CATALAN PATHS, QUASI-SYMMETRIC FUNCTIONS
AND SUPER-HARMONIC SPACES

J.-C. AVAL AND N. BERGERON

Abstract. We investigate the quotient ring \( R \) of the ring of formal power series \( \mathbb{Q}[[x_1, x_2, \ldots]] \) over the closure of the ideal generated by non-constant quasi-symmetric functions. We show that a Hilbert basis of the quotient is naturally indexed by Catalan paths (infinite Dyck paths). We also give a filtration of ideals related to Catalan paths from \((0,0)\) and above the line \( y = x - k \). We investigate as well the quotient ring \( R_n \) of polynomial ring in \( n \) variables over the ideal generated by non-constant quasi-symmetric polynomials. We show that the dimension of \( R_n \) is bounded above by the \( n \)-th Catalan number.

1. Introduction

The ring \( Qsym \) of quasi-symmetric functions was introduced by Gessel [19] as a source of generating functions for \( P \)-partitions [26]. Since then, quasi-symmetric functions have appeared in many combinatorial contexts [12, 26, 27]. The relation of \( Qsym \) to the ring of symmetric functions was first clarified by Malvenuto and Reutenauer [25] via a graded Hopf duality to the Solomon descent algebras, then Gelfand et. al. [18] defined the graded Hopf algebra \( NC \) of non-commutative symmetric functions and identified it with the Solomon descent algebra. In recent literature, we see a growing interest in quasi-symmetric functions and non-commutative symmetric functions as refinements of the ring of symmetric functions.

One unexplored avenue is as an analogue of the (symmetric) harmonic spaces. A classic combined result of Artin and Steinberg [1, 28] shows that the quotient ring of the polynomial ring \( \mathbb{Q}[x_1, x_2, \ldots, x_n] \) in \( n \) variables over the ideal \( \mathcal{I}_n \) generated by non-constant symmetric polynomials has dimension \( n! \). In fact, this space is a graded symmetric group module that affords the left regular representation. Moreover under the scalar product

\[
\langle P, Q \rangle = \left( P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})Q \right)(0,0,\ldots,0)
\]

the graded orthogonal complement \( H_n = \mathcal{I}_n^\perp \) is a set of representatives for the quotient spanned by all possible partial derivatives of the Vandermonde determinant. We see \( H_n \) as the set of polynomials in \( n \) variables that are killed by all symmetric partial derivative operators. In particular, the Laplacian \( \sum \partial_{x_i}^2 \) kills any such polynomial, thus the \( H - n \) if often called the space of harmonics. Refinement and generalization of this result has lead to an explosion of incredible results and conjectures, see [2, 3].

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for a small portion of this. The so-called $n!$-conjecture (Theorem) of Garsia and Haiman [16], just recently proven by Haiman [21], and its connection with Macdonald polynomials [21, 24] is a great achievement in this context.

Here we are interested in the quotient ring $\mathcal{R}$ of the ring of formal power series $\mathbb{Q}[\left[ x_1, x_2, x_3, \ldots \right]]$ over the closure of the homogeneous ideal $\mathcal{J}$ generated by all non-constant quasi-symmetric functions. That is the quotient

$$\mathcal{R} = \mathbb{Q}[\left[ x_1, x_2, x_3, \ldots \right]] / \mathcal{J}.$$  

This quotient is in fact a Hopf algebra. It will be interesting to study its structure in more detail in future work. Here we concentrate our attention on its linear structure only.

To every monomial $x_1^{\tilde{\alpha}_1}x_2^{\tilde{\alpha}_2}x_3^{\tilde{\alpha}_3} \cdots$ (of finite total degree) in $\mathbb{Q}[\left[ x_1, x_2, x_3, \ldots \right]]$ we associate a path in the plane as follow:

$$(0, 0) \rightarrow (\tilde{\alpha}_1, 0) \rightarrow (\tilde{\alpha}_1, 1) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2, 1) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2, 2) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3, 2) \rightarrow \cdots$$

If this path remains above the line $y = x$ we say that the path is a Catalan path (or infinite Dyck path). Our main result is

**Theorem 1.1.** A monomial Hilbert basis of $\mathcal{R}$ is given by the monomials of $\mathbb{Q}[\left[ x_1, x_2, x_3, \ldots \right]]$ corresponding to Catalan paths.

We also consider a special filtration of ideals $\mathcal{J}^{(e)}$ and their respective quotients, such that $\mathcal{J} = \mathcal{J}^{(0)}$ and $\mathcal{J}^{(e)} \subseteq \mathcal{J}^{(e+1)}$. The Hilbert basis of each quotient is indexed by paths above the line $y = x - e$.

This result relates to the harmonic polynomials in the following way. Consider the quotient of the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ over the ideal $\mathcal{I}_n$ generated by all non-constant quasi-symmetric polynomials. Since the ring of symmetric polynomials is a subring of the ring of quasi-symmetric polynomials, we have that $\mathcal{I}_n \subseteq \mathcal{J}_n$. We thus consider the quotient

$$\mathcal{R}_n = \mathbb{Q}[x_1, x_2, \ldots, x_n] / \mathcal{J}_n.$$  

The space $\mathcal{S}H_n = \mathcal{J}_n^\perp$ of representatives for the quotient is a subspace of harmonic polynomials as

$$\mathcal{S}H_n \subseteq \mathcal{H}_n.$$  

For this reason we call $\mathcal{S}H_n$ the space of super-harmonic polynomials. Recall that $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the famous Catalan numbers. The passage from infinitely many variables to finitely many variables is a priori non-trivial, and requires more work. Here we show:

**Theorem 1.2.**

$$\dim \mathcal{S}H_n = \dim \mathcal{R}_n \leq C_n.$$  

In fact we expect equality to hold in (1.3) and we are still in the process of completing a proof of this together with François Bergeron and Adriano Garsia, who have discovered the spaces $\mathcal{S}H_n$ and $\mathcal{R}_n$ completely independently and in the same period.
that we did. They have conjectured many of the results presented here, and much more. The results for $SH_n$ and the relations between $H_n$ and $SH_n$ are extremely interesting, and are the object of an ongoing collaboration with F. Bergeron and A. Garsia. We plan to write at least two papers: one dedicated to a proof of equality in (1.3), and another to investigate further properties of $SH_n$ and its generalization. In particular, the finite version of the successive quotients by the ideals $J_n^{(e)}$ is related to the work of [10]. Much of these results can be explained in a more general framework and will be the object of further study. We are convinced that these results are but the tip of a new iceberg. In particular, we would like to find any natural algebras acting on these spaces. What are the possible generalizations and specializations of the super-harmonics?

