Research on Probability of Detection of Submarine Time Limit Based on Continuous Time within Finite Region

ZHANG Chi*, CHEN Jianyong and TANG Shuai

Address: Navy Submarine Academy. Jinshui Road No. 1, Qingdao, Shandong, China, 266199

*E-mail address: troy100@qq.com

Abstract. The sonar detection of underwater submarines in a certain sea area was modeled by using the sonar in the sea area. The problem of continuous detection time of underwater large random targets was studied based on the fixed detection points. According to the characteristics of the target's motion, this paper puts forward the research method of detecting the probability of finding the time limit of the random moving target by fixed point detection, establishes the random constant velocity target distribution model under the target initial position distribution, uses this model to carry on the duration . The effective probing time and the redundant probing time exist in the probing and continuous probing times. The probabilistic trend of the probing probability under the probing limit time decreases with the increase of the distance between the probing point and the distribution center. The effective probing time length increases with the probing point The distance from the center generally increases, but there is no linear rule.

1. Introduction
In the study of the fixed detection point on the continuous detection of underwater large random targets, the probability model of continuous detection of fixed-point detection was studied in reference [1]-[3], and the researches on the initial distribution under the condition of a circular normal distribution, the time-varying discovery probability and the variation characteristics of integral domain are studied[4][5]. Modeling method for the problem of continuous Detection in a finite area of a moving Target based on Hellman and Foraker[6]-[9]. On the basis of finding probability calculation model in the fixed point finite region of moving target, a random constant velocity target distribution model is established under the initial position distribution of the target, and the model is used in the region. The duration detection is carried out at a detection point in the domain. The duration detection time tends to infinity. A method to calculate the time limit detection probability is proposed, and the limited detection time length included in the duration detection time is analyzed. In addition, the integral domain tends to be a definite finite field, and the integral time of the detection function on the characteristic trace is also limited at points outside the distribution center.

2. Time limit probability
Suppose for a simply connected convex domain $\Omega$, have $b(x,t) > 0$, if $x \in \Omega$; have $b(x,t) = 0$, Then it's called a probe region $x \not\in \Omega$. According to the probability model of continuous detection discovery in reference [1], The initial distribution density function of the target position is $\rho(x(t_0),t_0]$. From the
time of continuous detection $t_0$ to the moment of discovery time $t$, the expression of the discovery probability of continuous detection is as follows:

$$P(t) = \int_{\omega} \rho(x(t_0),t_0) \left[ 1 - e^{-\int_{t_0}^{t} b(x(s),s) \, ds} \right] dx(t_0)$$  \hspace{1cm} (1)

Where $\omega = \{x(t_0) : \int_{t_0}^{t} b(x(s),s) \, ds > 0\}$, $\Omega$ is the Collection of starting points for intersecting feature traces. When $t \to \infty$, $\omega$ with an index $-\int_{t_0}^{t} b(x(s),s) \, ds$ may be changed. A deterministic moving object with velocity mean motion $\mathbf{v}(x,t)$ in integral domain at the time $t_0$, During the time $t_0 \sim t$, one moment in the exploration zone at least.

With the initial distribution center as the origin, a coordinate system is established. For the diffusion random moving target, The characteristic trace equation $x(t) = G[x(t_0),t]$ obtained from the differential system $\frac{dx}{dt} = \mathbf{v}(x,t)$ and the initial point $x(t_0)$ \cite{10}. If $t > t_0$, then $|x(t)| = |G[x(t_0),t]| > |x(t_0)|$ can be launched, The characteristic trace from $x(t_0)$ always radiates outward away from the origin of the coordinates. As shown in Figure 1 and Figure 2. When $(0,0) \in \Omega$,have $x_0 \in \Omega\setminus\omega$; When $(0,0) \notin \Omega$, have $\Omega \subset \omega$;And for all of the point $x_\omega \in \Omega, x_\omega \in \omega \setminus \Omega$ on the Characteristic trace,have $|x_\omega| > |x_\omega - \omega|$. So, when $t \to \infty$,the $(0,0) \in \omega$,and $\omega$ is a finite field.

