Evolution from Bardeen–Cooper–Schrieffer to Bose–Einstein condensate superfluidity in the presence of disorder

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**Abstract.** We describe the effects of disorder on the critical temperature of s-wave superfluids from the Bardeen–Cooper–Schrieffer (BCS) to the Bose–Einstein condensate (BEC) regime, with direct application to ultracold fermions. We use the functional integral method and the replica technique to study Gaussian correlated disorder due to impurities, and we discuss how this system can be generated experimentally. In the absence of disorder, the BCS regime is characterized by pair breaking and phase coherence temperature scales that are essentially the same, allowing strong correlations between the amplitude and phase of the order parameter for superfluidity. As non-pair-breaking disorder is introduced, the largely overlapping Cooper pairs seek to maintain phase coherence such that the critical temperature remains essentially unchanged, and Anderson’s theorem is satisfied. However, in the BEC regime, the pair breaking and phase coherence temperature scales are very different such that non-pair-breaking disorder can dramatically affect phase coherence, and thus the critical temperature, without the requirement of breaking tightly bound fermion pairs simultaneously. In this case, Anderson’s theorem does not apply, and the critical temperature can be more easily reduced in comparison to the BCS limit. Lastly, we find that the superfluid is more robust against disorder in the intermediate region near unitarity between the two regimes.

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1. Introduction

Ultracold atoms are special systems for studying the superfluid phases of fermions or bosons at very low temperatures, because of their unprecedented tunability. In particular, ultracold fermions with tunable interactions were used to experimentally study so-called Bardeen–Cooper–Schrieffer-to-Bose–Einstein condensate (BCS-to-BEC) evolution, where fermion superfluids were investigated as a function of the interaction parameter (scattering length). The study of superfluidity in ultracold fermions has additional promising research directions that include BCS-to-BEC evolution in optical lattices [1–3], and the effects of disorder during BCS-to-BEC evolution, which allow the very important study of the simultaneous effects of interactions and disorder at zero [4, 5] and finite temperatures [6, 7].

In ordinary condensed matter systems, the control of interactions is not possible, and the control of disorder caused by impurities is very limited because their concentrations cannot be changed at the turn of a knob. However, in ultracold atoms it may be possible to create disordered impurity potentials, where control over the impurity concentration and potential could occur in conjunction with control over atom–atom interactions. Impurity-type potentials have very recently been created experimentally in the context of thermometry for spin-dependent optical lattices [8]. However, we envision the creation of another type of impurity potential where there are two types of atoms, e.g. a mixture of atoms with unequal masses [9]. The first type of atom (the heavier one) is tightly trapped by an optical lattice in a low filling fraction configuration, such that the locations of the atoms for any given realization are unknown (random). The second type of atom (the lighter one) is trapped by a harmonic potential, but does not feel the optical lattice potential of the first type that exists in the same spatial region. In this case, the lighter atoms only interact with randomly distributed heavier atoms that are then seen as randomly distributed impurities, and both the density of impurities (heavier atoms) and the scattering potential between impurities (heavier atoms) and lighter atoms can be controlled. This kind of situation is illustrated schematically in figure 1. The experimental proposal described here is related to earlier proposals of using two types of atoms to simulate disorder in the context of Anderson localization in one-dimensional (1D) BECs [10] or in the context of lattice spin models [11].

Another way of introducing disorder in ultracold atoms relies on the use of laser speckles or lasers with incommensurate wavelengths, which were used to study the phenomenon of
Anderson localization in ultracold Bose atoms [12, 13]. The disorder introduced in this way is very different from impurity disorder in two important aspects. Firstly, laser speckles or lasers with incommensurate wavelengths produce a disorder potential landscape that has many ups and downs (*mountains and valleys*) over a characteristic length scale comparable to the laser wavelength, while in the case of impurity disorder considered here, the disorder potential is purely repulsive, that is, has only ups (*mountains*) located at random positions. Secondly, in laser speckles, the probability measure used to calculate disorder averages depends exponentially on the disorder potential, while in the impurity problem the probability measure has a Gaussian dependence on the disorder potential. These two major differences may lead to results that are both quantitatively and qualitatively different, enforcing the idea that not all types of disorder produce the same results. Two interesting review articles have emerged recently covering mostly the effects of disorder in ultracold Bose atoms [14, 15].

In this paper, we describe the critical temperature of 3D s-wave Fermi superfluids from the BCS to the BEC limit as a function of disorder, which is independent of the hyperfine states of the atoms and is created by randomly distributed impurities. However, the effects of disorder during the BCS-to-BEC evolution in higher angular momentum pairing [16] or for spin-dependent (hyperfine-state-dependent) impurity potentials [6] can also be explored.

Our main results are as follows. Firstly, in the BCS limit the amplitude and the phase of the order parameter are strongly coupled, such that pair breaking and loss of phase coherence occur simultaneously. In this case, the critical temperature is essentially unaffected by weak disorder, since the disorder potential is not pair-breaking and phase coherence is not easily destroyed in accordance with Anderson’s theorem [17]. Secondly, in the BEC limit the breaking of local pairs and the loss of phase coherence occur at very different temperature scales. In this case, the critical temperature is strongly affected by weak disorder (in comparison to the BCS regime), since phase coherence is more easily destroyed without the need to break local pairs simultaneously, and Anderson’s theorem does not apply. Thirdly, we find that superfluidity is more robust to disorder in the intermediate region near unitarity between the BCS and BEC regimes.

2. Hamiltonian

In order to study the effects of disorder in Fermi superfluids from the BCS to the BEC regime, we start with the real-space Hamiltonian density for 3D s-wave superfluids (set $\hbar = 1$),

$$
\mathcal{H}(\mathbf{x}) = \sum_{\sigma} \psi_{\sigma}^{\dagger}(\mathbf{x}) \left( -\frac{\nabla^2}{2m} - \mu + V_{\text{dis}}(\mathbf{x}) \right) \psi_{\sigma}(\mathbf{x}) + \hat{U}(\mathbf{x}),
$$

(1)
where $\psi^\dagger_\sigma(x)$ represents the creation of fermions with mass $m$ and hyperfine state (spin) $\sigma$ at $x$, $V_{\text{dis}}(x)$ is the disorder potential and $\mu$ is the chemical potential. In addition, the interaction term

$$\hat{U}(x) = \int dx' V(x, x')\psi^\dagger_\sigma(x')\psi^\dagger_\sigma(x)\psi_\sigma(x)\psi_\sigma(x')$$

contains the contact interaction potential $V(x, x') = -g\delta(x - x')$.

