Application of GMLS-based numerical manifold method in buckling analysis of thin plates

Hongwei Guo\textsuperscript{1,3}, Hong Zheng\textsuperscript{2}

\textsuperscript{1}State Key Laboratory of Geomechanics and Geotechnical Engineering, Institute of Rock and Soil Mechanics, Chinese Academy of Sciences, Wuhan, China.

\textsuperscript{2}Key Laboratory of Urban Security and Disaster Engineering (Beijing University of Technology), Ministry of Education, Beijing 100124, China

\textsuperscript{3}University of Chinese Academy of Sciences, Beijing 100049, China

E-mail: ghway0703@163.com

Abstract. This paper applies the GMLS-based numerical manifold method (NMM) in analyzing the buckling behaviors of thin plates. The numerical manifold method (NMM), deploying two cover systems, namely the mathematical cover (MC) and physical cover (PC), has been successfully employed in the continuum and discontinuum deformation analysis. In this study, the MC system is constructed with influence domains in generalized moving least squares (GMLS) interpolation rather than the commonly-used finite element meshes, which is especially fitted for the fourth-order problems with rotational degrees of freedom and simplifies the possess of generating physical cover system. Moreover, the GMLS-based NMM can be readily utilized to treat plates with arbitrary complex geometry and openings without resorted to those complicated criteria and tailored operations. A series of numerical experiments concerning buckling analysis for thin plates with various shapes and cutouts have been performed, thus validating the effectiveness and accuracy of the proposed GMLS-based numerical manifold method.

1. Introduction

Thin plates are structural components commonly used in all branches of engineering, and its mechanical behaviors as bending, vibration and buckling are studied by various brilliant engineers. In most circumstance, the plates are not watertight, and they are often arranged with different types of cutouts for the reasons including weight reduction, venting, altering the critical load, outfitting and providing accessibility to other parts of the structure. So the treatment of those plate types need to be studied and a method which can accurately analysis their mechanical behaviors is becoming important. Analytical solutions of those simply-shaped thin plate problems such as elliptical or square plates have been obtained by those early brilliant researchers\textsuperscript{[1]}. However, in most cases, the exact analysis of thin plate structures can hardly be performed. The ushering in of various numerical techniques, such as finite element method (FEM) \textsuperscript{[2, 3]}, the boundary element method (BEM) \textsuperscript{[4, 5]}, the meshfree method \textsuperscript{[6-8]}, the Isogeometric analysis (IGA) \textsuperscript{[9-11]}, and the numerical manifold method (NMM) \textsuperscript{[12-14]}, have been widely tapped in analyzing the mechanical behaviors of thin plate structures.

With two set of covers, namely the mathematical cover (MC) and physical cover (PC), NMM aims at, on one hand, solving continuum and discontinuum deformation problems in a unified way, on the other hand obtaining the superb approximation to the primal field variables \textsuperscript{[12, 13]}. Invariably, a
uniform mesh is always employed to construct the MC, which largely determines the accuracy of approximation [14]. Additionally, NMM is considered as a resourceful but easily means of tackling physical puzzles by adjusting its local approximations on physical patches constituting the PC, rather than those complex approaches in resorting to the amendment of shape functions in FEM, which can be one of the major differences between NMM and FEM [13]. Due to those advantages, NMM has been widely applied to treat those knotty problems in civil engineering [13-16].

However, the previous numerical manifold method mainly adopted the finite element meshes as mathematical cover. Other methods of constructing mathematic cover system can be borrowed from MLS [17, 18] and Isogeometric analysis (IGA) [19], which demonstrate the flexibility for the construction of MC. The moving least square interpolation is widely used to generate shape functions for various meshfree method, due to its favorable features such as it creates a continuous and smooth approximation function in the field domain and any desired levels of consistency can be obtained for the field function. So if MLS interpolation is formulated in the setting of NMM, it will be helpful to solving a series of continuum and discontinuum problems in a unified way. Moreover, it can be naturally employed to deal with fourth-order problems which requires the displacement to be $H_2$ regular. This is however by no means essential, other interpolation methods in meshfree method can also be borrowed here as long as the partition of unity is satisfied.

