Convergence of Deep Neural Networks with General Activation Functions and Pooling

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Abstract

Deep neural networks, as a powerful system to represent high dimensional complex functions, play a key role in deep learning. Convergence of deep neural networks is a fundamental issue in building the mathematical foundation for deep learning. We investigated the convergence of deep ReLU networks and deep convolutional neural networks in two recent researches (arXiv:2107.12530, 2109.13542). Only the Rectified Linear Unit (ReLU) activation was studied therein, and the important pooling strategy was not considered. In this current work, we study the convergence of deep neural networks as the depth tends to infinity for two other important activation functions: the leaky ReLU and the sigmoid function. Pooling will also be studied. As a result, we prove that the sufficient condition established in arXiv:2107.12530, 2109.13542 is still sufficient for the leaky ReLU networks. For contractive activation function such as the sigmoid function, we establish a weaker sufficient condition for uniform convergence of deep neural networks.

Keywords: deep learning, convergence, neural networks, the sigmoid activation function, leaky ReLU, pooling

1 Introduction

Deep neural networks (DNNs) have achieved great successes in a wide range of machine learning tasks including face recognition, speech recognition, game intelligence, natural language processing, and autonomous navigation [13, 8]. Yet mathematical study of DNNs is still at its infancy. So far most of such efforts have been focusing on the expressive power of DNNs [3, 4, 6, 15, 16, 17, 19, 20, 21, 26, 27] in representing certain smooth functions.

The impressive successes in real applications indicate that DNNs constitute a highly efficient function representing system. Different from the classical linear systems such as polynomials, Fourier series, and wavelets, DNNs approximate a given function in a nonlinear manner and the parameters involved are massive. For the classical linear representation systems of functions, the convergence of an expansion such as Fourier series [22] and wavelet series [2] in terms of the coefficients is fundamental and

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extensively studied. The mechanism of why a DNN system with so many parameters can eventually converge to a meaningful function has not been well-understood. This question motivates our study.

As a mathematical research, we set aside the training process of DNNs and focus on understanding the convergence of a DNN system in terms of its parameters. We continue our investigation of this fundamental theoretical problem after the two pioneering works [24, 25]. To explain important issues not resolved in [24, 25], we briefly introduce the results therein.

We considered a general fully connected feed-forward neural network system with increasing depth in [24, 25]. Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded input domain and \( \mathbb{R}^d \) be the output space. Let \( m_i \in \mathbb{N}, i \geq 0 \) with \( m_0 = d \) be the widths of the neural networks. Then after choosing an activation function \( \sigma \) and a sequence of weight matrices \( W_i \in \mathbb{R}^{m_i \times m_{i-1}} \) and bias vectors \( b_i \in \mathbb{R}^{m_i} \), we obtain a system of neural networks illustrated by:

\[
\begin{align*}
  x &\in \Omega \xrightarrow{\sigma} W_1 b_1 \xrightarrow{\sigma} x^{(1)} &\xrightarrow{\sigma} \cdots &\xrightarrow{\sigma} W_n b_n \xrightarrow{\sigma} x^{(n)} &\xrightarrow{\sigma} W_o b_o \xrightarrow{\sigma} y \in \mathbb{R}^{d'}.
\end{align*}
\]

(1.1)

where \( w_o, b_o \) are the weight matrix and bias vector of the output layer. In the above, \( x^{(k)} := \sigma(W_k x^{(k-1)} + b_k), \ 1 \leq k \leq n \) with \( x^{(0)} = x \),

\[
x^{(k)} := \sigma(W_k x^{(k-1)} + b_k), \ 1 \leq k \leq n \quad \text{with} \quad x^{(0)} = x,
\]

(1.2)

\( y := W_o x^{(n)} + b_o \), and the activation function is applied to a vector component-wise. Since the output layer is simply an affine transformation, we do not consider it in the convergence. Therefore, we are interested in the convergence of the neural networks \( \mathcal{N}_n : x \rightarrow x^{(n)} \). We may have an explicit expression of \( \mathcal{N}_n \) with the notation of consecutive composition of functions. Let \( f_1, f_2, \ldots, f_n \) be a finite sequence of functions such that the range of \( f_i \) is contained in the domain of \( f_{i+1} \), \( 1 \leq i \leq n-1 \), the consecutive composition of \( \{f_i\}_{i=1}^n \) is defined to be the function

\[
\bigcirc_{i=1}^n f_i := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1.
\]

With this notation, we see that

\[
\mathcal{N}_n(x) := \left( \bigcirc_{i=1}^n \sigma(W_i x + b_i) \right)(x), \quad x \in \mathbb{R}^d.
\]

(1.3)

Let \( \| \cdot \| \) be a vector norm. The matrix norm induced from this vector norm is still denoted by \( \| \cdot \| \). The activation function in [24, 25] is the widely-used ReLU (rectified linear unit) function

\[
\text{ReLU} \ (x) := \max(x, 0), \quad x \in \mathbb{R}.
\]

