Analyzing collisions in classical mechanics using mass-momentum diagrams

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Abstract

We show the value of mass-momentum diagrams for analyzing collision problems in classical mechanics in one dimension. Collisions are characterized by the coefficient of restitution and the momentum of the interacting particles both before and after the collision. All those quantities are presented in the mass-momentum diagrams without the need to do any calculations. We also show that the same diagrams can be used to investigate collisions with respect to the center-of-mass system. In this case, also, we do not need to do any calculations to obtain the momentum. Since we give an alternative way of looking at collisions in classical mechanics, this article is aimed at undergraduate-level students.
1. **Introduction**

Collisions between interacting particles are fundamentally important in physics, not only Newtonian mechanics but also relativistic mechanics. In many classroom problems concerning collisions, the masses, the velocities before collision and the coefficient of restitution of the colliding particles are given. The students are required to find the velocities of the particles after the collision. For relativistic mechanics, collisions are well described in an intuitive and instructive way using Minkowsky diagrams in momentum space [1, 2, 3].

In this article, we use this intuitive and instructive way for collisions in Newtonian mechanics. For this purpose, we introduce a mass-momentum diagram [4] with mass $m$ along the vertical axis and momentum $p$ represented on the horizontal axis. In this diagram, a point $(m, p)$ expresses the state of a particle. This diagram presents us with all the information we need about the collision without any calculations.

Moreover, the center-of-mass system is also used in collision problems. Using this makes equations simpler and calculations easier. Even in this case, we can obtain the momenta of the colliding particles from the mass-momentum diagrams without any calculations.

This paper is organized in the following way. In Section 2, we recall one dimensional collisions with calculations. In Section 3, we introduce the mass-momentum diagrams and deduce all the equations from the diagrams without requiring any calculations. Next, we move to the center-of-mass system. We obtain all the quantities given in the center-of-mass system from the same mass-momentum diagram. In Section 4, we apply this diagram to the laboratory system. Section 5 is devoted to a summary.

2. **Collisions between particles in one dimension with calculations**

Let us recall here the equations for one dimensional collisions. We consider that two particles, whose mass are $m_A$ and $m_B$, have velocities $v_A$ and $v_B (v_A > v_B)$ before the collision. We distinguish by primes the variables after the collisions.

We write down the conservation of momentum and the definition of the coefficient of restitution $e$,

$$
\begin{align*}
&m_A v_A + m_B v_B = m_A v'_A + m_B v'_B, \\
&e = \frac{v'_A - v'_B}{v_A - v_B},
\end{align*}
$$

where the coefficient of restitution $e$ takes the value from 0 to 1. $e = 0$ signifies a perfectly inelastic collision in which two particles are combined after collision, while $e = 1$ a perfectly elastic collision in which the total energy is conserved. $0 < e < 1$ signifies an inelastic collision.
Eqs. (1) give the velocities after collision

\[
\begin{align*}
v'_A &= v_A - (1 + e) \frac{m_B}{m_A + m_B} (v_A - v_B), \\
v'_B &= v_B + (1 + e) \frac{m_B}{m_A + m_B} (v_A - v_B),
\end{align*}
\] (2)

and the momenta of the particles after collision

\[
\begin{align*}
p'_A &= p_A - (1 + e) \frac{m_A m_B}{m_A + m_B} (v_A - v_B), \\
p'_B &= p_B + (1 + e) \frac{m_A m_B}{m_A + m_B} (v_A - v_B),
\end{align*}
\] (3)

where the second term is the impulse experienced by each particles, which are equal and opposite, as fundamentally derived from Newton’s third law. We clearly see the conservation of momentum \( p_A + p_B = p'_A + p'_B \) from this expression for momentum. Note that \( \frac{m_A m_B}{m_A + m_B} \) is a reduced mass and \( v_A - v_B (> 0) \) is a relative velocity before the collision.

