Abstract

New method of solution of the motion equation of the mechanical system with higher degree of freedom is applied on the investigation of the vertical vibration of the vehicle. Mechanical model is composed from three spatially elastically supported and bounded bodies. Model represents the chassis of railroad vehicle with elastic and dissipative elements. Proposed method enables to determine the vertical displacement of the arbitrary point of the system.

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1. Introduction

The rail vehicles represents complicated system composed of many mutually connected bounded bodies with higher values of degree of freedom. Design of the new rail vehicle or optimization of the contemporary one with respect to its driving characteristics (stability) represents the challenging task for the mechanical engineers equipped by modern computational technique. Timoshenko [1] was the pioneer investigator of the dynamics of the rail vehicles. The most common method for investigation of the rail vehicle is application of the simplified numerical
model of the system composed by several mutually connected bodies, where the vertical vibration of the specific point is investigated. The system is excited by exactly defined external force in the certain point or it is being excited by vehicle movement over road obstacles or jumps on the vehicle track. The summary of the computation methods of the mechanical system dynamic gives Shabana et al. [15]. For the model of the rail vehicle the standard methods are commonly applied, e.g. FEM used by Stribersky et al. [2] or neural networks by Sayyaadi [3]. Cheli et al. [4] solved the system with 68-degree-of-freedom, where FEM software was utilized. Another approach to solve the vehicle vibration and thus the ride quality investigation was performed by the synthesized performance analysis with modal parameters (SPAMP) method, Zhou et al. [10]. Comfort quality of the high speed trains was also investigated by Diana et al. [11]. Currently, the investigation of the vehicle vibration turn over to the effect of the interaction between vehicle wheels and track surface, where model contents the model of the track as well Gong et al. [5], Hou et al. [6], Zhai et al. [9] and Lombaert et al. [16]. Popp et al. [14] consider the elasticity of wheelsets and track in his analytical model. In common cases the track has high values of degree of freedom, e.g. 569 Nielsen et al. [7]. A review article of track and vehicle/track interaction at high frequencies was performed by Knothe et al. [8]. Also the type of the suspension of the vehicle and the effect on the dynamics behavior of wagon was investigated; the pneumatic suspension Docquier et al. [13] semi-active suspensions Wu et al.[12]. The article presents the algebraic method for solution of the vertical vibration of the bodies system with higher degree of freedom.

2. Materials and methods

The proposed method of solution of the motion equation of the mechanical systems can be applied on the systems with higher number of degree of freedom \( p > 3 \). Due to extension of the theme, it is impossible to submit the method application for the general case of motion equation of the mechanical systems. But the proposed method is applied with limitation for the solution of vertical vibration of the vehicle model. The solution progress is presented on the mechanical system composed with three bodies. The bodies are spatially elastically supported and bounded. The model consists of the rail chassis vehicle with elastic and dissipative elements. The vehicle frame has determined mass \( m \) and inertia moments \( J_x, J_y \) and deviation moments \( D_{xy} = D_{yx} \) to axes passing through the center of gravity \( T \). The model also consists of two wheelsets with masses \( m_1 \) and \( m_2 \) and inertia moments \( J_{x1}, J_{x2}, J_{y1}, J_{y2} \) and deviation moments \( D_{xy1} = D_{yx1}, D_{xy2} = D_{yx2} \) to axes passing through the center of gravity of the wheelsets \( T_1, T_2 \).

This limitation of the number of bodies and number of degree of freedom (three for particular bodies of the system) is not the detriment of the method, i.e. it could be possible to present the method for the mechanical system with higher number of bodies and higher degree of freedom \( p \leq 6 \).

For the assume case with nine degree of freedom \( p = 9 \), the nine coordinates are need to be determined: vertical displacement of center of gravity for each body \( w_T, w_{T1}, w_{T2}, \phi_{x1}, \phi_{y1}, \phi_{x2}, \phi_{y2} \), rotation angle of each bodies around the axes \( x, y, x_1, y_1, x_2, y_2 \) passing through the center of gravity of particular bodies \( \phi_\alpha, \phi_\beta, \phi_{\alpha1}, \phi_{\beta1}, \phi_{\alpha2}, \phi_{\beta2} \) with relation to the general displacement \( q_j(t) \).

\[
q_j(t) = \left[ w_T, w_{T1}, w_{T2}, \phi_{x1}, \phi_{y1}, \phi_{x2}, \phi_{y2} \right], \quad \text{for } j = 1, \ldots, q. \tag{1}
\]

