A note on the computation of mode sums

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(September 11, 2018)

The computation of mode sums of the types encountered in basic quantum field theoretic applications is addressed with an emphasis on their expansions into functions of distance that can be interpreted as potentials. We show how to regularize and calculate the Casimir energy for the continuum Nambu-Goto string with massive ends as well as for the discrete Isgur-Paton non-relativistic string with massive ends. As an additional example, we examine the effect on the interquark potential of a constant Kalb-Ramond field strength interacting with a QCD string.
I. INTRODUCTION

The determination of the Casimir, or zero-point, energy for a quantum system through the evaluation of the zero mode energy sum is a problem of quite practical importance. Casimir originally considered the effect of conducting plates on the QED vacuum energy due to the changed boundary conditions \[1\]. A review of the Casimir effect can be found in the Physics Report by Plunien, Müller, and Greiner \[2\].

The zero-point energy sums are almost always divergent and need regularization in order to be interpreted sensibly. The resulting Casimir energy may be interpreted as a background energy or potential. This is made most evident if the sum can be expressed in terms of simple functions of distance, which might be put into one-to-one correspondence with simple potential terms such as Coulombic or centripetal potential components. The purpose of this paper is to illustrate how this can be done for several classes of mode sums encountered recently in the literature.

We begin in section II by examining the computation of the Casimir energy for a static string with massive ends that is allowed to vibrate transversely. This problem was solved in closed integral form by Lambiase and Nesterenko \[3\]. We review the derivation of the integral form and then find an approximate series representation for large quark end masses.

In section III, we find useful series approximation for a sum of Bessel functions that appears throughout this paper, and elsewhere in physics literature.

In section IV we examine the effect of a particular background Kalb-Ramond field upon the Casimir energy. The Kalb-Ramond field we examine is one giving rise to a constant three-form field strength. The Kalb-Ramond field strength may be viewed as the dual of the abelian monopole current found in lattice QCD that likely leads to the formation of the chromoelectric flux tubes. The effect on the Casimir energy of string self-interaction mediated by the Kalb-Ramond field and the dilaton is to renormalize the string tension \[4\], which also happens in a background Kalb-Ramond field. We find additional non-linear corrections to the static potential in a background Kalb-Ramond field.

Finally, we examine the Casimir energies in the discretized string model of Isgur and Paton \[12\]. We find the interesting result that although the Casimir energies of the continuum and discrete strings agree for fixed-end strings, the finite quark mass corrections are of a completely different form.

II. CONTINUUM QCD STRING WITH MASSIVE ENDS

The eigenfrequencies for a vibrating Nambu-Goto string of length \(r\) and tension \(a\), terminated by two massive quarks of mass \(M\) that are allowed to move only transversely has been given by Lambiase and Nesterenko \[3\] as the roots of the transcendental equation

\[
f(\omega) = (M^2 \omega^2 - a^2) \sin \omega r - 2M \omega \cos \omega r = 0 .
\]

The Casimir energy, which is half the sum of these eigenfrequencies, can be calculated with aid of the Watson-Sommerfeld transform

\[
E_{\text{Casimir}} = \frac{1}{2} \sum_n \omega_n = \frac{1}{4\pi i} \oint_C \omega \, d\omega \, \frac{d}{d\omega} \ln f(\omega) ,
\]

where the contour \(C\) encloses the roots of \(f(\omega)\), all of which lie on the positive real axis for physically meaningful frequencies. The energy computed in Eq. (2) is the energy per transverse degree of freedom. In \(D\) dimensions, there would be an extra factor of \(D - 2\). The sum in Eq. (3) is formally infinite and must be regularized and renormalized. We deform the contour to include the entire left half-plane and keep only the regular portion

\[
\frac{1}{2} \sum_n \omega_n^{\text{reg}} = -\frac{1}{2\pi} \int_0^\infty y \, dy \, \frac{d}{dy} \ln \left[ (M^2 y^2 + a^2) \sinh ry + 2M ay \cosh ry \right] .
\]

We renormalize by subtracting off the part of this integral for which \(r \to \infty\), to arrive at

\[
\frac{1}{2} \sum_n \omega_n^{\text{reg,ren}} = -\frac{1}{2\pi} \int_0^\infty y \, dy \, \frac{d}{dy} \ln \frac{[(M^2 y^2 + a^2) \sinh ry + 2M ay \cosh ry]}{\frac{1}{2} e^{ry} [M^2 y^2 + a^2 + 2M y]} ,
\]

