Exact solutions of the $C_n$ quantum spin chain

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Abstract

We study the exact solutions of quantum integrable model associated with the $C_n$ Lie algebra, with either a periodic or an open one with off-diagonal boundary reflections, by generalizing the nested off-diagonal Bethe ansatz method. Taking the $C_3$ as an example we demonstrate how the generalized method works. We give the fusion structures of the model and provide a way to close fusion processes. Based on the resulted operator product identities among fused transfer matrices and some necessary additional constraints such as asymptotic behaviors and relations at some special points, we obtain the eigenvalues of transfer matrices and parameterize them as homogeneous $T-Q$ relations in the periodic case or inhomogeneous ones in the open case. We also give the exact solutions of the $C_n$ model with an off-diagonal open boundary condition. The method and results in this paper can be generalized to other high rank integrable models associated with other Lie algebras.

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1 Introduction

Quantum integrable models have many applications in the fields of quantum field theory, condensed matter physics, string theory and mathematical physics. The algebraic/coordinate Bethe ansatz and $T-Q$ relations are the very powerful methods to obtain the exact solutions of integrable models with periodic or diagonal open boundary conditions [1–5]. Focusing on the boundary integrable models, it is well-known that some reflection matrices including the off-diagonal elements also satisfy the reflection equations, which implies that the systems are still integrable even with off-diagonal boundary reflections. However, due to the existence of off-diagonal elements, it is quite difficult to calculated the exact solutions of this kind of systems because that the reflection matrices at two boundaries cannot be diagonalized simultaneously. We also note that the models with off-diagonal boundary reflections are very important and have many applications in many issues such as the open AdS/CFT theory, edge states and topological physics. Therefore, many interesting methods such as q-Qnsager algebra [6–9], separation of variables [10–12], modified algebraic Bethe ansatz [13–16] and off-diagonal Bethe ansatz (ODBA) [17, 18] are proposed to study this kind of systems.

The ODBA is an universal method to solve the models with generic integrable boundary conditions. With the help of proposed inhomogeneous $T-Q$ relations, the exact solutions of some typical models with off-diagonal boundary reflections are obtained [18]. Furthermore, in order to solve the models with high ranks [19–25], the nested ODBA is proposed and the exact solutions of models associated with $A_n$ [26–27], $A_2^{(2)}$ [28], $B_2$ [29], $C_2$ [30] and $D_3^{(1)}$ [29] Lie algebras were obtained. One important property of high rank integrable models is that the eigenvalue of transfer matrix is a polynomial where the degree is higher, thus we need more functional relations to determine it completely. Meanwhile, due to the different algebraic structures, the closing conditions of these functional relations are different.

In this paper, we study the functional relations of the integrable $C_n$ vertex model by using the fusion technique [31–38] and the nested ODBA [18]. We systemically analyze the fusion behaviors and obtain recursive fusion relations among the fused transfer matrices. The fusion relations with periodic boundary conditions are different from those with open boundaries. We provide a way to close these recursive fusion relations. Based on them and asymptotic behaviors as well as values at certain points, we obtain the eigenvalues of transfer matrices and parameterize them as the homogeneous or inhomogeneous $T-Q$ relations. The
associated Bethe ansatz equations are also given. Then we generalize these results to the $C_n$ model with off-diagonal open boundary condition. We expect that the method and results provided in this work can be applied to other high rank integrable models associated with other Lie algebras.

The plan of the paper is as follows. In section 2, we study the model with periodic boundary condition. The fusion structures of integrable $C_3$ vertex model is shown in detailed. The closed recursive fusion relations among fused transfer matrices are given. By constructing the $T-Q$ relations, we obtain the eigenvalues and associated Bethe ansatz equations of the system. In section 3, we diagonalize the model with off-diagonal boundary reflections. The reflection matrices with off-diagonal elements and corresponding fusion behavior are introduced. Based on the closed operators product identities, we obtain the eigenvalues of transfer matrices and expressed them as the inhomogeneous $T-Q$ relations. These results are also generalized to the $C_n$ model, which are listed in section 4. The summary of main results and some concluding remarks are presented in section 5.

2 $C_3$ model with periodic boundary condition

2.1 Integrability

Through this paper, we adopt following standard notations. Let $V$ denote a 6-dimensional linear space with orthogonal bases $\{|i\rangle | i = 1, \cdots, 6\}$. For any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as an identity on the other factor spaces. For a matrix $R \in \text{End}(V \otimes V)$, $R_{ij}$ is an embedding operator defined in the same tensor space, which acts as an identity on the factor spaces except for the $i$-th and $j$-th ones.

The quantum integrable system associated with $C_3$ Lie algebra is described by a $36 \times 36$ $R$-matrix $R_{12}(u)$ with the elements \[22\]

$$R_{12}(u)_{kl}^{ij} = u(u + 4)\delta_{ik}\delta_{jl} + (u + 4)\delta_{il}\delta_{jk} - u\xi_i\xi_k\delta_{ji}\delta_{kl},$$

(2.1)

where $u$ is the spectral parameter, $i + \bar{i} = 7$, $\xi_i = 1$ if $i \in [1, 3]$ while $\xi_i = -1$ if $i \in [4, 6]$. For the simplicity, we introduce following notations

$$a(u) = R(u)_{ii}^{ii} = (1 + u)(u + 4), \quad b(u) = R(u)_{ij}^{ij} = u(u + 4), \quad (i \neq j, \bar{j}),$$

$$c(u) = R(u)_{ii}^{\bar{i}i} = 2u + 4, \quad d(u) = \xi_i\xi_j R(u)_{ij}^{\bar{i}j} = -u, \quad (i \neq j, \bar{j}),$$

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\[ e(u) = R(u)^{i,j}_{i,j} = u(u + 3), \quad g(u) = R(u)^{i,j}_{j,i} = u + 4, \quad (i \neq j, j). \tag{2.2} \]

The \( R \)-matrix (2.1) has following properties

- regularity: \( R_{12}(0) = \rho_v(0) \tilde{P}_{12} \),
- unitarity: \( R_{12}(u)R_{21}(-u) = \rho_v(u) \),
- crossing − unitarity: \( R_{12}(u)^{t_1}R_{21}(-u - 8)^{t_1} = \tilde{\rho}_v(u) = \rho_v(u + 4) \). \tag{2.3} \]

where \( \rho_v(u) = a(u)a(-u) \), \( \tilde{P}_{12} \) is the permutation operator with the matrix elements \( \tilde{P}_{12}^{ij}_{kl} = \delta_{il}\delta_{jk} \), \( t_i \) denotes the transposition in the \( i \)-th space, and \( R_{21} = \tilde{P}_{12}R_{12}\tilde{P}_{12} \). The \( R \)-matrix (2.1) satisfies the Yang-Baxter equation

\[ R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \tag{2.4} \]

The monodromy matrix of the system is constructed by the \( R \)-matrix (2.1) as

\[ T_0(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2) \cdots R_{0N}(u - \theta_N), \tag{2.5} \]

where the subscript 0 means the auxiliary space, the other tensor space \( V^k \otimes N \) is the physical or quantum space, \( N \) is the number of sites and \( \{\theta_j | j = 1, \cdots, N\} \) are the inhomogeneous parameters. The monodromy matrix satisfies the Yang-Baxter relation

\[ R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \tag{2.6} \]

Taking the partial trace of monodromy matrix in the auxiliary space, we arrive at the transfer matrix of the system with periodic boundary condition

\[ t^{(p)}(u) = tr_0 T_0(u). \tag{2.7} \]

From the Yang-Baxter relation (2.6), one can prove that the transfer matrices with different spectral parameters commutate with each other, i.e., \([t^{(p)}(u), t^{(p)}(v)] = 0\). Therefore, \( t^{(p)}(u) \) serves as the generating function of all the conserved quantities of the system. The model Hamiltonian with \( C_3 \)-invariant is given by

\[ H_p = \frac{\partial \ln t^{(p)}(u)}{\partial u} \bigg|_{u=0,\{\theta_j\}=0}. \tag{2.8} \]
2.2 Fusion

One wonderful property of \( R \)-matrix is that the \( R \)-matrix may degenerate into the projection operators at some special points, which makes it possible for us to do the fusion. Focus on the \( C_3 \) model, the elements of \( R \)-matrix (2.1) are the polynomials of \( u \) with degree two. Thus there are two degenerate points. One is \( u = -4 \). At which we have

\[
R_{12}(-4) = P_{12}^{(1)} \times S'_{12}.
\]

(2.9)

Here \( P_{12}^{(1)} \) is a one-dimensional projection operator with the form

\[
P_{12}^{(1)} = |\psi_0\rangle\langle\psi_0|,
\]

(2.10)

where \( |\psi_0\rangle = \frac{1}{\sqrt{6}}(|16\rangle + |25\rangle + |34\rangle - |43\rangle - |52\rangle - |61\rangle) \) is a one-dimensional vector in the product space \( V_1 \otimes V_2 \) and \( S'_{12} \) is a constant matrix (we omit its expression because we do not need it). Obviously, \( P_{21}^{(1)} = P_{12}^{(1)} \). From the Yang-Baxter equation (2.4), the one-dimensional fusion associated with projector (2.10) leads to

\[
P_{21}^{(1)} R_{13}(u) R_{23}(u - 4) P_{21}^{(1)} = a(u) e(u - 4) P_{21}^{(1)} \times \text{id}.
\]

(2.11)

We see that the result is also a one-dimensional vector.

The other degenerate point of \( R \)-matrix (2.1) is \( u = -1 \). At which we have

\[
R_{12}(-1) = P_{12}^{(14)} \times S_{12}.
\]

(2.12)

Here \( S_{12} \) is a constant matrix and \( P_{12}^{(14)} \) is a 14-dimensional projection operator with the form of

\[
P_{12}^{(14)} = \sum_{i=1}^{14} |\psi_i^{(14)}\rangle\langle\psi_i^{(14)}|, \quad P_{21}^{(14)} = P_{12}^{(14)},
\]

(2.13)

where the corresponding vectors are

\[
|\psi_1^{(14)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \quad |\psi_2^{(14)}\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle), \quad |\psi_3^{(14)}\rangle = \frac{1}{\sqrt{2}}(|14\rangle - |41\rangle),
\]

\[
|\psi_4^{(14)}\rangle = \frac{1}{\sqrt{2}}(|15\rangle - |51\rangle), \quad |\psi_5^{(14)}\rangle = \frac{1}{2}(|16\rangle - |61\rangle + |43\rangle - |34\rangle),
\]

\[
|\psi_6^{(14)}\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \quad |\psi_7^{(14)}\rangle = \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle),
\]

\[
|\psi_8^{(14)}\rangle = \frac{1}{\sqrt{12}}(-|16\rangle + |61\rangle + |43\rangle - |34\rangle + 2|25\rangle - 2|52\rangle),
\]
From the 14-dimensional fusion associated with the projector $[2,13]$, we obtain a new fused $R$-matrix
\[ R_{(12)3}(u) = \tilde{\rho}_0^{-1}(u + \frac{1}{2}) P_{21}^{(14)} R_{13}(u + \frac{1}{2}) R_{23}(u - \frac{1}{2}) P_{21}^{(14)} \equiv R_{13}(u), \quad (2.14) \]
where $\tilde{\rho}_0(u) = (u - 1)(u + 4)$. We note that the dimension of fused space $V_{(12)} = V_1$ is 14.

The fused $R$-matrix $[2,14]$ has the properties
\[ R_{12}(u) R_{21}(-u) = \rho_v(u) \times \text{id}, \]
\[ R_{12}(u)^{t_i} R_{21}(-u - 8)^{t_i} = \tilde{\rho}_v(u) \times \text{id}, \]
\[ R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v), \quad (2.15) \]
where $\rho_v(u) = (u + \frac{7}{2})(u - \frac{7}{2})(u + \frac{3}{2})(u - \frac{3}{2})$ and $\tilde{\rho}_v(u) = (u + \frac{1}{2})(u + \frac{3}{2})(u + \frac{11}{2})(u + \frac{15}{2})$.

The elements of fused $R$-matrix $[2,14]$ are the polynomials of $u$ with degree two. Thus there are two degenerate points. One is $u = -7/2$, at which the fused $R$-matrix $R_{12}(u)$ degenerates into a 6-dimensional projector
\[ R_{12}(-\frac{7}{2}) = P_{12}^{(6)} \times S_{12}, \quad (2.16) \]
where $S_{12}$ is not relevant here and we do not present its expression for simplicity, $P_{12}^{(6)}$ is a 6-dimensional projector
\[ P_{12}^{(6)} = \sum_{i=1}^{6} |\psi_i^{(6)}\rangle \langle \psi_i^{(6)}|, \quad (2.17) \]

and the corresponding bases are
\[ |\psi_1^{(6)}\rangle = \sqrt{\frac{3}{14}} \left( -|15\rangle - |24\rangle + |33\rangle + |32\rangle + \sqrt{\frac{1}{2}} |51\rangle - \sqrt{\frac{1}{6}} |81\rangle \right), \]
\[ |\psi_2^{(6)}\rangle = \sqrt{\frac{3}{14}} \left( |16\rangle - |64\rangle + |73\rangle + |91\rangle + \sqrt{\frac{2}{3}} |82\rangle \right), \]
\[ |\psi_3^{(6)}\rangle = \sqrt{\frac{3}{14}} \left( |26\rangle + |65\rangle + |10, 2\rangle + |11, 1\rangle - \sqrt{\frac{1}{2}} |53\rangle - \sqrt{\frac{1}{6}} |83\rangle \right), \]
\[ |\psi_4^{(6)}\rangle = \sqrt{\frac{3}{14}} \left( |36\rangle + |75\rangle + |12, 2\rangle + |13, 1\rangle - \sqrt{\frac{1}{2}} |54\rangle - \sqrt{\frac{1}{6}} |84\rangle \right), \]
\[ |\psi_i^{(6)}\rangle = \frac{\sqrt{3}}{14}(|\bar{46}\rangle + |\bar{10}, 4\rangle - |\bar{12}, 3\rangle + |\bar{14}, 1\rangle + \sqrt{\frac{2}{3}}|\bar{85}\rangle) \]
\[ |\psi_6^{(6)}\rangle = \sqrt{\frac{3}{14}}(|\bar{95}\rangle + |\bar{11}, 4\rangle - |\bar{13}, 3\rangle - |\bar{14}, 2\rangle + \frac{1}{2}|\bar{56}\rangle - \frac{1}{6}|\bar{86}\rangle) \]

The projector \( P_{21}^{(6)} \) can be obtained from \( P_{12}^{(6)} \) by exchanging the bases of \( V_1 \) and \( V_2 \). The projector (2.17) shows that we can fuse the spaces \( V_1 \) and \( V_2 \), and the result is that we obtain a new fused \( R \)-matrix,

\[ R_{(12)3}(u) = \tilde{\rho}_0^{-1}(u + 3)P_{12}^{(6)}R_{23}(u + 3)R_{13}(u - \frac{1}{2})P_{12}^{(6)}. \]  

We note the dimension of the fused space \( V_{(12)} \) is 6. Thus fused \( R \)-matrix (2.18) is a 36 \( \times \) 36 one. Taking the correspondence

\[ |\psi_i^{(6)}\rangle \rightarrow |i\rangle, \quad i = 1, \cdots, 6, \]  

we find that the fused \( R \)-matrix (2.18) is the same as the original one (2.1), i.e.,

\[ R_{(12)3}(u) = R_{13}(u). \]  

We remark that from the way of above fusion, the auxiliary space cannot be enlarged anymore. However, both the orders of elements of \( R \)-matrix (2.1) and that of the fused one (2.14) are two. Therefore, the above fusion processes indeed are not closed and we should go further.