3. **Mass-momentum diagrams**

In order to analyze the collisions in the previous section, we introduce the mass-momentum diagram which has the mass \( m \) along the vertical axis and momentum \( p \) represented by the horizontal axis \([4]\). If we extend this diagram to the special relativity, the vertical axis switches to energy instead of mass \([1, 2]\). In this diagram, a point \((m, p)\) expresses the state of a particle. The interacting particles before a collision are represented by two vectors

\[
\begin{align*}
\vec{\varepsilon}_A &= (m_A, p_A), \\
\vec{\varepsilon}_B &= (m_B, p_B),
\end{align*}
\] (4)

as shown in figure [1].

Now we consider the addition of these two vectors,

\[
\vec{\varepsilon} = \vec{\varepsilon}_A + \vec{\varepsilon}_B = (m_A + m_B, p_A + p_B),
\] (5)

which is depicted in figure [1]. This is obtained using the parallelogram law. This vector \( \vec{\varepsilon} \) has three characteristics which are described separately in following subsections.

3.1 **Perfectly inelastic collision** (\( e = 0 \) case)

The vector \( \vec{\varepsilon} \) presents a perfectly inelastic collision (\( e = 0 \)) between the two particles A and B. In this case, the two particles combine into a single particle which has mass \( m_A + m_B \) and momentum \( p_A + p_B \).
Figure 1: The two vectors before collision. The case \((m_A, p_A) = (1, 4)\) and \((m_B, p_B) = (4, 1)\) means \(v_A = 4\) and \(v_B = 1\). In a perfectly inelastic collision \((e = 0)\), the two particles combine into a single particle which has mass \(m_A + m_B = 5\) and momentum \(p_A + p_B = 5\). Eq. (6) shows \(AA' = BB' = 3\).

Since the masses are not altered after the collision in Newtonian mechanics, the tip of vector \(\vec{\varepsilon}_A\) slides from point A to \(A'\), as does \(\vec{\varepsilon}_B\) from B to \(B'\). Therefore, \(\overrightarrow{OA'}\) and \(\overrightarrow{OB'}\) indicate the vectors after collision and their addition \(\overrightarrow{OA'} + \overrightarrow{OB'}\) becomes \(\vec{\varepsilon}\). In figure 1, the lengths \(AA'\) and \(BB'\) describe the momentum lost by particle A and the momentum gained by particle B respectively, which are equal in length because of the conservation of the momentum. We obtain

\[
AA' = BB' = \frac{m_A m_B}{m_A + m_B} (v_A - v_B),
\]

from (3) with \(e = 0\) case.

### 3.2 Inelastic and elastic collision \((0 < e \leq 1\) case)

The vector \(\vec{\varepsilon}\) stays the same after collision in the case of \(e \neq 0\), since the masses of the colliding particles do not change after collision and the momentum is conserved, that is,

\[
\vec{\varepsilon} = (m_A + m_B, p_A + p_B) = (m_A + m_B, p'_A + p'_B) = \vec{\varepsilon}''.
\]

We show the cases for \(e = 0.5\) and \(e = 1\) in figures 2 and 3. The dashed vectors \(\overrightarrow{OA''} = p'_A\) and \(\overrightarrow{OB''} = p'_B\) show the mass-momentum vectors after collision with \(e \neq 0\). We see from the figures that the addition of these dashed vectors gives the original vector \(\vec{\varepsilon}\). In figures 2 and 3 we can extract the following relationships between the lengths;

\[
AA'' = BB'',
\]

from (3) with \(e = 0\) case.
\[ A'A'' = e \times AA', \quad B'B'' = e \times BB'. \tag{9} \]

Equation (8) signifies the momentum conservation. \( AA'' \) is the momentum lost of particle A and \( BB'' \) is the momentum gain of particle B. Equation (9) are understood by the second term in the right hand side of (3). A first part of it shows \( AA' = BB' = \frac{m_A m_B}{m_A + m_B} (v_A - v_B) \), which is already described by (6). In addition to this, another part \( A'A'' = B'B'' = e \frac{m_A m_B}{m_A + m_B} (v_A - v_B) \) signifies \( AA' = BB' \) times the coefficient of restitution \( e \), which takes the value from 0 to 1. So the length \( A'A'' = B'B'' \) are changed according to the value of \( e \). When the collision problem we concern gives the value of \( e \), we can immediately fix the points \( A'' \) and \( B'' \) according to (9).

