Quasihyperbolic Metric Universal Covers

David A. Herron and Gaven J. Martin

Abstract

We present a simple analytical proof that the natural metric universal cover of a quasihyperbolic planar domain is a complete Hadamard metric space.

1 Introduction

In an early paper [14], Pekka Koskela wrote about old and new results regarding the quasihyperbolic metric. Since then this metric has been a key player in much of Pekka’s work, either as a fundamental geometric tool or as the focus of his research. We anticipate that Pekka will find the following of interest.

Throughout this article \( \Omega \) denotes a quasihyperbolic plane domain in the complex plane \( \mathbb{C} \). Thus \( \Omega \) is open and connected and \( \Omega \neq \mathbb{C} \). Each such \( \Omega \) carries a quasihyperbolic metric \( \delta^{-1}ds = \delta_{\Omega}^{-1}ds \) whose length distance \( k = k_{\Omega} \) is called quasihyperbolic distance in \( \Omega \); here \( \delta(z) = \delta_{\Omega}(z) := \text{dist}(z, \partial \Omega) \) is the Euclidean distance from \( z \) to the boundary of \( \Omega \). See 2.2.1 below for more details.

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Quasihyperbolic distance in Euclidean domains was introduced by Gehring and Palka in [6] and has since become an indispensable tool in the program of metric space analysis. This paper is a companion to [3], [11], [12] where our program is to compare and contrast hyperbolic distance versus quasihyperbolic distance in planar domains.

A powerful tool available to hyperbolic geometers is the well-known fact that for any hyperbolic plane domain Ω, there is a holomorphic universal covering \((\mathbb{H}^2, \tilde{h}) \to (\Omega, h)\) which is a local isometry, where \((\mathbb{H}^2, \tilde{h})\) is the standard hyperbolic plane and \(h\) is hyperbolic distance in \(\Omega\). In [11, Theorem A] the first author established an analogous result for quasihyperbolic geometry. Let \(\tilde{\Omega} \xrightarrow{\Phi} \Omega\) be a holomorphic universal cover of \(\Omega\); topologically, \(\tilde{\Omega}\) is a simply connected surface. When \(\Omega\) is the once punctured plane \(\Omega = \mathbb{C} \setminus \{z_0\}\), \(\tilde{\Omega} = \mathbb{C}\) and \(\Phi(z) = e^z + z_0\); otherwise, we can take \(\tilde{\Omega}\) to be the unit disk. There is a unique length distance \(\tilde{k}\) on \(\tilde{\Omega}\) such that \((\tilde{\Omega}, \tilde{k}) \xrightarrow{\Phi} (\Omega, k)\) is a local isometry (see [2, Prop. 3.25, p.42] or [4, p.80]) and \(\tilde{k}\) is given by

\[
\tilde{k}(a, b) := \inf_{\gamma} \ell_{\tilde{k}}(\gamma) \quad \text{where} \quad \ell_{\tilde{k}}(\gamma) := \ell_k(\Phi \circ \gamma)
\]

and the infimum is taken over all rectifiable paths \(\gamma\) in \(\tilde{\Omega}\) with endpoints \(a, b\).

We give a simple proof of the following theorem that, essentially, relies solely on the fact that \(-\log \delta\) is always subharmonic in \(\Omega\).

**Theorem 1.1** For any quasihyperbolic plane domain \(\Omega\), \((\tilde{\Omega}, \tilde{k})\) is a Hadamard space.

Recall that a Hadamard space is a complete geodesic CAT(0) metric space.

As described in [10, Corollary B], there are some immediate consequences of the above.

**Corollary 1.2** Let \(\Omega\) be a simply connected quasihyperbolic plane domain. Then

1. \(\Omega, (\Omega, k)\) is Hadamard.

2. For each pair of points \(a, b\) in \(\Omega\), there is a unique quasihyperbolic geodesic with endpoints \(a, b\).

3. For each pair of points \(a, b\) in any quasihyperbolic plane domain, each homotopy class of paths in \(\Omega\) with endpoints \(a, b\) contains a unique quasihyperbolic geodesic.
In [10, Theorem C] we proved that the injectivity radius of \( (\tilde{\Omega}, \tilde{k}) \rightarrow (\Omega, k) \) is at least \( \pi \). Here, comparison theorems give quick bounds, but achieving the sharp bound requires more effort, so we do not attempt it here. We mention that already in [16, Corollary 3.6, p.44] Martin and Osgood proved that quasihyperbolic plane domains have non-positive generalized quasihyperbolic Gaussian curvature, and therefore the above Theorem is not so surprising.

