Title: Congruence Preservation, Lattices and Recognizability
Patrick Cegielski & Serge Grigorieff & Irène Guessarian

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References
Congruence Preservation, Lattices and Boolean Algebras

Patrick Cégielski\textsuperscript{1}  
Serge Grigorieff\textsuperscript{2}  
Irène Guessarian\textsuperscript{2,3}

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Abstract

We study in general algebras Grätzer’s notion of congruence preserving function, characterizing functions in terms of stability under inverse image of particular Boolean algebras of subsets generated from any subset of the algebra. Weakening Grätzer’s notion to only finite index congruences, a similar result holds with lattices of sets. Generalizing the notion to that of stable preorder preserving function, we extend these characterizations to Boolean algebras and lattices generated from any recognizable subset of the algebra. Our starting point is a result with related flavor on the additive algebra of natural integers which was obtained some years ago. All these results can be visualized in the diagram of Table 1. We finally consider some simple particular conditions on the algebra allowing to get a richer diagram.

Keywords: Congruence preservation, Lattice, Recognizable set.

1 Introduction

The aim of this paper is to tackle the following question.

Question 1.1. Is it possible to view a particular result about the particular semiring \((\mathbb{N}; +, \times)\) (stated as Theorem 5.1 in [2]) as an instance of a result about general algebraic structures?

1.1 Simple notions involved in the particular result on \((\mathbb{N}; +, \times)\)

The notion of recognizability was originally introduced in the context of monoids of words, cf. Schützenberger, 1956 [19] p. 11, then in general algebras by Mezei & Wright, 1967 [15] (Definition 4.5, p. 22).

Definition 1.2. A subset \(X\) of an algebra \(A\) is \(A\)-recognizable if \(X = \varphi^{-1}(Z)\) for some morphism \(\varphi: A \to B\) into a finite algebra and some subset \(Z \subseteq B\).

\textsuperscript{1}Emeritus at LACL, EA 4219, Université Paris-Est Créteil, IUT Sénart-Fontainebleau, France cegiel@u-pec.fr.

\textsuperscript{2}Retired from Université Paris 7 Denis Diderot, France, seg@irif.fr, corresponding author.

\textsuperscript{3}Emeritus at Sorbonne Université ig@irif.fr.
Other used notions are about lattices.

**Definition 1.3.** 1. A lattice \( \mathcal{L} \) of subsets of a set \( E \) is a family of subsets of \( E \) such that \( L \cap M \) and \( L \cup M \) are in \( \mathcal{L} \) whenever \( L, M \) are in \( \mathcal{L} \).

2. A lattice \( \mathcal{L} \) of subsets of \( E \) is **standardly bounded** if \( \emptyset \) and \( E \) are in \( \mathcal{L} \).

**Definition 1.4.** A lattice \( \mathcal{L} \) is closed under inverse image by a function \( g: E \to E \) if whenever it contains a set \( X \) it also contains the set \( g^{-1}(X) = \{ z \in E \mid g(z) \in X \} \). In other words, \( \mathcal{L} \) is closed under the function \( g^{-1}: \mathcal{P}(E) \to \mathcal{P}(E) \), where \( \mathcal{P}(E) \) denotes the family of all subsets of \( E \).

**Notation 1.5.** Given a set \( L \subseteq \mathbb{N} \), we denote \( \mathcal{L}_{\text{Suc}}(L) \) the smallest lattice of subsets of \( \mathbb{N} \) which is closed under the decrement function \( \text{Suc}^{-1} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \) and such that \( L \in \mathcal{L} \).

We can now restate as Theorem A the result of our paper [2] (Theorem 5.1) with two minor modifications.

1. First, we replace the notion of rational subset of \( \mathbb{N} \) used in [2] by that of recognizable subset of \( \mathbb{N} \), the two notions being equivalent on free monoids (Kleene’s theorem, which we recall below in [3.3]).

2. We add an extra fifth condition \((\mathcal{L}_{\text{Suc}}(L))^{\text{rec}}\) which is a straightforward avatar of the first condition \((\text{Latt}_{\text{Suc}}(\forall L))^{\text{rec}}\) and is similar to those dealing with finite sets and with arithmetic progressions.

**Theorem A.** Let \( f: \mathbb{N} \to \mathbb{N} \). The following conditions are equivalent.

\[
\begin{align*}
(\text{Latt}_{\text{Suc}})_{\mathbb{N}}^{\text{rec}} & \quad \text{ Every lattice } \mathcal{L} \text{ of recognizable subsets of } \mathbb{N} \text{ closed under decrement, i.e. under } \text{Suc}^{-1}, \text{ is also closed under } f^{-1} \\
(\mathcal{L}_{\text{Suc}}(\forall L))_{\mathbb{N}}^{\text{rec}} & \quad \mathcal{L}_{\text{Suc}}(L) \text{ is closed under } f^{-1} \text{ for every recognizable subset } L \subseteq \mathbb{N} \\
(\mathcal{L}_{\text{Suc}}(\forall L))_{\mathbb{N}}^{\text{lin}} & \quad \mathcal{L}_{\text{Suc}}(L) \text{ is closed under } f^{-1} \text{ for every finite subset } L \subseteq \mathbb{N} \\
(\mathcal{L}_{\text{Suc}}(\forall L))_{\mathbb{N}}^{\text{arith}} & \quad \begin{cases} 
\mathcal{L}_{\text{Suc}}(L) \text{ is closed under } f^{-1} \text{ for every arithmetic progression } L = q + r\mathbb{N} \text{ with } r > 0 \\
(a) \ y - x \text{ divides } f(y) - f(x) \text{ for all } x, y \in \mathbb{N}, \\
(b) \ f(x) \geq x \text{ for all } x \in \mathbb{N}, \\
(c) \ f \text{ is monotone non-decreasing}
\end{cases}

\text{(Arith)}^{(abc)} & \quad \text{Note.} \text{ The reason for the a priori strange notation } (\text{Arith})^{(abc)} \text{ is because we also consider variant of clause (b), namely } (\text{Arith})^{(ab)} \text{ and } (\text{Arith})^{(abc)}. \end{align*}
\]

Let’s just prove the equivalence between the first condition – which comes from [2] – and the added fifth condition \((\mathcal{L}_{\text{Suc}}(\forall L))_{\mathbb{N}}^{\text{rec}}\).

**Proof of** \((\text{Latt}_{\text{Suc}})_{\mathbb{N}}^{\text{rec}} \iff (\mathcal{L}_{\text{Suc}}(\forall L))_{\mathbb{N}}^{\text{rec}}\).

For the left to right implication, observe that \( \mathcal{L}_{\text{Suc}}(\forall L) \) consists of recognizable sets since they are \((\cup, \cap)\)-combinations of sets \( \text{Suc}^{-n}(L) \), with \( n \in \mathbb{N} \), and recognizability is preserved under \( \text{Suc}^{-1} \), union and intersection.

For the right to left implication, let \( \mathcal{L} \) be as in condition \((\text{Latt}_{\text{Suc}})_{\mathbb{N}}^{\text{rec}} \) and \( L \in \mathcal{L} \). By condition \((\mathcal{L}_{\text{Suc}}(\forall L))_{\mathbb{N}}^{\text{rec}} \), we get \( f^{-1}(L) \in \mathcal{L}_{\text{Suc}}(L) \), and inclusion \( \mathcal{L}_{\text{Suc}}(L) \subseteq \mathcal{L} \) insures that \( f^{-1}(L) \in \mathcal{L} \). \( \square \)
Remark 1.6. The successor function being such an elementary function, it is tempting to imagine that the equivalent conditions of Theorem A entail that \( f \) should itself be quite simple... but it turns out that this is far to be the case.

Indeed, functions satisfying clause \((a)\) of condition \((\text{Arith}^{(abc)})\) in Theorem A are characterized in our paper [3], and some of these functions \( \mathbb{N} \rightarrow \mathbb{N} \) are quite surprising, cf. Theorems 3.1 & 3.13 & Example 3.14 in [3] (as usual, \( \lfloor z \rfloor \) is the integral part of a real \( z \), and \( e \) is Euler number, and \( \cosh \) and \( \sinh \) are the hyperbolic cosine and sine functions), for instance

\[
x \mapsto \begin{cases} 
1 & \text{if } x = 0 \\
\lfloor e^{x!} \rfloor & \text{if } x \geq 1
\end{cases}
\]

\[
x \mapsto \begin{cases} 
\lfloor \text{cosh}(1/2)2^x x! \rfloor & \text{if } x \in 2\mathbb{N} \\
\lfloor \text{sinh}(1/2)2^x x! \rfloor & \text{if } x \in 2\mathbb{N} + 1 
\end{cases}
\]

\[
x \mapsto \lfloor e^{1/a} a^x x! \rfloor \quad \text{with } a \in \mathbb{Z} \setminus \{0, 1\}
\]

In particular, since \( e = 2.718... \), the first of these functions is monotone non-decreasing and also satisfies clause \((b)\) of condition \( \text{Arith} \), hence it satisfies all conditions in Theorem A.

In the ring \( (\mathbb{Z}; +, \times) \) the analog of condition \((\text{Arith})\) is also be satisfied by very intricate functions, for instance, cf. Theorem 26 in our paper [4]:

\[
f(n) = \begin{cases} 
\sqrt{\pi} \times 2 \times 4^n \times n! & \text{for } n \geq 0 \\
f(\lfloor n/2 \rfloor) & \text{for } n < 0
\end{cases}
\]

1.2 What is in the paper?

§ 2 presents tools used in this paper, most of them being classical ones.

Condition \((\text{Arith}^{(abc)})\) of Theorem A being specific to arithmetic on \( \mathbb{N} \), in order to get a version of this theorem expressible in general algebras, we need to modify it via some general algebraic notions. It turns out that convenient notions are those of congruence preservation, stable-preorder preservation and recognizability. The first two notions are recalled in § 2.1.

Recognizability is recalled in § 2.2.

We explicit in § 2.3 the freezeification of operations in a general algebra. This essentially reduces the operations of the algebra to unary ones as concerns what is studied in this paper. This allows to introduce in § 2.4 the syntactic congruence and the syntactic stable preorder of a subset of any general algebra, not only monoids contrarily to what is always done, cf. [11 17 18].

§ 3 illustrates the notions of § 2 in the algebras \( (\mathbb{N}, \text{Suc}, \{0,+\}) \), and \( (\mathbb{N}, +, \times) \).

Folklore characterization of congruences and recognizability are recalled in § 3.1 and § 3.2. Kleene’s celebrated theorem about recognizability in free monoids (which we use to go from our result in [2] to Theorem A) is recalled in § 3.3.

In § 3.4 and § 3.5 condition \((\text{Arith}^{(abc)})\) of Theorem A – which is obviously specific to arithmetic on \( \mathbb{N} \) – is related to the notions of congruence and stable preorder preservation in \( \mathbb{N} \), (cf. Theorem 3.7 and Theorem 3.10). These results give the key idea for extensions of Theorem A in a general algebra: replace condition \((\text{Arith}^{(abc)})\)
which is specific to \( \mathbb{N} \) – by a condition of preservation of congruences and stable preorders which makes sense in any algebra. These results also illustrate the general algebraic fact that stable-preorder preservation may be strictly more demanding than congruence preservation, cf. §3.6.

In §4 the notions of congruence and stable preorder preservation in an algebra are related to conditions on lattices and Boolean algebras much similar to those of Theorem A. This is done via an explicit representation of \( f^{-1}(L) \) – in terms of the inverse images \( \gamma^{-1}(L) \)'s of the freezifications – which leads to a lattice condition for stable preorder preservation, and to a Boolean algebra condition for congruence preservation cf. §4.2 and §4.3 and Theorem 4.9. These conditions involve lattices and Boolean algebras which are bounded when \( L \) is recognizable (which is always the case for Boolean algebras) and have to be complete when \( L \) is not recognizable.

In §5 we consider diverse hypothesis on the algebra to improve Theorem 4.9, replacing some or all implications by logical equivalences in the figure of Table 1.

2 Algebraic tools used in this paper

2.1 Grätzer notion of congruence preservation in general algebras

The first condition of Theorem A is related to the algebraic notion of congruence preservation introduced by Grätzer, 1962 [8] (under the name substitution property).

Let’s recall this notion, together with a natural extension to stable-preorder preservation which, for non-constant functions on \( \mathbb{N} \), happens to be equivalent to condition (\( \text{Arith} \)) in Theorem A, cf. Theorem 3.7 and Theorem 3.10.

