The Riordan Group and Symmetric Lattice Paths

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Abstract: In this paper, we study symmetric lattice paths. Let \(d_n\), \(m_n\), and \(s_n\) denote the number of symmetric Dyck paths, symmetric Motzkin paths, and symmetric Schröder paths of length \(2n\), respectively. By using Riordan group methods we obtain six identities relating \(d_n\), \(m_n\), and \(s_n\) and also give two of them combinatorial proofs. Finally, we investigate some relations satisfied by the generic element of some special Riordan arrays and get the average mid-height and the average number of points on the \(x\)-axis of symmetric Dyck paths of length \(2n\).

Key words: Symmetric Dyck paths; Symmetric Motzkin paths; Symmetric Schröder paths; Riordan group; Combinatorial identities

1 Introduction

Lattice paths have been widely studied from various points of view. A surprisingly large number of \(1\)-\(1\) correspondences are known that relate these lattice paths to other classes of objects, such as trees, polygon triangulations, 213 avoiding permutations, \(2 \times n\) standard Young tableaux, and so on. A large number of references can be found in \([1,2,12]\). In the present paper we will study some symmetric lattice paths.

Let \(k\) be any fixed positive integer. In the plane \(\mathbb{Z} \times \mathbb{Z}\), we consider lattice paths with three step types: an up step \(U = (1,1)\), a down step \(D = (1,-1)\) and a \(k\)-horizontal step \((k,0)\). Usually a 1-horizontal step is simply called a horizontal step. For convenience, we denote \(h = (1,0)\), and \(H = (2,0)\).

A generalized Motzkin path of length \(n\) is a lattice path from origin \((0,0)\) to \((n,0)\) consisting of up steps, down steps and \(k\)-horizontal steps that never goes below the \(x\)-axis. In the general setting of this paper, when \(k = 0,1,2\), we call it a Dyck path, a Motzkin path and a Schröder path, respectively. Recall that the generating functions for the number of Dyck paths, Motzkin paths and Schröder paths are \(C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}\), \(M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}\) and \(R(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}\).
Another generating function that will occur often is 

\[ B(x) = \frac{1}{\sqrt{1 - 4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots. \]

Since \( B(x^2) \) is the generating function for paths from \((0,0)\) and ending on the \(x\)-axis using just up and down steps, we see that 

\[ B(x^2) = \frac{1}{1 - 2x^2 C(x^2)} = \frac{C(x^2)}{1 - x^2 C(x^2)}. \]

For the first equality the 2 comes since after each return to the \(x\)-axis the next step can be either up or down. The second equality follows from starting a new component each time the path goes from the \(x\)-axis down to the line \(y = -1\).

A closely related generating function identity, perhaps of independent interest, is 

\[ \frac{1}{1 - 2x} = \frac{C(x^2)}{(1 - xC(x^2))^2}. \]

A generalized symmetric Motzkin path of length \(2n\) is a lattice path such that if the \(i\)-th step of generalized Motzkin path is an up step (down step or \(k\)-horizontal step) then the \(2n - i\)-th step is down step (up step or \(k\)-horizontal step). Basically the path is the same when read from left to right as when read from right to left with a vertex in the middle. In this paper, when \(k = 0, 1, 2\), we will call it a symmetric Dyck path, a symmetric Motzkin path and a symmetric Schröder path, respectively.

The number of symmetric Dyck paths of length \(2n\) equals the \(n\)-th Central binomial coefficient (one version, this one is sequence A001405 in [8]), denoted by \(d_n\). The first few numbers are 1, 1, 2, 3, 6, 10, 20, 35, \(\cdots\) and the generating function of \((d_n)_{n \in \mathbb{N}}\) is 

\[ d(x) = \frac{1}{2x} \left( \sqrt{\frac{1 + 2x}{1 - 2x}} - 1 \right) = B(x^2) + xB(x^2) C(x^2) = \frac{C(x^2)}{1 - xC(x^2)}. \]