We underline here that F. Hivert [22] has developed an action of the Hecke algebra for which a polynomial is invariant if and only if it is quasi-symmetric. Unfortunately Hivert’s action is not compatible with multiplication and does not preserve the ideal $J_n$, hence it does not induce the desired action on the quotient. It is still interesting to note that Hivert’s action is also related to Catalan numbers. One way to reformulate his result in [22] is as follows. Consider the generator $e_i = \frac{q-T_i}{(1+q)}$ of the Hecke algebra, where $T_i$ are the standard generators. Then

$$e_i e_{i+1} e_i - \frac{q}{(1+q)^2} e_i$$

acts, via Hivert’s action, as zero on the polynomial ring. Hence, the quotient of the Hecke algebra by Relation (1.4), classically known as the Temperley-Lieb algebra [23], naturally acts on polynomials. This algebra is known to have dimension equal to $C_n$.

In Section 2 we recall appropriate definitions. In Section 3 we introduce a special family of generators for the ideal $J$ and the associated filtration $J^{(e)}$. In Section 4 we use these generators to show that the monomials corresponding to Catalan paths span our quotient, as well as the analogous result for $J^{(e)}$. Theorem 1.2 follows from this section. To complete the proof Theorem 1.1 we use a Gröbner basis argument in Section 5 to show independence.

## 2. Basic definitions

A composition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ of a positive integer $d$ is an ordered list of positive integers whose sum is $d$. We denote this by $\alpha \models d$. We call the integers $\alpha_i$ the parts of $\alpha$, and denote the number of parts in $\alpha$ by $\ell(\alpha)$. Given two compositions $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ and $\beta = [\beta_1, \beta_2, \ldots, \beta_l]$, we denote by $\alpha \beta$ the concatenation product $[\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_l]$. Also, there exists a natural one-to-one correspondence between compositions of $d$ and subsets of \{1, 2, \ldots, d-1\}. If $A = \{a_1, a_2, \ldots, a_{k-1}\} \subset [d-1]$, where $a_1 < a_2 < \ldots < a_{k-1}$, then $A$ corresponds to the composition $\alpha = [a_1-a_0, a_2-a_1, \ldots, a_k-a_{k-1}]$, where $a_0 = 0$ and $a_k = d$. For ease of notation, we shall denote the set corresponding to a given composition $\alpha$ by $D(\alpha)$. For compositions $\alpha$ and $\beta$ we say that $\alpha$ is a refinement of $\beta$ if $D(\beta) \subset D(\alpha)$, and denote this by $\alpha \triangleleft \beta$. 
For any composition \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k] \) of \( d \) we denote by \( M_\alpha \) the monomial quasi-symmetric function \[ M_\alpha(x_1, x_2, \ldots) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}. \]

This is a homogeneous infinite series of degree \( d \). We define \( M_0 = 1 \), where 0 denotes the unique empty composition of 0. It is known from the work of Gessel that the monomial quasi-symmetric functions form a linear basis of a ring (in fact a Hopf algebra) \( Qsym \) of quasi-symmetric functions.

An other useful basis of the ring \( Qsym_n \) is given by the fundamental quasi-symmetric function \[ F_\alpha(x_1, x_2, \ldots) = \sum_{\alpha \triangleright \beta} M_\beta(x_1, x_2, \ldots) = \sum_{j_1 \leq j_2 \leq \cdots \leq j_d} \sum_{i \in D(\alpha)} x_{j_1} x_{j_2} \cdots x_{j_d}. \]

Fundamental quasi-symmetric functions satisfy the following obvious, but crucial, relations. For \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k] \models d \), if \( \alpha_1 > 1 \), then
\[
(2.1) \quad F_\alpha(x_1, x_2, \ldots) = x_1 F_{[\alpha_1 - 1, \alpha_2, \ldots, \alpha_k]}(x_1, x_2, \ldots) + F_\alpha(x_2, x_3, \ldots),
\]
and if \( \alpha_1 = 1 \), then
\[
(2.2) \quad F_\alpha(x_1, x_2, \ldots) = x_1 F_{[\alpha_2, \alpha_3, \ldots, \alpha_k]}(x_2, x_3, \ldots) + F_\alpha(x_2, x_3, \ldots).
\]

Here \( F_\alpha(x_2, x_3, \ldots) \) is the function \( F_\alpha(x_1, x_2, \ldots) \) in which the variable \( x_i \) is replaced by \( x_{i+1} \). We will see that these relations are the key ingredients in our proof.

In the following we have to consider generalized (infinite) compositions. That is a sequence \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots) \) such that the parts \( \tilde{\alpha}_j \geq 0 \) for \( j \geq 1 \) (we allow some parts to be zero) and the sum of the parts \( d(\tilde{\alpha}) = \sum \tilde{\alpha}_i < \infty \). We say that \( \tilde{\alpha} \) is a generalized composition of \( d(\tilde{\alpha}) < \infty \). We use a “” to indicate that we have a generalized composition and no “” if the composition is standard, that is without zeros. We also consider generalized compositions of finite length and denote by \( \ell(\tilde{\alpha}) \) the number of parts of \( \tilde{\alpha} \). The concatenation of a finite length generalized composition \( \tilde{\alpha} \) with an infinite one \( \tilde{\beta} \) is denoted by \( \tilde{\alpha} \tilde{\beta} \). We also write \( \tilde{\alpha} + \tilde{\beta} \) and \( \tilde{\alpha} \leq \tilde{\beta} \) to denote the componentwise sum and componentwise inequalities, respectively. For an infinite generalized composition \( \tilde{\alpha} \), since \( d(\tilde{\alpha}) < \infty \), only finitely many parts of \( \tilde{\alpha} \) are non-zero. Thus there is always a finite generalized composition \( \tilde{\nu} \) such that \( \tilde{\alpha} = \tilde{\nu} 0 0 \cdots \).

