Remarkable interaction effects in quadratic and cubic nodal line fermion systems

Jing-Rong Wang,1 Wei Li,2∗ and Chang-Jin Zhang1,3†

1 Anhui Province Key Laboratory of Condensed Matter Physics at Extreme Conditions, High Magnetic Field Laboratory of Anhui Province, Chinese Academy of Sciences, Hefei 230031, China
2 Key Laboratory of Materials Physics, Institute of Solid State Physics, Chinese Academy of Sciences, Hefei 230031, China
3 Institute of Physical Science and Information Technology, Anhui University, Hefei 230601, China

Influence of short-range four-fermion interactions on quadratic and cubic nodal line fermion systems is studied by renormalization group theory. It is found that arbitrarily weak four-fermion interaction could drive quadratic or cubic nodal line fermion system to a new phase. According to the initial conditions and value of fermion flavor, the system may appear three kinds of instabilities. First, quadratic or cubic nodal line is split into conventional nodal lines, thus the system becomes nodal line semimetal. Second, finite excitonic gap is generated, and the system becomes an excitonic insulator. Third, the system is driven into superconducting phase. Thus, quadratic and cubic nodal line fermion systems are rare strong correlated fermion systems in three dimension under the influence of four-fermion interactions. These theoretical results may be verified in the candidates for quadratic and cubic nodal line fermion systems.

I. INTRODUCTION

Study about topological semimetals (SMs) is one of the most important fields of modern condensed matter physics [1–15]. On the one hand, some SMs have wide potential industrial applications due to their fantastic properties, such as large thermoelectric power [16, 17]. On the other hand, some SMs provide a platform to verify certain important concepts in high energy physics, due to their low-energy fermion excitations resemble the elementary particles [1–15].

According to energy dispersion of the fermion excitations and topological property of the system, SMs can be classified as Dirac SM (DSM), Weyl SM (WSM), multi-WSM, semi-DSM, Luttinger SM, nodal line SM (NLSM) etc. Graphene is a prototypical two dimensional (2D) DSM. Angle-resolved photoemission spectroscopy (ARPES) experiments have confirmed that Cd3As2 and Na3Bi are 3D DSM [18, 19], and TaAs, TaP, NbAs, NbP are WSMs [20, 21]. NLSM has been realized in PbTaSe2, ZrSIS, HfSIS, and TiB2 according to ARPES measurements [22–27].

In SMs, the dimension of Fermi surface is at least two less than the dimension of the system. This characteristic is different from conventional metals, in which the dimension of Fermi surface is one less than the dimension of the system [28]. In DSM, WSM, multi-WSM, semi-DSM, and Luttinger SM, the Fermi surface is composed by discrete points, where conduction and valence bands touch each other in the Brillouin zone. Whereas, the Fermi surface of NLSM is a line in the 3D Brillouin zone. Due to the abovementioned characteristic of SMs, density of states (DOS) of SMs vanishes at the Fermi level.

Influence of interactions on SMs is an important and nontrivial question, which attracts much attentions [1–15, 56–60]. Due to the vanishing DOS, short range four-fermion interaction is irrelevant in SM [20], but may drive a quantum phase transition (QPT) to a new phase if the interaction strength is strong enough [57, 58, 59]. There have been studies on the effects of four-fermion interactions in SMs including 2D DSM [51, 52], 3D DSM [53], WSM [54, 55], multi-WSM [55], semi-DSM [56, 57], Luttinger SM [58], and NLSM [59, 60]. These studies showed that the SMs may be driven to different phases according to the types of four-fermion interactions. Additionally, the influence of four-fermion interactions is closely related to the fermion dispersion and topological property of the system.

In NLSM, the fermion dispersion is linear within the $x - y$ plane and also linear along the $z$ axis [43, 44, 59–61]. For NLSM, DOS takes the form $\rho(\omega) \sim \omega$, which vanishes at the Fermi level, i.e., $\rho(0) = 0$. Roy has analyzed the possible QPTs in NLSM under the influence of four-fermion interactions [60].

Recently, Yu et al. [43] proposed that quadratic and cubic nodal line fermion (NLF) systems could be realized in some materials [61]. In these materials, the Fermi surface is also a line in the 3D Brillouin zone, which is same as NLSM. However, the fermion dispersion is quadratic (cubic) within the $x - y$ plane and also quadratic (cubic) along the $z$ axis. Accordingly, DOS satisfies $\rho(\omega) \sim \omega^0$ in quadratic NLF system, and $\rho(\omega) \sim \omega^{-1/3}$ in cubic NLF system. Thus, the influence of four-fermion interactions on quadratic and cubic NLF systems could be substantially different from the one in NLSM. This is an interesting and urgent question, which needs comprehensive study.

In this article, we resolve this question through renormalization group (RG) theory [62]. We find that quadratic and cubic NLF fermion systems are unsta-
able to short-range four-fermion interactions. We show that arbitrarily weak four-fermion interaction could drive quadratic or cubic NLF system to a new phase. According to the initial conditions and value of fermion flavor, the system may appear three kinds of instabilities. First, the quadratic or cubic nodal line is split into conventional nodal lines, and the system becomes a NLSM. Second, finite excitonic gap is generated, then the system becomes an excitonic insulator. Third, the system is driven into a superconducting phase.

The rest of paper is structured as follows. The model is presented in Sec. II. In Sec. III we analyze the physical meaning of various fermion bilinears, which may be generated by the four-fermion interactions. In Sec. IV we show RG equations of the model parameters and numerical results of the RG equations. The behaviors of observable quantities in different phases are discussed in Sec. V. The main results are summarized in Sec. VI. The detailed derivation for the RG equations is given in the Appendices.

II. MODEL

The Hamiltonian density for free quadratic NLF system is given by

\[ \mathcal{H}_0^q(k) = A \left[ (k_x^2 - k_y^2) \sigma_1 + 2k_xk_y \sigma_2 \right], \]

where \( k_r = k_\perp - k_F \) with \( k_\perp = \sqrt{k_x^2 + k_y^2} \). \( A \) is a model parameter. \( \sigma_{1,2,3} \) are the standard Pauli matrices. The energy spectrum for quadratic NLF is

\[ E_q(k) = \pm A \left( k_x^2 + k_y^2 \right). \]

For simplicity, here we do not consider anisotropy of the fermion dispersion along \( k_r \) and \( k_\perp \). The qualitative conclusions will not be changed if this anisotropy is incorporated.

The Hamiltonian density for cubic NLF system can be written as

\[ \mathcal{H}_0^c(k) = B \left[ (k_x^3 - 3k_xk_y^2) \sigma_1 + (k_y^3 - 3k_yk_x^2) \sigma_2 \right]. \]

The energy dispersion for cubic NLF takes the form

\[ E_c(k) = \pm B \left( k_x^2 + k_y^2 \right)^{3/2}. \]

Both the Hamiltonian densities \( \mathcal{H}_0^q \) and \( \mathcal{H}_0^c \) satisfy the chiral symmetry \{\( \mathcal{H}_0^q, \sigma_3 \) = 0. Once a term \( \mathcal{H}_{\Delta_3} = \Delta_3 \sigma_3 \) is generated, the fermions become gapped and the chiral symmetry is broken [63].

We consider the four-fermion interactions described by the action

\[ S_{\psi^4} = \frac{1}{N} \sum_{i=0}^3 \lambda_i \left[ \frac{d\omega_1}{2\pi} \frac{d^3k_1}{(2\pi)^3} \frac{d\omega_2}{2\pi} \frac{d^3k_2}{(2\pi)^3} \frac{d\omega_3}{2\pi} \frac{d^3k_3}{(2\pi)^3} \right. \]

\[ \times \psi^\dagger(\omega_1, k_1) \sigma_i \psi(\omega_2, k_2) \psi^\dagger(\omega_3, k_3) \sigma_i \psi(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3), \]

where \( \lambda_i \) with \( i = 0, 1, 2, 3 \) are the four-fermion coupling parameters, and \( N \) is the fermion flavor. \( \sigma_0 \) is the identity matrix. In the following, we are only interested in the case that the initial value \( \lambda_{i,0} \) satisfies \( \lambda_{i,0} > 0 \), namely the interaction is repulsive initially.

