Product structure of graphs with an excluded minor

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Abstract

This paper shows that $K_t$-minor-free (and $K_{s,t}$-minor-free) graphs $G$ are subgraphs of products of a tree-like graph $H$ (of bounded treewidth) and a complete graph $K_m$. Our results include optimal bounds on the treewidth of $H$ and optimal bounds (to within a constant factor) on $m$ in terms of the number of vertices of $G$ and the treewidth of $G$. These results follow from a more general theorem whose corollaries include a strengthening of the celebrated separator theorem of Alon, Seymour, and Thomas [J. Amer. Math. Soc. 1990] and the Planar Graph Product Structure Theorem of Dujmović et al. [J. ACM 2020].

1 Introduction

Graph Product Structure Theory is a body of research which describes complicated graphs as subgraphs of products of simpler graphs. Typically, the simpler graphs are tree-like, in the sense that they have bounded treewidth, which is the standard measure of how similar a graph is to a tree. (We postpone the definition of treewidth and other standard graph-theoretic concepts until Section 2.) This area has recently received a lot of attention [2, 6, 7, 10, 15, 17, 19, 20, 25–27, 40] with highlights including the Planar Graph Product Structure Theorem of Dujmović et al. [15]; see Theorem 7 below.

Our main contribution is a powerful general result, Theorem 12, that essentially converts a tree-decomposition of a graph excluding a particular minor into a product that inherits some of the properties of the decomposition. Its applications include a strengthening of the celebrated Alon–Seymour–Thomas separator theorem as well as the Planar Graph Product Structure Theorem.

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Throughout the paper we work with strong products of graphs. The strong product $A \boxtimes B$ of graphs $A$ and $B$ has vertex-set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$. This paper focuses on products of the form $H \boxtimes K_m$ and $H \boxtimes P \boxtimes K_m$ where $H$ is a graph of bounded treewidth, $P$ is a path and $m$ is some function of the original graph. An alternative view of the product $H \boxtimes K_m$ is as a ‘blow-up’ of the graph $H$, obtained by replacing each vertex of $H$ be a copy of the complete graph $K_m$ and each edge of $H$ by a copy of the complete bipartite graph $K_{m,m}$.

In one of the cornerstone results of Graph Minor Theory, Alon, Seymour, and Thomas [1] proved that every $K_t$-minor-free graph has a balanced separator of size at most $t^{3/2}n^{1/2}$. In fact, they proved the following stronger result.\footnote{The balanced separator result follows from \textbf{Theorem 1} and the separator lemma of Robertson and Seymour [37, (2.6)].}

**Theorem 1** ([1]). Every $n$-vertex $K_t$-minor-free graph $G$ has treewidth $\text{tw}(G) < t^{3/2}n^{1/2}$.

Our first result is the following strengthening of \textbf{Theorem 1} that describes $K_t$-minor-free graphs as blow-ups of simpler graphs, namely graphs with bounded treewidth.

**Theorem 2.** For any integer $t \geq 4$, every $n$-vertex $K_t$-minor-free graph $G$ is

(a) isomorphic to a subgraph of $H \boxtimes K_{\lceil m \rceil}$, where $\text{tw}(H) \leq t - 1$ and $m := \sqrt{(t-3)n}$;

(b) isomorphic to a subgraph of $H \boxtimes K_{\lceil m \rceil}$, where $\text{tw}(H) \leq t - 2$ and $m := 2\sqrt{(t-3)n}$.

\textbf{Theorem 2(a)} immediately implies \textbf{Theorem 1}, since

\[ \text{tw}(G) \leq \text{tw}(H \boxtimes K_{\lceil m \rceil}) \leq (\text{tw}(H) + 1)m - 1 < t\sqrt{(t-3)n}. \]

The dependence on $n$ in the blow-up factor $m$ is best possible since the $n^{1/2} \times n^{1/2}$ planar grid graph $G$ is $K_5$-minor-free and has treewidth $n^{1/2}$. If $G$ is isomorphic to a subgraph of $H \boxtimes K_m$ where $H$ has bounded treewidth, then $n^{1/2} \leq \text{tw}(G) \leq (\text{tw}(H) + 1)m - 1$ and so $m = \Omega(n^{1/2})$. The dependence on $t$ is discussed in \textbf{Section 6}; see \textbf{Q1} there.

While our proof of \textbf{Theorem 2} uses some ideas from the proof of \textbf{Theorem 1} (in particular, \textbf{Lemma 10} below), it is in fact significantly simpler, avoiding the use of havens or any form of treewidth duality. Instead, the proof directly constructs an isomorphism from $G$ to $H \boxtimes K_{\lceil m \rceil}$ where $H$ is a graph obtained by repeated clique-sums (which implies the desired treewidth bound).

We also prove the following analogous theorem for excluded complete bipartite minors. Let $K^*_s,t$ be the graph whose vertex-set can be partitioned $A \cup B$, where $|A| = s, |B| = t$, $A$ is a clique, $B$ is an independent set, and every vertex in $A$ is adjacent to every vertex in $B$, that is, $K^*_s,t$ is obtained from $K_{s,t}$ by adding all the edges inside the part of size $s$.\footnote{The balanced separator result follows from \textbf{Theorem 1} and the separator lemma of Robertson and Seymour [37, (2.6)].}
Theorem 3. For all integers \( s, t \geq 2 \), every \( n \)-vertex \( K_{s,t}^\ast \)-minor-free graph \( G \) is isomorphic to a subgraph of \( H \boxtimes K_{m} \), where \( \text{tw}(H) \leq s \) and \( m := 2\sqrt{(s - 1)(t - 1)n} \).

Again the \( n^{1/2} \times n^{1/2} \) planar grid (which is \( K_{3,3} \)-minor-free) shows the dependence on \( n \) in the blow-up factor is best possible—we must have \( m = \Omega(n^{1/2}) \).

In light of Theorem 1, it is natural to try to qualitatively strengthen Theorems 2 and 3 by bounding the blow-up factor by a function of the treewidth of \( G \), and ideally by a linear function of \( \text{tw}(G) \) since if \( G \subseteq H \boxtimes K_{m} \) and \( \text{tw}(H) = O(1) \), then \( m = \Omega(\text{tw}(G)) \).

In this direction, Campbell et al. [7, Thm. 18] proved that every \( K_{t} \)-minor-free graph \( G \) is isomorphic to a subgraph of \( H \boxtimes K_{m} \) where \( \text{tw}(H) \leq t - 2 \) and \( m = O(tw(G)^2) \).

Similarly, they proved [7, Thm. 19] that every \( K_{s,t} \)-minor-free graph \( G \) is isomorphic to a subgraph of \( H \boxtimes K_{m} \) where \( \text{tw}(H) \leq s \) and \( m = O_{s,t}(\text{tw}(G)^2) \). Here \( O_{s,t}(\cdot) \) and \( \Omega_{s,t}(\cdot) \) hide dependence on \( s \) and \( t \).

We achieve a blow-up factor that is linear in \( \text{tw}(G) \), and is independent of \( t \) for \( K_{t} \)-minor-free graphs.

Theorem 4. For any integer \( t \geq 2 \), every \( K_{t} \)-minor-free graph \( G \) is isomorphic to a subgraph of \( H \boxtimes K_{m} \), where \( \text{tw}(H) \leq t - 2 \) and \( m := tw(G) + 1 \).