1
which we simplify to
\[ \frac{1}{2} \sum_n \omega_n^{\text{reg,ren}} = -\frac{1}{2\pi} \int_0^\infty y \, dy \, \frac{d}{dy} \ln \left[ 1 - e^{-2ry} \left( \frac{My - a}{My + a} \right)^2 \right], \tag{5} \]
and integrate by parts to arrive at
\[ \frac{1}{2} \sum_n \omega_n^{\text{reg,ren}} = \frac{1}{2\pi} \int_0^\infty dy \ln \left[ 1 - e^{-2y} \left( \frac{My - a}{My + a} \right)^2 \right], \tag{6} \]
\[ = \frac{1}{2\pi r} \int_0^\infty dy \ln \left[ 1 - e^{-2y} \left( \frac{y - s}{y + s} \right)^2 \right], \tag{7} \]
with the dimensionless quantity
\[ s = \frac{ar}{M}. \tag{8} \]

Defining the quantity \( I(s) \), we have the exact result \[ I(s) = 2\pi r E_{\text{Casimir}} = \int_0^\infty dy \ln \left[ 1 - e^{-2y} \left( \frac{y - s}{y + s} \right)^2 \right]. \tag{9} \]
The evaluation of the Casimir energy for small \( s \) can be performed as follows. For very small \( s \) (large masses \( M \)) the approximation,
\[ (x - s)^{2n} \approx e^{-4\pi s x} e^{-4\pi s^3 x} + \cdots, \tag{10} \]
is extremely accurate. The logarithm can be expanded
\[ I(s) = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty \frac{(x - s)^{2n}}{x + s} e^{-2nx} \, dx \approx -\sum_{n=1}^\infty \frac{1}{n} \int_0^\infty e^{-2nx - \frac{4\pi s}{x}} \, dx, \tag{11} \]
and used with the integral representation of the modified Bessel function
\[ \int_0^\infty x^{\nu-1} e^{-\beta x - \gamma x} \, dx = 2 \left( \frac{\beta}{\gamma} \right)^\frac{\nu}{2} K_\nu(2\sqrt{\beta\gamma}), \tag{13} \]
to obtain
\[ I(s) \approx -\sqrt{8s} \sum_{n=1}^\infty \frac{1}{n} K_1(2n \sqrt{8s}) \tag{14} \]
as an excellent approximation for \( s \) small. Using the results of section \[ \text{III} \] below, we find
\[ I(s) = -\frac{\pi^2}{12} + \sqrt{2\pi^2 s} + (4\gamma - 2)s - 4s \sinh^{-1} \left( \frac{\pi}{\sqrt{8s}} \right) + \cdots, \tag{15} \]
which agrees extremely well with numerical evaluations of the integral for small \( s \), as can be seen in Fig. \[ \text{I} \].

A comparison of the series \[ (13) \] and the sum \[ (14) \] is shown in Fig. \[ \text{II} \]. If more accuracy is needed for larger values of \( s \), higher terms in \( s \) in the exponent of Eq. \[ (10) \] can be retained and those contributions to the exponent expanded in a power series.

If the quark mass \( M \) is large, we can use the approximation Eq. \[ (13) \] to calculate the lowest order corrections in \( \frac{1}{M} \) to the Casimir energy. From the definition Eq. \[ (8) \], we find the Casimir energy per transverse degree of freedom
\[ E_{\text{Casimir}} = -\frac{\pi}{24r} + \sqrt{\frac{a}{2Mr}} + O(\frac{a}{M}). \tag{16} \]
III. EVALUATION OF A SUM OF BESSEL FUNCTIONS

In this section we evaluate the sum

\[ G(x) = \sum_{n=1}^{\infty} \frac{1}{n} K_1(nx), \]

for small \( x \). This sum appears in the evaluation of functional determinants \[1\] as well as Casimir energies \[2,4,5\]. Related sums of Bessel functions appear in generic one-loop finite temperature effective potentials for GUTs \[8\], and also appear in the mathematics literature \[9\].