In this paper, we devote our attention to the ideal \( J = \langle F_\alpha(x_1, x_2, \ldots) \rangle_{\alpha \models d > 0} \) of \( \mathbb{Q}[x_1, x_2, \ldots] \) generated by the non-constant quasi-symmetric functions, and consider the quotient
\[
(2.3) \quad R = \mathbb{Q}[x_1, x_2, \ldots] / J,
\]
where \( J \) denotes the closure (with respect to the standard topology with formal power series) of \( J \) in \( \mathbb{Q}[x_1, x_2, \ldots] \).
3. The generators \( G_{\tilde{\alpha}} \)

In the previous section we noted the relations \( (2.1) \) and \( (2.2) \). From the first one we deduce that if \( \alpha_1 > 1 \), then

\[
F_\alpha(x_2, x_3, \ldots) = F_\alpha(x_1, x_2, \ldots) - x_1 F_{[\alpha_1-1, \alpha_2, \ldots, \alpha_k]}(x_1, x_2, \ldots).
\]

Since both \( F_\alpha(x_1, x_2, \ldots) \) and \( F_{[\alpha_1-1, \alpha_2, \ldots, \alpha_k]}(x_1, x_2, \ldots) \) are in \( \mathcal{J} \), we conclude that \( F_\alpha(x_2, x_3, \ldots) \in \mathcal{J} \). We want to exploit these properties to a maximum. For this we construct a set \( \{ G_{\tilde{\alpha}} \} \subseteq \mathcal{J} \) indexed by the generalized (infinite) composition \( \tilde{\alpha} \) such that there exists a factorization \( \tilde{\alpha} = \pi \tilde{\rho} \) where

\[
d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq 0.
\]

For this, we first define recursively the functions \( G_{\tilde{\alpha}} \) for all infinite generalized composition \( \tilde{\alpha} \). Then in Lemma 3.1 we characterize the \( \tilde{\alpha} \) obtained from the transitive closure of the \( G_{\tilde{\alpha}} \in \mathcal{J} \). Let \( \tilde{\alpha} = \tilde{\nu} 0 0 \cdots \) where \( \ell(\tilde{\nu}) < \infty \) and the last part of \( \tilde{\nu} \) is non-zero, or \( \tilde{\alpha} = 0 0 \cdots \). Our definition is recursive on \( n = \ell(\tilde{\nu}) \). If \( \tilde{\nu} = \nu \) is a standard composition, then let

\[
G_{\tilde{\alpha}} = F_\nu(x_1, x_2, \ldots).
\]

If \( \ell(\tilde{\nu}) = 0 \), then this formula gives \( G_{00\cdots} = 1 \). Assume now that \( \tilde{\nu} \) is non-standard and let \( \tilde{\nu} = \tilde{\gamma} 0 a \beta \) be the unique factorization of \( \tilde{\nu} \) such that \( a > 0 \) is a positive integer, \( \beta \) is a (possibly empty) standard composition and \( \tilde{\gamma} \) is a (possibly empty) generalized composition. For \( \tilde{\alpha} = \tilde{\gamma} 0 a \beta 0 0 \cdots \) and \( k = \ell(\tilde{\gamma}) = \ell(\tilde{\gamma}) + 1 \), we define

\[
G_{\tilde{\alpha}} = G_{\tilde{\gamma} a \beta 0 \cdots} - x_k G_{\tilde{\gamma}(a-1) \beta 0 \cdots}.
\]

Both term on the right are well defined by induction since \( \ell(\tilde{\gamma} a \beta) = \ell(\tilde{\nu}) - 1 < n \).

We now characterize the transitive closure of the definition \( (3.2) \) and \( (3.3) \) within \( \mathcal{J} \). At this point it is useful to introduce the following family of ideals. For any \( e \geq 0 \), let

\[
\mathcal{J}^{(e)} = \{ F_\alpha : \exists \pi \rho = \alpha, \ d(\pi) - \ell(\pi) \geq e \}.
\]

This is a filtration \( \mathcal{J}^{(e)} \subseteq \mathcal{J}^{(e+1)} \) such that \( \mathcal{J} = \mathcal{J}^{(0)} \). For a generalized composition \( \tilde{\alpha} \), we say that it reaches level \( e \) if there exists a factorization \( \tilde{\alpha} = \pi \tilde{\rho} \) such that

\[
d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e.
\]

**Lemma 3.1.**

1. If \( \tilde{\alpha} \) reaches level \( e \), then \( G_{\tilde{\alpha}} \in \mathcal{J}^{(e)} \).
2. Conversely, in \( (3.3) \), if \( \tilde{\gamma} (a-1) \beta 0 0 \cdots \) reaches level \( e \), then \( \tilde{\alpha} \) reaches level \( e \).

**Proof.** For the first statement we proceed by induction on \( \ell(\tilde{\nu}) \) where \( \tilde{\alpha} = \tilde{\nu} 0 0 \cdots \) and the last part of \( \tilde{\nu} \) is non-zero. If \( \ell(\tilde{\nu}) = 0 \), then \( G_{00\cdots} = 1 \) is not in any of the ideals \( \mathcal{J}^{(e)} \). Assume that \( \ell(\tilde{\nu}) > 0 \).

We first consider the case when \( \tilde{\nu} = \nu \) is a standard composition. If \( \tilde{\alpha} \) reaches level \( e \), then so is \( \nu \) and we have \( G_{\tilde{\alpha}} = F_{\nu}(x_1, x_2, \ldots) \in \mathcal{J}^{(e)} \). If \( \tilde{\nu} \) is a non-standard generalized composition, then let \( \tilde{\alpha} = \pi \tilde{\rho} \) be the factorization such that \( d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e \), and let \( \tilde{\alpha} = \tilde{\nu} 0 0 \cdots = \tilde{\gamma} 0 a \beta 0 0 \cdots \) be the factorization used in \( (3.3) \). If \( \tilde{\pi} \) is an initial factor of \( \tilde{\gamma} \), then it is clearly an initial factor of both \( \tilde{\gamma} a \beta 0 0 \cdots \) and
\( \tilde{\gamma} (a - 1) \beta 0 0 \cdots \) and they both reach level \( e \). By the induction hypothesis, both \( G_{\tilde{\gamma} a \beta 0} \) and \( G_{\tilde{\gamma} (a-1) \beta 0} \) are in \( J(e) \) and in turn \( \tilde{G}_{\tilde{\alpha}} \in J(e) \). If we now assume that \( \tilde{\pi} = \tilde{\gamma} 0 \), then
\[
d(\tilde{\gamma}) - \ell(\tilde{\gamma}) = d(\tilde{\pi}) - (\ell(\tilde{\pi}) - 1) \geq e + 1 > e
\]
and again the induction hypothesis can be applied to \([3.3]\) to show that \( G_{\tilde{\alpha}} \in J(e) \).