The value of \( m \) in Theorem 4 is within a factor \( t - 1 \) of best possible, since

\[
\text{tw}(G) \leq \text{tw}(H \boxtimes K_{m}) \leq (\text{tw}(H) + 1)m - 1 < (t - 1)m.
\]

Furthermore, the \( t - 2 \) bound on the treewidth of \( H \) is best possible, since Campbell et al. [7, Thm. 18] proved that, for any function \( f \) and for all \( t \), there is a \( K_{t} \)-minor-free graph \( G \) that is not a subgraph of \( H \boxtimes K_{f(\text{tw}(G))} \) for any graph \( H \) with treewidth at most \( t - 3 \).

For \( K_{s,t}^\ast \)-minor-free graphs we also obtain a blow-up factor that is linear in \( \text{tw}(G) \).

Theorem 5. For all integers \( s, t \geq 2 \), every \( K_{s,t}^\ast \)-minor-free graph \( G \) is isomorphic to a subgraph of \( H \boxtimes K_{m} \), where \( \text{tw}(H) \leq s \) and \( m := (t - 1)(\text{tw}(G) + 1) \).

Here the value of \( m \) is within a factor \( (s + 1)(t - 1) \) of best possible and the \( \text{tw}(H) \leq s \) bound is best possible [7, Thm. 19].

An attraction of Theorems 3 and 5 is that \( \text{tw}(H) \) depends on \( s \) and not on the size of the excluded minor. This is particularly relevant for graphs of Euler genus\(^2\) \( g \), since these contain no \( K_{3,2g+3} \)-minor. Thus the next result follow from Theorems 3 and 5.

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\(^2\)The Euler genus of a surface with \( h \) handles and \( c \) cross-caps is \( 2h + c \). The Euler genus of a graph \( G \) is the minimum integer \( g \geq 0 \) such that \( G \) embeds in a surface of Euler genus \( g \); see [33] for more about graph embeddings in surfaces.
Corollary 6. For any integer \( g \geq 0 \), every \( n \)-vertex graph \( G \) of Euler genus \( g \) is isomorphic to a subgraph of \( H \boxtimes K_{\lfloor m \rfloor} \), where \( \text{tw}(H) \leq 3 \) and

\[
m := \min \{4\sqrt{(g+1)n}, \ 2(g+1)(\text{tw}(G)+1)\}.
\]

Corollary 6 is a product strengthening of results about balanced separators (equivalently, about treewidth) in graphs embeddable on surfaces of genus \( g \), independently due to Djidjev [11] and Gilbert, Hutchinson, and Tarjan [23]. In particular, Corollary 6 implies that \( \text{tw}(G) \leq (\text{tw}(H) + 1)m - 1 = 4m - 1 < 16\sqrt{(g+1)n} \) and that \( G \) has a balanced separator of size at most \( 4m \leq 16\sqrt{(g+1)n} \). Both these bounds are tight up to the multiplicative constant.

Theorems 4 and 5 are in fact special cases of a more general result, Theorem 12, that essentially converts any tree-decomposition of a graph excluding a particular minor into a strong product. The starting tree-decomposition may be chosen to suit one’s needs. Making use of this flexibility, we deduce the Planar Graph Product Structure Theorem, Theorem 7(b).

Theorem 7 ([15]). Every planar graph is isomorphic to a subgraph of:

(a) \( H \boxtimes P \) for some graph \( H \) of treewidth 8 and for some path \( P \).

(b) \( H \boxtimes P \boxtimes K_3 \) for some graph \( H \) of treewidth 3 and for some path \( P \).

Theorem 7 has been the key tool to resolve several open problems regarding queue layouts [15], nonrepetitive colouring [14], \( p \)-centred colouring [12], adjacency labelling [4, 13, 22], infinite graphs [28], twin-width [2, 5], and comparable box dimension [18].

The bound of 3 on the treewidth of \( H \) in (b) is tight [15] even if \( K_3 \) is replaced by any constant-sized complete graph. Note that \( \text{tw}(H \boxtimes K_3) \leq 3\text{tw}(H) + 2 \) for any graph \( H \), so (b) implies (a) but with 8 replaced by 11. Our proof of Theorem 7(b) removes much of the topology from the original proof, avoiding the use of Sperner’s planar triangulation lemma. This allows us to prove a more general \( H \boxtimes P \boxtimes K_m \) structure theorem, Theorem 16, which we apply in the more general setting of apex-minor-free graphs, Theorem 20. This in turn has applications for \( p \)-centred colourings.

2 Preliminaries

We consider simple finite undirected graphs \( G \) with vertex-set \( V(G) \) and edge-set \( E(G) \). For each vertex \( v \in V(G) \), let \( N_G(v) = \{w \in V(G): vw \in E(G)\} \). For \( S \subseteq V(G) \), let \( N_G(S) = \bigcup\{N_G(v): v \in S\} \setminus S \).
A graph $H$ is a **minor** of a graph $G$ if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. Say $G$ is **$H$-minor-free** if $H$ is not a minor of $G$. A $K_r$-model in a graph $G$ consists of pairwise-disjoint vertex-sets $(U_1, \ldots, U_r)$ such that, for each $i$, the induced subgraph $G[U_i]$ is connected and, for all distinct $i, j$, there is an edge between $U_i$ and $U_j$. Clearly $K_r$ is a minor of a graph $G$ if and only if $G$ contains a $K_r$-model.

### 2.1 Tree-decompositions and treewidth

A **tree-decomposition** $(T, \mathcal{W})$ of a graph $G$ consists of a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$, called **bags**, indexed by the nodes of a tree $T$, such that:

- for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a non-empty (connected) subtree of $T$; and
- for each edge $vw \in E(G)$, there is a node $x \in V(T)$ for which $v, w \in W_x$.

The **width** of such a tree-decomposition is $\max\{|W_x| : x \in V(T)\} - 1$. The **treewidth** $\text{tw}(G)$ of a graph $G$ is the minimum width of a tree-decomposition of $G$. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [3, 24, 36] for surveys.

We use the following property to prove treewidth upper bounds. A graph $G$ is a **clique-sum** of graphs $G_1$ and $G_2$, if for some clique $\{v_1, \ldots, v_k\}$ in $G_1$ and for some clique $\{w_1, \ldots, w_k\}$ in $G_2$, $G$ is obtained from the disjoint union of $G_1$ and $G_2$ by identifying $v_i$ and $w_i$ for each $i$.\(^3\) In this case, it is well known and easily seen that $\text{tw}(G) = \max\{\text{tw}(G_1), \text{tw}(G_2)\}$.

### 2.2 Partitions

Instead of working with products, it is convenient to present our proofs using the following definition. A **partition** of a graph $G$ is a graph $H$ such that:

- each vertex of $H$ is a set of vertices of $G$,
- each vertex of $G$ is in exactly one vertex of $H$, and
- for each edge $vw$ of $G$, if $v \in X \in V(H)$ and $w \in Y \in V(H)$ then $XY \in E(H)$ or $X = Y$.

\(^3\)It is common in the literature for clique-sums to allow the deletion of edges after the identification. In this paper we do not allow this.
We call the vertices of $H$ the *parts* of the partition. The *width* of a partition is the size of its largest part. The *treewidth* of a partition $H$ is $\text{tw}(H)$. The next observation follows from the definitions and gives a useful characterisation of when a graph is isomorphic to a subgraph of a product of the form $H \boxtimes K_m$.

**Observation 8.** A graph $G$ has a partition $H$ of width at most $m$ if and only if $G$ is isomorphic to a subgraph of $H \boxtimes K_{\lfloor m \rfloor}$.