Next, recent progress of MLS-based NMM, which is the major concerns in this study, is introduced. The following authors have contributed to the early development of the MLS-based cover NMM. Tian et al. [20] originally developed the finite-Cover based element free method, which combines the numerical manifold method and element-free Galerkin method, and successfully applied it in geotechnical Engineering. Liu et al. [21] studied the meshless method based on manifold cover ideas. Luan et al. [22, 23] successfully applied the Finite-cover-based element-free method in the computations of stress intensity factor and numerical simulations of crack expansion and simulated the fracture and damage evolution process of geo-materials. Li and Cheng [24] simulated the two-dimensional crack problems with the enriched meshless manifold method. Recent works on the meshfree finite-cover based manifold method mainly focused on the follows. Su and Cai [25] proposed a meshless Shepard interpolation method (MSIM) interpolation and applied it for the simulation of crack growth. Zheng [17, 18] employed the MLS-based numerical manifold method for the crack analysis. Zheng et al. [26] studied the primal mixed solution to unconfined seepage flow in porous media with MLS-based NMM. Liu proposed a structured mesh refinement for crack problems within the framework of MLS-based NMM without introducing any extra degree of freedom.

In this study, it should be noted that the moving least squares interpolation is based only on information of function values (fictitious values) of the variables at some scattering points and the approximation only requires the minimizing of the weighted least squares measure of field function at the node. However, information concerning the derivatives of variables at those scattering points, as in the cases of Kirchhoff plate, may produce a better approximation result if they are included in the interpolation than those without, which has been proved by Atluri [27] who proposed the generalized MLS (GMLS) to formulate the shape functions on the basis of the MLPG method. Valencia used the GMLS in the EFG method to analyze the influence of the selectable parameters by a one-dimensional beam in bending problems. Thus the generalized moving least square (GMLS) interpolation [27, 28] is introduced and embedded in the setting of NMM so as to deal with thin plate problems using Kirchhoff-Love model.

This paper is organized as follows: in Section 2, the weak form of thin plate buckling problems is briefly retrieved. Section 3 gives a brief description of the GMLS-based NMM. In Section 4, the discrete equations of buckling analysis are introduced. In Section 5, a series of numerical examples are performed and compared with the analytical solutions, which demonstrate the favorable features of GMLS-based NMM in analysis of thin plate with completed shape.

2. Weak form of thin plate buckling problems
A thin plate in the oxy plane, occupying the domain $\Omega$, is studied here. Based on the Kirchhoff-Love assumptions [1], the vertical deflection $w(x, y)$ of the middle plane is considered as the unique primal variable.

The Galerkin weak form for the Kirchhoff’s plate buckling problems is to find the deflection function $w$, such that the virtual work equation holds for any function $\delta w$ in the function space $W$ to be defined below.

$$a(\delta w, w) - \lambda b(\delta w, w) = 0,$$  \hspace{1cm} (1)

The boundary conditions for the thin plate problems concerned in this study are clamped boundary $\Gamma_C$, simply-supported boundary $\Gamma_S$ and the rest free boundary $\Gamma_F$. The three boundary portions do not overlap and make up the boundary $\partial \Omega$ of $\Omega$, anyone of which however can be empty.

Accordingly, the function space $W$ is defined as:

$$W = \{ v \mid v = 0, v_n \equiv \frac{\partial v}{\partial n} = 0, \text{ on } \Gamma_C; v = 0 \text{ on } \Gamma_S \},$$

with $n$ being the unit outward normal of $\partial \Omega$. Thus, $w(x, y)$ considered to be a function of $x$ and $y$ belongs to $W$.

In equation (1), the first item, $$a(\delta w, w) = \int_{\Omega} (\delta \kappa)^T m d\Omega,$$  \hspace{1cm} (3)

is the virtual work done by generalized stress vector $m$ to be explained shortly; and the geometric strain energy $b(\delta w, w)$ is defined as

$$b(\delta w, w) = \int_{\Omega} (\delta \theta)^T \Sigma \theta d\Omega,$$  \hspace{1cm} (4)

with $\theta = [\theta_x, \theta_y] = [w_x, w_y]$, $\Sigma = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix}$. $N_x$, $N_{xy}$, $N_y$ are the in-plane pre-buckling force resultants.