The number of symmetric Motzkin paths of length \(2n\) equals the number of directed animals of size \(n + 1\) (sequence A005773 in [8]), denoted by \(m_n\). The first few numbers are 1, 2, 5, 13, 35, 96, 267, 750 \(\cdots\) and the generating function of \((m_n)_{n \in \mathbb{N}}\) is 

\[ m(x) = \frac{1}{2x} \left( \sqrt{\frac{1 + x}{1 - 3x}} - 1 \right). \]
Let $s_n$ denote the number of symmetric Schröder paths of length $2n$. An equation for the generating function $s(x) = \sum_{n \geq 0} s_n x^n$ is obtained from the “first return decomposition” of a symmetric Schröder path $P$. If the first step of $P$ is a 2-horizontal step, then the contribution of this case gives $x^2 s(x)$. If the first step of $P$ is an up step, then the contribution of this case gives $x s(x) + x^2 s(x) R(x^2)$, where $R(x)$ is the generating function for Schröder paths. The term $x s(x)$ accounts for those paths that do not touch the $x$-axis until the last step while the term $x^2 s(x) R(x^2)$ accounts for paths that do return. $x R(x^2) x$ gives the first return and then $s(x)$ for the part between the first and last returns. Hence

$$s(x) = 1 + x s(x) + x^2 s(x) + x^2 s(x) R(x^2)$$

which implies that

$$s(x) = \frac{1}{2x} \left( \sqrt{x^2 - 2x - 1} \frac{x^2 - 2x - 1}{x^2 + 2x - 1} - 1 \right).$$

In fact, $s_n$ is the sequence A026003 in [8]. The first few numbers are 1, 1, 3, 5, 13, 25, 63, 129. These three sequences are all mentioned in [8] and were contributed in 2002-3 by Emeric Deutsch although without proofs. An equivalent way to view these sequences is as left factors of Dyck (Motzkin, Schröder) paths. By taking the mirror image of the left factor as the right factor you obtain the symmetric path.

Another natural way to define Symmetric paths would be as a path that is symmetric about the line, $x = n$ or as a path that looks the same whether going from left to right or from right to left. Call such paths palindromic. There exist palindromic Motzkin paths of odd length and palindromic Schröder paths with a 2-horizontal step from $x = n - 1$ to $n + 1$. Our definition however requires a lattice point on the path when $x = n$. Palindromic Motzkin paths of odd length, $2n + 1$, must have a horizontal step in the middle and removing it gives a symmetric Motzkin paths of length $2n$. Similarly palindromic Schröder paths with a 2-horizontal middle step can map to a symmetric Schröder path by removing the middle step. Thus our results are easily translated over to the palindromic version.

The sequences above occur in other combinatorial structures such as symmetric ordered trees. However in the this paper we describe them in terms of symmetric lattice paths. In Section 3 we will give six identities related to them by using Riordan group techniques which will be reviewed in Section 2. In Section 4 we will study some relations satisfied by the generic element of some special Riordan arrays and get the average mid-height and the average number of points on the $x$-axis of symmetric Dyck paths.

2 Riordan Group
In 1978, Rogers [6] introduced the renewal array, which is a generalization of the Pascal, and Motzkin triangles. Kettle [5] used the theory of renewal arrays to study other types of combinatorial triangles, especially those found in walk problems. Shapiro et al. [9] and Sprugnoli [11] generalized these kinds of arrays to Riordan arrays and the Riordan group. Riordan arrays constitute a practical device for solving combinatorial sums by means of composition of generating functions.

A Riordan array is an infinite lower triangular array \( R = \{ r_{n,k} \}_{n \geq k \geq 0} \) generated by a pair of analytic functions \( g(x) = 1 + g_1 x + g_2 x^2 + \cdots \) and \( f(x) = f_1 x + f_2 x^2 + \cdots \). If also \( f_1 \neq 0 \) then we have an element of the Riordan group. The array is defined by
\[
 r_{n,k} = [x^n] g(x) f(x)^k,
\]
where the notion \([x^n]\) denotes the “coefficient operator” that extracts the coefficient of \( x^n \). We often denote a Riordan array as \( R = (g(x), f(x)) \) or even as \((g, f)\).

