LOWER-MODULAR ELEMENTS
OF THE LATTICE OF SEMIGROUP VARIETIES. III

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ABSTRACT. We completely determine all lower-modular elements of the lattice of all semigroup varieties. As a corollary, we show that a lower-modular element of this lattice is modular.

1. INTRODUCTION AND SUMMARY

The collection \( \text{SEM} \) of all semigroup varieties forms a lattice with respect to the class-theoretical inclusion. Special elements of different types in this lattice have been studied in several articles. An overview of results obtained in these articles is given in the recent survey [6, Section 14]. Recall the definitions of special elements mentioned in this paper. An element \( x \) of a lattice \( \langle L; \lor, \land \rangle \) is called modular if

\[
\forall y, z \in L: \quad y \leq z \rightarrow (x \lor y) \land z = (x \land z) \lor y,
\]

lower-modular if

\[
\forall y, z \in L: \quad x \leq y \rightarrow x \lor (y \land z) = y \land (x \lor z),
\]

distributive if

\[
\forall y, z \in L: \quad x \lor (y \land z) = (x \lor y) \land (x \lor z).
\]

Upper-modular elements are defined dually to lower-modular ones. It is evident that a distributive element is lower-modular.

We call a semigroup variety modular [lower-modular, distributive] if it is a modular [lower-modular, distributive] element of the lattice \( \text{SEM} \). Distributive varieties are completely determined by the authors in [9, Theorem 1.1]. Here we consider wider class of lower-modular varieties. These varieties were mentioned for the first time in [10] (see Lemma 2.4 below) and examined systematically in [7, 8]. Here we complete this examination. The main result of this article gives a complete classification of lower-modular varieties. To formulate this result, we need a few definitions and notation.

A pair of identities \( wx = xw = w \) where the letter \( x \) does not occur in the word \( w \) is usually written as the symbolic identity \( w = 0 \). (This notation is justified because a semigroup with such identities has a zero element and all

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values of the word \( w \) in this semigroup are equal to zero.) Identities of the form \( w = 0 \) as well as varieties given by such identities are called 0-reduced. By \( \mathcal{T} \), \( \mathcal{SL} \), and \( \mathcal{SEM} \) we denote the trivial variety, the variety of all semilattices, and the variety of all semigroups, respectively. The main result of the article is the following

**Theorem 1.1.** A semigroup variety \( V \) is lower-modular if and only if either \( V = \mathcal{SEM} \) or \( V = M \lor N \) where \( M \) is one of the varieties \( \mathcal{T} \) or \( \mathcal{SL} \), while \( N \) is a 0-reduced variety.

Theorem 1.1, together with [4, Proposition 1.1] (see also Lemmas 2.4 and 2.5 below), immediately implies

**Corollary 1.2.** A lower-modular semigroup variety is modular. □

Theorem 1.1 and Corollary 1.2 give answers to Question 14.2 from [6] and Questions 1 and 2 from [8]. Besides, Theorem 1.1 solves Problems 3 and 4 from [8]. It is verified in [9, Corollary 1.2] that every distributive variety is modular. Clearly, this claim is generalized by Corollary 1.2.

The article consists of three sections. Section 2 contains some auxiliary results, while Section 3 is devoted to the proof of Theorem 1.1.

## 2. Preliminaries

### 2.1. Some properties of lower-modular, upper-modular and modular elements in abstract lattices and the lattice \( \mathcal{SEM} \)

We start with two easy lattice-theoretical observations. If \( L \) is a lattice and \( a \in L \) then \([a]\) stands for the principal coideal generated by \( a \), that is, the set \( \{x \in L \mid x \geq a\} \).

**Lemma 2.1.** If \( x \) is a lower-modular element of a lattice \( L \) and \( a \in L \) then the element \( x \lor a \) is a lower-modular element of the lattice \( [a] \).