In light of Observation 8, to prove our results it suffices to find a suitable partition. The following definition enables inductive proofs. A partition $H$ of a graph $G$ is *rooted* at a $K_r$-model $(U_1, \ldots, U_r)$ in $G$ if $U_1, \ldots, U_r$ are vertices of $H$. Note that $U_1, \ldots, U_r$ must be the vertices of an $r$-clique in $H$.

Finally, it will be useful to measure the ‘complexity’ of a vertex-set with respect to a tree-decomposition $(T, \mathcal{W})$ of $G$. For a vertex-set $S \subseteq V(G)$, the *$\mathcal{W}$-width* of $S$ is the minimum number of bags of $\mathcal{W}$ whose union contains $S$. The *$\mathcal{W}$-width* of a collection of vertex-sets is the maximum $\mathcal{W}$-width of one of its sets. In a slight abuse of terminology, the *$\mathcal{W}$-width* of a partition $H$ of $G$ is the maximum $\mathcal{W}$-width of one of the vertices of $H$.

### 2.3 Hitting sets

Our proofs use results that say a collection of connected subgraphs of a graph (satisfying certain conditions) either has a small ‘hitting set’ (a small set of vertices that meets every subgraph in the collection) or contains some suitable graphs. The following folklore lemma (see [38, (8.7)]) essentially says that complements of chordal graphs are perfect. We include the proof for completeness.

**Lemma 9.** For any integer $\ell \geq 0$ and any collection $\mathcal{F}$ of subtrees of a tree $T$, either:

(a) there are $\ell + 1$ vertex-disjoint trees in $\mathcal{F}$, or
(b) there is set $S$ of at most $\ell$ vertices such that $S \cap V(T') \neq \emptyset$ for all $T' \in \mathcal{F}$.

**Proof.** We proceed by induction on $|V(T)|$. The $|V(T)| = 1$ case is immediate. Let $x$ be a leaf of $T$ and $y$ its unique neighbour. Let $T' := T - x$.

First suppose that $T[\{x\}]$ is not in $\mathcal{F}$. Let $\mathcal{F}'$ be obtained by removing $x$ from every tree in $\mathcal{F}$. By induction, either (a) or (b) occurs for $\mathcal{F}'$ and $T'$. If (a) occurs, then the corresponding trees in $\mathcal{F}$ are also vertex-disjoint (since if two trees of $\mathcal{F}$ contain $x$, then they also both contain $y$). If (b) occurs, then the set obtained also meets every tree in $\mathcal{F}$.
Second suppose that $T[\{x\}]$ is in $\mathcal{F}$. Let $\mathcal{F}''$ be the set of all trees in $\mathcal{F}$ that do not contain $x$. So $\mathcal{F}''$ is a collection of subtrees of $T'$. Now apply induction to $\mathcal{F}''$ and $T'$ with $\ell - 1$ in place of $\ell$. If (a) occurs, then these trees together with $T[\{x\}]$ are $\ell + 1$ vertex-disjoint trees in $\mathcal{F}$. If (b) occurs, then this set together with $x$ meets every tree in $\mathcal{F}$.

In the setting of $O(\sqrt{n})$ blow-ups we need the following hitting set lemma due to Alon, Seymour, and Thomas [1]. Let $\mathcal{F}$ be the collection of connected subgraphs of $G$ that intersect all of $A_1, \ldots, A_k$. Lemma 10 says that $\mathcal{F}$ either contains a small graph or has a small hitting set.

**Lemma 10 ([1, (2.1)])**. Let $G$ be a graph, $A_1, \ldots, A_k$ be non-empty subsets of $V(G)$, and $x \geq 1$ be a real. Then either:

1. there is a subtree $X$ of $G$ with $|V(X)| \leq x$ such that $V(X) \cap A_i \neq \emptyset$ for each $i$, or
2. there is a set $Y$ of at most $(k - 1)|V(G)|/x$ vertices such that no component of $G - Y$ intersects all of $A_1, \ldots, A_k$.

The next result is a straightforward extension of Lemma 10.

**Lemma 11.** Let $G$ be a graph, $A_1, \ldots, A_k$ be non-empty subsets of $V(G)$, $x \geq 1$ be a real, and $\ell \geq 1$ be an integer. Then either:

1. there are pairwise disjoint trees $X_1, \ldots, X_\ell$ in $G$ with $|V(X_j)| \leq x$ and such that $V(X_j) \cap A_i \neq \emptyset$ for each $i$ and $j$, or
2. there is a set $Y$ of at most $(\ell - 1)x + (k - 1)|V(G)|/x$ vertices such that no component of $G - Y$ intersects all of $A_1, \ldots, A_k$.

**Proof.** We proceed by induction on $\ell$. Lemma 10 proves the result if $\ell = 1$. Now assume that $\ell \geq 2$ and the result holds for $\ell - 1$. If outcome (b) holds for $\ell - 1$, then the same set $Y$ satisfies outcome (b) for $\ell$. So assume that (a) holds for $\ell - 1$. That is, there are pairwise disjoint trees $X_1, \ldots, X_{\ell-1}$ in $G$ with $|V(X_j)| \leq x$ and such that $V(X_j) \cap A_i \neq \emptyset$ for each $i$ and $j$. Apply Lemma 10 to $G' := G - V(X_1 \cup \cdots \cup X_{\ell-1})$. If there is a tree $X_\ell$ in $G'$ with $|V(X_\ell)| \leq x$ such that $V(X_\ell) \cap A_i \neq \emptyset$ for each $i$, then $X_1, \ldots, X_\ell$ are the desired set of trees, and outcome (a) holds. Otherwise there exists $Y' \subseteq V(G')$ with $|Y'| \leq (k - 1)|V(G)|/x$ such that no component of $G' - Y'$ intersects all of $A_1, \ldots, A_k$. Let $Y := V(X_1 \cup \cdots \cup X_{\ell-1}) \cup Y'$. Thus $|Y| \leq (\ell - 1)x + (k - 1)|V(G)|/x$ and no component of $G - Y$ intersects all of $A_1, \ldots, A_k$ (since $G' - Y' = G - Y$). That is, $Y$ satisfies (b). \hfill \square
3 Main theorem and $\mathcal{O}(\text{tw}(G))$ blow-up

We now prove our main technical theorem and deduce Theorems 4 and 5 from it.

The following definition allows the $K_t$-minor-free and $K_{s,t}^*$-minor-free cases to be combined. Let $\mathcal{J}_{s,t}$ be the class of graphs $G$ whose vertex-set has a partition $A \cup B$, where $|A| = s$ and $|B| = t$, $A$ is a clique, every vertex in $A$ is adjacent to every vertex in $B$, and $G[B]$ is connected. A graph is $\mathcal{J}_{s,t}$-minor-free if it contains no graph in $\mathcal{J}_{s,t}$ as a minor. The following is our main theorem.

**Theorem 12.** Let $s,t \geq 2$ be integers, $G$ be a $\mathcal{J}_{s,t}$-minor-free graph, and $(T, \mathcal{W})$ be a tree-decomposition of $G$. Then $G$ has a partition of $\mathcal{W}$-width at most $t - 1$ and treewidth at most $s$.

This says that, given a $\mathcal{J}_{s,t}$-minor-free $G$ and a tree-decomposition $(T, \mathcal{W})$ of $G$, there is a simple (low treewidth) partition that is also simple with respect to $\mathcal{W}$. Theorem 12 follows immediately from the next lemma (for example, by taking $r = 1$ and $U_1$ to consist of a single vertex).