The vertical displacement arbitrary point \( w_j(t) \) of the system can be calculated by determination of these nine components of the general displacement. The vertical displacement is the function of the point coordinates, system coordinates and components of the general function

\[
w_j(t) = w_j\left(w_T, w_{T1}, w_{T2}, \phi_{x1}, \phi_{y1}, \phi_{x21}, \phi_{y21}, x_1, y_1, \ldots \right). \tag{2}
\]

It is possible to perform the derivation of the motion equation by second order Lagrange equations, which for our case can be described
\[
\frac{d}{dt} \left( \frac{\partial E_k}{\partial q_j} \right) - \frac{\partial E_k}{\partial q_j} + \frac{\partial E_p}{\partial q_j} + \frac{\partial R_d}{\partial q_j} = Q_j, \quad \text{for } j = 1, \ldots, p = 9. \quad (3)
\]

The kinetic energy appears in the equation
\[
E_k = \frac{1}{2} \left( m \dot{w}_r^2 + m_t \dot{W}_{T1}^2 + m_s \dot{W}_{T2}^2 + J_x \dot{\phi}_x^2 + J_y \dot{\phi}_y^2 - 2D_{xy} \dot{\phi}_x \dot{\phi}_y + J_{x1} \dot{\phi}_{x1}^2 + J_{y1} \dot{\phi}_{y1}^2 - 2D_{xy1} \dot{\phi}_{x1} \dot{\phi}_{y1} \right). \quad (4)
\]

And the potential energy
\[
E_p = \frac{1}{2} \sum_{j=1}^{m_k} k_j w_j^2, \quad (5)
\]

where \( w_j \) is vertical displacement of the elastically support or joint in the point \( j \), i.e. spring deformation by length change of the \( j \) spring with stiffness \( k_j \) for \( j = 1, \ldots, m_k \).

Rayleigh’s function of the dissipative energy
\[
R_d = \frac{1}{2} \sum_{j=1}^{m_b} b_j \dot{w}_j, \quad (6)
\]

where \( \dot{w}_j \) is vertical component of velocity of the point \( j \), linear viscosity dumping \( j = 1, \ldots m_b \) has coefficient of the dumping intensity \( b_j \).

The vector of the generalized function of the excitation forces \( Q_j(q_j, \dot{q}_j, t) \) has the vector of the generalized coordinates according to (1)
\[
q_j = \{ w_r, W_{T1}, W_{T2}, \phi_x, \phi_y, \phi_{x1}, \phi_{y1}, \phi_{x2}, \phi_{y2} \}^T . \quad (7)
\]

After substituting the equation for \( w_j \) expressing vertical displacement in the point \( j \) with dependency on the components of the generalized coordinate vector \( \dot{q}_j(t) \) according to (2) and after implementation of the relevant derivation and after arrangement, the system of the differential equations, which can be expressed in the matrix form (8) is obtained
\[
M_0 \ddot{q}_j(t) + M_v \dot{q}_j(t) + M_k q_j(t) = Q_j(q_j, \dot{q}_j, t), \quad \text{for } j = 1, \ldots 9. \quad (8)
\]

The matrix element of the mass \( M_k \) are \( \tilde{a}_{ij} \), the matrix elements of the dumping \( M_v \) are \( \tilde{b}_{ij} \), the matrix element of the stiffness \( M_k \) are \( \tilde{a}_{ij} \). We get the equation by multiplying the equation (8) by diagonal matrix \( D(d_{ii}) \) from left side
\[
DM_0 \ddot{q}_j(t) + DM_v \dot{q}_j(t) + DM_k q_j(t) = DQ_j(q_j, \dot{q}_j, t), \quad (9)
\]

where diagonal elements of the matrix \( D \) are \( d_{ii} = \tilde{a}_{ii}^{-1} \), and \( \tilde{a}_{ii} \) are diagonal elements of the matrix \( M_k \). After arrangement we get the equation
\[
M \ddot{q}_j(t) + B \dot{q}_j(t) + K q_j(t) = F_j(t), \quad \text{for } j = 1, \ldots, 9, \tag{10}
\]

where mass matrix \( M \) is possible to express by form

\[
M = E + H, \tag{11}
\]

where \( E \) is unit diagonal matrix and matrix elements \( H = (h_{ij}) \) are given by

\[
h_{ij} = 0, \quad \text{for } i = j, \quad 1, 2, \ldots, 9 \tag{12}
\]

\[
h_{ij} = d_i \alpha_{ij} = \alpha_{ij} \alpha_{ii}^{-1}, \quad \text{for } i \neq j. \tag{13}
\]

