Monopoles, Antimonopoles and Vortex Rings

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(Dated: 27th March 2022)

We present a new class of static axially symmetric solutions of SU(2) Yang-Mills-Higgs theory, where the Higgs field vanishes on rings centered around the symmetry axis. Associating a magnetic dipole moment with each Higgs vortex ring, the dipole moments add for solutions in the trivial topological sector, whereas they cancel for magnetically charged solutions.

PACS numbers: 14.80.Hv,11.15Kc

Introduction Defects, classical solutions of spontaneously broken gauge theories, where the Higgs field vanishes at points, lines or surfaces, are relevant in particle physics and cosmology. Monopoles, for instance, represent zero-dimensional defects, vortex solutions or strings are associated with one-dimensional defects, domain walls represent two-dimensional defects.

Here we present new classical solutions of SU(2) Yang-Mills-Higgs (YMH) theory with the Higgs field in the adjoint representation, where the Higgs field vanishes either at discrete points, as in single monopoles, or at rings, as in vortex loops, or at rings and at a point.

Configuration space of YMH theory consists of sectors, characterized by the topological charge of the Higgs field. The ’t Hooft-Polyakov monopole \( \mathbb{R}^3 \) carries unit topological charge and possesses spherical symmetry. Multimonopoles with higher topological charge possess at most axial symmetry \( \mathbb{R}^3 \), or no rotational symmetry at all \( \mathbb{R}^3 \). The magnetic charge of the (multi-)monopoles is proportional to their topological charge.

In the Bogomol’nyi-Prasad-Sommerfield (BPS) limit of vanishing Higgs potential monopoles and multimonopoles are obtained as solutions of the first order Bogomol’nyi equations \( \mathbb{R}^3 \). The energy of these solutions satisfies exactly the lower energy bound given by the topological charge. Since the repulsive and attractive forces between monopoles exactly compensate, monopoles experience no net interaction.

As shown by Taubes \( \mathbb{R}^3 \), each topological sector contains further smooth, finite energy solutions, which do not satisfy the Bogomol’nyi equations, but only the second order Euler-Lagrange equations. These solutions form saddlepoints of the energy functional. Their energy exceeds the Bogomol’nyi bound.

In the topologically trivial sector the simplest such solution is axially symmetric, and corresponds to an equilibrium state of a monopole-antimonopole pair \( \mathbb{R}^3 \). Here the forces acting on the monopole and antimonopole are balanced, resulting in this (unstable) state, which carries an Abelian magnetic dipole moment. The Abelian magnetic field resembles the field of a physical dipole with magnetic charges localized on the symmetry axis at the equilibrium distance.

Recently, more general static equilibrium solutions have been constructed, representing chains, where monopoles and antimonopoles alternate along the symmetry axis \( \mathbb{R}^3 \). m-chains in the topologically trivial sector carry a magnetic dipole moment and no charge, whereas m-chains in the sector with topological charge one carry charge and no magnetic dipole moment \( \mathbb{R}^3 \).

Here we address the question, whether chains of m multimonopoles and antimonopoles, each with charge n, can exist in static equilibrium. The simplest such generalization, an equilibrium state of a charge 2 monopole and a charge –2 antimonopole was obtained recently \( \mathbb{R}^3 \). We find that, beyond charge \( n = 2 \), no such equilibrium configurations of localized point charges are possible. Instead a new type of equilibrium solution appears.

Ansatz and boundary conditions We consider SU(2) YMH theory in the BPS limit, \( \mathbb{R}^3 \), with su(2) gauge potential \( A_\mu = A_\mu^a r^a / 2 \), field strength tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu,A_\nu] \), and covariant derivative of the Higgs field \( D_\mu \Phi = \partial_\mu \Phi + i[A_\mu, \Phi] \).