Notation 2.1. We denote \( \mathcal{A} = \langle A; \Xi \rangle \) the algebra consisting of the nonempty carrier set \( A \) equipped with a set of operations \( \Xi \), each \( \xi \in \Xi \) being a mapping \( \xi: A^{ar(\xi)} \rightarrow A \), where \( ar(\xi) \in \mathbb{N} \) is the arity of \( \xi \).

Definition 2.2. Let \( \mathcal{A} = \langle A; \Xi \rangle \) be an algebra.
1. A binary relation \( \rho \) on \( A \) is said to be compatible with a function \( f: A^p \rightarrow A \) if, for all elements \( x_1, \ldots, x_p, y_1, \ldots, y_p \) in \( A \), we have
   \[
   (x_1 \rho y_1 \wedge \cdots \wedge x_p \rho y_p) \implies f(x_1, \ldots, x_p) \rho f(y_1, \ldots, y_p).
   \] (1)

2. A binary relation \( \rho \) on \( A \) is said to be \( \mathcal{A} \)-stable if it is compatible with each operation \( \xi \in \Xi \), i.e., (1) holds with \( \xi \) in place of \( f \) for every \( \xi \in \Xi \). In particular, \( \mathcal{A} \)-stable equivalence relations on \( A \) are the \( \mathcal{A} \)-congruences.

3. A function \( f: A^p \rightarrow A \) is \( \mathcal{A} \)-congruence preserving if all \( \mathcal{A} \)-congruences are compatible with \( f \), i.e. (1) holds for every \( \mathcal{A} \)-congruence \( \rho \).

4. Recall that a preorder is a reflexive and transitive relation. A function \( f: A^p \rightarrow A \) is \( \mathcal{A} \)-stable-preorder preserving if all \( \mathcal{A} \)-stable-preorders are compatible with \( f \), i.e. (1) holds for every \( \mathcal{A} \)-stable-preorder \( \rho \).

As usual, when the algebra \( \mathcal{A} \) is clear from the context, \( f \) is simply said to be congruence preserving (resp. preorder preserving).
Remark 2.3. Clearly, any composition of functions in $\Xi$ preserves all $A$-stable preorders of $A = \langle A; \Xi \rangle$.

Since every congruence is a stable preorder, we get.

Lemma 2.4. If $f$ is $A$-stable preserving then it is also $A$-congruence preserving.

Remark 2.5. The converse of Lemma 2.4 is false: stable-preorder preservation is a strictly stronger condition than congruence preservation since Theorem 3.10 and Proposition 3.12 below prove that there exist functions which preserve $\langle \mathbb{N}; + \rangle$-congruences but not all $\langle \mathbb{N}; + \rangle$-stable-preorders.

We shall also need the following classical notion.

Definition 2.6. The index of an $A$-congruence is the number (finite or infinite) of equivalence classes. The index of an $A$-stable-preorder $\sqsubseteq$ is that of the associated $A$-congruence $\{(x, y) \mid x \sqsubseteq y \text{ and } y \sqsubseteq x\}$.

2.2 Recognizable subsets of an algebra

Since congruences are kernels of morphisms, Definition 1.2 can be expressed in terms of congruences.

Lemma 2.7. A subset $L$ of an algebra $A$ is $A$-recognizable if and only if it is saturated for some finite index $A$-congruence (i.e. $L$ is a union of equivalence classes).

We shall use the following straightforward result.

Lemma 2.8. 1. The family of recognizable subsets of $A$ is a Boolean algebra.

2. If $L \subseteq B$ is $B$-recognizable and $\varphi : A \to B$ is a morphism then $\varphi^{-1}(L)$ is $A$-recognizable.

We shall also use the following simple result.

Lemma 2.9. Let $\sqsubseteq$ be a finite index $A$-stable preorder on an algebra $A$. Every $\sqsubseteq$-initial segment $L$ (i.e. whenever $a \in L$ and $b \sqsubseteq a$ then $b \in L$) is $A$-recognizable.

Proof. Observe that any initial segment of $\sqsubseteq$ is saturated for the associated congruence which has finite index. \qed

2.3 Freezifications

The notion of 1-freezification was not introduced for our immediate problem but to characterize congruence preservation of a function $f$ of arbitrary arity via congruence preservation of suitably chosen unary functions related to the function $f$. It goes back to Slomiński, 1974, cf. Definition 1 in [20].

This notion allows to show that, relative to congruences, stable preorders and surjective homomorphisms, one can replace the operations of an algebra by unary operations, namely their freezifications.
Definition 2.10. 1. Let \( f : A^n \to A \). If \( n = 1 \) the sole 1-freezification of \( f \) is \( f \) itself. If \( n \geq 2 \) the 1-freezifications of \( f : A^n \to A \) are the unary functions

\[
f^{[i,\bar{c}]}(x) = f(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n)
\]

for \( i \in \{1, \ldots, n\} \) and \( \bar{c} = (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \in A^{n-1} \), i.e., all but one argument of \( f \) are frozen. The set of 1-freezifications of \( f \) is denoted \( \text{Frz}(f) \).

2. If \( \mathcal{F} \) is a family of operations over a set \( A \) with arbitrary arities, the 1-freezification \( \text{Frz}(\mathcal{F}) \) of \( \mathcal{F} \) is the family of all 1-freezifications of operations in \( \mathcal{F} \).

3. A composition of a finite sequence of 1-freezifications is called a \( * \)-freezification; in particular, the composition of an empty sequence is the identity on \( A \). The family of all \( * \)-freezifications of all operations in \( \mathcal{F} \) is denoted by \( \text{Frz}^*(\mathcal{F}) \).

4. To \( \mathcal{A} = (A; \Sigma) \) we associate two related algebras \( \text{Frz}(\mathcal{A}) \) and \( \text{Frz}^*(\mathcal{A}) : \)

\[
\text{Frz}(\mathcal{A}) = (A; \text{Frz}(\Sigma)) \quad \text{and} \quad \text{Frz}^*(\mathcal{A}) = (A; \text{Frz}^*(\Sigma)).
\]

![Figure 1: Examples of \( \text{Frz}(\Sigma) \), \( \text{Frz}^*(\Sigma) \)]

Example 2.11. Figure 1 gives algebras \( \text{Frz}(\mathcal{A}) \) and \( \text{Frz}^*(\mathcal{A}) \) for some algebras with carrier set \( \mathbb{N} \) and for free monoids with the operation \( \cdot \) of concatenation.

Remark 2.12. In general, neither \( \text{Frz}(\mathcal{A}) \) nor \( \text{Frz}^*(\mathcal{A}) \) coincide with what are usually called “affine” or “polynomial” functions of the algebra (which should more adequately be called “term” functions).

For instance, with \( (\mathbb{N}; +), (\mathbb{N}; \cdot), \) and \( (\mathbb{N}; +, \cdot) \) the usual “affine” or “polynomial” functions are respectively the usual affine functions, monomial functions \( x \mapsto nx^p \), and the polynomial functions in \( \mathbb{N}[x] \).

With \( (\Sigma^*, \cdot) \) the usual “polynomial” functions are the functions of the form \( x \mapsto u_0xu_1xu_2 \ldots xu_m \), the \( u_i \)'s being arbitrary words. If there are several occurrences of \( x \) in such a polynomial then it is neither in \( \text{Frz}(\Sigma^*, \cdot) \) nor in \( \text{Frz}^*((\Sigma^*, \cdot)) \).

The following result is inspired by the core of Theorem 1 in Słomiński [20] (trivially extending it from equivalences to preorders). It shows that, relative to congruences, stable preorders, surjective homomorphisms, recognizability, one can replace the operations of the algebra by unary operations, namely their freezifications.
Lemma 2.13. A preorder \( \leq \) on \( A \) is compatible with \( f : A^n \to A \) if and only if it is compatible with all its 1-freezifications. In particular, the three algebras \( \mathcal{A}, \text{Frz}(\mathcal{A}) \), and \( \text{Frz}^*(\mathcal{A}) \) have the same congruences and the same stable preorders, hence the same functions preserving congruences or stable preorders and the same recognizable sets.

Remark 2.14. This possibility of reduction to arity 1 algebraic operations for key notions relative to algebras somewhat reminds of the fact that in the framework of real analysis unary continuous functions and addition suffice to get all continuous functions of any arity, more precisely \( f \) of real analysis unary continuous functions and addition suffice to get all continuous functions preserving congruences or stable preorders and the same recognizable sets.

Proof of Lemma 2.13. The left to right implication is a trivial use of the reflexivity of \( \leq \). The right to left implication uses the transitivity of \( \leq \). Indeed, if \( \leq \) is compatible with the 1-freezifications

\[
\begin{align*}
&f([1,a_2,\ldots,a_n], f[2,b_1,a_2,\ldots,a_n], f[3,b_1,b_2,a_4,\ldots,a_n], \ldots, f[n-1,b_1,b_2,\ldots,a_n], f[n,b_1,b_2,\ldots,b_{n-1}])
\end{align*}
\]

and if \( a_1 \leq b_1, \ldots, a_n \leq b_n \) we have

\[
\begin{align*}
f(a_1, a_2, \ldots, a_n) &= f[1,a_2,\ldots,a_n](a_1) \leq f[1,a_2,\ldots,a_n](b_1) = f(b_1, a_2, \ldots, a_n) \\
&= f[2,b_1,a_3,\ldots,a_n](a_2) \leq f[2,b_1,a_3,\ldots,a_n](b_2) = f(b_1, b_2, a_3, \ldots, a_n) \\
& \vdots \\
&= f[n,b_1,b_2,\ldots,b_{n-1}](a_n) \leq f[n,b_1,b_2,\ldots,b_{n-1}](b_n) = f(b_1, b_2, \ldots, b_{n-1}, b_n)
\end{align*}
\]

hence \( \leq \) is compatible with \( f \).

Finally, let’s complete Lemma 2.8 as follows.

Lemma 2.15. The family of recognizable subsets of \( \mathcal{A} \) is closed under the \( \gamma^{-1}(L) \)’s for \( \gamma \in \text{Frz}^*(+A) \).

Proof. If \( L \) is saturated for a finite index congruence \( \sim \), and \( x \in \gamma^{-1}(L) \), i.e. \( \gamma(x) \in L \), the condition \( x \sim y \) entails \( \gamma(x) \sim \gamma(y) \) hence \( \gamma(y) \in L \) since \( L \) is saturated, i.e. \( y \in \gamma^{-1}(L) \). Thus, \( \gamma^{-1}(L) \) is also saturated for \( \sim \). Therefore it is also recognizable.

2.4 Syntactic preorder and congruence of a subset of an algebra

The notion of syntactic congruence first appeared in Schützenberger, 1956 [19] p. 11, in the framework of the monoid of words with concatenation over some alphabet (see also Mezei & Wright, 1967 [15], Słomiński [20]). It was extended to general monoids in an obvious way (see Almeida [1], Pin, 2022 [18]).

The syntactic preorder on monoids appears as such in Pin, 1995 [17].

Using freezifications, we present these notions in the framework of any algebra.
Definition 2.16. Let $A = \langle A; \Xi \rangle$ be an algebra. With any $L \subseteq A$ are associated two relations $\leq_L$ and $\sim_L$, called syntactic preorder and syntactic congruence of $L$:

\[
\begin{align*}
\text{x} \leq_L \text{y} & \quad \text{if and only if} \; \forall \gamma \in Frz^*(\mathcal{A}) \; (\gamma(y) \in L \implies \gamma(x) \in L) \\
\text{x} \sim_L \text{y} & \quad \text{if and only if} \; \forall \gamma \in Frz^*(\mathcal{A}) \; (\gamma(y) \in L \iff \gamma(x) \in L)
\end{align*}
\]

Lemma 2.17. 1. Relations $\leq_L$ and $\sim_L$ are an $A$-stable preorder and an $A$-congruence.

2. The congruence $\{ (x, y) \mid x \leq_L y \text{ and } y \leq_L x \}$ is $\sim_L$.

Proof. 1. Apply Lemma 2.16 to all operations in $\Xi$. Item 2 is obvious.

Example 2.18. Let $\mathcal{N} = \langle \mathbb{N}, \text{Suc} \rangle$. The syntactic congruence of $L = a + k\mathbb{N}$ satisfies

\[
x \sim_L y \iff \forall n \in \mathbb{N} \; (x + n \in (a + k\mathbb{N}) \iff y + n \in (a + k\mathbb{N}))
\]

Observe that $x \sim_L y$ implies $x \equiv y \mod k$ since, letting $n$ be such that $x + n \in a + k\mathbb{N}$, we then have $y + n \in a + k\mathbb{N}$ hence $y - x \in (a + k\mathbb{N}) - (a + k\mathbb{N}) = k\mathbb{Z}$.

