Twist and Spin-Statistics Relation in Noncommutative Quantum Field Theory

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Abstract

The twist-deformation of the Poincaré algebra as symmetry of the field theories on noncommutative space-time with Heisenberg-like commutation relation is discussed in connection to the relation between a sound approach to the twist and the quantization in noncommutative field theory. The recent claims of violation of Pauli’s spin-statistics relation and the absence of UV/IR mixing in such theories are shown not to be founded.
1 Introduction

Quantum field theories on noncommutative space-time with Heisenberg-like commutation relation

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \]  

where \( \theta_{\mu\nu} \) is a antisymmetric constant matrix, have been thoroughly investigated during the past years, after it had been shown that they appear as low-energy limits of string theory in a constant antisymmetric background field [1] (for a review, see [2]). Two features of such noncommutative quantum field theories (NC QFTs) render them specially interesting: the nonlocal interaction and the violation of Lorentz invariance. The latter feature is obvious by inspecting the commutation relation (1.1), in which \( \theta_{\mu\nu} \) does not transform under Lorentz transformations. However, translational invariance is preserved. The nonlocality becomes apparent when we write the product of functions on the commutative counterpart of the above-defined noncommutative space time, by the well-known Weyl-Moyal correspondence, as a \( \star \)-product:

\[ \phi(x) \star \psi(x) = \phi(x) e^{i \oint \theta_{\mu} \theta^{\mu\nu} \nabla^\nu \psi(x)}. \]  

The infinite number of derivatives involved in the \( \star \)-product induces the nonlocality in the interaction terms of the Lagrangean. One of the most interesting effects of the nonlocality is that the high-energy (short-distance) behaviour is influenced by the topology (long-distance effect) of the space-time [3] and, as a by-product, the UV/IR mixing [4] (an effect specific to string theory) appears in the NC QFT on noncompact spaces.

Since NC QFTs are peculiar in many ways, it seemed natural to investigate whether Pauli’s spin-statistics relation is violated. The idea was first mentioned in [5] and investigated in more detail in [6], based on the equal-time commutation relation of observables of free fields. The conclusion was that, at least in NC QFTs with commutative time, i.e. \( \theta_{0i} = 0 \), the spin-statistics relation holds. The only hint to a possible violation [6] was in theories with light-like noncommutativity, which are well-defined low-energy limits of string theory, without unitarity problems [7], but still acausal due to the nonlocality in time [8]. Later on, in [9] the spin-statistics relation was shown to hold also in the axiomatic formulation of NC QFT with commutative time (\( \theta_{0i} = 0 \)). In all these investigations, and indeed in the whole
literature on the subject, it was assumed that the fields are still in the representations of the Lorentz group, though it was obvious the Lorentz symmetry was violated. The justification of this treatment came when it was realized that NC QFT has twisted-Poincaré symmetry and as such it has exactly the same representation content as the usual Poincaré-invariant QFT \[\text{[10]}\] (since the twist does not affect the algebra of the generators of the Poincaré symmetry, but only their action in a tensor product of representations, i.e. their co-product, see also \[\text{[11]}\].)

Recently, however, the twisted Poincaré symmetry of NC QFT was exploited in a new manner, leading to claims that the spin-statistics relation does not hold in NC QFT \[\text{[12]}\] and the UV/IR mixing disappears \[\text{[13]}\]. In short, it was stated that in order to be ”compatible with the deformed action of the Poincaré group” \[\text{[13]}\], the standard commutation relations of creation and annihilation operators have to be also deformed, and not identical with those of the usual QFT, as it is taken in the ”traditional NC QFT”\[^*\].

In this letter we shall prove that the truly twisted-Poincaré compatible approach to NC QFT is the traditional one, in which the spin-statistics relation holds and UV/IR mixing remains, while the new approach \[\text{[12, 13]}\] is in effect a commutative theory.

