POWER PARTITIONS

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Abstract. In 1918, Hardy and Ramanujan published a seminal paper which included an asymptotic formula for the partition function. In their paper, they also claim without proof an asymptotic equivalence for \( p^k(n) \), the number of partitions of a number \( n \) into \( k \)-th powers. In this paper, we provide an asymptotic formula for \( p^k(n) \), using the Hardy-Littlewood Circle Method. We also provide a formula for the difference function \( p^k(n+1) - p^k(n) \). As a necessary step in the proof, we obtain a non-trivial bound on exponential sums of the form \( \sum_{m=1}^{q} e(\frac{am^k}{q}) \).

1. Introduction

A partition of number \( n \) is a non-increasing sequence of positive integers whose sum is equal to \( n \). Fix an integer \( k \geq 2 \). Define \( p^k(n) \) to be the number of partitions of \( n \) in which all parts are perfect \( k \)-th powers. This sequence has the generating function

\[
\Psi_k(z) := \sum_{n=0}^{\infty} p^k(n)z^n = \prod_{n=1}^{\infty} (1 - z^{nk})^{-1}.
\]

In 1918, Hardy and Ramanujan \([1]\) published a seminal paper introducing a new method for computing asymptotic formulae for integer sequences, called the Circle Method. In their paper, they list a number of problems to which their methods can be applied. In particular, they state (without proof) the following asymptotic equivalence for the number of partitions of \( n \) into \( k \)-th powers:

\[
\log p^k(n) \sim (k + 1) \left( \frac{1}{k} \Gamma \left( 1 + \frac{1}{k} \right) \zeta \left( 1 + \frac{1}{k} \right) \right)^{k/(k+1)} n^{1/(k+1)}.
\]

In 1934, E. Maitland Wright \([5]\) gave a precise asymptotic formula for this restricted partition function. His proof requires a number of complicated objects including generalized Bessel functions. In this paper, we provide a new asymptotic formula for the number of partitions into \( k \)-th powers, using a relatively simple implementation of the Hardy-Littlewood Circle Method. The special case \( k = 2 \) is treated by R.C. Vaughan \([4]\). This work is a generalization of Vaughan’s result.
**Theorem 1.** Let $n$ be a sufficiently large natural number, and choose positive numbers $X$ and $Y$ satisfying

\[
n = X \left( \frac{1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} - \frac{1}{2} - \frac{1}{2}\zeta(-k)X^{-1}\right),
\]

\[
Y = \frac{k+1}{2k^3} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} - \frac{1}{4}.
\]

Then, for each $J \in \mathbb{N}$, there are real numbers $c_1, c_2, \ldots, c_J$ (independent of $n$), so that

\[
p^k(n) = \exp\left(\frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{\frac{k}{k+1}} - \frac{1}{2}\right) \left(\pi^{\frac{k}{2}} + \sum_{j=1}^{J} c_j Y^{-j} + O(Y^{-J})\right).
\]

Note that

\[
X \sim \left(\frac{1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)\right)^{\frac{k}{k+1}} n^{\frac{1}{k+1}}, \quad Y \sim \frac{k+1}{2k^3} \left(\frac{1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)\right)^{\frac{k}{k+1}} n^{\frac{1}{k+1}}.
\]

If we take the logarithm of both sides of (2), we obtain the asymptotic equivalence

\[
\log p^k(n) \sim \frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{\frac{k}{k+1}}.
\]

Re-writing the right-hand side to be in terms of $n$ yields

\[
\log p^k(n) \sim (k+1) \left(\frac{1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)\right)^{\frac{k}{k+1}} n^{\frac{1}{k+1}},
\]

which is the same as Hardy and Ramanujan’s claim.

It is worthwhile to remark that it is possible to compute the coefficients $c_j$ that appear in (2). However, their closed form is sufficiently complicated to make the statement of the theorem unreadable. Thus we have omitted the closed form for $c_j$ from this paper. Section 5 provides an outline of how one might go about computing the values of $c_j$, given a particular choice of $J$.

The methods used to find the asymptotic formula in Theorem 1 can also be used to estimate the growth of $p^k(n)$. This yields the following:

**Theorem 2.** Let $n, X,$ and $Y$ be as above. Then there are real numbers $d_1, d_2, \ldots, d_k$ (independent of $n$), so that

\[
p^k(n + 1) - p^k(n) = \exp\left(\frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{\frac{k}{k+1}} - \frac{1}{2}\right) \left(\pi^{\frac{k}{2}} + \sum_{j=1}^{k-1} d_j Y^{-j} + O(Y^{-k})\right).
\]

From Theorem 2 we can immediately deduce an asymptotic equivalence:

**Corollary 1.** Let $n$ and $X$ be as above. Then

\[
p^k(n + 1) - p^k(n) \sim \frac{p^k(n)}{X}
\]
as $n \to \infty$. 

Before proceeding with the proof of the main result, we will need a few definitions. Let
\[
\Phi_k(z) := \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j} z^{jn^k}.
\]
We then have
\[
\Psi_k(z) = \exp(\Phi_k(z)).
\]
For convenience, we also define
\[
(5) \quad \rho := e^{-1/X} \quad \text{and} \quad \Delta := (1 + 4\pi^2 X^2 \Theta^2)^{-1/2}
\]
for \(X, \Theta \in \mathbb{R}\) with \(X \geq 1\).

We will prove Theorem 1 using the Hardy-Littlewood Method. In dealing with the major arcs, we require a bound on exponential sums of the form
\[
(6) \quad S_k(r, b) := \sum_{m=1}^{r} e\left(\frac{bm^k}{r}\right).
\]
Here we use the standard notation \(e(\alpha) = e^{2\pi i \alpha}\). The proof of the bound is quite simple, yet doesn’t appear to exist in the literature. We state it here as a lemma, as it may be of interest beyond the scope of this paper.

**Lemma 1.** For each \(k \geq 2\) there exists a positive constant \(\delta_k\) such that \(|S_k(r, b)| \leq (1 - \delta_k)r\) for all \(r > 1\) and \(b \in \mathbb{Z}\) with \((r, b) = 1\).

**Proof** By Theorem 4.2 of [3] we have that \(S_k(r, b) \ll r^{1-1/k}\) for \((r, b) = 1\). Thus there exists \(C_k\) such that \(|S_k(r, b)| \leq C_k r^{1-1/k}\). So, we can find \(R\) sufficiently large and \(\nu_k > 0\) such that \(|S_k(r, b)| \leq (1 - \nu_k)r\) for all \(r \geq R, b \in \mathbb{Z}\) with \((r, b) = 1\).

If \(1 < r < R\), then there is at least one term in \(S_k(r, b)\) that is not equal to 1. This term is of the form \(e(bm^k/r)\). Therefore
\[
S_k(r, b) \leq |r - 1 + e(bm^k/r)| \leq |r - 1 + e(1/r)| \leq |r - 1 + e(1/R)| < (1 - \eta_k)r,
\]
for some \(\eta_k > 0\).