We choose the disorder potential $V_{\text{dis}}(x)$ to describe the presence of random impurities and to be independent of the hyperfine state, a choice that can be easily relaxed. As discussed in the introduction, this choice is quite different from the cases of disorder introduced by laser speckles or incommensurate lattice potentials. In the present case, we consider impurity atoms to be heavier than the other type of atoms in the superfluid state. It is also more useful to have the impurity atoms as fermions to prevent any possibility of low-density superfluid behavior, as would occur for bosons in optical lattices at low filling factors. So, for instance, one could choose $^{40}\text{K}$ to be the impurity atoms randomly distributed in a 3D lattice that overlaps with a cloud of $^6\text{Li}$ atoms, such that $^4\text{Li}$ and $^{40}\text{K}$ only interact with each other without feeling the presence of the other atom’s trapping potential. This means that the lighter atoms (e.g. $^4\text{Li}$) do not feel the presence of the optical lattice trapping the heavier atoms (e.g. $^{40}\text{K}$), and conversely the heavier atoms do not feel the presence of the harmonic potential confining the lighter atoms, but the lighter and heavier atoms interact with each other. In this case, the interaction between the heavier impurity atoms and the lighter atoms can be expressed by an effective impurity potential,

$$V_{\text{dis}}(x) = \sum_i v_d F(x - r_i),$$

where $r_i$ are the locations of the impurities, and $F(R) = \pi^{-3/2}\ell_d^{-3}\exp(-R^2/\ell_d^2)$ is a Gaussian of width $\ell_d$ describing the range of the impurity potential. It is important to emphasize that this impurity potential acts only on the lighter atoms. In addition, notice that $v_d$ is a measure of the interaction between heavier impurity atoms and lighter atoms, and $F(x - r_i)$ is a measure of the square of the Wannier wavefunction $|\varphi(x - r_i)|^2$ of the heavier impurity atom. If the interactions between every heavier impurity atom and every lighter atom are the same and always repulsive, the effective impurity potential is smooth and strictly repulsive.

The average over impurity positions leads to the correlator

$$\langle V_{\text{dis}}(x)V_{\text{dis}}(x') \rangle = \kappa K(x - x'),$$

where $\kappa = n_iv_d^2$ with $n_i$ being the impurity concentration, and the Kernel

$$K(x - x') = \frac{1}{(2\pi)^{3/2}\ell_d^3}\exp\left[-(x - x')^2/2\ell_d^2\right].$$

By inspection, only in the limit of $\ell_d \to 0$, the correlator becomes $\langle V_{\text{dis}}(x)V_{\text{dis}}(x') \rangle = \kappa \delta(x - x')$, reflecting white-noise correlations.

Given that the Hamiltonian is fully described, we next discuss the effective action of the system, and the corresponding approximations to obtain the critical temperature of superfluid fermions from the BCS to BEC regime in the presence of disorder.

3. Effective action in the presence of disorder

The effective action during the evolution from BCS to BEC superfluidity and the calculation of the critical temperature in the absence of disorder were obtained long ago [18], and they
are briefly reviewed here to provide background for the derivation of the effective action in the presence of disorder.

We begin our discussion by recalling that the effective action in the absence of disorder can be derived by introducing the pairing field \( \Delta(x) = g \langle \psi(x) \psi^\dagger(x) \rangle \) as a thermal average, where \( x = (x, \tau) \), with \( \tau \) being the imaginary time. On integration of the residual fermionic degrees of freedom, the resulting action up to Gaussian order in the pairing field \( \Delta(x) \) has two parts,

\[
S_{\text{eff}} = S_0 + S_{\text{Gau}}.
\]

The first term describes the action of unbound fermions,

\[
S_0 = -2 \sum_k \ln [1 + e^{-\xi(k)/T}],
\]

with \( \xi(k) = \epsilon(k) - \mu \), where \( \epsilon(k) = k^2/2m \) is the kinetic energy of a fermion of momentum \( k \). The second term is the Gaussian action

\[
S_{\text{Gau}} = T^{-1} \sum_q \tilde{\Delta}(q) \Gamma^{-1}(q) \Delta(q)
\]

due to the pairing field \( \Delta(q) \) and its complex conjugate \( \tilde{\Delta}(q) \) in momentum space; the former is just the Fourier transform of \( \Delta(x) \). Here,

\[
\Gamma^{-1}(q) = \frac{V}{g} + \sum_k \frac{X_1 + X_2}{2(iq \xi_1 - \xi_1 - \xi_2)}
\]

is the fluctuation propagator, \( V \) is the volume and \( q = (q, iq_\ell) \) is the four-momentum with \( q_\ell = 2\pi \ell T \) being the bosonic Matsubara frequencies. Here, the function \( X_1 = \tanh[\xi_1/2T] \) describes the occupation of fermions with energy \( \xi_1 = \xi(k - q/2) \), and the function \( X_2 = \tanh[\xi_2/2T] \) describes the occupation of fermions with energy \( \xi_2 = \xi(k + q/2) \).

To derive the effective action for a fixed configuration of disorder, we define the local chemical potential \( \mu(x) = \mu - V_{\text{dis}}(x) \). This corresponds to the local density approximation and treats the effects of the disorder potential semiclassically. In the case of the impurity potential originating from heavier atoms randomly distributed, as described above, this is expected to be a reasonable approximation, as the impurity potential is purely repulsive and non-confining, such that bound states of the impurity potential do not exist. The situation for speckle potentials or other random potentials that possess bound states is quite different, and will be briefly discussed later.