**Proof.** Let \( y, z \in [a] \) and \( x \lor a \leq y \). Then

\[
(x \lor a) \lor (y \land z) = a \lor (x \lor (y \land z))
\]

because \( x \) is lower-modular

\[
= a \lor (y \land (x \lor z)) \quad \text{and} \quad x \leq x \lor a \leq y
\]

\[
= y \land (x \lor z) \quad \text{because} \ a \leq y \land (x \lor z)
\]

\[
= y \land (x \lor (a \lor z)) \quad \text{because} \ a \leq z
\]

\[
= y \land ((x \lor a) \lor z).
\]

Thus \( (x \lor a) \lor (y \land z) = y \land ((x \lor a) \lor z) \), and we are done. □

**Lemma 2.2.** Let \( L \) be a lattice and \( \varphi \) a surjective homomorphism from \( L \) onto a lattice \( L' \). If \( x \) is an upper-modular element of \( L \) then \( \varphi(x) \) is an upper-modular element of \( L' \).

**Proof.** Let \( x' = \varphi(x) \) and let \( y', z' \) be elements of \( L' \) with \( y' \leq x' \). Then there are \( y, z \in L \) such that \( y' = \varphi(y) \) and \( z' = \varphi(z) \). We may assume that \( y \leq x \).

Indeed, if this is not the case then we may consider the element \( x \land y \) rather than \( y \) because \( \varphi(x \land y) = \varphi(x) \land \varphi(y) = x' \land y' = y' \). Since the element \( x \)
is upper-modular in $L$, we have $(z \land x) \lor y = (z \lor y) \land x$. This implies that $(z' \land x') \lor y' = (z' \lor y') \land x'$ that completes the proof.

Now we provide some known partial results about lower-modular varieties. It is well-known that if a semigroup variety $V$ is periodic (that is, consists of periodic semigroups) then it contains the greatest nil-subvariety. We denote this subvariety by $\text{Nil}(V)$. A semigroup variety $V$ is called proper if $V \neq \text{SEM}$. 

**Lemma 2.3 ([7, Theorem 1]).** If a proper semigroup variety $V$ is lower-modular then $V$ is periodic and the variety $\text{Nil}(V)$ is 0-reduced.

**Lemma 2.4 ([10, Corollary 3]).** A 0-reduced semigroup variety is modular and lower-modular.

Note that the ‘modular half’ of Lemma 2.4 was rediscovered (in some other terms) in [4, Proposition 1.1].

**Lemma 2.5.** A semigroup variety $V$ is [lower-]modular if and only if the variety $V \lor \mathcal{SL}$ is such.

This fact was proved in [11, Corollary 1.5(i)] for modular varieties and in [7, Corollary 1.3] for lower-modular ones.

2.2. **Decomposition of some varieties into the join of subvarieties.** We denote by $\mathcal{LZ}$ [respectively $\mathcal{RZ}$] the variety of all left [right] zero semigroups. If $\Sigma$ is a system of semigroup identities then $\text{var} \Sigma$ stands for the semigroup variety given by $\Sigma$. Put

$$
\mathcal{P} = \text{var}\{xy = x^2y, x^2y^2 = y^2x^2\},
$$

$$
\hat{\mathcal{P}} = \text{var}\{xy = xy^2, x^2y^2 = y^2x^2\}.
$$

Lemma 2 of the article [12] and the proof of Proposition 1 of the same article imply the following

**Lemma 2.6.** If a periodic semigroup variety $V$ contains none of the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\hat{\mathcal{P}}$ then $V = \mathcal{M} \lor N$ where the variety $\mathcal{M}$ is generated by a monoid and $N = \text{Nil}(V)$.

For any natural $m$, we put $\mathcal{C}_m = \text{var}\{x^m = x^{m+1}, xy = yx\}$. In particular, $\mathcal{C}_1 = \mathcal{SL}$. For notation convenience, we define also $\mathcal{C}_0 = \mathcal{T}$.

**Lemma 2.7 ([3]).** If a semigroup variety $\mathcal{M}$ is generated by a commutative monoid then $\mathcal{M} = \mathcal{G} \lor C_m$ for some Abelian periodic group variety $\mathcal{G}$ and some $m \geq 0$.

