NONCOMPACT $L_p$-MINKOWSKI PROBLEMS

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Abstract. In this paper we prove the existence of complete, noncompact convex hypersurfaces whose $p$-curvature function is prescribed on a domain in the unit sphere. This problem is related to the solvability of Monge-Ampère type equations subject to certain boundary conditions depending on the value of $p$. The special case of $p = 1$ was previously studied by Pogorelov [28] and Chou-Wang [10]. Here, we give some sufficient conditions for the solvability for general $p \neq 1$.

1. Introduction

Let $M$ be a compact, strictly convex $C^2$-hypersurface in $\mathbb{R}^{n+1}$. Since the Gauss map is a bijection between $M$ and the unit sphere $S^n$, $M$ can be parametrised by the inverse of the Gauss map, and consequently the Gauss curvature $K$ of $M$ can be regarded as a function on $S^n$. Let $H$ be the support function of $M$ (see definitions in §2). For $p \in \mathbb{R}$, $K_p := KH^{p-1}$ is called the $p$-curvature of $M$. The $L_p$-Minkowski problem introduced by Lutwak [20] asks that whether a given function on $S^n$ is the $p$-curvature of a unique compact convex hypersurface. This problem is related to the solvability of the following Monge-Ampère type equation

\begin{equation}
\det (\nabla_{ij} H + H \delta_{ij}) = f H^{p-1} \quad \text{on} \quad S^n,
\end{equation}

where $\nabla$ is the covariant differentiation with respect to an orthonormal frame on $S^n$. When $p = 1$, one has the classical Minkowski problem [9, 27]. For general $p$, the $L_p$-Minkowski problem has been intensively studied in recent decades, for example, in [5, 11, 15, 18, 19, 20, 21, 23, 34, 35, 36] and many others. We refer the reader to the newly expanded book [29] by Schneider for a comprehensive introduction on related topics.

The same problem makes perfectly sense for complete, noncompact, convex hypersurfaces. In that case, by a suitable rotation, the spherical image of such a hypersurface is an open convex subset contained in the hemisphere $S^n_- := \{X \in S^n : X_{n+1} < 0\}$. The corresponding problem is then: Given an open convex subset $D$ of $S^n_-$ and a positive function $K_p$ in

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does there exist a (unique) complete convex hypersurface with spherical image $D$ and $p$-curvature $K_p$?

When $p = 1$, Pogorelov \[28\] firstly proved the existence of such a hypersurface under certain decay conditions on $K$ near the boundary of $D$. Chou and Wang \[10\] considered it in more general cases. For $p \neq 1$, this problem becomes much more complicated, partly because the $p$-curvature $K_p$ involves the support function $H$, which depends on the position of hypersurface $M$ and thus is not translation-invariant. In this paper we give sufficient conditions for the solvability for general $p \neq 1$, and extend Chou-Wang’s results in \[10\] for $p = 1$.

Similarly as above, the problem in noncompact setting is related to the solvability of Equation (1.1) in $D$ associated with certain compatible boundary conditions, where $f = K_p^{-1}$ is prescribed. One can see clearly from Equation (1.1) that whether $p > 1$ or $p < 1$ makes a big difference, as the right hand side of equation goes to degenerate or singular when $H \to 0$, respectively. Correspondingly, in the subsequent context we shall consider these two cases separately.

When $p < 1$, from a geometric observation we show that if $f \geq 0$, there does not exist such a complete, noncompact, convex hypersurface (with $H \geq 0$) satisfying Equation (1.1) (see Lemma \[3.1\]). Instead, we consider $H < 0$, namely the origin lies in the concave side of $M$, and the hypersurface $M$ satisfies

\[(1.2) \quad H = -\hat{f} \frac{1}{K_p^{1/p}} \quad (p < 1)\]

for a given positive function $\hat{f}$ in $D$. We remark that when $p < 1$, it is necessary to have $D$ strictly contained in $S^n$, see §3. Our first result is

**Theorem 1.1.** Let $D$ be a uniformly convex $C^2$-domain strictly contained in $S^n$, $\hat{f}$ a positive function in $C^\alpha(D) \cap L^{1-p}(D)$, where $\alpha \in (0, 1)$ and $p < 1$. Suppose there exists two positive functions $g$ and $h$ defined in $(0, r_0]$, $r_0 > 0$, satisfying

\[
\begin{align*}
(a) & \quad \int_0^{r_0} \left( \int_0^{r_0} g^{1-p}(t)dt \right)^{1/n} ds < \infty, \\
(b) & \quad \int_0^{r_0} h^{1-p}(t)dt = \infty,
\end{align*}
\]

so that $C^{-1}h(\text{dist}(X, \partial D)) \leq \hat{f}(X) \leq Cg(\text{dist}(X, \partial D))$ near $\partial D$ for some constant $C > 0$. Then there exists a unique complete, noncompact, strictly convex hypersurface $M$ such that the support function $H \in C^{2,\alpha}(D) \cap C(\overline{D})$ satisfies (1.2) in $D$, and $H = 0$ on $\partial D$.

We remark that the assumption $\hat{f} \in L^{1-p}(D)$ ensures $H = 0$ on $\partial D$ and $M$ approaches to an asymptotic convex cone. Without this assumption, one can also obtain the existence of $M$ with a general boundary condition $H = \Phi$ for a function $\Phi \in C^2(\partial D)$, but to show $M$ is complete, one needs a stronger assumption that $h(\text{dist}(X, \partial D))/\hat{f}(X) \to 0$ as $X \to \partial D$. More details are contained in §3.
When \( p > 1 \), depending on the relative position of \( D \) there are multiple cases for discussion. By suitably rotating axes, we may assume \( D \) satisfies one and exactly one of the following conditions:

(I) \( D \) is strictly contained in \( S^n \),

(II) \( D = S^n \),

(III) \( D \) is a proper subset of \( S^n \) and it is not strictly contained in any hemisphere.

We shall say \( M \) is of type I, II, or III when its spherical image \( D \) satisfies (I), (II), or (III), respectively. Notice that by our choice of coordinates, \( M \) is the graph of a convex function over a convex domain in the \((x_1, \cdots, x_n)\)-space.

For type I hypersurfaces, it is clear that \( M \) is complete if and only if \( M \) is a graph over \( \mathbb{R}^n \). Correspondingly, we impose a boundary condition to the support function \( H = \Phi \) on \( \partial D \), where \( \Phi \) is a prescribed function.

**Theorem 1.2.** Let \( D \) be a uniformly convex \( C^2 \)-domain satisfying condition (I), \( p \geq 1 \) and \( p \neq n + 1 \). Assume \( K_p \) is a positive function in \( C^\alpha(D) \) and \( \Phi \) is a function in \( C^2(\partial D) \).

Suppose there exists two positive functions \( g \) and \( h \) defined in \((0, r_0], r_0 > 0\), satisfying

\[
\begin{align*}
(a) & \quad \int_0^{r_0} \left( \int_0^{r_0} g(t)dt \right)^{1/n} ds < \infty, \\
(b) & \quad \int_0^{r_0} h(t)dt = \infty,
\end{align*}
\]

so that \( K_p(X)g(dist(X, \partial D)) \geq C^{-1} \) and \( K_p(X)h(dist(X, \partial D)) \leq C \) near \( \partial D \) for some constant \( C > 0 \). Then there exists a unique complete, noncompact, strictly convex hypersurface \( M \) such that \( K_p \) is the \( p \)-curvature of \( M \) and \( H = \Phi \) on \( \partial D \).

In fact, for this reconstruction of complete convex hypersurface, Aleksandrov \cite{Aleksandrov} firstly formulated the geometric problem with prescribed area of Gaussian mapping and its asymptotic cone. It amounts to the solvability of the boundary value problem for a Monge-Ampère equation, see \cite{Aleksandrov, Bakelman}. Bakelman \cite{Bakelman} established the existence and uniqueness of convex generalised solutions for the second boundary value problem of the Monge-Ampère equation

\[
(1.3) \quad \det D^2 u = \frac{T(x)}{Q(Du)}.
\]

It has been convinced that the necessary and sufficient condition for the existence of those complete hypersurfaces project one-to-one on \( \mathbb{R}^n \) with prescribed asymptotic cone \( \mathcal{C} \), is

\[
\int_{\mathbb{R}^n} T(x)dx = \int_{\mathcal{N}(\mathbb{R}^n)} Q(p)dp,
\]

where \( \mathcal{N}(\mathbb{R}^n) \) is the normal image of \( \mathcal{C} \). See also Pogorelov \cite{Pogorelov, Pogorelov2}. Moreover, Oliker \cite{Oliker} constructed a minimisation problem associated with the Monge-Kantorovich optimal mass transfer problem for this kind of reconstruction geometric problem.
For type II hypersurfaces, we prove the existence of such a complete hypersurface $M$ when $p > n + 1$, under an asymptotic growth assumption on the prescribed function $K_p$.

