SPLITTING THEOREMS ON COMPLETE MANIFOLDS WITH BAKRY-ÉMERY CURVATURE

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Abstract. In this paper we study some splitting properties on complete non-compact manifolds with smooth measures when $\infty$-dimensional Bakry-Émery Ricci curvature is bounded from below by some negative constant and spectrum of the weighted Laplacian has a positive lower bound. These results extend the cases of Ricci curvature and $m$-dimensional Bakry-Émery Ricci curvature.

1. Introduction and main results

Let $(M,g)$ be an $n$-dimensional complete Riemannian manifold and $\varphi$ be a smooth function on $M$. We define the weighted Laplacian

$$\Delta_\varphi := \Delta - \nabla \varphi \cdot \nabla,$$

which is the infinitesimal generator of the Dirichlet form

$$\mathcal{E}(\phi_1, \phi_2) = \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle d\mu, \quad \forall \phi_1, \phi_2 \in C^\infty_0(M),$$

where $\mu$ is an invariant measure of $\Delta_\varphi$ given by $d\mu = e^{-\varphi} dv(g)$. Clearly, the weighted Laplacian is self-adjoint with respect to the weighted measure $d\mu$ and $(M,g,e^{-\varphi} dv)$ is a smooth metric measure spaces. A smooth function $u$ is called weighted harmonic if $\Delta_\varphi u = 0$. The first nontrivial eigenvalue of the weighted Laplacian in $(M,g,e^{-\varphi} dv)$ is defined by

$$\lambda_1(M) := \inf_{f \neq 0} \left\{ \mathcal{E}(f, f) : \int_M f^2 d\mu = 1, \int_M f d\mu = 0 \right\}.$$

The above infimum can be achieved by some smooth eigenfunction $f$, which satisfies the following Euler-Lagrange equation

$$\Delta_\varphi f = -\lambda_1 f.$$

The $m$-dimensional Bakry-Émery Ricci curvature (see also [1, 2, 3, 14]) is linked with the smooth metric measure space $(M,g,e^{-\varphi} dv)$, which is defined by

$$Ric_{m,n} := Ric + Hess(\varphi) - \frac{\nabla \varphi \otimes \nabla \varphi}{m-n},$$

where $Ric$ and $Hess$ denote the Ricci curvature and the Hessian of the metric $g$, respectively. Here $m \geq n$ is a constant, and $m = n$ if and only if $\varphi$ is a constant function.
constant \cite{14,15}. If we let $m$ be infinite, then the $m$-dimensional Bakry-Émery Ricci curvature becomes the $\infty$-dimensional Bakry-Émery Ricci curvature

$$Ric_\infty := \lim_{m \to \infty} Ric_{m,n} = Ric + Hess(\varphi).$$

This curvature is closely related to the gradient Ricci soliton

$$Ric_\infty = \lambda g$$

for some constant $\lambda$. The soliton is called expanding, steady and shrinking, accordingly, if $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, which plays an important role in the theory of Ricci flows \cite{3}. The gradient Ricci solitons are special solutions of Ricci flows and often arise from the blow up analysis of the singularities of Ricci flows \cite{9}.

If $\varphi$ is constant, the above weighted geometric quantities all return to the classical case. Taking this point of view, it is natural to ask if classical results involving Ricci curvature in the geometric analysis remain valid in the Bakry-Émery curvature case. Along this direction, a lot of excellent work actually obtained in the past several years. Interesting generalized results concerning Cheeger-Gromoll splitting theorems \cite{5,6} are Cheeger-Gromoll type splitting theorems with $Ric_\infty \geq 0$ showed by Wei and Wylie \cite{21} that is originally due to Lichnerowicz. This result was then improved by Fang, Li and Zhang \cite{8}. Recently Munteanu and Wang \cite{16,17} studied function theoretic and spectral properties on complete noncompact smooth metric measure space with $\infty$-dimensional Bakry-Émery Ricci curvature bounded below. In particular, they obtained some splitting results on complete noncompact gradient steady and expanding Ricci solitons.

In \cite{20}, Wang proved a splitting theorem on complete metric measure spaces with $m$-dimensional Bakry-Émery Ricci curvature bounded below by a negative multiple of the lower bound of the weighted spectrum. In particular, he proved that

**Theorem A.** Let $(M^n, g, e^{-\varphi} dv)$ be a complete smooth metric measure space of dimension $n \geq 3$ with $Ric_{m,n} \geq -(m-1)$. If $\lambda_1(M) \geq (m-2)$, then either

1. $M$ has only one end with infinite weighted volume; or
2. $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \cosh^2 tds_N^2,$$

where $N$ is an $(n-1)$-dimensional compact Riemannian manifold. In this case, $\lambda_1(M) = m-2$.

This result extended Li-Wang’s theorem \cite{12} to the weighted measure case. If $\lambda_1(M) > (m-2)$, Theorem A asserts that $M$ has only one end with infinite weighted volume. Naturally, we may ask if finite weighted volume ends can be ruled out in this case. In \cite{24}, the author proved that there exists an example to indicate that finite weighted volume ends can exist in Theorem A. Moreover this example seems to be the only case when $M$ has a finite weighted volume end if the weighted eigenvalue has an optimal positive lower bound and $m$-dimensional Bakry-Émery Ricci curvature is bounded below by some negative constant. Precisely,

**Theorem B.** Let $(M^n, g, e^{-\varphi} dv)$ be a complete smooth metric measure space of dimension $n \geq 3$ with $Ric_{m,n} \geq -(m-1)$. If $\lambda_1(M) \geq \frac{(m-1)^2}{4}$, then either

1. $M$ has only one end; or
(2) $M = \mathbb{R} \times N$ with the warped product metric
\[ ds_M^2 = dt^2 + \exp(-2t)ds_N^2 \]
where $N$ is an $(n-1)$-dimensional compact manifold. Moreover,
\[ \varphi(t, x) = \varphi(0, x) + (m - n)t \]
for all $(t, x) \in \mathbb{R} \times N$.

