GRADED LOCAL COHOMOLOGY OF MODULES OVER SEMIGROUP RINGS

LAURA FELICIA MATUSEVICH AND BYEONGSU YU

ABSTRACT. We give a combinatorial description of local cohomology modules of a graded module over a semigroup ring, with support at the graded maximal ideal. This combinatorial framework yields Hochster-type formulas for the Hilbert series of such local cohomology modules in terms of the homology of finitely many polyhedral cell complexes. A Cohen–Macaulay criterion immediately follows. We also provide an alternative proof of a result of [13] characterizing Cohen–Macaulay affine semigroup rings.

1. INTRODUCTION

An affine semigroup ring is a subalgebra of the Laurent polynomial ring (over a field) that is finitely generated by monomials. Such rings are the coordinate rings of affine toric varieties, and have received much attention for this reason.

Local cohomology is a fundamental notion in homological commutative algebra. In this article, we make a detailed study of the graded structure of local cohomology (supported at the graded maximal ideal) for modules over affine semigroup rings. In this case, one computes local cohomology using the Ishida complex, which takes into account the combinatorics of the underlying affine semigroup.

For a given graded module (over an affine semigroup ring), we need to study all localizations by monomials. To do this, we generalize a tool introduced in the context of monomial ideals in polynomial rings, namely the standard pairs of Sturmfels, Trung and Vogel [14,17]. In the general context of graded modules, we use the name degree pairs. These pairs organize the supporting degrees of a module according to the faces of the underlying semigroup. Since localization is also controlled by faces, the degree pairs naturally control the information associated to localization.

Putting all degree pairs of all localizations together, and suitably topologizing, we can partition the relevant supporting degrees that appear in the Ishida complex. The parts consist of lattice points in carefully constructed polyhedra. Understanding these polyhedra (and associated polyhedral cell complexes) we can classify the graded pieces of the Ishida complex.

The key point of these constructions is that there are only finitely many polyhedra involved. In particular, we may follow foundational ideas from Stanley–Reisner rings, and write Hilbert series for local cohomology as finite sums involving lattice point generating functions. When the semigroup is pointed, we write these Hilbert series as finite sums of rational functions, thus obtaining Hochster-type formulas in this case. Cohen–Macaulayness criteria directly follow.

Our tools also provide alternative proofs of celebrated results for affine semigroup rings, namely Hochster’s theorem that normal affine semigroup rings are Cohen–Macaulay, and the Cohen–Macaulayness criterion for affine semigroup rings by Trung and Hoa [18].

2020 Mathematics Subject Classification. Primary 13D45, 13F65, 05E40, 20M25; Secondary 13C14, 13F55, 14M25, 52B20.
Outline. Polyhedral geometry, affine semigroups, hyperplane arrangements, and the Ishida complex are discussed in Section 2. Section 3 introduces degree pairs and develops an injective map from the set of overlap classes of degree pairs of a localization to that of an original module. Section 4 introduces the concept of degree space, which is composed of all degrees of nonzero elements in any localizations of an affine semigroup. The degree space is endowed with a special topology called degree pair topology. Section 5 derives a Hochster-type formula for the local cohomology from the degree pair topology and proposes combinatorial Cohen–Macaulay criteria for quotients of affine semigroup rings by monomial ideals. Finally, Section 6 proves the combinatorial Cohen–Macaulay condition for affine semigroup rings in a different way.

Acknowledgments. We are grateful to Aida Maraj, Aleksandra Sobieska, Alexander Yong, Bernd Siebert, Catherine Yan, Christine Berkesch, Christopher Eur, Erika Ordog, Ezra Miller, Frank Sotille, Galen Dornalen-Barry, Heather Harrington, Jaeho Shin, Jennifer Kenkel, Jonathan Montaño, Joseph Gubeladze, Kenny Easwaran, Mahrud Sayrafi, Mateusz Michałek, Melvin Hochster, Patricia Klein, Sarah Witherspoon, Semin Yoo, Serkan Hoşten, Yupeng Li for inspiring conversations we had while working on this project.

Notation. We adopt the convention that $\mathbb{N} = \{0, 1, 2, \cdots \}$ is the set of nonnegative integers, $\mathbb{k}$ is an arbitrary infinite field, and $\mathbb{R}$ is a field of real numbers. Throughout this article, $a, b, c, \cdots$ refer to integers, while $a, b, c, \cdots$ refer to integer vectors. For any set $S$, $S^c$ is the complement of the set. Given a Laurent polynomial ring $\mathbb{k}[[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}]]$, let $t^a := t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$. When the context is obvious, $x, y$ and $z$ denote, respectively, $t_1, t_2$, and $t_3$.

2. Preliminaries

2.1. Polyhedral geometry. A polyhedron is a bounded polyhedron. Equivalently, a convex polytope is a bounded polyhedron. A polyhedron $P$ which is also a cone, meaning that $\lambda x \in P$ for all nonnegative real $\lambda$ and all $x \in P$.

A nonempty subset $F$ of a polyhedron $P$ in $\mathbb{R}^d$ is called a face of $P$ if there exists $c \in \mathbb{R}^d$ such that $F$ is the set where the dot product with $c$ over $P$ achieves its maximum value. In other words, $F = \text{face}_c(P) := \{a \in P \mid c \cdot a \geq \langle c, x \rangle \text{ for all } x \in P\}$. The faces of a polyhedron are also polyhedra. A face of $P$ is proper if it is a proper subset of $P$. By convention, $\emptyset$ is a face of every polyhedron.

The affine hull of a finite set $a_1, a_2, \cdots, a_m \in \mathbb{R}^d$ is the set of all real linear combinations $\sum_{i=1}^m \lambda_i a_i$ for which $\sum_{i=1}^m \lambda_i = 1$. The relative interior $\text{RelInt}(P)$ of a polyhedron $P$ is the interior of $P$ with respect to its affine hull. If $F$ is a face of $P$ then $\text{RelInt}(F)$ can also be described as the set of all points in $F$ that do not lie in any other proper face of $P$. The dimension of a nonempty polyhedron is defined to be the dimension of its affine hull; we set $\text{dim}(\emptyset) = -1$.

A polyhedron is pointed if it has a unique zero-dimensional face. It is easy to see that a pointed polyhedron whose unique zero-dimensional face is the origin is a (pointed) polyhedral cone.

The collection $F(P)$ of all faces in $P$ is called the face lattice of $P$. It is a lattice with respect to the partial order given by inclusion. Two polyhedra are combinatorially equivalent if their face lattices are order-isomorphic. If $P$ is a pointed polyhedral cone, then there exists a convex polytope whose face lattice is order-isomorphic to $F(P) \setminus \{\emptyset\}$ [19, Proposition 1.12, Exercise 2.19].
A polyhedral complex $\Delta$ is a collection of polyhedra satisfying

- for $F \in \Delta$ and $G \in \mathcal{F}(F)$, we have $G \in \Delta$, and
- for $F, G \in \Delta$, $F \cap G$ is a common face of both $F$ and $G$.

The face lattice of a polyhedron is an example of a polyhedral complex. Any simplicial complex on $n$ vertices can be realized as a polyhedral subcomplex of the simplex obtained by taking the convex hull of the standard basis vectors in $\mathbb{R}^n$.

2.2. Affine semigroups. An affine semigroup is a finitely generated submonoid of $\mathbb{Z}^d$. Throughout this article, we denote $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d \setminus \{0\}$ and work with the affine monoid $Q = \mathbb{N}A$ consisting of all nonnegative linear combinations of the elements of $A$. Where it causes no confusion, we also denote by $A$ the $d \times n$ integer matrix with columns $a_1, \ldots, a_n$. The set $\mathbb{R}_{>0}A = \mathbb{R}_{>0}A$ of nonnegative real combinations of elements of $Q$ (or $A$) is a polyhedral cone, called the underlying cone of $Q$. The dimension of an affine semigroup $Q$ is defined to be dimension of its underlying cone.

A subset $T$ of an affine semigroup $Q$ is called an ideal if $Q + T \subseteq T$. For any subset $S$ of $Q$, the ideal $\langle S \rangle$ generated by $S$ is the smallest ideal in $Q$ that contains $S$. An ideal $T$ is prime if for any two elements $a, b \in Q$, $a + b \in T$ implies $a \in T$ or $b \in T$. A subset of $Q$ is called a face of $Q$ if its complement is a prime ideal; the collection of faces of $Q$ is denoted by $\mathcal{F}(Q)$. It can be shown [13 Lemma 7.12] that there is a one to one correspondence between $\mathcal{F}(\mathbb{R}_{>0}Q)$ and $\mathcal{F}(Q)$, given by intersecting the faces of $\mathbb{R}_{>0}Q$ with $Q$. The relative interior of a face of $Q$ is defined to be the intersection of the relative interior of the corresponding face of $\mathbb{R}_{>0}Q$ with $Q$. If the context is clear, we may abuse notation and refer to a subset $F \subseteq A$ as a face of $\mathbb{N}A$, to indicate that $\mathbb{N}F$ is a face of $\mathbb{N}A$.

The set $\mathcal{H}(Q) := (\mathbb{Z}Q \cap \mathbb{R}_{>0}Q) \setminus Q$ is called the set of holes of $Q$. Here $\mathbb{Z}Q$ denotes the set of integer combinations of the elements of $Q$. If an affine semigroup contains no holes, it is said to be normal.

An affine semigroup ring $\mathbb{K}[Q] = \mathbb{K}[t^{a_1}, \ldots, t^{a_n}]$ is a subring of the Laurent polynomial ring $\mathbb{K}[t^{\pm}] = \mathbb{K}[t_{d1}^{\pm 1}, \ldots, t_{dn}^{\pm 1}]$. There is a natural bijection between the elements of an affine semigroup $Q$ and the monomials of the corresponding affine semigroup ring $\mathbb{K}[Q]$. This establishes a one to one correspondence between monomial ideals of $\mathbb{K}[Q]$ and ideals of $Q$. If $T$ is an ideal of $Q$, we denote the corresponding monomial ideal of $\mathbb{K}[Q]$ by $I$; more precisely, $I = \langle t^a | a \in T \rangle$. If $F$ is a face of $Q$, the ideal $P_F := \langle t^a | a \notin F \rangle$ is a corresponding to the complement of $F$ is a prime monomial ideal of $\mathbb{K}[Q]$; all prime monomial ideals of $\mathbb{K}[Q]$ arise in this way.

A set $S \subseteq Q$ is called additively closed if it contains 0 and is closed under addition. The localization $Q - \mathbb{N}S$ of $Q$ by an additively closed set $S$ is defined as $Q - \mathbb{N}S := Q + \mathbb{Z}S$. The localization of $Q$ by $S$ is equal to the localization of $Q$ by the minimal additively closed set containing $S$ whose complement is a prime ideal [6 Lemma 1.1].