**Lemma 13.** Let $s,t \geq 2$ be integers, $G$ be a $\mathcal{J}_{s,t}$-minor-free graph, and $(T, \mathcal{W})$ be a tree-decomposition of $G$. Suppose that $(U_1, \ldots, U_r)$ is a $K_r$-model of $\mathcal{W}$-width at most $t - 1$ where $r \leq s$. Then $G$ has a partition of $\mathcal{W}$-width at most $t - 1$ and treewidth at most $s$ that is rooted at $(U_1, \ldots, U_r)$.

**Proof.** Let $U := U_1 \cup \cdots \cup U_r$. We proceed by induction on $|V(G)|$. If $V(G) = U$, then $(U_1, \ldots, U_r)$ is the desired partition $H$ where $H = K_r$ has treewidth $r - 1 \leq s$. Now assume that $V(G) \setminus U \neq \emptyset$. Let $A_i := N_G(U_i) \setminus U$ for each $i$.

First suppose that some $A_i$ is empty, say $A_1 = \emptyset$. By induction, $G - U_1$ has a partition $H_1$ of $\mathcal{W}$-width at most $t - 1$ and treewidth at most $s$ that is rooted at $(U_2, \ldots, U_r)$. Add a new part $U_1$ adjacent to each of $U_2, \ldots, U_r$ to obtain the desired $H$-partition of $G$. The neighbourhood of $U_1$ is a clique on $r - 1$ vertices, so $\text{tw}(H) = \max\{\text{tw}(H_1), r - 1\} \leq s$. Thus we may assume that $A_i$ is non-empty for all $i$.

Next suppose that $G - U$ is disconnected. Then there is a partition $U, V_1, V_2$ of $V(G)$ into three non-empty sets such that there is no edge between $V_1$ and $V_2$. Let $G_1 := G[U \cup V_1]$ and $G_2 := G[U \cup V_2]$. For $j \in \{1, 2\}$, let $\mathcal{W}_j$ be the tree-decomposition of $G_j$ obtained from $\mathcal{W}$ by deleting all the vertices of $G$ not in $G_j$. By induction, each $G_j$ has a partition $H_j$ of $\mathcal{W}_j$-width at most $t - 1$ and treewidth at most $s$ that is rooted at $(U_1, \ldots, U_r)$. Let $H$ be the partition of $G$ obtained from $H_1$ and $H_2$ by identifying the vertex $U_i$ in $H_1$ with the vertex $U_i$ in $H_2$ for each $i$. The graph $H$ is a clique-sum of $H_1$ and $H_2$, so $\text{tw}(H) = \max\{\text{tw}(H_1), \text{tw}(H_2)\} \leq s$. Since every bag of $\mathcal{W}_1$ and $\mathcal{W}_2$ is a subset of a bag
of $\mathcal{W}$, the partition $H$ has $\mathcal{W}$-width at most $t - 1$. Thus we may assume that $G - U$ is connected.

We now show there exists a set $Y \subseteq V(G) \setminus U$ of $\mathcal{W}$-width at most $t - 1$ such that

$$\text{no component of } G - U - Y \text{ meets every } A_i.$$  \hfill (†)

Let $\mathcal{F}$ be the collection of all connected subgraphs $F$ of $G - U$ such that $V(F) \cap A_i \neq \emptyset$ for all $i$. For each $F \in \mathcal{F}$, let $T_F := T[\{x \in V(T) : W_x \cap V(F) \neq \emptyset\}]$. Since $F$ is connected, $T_F$ is a (connected) subtree of $T$.

First consider the case $r \leq s - 1$.

First suppose there exists $F_1, F_2 \in \mathcal{F}$ such that $T_{F_1}$ and $T_{F_2}$ are disjoint. Let $xy$ be any edge of $T$ on the shortest path between $T_{F_1}$ and $T_{F_2}$. Then $W_x \cap W_y$ separates$^4$ $V(F_1)$ and $V(F_2)$. Let $S$ be a minimal subset of $W_x \cap W_y$ that separates $V(F_1)$ and $V(F_2)$.

By construction, $S$ has $\mathcal{W}$-width 1, $S \cap V(F_1) = \emptyset$, and $S \cap V(F_2) = \emptyset$. Then there is a partition $S \cup V_1 \cup V_2$ of $V(G) \setminus U$ such that $V(F_1) \subseteq V_1$, $V(F_2) \subseteq V_2$ and there is no edge between $V_1$ and $V_2$. We now show that $G[S \cup V_1]$ and $G[S \cup V_2]$ are connected. Consider some $s \in S$. Since $S$ is minimal, there is a path from $s$ to $V(F_1)$ internally disjoint from $S \cup V(F_2)$. Since there is no edge between $V_1$ and $V_2$, this path must lie entirely inside $S \cup V_1$. Since $F_1$ is connected, between any two vertices of $S$ there is a path entirely inside $S \cup V_1$. Since $G - U$ is connected, there is a path from any vertex of $V_1$ to $S$ inside $S \cup V_1$. Hence $G[S \cup V_1]$ is connected. Similarly for $G[S \cup V_2]$. For $j \in \{1, 2\}$, let $G_j$ be the graph obtained from $G$ by contracting all of $S \cup V_j$ into a single vertex $v_j$. Each $G_j$ is a minor of $G$ and thus is $\mathcal{J}_{s,t}$-minor-free. Furthermore, since $V(F_j) \subseteq V_j$, $(U_1, \ldots, U_r, \{v_j\})$ is a $K_{r+1}$-model in $G_j$. Let $\mathcal{W}_j$ be the tree-decomposition of $G_j$ obtained from $\mathcal{W}$ by replacing every instance of a vertex in $S \cup V_j$ by $v_j$. By induction, each $G_j$ has a partition $H_j$ of $\mathcal{W}_j$-width at most $t - 1$ and treewidth at most $s$ that is rooted at $(U_1, \ldots, U_r, \{v_j\})$. Let $H$ be obtained from the disjoint union of $H_1$ and $H_2$ by identifying corresponding $U_i$, and identifying $v_1$ and $v_2$ into a single vertex $S$. If $X \subseteq V(G_j) \setminus \{v_j\}$ is a subset of a bag of $\mathcal{W}_j$, then $X$ is a subset of a bag of $\mathcal{W}$. So if $X \subseteq V(G_j) \setminus \{v_j\}$ has $\mathcal{W}_j$-width at most $t - 1$, then $X$ has $\mathcal{W}$-width at most $t - 1$. Since $S$ also has $\mathcal{W}$-width $1 \leq t - 1$, the partition $H$ has $\mathcal{W}$-width at most $t - 1$. The graph $H$ is a clique-sum of $H_1$ and $H_2$, so $\text{tw}(H) \leq \max\{\text{tw}(H_1), \text{tw}(H_2)\} \leq s$ and the partition has all the required properties.

Now assume that $T_{F_1}$ and $T_{F_2}$ intersect for all $F_1, F_2 \in \mathcal{F}$. By the Helly property, there is a node $x \in V(T)$ such that $x \in V(T_F)$ for all $F \in \mathcal{F}$. Let $Y := W_x$. Then $Y$ has $\mathcal{W}$-width 1 and intersects every $F \in \mathcal{F}$. Thus $G - U - Y$ contains no graph of $\mathcal{F}$ and

$^4$Given a graph $G$ and $V_1, V_2 \subseteq V(G)$, a set $S$ separates $V_1$ and $V_2$ if no connected component of $G - S$ contains a vertex of both $V_1$ and $V_2$. 

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so every component of \( G - U - Y \) avoids some \( A_i \). This \( Y \) satisfies (\dag).

Now consider the case \( r = s \).