Theorem B was also independently proved by Su and Zhang [18]. This result can also be viewed as a weighted version of Li-Wang’s theorem [13]. In [22] (see also [19]) we know that if $\text{Ric}_m, n \geq -(m - 1)$, then $\lambda_1(M) \leq \frac{(m-1)^2}{4}$. Hence the eigenvalue assumption in Theorem B is, in fact that $\lambda_1(M) = \frac{(m-1)^2}{4}$.

In this paper, we continue our study of splitting type theorems for a smooth metric measure space, now under slight different assumption that its $\infty$-dimensional Bakry-Émery Ricci curvature is bounded from below. The main result is

**Theorem 1.1.** Let $(M^n, g, e^{-\varphi} dv)$ be a complete smooth metric measure space of dimension $n \geq 3$ with $\text{Ric}_\infty \geq -(n - 1)$. Assume that $|\nabla \varphi| \leq \theta$ for some positive constant $\theta$. For any $k > n$, if
\[ \lambda_1(M) \geq k - 2 \left( \frac{\theta^2}{k-n} + n - 1 \right), \]
then either
1. $M$ has only one end with infinite weighted volume; or
2. $M = \mathbb{R} \times N$ with the warped product metric
\[ ds_M^2 = dt^2 + \cosh^2 \left[ \sqrt{\frac{\theta^2}{(k-1)(k-n)} + \frac{n-1}{k-1}} \right] ds_N^2, \]
where $N$ is an $(n-1)$-dimensional compact Riemannian manifold. In this case, $\lambda_1(M) = k - 2 \left( \frac{\theta^2}{k-n} + n - 1 \right)$.

In Theorem 1.1 if $\theta = 0$, by letting $k \to n$, we then have Li-Wang’s theorem [12]. In particular, if we let $k = n + \theta$, then we have a explicit version.

**Corollary 1.2.** Let $(M^n, g, e^{-\varphi} dv)$ be a complete smooth metric measure space of dimension $n \geq 3$ with $\text{Ric}_\infty \geq -(n - 1)$. Assume that $|\nabla \varphi| \leq \theta$ for some positive constant $\theta$. If $\lambda_1(M) \geq n - 2 + \theta$, then either
1. $M$ has only one end with infinite weighted volume; or
2. $M = \mathbb{R} \times N$ with the warped product metric
\[ ds_M^2 = dt^2 + \cosh^2 tds_N^2, \]
where $N$ is an $(n-1)$-dimensional compact Riemannian manifold. In this case, $\lambda_1(M) = n - 2 + \theta$.

A natural question as above discussions is that if finite weighted volume ends can be excluded in Corollary 1.2. Similar to Theorem 1.1 we assert that finite weighted volume ends in Corollary 1.2 may occur by the following example.
Example 1.3. Consider the $n$-dimensional complete manifold $M = \mathbb{R} \times N$ endowed with the warped product metric

$$ds_M^2 = dt^2 + \exp(-2t)ds_N^2.$$ 

If $\{\bar{e}_\alpha\}$ for $\alpha = 2, \ldots, n$ form an orthonormal basis of the tangent space of $N$, then $e_1 = \frac{\partial}{\partial t}$ together with $\{e_\alpha = \exp(-t)\bar{e}_\alpha\}$ form an orthonormal basis for the tangent space of $M$. By the standard computation,

$$\text{Ric}_{M,1j} = -(n-1)\delta_{1j}$$

and

$$\text{Ric}_{M,\alpha\beta} = \exp(2t)\text{Ric}_{N,\alpha\beta} - (n-1)\delta_{\alpha\beta}.$$ 

If we let the weighted function $\varphi := \theta t$, then the $\infty$-dimensional Bakry-Émery Ricci curvature of $M$ is

$$\text{Ric}_{\infty,1j} = \text{Ric}_{M,1j} + \varphi_{1j} = -(n-1)\delta_{1j}$$

and

$$\text{Ric}_{\infty,\alpha\beta} = \exp(2t)\text{Ric}_{N,\alpha\beta} - (n-1)\delta_{\alpha\beta}.$$ 

Clearly, $\text{Ric}_N \geq 0$ if and only if $\text{Ric}_{\infty} \geq -(n-1)$. Moreover, we claim that

$$\lambda_1(M) = \frac{(n-1+\theta)^2}{4}.$$ 

Indeed, we may choose the function

$$f = \exp\left(\frac{n-1+\theta}{2}t\right).$$

A direct computation yields that

$$\Delta_{\varphi}f = \frac{d^2f}{dt^2} - (n-1)\frac{df}{dt} + \frac{df}{dt} \cdot \frac{df}{dt}$$

$$= -\frac{(n-1+\theta)^2}{4}f,$$

since $\Delta = \frac{d^2}{dt^2} - (n-1)\frac{df}{dt} + \exp(2t)\Delta_N$. So we have $\lambda_1(M) \geq \frac{(n-1+\theta)^2}{4}$ by Proposition 1.4 of [23]. Combining this and Theorem C, we immediately conclude that $\lambda_1(M) = \frac{(n-1+\theta)^2}{4}$ as claimed.

We can show that the above result is the only case (possible with different weighted function) when $M$ has a finite weighted volume end if $M$ achieves an optimal positive lower bound of the weighted spectrum.

Theorem 1.4. Let $(M^n, g, e^{-\varphi}dv)$ be a complete smooth metric measure space of dimension $n \geq 3$ with $\text{Ric}_{\infty} \geq -(n-1)$. Assume that $|\nabla \varphi| \leq \theta$ for some positive constant $\theta$. If $\lambda_1(M) \geq \frac{(n-1+\theta)^2}{4}$, then either

1. $M$ has only one end; or
2. $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \exp(-2t)ds_N^2,$$

where $N$ is an $(n-1)$-dimensional compact manifold with nonnegative Ricci curvature. Moreover,

$$\varphi(t, x) = \varphi(0, x) + \theta t$$

for all $(t, x) \in \mathbb{R} \times N$. 