**Theorem 1.3.** Assume that $K_p$ satisfies an asymptotic growth condition

\begin{equation}
K_p(X) \sim |X_{n+1}|^{2q},
\end{equation}

for some constant $q \in (0, 1)$. When $p > n + 1$, there exists a complete noncompact convex hypersurface $M$ such that $K_p$ is the $p$-curvature of $M$.

We further remark that when $M$ is a graph over a bounded domain $\Omega^*$, (1.4) is necessary for the solvability, see §4.2.

For type III hypersurfaces, we have a similar result as Theorem 1.2, however, the boundary condition is imposed on part of $\partial D$ that is away from $\partial S^n$. The corresponding statement is postponed to Theorem 4.2 in §4.3.

Last, we point out that the $L_p$-Minkowski problem is related to the expanding Gauss curvature flow when $p > 1$, and the contracting Gauss curvature flow when $p < 1$, as Equation (1.1) describes homothetic solutions in each case, respectively. For complete, noncompact hypersurfaces, one may consult Urbas [31, 32] for works in this direction, also [10, 14] and references therein.

This paper is organised as follows: In §2, we derive the Monge-Ampère equation (1.1). Although this has been done in some literatures, we would like to provide a more general and unified treatment, which includes (1.1) as a special case when the metric on $S^n$ is orthonormal. In §3, we consider the case of $p < 1$ and prove Theorem 1.1. Moreover, when the prescribed function $\hat{f}$ satisfies certain boundedness conditions, we can give a different proof without the uniform convexity assumption on $D$. In §4, we consider the case of $p > 1$, Theorems 1.2, 1.3 and 4.2 are proved in §4.1–§4.3, respectively.

## 2. Preliminaries

2.1. **Support function.** Let $M$ be a strictly convex $C^2$-hypersurface in $\mathbb{R}^{n+1}$ and let $D \subset S^n$ be its spherical image. Assume that $M$ is parametrised by the inverse Gauss map $X : D \rightarrow M \subset \mathbb{R}^{n+1}$. The support function of $M$ is defined by

\begin{equation}
H(\xi) = \langle \xi, X(\xi) \rangle, \quad \xi \in D,
\end{equation}

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{n+1}$.

The metric and the second fundamental form of $M$ can be represented in terms of its support function $H$. To see that, let $(u_1, \cdots, u_n)$ be a local coordinate chart and $\{e_i = \partial_i \xi\}$
be the local frame field on $D$, where $\partial_i = \partial/\partial u_i$, $i = 1, \ldots, n$. By differentiating (2.1) we obtain that
\[(2.2) \quad \partial_i H = \langle \partial_i \xi, X \rangle + \langle \xi, \partial_i X \rangle = \langle e_i, X \rangle,\]
since $\partial_i X(\xi)$ is tangential to $M$ at $X(\xi)$, and $\xi$ is the normal to $M$ at $X(\xi)$. Differentiating once again, we have
\[(2.3) \quad \partial_{ij} H = \langle \partial_j e_i, X \rangle + \langle e_i, \partial_j X \rangle = \langle \partial_j e_i, X \rangle + h_{ij},\]
where $h_{ij}$ is the second fundamental form of $M$.

To compute $\langle \partial_j e_i, X \rangle$, we use the Gauss derivation formulas to get
\[(2.4) \quad \partial_j e_i = \Gamma^k_{ij} e_k - \sigma_{ij} \xi,\]
where $\Gamma^k_{ij}$ are the Christoffel symbols, and $\sigma_{ij} = \langle e_i, e_j \rangle$ is the metric on $D \subset S^n$, respectively. Combining (2.2), (2.3) and (2.4), we can obtain
\[(2.5) \quad h_{ij} = \partial_{ij} H - \Gamma^k_{ij} \partial_k H + H \sigma_{ij} + h_{ij},\]
and thus
\[(2.6) \quad g_{ij} = h_{ik} h_{jl} \sigma^{kl}.\]

The principal radii of curvature are the eigenvalues of the matrix $b_{ij} = h^{ik} g_{jk}$, which, by virtue of (2.5) and (2.7), is given by
\[(2.8) \quad b_{ij} = h_{ik} \sigma^{kj} = (\nabla_{ik} H + H \sigma_{ik}) \sigma^{kj}.\]

Therefore, the Gauss curvature $K$ of $M$ is given by
\[(2.9) \quad \frac{1}{K} = \det b_{ij} = \frac{\det (\nabla_{ij} H + H \sigma_{ij})}{\det (\sigma_{ij})}.\]

For a general $p \in \mathbb{R}$, the $p$-curvature of $M$ is defined by $K_p := KH^{p-1}$. When $p = 1$, $K_1 = K$ is the Gauss curvature. When $p \neq 1$, $K_p$ involves the support function $H$, and thus is not intrinsic. The $L_p$-Minkowski problem asks for the existence of $M$ with a prescribed
\( p \)-curvature \( K_p \). Let \( f := K_p^{-1} = H^{1-p}/K \), from \([22]\) one can obtain the Monge-Ampère equation satisfied by \( H \),

\[
(2.10) \quad \det (\nabla_{ij} H + H \sigma_{ij}) = f H^{p-1} \det (\sigma_{ij}) \quad \text{on } D.
\]

In particular, under a smooth local orthonormal frame field on \( S^n \), namely \( \sigma_{ij} = \delta_{ij} \), the above equation becomes \([1.1]\), namely

\[
(2.11) \quad \det (\nabla_{ij} H + H \delta_{ij}) = f H^{p-1} \quad \text{on } D.
\]

### 2.2. Homogeneous extension.

In order to study the solvability of \( L_p \)-Minkowski problems, it is convenient to express the above equations for \( H \) in a local coordinate chart. Extend \( H \) to be a function of homogeneous degree one over the cone \( \{ \lambda \xi : \xi \in D, \lambda > 0 \} \), let \( \Omega := \{ \lambda \xi : \xi \in D, \lambda > 0 \} \cap \{ x_{n+1} = -1 \} \), and \( u(x) = H(x, -1) \), where \( x = (x_1, \cdots, x_n) \).

Denote

\[
(2.12) \quad \mu(x) = \left( 1 + \sum_{i=1}^{n} x_i^2 \right)^{1/2}.
\]

By the homogeneity,

\[
(2.13) \quad u(x) = \mu(x) H(\xi(x)), \quad x \in \Omega,
\]

where \( \xi(x) \in D \) is given by

\[
(2.14) \quad \mu(x) \xi(x) = (x, -1).
\]

In order to rewrite equation \((2.10)\) in terms of \( u \), we adopt the following computations from \([22]\). Differentiating \((2.14)\) we have

\[
(2.15) \quad (\partial_i \mu) \xi + \mu \partial_i \xi = (0, \cdots, 0, 1, 0, \cdots, 0).
\]

Differentiating once again yields

\[
(2.16) \quad (\partial_{ij} \mu) \xi + (\partial_i \mu) \partial_j \xi + (\partial_j \mu) \partial_i \xi + \mu \partial_{ij} \xi = 0.
\]

By the Gauss derivation formulas,

\[
(2.17) \quad \partial_{ij} \xi = \Gamma_{ik}^j \partial_k \xi - \sigma_{ij} \xi,
\]

where \( \sigma_{ij} = \langle \partial_i \xi, \partial_j \xi \rangle \) is the metric of \( S^n \). Taking the inner product of \((2.16)\) with \( \partial_k \xi \), and noting that \( \langle \partial_k \xi, \xi \rangle = 0 \), we get

\[
(2.18) \quad \partial_i \mu \sigma_{sj} + \partial_j \mu \sigma_{si} + \mu \Gamma_{ij}^k \sigma_{ks} = 0.
\]

While taking the inner product of \((2.16)\) with \( \xi \), we also get

\[
(2.19) \quad \partial_{ij} \mu = \mu \sigma_{ij}.
\]
Now, differentiating (2.13) we obtain

$$\partial_i u = \partial_i \mu H + \mu \partial_i H,$$

then by (2.18) and (2.19)

$$\partial_{ij} u = \partial_{ij} \mu H + \partial_i \mu \partial_j H + \mu \partial_{ij} H,$$

(2.20)

$$= \partial_{ij} \mu H - \mu \Gamma_{ij} \partial_i H + \mu \partial_{ij} H$$

$$= \partial_{ij} \mu H + \mu \nabla_{ij} H$$

$$= \mu (\nabla_{ij} H + H \sigma_{ij}).$$

On the other hand, by straightforward computations we have

$$\sigma_{ij} = (1 + |x|^2)^{-1} \left( \delta_{ij} - \frac{x_i x_j}{1 + |x|^2} \right),$$

(2.21)

and

$$\det (\sigma_{ij}) = (1 + |x|^2)^{-n+1} = \mu^{-2(n+1)}.$$

(2.22)

Substituting (2.13), (2.20) and (2.22) into (2.10), we obtain the standard Monge-Ampère equation satisfied by $u$,

$$\det D^2 u = (1 + |x|^2)^{-\frac{n+p+1}{2}} \left( \frac{x_i - 1}{\sqrt{1 + |x|^2}} \right)^{p-1} f^{-1}, \quad x \in \Omega,$$

where $D^2 u = (\partial_{ij} u)$ is the Hessian matrix of $u$. Therefore, the solvability of $L_p$-Minkowski problems is equivalent to the solvability of the Monge-Ampère equation (2.23).