In equation (3), $\kappa$ is the 3-dimensional generalized strain vector related to deflection $w$ by

$$\kappa = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \end{bmatrix} = L w,$$  \hspace{1cm} (5)

in the Cartesian oxy system, with $L$ the differential operator

$$L^T = \left( \begin{array}{ccc} -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -2 \frac{\partial^2}{\partial x \partial y} \end{array} \right);$$

$m$ is the 3-dimensional generalized stress vector dual to $\kappa$ through the constitution law

$$m = D \kappa,$$  \hspace{1cm} (7)

where $D$ is the $3 \times 3$ elasticity matrix defined by

$$D = D_0 \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$

Here $D_0 = \frac{E h^3}{12(1-\nu^2)}$ denotes the bending rigidity; $E$ and $\nu$ are Young’s modulus and Poisson’s ratio, respectively.

By using a uniform mesh as MC, NMM allows the mesh not to match the problem boundary with the aim of achieving the highest interpolation precision, which cannot enforce the essential boundary condition directly as in FEM, the following penalty formulation is used instead.
\[ a_p(\delta w, w) - \lambda b(\delta w, w) = 0, \tag{9} \]

with
\[ a_p(\delta w, w) = a(\delta w, w) + \int_{\Gamma_c \cup \Gamma_s} k_d w \delta w d\Omega + \int_{\Gamma_c} k_a w_n \delta w_n d\Omega, \tag{10} \]

where \( k_d \) and \( k_a \) are user-specified penalties, which work well for most problems as \( 10^{4-6} \) times of the maximum values of the diagonal elements in the stiffness matrix, and are often used as \( 10^{5-8} \) times of the Young’s modulus [29]. In the variational formulation (10), neither \( w \) nor \( \delta w \) is constrained except for the smoothness requirement.

3. A brief review of GMLS-NMM
Zheng and Liu have constructed the mathematical cover with the mathematical patches being the influence domains of scattered nodes in MLS which entitles NMM with more excellent numerical properties [17, 18, 26, 30] in solving the second order problems. In this study, the mathematical patches are replaced by the influence domains of GMLS nodes, and the global approximations by the GMLS approximations that compose the Hermitian NMM space for the fourth order problems. GMLS is helpful if the continuity of field derivatives is required.

![Fig. 1. Plate \( \Omega \), nodes \( r_i \), influence domain \( \Omega_i \) and its radius \( R_i \).](image)

To begin with, let \( \Omega \) represents the problem domain; and the \( m \) distinct nodes \( r_i, i = 1, ..., m \), be scattered inside or outside \( \Omega \), as shown in Fig. 1. Associated with node \( r_i \) is a circular domain with radius of \( R_i \), called the influence domain of node \( r_i \) and denoted by \( \Omega_i \). By convention, it is said that node \( r_i \) only influences points in \( \Omega_i \). As long as node \( r_i \) can influence \( \Omega \), it can be deployed outside \( \Omega \) in the context of NMM [18]. \( \Omega_i \) can also be a regular domain, but this causes no essential difference.

In GMLS-NMM, each influence domain \( \Omega_i \) of node \( r_i \) serves as a mathematical patch, all the \( m \) influence domains constitutes the mathematical cover of \( \Omega \), denoted by \( \{ \Omega_i \} \).

By cutting each mathematical patch \( \Omega_i \) with the internal and external boundary of \( \Omega \) and abandoning those portions outside \( \Omega \), we obtain the physical patches \( \Omega_{j,i}, j = 1, ..., n_i \). All the physical patches are still indexed using single subscripts \( \Omega_i, i = 1, ..., n \), and they constitute the physical cover \( \{ \Omega_i \} \). Here, \( n \) is the number of all the physical patches, with \( n \geq m \).

