Maximally transitive semigroups of $n \times n$ matrices

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Abstract

We prove that, in both real and complex cases, there exists a pair of matrices that generates a dense subsemigroup of the set of $n \times n$ matrices.

1 Introduction

Kronecker’s approximation theorem. The one-dimensional version of Kronecker’s approximation theorem [1] states that, given an irrational number $\theta$, a real number $r$, and a positive number $\epsilon$, there exist integers $m$ and $n$ such that

$$|r - m\theta - n| < \epsilon.$$ 

In other words, the set $\{m\theta + n : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, if (and only if) $\theta$ is irrational. From this, one can show that the semigroup generated by real numbers $a$ and $b$, defined by

$$\langle a, b \rangle = \{a^mb^n : m, n \in \mathbb{N}\},$$

is dense in $\mathbb{R}$, if $\ln(-a)/\ln b$ is an irrational negative number. In this paper, we are interested in the following $n$-dimensional generalization of this density
Observation:

Question: Let $K = \mathbb{C}$ or $\mathbb{R}$. Does there exist a pair $(A, B)$ of $n \times n$ matrices with entries in $K$ such that the semigroup generated by $A$ and $B$, defined by

$$\langle A, B \rangle = \{ A^{m_1} B^{n_1} \cdots A^{m_k} B^{n_k} : k \geq 1, \forall i m_i, n_i \geq 0 \},$$

is dense in the set of all $n \times n$ matrices?

The main results of this paper (Theorems 3 and 6) answer this question in the affirmative.

Hypercyclic operators. Given the action of a semigroup $G$ on a topological space $\mathcal{X}$, we say the action is hypercyclic, if there exists $x \in \mathcal{X}$ so that the $G$-orbit of $x$, defined by $\{ f(x) : f \in G \}$, is dense in $\mathcal{X}$. In [5], Feldman proved that there exists a hypercyclic semigroup generated by $n + 1$ diagonalizable matrices in dimension $n$. In the non-diagonalizable case, Costakis et al. [4] showed that one can find a hypercyclic abelian semigroup of $n$ matrices in dimension $n \geq 2$ (and that $n$ is the minimum number of generators of a hypercyclic abelian semigroup). Ayadi [2] has recently proved that the minimum number of matrices with entries in $\mathbb{C}$ that form a hypercyclic abelian semigroup is $n + 1$.

In the non-abelian case, it was shown in [7] that there exists a 2-generator hypercyclic semigroup in any dimension in both real and complex cases. In this paper, we prove the much stronger result that, in fact, there exists a dense 2-generator semigroup in any dimension in both real and complex cases. Since powers of a single matrix can never be dense [8], this result is optimal.

Topologically $k$-transitive actions. The action of a semigroup $G$ on a topological space $\mathcal{X}$ is called topologically transitive, if for every pair of nonempty open sets $\mathcal{U}$ and $\mathcal{V}$, there exists $f \in G$ so that $f(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. The action is called topologically $k$-transitive, if the induced action on $\mathcal{X}^k$ (cartesian product) is topologically transitive. Ayadi [2] proved that the action of an abelian semigroup of $n \times n$ matrices can never be $k$-transitive for $k \geq 2$ on $\mathbb{R}^n$ or $\mathbb{C}^n$. In the non-abelian case, Theorems 3 and 6 of this paper show that a 2-generator dense subsemigroup of $n \times n$ matrices can be constructed whose action on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is topologically $n$-transitive. These results are also optimal in the sense that the action of the entire set of $n \times n$ matrices
is not topologically \((n + 1)\)-transitive.

## 2 Preliminary results

Let \( \mathcal{M}_{n \times k}(\mathbb{K}) \) denote the set of all \( n \times k \) matrices with entries in \( \mathbb{K} \), where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). For a matrix \( M \in \mathcal{M}_{n \times k}(\mathbb{K}) \), its transpose is denoted by \( M^T \) and its inverse (if exists) is denoted by \( M^{-1} \). Also the entry on the \( i \)'th row and the \( j \)'th column of \( M \) is denoted by \( M_{ij} \). Finally, let \( 0_{n \times k} \) be the \( n \times k \) zero matrix.

**Lemma 1.** Suppose \( P_0, Q_0, P, Q \in \mathcal{M}_{n \times 1}(\mathbb{K}) \setminus \{0_{n \times 1}\}, n > 1, \) and 

\[ Q_0^T P_0 = Q^T P \neq 0. \]

Then there exists an invertible \( M \in \mathcal{M}_{n \times n}(\mathbb{K}) \) such that

\[
\begin{align*}
M^{-1}P_0 &= P, \\
M^TQ_0 &= Q, \\
\end{align*}
\]

(2.1)

Moreover, if \( \mathbb{K} = \mathbb{R} \), we can arrange for \( M \) to have positive determinant.