2.3. **Identities of certain semigroup varieties.** In the course of proving our results it will be convenient to have at our disposal a description of the identities of several concrete semigroup varieties. We denote by $F$ the free semigroup over a countably infinite alphabet. The equality relation on $F$ is denoted by $\equiv$. If $u$ is a word and $x$ is a letter then $c(u)$ stands for the set of all letters occurring in $u$, $\ell(u)$ is the length of $u$, $\ell_x(u)$ denotes the number of occurrences of $x$ in $u$, while $t(u)$ is the last letter of $u$. The statements (i) and (ii) of the following lemma are well-known and can be easily verified. The statement (iii) was proved in [2, Lemma 7].
Lemma 2.8. The identity \( u = v \) holds in the variety:

(i) \( \mathcal{RZ} \) if and only if \( t(u) \equiv t(v) \);

(ii) \( \mathcal{C}_2 \) if and only if \( c(u) = c(v) \) and, for every letter \( x \in c(u) \), either \( \ell_x(u) > 1 \) and \( \ell_x(v) > 1 \) or \( \ell_x(u) = \ell_x(v) = 1 \);

(iii) \( \mathcal{P} \) if and only if \( c(u) = c(v) \) and either \( \ell_{t(u)}(u) > 1 \) and \( \ell_{t(v)}(v) > 1 \) or \( \ell_{t(u)}(u) = \ell_{t(v)}(v) = 1 \) and \( t(u) \equiv t(v) \). \( \square \)

2.4. Verbal subsets of free groups. Similarly to the articles [7–9], we need here the technique developed by Sapir in [5]. We introduce the basic notation from that paper. Let \( G \) be a periodic group variety and \( \{v_i = 1 \mid i \in I\} \) a basis of identities of \( G \) (as a variety of groups) where \( v_i \) are semigroup words. Let \( r = \exp(G) \) where \( \exp(G) \) stands for the exponent of the variety \( G \). For a letter \( x \), put \( x^0 = x^{r+1} \). Let

\[
S(G) = \text{var}\{xyz = xy^{r+1}z, x^0y^0 = y^0x^0, x^2 = x^{r+2}, xuv^2y = xv_1y \mid i \in I\}.
\]

As it is shown in [5], the variety \( S(G) \) does not depend on the particular choice of the basis \( \{v_i = 1 \mid i \in I\} \) (see Remark 2.10 below). Furthermore, let \( F(G) \) be the free group of countably infinite rank in \( G \). A subset \( X \) of \( F(G) \) is called verbal if it is closed under all endomorphisms of \( F(G) \). Clearly, a verbal subset \( X \) of \( F(G) \) is a set of all values in \( F(G) \) of some set \( W \) of non-empty words; in this case we write \( X = G(W) \). If \( X \) is a verbal subset in \( F(G) \) and \( X = G(W) \) then we put

\[
S(G, X) = S(G) \land \text{var}\{xwx = (xwx)^{r+1} \mid w \in W\}.
\]

If \( X = \{1\} \) where 1 is the unit element of \( F(G) \) then we will write \( S(G, 1) \) rather than \( S(G, \{1\}) \). It is convenient to consider the empty set as a verbal subset in \( F(G) \) and put \( S(G, \varnothing) = S(G) \). If \( H \) is a subvariety of \( G \) and \( X \) is a verbal subset of \( F(G) \) then we put

\[
S(H, X) = S(H) \land S(G, X).
\]

(1)

To avoid a possible confusion, we note that the paper [5] does not contain an explicit definition of the variety \( S(H, X) \) where \( X \) is a verbal subset of \( F(G) \) with \( G \neq \mathcal{X} \). But one can trace the argument of [5] to see that the equality (1) is what Sapir tacitly meant by this definition but failed to explicitly define.

As usual, if \( \mathcal{X} \) is a variety then \( L(\mathcal{X}) \) stands for the subvariety lattice of \( \mathcal{X} \). To prove Theorem 1.1, we need the following

Lemma 2.9 ([5]). Let \( G \) be a variety of periodic groups. The interval \([S(T, 1), S(G)]\) of the lattice \( L(S(G)) \) consists of all varieties of the form \( S(H, X) \) where \( H \subseteq G \) and \( X \) is a (possibly empty) verbal subset of \( F(G) \). Here, for varieties \( S(H, X) \) and \( S(H', X') \) from the interval \([S(T, 1), S(G)]\), the inclusion \( S(H', X') \subseteq S(H, X) \) holds if and only if \( H' \subseteq H \) and there exists a set of words \( W \) such that \( X = H(W) \) and \( H'(W) \subseteq X' \).