If \( D_{xy} = D_{yx} = 0; D_{xy1} = D_{xy2} = 0; D_{xy2} = D_{xy2} = 0, \) the elements are \( h_{ij} = 0 \) for \( i \neq j, \) the matrix \( H = 0. \) The matrix \( H = (h_{ij}) \) defines distribution of the mass of the particular bodies relative to central axes of the body, i.e. for \( h_{ij} \neq 0 \) the mass is asymmetrically distributed, for \( h_{ij} = 0 \) the mass is symmetrically distributed and central inertia axes of the body are main axes.

The elements of the dumping matrix \( B \) are given by

\[
b_{ij} = d_i \beta_{ij}. \tag{14}
\]

The elements of the stiffness matrix \( K \) are given by

\[
a_{ij} = d_i \alpha_{ij}. \tag{15}
\]

The components of the generalized excitation function

\[
F_j(t) = d_i Q_j(t). \tag{16}
\]

It is possible to solve the system of differential equations (10); and the system of the arranged motion equations (8) by application of the Laplace’s integration transformation \( (s – \text{parameter of the transformation}), \) by which the linear algebraic equation is obtained

\[
G \ddot{q}_j(s) = F_j(s), \tag{17}
\]

where matrix elements \( G \) are given by expressions

\[
g_{ij} = s^2 + b_j s + a_{ij}, \quad \text{for } i = j \tag{18}
\]

\[
g_{ij} = b_j s + a_{ij}, \quad \text{for } i \neq j \tag{19}
\]

but for \( i = 4, 6, 8 \) and \( j = i + 1, \) or \( i = 5, 7, 9 \) and \( j = i – 1 \) is valid

\[
g_{ij} = h_{ij} s^2 + b_j s + a_{ij}, \quad \text{where according to (28) is} \tag{20}
\]
\[ h_{ij} = \overline{a}_{ij} a_{ii}^{-1} \]  \hspace{1cm} (21)

It is possible to obtain the solution of the algebraic equations (17) by various methods. The Cramerius rule best fits to our case, with respect to the reverse transformation of the images \( \overline{q}_j(t) \), to determination of the frequency and amplitude analysis of resulting time curve of the original component of the coordinate \( q_j(t) \) and with respect to relatively low degrees of freedom.

\[
q_j(s) = \frac{D_j(s)}{D(s)} = \sum_{i=1}^{p=q} (-1)^{i+j} \overline{F}_i(s) \frac{D_{ji}(s)}{D(s)},
\]  \hspace{1cm} (22)

where determinant of the system \( D(s) \) is given by polynomial expression

\[ D(s) = \sum_{l=0}^{n} A_{n-l}s^{n-l} \text{, where } n = 2p = 18. \]  \hspace{1cm} (23)

The determinant \( D_j(s) \) arise by substitution of the \( j \) column of the determinant \( D(s) \) by vector \( \overline{F}_j(s) \). The determinant \( D_{ji}(s) \) is algebraic supplement of the determinant development \( D_j(s) \) according to \( i \) element and \( j \) column of the vector \( \overline{F}_j(s) \). The ratio of the determinants \( D_j(s) \) and \( D(s) \) is necessary to arrange due to reverse transformation of the image \( \overline{q}_j(s) \). The ratio and its product with \( \overline{F}_j(s) \) have to meet the conditions, which are in compliance with theorem of the image of convolution. The determinant \( D_{ji}(s) \) is possible to rewrite to polynomial form

\[ D_{ji}(s) = \sum_{l=0}^{m} d_{ji,m-l}s^{m-l} \]  \hspace{1cm} (24)

For the Laplace’s transformation it is suitable to decompose the determinant ratio \( D_{ji}(s)/D(s) \) i.e., purely fractional ration function decompose to partial fraction by method of the indeterminate coefficients. The zero point of the polynomial has to be evaluated (23) from this reason. Some software allows it, e.g. Mathematica, Maple, Matlab etc. For solution of the frequency equations of undumped and dumped mechanical systems, the own method of solution of zero points was proposed.

There are several combination of zero point during solution of frequency equations \( D(s) = 0 \), according to (23). The zero points can have the form of the complex conjugated numbers or real numbers less than zero for dumped mechanical system. The zero points are conjugated imaginary numbers when the mechanical system is undumped.