2 Preliminaries

2.1 General Information

We view the Euclidean plane as the complex number field \( \mathbb{C} \). Everywhere \( \Omega \) is a quasihyperbolic plane domain. The open unit disk is \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \). The quantity \( \delta(z) = \delta_\Omega(z) := \text{dist}(z, \partial \Omega) \) is the Euclidean distance from \( z \in \mathbb{C} \) to the boundary of \( \Omega \), and \( 1/\delta \) is the scaling factor (that is the metric-density) for the so-called quasihyperbolic metric \( \delta^{-1} ds \) on \( \Omega \subset \mathbb{C} \); see §2.2.1.

2.2 Conformal Metrics

Each positive continuous function \( \rho : \Omega \to (0, +\infty) \) induces a length distance \( d_\rho \) on \( \Omega \) defined by

\[
d_\rho(a, b) := \inf_{\gamma} \ell_\rho(\gamma) \quad \text{where} \quad \ell_\rho(\gamma) := \int_\gamma \rho \, ds
\]

and where the infimum is taken over all rectifiable paths \( \gamma \) in \( \Omega \) that join the points \( a, b \). We describe this by calling \( \rho \, ds = \rho(z)|dz| \) a conformal metric on \( \Omega \). The geodesic regularity theorem of [15] shows that as soon as \( \rho \) is continuous and the metric is complete the infimum is attained by a curve with Lipschitz continuous first derivatives - a \( \rho \) geodesic.

When \( \rho \) is sufficiently differentiable, say \( C^2(\Omega) \), the Gaussian curvature of \((\Omega, d_\rho)\) is given by

\[
K_\rho := -\rho^{-2} \Delta \log \rho.
\]

In the plane, this curvature can also be defined for continuous metric densities through an integral formula for the Laplacian - though it is not always finite, see [10, §3]. This idea was first observed by Heins [8] who proved a version of the Schwarz lemma when the weak curvature was \( \leq -1 \).
2.2.1 The QuasiHyperbolic Metric & Distance

The quasi-hyperbolic distance \( k = k_\Omega \) in \( \Omega \) is the length distance \( k_\Omega := d_{\delta^{-1}} \) induced by the quasi-hyperbolic metric \( \delta^{-1}ds \) on \( \Omega \) where \( \delta(z) = \delta_\Omega(z) := \text{dist}(z, \partial \Omega) \) is the Euclidean distance from \( z \) to the boundary of \( \Omega \). Thanks to the Hopf-Rinow Theorem (see [2, p.35], [4, p.51], [17, p.62]) we know that \((\Omega, k)\) is a geodesic metric space. Indeed, any rectifiably connected non-complete locally compact metric space admits a quasi-hyperbolic metric that is geodesic. See also [5, Lemma 1] for Euclidean domains. Further, the basic estimates for quasi-hyperbolic distance were established by Gehring and Palka [6, 2.1]: For all \( a, b \in \Omega \),

\[
k(a, b) \geq \log \left( 1 + \frac{l(a, b)}{\delta(a) \wedge \delta(b)} \right) \geq \log \left( 1 + \frac{|a - b|}{\delta(a) \wedge \delta(b)} \right) \geq \left| \log \frac{\delta(a)}{\delta(b)} \right|
\]

where \( l(a, b) = l_\Omega(a, b) \) is the (intrinsic) Euclidean length distance between \( a \) and \( b \) in \( \Omega \). The first inequality above is a special case of the more general (and easily proven) inequality

\[
\ell_k(\gamma) \geq \log \left( 1 + \ell(\gamma)/\text{dist}(\gamma, \partial \Omega) \right)
\]

which holds for any rectifiable path \( \gamma \) in \( \Omega \). Using these estimates, we deduce that if \( k(a, b) \leq 1 \) or \( |a - b| \leq \frac{1}{2} \delta(a) \), then

\[
\frac{1}{2} k(a, b) \leq \frac{|a - b|}{\delta(a)} \leq 2k(a, b).
\]

These estimates reveal that \((\Omega, k)\) is complete.