The main result on the convergence of DNNs in [24] is that under the fixed width condition \( w_i \equiv m \) for all \( i \geq 1 \), if the weight matrices \( W_n, n \geq 2 \), satisfy

\[
W_n = I + P_n, \quad n \geq 2, \quad \sum_{n=2}^{\infty} \|P_n\| < +\infty,
\]

(1.4)

where \( I \) is the identity matrix, and the bias vectors \( b_i, i \in \mathbb{N} \), satisfy

\[
\sum_{n=1}^{\infty} \|b_n\| < +\infty,
\]

(1.5)
then the ReLU neural networks $N_n$ converge pointwise on $\Omega$. The result was extended in [25] to the case when the widths are increasing $m_i \leq m_{i+1}$ to deal with the important convolutional neural networks (CNNs).

The above result is the first one in the literature that characterizes the convergence of DNNs in terms of the parameters. It opens up the study of harmonic analysis of DNNs. However, there are two important issues unresolved in [24, 25]:

1. Only the ReLU activation function was considered. There are two other widely-used activation functions, which are the Leaky ReLU

$$\sigma_r(x) := \begin{cases} x, & x > 0, \\ rx, & x \leq 0, \end{cases} \quad (1.6)$$

where $r$ is a small positive constant like $r = 0.0001$, and the sigmoid function

$$S(x) := \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}. \quad (1.7)$$

The leaky ReLU activation function can avoid the dying ReLU phenomenon in training a ReLU neural network. Dying ReLU refers to a problem when a ReLU unit only outputs zero for any given inputs. The leaky ReLU is popular in training generative adversarial networks [14, 18]. The sigmoid function is infinitely differentiable, which is a favorable mathematical property. The output of the sigmoid function is naturally between 0 and 1, which makes it welcome in classification problems. When the ReLU activation function is replaced by the leaky ReLU or the sigmoid function, we are wondering the convergence of the resulting neural networks.

2. Poolings were not considered in [24, 25] either. Poolings, typically including the max pooling and the average pooling, are an important engineering technique in DNNs. They are used to downsample the features and thus help to reduce the dimensions. Poolings are indispensable in training DNNs.

We aim at resolving the above two issues, namely, we shall study the convergence of the DNNs with poolings and with the leaky ReLU or sigmoid activation functions. To this end, we shall first prove in Section 2 that the conditions (1.4) and (1.5) are still sufficient for the convergence of DNNs with the leaky ReLU activation function. Our emphasis in the paper is on the sigmoid activation function. We shall establish in Section 3 a sufficient condition for uniform convergence of DNNs with a general contractive activation function, thus including the sigmoid activation function as a special case. The condition will be much weaker than the one (1.4) in [24, 25]. The average pooling and max pooling will also be considered in Section 4.

## 2 DNNs with Leaky ReLU Activation Function

In this section, we let $\sigma = \sigma_r$, where $\sigma_r$ is the leaky ReLU activation function (1.6) and consider convergence of the deep neural network $N_n$ (1.3) as the number $n$ of layers goes to infinity. The widths of the neural network will be fixed. Thus, we suppose $W_1 \in \mathbb{R}^{m \times d}$, $W_i \in \mathbb{R}^{m \times m}$ for $i \geq 2$, and $b_i \in \mathbb{R}^m$ for $i \in \mathbb{N}$.

A key observation in [24] is that the application of the ReLU function to a vector can be replaced by multiplication of the vector with a certain activation matrix, which is a diagonal matrix with diagonal
entries being either 0 or 1. In other words, for each \( y \in \mathbb{R}^m \) there exists a diagonal matrix \( J \in \mathbb{R}^{m \times m} \) with \( J_{ii} \in \{0, 1\} \), \( 1 \leq i \leq m \), such that

\[
\text{ReLU} (y) = (\text{ReLU} (y_1), \text{ReLU} (y_2), \ldots, \text{ReLU} (y_m))^T = Jy.
\]

This observation leads to an explicit expression of the ReLU network in terms of product of matrices in [24]. For leaky ReLU networks, a similar observation holds.

Let \( \sigma \) be the leaky ReLU activation function \((1.6)\) with \( 0 < r < 1 \). Introduce the set of activation matrices

\[
\mathcal{D}_m := \{ \text{diag} (a_1, a_2, \ldots, a_m) : a_i \in \{r, 1\}, 1 \leq i \leq m \}.
\]

We have the following useful expression of \( \mathcal{N}_n \). Let \( \{A_i\} \) be a sequence of matrices in \( \mathbb{R}^{m \times m} \). For \( n, k \in \mathbb{N} \), we shall write

\[
\prod_{i=k}^n A_i = A_n A_{n-1} \cdots A_k, \quad \text{for } n \geq k, \quad \text{and} \quad \prod_{i=k}^n A_i = I, \quad \text{for } n < k,
\]

where \( I \) denotes the \( m \times m \) identity matrix.

