Pseudospherical surfaces on time scales

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Abstract

We define and discuss the notion of pseudospherical surfaces in asymptotic coordinates on time scales. Two special cases, namely discrete pseudospherical surfaces and smooth pseudospherical surfaces are consistent with this description. In particular, we define the Gaussian curvature in the discrete case.

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1 Introduction

A time scale is an arbitrary nonempty closed subset of the real numbers [11]. Typical examples are $\mathbb{R}$ and $\mathbb{Z}$. The time scales were introduced in order to unify differential and difference calculus [11, 12]. Partial differentiation, tangent lines and tangent planes on time scales have been introduced recently [4].

On the other hand, besides the differential geometry, there exists also the difference geometry [13]. In the last years one can observe a fast development of the integrable difference geometry (see, for instance, [3, 5, 6, 7, 8, 14])

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related closely to the classical differential geometry based on explicit constructions and transformations [9, 10]. A natural idea is to unify the difference and differential geometries and to formulate the integrable geometry on time scales.

In this paper we propose such formulation for pseudospherical immersions (surfaces of constant negative Gaussian curvature). The discrete pseudospherical surfaces have been introduced a long time ago [13, 17], and studied intensively in the last years [2]. The discrete pseudospherical surfaces have been recently expressed in terms of time scales [16]. However, the assumption that all points are isolated was essential and the Gaussian curvature was not discussed at all. In the present paper we use a different starting point and all cases are described in a unified framework.

2 Differentiation on time scales

This section collects basic notions and results concerning the differential calculus on time scales, compare [4]. To avoid some unimportant complications we confine ourselves to time scales which are not bounded neither from above nor from below (more general case is considered in [4], as far as partial derivatives are concerned).

Definition 1 ([11]). Let a time scale \( T \) is given. The maps \( \sigma : T \rightarrow T \) and \( \rho : T \rightarrow T \), defined by

\[
\begin{align*}
\sigma (u) & := \inf \{ v \in T : v > u \} , \\
\rho (u) & := \sup \{ v \in T : v < u \} ,
\end{align*}
\]

are called jump operator and backward jump operator, respectively.

Definition 2 ([11]). The points \( u \in T \) can be classified as follows

- \( \sigma (u) > u \implies u \text{ is right-scattered} \),
- \( \sigma (u) = u \implies u \text{ is right-dense} \),
- \( \rho (u) < u \implies u \text{ is left-scattered} \),
- \( \rho (u) = u \implies u \text{ is left-dense} \),
- \( \rho (u) < u < \sigma (u) \implies u \text{ is isolated} \).
Definition 3 ([4]). The delta derivative is defined as
\[
\frac{\partial f(t)}{\Delta t} = \lim_{s \to t, s \neq \sigma(t)} \frac{f(\sigma(t)) - f(t)}{\sigma(t) - s},
\]
and the nabla derivative is defined by
\[
\frac{\partial f(t)}{\nabla t} = \lim_{s \to t, s \neq \rho(t)} \frac{f(\rho(t)) - f(t)}{\rho(t) - s}.
\]

Definition 4 ([4]). We say that a function \( f : T \to \mathbb{R} \) is completely delta differentiable at a point \( t_0 \in T \), if there exist a number \( A \) such that
\[
f(t) - f(t_0) = A(t - t_0) + (t - t_0) \alpha(t_0, t),
\]
\[
f(t) - f(\sigma(t_0)) = A(t - \sigma(t_0)) + (t - \sigma(t_0)) \beta(t_0, t),
\]
where \( \alpha(t_0, t_0) = 0, \beta(t_0, t_0) = 0, \lim_{t \to t_0} \alpha(t_0, t) = 0, \) and \( \lim_{t \to t_0} \beta(t_0, t) = 0. \)

Proposition 1 ([4]). If the function \( f \) is completely delta differentiable at \( t_0 \), then the graph of this function has the uniquely determined delta tangent line at the point \( P_0 = (t_0, f(t_0)) \) specified by the equation
\[
y - f(t_0) = \frac{\partial f(t_0)}{\Delta t}(x - t_0)
\]