Let \(\delta_k = \min(\nu_k, \eta_k)\). Then \(|S_k(r, b)| \leq (1 - \delta_k)r\) for all \(r > 1, b \in \mathbb{Z}\) with \((r, b) = 1\). \(\square\)

2. **Auxiliary Lemmas**

At several points in the proof of Theorem 1 we will need to estimate the value of \(\Phi_k(\rho e(\Theta))\). So, before proving the theorem, we introduce two estimates for this expression. Lemma 2 provides a very precise estimate that will be used for \(|\Theta| \leq \frac{2}{38\pi X}\) and will establish the main term of (2). Lemma 3 provides a less precise estimate that will be used to deal with the major arcs (excluding \(|\Theta| \leq \frac{3}{38\pi X}\)). The estimate needed for the minor arcs is provided in the body of the proof of Theorem 1.
Lemma 2. Suppose $\Theta \in \mathbb{R}$ and $X \geq 1$. If $X \Delta^3 \geq 1$, then
\[
\Phi_k(\rho e(\Theta)) = \frac{1}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) \left( \frac{X}{1-2\pi i X \Theta} \right)^{1/k} - \frac{1}{2} \log \left( \frac{(2\pi)^k X}{1-2\pi i X \Theta} \right) + \frac{1}{2} \zeta(-k) \left( \frac{1-2\pi i X \Theta}{X} \right) + O \left( \Delta^{-1/2} \exp \left( - \frac{1}{k} (2(\pi \Delta)^{k+1} X)^{1/k} \right) \right).
\]

Proof. We have
\[
\Phi_k(\rho e(\Theta)) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j} \exp \left( -jn^k \left( \frac{1}{X} - 2\pi i \Theta \right) \right).
\]

Using a Mellin Transform (Theorem C.4 of [2]), this is equal to
\[
\frac{1}{2\pi i} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j} \int_{c-i\infty}^{c+i\infty} \Gamma(s) j^{-s} n^{-ks} \left( \frac{X}{1-2\pi i X \Theta} \right)^s ds
\]
for $c > 0$. Here $\left( \frac{X}{1-2\pi i X \Theta} \right)^s$ denotes $\exp \left( s \log \left( \frac{X}{1-2\pi i X \Theta} \right) \right)$ where the logarithm is defined by continuous variation of $\log \left( \frac{X}{1-2\pi i X \Theta} \right)$ as $\theta$ varies continuously from 0 to $\Theta$ through real values. The series $\zeta(s+1)$ and $\zeta(ks)$ converge absolutely and uniformly for $\Re s \geq 1/k + \delta$ where $\delta$ is any positive number. Hence, for any real $c > 1/k$ we have,
\[
\Phi_k(\rho e(\Theta)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+1) \zeta(ks) \left( \frac{X}{1-2\pi i X \Theta} \right)^s \Gamma(s) ds.
\]

Since $\Gamma \left( \frac{1}{k} \right)$ is well-defined, the integrand has a simple pole at $s = 1/k$ with residue
\[
\frac{1}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) \left( \frac{X}{1-2\pi i X \Theta} \right)^{1/k}.
\]
The integrand also has a double pole at $s = 0$ from $\zeta(s+1) \Gamma(s)$. The Laurent expansion of $\zeta(s+1) \Gamma(s)$ at $s = 0$ is of the form
\[
\frac{1}{s^2} + \sum_{j=0}^{\infty} a_j s^j,
\]
so the residue of the integrand at $s = 0$ is
\[
\zeta(0) \log \left( \frac{X}{1-2\pi i X \Theta} \right) + k \zeta'(0).
\]
We recall (see, for example, [2]) that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} \log 2\pi$, so the residue of the integrand at $s = 0$ is
\[
-\frac{1}{2} \log \left( \frac{(2\pi)^k X}{1-2\pi i X \Theta} \right)
\]
The $\Gamma$-function has simple poles at the negative integers, but the $\zeta$-function also has zeros at the negative even integers. If $s \leq -2$ is an integer, then either $ks$ or $s+1$ is an
even integer, so the poles of \( \Gamma \) are cancelled by the zeros of \( \zeta(ks)\zeta(s+1) \). This leaves one simple pole at \( s = -1 \). The residue here is

\[
-\zeta(0)\zeta(-k) \left( \frac{1 - 2\pi i X \Theta}{X} \right) = \frac{1}{2} \zeta(-k) \left( \frac{1 - 2\pi i X \Theta}{X} \right).
\]

By the functional equation for the zeta function in its asymmetrical form, we have

\[
\zeta(s+1)\zeta(ks)\Gamma(s) = 2^{(k+1)s} + 1 \zeta(s)\zeta(1-ks)\Gamma(-s)\Gamma(s)\Gamma(1-k). 
\]

Moreover, by the reflection formula for the gamma function,

\[
\Gamma(-s)\Gamma(s) = \frac{\pi}{-s \sin(\pi s)}.
\]

Combining this with the fact that \( \Gamma(1-ks) = -ks\Gamma(-ks) \), we have

\[
\zeta(s+1)\zeta(ks)\Gamma(s) = (2k)(2\pi)^{(k+1)s} \zeta(-s)\zeta(1-ks)\Gamma(-k)\cos(\pi s/2) \sin(k\pi s/2). 
\]

Note that

\[
\frac{\cos(\pi s/2) \sin(k\pi s/2)}{\sin(\pi s)} = \frac{\sin(k\pi s/2)}{2\sin(\pi s/2)} \ll e^{(k-1)|t|/2}.
\]

Therefore, by Stirling’s formula, when \( \sigma = \Re s \leq -3/2 \),

\[
\zeta(s+1)\zeta(ks)\Gamma(s) \ll (2\pi)^{(k+1)s} k^{-ks} |s|^{-1/2-k\sigma} e^{-\pi|t|/2}. 
\]

From the argument in [4], we see that

\[
\left| \left( \frac{X}{1 - 2\pi i X \Theta} \right)^s \right| \leq (X\Delta)^\sigma \exp \left( |t| \left( \frac{\pi}{2} - \Delta \right) \right).
\]

Therefore the integrand is

\[
\ll (2\pi)^{(k+1)\sigma} k^{-ks} |s|^{-1/2-k\sigma} (X\Delta)^\sigma e^{-\Delta|t|}.
\]

Let \( R \geq \frac{3}{2} \), and move the vertical line of integration to the line \( \Re s = -R \). On this line the integrand is

\[
\ll \left( \frac{1}{(2\pi)^{(k+1)} X \Delta} \right)^R k^{kR} |R + it|^{-1/2 + kR} e^{-\Delta|t|}.
\]

On the pieces with \( |t| > R \), the integrand is

\[
\ll \left( \frac{1}{2(\pi^{(k+1)} X \Delta)} \right)^R k^{kR} |t|^{-1/2 + kR} e^{-\Delta|t|}.
\]

After a change of variable \( y = \Delta|t| \), this contributes

\[
\ll \left( \frac{k^k}{2(\pi \Delta)^{(k+1)} X} \right)^R \Delta^{-1/2} \Gamma(kR + 1/2),
\]
which by Stirling’s formula is
\[ \ll \left( \frac{k^{2k}e^{-k}}{2(\pi \Delta)^{(k+1)}X} \right)^R R^{kR} \Delta^{-1/2}. \]

Meanwhile, the part of the integral with \(|t| \leq R\) contributes
\[ \ll \left( \frac{1}{2\pi(k+1)X\Delta} \right)^R k^{kR} R^{-1/2+kR} \Delta^{-1}. \]

Combining the estimates we see that the integral is
\[ \ll \left( \frac{k^{2k}}{2e^{k}(\pi \Delta)^{(k+1)}X} \right)^R R^{kR} \Delta^{-1/2}. \]

The choice of \(R\) which minimizes the expression above is
\[ R_0 = \left( \frac{2(\pi \Delta)^{(k+1)}X}{k^{2k}} \right)^{1/k}. \]

Let \(R = \max(R_0, 3/2)\). Then the integral is
\[ \ll \exp \left( -\frac{1}{k} \left( \frac{2(\pi \Delta)^{(k+1)}X}{k^{2k}} \right)^{1/k} \right) \Delta^{-1/2}. \]

Applying the Cauchy residue theorem, we obtain the result. \(\square\)

**Lemma 3.** Suppose that \(X \in \mathbb{R}, X > 1, \Theta \in \mathbb{R}, a \in \mathbb{Z}, q \in \mathbb{N}, (a,q) = 1\) and \(\theta = \Theta - a/q\). Then
\[ \Phi_k(\rho e(\Theta)) = \Gamma \left( \frac{k+1}{k} \right) \left( \frac{X}{1 - 2\pi i X \Theta} \right)^{1/k} \sum_{j=1}^{\infty} \frac{S_k(q_j, a_j)}{j^{k+1} q_j} + O(q^{1/2+\epsilon} \log X (1 + X^{1/2} |\theta|^{1/2})) \]
where \(q_j = q/(q,j)\) and \(a_j = a_j/(q,j)\).