We show that $\sim_L$ coincides with $\equiv_{\max(0,a-k+1),k}$ (defined by $\equiv$ in Lemma 3.4). We reduce to prove that they coincide on pairs $(x, y)$ such that $x < y$.

- Assume $x \equiv_{\max(0,a-k+1),k} y$. We prove that $x \sim_L y$. Since $x < y$, we have $max(0,a-k+1) \leq x$ and $x \equiv y \mod k$. Implication $x + n \in (a + k\mathbb{N}) \Rightarrow y + n \in (a + k\mathbb{N})$ is clear since $x \equiv y \mod k$ and $x < y$. In case $y + n \in (a + k\mathbb{N})$, since $x \equiv y \mod k$ we have $x + n = a + kZ$ with $z \in \mathbb{Z}$. Now, $x + n \geq max(0,a-k+1)$ hence $a + kZ \geq a - k + 1$, i.e. $k(z + 1) \geq 1$ hence $k \in \mathbb{N}$ and $x + n \in a + k\mathbb{N}$. This proves $x \sim_L y$.

- Assume $x \not\equiv_{\max(0,a-k+1),k} y$. We prove that $x \not\sim_L y$. Since $x < y$ there are two possible reasons for $x \not\equiv_{\max(0,a-k+1),k} y$.

Case $x \equiv y \mod k$ and $x < max(0,a-k+1)$. Let $p \geq 1$ be such that $y = x + kp$. Since $x \geq 0$ we must have $a - k + 1 \geq 1$, i.e. $a \geq k$, and also $x < a - k + 1$, i.e. $a - x > k$. Let $n = a - x - k \in \mathbb{N}$. We have $x + n = a - k \not\in a + k\mathbb{N}$. Now, $y + n = x + n + pk = a + k(p-1) \in a + k\mathbb{N}$. Thus, $x \not\sim_L y$. This concludes the proof that $\sim_L$ and $\equiv_{max(0,a-k+1),k}$ coincide.

Proposition 2.19. Let $L$ be a subset of an algebra $\mathcal{A}$.

1. $L$ is $\sim_L$-saturated and is an $\leq_L$-initial segment.

2. If $L$ is saturated for a congruence $\equiv$ of $\mathcal{A}$ then $\equiv$ refines the syntactic congruence $\sim_L$ of $L$, i.e., $x \equiv y$ implies $x \sim_L y$. In particular, $L$ is recognizable if and only if $\sim_L$ has finite index and if and only if $\leq_L$ has finite index.

3. If $L$ is an initial segment of a stable preorder $\leq$ of $\mathcal{A}$ then $\leq$ refines the syntactic preorder $\leq_L$ of $L$, i.e., $x \leq y$ implies $x \leq_L y$.

Proof. 1. Let $\gamma$ be the identity in $(\mathcal{I})$.

2. The usual proof in textbooks for the case of monoids (e.g. Almeida [1] Sections 0.2, 3.1, or Pin [2], chap. IV, §4.1, 5.3) extends to any algebra using freezifications. Suppose $x \equiv y$. Since $\equiv$ is an $\mathcal{A}$-congruence, for all $\gamma \in Frz^*(\mathcal{A})$ we have $\gamma(x) \equiv \gamma(y)$. Since $L$ is $\equiv$-saturated we have $\gamma(y) \in L \iff \gamma(x) \in L$. Being true for all $\gamma \in Frz^*(\mathcal{A})$, this means $x \sim_L y$. 3. Similar, initial segments replacing saturated sets.
Let’s state a straightforward consequence of item 1 of Proposition 2.19.

**Lemma 2.20.** Let is recognizable if and only if \( \overline{L} \) has finite index.

### 3 Case of the algebra \( \langle \mathbb{N}, \text{Suc}\rangle \) or \( \langle \mathbb{N}, + \rangle \) or \( \langle \mathbb{N}, +, \times \rangle \)

#### 3.1 Folklore characterization of congruences on \( \mathbb{N} \)

To show how Theorem A is an instance of our general results in §4, we need the classical characterization of congruences on \( \mathbb{N} \).

First, a straightforward observation.

**Lemma 3.1.** The three algebras \( \langle \mathbb{N}; \text{Suc}\rangle \), \( \langle \mathbb{N}; +\rangle \) and \( \langle \mathbb{N}; +, \times \rangle \) have the same stable preorders and the same congruences. A fortiori, they admit the same stable-preorder-preserving functions and the same congruence-preserving functions.

**Proof.** Stability of a relation \( \rho \subseteq \mathbb{N} \times \mathbb{N} \) under successor implies stability under addition: to deduce \( (x + z)\rho (y + z) \) from \( x\rho y \) merely apply \( z \) times stability under successor. Stability of a relation \( \rho \subseteq \mathbb{N} \times \mathbb{N} \) under addition implies stability under multiplication: straightforward induction on \( t \) since from \( x\rho y \) and \( (xt)\rho (yt + y) \) we deduce \( (xt + x)\rho (yt + y) \), that is \( x(t + 1)\rho (yt + 1) \).

**Remark 3.2.** A priori, \( \mathcal{A} \)-congruences and \( \mathcal{A} \)-stable-preorders strongly depend upon the signature of \( \mathcal{A} \). However, Lemma 3.1 witnesses that this is not always the case.

**Definition 3.3.** A congruence \( \sim \) is left cancellable if \( \xi(a, x) \sim \xi(a, y) \) implies \( x \sim y \) for each binary operation \( \xi \) of the algebra. Right cancellability and cancellability are similarly defined.

Congruences for the usual structures on \( \mathbb{N} \) are simply characterized, cf. Preston, 1963, [11] (Lemma 2.15 & Remark pp. 28-29), Grillet, 1995 [9] (Lemma 5.6 p. 19).

**Lemma 3.4.**

1. The congruences of the algebras \( \langle \mathbb{N}; \text{Suc}\rangle \), \( \langle \mathbb{N}; +\rangle \), and \( \langle \mathbb{N}; +, \times \rangle \) are the identity relation and the \( \equiv_{a,k} \)'s for \( a, k \in \mathbb{N}, k \geq 1 \), where

\[
\begin{align*}
x \equiv_{a,k} y \text{ if and only if } & \left\{ \begin{array}{l}
either x = y \\
or a \leq x \text{ and } a \leq y \text{ and } x \equiv y \pmod{k}
\end{array} \right.
\end{align*}
\]

Congruence \( \equiv_{a,k} \) has finite index \( a + k \), i.e. has \( a + k \) equivalence classes. It is cancellable if and only if \( a = 0 \).

2. Up to isomorphism, the quotient algebra \( \langle \mathbb{N}; \text{Suc}, +, \times \rangle / \equiv_{a,k} \) can be seen as the algebra \( \langle \{0, \ldots, a + k - 1\}; \equiv_{a,k}, \oplus, \otimes \rangle \) where

\[
\begin{align*}
\text{Suc}(x) &= \text{IF } \text{Suc}(x) < a + k \text{ THEN } \text{Suc}(x) \text{ ELSE } a \\
x \oplus y &= \text{IF } x + y < a + k \text{ THEN } x + y \text{ ELSE } a + ((x + y - a) \pmod{k}) \\
x \otimes y &= \text{IF } x \times y < a + k \text{ THEN } x \times y \text{ ELSE } a + ((x \times y - a) \pmod{k})
\end{align*}
\]
3.2 Folklore characterization of recognizable subsets of \( \mathbb{N} \)

First, observe the following immediate consequence of Lemma 3.1.

Lemma 3.5. The three algebras \( \langle \mathbb{N}; \text{Suc} \rangle, \langle \mathbb{N}; + \rangle, \) and \( \langle \mathbb{N}; +, \times \rangle \) have the same recognizable subsets.

Proof. By Lemma 3.1 they have the same congruences hence the same subsets saturated for some finite index congruence.

In case the monoid is \( \langle \mathbb{N}; + \rangle \), the characterization of recognizable sets dates back to Myhill, 1957 [10], cf. also Eilenberg [7], p. 101 Proposition 1.1. The published proofs are (as far as we know) always presented in terms of automata, we present it in terms of congruences.

Lemma 3.6. Let \( L \subseteq \mathbb{N} \). The following conditions are equivalent.

1. \( L \) is a recognizable subset of the monoid \( \langle \mathbb{N}, + \rangle \).
2. \( L \) is the union of a finite set with a finite (possibly empty) family of arithmetic progressions.
3. \( L \) is finite or \( L = A \cup (a + \Delta) + k\mathbb{N} \) with \( k \geq 2 \) and \( A \) a possibly empty subset of \( \{0, \ldots, k-1\} \) and \( \Delta \) a non-empty strict subset of \( \{0, \ldots, k-1\} \).

Proof. (1) \( \Rightarrow \) (3). Any recognizable subset \( L \) of \( \mathbb{N} \) is \( \sim_{a,k} \)-saturated for some \( a \in \mathbb{N} \) and \( k \geq 1 \), hence is of the form \( L = A \cup ((a + \Delta) + k\mathbb{N}) \) for some possibly empty sets \( A \subseteq \{0, \ldots, a-1\} \) and \( \Delta \subseteq \{0, \ldots, k-1\} \). If \( k = 1 \) we get \( L = A \) or \( L = A \cup (a + \mathbb{N}) \). If \( k \geq 2 \) observe that the equality \( \Delta = \{0, \ldots, k-1\} \) implies \( (a + \Delta) + k\mathbb{N} = a + \mathbb{N} \).

(3) \( \Rightarrow \) (2) is trivial.

(2) \( \Rightarrow \) (1). An arithmetic progression \( a + k\mathbb{N} \) is the \( \sim_{a,k} \) equivalence class of \( a \), hence is recognizable. A singleton set \( \{b\} \) is \( \sim_{a,1} \)-saturated for any \( a > b \), hence is recognizable. To conclude use closure under union of recognizability.

3.3 Kleene’s theorem: recognizable and rational sets in free monoids

First, recall a classical notion from theoretical computer science, cf. for instance Eilenberg & Schützenberger, 1969 [6], or Eilenberg, 1974 [7] p. 159 sq.).

Definition. Let \( \langle M, \cdot \rangle \) be a monoid with neuter element \( \varepsilon \). The family of rational (also called regular) subsets of \( M \) is the smallest family \( \mathcal{R} \) of subsets of \( M \) such that (1) the empty set and every singleton set is in \( \mathcal{R} \), (2) if \( P \) and \( Q \) are in \( \mathcal{R} \) then so are the union \( P \cup Q \), the product \( PQ = \{ u \cdot v \mid u \in P, v \in Q \} \), and the infinite union of powers (also called Kleene’s star) \( P^* = \bigcup_{n \in \mathbb{N}} P^n \), where \( P^0 = \{ \varepsilon \} \) and \( P^{n+1} = P^n \).

A celebrated theorem due to Kleene, 1956 [13], relates rationality and recognizability in a free monoid, i.e. (up to isomorphism) an algebra of words over some alphabet endowed with the concatenation operation, cf. Eilenberg [7] (p. 175 Theorem 5.1) or Almeida [1] or Pin [18] (p. 70).
Proposition (Kleene’s theorem). In free monoids, the notion of rational subset coincides with that of recognizable subset. In particular, this is true in the monoid \((\mathbb{N}, +)\) (integers being seen as words on a singleton alphabet).

3.4 \(\mathcal{N}\)-congruence preservation and arithmetic

Congruence preservation on \((\mathbb{N}, \text{Suc}), (\mathbb{N}, +), (\mathbb{N}, +, \times)\) can be simply characterized as follows with a weakening \((\text{Arith}^{(ab)})\) of condition \((\text{Arith}^{(abc)})\) of Theorem A in §111 which gives up the monotony of \(f\) (clause (c)) and weakens clause (b) by allowing \(f\) to be constant.

Theorem 3.7. For any \(f: \mathbb{N} \to \mathbb{N}\), the following conditions are equivalent

\[
\begin{align*}
(\text{CongPres})_\mathbb{N} & : f: \mathbb{N} \to \mathbb{N} \text{ is } \mathcal{N}\text{-congruence preserving}, \\
(\text{Arith}^{(ab)}) & : \\
& \begin{cases} 
(a) & |y - x| \text{ divides } |f(y) - f(x)| \text{ for all } x, y \in \mathbb{N} \\
(b)^1 & \text{either } f \text{ is constant or } f(x) \geq x \text{ for all } x
\end{cases}
\end{align*}
\]

Proof. \((\text{CongPres})_\mathbb{N} \Rightarrow (\text{Arith}^{(ab)})\). Suppose \(f\) is congruence preserving. Let \(x < y\) and consider the congruence modulo \(y - x\). As \(y \equiv x \mod y - x\) we have \(f(y) \equiv f(x) \mod y - x\), hence \(y - x\) divides \(f(y) - f(x)\). This proves clause (a) of condition \((\text{Arith}^{(ab)})\).