**Lemma 2.1** ([6 Lemma 1.1]). Let $S \subseteq \mathbb{K}[Q]$ be a set of monomials that is multiplicatively closed and let $\mathbb{N}F$ be the minimal face of $Q$ containing $\{a \in Q \mid x^a \in S\}$. Then,

$$S^{-1}\mathbb{K}[Q] \cong \mathbb{K}[Q - \mathbb{N}F].$$

If $T \subset Q$ is an ideal corresponding to the monomial ideal $I$ in $\mathbb{K}[Q]$, and $F$ is a face of $Q$, the localization of $I$ at the prime ideal $P_F$, denoted $I_F$ corresponds to the ideal $T_F := T - \mathbb{N}F$ of the semigroup $Q - \mathbb{N}F$. 

A graded module $M = \bigoplus_{a \in \mathbb{Z}^d} M_a$ is finely graded if $\dim_k M_a \leq 1$ for all degrees $a \in \mathbb{Z}^d$. For any face $F \in \mathcal{F}(Q)$, $k[Q - NF]$ is finely $\mathbb{Z}^d$-graded as follows.

**Lemma 2.2.** $\dim_k (k[Q - NF])_a = 1$ if $a \in Q - NF$. Otherwise, $\dim_k (k[Q - NF])_a = 0$.

**Proof.** As $k[Q - NF] \subseteq k[ZQ]$, it suffices to show that $\dim_k (k[ZQ])_a \leq 1$. If $a \notin ZQ$, then $k[ZQ]_a = \{0\}$, otherwise $k[ZQ]_a = \text{Span}_k(t^a)$. \hfill \Box

An affine semigroup $Q$ is pointed if its corresponding cone $\mathbb{R}_{\geq 0}Q$ is a pointed polyhedron.

**Example 2.3.**

1. (Monomial curves) Let $Q = NA$ with $A = \left[ \begin{array}{c} 1 \\ 0 \\ a_1 \\ \cdots \\ a_{n-1} \end{array} \right]$ such that $0 < a_1 < \cdots < a_{n-1}$ are relatively prime integers. If $0, a_1, a_2, \ldots, a_{n-1}$ are consecutive integers, then $k[Q]$ is the coordinate ring of a rational normal curve; otherwise, $k[Q]$ is not normal.

   Figure 1a illustrates the example where $a_1 = 1, a_2 = 3, a_3 = 4$. Elements of the semigroup $Q$ are represented by filled dots. Since $\left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$ is a hole of $Q$ (in fact, it is the only hole of $Q$), it is depicted as an empty circle. Let $T = \langle \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \rangle$. The elements of $T$ are colored black, while the elements in $Q$ but not in $T$ are colored blue. This includes $\left[ \begin{array}{c} 3 \\ 2 \end{array} \right]$, which is not in $T$ because $\left[ \begin{array}{c} 1 \\ 2 \end{array} \right]$ is a hole of $Q$.

2. (Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$) Let $Q = NA$ with $A = \left[ \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{array} \right]$. The affine semigroup ring $k[Q]$ is isomorphic to $k[z, xz, yz, xy^2]/(ac - bd)$. The exponent vectors of monomials in the ideal $I = \langle x^2z^2, x^2y^2z^2, x^2y^3z^3, x^3y^3z^3 \rangle \subset k[Q]$ are depicted by black dots in Fig. 1b.

2.3. **Hyperplane arrangements.** A polyhedron $\mathcal{P}$ can be expressed as an intersection of finitely many open half-spaces $\{\mathcal{H}_i^{(+)}\}_{i=1}^n$, where $\mathcal{A} := \{\mathcal{H}_i\}_{i=1}^n$ is a collection of hyperplanes and $n \in \mathbb{N}$. This is known as the $\mathcal{H}$-representation of $\mathcal{P}$. The collection $\mathcal{A}$ is called the hyperplane arrangement of $\mathcal{P}$ in $\mathbb{R}^d$. Throughout this article, we assume that $\mathcal{A}$ is linear, meaning that all hyperplanes in $\mathcal{A}$ contain the origin. A region is a connected component $r$ of $\mathbb{R}^d - \bigcup_{\mathcal{H} \in \mathcal{A}} \mathcal{H}$. Let
\( \tau(A) \) be a set containing all regions over \( A \). These are standard notions in combinatorics, see for instance [16].

Since our arrangement is linear, all regions in \( \tau(A) \) are unbounded [16], constituting rational polyhedral cones. Moreover, any region \( r \) in \( \tau(A) \) can be expressed as

\[
\tau_S := \left( \bigcap_{i \in S} \mathcal{H}_i^{(+)} \right) \cap \left( \bigcap_{i \in [n] \setminus S} \mathcal{H}_i^{(-)} \right)
\]

for a subset \( S \subseteq [n] \), where \( \mathcal{H}_i^{(-)} \) is the complement of \( \mathcal{H}_i^{(+)} \cup \mathcal{H}_i \). The collection \( \tau(A) \) is partially ordered by reverse inclusion; \( \tau_{S_1} \subseteq \tau_{S_2} \) if \( S_1 \supseteq S_2 \). The poset \( \tau(A) \) is called the poset of region \( \tau(A) \). Under the order given by inclusion, this object has received much attention [3, 7].

We can label faces of the affine semigroup \( Q \) using the corresponding indices of its supporting hyperplanes. This gives rise to a natural embedding \( F(Q) \rightarrow \tau(A) \) [7, Lemma 1.3]. To partition \( \mathbb{R}^n \), we modify the definition of \( \tau_S \) as follows

\[
\tau_S := \overline{\left( \bigcap_{i \in S} \mathcal{H}_i^{(+)} \right) \cap \left( \bigcap_{i \in [n] \setminus S} \mathcal{H}_i^{(-)} \right)},
\]

where \( \overline{(-)} \) denotes the closure of the set in the standard \( \mathbb{R}^n \)-topology.

Similarly, given a region \( \tau_S \in \tau(A) \), a cumulative region \( \mathcal{R}_S \) is the closure of the union of all regions less than \( \tau_S \) in the partial order. In other words,

\[
\mathcal{R}_S := \bigcup_{S' \supseteq S, \tau_{S'} \in \tau(A)} \tau_{S'} = \overline{\left( \bigcap_{i \in S} \mathcal{H}_i^{(+)} \right)}.
\]

The poset of cumulative regions \( \mathcal{R}(A) \) is the collection of all cumulative regions ordered by inclusion. As posets, \( \mathcal{R}(A) \cong \tau(A) \). It is worth noting that every element in \( \mathcal{R}(A) \) is a rational polyhedral cone. The following result shows that \( \mathcal{R}(A) \) is the set of all polyhedral cones corresponding to localizations of \( Q \).

**Proposition 2.4.** Given a face \( F \in F(Q) \), let \( S \) be a set of indices of hyperplanes whose half-space contains \( F \). Then, \( \mathbb{R}_{\geq 0}(Q - NF) = \mathcal{R}_S \).

**Proof.** From \( \mathbb{Z}F \cup Q \subseteq \mathcal{R}_S \), \( \mathbb{R}_{\geq 0}(Q - NF) \subseteq \mathcal{R}_S \). Conversely, for any \( x \in \mathcal{R}_S \), \( \langle c, x \rangle \geq 0 \) when \( i \in S \). Pick \( f \in \text{RelInt}(NF) \) and let \( x' = x + \left( \sum_{i \in S} a_i \right) f \) where \( a_i \) is a non-negative real number such that \( \langle c, x + a_i f \rangle \geq 0 \). Then, \( x = x' + (x - x') \) with \( x' \in \mathbb{R}_{\geq 0}Q \) and \( (x - x') \in \text{span}(F) \). Thus, \( \mathbb{R}_{\geq 0}(Q - NF) = \mathbb{R}_{\geq 0}(\mathbb{Z}F \cup Q) \supseteq \mathcal{R}_S \). \( \square \)

Let \( \text{Cat}_Q \) be the poset containing all localizations of \( Q \) with an order by inclusion. Below, we describe all posets that arise in this section using a commutative diagram.

\[
F(Q) \xrightarrow{Q - \mathbb{N}(-)} \text{Cat}_Q \xrightarrow{\text{RelInt}(\mathbb{N})} \mathcal{R}(A) \xrightarrow{\cong} \tau(A) \xrightarrow{\phi} (2^A)^{\text{op}}
\]
Note that the embedding $\mathcal{F}(Q) \to \tau(\mathcal{A})$ in [7] Lemma 1.3] is split into the diagram above. Moreover, all posets are indexed by a subposet of $(2^\mathcal{A})^\text{op}$, a poset of subsets of $\mathcal{A}$ by reverse inclusion. The inclusion map $\phi$ returns the set of indices of positive half-spaces containing the given element.

**Example 2.5 (Continuation of Example 2.3).**

1. Let $\mathcal{P} = \mathbb{R}_{\geq 0}Q$ with $Q = \mathbb{N} \left[ \begin{smallmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right]$. Then $\mathcal{P}$ is a 2-dimensional cone with facets (rays) $\mathbb{R}_{\geq 0} \left[ \begin{smallmatrix} 0 \\ 1 \\ 1 \\ 1 \end{smallmatrix} \right]$ and $\mathbb{R}_{\geq 0} \left[ \begin{smallmatrix} 1 \\ a_1 \\ a_2 \\ \cdots \end{smallmatrix} \right]$. Hence $\mathcal{A} = \{ \mathcal{H}_1 := \mathbb{R} \left[ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right], \mathcal{H}_2 := \mathbb{R} \left[ \begin{smallmatrix} 1 \\ a_{n-1} \end{smallmatrix} \right] \}$. Since $\mathcal{P}$ is a homogenization of the 1-simplex, $\tau(\mathcal{A}), \mathcal{R}(\mathcal{A}), \mathcal{F}(Q)$ and $\mathbf{Cat}_Q$ are all isomorphic as posets.

| $\mathcal{F}(Q)$ | $\mathbf{Cat}_Q$ | $\mathcal{R}(\mathcal{A})$ | $\tau(\mathcal{A})$ | $2^\mathcal{A}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0               | $Q$             | $\mathcal{P}$   | $\mathcal{P} = \mathcal{H}_1^{(+)} \cap \mathcal{H}_2^{(+)}$ | $\{ \mathcal{H}_1, \mathcal{H}_2 \}$ |
| $F_1$           | $Q - NF_1$      | $\mathcal{H}_1^{(+)} = \{ y \geq 0 \}$ | $\mathcal{H}_1^{(+)} \cap \mathcal{H}_2^{(+)}^{-} = \{ y \geq 0, y > a_{n-1}x \}$ | $\{ \mathcal{H}_1 \}$ |
| $F_2$           | $Q - NF_2$      | $\mathcal{H}_2^{(+)} = \{ y \leq a_{n-1}x \}$ | $\mathcal{H}_1^{(-)} \cap \mathcal{H}_2^{(+)} = \{ y < 0, y \leq a_{n-1}x \}$ | $\{ \mathcal{H}_2 \}$ |
| $Q$             | $\mathbb{Z}^2$  | $\mathbb{R}^2$  | $\mathcal{H}_1^{(-)} \cap \mathcal{H}_2^{(-)} = \{ y < 0, y > a_{n-1}x \}$ | $\varnothing$ |

2. Given $Q = \mathbb{N} \left[ \begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right]$, denote $a_i$ for the $i$-th column of $\left[ \begin{smallmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right]$. Then, as seen below, we may represent facets $F_i$ and the hyperplane arrangement $\mathcal{A} = \{ \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \}$.

