THE SPACE OF MERGING SUBMANIFOLDS IN $\mathbb{R}^n$

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Abstract. We compute the homotopy type of the space of possibly empty proper $d$-dimensional submanifolds of $\mathbb{R}^n$ with a topology coming from a Hausdorff distance. Our methods give also a different proof of the Galatius–Randal-Williams theorem on the homotopy type of their space of submanifolds.

1. Introduction

In [Galatius et al., 2009], the classifying space of the $d$-dimensional cobordism category was found to be homotopy equivalent to a delooping of the infinite loop space of the affine Grassmannian $\gamma_{d,n}^\perp$ of $d$-planes in $\mathbb{R}^n$, seen as a vector bundle over the linear Grassmannian $Gr_d(\mathbb{R}^n)$. This was proven again with different methods by [Galatius and Randal-Williams, 2010], who introduced the space $\Psi_d(\mathbb{R}^n)$ of submanifolds of $\mathbb{R}^n$. A crucial step in their proof was the following

Theorem. The inclusion $\Theta(\gamma_{d,n}^\perp) \hookrightarrow \Psi_d(\mathbb{R}^n)$ is a homotopy equivalence.

$\Psi_d(\mathbb{R}^n)$ is a topological space whose points are possibly empty proper $d$-submanifolds of $\mathbb{R}^n$ (in the sense that the intersection of such submanifold with a compact subset of $\mathbb{R}^n$ is compact). Before describing its topology, let us discuss what natural topologies one can put in its underlying set $\psi_d(\mathbb{R}^n)$:

If $W$ is a proper submanifold of $\mathbb{R}^n$, and we identify $\mathbb{R}^n \cup \{\infty\} \cong S^n$, then $W \cup \{\infty\}$ is a compact subset of $S^n$, so one has an inclusion

$\psi_d(\mathbb{R}^n) \hookrightarrow K(S^n)$

into the space of non-empty compact subsets of $S^n$ with the Hausdorff distance. The induced topology on $\psi_d(\mathbb{R}^n)$ is coarser than the one defined by Galatius and Randal-Williams, in part because it does not take into account the smoothness of the submanifolds. Instead one can include

$\psi_d(\mathbb{R}^n) \hookrightarrow K(S^n \times Gr_d(\mathbb{R}^n)/\{\infty\} \times Gr_d(\mathbb{R}^n))$ \hspace{1cm} (1.1)

using the affine Gauss map, that sends a submanifold to the collection of pointed affine tangent planes to it together with the point at infinity. This takes into account the $C^1$-information of the submanifolds. One could also improve the target of this inclusion to account for the complete $C^\infty$-information of the submanifold. We denote by $\tilde{\Psi}_d(\mathbb{R}^n)$ the set $\psi_d(\mathbb{R}^n)$ endowed with the topology induced by the Hausdorff distance on the right hand-side, and we refer to it as the space of merging submanifolds of $\mathbb{R}^n$.

This topology is still coarser than the topology in $\Psi_d(\mathbb{R}^n)$, but it is very close to it: A sequence of compact submanifolds in $\tilde{\Psi}_d(\mathbb{R}^n)$ converging to a compact submanifold $W$ eventually takes values in covering spaces of $W$ for non-compact

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submanifolds there is an analogous condition). If we refine the topology $\tilde{\Psi}_d(\mathbb{R}^n)$ imposing these covering spaces to be single-sheeted (i.e., diffeomorphisms), then we arrive to the topology $\Psi_d(\mathbb{R}^n)$ defined by Galatius and Randal-Williams. All the topologies described so far give rise to topological sheaves over $\mathbb{R}^n$.

The purpose of this note is to find the homotopy type of the space $\tilde{\Psi}_d(\mathbb{R}^n)$.

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2. Spaces of submanifolds

We begin by recalling the definition of the topology in $\Psi_d(U)$, when $U$ is an open subset of $\mathbb{R}^n$ (Galatius and Randal-Williams, 2010 §2). For the sake of clearness, we will give a $C^1$-version of this topology.

Recall, for a submanifold $W$ of $U$, the partially defined function $\exp_W : NW \rightarrow U$ defined in a neighbourhood $B$ of the zero section $z$, where $NW$ is the normal bundle of $W$.

