ENDPOINT EIGENFUNCTION BOUNDS FOR THE HERMITE OPERATOR

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Abstract. We establish the optimal $L^p$, $p = 2(d + 3)/(d + 1)$, eigenfunction bound for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^d$. Let $\Pi_\lambda$ denote the projection operator to the vector space spanned by the eigenfunctions of $H$ with eigenvalue $\lambda$. The optimal $L^2-L^p$ bounds on $\Pi_\lambda$, $2 \leq p \leq \infty$, have been known by the works of Karadzhov and Koch-Tataru except $p = 2(d + 3)/(d + 1)$. For $d \geq 3$, we prove the optimal bound for the missing endpoint case. Our result is built on a new phenomenon: improvement of the bound due to asymmetric localization near the sphere $\sqrt{\lambda} S^{d-1}$.

1. Introduction

The Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^d$ has a discrete spectrum $\lambda \in 2\mathbb{N}_0 + d$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\alpha \in \mathbb{N}_0^d$, we denote by $\Phi_\alpha$ the $L^2$–normalized Hermite function, which is an eigenfunction of $H$ with eigenvalue $2|\alpha| + d$. The set $\{\Phi_\alpha : \alpha \in \mathbb{N}_0^d\}$ forms an orthonormal basis in $L^2$. Let $\Pi_\lambda$ denote the spectral projection operator to the vector space spanned by the eigenfunctions with eigenvalue $\lambda$, i.e.,

$$\Pi_\lambda f = \sum_{\alpha : d+2|\alpha| = \lambda} \langle f, \Phi_\alpha \rangle \Phi_\alpha.$$

In this paper, we are concerned with bounds on the operator norm $\|\Pi_\lambda\|_{2 \to q}$ for $2 \leq q \leq \infty$, where $\|T\|_{s \to r}$ denotes the norm of an operator $T$ from $L^s$ to $L^r$. The sharp bound in terms of $\lambda$ has been of interest in connection to Bochner-Riesz summability of the Hermite expansion. See, Askey and Wainger [1], Karadzhov [10], and Thangavelu [24] (also, see [2] [3] and [16] for recent developments). The bounds independent of $\lambda$ have applications to the strong unique continuation problem for parabolic equations [4, 5, 13, 7].

The bounds on $\|\Pi_\lambda\|_{2 \to q}$ have been almost completely understood. When $d = 1$, the sharp bounds follow from those on $L^p$ norm of the Hermite functions in $\mathbb{R}$ ([23]). Let

$$q_0 = \frac{2(d + 3)}{d + 1}.$$
In higher dimensions \(d \geq 2\), by the works of Karadzhov [10] and Koch-Tataru [12] it is known that, for \(q \in [2, \infty] \setminus \{q_0\}\),
\[
\|\Pi_\lambda\|_{2 \to q} \sim B_q(\lambda) := \max(\lambda^{-\frac{1}{2} + \frac{d}{2} \delta(2,q)}, \lambda^\frac{d}{2} \delta(2,q) - \frac{1}{2}, \lambda^{-\frac{1}{2} \delta(2,q)}),
\]
where \(\delta(r, s) = r^{-1} - s^{-1}\). (See Notation below for the precise meaning of \(\sim\).)
The bound for \(q = 2\) is clear from Bessel’s inequality, and that for \(q = \infty\) is a consequence of the estimate for the kernel of \(\Pi_\lambda\) \((\text{see [23, Lemma 3.2.2]}\)).
Karadzhov [10] showed \(\|\Pi_\lambda\|_{2 \to 2d/(d-2)} \leq C\) for a constant \(C\).
Thangavelu [24] considered a local estimate over a compact set \(K\) and he obtained a sharp bound on \(\|\chi_\lambda \Pi_\lambda\|_{2 \to 2(d+1)/(d-1)}\).
A systematic study was carried out by Koch and Tataru and they almost completely characterized \(L^2\) bounds (\((12, \text{Corollary 3.2})\)) including the lower bounds \(\|\Pi_\lambda\|_{2 \to -q} \geq CB_q(\lambda)\) for some constant \(C > 0\) when \(2 < q < \infty\) \((\text{see [12, Section 5]}\)).
However, prior to the present work, the optimal estimate remains unsettled for \(q = q_0\). This contrasts with the spectral projections of other related differential operators whose optimal \(L^2\) bounds are well understood [18, 20, 11, 9]. By virtue of localized estimates over annuli \((\text{see (1.2) below})\), it was known [14] that
\[
\|\Pi_\lambda\|_{2 \to q_0} \leq C\lambda^{-\frac{1}{2} \delta(2,q)} (\log \lambda)^{1/q_0}.
\]
When \(d = 1\), the estimate fails without the logarithmic factor. However, when \(d \geq 2\), it was conjectured in [12] that the natural bound [14] extends to the missing endpoint \(q = q_0\). This case is the most significant since interpolation recovers the sharp bounds for \(2 < q < 2d/(d-2)\).

We prove the conjecture is true for every \(d \geq 3\).

**Theorem 1.1.** Let \(d \geq 3\). Then,
\[
\|\Pi_\lambda\|_{2 \to q_0} \sim \lambda^{-\frac{1}{2} \delta(2,q)}.
\]
It is likely that the theorem continues to be true for \(d = 2\) but our argument in this paper is not enough to prove this case.

**Localized estimate.** For \(\mu \in \mathbb{D}^- := \{2^k : -k \in \mathbb{N}\}\), we set
\[
A^\pm_\mu = \{x : \pm(1 - |x|) \in [\mu, 2\mu]\}, \quad A^\pm_{\lambda, \mu} = \{x : \lambda^{\pm \frac{1}{2}} x \in A^\pm_\mu\},
\]
respectively. For simplicity, we also denote
\[
\chi^\pm_\mu = \chi_{A^\pm_\mu}, \quad \chi^\pm_{\lambda, \mu} = \chi_{A^\pm_{\lambda, \mu}}.
\]
Of special interest is the estimate over the region near the sphere \(\sqrt{\lambda}S^{d-1} = \{x : |x| = \sqrt{\lambda}\}\), across which the kernel of \(\Pi_\lambda\) exhibits different behaviors.
Koch and Tataru [12] considered the localized operator \(\chi^\pm_{\lambda, \mu} \Pi_\lambda\). They proved the following sharp bounds:
\[
(1.2) \quad \|\chi^\pm_{\lambda, \mu} \Pi_\lambda\|_{2 \to q} \leq C \begin{cases} (\lambda^{\frac{1}{2} \delta(2,q)} \mu^{1 - \frac{d+1}{4} \delta(2,q)})^{1/2}, & 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ (\lambda \mu)^{\frac{d}{2} \delta(2,q) - 1/2}, & \frac{2(d+1)}{d-1} \leq q \leq \infty, \end{cases}
\]
for $\lambda^{-\frac{2}{3}} \leq \mu \leq 1/4$. Summation over $\mu$ and interpolation with the previously known bound give (1.1) except for $q = q_0$. Meanwhile, the estimates for $\chi_{\lambda,\mu}^+ \Pi_{\lambda}$ are of less interest, since $\chi_{\lambda,\mu}^- \Pi_{\lambda}$ has much smaller bounds thanks to rapid decay of its kernel (e.g., see (2.36) below).

Let $\mu_0 = \lambda^{-2/3}$ and $c_0 = (100d)^{-2}$. Thanks to the estimates in (12) which are mentioned above, we already have the desired $L^2 - L^{q_0}$ bounds on $\sum_{c_0 \leq \mu < 1} \chi_{\lambda,\mu}^+ \Pi_{\lambda}$ and $(1 - \sum_{\mu_0 < \mu < 1} \chi_{\lambda,\mu}^+) \Pi_{\lambda}$. (We refer the reader forward to Section 2.7 for the detail.) Therefore, to prove Theorem 1.1 it is sufficient to consider $L^2 - L^{q_0}$ bound on the operator

$$\Pi'_\lambda := \sum_{\mu_0 < \mu < c_0} \chi_{\lambda,\mu}^+ \Pi_{\lambda}.$$

By duality, $\|\Pi'_\lambda\|_{L^2 \to q_0} = \|\Pi'_\lambda(\Pi''_\lambda)^*\|_{q_0' \to q_0}$. So, if one shows $\|\Pi'_\lambda(\Pi''_\lambda)^*\|_{q_0' \to q_0} \leq C\lambda^{-\delta(q_0',q_0)/2}$, (1.1) follows since the lower bound is already shown in (12).

Since $\Pi'_\lambda = \Pi_{\lambda}$ and $\Pi''_\lambda = \Pi_{\lambda}$, we can write

$$\Pi'_\lambda(\Pi''_\lambda)^* = \sum_{\mu_0 < \mu, \tilde{\mu} < c_0} \chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda} \chi_{\lambda,\mu}^+.$$

This naturally leads us to consider $L^p - L^q$ bounds on the operators $\chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda} \chi_{\lambda,\mu}^+$ for general exponents $p, q$, not necessarily restricted to the case $p = q'$. Note that $\|\chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda} \chi_{\lambda,\mu}^+\|_{p \to q} \leq \|\chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda}\|_{2 \to q} \|\chi_{\lambda,\mu}^+ \Pi_{\lambda}\|_{2 \to q'}$. The bound (1.2) yields

$$\|\chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda} \chi_{\lambda,\mu}^+\|_{p \to q} \leq C\lambda^{-\frac{3}{2} \delta(p,q)(\mu \tilde{\mu})^{\frac{1}{2}} - \frac{3}{4} \delta(p,q)}$$

for $2(d + 1)/(d + 3) \leq p \leq 2 \leq q \leq 2(d + 1)/(d - 1)$. Attempting to add those bounds with some interpolation trick does not seem feasible to recover the missing endpoint case. Due to the optimality of (1.2), the bound (1.3) with $p = q'$ can not be improved if $\mu \sim \tilde{\mu}$. However, this does not exclude the possibility of an improved bound when $\mu \not\sim \tilde{\mu}$.

Asymmetric localization. Our main novelty is in the following theorem which shows improvement of the bounds thanks to the asymmetric localization near $\sqrt{\lambda} \mathbb{S}^{d-1}$. In other words, the bound on the operator $\chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda} \chi_{\lambda,\mu}^+, \tilde{\mu} \leq \mu$, compared with (1.3), significantly improves as $\tilde{\mu}/\mu$ gets smaller.

**Theorem 1.2.** Let $d \geq 3$, $\mu, \tilde{\mu} \in \mathbb{D}^-$, and $\lambda^{-2/3} \leq \tilde{\mu} \leq \mu \leq c_0 = (100d)^{-2}$. If $2 < q \leq 2(d + 1)/(d - 1)$, then there are positive constants $C, \delta, c$, independent of $\mu, \tilde{\mu}$, and $\lambda$, such that

$$\|\chi_{\lambda,\tilde{\mu}}^+ \Pi_{\lambda} \chi_{\lambda,\mu}^+\|_{q' \to q} \leq C\lambda^{-\delta(q',q)(\mu \tilde{\mu})^{\frac{1}{2}} - \frac{d+3}{8} \delta(q',q)(\mu/\tilde{\mu})^c}.$$

This is a new phenomenon which has not been observed before. Our approach in this paper provides an elementary alternative proof of the estimate (1.2) which corresponds to the case $\mu \sim \tilde{\mu}$. However, as we shall see later, to obtain the improved bound (1.4) is far less trivial (Section 3 and 4). The estimate (1.4) can also be extended to some other $p, q$ other than $p = q'$ by interpolation (see also [5]).

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*This is what was proved in (12), where some care with the notation $\ell^\infty L^p$ seems necessary.*
We briefly explain how one can obtain the missing endpoint bound from the estimate (1.4). More details are to be provided in Section 2.7. We write
\[ \Pi_\lambda^*(\Pi_\lambda')^* = \sum_k \mathcal{T}_k := \sum_k \sum_{(\mu, \tilde{\mu}) \in \mathcal{D}_k} \chi_\lambda^+ \mu \Pi_\lambda \chi_\lambda^+ \tilde{\mu}, \]
where \( \mathcal{D}_k = \{(\mu, \tilde{\mu}) : \tilde{\mu}/\mu \in [2^k, 2^{k+1}], \mu, \tilde{\mu} \in (\mu_0, c_0)\} \) and \( \mu, \tilde{\mu} \in \mathbb{D}^- \). Considering the adjoint operators, we also have improved bounds with the additional factor \((\mu/\tilde{\mu})^c\) when \( \mu < \tilde{\mu} \). Thus, applying Theorem 1.2 with \( q = q_0 \), one gets \( \|T_k\|_{q_0 \to q_0} \leq C 2^{-c|k|\lambda^{-\delta(q_0, q_0)/2}} \), which consequently shows the desired endpoint bound \( \|\Pi_\lambda^2\|_{2 \to q_0} \lesssim \lambda^{-1/(2d+3)} \).

**Weighted estimates.** Through the same argument we can obtain a more general result which contains the endpoint bound in Theorem 1.1. In fact, using Theorem 1.2 we prove the following weighted estimates which were conjectured in [12, Remark 3.1]. Let us set
\[ w_\pm(x) = 1 + \lambda^{-1/3}(\lambda - |x|^2)_\pm. \]

**Corollary 1.3.** Let \( d \geq 3 \) and \( 2 < q \leq \infty \). Set \( \gamma = \gamma(q) := \min\left(\frac{d+3}{4} - \frac{d}{2}, \frac{d+3}{4}, \frac{d+3}{2}\right) \). Then, for \( N > 0 \), there is a constant \( C = C(N) \) such that
\[ \|w_\pm^\gamma w_\pm^N \Pi_\lambda f\|_q \leq C \lambda^{\frac{d}{4} - \frac{d}{2}} \|f\|_2. \]

In particular, if we take \( q = q_0 \), \( \gamma = 0 \) and hence \( w_+^\gamma w_+^N \geq 1 \). So, the endpoint bound \( \|\Pi_\lambda\|_{2 \to q_0} \leq C \lambda^{-1/(2d+3)} \) follows from the estimate (1.5).

**Organization.** In Section 2, we obtain some preparatory results, and we prove Corollary 1.3 while assuming Theorem 1.2. In Section 3, we reduce the proof of Theorem 1.2 to that of an \( L^2 \) estimate, which we show in Section 4.

**Notation.** For nonnegative quantities \( A \) and \( B \), \( B \lesssim A \) means there is a constant \( C \), depending only on dimensions, such that \( B \leq CA \). Likewise, \( A \sim B \) if and only if \( B \lesssim A \) and \( A \lesssim B \). By \( D = O(A) \) we denote \( |D| \lesssim A \). Additionally, we denote \( A \gg B \) if there is a sufficiently large constant \( C > 0 \) such that \( A \geq CB \). By \( c \) and \( \varepsilon_0 \) we denote positive constants which are chosen to be small enough.

2. The Projection Operator \( \Pi_\lambda \)

In this section, we make reductions toward the proof of Theorem 1.2. We also obtain some estimates for the projection operator \( \Pi_\lambda \), which are to be used in Section 3 and 4. At the end of this section, we provide the proof of Corollary 1.3 while assuming Theorem 1.2.

2.1. The kernel of \( \Pi_\lambda \). The Hermite-Schrödinger propagator \( e^{-it\mathcal{H}} f \) is given by
\[ e^{-it\mathcal{H}} f = \sum_{\lambda \in 2\mathbb{N}_0 + d} e^{-it\lambda} \Pi_\lambda f, \quad f \in \mathcal{S}(\mathbb{R}^d). \]
Here, \( \mathcal{S}(\mathbb{R}^d) \) denotes the Schwartz class on \( \mathbb{R}^d \). It is easy to see the series converges uniformly. Since \( \mathcal{H}^N \Phi_\alpha = (2|\alpha| + d)^N \Phi_\alpha \), integration by parts gives
\[ \langle f, \Phi_\alpha \rangle = (d+2|\alpha|)^{-N} \langle \mathcal{H}^N f, \Phi_\alpha \rangle. \] Note \( \|\Phi_\alpha\|_p \leq C(1+|\alpha|)^{d/4} \) for \( 1 \leq p \leq \infty \) (e.g., see \cite{[23]}). Thus, \( \|\langle f, \Phi_\alpha \rangle\| \leq C(d+2|\alpha|)^{N+d/4} \) since \( \mathcal{H}^N f \in \mathcal{S}(\mathbb{R}^d) \).

Therefore, taking \( N \) large enough, we see the series converges uniformly.

Note \( \frac{1}{2\pi} \int_{-\pi}^\pi e^{i\frac{1}{2}(\lambda - \lambda')}dt = \delta(\lambda - \lambda'), \lambda, \lambda' \in 2\mathbb{N}_0 + d \). So, we have

\[ \Pi_\lambda f = \frac{1}{2\pi} \int_{-\pi}^\pi \sum_{\lambda' \in 2\mathbb{N}_0 + d} e^{i\frac{1}{2}(\lambda - \lambda')} \Pi_\lambda f dt, \quad f \in \mathcal{S}(\mathbb{R}^d), \]

since the series converges uniformly. By \( (2.1) \) it follows that

\[ \Pi_\lambda f = \frac{1}{2\pi} \int_{-\pi}^\pi e^{i\frac{1}{2}(\lambda - \lambda')} f dt, \quad f \in \mathcal{S}(\mathbb{R}^d). \]

Now, combining this and Mehler’s formula which expresses the kernel of \( e^{-it\mathcal{H}} \) (e.g., see \cite{[22], p.11} and \cite{[17]}), we get the following.

**Lemma 2.1.** Let \( \lambda \in 2\mathbb{N}_0 + d \). Set \( a(t) = (2\pi i \sin t)^{-d/2}e^{i\pi d/4} \) and

\[ \phi_\lambda(x, y, t) = \frac{\lambda t}{2} + \frac{|x|^2 + |y|^2}{2} \cot t - \langle x, y \rangle \csc t. \]

Then for \( f \in \mathcal{S}(\mathbb{R}^d) \), we have

\[ (2.2) \quad \Pi_\lambda f(x) = \frac{1}{2\pi} \int_{-\pi}^\pi a(t) \int e^{i\phi_\lambda(x, y, t)} f(y) dy dt. \]

In what follows, by \( T(x, y) \) we denote the kernel of an operator \( T \). Note

\[ (2.3) \quad \phi_\lambda(Ux, Uy, t) = \phi_\lambda(x, y, t), \quad U \in \mathcal{O}(d), \]

where \( \mathcal{O}(d) \) denotes the orthonormal group in \( \mathbb{R}^d \). Obviously, \( (2.3) \) implies

\[ \Pi_\lambda(Ux, Uy) = \Pi_\lambda(x, y). \]

Kochneff \cite{[15]} made the same observation by using the properties of the Hermite functions.