$F_1 := \langle a_1, a_2 \rangle$, $F_2 := \langle a_2, a_3 \rangle$, $F_3 := \langle a_3, a_4 \rangle$, $F_4 := \langle a_4, a_1 \rangle$

$\mathcal{H}_1^{(+)} := \{ y > 0 \}$, $\mathcal{H}_2^{(+)} := \{ z > x \}$, $\mathcal{H}_3^{(+)} := \{ z > y \}$, $\mathcal{H}_4^{(+)} := \{ x > 0 \}$.

Observe that $\mathbb{R}_{\geq 0}Q - F_i = \mathcal{R}_{\{i\}}$ for any $i \in [n]$ and

$\mathbb{R}_{\geq 0}(Q - \langle a_1 \rangle) = \mathcal{R}_{1,4}$, $\mathbb{R}_{\geq 0}(Q - \langle a_2 \rangle) = \mathcal{R}_{1,2}$,

$\mathbb{R}_{\geq 0}(Q - \langle a_3 \rangle) = \mathcal{R}_{2,3}$, $\mathbb{R}_{\geq 0}(Q - \langle a_4 \rangle) = \mathcal{R}_{3,4}$.

Thus, $\mathcal{R}(\mathcal{A}) \supsetneq \mathbb{R}_{\geq 0}(\mathbf{Cat}_Q)$, for example, because localization cannot generate affine semigroups in $\mathcal{R}_{1,2,3}$. This illustrates the nontrivial injection $\tau(\mathcal{A}) \cong \mathcal{R}(\mathcal{A}) \supsetneq \mathcal{F}(Q) \cong \mathbf{Cat}_Q$, described in Fig. 2.
2.4. Ishida complex. Using the polyhedral cone structure of an affine semigroup, the Ishida complex was developed to compute the local cohomology of modules over pointed affine semigroup rings supported on the graded maximal ideal \([10]\). Given a pointed affine semigroup \(Q\), \(\mathbb{R}_{\geq 0}Q\) has a transverse section \(K\) \([19]\) Exercise 2.19], which is a closed polytope generated by intersecting \(\mathbb{R}_{\geq 0}Q\) with a hyperplane \(\mathcal{H}\) that intersects all unbounded faces of \(\mathbb{R}_{\geq 0}Q\). This results in the following canonical isomorphism \(\hat{\sim} : \mathcal{F}(K) \rightarrow \mathcal{F}(Q)\) where \(\hat{\mathcal{F}}\) is the minimal face of \(Q\) such that \(\mathbb{R}_{\geq 0}\hat{\mathcal{F}} \supseteq F\). The vertex of \(Q\) corresponds to the (-1)-dimensional face \(\emptyset\) of \(K\). \(K\) is a CW complex with an incidence function

\[
\epsilon : \bigoplus_{i=-1}^{d-1} \mathcal{F}(K)^i \times \mathcal{F}(K)^{i-1} \rightarrow \{0, \pm 1\},
\]

in which \(\epsilon\) is determined by an orientation of \(K\) and has a nonzero value when two faces are incident \([12]\) IV. § 5.

**Definition 2.6** (Ishida complex \([10]\)). Let \(m\) be the maximal monomial ideal of \(\mathbb{k}[Q]\). The set of all \(k\)-dimensional faces in \(\mathcal{F}(K)\) is denoted by \(\mathcal{F}(K)^k\). Let \(L^\bullet\) be the chain complex

\[
L^\bullet : 0 \rightarrow L^0 \xrightarrow{\partial} L^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L^d \xrightarrow{\partial} 0,
\]

where the differential \(\partial : L^k \rightarrow L^{k+1}\) is induced by componentwise map \(\partial_{F,G}\) with \(F \in \mathcal{F}(K)^{k-1}\), \(G \in \mathcal{F}(K)^k\) such that

\[
\partial_{F,G} : \mathbb{k}[Q - \hat{\mathcal{F}}] \rightarrow \mathbb{k}[Q - \hat{\mathcal{G}}] \text{ to be } \begin{cases} 0 & \text{if } F \not\subset G \text{ nat} \\ \epsilon(F, G) \cdot \text{nat} & \text{if } F \subset G \end{cases}
\]

with nat, the canonical injection \(\mathbb{k}[Q - \hat{\mathcal{F}}] \rightarrow \mathbb{k}[Q - \hat{\mathcal{G}}]\) when \(F \subseteq G\). We say that \(L^\bullet \otimes_{\mathbb{k}[Q]} M\) is the Ishida complex of a \(\mathbb{k}[Q]\)-module \(M\) supported at the maximal monomial ideal.

The Ishida complex indeed computes local cohomology as follows.

**Theorem 2.7** \([10]\) Theorem 6.2.5]. For any graded \(\mathbb{k}[Q]\)-module \(M\), and all \(k \geq 0\),

\[
H^k_m(M) \cong H^k(L^\bullet \otimes_{\mathbb{k}[Q]} M).
\]

**Example 2.8** (Continuation of Example 2.3).

1. Given \(Q/T = \mathbb{N} \left[ \begin{array}{c} 0 \ 1 \ 1 \ 1 \\ 1 \ 0 \ 1 \ 1 \\ 1 \ 1 \ 0 \ 0 \\ 1 \ 1 \ 1 \ 0 \end{array} \right] / \langle \{1\} \rangle\), let \(S = \mathbb{k}[Q]/I\), where \(I\) is a monomial ideal corresponding to \(T\). The transverse section of \(\mathbb{R}_{\geq 0}Q\) is a line segment with vertices \(F_1 = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\
\end{array} \right]\) and \(F_2 = \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\
\end{array} \right]\) respectively. Thus, the corresponding Ishida complex of \(S\) with the maximal ideal support is

\[
L^\bullet : 0 \rightarrow S \rightarrow S_x \oplus S_{xy^4} \rightarrow 0 \rightarrow 0
\]

2. Given \(Q/T = \mathbb{N} \left[ \begin{array}{c} 0 \ 1 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \\ 1 \ 0 \ 1 \ 1 \\ 1 \ 1 \ 0 \ 0 \end{array} \right] / \langle \left[ \begin{array}{c} 0 \ 0 \ 1 \\ 1 \ 1 \\ 0 \ 0 \ 1 \ 1 \\
\end{array} \right]\rangle\), let \(S = \mathbb{k}[Q]/I\), where \(I\) is a monomial ideal corresponding to \(T\). \(\mathbb{R}_{\geq 0}Q\)’s transverse section is rectangular. Due to the fact that all other localizations of \(S\) are zero except for the localizations by monomial prime ideals corresponding to \(\hat{\mathcal{A}}_1 := \left[ \begin{array}{c} 0 \\ 0 \\
\end{array} \right]\), \(\hat{\mathcal{A}}_4 := \left[ \begin{array}{c} 1 \\ 0 \\
\end{array} \right]\), and \(F_4 := \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\
\end{array} \right]\), the Ishida complex of \(S\) with the maximal ideal support is as follows:

\[
L^\bullet : 0 \rightarrow S \rightarrow S_z \oplus S_{y^z} \rightarrow S_{z,y^z} \rightarrow 0
\]
Let $Q = \mathbb{N}A$ be an affine semigroup. We know that the faces of $Q$ govern the localizations of any $\mathbb{Z}^d$-graded $k[Q]$-module $M$ by monomials. In this section, we examine the effect of localization on the supporting multidegrees of $M$, defined as follows.

**Definition 3.1.** Let $M$ be a $\mathbb{Z}^d$-graded $k[Q]$-module.

1. The **degree set** of $M$ is defined to be 
   $$\text{deg}(M) := \{a \in \mathbb{Z}^d \mid M_a \neq 0\}.$$ 
2. A **proper pair** of $M$ is a pair $(a, F)$ where $a \in Q$ and $F \in \mathcal{F}(Q)$ such that $a + NF \subseteq \text{deg}(M)$.
3. If $(a, F)$ and $(b, G)$ are proper pairs, we say $(a, F) < (b, G)$ if $a + NF \subseteq b + NG$. A proper pair $(a, F)$ of $M$ is called a **degree pair** of $M$ if it is maximal among proper pairs in this partial order.

**Lemma 3.2.** Any finitely generated $\mathbb{Z}^d$-graded $k[Q]$-module $M$ has finitely many degree pairs.

**Proof.** Let $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ be a chain of submodules of $M$ such that $M_{i-1}/M_i \cong k[\mathbb{N}A]/P_i$ where $P_i$ is a graded prime ideal of $k[\mathbb{N}A]$.

Thus, $\text{deg}(M_i/M_{i-1}) = a_i + NF_i$ for some $a_i \in \mathbb{Z}^d$ and a face $F_i$ corresponding to $P_i$. To see that 
$$\text{deg}(M) = \text{deg}(M_0/M_0 \oplus \cdots \oplus M_l/M_{l-1}) = \bigcup_{i=1}^{l} a_i + NF_i,$$

note that for any homogeneous element $m \in M$, there exists $i$ such that $\overline{m} \in M_i/M_{i-1}$ is nonzero. Hence, $\text{deg}(m) \in a_i + NF_i$. Conversely, we may lift any graded element in the direct sum to $M$. This says we have a finite pair cover, from here to finitely many standard pairs as demonstrated by [14, Theorem 3.16].

Two degree pairs $(a, F)$ and $(b, F)$ with the same face $F$ **overlap** if the intersection $(a + NF) \cap (b + NF)$ is nonempty. Overlapping is an equivalence relation. The **overlap class** $[a, F]$ is the equivalence class containing the degree pair $(a, F)$. We define $\text{deg}, p(M)$ (resp. $\overline{\text{deg}}, \overline{p(M)}$) as the set of all (resp. overlap classes of) degree pairs of $M$. These definitions are based on [14, Definition 3.1 and 3.2], which apply to general (not necessarily pointed) affine semigroup rings, but only in the monomial ideal case.