To prove clause \((b)^1\) of condition \((\text{Arith}^{(ab)})\), we assume that condition \(f(x) \geq x\) is not satisfied and show that \(f\) is constant. Let \(a\) be least such that \(f(a) < a\). Consider the congruence \(\equiv_{a, 1}\). We have \(a \equiv_{a, 1} y\) for all \(y \geq a\), hence \(f(a) \equiv_{a, 1} f(y)\). As \(f(a) < a\), this implies \(f(y) < a\) and \(f(a) = f(y)\). Thus, \(f(a) = f(y)\) for all \(y \geq a\).

Let \(z < a\). By clause (a) (already proved) we know that \(p\) divides \(f(z) - f(z + p)\) for all \(p\). Now, \(f(z + p) = f(a)\) when \(z + p \geq a\). Thus, \(f(z) - f(a)\) is divisible by all \(p \geq a - z\). This shows that \(f(z) = f(a)\). Summing up, we have proved that \(f\) is constant with value \(f(a)\).

\((\text{Arith}^{(ab)}) \Rightarrow (\text{CongPres})_\mathbb{N}\). Constant functions are trivially congruence preserving. We thus assume that \(f\) is not constant, hence \(f\) satisfies clause (a) of condition \((\text{Arith}^\flat)\) and \(f(x) \geq x\) for all \(x\). Consider a congruence \(\sim\) and suppose \(x \sim y\). If \(\sim\) is the identity relation then \(x = y\) hence \(f(x) = f(y)\) and \(f(x) \sim f(y)\). Else, by Lemma 3.4 the congruence \(\sim\) is some \(\equiv_{a,k}\) with \(a \in \mathbb{N}\) and \(k \geq 1\). In case \(y < a\) then condition \(x \equiv_{a,k} y\) implies \(x = y\), hence \(f(x) = f(y)\) and \(f(x) \sim f(y)\). In case \(y \geq a\) then condition \(x \equiv_{a,k} y\) implies \(x \geq a\) and \(x \equiv y \mod k, \text{ hence } k\) divides \(y - x\). Since clause (a) of condition \((\text{Arith}^{(ab)})\) ensures that \(y - x\) divides \(f(y) - f(x)\), we see that \(k\) also divides \(f(y) - f(x)\). Also, our hypothesis yields \(f(x) \geq x\) and \(f(y) \geq y\). As \(x, y \geq a\), we get \(f(x), f(y) \geq a\). Since \(k\) divides \(f(y) - f(x)\), we conclude that \(f(x) \equiv_{a,k} f(y)\), whence \((\text{CongPres})_\mathbb{N}\). \(\square\)

Remark 3.8. Clause (a) of condition \((\text{Arith}^{(ab)})\) does not imply clause \((b)^1\), hence cannot be removed in Theorem 3.7. For instance, let \(F: \mathbb{N} \to \mathbb{N}\) be the polynomial \(F(x) = x(x - 1) \ldots (x - k)\) with \(k \geq 1\). Since \(y - x\) divides \(y^n - x^n\) for any \(n \geq 1\),
developing \( F(x) \) in a sum of monomials, we see that it also divides \( F(y) - F(x) \). However, \( F \) is neither constant nor overlinear as \( F(i) = 0 < i \) for \( i = 1, \ldots, k \).

### 3.5 \( \mathcal{N} \)-stable-preorder preservation and arithmetic

We now look at what monotonicity adds to congruence preservation in \( \langle \mathbb{N}; \text{Suc} \rangle \), \( \langle \mathbb{N}; + \rangle \), and \( \langle \mathbb{N}; +, \times \rangle \), namely, it insures stable-preorder preservation. First, a simple observation.

**Lemma 3.9.** Let \( \preceq \) be a \( \langle \mathbb{N}; + \rangle \)-preorder and, for \( a \in \mathbb{N} \), let \( M^+_a = \{ x \mid a \preceq a + x \} \) and \( M^-_a = \{ x \mid a + x \preceq a \} \).

1. \( M^+_a \) and \( M^-_a \) are submonoids of \( \langle \mathbb{N}; + \rangle \).
2. If \( a \leq b \) then \( M^+_a \subseteq M^+_b \) and \( M^-_a \subseteq M^-_b \).

**Proof.**

1. Clearly, \( 0 \preceq a + x \) and \( a \preceq a + y \). By stability, the first inequality yields \( a + y \preceq a + x + y \) and, by transitivity, the second inequality gives \( a \preceq a + x + y \). Thus, \( M^+_a \) is a submonoid. Idem with \( M^-_a \).
2. Since \( b - a \in \mathbb{N} \), if \( a \preceq a + x \) then by stability \( a + (b - a) \preceq a + x + (b - a) \), i.e., \( b \preceq b + x \). Thus, \( M^+_a \subseteq M^+_b \). Similarly, \( M^-_a \subseteq M^-_b \). □

**Theorem 3.10.** Let \( \mathcal{N} \) be any of the algebras \( \langle \mathbb{N}; \text{Suc} \rangle \) or \( \langle \mathbb{N}; + \rangle \) or \( \langle \mathbb{N}; +, \times \rangle \). For any \( f : \mathbb{N} \to \mathbb{N} \), the following conditions are equivalent.

\[
\begin{align*}
\text{(PreordPres)}_{\mathcal{N}} & \quad f \text{ is } \mathcal{N}\text{-stable-preorder preserving} \\
\text{(CongPres)}_{\mathcal{N}} & \quad f \text{ is monotone non-decreasing and } \mathcal{N}\text{-congruence preserving} \\
\text{(Arith(ab,c))} & \quad \begin{cases} (a) & \text{for all } x,y \in \mathbb{N}, \ y - x \text{ divides } f(y) - f(x) \\ (b) & f \text{ is either constant or } f(x) \geq x \text{ for all } x \in \mathbb{N} \\ (c) & f \text{ is monotone non-decreasing.} \end{cases}
\end{align*}
\]

**Proof.** \( \text{(PreordPres)}_{\mathcal{N}} \Rightarrow \text{(CongPres)}_{\mathcal{N}} \). If \( f \) is stable preorder preserving then it is a fortiori congruence preserving. As the usual order \( \preceq \) is \( \mathcal{N}\)-stable, it is preserved by \( f \), hence \( f \) is monotone non-decreasing.

\( \text{(CongPres)}_{\mathcal{N}} \Rightarrow \text{(Arith(ab,c))} \). This is Theorem 3.7 for monotone non-decreasing \( f \)’s.

\( \text{(Arith(ab,c))} \Rightarrow \text{(PreordPres)}_{\mathcal{N}} \). Assume \( \text{(Arith(ab,c))} \). We prove that \( f \) preserves every stable preorder \( \preceq \). The case where \( f \) is constant is trivial. We now suppose \( f \) is not constant. Suppose \( a \leq b \), we argue by distinguishing two cases.

**Case** \( a \leq b \). Then \( b - a \in M^+_a = \{ x \mid a \leq a + x \} \). By clause \( a \) of condition \( \text{(Arith(ab,c))} \) we know that \( b - a \) divides \( f(b) - f(a) \). Now, \( f(b) - f(a) \geq 0 \) since \( f \) is monotone non-decreasing. As a consequence, \( f(b) - f(a) = n(b - a) \) for some \( n \geq 0 \). As \( b - a \in M^+_a \) and \( M^+_a \) is a monoid, \( f(b) - f(a) \) is also in \( M^+_a \). Since \( f(x) \geq x \) for all \( x \), we have \( f(a) \geq a \) hence, by Lemma 3.9, \( M^+_a \subseteq M^+_{f(a)} \). Thus, \( f(b) - f(a) \) is in \( M^+_{f(a)} \) implying \( f(a) \leq f(b) \).

**Case** \( b \leq a \). Similar proof, observe that \( a - b \in M^-_b = \{ x \mid b + x \leq b \} \). □
3.6 Preservation of congruences may not extend to stable preorders

Condition \((Arith^{abc})\) of Theorem A in Proposition 3.11 and condition \((CongPres^1)\)_\(N\) of Theorem 3.10 ask for \(f\) to be monotone non-decreasing. We show that this requirement cannot be removed: there does exist congruence preserving functions over \(\mathbb{N}\) which fail monotonicity, cf. Proposition 3.12 below.

First, recall a result, cf. Theorem 3.15 in our paper [3].

**Proposition 3.11** ([3]). For every \(f : \mathbb{N} \to \mathbb{N}\) there exists a function \(g : \mathbb{N} \to \mathbb{Z}\) such that, letting \(\text{lcm}(x)\) be the least common multiple of \(1, \ldots, x\),

(i) \(x - y\) divides \(g(x) - g(y)\) for all \(x, y \in \mathbb{N}\),

(ii) \(f(x) - 2^s\text{lcm}(x) \leq g(x) \leq f(x)\).

Such a \(g\) is \(g(x) = \sum_{k \in \mathbb{N}} a_k \binom{x}{k}\), where \(f(x) = \sum_{k \in \mathbb{N}} a_k \binom{x}{k}\) is the Newton representation of \(f\) (where the \(a_k\)'s are in \(\mathbb{Z}\)) and \(a_k = \text{lcm}(k)q_k\), where \(a_k = \text{lcm}(k)q_k + b_k\) with \(0 \leq b_k < \text{lcm}(k)\).

**Proposition 3.12.** There exists an \((\mathbb{N};+)-congruence preserving function \(g : \mathbb{N} \to \mathbb{N}\) which is non-monotone, hence is not \((\mathbb{N};+)-stable preorder preserving.\)

**Proof.** Consider the function \(f\) such that

\[
f(x) = \sum \{2^{y+1}\text{lcm}(y + 2) \mid y \leq x, \text{ } x \text{ even}\} - \sum \{2^{z+1}\text{lcm}(z) \mid z \leq x, \text{ } z \text{ odd}\}
\]

We have \(f(0) = 8\text{lcm}(2) = 16\) and, for \(p \in \mathbb{N}\),

\[
\begin{align*}
    f(2p + 2) - f(2p + 1) &= 2^{2p+5}\text{lcm}(2p + 4) \\
    f(2p) &\geq 2^{2p+3}\text{lcm}(2p + 2) > 0 \\
    f(2p + 1) - f(2p) &= -2^{2p+2}\text{lcm}(2p + 1) \\
    f(2p + 1) &\geq 2^{2p+3}\text{lcm}(2p + 2) - 2^{2p+2}\text{lcm}(2p + 1) \\
    &\geq 2^{2p+2}\text{lcm}(2p + 1) > 0
\end{align*}
\]

(8) and (10) show that \(f\) maps \(\mathbb{N}\) into \(\mathbb{N}\) and \(f(x) \geq 2^{x+1}\text{lcm}(x)\)

Let \(g : \mathbb{N} \to \mathbb{Z}\) be the function associated to \(f\) by Proposition 3.11. We first prove that \(g\) also maps \(\mathbb{N}\) into \(\mathbb{N}\). Clause (ii) in Proposition 3.11 and inequality (11) insure that \(g(x) \geq f(x) - 2^s\text{lcm}(x) > 2^s\text{lcm}(x)\). This proves that \(g(x) \in \mathbb{N}\) and moreover \(g(x) > x\) for all \(x\).

We next prove that \(g\) is non monotone. Using clause (ii) and (11), we get

\[
f(x + 1) - f(x) - 2^{x+1}\text{lcm}(x + 1) \leq g(x + 1) - g(x) \leq f(x + 1) - f(x) + 2^s\text{lcm}(x) \tag{12}
\]

Using (12) and (9) and (7), we get

\[
g(2p + 1) - g(2p) \leq f(2p + 1) - f(2p) + 2^{2p}\text{lcm}(2p)
\]
Theorem 3.7 hold, hence clause (i) of Proposition 3.11, both clauses (iii) of Definition 1.3).

Theorem A in 3.7 A strengthening of Theorem 3.10 relates stable preorder preservation of \( f \) with an arithmetic condition \((\text{Arith}^{(ab)})\) on \( f \). On its side Theorem A or an implication from Theorem 4.9 which is proved later in the paper makes no use neither of Theorem A nor of Theorem 3.14 which we are going to prove, we see that there is no vicious circle and the proofs of Theorem 4.9 and Theorem 3.14 are indeed self-contained.

Notation 3.13. \( \mathcal{N} \subseteq \mathcal{P}(\mathbb{N}) \) denotes the smallest sublattice of \( \mathbb{N} \) which is closed under \( \mathcal{S}^{-1} \) is standardly bounded in the sense of Definition 1.3).