Within the local density approximation, the effective action in the presence of impurity disorder becomes

\[
S_{\text{eff}}[V_{\text{dis}}] = S_0[V_{\text{dis}}] + S_{\text{pair}}[V_{\text{dis}}],
\]

where \( S_0[V_{\text{dis}}] \) is the action of unbound fermions in the presence of disorder given by

\[
S_0[V_{\text{dis}}] = -2 \int \frac{dx}{V} \sum_k \ln [1 + e^{-\xi(k,x)/T}],
\]

with \( \xi(k, x) = \epsilon(k) - \mu(x) \). The second contribution to \( S_{\text{eff}} \) has two terms, that is \( S_{\text{pair}} = S_{\text{Gau}} + S_4 \). The first term corresponds to Gaussian pairing fluctuations,

\[
S_{\text{Gau}}[V_{\text{dis}}] = \int \frac{dx}{T V} \sum_q \tilde{\Delta}(q, x) \Gamma^{-1}(q, V_{\text{dis}}) \Delta(q, x),
\]
where \( \Delta(q, x) \) is the pairing field, and

\[
\Gamma^{-1}(q, V_{\text{dis}}) = \frac{V}{g} + \sum_{k} \frac{\tilde{X}_1 + \tilde{X}_2}{[2(iq \xi_1 - \xi_1 - \xi_2 - 2V_{\text{dis}}(x))]} \tag{13}
\]

is the pair correlation function in the presence of disorder. Here, the atom–atom interaction \( g \) can be expressed in terms of the scattering length \( a_s \) as

\[
\frac{V}{g} = -\frac{mV}{4\pi a_s} + \sum_{k} \frac{1}{2\epsilon_k}. \tag{14}
\]

In addition, the notation \( q = (q, iq\ell) \) represents the four-momentum, the function \( \tilde{X}_1 = \tanh [(\xi_1 + V_{\text{dis}}(x))/2T] \) describes the occupation of fermions with energy \( \xi_1 = \xi(k - q/2) \), and the function \( \tilde{X}_2 = \tanh [(\xi_2 + V_{\text{dis}}(x))/2T] \) describes the occupation of fermions with energy \( \xi_2 = \xi(k + q/2) \).

The interaction between the pairing fields is derived by taking into account non-Gaussian corrections as described in [18] leading to the action

\[
S_4 = V^{-1} \int dx \, d\tau \frac{b}{2} |\Delta(x, \tau)|^4, \tag{15}
\]

where we used the definition \( \Delta(x, \tau) = \Delta(q = 0, x, \tau) \). Here, the coefficient

\[
b = \sum_{k} \left[ \frac{X}{4\xi_k^3} - \frac{Y}{8T\xi_k^2} \right] \tag{16}
\]

represents the strength of the effective interaction between pairing fields, and the notations \( X = \tanh(\xi_k/2T) \) and \( Y = \text{sech}^2(\xi_k/2T) \) have been used.

An expansion of the pair correlation function \( \Gamma^{-1}(q, V_{\text{dis}}) \) in the limit of low energy (small \( iq\ell \), with the analytic continuation \( iq\ell \to \omega + i\delta \), long wavelength (small \( |q| \) and small disorder potential leads to

\[
\Gamma^{-1}(q, V_{\text{dis}}) = a + c \frac{|q|^2}{2m} + eV_{\text{dis}}(x) + d\omega. \tag{17}
\]

The coefficients of the Taylor expansion are

\[
a(\mu, T) = -\frac{mV}{4\pi a_s} + \sum_{k} \left[ \frac{1}{2\epsilon_k} - \frac{X}{2\xi_k} \right], \tag{18}
\]

describing the order parameter equation in the absence of disorder when \( a(\mu, T) = 0 \),

\[
c(\mu, T) = \sum_{k} \left[ \frac{X}{8\xi_k^2} - \frac{Y}{16\xi_k T} + \frac{XY}{16} \right] \tag{19}
\]

describing the kinetic energy of the pairing field, \( d(\mu, T) = d_R + id_l \), with \( d_R = \sum_k X/4\xi_k^2 \) and \( d_l = [\pi N(\epsilon_f)\sqrt{\mu}/(8T\sqrt{\epsilon_f})]\Theta(\mu) \) describing the frequency dependence, and \( e(\mu, T) = -\delta a/\delta \mu \) describing the strength of effective disorder potential.

It is very convenient to rewrite the effective action for the pairing field \( S_{\text{pair}} \) into a more familiar form. Thus, next, we put equation (17) back into the action of equation (12), set \( |q|^2 \to -\nabla^2 \), \( \Delta(q, x) \to \Delta(q = 0, iq\ell, x) \), and scale the pairing field \( \Delta \sqrt{d_R} \to \Psi \) such that the coefficient of the real part of \( d\omega \) becomes one. A subsequent Fourier transformation into
imaginary time, and the inclusion of the fourth-order term in \( \Psi \), leads to the pairing field action

\[
S_{\text{pair}}[V_{\text{dis}}] = S_{\text{Gau}}[V_{\text{dis}}] + S_4[V_{\text{dis}}],
\]

where \( S_{\text{Gau}}[V_{\text{dis}}] \) is defined in equation (12) and

\[
S_4[V_{\text{dis}}] = V^{-1} \int dx \, d\tau \frac{g_s}{2} |\Psi(x, \tau)|^4,
\]

where \( g_s = b/d_R^2 \) is the effective interaction. In order to make explicit the dissipative and non-dissipative temporal fluctuations, the pairing action is rewritten as

\[
S_{\text{pair}}[\bar{\Psi}, \Psi, V_{\text{dis}}] = V^{-1} \int dx \, d\tau \mathcal{L}_{\text{pair}},
\]

where \( \mathcal{L}_{\text{pair}} = \mathcal{L}_{\text{ND}} + \mathcal{L}_D \) is the Lagrangian density that contains two contributions. The first term is a non-dissipative part