**Dyadic decomposition.** We dyadically decompose the integral \( (2.2) \) away from the singularities \( 0, \pm \pi \) of \( a \). To do so, let \( \psi \in C_c^\infty([\frac{1}{4}, 1]) \) such that \( \sum_{j \in \mathbb{Z}} \psi(2^j t) = 1 \) for \( t > 0 \), and then define \( \psi_0 \) by

\[ (2.4) \quad \psi_0(t) + \sum_{j \geq 4} \left( \psi(2^j t) + \psi(-2^j t) + \psi(2^j (t + \pi)) + \psi(2^j (\pi - t)) \right) = 1 \]

for \( t \in (-\pi, \pi) \setminus \{0\} \). So, \( \text{supp} \psi_0 \subset (-\pi, \pi) \setminus \{0\} \). For a bounded function \( \eta \) supported in \([-\pi, \pi]\), we consider

\[ (2.5) \quad \Pi_\lambda[\eta] = \int \eta(t)e^{i\frac{1}{2}(\lambda - \lambda')}dt. \]

Since \( \|e^{-it\mathcal{H}}\|_2 = \|f\|_2 \) for \( t \in \mathbb{R} \), we have

\[ (2.6) \quad \|\Pi_\lambda[\eta]\|_{2\to 2} \leq \|\eta\|_1. \]

For simplicity, let us denote

\[ \psi_j^\pm(t) = \psi(\pm 2^j t), \quad \psi_j^\pm(t) = \psi(2^j (\pi - t)), \quad \psi_j^0 = \begin{cases} \psi_0, & \text{if } j = 4 \\ 0, & \text{otherwise} \end{cases}, \]

for \( j > 4 \).
Then, using (2.4) we decompose

(2.7) \[ \Pi_\lambda f = \sum_{\kappa = 0, \pm, \pm \pi} \sum_{j \geq 4} \Pi_\lambda [\psi_j^\kappa] f. \]

Validity of the decomposition is clear since the sum on the right hand side converges to \( \Pi_\lambda \) as a bounded operator on \( L^2 \). Indeed, (2.6) gives \( \| \Pi_\lambda [\psi_j^\kappa] \|_{2 \to 2} \lesssim 2^{-j}, \kappa = \pm, \pm \pi \), so the convergence follows.

We obtain the estimate for \( \Pi_\lambda \) by considering each \( \Pi_\lambda [\psi_j^\kappa] \). However, thanks to symmetric properties of the kernels \( \Pi_\lambda [\psi_j^\kappa](x, y) \), the matter reduces to showing the estimates for \( \Pi_\lambda [\psi_j^+ \pi] \) and \( \Pi_\lambda [\psi_j^0 \pi] \). Indeed, observe \( \phi_\lambda(x, y, -t) = -\phi_\lambda(x, y, t) \) and \( \phi_\lambda(x, y, \pm(\pi - t)) = \pm \lambda^{\frac{d}{2}} \mp \phi_\lambda(x, y, -t) \). Then, changes of variables give

(2.8) \[ \Pi_\lambda [\psi_j^- \pi](x, y) = C_d \Pi_\lambda [\psi_j^+ \pi](x, y), \]

(2.9) \[ \Pi_\lambda [\psi_j^\pm \pi](x, y) = C'_d \Pi_\lambda [\psi_j^\mp \pi](x, -y), \]

where \( C_d, C'_d \) are constants satisfying \( |C_d| = |C'_d| = 1 \). This implies

\[ \| \sum_j \Pi_\lambda [\psi_j^\kappa] \|_{p \to q} = \| \sum_j \Pi_\lambda [\psi_j^\kappa] \|_{p \to q}, \quad \kappa = -, \pm \pi. \]

Rescaled operators. Instead of \( \Pi_\lambda \) and \( \Pi_\lambda [\eta] \), it is more convenient to work with the rescaled operators \( \Psi_\lambda, \Psi_\lambda [\eta] \) whose kernels are given by

\[ \Psi_\lambda(x, y) = \Pi_\lambda(\sqrt{\lambda}x, \sqrt{\lambda}y), \quad \Psi_\lambda [\eta](x, y) = \Pi_\lambda [\eta](\sqrt{\lambda}x, \sqrt{\lambda}y), \]

respectively. By rescaling we have, for any measurable sets \( E, F \),

(2.10) \[ \| \chi_E \Psi_\lambda [\eta] \chi_F \|_{p \to q} = \lambda^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q} - 1)} \| \chi_E \chi_F \Pi_\lambda [\eta] \|_{p \to q}. \]

To prove Theorem 1.2 we need only to consider the case

(2.11) \[ \lambda^{-2/3} \lesssim \hat{\mu}, \mu \leq c_\circ, \quad \hat{\mu}/\mu \leq \varepsilon_\circ \]

for a small constant \( \varepsilon_\circ > 0 \), since (1.4) follows from (1.2) if \( \mu \sim \hat{\mu} \).

**Proposition 2.2.** Let \( d \geq 3 \) and \( \mu, \hat{\mu} \) satisfy (2.11). Set

\[ B_p(\mu, \hat{\mu}) = \lambda^{\frac{d-1}{2} \delta(p, p') - \frac{1}{2} (\mu \hat{\mu})^{\frac{1}{2} - \frac{d-1}{2} \delta(p, p')}}. \]

Suppose \( 0 < \delta(p, p') < \min(2/(d-1), 2/3) \). Then, for a positive constant \( c \) independent of \( \mu, \hat{\mu} \), we have

(2.12) \[ \| \chi_{\hat{\mu}^{-1}} \Psi_\lambda \chi_{\hat{\mu}} \|_{p \to p'} \lesssim B_p(\mu, \hat{\mu})(\hat{\mu}/\mu)^c. \]

Note that \( \min(2/(d-1), 2/3) > 2/(d+1) \) when \( d \geq 3 \). Thus, Theorem 1.2 follows. In fact, (1.4) holds in a slightly bigger range. The rest of the paper is devoted to the proof of Proposition 2.2.
2.2. The phase function $\mathcal{P}$. Let us set
\begin{equation}
\mathcal{P}(x, y, s) = \frac{s}{2} + \frac{|x|^2 + |y|^2}{2} \cot s - \langle x, y \rangle \csc s.
\end{equation}

Note $\mathcal{P}(x, y, s) = \phi_1(x, y, s)$. By Lemma 2.1 we have
\begin{equation}
\mathfrak{P}_\lambda[\eta](x, y) = \int (\eta a)(s)e^{i\lambda \mathcal{P}(x, y, s)}ds,
\end{equation}
which we shall extensively make use of throughout the paper. To estimate the kernels, we have a close look at the phase function $\mathcal{P}$. A computation together with an elementary trigonometric identity gives
\begin{equation}
\partial_s \mathcal{P}(x, y, s) = -\frac{\mathcal{Q}(x, y, \cos s)}{2 \sin^2 s},
\end{equation}
where
\begin{equation}
\mathcal{Q}(x, y, \tau) := (\tau - \langle x, y \rangle)^2 - \mathcal{D}(x, y),
\end{equation}
\begin{equation}
\mathcal{D}(x, y) := 1 + \langle x, y \rangle^2 - |x|^2 - |y|^2.
\end{equation}

The stationary points of $\mathcal{P}(x, y, \cdot)$ are given by the zeros of $\mathcal{Q}(x, y, \cos \cdot)$. So, $\mathcal{D}(x, y)$, which determines the nature of those stationary points, plays a significant role in estimating the kernel $\mathfrak{P}_\lambda[\eta](x, y)$. Note
\begin{equation}
\mathcal{D}(x, y) = -|x|^2|y|^2 \sin^2 \angle(x, y) + (1 - |x|^2)(1 - |y|^2),
\end{equation}
where $\angle(x, y)$ denotes the angle between $x$ and $y$. Since $(1 - |x|^2)(1 - |y|^2) \sim \mu \tilde{\mu}$ for $(x, y) \in A^+ \times A^+$, one can control $\mathcal{D}(x, y)$ by the relative size of $\angle(x, y)$ to $(\mu \tilde{\mu})^{1/2}$.

2.3. Preliminary decomposition. Fix a constant $c > 0$ such that $1/(20d) \leq c \leq 1/(10d)$. We partition the unit sphere $S^{d-1}$ into finite disjoint subsets $\{S_j\}$ of diameter less than $c$. Then, set $A_j = \{x \in A^+ : |x|^{-1}x \in S_j\}$ and $\tilde{A}_j = \{x \in A^+ : |x|^{-1}x \in S_j\}$, which respectively partition the annuli $A^+$ and $A^+$ into disjoint sets of diameter $\leq c$. So, we have
\begin{equation}
A^+_\mu = \bigcup_k A_k, \quad A^+_{\tilde{\mu}} = \bigcup_l \tilde{A}_l.
\end{equation}
This and (2.7) give
\begin{equation}
\chi^+_\mu \mathfrak{P}_\lambda \chi^+_\mu f = \sum_{k, l} \sum_{\kappa=0, \pm \pi} \sum_{j \geq 4} \chi_{A_k} \mathfrak{P}_\lambda[\psi^+_j]\chi_{\tilde{A}_l} f.
\end{equation}

Using this coarse decomposition, we can distinguish minor parts whose contributions are negligible. More precisely, we have the following.

Lemma 2.3. Let $1 \leq p \leq 2$. Let $j \geq 4$, $\lambda^{-2/3} \lesssim \mu, \tilde{\mu} \lesssim c_2$, and $(20d)^{-1} \leq c \leq (10d)^{-1}$. Suppose $\text{dist}(A_k, \tilde{A}_l) \geq c$ and $\text{dist}(A_k, -\tilde{A}_l) \geq c$. Then,
\begin{equation}
\|\chi_{A_k} \mathfrak{P}_\lambda[\psi^+_j]\chi_{\tilde{A}_l} f\|_{p'} \lesssim \lambda^{-N} 2^{-Nj}(\mu \tilde{\mu})^N\|f\|_p, \quad \forall N > 0
\end{equation}
for $\kappa = 0, \pm \pi$. Moreover, (i) if $\text{dist}(A_k, \tilde{A}_l) < c$, we have (2.20) for $\kappa = \pm \pi, 0$; (ii) if $\text{dist}(A_k, -\tilde{A}_l) < c$, we have (2.20) for $\kappa = \pm, 0$. 

To show Lemma 2.3, we use the following elementary lemmas.

Lemma 2.4. Let $E$, $F$ be measurable sets. If $|T(x, y)| \leq D \chi_E(x) \chi_F(y)$ for a constant $D$, then $\|T\|_{p \to q} \leq D|E|^{1/q} |F|^{1/p}$ for $1 \leq p, q \leq \infty$.

Lemma 2.5. Let $N \in \mathbb{N}_0$, $0 < \mu \leq 1$, and $L, \lambda > 0$. Suppose $a$ is a smooth function supported in an interval $I$ of length $\sim \mu$ and $\phi$ is smooth on $I$. If $|\phi'| \gtrsim L$, $(d/ds)^{k+1} \phi \lesssim L \mu^{-k}$, and $(d/ds)^k a \lesssim \mu^{-k}$, $0 \leq k \leq N$ on $I$, then

$$
(2.21) \quad \int e^{i\lambda \phi(s)} a(s) ds \leq C \mu(1 + \lambda \mu L)^{-N}
$$

for a constant $C = C(N)$ independent of $\lambda, L$ and $\mu$.

One can show Lemma 2.5 by repeated integration by parts. It can also be shown by changing variables $s \to \mu s + s_0$ where $s_0 \in \text{supp} a$. Setting $\tilde{\phi}(s) = (\mu L)^{-1}\phi(\mu s + s_0)$ and $\tilde{a}(s) = a(\mu s + s_0)$, we see $|\tilde{\phi}'| \gtrsim 1$, $(d/ds)^k \tilde{a} \lesssim 1$, and $(d/ds)^{k+1} \tilde{\phi} \lesssim 1$, $0 \leq k \leq N$, on $\text{supp} \tilde{a}$. Since the integral equals $\mu \int e^{i\lambda \mu L \tilde{\phi}(s)} \tilde{a}(s) ds$, routine integration by parts yields (2.21).

Proof of Lemma 2.3. We first prove (2.20) for $\kappa = 0, \pm, \pm \pi$ when $\text{dist}(A_k, \hat{A}_l) \geq c$ and $\text{dist}(A_k, -\hat{A}_l) \geq c$.

Let $(x, y) \in A_k \times \hat{A}_l$. Note $|x|^2 |y|^2 \sin^2 \angle(x, y) \geq 2^{-1} \pi^2$ and $0 < \mu, \tilde{\mu} \leq 2^{-3} \pi^2$. By (2.17) and (2.16), $-D(x, y) \sim \cos 2 \theta$ and $Q(x, y, \tau) \sim \cos 2 \tau$ for $\tau \in \mathbb{R}$.

By (2.15), we have $|\partial_s P| \gtrsim 2^{2j}$ and also $|\partial_s^l P| \lesssim 2^{l+1-2j}$ on $\text{supp} \phi_j^\kappa$. Note $(\partial_s/ds)^m (a\phi_j^\kappa) = O(2^{d/2+m}j)$. Thus, using Lemma 2.3, we get

$$
(2.22) \quad \|\chi_{A_k} \phi_j^\kappa(x, y)\| \lesssim 2^{(d/2-1)j} (\lambda 2^l + 1)^{-M}, \quad (x, y) \in (A_k, \hat{A}_l)
$$

for any $M$. Since $\lambda^{-2/3} \lesssim \mu, \tilde{\mu} \leq c_0$, (2.20) follows by Lemma 2.4.

To prove (i) and (ii), it is sufficient to show (ii) only thanks to (2.9) and the change of variables $y \to -y$. Let $(x, y) \in A_k \times \hat{A}_l$. Since $\text{dist}(A_k, -\hat{A}_l) < c$, $|1 + (x, y)| \leq 3c$ and $|D(x, y)| \leq 2c$ (by (2.17)). Note $\cos s \geq -\cos 2^{-3}$ on $\text{supp} \phi_j^\kappa$, $\kappa = \pm, 0$. Thus, using (2.16), we have $|Q(x, y, \cos s)| \sim 1$ and $|\partial_s P(x, y, s)| \gtrsim 2^{2j}$. As before, we also have $|\partial_s^l P| \lesssim 2^{l+1} j$ for $l \geq 1$ if $s \in \text{supp} \phi_j^\kappa$, $\kappa = \pm, 0$. Hence, Lemma 2.3 yields (2.22). Consequently, the estimate (2.20) follows in the same manner as above.

The bounds in Lemma 2.3 are much smaller than what we need to obtain for $\mathfrak{P}_\lambda$. Recalling (2.18) and (2.19), and discarding the harmless small contributions, we need only to consider $\chi_A \mathfrak{P}_\lambda \phi_j^\kappa \chi_{\hat{A}}$ when $A \subset A_{\mu}^\kappa$, $\hat{A} \subset A_{\mu}^\kappa$ are of diameter $c$ and

$$\text{dist}(A, \hat{A}) \leq c, \kappa = \pm, \quad \text{or} \quad \text{dist}(A, -\hat{A}) \leq c, \kappa = \pm \pi.$$

By (2.9) and changing variables $y \to -y$, the estimate for the second case can be deduced from that for the first, so it suffices to consider the first case only.
Moreover, the estimate for $\chi_A \mathcal{P}_\lambda [\psi_j^+] \chi_A$ follows from that for $\chi_A \mathcal{P}_\lambda [\psi_j^+] \chi_A$ thanks to (2.8). Therefore, the matter is reduced to showing

\begin{equation}
(2.23) \quad \left\| \sum_{j \geq 4} \chi_A \mathcal{P}_\lambda [\psi_j^+] \chi_A \right\|_{p \to p'} \lesssim B_p(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c
\end{equation}

when $A \subset A_\mu^+, \tilde{A} \subset A_{\tilde{\mu}}^+$, and $\text{dist}(A, \tilde{A}) \leq c$. Henceforth, we denote $\psi_j = \psi_j^+$. By (2.3) and (2.14), it follows that

\begin{equation}
\nu \geq \nu_k \chi_A \mathcal{P}_\lambda [\psi_j] \chi_A \mathcal{P}_\lambda [\psi_j](Ux, Uy), \quad U \in O(d).
\end{equation}

Set $S = \{e' \in \mathbb{S}^{d-1} : |e' - e_1| < 1/(25d)\}$. By rotation we may also assume $A, \tilde{A} \subset A_0 := \{x : |x|^{-1} x \in S\}$.

2.4. Sectorial decomposition of annuli. We decompose $A \times \tilde{A}$ in a way that we can conveniently control the angle between $x$ and $y$. To do so, we use a Whitney type decomposition of $S \times S$ away from its diagonal.

Following the typical dyadic decomposition process, for each $\nu \geq 0$ we partition $S$ into spherical caps $\Theta_k^\nu$ such that $\Theta_k^\nu \subset \Theta_{k'}^\nu$ for some $k'$ whenever $\nu \geq \nu'$ and $c_d 2^{-\nu} \leq \text{diam}(\Theta_k^\nu) \leq C_d 2^{-\nu}$ for some constants $c_d, C_d > 0$. Let $\nu_0(\mu, \tilde{\mu})$ denote the integer $\nu_0$ such that

$$
\mu \tilde{\mu}/2 < 2^6 C_d^2 2^{-2\nu_0} \leq \mu \tilde{\mu}.
$$

Then, we can write

$$
S \times S = \bigcup_{\nu_0 - 2^{-\nu} \leq 2^{-\nu} \leq 1} \bigcup_{k \sim \nu k'} \Theta_k^\nu \times \Theta_{k'}^\nu,
$$

where $k \sim \nu k'$ means $\text{dist}(\Theta_k^\nu, \Theta_{k'}^\nu) \sim 2^{-\nu}$ if $\nu \geq \nu_0$ and $\text{dist}(\Theta_k^\nu, \Theta_{k'}^\nu) \lesssim 2^{-\nu}$ if $\nu = \nu_0$ (e.g., see [21, p.971]). The sets $\Theta_k^\nu$ and $\Theta_{k'}^\nu$, $k \sim \nu_0 k'$ are not necessarily distanced from each other since the decomposition process terminates at $\nu = \nu_0$. Then, it follows that

$$
A \times \tilde{A} \subset \bigcup_{\nu_0 - 2^{-\nu} \leq 2^{-\nu} \leq 1} \bigcup_{k \sim \nu k'} A_k^\nu \times \tilde{A}_{k'}^\nu,
$$

where

$$
A_k^\nu = \{x \in A_\mu^+ : |x|^{-1} x \in \Theta_k^\nu\}, \quad \tilde{A}_{k'}^\nu = \{x \in A_{\tilde{\mu}}^+ : |x|^{-1} x \in \Theta_{k'}^\nu\}.
$$

Let $\chi_k^\nu = \chi_{A_k^\nu}$ and $\tilde{\chi}_{k'}^\nu = \chi_{\tilde{A}_{k'}^\nu}$. The estimate (2.23) follows once we obtain

\begin{equation}
(2.25) \quad \left\| \sum_{j \geq 4} \sum_{\nu_0 \leq \nu \leq \nu_0 k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j^+] \chi_k^\nu \right\|_{p \to p'} \lesssim B_p(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c,
\end{equation}

which we prove in Section 3 for $0 < \delta(p, p') < \min(2/(d-1), 2/3)$.

We occasionally use the next elementary lemma.

**Lemma 2.6.** Let $1 \leq p \leq q \leq \infty$. Suppose $\|\chi_k^\nu \mathcal{P}_\lambda [\psi_j^+] \chi_k^\nu\|_{p \to q} \leq B$ holds whenever $k \sim \nu k'$. Then for a constant $C$,

$$
\|\sum_{k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j^+] \chi_k^\nu\|_{p \to q} \leq CB.
$$
2.5. The kernel of $\mathfrak{P}_\lambda[\psi_j]$. In this subsection, we obtain estimates for the kernel $\mathfrak{P}_\lambda[\psi_j](x, y)$, which we use later.

