Analysing bullwhip effect in supply networks under information sharing and exogenous uncertainty

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Abstract: This paper analyses the bullwhip effect in single product supply network topologies, operated with linear and time-invariant inventory management policies and shared supply network information, considering exogenous uncertainty. Information sharing is determined as the degree of coordination across the supply network. Exogenous uncertainty (e.g., transportation delay) cannot be governed by any supply network members. We characterise the stream of orders placed at any stage of the network assuming the customer demand is ergodic. In fact, this paper gives exact formulae to predict the magnitude of bullwhip effect in any shared supply network information topologies. The mentioned formulae is explored by means of mathematical method called frequency domain analysis (FDA) and the relevant analyses are progressed by Fourier transform method. The main contribution of the present work is defined as considering information sharing and exogenous uncertainty simultaneously in supply networks and using Fourier transforms.

Keywords: bullwhip effect; BWE; supply networks; information sharing; frequency domain analysis; FDA; exogenous uncertainty.

Reference to this paper should be made as follows: Seifbarghy, M., Darvish, M. and Akbari, F. (2017) ‘Analysing bullwhip effect in supply networks under information sharing and exogenous uncertainty’, Int. J. Industrial and Systems Engineering, Vol. 26, No. 3, pp.291–317.

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1 Introduction

Supply chain includes a set of organisations which cooperate together to provide final products and services for end customers and creating value in the chain (Kouvelis et al., 2006). The concept of bullwhip effect (BWE) first is presented by Forrester (1961). Burbidge (1991) discussed problems with causes in detail and introduced an inventory control model regarding demand amplification. BWE is considered as a phenomenon in supply chains where the amplification in the order sequences is usually greater than that of the downstream of a chain (Lee et al., 1997). BWE was presented by macroeconomic data (Blinder, 1986; Blanchard, 1983; Kahn, 1987). They showed effect increases the cost of operating the supply chain network. Regarding BWE literature review and observations on huge extra operational costs records for suppliers, analysing BWE and trying to reduce it, is so important (Dejonckheere et al., 2003). Refer to BWE literature, the research areas are classified in seven branches, detecting the relevant causes and recommending some solutions (Dejonckheere et al., 2003, 2004; Geary et al., 2004; Kim, 2008; Su and Wong, 2008; Zarandi et al., 2008), drawing demand models (Bayraktar et al., 2008; Hien Duc et al., 2008; Zarandi et al., 2008), detecting the effects of demand amplifications (Haughton, 2009), analysing the BWE (Kouvelis et al., 2006; Ouyang and Li, 2010; Sucky, 2008), information sharing (IS) (Hsieh et al., 2007; Jaksic and Rousjan, 2008), channel alignment (Zhang, 2004) and operational efficiency (Bayraktar et al., 2008; Hsieh et al., 2007; Jaksic and Rousjan, 2008; Miragliotta, 2006; Ozelkan and Cakanyildirim, 2009).

Dejonckheere et al. (2004) studied the BWE avoiding solutions and characterised a model for measuring it. A supply chain includes multiple customers and multiple markets, so serial supply chains cannot accommodate these issues and as a result network structure is necessary for interactions potential competitions or collaborations among suppliers and customers. In network topology of a supply chain each supplier’s ordering decisions may be influenced directly by orders from multiple ‘neighbours’, or indirectly via network wide IS. It is considered that any member in supply network may undertake the role of being the customer of upstream members or the supplier of downstream members or the both (Ouyang and Li, 2010).

There is a large number of research related to the BWE in serial supply chain network (Table 1). This table includes techniques for reducing BWE, inventory management policies, BWE measurement and type of analysis BWE.
Table 1  The BWE in serial supply network

| Authors          | Year | Type of analysis | Bullwhip effect measurement | Inventory management policies | Techniques for reducing bullwhip effect |
|------------------|------|------------------|------------------------------|------------------------------|----------------------------------------|
| Ouyang           | 2007 | Frequency domain | RMSE amplification          | Linear and time-invariant inventory | Information sharing |
| Ouyang and Li    | 2010 | Frequency domain | RMSE amplification          | Linear and time-invariant inventory | Information sharing (past order sequences and inventory records at) |
| Wang et al.      | 2012 | Simulation+ Taguchi experiments | Deviation of demand orders/ deviation of end customer demand | (s, Q) pull-based replenishment policy | Information sharing (demand) |
| Chen et al.      | 2012 | Vendor management inventory (VMI) | Demand variance | Classical out | Information sharing(demand) |
| Hall and Saygin  | 2012 | Simulation       | Demand variance | Classical out | Information sharing (capacity tightness, resource reliability, demand) |
| Dominguez et al. | 2014 | Simulation       | Order rate variance | Smoothing replenishment a (S, R) policy | Information sharing (demand) |

IS is the practice of making strategic and operation information available for other partners of the network and as some degree of coordination across the supply network, and it creates visibility along the network and helps suppliers to plan their policies (Prajogo and Olhager, 2012). So, one of the BWE causes can be realised as decentralised supply network where information is not shared among suppliers. Miragliotta (2006) showed IS can decrease BWE from 0% to 35% dependent on the supply network stages interest and customer demand scenario. One of the particular IS strategies in supply networks can be pointed as sharing customer demand information [e.g., point of sales (POS) data] across the network to increase supply network visibility.

One of the other BWE causes which makes the propagation of order amplification strengthen is known as uncertainty. The exogenous uncertainty instances can be transportation delays, disasters, and natural accidents whose occurrence is not under control of the supply network members. This paper focuses on exogenous uncertainties (e.g., transportation delay) which cannot be controlled by any network members and are independent of the suppliers’ inventory management policy.

Darvish et al. (2014) explained a model for analysing bullwhip effect for a single-product supply network topology considering exogenous uncertainty and time-invariant inventory. Miragliotta (2006) applied variance ratio greater than 1 to detect BWE which is defined as the ratio between the supply variance at the upstream and the demand variance at the downstream. Ouyang and Daganzo (2006) showed how to reduce BWE by introducing advance demand information (ADI) into the ordering schemes of
supply chains. Ouyang and Daganzo (2008) analysed the BWE in single-echelon supply chain driven by arbitrary customer demands, considering exogenous uncertainty introducing the concept of uncertainty as exogenous to the suppliers. Ouyang (2007) analysed the BWE in multi-stage supply chains operated with linear and time-invariant inventory management policies and shared supply chain information. Such information includes past order sequences and inventory records at all supply chain stages.

Ouyang and Li (2010) analysed BWE in supply chain networks operated with linear and time-invariant inventory policies management which can be applied for any arbitrary supply network and stationary customer demand process.

The few studies based on the non-serial SCN or network supply chain modelling assumption investigating the dynamics of IS and demand amplification phenomenon. The best of our knowledge, there is a lack of consistent studies for evaluating the BWE with the IS and exogenous uncertainty, in general, in supply network. The aim of this paper is analysing the impact of bullwhip reduction strategies on a supply network with IS and exogenous uncertainty and comparing this impact with the effect of these techniques with conditions when there is no IS. In fact, the paper proposes a model for supply network as a Markovian chain system under stochastic dynamic parameters and exogenous uncertainty to diagnose BWE and presents network members complex reactions. The presented model provides the basis for predicting the presence of BWE and bound its magnitude in any network topologies operated with linear and time-invariant inventory management policies.