III. PHYSICAL MEANING OF FERMION BILINEARS

Decoupling of the four-fermion interactions, we could get four different fermion bilinears \( \psi^\dagger \sigma_0 \psi, \psi^\dagger \sigma_1 \psi, \psi^\dagger \sigma_2 \psi, \) and \( \psi^\dagger \sigma_3 \psi \). The expectation values of these bilinears are given by

\[ \Delta_0 = \langle \psi^\dagger \sigma_0 \psi \rangle, \]
\[ \Delta_1 = \langle \psi^\dagger \sigma_1 \psi \rangle, \]
\[ \Delta_2 = \langle \psi^\dagger \sigma_2 \psi \rangle, \]
\[ \Delta_3 = \langle \psi^\dagger \sigma_3 \psi \rangle. \]

(\( \langle \ldots \rangle \) represents taking mean value on the ground state of total Hamiltonian. They have different physical meanings. If \( \Delta_0 \) becomes finite, the Fermi level is modified, and the Fermi surface is changed from 1D nodal line to 2D tube, since \( \Delta_0 \) represents the chemical potential.

The original nodal line is gapless for \( (k_r, k_\perp) = (0, 0) \) For quadratic NLF system, if \( \Delta_1 \) becomes finite, the original nodal line with quadratic dispersion is split into two conventional nodal lines with linear dispersion. These two conventional nodal lines are gapless for the cases

\[ (k_{ar}, k_{az}) = \left( 0, (\Delta_1/A)^{1/2} \right), \]

and

\[ (k_{br}, k_{bz}) = \left( 0, - (\Delta_1/A)^{1/2} \right). \]

Around these two nodal lines, the fermion dispersion can be written as

\[ E = \pm \sqrt{4A\Delta_1 (K_r^2 + K_\perp^2)}, \]

where \( K_r \) and \( K_\perp \) are the momentum components relative to the nodal lines.

For cubic NLF system, if \( \Delta_1 \) becomes finite, one cubic nodal line is split into three conventional nodal lines, which are determined by

\[ (k_{ar}, k_{az}) = \left( - (\Delta_1/B)^{1/3}, 0 \right), \]
\[ (k_{br}, k_{bz}) = \left\{ \frac{1}{2} (\Delta_1/B)^{1/3}, \frac{\sqrt{3}}{2} (\Delta_1/B)^{1/3} \right\}, \]
\[ (k_{cr}, k_{cx}) = \left\{ \frac{1}{2} (\Delta_1/B)^{1/3}, -\frac{\sqrt{3}}{2} (\Delta_1/B)^{1/3} \right\}. \]

The energy dispersion around these three nodal lines can be expressed as

\[ E = \pm \sqrt{9B^{2/3} \Delta_1^{4/3} (K_r^2 + K_\perp^2)}. \]
For quadratic NLF system, if $\Delta_2$ acquires finite value, one quadratic nodal line is split into two conventional nodal lines, which satisfy
\[
(k_{ar}, k_{az}) = \left( (\Delta_2/(2A))^{1/2}, -(\Delta_2/(2A))^{1/2} \right),
\]
and
\[
(k_{br}, k_{bz}) = \left( -\Delta_2/(2A)^{1/2}, (\Delta_2/(2A))^{1/2} \right).
\]
The energy dispersion of the fermions around these two nodal lines reads as
\[
E = \pm \sqrt{4A\Delta_2(K_x^2 + K_y^2)}.
\]

For cubic NLF system, if $\Delta_2$ becomes finite, one cubic nodal line is split into three conventional nodal lines, which are decided by
\[
(k_{ar}, k_{az}) = \left( 0, -(\Delta_2/B)^{1/3} \right),
\]
\[
(k_{br}, k_{bz}) = \left( \frac{\sqrt[3]{3}}{2} (\Delta_2/B)^{1/3}, \frac{1}{2} (\Delta_2/B)^{1/3} \right),
\]
and
\[
(k_{cr}, k_{cz}) = \left( -\frac{\sqrt[3]{3}}{2} (\Delta_2/B)^{1/3}, \frac{1}{2} (\Delta_2/B)^{1/3} \right).
\]

Around these three nodal lines, the fermion dispersion can be expressed as
\[
E = \pm \sqrt{9B^{2/3}\Delta_2^{4/3}(K_x^2 + K_y^2)}.
\]

If $\Delta_3$ acquires finite value, the energy dispersions for quadratic and cubic NLF systems can be written as
\[
E = \pm \sqrt{A^2(k_x^2 + k_y^2)^2 + \Delta_3^2},
\]
and
\[
E = \pm \sqrt{B^2(k_x^2 + k_y^2)^3 + \Delta_3^2},
\]
respectively. We can find that the fermion dispersion becomes gapped once $\Delta_3$ becomes finite. Physically, it suggests that the system is driven into excitonic insulating phase.

For convenience, we show the splitting of quadratic and cubic nodal lines to conventional nodal lines in the presence of $\Delta_1$ or $\Delta_2$ by the schematic diagrams in Fig. 1.

For a fermion system under the influence of four-fermion interaction $\lambda (\psi^\dagger \Gamma \psi)^2$ where $\Gamma$ is a matrix, if the RG analysis shows that the four-fermion coupling strength $\lambda$ approaches to infinity at a finite running parameter $\ell_c$, it is usually considered that a finite expectation value $\Delta_\Gamma = \langle \psi^\dagger \Gamma \psi \rangle$ is generated. The magnitude of $\Delta_\Gamma$ can be estimated by
\[
\Delta_\Gamma \sim \Lambda e^{-\ell_c},
\]
where $\Lambda$ is an energy cutoff. This method has been usually adopted in the RG studies about the influence of four-fermion interactions on various fermion systems [51, 60, 68, 69].

If the four-fermion coupling parameter flows to negative infinity finally, we consider that the four-fermion interaction becomes attractive in the low-energy regime. Accordingly, system is unstable to pairing in the particle-particle channel, namely the generation of superconducting gap.

**IV. RENORMALIZATION GROUP ANALYSIS**

In this section, we present the RG results of the influence of four-fermion interactions on quadratic and cubic NFL systems. The detailed derivation for the RG equations are shown in the Appendices.
A. Quadratic NLF

For quadratic NLF system in the presence of four-fermion interactions, the RG equations for the coupling parameters are given by

\[
\frac{d\lambda_0}{d\ell} = (\lambda_0 \lambda_1 + \lambda_0 \lambda_2) \frac{1}{N},
\]

\[
\frac{d\lambda_1}{d\ell} = \left[ (\lambda_0^2 + \lambda_2^2 + \lambda_3^2 + \lambda_0 \lambda_1 + 2\lambda_1 \lambda_2 + \lambda_1 \lambda_3 - 2\lambda_2 \lambda_3) \frac{1}{N} + \lambda_1^2 \right],
\]

\[
\frac{d\lambda_2}{d\ell} = \left[ (\lambda_0^2 + \lambda_1^2 + \lambda_3^2 + \lambda_0 \lambda_2 + 2\lambda_1 \lambda_2 - 2\lambda_1 \lambda_3 + \lambda_2 \lambda_3) \frac{1}{N} + \lambda_2^2 \right],
\]

\[
\frac{d\lambda_3}{d\ell} = \left[ (-2\lambda_0^2 + 2\lambda_0 \lambda_3 - 2\lambda_1 \lambda_3 + 3\lambda_1 \lambda_3 + 3\lambda_2 \lambda_3) \frac{1}{N} + 2\lambda_3^2 \right].
\]

The transformations \(\frac{d\lambda_i}{d\ell} = \lambda_i\) with \(i = 0, 1, 2, 3\) have been employed in the derivation of the RG equations. We notice that \(\lambda_0\) does not get generated if the initial value \(\lambda_{0,0} = 0\).

If the initial value \(\lambda_{0,0}\) is finite, the flows of \(\lambda_0, \lambda_1, \lambda_3, \lambda_0/\lambda_1, \lambda_0/|\lambda_3|, \lambda_1/|\lambda_3|\) are shown in Fig. 2. As shown in Figs. 2(a)-2(c), \(\lambda_0, \lambda_1, \lambda_3\) approach to infinity, and \(\lambda_3\) flows to negative infinity at same finite energy scale. According to Figs. 2(d)-2(f), \(\lambda_0/\lambda_1\) and \(\lambda_0/|\lambda_3|\) approach to zero. Additionally, \(\lambda_1/|\lambda_3|\) flows to a constant smaller than 1 for \(N = 1\), but approaches to a constant larger than 1 for \(N \geq 2\). \(\lambda_1/\lambda_2\) is always equal to 1, which is not shown in Fig. 2. These results indicate that arbitrarily weak four-fermion interaction induces the system to be unstable. For \(N = 1\), generation of superconducting gap is the leading instability. However, splitting of quadratic nodal line with generation of \(\Delta_1\) or \(\Delta_2\) is the subleading instability. For \(N \geq 2\), splitting of quadratic nodal line with generation of \(\Delta_1\) or \(\Delta_2\) becomes the leading instability.