Let the scalar function \( w : \Omega \rightarrow \mathbb{R} \) be approximated. Suppose that the averages of the local approximations to \( w \), \( \frac{\partial w}{\partial x} \) and \( \frac{\partial w}{\partial y} \), over \( \Omega_i, i = 1, ..., n \), are the unknown constants \( w_i, \theta_{x_i} \) and \( \theta_{y_i} \), respectively. The major task in GMLS is to construct a global approximation \( Gw \) to \( w \), in terms of \( (r_i, w_i, \theta_{x_i}, \theta_{y_i}) \), where \( r_i \) is the center of the physical patch \( \Omega_i \).
To construct $Gw$, at each point $\hat{r} = (\hat{x}, \hat{y}) \in \Omega$, GMLS uses a local approximation $w_\hat{r}$ to $w$ in a local domain $\Omega_\hat{r}$ around $\hat{r}$. The local approximation $w_\hat{r}$ is spanned by the basis of predefined functions \( \{b^i(\hat{r})\}_{i=1}^d, d \leq m \).

\[
w_\hat{r}(r) = p^T(\hat{r})a(\hat{r}), \quad \forall r \in \Omega_\hat{r}
\]

with $p(\hat{r})$ is a vector-valued function composed by $b^i(\hat{r})$, $i = 1, \ldots, d$, with $d$ the number of basis functions; and $a(\hat{r})$ is a vector-valued function with its $d$ members constituted by $a_j(\hat{r}), j = 1, \ldots, d$.

Usually, polynomials are chosen for the basis functions. In order to reproduce the constant strain mode in the Kirchhoff plate, at least the complete quadratic monomials are selected for $b^i(\hat{r})$, namely,

\[
p^T(\hat{r}) = (1, x, y, x^2, xy, y^2), \text{ with } d = 6,
\]

Also, more complicated items can be added in the basis in order to capture the local behaviors of the solution [18].

The distance from local approximation $w_\hat{r}$ at $\hat{r}$ to $w$ is measured by a weighted discrete and $\hat{r}$-dependent norm, defined by

\[
J_\hat{r}(a) = \sum_{i=1}^n \omega_i(\hat{r}) \left[ (w_\hat{r}(r_i) - w_i)^2 + \left( \frac{\partial w_\hat{r}}{\partial x}(r_i) - \theta_{x\hat{r}} \right)^2 + \left( \frac{\partial w_\hat{r}}{\partial y}(r_i) - \theta_{y\hat{r}} \right)^2 \right]
\]

\[
= \sum_{i=1}^n \omega_i(\hat{r}) \left[ p^T(\hat{r})a(\hat{r}) - w_i \right]^2 + \left( \frac{\partial p^T}{\partial x}(r_i) - \theta_{x\hat{r}}a(\hat{r}) - \theta_{x\hat{r}} \right)^2 + \left( \frac{\partial p^T}{\partial y}(r_i) - \theta_{y\hat{r}}a(\hat{r}) - \theta_{y\hat{r}} \right)^2 \right]
\]

Here, $\omega_i(\hat{r})$ is known as the local weight function vanishing out $\Omega_i$. $\omega_i(\hat{r})$ must be continuous at least up to the second order over the whole problem domain $\Omega$. In this study, $\omega_i$ takes on the form

\[
\omega_i(r) = \omega(r) = \begin{cases} 
2/3 - 4r^2 + 4r^3, & r \leq 1/2 \\
4/3 - 4r + 4r^2 - 4r^3/3, & 1/2 < r \leq 1, \\
0, & r > 1 
\end{cases}
\]

where $r = ||r - r_i||/R_i$, and $R_i$ is the radius of $\Omega_i$. It is easy to confirm that $\omega_i(r)$ is continuous up to the second order with regard to $r$.

Invariably, the radius of influence domain is chosen to ensure a proper area of coverage for interpolation as

\[
d_m = \alpha_s d_c,
\]

with $d_c$ commonly defined as an average nodal spacing in the supported of the investigated point and $\alpha_s$ the dimensionless size of the supported domain. $\alpha_s$ is often given by computational experience, and $\alpha_s = 1.4$ to 4.0 yields good results [6].