In another direction, if \( \gamma : [0, L] \to \Omega \) is a quasi-hyperbolic geodesic in \( \Omega \) parameterized with respect to Euclidean arclength, then \( \gamma \) is differentiable (even at its endpoints) and for all \( s, t \in [0, L] \),

\[
|\gamma'(s) - \gamma'(t)| \leq 2 \frac{|\gamma(s) - \gamma(t)|}{\delta(\gamma(s))} \leq 2e \frac{1}{\delta(a)} |s - t|.
\]

Uniform estimates such as this, along with comparison principles, establish the convergence of geodesics in \( C^{1,1} \) for locally uniformly convergent sequences of complete continuous metric densities, in particular for our smoothing approach via mollifiers given below. We leave the reader to explore this.

As a well known example, well note that \((\mathbb{C}^*, k_*)\) is (isometric to) the Euclidean cylinder \( S^1 \times \mathbb{R} \) with its Euclidean length distance inherited from
its standard embedding into \( \mathbb{R}^3 \). One way to realize this is via the holomorphic covering \( \exp : \mathbb{C} \to \mathbb{C}^* \) which pulls back the quasihyperbolic metric \( \delta^{-1}_{-1} ds \) on \( \mathbb{C}^* \) to the Euclidean metric on \( \mathbb{C} \), as explained in [16]. In particular, quasihyperbolic geodesics in \( \mathbb{C}^* \) are logarithmic spirals, for all \( a, b \in \mathbb{C}^* \),

\[
k_*(a, b) = |\log(b/a)| = |\log |b/a| + i \arg(b/a)|,
\]

and \( \exp : (\mathbb{C}, |\cdot|) \to (\mathbb{C}^*, k_*) \) is a metric universal cover. Thus our Theorem holds when \( \Omega \) is a once punctured plane.

2.3 CAT(0) Metric Spaces

Here our terminology and notation conforms exactly with that in [2] and we refer the reader to this delightful trove of geometric information about non-positive curvature, and also see [4]. We recall a few fundamental concepts, mostly copied directly from [2]. Throughout this subsection, \( X \) is a geodesic metric space; for example, \( X \) could be a quasihyperbolic plane domain with its quasihyperbolic distance, or a closed rectifiably connected plane set with its intrinsic length distance.

2.3.1 Geodesic and Comparison Triangles

A geodesic triangle \( \Delta \) in \( X \) consists of three points in \( X \), say \( a, b, c \in X \), called the vertices of \( \Delta \) and three geodesics, say \( \alpha : a \bowtie b, \beta : b \bowtie c, \gamma : c \bowtie a \) (that we may write as \([a, b], [b, c], [c, a] \)) called the sides of \( \Delta \). We use the notation

\[
\Delta = \Delta(\alpha, \beta, \gamma) \quad \text{or} \quad \Delta = [a, b, c] := [a, b] \star [b, c] \star [c, a] \quad \text{or} \quad \Delta = \Delta(a, b, c)
\]

depending on the context and the need for accuracy.

A Euclidean triangle \( \bar{\Delta} = \Delta(\bar{a}, \bar{b}, \bar{c}) \) in \( \mathbb{C} \) is a comparison triangle for \( \Delta = \Delta(a, b, c) \) provided \( |a - b| = |\bar{a} - \bar{b}|, |b - c| = |\bar{b} - \bar{c}|, |c - a| = |\bar{c} - \bar{a}| \). We also write \( \Delta = \Delta(a, b, c) \) when a specific choice of \( \bar{a}, \bar{b}, \bar{c} \) is not required. A point \( \bar{x} \in [\bar{a}, \bar{b}] \) is a comparison point for \( x \in [a, b] \) when \( |x - a| = |\bar{x} - \bar{a}| \).

2.3.2 CAT(0) Definition

A geodesic triangle \( \Delta \) in \( X \) satisfies the CAT(0) distance inequality if and only if the distance between any two points of \( \Delta \) is not larger than the
Euclidean distance between the corresponding comparison points; that is,
\[ \forall x, y \in \Delta \text{ and corresponding comparison points } \bar{x}, \bar{y} \in \bar{\Delta}, \quad |x - y| \leq |\bar{x} - \bar{y}|. \]

A geodesic metric space is CAT(0) if and only if each of its geodesic triangles satisfies the CAT(0) distance inequality.