Suppose that \( \mathcal{F} \) contains \( t \) vertex-disjoint graphs \( F_1, \ldots, F_t \). Since \( G - U \) is connected, there is a partition \( Q_1, \ldots, Q_t \) of \( V(G) \setminus U \) such that \( V(F_i) \subseteq Q_i \) and \( G\{Q_i\} \) is connected, for all \( i \). Contract each \( Q_i \) to a single vertex \( q_i \) and each \( U_i \) to a single vertex \( u_i \) to get a graph \( G' \) with vertex-set \( \{u_1, \ldots, u_s, q_1, \ldots, q_t\} \). Since \( G - U \) is connected, \( G'\{q_1, \ldots, q_t\} \) is connected and so \( G' \in \mathcal{J}_{s,t} \), a contradiction. Hence, there are no \( t \) vertex-disjoint graphs in \( \mathcal{F} \). For any \( F_1, F_2 \in \mathcal{F} \), if \( T_{F_1} \) and \( T_{F_2} \) are disjoint, then \( F_1 \) and \( F_2 \) are disjoint. So \( \{T_F : F \in \mathcal{F}\} \) contains no \( t \) pairwise disjoint subtrees. Thus, by Lemma 9, there is a set \( S \subseteq V(T) \) of size at most \( t - 1 \) that meets every \( T_F \). Let \( Y := \bigcup_{x \in S} W_x \). Then \( Y \) has \( \mathcal{W} \)-width at most \( t - 1 \) and intersects every \( F \in \mathcal{F} \). This \( Y \) satisfies (\dag).

We have shown in all cases that there exists \( Y \subseteq V(G) \setminus U \) satisfying (\dag). Take a minimal such \( Y \). Let \( G_1, \ldots, G_r \) be unions of components of \( G - U - Y \) such that \( V(G_1), \ldots, V(G_r) \) is a vertex-partition of \( V(G) - U - Y \) and \( V(G_j) \cap A_j = \emptyset \) for each \( j \). Some \( G_j \) may be empty; ignore such indices henceforth. Fix \( j \) and consider \( w \in Y \). Since \( Y \) is minimal, there is a component of \( G - U - (Y \setminus \{w\}) \) that meets every \( A_i \). Since \( Y \) satisfies (\dag), this component contains \( w \). In particular, there is a path \( P_w \) from \( w \) to \( A_j \) in \( G - U - (Y \setminus \{w\}) \). \( P_w \) cannot meet \( G_j \) otherwise \( G - U - Y \) has a component meeting \( A_j \) and \( G_j \). Hence, for every \( w \in Y \), there is a path \( P_w \) from \( w \) to \( A_j \) that avoids \( V(G_j) \cup U \). Let \( Z_j \) be the subgraph induced by the union of \( U_j \) and all \( P_w \) (where \( w \in Y \)). By construction, \( Z_j \) is connected and disjoint from \( V(G_j) \cup (U \setminus U_j) \).

Take the subgraph of \( G \) induced by \( V(G_j) \cup Z_j \cup U \) and contract \( Z_j \) into a new vertex \( z_j \). Call the graph obtained \( G'_j \), which has vertex-set \( V(G_j) \cup (U \setminus U_j) \cup \{z_j\} \). Now \( (U_i : i \neq j, \{z_j\}) \) is a \( K_r \)-model in \( G'_j \). Let \( \mathcal{W}_j \) be the tree-decomposition of \( G'_j \) obtained from \( \mathcal{W} \) by deleting vertices of \( G \) not in \( V(G_j) \cup Z_j \cup U \), and then replacing each vertex in \( Z_j \) by \( z_j \). By induction, \( G'_j \) has a partition \( H_j \) of \( \mathcal{W}_j \)-width at most \( t - 1 \) and treewidth at most \( s \) that is rooted at \( (U_i : i \neq j, \{z_j\}) \). Add to \( H_j \) the vertex \( U_j \) adjacent to all other \( U_i \) and to \( \{z_j\} \). Since the neighbourhood of this added vertex is a clique of order \( r \leq s \), \( H_j \) still has treewidth at most \( s \). Let \( H \) be obtained from the disjoint union of \( H_1, \ldots, H_r \), by identifying corresponding \( U_i \), and identifying \( z_1, \ldots, z_r \) into a single vertex \( Y \). Note that if \( X \subseteq V(G_j) \setminus \{z_j\} \) is a subset of a bag of \( \mathcal{W}_j \), then \( X \) is a subset of a bag of \( \mathcal{W} \). So if \( X \subseteq V(G_j) \setminus \{z_j\} \) has \( \mathcal{W}_j \)-width at most \( t - 1 \), then \( X \) has \( \mathcal{W} \)-width at most \( t - 1 \). Since \( Y \) has \( \mathcal{W} \)-width at most \( t - 1 \), the partition \( H \) has \( \mathcal{W} \)-width at most \( t - 1 \). The graph \( H \) is a clique-sum of \( H_1, \ldots, H_r \), so \( \text{tw}(H) \leq \max_j \text{tw}(H_j) \leq s \).

We finally check that \( H \) is a partition of \( G \). The vertices \( U_1, \ldots, U_r, Y \) form a clique in \( H \), so all edges of \( G \) inside \( Y \cup U \) appear in \( H \). Every edge inside \( G_j \) appears in \( G'_j - z_j \),
thus appears in \( H_j \) and hence in \( H \). Any edge between \( U \) and \( G_j \) is, by definition of \( G_j \), an edge between \( G_j \) and \( U \setminus U_j \) so appears in \( G_j' - z_j \) and hence in \( H \). Finally consider edges between \( Y \) and \( G_j \). Let \( vw \) be an edge with \( v \in V(G_j) \) and \( w \in Y \). Note that \( w \in Z_j \) and so the edge \( vz_j \) is present in \( G_j' \) and hence in \( H_j \). Since \( z_j \) is replaced by \( Y \), the edge \( vw \) is in \( H \).

Applying Theorem 12 to a tree-decomposition of minimum width gives the following.

**Theorem 14.** For all integers \( s, t \geq 2 \), every \( J_{s,t} \)-minor-free graph \( G \) is isomorphic to a subgraph of \( H \boxtimes K_m \), where \( \text{tw}(H) \leq s \) and \( m := (\text{tw}(G) + 1)(t - 1) \).

*Proof.* Let \( G \) be a \( J_{s,t} \)-minor-free graph. Fix a tree-decomposition \( (T, \mathcal{W}) \) of \( G \) in which every bag has size at most \( \text{tw}(G) + 1 \). By Theorem 12, \( G \) has a partition \( H \) of \( \mathcal{W} \)-width at most \( t - 1 \) where \( \text{tw}(H) \leq s \). Since each bag of \( \mathcal{W} \) has size at most \( \text{tw}(G) + 1 \), the partition has width at most \( (t - 1)(\text{tw}(G) + 1) = m \). Hence, by Observation 8, \( G \) is isomorphic to a subgraph of \( H \boxtimes K_m \).

Observe that \( J_{t-2,2} = \{K_1\} \) so every \( K_t \)-minor-free graph is \( J_{t-2,2} \)-minor-free. Hence Theorem 14 implies Theorem 4. Clearly, \( K_{s,t}^* \) is a subgraph of every graph in \( J_{s,t} \) and so every \( K_{s,t}^* \)-minor-free graph is \( J_{s,t} \)-minor-free. Hence, Theorem 14 implies Theorem 5.

## 4 Layered treewidth: planar and apex-minor-free graphs

A *layering* of a graph \( G \) is a partition \( \mathcal{L} = (V_1, V_2, \ldots) \) of \( V(G) \) such that for each edge \( vw \in E(G) \), if \( v \in V_i \) and \( w \in V_j \), then \(|i - j| \leq 1 \). A layering of \( G \) is equivalent to a partition \( P \) of \( G \) where \( P \) is a path. The next observation, first noted in [15], gives a useful characterisation of when a graph is isomorphic to a subgraph of a product of the form \( H \boxtimes P \boxtimes K_m \).