Remark 2.10. Lemma 2.9 shows that the construction of the variety \( S(G, X) \) is in fact independent of the actual choice of the ‘generator’ \( W \) of the verbal subset \( X \); it is only \( X \) that really matters, as different choices of \( W \) will result in the same variety. In particular, by the definition of the variety \( S(G, X) \), it satisfies the identity \( xwx = (xwx)^{r+1} \) whenever \( w \in W \). In view of Lemma 2.9, this
identity holds in $S(G, X)$ not only for $w \in W$ but for any word $w$ representing an element of $X$.

2.5. **Overcommutative varieties.** We denote by $\mathcal{COM}$ the variety of all commutative semigroups. A semigroup variety $\mathcal{V}$ is called overcommutative if $\mathcal{V} \supseteq \mathcal{COM}$. The lattice of all overcommutative varieties is denoted by $\mathcal{OC}$. The description of this lattice was clarified by Volkov in [13]. It turns out that the lattice $\mathcal{OC}$ admits a concise and transparent description in terms of congruence lattices of unary algebras of some special type, called $G$-sets. This description plays an essential role in the proof of Theorem 1.1. To reproduce the result from [13], we need some new definitions and notation.

Let $A$ be a non-empty set. We denote by $S_A$ the group of all permutations on $A$. If $A = \{1, 2, \ldots, m\}$ then we will write $S_m$ rather than $S_{\{1, 2, \ldots, m\}}$. A $G$-set is a unary algebra on a set $A$ where the unary operations form a group of permutations on $A$ (that is, the subgroup of $S_A$). The congruence lattice of a $G$-set $A$ is denoted by $\operatorname{Con}(A)$.

Let $m$ and $n$ be positive integers with $2 \leq m \leq n$. A sequence $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$ of positive integers such that

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \text{ and } \sum_{i=1}^{m} \ell_i = n$$

is said to be a **partition of the number $n$ into $m$ parts**. For a word $u$, we put $\operatorname{part}(u) = (\ell_{x_1}(u), \ell_{x_2}(u), \ldots, \ell_{x_m}(u))$ where $m = \max\{i \mid x_i \in c(u)\}$. Let us fix positive integers $m$ and $n$ with $2 \leq m \leq n$ and a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$ of the number $n$ into $m$ parts. Put

$$W_\lambda = \{u \in F \mid \ell(u) = n, c(u) = \{x_1, x_2, \ldots, x_m\} \text{ and } \operatorname{part}(u) = \lambda\},$$

$$S_\lambda = \{\sigma \in S_m \mid \ell_i = \ell_{i\sigma} \text{ for all } i = 1, 2, \ldots, m\}.$$ 

Clearly, $S_\lambda$ is a subgroup in $S_m$.

If $u \equiv x_{i_1}x_{i_2} \cdots x_{i_n}$, where $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ are (not necessarily different) letters and $\pi \in S_{c(u)}$ then we denote by $u\pi$ the word $x_{i_1\pi}x_{i_2\pi} \cdots x_{i_n\pi}$. It is clear that if $u \in W_\lambda$ and $\pi \in S_\lambda$ then $u\pi \in W_\lambda$. For every $\pi \in S_\lambda$, we define the unary operation $\pi^*$ on $W_\lambda$ by letting $\pi^*(u) \equiv u\pi$ for any word $u \in W_\lambda$. Obviously, the set $W_\lambda$ with the collection of unary operations $\{\pi^* \mid \pi \in S_\lambda\}$ is an $S_\lambda$-set. The description of the lattice $\mathcal{OC}$ mentioned above is given by the following

**Proposition 2.11** ([13]). The lattice $\mathcal{OC}$ is anti-isomorphic to a subdirect product of congruence lattices $\operatorname{Con}(W_\lambda)$ where $\lambda$ runs over the set of all partitions. 

3. **Proof of Theorem 1.1**

* Sufficiency immediately follows from Lemmas 2.4 and 2.5 and the evident fact that the variety $\mathcal{SEM}$ is lower-modular.