![Figure 1](image1.png) ![Figure 2](image2.png) ![Figure 3](image3.png)

**Figure 1.** If $(0,0) \in \Omega$, Schematic diagram of $\omega$.

**Figure 2.** If $(0,0) \notin \Omega$, Schematic diagram of $\omega$.

**Figure 3** Detection probabilistic integral domain without distribution center $\omega$.

Assuming initial position distribution and velocity distribution are both circular normal distribution \cite{11}, where:

$$\rho(x_0) = \frac{1}{2\pi\sigma^2} e^{-\frac{x_0x_0^T}{2\sigma^2}} , \quad w(v,t_0) = \frac{1}{2\pi\mu^2} e^{-\frac{x_0x_0^T}{2\mu^2}}. \hspace{1cm} (2)$$

Set time-independent detection rate function: $b(x,t;a) = \begin{cases} B, & |x-a| \leq R \\ 0, & |x-a| > R \end{cases}$, where $B$ is a constant, The radius of the circular detection field ($\Omega$) is $R$, where centre point is located at $a$.

Any time $t > t_0$,The expected value of the spatial conditional distribution of velocity is:

$$\mathbf{v}(x) = \frac{\sigma_1^2(t-t_0)^2}{\sigma_1^2 + \sigma_2^2(t-t_0)^2} \cdot \frac{x}{t-t_0} , \quad \text{characteristic line equation is} \quad x(t) = \frac{\sqrt{\sigma_1^2 + \sigma_2^2(t-t_0)^2}}{\sigma_1} x_0.$$
For circumstances \((0,0) \in \Omega\), the discovery probability integral domain \(\omega = \{x_0 : |x_0 - a| \leq R\}\), according to characteristic line equation is:
\[
|x_0 - a| = R
\]
The only positive solution can be obtained: \(t_1 - t_0 = F_1(x_0, a, R)\).

The time limit found probability is:
\[
P_c = \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x_0 = \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x_0
\]
\[
(3)
\]

For circumstances \((0,0) \notin \Omega\), as shown in Figure 3, the shadow region is the domain of probabilistic integral of discovery \(\omega\). When \(t_1 \rightarrow \infty\), \(\omega\) extends along the fan toward the origin to include the origin.

When \(|x_0 - a| \leq R\), then \(x_0 \in \Omega\). There is also a unique positive solution: \(t_1 - t_0 = F_1(x_0, a, R)\).

When \(x_0 \notin \omega\), then \(x_0 \in \omega\) and \(|x_0 - a| > R\), with \(|a - \sqrt{\sigma^2 + \mu^2(t_1 - t_0)^2}\) is a solution, then:

Two positive solutions of \(t_1 - t_0\) are:
\[
F_2(x_0, a, R) > F_2(x_0, a, R)
\]

Take \(t_1 - t_2 = F_2(x_0, a, R) - F_2(x_0, a, R)\). Then the discovery probability of time limit is:
\[
P_a = \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x_0 + \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x_0
\]
\[
(4)
\]

The analysis of the solution process of the concrete problem is given by using the concrete expression.

First of all, let the calculation and analysis of the probability of time limit be divided into two cases, and take the limit of time \(t \rightarrow \infty\) [12].

First case: If \(\omega = \Omega\), origin in the detection domain, where point \((0,0) \in \Omega\). The detection probability is:
\[
P_a = \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x_0 = \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x_0
\]
\[
(5)
\]

Because of \(1 - e^{-\int_0^{b(x,a,t)} \text{d}s} < 1\), probe \(P_a < \int_\Omega \rho(x_0) \text{d}x_0\), it indicates that an infinitely long probe in an area containing a distribution center has a probability of finding that is less than the probability that the target exists in the probe domain at the initial time:
\[
P_a = \int_\Omega \rho(x_0)[1 - e^{-\int_0^{b(x,a,t)} \text{d}s}] \text{d}x(t_0)
\]
\[
(6)
\]

Where \(t_1\) is the time of the characteristic line leaves \(\Omega\), the one point \(x_0\) is on it.