Definition. The space $\Psi_d(U)$ has as underlying set the collection of all proper $d$-dimensional submanifolds of $U$, together with the empty submanifold. Its topology is given by the following neighbourhood basis of any proper submanifold $W$:.

- If $W \neq \emptyset$, then every compact subset $K \subset U$ and every $\epsilon > 0$ define a basic neighbourhood $(K, \epsilon)^\Psi$ of $W$: a submanifold $W'$ belongs to $(K, \epsilon)^\Psi$ if there is a section $f$ of the normal bundle $NW \rightarrow W$ such that $\exp_W(f(W)) \cap K = W' \cap K$ and $\|f(x)\| + \|D(f-z)(x)\| < \epsilon$ for all $x \in W$ such that $\exp_W \circ f(x) \in W' \cap K$.

- If $W = \emptyset$, then every compact subset $K \subset \mathbb{R}^n$ defines a basic neighbourhood $(K)^\Psi$ of $\emptyset$: and a submanifold $W'$ belongs to $(K)^\Psi$ if $W' \cap K = \emptyset$. We now define the topology in $\tilde{\Psi}_d(\mathbb{R}^n)$, the only difference being that instead of requiring $W' \cap K$ to be the image of a global section of $NW$, we only ask it to be the union of images of local sections of $NW$ whose domains cover $W$. The fact that this topology agrees with the one coming from the Hausdorff distance in (1,1) is shown in Cantero, 2013 §4.3.

Definition. The space $\tilde{\Psi}_d(U)$ has the same underlying set as $\Psi_d(U)$, with neighbourhood basis of a proper submanifold $W$:

- If $W \neq \emptyset$, then every compact subset $K \subset U$ and every $\epsilon > 0$ define a basic neighbourhood $(K, \epsilon)^\tilde{\Psi}$ of $W$: a submanifold $W'$ belongs to $(K, \epsilon)^\tilde{\Psi}$ if there is a subset $Q \subset NW$ such that the composite $Q \subset NW \rightarrow W$ is a covering map and $\exp_W(Q) \cap K = W' \cap K$.

- If $W = \emptyset$, then every compact subset $K \subset \mathbb{R}^n$ defines a basic neighbourhood $(K)^\tilde{\Psi}$ of $\emptyset$: and a submanifold $W'$ belongs to $(K)^\tilde{\Psi}$ if $W' \cap K = \emptyset$.

Remarks. (1) One can also define the topology in $\tilde{\Psi}(U)$ following the definition of the topology in $\Psi(U)$, but replacing the normal bundle of $W$ by the fibrewise Ran space on the normal bundle of $W$ and imposing the cardinality of the image of the section to be locally constant.
Let $\text{Gr}_d(\mathbb{R}^n)$ be endowed with its natural metric. Then from the definitions above it follows that if $W'' \in (K, \varepsilon)^\Psi$ (respectively, $W'' \in (K, \varepsilon)^\Psi$) and $f$ is a global (resp. local) section defining $W''$, then

\[ d(x, f(x)) + d(T_xW, T_{f(x)}W'') < \varepsilon \quad (2.1) \]

for all $x \in W$ such that $\exp_W \circ f(x) \in W' \cap K$.

(3) Condition (2) in both definitions says that $f$ is $\varepsilon$-close to the zero section in the $C^1$-topology. One can instead impose that condition in the $C^\infty$-topology. This would give, in the case of $\Psi_d(U)$, the actual definition given by Galatius and Randal-Williams. Both definitions give homotopy equivalent spaces.

(4) When $d = 0$, the subspace of $\Psi_0(\mathbb{R}^n)$ consisting of 0-submanifolds contained in the unit ball, is the unordered configuration space on the unit ball, whereas that subspace in $\Psi_0(\mathbb{R}^n)$ is the Ran space of the unit ball.

Theorem. Let $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$ be the subspace of $\tilde{\Psi}_d(\mathbb{R}^n)$, consisting only on unions of parallel affine planes, together with the empty set. The inclusion $i: \tilde{\mathcal{L}}_d(\mathbb{R}^n) \to \Psi_d(\mathbb{R}^n)$ is a homotopy equivalence and the space $\tilde{\mathcal{L}}_d(\mathbb{R}^n)$ is weakly contractible.