Evaluation of this sum using zeta functions quickly becomes problematic because no matter how small \( x \) is, for sufficiently large \( n \) values, \( n x \to \infty \) and small \( x \) expansions of \( K_1(x) \) cannot be used. However, we can use another integral representation of the Bessel function,

\[ K_1(nx) = \frac{2\pi x}{n} \int_0^\infty \frac{\cos(2\pi nt) \, dt}{((2\pi t)^2 + x^2)^{\frac{3}{2}}}, \]

(18)
together with the sum

\[ \sum_{n=1}^{\infty} \frac{\cos(2\pi nt)}{n^2} = \pi^2 B_2(t - [t]), \]

(19)
where \([t]\) is the greatest integer less than \( t \) and \( B_2(t) \) is the second Bernoulli polynomial,

\[ B_2(t) = \frac{1}{6} - t + t^2. \]

(20)
We obtain

\[ \sum_{n=1}^{\infty} \frac{K_1(nx)}{n} = 2\pi x \int_0^\infty \frac{dt}{((2\pi t)^2 + x^2)^{\frac{3}{2}}} \left( \sum_{n=1}^{\infty} \frac{\cos(2\pi nt)}{n^2} \right) \]
\[ = 2\pi x \int_0^\infty \frac{\pi^2 B_2(t - [t]) \, dt}{((2\pi t)^2 + x^2)^{\frac{3}{2}}}. \]

(21)
We may split this expression into the first term and a remainder

\[ G(x) = 2\pi x \int_0^1 \frac{\pi^2 B_2(t)}{(x^2 + (2\pi t)^2)^{\frac{3}{2}}} \, dt + 2\pi x \int_1^\infty \frac{\pi^2 B_2(t - [t])}{(x^2 + (2\pi t)^2)^{\frac{3}{2}}} \, dt. \]

(22)
The remainder may be written as a sum,

\[ 2\pi x \int_1^\infty \frac{\pi^2 B_2(t - [t])}{(x^2 + (2\pi t)^2)^{\frac{3}{2}}} \, dt = 2\pi x \sum_{m=1}^{\infty} \int_0^1 \frac{\pi^2 B_2(t)}{(x^2 + (2\pi(t + m))^2)^{\frac{3}{2}}} \, dt, \]

(23)
which may then be expanded for small \( x \) in a binomial series;

\[ 2\pi x \sum_{m=1}^{\infty} \int_0^1 \frac{\pi^2 B_2(t)}{(x^2 + (2\pi(t + m))^2)^{\frac{3}{2}}} \, dt = 2\pi^3 x \sum_{k=0}^{\infty} \frac{\left( -\frac{x}{2} \right)^k}{k!} \int_0^1 \frac{B_2(t)}{(2\pi)^{3+2k}} \, dt. \]

(24)
Finally, we rewrite this expression in term of the generalized zeta function

\[ G(x) = 2\pi x \int_0^1 \frac{\pi^2 B_2(t)}{(x^2 + (2\pi t)^2)^{\frac{3}{2}}} \, dt + 2\pi^3 x \sum_{k=0}^{\infty} \frac{\left( -\frac{x}{2} \right)^k}{k!} \int_0^1 \frac{B_2(t)\zeta(3+2k,t)}{(2\pi)^{3+2k}} \, dt. \]

(25)
All of these integrals are strongly convergent, which is obvious from simple power counting for all but the \( k = 0 \) term, which we will explicitly evaluate. The integral in the first term, which is dominant for \( x \approx 0 \), is elementary.
\[
\int_0^1 \frac{B_2(t)}{(x^2 + (2\pi t)^2)^{\frac{3}{2}}} = \frac{1}{6x^2\sqrt{x^2 + 4\pi^2}} - \frac{1}{4\pi^2|x|} + \frac{1}{8\pi^3} \sinh^{-1} \frac{2\pi}{|x|}.
\] (26)

For the \(k = 0\) term, we replace the sum of integrals by its original form to obtain
\[
\int_1^\infty \frac{B_2(t - [t])}{t^3} dt = \frac{1}{6} \int_1^\infty \frac{dt}{t^3} + \lim_{N \to \infty} \left( - \int_1^N \frac{(t - [t])}{t^3} dt + \int_1^N \frac{(t - [t])^2}{t^3} dt \right).
\] (27)

We evaluate each of the integrals separately. The first integral is trivial and gives \(\frac{1}{12}\). Using the notation \(F_2(m) = m^{-2}\), we find that the second integral is
\[
\lim_{N \to \infty} \int_1^N \frac{1}{t^3} dt = - \lim_{N \to \infty} \left( \int_1^N \frac{dt}{t^3} - \sum_{m=1}^{N-1} m \int_0^1 \frac{dt}{(t + m)^3} \right) = - \lim_{N \to \infty} \left( 1 - \frac{1}{N} + \frac{1}{2} \sum_{m=1}^{N-1} m (F_2(m+1) - F_2(m)) \right) = - \lim_{N \to \infty} \left( 1 - \frac{1}{N} + \frac{1}{2} \left( - \sum_{m=1}^{N} F_2(m) + N F_2(N) \right) \right) \approx \frac{\pi^2}{12} - 1.
\] (28)

