Exponential Ergodicity for Singular Reflecting McKean-Vlasov SDEs

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March 10, 2023

Abstract

By refining a recent result of Xie and Zhang [27], we prove the exponential ergodicity under a weighted variation norm for singular SDEs with drift containing a local integrable term and a coercive term. This result is then extended to singular reflecting SDEs as well as singular McKean-Vlasov SDEs with or without reflection. The exponential ergodicity in the relative entropy and (weighted) Wasserstein distances are also studied for reflecting McKean-Vlasov SDEs. The main results are illustrated by non-symmetric singular granular media equations.

AMS subject Classification: 60H10, 60G65.
Keywords: Exponential ergodicity, reflecting McKean-Vlasov SDEs, weighted variation norm, non-symmetric singular granular media equations.

1 Introduction

Let $D \subset \mathbb{R}^d$ be a connected open domain including the global situation $D = \mathbb{R}^d$, and let $\mathcal{P}$ denote the space of probability measures on $\bar{D}$, the closure of $D$. Consider the following distribution dependent (i.e. McKean-Vlasov) SDE on $\bar{D}$ with reflection if $D \neq \mathbb{R}^d$:

\begin{equation}
\text{d}X_t = b(X_t, \mathcal{L}_{X_t}) \text{d}t + \sigma(X_t) \text{d}W_t + n(X_t) \text{d}l_t, \quad t \geq 0,
\end{equation}

where $(W_t)_{t \geq 0}$ is an $m$-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\mathcal{L}_{X_t}$ is the distribution of $X_t$,

$$
 b : D \times \mathcal{P} \to \mathbb{R}^d, \quad \sigma : D \to \mathbb{R}^d \otimes \mathbb{R}^m
$$

*Supported in part by NNSFC (11771326, 11831014, 11921001) and the DFG through CRC 1283.
are measurable, and when \( D \neq \mathbb{R}^d \), \( n \) is the inward unit normal vector field of the boundary \( \partial D \), and \( l_t \) is an adapted continuous increasing process which increases only when \( X_t \in \partial D \).

In the case that \( D = \mathbb{R}^d \), we have \( l_t = 0 \) so that (1.1) becomes the distribution dependent SDE (DDSDE)

\[
    dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t)dW_t, \quad t \geq 0.
\]

If moreover \( b(x, \mu) = b(x) \) does not depend on \( \mu \), it reduces to the classical Itô’s SDE

\[
    dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0.
\]

In the recent work [25], the well-posedness and regularity estimates have been studied for solutions to (1.1) with \( b \) containing a locally integrable term and a Lipchitz continuous term. However, the ergodicity was only investigated under monotone or Lyapunov conditions excluding this singular situation. See also [5, 9, 10, 11, 13, 16, 19, 24] and references within for results on the ergodicity of McKean-Vlasov SDEs without reflection under monotone or Lyapunov conditions. On the other hand, by using Zvokin’s transform, the exponential ergodicity was proved by Xie and Zhang [27] for the singular SDE (1.3). In this paper, we aim to refine the result of [27] and make extensions to singular SDEs with reflection and distribution dependent drift.

When the SDE (1.1) is well-posed, let \( P^*_t \nu = \mathcal{L}_{X_t} \) for the solution with initial distribution \( \nu \in \mathcal{P} \). We will study the exponential convergence of \( P^*_t \) under the weighted measurable function \( V \):

\[
    \|\mu - \nu\|_V := |\mu - \nu|(V) = \sup_{|f| \leq V} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathcal{P},
\]

where \( |\mu - \nu| \) is the total variation of \( \mu - \nu \) and \( \mu(f) := \int f \, d\mu \) for a measure \( \mu \) and \( f \in L^1(\mu) \). When \( V = 1 \), \( \| \cdot \|_V \) reduces to the the total variation norm \( \| \cdot \|_{\text{var}} \).

We will consider \( b(x, \mu) = b^{(0)}(x) + b^{(1)}(x, \mu) \), where \( b^{(0)} \) is the singular term satisfying

\[
    \sup_{z \in \mathbb{R}^d} \int_{B(z,1) \cap D} |b^{(0)}(x)|^p \, dx < \infty
\]

for some \( p > d \vee 2 \), and \( b^{(1)}(\cdot, \mu) \) is a coercive term such that

\[
    \limsup_{x \to \partial D, |x| \to \infty, \mu \in \mathcal{P}} \left\langle b^{(1)}(x, \mu), \nabla V(x) \right\rangle = -\infty
\]

holds for some compact function \( V \in C^2(\mathbb{R}^d) \) (i.e. \( \{ V \leq r \} \) is compact for any \( r > 0 \)). The later condition is trivial for bounded \( D \) by taking \( V = 1 \) and the convention that \( \sup \emptyset = -\infty \).

To conclude this section, we present below an example for the \( L^1 \)-exponential convergence of non-symmetric singular granular media equations, see [6, 10, 16] for the study of regular and symmetric models for \( D = \mathbb{R}^d \).
Example 1.1. Let $D = \mathbb{R}^d$ or be a $C^{2,L}$-domain (see Definition 2.1 below). Consider the following nonlinear PDE for probability density functions on $D$:

\begin{equation}
\partial_t \rho_t = \Delta \rho_t - \text{div}\{\rho_t b + \rho_t (W * \rho_t)\}, \quad \nabla_n \rho_t |_{\partial D} = 0 \text{ if } \partial D \neq \emptyset,
\end{equation}

where

(i) $W$ is a bounded measurable function on $\bar{D} \times \bar{D}$, and

\[(W * \rho_t)(x) := \int_{\mathbb{R}^d} W(x,z) \rho_t(z) dz;\]

(ii) $b = b^{(0)} + b^{(1)}$ is a vector field such that (1.4) holds for some $p > d \lor 2$, and $b^{(1)}$ is locally bounded with $b^{(1)}(x) = -\phi(|x|^2)x$ for larger $|x|$ and some increasing function $\phi : [0, \infty) \to [1, \infty)$ with $\int_1^\infty \frac{ds}{s\phi(s)} < \infty$.

In physics, $\rho$ stands for the distribution density of particles, $W$ describes the interaction among particles, and $b$ refers to the potential of individual particles. When $b$ and $W$ are not of gradient type, the associated mean field particle systems are non-symmetric.

To characterize (1.5) using (1.1), let

\[b(x,\mu) = b(x) + (W * \mu)(x), \quad \sigma(x) = \sqrt{2}I_d,\]

where $I_d$ is the $d \times d$ identity matrix, and $(W * \mu)(x) := \int_{\bar{D}} W(x,z) \mu(dz)$.

By (i) and (ii), (A1) holds for $V(x) := |x|^2$ when $D = \mathbb{R}^d$, while (A2) holds for $V = 1$ when $D$ is a bounded $C^{2,L}$ domain. So, by Theorem 3.1, (1.1) is well-posed, and by Itô’s formula, $\rho_t(x) := \frac{dP^*_t}{dx}$ solves (1.5) for $\rho_0(x) := \frac{d\nu}{dx}$, see Subsection 1.2 in [25]. On the other hand, when $D = \mathbb{R}^d$ the superposition principle in [2] says that a solution of (1.5) is the distribution density of a weak solution to (1.1), such that (1.5) is well-posed as well. Moreover:

(a) By Theorem 3.1, when $\|W\|_{\infty}$ is small enough, $P^*_t$ has a unique invariant probability measure $\mu$ satisfying (3.2), so that the solution $\rho_t := \frac{dP^*_t}{dx}$ of (1.5) satisfies

\[\|\rho_t - \rho\|_{L^1} = \|P^*_t \nu - \mu\|_{\text{var}} \leq ce^{-\lambda t} \|\rho_0 - \rho\|_{L^1}, \quad t \geq 0\]

for some constants $c, \lambda > 0$, where $\rho$ is the density function of $\mu$.

(b) Let $D = \mathbb{R}^d$ or $D$ be convex. If there exists a constant $K > 0$ such that

\begin{equation}
(b(x) - b(y), x - y) \leq -K|x - y|^2, \quad x, y \in D
\end{equation}

holds, by Theorem 4.1 when $\|\nabla^2 W\|_{\infty}$ is small enough $P^*_t$ is exponential ergodic in the relative entropy and the quadratic Wasserstein distance $W_2$. If (1.6) only holds for large $|x - y|$, according to Theorem 4.3, $P^*_t$ is exponential ergodic under a weighted Wasserstein distance provided $\|\nabla^2 W\|_{\infty}$ is small enough.

In the remainder of the paper, we study in Section 2 the exponential ergodicity for singular reflecting SDEs, then prove the uniform ergodicity for singular reflecting McKean-Vlasov SDEs in Section 3, and finally investigate in Section 4 the exponential ergodicity for reflecting McKean-Vlasov SDEs in relative entropy and (weighted) Wasserstein distances.
2 Exponential ergodicity for singular reflecting SDEs

To measure the singularity of the SDE, we introduce some functional spaces used in [26]. For any $p \geq 1$, let $L^p$ be the class of measurable functions $f$ on $D$ such that

$$\|f\|_{L^p} := \left(\int_D |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$ 

For any $\epsilon > 0$ and $p \geq 1$, let $H^{\epsilon,p} := (1 - \Delta)^{-\frac{\epsilon}{2}} L^p$ with

$$\|f\|_{H^{\epsilon,p}} := \| (1 - \Delta)^{\frac{\epsilon}{2}} f \|_{L^p} < \infty, \quad f \in H^{\epsilon,p},$$

where $\Delta$ is the (Neumann if $\partial D \neq \emptyset$) Laplacian. For any $z \in \mathbb{R}^d$ and $r > 0$, let

$$B(z, r) := \{x \in \mathbb{R}^d : |x - z| \leq r\}$$

be the closed ball centered at $z$ with radius $r$. We will simply denote $B_r = B(0, r)$ for $r > 0$. We write $f \in \tilde{L}^p$ if

$$\|f\|_{\tilde{L}^p} := \sup_{z \in \bar{D}} \|1_{B(z,1)} f\|_{L^p} < \infty.$$ 

Moreover, let $g \in C_0(\bar{D})$ with $g|_{B_1} = 1$ and the Neumann boundary condition $\nabla_n g|_{\partial D} = 0$ if $\partial D$ exists. We denote $f \in \tilde{H}^{\epsilon,p}$ if

$$\|f\|_{\tilde{H}^{\epsilon,p}} := \sup_{z \in \bar{D}} \|g(z + \cdot) f\|_{H^{\epsilon,p}} < \infty.$$ 

We note that the space $\tilde{H}^{\epsilon,p}$ does not depend on the choice of $g$. If a vector or matrix valued function has components in one of the above introduced spaces, then it is said in the same space with norm defined as the sum of components’ norms.

In the following subsections, we first state the main results, then present some lemmas, and finally prove the main results.

2.1 Main results

We first consider the ergodicity of SDE (1.3) under the following assumption, where by the Sobolev embedding theorem $\sigma$ (hence $\sigma\sigma^*$) is Hölder continuous by the boundedness of $\sigma$ and $\|\nabla \sigma\| \in \tilde{L}^p$ for some $p > d$.

(A1) $\sigma$ is weakly differentiable, $\sigma\sigma^*$ is invertible, and $b = b^{(0)} + b^{(1)}$ such that the following conditions hold.

1. There exists $p > d \lor 2$ such that

$$\|\sigma\|_\infty + \| (\sigma\sigma^*)^{-1} \|_\infty + \| b^{(0)} \|_{L^p} + \| \nabla \sigma \|_{\tilde{L}^p} < \infty.$$
(2) \( b^{(1)} \) is locally bounded, there exist constants \( K > 0, \varepsilon \in (0,1) \), some compact function \( V \in C^2(\mathbb{R}^d; [1, \infty)) \), and a continuous increasing function \( \Phi : [1, \infty) \to [1, \infty) \) with \( \Phi(n) \to \infty \) as \( n \to \infty \), such that

\[
\langle b^{(1)}, \nabla V \rangle(x) + \varepsilon |b^{(1)}(x)| \sup_{B(x, \varepsilon)} \{|\nabla V| + |\nabla^2 V|\} \leq K - \varepsilon(\Phi \circ V)(x),
\]

\[
\lim_{|x| \to \infty} \sup_{B(x, \varepsilon)} \{|\nabla^2 V| + |\nabla V|\} = 0.
\]

Theorem 2.1. Assume (A1). Then (1.3) is well-posed, the associated Markov semigroup \( P_t \) has a unique invariant probability measure \( \mu \) such that \( \mu(\Phi(\varepsilon_0 V)) < \infty \) for some \( \varepsilon_0 \in (0,1) \), and

\[
\lim_{t \to \infty} \| P^*_t \nu - \mu \|_{\text{var}} = 0, \quad \nu \in \mathcal{P}.
\]

Moreover:

1. If \( \Phi(r) \geq \delta r \) for some constant \( \delta > 0 \) and all \( r \geq 0 \), then there exist constants \( c > 1, \lambda > 0 \) such that

\[
\| P^*_t \mu_1 - P^*_t \mu_2 \| \leq c e^{-\lambda t} \| \mu_1 - \mu_2 \|, \quad \mu_1, \mu_2 \in \mathcal{P}, t \geq 0.
\]

In particular,

\[
\| P^*_t \nu - \mu \| \leq c e^{-\lambda t} \| \nu - \mu \|, \quad \nu \in \mathcal{P}, t \geq 0.
\]

2. Let \( H(r) := \int_0^r \frac{ds}{\Phi(s)} < \infty \) for \( r \geq 0 \). If \( \Phi \) is convex, then there exist constants \( k > 1, \lambda > 0 \) such that

\[
\| P^*_t \delta_x - \mu \| \leq k \{ 1 + H^{-1}(H(V(x)) - k^{-1} t) \} e^{-\lambda t}, \quad x \in \mathbb{R}^d, t \geq 0,
\]

where \( H^{-1} \) is the inverse of \( H \) with \( H^{-1}(r) := 0 \) for \( r \leq 0 \). Consequently, if \( H(\infty) < \infty \) then there exist constants \( c, \lambda, t^* > 0 \) such that

\[
\| P^*_t \mu_1 - \mu_2 \| \leq c e^{-\lambda t} \| \mu_1 - \mu_2 \|_{\text{var}}, \quad t \geq t^*, \mu_1, \mu_2 \in \mathcal{P}.
\]

To illustrate this result, we present below a consequence which covers the situation of [27, Theorem 2.10] where

\[
\langle b^{(1)}(x), x \rangle \leq c_1 - c_2 |x|^{1+p}, \quad |b^{(1)}(x)| \leq c_1 (1 + |x|)^p
\]

holds for some constants \( c_1, c_2 > 0 \) and \( p \geq 1 \). Indeed, Corollary 2.2 implies the exponential ergodicity under the weaker condition

\[
\langle b^{(1)}(x), x \rangle \leq c_1 - c_2 |x|^{1+p}, \quad |b^{(1)}(x)| \leq c_1 (1 + |x|)^{p+1}
\]

for some constants \( p, c_1, c_2 > 0 \) (\( p \) may be smaller than 1, \( |b^{(1)}| \) may have higher order growth), since in this case, (2.7) and (2.8) hold for \( \phi(r) := (1 + r)^{1+p} \), and (2.9) holds for \( \psi(r) := (1 + r^2)^q \) for any \( q > 0 \) when \( p \geq 1 \).
Corollary 2.2. Assume (A1)(1) and let $b^{(1)}$ satisfy

$$(2.7) \langle b^{(1)}(x), x \rangle \leq c_1 - c_2 \phi(|x|^2), \quad |b^{(1)}(x)| \leq c_1 \phi(|x|^2), \quad x \in \mathbb{R}^d$$

for some constants $c_1, c_2 > 0$ and increasing function $\phi : [0, \infty) \to [1, \infty)$ with

$$(2.8) \alpha := \liminf_{r \to \infty} \frac{\log \phi(r)}{\log r} > \frac{1}{2},$$

Then

(1) $[13]$ is well-posed, $P_t$ has a unique invariant probability measure $\mu$ such that $\mu(V) < \infty$ and $[2.3]$ hold for $V := e^{(1+|\cdot|^2)^{\theta}}$ with $\theta \in ((1-\alpha)^+, \frac{1}{2})$. In general, for any increasing function $1 \leq \psi \in C^2([1, \infty))$ satisfying

$$(2.9) \liminf_{r \to \infty} \frac{\psi'(r)\phi(r)}{\psi(r)} > 0, \quad \lim_{r \to \infty} \frac{\psi''(r)}{\psi(r)} = 0,$$

$\mu(V) < \infty$ and $[2.3]$ hold for $V := \psi(|\cdot|^2)$.