**Lemma 2.1** For each \( x \in \mathbb{R}^{d} \), there exists a sequence of activation matrices \( I_n \in \mathcal{D}_m, n \in \mathbb{N} \) such that

\[
\mathcal{N}_n(x) = \left( \prod_{i=1}^n I_i W_i \right) x + \sum_{i=1}^n \left( I_i W_i \right) I_i b_i.
\]  

**Proof:** We first point out that for each \( y \in \mathbb{R}^m \) there exists an activation matrix \( J \in \mathcal{D}_m \) such that

\[
\sigma(y) = (\sigma(y_1), \sigma(y_2), \ldots, \sigma(y_m))^T = Jy.
\]

Then the lemma can be proved by induction. By the above fact, there exists some \( I_1 \in \mathcal{D}_m \) such that

\[
\mathcal{N}_1(x) = \sigma(W_1 x + b_1) = I_1 (W_1 x + b_1) = I_1 W_1 x + I_1 b_1.
\]

Thus, (2.1) holds when \( n = 1 \). Suppose it is true for \( n - 1 \). For the \( n \)-th layer, there exists some \( I_n \in \mathcal{D}_m \) such that

\[
\mathcal{N}_n(x) = \sigma(W_n \mathcal{N}_{n-1}(x) + b_n)
\]

\[
= I_n (W_n \mathcal{N}_{n-1}(x) + b_n)
\]

\[
= I_n W_n \left( \prod_{i=1}^{n-1} I_i W_i \right) x + I_n W_n \sum_{i=1}^{n-1} \left( \prod_{j=i+1}^{n-1} I_j W_j \right) I_i b_i + I_n b_n
\]

\[
= \left( \prod_{i=1}^{n} I_i W_i \right) x + \sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} I_j W_j \right) I_i b_i,
\]

which proves the lemma.

One sees from the expression (2.1) that the convergence of leaky ReLU networks is closely related to the infinite product of matrices

\[
\prod_{n=1}^{\infty} I_n W_n.
\]
In the following, we show the above product converges under the condition (1.4). To this end, we let $\| \cdot \|$ be a norm on $\mathbb{R}^m$ that is nondecreasing on the modules of vector components:

$$\|a\| \leq \|b\| \text{ whenever } |a_i| \leq |b_i|, \text{ for } a = (a_1, a_2, \ldots, a_m), b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m. \quad (2.2)$$

It induces a norm on $\mathbb{R}^{m \times m}$, also denoted by $\| \cdot \|$, by

$$\|A\| = \sup_{x \in \mathbb{R}^m, x \neq 0} \frac{\|Ax\|}{\|x\|}, \text{ for } A \in \mathbb{R}^{m \times m}.$$ 

Clearly, this matrix norm has the property that

$$\|AB\| \leq \|A\|\|B\| \text{ for all matrices } A, B \quad (2.3)$$

and

$$\|I_i\| \leq 1 \text{ for each } I_i \in \mathcal{D}_m \quad (2.4)$$
as the parameter $r$ in the leaky ReLU satisfies $0 < r < 1$.

The main result of this section is as follows. It provides a sufficient condition for pointwise convergence of leaky ReLU neural networks.

**Theorem 2.2** Let $\| \cdot \|$ be a norm on $\mathbb{R}^m$ that satisfies (2.2) and $\| \cdot \|$ be its induced matrix norm. If the weight matrices $W_n, n \geq 2$, satisfy

$$W_n = I + P_n, \quad n \geq 2, \quad \sum_{n=2}^{\infty} \|P_n\| < +\infty \quad (2.5)$$

and the bias vectors $b_i, i \in \mathbb{N}$, satisfy

$$\sum_{n=1}^{\infty} \|b_n\| < +\infty, \quad (2.6)$$

then the leaky ReLU neural network $N_n$ converges pointwise on $\mathbb{R}^d$.

**Proof:** Let $x \in \mathbb{R}^d$. By Lemma 2.1, there exists a sequence of matrices $I_n \in \mathcal{D}_m, n \in \mathbb{N}$ so that (2.1) holds. We first observe that

$$\lim_{n \to \infty} \prod_{i=j}^{n} I_i \text{ exists for every } j \geq 1. \quad (2.7)$$

This is true as for $j \in \mathbb{N},$

$$A_i = \prod_{i=j}^{n} I_i, \quad i \geq j$$
is a diagonal matrix and for $1 \leq k \leq m$, the $k$-th diagonal entry $(A_i)_{kk}$ of $A_i$ is non-increasing as $i$ increases.

We shall also use the inequality that if a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < +\infty$, then for all $p \in \mathbb{N}$ and $n > p$

$$\sum_{i=p+1}^{n} a_i + \sum_{l=2}^{n} \sum_{1 \leq i_2 < \cdots < i_l \leq n \atop i_l > p} \prod_{k=1}^{l} a_{i_k} \leq \left( \sum_{i=p+1}^{n} a_i \right) \exp \left( \sum_{i=1}^{\infty} a_i \right). \quad (2.8)$$
The proof of this inequality can be found in [24].

