ON SETS WITH FEW DISTINCT DISTANCES

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ABSTRACT. It is widely believed that point sets in the plane which determine few distinct distances must have some special structure. In particular, such sets are believed to be similar to a lattice. This note considers two different ways to quantify this idea. Firstly, improving on a result of Hanson [3], it is proven that if $P = A \times A$ with $A \subset \mathbb{R}$ and $P$ determines $O(|A|^2)$ distinct distances, then $|A - A| = O\left(|A|^2 - \frac{1}{11}\right)$. This result gives further evidence that cartesian products which determine few distinct distances have some additive structure. Secondly, it is shown that if a set $P \subset \mathbb{R}^2$ of $N$ points determines $O(N/\sqrt{\log N})$ distinct distances, then there exists a reflection $R$ and a set $P' \subset P$ with $|P'| = \Omega(\log^{3/2} N)$ such that $R(P') \subset P$. In other words, sets with few distinct distances have some degree of reflexive symmetry.

1. Introduction

Given a set $P$ of $N$ points in $\mathbb{R}^2$, let $d(P) := \{|p - q| : p, q \in P\}$ be the set of distances determined by $P$. A classical and beautiful problem in discrete geometry is the Erdős distinct distance conjecture, which states that $|d(P)| = \Omega(N/\sqrt{\log N})$ for all finite $P \subset \mathbb{R}^2$. The problem was resolved up to logarithmic factors in a landmark work of Guth and Katz [2].

A question that remains wide open concerns the possible structure of point sets which determine few distances. It was suggested by Erdős [1] that such a set should have “lattice structure”. More precisely, he suggested the conjecture that if $|d(P)|$ is minimal then there exists a line which contains $\Omega(N^{1/2})$ points of $P$. The conjecture remains wide open; the current best known result establishes that such a set contains $\Omega(\log N)$ points which are supported on a single line.$^2$

This paper seeks to find other ways to quantify the qualitative idea that sets with few distinct distances are similar to a lattice. One property of lattice point sets is that they are additively structured. A recent paper of Hanson [3] considered the additive properties of points sets of the form $P = A \times A$ with few distinct distances. The main result in [3] was that

$$|d(A \times A)| = \Omega(|A - A||A|^{1/8}),$$

1Here $|p - q|$ denotes the Euclidean distance between points $p$ and $q$.

2See the blog post of Adam Sheffer https://adamsheffer.wordpress.com/2014/10/07/few-distinct-distances-implies-many-points-on-a-line/
where
\[ A - A := \{ a - b : a, b \in A \} \]
is the difference set of \( A \).

In particular, Hanson’s result shows that if \( |d(A \times A)| = O(|A|^2) \) then \( |A - A| = O(|A|^{2 - \frac{1}{8}}) \). This says that if the set \( d(A \times A) \) is small then \( A \) must have some degree of additive structure. It appears plausible to conjecture that the exponent \( 2 - \frac{1}{8} \) could be replaced by \( 1 + o(1) \). The first aim of this paper is to improve the result of Hanson and make a step in this direction.

**Theorem 1.** Let \( A \subset \mathbb{R} \) be a finite set and let \( P = A \times A \). Then
\[ |d(P)| = \Omega(|A - A|^{11/10}). \]

In particular,
\[ |d(P)| = O(|A|^2) \Rightarrow |A - A| = O(|A|^{2 - \frac{1}{10}}). \]

Although this result is stated in terms of discrete geometry and distance problems, it should perhaps be viewed as a result in additive combinatorics. This is reflected in the notation and tools used for the problem. One can view the Guth-Katz theorem for direct product sets as a sum-product type result. It says that \( 3 \)
\[ |(A - A)^2 + (A - A)^2| = \Omega \left( \frac{|A|^2}{\log |A|} \right). \]

Theorem 1 and [3] show that this bound being close to tight implies some additive structure. There is a similarity here with the work of Shkredov [7], who proved the following inverse sum-product result:
\[ |(A - A)(A - A)| = O(|A|^2) \Rightarrow |A - A| = O \left( |A|^{2 - \frac{1}{8} + o(1)} \right). \]

Another feature of lattice point sets is that they are highly symmetric; there exists a reflection which maps \( P \) to itself. Therefore, we might expect that any set which determines few distinct distances is in some sense highly symmetric. More precisely, we might expect that such a set \( P \) has the property that there is a large subset \( P' \subset P \) and some reflection \( \mathcal{R} \) such that \( \mathcal{R}(P') \subset P \). The second aim of this paper is to prove the following result in this direction.