**Example 3.3** (Standard pairs and void pairs).

1. Let $I$ be a monomial ideal of $k[\mathbb{N}A]$. The degree pairs of $M = k[\mathbb{N}A]/I$ are the **standard pairs** of $I$ introduced in [17] and generalized in [14, Definition 3.1 and 3.2].
2. Given an affine semigroup $Q := \mathbb{N}A$, the **saturation of** $Q$ is $Q_{\text{sat}} = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}Q$. It is known that the affine semigroup ring corresponding to the saturation of of $Q$ is the normalization of $k[Q]$. The set of **holes** of $Q$ is defined to be the difference $Q_{\text{sat}} \setminus Q$.

The set of holes of $Q$ is also the degree set $\text{deg}(k[Q_{\text{sat}}]/k[Q])$. As the $k[Q]$-module $M = k[Q_{\text{sat}}]/k[Q]$ is finitely generated by Noether’s normalization lemma, applying Lemma 3.2 provides an alternative algebraic proof of the well-known combinatorial result [8]. Namely, the set of holes of a semigroup $Q$ is a finite union of translates of faces of $Q$. Later on, we refer to degree pairs of $M$ as **void pairs**.
Example 3.4 (Continuation of Example 2.8).

(1) Degree pairs of $M := \mathbb{k}[x, xy, xy^3, xy^4]/\langle xy \rangle$ are

(green) $\left( \left[ \begin{smallmatrix} 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\
4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 
\end{smallmatrix} \right], \emptyset \right)$, (blue) $\left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_1 \right)$, (red) $\left( \left[ \begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right], F_2 \right)$, $\left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_2 \right)$.

In Fig. 3a these are indicated by a green (dotted) circle, a blue (dashed) line, and red (straight) lines. Each of the standard pairs forms an overlap class.

(2) Let $A := \left[ \begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right]$. Degree pairs of $M := \mathbb{k}[N A] / I$ where $I = \langle x^2 z^2, x^2 y z^2, x^2 y^3 z^3, x^3 y^3 z^3 \rangle$ is a monomial ideal of $\mathbb{k}[z, xz, xyz, yz] \cong \mathbb{k}[N A]$ are

(green) $\left( \left[ \begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\
2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 
\end{smallmatrix} \right], \emptyset \right)$, (blue) $\left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_4 \right)$, (yellow) $\left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_4 \right)$, (red) $\left( \left[ \begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right], F_4 \right)$.

In Fig. 3b these are indicated by a green circle, a blue triangle (in $zy$-plane), a yellow triangle, and a red triangle (in $x = 1$ plane), respectively. As illustrated in Fig. 3b the yellow and red triangles represented by $\left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_4 \right)$ and $\left( \left[ \begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right], F_4 \right)$ overlap. Hence, the overlap classes of $I$ are

$\left\{ \left( \left[ \begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\
2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 
\end{smallmatrix} \right], \emptyset \right) \right\}$, $\left\{ \left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_4 \right) \right\}$, $\left\{ \left( \left[ \begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 
\end{smallmatrix} \right], F_4 \right), \left( \left[ \begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right], F_4 \right) \right\}$.

Notably, the union $\left( \left[ \begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right] + NF_4 \right) \cup \left( \left[ \begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 
\end{smallmatrix} \right] + NF_4 \right)$ is a subset of translates of faces represented by standard pairs of $I_{F_4}$, the localization of $I$ by a face $F_4$.

We assert that localization by a monomial prime ideal $P_F$ generates an injective map between sets of overlap classes of degree pairs, $\text{deg}_{p(M)}$ and $\text{deg}_{p(M_{P_F})}$. For economy of notation, let $M_{F}$ be the localization of $M$ by a monomial prime ideal $P_F$ corresponding to a face $F \in \mathcal{F}(Q)$.

To begin, we see that the face lattice $\mathcal{F}(Q - NF)$ of the affine semigroup $Q - NF$ arising from localization by the face $F$ can be identified as a subset of the face lattice $\mathcal{F}(Q)$ as follows.
Lemma 3.5. The maps
\[ F(Q - NF) \to \{ G \in F(Q) \mid G \supseteq F \} \quad \text{given by} \quad NG' \mapsto N(G' \cap A) \]
\[ \{ G \in F(Q) \mid G \supseteq F \} \to F(Q - NF) \quad \text{given by} \quad NG \mapsto NG - NF \]
are bijective.

Proof. It suffices to show that \( F(Q - NF) \) and \( \{ G \in F(Q) \mid G \supseteq F \} \) are in bijection. Fix \( G \in F(Q) \). Let \( c \) be an outer normal vector so that \( G = \text{face}_c(Q) \). Recall that the absolute maximum of the functional \( \langle c, - \rangle \) on \( \mathbb{R}_{\geq 0}Q \) is zero since every face of \( Q \) contains the origin.

We claim
\[ \text{face}_c(Q - NF) = \begin{cases} \emptyset & \text{if } F \nsubseteq G \\ G \cup (-F) & \text{if } F \subseteq G. \end{cases} \]

If \( F \) is not a face of \( G \), there exists a nonzero element \( f \in F \setminus G \) such that \( \langle c, f \rangle < \langle c, g \rangle = 0 \) for any \( g \in G \). Since \( \langle c, -mf \rangle \) diverges when \( m \to \infty \), \( \text{face}_c(Q - NF) = \emptyset \). If \( F \) is a face of \( G \), \( G \cup (-F) \subseteq \text{face}_c(Q - NF) \). Pick \( b \in \text{face}_c(Q - NF) \cap (Q - NF) \). Then, \( b = a - f \) for some \( a \in Q \) and \( f \in NF \). Since \( 0 = \langle c, b \rangle = \langle c, a \rangle, a \in NG \). Thus, \( b \in G \cup (-F) \).

As a consequence of this result, we can express any face of \( Q - NF \) as \( NG - NF \) for some face \( G \in F(Q) \) such that \( G \supseteq F \). Likewise, \( (a, G \cup (-F)) \) denotes a degree pair of a \( \mathbb{k}[Q - NF] \)-module \( M_F \).

Our next step is to show that each degree pair of a localization of \( M \) can be lifted to a degree pair of \( M \).

Lemma 3.6. Suppose \( G \supseteq F \in F(Q) \). Given a degree pair \( (a, G \cup (-F)) \) of \( M_F \), there exists \( a' \in \deg(M) \) such that \( (a', G \cup (-F)) = (a, G \cup (-F)) \) and \( (a', G) \) is a degree pair of \( M \).

By abuse of notation, let \( t^{-\infty} = 0 \in \mathbb{k}[Q] \).

Proof. Assume that \( \{m_1, m_2, \ldots, m_l\} \) is a minimal generating set of \( M \) with \( \deg(m_i) = a_i \). Select an appropriate \( f \in NF \) so that \( a + f \in Q \). Let \( c_i = a + f - a_i \) if \( c_i \in \deg(Q) \) and \( t^{c_i}m_i \neq 0 \) or \( c_i = -\infty \) otherwise. Set \( m := \left( \sum_{i=1}^{l} t^{c_i}m_i \right) \in M \). Then, \( m/t^f \in M_F \) is a nonzero homogeneous element of degree \( a \); otherwise no element of \( M_F \) with degree \( a \) can be generated. Hence, \( m \) is a nonzero homogeneous element of degree \( a + f \). Also, \( (\deg(m), G) \) is a proper pair of \( M \), otherwise, if \( a + f + g \notin \deg(M) \), then no element of \( M_F \) with degree \( a + f + g \) exists. Thus, a degree pair \( (a', G') \) exists that contains \( (\deg(m), G) \) with \( G' \supseteq G \). We may assume that \( m' = \sum_{i=1}^{l} t^{c_i'}m_i \) is of order \( a' \) with \( c_i = a' - a_i \in Q \) or \( c_i = -\infty \). Since \( \deg(m) = a' + g' \) for some \( g' \in NG' \) and \( c_i' \neq -\infty \) if \( c_i \neq -\infty \), \( t^{g'm'} = m \).

Furthermore, we claim \( G' = G \). Suppose not, then we can have \( c \in NG' \setminus NG \) such that \( a + c \notin \deg(M_F) \) by the maximality of \( (a, G \cup (-F)) \). Thus, \( t^{c}m/t^f = 0 \), which implies \( t^{c+g'} \cdot m' = 0 \), contradicting the fact that \( c + g' + a' \in \deg(M) \). Hence, \( g' \in NG \).

Finally, we assert \( (a', G \cup (-F)) = (a, G \cup (-F)) \). Indeed \( a = a' + g' - f \) indicates that \( a \notin a' + N(G \cup (-F)) \), implying \( (a', G \cup (-F)) > (a, G \cup (-F)) \). Also, \( (a', G \cup (-F)) \) is a proper pair; otherwise, we would not have an element whose degree is in \( a + N(G \cup (-F)) \), contradiction. These two degree pairs are same due to the maximality of \( (a, G \cup (-F)) \).
The choice of \( \alpha' \) is not unique; see Example 3.10(2). Fortunately, their overlap class is uniquely determined.

**Lemma 3.7.** Suppose \( G \supseteq F \in \mathcal{F}(Q) \). Given two overlapping degree pairs \( (a, G \cup (-F)) \) and \( (b, G \cup (-F)) \) of \( M_F \), let \( \alpha \) and \( \beta \) be degrees of \( \deg(M) \) chosen by Lemma 3.6. Then, \( (\alpha', G) \) and \( (\beta', G) \) overlap.

**Proof.** From \( (a, G \cup (-F)) = (\alpha', G \cup (-F)) \) and \( (b, G \cup (-F)) = (\beta', G \cup (-F)) \), there exists \( g_\alpha, g_\beta \in NG, f_\alpha, f_\beta \in NF \) such that \( \alpha' + g_\alpha - f_\alpha = a + g_\beta - f_\beta \). Hence, \( \alpha' + g_\alpha + f_\beta = a + g_\beta + f_\alpha \in \deg(M) \). Again, \( \alpha' + g_\alpha + f_\beta \in \deg(M) \) when a set of minimal generators is fixed and a similar construction in the proof of Lemma 3.6 is used.

As a consequence of the lemma above, we obtain the desired injective map between sets of overlap classes under localization. We provide a new notation to describe this map. Given an overlap class \([a, F] \in \deg \cdot p(M) \), let \( \bigcup [a, F] := \bigcup_{(b, F) \in [a, F]} b + NF \). In other words, \( \bigcup [a, F] \) is the union of all translates of faces represented by degree pairs in \([a, F]\).

