ON THE EXPECTED NUMBER OF ZEROS OF NONLINEAR EQUATIONS

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ABSTRACT. This paper investigates the expected number of roots of nonlinear equations. Those equations are assumed to be analytic, and to belong to certain inner product spaces. Those spaces are then endowed with the Gaussian probability distribution.

The root count on a given domain is proved to be ‘additive’ with respect to a product operation of functional spaces. This allows to deduce a general theorem relating the expected number of roots for unmixed and mixed systems. Examples of root counts for equations that are not polynomials nor exponential sums are given at the end.

1. INTRODUCTION

We consider systems of analytic equations of the form

\[ f_1(x) = \cdots = f_n(x) = 0 \]

where \( x \) is assumed to belong to a complex \( n \)-dimensional manifold \( M \). Each \( f_i \) belongs to a certain complex inner product space \( \mathcal{F}_i \). Those will be called spaces of complex fewnomials, or fewspaces for short (see definition 3). Let \( n_M(f) \) denote the number of isolated roots of the system above. More generally, let \( n_K(f) \) be the number of isolated roots in a set \( K \). A consequence of Brouwer’s degree theorem is that when \( K \) is open, the number \( n_K(f) \) is lower semi-continuous as a function of \( f \) (details in [22 Ch.3]).

When the \( \mathcal{F}_i \) are spaces of polynomials (resp. Laurent polynomials) and \( M = \mathbb{C}^n \) (resp. \( M = (\mathbb{C} \setminus \{0\})^n \)), the number \( n_M(f) \) is known to be equal to its maximum generically, that is for all \( f \) except in a codimension 1 (hence measure zero) variety. Bounds for this maximum are known, and some of them are exact.
For instance, let $\mathcal{F}_A$ be the set of Laurent polynomials with support $A$, viz.
\[
f(x) = \sum_{a \in A} f_a x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},
\]
where $A \subset \mathbb{Z}^n$ is assumed to be finite and $f_a \in \mathbb{C}$. The inner product in $\mathcal{F}_A$ is arbitrary. Let $\mathcal{A}$ denote the convex hull of $A$.

**Theorem 1** (Kushnirenko [19]). Let $f_1, \ldots, f_n \in \mathcal{F}_A$. For a generic choice of coefficients $f_{i_a} \in \mathbb{C}$,
\[
n_{(\mathbb{C}\setminus\{0\})^n}(f) = n! \Vol(A).
\]

The case $n = 1$ was known to Newton, and $n = 2$ was published by Minding [31] in 1841. A system as above, where all the equations have same support $A$ is said to be *unmixed*. Otherwise, the system is said to be *mixed*. The following root count for mixed polynomial systems was published by Bernstein [4] and is known as the BKK bound (for Bernstein, Kushnirenko and Khovanskii) [5]:

**Theorem 2** (Bernstein). Let $A_1, \ldots, A_n \subset \mathbb{Z}^n$ be finite sets. Let $\mathcal{A}_i$ be the convex hull of $A_i$. For a generic choice of coefficients $f_{i_a} \in \mathbb{C}$,
\[
n_{(\mathbb{C}\setminus\{0\})^n}(f)
\]
is $n!$ times the coefficient $V$ of $\lambda_1 \cdots \lambda_n$ in the polynomial
\[
\frac{1}{n!} \Vol(\lambda_1 A_1 + \cdots + \lambda_n A_n).
\]

This number $V$ is known as the *mixed volume* of the tuple of convex bodies $(A_1, \ldots, A_n)$.

The objective of this paper is to extend the results above to more general spaces of analytic equations. For instance, we would like to count zeros of equations such as
\[
f_{00} + f_{01} x + f_{02} x^2 + \cdots + f_{0d} x^d +
+f_{10} e^x + f_{01} x e^x + f_{02} x^2 e^x + \cdots + f_{0d} x^d e^x = 0.
\]

It is easy to see that the number of solutions in $\mathbb{C}$ for (say) $d = 0$ is infinite. However, we can inquire about the number of solutions in a smaller set, like the disk $\mathcal{D} = \{x \in \mathbb{C} : |x| < 1\}$.

Instead of counting the generic number of zeros (that exists no more), we endow the space of equations with a probability measure (zero average, unit variance normal distribution) and compute the expected number of isolated roots.