If the parameter \(\lambda_{1,0}\) takes finite value, the relations between \(\lambda_1, \lambda_2, \lambda_3, \lambda_1/\lambda_2, \lambda_1/|\lambda_3|, \lambda_2/\lambda_3\) and running parameter \(\ell\) are shown in Fig. 3. \(\lambda_0\) always equals to zero, if only \(\lambda_{1,0}\) is finite initially. We can find that \(\lambda_1\) always approaches to infinity at some finite value \(\ell_c\). As shown in Fig. 3(b), \(\lambda_2\) flows from zero to infinity at the same \(\ell_c\). According to Fig. 3(c), \(\lambda_3\) is generated from zero and approaches to negative infinity finally. As depicted in Figs. 3(d)-3(f), \(\lambda_1/\lambda_2\) flows to 1, and \(\lambda_{1,2}/|\lambda_3|\) flows to a constant smaller than 1 for \(N = 1\) but flows to a constant larger than 1 for \(N \geq 2\). These results represent that transition into superconducting phase is the leading instability for \(N = 1\), but the generation of \(\Delta_1\) or \(\Delta_2\) and splitting of quadratic nodal line into conventional nodal lines becomes the leading instability of \(N \geq 2\).

If the parameter \(\lambda_{2,0}\) takes finite value, we will obtain qualitatively similar results comparing the ones in the case that only \(\lambda_{1,0}\) is finite.

If the initial value \(\lambda_{3,0}\) is finite, the flows of \(\lambda_1, \lambda_2, \lambda_3, \lambda_1/\lambda_2, \lambda_1/|\lambda_3|, \lambda_2/\lambda_3\) are shown in Fig. 4. According to Figs. 4(a)-4(c), \(\lambda_3\) approaches to infinity at a finite \(\ell_c\), and \(\lambda_1\) or \(\lambda_2\) flows from zero and approaches to infinity at the same \(\ell_c\). As shown in Fig. 4(d), \(\lambda_1/\lambda_2\) equals to 1. As displayed in Figs. 4(e) and 4(f), \(\lambda_{1,2}/|\lambda_3|\) always flows to a constant smaller than 1 for any fermion flavor. It represents that generation of excitonic gap is always the leading instability for any fermion flavor, if only \(\lambda_{3,0}\) takes finite value.

If the initial values of two coupling parameters are finite, the flows of \(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_0/|\lambda_3|, \lambda_1/\lambda_2, \lambda_1/|\lambda_3|\),
The flows of $\lambda_0$, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_0/|\lambda_1|$, $\lambda_1/|\lambda_2|$, and $\lambda_2/|\lambda_3|$ in the quadratic NLF system are displayed in Fig. 5. According to these results, we find that the system could be driven to NLSM, excitonic insulator, or superconducting phase, which is determined by the concrete initial conditions and fermion flavor sensitively.

### B. Cubic NLF

For cubic NLF system in the presence of four-fermion interactions, the RG equations for the four-fermion coupling parameters are given by

$$
\frac{d\lambda_0}{d\ell} = \frac{1}{3} \lambda_0 + (\lambda_1 \lambda_3 + \lambda_2 \lambda_3) \frac{1}{N},
$$

(31)

$$
\frac{d\lambda_1}{d\ell} = \frac{1}{3} \lambda_1 + \left[ (-\lambda_2^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 \
-2 \lambda_2 \lambda_3) \frac{1}{N} + \lambda_1^2 \right],
$$

(32)

$$
\frac{d\lambda_2}{d\ell} = \frac{1}{3} \lambda_2 + \left[ (-\lambda_3^2 + \lambda_0 \lambda_2 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 - 2 \lambda_1 \lambda_3 \
+ \lambda_2 \lambda_3) \frac{1}{N} + \lambda_2^2 \right],
$$

(33)

$$
\frac{d\lambda_3}{d\ell} = \frac{1}{3} \lambda_3 + \left[ (-2 \lambda_3^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_2 + \lambda_0 \lambda_3 - 2 \lambda_1 \lambda_2 \
+ 2 \lambda_1 \lambda_3 + 2 \lambda_2 \lambda_3) \frac{1}{N} + 2 \lambda_3^2 \right].
$$

(34)

The transformations $\frac{d\lambda_i}{d\ell} = \lambda_i$ with $i = 0, 1, 2, 3$ have been utilized. One could find that one type of four-fermion interaction can exist solely.

If only consider the four-fermion interaction $\lambda_0 (\psi^\dagger \sigma_0 \psi)^2$, the RG equation for the coupling strength takes the form

$$
\frac{d\lambda_0}{d\ell} = \frac{1}{3} \lambda_0.
$$

(35)

The solution is

$$
\lambda_0 = \lambda_{0,0} e^{\frac{1}{3} \ell}.
$$

(36)

It is easy to find that $\lambda_0$ does not divergent at a finite energy scale, but only becomes divergent in the lowest energy limit $\ell \to \infty$. We believe that divergence of $\lambda_0$ at $\ell \to \infty$ does not represent the generation of a finite expectation value $\Delta_0 = \langle \psi^\dagger \psi \rangle$.

If only the four-fermion interaction $\lambda_0 (\psi^\dagger \sigma_1 \psi)^2$ is considered, the RG equation for $\lambda_1$ is given by

$$
\frac{d\lambda_1}{d\ell} = \frac{1}{3} \lambda_1 + \left( 1 - \frac{1}{N} \right) \lambda_1^2.
$$

(37)

For the fermion flavor $N = 1$, the RG equation becomes

$$
\frac{d\lambda_1}{d\ell} = \frac{1}{3} \lambda_1.
$$

(38)

The corresponding solution reads as

$$
\lambda_1 = \lambda_{1,0} e^{\frac{1}{3} \ell}.
$$

(39)
This result implies that a finite expectation value \( \Delta_1 \) can be estimated by
\[
\Delta_1 \sim \Lambda e^{-\ell_{1c}},
\]
where \( \Lambda \) is an energy cutoff.

Similarly, if only the four-fermion interaction \( \lambda_2 \left( \psi_1 \sigma_2 \psi_2 \right)^2 \) is considered, we could find that a finite expectation value \( \Delta_2 = \left\langle \psi_1 \sigma_2 \psi_2 \right\rangle \) is generated for \( N > 1 \).

Considering only the four-fermion interaction \( \lambda_3 \left( \psi_1 \sigma_3 \psi_3 \right)^2 \), the RG equation for the coupling strength can be written as
\[
\frac{d\lambda_3}{d\ell} = \frac{1}{3} \lambda_3 + 2 \left( 1 - \frac{1}{N} \right) \lambda_3, \tag{43}
\]
which becomes divergent in the lowest energy limit \( \ell \to \infty \). In this case, divergence of \( \lambda_1 \) does not indicate the generation of long-range order parameter \( \Delta_1 = \left\langle \psi^\dagger \sigma_1 \psi \right\rangle \).

For the case \( N > 1 \), solving Eq. (43) gives rise to
\[
\lambda_3 = \lambda_{3,0} e^{\frac{1}{3} \ell_{3c}}, \tag{46}
\]
where \( \ell_{3c} = 3 \ln \left[ 1 + \frac{1}{3(1 - \frac{1}{N})} \lambda_{3,0} \right] \).

Therefore, for \( N > 1 \), finite expectation value \( \Delta_3 = \left\langle \psi^\dagger \sigma_3 \psi \right\rangle \) should not be generated. For \( N > 1 \), solving Eq. (43) gives rise to
\[
\lambda_3 = \lambda_{3,0} e^{\frac{1}{3} \ell_{3c}}, \tag{46}
\]
where
\[
\ell_{3c} = 3 \ln \left[ 1 + \frac{1}{6(1 - \frac{1}{N})} \lambda_{3,0} \right]. \tag{47}
\]
parameters which vanishes initially are generated. The absolute values of $\lambda_0$, $\lambda_1$, $\lambda_2$, and $\lambda_3$ all approach to infinity at a finite RG running parameter $\ell_c$. The ratios between the coupling parameters in the limit $\ell \to \ell_c$ are determined by the initial conditions. After checking these ratios, we find that the system would become to NLSM, excitonic insulator, or superconducting phase, according to the concrete initial conditions and the value of $N$.

V. OBSERVABLE QUANTITIES

For convenience, we compare the observable quantities in different phases.