In this paper we fix our attention on functions defined on two-dimensional time scales, i.e., on \( T_1 \times T_2 \), where \( T_1, T_2 \) are given time scales. The extension on \( n \)-dimensional time scales is usually straightforward. We denote:
\[
t \equiv (t_1, t_2) \in T_1 \times T_2,
\]
\[
\sigma_1(t) = (\sigma(t_1), t_2), \quad \sigma_2(t) = (t_1, \sigma(t_2)),
\]
\[
\rho_1(t) = (\rho(t_1), t_2), \quad \rho_2(t) = (t_1, \rho(t_2)).
\]

Remark 1. In the discrete case (\( T_1 = T_2 = \mathbb{Z} \)) we have \( \sigma_j(u) = T_j u \) and \( \rho_j(u) = T_j^{-1} u \), where \( T_j \) mean usual shift operators. Therefore delta and nabla differentiation can be associated with forward and backward data, respectively [7].
Definition 5 (\[4\]). The partial delta derivative is defined as
\[
\frac{\partial f(t)}{\Delta t_j} = \lim_{s_j \to t_j, s_j \neq \sigma(t_j)} \frac{f(\sigma_j(t)) - f(t)}{\sigma(t_j) - s_j}.
\] (5)

The definition of the partial nabla derivative is analogical.

In the continuous case (e.g., \(T_1 = T_2 = \mathbb{R}\)) the delta derivative coincides with the right-hand derivative, while the nabla derivative coincides with the left-hand derivative. Note that all results and definitions in terms of delta derivatives have their nabla derivatives analogues.

Proposition 2 (\[4\]). If the mixed partial delta derivatives exist in a neighbourhood of \(t_0 \in T_1 \times T_2\) and are continuous at \(t = t_0\), then \[
\frac{\partial^2 f(t_0)}{\Delta t_1 \Delta t_2} = \frac{\partial^2 f(t_0)}{\Delta t_2 \Delta t_1}.
\]

The definition of the complete delta differentiability is similar to Definition 4, see \[4\], Definition 2.1. Instead of this definition we present here an important sufficient condition.

Proposition 3 (\[4\]). Let a function \(f : T_1 \times T_2 \to \mathbb{R}\) be continuous and have first order partial derivatives in a neighbourhood of \(t_0\). If these derivatives are continuous at \(t_0\), then \(f\) is completely delta differentiable at \(t_0\).

Definition 6 (\[4\]). Let \(z = f(x, y) (x \in T_1, y \in T_2)\) be a given surface (on the time scale) in \(\mathbb{R}^3\). A plane \(\Omega_0\) passing through \(P_0 = (t_0, s_0, f((t_0, s_0)))\) (where \(t_0 \in T_1, s_0 \in T_2\)) is called the delta tangent plane to the surface \(S\) at the point \(P_0\) if

1. \(\Omega_0\) passes also through the points \(P_0^{a_1} = (\sigma_1(t_0), s_0, f(\sigma_1(t_0), s_0))\) and \(P_0^{a_2} = (t_0, \sigma_2(s_0), f(t_0, \sigma_2(s_0)))\);

2. if \(P_0\) is not isolated point of \(S\) then \[
\lim_{P \to P_0, P \neq P_0} \frac{d(P, \Omega_0)}{d(P, P_0)} = 0 ,
\]
where \(P\) is a moving point of the surface \(S\), \(d(P, \Omega_0)\) is the distance from \(P\) to the plane \(\Omega_0\) and \(d(P, P_0)\) is the distance between \(P\) and \(P_0\).
Delta tangent line is defined in an analogous way. If $P_0$ is an isolated point of the curve $\Gamma$ (hence $P_0 \neq P_0^\sigma$), then the delta tangent line to $\Gamma$ at $P_0$ coincides with the unique line through the points $P_0$ and $P_0^\sigma$.

Similarly, if $P_0 \neq P_0^{\sigma_1}$ and $P_0^{\sigma_2} \neq P_0$ (hence also $P_0^{\sigma_1} \neq P_0^{\sigma_2}$), then the delta tangent plane to the surface $S$ at $P_0$ (if exists) coincides with the unique plane through $P_0$, $P_0^{\sigma_1}$ and $P_0^{\sigma_2}$.