**Observation 15** ([15]). A graph \( G \) has a layering \( \mathcal{L} \) and a partition \( H \) such that each layer of \( \mathcal{L} \) and each part of \( H \) intersect in at most \( m \) vertices if and only if \( G \) is isomorphic to a subgraph of \( H \boxtimes P \boxtimes K_m \) for some path \( P \).

*Proof.* Suppose that \( G \) is isomorphic to a subgraph of \( H \boxtimes P \boxtimes K_m \) where \( V(H) = \{x_1, \ldots, x_h\} \), \( V(P) = \{y_1, y_2, \ldots\} \), and \( V(K_m) = \{z_1, \ldots, z_m\} \). Then the isomorphism maps each vertex \( v \) of \( G \) to \((x_{a(v)}, y_{b(v)}, z_{c(v)})\) where \( v \mapsto (a(v), b(v), c(v)) \) is injective. Let \( \mathcal{L} \) have layers \( V_i = \{v: b(v) = i\} \) and the partition \( H \) have parts \( \{v: a(v) = j\} \) for \( j \in \{1, \ldots, h\} \). Since \( c(v) \) takes at most \( m \) values, each layer and part have at most \( m \) vertices in common.
Reversing this identification converts a suitable layering $\mathcal{L}$ and partition $H$ into an isomorphism from $G$ to a subgraph of $H \boxtimes P \boxtimes K_m$. \hfill $\square$

The \textit{layered treewidth} $\text{ltw}(G)$ of a graph $G$ is the minimum integer $k$ for which $G$ has a layering $\mathcal{L}$ and tree-decomposition $(T, W)$ such that $|L \cap W| \leq k$ for each layer $L \in \mathcal{L}$ and each bag $W \in W$. This notion was independently introduced by Dujmović, Morin, and Wood [16] and Shahrokhi [39]. Theorem 12 has the following corollary.

\textbf{Theorem 16.} For all integers $s, t \geq 2$, every $J_{s,t}$-minor-free graph $G$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where $P$ is a path, $\text{tw}(H) \leq s$, and $m := (t-1) \text{ltw}(G)$.

\textit{Proof.} Let $G$ be a $J_{s,t}$-minor-free graph. Fix a layering $\mathcal{L}$ and tree-decomposition $(T, W)$ of $G$ such that $|L \cap W| \leq \text{ltw}(G)$ for every layer $L \in \mathcal{L}$ and each bag $W \in W$. By Theorem 12, $G$ has a partition $H$ of $\mathcal{W}$-width at most $t-1$ where $\text{tw}(H) \leq s$.

Let $X \subset V(H)$ be a part and $L \in \mathcal{L}$ be a layer. Since the partition has $\mathcal{W}$-width at most $t-1$, there are bags $W_1, \ldots, W_{t-1} \in \mathcal{W}$ such that $X \subset \bigcup_{i=1}^{t-1} W_i$. Since $|L \cap W_i| \leq \text{ltw}(G)$ for each $i$, $|X \cap L| \leq (t-1) \text{ltw}(G)$. The result now follows from Observation 15. \hfill $\square$

Again, since $J_{t-2,2} = \{K_t\}$ and $K^*_{s,t}$ is a subgraph of every graph in $J_{s,t}$, Theorem 16 has the following corollaries.

\textbf{Theorem 17.} For any integer $t \geq 2$, every $K_t$-minor-free graph $G$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where $P$ is a path, $\text{tw}(H) \leq t-2$, and $m := \text{ltw}(G)$.

\textbf{Theorem 18.} For all integers $s, t \geq 2$, every $K^*_{s,t}$-minor-free graph $G$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where $P$ is a path, $\text{tw}(H) \leq s$, and $m := (t-1) \text{ltw}(G)$.

The Planar Graph Product Structure Theorem (Theorem 7(b)) follows from Theorem 17 (with $t = 5$) and the fact that every planar graph has layered treewidth at most 3, as proved by Dujmović et al. [16]. We sketch the proof for completeness.

\textbf{Theorem 19 ([16, Thm. 12]).} Every planar graph has layered treewidth at most 3.

\textit{Proof Sketch.} We may assume that $G$ is a planar triangulation. Let $T$ be a breadth-first-search spanning tree rooted at an arbitrary vertex $r$. Let $G^*$ be the dual of $G$ and $T^*$ be the spanning subgraph of $G^*$ consisting of those edges not dual to edges in $T$. Von Staudt [41] showed that $T^*$ is a spanning tree of $G^*$. For each vertex $x$ of $T^*$, corresponding to face $uvw$ of $G$, let $W_x$ be the union of the $wr$-path in $T$, the $vr$-path in $T$, and the $rw$-path in $T$. Eppstein [21] showed that $(W_x : x \in V(T^*))$ is a tree-decomposition of $G$. Let $V_i := \{v \in V(G) : \text{dist}_G(v, r) = i\}$ and so $(V_0, V_1, \ldots)$ is a layering of $G$. Since $T$ is a breadth-first-search spanning tree, each bag $W_x$ has at most three vertices in each layer $V_i$. Hence $\text{ltw}(G) \leq 3$. \hfill $\square$
We now show that the bound in Theorem 19 is tight. Suppose on the contrary that $\text{ltw}(G) \leq 2$ for every planar graph $G$. Then each layer induces a subgraph with treewidth 1, which is thus a forest. Taking alternate layers, $G$ has a vertex-partition into two induced forests (which would imply the 4-colour theorem). Chartrand and Kronk [8] constructed planar graphs $G$ that have no vertex-partition into two induced forests, implying $\text{ltw}(G) \geq 3$.

Theorem 7 is generalised as follows. The vertex-cover number $\tau(G)$ of a graph $G$ is the size of a smallest set $S \subseteq V(G)$ such that every edge of $G$ has at least one end-vertex in $S$. By definition, $G$ is a subgraph of every graph in $J_{\tau(G),|V(G)|-\tau(G)}$. A graph $X$ is apex if $X - v$ is planar for some vertex $v \in V(X)$. Dujmović et al. [16] showed that for any graph $X$, the class of $X$-minor-free graphs has bounded layered treewidth if and only if $X$ is apex. Thus, the next result follows from Theorem 18.

**Theorem 20.** For every apex graph $X$ there exists $m \in \mathbb{N}$, such that every $X$-minor-free graph is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_m$, where $P$ is a path and $\text{tw}(H) \leq \tau(X)$.

Dujmović et al. [15] proved a similar result to Theorem 20, but with a much larger bound on $\text{tw}(H)$ (depending on constants from the Graph Minor Structure Theorem).