For conventional NLF system, the DOS satisfies

$$\rho(\omega) = \frac{Nk_F |\omega|}{2\pi v_F v_z},$$

where $v_F$ and $v_z$ are the fermion velocities within $x - y$ plane and along $z$ axis. The specific heat and compressibility depend on temperature as

$$C_v(T) = \frac{9\zeta(3)Nk_F T}{\pi \nu_F v_z^2},$$

$$\kappa(T) = \frac{2 \ln(2) Nk_F}{\pi \nu_F v_z T}.$$ (50)

For quadratic and cubic NLF systems with an excitonic gap $\Delta_3$, the retarded fermion propagator takes the form

$$G_{q,c}^{ret}(\omega, k) = \frac{1}{-(\omega + i\eta) + \mathcal{H}_0^{q,c}(k) + \Delta_3 \sigma_3},$$ (52)

where $\eta$ is infinitesimal. The spectral function is given by

$$A_{q,c}(\omega, k) = \frac{1}{\pi} \text{Tr} \left[ \text{Im} \left[ G_{q,c}^{ret}(\omega, k) \right] \right] = 2|\omega| \delta \left( \omega - (E_{q,c}^2(k) + \Delta_3^2) \right).$$ (53)

The DOS can be written as

$$\rho_q(\omega) = N \int \frac{d^3k}{(2\pi)^3} A_{q,c}(\omega, k)$$

$$\approx Nk_F \int \frac{dk_x dk_z}{(2\pi)^2} A_{q,c}(\omega, k).$$ (54)

Substituting Eq. (2) into Eqs. (53) and (54), we can get the DOS for quadratic NLF system

$$\rho_q(\omega) = \frac{Nk_F |\omega|}{4\pi A \sqrt{\omega^2 - \Delta_3^2}} \theta(|\omega| - |\Delta_3|).$$ (55)

Substituting Eq. (1) into Eqs. (53) and (54), the DOS for cubic NLF system can be written as

$$\rho_c(\omega) = \frac{Nk_F |\omega|}{6\pi B^{2/3} (\omega^2 - \Delta_3^2)^{2/3}} \theta(|\omega| - |\Delta_3|).$$ (56)

If $\Delta_3 = 0$, $\rho_q$ and $\rho_c$ become

$$\rho_q(\omega) = \frac{Nk_F}{4\pi A},$$

and

$$\rho_c(\omega) = \frac{Nk_F}{6\pi B^{2/3} |\omega|^{1/3}},$$ (57)

respectively.

For quadratic and cubic NLF systems with finite excitonic gap $\Delta_3$ and finite chemical potential $\mu$, the propagator of fermions in Matsubara formalism can be written as

$$G_{q,c}(\omega_n, k) = \int \frac{dk_x dk_z}{(2\pi)^2} \left( E_{q,c}(k) + \Delta_3^2 + \sum_{\alpha = \pm} \frac{\omega_n + \mu + \mathcal{H}_0^{q,c}(k) + \Delta_3 \sigma_3}{(\omega_n - i\mu)^2 + E_{q,c}^2(k) + \Delta_3^2} \right),$$ (59)

where $\omega_n = (2n + 1)\pi T$ with $n$ being integers. The parameter chemical potential $\mu$ is introduced to calculate the compressibility subsequently. The free energy of the fermions is given by

$$F_f(T, \mu) = -2NT \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \ln \left[ \left( (\omega_n - i\mu)^2 + E_{q,c}^2(k) \right) \right].$$ (60)

Carrying out the frequency summation, we obtain

$$F_f(T, \mu) = -2N \sum_{\alpha = \pm} \int \frac{d^3k}{(2\pi)^3} \left[ E_{q,c}(k) \right.$$

$$+ T \ln \left( 1 + e^{-E_{q,c}^2(k) + \alpha \mu \frac{n \pi}{T}} \right)],$$ (62)

which is clearly divergent. In order to get a finite free energy, we redefine $F_f(T) - F_f(0)$ as $F_f(T)$, and get

$$F_f(T, \mu) = -2NTk_F \sum_{\alpha = \pm} \int \frac{dk_x dk_z}{(2\pi)^2} \left[ E_{q,c}(k) \right.$$

$$\times \ln \left( 1 + e^{-E_{q,c}^2(k) + \alpha \mu \frac{n \pi}{T}} \right).$$ (63)

Taking the limit $\mu = 0$, we have

$$F_f(T) = -4NTk_F \int \frac{dk_x dk_z}{(2\pi)^2} \ln \left( 1 + e^{-E_{q,c}^2(k)} \right).$$ (64)
The specific heat is defined as
\[ C_v(T) = -T \frac{\partial^2 F_f(T)}{\partial T^2}. \] (65)

Substituting Eq. (2) into Eqs. (64) and (65), we find that for quadratic NLF system, if \( \Delta_3 = 0 \), the specific heat reads as
\[ C_v(T) = \frac{\pi N k_F}{24 A} T; \] (66)

If \( \Delta_3 \) is finite, the specific heat satisfies
\[ C_v(T) \approx \frac{N k_F \Delta_3^3}{\pi A T^2} e^{-\frac{\Delta_3}{T}}, \] (67)
in the limit \( \Delta_3 \gg T \).

For cubic NLF system, substituting Eq. (4) into Eqs. (64) and (65), for \( \Delta_3 = 0 \), we obtain
\[ C_v(T) = \frac{20 a_1 N T^{2/3} k_F}{9 \pi B^{2/3}}, \] (68)
where
\[ a_1 = \int_0^{+\infty} dxx \ln \left(1 + e^{-x^3}\right) \approx 0.3547; \] (69)

For finite \( \Delta_3 \), we have
\[ C_v(T) \approx \frac{N k_F \Delta_3^{8/3}}{\pi B^{2/3} T^2} e^{-\frac{\Delta_3}{T}}, \] (70)
in the limit \( \Delta_3 \gg T \).

The compressibility is defined as
\[ \kappa(T) = -\frac{\partial^2 F_f(T, \mu)}{\partial \mu^2}. \] (71)

Substituting Eq. (63) into Eq. (71) and then taking \( \mu = 0 \), we can get the expressions of compressibility for quadratic and cubic NLF systems. Concretely, for quadratic NLF system, the compressibility is given by
\[ \kappa(T) = \frac{N k_F}{2 \pi A}, \] (72)
for the case \( \Delta_3 = 0 \), and
\[ \kappa(T) = \frac{N k_F \Delta_3^2}{\pi A T^2} e^{-\frac{\Delta_3}{T}}, \] (73)
for finite \( \Delta_3 \) in the limit \( T \ll \Delta_3 \). For cubic NLF system, in the case \( \Delta_3 = 0 \), the compressibility reads as
\[ \kappa(T) = \frac{2 a_2 N k_F}{\pi B^{2/3} T^{1/3}}, \] (74)
in the case \( \Delta_3 = 0 \), where
\[ a_2 = \int_0^{+\infty} dxx \frac{e^{x^3}}{(1 + e^{x^3})^2} \approx 0.1903; \] (75)

For finite \( \Delta_3 \),
\[ \kappa(T) \approx \frac{N k_F \Delta_3^{2/3}}{\pi B^{2/3} T} e^{-\frac{\Delta_3}{T}}, \] (76)
in the limit \( T \ll \Delta_3 \).

VI. SUMMARY AND DISCUSSION

In summary, we study the influence of four-fermion interactions on the quadratic and cubic NLF systems by RG theory. We find that arbitrarily weak four-fermion interactions could drive the system to NLSM, excitonic insulator, or superconducting phase, which is determined by the concrete initial conditions and value of fermion flavor. The remarkable interaction effects in quadratic and cubic NLF systems are closely related to the dispersion of fermion excitations.

Yu et al. predicted that quadratic NLF system may be realized in the candidate materials including ZrPtGa, V12P7, ZrRuAs, and cubic NLF system may be realized in CaAgBi. We expect our theoretical predictions may be verified experimentally in these candidate materials for quadratic and cubic NLF systems in future.

Recently, Volkov and Moroz found that nodal surface fermion system is another strong correlated system in three dimension, since it would be driven to excitonic insulator phase under arbitrarily weak Coulomb interaction.

ACKNOWLEDGEMENTS

J.R.W. is grateful to Prof. G.-Z. Liu for the valuable discussions. We acknowledge the support from the National Key R&D Program of China under Grants 2017YFA0403600 and 2016YFA0300404, and that from the National Natural Science Foundation of China under Grants 11504379, 11674327, 11974356, and U1832209. A portion of this work was supported by the High Magnetic Field Laboratory of Anhui Province.

FIG. 9: Feynman diagrams for the self-energies of fermions induced by four-fermion interactions. Solid line represents the fermion propagator, and dashed line stands for the four-fermion interaction.