In the complete, noncompact case with certain boundary conditions, whenever a convex solution of (2.23) is given, as a rescaled support function it determines the hypersurface $M$ in the following way (see §9 [27]): Let $\Omega^* = Du(\Omega)$ and

$$u^*(y) = \sup \{ \langle x, y \rangle - u(x) : x \in \Omega \}, \quad y \in \Omega^*.$$

Then $M$ is the graph $\{(y, u^*(y)) : y \in \Omega^*\}$, and its $p$-curvature is equal to $f^{-1} = K_p$ as prescribed.

By straightforward computations, the dual function $u^*$ satisfies

$$\det D^2 u^* = \left(1 + |Du^*|^2\right)^{-\frac{n+p+1}{2}} \left( y \cdot Du^* - u^* \right)^{p-1} f^{-1}(\gamma), \quad y \in \Omega^*,$$

(2.25)

where $\gamma = \frac{(Du^* - 1)}{\sqrt{1 + |Du^*|^2}}$ is the unit normal of $M$ at the point $(y, u^*(y))$. 

3. The case of $p < 1$

In this section we first show a nonexistence result for hypersurface $M$ with support function $H \geq 0$. Then alternatively, we consider the hypersurface satisfying (1.2) with $H < 0$ and prove Theorem 1.1. When the prescribed function $\hat{f}$ satisfies further boundedness conditions, we also give some independent and interesting results for the existence and completeness. Throughout this section we assume $p < 1$.

Originally, one asks for a strictly convex $C^2$-hypersurface $M$ in $\mathbb{R}^{n+1}$ such that

$$H = (fK)^{\frac{1}{1-p}},$$

for some prescribed function $f \geq 0$ on the spherical image $D \subset S^n$, where $H$ is the support function of $M$.

**Lemma 3.1 (Nonexistence).** If $f \in L^1(D)$ is a nonnegative function, there does not exist any complete, noncompact, strictly convex hypersurface $M$ satisfying (3.1).

**Proof.** Since $M$ is convex, $K \geq 0$. From assumption $f \geq 0$, the support function $H \geq 0$ by (3.1). If $H = 0$ on $\partial D$, then either $M$ is a cone or $M$ is degenerate with zero $n$-dimensional Hausdorff measure, $\mathcal{H}^n(M) = 0$. Therefore, we assume that $\sup_{\partial D} H \geq \delta$ for some positive constant $\delta$. By continuity of $H$, there exists a subset $E \subset D$ such that $H \geq \delta/4$ in $E$.

Let $G := X(E) \subset M$, where $X$ is the inverse Gauss map, see (2.1). Since $M \in C^2$ is strictly convex, the map $X : D \to M$ is a bijection. Since $M$ is complete noncompact and $E \cap \partial D \neq \emptyset$, we have $\mathcal{H}^n(G) = \infty$ [33]. Then by integration we obtain

$$\infty = \int_G (\delta/4)^{1-p} \, d\mathcal{H}^n \leq \int_E \frac{H^{1-p}}{K} \, dx$$

$$\leq \int_D \frac{H^{1-p}}{K} \, dx = \int_D f \, dx < \infty,$$

where $dx$ is the spherical measure of $S^n$. The last equality is due to (3.1). This is a contradiction to the assumption $f \in L^1(D)$, and thus Lemma 3.1 is proved. \hfill $\square$

We remark that in proving the above lemma, one can in fact show that the set $\{ \xi \in D : H(\xi) = 0 \}$ has zero $\mathcal{H}^n$ measure, where $H \geq 0$ is the support function of a complete, noncompact, strictly convex hypersurface $M$. Hence, the contradiction (3.2) will occur under the assumption of Lemma 3.1.

Therefore, in the case of $p < 1$, it is reasonable to consider the existence of hypersurface $M$ satisfying (1.2) and $H \leq 0$, that is

$$H = -\hat{f}K^{\frac{1}{1-p}}.$$
where \( \hat{f} \geq 0 \) is a prescribed function on the spherical image \( D \). Then one’s aim is to look for a complete, noncompact, strictly convex hypersurface \( M \) satisfying (3.3). And in such cases, \( K_p = \hat{f}^{p-1} = K(-H)^{p-1} \) is the \( p \)-curvature of \( M \).

Note that \( H \leq 0 \) implies that any tangent hyperplane \( T \) to \( M \) either contains the origin, or else, the origin lies on the opposite side of \( T \) from \( M \). If \( 0 \notin M \), it follows that every tangent hyperplane to \( M \) must contain \( 0 \) and \( H \equiv 0 \), which implies that \( M \) is a hyperplane containing the origin or a cone with vertex at the origin, but this contradicts with the strict convexity of \( M \). Now assume that \( 0 \notin M \), the origin lies on the opposite sides of all tangent hyperplanes from \( M \), or equivalently, \( H < 0 \) in \( D \).

Let \( C \) be the intersection of all closed halfspaces \( \mathcal{P} \) of \( \mathbb{R}^{n+1} \) with \( 0 \in \partial \mathcal{P} \) and \( M \subset \mathcal{P} \). Then \( C \) is a closed convex cone with vertex at the origin with nonempty interior. Moreover, \( \partial C \) can be represented as the graph of a convex degree one homogeneous function \( \psi \) with \( \psi \geq 0 \) in \( \mathbb{R}^n \setminus \{0\} \) and \( |D\psi| \) bounded. Recall that \( M \) is a graph of \( u^* \) over \( \Omega^* \subset \mathbb{R}^n \), from the construction of \( C \), \( Du^*(\Omega^*) \subset D\psi(\mathbb{R}^n) \). Because \( M \) is complete and \( Du^* \) is bounded, we must have \( \Omega^* = \mathbb{R}^n \), namely \( M \) is an entire graph over \( \mathbb{R}^n \). By parallel translating supporting hyperplanes between \( M \) and \( C \), one also has

\[
Du^*(\mathbb{R}^n) = D\psi(\mathbb{R}^n),
\]

see [31] for more geometric details. Therefore, the spherical image of \( M \),

\[
D = \frac{(Du^*,-1)}{\sqrt{1 + |Du^*|^2}}(\mathbb{R}^n)
\]

must be strictly contained in \( S^n \), and its projection image \( \Omega = \{\lambda \xi : \xi \in D, \lambda > 0\} \cap \{x_{n+1} = -1\} \) must be a bounded, convex domain in \( \mathbb{R}^n \).

Next lemma shows that under hypotheses of Theorem 1.1, \( M \) is asymptotically approaching to \( C \) in the sense that \( H(X) \to 0 \) as \( X \to \partial D \).

**Lemma 3.2 (Asymptotic).** Under the hypotheses of Theorem 1.1, let \( M \) be a solution satisfying (3.3). If \( \hat{f} \in L^{1-p}(D) \), then \( H(X) \to 0 \) as \( X \to \partial D \).

**Proof.** Suppose if not true, by continuity of \( H \), there exists some \( X_0 \in \partial D \) and a positive constant \( \delta \) such that \( -H \geq \delta \) in a neighborhood of \( X_0 \), \( E \subset D \). Let \( G := X(E) \subset M \), where \( X \) is the inverse Gauss map, which is a bijection from \( D \) to \( M \). As \( X_0 \in E \cap \partial D \), \( E \cap \partial D \neq \emptyset \). Since \( M \) is complete noncompact, one has \( \mathcal{H}^n(G) = \infty \). Then similarly to Lemma 3.1 by integration we have

\[
\infty = \int_G (\delta/4)^{1-p} d\mathcal{H}^n \leq \int_E \frac{(-H)^{1-p}}{K} dx
\]

\[
\leq \int_D \frac{(-H)^{1-p}}{K} dx = \int_D \hat{f}^{1-p} dx < \infty,
\]

(3.4)
where $dx$ is the spherical measure of $S^n$. The last equality is due to (3.3). By assumption $\hat{f} \in L^{1-p}(D)$, this contradiction thus implies that $H(X) \to 0$ as $X \to \partial D$.