A complete CAT(0) metric space is called a Hadamard space. A geodesic metric space \( X \) has non-positive curvature if and only if it is locally CAT(0), meaning that for each point \( a \in X \) there is an \( r > 0 \) (that can depend on \( a \)) such that the metric ball \( B(a; r) \) (endowed with the distance from \( X \)) is CAT(0).

Each sufficiently smooth Riemannian manifold has non-positive curvature if and only if all of its sectional curvatures are non-positive; see [2, Theorem 1A.6, p.173]. In particular, if \( \rho ds \) is a smooth conformal metric on \( \Omega \) with \( K_\rho \leq 0 \), then \( (\Omega, d_\rho) \) has non-positive curvature.

### 2.4 Smoothing

As usual, we start with a \( C^\infty(\Omega) \) smooth \( \eta : \mathbb{C} \to \mathbb{R} \) with \( \eta \geq 0, \eta(z) = \eta(|z|) \), the support of \( \eta \) lies in \( \mathbb{D} \), and \( \int_{\mathbb{C}} \eta = 1 \). For each \( \varepsilon > 0 \) we set \( \eta_\varepsilon(z) := \varepsilon^{-2} \eta(z/\varepsilon) \). The regularization (or mollification) of an \( L^1_{loc}(\Omega) \) function \( u : \Omega \to \mathbb{R} \) are the convolutions \( u_\varepsilon := u \ast \eta_\varepsilon \), so
\[
    u_\varepsilon(z) := \int_{\mathbb{C}} u(w) \eta_\varepsilon(z - w) dA(w),
\]

which are defined in \( \Omega_\varepsilon := \{ z \in \Omega : \delta(z) > \varepsilon \} \). It is well known that \( u_\varepsilon \in C^\infty(\Omega_\varepsilon) \) and \( u_\varepsilon \to u \) as \( \varepsilon \to 0^+ \) where this convergence is:

- pointwise at each Lebesgue point of \( u \), locally uniformly in \( \Omega \) if \( u \) is continuous in \( \Omega \), and in \( L^p_{loc}(\Omega) \) if \( u \in L^p_{loc}(\Omega) \). Moreover, if \( u \) is subharmonic in \( \Omega \), then so is each \( u_\varepsilon \). See for example [7, Proposition I.15, p.235] or [18, Theorem 2.7.2, p.49].

### 3 Proof of Theorem

Let \( \Omega \subset \mathbb{C} \) be a planar domain. The main idea is to approximate \((\Omega, k)\) by metric spaces \((\Omega_\varepsilon, d_\varepsilon)\) that all have non-positive curvature. Then a limit argument, exactly like that used in the proof of [10, Theorem A], gives the asserted conclusion.
We start with the well known fact that \( u(z) := -\log \delta(z) \) is subharmonic in \( \Omega \). This is easy to see. Evidently, \( u \) is continuous. For each fixed \( \zeta \in \partial \Omega \), \( z \mapsto -\log |z - \zeta| \) is harmonic in \( \mathbb{C} \setminus \{ \zeta \} \supset \Omega \) and so has the mean value property in \( \Omega \). It follows that \( u \) has the submean value property in \( \Omega \).

Let \( u_\varepsilon := u \ast \eta_\varepsilon \) be the regularization of \( u \) as described at (2.4). Thus \( u_\varepsilon \) is defined, \( C^\infty \) smooth, and subharmonic (so, \( \Delta u_\varepsilon \geq 0 \)) in \( \{ z \in \Omega : \delta(z) > \varepsilon \} \). Moreover, \( u_\varepsilon \to u \) as \( \varepsilon \to 0^+ \) locally uniformly in \( \Omega \).

We assume that \( \Omega \) is not a once punctured plane and that the origin lies in \( \Omega \). Put \( \varepsilon_n := \delta(0)/n, u_n := u_{\varepsilon_n} \), and let \( \Omega_n \) be the component of \( \{ z \in \Omega : \delta(z) > \varepsilon_n \} \) that contains the origin. Then \( \{ \Omega_n \}_{n=1}^\infty \) Carathéodory kernel converges to \( \Omega \) with respect to the origin.