**Proof** From the definition, we have
\[ \Phi_k(\rho e(\Theta)) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j} e^{-nkj/X} e(j n^k \Theta). \]

We write
\[ e^{-nkj/X} = \int_n^{\infty} kx^{k-1} jX^{-1} e^{-xkj/X} \, dx. \]

Thus
\[ \Phi_k(\rho e(\Theta)) = \sum_{j=1}^{\infty} \frac{1}{j} \int_0^{\infty} kx^{k-1} jX^{-1} e^{-xkj/X} \sum_{n \leq x} e(j n^k \Theta) \, dx. \]
It is useful to observe the crude bound
\[
\int_0^\infty kx^{k-1}jX^{-1}e^{-xjX/X}e^{jn_k}dx \ll \int_0^\infty kx^{k}jX^{-1}e^{-xj/X}dx.
\]
Using integration by parts, this is
\[
\int_0^\infty e^{-xj/X}dx = \left(\frac{X}{j}\right)^{1/k} \int_0^\infty e^{-y^{1/k}}dx \ll \left(\frac{X}{j}\right)^{1/k}.
\]
Let \( J \) be a parameter at our disposal. Then
\[
\sum_{j=J+1}^\infty \frac{1}{j} \int_0^\infty kx^{k-1}jX^{-1}e^{-xjX/X}e^{jn_k}dx \ll \sum_{j=J+1}^\infty \frac{1}{j} \left(\frac{X}{j}\right)^{1/k} \ll \left(\frac{X}{J}\right)^{1/k}.
\]
It remains to consider
\[
\sum_{j=1}^J \frac{1}{j} \int_0^\infty kx^{k-1}jX^{-1}e^{-xjX/X}e^{jn_k}dx.
\]
By Theorem 4.1 of [3], we have
\[
\sum_{n \leq x} e(jn_k\Theta) = q_j^{-1}S_k(q_j, a_j) \int_0^x e(j\theta\gamma^k) d\gamma + O \left(q_j^{1/2+\epsilon} \left(1 + x^k j|\theta|\right)^{1/2}\right).
\]
Hence the expression in (8) is equal to
\[
\sum_{j=1}^J \frac{1}{j} \int_0^\infty kx^{k-1}jX^{-1}e^{-xjX/X}e^{jn_k}dx + E_1,
\]
where
\[
E_1 \ll \sum_{j=1}^J \frac{q_j^{1/2+\epsilon}}{j} \int_0^\infty kx^{k-1}jX^{-1}e^{-xjX/X} \left(1 + x^k j|\theta|\right)^{1/2} dx.
\]
By integration by parts we have
\[
E_1 \ll \sum_{j=1}^J \frac{q_j^{1/2+\epsilon}}{j} \left(1 + \frac{k}{2} (j|\theta|)^{1/2} \right) \int_0^\infty x^{k/2-1}e^{-xjX/X} dx.
\]
We now make the substitution \( y = x^{k}jX^{-1}. \) Then we have \( x = (yX/j)^{1/k}, \) and \( dx = \frac{1}{k} y^{1/k-1} (X/j)^{1/k} dy. \) So the integrand becomes
\[
\left(\frac{yX}{j}\right)^{1/2-1/k} e^{-y} \frac{1}{k} y^{1/k-1} \left(\frac{X}{j}\right)^{1/k} = \frac{1}{k} \left(\frac{X}{j}\right)^{1/2} y^{-1/2} e^{-y}.
\]
Thus
\[ E_1 \ll \sum_{j=1}^{J} \frac{q_j^{1/2+\varepsilon}}{j} \left( 1 + \frac{1}{2} (X|\theta|)^{1/2} \int_{0}^{\infty} y^{-1/2} e^{-y} \, dy \right) \]
\[ \ll q^{1/2+\varepsilon} \log J \left( 1 + |\theta|^{1/2} X^{1/2} \right). \]

We now turn our attention to the main term of the expression in (9). By integration by parts, this is
\[ \sum_{j=1}^{J} S_k(q_j, a_j) \frac{1}{jq_j} \int_{0}^{\infty} e^{-x^{k/j}X} e(x^{k/j} \theta) \, dx. \]

The integral here is
\[ \int_{0}^{\infty} \exp(-x^{k/j}X^{-1}(1 - 2\pi i X\theta)) \, dx. \]

We make the substitution \( z = \left( jX^{-1}(1 - 2\pi i X\theta) \right)^{1/k} x \). Choose \( \phi \) so that \( |\phi| < \pi \) and
\[ \frac{1 - 2\pi i X\theta}{|1 - 2\pi i X\theta|} = e^{i\phi}. \]

We thus obtain
\[ z = \left( jX^{-1}|1 - 2\pi i X\theta| \right)^{1/k} e^{i\phi/k} x. \]

This gives
\[ \int_{0}^{\infty} \exp(-x^{k/j}X^{-1}(1 - 2\pi i X\theta)) \, dx = \left( \frac{X}{j(1 - 2\pi i X\theta)} \right)^{1/k} \int_{\mathcal{L}} e^{-z^k} \, dz, \]

where \( \mathcal{L} \) is the ray \( \{ z = xe^{i\phi/k} : 0 \leq x < \infty \} \). By Cauchy’s theorem, the integral here is \( \Gamma \left( \frac{k+1}{k} \right) \). Putting everything together, we have
\[ \Phi_k(\rho e(\Theta)) = \Gamma \left( \frac{k+1}{k} \right) \left( \frac{X}{1 - 2\pi i X\theta} \right)^{1/k} \sum_{j=1}^{J} S_k(q_j, a_j) \frac{1}{jq_j} \]
\[ + O \left( q^{1/2+\varepsilon} \log J \left( 1 + |\theta|^{1/2} X^{1/2} \right) \left( \frac{X}{j} \right)^{1/k} \right). \]

Since \( |S_k(q_j, a_j)| \leq q_j \), we have
\[ \Gamma \left( \frac{k+1}{k} \right) \left( \frac{X}{1 - 2\pi i X\theta} \right)^{1/k} \sum_{j=J+1}^{\infty} \frac{S_k(q_j, a_j)}{j^{1/k} q_j} \ll \left( \frac{X}{j} \right)^{1/k}. \]
Thus we can extend the sum in the main term to infinity. Setting $J = X$ gives
\[
\Phi_k(\rho \varepsilon(\Theta)) = \Gamma \left( \frac{k + 1}{k} \right) \left( \frac{X}{1 - 2\pi i X \theta} \right)^{1/k} \sum_{j=1}^{\infty} \frac{S_k(q_j, a_j)}{j^{k+1} q_j^{k+1}} \frac{X}{1 - 2\pi i X \theta}
+ O \left( q^{1/2+\varepsilon} \log X \left( 1 + |\theta|^{1/2} X^{1/2} \right) \right),
\]
as desired. □

3. Proofs of Theorem I

We prove the theorem using the Hardy-Littlewood circle method. From Cauchy’s theorem, we have
\[
p^k(n) = \int_{0}^{1} \rho^{-n} \exp(\Phi_k(\rho \varepsilon(\Theta)) - 2\pi i n \Theta) \, d\Theta.
\]
By the periodicity of the integrand, we may replace the unit interval by any interval of length 1. We will use the interval $U = (-X^{1/k} - 1, 1 - X^{1/k})$. This is a convenient choice because the main contribution to the integral comes from $\Theta$ near the origin. Using $U$ instead of $[0, 1]$ prevents that region from being split in two.