We are ready to prove the convergence of $N_n(x)$. By (2.1), it suffices to show that under conditions (2.5) and (2.6), the two limits

$$\lim_{n \to \infty} \prod_{i=1}^{n} I_iW_i$$

and

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} I_jW_j \right) I_ib_i$$

exist in $\mathbb{R}^{m \times m}$ and $\mathbb{R}^m$, respectively. We prove the first one only as its proof is essentially different from that in [24] while the proof of the second one is much like to the proof of Theorem 4.5 in [24].

To prove the existence of limit (2.9), it suffices to prove that the infinite product of matrices

$$\prod_{n=2}^{\infty} I_nW_n$$

converges under the assumed condition (2.5). Let $\varepsilon > 0$. By (2.5), there exists $p \in \mathbb{N}$ such that

$$\sum_{n=p+1}^{\infty} \|P_n\| < \varepsilon.$$  

(2.12)

We desire to prove that

$$\prod_{i=2}^{n} I_iW_i = \prod_{i=2}^{n} (I_i + I_iP_i)$$

forms a Cauchy sequence in $\mathbb{R}^{m \times m}$. To this end, we expand the product on the right hand side above to get

$$\prod_{i=2}^{n} I_iW_i = \prod_{i=2}^{n} I_i + \sum_{l=1}^{n-1} \sum_{2 \leq j_1 < j_2 < \ldots < j_l \leq n} \left( \prod_{k=j_{i-1}+1}^{n} I_k \right) \left( \prod_{i=2}^{l} \left( I_{j_i}P_{j_i} \prod_{k=j_{i-1}+1}^{j_{i-1}} I_k \right) \right) I_{j_1}P_{j_1} \left( \prod_{k=2}^{j_{i-1}} I_k \right).$$  

(2.13)

By (2.7), there exists some $N \in \mathbb{N}$ such that for all $n' > n > N$ and $2 \leq j \leq p + 1$

$$\left\| \prod_{i=j}^{n} I_i - \prod_{i=j}^{n'} I_i \right\| < \varepsilon.$$

(2.14)
We now let \( n' > n > \max(N, p + 1) \) and estimate by equations (2.13) that
\[
\left\| \prod_{i=2}^{n} I_i^* W_i - \prod_{i=2}^{n'} I_i^* W_i \right\| \leq \left\| \prod_{i=2}^{n} I_i^* W_i - \prod_{i=2}^{n'} I_i^* \right\| + \sum_{l=1}^{n-1} \sum_{2 \leq j_1 < j_2 < \cdots < j_l \leq p} I_{k=1}^{l} \left\| P_{j_1} \right\| \left\| P_{j_2} \right\| + \cdots + \left\| P_{j_l} \right\| \exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right)
\]
By equations (2.14) and properties (2.3) and (2.4), we further have
\[
\left\| \prod_{i=2}^{n} I_i^* W_i - \prod_{i=2}^{n'} I_i^* W_i \right\| \leq \varepsilon + \varepsilon \sum_{l=1}^{n-1} \sum_{2 \leq j_1 < j_2 < \cdots < j_l \leq p} I_{k=1}^{l} \left\| P_{j_1} \right\| \left\| P_{j_2} \right\| + \cdots + \left\| P_{j_l} \right\| \exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right) + \sum_{l=1}^{n-1} \sum_{2 \leq j_1 < j_2 < \cdots < j_l \leq p} I_{k=1}^{l} \left\| P_{j_1} \right\| \left\| P_{j_2} \right\| + \cdots + \left\| P_{j_l} \right\| \exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right).
\]
Finally, we engage inequalities (2.8) and (2.12) to obtain
\[
\left\| \prod_{i=2}^{n} I_i^* W_i - \prod_{i=2}^{n'} I_i^* W_i \right\| \leq \varepsilon + \varepsilon \left( \left\| P_2 \right\| + \left\| P_2 \right\| + \cdots + \left\| P_p \right\| \right) \exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right) + \left( \sum_{k=p+1}^{n} \left\| P_k \right\| \right) \exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right)\exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right)
\]
\[
\leq C \varepsilon,
\]
where
\[
C = 1 + \left( \sum_{i=2}^{\infty} \left\| P_i \right\| + 2 \right) \exp \left( \sum_{i=2}^{\infty} \left\| P_i \right\| \right).
\]
Therefore, the infinite product of matrices (2.11) converges. The proof is complete. \( \square \)

3 Sigmoid Neural Networks

In this section, we consider neural networks with the sigmoid activation function (1.7). The sigmoid function satisfies
\[
S(x) \in (0, 1) \text{ and } S'(x) = S(x)(1 - S(x)), \ x \in \mathbb{R}.
\]
Thus it holds
\[ |S(x) - S(y)| \leq \frac{1}{4}|x - y|, \; x, y \in \mathbb{R}. \]
In other words, the sigmoid function is a contractive mapping. This is a useful and well-known property. For instance, it was used in investigating the Lipschitz properties of neural networks [1].