Suppose we multiply the matrix \( R = (g, f) \) by a column vector \((a_0, a_1, \cdots)^T\) and get a column vector \((b_0, b_1, \cdots)^T\). Let \( A(x) \) and \( B(x) \) be the generating functions for the sequence \((a_0, a_1, \cdots)\) and \((b_0, b_1, \cdots)\) respectively. Then it follows quickly that
\[
 B(x) = (g(x), f(x)) \ast A(x) = g(x)A(f(x)).
\]
This is the essential fact sometimes referred to as “The Fundamental Theorem of Riordan Arrays” or even as the FTRA. Many examples and properties of the Riordan group are described in [9,10,11] along with the connection to the Lagrange inversion formula.

The Riordan group \( \mathcal{R} = \{ R | R = (g(x), f(x)) \} \) consists of the Riordan arrays with \( g_0 = 1, f_0 = 0, f_1 \neq 0 \). The multiplication in \( \mathcal{R} \) is just matrix multiplication and is given by
\[
 (g_1(x), f_1(x)) \ast (g_2(x), f_2(x)) = (g_1(x)g_2(f_1(x)), f_2(f_1(x))).
\]
The identity is \( I = (1, x) \), and the inverse matrix is specified by
\[
 (g(x), f(x))^{-1} = \left( \frac{1}{g(f(x))}, \bar{f}(x) \right),
\]
where \( \bar{f} \) is the compositional inverse of \( f(x) \). That is
\[
 f(\bar{f}(x)) = \bar{f}(f(x)) = x.
\]
It is easy to see that the Riordan group can used to study inverse relations. From \( R \) and \( R^{-1} \), an inverse relation can be established and thus we have
a systematic way to find inverse relations and sums. The paper [9] and [10] provide many examples. It should be noticed that earlier Riordan [7] and Gould and Hsu [4] studied many inverse relations.

3 Combinatorial Identities

Deng and Yan [3] obtained some identities involving the Catalan, Motzkin and Schröder numbers using Riordan group methods. In this section we will give six identities among \( d_n, m_n, \) and \( s_n \) by using the Riordan group and give combinatorial proofs for two of them. We start by noting a very suggestive equation connecting the first few terms of \( m_n \) and \( d_n \).

\[
\begin{bmatrix}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 \\
2 \\
3 \\
6 \\
\vdots \\
\end{bmatrix}
=
\begin{bmatrix}
1 \\
5 \\
13 \\
35 \\
\vdots \\
\end{bmatrix}
\]

Since the first matrix above is the Pascal matrix

\[
P = \left( \frac{1}{1-x}, \frac{x}{1-x} \right),
\]

we can use the fundamental theorem as follows.

**Theorem 3.1.** For \( n \geq 0 \), we have the following formula

\[
m_n = \sum_{k=0}^{n} \binom{n}{k} d_k.
\]  

**Proof.** Since the generating function of sequence \((d_n)_{n\in\mathbb{N}}\) is \(d(x)\), it follows from the fundamental theorem that

\[
\left( \frac{1}{1-x}, \frac{x}{1-x} \right) \ast d(x) = \frac{1}{1-x} \cdot d \left( \frac{x}{1-x} \right) = m(x).
\]

So we get

\[
m_n = \sum_{k=0}^{n} \binom{n}{k} d_k.
\]

**Combinatorial proof.** Suppose that a symmetric Motzkin path of length \( 2n \) contains \( 2k \) horizontal steps. We can reduce it to a symmetric Dyck path of length \( 2(n-k) \) by removing all the horizontal steps. Conversely, given a symmetric Dyck path of length \( 2(n-k) \), we can reconstruct \( \binom{n}{k} \) symmetric Motzkin paths of length \( 2n \) by inserting \( 2k \) horizontal steps. The symmetric
condition means placing \( k \) horizontal steps into the left half of the Dyck path. There are \( n - k + 1 \) vertices in which we can place \( k \) horizontal steps with repetition allowed so we have \( \binom{n-k+1+k-1}{k} = \binom{n}{k} \) possibilities. Therefore we do have the relation

\[
m_n = \sum_{k=0}^{n} \binom{n}{k} d_k. \quad \blacksquare
\]

Now we get the inverse of the identity (1) by multiplying the Riordan matrix inverse

\[
P^{-1} = \left( \frac{1}{1-x}, \frac{x}{1+x} \right) = \left( \frac{1}{1-x}, \frac{x}{1+x} \right) = \left[ \begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0 & \cdots \\ -1 & 4 & -6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \end{array} \right].
\]

So we have

**Theorem 3.2.** For \( n \geq 0 \), we have the following formula

\[
d_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} m_k.
\] (2)

**Remark:** In fact, the formulas (1) and (2) are a special case of the inverse transformation

\[
b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \iff a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k.
\]

The transformation is widely used in the study of integer sequences where it is called the binomial transform. It is also called the Euler transform as it was introduced by Euler as a tool to accelerate the speed of convergence of sequences.