**Theorem 3.8.** Suppose \( F \subseteq G \subseteq H \) are faces of \( Q \). Given an overlap class \([a, H \cup (-F)] \in \deg \cdot p(M_G) \), there exists a unique overlap class \([\alpha', H \cup (-F)] \in \deg \cdot p(M_F) \) such that

\[
\bigcup [\alpha', H \cup (-F)] = \left( \bigcup [a, H \cup (-F)] \right) \cap \deg(M_F).
\]

We denote this injection \([\alpha', H \cup (-F)] = \res_{G,F} ([a, H \cup (-F)]) \), and call it restriction. These restriction maps satisfy that for any \( F \subseteq G, G' \subseteq H, \res_{H,G} \circ \res_{G,F} = \res_{H,F} = \res_{H,G'} \circ \res_{G',F} \).

**Proof of Theorem 3.8.** It is sufficient to show that the map is defined in the case \( F = 0 \). For a given overlap class \([a, H] \in \deg \cdot p(M_G) \), let \( \res_{G,0} ([a, H]) := [\alpha', H] \) where \( \alpha' \) is determined by Lemma 3.6. As demonstrated in Lemma 3.7, \( \res_{G,0} \) is well-defined and injective. Moreover, by analogy to the construction of elements of \( M_F \) with degrees in \( \bigcup [a, H] \) in the proof of Lemma 3.6, \( \bigcup [a, H] = (\bigcup [\alpha', H]) \cap \deg(M) \). Associativity is clear from the definition.

Finally, we provide a statement about void pairs, which is used in Section 6.

**Corollary 3.9.** Let \( \{F_i\}_{i=1}^m \) be the set of all facets of a (not-necessarily pointed) affine semigroup \( Q \). Let \( M := \mathbb{k}[Q_{sat}] / \mathbb{k}[Q] \). If \( Q \neq \bigcap_{i=1}^m Q - NF_i \), then there exists a void pair \( (a, F) \) such that \( F \) is not a facet.

**Proof.** By the previous results, if the only void pairs \( Q \) arise from facets, then \( Q = \bigcap_{i=1}^m Q - NF_i \).

**Example 3.10** (Continuation of Example 3.4).

1. Given \( M = \mathbb{k} \left[ \mathbb{N} \left[ \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \right] \right] / \langle t_1^1 \rangle \), all overlap classes of \( \deg(M) \) are singletons. Indeed,

\[
\deg \cdot p(M) = \{ (\text{green}) ([\frac{1}{2}], \emptyset), (\text{blue}) ([\frac{1}{2}], F_1), (\text{red}) ([\frac{2}{3}], F_2), ([\frac{3}{4}], F_2) \}
\]
\[
\deg \cdot p(M_{F_1}) = \{ (\text{blue}) ([\frac{1}{3}], F_1 \cup (-F_1)) \}
\]
\[
\deg \cdot p(M_{F_2}) = \{ (\text{red}) ([\frac{2}{3}], F_2 \cup (-F_2)), ([\frac{3}{4}], F_2 \cup (-F_2)) \}.
\]

This shows two injections \( \overrightarrow{\deg \cdot p(I_{F_1})} \rightleftharpoons \overrightarrow{\deg \cdot p(I)} \leftrightarrow \overrightarrow{\deg \cdot p(I_{F_2})} \).
(2) Given \( M = \mathbb{k}[\mathbb{N} \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right]] / \left\langle \left[ \begin{array}{ccc} 2 & 2 & 3 \\ 0 & 1 & 3 \\ 3 & 2 & 3 \end{array} \right] \right\rangle \), the set of degree pairs of ideals in each localizations are as follows.

\[
\deg \cdot p(M) = \left\{ \left( \text{green} \left[ \begin{array}{c} 2 \\ 2 \end{array} \right], \emptyset \right), \left( \text{blue} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], F_4 \right), \left( \text{yellow} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], F_4 \right), \left( \text{red} \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], F_4 \right) \right\}
\]

\[
\deg \cdot p(M_G) = \left\{ \left( \text{blue} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], F_4 \cup (-G) \right), \left( \text{orange} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], F_4 \cup (-G) \right) \right\}
\]

for any \( G \in \{ a_1, a_4, F_4 \} \). Indeed,

\[
\left( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right], F_4 \cup (-F_4) \right) = \left( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], F_4 \cup (-F_4) \right) = \left( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], F_4 \cup (-F_4) \right).
\]

This is an example of how the selection of \( \alpha' \) in Lemma 3.6 is not unique. Nonetheless, we have an injection \( \deg \cdot p(I_G) \rightarrow \deg \cdot p(I) \) by sending the orange overlap class to the overlap class consisting of yellow and red standard pairs.

4. Degree Space with Degree Pair Topology

We now construct the degree space of \( M, \bigcup \deg(M) := \bigcup_{F \in \mathcal{F}(Q)} \deg(M_F) \), a topological space formed by gluing all degree pairs of localizations of a module. This structure enables us to simultaneously record all degrees resulting from localizations. Moreover, the minimal open sets of \( \bigcup \deg(M) \), called \textit{grains}, partition all degrees belonging to a fixed collection of localizations. A grain’s \textit{chaff} is the poset of all localizations that contain the given grain. Together with Section 2.4 these tools yield a Hochster-type formula for the Hilbert series of the local cohomology of \( M \) in Section 5.

**Definition 4.1** (Degree space and degree pair topology). The degree space \( \bigcup \deg(M) \) of a finitely generated \( \mathbb{k}[Q] \)-module \( M \) is the union of \( \deg(M_F) \) for all faces \( F \) of \( \mathcal{F}(Q) \). The degree pair topology is the smallest topology on \( \bigcup \deg(M) \) such that for any face \( F \in \mathcal{F}(Q) \) and for an overlap class \( [a, G \cup (-F)] \in \deg \cdot p(M_F) \), the set \( \bigcup [a, G \cup (-F)] \) is both open and closed.

**Definition 4.2** (Grain and chaff). In the degree pair topology, we refer to a minimal nonempty open set as a \textit{grain} of \( \bigcup \deg(M) \). Let \( \mathcal{G}(M) \) be the set of all such grains. The \textit{chaff} of a grain \( G, D_G \), is defined as the collection of all localizations of \( M \) containing \( G \).

These names are inspired by the agricultural metaphors of Grothendieck; we bundle degree pairs on \( \bigcup \deg(M) \) and thresh (topologize) them in order to obtain grains. Chaff is a layer of grain that provides information about the grain’s containment in certain localizations.

**Remark 4.3.** The sectors and sector partition introduced in [13] are almost the same the grains and chaff used in this article. The main differences are the topological context, and that grains actually refine the sector partition.

**Lemma 4.4.** The degree pair topology has finitely many open sets.

**Proof.** From Lemma 3.2, \( \deg \cdot p(M_F) \) is finite. Also, \( \mathcal{F}(Q) \) is finite. Finally, a subbase including all overlap classes and their complements over all localizations is used to generate the topology. Hence the topology has finitely many open sets.

Our next result is that the grain set \( \mathcal{G}(Q/T) \) partitions the degrees of a module.
Proposition 4.5. \( \mathcal{G}(M) \) partitions \( \bigcup \deg(M) \) and is therefore a basis of the degree pair topology.

Proof. It suffices to show that \( \mathcal{G}(M) \) partitions \( \bigcup \deg(M) \). First of all, for any two elements \( S \) and \( S' \) of \( \mathcal{G}(M) \), \( S \cap S' = \emptyset \). Otherwise, \( S \) and \( S' \) cannot be minimal nonempty opens, a contradiction.

To see that \( \mathcal{G}(M) \) covers \( \bigcup \deg(M) \), suppose \( a \in \deg(M_F) \) for some face \( F \). Let

\[
S := \left( \bigcap_{a \in [b,G] \in \deg (M_F)} \left( \bigcup [b,G] \right) \right) \cap \left( \bigcap_{a \in [b,G] \in \deg (M_F)} \left( \bigcup [b,G]^c \right) \right).
\]

This is a nonempty open set since \( a \in S \). We claim that \( S \in \mathcal{G}(M) \). Suppose not; then there exists an open set \( S' \subseteq S \). By the property of the subbase, we may let \( S' \subseteq S \cap \left( \bigcup [b,G] \right) \subseteq S \) or \( S' \subseteq S \cap \left( \bigcup [b,G]^c \right) \subseteq S \) for some overlap class \([b,G] \). If \( S' \subseteq S \cap \left( \bigcup [b,G] \right) \) holds, then \( \left( \bigcup [b,G]^c \right) \) contains \( a \), thus \( \left( \bigcup [b,G]^c \right) \cap S = S \) by the construction of \( S \). This implies that \( S' = \emptyset \). If \( S' \subseteq S \cap \left( \bigcup [b,G]^c \right) \) holds, then \( S \cap \left( \bigcup [b,G] \right) = S \) implies \( S' = \emptyset \). In both cases, \( S' \) is empty, a contradiction. \( \square \)

Example 4.6 (Continuation of Example 3.10).

1. Given \( M = \mathbb{k}[N \{1, 1, 1, 1\}] / \langle xy \rangle \), \( \bigcup \deg(M) \) is the union of integral points in \( y = 4x - 1, y = 4x - 2, y = 0 \) and \( \{ (2,2) \} \). Moreover, 10 grains

   (red) \([\frac{1}{4}] + \mathbb{N} F_2, \frac{1}{3} + \mathbb{N} F_2, \frac{2}{7} + \mathbb{N} F_1 \),
   (blue) \( \frac{1}{6} + \mathbb{N} F_1, \frac{2}{3} + \mathbb{N} (-F_1) \),
   (cyan) \( \frac{1}{4} + \mathbb{N} (-F_2), \frac{1}{2} + \mathbb{N} (-F_2) \),
   (yellow) \( \frac{5}{6}, \frac{3}{2} \)
are depicted in Fig. 4a. Two grains with the same color have the same chaff. Indeed, for the given grain $G$ with color from Fig. 4a

- (red) $D_G := \{0, F_2\}$
- (blue) $D_G := \{0, F_1\}$
- (cyan) $D_G := \{F_1\}$
- (orange) $D_G := \{F_2\}$
- (green) $D_G := \{0\}$
- (yellow) $D_G := \{0, F_1, F_2\}$

Note that $\lfloor \frac{1}{2} \rfloor$ is a hole filled by the localization with respect to $F_2$, and therefore lies only in the degree pair of $M_{F_2}$.