\section{Twisted Poincaré symmetry and canonical quantization}

We shall not repeat the construction of the twisted Poincaré algebra, but refer the reader to \[\text{[10, 14]}\] and references therein. It suffices here to say that an Abelian twist element is introduced

\[\mathcal{F} = e^{\frac{i}{2} \theta^\mu P_\mu \otimes P_\nu}, \tag{2.3}\]

where \(P_\mu\) is the generator of the (Abelian) translation subalgebra of the Poincaré algebra and \(P_\mu \otimes P_\nu\) is the tensor product of generators. This twist element does not affect the

\[^*\] Though NC QFT is a relatively new field of research, we shall call the well-known approach based on the Weyl-Moyal correspondence ”traditional NC QFT”, to differentiate it from the ”deformed-statistics” approach of \[\text{[12, 13]}\].
algebra of the Poincaré generators, but deforms the action of the Lorentz generators \( M_{\mu\nu} \) in the tensor product of representations. Moreover, the twist changes the multiplication in the algebra of representations (in the case of field theory, the algebra of representations is the algebra of the fields). \( \mathcal{A} \) is the algebra of functions of the coordinates of the Minkowski space, which carries the following representation of the Poincaré algebra:

\[
[P_{\mu}, \phi(x)] = (P_{\mu} \phi)(x) = i \partial_{\mu} \phi(x) ,
\]

\[
[M_{\mu\nu}, \phi(x)] = (M_{\mu\nu} \phi)(x) = i(x_{\mu} \partial_{\nu} - x_{\nu} \partial_{\mu}) \phi(x) .
\] (2.4)

The product of the elements of the algebra \( \mathcal{A} \) (the product of fields) is deformed upon twisting as

\[
\phi(x) \ast \psi(y) = m \left( e^{-\frac{i}{2} \theta_{\mu\nu} P_{\mu} \otimes P_{\nu} \phi(x) \otimes \psi(x)} \right)
= e^{-\frac{i}{2} \theta_{\mu\nu} P_{\mu} \otimes P_{\nu} \phi(x) \psi(y)} = e^{\frac{i}{2} \theta_{\mu\nu} \partial_{\mu} \partial_{\nu} \phi(x) \psi(y)} ,
\] (2.5)

which is the generalization of the \( \ast \)-product (1.2) defined through Weyl-Moyal correspondence (see also [2]). By taking in (2.5) \( \phi(x) = x_{\mu} \) and \( \psi(x) = x_{\nu} \), one obtains immediately the Moyal bracket of coordinates, \([x_{\mu}, x_{\nu}]_{\ast} = i \theta_{\mu\nu} \). The \( \ast \)-product of two functions written as Fourier expansions

\[
\phi(x) = \int d^4 p \ \tilde{\phi}(p)e^{-ipx} , \quad \psi(y) = \int d^4 k \ \tilde{\psi}(k)e^{-iky}
\] (2.6)

is the one induced also by the Weyl-Moyal correspondence

\[
\phi(x) \ast \psi(y) = \int d^4 p \ d^4 k \ \tilde{\phi}(p)\tilde{\psi}(k)e^{-ipx} \ast e^{-iky}
= \int d^4 p \ d^4 k \ \tilde{\phi}(p)\tilde{\psi}(k)e^{-\frac{i}{2} \theta_{\mu\nu} \partial_{\mu} \partial_{\nu}} e^{-i(px + ky)} .
\] (2.7)

where \( \tilde{\psi}(k)\tilde{\phi}(p) = \tilde{\phi}(p)\tilde{\psi}(k) \).

The Lagrangean of a noncommutative field theory, e.g., the NC \( \lambda \Phi^4 \) theory, is built therefore with \( \ast \)-products instead of usual products of field:

\[
\mathcal{L}_{\ast}(x) = \frac{1}{2} \partial^\mu \Phi(x) \ast \partial_\mu \Phi(x) - \frac{1}{2} m^2 \Phi(x) \ast \Phi(x) - \frac{\lambda}{4!} \Phi(x) \ast \Phi(x) \ast \Phi(x) \ast \Phi(x) ,
\] (2.8)

and it is twisted-Poincaré invariant.
Thus by the deformation of Poincaré algebra with the twist \[2.3\], one reproduces the construction of NC field theory by Weyl-Moyal correspondence. It is well-known that under the integration over the whole space-time one ⋆-product drops out, therefore the action of the free NC field is the same as the action of the corresponding commutative field. They also satisfy the same equation of motion, therefore one can use the same mode-expansion for the free NC hermitian scalar field as for the commutative one:

\[
\Phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}2E_p} \left[ c(p)e^{-ipx} + c^\dagger(p)e^{ipx} \right], \quad p_0 = E_p = \sqrt{p^2 + m^2}.
\] (2.9)