**Proof.** We first prove the claim for \( P_0 = Q_0 = V \), where

\[ V_{ii} = \begin{cases} 
1 & i = 1, \\
0 & i \neq 1.
\end{cases} \]  

(2.2)

Equivalently, we need to show that if \( P \) and \( Q \) are such that \( Q^T P = 1 \), then there exists an invertible matrix \( M \) such that the first column of \( M^{-1} \) is given by \( P \) and the first column of \( M^T \) is given by \( Q \). We construct the remaining columns of \( M^{-1} \) and \( M^T \) by induction. Suppose that we have constructed the linearly independent columns \( P_1 = P, \ldots, P_k, k \geq 1, \) so that for \( 1 \leq i, j \leq k, \)

\[ Q_i^T P_j = \begin{cases} 
1 & i = j, \\
0 & i \neq j.
\end{cases} \]

If \( k < n \), choose a vector \( Q_{k+1} \in \mathcal{M}_{n \times 1}(\mathbb{K}) \) such that \( Q_{k+1}^T P_i = 0 \) for all \( 1 \leq i \leq k \). Then \( Q_{k+1} \) is linearly independent of \( Q_1, \ldots, Q_k \). Let \( V_k \) be the subspace of vectors \( Z \) with \( Q_i^T Z = 0 \) for all \( 1 \leq i \leq k \). If \( k < n \), then
$Q^T_{k+1}Z$ cannot be zero for all $Z \in V_k$, and so there exists $P_{k+1} \in V_k$ such that $Q^T_{k+1}P_{k+1} = 1$. The vector $P_{k+1}$ is then linearly independent of $P_1, \ldots, P_k$. When we reach $k = n$, we have found $P_1, \ldots, P_n$, which form the columns of $M_{\pm}^{-1}$, and $Q_1, \ldots, Q_n$, which form the columns of $M^T$. If $\mathbb{K} = \mathbb{R}$, by replacing $Q_n$ with $-Q_n$ and $P_n$ with $-P_n$, if necessary, we can have $\det(M) > 0$.

Now suppose that $P, Q, P_0, Q_0$ are arbitrary vectors with $Q^T P = Q_0^T P_0 = d \neq 0$. By rescaling the vectors, if necessary, we can assume that $d = 1$. By the first part of this proof, there exist matrices $M_1$ and $M_2$ such that $M_1^{-1} V = P_0$, $M_1^T V = Q_0$, $M_2^{-1} V = P$, $M_2^T V = Q$.

Then by setting $M = M_1^{-1} M_2$, we get an invertible solution of (2.1).

Let $\mathcal{I}_n(\mathbb{K})$ denote the set of $(n + 1) \times (n + 1)$ matrices with entries in $\mathbb{K}$ that are of the form

$$G = \begin{pmatrix} F & X \\ Y^T & \eta \end{pmatrix},$$

(2.3)

where $F$ is an invertible $n \times n$ matrix, $X, Y \in \mathcal{M}_{n \times 1}(\mathbb{K})$ with $Y^T F^{-1} X \neq 0$, and $\eta \in \mathbb{K}$.

Let $\mathcal{I}_n^+(\mathbb{R})$ (respectively, $\mathcal{I}_n^-(\mathbb{R})$) denote the subset of $\mathcal{I}_n(\mathbb{R})$ consisting of matrices of the form (2.3), with $\det(F) > 0$ (respectively, $\det(F) < 0$). Also, let $S_n(\mathbb{K}) \subseteq \mathcal{I}_n(\mathbb{K})$ denote the set of matrices of the form (2.3) with $X = Y = 0_{n \times 1}$ and $\eta = 1$. Finally, we set

$$S_n^+ = S_n(\mathbb{R}) \cap \mathcal{I}_n^+(\mathbb{R}).$$

We define a map $\Upsilon : \mathcal{I}(\mathbb{K}) \to \mathbb{K}^2$, by setting

$$\Upsilon(G) = (Y^T F^{-1} X, \eta).$$

(2.4)

We use the notation $\bar{\Upsilon}$ to denote the extension of $\Upsilon$ to the set of matrices of the form (2.3), where $F$ is invertible.

**Lemma 2.** Suppose that $G_1, G_2 \in \mathcal{I}(\mathbb{K})$ such that $\Upsilon(G_1) = \Upsilon(G_2)$, then $G_2 \in \langle G_1, S_n(\mathbb{K}) \rangle$, where $\langle G_1, S_n(\mathbb{K}) \rangle$ denotes the semigroup generated by $G_1$ and $S_n(\mathbb{K})$. Moreover, if in addition to $\Upsilon(G_1) = \Upsilon(G_2)$, we have $G_1, G_2 \in \mathcal{I}_n^+(\mathbb{R})$ or $G_1, G_2 \in \mathcal{I}_n^-\mathbb{R})$, then $G_2 \in \langle G_1, S_n^+ \rangle$.