The strategy to construct a homotopy inverse to $i$ will be the following: We continuously change $W$ through a homotopy $H_t$ that ends in a submanifold $V = H_0(W)$ whose tangent planes are close to each other, seen as points in $\text{Gr}_d(\mathbb{R}^n)$. To the submanifold $V$ we can continuously assign a plane $\mu(V)$ obtained by averaging all tangent planes of $V$. The plane $\mu(V)$ will have another property: every tangent plane to $V$ is transverse to $\mu(V)^{\perp}$. Finally, we stretch out $V$ in the direction $\mu(V)$, obtaining several parallel copies of $\mu(V)$, one for each point in $\mu(V)^{\perp} \cap V$.

Remarks. (1) The argument here applies as well to give a different proof of the Galatius–Randal-Williams theorem, as the analogous subspace $\mathcal{L}_d(\mathbb{R}^n) \subset \Psi_d(\mathbb{R}^n)$ is easily seen to be homotopy equivalent to $\text{Th}(\gamma_d, n)$.

(2) The arguments in Galatius and Randal-Williams, 2010 §3 are enough to show that if one topologises the cobordism category in that paper using the topology $\tilde{\Psi}_d(\mathbb{R}^n)$ instead of $\Psi_d(\mathbb{R}^n)$, then its classifying space is contractible.

(3) The assignment $\tilde{\Psi}$ defines a sheaf on the site of manifolds and open embeddings, and in Section 3 we show that this sheaf is not microflexible, so the methods of Randal-Williams, 2011 §3-6 do not generalize to this situation.

3. Proof

3.1. The action of a embedding space on the space of merging submanifolds. The next lemma can be proven following Galatius and Randal-Williams, 2010 §2.2 and making the appropriate modifications. Instead, we will take advantage of knowing that the topology of $\tilde{\Psi}_d(U)$ is induced by a Hausdorff metric.

Lemma. Let $U, V$ be open subsets of $\mathbb{R}^n$. Then the map

\[ \text{Emb}(U, V) \times \tilde{\Psi}_d(V) \to \tilde{\Psi}_d(U) \quad (3.1) \]

given by sending a pair $f, W$ to $f^{-1}(W)$ is continuous.

Here we take the $C^1$ compact-open topology on $\text{Emb}(U, V)$.

Proof. If we let $\overline{X}$ denote the one-point compactification of a locally compact space $X$, then there is a continuous map

\[ \text{OEmb}(X, Y) \to \text{map}(\overline{Y}, \overline{X}) \]
from the space of open embeddings of $X$ into $Y$ to the mapping space between $\overline{Y}$ and $\overline{X}$, both endowed with the compact-open topology (this is an adaptation of the second part of the proof of Theorem 4 in [Arens, 1946], c.f. [Cantero, 2014a]). It is given by sending an embedding $e$ to the map that sends a point $y$ to $e^{-1}(y)$ if the latter exists and to $\infty$ otherwise. Denote by $K(\overline{X})$ the set of non-empty compact subsets of $\overline{X}$ with the Hausdorff metric. There is a continuous map [Cantero, 2014b]

$$\text{map}(\overline{Y}, \overline{X}) \rightarrow \text{map}(K(\overline{Y}), K(\overline{X}))$$



to the mapping space between $K(\overline{Y})$ and $K(\overline{X})$ with the compact-open topology. Composing these two maps with the evaluation map, we obtain a continuous map

$$\text{Emb}(X, Y) \times K(\overline{Y}) \rightarrow K(\overline{X}).$$

Finally, if $B \subset K(\overline{Y})$ is a subspace and the latter map restricted to this subspace takes values in a subspace $A \subset K(\overline{X})$, then

$$\text{Emb}(X, Y) \times B \rightarrow A$$

is continuous as well. We now take $X = U \times \text{Gr}_d(\mathbb{R}^n), Y = V \times \text{Gr}_d(\mathbb{R}^n), A = \tilde{\Psi}_d(U)$ and $B = \tilde{\Psi}_d(V)$, obtaining a continuous map

$$\text{Emb}(U \times \text{Gr}_d(\mathbb{R}^n), V \times \text{Gr}_d(\mathbb{R}^n)) \times \tilde{\Psi}_d(V) \rightarrow \tilde{\Psi}_d(U)$$

and the lemma follows by precomposing with the map

$$\text{Emb}(U, V) \rightarrow \text{Emb}(U \times \text{Gr}_d(\mathbb{R}^n), V \times \text{Gr}_d(\mathbb{R}^n))$$

that sends an embedding $e$ to the embedding given by $(x, L) \mapsto (e(x), De(L))$. □