**Proposition 4 ([4]).** If the function $f : T_1 \times T_2 \rightarrow \mathbb{R}$ is completely delta differentiable at $(t_0, s_0)$, then the surface represented by this function has the uniquely determined delta tangent plane at the point $P_0 = (t_0, s_0, f(t_0, s_0))$ specified by the equation

$$z = f(t_0, s_0) + \frac{\partial f(t_0, s_0)}{\Delta t}(x - t_0) + \frac{\partial f(t_0, s_0)}{\Delta s}(y - s_0)$$

where $(x, y, z)$ is the current point of the plane.

In the following sections of this paper we define pseudospherical surfaces on time scales in terms of delta derivatives. In order to simplify the notation the delta derivatives will be denoted by

$$D_j f = \frac{\partial f(t)}{\Delta t_j}.$$  

(7)

Proposition [4] suggests that in geometrical contexts it is more natural to use complete delta differentiability rather than delta differentiability.

### 3 Pseudospherical surfaces

Let us consider a surface immersed in $\mathbb{R}^3$ explicitly described by a position vector $\vec{r} = \vec{r}(u, v)$. Denoting the normal vector by $\vec{n}$ we define the so called fundamental forms:

$$I := d\vec{r} \cdot d\vec{r}, \quad II := -d\vec{r} \cdot d\vec{n},$$

where the center dot denotes the standard scalar product in $\mathbb{R}^3$. We denote the coefficients of the fundamental forms in a traditional way:

$$I = Edu^2 + 2Fdudv + Gdv^2,$$

$$II = Ldu^2 + 2Mdudv + Ndv^2.$$  

(8)
Hence,

\[
E = \vec{r}_u \cdot \vec{r}_u, \quad F = \vec{r}_u \cdot \vec{r}_v, \quad G = \vec{r}_v \cdot \vec{r}_v, \\
L = -\vec{n}_u \cdot \vec{r}_u, \quad M = -\vec{n}_u \cdot \vec{r}_v, \quad N = -\vec{n}_v \cdot \vec{r}_v.
\]  

(9)

The Gaussian curvature \( K \) and the mean curvature \( H \) are given by:

\[
K = \frac{LN - M^2}{W^2}, \quad H = \frac{EN - 2FM + GL}{2W^2},
\]  

(10)

where \( W = EG - F^2 \) (by assumption \( W \neq 0 \), i.e., the first fundamental form is not degenerated). The coefficients \( E, F, G, L, M, N \) satisfy the Gauss equation \([1]\)

\[
K = -\frac{1}{4W^2} \left| \begin{array}{ccc}
E & E_u & E_v \\
F & F_u & F_v \\
G & G_u & G_v
\end{array} \right| + \frac{1}{2W} \left( \frac{\partial}{\partial u} \left( \frac{F_v - G_u}{W} \right) + \frac{\partial}{\partial v} \left( \frac{F_u - E_v}{W} \right) \right),
\]  

(11)

and two Peterson-Mainardi-Codazzi equations

\[
L_v - M_u - H(E_v - F_u) + \frac{1}{2W^2} \left| \begin{array}{ccc}
E & E_u & L \\
F & F_u & M \\
G & G_u & N
\end{array} \right| = 0,
\]  

\[
M_v - N_u - H(F_v - G_u) + \frac{1}{2W^2} \left| \begin{array}{ccc}
E & E_v & L \\
F & F_v & M \\
G & G_v & N
\end{array} \right| = 0.
\]  

(12)

The Bonnet theorem says that any solution of the system \([11], [12]\) implicitly defines a surface immersed in \( \mathbb{R}^3 \) (provided that \( E > 0 \) and \( W > 0 \)) \([1]\).

**Proposition 5.** Let asymptotic lines on a surface admit parameterization by Chebyshev coordinates, i.e., the fundamental forms are expressed in terms of two real functions \( F, M : \mathbb{R}^2 \supset \Omega \to \mathbb{R} \) as follows

\[ I = du^2 + 2F(u, v)du dv + dv^2, \quad II = 2M(u, v)du dv, \]  

(13)

then the surface \( \vec{r} = \vec{r}(u, v) \) \((u, v \in \Omega)\), implicitly defined by the fundamental forms \([13]\) has a constant negative Gaussian curvature.