In a local coordinate chart, by (2.13),

$$u = \mu H < 0 \quad \text{in } \Omega.$$ 

Since $M$ is enclosed by the asymptotic cone $C$, $u = 0$ on $\partial \Omega$. It is also clear that $M$ is complete if and only if $Du(\Omega) = \mathbb{R}^n$.

From the computations in §2, the above question is related to a variant of Monge-Ampère equation

$$\mu^{-1}u = -\hat{f}\mu^{-\frac{n+2}{1-p}}(\det D^2u)^{-\frac{1}{1-p}} \quad \text{in } \Omega.$$ 

Hence, by Lemma 3.2 we pose the following boundary value problem

\begin{align*}
\det D^2u &= \mu^{-(n+p+1)}\hat{f}^{1-p}\left(\frac{1}{u}\right)^{1-p} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
|Du(x)| &\to \infty \quad \text{as } x \to \partial \Omega.
\end{align*}

Here, $\hat{f}(x)$ is interpreted as $\hat{f}\left(\frac{x,-1}{\sqrt{1+|x|^2}}\right)$ for $x \in \Omega$.

Therefore, in order to prove Theorem 1.1, it suffices to prove the following result.

**Theorem 3.1.** Let $\Omega$ be a bounded, uniformly convex smooth domain in $\mathbb{R}^n$, and $\hat{f} \geq 0$ be a smooth function in $\Omega$. Suppose there exists two positive functions $g$ and $h$ satisfying conditions (a) and (b) in Theorem 1.1 such that $C^{-1}h(\text{dist}(x,\partial \Omega)) \leq \hat{f}(x) \leq Cg(\text{dist}(x,\partial \Omega))$ near $\partial \Omega$ for some constant $C > 0$. Then (3.5)–(3.7) has a unique smooth solution $u$.

The Dirichlet problem (3.5)–(3.6) was previously studied by Cheng-Yau [9]. They proved that if $\Omega$ satisfies a uniform enclosing sphere condition and $\hat{f}(x) \leq C\text{dist}(x,\partial \Omega)^{\beta-n-1}$, $\beta > 0$, then there admits a unique solution. If $\hat{f} \equiv 1$, while $\Omega$ is merely a bounded convex domain, Urbas [31] also obtained the unique existence of convex solution. One can easily check that the assumption on $\hat{f}$ in [9] is contained in condition (a) of Theorem 3.1. We divide the proof of Theorem 3.1 into two parts: Prove the solvability of Dirichlet problem (3.5)–(3.6), and verify the solution satisfies boundary condition (3.7).

### 3.1. **Existence.**

For the existence part, we consider a general Dirichlet boundary condition

$$u = \phi \quad \text{on } \partial \Omega,$$

where $\phi \leq 0$ is a convex function in $\Omega$. Write

$$R(x) = \mu^{-(n+p+1)}\hat{f}^{1-p}(x), \quad x \in \Omega.$$
Equation (3.5) can be simplified to

\begin{equation}
\det D^2 u = R(x) \left( \frac{-1}{u} \right)^{1-p}.
\end{equation}

Let \( \Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \} \). When \( \Omega \) is uniformly convex, for \( r_0 > 0 \) small depending on the geometry of \( \Omega \), \( \Omega_r \) is still uniformly convex. For \( x \in \Omega \setminus \Omega_{r_0} \), \( x \) can be represented uniquely by \( x_b + dn(x_b) \), where \( x_b \in \partial \Omega \), \( d = \text{dist}(x, \partial \Omega) \), and \( n(x_b) \) is the unit inner normal at \( x_b \). For a function \( f \) defined near \( \partial \Omega \) we write \( f(x) = f(x_b, d) \).

**Lemma 3.3.** Let \( \Omega \) be a bounded, uniformly convex \( C^2 \)-domain. Suppose there exists a positive function \( g \) in \( (0, r_0] \) satisfying

\begin{equation}
\int_0^{r_0} \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds < \infty
\end{equation}

such that

\[ R(x_b, d) \leq g(d). \]

Then (3.10) admits a unique generalised solution \( u \) satisfying (3.8).

**Proof.** Our proof is inspired by [10]. For \( x = x_b + dn(x_b) \) in \( \Omega \setminus \Omega_{r_0} \), we define

\begin{equation}
v(x) = \tilde{\rho}(d) := -(\rho(d))^\varepsilon,
\end{equation}

where \( \varepsilon \in (0, 1) \) is a constant to be determined and

\[ \rho(d) := -\int_0^d \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds. \]

By computations, see [10] Lemma 1,

\[ \det D^2 v(x) = \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b)d} (-\tilde{\rho}'(d))^{n-1} \tilde{\rho}''(d) \]

in \( \Omega \setminus \Omega_{r_0} \), where \( k_i(x_b), i = 1, \cdots, n - 1 \), are the principal curvatures of \( \partial \Omega \) at \( x_b \).

By differentiation

\[ \tilde{\rho}' = \varepsilon(\rho)^{\varepsilon-1} \rho', \]

\[ \tilde{\rho}'' = \varepsilon(\rho)^{\varepsilon-1} \rho'' + \varepsilon (1 - \varepsilon)(\rho)^{\varepsilon-2}(\rho')^2. \]

Hence,

\begin{equation}
\det D^2 v(x) 
\geq \varepsilon^n \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b)d} \left((-\rho)^{\varepsilon-1} \rho' \right)^{n-1} \left((-\rho)^{\varepsilon-1} \rho'' + (1 - \varepsilon)(\rho)^{\varepsilon-2}(\rho')^2 \right)
\geq \varepsilon^n C(n, \Omega) g(d) (-\rho)^{n(\varepsilon-1)}.
\end{equation}
By setting \(\varepsilon = \frac{p}{n+1-p}\) and rescaling \(v\) to \(bv\) for a constant \(b\) satisfying \(b^{n+1-p} \varepsilon^n C(n, \Omega) \geq 1\), one can see that \(\det D^2v \geq R(x)(-1/v)^{1-p}\) in \(\Omega \setminus \Omega_{r_0}\).

Observing that \(v\) is a negative constant on \(\partial \Omega_r\) for \(r \in (0, r_0)\), we can extend \(v\) to \(\Omega_{r_0}\) so that \(\det D^2v = \varepsilon (-1/v)^{1-p}\) in \(\Omega_{r_0}\). For \(\varepsilon\) small, \(v\) is uniformly convex in \(\Omega\). Similarly, by a rescaling we have \(v\) is a subsolution of (3.10) in \(\Omega\) and \(v = 0\) on \(\partial \Omega\).

Last step is to use the Perron method as in [11]. Denote \(\Phi\) by the set of all subsolutions of (3.10) and (3.8), and let \(u(x) = \sup \{\tilde{u}(x) : \tilde{u} \in \Phi\}\). One can easily verify that \(u\) is a generalised solution of (3.10). Since \(w \in \Phi\) we conclude that \(u = \phi\) on \(\partial \Omega\). The uniqueness is due to the comparison principle [9, 12].

One example for \(g\) satisfying (3.11) is that \(g(d) \leq Cd^{3-n-1}\) for some \(\beta > 0\), which is also the case considered in [9]. Notice that when \(\beta \geq n+1\), \(g\) is bounded in \((0, r_0]\) and thus \(R\) in (3.10) is bounded in \(\Omega\). In such a case, we can reduce the uniform convexity assumption on \(\Omega\) in Lemma 3.3 following the work in [31].

**Lemma 3.4.** If \(R\) in (3.10) is bounded from above and \(\Omega\) is a bounded convex domain, the Dirichlet problem (3.10) and (3.8) admits a unique generalised solution.

**Proof.** First, we consider the zero boundary condition \(\phi \equiv 0\). Let \(\{\Omega_k\}\) be an increasing sequence of smooth uniformly convex subdomains of \(\Omega\) with \(\Omega = \bigcup \Omega_k\). Let \(\{v_k\}\) be the sequence of convex solution of (3.10) in \(\Omega_k\), and \(v_k = 0\) on \(\partial \Omega_k\). Since \(\{\Omega_k\}\) is an increasing sequence, \(\{v_k\}\) is a decreasing sequence, by the comparison principle [9, 12]. We will show that \(v^* = \lim_{k \to \infty} v_k\) exists, and \(v^* = 0\) on \(\partial \Omega\).