For $a, q \in \mathbb{N}$ with $(a, q) = 1$, define
\[
\mathcal{M}(q, a) = \{ \Theta \in U : |\Theta - \frac{a}{q}| \leq q^{-1} X^{1/k} \},
\]
and let
\[
\mathcal{M} = \bigcup_{1 \leq a \leq q \leq X^{1/k}} \mathcal{M}(q, a).
\]
We refer to these disjoint intervals as the major arcs, and we define the minor arcs to be the complement of the major arcs, namely
\[
\mathcal{m} = U \setminus \mathcal{M}.
\]

In a typical implementation of the circle method, one would split the integral into the major arcs and the minor arcs, with the major arcs making up the main term of the asymptotic formula. However, in our case, the contribution from $\mathcal{M}(1, 0)$ is significantly greater than the contribution from the rest of the major arcs. So we will split the integral into three main parts, namely
\[
p^k(n) = \left\{ \int_{\mathcal{M}(1, 0)} + \int_{\mathcal{M} \setminus \mathcal{M}(1, 0)} + \int_{\mathcal{m}} \right\} \rho^{-n} \exp(\Phi_k(\rho \varepsilon(\Theta)) - 2\pi i n \Theta) \, d\Theta.
\]
We will treat $\mathcal{M} \setminus \mathcal{M}(1, 0)$ and $\mathcal{m}$ in the way one would traditionally treat the major and minor arcs, respectively, but the major arcs with $q > 1$ will not contribute to the main term of the asymptotic formula. Rather they will be “thrown away” into the error term. The main term of the asymptotic formula will come from the first part of the integral, when $\Theta$ is close to the origin. We examine that piece first.
We first consider
\[ \int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi i n \Theta) \, d\Theta. \]

When \(|\Theta| \leq 1/X\),
\[ (1 + 4\pi^2)^{-1/2} \leq \Delta \leq 1 \]
and by Lemma \(2\) we have
\[
\Phi_k(\rho e(\Theta)) = \frac{1}{k} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) \left(\frac{X}{1 - 2\pi i \Theta}\right)^{1/k} - \frac{1}{2} \log \left(\frac{(2\pi)^k X}{1 - 2\pi i X \Theta}\right) + \frac{1}{2} \zeta(-k) \frac{1 - 2\pi i X \Theta}{X} + O\left(\Delta^{-1/2} \exp\left(-\frac{1}{k} \left(2(\pi \Delta)^{k+1} X\right)^{1/k}\right)\right). \]

Thus we have
\[ \rho^{-n} \Psi_k(\rho e(\Theta)) = \rho^{-n} \exp(\Xi_k(\rho e(\Theta))) \left(1 + O\left(\Delta^{-1/2} \exp\left(-\frac{1}{k} \left(2(\pi \Delta)^{k+1} X\right)^{1/k}\right)\right)\right) \]
where
\[
\Xi_k(\rho e(\Theta)) = \frac{1}{k} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) \left(\frac{X}{1 - 2\pi i \Theta}\right)^{1/k} - \frac{1}{2} \log \left(\frac{(2\pi)^k X}{1 - 2\pi i X \Theta}\right) + \frac{1}{2} \zeta(-k) \frac{1 - 2\pi i X \Theta}{X}. \]

We also write
\[ \frac{X}{1 - 2\pi i X \Theta} = X \Delta e^{i\phi} \]
where \(\phi = \arg(1 + 2\pi i X \Theta)\). Note that 0 < \(|\phi| < \pi/2\), so 0 < \(\cos(\phi/k) < 1\). Hence
\[ \left|\left(\frac{X}{1 - 2\pi i X \Theta}\right)^{1/k}\right| = (X \Delta)^\frac{1}{k}. \]

So the \(O\)-term in (12) contributes
\[ \ll X^{-1/2} \exp\left(\frac{n}{X} + \frac{1}{k} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) (X \Delta)^{\frac{1}{k}} + \frac{1}{2} \zeta(-k) (X \Delta)^{-1} - \frac{1}{k} \left(2(\pi \Delta)^{k+1} X\right)^{\frac{1}{k}}\right). \]

As a function of \(\Delta\), the expression here has a unique local maximum at
\[ \Delta_0 = \frac{\zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right)}{(k+1)2\pi \pi^{\frac{1}{k}}}. \]
Direct computation shows that $\Delta_0 < 1$ for $k \leq 5$ and $\Delta_0 > 1$ for $k \geq 6$. We will address these cases separately. It will help to recall that

$$
\frac{n}{X} = \frac{1}{k^2} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k} - \frac{1}{2} - \frac{1}{2} \zeta(-k) X^{-1}.
$$

If $k \geq 6$, then $\Delta_0 > 1$, so the expression in (13) is monotonically increasing in $\Delta$ for $\Delta \leq 1$. Thus we may replace $\Delta$ by 1 to obtain that (13) is

$$
\ll X^{-1/2} \exp \left( \frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k} - \frac{1}{2} - \frac{1}{k} \left(2\pi^{-k+1}X\right)^{1/k} \right).
$$

If $k = 2$ or $k = 4$, then $\zeta(-k) = 0$ and $0 < \Delta_0 < 1$, so the expression in (13) is

$$
\ll X^{-1/2} \exp \left( \frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k} - \frac{1}{2} - \frac{1}{k} \left(2\pi \Delta_0 (k+1) X\right)^{1/k} \right).
$$

If $k = 3$ or $k = 5$, then $\frac{2}{3} < \Delta_0 < 1$. Note that $\zeta(-3) = \frac{1}{120}$, $\zeta(-5) = \frac{-1}{252}$. Thus the terms involving $\zeta(-k)$ in (13) have absolute value less than 1, and certainly less than $\left| \frac{1}{k} \left(2\pi \Delta (k+1) X\right)^{1/k} \right|$. So, there exists some fixed $\delta > 0$ such that for any $k \geq 2$, the expression in (13) is

$$
\ll X^{-1/2} \exp \left( \frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k} - \frac{1}{2} - \delta X^{1/k} \right).
$$

Hence the integral in (10) is

$$
\int_{3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Xi_k(\rho e(\Theta)) - 2\pi n \Theta) d\Theta + O \left( X^{-1/2} \exp \left( \frac{k+1}{k^2} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k} - \frac{1}{2} - \delta X^{1/k} \right) \right).
$$