2.1 Traditional approach in the light of twist

Quantization in operator formulation versus path-integral formulation

When one attempts the canonical quantization of a free NC field, one has to be careful with the canonical commutation relation imposed on the field \(\Phi(x)\) and its conjugated momentum \(\Pi(x) = \frac{\delta L}{\delta \dot{\Phi}(x)}\), since such a commutation relation involves \textit{products} and thus the multiplication, to be compatible with the twisted Poincaré symmetry of the NC space-time, has to be effected by the ⋆-product. The ⋆-product induces an infinite nonlocality in the noncommuting directions, i.e. an infinite speed of propagation of signal in these directions, leading to an alteration of the causality condition, for example, which should be formulated as non-correlation of the events out of each other’s light-wedge \[15\] and not light-cone. Therefore, the equal-time (canonical) commutation relation of the commutative case

\[
[\Phi(t, x), \Pi(t, y)] = i\delta(x - y)
\] (2.10)

has to be also suitably modified to take into account the nonlocality in the NC directions.

However, one can bypass many difficulties of the operator formulation and quantize the theory using the path integral approach, subsequently drawing also conclusions on the canonical quantization procedure.

The straightforward generalization of the generating functional \(W(J)\) to the NC case is:

\[
W(J) = \int \prod_x D\mu(u_1(x), \ldots, u_n(x))
\]
\[
\times \exp \left[ i \int d^4x \left( \mathcal{L}_i(x) + u_1(x) \star J_1(x) + \ldots + u_n(x) \star J_n(x) \right) \right]
\]
\[
= \int \prod_x \mathcal{D}_\mu(u_1(x), \ldots, u_n(x))
\times \exp \left[ i \int d^4x \left( \mathcal{L}_0(x) + \mathcal{L}_i(x) + u_1(x)J_1(x) + \ldots + u_n(x)J_n(x) \right) \right],
\]
where \( u_i(x), i = 1, \ldots, n \) are the fields entering the noncommutative Lagrangian density \( \mathcal{L}_i \), which is obtained from its commutative counterpart by replacing the usual products by \( \star \)-products, as indicated in the previous section. The fact that the \( \star \)-product drops out in the quadratic terms under the integration over the whole space-time is taken into account.

The expression (2.11) is twisted Poincaré covariant, since the integration measure does not change under twist.

It is common knowledge in the traditional approach to NC QFT that by using the generating functional (2.11) one obtains an S-matrix expansion equivalent to the Dyson expansion obtained in the operator formulation by using noncommutative form of the interaction Lagrangean, but the usual commutation relations of the creation and annihilation operators in the in-fields, e.g.

\[
[c(p), c^\dagger(q)] = 2E_p \delta(p - q)
\]
\[
[c(p), c(q)] = [c^\dagger(p), c^\dagger(q)] = 0,
\]
for the scalar field (2.9).

In this quantization framework for the noncommutative fields, the requirement of commutative time, \( \theta_{0i} = 0 \), is natural, since this is the situation which does not violate unitarity [7], nor causality [8]. In this case, one can always choose a frame of reference (the inertial frames of reference are not equivalent, since Lorentz invariance is violated) in which only two space directions are noncommutative, e.g. \( \theta_{12} = -\theta_{21} \neq 0 \), all the other components of the matrix \( \theta_{ij} \) being zero. In this configuration, which we shall adopt throughout, noncommutativity is in the plane \((x_1, x_2)\), while the coordinates \( x_0 \) and \( x_3 \) commute among themselves and with the others.

With the hint that the commutation relation of the creation and annihilation operators remains the same as in the commutative case (2.12), we turn to the twisted-Poincaré analog
of (2.10), to see which alterations the noncommutativity induce on it. The simple equal-time \(\ast\)-commutator of the free field (2.13) with its canonically conjugated momentum

\[
[\Phi(t, x), \Pi(t, y)]_\ast,
\tag{2.13}
\]

turns out to be operator valued on the Hilbert space of quantum states. However, the matrix elements of (2.13) between two states of equal total momentum \(|\Psi_i(P_\mu)\rangle\) and \(|\Psi_f(P_\mu)\rangle\) (i.e., the matrix elements corresponding to physical transitions) have the familiar form

\[
\langle \Psi_f(P_\mu) | [\Phi(t, x), \Pi(t, y)]_\ast | \Psi_i(P_\mu) \rangle = i\delta(x - y)
\tag{2.14}
\]

if \(|\Psi_i(P_\mu)\rangle = |\Psi_f(P_\mu)\rangle\) (diagonal matrix elements). If \(|\Psi_i(P_\mu)\rangle \neq |\Psi_f(P_\mu)\rangle\), then the r.h.s. of (2.14) vanishes, just as in the commutative case.

