We formulate the conformal bootstrap approach to four–fermion theory at its strong coupling fixed point in dimensions $2 < d < 4$. We present a solution of the bootstrap equations in the five–vertex approximation. We show that the bootstrap approach gives a particularly simple way to obtain next to leading order corrections to critical exponents in the large $N$ expansion and present the values of the anomalous dimensions of the fermion field $\psi$ and the composite $\bar{\psi}\psi$ to order $1/N^2$. 

Abstract

We formulate the conformal bootstrap approach to four–fermion theory at its strong coupling fixed point in dimensions $2 < d < 4$. We present a solution of the bootstrap equations in the five–vertex approximation. We show that the bootstrap approach gives a particularly simple way to obtain next to leading order corrections to critical exponents in the large $N$ expansion and present the values of the anomalous dimensions of the fermion field $\psi$ and the composite $\bar{\psi}\psi$ to order $1/N^2$. 

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Conformal symmetry describes the critical behavior of quantum field theories or statistical systems at points of second order phase transition. It is widely used in two dimensions where it provides exact solutions to a variety of non-trivial interacting field theories. In this Letter we shall demonstrate that it can also be useful in higher dimensions. If a field theory has a fixed point of the renormalization group flow, the dynamics at that point is conformally invariant. In any spacetime dimensions, conformal symmetry determines the form of the two and three-point Green functions of the field theory up to a few constants, the scaling dimension of the field operators and the value of the coupling constant at the fixed point. This information, when combined with Dyson–Schwinger equations can provide a powerful tool for the analysis of field theories.

In fact, in a field theory with only three-point coupling constants, the Dyson–Schwinger equations which relate two and three point functions could in principle be used to determine all anomalous dimensions in terms of the coupling constants, which are then left to be determined by higher order equations. Unfortunately, this direct approach is plagued by technical difficulties involving indeterminate expressions. This problem was partially solved by Parisi [1] who showed how to use the information of the lower order Dyson–Schwinger equations to derive bootstrap equations for the effective coupling constants. The resulting equations can not be solved exactly, but can be used to find interesting approximate solutions of a field theory.

In this Letter, we shall consider the example of a fermionic field theory with a four-fermion interaction, $-\frac{\lambda^2}{2} \left( \sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2$ in spacetime dimensions $2 < d < 4$. By introducing an auxiliary scalar field $\phi = \lambda \sum_{i=1}^{N} \bar{\psi}_i \psi_i$ (with no kinetic term), this is equivalent to a model with a Yukawa vertex $-\lambda \phi \sum_{i=1}^{N} \bar{\psi}_i \psi_i + \phi^2/2$. The renormalization group flow in this model has an infrared stable fixed point at $\lambda = 0$ and there is a long-standing conjecture that it also has an ultraviolet stable fixed point at some $\lambda_* > 0$. This fixed point is seen both in the large $N$ and in the $\epsilon$ expansion. Also, as $d \to 2^+$, $\lambda_* \to 0$ to obtain the asymptotic freedom of the Gross-Neveu model. Recently, renormalizability of the large $N$ expansion of $d = 3$ four-Fermi theory has been proven using the rigorous renormalization group technique of constructive field theory [3].
In this Letter we shall assume that $\lambda_*$ exists and analyze the resulting conformal field theory. We find a solution of the bootstrap equations in the three and five vertex approximation. In particular, we show that these give an easy way to find the anomalous dimensions of $\psi$ and $\bar{\psi}\psi$ in the large $N$ limit to order $1/N^2$.

Under rescaling of the coordinate, $x \to \rho x$, the fields transform as

$$\psi'(\rho x) = \rho^{-l} \psi(x), \quad \bar{\psi}'(\rho x) = \rho^{-l} \bar{\psi}(x), \quad \phi'(\rho x) = \rho^{-b} \phi(x),$$

(1)

with $l$ and $b$ the scaling dimensions of the fermion (anti–fermions) and the boson, respectively. Under a special conformal transformation parameterized with a constant vector $t_{\mu}$,

$$x_{\mu} \to x'_{\mu} = \frac{x_{\mu} + t_{\mu} x^2}{1 + 2 t \cdot x + t^2 x^2},$$

(2)

where $\sigma_x \equiv \sigma_{t}(x) = \sqrt{-1/d} = 1 + 2 t \cdot x + t^2 x^2$, they transform as

$$\psi(x) \to \psi'(x') = \sigma_x^{-1/2} (1 + \hat{x} \hat{x}) \psi(x),$$

$$\bar{\psi}(x) \to \bar{\psi}'(x') = \sigma_x^{-1/2} \bar{\psi}(x) (1 + \hat{x} \hat{t}),$$

$$\phi(x) \to \phi'(x') = \sigma_x^b \phi(x),$$

(3) \hspace{1cm} (4) \hspace{1cm} (5)

where $\hat{x} = \gamma_\mu x_\mu$ ($\gamma_\mu$ are the Dirac Matrices).