In the example above, the expected root count is
\[
\mathbb{E}(n_\mathcal{D}(f)) = d/2 + 0.202, 918, 921, 282 \cdots
\]
(see Section 4 for the precise inner product we are using). The constant $0.202\cdots$ was obtained numerically. I would like to thank Steven Finch for pointing out an error in the 4-th decimal of a previous computation, and giving the correct decimal expansion.

This and other examples are worked out in Section 6.

It turns out that complex fewnomial spaces are reproducing kernel spaces. A meaningful multiplication operation between reproducing kernel spaces was studied by Aronszajn [2] (see Section 4). We denote the product space of $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}\mathcal{G}$, and the $\lambda$-th power of $\mathcal{F}$ by $\mathcal{F}^\lambda$. The main result in this paper is an analogous to Bernstein’s theorem. However, there is no more an interpretation of the number of roots in terms of a volume of a convex body (Minding and Kushnirenko) or in terms of mixed volume. But the relation between root counts in mixed and unmixed systems is preserved.

**Theorem 3.** Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite dimensional fewspaces of functions of $M$, endowed with the zero average unit variance normal probability distribution. Let $\mathcal{X} \subseteq M$ be measurable. Then,

$$E_{f_1 \in \mathcal{F}_1, \ldots, f_n \in \mathcal{F}_n} (n_{\mathcal{X}}(f))$$

is the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in the $n$-th degree homogeneous polynomial

$$\frac{1}{n!} E_{g_1, \ldots, g_n \in \mathcal{F}_1^{\lambda_1} \mathcal{F}_2^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}} (n_{\mathcal{X}}(g))$$

where zero average and unit variance normal probability distribution is assumed in each $\mathcal{F}_1^{\lambda_1} \mathcal{F}_2^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}$.

In the setting of Bernstein’s theorem, one may identify $\mathcal{F}_{\lambda_1 A_1 + \cdots + \lambda_n A_n}$ to $\mathcal{F}_{\lambda_1 A_1}^{\lambda_2 A_2} \cdots \mathcal{F}_{\lambda_n A_n}^{\lambda_n}$. With this identification, Bernstein’s theorem follows immediately from Kushnirenko’s theorem and Theorem 3.

The basic idea for the proof of Theorem 3 is:

**Lemma 4.** Let $\mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ be finite dimensional fewspaces and let $\lambda \geq 0$ be integer. Then,

$$E_{f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2, \ldots, f_n \in \mathcal{F}_n} (n_{\mathcal{X}}(f)) = E_{f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2, \ldots, f_n \in \mathcal{F}_n} (n_{\mathcal{X}}(f)) + \lambda E_{f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2, \ldots, f_n \in \mathcal{F}_n} (n_{\mathcal{X}}(f)).$$

Above, all fewspaces are assumed with the zero average, unit variance normal probability distribution.
2. Related work

Random polynomial systems constitute a classical subject of studies, and received a lot of attention lately (See for instance the book by Azaïs and Wschebor [3] and references). Part of the interest comes from the study of algorithms for solving polynomial systems such as in [26,30]. The running time of algorithms can be estimated in terms of certain invariants, such as the number of real or complex zeros, and the condition number. While the number of real zeros of real polynomial systems and the condition number depend on the input system, it is possible to obtain probabilistic complexity estimates by endowing the space of polynomials with a probability distribution, and then treating those quantities as random variables. For the full picture, see the book [6] and two forthcoming books [8,22]. Recent papers on the subject include [1,10,12]. The extension of this theory to systems of sparse polynomial systems started with [23,24] (see below) and is still a research subject (see [22]).

Another source of interest comes from classical asymptotic estimates such as in Littlewood-Offord [20,21] and Kac [14,15]. Asymptotic formulas for the number of roots of sparse polynomial systems can be obtained by scaling the supports. For instance, one looks at systems of Laurent polynomials such as

$$f_i(x) = \sum_{a \in A_i} f_{ia} x^{ta}$$

where $t$ is a scaling parameter. A random variable of interest in the zero-dimensional case is $t^{-n} n_M(f)$. In [25], Shiffman and Zelditch gave asymptotic formulas for the root density in terms of the mixed volume form.