Appendix A: Fermion propagator

The fermion propagator for quadratic NLF system takes the form
\[ G_q(\omega, k) = \frac{1}{i \omega - H_0^q(k)}. \] (A1)
FIG. 10: One-loop Feynman diagrams for the corrections to
the four-fermion couplings.

where \( H_0 \) is given by Eq. (1). The fermion propagator
for cubic NLF system can be written as

\[
G_{c,0}(\omega, k) = \frac{1}{i\omega - H_{0c}(k)},
\]

where \( H_{0c} \) is expressed by Eq. (3).

Appendix B: Self-energy of the fermions

The self-energy of fermions induced by Fig. 9(a) is de-
fin ted as

\[
\Sigma_a = \sum_{i=0}^{3} \frac{\lambda_i}{N} \int \frac{d\omega}{2\pi} \int' \frac{d^3k}{(2\pi)^3} G_{q,c,0}(\omega, k),
\]

where \( \int' \) represents that a momentum shell will be prop-
erly taken. Figure 9(b) results in the self-energy of
fermions as following

\[
\Sigma_b = N \sum_{i=0}^{3} \frac{\lambda_i}{N} \int \frac{d\omega}{2\pi} \int' \frac{d^3k}{(2\pi)^3} \text{Tr} [G_{q,c,0}(\omega, k)].
\]

Substituting Eq. (A1) or Eq. (A2) into Eqs. (B1) and
(B2), we obtain

\[
\Sigma_a = 0,
\]

\[
\Sigma_b = 0,
\]

for both of quadratic and cubic NLF systems. Thus
the fermion propagator is not renormalized by the four-
ermion interactions to one-loop order.

Appendix C: One-loop order corrections for the four-fermion couplings

1. General expressions for the one-loop order corrections

The correction contributed by Fig. 10(a) is given by

\[
W^{(1)} = \sum_{i=0}^{3} W_i^{(1)},
\]

where

\[
W_i^{(1)} = \sum_{j=0}^{3} \frac{4\lambda_i \lambda_j}{N} \left( \psi^\dagger \sigma_i \psi \right) \int \frac{d\omega}{2\pi} \int' \frac{d^3k}{(2\pi)^3} \left[ \psi^\dagger \sigma_j G_{q,c,0}(\omega, k) \sigma_i G_{q,c,0}(\omega, k) \sigma_j \psi \right].
\]

The diagrams as shown in Figs. 10(b) and 10(c) lead to the correction for the four-fermion couplings as following

\[
W^{(2)+(3)} = \sum_{i=0}^{3} \sum_{i \leq j \leq 3} W_{ij}^{(2)+(3)},
\]

where

\[
W_{ij}^{(2)+(3)} = \frac{4\lambda_i \lambda_j}{N} \int \frac{d\omega}{2\pi} \int' \frac{d^3k}{(2\pi)^3} \left( \psi^\dagger \sigma_i G_{q,c,0}(\omega, k) \sigma_j \psi \right) \left\{ \psi^\dagger \left[ \sigma_j G_{q,c,0}(\omega, k) \sigma_i + \sigma_i G_{q,c,0}(\omega, k) \sigma_j \right] \psi \right\}.
\]

The correction for the four-fermion couplings resulting from Fig. 10(d) can be written as

\[
W^{(4)} = \sum_{i=0}^{3} W_i^{(4)},
\]
where

\[ W_i^{(4)} = -2\lambda_i^2 (\psi^\dagger \sigma_i \psi) (\psi^\dagger \sigma_i \psi) \int \frac{d\omega}{2\pi} \int d^3k \frac{1}{(2\pi)^3} \text{Tr} [\sigma_i G_{q,0}(\omega, \mathbf{k}) \sigma_i G_{q,0}(\omega, \mathbf{k})]. \tag{C6} \]

A momentum shell \( b \Lambda < \sqrt{k_x^2 + k_y^2} < \Lambda \) with \( b = e^{-\ell} \) will be utilized in the derivation, where \( \ell \) stands for the running parameter.

2. Results for quadratical NLF

Substituting Eq. (A1) into Eqs. (C1) - (C6), we obtain

\[
W^{(1)} = \lambda_1 (\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \frac{1}{N} \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_1 \psi)^2 + \lambda_2 (\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) \frac{1}{N} \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_2 \psi)^2 + \lambda_3 (\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) \frac{1}{N} \frac{k_F}{\pi A} \ell (\psi^\dagger \sigma_3 \psi)^2, \tag{C7}
\]

\[
W^{(2)+(3)} = (\lambda_0 \lambda_1 + \lambda_0 \lambda_2) \frac{1}{N} \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_0 \psi)^2 + (\lambda_0 \lambda_1 + \lambda_1 \lambda_2 - 2\lambda_1 \lambda_3) \frac{1}{N} \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_2 \psi)^2 + (\lambda_1 \lambda_2 + \lambda_0 \lambda_3 - 2\lambda_2 \lambda_3) \frac{1}{N} \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_3 \psi)^2, \tag{C8}
\]

\[
W^{(4)} = \lambda_1^2 \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_1 \psi)^2 + \lambda_2^2 \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_2 \psi)^2 + \lambda_3^2 \frac{k_F}{2\pi A} \ell (\psi^\dagger \sigma_3 \psi)^2. \tag{C9}
\]

From Eqs. (C7) - (C9), we get

\[
W = W^{(1)} + W^{(2)+(3)} + W^{(4)} = \sum_{i=0}^{3} \delta \lambda_i (\psi^\dagger \sigma_i \psi)^2, \tag{C10}
\]

where

\[
\delta \lambda_0 = (\lambda_0 \lambda_1 + \lambda_0 \lambda_2) \frac{1}{N} \frac{k_F}{2\pi A} \ell, \tag{C11}
\]

\[
\delta \lambda_1 = \left[ (\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_0 \lambda_1 + 2\lambda_1 \lambda_2 + \lambda_1 \lambda_3 - 2\lambda_2 \lambda_3) \frac{1}{N} + \lambda_3^2 \right] \frac{k_F}{2\pi A} \ell, \tag{C12}
\]

\[
\delta \lambda_2 = \left[ (\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_0 \lambda_2 + 2\lambda_1 \lambda_2 - 2\lambda_1 \lambda_3 + 2\lambda_2 \lambda_3) \frac{1}{N} + \lambda_3^2 \right] \frac{k_F}{2\pi A} \ell, \tag{C13}
\]

\[
\delta \lambda_3 = \left[ (-2\lambda_3^2 + 2\lambda_0 \lambda_3 - 2\lambda_1 \lambda_2 + 3\lambda_1 \lambda_3 + 3\lambda_2 \lambda_3) \frac{1}{N} + 2\lambda_3^2 \right] \frac{k_F}{2\pi A} \ell. \tag{C14}
\]

3. Results for cubic NLF

Substituting Eq. (A2) into Eqs. (C1) - (C6), we get

\[
W^{(1)} = \lambda_1 (\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \frac{1}{N} \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_1 \psi)^2 + \lambda_2 (\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) \frac{1}{N} \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_2 \psi)^2 + \lambda_3 (\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) \frac{1}{N} \frac{k_F}{\pi B_A} \ell (\psi^\dagger \sigma_3 \psi)^2, \tag{C15}
\]

\[
W^{(2)+(3)} = (\lambda_1 \lambda_3 + \lambda_2 \lambda_3) \frac{1}{N} \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_0 \psi)^2 + (\lambda_0 \lambda_3 - 2\lambda_2 \lambda_3) \frac{1}{N} \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_1 \psi)^2 + (\lambda_0 \lambda_3 - 2\lambda_1 \lambda_3) \frac{1}{N} \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_2 \psi)^2 + (\lambda_0 \lambda_1 - 2\lambda_1 \lambda_3 - 2\lambda_2 \lambda_3) \frac{1}{N} \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_3 \psi)^2, \tag{C16}
\]

\[
W^{(4)} = \lambda_1^2 \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_1 \psi)^2 + \lambda_2^2 \frac{k_F}{2\pi B_A} \ell (\psi^\dagger \sigma_2 \psi)^2 + \lambda_3^2 \frac{k_F}{\pi B_A} \ell (\psi^\dagger \sigma_3 \psi)^2. \tag{C17}
\]
From Eqs. (C15)-(C17), we find

\[ W = W^{(1)} + W^{(2)+(3)} + W^{(4)} = \sum_{i=0}^{3} \delta \lambda_i \left( \psi_i \sigma_i \psi \right)^2, \]  

where

\[ \delta \lambda_0 = \left( \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \right) \frac{1}{N} \frac{k_F}{2\pi BA} \ell, \]  

\[ \delta \lambda_1 = \left[ -\lambda_1^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 - 2\lambda_2 \lambda_3 \right] \frac{1}{N} \frac{k_F}{2\pi BA} \ell, \]  

\[ \delta \lambda_2 = \left[ -\lambda_2^2 + \lambda_0 \lambda_2 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 - 2\lambda_1 \lambda_3 + \lambda_2 \lambda_3 \right] \frac{1}{N} \frac{k_F}{2\pi BA} \ell, \]  

\[ \delta \lambda_3 = \left[ -2\lambda_3^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_2 + 2\lambda_0 \lambda_3 - 2\lambda_1 \lambda_2 + 2\lambda_1 \lambda_3 + 2\lambda_2 \lambda_3 \right] \frac{1}{N} \frac{k_F}{2\pi BA} \ell. \]  

