Z^0] = \mathbb{E}_{\varphi} \mathbb{E}_{\xi}[d^t(a' + \xi, b' + \xi) - d(a' + \xi, b' + \xi)]. \]

However, reusing some elements of the proof of Lemma 1 and considering \( \varphi \) fixed,

\[ \mathbb{E}_{\xi}|d^t(a' + \xi, b' + \xi) - d(a' + \xi, b' + \xi)| \]

\[ \leq \delta \sum_{k \in \mathbb{Z}} \mathbb{E}_{\xi} \mathbb{I}[\{|a' + \xi - k\delta| \leq |t|\} \cup \{|b' + \xi - k\delta| \leq |t|\}], \]

\[ \leq \delta \sum_{k \in \mathbb{Z}} \mathbb{E}_{\xi} \mathbb{I}[\{|a' + \xi - k\delta| \leq |t|\}] + \delta \sum_{k \in \mathbb{Z}} \mathbb{E}_{\xi} \mathbb{I}[\{|b' + \xi - k\delta| \leq |t|\}] . \]
Moreover, since $\xi \sim \mathcal{U}([0, \delta])$,
\[
\delta \sum_{k \in \mathbb{Z}} \mathbb{E}_\xi \mathbb{I}[\{|a'| + \xi - k\delta| \leq |t|\}] = \sum_{k \in \mathbb{Z}} f_0^\delta \mathbb{I}[\{|a'| + s - k\delta| \leq |t|\}] ds = \int_0^t \mathbb{I}[\{|a'| + s| \leq |t|\}] ds = 2|t|,
\]
which provides also $\delta \sum_{k \in \mathbb{Z}} \mathbb{E}_\xi \mathbb{I}[\{|b'| + \xi - k\delta| \leq |t|\}] = 2|t|$. Consequently, since these two quantities do not depend on $\varphi$, we find $|E \mathbb{Z}' - E \mathbb{Z}'_0| \lesssim |t|$. Finally, if $\varphi \sim \mathcal{N}(0, 1)$, $Z^0 \sim \mathcal{N}(0, \|x - y\|^2)$, and $\mathbb{E}|Z^0| = (\frac{2}{\sqrt{\pi}})^2 \|x - y\|.$

### D Proof of Lemma 3

We adapt the proof of Lemma 5.5 in [21] to both $\ell_2$-perturbations (instead of $\ell_1$ ones) of $x_0$ and $y_0$, and to the context of uniform dithered quantization instead of 1-bit (sign) quantization. By assumption, we have $\|\Phi x'\| \leq \eta \sqrt{M}$ and $\|\Phi y'\| \leq \eta \sqrt{M}$. Therefore, the set
\[
T := \{i \in [M] : \|\Phi x'_i\| \leq \eta \sqrt{P}, \|\Phi y'_i\| \leq \eta \sqrt{P}\}
\]
is such that $|T^c| \leq 2M/P$ as $2\eta^2 M \geq \|\Phi x'\|^2 + \|\Phi y'\|^2 \geq \|\Phi x'\|^2 + \|\Phi y'\|^2 \geq |T^c| P \eta^2$. Considering the definition of $F^t$ in (25), we have, for all $i \in T$ and any $\lambda \in \mathbb{R}$,
\[
F^{t+\eta \sqrt{P}}_i(x_0, y_0, \lambda) \subset F^t_i(x_0 + x', y_0 + y', \lambda) \subset F^{t-\eta \sqrt{P}}_i(x_0, y_0, \lambda),
\]
with $F^t_i(x_0, y_0, \lambda) := F^t(\varphi^T_i x_0 + \xi_i - \lambda, \varphi^T_i y_0 + \xi_i - \lambda)$.