Kazarnovskii [17] obtained more general formulas. He considered fewnomials that are (after multiplying variables by $\sqrt{-1}$) Fourier transforms of distributions supported by real compact sets. For instance, (1) is the Fourier transform of a distribution with support $\{0, 1\}$, namely

$$\sum_{i=0,1} \sum_{j=0,\ldots,d} f_{ij} \frac{(-1)^j}{j!} \delta_i^{(j)}(y).$$

The convex bodies that appear in the Kushnirenko and Bernstein theorems are replaced by the convex hull of the support of the distributions. In this sense, he generalized Bernstein’s theorem to non-polynomials and non-exponential-sums. However, his bounds for (say)
do not take into account different values of \( d \). That is why those bounds must be asymptotic.

3. Spaces of complex fewnomials

Let \( M \) be an \( n \)-dimensional complex manifold. In this section we review part of the theory of spaces of complex fewnomials in \( M \). This theory is developed in more details in [22]. Canonical references for analytic functions of several variables and for reproducing kernel spaces are, respectively, [18] and [2].

**Definition 5.** A complex fewnomial space (or fewspace for short) of functions over a complex manifold \( M \) is a Hilbert space of holomorphic functions from \( M \) to \( \mathbb{C} \) such that the following holds. Let \( V : M \to \mathcal{F}^* \) denote the evaluation form \( V(x) : f \mapsto f(x) \). For any \( x \in M \),

1. \( V(x) \) is a continuous linear form.
2. \( V(x) \) is not the zero form.

In addition, we say that the fewspace is non-degenerate if and only if, for any \( x \in M \),

3. \( P_{V(x)}DV(x) \) has full rank,

where \( P_W \) denotes the orthogonal projection onto \( W^\perp \). (The derivative is with respect to \( x \)). In particular, a non-degenerate fewspace has dimension \( \geq n + 1 \).

**Example 6.** Let \( M \) be an open connected subset of \( \mathbb{C}^n \). Bergman space \( \mathcal{A}(M) \) is the space of holomorphic functions defined in \( M \) with finite \( \mathcal{L}^2 \) norm. The inner product is the \( \mathcal{L}^2 \) inner product. When \( M \) is bounded, \( \mathcal{A}(M) \) contains constant and linear functions, hence it is a non-degenerate fewspace.

**Remark 7.** Condition 1 holds trivially for any finite dimensional fewnomial space, and less trivially for subspaces of Bergman space.

To each fewspace \( \mathcal{F} \) we associate two objects: The reproducing kernel \( K(x,y) = K_{\mathcal{F}}(x,y) \) and a possibly degenerate Kähler form \( \omega = \omega_{\mathcal{F}} \) on \( M \).

Item (1) in the definition makes \( V(x) \) an element of the dual space \( \mathcal{F}^* \) of \( \mathcal{F} \) (more precisely, the space of continuous functionals \( \mathcal{F} \to \mathbb{C} \)).

Riesz-Fréchet representation Theorem (e.g. [7] Th.V.5 p.81) allows to identify \( \mathcal{F} \) and \( \mathcal{F}^* \), whence the Kernel \( K(x,y) = (V(x)^*)(y) \).

For fixed \( x \), \( K(x,y) \in \mathcal{F} \) as a function of \( y \).

By construction, for \( f \in \mathcal{F} \),

\[ f(y) = \langle f(\cdot), K(\cdot, y) \rangle. \]
There are two consequences. First of all,
\[ K(y, x) = \langle K(\cdot, x), K(\cdot, y) \rangle = \langle K(\cdot, y), K(\cdot, x) \rangle = K(x, y) \]
and in particular, for any fixed \( y \), \( x \mapsto K(x, y) \) is also an element of \( \mathcal{F} \). Thus, \( K(x, y) \) is analytic in \( x \) and in \( y \). Moreover, \( \|K(x, \cdot)\|^2 = K(x, x) \).

Secondly, \( Df(y) \dot{y} = \langle f(\cdot), D_y K(\cdot, y) \dot{y} \rangle \) and the same holds for higher derivatives.

Because of Definition 5(2), \( K(\cdot, y) \neq 0 \). Thus, \( y \mapsto K(\cdot, y) \) induces a map from \( M \) to \( \mathbb{P}(\mathcal{F}) \). The differential form \( \omega \) is defined as the pull-back of the Fubini-Study form
\[ \omega_f = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \|f\|^2 \]
of \( \mathbb{P}(\mathcal{F}) \) by \( y \mapsto K(\cdot, y) \).