We are left to check the case where
\[
\tilde{\pi} = \tilde{\gamma} 0 a \tilde{\mu}.
\]
For the first term in \([3.3]\), \( \tilde{\gamma} a \tilde{\mu} \) is an initial factor and
\[
d(\tilde{\mu} a \tilde{\gamma}) - \ell(\tilde{\gamma} a \tilde{\mu}) = d(\tilde{\pi}) - (\ell(\tilde{\pi}) - 1) \geq e + 1 > e.
\]
The induction hypothesis gives that \( G_{\tilde{\gamma} a \beta 0} \in J(e) \). For the second term indexed by \( \tilde{\gamma} (a - 1) \beta 0 0 \cdots \) we have
\[
d(\tilde{\gamma}(a - 1)\tilde{\mu}) - \ell(\tilde{\gamma}(a - 1)\tilde{\mu}) = (d(\tilde{\pi}) - 1) - (\ell(\tilde{\pi}) - 1) \geq e.
\]
Again the induction hypothesis gives us that \( G_{\tilde{\gamma} (a-1) \beta 0} \in J(e) \), concluding the proof that \( G_{\tilde{\alpha}} \in J(e) \).

For the second statement of the lemma let \( \tilde{\gamma} (a - 1) \beta 0 0 \cdots = \tilde{\pi} \tilde{\rho} \) be a factorization such that \( d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e \). If \( \tilde{\pi} \) is an initial factor of \( \tilde{\gamma} \) then it is clear that \( \tilde{\alpha} \) reaches level \( e \). On the other hand if \( \tilde{\pi} = \tilde{\gamma} (a - 1) \tilde{\mu} \), then we have
\[
d(\tilde{\gamma} 0 a \tilde{\mu}) - \ell(\tilde{\gamma} 0 a \tilde{\mu}) = (d(\tilde{\gamma} (a - 1) \tilde{\mu}) + 1) - (\ell(\tilde{\gamma} (a - 1) \tilde{\mu}) + 1) \geq e.
\]
Thus \( \tilde{\alpha} \) reaches level \( e \) which concludes our proof. \( \Box \)

In light of the previous lemma, let \( G^{(e)} \) denote the set of all generalized infinite compositions \( \tilde{\alpha} \) reaching level \( e \), that satisfy \([3.4]\). We remark that the set \( \{ G_{\tilde{\alpha}} \}_{\tilde{\alpha} \in G^{(e)}} \) constructed above is contained in \( J^{(e)} \) and contains \( \{ F_{\alpha}(x_1, x_2, \ldots) : \exists \alpha = \pi \rho, \ d(\pi) - \ell(\pi) \geq e \} \). Hence we have

**Lemma 3.2.**
\[
J^{(e)} = \langle G_{\tilde{\alpha}} \rangle_{\tilde{\alpha} \in G^{(e)}}.
\]

\( \Box \)

Our next task is to characterize the leading monomial of each function \( G_{\tilde{\alpha}} \). Before this we need to specify which monomial order we use. Let \( X^{\tilde{\alpha}} = x_1^{\tilde{\alpha}_1} x_2^{\tilde{\alpha}_2} \cdots \) and \( X^{\tilde{\beta}} = x_1^{\tilde{\beta}_1} x_2^{\tilde{\beta}_2} \cdots \) be any two monomials where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are two generalized infinite compositions. We say that \( X^{\tilde{\alpha}} \leq_{\text{lex}} X^{\tilde{\beta}} \) if and only if \( d(\tilde{\alpha}) > d(\tilde{\beta}) \), or \( d(\tilde{\alpha}) = d(\tilde{\beta}) \) and the leftmost non-zero entry in \( [\tilde{\beta}_1 - \tilde{\alpha}_1, \tilde{\beta}_2 - \tilde{\alpha}_2, \ldots] \) is positive. The order \( \leq_{\text{lex}} \) is a classical monomial order in the sense that it is a total order and if \( X^{\tilde{\alpha}} \leq_{\text{lex}} X^{\tilde{\beta}} \), then \( X^{\tilde{\alpha}} X^{\tilde{\gamma}} = X^{\tilde{\alpha} + \tilde{\gamma}} \leq_{\text{lex}} X^{\tilde{\beta} + \tilde{\gamma}} = X^{\tilde{\beta} \cdot X^{\tilde{\gamma}}} \). Here the sum of generalized compositions is componentwise.

For any formal power series \( P = P(x_1, x_2, \ldots) \in \mathbb{Q}[[x_1, x_2, \ldots]] \) we let \( LM(P) \) denote the leading monomial of \( P \). That is \( LM(P) \) is the monomial of \( P \) with non-zero coefficient of smallest degree and largest in lexicographic order. In other words,
the leading monomial for the order $\leq_{lex}$. We let $LC(P)$ denote the coefficient of $LM(P)$ in $P$. Remark that for any two functions $P$ and $Q$, we have

$$LM(PQ) = LM(P)LM(Q).$$

We need the following result which is the extension for the $G$-functions of Relations (2.1) and (2.2) for the $F$-functions.

Lemma 3.3. Let $\tilde{\alpha} = b \tilde{\rho}$ be any generalized infinite composition and $b \geq 0$.

1. If $b = 0$, then $G_{\tilde{\alpha}}$ is a functions with no variable $x_1$. More precisely, we have

$$G_{\tilde{\alpha}}(x_1, x_2, \ldots) = G_{\tilde{\rho}}(x_2, x_3, \ldots).$$

2. If $b > 0$, then

$$G_{\tilde{\alpha}} = x_1 G_{(b-1)\tilde{\rho}} + M_{\tilde{\alpha}}(x_2, x_3, \ldots),$$

where $M_{\tilde{\alpha}} \in \mathcal{J}^{(c)}(x_2, x_3, \ldots)$ whenever $G_{\tilde{\alpha}} \in \mathcal{J}^{(c)}$.

Proof. Remark that in (3.5), the generalized composition $\tilde{\rho}$ is also infinite and $G_{\tilde{\rho}}(x_2, x_3, \ldots)$ is the function $G_{\tilde{\rho}}(x_1, x_2, \ldots)$ in which the variable $x_i$ is replaced by $x_{i+1}$. Similarly, $\mathcal{J}^{(c)}(x_2, x_3, \ldots)$ is the ideal $\mathcal{J}^{(c)}$ where each variable $x_i$ is replaced by $x_{i+1}$.

