Thermal Yang–Mills theory in the Einstein universe

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Abstract
We study the stability of a non-Abelian chromomagnetic vacuum in Yang–Mills theory in Euclidean Einstein universe $S^1 \times S^3$. We assume that the gauge group is a simple compact group $G$ containing the group $SU(2)$ as a subgroup and consider static covariantly constant gauge fields on $S^3$ taking values in the adjoint representation of the group $G$ and forming a representation of the group $SU(2)$. We compute the heat kernel for the Laplacian acting on fields on $S^3$ in an arbitrary representation of $SU(2)$ and use this result to compute the heat kernels for the gluon and the ghost operators and the one-loop effective action. We show that the only configuration of the covariantly constant Yang–Mills background that is stable is the one that contains only spinor (fundamental) representations of the group $SU(2)$; all other configurations contain negative modes and are unstable. For the stable configuration we compute the asymptotics of the effective action, the energy density, the entropy and the heat capacity in the limits of low/high temperature and small/large volume and show that the energy density has a non-trivial minimum at a finite value of the radius of the sphere $S^3$.

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1. Introduction
A deep understanding of the physics of quantum Yang–Mills gauge field theory at low energies is lacking due to the failure of perturbation theory. At low energies the interaction becomes strong which leads to the phenomenologically observed property of confinement in quantum chromodynamics. However, the precise nature of the physical mechanism responsible for confinement is still not known. From the field-theoretical point of view this means that the vacuum of Yang–Mills theory at low energies has a far more complicated structure than the perturbative one. One of the models of a non-perturbative Yang–Mills vacuum is the chromomagnetic vacuum that has been proposed by Savvidy [30]. Savvidy considered a
constant $SU(2)$ gauge field (which was necessarily Abelian) in four-dimensional flat spacetime and showed that the trivial zero-field perturbative vacuum is unstable under creation of a constant chromomagnetic field. Further, Nielsen and Olesen [27] showed that the Savvidy vacuum is itself unstable, meaning that the physical non-perturbative vacuum has an even more complicated structure. It has been suggested [28] that the real vacuum is likely to have a small domain structure with random constant chromomagnetic fields.

One way to stabilize the chromomagnetic vacuum is to increase the dimension of the spacetime. In our papers [5, 6] we studied the chromomagnetic vacuum (for general compact gauge groups) in higher dimensions in flat spacetime and showed the existence of stable chromomagnetic configurations in dimensions greater than 4. Another way to stabilize the vacuum is to consider curved manifolds. It is easy to see that a positive space curvature can provide an effective mass term for the gauge fields on the chromomagnetic vacuum, thus, making the vacuum stable. Of course, on curved manifolds the notion of constant chromomagnetic gauge fields has to be replaced with covariantly constant fields.

In our recent paper [11] we considered the finite temperature Yang–Mills theory on $S^1 \times S^1 \times S^2$ with an Abelian covariantly constant background on $S^2$. We showed that despite the positive spatial curvature the gluon operator still has negative modes for any compact semi-simple gauge group, which means that the vacuum with covariantly constant chromomagnetic fields on $S^2$ does not represent the true vacuum of Yang–Mills theory. This happens because on spheres, and, more generally, on symmetric spaces, covariantly constant gauge fields cannot have an arbitrary value of the chromomagnetic field independent of the spatial curvature; there are severe algebraic constraints that force the strength of the chromomagnetic field to be of the same order as the spatial curvature. This means that the stabilizing role of the spatial curvature is depressed by the destabilizing role of the chromomagnetic field, even on curved manifolds with constant positive curvature.

This paper is a continuation of the study of Yang–Mills theory on spheres; our primary goal here is to extend this analysis to the Einstein universe $S^1 \times S^1 \times S^3$ with a non-Abelian covariantly constant background on $S^3$ with a compact simple gauge group $G$ that has the group $SU(2)$ as a subgroup. We refer the reader to [11] for general introduction and notation.

This paper is organized as follows. In section 2 we briefly describe our model and fix notation. In section 3 we derive the one-loop effective action in terms of the heat kernels of relevant operators. In section 4 we compute the heat kernel on $S^1$. In section 5 we describe in detail the geometry of $S^3$ and compute the scalar heat kernel on the group $SU(2)$, which is used later for the calculation of the heat kernel for arbitrary fields on $S^3$ in section 6. In section 7 we use these results to compute the trace of the Yang–Mills heat kernel (which is the difference between the gluon heat kernel and the ghost one) and the effective action.

The main results of this paper are the calculation of the heat kernel for arbitrary fields on $S^3$ and the proof that the minimal eigenvalue of the gluon operator is positive only in a very specific case when the representation of the gauge fields does not contain any higher spin, $j \geq 1$, representations of the group $SU(2)$ but contains only the spinor (fundamental) representation, $j = 1/2$, of $SU(2)$. In all other cases, that is, when the representation of the gauge fields contains at least one representation of $SU(2)$ with $j \geq 1$, the minimal eigenvalue is negative and the heat kernel grows exponentially at infinity leading to the infrared instability of the chromomagnetic vacuum. Of course, the gauge fields are taken in the adjoint representation of the gauge group $G$. The question whether the adjoint representation of a compact simple group $G$, that has the group $SU(2)$ as a subgroup, can contain the spinor representation of $SU(2)$, is a representation-theoretic problem that we do not solve in this paper. If this is impossible then our results indicate the instability of any chromomagnetic background in Yang–Mills theory in the Einstein universe.
We assume that this is possible, that is, the adjoint representation of the gauge group $G$ may contain the spinor representation of $SU(2)$, and compute the heat kernel and the effective action for this specific case. Of course, it does not make any sense to study thermodynamics of an unstable theory. Therefore, in section 8 we study thermodynamics of the model for this specific case; we compute the entropy, the heat capacity and the pressure.

The quantum field theory on homogeneous spaces has an extensive bibliography. We list only some papers that had an influence on our own approach. First of all, we would like to mention the early important papers by Dowker [18–22] as well his papers with co-workers [23, 1]. A very good source is the excellent review by Camporesi [12] and the references therein as well as [29, 2, 13–15, 24]. A thermal Yang–Mills theory in Einstein universe was also studied in [31]. More recent papers [25, 16, 26] have some overlap with our work since they also studied the heat kernel on $S^3$.

We would like to stress from the very beginning the differences from these papers. Our approach is completely different from the papers mentioned above; it is based on a fundamental integral representation of the heat semi-group on homogeneous bundles as an integral over the isotropy group (see our previous work [4, 7, 8]; for application of this approach to quantum gravity see [9]). Also, while most of the authors deal with the heat kernel for irreducible (that is, traceless divergence free) tensor fields, we compute it for arbitrary fields.

2. Yang–Mills theory

We consider the manifold $M = S^1 \times S^3$ equipped with the standard product metric $g_{\mu \nu}$, and a compact simple gauge group $G$. We denote the tensor components with respect to the coordinate basis by Greek indices. Let $x^\alpha$ be some local coordinates, $A = A_\mu dx^\mu$ be the Yang–Mills connection one-form taking values in the adjoint representation of the Lie algebra of the group $G$ and $F = -\frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu = dA + A \wedge A$ be its curvature two-form. Then the classical action of the pure Yang–Mills theory is

$$S = \int_M dvol \, \frac{1}{2e^2} |F|^2,$$

(2.1)

where $dvol = dx g^{1/2}$ is the Riemannian volume element with $g = det g_{\mu \nu}$, $e$ is the Yang–Mills coupling constant and $|F|^2 = -\frac{1}{4} tr g^{\mu \alpha} g^{\nu \beta} F_{\mu \alpha} F_{\nu \beta}$.

We choose a local orthonormal frame on the tangent bundle $TM$ and denote the tensor components with respect to the orthonormal frame by lower case Latin indices from the beginning of the alphabet. The components of tensors on $S^3$ are denoted by Latin indices from the middle of the alphabet. We introduce the projection tensor $h_{ab}$ to the tangent space of $S^3$ and the Lévi-Civitá symbol $\epsilon_{abc}$ on $S^3$.

We assume that the background Yang–Mills curvature is parallel $\nabla_a F_{bc} = 0$. Such background on $S^1 \times S^3$ has to satisfy the constraint

$$[F_{ab}, F_{cd}] = -\frac{4}{a^2} h^{[a} [c, F^{b]} d],$$

(2.2)

where $a$ is the radius of $S^3$. To satisfy this constraint we assume that the gauge group $G$ has the group $SU(2)$ as a subgroup. Then there exist generators $X_i$ taking values in the adjoint representation of the Lie algebra of the group $G$ satisfying the algebra $su(2)$,

$$[X_i, X_j] = -\epsilon^{k}_{ij} X_k,$$

(2.3)

that is, $X_i$ form a representation $X$ of the algebra $su(2)$. Now, the above constraint is satisfied by

$$F_{ab} = \frac{1}{a^2} \epsilon^{c}_{ab} X_c.$$  

(2.4)
This means that the magnitude of a covariantly constant gauge field cannot be assigned arbitrarily but it is determined by the radius of the sphere \( S^3 \), which also means that the spatial curvature and the chromomagnetic field are of the same order, \( 1/a^2 \).