In order to obtain the closed fusion relations among fused \( R \)-matrices, we have to consider the degenerations of fused \( R \)-matrix (2.14) at the other degenerate point, \( u = -3/2 \). At which, the fused \( R \)-matrix (2.14) has a 14-dimensional projected subspace, which can be seen from the identity

\[ R_{12}(-\frac{3}{2}) = P_{12}^{(14)} \times S'_{12}, \]

where \( S'_{12} \) is an irrelevant constant matrix, \( P_{12}^{(14)} \) is the 14-dimensional projector

\[ P_{12}^{(14)} = \sum_{i=1}^{14} |\bar{\psi}_i^{(14)}\rangle\langle \bar{\psi}_i^{(14)}|, \]  

and the corresponding bases are

\[ |\bar{\psi}_1^{(14)}\rangle = \frac{1}{\sqrt{3}}(|\bar{13}\rangle - |\bar{22}\rangle + |\bar{61}\rangle), \quad |\bar{\psi}_2^{(14)}\rangle = \frac{1}{\sqrt{3}}(|\bar{14}\rangle - |\bar{32}\rangle + |\bar{71}\rangle), \]
It is obvious that the projector $P_{12}^{(14)}$ can be obtained from $P_{12}^{(14)}$ by exchanging the bases of $V_1$ and $V_2$. The projector (2.21) is survived in the tensor space $V_1 \otimes V_2 \otimes V_3$. By carefully analyzing the fusion structure, We find that the 14-dimensional projected space defined by (2.21) can also be obtained from the product of three $R$-matrices (2.1) at certain points with the following way

$$R_{12}(-1) R_{13}(-2) R_{23}(-1) = P_{123}^{(14)} \times S_{123},$$

(2.22)

where $S_{123}$ is constant matrix, $P_{123}^{(14)}$ is a 14-dimensional projector defined in the spaces $V_1 \otimes V_2 \otimes V_3$

$$P_{123}^{(14)} = \sum_{i=1}^{14} |\phi_i^{(14)}\rangle \langle \phi_i^{(14)}|, \quad P_{321}^{(14)} = P_{123}^{(14)}$$

(2.23)

and the corresponding bases are

$$|\phi_1^{(14)}\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle - |213\rangle + |231\rangle + |312\rangle - |321\rangle),$$

$$|\phi_2^{(14)}\rangle = \frac{1}{\sqrt{6}}(|124\rangle - |142\rangle - |214\rangle + |241\rangle + |412\rangle - |421\rangle),$$

$$|\phi_3^{(14)}\rangle = \frac{1}{\sqrt{12}}(|125\rangle - |152\rangle - |215\rangle + |251\rangle + |512\rangle - |521\rangle - |134\rangle + |143\rangle + |314\rangle - |341\rangle - |413\rangle + |431\rangle),$$

$$|\phi_4^{(14)}\rangle = \frac{1}{\sqrt{12}}(|126\rangle - |162\rangle - |216\rangle + |261\rangle + |612\rangle - |621\rangle)$$
the fusion results with the projector (2.23). We note that the projectors (2.21) and (2.23) give the same subspace, and the only difference we have used the relation (2.22). The fused space \( V \),

\[ \langle 4 \rangle_{\rho} = \langle 4 \rangle \]

Taking the fusion with projector (2.23), we construct another fused \( R \)-matrix

\[ R_{(123)4}(u) = \left[ \tilde{\rho}_0(u + 1)\tilde{\rho}_0(u)(u + 2) \right]^{-1} P_{321}^{(14)} R_{14}(u + 1) R_{24}(u) R_{34}(u - 1) P_{321}^{(14)} \]

\[ \equiv R_{14}(u) . \]

(2.24)

We note that the dimension of the fused space \( V_{(123)} = V_1 \) is 14. In the above construction, we have used the relation (2.22). The fused \( R \)-matrix (2.24) has following properties

\[ R_{12}(u) R_{21}(-u) = \rho_{\tilde{v}}(u) \times \text{id}, \]

\[ R_{12}(u)^{t_1} R_{21}(-u - 8)^{t_1} = \tilde{\rho}_{\tilde{v}}(u) \times \text{id}, \]
\[ R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \]  

(2.25)

where \( \rho_6(u) = -(u + 3)(u - 3) \) and \( \tilde{\rho}_6(u) = -(u + 1)(u + 7) \).

The elements of fused \( R \)-matrix \((2.24)\) are the polynomials of \( u \) with degree one. Thus there is only one degenerate point \( u = -3 \). At which, we have

\[ R_{12}(-3) = P_{12}^{(14)} \times S_{12}, \]

(2.26)

where \( S_{12} \) is an irrelevant constant matrix omitted here, \( P_{12}^{(14)} \) is a 14-dimensional projector

\[ P_{12}^{(14)} = \sum_{i=1}^{14} |\varphi_i^{(14)}\rangle \langle \varphi_i^{(14)}|, \]

(2.27)

and the corresponding bases are

\[
\begin{align*}
|\varphi_1^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|14\rangle + \sqrt{2}|23\rangle + |32\rangle + |41\rangle), \\
|\varphi_2^{(14)}\rangle &= \sqrt{\frac{1}{6}} (\sqrt{2}|15\rangle + \sqrt{2}|52\rangle - |33\rangle + |61\rangle), \\
|\varphi_3^{(14)}\rangle &= \sqrt{\frac{1}{6}} (\sqrt{2}|25\rangle + \sqrt{2}|72\rangle - |34\rangle + |81\rangle), \\
|\varphi_4^{(14)}\rangle &= \sqrt{\frac{1}{6}} (\sqrt{2}|54\rangle - \sqrt{2}|73\rangle + |35\rangle + |91\rangle), \\
|\varphi_5^{(14)}\rangle &= \sqrt{\frac{1}{12}} (2|45\rangle - 2|92\rangle - |36\rangle + |64\rangle - |83\rangle + |12,1\rangle), \\
|\varphi_6^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|16\rangle + \sqrt{2}|10,1\rangle + |43\rangle - |62\rangle), \\
|\varphi_7^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|26\rangle + \sqrt{2}|11,1\rangle + |44\rangle - |82\rangle), \\
|\varphi_8^{(14)}\rangle &= \frac{1}{2} (-|36\rangle - |64\rangle + |83\rangle + |12,1\rangle), \\
|\varphi_9^{(14)}\rangle &= \sqrt{\frac{1}{6}} (\sqrt{2}|10,4\rangle - \sqrt{2}|11,3\rangle - |46\rangle - |12,2\rangle), \\
|\varphi_{10}^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|56\rangle + \sqrt{2}|13,1\rangle + |65\rangle - |93\rangle), \\
|\varphi_{11}^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|10,5\rangle - \sqrt{2}|13,2\rangle - |66\rangle + |12,3\rangle), \\
|\varphi_{12}^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|76\rangle + \sqrt{2}|14,1\rangle + |85\rangle - |94\rangle), \\
|\varphi_{13}^{(14)}\rangle &= \sqrt{\frac{1}{6}} (-\sqrt{2}|11,5\rangle - \sqrt{2}|14,2\rangle - |86\rangle + |12,4\rangle),
\end{align*}
\]

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\[ |\varphi_{14}^{(14)}\rangle = \sqrt{\frac{1}{6}}(-\sqrt{2}|\tilde{13}, 4\rangle + \sqrt{2}|\tilde{14}, 3\rangle - |\tilde{96}\rangle - |\tilde{12}, 5\rangle). \]

Again, the projector \( P_{21}^{(14)} \) can be obtained from \( P_{12}^{(14)} \) by exchanging the bases of \( V_1 \) and \( V_2 \).

Taking the fusion of \( R \)-matrix (2.24) in the auxiliary space by using the 14-dimensional projector \( P_{12}^{(14)} \), we obtain a fused \( R \)-matrix
\[
R_{(12)3}(u) = (u + \frac{13}{2})^{-1}P_{12}^{(14)}R_{23}(u + \frac{5}{2})R_{13}(u - \frac{1}{2})P_{12}^{(14)}. \tag{2.28}
\]
The dimension of fused space \( V_{(12)} \) is 14, which equals to the dimension of fused space \( V_1 \).

After taking the correspondence
\[
|\varphi_{i}^{(14)}\rangle \rightarrow |\psi_{i}^{(14)}\rangle, \quad i = 1, \cdots, 14,
\]
we find the fused \( R \)-matrix (2.28) is the same as the fused one (2.14), i.e.,
\[
R_{(12)3}(u) = R_{13}(u). \tag{2.30}
\]

Eq. (2.30) gives another intrinsic relation to close the fusion processes.

Taking the fusion of \( R \)-matrix (2.24) in the quantum space by using the 14-dimensional projector \( P_{23}^{(14)} \) given by (2.13), we obtain a fused \( R \)-matrix
\[
R_{1(23)}(u) = P_{23}^{(14)}R_{12}(u + \frac{1}{2})R_{13}(u - \frac{1}{2})P_{23}^{(14)} \equiv R_{12}(u). \tag{2.31}
\]
The fused \( R \)-matrix (2.31) is defined in the tensor space \( V_1 \otimes V_2 \) and has following properties
\[
R_{12}(u)R_{21}(-u) = \rho_{\tilde{v}\tilde{v}}(u) \times \text{id},
R_{12}(u)^{t_2}R_{21}(-u - 8)^{t_2} = \tilde{\rho}_{\tilde{v}\tilde{v}}(u) \times \text{id},
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \tag{2.32}
\]

where \( \rho_{\tilde{v}\tilde{v}}(u) = (u + \frac{5}{2})(u - \frac{5}{2})(u + \frac{7}{2})(u + \frac{7}{2}) \) and \( \tilde{\rho}_{\tilde{v}\tilde{v}}(u) = (u + \frac{3}{2})(u + \frac{13}{2})(u + \frac{13}{2})(u + \frac{15}{2}). \)

Last, we remark that the following identity holds
\[
R_{12}(-1)R_{13}(-2)R_{14}(-3)R_{23}(-1)R_{24}(-2)R_{34}(-1) = 0, \tag{2.33}
\]
which can be checked by direct calculation. Eq. (2.33) implies that we can not obtain more nontrivial fused \( R \)-matrix if we take fusion only in the auxiliary spaces.
2.3 Operator product identities

Based on the obtained fused $R$-matrices, we define the fused monodromy matrices

$$T_0(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2)\cdots R_{0N}(u - \theta_N),$$

$$T_\bar{0}(u) = R_{\bar{0}1}(u - \bar{\theta}_1)R_{\bar{0}2}(u - \bar{\theta}_2)\cdots R_{\bar{0}N}(u - \bar{\theta}_N).$$

(2.34)

We note that the quantum spaces of the above monodromy matrices are the same, which is $V^\otimes N$, and the corresponding auxiliary spaces are $V_0$ and $\bar{V}_0$ with dimension 14. The fused monodromy matrices (2.34) satisfy the Yang-Baxter relations

$$R_{1\bar{2}}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{1\bar{2}}(u - v),$$

$$R_{\bar{1}2}(u - v)T_{\bar{1}}(u)T_2(v) = T_2(v)T_{\bar{1}}(u)R_{\bar{1}2}(u - v),$$

$$R_{1\bar{2}}(u - v)T_1(u)T_\bar{2}(v) = T_\bar{2}(v)T_1(u)R_{1\bar{2}}(u - v).$$

(2.35)

where $R_{1\bar{2}}(u)$ is the fused $R$-matrix defined in the fused space $V_1 \otimes V_2$, which can be determined by the first equation in (2.32). Besides the transfer matrix $t^{(p)}(u)$, let us introduce two fused transfer matrices

$$t^{(p)}_2(u) = tr_0 T_0(u), \quad t^{(p)}_3(u) = tr_\bar{0} T_\bar{0}(u).$$

(2.36)

From above Yang-Baxter relations (2.6) and (2.35), we can prove these transfer matrices commutate with each other, namely,

$$[t^{(p)}_1(u), t^{(p)}_2(u)] = [t^{(p)}_2(u), t^{(p)}_3(u)] = [t^{(p)}_3(u), t^{(p)}_1(u)] = 0.$$

(2.37)

Therefore, they have common eigenstates and can be diagonalized simultaneously.