To calculate the probability of detection of limit time, we first need to find out the intersection point \(x_0\) of the point along the feature curve \(x_0\) with the detection area at time \(t_0\), the circle equation of the joint detection area and the characteristic equation of the line is:
\[
\begin{cases}
(x - a)^2 + (y - b)^2 = R^2 \\
y = \frac{y_0}{x_0}x
\end{cases}
\]
\[
(7)
\]

Get the intersection point \(x_1, x_2\):
\[ x_{1,2} = ax_0 + by_0 \pm \frac{\sqrt{(x_0^2 + y_0^2)R^2 - (ay_0 - bx_0)^2}}{x_0^2 + y_0^2} \]  
\[ (8) \]

Because in the trace equation:

\[ x(t) = \frac{\sqrt{\sigma_1^2 + \sigma_2^2 (t-t_0)^2}}{\sigma_1} x(t_0), \quad \frac{\sqrt{\sigma_1^2 + \sigma_2^2 (t-t_0)^2}}{\sigma_1} \geq 1, \]

Positive and negative solutions for coefficients in eq. (8), put Negative root, Preserve only the positive root:

\[ \begin{cases} 
   x_0 = \frac{ax_0 + by_0 + \sqrt{\left(x_0^2 + y_0^2\right)R^2 - \left(ay_0 - bx_0\right)^2}}{x_0^2 + y_0^2} \\
   y_0 = \frac{ax_0 + by_0 + \sqrt{\left(x_0^2 + y_0^2\right)R^2 - \left(ay_0 - bx_0\right)^2}}{x_0^2 + y_0^2}
\end{cases} \]

\[ (9) \]

Make \( \gamma(x_0, y_0) = \frac{ax_0 + by_0 + \sqrt{\left(x_0^2 + y_0^2\right)R^2 - \left(ay_0 - bx_0\right)^2}}{x_0^2 + y_0^2} \). By this definitions eq. (9) can be rewritten in the following form:

\[ \gamma(x_0, y_0) = \frac{\sqrt{\sigma_1^2 + \sigma_2^2 (t-t_0)^2}}{\sigma_1} \]

\[ (10) \]

The result is:

\[ t_1 - t_0 = \frac{\sigma_1 \sqrt{\gamma^2(x_0, y_0) - 1}}{\sigma_2} \]

\[ P_x = \int_{\Omega} \rho(x_0, y_0) [1 - e^{\frac{-\gamma}{\sigma_2}}] \, dx_0 \, dy_0 \]

\[ (11) \]

The second case: \( w_u \neq \Omega \), The origin is detecting beyond the border, \((0,0) \notin \Omega\), The detection probability at the limit time is:

\[ P_x = \int_{\Omega} \rho(x_0) [1 - e^{\frac{-\gamma}{\sigma_2} + \frac{-b \gamma}{\sigma_2} \gamma^2(x_0, y_0) - 1}] \, dx_0 \]

\[ = \int_{\Omega} \rho(x_0) [1 - e^{\frac{-\gamma}{\sigma_2} + \frac{-b \gamma}{\sigma_2} \gamma^2(x_0, y_0) - 1}] \, dx_0 \]

\[ = \int_{\Omega} \rho(x_0) [1 - e^{\frac{-\gamma}{\sigma_2} + \frac{-b \gamma}{\sigma_2} \gamma^2(x_0, y_0) - 1}] \, dx_0 \]

\[ = P_x + P_{\omega - \Omega} \]

Where the time \( t_1 \) in \( P_x \Omega \) account form is the same way as in the first case.