Denoting $a_i = \max(|\varphi^T_i x'|, |\varphi^T_i y'|)$, we find
\[
D^{t+\eta \sqrt{P}}(x_0, y_0) = \frac{\delta}{M} \sum_{i=1}^M \sum_{k \in \mathbb{Z}} \mathbb{I}[|F^{t+\eta \sqrt{P}}_i(x_0, y_0, k\delta)|]
\leq \frac{\delta}{M} \sum_{i \in T^c} \sum_{k \in \mathbb{Z}} \mathbb{I}[|F^t_i(x_0 + x', y_0 + y', k\delta)|] \leq \frac{\delta}{M} \sum_{i \in T^c} \sum_{k \in \mathbb{Z}} \mathbb{I}[|F^{t+\eta \sqrt{P}}_i(x_0 + x', y_0 + y', k\delta)|]
\leq \frac{\delta}{M} \sum_{i \in T^c} \sum_{k \in \mathbb{Z}} \mathbb{I}[|F^t_i(x_0 + x', y_0 + y', k\delta)|] + \frac{\delta}{M} \sum_{i \in T^c} \sum_{k \in \mathbb{Z}} \mathbb{I}[|F^t_i(x_0 + x', y_0 + y', k\delta)|].
\]
Using (28) to bound the last sum of the last expression and since, by definition of $T$, $a_i \geq \eta \sqrt{P}$ for $i \in T^c$, we find
\[
D^{t+\eta \sqrt{P}}(x_0, y_0) \leq D^t(x_0 + x', y_0 + y') + \frac{\delta}{M} \sum_{i \in T^c} \mathbb{I}[|F^t_i(x_0 + x', y_0 + y', k\delta)|] \leq D^t(x_0 + x', y_0 + y') + \frac{\delta}{M} \sum_{i \in T^c} (a_i - \eta \sqrt{P}) \leq D^t(x_0 + x', y_0 + y') + \frac{\delta}{M} \sum_{i \in T^c} a_i - \frac{4\sqrt{\mathbb{T}^c}}{M^2} \eta \sqrt{P}.
\]

However,
\[
\frac{1}{M} \sum_{i \in T^c} a_i \leq \frac{1}{M} (\|\Phi x'\| T^c + \|\Phi y'\| T^c) \leq \frac{\sqrt{\mathbb{T}^c}}{M} (\|\Phi x'\| T^c + \|\Phi y'\| T^c) \leq 2\eta \sqrt{\mathbb{T}^c} / M,
\]
and since $f(t) = 2t - t^2 \sqrt{P} \leq 1/\sqrt{P}$ for all $t \in \mathbb{R}$, we find
\[
D^{t+\eta \sqrt{P}}(x_0, y_0) \leq D^t(x_0 + x', y_0 + y') + \frac{4\delta}{P} + 4\eta (2\sqrt{\mathbb{T}^c} / M - \frac{\mathbb{T}^c}{M^2} \sqrt{P}) \leq D^t(x_0 + x', y_0 + y') + \frac{4\delta}{P} + 4\eta \sqrt{P},
\]
which provides the lower bound of (39).
For the upper bound,
\[
D^{l-\eta\sqrt{F}}(x_0, y_0) = \frac{\delta}{\lambda} \sum_{i=1}^{M} \sum_{k \in \mathbb{Z}} \mathbb{I}[F^l_i(x_0, y_0, k\delta)]
\geq \frac{\delta}{\lambda} \sum_{i \in T} \sum_{k \in \mathbb{Z}} \mathbb{I}[F^l_i(x_0 + x', y_0 + y', k\delta)] + \frac{\delta}{\lambda} \sum_{i \in T^c} \sum_{k \in \mathbb{Z}} \mathbb{I}[F^l_i-\eta\sqrt{P+a_i}(x_0 + x', y_0 + y', k\delta)]
\geq D^l(x_0 + x', y_0 + y')
- \frac{1}{\lambda} \sum_{i \in T^c} \delta \sum_{k \in \mathbb{Z}} \mathbb{I}[F^l_i(x_0 + x', y_0 + y', k\delta)] - \mathbb{I}[F^l_i(x_0 + x', y_0 + y', k\delta)]
\]
and, as above, the last sum can be upper bounded by \(\frac{2\delta}{\lambda} + \frac{4n}{\lambda^2}\) using (28).

E Proof of Lemma 4

We use here a similar proposition of Mendelson et al. in [36] for subsets of \(\mathbb{S}^{N-1}\) that we lift in the subsets of \(\mathbb{R}^{N+1}\) thank to some tools developed in [37] for other purposes.