IS is supposed as centralised supply network definition where information is shared among network members. Exogenous uncertainty is defined as transportation delay which any network member cannot control it. Supply network is allowed to include any member which can play the role of being the customer of up-streams or the role of being the supplier of down-streams or the both. The customer demand process is known and ergodic. The mentioned formula is come by the means of frequency domain analysis (FDA) method (Ouyang and Daganzo, 2008).

FDA has been used to study properties of time series for a long time (Brockwell and Davis, 1998). A system control framework was recently introduced to study the BWE in the frequency domain (Ouyang and Li, 2010). As a result, this paper aims at studying the supply network system in equilibrium state and then analysing different equations (as state deviation) to propose a robust field for measuring the BWE by the means of FDA and significantly, by Fourier Transform method, under IS. The rest of the paper is organised as follows: Section 2 presents a review of supply network concept and system dynamic parameters; Section 3 describes the steady state in supply network with IS and exogenous uncertainty; Section 4 presents the model of supply chain network with IS and exogenous uncertainty; Section 5 presents the BWE and the FDA for the BWE metric; Section 6 demonstrates numerical examples and in Section 7 the conclusions and future research lines are pointed out.

2 System dynamics of supply network

A general supply network is illustrated in Figure 1. Network members consist of three sets of various members, including a set of primitive suppliers \( A_1 \), a set of intermediate suppliers \( A_2 \), and a set of final customers \( A_3 \). The set of intermediate suppliers serves the set of final customers with determined product and services, and the set of primitive
suppliers serves the set of intermediate suppliers with products and services. Inventory stream starts from the set of primitive suppliers to the set of intermediate suppliers and then the set of final customers, in supply network. It is clear that the order stream is reverse in supply network. The concept of primitive suppliers is pointed not to have anymore up-streams in supply network and the concept of final customers is applied not to have anymore down-streams. The set of intermediate suppliers is introduced as system main part in which any member of the set of intermediate suppliers plays the role of both order placer (customer of up-streams) and order receiver (supplier of down-streams). While primitive suppliers just play the role of order receivers (as system output) and final customers just play the role of order placers (as system input). A directed arc \((i, k) \in N\) is applied to show that partner \(i \in A_2 \cup A_3\) orders from \(k \in A_1 \cup A_2\).

This paper assumes that \(A_1, A_2\) and \(A_3\) are disjoint and the entire network includes four disjoint subsets of arcs (Ouyang and Li, 2010):

\[
N_1 = \{(i, k) \in N | i \in A_1, k \in A_2\} \quad (1)
\]

\[
N_2 = \{(i, k) \in N | i \in A_2, k \in A_2\} \quad (2)
\]

\[
N_3 = \{(i, k) \in N | i \in A_2, k \in A_3\} \quad (3)
\]

\[
N_4 = \{(i, k) \in N | i \in A_3, k \in A_3\} \quad (4)
\]

Figure 1  Supply network topology
Refer to have part $A_2$ as the system main part, this paper try to identify system dynamics on part $A_2$ basis. So, for a generic supplier $i \in A_2$, its inventory position $x_i(t)$ (including in-transit inventory) and in-hand inventory $y_i(t)$ satisfy the orders placed and received. Linear and time-invariant policies definition comes as follow: if $(i, k) \in N$, supplier $i$ orders $u_{ik}(t)$ items from $k$ at discrete times $t = ..., -2, -1, 0, 1, 2, ...$, and receives the items after a constant lead time, $l_{ik} = 0, 1, 2, ...$ Some researches assuming the up-streams always have in-stock items (Lee et al., 2000; Chen et al., 2012; Ouyang and Daganzo, 2006, 2008; Ouyang, 2007; Ouyang and Li, 2010).

Equations (5) to (6) define the system dynamics for the supply network. The order quantity by each member is related on member’s order policy. Consider situations that complete information are shared between members. At time $t$, the complete information set for the entire network includes the inventory records $x_i, y_i \forall i \in A_2$ up to period $t$, and orders $u_{rs} \forall (r, s) \in N$ up to period $t - 1$ (Ouyang and Li, 2010):

$$\beta(t) = \left[ \bigcup \{ x_i(t), x_i(t - 1), ..., x_i(-\infty); y_i(t), y_i(t - 1), ..., y_i(-\infty) \mid i \in A_2 \} \right]$$

$$\bigcup \left[ \{ u_{rs}(t - 1), u_{rs}(t - 2), ..., u_{rs}(-\infty) \mid (r, s) \in N \} \right]$$

If information is shared across the network, every supplier may determine its order quantities based on any subset of $\beta(t)$. So, the impressive role of order policies in supply network order quantities introduces them as system dynamic. As in Ouyang and Daganzo (2006, 2008), Ouyang (2007) and Ouyang and Li (2010), this paper focuses on linear and time-invariant policies (Wu and Katok, 2006; Zarandi et al., 2008; Zhang, 2005, 2004). So, the most general linear and time-invariant expression of policy for $u_{ik}(t)$, $(i, k) \in N_3 \cup N_4$ is:

$$u_{ik}(t) = \lambda_{ik} + \sum_{r \in A \cup B} \left[ A_{ik}^r(P)x_i(t) + B_{ik}^r(P)y_i(t) \right]$$

$$+ \sum_{r \in A \cup B} \left[ C_{ik}^r(P)u_{rs}(t - 1) \right], \forall (i, k) \in N_3 \cup N_4$$

where unit shift operator $P$ satisfies $P^n x_i(t) = x_i(t - m)$, $\forall t$ and $\forall m = 0, 1, ...$; $\lambda_{ik}$ is a real number; and $A_{ik}^r(P)$, $B_{ik}^r(P)$ and $C_{ik}^r(P)$ are polynomials with real coefficients. Polynomials $A_{ik}^r(P)$, $B_{ik}^r(P)$ and $C_{ik}^r(P)$ respectively indicate how $u_{ik}(t)$ is determined based on supplier $r$’s inventory history, $x_i, y_i$, and its past orders $u_{rs}$ (Zhang, 2004).

A general definition of $A_{ik}^r(P)$, $B_{ik}^r(P)$ and $C_{ik}^r(P)$ can represent any possible linear and time-invariant ordering policies. A general definition of $A_{ik}^r(P)$, $B_{ik}^r(P)$ and $C_{ik}^r(P)$ shows that these equations may improve any shared (or local) information so can represent any possible LTI ordering polices and these equations are denoted by:
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\[ B'_s(p) = 0 \forall r; C'^s_s(p) = \begin{cases} \frac{1}{m} (1 + p + \ldots + p^{m-1}) , & \text{if } s = i0 \\ 0, & \text{otherwise} \end{cases} \]

\[ A'_s(p) = \begin{cases} -1, & \text{if } k = i \\ 0, & \text{otherwise} \end{cases} \]

In IS as a remedy to reduce the BWE has been mentioned by several authors notably (Lee et al., 1997). We consider situation that information are shared between suppliers; therefore supplier may instead use the moving average of customer demand to forecast future demand. Therefore the policy for calculation of \( A'_s(p) \) \( B'_s(p) \) and is same as before, but for calculation of \( C'^s_s(p) \) we have:

\[ C'^s_s(p) = \begin{cases} \frac{1}{m} (1 + p + \ldots + p^{m-1}) , & \text{if } k = 1 \\ 0, & \text{otherwise} \end{cases} \]

This part shows system dynamics (5) to (8) as a basis for model introduced in next part.