We remark that the assumption on $\lambda_1(M)$ in Theorem 1.4 in fact only occurs on the equality case due to the following result (see Theorem 1.1 in [23]), which is a generalization of Cheng’s eigenvalue estimate [7].

**Theorem C.** Let $(M^n, g, e^{-\varphi} dv)$ be a complete smooth metric measure space of dimension $n \geq 2$ with $\text{Ric}_{\infty} \geq -(n-1)$. Assume that $|\nabla \varphi| \leq \theta$ for some positive constant $\theta$. Then

$$\lambda_1(M) \leq \frac{(n-1+\theta)^2}{4}.$$  

Moreover, if $f$ be a positive function satisfying

$$\Delta \varphi f = -\lambda f$$

for some constant $\lambda \geq 0$, then $f$ must satisfy the gradient estimate

$$|\nabla \ln f|^2 \leq \frac{(n-1+\theta)^2}{2} - \lambda + \frac{(n-1+\theta)^4}{4} - (n-1+\theta)^2 \lambda.$$  

After we have finished the proof of Theorem 1.1, we saw that Su-Zhang [18] and Munteanu-Wang [17] have independently proved Theorem 1.4. We include it for completeness. Our proof may be slight different from theirs. Here our approach mainly follows the same spirit of proving Theorem 1.5 in [24].

2. Basic definitions and lemmas

In this section, we will summarize some definitions and basic results on a complete smooth metric measure space $(M^n, g, e^{-\varphi} dv)$. For proofs and further details we refer to [20] and [24].

**Definition 2.1.** Let $(M^n, g, e^{-\varphi} dv)$ be a complete smooth metric measure space. A weighted Green’s function $G_\varphi(x, y)$ is a function defined on $(M \times M) \setminus \{(x, x)\}$ satisfying $G_\varphi(x, y) = G_\varphi(y, x)$ and $\Delta_{\varphi,y} G(x, y) = -\delta_{\varphi,x}(y)$ for all $x \neq y$, where $\delta_{\varphi,x}(y)$ is defined by

$$\int_M \psi(y) \delta_{\varphi,x}(y) d\mu = \psi(x)$$

for every compactly supported function $\psi$.

**Definition 2.2.** A complete smooth metric measure manifold $(M, g, e^{-\varphi} dv)$ is said to be weighted non-parabolic if it admits a positive weighted Green’s function. Otherwise, it is said to be weighted parabolic.

**Definition 2.3.** An end, $E$, with respect to a compact subset $\Omega \subset M$ is an unbounded connected component of $M \setminus \Omega$. The number of ends with respect of $\Omega$, denoted by $N_\Omega(M)$, is the number of unbounded connected component of $M \setminus \Omega$.

In general, when we say that $E$ is an end we mean that it is an end with respect to some compact subset $\Omega$. In particular, its boundary $\partial E$ is given by $\partial \Omega \cap \bar{E}$.

**Definition 2.4.** An end $E$ is said to be weighted non-parabolic if it admits a positive weighted Green’s function with Neumann boundary condition on $\partial E$. Otherwise, it is said to be weighted parabolic.
Following similar arguments of Li-Tam [11], we can verify that a complete measure manifold is weighted non-parabolic if and only if it has a weighted non-parabolic end. Of course, it is possible for a weighted non-parabolic measure manifold to have many weighted parabolic ends.

As similar discussions in Li-Tam [11], we know that an end $E$ with respect to the compact set $B_p(R_0)$ is weighted non-parabolic if and only if there exists a sequence of positive weighted harmonic functions $f_i$, defined on $E_p(R_i) = E \cap B_p(R_i)$ for $R_0 < R_1 < R_2 < \cdots \to \infty$, satisfying $f_i = 1$ on $\partial E$ and $f_i = 0$ on $\partial B_p(R_i) \cap E$, where $f_i$ is called to be a barrier function of $E$. Moreover the sequence $f_i$ converges uniformly on compact subsets of $E \cup \partial E$ to a minimal barrier function $f$, and $f$ has finite weighted Dirichlet integral on $E$.

If $E$ be a weighted parabolic end with respect to $B_p(R_0)$, then there exists a sequence of positive weighted harmonic functions $g_i$, defined on $E_p(R_i)$ and the sequence of constants $C_i \to \infty$, satisfying $g_i = 0$ on $\partial E$ and $g_i = C_i$ on $\partial B_p(R_i) \cap E$. Moreover the sequence $g_i$ converges uniformly on compact subsets of $E \cap \partial E$ to a positive weighted function $g$, satisfying $g = 0$ on $\partial E$ and $\sup_{\theta \in E} g = \infty$.

Now we give a decay estimate for weighted harmonic functions on a weighted non-parabolic end of a smooth metric measure space, which is a slight generalization of Lemma 1.1 in [12].

**Lemma 2.5.** Let $(M^n, g, e^{-\varphi}dv)$ be a complete smooth metric measure space of dimension $n$ with $\lambda_1(M) > 0$. Assume that $E$ is a weighted non-parabolic end of $M$. Then any weighted harmonic function $f$ on $E$ satisfies the decay estimate

$$\int_{E(R+1) \setminus E(R)} f^2 d\mu \leq C \exp(-2\sqrt{\lambda_1(M)} R)$$

for some constant $C > 0$ depending on $f$, $\lambda_1(M)$ and $n$, where $B_p(R)$ denotes a geodesic ball centered at some fixed point $p \in M$ with radius $R > 0$, and $E(R) = B_p(R) \cap E$.

The following lemma is a characterization for an end by its weighted volume. Here we let $E$ be an end of $M$, and let $V_\varphi(E)$ be the simply weighted volume of end $E$. We denote the weighted volume of the set $E(R)$ by $V_\varphi(E(R))$.

**Lemma 2.6.** Let $(M^n, g, e^{-\varphi}dv)$ be a complete smooth metric measure space of dimension $n \geq 2$, satisfying $|\nabla \varphi| \leq \theta$ for some positive constant $\theta$. Assume that $\lambda_1(M) \geq \frac{(n-1+\theta)^2}{4}$.