These, together with translation and Lorentz invariance, form the global finite dimensional conformal group. Conformal invariance determines the form of two-point correlation functions. In momentum space,

$$G(p) = \frac{1}{i \hat{p}} (p^2)^{l-h+1/2}, \quad D(p) = \frac{1}{\hat{p}} (p^2)^{b-h+1}.$$

(6)

A convenient normalization has been chosen. (This can always be done using a finite rescaling of $\phi$ and $\psi$.) Moreover, conformal invariance fixes the three point vertex up to a dimensionless constant factor, the effective coupling $\lambda$,

$$\Gamma(p_1, p_2) = \lambda \frac{N(\gamma)}{N(b)N(l-b/2)} \int \frac{d^d k}{\pi^h} \frac{\hat{k} + \hat{p}_1}{[(k + p_1)^2]^{b/2+1/2}} \frac{\hat{k} + \hat{p}_2}{[(k + p_2)^2]^{b/2+1/2}} \frac{1}{[k^2]^{l-b/2}},$$

(7)

where $N(\tau) \equiv \frac{\Gamma(h-\tau)}{\Gamma(\tau)}$; and

$$\gamma = l + b/2 - h,$$

(8)
which is defined as the index of the vertex. By dimensional analysis, it is easy to check that the (anomalous) dimension of the vertex \( \Gamma \) in momentum units is \(-2\gamma\).

Our task is to determine the scaling dimensions \( l \) and \( b \), which describe the critical behavior of the model, and the critical coupling constant \( \lambda = \lambda_* \). We shall use the bootstrap equations. These are derived from the Dyson-Schwinger equations for the fermion and boson self-energies and for the Yukawa vertex. We present the one for the vertex first. Graphically, the bootstrap equation of the Yukawa vertex is [3][1]

\[
\Gamma = k p_1 p_2 + q p_1 p_2 + \ldots
\]

Fig. 1 The bootstrap equation for the Yukawa vertex. The dark spots represent the conformal Yukawa vertex, the solid and dashed lines represent the conformal fermion and boson propagators, respectively.

It is worth noting that the contribution of the bare vertex which would normally appear in the right-hand-side of this equation is absent. This is a result of the infinite renormalization (and therefore zero of the multiplicative renormalization constant) which is necessary to make the theory finite. Moreover, it is required by the conformal ansatz, since the bare vertex does not have the conformally invariant form given in (7) [3]. The Dyson–Schwinger equation can be rewritten in a neat form by factoring the momentum dependence from both sides. To see this, let the momentum carried by the external boson field be zero. Then the l.h.s. is readily calculated. Performing the integration in (7) with \( p_1 = p_2 = p \), we obtain

\[
\Gamma(p, p) = \lambda \frac{1}{|p^2|^{\gamma}}. \tag{9}
\]

The r.h.s. of the bootstrap vertex equation is an infinite series. By dimensional
analysis, each term is proportional to $\frac{1}{|p|^\gamma}$. More precisely, the n-th term is

$$\lambda^{2n+1} f_{(2n+1)}(l, l, b) \frac{1}{|p|^\gamma},$$

where the function $f_{(2n+1)}(l, l, b)$ is given by the diagrams with $2n + 1$ conformal vertices. The arguments $l$, $l$, and $b$ refer to the scaling dimensions of the external fermion, anti-fermion, and boson, respectively. The bootstrap equation for the vertex is

$$1 = \lambda^2 f(l, l, b; \lambda_*) ,$$

$$f(l, l, b; \lambda_*) = \sum_{n=1}^{\infty} \lambda_n^{2(n-1)} f_{(2n+1)}(l, l, b).$$

We call $f(l, l, b; \lambda_*)$ the vertex function.