(2) If $\int_0^\infty \frac{d\pi}{\phi(s)} < \infty$, then $[2.5]$ holds $V := (1+|\cdot|^2)^\theta(g > 0)$ and some constants $c, \lambda, t^* > 0$.

Remark 2.1. We have the following assertions on the invariant probability measure $\mu$ and the ergodicity in Wasserstein distance and relative entropy.

(1) According to $[13]$ Corollary 1.6.7 and Theorem 3.4.2, (A1) implies that $\mu$ has a strictly positive density function $\rho \in H^{1,p}_{\text{loc}}$, the space of functions $f$ such that $fg \in H^{1,2}$ for all $g \in C^\infty_0(\mathbb{R}^d)$. Moreover, by $[13]$ Theorem 3.1.2, when $\sigma$ is Lipschitz continuous and $\mu(|b|^2) < \infty$, we have $\sqrt{\rho} \in H^{1,2}$. So, when $[2.7]$ holds for $\phi(r) \sim r^p$ for some $p > \frac{1}{2}$ and large $r > 0$, Corollary 2.2(1) implies that $\mu$ has density with $\sqrt{\rho} \in H^{1,2}$. See also $[22]$ and $[23]$ for different type global regularity estimates on $\rho$ under integrability conditions.

(2) Let $V := (1+|\cdot|^2)^\theta$ for some $p \geq 1$. By $[13]$ Theorem 6.15], there exists a constant $c(p) > 0$ such that

$$\mathbb{W}_p(\mu, \nu)^p \leq c(p)\|\mu - \nu\|, \quad \text{where}$$

$$\mathbb{W}_p(\mu_1, \mu_2) := \inf_{\pi \in C(\mu_1, \mu_2)} \left( \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} |x-y|^p \pi(dx, dy) \right)^{\frac{1}{p}},$$

for $C(\mu_1, \mu_2)$ being the set of couplings for $\mu_1$ and $\mu_2$. So, by Corollary 2.2 if (A1) holds with $\Phi(r) \geq \delta r$ for some $\delta > 0$, then there exist constants $c, \lambda > 0$ such that

$$\mathbb{W}_p(P_t^*\nu, \mu)^p \leq c(1+\nu(|\cdot|^p))e^{-\lambda t}, \quad t \geq 0, \nu \in \mathcal{P};$$

and if moreover $\Phi$ is convex with $\int_0^\infty \frac{d\pi}{\phi(s)} < \infty$, then there exist constants $c, \lambda, t^* > 0$ such that

$$\mathbb{W}_p(P_t^*\nu, \mu)^p \leq ce^{-\lambda t}\|\mu - \nu\|_{\text{var}}, \quad t \geq t^*, \nu \in \mathcal{P}. $$
(3) When \( b^{(1)} \) is Lipschitz continuous, the log-Harnack inequality in [28, Theorem 4.1] implies
\[
\text{Ent}(P^*_{t} \nu | \mu) \leq \frac{c'}{1 + t} W_2(\nu, \mu)^2, \quad \nu \in \mathcal{P}, t > 0
\]
for some constant \( c' > 0 \), where \( \text{Ent}(\nu | \mu) \) is the relative entropy. Thus, by Corollary 2.2 if (A1) holds for \( V(x) := 1 + |x|^2 \) and \( \Phi(r) \geq \delta r \) for some constant \( \delta > 0 \), then there exist constants \( c, \lambda > 0 \) such that
\[
\text{Ent}(P^*_{t} \nu | \mu) \leq c(1 + \nu(\cdot)^2) e^{-\lambda t}, \quad t \geq 1, \nu \in \mathcal{P};
\]
and if moreover \( \Phi \) is convex with \( \int_0^\infty ds \Phi'(s) < \infty \), then there exist \( c, \lambda, t^* > 0 \) such that
\[
\text{Ent}(P^*_{t} \nu | \mu) \leq ce^{-\lambda t} \| \mu - \nu \|_{\text{var}}, \quad t \geq t^*, \nu \in \mathcal{P}.
\]

Next, consider the following reflecting SDE on \( D \neq \mathbb{R}^d \):
\[
(2.10) \quad dX_t = b(X_t)dt + \sigma(X_t)dW_t + n(X_t)dl_t, \quad t \geq 0,
\]
where \( \partial D \in C^{2,L}_b \) which is defined as follows.

**Definition 2.1.** Let \( \rho_\partial \) be the distance function to \( \partial D \). For any \( k \in \mathbb{N} \), we write \( \partial D \in C^k_b \) if there exists a constant \( r_0 > 0 \) such that the polar coordinate around \( \partial D \)
\[
\partial D \times [-r_0, r_0) \ni (\theta, r) \mapsto \theta + r n(\theta) \in B_{r_0}(\partial D) := \{ x \in \mathbb{R}^d : \rho_\partial(x) \leq r_0 \}
\]
is a \( C^k \)-diffeomorphism. We write \( \partial D \in C^{k,L}_b \), if it is \( C^k_b \) with \( \nabla^k \rho_\partial \) being Lipschitz continuous on \( B_{r_0}(\partial D) \).

We also need heat kernel estimates for the Neumann semigroup \( \{P^*_t\}_{t \geq 0} \) generated by
\[
L^* := \frac{1}{2} \text{tr} (\sigma \sigma^* \nabla^2).
\]
For any \( \varphi \in C^2_b(\bar{D}) \), let \( P^*_t \varphi \) be the solution of the PDE
\[
(2.11) \quad \partial_t u_t = L^* u_t, \quad \nabla_n u_t |_{\partial D} = 0 \text{ for } s > 0, u_0 = \varphi.
\]
We will prove the exponential ergodicity of (2.10) under the following assumption.

**(A2)** \( \partial D \in C^{2,L}_b \) and the following conditions hold.

1. **(A1)** holds for \( \bar{D} \) replacing \( \mathbb{R}^d \), and there exists \( r_0 > 0 \) such that
   \[
   \nabla_n(x) V(y) \leq 0, \quad x \in \partial D, |y - x| \leq r_0.
   \]

2. For any \( \varphi \in C^2_b(\bar{D}) \), the PDE (2.11) has a unique solution \( P^*_t \varphi \in C^{1,2}_b(\bar{D}) \), such that for some constant \( c > 0 \) we have
   \[
   \| \nabla^i P^*_t \varphi \|_{\infty} \leq c(1 + t)^{-\frac{i}{2}} \| \nabla^{i-1} \varphi \|_{\infty}, \quad t > 0, \quad i = 1, 2, \varphi \in C^2_b(\bar{D}),
   \]
   where \( \nabla^0 \varphi := \varphi \).
As explained in [25, Remark 2.2(2)] that, (A2) holds if $D$ is bounded and $\sigma$ is Hölder continuous. Moreover, (2.1) is trivial when $\partial D$ is bounded, since in this case we may take $1 \leq \tilde{V} \in C^2(\mathbb{R}^d)$ such that $\tilde{V} = 1$ on $\partial_D(\partial D)$ and $\tilde{V} = V$ outside a compact set, so that (2.1) remains true for $\tilde{V}$ replacing $V$. Similarly, (2.12) holds for $V(x_1, x_2) := V_1(x_1) + V_2(x_2)$ and $D = D_1 \times \mathbb{R}^l$ where $l \in \mathbb{N}$ is less than $d$, $\partial D_1 \subset \mathbb{R}^{d-l}$ is bounded, and $V_1 = 1$ in a neighborhood of $\partial D_1$.

**Theorem 2.3.** Assume (A2). Then all assertions in Theorem 2.1 hold for the reflecting SDE (2.10).

### 2.2 Some lemmas

We first consider the following time dependent SDE with reflection when $\partial D$ exists:

$$
(2.13) \quad dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t + n(X_t)dl_t, \quad t \geq 0.
$$

For any $T > 0$ and $p, q > 1$, let $\tilde{L}^p_q(T)$ denote the class of measurable functions $f$ on $[0, T] \times D$ such that

$$
\|f\|_{\tilde{L}^p_q(T)} := \sup_{z \in D} \left( \int_0^T \|1_{B(z, 1)}f_t\|_{L^p}^q dt \right)^{\frac{1}{q}} < \infty.
$$

For any $\epsilon > 0$, let $\tilde{H}^\epsilon_{q,p}(T)$ be the space of $f \in \tilde{L}^p_q$ with

$$
\|f\|_{\tilde{H}^\epsilon_{q,p}(T)} := \sup_{z \in D} \left( \int_0^T \|f_t\|_{H^\epsilon_{q,p}}^q dt \right)^{\frac{1}{q}} < \infty.
$$

We will study the well-posedness, strong Feller property and irreducibility under the following assumptions for $D = \mathbb{R}^d$ and $D \neq \mathbb{R}^d$ respectively.

**A3** Let $T > 0$, $D = \mathbb{R}^d$, $a_t(x) := (\sigma_t \sigma^*_t)(x)$ and $b_t(x) = b_t^{(0)}(x) + b_t^{(1)}(x)$.

1. $a$ is invertible with $\|a\|_{\infty} + \|a^{-1}\|_{\infty} < \infty$ and

   $$
   \lim_{\epsilon \to 0} \sup_{|x-y| \leq \epsilon, t \in [0,T]} \|a_t(x) - a_t(y)\| = 0.
   $$

2. There exist $l \geq 1$, $\{(p_i, q_i)\}_{0 \leq i \leq l} \in \mathcal{K} := \{(p, q) : p, q \in (2, \infty), \frac{d}{p} + \frac{2}{q} < 1\}$ and $1 \leq f_i \in L^p_{q_i}$ such that

   $$
   |b_t^{(0)}| \leq f_0, \quad \|\nabla \sigma\| \leq \sum_{i=1}^l f_i.
   $$

3. $b^{(1)}$ is locally bounded, there exist constants $K, \epsilon > 0$, increasing $\phi \in C^1([0, \infty); [1, \infty))$ with $\int_0^\infty \frac{ds}{r + \phi(s)} = \infty$, and a compact function $V \in C^2(\mathbb{R}^d; [1, \infty))$ such that

   $$
   \sup_{B(x, \epsilon)} \left\{ |\nabla V| + \|\nabla^2 V\| \right\} \leq KV(x),
   $$

   $$
   \langle b_t^{(1)}(x), \nabla V(x) \rangle + \epsilon |b_t^{(1)}(x)| \sup_{B(x, \epsilon)} \|\nabla^2 V\| \leq K\phi(V(x)), \quad (t, x) \in [0, T] \times \mathbb{R}^d.
   $$

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When $D \neq \mathbb{R}^d$, we consider the following time dependent differential operator on $\bar{D}$:

\begin{equation}
L^\sigma_t := \frac{1}{2} \text{tr}(\sigma_t \sigma_t^* \nabla^2), \quad t \in [0, T].
\end{equation}

Let $\{P^\sigma_{s,t}\}_{T \geq t_1 \geq t_2 \geq s \geq 0}$ be the Neumann semigroup on $\bar{D}$ generated by $L^\sigma_t$; that is, for any $\varphi \in C_b^2(\bar{D})$, and any $t \in (0, T]$, $(P^\sigma_{s,t}\varphi)_{s \in [0, t]}$ is the unique solution of the PDE

\begin{equation}
\partial_s u_s = -L^\sigma_s u_s, \quad \nabla_n u_s|_{\partial D} = 0 \quad \text{for } s \in [0, t), \ u_t = \varphi.
\end{equation}

For any $t > 0$, let $C_b^{1,2}([0, t] \times \bar{D})$ be the set of functions $f \in C_b([0, t] \times \bar{D})$ with bounded and continuous derivatives $\partial_t f, \nabla f$ and $\nabla^2 f$.

\textbf{(A4)} $D \in C_b^{2,1}$, \textbf{(A3)} holds with $V$ satisfying \eqref{eq:2.12} holds for some $r_0 > 0$. Moreover, for any $\varphi \in C_b^{2}(\bar{D})$ and $t \in (0, T]$, the PDE \eqref{eq:2.15} has a unique solution $P^\sigma_{s,t}\varphi \in C_b^{1,2}([0, t] \times \bar{D})$, such that for some constant $c > 0$ we have

\begin{equation}
\|\nabla^i P^\sigma_{s,t}\varphi\|_\infty \leq c(t - s)^{-\frac{1}{2}}\|\nabla^{i-1}\varphi\|_\infty, \quad 0 \leq s < t \leq T, \ i = 1, 2, \varphi \in C_b^{2}(\bar{D}).
\end{equation}

We have the following result, where the well-posedness for $D = \mathbb{R}^d$ has been addressed in [15].

**Lemma 2.4.** Assume \textbf{(A3)} for $D = \mathbb{R}^d$ and \textbf{(A4)} for $D \neq \mathbb{R}^d$. Then \eqref{eq:2.13} is well-posed up to time $T$. Moreover, for any $t \in (0, T]$, \eqref{eq:2.17}

\begin{equation}
\lim_{D \ni y \to x} \|P^\sigma_t \delta_x - P^\sigma_t \delta_y\|_{\text{var}} = 0, \quad t \in (0, T], x \in D,
\end{equation}

and $P_t$ has probability density (i.e. heat kernel) $p_t(x, y)$ such that

\begin{equation}
\inf_{x, y \in \bar{D} \cap B_N, \ \varrho(y) \geq N^{-1}} p_t(x, y) > 0, \quad N > 1, t \in (0, T],
\end{equation}

where $\inf \emptyset := \infty$.