To make our study more applicable, we assume in this section that the activation function \( \sigma \) chosen is a contractive mapping, and satisfies for some constant \( \gamma > 1 \) that
\[ |\sigma(x) - \sigma(y)| \leq \frac{1}{\gamma}|x - y|, \; x, y \in \mathbb{R}. \] (3.1)

For a neural network (1.3) with such an activation function, we are able to derive a sufficient condition for its convergence that is much weaker than the one (1.4) and (1.5) in Theorem 2.2.

We still let \( W_1 \in \mathbb{R}^{m \times d} \), \( W_n \in \mathbb{R}^{m \times m} \) for \( n \geq 2 \), and \( b_n \in \mathbb{R}^m \) for \( n \in \mathbb{N} \). Also let \( \| \cdot \| \) be a vector norm that satisfies (2.2) and let \( \| \cdot \| \) be its induced matrix norm. Then it holds
\[ \| \sigma(x) - \sigma(y) \| \leq \frac{1}{\gamma} \| x - y \|, \; x, y \in \mathbb{R}^m. \] (3.2)

The neural network \( N_n(x) \) is defined by (1.3). We shall add the pooling operator in the next section.

**Theorem 3.1** Let \( \sigma \) be an activation function satisfying (3.1), where \( \gamma > 1 \). If \( \{W_n\}_{n=2}^\infty \) and \( \{b_n\}_{n=1}^\infty \) form a Cauchy sequence in \( \mathbb{R}^{m \times m} \) and \( \mathbb{R}^m \), respectively, and \( \{W_n\} \) satisfies
\[ \lim_{n \to \infty} \| W_n \| < \gamma \] (3.3)
then the neural network (1.3) converges uniformly on any bounded domain \( \Omega \subseteq \mathbb{R}^d \).

**Proof:** By (3.3), there exists a constant \( \alpha < 1 \), \( N_0 \in \mathbb{N} \) and \( C_1 > 1 \) such that
\[ \prod_{i=j}^n \frac{\| W_i \|}{\gamma} < C_1 \alpha^{n-j+1}, \; \text{for all } n \geq N_0. \] (3.4)

For convenience, we shall define
\[ \prod_{i=j}^n \frac{\| W_i \|}{\gamma} = 1 \text{ when } j > n. \]

Since the input space \( \Omega \subseteq \mathbb{R}^d \) is bounded and \( \sigma \) satisfying (3.1) must be bounded on \( \Omega \), there exists a constant \( C_2 > 0 \) such that
\[ \| N_n(x) \| \leq C_2 \text{ and } \| W_{n+1}N_n(x) - W_1x \| \leq C_2, \; \text{for all } n \in \mathbb{N}, x \in \Omega. \] (3.5)

Let \( \varepsilon > 0 \) be fixed. Since \( \{W_n\} \) and \( \{b_n\} \) are both Cauchy sequences, there exists a positive integer \( N_1 \) and a constant \( C_3 > 0 \) such that
\[ \| W_{n+p} - W_n \| < \varepsilon, \; \| b_{n+p} - b_n \| < \varepsilon, \; \text{for all } n \geq N_1 \text{ and } p \in \mathbb{N}, \]
\[ \| W_n \| \leq C_3, \; \| b_n \| \leq C_3, \; \text{for all } n \in \mathbb{N}. \] (3.6)

Last, there exists \( N_2 \in \mathbb{N} \) such that
\[ \alpha^n < \varepsilon, \; \text{for all } n \geq N_2. \] (3.7)
We have made enough preparations. Write $N_0(x) = x$. By (3.2) and equations (3.4), we estimate for $n \geq 2 \max(N_0, N_1, N_2)$, $p \in \mathbb{N}$, and $x \in \Omega$ that

$$
\|N_{n+p}(x) - N_n(x)\| = \|\sigma(W_{n+p}N_{n+p-1}(x) + b_{n+p}) - \sigma(W_nN_{n-1}(x) + b_n)\|
$$

$$
\leq \frac{1}{\gamma} \|W_{n+p}N_{n+p-1}(x) + b_{n+p} - W_nN_{n-1}(x) - b_n\|
$$

$$
\leq \frac{1}{\gamma} \|b_{n+p} - b_n\| + \frac{1}{\gamma} \|W_{n+p}\| \|N_{n+p-1}(x) - N_{n-1}(x)\| + \frac{1}{\gamma} \|W_{n+p} - W_n\| \|N_{n-1}(x)\|
$$

$$
\leq \sum_{i=0}^{n-1} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|b_{n+p-i} - b_{n-i}\| + \left( \prod_{i=0}^{n-2} \frac{\|W_{n+p-i}\|}{\gamma} \right) \frac{1}{\gamma} \|W_{n+p-i} - W_{n-i}\| \|N_{n-i-1}(x)\|
$$