(2) Given $M = k \left[ \mathbb{N} \left[ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \right] \right] \langle x^2z^2, x^2yz^2, x^3y^2z^3, x^3y^3z^3 \rangle \cup \deg(M)$ consists of the $yz$-plane ($x = 0$), its translation $x = 1$, and the point $(2, 2, 2)$. 10 grains

- (red) $\left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) + NF_4, \left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right) + NF_4$,
- (blue) $\left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$
- (green) $\left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$, $\left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$

are depicted in $x = 1$ and $x = 0$ planes of Fig. 4b except $\left( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)$. Note that $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$ is not a monomial of $Q$ but that of $Q - Na_1$ or $Q - Na_4$, and therefore it lies in the intersection of two degree pairs that came from $M_{a_1}$ and $M_{a_4}$ respectively. For the given grain $G$ with color as in the figures, the chaffs are as follows.

- (red) $D_G := \{0, a_1, a_4, F_4\}$
- (blue) $D_G := \{a_1, F_4\}$
- (cyan) $D_G := \{a_1, F_4\}$
- (orange) $D_G := \{F_4\}$
- (green) $D_G := \{0\}$
- (violet) $D_G := \{a_1, a_4, F_4\}$

5. Hilbert Series via Grains

We derive a Hochster-type formula for the Hilbert series of the local cohomology of a finitely generated $\mathbb{Z}^d$-graded $k[Q]$-module $M$. Quotients of affine semigroup rings by monomial ideals are an important example.

**Definition 5.1** (Degree $a$-piece of the transverse section and its reduced chain complex). Let $M$ be a finitely generated graded $k[Q]$-module. Given an element $a \in \mathbb{Z}^d$, let $M_a$ be the degree-$a$ graded piece of $M$. Let $K$ be the transverse section of $Q$ from Section 2.4. The degree-$a$ graded piece of $K$ over $M$ is the subset of the face lattice of $K$ given by

$$K_a := \{ F \in \mathcal{F}(K) \mid a \in \deg(M_F) \}.$$  

Likewise, denote $\hat{K}_a := \{ \hat{F} \mid F \in K_a \}$ the degree-$a$ graded piece of the affine semigroup $Q$ over $M$. To align with the Ishida complex, the homological degree of the reduced chain complex of $K_a$ must be shifted as follows:

$$\hat{C}(K_a) : 0 \longrightarrow C^d \xrightarrow{\partial} C^{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^0 \xrightarrow{\partial} 0, \quad C^k := \bigoplus_{F \in K_a, \dim F = k} kF$$

with the differential

$$\partial(F) := \sum_{\substack{G \in K_a, \dim G = k-1 \atop \epsilon(G, F)}} \epsilon(F, G)G$$

for $\dim F = k$, where $\epsilon$ is an incidence function inherited from $K$ and $kF$ is a 1-dimensional $k$-vector space having $\{F\}$ as a basis.
Lemma 5.2. \( \tilde{C}(K_a) \) is well-defined and \( (L^* \otimes_{\mathbb{k}[Q]} M) = \text{Hom}_k(\tilde{C}(K_a), \mathbb{k}) \).

Proof. For any \( k \in \mathbb{N} \),

\[
\left( L^k \otimes_{\mathbb{k}[Q]} M \right)_a = \left( \bigoplus_{F \in \mathcal{F}(K)^{k-1}} M_F \right)_a = \bigoplus_{F \in \mathcal{F}(K)^{k-1}} \left( M_F \right)_a \cong \bigoplus_{a \in \deg(M_F)} \mathbb{k}F = \bigoplus_{\dim F = k-1} \mathbb{k}F,
\]

which is equal to \( C^k \). Apply the functor \( \text{Hom}_k(\cdot, \mathbb{k}) \) to obtain \( \tilde{C}(K_a) \). Since differentials in \( L^* \otimes_{\mathbb{k}[Q]} M \) are \( \mathbb{k} \)-linear, their images under \( \text{Hom}_k(\cdot, \mathbb{k}) \) agree with differentials in \( \tilde{C}(K_a) \). \( \square \)

Furthermore, if two elements of \( \mathbb{Z}^d \) are in the same grain, their graded pieces of the Ishida complex coincide.

Lemma 5.3. For any \( a \in G \in \mathcal{G}(M) \), \( \hat{K}_a = D_G \). Thus, \( \tilde{C}(K_a) = \tilde{C}(K_b) \) if \( a, b \in G \). If there is no grain containing \( a \), then \( \tilde{C}(K_a) = 0 \).

Proof. By Proposition 4.5, if there is no grain containing \( a \), then \( a \notin \deg(M_F) \) for any \( F \in \mathcal{F}(Q) \), so \( \tilde{C}(K_a) = 0 \). \( \hat{K}_a = D_G \) is clear from the definition of chaff. \( \square \)

As a consequence of the previous result, we may use the notation \( \tilde{C}(K_G) := \tilde{C}(K_a) \) for the grain \( G \) containing \( a \). \( \tilde{C}(K_G) \) coincides with the chain complex of \( K_G = D_G \). Since \( \mathcal{G}(M) \) is finite, according to Lemma 4.4, the Hilbert series of the local cohomology of \( M \) is a finite sum over cohomologies of chaffs as follows.

Theorem 5.4 (Hochster-type formula for the Ishida complex). The multi-graded Hilbert series for the local cohomology of a graded module \( M \) with support at the maximal ideal \( m \) is

\[
\text{Hilb}(H^i_m(M), t) = \sum_{G \in \mathcal{G}(M)} \dim_k H^i(\text{Hom}_k(\tilde{C}(K_G), \mathbb{k})) \sum_{a \in G} t^a.
\]

Proof.

\[
\text{Hilb}(H^i_m(M), t) = \sum_{a \in \mathbb{Z}^d} \dim_k \left( H^i_m(M) \right)_a t^a = \sum_{a \in \mathbb{Z}^d} \dim_k \left( H^i(L^* \otimes_{\mathbb{k}[Q]} M) \right)_a t^a
\]

\[
= \sum_{a \in \bigcup \deg(M)} \dim_k H^i(\text{Hom}_k(\tilde{C}(K_a), \mathbb{k})) t^a \quad \text{(Lemma 5.2)}
\]

\[
= \sum_{G \in \mathcal{G}(M)} \sum_{a \in G} \dim_k H^i(\text{Hom}_k(\tilde{C}(K_G), \mathbb{k})) t^a \quad \text{(Lemma 5.3)}
\]

\[
= \sum_{G \in \mathcal{G}(M)} \dim_k H^i(\text{Hom}_k(\tilde{C}(K_G), \mathbb{k})) \left( \sum_{a \in G} t^a \right)
\]

The Hilbert series in Theorem 5.4 is a finite sum involving generating functions of lattice points in polyhedra. To conclude these generating functions are rational, the underlying cone must be pointed \([1, 2]\).
Corollary 5.5. If \( Q \) is pointed, the Hilbert series in Theorem 5.4 can be expressed as a (formal) sum of rational functions.

In the non-pointed case, Hochster-type formulas are not necessarily given by rational functions. For example, the Hilbert series of the Laurent polynomial ring \( \mathbb{k}[x, x^{-1}] \) cannot be expressed as a rational function, since \( \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}} = 0 \), which is different from the formal sum \( \sum_{i \in \mathbb{Z}} x^i \).

Vanishing of local cohomology is a standard way to detect whether a ring is Cohen–Macaulay.

**Theorem 5.6** (Combinatorial Cohen–Macaulay criterion). Given a pointed affine semigroup ring \( \mathbb{k}[Q] \) and a monomial ideal \( I \), \( \mathbb{k}[Q]/I \) is Cohen–Macaulay ring if and only if every chaff of grains in \( G(\mathbb{k}[Q]/I) \) is either acyclic or (-1)-dimensional in homological index \( \ell := \dim \mathbb{k}[Q]/I \). □

**Example 5.7** (Continuation of Example 2.8).

1. As illustrated in Example 4.6(1), \( Q/I = \mathbb{N} \{ \begin{smallmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{smallmatrix} \} / \langle \{ 1 \} \rangle \) has six distinct colored chaffs. We take the unions of grains of the same color and color-code these unions. The following table summarizes their rational generating functions.

\[
\begin{align*}
\text{(red)} \ f_r & := \frac{xy^2 + xy^3 + x^2y^6}{1 - xy^4} \\
\text{(blue)} \ f_b & := \frac{x}{1 - x} \\
\text{(cyan)} \ f_c & := \frac{1}{x - 1} \\
\text{(orange)} \ f_o & := \frac{1 + xy^3 + x^2y^6}{xy^4 - 1} \\
\text{(green)} \ f_g & := x^2y^3 \\
\text{(yellow)} \ f_y & := 1
\end{align*}
\]

where \( x := t[1] \) and \( y := t[1] \). Thus, \( \tilde{C}(K_G) \) is a member of one of three chain complexes below.

\[
\begin{align*}
\tilde{C}(K_g) :0 \to \mathbb{k} \to \mathbb{k}^2 \to 0 & \quad \tilde{C}(K_r), \tilde{C}(K_b) :0 \to \mathbb{k} \to \mathbb{k} \to 0 \\
\tilde{C}(K_g) :0 \to \mathbb{k} \to 0 \to 0 & \quad \tilde{C}(K_c), \tilde{C}(K_o) :0 \to 0 \to \mathbb{k} \to 0
\end{align*}
\]

As a result,

\[
\text{Hilb}(H^0_m(S), \{x, y\}) = f_g \quad \text{Hilb}(H^1_m(S), \{x, y\}) = f_y + f_c + f_o.
\]

2. As illustrated in Example 4.6(2), \( Q/I = \mathbb{N} \{ \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \} / \langle \{ 1 \} \rangle \) has six distinct colored chaffs. As before, we take unions of grains of the same color and index these unions according to their color. Their rational generating functions are as follows:

\[
\begin{align*}
\text{(red)} \ f_r & := \frac{1 + x}{(1 - z)(1 - yz)} = x \\
\text{(blue)} \ f_b & := \frac{1 + x}{(z - 1)(1 - yz)}, \\
\text{(green)} \ f_g & := (xyz)^2 \\
\text{(orange)} \ f_o & := \frac{1 + x}{(z - 1)(yz - 1)}, \\
\text{(cyan)} \ f_c & := \frac{1 + x}{(1 - z)(yz - 1)}, \\
\text{(violet)} \ f_v & := x
\end{align*}
\]

where \( x := t[1], \ y := t[1], \) and \( z := t[1] \). We may classify \( \tilde{C}(K_G) \) as follows:

\[
\begin{align*}
\tilde{C}(K_r) :0 \to \mathbb{k} \to \mathbb{k}^2 \to \mathbb{k} \to 0 & \quad \tilde{C}(K_b), \tilde{C}(K_o) :0 \to 0 \to \mathbb{k} \to \mathbb{k} \to 0 \\
\tilde{C}(K_g) :0 \to \mathbb{k} \to 0 \to 0 \to 0 & \quad \tilde{C}(K_c) :0 \to 0 \to \mathbb{k}^2 \to \mathbb{k} \to 0 \\
\tilde{C}(K_o) :0 \to 0 \to 0 \to \mathbb{k} \to 0 & \quad \tilde{C}(K_v) :0 \to 0 \to \mathbb{k}^2 \to \mathbb{k} \to 0
\end{align*}
\]

Hence,

\[
\text{Hilb}(H^0_m(S), \{x, y\}) = f_g \quad \text{Hilb}(H^1_m(S), \{x, y\}) = f_v \quad \text{Hilb}(H^2_m(S), \{x, y\}) = f_o.
\]
In this section we concentrate on the special case when $I = 0$. We give a new criterion using grains to detect whether $k[Q]$ is Cohen–Macaulay and give an alternative proof of the Cohen–Macaulay condition in [13]. To begin, we recall a celebrated theorem of Hochster [9] when $Q$ is normal, and prove it yet again with our methods.