**Proof.** (a) The well-posedness. For any $n \geq 1$, let

\[ b^n := 1_{B_n} b^{(1)} + b^{(0)}. \]

Since $b^{(1)}$ is locally bounded, by [26, Theorem 1.1] for $D = \mathbb{R}^d$ and [25, Theorem 2.2] for $D \neq \mathbb{R}^d$, for any $x \in D$, the following SDE is well-posed:

\[ dX^{x,n}_t = b^n (X^{x,n}_t) dt + \sigma(X_t^{x,n}) dW_t + n(X_t^{x,n}) d\tau^{x,n}_t, \quad X_0^{x,n} = x. \]

Let $\tau^{x}_n := \inf \{ t \geq 0 : |X_t^{x,n}| \geq n \}$. Then $X_t^{x,n}$ solves \eqref{eq:1.3} up to time $\tau^{x}_n$, and by the uniqueness we have

\[ X_t^{x,n} = X^{x,m}_t, \quad t \leq \tau^{x}_n \wedge \tau^{x}_m, \ n, m \geq 1. \]

So, it suffices to prove that $\tau^{x}_n \to \infty$ as $n \to \infty$. 

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Let $L^0_t := L^0_t + \nabla b^0_t$. By [26] Theorem 3.1 for $D = \mathbb{R}^d$ and [25] Lemma 2.6] for $D \neq \mathbb{R}^d$, (A3) implies that for any $\lambda \geq 0$, the PDE

\[(2.21) \quad (\partial_t + L^0_t)u_t = \lambda u_t - b^0_t, \quad t \in [0, T], u_T = 0, \nabla_n u_t|_{\partial D} = 0\]

has a unique solution $u \in \tilde{H}^{p_0}(T)$, and there exist constants $\lambda_0, c, \theta > 0$ such that

\[(2.20) \quad \lambda^\theta (\|u\|_\infty + \|\nabla u\|_\infty) + \|\partial_t u\|_{L^p_0(T)} + \|\nabla^2 u\|_{L^p_0(T)} \leq c, \quad \lambda \geq \lambda_0.\]

So, we may take $\lambda \geq \lambda_0$ such that

\[(2.21) \quad \|u\|_\infty + \|\nabla u\|_\infty \leq \varepsilon,\]

where we take $\varepsilon \leq r_0$ when $\partial D$ exists. Let $\Theta_t(x) = x + u_t(x)$. By (2.12) and (2.21) for $\varepsilon \leq r_0$ when $\partial D$ exists, we have

\[\langle \nabla V(Y_{t,x}^n), n(X_{t,x}^n) \rangle d\tilde{t}_t^n \leq 0.\]

So, by Itô’s formula, $Y_{t,x}^n := \Theta_t(X_{t,x}^n)$ satisfies

\[(2.22) \quad dY_{t,x}^n = \{1_{B_n}b^{(1)}(t) + \lambda u_t + 1_{B_n}b^{(1)}_n u_t\}(X_{t,x}^n)dt + \{(\nabla \Theta_t)\sigma_t\}(X_{t,x}^n)dw_t + n(X_{t,x}^n)dl_t^n,\]

where we have used the fact that $\nabla_n u_t|_{\partial D} = 0$ implies that $\{(\nabla \Theta_t)\}n = n$ holds on $\partial D$. By (2.21) and (A3)(3) with (2.12) when $\partial D \neq \emptyset$, there exists a constant $c_0 > 0$ such that for some martingale $M_t$,

\[d\{V(Y_{t,x}^n) + M_t\}\]

\[\leq \left[\left\{b^{(1)} + \nabla b^{(1)}u_t\right\}(X_{t,x}^n), \nabla V(Y_{t,x}^n)\right\} + c_0(\|\nabla V(Y_{t,x}^n)\| + \|\nabla^2 V(Y_{t,x}^n)\|)\right]dt \]

\[\leq \left\{b^{(1)}(X_{t,x}^n), \nabla V(Y_{t,x}^n)\right\} + \varepsilon \sup_{B(X_{t,x}^n, \varepsilon)} \|\nabla^2 V\| + c_0 K V(Y_{t,x}^n)\right\}dt \]

\[\leq \left\{K \phi(V(Y_{t,x}^n)) + c_0 K V(Y_{t,x}^n)\right\}dt \leq K \left\{\phi((1 + \varepsilon K)V(Y_{t,x}^n)) + c_0 V(Y_{t,x}^n)\right\}dt, \quad t \leq T_n^x.\]

Letting $H(r) := \int_0^r \frac{ds}{r + \phi((1+\varepsilon K)s)}$, by Itô’s formula and noting that $\phi' \geq 0$, we find a constant $c_1 > 0$ such that

\[dH(V(Y_{t,x}^n)) \leq c_1 dt + d\tilde{M}_t, \quad t \in [0, T_n^x]\]

holds for some martingale $\tilde{M}_t$. Thus,

\[\mathbb{E}[V(Y_{t,x}^n)] \leq V(x + u(x)) + c_1 t, \quad t \geq 0, n \geq 1.\]

Since (2.21) and $|z| \geq n$ imply $|\Theta_t(z)| \geq |z| - |u(z)| \geq n - \varepsilon$, we derive

\[(2.23) \quad P(\tau_n^x \leq t) \leq \frac{V(x + \Theta_0(x)) + c_1 t}{\inf_{|y| \geq n - \varepsilon} H(V(y))} =: \varepsilon_{t,n}(x), \quad t > 0.\]

Since $\lim_{|x| \to \infty} H(V)(x) = \int_0^\infty \frac{ds}{s + \phi((1+\varepsilon K)s)} = \infty$, we obtain $\tau_n^x \to \infty (n \to \infty)$ as desired.
(b) Proof of (2.17). By [21, Proposition 1.3.8], the log-Harnack inequality

\[ P_t \log f(y) \leq \log P_tf(x) + c|x - y|^2, \quad x, y \in \tilde{D}, 0 < f \in \mathcal{B}_b(\tilde{D}) \]

for some constant \( c > 0 \) implies the gradient estimate

\[ |\nabla P_tf|^2 \leq 2cP_t|f|^2, \quad f \in \mathcal{B}_b(\tilde{D}), \]

and hence

\[ \lim_{y \to x} \| P_t^* \delta_x - P_t^* \delta_y \|_{\text{var}} = 0, \quad x \in \tilde{D}. \]

Let \( P_t^n \) be the Markov semigroup associated with \( X_t^n \). Thus, by the log-Harnack inequality in [28, Theorem 4.1] for \( D = \mathbb{R}^d \) and in [25, Theorem 4.1] for \( D \neq \mathbb{R}^d \), we have

\[ \lim_{y \to x} \| (P_t^n)^* \delta_x - (P_t^n)^* \delta_y \|_{\text{var}} = 0, \quad t \in (0, T). \]

On the other hand, by (2.23) and \( X_t = X_t^n \) for \( t \leq \tau_n \), we obtain

\[ \lim_{n \to \infty} \sup_{y \in \bar{D} \cap B(x,1)} \| P_t^* \delta_y - (P_t^n)^* \delta_y \|_{\text{var}} = \lim_{n \to \infty} \sup_{|f| \leq 1, y \in \bar{D} \cap B(x,1)} |P_t f(y) - P_t^n f(y)| \]

\[ \leq 2 \lim_{n \to \infty} \sup_{y \in \bar{D} \cap B(x,1)} \mathbb{P}(\tau^n \leq t) = 0. \]

Combining this with (2.24) and the triangle inequality, we prove (2.17).

(c) Finally, let \( L_t := L_t^s + \nabla b_t \). By Itô’s formula, for any \( f \in C^2_0((0, T) \times D) \) we have

\[ df_t(X_t) = (\partial_t + L_t)f_t(X_t)dt + dM_t \]

for some martingale \( M_t \), so that \( f_0 = f_T = 0 \) yields

\[ \int_{(0,T)} P_t \{ (\partial_t + L)f_t \} dt = 0, \quad f \in C^\infty_0((0, T) \times D). \]

This implies that the heat kernel \( p_t(x, \cdot) \) of \( P_t \) solves the following PDE on \((0, T) \times D\) in the weak sense:

\[ \partial_t u_t = L_t^* u_t = \text{div} \mathcal{A}(t, \cdot, u_t, \nabla u_t) + \mathcal{B}(t, \cdot, \nabla u_t), \]

where \( \mathcal{A} := (\mathcal{A}_1, \ldots, \mathcal{A}_d) \) and \( \mathcal{B} \) are defined as

\[ \mathcal{A}_i(t, \cdot, u, \nabla u) := \frac{1}{2} \sum_{j=1}^d (\sigma_i \sigma_i^*)_{ij} \partial_j u + \sum_{j=1}^d \left\{ \frac{1}{2} \partial_j (\sigma_i \sigma_i^*)_{ij} - b_i \right\} u, \]

\[ \mathcal{B}(t, \cdot, \nabla u) := -\sum_{i,j=1}^d \{ \partial_j (\sigma_i \sigma_i^*)_{ij} \} \partial_i u. \]

By the Harnack inequality as in [1, Theorem 3] (see also [17]), under the given conditions, for any \( 0 < s < t \leq T \) and \( N > 1 \) with

\[ \tilde{B}_N := \{ x \in \tilde{D} \cap B_N : \rho_\delta(x) \geq N^{-1} \} \]
having positive volume, there exists a constant $c(s, t, N) > 0$ such that satisfies

$$\sup_{\bar{B}_N} p_s(x, \cdot) \leq c(s, t, N) \inf_{\bar{B}_N} p_t(x, \cdot), \quad x \in \bar{D}.$$  

Since $\int_{\bar{B}_N} p_s(x, y)dy \to 1$ as $N \to \infty$, this implies $p_t(x, y) > 0$ for any $(t, x, y) \in (0, T] \times \bar{D} \times D$. In particular, $P_t1_{\bar{B}_N} > 0$. On the other hand, (2.17) implies that $P_t1_{\bar{B}_N}$ is continuous, so that

$$\inf_{x \in D \cap B_N} P_t1_{\bar{B}_N}(x) > 0, \quad t \in (0, T].$$

This together with (2.25) gives

$$\inf_{(D \cap B_N) \times B_N} p_t \geq \frac{1}{c(s, t, N)} \inf_{x \in D \cap B_N} P_11_{\bar{B}_N}(x) > 0, \quad 0 < s < t \leq T.$$  

Therefore, (2.18) holds.

To make Zvonkin’s transform to kill the singular drift, we present the lemma which extends Theorem 2.10 in [27] for $D = \mathbb{R}^d$.

(A5) $D = \mathbb{R}^d$, $\sigma$ and $b^{(0)}$ satisfy the following conditions.

1. $a := \sigma\sigma^*$ is invertible and uniformly continuous with $\|a\|_{\infty} + \|a^{-1}\|_{\infty} < \infty$.
2. $|b^{(0)}| \in \tilde{L}^p$ for some $p > d$.

(A6) $\partial D \in C^{2,L}_b$, (A5) holds for $\bar{D}$ replacing $\mathbb{R}^d$, and (A2)(2) holds.

Lemma 2.5. Assume (A5) for $D = \mathbb{R}^d$ and (A6) for $D \neq \mathbb{R}^d$. Let $L^0 = \frac{1}{2}\text{tr}\{\sigma\sigma^*\nabla^2\} + \nabla b^{(0)}$. Then there exist constants $\lambda_0 > 0$ increasing in $\|b^{(0)}\|_{L^p}$ such that for any $\lambda \geq \lambda_0$ and any $f \in \tilde{L}^k$ for some $k \in (1, \infty)$, the elliptic equation

$$\begin{align*}
(L^0 - \lambda)u &= f, \quad \nabla_n u|_{\partial D} = 0 \quad \text{if} \ D \neq \mathbb{R}^d
\end{align*}$$  

has a unique solution $u \in \tilde{H}^{2,k}$. Moreover, for any $p' \in [k, \infty]$ and $\theta \in [0, 2 - \frac{d}{k} + \frac{d}{p'})$, there exists a constant $c > 0$ increasing in $\|b^{(0)}\|_{L^p}$ such that

$$\lambda^{\frac{1}{2}(2-\theta+\frac{d}{p'}-\frac{d}{2})}\|u\|_{\tilde{H}^{\theta,p'}} + \|u\|_{\tilde{H}^{2,k}} \leq c\|f\|_{\tilde{L}^k}, \quad f \in \tilde{L}^k.$$  

Proof. (a) Let us verify the priori estimate (2.27) for a solution $u$ to (2.26), which in particular implies the uniqueness, since the difference of two solutions solves the equation with $f = 0$.

For $u \in \tilde{H}^{2,k}$ solving (2.26), let

$$\tilde{u}_t = u(1-t), \quad t \in [0, 1].$$

By (2.26) we have

$$\begin{align*}
(\partial_t + L^0 - \lambda)\tilde{u}_t &= f(1-t) - u, \quad t \in [0, 1], \tilde{u}_1 = 0, \quad \nabla_n \tilde{u}_t|_{\partial D} = 0 \quad \text{if} \ D \neq \mathbb{R}^d.
\end{align*}$$
By Theorem 2.1 with \( q = q' = 2 \) in [28] for \( D = \mathbb{R}^d \), and Lemma 2.6 in [25] for \( D \neq \mathbb{R}^d \), there exist constants \( \lambda_1, c_1 > 1 \) increasing in \( \|b(0)\| \tilde{L}p \) and sufficient large \( q > 2 \) such that

\[
\lambda_1^{\frac{1}{2} (2 - \theta + \frac{d}{p} - \frac{d}{k})} \|\bar{u}\| \tilde{H}_q^{\theta, p'} + \|\bar{u}\| \tilde{H}_q^{2, k} \leq c_1 \|f(1 - t) - u\| \tilde{L}_k \leq c_1 \|f\| \tilde{L}_k + c_1 \|u\| \tilde{L}_k. \tag{2.28}
\]

Taking \( \theta = 0, p = p' \) and \( c_2 = |1 - |L^p(\{0,1\})| \), we obtain

\[
\lambda_1^{\frac{1}{2} (2 - \theta)} \|u\| \tilde{L}_p \leq \frac{c_1}{c_2} (\|f\| \tilde{L}_p + \|u\| \tilde{L}_p), \quad \lambda \geq \lambda_1.
\]

Letting \( \lambda_0 > \lambda_1 \) such that

\[
\lambda_0^{\frac{1}{2} (2 - \theta)} \geq 2 \frac{c_1}{c_2},
\]

we obtain

\[
\|u\| \tilde{L}_k \leq \|f\| \tilde{L}_k, \quad \lambda \geq \lambda_0.
\]

Combining this with (2.28) implies (2.27) for some constant \( c > 0 \).