$$
< \sum_{i=0}^{n-1} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|b_{n+p-i} - b_{n-i}\| + \frac{1}{\gamma} C_1 C_2 \varepsilon
$$

$$
+ C_2 \sum_{i=0}^{n-2} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|W_{n+p-i} - W_{n-i}\|
$$

$$
= \frac{1}{\gamma} C_1 C_2 \varepsilon + \sum_{i=n-N+1}^{n-1} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|b_{n+p-i} - b_{n-i}\|
$$

$$
+ \sum_{i=0}^{n-N} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|b_{n+p-i} - b_{n-i}\|
$$

$$
+ C_2 \sum_{i=n-N+1}^{n-2} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|W_{n+p-i} - W_{n-i}\|
$$

$$
+ C_2 \sum_{i=0}^{n-N} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|W_{n+p-i} - W_{n-i}\|,
$$

where $N = \max(N_0, N_1, N_2, 3)$. Since $n - N + 1 > N = \max(N_0, N_1, N_2, 3)$, we have

$$
\sum_{i=n-N+1}^{n-1} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|b_{n+p-i} - b_{n-i}\| < C_1 \frac{2C_3}{\gamma} \sum_{i=n-N+1}^{n-1} \alpha^i < \frac{2C_1 C_3}{\gamma(1 - \alpha)} \alpha^{n-N+1} < \frac{2C_1 C_3}{\gamma(1 - \alpha)} \varepsilon,
$$

(3.9)

$$
\sum_{i=0}^{n-N} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|b_{n+p-i} - b_{n-i}\| < C_1 \frac{\varepsilon}{\gamma} \sum_{i=0}^{n-N} \alpha^i < \frac{C_1}{\gamma(1 - \alpha)} \varepsilon,
$$

(3.10)

$$
C_2 \sum_{i=n-N+1}^{n-2} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|W_{n+p-i} - W_{n-i}\| < C_1 C_2 \frac{2C_3}{\gamma} \sum_{i=n-N+1}^{n-2} \alpha^i < \frac{2C_1 C_2 C_3}{\gamma(1 - \alpha)} \varepsilon
$$

(3.11)

and

$$
C_2 \sum_{i=0}^{n-N} \left( \prod_{k=0}^{i-1} \frac{\|W_{n+p-k}\|}{\gamma} \right) \frac{1}{\gamma} \|W_{n+p-i} - W_{n-i}\| < C_2 C_1 \frac{\varepsilon}{\gamma} \sum_{i=0}^{n-N} \alpha^i < \frac{C_1 C_2}{\gamma(1 - \alpha)} \varepsilon.
$$

(3.12)
By inequalities (3.8)-(3.12), we obtain for \( n > 2N \) that
\[
\| N_n + p(x) - N_n(x) \| \leq \left( \frac{C_1 C_2}{\gamma} + \frac{2C_1 C_3 + C_1 + 2C_1 C_2 C_3 + C_1 C_2}{\gamma(1 - \alpha)} \right) \varepsilon,
\]
which proves that \( \{ N_n(x) \} \) converges uniformly on \( \Omega \).

We remark that conditions (1.4) and (1.5) immediately imply the assumptions in Theorem 3.1. In particular, if (1.4) is true then (3.3) holds with
\[
\lim_{n \to \infty} \| W_n \| = 1.
\]
Therefore, the assumptions in Theorem 3.1 are much weaker than (1.4) and (1.5).

4 Neural Networks with Pooling

We consider neural networks with pooling in this section. Poolings are useful in enhancing features and reducing the dimension of features. Two popular poolings in the machine learning are the max pooling and the average pooling.

Let \( s \geq 2 \) be a positive integer. The average pooling \( P_A \) and the max pooling \( P_M \) are two operators from \( \mathbb{R}^{m+s} \) to \( \mathbb{R}^m \) defined by
\[
(P_A x)_i := \frac{x_i + x_{i+1} + \cdots + x_{i+s}}{s+1}, \quad 1 \leq i \leq m, \quad x \in \mathbb{R}^{m+s} \tag{4.1}
\]
and
\[
(P_M x)_i := \max(x_i, x_{i+1}, \ldots, x_{i+s}), \quad 1 \leq i \leq m, \quad x \in \mathbb{R}^{m+s}. \tag{4.2}
\]
We shall need a simple property about these two pooling operators. It should be well-known. But we still include the proof for the completeness of the paper.