The \(\ast\)-commutator of free scalar fields at two space-time points is also operator-valued on the Hilbert space of states, but its diagonal matrix elements are

\[
\langle \Psi_f(P_\mu) | [\Phi(x), \Phi(y)]_\ast | \Psi_i(P_\mu) \rangle = i\Delta_c(x - y), \quad |\Psi_i(P_\mu)\rangle = |\Psi_f(P_\mu)\rangle
\tag{2.15}
\]

where \(\Delta_c(x - y)\) is the causal function:

\[
\Delta_c(x - y) = -\frac{i}{2(2\pi)^3} \int d^4k \epsilon(k_0)\delta(k^2 - m^2)e^{-ikx},
\tag{2.16}
\]

implying

\[
\langle \Psi_f(P_\mu) | [\Phi(x), \Phi(y)]_\ast | \Psi_i(P_\mu) \rangle = 0, \quad \text{for} \quad (x_0 - y_0)^2 - (x - y)^2 < 0, \quad |\Psi_i(P_\mu)\rangle = |\Psi_f(P_\mu)\rangle.
\tag{2.17}
\]

Again, for \(|\Psi_i(P_\mu)\rangle \neq |\Psi_f(P_\mu)\rangle\), the r.h.s. of (2.17) vanishes, as expected by comparison with the commutative case.

Thus, for the NC free scalar hermitian field, the physically meaningful matrix elements of the \(\ast\)-equal time commutation relation (2.14) or \(\ast\)-commutator of fields (2.17) have the same expressions as in the commutative case. As a consequence, when deriving the Feynman rules for the NC case, one obtains the same propagator as in the corresponding commutative theory.

The situation becomes different when considering NC interactions. A similar matrix element has been calculated for the Heisenberg field of an interacting scalar field theory, in
one loop, but the result was generalized to any order in perturbation theory [16]. The only difference was that the commutator was calculated without \( \star \)-product between the fields, but, as it will be shown later, for Heisenberg fields it is not important whether one takes their commutator with \( \star \)-product (as one rigorously should) or without. Thus, according to [16] and having in view the above comment, for scalar Heisenberg fields one obtains the diagonal matrix elements

\[
\langle \Psi(P_\mu) | [\Phi_H(x), \Phi_H(y)]_\star | \Psi(P_\mu) \rangle = i \Delta(x_0 - y_0, x_3 - y_3) F(x_1 - y_1, x_2 - y_2, \theta_{12}), \tag{2.18}
\]

where \( F(x_1 - y_1, x_2 - y_2, \theta_{12}) \) is a function which may vanish in a finite number of points and

\[
\Delta_c(x_0 - y_0, x_3 - y_3) = -\frac{i}{2(2\pi)^3} \int dk^0 dk^3 \epsilon(k_0) \delta(k_0^2 - k_3^2 - m^2)e^{-ik_0(x_0 + k^3 x_3)}
\]

is the analog of the causal function \( \Delta_c(x - y) \), implying the "light-wedge" causality condition (see [15])

\[
\langle \Psi(P_\mu) | [\Phi_H(x), \Phi_H(y)]_\star | \Psi(P_\mu) \rangle = 0, \quad \text{for } (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \tag{2.19}
\]

Correspondingly, the diagonal matrix elements of the \( \star \)-commutator of a scalar Heisenberg field with its canonically conjugated momentum shall read

\[
\langle \Psi(P_\mu) | [\Phi_H(t, x), \Pi_H(t, y)]_\star | \Psi(P_\mu) \rangle = i \delta(x_3 - y_3) G(x_1 - y_1, x_2 - y_2, \theta_{12}), \tag{2.20}
\]

where \( G(x_1 - y_1, x_2 - y_2, \theta_{12}) \) has similar properties with \( F(x_1 - y_1, x_2 - y_2, \theta_{12}) \). Again, for \( |\Psi_i(P_\mu)\rangle \neq |\Psi_f(P_\mu)\rangle \), the r.h.s. of both (2.19) and (2.20) are zero.