**Theorem 3.3.** For \( n \geq 0 \), we have the following formula

\[
s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{n-2k} d_{n-2k}.
\] (3)

**Proof.** Consider the Riordan array
\[ D = \left( \frac{1}{1-x^2}, \frac{x}{1-x^2} \right) = \begin{bmatrix}
1 \\
0 1 \\
1 0 1 \\
0 2 0 1 \\
1 0 3 0 1 \\
0 3 0 4 0 1 \\
\vdots
\end{bmatrix} \]

Since the generic element of the Riordan array \( \left( \frac{1}{1-x^2}, \frac{x}{1-x^2} \right) \) is

\[ [x^n] \frac{x^j}{(1-x^2)^{j+1}} = \begin{cases} 
\left( \frac{n-j}{2} \right) & n-j \text{ even} \\
0 & n-j \text{ odd} 
\end{cases}, \]

we finish the proof by setting \( \frac{n-j}{2} = k \).

**Combinatorial proof.** Suppose that a symmetric Schröder path ending at \((2n,0)\) contains \( 2k \) 2-horizontal steps. We can reduce it to a symmetric Dyck path of length \( 2(n-2k) \) by deleting all 2-horizontal steps. Conversely, given a symmetric Dyck path of length \( 2(n-2k) \), we can construct \( \binom{n-k}{k} \) symmetric Schröder paths of length \( 2n \) by inserting \( 2k \) 2-horizontal steps. So the numbers \( s_n \) relate to the numbers \( d_n \) by

\[ s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{n-2k} d_{n-2k}. \]

**Theorem 3.4.** For \( n \geq 0 \), we have the following formula

\[ d_n = \sum_{k=0}^{n} (-1)^{(n-k)/2} d_{n,k} s_k, \tag{4} \]

where \( d_{n,k} = \begin{cases} 
\frac{k+1}{n+1} \binom{n+1}{n-k} & n-k \text{ even} \\
0 & n-k \text{ odd} 
\end{cases} \), and the generating function for \( d_{n,k} \) is \( x^k C^{k+1}(x^2) \), \( k = 0, 1, 2, \ldots \).

**Proof.** Since

\[ D^{-1} = \left( \frac{1}{1-x^2}, \frac{x}{1-x^2} \right)^{-1} = \left( \frac{\sqrt{1+4x^2} - 1}{2x^2}, \frac{\sqrt{1+4x^2} - 1}{2x} \right) \]
The generic element of $D^{-1}$ is

$$[x^n]x^k\left(\frac{\sqrt{1 + 4x^2 - 1}}{2x^2}\right)^{k+1} = \begin{cases} (-1)^{(n-k)/2}k + 1\left(\frac{n+1}{n-k}\right) & n-k \text{ even;} \\ 0 & \text{otherwise.} \end{cases}$$

Set $d_{n,k} = \frac{k+1}{n+1}(\frac{n+1}{n-k})^{k+1}$, then we obtain the formula (4). It is easy to see that $d_{n,k}$ is the $(n,k)$ entry of the Riordan array

$$D^* = \left(\frac{1 - \sqrt{1 - 4x^2}}{2x^2}, \frac{1 - \sqrt{1 - 4x^2}}{2x}\right) = \left(C(x^2), xC(x^2)\right) \quad \Box$$

**Theorem 3.5.** For $n \geq 0$, we have the following formula

$$s_n = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} m_k,$$

where $s_{n,k} = \sum_{j=0}^{n-k} \binom{k+j}{k} \binom{j}{n-j-k}$, and the generating function of $(s_{n,k})_{n \in \mathbb{N}}$ is $x^k \left(\frac{1}{1-x-x^2}\right)^{k+1}, k = 0, 1, 2, \cdots$.