**Theorem 6.1** ([9]). If $Q$ is normal, $k[Q]$ is Cohen–Macaulay.

To prove this, we recall concepts of [5]. Given a polyhedron $P$ in $\mathbb{R}$-vector space $V$, let $a, b \in V$ two distinct points. If $[a, b]$ does not contain a point $b' \in P$ with $b' \neq b$, we say that $b$ is visible from $a$. A subset $S$ is visible if every $b \in S$ is visible. Given a polytope $P'$, a contractible polyhedral subcomplex is formed by the set of all visible points from $a \in V \setminus P'$. We refer to this polyhedral subcomplex as the a-visible subpolytope of $P'$.

**Proof.** Let $A$ be the hyperplane arrangement generated by hyperplanes in the $H$-representation of $\mathbb{R}_{\geq 0}Q$. We claim that

$$G(k[Q]) = \{\tau_S \cap Q | \tau_S \in \tau(A)\}.$$  

If the equality (1) holds, for the given nonempty $S$ and a point $a \in \tau_S \cap Q$, construct a hyperplane $H_i$ containing $a$ and transversally intersecting $\mathbb{R}_{\geq 0}Q$. The transverse section $K$ is then realized as a polytope $H_i \cap \mathbb{R}_{\geq 0}Q$. Thus, $H_i$ is a visible subpolytope of $K$, the chain complex over the contractible via the long exact sequence of cohomology. Thus, except the top dimension, the chaff of any grain has vanishing homology. This argument essentially paraphrases the proof of [5] Theorem 6.3.4).

To prove (1) recall that the poset of regions $\tau(A)$ partitions $\bigcup \deg(k[Q]) = Q$. We may use induction over the cardinality of $S$ to determine that each grain is of the form $\tau_S \cap Q$. Assume that the hyperplane arrangement $A$ consists of the elements $H_1, H_2, \ldots, H_m$, and that $F_i$ is the facet supported by $H_i$ for $i \leq m$. Start with $|S| = m$; $\tau_S \cap Q$ is a grain since $(0, Q)$ is the unique degree pair of $Q$. Indeed, $\tau_S \cap Q = k[S] \cap \mathbb{R}_{\geq 0}Q$.

To use induction, suppose we showed that $\tau_S \cap Q$ with $|S| \leq m - i$ is a grain. Then, for any $S$ with cardinality $m - (i + 1)$, we claim $\tau_S \cap Q = (\bigcap_{i \in S} (Q - NF_i)) \setminus \bigcup_{T \supseteq S} \tau_T \cap Q$. Indeed, $(\bigcap_{i \in S} (Q - NF_i)) \cap \bigcup_{T \supseteq S} \tau_T \cap Q$ is nothing more than the construction of the cumulative regions and the normality of $Q$. Then, the right-hand side $(\bigcap_{i \in S} (Q - NF_i)) \setminus \bigcup_{T \supseteq S} \tau_T \cap Q$ is a grain by inductive hypothesis. This shows the proposed one-to-one correspondence between grains and regions in $\tau(A)$.

Now pick a grain $\tau_S \cap Q$. Then $\tau_S \subseteq \tau_T \cap Q$ if and only if $T \subseteq S$. Hence, its chaff can be identified as a subset of faces in $\mathcal{F}(Q)$ whose corresponding localizations contain $\tau_S \cap Q$.

When $Q$ is not normal, we need the chaffs and grains of the module $k[Q_{sat}] / k[Q]$ to determine the chaffs of grains of $k[Q]$. To distinguish two chaffs and grains from different modules, we refer to the grains and chaffs of the module $k[Q_{sat}] / k[Q]$ as void grains and void chaffs, respectively, in accordance with the conventions in Example 3.3.2.

**Theorem 6.2.** $k[Q]$ is Cohen–Macaulay if and only if every grain consisting of void grains has vanishing homology except in top dimension.
Proof. If a grain $G$ has a degree which exists in $\bigcup \deg(Q_{\text{sat}})$, then we may apply the same argument of Theorem 5.6 to show that the homology of $D_G$ vanishes except for the top dimension. Hence, the only grains we need to investigate the homology of their chaff are a grain consisting of holes. □

Now we are prepared to give an alternative proof of the main result of [18]. Let $F_1, \ldots, F_m$ be facets of a pointed affine semigroup $Q$. Let $\tilde{Q} := \bigcap_{i=1}^m (Q - NF_i)$. For any nonempty subset $S$ of $\{1, 2, \ldots, m\}$, let $G_S := \bigcap_{i \in S} (Q - NF_i) \setminus \bigcup_{j \not\in S} (Q - NF_j)$. Let $\pi_S$ be the simplicial complex of nonempty subsets $I$ of $S$ such that $\bigcap_{i \in I} F_i$ is a nonempty face of $Q$. By abuse of notation, we identify the face lattice $\mathcal{F}(\pi_S)$ of $\pi_S$ as a subset $\{\bigcap_{i \in I} F_i \in \mathcal{F}(Q) \setminus \{\emptyset\} \mid I \subset \pi_S \cup \{\emptyset\} \}$ of $\mathcal{F}(Q)$. We say $\pi_S$ is acyclic if its reduced homology group is zero for all indices.

**Theorem 6.3** (Main theorem in [18]). $k[Q]$ is a Cohen–Macaulay ring if and only if (1) $\tilde{Q} = Q$ and (2) for every $S \subseteq \{1, 2, \ldots, m\}$ with $\mathcal{R}_S \in \tau(A)$, $\pi_S$ is acyclic.

Proof. Note that for any $a \in \bigcup \deg(k[Q]) \setminus \bigcup_{i=1}^m \deg(Q - NF_i)$, $K_a = \{Q\}$, which therefore only contributes to the $(\dim Q)$-th local cohomology. Thus, to prove the conditions above imply Cohen–Macaulayness, it suffices to show that for any $a \in \bigcup_{i=1}^m Q - NF_i$, $\tilde{C}(K_a)$ is exact. Since $G_S$ partitions $\bigcup \deg(k[Q])$, assume $a \in G_S$ for some proper subset of $\{1, 2, \ldots, m\}$. Then, for any $F \in \mathcal{F}(\pi_S)^c$, $F \not\subseteq F_i$ for any $i \in S$, since $F_i \in \pi_S$. Thus, $F = \bigcap_{i \in J} F_i$ for some $J \subset \{1, 2, \ldots, m\} \setminus S$ implies that $Q - NF$ contains $a$. Conversely, for any face $G$ of $\bigcap_{i \in S} F_i$, $a \not\subseteq Q - NG$. Hence $K_a = \mathcal{F}(\pi_S)^c$ by identifying $\mathcal{F}(\pi_S)$ as a subset of $\mathcal{F}(Q)$. In this identification, $\pi_S$ is isomorphic to a polyhedral subcomplex of the transverse section $K$ of $\mathbb{R}_{\geq 0}Q$. Hence, the complements $\mathcal{F}(\pi_S)^c$ form a polyhedral subcomplex of the dual polytope $K^{\text{dual}}$. Apply Alexander duality to conclude that $K_a$ is acyclic. This proves that $k[Q]$ is Cohen–Macaulay in accordance with Theorem 5.6.

Conversely, suppose $\pi_S$ is not acyclic. Pick $a \in G_S$ such that $K_a = \mathcal{F}(\pi_S)^c$. Now Alexander duality ensures that $\tilde{C}(\mathcal{F}(\pi_S)^c)$ has nontrivial cohomology at $i$-th index which is less than $(\dim Q)$. Also, if $Q' \neq Q$, Corollary 5.9 gives a void pair $(b, F)$ with $\dim F \leq \dim Q - 2$. Let $S$ be a set of indices of hyperplanes containing $F$. By Proposition 2.4 there exists $a \in (b + N(F \cup (-F))) \cap \tau_S$. Hence, $K_a = (Q/F) \setminus F := \{G \in \mathcal{F}(Q) \mid Q \supseteq G \supseteq F\}$ is combinatorially equivalent to the polytope $(Q/F)^{\text{dual}}$ without its relative interior. Hence, $(\dim F)$-th homology of $K_a$ is nonzero, as is the $(\dim Q - \dim F)$-th local cohomology. Thus, the $a$-graded part of the Ishida complex admits nonzero local cohomology with an index less than $\dim Q$, indicating that the semigroup ring is not Cohen–Macaulay.

**Example 6.4** (Continuation of Example 5.7). Both cases are not Cohen–Macaulay due to the presence of nonzero 0-th local cohomology.