**Proof.** Suppose that $G_1$ and $G_2$ are given by

$$G_1 = \begin{pmatrix} F_1 & X_1 \\ Y_1^T & \eta \end{pmatrix}, \quad G_2 = \begin{pmatrix} F_2 & X_2 \\ Y_2^T & \eta \end{pmatrix}.$$  

(2.5)
Consider the following system of equations:

\[
\begin{cases}
R^{-1}(F_1^{-1}X_1) = F_2^{-1}X_2 \\
R^TY_1 = Y_2.
\end{cases}
\] (2.6)

Since \(G_1, G_2 \in \mathcal{I}_n(\mathbb{K})\), the vectors \(F_1^{-1}X_1, F_2^{-1}X_2, Y_1,\) and \(Y_2\) are all nonzero and \(Y_2^TF_2^{-1}X_2 = Y_1^TF_1^{-1}X_1 \neq 0\). Therefore, by Lemma 1, there exists an invertible solution \(R\) to the system (2.6), and so

\[
\begin{pmatrix}
F_2 \\
X_2
\end{pmatrix}
= \begin{pmatrix}
F_2R^{-1}F_1^{-1} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
F_1 \\
X_1
\end{pmatrix}
\begin{pmatrix}
R & 0 \\
0 & 1
\end{pmatrix}
\in \langle G_1, \mathcal{S}_n(\mathbb{K}) \rangle.
\]

For the second part of the lemma, note that by Lemma 1 we can arrange for \(R\) to have positive determinant. Since \(G_1, G_2 \in \mathcal{I}_n^+(\mathbb{R})\) or \(G_1, G_2 \in \mathcal{I}_n^-((R))\), we also have \(F_2R^{-1}F_1^{-1} \in \mathcal{S}^+\), which implies that \(G_2 \in \langle G_1, \mathcal{S}^+ \rangle\).

\(\square\)

### 3 The complex case

For \(n \geq 2\), let \(\mathcal{C}_n\) denote the set of \(n \times n\) matrices with entries in \(\mathbb{C}\) that in some basis can be written as (hence, are similar to)

\[
A = \begin{pmatrix}
Z_1 & 0 & \ldots & 0 \\
0 & Z_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & Z_k
\end{pmatrix},
\] (3.1)

where \(Z_k = 1\) and \(Z_i\) is a root of unity for each \(i = 2, \ldots, k\).

**Theorem 3.** For any \(n \geq 1\), there exists a pair of matrices in \(\mathcal{M}_{n \times n}(\mathbb{C})\) that generates a dense subsemigroup of \(\mathcal{M}_{n \times n}(\mathbb{C})\). Moreover, for \(n \geq 2\), we can arrange for one of the matrices in the pair to belong to \(\mathcal{C}_n\).

**Proof.** Proof is by induction on \(n \geq 1\). For the cases \(n = 1\) and \(n = 2\), see [6]. The inductive step is proved in Lemma 5. \(\square\)

**Lemma 4.** Let \(\Upsilon\) be the map defined by (2.4) and let \(\Omega\) be a closed subset of \(\mathcal{M}_{n \times n}(\mathbb{C})\). Suppose that \((a_i, \epsilon_i) \in \mathbb{C}^2, i \geq 1,\) such that \(\Upsilon^{-1}(a_i, \epsilon_i) \subseteq \Omega\). If \(G \in \mathcal{M}_{n \times n}(\mathbb{C})\) such that \(\Upsilon(G) = \lim_{i \to \infty}(a_i, \epsilon_i)\), then \(G \in \Omega\).
Proof. Let $\tilde{\Upsilon}(G) = (a, \epsilon)$ and $G = [F, X; Y^T, \epsilon]$ so that $Y^T F^{-1} X = a$. Choose $W \in \mathcal{M}_{n \times 1}$ so that $W^T F^{-1} W \neq 0$. We define

$$G_i^\pm(t) = \begin{pmatrix} F & X \pm tW \\ Y^T + tW^T & \epsilon_i \end{pmatrix}.$$ 

Then $G_i^\pm(t) \in \mathcal{I}_n(\mathbb{C})$ and $G_i^\pm(t) \to G$ as $i \to \infty$ and $t \to 0$. Next, we set

$$\tilde{\Upsilon}(G_i^\pm(t)) = (g^\pm(t), \epsilon_i),$$

where $g^\pm(t) = Y^T F^{-1} X + t(\pm Y^T F^{-1} W + W^T F^{-1} X) \pm t^2 W^T F^{-1} W$.