**Proof.** By multiplying in the Riordan group, we immediately obtain

$$E = D^*P = \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) * \left(\frac{1}{1+x}, \frac{x}{1+x}\right) = \left(\frac{1}{1-x-x^2}, \frac{1}{1-x-x^2}\right)$$

However

$$[x^n]x^k\left(1 - x - x^2\right)^{k+1} = (-1)^{n-k} \sum_{j=0}^{n-k} \binom{k+j}{k} \binom{j}{n-j-k}$$

Let $s_{n,k} = \sum_{j=0}^{n-k} \binom{k+j}{k} \binom{j}{n-j-k}$, we obtain the formula (5) and $s_{n,k}$ is the entry of the Riordan array

$$E^* = \left(\frac{1}{1-x-x^2}, \frac{x}{1-x-x^2}\right). \quad \Box$$

**Remark.** The matrices $E$ and $E^*$ could be called Fibonacci matrices.

**Theorem 3.6.** For $n \geq 0$, we have the following formula

$$m_n = \sum_{k=0}^{n} t_{n,k} s_k,$$

where the generating function of $(t_{n,k})_{n \in \mathbb{N}}$ is $\frac{1}{x} \left(\frac{\sqrt{1-2x+3x^2+x-1}}{2x}\right)^{k+1}, k = 0, 1, 2, \cdots$. 8
Proof. The inverse of Riordan array $E$ is

$$E^{-1} = \left( \frac{\sqrt{1 - 2x + 5x^2 + x - 1}}{2x^2}, \frac{\sqrt{1 - 2x + 5x^2 + x - 1}}{2x} \right)$$

Giving the result. ■

4 Relations of the Generic Element

In Section 3, we introduced two arrays $d_{n,k}$ and $s_{n,k}$. In this section, we will discuss some identities related to them. Let review them as follows firstly.

For the Riordan array

$$D^* = (C(x^2), xC(x^2)) = \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 5 & 0 & 4 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

dn,k is the generic element of $D^*$ and satisfies the following recurrence relation

$$d_{n,k} = d_{n-1,k-1} + d_{n-1,k+1}. \quad (7)$$

For the Riordan array

$$E^* = \left( \frac{1}{1-x-x^2}, \frac{x}{1-x-x^2} \right) = \begin{bmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 3 & 5 & 3 & 1 \\ 5 & 10 & 9 & 4 & 1 \\ 8 & 20 & 22 & 14 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$s_{n,k}$ is the generic element of $E^*$ and satisfies the following recurrence relation

$$s_{n,k} = s_{n-1,k-1} + s_{n-1,k} + s_{n-2,k}. \quad (8)$$

We need the combinatorial interpretation of entries in the above matrices in terms of free symmetric lattice paths. To be precise, a generalized free symmetric Motzkin path of length $2n$ is a path that the $i$-th step is up step (down step, $k$-horizontal step) then the $2n - i$-th step is down step (up step, $k$-horizontal step) without the restriction that it cannot go below the $x$-axis.
In this paper, when \( k = 0, 1, 2 \), we call such a path a free symmetric Dyck path, a free symmetric Motzkin path and a free symmetric Schröder path, respectively. Let \( \mathcal{D}_n, \mathcal{M}_n, \mathcal{S}_n \) denote the set of free symmetric Dyck paths, free symmetric Motzkin paths, free symmetric Schröder paths of length \( 2n \), respectively. Note that \( |\mathcal{D}_n| = 2^n \) while \( |\mathcal{S}_n| = p_{n+1} \), the Pell numbers, (sequence A000129 in [8]) whose generating function is \( \frac{1}{1-2x-x^2} \).

A free symmetric MS path is one that can have level steps both of length one and two. Let \( \mathcal{MS}_n \) denote the set of free symmetric MS paths ending at \( (2n, 0) \). Moreover, \( |\mathcal{MS}_n| = h_{n+1} \) (sequence A006190 in [8]) whose generating function is \( \frac{1}{1-3x-x^2} \).

Now we go back to the paths that don’t go below the \( x \)-axis.

**Theorem 4.1.** Let \( \mathcal{D}_{n,k} \) denote the set of symmetric Dyck paths of length \( 2n \) with the mid-height \( k \). (The mid-height is the \( y \)-coordinate of the middle point.) Then \( d_{n,k} \) is the cardinality of \( \mathcal{D}_{n,k} \).