Theorem 3.14. Let \( \mathcal{N} \) be \( \langle \mathbb{N}; +\rangle \), \( \langle \mathbb{N}; \times\rangle \) or \( \langle \mathbb{N}; +, \times\rangle \). The following conditions are equivalent for any function \( f: \mathbb{N} \rightarrow \mathbb{N} \),

\[
= -2^{2p+2}\text{lcm}(2p+1) + 2^{2p}\text{lcm}(2p) < 0 \tag{13}
\]

\[
g(2p+2) - g(2p+1) \geq f(2p+2) - f(2p+1) - 2^{2p+2}\text{lcm}(2p+2) = 2^{2p+5}\text{lcm}(2p+4) - 2^{2p+2}\text{lcm}(2p+2) > 0 \tag{14}
\]

Inequalities (13) and (14) show that \( g \) is a non-monotone zigzag function.

Finally, as \( g(x) > x \) for all \( x \) and \( x - y \) divides \( g(x) - g(y) \) for all \( x, y \in \mathbb{N} \) (by clause (i) of Proposition 3.11), both clauses \((a)\) and \((b)\) of condition \((\text{Arith}^{(ab)})\) of Theorem 3.7 hold, hence \( g \) is congruence preserving.

3.7 A strengthening of Theorem 3.10

Theorem A in §1.1 relates conditions involving particular lattices of subsets of \( \mathbb{N} \) closed under \( \mathcal{S}^{-1} \) with an arithmetic condition \((\text{Arith}^{(abc)})\) on \( f \). On its side Theorem 3.10 relates stable preorder preservation of \( f \) with an arithmetic condition \((\text{Arith}^{(ab/c)})\) which is equivalent to the disjunction of condition \((\text{Arith}^{(abc)})\) of Theorem A and the condition that \( f \) is constant.

We strengthen Theorem 3.10 in order to also relate \((\text{Arith}^{(ab/c)})\) to conditions involving lattices closed under \( \mathcal{S}^{-1} \).

A part of the proof of this strengthening has to use either an implication from Theorem A or an implication from Theorem 4.9 which is proved later in the paper in the framework of general algebras. For self-containment, we use the implication from Theorem 4.9. Observing that the proof of Theorem 4.9 makes no use neither of Theorem A nor of Theorem 3.14 which we are going to prove, we see that there is no vicious circle and the proofs of Theorem 4.9 and Theorem 3.14 are indeed self-contained.

Notation 3.13. \( \mathcal{N} \subseteq \mathcal{P}(\mathbb{N}) \) denotes the smallest sublattice of \( \mathbb{N} \) which is closed under \( \mathcal{S}^{-1} \) and contains \( L \subseteq \mathbb{N} \) and \( \varnothing, \mathbb{N} \) (i.e. is standardly bounded in the sense of Definition 1.3).

Theorem 3.14. Let \( \mathcal{N} \) be \( \langle \mathbb{N}; \mathcal{S}\rangle \), \( \langle \mathbb{N}; +\rangle \) or \( \langle \mathbb{N}; +, \times\rangle \). The following conditions are equivalent for any function \( f: \mathbb{N} \rightarrow \mathbb{N} \),

\[
(1) \quad \text{(PreorderPres)}_{\mathcal{N}} \quad \text{f is } \mathcal{N}-\text{stable-preorder preserving}
\]

\[
(2) \quad \text{(PreorderPres)}_{\mathcal{N}} \quad \text{f preserves every finite index } \mathcal{N}-\text{stable-preorder}
\]

\[
(3) \quad \text{(CongPres)}_{\mathcal{N}} \quad \{ \text{f is monotone non-decreasing and } \mathcal{N}-\text{congruence preserving} \}
\]

\[
(4) \quad \text{(CongPres)}_{\mathcal{N}} \quad \text{for all recognizable } L \subseteq \mathbb{N} \quad \{ \text{f is constant for all recognizable } L \subseteq \mathbb{N} \quad \text{and } \mathcal{S}^{-1} \text{-closed under } f^{-1} \}
\]

\[
(5) \quad \text{(CongPres)}_{\mathcal{N}} \quad \text{for all recognizable } L \subseteq \mathbb{N} \quad \{ \text{f is constant for all } x, y \in \mathbb{N} \quad \text{and } |y - x| \text{ divides } |f(y) - f(x)| \}
\]

\[
\{ \text{f is constant or } f(x) \geq x \text{ for all } x \in \mathbb{N} \quad \text{and } \mathcal{S}^{-1} \text{-closed under } f^{-1} \}
\]

\[
\{ \text{f is monotone non-decreasing} \}
\]

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Claim 1. If \( f \in L \) then \( \text{Frz}^*(L) \) is either equivalent or a reduction is convenient one since then \( \text{Frz}^*(L) \) is then the family of iterations of the sole \( \text{Suc} \) function.

Proof. Due to Lemma 3.5, Lemma 3.11 it suffices to consider the case \( \mathcal{N} = \langle \mathbb{N}, \text{Suc} \rangle \). Such a reduction is a convenient one since then \( \text{Frz}^*(L) \) is then the family of iterations of the sole \( \text{Suc} \) function.

Theorem 3.10 insures that conditions (1), (3), and (6) are equivalent.

Theorem 3.9 insures equivalence of conditions in the framework of any algebra \( \mathcal{A} \). Applying it to the algebra \( \mathcal{N} \), it insures that condition (2) is equivalent to the condition \( (\mathcal{L}^{\phi, \mathcal{A}}_\text{Frz}^*(\forall L))_\mathcal{N} \) which expresses that \( f^{-1}(L) \in \mathcal{L}^{\phi, \mathcal{N}}_\text{Suc}(L) \) for every \( \mathcal{N} \)-recognizable \( L \in \mathbb{N} \). Now, \( (\mathcal{L}^{\phi, \mathcal{A}}_\text{Frz}^*(\forall L))_\mathcal{N} \) is equivalent to condition (5) since we reduced to \( \mathcal{N} = \langle \mathbb{N}, \text{Suc} \rangle \) and closure of a lattice under \( \text{Suc}^{-1} \) yields closure under its iterations (which constitute \( \text{Frz}^*(\text{Suc}) \)). Thus, conditions (2), (3), and (5) are equivalent.

Having the equivalences (1) \( \iff \) (3) \( \iff \) (6) and (2) \( \iff \) (5), it suffices to prove the equivalences (1) \( \iff \) (2) and (4) \( \iff \) (5).

- The implication (1) \( \implies \) (2) is trivial.
- To prove (2) \( \implies \) (1) assume (2) and let \( \mathcal{E} \subseteq \mathcal{N} \)-stable preorder, and suppose \( x \subseteq y \). Let’s get (4) \( \implies \) (5). Assume (4). If \( f \) is constant with value \( c \) then \( f^{-1}(L) \) is either \( \emptyset \) or \( \mathbb{N} \) (depending if \( c \notin L \) or \( c \in L \) hence \( f^{-1}(L) \in \mathcal{L}^{\emptyset, \mathcal{N}}_\text{Suc}(L) \). If \( f \) is not constant and \( L \notin \{\emptyset, \mathbb{N}\} \) then (4) insures that \( f^{-1} \) is in \( \mathcal{L}_\text{Suc}(L) \) hence also in \( \mathcal{L}^{\emptyset, \mathcal{N}}_\text{Suc}(L) \).
- Finally, (5) \( \implies \) (4) is proved through a series of seven claims (for readability).

Claim 1. If \( L = A \cup \{B + k \mathbb{N} \} \) with \( A, B \) finite and \( k \geq 2 \) then \( \emptyset \in \mathcal{L}_\text{Suc}(L) \).

Proof of Claim 1. If \( t > \max(A) \) then \( \text{Suc}^{-t}(L) = \emptyset \). If \( t > \max(A \cup B) \) and \( L = A \cup (B + k \mathbb{N}) \) then \( \text{Suc}^{-t}(L) = C + k \mathbb{N} \) for some \( C \subseteq \{0, \ldots, k-1\} \). If \( i \in \{0, \ldots, k-1\} \setminus C \) then, setting \( L' = C + k \mathbb{N} \), we have \( i - p \notin L' \) for \( p = 0, \ldots, i \) and \( k + i - p \notin L' \) for \( p = i + 1, \ldots, k-1 \). Thus, \( \bigcap \{ \text{Suc}^{-q}(L) \mid q = t, \ldots, t + k - 1 \} = \emptyset \in \mathcal{L}_\text{Suc}(L) \).

Claim 2. If \( L = A \cup (B + k \mathbb{N}) \) with \( B \) non-empty and \( k \geq 2 \) then \( \emptyset \in \mathcal{L}_\text{Suc}(L) \).

Proof of Claim 2. If \( i \in B \) then \( 0 + k \mathbb{N} \subseteq \text{Suc}^{-i}(L) \). Since \( \text{Suc}^{-n}(k \mathbb{N}) = k - n + k \mathbb{N} \) for \( n = 1, \ldots, k-1 \), we see that \( \bigcup_{n=0, \ldots, k-1} \text{Suc}^{-|k-n|}(L) = k \mathbb{N} \).

Since \( \mathcal{L}^{\emptyset, \mathbb{N}}_\text{Suc}(L) = \mathcal{L}_\text{Suc}(L) \cup \{\emptyset, \mathbb{N}\} \), applying Claim 1 and Claim 2 we get
Claim 3. \( x_{L,\emptyset}^N(L) = \mathcal{L}_{\text{Suc}}(L) \) if \( L = A \cup (B + k\mathbb{N}) \) with \( A, B \) finite, \( B \neq \emptyset \), and \( k \geq 2 \).

We now assume the condition \( \mathcal{L}_{\text{Suc}}(\forall L)_{\mathbb{N}} \).

Claim 4. If \( f \) satisfies (5) then \( f \) is monotonous non-decreasing.

Proof of Claim 4. Fix some \( a \in \mathbb{N} \) and let \( L = \{ z \mid z \geq f(a) \} \). Being a final segment of \( \mathbb{N} \), the set \( L \) is recognizable. Observe that the \( \text{Suc}^{-t}(L) \)'s, for \( t \in \mathbb{N} \), are the final segments \( X_b = \{ z \mid z \geq b \} \) where \( b \leq f(a) \). Condition (5) insures that \( f^{-t}(L) \) is in the lattice \( \mathcal{L}_{\text{Suc}}^\emptyset(L) \), i.e. is a \((\cup, \cap)\) combination of \( \emptyset, \mathbb{N} \) and the \( X_b \)'s. In particular, it is a final segment. Since obviously \( a \in f^{-1}(L) \) this segment is not empty and is of the form \( f^{-1}(L) = \{ z \mid z \geq c \} \) for some \( c \leq a \). Finally, if \( x \geq a \) then \( x \geq c \) hence \( f(x) \in L \), which means that \( f(x) \geq f(a) \).

Claim 5. If \( f \) satisfies (5) and is not constant then it takes infinitely many values.

Proof of Claim 5. By way of contradiction, suppose that the range of \( f \) is a finite set \( \{b_1, \ldots, b_n\} \) with \( n \geq 2 \) and \( b_1 < \ldots < b_n \). By monotony of \( f \) the inverse image of \( \{b_n\} \) is a final segment of \( \mathbb{N} \) and since \( n \geq 2 \) this is a strict final segment of \( \mathbb{N} \), say \( f^{-1}(b_n) = p + \mathbb{N} \) with \( p \geq 1 \). Let \( L = b_n + k\mathbb{N} \) with \( k > b_n \). Then \( f^{-1}(L) = f^{-1}(b_n) = p + \mathbb{N} \). Observe that the \( \text{Suc}^{-t}(L) \)'s, with \( t \in \mathbb{N} \), are the sets

\[
\text{Suc}^{-t}(L) = \begin{cases} 
   b_n - t + k\mathbb{N} & \text{if } t = 0, \ldots, b_n - 1, \\
   ((b_n - t) \mod k) + k\mathbb{N} & \text{if } t \geq b_n.
\end{cases}
\]

Thus, the \( \text{Suc}^{-t}(L) \)'s are all the sets \( a + k\mathbb{N} \) with \( a \leq \max(b_n, k - 1) = k - 1 \) since we choose \( k > b_n \). Considering intersections of \( \emptyset, \mathbb{N} \), and the \( \text{Suc}^{-t}(L) \)'s does not produce any set outside these ones. Since \((\cup, \cap)\)-combinations can be normalized to finite non-empty unions of finite non-empty intersections, we see that sets in the lattice \( \mathcal{L}_{\text{Suc}}^\emptyset(L) \) are finite non-empty unions of \( \emptyset, \mathbb{N} \), and the \( \text{Suc}^{-t}(L) \)'s.