Namely,
\[ (\omega_f)_x = \omega_x = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log K(x, x). \]

When the form \( \omega \) is non-degenerate for all \( x \in M \), it induces a Hermitian structure on \( M \). This happens if and only if the fewspace is a non-degenerate fewspace.

**Remark 8.** If \( \mathcal{F} = \mathcal{A}(M) \) is the Bergman space, the kernel obtained above is known as the **Bergman Kernel** and the metric induced by \( \omega \) as the **Bergman metric**.

**Remark 9.** If \( \phi_i(x) \) denotes an orthonormal basis of \( \mathcal{F} \) (finite or infinite), then the kernel can be written as
\[ K(x, y) = \sum \phi_i(x) \overline{\phi_i(y)}. \]

Let \( n_{\mathcal{K}}(f) \) be the number of isolated zeros of \( f \) that belong to a measurable set \( \mathcal{K} \). The following result is well-known. It appears in [16 Prop.3] and [13 Prop-Def.1.6A]. It is a consequence of Crofton’s formula, also known as Rice formula or coarea formula.

**Theorem 10** (Root density). Let \( \mathcal{K} \) be a locally measurable set of an \( n \)-dimensional manifold \( M \). Let \( \mathcal{F}_1, \ldots, \mathcal{F}_n \) be fewspaces. Let \( \omega_1, \ldots, \omega_n \) be the induced symplectic forms on \( M \). Assume that \( \mathbf{f} = f_1, \ldots, f_n \) is a zero average, unit variance variable in \( \mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n \). Then,
\[ \mathbb{E}(n_{\mathcal{K}}(\mathbf{f})) = \frac{1}{\pi^n} \int_{\mathcal{K}} \omega_1 \wedge \cdots \wedge \omega_n. \]
As the formulation in terms of reproducing kernel spaces is not standard, we sketch the proof below (more details are available in [22 Th.5.11]).

Proof. First of all, let \( V = \{(f, x) \in \mathcal{F} \times \mathcal{K} : f(x) = 0\} \) be the incidence variety, and \( \pi_1 : V \to \mathcal{F}, \pi_2 : V \to \mathcal{K} \) be the canonical projections.

In a neighborhood of each regular point \((f_0, x_0)\) of \( V \), it is possible to parametrize \( V \) by an implicit function \((G(f), x)\) with \( G(f_0) = x_0 \) and

\[
DG(x_0) = -Df(x_0)^{-1} (K_1(x_0, \cdot)^* \otimes \cdots \otimes K_n(x_0, \cdot)^*)
\]

where \( K_i \) is the reproducing kernel of \( \mathcal{F}_i \).

Let \( \mathcal{F}_x \) denote the product \( K_1(x, \cdot)^- \times \cdots \times K_n(x, \cdot)^- \).

The coarea formula is now

\[
\mathbb{E}(n_\mathcal{K}(f)) = \frac{1}{(2\pi)^{\dim \mathcal{F}}} \int_{\mathcal{F}} \#\pi_2 \circ \pi_1^{-1}(\{\}) e^{-\|f\|^2/2} \, dV_{\mathcal{F}}(f)
\]

\[
= \frac{1}{(2\pi)^{\dim \mathcal{F}}} \int_{\mathcal{K}} dV_M(x) \int_{\mathcal{F}_x} NJ(f, x)^2 e^{-\|f\|^2/2} \, dV_{\mathcal{F}_x}(f)
\]

with \( NJ = \det (DG(x)DG(x)^*)^{1/2} = |\det Df(x)|^{-1} \prod(K_i(x, x))^{1/2} \).

The reader may check that

\[
\det |Df(x)|^2 dV = \bigwedge_{i=1}^n \sum_{j,k=1}^n \frac{\partial f_i(x)}{\partial x_j} \frac{\partial f_i(x)}{\partial x_k} \frac{\sqrt{-1}}{2} \, dx_j \wedge dx_k
\]

(otherwise, this is Lemma 4.3 in [22]). At this point,

\[
\mathbb{E}(n_\mathcal{K}(f)) = \frac{1}{(2\pi)^n} \int_{\mathcal{K}} dV_M(x) \bigwedge_{i=1}^n \Omega_i
\]

with

\[
\Omega_i = \int_{K_i(x, \cdot)^-} \frac{\frac{\partial f_i(x)}{\partial x_j} \frac{\partial f_i(x)}{\partial x_k} \frac{\sqrt{-1}}{2} \, dx_j \wedge dx_k}{(2\pi)^{\dim \mathcal{F}_i} - 1} \, e^{-\|f_i\|^2/2} \, dV_{K_i(x, \cdot)^-}(f_i).
\]

The result below implies that \( \Omega_i = 2\omega_i \), concluding the proof of the density theorem.