It appears thus that the physical matrix elements of commutators of nonlocal operators, like (2.19), vanish outside the light-wedge, as expected from nonlocality considerations.

In brief, this is the essence of the traditional approach to NC QFT, leading to the usual spin-statistics relation [6, 9] and UV/IR mixing [4].

### 2.2 Deformed-statistics approach

In [12, 13], it was argued that, since the Fourier transform \( \tilde{\phi}(p) \) of a function \( \phi(x) \) is also a linear representation of the momentum generator \( P_\mu \) of the Poincaré algebra, the product of
Fourier transforms should be also deformed upon twisting the Poincaré algebra. Pursuing this line of thought, it was claimed that the creation and annihilation operators satisfy deformed commutation relations. We do not repeat the argumentation here, but merely cite their outcome, using the common convention for Fourier expansion (2.6).†

A free quantum scalar field of mass \( m \) is expanded as:

\[
\Phi(x) = \int d\mu(p) \left[ a(p)e^{-ipx} + a^\dagger(p)e^{ipx} \right],
\]

where \( d\mu(p) = \frac{d^3p}{(2\pi)^{3/2}2E_p} \) and \( p_0 = E_p = \sqrt{P^2 + m^2} \). The deformed creation and annihilation operators \( a^\dagger(p) \) and \( a(p) \) were represented in [12] in terms of the nondeformed ones, \( c^\dagger(p) \) and \( c(p) \), as

\[
a(p) = c(p)e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu},
\]

\[
a^\dagger(p) = e^{\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu}c^\dagger(p),
\]

where \( c^\dagger(p) \) and \( c(p) \) satisfy the usual commutation relations (2.12) and

\[
P_\mu = \int d\mu(p)p_\mu c^\dagger(p)c(p) = \int d\mu(p)p_\mu a^\dagger(p)a(p)
\]

is the quantum momentum operator, generating a linear representation on the creation and annihilation operators:

\[
[P_\mu, a(p)] = -p_\mu a(p), \quad [P_\mu, c(p)] = -p_\mu c(p)
\]

\[
[P_\mu, a^\dagger(p)] = p_\mu a(p), \quad [P_\mu, c^\dagger(p)] = p_\mu c(p).
\]

Using (2.22) and (2.24), we can write down the deformed commutation relations of the creation and annihilation operators \( a^\dagger(p) \) and \( a(p) \):

\[
a(p)a(q) = e^{-iq\theta p}a(q)a(p),
\]

\[
a^\dagger(p)a^\dagger(q) = e^{-iq\theta p}a^\dagger(q)a^\dagger(p),
\]

\[
a(p)a^\dagger(q) = e^{iq\theta p}a^\dagger(q)a(p) + 2E_p\delta(p - q),
\]

†The conventions used in [12, 13] are different from the conventions usually used in the literature and in this letter, but amounting, essentially, to taking the Fourier expansion as \( \phi(x) = \int d^4p\phi(p)e^{ipx} \) and \( \theta_{\mu\nu} \rightarrow -\theta_{\mu\nu} \) in (1.1).
\[ a^\dagger(p)a(q) = e^{iq\theta p}a(q)a^\dagger(p) - 2E_p\delta(p - q), \tag{2.25} \]

where the notation \( q\theta p = q_\mu \theta^{\mu\nu} p_\nu \) is used. A typical term of the product of fields \( \Phi \star \Phi \) is

\[ a(p)a(q)e^{-ipx} \ast e^{-iqy}. \tag{2.26} \]

It is obvious that the multiparticle-states \( |n\rangle = a^\dagger(p_1)a^\dagger(p_2) \cdots a^\dagger(p_n)|0\rangle \) described by the scalar field with the above quantization are not symmetric, therefore they do not satisfy the Bose-Einstein statistics. Therefore, it was argued \[12\] that the spin-statistics relation, in this case, does not hold. Moreover, with the commutation rules of the creation and annihilation operators (2.25) it was concluded in \[13\] that the S-matrix in this case turns out to be identical to the corresponding one in the commutative case, consequently the UV/IR mixing does not appear.

