Flop Transitions in Cuprate and Color Superconductors: From $SO(5)$ to $SO(10)$ Unification?

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The phase diagrams of cuprate superconductors and of QCD at non-zero baryon chemical potential are qualitatively similar. The Néel phase of the cuprates corresponds to the chirally broken phase of QCD, and the high-temperature superconducting phase corresponds to the color superconducting phase. In the $SO(5)$ theory for the cuprates the $SO(3)$ spin rotational symmetry and the $U(1)_{em}$ gauge symmetry of electromagnetism are dynamically unified. This suggests that the $SU(2)_L \otimes SU(2)_R \otimes U(1)_B$ chiral symmetry of QCD and the $SU(3)_c$ color gauge symmetry may get unified to $SO(10)$. Dynamical enhancement of symmetry from $SO(2)_s \otimes Z(2)$ to $SO(3)_s$ is known to occur in anisotropic antiferromagnets. In these systems the staggered magnetization flops from an easy 3-axis into the 12-plane at a critical value of the external magnetic field. Similarly, the phase transitions in the $SO(5)$ and $SO(10)$ models are flop transitions of a “superspin”. Despite this fact, a renormalization group flow analysis in $4-\epsilon$ dimensions indicates that a point with full $SO(5)$ or $SO(10)$ symmetry exists neither in the cuprates nor in QCD.

Understanding QCD at non-zero baryon chemical potential $\mu$ is very important in the context of both heavy ion and neutron star physics. While asymptotically large values of $\mu$ are accessible in perturbative QCD calculations, such values are not realized in actual physical systems. Investigations of the phenomenologically relevant regime at intermediate $\mu$ require the use of nonperturbative methods. Unfortunately, first-principles lattice calculations in this regime are presently prevented by the notorious complex action problem. Conjectures for the QCD phase diagram at non-zero $\mu$ are thus based on model calculations. These calculations reveal interesting phenomena such as color superconductivity\textsuperscript{a} but one cannot expect the results to be quantitatively correct.

While it is very important to develop quantitative methods to understand QCD at non-zero $\mu$, here we ask if further qualitative insight can be gained through analogies with related condensed matter systems. In particular, the phase diagram of high-temperature cuprate superconductors is qualitatively similar to the one conjectured for two flavor QCD. The ordinary hadronic phase of QCD at small $\mu$ in which the chiral $SO(4) = SU(2)_L \otimes SU(2)_R$ symmetry is spontaneously broken down to $SO(3) = SU(2)_{L=R}$ corresponds to the antiferromagnetic Néel phase of the undoped cuprates in which the $SO(3)_s$ spin rotational symmetry is broken down to $SO(2)_s$ due to the spontaneous generation of a staggered magnetization. The high-temperature superconducting phase of the doped cuprates with spontaneous $U(1)_{em}$ breaking corresponds to the color superconducting phase of two flavor QCD in which the $SU(3)_c$ gauge symmetry is expected to break down to $SU(2)_c$. Finally, the quark-gluon plasma corresponds to the high-temperature metallic phase of the cuprates.

QCD in the color superconducting phase is a genuine high-temperature superconductor. The mechanism that leads to quark Cooper pair binding is direct one-gluon exchange in the attractive color anti-triplet channel. This is in contrast to ordinary (low-temperature) superconductors in which the direct electron-electron Coulomb interaction (mediated by one-photon exchange) is repulsive. Ordinary superconductivity is due to
an indirect phonon-mediated attraction which occurs at rather low energies and thus gives rise to small transition temperatures. The mechanism that leads to high-temperature superconductivity in the cuprates is presently not understood, but is expected to be due to processes that happen at a rather high energy scale unrelated to phonon exchange. Both in the cuprates and in QCD the phase transition that separates the phases of broken global and local symmetries is driven by a chemical potential (for electrons and quarks, respectively).

