A generalization of Scheunert’s Theorem on cocycle twisting of color Lie algebras

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March 31, 2022
preliminary version

Abstract

A classical theorem of Scheunert on $G$-color Lie algebras, asserts in the case of finitely generated abelian groups, one can twist the algebra structure and the commutation bicharacter on $G$ by a 2-cocycle twist to a super-Lie $G$ graded, algebra. In this paper we show that this can be done for an arbitrary group.

Introduction and notation

We recall first the following definitions (see [Sch] and [Mo]): Let $G$ be a group and $\chi : G \times G \to k^*$ a bicharacter on $G$, i.e. a bimultiplicative morphism. We assume that $\chi$ is symmetric, i.e. $\chi(h, g) \chi(g, h) = 1$ for all $h, g \in G$. Since $k$ is abelian it follows that $\chi$ is trivial for every commutator in $G$ so it factors through $G^{ab} \times G^{ab} \to k^*$. Therefore from now on we assume $G$ to be abelian.

In this paper we assume that $k$ is an algebraically closed field, $e$ will denote the the neutral element of $G$.

We call $L$ a $G$-color Lie algebra over $k$ with commutation factor $\chi$ if $L$ is a $G$-graded $k$-vector space and the bracket $[\ , ] : L \times L \to L$ satisfies:

$[a , b] = -\chi(h, g)[b , a]$

$\chi(g, k)[[a , [b , c]] + \chi(k, h)[c , [a , b]] + \chi(h, g)[b , [c , a]]$ for all $a \in L_g, b \in L_h, c \in L_k$

We say that $\sigma$ is a 2 cocycle on $G$ if $\sigma : G \times G \to k^*$ , satisfies

$\sigma(a, bc)\sigma(b, c) = \sigma(a, b)\sigma(ab, c)$. Then we can define a new bracket $[\ , ]_\sigma : L \times L \to L$ by:

$[a , b]_\sigma = \sigma(g, h)[a , b]$ for all $a \in L_g, b \in L_h$.

If $\sigma$ is a 2 cocycle then $\chi_\sigma(g, h) = \chi(g, h)\sigma(g, h)\sigma^{-1}(h, g)$ is a bicharacter. We denote by $L^\sigma$ the (new) $G$-color Lie algebra structure on $L$ for this new bracket $[\ , ]_\sigma$ and the commutation factor given by the twisted bicharacter $\chi_\sigma$. 

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Let \( G_+ = \{ g|\chi(g, g) = 1 \} \), this is a subgroup of \( G \) of index at most 2, we call these the even elements in \( G \). Define the odd elements by: \( G_- = \{ g|\chi(g, g) = -1 \} \), then \( G = G_+ \cup G_- \).

We may define now \( \chi_o(g|g) = 1 \) iff at least one of \( g \) or \( h \) is even else if both \( G \) and \( H \) are odd \( \chi_o(g|h) = -1 \).

Scheunert’s theorem [Sch] shows that for a \( G \) color Lie algebra \( L \) with bicharacter \( \chi \) and \( G \) a finitely generated abelian group there exists a 2-cocycle \( \sigma \) on \( G \) such that the bicharacter \( \chi \sigma = \chi_o \). Thus \( L^\sigma \) can be regarded as a \( \mathbb{Z}_2 \) graded Lie algebra with the \( \mathbb{Z}_2 \) (super)bicharacter \( \chi_o \).

**The proof for an arbitrary abelian group \( G \)**

In this section we prove:

**Theorem** Let \( G \) be any abelian group and let \( L \) be a \( G \) color Lie algebra with commutation factor \( \chi \) Then there exists a bimultiplicative 2-cocycle \( \sigma \) on \( G \) such that the twisted color Lie algebra \( L^\sigma \) is a super-Lie algebra with commutation factor \( \chi_o \).

Proof.

Like in the original proof in [Sche] we may change \( \chi \) to \( \chi \chi_o \) so that we may assume that \( \chi(g, g) = 1 \) for all \( g \in G \). We show then that if \( \chi(g, g) = 1 \) for all \( g \in G \) then there is a cocycle \( \sigma \) on \( G \) with \( \chi(g, h) = \sigma(g, h)\sigma^{-1}(h, g) \) for all \( g, h \in G \). Note that any bimultiplicative map is automatically a 2-cocycle.

To do that we shall use Zorn’s lemma. Define a family of subgroups of \( G \)

\[ \mathcal{F} = \{(H, \sigma_H)|H \text{subgroup of } G, \sigma_H \text{ bilinear } 2\text{-cocycle on } H, \chi(g, h) = \sigma(g, h)\sigma^{-1}(h, g)\ni g, h \in H\} \]

We order this family by \( (H', \sigma_{H'}) \preceq (H'', \sigma_{H''}) \) iff \( H' \subseteq H'' \) and \( \sigma_{H''}|_{H'} = \sigma_{H'} \). It is clear that \( e \in \mathcal{F} \) so that \( \mathcal{F} \) is non-void. When in the sequel it is clear on what subgroup \( \sigma \) is defined, we shall not show any more the indices.

This way \( \mathcal{F} \) is inductively ordered and by Zorn’s lemma there exists a maximal element of \( \mathcal{F} \), say \( (K, \sigma_K) \). Assume \( K \neq G \). We shall prove this contradicts the maximality of \( K \).

Let \( t \) be an element in \( G \) that does not belong to \( K \). We look at the subgroup \( < t > \) generated by \( t \).

If \( < t > \cap K = \{ e \} \) then let \( L = < t > \times K \). Define \( \sigma(k, t) = \chi(k, t) \) and \( \sigma(t, k) = 1 \) for all \( k \in K \) and extend \( \sigma \) bimultiplicatively. Since there are no new relations this is well defined and one can see that \( \chi(g, h) = \sigma(g, h)\sigma^{-1}(h, g) \) holds on \( L \).