We fix \(t = \|R\|/\sqrt{6}\) and we form the set \(R' := \{u/\|u\| : u \in R \oplus t\} \) with \(R \oplus t := \{(\tau) : x \in R\} \subset \mathbb{R}^{N+1}\). As \(R' \subset \mathbb{S}^{N}\), we know from [36] Theorem 2.1 that for \(0 < \epsilon < 1\),
\[
M \geq \frac{\alpha^4}{\epsilon^2} \omega(R')^2
\]
and \(\Phi' \sim \mathcal{N}_{\alpha^4}^M(N+1)(0, 1)\),
\[
P\left[ \sup_{x' \in \mathcal{R}} \left| \frac{1}{\|R\|} \|\Phi' x'\|^2 - 1 \right| \geq \epsilon \right] \leq \exp(-c\alpha^2 M).
\]
However, for \(g \sim \mathcal{N}^M(0, 1)\) and \(\gamma \sim \mathcal{N}(0, 1)\), as observed similarly in [37],
\[
w(R') = \mathbb{E} \sup_{x \in \mathcal{R}} (\|x\|^2 + t^2)^{-1/2} |\langle g, x \rangle| + t\gamma| \leq \frac{1}{t} (\mathbb{E} \sup_{x \in \mathcal{R}} |\langle g, x \rangle| + t(\frac{\gamma}{t})^{1/2})
\leq \sqrt{\frac{n}{\|R\|}} w(R) + (\frac{\gamma}{t})^{1/2} \leq 4 \frac{w(R)}{\|R\|},
\]
since, for all \(x \in \mathcal{R}, w(R) \geq (\frac{\gamma}{t})^{1/2}\|x\|\), i.e., \(w(R) \geq (\frac{\gamma}{t})^{1/2}\|R\|\). Therefore, fixing \(\epsilon = 1/2\), if \(M \geq \alpha^4 w(R)^2/|\mathcal{R}|^2\), with probability at least \(1 - e^{-c\alpha^4 M}\), we have, for all \(x \in \mathcal{R}\),
\[
\sqrt{\frac{2}{\|R\|}} \geq \frac{1}{\sqrt{\|R\|}} \|\Phi' x'\| \geq \frac{1}{\sqrt{\|R\|}} \|\Phi x + t\phi\| \geq \frac{1}{\sqrt{\|R\|}} (\|\Phi x\| - t\|\Phi'(0)\|) \geq \frac{1}{\sqrt{\|R\|}} \|\Phi x\| - \sqrt{\frac{2}{\|R\|}},
\]
where \(x' = ((\frac{\gamma}{t})^{-1} (\frac{\gamma}{t}) \in \mathcal{R}', \phi \in \mathcal{R}\) is the last column of \(\Phi'\) and using the fact that \((\frac{\gamma}{t}) \in \mathcal{R}'\)
since \(0 \in \mathcal{R}\). Therefore, replacing \(t\) by its value, we find with the same probability,
\[
\frac{1}{\sqrt{\|R\|}} \|\Phi x\| \leq \|R\|,
\]
for all \(x \in \mathcal{R}, i.e., \|\Phi R\| \leq \sqrt{\|R\|}\|R\|\).

F Proof of Lemma 5

From the relation \(D^l(u, v) = \frac{1}{\lambda} \sum_{i=1}^{M} d_i(\Phi_i^l(u), \Phi_i^l(v))\) established in Sec. 3 between \(D^l\) and \(d^l \in \delta N\) defined in [26], and associated to the vectorial mapping \(u \in \mathbb{R}^N \rightarrow \Phi(u) = \Phi u + \xi\) whose
\[\text{Where a totally equivalent sub-Gaussian norm is used, i.e., } \|X\|_{\Omega_2}^\text{(Mend.)} := \inf \{s : \mathbb{E} \exp(X^2/s^2) \leq 2\} \text{ with } \|X\|_{\Omega_2}^\text{(Mend.)} \approx \|X\|_{\Omega_2} \text{ [40].} \]
components are independent, we reach the bound \[^{[49]}\] with the cdf of a binomial distribution: since

\[
P\left[ \frac{M}{\delta} D^t(u, v) \leq r \right] \leq P\left[ \sum_{i=0}^{M-1} \left( \frac{M}{k} \right) p^k (1 - p)^{M-k} \right] \leq r
\]

Chernoff’s inequality can upper bound this binomial cdf with

\[
P\left[ \frac{M}{\delta} D^t(u, v) \leq r \right] \leq \exp(-\frac{(Mp-r)^2}{2Mp})
\]

Let us now lower bound \(p\). Defining \(w = u - v \in \mathcal{U}_0\) and \(\hat{w} = w/\|w\|\), the action of dithering \(\xi \sim \mathcal{U}([0, \delta])\) allows us to compute easily that,

\[
p = \mathbb{E}_\varphi P_\xi[d^t(\varphi^T u + \xi, \varphi^T v + \xi) \neq 0] = \mathbb{E} \min(1, \delta^{-1}(\|\varphi^T w\| - 2t)_+)
\]