Since $W^T F^{-1} W \neq 0$ and $g(0) = a$, for $i$ large enough, there exists $t_i$ such that $g^\pm(t_i) = a_i$ (for a choice of $+$ or $-$). Therefore, $G_i^\pm(t_i) \in \Omega$ (for the same choice of sign), which implies that $G \in \Omega$. \hfill $\square$

**Lemma 5.** Let $A, E \in \mathcal{M}_{n \times n}(\mathbb{C})$ such that $A \in \mathcal{C}_n$ and $\langle A, E \rangle$ is dense in $\mathcal{M}_{n \times n}(\mathbb{C})$. Then there exist matrices $C$ and $D$ such that $C \in \mathcal{C}_{n+1}$ and $\langle C, D \rangle$ is dense in the set of $(n + 1) \times (n + 1)$ matrices with entries in $\mathbb{C}$.

**Proof.** The proof is divided into several steps.

**Step 1.** Since $A \in \mathcal{C}_n$, we can assume, by a change of basis if necessary, that $A$ is given by (3.1). Then, define

$$C = \begin{pmatrix} Z_1' & 0 & \ldots & 0 \\ 0 & Z_2' & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Z_k' \end{pmatrix}, \quad D = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix},$$

where $Z_i' = \sqrt{Z_i}$ for $1 \leq i < k$, and

$$Z_k' = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}.$$ 

Note that $C \in \mathcal{C}_{n+1}$, since $Z_k'$ is similar to $[-1, 0; 0, 1]$. Moreover, by this construction, we have

$$C^2 = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Let $\Lambda$ denote the closure of the semigroup generated by $C$ and $D$ in the set of $(n + 1) \times (n + 1)$ matrices with complex entries. Since $\langle A, E \rangle$ is dense in $\mathcal{M}_{n \times n}(\mathbb{C})$, by equations (3.2) and (3.3), we conclude that

$$\mathcal{S}_n(\mathbb{C}) \subseteq \Lambda.$$
Step 2. Let \( \mathcal{L} = \Upsilon(\mathcal{I}_n(\mathbb{C}) \cap \Lambda) \) denote the image of \( \mathcal{I}_n(\mathbb{C}) \cap \Lambda \) under the map \( \Upsilon \). By Lemma 2 and (3.4), we have
\[
\Upsilon^{-1}(\mathcal{L}) \subseteq \Lambda. \tag{3.5}
\]
It is then left to show that \( \mathcal{L} \) is dense in \( \mathbb{C}^2 \). In this step, we first prove that if \((a, \epsilon), (b, \delta) \in \mathcal{L}\), then
\[
\left( \frac{(z + a\delta)(z + be)}{z + ab}, z + \epsilon\delta \right) \in \mathcal{L}, \tag{3.6}
\]
for every \( z \in \mathbb{C} \setminus \{0, -ab, -a\delta, -be\} \). Since \((a, \epsilon), (b, \delta) \in \mathcal{L}\), it follows from (3.5) that for every \( r, s \neq 0 \), we have
\[
\left( I_{n \times n} \frac{(a/r)V}{\epsilon}, I_{n \times n} \frac{(b/s)V^T}{\delta} \right) \in \Lambda. \tag{3.7}
\]
By multiplying the two matrices in (3.7), and computing \( \Upsilon \) on the resulting matrix, we obtain (3.6) for \( z = rs \).

Step 3. We prove that \( \mathcal{L} \) is dense in \( \mathbb{C}^2 \). It follows from (3.5) and Lemma 4 that
\[
\overline{\Upsilon^{-1}(\mathcal{L})} \subseteq \Lambda. \tag{3.8}
\]
Since \( C \in \mathcal{I}_{n+1} \cap \Lambda \), one has \( \Upsilon(C) = (\sqrt{2}/2, -\sqrt{2}/2) \in \mathcal{L} \). By taking \((a, \epsilon) = (b, \delta) = (\sqrt{2}/2, -\sqrt{2}/2) \in \mathcal{L}\) in Step 2, we obtain:
\[
\left( \frac{(z - 1/2)^2}{z + 1/2}, z + 1/2 \right) \in \mathcal{L}, \tag{3.9}
\]
for all \( z \in \mathbb{C} \setminus \{0, \pm 1/2\} \). In particular, by letting \( z \to 1/2 \), we obtain \((0, 1) \in \overline{\mathcal{L}} \). It follows from (3.8) and (3.6) with \((a, \epsilon) = (0, 1) \) and \((b, \delta) = (\sqrt{2}/2, -\sqrt{2}/2) \) that
\[
\left( z + \sqrt{2}/2, z - \sqrt{2}/2 \right) \in \overline{\mathcal{L}},
\]
for all \( z \). For a given pair \((u, v) \in \mathbb{C}^2 \) with \( u - v + 1 \neq 0 \), set
\[
y = \frac{(v - 1)^2 - uv}{\sqrt{2}(u - v + 1)}.\]
By using (3.6) again, this time with \((a, \epsilon) = (\sqrt{2}/2, -\sqrt{2}/2) \) and \((b, \delta) = (y + \sqrt{2}/2, y - \sqrt{2}/2) \), and \( z = v + \sqrt{2}y/2 - 1/2 \), we obtain \((u, v) \in \overline{\mathcal{L}} \) i.e., \( \mathcal{L} \) is dense, and proof is completed. \( \square \)
4 The real case