3 Steady state with IS and exogenous uncertainty

This paper assumes that linear and time-invariant policies are stable. All suppliers of a supply network use proper policies where the place orders of suppliers have the same sizes over time, so the supplier inventories tend to equal equilibrium values that are independent of the initial conditions. For this condition we have:

1 place orders:
\[ u_i(t) = u_i, \quad \forall (r, i) \in N, \]
\[ u_i(t) = u_i, \quad \forall (i, s) \in N. \]

2 inventory position: \( x^i \).

3 in-stock inventory: \( y^i \).

In the steady state the whole orders received by supplier \( i \) is equal with the whole orders:

\[ \sum_{i} u_i^w = \sum_{i} u_i^w, \quad \forall i \in A_2 \]

Refer to above sentences, system dynamics with complete information with relationship steady-state inventory positions and the equilibrium order quantity can be written as:

\[ 0 = \sum_{i \in A_2} u_i^w - \sum_{r \in A_2} u_r^w, \quad \forall i \in A_2 \]
The target of this model is formulated according to studied exogenous uncertainty which can be transportation delays, disasters, and natural accidents whose occurrence is not under the control of the supply network members. We define supply chain uncertainty as the standard deviation of the difference between the actual and expected amount. Therefore, when exogenous uncertainty is occurred, there is a deviation from steady state conditions. The major parameters can be expressed in terms of deviations from their value at the equilibrium condition that which are on-hand inventory, inventory position and order policy. In order to show order sequences, the inventories and orders equations can be expressed in terms of deviations from their value at equilibrium:

\[ u_{i}^{t} = \lambda_{i} + \sum_{r \in I_{i} \cup A_{i}} \left[ A_{i}^{r}(P)\chi_{r}^{t} + B_{i}^{r}(P)\gamma_{r}^{t} \right] \]
\[ + \sum_{r \in A_{i} \cup A_{i} \cup r \in A_{k}} \left[ C_{i}^{r}(P)\mu_{r}^{t} \right] , \forall (i, k) \in N_{3} \cup N_{4} \] (13)

The system dynamics with complete information and exogenous uncertainty by subtracting equations (5) to (8) and (14) to (16) can be represented by the following equations:

\[ \tilde{x}_{i}(t) = x_{i}(t) - x_{i}^{e} \] (14)
\[ \tilde{y}_{i}(t) = y_{i}(t) - y_{i}^{e} \] (15)
\[ \tilde{u}_{r}(t) = u_{r}(t) - u_{r}^{e} \] (16)

As a result by considering this assumption, for downstream demand and upstream order sequence we have:

1 Downstream demand \( \equiv \{ \bar{u}_{i}(t) : (r, s) \in N_{1} \cup N_{2} \} \) (20)
2 Upstream order sequence \( \equiv \{ \bar{u}_{i}(t) : (i, k) \in N_{3} \cup N_{4} \} \) (21)

Figure 2 illustrates summary of consequence for constructions a supply network with IS and exogenous uncertainty.
4 Model representation

In previous section system dynamic for a network supply with IS and all order sequences in the network is presented. Regarding to equations (17) to (21), the system is represented by following equations:
\[ \bar{u}_a(t+1) - \bar{u}_{rs}(t) = \lambda_{ik} + \sum_{r \in R_a \cup A_1} \left[ A_{ik}^r(P) \left( \bar{x}_r(t+1) - \bar{x}_r(t) \right) \right] + \sum_{r \in R_a \cup A_1, s \in R_s \cup A_2} \left[ C_{ik}^{rs}(P) \left( \bar{u}_{rs}(t) \right) \right] , \forall(i,k) \in N_3 \cup N_4 \] (22)

Then using equations (18) and (19) to eliminate \( \bar{x}_r(t+1) - \bar{x}_r(t) \) and \( \bar{y}_s(t+1) - \bar{y}_s(t) \), the equation can be rewritten as follows.

\[ \bar{u}_a(t+1) = \Psi(P) \bar{u}_a(t) + \Phi(P) \bar{u}_{rs}(t). \] (23)

where \( \Psi(P) \) and \( \Phi(P) \) are polynomials respectively referred to follow equations:

1. \[ \Psi(P) = \sum \left[ A_{ik}^r(P) + B_{ik}^r(P) \right] \] (24)
2. \[ \Phi(P) = \sum \left[ C_{ik}^{rs}(P) \right] \] (25)

Equation (25) shows the effect of deviation of order placed by supplier \( i \) from \( k \) \( \{(i,k) \in N_3 \cup N_4\} \) and by supplier \( r \) from \( s \) \( \{(r,s) \in N_1 \cup N_2\} \) at time (state) \( t \), on deviation of order placed by supplier \( i \) from \( k \) \( \{(i,k) \in N_3 \cup N_4\} \) at time (state) \( t+1 \).

So, equation (25) is just represented at state \( t \).

To identify system dynamics, the paper satisfies relevant equation in terms of multiple states, i.e., \( \{t, t-1, t-2, \ldots\} \). At first, defines \((K+1) \times 1\) column vector for two unique members \( \{i,k\} \) in supply network. These vectors are calculated as:

\[ u_{ik}(t+1) = \left[ \bar{u}_a(t+1), \bar{u}_a(t), \ldots, \bar{u}_a(t-K-1) \right]^T \text{ for two unique members} \] (26)

\[ u_{ik}(t) = \left[ \bar{u}_a(t), \bar{u}_a(t-1), \ldots, \bar{u}_a(t-K) \right]^T \text{ for two unique members} \] (27)

\[ u_{rs}(t) = \left[ \bar{u}_{rs}(t), \bar{u}_{rs}(t-1), \ldots, \bar{u}_{rs}(t-K) \right]^T \text{ for two unique members} \] (28)

The system dynamics of a supply network with IS and exogenous uncertainty with matrix presentation can now be written in terms of multiple states:

\[ u_{ik}(t+1) = R_{ik} u_{ik}(t) + S_{rs} u_{rs}(t) \] (29)

where \( R_{ik} \) and \( S_{rs} \) are matrix including the effect of multiple states introduced in (25). To augment the state, the paper represents system dynamics (25) for entire supply network not just for two unique members of supply network, it obtains:

\[ U_{ik}(t) = \left[ u_{ik}(t), u_{ik}(t+1), \ldots, u_{ik}(t+K-1) \right]^T \] \( \forall (I,K) \in N_3 \cup N_4 \) (30)

\[ U_{rs}(t) = \left[ u_{rs}(t), u_{rs}(t+1), \ldots, u_{rs}(t+K-1) \right]^T \] \( \forall (R,S) \in N_1 \cup N_2 \) (31)

\[ U_{ik}(t+1) = R_{ik} U_{ik}(t) + S_{rs} U_{rs}(t) \] (32)
where $R_{IK}$ and $S_{RS}$ are square matrices $\{(K + 1) \times (K + 1)\}$ including some sub-matrices, i.e., $\{R_{IK}, S_{RS}\}$ as their elements. The paper assumes now that the state space of Markovian chain has multiple dimensions that capture the stochastic status at all supplier stages, represented in matrix pairs:

$$M = \{(R_{IK}, S_{RS}), (R_{I-1,K-1}, S_{R-1,S-1}), \ldots, (R_{I-M,K-M}, S_{R-M,S-M})\}$$

It should be mentioned that the number of state space members is equivalent to the number of matrix pairs in supply network which affect on order relationships.