* Necessity. Let $\mathcal{V}$ be a proper lower-modular semigroup variety. Lemma 2.1 implies that the variety $\mathcal{W} = \mathcal{V} \vee \mathcal{COM}$ is a lower-modular element of the lattice $\mathcal{OC}$. The variety $\mathcal{W}$ is proper because the variety $\mathcal{SEM}$ is not decomposable into the join of any two proper varieties [1].
Recall that an identity \( u = v \) is called balanced if each letter occurs in \( u \) and \( v \) the same number of times. It is well-known that if an overcommutative variety satisfies some identity then this identity is balanced.

Being proper, the variety \( \mathcal{W} \) satisfies a non-trivial balanced identity \( u = v \). Let \( |c(u)| = m \). We may assume that \( c(u) = \{x_1, x_2, \ldots, x_m\} \) and \( \ell_{x_i}(u) \geq \ell_{x_{i+1}}(u) \) (otherwise we may rename letters). Put \( \ell_i = \ell_{x_i}(u) \) for all \( i = 1, 2, \ldots, m \). Then \( \text{part}(u) = \text{part}(v) = (\ell_1, \ell_2, \ldots, \ell_m) \). We may assume that \( \ell_1 > \ell_2 > \cdots > \ell_m > 1 \) (if it is not the case, we may multiply \( u = v \) by an appropriate word on the right). Let \( x \) and \( y \) be arbitrary letters with \( x, y \not\in c(u) \) and \( \lambda = \text{part}(xyu) = (\ell_1, \ell_2, \ldots, \ell_m, 1, 1) \) (we identify here \( x \) and \( y \) with \( x_{m+1} \) and \( x_{m+2} \) respectively). We denote by \( \nu \) the fully invariant congruence on \( F \) corresponding to the variety \( \mathcal{W} \) and by \( \alpha \) the restriction of \( \nu \) to \( W_\lambda \). Then

\[
xyu \alpha xyv, \ ywu \alpha yxv, \ xuy \alpha xvy, \ yux \alpha yvx.
\]

Proposition 2.11 implies that there is a surjective homomorphism from the lattice dual to \( \mathbf{OCl} \) onto \( \text{Con}(W_\lambda) \). Now Lemma 2.2 applies with the conclusion that \( \alpha \) is an upper-modular element of the lattice \( \text{Con}(W_\lambda) \). Since \( \ell_1 > \ell_2 > \cdots > \ell_m > 1 \), the group \( S_\lambda \) consists of two elements. Let \( \beta \) be the equivalence relation on \( W_\lambda \) with only two non-singleton classes \( \{xyu, xyv\} \) and \( \{ywu, yxv\} \), and let \( \gamma \) be the equivalence relation on \( W_\lambda \) with only four non-singleton classes \( \{xyu, xuy\} \), \( \{xyv, xvy\} \), \( \{yuw, yux\} \), and \( \{yxv, yvx\} \). It is evident that \( \beta \) and \( \gamma \) are congruences on \( W_\lambda \) and \( \beta \subseteq \alpha \). Therefore \( (\gamma \lor \beta) \land \alpha = (\gamma \land \alpha) \lor \beta \).

Note that \( xuy \gamma xyu \beta xyv \gamma xvy \). Therefore \( (xuy, xyv) \in (\gamma \lor \beta) \land \alpha = (\gamma \land \alpha) \lor \beta \), whence

\[
(xuy, xyv) \in (\gamma \lor \beta) \land \alpha = (\gamma \land \alpha) \lor \beta.
\]

This means that there is a word \( w \in W_\lambda \) such that \( xuy \not\equiv w \) and either \( (xuy, w) \in \gamma \land \alpha \) or \( xuy \beta w \). But the latter contradicts the choice of the congruence \( \beta \). Hence \( (xuy, w) \in \gamma \land \alpha \). In particular, \( xuy \gamma w \). The definition of \( \gamma \) implies that \( w \equiv xyu \). Thus \( (xuy, xyv) \in (\gamma \land \alpha) \). In particular, we have verified that \( xuy \alpha xyu \).