Under a suitable coordinate we may assume that \(0 \in \partial \Omega\) and \(\Omega \subset \{x_n > 0\}\). Since \(\Omega\) is bounded, there exists a large \(K > 0\) such that \(\Omega \subset B_K^+(0) = B_K(0) \cap \{x_n > 0\}\). Define the barrier function

\[w(x) = (|x'|^2 - A)x^\delta_n,\]

where \(x' = (x_1, \cdots, x_{n-1})\) and \(A > R^2, \delta \in (0, 1)\) are to be fixed. One can see that \(w\) is convex, and by computation [31]

\[
\det D^2w = \{2^{n-1} \delta(1 - \delta)(A - |x'|^2) - 2^n \delta^2 |x'|^2 \} \\
	\times (A - |x'|^2)^{\delta-n} \left( \frac{-1}{w} \right)^{\delta-n}.
\]
If $n \geq 2$ we choose $\delta = \frac{2}{n+1-p} \in (0, 1)$, and then fix $A > K^2$ sufficiently large, so that
\[
\det D^2 w \geq R(x) \left( \frac{1}{w} \right)^{1-p} \quad \text{in } B^+_K(0).
\]
When $n = 1$, if $p < 0$ we obtain a similar inequality with $\delta = \frac{2}{2-p} < 1$, and if $0 \leq p < 1$ we can choose any $\delta \in (0, 1)$. Since $\Omega_k \subset B^+_K$ and $v_k = 0$ on $\partial \Omega_k$, by the comparison principle we have $w \leq v_k$ in $\Omega_k$ for each $k$. By a similar argument at any boundary point of $\Omega$ we see that $v^* = \lim_{k \to \infty} v_k$ is well defined and is a convex generalised solution of (3.10) satisfying $v^* = 0$ on $\partial \Omega$.

For general boundary value (3.8), let $w^* = v^* + \phi$, where $\phi \leq 0$ is convex in $\Omega$. Then $w^*$ is a subsolution of (3.10) and $w = \phi$ on $\partial \Omega$. Using the Perron method as in Lemma 3.3 we then obtain the generalised solution of (3.10) and (3.8). The uniqueness is due to the comparison principle [9, 12].

3.2. Completeness. Since the spherical image of $M$ is strictly contained in $S^n$, in order to be complete, $M$ must be an entire graph, namely the scaled support function $u$ must satisfy $|Du(x)| \to \infty$ as $x \to \partial \Omega$.

Lemma 3.5. Let $\Omega$ be a bounded, uniformly convex $C^2$-domain. Suppose that there exists a positive function $h$ in $(0, r_0]$ satisfying
\[
(3.14) \quad \int_0^{r_0} h(t)dt = \infty,
\]
such that
\[
(3.15) \quad \frac{h(d)}{R(x_b, d)} = o(1) \quad \text{as } d \to 0.
\]
Then the solution $u$ of (3.10) and (3.8), produced by Lemma 3.3, satisfies $|Du(x)| \to \infty$ as $x \to \partial \Omega$. In particular, if $\phi = 0$ on $\partial \Omega$, condition (3.15) can be reduced to
\[
(3.16) \quad h(d) \leq CR(x_b, d)
\]
for a constant $C > 0$.

Proof. Adopting the notations from §3.1. In $\Omega \setminus \Omega_{r_0}$ we define
\[
(3.17) \quad w(x) = -a\rho(d) + \phi = -a \int_0^d \left( \int_0^s h(t)dt \right)^{1/n}ds + \phi,
\]
which is uniformly convex for $a > 0$, $w = \phi$ on $\partial \Omega$ and $|Dw(x)| \to \infty$ as $x \to \partial \Omega$. Since $\partial \Omega \in C^2$, for sufficiently large $a > 0$ we have an estimate
\[
(3.18) \quad \det D^2 w \leq (2a)^n \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1-k_i(x_b)d}(-\rho'(d))^{n-1}\rho''(d)
\]
\[
\leq (2a)^n Ch(d),
\]
where $C$ is a constant depending on $n$ and $\partial \Omega$. 

Recall that $\phi \leq 0$. Let $\Omega' := \{ x \in \Omega : w(x) < \inf_{\partial \Omega} \phi - \varepsilon \}$ be a sub-level set of $w$, which is uniformly convex. Choose $\varepsilon > 0$ sufficiently small such that $\Omega \setminus \Omega'$ and $\Omega' \subset \Omega$. Note that $w$ is constant on $\partial \Omega'$, we can extend $w$ inside $\Omega'$ similarly as before and then modify $w$ to get a uniformly convex function $\tilde{w} \in C^2(\Omega)$ such that $\tilde{w} = w$ in $\Omega \setminus \Omega'$. Near $\partial\Omega$, observe that in $\Omega \setminus \Omega' \setminus R(x_b, d)(-\frac{1}{w})^{1-p} \geq R(x_b, d)C_1,$

where $C_1 = (-\inf_{\partial\Omega} \phi + \varepsilon)^{p-1} > 0$ is a finite constant. Therefore, $\tilde{w}$ is a supersolution of (3.10)–(3.8), and by the comparison principle, $\tilde{w} \geq u$ in $\Omega$. Hence, $|Du(x)| \to \infty$ as $x \to \partial\Omega$.

In the special case $\phi \equiv 0$, we set $a = 1$ in (3.17), and at the end, replace $\tilde{w}$ by $b\tilde{w}$ with $b > 0$ sufficiently small, so that we can obtain a supersolution.

An example for $h$ satisfying (3.14) is that $h(t) = t^{-\alpha}$ for some $\alpha > 1$. Alternatively, when studying the homothetic solutions to Gauss curvature flow, Urbas [31] proved that if $\hat{f} \equiv 1$ in $\Omega$, $\phi = 0$ on $\partial\Omega$, then for a range of $p$, $|Du(x)| \to \infty$ as $x \to \partial\Omega$. Inspired by that, we have the following results.

**Lemma 3.6.** Let $\Omega$ be a bounded convex domain and $\partial\Omega \in C^{1,1}$. Assume that $\phi = 0$ on $\partial\Omega$, $\hat{f} > 0$ in $\overline{\Omega}$. Then, when $p \leq 0$, the solution $u$ of (3.10)–(3.8) satisfies $|Du(x)| \to \infty$ as $x \to \partial\Omega$.

**Proof.** The proof follows from [31]. Let $x_0 \in \partial\Omega$ and let $B$ be an interior ball at $x_0$, i.e., $B \subset \Omega$ and $\partial B \cap \partial\Omega = \{x_0\}$. From assumptions, $R = \inf_{x \in B} R(x)$ is a positive constant, where $R$ is defined in (3.9). Let $w$ be the unique convex solution of the Dirichlet problem

(3.19) $\det D^2 w = R \left( \frac{-1}{w} \right)^{1-p}$ in $B$, $w = 0$ on $\partial B$,

The solvability is due to Lemma 3.3 and the solution $w$ is radially symmetric since the above problem has at most one convex solution. Hence, $w$ is a supersolution of (3.10) and (3.8), and $w \geq u$ in $B$ by the comparison principle. So, it suffices to show that $|Dw(x)| \to \infty$ as $x \to x_0$. Suppose on the contrary that $N = \sup_B |Dw| < \infty$, then $|w| \leq N d$ where
\( d = \text{dist}(x, \partial B) \), and
\[
\omega_n N^n = |Dw(B)| = \int_B \det D^2 w = R \int_B \left( \frac{-1}{w} \right)^{1-p} \geq RN^{p-1} \int_B d^{p-1}.
\]

(3.20)

When \( p \leq 0 \), the last integral is infinite, which gives a contradiction. Therefore, \( N = \infty \), and Lemma 3.6 is proved. \( \square \)

The following lemma shows that in order for \( M \) to be complete, it is necessary to have \( \hat{f}(\xi) \to \infty \) as \( \xi \to \partial D \), where \( D \subset S^n_- \) is the spherical image of \( M \).

**Lemma 3.7.** If \( 0 < p < 1 \), \( \phi = 0 \) on \( \partial \Omega \), \( \hat{f} \) is bounded above in \( \Omega \), and \( \Omega \) satisfies a uniform enclosing sphere condition (namely, there exists a \( K > 0 \) such that for each \( x_0 \in \partial \Omega \) there is a ball \( B \) of radius \( K \) with \( \Omega \subset B \) and \( \partial B \cap \partial \Omega = \{x_0\} \)), then the solution \( u \) of (3.10)–(3.8) satisfies \( \sup_{\Omega} |Du| \leq C \).