(b) Existence of solution for \( f \in \tilde{L}^k \). Let \( \{f_n\}_{n \geq 1} \subset C^\infty(\bar{D}) \) with \( n f_n|_{\partial D} = 0 \) is \( \partial D \neq \emptyset \) such that \( \|f_n - f\| \tilde{L}_k \to 0 \) as \( n \to \infty \). Let \( P^0_t \) be the Markov semigroup generated by \( L^0 \) and let

\[
u_n = \int_0^\infty e^{-\lambda t} P^0_t f_n \, dt.
\]

By Kolmogorov equation we have

\[
\partial_t P^0_t f_n = L^0 P^0_t f_n = P^0_t L^0 f_n
\]

so that

\[
L^0 \nu_n = \int_0^\infty e^{-\lambda t} L^0 P^0_t f_n \, dt = \int_0^\infty e^{-\lambda t} \partial_t P^0_t f_n \, dt = \lambda \nu_n - f_n.
\]

Then

\[
(L^0 - \lambda)(u_n - u_m) = f_n - f_m, \quad n, m \geq 1.
\]

By (2.27),

\[
\lim_{n, m \to \infty} \{\|u_n - u_m\| \tilde{H}^{\theta, p'} + \|\nabla^2 (u_n - u_m)\| \tilde{L}_k\} = 0,
\]

so that \( u := \lim_{n \to \infty} u_n \) exists in \( \tilde{H}^{\theta, p'} \cap \tilde{H}^{2, k} \), which solves (2.26).

\[
2.3 \quad \text{Proofs of Theorems 2.1, 2.3 and Corollary 2.2}
\]

Proofs of Theorems 2.1, 2.3. It is easy to see that (2.1) implies (A3) (3) for any \( T > 0 \) and \( \phi(r) = 1 \), by Lemma 2.4. (A1) and (A2) imply the well-posedness, strong Feller property and irreducibility of (1.3) and (2.10) respectively. According to [17, Theorem 4.2.1], the strong Feller property and the irreducibility imply the uniqueness of invariant probability measure. So, it remains to prove the existence of the invariant probability measure \( \mu \) and the claimed assertions on the ergodicity.
(a) Let \( u \) solve (2.24) for \( b = -b^{(0)} \) and large enough \( \lambda > 0 \) such that (2.24) implies (2.21). Moreover, for \( \Theta(x) := x + u(x) \), let \( \bar{P}_t \) be the Markov semigroup associated with \( Y_t := \Theta(X_t) \), so that

\[
\bar{P}_t f(x) = \{P_t(f \circ \Theta)\}(\Theta^{-1}(x)), \quad t \geq 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d).
\]

(2.29)

Since \( \lim_{|x| \to \infty} \sup_{|y-x| \leq \varepsilon} |\nabla \bar{V}(y)| = 0 \), by (2.21) and \( V \geq 1 \) we find a constant \( \theta \in (0, 1) \) such that

\[
\|\nabla u(x)\| \vee |\Theta(x) - x| \leq \varepsilon, \quad \theta V(\Theta(x)) \leq V(x) \leq \theta^{-1} V(\Theta(x)), \quad x \in \bar{D}.
\]

Thus, it suffices to prove the desired assertions for \( \bar{P}_t \) replacing \( P_t \), where the unique invariant probability measure \( \bar{\mu} \) of \( \bar{P}_t \) and that \( \mu \) of \( P_t \) satisfies

\[
\bar{\mu} = \mu \circ \Theta^{-1}.
\]

(b) Let \( X^n_t, Y^n_t \) and \( \tau_n \) be in the proof of Lemma 2.4 for the present time-homogenous setting. Since \( Y^n_t = Y_t \) and \( 1_{B_n}(X^n_t) = 1 \) for \( t \leq \tau_n \), and since \( \tau_n \to \infty \) as \( n \to \infty \), (2.22) implies

\[
d\bar{Y}_t = \{b^{(1)} + \lambda u + \nabla_{b^{(1)}} u\}(X_t)dt + \{(\nabla \Theta)\sigma\}(X_t)dW_t + n(X_t)dt,
\]

so that for any \( \varepsilon \in (0, 1 \wedge r_0) \), where \( r_0 > 0 \) is in (2.12) when \( \partial D \neq \emptyset \), by Itô’s formula and (2.12), we find a constant \( c_{\varepsilon} > 0 \) such that

\[
d\{\bar{V}(Y_t) + M_t\} \leq \left\{b^{(1)} + \nabla_{b^{(1)}} u\}(X_t), \nabla \bar{V}(Y_t)\right\} + c_{\varepsilon}(\|\nabla \bar{V}(Y_t)\| + \|\nabla^2 \bar{V}(Y_t)\|)\right\}dt
\]

\[
\leq \left\{(b^{(1)}(X_t), \nabla \bar{V}(X_t)) + \varepsilon |b^{(1)}(X_t)| \sup_{B(X_t, \varepsilon)} (|\nabla \bar{V}| + \|\nabla^2 \bar{V}\|) + c_{\varepsilon} \sup_{B(X_t, \varepsilon)} (|\nabla \bar{V}| + \|\nabla^2 \bar{V}\|)\right\}dt.
\]

Combining this with (2.21) and (2.12) for \( D \neq \mathbb{R}^d \), when \( \varepsilon > 0 \) is small enough we find constants \( c_1, c_2 > 0 \) such that

\[
d\{\bar{V}(Y_t) + M_t\} \leq \{c_1 - c_2 \Phi(\bar{V}(X_t))\}dt.
\]

By (2.30), this implies that for some constant \( c_4 > 0 \),

\[
d\bar{V}(Y_t) \leq \{c_4 - c_2 \Phi(\theta V(\bar{Y}_t))\}dt - dM_t.
\]

Thus,

\[
\int_0^t \mathbb{E}\Phi(\theta V(Y_s))ds \leq \frac{c_4 + V(x)}{c_2} < \infty, \quad t > 0, Y_0 = x \in \Theta(\bar{D}).
\]

Since \( \Phi(\theta V) \) is a compact function, this implies the existence of invariant probability \( \bar{\mu} \) according to the standard Bogoliubov-Krylov’s tightness argument. Moreover, (2.32) implies \( \bar{\mu}(\Phi(\theta V)) < \infty \), so that by (2.30) and (2.31), \( \mu(\Phi(\varepsilon_0 V)) < \infty \) holds for \( \varepsilon_0 = \theta^2 \).

(c) By (2.18), (2.29) and (2.30), any compact set \( K \subset \Theta(\bar{D}) \) is a petite set of \( \bar{P}_t \), i.e. there exist \( t > 0 \) and a nontrivial measure \( \nu \) such that

\[
\inf_{x \in K} \bar{P}_t^* \delta_x \geq \nu.
\]
When \( \Phi(r) \geq kr \) for some constant \( k > 0 \), (2.32) implies

\[
\hat{P}_t V(x) \leq \frac{k_1}{k_2} + e^{-k_2 t} V(x), \quad t \geq 0, x \in \Theta(D)
\]

for some constants \( k_1, k_2 > 0 \). Since \( \lim_{|x| \to \infty} V(x) = \infty \) and as observed above that any compact set is a petite set for \( \hat{P}_t \), by Theorem 5.2(c) in [8], we obtain

\[
\| \hat{P}_t^* \delta_x - \hat{\mu} \|_V \leq c e^{-\lambda t} V(x), \quad x \in \Theta(D), t \geq 0
\]

for some constants \( c, \lambda > 0 \). Thus,

\[
\| \hat{P}_t^* \delta_x - \hat{P}_t^* \delta_y \|_V \leq c e^{-\lambda t} (V(x) + V(y)), \quad t \geq 0, x, y \in \Theta(D).
\]

Therefore, for any probability measures \( \mu_1, \mu_2 \) on \( \Theta(D) \),

\[
\| \hat{P}_t^* \mu_1 - \hat{P}_t^* \mu_2 \|_V = \| \hat{P}_t^* (\mu_1 - \mu_2)^+ - \hat{P}_t^* (\mu_1 - \mu_2)^- \|_V
\]

\[
= \frac{1}{2} \| \mu_1 - \mu_2 \|_{\text{var}} \left\| \hat{P}_t^* \frac{2(\mu_1 - \mu_2)^+}{\| \mu_1 - \mu_2 \|_{\text{var}}} - \hat{P}_t^* \frac{2(\mu_1 - \mu_2)^-}{\| \mu_1 - \mu_2 \|_{\text{var}}} \right\|_V
\]

\[
\leq \frac{c}{2} e^{-\lambda t} \| \mu_1 - \mu_2 \|_{\text{var}} \left( \frac{2(\mu_1 - \mu_2)^+}{\| \mu_1 - \mu_2 \|_{\text{var}}} + \frac{2(\mu_1 - \mu_2)^-}{\| \mu_1 - \mu_2 \|_{\text{var}}} \right)(V)
\]

\[
\leq c e^{-\lambda t} \| \mu_1 - \mu_2 \|_V.
\]

This together with (2.29) and (2.30) implies (2.3) for some constants \( c, \lambda > 0 \).

(d) Let \( \Phi \) be convex. By Jensen’s inequality and (2.32), \( \gamma_t := \theta E[V(Y_t)] \) satisfies

\[
\frac{d}{dt} \gamma_t \leq \theta c_4 - \theta c_2 \Phi(\gamma_t), \quad t \geq 0.
\]

Let

\[
H(r) := \int_0^r \frac{ds}{\Phi(s)}, \quad r \geq 0.
\]

We aim to prove that for some constant \( k > 1 \)

\[
\gamma_t \leq k + H^{-1}(H(\gamma_0) - tk^{-1}), \quad t \geq 0,
\]

where \( H^{-1}(r) := 0 \) for \( r \leq 0 \). We prove this estimate by considering three situations.

(1) Let \( \Phi(\gamma_0) \leq \frac{c_4}{c_2} \). Since (2.34) implies \( \gamma'_t \leq 0 \) for \( \gamma_t \geq \Phi^{-1}(\frac{c_4}{c_2}) \), so

\[
\gamma_t \leq \Phi^{-1}(\frac{c_4}{c_2}), \quad t \geq 0.
\]

(2) Let \( \frac{c_4}{c_2} < \Phi(\gamma_0) \leq \frac{2c_4}{c_2} \). Then (2.34) implies \( \gamma'_t \leq 0 \) for all \( t \geq 0 \) so that

\[
\gamma_t \leq \Phi^{-1}(2c_4/c_2), \quad t \geq 0.
\]
(3) Let \( \Phi(\gamma_0) > \frac{2c_4}{c_2} \). If 
\[
t \leq t_0 := \inf \left\{ t \geq 0 : \Phi(\gamma_t) \leq \frac{2c_4}{c_2} \right\},
\]
then (2.34) implies
\[
\frac{dH(\gamma_t)}{dt} = \frac{\gamma'_t}{\Phi(\gamma_t)} \leq -\frac{\theta c_2}{2},
\]
so that
\[
H(\gamma_t) \leq H(\gamma_0) - \frac{\theta c_2}{2} t, \quad t \in [0, t_0],
\]
which implies
\[
\gamma_t \leq H^{-1}(H(\gamma_0) - \frac{\theta c_2}{2} t), \quad t \in [0, t_0].
\]
Noting that when \( t > t_0 \), \( (\gamma_t)_{t \geq t_0} \) satisfies (2.34) with \( \gamma_{t_0} \) satisfies \( \frac{c_4}{c_2} < \Phi(\gamma_{t_0}) \leq \frac{2c_4}{c_2} \), so that (2.36) holds, i.e.
\[
\gamma_t \leq \Phi^{-1}(2c_4/c_2).
\]
In conclusion, we obtain
\[
\gamma_t \leq \Phi^{-1}(2c_4/c_2) + H^{-1}(H(\gamma_0) - \frac{\theta c_2}{2} t), \quad t \geq 0.
\]
Combining this with (1) and (2), we prove (2.35) for some constant \( k > 1 \).

(e) Since \( 1 \leq \Phi(r) \to \infty \) as \( r \to \infty \), when \( \Phi \) is convex we find a constant \( \delta > 0 \) such that 
\[
\Phi(r) \geq \delta r, r \geq 0.
\]
So, by step (b), (2.3) holds. Combining this with (2.35) and applying the semigroup property, we derive
\[
\| \hat{P}_t \delta_x - \mu \|_V \leq \sup_{f \leq V} | \hat{P}_t/2(\hat{P}_t/2f - \hat{\mu}(f))(x) | \leq ce^{-\lambda t/2} \hat{P}_t/2V(x) \leq c\{k + H^{-1}(H(\theta V(x)) - (2k)^{-1}t)\} e^{-\lambda t/2}.
\]
Combining this with (2.29), (2.30) and (2.31), we prove (2.4) for some constants \( k, \lambda > 0 \).

Finally, if \( H(\infty) < \infty \), we take \( t^* = kH(\infty) \) in (2.4) to derive
\[
\sup_{x \in D} \| \hat{P}_t \delta_x - \mu \|_V \leq ce^{-\lambda t}, \quad t \geq t^*
\]
for some constants \( c, \lambda > 0 \), which implies (2.5) by the argument leading to (2.3) in step (c).

Proof of Corollary 2.2. By (2.8), for any \( \theta \in ((1 - \alpha)^+, \frac{1}{2}) \) there exists a constant \( c_3 > 0 \) such that
\[
\phi(r) \geq c_3(1 + r)^{1-\theta}, \quad r \geq 0.
\]
Then (2.1) holds for \( V := e^{(1+\| \cdot \|^2)\theta} \) and \( \Phi(r) = r \). So the first assertion in (1) follows from Theorem 2.1(1).

Next, (2.7) and (2.9) imply (2.1) for \( V := \psi(| \cdot |^2) \) and \( \Phi(r) = r \), so that the second assertion in (1) holds by Theorem 2.1(1).

Finally, if \( \int_0^\infty \frac{ds}{\phi(s)} < \infty \), then for any \( q > 0 \), (2.1) holds for \( V := (1 + | \cdot |^2)^q \) and \( \Phi(r) = (1 + r)^{1-\frac{q}{2}}\phi(r^{\frac{1}{2}}) \), so that \( \int_0^\infty \frac{ds}{\phi(s)} < \infty \). Then the proof is finished by Theorem 2.1(2).
3 Uniform ergodicity for singular reflecting McKean-Vlasov SDEs

We now consider the SDE \((\text{1.1})\) for \(D = \mathbb{R}^d\) or \(D\) being a \(C^2_{b,1}\) domain. To prove the uniform ergodicity, compare \((\text{1.1})\) with the following classical SDE with fixed distribution parameter \(\gamma \in \mathcal{P}\):

\[
dX_t^\gamma = b(X_t^\gamma, \gamma) + \sigma(X_t^\gamma)dW_t + n(X_t^\gamma)dl_t^\gamma,
\]

where we set \(l_t^\gamma = 0\) if \(D = \mathbb{R}^d\). If for any \(\nu \in \mathcal{P}\), \((A1)\) for \(D = \mathbb{R}^d\) or \((A2)\) for \(D \neq \mathbb{R}^d\) holds for \(b(\cdot, \nu)\) replacing \(b\), then the well-posedness follows from that of \((3.1)\) and [25, Theorem 3.2] for \(k = 0\). Let \((P_t^\gamma)^*\nu = \mathcal{L}_{X_t^\gamma}\) for \(\mathcal{L}_{X_t^\gamma} = \nu\).

Let \(\zeta(\gamma_1, \gamma_2) := \sigma^*(\sigma\sigma^*)^{-1}[b(\cdot, \gamma_2) - b(\cdot, \gamma_1)]\), \(\gamma_1, \gamma_2 \in \mathcal{P}\). We make the following assumption on the dependence of distribution.