\[
\mathcal{L}_{\text{ND}}(x, \tau) = \bar{\Psi} \left[ \partial_\tau - \frac{\nabla^2}{2m_*} - \mu_* + \gamma V_{\text{dis}}(x) \right] \Psi + \frac{g_s}{2} |\Psi|^4,
\]

and describes a generalized Gross–Pitaevskii Lagrangian for the scaled pairing field \( \Psi = \Psi(x, \tau) \). The term containing the effective mass \( m_* = d_R m/c \) is the kinetic energy of the pairing field, \( \mu_* = -a/d_R \) plays the role of the pairing field chemical potential, and \( \gamma V_{\text{dis}}(x) \) is the effective disorder potential, with \( \gamma = e/d_R \). The second term is the dissipative contribution that comes from the term containing \( i \hbar \omega \) and reflects the decay of fermion pairs into unbound fermions (Landau damping), which in imaginary time takes the Caldeira–Leggett form

\[
\mathcal{L}_D(x, \tau) = \frac{\lambda}{2\pi} \int d\tau' \frac{|\Psi(x, \tau) - \Psi(x, \tau')|^2}{(\tau - \tau')^2},
\]

where \( \lambda = d_t/d_R \). In our present discussion, \( \lambda \neq 0 \) for \( \mu < 0 \) and \( \lambda = 0 \) for \( \mu > 0 \), since \( d_t \) is non-zero for positive \( \mu \), but vanishes for negative \( \mu \). This indicates that on the BCS side (\( \mu > 0 \)) the lifetime of pairs is short (pairs break) near the critical temperature, while on the BEC side (\( \mu < 0 \)) the lifetime of pairs is infinite or, better said, very large, such that stable fermion pairs exist even above the critical temperature.

It is important to emphasize that the long-wavelength, low-frequency effective action derived in equation (22) is valid for the entire evolution from BCS to BEC superfluidity near the critical temperature \( T_c \), so long as the local density approximation is applicable. Now that we have derived the effective action for the pairing field in the presence of disorder, i.e. a purely repulsive random impurity potential, we discuss next the self-consistency equations that output the critical temperature as a function of disorder for a given interaction parameter.

4. Critical temperature in the presence of disorder

In order to determine the critical temperature in the presence of disorder, we need to derive the corresponding order parameter and number equations. For this purpose, a natural choice of dimensionless parameters is \( 1/(k_F a_s) \) for interactions, \( \eta = \kappa n_F/\epsilon_F^2 \) for disorder and \( \tilde{T} = T/\epsilon_F \) for temperature, where \( k_F \) is the Fermi momentum, \( n_F = k_F^3/3\pi^2 \) is the fermion density and \( \epsilon_F = k_F^2/2m \) is the Fermi energy. Note that our dimensionless parameter \( \eta \) that characterizes the degree of disorder is directly obtained from the original Hamiltonian density in equation (1)
and from the disorder correlator $\langle V_{\text{dis}}(x)V_{\text{dis}}(x') \rangle$, and is thus an input parameter of the theory. Since $\kappa = n_i \ell_d^3$, and $v_d$ has dimensions of energy times volume, it can be written as the product of the impurity volume $\ell_d^3$ and the characteristic amplitude of the disorder potential $V_{\text{dis}}(x)$, i.e. $v_d = \tilde{v}_d \ell_d^3$. Therefore, we may write $\eta = (n_i \ell_d^3)(n_F \ell_d^3)(\tilde{v}_d/\epsilon_F)^2$, which reveals that $\eta$ is small when the range of the impurity potential $\ell_d$ is much smaller than the average separation between impurities $n_i^{1/3}$ ($n_i^{1/3} \ell_d \ll 1$) or between fermions $n_F^{1/3}$ ($n_F^{1/3} \ell_d \ll 1$), and also when the amplitude of the disorder potential $\tilde{v}_d$ is much smaller than the Fermi energy $\epsilon_F$ ($\tilde{v}_d/\epsilon_F \ll 1$).

In some of the older literature discussing the effects of disorder for non-interacting fermions, the dimensionless parameter is typically chosen to be $1/(k_F \ell)$, where $\ell$ is the mean free path, which is a derived quantity, and not an input parameter of the theory. Even though the mean free path is a useful concept, when the Fermi system is strongly interacting, it becomes increasingly more difficult to define it and also to calculate it. Thus we prefer to use our input parameter $\eta$ to describe the degree of disorder of the system. Later in our discussion, we will return to the connection between our dimensionless parameter $\eta$ and the mean free path of paired and unpaired fermions.

The order parameter equation can be immediately read from equation (23) and is given by the condition
\[ \mu_s(\tilde{T}, 1/(k_F a_s)) = 0, \tag{25} \]
corresponding precisely to the order parameter equation in the absence of disorder $a(\mu, T) = 0$ given in equation (18), indicating that this equation is not explicitly affected by weak disorder, as required by Anderson’s theorem [17]. However, the determination of the critical temperature $T_c(\eta)$ and the chemical potential $\mu(\eta)$ as a function of dimensionless disorder $\eta$ for fixed scattering parameter $1/(k_F a_s)$ requires simultaneous solution of the order parameter equation (25) and the number equation, which is discussed next.