Generalizing both the Ansatz for the monopole-antimonopole pairs and chains \( \mathbb{R}^3 \), \( \mathbb{R}^3 \), and the axially symmetric multimonopole ansatz \( \mathbb{R}^3 \), \( \mathbb{R}^3 \), we parametrize the gauge potential and the Higgs field by:

\[
A_\mu dx^\mu = \left( \frac{K_1}{r} \right) d\tau + (1 - K_2) d\theta \left( \frac{\tau_{\phi}}{2} \right) \]

\[
\Phi = \Phi_1 \tau_1^{(n,m)} + \Phi_2 \tau_2^{(n,m)}
\]

with su(2) matrices \( \tau_{\phi}^{(n,m)} = \sin(m\theta) \tau_\rho + \cos(m\theta) \tau_\tau \), \( \tau_{\rho}^{(n,m)} = \cos(m\theta) \tau_\rho - \sin(m\theta) \tau_\phi \), \( \tau_{\tau}^{(n,m)} = -\sin(n\varphi) \tau_\tau + \cos(n\varphi) \tau_y + \sin(n\varphi) \tau_x \). We refer to the integers \( m \) and \( n \) as \( \theta \) winding number and \( \varphi \) winding number, respectively. The profile functions \( K_1 - K_4 \) and \( \Phi_1, \Phi_2 \) depend on the coordinates \( r \) and \( \theta \), only. The ansatz possesses a residual U(1) gauge symmetry. To fix
the gauge we impose the condition \( r \partial_r K_1 - \partial_\theta K_2 = 0 \) \[10, 12\].

To obtain regular solutions with finite energy and energy density we have to impose appropriate boundary conditions. Regularity at the origin requires \( K_1 = K_2 = K_4 = 0 \), \( K_3 = 1 \), \( \sin(m \theta) \Phi_1 + \cos(m \theta) \Phi_2 = 0 \), \( \partial_r [\cos(m \theta) \Phi_1 - \sin(m \theta) \Phi_2] = 0 \). At infinity we require the solutions in the vacuum sector \( (m = 2k) \) to tend to a gauge transformed trivial solution,

\[
\Phi \longrightarrow U \tau_r U^\dagger, \quad A_\mu \longrightarrow i \partial_\mu U U^\dagger,
\]

and the solutions in the topological charge \( n \) sector \( (m = 2k + 1) \) to tend to

\[
\Phi \longrightarrow U \Phi^{(1,n)}_\infty U^\dagger, \quad A_\mu \longrightarrow U A^{(1,n)}_\mu U^\dagger + i \partial_\mu U U^\dagger,
\]

where

\[
\Phi^{(1,n)}_\infty = \tau_r^{(1,n)}, \quad A^{(1,n)}_\mu dx^\mu = \frac{\tau_r^{(n)}}{2} d\theta - n \sin \theta \frac{\tau^{(1,n)}}{2} d\varphi
\]

is the asymptotic solution of a charge \( n \) monopole, and \( U = \exp \{-ik \tau_r^{(n)} \} \), both for even and odd \( m \). Consequently, solutions with even \( m \) have vanishing magnetic charge, whereas solutions with odd \( m \) possess magnetic charge.

In terms of the functions \( K_1 - K_4, \Phi_1, \Phi_2 \) these boundary conditions read \( K_1 = 0, K_2 = 1 - m, K_3 = (\cos \theta - \cos(m \theta))/\sin \theta \) for odd \( m \) and \( K_3 = (1 - \cos(m \theta))/\sin \theta \) for even \( m \), \( K_4 = 1 - \sin(m \theta)/\sin \theta, \Phi_1 = 1 \), and \( \Phi_2 = 0 \).

Regularity on the \( z \)-axis, finally, requires \( K_1 = K_3 = \Phi_2 = 0, \partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta \Phi_1 = 0, \) for \( \theta = 0 \) and \( \theta = \pi \). Defining the Abelian magnetic field via the ’t Hooft tensor with normalized Higgs field \( \hat{\Phi} \)

\[
\mathcal{F}_{\mu\nu} = \text{Tr} \left\{ \hat{\Phi} F_{\mu\nu} - \frac{i}{2} \hat{\Phi} D_\mu \hat{\Phi} D_\nu \hat{\Phi} \right\}
\]

we note, that only solutions with even \( m \) possess an Abelian magnetic dipole moment \[10\].

With this Ansatz the general field equations reduce to six PDEs in the coordinates \( r \) and \( \theta \), which are solved numerically, subject to the above boundary conditions.