**Proposition 11.** Let \( \langle u, w \rangle_{i,x} = \omega_{i,x}(u, Jw) \) be the (possibly degenerate) Hermitian product associated to \( \omega_i \). Then,

\[
\langle u, w \rangle_{i,x} = \frac{1}{2} \int_{K_i(x, \cdot)^-} \frac{(Df_i(x)u)(Df_i(x)w)}{K_i(x, x)} \frac{e^{-\|f_i\|^2}}{(2\pi)^{\dim \mathcal{F}_i} - 1} \, dV_{K_i(x, \cdot)^-}(f_i).
\]

**Proof of Proposition** [11] Let

\[
P_x = I - \frac{K_i(x, \cdot)K_i(x, \cdot)^*}{K_i(x, x)}
\]
be the orthogonal projection. Since the inner product \( \langle \cdot, \cdot \rangle_i \) is the pull-back of Fubini-Study by \( x \mapsto K_i(x, \cdot) \), we can write the left-hand-side as:

\[
\langle u, w \rangle_{i,x} = \frac{\langle P_x DK_i(x, \cdot)u, P_x DK_i(x, \cdot)w \rangle}{K_i(x, x)}
\]

For the right-hand-side, note that

\[D f_i(x) u = (f_i(\cdot), DK_i(\cdot, x)u) = (f_i(\cdot), P_x DK_i(\cdot, x)u).\]

Let \( U = \frac{1}{\|K_i(x, x)\|} P_x DK_i(\cdot, x)u \) and \( W = \frac{1}{\|K_i(x, x)\|} P_x DK_i(\cdot, x)w \). Both \( U \) and \( W \) belong to \( \mathcal{F}_x \). The right-hand-side is

\[
\frac{1}{2} \int_{K_i(x, x)^\perp} \frac{(DF_i(x)u)(DF_i(x)w)}{\|K_i(x, x)\|^2} e^{-\|f_i\|^2} \frac{dV_{K_i(x, x)^\perp}}{(2\pi)^{\dim \mathcal{F}}-1} f_i = \frac{1}{2} \int_{K_i(x, x)^\perp} \frac{\langle f_i, U \rangle \langle f_i, W \rangle}{(2\pi)^{\dim \mathcal{F}}} \frac{e^{-\|f_i\|^2}}{\|f_i\|^2} dV_{K_i(x, x)^\perp}
\]

\[
= \frac{1}{2} \int 1 \frac{1}{2\pi} |z|^2 e^{-|z|^2/2} \ dz
\]

which is equal to the left-hand-side. \( \square \)

4. Product Spaces

Let \( E \) and \( F \) be complex inner product spaces. If \( e \in E \) and \( f \in F \), we denote by \( e \otimes f \) the class of equivalence of pairs \((e, f) \sim (\lambda e, f)\). The tensor product of \( E \) and \( F \) is the space of all linear combinations of elements of the form \( e \otimes f \). In the case \( E \) and \( F \) are finite dimensional, \( E \otimes F \) can be assimilated to the space of bilinear maps \( E \times F \to \mathbb{C} \).

The canonical inner product for the tensor product of two spaces is given by

\[
\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes F} = \langle e_1, e_2 \rangle_E \langle f_1, f_2 \rangle_F.
\]

Now, let \( E \) and \( F \) be fewnomial spaces on some complex manifold \( M \). Then, \( E \otimes F \) is a fewnomial space on the product \( M \times M \), where we interpret \( (e \otimes f)(x_1, x_2) = e(x_1) f(x_2) \). A classical fact on reproducing kernel spaces allows to recover the kernel of the tensor product:

**Theorem 12** (Aronszajn). The direct (=tensor) product \( E \otimes F \) possesses the reproducing kernel

\[
K_{E \otimes F}((x_1, x_2), (y_1, y_2)) = K_E(x_1, y_1) K_F(x_2, y_2)
\]