**Proof.** Let \( Q \in \mathcal{D}_{n,k} \). If the \( n \)-th step of \( Q \) is \( U \), then we can obtain a subpath \( Q_1 \in \mathcal{D}_{n-1,k-1} \) by deleting the \( n \)-th and \( n+1 \)-th steps of \( Q \). If the \( n \)-th step of \( Q \) is \( D \), then we can obtain a subpath \( Q_2 \in \mathcal{D}_{n-1,k+1} \) by deleting the \( n \)-th and \( n+1 \)-th steps of \( Q \).

So \( |\mathcal{D}_{n,k}| = |\mathcal{D}_{n-1,k-1}| + |\mathcal{D}_{n-1,k+1}| \). Combining this with relation (7), we see that \( d_{n,k} = |\mathcal{D}_{n,k}| \). □

**Theorem 4.2.** For \( n \geq 0 \) the sequence \( d_{n,k} \) satisfies

\[
\sum_{k=0}^{n} d_{n,k} = \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right). \quad (9)
\]

**Proof.** Consider the Riordan array \( D^* \) and the sequence \( (1, 1, 1, \ldots) \) whose generating function is \( A(x) = \frac{1}{1-x} \), then

\[
\sum_{k=0}^{n} d_{n,k} = [x^n] C(x^2) A(x) C(x^2) = [x^n] \frac{C(x^2)}{1-xC(x^2)} = \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right)
\]

**Combinatorial proof.** Noting that the number of symmetric Dyck paths of length \( 2n \) equals \( d_n = \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right) \), we have an immediate proof the theorem. □

**Theorem 4.3.** For \( n \geq 0 \) the sequence \( d_{n,k} \) satisfies

\[
\sum_{k=0}^{n} (k+1)d_{n,k} = 2^n. \quad (10)
\]

**Proof.** Consider the Riordan array \( D^* \) and sequence \( (1, 2, 3, \ldots) \) whose gen-
Generating function is \( A(x) = \frac{1}{(1-x)^2} \), then
\[
\sum_{k=0}^{n} (k+1)d_{n,k} = [x^n]C(x^2)A(xC(x^2)) = [x^n]\frac{C(x^2)}{(1-xC(x^2))^2} = 2^n
\]

**Combinatorial proof.** To prove the theorem, we will present a bijection between the set of restricted \( \mathcal{D}_{n,k} \) and the set \( \mathcal{D}_n \). Clearly, \( |\mathcal{D}_n| = 2^n \). Suppose \( Q \in \mathcal{D}_{n,k} \). We will present a set of maps \( \phi_i, i = 0, 1, 2, \ldots, k \) so that the left hand side of identity (10) is the cardinality of the set
\[
\bigcup_{k=0}^{n} \bigcup_{Q \in \mathcal{D}_{n,k}} \{ \phi_i(Q) : 0 \leq i \leq k \}.
\]
\( \phi_i(Q) \) denotes the path obtained by changing each of the last ascents to height 1, 2, 3, \ldots, \( i \) to down steps. The last ascent to height \( i \) of \( Q \) is the last up step (going from the first step to the midpoint at the \( n \)-th step) which starts at height \( i - 1 \) and ends at height \( i \).

Since \( Q \) has the mid-height \( k \), it contains exactly \( k \) last ascents of the left \( n \) steps. By replacing the first \( i \) last ascents in \( Q \) with \( D \)'s, we ensure that, to some point, the number of \( D \)'s exceed the number of preceding \( U \)'s, so that \( \phi_i(Q) \) necessarily goes below the \( x \)-axis, eventually the \( n \)-th step ending at height \( k - 2i \).

See Figure 1 and 2 for an illustration of this map.

![Figure 1](image1.png)

**Figure 1:** \( Q = U D U U D U U D D U U D D U D U D D U D U D U U D D U D U D U D U D U D U D U U D \), a restricted symmetric Dyck path of length \( 2n = 26 \) and the mid-height \( k = 3 \). Last ascents are indicated with a bold \( U \) and marked above with an single star.