To determine the number equation in the presence of disorder, it is necessary to average the thermodynamic potential $\Omega(V_{\text{dis}})$ rather than the partition function [19]. Thus, to compute the disorder average $\langle \cdots \rangle = \int \mathcal{D}V_{\text{dis}} P[V_{\text{dis}}] (\cdots)$, where the probability measure is
\[ P[V_{\text{dis}}] = \exp \left[ -\frac{1}{2\kappa} \int x \frac{dx'}{V} V_{\text{dis}}(x) K^{-1}(x-x') V_{\text{dis}}(x') \right], \tag{26} \]
we just need the disorder correlator $K(x-x')$. The thermodynamic potential for a fixed configuration is $\Omega(V_{\text{dis}}) = -T \ln Z(V_{\text{dis}})$ and can be expressed in terms of the partition function
\[ Z(V_{\text{dis}}) = Z_0(V_{\text{dis}}) \times Z_{\text{pair}}(V_{\text{dis}}), \tag{27} \]
which is the product of two contributions. The first is the partition function for unbound fermions,
\[ Z_0(V_{\text{dis}}) = \exp \left[ -S_0(V_{\text{dis}}) \right], \tag{28} \]
and the second is the partition function for paired fermions,
\[ Z_{\text{pair}}(V_{\text{dis}}) = \int \mathcal{D}[\tilde{\Psi}, \Psi] \exp \left[ -S_{\text{pair}}(\tilde{\Psi}, \Psi, V_{\text{dis}}) \right]. \tag{29} \]
Thus, $\Omega(V_{\text{dis}}) = \Omega_0(V_{\text{dis}}) + \Omega_{\text{pair}}(V_{\text{dis}})$, where the unbound fermion thermodynamic potential is $\Omega_0(V_{\text{dis}}) = -T \ln Z_0(V_{\text{dis}}) = TS_0(V_{\text{dis}})$, while the contribution due to paired fermions is

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Figure 2. Two types of interaction lines between replicated paired fields. The diagram in (a) is the direct intra-replica interaction, containing $g_*$, whereas the diagram in (b) is the inter-replica interaction containing $\kappa$.

$$\Omega_{\text{pair}}(V_{\text{dis}}) = -T \ln Z_{\text{pair}}(V_{\text{dis}}).$$ However, the calculation of the average $\langle \Omega \rangle = -T \langle \ln Z \rangle$ can be performed by using the standard replica trick [19] for statistical averages, where

$$\langle \ln Z \rangle = \lim_{M \to 0} \ln \langle Z_M \rangle^{1/M}. \tag{30}$$

This trick was first introduced in the context of spin glasses [20], and can be applied to both unbound and paired fermion parts of the effective action. Here, however, we discuss explicitly the application of the replica trick only to the more complex case of the pairing field, and just quote the result for the thermodynamic potential of unbound fermions. In the case of the pairing field, the replicated ($M$-copies) partition function is

$$Z_M^\text{pair} = \int \prod_{i=1}^M D[\bar{\Psi}_i, \Psi_i] \exp \left\{ -\sum_{i=1}^M S_{\text{pair}}[\bar{\Psi}_i, \Psi_i, V_{\text{dis}}] \right\}, \tag{31}$$

with $S_{\text{pair}}$ defined in equation (22). Taking the configurational average $\langle Z_M^\text{pair} \rangle$ with the probability measure $P[V_{\text{dis}}]$ amounts solely to a Gaussian integral over $V_{\text{dis}}$, which can be easily performed leading to

$$\langle Z_M^\text{pair} \rangle = \int D\Psi \exp \left\{ -\sum_{i=1}^M S_{\text{pair}}(i, 0) + \sum_{i,j=1}^M S_{\text{dis}}(i, j) \right\}. \tag{32}$$

Here, we used the notation $D\Psi = \prod_{i=1}^M D[\bar{\Psi}_i, \Psi_i]$ with $S_{\text{pair}}(i, 0) = S_{\text{pair}}[\bar{\Psi}_i, \Psi_i, V_{\text{dis}} = 0]$ corresponding to the pair action without disorder and with the replacement of $\Psi \to \Psi_i$ in equation (22). The second term in equation (32) corresponds to a density–density interaction between the replicated fields given by

$$S_{\text{dis}}(i, j) = -\frac{\kappa_{\text{pair}}}{2T} \sum_{ij} \int \frac{dx}{V} \frac{dx'}{V} \rho_i(x) K(x - x') \rho_j(x'), \tag{33}$$

where $\kappa_{\text{pair}} = \gamma^2 \kappa$. The densities $\rho_i(x) = \bar{\Psi}_i(x, \tau) \Psi_i(x, \tau)$ and $\rho_j(x') = \bar{\Psi}_j(x', \tau) \Psi_j(x', \tau)$ appear for equal time $\tau$, while $K(x - x')$ is the disorder correlator defined previously. Thus, the replica trick transforms the problem of the pairing field $\Psi$ in a disorder potential, into the problem of interacting replicated pairing fields $\Psi_i$ with interaction strength $-[\kappa_{\text{pair}}/(2T)]K(x - x')$, in addition to the standard direct interaction term characterized by the interaction $g_*/2$. There are now two types of interactions between the replicated pairing fields, as indicated in figure 2.

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The calculation of the averaged thermodynamic potential for paired fermions
\[
\langle \Omega_{\text{pair}} \rangle = -T \lim_{M \to 0} \ln \langle Z_{\text{pair}}^M \rangle^{1/M}
\tag{34}
\]
can now be performed perturbatively as a function of the disorder parameter $\kappa_{\text{pair}}$ and the interaction $g_s$. For this purpose it is better to work in four-momentum space $q = (q, i\ell)$ and obtain the replica Green’s function,
\[
G^{-1}_{\text{pair}}(q) = -iq + E_{\text{pair}}(q),
\tag{35}
\]
where $E_{\text{pair}}(q) = \lambda |q| + |q|^2/2m_s - \mu + \Sigma_{\text{pair}}(q)$. Here, $\Sigma_{\text{pair}}(q)$ is Dyson’s self-energy for paired fermions containing all possible Feynman diagrams to the desired order in powers of $g_s$ and $\kappa_{\text{pair}}$. These types of diagrams are standard in diagrammatic perturbation theory \cite{21}. For instance, to first order in $g_s$ and $\kappa_{\text{pair}}$, the self-energy has only $2 \times 2 = 4$ Feynman diagrams, as shown in figure 3, while to second order in $g_s$ and $\kappa_{\text{pair}}$, there are $8 \times 4 = 32$ diagrams, as seen in figure 4.