In order to avoid any future singularity when \(\delta \to 0\), we can benefit of the fact that \(p \geq 1\) and work with this slightly looser bound:

\[
p \geq \mathbb{E} \min(1, (\epsilon_0 + \delta)^{-1}(\|\varphi^T w\| - 2t)_+)
\]

Moreover, with \(\alpha = \|u - v\|/(\delta + \epsilon_0)\),

\[
p \geq \mathbb{E} \min(1, \alpha^t \hat{w}) - \frac{2\alpha}{\delta + \epsilon_0} \geq \mathbb{E} \min(1, \alpha^t \hat{w}) - \frac{2\alpha}{\delta + \epsilon_0} = \mathbb{E} \min(1, \alpha^t \hat{w}) - \frac{2\alpha}{\delta + \epsilon_0}
\]

so that

\[
p \geq \mathbb{E} \min(1, \alpha^t \hat{w}) - \frac{2\alpha}{\delta + \epsilon_0} - A
\]

where \(g \sim \mathcal{N}(0, 1)\) and \(A := \mathbb{E} \min(1, \alpha^t \hat{w}) - \mathbb{E} \min(1, \alpha^t |g|)\).

We can upper bound \(A\) from our assumptions made on the sub-Gaussian vector \(\varphi \sim \mathcal{N}^N_{sg, \alpha}(0, 1)\):

\[
A = \max_{\|\omega\| = 1} \mathbb{E}(\min(1, \alpha^t \hat{w}) \geq u) - \mathbb{E}(\min(1, \alpha^t |g| \geq u))\]

\[
= \max_{\|\omega\| = 1} \mathbb{E}(\min(1, \alpha^t \hat{w}) \geq u) - \mathbb{E}(\min(1, \alpha^t |g| \geq u))\]

\[
\leq \alpha \int_0^{\infty} \mathbb{E}(\min(1, \alpha^t \hat{w}) \geq u) - \mathbb{E}(\min(1, \alpha^t |g| \geq u))\]

\[
\leq \frac{\alpha}{\delta + \epsilon_0} \|\omega\| \leq \frac{\alpha}{\delta + \epsilon_0}
\]

where the last inequalities relies on assumption \[^{[4]}\] (setting \(u = \hat{w}\)) and on the fact that \(w \in \mathcal{U}_0\).

Moreover, for lower bounding \(\mathbb{E} \min(1, \alpha^t |g|)\) in \[^{[58]}\], we observe that \(\min(1, \alpha x) = \alpha x - \alpha(x - 1/\alpha)_+\) for \(x \in \mathbb{R}\). Therefore, defining \(F(x) := \frac{\alpha}{2} x^2 - \frac{\alpha}{2} (x - 1/\alpha)_+^2 = \int_0^x \min(1, \alpha^t |u|)|du\) and integrating by part, we find

\[
\mathbb{E} \min(1, \alpha^t |g|) = \mathbb{E}(\|g\| F(|g|)) \geq \left(\frac{2}{\pi}\right)^{\alpha} \mathbb{E}(\|g\| F(|g|))
\]

where in the last inequality we used Jensen’s inequality and the convexity of \(x \in \mathbb{R}_+ \mapsto x F(x)\). It is easy to see that \(2 F(x) \geq \alpha x^2/(1 + \alpha x)\) so that

\[
\mathbb{E} \min(1, \alpha^t |g|) \geq \frac{1}{2} \left(\frac{2}{\pi}\right)^{\alpha} \frac{2}{1 + (\frac{2}{\pi})^{\alpha}} \geq \frac{4}{1 + \alpha}
\]

Finally,

\[
p \geq \frac{4}{1 + \alpha} - \frac{2\alpha}{\delta + \epsilon_0} - \frac{\alpha}{\delta + \epsilon_0} \geq \frac{1}{2} \left(\frac{2}{\pi}\right)^{\alpha} \frac{2}{1 + (\frac{2}{\pi})^{\alpha}} \geq \frac{4}{1 + \alpha} \frac{1}{4 \sqrt{\kappa_0}} - \frac{2\alpha}{\delta + \epsilon_0} - \frac{\alpha}{\delta + \epsilon_0}
\]

the last expression providing \[^{[50]}\] if \(\sqrt{\kappa_0} \geq 16 \kappa_0\).
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