\((H1)\) \((A1)\) for \(D = \mathbb{R}^d\) or \((A2)\) for \(D \neq \mathbb{R}^d\) holds with \(b(\cdot, \nu)\) replacing \(b\) uniformly in \(\nu \in \mathcal{P}\).

\((H2)\) There exist constants \(q \geq 2\) and \(k > 0\) such that for any \(\gamma \in \mathcal{P}\) and \(\nu \in C([0, 1]; \mathcal{P})\),

\[
\begin{align*}
\int_0^t ds \int_D |\zeta(\gamma, \nu_s)|^2 d(P_s^\gamma)^*\nu_0 & \leq k^2 \left( \int_0^t \|\nu_s - \gamma\|_{\text{var}}^q ds \right)^{\frac{2}{q}}, \\
\int_D e^{s\int_D |\zeta(\gamma, \nu_s)|^2} d(P_s^\gamma)^*\nu_0 & < \infty, \quad t \geq 0.
\end{align*}
\]

Remark 3.1. (1) Obviously, \((H2)\) holds for \(q = 2\) if there exists a constant \(\kappa > 0\) such that

\[
|b(x, \gamma_1) - b(x, \gamma_2)| \leq \kappa \|\gamma_1 - \gamma_2\|_{\text{var}}, \quad x \in D, \gamma_1, \gamma_2 \in \mathcal{P}.
\]

By a standard fixed point argument, this together with \((H1)\) implies the well-posedness of \((\text{1.1})\) for any initial value, see [25].

(2) In general, \((H2)\) follows from Krylov’s estimate for \(p, q > 2\) with \(\frac{4}{p} + \frac{2}{q} < 1\) and the condition

\[
\|\zeta(\gamma_1, \gamma_2)\|_{L^p} \leq k \|\gamma_1 - \gamma_2\|_{\text{var}}, \quad \gamma_1, \gamma_2 \in \mathcal{P}.
\]

Note that under \((H1)\), the Krylov’s estimate holds when \(b\) contains an \(\bar{L}^p\) term for \(p > d\) and a Lipschitz continuous term, see [25] for details.

Theorem 3.1. Assume \((H1)\) and \((H2)\) and let \((\text{1.1})\) be well-posed. If \(k\) is small enough and \(\Phi\) is convex with \(\int_0^\infty \frac{ds}{\Phi(s)} < \infty\), then \(P_t^\gamma\) has a unique invariant probability measure \(\mu\), \(\mu(\Phi(\xi_0 V)) < \infty\) holds for some constant \(\xi_0 > 0\), and there exist constants \(c, \lambda > 0\) such that

\[
\|P_t^\gamma \nu - \mu\|_{\text{var}} \leq ce^{-\lambda t}\|\mu - \nu\|_{\text{var}}, \quad t \geq 0, \nu \in \mathcal{P}.
\]

To prove this result, we first present a general result deducing the uniform ergodicity of McKean-Vlasov SDEs from that of classical ones.

The following result says that if \((3.1)\) is uniformly ergodic uniformly in \(\gamma\), and if the dependence of \(b(x, \mu)\) on \(\mu\) is weak enough, then \((\text{1.1})\) is uniformly ergodic.
Lemma 3.2. Assume \((H_1)\) and that for any \(\gamma \in \mathcal{P}^\gamma\), \((P_t^\gamma)^*\) has a unique invariant probability measure \(\mu_{\gamma}\) such that

\[
\| (P_t^\gamma)^* \mu - \mu_{\gamma} \|_{\text{var}} \leq ce^{-\lambda t} \| \mu - \mu_{\gamma} \|_{\text{var}}, \quad t \geq 0, \gamma, \mu \in \mathcal{P}
\]

holds for some constants \(c, \lambda > 0\). Then \((1.1)\) is well-posed and the following assertions hold.

(1) If there exists a constant \(\kappa \in (0, \kappa_1)\) for

\[
\kappa_1 := \sup_{t > (\log c)/\lambda} \frac{1 - ce^{-\lambda t}}{\sqrt{t}},
\]

such that

\[
\int_D |\zeta(\gamma_1, \gamma_2)|^2 d\mu_{\gamma} \leq \kappa^2 \| \gamma_1 - \gamma_2 \|_{\text{var}}^2, \quad \gamma_1, \gamma_2 \in \mathcal{P},
\]

then \(P_t^*\) associated with \((1.1)\) has a unique invariant probability measure \(\mu\).

(2) Let \(\mu\) be \(P_t^*\)-invariant. If there exist constants \(q \geq 2\) and \(k \in (0, k_q)\), where

\[
k_q := \sup \left\{ k > 0 : \frac{4q^{-1}(ck)^q e^{2q-1k^q t}}{q\lambda + 2q-1k^q} \leq 1 \right\},
\]

such that for \(\hat{t} := \frac{\log(2c)}{\lambda}\),

\[
\mathbb{E} \int_0^{\hat{t}} |\zeta(\mu, P_s^\nu)(X_s^\nu)|^2 ds \leq k^2 \left( \int_0^{\hat{t}} \| \mu - P_s^\nu \|_{\text{var}}^q ds \right)^{\frac{2}{q}}, \quad t \in (0, \hat{t}), \nu \in \mathcal{P},
\]

then there exists a constant \(c' > 0\) such that

\[
\| P_t^\nu \nu - \mu \|_{\text{var}}^q \leq c' e^{-\lambda' t} \| \nu - \mu \|_{\text{var}}^q, \quad t \geq 0, \nu \in \mathcal{P}
\]

holds for

\[
\lambda' := -\frac{\lambda}{\log(2c)} \log \left( \frac{1}{2} + \frac{4q^{-1}(ck)^q e^{2q-1k^q t}}{q\lambda + 2q-1k^q} \right) > 0.
\]

Proof. (a) Existence and uniqueness of \(\mu\). For any \(\gamma \in \mathcal{P}\), \((3.3)\) implies that \(P_t^\gamma\) has a unique invariant probability measure \(\mu_{\gamma}\). It suffices to prove that the map \(\gamma \mapsto \mu_{\gamma}\) has a unique fixed point \(\mu\), which is the unique invariant probability measure of \(P_t^*\).

For \(\gamma_1, \gamma_2 \in \mathcal{P}\), \((3.3)\) implies

\[
\| (P_t^{\gamma_1})^* \mu_{\gamma_2} - \mu_{\gamma_1} \|_{\text{var}} \leq ce^{-\lambda t} \| \mu_{\gamma_2} - \mu_{\gamma_1} \|_{\text{var}}, \quad t \geq 0.
\]

On the other hand, let \((X_t^1, X_t^2)\) solve the SDEs

\[
dX_t^i = b(X_t^i, \gamma_t) dW_t + \sigma(X_t^i) d\lambda_t, \quad i = 1, 2
\]

with \(X_0^1 = X_0^2\) having distribution \(\mu_{\gamma_2}\). Since \(\mu_{\gamma_2}\) is \((P_t^{\gamma_2})^*\)-invariant, we have

\[
\mathcal{L}X_t^2 = (P_t^{\gamma_2})^* \mu_{\gamma_2} = \mu_{\gamma_2}, \quad \mathcal{L}X_t^1 = (P_t^{\gamma_1})^* \mu_{\gamma_2}, \quad t \geq 0.
\]
By \((H_2)\),
\[
R_t = e^{\int_0^t \langle \zeta(\gamma_1, \gamma_2)(X^1_s), dW_s \rangle - \frac{1}{2} \int_0^t \langle \zeta(\gamma_1, \gamma_2)(X^1_s) \rangle^2 ds}, \quad t \geq 0
\]
is a martingale, and by Girsanov’s theorem, for any \(t > 0\),
\[
\tilde{W}_r := W_r - \int_0^r \zeta(\gamma_1, \gamma_2)(X^1_s) ds, \quad r \in [0, t]
\]
is a Brownian motion under \(Q_t := R_t \mathbb{P}\). Reformulating the SDE for \(X^1_t\) as
\[
dX^1_r = b(X^1_r, \gamma_2) dr + \sigma(X^1_r) d\tilde{W}_r + n(X^1_r) dl^1_r, \quad r \in [0, t],
\]
by \(X^1_0 = X^2_0\) and the weak uniqueness, the law of \(X^1_t\) under \(Q_t\) satisfies
\[
\mathcal{L}_{X^1_t|Q_t} = \mathcal{L}_{X^2_t} = (P^T_\gamma)^* \mu_{\gamma_2} = \mu_{\gamma_2}.
\]
Combining this with (3.8) and Pinsker’s inequality, we obtain
\[
\|(P^T_\gamma)^* \mu_{\gamma_2} - \mu_{\gamma_2}\|_{\text{var}}^2 \leq \|(P^T_\gamma)^* \mu_{\gamma_2} - (P^T_\gamma)^* \mu_{\gamma_2}\|_{\text{var}}^2 = \sup_{|f| \leq 1} \left| \mathbb{E}[f(X^1_t)] - \mathbb{E}[f(X^1_t)] R_t \right|^2 \leq (\mathbb{E}[R_t - 1])^2 \leq 2 \mathbb{E}[R_t \log R_t]
\]
\[
= 2 \mathbb{E}_{Q_t} \log R_t = \mathbb{E}_{Q_t} \int_0^t |\zeta(\gamma_1, \gamma_2)|^2(X^1_s)^2 |d\mu_{\gamma_2}.\tag{3.9}
\]
Then (3.4) implies
\[
\|(P^T_\gamma)^* \mu_{\gamma_2} - \mu_{\gamma_2}\|_{\text{var}}^2 \leq \kappa^2 t \|\gamma_1 - \gamma_2\|_{\text{var}}^2.
\]
Combining this with (3.7) and taking \(t = \frac{\log(2c)}{\lambda}\), we derive
\[
\|\mu_{\gamma_1} - \mu_{\gamma_2}\|_{\text{var}} \leq \|(P^T_\gamma)^* \mu_{\gamma_2} - \mu_{\gamma_1}\|_{\text{var}} + \|(P^T_\gamma)^* \mu_{\gamma_2} - \mu_{\gamma_2}\|_{\text{var}} \leq ce^{-\lambda t} \|\mu_{\gamma_1} - \mu_{\gamma_2}\|_{\text{var}} + \kappa \sqrt{t} \|\gamma_1 - \gamma_2\|_{\text{var}}, \quad t > 0.
\]
Thus,
\[
\|\mu_{\gamma_1} - \mu_{\gamma_2}\|_{\text{var}} \leq \inf_{t > \log(c)/\lambda} \frac{\kappa \sqrt{t}}{1 - ce^{-\lambda t}} \|\gamma_1 - \gamma_2\|_{\text{var}} = \frac{\kappa}{\kappa_1} \|\gamma_1 - \gamma_2\|_{\text{var}}.
\]
Since \(\kappa < \kappa_1\), \(\mu_{\gamma}\) is contractive in \(\gamma\), hence has a unique fixed point.

(b) Uniform ergodicity. Let \(\mu\) be the unique invariant probability measure of \(P^*_t\), and for any \(\nu \in \mathcal{P}\) let \((\bar{X}_0, X_0)\) be \(\mathcal{F}_0\)-measurable such that
\[
\mathbb{P}(\bar{X}_0 \neq X_0) = \frac{1}{2} \|\mu - \nu\|_{\text{var}}, \quad \mathcal{L}_{\bar{X}_0} = \mu, \quad \mathcal{L}_{X_0} = \nu.
\]
Let \(\bar{X}_t\) and \(X_t\) solve the following SDEs with initial values \(\bar{X}_0\) and \(X_0\) respectively:
\[
d\bar{X}_t = b(\bar{X}_t, \mu) dt + \sigma(\bar{X}_t) dW_t + n(\bar{X}_t) dl^1_t, \quad dX_t = b(X_t, P^*_t \nu) dt + \sigma(X_t) dW_t + n(X_t) dl^1_t.
\]
Since \( \mu \) is \( P_t^* \)-invariant, we have
\[
(3.10) \quad \mathcal{L}_{X_t} = (P_t^\mu)^* \mu = P_t^* \mu = \mu.
\]
Moreover, \( \mathcal{L}_{X_t} = P_t^\nu \) by the definition of \( P_t^* \). Let
\[
\bar{R}_t := \exp \int_0^t \langle (P_t^* \nu - \mu)(X_s), dW_s \rangle - \frac{1}{2} \int_0^t \langle (P_t^* \nu - \mu)(X_s) \rangle^2 ds.
\]
Similarly to (3.9), by Girsanov’s theorem we have \( \mathcal{L}_{X_t | \bar{R}_t} = (P_t^\mu)^* \nu \), so that Pinsker’s inequality and (3.5) yield
\[
\| (P_t^* \nu - P_t^s \nu) \|^2_{\text{var}} = \sup_{\| f \|_1} \| E[f(X_t) \bar{R}_t] - E[f(X_t)] \|^2 \\
\leq k^2 \left( \int_0^t \| \mu - P_s^\nu \|^q_{\text{var}} \right) ds, \quad t \in [0, \tilde{t}].
\]
This together with (3.7) for \( \gamma_1 = \mu \) and (3.10) gives
\[
\| P_t^* \nu - \mu \|^q_{\text{var}} \leq 2^{q-1} \left( \| P_t^* \nu - (P_t^\mu)^* \nu \|^q_{\text{var}} + \| (P_t^\mu)^* \nu - \mu \|^q_{\text{var}} \right) \\
\leq 2^{q-1} k^q \left( \int_0^t \| \mu - P_s^\nu \|^q_{\text{var}} ds + 2^{q-1} c^q e^{-q \lambda t} \| \nu - \mu \|^q_{\text{var}} \right), \quad t \in [0, \tilde{t}].
\]
By Gronwall’s inequality we obtain
\[
\| P_t^* \nu - \mu \|^q_{\text{var}} \leq \left( 2^{q-1} c^q e^{-q \lambda t} + 4^{q-1} k^q c^q \int_0^t e^{-q \lambda s + 2^{q-1} k q (t-s)} ds \right) \| \mu - \nu \|^q_{\text{var}}, \quad t \in [0, \tilde{t}].
\]
Taking \( t = \tilde{t} := \frac{\log(2c)}{\lambda} \), we arrive at
\[
\| P_t^* \nu - \mu \|^2_{\text{var}} \leq \delta_k \| \mu - \nu \|^2_{\text{var}}, \quad \nu \in \mathcal{P}
\]
for
\[
\delta_k := \left( 1 + \frac{4^{q-1} (ck)^q e^{2^{q-1} k q t}}{q \lambda + 2^{q-1} k q} \right) < 1, \quad k \in (0, k_p).
\]
So, (3.6) holds for some constant \( c' > 0 \) due to the semigroup property \( P_{t+s}^* = P_t^* P_s^* \). \( \square \)

To verify condition (3.3), we present below a Harris type theorem on the uniform ergodicity for a family of Markov processes.