For \( 1 \leq q \leq +\infty \), we denote by \( \| \cdot \|_q \) the \( q \)-norm on the Euclidean space. Namely,
\[
\| x \|_\infty = \max(|x_1|, |x_2|, \ldots, |x_m|), \quad \text{and for } 1 \leq q < +\infty, \quad \| x \|_q = \left( \sum_{i=1}^{m} |x_i|^q \right)^{1/q}, \quad x \in \mathbb{R}^m.
\]

Lemma 4.1 It holds for all \( 1 \leq q \leq +\infty \) and \( x, y \in \mathbb{R}^{m+s} \)
\[
\| P_A x \|_q \leq \| x \|_q \tag{4.3}
\]
and
\[
\| P_M x - P_M y \|_q \leq (s + 1)^{1/q} \| x - y \|_q. \tag{4.4}
\]

Proof: We deal with the average pooling first. Let \( x \in \mathbb{R}^{m+s} \). Then (4.3) is obvious when \( q = +\infty \). Let \( 1 \leq q < +\infty \). By the Hölder inequality,
\[
\| P_A x \|_q = \frac{1}{s+1} \left( \sum_{i=1}^{m} \sum_{j=i}^{i+s} x_j^q \right)^{1/q} \leq \frac{1}{s+1} \left( \sum_{i=1}^{m} (s+1)^{q(1-1/q)} \sum_{j=i}^{i+s} x_j^q \right)^{1/q} \leq \frac{1}{(s+1)^{1/q}} \left( \sum_{i=1}^{m+s} (s+1) |x_i|^q \right)^{1/q} = \| x \|_q.
\]
We next consider the max pooling. By drawing the four points involved in the real axis, one sees

$$\left| \max(x_1, x_2) - \max(y_1, y_2) \right| \leq \max(|x_1 - y_1|, |x_2 - y_2|).$$

Based on this observation, it can be shown by induction on \(s\) that for all \(x_i, y_i \in \mathbb{R}, 1 \leq i \leq s + 1,$$\n
$$\left| \max(x_1, x_2, \cdots, x_{s+1}) - \max(y_1, y_2, \cdots, y_{s+1}) \right| \leq \max(|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_{s+1} - y_{s+1}|).$$

Inequality (4.1) follows immediately from the above equation. \(\square\)

We next describe a neural network with pooling. Let \(\mathcal{P}\) be the average pooling (4.1) or the max pooling (4.2). The width of the neural network will be fixed. Thus, we let \(W_1 \in \mathbb{R}^{(m+s) \times d}, W_n \in \mathbb{R}^{(m+s) \times m}\) for \(n \geq 2, b_n \in \mathbb{R}^m\) for \(n \in \mathbb{N}\). With a chosen activation function \(\sigma\), we illustrate the neural network with pooling \(\mathcal{P}\) as follows:

\[
\begin{align*}
\mathbf{x} & \in \mathbb{R}^d \xrightarrow{\mathcal{P}, \sigma} \mathbf{x}^{(1)} \quad \text{input} \quad \text{1st layer} \\
& \xrightarrow{\mathcal{P}, \sigma} \mathbf{x}^{(2)} \quad \text{2nd layer} \\
& \quad \cdots \\
& \xrightarrow{\mathcal{P}, \sigma} \mathbf{x}^{(n)} \quad \text{n-th layer}
\end{align*}
\]

In the above,

\[
\mathbf{x}^{(k)} := \sigma(\mathcal{P}(W_k \mathbf{x}^{(k-1)}) + b_k), \quad 1 \leq k \leq n \quad \text{with} \quad \mathbf{x}^{(0)} = \mathbf{x}.
\]

With the notation of consecutive composition, the function \(\tilde{\mathcal{N}}_n\) determined by (4.5) is

\[
\tilde{\mathcal{N}}_n(\mathbf{x}) := \left( \bigotimes_{i=1}^n \sigma(\mathcal{P}(W_i \cdot) + b_i) \right) (\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.
\]

By Lemma 4.1, the average pooling and the max pooling are non-expansive with respect to the \(q\)-norm \((1 \leq q \leq +\infty)\) and the \(\infty\)-norm, respectively. Based on this fact, it is straightforward to extend the result in Theorem 3.1 to the neural network (4.7) with pooling.

**Theorem 4.2** Let \(\sigma\) be a contractive activation function satisfying (3.1) with \(\gamma > 1\). Suppose \(\{W_n\}_{n=2}^\infty\) and \(\{b_n\}_{n=1}^\infty\) form a Cauchy sequence in \(\mathbb{R}^{m \times m}\) and \(\mathbb{R}^m\), respectively. The followings hold true:

\[
\begin{align*}
\text{• if } \{W_n\} \text{ satisfies for some } q \in [1, +\infty] \text{ that } \\
& \quad \lim_{n \to \infty} \|W_n\|_q < \gamma \quad (4.8) \\
\text{then the neural network (4.7) with average pooling } \mathcal{P} = \mathcal{P}_A \text{ converges uniformly on any bounded domain } \Omega \subseteq \mathbb{R}^d.
\end{align*}
\]