Results The \( m \)-chains constructed in \[10\] are characterized by \( \theta \) winding number \( m > 1 \) and \( \varphi \) winding number \( n = 1 \). In these solutions \( m \) monopoles and antimonopoles are located on the symmetry axis, with (roughly) equal distance between them. Their energy increases (approximately) linearly with \( m \), and likewise does the Abelian magnetic moment of \( m \)-chains with even \( m \) \[10\].

Let us now consider chains consisting of multimonopoles with \( \varphi \) winding number \( n = 2 \). For such chains with \( m \leq 5 \) the energies, magnetic moments and locations of the Higgs zeros are shown in Table 1. Their energy increases (approximately) still linearly with \( m \), and so does the magnetic moment of the chains with even \( m \).

Identifying the locations of the Higgs zeros on the symmetry axis with the locations of the monopoles and antimonopoles, we observe that when each pole carries charge \( n = 2 \), the zeros form pairs, when possible, where the distance between the monopole and the antimonopole of a pair is less than the distance to the neighboring monopole or antimonopole, belonging to the next pair.

| \( m/n \) | \( E[4\pi n] \) | \( \mu/n \) | \( (\rho_i, z_i) \) |
|-------|-------|-------|-------|
| \( 1 \) | 1.00, 2.00, 3.00, 4.00 | 0.0, 0.0, 0.0, 0.0 | (0, 0), (0, 0), (0, 0), (0, 0) |
| \( 2 \) | 1.70, 2.96, 4.03, 5.01 | 2.36, 2.38, 2.6, 2.87 | (0, 2.1), (0, 0.9), (3.0, 0), (4.9, 0) |
| \( 3 \) | 2.44, 4.17, 5.62, 6.96 | 0.0, 0.0, 0.0, 0.0 | (0, 0), (0, 0), (0, 0), (0, 0) |
| \( 4 \) | 3.12, 5.07, 6.63, 8.00 | 4.93, 4.81, 5.20, 5.42 | (0, 2.4), (0, 2.0), (3.0, 3.0), (5.4, 2.8) |
| \( 5 \) | 3.78, 6.11, 7.96, 9.59 | 0.0, 0.0, 0.0, 0.0 | (0, 0), (0, 0), (0, 0), (0, 0) |

Table 1 The dimensionless energy, the dipole moment per winding number \( \mu/n \) and the coordinates of the zeros of the Higgs field are given for several values of \( m \) and \( n \).

We observe furthermore, that the equilibrium distance of the monopole-antimonopole pair composed of \( n = 2 \)
multimonopoles is smaller than the equilibrium distance of the monopole-antimonopole pair composed of single monopoles. Thus the higher attraction between the poles of a pair with charge \( n = 2 \) is balanced by the repulsion only at a smaller equilibrium distance.

When increasing the charge of the poles to \( n > 2 \), we expect this trend to continue. The monopoles and antimonopoles of the pairs should approach each other further, settling at a still smaller equilibrium distance.

Constructing solutions with charge \( n = 3 \), however, we do not find chains at all. Now there is no longer sufficient repulsion to balance the strong attraction between the 3-monopoles and 3-antimonopoles. Instead of chains, we now observe solutions with vortex rings, where the Higgs field vanishes on closed rings around the symmetry axis.

To better understand these findings let us consider unphysical intermediate configurations, where we allow the \( \varphi \) winding number \( n \) to continuously vary between the physical integer values. Beginning with the simplest such solution, the \( m = 2 \) solution, we observe, that the zeros of the solution with winding number \( n \) continue to approach each other when \( n \) is increased beyond 2, until they merge at the origin. Here the pole and antipole do not annihilate, however. We conclude, that this is not allowed by the imposed symmetries and boundary conditions. Instead the Higgs zero changes its character completely, when \( n \) is further increased. It turns into a ring with increasing radius for increasing \( n \). The physical 3-monopole-3-antimonopole solution then has a single ring of zeros of the Higgs field and no point zeros. A further increase of \( n \) only increases the radius of the ring further.