**Appendix D: Derivation of the RG equations**

1. Quadratic NLF

The action for the quadratic NLF is

\[ S_\psi = \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \psi^\dagger(\omega, k) \left[ -i\omega + A \left( k_r^2 - k_z^2 \right) \sigma_1 + 2 A k_r k_z \sigma_2 \right] \psi(\omega, k), \]  

where \( k_r \equiv k_{\perp} - k_F \equiv \sqrt{k_x^2 + k_y^2} - k_F \). In the low-energy regime, the action for quadratic NLF can be also written as

\[ S_\psi \approx k_F \int \frac{d\omega}{2\pi} \frac{dk_r}{2\pi} \frac{dk_z}{2\pi} \psi^\dagger(\omega, k) \left[ -i\omega + A \left( k_r^2 - k_z^2 \right) \sigma_1 + 2 A k_r k_z \sigma_2 \right] \psi(\omega, k). \]  

Employing the transformations

\[ k_r = k_r' e^{-\frac{k_z}{k_F}}, \]  

\[ k_z = k_z' e^{-\frac{k_r}{k_F}}, \]  

\[ \omega = \omega' e^{-\ell}, \]  

\[ \psi = \psi' e^{2\ell}, \]  

\[ A = A', \]  

the action for the quadratic NLF becomes

\[ S_{\psi'} = k_F \int \frac{d\omega'}{2\pi} \frac{dk_r'}{2\pi} \frac{dk_z'}{2\pi} \psi'^\dagger(\omega', k') \left[ -i\omega' + A' \left( k_r'^2 - k_z'^2 \right) \sigma_1 + 2 A' k_r' k_z' \sigma_2 \right] \psi'(\omega', k'), \]  

which recovers the form of the original action.

The action for the four-fermion interactions between quadratic NLFs is given by

\[ S_{\psi^4} = \frac{1}{N} \sum_{i=0}^{3} \int \frac{d\omega_1}{2\pi} \frac{d^3k_1}{(2\pi)^3} \frac{d\omega_2}{2\pi} \frac{d^3k_2}{(2\pi)^3} \frac{d\omega_3}{2\pi} \frac{d^3k_3}{(2\pi)^3} \psi^\dagger(\omega_1, k_1) \sigma_i \psi(\omega_2, k_2) \psi^\dagger(\omega_3, k_3) \sigma_i \psi(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3) \]

\[ \approx \frac{1}{N} \sum_{i=0}^{3} \lambda_i k_F^2 \int \frac{d\omega_1}{2\pi} \frac{dk_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{dk_2}{2\pi} \frac{d\omega_3}{2\pi} \frac{dk_3}{2\pi} \frac{d\omega_4}{2\pi} \frac{dk_4}{2\pi} \frac{d\omega_5}{2\pi} \frac{dk_5}{2\pi} \psi^\dagger(\omega_1, k_1) \sigma_i \psi(\omega_2, k_2) \psi^\dagger(\omega_3, k_3) \sigma_i \psi(\omega_4 - \omega_2 + \omega_3, k_1 - k_2 + k_3) \times \psi(\omega_5 - \omega_2 + \omega_3, k_1 - k_2 + k_3). \]
Incorporating the one-loop order corrections, the action becomes
\[
S_{\psi^4} = \frac{1}{N} \sum_{i=0}^{3} (\lambda_i + \delta \lambda_i) k_F^2 \int \frac{d\omega_1}{2\pi} \frac{dk_{1r}}{2\pi} \frac{dk_{1z}}{2\pi} \frac{d\omega_2}{2\pi} \frac{dk_{2r}}{2\pi} \frac{dk_{2z}}{2\pi} \frac{d\omega_3}{2\pi} \frac{dk_{3r}}{2\pi} \frac{dk_{3z}}{2\pi} \psi^\dagger(\omega_1, k_1) \sigma_i \psi(\omega_2, k_2) \psi^\dagger(\omega_3, k_3) \sigma_i \times \psi(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3).
\]
\[
(D10)
\]
Adopting the transformations shown in Eqs. (D3)-(D6), and which recovers the original form the action. From Eq. (D11), we obtain
\[
\frac{d\lambda_i}{dt} = \frac{d\delta \lambda_i}{dt},
\]
which recovers the original form the action. From Eq. (D11), we obtain
\[
\frac{d\lambda_i}{dt} = \frac{d\delta \lambda_i}{dt}.
\]
Substituting Eqs. (C11)-(C14) into Eq. (D11), we get the RG equations
\[
\frac{d\lambda_0}{dt} = (\lambda_0 \lambda_1 + \lambda_0 \lambda_2) \frac{1}{N},
\]
\[
(D14)
\]
\[
\frac{d\lambda_1}{dt} = \left[ \left( \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_0 \lambda_1 + 2 \lambda_1 \lambda_2 + \lambda_1 \lambda_3 - 2 \lambda_2 \lambda_3 \right) \frac{1}{N} + \lambda_1^2 \right],
\]
\[
(D15)
\]
\[
\frac{d\lambda_2}{dt} = \left[ \left( \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_0 \lambda_1 + 2 \lambda_1 \lambda_2 - 2 \lambda_1 \lambda_3 + 3 \lambda_2 \lambda_3 \right) \frac{1}{N} + \lambda_2^2 \right],
\]
\[
(D16)
\]
\[
\frac{d\lambda_3}{dt} = \left[ (-2 \lambda_0^2 + 2 \lambda_0 \lambda_3 + 2 \lambda_1 \lambda_2 - 3 \lambda_1 \lambda_3 + 3 \lambda_2 \lambda_3) \frac{1}{N} + 2 \lambda_3^2 \right].
\]
\[
(D17)
\]
The transformations
\[
\frac{k_F}{2\pi A} \lambda_i \rightarrow \lambda_i
\]
with \( i = 0, 1, 2, 3 \) have been used.

2. Cubic NLF

The action for the cubic NLF takes the form
\[
S_\psi = \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \psi^\dagger(\omega, k) \left[ -i \omega + B (k_0^3 - 3k_r k_z^2) \sigma_1 + B (k_0^3 - 3k_r k_z^2) \sigma_2 \right] \psi(\omega, k),
\]
\[
(D19)
\]
which is equivalent to
\[
S_\psi \approx k_F \int \frac{d\omega}{2\pi} \frac{dk_r}{2\pi} \frac{dk_z}{2\pi} \psi^\dagger(\omega, k) \left[ -i \omega + B (k_0^3 - 3k_r k_z^2) \sigma_1 + B (k_0^3 - 3k_r k_z^2) \sigma_2 \right] \psi(\omega, k).
\]
\[
(D20)
\]
Using the transformations
\[
k_r = k'_r e^{-\frac{4}{3}t},
\]
\[
(k'_r e^{-\frac{4}{3}t})
\]
\[
k_z = k'_z e^{-\frac{4}{3}t},
\]
\[
(D21)
\]
\[
\omega = \omega' e^{-t},
\]
\[
(D22)
\]
\[
\psi = \psi' e^{\frac{2}{3}t},
\]
\[
(D23)
\]
\[
B = B'
\]
\[
(D24)
\]
\[
B = B'
\]
\[
(D25)
\]
the action can be written as
\[ S_{\psi'} = k_F \int \frac{d\omega'}{2\pi} \frac{dk'_y}{2\pi} \frac{dk'_z}{2\pi} \psi'^\dagger(\omega', k') \left[ -i\omega' + B' \left( k'^2_x - 3k'^2_yk'^2_z \right) \sigma_1 + B' \left( k'^2_y - 3k'^2_z \right) \sigma_2 \right] \psi'(\omega', k'), \] (D26)

which has the same form as the original action.