Let \( \gamma' \) be the equivalence relation on \( W_\lambda \) with only four non-singleton classes \( \{xyu, yuw\} \), \( \{xyv, yvx\} \), \( \{yuw, yux\} \), and \( \{yxv, yvx\} \). Clearly, \( \gamma' \) is a congruence on \( W_\lambda \). Now we may repeat almost literally arguments from the previous paragraph with using \( \gamma' \) rather than \( \gamma \). As a result, we obtain that \( xuy \alpha yux \). Thus \( xuy \alpha xyu \alpha yux \). Since \( \alpha \subseteq \nu \), we have \( xuy \nu yux \). This means that the identity

\[
(2) \quad xuy = yux
\]

holds in the variety \( \mathcal{W} \). Therefore this identity holds in the variety \( \mathcal{V} \) as well.

Lemma 2.8 and its dual imply that the identity \((2)\) fails in the varieties \( \mathcal{LZ}, \mathcal{RZ}, \mathcal{P}, \) and \( \mathcal{P}' \). Thus \( \mathcal{V} \) does not contain these varieties. By Lemma 2.3 the variety \( \mathcal{V} \) is periodic. Now Lemma 2.6 applies with the conclusion that \( \mathcal{V} = \mathcal{M} \lor \mathcal{N} \) where the variety \( \mathcal{M} \) is generated by a monoid and \( \mathcal{N} = \text{Nil}(\mathcal{V}) \). Lemma 2.3 implies that the variety \( \mathcal{N} \) is 0-reduced. It remains to verify that \( \mathcal{M} \) is one of the varieties \( \mathcal{T} \) or \( \mathcal{SL} \).

Since \( \mathcal{M} \) is generated by a monoid, the set of its identities is closed for deleting letters; therefore, by deleting of all letters from \( c(u) \) in \((2)\), we obtain
that \( \mathcal{M} \) is commutative. Now we can apply Lemma 2.7 and conclude that 
\( \mathcal{M} = \mathcal{G} \lor C_m \) for some Abelian periodic group variety \( \mathcal{G} \) and some \( m \geq 0. \) Suppose that \( m \geq 2. \) Then \( \mathcal{V} \supseteq C_2. \) It is easy to deduce from Lemma 2.8 that 
\( C_2 \lor RZ \supseteq \mathcal{P}. \) Hence \( \mathcal{V} \lor RZ \supseteq C_2 \lor RZ \supseteq \mathcal{P}. \) Put \( \mathcal{U} = \mathcal{V} \lor \mathcal{P}. \) Note that 
\( \mathcal{V}, \mathcal{P} \not\supset RZ. \) As is well-known, the variety \( RZ \) is an atom of the lattice \( \text{SEM}. \) Therefore \( \mathcal{V} \lor RZ = \mathcal{P} \lor RZ = \mathcal{T}. \) It is well-known also that the lattice \( \text{SEM} \) is 0-
distributive, that is, satisfies the condition

\[
\forall x, y, z : \quad x \land z = y \land z = 0 \Rightarrow (x \lor y) \land z = 0
\]

(see [6, Section 1], for instance). Therefore \( \mathcal{U} \lor RZ = (\mathcal{V} \lor \mathcal{P}) \lor RZ = \mathcal{T}. \) Combining these observations, we have

\[
\mathcal{V} = \mathcal{V} \lor \mathcal{T} = \mathcal{V} \lor (\mathcal{U} \lor RZ)
\]

because \( \mathcal{U} \lor RZ = \mathcal{T} \)

\[
= \mathcal{U} \land (\mathcal{V} \lor \mathcal{RZ})
\]

because \( \mathcal{V} \) is lower-modular and \( \mathcal{V} \subseteq \mathcal{U} \)

\[
\supseteq \mathcal{P}
\]

because \( \mathcal{U} = \mathcal{V} \lor \mathcal{P} \supseteq \mathcal{P} \) and \( \mathcal{V} \lor \mathcal{RZ} \supseteq \mathcal{P}. \)

Thus \( \mathcal{V} \supseteq \mathcal{P}. \) A contradiction shows that \( m \leq 1, \) whence \( \mathcal{M} = \mathcal{G} \lor K \) where \( K \) is one of the varieties \( \mathcal{T} \) or \( \mathcal{SL}. \) It remains to check that \( \mathcal{G} = \mathcal{T}. \)

The remaining part of the proof is based on Lemma 2.9. Note that in what follows we combine (with slight modifications) arguments from the proofs of [8, Lemma 2.2] and [9, Proposition 3.1].