There are two similar Dyson-Schwinger equations for the fermion and boson self-energies, respectively,

$$\Sigma = \begin{array}{c}
\hline
\hline
\hline
\end{array} \quad (a)$$

$$\Pi = \begin{array}{c}
\hline
\hline
\hline
\end{array} \quad (b)$$

Fig. 2 Dyson-Schwinger equations for the fermion (a) and boson (b) self-energies. The vertex without a dark spot stands for the bare Yukawa vertex, associated with the bare coupling $\lambda_0$.

As one might expect for primitively divergent vertices in a scale invariant theory, these are ill-defined. The l.h.s. is finite. The r.h.s. is a diverging integral times a multiplicative renormalization constant and is the indeterminate $0 \cdot \infty$. A way to
resolve this ambiguity was suggested by Parisi [1]. It involves taking derivatives of each side by external momenta and by the scale dimensions of the operators. The result is the bootstrap equations for the vertex function. In the present model, we obtain [4]

\[
1 = -\frac{\lambda^4}{(4\pi)^b} \frac{N(\gamma)\tilde{N}(d-l)\tilde{N}(l)\tilde{N}(b/2)}{N(2b)N(l-b/2)} \frac{\partial f(l',l,b;\lambda_*)}{\partial l'/2} \bigg|_{l'=l}, \quad (13)
\]

\[
1 = N \text{Tr} 1 \frac{\lambda^4}{(4\pi)^b} \frac{N(\gamma)\tilde{N}^2(b/2)N(d-b)}{N(b)N(l-b/2)} \frac{\partial f(l,l,b';\lambda_*)}{\partial b'/2} \bigg|_{b'=b}, \quad (14)
\]

where \(N(\tau)\) is defined after (7) and \(\tilde{N}(\tau) \equiv \frac{\Gamma(h-\tau+1/2)}{\Gamma(\tau+1/2)}\). Also, \(N\) is the number of fermion species and \(\text{Tr} 1\) is the dimension of the Dirac Matrices.

It is remarkable that, in all three equations (13), (14) and (11), the only unknown is the vertex function \(f(l,l,b;\lambda_*)\), which contains all the necessary information to determine the three parameters. In particular, there is no need to calculate any Feynman diagrams for the self-energies.

Unfortunately, calculating the vertex function \(f(l,l,b;\lambda_*)\) is a difficult problem, as it involves an infinite number of Feynman diagrams. On the other hand (12) is a power series in the coupling constant and if the critical coupling is small, the first few terms are expected to give good approximate results. One limit in which \(\lambda_*\) is small is the limit of large number of fermion species, \(N\). There, \(\lambda_*^2 \sim 1/N + \cdots\) and, as we shall demonstrate below, further approximation to the bootstrap equations can be solved analytically to order \(1/N^2\). Quantities which are also small in the large \(N\) limit are the anomalous dimensions defined by

\[
\gamma_\psi = l - h + 1/2, \quad (15)
\]

\[
\gamma_\phi = b - 1; \quad (16)
\]

which are both of order \(1/N\). In terms of them, the index of the vertex is

\[
\gamma = l + b/2 - h = \gamma_\psi + \gamma_\phi/2. \quad (17)
\]

The integral on the r.h.s. of the vertex (7) is related to the Appel function \(F_4\) [3]. It takes a simple form, when \(p_1^2 \ll p_2^2 = p^2\). In this case,

\[
\Gamma(p_1 = 0, p_2 = p) = \frac{\lambda}{[p^2]^\gamma} \frac{\tilde{N}(b/2)\tilde{N}(l)}{N(b)N(l-b/2)}. \quad (18)
\]
It is easy to check that, at the lowest order in $\gamma$, (18) coincides with (3).

To solve the set of conformal bootstrap equations, we start with calculating the three-vertex correction to the vertex function. Using the conformal propagators and vertex, and setting the momentum carried by the external boson zero (i.e. $p_1 = p_2$), the three-vertex diagram in Fig. 1 is

$$-\lambda^3 \int \frac{d^d k}{(2\pi)^d} \Gamma(p, k) \frac{\hat{k}}{[k^2]^{h-1/2}} \Gamma(k, k) \frac{\hat{k}}{[k^2]^{h-1/2}} \Gamma(k, p) \frac{1}{[(k - p)^2]^{h-b}} \frac{1}{\pi^2} \left[ N(b) N\left( l - b/2 \right) \right]^2 N(h) = -\lambda^3 \left( \frac{p^2}{2} \right)^{2h} \left( \frac{1}{4\pi} \right)^h \frac{1}{\pi^2} \left[ N(b) N\left( l - b/2 \right) \right]^2 N(h) \gamma.$$  