Then system dynamics (32), with exogenous uncertainty is as follow.

$$U_{IK}(t + 1) = R_{IK}(t)U_{IK}(t) + S_{RS}(t)U_{RS}(t)$$

Regarded introduced order state, $R_{IK}(t)$ and $S_{RS}(t)$ including the matrices of $R_{IK}$ and $S_{RS}$ at multiple stages of state space. In other hands, the transaction probability matrix of supply network is defined as following:

$$\rho = [P_{mn}]_{M \times M} ; \quad P_{mn} = \Pr\{t + 1 = n \mid t = m\}, \quad \forall m, n \in M$$

Which its degree can be represented as:

$$A = \{(i, k) \mid (i, k) \in N_1 \cup N_2\}$$

$$B = \{(r, s) \mid (r, s) \in N_1 \cup N_2\}$$

Transaction probability matrix degree

$$= \text{Max} \{n(A), n(B)\}$$

The stochastic order relationship equation (34) is useful to model any supply network topologies under exogenous uncertainty. Figure 3 shows matrix presentation of system dynamic a supply network with exogenous uncertainty.

In previous section, the inventories and orders equations can be expressed in terms of deviations from their value at equilibrium. A general definition of $A^\prime (\rho)$, $B^\prime (\rho)$ and $C^\prime (\rho)$ shows that these equations may improve any shared (or local) information.

In this section it is defined that $R_{IK}(t)$ and $S_{RS}(t)$ are square matrix which including the effect of multiple states for these deviations from their value at equilibrium.

Therefore and $S_{RS}(t)$ come as following:

$$a$$

With IS and exogenous uncertainty:

$$R_{IK}(t) = \begin{bmatrix}
I & & & & & & & & & & \\
& R_{IK} & R_{I,K-1} & \ldots & \ldots & R_{I,K-n} \\
& R_{I-1,K} & R_{I-1,K-1} & \ldots & \ldots & R_{I-1,K-n} \\
& & & & & & & & & & \\
& & & & & & & & & & \\
I - n & R_{I-n,K} & R_{I-n,K-1} & \ldots & \ldots & R_{I-n,K-n}
\end{bmatrix}_{2n+1}$$


\[
\begin{bmatrix}
S & S-1 & \cdots & S-n \\
R & S_{RS} & S_{R,S-1} & \cdots & S_{R,S-n} \\
R^{-1} & S_{R,S-1} & S_{R,S-2} & \cdots & S_{R,S-n} \\
\end{bmatrix}
\]

\[
S_{RS(i)} = \begin{bmatrix}
S_{R,S} & S_{R,S-1} & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{bmatrix}
\]

\[
R^{-n} = \begin{bmatrix}
S_{R,S-n} & S_{R,S-n-1} & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{bmatrix}
\]

\[
(40)
\]

\[
\begin{bmatrix}
K & K-1 & \cdots & K-n \\
I & R_{IK} & R_{I,K-1} & \cdots & R_{I,K-n} \\
I^{-1} & 0 & 0 & \cdots & 0 \\
R_{IK(i)} = \begin{bmatrix}
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{bmatrix}
\]

\[
(41)
\]

\[
\begin{bmatrix}
S & S-1 & \cdots & S-n \\
R & S_{RS} & 0 & \cdots & 0 \\
R^{-1} & S_{R,S} & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
S_{RS(i)} = \begin{bmatrix}
S_{R,S} & 0 & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{bmatrix}
\]

\[
R^{-n} = \begin{bmatrix}
S_{R,S-n} & 0 & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{bmatrix}
\]

\[
(42)
\]

where the relations of any columns and rows are shown by out-space symbols of each matrices. The element in which across first row and second column (\(S_{RS,1}\)), shows the effect amount of deviation of orders placed by supplier \(R\) from \(S-1\) on the mentioned supplier (i.e., supplier \(i\)) order deviation at relevant state. In this paper, we consider situations with IS.

Respectively, based on previous points, any elements of matrices (39) to (42) can be expressed as:

\[
\begin{bmatrix}
t & t-1 & \cdots & t-K \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha_{00} & \alpha_{01} & \cdots & \alpha_{0K} \\
\alpha_{10} & \alpha_{11} & \cdots & \alpha_{1K} \\
\alpha_{K0} & \alpha_{K1} & \cdots & \alpha_{KK} \\
\end{bmatrix}
\]

\[
R_{IK} = \begin{bmatrix}
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\end{bmatrix}
\]

\[
(43)
\]
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The matrices (43) and (44) can play role as any elements of matrices (39) to (42) at multiple defined states (focus on one supplier to analyse the BWE magnitude in that stage). Figure 4 illustrates stochastic order relationship equation for supply network with Exogenous uncertainty and IS.

**Figure 3** Matrix presentation of system dynamics for supply network with information sharing and with exogenous uncertainty

\[
\begin{align*}
\vec{u}_{rs}(t) &= \beta_{00} \beta_{01} \ldots \beta_{0K} \\
\vec{u}_{rs}(t-1) &= \beta_{10} \beta_{11} \ldots \beta_{1K} \\
S_{RS} &= \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\end{align*}
\]

The matrices (43) and (44) can play role as any elements of matrices (39) to (42) at multiple defined states (focus on one supplier to analyse the BWE magnitude in that stage). Figure 4 illustrates stochastic order relationship equation for supply network with Exogenous uncertainty and IS.
Figure 4 Stochastic order relationship equation for supply network

\[ M = \{(R_{IK}, S_{RS}),(R_{I,K-I}, S_{R,S-I}),(\ldots),(R_{I,M-K-M}, S_{R,M,S-M,M})\} \quad n(M) = M \]

\[
\begin{align*}
K & \quad K-1 & \quad \cdots & \quad K-n \\
I-1 & \quad 0 & \quad 0 & \quad \cdots & \quad 0 \\
R_{IK(i)} & = & \cdot & \quad \cdots & \quad \cdots & \quad \cdot \\
I-1 & \quad 0 & \quad 0 & \quad \cdots & \quad 0 \\
S_{RS(i)} & = & \cdot & \quad \cdots & \quad \cdots & \quad \cdot \\
\end{align*}
\]

5 Bullwhip effect

5.1 BWE measurement

In this section a metric is defined for measuring BWE. Lee et al. (1997) presented an amplification effect as the phenomenon where orders to the suppliers tend to have larger variance than the sales to the buyers. Authors proposed a quantitative measurement for BWE as the ratio of variance of order quantity at the echelon under consideration to the variance of demand of the end customer. Five roots are defined for these measurement tools for amplification effect in supply chains and BWE that they are demand forecast updating, lead time, batch ordering, supply shortages and price variations.

This paper, for robust analysis, adopts the worst case expected root mean square errors (RMSE) amplification factor (\( W \)) across all possible customer demand sequences (Ouyang and Daganzo, 2006, 2008; Ouyang, 2007; Ouyang and Li, 2010). This RMSE is calculated by:

\[
RMSE = \left[ \frac{\sum(x-\mu)^2}{n} \right]^{1/2}
\]

To have order deviation of down streams (46) and up streams (47):

\[
\bar{u}D(t) = \sum_{(r,s) \in N \cup N_t} \bar{u}_{RS}(t), \quad \forall t
\]
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The BWE magnitude can be represented by Dejonckheere et al. (2003), Gaalman and Disney (2006), Geary et al. (2004) and Kouvelis et al. (2006):

\[ W = \left[ \sum_{t=0}^{\infty} \overline{u}_V(t) - \overline{u}_D(t) \right]^{1/2}, \quad \forall \left\{ \overline{u}_D(t) \neq 0 \right\} \]  

By this mean, the condition \( W \leq 1 \) guarantees that the RMSE is not amplified (i.e., BWE dose not arise) under any customer ordering scenario (Ogata, 1987; Ouyang et al., 2010).