Let $\mathcal{R}_n$ denote the set of $n \times n$ matrices that in some basis can be written as (hence, are similar to)

\[
A = \begin{pmatrix}
Z_1 & 0 & \ldots & 0 \\
0 & Z_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & Z_k \\
\end{pmatrix},
\]

so that

i) $Z_1 > 0$.

ii) $Z_k = |\det(A)|/\det(A)$.

iii) For $1 < i < k$, either $Z_i = 1$ or $Z_i$ is a $2 \times 2$ block of the form

\[
\begin{pmatrix}
\cos(2^{-m}\pi) & -\sin(2^{-m}\pi) \\
\sin(2^{-m}\pi) & \cos(2^{-m}\pi) \\
\end{pmatrix}, \ m \in \mathbb{N} \cup \{0\}.
\]

iv) $A$ has at least one eigenvalue equal to 1.

In this section, we prove the following theorem.

**Theorem 6.** In any dimension $n \geq 1$, there exists a pair of $n \times n$ real matrices that generates a dense subsemigroup of $n \times n$ real matrices. Moreover, for $n \geq 3$, we can arrange for one of the matrices to belong to $\mathcal{R}_n$.

**Proof.** For $n = 1$ and $n = 2$, see [6]. For $n \geq 3$, we prove the claim by induction. The case of $n = 3$ is proved in Lemma, while the inductive step is proved in Lemma. \[ \square \]

**Lemma 7.** Let

\[
a = -2^{3/5}, \ b = \frac{8}{3}, \ e = -2^{-4/5}.
\]

Then the semigroup generated by the real matrices

\[
A = \begin{pmatrix}
a & e \\
1 & 0 \\
\end{pmatrix}, \ B = \begin{pmatrix}
1 & 0 \\
0 & b \\
\end{pmatrix},
\]

is dense in the set of all $2 \times 2$ real matrices with positive determinant.
Proof. Let \( \mathcal{T} \) denote the closure of the semigroup generated by \( A \) and \( B \). Then, for \( c = 4/9 \) and 

\[
C = \begin{pmatrix} 4/9 & 0 \\ 0 & 1 \end{pmatrix},
\]

we have \( C = ABA^3BA \in \mathcal{T} \). Next, we show that \( dI_{2 \times 2} \in \mathcal{T} \) for every \( d \in \mathbb{R} \). First suppose \( d < 0 \), and choose sequences of positive integers \( k_i, l_i \) so that 

\[
b_{k_i}c_{l_i} \to d/e.
\]

Then 

\[
\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix} c_{l_i} & b_{k_i}c_{l_i}e \\ 1 & 0 \end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix} c_{l_i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & e \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b_{k_i} \end{pmatrix} \in \mathcal{T},
\]

and so \( dI_{2 \times 2} = \begin{bmatrix} 0, d; 1, 0 \end{bmatrix}^2 \in \mathcal{T} \). If \( d > 0 \), we have \( dI_{2 \times 2} = (-\sqrt{d}I_{2 \times 2})^2 \in \mathcal{T} \). Therefore, we need to show that \( \mathcal{T} = \langle A, B, C, dI_{2 \times 2} : d \in \mathbb{R} \rangle \) is dense in the set of all \( 2 \times 2 \) real matrices with positive determinant. Equivalently, we show that \( \mathcal{T}' = \langle MAM^{-1}, MBM^{-1}, MCM^{-1}, M(dI_{2 \times 2})M^{-1} : d \in \mathbb{R} \rangle \) is dense, where 

\[
M = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.
\]

We have \( MBM^{-1} = B, MCM^{-1} = C, \) and \( M(dI_{2 \times 2})M^{-1} = dI_{2 \times 2} \). Moreover, 

\[
\begin{pmatrix} 1 & -1/4 \\ 1 & 0 \end{pmatrix} = (a^{-1}I_{2 \times 2})MAM^{-1} \in \mathcal{T}'.
\]