At this stage it is interesting to see what is the equal-time commutation relation of fields and conjugated momenta and what is the causality condition analog to (2.19) to which these new commutation rules for creation and annihilation operators (2.25) lead. A straightforward calculation gives:

\[ [\Phi(t,x), \Pi(t,y)]_\star = i\delta(x - y) \tag{2.27} \]

and

\[ [\Phi(x), \Phi(y)]_\star = i\Delta_c(x - y), \tag{2.28} \]

where \( \Delta_c(x - y) \) is the usual four-dimensional causal function (2.16), leading to

\[ [\Phi(x), \Phi(y)]_\star = 0, \quad \text{for} \quad (x_0 - y_0)^2 - (\vec{x} - \vec{y})^2 < 0. \tag{2.29} \]

It is obvious from the calculations that the \( \star \)-product of the exponentials from the mode expansion (2.21) is exactly canceled by the deformed commutation rules of the creation and annihilation operators (2.25). In effect, for any two functions \( \phi(x) = \int d^4p \tilde{\phi}(p)e^{-ipx} \) and \( \psi(y) = \int d^4q \tilde{\psi}(q)e^{-iqy} \), with \( \tilde{\phi}(p)\tilde{\psi}(q) = e^{-iq\theta p}\tilde{\psi}(q)\tilde{\phi}(p) \), according to \[12\], one has:

\[
\phi(x) \star \psi(y) = \int d^4p d^4q \tilde{\phi}(p)\tilde{\psi}(q)e^{-ipx} \ast e^{-iqy} = \int d^4p d^4q \tilde{\phi}(p)\tilde{\psi}(q)e^{-i\frac{1}{2}q\theta p}e^{-ipx - iqy} = \int d^4p d^4q e^{-i\frac{1}{2}q\theta p}\tilde{\psi}(q)\tilde{\phi}(p)e^{i\frac{1}{2}q\theta p}e^{-ipx - iqy}
\]

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Thus the $\ast$-product of two functions of $x$ is commutative, thereby rendering any $\ast$-product of quantum fields in the S-matrix expansion of an interacting theory precisely as in the corresponding commutative theory. With the same argument, the commutator of Heisenberg fields of a theory with interactions will be the same as in the corresponding commutative case:

$$[\Phi_H(x), \Phi_H(y)]_\ast = 0 \quad \text{for} \quad (x_0 - y_0)^2 - (x - y)^2 < 0.$$  \hfill (2.31)

Now the reason for the absence of the UV/IR mixing is cleared up: the theory constructed in \cite{12,13} is a local one, in spite of the nonlocal $\ast$-product specific to noncommutative field theories.

*Spin-statistics relation*

As for the spin-statistics relation, the situation is slightly more subtle. Indeed, the deformation of the commutation relations (2.12) into the form (2.25) implies a deformed statistics for a scalar field, which might be interpreted as a violation of the spin-statistics relation. However, since the theory under consideration in \cite{12} is a local relativistic scalar field theory, as is clearly seen from (2.31), it does fulfill all the requirements of Pauli’s spin-statistics theorem \cite{17}, but it contradicts its conclusion, the spin-statistics relation. This is an obvious indication that the corresponding NC theory is not properly quantized and raises the suspicion whether the deformed statistics (2.25) as a quantization procedure is introduced in a manner compatible with the twist, investigated in the next section.

### 2.3 Twisted multiplication of representations of $P_\mu$

To require that the Fourier transforms, as linear representations of the momentum generator $P_\mu$ undergo the action of the twist in a product just like any other representations of $P_\mu$, e.g. the exponentials $e^{-ipx}$, is certainly legitimate. However, in this case one should require that the product of Fourier transforms and exponentials corresponding to different momenta be also deformed. Therefore, the usual product between elements of different algebras of
representation of $P_\mu$, e.g.
\[ \tilde{\phi}(p)e^{-iqx} = e^{-iqx}\tilde{\phi}(p), \] (2.32)
as used in [12, 13], is not compatible with the concept of twist as a general abstract operation.