**Lemma 2.7.** Let $0 < \bar{\mu} \leq \mu \leq 1/100$, and $(x, y) \in A_k^\nu \times \tilde{A}_{k'}^\nu$, $k \sim_\nu k'$. Then for any $N > 0$, we have

$$\| \mathfrak{P}_\lambda[\psi_j](x, y) \| \lesssim 2^{-d/2} (\lambda 2^j \max(2^{-2\nu}, 2^{-4j}) + 1)^{-N}. \tag{2.26}$$

**Proof.** Note $A(x, y) \sim 2^{-\nu}$ for $(x, y) \in A_k^\nu \times \tilde{A}_{k'}^\nu$, $k \sim_\nu k'$. Since $\mu \ll 2^{-\nu}$, it is easy to see that $|x - y| \sim 2^{-\nu}$. So, $\sim D(x, y) \sim 2^{-2\nu}$ by (2.17). Note $|x - y|^2 + D(x, y) = (1 - (x, y))^2$, hence $|1 - (x, y)| \lesssim 2^{-\nu}$. Combining these observations with (2.15) and (2.16), we have

$$|\partial_s \mathcal{P}(x, y, s)| \gtrsim 2^{2j} \max(2^{-2\nu}, 2^{-4j}), \quad s \in \text{supp } \psi_j.$$

By (2.15), it also follows that $|\partial_s^k \mathcal{P}(x, y, s)| \lesssim 2^{(1 + k)j} \max(2^{-2\nu}, 2^{-4j})$ for $s \in \text{supp } \psi_j$. Thus, using Lemma 2.5 with $L = 2^{2j} \max(2^{-2\nu}, 2^{-4j})$ and $\mu = 2^{-j}$, we get (2.26) in the same manner as in the proof of Lemma 2.3. \hfill \square

**Lemma 2.8.** Let $0 < \bar{\mu} \ll \mu \leq 1/100$ and $(x, y) \in A_k^\nu \times \tilde{A}_{k'}^\nu$, $k \sim_\nu k'$. Suppose $2^{-\nu} \leq 2^{-\nu} \lesssim \mu$. Then for any $N > 0$, we have

$$\| \mathfrak{P}_\lambda[\psi_j](x, y) \| \lesssim \begin{cases} 2^{-d/2} (\lambda 2^j \mu^2 + 1)^{-N}, & 2^{-j} \ll \sqrt{\mu}, \\
2^{-d/2} (\lambda 2^{-3j} + 1)^{-N}, & 2^{-j} \gg \sqrt{\mu}. \end{cases} \tag{2.27}$$

Additionally, if $2^{-j} \sim \sqrt{\mu}$, and $D(x, y) \sim \bar{\mu}$ or $D(x, y) < 0$, then

$$\| \mathfrak{P}_\lambda[\psi_j](x, y) \| \lesssim \lambda^{-1/2} \mu^{1/4} |D(x, y)|^{-1/4}. \tag{2.28}$$

**Proof.** We consider (2.27) first. To this end, we claim

$$|\partial_s \mathcal{P}(x, y, s)| \gtrsim \begin{cases} \mu^2 2^{2j}, & 2^{-j} \ll \sqrt{\mu}, \\
2^{-2j}, & 2^{-j} \gg \sqrt{\mu}. \end{cases} \tag{2.29}$$

Note $2(1 - (x, y)) = 1 - |x|^2 + 1 - |y|^2 - |x - y|^2$. Since $|x - y| \sim \mu$, $|1 - (x, y)| \sim \mu$. So, $|\partial_s Q(x, y, \tau)| = 2|\tau - (x, y)| \lesssim \mu$ if $\tau \in [1 - c\mu, 1]$ for some $c > 0$. If $2^{-j} \ll \sqrt{\mu}$, by the mean value theorem $|Q(x, y, \cos s) - Q(x, y, 1)| \ll \mu^2$ for $s \in \text{supp } \psi_j$ because $|1 - \cos s| \leq s^2/2$. Observing $Q(x, y, 1) = |x - y|^2 \sim \mu^2$, we see $Q(x, y, \cos s) \sim \mu^2$ for $s \in \text{supp } \psi_j$ if $2^{-j} \ll \sqrt{\mu}$. Thus, by (2.15) we have the first case in (2.29). For the second case, note $|D(x, y)| \lesssim 2^{-2\nu} \lesssim \mu^2$ and $1 - \tau \sim 2^{-2j}$ for $\tau \in \text{supp } \psi_j$. Recalling (2.16), we have $Q(x, y, \tau) = (\tau - (x, y))^2 - D(x, y) \sim 2^{-4j}$ if $2^{-j} \gg \sqrt{\mu}$. So, by (2.15) we obtain the second case in (2.29).
A computation using (2.15) shows

\[(2.30) \quad |\partial_s^k \mathcal{P}(x, y, s)| \lesssim \begin{cases} \mu^2 2^{(1+k)j}, & 2^{-j} \ll \sqrt{\mu}, \\ 2^{-j} 2^{1+(1+k)j}, & 2^{-j} \gg \sqrt{\mu}, \end{cases} \quad s \in \text{supp} \psi_j.\]

Therefore, combining (2.29) and (2.30), we obtain (2.27) by Lemma 2.5.

We now turn to (2.28) and consider the case \(D(x, y) \sim \mu \tilde{\mu}\) first. Since \(|1 - \langle x, y \rangle| \sim \mu, \mathcal{Q}(x, y, \cdot)\) has two distinct zeros \(r_1 > r_2\), which are close to 1. Let \(s_1, s_2 \in (0, \pi/2)\) be numbers such that \(\cos s_i = r_i, i = 1, 2, \) and \(s_1 < s_2\).

By (2.15) and (2.16) we have

\[L_{k, j} = \sum_{k \geq k_0} I_{k, j}^{k, +} \quad L_{k, j} = \sum_{k \geq k_0} I_{k, j}^{k, -}, \quad L_{k, j} = \sum_{k \geq k_0} I_{k, j}^{k, +} \quad L_{k, j} = \sum_{k \geq k_0} I_{k, j}^{k, -}, \quad L_{k, j} = \sum_{k \geq k_0} I_{k, j}^{k, +}.\]

By van der Corput’s lemma, it follows that \(|\partial_s \mathcal{P}(x, y, s)| \gtrsim \mu\) for \(s \in \text{supp} \psi_j\). Since \(\|\psi \partial_s \psi\|_1 \lesssim 2^{d/2}\), van der Corput’s lemma (e.g., [10, p. 334]) gives \(|I_1| \lesssim \min(\lambda^{-1} 2^{d+1} D^{-1}, 2^{d+2} D^{1/2})\), which yields \(|I_1| \lesssim \lambda^{-1/2} 2^{(1-d)/2} D^{-1/4} \). Therefore, we have only to show

\[|I_1^{\pm}| \lesssim \lambda^{-1/2} 2^{(1-d)/2} D^{-1/4}, \quad l = 1, 2.\]

We consider \(I_1^{\pm}\) only. The estimates for the others can be shown in a similar manner. Since \(s_2 - s_1 \sim \mu^{1/2}\), by (2.31) we have \(|\partial_s \mathcal{P}(x, y, s)| \gtrsim 2^{j/2 - k} D^{1/2}\) for \(s \in \text{supp} \psi_j^{k, -}\). By van der Corput’s lemma, \(|I_1^{k, -}| \lesssim 2^{d/2} \min(\lambda^{-1} 2^{d-k} D^{-1/2}, 2^{-k})\). Summation over \(k\) gives \(|I_1^{\pm}| \lesssim \lambda^{-1/2} 2^{(d-1)/2} D^{-1/4}\) as desired.

Following the previous argument closely, we show (2.28) when \(D < 0\). From (2.15) and (2.16), we have

\[(2.32) \quad |\partial_s \mathcal{P}(x, y, s)| \sim ((\cos s - \langle x, y \rangle)^2 + |D|) \mu^{-1}, \quad s \in \text{supp} \psi_j.\]

Let \(s_* \in (0, \pi/2)\) denote the point such that \(\cos s_* = \langle x, y \rangle\), and let \(k_*\) be the smallest number satisfying \(2^{-k_*} \leq |D|^{1/2} \mu^{-1/2}\). We decompose the integral
We begin by showing
$$
I^k := \int (\psi^k \psi_j(s))e^{i\lambda P(x,y,s)} ds,
$$
we break $I_j = \sum_{k \leq k_*} I^k$. From (2.32) we see $|\partial_s P(x,y,s)| \gtrsim 2^{-2k}$ for $s \in \text{supp} \psi^k$. Thus, the van der Corput lemma gives $|I^k| \lesssim \mu^{-d/4} \min(\lambda^{-1} 2^k, 2^{-k}) \lesssim \mu^{-d/4} \lambda^{-1/2} 2^{k/2}$. Taking sum over $k \leq k_*$, we get (2.28).

2.6. **An $L^2$-estimate for $\mathfrak{P}_\lambda \eta_j$**. We denote $A_\mu^0 = \{ x : |1 - |x|| \leq 2 \mu \}$ and $A_{\lambda, \mu}^0 = \{ x : \lambda^{-1/2} x \in A_\mu^0 \}$. We also set
$$
\chi_\mu^o = \chi A_\mu^0, \quad \chi_{\lambda, \mu}^o = \chi A_{\lambda, \mu}^0.
$$

**Lemma 2.9.** Let $\lambda^{-\frac{d}{2}} \lesssim \tilde{\mu}, \mu \leq 1/4$ and $2^{-j} \gtrsim (\lambda \mu)^{-1}$. Suppose $\eta_j \in C_c^\infty(-\pi, \pi)$ supported in an interval of length $\sim 2^{-j}$ satisfies $|\eta_j^{(k)}| \lesssim 2^k$ for any $k$.

$$
\| \chi^0_{\lambda, \mu} \mathfrak{P}_\lambda \eta_j \chi^0_{\lambda, \mu} \|_{2 \to 2} \lesssim \lambda^{-\frac{d}{2}} (\mu \tilde{\mu})^{\frac{1}{2}}.
$$

To prove (2.33), we instead show $\| \chi_{\lambda, \mu}^0 \Pi \lambda \eta_j \xi_{\lambda, \tilde{\mu}} \|_{2 \to 2} \lesssim (\mu \tilde{\mu})^{1/4}$ which is equivalent to (2.33) (see (2.11)). In fact, we can show a stronger estimate
$$
\| \chi_{\lambda, \mu}^0 \Pi \lambda \eta_j \xi_{\lambda, \tilde{\mu}} \|_{2 \to 2} \lesssim (\mu \tilde{\mu})^{\frac{1}{4}}
$$
for $\lambda^{-\frac{d}{2}} \lesssim \tilde{\mu}, \mu \leq 1/4$, where
$$
\chi_{\lambda, \mu}^e = \chi A_{\lambda, \mu}^0, \quad A_{\lambda, \mu}^0 = \{ x : |x| \geq \lambda^{1/2} (1 - \mu) \}.
$$

We now recall the next estimates which follow from [12] Theorem 3:

$$
\| \chi_{\lambda, \mu}^0 \Pi \lambda \eta_j \xi_{\lambda, \mu} \|_{p \to p'} \lesssim \lambda^{\frac{d}{2}} (\lambda \mu)^{1/4} \lesssim \lambda^{d/2} (\lambda \mu)^{1/4}
$$

for $1 \leq p \leq 2$. The estimate (2.35) holds for any $M > 0$. In particular, the $L^2-L^\infty$ estimate in [12] Theorem 3 implies (2.36) for $p = 1$. Interpolation with $L^2$ estimate shows (2.36) for $1 \leq p \leq 2$. One can also show (2.35) and (2.36) in an elementary manner using estimates for the kernels of $\Pi \lambda$ (e.g., see [8]).

**Proof of (2.34).** Considering the adjoint operator, we may assume $\tilde{\mu} \leq \mu$. We begin by showing
$$
\| \chi_{\lambda, \mu}^e \Pi \lambda \eta_j \chi_{\lambda, \mu}^e \|_{2 \to 2} \lesssim (\mu \tilde{\mu})^{\frac{1}{4}},
$$

Note $\| \chi_{\lambda, \mu}^e \Pi \lambda \eta_j \chi_{\lambda, \mu}^e \|_{2 \to 2} \leq \| \chi_{\lambda, \mu}^e \Pi \lambda \|_{2 \to 2} \Pi \lambda \chi_{\lambda, \mu}^e \|_{2 \to 2}$. By duality it is enough to show $\| \chi_{\lambda, \mu}^e \Pi \lambda \|_{2 \to 2} \lesssim \mu^{1/4}$ for $\lambda^{-2/3} \lesssim \mu \leq 1$. To do this, recalling that $\mu, \tilde{\mu}$ denote dyadic numbers, we decompose
$$
\chi_{\lambda, \mu}^e = \sum_{\lambda^{-2/3} \leq \tilde{\mu} \leq \mu} \chi_{\lambda, \mu}^e + \chi_{\lambda, \lambda^{-2/3}} + \sum_{\lambda^{-2/3} < \tilde{\mu} < 1} \chi_{\lambda, \tilde{\mu}} + \sum_{1 \leq \tilde{\mu} \leq \mu} \chi_{\lambda, \tilde{\mu}}.
$$
By (2.36), it follows that \( \sum_{1 \leq \tilde{\mu}} \| \chi_{\lambda, \tilde{\mu}} \Pi_{\lambda} \|_{2 \rightarrow 2} \lesssim \lambda^{-N} \) for any \( N \). The estimates (1.2), (2.35), and (2.36) yield \( \sum_{\lambda^{-2/3} \leq \mu \leq \lambda} \| \chi_{\lambda, \mu} \Pi_{\lambda} \|_{2 \rightarrow 2} \lesssim \mu^{1/4} \), \( \| \chi_{\lambda, \lambda^{-2/3}} \Pi_{\lambda} \|_{2 \rightarrow 2} \lesssim \lambda^{-1/6} \), and \( \sum_{\lambda^{-2/3} \leq \mu \leq 1} \| \chi_{\lambda, \mu} \Pi_{\lambda} \|_{2 \rightarrow 2} \lesssim \lambda^{-1/6} \), respectively. Therefore, we get the desired estimate since \( \mu \gtrsim \lambda^{-2/3} \).

Now, observe \( \Pi_{\lambda} \eta_{j} = \sum_{\lambda'} \tilde{\eta}_{j}(2^{-1/2}(\lambda' - \lambda)) \Pi_{\lambda'}, \) which follows by (2.5) and (2.11). Since \( \| \tilde{\eta}_{j}(\tau) \|_{2^{-1/2}(1 + 2^{-1/2}|\tau|)^{-N}} \), we have
\[
\| \chi_{\lambda, \tilde{\mu}} \Pi_{\lambda} \eta_{j} \chi_{\lambda, \tilde{\mu}} \|_{2 \rightarrow 2} \lesssim 2^{-j} \sum \lambda' (1 + 2^{-j}|\lambda - \lambda'|)^{-N} \| \chi_{\lambda, \tilde{\mu}} \Pi_{\lambda'} \chi_{\lambda, \tilde{\mu}} \|_{2 \rightarrow 2}.
\]
However, we cannot directly apply (2.37) since \( \lambda \neq \lambda' \). We get around the problem by enlarging the sets \( A_{\lambda, \tilde{\mu}}, A_{\lambda', \tilde{\mu}} \). Let
\[
\ell_{\lambda}(\rho) = \begin{cases} 
(\lambda/\lambda')^{1/2} \rho, & \lambda \geq \lambda', \\
(\lambda' - \lambda)/\lambda' + \rho, & \lambda < \lambda'.
\end{cases}
\]
Note that \( \ell_{\lambda}(\rho) \gtrsim (\lambda')^{-2/3} \) for \( \lambda^{-2/3} \leq \rho \). Since \( (\lambda')^{1/2}(1 - \ell_{\lambda}(\mu)) \leq \lambda^{1/2}(1 - \mu) \), i.e., \( A_{\lambda, \mu} \subset A_{\lambda', \ell_{\lambda}(\mu)} \), \( \| \chi_{\lambda', \tilde{\mu}} \Pi_{\lambda'} \chi_{\lambda, \tilde{\mu}} \|_{2 \rightarrow 2} \leq \| \chi_{\lambda', \ell_{\lambda}(\mu)} \Pi_{\lambda'} \chi_{\lambda, \ell_{\lambda}(\mu)} \|_{2 \rightarrow 2} \).

Using (2.37), we have \( \| \chi_{\lambda, \tilde{\mu}} \Pi_{\lambda} \chi_{\lambda, \tilde{\mu}} \|_{2 \rightarrow 2} \lesssim (\ell_{\lambda}(\mu) \ell_{\lambda}(\tilde{\mu}))^{1/4} \). Therefore, it suffices for (2.34) to show
\[
2^{-j} \sum \lambda' (1 + 2^{-j}|\lambda - \lambda'|)^{-N} (\ell_{\lambda}(\mu) \ell_{\lambda}(\tilde{\mu}))^{1/4} \lesssim (\mu \tilde{\mu})^{-1/4}.
\]
This can be shown by a simple computation because \( 2^{-j} \gtrsim (\lambda \mu)^{-1} \) and \( \mu, \tilde{\mu} \gtrsim \lambda^{-2/3} \). We omit the detail. \( \square \)

2.7. Proof of Corollary 1.3. Before we conclude this section, assuming Theorem 1.2 we prove Corollary 1.3. We follow the lines of argument for the endpoint bound which is sketched in the introduction.