We turn our attention to the main term in (14). When $|\Theta| < 1/(2\pi X)$, the function

$$
F(\Theta) := \Xi_k(\rho e(\Theta)) - 2\pi n \Theta
$$

can be expanded as

$$
F(\Theta) = \frac{1}{k} \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k} + \frac{1}{2} \zeta(-k) X^{-1} - \frac{1}{2} \log \left( (2\pi)^k X \right) - Y(2\pi X \Theta)^2 + G(\Theta)
$$

where

$$
G(\Theta) = \sum_{j=3}^{\infty} \left( \frac{\zeta(k+1) \Gamma\left(j+\frac{1}{k}\right) X^{1/k}}{j! k} - \frac{1}{2j} \right) (2\pi i X \Theta)^j.
$$
Thus the integral in the main term of (14) becomes

$$\exp\left(\frac{n}{X} + \frac{1}{k}\zeta(k+1)\Gamma\left(\frac{1}{k}\right)X^{\frac{1}{k}} + \frac{1}{2}\zeta(-k)X^{-1}\right) \int_{-3/(8\pi X)}^{3/(8\pi X)} \exp (-Y(2\pi X\Theta)^2 + G(\Theta)) \, d\Theta$$

We rewrite the coefficients in the series for $G(\Theta)$ as

$$\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{j+1}{k}\right)X^{\frac{1}{k}} - \frac{1}{2j} = a_j Y + b_j$$

where

$$a_j = \frac{2k^2}{k+1} \frac{\Gamma(j + \frac{1}{k})}{j!} \Gamma\left(\frac{1}{k}\right), \quad b_j = \frac{1}{2} \left(\frac{2k^2}{k+1} \frac{\Gamma(j + \frac{1}{k})}{j!} \Gamma\left(\frac{1}{k}\right) - \frac{1}{2j}\right)$$

We then rewrite the integral on the right-hand side of (16) as

$$\int_0^{3/(8\pi X)} (\exp(G(\Theta)) + \exp(G(-\Theta))) \exp (-Y(2\pi X\Theta)^2) \, d\Theta$$

We now make the change of variable $\phi = (2\pi X\Theta)^2 Y$, so that the right-hand side of (16) becomes

$$\int_0^{9Y/16} \Re \left(\exp (-\phi + H(\phi))\right) \phi^{-1/2} \, d\phi$$

where

$$H(\phi) = \sum_{j=3}^{\infty} i^j (a_j + b_j Y^{-1}) \phi^{\frac{j}{2}} Y^{1-\frac{j}{2}}.$$

We have

$$\Re \left(\exp (-\phi + H(\phi))\right) \leq e^{-\phi} \left|e^{H(\phi)}\right| = e^{-\phi} \Re H(\phi)$$

and

$$\Re H(\phi) = Y \sum_{j=2}^{\infty} (-1)^j (a_{2j} + b_{2j} Y^{-1}) \phi^j Y^{-j}.$$
For \( j \geq 2 \), we have \( 0 < a_{2j} \leq a_4 = \frac{6k^2 + 5k + 1}{12k^2} \) and \( 0 \leq b_{2j} \leq a_{2j} \). Hence, when \( 0 \leq \phi \leq 9Y/16 \), we have

\[
|\Re H(\phi)| \leq \left( \frac{6k^2 + 5k + 1}{12k^2} \right) Y(1 + Y^{-1}) \sum_{j=2}^{\infty} \left( \frac{\phi}{Y} \right)^j
\]

\[
= \left( \frac{6k^2 + 5k + 1}{12k^2} \right) Y(1 + Y^{-1})(\phi/Y)^2
\]

\[
\leq \frac{9}{7} \left( \frac{6k^2 + 5k + 1}{12k^2} \right) (1 + Y^{-1}) \phi < \left( 1 - \frac{1}{56k^2} \right) \phi,
\]

since \( n \) is sufficiently large. Hence for \( Z > 0 \),

\[
\int_{\phi=0}^{9Y/16} \Re (\exp (-\phi + H(\phi))) \phi^{-1/2} d\phi \leq \int_{\phi=0}^{9Y/16} e^{-\phi} \exp \left( \left( 1 - \frac{1}{56k^2} \right) \phi \right) \phi^{-1/2} d\phi
\]

\[
\ll Z^{-1/2} \int_{\phi=0}^{\phi=0} e^{-\phi/(56k^2)} d\phi \ll Z^{-1/2} e^{Z/(56k^2)}.
\]

Take \( Z = 56k^2 J \log Y \). Then

\[
\int_{\phi=0}^{9Y/16} \Re (\exp (-\phi + H(\phi))) \phi^{-1/2} d\phi \ll Y^{-J}.
\]

The integral in (14) now becomes

\[
\frac{\exp \left( \frac{k+1}{k^2} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{k+1}{k} \right) \frac{1}{2} \right)}{(2\pi)^{k+1} X^{\frac{k+1}{2}} Y^{\frac{k+3}{2}}} \left( \int_{\phi=0}^{\phi=0} \Re (\exp (-\phi + H(\phi))) \phi^{-1/2} d\phi + O(Y^{-J}) \right).
\]

When \( 0 \leq \phi \leq Z \), we have

\[
\sum_{j=2J+3}^{\infty} i^j (a_j + b_j Y^{-1}) \phi^{\frac{j+3}{2}} Y^{-\frac{j+1}{2}} \ll \frac{\phi^{J+\frac{3}{2}} Y^{-\frac{1}{2} - J}}{1 - (\phi/Y)^{\frac{1}{2}}} \ll \phi^{J+\frac{3}{2}} Y^{-\frac{1}{2} - J}.
\]

Hence,

\[
\exp \left( \sum_{j=2J+3}^{\infty} i^j (a_j + b_j Y^{-1}) \phi^{\frac{j+3}{2}} Y^{-\frac{j+1}{2}} \right) = 1 + O(\phi^{J+\frac{3}{2}} Y^{-\frac{1}{2} - J}).
\]

Let

\[
H_J(\phi) = \sum_{j=3}^{2J+2} i^j (a_j + b_j Y^{-1}) \phi^{\frac{j}{2}} Y^{-\frac{j-1}{2}}.
\]

Then

\[
\int_{\phi=0}^{\phi=0} \Re (\exp (-\phi + H(\phi))) \phi^{-1/2} d\phi
\]