1. If $E$ is a weighted parabolic end, then

$$V_\varphi(E) - V_\varphi(E(R)) \leq C \exp(-(n - 1 + \theta) R)$$

for some constant $C > 0$ depending on $E$, where $R > 0$ is large enough.

2. If $E$ is a weighted non-parabolic end, then

$$V_\varphi(E(R)) \geq C \exp((n - 1 + \theta) R)$$

for some constant $C > 0$ depending on $E$, where $R > 0$ is large enough.

**Remark 2.7.** Lemma 2.6 can be viewed as a refined version of eigenvalue estimate in Theorem 1. Indeed, if $\text{Ric} \geq -(n - 1)$ and $|\nabla \varphi| \leq \theta$ for constant $\theta > 0$, then the weighted Bishop volume comparison theorem (Theorem 1.2 in [21]) asserts that

$$V_\varphi(B_p(R)) \leq C \exp(\theta R) \cdot V_\varphi(B_p(R)) \leq C \exp((n - 1 + \theta) R).$$
Combining this and Lemma 2.6, we conclude that
\[ \lambda_1(M) \leq \frac{(n-1+\theta)^2}{4}, \]
as asserted in Theorem C. Regarding this, we think that this trick is an effective method for the first eigenvalue upper estimate on complete manifolds.

On the other hand, if \( \text{Ric} \geq -(n-1) \) and \( |\nabla\phi| \leq \theta \) for constant \( \theta > 0 \), then the weighted Bishop volume comparison theorem, asserts that for any \( x \in M \) and \( R_1 < R_2 \),
\[ \frac{V_{\phi}(B_x(R_2))}{V_{\phi}(B_x(R_1))} \leq e^{\theta R_2} \frac{V_{H^n}(B(R_2))}{V_{H^n}(B(R_1))}, \]
where \( V_{\phi}(B_x(R)) = \int_{B_x(R)} e^{-\phi} dv(g) \) denotes the weighted volume of the geodesic ball \( B_x(R) \), and \( V_{H^n}(B(R)) \) denotes the volume of a geodesic ball of radius \( R \) in the \( n \)-dimensional hyperbolic space form \( H^n \) with constant curvature \(-1\). In particular, if we let \( x = p \), \( R_1 = 0 \) and \( R_2 = R \), then
\[ V_{\phi}(B_p(R)) \leq C \exp((n-1+\theta)R) \]
for sufficiently large \( R \). If we let \( x \in \partial B_p(R) \), \( R_1 = 1 \) and \( R_2 = R + 1 \), then
\[ V_{\phi}(B_x(1)) \geq C V_{\phi}(B_x(R + 1)) \exp(-\theta) \exp(-(n-1+\theta)R) \]
\[ \geq C \exp(-\theta)V_{\phi}(B_p(1)) \exp(-(n-1+\theta)R). \]
Combining (3), (4) and Lemma 2.6 immediately yields that

**Corollary 2.8.** Let \((M^n, g, e^{-\phi} dv)\) be a complete smooth metric measure space of dimension \( n \geq 2 \), satisfying \( |\nabla\phi| \leq \theta \) for some positive constant \( \theta \). Assume that \( \lambda_1(M) \geq \frac{(n-1+\theta)^2}{4} \).

1. If \( E \) is a weighted-parabolic end, then \( E \) must have exponential weighted volume decay given by
   \[ C_4 \exp(-(n-1+\theta)R) \leq V_{\phi}(E) - V_{\phi}(E(R)) \leq C_1 \exp(-(n-1+\theta)R) \]
   for some constant \( C_1 \geq C_4 > 0 \) depending on \( E \) and \( \theta \), where \( R > 0 \) is large enough.

2. If \( E \) is a weighted-non-parabolic end, then \( E \) must have exponential volume growth given by
   \[ C_3 \exp((n-1+\theta)R) \geq V_{\phi}(E) \geq C_2 \exp((n-1+\theta)R) \]
   for some constant \( C_3 \geq C_2 > 0 \) depending on \( E \), where \( R > 0 \) is large enough.

Besides the above properties, we can also confirm that if the spectrum of the weighted Laplacian has a positive lower bound, then an end is weighted non-parabolic if and only if its weighted volume is infinite (see also [20]).

### 3. Generalized Bochner Formula

In this section, we will give an improved version of the Bochner formula for weighted harmonic functions, which is a mild generalization of the case of harmonic functions due to Yau [25].
Theorem 3.1. Let \((M^n, g, e^{-\varphi}dv)\) be a complete smooth metric measure space of dimension \(n \geq 2\) with \(\text{Ric}_\infty \geq -(n-1)\). Assume that \(|\nabla \varphi| \leq \theta\) for some positive constant \(\theta\) and \(f\) is a weighted harmonic function. For any constants \(k > n\) and \(p\),

\[
\Delta \varphi |\nabla f|^p \geq \frac{1}{p} \left( \frac{k}{k-1} + p - 2 \right) |\nabla f|^{-p} (|\nabla f|^p)^2 - p \left( \frac{\theta^2}{k-n} + n - 1 \right) |\nabla f|^p.
\]

In particular, if \(p = \frac{k-2}{k-n}\), then

\[
(\Delta \varphi |\nabla f|^p) \geq \frac{k - 2}{k - 1} \left( \frac{\theta^2}{k-n} + n - 1 \right) |\nabla f|^p.
\]

Proof of Theorem 3.1. The proof is similar to the proof of Lemma 7.2 in [10]. Choose a local orthogonal frame \(\{e_1, e_2, \ldots, e_n\}\) near any given point so that at the given point \(\nabla f = |\nabla f| e_1\). Since \(f\) is a weighted harmonic function and \(\text{Ric}_\infty \geq -(n-1)\), we compute that

\[
(\Delta \varphi |\nabla f|^2) = \Delta |\nabla f|^2 - \langle \nabla \varphi, \nabla |\nabla f|^2 \rangle
\]

\[
= 2f_{ij}^2 + 2(R_{ij} + \nabla^2 \varphi) f_i f_j
\]

\[
\geq 2f_{ij}^2 - 2(n-1)|\nabla f|^2.
\]