The action describing the four-fermion interactions between cubic NLFs is given by
\[ S_{\psi^4} = \frac{1}{N} \sum_{i=0}^{3} \lambda_i k_F^3 \int \frac{d\omega_1}{2\pi} \frac{dk_{1y}}{2\pi} \frac{dk_{1z}}{2\pi} \frac{d\omega_2}{2\pi} \frac{dk_{2y}}{2\pi} \frac{dk_{2z}}{2\pi} \frac{d\omega_3}{2\pi} \frac{dk_{3y}}{2\pi} \frac{dk_{3z}}{2\pi} \psi'^\dagger(\omega_1, k_1) \sigma_i \psi(\omega_2, k_2) \psi'^\dagger(\omega_3, k_3) \sigma_i \times \psi(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3). \] (D27)

Including one-loop order corrections, the action becomes
\[ S_{\psi^4} = \frac{1}{N} \sum_{i=0}^{3} (\lambda_i + \delta\lambda_i) k_F^3 \int \frac{d\omega_1}{2\pi} \frac{dk_{1y}}{2\pi} \frac{dk_{1z}}{2\pi} \frac{d\omega_2}{2\pi} \frac{dk_{2y}}{2\pi} \frac{dk_{2z}}{2\pi} \frac{d\omega_3}{2\pi} \frac{dk_{3y}}{2\pi} \frac{dk_{3z}}{2\pi} \psi'^\dagger(\omega_1, k_1) \sigma_i \psi(\omega_2, k_2) \psi'^\dagger(\omega_3, k_3) \sigma_i \times \psi(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3). \] (D28)

Utilizing the transformations shown in Eqs. (D21)-(D24), and
\[ \lambda'_i = (\lambda_i + \delta\lambda_i) e^{\frac{1}{3}\ell} \approx \lambda_i + \lambda_i \frac{1}{3} \ell + \delta\lambda_i, \] (D29)

the action becomes
\[ S_{\psi^4} = \sum_{i=0}^{3} \lambda'_i k_F^3 \int \frac{d\omega'_1}{2\pi} \frac{dk'_{1y}}{2\pi} \frac{dk'_{1z}}{2\pi} \frac{d\omega'_2}{2\pi} \frac{dk'_{2y}}{2\pi} \frac{dk'_{2z}}{2\pi} \frac{d\omega'_3}{2\pi} \frac{dk'_{3y}}{2\pi} \frac{dk'_{3z}}{2\pi} \psi'^\dagger(\omega'_1, k'_1) \sigma_i \psi'(\omega'_2, k'_2) \psi'^\dagger(\omega'_3, k'_3) \sigma_i \times \psi'(\omega'_1 - \omega'_2 + \omega'_3, k'_1 - k'_2 + k'_3), \] (D30)

which has the same form of the original action. According to Eq. (D29), the RG equation for \( \lambda_i \) is given by
\[ \frac{d\lambda_i}{d\ell} = \frac{1}{3} \lambda_i + \frac{d\delta\lambda_i}{d\ell}. \] (D31)

Substituting Eqs. (C19)-(C22) into Eq. (D31), the RG equations for \( \lambda_i \) can be written as
\[ \frac{d\lambda_0}{d\ell} = \frac{1}{3} \lambda_0 + (\lambda_1 \lambda_3 + \lambda_2 \lambda_3) \frac{1}{N}, \] (D32)
\[ \frac{d\lambda_1}{d\ell} = \frac{1}{3} \lambda_1 + \left[ (-\lambda_1^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 - 2\lambda_2 \lambda_3) \frac{1}{N} + \lambda_1^2 \right], \] (D33)
\[ \frac{d\lambda_2}{d\ell} = \frac{1}{3} \lambda_2 + \left[ (-\lambda_2^2 + \lambda_0 \lambda_2 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 - 2\lambda_1 \lambda_3 + \lambda_2 \lambda_3) \frac{1}{N} + \lambda_2^2 \right], \] (D34)
\[ \frac{d\lambda_3}{d\ell} = \frac{1}{3} \lambda_3 + \left[ (-2\lambda_3^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_2 + 2\lambda_0 \lambda_3 - 2\lambda_1 \lambda_2 + 2\lambda_1 \lambda_3 + 2\lambda_2 \lambda_3) \frac{1}{N} + 2\lambda_3^2 \right]. \] (D35)

The transformations
\[ \frac{k_F}{2\pi BA} \lambda_i \rightarrow \lambda_i \] (D36)

with \( i = 0, 1, 2, 3 \) have been adopted.

\[ \text{[1] V. N. Kotov, B. Uchoa, V. M. Pereira, F. Guinea, and A. H. Castro Neto, Electron-electron interactions in graphene: Current status and perspectives, Rev. Mod.} \]
H. Wang and J. Wang, Electron transport in Dirac and B. Lv, T. Qian, and H. Ding, Angle-resolved photoemission spectroscopy and its application to topological materials using symmetry indicators, Nature 566, 486 (2019).

T. Zhang, Y. Jiang, Z. Song, H. Huang, Y. He, Z. Fang, H. Weng, and C. Fang, Catalogue of topological electronic materials, Nature 566, 475 (2019).

M. G. Vergniory, L. Elcoro, C. Felser, N. Regnault, B. A. Bernevig, and Z. Wang, A complete catalogue of high-quality topological materials, Nature 566, 480 (2019).

B. Skinner and L. Fu, Large, nonsaturating thermopower in a quantizing magnetic field, Sci. Adv. 4, eaat2621 (2018).

M. Markov, S. Emad Razaei, S. N. Sadeghi, K. Esfarjani, and M. Zebarjadi, Thermoelectric properties of semimetals, Phys. Rev. Materials 3, 095401 (2019).

M. Neupane, S.-Y. Xu, R. Sankar, N. Alidoust, G. Bian, C. Liu, I. Belopolski, T.-R. Chang, H.-T. Jeng, H. Lin, A. Bansil, F. Chou, and M. Z. Hasan, Observation of a three-dimensional topological Dirac semimetal phase in high-mobility CdAs2, Nat. Commun. 5, 3786 (2014).

Z. K. Liu, B. Zhou, Y. Zhang, Z. J. Wang, H. M. Weng, D. Prabhakaran, S.-K. Mo, Z. X. Shen, Z. Fang, X. Dai, Z. Hussain, and Y. L. Chen, Discovery of a three-dimensional Topological Dirac Semimetal Na3Bi, Science 343, 864 (2014).

S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia, and M. Z. Hasan, Discovery of a Weyl fermion semimetal and topological Fermi arcs, Science 349, 613 (2015).

B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Experimental Discovery of Weyl Semimetal TaAs, Phys. Rev. X 5, 031013 (2015).

G. Bian, T.-R. Chang, R. Sankar, S.-Y. Xu, H. Zheng, T. Neupert, C.-K. Chiu, S.-M. Huang, G. Chang, I. Belopolski, D. S. Sanchez, M. Neupane, N. Alidoust, C. Liu, B. Wang, C.-C. Lee, H.-T. Jeng, C. Zhang, Z. Yuan, S. Jia, A. Bansil, F. Chou, H. Lin, and M. Z. Hasan, Topological nodal-line fermions in spin-orbit metal PbTaSe2, Nat. Commun. 7, 10556 (2016).

M. Neupane, I. Belopolski, M. M. Hosen, D. S. Sanchez, R. Sankar, M. Sadowski, S.-Y. Xu, K. Dimitri, D. Dhakal, P. Maldonado, P. M. Oppeneer, D. Kaczorowski, F. Chou, M. Z. Hasan, and T. Durakiewicz, Observation of topological nodal fermion semimetal phase in ZrSiS, Phys. Rev. B 93, 201104(R) (2016).

L. M. Schoop, M. N. Ali, C. Strasser, A. Topp, A. Varykhalov, D. Marchenko, V. Duppel, S. S. P. Parkin, B. V. Lotsch, and C. R. Ast, Dirac cone protected by non-symmetric symmetry and three-dimensional Dirac line node in ZrSiS, Nat. Commun. 7, 11696 (2016).

T. Takane, Z. Wang, S. Souma, K. Nakayama, C. X. Trang, T. Sato, T. Takahashi, and Y. Ando, Dirac-node arc in the topological line-node semimetal HSiS, Phys. Rev. B 94, 121108(R) (2016).

C.-J. Yi, B. Q. Lv, Q. S. Wu, B.-B. Fu, X. Gao, M. Yang, X.-L. Peng, M. Li, Y.-B. Huang, P. Richard, M. Shi, G. Li, O. V. Yazeyev, Y.-G. Shi, T. Qian, and H. Ding, Observation of a nodal chain with Dirac surface states in TiB2, Phys. Rev.B 97, 201107(R) (2018).