![Figure 2](image2.png)

**Figure 2:** \( \phi_2(Q) = U D D U D D D U D U D U D U D U D U D \), the unrestricted Dyck
path of length $n = 13$ and terminal height $j = k - 2i = 3 - 2(2) = -1$ which is obtained by changing the last ascents 1 and 2 to down steps. Premier descents are indicated with a bold $D$ and marked with a single star.

From the construction, we see directly that $\phi_i$ is one to one. In order to show that the set $\{Q, \phi_0(Q), \ldots \phi_k(Q) : Q \in D_{n,k}, 0 \leq k \leq n\}$ is indeed equal to the set $D_n$, we need only show that it is possible to take a path $p \in D_n$ and recover its unique preimage $Q$ under $\phi_i$ for some $i$.

Let $p \in D_n$ with mid-height $j$. We consider the left $n$ steps. Since $p$ goes below the $x$-axis there exists a first step down from height $m$ to $m - 1$ from $m = 0, -1, -2, \ldots$ as one proceeds from left to right. We call these premier steps. Assuming that the number of the first step below the $x$-axis of $p$ is $i (> 0)$, then changing each of the $i$ premier steps to up steps will create a new path $Q$ which stays above the $x$-axis and ends at height $k = j + 2i$.

So this map is a bijection. ■

**Corollary 4.4.** The average mid-height of the symmetric Dyck paths from $(0,0)$ to $(2n,0)$ is

$$
\begin{align*}
\frac{2^{2m} - \binom{2m}{m}}{\binom{2m+1}{m}} & \sim \sqrt{\pi m} - 1 \sim \sqrt{\pi m} & \text{if } n = 2m \\
\frac{2^{2m+1} - \binom{2m+1}{m}}{\binom{2m+1}{m}} & \sim \sqrt{\pi m} - 1 \sim \sqrt{\pi m} & \text{if } n = 2m + 1
\end{align*}
$$

**Proof.** To compute the total height we multiply the matrix $D^*$ by the column vector $(0, 1, 2, 3, \cdots)^T = (1, 2, 3, \cdots)^T - (1, 1, 1, \cdots)^T$. We consider the case $n = 2m$, the other being quite similar. Combining the results of Theorem 4.2 and 4.3 and $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(n \to +\infty)$ gives the result. ■

Thus the average mid-height tends to infinity as $n$ gets large. How about the number of points where the path touches the $x$-axis?

**Theorem 4.5.** The generating function for the total number of points on the $x$-axis for paths from $(0,0)$ to $(2n,0)$ is

$$
2 \left[ C(x^2) \cdot \frac{1}{2x} \left( \sqrt{1 + 2x} - 1 \right) \right] - C(x^2) = 1 + 2x + 5x^2 + 8x^3 + 18x^4 + 30x^5 + 65x^6 + 112x^7 + \cdots.
$$

For $n = 2m$ the generating function of total number of points on the $x$-axis is

$$
C(x^2) [2B(x^2) - 1] = 1 + 5x^2 + 18x^4 + 65x^6 + 238x^8 + \cdots.
$$

and thus the average number of points on the $x$-axis is

$$
\frac{\binom{2m+2}{m+1} - \binom{2m}{m}}{\binom{2m+1}{m}} = \frac{4m+1}{m+1} \to 4
$$

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while if $n = 2m + 1$ then the corresponding results are

$$2x B \left( x^2 \right) C \left( x^2 \right)^2 = 2x + 8x^3 + 30x^5 + 112x^7 + \cdots$$

and the average number of points is

$$2 \frac{(2m+2)}{(2m+1)} = 2 \frac{(2m+2)}{m} = \frac{4}{2} \frac{(2m+2)}{m} = 4 \cdot \frac{n + 1}{n + 2} \to 4. \quad \blacksquare$$

**Remark 1.** When $n$ is even the total number of points on the $x$-axis is $(4m + 1) C_m$.

**Remark 2.** Without the symmetric condition the classical theorem of Der- showitz and Zaks [2], stated for ordered trees, gives the total number of points on the $x$-axis is $C_{n+1}$ and thus the average value becomes $\frac{4(n+2)}{n+2} \to 4$

**Theorem 4.6.** Let $S_{n,k}$ denote the set of all free symmetric Schröder paths of length $2n$ that the left $n$ steps contain $k$ up steps. Then $s_{n,k}$ enumerates the cardinality of $S_{n,k}$.