This is [2, Theorem I p.361]. Theorem II ibid gives us a convenient notion of 'product' for reproducing kernel spaces with same domain:
Theorem 13 (Aronszajn). The kernel $K_g(x, y) = K_E(x, y)K_F(x, y)$ is the reproducing kernel of the class $G$ of restrictions of all functions of the direct (=tensor) product $E \otimes F$ to the diagonal set $M_1 = \{(x, x) : x \in M\} \simeq M$. For any such restriction, $\|g\| = \min \|g'\|_{E \otimes F}$, the restriction of which to the diagonal set $M_1$ is $g$.

If $E$ and $F$ are spaces of fewnomials on $M$, we denote by $\mathcal{E}\mathcal{F}$ the class $G$ described above. As an inner product space, $G$ is just the orthogonal complement of the kernel of the restriction operator

$$
\Delta : \mathcal{E} \otimes \mathcal{F} \to \mathcal{O}(M),

\quad g' \mapsto g = g'|_M.
$$

The inner product of $G$ is by definition the inner product of $E \otimes F$ restricted to $(\ker \Delta)^\perp$.

Given orthonormal bases $(e_a)_{a \in A}$ and $(f_b)_{b \in B}$ of $E$ and $F$, we can produce an orthonormal basis of $G$ as follows.

First, we notice that $(e_a \otimes f_b)_{(a, b) \in A \times B}$ is an orthonormal basis of $E \otimes F$.

Let $\sim$ be the equivalence relation of $A \times B$ given by

$$(a, b) \sim (a', b') \text{ if and only if } e_a(x)f_b(x) \equiv \pm e_{a'}(x)f_{b'}(x).$$

Let $C = \frac{A \times B}{\sim}$. For each equivalence class $c \in C$ (as a subset of $A \times B$), choose a pair $(a, b) \in c$ and set

$$g_c = \sqrt{c} e_a f_b.$$

Lemma 14. $(g_c)_{c \in C}$ is an orthonormal basis of $G$.

Proof. Define

$$G_c = \frac{1}{\sqrt{\#c}} \sum_{(a, b) \in c} \sigma_{ab} e_a \otimes f_b$$

where $\sigma_{ab} = \pm 1$ with sign chosen so that

$$\sqrt{\#c} \sigma_{ab} e_a f_b = g_c.$$

The $(G_c)_{c \in C}$ are linearly independent, hence $(g_c)_{c \in C}$ is a linearly independent set.

In order to show that the span of $(g_c)_{c \in C}$ is ker $\Delta^\perp$, let

$$H = \sum_{(a, b) \in A \times B} H_{ab} e_a \otimes f_b \in \ker \Delta.$$

This implies that

$$\sum_{c \in C} \frac{1}{\sqrt{\#c}} \sum_{(a, b) \in c} H_{ab} \sigma_{ab} g_c = 0$$

as required.
and hence for all $c$,

$$\langle H, G_c \rangle = \frac{1}{\#c} \sum H_{ab} \sigma_{ab} = 0.$$ 

Reciprocally, let $G \in \ker \Delta \perp$. Write

$$G = \sum_{(a,b) \in A \times B} e_a \otimes f_b.$$ 

By hypothesis, $G \perp \sigma_{ab} e_a \otimes f_b - \sigma_{a'b'} e_{a'} \otimes f_{b'}$ whenever $(a, b) \sim (a', b')$. Therefore, $G$ is a linear combination of the $G_c$.

Finally, it is easy to check that

$$\langle G_c, G_{c'} \rangle = \begin{cases} 0 & \text{when } c \neq c' \\ 1 & \text{when } c = c'. \end{cases}$$

□

Lemma 15. Let $M$ be fixed. The product of fewspaces of $M$ is associative and comutative. If one introduces the ‘constant’ fewspace $I = \{1\}$, then fewspaces on $M$ are a semigroup.