In general, this representation is reducible. It decomposes into a sum of irreducible representations \( X_i = \text{diag}\{Y_j^1 \oplus \cdots \oplus Y_j^{n_i}\} \), where \( j_i \) are the standard non-negative integer or half-integer labels of the representations of \( su(2) \). The Casimir operator \( X^2 = X_i X^i \) is a diagonal matrix,

\[
X^2 = \text{diag}\{-j_1(j_1+1)I_{2j_1+1} \oplus \cdots \oplus -j_N(j_N+1)I_{2j_N+1}\},
\]

where \( I_{2j+1} \) is the identity matrix of dimension \((2j+1)\). The labels \( j_i \) are uniquely determined from the adjoint representation of the group \( G \). It is an interesting representation-theoretic problem to determine all labels \( j_i \) given a compact simple group \( G \). In particular, we will be interested in the question whether any of the labels \( j_i \) can be equal to \( 1/2 \), that is, whether the adjoint representation of the group \( G \) contains the spinor representation of the group \( SU(2) \).

We will show below in section 7 that the stability of the model depends on the answer to this question. We do not solve this question in this paper. We simply carry out the calculations for arbitrary values of the labels \( j_i \).

We will consider spin-tensor fields taking values in the Lie algebra of the gauge group \( G \). We consider a spin-tensor representation \( \Sigma \) of the spin group \( \text{Spin}(4) \) with generators \( \Sigma_{ab} \) satisfying the algebra

\[
[\Sigma^{ab}, \Sigma_{cd}] = -4\delta^{[a}_{c} \Sigma^{b]d]},
\]

Recall that \( \text{Spin}(4) = SU(2) \times SU(2) \). That is why the generators

\[
T_i = \frac{1}{2} \epsilon_{ijk} \Sigma^{jk}
\]

form a representation \( T \) of the subalgebra \( su(2) \),

\[
[T_i, T_j] = -\epsilon^{k}_{ij} T_k.
\]

Then the matrices

\[
G_i = T_i \otimes I_X + I_T \otimes X_i
\]

form the twisted representation \( T \otimes X \) of the algebra \( su(2) \) and satisfy the same algebra

\[
[G_i, G_j] = -\epsilon^{k}_{ij} G_k.
\]

The covariant derivative acting on spin-tensor fields taking values in the Lie algebra of the group \( G \) is defined as follows. Let \( T \) be a general spin-tensor representation of the group \( \text{SO}(4) \), \( X \) be a representation of the gauge group \( G \) and \( \omega^{\mu\nu} \) be the spin connection corresponding to the chosen orthonormal frame. Then the covariant derivative is defined by

\[
\nabla^{T \otimes X}_\mu = \partial_\mu + \frac{1}{2} \omega^{ab}_\mu T (\Sigma_{ab}) + X (A_\mu).
\]

3. Effective action

There exists a gauge such that the one-loop effective action of the Yang–Mills theory is given by \([5, 6]\)

\[
\Gamma_{(1)} = \frac{1}{2} \left( \log \text{Det} L_1 - 2 \log \text{Det} L_0 \right),
\]

where \( \text{Det} \) is the functional determinant, the operators \( L_0 \) and \( L_1 \) have the form

\[
L_0 = -\Delta_X, \quad L_1 = -\Delta_{T \otimes X} + Q,
\]
The most important observation that should be made at this point is that the positive curvature acts as a mass (or positive potential) term in the Yang–Mills operator $L$. While the magnetic field reduces the eigenvalues of the Yang–Mills operator the positive Ricci tensor increases them. Roughly speaking, it is the balance of these two terms that determines whether or not the Yang–Mills operator is positive. However, as we have seen above, the relative value of these two terms is not arbitrary; one cannot increase the spatial curvature without increasing the chromomagnetic field at the same time. It is precisely this feature that leads to the instability of the model, as shown below.

In order to study the infrared behavior of the system, one has to introduce an infrared regularization as in [6]. That is why we introduce a sufficiently large mass parameter $z$ so that all operators are positive, which is equivalent to replacing the operators $L$ by $L + z^2$. The determinants of positive elliptic operators can be regularized as follows [3, 10, 17]. We denote the heat kernel diagonal of the operator $L$ by $U_L(t)$ and its fiber trace $\Theta_L(t) = tr U_L(t)$. We introduce an arbitrary mass parameter $\lambda$ and define the coefficients $A_k^L(\lambda)$ by the expansion as $t \to 0$

$$\Theta_L(t) \sim (4\pi t)^{-\frac{d}{2}} e^{-\frac{\lambda^2}{2}} \sum_{k=0}^{\infty} A_k^L(\lambda) t^k.$$  

Then one can define the renormalized determinant by

$$\log \det_{\text{ren}}(L + z^2) = - \text{vol}(M) \int_0^\infty dt \frac{e^{-\lambda^2}}{t} \Theta^\text{ren}_L(t),$$

where

$$\Theta^\text{ren}_L(t) = \Theta_L(t) - (4\pi t)^{-\frac{d}{2}} e^{-\frac{\lambda^2}{2}} \left[ A_k^L(\lambda) + A_0^L(\lambda) t + A_1^L(\lambda) t^2 \right].$$

Next, let

$$\Theta_{YM}(t) = \Theta_L(t) - 2 \Theta_{L_0},$$

$$A_k^{YM}(\lambda) = A_k^L(\lambda) - 2 A_k^{L_0}(\lambda).$$

Then the effective action has the form [11]

$$\Gamma(t) = - \frac{1}{2} \text{vol}(M) \left\{ \beta \log \frac{\lambda^2}{t^2} + \int_0^\infty dt \frac{e^{-\lambda^2}}{t} \Theta^\text{ren}_{YM}(t) \right\},$$

where $\beta = A_2^{YM}(0) - \zeta A_1^{YM}(0) + \frac{3}{4} A_0^{YM}(0)$,

$$\Theta^\text{ren}_{YM}(t) = \Theta_{YM}(t) - (4\pi t)^{-\frac{d}{2}} e^{-\frac{\lambda^2}{2}} \left[ A_0^{YM}(\lambda) + A_1^{YM}(\lambda) t + A_2^{YM}(\lambda) t^2 \right].$$

Finally, there are two simplifications we can make. First, the heat kernel for a Laplace-type operator $L = -\Delta + Q$ with a covariantly constant potential $Q$ factorizes

$$\exp(-tL) = \exp(-tQ) \exp(t\Delta).$$

Second, for the product manifold $M = S^1 \times S^1$ the heat kernel for the Laplacian factorizes accordingly,

$$\exp(t\Delta) = \exp(t\partial_{S^1}^2) \exp(t\Delta_{S^1}).$$
\[ UL(t) = US^1(t) \exp(-tQ)US^3(t), \]  
\[ (3.13) \]
where \( US^1(t) \) and \( US^3(t) \) are the heat kernel diagonals for the Laplacians on \( S^1 \) and \( S^3 \). Thus, we finally obtain
\[ \Theta_{YM}(t) = US^1(t) \text{tr} \left[ \text{tr}_1 \exp(-tQ)US^3_{T_1 \otimes X}(t) - 2US^3_X(t) \right], \]  
\[ (3.14) \]
where \( US^3_{T_1 \otimes X}(t) \) and \( US^3_X(t) \) are the heat kernel diagonals of the Laplacians acting on vectors and scalars on \( S^3 \) in the representation \( X \).

4. Heat kernel on \( S^1 \)

We denote the radius of the circle \( S^1 \) by \( a_1 \). There are two dual representations of the heat kernel diagonal: the spectral one
\[ US^1(t) = \frac{1}{2\pi a_1} \sum_{n=-\infty}^{\infty} \exp \left( -\frac{t}{a_1^2} n^2 \right), \]  
\[ (4.1) \]
and the geometric one
\[ US^1(t) = (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp \left( -\frac{a_1^2 \pi^2}{t} n^2 \right). \]  
\[ (4.2) \]
That is why we will write it in the form
\[ US^1(t) = (4\pi t)^{-1/2} \Omega \left( \frac{t}{a_1^2} \right), \]  
\[ (4.3) \]
where the function \( \Omega(t) \) is defined by
\[ \Omega(t) = \sum_{n=-\infty}^{\infty} \exp \left( -\frac{n^2 \pi^2}{t} \right) = \theta_3(0, e^{-\pi^2/t}) \]  
\[ (4.4) \]
and \( \theta_3(v, q) \) is the third Jacobi theta function. This function satisfies the Poisson duality formula
\[ \Omega(t) = \sqrt{\frac{t}{\pi}} \Omega \left( \frac{\pi^2}{t} \right), \]  
\[ (4.5) \]
and has the following asymptotics: as \( t \to 0 \)
\[ \Omega(t) = 1 + 2e^{-\pi^2/t} + O(e^{-4\pi^2/t}) \]  
\[ (4.6) \]
and as \( t \to \infty \)
\[ \Omega(t) = \sqrt{\frac{t}{\pi}} \left[ 1 + 2e^{-t} + O(e^{-4t}) \right]. \]  
\[ (4.7) \]

5. Geometry of \( S^3 \)

We consider the sphere \( S^3 \) of radius \( a \). However, to simplify notation we set the radius \( a = 1 \). It can be easily reintroduced later from dimensional arguments. Let \( x' \) be the normal geodesic coordinates on \( S^3 \) with the origin at the North pole and ranging over \((-\pi, \pi)\). In this section all indices denote tensor components on \( S^3 \). The position of the indices on the coordinates will be irrelevant, that is, \( x_i = x' \). We introduce the radial coordinate \( r = |x| = \sqrt{x_i x^i} \) and
the angular coordinates (the coordinates on $S^2$) $\theta^i = \xi^i / r$ (so that $\theta^i \theta^i = 1$). The round metric on $S^3$ in geodesic coordinates is
\[
dx^2 = dr^2 + \sin^2 r \, d\theta^i d\theta_i.
\]
That is, the metric tensor is
\[
g_{ij} = \delta_{ij} - \theta_i \theta_j,
\]
where
\[
\Pi_{ij} = \delta_{ij} - \theta_i \theta_j.
\]
The sphere $S^3$ can be viewed as the homogeneous space $S^3 = SO(4)/SO(3)$ or $S^3 = [SU(2) \times SU(2)]/SU(2) = SU(2)$. The geodesic coordinates on $S^3$ are exactly the canonical coordinates on the group $SU(2)$. Let $F(q, p)$ be the group multiplication map in canonical coordinates. This map has a number of important properties. In particular, $F(0, p) = F(p, 0) = p$ and $F(p, -p) = 0$. Another obvious but very useful property of the group map is that if $q = F(\omega, p)$, then $\omega = F(q, -p)$ and $p = F(-\omega, q)$. Also, there is the associativity property $F(\omega, F(p, q)) = F(F(\omega, p), q)$ and the inverse property $F(-p, -\omega) = -F(\omega, p)$.