**Proof.** Let \( B \) be an enclosing ball at any point \( x_0 \in \partial \Omega \). From assumptions, \( \overline{R} = \sup_{x \in \Omega} R(x) \) is a positive constant. Replacing \( R \) in (3.19) by \( \overline{R} \), the convex solution \( w \) will be a subsolution of (3.10) and (3.8), and a gradient bound for \( u \) follows if we can prove \( N = \sup_{\Omega} |Dw| < \infty \). Since \( w \) is convex, \( w \neq 0 \) and \( w = 0 \) on \( \partial B \), we have \( |w| \geq \theta d \) for some positive constant \( \theta \), where \( d = \text{dist}(x, \partial B) \). Proceeding as above, we now obtain
\[
\omega_n N^n \leq \overline{R} \theta^{p-1} \int_B d^{p-1}.
\]
The last integral is finite if \( p > 0 \). Therefore, we have a gradient bound for \( w \), and hence also for the solution \( u \). This completes the proof of Lemma 3.7. \( \square \)

4. The case of \( p > 1 \)

Recall that in §1 we know that for a complete, noncompact, convex hypersurface \( M \) in \( \mathbb{R}^{n+1} \), by a suitable rotation of coordinates its spherical image \( D \subset S^n_- \) satisfies one and exactly one of three cases (I), (II) and (III). Given a function \( f \) on \( D \), we investigate the existence of \( M \) such that \( f = H^{1-p}/K \) is the \( p \)-curvature function of \( M \), where \( H \) is the support function and \( K \) is the Gauss curvature of \( M \). When \( M \) is \( C^2 \) smooth, a function \( f \) is the \( p \)-curvature function of \( M \) if it satisfies equation (2.10), or (2.11) under an orthonormal frame field on \( S^n \). By the homogeneous extension (2.13), one has \( u \) satisfies equation (2.23)
in the domain $\Omega$, and $u^*$ satisfies (2.25) in $\Omega^*$. The hypersurface $M$ is then the graph of $u^*$ over $\Omega^*$.

Notice that by the convexity of $M$, $K$ is always nonnegative. However, the sign of the support function $H$ depending on the relative position of $M$ and the origin. As seen in §3, the above problem is equivalent to (3.1) that

\[(4.1)\]

\[\frac{1}{H} = \tilde{f} K_{n-1},\]

where $\tilde{f}$ is the given function on $D$. Similar to the nonexistence result in the case of $p < 1$, i.e. Lemma 3.1, when $p > 1$ we have the following analogous result.

**Lemma 4.1 (Nonexistence).** If $\tilde{f} \leq 0$ is a nonpositive function on $D$, satisfying $\tilde{f} \in L^{p-1}(D)$, there does not exist any complete, noncompact, strictly convex hypersurface $M \in C^2$ satisfying (4.1).

**Proof.** Since $M$ is in the class $C^2$, by (4.1) we have $H < 0$ and $0 \notin M$. Hence, every tangent hyperplane $T$ of $M$ must pass between 0 and $M$, so $\text{dist}(0, T) \leq \text{dist}(0, M) =: d$. Thus, $-H^{-1} \geq d^{-1}$. Since $M$ has infinite $n$-dimensional Hausdorff measure $\mathcal{H}^n$, by integrating we have

\[
\int_D |\tilde{f}|^{p-1} dx = \int_M (-\tilde{f})^{p-1} K d\mathcal{H}^n \\
= \int_M \left( \frac{1}{H} \right)^{p-1} d\mathcal{H}^n \\
\geq \int_M \left( \frac{1}{d} \right)^{p-1} d\mathcal{H}^n = \infty,
\]

where $dx$ is the spherical measure of $\mathbb{S}^n$. The above inequality contradicts the assumption and thus completes the proof of Lemma 4.1. \qed

In the subsequent context, we assume $\tilde{f} \geq 0$. Let $f = \tilde{f}^{p-1}$, we consider the equation

\[(4.2)\]

\[\frac{1}{K} = f H^{p-1},\]

where $p > 1$. This is a counterpart of Equation (3.3) in the case of $p < 1$. By the rescaling (2.13), Equation (4.2) is equivalent to (2.23), namely

\[(4.3)\]

\[\det D^2 u = (1 + |x|^2)^{\frac{n+1}{2}} f \left( \frac{x_n - 1}{\sqrt{1 + |x|^2}} \right) u^{p-1}, \quad x \in \Omega,
\]

where $\Omega = \{ \lambda \xi : \xi \in D, \lambda > 0 \} \cap \{ x_n+1 = -1 \}$, $u(x) = H(x, -1)$, and $x = (x_1, \cdots, x_n)$. Depending on the spherical image $D \subset S^n_-$ of $M$, let’s consider the cases (I), (II) and (III) separately in the following.
### 4.1. Type I

For type I hypersurfaces, $D$ is strictly contained in $S^n$, $\Omega$ is a bounded convex domain in $\mathbb{R}^n$. It is clear that $M$ is complete if and only if $\Omega^* = \mathbb{R}^n$, where $\Omega^* = Du(\Omega)$. Thus we pose the following boundary conditions associated with equation (4.3)

\begin{align}
& (4.4) \quad |Du(x)| \to \infty \quad \text{as} \quad x \to \partial \Omega, \nonumber \\
& (4.5) \quad u(x) = \phi(x) \quad x \in \partial \Omega, 
\end{align}

where $\phi$ is assumed to be a positive, convex function in $\Omega$. We remark that in [32], when studying homothetic solutions of negative powered Gauss curvature flows, Urbas considered the above boundary value problem with $f \equiv 1$ and $\phi = \infty$ on $\partial \Omega$.

The solvability of Dirichlet problem (4.3) and (4.5) has been previously obtained in [7, §7], [13] and [17] under appropriate assumptions, especially $f$ is required to be positive and bounded. However, if the solution $u$ satisfies (4.4), by integrating equation (4.3),

\begin{align}
\infty = |Du(\Omega)| = \int_{\Omega} \det D^2 u \leq C \int_{\Omega} fu^{p-1}.
\end{align}

Notice that $0 < u \leq \sup_{\partial \Omega} \phi$. If $\sup_{\partial \Omega} \phi < \infty$, then it is necessary to have $f(x) \to \infty$ as $x \to \partial \Omega$. In the following, due to some technical differences, we consider two cases $p \in (1, n + 1)$ and $p \in (n + 1, \infty)$ separately. In each case, we first show the solvability of Dirichlet problem (4.3) and (4.5), and then prove that such an obtained solution $u$ satisfies (4.4). Hence, Theorem 1.2 is proved. Our approach is inspired by the work of Chou-Wang [10], in which they considered the special case $p = 1$, see also [28] and references therein.

#### 4.1.1. The case of $1 < p < n + 1$

Write

\begin{align}
R(x) = (1 + |x|^2)^{-\frac{n+p-1}{2}} f \left( \frac{x, -1}{\sqrt{1 + |x|^2}} \right), \quad x \in \Omega.
\end{align}

Equation (4.3) can be reduced to

\begin{align}
(4.6) \quad \det D^2 u = u^{p-1} R(x), \quad x \in \Omega.
\end{align}

As in §3, let $\Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \}$. For $x \in \Omega \setminus \Omega_{r_0}$, $r_0 > 0$ small, $x$ can be represented uniquely by $x_b + dn(x_b)$, where $x_b \in \partial \Omega$, $d = \text{dist}(x, \partial \Omega)$, and $n(x_b)$ is the unit inner normal at $x_b$. For a function $f$ defined near $\partial \Omega$ we write $f(x) = f(x_b, d)$.

**Lemma 4.2.** Assume that $1 \leq p < n + 1$ and $\Omega \in C^2$ is uniformly convex. Suppose there exists a positive function $g$ in $(0, r_0]$, satisfying

\begin{align}
(4.7) \quad \int_0^{r_0} \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds < \infty,
\end{align}

such that

\begin{align}
R(x_b, d) \leq g(d).
\end{align}

Then (4.6) admits a unique generalised solution $u$ in $C(\Omega)$ and $u = \phi$ on $\partial \Omega$. 

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Proof. Similarly as in [10], for \( x = x_b + d_n(x_b) \) in \( \Omega \setminus \Omega_{r_0} \) we define
\[
(4.8) \quad v(x) = \rho(d) = -\int_0^d \left( \int_s^{r_0} g(t) \, dt \right)^{1/n} \, ds,
\]
and have
\[
(4.9) \quad \det D^2 v(x) = n^{-1} \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b)d} (-\rho'(d))^{n-1} \rho''(d)
\]
in \( \Omega \setminus \Omega_{r_0} \), where \( k_i(x_b), \ i = 1, \cdots, n-1 \), are the principal curvatures of \( \partial \Omega \) at \( x_b \).