**Lemma 3.3.** Let \((E, \rho)\) be a metric space and let \( \{ (P_t^i)_{t \geq 0} : i \in I \} \) be a family of Markov semigroups on \( \mathcal{B}(E) \). If there exist \( t_0, t_1 > 0 \) and measurable set \( B \subset E \) such that
\[
(3.11) \quad \alpha := \inf_{i \in I, x \in E} P_{t_0}^i 1_B(x) > 0,
\]
\begin{equation}
\beta := \sup_{i \in I, x,y \in B} \|(P_{t_1}^{i})^* \delta_x - (P_{t_1}^{i})^* \delta_y\|_{\text{var}} < 2,
\end{equation}

then there exists \( c > 0 \) such that
\begin{equation}
\sup_{i \in I, x,y \in E} \|(P_{t_1}^{i})^* \delta_x - (P_{t_1}^{i})^* \delta_y\|_{\text{var}} \leq c e^{-\lambda t}, \quad t \geq 0
\end{equation}
holds for
\[ \lambda := \frac{1}{t_0 + t_1} \log \frac{2}{2 - a^2(2 - \beta)} > 0. \]

Proof. The proof is more or less standard. By the semigroup property, we have
\[
\|(P_{t_0+t_1}^{i})^* \delta_x - (P_{t_0+t_1}^{i})^* \delta_y\|_{\text{var}} \\
= \sup_{|f| \leq 1} \left| \int_{E \times E} \left( (P_{t_1}^i f(x') - P_{t_1}^i f(y')) \{(P_{t_0}^i)^* \delta_x\} \{(P_{t_0}^i)^* \delta_y\} \right) \text{d}y' \right| \\
\leq \int_{B \times B} \|(P_{t_1}^{i})^* \delta_{x'} - (P_{t_1}^{i})^* \delta_{y'}\|_{\text{var}} \{(P_{t_0}^i)^* \delta_x\} \{(P_{t_0}^i)^* \delta_y\} \text{d}(x', y') \\
+ 2 \int_{B \times B} \{(P_{t_0}^i)^* \delta_x\} \{(P_{t_0}^i)^* \delta_y\} \text{d}(x', y') \\
\leq \beta \{P_{t_0}^i 1_B(x)\} P_{t_0}^i 1_B(y) + 2 \int_{B \times B} \{(P_{t_0}^i)^* \delta_x\} \{(P_{t_0}^i)^* \delta_y\} \text{d}(x', y') \\
\leq 2 - \alpha^2(2 - \beta).
\]

Thus, for \( \delta := \frac{2 - \alpha^2(2 - \beta)}{2} < 1 \), we have
\[
\|(P_{t_0+t_1}^{i})^* \delta_x - (P_{t_0+t_1}^{i})^* \delta_y\|_{\text{var}} \leq \delta \|(\delta_x - \delta_y)\|_{\text{var}}, \quad x, y \in E.
\]

Combining this with the semigroup property, we find constants \( c > 0 \) such that (3.13) holds for the claimed \( \lambda > 0 \).

Proof Theorem 3.7. According to Theorems 2.1-2.3 and Lemma 3.2, it suffices to prove (3.3) for small \( k > 0 \). By Lemma 3.3, we only need to prove (3.11) and (3.12) for the family \( \{P_{t_0}^i : \gamma \in \mathcal{P}\} \), where \( P_{t_0}^i f(x) := \int_{\mathcal{D}} f d(P_{t_1}^i)^* \delta_x \).

(a) Proof of (3.12). Let us fix \( \gamma \in \mathcal{P} \), and let \( X_{t}^{x,\gamma} \) solve (3.1) with \( X_0^{\gamma} = x \). For any \( \nu \in \mathcal{P} \), by Girsanov’s theorem we have
\[
P_{t_0}^\nu f(x) = \mathbb{E}[f(X_{t}^{x,\gamma} R_{t_0}^{x,\gamma,\nu})], \quad t \geq 0,
\]
where
\[
R_{t_0}^{x,\gamma,\nu} := e^{\int_0^t \langle \zeta(\gamma, \nu) (X_{s}^{x,\gamma}), dW_s \rangle - \frac{1}{2} \gamma(\zeta(\gamma, \nu) (X_{s}^{x,\gamma}))^2 ds}.
\]

By (H2), Girsanov’s theorem and Pinsker’s inequality, we obtain
\[
\|(P_{t_0}^{\gamma})^* \delta_z - (P_{t_0}^{\gamma})^* \delta_z\|_{\text{var}} \leq (\mathbb{E}[R_{t_0}^{x,\gamma,\nu} - 1])^2 \leq k^2 t^{\lambda/2} \|\gamma - \nu\|_{\text{var}}^2 \leq 4 k^2 t^{\lambda/2}, \quad t \geq 0, z \in \mathcal{D}, \nu \in \mathcal{P}.
\]

Taking \( t = t_1 = (4k)^{-\alpha} \), we obtain
\begin{equation}
\sup_{\nu \in \mathcal{P}} \|(P_{t_1}^{\gamma})^* \delta_z - (P_{t_1}^{\gamma})^* \delta_z\|_{\text{var}} \leq \frac{1}{2}, \quad z \in \mathcal{D}, \nu \in \mathcal{P}.
\end{equation}
On the other hand, by (2.17), there exists \( x_0 \in D \) and a constant \( \varepsilon > 0 \) such that \( B(x_0, \varepsilon) \subset D \) and
\[
\| (P^\gamma_{t_1})^\# \delta_x - (P^\gamma_{t_1})^\# \delta_y \|_{\text{var}} \leq \frac{1}{4}, \quad x, y \in B(x_0, \varepsilon).
\]
Combining this with (3.14) we derive
\[
\sup_{\nu \in \mathcal{P}} \| (P^\nu_{t_1})^\# \delta_x - (P^\nu_{t_1})^\# \delta_y \|_{\text{var}} \leq \frac{3}{2} < 2, \quad x, y \in B(x_0, \varepsilon).
\]
So, (3.12) holds for \( B = B(x_0, \varepsilon) \).

(b) Let \( u \) solve (2.26) for \( f = -b(0) \) and large \( \lambda > 0 \) such that (2.21) holds, and let \( \Theta(x) = x + u(x) \). By (H1), we see that (2.32) holds with \( Y_{t,\nu} := \Theta(X_{t,\nu}) \) replacing \( Y_t \) for all \( \nu \in \mathcal{P} \). So, by \( H(\infty) < \infty \) and the argument leading to (2.35), we obtain
\[
\sup_{\nu \in \mathcal{P}, x \in \bar{D}} \mathbb{E}[V(Y_{t,\nu})] \leq \theta^{-1} k, \quad t \geq kH(\infty) =: t_2.
\]
This together with (2.30) implies
\[
\sup_{\nu \in \mathcal{P}, x \in \bar{D}} \mathbb{E}[V(X_{t,\nu})] \leq \theta^{-2} k, \quad t \geq t_2.
\]
Letting \( K := \{ V \leq 2\theta^{-2}k \} \), we derive
\[
\inf_{\nu \in \mathcal{P}, x \in \bar{D}} P^\nu_{t_2}1_{K}(x) \geq \frac{1}{2}.
\]

On the other hand, by Girsanov’s theorem and Schwartz’s inequality, we find a constant \( c_0 > 0 \) such that
\[
P^\nu_{t_1}1_{B(x_0, \varepsilon)}(x) = \mathbb{E}[1_{B(x_0, \varepsilon)}(X_{t_1,\nu})] \geq \frac{\mathbb{E}[1_{B(x_0, \varepsilon)}(X_{t_1,\nu})]^2}{\mathbb{E}[R_{t_1,\nu}]} \geq c_0 (P^\gamma_{t_1}1_{B(x_0, \varepsilon)}(x))^2.
\]
Since \( K \) is bounded, combining this with Lemma 2.4 for \( P^\gamma_{t_1} \), we find a constant \( c_1 > 0 \) such that
\[
\inf_{\nu \in \mathcal{P}, x \in K} P^\nu_{t_1}1_{B(x_0, \varepsilon)}(x) \geq c_1.
\]
This together with (3.15) and the semigroup property yields
\[
P^\nu_{t_2 + 1}1_{B(x_0, \varepsilon)}(x) \geq P^\nu_{t_2} \{ 1_K P^\nu_{t_1}1_{B(x_0, \varepsilon)} \}(x) \geq c_1 P^\nu_{t_2}1_{K}(x) \geq \frac{c_1}{2} > 0, \quad x \in \bar{D}, \nu \in \mathcal{P}.
\]
Therefore, (3.11) holds for \( t_0 = t_2 + 1 \).

4 Exponential ergodicity in entropy and Wasserstein distance

In this section, we consider the following reflecting McKean-Vlasov SDE where the noise may also be distribution dependent:
\[
dX_t = b(X_t, \mathcal{L}_X)dt + \sigma_t(X_t, \mathcal{L}_X)dW_t + n(X_t)dl_t, \quad t \geq 0,
\]
where
\[ b : [0, \infty) \times \bar{D} \times \mathcal{P} \rightarrow \mathbb{R}^d, \quad \sigma : [0, \infty) \times \bar{D} \times \mathcal{P} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^n \]
are measurable. We study the exponential ergodicity under entropy and weighted Wasserstein distance for dissipative and partially dissipative cases respectively, such that the corresponding results in [16] and [24] are extended to the reflecting setting. For simplicity, we only consider convex \( D \), for which the local time on boundary does not make trouble in the study.

**A7** Let \( k > 1, \mathcal{P}_k := \{ \mu \in \mathcal{P} : \mu(|\cdot|^k) < \infty \} \). \( D \) is convex, \( b \) and \( \sigma \) are bounded on bounded subsets of \([0, \infty) \times \bar{D} \times \mathcal{P}_k(\bar{D})\), and the following two conditions hold.

1. For any \( T > 0 \) there exists a constant \( K > 0 \) such that
   \[
   \| \sigma_t(x, \mu) - \sigma_t(y, \nu) \|_{HS}^2 + 2\langle x - y, b_t(x, \mu) - b_t(y, \nu) \rangle^+
   \leq K \{ |x - y|^2 + |x - y| \mathbb{W}_k(\mu, \nu) + 1_{\{k \geq 2\}} \mathbb{W}_k(\mu, \nu)^2 \}, \quad t \in [0, T], x, y \in \bar{D}, \mu, \nu \in \mathcal{P}_k(\bar{D}).
   \]

2. There exists a subset \( \tilde{\partial}D \subset \partial D \) such that
   \[
   \langle y - x, n(x) \rangle \geq 0, \quad x \in \partial D \setminus \tilde{\partial}D, \quad y \in \bar{D},
   \]
   and when \( \tilde{\partial}D \neq \emptyset \), there exists \( \tilde{\rho} \in C^0_b(\bar{D}) \) such that \( \tilde{\rho}|_{\partial D} = 0, \langle \nabla \tilde{\rho}, n \rangle|_{\partial D} \geq 1_{\partial D} \) and
   \[
   \sup_{(t, x) \in [0, T] \times \bar{D}} \{ \| (\sigma_t^\mu)^* \nabla \tilde{\rho} \|^2(x) + \langle b_t^\mu, \nabla \tilde{\rho} \rangle^-(x) \} < \infty, \quad \mu \in C([0, T]; \mathcal{P}_k(\bar{D})).
   \]

According to [25], this assumption implies the well-posedness of (4.1) for distributions in \( \mathcal{P}_k \). Let \( P_t^* \mu = \mathcal{L}_X \), for the solution with \( \mathcal{L}_{X_0} = \mu \in \mathcal{P}_k \).

### 4.1 Dissipative case: exponential convergence in entropy and \( \mathbb{W}_2 \)

In this part, we study the exponential ergodicity of \( P_t^* \) in entropy and \( \mathbb{W}_2 \). For probability measures \( \mu_1, \mu_2 \) on \( \bar{D} \), let
\[
\text{Ent}(\mu_1|\mu_2) := \begin{cases} 
\mu_2(f \log f), & \text{if } d\mu_1 = f d\mu_2, \\
\infty, & \text{if } \mu_1 \text{ is not absolutely continuous w.r.t. } \mu_2
\end{cases}
\]
be the relative entropy of \( \mu_1 \) w.r.t. \( \mu_2 \), and let
\[
\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \inf_{D \times \bar{D}} \int_{D \times \bar{D}} |x - y|^2 \pi(dx, dy) \right)^{1/2}
\]
be the quadratic Wasserstein distance, where \( \mathcal{C}(\mu_1, \mu_2) \) is the set of all couplings for \( \mu_1 \) and \( \mu_2 \). The following result extends the corresponding one derived in [16] for McKean-Vlasov SDEs without reflection.
Theorem 4.1. Let $D$ be convex and $(\sigma, b)$ satisfy (A7) with $k = 2$. Let $K_1, K_2 \in L^1_{loc}([0, \infty); \mathbb{R})$ such that
\begin{equation}
2\langle b_t(x, \mu) - b_t(y, \nu), x - y \rangle + \|\sigma_t(x, \mu) - \sigma_t(y, \nu)\|_{HS}^2 \\
\leq K_1(t)|x - y|^2 + K_2(t)\mathbb{W}_2(\mu, \nu)^2, \quad t \geq 0.
\end{equation}
Then and $P^*_t$ satisfies
\begin{equation}
\mathbb{W}_2(P^*_t \mu, P^*_t \nu)^2 \leq e^{t(K_1 + K_2)}\mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathcal{P}_2(\bar{D}), \quad t \geq 0.
\end{equation}
Consequently, if $(b_t, \sigma_t)$ does not depend on $t$ and $\lambda := -(K_1 + K_2) > 0$, then $P^*_t$ has a unique invariant probability measure $\bar{\mu}$ such that
\begin{equation}
\mathbb{W}_2(P^*_t \mu, \bar{\mu})^2 \leq e^{-\lambda t}\mathbb{W}_2(\mu, \bar{\mu})^2, \quad \mu \in \mathcal{P}_2(\bar{D}), \quad t \geq 0,
\end{equation}
and the following assertions hold:

1. When $\sigma_t(x, \mu) = \sigma_t(x)$ does not depend on $\mu$ and $\sigma^*$ is invertible with $\|\sigma\|_\infty + \|\sigma^*\|^{-1}_\infty < \infty$, there exists a constant $c > 0$ such that
\begin{equation}
\text{Ent}(P^*_t \mu | \bar{\mu}) \leq ce^{-\lambda t}\mathbb{W}_2(\mu, \bar{\mu})^2, \quad t \geq 1, \quad \mu \in \mathcal{P}_2(\bar{D}).
\end{equation}

2. When $\sigma(x, \mu) = \sigma(\mu)$ does not depend on $x$, there exists a constant $c > 0$ such that $\bar{\mu}$ satisfies the following log-Sobolev inequality and Talagrand inequality:
\begin{equation}
\bar{\mu}(f^2 \log f^2) \leq c\bar{\mu}(|\nabla f|^2), \quad f \in C^1_b(\mathbb{R}^d), \quad \bar{\mu}(f^2) = 1,
\end{equation}
\begin{equation}
\mathbb{W}_2(\mu, \bar{\mu})^2 \leq c\text{Ent}(\mu | \bar{\mu}), \quad \mu \in \mathcal{P}_2.
\end{equation}

3. When $\sigma(x, \mu) = \sigma$ is constant with $\sigma^*$ invertible, there exists a constant $c > 0$ such that
\begin{equation}
\mathbb{W}_2(P^*_t \mu, \bar{\mu})^2 + \text{Ent}(P^*_t \mu | \bar{\mu}) \leq ce^{-\lambda t}\min\left\{\mathbb{W}_2(\mu, \bar{\mu})^2, \text{Ent}(\mu | \bar{\mu})\right\}, \quad t \geq 1, \mu \in \mathcal{P}_2(\bar{D}).
\end{equation}

Proof. The well-posedness is ensured by [24, Theorem 3.3]. Since $D$ is convex, by Remark 2.1 in [24],
\begin{equation}
\langle y - x, n(x) \rangle \geq 0, \quad y \in \bar{D}, x \in \partial D, n(x) \in N_x.
\end{equation}
For any $\mu, \nu \in \mathcal{P}_2(\bar{D})$, let $X^\mu_0$ and $X^\nu_0$ be $\mathcal{F}_0$-measurable such that
\begin{equation}
\mathcal{L}X^\mu_0 = \mu, \quad \mathcal{L}X^\nu_0 = \nu, \quad \mathbb{E}|X^\mu_0 - X^\nu_0|^2 = \mathbb{W}_2(\mu, \nu)^2.
\end{equation}
By (4.1), (4.11), and applying Itô’s formula to $|X^\mu_t - X^\nu_t|^2$, where $(X^\mu_t)_{t \geq 0}$ and $(X^\nu_t)_{t \geq 0}$ solve (4.1), we obtain
\begin{align*}
d|X^\mu_t - X^\nu_t|^2 &\leq \{K_1(t)|X^\mu_t - X^\nu_t|^2 + K_2(t)\mathbb{W}_2(P^*_t \mu, P^*_t \nu)^2\}dt + dM_t.
\end{align*}
for some martingale $M_t$. Combining this with (4.12), $\mathbb{W}_2(P_t^*\mu, P_t^*\nu)^2 \leq \mathbb{E}|X_t^\mu - X_t^\nu|^2$, and Gronwall’s lemma, we prove (4.5).