Notice that

\[
|\nabla |\nabla f|^2| = 4 \sum_{j=1}^n \left( \sum_{i=1}^n f_i f_{ij} \right)^2 = 4f_{11}^2 \cdot \sum_{i=1}^n f_{ii}^2 = 4|\nabla f|^2 \cdot \sum_{i=1}^n f_{ii}^2.
\]

and

\[
f_{ij}^2 \geq f_{11}^2 + 2 \sum_{\alpha=2}^n f_{1\alpha}^2 + \sum_{\alpha=2}^n f_{\alpha\alpha}^2
\]

\[
\geq f_{11}^2 + 2 \sum_{\alpha=2}^n f_{1\alpha}^2 + \frac{1}{n-1} \left( \sum_{\alpha=2}^n f_{\alpha\alpha} \right)^2
\]

\[
= f_{11}^2 + 2 \sum_{\alpha=2}^n f_{1\alpha}^2 + \frac{1}{n-1} (\Delta f - f_{11})^2
\]

\[
= f_{11}^2 + 2 \sum_{\alpha=2}^n f_{1\alpha}^2 + \frac{1}{n-1} (\varphi f_i - f_{11})^2
\]

\[
\geq f_{11}^2 + 2 \sum_{\alpha=2}^n f_{1\alpha}^2 + \frac{1}{n-1} \left[ \frac{f_{11}^2}{1 + \frac{\varphi}{\theta^2}} + \frac{(\varphi f_i)^2}{\frac{k-n}{n-1}} \right]
\]

\[
\geq \frac{k}{k-1} \sum_{j=1}^n f_{1j}^2 - \frac{\theta^2}{k-n} |\nabla f|^2
\]

for any constant \(k > n\), where we used the following inequality:

\[
(a + b)^2 \geq \frac{a^2}{1 + \delta} - \frac{b^2}{\delta}
\]
for any $\delta > 0$. Therefore

$$\Delta_\varphi |\nabla f|^2 \geq \frac{k}{2(k-1)} |\nabla f|^2 |\nabla |\nabla f|^2|^2 - 2 \left( \frac{\theta^2}{k-n} + n - 1 \right) |\nabla f|^2.$$ 

Since

$$\nabla |\nabla f|^p = \frac{p}{2} |\nabla f|^{p-2} \nabla |\nabla f|^2 \quad \text{and} \quad |\nabla f|^2 = 2 |\nabla f| \cdot \nabla |\nabla f|,$$

we have that

$$\Delta_\varphi |\nabla f|^p = \frac{1}{p} (p-2) |\nabla f|^{-p} (\nabla |\nabla f|^p)^2 + \frac{p}{2} |\nabla f|^{p-2} \Delta_\varphi |\nabla f|^2$$

$$\geq \frac{1}{p} (p-2) |\nabla f|^{-p} (\nabla |\nabla f|^p)^2$$

$$+ \frac{p}{2} |\nabla f|^{p-2} \left[ \frac{2k}{p^2(k-1)} |\nabla |\nabla f|^p|^2 - 2 \left( \frac{\theta^2}{k-n} + n - 1 \right) |\nabla f|^2 \right]$$

$$= \frac{1}{p} \left( \frac{k}{k-1} + p - 2 \right) |\nabla f|^{-p} (\nabla |\nabla f|^p)^2 - p \left( \frac{\theta^2}{k-n} + n - 1 \right) |\nabla f|^p.$$ 

Letting $p = \frac{k-2}{k-1}$ in the above inequality yields (5).

4. **Proof of Theorem 1.1**

In this section, we will prove Theorem 1.1 stated in introduction. Here we mainly follow the arguments of Li-Wang [12].

**Proof of Theorem 1.1** Assume that $M$ has at least two weighted non-parabolic ends. By the construction described in Section 2, there exists a nonconstant weighted harmonic function $f$ with finite weighted Dirichlet integral on $M$. Let $g$ be the function defined by

$$g := |\nabla f|^{\frac{1}{k-1}},$$

where $k > n$ is a constant. By Theorem 5.1 we have

$$\Delta_\varphi g \geq \frac{k-2}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right) g.$$ 

In the following we **claim** that the function $g$ satisfies

$$\int_{B_{p}(2R)\backslash B_{p}(R)} g^2 d\mu \to 0$$

for $R$ large enough. To prove this claim, firstly, we see that

$$\int_{B_{p}(2R)\backslash B_{p}(R)} g^2 d\mu \leq \left( \int_{B_{p}(2R)\backslash B_{p}(R)} \exp \left( 2 \sqrt{\lambda_1 r} |\nabla f|^2 d\mu \right) \right)^{\frac{1}{k-1}}$$

$$\times \left( \int_{B_{p}(2R)\backslash B_{p}(R)} \exp \left( -2(k-2) \sqrt{\lambda_1 r} \right) d\mu \right)^{\frac{1}{k-1}},$$

where $\lambda_1$ is the smallest eigenvalue of the Laplacian on $M$. This completes the proof of Theorem 1.1.
where we used the Holder inequality. In the following, we shall estimate the right hand side of (11). On one hand, since the weighted volume comparison theorem [21] asserts that

\[ A_p(B_p(r)) \leq \exp(\theta r) \cdot A_{\mathbb{R}^n}(B_p(r)) \]

using this, we have

\[
\int_{B_p(2R) \setminus B_p(R)} \exp\left(-2(k-2)\sqrt{\lambda_1} r\right) d\mu \\
\leq C \int_{R}^{2R} \exp\left(-2(k-2)\sqrt{\lambda_1} r\right) \exp(\theta r) \cdot \exp((n-1) r) dr \\
= C \int_{R}^{2R} \exp\left[(n-1 + \theta - 2(k-2)\sqrt{\lambda_1}) r\right] dr
\]

Noticing

\[ \lambda_1(M) \geq \frac{k-2}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right), \]

and \( k > n \geq 3 \), by Appendix [6] we conclude that

\[ n-1 + \theta - 2(k-2)\sqrt{\lambda_1} < 0 \]

and therefore the right hand side of (12) exponentially decays to 0. On the other hand, since we have assume that \( M \) has at least two weighted non-parabolic ends, by Lemma 2.4 in [20], we have the following decay estimate

\[
\int_{B_p(2R) \setminus B_p(R)} \exp\left(2\sqrt{\lambda_1} r\right) |\nabla f|^2 d\mu \leq CR.
\]

Combining this and (12), we confirm our claim (10).