The self-energy can be written as the expansion $\Sigma_{\text{pair}} = \Sigma^{(1)}_{\text{pair}} + \Sigma^{(2)}_{\text{pair}} + \cdots$, where $\Sigma^{(1)}_{\text{pair}}$ corresponds to the diagrams in figure 3, and $\Sigma^{(2)}_{\text{pair}}$ to the diagrams in figure 4. The replicated partition function is $\langle Z_{\text{pair}}^M \rangle = |\prod_q G^{-1}_{\text{pair}}(q)/T|^{-M}$, leading to the pair thermodynamic potential $\langle \Omega_{\text{pair}} \rangle = T \sum_q \ln[G^{-1}_{\text{pair}}(q)/T]$, from which the contribution to the number of paired fermions can be obtained via the thermodynamic identity $N_{\text{pair}} = -\partial \langle \Omega_{\text{pair}} \rangle / \partial \mu$. 

**Figure 3.** Two possible first-order self-energy diagrams side by side for each interaction line of type $\alpha$. Here, $\alpha$ describes either the intra-replica or inter-replica interaction line.

**Figure 4.** Eight possible second-order self-energy diagrams side by side for fixed interaction lines labeled $\alpha$ and $\beta$. Here, $\alpha$ and $\beta$ describe either an intra-replica or inter-replica interaction line.
The disorder averaged thermodynamic potential is \( \langle \Omega \rangle = \langle \Omega_0 \rangle + \langle \Omega_{\text{pair}} \rangle \), where the first term (also calculated using the replica trick) corresponds to \( \langle \Omega_0 \rangle = -T \sum_k \ln[G_0^{-1}(k)/T] \) and \( G_0^{-1}(k) = -ik_m + E_0(k) \) is the fermion Green’s function for unbound fermions with four-momentum \( k = (k, ik_m) \) in the presence of the disorder potential. Here, \( E_0(k) = k^2/2m - \mu + \Sigma_0(k) \), where \( \Sigma_0(k) \) is the self-energy for unbound fermions in the presence of the disorder potential. This self-energy can be written as the expansion \( \Sigma_0 = \kappa \partial \Sigma_0/\partial \kappa + \frac{1}{2} \kappa^2 \partial^2 \Sigma_0/\partial \kappa^2 + O(\kappa^3) \). The number of unbound fermions is then given by \( N_0 = -\partial \langle \Omega_0 \rangle/\partial \mu \), and the full number equation is \( N = -\partial \langle \Omega \rangle/\partial \mu \), where

\[
N(\kappa) = N_0(\kappa) + N_{\text{pair}}(\kappa)
\]

contains the contributions \( N_0(\kappa) = T \sum_k G_0(k) \partial E_0(k)/\partial \mu \) for unpaired fermions and \( N_{\text{pair}}(\kappa) = -T \sum_q G_{\text{pair}}(q) \partial E_{\text{pair}}(q)/\partial \mu \) for paired fermions.

In our approximation, the solution to equations (25) and (36) produces the critical temperature \( T_c \) as a function of both disorder \( \eta \) and interaction parameter \( 1/(k_Fa_s) \), as shown in figures 5 and 6. Our phase diagram for superfluidity is obtained at finite \( T \) and to second order in \( \kappa(\eta) \), in contrast to \( T = 0 \) properties [4, 5] (such as superfluid density or condensate fraction) obtained to linear order in \( \kappa(\eta) \). Although the replica technique can be used to all orders in \( \kappa(\eta) \), we perform numerical calculations only to second order in \( \kappa(\eta) \), because it becomes impractical to calculate higher-order contributions to the self-energies. Therefore, our
critical temperatures can be expressed as \( \tilde{T}_c(\eta) = \tilde{T}_c(0)[1 - \alpha \eta - \beta \eta^2] \), where \( \tilde{T}_c(0) \), \( \alpha \) and \( \beta \) are functions of \( 1/(k_Fa_s) \) only. In figure 5, we show \( T_c/\epsilon_F \) versus \( \eta \) for three different values of the interaction parameter \( 1/(k_Fa_s) \) corresponding to the BCS side at \( 1/(k_Fa_s) = -1.2 \), to unitarity at \( 1/(k_Fa_s) = 0 \), and to the BEC side at \( 1/(k_Fa_s) = 1.2 \). In figure 6, we show a 3D plot of \( T_c/\epsilon_F \) as a function of \( 1/(k_Fa_s) \) and \( \eta \). Two important points can be inferred from figures 5 and 6 describing the critical temperature. Firstly, impurity disorder effects seem to be more detrimental in the BEC regime, when compared to the BCS regime. Secondly, it is near unitarity \( (k_Fa_s)^{-1} \approx -0.3 \) that the superfluid is most robust to impurity disorder. We attribute this qualitative difference to the fact that in the BCS regime, phase coherence can only be destroyed through the simultaneous breaking of pairs, which does not occur since the impurity scattering is elastic, and impurities can provide the required energy to break the pairs. On the other hand, in the BEC regime, the pairing field corresponds to molecular bosons, and phase coherence can be more easily destroyed without the requirement of simultaneously breaking pairs.

Even though numerical results are useful, analytical insight into the evolution from BCS to BEC superfluidity in the presence of disorder can be obtained through calculations of the critical temperature to first order in \( \eta \), as discussed next.

5. Analytical results for weak disorder

Analytical results to first order in the dimensionless parameter \( \eta \) can be obtained in both the BCS and the BEC limits, where simplifications occur and calculations are easier, but yet non-trivial.

In the BCS limit, the interaction parameter is large and negative \( (1/(k_Fa_s) \ll -1) \), and the number equation is dominated by the contribution due to unbound fermions, \( N_0 \gg N_{\text{pair}} \). Given that the order parameter equation described in equation (25) is unchanged, any change in the critical temperature \( T_c \) has to come from a change in the chemical potential due to the presence of impurities. This change can be computed through the number equation defined in equation (36) by including only the dominant contribution \( N_0 \) due to unbound fermions.