3.2. A more convenient assumption. Let $B^n$ be the open unit ball in $\mathbb{R}^n$. By the previous lemma, the diffeomorphism $f: B^n \rightarrow \mathbb{R}^n$ given by $f(x) = \frac{x}{\|x\|}$ induces a homeomorphism $\tilde{\Psi}_d(B^n) \rightarrow \tilde{\Psi}_d(\mathbb{R}^n)$. Therefore our theorem will be proven by showing that the inclusion $\tilde{L}_d(B^n) \rightarrow \tilde{\Psi}_d(B^n)$ is a homotopy equivalence, where $\tilde{L}_d(B^n) \cong f^{-1}(\tilde{L}_d(\mathbb{R}^n))$ consists on intersections of affine planes with the unit ball. The reason for this change is that it will be convenient to make use of conformal maps between $B^n$ and balls of smaller radius, and this is not possible if we take $\mathbb{R}^n$ instead.

3.3. Averaging the tangent planes of a submanifold of $B^n$. Recall that, for a submanifold $W$ of $B^n$, the Gauss map $W \rightarrow \text{Gr}_d(\mathbb{R}^n)$ sends a point to its tangent plane.

Let us fix once and for all a Riemannian metric on $\text{Gr}_d(\mathbb{R}^n)$ and write diam for diameter. A submanifold $V \in \tilde{\Psi}_d(B^n)$ is compactifiable if it is of the form $e^{-1}(W)$ for some relatively compact embedding $e: B^n \rightarrow B^n$ and some $W \in \tilde{\Psi}_d(B^n)$. A compactifiable submanifold has finite volume, so it makes sense to integrate over it.

For each non-empty compactifiable $W \in \tilde{\Psi}_d(B^n)$ and each plane $L \in \text{Gr}_d(\mathbb{R}^n)$, define

$$\lambda(W, L) = \frac{1}{2} \int_W (1 - \|x\|)dx \int_W (1 - \|x\|)d(L, T_xW)^2dx.$$ 

By [Karcher, 1977], Theorem 1.2, there exists a $\delta > 0$ (that depends only on the metric of $\text{Gr}_d(\mathbb{R}^n)$) such that if

$$\text{Gauss}(W) \subset B_\delta \quad \text{(for some ball $B_\delta$ of radius $\delta$)},$$

then the above function is convex in $B_\delta$. In this case, we define $\mu(W)$ to be the $L \in B_\delta$ that minimizes $\lambda(W, -)$, and observe that

$$d(\mu(W), T_xW) \leq \text{diam} \circ \text{Gauss}(W) \quad \text{for all } x \in W.$$  (3.3)
Proposition. The assignment \( \mu \) defines a continuous map from the subspace \( \tilde{\Psi}_d(B^n) \) of compactifiable submanifolds satisfying (3.2) to the space \( \text{Gr}_d(\mathbb{R}^n) \).

Proof. We will prove that the function \( \lambda(L, -) \) is continuous in the second variable. Since in our case the functions \( \lambda(-, W) \) have a unique minimum, it follows that \( \min_\lambda(L, -) \) is also continuous in the second variable.

Let \( W' \in (K_t, \epsilon) \tilde{\Psi} \), a neighbourhood of \( W \), where \( K_t \) is a disc of radius \( t \). Then there is a covering map \( q: Q \subset NW \to W \) such that

1. \( \exp_{W'}(Q) \cap K \subset W' \cap K \),
2. \( ||f(x)|| + ||D(f - Id)(x)|| < \epsilon \) for each local section \( f \) of the covering map.