By using the above fusion relations of $R$-matrices and the definitions (2.6) and (2.34), we obtain the fusion behavior of monodromy matrices

$$P_{21}^{(1)} T_1(u) T_2(u - 4) P_{21}^{(1)} = T_1(u) T_2(u - 4) P_{21}^{(1)}$$

$$= \prod_{i=1}^{N} a(u - \theta_i) e(u - \theta_i - 4) P_{21}^{(1)} \times \text{id},$$

$$P_{21}^{(14)} T_1(u) T_2(u - 1) P_{21}^{(14)} = T_1(u) T_2(u - 1) P_{21}^{(14)}$$

$$= T_{12}(u) = \prod_{i=1}^{N} \rho_0(u - \theta_i) T_1(u - \frac{1}{2}),$$

$$P_{321}^{(14)} T_1(u) T_2(u - 1) T_3(u - 2) P_{321}^{(14)} = T_1(u) T_2(u - 1) T_3(u - 2) P_{321}^{(14)}$$

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\[ P_{12}^{(14)} T_2(u) T_1(u - 3) = T_2(u) T_1(u - 3) P_{12}^{(14)} = \prod_{i=1}^{N} (u + \theta_i) T_1(u - 3), \]

Here the subscripts 1 and 2 mean the original 6-dimensional auxiliary spaces \( V_1 \) and \( V_2 \), the \( \bar{V} \) means the 14-dimensional fused auxiliary space \( V_{1} \) by the operators \( P_{21}^{(14)} \) or \( P_{12}^{(14)} \), and \( \bar{V} \) means the 14-dimensional fused auxiliary space \( V_{1} \) by the operator \( P_{321}^{(14)} \).

Next, we calculate the products of two monodromy matrices with special spectral parameters. By using the property of permutation operator, we obtain

\[ T_a(\theta_j) T_b(\theta_j + \delta) = R_{a1}(\theta_j - \theta_1) \cdots R_{a\jmath-1}(\theta_j - \theta_{\jmath-1}) R_{a\jmath}(0) R_{a\jmath+1}(\theta_j - \theta_{\jmath+1}) \cdots \]

\[ \times R_{aN}(\theta_j - \theta_N) R_{b1}(\theta_j - \theta_1 + \delta) \cdots R_{b\jmath-1}(\theta_j - \theta_{\jmath-1} + \delta) R_{b\jmath}(\delta) \]

\[ \times R_{a\jmath}(0) R_{ja}(0) \rho_v(0)^{-1} R_{b\jmath+1}(\theta_j - \theta_{\jmath+1} + \delta) \cdots R_{bN}(\theta_j - \theta_N + \delta) \]

\[ = R_{jj+1}(\theta_j - \theta_{\jmath+1}) \cdots R_{jN}(\theta_j - \theta_N) R_{a1}(\theta_j - \theta_1) \cdots R_{a\jmath-1}(\theta_j - \theta_{\jmath-1}) \]

\[ \times R_{b1}(\theta_j - \theta_1 + \delta) \cdots R_{b\jmath-1}(\theta_j - \theta_{\jmath-1} + \delta) \]

\[ \times P_{ba}^{(d)} S_{ba} R_{ja}(0) R_{b\jmath+1}(\theta_j - \theta_{\jmath+1} + \delta) \cdots R_{bN}(\theta_j - \theta_N + \delta) \]

\[ = P_{ba}^{(d)} R_{a1}(\theta_j - \theta_1) \cdots R_{a\jmath-1}(\theta_j - \theta_{\jmath-1}) R_{a\jmath}(0) R_{ja}(0) \rho_v(0)^{-1} R_{jj+1}(\theta_j - \theta_{\jmath+1}) \cdots \]

\[ \times R_{jN}(\theta_j - \theta_N) R_{b1}(\theta_j - \theta_1 + \delta) \cdots R_{b\jmath-1}(\theta_j - \theta_{\jmath-1} + \delta) \]

\[ \times R_{ba}(\delta) R_{ja}(0) R_{b\jmath+1}(\theta_j - \theta_{\jmath+1} + \delta) \cdots R_{bN}(\theta_j - \theta_N + \delta) \]

\[ = P_{ba}^{(d)} T_a(\theta_j) T_b(\theta_j + \delta), \] (2.39)

where \( \delta \) is the degenerate point of \( R_{ab}(u) \) and \( P_{ba}^{(d)} \) is the corresponding \( d \)-dimensional project operator. The product of three monodromy matrices at fixed points is

\[ T_{\prime\prime 1}(\theta_j) T_{\prime\prime 2}(\theta_j - 2) = T_{\prime\prime 1}(\theta_j) P_{32}^{(14)} T_{\prime\prime 2}(\theta_j - 2) P_{32}^{(14)} \]

\[ = R_{\prime\prime 1}(\theta_j - \theta_1) \cdots R_{\prime\prime j-1}(\theta_j - \theta_{j-1}) R_{\prime\prime j}(0) R_{\prime\prime j+1}(\theta_j - \theta_{j+1}) \cdots R_{\prime\prime N}(\theta_j - \theta_N) \]

\[ \times R_{\prime\prime 2}(\theta_j - \theta_1 - 1) \cdots R_{\prime\prime j-1}(\theta_j - \theta_{j-1} - 1) R_{\prime\prime j}(\theta_j - \theta_{j+1} - 1) \cdots \]
\begin{align*}
\times & R_{2'N}(\theta_j - \theta_N - 1)R_{3'1}(\theta_j - \theta_1 - 2) \cdots R_{3'j-1}(\theta_j - \theta_{j-1} - 2)R_{3'j}(-2) \\
\times & [R_{1'j}(0)R_{j'1}(0)\rho(0)^{-1}]R_{3'j+1}(\theta_j - \theta_{j+1} - 2) \cdots R_{3'N}(\theta_j - \theta_N - 2)P_{3'2'}^{(14)} \\
= & R_{jj+1}(\theta_j - \theta_{j+1}) \cdots R_{jN}(\theta_j - \theta_N)R_{1'j}(\theta_j - \theta_1) \cdots R_{1'j-1}(\theta_j - \theta_{j-1}) \\
\times & R_{2'1}(\theta_j - \theta_1 - 1) \cdots R_{2'j-1}(\theta_j - \theta_{j-1} - 1)R_{3'1}(\theta_j - \theta_1 - 2) \cdots \\
\times & R_{3'j-1}(\theta_j - \theta_{j-1} - 2)R_{2'1}(-1)R_{3'1}(-2)R_{j'1}(0)P_{3'2'}^{(14)} R_{2'j+1}(\theta_j - \theta_{j+1} - 1) \cdots \\
\times & R_{2'N}(\theta_j - \theta_N - 1)R_{3'j+1}(\theta_j - \theta_{j+1} - 2) \cdots R_{3'N}(\theta_j - \theta_N - 2)P_{3'2'}^{(14)} \\
= & R_{jj+1}(\theta_j - \theta_{j+1}) \cdots R_{jN}(\theta_j - \theta_N)R_{1'j}(\theta_j - \theta_1) \cdots R_{1'j-1}(\theta_j - \theta_{j-1}) \\
\times & R_{2'1}(\theta_j - \theta_1 - 1)R_{2'2}(\theta_j - \theta_2 - 1) \cdots R_{2'j-1}(\theta_j - \theta_{j-1} - 1) \\
\times & R_{3'1}(\theta_j - \theta_1 - 2)R_{3'2}(\theta_j - \theta_2 - 2) \cdots R_{3'j-1}(\theta_j - \theta_{j-1} - 2)P_{3'2'}^{(14)}S_{3'2'}S_{3'2'}^{-1} \\
\times & R_{j'1}(0)R_{2'j+1}(\theta_j - \theta_{j+1} - 1) \cdots R_{2'N}(\theta_j - \theta_N - 1) \\
\times & R_{3'j+1}(\theta_j - \theta_{j+1} - 2) \cdots R_{3'N}(\theta_j - \theta_N - 2)P_{3'2'}^{(14)} \\
= & P_{3'2'2'}^{(14)}R_{1'j}(\theta_j - \theta_1) \cdots R_{1'j-1}(\theta_j - \theta_{j-1})R_{1'j}(0)R_{j'1}(0)\rho(0)^{-1} \\
\times & R_{jj+1}(\theta_j - \theta_{j+1}) \cdots R_{jN}(\theta_j - \theta_N)R_{2'1}(\theta_j - \theta_1 - 1) \cdots R_{2'j-1}(\theta_j - \theta_{j-1} - 1) \\
\times & R_{2'1}(-1)R_{2'j+1}(\theta_j - \theta_{j+1} - 1) \cdots R_{2'N}(\theta_j - \theta_N - 1) \\
\times & R_{3'1}(\theta_j - \theta_1 - 2) \cdots R_{3'j-1}(\theta_j - \theta_{j-1} - 2)R_{2'1}(-2)R_{j'1}(0) \\
\times & R_{3'j+1}(\theta_j - \theta_{j+1} - 2) \cdots R_{3'N}(\theta_j - \theta_N - 2)P_{3'2'}^{(14)} \\
= & P_{3'2'2'}^{(14)}T_{1'j}(\theta_j)T_{2'2'}(\theta_j - 1)
\end{align*}
Substituting \( \delta = \{-4, -1, -7/2, -3\} \) into Eq. (2.39) and using the relations (2.38) and (2.40), we obtain

\begin{align*}
T_1(\theta_j)T_2(\theta_j - 4) &= P_{21}^{(1)} T_1(\theta_j)T_2(\theta_j - 4), \\
T_1(\theta_j)T_2(\theta_j - 1) &= P_{21}^{(14)} T_1(\theta_j)T_2(\theta_j - 1), \\
T_1(\theta_j)T_{2(3)}(\theta_j - 1) &= T_1(\theta_j)P_{32}^{(14)} T_2(\theta_j - 1)T_3(\theta_j - 2)P_{32}^{(14)} \\
&= P_{321}^{(14)} T_1(\theta_j)T_{(23)}(\theta_j - 1), \\
T_2(\theta_j)T_1(\theta_j - \frac{7}{2}) &= P_{12}^{(6)} T_2(\theta_j)T_1(\theta_j - \frac{7}{2}),
\end{align*}
\[ T_2(\theta_j)T_1(\theta_j - 3) = P^{(14)}_{12} T_2(\theta_j) T_1(\theta_j - 3). \] (2.41)

Taking the partial traces of Eq. (2.41) in the auxiliary spaces and using the correspondences (2.20) and (2.30), we obtain the closed operator product identities among transfer matrices

\[
\begin{align*}
t^{(p)}(\theta_j) t^{(p)}(\theta_j - 4) &= \prod_{i=1}^{N} a(\theta_j - \theta_i) e(\theta_j - \theta_i - 4) \times \text{id}, \\
t^{(p)}(\theta_j) t^{(p)}(\theta_j - 1) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t^{(p)}_2(\theta_j - 1), \\
t^{(p)}(\theta_j) t^{(p)}_2(\theta_j - \frac{3}{2}) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) (\theta_j - \theta_i + 1) \tilde{t}^{(p)}_3(\theta_j - 1), \\
t^{(p)}(\theta_j) t^{(p)}_2(\theta_j - \frac{7}{2}) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t^{(p)}(\theta_j - 3), \\
t^{(p)}(\theta_j) \tilde{t}^{(p)}_3(\theta_j - 3) &= \prod_{i=1}^{N} (\theta_j - \theta_i + 4) \tilde{t}^{(p)}_2(\theta_j - \frac{5}{2}), \quad j = 1, \ldots, N. \quad (2.42)
\end{align*}
\]

In the derivation, we have used the property of projector

\[ P^{(14)}_{32} P^{(14)}_{321} = P^{(14)}_{32} P^{(14)}_{321} S_{321}^{-1} = P^{(14)}_{32} R_{321}(-1) R_{31}(-2) R_{21}(-1) S_{321}^{-1} \]
\[ = P^{(14)}_{32} P^{(14)}_{32} S_{32}(-2) R_{21}(-1) S_{321}^{-1} \]
\[ = R_{32}(-1) R_{31}(-2) R_{21}(-1) S_{321}^{-1} = P^{(14)}_{321} S_{321}^{-1} S_{321}^{-1} = P^{(14)}_{321}. \] (2.43)

The asymptotic behaviors of the fused transfer matrices can be calculated directly

\[
\begin{align*}
t^{(p)}(u)|_{u \to \pm \infty} &= 6u^{2N} \times \text{id} \times \cdots, \\
t^{(p)}_2(u)|_{u \to \pm \infty} &= 14u^{2N} \times \text{id} \times \cdots, \\
t^{(p)}_3(u)|_{u \to \pm \infty} &= 14u^{N} \times \text{id} \times \cdots. \quad (2.44)
\end{align*}
\]

Denote the eigenvalues of the transfer matrices \( t^{(p)}(u) \), \( t^{(p)}_2(u) \), and \( t^{(p)}_3(u) \) as \( \Lambda^{(p)}(u) \), \( \Lambda^{(p)}_2(u) \), and \( \Lambda^{(p)}_3(u) \), respectively. From the operator product identities (2.42), we obtain the functional relations among the eigenvalues

\[
\begin{align*}
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - 4) &= \prod_{i=1}^{N} a(\theta_j - \theta_i) e(\theta_j - \theta_i - 4), \\
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - 1) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda^{(p)}_2(\theta_j - \frac{1}{2}),
\end{align*}
\]
By using above functions, the eigenvalues of transfer matrices can be expressed as
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \bar{\rho}_0(\theta_j - \theta_i)(\theta_j - \theta_i + 1) \Lambda^{(p)}(\theta_j - 1), \]
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - \frac{7}{2}) = \prod_{i=1}^{N} \bar{\rho}_0(\theta_j - \theta_i) \Lambda^{(p)}(\theta_j - 3), \]
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - 3) = \prod_{i=1}^{N} (\theta_j - \theta_i + 4) \Lambda^{(p)}(\theta_j - \frac{5}{2}), \quad j = 1, \ldots, N. \] (2.45)

The asymptotic behaviors (2.44) of the fused transfer matrices lead to
\[ \Lambda^{(p)}(u)|_{u \to \pm \infty} = 6u^{2N} + \cdots, \quad \Lambda^{(p)}_2(u)|_{u \to \pm \infty} = 14u^{2N} + \cdots, \]
\[ \Lambda^{(p)}_3(u)|_{u \to \pm \infty} = 14u^{N} + \cdots. \] (2.46)

From the definitions (2.7) and (2.36), we know that the eigenvalues \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}_2(u) \) are the polynomials of \( u \) with degree \( 2N \), while \( \Lambda^{(p)}_3(u) \) is a polynomial of \( u \) with degree \( N \). Hence the \( 5N \) functional relations (2.45) and 3 asymptotic behaviors (2.46) can completely determine the eigenvalues of \( \Lambda^{(p)}(u) \), \( \Lambda^{(p)}_2(u) \) and \( \Lambda^{(p)}_3(u) \).