**Proof.** Let $Q \in S_{n,k}$. If the middle step is $U$, then we can obtain a subpath $Q_1 \in S_{n-1,k-1}$ by deleting the middle step and the following step of $Q$. If the middle step is $D$, then we can obtain a subpath $Q_2 \in S_{n-1,k}$ by deleting the middle step and the following step of $Q$. If the middle step is $H$, then we can obtain a subpath $Q_3 \in S_{n-2,k}$ by deleting the middle step and the following step of $Q$.

So $|S_{n,k}| = |S_{n-1,k-1}| + |S_{n-1,k}| + |S_{n-2,k}|$. Combined with the relation (8), we can see that $s_{n,k} = |S_{n,k}|$. \quad \blacksquare

**Theorem 4.7.** For $n \geq 0$ the sequence $s_{n,k}$ satisfies

$$\sum_{k=0}^{n} s_{n,k} = p_{n+1}. \quad (11)$$

**Proof.** By FTDA and $A(x) = \frac{1}{1-x}$ we have

$$\sum_{k=0}^{n} s_{n,k} = [x^n] \left( \frac{1}{1-x-x^2}, \frac{x}{1-x-x^2} \right) \ast \frac{1}{1-x} = [x^n] \frac{1}{1-2x-x^2} = p_{n+1}$$

This completes the proof.

**Combinatorial proof.** Clearly, the term $\sum_{k=0}^{n} s_{n,k}$ suggests the number of the free symmetric Schröder paths of length $2n$. This observation gives an immediate proof of the identity. \quad \blacksquare
Theorem 4.8. For \( n \geq 0 \) the sequence \( s_{n,k} \) satisfies
\[
\sum_{k=0}^{n} 2^k s_{n,k} = h_{n+1}.
\] (12)

Proof. By FTRA and \( A(x) = \frac{1}{1-2x} \) we have
\[
\sum_{k=0}^{n} 2^k s_{n,k} = [x^n] \left( \frac{1}{1-x-x^2}, \frac{x}{1-x-x^2} \right) = [x^n] \frac{1}{1-2x} = [x^n] \frac{1}{1-3x-x^2} = h_{n+1}
\]
This completes the proof.

Combinatorial proof. Suppose \( Q \in S_{n,k}(k > 0) \). We will present a set of maps \( \phi_{j_1,j_2,...,j_k}(Q) : j_i = 0 \text{ or } 1 \) so that the left hand side of (12) is the cardinality of the set
\[
S_{n,0} \bigcup_{k=1}^{n} \bigcup_{Q \in S_{n,k}} \{ \phi_{j_1,j_2,...,j_k}(Q) : j_i = 0 \text{ or } 1 \}.
\]
\( \phi_{j_1,j_2,...,j_k}(Q) \) denote the path obtained by changing each of the \( k \) up steps to horizontal steps \( h \) or remain \( U \). If we change the up step \( i \) to horizontal step, we have \( j_i = 1 \). If the up step \( i \) remains we have \( j_i = 0 \). See the following for an illustration of this map.

Figure 1 : \( Q = U D D U H U D U D H D U U D \), a free symmetric Schröder path of length \( 2n = 16 \) such that the left \( n \) steps contain \( k = 3 \) up steps. The \( k \) up steps are indicated with a bold \( U \) and marked above with an star.

Figure 2: \( \phi_{1,0,1}(Q) = h D D U H h D U h H D U U h \), the free generalized Schröder path of length \( 2n = 16 \) which obtained by changing the up steps 1 and 3 to horizontal steps \( h \). The changed horizontal steps are indicated with a bold \( h \) and marked above with an star.

Let \( P \in M S_n \). We concentrate on the left half of the path which has \( j \) horizontal steps and \( i \) up steps. Then changing each of the \( j \) horizontal steps
$h$ to up steps will get a free symmetric Schröder path of length $2n$ such that the left $n$ steps contain $k = i + j$ up steps.

From above, the set $S_{n,0} \bigcup_{k=1}^{n} \bigcup_{Q \in S_{n,k}} \{ \phi_{j_1,j_2,\ldots,j_k} : j_i = 0 \text{ or } 1 \}$ is indeed equal to the set $MS_n$. 

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