The stationary of the function of the weighted residual $J_\hat{r}(a)$ with respect to $a$ leads to the following linear relation

\[
A(\hat{r})a(\hat{r}) = P(\hat{r})d,
\]

between $a(\hat{r})$ and $d^T = (w_1, \theta_{x1}, \theta_{y1}, \ldots, w_m, \theta_{xm}, \theta_{ym})$, where

\[
A(\hat{r}) = \sum_{i=1}^n \omega_i(\hat{r}) \left[ p(r_i)p^T(r_i) + \frac{\partial p(r_i)}{\partial x} \frac{\partial p^T(r_i)}{\partial x} + \frac{\partial p(r_i)}{\partial y} \frac{\partial p^T(r_i)}{\partial y} \right]
\]

\[
a d \times d \text{ symmetric matrix; and }
\]

\[
P(\hat{r}) = [g_1, g_2, g_3, \ldots, g_{3n-2}, g_{3n-1}, g_{3n}],
\]

\[
a d \times 3n \text{ matrix with }
\]

\[
g_{3i-2}(\hat{r}) = \omega_i(\hat{r})p(r_i)
\]

\[
g_{3i-1}(\hat{r}) = \omega_i(\hat{r})\frac{\partial p(r_i)}{\partial x}, \quad i = 1, \ldots, n
\]

\[
g_{3i}(\hat{r}) = \omega_i(\hat{r})\frac{\partial p(r_i)}{\partial y}
\]

The GMLS approximation defined in equation (11) is justified only when the moment matrix $A(\hat{r})$ is nonsingular as in the MLS approximation. Each item in the square brackets of equation (16) is a rank one matrix. Hence, to make matrix $A(\hat{r})$ invertible, $3k \geq d$ must be satisfied, with $k$ the number of physical patches covering point $\hat{r}$. In the setting of the MLS approximation, the condition for matrix
This page contains a complex mathematical derivation involving matrices and functions, specifically focusing on the formulation of discrete equations for buckling analysis in the context of geometrically non-linear problems. The text describes how to formulate equations for buckling analysis, including the use of shape functions and the construction of stiffness and mass matrices. The derivation involves the use of GMLS (Generalized Moving Least Squares) and MLS (Moving Least Squares) methods, with a focus on ensuring the matrices used are nonsingular.

The text includes the following key points:

1. **Formulation of Equations**: The paper formulates discrete equations for buckling analysis, focusing on the construction of matrices that ensure the system is well-posed.
2. **GMLS and MLS**: It explains how GMLS and MLS are used in the analysis, highlighting the importance of nonsingularity in the matrices.
3. **Matrices and Their Properties**: The paper discusses the properties of the stiffness matrix (\( K \)) and the mass matrix (\( M \)), and how they are constructed using shape functions.

Mathematically, the paper explores the use of shape functions in the context of buckling analysis, relating them to the differential operator defined in equation (6). The stiffness matrix is shown to be related to the global approximation (20) through the use of shape functions, and the mass matrix is associated with the action integral (21). The nonsingularity of the stiffness matrix is a critical aspect of the formulation, ensuring the stability and convergence of the solution.

The derivations are grounded in the analysis of nonlinear problems, with a focus on the geometrically non-linear aspects. The text is rich with mathematical expressions, including integrals, partial derivatives, and matrix operations, typical of advanced engineering and applied mathematics literature.
\[ K_{Gy} = \int_{\Omega} \frac{\partial N_i^T}{\partial y} \frac{\partial N_j}{\partial y} d\Omega, \]
\[ K_{Gxy} = 2 \int_{\Omega} \frac{\partial N_i^T}{\partial x} \frac{\partial N_j}{\partial y} d\Omega, \]

The buckling analysis of thin plates to the following generalized eigenvalue problem [3]
\[ Kd = \lambda K_G d, \]
where \(\lambda\) is a critical buckling load and \(d\) the eigenvector associated with \(\lambda\), with \((\lambda, d)\) known as a buckling mode shape.