### 2.4 \( T - Q \) relations

For the simplicity, let us introduce some functions
\[ Z_1^{(p)}(u) = A^{(p)}(u) \frac{Q^{(1)}_p(u - 1)}{Q^{(1)}_p(u)}, \quad Z_2^{(p)}(u) = B^{(p)}(u) \frac{Q^{(1)}_p(u + 1)Q^{(2)}_p(u - 1)}{Q^{(1)}_p(u)Q^{(2)}_p(u)}, \]
\[ Z_3^{(p)}(u) = B^{(p)}(u) \frac{Q^{(2)}_p(u + 1)Q^{(3)}_p(u - \frac{3}{2})}{Q^{(2)}_p(u)Q^{(3)}_p(u + \frac{1}{2})}, \quad Z_4^{(p)}(u) = B^{(p)}(u) \frac{Q^{(2)}_p(u + 1)Q^{(3)}_p(u + \frac{5}{2})}{Q^{(2)}_p(u + 2)Q^{(3)}_p(u + \frac{1}{2})}, \]
\[ Z_5^{(p)}(u) = B^{(p)}(u) \frac{Q^{(1)}_p(u + 2)Q^{(2)}_p(u + 3)}{Q^{(1)}_p(u + 3)Q^{(2)}_p(u + 2)}, \quad Z_6^{(p)}(u) = V^{(p)}(u) \frac{Q^{(1)}_p(u + 4)}{Q^{(1)}_p(u + 3)}, \] (2.47)

where
\[ A^{(p)}(u) = \prod_{j=1}^{N} a(u - \theta_j), \quad B^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j), \quad V^{(p)}(u) = \prod_{j=1}^{N} e(u - \theta_j), \]
\[ Q^{(m)}_p(u) = \prod_{k=1}^{L_m} (u - \mu^{(m)}_k + \frac{m}{2}), \quad m = 1, 2, 3. \] (2.48)

By using above functions, the eigenvalues of transfer matrices can be expressed as the \( T - Q \) relations
\[ \Lambda^{(p)}(u) = \sum_{l=1}^{6} Z_l^{(p)}(u), \]
\( \Lambda^{(p)}_2(u) = \prod_{i=1}^{N} \rho_0^{-1}(u - \theta_i + \frac{1}{2}) \left\{ \sum_{i<j}^6 Z_i^{(p)}(u + \frac{1}{2})Z_j^{(p)}(u - \frac{1}{2}) - Z_3^{(p)}(u + \frac{1}{2})Z_3^{(p)}(u - \frac{1}{2}) \right\} \)

\( \Lambda^{(p)}_3(u) = \prod_{i=1}^{N} [\tilde{\rho}_0(u - \theta_i + 1)\tilde{\rho}_0(u - \theta_i)(u - \theta_i + 2)]^{-1} \times \left\{ \sum_{i<j<k}^6 Z_i^{(p)}(u + 1)Z_j^{(p)}(u)Z_k^{(p)}(u - 1) - \sum_{k=5}^{6} Z_3^{(p)}(u + 1)Z_4^{(p)}(u)Z_k^{(p)}(u - 1) - \sum_{j=3}^{2} Z_3^{(p)}(u + 1)Z_j^{(p)}(u)Z_5^{(p)}(u - 1) \right\}. \tag{2.49} \)

All the eigenvalues are polynomials, thus the residues of right hand sides of Eq. (2.49) should be zero, which gives that the Bethe roots \( \{\mu_k^{(m)}\} \) in (2.49) should satisfy the Bethe ansatz equations

\[
\frac{Q_p^{(1)}(\mu_k^{(1)} + \frac{1}{2})Q_p^{(2)}(\mu_k^{(1)} - \frac{3}{2})}{Q_p^{(1)}(\mu_k^{(1)} + \frac{1}{2})Q_p^{(2)}(\mu_k^{(1)} - \frac{3}{2})} = -\prod_{j=1}^{N} \frac{\mu_k^{(1)} + \frac{1}{2} - \theta_j}{\mu_k^{(1)} - \frac{1}{2} - \theta_j}, \quad k = 1, \cdots, L_1,
\]

\[
\frac{Q_p^{(1)}(\mu_l^{(2)} - \frac{1}{2})Q_p^{(1)}(\mu_l^{(2)} - \frac{1}{2})}{Q_p^{(1)}(\mu_l^{(2)} - \frac{1}{2})Q_p^{(1)}(\mu_l^{(2)} - \frac{1}{2})} = -1, \quad l = 1, \cdots, L_2,
\]

\[
\frac{Q_p^{(2)}(\mu_l^{(3)} + \frac{1}{2})Q_p^{(3)}(\mu_l^{(3)} - \frac{3}{2})}{Q_p^{(2)}(\mu_l^{(3)} + \frac{1}{2})Q_p^{(3)}(\mu_l^{(3)} - \frac{3}{2})} = -1, \quad l = 1, \cdots, L_3. \tag{2.50} \]

We note that the Bethe ansatz equations obtained from the regularities of \( \Lambda^{(p)}(u) \) are the same as those obtained from \( \Lambda_2^{(p)}(u) \) and \( \Lambda_3^{(p)}(u) \). Meanwhile, any of these eigenvalues can give the complete set of Bethe ansatz equations.

It is easy to check that the \( T - Q \) relations (2.49) satisfy the functional relations (2.45) and the asymptotic behaviors (2.46). Therefore, we conclude that the \( \Lambda^{(p)}(u) \), \( \Lambda_2^{(p)}(u) \) and \( \Lambda_3^{(p)}(u) \) are the eigenvalues of the transfer matrices \( t^{(p)}(u) \), \( t_2^{(p)}(u) \) and \( t_3^{(p)}(u) \), respectively, provided that the Bethe roots satisfy the Bethe ansatz equations (2.50). It is remarked that the \( T - Q \) relations (2.49) and the associated Bethe ansatz equations (2.50) (after taking the homogeneous limit \( \{\theta_j \to 0|j = 1, 2, \cdots, N\} \)) coincide with the previous results [21][22]. Then the eigenvalues of the Hamiltonian (2.8) reads

\[
E_p = \left. \frac{\partial \ln \Lambda^{(p)}(u)}{\partial u} \right|_{u=0, \{\theta_j \}=0}. \tag{2.51} \]
3  $C_3$ model with open boundary condition

3.1 Boundary integrability

Now, we consider the system with open boundary condition. The boundary reflections are quantified by the reflection matrix $K^-$ at one side and dual one $K^+$ at the other side. The integrable requires that $K^-$ satisfies the reflection equation

$$R_{12}(u - v)K_1^-(u)R_{21}(u + v)K_2^-(v) = K_2^-(v)R_{12}(u + v)K_1^-(u)R_{21}(u - v),$$

(3.1)

while $K^+$ satisfies the dual reflection equation

$$R_{12}(-u + v)K_1^+(u)R_{21}(-u - v - 8)K_2^+(v) = K_2^+(v)R_{12}(-u - v - 8)K_1^+(u)R_{21}(-u + v).$$

(3.2)

In this paper, we consider the case that the reflection matrices have off-diagonal elements, thus the numbers of quasi-particles with different intrinsic degrees of freedom are not conserved during the reflection processes. The reflection matrix $K_0^-(u)$ defined in the space $V_0$ takes the form of [39-41]

$$K_0^-(u) = \zeta + Mu, \quad M = \begin{pmatrix}
-1 & 0 & 0 & c_1 & 0 & 0 \\
0 & -1 & 0 & 0 & c_1 & 0 \\
0 & 0 & -1 & 0 & 0 & c_1 \\
c_2 & 0 & 0 & 1 & 0 & 0 \\
0 & c_2 & 0 & 0 & 1 & 0 \\
0 & 0 & c_2 & 0 & 0 & 1 \\
\end{pmatrix},$$

(3.3)

where $\zeta$, $c_1$ and $c_2$ are the arbitrary boundary parameters. The dual reflection matrix $K_0^+(u)$ is defined as

$$K_0^+(u) = K_0^-(u - 4)|_{\zeta,c_i \rightarrow \tilde{\zeta},\tilde{c}_i},$$

(3.4)

where $\tilde{\zeta}$ and $\tilde{c}_i (i = 1, 2)$ are the boundary parameters.

Due to the boundary reflection, besides the monodromy matrix $T_0(u)$ given by (2.5), we also need the reflecting monodromy matrix

$$\hat{T}_0(u) = R_{N0}(u + \theta_N) \cdots R_{20}(u + \theta_2)R_{10}(u + \theta_1),$$

(3.5)
which satisfies the Yang-Baxter relation

\[ R_{12}(u - v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u - v). \]  
(3.6)

The transfer matrix \( t(u) \) of the system with open boundary condition is

\[ t(u) = \text{tr}_0\{K^+_0(u)T_0(u)K^-_0(u)\hat{T}_0(u)\}. \]  
(3.7)

From the Yang-Baxter relations (2.6) and (3.6), reflection equation (3.1) and dual one (3.2), we can prove that the transfer matrices (3.7) with different spectral parameters commute with each other, i.e., \([t(u), t(v)] = 0\). Therefore, \( t(u) \) serves as the generating function of all the conserved quantities of the system. The Hamiltonian is constructed as the derivative of logarithm of the transfer matrix

\[ H = \frac{\partial \ln t(u)}{\partial u}\bigg|_{u=0,\{\theta_j\}=0}. \]  
(3.8)

In the Hamiltonian (3.8), because two boundary reflection matrices \( K^-_0(u) \) (3.3) and \( K^+_0(u) \) (3.4) do not commute with each other, i.e., \([K^-_0(u), K^+_0(v)] \neq 0\), the \( K^+_0(u) \) cannot be diagonalized simultaneously. Then it is quite hard to derive the exact solutions of the system via the conventional Bethe Ansatz due to the absence of a proper reference state. We will generalize the method developed in section 2 to calculate the eigenvalues of transfer matrix (3.7) and that of Hamiltonian (3.8) in the following subsections.

### 3.2 Fusion

Because the reflection matrices are defined in the auxiliary spaces and we have fused the auxiliary spaces into different forms with different dimensions, we should fuse the reflection matrices correspondingly. All the fusion relations with boundary reflections can be obtain from the degeneration properties of \( R \)-matrix and the (dual) reflection equation. The related projectors are \( P_{12}^{(1)} \), \( P_{12}^{(14)} \), \( P_{123}^{(14)} \), \( P_{12}^{(6)} \) and \( P_{12}^{(14)} \) defined above. The fusion of reflection matrices with one-dimensional projector gives

\[
P_{21}^{(1)}K^-_1(u)R_{21}(2u - 4)K^-_2(u - 4)P_{12}^{(1)} = \text{Det}_q(K^-(u)) P_{12}^{(1)},
\]

\[
P_{12}^{(1)}K^+_2(u - 4)R_{12}(-2u - 4)K^+_1(u)P_{21}^{(1)} = \text{Det}_q(K^+(u)) P_{21}^{(1)}, \]  
(3.9)

where \( \text{Det}_q(K^\pm(u)) \) are the quantum determinants of reflection matrices \( K^\pm(u) \),

\[
\text{Det}_q(K^-(u)) = (u - \frac{3}{2})(u - 4)h_1(u)h_2(u),
\]
\[ \text{Det}_q(K^+(u)) = (u + \frac{3}{2})(u + 4)\tilde{h}_1(u)\tilde{h}_2(u), \]
\[ h_1(u) = 2(\sqrt{1 + c_1c_2}u + \zeta), \quad h_2(u) = 2(\sqrt{1 + c_1c_2}u - \zeta), \]
\[ \tilde{h}_1(u) = -2(\sqrt{1 + \tilde{c}_1\tilde{c}_2}u + \tilde{\zeta}), \quad \tilde{h}_2(u) = -2(\sqrt{1 + \tilde{c}_1\tilde{c}_2}u - \tilde{\zeta}). \] (3.10)

We note that the reflection equation and dual one require that the inserted \( R \)-matrices in (3.9) with determined spectral parameters are necessary.

Using the 14-dimensional projector \( P_{12}^{(14)} \), we construct the 14 \( \times \) 14 fused \( K \)-matrices

\[ K_{(12)}^- (u + \frac{1}{2}) = \frac{1}{2(u - \frac{1}{2})(u + 2)} P_{21}^{(14)} K_1^-(u + \frac{1}{2}) R_{21}(2u) K_2^- (u - \frac{1}{2}) P_{12}^{(14)} \equiv K_1^-(u), \]
\[ K_{(12)}^+ (u + \frac{1}{2}) = \frac{1}{2(u + 2)(u + \frac{3}{2})} P_{12}^{(14)} K_2^+ (u - \frac{1}{2}) R_{12}(-2u - 8) K_1^+(u + \frac{1}{2}) P_{21}^{(14)} \equiv K_1^+(u). \] (3.11)

The fused reflection matrices (3.11) satisfy the reflection equations

\[ R_{12}(-u - v)K_1^{-}(u) R_{21}(u + v) K_2^{-} (v) = K_2^{-} (v) R_{12}(u + v) K_1^{-} (u) R_{21}(u - v), \]
\[ R_{12}(-u + v)K_1^{+}(u) R_{21}(-u - v - 8) K_2^{+} (v) \]
\[ = K_2^{+} (v) R_{12}(-u - v - 8) K_1^{+} (u) R_{21}(-u + v), \] (3.12)

which means that the fusion does not break the integrability.