(19)

Above, we have used (18) for $\Gamma(p, k)$, as the integration over the region of large internal momentum dominates. Then we have

$$f_3(l, l, b) = -\frac{1}{(4\pi)^h \Gamma(h)} \frac{1}{\gamma} \left[ 1 - \frac{\gamma}{h - 1} + \ldots \right].$$  

(20)

Substituting the first term of (20) into the set of conformal bootstrap equations (11), (13), and (14), we obtain the anomalous dimensions of the fermion field $\psi$ and composite operator $\phi = \bar{\psi}\psi$ at the leading order $1/N$

$$\gamma^{(1)}_\psi = -\frac{1}{NTr1} \frac{\Gamma(2h - 1) \sin(\pi h)}{\pi \Gamma(h) \Gamma(h - 1)} \Gamma(h + 1),$$  

(21)

$$\gamma^{(1)}_\phi = -\frac{2(2h - 1)}{h - 1} \gamma^{(1)}_\psi,$$  

(22)

which reproduce the known results, with

$$(\lambda_2^{(1)})^{(1)} = -\frac{1}{NTr1} \left( \frac{4\pi}{\Gamma(h)} \right)^{2h} \frac{\sin(\pi h)}{\pi \Gamma(h)},$$  

(23)

$$\gamma^{(1)} = \frac{1}{NTr1} \frac{\Gamma(2h - 1) \sin(\pi h)}{\pi \Gamma^2(h)}.$$  

(24)

Moreover, to the next to leading order, as we shall see below, the vertex function $f$ takes a same form as that in (20). This implies to the order

$$\left. \frac{\partial f(l', l, b; \lambda_*)}{\partial l'/2} \right|_{l' = l} = \left. \frac{\partial f(l, l, b'; \lambda_*)}{\partial b'/2} \right|_{b' = b}. $$  

(25)

Then, with no need of further details of the vertex function $f(l, l, b)$, by using the bootstrap equations (13) and (14), one can determine the fermion anomalous dimension...
to order $1/N^2$. Dividing one of them by the other, we have

$$1 = -\frac{1}{NTr \tilde{N}(l)\tilde{N}(d-l)}. \tag{26}$$

Expand the r.h.s. of the above equation over $\gamma_\psi$ and $\gamma_\phi$ (both $\sim 1/N + ...$), we obtain

$$\gamma_\psi = \gamma_\psi^{(1)}[1 + \frac{\gamma_\psi^{(1)}}{h} - \gamma_\phi^{(1)}(\frac{1}{h-1} + \psi(2h-1) - \psi(1) + \pi c tg(h\pi)) + ...], \tag{27}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$, and $\gamma_\psi^{(1)}$ and $\gamma_\phi^{(1)}$ are given in (21) and (22).

To obtain the anomalous dimension of $\phi = \bar{\psi}\psi$ to order $1/N^2$, one needs the vertex function $f$ to the same order. Besides the second term in the three-vertex correction (20), it involves the leading order of the five-vertex correction, and also the leading order of the seven-vertex diagrams with a fermion loop.

Notice that only one five-vertex diagram, depicted in Fig. 1, contributes to $f_5(l,l,b)$. (Another diagram which also contains five conformal vertices has a subdiagram with a fermion loop attached to three boson lines and is zero by parity symmetry). The Feynman integral (setting $p_1 = p_2$) is

$$\int \frac{d^dq}{(2\pi)^d} \int \frac{d^dk}{(2\pi)^d} \Gamma(p,k)G(k,k+q+p)\Gamma(k+q-p,k+q-p)G(k+q-p)\Gamma(k+q-p,q)G(q,p)D(k-p)D(k-q). \tag{28}$$

Performing the integral, to the leading logarithmic term, we obtain

$$f_5(l,l,b) = -\frac{1}{(4\pi)^d\Gamma^2(h)(h-1)}\frac{1}{\gamma} + ... \tag{29}$$

To order $1/N^2$, the seven-vertex diagrams with a fermion loop attached to four boson lines contribute, as a fermion loop carries a factor $N$. This sort of seven-vertex diagrams are given in Fig. 3.
The diagrams of seven-vertex correction with a fermion loop. Each fermion loop carries a factor $N$.