Proof of Corollary 1.3. We first prove (1.5) for \( 2 < q \leq q_{*} := 2(d+1)/(d-1) \). Let us set
\[
\mathcal{W} = w_{+}^{\delta(2,q)-1/2}.
\]
Note that \( \mathcal{W} = w_{+}^{\gamma} \) if \( 2 < q \leq q_{*} \). Recall \( \mu_{o} = \lambda^{-2/3} \) and \( c_{o} = (100d)^{-2} \). By the triangle inequality, \( \| \mathcal{W}w_{+}^{N} \Pi_{\lambda} \|_{2 \rightarrow q} \) is bounded by
\[
\| \sum_{\mu \geq \mu_{o}} \mathcal{W} \chi_{\lambda, \mu} \Pi_{\lambda} \|_{2 \rightarrow q} + \| \chi_{\lambda, \mu_{o}} \Pi_{\lambda} \|_{2 \rightarrow q} + \sum_{\mu \geq \mu_{o}} (\lambda^{\delta} 2^{k} (2^{k} + 1))^{N} \| \chi_{\lambda, 2^{k}} \Pi_{\lambda} \|_{2 \rightarrow q},
\]
where \( \mu \in \mathcal{D}^{-} \). By the estimates (2.35) and (2.36), the last two are bounded by \( C \lambda^{(\delta(2,q)-1/6)} \). It follows by (1.2) that \( \sum_{\mu \geq \mu_{o}} \| \mathcal{W} \chi_{\lambda, \mu} \Pi_{\lambda} \|_{2 \rightarrow q} \lesssim \lambda^{(\delta(2,q)-1/6)} \). Therefore, the matter is reduced to showing
\[
\| \mathcal{W} \|_{2 \rightarrow q} \lesssim \lambda^{\delta(2,q)-1/2},
\]
where \( \mathcal{W} = \sum_{\mu_{o} < \mu < c_{o}} \mathcal{W} \chi_{\lambda, \mu} \Pi_{\lambda} \). Equivalently, we need to show
\[
(2.38) \quad \| \mathcal{W} \mathcal{W} \mathcal{W}^{*} \|_{q' \rightarrow q} \lesssim \lambda^{\delta(q',q)-1/2}.
\]
To this end, we write
\[ \mathfrak{M} = \sum_k (\mathfrak{M})_k := \sum_k \sum_{(\mu, \bar{\mu}) \in \mathfrak{D}} \mathcal{W}_{\lambda, \mu}^+ \Pi \mathcal{W}_{\lambda, \bar{\mu}}^+. \]
Recall that \( \mathfrak{D} = \{(\mu, \bar{\mu}) : \bar{\mu}/\mu \in [2^k, 2^{k+1}) \}, \mu, \bar{\mu} \in (\mu_0, c_0) \} \) and \( \mu, \bar{\mu} \in \mathfrak{D} \).
Since \( \text{supp} \chi_{\lambda, \mu}^+ \) are almost disjoint, for each \( k \) we have
\[ \| (\mathfrak{M})_k f \|_q \lesssim (\sum_{(\mu, \bar{\mu}) \in \mathfrak{D}} \| \mathcal{W}_{\lambda, \mu}^+ \Pi \mathcal{W}_{\lambda, \bar{\mu}}^+ f \|_q^{q})^{1/q}. \]
Note \( \mathcal{W}_{\lambda, \mu}^+ \sim \lambda^{d_{\lambda,q}(q') - \frac{1}{2}} \mu^{\frac{d_{\lambda,q}(q')}{2} - \frac{1}{4}} \chi_{\lambda, \mu}^+. \) Using (1.4), we get
\[ \| \mathcal{W}_{\lambda, \mu}^+ \Pi \mathcal{W}_{\lambda, \bar{\mu}}^+ f \|_q \lesssim \lambda^{\frac{d_{\lambda,q}(q') - \frac{1}{2}}{2} - c|k|} \chi_{\lambda, \mu}^+ f \|_{q'}, \ \bar{\mu}/\mu \sim 2^k. \]
Indeed, when \( 2^k > 1 \), we consider the adjoint operator \( T^* = \mathcal{W}_{\lambda, \mu}^+ \Pi \mathcal{W}_{\lambda, \mu}^+ \) of \( T := \mathcal{W}_{\lambda, \mu}^+ \Pi \mathcal{W}_{\lambda, \bar{\mu}}^+ \) and then we may use the estimate (1.4) thanks to the fact that \( \| T \|_{q' \rightarrow q} = \| T^* \|_{q' \rightarrow q} \). Since \( \sum_{\bar{\mu}} \| \chi_{\lambda, \mu}^+ f \|_{q'}^{q} \lesssim \| f \|_{q'}^{q} \), we obtain
\[ \| (\mathfrak{M})_k f \|_q \lesssim 2^{-c|k|} \lambda^{\frac{d_{\lambda,q}(q') - \frac{1}{2}}{2}} \| f \|_{q'}. \]
Summation over \( k \) gives the desired estimate (2.38).

We now prove (1.5) for \( 2(d+1)/(d-1) < q \leq \infty \). Note \( \gamma = \frac{1}{2} - \frac{d}{2} \delta(2,q) \). By decomposing the operator in the same way as above, it suffices to show
\[ \sum_{\mu_0 < \mu < c_0} \| \mathcal{W}_{\lambda, \mu}^+ \Pi \mathcal{W}_{\lambda, \mu}^+ f \|_{2 \rightarrow q} \lesssim \lambda^{\frac{d_{\lambda,q}(2,q) - \frac{1}{2}}{2}} \| f \|_{q'} \]
for \( 2(d+1)/(d-1) < q \leq \infty \). The other parts can be handled similarly as before. By interpolation, we need only to show (2.39) for \( q = \infty \), \( 2(d+1)/(d-1) \). Thanks to disjointness of the annuli, (1.2) for \( q = \infty \) gives (2.39) for \( q = \infty \), while (2.38) is equivalent to (2.39) when \( q = 2(d+1)/(d-1) \). □

3. Asymmetric improvement: Proof of Proposition 2.2

We prove Proposition 2.2 by establishing the estimate (2.25). To this end, we separately consider some cases. The desired estimates can be shown by the kernel estimates (in the previous section) except for the case \( 2^{-j} \sim \sqrt{\mu} \), \( 2^{-\nu} \sim (\mu \bar{\mu})^{1/2} \), and \( |\mathfrak{D}| < \varepsilon \mu \bar{\mu} \), which requires a different approach. We handle this case in the next section.

To show (2.25), we distinguish the cases \( \nu \in \mathcal{N}_e \) and \( \nu \in \mathcal{N}_c \), where
\[ \mathcal{N}_e = \{ \nu : 2^{-\nu} \gg (\mu \bar{\mu})^{1/2} \text{ or } \nu = \nu_0 \}, \quad \mathcal{N}_c = \{ \nu : (\mu \bar{\mu})^{1/2} \gtrsim 2^{-\nu} > 2^{-\nu_0} \}. \]

3.1. The sum over \( \nu \in \mathcal{N}_e \). In this case, the desired estimates are easier.

**Proposition 3.1.** Let \( d \geq 2 \) and \( \mu, \bar{\mu} \) satisfy (2.11). If \( 2/(d+3) < \delta(p, p') < 2/(d-1) \), then for some \( c > 0 \), we have
\[ \| \sum_{\nu \in \mathcal{N}_e} \sum_{j \geq 1} \sum_{k \sim \nu \bar{\nu}} \chi_{\nu}^j \mathcal{W}_{\lambda} \mathcal{W}_{\lambda}^\nu \|_{p \rightarrow p'} \lesssim \mathcal{B}_p(\mu, \bar{\mu})(\mu/\bar{\mu})^c. \]
Proof. We further divide $\mathcal{N}_e = \mathcal{N}_e^1 \cup \mathcal{N}_e^2$, where

$$\mathcal{N}_e^1 = \{ \nu : 2^{-\nu} \gg \mu \}, \quad \mathcal{N}_e^2 = \{ \nu : \mu \geq 2^{-\nu} \gg (\bar{\mu})^{1/2} \text{ or } \nu = \nu_o \}.$$

We first consider $\nu \in \mathcal{N}_e^1$. Using Lemma 2.4 we get

$$\| \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'} \lesssim \left\{ \begin{array}{ll}
\lambda^{d+\frac{3}{2}\delta(p,p') - \frac{d}{2}} 2^{(d+\frac{3}{2}\delta(p,p') - 1)j} 2^{2\nu \delta(p,p')}, & 2^{-2j} \lesssim 2^{-\nu} \\
\lambda^{d+\frac{1}{2}\delta(p,p') - \frac{d}{2}} 2^{(d+\frac{3}{2}\delta(p,p') - 1)j}, & 2^{-2j} \gg 2^{-\nu},
\end{array} \right.$$  

for $k \sim \nu k'$ and $1 \leq p \leq 2$. Indeed, taking $N = 1/2$ in (2.26), we get the $L^1$–$L^\infty$ bounds, and then interpolation with $\| \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{2 \to 2} \lesssim \lambda^{-d/2} 2^{-j}$, which follows from (2.6) and (2.10), gives the estimates for $1 \leq p \leq 2$. By Lemma 2.6 we have the same bounds on $\| \sum_{k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'}$ as above. Since $2/(d+3) < \delta(p,p') < 2/(d-1)$, summation over $j$ gives

$$\sum_j \| \sum_{k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'} \leq C \lambda^{\frac{d+1}{2}\delta(p,p') - \frac{d}{4} 2^{(d+\frac{3}{2}\delta(p,p') - \frac{1}{4})\nu}}.$$

Thus, taking sum over $\nu : 2^{-\nu} \gg \mu$, we obtain

$$\sum_{\nu \in \mathcal{N}_e^1} \sum_j \| \sum_{k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'} \lesssim B_p(\mu, \bar{\mu})(\mu/\bar{\mu})^{\frac{d+3}{4}\delta(p,p') - \frac{1}{4}}.$$

We turn to the case $\nu \in \mathcal{N}_e^2$. As above, by (2.27) and (2.28) we have

$$\| \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'} \lesssim \left\{ \begin{array}{ll}
\lambda_p 2^{(\frac{d+3}{2}\delta(p,p') - \frac{d}{2})j} (\mu/\bar{\mu})^{-\delta(p,p')}, & 2^{-j} \ll \sqrt{\mu} \\
\lambda_p 2^{(\frac{d+3}{2}\delta(p,p') - \frac{d}{4})j}, & 2^{-j} \sim \sqrt{\mu} \\
\lambda_p 2^{(\frac{d+3}{2}\delta(p,p') - \frac{d}{2})j}, & 2^{-j} \gg \sqrt{\mu},
\end{array} \right.$$  

for $k \sim \nu k'$ and $1 \leq p \leq 2$ when $\nu \in \mathcal{N}_e^2$. Here, $\lambda_p$ denotes $\frac{1}{\sqrt{\lambda} \lambda^p}$.

For the first and third cases, we take $N = 1/2$ in (2.27) to get the $L^1$–$L^\infty$ estimates and interpolate them with $\| \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{2 \to 2} \lesssim \lambda^{-d/2} 2^{-j}$. We get the second case using $\| \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{2 \to 2} \lesssim \lambda^{-d/2} (\mu\bar{\mu})^{1/4}$ and $\| \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{1 \to \infty} \lesssim \lambda^{-1/2} \mu^{1/d} 4^{2\nu/2}$, which follow from (2.33) and (2.28), respectively.

Since $2/(d+3) < \delta(p,p') < 2/(d-1)$, by Lemma 2.6 we get

$$\sum_j \| \sum_{k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'} \lesssim B_p(\mu, \bar{\mu}) B_*(\mu, \bar{\mu}), \quad \nu \in \mathcal{N}_e^2,$$

where

$$B_*(\mu, \bar{\mu}) = (\mu/\bar{\mu})^{\frac{d+3}{4}\delta(p,p') - \frac{1}{4}} + \mu^{-\frac{d+3}{8}\delta(p,p')} \bar{\mu}^{-\frac{d+3}{8}\delta(p,p')} 2^{\frac{d}{4}\delta(p,p')}.$$

Thus, $\sum_{\nu \in \mathcal{N}_e^2} \sum_j \| \sum_{k \sim \nu k'} \chi_k^\nu \mathcal{P}_\lambda [\psi_j]\tilde{\chi}_{k'}^\nu \|_{p \to p'}$ is bounded by

$$C B_p(\mu, \bar{\mu}) ((\mu/\bar{\mu})^{\frac{d+3}{4}\delta(p,p') - \frac{1}{4}} \log (\mu/\bar{\mu}) + (\mu/\bar{\mu})^{\frac{d+3}{4}\delta(p,p')}).$$

Combining this and (3.2), we obtain (3.1) for some $c > 0$. 

\hfill $\square$
3.2. The sum over $\nu \in \mathcal{N}_c$. Since there are only as many as $O(1) \, \nu$, it suffices to consider a single $\nu$.

**Proposition 3.2.** Let $d \geq 3$, $\mu, \tilde{\mu}$ satisfy (2.11), and $\nu \in \mathcal{N}_c$. If $2/(d+3) < \delta(p, p') < \min(2/(d-1), 2/3)$, then for some $c > 0$, we have

$$\| \sum_{j} \sum_{k \sim \nu, k'} \chi_{k}^* \mathcal{P}_{\lambda}[\psi_{j}] \hat{\chi}_{k'}^\nu \|_{p \rightarrow p'} \lesssim B_{p}(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c.$$ 

Combining Proposition 3.1 and 3.2, we prove Proposition 2.2.

**Proof of Proposition 2.2.** Proposition 3.1 and 3.2 give (2.25), from which (2.12) follows for $2/(d+3) < \delta(p, p') < \min(2/(d-1), 2/3)$. Interpolating the estimate with

$$\| \chi_{k}^\nu \mathcal{P}_{\lambda} \chi_{k}^\nu \|_{2 \rightarrow 2} \lesssim \lambda^{-d/2}(\mu \tilde{\mu})^{1/4},$$

which follows from (2.37) after scaling, we obtain (2.12) for $0 < \delta(p, p') < \min(2/(d-1), 2/3)$.

Since $\nu \in \mathcal{N}_c$, the estimates in (2.27) remain valid. So, we have the first and third estimates in (3.3). By the same argument as before we see

$$(\sum_{2^{-j} \ll \sqrt{n}} + \sum_{2^{-j} \gg \sqrt{n}}) \| \sum_{k \sim \nu, k'} \chi_{k}^* \mathcal{P}_{\lambda}[\psi_{j}] \hat{\chi}_{k'}^\nu \|_{p \rightarrow p'} \lesssim B_{p}(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c$$

for some $c > 0$ because $2/(d+3) < \delta(p, p') < 2/(d-1)$. Thus, by Lemma 2.6 the proof of Proposition 3.2 is now reduced to showing

$$(3.4) \quad \| \chi_{k}^\nu \mathcal{P}_{\lambda}[\psi_{j}] \hat{\chi}_{k'}^\nu \|_{p \rightarrow p'} \lesssim B_{p}(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c, \quad 2^{-j} \sim \sqrt{n}, \quad k \sim \nu, k'.$$

In what follows, we make further reductions to prove (3.4). By (2.24), we may assume that

$$A_{k}^\nu \subset \mathcal{R} := \{ x : |x_1 - 1| \sim \mu, |\bar{x}| \lesssim (\mu \tilde{\mu})^{1/2} \},$$

$$\hat{A}_{k'}^\nu \subset \hat{\mathcal{R}} := \{ y : |y_1 - 1| \sim \tilde{\mu}, |\bar{y}| \lesssim (\mu \tilde{\mu})^{1/2} \},$$

where $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. For $x^* \in \mathbb{R}^d$ and $r > 0$, we denote

$$\mathcal{R}_{\mu, \tilde{\mu}}(x^*, r) = \{ (x_1, \bar{x}) : |x_1 - x_1^*| \leq r \mu, |\bar{x} - \bar{x}^*| \leq r(\mu \tilde{\mu})^{1/2} \}.$$ 

If $(x, y) \in \mathcal{R} \times \hat{\mathcal{R}}$, unlike the previous cases, $\mathcal{D}(x, y)$ may vanish. To handle this, we further localize the value of $\mathcal{D}$ by decomposing $\mathcal{R} \times \hat{\mathcal{R}}$. To this end, we break $\mathcal{R} \times \hat{\mathcal{R}}$ into finitely many disjoint rectangles $\mathcal{R}_{\mu, \tilde{\mu}}(x', \epsilon) \times \mathcal{R}_{\tilde{\mu}, \mu}(y', \epsilon)$ so that $|\mathcal{D}| \ll \epsilon \mu \tilde{\mu}$, or $|\mathcal{D}| \gtrsim \epsilon \mu \tilde{\mu}$ holds on each of those rectangles for a small $\epsilon > 0$. This particular form of decomposition shall be important later.

**Lemma 3.3.** Let $\mu, \tilde{\mu}$ satisfy (2.11) and $0 < \epsilon \leq \epsilon_0$. Let $(x^*, y^*) \in \mathcal{R} \times \hat{\mathcal{R}}$. If $x' \in \mathcal{R}_{\mu, \tilde{\mu}}(x^*, \epsilon)$ and $y' \in \mathcal{R}_{\tilde{\mu}, \mu}(y^*, \epsilon)$, then for a constant $C$ we have

$$(3.5) \quad |\mathcal{D}(x', y') - \mathcal{D}(x^*, y^*)| \leq C \epsilon \mu \tilde{\mu}.$$
Proof. Denote \( z' := (z'_1, \ldots, z'_{2d}) = (x', y') \) and \( z^* := (z^*_1, \ldots, z^*_d) = (x^*, y^*) \), then set \( D_0 = D(z') \) and \( D_k = D(z'_1, \ldots, z^*_k, z'_{k+1}, \ldots, z'_{2d}) \), \( 1 \leq k \leq 2d \). So, 
\[
|D(x', y') - D(x^*, y^*)| = \sum_{k=0}^{2d-1} (D_k - D_{k+1}) .
\]
Thus, (3.5) follows if we show 
\[
|D_k - D_{k+1}| \leq C\epsilon \lambda, \quad 1 \leq k \leq 2d .
\]

Note \( |x'_1 - x^*_1| \leq \epsilon \lambda, |y'_1 - y^*_1| \leq \epsilon \lambda, \) and \( |x'_j - x^*_j|, |y'_j - y^*_j| \leq \epsilon (\mu \lambda)^{1/2}, j \geq 2 \). By the mean value theorem we only need to show 
\[
|\partial_{x_1} D| \preceq \lambda, \quad |\partial_{y_1} D| \preceq \lambda, \quad |\partial_{x_j} D|, |\partial_{y_j} D| \preceq (\mu \lambda)^{1/2}
\]
on \( \mathcal{R} \times \tilde{\mathcal{R}} \). Since \( \partial_{x_1} D(x, y) = 2(x, y)y - 2x \) and \( \partial_{y_1} D(x, y) = 2(x, y)x - 2y \), it is clear that \( |\partial_{x_1} D|, |\partial_{y_1} D| \leq (\mu \lambda)^{1/2} \). Writing \( \partial_{x_1} D(x, y) = 2x_1(y^2 - 1) + 2(x, y)y_1 \), we see \( |\partial_{x_1} D| \preceq \lambda \). Similarly, we get \( |\partial_{y_1} D| \preceq \lambda \). \qed

Let \( \epsilon = c\epsilon_0 \) for a small enough \( c > 0 \). Let \( \mathcal{R}_{\mu, \lambda}(x_k, \epsilon/2), 1 \leq k \leq K, \) and \( \mathcal{R}_{\mu, \lambda}(y_l, \epsilon/2), 1 \leq l \leq L \) be almost disjoint rectangles which cover the rectangles \( \mathcal{R}, \tilde{\mathcal{R}} \), respectively. We denote 
\[
\mathcal{B} = \mathcal{R}_{\mu, \lambda}(x_k, \epsilon), \quad \tilde{\mathcal{B}} = \mathcal{R}_{\mu, \lambda}(y_l, \epsilon).
\]
Taking \( c > 0 \) small enough, by Lemma 3.3 we may assume that one of the following holds:
\[
|D(x, y)| \gtrsim \epsilon \lambda \mu \lambda, \quad (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}},
\]
\[
|D(x, y)| \ll \epsilon \lambda \mu \lambda, \quad (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}.
\]

Let \( \tilde{x}_B, \tilde{\chi}_B \) be smooth functions adapted to the rectangles \( \mathcal{B}, \tilde{\mathcal{B}} \), respectively, i.e., \( \text{supp} \tilde{x}_B \subset \mathcal{B}, \text{supp} \tilde{\chi}_B \subset \tilde{\mathcal{B}} \), and
\[
\partial^a_{x_1} \partial^a_{y_1} \tilde{x}_B = O(\mu^{-a_1}(\mu \lambda)^{-|a|/2}), \quad \partial^a_{y_1} \partial^b_{y_1} \tilde{\chi}_B = O(\mu^{-a_1}(\mu \lambda)^{-|a|/2}).
\]
For the proof of (3.4) it is enough to show 
\[
\|\tilde{x}_B \mathfrak{P}_\lambda [\psi_j] \tilde{\chi}_B\|_{p' \rightarrow p'} \lesssim B_p(\mu, \lambda \mu \lambda)^{\epsilon}, \quad 2^{-j} \sim \sqrt{\mu}
\]
for some \( c > 0 \) while either (3.6) or (3.7) holds. When (3.6) holds, one can show (3.9) in the same manner as in the proof of Proposition 3.1. Indeed, since \( 2^{-\nu} \sim (\mu \lambda)^{1/2}, 2^{-j} \sim \sqrt{\mu} \), and \( |D| \sim \mu \lambda \) on \( \mathcal{B} \times \tilde{\mathcal{B}} \), we have the second estimate in (3.3), which gives (3.9) for \( 1 \leq p < 2 \).