5. Benchmark tests
In the benchmark tests, those typical numerical examples concerning thin plate of complicated shape are mainly studied, which help to unveil the favorable features of the GMLS-based NMM. And for numerical examples presented in this section, the material properties are given as: Young’s modulus \(E = 2 \times 10^{11} \text{ N/m}^2\), Poisson’s ratio \(\nu = 0.3\). The relative values of the buckling load are needed for convergence evaluation as
\[ \varepsilon_w = \left| \frac{k - k_{anal}}{k_{anal}} \right|, \]

5.1. Simply-supported circular plate
A simply-supported circular plate shown in subjected to radial compressive forces is considered. The geometry parameters are: \(R = 5\) and thickness \(h = 0.05\). The analytical critical buckling factor is defined as \(k = \lambda_{cr} R^2/D_0\), and for simply-supported circular plate, \(k = 4.196\). The convergence behavior of the critical buckling load is investigated by calculating the relative error of the buckling factor versus the increased number of physical nodes in Fig. 3. And the first eight buckling mode shapes are depicted in Fig. 2. Seen from the Fig. 3, as the number of physical nodes increases, the buckling factor converges to the analytical solution. And the buckling mode shapes in Fig. 4 predicted by proposed GMLS-based NMM agrees well with those predicted by Wang in [31].

Fig. 2 The first eight buckling mode shapes for the circular plate assessed by GMLS-based NMM.
5.2. Annular plate with outer edge clamped and inner edge simply-supported
The presence of a hole in a plate panel changes the stress distribution in the member, alters its elastic buckling characteristics and reduces its ultimate load carrying capacity, so the buckling of plate with cutouts needs specific attentions. In this article we studied the circular with a concentric circular hole, whose analytical solution was obtained by Yamaki [32, 33]. The buckling factor $k = \lambda_{cr} a^2 / D_0$ is calculated, with $D_0 = Eh^3 / [12(1 - \nu^2)]$ the flexural rigidity. The geometry parameters: outer edge radius $a = 5$, inner edge $b = 2.5$. And the analytical critical buckling factor is 88.7364 [32]. Thickness is $h = 0.05$.

Accordingly, the relative error of the critical buckling factor is Fig. 5. It is concluded that as the physical nodes number increases, the critical buckling factor converges to the analytical buckling factor fast. And it can be seen that the proposed method produces favorable solution accuracy. The first eight buckling mode shapes for the annular plate assessed by GMLS-based NMM are plotted in Fig. 4.

---

**Fig. 3** The relative error of critical buckling factor $k$ versus the selected number of physical nodes.

**Fig. 4** The first eight buckling mode shapes for the annular plate assessed by GMLS-based NMM.
Fig. 5 The relative error of critical buckling factor $k$ versus the selected number of physical nodes.

6. Discussions and conclusions
The application of GMLS-based NMM method have been investigated with a series of numerical examples in buckling analysis. The numerical results thus validate the effectiveness and accuracy of the proposed method in solving the buckling of thin plate problems. The numerical results obtained demonstrate that the GMLS-NMM method gains favorable and agreeable solutions for the thin plate analysis, which will benefit the engineering analysis of complicated shape plates free from auto-refinement and plate with cutouts.

Acknowledge
This study is supported by the National Basic Research Program of China (973 Program), under the Grant No. 2014CB047100; and the National Natural Science Foundation of China, under the Grant Nos. 11172313 and 51538001.