**Theorem 2.** Let \( P \) be a set of \( N \) points in \( \mathbb{R}^2 \) such that \( |d(P)| \leq N/K \), where \( K > 1 \) is some parameter. Then, there exists a subset \( P' \subset P \) with \( |P'| = \Omega(K^3) \), and some reflection \( \mathcal{R} \) such that \( \mathcal{R}(P') \subset P \).

In particular, if \( |d(P)| = O(N/\sqrt{\log N}) \), then, there exists a subset \( P' \subset P \) with \( |P'| = \Omega(\log^{3/2} N) \), and some reflection \( \mathcal{R} \) such that \( \mathcal{R}(P') \subset P \).

\(^3\)See the notation in the next section.
2. Notation and Preliminary results

Throughout this paper, for positive values $X$ and $Y$ the notation $X \gg Y$ is used as a shorthand for $X \geq cY$, for some absolute constant $c > 0$.

Similar to the difference set, the sum set of $A$ and the product set of $A$ are defined respectively as

$$A + A := \{a + b : a, b \in A\}, \quad AA = \{ab : a, b \in A\}.$$ 

The shorthand $2A$ is sometimes used for $A + A$. Similar notation is used for longer combinations of sum and difference set; for example $A + A + A - A - A$ is denoted $3A - 2A$. Sets formed by a combination of additive and multiplicative operations on different sets are also considered. For example, if $A, B$ and $C$ are sets of real numbers, then

$$AB + C := \{ab + c : a \in A, b \in B, c \in C\}.$$ 

Let $A \subset \mathbb{R}$ be finite and $\lambda \in \mathbb{R}$. The set of all dilates of $A$ by $\lambda$ is denoted $\{\lambda\}A$. That is,

$$\{\lambda\}A = \{\lambda a : a \in A\}.$$ 

The curly brackets here are used to distinguish the dilate $\{2\}A$ from the sum set $2A$. Also, $A^2$ denotes the set of all squares of $A$. That is $A^2 := \{a^2 : a \in A\}$.

The proof of Theorem 1 is a modification of the argument of Hanson [3]. The key new idea in [3] was the following lemma:

**Lemma 1.** Let $A \subset \mathbb{R}$ and let $D = A - A$. Then

$$\{2\}DD \subset 2D^2 - 2D^2.$$ 

As in [3], we use the following version of Plünecke-Ruzsa Theorem (see [6]).

**Lemma 2.** Suppose $A$ is a finite subset of an additive abelian group. Then

$$|mA - nA| \leq \left(\frac{|A + A|}{|A|}\right)^{m+n} |A|.$$ 

We also utilise the following sum-product type result, which follows from the Szemerédi-Trotter. See Exercise 8.3.3 in Tao-Vu [5].

**Lemma 3.** Let $A, B, C \subset \mathbb{R}$ be finite sets. Then

$$|AB + C| \gg (|A||B||C|)^{1/2}.$$ 

To prove Theorem 2 we require the following weighted version of the Szemerédi-Trotter Theorem. The result can be found in the literature, see for example [4].