Furthermore, the solution of microscopic models for the cuprates (such as the Hubbard model) is prevented by a severe sign problem. Still, there are interesting attempts to understand superconductivity in the cuprates in analogy to simpler condensed matter systems. In particular, Zhang has conjectured that a transition that separates the antiferromagnetic Néel phase from the high-temperature superconducting phase is analogous to the spin flop transition of the staggered magnetization in a 3-d anisotropic antiferromagnet. This transition is driven by an external uniform magnetic field \( B \) which plays the role of a chemical potential. Due to the anisotropy, such an antiferromagnet has only a \( \mathbb{Z}(2) \otimes SO(2) \) spin rotational symmetry. At small \( B \), the staggered magnetization \( \vec{n} = (n_1, n_2, n_3) \) points along the easy 3-axis, while at large \( B \) it flops into the 12-plane. The spin flop transition is illustrated in figure 1. The flop transition is a first order phase transition line that ends in a bicritical point from which two second order lines emerge — one in the 3-d Ising and one in the 3-d XY model universality class. Zhang has argued that the bicritical point has a dynamically enhanced \( SO(5) \) symmetry although the microscopic Hamiltonian is only \( SO(3)_s \otimes U(1)_{em} \) invariant. The expected phase diagram of an \( SO(N) \otimes SO(M) \) invariant theory with potential dynamical symmetry enhancement to \( SO(N+M) \) is shown in figure 2. For the cuprates \( N = 2 \) and \( M = 3 \), while for anisotropic antiferromagnets \( N = 2 \) and \( M = 1 \). As we will see next, for QCD \( N = 6 \) and \( M = 4 \).

Recently, we have generalized the \( SO(5) \) unified theory of high-temperature superconductivity and antiferromagnetism to an \( SO(10) \) unified description of color superconductivity and chiral symmetry breaking in QCD. Although the unifying group is the same as in a grand unified
theory, the unification scale would now be around 10 MeV. We consider left and right-handed quark fields \( \Psi_L^f \) and \( \Psi_R^f \) with two flavors \( f = 1, 2 \) and three colors \( c = 1, 2, 3 \). The chiral symmetry breaking order parameter

\[
(\bar{\Psi} \Psi)^f_g = \sum_c \bar{\Psi}_L^f c \Psi_R^g c
\]

(1)

is a color singlet, \( SU(2)_L \) and \( SU(2)_R \) doublet, with baryon number zero. The color symmetry breaking order parameter

\[
(\bar{\Psi} \Psi)^c = \sum_{f,g,a,b} \epsilon_{fgcba} (\bar{\Psi}_{L,R}^f c \Psi_{L,R}^g b)^T C \Psi_{L,R}^g b
\]

(2)

on the other hand, is a color anti-triplet, \( SU(2)_L \otimes SU(2)_R \) singlet, with baryon number 2/3. Similarly, \( (\bar{\Psi} \Psi)^3 \) is a color triplet, \( SU(2)_L \otimes SU(2)_R \) singlet, with baryon number −2/3. The group \( SO(10) \) contains \( SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_R \) as a subgroup. The 10-dimensional vector representation of \( SO(10) \) decomposes into

\[
\{10\} = \{1, 2, 2\}_{0} \oplus \{\bar{3}, 1, 1\}_{2/3} \oplus \{3, 1, 1\}_{-2/3}, \quad (3)
\]

and thus naturally combines the order parameters for chiral symmetry breaking and color superconductivity to a 10-component “supervector” \( \vec{n} = (n^1, n^2, ..., n^{10}) \) with

\[
\begin{align*}
n^c &= (\bar{\Psi} \Psi)^c + (\bar{\Psi} \Psi)^c, \\
n^{c+3} &= -i[(\bar{\Psi} \Psi)^c - (\bar{\Psi} \Psi)^c], \quad c \in \{1, 2, 3\}, \\
n^7 &= (\bar{\Psi} \Psi)^{11} + (\bar{\Psi} \Psi)^{22}, \\
n^8 &= -i[(\bar{\Psi} \Psi)^{12} + (\bar{\Psi} \Psi)^{21}], \\
n^9 &= (\bar{\Psi} \Psi)^{12} - (\bar{\Psi} \Psi)^{21}, \\
n^{10} &= -i[(\bar{\Psi} \Psi)^{11} - (\bar{\Psi} \Psi)^{22}].
\end{align*}
\]

(4)