5.2 FDA for BWE metric

FDA has been used to study properties of time series for a long time (Brockwell and Davis, 1998). There are some techniques for FDA which expressively which they are orthogonal decomposition, transformation and superposition (Ouyang and Daganzo, 2009). This paper presents relevant transformation techniques beyond the exogenous uncertainty and also considering IS. The supply network can be regarded as multiple-input such as final customer demands and multiple-output system, such as supplier orders.

The BWE magnitude (\( W \)) can be expressed equivalently in the frequency domain. Any realisation of customer demands \( \{ \overline{u}_{rs}(t) \}, \forall (r,s) \in N_1 \cup N_2 \) can be decomposed by a discrete Fourier transform into a set of pure harmonic components as \( A_{rs}(w)e^{iw} \), \( w \in [-\pi, \pi] \) (i.e., \( i = \sqrt{-1} \)).

With regarding to Fourier transform definition and also by considering that system dynamics are linear and time-invariant, it is known that for any harmonic component of any customer demand (as system input), \( \{ A_{rs}(w)e^{iw} \} \), the resulting orders placed by each supplier of the linear and time-invariant network (as system output) \( \{ A_{rs}(w)e^{iw} \} \) are also harmonic. Fourier transform presentation of all received orders by \( A_1 \) can be written as follows.

\[
\begin{align*}
\{ A_{rs}^{(1)}(w)e^{iw} \} & \forall (i,k) \in N_4, \forall (r,s) \in N_1 \cup N_2, w \in [-\pi, \pi] \\
\cup \{ A_{rs}(w)e^{iw} \} & \forall (r,s) \in N_1, w \in [-\pi, \pi]
\end{align*}
\]

(49)

where first set shows received orders by \( A_1 \) from \( A_2 \) which they have received from \( A_3 \) in one back-ward step and second set shows received orders by \( A_1 \) from \( A_3 \) directly. So, it is clear that the union of two mentioned sets represents all received orders by \( A_1 \). Also, Fourier transform representation of all placed orders by \( A_3 \) is introduced by:

\[ A_{rs}(w)e^{iw}, \quad \forall (r,s) \in N_1 \cup N_2 , w \in [-\pi, \pi] \]  

(50)
Therefore, regarding to system matrix representation and order variances, in order to define transfer function matrix \((T)\), the ratio of output equations and input equations is just needed and this ratio is defined as:

\[
T_{ik}^{rs}(w) = \frac{A_{ik}^{rs}(w)}{A_{rs}(w)}, \quad \forall (i, k) \in N_i \cup N_k, \forall (r, s) \in N_r \cup N_s
\]  

(51)

where transfer function matrix is introduced by focusing on \(A_2\) as multiple-input, multiple-output system. As Ouyang and Daganzo (2006, 2008), Ouyang (2007) and Ouyang and Li (2010) methodology, this paper uses z-transform in system dynamics to obtain transfer function matrix in its own way. Denote the z-transforms of different equations \((X_i(z) = z\{\bar{x}_i(t)\}, \forall i \in A_2, Y_i(z) = z\{\bar{y}_i(t)\}, \forall i \in A_2\) and \(U_{ik}(z) = z\{\bar{u}_{ik}(t)\}, \forall (i, k) \in N\) for all polynomials introduced in (8):

\[
z\left\{A_{ik}^{rs}(P)\bar{x}_i(t)\right\} = A_{ik}^{rs}(z^{-1})X_i(z)
\]

(52)

\[
z\left\{B_{ik}^{rs}(P)\bar{y}_i(t)\right\} = B_{ik}^{rs}(z^{-1})Y_i(z)
\]

(53)

\[
z\left[C_{ik}^{rs}(P)\bar{u}_{ik}(t-1)\right] = C_{ik}^{rs}(z^{-1})z^{-1}U_{ik}(z)
\]

(54)

Then, we apply z-transform to both sides of system dynamics equation which is defined for a supply network with IS:

\[
U_{ik}(z) = \sum \left[A_{ik}^{rs}(z^{-1})X_i(z) + B_{ik}^{rs}(z^{-1})Y_i(z)\right] + z^{-1} \sum \left[C_{ik}^{rs}(z^{-1})U_{ij}(z)\right], \quad \forall (r, s) \in N, \forall r \in A_2
\]

(55)

\[
(z-1)X_i(z) = \sum U_{ik}(z) + \sum U_{ij}(z) + z\bar{x}_i(0)
\]

(56)

\[
(z-1)Y_i(z) = \sum z^{-1}U_{ik}(z) - \sum U_{ij}(z) + z\bar{y}_i(0)
\]

(57)

The paper assumes the system state from the equilibrium state at \(t = 0\), thus we have:

\[
\bar{x}_i(0) = \bar{y}_i(0) = \bar{u}_{ik}(0) = 0, \forall i, \forall (r, s).
\]

As the paper aims at analysing BWE, it satisfies the stochastic order relationship equation (34) by z-transform. So, in order to apply z-transform, relevant steps are represented as following:

a) The elements of (23) represent by z-transform:

\[
z\{\bar{u}_{ik}(t+1)\} = (z^{-1})U_{ik}(z)
\]

(58)

\[
z\{\Psi(P)\bar{u}_{ik}(t)\} = \Psi(z^{-1})U_{ik}(z)
\]

(59)

\[
z\{\Phi(P)\bar{u}_{ik}(t)\} = \Phi(z^{-1})U_{ik}(z)
\]

(60)

So, equation (23) comes out as:

\[
(z-1)U_{ik}(z) = \Psi(z^{-1})U_{ik}(z) + \Phi(z^{-1})U_{ik}(z)
\]

(61)
b. Then, regarded the definition of matrices (26) to (28), the equation (29) is expressed by $z$-transform as in (62):

$$(z-1)U_{jk}(z)=R_{jk} \cdot U_{jk}(z)+S_{RS} \cdot U_{RS}(z)$$

(62)

c. Thus, the $z$-transform representation of equation (34) as the second step is:

$$(z-1)U_{jk}(z)=R_{jk(z)} \cdot U_{jk}(z)+S_{RS(z)} \cdot U_{RS}(z)$$

(63)

where $M$ is an identity matrix:

$$(z-1)M \cdot U_{jk}(z)=R_{jk(z)} \cdot U_{jk}(z)+S_{RS(z)} \cdot U_{RS}(z)$$

(64)

$$[(z-1)M-R_{jk(z)}]U_{jk}(z)=S_{RS(z)} \cdot U_{RS}(z)$$

(65)

where $[(z-1)M-R_{jk(z)}]$ is invertible, it will be:

$$U_{jk}(z)=\left(\frac{S_{RS(z)}}{(z-1)M-R_{jk(z)}}\right)U_{RS}(z)$$

(66)

where $\left(\frac{S_{RS(z)}}{(z-1)M-R_{jk(z)}}\right)$ is represented as $T(z)$, it can be written:

$$U_{jk}(z)=T(z) \cdot U_{RS}(z)$$

(67)

It should be mentioned that each element in transfer function matrix is applied for showing how customer demands transform to supplier orders. Refer to Parseval’s Theorem² (Ogata, 1987) the order flow variance of two specified members is presented as follows.