References
[1] Ventsel E, Krauthammer T. Thin plates and shells: theory: analysis, and applications: CRC press; 2001.
[2] Bathe K-J. Finite element procedures: Klaus-Jurgen Bathe; 2006.
[3] Hughes TJ. The finite element method: linear static and dynamic finite element analysis: Courier Corporation; 2012.
[4] Brebbia CA, Dominguez J. Boundary elements: an introdutory course: WIT press; 1994.
[5] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques: theory and applications in engineering: Springer Science & Business Media; 2012.
[6] Liu GR. Meshfree Methods: Moving Beyond the Finite Element Method, Second Edition. Crc Press 2009.
[7] Li S, Liu WK. Meshfree particle methods: Springer Science & Business Media; 2007.
[8] Belytschko T, Krongauz Y, Organ D, Fleming M, Krysl P. Meshless methods: An overview and recent developments. Computer Methods in Applied Mechanics and Engineering 1996; 139:3-47.
[9] Hughes TJR, Cottrell JA, Bazilevs Y. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. Computer Methods in Applied Mechanics and Engineering 2005; 194:4135-95.
[10] Nguyen VP, Anitescu C, Bordas SPA, Rabczuk T. Isogeometric analysis: An overview and computer implementation aspects. Mathematics and Computers in Simulation 2015; 117:89-
116.
[11] Beer G, Bordas S. Isogeometric Methods for Numerical Simulation: Springer; 2015.
[12] SHI G. Manifold Method of Material Analysis. Transactions of the 9th Army Conference On Applied Mathematics and Computing, 1991: US Army Research Office; 1991.
[13] Zheng H, Li W, Du X. Exact imposition of essential boundary condition and material interface continuity in Galerkin-based meshless methods. International Journal for Numerical Methods in Engineering 2017; 110:637-66.
[14] Ma G, An X, He L. The numerical manifold method: a review. International Journal of Computational Methods 2010; 7:1-32.
[15] Zheng H, Xu D. New strategies for some issues of numerical manifold method in simulation of crack propagation. International Journal for Numerical Methods in Engineering 2014; 97:986-1010.
[16] Yang Y, Zheng H. Direct Approach to Treatment of Contact in Numerical Manifold Method. International Journal of Geomechanics 2016; E4016012.
[17] Zheng H, Liu F, Du X. Complementarity problem arising from static growth of multiple cracks and MLS-based numerical manifold method. Computer Methods in Applied Mechanics and Engineering 2015; 295: 150-71.
[18] Zheng H, Liu F, Li C. The MLS-based numerical manifold method with applications to crack analysis. International Journal of Fracture 2014; 190:147-66.
[19] Zhang Y, Liu D, Tan F. Numerical manifold method based on isogeometric analysis. Science China Technological Sciences 2015; 58:1520-32.
[20] Rong T. Finite-cover based element-free method for continuous and discontinuous deformation analysis with applications in Geotechnical Engineering: Ph. D. Thesis. Dalian University of Technology, Dalian, China Google Scholar; 2001.
[21] Liu X, Zhu DM, Lu MW, Zhang X. Study on meshless method based on manifold cover ideas. Chinese Journal of Computational Mechanics 2001;18.
[22] Luan M-t, Zhang D-l, Yang Q, Tian R. Applications of the finite-cover element-free method in numerical analyses of crack-expansion. Chinese journal of geotechnical engineering-Chinese edition, 2003;25:527-31.
[23] Luan M, Yang X, Tian R, Yang Q. Numerical analysis on progressive fracture behavior by using element-free method based on finite covers. International Journal of Computational Methods 2005; 2:543-53.
[24] Li SC, Li SC, Cheng YM. Enriched meshless manifold method for two-dimensional crack modeling. Theoretical & Applied Fracture Mechanics 2005; 44:234-48.
[25] Su F, Cai Y. Simulation of crack propagation by MSIM method. Zhongnan Daxue Xuebao 2014; 45:2360-8.
[26] Zheng H, Liu F, Li C. Primal mixed solution to unconfined seepage flow in porous media with numerical manifold method. Applied Mathematical Modelling 2015; 39:794-808.
[27] Atluri NS, Cho YJ, Kim H-G. Analysis of thin beams, using the meshless local Petrov–Galerkin method, with generalized moving least squares interpolations. Computational Mechanics 1999; 24:334-47.
[28] Xie D, Jian K, Wen W. Global interpolating meshless shape function based on generalized moving least-square for structural dynamic analysis. Applied Mathematics and Mechanics-english Edition 2016; 37:1153-76.
[29] Liu GR, Quek SS. The finite element method: A practical course: Butterworth-Heinemann; 2003.
[30] Liu F, Xia K. Structured mesh refinement in MLS-based numerical manifold method and its application to crack problems. Engineering Analysis with Boundary Elements 2017; 84:42-51.
[31] D. W, H. P. A Hermite reproducing kernel Galerkin meshfree approach for buckling analysis of thin plates. Comput Mech 2013; 51:1013.
[32] Yamaki N. Buckling of a thin annular plate under uniform compression. Journal of Applied Mechanics 1958; 25:267-73.
[33] Wang C, Wang CY. Exact solutions for buckling of structural members: CRC press; 2004.