If \( f^{-1}(L) = p + \mathbb{N} \) were in the lattice \( \mathcal{L}_{\text{Suc}}^\emptyset(L) \), it would be a finite non-empty union of sets \( a + k\mathbb{N} \) with \( a \leq k - 1 \) since \( \emptyset \) is useless in such a union and \( \mathbb{N} \) would give a strict superset of \( p + \mathbb{N} \) because \( p \geq 1 \). Thus, we would have \( f^{-1}(L) = \bigcup_{a \in I} a + k\mathbb{N} \) with \( I \subseteq \{0, \ldots, k - 1\} \).

Since \( f^{-1}(L) = p + \mathbb{N} = \bigcup_{i=0 \ldots, k-1} p + i + k\mathbb{N} \) with \( p \geq 1 \), we would have

\[
\bigcup_{a \in I} a + k\mathbb{N} = \bigcup_{i=0 \ldots, k-1} p + i + k\mathbb{N} \quad \text{where } I \subseteq \{0, \ldots, k - 1\}. \quad (15)
\]

The right side is a union of \( k \) many pairwise disjoint sets \( p + i + k\mathbb{N} \) for \( i = 0, \ldots, k - 1 \). Equality (15) implies that \( I = \{p, \ldots, p + k - 1\} \). Contradiction since \( I \subseteq \{0, \ldots, k - 1\} \) and \( p \geq 1 \).

Claim 6. If \( f \) satisfies (5) and \( L \) is finite then \( f^{-1}(L) \) is finite and \( f^{-1}(L) \in \mathcal{L}_{\text{Suc}}(L) \).
Proof of Claim 6. Since $f$ takes infinitely many values and is monotone, the inverse images of singletons are finite (possibly empty) segments of $\mathbb{N}$. In particular, $f^{-1}(L)$ is finite. Assumption (5) insures that $f^{-1}(L)$ is a finite non-empty union of finite non-empty intersections of $\emptyset$, $\mathbb{N}$, and the $\text{Suc}^{-1}(L)$’s with $s \in \mathbb{N}$. Since $f^{-1}(L)$ is finite we can exclude $\mathbb{N}$ from this combination, and using Claim 1 we can also exclude $\emptyset$. Thus, $f^{-1}(L)$ is in fact in $\mathcal{L}_{\text{Suc}}(L)$.

Claim 7. If $f$ satisfies (5) and if $L = A \cup (B + \mathbb{N})$ with $A, B$ finite and $B$ not empty then $f^{-1}(L)$ is cofinite and $f^{-1}(L) \in \mathcal{L}_{\text{Suc}}(L)$.

Proof of Claim 7. Using assumption (5), consider a finite non-empty union of finite non-empty intersections of $\emptyset$, $\mathbb{N}$, and the $\text{Suc}^{-1}(L)$’s with $t \in \mathbb{N}$ which gives $f^{-1}(L)$. Claim 2 insures that $\mathbb{N} \in \mathcal{L}_{\text{Suc}}(L)$ hence we can exclude $\mathbb{N}$ and only use $\emptyset$ and the $\text{Suc}^{-1}(L)$’s. Now, since $L$ is cofinite, $\mathbb{N} \setminus L$ is finite and by Claim 6 $f^{-1}(\mathbb{N} \setminus L)$ is also finite. Hence $f^{-1}(L)$ is cofinite. In particular, it is not empty and we can exclude $\emptyset$ from the combination giving $f^{-1}(L)$. This shows that $f^{-1}(L) \in \mathcal{L}_{\text{Suc}}(L)$. By Lemma 3.6 and Claims 3, 6, 7, in all cases $f^{-1}(L) \in \mathcal{L}_{\text{Suc}}(L)$, proving (4).

4 Lattice and Boolean algebra conditions for stable preorder and congruence preservation in an algebra

4.1 Lattices $\mathcal{L}_{A}^{\emptyset,A}(L)$, $\mathcal{L}_{A}^{\infty}(L)$, Boolean algebras $\mathfrak{B}_{A}^{\emptyset,A}(L)$, $\mathfrak{B}_{A}^{\infty}(L)$

As witnessed by the lattice of closed subsets of a topological space such as $\mathbb{R}$, the supremum of an infinite family in a lattice of sets may strictly contain its set theoretic union. This is why we recall the following notion.

Definition 4.1. 1. A lattice $\mathcal{L}$ of subsets of $A$ is set-complete if for every (possibly empty or infinite) $\mathcal{F} \subseteq \mathcal{L}$ the union $\bigcup_{X \in \mathcal{F}} X$ and the intersection $\bigcap_{X \in \mathcal{F}} X$ are in $\mathcal{L}$ (hence are the supremum and infimum of $\mathcal{F}$). In particular, such a lattice is standardly bounded (cf. Definition 1.3).

2. A standardly bounded lattice closed under complementation is called a standardly bounded Boolean algebra.

3. A set-complete (hence standardly bounded) lattice closed under complementation is called a set complete Boolean algebra.

Quitting the rich framework of algebras $\langle \mathbb{N}, \text{Suc} \rangle$, $\langle \mathbb{N}, + \rangle$, $\langle \mathbb{N}, +, \times \rangle$, and considering general algebras, a key tool is the notion of freezification introduced in §2.3.

Definition 4.2. For $\mathcal{A} = (A; \Xi)$ an algebra and $L \subseteq A$, among the families of subsets of $A$ which contain $L$ and are closed under all the $\gamma^{-1}$'s for $\gamma \in \text{Frz}^{*}(\mathcal{A})$, we respectively denote...
\( \mathcal{L}_A(L) \) the smallest lattice,
\( \mathcal{L}_A^\emptyset(L) \) the smallest standardly bounded lattice (i.e. \( \mathcal{L}_A(L) \cup \{ \emptyset, A \} \)),
\( \mathcal{L}_A^\infty(L) \) the smallest set complete (hence standardly bounded) lattice,
\( \mathcal{B}_A^\emptyset(L) \) the smallest standardly bounded Boolean algebra,
\( \mathcal{B}_A^\infty(L) \) the smallest set complete (hence standardly bounded) Boolean algebra.

The following Lemma is straightforward.

**Lemma 4.3.** If \( \mathcal{L}_A(L) \) is finite then
\[
\mathcal{L}_A^\emptyset(L) = \mathcal{L}_A^\infty(L) \quad \text{and} \quad \mathcal{B}_A^\emptyset(L) = \mathcal{B}_A^\infty(L).
\]

| \( \mathbb{N}; \text{Suc} \) | \( \mathbb{N}; + \) | \( \mathbb{N}; \times \) | \( \mathbb{N}^*; +, \times \) | \( \Sigma^*; \cdot \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \mathbb{N}; \text{Suc} \) | \( \mathbb{N}; + \) | \( \mathbb{N}; \times \) | \( \mathbb{N}^*; +, \times \) | \( \Sigma^*; \cdot \) |
| \( {\text{X \mapsto X - n}} \) | \( {\text{X \mapsto X/n}} \) | \( {\text{X \mapsto (X - m)/n}} \) | \( {\text{X \mapsto u^{-1}Xv^{-1}}} \) |
| \( \{ y \mid n + y \in X \} \) with \( n \in \mathbb{N} \) | \( \{ y \mid ny \in X \} \) with \( n \in \mathbb{N} \) | \( \{ y \mid ny + m \in X \} \) with \( m, n \in \mathbb{N} \) | \( \{ x \mid uxv \in X \} \) with \( u, v \in \Sigma^* \) |

Figure 2: Maps \( X \mapsto \gamma^{-1}(X) \) for \( \gamma \in \text{Frz}^*(A) \)

**Note 4.4.** Maps \( X \in \mathcal{P}(A) \mapsto \gamma^{-1}(X) \) are called cancellations in \([1\), §3.1, p.55. See Figure 2 for examples.

**Lemma 4.5.** Let \( \sim \) be an \( A \)-congruence. If \( L \subseteq A \) is \( \sim \)-saturated then so is every set in \( \mathcal{B}_A^\infty(L) \). In particular, this is true for the syntactic congruence \( \sim_L \).

**Proof.** The family of \( \sim \)-saturated subsets of \( A \) is closed under arbitrary unions, complementation and preimage by maps in \( \text{Frz}^*(A) \), hence it contains \( \mathcal{B}_A^\infty(L) \).

**Lemma 4.6.** Let \( A \) be an algebra and \( \preceq \) an \( A \)-stable preorder. If \( L \) is a \( \preceq \)-initial segment (i.e. \( a \in L \) and \( b \preceq a \) imply \( b \in L \)) then so is every set in \( \mathcal{L}_A^\infty(L) \).

**Proof.** The family of \( \preceq \)-initial segments of \( A \) is closed under arbitrary unions and intersections and preimage by maps in \( \text{Frz}^*(A) \), hence it contains \( \mathcal{L}_A^\infty(L) \).

**Lemma 4.7.** If \( L \) is \( A \)-recognizable then \( \mathcal{B}_A(L) \) is finite, (so that equalities of Lemma 4.3 hold) and every set in \( \mathcal{B}_A(L) \) is also \( A \)-recognizable.

**Proof.** By Proposition 2.19 \( L \) is \( \sim_L \)-saturated and since \( L \) is \( A \)-recognizable the congruence \( \sim_L \) has finite index, say \( k \), and the \( 2^k \) many \( \sim_L \)-saturated subsets of \( A \) are all \( A \)-recognizable. By Lemma 4.5 all sets in \( \mathcal{B}_A^\infty(L) \) are \( \sim_L \)-saturated, hence this algebra is finite with cardinal at most \( 2^k \), constituted of \( A \)-recognizable sets.
4.2 Representing $f^{-1}(L)$ when $f$ preserves $\sim_L$ or $\preceq_L$

**Lemma 4.8.** Consider an algebra $A = \langle A; \Xi \rangle$, a function $f: A \to A$ and a set $L \subseteq A$.

Let $X_{aL} = \{ \gamma \in \Frz^*(\Xi) \mid a \in \gamma^{-1}(L) \}$ and $Y_{aL} = \{ \gamma \in \Frz^*(\Xi) \mid a \notin \gamma^{-1}(L) \}$.

1. If $f$ preserves the syntactic congruence $\sim_L$ then

\[
\left(\gamma \in X_{aL}\right) \Rightarrow \left(\gamma \in Y_{aL}\right) \quad \forall \gamma \in \Frz^*(\Xi)
\]

2. If $f$ preserves the syntactic preorder $\preceq_L$ then $f^{-1}(L) \in \Sigma^\infty_A(L)$ and

\[
f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} \{ x \mid x \sim_L a \} = \bigcup_{a \in f^{-1}(L)} \bigcap_{\gamma \in X_{aL}} \gamma^{-1}(L)
\]

3. If $f$ preserves the syntactic preorder $\preceq_L$ then $f^{-1}(L) \in \Sigma^\infty_A(L)$ and

\[
f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} \{ x \mid x \preceq_L a \} = \bigcup_{a \in f^{-1}(L)} \bigcap_{\gamma \in X_{aL}} \gamma^{-1}(L)
\]

**Proof.** 1. If $f^{-1}(L) = \emptyset$ then $\bigcup_{a \in f^{-1}(L)} \{ x \mid x \sim_L a \}$ is empty as union of an empty family, hence is equal to $f^{-1}(L)$. Assume now $f^{-1}(L) \neq \emptyset$. Since $a \sim_L a$ we have $f^{-1}(L) \subseteq \bigcup_{a \in f^{-1}(L)} \{ x \mid x \sim_L a \}$. Conversely, if $x \sim_L a$ where $a \in f^{-1}(L)$ then, as $f$ preserves $\sim_L$, we also have $f(x) \sim_L f(a)$, which means that, for every $\gamma \in \Frz^*(A)$, we have the equivalence $\gamma(f(a)) \in L \iff \gamma(f(x)) \in L$. For $\gamma = \text{id}$, since $f(a) \in L$ this gives $f(x) \in L$, i.e. $x \in f^{-1}(L)$. Thus, $\{ x \mid x \sim_L a \} \subseteq f^{-1}(L)$ for every $a \in f^{-1}(L)$, which proves $\bigcup_{a \in f^{-1}(L)} \{ x \mid x \sim_L a \} \subseteq f^{-1}(L)$, giving the first equality in (16). The second equality simply translates the definition of $\sim_L$ (cf. Definition 2.16).

Finally, the second equality in (16) yields $f^{-1}(L) \in \mathcal{B}_A^\infty(L)$, the case of empty union (in case $f^{-1}(L)$ is empty) and empty intersections (in case some $X_{aL}$’s or $Y_{aL}$’s are empty) justifying the requirement that $\emptyset$ and $A$ be forcibly put in this Boolean algebra.