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |U_{jk}(e^{iw})|^2 \, dw$$

(68)

where $e^{iw}$ can be placed by $z$. For calculating the ratio of supplier order sequences and customer demand sequences (BWE magnitude or $W$) related on entire network (not for two specified members), the equation (68) can be applied as following:

$$W = \left[ \left( \int_{-\pi}^{\pi} |U_{jk}(e^{iw})|^2 \, dw \right) \left( \int_{-\pi}^{\pi} |U_{RS}(e^{iw})|^2 \, dw \right) \right]^{-1/2}$$

(69)

where $W$ is the ration of order received by up-streams and order placed by down-streams, represented by Fourier transform. After calculating the BWE, it can be ignored if the amount of $W$ get lesser than 1. In order to present better sense of paper for applying these transforms functions Figure 5 illustrates transfer function matrix for system dynamics of supply network with IS and with exogenous uncertainly.
Figure 5  Transfer function matrix for system dynamics of supply network

Extracting transfer function matrix for system dynamics of supply network with information sharing and with exogenous uncertainty

\[(x - 1)U_{ik}(x) = R_{ik} \cdot U_{ik}(x) + S_{RS} \cdot U_{RS}(x)\]

\[\left((x - 1)M - R_{ik}(1)\right)U_{ik}(z) = S_{RS(1)} \cdot U_{RS}(z)\]

\[U_{ik}(z) = \frac{S_{RS(1)}}{(x - 1)M - R_{ik}(1)} \cdot U_{RS}(z)\]

\[T(z) = \frac{S_{RS(1)}}{(z - 1)M - R_{ik}(1)}\]

\[U_{ik}(z) = T(z) \cdot U_{RS}(z)\]

\[W = \left[\int_{-\infty}^{\infty} \left|U_{ik}(e^{j\omega})\right|^2 dw\right]^{1/2} \div \left[\int_{-\infty}^{\infty} \left|U_{RS}(e^{j\omega})\right|^2 dw\right]^{1/2}\]

6  Numerical examples

In order to analyse the characteristics of various expressions of the order quantity and on-hand inventory, as well as the expressions regarding the impact of IS in a supply network, developed in previous sections, the following numerical example is studied. Ouyang (2010) considered one of the most common network topologies. This topology involves tree-shaped ‘distribution system structure’, where one central warehouse serves multiple downstream distributors and even more retailers. In this paper we consider a simple distribution network with the shape of a binary tree that is shown in Figure 6. The assumed network topology includes six members: two final customers, three intermediate suppliers and a primitive supplier.
The model framework and analysis results presented can be applied to supply networks with any general topologies.

In this example $A_1, A_2$ and $A_3$ are disjoint and the entire network includes four disjoint subsets of arcs:

$$N_1 = \{\}$$

$$N_2 = \{(1, 3), (2, 4)\}$$

$$N_3 = \{(3, 5), (4, 5)\}$$

$$N_4 = \{(5, 6)\}$$

In order to analyse BWE and for comparison effectiveness of model the paper introduces the assumed topologies in two main following categories:

1. with IS and exogenous uncertainty
2. without IS while considering exogenous uncertainty

6.1 With IS and exogenous uncertainty

In this section, we examine how the sharing of information reduces the variance of the order quantity at the upstream level or reduce BWE. It is supposed that electronic data interchange (EDI) is elected as IS strategy in entire network. So, the relevant information can be shared from POS to all over the supply network. Regarding to paper assumption, the lead-time is considered as constant amount ($m$). It is assumed that all network
members use (S, s) ordering policy with IS and the demand prediction methodology is
determined by moving-average of orders received in the two most recent periods. And it
is considered that all the analysis and results obtained by focusing on intermediate
supplier 3 (in this case). The relevant order sequence equations mathematically can be
written as:

\[
\tilde{u}_{35}(t+1) = 0.52\tilde{u}_{35}(t) + 0.65\tilde{u}_{35}(t-1) + 0.73u_{13}(t) + 0.90u_{13}(t-1) \tag{74}
\]

\[
\tilde{u}_{45}(t+1) = 0.6\tilde{u}_{45}(t) + 0.78\tilde{u}_{45}(t-1) + 0.90u_{24}(t) + 0.66u_{24}(t-1) \tag{75}
\]

\[
\tilde{u}_{56}(t+1) = 0.32\tilde{u}_{56}(t) + 0.30u_{56}(t-1) \tag{76}
\]

where they are mathematically obtained from following system dynamic metrics:

\[
x_3(t+1) = x_3(t) + u_{35}(t) - u_{13}(t)
\]

\[
x_4(t+1) = x_4(t) + u_{45}(t) - u_{24}(t)
\]

\[
x_5(t+1) = x_5(t) + u_{56}(t) - u_{35}(t) - u_{45}(t)
\]

\[
u_{35}(t) = \lambda_{35} - x_3(t) + \left(\frac{2 + 2P}{2}\right)u_{13}(t-1)
\]

\[
u_{35}(t) = \lambda_{35} - x_3(t) + u_{13}(t-1) + P \cdot u_{13}(t)
\]

\[
u_{45}(t) = \lambda_{45} - x_4(t) + u_{13}(t-1) + u_{13}(t-2)
\]

\[
u_{45}(t) = \lambda_{45} - x_4(t) + u_{24}(t-1) + u_{24}(t-2)
\]

\[
u_{56}(t) = \lambda_{56} - x_5(t) + u_{35}(t-1) + u_{35}(t-2) + u_{45}(t-1) + u_{45}(t-2)
\]

To achieve the main purpose of paper (analysing BWE), we progress relevant steps.
Refer to assumed network topology, relevant matrices of relations equations (30) to (31)
are represented as following:

\[
U_{IK}(t) = \begin{bmatrix} u_{35}(t)^T, u_{45}(t)^T, u_{56}(t)^T \end{bmatrix}_{t=3}^T (I, K) \in N_3 \cup N_4 \tag{77}
\]

\[
U_{RS}(t) = \begin{bmatrix} u_{13}(t)^T, u_{24}(t)^T \end{bmatrix}_{t=2}^T (R, S) \in N_2 \tag{78}
\]

It should be said that matrices (77) and (78) are including some sub-matrices in
which show the amount of order deviation from equilibrium state effect (i.e.,
\[
u_{35}(t) = \begin{bmatrix} u_{35}(t), u_{35}(t-1), \ldots \end{bmatrix}^T \). In order to progress the case, the paper needs to define
\[R_{IK(s)} \text{ and } S_{RS(s)} \text{ (the effect of exogenous uncertainty):}
\]

\[
R_{IK(s)} = \begin{bmatrix} 5 & 6 & \# \\
3 & R_{35} & 0 & 0 \\
4 & R_{45} & 0 & 0 \\
5 & 0 & R_{56} & 0 \end{bmatrix}_{t=3}^{3x3} \tag{79}
\]
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\[
S_{RS}(z) = 2 \begin{bmatrix}
0 & 0 & 0 \\
0 & S_{24} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

where the symbol (#) is used to control the amount of matrices rows and columns not to front any problems in calculations. Regarding literature review and equations (78) to (80), following matrices are mathematically written:

\[
R_{35} = \begin{bmatrix}
0.52 & 0 \\
0 & 0.63 \\
\end{bmatrix}
\]

(81)

\[
R_{45} = \begin{bmatrix}
0.61 & 0 \\
0 & 0.78 \\
\end{bmatrix}
\]