To show the two relations, we let $\tilde{\alpha} = \tilde{\nu}0\ldots$ and proceed by induction on $\ell(\tilde{\nu})$. If $\ell(\tilde{\nu}) = 0$ then we have $G_{0\tilde{\nu}0\ldots} = 1$ and (3.3) is valid. In the following we extensively use the two relations (2.1) and (2.2). If $\tilde{\alpha} = b \rho 0\ldots$, then Equation (3.2) gives us $G_{\tilde{\alpha}} = F_{b\rho}(x_1, x_2, \ldots)$. If $b > 1$, then we get

$$G_{b\rho0\ldots} = F_{b\rho}(x_1, x_2, \ldots) = x_1 F_{(b-1)\rho}(x_1, x_2, \ldots) + F_{b\rho}(x_2, x_3, \ldots)$$

$$= x_1 G_{(b-1)\rho0\ldots} + M_{b\rho0\ldots}(x_2, x_3, \ldots),$$

and (3.6) follows for this case with $M_{\tilde{\alpha}}(x_2, x_3, \ldots) = F_{b\rho}(x_2, x_3, \ldots)$.

For $b = 1$, we first need to understand (3.3) in the case $G_{0\rho0\ldots}$. For this assume that $\rho = a\beta$. If $a > 1$, then the Definitions (3.2) and (3.3) give

$$G_{0a\beta0\ldots} = G_{a\beta0\ldots} - x_1 G_{(a-1)\beta0\ldots} = F_{a\beta}(x_1, x_2, \ldots) - x_1 F_{(a-1)\beta}(x_1, x_2, \ldots)$$

$$= F_{a\beta}(x_2, x_3, \ldots) = G_{a\beta0\ldots}(x_2, x_3, \ldots).$$

If $a = 1$, then we use the induction hypothesis on $\ell(0\beta) = \ell(01\beta) - 1 < \ell(\tilde{\nu})$ to get

$$G_{01\beta0\ldots} = G_{1\beta0\ldots} - x_1 G_{0\beta0\ldots} = F_{1\beta}(x_1, x_2, \ldots) - x_1 F_{\beta}(x_2, x_3, \ldots)$$

$$= F_{1\beta}(x_2, x_3, \ldots) = G_{1\beta0\ldots}(x_2, x_3, \ldots).$$

Now we can go back to (3.4) in the case of $\tilde{\alpha} = 1\rho0\ldots$:

$$G_{1\rho0\ldots} = F_{1\rho}(x_1, x_2, \ldots) = x_1 F_{\rho}(x_2, x_3, \ldots) + F_{1\rho}(x_2, x_3, \ldots)$$

$$= x_1 G_{0\rho0\ldots} + M_{1\rho0\ldots}(x_2, x_3, \ldots).$$

We remark that in both cases, for $\tilde{\alpha} = \nu0\ldots$, we have

$$M_{\nu0\ldots}(x_2, x_3, \ldots) = F_{\nu}(x_2, x_3, \ldots) \in \mathcal{J}^{(c)}(x_2, x_3, \ldots)$$

whenever $G_{\nu0\ldots} \in \mathcal{J}^{(c)}(x_2, x_3, \ldots)$.

We then consider when $\tilde{\alpha} = \tilde{\gamma}0a\beta0\ldots$. This is the factorization needed to use (3.3) with $k = \ell(\tilde{\gamma}) + 1$. If $\tilde{\gamma}$ is empty, then we have $b = 0$ and we are in the
case considered above. Assume that $\tilde{\gamma} = b \tilde{\mu}$. If $b = 0$, then applying the induction hypothesis we have

$$
G_{0 \tilde{\mu} 0 \alpha \beta 0 \ldots}(x_1, x_2, \ldots) = G_{0 \tilde{\mu} a \beta 0 \ldots}(x_1, x_2, \ldots) - x_k G_{0 \tilde{\mu} (a-1) \beta 0 \ldots}(x_1, x_2, \ldots)
$$

$$
= G_{\mu a \beta 0 \ldots}(x_2, x_3, \ldots) - x_{(k-1)+1} G_{\mu (a-1) \beta 0 \ldots}(x_2, x_3, \ldots)
$$

$$
= G_{\mu 0 a \beta 0 \ldots}(x_2, x_3, \ldots).
$$

Here remark that even though $\ell(\tilde{\mu}) + 1 = k - 1$, we have to replace $x_{k-1}$ by $x_{(k-1)+1} = x_k$ in the defining recurrence for $G_{\mu 0 a \beta 0 \ldots}(x_2, x_3, \ldots)$. If $b > 0$, the induction hypothesis now gives

$$
G_{b \tilde{\mu} 0 a \beta 0 \ldots}(x_2, x_3, \ldots) = G_{b \tilde{\mu} a \beta 0 \ldots} - x_k G_{b \tilde{\mu} (a-1) \beta 0 \ldots}
$$

$$
= x_1 G_{(b-1) \tilde{\mu} a \beta 0 \ldots} + M_{b \tilde{\mu} a \beta 0 \ldots}(x_2, x_3, \ldots)
$$

$$
- x_k \left( x_1 G_{(b-1) \tilde{\mu} (a-1) \beta 0 \ldots} + M_{b \tilde{\mu} (a-1) \beta 0 \ldots}(x_2, x_3, \ldots) \right)
$$

$$
= x_1 \left( G_{(b-1) \tilde{\mu} a \beta 0 \ldots} - x_k G_{(b-1) \tilde{\mu} (a-1) \beta 0 \ldots} \right)
$$

$$
+ M_{b \tilde{\mu} a \beta 0 \ldots}(x_2, x_3, \ldots) - x_k M_{b \tilde{\mu} (a-1) \beta 0 \ldots}(x_2, x_3, \ldots)
$$

$$
= x_1 G_{(b-1) \tilde{\mu} 0 a \beta 0 \ldots} + M_{b \tilde{\mu} 0 a \beta 0 \ldots}(x_2, x_3, \ldots),
$$

where the function

$$
(3.8) \quad M_{b \tilde{\mu} 0 a \beta 0 \ldots}(x_2, x_3, \ldots) = \left( M_{b \tilde{\mu} a \beta 0 \ldots} - x_k M_{b \tilde{\mu} (a-1) \beta 0 \ldots} \right)(x_2, x_3, \ldots)
$$

contains no variable $x_1$. Using the argument in Lemma 3.1, if $G_{b \tilde{\mu} 0 a \beta 0 \ldots} \in J^{(e)}$, then both $G_{b \tilde{\mu} a \beta 0 \ldots}$ and $G_{b \tilde{\mu} (a-1) \beta 0 \ldots}$ are in $J^{(e)}$. The induction hypothesis gives us that $M_{b \tilde{\mu} 0 a \beta 0 \ldots}(x_2, x_3, \ldots) \in J^{(e)}(x_2, x_3, \ldots)$ and this completes the proof of the lemma.