Let $C_i$ be the matrices of the adjoint representation of $su(2)$ defined by
\[
(C_i)^k_l = -2e^{i k}_l;
\]
they satisfy the algebra
\[
[C_i, C_j] = -2e^{k}_j C_k.
\]
Note that the Casimir operator in this normalization is $C_a C_a = -8 I$. Let $C = C(x)$ be the matrix defined by
\[
C = C_{i x}^i,
\]
and let $R, Y$ and $D$ be the matrices defined by
\[
Y = \frac{I - \exp(-C)}{C}, \quad R = \frac{C}{I - \exp(-C)}, \quad D = \exp C.
\]
Obviously, $R Y = Y R = I$ and $D = Y^T R$.

To compute these matrices explicitly we note that
\[
C^2 = -4r^2 \Pi,
\]
and, therefore,
\[
C^{2n} = (-1)^n (2r)^{2n} \Pi, \quad C^{2n+1} = (-1)^n (2r)^{2n} C.
\]
Therefore, the eigenvalues of the matrix $C$ are $(2i r, -2i r, 0)$, and for any analytic function
\[
f(C) = f(0)(I - \Pi) + \Pi \frac{1}{2} \left[ f(2i r) + f(-2i r) \right] + \frac{1}{4i r} C \left[ f(2i r) - f(-2i r) \right],
\]
and, therefore,
\[
\text{tr} f(C) = f(0) + f(2i r) + f(-2i r),
\]
\[
\text{det} f(C) = f(0) f(2i r) f(-2i r).
\]
This enables one to compute
\[
Y = I - \Pi + \frac{\sin r}{r} \cos r \Pi - \frac{1}{2} \frac{\sin^2 r}{r^2} C,
\]
\[
R = I - \Pi + r \cot r \Pi + \frac{1}{2} C.
\]
\[ D = I - \Pi + \Pi \cos(2r) + \frac{\sin(2r)}{2r} C. \] (5.15)

These matrices satisfy the identities
\[ g_{ij} = \delta_{ab} Y^a_i Y^b_j, \quad g^{ij} = \delta^{ab} R^a_i R^b_j. \] (5.16)

Therefore, both sets of one-forms
\[ \sigma^a_+ = Y^a_i(x) \, dx^i, \quad \sigma^a_- = Y^a_i(x) \, dx^i \] (5.17)

form orthonormal bases on the cotangent bundle of \( S^3 \) and the standard metric on \( S^3 \) is bi-invariant. These are the left-invariant and the right-invariant one-forms satisfying the identities
\[ d\sigma^a_\pm = \pm \varepsilon_{abc} \sigma^b_\pm \wedge \sigma^c_\pm. \] (5.18)

To avoid any confusion we will always use the right-invariant orthonormal basis \( \sigma^a_+ \).

The right-invariant and the left-invariant vector fields are defined by
\[ K^+_a = R^+_a(x) \frac{\partial}{\partial x^i}, \quad K^-_a = R^-_a(x) \frac{\partial}{\partial x^i}. \] (5.19)

They satisfy the algebra
\[ [K^+_a, K^+_b] = -2 \varepsilon_{abc} K^+_c, \] (5.20)
\[ [K^-_a, K^-_b] = 2 \varepsilon_{abc} K^-_c, \] (5.21)
\[ [K^+_a, K^-_b] = 0. \] (5.22)

and form two mutually commuting representations of the group \( SU(2) \). The Casimir operators of these representations are equal to the scalar Laplacian
\[ \Delta_0 = \delta^{ab} K^+_a K^-_b = \delta^{ab} K^+_a K^+_b. \] (5.23)

The left-invariant and the right-invariant vector fields are the Killing vectors of the metric generating the whole isometry group of the sphere \( S^3 \). They provide the orthonormal bases for the tangent bundle. We will always use the right-invariant orthonormal basis \( K^+_a \) dual to \( \sigma^a_+ \).

Note that the geodesic distance between the points \( p \) and \( q \) is just equal to \( |F(p, q)| \).

The Riemannian volume elements of the metric \( g \) are defined as usual
\[ d\text{vol}(x) = g^{1/2}(x) \, dx^1 \wedge dx^2 \wedge dx^3 = \sin^2 r \, dr \, d\theta, \] (5.24)
where \( d\theta \) is the volume element on \( S^2 \). The invariance of the volume element means that for any fixed \( p \)
\[ d\text{vol}(x) = d\text{vol}(-x) = d\text{vol}(F(x, p)) = d\text{vol}(F(p, x)). \] (5.25)

We denote the covariant derivative with respect to \( K^+_a \) simply by \( \nabla_a \). The Lévi-Civitá connection of the bi-invariant metric in the right-invariant basis is defined by
\[ \nabla_a K^+_b = \varepsilon^e_{bc} K^+_e, \] (5.26)
so that the coefficients of the affine connection are
\[ \omega^{ab}_c = \sigma^a_c (\nabla_b K^+_c) = \varepsilon^{ab}_c. \] (5.27)

Then
\[ \nabla_a K^-_b = -\varepsilon^b_{ca} D^c_d K^+_d. \] (5.28)

Now, the Riemann curvature tensor is
\[ R^{ab}_{bcd} = -\varepsilon^{b}_{ef} \varepsilon^{a}_{fb}, \] (5.29)
and the Ricci curvature tensor and the scalar curvature are

\[ R_{ab} = 2\delta_{ab}, \quad R = 6. \] (5.30)

Let \( \nabla \) be the total connection on a vector bundle \( V \) realizing the representation \( G \) of the group \( SU(2) \). Then the Yang–Mills connection on this bundle and its curvature are

\[ A = \sigma^{a}_{\,a} G_{a}, \quad \mathcal{F} = \frac{1}{2} F_{ab} \sigma^{a}_{\,+} \wedge \sigma^{b}_{\,+} = \frac{1}{2} \varepsilon_{ab} G_{a} \sigma^{a}_{\,+} \wedge \sigma^{b}_{\,+}. \] (5.31)

The covariant derivative of a section of the bundle \( V \) is then

\[ \nabla_{a} \phi = (K_{a}^{+} + G_{a}) \phi, \] (5.32)

and the Laplacian takes the form

\[ \Delta = \nabla_{a} \nabla^{a} = (K_{a}^{+} + G_{a}) (K_{a}^{+} + G_{a}) = \Delta_{0} + 2G^{2} K_{a}^{+} + G^{2}, \] (5.33)

where \( G^{2} = G_{a} G^{a} \). Also, the derivatives along the left-invariant vector fields are

\[ \nabla_{K_{a}} \phi = (K_{a}^{-} + B_{a}) \phi, \] (5.34)

where \( B_{a} = D_{a}^{b} G_{b} \).

We want to rewrite the Laplacian in terms of Casimir operators of some representations of the group \( SU(2) \). The covariant derivatives \( \nabla_{a} \) do not form a representation of the algebra \( SU(2) \). The operators that do are the covariant Lie derivatives. The covariant Lie derivatives along a Killing vector \( \xi \) of sections of this vector bundle are defined by

\[ \mathcal{L}_{\xi} = \nabla_{\xi} - \frac{1}{2} \sigma^{a}_{\,+} (\nabla_{\xi} \sigma^{a}_{\,+}) G_{a}. \] (5.35)

By denoting the Lie derivatives along the Killing vectors \( K_{a}^{\pm} \) by \( K_{a}^{\pm} \), this gives for the right-invariant and the left-invariant bases

\[ K_{a}^{+} = \mathcal{L}_{K_{a}^{+}} = \nabla^{a} + G_{a} = K_{a}^{+} + 2G_{a}, \] (5.36)

\[ K_{a}^{-} = \mathcal{L}_{K_{a}^{-}} = \nabla_{K_{a}^{-}} - B_{a} = K_{a}^{-}. \] (5.37)

It is easy to see that these operators form a representation of the isometry algebra \( su(2) \times su(2) \),

\[ [K_{a}^{+}, K_{b}^{+}] = - 2\varepsilon_{ab} K_{c}^{+}, \] (5.38)

\[ [K_{a}^{-}, K_{b}^{-}] = 2\varepsilon_{ab} K_{c}^{-}, \] (5.39)

\[ [K_{a}^{+}, K_{b}^{-}] = 0. \] (5.40)

The Laplacian is now given by the sum of the Casimir operators

\[ \Delta = \frac{1}{2} K_{a}^{2} + \frac{1}{2} K_{c}^{2} - G^{2}, \] (5.41)

where \( K_{c}^{2} \) is the sum of the Casimir operators.