The 14-dimensional projector \( P_{12}^{(14)} \) allows us to construct the 14 \( \times \) 14 fused \( K \)-matrices

\[ K_{(123)}^-(u + 1) = [2^3(u + \frac{5}{2})(u + \frac{3}{2})(u - \frac{1}{2})u(u - 1)(u + 2)]^{-1} \]
\[ \times P_{321}^{(14)} K_1^-(u + 1) R_{21}(2u + 1) R_{31}(2u) K_2^-(u) R_{32}(2u - 1) K_3^- (u - 1) P_{123}^{(14)} \equiv K_1^-(u), \]
\[ K_{(123)}^+(u + 1) = [2^3(u + \frac{3}{2})(u + \frac{5}{2})(u + 2)(u + 4)(u + 5)]^{-1} P_{123}^{(14)} K_3^+(u - 1) \]
\[ \times R_{23}(-2u - 7) R_{13}(-2u - 8) K_2^+(u) R_{12}(-2u - 9) K_1^+(u + 1) P_{321}^{(14)} \equiv K_1^+(u). \] (3.13)

The fused reflection matrix (3.13) satisfy the reflection equations

\[ R_{12}(-u - v)K_1^{-}(u) R_{21}(u + v) K_2^{-} (v) = K_2^{-} (v) R_{12}(u + v) K_1^{-} (u) R_{21}(u - v), \]
\[ R_{12}(-u + v)K_1^{+}(u) R_{21}(-u - v - 8) K_2^{+} (v) \]
\[ = K_2^{+} (v) R_{12}(-u - v - 8) K_1^{+} (u) R_{21}(-u + v), \]
\[ R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \]

\[ R_{12}(-u+v)K_1^+(u)R_{21}(-u-v-8)K_2^+(v) = K_2^+(v)R_{12}(-u-v-8)K_1^+(u)R_{21}(-u+v). \]  

(3.14)

Using the 6-dimensional projectors \( P_{12}^{(6)} \) and the correspondence (2.19), we have

\[ K_{(12)}^-(u+3) = \frac{P_{12}^{(6)}K_2^-(u+3)R_{12}(2u+\frac{5}{2})K_1^-(u-\frac{1}{2})P_{21}^{(6)}}{2(u+2)(u-\frac{1}{2})h_1(u+3)h_2(u+3)} \equiv K_1^-(u), \]

(3.15)

\[ K_{(12)}^+(u+3) = \frac{P_{21}^{(6)}K_1^+(u-\frac{1}{2})R_{21}(-2u-\frac{21}{2})K_2^+(u+3)P_{12}^{(6)}}{2(u+7)(u+\frac{9}{2})h_1(u+3)h_2(u+3)} \equiv K_1^+(u). \]

We note the fused reflection matrices (3.15) are the same as the original ones given by (3.3) and (3.4). Similarly, with the help of the 14-dimensional projectors \( P_{12}^{(14)} \) and the correspondence (2.29), we have

\[ K_{(12)}^-(u+\frac{5}{2}) = \frac{P_{12}^{(14)}K_2^-(u+\frac{5}{2})R_{12}(2u+2)K_1^-(u-\frac{1}{2})P_{21}^{(14)}}{2(u-\frac{1}{2})h_1(u+\frac{5}{2})h_2(u+\frac{5}{2})} \equiv K_1^-(u), \]

(3.16)

\[ K_{(12)}^+(u+\frac{5}{2}) = \frac{P_{21}^{(14)}K_1^+(u-\frac{1}{2})R_{21}(-2u-10)K_2^+(u+\frac{5}{2})P_{12}^{(14)}}{2(u+\frac{19}{2})h_1(u+\frac{5}{2})h_2(u+\frac{5}{2})} \equiv K_1^+(u). \]

We note that the fused reflection matrices (3.16) are the same as the fused ones (3.11). Now we have obtained all the necessary fused reflection matrices, which are used to construct the conserved quantities and fusion relations of the system with open boundary conditions.

### 3.3 Operator product identities

The fused reflecting monodromy matrices are defined as

\[ \hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1), \]

\[ \hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1). \]  

(3.17)

where \( R_{21}(u) \) and \( R_{21}(u) \) can be obtained from the first relations in (2.15) and (2.25), respectively. The fused reflecting monodromy matrices satisfy the Yang-Baxter relations

\[ R_{12}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u-v), \]

\[ R_{12}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u-v), \]

where the R matrices are defined in (2.15) and (2.25).
Meanwhile, the products of reflecting monodromy matrices at two special points satisfy
\[ \hat{T}_1(-\theta)\hat{T}_2(-\theta - 4) = P_{12}^{(1)} \hat{T}_1(-\theta)\hat{T}_2(-\theta - 4), \]
\[ \hat{T}_1(-\theta)\hat{T}_2(-\theta - 1) = P_{12}^{(14)} \hat{T}_1(-\theta)\hat{T}_2(-\theta - 1), \]
\[ \hat{T}_1(-\theta)\hat{T}_{(23)}(-\theta - 1) = P_{123}^{(14)} \hat{T}_1(-\theta)\hat{T}_{(23)}(-\theta - 1), \]

The fused transfer matrices are the partial traces of fused monodromy matrices
\[ t_2(u) = tr_\theta \{ K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u) \}, \]
\[ t_3(u) = tr_\theta \{ K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u) \}, \]
where the fused reflection matrices \( K_0^\pm(u) \) and \( K_2^\pm(u) \) are given by (3.11) and (3.13), respectively. From the Yang-Baxter relations (2.35), (3.18) and reflection equations (3.12), (3.14), one can prove that the transfer matrices \( t(u) \), \( t_2(u) \) and \( t_3(u) \) commute with each other
\[ [t(u), t_2(u)] = [t(u), t_3(u)] = [t_2(u), t_3(u)] = 0. \]
Thus these transfer matrices have common eigenstates and can be diagonalized simultaneously.

In order to solve these transfer matrices, we should seek the constraints they satisfied. The method is fusion. The fusions of reflecting monodromy matrices read
\[ P_{12}^{(3)} \hat{T}_1(u)\hat{T}_2(u - 4) P_{12}^{(1)} = \prod_{i=1}^{N} a(u + \theta_i)e(u + \theta_i - 4) P_{12}^{(1)} \times \text{id}, \]
\[ P_{12}^{(14)} \hat{T}_1(u)\hat{T}_2(u - 1) P_{12}^{(14)} = \hat{T}_{(12)}(u) = \prod_{i=1}^{N} \hat{\rho}_0(u + \theta_i)\hat{T}_1(u - \frac{1}{2}), \]
\[ P_{123}^{(14)} \hat{T}_1(u)\hat{T}_2(u - 1)\hat{T}_3(u - 2) P_{123}^{(14)} = \prod_{i=1}^{N} \hat{\rho}_0(u + \theta_i)\hat{\rho}_0(u + \theta_i - 1)(u + \theta_i + 1)\hat{T}_1(u - 1), \]
\[ P_{21}^{(6)} \hat{T}_2(u)\hat{T}_1(u - \frac{7}{2}) P_{21}^{(6)} = \prod_{i=1}^{N} \hat{\rho}_0(u + \theta_i)\hat{T}_1(u - 3), \]
\[ P_{21}^{(14)} \hat{T}_2(u)\hat{T}_1(u - 3) P_{21}^{(14)} = \prod_{i=1}^{N} (u + \theta_i + 4)\hat{T}_1(u - \frac{5}{2}). \]

Meanwhile, the products of reflecting monodromy matrices at two special points satisfy
\[ R_{11}(u - v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u - v). \]
Then, we are ready to consider the constraints of transfer matrices. Direct calculation shows

\[
t_a(u)t_b(u + \delta) = tr_a\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)\} \\
times tr_b\{K_b^+(u + \delta)T_b(u + \delta)K_b^-(u + \delta)\hat{T}_b(u + \delta)\}^t \\
= tr_{ab}\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)[T_b(u + \delta)K_b^+(u + \delta)\hat{T}_b(u + \delta)]^t\} \\
= [\tilde{\rho}_{ab}(2u + \delta)]^{-1}tr_{ab}\{[K_b^+(u + \delta)R_{ab}(2u + \delta)K_a^-(u)\hat{T}_a(u)]T_b(u + \delta)K_b^-(u + \delta)\hat{T}_b(u + \delta)]^t\} \\
= [\tilde{\rho}_{ab}(2u + \delta)]^{-1}tr_{ab}\{[K_b^+(u + \delta)R_{ab}(2u + \delta)K_a^-(u)\hat{T}_a(u)]T_b(u + \delta)K_b^-(u + \delta)\hat{T}_b(u + \delta)]^t\} \\
= [\tilde{\rho}_{ab}(2u + \delta)]^{-1}tr_{ab}\{[K_b^+(u + \delta)R_{ab}(2u + \delta)K_a^-(u)\hat{T}_a(u)]T_b(u + \delta)K_b^-(u + \delta)\hat{T}_b(u + \delta)]^t\} \\
= tr_{ab}\{A_{ab}^tB_{ab}^t\} = tr_{ab}\{A_{ab}^tB_{ab}\} = tr_{ab}\{A_{ab}B_{ab}\}, \\
\hat{T}_a(u)R_{ba}(2u + \delta)T_b(u + \delta) = T_b(u + \delta)R_{ba}(2u + \delta)\hat{T}_a(u), \\
R_{ba}^t(2u + \delta)R_{ab}^t(-2u - 8 - \delta) = \tilde{\rho}_{ab}(2u + \delta).
\]

In the derivation, we have used the relations

The following relation also holds

\[
t_1(u)t_{(23)}(u - 1) = t_1(u)tr_{23}\{P_{23}^{(14)}K_3^+(u - 2)R_{23}(-2u - 5)K_2^+(u - 1) \\
\times T_2(u - 1)T_3(u - 2)K_2^-(u - 1)R_{32}(2u - 3)K_3^-(u - 2)\hat{T}_2(u - 1)\hat{T}_3(u - 2)P_{23}^{(14)}\} \\
= \tilde{\rho}_v(2u - 1)\tilde{\rho}_v(2u - 2)tr_{123}\{K_1^+(u - 2)R_{23}(-2u - 5)K_2^+(u - 1)R_{13}(-2u - 6) \\
\times R_{12}(-2u - 7)K_1^+(u)T_1(u)T_{(23)}(u - 1)K_1^-(u)R_{23}(2u - 1)R_{31}(2u - 2)
\]
\[ \times K_2^-(u - 1) R_{32}(2u - 3) K_3^-(u - 2) \hat{T}_1(u) \hat{T}_{(23)}(u - 1) \].

(3.24)

Substituting \( u = \pm \theta_j, \delta = \{-4, -1, -7/2, -3\} \) into Eqs. (3.23) and (3.24), and using the relations (2.38), (2.41), (3.21), (3.22) and the forms of reflection matrices, we obtain the closed operator product identities among fused transfer matrices

\[
\begin{align*}
t(\pm \theta_j)t(\pm \theta_j - 4) &= \frac{1}{24} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j - 4)(\pm \theta_j + 4)}{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j + 2)(\pm \theta_j + 2)} \\
&\times H_1(\pm \theta_j) H_2(\pm \theta_j) \rho(\pm \theta_j), \\
t(\pm \theta_j)t(\pm \theta_j - 1) &= \frac{1}{22} \frac{(\pm \theta_j - 1)(\pm \theta_j + \frac{3}{2})(\pm \theta_j + 4)}{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j + 2)(\pm \theta_j + 2)} \\
&\times \rho(\pm \theta_j) \rho(\pm \theta_j), \\
t(\pm \theta_j)t_2(\pm \theta_j - \frac{3}{2}) &= \frac{1}{24} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + 1)(\pm \theta_j + 4)}{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j + 2)(\pm \theta_j + 2)} \\
&\times \rho(\pm \theta_j) \prod_{i=1}^{N} (\pm \theta_j - \theta_i + 1)(\pm \theta_j + \theta_i + 1)t_3(\pm \theta_j - 1), \\
t(\pm \theta_j)t_2(\pm \theta_j - \frac{7}{2}) &= \frac{1}{22} \frac{(\pm \theta_j - 1)(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + 4)}{(\pm \theta_j - 1)(\pm \theta_j - \frac{3}{2})(\pm \theta_j + 1)(\pm \theta_j + 2)} \\
&\times H_1(\pm \theta_j) H_2(\pm \theta_j) \rho(\pm \theta_j) t(\pm \theta_j - 3), \\
t(\pm \theta_j)t_3(\pm \theta_j - 3) &= \frac{(\pm \theta_j - 3)(\pm \theta_j + 4)}{(\pm \theta_j - 1)(\pm \theta_j + 2)} \prod_{i=1}^{N} (\pm \theta_j - \theta_i + 4)(\pm \theta_j + \theta_i + 4) \\
&\times H_1(\pm \theta_j) H_2(\pm \theta_j) t_2(\pm \theta_j - \frac{5}{2}), \quad j = 1, \cdots, N, \\
\end{align*}
\]

(3.25)

where

\[ H_1(u) = h_1(u) \tilde{h}_1(u), \quad H_2(u) = h_2(u) \tilde{h}_2(u), \quad \rho(u) = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) \rho_0(u + \theta_i). \]

The asymptotic of transfer matrices can be derived directly

\[
\begin{align*}
t(u)_{|u \to \pm \infty} &= -3(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)u^{4N+2} \times \text{id} + \cdots, \\
t_2(u)_{|u \to \pm \infty} &= 2^2[3(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)^2 + 2(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)]u^{4N+4} \times \text{id} + \cdots, \\
t_3(u)_{|u \to \pm \infty} &= -2^6(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)[(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)^2 + 3(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)]u^{2N+6} \times \text{id} + \cdots.
\end{align*}
\]