Let $(b_t, \sigma_t)$ do not depend on $t$ and $\lambda := -(K_1 + K_2) > 0$. Then (4.5) implies the uniqueness of $P_t^*$-invariant probability measure $\bar{\mu} \in \mathcal{P}_2(\bar{D})$ and (4.6).

The existence of $\bar{\mu}$ follows from a standard argument by showing that for $x_0 \in D$, $\{P_t^*\delta_{x_0}\}_{t \geq 0}$ is a $\mathbb{W}_2$-Cauchy family as $t \to \infty$. Since the term of local time does not make trouble due to (4.11), the proof is completely similar to that of [23, Theorem 3.1] for the case $D = \mathbb{R}^d$, so we skip the details to save space. Below we prove statements (1)-(3) respectively.

(1) When $\sigma_t(x, \mu) = \sigma_t(x)$ and $\sigma\sigma^*$ is invertible with $\|\sigma\|_\infty + \|\sigma\sigma^*\|\|^{-1}_\infty < \infty$, by Theorem 4.2 in [23], (A7) with $k = 2$ implies the log-Harnack inequality

$$\text{Ent}(P_t^*\mu|\bar{\mu}) \leq c\mathbb{W}_2(\mu, \bar{\mu})^2, \quad \mu \in \mathcal{P}_2(\bar{D})$$

for some constant $c > 0$. So, (4.7) follows from (4.6) and $P_t^* = P_t^*P_{t-1}^*$ for $t \geq 1$.

(2) Let $\sigma(x, \mu) = \sigma(\mu)$ be independent of $x$. Consider the SDE

$$d\bar{X}_t^x = b(\bar{X}_t^x, \bar{\mu})dt + \sigma(\bar{\mu})dW_t + n(\bar{X}_t^x)dt, \quad t \geq s, \bar{X}_0^x = x \in \bar{D}.$$

The associated Markov semigroup $\{P_t\}_{t \geq 0}$ is given by

$$\bar{P}_t f(x) := \mathbb{E}f(\bar{X}_t^x), \quad t \geq 0, f \in \mathcal{B}(\bar{D}), x \in \bar{D}.$$

Let $\bar{P}_t^*$ be given by

$$(\bar{P}_t^*\mu)(f) := \mu(\bar{P}_t f), \quad \mu \in \mathcal{P}, t \geq 0, f \in \mathcal{B}(\bar{D}).$$

Since (4.4) with $x = y$ implies $K_2 \geq 0$, we have

$$K_1 \leq -\lambda < 0.$$

As explained in the above proofs of (4.5) and (4.6), this implies that $\bar{P}_t^*$ has a unique invariant probability measure $\bar{\mu}$ such that

$$\lim_{t \to \infty} \bar{P}_t f(x) = \bar{\mu}(f), \quad f \in C_b(\bar{D}), x \in \bar{D}.$$

Since $\bar{\mu}$ is the unique invariant probability measure of $P_t^*$, and when the initial distribution is $\bar{\mu}$, the SDE (4.13) coincides with (4.11), we conclude that $\bar{\mu} = \bar{\mu}$. Hence, (4.15) yields

$$\bar{\mu}(f) = \lim_{t \to \infty} P_t f(x_0), \quad f \in C_b(\bar{D}), x_0 \in D.$$

Now, by Itô's formula, (4.11) and (4.4) with $(b_t, \sigma_t)$ independent of $t$, we obtain

$$|\bar{X}_t^x - \bar{X}_t^y|^2 \leq e^{K_1t}|x - y|^2, \quad x, y \in \bar{D}, t \geq 0.$$

This and (4.14) imply

$$|\nabla \bar{P}_t f(x)| := \limsup_{y \to x} \frac{|\bar{P}_t f(x) - \bar{P}_t f(y)|}{|x - y|} \leq \limsup_{y \to x} \frac{\mathbb{E}|f(\bar{X}_t^x) - f(\bar{X}_t^y)|}{|x - y|}$$

$$\leq e^{-\lambda t/2} \limsup_{y \to x} \mathbb{E} \frac{|f(\bar{X}_t^x) - f(\bar{X}_t^y)|}{|\bar{X}_t^x - \bar{X}_t^y|} = e^{-\lambda t/2} \bar{P}_t|\nabla f|(x), \quad t \geq 0, f \in C_b^1(\bar{D}),$$

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We have
\[ \partial_t \bar{P}_t f = \bar{L} \bar{P}_t f, \quad \langle n, \nabla \bar{P}_t f \rangle |_{\partial D} = 0, \quad t \geq 0, \ f \in C^2_N(\bar{D}), \]

where \( C^2_N(\bar{D}) \) is the set of \( f \in C^2(\bar{D}) \) satisfying with \( \langle n, \nabla f \rangle |_{\partial D} = 0, \) and
\[ \bar{L} := \frac{1}{2} \text{tr} \{ (\bar{\sigma}^* \bar{\sigma}) \nabla^2 \} + \nabla_{b(\cdot, \bar{\alpha})}, \quad \bar{\sigma} := \sigma(\bar{\mu}), \quad s \geq 0. \]

On the other hand, we have
\[ \bar{L} := \frac{1}{2} \text{tr} \{ (\bar{\sigma}^* \bar{\sigma}) \nabla^2 \} + \nabla_{b(\cdot, \bar{\alpha})}, \quad \bar{\sigma} := \sigma(\bar{\mu}), \quad s \geq 0. \]

So, by Itô’s formula, for any \( \varepsilon > 0 \) and \( f \in C^2_N(\bar{D}), \)
\[ \text{d}\{ (\bar{P}_{t-s}(\varepsilon + f^2)) \log (\bar{P}_{t-s}(\varepsilon + f^2)) \} \langle \bar{X}_s \rangle = \left\{ \frac{\bar{\sigma}^* \bar{\sigma} \bar{P}_{t-s} f^2}{\varepsilon + \bar{P}_{t-s} f^2} \right\} \text{d}t + \text{d}M^\varepsilon_s, \quad s \in [0, t] \]
holds for some martingale \( (M^\varepsilon_s)_{s \in [0, t]} \). Combining this with \( (4.17) \), we find a constant \( c > 0 \) such that for any \( f \in C^2_N(\bar{D}), \)
\[ \bar{P}_t \{ (\varepsilon + f^2) \log (\varepsilon + f^2) \} - (\varepsilon + \bar{P}_t f^2) \log (\varepsilon + \bar{P}_t f^2) \]
\[ = \int_0^t \bar{P}_s \frac{\bar{\sigma}^* \nabla \bar{P}_{t-s} f^2}{\varepsilon + \bar{P}_{t-s} f^2} \text{d}s \leq 4(c_1 \| \bar{\sigma} \|_\infty)^2 \int_0^t e^{-\lambda(t-s)} \bar{P}_s \bar{P}_{t-s} |\nabla f|^2 \text{d}s \]
\[ = 4(c_1 \| \bar{\sigma} \|_\infty)^2 (\bar{P}_t |\nabla f|^2) \int_0^t e^{-\lambda(t-s)} \text{d}s \leq c \bar{P}_t |\nabla f|^2, \quad t \geq 0, \varepsilon > 0. \]

By letting first \( \varepsilon \downarrow 0 \) then \( t \to \infty, \) we deduce from this and \( (4.16) \) that
\[ \bar{\mu}(f^2 \log f^2) \leq c_2 \bar{\mu}(|\nabla f|^2), \quad f \in C^2_N(\bar{D}), \bar{\mu}(f^2) = 1 \]
holds for some constant \( c_2 > 0. \) This implies \( (4.8) \) by an approximation argument, indeed the inequality holds for \( f \in H^{1,2}(\bar{\mu}) \) with \( \bar{\mu}(f^2) = 1. \) According to Lemma 4.2 below, \( (4.9) \) holds.

(3) Let \( \sigma \) be constant with \( \sigma \sigma^* \) invertible. Then \( (4.10) \) follows from \( (4.6), (4.7) \) and \( (4.9). \]

The following result on the Talagrand inequality is known by [3] when \( \bar{\mu}(dx) = e^{V(x)}dx \) for some \( V \in C(\mathbb{R}^d), \) which is first proved in [14] on Riemannian manifolds under a curvature condition, see also [20] for more general results. We extend it to general probability measures for the above application to \( \bar{\mu} \) which is supported on \( \bar{D} \) rather than \( \mathbb{R}^d. \)

**Lemma 4.2.** Let \( c > 0 \) be a constant and \( \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d). \) Then the log-Sobolev inequality \( (4.8) \) implies \( (4.9). \)

**Proof.** By an approximation argument, we only need to prove for \( \mu = \varrho \bar{\mu} \) for some density \( \varrho \in C_b(\mathbb{R}^d) \) Let \( P_{t(0)} \) be the Ornstein-Uhlenbeck semigroup generated by \( \Delta - x \cdot \nabla \) on \( \mathbb{R}^d. \) We have
\[ |\nabla P_{t(0)} f| \leq P_{t(0)} |\nabla f|, \quad P_{t(0)} (f^2 \log f^2) \leq t P_{t(0)} |\nabla f|^2 + (P_{t(0)} f^2) \log P_{t(0)} f^2, \quad f \in C^1_b(\mathbb{R}^d). \]
Combining this with (4.8), we see that \( \tilde{\mu}_t := (P_t^{(0)})^* \mu \) satisfies
\[
\tilde{\mu}_t(f^2 \log f^2) = \tilde{\mu}(P_t^{(0)}(f^2 \log f^2)) \leq t \tilde{\mu}_t(\|\nabla f\|^2 + \tilde{\mu}(P_t^{(0)}f^2) \log P_t^{(0)}f^2)
\leq t \tilde{\mu}_t(\|\nabla f\|^2) + c\tilde{\mu}(\sqrt{P_t^{(0)}f^2})^2 + \tilde{\mu}_t(f^2) \log \tilde{\mu}_t(f^2)
\leq (t + c)\tilde{\mu}_t(\|\nabla f\|^2) + \tilde{\mu}_t(f^2) \log \tilde{\mu}_t(f^2), \quad f \in C_b^1(\mathbb{R}^d), \quad t > 0,
\]
where the last step follows from the gradient estimate \( \|\nabla P_t^{(0)}f\| \leq P_t^{(0)}\|\nabla f\| \), which and the Schwarz inequality imply
\[
\left|\nabla \sqrt{P_t^{(0)}f^2}\right|^2 = \frac{|\nabla P_t^{(0)}f^2|^2}{4P_t^{(0)}f^2} \leq \frac{P_t^{(0)}(\|f \nabla f\|))^2}{P_t^{(0)}f^2} \leq P_t^{(0)}\|\nabla f\|^2.
\]
Therefore, \( \tilde{\mu}_t \) satisfies the log-Sobolev inequality with constant \( t + c \) and has smooth strictly positive density. According to [3], we have
\[
\mathbb{W}_2(\mu, \tilde{\mu}_t)^2 \leq (t + c)\text{Ent}(\mu|\tilde{\mu}_t), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d).
\]
Since \( \mathbb{W}_2(\tilde{\mu}_t, \mu) \to 0 \) as \( t \to 0 \), and \( \mu = \varrho \tilde{\mu} \) with \( \varrho \in C_b(\mathbb{R}^d) \), this implies
\[
\mathbb{W}_2(\mu, \tilde{\mu})^2 = \lim_{t \downarrow 0} \mathbb{W}_2(\mu, \tilde{\mu}_t)^2 \leq \lim_{t \downarrow 0} (t + c)\text{Ent}(\mu|\tilde{\mu}_t)
= \lim_{t \downarrow 0} (t + c)\tilde{\mu}(\|P_t^{(0)}\varrho\| \log P_t^{(0)}\varrho) = c\tilde{\mu}(\varrho \log \varrho).
\]
Therefore, (4.9) holds.

\[\square\]

4.2 Partially dissipative case: exponential convergence in \( \mathbb{W}_\psi \)

In this part, we consider the partially dissipative case such that [24, Theorem 3.1] is extended to the reflecting setting. For any \( \kappa > 0 \), let
\[
\Psi_\kappa := \{\psi \in C^2((0, \infty)) \cap C^1([0, \infty)) : \psi(0) = 0, \ \psi'|_{[0, \infty)} > 0, \ ||\psi'||_\infty < \infty, \ \ r\psi'(r) + r^2\{\psi''(r)\}^+ \leq \kappa \psi(r) \quad \text{for } r > 0\}.
\]

For \( \psi \in \Psi_\kappa \), we introduce the associated Wasserstein “distance” (also called transportation cost)
\[
\mathbb{W}_\psi(\mu, \nu) := \inf_{\pi \in \mathcal{P}(\mu, \nu)} \int_{D \times D} \psi(|x - y|) \pi(dx, dy), \quad \mu, \nu \in \mathcal{P}_\psi.
\]
Then \( \mathbb{W}_\psi \) is a complete quasi-metric on the space
\[
\mathcal{P}_\psi := \{\mu \in \mathcal{P} : \mu(\psi(\cdot |)) < \infty\}.
\]