Z. Liu, R. Luo, P. Guo, Q. Wang, S. Sun, C. Li, S. Thirupathaiah, A. Fedorov, D. Shen, K. Liu, H. Lei, and S. Wang, Experimental observation of Dirac nodal link in centrosymmetric semimetal TiB2, Phys. Rev. X 8, 031044 (2018).

G. F. Giuliani and G. Vignale, Quantum Theory of the Electron Liquid (Cambridge University Press, 2005).

J. González, F. Guinea, and M. A. H. Vozmediano, Marginal-Fermi-liquid behavior from two-dimensional Coulomb interaction, Phys. Rev. B 59, R2474(R) (1999).

J. Hofmann, E. Barnes, and S. Das Sarma, Why does graphene behave as a weakly interacting system?, Phys. Rev. Lett. 113, 105502 (2014).

P. Goswami and S. Chakravarty, Quantum criticality between topological and band insulators in 3+1 dimensions, Phys. Rev. Lett. 107, 196803 (2011).

P. Horus, S. A. Parameswaran, and A. Vishwanath, Charge transport in Weyl semimetals, Phys. Rev. Lett. 108, 046602 (2012).

R. E. Throckmorton, J. Hofmann, E. Barnes, and S. Das Sarma, Many-body effects and ultraviolet renormalization in three-dimensional Dirac materials, Phys. Rev. B 92, 115101 (2015).

E.-G. Moon, C. Xu, Y. B. Kim, and L. Balents, Non-Fermi-liquid and topological states with strong spin-orbit coupling, Phys. Rev. Lett. 111, 206401 (2013).

I. F. Herbut and L. Janssens, Topological Mott insulator in three-dimensional systems with quadratic band touching, Phys. Rev. Lett. 113, 106401 (2014).

B.-J. Yang, E.-G. Moon, H. Isole, and N. Nagaosa,
Quantum criticality of topological phase transitions in three-dimensional interacting electronic systems, Nat. Phys. 10, 774 (2014).

[37] H. Isobe, B.-J. Yang, A. Chubukov, J. Schmalian, and N. Nagaosa, Emergent non-Fermi-liquid at the quantum critical point of a topological phase transition in two dimensions, Phys. Rev. Lett. 116, 076803 (2016).

[38] J.-R. Wang, G.-Z. Liu, and C.-J. Zhang, Excitonic pairing and insulating transition in two-dimensional semi-Dirac semimetals, Phys. Rev. B 95, 075129 (2017).

[39] S.-K. Jian and H. Yao, Correlated double-Weyl semimetals with Coulomb interactions: Possible applications to HgCr$_2$Se$_2$ and SrSi$_2$, Phys. Rev. B 92, 045121 (2015).

[40] J.-R. Wang, G.-Z. Liu, and C.-J. Zhang, Quantum phase transition and unusual critical behavior in multi-Weyl semimetals, Phys. Rev. B 96, 165142 (2017).

[41] S.-X. Zhang, S.-K. Jian, and H. Yao, Correlated triple-Weyl semimetals with Coulomb interactions, Phys. Rev. B 96, 241111(R) (2017).

[42] J.-R. Wang, G.-Z. Liu, and C.-J. Zhang, Breakdown of Fermi liquid theory in topological multi-Weyl semimetals, Phys. Rev. B 98, 205113 (2018).

[43] Y. Huh, E.-G. Moon, and Y. B. Kim, Long-range Coulomb interaction in nodal-ring semimetals, Phys. Rev. B 93, 035138 (2016).

[44] Y. Wang and R. Nandkishore, Interplay between short-range correlated disorder and Coulomb interaction in nodal-line semimetals, Phys. Rev. B 96, 115130 (2017).

[45] J.-R. Wang, G.-Z. Liu, and C.-J. Zhang, Topological quantum critical point in a triple-Weyl semimetal: Non-Fermi-liquid behavior and instabilities, Phys. Rev. B 99, 195119 (2019).

[46] S. Han, C. Lee, E.-G. Moon, and H. Min, Emergent Anisotropic Non-Fermi Liquid at a Topological Phase Transition in Three Dimensions, Phys. Rev. Lett. 122, 187601 (2019).

[47] S.-X. Zhang, S.-K. Jian, and H. Yao, Quantum criticality preempted by nematicity, [arXiv:1809.10686](https://arxiv.org/abs/1809.10686)

[48] S. Han and E.-G. Moon, Long-range Coulomb interaction effects on the topological phase transitions between semimetals and insulators, Phys. Rev. B 97, 241101(R) (2018).

[49] S. Han, G. Y. Cho, and E.-G. Moon, Quantum criticality with infinite anisotropy in topological phase transitions between Dirac and Weyl semimetals, Phys. Rev. B 98, 085149 (2018).

[50] B. Roy, M. P. Kennett, K. Yang, and V. Jurićić, From birefringent electrons to a marginal or non-Fermi liquid of relativistic spin-1/2 fermions: An emergent superuniversality, Phys. Rev. Lett. 121, 157602 (2018).

[51] I. F. Herbut, Interactions and phase transitions on graphenes honeycomb lattice, Phys. Rev. Lett. 97, 146401 (2006).

[52] I. F. Herbut, V. Jurićić, and B. Roy, Theory of interacting electrons on the honeycomb lattice, Phys. Rev. B 79, 085116 (2009).

[53] B. Roy and S. Das Sarma, Quantum phases of interacting electrons in three-dimensional dirty Dirac semimetals, Phys. Rev. B 94, 115137 (2016).

[54] J. Maciejko and R. Nandkishore, Weyl semimetals with short-range interactions, Phys. Rev. B 90, 035126 (2014).

[55] B. Roy, P. Goswami, and B. Roy, Interacting Weyl fermions: Phases, phase transitions, and global phase diagram, Phys. Rev. B 95, 201102(R) (2017).

[56] B. Roy and M. S. Foster, Quantum multicriticality near the Dirac-semimetal to band-insulator critical point in two dimensions: A controlled ascent from one dimension, Phys. Rev. X 8, 011049 (2018).

[57] J. Wang, Role of four-fermion interaction and impurity in the states of two-dimensional semi-Dirac materials, J. Phys.: Condens. Matter 30, 125401 (2018).

[58] A. L. Szabó, R. Moessner, and B. Roy, Interacting spin-3/2 fermions in a Luttinger (semi)metal: competing phases and their selection in the global phase diagram, [arXiv:1811.12415](https://arxiv.org/abs/1811.12415)

[59] S. Sur and R. Nandkishore, Instabilities of Weyl loop semimetals, New J. Phys. 18, 115006 (2016).

[60] B. Roy, Interacting nodal-line semimetal: Proximity effect and spontaneous symmetry breaking, Phys. Rev. B 96, 041113(R) (2017).

[61] S. V. Syzranov and B. Skinner, Electron transport in nodal-line semimetals, Phys. Rev. B 96, 161105(R) (2017).

[62] C. Li, C. M. Wang, B. Wan, X. Wan, H.-Z. Lu, and X. C. Xie, Rules for phase shifts of quantum oscillations in topological nodal-line semimetals, Phys. Rev. Lett. 120, 146602 (2018).

[63] W. Chen, H.-Z. Lu, and O. Zilberberg, Weak localization and antilocalization in nodal-line semimetals: Dimensionality and topological effects, Phys. Rev. Lett. 122, 196603 (2019).

[64] Z.-M. Yu, W. Wu, X.-L. Sheng, Y. X. Zhao, and S. A. Yang, Quadratic and cubic nodal lines stabilized by crystalline symmetry, Phys. Rev. B 99, 121106(R) (2019).

[65] R. Shankar, Renormalization-group approach to interacting fermions, Rev. Mod. Phys. 66, 129 (1994).

[66] R. Nandkishore, L. S. Levitov, and A. V. Chubukov, Chiral superconductivity from repulsive interactions in doped graphene, Nat. Phys. 8, 158 (2012).

[67] M. A. Metlitski, D. F. Mross, S. Sachdev, and T. Senthil, Cooper pairing in non-Fermi liquids, Phys. Rev. B 91, 115111 (2015).

[68] O. Vafek and K. Yang, Many-body instability of Coulomb interacting bilayer graphene: Renormalization group approach, Phys. Rev. B 81, 041401(R) (2010).

[69] F. Zhang, H. Min, M. Polini, and A. H. MacDonald, Spontaneous inversion symmetry breaking in graphene bilayers, Phys. Rev. B 81, 041402(R) (2010).

[70] P. A. Volkov and S. Moroz, Coulomb-induced instabilities of nodal surfaces, Phys. Rev. B 98, 241107(R) (2018).