**Corollary 3.4.** Let $\tilde{\alpha}$ be any generalized infinite composition. We have

$$
LM(G_{\tilde{\alpha}}) = X^{\tilde{\alpha}}.
$$

**Proof.** Let $\tilde{\alpha} = \tilde{\nu} 0 0 \cdots$. We proceed by induction on $\ell(\tilde{\nu})$ and the degree $d = d(\tilde{\alpha}) = \sum \alpha_i$. If $\ell(\tilde{\nu}) = 0$ we have $G_{0 0 \ldots} = 1 = X^{0 0 \ldots}$. If $\ell(\tilde{\nu}) \geq 1$, then let $\tilde{\alpha} = b \tilde{\rho}$ as in Lemma 3.3. If $b = 0$, then the induction hypothesis on $\ell(\tilde{\nu})$ gives

$$
LM(G_0 \tilde{\rho}) = LM\left( G_{\tilde{\rho}}(x_2, x_3, \ldots) \right) = x_1^0 x_2^{\tilde{\rho}} x_3^{\tilde{\rho}} \cdots = X^{\tilde{\alpha}}.
$$

Now if $b > 0$ we use the second part of Lemma 3.3 and the induction hypothesis on $d$, and get

$$
LM \left( x_1 G_{(b-1) \tilde{\rho}} + M_{\tilde{\alpha}}(x_2, x_3, \ldots) \right) = x_1 LM(G_{(b-1) \tilde{\rho}}) = X^{\tilde{\alpha}}.
$$

\[\square\]

**Remark 3.5.** From the above corollary, by triangularity, it is clear that the set $\{G_{\tilde{\alpha}}\}$ for all $\tilde{\alpha}$ forms a Hilbert basis of $\mathbb{Q}[[x_1, x_2 \ldots]]$. We will see in Section 5 that in fact $\{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathbb{G}^{(e)}}$ forms a Hilbert basis of $\mathcal{J}^{(e)}$. 

4. It is at most Catalan.

Let \( Q^{(e)} = \{ G_{\tilde{\alpha}} \}_{\tilde{\alpha} \in G^{(e)}} \) be the generating set of \( J^{(e)} \) constructed in Lemma 3.2. In this section we show that after reduction, at most the monomials corresponding to Catalan paths form a Hilbert basis of \( R = Q[[x_1, x_2, \ldots]]/J \). For this we reduce every other monomial to these. In fact for \( R^{(e)} = Q[[x_1, x_2, \ldots]]/J^{(e)} \), we show that at most the monomials corresponding to paths above the line \( y = x - e \) form a Hilbert basis of \( R^{(e)} \), for all \( e \geq 0 \). We conclude this section with the corresponding result for finitely many variables, \( R^{(e)}_n = Q[x_1, x_2, \ldots, x_n]/J^{(e)}_n \), which is a generalization of Theorem 1.2.

Given any generalized infinite composition \( \tilde{\alpha} \) we associate a unique path in the plane with steps going north or east. More precisely, for \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \ldots) \), we construct the path that starts at \((0,0)\), then moves \( \tilde{\alpha}_1 \) steps east to \((\tilde{\alpha}_1,0)\); then one step north to \((\tilde{\alpha}_1,1)\), and then \( \tilde{\alpha}_2 \) steps east to \((\tilde{\alpha}_1 + \tilde{\alpha}_2,1)\); then one step north to \((\tilde{\alpha}_1 + \tilde{\alpha}_2,2)\), and then \( \tilde{\alpha}_3 \) steps east to \((\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3,2)\); and so on. For example for \( \tilde{\alpha} = 002100300\cdots \) we have the path

\[
\begin{align*}
&x_7 x_7 x_7 \\
&x_4 \\
&x_3 x_3 \\
&\vdots
\end{align*}
\]

For every east step at height \( i - 1 \) we associate a variable \( x_i \). The product of all the variables associate to a path encoded by \( \tilde{\alpha} \) is denoted \( X^{\tilde{\alpha}} \). We now remark that for any factorization \( \tilde{\alpha} = \tilde{\pi} \tilde{\rho} \), the rightmost coordinate of the path at height \( \ell(\tilde{\pi}) \) is \( (d(\tilde{\pi}), \ell(\tilde{\pi}) - 1) \).

Definition 4.1. For an integer \( e \geq 0 \), we say that a generalized composition \( \tilde{\alpha} \) is of type \( e \)-Catalan if its associated path remains above the line \( y = x - e \). That is, every coordinate \((x_i, y_i)\) of the path is such that \( x_i - y_i \leq e \).

Lemma 4.2. The monomials of \( Q[[x_1, x_2, \ldots]] \) corresponding to paths remaining above the line \( y = x - e \) contains a Hilbert basis of the quotient \( R^{(e)} \).

Proof. Let \( X^{\tilde{\alpha}} \) be any monomial of degree \( d \). If the path corresponding to \( \tilde{\alpha} \) goes under the line \( y = x - e \), then let \( \tilde{\alpha} = \tilde{\pi} \tilde{\rho} \) be any factorization such that the coordinate \((d(\tilde{\pi}), \ell(\tilde{\pi}) - 1)\) is under the line \( y = x - e \). That is

\[
d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e.
\]

From Lemma 3.1 we conclude that the function \( G_{\tilde{\alpha}} \) with leading monomial \( X^{\tilde{\alpha}} \) is in \( J^{(e)} \). This monomial can thus be replaced by monomials of degree \( d \) but strictly smaller with respect to \( <_{\text{lex}} \). Repeating this step (possibly countably many times) with the next largest monomial going under the line \( y = x - e \), any monomial \( X^{\tilde{\alpha}} \) can
be reduced modulo the ideal $\overline{J^{(e)}}$ to a series containing only monomials $X^{\tilde{\beta}}$ where $\tilde{\beta}$ is of type $e$-Catalan.