**Proof:** Substituting \( E = G = 1 \) and \( L = N = 0 \) to \([10] \) and \([12] \), we get

\[
K = -\frac{M^2}{1 - F^2}, \quad H = -\frac{FM}{1 - F^2}, \quad M_u - HF_u = 0, \quad M_v - HF_v = 0.
\]
Hence,
\[ MM_u(1 - F^2) + FF_uM^2 = 0, \quad MM_v(1 - F^2) + FF_vM^2 = 0, \]
which means \( \frac{M^2}{1 - F^2} = \text{const} > 0 \). Therefore, \( K = \text{const} < 0 \).

\[ \square \]

**Remark 2.** The assumptions of the Lemma 5 can be rewritten as
\[ (\vec{r}_u)^2 = (\vec{r}_v)^2 = 1, \quad \vec{n}_u \cdot \vec{r}_u = \vec{n}_v \cdot \vec{r}_v = 0, \quad (14) \]
and the conclusion of Lemma 5 states
\[ K \equiv \frac{-M^2}{1 - F^2} = \text{const} < 0. \quad (15) \]

We recall that asymptotic lines are characterized by \( L = N = 0 \), i.e., the second fundamental form is given by \( (13) \). Having Chebyshev coordinates \( u, v \) we can consider more general parameterization of asymptotic lines, namely: \( \tilde{u} = f(u), \tilde{v} = g(v) \). They are called weak Chebyshev coordinates.

**4 Discrete pseudospherical surfaces**

In the discrete case the time scale \( T_1 \times T_2 \) contains only isolated points. We confine ourselves to the case \( T_1 = T_2 = a\mathbb{Z} \), where \( a \) is a fixed constant (the mesh size).

**Remark 3.** Let \( T_1 = T_2 = a\mathbb{Z} \) and \( f : T_1 \times T_2 \to \mathbb{R} \), then we denote
\[ \Delta_j f = \frac{\partial f(t)}{\Delta_j t} = \frac{T_j f - f}{a}. \quad (16) \]

Therefore, in the discrete case \( D_j = \Delta_j \). In particular, for \( a = 1 \) we have \( \Delta_j = T_j - 1 \).

The discrete analogue of pseudospherical surfaces endowed with Chebyshev coordinates \( (13) \), i.e., discrete Chebyshev net, is defined as follows, compare \( [17] \).

**Definition 7** \( ([2]) \). Discrete Chebyshev net (discrete K-surface) is an immersion \( \vec{r} : a\mathbb{Z} \times a\mathbb{Z} \ni (am, an) \to \vec{r}(am, an) \in \mathbb{R}^3 \) such that for any \( m, n \)

- \( |\Delta_1 \vec{r}| = |\Delta_2 \vec{r}| = 1 \),
• the points $\vec{r}, T_1\vec{r}, T_2\vec{r}, T_1^{-1}\vec{r}, T_2^{-1}\vec{r}$ are coplanar (we denote this plane by $\pi(\vec{r})$).

By the discrete immersion we mean that $\Delta_1\vec{r}$ and $\Delta_2\vec{r}$ are linearly independent for any $m,n$.

Similarly one can discretize weak Chebyshev coordinates [2]. However, in this paper we confine ourselves only to discrete Chebyshev nets.

The plane $\pi(\vec{r})$ can be interpreted, obviously, as the discrete analogue of the tangent plane. Therefore

$$\vec{n} := \frac{\Delta_1\vec{r} \times \Delta_2\vec{r}}{|\Delta_1\vec{r} \cdot \Delta_2\vec{r}|} = \frac{\Delta_1\vec{r} \times \Delta_2\vec{r}}{\sqrt{1 - \Delta_1\vec{r} \cdot \Delta_2\vec{r}}},$$

(17)

is the discrete analogue of the normal vector (here the cross means the vector product).