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**Fig. 1** The field lines of the Abelian magnetic field are shown for \( m = 2, n = 3 \) (a), \( m = 2, n = 3 \) (b), \( m = 4, n = 3 \) (c), \( m = 4, n = 3 \) (d), \( m = 3, n = 3 \) (e) and \( m = 3, n = 4 \) (f).
Considering the magnetic moment of the \( m = 2 \) solutions, we observe, that it is (roughly) proportional to \( n \). The pair of poles on the \( z \)-axis for \( n = 2 \) clearly gives rise to the magnetic dipole moment of a physical dipole, as illustrated in Fig. 1a, where we show the field lines of the magnetic field, obtained from the \('t\) Hooft tensor. As seen in Fig. 1b, the ring of zeros also gives rise to a magnetic dipole field, which however looks like the field of a ring of mathematical dipoles. This corresponds to the simple picture that the positive and negative charges have merged but not annihilated, and then spread out on a ring.

The solutions with even \( \theta \) winding number reside in the vacuum sector. For \( m = 2k > 2 \) solutions it is now clear how they evolve, when the \( \varphi \) winding number is increased beyond \( n = 2 \). Starting from \( k \) pairs of physical dipoles, the pairs merge and form \( k \) vortex rings, which carry the dipole strength of the solutions. This is illustrated in Fig. 1c for \( m = 4, n = 2 \), and in Fig. 1d for \( m = 4, n = 3 \). As seen in Table 1, the total dipole moment increases (roughly) linearly both with \( m \) and \( n \), since there are \( m/2 \) rings, each formed from charges \( \pm z \).

The solutions with odd \( \theta \) winding number reside in the topological sector with charge \( n \). For \( m = 4k + 1 \) the situation is somewhat similar to the above. Here a single \( n \)-monopole remains at the origin, whereas all other zeros form pairs, which for \( n > 2 \) approach each other, merge and form rings carrying dipole strength. Since, however, a dipole on the positive axis and its respective counterpart on the negative axis have opposite orientation, their contributions cancel in the total magnetic moment. Thus the magnetic moment remains zero, as it must, because of symmetry.

For \( m = 4k - 1 \), on the other hand, the situation is more complicated, because in this case there are for \( n = 2 \) always 3 poles on the \( z \)-axis, which cannot form pairs, such that all zeros belong to a pair, symmetrically located around the origin. For the simplest case, \( m = 3 \), we observe, that two vortices appear in the charge \( n = 3 \) solution, emerging from the upper and lower unpaired zero, respectively, carrying opposite dipole strength. For \( n = 4 \) a third ring appears, emerging from the zero at the origin, which however does not carry a dipole moment. The magnetic field lines of the \( n = 3 \) and \( 4 \) solutions are shown in Figs. 1e and f, respectively. For \( n = 5 \) finally all three rings merge to form a single ring. Further details of these solutions will be given elsewhere.

Concluding, we have found new static axially symmetric solutions of SU(2) YMH theory, characterized by two winding numbers, \( m \) and \( n \). For \( n < 2 \) the Higgs field vanishes on \( m \) discrete points on the \( z \)-axis, for \( n > 2 \) it vanishes on \( m/2 \) rings centered around the \( z \)-axis for even \( m \), while for odd \( m \), it vanishes on one or more rings and at the origin. Solutions with even \( m \) reside in the topologically trivial sector. They carry no magnetic charge but a magnetic dipole moment, (roughly) proportional to the product \( m \times n \). In contrast, solutions with odd \( m \) reside in sectors with non-trivial topology. They carry magnetic charge \( n \) and possess no magnetic dipole moment. Analogous results holds for finite Higgs self-coupling.

We expect that solutions of similar structure might exist in Weinberg-Salam theory \([14]\), where so far only the sphaleron \((m = 1, n = 1)\) is known, the multisphalerons \((m = 1, n = > 1)\), and the sphaleron \( S^* \) \((m = 2, n = 1)\) are known.

Rings of vanishing or small Higgs field are also present in Alice electrodynamics, where they carry magnetic Cheshire charge \([18]\), while closed knotted vortices can arise in theories, allowing for solutions with non-trivial Hopf number \([19]\).

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