Suppose that \( \mathcal{G} \neq \mathcal{T}. \) Put \( \mathcal{Y} = \mathcal{V} \lor S(\mathcal{G}, 1) \) and \( \mathcal{Z} = S(\mathcal{T}). \) Further considerations are illustrated by Fig. 1.

![Figure 1. A fragment of the lattice \( L(\mathcal{V} \lor S(\mathcal{G})) \)](image)

Lemma 2.9 implies that \( S(\mathcal{G}) = S(\mathcal{T}) \lor \mathcal{G} \) (see Fig. 1). Using this equality and the inclusion \( \mathcal{G} \subseteq \mathcal{V}, \) we have

\[
\mathcal{Y} = S(\mathcal{G}, 1) \lor \mathcal{V} \subseteq S(\mathcal{G}) \lor \mathcal{V} = S(\mathcal{T}) \lor \mathcal{G} \lor \mathcal{V} = S(\mathcal{T}) \lor \mathcal{V} = \mathcal{Z} \lor \mathcal{V}.
\]

Therefore \( (\mathcal{Z} \lor \mathcal{V}) \land \mathcal{Y} = \mathcal{Y}. \) Since the variety \( \mathcal{V} \) is lower-modular and \( \mathcal{V} \subseteq \mathcal{Y}, \) we have \( (\mathcal{Z} \land \mathcal{Y}) \lor \mathcal{V} = (\mathcal{Z} \lor \mathcal{V}) \land \mathcal{Y}, \) whence

\[
(3) \quad (\mathcal{Z} \land \mathcal{Y}) \lor \mathcal{V} = \mathcal{Y}.
\]
Furthermore, Lemma 2.9 implies that $S(\mathcal{T}, 1) = S(\mathcal{T}) \land S(\mathcal{G}, 1)$ (see Fig. 1). Therefore

$$S(\mathcal{T}, 1) = S(\mathcal{T}) \land S(\mathcal{G}, 1) \subseteq S(\mathcal{T}) \land (\mathcal{V} \lor S(\mathcal{G}, 1)) = \mathcal{Z} \land \mathcal{Y} \subseteq \mathcal{Z} = S(\mathcal{T}),$$

that is, $S(\mathcal{T}, 1) \subseteq \mathcal{Z} \land \mathcal{Y} \subseteq S(\mathcal{T})$. It is evident that the group $F(\mathcal{T})$ contains only two verbal subsets, namely $\emptyset$ and $\{1\}$. Therefore Lemma 2.9 implies that the interval $[S(\mathcal{T}, 1), S(\mathcal{T})]$ of the lattice $L(S(\mathcal{T}))$ consists of the varieties $S(\mathcal{T}, 1)$ and $S(\mathcal{T})$ only. Thus either $\mathcal{Z} \land \mathcal{Y} = S(\mathcal{T}, 1)$ or $\mathcal{Z} \land \mathcal{Y} = S(\mathcal{T})$. Let us consider these two cases separately. Let $\exp(\mathcal{G}) = r$.

**Case 1:** $\mathcal{Z} \land \mathcal{Y} = S(\mathcal{T}, 1)$. Lemma 2.9 implies that $S(\mathcal{T}, 1) \lor \mathcal{G} = S(\mathcal{G}, S(\mathcal{T}))$ (see Fig. 1). Using the equality (3) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$S(\mathcal{G}, S(\mathcal{T})) \lor \mathcal{V} = S(\mathcal{T}, 1) \lor \mathcal{G} \lor \mathcal{V} = S(\mathcal{T}, 1) \lor \mathcal{V} = (\mathcal{Z} \land \mathcal{Y}) \lor \mathcal{V} = \mathcal{Y} = S(\mathcal{G}, 1) \lor \mathcal{V},$$

that is,

$$S(\mathcal{G}, S(\mathcal{T})) \lor \mathcal{V} = S(\mathcal{G}, 1) \lor \mathcal{V}. \quad (4)$$