Next step, we consider a smooth compactly supported function \( \psi \) on \( M \), satisfying

\[
\psi(x) = \begin{cases} 
1 & x \in B_p(R) \\
\frac{C}{R} & x \in B_p(2R) \setminus B_p(R) \\
0 & x \notin B_p(2R).
\end{cases}
\]

Then we have

\[
\int_M |\nabla (\psi g)|^2 d\mu = \int_M |\nabla \psi|^2 g^2 d\mu + \int_M \psi^2 |\nabla g|^2 d\mu + 2 \int_M \psi g \nabla \psi \nabla g d\mu.
\]

Since

\[ 2 \int_M \psi g \nabla \psi \nabla g d\mu = - \int_M \psi^2 |\nabla g|^2 d\mu - \int_M \psi^2 g \Delta \psi g d\mu, \]

by (14), we have

\[
\int_M |\nabla (\psi g)|^2 d\mu = \int_M |\nabla \psi|^2 g^2 d\mu - \int_M \psi^2 g \Delta \psi g d\mu
\]

\[
= \int_M |\nabla \psi|^2 g^2 d\mu + \int_M \left( \frac{k-2}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right) \right) \psi^2 g^2 \\
- \int_M \psi^2 \left[ \frac{k-2}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right) \right] g + \Delta g d\mu.
\]
Since
\[ \lambda_1(M) \geq \frac{k - 2}{k - 1} \left( \frac{\theta^2}{k - n} + n - 1 \right), \]
the variational property of \( \lambda_1(M) \) gives
\[ \frac{k - 2}{k - 1} \left( \frac{\theta^2}{k - n} + n - 1 \right) \int_M \psi^2 g^2 d\mu \leq \int_M |\nabla (\psi g)|^2 d\mu. \]
Substituting this into (15) yields
\[
(16) \int_M \psi^2 g \left[ \frac{k - 2}{k - 1} \left( \frac{\theta^2}{k - n} + n - 1 \right) g + \Delta \varphi g \right] d\mu \leq \int_M |\nabla \psi|^2 g^2 d\mu.
\]
We also note that the cut-function \( \psi \) is the form of (13), and the right hand side of (16) can be estimated by
\[
\int_M |\nabla \psi|^2 g^2 d\mu \leq C R^2 \int_{B_p(2R) \setminus B_p(R)} g^2 d\mu.
\]
By our claim (10), we know the right hand side of integral tends to 0 as \( R \to \infty \). Combining this with (9) we conclude that
\[ g \]
\[ \text{either be to 0 or it satisfies } \Delta \varphi g = -\frac{k - 2}{k - 1} \left( \frac{\theta^2}{k - n} + n - 1 \right) g. \]
By the construction of the weighted harmonic function \( f \), if \( M \) has more than one weighted non-parabolic end, then function \( f \) must be non-constant, hence \( g \neq 0 \). So all the inequalities used to derive (19) are equalities. In particular, from (8) we conclude that
\[ f_{ij} = 0, \quad i \neq j \quad \text{and} \quad f_{\alpha\alpha} = \rho, \quad 2 \leq \alpha \leq n. \]
The equality case of (8) also implies that
\[ |\nabla \varphi| = \theta \quad \text{and} \quad f_{11} = \frac{k - 1}{k - n} (\nabla f, \nabla \varphi). \]
Since \( \Delta \varphi f = 0 \), we also have
\[ f_{11} + (n - 1) \rho = \langle \nabla f, \nabla \varphi \rangle. \]
From above equalities, we derive that
\[
(17) \quad f_{11} = -(k - 1) \rho \quad \text{and} \quad \langle \nabla f, \nabla \varphi \rangle = -(k - n) \rho.
\]
Using the fact that \( f_{\alpha\alpha} = 0 \) for all \( \alpha \neq 1 \), we conclude that \( |\nabla f| \) is identically constant along the level set of \( f \). In particular, the level sets of \( |\nabla f| \) and \( f \) coincide. Moreover,
\[ \rho \delta_{\alpha\beta} = f_{\alpha\beta} = \Pi_{\alpha\beta} f_{11}, \]
where \( (\Pi_{\alpha\beta}) \) is the second fundamental form of the level sets of \( f \). From this, we also have
\[
(18) \quad f_{11} = -\frac{k - 1}{n - 1} H f_1,
\]
where \( H \) denotes the mean curvature of the level sets of \( f \). Applying the same computation to the function \( g \), we have
\[
(19) \quad -\frac{k - 2}{k - 1} \left( \frac{\theta^2}{k - n} + n - 1 \right) g = \Delta \varphi g = g_{11} + H g_1 - \langle \nabla g, \nabla \varphi \rangle.
\]
Here since $g = |\nabla f|^{\frac{k-2}{k-1}}$, we have
$$g_1 = \left( |\nabla f|^{\frac{k-2}{k-1}} \right)_1 = \frac{k-2}{k-1} |\nabla f|^{\frac{k-2}{k-1}} f_j f_{j1} = \frac{k-2}{k-1} f_1 f_{11}.$$
Combining this with (15) yields
$$(20)\quad H = -\frac{n-1}{k-2} g_1 g^{-1}.$$
Meanwhile, by the above equality, (17) and (18) we have
$$\varphi_1 = \frac{k-n}{k-2} g_1 g^{-1}.$$
Plugging the above two relations into (19), we conclude that
$$g_{11} - \frac{k-1}{k-2} g_1^2 g^{-1} + \frac{k-2}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right) g = 0.$$
Letting
$$u := g^{-\frac{1}{\lambda-1}} = |\nabla f|^{-\frac{1}{\lambda-1}},$$
the above equation becomes
$$u_{11} - \frac{1}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right) u = 0.$$
This equation can be regarded as an ordinary differential equation along the integral curve generated by the vector field $e_1$. Hence
$$u(t) = A \exp \left[ \sqrt{K(k, n, \theta)} t \right] + B \exp \left[ -\sqrt{K(k, n, \theta)} t \right],$$
where $A$ and $B$ are nonnegative constants, and
$$K(k, n, \theta) := \frac{1}{k-1} \left( \frac{\theta^2}{k-n} + n - 1 \right).$$
Since $M$ has at least two weighted non-parabolic ends by the assumption of theorem, we claim that any fixed level set $N$ of $|\nabla f|$ must be compact. Indeed, The facts that $f$ has no critical points and that the level set of $f$ coincides with the level set of $|\nabla f|$ lead to the splitting of $M$ as a warped product $\mathbb{R} \times N$. Here the manifold $N$ is necessarily compact due to the fact that $M$ is assumed to have at least two ends. Since
$$\int_M |\nabla f|^2 d\mu < +\infty,$$
we conclude that $|\nabla f|$ must have an interior maximum, saying $\nabla f = 1$. Now we fix $N = \{ |\nabla f| = 1 \}$ and then $u$ must have its minimum along $N$. By reparameterizing, we assume $N$ is given by $t = 0$. So
$$0 = u'(0) = A - B \quad \text{and} \quad 1 = u(0) = A + B.$$
This gives us
$$u(t) = \cosh \left[ \sqrt{K(k, n, \theta)} t \right]$$
and
$$g(t) = \cosh^{-1} \left( k-2 \right) \left[ \sqrt{K(k, n, \theta)} t \right].$$
Substituting this into (20), we have
$$H(t) = (n-1) \sqrt{K(k, n, \theta)} \tanh \left[ \sqrt{K(k, n, \theta)} t \right]$$
and
$$h_{\alpha\beta}(t) = \sqrt{K(k, n, \theta)} \tanh \left[ \sqrt{K(k, n, \theta)} t \right] \delta_{\alpha\beta}.$$
This implies that the metric on \( M = \mathbb{R} \times N \) must be of the form (11).