\[
= \int_{\phi=0}^{\phi=0} \Re (\exp (-\phi + H_J(\phi))) \left( 1 + O(\phi^{J+\frac{3}{2}} Y^{-\frac{1}{2} - J}) \right) \phi^{-1/2} d\phi.
\]
Similar to the argument for \(H\), we have \(\Re H_J(\phi) < \left(1 - \frac{1}{36k^2}\right) \phi\) and so the error term here contributes
\[
Y^{-\frac{1}{2} - J} \int_0^\infty e^{-\phi/(56k^2)} \phi^{J+1} d\phi \ll Y^{-\frac{1}{2} - J}.
\]
Hence, by (19), the integral in (14) is now
\[
\exp\left(\frac{k+1}{k^2} \frac{\zeta(\frac{k+1}{k}) \Gamma(\frac{k}{k})}{(2\pi \frac{k+2}{2}) \frac{X^\frac{k}{2}}{Y^\frac{k}{2}}} \left( \int_Z^\infty \Re(\exp(-\phi + H_J(\phi))) \phi^{-1/2} d\phi + O(Y^{-J}) \right) \right).
\]
We have
\[
\Re(H_J(\phi)) = \sum_{j=0}^\infty \frac{H_J(\phi)^j}{j!}.
\]
The method of estimation of \(\Re(H(\phi))\) can be used again to show that \(|H_J(\phi)| \ll Y^{-\frac{1}{4}} \phi^{\frac{3}{2}}\).
Thus \(|H_J(\phi)| \leq Y^{-\frac{1}{4}}\). Therefore
\[
\left| \int_0^Z e^{-\phi} \Re(H_J(\phi)^j) \phi^{-\frac{1}{2}} d\phi \right| \leq Y^{-\frac{1}{4}}
\]
and so
\[
\sum_{j=0}^\infty \int_0^Z e^{-\phi} \Re\left(\frac{H_J(\phi)^j}{j!}\right) \phi^{-\frac{1}{2}} d\phi \ll Y^{-J-1}.
\]
We are left to deal with
\[
\int_0^Z e^{-\phi} \sum_{j=0}^{4J+3} \Re\left(\frac{H_J(\phi)^j}{j!}\right) \phi^{-\frac{1}{2}} d\phi.
\]
If \(Y\) is fixed, then \(H_J(\phi)\) is a polynomial in \(i\phi^{1/2}\) of degree \(2J+2\) with real coefficients. Moreover, the coefficient of \((i\phi^{1/2})^j\) in \(H_J(\phi)\) is given by
\[
(a_j + b_j Y^{-1}) Y^{-\frac{1}{2}} = a_j (Y^{-\frac{1}{2}})^{j-2} + b_j (Y^{-\frac{1}{2}})^j,
\]
which is itself a real polynomial in \(Y^{-\frac{1}{2}}\) of degree \(j\) with a zero of order \(j-2\). The expression \(\sum_{j=0}^{4J+3} \left(\frac{H_J(\phi)^j}{j!}\right)\) is therefore a real polynomial in \(i\phi^{1/2}\) of degree at most \(L = (2J+2)(4J+3)\). This polynomial can be written as
\[
\sum_{h=0}^L p_h(Y^{-\frac{1}{2}})(i\phi^{\frac{1}{2}})^h
\]
where the coefficients \(p_h(\phi)\) are polynomials in \(\phi\) of degree at most \(h\). In particular, we have \(p_0(\phi) = 1\), \(p_1(\phi) \equiv p_2(\phi) \equiv 0\), and for \(h \geq 3\), \(p_h(0) = 0\). The polynomials \(p_h(\phi)\) have parity that agrees with the parity of \(h\).
For \(0 \leq h \leq L\), we have
\[
\int_Z^\infty e^{-\phi} \phi^{\frac{h-1}{2}} d\phi \leq Y^{-J} \int_0^\infty e^{-\phi} \phi^{\frac{h-1}{2}} d\phi \ll Y^{-J}.
\]
Therefore, by (20),

\[
\int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta) \, d\Theta 
= \frac{\exp \left( \frac{k+1}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) X^{\frac{k}{2}} - \frac{1}{2} \right)}{(2\pi)^{\frac{k+2}{2}} X^{\frac{3}{2}} Y^{\frac{1}{2}}} \left( I + O(Y^{-J}) \right)
\]

(22)

where

\[
I = \int_0^Z \Re \left( \exp \left( -\phi + H_J(\phi) \right) \right) \phi^{-\frac{k}{2}} \, d\phi
\]

\[
= \sum_{h=0}^L \sum_{h \text{ even}} p_h(Y^{-\frac{k}{2}}) \int_0^Z e^{-\phi \phi^{-\frac{h-1}{2}}} \, d\phi + O(Y^{-J})
\]

(23)

Recall that \( p_{2h} \) is an even polynomial, so the sum above is in fact a polynomial in \( Y^{-1} \). Let \( c_j \) denote the coefficient of \( Y^{-j} \) in that polynomial. Then

\[
\int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta) \, d\Theta =
\]

\[
= \frac{\exp \left( \frac{k+1}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) X^{\frac{k}{2}} - \frac{1}{2} \right)}{(2\pi)^{\frac{k+2}{2}} X^{\frac{3}{2}} Y^{\frac{1}{2}}} \left( \frac{1}{2} \pi^2 + \sum_{j=1}^{J-1} c_j Y^{-j} + O(Y^{-J}) \right).
\]

(24)

If we replace \( J \) by \( J + 1 \), this is the expression given in (2). The remainder of the proof consists of showing that

\[
\int_{U \cap [3/(8\pi X), 3/(8\pi X)]} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta) \, d\Theta
\]

\[
\ll \frac{\exp \left( \frac{k+1}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) X^{\frac{k}{2}} - \frac{1}{2} \right)}{(2\pi)^{\frac{k+2}{2}} X^{\frac{3}{2}} Y^{\frac{1}{2}}} Y^{-J}.
\]

Suppose that \( \Theta \in \mathfrak{M}(1, 0) \setminus \left[ -3/(8\pi X), 3/(8\pi X) \right] \). Then \( \Delta \), given by (5), is less than \( \frac{4}{X} \).

Applying Lemma 3 with \( q = 1, a = 0 \), we see that

\[
\Re \Phi_k(\rho e(\Theta)) \leq \Gamma \left( \frac{k+1}{k} \right) \zeta \left( \frac{k+1}{k} \right)(X \Delta)^{\frac{1}{2}} + O(X^{\frac{1}{2}} |\theta|^{\frac{1}{2}} \log X)
\]

\[
\leq \frac{1}{k} \Gamma \left( \frac{1}{k} \right) \zeta \left( \frac{k+1}{k} \right) \left( \frac{4X}{5} \right)^{\frac{1}{2}} + O(X^{\frac{1}{2} + \epsilon}).
\]
Note that for Θ ∈ ℝ(1, 0) we have |Θ| = |Θ| ≤ X^{1/k}. Therefore,

\begin{align}
(25) \quad & \left| \int_{\mathbb{R}(1,0)[\{-3/(8\pi X),3/(8\pi X)\}]} \rho^{-n} \exp\left(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta\right) d\Theta \right| \\
& \leq \exp\left(\frac{k+1}{k} \frac{\zeta(\frac{k+1}{k}) \Gamma\left(\frac{1}{2}\right)}{2} X^{\frac{1}{2}} \frac{(2\pi)^{\frac{1}{2}}}{2} X^{\frac{1}{2}} Y^{\frac{1}{2}} \right)^{1/2}. 
\end{align}

We next study the integral on the remaining major arcs. Suppose that Θ ∈ ℝ(q, a) with q > 1. Then we have q ≤ X^{1/k} and θ = Θ - a/q satisfies |θ| ≤ q^{-1}X^{1/k-1}. Thus

\[ q^{1/2+\varepsilon} \log X \left(1 + X^{\frac{1}{k}}|\theta|^\frac{1}{2}\right) \ll X^{\frac{1}{k}+\varepsilon}, \]

and by Lemma 3 we have

\[ \Phi_k\left(e^{-1/X} e(\Theta)\right) = \Gamma\left(\frac{k+1}{k}\right) \left(\frac{X}{1 - 2\pi i X \Theta}\right)^{1/k} \sum_{j=1}^{\infty} \frac{S_k(q_j,a_j)}{j^{\frac{k+1}{k}} q_j} + O\left(X^{1/(2k)+\varepsilon}\right). \]

In order to obtain an estimate for the integral over the major arcs, we will first need a bound on the sum

\[ \sum_{j=1}^{\infty} \frac{|S_k(q_j,a_j)|}{j^{\frac{k+1}{k}} q_j}. \]

If q | j, then we have q = (q, j), i.e. q_j = 1 and S_k(q_j,a_j) = 1. On the other hand, if q \nmid j, then q_j > 1 and Lemma 1 tells us that there is a constant δ_k > 0 such that

\[ |S_k(q_j,a_j)| \leq q_j(1 - \delta_k). \]