(82)

\[
R_{56} = \begin{bmatrix}
0.32 & 0 \\
0 & 0.30 \\
\end{bmatrix}
\]

(83)

\[
S_{13} = \begin{bmatrix}
0.73 & 0 \\
0 & 0.90 \\
\end{bmatrix}
\]

(84)

\[
S_{24} = \begin{bmatrix}
0.90 & 0 \\
0 & 0.66 \\
\end{bmatrix}
\]

(85)

The frequency domain representation of entire network orders can be stacked into the following matrices:

\[
U_{RS}(z) = \begin{bmatrix}
U_1(z) \\
U_2(z) \\
DP \\
\end{bmatrix}
\]

(86)

\[
U_{IK}(z) = \begin{bmatrix}
U_3(z) \\
U_4(z) \\
U_5(z) \\
\end{bmatrix}
\]

(87)

The matrix (85) is assumed as (Dejonckheere et al., 2004; Gaalman, 2006; Zhang, 2005):

\[
U_{RS}(z) = \begin{bmatrix}
1.2 & 1 \\
1.6 & 0.7 \\
0 & 0 \\
\end{bmatrix} 
\]

(88)

So, regarding to previous sections and refer to equation (70), the matrix \(U_{IK}(z)\) and the probability function matrix \(T(z)\) can be obtained:
It should be considered that $z$ is equivalent to $e^{i\omega}$ and in order to maintain the system integrity, it is supposed as the average of $e^{i\omega}$ ($z = -0.001$). Thus, focusing on intermediate supplier 3, the BWE magnitude is calculated as:

$$W = \frac{\int_{-\pi}^{\pi} |U_{IK}(e^{i\omega})|^2 d\omega}{\int_{-\pi}^{\pi} |U_{RS}(e^{i\omega})|^2 d\omega} = 0.48$$

(91)

where the amount of $W^2$ determines not to face with BWE in supply network ($W$ get lesser than 1) and there is BWE.

6.2 Without IS while considering exogenous uncertainty

Regarding model assumptions and the previous section, the lead-time is either considered as constant amount ($m$) in this mode. As before, it is assumed that all network members use $(S, s)$ ordering policy, but with no IS and the demand prediction methodology is determined by moving-average of orders received in the two most recent periods. And it is considered that all the analysis and results obtained by focusing on intermediate supplier 3 (in this case). The order sequence equation mathematically can be written as:

$$\ddot{u}_{35}(t + 1) = 2\ddot{u}_{35}(t) + 1.5\ddot{u}_{35}(t - 1) + 3.1\ddot{u}_{13}(t) + 3\ddot{u}_{13}(t - 1)$$

(92)

where, it is simply obtained from following system dynamic metrics:

$$\begin{align*}
x_3(t + 1) &= x_3(t) + u_{35}(t) - u_{13}(t) \\
u_{35}(t) &= \lambda_{35} - x_3(t) + \left(\frac{2 + 2P}{2}\right)u_{13}(t - 1) \\
u_{35}(t) &= \lambda_{35} - x_3(t) + u_{13}(t - 1) + P\cdot u_{13}(t) \\
u_{35}(t) &= \lambda_{35} - x_3(t) + u_{13}(t - 1) + u_{13}(t - 2)
\end{align*}$$

The matrices $R_{IK(0)}$ and $S_{RS(0)}$ (the effect of exogenous uncertainty) come:
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\[
\begin{bmatrix}
5 & 6 \\
3 & R_{35}
\end{bmatrix}
\]

\[
R_{jk(z)} = \begin{bmatrix}
4 & 0 & 0 \\
5 & 0 & 0 & \_j\_k\_c_3
\end{bmatrix}
\]

\[
3 & 4 \\
1 & S_{13}
\]

\[
S_{RS(z)} = \begin{bmatrix}
2 & 0 & 0 \\
\_0 & \_0 & \_j\_c_3
\end{bmatrix}
\]

where the symbol (#) is used to control the amount of matrices rows and columns not to front any problems in calculations. Considering equation (92), matrices \(R_{35}\) and \(S_{13}\) can be mathematically written:

\[
R_{35} = \begin{bmatrix}
2 & 0 \\
0 & 1.5
\end{bmatrix}
\]

\[
S_{13} = \begin{bmatrix}
3.1 & 0 \\
0 & 3
\end{bmatrix}
\]

The frequency domain representation of entire network orders can be stacked into the following matrices:

\[
U_{RS}(z) = \begin{bmatrix}
U_1(z) \\
U_2(z) \\
DP
\end{bmatrix}
\]

\[
U_{IK}(z) = \begin{bmatrix}
U_3(z) \\
U_4(z) \\
U_5(z)
\end{bmatrix}
\]

The matrix (97) is assumed as:

\[
U_{RS}(z) = \begin{bmatrix}
[2 & 5]^T, [1 & 2.5]^T, [0 & 0]^T
\end{bmatrix}
\]

So, regarding to previous sections and refer to equation (70), the matrix \(U_{IK}(z)\) and the probability function matrix \(T(z)\) can be obtained:

\[
T(z) = \begin{bmatrix}
-1.03 & 0 & 0 & 0 & 0 \\
0 & -1.20 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Here, \( z \) is equivalent to \( e^{i\omega} \) and in order to maintain the system integrity, it is supposed as the average of \( e^{i\omega} \) (\( z = -0.001 \)). The magnitude of the BWE is calculated from formula (101) and estimated from simulations for a range of parameters. Thus, focusing on intermediate supplier 3, the BWE magnitude is calculated as:

\[
W = \left[ \frac{\int_{-\infty}^{\infty} \left| U_{ik} \left( e^{i\omega} \right) \right|^2 d\omega}{\int_{-\infty}^{\infty} \left| U_{rs} \left( e^{i\omega} \right) \right|^2 d\omega} \right]^{1/2} = 1.03
\]

where the amount of \( W^3 \) determines the presence of BWE in supply network (\( W \) get greater than 1) and its magnitude is equivalent to 1.03. This result implies that IS tend to reduce the BWE. The formula of this research turns out using a mathematical method called FDA for a supply network with exogenous uncertainty. The major target of this paper is analysing BWE considering exogenous uncertainty in supply networks with using Fourier transform in order to simplify the relevant calculations and applying IS and compare results with conditions that information are not shared. Formula in reference (Ouyang and Li, 2010) for calculating BWE has a complex structure so a simplification equation is used for calculating the BWE. According the results shows that IS reduces the BWE and output of the methodology confirms the efficacy of the IS rule in terms of bullwhip reduction in the supply chain network. This methodology gives exact formula for calculating the BWE. Therefore, this comparison shows that the approach which can derive exact solution with Fourier transform for calculation the BWE is superior to Ouyang and Li (2010) approach.

7 Conclusions and further research

This paper has presented a system control framework for analysing order sequences deviation and BWE in supply networks under IS and exogenous uncertainty. By presenting supply network concept and formulating it as Markovian chain, the paper derived robust analytical conditions that diagnose the system under Markovian uncertainties in which are assumed to be as exogenous to the suppliers and are independent of the suppliers’ state while the information is shared among entire network. The presented model provides the basis for developing exact formula for analysing the propagation of order amplification in any single-product supply network topologies operated with linear and time-invariant inventory management policies and shared supply network information considering exogenous uncertainty. The mentioned formula is explored by the means of FDA. The numerical examples which were divided into two main categories have shown how the presented framework enables supply managers (or any practitioners) to study the effect of various factors (e.g., network structure) under robust conditions (exogenous uncertainty) and IS, on the BWE in supply network. Also, refer to numerical examples analysis, it can be noted that IS reduces BWE even while considering exogenous uncertainty model in entire supply network. The analysis
framework presented in this paper can be further extended by focusing on endogenous uncertainty (controllable conditions by network suppliers, e.g., ordering policy) as first direction. Second one is to consider variant lead-times in supply network and the third is to present nonlinear ordering policies for determining system control framework.