We need to compute the action of isometries on \( SU(2) \). Let \( T_{a} \) be the generators of some representation of the group \( SU(2) \) satisfying the algebra \( su(2) \),

\[ [T_{a}, T_{b}] = - 2\varepsilon_{ab} T_{c}, \] (5.42)

and \( T(x) = T_{a} x^{a} \). First, one can derive a useful commutation formula

\[ \exp{T(x) T_{b}} \exp[-T(x)] = D^{c}_{b}(x) T_{c}. \] (5.43)

Next, one can show that

\[ K_{a}^{+} \exp{T(x)} = \exp{T(x)} T_{a}, \] (5.44)

\[ K_{a}^{-} \exp{T(x)} = T_{a} \exp{T(x)}. \] (5.45)
Therefore,
\[ \Delta_0 \exp[T(x)] = \exp[T(x)]T^2 = T^2 \exp[T(x)]. \] (5.46)

In particular, for the adjoint representation these formulas take the form
\[ DC_b = D'\alpha C_\alpha D, \quad K^+ \alpha D = DC_\alpha, \quad K^- \alpha D = C_\alpha D, \] (5.47)
\[ \Delta_0 D = -8D. \] (5.48)

This immediately leads to further important equations
\[ \exp[K^+(p)] \exp[T(x)] = \exp[T(x)] \exp[T(p)] = \exp[T(F(x, p))], \] (5.49)
\[ \exp[K^-(q)] \exp[T(x)] = \exp[T(q)] \exp[T(x)] = \exp[T(F(q, x))], \] (5.50)
where \( K^\pm(p) = p^\pm K^\pm \). More generally,
\[ \exp[K^+(p) + K^-(q)] \exp[T(x)] = \exp[T(q)] \exp[T(x)] \exp[T(p)] \]
\[ = \exp[T(F(q, F(x, p)))] \]. (5.51)

Then for a scalar function \( f(\omega) \) the action of the right-invariant and left-invariant vector fields is simply
\[ \exp[K^+(p)] f(x) = f(F(x, p)), \quad \exp[K^-(q)] f(x) = f(F(q, x)), \] (5.52)
more generally,
\[ \exp[K^+(p) + K^-(q)] f(x) = f(F(q, F(x, p))). \] (5.53)

We introduce the average of the group elements over \( S^2 \) by
\[ \Lambda_T(r) = \int_{S^2} \frac{d\theta}{4\pi} \exp[rT(\theta)]. \] (5.54)

One can show that this is a group invariant. Therefore, it can be only a function of the Casimir operator \( T^2 \). It is determined by the characters of the irreducible representations. For an irreducible representation \( j \), this is equal to
\[ \Lambda_j(r) = \frac{1}{2j + 1} \sum_{|\mu| < j} \cos(2\mu r)I_j, \] (5.55)
where the sum goes over integer \( \mu \) for integer \( j \) and over half-integer \( \mu \) for half-integer \( j \). Note that the function \( \Lambda_j(r) \) is periodic with period \( \pi \) for integer \( j \) and antiperiodic for half-integer \( j \), that is, for all integer \( j \):
\[ \Lambda_j(t, r - \pi n) = \Lambda_j(t, r), \] (5.56)
and for all half-integer \( j \):
\[ \Lambda_j(t, r - \pi n) = (-1)^n \Lambda_j(t, r). \] (5.57)

We can combine these formulas by writing
\[ \Lambda_j(t, r - \pi n) = (-1)^{2\mu} \Lambda_j(t, r). \] (5.58)

For a general reducible representation it is the direct sum of the irreducible ones,
\[ \Lambda_T(r) = \text{diag}\{\Lambda_{j_1} \oplus \cdots \oplus \Lambda_{j_n}\}. \] (5.59)

Next, we show that for any representation \( T \) of \( SU(2) \) we have the identity
\[ \exp(iT^2) = (4\pi t)^{-3/2} e^{\int_0^\infty dr \int_{S^2} d\theta \ r \sin r \exp\left(-\frac{r^2}{4t}\right) \exp[rT(\theta)]}, \] (5.60)
which can also be written as an integral over $\mathbb{R}^3$,
\[
\exp(tT^2) = (4\pi t)^{-3/2} e^{\frac{t}{24}} \int_{\mathbb{R}^3} dx \ r \sin r \exp\left(\frac{-r^2}{4t}\right) \exp[T(x)].
\] (5.61)
where, as usual, $r = |x|$. First, we compute the following Gaussian integral:
\[
(4\pi t)^{-3/2} e^{\frac{t}{24}} \int_0^\infty dr \ r \sin r \exp\left(\frac{-r^2}{4t}\right) 4\pi \cos(2\mu r)
= \left(\frac{1}{2} + \mu\right) e^{-4\mu(\mu+1)t} + \left(\frac{1}{2} - \mu\right) e^{-4\mu(\mu-1)t}.
\] (5.62)
Therefore, for any irreducible representation $j$ of $SU(2)$ we have
\[
(4\pi t)^{-3/2} e^{\frac{t}{24}} \int_0^\infty dr \ r \sin r \exp\left(\frac{-r^2}{4t}\right) 4\pi \Lambda_j(r)
= I_j \left\{ \frac{1}{2j + 1} \sum_{|\mu| \leq j} \left(\left(\frac{1}{2} + \mu\right) e^{-4\mu(\mu+1)t} + \left(\frac{1}{2} + \mu\right) e^{-4\mu(\mu-1)t}\right) \right\}
= \exp[-4j(j+1)t]I_j.
\] (5.63)
This is nothing but equation (5.60) for an irreducible representation. The general case follows from this trivially.

We note that the above identity (5.60) can also be written as an integral over the group $SU(2)$.
\[
\exp(tT^2) = \int_{SU(2)} d\text{vol}(x) U_0(t, x) \exp[T(x)],
\] (5.64)
where
\[
U_0(t, x) = \sum_{n=-\infty}^{\infty} (4\pi t)^{-3/2} e^{\frac{t}{24}} \frac{2\pi n}{\sin r} \exp\left[-\frac{(r + 2\pi n)^2}{4t}\right].
\] (5.65)
One can show that $U_0(t, x)$ is nothing but the scalar heat kernel on the group $SU(2)$. First of all, it satisfies the initial condition
\[
U_0(0, x) = \delta_S(x).
\] (5.66)
Second, it satisfies the heat equation. This can be shown either by a direct computation or, more elegantly, as follows. We have
\[
0 = (\partial_t - T^2) \exp(tT^2)
= \int_{SU(2)} d\text{vol}(x) (\partial_t - T^2) U_0(t, x) \exp[T(x)].
\] (5.67)
Now, by using the fact that $T^2 \exp[T(x)] = \Delta_0 \exp[T(x)]$ and by integrating by parts we obtain
\[
\int_{SU(2)} d\text{vol}(x) \exp[T(x)] (\partial_t - \Delta_0) U_0(t, x) = 0,
\] (5.68)
which gives
\[
\partial_t U_0 = \Delta_0 U_0.
\] (5.69)
6. Heat kernel on $S^3$

Our goal is to evaluate the heat kernel diagonal. Since it is constant we can evaluate it at any point, say, at the origin. That is why we will evaluate the heat kernel when one point is fixed at the origin, which we will denote simply by $U(t; x)$. By using equation (5.41) we obviously have

$$\exp(t\Delta) = \exp(-tG^2) \exp\left[\frac{t}{2} (\mathcal{K}_+^2 + \mathcal{K}_-^2)\right].$$

(6.1)

Therefore, the heat kernel is equal to

$$U(t, x) = \exp(-tG^2)\Psi\left(\frac{t}{2}, x\right),$$

(6.2)

where $\Psi(t, x) = \exp\left[\frac{1}{t} (\mathcal{K}_+^2 + \mathcal{K}_-^2)\right] \delta(x)$ is the heat kernel of the operator $(\mathcal{K}_+^2 + \mathcal{K}_-^2)$. First of all, we note that the scalar heat kernel has two important properties: the invariance property

$$U_0(t, F(p, q)) = U_0(t, F(q, p)),$$

(6.3)

and the semi-group property

$$\int_{SU(2)} \text{dvol}(p) U_0(t, F(p, z)) U_0(s, p) = U_0(t + s, z).$$

(6.4)

By using the above method, equation (5.64), with $T = \mathcal{K}_+$ and $T = -\mathcal{K}_-$ we obtain

$$\exp(t\mathcal{K}_+^2 + t\mathcal{K}_-^2) = \int_{SU(2)\times SU(2)} \text{dvol}(q) \text{dvol}(p) U_0(t, q) U_0(t, p) \exp[\mathcal{K}_+^2(q) - \mathcal{K}_-^2(p)].$$

(6.5)