The self-energy to linear order in \( \kappa \) is \( \Sigma_0 = \kappa V^{-1} \sum_k (i \omega - \xi_k)^{-1} = \eta S(\mu)\Lambda_c - i\pi \text{sgn}(\omega) \), where \( S(\mu) = (3/2)\sqrt{\epsilon_F\mu} \), \( \Lambda_c = F \int_0^{\epsilon_d/\epsilon_F} dy \sqrt{\gamma/(1-y)} \), and \( \epsilon_d/\epsilon_F = 1/(k_Fa_d)^2 \) takes into account the range \( \ell_d \) of the disorder correlator \( K(x-x') \). Replacing \( \Sigma_0 \) into the number equation, and performing the integrations over four-momentum, leads to a fermion density \( n = N/V \approx n_0 + \delta n \), where \( n_0 = 2 \sum_k f(\xi_k) \approx C\mu^{3/2} \) and \( \delta n = 3\eta C\Lambda_c\sqrt{\epsilon_F\mu} \) with \( C = (4/3)\sqrt{V(N(\epsilon_F)\epsilon_F)}^{-1/2} \). In the case of perfect particle–hole symmetry, \( \delta n \) can be neglected and the chemical potential is pinned to the Fermi energy \( (\mu = \epsilon_F) \) leading to unchanged \( T_c \) given by

\[
T_c = \epsilon_F(e^{\gamma}/\pi)8e^{-2} \exp[-\pi/(2k_F|a_s|)] .
\]

where \( \gamma \) here is the Euler constant. This reflects Anderson’s theorem, as changes in \( T_c \) occur only via the disorder-dependent shift in the chemical potential. Relaxing the condition of perfect particle–hole symmetry leads to \( \mu = \epsilon_F[1 - D\eta] \), where \( D = 2\Lambda_c = 4(\epsilon_d/\epsilon_F)^{1/2} \). The change in \( \mu \) produces a corresponding change in \( \tilde{T}_c(\eta) = \tilde{T}_c(0)[1 - \alpha \eta] \), where \( \alpha = \pi D/(4|k_Fa_s|) \).

In the BEC regime \( (1/(k_Fa_s) \gg 1) \), the number equation is dominated by \( N_{\text{pair}} \gg N_0 \) given that all fermions are paired into molecular bosons. To linear order in \( \kappa \), the pair self-energy is \( \Sigma_{\text{pair}}(q) = g_+n_{\text{pair}}/2 + \kappa_{\text{pair}}A(q) \), where \( A(q) = \Lambda(q_c) - 2m_s^{3/2}/(4\pi)\sqrt{|\mu_\epsilon| - i\epsilon_\ell} \), with \( \Lambda(q_c) = \sum_{|q|<q_c} m_s/(|q|^2 V) \), and the renormalized chemical potential is \( \tilde{\mu}_\epsilon = \mu_\epsilon + g_+n_{\text{pair}}/2 + \kappa_{\text{pair}}\Lambda(q_c) \), with \( n_{\text{pair}} = N_{\text{pair}}/V \). Noticing that \( \partial E_{\text{pair}}/\partial \mu \to -2 \) in the BEC limit,
and on summation over Matsubara frequencies $i\omega_n$, we arrive at $n_{\text{pair}} = 2n_B$, with $n_B = \zeta(3/2)(m_\star T_c/2\pi)^{3/2} + \kappa_{\text{pair}} T_c m_\star^3/4\pi^2$, when $\bar{\mu}_\star = 0$. The solution of the number equation gives the critical temperature

$$T_c(\kappa) = T_c(0) \left[ 1 - \kappa_{\text{pair}} T_c(0) m_\star^3/(6\pi^2 n_B) \right],$$

where $T_c(0) = 2\pi [n_B/\zeta(3/2)]^{2/3}/m_\star$ is the Bose–Einstein condensation temperature for a gas of molecular bosons, which in terms of $\eta$ leads to $T_c(\eta) = T_c(0) [1 - \alpha \eta]$. When the BEC limiting values $T_c(0) \rightarrow 0.218\epsilon_F$, $m_\star \rightarrow 2m$ and $n_B \rightarrow n_F/2$ are used, the coefficient $\alpha = 12\pi^2(T_c(0)/\epsilon_F) \approx 25.8$ is large. It is very important to emphasize that if temporal fluctuations were neglected in the self-energy $\Sigma_{\text{pair}}(q)$ by setting $i\omega_n = 0$, then one would have come to the conclusion that the critical temperature $T_c$ for the paired fermions (molecular bosons) is essentially unaffected by disorder. However, our calculations show that disorder affects the phase coherence of the molecular bosons via the incoherent part of $G_{\text{pair}}(q)$ manifested in the branch cut $\sqrt{|\mu_\star - i\omega_n|}$ of the self-energy $\Sigma_{\text{pair}}(q)$. It is these quantum (temporal) phase fluctuations that lead to the suppression of $T_c$ in the BEC limit (in comparison to the BCS regime), where fermions are largely non-degenerate, particle–hole symmetry is absent, and Anderson’s theorem is not applicable.

Now that we have established analytically and numerically the critical temperature of a disordered superfluid from the BCS to the BEC limit as a function of the dimensionless impurity parameter $\eta$, which measures the disorder correlation energy with respect to the Fermi energy, we would like to comment briefly on the relation between $\eta$ and the unbound fermion mean free path $\ell_F$ and the relation between $\eta$ and the pair mean free path $\ell_{\text{pair}}$.

6. Mean free paths

The concept of mean free path $\ell$ is often used in connection with the classical idea of the average distance traveled by a particle between collisions with impurities. For quantum particles, however, it is necessary to formulate a more precise meaning for the mean free path in terms of the details of the impurity potential. For instance, consider the transition amplitude $U(x, y, t) = \langle y | \exp(-i\hat{H}t) | x \rangle$ for a quantum particle to propagate from position $x$ to position $y$ for a given realization of the impurity potential contained in the Hamiltonian operator $\hat{H}$. This amplitude can be thought of as the sum of all Feynman paths connecting the two positions, which implies that the action for each path depends strongly on each impurity configuration. Taking the impurity average of the transition amplitude $\langle U(x, y, t) \rangle_{\text{dis}}$ leads to the averaging of random scattering phases, and the expectation that translational invariance is restored by the averaging process, followed by a rapidly decaying amplitude: $\langle U(x, y, t) \rangle_{\text{dis}} \sim \exp(-|x - y|/\ell)$. The decay constant $\ell$ is called the elastic mean free path. Notice that the very definition of $\ell$ requires an exponential decay of $\langle U(x, y, t) \rangle_{\text{dis}}$. In cases where this is not true, it is more difficult to define the elastic mean free path and thus to use it as a measure of the strength of the impurity potential. It is important to emphasize that the concept is only useful if indeed one can show that this exponential behavior exists.