Theorem 20 has applications to $p$-centred colouring, as we now explain. For $p \in \mathbb{N}$, a vertex colouring of a graph $G$ is $p$-centred if for every connected subgraph $X$ of $G$, $X$ receives more than $p$ colours or some vertex in $X$ receives a unique colour. The $p$-centred chromatic number $\chi_p(G)$ is the minimum number of colours in a $p$-centred colouring of $G$. Centreed colourings are important within graph sparsity theory as they characterise graph classes with bounded expansion [34]. A result of Dębski, Felsner, Micek, and Schröder [12, Lem. 8] implies that $\chi_p(H \boxtimes P \boxtimes K_m) \leq m(p+1)\chi_p(H)$ for every graph $H$. Pilipczuk and Siebertz [35, Lem. 15] proved that every graph of treewidth at most $t$ has $p$-centred chromatic number at most $\binom{p+t}{t} \leq (p+1)^t$. In particular, Theorem 20 implies:

**Theorem 21.** For every apex graph $X$ with $\tau(X) \leq t$ there exists $m \in \mathbb{N}$ such that for every $X$-minor-free graph $G$,

$$\chi_p(G) \leq m(p+1)^{t+1}.$$ 

Pilipczuk and Siebertz [35] proved that for every graph $X$ there exists $c$ such that every $X$-minor-free graph has $p$-centred chromatic number $O(p^c)$. However, the known bounds on $c$ are huge (depending on the Graph Minor Structure Theorem). Theorem 21 provides much improved bounds in the case of apex-minor-free graphs. As an example, since $K^*_3,4$ is apex with $\tau(K^*_3,4) \leq 3$, Theorem 21 implies there exists $m = m(t)$ such that $\chi_p(G) \leq m(p+1)^4$ for every $K^*_3,4$-minor-free graph $G$. This bound is only slightly
greater than the best bound for planar graphs of $O(p^3 \log p)$, and for graphs of Euler genus $g$ (which are $K_{3,2g+3}$-minor-free) of $O(gp + p^3 \log p)$, both due to Dębski et al. [12].

5 Blow-up $O(\sqrt{n})$

In this section we employ a similar proof strategy but with a different hitting result (Lemma 11 in place of Lemma 9) to prove Theorems 2 and 3.

**Theorem 22.** Let $s, t, n$ be positive integers and define

$$m := \begin{cases} 
\max\{t-1, 1\} & \text{if } s = 1 \text{ or } 2, \\
\sqrt{(s-2)n} & \text{if } s \geq 3 \text{ and } t = 1, \\
2\sqrt{(s-1)(t-1)n} & \text{otherwise.}
\end{cases}$$

Then every $\mathcal{J}_{s,t}$-minor-free graph $G$ on $n$ vertices is isomorphic to a subgraph of $H \boxtimes K_{\lceil m \rceil}$ for some graph $H$ of treewidth at most $s$.

Theorem 22 implies Theorems 2 and 3 since $\mathcal{J}_{t-1,1} = \mathcal{J}_{t-2,2} = \{K_t\}$ and $K^*_{s,t}$ is a subgraph of every graph in $\mathcal{J}_{s,t}$. Theorem 22 is implied by Observation 8 and the following lemma.

**Lemma 23.** Let $s, t, n$ be positive integers and define $m$ as in Theorem 22. Suppose $G$ is a $\mathcal{J}_{s,t}$-minor-free graph on $n$ vertices and $(U_1, \ldots, U_r)$ is a $K_r$-model in $G$ where $r \leq s$ and $|U_i| \leq m$ for all $i$. Then $G$ has a partition of width at most $m$ and treewidth at most $s$ that is rooted at $(U_1, \ldots, U_r)$.

*Proof.* Let $U := U_1 \cup \cdots \cup U_r$. We proceed by induction on $n$. If $n \leq r + m$, then the partition $(U_1, \ldots, U_r, V(G) \setminus U)$ is the desired partition $H$ where $H = K_{r+1}$ has treewidth $r \leq s$. Now assume that $n > r + m$. Note that if $n \leq t - 1$, then $n \leq m$ in all cases and so we may assume that $n > t - 1$. Let $A_i := N_G(U_i) \setminus U$ for each $i$.

By the same argument used in the proof of Lemma 13, we may assume that $A_i$ is non-empty for all $i$ and that $G - U$ is connected.

If $r \leq s - 1$ and there is some $U_{r+1}$ of size at most $m$ such that $(U_1, \ldots, U_{r+1})$ is a $K_{r+1}$-model in $G$, then Lemma 23 for $U_1, \ldots, U_{r+1}$ would imply it is also true for $U_1, \ldots, U_r$ (with the same partition). In particular, if $r \leq s - 1$, then we may assume there is no $U_{r+1}$ of size at most $m$ such that $(U_1, \ldots, U_{r+1})$ is a $K_{r+1}$-model in $G$. Call this property the ‘maximality of $r$’.
We now show there exists a set \( Y \subseteq V(G) \setminus U \) of size at most \( m \) such that

\[
\text{no component of } G - U - Y \text{ meets every } A_i. \quad (\dagger)
\]

First suppose that \( s = 1 \) and so \( U = U_1 \). Suppose that \( |A_1| \geq t \). Let \( v_1, \ldots, v_t \) be distinct vertices in \( A_1 \). Since \( G - U \) is connected, it is possible to partition \( V(G) \setminus U \) into vertex-sets \( Q_1, \ldots, Q_t \) such that for all \( i \), \( v_i \in Q_i \) and \( G[Q_i] \) is connected. Now contract each \( Q_i \) into a single vertex \( q_i \) and \( U_1 \) into a single vertex \( u_1 \) to get a graph \( G' \) on vertex-set \( \{u_1, q_1, \ldots, q_t\} \). Since \( G - U \) is connected, \( G' [\{q_1, \ldots, q_t\}] \) is connected and so \( G' \in \mathcal{J}_{1,t} \), a contradiction. Hence \( |A_1| \leq t - 1 \leq m \). Then \( Y = A_1 \) satisfies \( (\dagger) \).

Next suppose that \( s = 2 \). If \( r = 1 \), then for any \( x \in A_1 \), the pair \( (U_1, \{x\}) \) is a \( K_2 \)-model in \( G \), which contradicts the maximality of \( r \). Hence \( r = 2 \) and \( U = U_1 \cup U_2 \). Suppose \( G - U \) contains \( t \) pairwise vertex-disjoint paths \( P_1, \ldots, P_t \) from \( A_1 \) to \( A_2 \). Since \( G - U \) is connected, there is a partition of \( V(G) \setminus U \) into vertex-sets \( Q_1, \ldots, Q_t \) such that, for all \( i \), \( V(P_i) \subseteq Q_i \) and \( G[Q_i] \) is connected. Now contract each \( Q_i \) to a single vertex \( q_i \) and each \( U_i \) to a single vertex \( u_i \) to get a graph \( G' \) on vertex-set \( \{u_1, u_2, q_1, \ldots, q_t\} \). Since \( G - U \) is connected, \( G' [\{q_1, \ldots, q_t\}] \) is connected and so \( G' \in \mathcal{J}_{2,t} \), a contradiction. Thus, by Menger’s theorem, there is a set \( Y \subseteq V(G) \setminus U \) of size at most \( t - 1 \leq m \) such that there is no path from \( A_1 \) to \( A_2 \) in \( G - U - Y \). In particular, no component of \( G - U - Y \) meets both \( A_1 \) and \( A_2 \) and so \( Y \) satisfies \( (\dagger) \). Thus we may assume that \( s \geq 3 \).

Suppose that \( r \leq s - 1 \). Apply Lemma 10 to \( G - U \) with \( x = \sqrt{(s - 2)n} \geq 1 \) and \( k = r \). If (a) occurs, then there is a tree \( T \) on at most \( x \leq m \) vertices intersecting each \( A_i \). Then \( (U_1, \ldots, U_r, T) \) is a \( K_{r+1} \)-model in \( G \) with all parts of size at most \( m \), which contradicts the maximality of \( r \). Hence, (b) occurs. That is, there is a vertex-set \( Y \) of size at most \((r - 1)n/x \leq (s - 2)n/x = x \leq m \) such that no component of \( G - U - Y \) meets every \( A_i \). This \( Y \) satisfies \( (\dagger) \).