By using the form of the operators $\mathcal{K}_u^\pm$, equations (5.36) and (5.37), we obtain

$$\exp(t\mathcal{K}_+^2 + t\mathcal{K}_-^2) = \int_{SU(2)\times SU(2)} \text{dvol}(q) \text{dvol}(p) U_0(t, q) U_0(t, p) \times \exp[2G(q)] \exp[\mathcal{K}_+^2(q) - \mathcal{K}_-^2(p)].$$

(6.6)

Therefore, by acting on the function $\Psi(s; x)$ we obtain

$$\Psi(t + s; x) = \int_{SU(2)\times SU(2)} \text{dvol}(q) \text{dvol}(p) U_0(t, q) U_0(t, p) \times \exp[2G(q)] \Psi(s; F(-p, F(x, q))).$$

(6.7)

We change the variable $p$ by $p = F(x, F(q, -z))$ so that $F(-p, F(x, q)) = z$. Then,

$$\Psi(t + s; x) = \int_{SU(2)\times SU(2)} \text{dvol}(z) \text{dvol}(q) U_0(t, q) U_0(t, F(x, F(q, -z))) \exp[2G(q)] \Psi(s; z).$$

(6.8)

Now, we take the limit $s \to 0$ and use the fact that $\Psi(0, z) = \delta'_{\mathcal{S}}(z)$ to obtain

$$\Psi(t; x) = \int_{SU(2)} \text{dvol}(q) U_0(t, q) U_0(t, F(q, x)) \exp[2G(q)].$$

(6.9)

Finally, we compute the diagonal by setting $x = 0$,

$$\Psi(t; 0) = \int_{SU(2)} \text{dvol}(q) U_0(t, q) U_0(t, q) \exp[2G(q)].$$

(6.10)

Now, by using equations (6.2) and (5.65) we obtain the heat kernel diagonal

$$U^S(t) = (4\pi t)^{-3/2} e^t \exp[-tG^2] S(t),$$

(6.11)
where
\[
S(t) = \sum_{n,m=-\infty}^{\infty} 32\pi \int_0^\pi dr \frac{(r+2\pi n)(r+2\pi m)\Lambda_G(r)}{r^2} \times \exp \left\{ -\frac{1}{2t} [(r+2\pi n)^2 + (r+2\pi m)^2] \right\},
\]
and \(\Lambda_G(r)\) is the average of \(\exp[2rG(\theta)]\) over the sphere \(S^2\) introduced above, equation (5.54). Note that \(\Lambda_G(r)\) is an even function of \(r\) which is periodic with the period \(2\pi\). Therefore, the integral can be extended to the interval \([-\pi, \pi]\). Thus, the integrand is a periodic function with the period \(2\pi\). Therefore, by changing the variables by \(r \mapsto r - 2\pi m\) and \(n \mapsto n + m\) one summation results in the integration over the whole \(\mathbb{R}\) and we obtain
\[
S(t) = 2r^{-3/2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dr}{\sqrt{\pi}} (r+2\pi n) r \Lambda_G(r) \exp \left\{ -\frac{1}{2t} [(r+2\pi n)^2 + r^2] \right\}.
\]
(6.13)

Next, we change variable by \(r \mapsto r\sqrt{t} - \pi n\) to obtain
\[
S(t) = \sum_{n=-\infty}^{\infty} \exp \left( -\frac{\pi^2 n^2}{t} \right) \int_{-\infty}^{\infty} \frac{dr}{\sqrt{\pi}} e^{-r^2} \left( 2r^2 - 2\pi^2 n^2 \right) \Lambda_G(r\sqrt{t} - \pi n).
\]
(6.14)

Let us compute the heat kernel diagonal for an irreducible representation \(j\) of \(SU(2)\). By using the explicit form of the function \(\Lambda_j\) we obtain
\[
U_j^S(t) = \frac{1}{2j+1} \sum_{[\mu]|J\leq j} (4\pi t)^{-3/2} e^{j(j+1) + \mu^2} \sum_{n=-\infty}^{\infty} (-1)^{2m} \exp \left( -\frac{\pi^2 n^2}{t} \right)
\times \int_{-\infty}^{\infty} \frac{dr}{\sqrt{\pi}} e^{-r^2} \left( 2r^2 - 2\pi^2 n^2 \right) \cos(2\mu r\sqrt{t}).
\]
(6.15)

This integral is Gaussian and can be easily computed,
\[
U_j^S(t) = \frac{1}{2j+1} \sum_{[\mu]|J\leq j} (4\pi t)^{-3/2} e^{j(j+1) + \mu^2} \sum_{n=-\infty}^{\infty} (-1)^{2m} \exp \left( -\frac{\pi^2 n^2}{t} \right)
\times \left\{ 1 - 2\mu^2 t - 2\pi^2 n^2 \right\}.
\]
(6.16)

This formula is one of the main results of this paper. It gives the heat kernel diagonal of the Laplacian on \(S^3\) for an arbitrary irreducible representation of the group \(SU(2)\).

At this point it is convenient to introduce a new function
\[
\Omega_j(t) = \sum_{n=-\infty}^{\infty} (-1)^{2m} \exp \left[ -\frac{\pi^2 n^2}{t} \right].
\]
(6.17)

Then,
\[
U_j^S(t) = \frac{1}{2j+1} \sum_{[\mu]|J\leq j} (4\pi t)^{-3/2} e^{j(j+1) + \mu^2} \left\{ (1 - 2\mu^2 t)\Omega_j(t) - 2\Omega_j'(t) \right\}.
\]
(6.18)

Note that for integer \(j\)
\[
\Omega_j(t) = \Omega(t) = \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{\pi^2 n^2}{t} \right] = \theta_j(0, e^{-\pi^2/t}),
\]
(6.19)
and for half-integer \( j \)

\[
\Omega_j(t) = \hat{\Omega}(t) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \left[ -\frac{\pi^2 n^2}{t} \right] = \theta_4(0, e^{-\pi^2/t}).
\]  

By using the duality of the theta functions we can rewrite these functions in the form

\[
\Omega(t) = \sqrt{\frac{t}{\pi}} \theta_3(0, e^{-t/\pi}) = \sqrt{\frac{t}{\pi}} \sum_{n \in \mathbb{Z}} e^{-tn^2},
\]

\[
\tilde{\Omega}(t) = \sqrt{\frac{t}{\pi}} \theta_2(0, e^{-t/\pi}) = \sqrt{\frac{t}{\pi}} \sum_{v \in \mathbb{Z} + \mathbb{Z}/2} e^{-t\nu^2},
\]

where the first sum goes over all integers and the second sum goes over all half-integers. Here, of course, \( \theta_i(v, q) \) are Jacobi theta functions.

7. Yang–Mills heat trace

Now we can compute the heat trace for the Yang–Mills theory. By using the results for the heat kernel (and restoring the radius \( a \) of the sphere \( S^3 \)) we obtain

\[
\Theta_{YM}(t) = (4\pi t)^{-2} \Omega \left( \frac{t}{a^2} \right) e^{\frac{1}{2} t \boxtimes X^2} \left\{ \text{tr}_T \exp \left[ -\frac{t}{a^2} \left( G^2 + a^2 Q \right) \right] S_G \left( \frac{t}{a^2} \right) -2 \exp \left( -\frac{t}{a^2} X^2 \right) S_X \left( \frac{t}{a^2} \right) \right\}.
\]

(7.1)

For the vector representation \( T_1 \) the generator \( G_a \) has the form

\[
(G_a)^b_c = -\epsilon^{bc}_{~a} \otimes I_X + h^b_c X_a.
\]

(7.2)

and the potential term \( Q \) has the form

\[
(a^2 Q)^b_c = 2h^b_c - 2\epsilon^{bc}_{~a} X_a.
\]

(7.3)

The Casimir operator is then

\[
(G^2)^b_c = -2h^b_c + h^b_c X^2 + 2\epsilon^{bc}_{~a} X_a.
\]

(7.4)

Therefore,

\[
(G^2 + a^2 Q)^b_c = h^b_c X^2.
\]

(7.5)

Thus, the above expression simplifies to

\[
\Theta_{YM}(t) = (4\pi t)^{-2} \Omega \left( \frac{t}{a^2} \right) W \left( \frac{t}{a^2} \right).
\]

(7.6)

where

\[
W(t) = e^{2 t X^2} \left\{ \text{tr}_T S_G(t) - 2 S_X(t) \right\}.
\]

(7.7)

To compute the function \( S_G \) we need to compute the function \( \Lambda_G \), that is, the trace \( \text{tr}_T \exp[2G(q)] \), where

\[
2G(q) = C(q) \otimes I_X + I_C \otimes 2X(q).
\]

(7.8)

Obviously, these two matrices commute. Therefore,

\[
\text{tr}_T \exp[2G(q)] = \exp[2X(q)] \text{tr}_T \exp[C(q)].
\]