Figure 2: The dashed vectors \( \varepsilon''_A = (1, -0.5) \) and \( \varepsilon''_B = (4, 5.5) \) are the mass-momentum vectors after collision with \( e = 0.5 \)

Figure 3: The dashed vectors \( \varepsilon''_A = (1, -2) \) and \( \varepsilon''_B = (4, 7) \) are the mass-momentum vectors after collision with \( e = 1 \): elastic collision.

Figure 3 presents the complete picture of a one dimensional collision without any calculation. During the collision, the vectors \( \varepsilon_A \) and \( \varepsilon_B \) slide in parallel to the horizontal line because of the invariance of the masses. After the collision, the tips of the vectors lie between \( A'A'' \) and \( B'B'' \) according to (9). Once we fix the value of the coefficient of restitution \( e \), we write down the two vectors \( \varepsilon''_A \) and \( \varepsilon''_B \) immediately. The addition of these vectors gives the original vector \( \varepsilon \).

As a homework or exam problems to these diagrams, let us consider a moving multistage rocket which gets rid of a stage. The remaining rocket goes to the same direction, while the stage goes to the opposite direction. This problem is depicted like figure 3. The original rocket is assigned to \( \varepsilon \), the stage is for \( A'' \) and the remaining rocket is for \( B'' \). This disintegration can be viewed the time-reverse process of an \( e = 0 \) collision.
3.3 Center-of-mass systems

In collision problems, we often see the collisions from the point of view of the center-of-mass system. Since the sum of the momentum is zero before and after collision, the equations become simpler, compared to the laboratory system.

The speed of the center-of-mass system $V$ is given by

$$V = \frac{p_A + p_B}{m_A + m_B} = \frac{m_A v_A + m_B v_B}{m_A + m_B} = \tan \theta,$$  \hspace{1cm} (10)

where $\theta$ is depicted in figure 4 and describes the slope of the vector $\vec{\varepsilon}$ relative to the vertical axis $m$.

Figure 4: We draw a line from the tips of the vectors to the horizontal axis parallel to the vector $\vec{\varepsilon}$. The crossing points are the momenta in the center of mass system. Eqs. (12) show $p_A^* = -p_B^* = 3$.

Figure 5: The dashed vectors are the mass-momentum vectors after collision with $e = 1$: elastic collision case. Eqs. (14) show $-p_A^{*'} = p_B^{*'} = 3$.

We assign an * to the variables which describe collisions in the center-of-mass system. The velocities before collision are obtained

$$\begin{align*}
v_A^* &= v_A - V = + \frac{m_B}{m_A + m_B} (v_A - v_B), \\
v_B^* &= v_B - V = - \frac{m_A}{m_A + m_B} (v_A - v_B),
\end{align*}$$  \hspace{1cm} (11)

and the momenta are

$$\begin{align*}
p_A^{*'} &= m_A v_A^* = + \frac{m_A m_B}{m_A + m_B} (v_A - v_B), \\
p_B^{*'} &= m_B v_B^* = - \frac{m_A m_B}{m_A + m_B} (v_A - v_B).
\end{align*}$$  \hspace{1cm} (12)
From (2), we obtain the velocities and momenta after collision as

\[
\begin{align*}
    v_A' &= v_A - V = -e \frac{m_B}{m_A + m_B} (v_A - v_B), \\
    v_B' &= v_B - V = +e \frac{m_A}{m_A + m_B} (v_A - v_B),
\end{align*}
\]

and

\[
\begin{align*}
    p_A^* &= -e \frac{m_A m_B}{m_A + m_B} (v_A - v_B), \\
    p_B^* &= +e \frac{m_A m_B}{m_A + m_B} (v_A - v_B).
\end{align*}
\]