It follows that the matrices \([1, -1/4; 1, 0]\) and \([1, 0; 0, 8/3]\) and \([4/9, 0; 0, 1]\) all belong to \( \mathcal{T}' \). Now, by Proposition 4.1 of [6], the semigroup of linear fractional maps generated by the maps 

\[
1 - \frac{1}{4x}, \frac{3x}{8}, \text{ and } \frac{4x}{9}
\]

is dense in the set of all real linear fractional maps with positive determinant. From this and the fact that \( \mathcal{T}' \) contains all multiples of the identity matrix, it follows that \( \mathcal{T}' \) is dense in the set of real \( 2 \times 2 \) matrices with positive determinant. \( \square \)

Lemma 8. Let \( a \) and \( b \) be given by (4.3). Then the matrices 

\[
A = \begin{pmatrix} \sqrt{b} & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ e & a & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

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generate a dense subsemigroup of the set of all $3 \times 3$ real matrices. Moreover, $A \in \mathcal{R}_3$.

Proof. Let $\Lambda$ denote the closure of the subsemigroup generated by $A$ and $E$. We have

$$A^2 = \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so it follows from Lemma 7 that

$$S_2^+ \subseteq \Lambda.$$ (4.4)

Let $L^+ = \Upsilon(\Lambda \cap \mathcal{I}_2^+(\mathbb{R}))$. It follows from Lemma 2 and (4.4) that

$$\mathcal{I}_2^+(\mathbb{R}) \cap \Upsilon^{-1}(L^+) \subseteq \Lambda,$$ (4.5)

and so by Lemma 4

$$\{(G = [F, X; Y^T, \eta]) : \det(F) > 0 \text{ and } \bar{\Upsilon}(G) \in \overline{L^+}\} \subseteq \Lambda,$$ (4.6)

Step 2 of Lemma 5 implies that if $(a, \epsilon), (b, \delta) \in \mathcal{L}^+$ and if $z \in \mathbb{R}$ is such that $1 + ab/z > 0$, then (3.6) holds. Here the condition $1 + ab/z > 0$ is required to make sure that the product of the two matrices in (3.7) belongs to $\mathcal{I}_2^+(\mathbb{R})$.

Next, we show that $\mathcal{L}^+$ is dense in $\mathbb{R}^2$. Suppose $(a, \epsilon) \in \mathcal{L}^+$. By taking $(b, \delta) = (a, \epsilon)$ in (3.6), we obtain

$$\left(\frac{(z + ae)^2}{z + a^2}, z + \epsilon^2\right) \in \overline{L^+},$$ (4.7)

for all $z < -a^2$. By taking two pairs of the form (4.7) with $z$ replaced by $x, y \rightarrow -(a^2)^-$, and applying (3.6) again, we obtain:

$$((\epsilon^2 - a^2)^2, z + (\epsilon^2 - a^2)^2) \in \overline{L^+}, \forall z > 0.$$ (4.8)

Since $A \in \mathcal{I}_2^+(\mathbb{R}) \cap \Lambda$, we have $(\sqrt{2}/2, -\sqrt{2}/2) \in \mathcal{L}^+$. It then follows from (4.8) with $(a, \epsilon) = (\sqrt{2}/2, -\sqrt{2}/2)$ that $(0, z) \in \overline{L^+}$ for all $z > 0$. Applying (4.8) to $(a, \epsilon) = (0, z)$ implies that $(u, v) \in \overline{L^+}$ for all $v > u > 0$. Another application of (3.6) with $\epsilon > a > 0$ and $\delta > b > 0$, and with $a \rightarrow 0^+$, implies that $(z + be, z + \epsilon \delta) \in \overline{L^+}$ for all $z$. It follows that

$$\{(u, v) : v > u\} \subseteq \overline{L^+}.$$ (4.9)
By letting \((a, \epsilon) = (u, v)\) with \(v > u\), and \((b, \delta) = (\sqrt{2}/2, -\sqrt{2}/2)\) in (3.6), we obtain
\[
\left(\frac{z + v\sqrt{2}/2}{z + u\sqrt{2}/2}, z - v\sqrt{2}/2\right) \in \mathcal{L}^e,
\]
for all \(z\) with \(1 + \sqrt{2}u/(2z) > 0\). We let \(u \to 0\) to obtain
\[
\left(z + v\sqrt{2}/2, z - v\sqrt{2}/2\right) \in \mathcal{L}^e, \quad \forall \ z \forall v > 0.
\]
This together with (4.9) show that \(\mathcal{L}^e\) is dense in \(\mathbb{R}^2\).