Let \( W_t := W \cap K_t \) and let \( W'_t = q^{-1}(W_t) \subset W' \cap K_t \). Therefore we have that, if \( \nabla_W := \int_W (1 - ||x||)dx \), then

\[
\frac{1}{2\nabla_W} \int_{W'} (1 - ||x||)d(L, T_sW')^2dx = \frac{1}{2\nabla_W} \left( \int_{W'_t} (1 - ||x||)d(L, T_sW')^2dx + \int_{W'\setminus W_t} (1 - ||x||)d(L, T_sW')^2dx \right)
\]

\[
\frac{1}{2\nabla_W} \left( \int_{W'_t} (1 - ||x||)(d(L, T_q(x)W) + \alpha(x))^2dx + \beta(x) \right)
\]

\[
\frac{\#sheets of q}{2\nabla_W} \left( \int_{W'_t} (1 - ||x||)d(L, T_sW) + \alpha(x))^22\gamma(x)dx + \beta(x) \right)
\]

and \( |\alpha(x)| < d(T_sW', T_q(x)W) < \epsilon \) and \( 0 \leq \beta(x) \leq \nabla_W\setminus W_t(1 - t)\delta \), \( \gamma(\epsilon) = \det(Dq) \) and \( |\nu(x)| < \epsilon \). Therefore, when \( t \to 1 \) and \( \epsilon \to 0 \), the above integral converges to

\[
\frac{\#sheets \cdot \nabla_W}{2\nabla_W} \int_W (1 - ||x||)d(L, T_sW)^2dx
\]

and similarly \( \nabla_W \) tends to \( \#sheets \cdot \nabla_W \) as \( t \to 1 \) and \( \epsilon \to 0 \). \( \square \)

Definition. Let \( \alpha \) be smaller than the \( \delta \) provided by Karcher’s theorem, and also smaller than half the distance between any two planes \( P, Q \in \text{Gr}_d(\mathbb{R}^n) \) such that \( P' \cap Q' \neq 0 \).

Controlling the value \( \text{diam} \circ \text{Gauss}(W 
abla B(\epsilon)) \). Let \( F_{\geq}((0, 1)) \) be the set of non-decreasing functions from \([0, 1)\) to \([0, \infty)\) that preserve 0, and let \( C_{\geq}((0, 1)) \) be the subset of continuous increasing functions. We topologize this space as follows: For each pair of positive real numbers \((t, \epsilon)\) with \( \epsilon < t \), there is a basic neighbourhood \((t, \epsilon)^F\) of a function \( f \). Another function \( g \) is in the \((t, \epsilon)\)-neighbourhood of \( f \) if for all \( s < 1 - t \), \( f(s - \epsilon) - \epsilon \leq g(s) \leq f(s + \epsilon) + \epsilon \). The subspace \( C_{\geq}((0, 1)) \) inherits the compact-open topology.

Let us define

\[
\theta: \tilde{\Psi}_d(B^n) \to F_{\geq}((0, 1))
\]

as the adjoint of the map \( \tilde{\Psi}_d(B^n) \times [0, 1) \to [0, \infty) \) that sends a pair \((W, \epsilon)\) to \( \text{diam} \circ \text{Gauss}(W_\epsilon) \), where \( W_\epsilon = W \cap B(\epsilon) \) and \( B(\epsilon) \) is the ball of radius \( \epsilon \) centered at the origin.

Lemma. The map \( \theta \) is continuous.

Proof. Let \((t, \epsilon)^F\) be a neihbourhood of \( \psi(W) \). Let \( K(r) \subset \mathbb{R}^n \) be the closed disc with radius \( r = 1 - t \), and let \((K(r), \epsilon)\) be a neihbourhood of \( W \). If \( s < r \) and \( W' \in (D(s), \epsilon) \tilde{\Psi} \), then, by condition (1) and by (2.1) in the definition of the topology of \( \tilde{\Psi}_d(\mathbb{R}^n) \) the followig holds: for each \( y \in W'_t \), there is a \( x \in W \) at distance at
most $\epsilon$ (so $x \in W_{s+\epsilon}$), and such that $T_yW'$ and $T_xW$ are also at distance at most $\epsilon$. Therefore for all $s < r$:

$$\operatorname{diam} \circ \operatorname{Gauss}(W'_s) \leq \operatorname{diam} \circ \operatorname{Gauss}(W_{s+\epsilon}) + \epsilon.$$ 