**Lemma 4.** Let $P$ be a finite set of points in $\mathbb{R}^2$ and let $L$ be a set of weighted lines. Each line $l \in L$ is assigned a weight $w(l)$. Let $W_L = \sum_{l \in L} w(l)$ denote the total weight of $L$, and let $w_L = \max_{l \in L} w(l)$ be the maximum weight. Then, the number of weighted incidences $I_w(P, L)$ satisfies

$$I_w(P, L) := \sum_{p \in P, l \in L : p \in L} w(l) \ll (w_L)^{1/3}(|P||W_L|^{2/3} + W_L + w_L)|P|. $$  

(1)
3. PROOF OF THEOREM 1

Let $D = A - A$ and note that $d(A \times A) = D^2 + D^2$. By Lemma 1 and Lemma 2

$$|\{2\}D + D^2| \leq |3D^2 - 2D^2| \leq \left(\frac{|D^2 + D^2|}{|D^2|}\right)^5 |D^2|.$$ 

By Lemma 3

$$|\{2\}D + D^2| \gg |D|^{3/2}.$$ 

Combining these two estimates, we have

$$|d(A \times A)| = |D^2 + D^2| \gg |D|^{11/10}$$

as required.

4. PROOF OF THEOREM 2

The proof makes use of some observations from a recent paper of Lund, Sheffer and de Zeeuw [5], which considered structural properties of point sets which determine few distinct distances via studying the perpendicular bisectors determined by $P$.

We will double count the set of (ordered) isosceles triangles determined by $P$. That is, the proof proceeds by comparing an upper and lower bound for the quantity

$$T := \{(p, q, s) \in P \times P \times P : |p - s| = |q - s|, p \neq q\}.$$ 

For the upper bound we use Lemma 4. For two distinct points $p, q \in \mathbb{R}^2$, let $B(p, q)$ denote their perpendicular bisector. Let $L$ be the multiset of perpendicular bisectors determined by $P$. For $l \in L$, its weight is the number of (ordered) pairs of points from $P$ that determine $l$ as a bisector; that is,

$$w(l) := \{(p, q) \in P \times P : B(p, q) = l\}.$$ 

Note that $W_L = |P|^2 - |P| < |P|^2$. Note also that $T = I_w(P, L)$. Indeed $(p, q, s) \in T$ if and only if $s \in B(p, q)$. Therefore, it follows from Lemma 4 that

$$(2) \quad |T| \ll w_L^{1/3} N^2 + N^2 \ll w_L^{1/3} N^2.$$ 

On the other hand, if we denote by $C(s, r)$ the circle with radius $r$ and centre $s$, then

$$|T| = \sum_{s \in P} \sum_{r \in d(P)} 2 \binom{|C(s, r) \cap P|}{2}
\gg \sum_{s \in P} \sum_{r \in d(P)} |C(s, r) \cap P|^2 - \sum_{s \in P, r \in d(P)} \frac{1}{|C(s, r) \cap P| \leq 1}
\gg \sum_{s \in P} \sum_{r \in d(P)} |C(s, r) \cap P|^2 - \frac{N^2}{K}.$$

\footnote{The possibility that $r = 0$ is not excluded.}
Combining this information with (2), we deduce that

\[
\sum_{s \in P} \sum_{r \in d(P)} |C(s, r) \cap P|^2 = \ll w_L^{1/3} N^2 + \frac{N^2}{K} \ll w_L^{1/3} N^2.
\]

Note also that

\[
\sum_{s \in P} \sum_{r \in d(P)} |C(s, r) \cap P| = N^2.
\]

Therefore, by the Cauchy-Schwarz inequality and (3), we have

\[
N^4 = \left( \sum_{s \in P} \sum_{r \in d(P)} |C(s, r) \cap P| \right)^2 \\
\leq N|d(P)| \sum_{s \in P} \sum_{r \in d(P)} |C(s, r) \cap P|^2 \\
\ll \frac{N^4 w_L^{1/3}}{K}.
\]

This tells us that \(w_L = \Omega(K^3)\). This completes the proof, since there is some perpendicular bisector \(l\) such that \(l = B(p_i, q_i)\) for \(i = 1, \ldots, k, k = \Omega(K^3)\) and with the \(p_i\) all distinct. Therefore we can take \(P' = \{p_i : 1 \leq i \leq k\}\) and \(R\) to be reflection in the line \(l\), and observe that

\[R(P') = \{q_i : 1 \leq i \leq k\} \subset P.\]

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