Next, let \( \rho_n := e^{u_n} \). Then \( \rho_n > 0 \) and \( C^\infty \) in \( \Omega_n \). Let \( d_n := d_{\rho_n} \) be the length distance in \( \Omega_n \) associated to the conformal metric \( \rho_n \, ds \). Since \( u_n \) is subharmonic in \( \Omega_n \), \( \rho_n \, ds \) has Gaussian curvature

\[
K_{\rho_n} = -\rho_n^{-2} \Delta \log \rho_n \leq 0 \quad \text{in } \Omega_n.
\]

It follows that the metric spaces \( (\Omega_n, d_n) \) all have non-positive curvature; see [10] Theorem 1A.6, p.173.

Let \( \Phi : \mathbb{D} \to \Omega \) and \( \Phi_n : \mathbb{D} \to \Omega_n \) be holomorphic covering projections with \( \Phi(0) = 0 = \Phi_n(0) \) and \( \Phi'(0) > 0, \Phi'_n(0) > 0 \). Since \( \{ \Omega_n \}_{n=1}^\infty \) Carathéodory kernel converges to \( \Omega \) with respect to the origin, a theorem of Hejhal's [9, Theorem 1] (see also [1, Corollary 5.3]) asserts that \( \Phi_n \to \Phi \), so also \( \Phi'_n \to \Phi' \), locally uniformly in \( \mathbb{D} \).

Let \( \tilde{k}, \tilde{d}_n \) be the \( \Phi, \Phi_n \) lifts of the distances \( k, d_n \) on \( \Omega, \Omega_n \) respectively. That is, \( \tilde{k} \) and \( \tilde{d}_n \) are the length distances on \( \mathbb{D} \) induced by the pull backs

\[
\tilde{\rho} \, ds := \Phi^* [\delta^{-1} \, ds] \quad \text{and} \quad \tilde{\rho}_n \, ds := \Phi_n^* [\rho_n \, ds]
\]

of the metrics \( \delta^{-1} \, ds \) and \( \rho_n \, ds \) in \( \Omega \) and \( \Omega_n \) respectively. Thus, for \( \zeta \in \mathbb{D} \),

\[
\tilde{\rho}(\zeta) \, |d\zeta| = \frac{|\Phi'(\zeta)|}{\delta(\Phi(\zeta))} |d\zeta| \quad \text{and} \quad \tilde{\rho}_n(\zeta) \, |d\zeta| = |\Phi'_n(\zeta)| \rho_n(\Phi_n(\zeta)) |d\zeta|
\]

and \( \Phi : (\mathbb{D}, \tilde{k}) \to (\Omega, k), \Phi_n(\mathbb{D}, \tilde{d}_n) \to (\Omega_n, d_n) \) are metric universal coverings.

Note that as \( (\Omega_n, d_n) \) has non-positive curvature, the Cartan-Hadamard Theorem [2] Chapter II.4, Theorem 4.1, p.193] asserts that \( (\mathbb{D}, \tilde{d}_n) \) is CAT(0).

Using the locally uniform convergences of \( \rho_n \to \delta^{-1} \) and \( \Phi_n \to \Phi, \Phi'_n \to \Phi' \) (in \( \Omega \) and \( \mathbb{D} \) respectively) we deduce that \( \tilde{\rho}_n \, ds \to \tilde{\rho} \, ds \) locally uniformly in \( \mathbb{D} \).
This implies pointed Gromov-Hausdorff convergence of \((\mathbb{D}, \tilde{d}_n, 0)\) to \((\mathbb{D}, \tilde{k}, 0)\) (see the proof of [13, Theorem 4.4]) which in turn says that \((\mathbb{D}, \tilde{k})\) is a 4-point limit of \((\mathbb{D}, \tilde{d}_n)\) and hence, as each \((\mathbb{D}, \tilde{d}_n)\) is CAT(0), it follows that \((\mathbb{D}, \tilde{k})\) is CAT(0); see [2, Cor. 3.10, p.187; Theorem 3.9, p.186]. Finally, it is a routine matter to check that \((\mathbb{D}, \tilde{k})\) is complete; for instance, see [4, Exercise 3.4.8, p. 80].

\[\square\]

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DAH, Department of Mathematical Sciences, University of Cincinnati, OH 45221-0025, USA  
email: David.Herron@UC.edu

GJM, Institute for Advanced Study, Massey University, Auckland, New Zealand  
email: G.J.Martin@Massey.ac.NZ