Example 16. Let $M = \mathbb{C}^n$ and let $P_1$ be the space of affine functions in $n$ variables. To make it an inner product space, we assume that $(1, x_1, x_2, \ldots, x_n)$ is an orthonormal basis. We define inductively $P_{d+1} = P_d P_1$. Here is the orthonormal basis of $P_d$:

$$\left(\sqrt{\frac{d}{a_0a_1, \ldots, a_n}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right)_{a_0, \ldots, a_n \geq 0 \sum_0 \leq j \leq n}^\alpha_{a_0! \cdots a_n!}$$

Above, the multinomial coefficient

$$\binom{d}{a_0, a_1, \ldots, a_n} = \frac{d!}{a_0! \cdots a_n!}$$

is the number of ways to distribute $d = a_0 + \cdots + a_n$ balls into $n + 1$ numbered buckets of size $a_0, \ldots, a_n$. It is also the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in $(1 + x_1 + \cdots + x_n)^d$. This corresponds to the unitarily invariant inner product defined by Weyl [33], also known as Bombieri’s.

The reproducing kernel of $P_d$ is easily seen to be

$$K_d(\mathbf{x}, \mathbf{y}) = (1 + x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n)^d.$$ 

With the same formalism, we can also retrieve the multi-unitarily invariant inner product for the space of roots of multihomogeneous polynomial systems introduced by Rojas [32].
Example 17. Let $A \subseteq (\mathbb{Z})^n$ be finite, and $M = (\mathbb{C}_{\neq 0})^n$. Let $\mathcal{F}_A$ be the space of Laurent polynomials of the form

$$f(x) = \sum_{a \in A} x^a.$$  

Assume arbitrary weights $c_a > 0$ so that $\|x_a\|_A^2 = c_a$. Then,

$$K_A(x, y) = \sum_{a \in A} c_a x^a \bar{y}^a.$$  

The $\lambda$-th power $\mathcal{F}_A^\lambda$ of $\mathcal{F}_A$ is precisely $\mathcal{F}_B$ with $B = \lambda \text{Conv}(A) \cap \mathbb{Z}^n$ and weights

$$c_b = \sum_{a_1 + \cdots + a_\lambda = b} c_{a_1} c_{a_2} \cdots c_{a_\lambda}.$$  

An orthonormal basis is $(c_b^{-1/2} x^b)$.

5. PROOF OF THE MAIN RESULTS

Proof of Lemma 4. Let $\mathcal{E}$ and $\mathcal{F}_1$ be fewspaces on a complex manifold $M$, and let $\mathcal{G} = \mathcal{E}^{\lambda_1} \mathcal{F}_1^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}$. By Theorem 13

$$K_{\mathcal{G}}(x, y) = K_{\mathcal{E}}(x, y)^\lambda K_{\mathcal{F}_1}(x, y).$$  

By (2), we deduce that

$$\omega_{\mathcal{G}} = \lambda \omega_{\mathcal{E}} + \omega_{\mathcal{F}_1}.  \quad \square$$

Proof of Theorem 3. Let $\mathcal{G} = \mathcal{F}_1^{\lambda_1} \mathcal{F}_2^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}$. Let $\omega_i$ be the Kähler form associated to the space $\mathcal{F}_i$. The form associated to $\mathcal{G}$ is

$$\omega_{\mathcal{G}} = \lambda_1 \omega_{\mathcal{F}_1} + \cdots + \lambda_n \omega_{\mathcal{F}_n}.$$  

Because the $\omega_i$ are 2-forms, they commute. Theorem 10 implies that

$$\mathbb{E}_{g_1, \ldots, g_n \in \mathcal{F}_1^{\lambda_1} \mathcal{F}_2^{\lambda_2} \cdots \mathcal{F}_n^{\lambda_n}} (n_M(g)) =$$

$$= \frac{1}{\pi^n} \int_M \omega_{\mathcal{G}} \wedge \cdots \wedge \omega_{\mathcal{G}}$$

$$= \sum_{i_1, \ldots, i_n \in \{1, \ldots, n\}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} \frac{1}{\pi^n} \int_M \omega_{i_1} \wedge \cdots \wedge \omega_{i_n}$$

$$= \sum_{i_1, \ldots, i_n \in \{1, \ldots, n\}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} \mathbb{E}_{f_1 \in \mathcal{F}_1, \ldots, f_n \in \mathcal{F}_n} (n_M(f)).$$  

In the last line above, the monomial $\lambda_1 \lambda_2 \cdots \lambda_n$ appears $n!$ times. Its coefficient is therefore

$$n! \mathbb{E}_{f_1 \in \mathcal{F}_1, \ldots, f_n \in \mathcal{F}_n} (n_M(f)).$$
6. Explicit calculation of the number of zeros