We can evaluate the last integral similarly, this time using the notation \(F_1(m) = m^{-1}\).
\[
\lim_{N \to \infty} \int_1^N \frac{(t - [t])^2}{t^3} dt = \lim_{N \to \infty} \left( \ln N - 2 \sum_{m=1}^{N-1} \int_0^1 \frac{m dt}{(t + m)^2} + \sum_{m=1}^{N-1} \int_0^1 \frac{m^2 dt}{(t + m)^3} \right) = \lim_{N \to \infty} \left( \ln N + 2 \sum_{m=1}^{N-1} m (F_1(m+1) - F_1(m)) - \frac{1}{2} \sum_{m=1}^{N-1} m^2 (F_2(m+1) - F_2(m)) \right). \] (30)

Rearranging the series according to Eq. (28) and
\[
\sum_{m=1}^{N-1} m^2 (F_2(m+1) - F_2(m)) = - \sum_{m=1}^{N} (2m-1) F_2(m) + N^2 F_2(N),
\] (31)
we obtain
\[
\lim_{N \to \infty} \int_1^N \frac{(t - [t])^2}{t^3} dt = \lim_{N \to \infty} \left( \ln N - 2 \sum_{m=1}^{N} \frac{1}{m} - \frac{1}{2} \left( \frac{N^2}{2} - \sum_{m=1}^{N} \frac{(2m-1)}{m^2} \right) \right) = \lim_{N \to \infty} \left( \ln N - \sum_{m=1}^{N} \frac{1}{m} + \frac{3}{2} - \frac{1}{2} \sum_{m=1}^{N} \frac{1}{m^2} \right) = \frac{3}{2} - \gamma - \frac{\pi^2}{12}.
\] (32)

We put all the terms together from Eqs. (27), (29), and (33) to find
\[
2\pi^3 x \left( \frac{-3}{0} \right) \frac{1}{(2\pi)^3} \int_1^\infty \frac{B_2(t - [t])}{t^3} dt = \frac{x}{4} \left( \frac{7}{12} - \gamma \right),
\] (34)
which implies that
\[
G(x) = \sum_{n=1}^{\infty} \frac{1}{n} K_1(n x) = \frac{\pi^2}{6x} - \frac{\pi}{2} \text{sign}(x) + \left( \frac{1 - 2\gamma}{8} \right) x + \frac{x}{4} \sinh^{-1} \left( \frac{2\pi}{|x|} \right) + O(x^3).
\] (35)
The physics of charged particles moving in a plane under the influence of a constant magnetic field is a very beautiful and enlightening subject containing such surprises as the quantum Hall effect. The string analog of a charge moving in a constant magnetic field is the motion of a string in a constant Kalb-Ramond field strength

\[
H_{0ij}(x) = 0, \\
H_{ijk}(x) = \hbar \epsilon_{ijk}.
\]  

This system is a model of a QCD string interacting with a constant monopole density and, like the Landau problem, is also exactly solvable \(^1\). \(^2\) (The monopole current is proportional to the dual of \( H_{\alpha\beta\gamma} \) : \( J^\alpha_{\text{monopole}} \propto \epsilon^{\alpha\beta\gamma} H_{\alpha\beta\gamma} \).) The Kalb-Ramond potential \( B_{\mu\nu} \) corresponding to the field strength \((36)\) is

\[
B_{0i}(x) = 0, \\
B_{ij}(x) = \frac{\hbar}{3} \epsilon_{ijk} x^k.
\]

In vortex mechanics, the Kalb-Ramond field describes the Magnus forces on a vortex line moving in a fluid.