(3.26)
According to the definitions, we also know

\[ t(0) = 6\zeta \tilde{c} \prod_{l=1}^{N} \rho_l(-\theta_l) \times \text{id}, \quad t(-4) = 6\zeta \prod_{l=1}^{N} \rho_l(-\theta_l) \times \text{id}, \]

\[ t_2(0) = \frac{7}{2}(1 + c_1 c_2 - 4\zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \prod_{l=1}^{N} \rho_l(-\theta_l) \times \text{id}, \]

\[ t_2(-4) = \frac{7}{2}(1 + c_1 c_2 - 4\zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \prod_{l=1}^{N} \rho_l(-\theta_l) \times \text{id}, \]

\[ t_2(-\frac{1}{2}) = \frac{28}{3} \zeta \tilde{c} t(-1), \quad t_2(-\frac{7}{2}) = \frac{28}{3} \zeta \tilde{c} t(-3), \]

\[ t_3(0) = 2^7 \cdot 7\zeta \tilde{c}(1 + c_1 c_2 - 4\zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \prod_{l=1}^{N} \rho_l(-\theta_l) \times \text{id}, \]

\[ t_3(-4) = 2^7 \cdot 7\zeta \tilde{c}(1 + c_1 c_2 - 4\zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \prod_{l=1}^{N} \rho_l(-\theta_l) \times \text{id}, \]

\[ t_3(-1) = \frac{16\zeta \tilde{c}}{\prod_{l=1}^{N}(1 - \theta_l)(1 + \theta_l)} t_2(-\frac{3}{2}), \quad t_3(-3) = \frac{16\zeta \tilde{c}}{\prod_{l=1}^{N}(1 - \theta_l)(1 + \theta_l)} t_2(-\frac{5}{2}), \]

\[ t_3(-\frac{1}{2}) = -\frac{28(1 + c_1 c_2 - 4\zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2)}{\prod_{l=1}^{N}(\frac{3}{2} - \theta_l)(\frac{3}{2} + \theta_l)} t(-\frac{3}{2}), \]

\[ t_3(-\frac{7}{2}) = -\frac{28(1 + c_1 c_2 - 4\zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2)}{\prod_{l=1}^{N}(\frac{3}{2} - \theta_l)(\frac{3}{2} + \theta_l)} t(-\frac{5}{2}). \]

(3.27)

In the derivation, we have used the relations

\[ tr\{K^+(0)\} = 6\tilde{c}, \quad K^-(0) = \zeta \times \text{id}, \quad tr\{K^-(4)\} = 6\zeta, \quad K^+(4) = \tilde{\zeta} \times \text{id}, \]

\[ tr\{K_1^+(0)\} = 7(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2), \quad K_1^-(0) = \frac{1}{2}(1 + c_1 c_2 - 4\zeta^2) \times \text{id}, \]

\[ tr\{K_1^-(4)\} = 7(1 + c_1 c_2 - 4\zeta^2), \quad K_1^+(4) = \frac{1}{2}(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \times \text{id}, \]

\[ tr\{K_1^+(0)\} = 2^4 \cdot 7\zeta(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2), \quad K_1^-(0) = 8\zeta(1 + c_1 c_2 - 4\zeta^2) \times \text{id}, \]

\[ tr\{K_1^-(4)\} = 2^4 \cdot 7\zeta(1 + c_1 c_2 - 4\zeta^2), \quad K_1^+(4) = 8\zeta(1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \times \text{id}, \]

\[ tr_1\{R_{12}(-1)K_1^+(0)R_{21}(-7)\} = -2^4 \cdot 3 \cdot 7\zeta \times \text{id}, \]

\[ tr_2\{R_{21}(-7)K_2^-(4)R_{12}(-1)\}^{t_1 t_2} = -2^4 \cdot 3 \cdot 7\zeta \times \text{id}, \]

\[ tr_1\{R_{12}(-1)R_{13}(-2)K_1^+(0)R_{31}(-6)R_{21}(-7)\} = 2^6 \cdot 3^3 \cdot 7\zeta \times \text{id}, \]

\[ tr_3\{R_{31}(-6)R_{32}(-7)K_3^-(4)R_{23}(-1)R_{13}(-2)\}^{t_1 t_2 t_3} = 2^6 \cdot 3^3 \cdot 7\zeta \times \text{id}, \]

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\[ tr_{12}\{ R_{23}(-6)R_{13}(-7)K_2^+(-\frac{1}{2})R_{12}(-8)K_1^+(\frac{1}{2})P_{321}R_{32}(-1)R_{31}(-2)R_{21}(0) \} \]
\[ = 2^8 \cdot 3^4 \cdot 7^2 (1 + \tilde{c}_1 \tilde{c}_2 - 4\zeta^2) \times \text{id}, \]
\[ tr_{23}\{ R_{21}(-6)R_{31}(-7)K_2^-(\frac{7}{2})R_{32}(-8)K_1^-(\frac{9}{2})P_{123}R_{12}(-1)R_{13}(-2)R_{25}(0) \}^{t_1t_2t_3} \]
\[ = 2^8 \cdot 3^4 \cdot 7^2 (1 + c_1 c_2 - 4\zeta^2) \times \text{id}, \]
\[ K^{-}(\frac{1}{2})K^{-}(\frac{1}{2}) = \frac{1}{4} (1 + c_1 c_2 - 4\zeta^2) \times \text{id}, \quad K^{+}(\frac{7}{2})K^{+}(\frac{9}{2}) = \frac{1}{4} (1 + \tilde{c}_1 \tilde{c}_2 - 4\tilde{\zeta}^2) \times \text{id}. \]

From the construction of transfer matrices, we know that \( t(u), t_2(u) \) and \( t_3(u) \) are the operator polynomials of \( u \) with degrees \( 4N + 2, 4N + 4 \) and \( 2N + 6 \), respectively. Thus we need \( 10N + 15 \) independent conditions to determine their eigenvalues.

### 3.4 Functional relations

We have proved that the transfer matrices \( t(u), t_2(u) \) and \( t_3(u) \) have common eigenstates. Acting the transfer matrices on the common eigenstates, we obtain the corresponding eigenvalues. Denote the eigenvalues of \( t(u), t_2(u) \) and \( t_3(u) \) as \( \Lambda(u), \Lambda_2(u) \) and \( \Lambda_3(u) \), respectively. Acting the operators \( \{\Lambda_2, \Lambda_3\} \) on the common eigenstate, we obtain that these eigenvalues satisfy following closed functional relations

\[
\Lambda(\pm \theta_j)\Lambda(\pm \theta_j - 4) = \frac{1}{2^4} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j - 4)(\pm \theta_j + 4)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{2})(\pm \theta_j - 2)(\pm \theta_j + 2)} \times H_1(\pm \theta_j)H_2(\pm \theta_j)\varrho(\pm \theta_j)\varrho(\mp \theta_j),
\]
\[
\Lambda(\pm \theta_j)\Lambda(\pm \theta_j - 1) = \frac{1}{2^2} \frac{(\pm \theta_j - 1)(\pm \theta_j + \frac{3}{2})^2(\pm \theta_j + 4)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{2})(\pm \theta_j + 1)(\pm \theta_j + 2)} \times \varrho(\pm \theta_j)\Lambda_2(\pm \theta_j - \frac{1}{2}),
\]
\[
\Lambda(\pm \theta_j)\Lambda_2(\pm \theta_j - \frac{3}{2}) = \frac{1}{2^4} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + 1)(\pm \theta_j + 4)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{2})(\pm \theta_j + 2)(\pm \theta_j + 3)} \times \varrho(\pm \theta_j) \prod_{i=1}^{N}(\pm \theta_j - \theta_i + 1)(\pm \theta_j + \theta_i + 1)\Lambda_3(\pm \theta_j - 1),
\]
\[
\Lambda(\pm \theta_j)\Lambda_2(\pm \theta_j - \frac{7}{2}) = \frac{1}{2^2} \frac{(\pm \theta_j - 1)(\pm \theta_j - \frac{7}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + 4)}{(\pm \theta_j - \frac{3}{2})(\pm \theta_j - \frac{1}{2})(\pm \theta_j + 1)(\pm \theta_j + 2)} \times H_1(\pm \theta_j)H_2(\pm \theta_j)\varrho(\pm \theta_j)\Lambda(\pm \theta_j - 3),
\]
\[ \Lambda(\pm \theta_j) \Lambda_3(\pm \theta_j - 3) = \frac{(\pm \theta_j - 3)(\pm \theta_j + 4)}{(\pm \theta_j - 1)(\pm \theta_j + 2)} \prod_{i=1}^{N}(\pm \theta_j - \theta_i + 4)(\pm \theta_j + \theta_i + 4) \times H_1(\pm \theta_j)H_2(\pm \theta_j)\Lambda_2(\pm \theta_j - \frac{5}{2}), \quad j = 1, \cdots, N. \] (3.28)

The asymptotic behaviors (3.26) imply

\[ \Lambda(u)|_{u \to \pm \infty} = -3(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)u^{4N+2} \times \text{id} + \cdots, \]
\[ \Lambda_2(u)|_{u \to \pm \infty} = 2^2[3(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)^2 + 2(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)]u^{4N+4} \times \text{id} + \cdots, \]
\[ \Lambda_3(u)|_{u \to \pm \infty} = -2^6(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)[(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1)^2 + 3(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)]u^{2N+6} \times \text{id} + \cdots. \] (3.29)

Besides, from Eq. (3.27), we also have

\[ \Lambda(0) = 6 \zeta \tilde{\zeta} \prod_{l=1}^{N} \rho_1(-\theta_l), \quad \Lambda(-4) = 6 \zeta \tilde{\zeta} \prod_{l=1}^{N} \rho_1(-\theta_l), \]
\[ \Lambda_2(0) = \frac{7}{2}(1 + c_1 c_2 - 4 \zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4 \tilde{\zeta}^2) \prod_{l=1}^{N} \rho_1(-\theta_l), \]
\[ \Lambda_2(-4) = \frac{7}{2}(1 + c_1 c_2 - 4 \zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4 \tilde{\zeta}^2) \prod_{l=1}^{N} \rho_1(-\theta_l), \]
\[ \Lambda_2(-\frac{1}{2}) = \frac{28}{3} \zeta \tilde{\zeta} \Lambda(-1), \quad \Lambda_2(-\frac{7}{2}) = \frac{28}{3} \zeta \tilde{\zeta} \Lambda(-3), \]
\[ \Lambda_3(0) = 2^7 \cdot 7 \zeta \tilde{\zeta}(1 + c_1 c_2 - 4 \zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4 \tilde{\zeta}^2) \prod_{l=1}^{N} \rho_1(-\theta_l), \]
\[ \Lambda_3(-4) = 2^7 \cdot 7 \zeta \tilde{\zeta}(1 + c_1 c_2 - 4 \zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4 \tilde{\zeta}^2) \prod_{l=1}^{N} \rho_1(-\theta_l), \]
\[ \Lambda_3(-1) = \frac{16 \zeta \tilde{\zeta}}{\prod_{l=1}^{N}(1 - \theta_l)(1 + \theta_l)} \Lambda_2(-\frac{3}{2}), \quad \Lambda_3(-3) = \frac{16 \zeta \tilde{\zeta}}{\prod_{l=1}^{N}(1 - \theta_l)(1 + \theta_l)} \Lambda_2(-\frac{5}{2}), \]
\[ \Lambda_3(-\frac{1}{2}) = -\frac{28(1 + c_1 c_2 - 4 \zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4 \tilde{\zeta}^2)}{\prod_{l=1}^{N}(\frac{3}{2} - \theta_l)(\frac{3}{2} + \theta_l)} \Lambda(-\frac{3}{2}), \]
\[ \Lambda_3(-\frac{7}{2}) = -\frac{28(1 + c_1 c_2 - 4 \zeta^2)(1 + \tilde{c}_1 \tilde{c}_2 - 4 \tilde{\zeta}^2)}{\prod_{l=1}^{N}(\frac{3}{2} - \theta_l)(\frac{3}{2} + \theta_l)} \Lambda(-\frac{5}{2}). \] (3.30)

From above 10N functional relations (3.28), 3 asymptotic behaviors (3.29) and 12 constraints (3.30), we can completely determine the eigenvalues \( \Lambda(u) \), \( \Lambda_2(u) \) and \( \Lambda_3(u) \).
3.5 Inhomogeneous $T-Q$ relations

For the simplicity, we define some functions

\[
Z_1(u) = \frac{1}{2^2} \frac{(u + \frac{3}{2})(u + 4)}{(u + \frac{1}{2})(u + 2)} A(u) \frac{Q^{(1)}(u - 1)}{Q^{(1)}(u)} H_1(u),
\]

\[
Z_2(u) = \frac{1}{2^2} \frac{u(u + \frac{3}{2})(u + 4)}{(u + \frac{1}{2})(u + 1)(u + 2)} B(u) \frac{Q^{(1)}(u + 1)Q^{(2)}(u - 1)}{Q^{(1)}(u)Q^{(2)}(u)} H_2(u + 1),
\]

\[
Z_3(u) = \frac{1}{2^2} \frac{u(u + 4)}{B(u)} \frac{Q^{(2)}(u + 1)Q^{(3)}(u - \frac{3}{2})}{Q^{(2)}(u)Q^{(3)}(u + \frac{1}{2})} H_2(u + 1),
\]

\[
Z_4(u) = \frac{1}{2^2} \frac{u(u + 4)}{(u + 2)(u + 3)} B(u) \frac{Q^{(2)}(u + 1)Q^{(3)}(u + \frac{5}{2})}{Q^{(2)}(u + 2)Q^{(3)}(u + \frac{3}{2})} H_2(u + 3),
\]

\[
Z_5(u) = \frac{1}{2^2} \frac{u(u + \frac{5}{2})(u + 4)}{(u + 2)(u + 3)(u + \frac{5}{2})} B(u) \frac{Q^{(1)}(u + 2)Q^{(2)}(u + 3)}{Q^{(1)}(u + 3)Q^{(2)}(u + 2)} H_1(u + 3),
\]

\[
Z_6(u) = \frac{1}{2^2} \frac{u(u + \frac{5}{2})}{V(u)} \frac{Q^{(1)}(u + 4)}{Q^{(1)}(u + 3)} H_2(u + 4),
\]

\[
f_1(u) = \frac{1}{2^2} \frac{u(u + \frac{2}{2})(u + 4)}{u + 2} B(u) G(u + 1) \frac{Q^{(2)}(u - 1)}{Q^{(1)}(u)} x,\]

\[
f_2(u) = \frac{1}{2^2} u(u + 4) B(u) \frac{Q^{(2)}(u + 1)}{Q^{(3)}(u + \frac{1}{2})} x,
\]

\[
f_3(u) = \frac{1}{2^2} \frac{u(u + \frac{3}{2})(u + 4)}{u + 2} B(u) G(u + 3) \frac{Q^{(2)}(u + 3)}{Q^{(1)}(u + 3)} x,
\]


\[
\text{were } x = 8 \sqrt{1 + c_1 c_2 (1 + \tilde{c}_1 \tilde{c}_2)} - 4(2 + c_1 \tilde{c}_2 + c_2 \tilde{c}_1),
\]