Then it is possible to express the polynomial \( D(s) \) according to (23) in the form of product of the rooted factors, and when the \( A_n = 1 \) the following case are possible

1) \[ D(s) = \sum_{l=0}^{n} A_{n-l}s^{n-l} = \prod_{k=1}^{n} (s^2 + 2\beta_k s + \omega_k^2), \]  \hspace{1cm} (25)

or

2) \[ D(s) = \sum_{l=0}^{n} A_{n-l}s^{n-l} = \prod_{k=1}^{n} (s + 2\beta_k s + \omega_k) + \prod_{k=n+1}^{n} (s + \beta_k), \]  \hspace{1cm} (26)

or
3) \[ D(s) = \sum_{j=0}^{n} A_{n-j} s^{n-j} = \prod_{k=1}^{n}(s + \beta_k). \] (27)

The ratio of the determinants \( D_{ji}(s)/D(s) \) can be expressed in the form

\[ \frac{D_{ji}(s)}{D(s)} = \frac{\sum_{k=0}^{m} d_{ji,k} s^{m-k}}{\prod_{k=1}^{n/2} (s^2 + 2 \beta_k s + \alpha_k^2)} = \sum_{k=1}^{n/2} \left( K_{ji,k} s + L_{ji,k} \right), \] where \( m \leq n - 1. \) (28)

The fraction at the left side of the equation (28) is then summarized. The denominators on the both sides of the equation are equal. Then the denominator is given by equation

\[ \sum_{k=0}^{m} d_{ji,m-k} s^{m-k} = \sum_{k=1}^{n/2} \left( K_{ji,k} s + L_{ji,k} \right) \prod_{l=1}^{n/2} \frac{(s^2 + 2 \beta_k s + \alpha_k^2)}{s^2 + 2 \beta_k s + \alpha_k^2}, \] (29)

The right side of the equation can be arranged

\[ \prod_{l=1}^{n/2} \frac{(s^2 + 2 \beta_k s + \alpha_k^2)}{s^2 + 2 \beta_k s + \alpha_k^2} = \sum_{j=0}^{n} A_{n-j} s^{n-j} = \sum_{l=2}^{n/2} t_{k,n-l} s^{n-l}, \] (30)

where the coefficients of polynomial on the right side is given by

\[ t_{k,n-1} = 0, \quad t_{k,n-2} = 1 \quad (= A_n), \] (31, 32)

\[ t_{k,n-l} = A_{n-l+2} - 2 \beta_k t_{k,n-l+1} - \alpha_k^2 t_{k,n-l+2}, \quad \text{for} \quad l = 3, \ldots, n = 18. \] (33)

Then the equation (29) can be rewrote to

\[ \sum_{k=1}^{n/2} \left( K_{ji,k} t_{k,n-l-1} + L_{ji,k} t_{k,n-l-1} \right) = d_{ji,m-l}, \quad \text{for} \quad l = 1, \ldots, n = 18. \] (34)

where for \( l = n, \) \( t_{k,n-l} = t_{k-1} = 0 \) represents the remainder after division.

The system of the linear algebraic equations (34) for indeterminate coefficients \( K_{ji,k} \) and \( L_{ji,k} \) in the matrix form

\[ T\lambda = \delta, \] (35)

where elements of the matrix \( T(a_{zv}) \) of type \((n x n)\) are given by equations,

for \( z = 1,2,\ldots, n = 18, \quad v = 1,2,\ldots, n/2 = 9 \Rightarrow k = v, \) \( a_{zv} = t_{k,n-z-1} \)
where expressions $t_{k,n-z}$ are given by equations (33).

The vector of unknown coefficients

$$
\lambda = \lambda^T
$$

where for $z = 1, \ldots, n/2 = 9$, is $\lambda_z = K_{ji,k}$

for $z = n/2+1, \ldots, n = 18$; is $\lambda_z = L_{ji,k}$.

The vector $\delta = \delta^T$ of the right side of the equation (35) has elements which are the coefficient of polynomial $D_j(s)$ according to (24) for $z = 1, \ldots, n = 8$ is $\delta_z = d_{ji,n-z}$, where elements $d_{ji,n-z}$ are coefficient of polynomial $D_j(s)$, with $d_{ji,n-1} = 0$.

System is necessary to solve for each $j$ and $i$, therefore $j \times i$ times.