In effect, any tensor product of representations should be affected by the twist. In the above case, according to the general rule (2.10) of [10]:
\[ \tilde{\phi}(p)\star e^{-ipx} = m_t(\tilde{\phi}(p)\otimes e^{-iqx}) = m(e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu}\tilde{\phi}(p)\otimes e^{-iqx}) = e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu}\tilde{\phi}(p)e^{-iqx}, \] (2.33)
since
\[ [P_\mu, \tilde{\phi}(p)] = P_\mu\tilde{\phi}(p) = P_\mu\tilde{\phi}(p), \quad [P_\nu, e^{-iqx}] = P_\nu e^{-iqx} = q_\nu e^{-iqx}. \] (2.34)
Obviously, with this rule, the Fourier transform of a function is preserved as usual, since
\[ \tilde{\phi}(p)\star e^{-ipx} = \tilde{\phi}(p)e^{-ipx}. \]

Then the product of two functions defined on the Minkowski space, written as Fourier expansions (2.6), will be indeed given by the action of the twist in the product of four representations
\[ \tilde{\phi}(p)\star e^{-ipx} \star \tilde{\psi}(q) \star e^{-iqy} = m_t(\tilde{\phi}(p)\otimes e^{-ipx} \otimes \tilde{\psi}(q) \otimes e^{-iqy}). \]
The only nontrivial product under the twist is the one in the middle, $e^{ipx}\star\tilde{\psi}(q)$. Consequently
\[ \phi(x)\star\psi(y) = \int d^4p\, d^4q\, \tilde{\phi}(p)\star e^{-ipx} \star \tilde{\psi}(q) \star e^{-iqy} = \int d^4p\, d^4q\, \tilde{\phi}(p)\left(e^{-ipx} \star \tilde{\psi}(q)\right)e^{-iqy} \]
\[ = \int d^4p\, d^4q\, \tilde{\phi}(p)\tilde{\psi}(q)e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu e^{-ipx-iyq}} \]
\[ = \int d^4p\, d^4q\, \tilde{\phi}(p)\tilde{\psi}(q)e^{-ipx} \star e^{-iqy}, \] (2.35)
i.e., by considering the action of the twist on all multiplications of the representations of the momentum generator $P_\mu$, one obtains exactly the $\star$-product induced by the Weyl-Moyal correspondence (2.7). In other words, the traditional approach to NC QFT described in Subsection (2.1), with usual spin-statistics relation and the UV/IR mixing, is the one which is truly compatible with the twisted Poincaré symmetry.
3 Conclusions

In this letter we show that by consistently handling the twisted Poincaré algebra as the symmetry of NC field theory, namely the deformed multiplication in the tensor product of representations of the momentum generator $P_\mu$, one is led to a quantization procedure in operator formulation which preserves Pauli’s spin-statistics relation, for theories with commutative time. Moreover, the traditional approach based on the Weyl-Moyal correspondence for NC models is entirely recovered, together with its UV/IR mixing problems.

The claims of spin-statistics violation in theories with twisted Poincaré symmetry are thereby naturally rejected. By this approach one obtains essentially commutative, local quantum field theories with deformed statistics, thus contradicting Pauli’s spin-statistics theorem. The drawback of the construction leading to such claims is that the quantization of the NC field theory is performed by simply imposing (deformed) commutation relations between the creation and annihilation operators, instead of following a canonical quantization procedure. Technically, in a twist-deformed product of quantum fields $\Phi \ast \Phi$, the deformed commutation relations of creation and annihilation operators, e.g. $a(p)a(q) = e^{-iq\theta p}a(q)a(p)$, is not equivalent to the deformed product of those operators, $a(p) \ast a(q) = e^{-\frac{i}{2}p\theta q}a(p)a(q)$. Thus, at present, the only possible case in which violation of spin-statistics relation might appear within NC QFT seems to be in theories with light-like noncommutativity, $\theta_{\mu\nu}\theta^{\mu\nu} = 0$ [6]. The latter has no problems with unitarity and can be obtained as a low-energy limit of string theory.

NC QFT has an infinite range of nonlocality of interaction in the noncommuting directions, i.e. the nonlocality is not restricted to a finite range, such as the Planck scale. It is certainly desirable to find means of restricting this nonlocality, while still preserving some of its peculiarities which mimic stringy effects, in a more manageable context. However, the deformation of statistics (i.e. deformation of the commutation relations between creation and annihilation operators) in NC quantum field theories with commutative time is not the way to attain this scope.
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