In the eq. (12), \( t_1 \) is the time of the characteristic line leave \( \Omega \), the one point \( x_0 \) is in \( \Omega \), \( t_2 \) is the time of the characteristic line get into \( \Omega \), which one point \( x_0 \) is in \( \omega - \Omega \), \( t_3 \) is the time of the characteristic line leave \( \Omega \), which one point \( x_0 \) is in \( \omega - \Omega \). From the characteristic line equation and the boundary equation of \( \Omega \), we can know, \( t_1, t_2 \) and \( t_3 \) are functions of \( x(t_0) \).

3. Example analysis

Through the above discussion of the time limit probabilities of continuous detection of random moving targets, the proper detection function, simplified probabilistic model and target distribution model of submarine are selected and analyzed[13].

The following gives specific values to explore the detection of continuous detection of the time limit probabilities of changes with the detection of location.
Example 1: Hypothesized normal distribution of circles $\sigma_1 = 8nm, \sigma_2 = 8kn$. Probe radius $R = 6nm$, Detection rate constant $B = \frac{1}{6s}$. Let the distance from the center of the probe to the origin of the coordinates $|a|$ change from 0 to 50 n-mile, Figure 4 shows the time limit discovery probability for continuous detection.

$P_{\infty(\Omega)}$ is the calculation result of the first integral eq. (12) shown in Figure 4, Represents the time limit probability that a target in the probe domain was discovered at the time of detection, which is the result of the second integral formula in the eq. (12), Represents the time limit probability that a target outside the detection area enters and is discovered at the time of detection. Because the moving target is a diffusive distribution. When the detection region includes a diffusion center. That is, the distance between the center of the circle and the center of diffusion is not greater than $6nm$, which can be seen from Figure 4. discovery probability does not exist the probe $P_{\infty(\omega-\Omega)}$. With the increase of $|a|$, $P_{\infty(\Omega)}$ Continuous decline, And there is a maximum in $P_{\infty(\omega-\Omega)}$. The proportion of $P_{\infty(\omega-\Omega)}$ components gradually increased in time limit discovery probability even when $|a|$ is coming bigger. Contribution of $P_{\infty(\Omega)}$ in the probability of time limit discovery may be passed.

Example2: Study on the relationship between the effective detection time length and the distance between the probe point and the center. Make $t_0 = 0$, defined proportional coefficient [14] $\eta(t,a) = \frac{P(t,a)}{P_0(a)}$. For a given value, time $t$ is a function of $|a|$.

The calculation condition is the same as that of the example 1. Calculating the probability of continuous detection of time varying $P(t,a)$. Set ratio coefficient value $\eta = 0.95$, $|a|$ change from 0 to 50 n-mile, the calculated time values are shown in figure 5. The result of the calculation clearly shows, Detection time for a fixed ratio of discovery probability reaching the near time limit, when the detection domain comprises a distribution center or near the distribution center, almost invariable and when the probe is away from the center of the distribution, the detection time increases significantly with the increase of $|a|$.

It can be seen from the results of the example that the probability of finding the time limit of continuous detection decreases with the increase of the distance between the detection point and the distribution center, and the integral value in the non-detection region shows a peak value of probability. When the detection limit of 95% of the time is reached, the detection time tends to increase with the
increase of the distance between the detection point and the distribution center, which $|a|$ varies from about 20 to 40 n-mile fluctuations in detection time, The fluctuation of the detection time is due to the change of the time limit probability and the non-synchronization of the temporal change of the discovery probability with the change of $|a|$.

4. Conclusions
The probability of detecting the time limit of random moving targets by fixed point detection is discussed. And the method of numerical integration is used to calculate the example. The research results show that there are effective detection time and redundant detection time in the continuous detection time. The simulation results of the discovery probability under the limit time show that the results show a probability peak. From Case 2, it can be concluded that the effective detection time tends to increase as the distance between the detection point and the center increases, but there is no linear rule.

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