(A8) \( \sigma_t(x, \mu) = \sigma_t(x) \) does not depend on \( \mu \) so that (4.1) reduces to (1.1).
(1) (Ellipticity) There exist \( \alpha \in C([0, \infty); (0, \infty)) \) and \( \hat{\sigma} \in \mathcal{B}([0, \infty) \times \bar{D}; \mathbb{R}^d \otimes \mathbb{R}^d) \) such that
\[
\sigma_t(x)\sigma_t(x)^* = \alpha_t I_d + \hat{\sigma}_t(x)\hat{\sigma}_t(x)^*, \quad t \geq 0, x \in \bar{D}.
\]

(2) (Partial dissipativity) Let \( \psi \in \Psi_\kappa \) in (4.18) for some \( \kappa > 0, \gamma \in C([0, \infty)) \) with \( \gamma(r) \leq Kr \) for some constant \( K > 0 \) and all \( r \geq 0 \), such that
\[
2\alpha_t\psi''(r) + (\gamma \psi')(r) \leq -\zeta_t \psi(r), \quad r \geq 0, t \geq 0
\]
holds for some for some \( \zeta \in C([0, \infty); \mathbb{R}) \). Moreover, \( b \in C([0, \infty) \times \bar{D} \times \mathcal{P}_\psi) \), and there exists \( \theta \in C([0, \infty); [0, \infty)) \) such that
\[
(4.20) \quad 2\alpha_t\psi''(r) + (\gamma \psi')(r) \leq -\zeta_t \psi(r), \quad r \geq 0, t \geq 0
\]

\[
(4.21) \quad (b_t(x, \mu) - b_t(y, \nu), x - y) + \frac{1}{2}\|\hat{\sigma}_t(x) - \hat{\sigma}_t(y)\|^2_{\mathcal{H}S}
\leq |x - y| \{\theta_t \mathbb{W}_\psi(\mu, \nu) + \gamma(|x - y|)\}, \quad t \geq 0, x, y \in \bar{D}, \mu, \nu \in \mathcal{P}_\psi.
\]

**Theorem 4.3.** Let \( D \) be convex and assume (A8), where \( \psi'' \leq 0 \) if \( \hat{\sigma} \) is non-constant. Then (1.1) is well-posed for distributions in \( \mathcal{P}_\psi \), and \( P_t^* \) satisfies
\[
(4.22) \quad \mathbb{W}_\psi(P_t^* \mu, P_t^* \nu) \leq e^{-\int_0^t (\zeta - \hat{\theta}_t \|\psi\|_\infty) ds} \mathbb{W}_\psi(\mu, \nu), \quad t \geq 0, \mu, \nu \in \mathcal{P}_\psi.
\]

Consequently, if \( (b_t, \sigma_t, \zeta_t, \theta_t) \) do not depend on \( t \) and \( \zeta > \theta \|\psi\|_\infty \), then \( P_t^* \) has a unique invariant probability measure \( \bar{\mu} \in \mathcal{P}_\psi \) such that
\[
(4.23) \quad \mathbb{W}_\psi(P_t^* \mu, \bar{\mu}) \leq e^{-(\zeta - \theta \|\psi\|_\infty) t} \mathbb{W}_\psi(\mu, \bar{\mu}), \quad t \geq 0, \mu \in \mathcal{P}_\psi.
\]

**Proof.** Since \( D \) is convex, the proof is similar to that of [24, Theorem 3.1]. We outline it below for complement.

By Theorem 3.1 the well-posedness follows from (A8)(1) and (A8)(2). Next, according to the proof of Theorem 4.1(2) with \( \mathbb{W}_\psi \) replacing \( \mathbb{W}_2 \), the second assertion follows from the first. So, in the following we only prove (4.22).

For any \( s \geq 0 \), let \( (X_s, Y_s) \) be \( \mathcal{F}_s \)-measurable such that
\[
(4.24) \quad \mathcal{L}_{X_s} = P_s^* \mu, \quad \mathcal{L}_{Y_s} = P_s^* \nu, \quad \mathbb{W}_\psi(P_s^* \mu, P_s^* \nu) = \mathbb{E}_\psi(|X_s - Y_s|).
\]

Let \( W_t^{(1)} \) and \( W_t^{(2)} \) be two independent \( d \)-dimensional Brownian motions and consider the following SDE:
\[
(4.25) \quad dX_t = b_t(X_t, P_t^* \mu)dt + \sqrt{\alpha_t}dW_t^{(1)} + \hat{\sigma}_t(X_t)dW_t^{(2)} + n(X_t)dl_t^X, \quad t \geq s,
\]
where \( l_t^X \) is the local time of \( X_t \) on \( \partial D \). By Theorem 4.1 (A8)(1) and (A8)(2) imply that this SDE. By \( \sigma_t \sigma_t^* = \alpha_t I_d + \hat{\sigma}_t \hat{\sigma}_t^* \), we have
\[
\sigma_t^*(\sigma_t^*)^{-1}\{\alpha_t + \hat{\sigma}_t \hat{\sigma}_t^*\}(\sigma_t^*)^{-1} + \{I_m - \sigma_t^*(\sigma_t^*)^{-1} \}
\leq \sigma_t^*(\sigma_t^*)^{-1} \sigma_t + I_m - \sigma_t^*(\sigma_t^*)^{-1} \sigma_t = I_m.
\]
So, for an $m$-dimensional Brownian motion $W^{(3)}$ independent of $(W^{(1)}, W^{(2)})$

$$W_t := \int_0^t \left\{ \sigma_s^* (\sigma_s^*)^{-1} \right\}(X_s) \left\{ \sqrt{\alpha_t} dW^{(1)}_s + \hat{\sigma}_t(X_s) dW^{(2)}_s \right\} + \int_0^t \big\{ I_m - \sigma_s^* (\sigma_s^*)^{-1} \sigma_s \big\}(X_s) dW^3_s$$

is an $m$-dimensional Brownian motion such that

$$\sigma_t(X_t) dW_t = \sqrt{\alpha_t} dW^{(1)}_t + \hat{\sigma}_t(X_t) dW^{(2)}_t.$$ 

Thus, by the weak uniqueness of (1.11), we have $\mathcal{L}_{X_t} = P^*_t \mathcal{P}_t = P^*_t \mu$ for $t \geq s$, where for $\gamma \in \mathcal{P}_\psi$ we denote $P^*_s \gamma = \mathcal{L}_{X_s}$ for $X_t$ solving (1.25) with $\mathcal{L}_{X_t} = \gamma$.

To construct the coupling with reflection, let

$$\psi$$

we denote

$$\psi$$

and

$$\nu$$

so that

$$\psi$$

so that

$$\nu$$

By a standard argument and noting that $\mathcal{L}_{X_t} = P^*_t \psi$ and $\mathcal{L}_{Y_t} = P^*_t \nu$ by the same reason leading to $\mathcal{L}_{X_t} = P^*_t \mu$. Since $D$ is convex, (4.11) holds. So, by (A8)(1) and (A8)(2) for $\psi \in \Psi$ with $\psi'' \leq 0$ when $\hat{\sigma}_t$ is non-constant, and applying Itô’s formula, we obtain

$$d\psi(|X_t - Y_t|) \leq \left\{ \theta_t \psi(|X_t - Y_t|) \mathbb{W}_\psi(P^*_t \mu, P^*_t \nu) - \zeta_t \psi(|X_t - Y_t|) \right\} dt$$

$$+ \psi'(|X_t - Y_t|) \left[ 2\sqrt{\alpha_t} \left\langle u(X_t, Y_t), dW^{(1)}_t \right\rangle \right]$$

$$+ \left\langle u(X_t, Y_t), (\hat{\sigma}_t(X_t) - \hat{\sigma}_t(Y_t)) dW^{(2)}_t \right\rangle, s \leq t < \tau.$$

By a standard argument and noting that $\psi(|X_{t^\wedge \tau}, Y_{t^\wedge \tau}|)_{1_{\{\tau \leq t\}}} = 0$, this implies

$$e^{f_s^{1\wedge \tau}} \zeta_{s+} \mathbb{E}_\psi \left[ \psi(|X_{t^\wedge \tau} - Y_{t^\wedge \tau}|) \right] = \mathbb{E}_\psi \left[ e^{f_s^{1\wedge \tau}} \zeta_{s+} \psi(|X_{t^\wedge \tau} - Y_{t^\wedge \tau}|) \right]$$

$$\leq \mathbb{E}_\psi \left[ |X_s - Y_s| \right] + \|\psi''\|_{\infty} \int_s^{t^\wedge \tau} \theta_r e^{f_r^{1\wedge \tau}} \zeta_{r+} \mathbb{W}_\psi(P^*_r \mu, P^*_r \nu) dr, t \geq s.$$

Consequently,

$$\mathbb{E}_\psi \left[ |X_{t^\wedge \tau} - Y_{t^\wedge \tau}| \right]$$

$$\leq e^{-f_s^{1\wedge \tau}} \zeta_0 \mathbb{E}_\psi \left[ |X_s - Y_s| \right] + \|\psi''\|_{\infty} \int_s^{t^\wedge \tau} \theta_r e^{-f_r^{1\wedge \tau}} \zeta_{r+} \mathbb{W}_\psi(P^*_r \mu, P^*_r \nu) dr, t \geq s.$$
On the other hand, when $t \geq \tau$, by (A8)(2) and applying Itô’s formula for (4.25) and (4.27), we find a constant $C > 0$ such that

$$
\mathrm{d}\psi(|X_t - Y_t|) \leq \{C \psi(|X_t - Y_t|)dt + \theta_1 \|\psi\|_\infty \mathbb{W}_\psi(P_t^* \mu, P_t^* \nu)\}dt \\
+ \psi(|X_t - Y_t|)\{[\bar{\sigma}_t(X_t) - \bar{\sigma}_t(Y_t)]^*u(X_t, Y_t), dW_t^2\}.
$$

Noting that $\psi(|X_t - Y_t|) = 0$, we obtain

$$
\mathbb{E}\left[1_{\{t > \tau\}} \psi(|X_t - Y_t|)\right] \leq \|\psi\|_\infty e^{C(t-\tau)} e \int_{t\wedge \tau}^t \theta_\tau \mathbb{W}_\psi(P_{\tau}^* \mu, P_{\tau}^* \nu)dr, \quad t \geq s.
$$

Combining this with (4.29) and (4.24), we derive

$$
\mathbb{W}_\psi(P_t^* \mu, P_t^* \nu) \leq e^{-\int_\tau^t \zeta_r dr} \mathbb{W}_\psi(|X_s - Y_s|) + \|\psi\|_\infty e^{C(t-\tau)} \int_{\tau}^t \theta_r \mathbb{W}_\psi(P_r^* \mu, P_r^* \nu)dr \\
= e^{-\int_\tau^t \zeta_r dr} \mathbb{W}_\psi(P_s^* \mu, P_s^* \nu) + \|\psi\|_\infty e^{C(t-\tau)} \int_{\tau}^t \theta_r \mathbb{W}_\psi(P_r^* \mu, P_r^* \nu)dr, \quad t \geq s.
$$

Therefore,

$$
\frac{d^+}{ds} \mathbb{W}_\psi(P_s^* \mu, P_s^* \nu) := \lim_{t \downarrow s} \sup_{t < s} \frac{\mathbb{W}_\psi(P_t^* \mu, P_t^* \nu) - \mathbb{W}_\psi(P_s^* \mu, P_s^* \nu)}{t - s} \\
\leq - (\zeta_s - \theta_s \|\psi\|_\infty) \mathbb{W}_\psi(P_s^* \mu, P_s^* \nu), \quad s \geq 0.
$$

This implies (4.22). \hfill \square

As a consequence of Theorem 4.3, we consider the non-dissipative case where $\nabla b_t(\cdot, \mu)(x)$ is positive definite in a possibly unbounded set but with bounded “one-dimensional puncture mass” in the sense of (4.32) below.

Let $\mathbb{W}_1 = \mathbb{W}_\psi$ and $\mathcal{P}_1(\bar{D}) = \mathcal{P}_\psi$ for $\psi(r) = r$, and define

$$
S_b(x) := \sup \left\{ \langle \nabla v, b_t(\cdot, \mu)(x), v \rangle : t \geq 0, |v| \leq 1, \mu \in \mathcal{P}_1(\bar{D}) \right\}, \quad x \in \bar{D}.
$$

(A8) (3) There exist constants $\theta_0, \theta_1, \theta_2, \beta > 0$ such that

$$
\frac{1}{2} \|\sigma_t(x) - \sigma_t(y)\|^2_{HS} \leq \theta_0 |x - y|^2, \quad t \geq 0, x, y \in \bar{D};
$$

$$
S_b(x) \leq \theta_1, \quad |b_t(x, \mu) - b_t(x, \nu)| \leq \beta \mathbb{W}_1(\mu, \nu), \quad t \geq 0, x \in \bar{D}, \mu, \nu \in \mathcal{P}_1(\bar{D});
$$

$$
\zeta := \sup_{x, \nu \in \mathcal{D}, |v| = 1} \int_{\mathbb{R}} 1_{\{S_b(x + sv) > -\theta_2\}} ds < \infty.
$$

According to the proof of [24, Corollary 3.2], the following result follows from Theorem 4.3.
Corollary 4.4. Let $D$ be convex. Assume $(A8)(1)$ and $(A8)(3)$. Let
\begin{align}
\gamma(r) := (\theta_1 + \theta_2) \left\{ (r^2 - 1) \right\} - (\theta_2 - \theta_0)r, & \quad r \geq 0, \\
k := \frac{2\alpha}{\int_0^\infty te^{\frac{1}{2\alpha} \int_0^t \gamma(u) du} dt} - \frac{\beta(\theta_2 - \theta_0)}{2\alpha} \int_0^\infty te^{\frac{1}{2\alpha} \int_0^t \gamma(u) du} dt.
\end{align}
(4.33)

Then there exists a constant $c > 0$ such that
\[ \mathbb{W}_1(P_t^* \mu, P_t^* \nu) \leq ce^{-kt} \mathbb{W}_1(\mu, \nu), \quad t \geq 0, \mu, \nu \in \mathcal{P}_1(\bar{D}). \]

If $(b_t, \sigma_t)$ does not depend on $t$ and $\theta_2 > \theta_0$ with
\[ \beta < \frac{4\alpha^2}{(\theta_2 - \theta_0)(\int_0^\infty te^{\frac{1}{2\alpha} \int_0^t \gamma(u) du} dt)^2}, \]
then $k > 0$ and $P_t^*$ has a unique invariant probability measure $\bar{\mu} \in \mathcal{P}_1(\bar{D})$ satisfying
\[ \mathbb{W}_1(P_t^* \mu, \bar{\mu}) \leq ce^{-kt} \mathbb{W}_1(\mu, \bar{\mu}), \quad t \geq 0, \mu \in \mathcal{P}_1(\bar{D}). \]

Remark 4.1 We note that [24, Theorem 2.1] presents an ergodicity result for the non-dissipative case, which also holds for present setting with convex $D$. We drop the detailed statement to save space.

Acknowledgement. The author would like to thank the referee for helpful comments and corrections.

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