For item 2 argue similarly and for the case of recognizable $L$ use Lemma 4.7. □

4.3 $A$-congruence (resp. $A$-stable preorder) preservation and Boolean algebras (resp. lattices) of subsets of $A$

**Theorem 4.9** (Preservation in general algebras). Let $A = \langle A; \Xi \rangle$ be an algebra and $f: A \to A$. For each of the pairs of conditions detailed in Table 7, the two shown conditions are equivalent. The figure of Table 7 illustrates these equivalences and the straightforward implications.

**Proof.** All the simple implications (horizontal and vertical arrows) of the diagram are trivial. So, we are left with the four inclined bi-implications to prove.

- (1) $(\text{CongPres})_A \iff (\mathcal{B}_A^\infty(\forall L))_A$. Assume $f$ preserves all $A$-congruences. Then $f$ preserves the syntactic congruence $\sim_L$ of any $L$ and item 1 in Lemma 4.8 insures that $f^{-1}(L) \in \mathcal{B}_A^\infty$. 

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Four pairs of equivalent conditions

| (PreordPres)$_A$ | F is $A$-stable-preorder preserving |
|------------------|-----------------------------------|
| (CongPres)$_A$  | F preserves all finite index $A$-stable-preorders |
| (CongPres)$_A^{\text{FinInd}}$ | F preserves all finite index $A$-congruences |

Table 1: Implications and equivalent conditions of Theorem 4.9

\[(\mathcal{B}^\infty_{Frz}(\forall L))_A \implies (CongPres)_A.\] Let $L = \{x \mid x \sim f(a)\}$, where ~ is an $A$-congruence and $a \in A$. Condition $(B^\infty_{Frz}(\forall L))_A$ insures that $f^{-1}(L)$ is in $\mathcal{B}^\infty_A(L)$. Since $L$ is $\sim$-saturated so is also every set in $\mathcal{B}^\infty_A(L)$ by Lemma 4.5. In particular $f^{-1}(L)$ is $\sim$-saturated. Now, $a \in f^{-1}(L)$ since $f(a) \sim f(a)$, hence if $x \sim a$ then $x$ is also in $f^{-1}(L)$, hence $f(x) \in L$ which means $f(x) \sim f(a)$. Thus, $f$ preserves the congruence $\sim$.

- (2) $(CongPres)_A^{\text{FinInd}} \iff (\mathcal{B}^\infty_{Frz}(\forall L))_A^{\text{rec}}$. Since $L$ is recognizable if and only if $\sim_L$ has finite index (cf. Proposition 2.19), we can argue as in (1) above.
- (3) $(PreordPres)_A \iff (\mathcal{L}^\infty_{Frz}(\forall L))_A$. Argue as for congruences and Boolean algebras, using item 2 in Lemma 4.8.

\[(\mathcal{L}^\infty_{Frz}(\forall L))_A \implies (PreordPres)_A.\] Let $x \leq y$ be an $A$-stable preorder and $a \in A$ and $L = \{x \mid x \leq f(a)\}$. Condition $(\mathcal{L}^\infty_{Frz}(\forall L))_A$ insures that $f^{-1}(L)$ is in $\mathcal{L}^\infty_A(L)$. Since $L$ is a $\leq$-initial segment so is also every set in $\mathcal{L}^\infty_A(L)$ (by Lemma 4.6). In particular $f^{-1}(L)$ is a $\leq$-initial segment. Now, $a \in f^{-1}(L)$ since $f(a) \leq f(a)$, hence if $x \leq a$ then $x$ is also in $f^{-1}(L)$, hence $f(x) \in L$ which means $f(x) \leq f(a)$. Thus, $f$
preserves the stable preorder ≤.

• (4) \((\text{PreordPres})_{A}^\text{FinInd} \iff (\Sigma_{L \in \text{Frz}}^A (\forall L))^\text{rec}_A\). Since \(L\) is recognizable if and only if \(\leq_L\) has finite index (cf. Proposition 2.19), we can argue as in (3) above.

**Remark 4.10.** In case \(f\) is constant it obviously preserves congruences and stable preorders. If its value is \(a\) then the inverse image \(f^{-1}(L)\) is either the empty set if \(a \notin L\) or the whole set \(A\) if \(a \in L\). This illustrates why we have to put \(\emptyset\) and \(A\) in the lattices and Boolean algebras considered in Theorem 4.9.

## 5 Adding algebraic hypothesis to enrich Table 1

In this section we consider particular properties of algebras allowing to replace some implications in the figure of Table 1 by logical equivalences.

### 5.1 Algebras with a group operation and the right side-trapezium

For some algebras it happens that the notion of finite index stable preorder is equivalent to that of finite index congruence, cf. Propositions 5.4 and 5.6 below.

We first give a version involving semigroups and cancellable stable preorders.

**Definition 5.1.** 1) A semigroup \(S\) is said to be cancellable if \(xz = yz\) implies \(x = y\) and \(zx = zy\) implies \(x = y\).

2) A stable preorder \(\preceq\) on \(S\) is said to be cancellable if \(xz \preceq yz\) implies \(x \preceq y\) and \(zx \preceq zy\) implies \(x \preceq y\).

Let’s recall two folklore Lemmas.

**Lemma 5.2.** The only stable order of a finite group \(G\) is the identity relation.

**Proof.** Assuming \(x \preceq y\) we prove \(x = y\). Let \(e\) be the unit of \(G\). Stability under left product by \(x^{-1}\) and \((x^{-1})^n\) successively yield \(e \preceq x^{-1}y\) and then \((x^{-1})^n \preceq (x^{-1})^{n+1}\). By transitivity, \(e \preceq x^{-1}y \preceq (x^{-1})^n\). As the group is finite there exists \(k\) and \(n \geq 1\) such that \((x^{-1})^k = (x^{-1})^n\) hence \((x^{-1})^n = e\). Thus, \(e \preceq x^{-1}y \preceq e\) and by antisymmetry \(e = x^{-1}y\) and \(x = y\).

**Lemma 5.3.** Any finite cancellable semigroup \(S\) is a group.

**Proposition 5.4.** Let \(A = (A; \Xi)\) be an algebra where \(\Xi\) contains a semigroup operation. Every cancellable finite index stable preorder \(\preceq\) of \(A\) coincides with its associated congruence \(\sim\).

**Proof.** The semigroup operation on \(A\) induces a semigroup operation on the quotient algebra \(G = A/\sim\) with carrier set \(G = A/\sim\).

The cancellability property of the congruence \(\sim\) implies that of the semigroup operation on \(G\). Indeed, suppose \(X, Y, Z \in G\) satisfy \(XZ = YZ\) and let \(x, y, z \in A\) be representatives of the classes \(X, Y, Z\). Then we have \(xz \sim yz\) and cancellability yields \(x \sim y\), hence \(X = Y\). Idem if \(ZX = ZY\).
As \( \sim \) has finite index, \( G \) is finite and Lemma 5.3 insures that the semigroup operation on \( G \) is a group operation. As \( \sim \) is the congruence associated to the stable preorder \( \preceq \), it induces a quotient stable order \( \preceq /\sim \) on the finite quotient algebra \( G /\sim \).

As \( G \) is an expansion of a finite group, Lemma 5.2 ensures that \( \preceq /\sim \) is the identity relation on \( G \), hence \( \preceq \) coincides with \( \sim \) and is therefore a congruence on \( A \). □

In the framework of groups the requirement that a preorder be stable is quite a strong requirement as shown by the following corollary of Proposition 5.4.

**Proposition 5.5.** Let \( A = \langle A; \Xi \rangle \) be an algebra where \( \Xi \) contains a group operation.

1. Every stable preorder with finite index is a congruence.
2. A function \( f : A \to A \) is \( A \)-stable finite index preorder preserving if and only if it is finite index \( A \)-congruence preserving. In particular, the right side trapezium in Table 1 becomes a series of logical equivalences.
3. If \( L \subseteq A \) is recognizable then its syntactic preorder is equal to its syntactic congruence and \( L(A) = L(\Xi) \)

**Proof.** 1. Observe that stability of \( \preceq \) implies its cancellability: if \( xz \preceq yz \) then \( xz^{-1} \preceq yz^{-1} \) hence \( x \preceq y \). Idem if \( zx \preceq zy \). Then apply Proposition 5.4.
2. Obvious corollary of 1.
3. Finally, for 3 observe that the syntactic congruence of a recognizable set has finite index. □

### 5.2 Algebras expanding unit rings and the two side-trapeziums

The following easy folklore result is a kind of companion to Proposition 5.5.

**Proposition 5.6.** If the algebra \( A \) is an algebraic expansion of a unit ring then every \( A \)-stable preorder is an \( A \)-congruence. As a consequence, in the two side trapeziums in Table 1 all arrows become logical equivalences.

**Proof.** Let \( 1 \) be the unit of the unit ring multiplication and \(-1\) its opposite relative to the ring addition. If \( \preceq \) is an \( A \)-stable preorder and \( x \preceq y \) then \( (-1)x \preceq (-1)y \) hence \( y = x + (-1)x + y \preceq x + (-1)y + y = x \). Thus, \( x \preceq y \) and \( y \preceq x \), showing that \( \preceq \) is an \( A \)-congruence. □

### 5.3 Residually finite, residually c-finite and sp-finite algebras

In this section we review some notions of residually finite algebra stronger than the classical one, as in [1] (p. 23) and [12] (p. 102), that are used in the next §5.4.

**Definition 5.7 (Classical definition).** An algebra \( A \) is residually finite if morphisms from \( A \) to finite algebras separate points, i.e., if \( x \neq y \) then there exists a morphism \( \varphi \) into a finite algebra such that \( \varphi(x) \neq \varphi(y) \). In other words, \( A \) is residually finite if equality is the intersection of congruences having finite index.

**Lemma 5.8.** Free groups and free monoids (in particular \( \langle N; + \rangle \) and \( \langle Z; + \rangle \)) are residually finite whereas the algebra \( \langle Q; + \rangle \) is not residually finite.
Proof. Case of free groups. See Daniel E. Cohen’s book \[5\], pp. 7 and 11.

Case of free monoids. Let’s recall the easy folklore proof. Given words $u \neq v$ in the set $\Sigma^*$ of finite words over alphabet $\Sigma$, let $\Delta$ be the finite set of letters of $\Sigma$ used in the words $u, v$ and $k = \max(|u|, |v|)$ be the maximum of the lengths of $u$ and $v$. Consider the set $I$ of all words containing a letter in $\Sigma \setminus \Delta$ or having length strictly greater than $k$. This set $I$ is a two-sided ideal of the monoid $\Sigma^*$ and the quotient monoid $M = \Sigma^*/I$ is finite since it can be identified to $\{x \in \Delta^* \mid |x| \leq k\} \cup \{0\}$ where 0 is a zero element and the product of $x$ and $y$ is $xy$ if $|xy| \leq k$ and 0 otherwise. Clearly, the canonical morphism $\varphi : \Sigma^* \to M$ separates $u$ and $v$ since it maps the distinct elements $u$ and $v$ onto themselves.

Case of $\langle \mathbb{Q}; + \rangle$. Any congruence of a group is defined by the class of its neutral element which is a normal subgroup. It is known that in $\langle \mathbb{Q}; + \rangle$ the sole finite index subgroup is $\mathbb{Q}$ itself. Thus, the sole finite index congruence is the trivial one which identifies all elements of $\mathbb{Q}$ hence $\langle \mathbb{Q}; + \rangle$ is not residually finite.

Extending Definition 5.7 to all congruences – not only equality – we get the following notions.

Definition 5.9. 1) A congruence $\sim$ on an algebra $\mathcal{A}$ is said to be c-residually finite if it is the intersection of a family of finite index congruences. In other words, a congruence $\sim$ is c-residually finite if the quotient $\mathcal{A}/\sim$ is residually finite in the usual sense of Definition 5.7.

2) A stable preorder on an algebra $\mathcal{A}$ is said to be sp-residually finite if it is the intersection of a family of stable preorders all of which have finite index.

3) An algebra $\mathcal{A}$ is said to be c-residually finite (resp. sp-residually finite) if all congruences (resp. stable preorders) on $\mathcal{A}$ are c-residually finite (resp. sp-residually finite).

Every congruence being a preorder, Definition 5.9 gives a priori two notions of residual finiteness for a congruence. However, both notions coincide.

Lemma 5.10. 1. If a stable preorder $\preceq$ is sp-residually finite then its associated congruence $\sim$ is c-residually finite.

2. A congruence is c-residually finite if and only if, as a preorder, it is sp-residually finite. In particular, an sp-residually finite algebra is also c-residually finite.