We are now in position to generalize Theorem 1.2 and prove it. For this, note that the quasi-symmetric polynomials in $n$ variables are defined by setting $0 = x_{n+1} = x_{n+2} = \cdots$ in the quasi-symmetric functions. That is

$$F_\alpha(x_1, x_2, \ldots, x_n) = F_\alpha(x_1, x_2, \ldots, x_n, 0, 0, \ldots).$$

We then define

$$J^{(e)}_n = \langle F_\alpha(x_1, x_2, \ldots, x_n) : \alpha \text{ reaches level } e \rangle$$
and $R^{(e)}_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]/J^{(e)}_n$. Similarly we set

$$G_{\tilde{\alpha}}(x_1, x_2, \ldots, x_n) = G_{\tilde{\alpha}}(x_1, x_2, \ldots, x_n, 0, 0, \ldots).$$

It is clear that Lemma 3.1 holds for $J^{(e)}_n$ in the same way. Moreover if $\tilde{\alpha} = \tilde{\nu} \cdot 0 \cdot \cdots$ for $\ell(\tilde{\nu}) = n$ then

$$LM(G_{\tilde{\alpha}}(x_1, x_2, \ldots, x_n)) = LM(G_{\tilde{\alpha}}(x_1, x_2, \ldots, x_n, 0, 0, \ldots)) = x_{\tilde{\nu}_1} x_{\tilde{\nu}_2} \cdots x_{\tilde{\nu}_n}.$$

Let $C^{(e)}_n$ denotes the number of generalized compositions $\tilde{\alpha}$ of type $e$-Catalan such that $\tilde{\alpha} = \tilde{\nu} \cdot 0 \cdot \cdots$ and $\ell(\tilde{\nu}) = n$. These are in bijection with the paths from $(0, 0)$ to $(n + e, n)$ that remain above the line $y = x - e$. Indeed, if we have a path of type $e$-Catalan, it suffices to add a horizontal line from $(\tilde{\nu}_1 + \tilde{\nu}_2 + \cdots + \tilde{\nu}_n, n)$ to $(n + e, n)$. When $e = 0$, we have $C^{(0)}_n = C_n$ the $n$th Catalan number. This enumerates the classical Dyck path from $(0, 0)$ to $(n, n)$ remaining above the line $y = x$. See [26] for an extensive account on Catalan numbers. We have the following generalization to our Theorem 1.2.

**Corollary 4.3.** $\dim(R^{(e)}_n) \leq C^{(e)}_n$.

**Proof.** We use the same argument as in Lemma 1.2. For this we use the fact that for any monomial $x_{\tilde{\nu}_1} x_{\tilde{\nu}_2} \cdots x_{\tilde{\nu}_n}$ in $\mathbb{Q}[x_1, x_2, \ldots, x_n]$, if $\tilde{\nu}$ reaches level $e$, then $G_{\tilde{\nu} \cdot 0 \cdots}(x_1, x_2, \ldots, x_n) \in J^{(e)}_n$. Hence a basis of $R^{(e)}_n$ is contained in the monomials corresponding to paths of type $e$-Catalan and our result follows.

Again, we expect the equality to hold in Corollary 4.3, and we will address this question in [3].

5. It is a Hilbert basis

In the previous Section, the generating set $Q^{(e)} = \{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$ of $J^{(e)}$ is very useful to reduce every monomial to $e$-Catalan type generalized compositions. It is in fact a Hilbert basis for the given ideal. We use here ideas of Gröbner basis theory. This is crucial to complete the proof of Theorem 1.1. Let us recall a few basic facts about Gröbner bases, see [8, 14] for more details.
To show that a set $S$ is a Gröbner basis it is enough to show that all polynomial syzygies of that set are reducible in $S$. The polynomial syzygy of $P$ and $Q$ is defined by

$$S(P, Q) = LC(Q)M_1 P - LC(P)M_2 Q$$

where $lcm(LM(P), LM(Q)) = M_1 \cdot LM(P) = M_2 \cdot LM(Q)$. This shows that the given set contains all the generators of the leading monomials of the ideal.

To help us we use the classic Buchberger’s lemma (5.1):

**Lemma 5.1.** Given $P, Q \in S$. If there is an $R \in S$ such that $LM(R)$ divides $lcm(LM(P), LM(Q))$, and if both $S(R, Q)$ and $S(R, P)$ are reducible in $S$, then $S(P, Q)$ is reducible in $S$.

This result is easily adapted to our context. We first remark that our sets $Q^{(e)}$ are lattice. That is

**Lemma 5.2.**

$$G_{\tilde{\alpha}} \in Q^{(e)} \implies G_{\tilde{\rho}} \in Q^{(e)}$$

for all $\tilde{\alpha} \leq \tilde{\rho}$ componentwise.

**Proof.** If $\tilde{\alpha} = \tilde{\pi}\tilde{\nu}$ satisfies $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$, then let $r = \ell(\tilde{\pi})$ and consider $\tilde{\rho} = \tilde{\gamma}\tilde{\mu}$ where $\ell(\tilde{\gamma}) = r$. Since $d(\tilde{\pi}) \leq d(\tilde{\gamma})$, we have

$$d(\tilde{\gamma}) - \ell(\tilde{\gamma}) \geq d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e.$$ 

By Lemma 5.1, $G_{\tilde{\rho}} \in Q^{(e)}$. \hfill $\square$

We can now adapt the proof (see (5.4)) of Lemma 5.1 to our situation. For any pair $\tilde{\alpha}, \tilde{\pi} \in G^{(e)}$, we define $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$ as in (5.1) with our definition of $LM$ and $LC$. We show in this section that any such $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$ is reducible in $Q^{(e)}$. Let $\tilde{\rho} \in G^{(e)}$ be the unique element such that

$$X^{\tilde{\rho}} = lcm(X^{\tilde{\alpha}}, X^{\tilde{\pi}}) = M_1 X^{\tilde{\alpha}} = M_2 X^{\tilde{\pi}}.$$ 

We have

$$S(G_{\tilde{\alpha}}, G_{\tilde{\pi}}) = M_1 G_{\tilde{\alpha}} - M_2 G_{\tilde{\pi}}$$

(5.2)

$$= M_1 G_{\tilde{\alpha}} - G_{\tilde{\rho}} + G_{\tilde{\rho}} - M_2 G_{\tilde{\pi}}$$

$$= S(G_{\tilde{\alpha}}, G_{\tilde{\rho}}) + S(G_{\tilde{\rho}}, G_{\tilde{\pi}}).$$

If both $S(G_{\tilde{\alpha}}, G_{\tilde{\rho}})$ and $S(G_{\tilde{\rho}}, G_{\tilde{\pi}})$ are reducible in $Q^{(e)}$, then so is $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$. It is thus sufficient to show that all $S(G_{\tilde{\alpha}}, G_{\tilde{\rho}})$ are reducible in $Q^{(e)}$ for $\tilde{\alpha} \leq \tilde{\rho}$ componentwise.