For $f_7$ in the second order, to the leading logarithmic term again, we have

$$
\frac{N \text{Tr} 1}{(4\pi)^{3h}(h-1)^2}\Gamma(2h-2)\gamma \left[1 - \frac{3\Gamma^3(h/3)\Gamma(h)(2h/3 - 1)}{8h\Gamma^3(2h/3)}\right] + \ldots ,
$$

(30)

$$
\frac{N \text{Tr} 1}{(4\pi)^{3h}(h-1)^2}\Gamma(2h-2)\gamma \left[1 + \frac{1}{h-1} - \frac{\Gamma^3(h/3)\Gamma(h)}{2\Gamma^3(2h/3)}\right] + \ldots ,
$$

(31)

for the first and the next two diagrams in Fig. 3, respectively. The vertex function is given by

$$
f(l, l, b, \lambda) = f_3(l, l, b) + \lambda^2 f_5(l, l, b) + \lambda^4 f_7(l, l, b) + \ldots ,
$$

(32)

where $f_3$, $f_5$, and $f_7$, to the second order, are given in (20), (29), and (30) and (31). This is a systematic expansion procedure. To calculate the next order, $1/N^3$, for instance, requires the first three terms in the expansion of $f_3$, the first two terms of $f_5$, and the first two terms of $f_7$ (one with a factor $N$ carried by a fermion loop and the other without), and the first term of $f_9$ with an explicit $N$ factor. Similarly, as the multiplicative factors of the partial derivatives of $f(l, l, b; \lambda_*)$ of equations (13) and (14) are functions of $l$ and $b$, we expand these to the second order in $\gamma_\psi$ and $\gamma_\phi$ (or $\gamma$) as well. Substituting these expansions in the equation (13) and (14), we obtain

$$
\lambda_*^2 = (\lambda_*^2)^{(1)}[1 - \gamma_\phi^{(1)}(\pi\text{ctg}(\pi h) + \psi(2h - 1) - \psi(1)) + \ldots] ,
$$

(33)

$$
\gamma = \gamma^{(1)}[1 - \frac{\gamma^{(1)}}{h-1} + \frac{\lambda_*^{2(1)}}{(4\pi)^2h\Gamma(h)(h-1)} - \gamma_\phi^{(1)}(\pi\text{ctg}(\pi h) + \psi(2h - 1) - \psi(1))]
$$

$$
- \frac{(\lambda_*^2)^{(1)}N \text{Tr} 1}{(4\pi)^{2h}\Gamma(2h-2)^2(h-1)^2}\Gamma(2h-2) \left(\frac{2h-1}{h-1} - \frac{\Gamma^3(h/3)\Gamma(h)}{2\Gamma^3(2h/3)}\right) (1 + \frac{1}{2}(1 - \frac{3}{2h})) + \ldots .
$$

(34)

Then finally,

$$
\gamma_\phi = 2(\gamma - \gamma_\psi) ,
$$

(35)

where $\gamma$ and $\gamma_\psi$ are given in (34) and (27).

In particular, when $d = 2h = 3$ and Tr1 = 2, we have

$$
\gamma_\psi = \frac{2}{3\pi^2 N}[1 + \frac{148}{9\pi^2 N} + \ldots] ,
$$

(36)

$$
\gamma_\phi = -\frac{16}{3\pi^2 N}[1 + \frac{76 + 27\pi^2}{36\pi^2 N} + \ldots] .
$$

(37)
The $\psi$ anomalous dimension ($\gamma_\psi$) has been computed to this order by conventional diagrammatic methods in [6]. That calculation involves much more labor than the present one. Also, the present calculation gives the anomalous dimension of the composite operator $\phi = \bar{\psi}\psi$ which has not been computed elsewhere. Note that the result for $\gamma_\psi$ is not the same as in [6]. (The numerator of the second term quoted in [6] has 122 instead of 148.) We presently do not understand the source of this discrepancy.

Here, we see the power of the bootstrap method in performing approximate calculations, such as the large $N$ expansion to higher orders. It would be interesting to apply this approach to other field theories, such as gauge theory or theories with vector-like vertices. It would also be interesting to search for other, non-perturbative solutions in the present four-fermion, or in $\lambda\phi^4$ theory [7].

Note added: After this work was submitted for publication, J A Gracey informed the authors that the number 122 was a misprint in [6], his result would be 112, instead. In a recent publication [8] he also calculated the exponent $\gamma_\phi$ to order $1/N^2$.

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