Similarly, using (3.1),

$$\operatorname{diam} \circ \operatorname{Gauss}(W'_s) \geq \operatorname{diam} \circ \operatorname{Gauss}(W_{s-\epsilon}) - \epsilon.$$ 

As a consequence, if we write $f = \vartheta(W)$ and $g = \vartheta(W')$,

$$f(s - \epsilon) - \epsilon \leq g(s) \leq f(s + \epsilon) + \epsilon.$$ 

Hence $\vartheta((D(r), \epsilon) \Phi) \subset (t, \epsilon)^F$. \qed

The following lemma and its use was suggested to me by Abdó Roig.

**Lemma.** There is a continuous map $a: F_\geq(\mathbb{R}^+ \to C_\geq(\mathbb{R}^+)$ such that $a(f) \geq f$, $a(f)(x) \geq ax$ and $a(0)(x) = ax$.

**Proof.** The function $b(f)(x) = \frac{1}{x} \int_x^2 f(y)dy$ is continuous and non-decreasing. In addition, being $f$ non-decreasing, $b(f)(x) = \frac{1}{x}(f(x)(2x - x)) = f(x)$. To make it strictly increasing, first define $g(x) = ax$ and replace $f$ by $f + g$, which is always positive and bigger than $f$. Therefore $a(f) = b(f + g)$ gives the desired function. \qed

**Definition.** Let \( \varphi: \tilde{\Psi}_d(B^n) \to (0, \infty) \) the continuous function obtained applying \( \vartheta \), then \( a \), then taking the inverse of the resulting function, and finally evaluating that function on \( \alpha \). It has the following properties:

$$\operatorname{diam} \circ \operatorname{Gauss}(W_{\varphi(W)}) < \alpha \quad \text{for all } W \quad (3.4)$$
$$\varphi(W) = 1 \quad \text{if } W \text{ is a union of parallel planes.} \quad (3.5)$$
$$\varphi(W) < 1 \quad \text{if } W \text{ is not a union of parallel planes.} \quad (3.6)$$

**Proof of the first part.** Let \( f_{r,t} \in \operatorname{Emb}(B^n, B^n) \) be the isotopy of embeddings \( f_{r,t}(x) = (t + (1 - t)r)x \). Let \( H_t: \tilde{\Psi}_d(B^n) \to \tilde{\Psi}_d(B^n) \) be the induced homotopy that sends a submanifold \( W \) to \( f_{\varphi(W),t}^{-1}(W) \). This is a continuous map by (3.1).

Now, \( f_{r,t} \) is conformal, and this implies that \( \operatorname{Gauss}(W_{\varphi(W)}) = \operatorname{Gauss}(H_0(W)) \), and it follows from (3.4) that \( H_0(W) \) satisfies (3.2). In addition, \( H_0(W) \) is compactifiable as a consequence of (3.4) (observe that unions of affine planes are always compactifiable). Hence \( H_0(W) \in \tilde{\Psi}_d(B^n)^d \).

If \( p \in \operatorname{Gr}_d(\mathbb{R}^n) \), and \( \pi, \pi^\perp \) are the projections onto \( P \) and its orthogonal complement, then define an isotopy of embeddings \( g_{t,p}: B^n \to B^n \) by \( g_{t,p}(x) = t \cdot \pi(x) + \pi^\perp(x) \). Let \( G_t: \tilde{\Psi}_d(B^n)^d \to \tilde{\Psi}_d(B^n) \) be the induced homotopy that sends a submanifold \( W \) to \( g_{t,\mu(W)}^{-1}(W) \). Because \( \mu \) is continuous in \( \tilde{\Psi}_d(B^n)^d \) and because of (3.1), it follows that \( G_t \circ H_0 \) is continuous as well except possibly at \( t = 0 \).