So far, we have proved that \(\mathcal{L}^+\) is dense in \(\mathbb{R}^2\). It follows that, given any \(c, d, v \in \mathbb{R}\), we have
\[
\begin{pmatrix}
1 + d & 0 & c + 1 \\
0 & 1 & 0 \\
d + (v - 1)/c & 0 & v
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
(v - 1)/c & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & c \\
0 & 1 & 0 \\
d & 0 & 1
\end{pmatrix}
\in \mathcal{L}.
\]
By computing \(\Upsilon\) on this matrix, we conclude that if
\[
d = \frac{c(u - v - 1) + 1 - v}{c(c + 1 - u)} < -1,
\]
we have \((u, v) \in \mathcal{L}^e\). The inequality (4.10) can be guaranteed as \(c \to 0\) for any given values of \(u \neq 1\) and \(v \neq 1\). In other words, \(\mathcal{L}^-\) is also dense in \(\mathbb{R}^2\) and proof is completed.

**Lemma 9.** Let \(A, E \in M_{n \times n}(\mathbb{R})\) such that \(\langle A, E \rangle\) is dense in \(M_{n \times n}(\mathbb{R})\) and \(A \in \mathcal{R}_n\). Then there exist \((n + 1) \times (n + 1)\) real matrices \(C\) and \(D\) such that \(C \in \mathcal{R}_{n+1}\) and \(\langle C, D \rangle\) is dense in the set of \((n + 1) \times (n + 1)\) matrices with real entries.

**Proof.** We define
\[
F = \begin{pmatrix}
A & 0 \\
0 & \text{sgn}(\det(A))
\end{pmatrix}, \quad D = \begin{pmatrix}
E & 0 \\
0 & -\text{sgn}(\det(E))
\end{pmatrix},
\]
where \(\text{sgn}(x) = |x|/x\) for \(x \neq 0\). Let also
\[
C = \begin{pmatrix}
Z_1' & 0 & \ldots & 0 \\
0 & Z_2' & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & Z_k'
\end{pmatrix},
\]

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where we let $Z'_1 = \sqrt{Z_1}$, and if $Z_i$ is a block of the form (1.2), then we let
\[
Z'_i = \begin{pmatrix}
\cos(2^{-m-1} \pi) & -\sin(2^{-m-1} \pi) \\
\sin(2^{-m-1} \pi) & \cos(2^{-m-1} \pi)
\end{pmatrix}.
\]

In addition, if $Z_i = 1$ for $i < k$, then let $Z'_i = 1$. To define $Z'_k$, we have two cases.

Case 1. Suppose $\det(A) > 0$. In this case, we define
\[
Z'_k = \begin{pmatrix}
\sqrt{2}/2 & \sqrt{2}/2 \\
\sqrt{2}/2 & \sqrt{2}/2
\end{pmatrix}.
\]

By this construction, we have $C^2 = F$. Note that $C \in \mathcal{R}_{n+1}$, since $Z'_k$ is similar to $[1, 0; 0, -1]$. The rest of the proof in this case is the same as steps 2 and 3 of Lemma 5.

Case 2. Suppose that $\det(A) < 0$. In this case, we define
\[
Z'_k = \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}.
\]

Here again $C^2 = F$ and we have $C \in \mathcal{R}_{n+1}$ (since $A$ has an eigenvalue 1 and so does $C$; then by a change of basis, we can place 1 in lower-right corner of $C$; moreover, $Z'_k$ is similar to a block of the form (1.2) with $m = 1$). It is left to show that $\langle C, D \rangle$ is dense. Proof of Lemma 5 shows that we only need to check that $S_n(\mathbb{R}) \subseteq \langle C, D \rangle$. Since $C^2 = F$ and $\langle A, E \rangle$ is dense, we conclude that for every $n \times n$ real matrix $M$, there exists $\sigma(X) \in \{\pm 1\}$ such that
\[
\begin{pmatrix} M & 0 \\ 0 & \sigma(M) \end{pmatrix}
\]
belongs to the closure of $\langle C, D \rangle$. We need to show that for every $M$, we can have $\sigma(M) = 1$. Suppose that there exists an invertible $M$ such that $\sigma(M)$ could take both values of $\pm 1$. Then it follows that for every $N$, $\sigma(N) = \sigma(M)\sigma(M^{-1}N)$ can take both values of $\pm 1$. Therefore, suppose that
\[
\sigma : GL(n, \mathbb{R}) \to \{\pm 1\}
\]
is a well-defined function so that $[M, 0; 0, \sigma(M)] \in \Lambda$ but $[M, 0; 0, -\sigma(M)] \notin \Lambda$. It follows that $\sigma(I_{n\times n}) = 1$ and $\sigma$ is an onto group homomorphism. In
particular, the set \( \{ M \in SL(n, \mathbb{R}) : \sigma(M) = 1 \} \) is a normal subgroup of \( SL(n, \mathbb{R}) \) containing \( \{ N^2 : N \in SL(n, \mathbb{R}) \} \). It follows from Jordan-Dickson Theorem \([3]\) that
\[
SL(n, \mathbb{R}) \subseteq \sigma^{-1}(1).
\]
Given a matrix \( M \) with \( \det(M) > 0 \), we then have
\[
\sigma(M) = \sigma((\det(M))^{1/n} I_n) \cdot \sigma(\det(M)^{-1/n} M) = 1.
\]
Since \( \sigma(A) = -1 \), it follows that for every \( M \) with \( \det(M) < 0 \), we have \( \sigma(M) = -1 \). In other words:
\[
\sigma(M) = \text{sgn}(\det(M)),
\]
which is a contradiction, since \( \sigma(E) = -\text{sgn}(\det(E)) \).