(7.9)
The eigenvalues of the matrix $C(q)$ are $(0, 0, 2|q|, -2|q|)$; therefore, 
\[ \text{tr}_\gamma \exp[C(q)] = 2 + 2 \cos(2|q|), \]  
and we obtain the trace 
\[ \text{tr}_\gamma \exp[2G(q)] = 2 [1 + \cos(2|q|)] \exp[2X(q)]. \] 
Now, by using the form (6.14) of the function $S(t)$ we obtain 
\[
W(t) = e^t \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi^2 n^2}{t}\right) \int_{-\infty}^{\infty} \frac{dr}{\sqrt{\pi}} e^{-r^2} \left(2r^2 - 2\frac{\pi^2 n^2}{t}\right) 
\times 2 \cos(2r\sqrt{t}) \text{tr}_\gamma \exp(-tX^2) \Lambda_X(r\sqrt{t}).
\] 
Next, by using the explicit form of the function $\Lambda_X$ in terms of the irreducible representations (5.55) we obtain 
\[
W(t) = \sum_{i=1}^{N} \sum_{|\mu| \leq j} e^{i(j + 1)\mu} \sum_{n=-\infty}^{\infty} (-1)^{2\mu} \exp\left(-\frac{\pi^2 n^2}{t}\right) 
\times \int_{-\infty}^{\infty} \frac{dr}{\sqrt{\pi}} e^{-r^2} \left(2r^2 - 2\frac{\pi^2 n^2}{t}\right) 2 \cos(2r\sqrt{t}) \cos(2\mu r\sqrt{t}).
\] 
Now the integrals over $r$ are Gaussian and can be computed; we finally obtain 
\[
W(t) = \sum_{i=1}^{N} \sum_{|\mu| \leq j} e^{i(j + 1)\mu} \sum_{n=-\infty}^{\infty} (-1)^{2\mu} \exp\left(-\frac{\pi^2 n^2}{t}\right) 
\times \left[ \left(1 - 2(t\mu + 1)^2 - 2\frac{\pi^2 n^2}{t}\right) e^{-t(\mu + 1)^2} 
+ \left(1 - 2(t\mu - 1)^2 - 2\frac{\pi^2 n^2}{t}\right) e^{-t(\mu - 1)^2} \right].
\] 
Also, one can express the function $W$ in terms of the function $\Omega_j$ introduced above, 
\[
W(t) = \sum_{i=1}^{N} W_{j_i}(t),
\] 
where 
\[
W_{j_i}(t) = \sum_{|\mu| \leq j} e^{i(j + 1)\mu} [(1 - 2(t\mu + 1)^2)\Omega_j(t) - 2\Omega'_{j_1}(t)] e^{-t(\mu + 1)^2} 
+ [(1 - 2(t\mu - 1)^2)\Omega_j(t) - 2\Omega'_{j_1}(t)] e^{-t(\mu - 1)^2}.
\] 
We will need the asymptotics of the function $W_{j_i}(t)$ as $t \to 0$ and as $t \to \infty$. The asymptotics of the function $W_{j}(t)$ as $t \to 0$ are 
\[
W_{j_i}(t) \sim (4j + 2) - (8j + 4) t + \left(\frac{2j}{\pi} J^3 + 11 j^2 + \frac{11}{4} j^3 \right) t^2 + \cdots.
\] 
By using the dual representation (6.21), (6.22), of the functions $\Omega_j(t)$ we can rewrite the function $W_j$ in a form convenient for the calculation of the asymptotics as $t \to \infty$. After some straightforward but tedious cancellations the result takes the form 
\[
W_j(t) = \frac{2}{\sqrt{\pi}} e^{3j/2} \sum_{|\mu| \leq j} [(\nu + 1)^2 - (\mu + 1)^2] e^{-t(\nu + 1)^2 + (\mu + 1)^2} 
+ [(\nu - 1)^2 - (\mu - 1)^2] e^{-t(\nu - 1)^2 + (\mu - 1)^2},
\]
where the summation goes over integers \( \mu \) and \( \nu \) for integer \( j \) and over half-integers \( \mu \) and \( \nu \) for half-integer \( j \) (the sum over \( \mu \) is finite and the sum over \( \nu \) is infinite).

Thus, as \( t \to \infty \) we obtain for integer \( j \)
\[
W_j(t) \sim \frac{4}{\sqrt{\pi}} j^{3/2} e^{-\lambda_j^\text{min} t} ,
\]
(7.19)
where
\[
\lambda_j^\text{min} = -j - 1 ,
\]
(7.20)
and for half-integer \( j \geq 3/2 \)
\[
W_j(t) \sim \frac{4}{\sqrt{\pi}} \left( j^2 - \frac{1}{4} \right)^{3/2} e^{-\lambda_j^\text{min} t} ,
\]
(7.21)
where
\[
\lambda_j^\text{min} = -j - \frac{3}{4} .
\]
(7.22)
Thus, for integer \( j \geq 1 \),
\[
\lambda_j^\text{min} \leq -2 .
\]
(7.23)
Also, for any half-integer \( j \geq 3/2 \),
\[
\lambda_j^\text{min} \leq -\frac{9}{4} .
\]
(7.24)
The only case when the minimal eigenvalue is positive is when \( j = 1/2 \); we show below that in this case
\[
\lambda_{1/2}^\text{min} = \frac{19}{4} .
\]
(7.25)
This is one of the main results of this paper. It tells us that the minimal eigenvalue is positive only in the case when the representation \( X \) does not contain any higher spin representations of \( SU(2) \) with \( j \geq 1 \) but contains only the spinor (fundamental) representation of \( SU(2) \) with \( j = 1/2 \). In all other cases, that is, when the representation \( X \) contains at least one representation with \( j \geq 1 \), the minimal eigenvalue is negative and the heat kernel grows exponentially at infinity, at least as \( e^{2t} \), leading to the infrared instability of the chromomagnetic vacuum.

Now, we can compute the effective action. First, by using the asymptotics of the function \( W_j(t) \) as \( t \to 0 \) we obtain the asymptotics of the heat trace
\[
\Theta_{\text{YM}}(t) \sim (4\pi t)^{-2} \left\{ C_0 + C_1 \frac{t}{a^2} + C_2 \frac{t^2}{a^4} + \cdots \right\} ,
\]
(7.26)
where
\[
C_0 = \sum_{i=1}^{N} (4j_i + 2) ,
\]
(7.27)
\[
C_1 = \sum_{i=1}^{N} (-8j_i - 4) ,
\]
(7.28)
\[
C_2 = \sum_{i=1}^{N} \left( \frac{22}{3} j_i^3 + 11j_i^2 + \frac{11}{3} j_i \right) .
\]
(7.29)
Therefore, \( \beta = (4\pi)^{-2} \left[ C_2 - \frac{1}{2} a^2 C_1 + \frac{1}{2} a^4 C_0 \right] /a^4 \). Now, the renormalized heat trace becomes
\[
\Theta_{\text{YM}}^\text{ren}(t) = (4\pi t)^{-2} \left\{ \Omega \left( \frac{t}{a^2} \right) W \left( \frac{t}{a^2} \right) - R_{\text{YM}} \left( \frac{t}{a^2} \right) \right\} ,
\]
(7.30)
where
\[ R_{\text{YM}}(t) = e^{-\xi^t} \left\{ C_0 + (C_1 + C_0\lambda^2)t + (C_2 + C_1\lambda^2 + \frac{1}{2}C_0\lambda^4)t^2 \right\}. \]  
(7.31)

Thus, the one-loop effective action is
\[ \Gamma_{(1)} = -\frac{\pi}{8x} \left\{ C_2 - z^2 a^2 C_1 + \frac{1}{2}z^4 a^4 C_0 \right\} \log \frac{\mu^2}{\lambda^2} + \Phi(x), \]  
(7.32)
where \( x = a/a_1 \) and (after rescaling the integration variable \( t \to ta \))
\[ \Phi(x) = \int_0^{\infty} \frac{dt}{t^3} e^{-t\bar{z}^2} \{ \Omega(\bar{z}^2)W(t) - R_{\text{YM}}(t) \}. \]  
(7.33)
The total effective action (including the classical action) is then
\[ \Gamma = \frac{\pi}{8x} \left\{ \frac{8\pi^2\sigma}{e^2} - \left[ C_2 - z^2 a^2 C_1 + \frac{1}{2}z^4 a^4 C_0 \right] \log \frac{\mu^2}{\lambda^2} - \Phi(x) \right\}, \]  
(7.34)
where \( \sigma = -\text{tr}X^2 = \sum_{i=1}^N i_{ji}(j_i + 1) \). This formula is another important result of this paper. It gives an exact integral representation of the infrared regularized one-loop effective action for pure Yang–Mills theory in Einstein universe.

Recall that \( z \) is an infrared regularization parameter introduced to ensure convergence of the integral (7.33) at infinity. Eventually, we need to take off the infrared regularization, that is, to take the limit \( z \to 0 \). The convergence of this integral in this limit depends on the asymptotic properties of the function \( W(t) \) at infinity. Using the asymptotics of the function \( W(t) \) at infinity obtained above we see that as a function of the infrared regularization parameter \( z \) the effective action \( \Gamma(z) \) is analytic for \( \text{Re} \bar{z}^2 > -\lambda_{\text{min}}/a^2 \), where
\[ \lambda_{\text{min}} = \min_{1 \leq i \leq N} \lambda_i^{\text{min}}, \]  
(7.35)
and has a branch singularity at \( z = \sqrt{-\lambda_{\text{min}}/a} \). Therefore, if \( \lambda_{\text{min}} > 0 \) the effective action has a well-defined regular value as \( z \to 0 \), but if \( \lambda_{\text{min}} < 0 \) the effective action is singular in this limit and the model is unstable at low energies.