Therefore, to complete the proof of Proposition 3.2 it suffices to show the following.

**Proposition 3.4.** Let \( d \geq 3 \) and \( \mu, \lambda \) satisfy (2.11). Suppose that (3.7) holds. Then, if \( 0 < \delta(p, p') < 2/3 \), we have (3.9) for a constant \( c > 0 \).

**3.3 2nd-order derivative of \( \mathcal{P} \).** To prove Proposition 3.4 we shall dyadically decompose \( \mathfrak{P}_\lambda [\psi_j](x, y) \) away from the zero of \( \partial^2_{x} \mathcal{P} \) (see (2.14)). A computation shows 
\[
\partial^2_x \mathcal{P}(x, y, s) = -\langle x, y \rangle \mathcal{R}(x, y, \cos s) \frac{1}{\sin^3 s},
\]
where
where 
\[ R(x, y, \tau) = \tau^2 - \langle x, y \rangle^{-1}(|x|^2 + |y|^2)\tau + 1. \]

For \((x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}\), the equation \(R(x, y, \tau) = 0\) has two distinct zeros \(\tau^\pm(x, y)\):
\[
\tau^\pm(x, y) = \frac{|x|^2 + |y|^2 \pm |x + y||x - y|}{2\langle x, y \rangle}.
\]

Since \(\tau^+(x, y) > 1\), \(\tau^-(x, y)\) is more relevant for our purpose.

**Lemma 3.5.** Define a function \(S_c : \mathcal{B} \times \tilde{\mathcal{B}} \to (0, \pi/2)\) by
\[
\cos S_c(x, y) = \tau^-(x, y).
\]

Then, \(S_c\) is smooth and \(S_c(x, y) \sim |x - y|^{1/2}\) for \((x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}\).

We here record a few identities, which are to be useful later:
\[
\begin{align*}
\partial_s^2 \mathcal{P}(x, y, s) &= -(x, y) \frac{\cos S_c(x, y) - \cos s)(\tau^+(x, y) - \cos s)}{\sin^3 s}, \\
\tau^+ - \cos S_c &= \frac{|x + y||x - y|}{\langle x, y \rangle} = \frac{\sin^2 S_c}{\cos S_c}.
\end{align*}
\]

From now on, for simplicity we occasionally omit the arguments \((x, y)\) of \(S_c\) and related functions as long as there is no ambiguity.

**Proof of Lemma 3.5.** Let \((x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}\). From (3.10) we note
\[
1 - \tau^-(x, y) = 2^{-1}|x - y|(|x + y| - |x - y|)/\langle x, y \rangle.
\]

So, \(\tau^-(x, y) \in [1 - c_2\mu, 1 - c_1\mu]\) for some positive constants \(c_2 > c_1 > 0\). Thus, it is clear that \(S_c\) is smooth on \(\mathcal{B} \times \tilde{\mathcal{B}}\). Since \(1 - \cos S_c(x, y) \in [c_1\mu, c_2\mu]\), to see \(|x - y|^{1/2} \sim S_c(x, y)\) it suffices to observe that
\[
1 - \cos S_c(x, y) = \frac{2|x - y|}{|x + y| + |x - y|}.
\]

Note \(|x + y|^2 - |x - y|^2 = 4\langle x, y \rangle\). Thus, (3.15) follows by (3.14). \(\square\)

**3.4. Reduction to \(L^2\) estimates.** From (3.15) we note
\[
|\cos S_c(x, y) - \cos S_c(x^*, y^*)| \lesssim \varepsilon_0\mu, \quad (x, y), (x^*, y^*) \in \mathcal{B} \times \tilde{\mathcal{B}}.
\]

Consequently, \(S_c(\mathcal{B} \times \tilde{\mathcal{B}})\) is contained in an interval of length \(C_1\varepsilon_0\mu^{1/2}\) for a constant \(C_1\). Using this, we make a further localization.

Let \(c_\mathcal{B}\) denote the center of the rectangle \(\mathcal{B} \times \tilde{\mathcal{B}}\), and set
\[
\psi_\mathcal{B}(s) = \psi_0\left(\frac{s - S_c(c_\mathcal{B})}{C_1\varepsilon_0\mu^{1/2}}\right),
\]
where \(\psi_0 \in C^\infty_c(-4, 4)\) such that \(\psi_0 = 1\) on \([-2, 2]\). We decompose
\[
\tilde{\chi}_\mathcal{B}\mathcal{P}_\lambda[\psi_j]\tilde{\chi}_\mathcal{B} = \mathfrak{P}^0 + \mathfrak{P}^1 : = \tilde{\chi}_\mathcal{B}\mathcal{P}_\lambda[\psi_j\psi_\mathcal{B}]\tilde{\chi}_\mathcal{B} + \tilde{\chi}_\mathcal{B}\mathcal{P}_\lambda[\psi_j(1 - \psi_\mathcal{B})]\tilde{\chi}_\mathcal{B}.
\]

The operator \(\mathfrak{P}^1\) is easy to handle. In fact, one can show without difficulty
\[
\|\mathfrak{P}^1\|_{2 \to 2} \lesssim \lambda^{-\frac{d}{4}}(\mu\tilde{\mu})^{\frac{1}{4}}, \quad \|\mathfrak{P}^1\|_{1 \to \infty} \lesssim \lambda^{-\frac{d}{4}}\mu^{-\frac{d+1}{4}}.
\]
Interpolation gives \( \|\Psi^1\|_{p\to p'} \lesssim B_p(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c \) for \( 1 \leq p < 2 \). The \( L^2 \) bound follows from Lemma 2.9. For the \( L^1 - L^\infty \) bound, we note \( |\cos s - \cos S_c(x,y)| \sim \mu \) if \( s \in \text{supp}(\psi_j(1-\psi_B)) \) and \((x,y) \in B \times \tilde{B}\). We also have \( \tau^+(x,y) - \cos s \gtrsim \mu \) for \((x,y) \in B \times \tilde{B}\) since \( \tau^+(x,y) - 1 \sim |x - y| \). Thus, recalling (3.12), we see \( |\partial^2_x \mathcal{P}(x,y,s)| \gtrsim \mu^{1/2} \) for \( s \in \text{supp}(\psi_j(1-\psi_B)) \) if \((x,y) \in B \times \tilde{B}\). Applying the van der Corput lemma to the integral \( \mathcal{P}(x,y) \) (see (2.11)) gives the desired estimate \( \|\Psi^1(x,y)\| \lesssim \lambda^{-1/2} \mu^{-(d+1)/4} \).

**Proposition 3.6.** Let \( d \geq 3 \) and \( \mu, \tilde{\mu} \) satisfy (2.11). If \( 2/(d+1) < \delta(p,p') < 2/3 \), then for some \( c > 0 \) we have

\[
(3.16) \quad \|\Psi^0\|_{p\to p'} \lesssim B_p(\mu, \tilde{\mu})(\tilde{\mu}/\mu)^c.
\]

Now, the proof of Proposition 3.4 is straightforward.

**Proof of Proposition 3.4** Combining the estimates for \( \Psi^0 \) and \( \Psi^1 \), we obtain (3.9) for \( 2/(d+1) < \delta(p,p') < 2/3 \). Meanwhile, we have the estimate \( \|\tilde{\chi}_B \mathcal{P}_\lambda[\psi_j] \tilde{\chi}_B\|_{2 \to 2} \lesssim \lambda^{-\frac{d}{2}}(\tilde{\mu}/\mu)^{1/4} \) by Lemma 2.9. Interpolation yields (3.9) for \( 0 < \delta(p,p') < 2/3 \). □

If \((x,y) \in B \times \tilde{B}, \mathcal{D}(x,y) \) is no longer bounded away from the zero. To get the correct order of decay in \( \lambda \), i.e., \( O(\lambda^{-1/2}) \), we consider \( \partial^2_x \mathcal{P} \), which alone is not enough to give a favorable lower bound since it also vanishes at some point. Such difficulty is typically circumvented by considering \( \partial_y \mathcal{P} \) and \( \partial^2_y \mathcal{P} \) together. However, this is not viable in our situation since the zeros of \( \partial_x \mathcal{P} \) and \( \partial^2_y \mathcal{P} \) merge as \( \mathcal{D}(x,y) \to 0 \). This leads us to break the integral away from \( S_c \).

Inserting the cutoff functions \( \tilde{\psi}(2^l(s - S_c)) \), we decompose

\[
(3.17) \quad \Psi^0 = \sum_l \Psi_l := \sum_l \tilde{\chi}_B \mathcal{P}_\lambda[\psi_j \psi_B \tilde{\psi}(2^l(\cdot - S_c))] \tilde{\chi}_B,
\]

where \( \tilde{\psi} = \psi(|\cdot|) \). Clearly, we have

\[
(3.18) \quad 2^{-l} \leq \varepsilon_0 \mu^{1/2},
\]

since \( \Psi_l = 0 \) otherwise. As to be seen later, (3.18) makes it possible to render the minor contribution manageable if we take \( \varepsilon_0 \) small enough. We set

\[
\varepsilon_1 = C\varepsilon_0^{1/2}
\]

for a large constant \( C > 0 \).

We have different estimates for \( \Psi_l \) depending on \( l \).

**Lemma 3.7.** Let \( d \geq 2 \), and \( \mu, \tilde{\mu} \) satisfy (2.11). If \( 2^{-1} < \varepsilon_1 \tilde{\mu}^{1/2} \), then

\[
(3.19) \quad \|\Psi_l\|_{2 \to 2} \lesssim \lambda^{-\frac{d}{2}} 2^{-l}(\tilde{\mu}/\mu)^{-\frac{1}{2}}.
\]

If \( \varepsilon_1 \tilde{\mu}^{1/2} \leq 2^{-1} \), then we have

\[
(3.20) \quad \|\Psi_l\|_{2 \to 2} \lesssim \lambda^{-\frac{d}{2}} 2^{-l} \tilde{\mu}^{\frac{1}{2}} \mu^{\frac{1}{2}}.
\]
We postpone the proof of Lemma 3.7 until the next section. Assuming this for the moment, we prove Proposition 3.6. To do this, we set

$$S_c^l(x, y, s) = 2^{-l} s + S_c(x, y).$$

Then, changing variables $s \to S_c^l(x, y, s)$, we have

$$\Psi_l(x, y) = \chi_B(x) \chi_B(y) 2^{\frac{d}{2}} 2^{-l} \int (2^{-\frac{d}{2}} a \psi_j \psi_B)(S_c^l) \tilde{\psi}(s) e^{i\lambda p(x,y,S_c^l)} ds.$$  \hspace{1cm} (3.21)

Here, we also drop the arguments of $S_c^l$ for simplicity as before.

**Proof of Proposition 3.6.** Since $|\cos S_c^l - \cos S_c| \sim \mu^{1/2} 2^{-l}$ on the support of $(a \psi_j \psi_B) \circ S_c^l$, by (3.12) we have $|\partial^2 (P(x,y,S_c^l))| \sim 2^{-3l}$. Applying the van der Corput lemma, for $l$ satisfying (3.18) we get

$$\sum \mu |\partial^2 (P(x,y,S_c^l))| \leq B_p(\mu, \mu) \mu^{-\frac{d+1}{2}}.$$  \hspace{1cm} (3.22)

Interpolation with (3.20) gives

$$\|\Psi_l\|_{p \to p'} \lesssim \lambda^{\frac{d-1}{2}} \delta(p,p')^{-\frac{d}{2}} 2^l \mu^{-\frac{d+1}{2}} \delta(p,p') \mu^{-\frac{1}{2}} (1-\delta(p,p'))$$

when $\varepsilon_1 \mu^{1/2} \leq 2^{-l} \leq \varepsilon_2 \mu^{1/2}$. Hence, by summation over $l$ we have

$$\sum_{\varepsilon_1 \mu^{1/2} \leq 2^{-l} \leq \varepsilon_2 \mu^{1/2}} \|\Psi_l\|_{p \to p'} \lesssim B_p(\mu, \mu) < \mu^{-\frac{d+1}{2}} \delta(p,p').$$

By interpolation between the estimates (3.22) and (3.19) we get

$$\|\Psi_l\|_{p \to p'} \lesssim \lambda^{\frac{d+1}{2}} \delta(p,p')^{-\frac{d}{2}} 2^{-l(1-\frac{4}{d}) \delta(p,p')} \mu^{-\frac{d+1}{2}} \delta(p,p')} \mu^{-\frac{1}{2}} + \frac{d+1}{2} \delta(p,p')$$

for $2^{-l} < \varepsilon_1 \mu^{1/2}$, which yields

$$\sum_{\varepsilon_1 \mu^{1/2} \leq 2^{-l} \leq \varepsilon_2 \mu^{1/2}} \|\Psi_l\|_{p \to p'} \lesssim B_p(\mu, \mu) < \mu^{-\frac{d+1}{2}} \delta(p,p').$$

if $\delta(p,p') < 2/3$. Recalling (3.17), we combine the estimates above and obtain (3.16) for $2/(d+1) < \delta(p,p') < 2/3$. \hspace{1cm} \(\square\)

4. $L^2$ estimate: Proof of Lemma 3.7

For $a \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^d)$ and a smooth function $\phi$ on $\text{supp} \ a$, we define

$$O_\lambda[\phi, a] f(x) = \int e^{i\lambda \phi(x,y)} a(x,y) f(y) dy.$$\hspace{1cm}

We denote $\partial_x f = (\partial_{x_1} f, \ldots, \partial_{x_d} f)^\top$ and $\partial_\lambda f = (\partial_{x_1} f, \ldots, \partial_{x_d} f)$, so that $\partial_x \partial_\lambda^T = (\partial_{x_i} \partial_{y_j})_{1 \leq i,j \leq d}$. We now recall the following lemma.

**Lemma 4.1** ([6], [19] p.377]). Let $\lambda > 0$. If $\det(\partial_x \partial_\lambda^T) \neq 0$ on $\text{supp} \ a$, then there is a constant $C = C(\phi, a)$ such that

$$\|O_\lambda[\phi, a] f\|_2 \leq C \lambda^{-\frac{d}{2}} \|f\|_2.$$
From (3.21), we see
\begin{equation}
\mathcal{P}_l f = 2^{4l/2-l} \int \tilde{\psi}(s) O_\lambda [\Phi_s, A_s] f \, ds,
\end{equation}
where
\begin{align}
\Phi_s(x, y) &= \mathcal{P}(x, y, S^l_c), \\
A_s(x, y) &= \tilde{\chi}_B(x) \tilde{\chi}_B(y) (2^{-4j} a \psi_j \psi_{\mathcal{B}})(S^l_c).
\end{align}

To obtain the estimates (3.19) and (3.20), one may try to use Lemma 4.1 for $O_\lambda [\Phi_s, A_s]$. However, $O_\lambda [\Phi_s, A_s]$ exhibits different natures depending on $\mu, \tilde{\mu}$ and $l$. When $\mu \sim \tilde{\mu}$, via a suitable change of variables $O_\lambda [\Phi_s, A_s]$ can be handled by the estimate in Lemma 4.4. However, when $\tilde{\mu}/\mu$ gets smaller, Lemma 4.1 is not enough (see Lemma 4.9 below).

We set $\mathcal{J} = \{s : 1/4 \leq |s| \leq 1\}$. Since $\text{supp} \tilde{\psi} \subset \mathcal{J}$ and $2^{-j} \sim \sqrt{\mu}$, by (4.11) the estimate (3.19) follows by the next lemma.

**Lemma 4.2.** Let $d \geq 2$ and (2.11) hold. If $2^{-l} < \varepsilon_1 \tilde{\mu}^{1/2}$, then
\begin{equation}
\|O_\lambda [\Phi_s, A_s]\|_{2 \to 2} \leq C \lambda^{-\frac{d}{2}} \mu^{\frac{1}{4}} (\tilde{\mu}/\mu)^{-\frac{1}{2}}, \quad s \in \mathcal{J}.
\end{equation}

When $\varepsilon_1 \tilde{\mu}^{1/2} \leq 2^{-l} \leq \varepsilon_0 \mu^{1/2}$, in order to prove (3.20) we use a different expression of $\mathcal{P}_l$ so that we can exploit a lower bound on $\partial_s \Phi_s$. Recalling (3.11), by integration by parts in $s$ we have
\begin{equation}
\mathcal{P}_l f(x) = 2^{4l/2} 2^{-l} \int e^{i\lambda \Phi_s(x,y)} \tilde{A}_s(x, y) ds f(y) dy,
\end{equation}
where
\[\tilde{A}_s(x, y) = \tilde{\psi}(s) A_s(x, y) \frac{\partial^2 \Phi_s(x, y)}{i \lambda \partial_s \Phi_s(x, y)^2} - \frac{\partial_s (\tilde{\psi}(s) A_s(x, y))}{i \lambda \partial_s \Phi_s(x, y)}.
\]

Note $\partial_s \Phi_s \neq 0$ if $A_s \neq 0$. In fact, we have
\begin{equation}
|\partial_s \Phi_s(x, y)| \gtrsim 2^{-3l}, \quad (s, x, y) \in \mathcal{J} \times \mathcal{B} \times \tilde{\mathcal{B}}.
\end{equation}

Using (3.11) and (3.10), we see $|\langle x, y \rangle - \cos S_c(x, y)| \lesssim |\mathcal{D}(x, y)|/|x - y|$. Thus, by (3.7) it follows that
\begin{equation}
|\langle x, y \rangle - \cos S_c(x, y)| \leq \varepsilon_0 \tilde{\mu}, \quad (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}.
\end{equation}

Note $|\cos S^l_c - \cos S_c| \sim 2^{-l} \mu^{1/2}$, so $|\langle x, y \rangle - \cos S^l_c(x, y)| \gtrsim 2^{-l} \mu^{1/2} \gtrsim \varepsilon_1 (\mu \tilde{\mu})^{-1/2}$. By (2.16), (3.7), and our choice of $\varepsilon_1$, we get $Q(x, y, \cos S^l_c) \gtrsim 2^{-2l} \mu$. So, (4.5) follows by (2.15) since $\partial_s \Phi_s = 2^{-l} \partial_s \mathcal{P}(x, y, S^l_c)$.

The estimate (3.20) is an immediate consequence of the following.

**Lemma 4.3.** Let $d \geq 2$ and (2.11) hold. If $\varepsilon_1 \tilde{\mu}^{1/2} \leq 2^{-l} \leq \varepsilon_0 \mu^{1/2}$, then
\begin{equation}
\|O_\lambda [\Phi_s, A_s]\|_{2 \to 2} \leq C \lambda^{-\frac{d}{2}} \mu^{\frac{1}{4}} \tilde{\mu} \mu^{\frac{1}{2}} \mu^{\frac{1}{2}}, \quad s \in \mathcal{J}.
\end{equation}

For the rest of this section, we assume (2.11) and (3.18) hold, and $(x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}, s \in \mathcal{J}$, even if it is not made explicit.
4.1. **Bounds on \( \partial_x^\alpha \partial_y^\beta \Phi_s \) and \( \partial_x^\alpha A_s \).** In order to prove Lemma 4.4 and 4.5, we need estimates for the derivatives of \( \Phi_s \) and \( A_s \). For a given multi-index \( \alpha \in \mathbb{N}_0^d \), we write \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1} \).