Thus the sum satisfies

\[ \sum_{j=1}^{\infty} \frac{|S_k(q_j,a_j)|}{j^{\frac{k+1}{k}} q_j} = \sum_{j=1}^{\infty} \frac{|S_k(q_j,a_j)|}{j^{\frac{k+1}{k}} q_j} + \sum_{j=1}^{\infty} \frac{|S_k(q_j,a_j)|}{j^{\frac{k+1}{k}} q_j} \leq \sum_{j=1}^{\infty} \frac{1 - \delta_k}{j^{\frac{k+1}{k}}} + \sum_{j=1}^{\infty} \frac{1}{j^{\frac{k+1}{k}}} \\
= (1 - \delta_k) \left(1 - \frac{1}{q^{\frac{k}{k+1}}}\right) \zeta\left(\frac{k+1}{k}\right) + \frac{1}{q^{\frac{k}{k+1}}} \zeta\left(\frac{k+1}{k}\right) \\
= \left(1 - \delta_k + \frac{\delta_k}{q^{\frac{k}{k+1}}}\right) \zeta\left(\frac{k+1}{k}\right). \]

Here we used the readily verifiable fact that \( \sum_{q \in \mathbb{N} \setminus \mathbb{N}} q^{-\alpha} = q^{-\alpha} \zeta(\alpha) \). We now have

\[ \left| \mathbb{R} \left( \Gamma\left(\frac{k+1}{k}\right) \frac{X}{1 - 2\pi i X \Theta}^{1/k} \sum_{j=1}^{\infty} \frac{S_k(q_j,a_j)}{j^{\frac{k+1}{k}} q_j} \right) \right| \leq \frac{1}{k} \left(1 - \delta_k\right) \zeta\left(\frac{k+1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{1/k}. \]
Let \( \widetilde{\mathfrak{m}} = \mathfrak{m} \setminus \mathfrak{m}(1, 0) \). Then the above argument proves that there is a constant \( \delta \) with \( 0 < \delta < 1 \) such that

\[
\int_{\tilde{\mathfrak{m}}} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta) \, d\Theta \ll \exp \left( \frac{n}{X} + \frac{\delta}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) X^{1/k} \right) \exp \left( \frac{(k+1)}{k^2} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{1}{k} \right) X^{\frac{1}{k} - \frac{1}{2}} \right)^{Y-J}.
\]

(26)

Finally, we deal with the minor arcs. We will need one more estimate for \( \Phi_k(\rho e(\Theta)) \).

**Lemma 4.** Let \( \Theta \in \mathfrak{m} \). Then, with all definitions as above,

\[
\Phi_k(\rho e(\Theta)) \ll X^{\frac{1}{k} + \varepsilon - \frac{1}{k2^{k-1}}}. \]

**Proof** Let \( K \in \mathbb{Z} \) be a parameter at our disposal. As in the proof of Lemma 3, we have

(27) \[
\Phi_k(e^{-1/X} e(\Theta)) = \sum_{j=1}^{K} \frac{1}{j} \int_{1}^{\infty} jkx^{k-1} X^{-1} e^{-xk/j/X} \sum_{n \leq x} e(jn^k \Theta) \, dx + O \left( \left( \frac{X}{K} \right)^{1/k} \right).
\]

For each \( j \), we use Dirichlet’s Theorem to choose \( a_j \in \mathbb{Z}, q_j \in \mathbb{N} \), so that

\[
\left| j\Theta - \frac{a_j}{q_j} \right| \leq q_j^{-1} X^{\frac{1}{k} - 1}, \quad q_j \leq X^{1 - \frac{1}{k}}.
\]

We now use Weyl’s inequality to obtain

\[
\sum_{n \leq x} e(jn^k \Theta) \ll x^{1 + \varepsilon - 2^{-k(1-k)} - \frac{2}{k} + \varepsilon} q_j^{2^{-k(1-k)}} + x^{1 + \varepsilon} \left( \frac{q_j}{x^k} \right)^{2^{-k(1-k)}}.
\]

Note that for any \( \lambda > 0 \),

\[
\int_{1}^{\infty} x^{-\lambda} \left( jkx^{k-1} X^{-1} e^{-xk/j/X} \right) \, dx \ll \left( \frac{X}{j} \left( \frac{X}{j} \right)^{1/k} - \frac{1}{k2^{k-1}} \right)^{\lambda/k}.
\]

So, the main term of (27) is

\[
\ll \sum_{j=1}^{K} \frac{1}{j} \left( \left( \frac{X}{j} \right)^{1/k} q_j^{2^{-k(1-k)}} \right)
\]

Since \( q_j \leq X^{1 - \frac{1}{k}} \), the last term is

\[
\left( \frac{X}{j} \right)^{1/k} q_j^{2^{-k(1-k)}} \leq \left( \frac{X}{j} \right)^{1/k} \left( \frac{X}{j} \right)^{-\frac{1}{k2^{k-1}}}
\]

As \( \Theta \notin \mathfrak{m} \), we must have \( jq_j > X^{1/k} \). So we can replace the middle term as well:

\[
\left( \frac{X}{j} \right)^{1/k} q_j^{2^{-k(1-k)}} < \left( \frac{X}{j} \right)^{1/k} \left( \frac{X}{j} \right)^{1/k2^{k-1}}.
\]
Putting all of this together, we see that
\[
\Phi_k(e^{-1/X} e(\Theta)) \ll X^{1/k - \frac{1}{k^2 - 1}} \sum_{j=1}^{K} \left( \left( \frac{1}{j} \right)^{1 + \frac{1}{k} - \frac{1}{j^2 - 1}} + \left( \frac{1}{j} \right)^{1 + \frac{1}{k} - \frac{1}{k^2 - 1}} \right) + \left( \frac{X}{K} \right)^{1/k}.
\]

Letting \( K \to \infty \) gives the desired result. \( \square \)

For \( \Theta \in m \), we have by Lemma 4
\[
\Phi_k(\rho e(\Theta)) \ll X^{1/k - \frac{1}{k^2}}.
\]

Therefore
\[
\int_m \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta) d\Theta \ll \frac{\exp \left( \frac{1}{X} \zeta(\frac{k+1}{k}) \Gamma(\frac{1}{k}) X^{\frac{1}{k}} - \frac{1}{2} \right)}{(2\pi)^{1/2} X^{3/2} Y^{1/2}} Y^{-J}.
\]

Combining (25), (26), and (28) completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

We will use the notation from the previous section. Recall that
\[
p^k(n) = \int_U \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta) d\Theta.
\]

Hence we have
\[
p^k(n + 1) - p^k(n) = \int_U \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi in\Theta)(\rho^{-1} e^{-2\pi i \Theta} - 1) d\Theta.
\]

Since \( |\rho^{-1} e^{-2\pi i \Theta} - 1| \leq e^{1/X} + 1 \leq 4 \), the contribution from \( |\Theta| > 3/(8\pi X) \) is
\[
\ll \frac{\exp \left( \frac{1}{X} \zeta(\frac{k+1}{k}) \Gamma(\frac{1}{k}) X^{\frac{1}{k}} - \frac{1}{2} \right)}{(2\pi)^{1/2} X^{3/2} Y^{1/2}} X^{-2} Y^{-J}
\]

by the proof of Theorem 1. On the other hand, when \( |\Theta| \leq 3/(8\pi X) \), we have
\[
\rho^{-1} e^{-2\pi i \Theta} - 1 = \exp \left( \frac{1}{X} - 2\pi i \Theta \right) - 1 = \frac{1}{X} - 2\pi i \Theta + O(X^{-2}).
\]
We thus deduce that

\begin{equation}
    p^k(n + 1) - p^k(n) = -2\pi i \int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi i n\Theta) \Theta \, d\Theta
\end{equation}

\begin{equation}
    + \frac{\exp\left(\frac{1}{k} \zeta\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{\frac{1}{k}} - \frac{1}{2}\right)}{(2\pi)^{\frac{1}{k}} X^{\frac{1}{k}} Y^{\frac{1}{k}}} \left(\frac{1}{2} + \sum_{j=1}^{j-1} c_j Y^{-j} + O(Y^{-j})\right) (X^{-1} + O(X^{-2})).
\end{equation}