The first-quantized action for a bosonic string moving in a background Kalb-Ramond field is

\[
S = -\frac{a}{2} \int d\sigma d\tau \left( \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + B_{\mu\nu}(X) \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu \right). \quad (38)
\]

If we substitute the field \( B_{ij} \) above into the string action, and we choose orthonormal coordinates on the worldsheet, we find the action becomes

\[
S = -\frac{a}{2} \int d\sigma d\tau \left( \eta^{ab} \partial_a X^\mu \partial_b X^\nu + \frac{\hbar}{3} \epsilon_{ijk} \epsilon^{ab} X^i \partial_a X^j \partial_b X^k \right), \quad (39)
\]

and implies the equations of motion

\[
\ddot{X}^0 - X^{0\nu} = 0, \quad (40) \\
\dot{X} - X'' = -\hbar \dot{X} \times X'. \quad (41)
\]

Here we have used reparametrization invariance of the action to set coordinate time equal to laboratory time \( \tau = x^0 \) and to set the spatial parameter of the string equal to the distance in real space along the string, \( \sigma = z \). We wish to find the Casimir energy of a long straight string acting under the influence of this external Kalb-Ramond field. We assume the string lies along the \( z \) direction and has length \( r \). Thus we find that \( X' = \hat{z} \), and the linearized equations of motion for the small vibrations on the string become

\[
\ddot{X} - X'' = -\hbar \dot{Y}, \\
\dot{Y} - Y'' = \hbar \dot{X}.
\]

We define the right and left circularly polarized modes \( X^\pm \equiv X \pm iY \), and find that in terms of them the small amplitude vibration equations above reduce to

\[
\ddot{X}^\pm - X'^{\pm\nu} = \mp i \hbar \dot{X}^\pm. \quad (43)
\]

Since our string has fixed length, we know that the wavelength with either fixed Dirichlet or Neumann boundary conditions (infinitely massive or completely massless quarks at the end) lead to

\[
\dot{X}^\pm - i\hbar X^\pm = X'^{\pm\nu} = -\left( \frac{\pi n}{r} \right)^2 X^\pm, \quad n = 1, 2, 3, \ldots.
\]

or, if we assume that \( X = e^{i\omega t} X_0 \),

\[
\omega_\pm^2 = \hbar \omega_\pm^2 = \left( \frac{\pi n}{r} \right)^2 = k_n^2.
\]
The frequencies of small vibrations are then
\[ \omega_{n \pm} = \pm \frac{h}{2} + \sqrt{k_n^2 + \left( \frac{h}{2} \right)^2} = \pm \frac{h}{2} + \frac{\pi}{r} \sqrt{n^2 + \left( \frac{r h}{2 \pi} \right)^2}. \] (46)

Naively, the Casimir energy as a function of the Kalb-Ramond field strength \( h \) is half the sum of all the frequencies
\[ E_{\text{Casimir}}(h) = \frac{\pi}{r} \sum_{n=1}^{\infty} \sqrt{n^2 + \left( \frac{r h}{2 \pi} \right)^2}. \] (47)

Mode sums of this form,
\[ \sigma(\xi) = \sum_{n=1}^{\infty} \sqrt{n^2 + \xi^2}, \] (48)

appear in a large variety of problems (see the references following Eq. (5.33) in [3]). We now show the simple steps with which this sum can be expressed in a form very similar to Eq. (14).

First, we subtract out the divergence from the sum in order to regularize it
\[ \sigma_{\text{reg}}(\xi) = \sum_{n=1}^{\infty} \left( \sqrt{n^2 + \xi^2} - n \right) - \frac{1}{12}. \] (49)

We may rewrite this as
\[ \sigma_{\text{reg}}(\xi) = -\frac{1}{12} + \sum_{n=1}^{\infty} \int_{0}^{\xi} x \frac{dx}{\sqrt{n^2 + x^2}} \] (50)
\[ = -\frac{1}{12} + \sum_{n=1}^{\infty} \int_{0}^{\xi} x \frac{dx}{\sqrt{n^2 + x^2}} \int_{0}^{\infty} \frac{1}{\Gamma(\frac{1}{2})} t^{-1/2} e^{-t(n^2 + x^2)} dt. \] (51)

The sum may be done by Poisson resummation
\[ \sum_{n=1}^{\infty} e^{-\frac{t n^2}{2}} = \frac{1}{2} \sqrt{\frac{\pi}{t}} - \frac{1}{2} + \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} e^{-\frac{2 n^2}{t}}, \] (52)

leading to
\[ \sigma_{\text{reg}}(\xi) = -\frac{1}{12} + \int_{0}^{\xi} x \frac{dx}{\sqrt{\pi t}} \int_{0}^{\infty} \frac{dt}{\sqrt{t}} e^{-t x^2} \left( \frac{1}{2} \sqrt{\frac{\pi}{t}} - \frac{1}{2} + \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} e^{-\frac{2 n^2}{t}} \right). \] (53)