**Lemma 4.4.** Let \( s \in \mathbb{J} \) and \( (x, y) \in B \times \tilde{B} \). Suppose \((2.11)\) holds. Then,

\[
\begin{align*}
|\partial_x^\alpha \partial_y^\beta S_c(x, y)| & \lesssim \mu^{1/2-|\alpha|-|\beta|}, \\
|\partial_x^\alpha A_s(x, y)| & \lesssim \mu^{-\alpha_1}(\mu \tilde{\mu})^{-1/2}.
\end{align*}
\]

**Proof.** To show \((4.8)\), we note from \((3.15)\) that \( S_c(x, y) \) behaves like \( g(x, y) := |x - y|^{1/2} \), and that \((4.8)\) and \((4.9)\) are easy to show if \( S_c \) is replaced by \( g \).

We prove \((4.8)\) in an inductive way by making use of \((3.15)\). By Lemma 3.5 \((4.3)\) holds with \( \alpha = \beta = 0 \) since \( \text{dist}(B, \tilde{B}) \sim \mu \). We now assume \((4.8)\) is true for \(|\alpha| + |\beta| \leq N\). Applying \( \partial_x^\alpha \partial_y^\beta \) on both side of \((3.15)\) for \(|\alpha| + |\beta| = N + 1\), it is not difficult to see

\[
\sin S_c(x, y) \partial_x^\alpha \partial_y^\beta S_c(x, y) + O(\mu^{-N}) = O(\mu^{-N})
\]

for \((x, y) \in B \times \tilde{B} \). Indeed, the right hand side of \((3.15)\) behaves as if it were \(|x - y|\). On the left hand side of \((4.10)\), the terms other than the first one are given by a linear combination of the products of \( \sin S_c \), \( \cos S_c \), and \( \prod_{i=1}^l \partial_x^{a_i} \partial_y^{b_i} S_c \) with \( l \geq 2 \) and \( \sum_{i=1}^l (|a_i| + |b_i|) \leq N + 1 \). Our induction assumption shows those are \( O(\mu^{-N}) \), thus we get \((4.10)\). Since \(|x - y| \sim \mu \), \( S_c(x, y) \sim \sqrt{\mu} \) by Lemma 3.5. Therefore, \((4.10)\) gives \( \partial_x^\alpha \partial_y^\beta S_c(x, y) = O(\mu^{-N-1/2}) \) as desired.

Once we have \((4.8), (4.9)\) is easier to show. Recall \((1.3)\). Since we have \((3.3)\), it is sufficient to show

\[ \partial_x^\alpha \left( (2^{-jd/2} a \psi_j \psi_B) \circ S_c^l \right) = O(\mu^{-|\alpha|}). \]

Note \((d/ds)^k(2^{-jd/2} a \psi_j) = O(\mu^{-k/2}) \) and \((d/ds)^k \psi_B = O(\mu^{-k/2}) \). By the chain rule and \((4.8)\), we get \( \partial_x^\alpha \left( (2^{-jd/2} a \psi_j) \circ S_c^l \right) = O(\mu^{-|\alpha|}) \) and \( \partial_x^\alpha \psi_B \circ S_c^l = O(\mu^{-|\alpha|}) \). From those estimates the desired bound follows. \( \square \)

**Lemma 4.5.** Let \( s \in \mathbb{J} \) and \( (x, y) \in B \times \tilde{B} \). Suppose that \((2.11)\) and \((3.15)\) hold. Then,

\[(4.11) \quad \partial_x^\alpha \partial_y^\beta \Phi_s(x, y) = O(\mu^{3/2-|\alpha|-|\beta|}), \quad |\alpha|, |\beta| \geq 1.\]

**Proof.** Let us set

\[
\begin{align*}
u(t) &= \frac{t - \sin t}{2}, \\
w(t) &= \frac{2 \cos t - 2 + \sin^2 t}{2 \sin t}, \quad w(t) = \frac{\cos t}{2 \sin t}.
\end{align*}
\]

Recalling \((1.2)\) and \((2.13)\), we write

\[
\Phi_s(x, y) = u(S_c^l) + 2^{-1}(1 - \langle x, y \rangle) \sin S_c^l + \langle x, y \rangle v(S_c^l) + |x - y|^2 w(S_c^l).
\]

From \((4.8)\) it is clear that \( |\partial_x^\alpha \partial_y^\beta S_c^l(x, y)| \lesssim \mu^{1/2-|\alpha|-|\beta|} \). We note \((d/dt)^k u(t), (d/dt)^k v(t) = O(t^{3-k}), \) and \((d/dt)^k w(t) = O(t^{1-k}) \). Since \( S_c^l(x, y) \sim \sqrt{\mu} \), \( 1 - \langle x, y \rangle = O(\mu) \), and \(|x - y| \sim \mu \), a routine computation yields \((4.11)\). \( \square \)
Remark 1. Under the same assumption as in Lemma 4.3, we also have $|\partial_x^\alpha \partial_y^\beta \sin^n S_c|, |\partial_x^\alpha \partial_y^\beta \sin m S'_c| \lesssim \mu^{1-|\alpha|-|\beta|}$ for any $m \in \mathbb{Z}$, and $|\partial_x^\alpha \partial_y^\beta (1 - \cos S_c)|, |\partial_x^\alpha \partial_y^\beta (1 - \cos S'_c)| \lesssim \mu^{1-|\alpha|-|\beta|}$ for any multi-indices $\alpha, \beta$.

4.2. Estimate for $\partial_x \partial_y^T \Phi_s$. For a given matrix $N$, we denote by $N_{i,j}$ the $(i,j)$-th entry of $N$. We consider a $d \times d$ matrix $\mathcal{M}^0$ which is given as follows:

\[
\mathcal{M}^0_{i,i}(x,y) = 1 - \frac{(x_i - y_i)^2}{|x - y|^2}, \quad 1 \leq i \leq d,
\]

\[
\mathcal{M}^0_{i,j}(x,y) = -\frac{x_j - y_j}{2|x - y|^2} (2x_1 - (1 + \cos S_c)y_1), \quad j \geq 2,
\]

\[
\mathcal{M}^0_{i,1}(x,y) = \mathcal{M}^0_{i,i}(y,x), \quad i \geq 2,
\]

\[
\mathcal{M}^0_{i,j}(x,y) = -\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2}, \quad i, j \geq 2, \quad i \neq j.
\]

The following lemma shows the matrix $\partial_x \partial_y^T \Phi_s(x,y)$ is close to $\mathcal{M}^0$. Let

\[
E_k^\alpha = \begin{cases} 
\varepsilon_0 (\mu/\mu)^k + 2^{-l} \mu^{1-1}, & \alpha = 0, \\
((\mu/\mu)^k + 2^{-l} \mu^{1-1}) \mu^{-\alpha_1} (\mu\mu)^{-\beta_1}, & \alpha \neq 0,
\end{cases} \quad k = 1, 2.
\]

Lemma 4.6. Let $s \in \mathbb{J}$ and $(x,y) \in \mathcal{B} \times \tilde{\mathcal{B}}$. Set

\[
\mathcal{M}(x,y) = -\sin S'_c(x,y) \partial_x \partial_y^T \Phi_s(x,y).
\]

Suppose that (2.11) and (3.18) hold. Then, $\mathcal{M} = \mathcal{M}^0 + \mathcal{E}$ and $\mathcal{E}_{i,j}$ satisfies

\[
|\partial_x^\alpha \mathcal{E}_{i,j}(x,y)| \lesssim \begin{cases} 
E_1^\alpha, & (i,j) = (1,1) \text{ or } i,j \geq 2,
E_2^\alpha, & \text{otherwise}.
\end{cases}
\]

Note $E_1^\alpha \leq E_2^\alpha$. We postpone the proof of Lemma 4.6 until the end of this section. Instead, we deduce a couple of lemmas from it for later use. By $\tilde{\mathcal{M}}$ we denote the $(1,1)$ minor matrix of $\mathcal{M}$, i.e., $\tilde{\mathcal{M}} = (\mathcal{M}_{i+1,j+1})_{1 \leq i,j \leq d-1}$.

Lemma 4.7. Let $s \in \mathbb{J}$ and $(x,y) \in \mathcal{B} \times \tilde{\mathcal{B}}$. Suppose (2.11) and (3.18) hold. Then, we have

\[
\det \mathcal{M}(x,y) \sim \tilde{\mu}/\mu, \quad 2^{-l} \leq \varepsilon_0 \mu^{1/2},
\]

\[
\det \tilde{\mathcal{M}}(x,y) \sim 1, \quad 2^{-l} \leq \varepsilon_0 \mu^{1/2}.
\]

Proof. Note $\mathcal{M}^0_{i,i} = 1 + O(\tilde{\mu}/\mu)$ for $2 \leq i \leq d$ and $\mathcal{M}^0_{i,j} = O(\tilde{\mu}/\mu)$ for $2 \leq i \neq j \leq d$. Thus, by Lemma 4.6 we have $\tilde{\mathcal{M}} = I_{d-1} + O(\varepsilon_0 + \tilde{\mu}/\mu)$. Since $\tilde{\mu}/\mu \leq \varepsilon_0$, (1.14) follows if $\varepsilon_0$ is small enough.

For the proof of (1.13), we may assume

\[
x = (r,0,0,\ldots,0), \quad y = (\rho,h,0,\ldots,0).
\]

Indeed, observe $S_c(x,y) = S_c(Ux,Uy)$ and $\Phi_s(x,y) = \Phi_s(Ux,Uy)$ for $U \in O(d)$. The second identity and a computation show $\partial_x \partial_y^T \Phi_s(x,y) = U^\top \partial_x \partial_y^T \Phi_s(Ux,Uy)U$, i.e., $U\mathcal{M}(x,y)U^\top = \mathcal{M}(Ux,Uy)$. Since $\det \mathcal{M}(Ux,Uy)$
By Lemma 4.6 and (4.15), the matrix 
\[ M = det \begin{pmatrix} 2h^2 & (1 + \cos S_c) \rho h - 2rh \\ 2rh - (1 + \cos S_c) rh & 2(r - \rho)^2 \end{pmatrix}. \]

Let \( M \) be the 2 \times 2 matrix given by
\[ M = \frac{1}{2|x - y|^2} \begin{pmatrix} 2h^2 & (1 + \cos S_c) \rho h - 2rh \\ 2rh - (1 + \cos S_c) rh & 2(r - \rho)^2 \end{pmatrix}. \]

By Lemma 4.6 and (4.15), the matrix \( M \) is of the form
\[ M(x, y) = \begin{pmatrix} M & 0 \\ 0 & I_{d-2} \end{pmatrix} + \tilde{E}, \]
where \( \tilde{E}_{i,j} = O((\varepsilon_0 + \varepsilon_1)(\hat{\mu}/\mu)) \) if \( i, j = (1, 1) \) or \( i, j \geq 2 \), and \( \tilde{E}_{i,j} = O((\varepsilon_0 + \varepsilon_1)(\hat{\mu}/\mu)^{1/2}) \) otherwise. Since \( (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}} \), we have \( 1 - r \sim \mu, 1 - \rho \sim \hat{\mu}, \) and \( h \sim (\mu \hat{\mu})^{1/2} \). Hence, \( M_{1,1} \sim \hat{\mu}/\mu, M_{2,2} \sim 1, \) and \( M_{1,2}, M_{1,2} = O((\hat{\mu}/\mu)^{1/2}) \). Consequently, we see det \( M(x, y) = det M + O((\varepsilon_0 + \varepsilon_1)\hat{\mu}/\mu) \). Note \( r - \rho \sim \mu \) and \( 1 - \cos S_c \sim \mu \). So, we have
\[ \det M = \frac{h^2}{4|x - y|^2}(1 - \cos S_c)(2(r - \rho)^2 + r\rho(1 - \cos S_c)) \sim \frac{\hat{\mu}}{\mu}. \]

Therefore, (4.13) follows if \( \varepsilon_0 \) is sufficiently small.

The following is a straightforward consequence of Lemma 4.6.

**Lemma 4.8.** Let \( s \in J \) and \( (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}} \). Suppose (2.11) holds. If \( 2^{-l} \leq \varepsilon_1 \hat{\mu}^{1/2} \), then for \( \alpha \in \mathbb{N}_0^d \)
\[ |\partial_\alpha^s \mathcal{M}_{i,j}(x, y)| \lesssim \begin{cases} (\hat{\mu}/\mu)^{-\alpha_1} (\hat{\mu}\hat{\mu})^{-\frac{|\alpha|}{2}}, & i = j = 1, \\ (\hat{\mu}/\mu)^{\frac{1}{2}}^{-\alpha_1} (\hat{\mu}\hat{\mu})^{-\frac{|\alpha|}{2}}, & i \text{ or } j = 1, \\ \mu^{-\alpha_1} (\mu \hat{\mu})^{-\frac{|\alpha|}{2}}, & i, j \geq 2. \end{cases} \]

Furthermore, the last bound remains valid even if \( 2^{-l} \leq \varepsilon_0 \sqrt{\mu} \).

4.3. **Estimate for** \( \mathcal{O}_\Lambda [\Phi_\lambda, A_\lambda] \) **when** \( 2^{-l} < \varepsilon_1 \hat{\mu}^{1/2} \). In order to prove the estimate (4.4), we use the following lemma, which differs from the typical one (see Lemma 4.1), in that the phase and amplitude functions depend on a parameter. We denote \( \mathcal{B}_d(x, r) = \{ y \in \mathbb{R}^d : |y - x| < r \} \).

**Lemma 4.9.** Let \( 0 < \omega \leq 1 \). Let \( a \) be a smooth function supported in \( S_\omega := \mathbb{B}_d(0, 1) \times \mathbb{B}_1(0, \omega) \times \mathbb{B}_{d-1}(0, 1) \) such that \( |\partial^\alpha a| \lesssim 1 \) for \( |\alpha| \leq d + 1 \). Suppose that \( |\det \partial_x \partial_y^\beta \phi| \sim 1 \) and
\[ |\partial^\alpha_x \partial^\beta_y \phi| \lesssim \begin{cases} 1, & |\beta| = 1, \\ \omega^{-1+|\alpha|+|\beta|}, & |\beta| = 2, \end{cases} \]
for \( 1 \leq |\alpha| \leq d + 1 \) on \( S_\omega \). Then, we have
\[ \| \mathcal{O}_\Lambda [\phi, a] f \|_2 \lesssim \lambda^{-\frac{d}{2}} \| f \|_2. \]
Proof of Lemma 4.2. By finite decomposition and translation we may replace $S_\omega$ with $B_d(0, \epsilon_0) \times [-\epsilon_0, \epsilon_0] \times B_{d-1}(0, \epsilon_0)$ for a small enough $\epsilon_0 > 0$. We set

$$\Psi(x) = \phi(x, y) - \phi(x, y'), \quad A(x) = a(x, y)\tilde{a}(x, y'),$$

and consider the integral

$$I_\lambda(y, y') = \int e^{i\lambda x} A(x)dx,$$

which is the kernel of the operator $O_\lambda[\phi, a]^{\ast}O_\lambda[\phi, a]$. The estimate (4.17) follows by a standard argument if we show

$$|I_\lambda(y, y')| \leq C(1 + \lambda|y - y'|)^{-d-1} \tag{4.18}$$

Assuming, for the moment, that (4.18). Note $\partial_\alpha^a A = O(1)$ for $|\alpha| \leq d + 1$ since $|\partial_\alpha^a| \leq 1$. Using

$$(\nabla^{\lambda(x)}(\phi)) = (\nabla^{\lambda(x)}(\phi)) + \nabla^{\lambda(x)}(\phi)$$

we prove (4.19) and (4.20). For (4.19) it is sufficient to show

$$|\nabla_x \phi(x, y) - \nabla_x \phi(x, y')| \gtrsim |y - y'|$$

By Taylor series expansion, $\nabla_x \phi(x, y) - \nabla_x \phi(x, y')$ is equal to

$$\partial_x^2 \partial_y^2 \phi(x, y')(y - y') + \sum_{\beta, |\beta| = 2} O(E_{i,\beta}(y - y')),$$

where $E_{i,\beta} = \sup_{x, y \in S_\lambda} |\partial_x^2 \partial_y^2 \phi(x, y)|$. Since $|y_1 - y'_1| \leq \epsilon_0$, by (4.16) it is clear that $E_{i,\alpha}(y - y') \sim |y - y'|$ because $\partial_x^2 \partial_y^2 \phi(x, y'(y - y')) \sim 1$. Thus, $|\nabla_x \phi(x, y) - \nabla_x \phi(x, y')| \gtrsim |y - y'|$. This follows from (4.16). \hfill \square

We prove Lemma 4.2 by combining Lemma 4.7, 4.8, and 4.9.

Proof of Lemma 4.2. We first transform $O_\lambda[\Phi_s, A_s]$ via scaling and translation so that we can apply Lemma 4.9. Recalling that $c_B$ denotes the center of the rectangle $B \times B$, we set

$$L(x, y) = (L_1 x, L_2 y) + c_B := \left(\mu x_1, (\mu \tilde{\mu})^{1/2} \tilde{x}, \mu y_1, (\mu \tilde{\mu})^{1/2} \tilde{y}\right) + c_B.$$ 

Changing variables $(x, y) \to L(x, y)$, we have

$$\|O_\lambda[\Phi_s, A_s]\|_{2 \to 2} = \left(\mu / (\mu \tilde{\mu})^{1/4}\right)^{\frac{d}{2}} \|O_\sqrt{\mu \tilde{\mu}}[\tilde{\Phi}, \tilde{A}]\|_{2 \to 2}, \tag{4.21}$$

where $\tilde{\Phi}$ and $\tilde{A}$ are given by (1.21) and (1.2).
where
\[ \tilde{\Phi}(x, y) = (\sqrt{\mu \tilde{\mu}})^{-1} \Phi_s(\bar{L}(x, y)), \quad \tilde{A}(x, y) = A_s(\bar{L}(x, y)) \].

Since \( B, \tilde{B} \) are of dimensions about \( \varepsilon_0 \mu \times \varepsilon_0(\mu \tilde{\mu})^{1/2} \times \cdots \times \varepsilon_0(\mu \tilde{\mu})^{1/2}, \varepsilon_0 \mu \times \varepsilon_0(\mu \tilde{\mu})^{1/2} \times \cdots \times \varepsilon_0(\mu \tilde{\mu})^{1/2} \), respectively, \( \tilde{A} \) is supported in \( B_2(0, 1) \times B_1(0, \mu / \mu \times B_{d-1}(0, 1) \). From (4.29) it follows that \( \partial_{\alpha}^2 \tilde{A} = O(1) \) for all \( \alpha \). Since \( S_\varepsilon(x, y) \sim \sqrt{\mu} \), Lemma 4.7 gives \( |\det \partial_{x,y}^\alpha \tilde{\Phi}| \sim 1 \). Indeed, note that
\[ \partial_{x,y}^\alpha \tilde{\Phi} = -\frac{\partial}{\sin S_\varepsilon^2(L(x,y))} (\mu \tilde{\mu})^{-1} \mathfrak{M}(L(x,y)) L_2. \]

Besides, since \( \partial_{\alpha}^\alpha (\sqrt{\mu} / \sin S_\varepsilon^2) = (\mu^{-|\alpha|}) \), by Lemma 4.5 and 4.8 we also have
\[ \partial_{x,y}^\alpha \tilde{\Phi} = \begin{cases} O(1), & |\beta| = 1, \\ O((\mu \tilde{\mu})^{-1 + |\alpha| + |\beta|}), & |\beta| = 2. \end{cases} \]
for \( (x, y) \in \text{supp} \tilde{A} \) and \( \alpha \in \mathbb{N}_0^d \). In fact, we use Lemma 4.8 for \( |\beta| = 1 \) and Lemma 4.5 for \( |\beta| = 2 \). Therefore, we may use Lemma 4.9 with \( \omega = \mu / \mu \) for \( O_{\sqrt{\mu \tilde{\mu} \lambda}}(\tilde{\Phi}, \tilde{A}) \), and we hence get
\[ \|O_{\sqrt{\mu \tilde{\mu} \lambda}}(\tilde{\Phi}, \tilde{A})\|_{2 \to 2} \lesssim \lambda^{-\frac{d}{2}} \mu^{-\frac{d}{2}} \tilde{\mu}^{-\frac{d}{2}}. \]

By (4.21) the estimate (4.4) follows as desired. \( \square \)

4.4. Estimate for \( O_\lambda(\Phi_s, \bar{A}_s) \) when \( 2^{-l} \geq \varepsilon_1 \mu^{1/2} \). In this case, as seen in Lemma 4.6 the matrix \( \partial_{x,y}^\alpha \Phi_s \) can be singular. So, we need an approach different from that used for the case \( 2^{-l} < \varepsilon_1 \mu^{1/2} \). We consider an operator given by freezing \( x_1, y_1 \) (see Proof of Lemma 4.3 below) and then make use of (4.14).