\[
A(u) = \prod_{j=1}^{N} a(u - \theta_j) a(u + \theta_j), \quad B(u) = \prod_{j=1}^{N} b(u - \theta_j) b(u + \theta_j),
\]

\[
V(u) = \prod_{j=1}^{N} e(u - \theta_j) e(u + \theta_j), \quad G(u) = \prod_{j=1}^{N} (u - \theta_j)(u + \theta_j),
\]

\[
Q^{(m)}(u) = \prod_{k=1}^{L_m} (u - \lambda_k^{(m)} + \frac{m}{2})(u + \lambda_k^{(m)} + \frac{m}{2}), \quad m = 1, 2, 3,
\]

and the numbers of Bethe roots satisfy the constraints $L_1 = L_2 + N$ and $L_3 = L_2$. By using these functions, we construct the eigenvalues of transfer matrices as

\[
\Lambda(u) = \sum_{i=1}^{6} \tilde{Z}_i(u),
\]
\[
\Lambda_2(u) = 2^{-2}\left[(u - \frac{1}{2})(u + 2)^2(u + \frac{9}{2})\varrho(u + \frac{1}{2})\right]^{-1}\tilde{\rho}_v(2u) \\
\times \left\{ \sum_{i < j}^{6} \tilde{Z}_i(u + \frac{1}{2})\tilde{Z}_j(u - \frac{1}{2}) - \tilde{Z}_3(u + \frac{1}{2})\tilde{Z}_4(u - \frac{1}{2}) - f_1(u + \frac{1}{2})\tilde{Z}_2(u - \frac{1}{2}) - \tilde{Z}_5(u + \frac{1}{2})f_3(u - \frac{1}{2}) \right\}, \\
\Lambda_3(u) = 2^{-6}\left[(u + \frac{5}{2})^2(u + \frac{3}{2})^2(u - \frac{1}{2})(u + \frac{9}{2})u(u - 1)(u + 2)^2(u + 4)(u + 5) \right] \\
\times \varrho(u + 1)\varrho(u)\prod_{i=1}^{N}(u + \theta_i + 2)(u - \theta_i + 2)^{-1}\tilde{\rho}_v(2u + 1)\tilde{\rho}_v(2u)\tilde{\rho}_v(2u - 1) \\
\times \left\{ \sum_{i < j < k}^{6} \tilde{Z}_i(u + 1)\tilde{Z}_j(u)\tilde{Z}_k(u - 1) - \sum_{k=5}^{6} \tilde{Z}_3(u + 1)\tilde{Z}_4(u)\tilde{Z}_k(u - 1) - \sum_{i=1}^{2} \tilde{Z}_i(u + 1)\tilde{Z}_3(u)\tilde{Z}_4(u - 1) - \sum_{j=3}^{4} \tilde{Z}_2(u + 1)\tilde{Z}_j(u)\tilde{Z}_5(u - 1) - \sum_{j=3}^{6} f_1(u + 1)\tilde{Z}_2(u)\tilde{Z}_j(u - 1) - \sum_{i=1}^{4} \tilde{Z}_i(u + 1)\tilde{Z}_5(u)f_3(u - 1) \right\}, \\
(3.33)
\]

where
\[
\tilde{Z}_1(u) = Z_1(u) + f_1(u), \quad \tilde{Z}_2(u) = Z_2(u), \\
\tilde{Z}_3(u) = Z_3(u) + f_2(u), \quad \tilde{Z}_4(u) = Z_4(u), \\
\tilde{Z}_6(u) = Z_6(u) + f_3(u), \quad \tilde{Z}_5(u) = Z_5(u). \\
(3.34)
\]

All the eigenvalues are the polynomials, thus the residues of right hand sides of Eq. (3.33) should be zero, which gives the Bethe ansatz equations

\[
\begin{align*}
\lambda_k^{(1)} & = \frac{1}{\lambda_k^{(1)} - \frac{1}{2}} \prod_{j=1}^{N}(\lambda_k^{(1)} - \theta_j - \frac{1}{2})(\lambda_k^{(1)} + \theta_j - \frac{1}{2}) + \frac{Q^{(1)}(\lambda_k^{(1)} - \frac{3}{2})}{Q^{(1)}(\lambda_k^{(1)} + \frac{1}{2})} \\
\lambda_k^{(2)} & = \frac{1}{\lambda_k^{(2)} - \frac{1}{2}} \prod_{j=1}^{N}(\lambda_k^{(2)} - \theta_j + \frac{1}{2})(\lambda_k^{(2)} + \theta_j + \frac{1}{2}) + \frac{Q^{(2)}(\lambda_k^{(2)} + \frac{1}{2})}{Q^{(2)}(\lambda_k^{(2)} - \frac{1}{2})} \\
& = -x, \\
k & = 1, 2, \cdots, L_1, \\
Q^{(1)}(\lambda_l^{(2)})Q^{(2)}(\lambda_l^{(2)} - 3) & = \frac{Q^{(1)}(\lambda_l^{(2)} - 1)Q^{(2)}(\lambda_l^{(2)})Q^{(3)}(\lambda_l^{(2)} - \frac{5}{2})}{Q^{(1)}(\lambda_l^{(2)} - \frac{1}{2})Q^{(2)}(\lambda_l^{(2)} - \frac{3}{2})Q^{(3)}(\lambda_l^{(2)} - \frac{3}{2})} h_1(\lambda_l^{(2)})h_2(\lambda_l^{(2)})h_3(\lambda_l^{(2)}) = -\frac{\lambda_l^{(2)} - \frac{1}{2}}{\lambda_l^{(2)} + \frac{1}{2}}, \\
l & = 1, 2, \cdots, L_2,
\end{align*}
\]
\[
\begin{aligned}
&\frac{h_1(\lambda_m^{(3)} - 1)\tilde{h}_1(\lambda_m^{(3)} - 1) Q^{(3)}(\lambda_m^{(3)} - \frac{7}{2})}{\lambda_m^{(3)}(\lambda_m^{(3)} - 1) Q^{(2)}(\lambda_m^{(3)} - 2)} \\
&\quad + \frac{h_2(\lambda_m^{(3)} + 1)\tilde{h}_2(\lambda_m^{(3)} + 1) Q^{(3)}(\lambda_m^{(3)} + \frac{1}{2})}{\lambda_m^{(3)}(\lambda_m^{(3)} + 1) Q^{(2)}(\lambda_m^{(3)})} = -x, \quad m = 1, 2, \ldots, L. \quad (3.35)
\end{aligned}
\]

We note that from the regularity analysis of any \(\Lambda(u), \Lambda_2(u)\) or \(\Lambda_3(u)\), one can obtain the complete set of Bethe ansatz equations. The Bethe ansatz equations obtained from \(\Lambda(u)\) are the same as those obtained from \(\Lambda_2(u)\) and \(\Lambda_3(u)\). Meanwhile, the function \(Q^{(m)}(u)\) has two zero points, namely, \(\lambda_k^{(m)} - \frac{m}{4}\) and \(-\lambda_k^{(m)} - \frac{m}{4}\). These two zero points should give the same Bethe ansatz equations.

We have checked that the eigenvalues \(\Lambda(u), \Lambda_2(u)\) and \(\Lambda_3(u)\) given by (3.33) satisfy the closed fusion relations (3.28), asymptotic behaviors (3.29) and constraints (3.30). Therefore, we conclude that the eigenvalues constructed by the inhomogeneous \(T - Q\) relations are indeed the eigenvalues of transfer matrices, provided that the Bethe roots satisfy Bethe ansatz equations (3.35). The eigenvalue of Hamiltonian (3.8) can be expressed in terms of the Bethe roots as

\[
E = \left. \frac{\partial \ln \Lambda(u)}{\partial u} \right|_{u=0,\{\theta_j\}=0} . \quad (3.36)
\]

If \(c_1 = c_2 = \tilde{c}_1 = \tilde{c}_2 = 0\), the boundary reflection matrices degenerate into the diagonal ones and our results cover that obtained by the algebraic Bethe ansatz [42].

### 4 \(C_n\) model

In this section, we generalize above results to the \(C_n\) model. The \(C_n\) model with periodic boundary condition has been studied in reference [22]. Thus we focus on the open boundary conditions. The main idea is the same as before. Here we only list the results. The \(R\)-matrix of the \(C_n\) model is a \((2n)^2 \times (2n)^2\) one with the elements

\[
\tilde{R}^{ij}(u)_{kl} = u(u + n + 1)\delta_{ik}\delta_{jl} + (u + n + 1)\delta_{il}\delta_{jk} - u\xi_i\xi_k\delta_{ji}\delta_{kl}, \quad (4.1)
\]

where \(i, j, k, l = 1, \ldots, 2n, i + \tilde{i} = 2n + 1, \xi_i = 1\) if \(i \in [1, n]\) and \(\xi_i = -1\) if \(i \in [n + 1, 2n]\). The off-diagonal boundary reflection matrices is chosen as

\[
\tilde{K}_0^{-}(u) = \tilde{\zeta} + M_0 u, \quad M_0 = \begin{pmatrix} -1 & \tilde{c}_1 \\ \tilde{c}_2 & 1 \end{pmatrix} \otimes I, \quad (4.2)
\]
where $\zeta$, $\tilde{c}_1$ and $\tilde{c}_2$ are the free boundary parameters and $I$ is a $n \times n$ unitary matrix. The dual reflection matrix $\tilde{K}_0^+(u)$ is determined by the mapping

$$\tilde{K}_0^+(u) = \tilde{K}_0^-(u - n - 1)|_{\tilde{c}_i \rightarrow \tilde{c}_i}, \quad (4.3)$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are the boundary parameters.

From $R$-matrix (4.1) and reflection matrices (4.2)-(4.3), the transfer matrix of $C_n$ model is constructed as

$$\tilde{t}(u) = tr_0\{\tilde{K}_0^+(u)\tilde{T}_0(u)\tilde{K}_0^-(u)\tilde{T}_0(u)\}, \quad (4.4)$$

where

$$\tilde{T}_0(u) = \tilde{R}_{01}(u - \theta_1)\tilde{R}_{02}(u - \theta_2)\cdots\tilde{R}_{0N}(u - \theta_N),$$
$$\tilde{T}_0(u) = \tilde{R}_{N0}(u + \theta_N)\cdots\tilde{R}_{20}(u + \theta_2)\tilde{R}_{10}(u + \theta_1). \quad (4.5)$$

The transfer matrix (4.4) is the generating function of all the conserved quantities including the model Hamiltonian. The eigenvalues of the transfer matrix (4.4) read

$$\tilde{\Lambda}(u) = \sum_{l=1}^{2n} \tilde{Z}_l(u) + \sum_{j=1}^{n} \tilde{f}_j(u). \quad (4.6)$$

Here the functions $\tilde{Z}_l(u)$ are defined as

$$\tilde{Z}_1(u) = \frac{1}{2^2} (u + \frac{n}{2})(u + n + 1) \tilde{A}(u) \frac{Q^{(1)}(u - 1)}{Q^{(1)}(u)} \tilde{H}_1(u),$$

$$\tilde{Z}_{2n}(u) = \frac{1}{2^2} (u + \frac{n}{2})(u + n + 1) \tilde{V}(u) \frac{Q^{(1)}(u + n + 1)}{Q^{(1)}(u + n)} \tilde{H}_{2n}(u + n + 1),$$

$$\tilde{Z}_l(u) = \frac{1}{2^2} (u + \frac{n}{2})(u + n + 1) \frac{u + \frac{n}{2}}{(u + l - \frac{1}{2})(u + l - 1)(u + \frac{n}{2})} \tilde{B}(u) \frac{Q^{(l-1)}(u + 1)Q^{(l)}(u - 1)}{Q^{(l-1)}(u)Q^{(l)}(u)} \tilde{H}_l(u),$$

$$\tilde{Z}_{2n-l+1}(u) = \frac{1}{2^2} (u + \frac{n}{2} + 1)(u + n + 1) \frac{u + \frac{n}{2}}{(u + n - l + \frac{1}{2})(u + n - l + 1)(u + \frac{n}{2})} \tilde{B}(u) \frac{Q^{(l-1)}(u + n - l + 1)Q^{(l)}(u + n - l + 2)}{Q^{(l-1)}(u + n - l + 2)Q^{(l)}(u + n - l + 1)} \tilde{H}_{2n-l+1}(u), \quad l = 2, 3, \ldots, n - 1,$$

$$\tilde{Z}_n(u) = \frac{1}{2^2} (u + \frac{n}{2})(u + n + 1) \frac{u + \frac{n}{2}}{(u + \frac{n+1}{2})(u + \frac{n-1}{2})} \tilde{B}(u) \frac{Q^{(n-1)}(u + 1)Q^{(n)}(u - \frac{3}{2})}{Q^{(n-1)}(u)Q^{(n)}(u + \frac{1}{2})} \tilde{H}_n(u),$$

$$\tilde{Z}_{n+1}(u) = \frac{1}{2^2} (u + \frac{n+1}{2})(u + \frac{n+2}{2}) \frac{u + \frac{n}{2}}{(u + \frac{n+1}{2})(u + \frac{n+2}{2})} \tilde{B}(u) \frac{Q^{(n-1)}(u + 1)Q^{(n)}(u + \frac{5}{2})}{Q^{(n-1)}(u + 2)Q^{(n)}(u + \frac{1}{2})} \tilde{H}_{n+1}(u). \quad (4.7)$$
where
\[ A(u) = \prod_{j=1}^{N} (u - \theta_j + 1)(u - \theta_j + n + 1)(u + \theta_j + 1)(u + \theta_j + n + 1), \]
\[ \bar{B}(u) = \prod_{j=1}^{N} (u - \theta_j)(u - \theta_j + n + 1)(u + \theta_j)(u + \theta_j + n + 1), \]
\[ \bar{V}(u) = \prod_{j=1}^{N} (u - \theta_j)(u - \theta_j + n)(u + \theta_j)(u + \theta_j + n), \]