Next, we extend \( v \) inside \( \Omega_{r_0} \). Note that \( v = -G_0 \) is a constant on \( \partial \Omega_{r_0/2} \). We extend \( v \) to \( \Omega_{r_0/2} \) so that \( \det D^2 v = \varepsilon > 0 \) in \( \Omega_{r_0/2} \). For \( \varepsilon \) small, \( v \) is uniformly convex in \( \Omega \). By the uniform estimate, we have
\[
(4.10) \quad \sup |v| \leq G_0 + C|\Omega|^{2/n},
\]
for some constant \( C \) depending only on \( n, \varepsilon \).

Let
\[
(4.11) \quad w(x) = \phi(x) + Av(x), \quad x \in \Omega.
\]
Then \( w \) is convex in \( \Omega \), \( w = \phi \) on \( \partial \Omega \), and
\[
\sup |w| \leq |\phi|_0 + A|v|
\]
\[
\leq |\phi|_0 + A(G_0 + C|\Omega|^{2/n}).
\]

By computation we have the left hand side of equation (4.6)
\[
(4.12) \quad \det D^2 w \geq A^n \det D^2 v \geq \begin{cases} \ A^n\varepsilon & \text{in } \Omega_{r_0} \\ A^nC_1g & \text{in } \Omega \setminus \Omega_{r_0}, \end{cases}
\]
where \( C_1 \) is a constant depending on \( n, r_0 \) and \( \partial \Omega \). Meanwhile, the right hand side
\[
(4.13) \quad w^{p-1}R(x) \leq \sup |w|^{p-1}R(x) \leq \begin{cases} \sup |w|^{p-1}\bar{R} & \text{for } x \in \Omega_{r_0} \\ \sup |w|^{p-1}R(x) & \text{for } x \in \Omega \setminus \Omega_{r_0}, \end{cases}
\]
where \( \bar{R} = \sup_{x \in \Omega_{r_0}} R(x) \) is finite.

Since \( p < n + 1 \), we can choose \( A \) sufficiently large such that \( \det D^2 w \geq w^{p-1}R(x) \) in \( \Omega \).
This means that \( w \) is a subsolution of (4.6) and \( w = \phi \) on \( \partial \Omega \). Last step is to use the Perron method, which requires \( p \geq 1 \). Denote \( \Phi \) by the set of all subsolutions of (4.6) and (4.5), and let \( u(x) = \sup \{ \tilde{u}(x) : \tilde{u} \in \Phi \} \). One can easily verify that \( u \) is a generalised solution of (4.6). Since \( w \in \Phi \) we conclude \( u = \phi \) on \( \partial \Omega \). \( \square \)

Lemma 4.3. Suppose that there exists a positive function \( h \) in \( (0, r_0] \) satisfying
\[
(4.14) \quad \int_0^{r_0} h(t) \, dt = \infty,
\]
such that
\[
R(x_b, d) \geq h(d).
\]
Then the solution \( u \) produced by the above lemma satisfies (4.4).
Proof. For this proof we need an upper barrier function. Recall that $\phi > 0$ on $\partial \Omega$. Introduce the function $v$ as before, where $g$ in (4.8) is now replaced by $h$, namely
$$v(x) = \rho(d) = -\int_0^d \left( \int_s^{r_0} h(t) dt \right)^{1/n} ds, \quad x \in \Omega \setminus \Omega_{r_0}.$$ Then $|Dv(x)| \to \infty$ as $x \to \partial \Omega$. Extend $v$ to $\Omega_{r_0}$ as in the previous proof and then modify $v$ to get a uniformly convex function $\tilde{v} \in C^2(\Omega)$ so that $\tilde{v} = v$ in $\Omega \setminus \Omega_{r_0}/2$.

For any point $x_0 \in \partial \Omega$ we shall assume $x_0 = 0$ and the positive $x_n$-axis is in the inner normal direction. Define
$$\hat{v}(x) := \eta \tilde{v}(x) + \phi(0) + x \cdot D\phi(0) + Kx_n, \quad x \in \Omega,$$ where $K > 0$ is a constant. As $\phi \in C^2(\Omega)$ and $\partial \Omega \in C^2$ is uniformly convex, we can choose $K$ large enough such that $\hat{v} \geq \phi$ on $\partial \Omega$ and $\hat{v}(x_0) = \phi(x_0)$. Then by choosing $\eta > 0$ small enough, we also have $\hat{v} \geq v_0 > 0$ in $\Omega$.

Using similar computations as before, we have
$$\det D^2 \hat{v} \leq \eta^n \det D^2 \tilde{v} \leq \left\{ \begin{array}{ll}
\eta^n \varepsilon & \text{in } \Omega_{r_0} \\
\eta^n C_1 h & \text{in } \Omega \setminus \Omega_{r_0},
\end{array} \right.$$ where $C_1$ is a constant depending on $n, r_0$ and $\partial \Omega$. For the right hand side we have
$$\hat{v}^{p-1} R(x) \geq v_0^{p-1} R(x) \geq \left\{ \begin{array}{ll}
v_0^{p-1} R & \text{for } x \in \Omega_{r_0} \\
v_0^{p-1} R(x) & \text{for } x \in \Omega \setminus \Omega_{r_0},
\end{array} \right.$$ where $R = \inf_{x \in \Omega_{r_0}} R(x)$ is positive and finite.

Therefore, by choosing $K$ sufficiently large and $\eta$ sufficiently small, using the comparison principle [12] we obtain $\hat{v} \geq u$ in $\Omega$. Hence $|Du(x)| \to \infty$ as $x \to x_0$. \hfill \Box

4.1.2. The case of $p > n + 1$. To obtain existence, we adopt a different approach of constructing subsolutions. Let’s define

$$\rho(d) = -\int_0^d \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds.$$ Assume $\phi = \phi_0 > 0$ is a constant on $\partial \Omega$. Define
$$v(x) = (-A\rho(d) + \phi_0^{1/\delta})^{-\delta}, \quad \text{in } \Omega \setminus \Omega_{r_0},$$ where $\delta > 0$, $A > 0$ are constants to be determined.

Setting $\delta = \frac{n}{p-n-1}$, by computation we have
$$\det D^2 v \geq A^n \delta^n v^{\frac{n(\delta+1)}{\delta}} \prod_{i=1}^{n-1} \frac{\kappa_i}{1 - \kappa_i d}(\rho')^{n-1} \rho^n \geq A^n \delta^n C(n, \Omega) g(d) v^{p-1},$$
where $C$ is a constant depending only on $n$ and $\Omega$. Choosing $A$ sufficiently large and by extending $v$ inside $\Omega_{r_0}$ as before, we then obtain a subsolution. Note that $0 < v \leq \phi_0$ in $\Omega$ and $v = \phi_0$ on $\partial \Omega$. The existence of solution $u$ thus follows by the Perron process.

For a general $\phi > 0$ on $\partial \Omega$, we need modify $v$ in (4.17). For a point $x_0 \in \partial \Omega$, we may assume $x_0 = 0$ and the positive $x_n$-axis is in the inner normal direction. Let $\phi_0 = \phi(0)$.

Define

$$
(4.19) \quad v(x) = (-A\rho(d) + \phi_0 \cdot \frac{1}{\delta} \cdot \frac{1}{\delta - 1} \cdot x \cdot D\phi(0)) - \frac{1}{\delta - 1} \cdot \phi_0 \cdot \frac{1}{\delta - 1} \cdot x \cdot D\phi(0), \quad \text{in } \Omega \setminus \Omega_{r_0},
$$

where $K > 0$ is chosen sufficiently large such that $v \leq \phi$ on $\partial \Omega$ and $v = \phi$ at $x_0$. By choosing $\delta = \frac{n}{p-n-1}$ and $A$ sufficiently large as above, we have $v$ is a solution. Therefore, the existence of solution $u$ follows.

For completeness, Lemma 5.2 applies in this case, so we have $|Du(x)| \to \infty$ as $x \to \partial \Omega$, and obtain the completeness.

4.2. Type II. Next, we consider type II hypersurfaces. In this case, we investigate the entire solution of (4.3), i.e.

$$
(4.20) \quad \det D^2 u = (1 + |x|^2)^{-\frac{n+p+1}{2}} \cdot f \left( \frac{x, -1}{\sqrt{1 + |x|^2}} \right) u^{p-1} \quad \text{in } \Omega = \mathbb{R}^n.
$$

When $p > n + 1$, we prove the existence of a solution by constructing suitable upper and lower barriers.