To prove Lemma 4.3 we need the following.

Lemma 4.10. Let \( 2^{-l} \geq \varepsilon_1 \mu^{1/2} \) and \( (x, y) \in B \times \tilde{B} \). Then,
\[ |\partial_{x,y}^\alpha \bar{A}_s(x, y)| \lesssim (\lambda 2^{-3l})^{-1} (\mu \tilde{\mu})^{-|\alpha|/2}, \quad \bar{\alpha} \in \mathbb{N}_0^{d-1}. \]

Assuming this for the moment, we prove Lemma 4.3.

Proof of Lemma 4.3 Fixing \( s \in \mathbb{J} \), we set
\[ \tilde{\Phi}(x, y) = (\sqrt{\mu \tilde{\mu}})^{-1} \Phi_s(\bar{L}(x, y)), \quad \tilde{B}(x, y) = \lambda 2^{-3l} \bar{A}_s(\bar{L}(x, y)), \]
where
\[ \bar{L}(x, y) = (\mu x_1, (\mu \tilde{\mu})^{1/2} \bar{x}, \tilde{\mu} y_1, (\mu \tilde{\mu})^{1/2} \bar{y}) + c_s. \]
Changing variables \( (x, y) \to \bar{L}(x, y) \) gives
\[ \|O_\lambda(\Phi_s, \bar{A}_s)\|_{2 \to 2} = (\lambda 2^{-3l})^{-1} (\mu \tilde{\mu})^\frac{d}{2} \|O_{\sqrt{\mu \tilde{\mu} \lambda}}(\tilde{\Phi}, \tilde{B})\|_{2 \to 2}. \]

Freezing \( z_1 := (x_1, y_1) \), we define
\[ \tilde{O}_\lambda^{z_1} h(x) = \int e^{i \lambda \tilde{\Phi}(z_1) z_1(\bar{x}, \bar{y})} \tilde{B}(z_1(\bar{x}, \bar{y}) h(\bar{y}) d\bar{y}, \]

where \( \tilde{\Phi}(x, y) = (\sqrt{\mu \tilde{\mu}})^{-1} \Phi_s(\bar{L}(x, y)), \quad \tilde{A}(x, y) = A_s(\bar{L}(x, y)). \)
where \( \Phi_{x_1}(\bar{x}, \bar{y}) = \Phi(x, y) \) and \( B_{x_1}(\bar{x}, \bar{y}) = B(x, y) \). Note \( 2^{-l} \geq \epsilon_1 \bar{\mu}^{1/2} \) and \( \lambda \bar{\mu}^{3/2} \geq 1 \). Since \( \mathcal{O}_{\lambda}^\mu \Phi, B \) \( f = \int \mathcal{O}_{\lambda}^\mu \Phi(y_1, \cdot') dy_1 \), by (4.23) the estimate (4.7) follows if we show

\[
\| \mathcal{O}_{\lambda}^\mu \|_{2 \to 2} \leq C \lambda^{-\frac{d-1}{2}}, \quad \lambda > 0
\]

for a constant \( C \). Note \( \text{supp } \tilde{B}_{x_1} \subset \mathbb{B}_{d-1}(0,1) \times \mathbb{B}_{d-1}(0,1) \). Moreover, by (4.10) and (4.11) we have \( \partial_y^\beta \tilde{B}_{x_1} = O(1) \) and det \( \partial_y \partial_y^\beta \Phi_{x_1} \sim 1 \). Lemma 4.5 and 4.8 as before (cf. Proof of Lemma 4.2), give

\[
\partial_x^\alpha \partial_y^\beta \Phi_{x_1} = O(1), \quad |\alpha| \geq 1, |\beta| = 1, 2
\]

whenever \( (\bar{x}, \bar{y}) \in \mathbb{B}_{d-1}(0,1) \times \mathbb{B}_{d-1}(0,1) \). Therefore, applying Lemma 4.9 (with \( \omega = 1 \) and \( d \) replaced by \( d - 1 \)) to \( \mathcal{O}_{\lambda}^\mu \), we obtain (4.24). \( \Box \)

**Proof of Lemma 4.10** We first consider the case \( \bar{\alpha} = 0 \). Note that \( \tilde{\psi}(s)A_s(x, y) \) and \( \partial_x(\tilde{\psi}(s)A_s(x, y)) \) are uniformly bounded on \( \mathcal{B} \times \mathcal{B} \). Recall \( |\partial_x^\alpha \Phi_s(x, y)| \leq 2^{-3l} \) (see Proof of Proposition 4.6). Thus, by (4.5) we get (4.22) for \( \bar{\alpha} = 0 \).

Let \( \bar{\alpha} \neq 0 \). We have (4.19), and \( \partial_x^\alpha \partial_x \Phi_s(x, y) = O((\mu \bar{\mu})^{-|\alpha|/2}) \), which follows by (4.8) since \( 2^{-l} \leq \mu^{1/2} \). Therefore, for (4.22) we need only to show

\[
|\partial_x^\alpha \partial_x^\beta \Phi_s(x, y)| \leq 2^{-\lambda} |\alpha|, \\
|\partial_x^\alpha \partial_x \Phi_s(x, y)| \leq 2^{-\lambda}(\mu \bar{\mu})^{-\frac{|\alpha|}{2}}.
\]

We verify (4.25) first. Since \( \partial_x^\alpha \Phi_s = 2^{-2l} \partial_x^\alpha \mathcal{P}(x, y, S_l^c) \), it suffices to show

\[
|\partial_x^\alpha \partial_x^\beta \Phi_s(x, y, S_l^c)| \leq 2^{-l} |\alpha|, \quad \alpha \neq 0.
\]

Using (3.12), we write

\[
\frac{\partial_x^\alpha \partial_x^\beta \mathcal{P}(x, y, S_l^c)}{\langle x, y \rangle} = \frac{(\cos S_l^c - \cos S_c)^2}{\sin^3 S_l^c} + \frac{(\cos S_l^c - \cos S_c)(\cos S_c - \tau^+)}{\sin^3 S_l^c}.
\]

Note \( \partial_x^\alpha (\sin S_l^c)^{-3} = O(\mu^{-\frac{3}{2}} |\alpha|) \) (see Remark 1). By the Leibniz rule, (4.27) follows if we show

\[
|\partial_x^\alpha (\cos S_c - \tau^+)| \leq \mu^{-1} |\alpha|, \\
|\partial_x^\alpha (\cos S_l^c - \cos S_c)| \leq 2^{-l} |\alpha|.
\]

The estimate (4.28) is clear from (3.13) since \( \partial_x^\alpha (|x - y|) = O(\mu^{-1} |\alpha|) \) for \( (x, y) \in \mathcal{B} \times \mathcal{B} \). To show (4.29), we observe \( \partial_x^\alpha (\cos S_l^c - \cos S_c) \) is given by a linear combination of the terms

\[
(\sin S_l^c - \sin S_c) \prod_{m=1}^{2l-1} \partial_x^{2m} S_c, \quad (\cos S_l^c - \cos S_c) \prod_{m=1}^{2l-1} \partial_x^{2m} S_c, \quad l \geq 0,
\]

where \( a_1, \ldots, a_{2l-1}, b_1, \ldots, b_{2l} \neq 0; |a_1| + \cdots + |a_{2l-1}| = |b_1| + \cdots + |b_{2l}| = |\alpha| \).

Since \( \sin S_l^c - \sin S_c = O(2^{-l}) \) and \( \cos S_l^c - \cos S_c = O(2^{-l} \mu^{1/2}) \), (4.29) follows by (4.18).

We now show (4.26). Recalling \( \partial_x \Phi_s = 2^{-l} \partial_x \mathcal{P}(x, y, S_l^c) \), we have

\[
\partial_x^\beta \partial_x \Phi_s = 2^{-l} \partial_x^\beta \partial_x \mathcal{P}(x, y, S_l^c) + 2^l \partial_x^\alpha \Phi_s \partial_x^\beta S_c, \quad |\beta| = 1.
\]
Let $\tilde{\alpha}' + \tilde{\beta} = \tilde{\alpha}$. By (1.15) and (1.8), we have $\partial_{x}^{j} (2 \tilde{\alpha}^{2} \Phi_{s} \partial_{x}^{3} S_{c}) \lesssim 2^{-3l} (\mu \tilde{\mu})^{-|\tilde{\alpha}|/2}$ since $2^{-l} \geq \varepsilon_{1} \tilde{\mu}^{1/2}$. To handle the first term, from (2.15) we note

\begin{equation}
\partial_{x_{j}} \partial_{s} P(x, y, S_{c}^{l}) = \frac{y_{j} \cos S_{c}^{l} - x_{j}}{\sin^{2} S_{c}^{l}}.
\end{equation}

Thus, it suffices to show $|\partial_{x}^{3} (y_{j} \cos S_{c}^{l} - x_{j})| \lesssim (\mu \tilde{\mu})^{(1 - |\tilde{\beta}|)/2}$, $2 \leq j \leq d$. Since $|y_{j}|, |x_{j}| \lesssim (\mu \tilde{\mu})^{1/2}$, using (1.8) one can easily show the desired bounds (see Remark 1).

The rest of this section is dedicated to proving Lemma 4.6.

4.5. Proof of Lemma 4.6. In order to prove (4.12), we start by removing an insignificant part of $\mathcal{M}$. Differentiating (1.2), we have

\begin{equation}
\begin{aligned}
\partial_{x} \partial_{y}^{l} \Phi_{s}(x, y) &= \partial_{x} \partial_{y}^{l} P(x, y, S_{c}^{l}) + \partial_{x} S_{c} \partial_{y}^{l} \partial_{y} P(x, y, S_{c}^{l}) + \\
\partial_{x} \partial_{y} P(x, y, S_{c}^{l}) \partial_{y}^{l} S_{c} + \partial_{x}^{2} P(x, y, S_{c}^{l}) \partial_{x} S_{c} \partial_{y}^{l} S_{c} + \partial_{y} P(x, y, S_{c}^{l}) \partial_{x} \partial_{y}^{l} S_{c}.
\end{aligned}
\end{equation}

The last term on the right hand side is negligible.

Lemma 4.11. Let $s \in \mathcal{J}$ and $(x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}$. Suppose that (2.11) and (3.18) hold. Then, for $\alpha \in \mathbb{N}_{0}^{d}$

$$
\sin S_{c}^{l} \partial_{x}^{l} \partial_{y}^{l} \partial_{x} \partial_{y}^{l} S_{c} = O(E_{1}^{n}).
$$

To prove Lemma 4.11 we make use of the following identities:

\begin{equation}
\partial_{x} S_{c}(x, y) = \mathcal{G}(2x - \mathfrak{A} y), \quad \partial_{y}^{l} S_{c}(x, y) = \mathcal{G}(2y^{T} - \mathfrak{A} x^{T}),
\end{equation}

where

\begin{equation}
\mathfrak{A}(x, y) = \frac{|x|^{2} + |y|^{2}}{(x, y)}, \quad \mathcal{G}(x, y) = \frac{\cos S_{c}(x, y)}{\sin S_{c}(x, y)|x + y||x - y|}.
\end{equation}

Proof of (4.32). Differentiating (3.15), followed by a simple computation, yields

$$
\sin S_{c}(x, y) \partial_{x} S_{c}(x, y) = F(x, y)x - 2^{-1} \mathfrak{A}(x, y) F(x, y)y,
$$

where

$$
F(x, y) = \frac{|x|^{2} + |y|^{2} - |x + y||x - y|}{(x, y)|x + y||x - y|}.
$$

Here, we also use the identity $|x + y|^{2} |x - y|^{2} = (|x|^{2} + |y|^{2})^{2} - 4(x, y)^{2}$. By (3.11) and (3.10) we see $F(x, y) = 2 \cos S_{c}(x, y)/|x - y||x + y|$. We thus get the first identity in (4.32). Since $S_{c}(x, y) = S_{c}(x, y)$, the second one follows from the first by interchanging the roles of $x, y$.

Proof of Lemma 4.11. Since $|\cos S_{c}^{l} - \cos S_{c}| \sim 2^{-l} \mu^{1/2}$, (4.10) gives $|\langle x, y \rangle - \cos S_{c}(x, y)| \lesssim \varepsilon_{0} (\mu \tilde{\mu})^{1/2} + 2^{-l} \mu^{1/2}$. Thus, by (2.15) and (3.7) it follows that

$$
\partial_{s} P(x, y, S_{c}^{l}(x, y, s)) = O(\varepsilon_{0} \tilde{\mu} + 2^{-2l}).
$$

Combining this and (4.8), we get the desired estimate for $\alpha = 0$. 

We now assume $|\alpha| \geq 1$. Thanks to (1.8), it suffices to show
\begin{equation}
\partial_\alpha^\beta (\partial_\alpha P(x, y, S_c^j)) = O((\bar{\mu} + 2^{-2l})\mu^{-|\alpha|} (\mu\bar{\mu})^{-|\alpha|/2}).
\end{equation}
Since $\partial_{x_j}(\partial_\alpha P(x, y, S_c^j)) = \partial_{x_j} \partial_\alpha P(x, y, S_c^j) + \partial_\alpha^2 P(x, y, S_c^j)\partial_{x_j} S_c$, using (4.30), (3.12) and the first identity in (1.32), we write
\begin{align*}
\partial_{x_j}(\partial_\alpha P(x, y, S_c^j)) &= \bar{I}_j + \cos S_c^j - \cos S_c \
&= \frac{\bar{I}_j}{\sin^2 S_c^j} + \frac{\cos S_c^j - \cos S_c}{\sin^3 S_c^j} \bar{I}_j^2,
\end{align*}
where
\begin{align*}
\bar{I}_j^1 &= y_j \cos S_c - x_j, \\
\bar{I}_j^2 &= y_j \sin S_c - \mathcal{G}(x, y)(\cos S_c^j - \tau^+)(2x_j - \mathcal{A}y_j).
\end{align*}
We may assume $\alpha_j \neq 0$ for some $j$. Thus,
\begin{align*}
\partial_\alpha^\beta (\partial_\alpha P(x, y, S_c^j)) &= \partial_\alpha^\beta \left(\frac{\bar{I}_j^1}{\sin^2 S_c^j} + \frac{\cos S_c^j - \cos S_c}{\sin^3 S_c^j} \bar{I}_j^2\right),
\end{align*}
where $\beta = \alpha - e_j$. By the Leibniz rule, (4.29), and the bounds in Remark 1, the desired estimate (4.34) follows if we show
\begin{align}
|\partial_\alpha^\beta \bar{I}_j^1| &\lesssim \begin{cases} 
\bar{\mu}^{-|\alpha|} (\mu\bar{\mu})^{-|\beta|/2}, & j = 1, \\
(\mu\bar{\mu})^{1/2} - \beta_1 (\mu\bar{\mu})^{-|\beta|/2}, & j \neq 1,
\end{cases} \\
|\partial_\alpha^\beta \bar{I}_j^2| &\lesssim \begin{cases} 
(\bar{\mu}^{1/2} + 2^{-l})\mu^{-|\alpha|} (\mu\bar{\mu})^{-|\beta|/2}, & j = 1, \\
\mu^{1/2} - \beta_1 (\mu\bar{\mu})^{-|\beta|/2}, & j \neq 1.
\end{cases}
\end{align}
It should be noted that slightly weaker bounds are good enough for our purpose when $\alpha_j \neq 0$ for $j \neq 1$, i.e., $|\tilde{\alpha}| - 1 = |\beta|$, since there is an improvement of factor $(\mu\bar{\mu})^{1/2}$ thanks to the particular form of the estimate (1.31).

One can easily show (4.35) for $j \neq 1$ using (1.8) since $|x_j|, |y_j| \lesssim (\mu\bar{\mu})^{1/2}$. To show (4.35) for $j = 1$, we write $\bar{I}_1 = \frac{1}{x_1} (\langle x, y \rangle \cos S_c - |x|^2 + \langle \bar{x}, \bar{y} \rangle \cos S_c)$. By (3.11), we have
\begin{align*}
\bar{I}_1 &= \frac{1}{x_1} \left( \frac{2(|x, y|^2 - |x|^2|y|^2)}{|y|^2 - |x|^2 + |x + y||x - y|} + \langle \bar{x}, \bar{y} \rangle \cos S_c \right).
\end{align*}
Using (4.35) with $j \neq 1$, we have $\partial_\alpha^\beta (\langle \bar{x}, \bar{y} \rangle \cos S_c) = O((\mu\bar{\mu})\mu^{-|\alpha|} (\mu\bar{\mu})^{-|\beta|/2})$. One can easily check $\partial_\alpha^2 (\langle x, y \rangle |y|^2 - \langle x, y \rangle^2) = O((\mu\bar{\mu})^{-|\alpha|} (\mu\bar{\mu})^{-|\beta|/2})$. Also, note $\partial_\alpha^2 (|y|^2 - |x|^2 + |x + y||x - y|) = O(\mu^{-|\beta|/2})$. Since $|y|^2 - |x|^2 + |x + y||x - y| \sim \mu$, combining those estimates with the Leibniz rule, we get (4.35) with $j = 1$.

For the proof of (4.36), we first claim that we may replace $\bar{I}_1^2$ with $\tilde{I}_j^2 = y_j \sin S_c - \mathcal{G}(x, y)(\cos S_c - \tau^+)(2x_j - \mathcal{A}y_j)$.

In what follows, we frequently use this type of argument to replace less favorable terms by allowing acceptable errors. To show the claim, we make use of some easy estimates. We first note
\begin{align}
\partial_\alpha^\beta (\sin S_c^j - \sin S_c) \lesssim 2^{-l} \mu^{-|\beta|}, \quad \beta \in \mathbb{N}^d,
\end{align}
which one can show in the same manner as \((4.29)\). Using \((4.8)\) and the bounds in Remark 1, we also have
\[
\partial^2_x \Phi = O(\mu^{-\frac{3}{2} - |\beta|}), \quad \beta \in \mathbb{N}^d.
\]
Noting \(\mathfrak{A} = 2 + |x - y|^2/(x, y)\), via a routine computation we have
\[
|\partial^2_x (2x_i - \mathfrak{A}y_i)|, |\partial^2_x (2y_i - \mathfrak{A}x_i)| \lesssim \begin{cases} \mu^{-\beta_1} \mu^{-|\beta|/2}, & i = 1, \\ \mu^{-\beta_1} (\mu \tilde{\mu})^{-|\beta|/2}, & i \geq 2, \end{cases}
\]
for \(\beta \in \mathbb{N}^d\). Indeed, to see this one needs only to write \(2x_i - \mathfrak{A}y_i = (2\langle x, y \rangle x_i - (|x|^2 + |y|^2)\mathfrak{A}y_i)/(x, y)\), and \(2y_i - \mathfrak{A}x_i\) can be handled similarly. Putting together the estimates \((4.37), (4.29), (4.28), (4.38),\) and \((4.39)\), we now see
\[
\partial^2_x (\mathcal{I}^2_j - \mathcal{I}^2_j) = O(2^{-i} \mu^{-\beta_1} (\mu \tilde{\mu})^{-|\beta|/2}), \quad \beta \in \mathbb{N}^d.
\]
In view of \((4.36)\), the difference is an acceptable error. Therefore, as claimed above, we only have to show \((4.36)\) for \(\mathcal{I}^2_j\) replacing \(\mathcal{I}^2_j\).