Acknowledgements

This research is supported by Education and Research Institute for Information and Communication Technology (Iran). The helpful supporting of mentioned institute is gratefully acknowledged.

References

Bayraktar, E., Koh, S.C.L., Gunasekaran, A., Sari, K. and Tatoglu, E. (2008) ‘The role of forecasting on bullwhip effect for E-SCM applications’, International Journal of Production Economic, Vol. 113, No. 1, pp.193–204.
Blanchard, O.J. (1983) ‘The production and inventory behavior of the American automobile industry’, International Journal of Political Economy, Vol. 91, No. 1, pp.365–400.
Blinder, A.S. (1986) Can the Production Smoothing Model of Inventory Behavior be Saved?’, No. 1257, National Bureau of Economic Research.
Brockwell, P.J. and Davis, R.A. (1998) Time Series: Theory and Methods, 2nd ed., Springer, New York.
Burbidge, J.L. (1991) ‘Period batch control (PBC) with GT the way forward from MRP’, Paper presented at the BPICS, Vol. 1, No. 1, pp.25–33.
Chen, X., Hao, G., Li, X. and Yiu, K.F.C. (2012) ‘The impact of demand variability and transshipment on vendor’s distribution policies under vendor managed inventory strategy’, International Journal of Production Economics, Vol. 139, No. 1, pp.42–48.
Darvish, M., Seifabrghy, M., Sanieimonfared, M.A. and Akbari, F. (2014) ‘Analyzing bullwhip effect in supply networks under exogenous uncertainty’, International Journal of Supply and Operations Management, Vol. 1, No. 1, pp.81–107.
Dejonckheere, J., Disney, S.M., Lambrecht, M.R. and Towill, D.R. (2003) ‘Measuring and avoiding the bullwhip effect: a control theory approach’, European Journal of Operational Research, Vol. 147, No. 3, pp.567–590.
Dejonckheere, J., Disney, S.M., Lambrecht, M.R. and Towill, D.R. (2004) ‘The impact of information enrichment on the bullwhip effect in supply chains: a control engineering perspective’, European Journal of Operational Research, Vol. 153, No. 3, pp.727–750.
Dominguez, R., Cannella, S. and Framinan, J.M. (2014) ‘On bullwhip-limiting strategies in divergent supply chain networks’, Computers and Industrial Engineering, Vol. 73, No. 1, pp.85–95.
Forrester, J.W. (1961) ‘Industrial dynamics’, Journal of the Operational Research Society, Vol. 48, No. 10, pp.1037–1041.
Gaalmann, G. and Disney, S. (2006) ‘State space investigation of the bullwhip problem with ARMA (1, 1) demand processes’, International Journal of Production Economics, Vol. 104, No. 2, pp.327–339.
Geary, S., Disney, S.M. and Towill, D.R. (2004) ‘On bullwhip in supply chains-historical review, present practice and expected future impact’, Int. J. Production Economics, Vol. 101, No. 1, pp.2–18.
Hall, D.C. and Saygin, C. (2012) ‘Impact of information sharing on supply chain performance’, International Journal of Advanced Manufacturing Technology, Vol. 58, Nos. 1–4, pp.397–409.

Haughton, M.A. (2009) ‘Distortional bullwhip effects on carriers’, Transportation Research Part E: Logistics and Transportation Review, Vol. 45, No. 1, pp.172–185.

Hien Duc, T.T., Luong, H.T. and Kim, Y. (2008) ‘A measure of bullwhip effect in supply chains with a mixed autoregressive-moving average demand process’, European Journal of Operational Research, Vol. 187, No. 1, pp.243–256.

Hsieh, K., Chen, Y. and Shen, C. (2007) ‘Bootstrap confidence interval estimates of the bullwhip effect’, Simulation Modelling Practice and Theory, Vol. 15, No. 8, pp.908–917.

Jaksic, M. and Rousjan, B. (2008) ‘The effect of replenishment policies on the bullwhip effect: a transfer function approach’, European Journal of Operational Research, Vol. 184, No. 3, pp.946–961.

Kahn, J. (1987) ‘Inventories and the volatility of production’, American Economic Review, Vol. 77, No. 1, pp.667–679.

Lee, H.L., Padmanabhan, V. and Whang, S. (1997) ‘The bullwhip effect in supply chains’, Sloan Management Review, Vol. 38, No. 3, pp.93–102.

Lee, H.L., So, K.C. and Tang, C.S. (2000) ‘The value of information sharing in a two-level supply chain’, Management Science, Vol. 46, No. 5, pp.626–643.

Prajogo, D. and Olhager, J. (2012) ‘Supply chain integration and performance. The effects of long-term relationships, information technology and sharing, and logistics integration’, International Journal of Production Economics, Vol. 135, No. 1, pp.514–522.

Ouyang, Y. and Li, X. (2010) ‘The bullwhip effect in supply chain networks’, European Journal of Operational Research, Vol. 201, No. 3, pp.799–810.

Ozcelik, E.C. and Cakanyildirim, M. (2009) ‘Reverse bullwhip effect in pricing’, European Journal of Operational Research, Vol. 192, No. 1, pp.302–312.

Pradhan, A. and Olhager, J. (2012) ‘Supply chain integration and performance. The effects of long-term relationships, information technology and sharing, and logistics integration’, International Journal of Production Economics, Vol. 135, No. 1, pp.514–522.

Ouyang, Y. and Li, X. (2010) ‘The bullwhip effect in supply chain networks’, European Journal of Operational Research, Vol. 201, No. 3, pp.799–810.

Ozcelik, E.C. and Cakanyildirim, M. (2009) ‘Reverse bullwhip effect in pricing’, European Journal of Operational Research, Vol. 192, No. 1, pp.302–312.

Prajogo, D. and Olhager, J. (2012) ‘Supply chain integration and performance. The effects of long-term relationships, information technology and sharing, and logistics integration’, International Journal of Production Economics, Vol. 135, No. 1, pp.514–522.

Ouyang, Y. and Li, X. (2010) ‘The bullwhip effect in supply chain networks’, European Journal of Operational Research, Vol. 201, No. 3, pp.799–810.
Zarandi, M.H., Pourakbar, M. and Turksen, I.B. (2008) ‘A fuzzy agent-based model for reduction of bullwhip effect in supply chain systems’, *Expert Systems with Applications*, Vol. 34, No. 3, pp.1680–1691.

Zhang, X. (2004) ‘The impact of forecasting methods on the bullwhip effect’, *Int. J. Production Economics*, Vol. 88, No. 1, pp.15–27.

Zhang, X. (2005) ‘Delayed demand information and dampened bullwhip effect’, *Operations Research Letters*, Vol. 33, No. 3, pp.289–294.

**Notes**

1. $\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$.

2. The relevant analysis is calculated by MATLAB software.

3. The relevant analysis is calculated by MATLAB software.