5. **Proof of Theorem 1.4**

We now follow the lines of [24] and prove Theorem 1.4 given in introduction. The proof method belongs to Li-Wang [13].

**Proof of Theorem 1.4.** Since manifold \( M \) satisfies the hypothesis of Theorem 1.4, Corollary 1.2 asserted that \( M \) must have only one infinite weighted volume end because the warped product with the metric given by

\[
ds_M^2 = dt^2 + \cosh t \, ds_N^2
\]

has \( \lambda_1(M) = n - 2 + \theta \). Now we assume that manifold \( M \) has a finite weighted volume end. Since \( \lambda_1(M) > 0 \), \( M \) must also have an infinite weighted volume end.

By choosing the compact set \( D \) appropriately, we may assume that \( M \setminus D \) has one infinite weighted volume, weighted non-parabolic end \( E_1 \) and one finite weighted volume, weighted parabolic end \( E_2 \).

Following similar Li-Tam’s arguments [11], there exists a positive weighted harmonic function \( f \), satisfying

\[
\inf_{\partial E_1(R)} f \to 0 \quad \text{as} \quad R \to \infty,
\]

\[
\sup_{\partial E_2(R)} f \to \infty \quad \text{as} \quad R \to \infty
\]

and \( f \) has finite weighted Dirichlet integral on \( E_1 \). Meanwhile \( f \) satisfies the following gradient estimate of Theorem C

\[
|\nabla f|^2 \leq (n - 1 + \theta)^2 f^2.
\]

Since function \( f \) is weighted harmonic, we have

\[
\Delta \varphi f^{1/2} = -\frac{1}{4} f^{-3/2} |\nabla f|^2 \leq -\frac{(n - 1 + \theta)^2}{4} f^{1/2}.
\]

If we let \( h = f^{1/2} \), then for any nonnegative cut-off function \( \psi \), we have

\[
\int_M |\nabla (\psi h)|^2 d\mu = \int_M |\nabla \psi|^2 h^2 d\mu + \int_M \psi^2 |\nabla h|^2 d\mu + 2 \int_M \psi \nabla \psi \nabla hd\mu.
\]

Since

\[
\int_M \psi h \nabla \psi \nabla h d\mu = -\int_M \psi \nabla \psi h \nabla h d\mu - \int_M \psi^2 |\nabla h|^2 d\mu - \int_M \psi^2 h \Delta \varphi h d\mu,
\]

the integral equality (22) reduces to

\[
\int_M |\nabla (\psi h)|^2 d\mu = \int_M |\nabla \psi|^2 h^2 d\mu - \int_M \psi^2 h \Delta \varphi h d\mu
\]

\[
= \int_M |\nabla \psi|^2 h^2 d\mu + \frac{(n - 1 + \theta)^2}{4} \int_M \psi^2 h^2
\]

\[
- \int_M \psi^2 h \left[ \frac{(n - 1 + \theta)^2}{4} h + \Delta \varphi h \right] d\mu.
\]

Since \( \lambda_1(M) \geq \frac{(n-1+\theta)^2}{4} \), from the definition of \( \lambda_1(M) \), we have

\[
\frac{(n - 1 + \theta)^2}{4} \int_M \psi^2 h^2 d\mu \leq \int_M |\nabla (\psi h)|^2 d\mu.
\]