Below we show how we obtain these results from the mass-momentum diagram without requiring any calculations. Figure 4 shows the diagram before collision, which is the same diagram as figure 1. We draw straight lines from the tips of the vectors \( \vec{\varepsilon}_A \) and \( \vec{\varepsilon}_B \) to the horizontal axis \( p \), parallel to the vector \( \vec{\varepsilon} \). The points of intersection show the momenta in the center-of-mass system, \( p^*_A \) and \( p^*_B \). Moreover, \( p^*_A \) and \( p^*_B \) are always positioned symmetrically about the origin. So, \( p^*_A + p^*_B = 0 \) is understood at a glance. Figure 5 shows the elastic collision \( e = 1 \) case, which is the same diagram as in figure 3. For elastic collisions, the momenta of the particles are exchanged after the collisions. As we see from figure 5, \( p_A^* = p_B' \), \( p_B^* = p_A' \) and \( p_A^* + p_B^* = p_A'^* + p_B'^* = 0 \) are fulfilled.

The direction of \( \vec{\varepsilon} \) in figure 5 constructs an oblique coordinate system with the horizontal axis \( p \). This oblique coordinate system describes the state of the particles in the center-of-mass system. Oblique coordinates are often used in special relativity to describe events from different reference frames [4].

4. Laboratory systems

In this section, we utilize the foregoing methods for collisions in laboratory systems. We consider the system in which the particle B is at rest before the collision. This case is depicted in figure 6. The vector \( \vec{\varepsilon}_B \) is along the vertical axis. The vector \( \vec{\varepsilon} \) is the addition of \( \vec{\varepsilon}_A \) and \( \vec{\varepsilon}_B \) and describes the state of the particle in the case of a perfectly inelastic collision (\( e = 0 \)).

We also show the elastic collision (\( e = 1 \)) case in figure 6. The dashed vectors are the states of the colliding particles after collision and the adding them gives \( \vec{\varepsilon} \). Moreover, \( AA' = BB' \) due to the conservation of momentum.

We see this collision from the center-of-mass system depicted in figure 7. This figure is the same as figure 6. For the elastic collision (\( e = 1 \)), the momenta of the colliding particles are exchanged after collision. This can be seen in the figure at a glance.
Figure 6: Two vectors in the laboratory system. The case $\vec{\epsilon}_A = (m_A, p_A) = (1, 2)$ and $\vec{\epsilon}_B = (m_B, p_B) = (3, 0)$ means $v_A = 2$ and $v_B = 0$. The addition of $\vec{\epsilon}_A$ and $\vec{\epsilon}_B$ describes a perfectly inelastic collision. The dashed vectors are the mass-momentum vectors after collision with $\epsilon = 1$: elastic collision case.

Figure 7: We draw a line from the tips of the vectors to the horizontal axis parallel to $\vec{\epsilon}$. The points at which these cross the horizontal axis give the momenta in the center-of-mass system. Eqs. (12) and (14) show $p^*_A = -p^*_B = 1.5 = -p^*_A = p^*_B$. 
5. Summary

We introduce the mass-momentum diagram to analyze collision problems in Newtonian mechanics. This diagram serves as a powerful tool for obtaining the momenta of particles after a collision without the need for any calculations. The same diagram can also be used to analyze collisions from the point of view of the center-of-mass system. Students are often confusing in transposition between the laboratory system and the center-of-mass system. A single diagram can give students not only intuitive point of view, but also qualitative description of all features of the collisions without any calculations.

As a homework or exam problem, let us analyze a disintegration of a particle into two parts which move independently to opposite direction after the disintegration. The original particle is drawn along the vertical axis in which the particle is at rest before the disintegration. After the disintegration, the two parts are drawn symmetrically about the vertical axis. Even if the masses of the two parts are different, the addition of those vectors become the same as the original particle before the disintegration. This example is the same case with the rocket which is mentioned at the end of subsection 3.2. However, we should draw a different diagram according to the reference frame which we observe the parent body.

Finally, the use of these diagrams can be extended to two dimensional case in a future paper.

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