It remains to evaluate

\begin{equation}
    -2\pi i \int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi i n\Theta) \Theta \, d\Theta.
\end{equation}

The methods here are similar to those of Theorem 1, so we only outline the major differences here. The extra factor of \( \Theta \) in the integrand means that (14) becomes

\begin{equation}
    \int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Xi_k(\rho e(\Theta)) - 2\pi i n\Theta) \Theta \, d\Theta
\end{equation}

and the righthand side of (16) becomes

\begin{equation}
    \exp\left(\frac{1}{k} \zeta\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{\frac{1}{k}} - \frac{1}{2}\right) \left(\frac{1}{2} + \sum_{j=1}^{j-1} c_j Y^{-j} + O(Y^{-j})\right) (X^{-1} + O(X^{-2})).
\end{equation}

We rewrite the integral in (31) as

\begin{equation}
    \int_{0}^{3/(8\pi X)} \exp\left(G(\Theta)ight) - \exp\left(G(-\Theta)\right) \exp\left(-Y(2\pi X\Theta)^2 + G(\Theta)\right) \Theta \, d\Theta
\end{equation}

\begin{equation}
    = 2i\Im\int_{0}^{3/(8\pi X)} \exp\left(G(\Theta) - Y(2\pi X\Theta)^2\right) \Theta \, d\Theta
\end{equation}

\begin{equation}
    = \frac{i}{4\pi^2 X^2 Y} \Im\int_{0}^{9Y/16} \exp\left(H(\phi) - \phi\right) \, d\phi,
\end{equation}

using the change of variables \( \phi = (2\pi X\Theta)^2 Y \) with \( H \) defined by (18). Putting this into (30), we see that

\begin{equation}
    -2\pi i \int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(\rho e(\Theta)) - 2\pi i n\Theta) \Theta \, d\Theta
\end{equation}

\begin{equation}
    = \frac{\exp\left(\frac{1}{k} \zeta\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{k}\right) X^{\frac{1}{k}} - \frac{1}{2}\right)}{(2\pi)^{\frac{1}{2}} X^{\frac{1}{2}} Y^{\frac{1}{2}}} \left(\Im\int_{0}^{9Y/16} \exp\left(H(\phi) - \phi\right) \, d\phi + O(Y^{-J})\right).
\end{equation}
The computations from this point on are entirely similar to Theorem 1, and we end up with

\[ \Im \int_0^{9Y/16} \exp \left( H(\phi) - \phi \right) d\phi \]

\[ = \int_0^Z e^{-\phi} \sum_{j=0}^{4J+3} \Im \left( \frac{H_J(\phi)^j}{j!} \right) d\phi + O(Y^{-J}) \]

\[ = \sum_{h=1 \atop h \text{ odd}}^L \rho_h(Y^{-\frac{1}{2}}) \int_0^Z e^{-\phi} \frac{\rho_h}{\phi} d\phi + O(Y^{-J}) \]

\[ = \sum_{h=2}^{L/2} \Gamma \left( h + \frac{1}{2} \right) \rho_{2h-1}(Y^{-\frac{1}{2}}) + O(Y^{-J}) \]

(34)

Recall that \( p_{2h-1} \) is an odd polynomial in \( Y^{-1/2} \). For \( j \geq 1 \), let \( \tilde{c}_j \) denote the coefficient of \( (Y^{-1/2})^{2j-1} \) in the polynomial in (34). Then we have

\[ -2\pi i \int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi_k(p\epsilon(\Theta)) - 2\pi in\Theta) d\Theta \]

\[ = \frac{\exp \left( \frac{k+1}{k^2} \zeta(\frac{k+1}{k}) \Gamma(\frac{1}{k}) X^\frac{1}{k} - \frac{1}{2} \right) \left( \sum_{j=1}^{k-1} \tilde{c}_j Y^{-j} + O(Y^{-J-1}) \right)}{(2\pi)^{k+1} X^{\frac{1}{k} Y^{\frac{1}{k}}} \left( \pi^{\frac{1}{k}} + \sum_{j=1}^{k-1} d_j Y^{-j} + O(Y^{-k}) \right)} \]

Combining this with (29) and letting \( d_j = c_j + \tilde{c}_j \), we have

\[ p^k(n + 1) - p^k(n) = \exp \left( \frac{k+1}{k^2} \zeta(\frac{k+1}{k}) \Gamma(\frac{1}{k}) X^\frac{1}{k} - \frac{1}{2} \right) \left( \pi^{\frac{1}{k}} + \sum_{j=1}^{k-1} d_j Y^{-j} + O(Y^{-k}) \right) \]

The error term of \( Y^{-k} \) is due to the error of \( X^{-1} \) in (29).

5. Computing the Coefficients

In certain applications, it may be useful to know the values of the coefficients \( c_j \), which appear in (2). As seen in (23), these coefficients satisfy

(35)

\[ \sum_{j=1}^L c_j Y^{-j} = \sum_{h=2}^{L/2} \Gamma \left( h + \frac{1}{2} \right) p_{2h}(Y^{-1/2}), \]

where the polynomials \( p_{2h} \) are given by

(36)

\[ \sum_{h=0}^L p_{2h}(Y^{-1/2}) \phi^h = \Re \sum_{\ell=0}^{4J+3} \frac{H_J(\phi)^\ell}{\ell!} \]
with

\[ H_J(\phi) = \sum_{m=3}^{2J+2} i^m (a_m + b_m Y^{-1}) \phi^m Y^{1-m/2} \]

where

\[ a_m = \left( \frac{2k^2}{k+1} \right) \frac{\Gamma(m+\frac{1}{k})}{m! \Gamma(\frac{1}{k})}, \quad b_m = \frac{1}{2} \left( \frac{2k^2}{k+1} \right) \frac{\Gamma(m+\frac{1}{k})}{m! \Gamma(\frac{1}{k})} - \frac{1}{2m}. \]

Given a particular value of \( J \), one could input (37) and (36) into a computer algebra program, and expand it out into powers of \( \phi \) to obtain an expression for each of the polynomials \( p_h(z) \), which in turn could be put into (35) to obtain \( c_j \). Note that \( L = (4J+3)(2J+2) \) is the degree of \( \sum_{\ell=0}^{4J+3} \frac{H_J(\phi)^\ell}{\ell!} \) as a polynomial in \( \phi^{1/2} \). Because of the error term of \( Y^{-J-1} \) in (2), it is only useful to find \( c_j \) up to \( j = J \). However, it is necessary to compute all of the polynomials \( p_h, h = 0, \ldots, L \), in order to compute any of the coefficients \( c_j \). For \( J = 1 \), there are 29 polynomials to compute to obtain the first coefficient \( c_1 = -\frac{\sqrt{\pi}}{24k^2} (k^2 + \frac{5}{2}k + 1) \).

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