Now assume that \( r = s \). For \( t = 1 \) we are done: since \( G - U \) is connected, contracting each of \( U_1, \ldots, U_s \), \( G - U \) to a single vertex gives a \( K_{s+1} \)-minor in \( G \), which is a contradiction since \( K_{s+1} \notin \mathcal{J}_{s,1} \). Thus \( t \geq 2 \). Apply Lemma 11 to \( G - U \) with \( \ell = t \), \( k = r = s \) and \( x = \sqrt{\frac{s-1}{s+1}n} > 1 \). Suppose (a) occurs. Then there are pairwise disjoint trees \( T_1, \ldots, T_t \) in \( G - U \) such that each \( T_j \) meets each \( A_i \). Since \( G - U \) is connected, it is possible to partition \( V(G) \setminus U \) into vertex-sets \( Q_1, \ldots, Q_t \) such that, for all \( i \), \( V(T_i) \subseteq Q_i \) and \( G[Q_i] \) is connected. Now contract each \( Q_i \) to a single vertex \( q_i \) and each \( U_i \) to a single vertex \( u_i \) to get a graph \( G' \) on vertex-set \( \{u_1, \ldots, u_s, q_1, \ldots, q_t\} \). Since \( G - U \) is connected, \( G' [\{q_1, \ldots, q_t\}] \) is connected and so \( G' \in \mathcal{J}_{s,t} \), a contradiction. Hence, (b) occurs: there is a vertex-set \( Y \) of size at most \((t - 1)x + (s - 1)n/x = m \) such that no component of \( G - U - Y \) meets every \( A_i \). This \( Y \) satisfies \( (\dagger) \).

We have shown in all cases that there exists \( Y \subseteq V(G) \setminus U \) satisfying \( (\dagger) \). We may
now finish exactly as in the proof of Lemma 13 (with width instead of $W$-width, so the argument is even simpler).

Since $K_{2,t}^∗$ is planar and so $K_{2,t}^*$-minor-free graphs have bounded treewidth, one would expect a good bound (independent of $n$) on the blow-up factor. Campbell et al. [7] showed that every $K_{2,t}^*$-minor-free graph is isomorphic to a subgraph of $H \boxtimes K_{O(t^3)}$ where $\text{tw}(H) \leq 2$. They state as an open problem whether this $O(t^3)$ bound can be improved to $O(t)$. Theorem 22 for $s = 2$ gives an affirmative answer to this question.

**Theorem 24.** For every integer $t \geq 2$, every $K_{2,t}^*$-minor-free graph $G$ is isomorphic to a subgraph of $H \boxtimes K_{t-1}$, where $\text{tw}(H) \leq 2$.

Note that Theorem 24 implies $K_{2,t}^*$-minor-free graphs have treewidth $O(t)$, which was first proved by Leaf and Seymour [31, (4.4)].

### 6 Concluding Remarks

In the arXiv version of this paper [29] we show that Theorem 2(a), Theorem 3, Theorem 5, Corollary 6, Theorem 18, Theorem 20, and Theorem 24 can be slightly strengthened by replacing the bound on the treewidth of $H$ by the same bound on the simple treewidth of $H$. In particular, in Theorem 24, $H$ is outerplanar and, in Corollary 6, $H$ is planar with treewidth at most 3.

We conclude the paper by first discussing some possible ways in which Theorem 2 might be strengthened. Similar questions can be asked for $K_{s,t}^*$-minor-free graphs. Consider the following meta-theorem:

Every $K_t$-minor-free graph $G$ is isomorphic to a subgraph of $H \boxtimes K_{m(G)}$ for some function $m$ and some graph $H$ of treewidth at most $f(t)$.

Note that Theorem 2 says that ($\star$) holds for $m(G) = 2\sqrt{(t-3)n}$ where $n := |V(G)|$ and $f(t) = t - 2$ while Theorem 4 says it holds for $m(G) = \text{tw}(G) + 1$ and $f(t) = t - 2$.

**Q1.** Is it possible to improve $f(t)$ in Theorem 2 (possibly sacrificing some dependence on $t$ in $m(G)$)? In particular, can ($\star$) be proved with $m(G) = O_t(n^{1/2})$ and $f(t) = c$ for some constant $c$ independent of $t$? It follows from a result of Linial, Matoušek, Sheffet, and Tardos [32] that, even for planar graphs, $c \geq 2$. On the other hand, ($\star$) holds with $H$ a star ($c = 1$) and $m(G) = O_t(n^{2/3})$, and for any $\varepsilon > 0$ there exists $c$ such that ($\star$) holds with $f(t) \leq c$ and $m(G) = O_t(n^{1/2+\varepsilon})$; see [20]. The real interest is when $m(G) = O_t(n^{1/2})$. 

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As noted in Section 1, there is no corresponding improvement to Theorem 4 since \( f(t) = t - 2 \) is best possible when \( m(G) \) is a function of \( \text{tw}(G) \).

**Q2.** We highlight the \( t = 5 \) case of Q1: is every \( K_5 \)-minor-free \( n \)-vertex graph \( G \) isomorphic to a subgraph of \( H \boxtimes K_{O(\sqrt{n})} \) for some graph \( H \) of treewidth at most 2? Having treewidth at most 2 is equivalent to being \( K_4 \)-minor-free, so this problem is particularly appealing. It is open even when \( G \) is planar.

**Q3.** Optimising the dependence on \( t \) in Theorem 2 is an interesting question. Note that Kawarabayashi and Reed [30] proved that \( K_t \)-minor-free graphs have balanced separators of order \( O(t \sqrt{n}) \), which is best possible. Does \((\ast)\) hold with \( f(t) \cdot m(G) = O(t \sqrt{n}) \)?

Finally we mention a connection to row treewidth. Bose et al. [6] defined the row treewidth of a graph \( G \) to be the minimum treewidth of a graph \( H \) such that \( G \) is isomorphic to a subgraph of \( H \boxtimes P \) for some path \( P \). For example, Theorem 7(a) says that planar graphs have row treewidth at most 8, which was improved to 6 by Ueckerdt, Wood, and Yi [40]. It is easily seen that \( \text{ltw}(G) \leq \text{rtw}(G) + 1 \) for every graph \( G \). The next result, which provides a partial converse, follows from Theorem 17 since \( \text{tw}(H \boxtimes K_m) \leq (\text{tw}(H) + 1)m - 1 \).

**Corollary 25.** For every \( K_t \)-minor-free graph \( G \),

\[
\text{rtw}(G) \leq (t - 1) \text{ltw}(G) - 1.
\]

**Corollary 25** is in marked contrast to a result of Bose et al. [6] who constructed graphs with layered treewidth 1 and arbitrarily large row-treewidth. Thus the \( K_t \)-minor-free (or some other sparsity) assumption in Corollary 25 is necessary.

**Q4.** For what other graph classes \( \mathcal{G} \) (beyond those defined by an excluded minor) is row treewidth bounded by a function of layered treewidth for graphs in \( \mathcal{G} \)?

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**Note Added in Proof**

Following the initial release of this paper, there has been significant progress on some of the above questions. Distel, Dujmović, Eppstein, Hickingbotham, Joret, Micek, Morin, Seweryn, and Wood [9] answered Q1 in the affirmative by proving \((\ast)\) with
They also solved Q2 for planar graphs, and indeed for $K_{3,t}$-minor-free graphs. In particular, they showed that every $n$-vertex $K_{3,t}$-minor-free graph is isomorphic to a subgraph of $H \boxtimes K_{O(t,\sqrt{n})}$ for some graph $H$ of treewidth 2.

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