Recall that the stable configuration only occurs when the adjoint representation of the gauge group \( G \) contains only the spinor representation of \( SU(2) \). In principle, for any given group \( G \) containing the group \( SU(2) \) as a subgroup we should be able to determine all the labels \( j_i \) of the representation \( X \), which might be an interesting representation-theoretic problem. If one could show that among the labels \( j_i \) there must be at least one label \( j \geq 1 \), then we would simply conclude that the model is unstable for any compact simple group, a very strong assertion.

However, we do not do this in this paper. We will simply assume that this is possible and proceed with the calculation. That is why we will consider further the case when the representation \( X \) contains only spinor representations, that is, all \( j_i = 1/2 \). In this case the function \( W \) simplifies
\[ W(t) = N \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi^2t^2/4} \left[ \left( 2 - 9t - 4\pi^2n^2/t \right) e^{-t^2/4} + \left( 2 - t - 4\pi^2n^2/t \right) e^{3t/2} \right] \]  
(7.36)
Therefore, as \( t \to 0 \)
\[ W(t) \sim N \left( 4 - 8t + \frac{11}{2}t^2 + \cdots \right), \]  
(7.37)
so that
\[ C_0 = 4N, \quad C_1 = -8N, \quad C_2 = \frac{11}{2}N. \]  
(7.38)
Further, by using the dual representation of the function \( \tilde{\Omega} \) we obtain

\[
W(t) = N \frac{8}{\sqrt{\pi}} \sum_{n=0}^{\infty} e^{-t(n+1/2)^2} \left[ (n-1)(n+2) e^{-t/2} + n(n+1) e^{t^2/2} \right].
\]  
(7.39)

Therefore, as \( t \to \infty \)

\[
W(t) \sim 48 \sqrt{\pi} N \frac{e^{-19t^2/4}}{t^{3/2}}.
\]  
(7.40)

It is convenient to introduce the function

\[
\tilde{W}(t) = \frac{1}{N} e^{19t^2/4} W(t).
\]  
(7.41)

Then as \( t \to 0 \)

\[
\tilde{W}(t) \sim 4 + 11t + \frac{101}{8} t^2 + \ldots,
\]  
(7.42)

and as \( t \to \infty \)

\[
\tilde{W}(t) \sim 48 \sqrt{\pi} t^{3/2} + \ldots.
\]  
(7.43)

In this case the effective action is well defined even in the limit when the infrared regularization parameter \( z \) is set to zero. Therefore, by taking the limit \( z \to 0 \) and choosing the parameter \( \lambda \) by \( \lambda = 19\sqrt{\pi} / (2x) \) we obtain

\[
\Gamma = \frac{\pi}{8x} N \left\{ \frac{6}{} 2 \log \left( \frac{4}{11} \mu^2 \right) - \tilde{\Phi}(x) \right\},
\]  
(7.44)

where

\[
\tilde{\Phi}(x) = \int_{0}^{\infty} \frac{dt}{t^3} e^{-19t^2/4} \left[ \Omega(x^2) \tilde{W}(t) - 4 - 11t + \frac{101}{8} t^2 \right].
\]  
(7.45)

Note that as \( x \to 0 \) this function approaches a well-defined constant

\[
\Phi_0 = \int_{0}^{\infty} \frac{dt}{t^3} e^{-19t^2/4} \left[ \tilde{W}(t) - 4 - 11t + \frac{101}{8} t^2 \right].
\]  
(7.46)

Therefore, we can split the integral into two parts

\[
\tilde{\Phi}(x) = \Phi_0 + \Phi_1(x),
\]  
(7.47)

where (after rescaling \( t \to t/x \))

\[
\Phi_1(x) = x^2 \int_{0}^{\infty} \frac{dt}{t^3} \exp \left( -\frac{19}{4} \frac{t^2}{x} \right) [\Omega(x^2) - 1] \tilde{W} \left( \frac{t}{x} \right).
\]  
(7.48)

By using the asymptotics of the function \( [\Omega(x^2) - 1] \) it is not difficult to see that there is a critical value \( x_c = \frac{2\sqrt{\pi}}{\sqrt{19}} \).

If \( x < x_c \) then the integral \( \Phi_1(x) \) is exponentially small. Indeed, in the limit \( x \to 0 \) the function \( [\Omega(x^2) - 1] \) is determined by the first exponential term, (4.6); therefore,

\[
\Phi_1(x) \sim 2x^2 \int_{0}^{\infty} \frac{dt}{t^3} \exp \left( -\frac{19}{4} \frac{t^2}{x} \right) \tilde{W} \left( \frac{t}{x} \right).
\]  
(7.50)

The main contribution to this integral comes from the neighborhood of the point \( t_0 = 2\pi / \sqrt{19} \). Therefore, by using the asymptotics of the function \( \tilde{W}(t) \) (7.43) we obtain

\[
\Phi_1(x) \sim \frac{96}{\pi} x^{5/2} \exp \left( -\frac{\sqrt{19}\pi}{x} \right).
\]  
(7.51)
By rewriting the integral (7.45) in the form
\[
\Phi(x) = x^4 \int_0^\infty \frac{dt}{t^3} e^{-19/(4t^2)} \left\{ \Omega(t) \tilde{W} \left( \frac{t}{x^2} \right) - 4 - 11 \frac{t}{x^2} - \frac{101 t^2}{8 x^4} \right\},
\]
(7.52)
one can obtain the asymptotics of the function \( \Phi(x) \) as \( x \to \infty \),
\[
\Phi(x) \sim v x^4,
\]
(7.53)
where \( v \) is a positive constant defined by
\[
v = 4 \int_0^\infty \frac{dt}{t^3} [\Omega(t) - 1].
\]
(7.54)
It can be computed exactly \( v = \frac{8}{9} \zeta(4) = \frac{8}{90} \), with \( \zeta(s) \) the Riemann zeta function. Thus,
\[
\Phi(x) \sim \frac{8}{90} x^4.
\]
(7.55)
By expanding the function \( \tilde{W}(t) \) in a power series one can also obtain the expansion of the function \( \Phi(x) \) in inverse powers of \( x \).

8. Thermodynamics of Yang–Mills theory

In this section we investigate the entropy and the heat capacity of the model for the stable configuration of the background fields containing only one representation with \( j = 1/2 \). The temperature is related to the radius of the circle \( S^1 \) by \( T = 1/(2\pi aT) = x/(2\pi a) \). The volume of the space is the volume of the sphere \( S^3, V = 2\pi^2 a^3 \). For a canonical statistical ensemble with fixed \( T \) and \( V \), the free energy \( F = E - TS \) is a function of \( T \) and \( V \) defined by the total effective action \( F = T \Gamma \), with \( \Gamma = \Gamma_\text{cl} + \Gamma_\text{ren} \). Then the entropy, the heat capacity at constant volume and the pressure are determined by the derivatives of the free energy,
\[
S = -\frac{\partial(T\Gamma)}{\partial T}, \quad C_v = -T \frac{\partial^2(T\Gamma)}{\partial T^2}, \quad P = -\frac{\partial(T\Gamma)}{\partial V}.
\]
(8.1)
It is easy to see that neither the classical part \( \Gamma_\text{cl} \) nor the renormalization logarithmic term in the combination \( T\Gamma \) depends on the temperature. Therefore, the entropy and the heat capacity do not depend either on the classical term or on the renormalization parameter \( \mu \). Thus, the entropy and the heat capacity (per unit volume) are given by the derivatives of the function \( \Phi \),
\[
s = \frac{S}{V} = \frac{N}{16\pi a^3} \Phi(x), \quad c_v = \frac{C_v}{V} = \frac{N}{16\pi a^3} x \Phi'(x).
\]
(8.2)
By using the asymptotics of the function \( \Phi(x) \) we can now compute the entropy and the heat capacity of the gluon gas: at low temperature and small volume, as \( aT \to 0 \), we have
\[
s \sim 6\sqrt{\frac{3}{\pi}} \frac{Na^{-5/2}}{\sqrt{\pi} T^{1/2}} \exp \left( -\sqrt{\frac{19}{2aT}} \right),
\]
(8.3)
\[
c_v \sim \sqrt{\frac{2}{\pi}} \frac{114}{\sqrt{2\pi}} \frac{Na^{-7/2}}{T^{1/2}} \exp \left( -\sqrt{\frac{19}{2aT}} \right).
\]
(8.4)
And at high temperature and large volume, as \( aT \to \infty \), we obtain
\[
s \sim \frac{8}{45} \pi^2 NT^3, \quad c_v \sim \frac{8}{15} \pi^2 NT^3.
\]
(8.5)
One can also compute the energy density \( \varepsilon = E/V = T(\Gamma + S)/V \) to obtain
\[
\varepsilon = \frac{N}{32\pi^2 a^4} \left\{ \frac{6\pi^2}{e^2} - \frac{11}{2} \log \left( \frac{4}{19} a^2 \mu^2 \right) + x \Phi'(x) - \Phi(x) \right\},
\]
(8.6)
and the pressure

\[ P = \frac{N}{96\pi^2a^4} \left\{ \frac{6\pi^2}{e^2} - 11 - \frac{11}{2} \log \left( \frac{4}{19}a^2\mu^2 \right) + x\Phi'(x) - \Phi(x) \right\}. \tag{8.7} \]

Note that both the pressure and the energy density depend on the renormalization parameter \( \mu \). Further, this gives the equation of state of gluon gas

\[ P = \frac{1}{3} \varepsilon - \frac{11N}{96\pi^2a^4}. \tag{8.8} \]

In the limit of low temperature and small volume as \( aT \to 0 \) the energy density has the form

\[ \varepsilon \sim -\frac{N}{32\pi^2a^4} \left\{ \frac{6\pi^2}{e^2} - \frac{11}{2} \log \left( \frac{4}{19}a^2\mu^2 \right) - \Phi_0 \right\}. \tag{8.9} \]

In the limit of high temperature and large volume, as \( aT \to \infty \), the energy density is

\[ \varepsilon \sim -\frac{N}{32\pi^2a^4} \left\{ \frac{6\pi^2}{e^2} - \frac{11}{2} \log \left( \frac{4}{19}a^2\mu^2 \right) \right\} + \frac{2}{15}\pi^2NT^4. \tag{8.10} \]

Of course, in this limit the gluon gas behaves like the ‘colored’ photon gas that has \( 2N \) as many degrees of freedom. Recall that for the photon gas \( P = \frac{1}{\mu^2} \varepsilon = \frac{1}{3} \pi^2T^4 \) and \( s = \frac{2\pi^2}{3}T^3 \).