Being a nilvariety, $\mathcal{N}$ satisfies an identity $u = 0$ for some word $u$. Suppose that the variety $\mathcal{G}$ (considered as a variety of groups) satisfies the identity $u = 1$. Let $x$ be a letter with $x \notin c(u)$. Since the variety $\mathcal{G}$ is non-trivial, it does not satisfy the identity $ux = 1$. It is evident that $ux = 0$ in $\mathcal{N}$. Thus there is a word $w$ such that the variety $\mathcal{N}$ satisfies the identity $w = 0$ but the variety $\mathcal{G}$ does not satisfy the identity $w = 1$. Let $x$ be a letter with $x \notin c(w)$. Remark 2.10 implies that $S(\mathcal{G}, S(\mathcal{T}))$ satisfies the identity

$$xwx = (xwx)^{r+1}. \quad (5)$$

This identity holds in the variety $\mathcal{V}$ as well because $\mathcal{V} \subseteq \mathcal{G} \lor S\mathcal{L} \lor \mathcal{N}$. Therefore the variety $\mathcal{V} \lor S(\mathcal{G}, S(\mathcal{T}))$ satisfies the identity (5). But (5) fails in the variety $S(\mathcal{G}, 1)$ by the definition of this variety, whence (5) fails in the variety $\mathcal{V} \lor S(\mathcal{G}, 1)$. We have a contradiction with the equality (4).

**Case 2:** $\mathcal{Z} \land \mathcal{Y} = S(\mathcal{T})$. As we have already noted above, $S(\mathcal{G}) = S(\mathcal{T}) \lor \mathcal{G}$ (see Fig. 1). Taking into account the equality (3) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$S(\mathcal{G}, 1) \lor \mathcal{V} = \mathcal{Y} = (\mathcal{Z} \land \mathcal{Y}) \lor \mathcal{V} = S(\mathcal{T}) \lor \mathcal{V} = S(\mathcal{T}) \lor \mathcal{G} \lor \mathcal{V} = S(\mathcal{G}) \lor \mathcal{V}.$$  

We see that

$$S(\mathcal{G}, 1) \lor \mathcal{V} = S(\mathcal{G}) \lor \mathcal{V}. \quad (6)$$

Let $w$ be an arbitrary word such that the variety $\mathcal{G}$ satisfies (as a variety of groups) the identity $w = 1$. Being a nil-variety, $\mathcal{N}$ satisfies the identity $x^n = 0$ for some $n$. The variety $\mathcal{G}$ satisfies the identity $w^{2n} = 1$. Remark 2.10 implies that the variety $S(\mathcal{G}, 1)$ satisfies the identity

$$xwx^{2n} = (xwx^{2n})^{r+1}. \quad (7)$$

This identity holds in the varieties $\mathcal{M}$ and $\mathcal{N}$ as well, whence it holds in $\mathcal{V}$, and therefore in $S(\mathcal{G}, 1) \lor \mathcal{V}$. The equality (6) implies that (7) holds in $S(\mathcal{G}) \lor \mathcal{V}$, and therefore in $S(\mathcal{G})$. We always may include the identity $w = 1$ in the
identity basis of $G$. By the definition of the variety $S(G)$, it satisfies the identity $xwx = xw^2x$, and therefore the identities

$$xwx = xw^2x = xw^4x = x(w \cdot w^2 \cdot w)x = x(w \cdot w^4 \cdot w)x = xw^6x$$

$$\equiv x(w^2 \cdot w^2 \cdot w^2)x = x(w^2 \cdot w^4 \cdot w^2)x = xw^8x = \cdots = xw^{2n}x.$$  

Combining the identities $xwx = xw^{2n}x$ and (7), we have that the identities

$$xwx = xw^{2n}x = (xw^{2n}x)^{r+1} = (xwx)^{r+1}$$

hold in $S(G)$. Thus $S(G)$ satisfies the identity (5) whenever $G$ satisfies $w = 1$. Therefore $S(G) \subseteq S(G, 1)$ but this inclusion contradicts Lemma 2.9.

We have verified that $G = T$ and completed the proof of Theorem 1.1. \hfill $\Box$

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