\[ \bar{Q}^{(m)}(u) = \prod_{k=1}^{\bar{L}_m}(u - \lambda_k^{(m)} + \frac{m}{2})(u + \lambda_k^{(m)} + \frac{m}{2}), \quad m = 1, 2, \ldots, n, \]
\[ \bar{H}_1(u) = \begin{cases} \bar{h}_1(u + \frac{l-1}{2}), & l \text{ odd in } [1, n], \\ \bar{h}_2(u + \frac{l}{2}), & l \text{ even in } [1, n], \end{cases} \]
\[ \bar{H}_{2n-l+1}(u) = \begin{cases} \bar{h}_2(u + n + 1 - \frac{l-1}{2}), & l \text{ odd in } [1, n], \\ \bar{h}_1(u + n + 1 - \frac{l}{2}), & l \text{ even in } [1, n], \end{cases} \]
\[ \bar{h}_1(u) = -4(\sqrt{(1 + \tilde{c}_1\tilde{c}_2)u + \tilde{\zeta}})(\sqrt{1 + \tilde{c}_1\tilde{c}_2u + \tilde{\zeta}}), \]
\[ \bar{h}_2(u) = -4(\sqrt{(1 + \tilde{c}_1\tilde{c}_2)u - \tilde{\zeta}})(\sqrt{1 + \tilde{c}_1\tilde{c}_2u - \tilde{\zeta}}), \quad (4.8) \]

and the numbers of Bethe roots satisfy the constraints
\[ \bar{L}_1 = \bar{L}_2 + N, \quad \bar{L}_{2l-1} = \bar{L}_{2l-2} + \bar{L}_{2l}, \quad \bar{L}_n = \bar{L}_{n-1}, \quad l = 2, 3, \ldots, \frac{n-1}{2}, \quad (4.9) \]
if \( n \) is odd and the constraints
\[ \bar{L}_1 = \bar{L}_2 + N, \quad \bar{L}_{2l-1} = \bar{L}_{2l-2} + \bar{L}_{2l}, \quad \bar{L}_{n-1} = \bar{L}_{n-2} + 2\bar{L}_n, \quad l = 2, 3, \ldots, \frac{n-2}{2}, \quad (4.10) \]
if \( n \) is even. The inhomogeneous terms \( \bar{f}_i(u) \) with odd \( n \) are also different from that with even \( n \). If \( n \) is odd, we have
\[ \bar{f}_1(u) = \frac{1}{2^2} \frac{u(u + \frac{n}{2})(u + n + 1)}{u + \frac{n+1}{2}} \bar{B}(u) \bar{G}(u + 1) \bar{Q}^{(2)}(u - 1) \bar{Q}^{(1)}(u)^{-\bar{x}}, \]
\[ \bar{f}_n(u) = \frac{1}{2^2} \frac{u(u + \frac{n}{2} + 1)(u + n + 1)}{u + \frac{n+1}{2}} \bar{B}(u) \bar{G}(u + n) \bar{Q}^{(2)}(u + n) \bar{Q}^{(1)}(u + n)^{-\bar{x}}, \]
\[ \bar{f}_l(u) = \frac{1}{2^2} \frac{u(u + \frac{n}{2})(u + n + 1)}{u + \frac{n+1}{2}} \bar{B}(u) \bar{Q}^{(2l-2)}(u + 1) \bar{Q}^{(2l)}(u - 1) \bar{Q}^{(2l-1)}(u)^{-\bar{x}}, \]
\[ \bar{f}_{n-l+1}(u) = \frac{1}{2^2} \frac{u(u + \frac{n}{2} + 1)(u + n + 1)}{u + \frac{n+1}{2}} \bar{B}(u) \bar{Q}^{(2l)}(u - 1) \bar{Q}^{(2l-1)}(u)^{-\bar{x}}. \]
We note that if \( n \) equations are Bethe ansatz equations, which also depend on the parity of \( n \).

From the singularities analysis of inhomogeneous \( T - Q \) relations, we obtain the Bethe ansatz equations, which also depend on the parity of \( n \). If \( n \) is odd, the Bethe ansatz equations are

\[
\frac{\bar{h}_1(\lambda_k^{(i)} - \frac{1}{2})}{\lambda_k^{(i)}(\lambda_k^{(i)} - \frac{1}{2})} \cdot \frac{\bar{h}_2(\lambda_k^{(i)} + \frac{1}{2})}{\lambda_k^{(i)}(\lambda_k^{(i)} + \frac{1}{2})} \cdot \frac{Q^{(i)}(\lambda_k^{(i)} + \frac{1}{2})}{Q^{(i)}(\lambda_k^{(i)} - \frac{1}{2})} = -\bar{x},
\]

\( k = 1, 2, \ldots, L_1, \ldots, L_l, \) \( l \in \text{odd in } [2, n - 2], \)

and

\[
\bar{f}_{u+1}(u) = \frac{1}{2^u} \bar{B}(u) \frac{Q^{(u-2)}(u+1)Q^{(u)}(u+\frac{3}{2})}{Q^{(u)}(u)} \bar{x},
\]

where \( \bar{x} = 8\sqrt{(1 + c_1\bar{c}_2)(1 + \bar{c}_1\bar{c}_2)} - 4(2 + c_1\bar{c}_2 + \bar{c}_2c_1) \) and

\[
\bar{G}(u) = \prod_{j=1}^N (u - \theta_j)(u + \theta_j).
\]

We note that if \( n = 2 \), the functions \( \bar{f}_1(u) \) and \( \bar{f}_2(u) \) are defined by Eq. (4.13) instead of (4.11) because of the present parametrization.

From the singularities analysis of inhomogeneous \( T - Q \) relations, we obtain the Bethe ansatz equations, which also depend on the parity of \( n \). If \( n \) is odd, the Bethe ansatz equations are

\[
\frac{\bar{h}_1(\lambda_k^{(i)} - \frac{1}{2})}{\lambda_k^{(i)}(\lambda_k^{(i)} - \frac{1}{2})} \cdot \frac{\bar{h}_2(\lambda_k^{(i)} + \frac{1}{2})}{\lambda_k^{(i)}(\lambda_k^{(i)} + \frac{1}{2})} \cdot \frac{Q^{(i)}(\lambda_k^{(i)} + \frac{1}{2})}{Q^{(i)}(\lambda_k^{(i)} - \frac{1}{2})} = -\bar{x},
\]

\( k = 1, 2, \ldots, L_1, \ldots, L_l, \) \( l \in \text{odd in } [2, n - 2], \)

\[
\bar{f}_{u+1}(u) = \frac{1}{2^u} \bar{B}(u) \frac{Q^{(u-2)}(u+1)Q^{(u)}(u+\frac{3}{2})}{Q^{(u)}(u)} \bar{x},
\]

where \( \bar{x} = 8\sqrt{(1 + c_1\bar{c}_2)(1 + \bar{c}_1\bar{c}_2)} - 4(2 + c_1\bar{c}_2 + \bar{c}_2c_1) \) and

\[
\bar{G}(u) = \prod_{j=1}^N (u - \theta_j)(u + \theta_j).
\]

We note that if \( n = 2 \), the functions \( \bar{f}_1(u) \) and \( \bar{f}_2(u) \) are defined by Eq. (4.13) instead of (4.11) because of the present parametrization.

From the singularities analysis of inhomogeneous \( T - Q \) relations, we obtain the Bethe ansatz equations, which also depend on the parity of \( n \). If \( n \) is odd, the Bethe ansatz equations are

\[
\frac{\bar{h}_1(\lambda_k^{(i)} - \frac{1}{2})}{\lambda_k^{(i)}(\lambda_k^{(i)} - \frac{1}{2})} \cdot \frac{\bar{h}_2(\lambda_k^{(i)} + \frac{1}{2})}{\lambda_k^{(i)}(\lambda_k^{(i)} + \frac{1}{2})} \cdot \frac{Q^{(i)}(\lambda_k^{(i)} + \frac{1}{2})}{Q^{(i)}(\lambda_k^{(i)} - \frac{1}{2})} = -\bar{x},
\]

\( k = 1, 2, \ldots, L_1, \ldots, L_l, \) \( l \in \text{odd in } [2, n - 2], \)

\[
\bar{f}_{u+1}(u) = \frac{1}{2^u} \bar{B}(u) \frac{Q^{(u-2)}(u+1)Q^{(u)}(u+\frac{3}{2})}{Q^{(u)}(u)} \bar{x},
\]

where \( \bar{x} = 8\sqrt{(1 + c_1\bar{c}_2)(1 + \bar{c}_1\bar{c}_2)} - 4(2 + c_1\bar{c}_2 + \bar{c}_2c_1) \) and

\[
\bar{G}(u) = \prod_{j=1}^N (u - \theta_j)(u + \theta_j).
\]
\[
\frac{Q^{(l-1)}(\lambda_k^{(l)} - \frac{l}{2} + 1)Q^{(l)}(\lambda_k^{(l)} - \frac{l}{2} - 1)Q^{(l+1)}(\lambda_k^{(l)} - \frac{l}{2})}{Q^{(l-1)}(\lambda_k^{(l)} - \frac{l}{2})Q^{(l)}(\lambda_k^{(l)} - \frac{l}{2} + 1)Q^{(l+1)}(\lambda_k^{(l)} - \frac{l}{2} - 1)\tilde{h}_1(\lambda_k^{(l)})} = -\frac{\lambda_k^{(l)} - \frac{l}{2}}{\lambda_k^{(l)} + \frac{l}{2}},
\]
\[
k = 1, 2, \ldots, L_l, \quad l \in \text{even in } [2, n-2], \tag{4.15}
\]

and the rest two read
\[
\frac{Q^{(n-2)}(\lambda_k^{(n-1)} - \frac{n-1}{2} + 1)Q^{(n-1)}(\lambda_k^{(n-1)} - \frac{n-1}{2} - 1)Q^{(n)}(\lambda_k^{(n-1)} - \frac{n-1}{2})}{Q^{(n-2)}(\lambda_k^{(n-1)} - \frac{n-1}{2})Q^{(n-1)}(\lambda_k^{(n-1)} - \frac{n-1}{2} + 1)Q^{(n)}(\lambda_k^{(n-1)} - \frac{n-1}{2} + \frac{3}{2})} \times \frac{\tilde{h}_1(\lambda_k^{(n-1)})}{\tilde{h}_1(\lambda_k^{(n-1)})} = -\frac{\lambda_k^{(n-1)} - \frac{1}{2}}{\lambda_k^{(n-1)} + \frac{1}{2}}, \quad k = 1, 2, \ldots, L_{n-1},
\]
\[
\frac{\tilde{h}_1(\lambda_k^{(n-1)} - 1)}{\lambda_k^{(n-1)}(\lambda_k^{(n-1)} - 1)} \frac{Q^{(n)}(\lambda_k^{(n)} - \frac{n+1}{2} + \frac{3}{2})}{Q^{(n-1)}(\lambda_k^{(n)} - \frac{n+1}{2} - \frac{3}{2})} + \frac{\tilde{h}_2(\lambda_k^{(n)} + 1)}{\lambda_k^{(n)}(\lambda_k^{(n)} + 1)} \frac{Q^{(n)}(\lambda_k^{(n)} - \frac{n+1}{2} + \frac{5}{2})}{Q^{(n-1)}(\lambda_k^{(n)} - \frac{n+1}{2} + 2)} = -\bar{x},
\]
\[
k = 1, 2, \ldots, L_n. \tag{4.16}
\]

If \( n \) is even, besides (4.15), the rest two are
\[
\frac{\tilde{h}_1(\lambda_k^{(n-1)} - \frac{1}{2})}{\lambda_k^{(n-1)}(\lambda_k^{(n-1)} - \frac{1}{2})} \frac{Q^{(n-1)}(\lambda_k^{(n-1)} - \frac{n-1}{2} - 1)}{Q^{(n-2)}(\lambda_k^{(n-1)} - \frac{n-1}{2} + 1)Q^{(n)}(\lambda_k^{(n-1)} - \frac{n-1}{2} + \frac{3}{2})} + \frac{\tilde{h}_2(\lambda_k^{(n)} + \frac{1}{2})}{\lambda_k^{(n)}(\lambda_k^{(n)} + \frac{1}{2})} \frac{Q^{(n)}(\lambda_k^{(n)} - \frac{n+1}{2} + \frac{3}{2})}{Q^{(n-1)}(\lambda_k^{(n)} - \frac{n+1}{2} + 2)} = -\bar{x}Q^{(n)}(\lambda_k^{(n-1)} - \frac{n-1}{2} - \frac{1}{2}), \quad k = 1, 2, \ldots, L_{n-1}, \tag{4.17}
\]
\[
\frac{\tilde{h}_2(\lambda_k^{(n)} - \frac{1}{2})}{\lambda_k^{(n)}(\lambda_k^{(n)} - \frac{1}{2})} \frac{Q^{(n)}(\lambda_k^{(n)} - \frac{n+1}{2} - \frac{3}{2})}{Q^{(n-1)}(\lambda_k^{(n)} - \frac{n+1}{2})} + \frac{\tilde{h}_1(\lambda_k^{(n)} + \frac{1}{2})}{\lambda_k^{(n)}(\lambda_k^{(n)} + \frac{1}{2})} \frac{Q^{(n)}(\lambda_k^{(n)} - \frac{n+1}{2} + \frac{5}{2})}{Q^{(n-1)}(\lambda_k^{(n)} - \frac{n+1}{2} + 2)} = 0,
\]
\[
k = 1, 2, \ldots, L_n. \tag{4.18}
\]

We note that if \( n = 2 \), the Bethe roots are determined by Eq.(4.18) due to the parametrization we used.

5 Discussion

In this paper, we study the exact solutions of the \( C_n \) vertex model with either the periodic or the open boundary conditions corresponding to the \( K \)-matrices \((4.12)-(4.13)\) by using fusion and the nested off-diagonal Bethe ansatz. Taking the \( C_3 \) model as an example, we obtain its fusion structures and provide a way to close the recursive operator product identities among the transfer matrices. Based on them and some necessary additional information such as
the asymptotic behaviors and the relations at some special points, we obtain the eigenvalues \((3.33)\) of the system and give the associated Bethe ansatz equations \((3.35)\). Moreover, we also generalize these results \((4.6)-(4.18)\) to the \(C_n\) model with off-diagonal boundary reflections \((4.2)-(4.3)\). The method and results given in this paper can be generalized to other high rank quantum integrable systems.

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