Note that at a fixed temperature \( T \) as \( a \to 0 \) the energy density \( \varepsilon \to +\infty \) and as \( a \to \infty \) it goes to a positive constant. Also, both functions (8.9) and (8.10) have minima at some different values of the radius \( a \),

\[ a_{1,2} = \frac{1}{\mu} b_{1,2} \exp \left( \frac{12\pi^2}{11e^2} \right), \tag{8.11} \]

where

\[ b_1 = \sqrt{\frac{19}{4}} \exp \left( -\frac{1}{4} \right), \quad b_2 = \sqrt{\frac{19}{4}} \exp \left( -\frac{1}{4} - \Phi_0 \right). \tag{8.12} \]

Therefore, it is reasonable that the energy density has a minimum at a finite non-trivial value of the radius \( a_0(T) = \frac{1}{\mu} b(T) \exp \left( \frac{12\pi^2}{11e^2} \right) \) between \( a_1 \) and \( a_2 \), which can be determined numerically. Obviously, this is a purely non-perturbative effect; it indicates the existence of a non-trivial vacuum with the magnetic field and the curvature of the order \( \sim \mu^2 \exp \left( -\frac{12\pi^2}{11e^2} \right) \).

9. Conclusion

The primary goal of this paper was to study the low-energy structure of the Yang–Mills vacuum. We assumed the chromomagnetic nature of the vacuum with covariantly constant chromomagnetic fields, which was well known to be unstable in flat space. We noted that the potential term of the gluon operator \( L_1 = -\Delta_{T,\partial X} + Q \) has the form \( Q^a_b = R^a_{\delta} - 2F^a_{\delta b} \). Therefore, a large Ricci curvature increases the minimal eigenvalue and a large magnetic field decreases it. Therefore, to make the gluon operator positive one needs a large Ricci tensor and a small magnetic field. Moreover, one needs the ability to independently change the magnitudes of the curvature and the magnetic field to make this work.

Of course, to be able to carry out the calculations one needs some degree of symmetry. So, we assumed that the chromomagnetic field is covariantly constant and the spatial curvature is constant and positive. In four dimensions there are not many choices of spaces of non-negative constant curvature, only products of spheres and circles; the non-flat ones are \( S^1 \times S^3 \times S^3 \) and \( S^1 \times S^3 \). The case of \( S^1 \times S^3 \times S^3 \) was studied in our previous paper [11] and in this paper we studied the case of \( S^1 \times S^3 \).

One of our results is the calculation of the heat kernel of the Laplacian acting on arbitrary fields on \( S^3 \) in the arbitrary representation of \( SU(2) \). By using this result one can compute
the one-loop effective action of any field-theoretic model. We apply it to the calculation of the effective action of Yang–Mills theory on Euclidean Einstein universe, $S^1 \times S^3$. Our main result is the proof that generically the gluon operator almost always has negative eigenvalues. The only case in which the gluon operator does not have negative eigenvalues occurs when the fundamental (spinor) representation of the group $SU(2)$ can be embedded in the adjoint representation of the gauge group (that contains the group $SU(2)$ as a subgroup). If one can show that this is impossible, then our results prove the instability of the chromomagnetic vacuum of Yang–Mills theory in Einstein universe for any compact simple gauge group. We intend to study this representation-theoretic problem in a future work.

We confirm the conclusion of our previous work [11] that to stabilize the chromomagnetic vacuum at lower energies one should consider non-constant magnetic fields on non-compact spaces. Covariantly constant magnetic fields on compact symmetric spaces are too rigid, they are completely determined by the spin connection and are of the same order as the spatial curvature. This makes it impossible for the gluon operator to be strictly positive.

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References

[1] Altaie M B and Dowker J S 1978 Spinor fields in an Einstein universe: finite temperature effects Phys. Rev. D 18 3557–64
[2] Anderson A and Camporesi R 1990 Intertwining operators for solving differential equations, with applications to symmetric spaces Commun. Math. Phys. 130 61–82
[3] Avramidi I G 1991 A covariant technique for the calculation of the one-loop effective action Nucl. Phys. B 355 712–54
[4] Avramidi I G 1998 A covariant technique for the calculation of the one-loop effective action Nucl. Phys. B 509 557–8 (erratum)
[5] Avramidi I G 1993 A new algebraic approach for calculating the heat kernel in gauge theories Phys. Lett. B 305 27–34
[6] Avramidi I G 1995 Covariant algebraic calculation of the one-loop effective potential in non-Abelian gauge theory and a new approach to stability problem J. Math. Phys. 36 1557–71
[7] Avramidi I G 1999 A model of stable chromomagnetic vacuum in higher-dimensional Yang–Mills theory Fortschr. Phys. 47 433–55
[8] Avramidi I G 2008 Heat kernel on homogeneous bundles Int. J. Geom. Methods Mod. Phys. 5 1–23
[9] Avramidi I G 2009 Heat kernel on homogeneous bundles over symmetric spaces Commun. Math. Phys. 258 963–1006
[10] Avramidi I G 2010 Non-perturbative effective action in gauge theories and quantum gravity Adv. Theor. Math. Phys. 14 309–33
[11] Avramidi I G 2010 Mathematical tools for calculation of the effective action in quantum gravity New Paths Toward Quantum Gravity ed B Booss-Bavnbek, G Esposito and M Lesch (Berlin: Springer) pp 193–259
[12] Avramidi I G and Collropy S 2012 Effective action and phase transitions in thermal Yang–Mills theory on spheres Commun. Math. Phys. DOI:10.1007/s00220-012-1418-y (arXiv:1012.2414 [hep-th])
[13] Camporesi R 1990 Harmonic analysis and propagators on homogeneous spaces Phys. Rep. 196 1–134
[14] Camporesi R 1992 The spinor heat kernel in maximally symmetric spaces Commun. Math. Phys. 148 283–308
[15] Camporesi R 1994 Spectral functions and zeta functions in hyperbolic spaces J. Math. Phys. 35 4217–46
[16] David J R, Gaberdiel M R and Gopakumar R 2010 The heat kernel on $AdS_3$ and its applications J. High Energy Phys. JHEP04(2010)125
[17] De Witt B S 1965 Dynamical Theory of Groups and Fields (New York: Gordon and Breach)
[18] Dowker J S 1970 When is the ‘sum over classical paths’ exact? J. Phys. A: Gen. Phys. 3 451
[19] Dowker J S 1971 Quantum mechanics on group space and Huygens’ principle Ann. Phys. 62 361–82
[20] Dowker J S 1972 Quantum mechanics and field theory on multiply connected and on homogeneous spaces J. Phys. A: Gen. Phys. 5 936–43
[21] Dowker J S 1972 Propagators for arbitrary spin in an Einstein universe Ann. Phys. 71 577–602
[22] Dowker J S 1983 Arbitrary spin theory in the Einstein universe Phys. Rev. D 28 3013
[23] Dowker J S and Critchley R 1977 Vacuum stress tensor in an Einstein universe. Finite temperature effects Phys. Rev. D 15 1484
[24] Elizalde E, Lygren M and Vassilevich D V 1996 Anti-symmetric tensor fields on spheres: functional determinants and non-local counterterms J. Math. Phys. 37 3105–17
[25] Giombi S, Maloney A and Yin X 2008 One-loop partition functions of 3D gravity J. High Energy Phys. JHEP08(2008)007
[26] Gopakumar R, Gupta R K and Lal S 2011 The heat kernel on AdS arXiv:1103.3627
[27] Nielsen N K and Olesen P 1978 An unstable Yang–Mills mode Nucl. Phys. B 144 376–96
[28] Nielsen H B and Olesen P 1979 A quantum liquid model for the QCD vacuum: gauge and rotational invariance of domain and quantized homogeneous color fields Nucl. Phys. B 160 380–96
[29] Rubin M A and Ordóñez C R 1985 Symmetric-tensor eigenspectrum of the Laplacian on n-spheres J. Math. Phys. 26 65–7
[30] Savvidy G K 1977 Infrared instability of the vacuum state of gauge theories and asymptotic freedom Phys. Lett. B 71 133–4
[31] Volkov M S 1996 Computation of the winding number diffusion rate due to the cosmological sphaleron Phys. Rev. D 54 5014–30