5 Topologically \( n \)-transitive subsemigroups of \( n \times n \) matrices

As we noted in the introduction section, there are no abelian \( k \)-transitive subsemigroups of \( n \times n \) matrices for \( k \geq 2 \) and \( n \geq 1 \). In this section, we first prove that it is not possible for any semigroup action of matrices on \( \mathbb{K}^n \) to be \( (n+1) \)-transitive.

**Proposition 10.** Let \( G \) be a semigroup of linear maps on \( \mathbb{K}^n \). Then the action of \( G \) on \( \mathbb{K}^n \) is never \( (n+1) \)-transitive.

**Proof.** On the contrary, suppose the action of \( G \) is \( (n+1) \)-transitive, and so there exists \( X = (X_1, \ldots, X_{n+1}) \in (\mathbb{K}^n)^{n+1} \) so that the orbit of \( X \) under the induced action of \( G \) on \( (\mathbb{K}^n)^{n+1} \) is dense. Choose \( \alpha_1, \ldots, \alpha_{n+1} \in \mathbb{K} \) so that
\[
\sum_{i=1}^{n+1} \alpha_i X_i = 0.
\]
But then the orbit of \( X \) under the action of \( G \) stays within the linear subspace of \( (\mathbb{K}^n)^{n+1} \) given by the set of points \( (Y_1, \ldots, Y_{n+1}) \in (\mathbb{K}^n)^{n+1} \) satisfying the linear equation \( \sum_{i=1}^{n+1} \alpha_i Y_i = 0 \), and so it cannot be dense in \( (\mathbb{K}^n)^{n+1} \). This is a contradiction, and the proposition is proved.

On the other hand, Theorems \([3]\) and \([6]\) imply the following theorem.
Theorem 11. For any dimension \( n \geq 1 \), in both real and complex cases, there exists a topologically \( n \)-transitive subsemigroup generated by two \( n \times n \) matrices.

Proof. Let \( A \) and \( E \) be the \( n \times n \) matrices with entries in \( \mathbb{K} \) obtained by Theorem 3 (if \( \mathbb{K} = \mathbb{C} \)) or Theorem 6 (if \( \mathbb{K} = \mathbb{R} \)). We need to show that the subsemigroup action of \( \langle A, E \rangle \) on \( \mathcal{M}_{n \times n}(\mathbb{K}) \) is topologically transitive. Let \( \mathcal{U} \) and \( \mathcal{V} \) be a pair of nonempty open subsets of \( \mathcal{M}_{n \times n}(\mathbb{K}) \) and let \( M \in \mathcal{U} \) be an invertible matrix. It follows that the set \( \{FM : F \in \langle A, E \rangle\} \) is dense in \( \mathcal{M}_{n \times n}(\mathbb{K}) \), and so the orbit of \( M \) under the action of \( \langle A, E \rangle \) must intersect \( \mathcal{V} \).

References

[1] T.M. Apostol, Modular functions and Dirichlet series in number theory, Springer, 2nd ed. 1990.

[2] A. Ayadi, Hypercyclic abelian semigroup of matrices on \( \mathbb{C}^n \) and \( \mathbb{R}^n \) and \( k \)-transitivity \((k \geq 2)\), Appl. Gen. Topol., vol. 12, no. 1 (2011) 35–39.

[3] O. Bogopolski, Introduction to Group Theory (EMS Textbooks in Mathematics), European Mathematical Society (2008).

[4] G. Costakis, D. Hadjiloucas, and A. Manoussos, Dynamics of tuples of matrices, Proc. Amer. Math. Soc. 137 (2009), 1025–1034.

[5] N.S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, J. Math. Anal. Appl. 346 (2008), 82–98.

[6] M. Javaheri, Dense 2-generator subsemigroups of \( 2 \times 2 \) matrices, J. Math. Anal. Appl. 387 (2012) 103–113.

[7] M. Javaheri, Semigroups of matrices with dense orbits, Dyn. Syst. 26 (3) (2011), 235–243.

[8] S. Rolewicz, On orbits of elements, Studia Math. 32 (1969), 17–22.