Concentrating on the last term, we may use
\[ \int_{0}^{\infty} x^{\mu-1} e^{-x - \frac{x^2}{4}} dx = 2 \left( \frac{\mu}{2} \right)^{\nu} K_{-\nu}(\mu), \] (54)

where \( K_{\nu} \) is a modified Bessel function. Letting \( s = t x^2 \), we rewrite the last term in the sum as
\[ \int_{0}^{\xi} x \frac{dx}{\sqrt{\pi t}} \int_{0}^{\infty} \frac{dt}{\sqrt{t}} e^{-t x^2} \left( \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} e^{-\frac{2 n^2}{t}} \right) = \sum_{n=1}^{\infty} \int_{0}^{\xi} x \frac{dx}{\sqrt{\pi}} \int_{0}^{\infty} \frac{ds}{s} e^{-s - \frac{x^2}{s}} \] (55)
\[ = 2 \sum_{n=1}^{\infty} \int_{0}^{\xi} x dx K_{0}(2 n \pi x) \] (56)
\[ = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{\xi}{n \pi} K_{1}(2 n \pi |\xi|) \] (57)
Finally, we have

\[
\sigma_{\text{reg}}(\xi) = -\left|\xi\right| - \sum_{n=1}^{\infty} \frac{K_1(2n\pi|\xi|)}{n} + \frac{1}{2\sqrt{\pi}} \int_0^{\left|\xi\right|} x \, dx \int_0^{\infty} t^{-1/2} \left(\sqrt{\frac{\pi}{t}} - 1\right) e^{-x^2t} \, dt,
\]

(56)
in which the final integral may be expanded and evaluated in terms of Gamma functions as

\[
\frac{1}{2\sqrt{\pi}} \int_0^{\left|\xi\right|} x \, dx \int_0^{\infty} t^{-1/2} \left(\sqrt{\frac{\pi}{t}} - 1\right) e^{-x^2t} \, dt = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} t^{-1/2} \left(\sqrt{\frac{\pi}{t}} - \frac{1}{2t} \right) \left(1 - e^{-\xi^2t}\right) \, dt
\]

\[
= \frac{1}{4\sqrt{\pi}} \int_0^{\infty} \left(\sqrt{\frac{\pi}{t}} - \sqrt{\frac{\pi}{t}} e^{-\xi^2t} - 1 + e^{-\xi^2t}\right) \, dt
\]

\[
= \infty + \frac{1}{4\sqrt{\pi}} \int_0^{\infty} e^{-\xi^2t} t^{-3/2} \, dt
\]

\[
= \infty + \infty \xi^2 - \frac{|\xi|^2}{2}.
\]

(57)

Subtracting the infinite pieces above, we find the final regularized result

\[
\sigma_{\text{reg}}(\xi) = -\left|\xi\right| - \sum_{n=1}^{\infty} \frac{K_1(2n\pi|\xi|)}{n} = -\frac{|\xi|}{2} - \frac{|\xi|}{\pi} G(2\pi|\xi|).
\]

(58)

This and related results are found by a slightly different method in Ref. [7].

The Casimir energy corresponding to the zero-point frequencies (46),

\[
E_{\text{Casimir}} = \frac{\pi}{\hbar r} \sigma(hr/2\pi),
\]

(59)
can be evaluated using the results of section III. We find

\[
E_{\text{Casimir}} = -\frac{\pi}{12r} + \frac{\hbar^2 r}{16\pi} \left[(2\gamma - 1) - 2 \sinh^{-1} \left(\frac{2\pi}{\hbar r}\right)\right] + O(h^3r^2).
\]

(60)

For \(h \to 0\) this reduces to the fixed-end straight string result for two transverse degrees of freedom.