In the chirally broken phase at small \( \mu \) the 4-component vector \( (n^7, n^8, n^9, n^{10}) \) develops an expectation value, thus breaking \( SU(2)_L \otimes SU(2)_R \) spontaneously to \( SU(2)_{L=\bar{R}} \). The corresponding Goldstone pions are described by fields in the \( SU(2)_{L} \otimes SU(2)_{R}/SU(2)_{L=\bar{R}} = S^3 \) easy 3-sphere. In the color superconducting phase at larger \( \mu \), on the other hand, the 6-component vector \( (n^1, n^2, ..., n^6) \) gets an expectation value and the supervector flops into the 5-sphere \( SU(3)_c/SU(2)_c = S^5 \) that describes the quark Cooper pair condensate. In this case, one would expect that the first order supervector flop line ends at a bicritical point with dynamical symmetry enhancement to \( SO(10) \). However, a renormalization group flow analysis shows that both the \( SO(5) \) and \( SO(10) \) fixed point are unstable, at least in \( (4 - \epsilon) \) dimensions. We have simulated the flop transition in simple classical \( SO(N) \otimes SO(M) \) invariant models. Figure 3 shows the flop of the staggered magnetization from the 3-axis into the 12-plane for an anisotropic antiferromagnet. Similar studies for high-temperature and color superconductors are in progress. In particular, we investigate the endpoint of the flop transition in order to see if the perturbative results obtained in \( (4 - \epsilon) \) dimensions also apply to the nonperturbative 3-d case.

To illustrate the dynamics of the flop transition, we now construct a unified low-energy ef-

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**Figure 2.** Expected phase diagram of an \( SO(N) \otimes SO(M) \) invariant theory with potential dynamical symmetry enhancement to \( SO(N+M) \). The first order flop transition (solid) line ends in a bicritical point \( (T_{bc}, \mu_{bc}) \). Two second order (dashed) lines emerge vertically from this point.
Figure 3. **Flop transition in an anisotropic antiferromagnet.** The two wings correspond to the square of the staggered magnetization $n_3^2$ along the $3$-axis and $n_1^2 + n_2^2$ in the $12$-plane.

Effective Lagrangian for the Goldstone modes described by an $(N + M)$-component unit vector $\vec{n}$. In the absence of $SO(N + M)$ symmetry breaking terms (other than the chemical potential), the low-energy effective action takes the form

$$S[\vec{n}] = \int_0^{1/T} dt \int d^3x \ F_0^2 \sum_{\alpha} \nabla_\alpha n^\alpha \nabla_\alpha n^\alpha + \frac{1}{c^2} (\partial_0 n^\alpha + A_0^\alpha n^\beta)(\partial_0 n^\alpha + A_0^\beta n^\beta).$$  \hspace{1cm} (5)

The chemical potential $\mu$ couples as an imaginary non-Abelian constant vector potential

$$A_0^\alpha = i\mu \sum_{c=1,\ldots,N/2} (\delta^\alpha, c \delta^+, N/2, \beta - \delta^\alpha, c + N/2 \delta^-, \beta).$$  \hspace{1cm} (6)

in the Euclidean time direction. To account for explicit $SO(N + M)$ breaking to $SO(N) \otimes SO(M)$ we add a potential term $-V_0[(n^{N+1})^2 + \ldots + (n^{N+M})^2]$ to the action that favors the easy $(M - 1)$-sphere. In the case of QCD this leads to chiral symmetry breaking. The total potential for constant fields $\vec{n}$ then takes the form

$$V(\vec{n}) = -\frac{F_0^2}{2c^2} \mu^2 [(n^1)^2 + \ldots + (n^N)^2] - V_0[(n^{N+1})^2 + \ldots + (n^{N+M})^2].$$  \hspace{1cm} (7)

For $\mu < \mu_c = \sqrt{2V_0c^2/F_0^2}$, it is energetically favorable for the supervector to lie in the easy $(M - 1)$-sphere. For $\mu > \mu_c$, on the other hand, the supervector $\vec{n}$ flops into the $(N-1)$-sphere. In QCD, this induces a first order phase transition from the chirally broken to the color superconducting phase.

It is interesting to ask if the supervector can play a dynamical role in the real world. In particular, with the strange quark present, a new color superconducting phase with color-flavor locking arises \[3\]. This phase may be analytically connected to the ordinary hadronic phase \[6\]. Then there would be no supervector flop transition. However, when the strange quark is sufficiently heavy, a flop transition may exist.

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