Hence

\[
\int_M \psi^2 h \left[ \frac{(n - 1 + \theta)^2}{4} h + \Delta \varphi h \right] d\mu \leq \int_M |\nabla \psi|^2 h^2 d\mu.
\]
Integrating the gradient estimate of Theorem C along geodesics, we know that $f$ must satisfy the growth estimate

$$f(x) \leq C \exp((n-1+\theta) r(x)),$$

where $r(x)$ is the geodesic distance from $x$ to a fixed point $p \in M$. In particular, when restricted on the parabolic end $E_2$, together with the volume estimate of Lemma 2.6 we conclude that

$$(25) \quad \int_{E_2(R)} f \, d\mu \leq CR.$$ 

On the other hand, by Lemma 2.5, function $f$ must satisfy the decay estimate on $E_1$

$$\int_{E_1(R+1) \setminus E_1(R)} f^2 \, d\mu \leq C \exp(-(n-1+\theta) R)$$

for $R$ sufficiently large. By the Schwarz inequality, we have

$$\int_{E_1(R+1) \setminus E_1(R)} f \, d\mu \leq C \exp\left(-\frac{n-1+\theta}{2} R\right) V_{\phi E_1}(R+1)$$

where $V_{\phi E_1}(r)$ denotes the weighted volume of $E_1(r)$. Combining this with the volume estimate of Corollary 2.8 we have that

$$\int_{E_1(R+1) \setminus E_1(R)} f \, d\mu \leq C$$

for some constant $C$ independent of $R$. In particular, we have

$$\int_{E_1(R)} f \, d\mu \leq CR.$$ 

Combining this with (25), we conclude that

$$(26) \quad \int_{B_p(R)} f \, d\mu \leq CR.$$ 

Now we define the cut-off function $\psi$ on $M$ in (24) by

$$\psi(x) = \begin{cases} 
1 & x \in B_p(R) \\
\frac{2R-r}{R} & x \in B_p(2R) \setminus B_p(R) \\
0 & x \not\in B_p(2R).
\end{cases}$$

Hence the right hand side of (24) is given by

$$\int_M |\nabla \psi|^2 h^2 d\mu = R^{-2} \int_{B_p(2R) \setminus B_p(R)} h^2 d\mu$$

and (20) implies

$$\int_M |\nabla \psi|^2 h^2 d\mu \to 0$$

as $R \to \infty$. Therefore we obtain

$$\Delta \psi h = -\frac{(n-1+\theta)^2}{4} h$$

and inequality (21) used in the above argument is an equality. In particular,

$$|\nabla f| = (n-1+\theta) f$$
and hence
\[(27) \quad |\nabla (\ln f)|^2 = (n - 1 + \theta)^2.\]

Therefore inequalities used to prove the gradient estimate of Theorem [C] are all equalities. More precisely, we must have equality (2.8) in [23] since
\[\Delta \varphi |\nabla (\ln f)|^2 = \Delta |\nabla (\ln f)|^2 - \nabla \varphi \cdot \nabla |\nabla (\ln f)|^2 = \nabla |\nabla (\ln f)|^2 = 0.\]

Furthermore, inequalities used to derive (2.8) in [23] must all be equalities. More specifically, equality (2.5) in [23] implies
\[\begin{aligned}
\langle \nabla \varphi, \nabla \ln f \rangle &= (n - 1 + \theta)\theta \\
(\ln f)_{\alpha\beta} &= -(n - 1 + \theta)\delta_{\alpha\beta}
\end{aligned}\]
for all \(2 \leq \alpha, \beta \leq n\). Since \(e_1\) is the unit normal to the level set of \(\ln f\), the second fundamental form \(II\) of the level set is given by
\[II_{\alpha\beta} = (\ln f)_{\alpha\beta} (\ln f)_1 = -(n - 1 + \theta)\delta_{\alpha\beta}
\]
Moreover, (27) implies that if we set \(t = \frac{\ln f}{n - 1 + \theta}\), then \(t\) must be the distance function between the level sets of \(f\), hence also for \(\ln f\). Since \(II_{\alpha\beta} = (-\delta_{\alpha\beta})\), this implies that the metric on \(M\) can be written as
\[ds^2_M = dt^2 + \exp(-2t)ds^2_N.\]

By (28), we also have
\[\phi(t, x) = \phi(0, x) + \theta t,
\]
where \((t, x) \in \mathbb{R} \times N\). Since we assume that the manifold \(M\) has two ends, \(N\) must be compact. A direct computation shows that the condition \(\text{Ric}_M \geq -(n - 1)\) implies that \(\text{Ric}_N \geq 0\). This completes the proof of theorem. \(\square\)

6. Appendix

In this part we will prove the following fact:

**Proposition 6.1.** If \(k > n \geq 3\) and
\[\lambda_1(M) \geq \frac{k - 2}{k - 1} \left(\frac{\theta^2}{k - n} + n - 1\right),\]
then we have
\[n - 1 + \theta - 2(k - 2)\sqrt{\lambda_1} < 0.\]
Proof. Indeed, we only need to confirm that
\[ 2(k - 2)\sqrt{\lambda_1} > n - 1 + \theta. \]
That is,
\[ 4(k - 2)^2\lambda_1 > (n - 1 + \theta)^2. \]
We also notice that (29). Hence if we can prove
\[ 4(k - 2)^2 \frac{k - 2}{k - 1} \left( \frac{\theta^2}{k - n} + n - 1 \right) > (n - 1 + \theta)^2, \]
then the desired conclusion follows. For the above inequality, rearranging terms gives
\[ \left[ \frac{4(k - 2)^3}{(k - 1)(k - n)} - 1 \right] \theta^2 - 2(n - 1)\theta + \left[ \frac{4(k - 2)^3(n - 1)}{k - 1} - (n - 1)^2 \right] > 0. \]
This is a quadratic inequality in \( \theta \). We assert that this inequality is always true. Because when \( k > n \geq 3 \), we have
\[ \frac{4(k - 2)^3}{(k - 1)(k - n)} - 1 > 0 \]
and
\[ 4(n - 1)^2 - 4 \left[ \frac{4(k - 2)^3}{(k - 1)(k - n)} - 1 \right] \left[ \frac{4(k - 2)^3(n - 1)}{k - 1} - (n - 1)^2 \right] \]
\[ = \frac{16(k - 2)^3(n - 1)}{(k - 1)^2(k - n)} \left[ -4(k - 2)^3 + (k - 1)^2 \right] \]
\[ < 0. \]
This proves the proposition. \( \square \)

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