V. DISCRETE ISGUR-PATON STRING

In this section we consider the interquark potential of a static quark-antiquark pair bound by a flux tube, considered by Isgur and Paton [12] and Merlin and Paton [13] to be a discrete string consisting of \(N\) masses \(m\) a distance \(d\) apart. This string, when terminated by particles of mass \(M\), has equations of motion [12,13,14] with the secular equation

\[
1 - \frac{M}{m} \lambda = \frac{\sin(N\theta)}{\sin((N+1)\theta)},
\]

(61)

\[
\lambda = 2(1 - \cos \theta), \quad \lambda = \frac{\omega^2}{\beta},
\]

(62)

for the transverse vibrational frequencies \(\omega\). Here \(\beta\) is defined in terms of the masses \(m\), the bare tension \(a_0\) and interquark distance \(r\), as \(\beta = \frac{a_0 (N+1)}{mr}\). The infinite mass \(M\) limit results in

\[
\theta = \frac{p\pi}{(N+1)}, \quad p = 1, 2, \ldots, N.
\]

(63)

Assuming finite \(M\) corrections of the form

\[
\theta \to \frac{p\pi + \delta}{(N+1)},
\]

(64)
we discover that

\[
\omega \approx \sqrt{\beta} \left( \sin \left( \frac{p\pi}{2(N+1)} \right) + \frac{m}{4M(N+1)} \left( \frac{1}{\sin \left( \frac{p\pi}{2(N+1)} \right)} - \sin \left( \frac{p\pi}{2(N+1)} \right) \right) \right) .
\]  

(65)

The contribution of the first and third terms to the mode sum is simple, following from

\[
\sum_{p=1}^{N} \sqrt{\beta} \sin \left( \frac{p\pi}{2(N+1)} \right) = \sqrt{\beta} \frac{\sin \left( \frac{N\pi}{\pi(N+1)} \right)}{2} \sin \left( \frac{\pi}{4(N+1)} \right) .
\]  

(66)

The second term can be analyzed by use of the expansion

\[
\csc(x) = \frac{1}{x} + 2 \sum_{k=1}^{\infty} \frac{(2^{2k-1} - 1)}{(2k)!} |B_{2k}| x^{2k-1} ,
\]  

(67)

in which \( B_k \) are the Bernoulli numbers.

We begin with

\[
J(N) = \sum_{p=1}^{N} \csc \left( \frac{\pi p}{2(N+1)} \right)
\]

\[
= \frac{2(N+1)}{\pi} \sum_{p=1}^{N} \frac{1}{p} + \sum_{k=1}^{\infty} 2 \frac{(2^{2k-1} - 1)}{(2k)!} |B_{2k}| \left( \frac{\pi}{2(N+1)} \right)^{2k-1} \sum_{p=1}^{N} p^{2k-1} ,
\]  

(68)

and use

\[
\sum_{p=1}^{N} p^{2k-1} = \frac{B_{2k}(N+1) - B_{2k}}{2k} ,
\]  

(69)

in which the Bernoulli polynomials \( B_n(x) \) satisfy the formal recursion

\[
B_n(x) = (B + x)^n \quad \text{with} \quad B^n \to B_n .
\]  

(70)

We find that

\[
\sum_{p=1}^{N} p^{2k-1} = \frac{1}{2k}(N+1)^{2k} - \frac{1}{2}(N+1)^{2k-1} + \frac{2k-1}{12}(N+1)^{2k-2}
\]

\[
- \frac{(2k-1)(2k-2)(2k-3)}{30 \cdot 4!}(N+1)^{2k-4} + \cdots .
\]  

(71)

The second term in Eq. (65) thus contributes five terms to the mode sum

\[
J(N) = \sum_{p=1}^{N} \csc \left( \frac{\pi p}{2(N+1)} \right) = I_0 + I_1 + I_2 + I_3 + I_4
\]  

(72)

of leading orders in \( N \). The first is simple

\[
I_0 = \frac{2(N+1)}{\pi} \sum_{p=1}^{N} \frac{1}{p}
\]

\[
= \frac{2(N+1)}{\pi} \left( \gamma + \ln N + \frac{1}{2N} - \sum_{k=2}^{\infty} \frac{A_k}{N(N+1)(N+2) \cdots (N+k-1)} \right) ,
\]  

(73)
for which $A_2 = A_3 = \frac{1}{12}$, $A_4 = \frac{1}{30}$, and so forth. This is a standard expansion of the Digamma function [15].