**Example 6.5** (Examples of non-normal affine semigroup rings).

1. (A 3-dimensional non-Cohen–Macaulay affine semigroup ring) Let $Q := \mathbb{N}^{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}}$. Index the rays, facets, and hyperplanes as follows.

| $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|------|------|------|------|
| $\langle 1, 1 \rangle$ | $\langle 0, 1 \rangle$ | $\langle 1, 0 \rangle$ | $\langle 1, 1 \rangle$ |

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|------|------|------|------|
| $\langle a_1, a_2 \rangle$ | $\langle a_2, a_3 \rangle$ | $\langle a_3, a_4 \rangle$ | $\langle a_4, a_1 \rangle$ |

| $\mathcal{H}_1$ | $\mathcal{H}_2$ | $\mathcal{H}_3$ | $\mathcal{H}_4$ |
|------|------|------|------|
| $\{y = 0\}$ | $\{2x - 2y + z = 0\}$ | $\{x - y = 0\}$ | $\{x - z = 0\}$ |
The Hasse diagram for the region of posets is identical to that in Fig. 2. Moreover, the set of holes of the affine semigroup, \( \mathcal{H}(Q) \), is \( \left[ \frac{1}{1} \right] + \mathbb{N} \left[ \frac{0}{-2} \right] \) which lies in the \( xz \)-plane. This is because \( \left[ \frac{1}{1} \right] = \left[ \frac{1}{1} \right] + \mathbb{N} \left[ \frac{0}{-2} \right] \) acts as a barrier to the spread of holes in the relative interior of \( Q \). According to Theorem 3.8, \( Q \) and \( Q - \mathbb{N}a_2 \) are the only non-normal affine semigroups that arise as a result of localization. Thus, the space of holes \( \text{Holes}(Q) \) equals \( \left[ \frac{1}{1} \right] + \mathbb{Z} \left[ \frac{0}{-2} \right] \), which is consistent with the set of holes of \( Q - \mathbb{N}a_2 \). Using Fig. 2, \( \text{Holes}(Q) \) is decomposed into void grains below.

| \( \mathcal{H}_{1,2,3,4} \) | \( \mathcal{H}_{1,2,3} \) | \( \mathcal{H}_{1,2} \) |
|-----------------|-----------------|-----------------|
| \( \left[ \frac{1}{1} \right] + \mathbb{N} \left[ \frac{0}{-2} \right] \) | \( \left[ \frac{1}{1} \right] + \mathbb{Z} \left[ \frac{0}{-2} \right] \) |

\( \mathcal{H}_{1,2,3,4} = \bigcap_{i=1,3,4} (Q - \mathbb{N}a_i) \setminus Q \) and \( \mathcal{H}_{1,2} = (Q - \mathbb{N}F_1) \cap (Q - \mathbb{N}F_2) \setminus (Q - \mathbb{N}a_2) \), \( \mathcal{H}_{1,2,3,4} \) and \( \mathcal{H}_{1,2} \) form grains. On the other hand,

\( \mathcal{H}_{1,2,3} \subseteq G := (Q - \mathbb{N}a_3) \setminus (Q \cup (Q - \mathbb{N}a_2) \cup \mathcal{H}_{1,2,3,4}) \)

shows that \( \mathcal{H}_{1,2,3} \) is a part of \( G \), whereas the remaining elements of \( G \) come from the region \( r_{2,3} \). Thus,

\[
\mathcal{G}(Q/T) = \{ \mathcal{H}_{1,2,3,4}, \mathcal{H}_{1,2}, Q, (Q - \mathbb{N}a_2) \cap r_{1,2}, (r_{2,3} \cap (Q - \mathbb{N}a_3)) \cup \mathcal{H}_{1,2,3} \}
\]

\( \cup \{ \mathbb{Z}^3 \cap r_S \mid S \in \text{index}(r(A)) \text{ such that } S \neq \{1, 2, 3, 4\} \} \),

where \( \text{index}(r(A)) \) denotes the set of all indices of elements of \( r(A) \). Hence, it suffices to check whether chaffs

\[ D_{\mathcal{H}_{1,2,3,4}} = \{ a_i, F_j, Q \}_{i=1,3,4 \atop j=1,2,3,4} \text{ and } D_{\mathcal{H}_{1,2}} = \{ F_1, F_2, Q \} \]

have vanishing homology. Since \( D_{\mathcal{H}_{1,2}} \) produces a non-zero second homology, the affine semigroup ring \( k[Q] \) is not Cohen–Macaulay.

(2) (4-dimensional non-normal Cohen–Macaulay affine semigroup ring [4 Exercise 6.4]) Assume \( P \) is a simplex with the vertices \( (0,0,0), (2,0,0), (0,3,0), \) and \( (0,0,5) \). \( \mathbb{Z}^3 \) is the smallest lattice that contains vertices of \( P \). The polytopal affine monoid \( M(P) \) [4 Definition 2.18] associated \( P \) is the affine semigroup \( Q := \mathbb{N}A \) where \( A = \{(1, a) : a \in \mathbb{Z}^3 \cap P\} \). In this example,

\[ A = [a_1 \ a_2 \ldots a_{18}] = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \]

\( \mathbb{R}_{\geq 0}Q \) is a simplicial polyhedron with tetrahedral transverse section. We index its facets and hyperplanes as follows

\[ F_1 := \langle a_6, a_{13}, a_{18} \rangle \quad \quad F_2 := \langle a_1, a_7, a_{11}, a_{13}, a_{14}, a_{17}, a_{18} \rangle \]
\[ F_3 := \langle a_1, \ldots, a_6, a_{14}, \ldots, a_{16}, a_{18} \rangle \quad \quad F_4 := \langle a_1, \ldots, a_{13} \rangle \]
\[ \mathcal{H}_1 := \{(30, -15, -10, -6) \cdot t = 0 \} \quad \quad \mathcal{H}_2 := \{(0, 0, 0, 1) \cdot t = 0 \} \]
\[ \mathcal{H}_3 := \{(0, 0, 1, 0) \cdot t = 0 \} \quad \quad \mathcal{H}_4 := \{(0, 1, 0, 0) \cdot t = 0 \} \]

Since the transverse section \( K \) is a tetrahedron, we can index faces as intersections of facets uniquely. For example, each of the rays can be denoted as follows.

\[ F_{2,3,4} := \langle a_1 \rangle \quad \quad F_{1,3,4} := \langle a_6 \rangle \quad \quad F_{1,2,4} := \langle a_{13} \rangle \quad \quad F_{1,2,3} := \langle a_{18} \rangle. \]
According to the HASE package [11], the affine semigroup \( Q = \mathbb{N}A \) contains holes \( \mathcal{H}(Q) := [2 1 2 4]^I + \mathbb{N}F_1 \). Thus, the space of holes \( \text{Holes}(Q) \) is \([2 1 2 4]^I + \mathbb{Z}F_1\). We can decompose \( \text{Holes}(Q) \) into void grains using the hyperplane arrangement as follows:

\[
\mathcal{H}_S := \left\{ \left[ \begin{array}{c} a + b + c \\ 1 + 2a \\ 2 + 3b \\ 4 + 5c \\ \end{array} \right] \mid a \in \text{sgn}_1(S), b \in \text{sgn}_2(S), c \in \text{sgn}_3(S) \right\}
\]

where \( \text{sgn}_i(S) := \begin{cases} \mathbb{N} & \text{if } i \in S \\ \mathbb{Z} \smallsetminus \mathbb{N} & \text{if } i \notin S \end{cases} \) for all \( \{1\} \subseteq S \subseteq \{1, 2, 3, 4\} \). There are two types of grains indexed by \( 2^{\{1,2,3,4\}} \) that emerge from iterative intersections of affine semigroups. For every \( S \) with \( \{1\} \subseteq S \subseteq \{1, 2, 3, 4\} \), the union \( G_S := \mathcal{H}_S \cup (r_S \cap (Q - \mathbb{N}F_S \smallsetminus \{1\})) \) generates a grain of the first type, whereas \( G_S := r_S \cap (Q - \mathbb{N}F_S) \) generates a grain of the second type. Since there is no grain composed entirely of holes, all chaffs have vanishing homology except the top dimension. Therefore, \( k[Q] \) is Cohen–Macaulay.

REFERENCES

[1] Alexander Barvinok and James E. Pommersheim, An algorithmic theory of lattice points in polyhedra, New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 1999, pp. 91–147. MR1731815
[2] Alexander Barvinok and Kevin Woods, Short rational generating functions for lattice point problems, J. Amer. Math. Soc. 16 (2003), no. 4, 957–979. MR1992831
[3] Anders Björner, Paul H. Edelman, and Günter M. Ziegler, Hyperplane arrangements with a lattice of regions, Discrete Comput. Geom. 5 (1990), no. 3, 263–288. MR1036875
[4] Winfried Bruns and Joseph Gubeladze, Polytopes, rings, and \( K \)-theory, Springer Monographs in Mathematics, Springer, Dordrecht, 2009. MR2508056
[5] Winfried Bruns and Jürgen Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956
[6] Guillermo Cortiñas, Christian Haesemeyer, Mark E. Walker, and Charles Weibel, Toric varieties, monoid schemes and cdh descent, J. Reine Angew. Math. 698 (2015), 1–54. MR3294649
[7] Paul H. Edelman, A partial order on the regions of \( \mathbb{R}^n \) dissected by hyperplanes, Trans. Amer. Math. Soc. 283 (1984), no. 2, 617–631. MR737888
[8] Raymond Hemmecke, Akimichi Takemura, and Ruriko Yoshida, Computing holes in semi-groups and its applications to transportation problems, Contrib. Discrete Math. 4 (2009), no. 1, 81–91. MR2541989
[9] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. (2) 96 (1972), 318–337. MR304376
[10] Masa-Nori Ishida, The local cohomology groups of an affine semigroup ring, Algebraic geometry and commutative algebra, Vol. I, 1988, pp. 141–153. MR977758
[11] Florian Kohl, Yanxi Li, Johannes Rauh, and Ruriko Yoshida, Semigroups—a computational approach, The 50th anniversary of Gröbner bases, 2018, pp. 155–170. MR3839710
[12] William S. Massey, Singular homology theory, Graduate Texts in Mathematics, vol. 70, Springer-Verlag, New York-Berlin, 1980. MR569059
[13] Laura Felicia Matusevich and Ezra Miller, Combinatorics of rank jumps in simplicial hypergeometric systems, Proc. Amer. Math. Soc. 134 (2006), no. 5, 1375–1381. MR2199183
[14] Laura Felicia Matusevich and Byeongsu Yu, Standard pairs for monomial ideals in semigroup rings, J. Pure Appl. Algebra 226 (2022), no. 9, 107036.
[15] Ezra Miller and Bernd Sturmfels, Combinatorial commutative algebra, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR2110098
[16] Richard P. Stanley, An introduction to hyperplane arrangements, Geometric combinatorics, 2007, pp. 389–496. MR2383131
[17] Bernd Sturmfels, Ngô Viêt Trung, and Wolfgang Vogel, Bounds on degrees of projective schemes, Math. Ann. 302 (1995), no. 3, 417–432. MR1339920
[18] Ngô Viêt Trung and Lê Tuân Hoa, *Affine semigroups and Cohen-Macaulay rings generated by monomials*, Trans. Amer. Math. Soc. **298** (1986), no. 1, 145–167. MR857437

[19] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028

(Laura Felicia Matusevich) Department of Mathematics, Texas A&M University, College Station, TX 77843.

*Email address*, Laura Felicia Matusevich: matusevich@tamu.edu

(Byeongsu Yu) Department of Mathematics, Texas A&M University, College Station, TX 77843.

*Email address*, Byeongsu Yu: byeongsu.yu@tamu.edu