If \( < t > \cap K =< t^n > \) then there are some \( k_1, k_2 \ldots k_m \in K \) and some positive integers \( n_1, n_2 \ldots n_m \) such that \( t^n = k_1^{n_1} k_2^{n_2} \ldots k_m^{n_m} \). More than one such relations is possible but we just select one, say with a minimal \( m \).

Define now \( L =< t, K > \) to be the subgroup generated by \( K \) and \( t \). We need to extend \( \sigma \) to \( L \) in such a way that:

1) \( \sigma \) is well defined and bimultiplicative on \( L \)
2) $\chi(g, h) = \sigma(g, h)\sigma^{-1}(h, g)$ for all $g, h \in L$.

Because $t^n = k_1^{n_1}k_2^{n_2} \ldots k_m^{n_m}$ it is clear that for any $u \in L$ we have $\sigma(u, t^n) = \sigma(u, k_1^{n_1}k_2^{n_2} \ldots k_m^{n_m})$.

This means is $\sigma(u, t^n)$ is already determined, so loosely speaking we may say $\sigma(u, t) = \sqrt[n]{\prod_i \sigma(u, k_i^{n_i})}$

The problem is that while we have $n$-th roots, $k$ being algebraically closed, we do not have a uniform radical function (say like the real radical), so we need to make sure that we define $\sigma$ as a function multiplicative on both first and second variable.

We start by defining a multiplicative function $f(u) = \sigma(u, t)$, $f : K \rightarrow k^*$ (multiplicative in $u$) such that:

$$f(u)^n = \sigma(u, t^n) = \sigma(u, k_1^{n_1}k_2^{n_2} \ldots k_m^{n_m})$$ and $f(t^n) = 1$

We let $\mathcal{M}$ be the family of subgroups of $K$ that contain $< t^n >$, on which $f$ can be defined with the above properties, ordered by set inclusion and by the requirement that $f$ extends from the small subgroup to the bigger one.

Then $\mathcal{M}$ is non-void and inductively ordered hence it has a maximal element $M$. If this maximal element is not $K$ itself say $M \subset K$ and $M \neq K$ then we may contradict the maximality of $M$.

For an $w \in K - M$ we extend $f$ to $< w, M >$ by:

If $< w > \cap M = \{e\}$ then let $f(w)$ be any selection of $\sqrt[n]{\prod_i \sigma(w, k_i^{n_i})}$. This works since $< w > \cap M = \{e\} < w > \times M$ and contradicts the maximality of $M$ unless $M = K$.

Else if $< w > \cap M = < w^r >$ and $w^r = \prod z_i^{r_i}$, (a finite product) define:

$$f(w) = \sqrt[n]{\prod_i \sigma(z_i, k_i)^{n_i}}$$

This contradicts again the maximality of $M$ and it means there is a multiplicative mapping $f(u) = \sigma(u, t) : K \rightarrow k^*$ such that $f(u)^n = \sigma(u, t^n) = \sigma(u, k_1^{n_1}k_2^{n_2} \ldots k_m^{n_m})$ and $f(t^n) = 1$

We use now the required relation to move $u$ on the right side by defining now an analog of $f$ on the “right”:

$$\sigma(t, u) = \chi(t, u)\sigma(u, t)$$

This is multiplicative in the second variable, i.e. in $u$, because $\chi$ is bimultiplicative and also $f$ is multiplicative.

Since $\chi(g, g) = 1$ was granted we define $\sigma(t, t) = 1$, this is consistent with the previous definitions (and this was the reason we asked $f(< t^n >) = 1$).

Now we define $\sigma$ on all $< t, K >$ by

$$\sigma(t^\alpha u, t^\beta v) = \sigma(t, v)^\alpha \sigma(u, t)^\beta \sigma(u, v)$$

This is bimultiplicative because of the way it was defined. One can use the fact that $f$ is multiplicative, to show that $\sigma$ is well defined. One needs to show that $\sigma$ respects the relation: $t^n = k_1^{n_1}k_2^{n_2} \ldots k_m^{n_m}$, when substituted on either side. For this we look at reduced forms of $t^\alpha u$, with $\alpha < n$. The relation holds because of the way that $f(t)$ was defined. This way we contradict now the maximality of $K$ so we may conclude $K = G$ and the proof of our theorem.
Remark There is another really interesting instance of twisting in the paper by Artin-Schelter-Tate [AST]. It is proved there that the multiparametric quantum general linear group is a twist of the standard quantization of the general linear group.

In fact we are interested in the result of Proposition 1 in [AST], where \( G \) is a free abelian group of dimension \( n < \infty \), it is shown that any cocycle cohomology class in \( H^2(G, k^*) \) contains exactly one bicharacter on \( G \). We conjecture this is the case for an arbitrary abelian group somehow along a construction similar to that of \( \sigma \) in the proof above.

Remark In fact the proof here does not fully use the fact that \( k \) is algebraically closed. Assume that we use transfinite induction to find the following presentation for \( G \): \( G \) is given by a system of generators \( \{ t_\lambda \}_{\lambda \in \Lambda} \) such that for each generator \( t_\lambda \) there is a unique relation \( r(t_\lambda) : t^{n_\lambda} = k_1^{n_1} k_2^{n_2} \ldots k_m^{n_m} \).

In our proof we only used the fact that \( k \) was closed under radicals of orders equal to the numbers \( n_\lambda \).

Corollaries. The ones given in [Sch]: PBW bases and Ado’s Theorem.

References

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