The next term is

$$I_1 = \frac{2(N + 1)}{\pi} \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)}{(2k)!} \left| B_{2k} \right| \left( \frac{\pi}{2} \right)^{2k}$$

$$= \frac{2(N + 1)}{\pi} \int_{0}^{\pi} \left( \csc x - \frac{1}{x} \right) dx = \frac{2(N + 1)}{\pi} \left( \ln \frac{2}{\pi} - \ln \frac{1}{2} \right) ,$$

and subsequent terms can all be expressed as valuations of the derivatives of $\csc(x) - \frac{1}{x}$ at the point $x = \frac{\pi}{2}$.

$$I_2 = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)}{(2k)!} \left| B_{2k} \right| \left( \frac{\pi}{2} \right)^{2k-1}$$

$$= -\frac{1}{2} \left( \csc x - \frac{1}{x} \right) \bigg|_{x=\frac{\pi}{2}} = \frac{1}{\pi} - \frac{1}{2} ,$$

$$I_3 = \frac{1}{6(N + 1)} \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)}{(2k)!} \left| B_{2k} \right| \left( \frac{\pi}{2} \right)^{2k}$$

$$= \frac{1}{(24(N + 1))} x \frac{d}{dx} \left( \csc x - \frac{1}{x} \right) \bigg|_{x=\frac{\pi}{2}} = \frac{\pi}{24(N + 1)} ,$$

and finally

$$I_4 = -\frac{1}{30 \cdot 4! (N + 1)^3} x^3 \frac{d^3}{dx^3} \left( \csc x - \frac{1}{x} \right) \bigg|_{x=\frac{\pi}{2}} .$$

All of the parts can be assembled to give

$$J(N) = \sum_{p=1}^{N} \csc \left( \frac{\pi p}{2(N + 1)} \right) = \frac{2(N + 1)}{\pi} \left( \gamma + \ln \frac{4(N + 1)}{\pi} \right) - \frac{1}{2}$$

$$+ \frac{\pi}{24(N + 1)} + \left( \frac{1}{\pi} - \frac{1}{12} \right) \frac{1}{N(N + 1)} + O(N^{-3}) .$$

This expression agrees very well with the actual sum, plotted as a function of $N$. This is illustrated in Fig. 3.

For large quark masses $M$, we can again calculate the lowest order $\frac{1}{M}$ corrections to the Casimir energy, this time for the discrete string before the limit of link spacing going to zero ($d \rightarrow 0$) is taken. To make contact with Isgur and Paton [12], we take $\beta = \frac{1}{d^2}$ and then use $N = (\frac{r}{d}) - 1$ to arrive at the Casimir energy contribution per transverse degree of freedom

$$E_{\text{Casimir}} = \frac{2r}{\pi d^2} - \frac{1}{2d} - \frac{\pi}{24r} - \frac{a_0(1 - \gamma - \ln(4r^2))}{2\pi M} + O\left( \frac{1}{M^2} \right) .$$

We note that although the universal Lüsücher term $-\frac{\pi}{24r}$ is present in both (16) and (82), the $\frac{1}{M}$ corrections are completely different. In the former case, the correction is $O\left( \frac{1}{\sqrt{M}} \right)$, while in the latter it is $O\left( \frac{1}{M^2} \right)$. Of course, there is no reason to expect that the form of the Casimir energies will be the same for the Nambu-Goto string and the discrete string of Isgur and Paton. The former is relativistic while the later is not. In addition, it is not likely that the Casimir energy of a discrete string will go to the continuum result as the number of mass points is taken to infinity. Sums and limits generally depend upon the order in which they are done, especially when the sums are divergent and must be regularized.
VI. CONCLUSIONS

We have shown how to regularize and perform the zero-point energy mode sums for the discrete non-relativistic string studied by Isgur and Paton and the relativistic continuum Nambu-Goto string. Computation of these divergent sums is necessary in order to obtain the Casimir energy of the system to be interpreted as part of the static potential for quarks bound by a QCD string. We have shown that the lowest order corrections to the Casimir energies in the reciprocal of the mass of the terminating quark is different for the two cases. We have also shown how the addition of a background Kalb-Ramond field with a corresponding constant field strength affects the static potential. We hope to extend these results to the case of a string interacting with itself through a Kalb-Ramond field.

ACKNOWLEDGMENTS

We thank S. Dowker, R. Easther, and V.V. Nesterenko for useful comments on an earlier version of this paper. This work was supported in part by the US Department of Energy under Contract No. DE-FG02-95ER40896.

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FIG. 1. Comparison of exact result with series representation of Casimir energies of string with massive ends. Numerical evaluations of Eq. (9), dotted line, and Eq. (15), solid curve, for small values of $s$. 
FIG. 2. Comparison of sum of Bessel functions and series expansion. Numerical evaluations of Eq. (14), circles, and Eq. (15), solid curve, for small values of $s$. 
FIG. 3. Numerical evaluations of the sum $J(N)$. The solid curve is the function Eq. (81), while the diamonds are for the sum Eq. (68).