6.1. The example in the introduction. We start by the bound on the expected number of roots of (1) in the introduction. Let $E$ denote the fewspace of functions on the disk $D = \{z \in \mathbb{C} : |z| < 1\}$ spanned by 1 and $e^z$. We assume that 1 and $e^z$ form an orthonormal basis. Then

$$K_E(x, y) = 1 + e^{x+y}.$$ 

An easy computation is now

$$\omega_E = \sqrt{-1 \over 2} \partial \bar{\partial} \log K_E(z, z) = {e^{2\text{Re}(z)} \over (1 + e^{2\text{Re}(z)})^2}.$$ 

The following numerical approximation was obtained by Steven Finch using Mathematica. It was independently checked by this author using long double IEEE arithmetic.

$$E_{f \in E}(n_{f}(D)) = \pi^{-1} \int_D \omega = 0.202, 918, 921, 282 \cdots.$$ 

It is obvious that $E_{f \in P_d}(n_{P}(f)) = \pi^{-1} \int_D \omega_P = d/2$. Hence,

$$E_{f \in P_d}(n_{f}(D)) = \pi^{-1} \int_D \omega + \omega_P = d/2 + 0.202, 918, 921, 282 \cdots.$$ 

6.2. An $n$-dimensional example. We consider now systems where each equation is of the form

$$\sum f_{a_1, b_1} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} e^{b_1 x_1 + \cdots + b_n x_n}$$ 

and the sum is taken for all $0 \leq a_i \leq d$ and $b_i = 0, 1$. The corresponding domain will be the polydisc $D^n$.

The fewnomial space is

$$G = (EP_d) \otimes (EP_d) \otimes \cdots \otimes (EP_d).$$ 

Let $\omega = g(z) \sqrt{-1 \over 2} dz \wedge d\bar{z}$ be the Kähler form corresponding to $(EP_d)$. Then from Th.12 and (2), we deduce that

$$\omega_G = \sum_{i=1}^{n} g(z_i) \sqrt{-1 \over 2} dz_i \wedge d\bar{z}_i.$$ 

Hence,

$$E_{f_1, \ldots, f_n \in G}(n_{f}(D^n)) = \pi^{-n} \int_{D^n} \omega_G^n = n!(d/2 + 0.202, 918, 921, 282 \cdots)^n.$$
6.3. **An unmixed example.** We consider now the case where the first equation belongs to $\mathcal{G} = (\mathcal{E} \mathcal{P}_{d_1})^\otimes n$ as above, but the other equations polynomials of degree $d_2, \cdots, d_n$ in each variable (they belong to $\mathcal{P}_{d_j}^\otimes n$).

Then, let $\mathcal{H} = \mathcal{G}_{1}^{\lambda_1} \mathcal{P}_{d_2}^{\lambda_2} \cdots \mathcal{P}_{d_n}^{\lambda_n}$. Note that

$$\mathcal{H} = \mathcal{E}^{\lambda_1} \mathcal{P}_{1}^{\lambda_1 d_1 + \cdots + \lambda_n d_n}$$

From the previous example,

$$\frac{1}{n!} \mathbb{E}_{f_1, \ldots, f_n \in \mathcal{H}}(n_{\mathcal{F}}(\mathcal{D}^n)) =$$

$$= \left( \frac{\lambda_1 d_1 + \cdots + \lambda_n d_n + \lambda_1 0.202, 918, 921, 282 \cdots}{2} \right)^n$$

The coefficient in $\lambda_1 \lambda_2 \cdots \lambda_n$ is

$$n! \frac{d_1 d_2 \cdots d_n}{2^n} + n - 1! \frac{d_2 \cdots d_n}{2^{n-1}} 0.202, 918, 921, 282 \cdots$$

By Theorem 3

$$\mathbb{E}_{f_1 \in \mathcal{G}, f_2 \in \mathcal{P}_2, \ldots, f_n \in \mathcal{P}_n} n_{\mathcal{D}}(\mathcal{F}) =$$

$$= n! \frac{d_1 d_2 \cdots d_n}{2^n} + n - 1! \frac{d_2 \cdots d_n}{2^{n-1}} 0.202, 918, 921, 282 \cdots .$$

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