We can reduce further our problem as follows. Assume that $\tilde{\alpha}$ and $\tilde{\rho}$ in $G^{(e)}$ are generalized compositions of $d_1$ and $d_2$ respectively. If $\tilde{\alpha} \leq \tilde{\rho}$, then $d_1 \leq d_1$. If $d_2 - d_1 > 1$, we can select a generalized composition $\tilde{\alpha} \leq \tilde{\pi} \leq \tilde{\rho}$ and use (5.2) again. We can thus assume that $d_2 - d_1 = 1$. That is the two generalized compositions differ on one part only and by one unit.

**Lemma 5.3.** The set $Q^{(e)}$ is a Gröbner basis of $J^{(e)}$. 

Proof. From the discussion above it is sufficient to show that all the expressions of \( Q^{(e)} \) of the form
\[
S(G_{\tilde{\gamma} a \tilde{\beta}}, G_{\tilde{\gamma}(a-1) \tilde{\beta}}) = G_{\tilde{\gamma} a \tilde{\beta}} - x_k G_{\tilde{\gamma}(a-1) \tilde{\beta}}
\]
where \( k = \ell(\tilde{\gamma}) + 1 \), are reducible in \( Q^{(e)} \). Let us denote by \( m_{\tilde{\gamma} a \tilde{\beta}}(x_1, x_2, \ldots) \) the leading monomial of \( S(G_{\tilde{\gamma} a \tilde{\beta}}, G_{\tilde{\gamma}(a-1) \tilde{\beta}}) \).

Let \( \tilde{\beta} = \tilde{\nu} 0 0 \cdots \). We set up an induction on \( \ell(\tilde{\nu}) \). Assume first that \( \tilde{\nu} = \nu \) is a (possibly empty) standard composition. The second part of Lemma 3.4 and the recursive definition (3.3) give
\[
S(G_{\tilde{\gamma} a \nu 0 \cdots}, G_{\tilde{\gamma}(a-1) \nu 0 \cdots}) = G_{\tilde{\gamma} 0 a \nu 0 \cdots} \in Q^{(e)}.
\]
If \( \tilde{\nu} \) is not standard, then let \( \tilde{\beta} = \tilde{\pi} 0 b \mu 0 0 \cdots \) for \( b > 0 \) and \( \mu \) a (possibly empty) standard composition. Let \( \ell = \ell(\tilde{\gamma} a \tilde{\pi}) + 1 \). Using (3.3), we have
\[
S(G_{\tilde{\gamma} a \tilde{\pi} 0 b \mu 0 \cdots}, G_{\tilde{\gamma}(a-1) \tilde{\pi} 0 b \mu 0 \cdots}) = G_{\tilde{\gamma} a \tilde{\pi} 0 b \mu 0 \cdots} - x_k G_{\tilde{\gamma}(a-1) \tilde{\pi} 0 b \mu 0 \cdots} - x_k G_{\tilde{\gamma}(a-1) \tilde{\pi} (b-1) \mu 0 \cdots} = \ldots
\]
\[
-x_{\ell} S(G_{\tilde{\gamma} a \tilde{\pi} (b-1) \mu 0 \cdots}, G_{\tilde{\gamma}(a-1) \tilde{\pi} (b-1) \mu 0 \cdots}).
\]
We observe that
\[
m_{\tilde{\gamma} a \tilde{\pi} 0 b \mu 0 \cdots}(x_1, x_2, \ldots) = x_{\ell} m_{\tilde{\gamma} a \tilde{\pi} (b-1) \mu 0 \cdots}(x_1, x_2, \ldots)
\]
since \( m_{\tilde{\gamma} a \tilde{\pi} b \mu 0 \cdots}(x_1, x_2, \ldots) \) is distinct and lexicographically smaller than \( x_{\ell} m_{\tilde{\gamma} a \tilde{\pi} (b-1) \mu 0 \cdots}(x_1, x_2, \ldots) \). Thus the left hand side of (5.4) is resolved by two expressions such that \( \ell(\tilde{\pi} b \mu) = \ell(\tilde{\pi} (b-1) \mu) < \ell(\tilde{\nu}) \) and by the induction hypothesis the right hand side of (5.5) is reducible in \( Q^{(e)} \). This completes the proof of the lemma. \( \Box \)

Corollary 5.4. The set \( Q^{(e)} \) is a Hilbert basis of \( \overline{J}^{(e)} \).

Proof. Given an element \( P \in \overline{J}^{(e)} \) let \( X^{\tilde{\beta}} = LM(P) \). Since \( Q^{(e)} \) is a Gröbner basis it contains an element \( G_{\tilde{\alpha}} \) such that \( \tilde{\alpha} \leq \tilde{\beta} \) componentwise. Lemma 5.2 gives us that \( G_{\tilde{\beta}} \in Q^{(e)} \). The element \( P - LC(P) \cdot G_{\tilde{\alpha}} \in \overline{J}^{(e)} \) is such that \( LM(P - LC(P) \cdot G_{\tilde{\alpha}}) <_{\text{lex}} X^{\tilde{\beta}} \). If we repeat this process (possibly countably many times) we can express \( P \) as a series in the elements of \( Q^{(e)} \). \( \Box \)

We are now in position to conclude our investigation and prove the general version of our Theorem 1.1.

Corollary 5.5. The monomial Hilbert basis of \( R^{(e)} \) is given by the monomials of \( Q[[x_1, x_2, x_3, \ldots]] \) corresponding to the paths of type e-Catalan.

Proof. As noted in Remark 3.3, the set \( \{G_{\tilde{\alpha}}\} \) forms a Hilbert basis of \( Q[[x_1, x_2, x_3, \ldots]] \), and Corollary 5.3 gives that the set \( Q^{(e)} \) is a Hilbert basis of \( \overline{J}^{(e)} \). Thus a Hilbert basis of the quotient \( R^{(e)} = Q[[x_1, x_2, x_3, \ldots]]/\overline{J}^{(e)} \) is given by the set
\[
\{G_{\tilde{\alpha}}\} \setminus Q^{(e)} = \{G_{\tilde{\alpha}} \mid \tilde{\alpha} \text{ corresponds to a path of type e-Catalan}\}.
\]
The result follows by triangularity.

**Remark 5.6.** To show the equality in Equation (1.3), it appears that the set
\[ \{ G_{\tilde{\alpha}} \mid \ell(\tilde{\alpha}) = n, \text{ and } \tilde{\alpha} \text{ reaches level } e \} \]
forms a linear basis of \( J_n^{(e)} \). Unfortunately the argument of Section 5 is not sufficient to show this with finitely many variables and it requires more work. This is the object of our collaboration in \[\text{[5]}\].

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