The submanifold \( g_{t,\mu(W)}^{-1}(W) \) is well-defined for \( t = 0 \) if and only if \( W \) intersects \( \mu(W)^\perp \) transversally, in which case \( g_{0,\mu(W)}^{-1}(W) \) is a union of affine planes parallel to \( P \) whose origins (their closest points to the origin of \( B^n \)) are precisely \( \mu(W)^\perp \cap W \). On the other hand, the tangent planes of the submanifold \( V := H_0(W) \) are all very close because its diameter is bounded by \( \alpha \), and, by definition, close to \( \mu(V) \), and therefore they meet \( \mu(V)^\perp \) transversally: for any \( x \in W \) and any \( L \) with \( L \cap \mu(V)^\perp \neq 0 \), using (3.3) and (3.4)

$$d(T_yV, L) \geq d(\mu(V), L) - d(\mu(V), T_xV) \geq 2\alpha - \operatorname{diam} \circ \operatorname{Gauss}(V) = 2\alpha - \operatorname{diam} \circ \operatorname{Gauss}(W_{\varphi(W)}) > 2\alpha - \alpha = \alpha > 0.$$
Let $V$ meet $\mu(V)\perp$ transversally, $\gamma_0^{\mu(V)}(V)$ is well-defined, and so is $H_0 \circ G_0$.

Therefore $h := G_0 \circ H_0$ is well-defined and lands in $\tilde{L}_d(B^n)$, and $h \circ i$ is the identity (see [5]), whereas performing $H_t$ and then $G_t$ defines a homotopy between $i \circ h$ and the identity.

□

**Proof of the second part.** Let $C_d(R^n) \subset \tilde{L}_d(R^n)$ be the subspace of those non-empty unions of affine planes, all of whose origins are at distance at most 1 from the origin of $R^n$. This is a closed subset. Let $U \subset \tilde{L}_d(R^n)$ be the subspace of those (possibly empty) unions of planes that do not contain the origin. Then, there is a pushout square

$$
\begin{array}{ccc}
U \cap C_d(R^n) & \longrightarrow & U \\
\downarrow & & \downarrow \\
C_d(R^n) & \longrightarrow & \tilde{L}_d(R^n)
\end{array}
$$

which is also a homotopy pushout square because the upper horizontal arrow is a cofibration. Now, a point in $C_d(R^n)$ is a collection of parallel planes, and remembering the underlying linear plane defines a map $C_d(R^n) \to Gr_d(R^n)$ that is also a fibre bundle. Its fibre over a plane $P$ is the space $C_d(P^\perp)$, which is the Ran space of $P^\perp$ and is well-known to be weakly contractible [Gaitsgory, 2013, Appendix]. Therefore $C_d(R^n) \simeq Gr_d(R^n)$. The same argument proves that $U \cap C_d(R^n) \simeq Gr_d(R^n)$, and since the left vertical map is a map over $Gr_d(R^n)$, it is a weak homotopy equivalence. On the other hand, $U$ is contractible as the homotopy $(W,t) \mapsto \overline{\gamma(V \cup t)}$ defines a contraction of $U$ to the empty submanifold. As a consequence, the homotopy pushout $\tilde{L}_d(R^n)$ is weakly contractible as well. □

4. Microflexibility

The spaces $\Psi_d(U)$ and $\tilde{\Psi}_d(U)$ glue together to form sheaves $\Psi$ and $\tilde{\Psi}$ on $R^n$: If $U \subset U'$ is a pair of open subsets then the restriction map $\tilde{\Psi}_d(U') \to \tilde{\Psi}_d(U)$ sends a submanifold $W$ to the intersection $W \subset W'$. A sheaf of topological spaces in $R^n$ extends canonically to a sheaf of topological spaces in the site of manifolds and open embeddings [Randal-Williams, 2011, Theorem 3.3].

At this point one is tempted to use the methods in the latter article to extend our theorem to the space of merging submanifolds in an arbitrary open manifold: In that article, it was proven that the sheaf $\Psi$ on a manifold $M$ is $\text{Diff}(M)$-equivariant and that it is microflexible. By a theorem of Gromov [Gromov, 1986], this automatically implies that for connected non-compact manifolds $M$ a certain map

$$
\Psi(M) \longrightarrow \Gamma(\Psi^{\text{fib}}(TM) \to M)
$$

is a homotopy equivalence. The space on the right is the space of sections of the fibrewise space of submanifolds of the tangent bundle of $M$. By the Galatius–Randal-Williams theorem, the fibre over each point is homotopy equivalent to the Thom space $\text{Th}_{d,n}$. The sheaf $\tilde{\Psi}$ is in fact $\text{Diff}(M)$-equivariant, and if it were also microflexible, then one could deduce that a certain map

$$
\tilde{\Psi}(M) \longrightarrow \Gamma(\tilde{\Psi}^{\text{fib}}(TM) \to M)
$$

is a homotopy equivalence. Since the space of sections on the right has weakly contractible fibres by Theorem [2] one would have that $\tilde{\Psi}(M)$ would be weakly contractible when $M$ is connected and non-compact. But this is a castle in the sky:

**Proposition.** The sheaf $\tilde{\Psi}$ is not microflexible.

Let us recall first the definition of microflexibility.
Definition. A sheaf $\Phi$ on a manifold $M$ is microflexible if for each pair $C' \subset C$ of compact subspaces of $M$, and each pair $C' \subset U', C \subset U$ of open subsets of $M$ such that $U' \subset U$, and for each diagram

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{f} & \Phi(U) \\
\downarrow & & \downarrow r \\
P \times [0,1] & \xrightarrow{h} & \Phi(U')
\end{array}
$$

there exists an $\epsilon > 0$ and a pair of open subsets $C' \subset V' \subset U'$ and $C \subset V \subset U$ such that $V' \subset V$, and a dashed arrow

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{f} & \Phi(U) \\
\downarrow & & \downarrow h \\
P \times [0,\epsilon] & \longrightarrow & \Phi(V')
\end{array}
$$

Proof. Let $W'$ be a connected compact submanifold of $\mathbb{R}^n$, and let $C' = U'$ be a tubular neighbourhood of $W'$ (which we implicitly identify with $NW$ from now on).

Let $W'' \subset W \subset W'$ be codimension 0 submanifolds such that $W''$ is closed as a subset and $W$ is open, and the first inclusion is a homotopy equivalences.

Let $U$ and $C$ be the restrictions of $U'$ to $W$ and $W''$ respectively.

Let $C_k(NW)$ be the fibrewise configuration space of $k$ unordered points in the normal bundle of $W$. A section $f$ of this bundle defines

1. a $k$-sheeted covering of $W$ and
2. an element in $\tilde{\Psi}_d(NW) \subset \tilde{\Psi}_d(U)$.

Since the fibre of $NW$ is a vector space, we can multiply any subset of it by a real number. Then we can define a path

$$[0,1] \longrightarrow \tilde{\Psi}_d(NW)$$

by sending $t > 0$ to $t \cdot f(W)$ and $t = 0$ to $W$. If we have a family of sections of $C_k(NW)$ indexed by $P$, we obtain a map

$$g: P \times [0,1] \longrightarrow \tilde{\Psi}_d(NW)$$

whose restriction to $P \times \{0\}$ is constant. Suppose now that we have a microflexible solution for the diagram

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{c} & \tilde{\Psi}(U') \\
\downarrow & & \downarrow r \\
P \times [0,1] & \xrightarrow{g} & \tilde{\Psi}(U)
\end{array}
$$

where $c$ is the constant map with value $W'$. This means that we can find an $\epsilon > 0$ and an open subset $C \subset V \subset U$ such that

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{c} & \Phi(U) \\
\downarrow & & \downarrow h \\
P \times [0,\epsilon] & \longrightarrow & \Phi(V')
\end{array}
$$

Then, for small values of $\delta \in [0,\epsilon)$, the map $h$ takes values in the space of sections of $C_k(NW)$ (the $k$ is determined because $h$ is extending the section $g$ that takes values in $C_k(NW)$, and because the inclusion $W \subset W''$ induces an epimorphism in components), and therefore defines for each $p \in P$ a $k$-sheeted covering of $W$. As
a consequence, the above solution gives also a lift to the following diagram (where Cov(W) denotes the space of finite sheeted coverings of W):

\[
\begin{array}{ccc}
\text{Cov}(W') & \overset{h}{\rightarrow} & \text{Cov}(W) \\
\downarrow & & \downarrow \\
P & \overset{\varphi}{\rightarrow} & \text{Cov}(W)
\end{array}
\]

But this would mean that any family of coverings of W can be extended to a family of coverings of W' which is false (for instance, if W' = S^2 and W is an equatorial band in S^2).

\[\square\]

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