The ratio of determinants is possible to express by equation with determining the coefficients $K_{ji,k}$ and $L_{ji,k}$.

$$
\frac{D_j(s)}{D(s)} = \sum_{k=1}^{n/2=9} \frac{K_{ji,k} s + L_{ji,k}}{s^2 + 2 \beta_k s + \Omega_k^2},
$$

(36)

denominator of the fraction on the right side is possible to arrange into the form

$$
s^2 + 2 \beta_k s + \Omega_k^2 = (s + \beta_k)^2 + \Omega_k^2,
$$

(37)

where $\Omega_k = \sqrt{\omega_k^2 - \beta_k^2}$ is the eigenfrequency of the dumped oscillation, $\omega_k$ is the eigenfrequency of the undumped oscillation and $\beta_k$ is the coefficient of the linear viscous dumping. Then ratio of determinants is

$$
\frac{D_j(s)}{D(s)} = \sum_{k=1}^{9} \frac{K_{ji,k} s + L_{ji,k}}{(s + \beta_k)^2 + \Omega_k^2}.
$$

(38)

The equation for demanded imagine of the generalized coordinate $\bar{q}_j(s)$ according to (22)

$$
\bar{q}_j(s) = \sum_{i=1}^{n/2} (-1)^{i+j} \bar{F}_i(s) \sum_{k=1}^{n/2} \frac{K_{ji,k} s + L_{ji,k}}{(s + \beta_k)^2 + \Omega_k^2}
$$

(39)

and after arrangement

$$
\bar{q}_j(s) = \sum_{k=1}^{n/2} \left[ K_{ji,k} \frac{s + \beta_k}{(s + \beta_k)^2 + \Omega_k^2} + \frac{L_{ji,k} - \beta_k K_{ji,k}}{\Omega_k} \frac{\Omega_k}{(s + \beta_k)^2 + \Omega_k^2} \right].
$$

(40)

After reverse Laplace’s transformation we obtain the equation for generalized coordinate $\bar{q}_j(s)$ for $j = 1, 2, 3$ in the form of the summarization convolution integrals

$$
q_j(t) = \sum_{i=1}^{n/2} (-1)^{i+j} \sum_{k=1}^{n/2} \left[ K_{ji,k} \int_0^t F_i(\tau) e^{-\beta_k(t-\tau)} \cos \Omega_k (t-\tau) d\tau + \frac{\Omega_k}{(s + \beta_k)^2 + \Omega_k^2} \right].
$$
After evaluation of the \( q_j(t) \) it is possible according to (1) determine the time dependency of the vertical displacement \( w_i(t) \) of the arbitrary point in case of (21) variant of polynomial zero point \( D(s) \). For different variant of the zero points, i.e. according to (26) and (27) it is possible to determine by analogous manner the equations for calculation of the generalized coordinate \( q_j(t) \). In the second case (26) of the polynomial zero point \( D(s) \) we arise to

\[
q_j(t) = \sum_{i=1}^{\sqrt{2}} (-1)^{i+\sqrt{2}} \left[ \sum_{k=1}^{n_1} \left( K_{ji,k} \int_0^t F_i(\tau)e^{-\beta_k(t-\tau)} \cos \Omega_k (t-\tau) \, d\tau \right) + \right.
\]

\[
\left. + \frac{L_{ji,k} - \beta_k K_{ji,k}}{\Omega_k} \int_0^t F_i(\tau)e^{-\beta_k(t-\tau)} \sin \Omega_k (t-\tau) \, d\tau \right] + \sum_{k=n_1+1}^{n/2} \frac{L_{ji,k}}{\Omega_k} \int_0^t F_i(\tau)e^{-\beta_k(t-\tau)} \, d\tau \].
\]

In the third case (29) the equation for generalized coordinate is

\[
q_j(t) = \sum_{k=1}^{\sqrt{2}} (-1)^{\sqrt{2}} \left[ \sum_{k=1}^{\sqrt{2}} L_{ji,k} \int_0^t F_i(\tau)e^{-\beta_k(t-\tau)} \, d\tau \right] .
\]

Now it is possible to obtain the vertical displacement of the arbitrary point of the system by evaluation of the \( q_j(t) \) in the different cases (27), (28) and (29) of the \( D(s) \) frequency polynomial zero points.

3. Conclusion

The proposed method allows solving various vibration tasks for model of vehicle with higher degree of freedom. It allows analyzing the effect of the input values, e.g. body geometry and its support, properties of springs and damping, mass distribution, track imperfection; it is simplifying analysis of the resulting motion, including frequency analysis. The unicity of the solution can be in advance implemented into the standard commercial software (Mathematica, Matlab,…) to solve wide range of the similar described problems.

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