To do this, using the first equality in \((3.13)\) and \((4.33)\), we note
\[
\sin S_c \mathcal{I}^2_j = \mathcal{I}^2_j := y_j \sin^2 S_c + \cos S_c (2x_j - \mathfrak{A}y_j).
\]
Thus, the desired bound \((4.36)\) follows once we have
\[
|\partial^2_x \mathcal{I}_j| \lesssim \begin{cases} (\tilde{\mu} \mu)^{1/2} \mu^{-\beta_1} (\mu \tilde{\mu})^{-1/2}, & j = 1, \\ \mu^{-\beta_1} (\mu \tilde{\mu})^{-1/2}, & j \neq 1. \end{cases}
\]
Since \(|\tilde{x}|, |\tilde{y}| = O((\mu \tilde{\mu})^{1/2})\), \((4.40)\) for \(j \neq 1\) is easy. Indeed, it follows by \((4.39)\) and the bounds in Remark 1. To show \((4.40)\) with \(j = 1\), we break
\[
\tilde{I}_1 = \tilde{I}_{1,1} - \tilde{I}_{1,2},
\]
where
\[
\tilde{I}_{1,1} := x_i^{-1} (\langle x, y \rangle \sin^2 S_c + \cos S_c (2|x|^2 - \mathfrak{A}(x, y))), \\
\tilde{I}_{1,2} := x_i^{-1} (\langle \tilde{x}, \tilde{y} \rangle \sin^2 S_c + \cos S_c (2|\tilde{x}|^2 - \mathfrak{A}(\tilde{x}, \tilde{y}))).
\]
Since \(|\tilde{x}|, |\tilde{y}| = O((\mu \tilde{\mu})^{1/2})\), one can easily see \(|\partial^2_x \tilde{I}_{1,2}| \lesssim \mu \mu^{-\beta_1} (\mu \tilde{\mu})^{-1/2}\).

To handle \(\tilde{I}_{1,1}\), using \(\mathfrak{A}(x, y) = |x|^2 + |y|^2\), the second equality in \((3.13)\), and \(|x - y||x + y| = |x|^2 + |y|^2 - 2\langle x, y \rangle \cos S_c\), successively, we note
\[
\tilde{I}_{1,1} = 2x_i^{-1} \cos S_c (x, x - \cos S_c y).
\]
By \((4.35)\), we have \(\partial^2_x (x - \cos S_c y) = O((\mu \tilde{\mu})^{1/2} \mu^{-\beta_1} (\mu \tilde{\mu})^{-1/2})\). Therefore, we get \((4.40)\) for \(j = 1\) similarly as before (see Remark 1).

Let us set
\[
\mathbf{G}(x, y) = -\sin S_c \partial_x S_c \partial_y \mathcal{P}(x, y, S_c) + \partial_y S_c \mathcal{P}(x, y, S_c) \partial_x S_c,
\]
\[
\mathbf{F}(x, y) = -\sin S_c \partial^2_x \mathcal{P}(x, y, S_c) \partial_x S_c \partial_y S_c,
\]
\[
\mathbf{K}(x, y) = I_d + \mathbf{G}(x, y) + \mathbf{F}(x, y).
\]

In order to prove Lemma 4.6 by (1.31) and Lemma 4.11 it is sufficient to show that the entries \( \tilde{E}_{i,j} \) of the matrix

\[
\tilde{E} := K - M^0
\]
satisfy (4.12) in place of \( E_{i,j} \). A simple computation gives

\[
\partial_x \partial_y \mathcal{P}(x, y, S_c^l) = \frac{\cos S_c^l y - x}{\sin^2 S_c^l}, \quad \partial_y \partial_x \mathcal{P}(x, y, S_c^l) = \frac{\cos S_c^l x - y}{\sin^2 S_c^l}.
\]

Using those identities, we write

\[
(4.41) \quad G = -\frac{\delta}{\sin S_c^l} \left( (2 \cos S_c^l + \mathfrak{A})(xx^\top + yy^\top) - 4xy^\top - 2\mathfrak{A} \cos S_c^l yx^\top \right),
\]

\[
(4.42) \quad F = \sin S_c^l \partial_x^2 \mathcal{P}(x, y, S_c^l)^\top \left( 2\mathfrak{A}(xx^\top + yy^\top) - 4xy^\top - \mathfrak{A}^2 yx^\top \right).
\]

In what follows, for \( i = 1, 2 \), we denote \( E(x, y) = \mathcal{O}_a(E_i) \) if \( \partial_x^a E(x, y) = O(E_i^a) \) for any \( \alpha \) and \( (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}} \). We first show (4.12) for \( (i, j) = (1, 1) \), which is more involved than the others.

**Proof of (4.12) for \( (i, j) = (1, 1) \).** We consider \( G_{1,1} \) and \( F_{1,1} \), from which we discard some harmless parts. By (4.41),

\[
G_{1,1} = -\frac{\delta}{\sin S_c^l} \left( (\mathfrak{A} + 2)(x_1 - y_1)^2 + 2(\cos S_c^l - 1)(x_1^2 + y_1^2 - \mathfrak{A} x_1 y_1) \right).
\]

Using \( \mathfrak{A}(x, y) = |x|^2 + |y|^2 \), we have

\[
G_{1,1} = -\frac{\delta}{\sin S_c^l} \left( (\mathfrak{A} + 2)(x_1 - y_1)^2 + 2\delta \frac{(1 - \cos S_c^l)}{\sin S_c^l} \left( \mathfrak{A}(\bar{x}, \bar{y}) - |\bar{x}|^2 - |\bar{y}|^2 \right) \right).
\]

The second term, which we denote by \( \mathcal{G}_{1,1} \), is \( \mathcal{O}_a(E_1) \). Indeed, since \( \partial_x^a (\mathfrak{A}(\bar{x}, \bar{y}) - |\bar{x}|^2 - |\bar{y}|^2) = O(\mu \bar{\mu}^{\alpha-1}(\mu \bar{\mu})^{-|\alpha|/2}) \), the bounds in Remark 1 and (4.38) show \( \partial_x^a \mathcal{G}_{1,1} = O(\mu \bar{\mu}^{\alpha-1}(\mu \bar{\mu})^{-|\alpha|/2}) \). Thus, by the first identity in (4.33) we get

\[
G_{1,1} = -\frac{\cos S_c(x_1 - y_1)^2}{\sin S_c \sin S_c^l} |x + y| \frac{|x + y|}{\langle x, y \rangle} + \mathcal{O}_a(E_1).
\]

By writing \( (x_1 - y_1)^2 = |x - y|^2 - |\bar{x} - \bar{y}|^2 \) and using (3.13) (the second equality), the first term on the right hand side equals

\[
\frac{|\bar{x} - \bar{y}|^2}{|x - y|^2} \sin S_c - \sin S_c^l \frac{|x + y| |x - y| \cos S_c}{\langle x, y \rangle} \sin S_c^l \sin S_c.
\]

Denote the second term by \( \mathcal{G}_{1,1}(x, y) \). By (4.37) and (4.38), we have \( \partial_x^a \mathcal{G}_{1,1} = O(2^{-l} \mu \bar{\mu}^{-3/2} \mu \bar{\mu}^{\alpha-1}(\mu \bar{\mu})^{-|\alpha|/2}) \), so \( \mathcal{G}_{1,1} = \mathcal{O}_a(E_1) \) by (3.18). Thus, we obtain

\[
G_{1,1} - M^0_{1,1} + \mathcal{O}_a(E_1) = -\frac{|x + y| |x - y| \cos S_c}{\langle x, y \rangle} \frac{\sin S_c^l}{\sin S_c} = \frac{(\tau^+ - \cos S_c) \cos S_c}{\sin S_c^l \sin S_c}.
\]

We use (3.13) for the second equality.
From (1.42), we have
\[
F_{1,1} = \sin S_c^l \partial_x^2 \mathcal{P}(x, y, S_c^l) \Theta^2(2\mathcal{A}(|x|^2 + |y|^2) - (4 + A^2)(x, y)) \\
- \sin S_c^l \partial_x^2 \mathcal{P}(x, y, S_c^l) \Theta^2(2\mathcal{A}(|\bar{x}|^2 + |\bar{y}|^2) - (4 + A^2)(\bar{x}, \bar{y})).
\]

We denote the second term by \( \tilde{F}_{1,1} \). By (4.27) and (4.38), we see \( \partial_x^2 \tilde{F}_{1,1} = O(2^{-l} \mu^{-3/2} \mu^{-\alpha_1}(\mu \bar{\mu})^{-|\alpha|/2}) \), so \( \tilde{F}_{1,1} = O_c(E_1) \). Meanwhile, (4.33) gives
\[
2\mathcal{A}(|x|^2 + |y|^2) - (4 + A^2)(x, y) = \frac{|x - y|^2 |x + y|^2}{\langle x, y \rangle} = \Theta^{-2} \cos^2 S_c \langle x, y \rangle \sin^2 S_c.
\]
Combining this with (3.12) yields
\[
F_{1,1} = \frac{(\tau^+ - \cos S_c^l)(\cos S_c^l - \cos S_c) \cos^2 S_c}{\sin^2 S_c^l \sin^2 S_c} + O_c(E_1).
\]

We now set
\[
D_1 = (\tau^+ - \cos S_c^l)(\cos S_c^l - \cos S_c) \cos^2 S_c,
\]
\[
D_2 = (\tau^+ - \cos S_c)(\sin S_c^l - \sin S_c) \sin S_c^l \cos S_c,
\]
and
\[
A = \frac{D_1 + D_2}{\sin^2 S_c \sin^2 S_c^l}.
\]

Since \( K_{1,1} = 1 + F_{1,1} + G_{1,1} \), using the above equalities, we obtain
\[
K_{1,1} - \mathfrak{m}^0_{1,1} = A + O_c(E_1).
\]

Indeed, from (3.11) and (3.10) note \( \sin^2 S_c = (\tau^+ - \cos S_c) \cos S_c \), which then gives \( D_2 = \sin^2 S_c \sin^2 S_c^l - (\tau^+ - \cos S_c) \sin S_c \sin S_c^l \cos S_c \).

To complete the proof, it remains to show \( A = O_c(E_1) \). Let \( \tilde{D}_1 \) denote \( D_1 \) in which the first \( \cos S_c^l \) is replaced with \( \cos S_c \), and by \( \tilde{D}_2 \) we denote \( D_2 \) in which the second \( \sin S_c^l \) is replaced with \( \sin S_c \). Then, using (4.29) as before, we see \( \partial_x^2 (\tilde{D}_1 - \tilde{D}_1) = O(2^{-2l} \mu^{-\alpha_1}(\mu \bar{\mu})^{-|\alpha|/2}) \). Similarly, \( \partial_x^2 (\tilde{D}_2 - \tilde{D}_2) = O(2^{-2l} \mu^{-\alpha_1}(\mu \bar{\mu})^{-|\alpha|/2}) \) by (4.37) and (4.28).

So, \((\tilde{D}_1 - \tilde{D}_1 + \tilde{D}_2 - \tilde{D}_2)/(|\sin^2 S_c \sin^2 S_c^l|) = O_c(E_1) \). Thus, we have
\[
A = \frac{\tilde{D}_1 + \tilde{D}_2}{\sin^2 S_c \sin^2 S_c^l} + O_c(E_1).
\]

Elementary trigonometric identities give \( \tilde{D}_1 + \tilde{D}_2 = 2 \cos S_c (\cos S_c - \tau^+) \sin^2 ((S_c^l - S_c)/2) \). Therefore,
\[
A = 2 \cos S_c (\cos S_c - \tau^+) \sin^2 \left( \frac{S_c^l - S_c}{2} \right) + O_c(E_1).
\]

Since \( S_c^l - S_c = 2^{-l} s \), using (4.28) and the bounds in Remark 1, we conclude \( A = O_c(E_1) \). \( \square \)
Before we begin to prove \((4.12)\) for \((i, j) \neq (1, 1)\), we show that the contribution of \(F\) is negligible. By \((4.42)\) we have
\[
F_{i, j} = -\sin S^l_{c} P(x, y, S_{c}^l) \tilde{G}^2(2x_i - \mathfrak{A} y_i)(2y_j - \mathfrak{A} x_j).
\]
If \(i, j \geq 2\), using \((4.27)\), \((4.39)\), and \((4.38)\) (also see Remark \(1\)), we get
\[
\partial^2_{\alpha} F_{i, j} = O(2^{-l} \mu^{1/2} \mu^{-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2}),
\]
which shows \(F_{i, j} = O_{\epsilon}(E_1)\). If \(i = 1, j \geq 2\), or \(j = 1, i \geq 2\), we obtain
\[
\partial^2_{\alpha} F_{i, j} = O(2^{-l} \mu^{1/2} \mu^{-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2})
\]
similarly, so it follows that \(F_{i, j} = O_{\epsilon}(E_2)\).

Therefore, the following completes the proof of Lemma 4.6:
\[
(4.43) \quad G_{i, j} - \mathfrak{M}^0_{i, j} = \begin{cases} O_{\epsilon}(E_1), & i, j \geq 2, \\ O_{\epsilon}(E_2), & i = 1, j \geq 2, \text{ or } j = 1, i \geq 2. \end{cases}
\]

**Proof of (4.43).** We consider the case \(i, j \geq 2\) first. By \((4.41)\) we have
\[
G_{i, j} = -\frac{\mathfrak{G}}{\sin S_{c}} \tilde{G}((2 \cos S_{c} + \mathfrak{A})(x_i, x_j + y_i y_j) - 4x_i y_j - 2\mathfrak{A} \cos S_{c} y_i x_j).
\]
By \(G'_{i, j}\), we denote \(G_{i, j}\) in which we replace \(\mathfrak{A}\) with 2. Since \(\mathfrak{A} = 2 + |x - y|^2/(x, y)\), \(\partial^2_{\alpha}(\mathfrak{A} - 2) = O(\mu^{2-|\alpha|})\). Using \((4.38)\), we see
\[
\partial^2_{\alpha}(G_{i, j} - G'_{i, j}) = O(\mu \mu^{1-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2})
\]
because \(x_i, x_j, y_i, y_j = O((\mu \tilde{\mu})^{1/2})\), \(i, j \geq 2\). Thus, it is enough to consider \(G'_{i, j}\). Furthermore, by \((4.29)\) and \((4.38)\), \(\cos S_{c}\) in \(G'_{i, j}\) can similarly be replaced with \(\cos S_{c}\) if we allow an error of \(O_{\epsilon}(E_1)\). Thus,
\[
(4.44) \quad G_{i, j} = \tilde{G}_{i, j} + G_{i, j} + O_{\epsilon}(E_1), \quad i, j \geq 2,
\]
where
\[
\tilde{G}_{i, j} = -\frac{2\mathfrak{G}}{\sin S_{c}} (x_i, x_j + y_i y_j)(2x_i - (1 + \cos S_{c}) y_i),
\]
\[
G_{i, j} = \frac{2\mathfrak{G}}{\sin S_{c}} (x_i, x_j)(y_i - x_i)(1 - \cos S_{c}) x_j.
\]

It is easy to see \(\partial^2_{\alpha} \tilde{G}_{i, j} = O(\mu \mu^{1-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2})\) for \(i, j \geq 2\), thus \(\tilde{G}_{i, j} = O_{\epsilon}(E_1)\). Using \((4.31)\), by \(\sin S_{c}\) we may replace \(\sin S_{c}^l\) in the expression of \(\tilde{G}_{i, j}\) with an error of \(O_{\epsilon}(E_1)\). Then, applying \((4.33)\) and \((4.13)\), we get
\[
(4.45) \quad G_{i, j} = \frac{2(x, y)}{|x + y|^2} \frac{(x - y)(x_i, x_j) - (2x_i - (1 + \cos S_{c}) y_i)}{|x - y|^2} + O_{\epsilon}(E_1).
\]
As before, we may replace \(2(x, y)/|x + y|^2\) by \(1/2\), since \(\partial^2_{\alpha}(1/2 - 2(x, y)/|x + y|^2) = O(\mu^{2-|\alpha|})\), \(\partial^2_{\alpha}(2x_i - (1 + \cos S_{c}) y_i) = O((\mu \tilde{\mu})^{1/2} \mu^{-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2})\) and \(\partial^2_{\alpha}(x_i - y_i) = O((\mu \tilde{\mu})^{1/2} \mu^{-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2})\), \(i, j \geq 2\). Therefore, we have
\[
G_{i, j} = \mathfrak{M}^0_{i, j} + \frac{(x_i - y_i)(x_i, x_j) \mathfrak{A} S_{c} - y_i)}{2|x - y|^2} + O_{\epsilon}(E_1).
\]
Note \(\partial^2_{\alpha}((x_i - y_i)(1 - \cos S_{c}) y_i)/|x - y|^2) = O(\mu \mu^{-\alpha_1}(\mu \tilde{\mu})^{-|\alpha|/2})\). Thus, \((4.43)\) follows for \(i, j \geq 2\).

We now show \((4.43)\) when \(i = 1, j \geq 2\), or \(j = 1, i \geq 2\). We only consider the case \(i = 1, j \geq 2\), since the other one can be handled in the same manner.
Since $x_j, y_j = O((\mu \tilde{\mu})^{1/2})$ for $j \geq 2$, repeating the same argument used to show (4.41), we obtain $G_{1,j} = \tilde{G}_{1,j} + G_{1,j} + \mathcal{O}_s(E_2)$. Here we use the same notations as above. Note $\partial_2^x(2x_1 - (1 + \cos S_c)y_1) = O(\mu \mu^{-\alpha_1} (\mu \tilde{\mu})^{-|\alpha_1|/2})$. By this and (4.38), $\partial_2^x \tilde{G}_{1,j} = O(\mu^{1/2} \mu^{1/2} \mu^{-\alpha_1} (\mu \tilde{\mu})^{-|\alpha_1|/2})$, thus $G_{1,j} = \tilde{G}_{1,j} + \mathcal{O}_s(E_2)$. Allowing an error of $\mathcal{O}_s(E_2)$, as before, we may replace $\sin S_c$ with $\sin S_c$ in $G_{1,j}$. Consequently, using (4.33) and (3.13) (cf. (4.45)), we obtain

$$\tilde{G}_{1,j} = -\frac{2\langle x, y \rangle}{|x + y|^2} \frac{x_j - y_j}{|x - y|^2} (2x_1 - (1 + \cos S_c)y_1) + \mathcal{O}_s(E_2).$$

We may also replace, as above, the factor $2\langle x, y \rangle / |x + y|^2$ by $1/2$ with an error of $\mathcal{O}_s(E_2)$. Therefore, (4.43) follows for $i = 1, j \geq 2$. \hfill \square

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