Proof. 1) Let $(\preceq_i)_{i \in I}$ be a family of stable preorders having finite index and such that $\preceq = \bigcap_{i \in I} \preceq_i$. Let $\sim_i$ be the congruence associated to $\preceq_i$. We show that $\sim = \bigcap_{i \in I} \sim_i$. As $\sim_i$ is included in $\preceq_i$ we have $(\bigcap_{i \in I} \sim_i) \subseteq (\bigcap_{i \in I} \preceq_i) = \preceq$. As $\bigcap_{i \in I} \sim_i$ is a congruence, the last inclusion yields $(\bigcap_{i \in I} \sim_i) \subseteq \sim_i$. The inclusion of the congruence $\sim$ in the preorder $\preceq$ together with the inclusion $\preceq \subseteq \preceq_i$ imply the inclusion $\sim \subseteq \preceq_i$. As $\sim$ is a congruence, this last inclusion yields $\sim \subseteq \sim_i$. Thus, $\sim \subseteq (\bigcap_{i \in I} \sim_i)$.

2) If a congruence $\sim$ is c-residually finite and $\sim = \bigcap_{i \in I} \sim_i$ then the congruences $\sim_i$’s, being also preorders, witness that $\sim$ is sp-residually finite. Conversely, applying
Proposition 5.11. ⟨

of all modular congruences.

modular congruence hence has finite index. As for the identity, it is the intersection

Proof. ⟨

not sp-residually finite.

finite whereas

finite whereas

item 1 to a congruence ∼, we see that if ∼ is sp-residually finite then it is also c-residually finite.

The following proposition gives very simple examples.

Proposition 5.11. ⟨

(i.e. the free monoid with one generator) is sp-residually finite whereas \( \langle \mathbb{Z}; + \rangle \) (i.e. the free group with one generator) is c-residually finite but not sp-residually finite.

Proof. ⟨

\( \langle \mathbb{Z}; + \rangle \) is c-residually finite. Every congruence which is not the identity is a modular congruence hence has finite index. As for the identity, it is the intersection of all modular congruences.

\( \langle \mathbb{Z}; + \rangle \) is not sp-residually finite. Let’s see that the usual order \( \leq \) is not sp-residually finite. Indeed, let \( \leq \) be a stable preorder which strictly contains \( \leq \) (which is the case if \( \leq \) has finite index and contains \( \leq \)). We prove that \( \leq \) is trivial: it identifies all integers. Since \( \leq \) strictly contains \( < \), there exists \( a \in \mathbb{Z} \) and \( \ell \neq 0 \) such that \( a \leq a + \ell \) and \( a + \ell \leq a \). We can suppose \( \ell > 0 \) since otherwise it suffices to add \( -\ell \) to these inequalities. Adding \( -\ell + 1 \) to the second inequality we get \( a + 1 \leq a + 1 - \ell \). Since \( a + 1 - \ell \leq a \) and \( \leq \) contains \( \leq \) we also have \( a + 1 - \ell \leq a \) hence by transitivity \( a + 1 \leq a \). By stability we get \( x + 1 \leq x \) for all \( x \) and by transitivity \( x > y \Rightarrow x \leq y \) for all \( x, y \). Since \( \leq \) contains \( \leq \) hence also \( < \), this shows that \( \leq \) identifies all integers.

\( \langle \mathbb{N}; + \rangle \) is sp-residually finite. Let \( \leq \) be a stable preorder. If it has finite index it is trivially sp-residually finite. So, we now suppose that \( \leq \) has infinite index.

Claim. At least one of the preorders \( \leq \) and its reverse \( \geq \) is included in the natural order \( \leq \). In other words, at least one of the following two properties holds:

\[
\forall x, y \in \mathbb{N} \ (x \leq y \Rightarrow x \leq y), \quad \forall x, y \in \mathbb{N} \ (x \geq y \Rightarrow x \leq y). \quad (18)
\]

If both are true then \( \leq \) coincides with the identity relation on \( \mathbb{N} \).

Proof of Claim. We argue by way of contradiction. If the two stated properties both failed, let \( d, e > 0 \) be minimum such that there exist \( x, y \in \mathbb{N} \) satisfying \( x \leq x + d \) and \( y + e \leq y \). Let \( z = \max(x, y) \). Stability yields \( z + e \leq z \leq z + d \) (just add \( z - x \) and \( z - y \)).

Let \( f = \gcd(d, e) \). Applying Bachet-Bézout’s theorem, \( ad - \beta e = \gamma e - \delta d = f \) for some \( \alpha, \beta, \gamma, \delta \in \mathbb{N} \). Adding \( d \) and \( e \) again and again, we get

\[
z \leq z + d \leq z + 2d \leq \ldots \leq z + \alpha d = z + \beta e + f \quad \text{and} \quad z \geq z + e \geq z + 2e \geq \ldots \geq z + \beta e
\]

\[
z \leq z + d \leq z + 2d \leq \ldots \leq z + \alpha d = z + \gamma e - f \quad \text{and} \quad z \geq z + e \geq z + 2e \geq \ldots \geq z + \gamma e
\]

Thus, \( z + \beta e \leq z \leq z + \beta e + f \) and \( z + \gamma e \leq z \leq z + \gamma e - f \). Letting \( x' = z + \beta e \) and \( y' = z + \gamma e \), we get \( x' \leq x' + f \), and (adding \( f \)) \( y' + f \leq y' \). Minimality of \( d, e \) insures that \( f \geq d \) and \( f \geq e \). Letting \( z' = \max(x', y') \) stability yields \( z' + f \leq z' + f \). Since \( \leq \) has infinite index its associated congruence is the identity (cf. Lemma 3.1) so that \( z' = z' + f \) hence \( f = 0 \), contradicting inequalities \( d, e > 0 \), \( f \geq d \), and \( f \geq e \).
Using the above Claim, we can now finish the proof.

- **Case** \( \forall x, y \in \mathbb{N} \ (x \leq y \Rightarrow x \leq y) \) holds. For \( i \in \mathbb{N} \) let
  \[ \leq_i = \leq \cup \{(x, y) \mid y \geq i\} \]  

Let’s check that \( \leq_i \) is a stable preorder. Reflexivity is trivial. As for transitivity, if \( x \leq_i y \) and \( y \leq_i z \) then the Case assumption insures that \( x \leq y \leq z \). If \( z \geq i \) then (*) insures \( x \leq_i z \). If \( z < i \) then also \( y < i \) and (*) insures \( x \leq y \) and \( y \leq z \) hence \( x \leq z \) by transitivity of \( \leq \). Observe also that \( \leq_i \) has finite index (at most \( i + 1 \)).

Since \( \leq = \cap_{i \in \mathbb{N}} \leq_i \) we see that \( \leq \) is residually finite.

- **Case** \( \forall x, y \in \mathbb{N} \ (x \leq y \Rightarrow x \geq y) \) holds. Similar, letting \( \leq_i = \leq \cup \{(x, y) \mid x \geq i\} \).

Usual residual finiteness is strictly weaker than c-residual finiteness.

**Proposition 5.12.** Though free groups and free monoids are residually finite,

1. Free groups on at least two generators (possibly infinitely many) are not c-residually finite.

2. Free monoids with at least four generators (possibly infinitely many) are not c-residually finite.

**Proof.** 1. G. Baumslag proved (cited in Magnus, 1969 [14] pp. 307–308) that (denoting \( e \) the empty word) the quotient of the free group \( F_2 \) with the two generators \( a \) and \( b \) by the relation \( a^{-1}b^2ab^3 = e \) is not residually finite. Which means that the congruence \( \approx \) of \( F_2 \) generated by the relation \( a^{-1}b^2ab^3 = e \) is not c-residually finite. Hence \( F_2 \) is not c-residually finite.

   For free groups with more generators than the sole \( a \) and \( b \), simply add to the previous relation the relations equating to \( e \) all generators different from \( a \) and \( b \).

2. Recall that if \( \theta: \mathcal{A} \to \mathcal{B} \) is a surjective homomorphism between the algebras \( \mathcal{A} \) and \( \mathcal{B} \) and \( \bar{z} \) is a congruence on \( \mathcal{B} \) then \( \theta^{-1}(\bar{z}) \) is a congruence on \( \mathcal{A} \) and the quotient algebras \( \mathcal{A}/\theta^{-1}(\bar{z}) \) and \( \mathcal{B}/\bar{z} \) are isomorphic.

   Let \( \Sigma = \{\ell_1, \ell_2, \ell_3, \ell_4\} \cup \Delta \) and let \( \theta: \Sigma^* \to F_2 \) be the surjective semigroup morphism such that \( (a \text{ and } b \text{ being the two generators of } F_2) \theta(\ell_1) = a, \theta(\ell_2) = a^{-1}, \theta(\ell_3) = b, \theta(\ell_4) = b^{-1}, \text{ and } \theta(z) \) is arbitrarily chosen in \( F_2 \) for every \( z \in \Delta \). Using item 1, we can choose a non-residually finite congruence \( \bar{z} \) on \( F_2 \). Since \( F_2/\bar{z} \) is then not residually finite, neither is \( \Sigma^*/\theta^{-1}(\bar{z}) \). Thus, the congruence \( \theta^{-1}(\bar{z}) \) on \( \Sigma^* \) is not c-residually finite. \( \square \)

### 5.4 **sp/c-residually finite algebras and the upper/lower trapeziums**

The above notions of c-residually finiteness and sp-residually finiteness easily lead to the following version of Theorem 4.9.

**Theorem 5.13.** Let \( \mathcal{A} = (A; \Xi) \) be an algebra and \( f: A \to A \).

1. If \( \mathcal{A} \) is sp-residually finite and \( f \) preserves all finite index stable preorders then \( f \) is stable preorder preserving and congruence preserving. Thus,

\[
(\text{PreordPres})^\text{FinInd}_A \iff (\text{PreordPres})_A \text{ and } (\text{CongPres})^\text{FinInd}_A \iff (\text{CongPres})_A
\]
and the upper and lower horizontal trapeziums in Table 4 become series of logical equivalences.

2. If \( \mathcal{A} \) is c-residually finite and \( f \) preserves all finite index congruences then \( f \) is congruence preserving. Thus, \((\text{CongPres})^{\text{FInd}}_{\mathcal{A}} \iff (\text{CongPres})_{\mathcal{A}} \) and the lower trapezium in Table 4 becomes a series of logical equivalences.

Proof. 1) Let \( \preceq \) be a stable preorder. The hypothesis of sp-residual finiteness of \( \mathcal{A} \) ensures that \( \preceq \) is sp-residually finite: there exists a family of stable preorders \( (\preceq_i)_{i \in I} \) with associated congruences having finite indexes, such that \( \preceq = \bigcap_{i \in I} \preceq_i \). Thus, \( a \preceq b \) if and only if, for all \( i \in I \), \( a \preceq_i b \). The hypothesis ensures that \( f \) preserves the \( \preceq_i \)'s hence \( f(a) \preceq_i f(b) \) for all \( i \in I \) and therefore \( f(a) \preceq f(b) \). Lemma 2.4 leads to the stated result about congruence preservation. The proof of 2) is similar.

As a corollary of Theorem 5.13, we get a new proof of the equivalence of the first and third conditions in Theorem 3.14.

Theorem 5.14. Let \( \mathcal{N} = (\mathbb{N}; \text{Suc}), (\mathbb{N}; +) \) or \( (\mathbb{N}; +, \times) \). The following conditions are equivalent for any function \( f: \mathbb{N} \to \mathbb{N} \).

\[ \text{(PreorderPres)}_{\mathcal{N}} \quad f \text{ is } \mathcal{N}\text{-stable-preorder preserving} \]
\[ \text{(L}_\infty \text{Frz}^\ast (\forall L))_{\mathcal{A}} \quad f^{-1}(L) \in \Sigma_{\mathcal{A}}^L(L) \text{ for every recognizable } L \subseteq A \]

5.5 Residually finite algebra with a group operation

When there is a group operation in a c-residually finite algebra, Theorem 4.9 can be strengthened to the equivalence of all considered conditions.

Theorem 5.15. Let \( \mathcal{A} = (A; \Xi) \) be a c-residually finite algebra such that \( \Xi \) contains a group operation (for instance \( (\mathbb{Z}; +, \times) \)). Let \( f: A \to A \). The eight conditions of Theorem 4.9 are equivalent. In other words all arrows in the diagram of Table 4 are bi-implications.

Proof. If in the horizontal upper, lower trapeziums and the vertical right-sided trapezium the implication arrows can be replaced by bi-implication arrows, so also does the vertical left-sided trapezium.

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