In two simple limits, we can relate our dimensionless disorder parameter $\eta$ to an appropriately defined elastic mean free path $\ell$. For this purpose, we look at the imaginary time description of the transition probability.
In the BCS limit, the Fourier transformation of the unbound fermion Green’s function $G_0(k)$ discussed above into real space and imaginary time leads to $G_0(x, y, \tau) = G_{0,\text{no}}(x, y, \tau) \exp[-|x - y|/\ell_F]$, where $\ell_F$ is the elastic mean free path of unbound fermions and $G_{0,\text{no}}$ is the single-particle Green’s function in the absence of disorder. In this case, $\ell_F$ can be calculated analytically as $k_F\ell_F = 4/(3\pi\eta)$ to leading order in $\eta$. This result shows that when $\eta$ is small, $k_F\ell_F$ is large.

In the BEC limit, it becomes more appropriate to look at the paired fermion mean free path. In this case, we look at the Fourier transform of $G_{\text{pair}}(q)$, which becomes $G_{\text{pair}}(x, y, \tau) = G_{\text{pair, no}}(x, y, \tau) \exp[-|x - y|/\ell_{\text{pair}}]$, where $\ell_{\text{pair}}$ is generally a function of $\tau$ and $G_{\text{pair, no}}$ is the pair Green’s function in the absence of disorder. When only low-frequency contributions are included, we have $\ell_{\text{pair}} = \sqrt{4\pi}/[4m^2\pi^2\kappa_{\text{pair}}\sqrt{g_{\text{pair}}n_{\text{pair}}}/2]^{1/2}$, which in units of $k_F$ becomes $k_F\ell_{\text{pair}} = F(1/k_Fa_s)/\sqrt{\eta}$. Here, $F$ is just a function of $1/k_Fa_s$, which takes the limiting value of $F(1/k_Fa_s \to \infty) \approx 0.26$. Note the non-analytic behavior of $k_F\ell_{\text{pair}}$ when $\eta \to 0$.

In the BCS and BEC cases, the phase boundary between the superfluid state and the normal state occurs prior to the limit where $k_F\ell_F = 1$ and $k_F\ell_{\text{pair}} = 1$, respectively. However, the present estimates do not include the effects of Anderson localization, which may become important at low temperatures ($T \approx 0$) and at larger values of $\eta$.

Having made the connection between the dimensionless disorder parameter $\eta$ and the elastic mean free paths for repulsive impurity potentials, we would like to make some comparative remarks between speckle and impurity potentials.

### 7. Differences between speckle and impurity potentials

Before concluding, we would like to point out some key differences between speckle and impurity potentials to emphasize that not all disorders are equal.

The impurity potential described here is purely repulsive, meaning that it is non-confining, in such a way that for each configuration of disorder there are no confined single-particle states. Thus, the impurity potential represents essentially scattering centers that affect mostly phases of the wavefunctions of particles that are scattered from them. In particular in the BEC regime, phase fluctuations of the pairing field occur more easily than in the BCS regime, and thus impurity effects are more detrimental, relatively speaking.

In the case of speckle potentials, the situation is quite different, as there are many mountains and valleys in the disorder potential landscape, which allow for confinement and the existence of spatially confined states. While in the BCS regime these differences between speckle and impurity potential are less dramatic due to the robustness of Cooper pairs to phase fluctuations, the situation in the BEC regime is different. In the BEC limit, the speckle disorder potential seen by bound pairs allows Bose–Einstein condensation into the spatially confined states, creating many lakes (local condensates) near the minima of the disorder potential. As the average amplitude of the speckle potential increases, lakes are formed and become disconnected beyond a certain threshold. Once the connection between local condensates is lost, global phase coherence through the entire cloud is lost, and superfluidity disappears, leading to an insulating state. Thus, the problem of disorder due to speckle potentials is more like a percolation problem, and has a percolation threshold. While the confining regions of the speckle potential favor Bose–Einstein condensation, overlap between regions of local condensation determines global phase coherence and thus superfluidity in the system. Because of this qualitative difference that favors local Bose–Einstein condensation, which is a necessary but not sufficient condition for
Bose superfluidity in three dimensions, we expect that speckle potentials have less impact on the critical temperature for superfluidity (as found in quantum Monte Carlo simulations for atomic Bose systems in speckle potentials [23]) than the purely repulsive impurity potentials described here. A more quantitative comparison between the effects of disorder due to speckle potentials and repulsive impurity potentials requires modifications of the current approach in order to include the very important contributions of confining regions, which are, of course, absent in the repulsive impurity potential treated here. In addition, one has to be careful in comparing the sizes of molecular bosons (produced via Feshbach resonances) and the characteristic lengths of speckle potentials to determine the effective disorder potential felt by such molecules. These important theoretical comparisons between different types of disorder are left for a future publication, but it may also be interesting to explore experimentally the differences between these types of potentials.

8. Conclusions

In conclusion, we have analyzed the effects of repulsive impurity disorder potentials on the critical temperature for superfluidity of ultracold fermions during evolution from the BCS to the BEC regime. For s-wave superfluids, we showed that weak disorder does not substantially affect the critical temperature of a BCS superfluid with perfect particle–hole symmetry in accordance with Anderson’s theorem, as the breaking of fermion pairs and the loss of phase coherence occur at the same temperature. However, in the BEC regime, phase coherence is more easily destroyed by repulsive impurity disorder without the need of simultaneously breaking fermion pairs. Thus, in the BEC regime a more dramatic change in the critical temperature occurs, when compared with the BCS limit. Finally, we also showed that the superfluid is more robust to the presence of disorder in the intermediate region near unitarity.

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