\[
\begin{align*}
\text{• if } \{W_n\} \text{ satisfies } \\
& \quad \lim_{n \to \infty} \|W_n\|_\infty < \gamma \quad (4.9) \\
\text{then the neural network (4.7) with max pooling } \mathcal{P} = \mathcal{P}_M \text{ converges uniformly on any bounded domain } \Omega \subseteq \mathbb{R}^d.
\end{align*}
\]
Proof: Since any two norms on a common finite-dimensional vector space are equivalent. We may use the $q$-norm when $P = P_A$ and use the $\infty$-norm when $P = P_M$. In both cases, the pooling operator is non-expansive with respect to the chosen vector norm. Therefore,

\[
\left\| \tilde{N}_{n+p}(x) - \tilde{N}_n(x) \right\| = \left\| \sigma \left( P(W_{n+p}\tilde{N}_{n+p-1}(x)) + b_{n+p} \right) - \sigma \left( P(W_n\tilde{N}_{n-1}(x)) + b_n \right) \right\| \\
\leq \frac{1}{\gamma} \left\| P(W_{n+p}\tilde{N}_{n+p-1}(x)) + b_{n+p} - P(W_n\tilde{N}_{n-1}(x)) - b_n \right\| \\
\leq \frac{1}{\gamma} \left\| b_{n+p} - b_n \right\| + \frac{1}{\gamma} \left\| W_{n+p}\tilde{N}_{n+p-1}(x) - W_n\tilde{N}_{n-1}(x) \right\|.
\]

Thus the arguments in the proof of Theorem 3.1 can be directly applied here to prove the desired results.

In the final part of the paper, we discuss applications of the above results to convolutional neural networks (CNNs). In a CNN, the weight matrices in (4.5) are generated from convolutions. Specifically, let $w^{(n)} := (w_0^{(n)}, w_1^{(n)}, \ldots, w_s^{(n)}) \in \mathbb{R}^{s+1}$ be the filter mask at the $n$-th layer for $n \geq 2$. The corresponding weight matrix $W_n$ is determined by

\[
W_n x = x \ast w^{(n)}, \quad y \in \mathbb{R}^m,
\]

where the convolution $x \ast w$ of a vector $x := (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ with a filter mask $w := (w_0, w_1, \ldots, w_s)$ outputs a vector $y := (y_1, y_2, \ldots, y_{m+s}) \in \mathbb{R}^{m+s}$ is defined by

\[
y_i := \min_{j=max(0,i-m)} \sum_{j=max(0,i-m)} w_j x_{i-j}, \quad 1 \leq i \leq m + s.
\]

Thus, $W_n$ is an $(m + s) \times m$ Toeplitz type matrix given by

\[
(W_n)_{ij} := \begin{cases} 
0, & i < j, \\
w^{(n)}_{i-j}, & i \geq j,
\end{cases} \quad (4.10)
\]

In other words,

\[
W_n = \begin{bmatrix}
0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & w_0^{(n)} & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
w_s^{(n)} & w_{s-1}^{(n)} & \cdots & w_0^{(n)} & 0 & \cdots & 0 \\
0 & w_s^{(n)} & \cdots & w_1^{(n)} & w_0^{(n)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 0 & w_s^{(n)} & \cdots & w_0^{(n)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & w_s^{(n)} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & w_0^{(n)} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & w_{s-1}^{(n)} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & w_1^{(n)} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

We have the following result about convergence of CNNs with pooling.
Theorem 4.3 Let $W_n$ be the weight matrix defined by (4.10) via a mask $w^{(n)} := (w_0^{(n)}, w_1^{(n)}, \ldots, w_s^{(n)}) \in \mathbb{R}^{s+1}$ for $n \geq 2$. Also let $P$ be either the average pooling or the max pooling. If the following conditions are satisfied:

(i) the activation function $\sigma$ is a contractive mapping satisfying (3.1) with $\gamma > 1$,

(ii) the bias vectors $\{b_n\}_{n=1}^{\infty}$ form a Cauchy sequence in $\mathbb{R}^m$,

(iii) for each $0 \leq k \leq s$, $\{w_k^{(n)}\}_{n=1}^{\infty}$ converges in $\mathbb{R}$,

(iv) it holds

$$\lim_{n \to \infty} \sum_{k=0}^{s} |w_k^{(n)}| < \gamma,$$

then the convolutional neural network (4.7) converges uniformly on any bounded domain $\Omega \subseteq \mathbb{R}^d$.

Proof: It suffices to point out that conditions (4.8) and (4.9) are met under the above assumptions. By well-known facts about matrix norms,

$$\|W_n\|_1 = \|W_n\|_\infty = \sum_{k=0}^{s} |w_k^{(n)}|.$$  

By the Riesz-Thorin interpolation theorem (see, [7], page 200),

$$\|W_n\|_q \leq \|W_n\|_1^{\frac{1}{q}} \|W_n\|_\infty^{1-\frac{1}{q}}, \quad q \in [1, +\infty].$$

It follows immediately from the above two equations that (4.11) implies (4.8) and (4.9). $\square$

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