Assume that $f$ satisfies the asymptotic growth condition

$$
(4.21) \quad f(x) \sim (1 + |x|^2)^q \quad \text{as } x \to \infty,
$$

where $q \in (0, 1)$ is a constant. Note that this is equivalent to $f(X) \sim |X_{n+1}|^{-2q}$. We remark that it is necessary to have a growth condition on $f$. Otherwise, by integration one can see that when $\Omega^*$ is bounded, $f$ is bounded, there doesn’t exist a complete noncompact hypersurface $M$ satisfying (4.20), (see [32] for the case of $f \equiv 1$). To prove this claim, it is convenient to use Equation (2.25) for the dual function $u^*$. In that case, $M$ is a graph of $u^*$ over $\Omega^*$. If $\Omega^*$ is bounded, let $y_0 \in \partial \Omega^*$. Since $u^*$ is convex, there exists a constant $C_0 > 0$ such that

$$
\quad u^* \geq -C_0 \quad \text{on } \Omega^* \cap B_1(y_0).
$$

Let $P \in \tilde{M} := M \cap (B_1(y_0) \times \mathbb{R})$. Assuming $M$ is a complete, noncompact hypersurface, we compute its support function $H$ at $P$ and have

$$
H|_P = \frac{y \cdot Du^* - u^*}{\sqrt{1 + |Du^*|^2}} \leq |y| + C_0 \leq 1 + |y_0| + C_0.
$$
Consequently, $H^{-1} \geq c_0 > 0$ in a neighbourhood $G \subset \tilde{M}$. Similarly to the nonexistence Lemmas 3.1 and 4.1, by integrating (4.1) we obtain

$$c_0^{p-1} \mathcal{H}^n(G) \leq \int_D f K d\mu = \int_D f dx,$$

where $D \subset S^n$ is the spherical image of $G$, $d\mu = K^{-1} dx$ is the area measure, and $dx$ is the spherical measure. As $\mathcal{H}^n(G) = \infty$, so the function $f$ cannot be bounded.

From (4.20) and (4.21), an upper (or lower) barrier is a function satisfying

$$\det D^2 u \leq (1 + |x|^2)^{-\gamma} w^{p-1} \quad (\text{or } \geq)$$

as $x \to \infty$, where $\gamma := \frac{n+p+1}{2} - q$. In a bounded domain, one can always construct such a barrier by rescaling $u$ to $\lambda u$ for a suitable constant $\lambda$, provided $p \neq n + 1$.

Now, let’s consider the function $w(x) = (1 + |x|^2)^{\delta}$ where $\delta > 1/2$ is to be chosen. Clearly $w$ is a convex function.

By computations

$$D_{ij} w = 2\delta(1 + |x|^2)^{\delta-1} \delta_{ij} + 4\delta(\delta - 1)(1 + |x|^2)^{\delta-2} x_i x_j,$$

where $\delta_{ij}$ is the Kronecker delta. Hence,

$$\det D^2 w = (2\delta)^n (1 + |x|^2)^n(\delta-1) \left( \frac{1 + (2\delta - 1)|x|^2}{1 + |x|^2} \right)^{n(\delta-1)} = C(n, \delta) w^{\frac{n(\delta-1)+\gamma}{\delta}} (1 + |x|^2)^{-\gamma},$$

where $C(n, \delta)$ is a positive constant bounded by $C_1 \leq C(n, \delta) \leq C_2$, and

$$C_1 := (2\delta)^n \inf_{x \in \mathbb{R}^n} \left\{ \frac{1 + (2\delta - 1)|x|^2}{1 + |x|^2} \right\},$$

$$C_2 := (2\delta)^n \sup_{x \in \mathbb{R}^n} \left\{ \frac{1 + (2\delta - 1)|x|^2}{1 + |x|^2} \right\}.$$ Choose $\delta = (\gamma - n)/(p - n - 1)$ such that $\frac{n(\delta-1)+\gamma}{\delta} = p - 1$. One can see that as far as $q < 1$,

$$\delta > \frac{n+p+1}{2} - \frac{n-1}{p-n-1} = \frac{1}{2}.$$ By a rescaling we obtain that $w = \mu w$ is a convex supersolution of (4.20) for $\mu^{p-n-1} \geq C_1$, while $w = \mu w$ is a convex subsolution of (4.20) for $0 < \mu^{p-n-1} \leq C_2$. Since $\mu \leq \mu$, we have $w \leq \overline{w}$. Let $\phi$ be any smooth function such that $w \leq \phi \leq \overline{w}$ in $\mathbb{R}^n$. By [7] the Dirichlet problem

$$\det D^2 w_k = (1 + |x|^2)^{-\frac{n+p+1}{2}} f w_k^{p-1} \quad \text{in } B_{2^k}(0),$$

$$w_k = \phi \quad \text{on } \partial B_{2^k}(0),$$
has a unique convex solution \( w_k \in C^\infty(B_{2k}) \), \( \underline{w} \leq w_k \leq \bar{w} \) in \( B_{2k} \). From this there exists a subsequence of \( \{w_k\} \) converges locally in any \( C^l \) form to a convex solution \( u \in C^\infty(\mathbb{R}^n) \) of (4.20), and \( \underline{w} \leq u \leq \bar{w} \) in \( \mathbb{R}^n \). Thus \( u(x)/\sqrt{1 + |x|^2} \to \infty \) as \( |x| \to \infty \), \( \Omega^* = Du(\Omega) = \mathbb{R}^n \), and hence the corresponding hypersurface \( M \) is complete.

In fact, any admissible solution \( u \) of (4.20) with \( Du(\Omega) = \mathbb{R}^n \) must satisfies (4.22)
\[
\frac{u(x)}{\sqrt{1 + |x|^2}} \to \infty \quad \text{as} \quad |x| \to \infty.
\]

Otherwise, if this is not true, there exists a sequence \( \{z_k\} \subset \Omega \) such that \( |z_k| \to \infty \) and for each \( k \), \( u(z_k) \leq C|z_k| \) for some constant \( C \). By choosing a subsequence and making a rotation of coordinates if necessary we may assume that \( z_k/|z_k| \to e_n = (0, \cdots, 0, 1) \). Let
\[
x_{n+1} = a_0 + \langle a, x \rangle = a_0 + \sum_{i=1}^{n} a_i x_i
\]
be the graph of any tangent hyperplane to graph \( u \). Then
\[
a_0 + \langle a, z_k \rangle \leq C|z_k|
\]
for each \( k \), so dividing by \( |z_k| \) and letting \( k \to \infty \) we obtain \( a_n \leq C \). This implies that \( Du(\Omega) \cap \{x_n > C\} = \emptyset \), which contradicts with \( Du(\Omega) = \mathbb{R}^n \). This proves (4.22).

Therefore, we have the following existence result, which is equivalent to Theorem 1.3.

**Theorem 4.1.** When \( p > n+1 \), \( f \) satisfies the asymptotic growth condition (4.21), there exists a complete noncompact hypersurface \( M \) whose support function is a solution of (4.20).

### 4.3 Type III.
This case can be handled similarly as in [10]. We observe that for a type III hypersurface, \( \Omega \) is of the form \( \omega \times \mathbb{R}^m \) for some \( m < n \), where \( \omega \) is a bounded convex domain in \( \mathbb{R}^{n-m} \). Near \( \partial \omega \) we may write \( \tilde{x} = (x_1, \cdots, x_{n-m}) = \tilde{x}_b + d_n(\tilde{x}_b) \) analogously as before. Correspondingly the boundary conditions (4.4) and (4.5) are imposed on \( \partial \omega \)
\[
(Du(x)) \to \infty \quad \text{as} \quad x \to \partial \omega,
\]
\[
u(x) = \phi(x) \quad x \in \partial \omega,
\]
where \( \phi \) is prescribed on \( \partial \omega \). Then by following the lines in §4.1, we have

**Theorem 4.2.** Let \( p \geq 1, \neq n+1 \), \( \Omega = \omega \times \mathbb{R}^m \), where \( \omega \) is a uniformly convex \( C^2 \)-domain in \( \mathbb{R}^{n-m} \). Suppose that \( \phi \) can be extended to \( \Omega \) so that \( D^2 \phi(x) \geq \delta_0 I \) for some positive constant \( \delta_0 \), where \( I \) is the identity matrix. Suppose moreover there exist two positive functions \( g \) and \( h \) defined in \( (0, r_0) \), \( r_0 > 0 \), satisfying
\[
\int_0^{r_0} \left( \int_s^{r_0} g(t) dt \right)^{1/(n-m)} ds < \infty,
\]
\[
\int_0^{r_0} h(t) dt = \infty,
\]
such that
\begin{equation}
(4.27) \quad h(d) \leq R(\tilde{x}_b, d) \leq g(d), \quad \text{where } \tilde{x} = \tilde{x}_b + dn(\tilde{x}_b),
\end{equation}

near \( \partial \Omega \). Then there exists a unique solution \( u \) of (4.3), (4.23) and (4.24) in \( C(\Omega) \cap C^{2,\alpha}(\Omega) \).

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