**Proposition 6.** In the discrete case

$$\Delta_1\vec{n} \cdot \Delta_1\vec{r} = 0 \iff \Delta_1\vec{r}, T_1(\Delta_1\vec{r}), T_1(\Delta_2\vec{r})$$

are coplanar.

$$\Delta_2\vec{n} \cdot \Delta_2\vec{r} = 0 \iff \Delta_2\vec{r}, T_2(\Delta_1\vec{r}), T_2(\Delta_2\vec{r})$$

are coplanar.

**Proof:** From the definition of $\vec{n}$ it follows: $\vec{n} \cdot \Delta_1\vec{r} = 0$, $T_1\vec{n} \cdot T_1\Delta_1\vec{r} = 0$ and $T_1\vec{n} \cdot T_1\Delta_2\vec{r} = 0$. Then $\Delta_1\vec{n} \cdot \Delta_1\vec{r} = 0 \iff T_1\vec{n} \cdot \Delta_1\vec{r} = \vec{n} \cdot \Delta_1\vec{r}$. Hence, $T_1\vec{n} \cdot \Delta_1\vec{r} = 0$. Therefore, $\Delta_1\vec{r}, T_1\Delta_1\vec{r}$ and $T_1\Delta_2\vec{r}$ are co-planar. The proof of the second statement is similar. $\square$

**Corollary 1.** In the discrete case $\vec{r}, T_1\vec{r}, T_2\vec{r}, T_1^{-1}\vec{r}, T_2^{-1}\vec{r}$ are coplanar if and only if $\Delta_1\vec{n} \cdot \Delta_1\vec{r} = 0$ and $\Delta_2\vec{n} \cdot \Delta_2\vec{r} = 0$.

In the next part of this section we consider the tetrahedron $ABCD$:

$$\vec{r} \equiv A, \quad T_1\vec{r} \equiv B, \quad T_2\vec{r} \equiv D, \quad T_1T_2\vec{r} \equiv C.$$

The angle between $\Delta_1\vec{r}$ and $\Delta_2\vec{r}$ will be denoted by $\phi$ and the angle between $-\Delta_2\vec{r}$ and $T_2\Delta_1\vec{r}$ will be denoted by $\psi$. The tetrahedron ABCD is uniquely defined by specifying $a, \phi, \psi$.

**Proposition 7.** The angle $\theta_j$ between $\pi(\vec{r})$ and $T_j(\pi(\vec{r}))$ ($j = 1, 2$) is constant, i.e., $\theta_j(m,n) = \theta = \text{const.}$

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Proof: The transformation \( T_1 \vec{r} \leftrightarrow T_2 \vec{r} \) is an isometry of the tetrahedron \( ABCD \). Hence the angle between \( \pi(\vec{r}) \), \( \pi(T_1 \vec{r}) \) is equal to the angle between \( \pi(\vec{r}) \), \( \pi(T_2 \vec{r}) \). The transformation \( \vec{r} \leftrightarrow T_1 T_2 \vec{r} \), \( T_1 \vec{r} \leftrightarrow T_2 \vec{r} \) is another isometry of this tetrahedron. Thus \( \pi(\vec{r}) \leftrightarrow \pi(T_1 T_2 \vec{r}) \), \( \pi(T_1 \vec{r}) \leftrightarrow \pi(T_2 \vec{r}) \). Hence, the angle between \( \pi(\vec{r}) \), \( \pi(T_j \vec{r}) \) is equal to the angle between \( \pi(T_k \vec{r}) \), \( \pi(T_k T_j \vec{r}) \), which means that this angle does not depend on \( m, n \).

\[ ✷ \]

**Proposition 8.** In the discrete case \( K \) defined by

\[
K = \frac{-(\Delta_1 \vec{n} \cdot \Delta_2 \vec{r})(\Delta_2 \vec{n} \cdot \Delta_1 \vec{r})}{1 - (\Delta_1 \vec{r} \cdot \Delta_2 \vec{r})^2} \quad (18)
\]

is constant (i.e., does not depend on \( m, n \)). Moreover

\[
K = -\frac{\sin^2 \theta}{a^2} . \quad (19)
\]

Proof: Taking into account \(|\vec{AB}| = |\vec{AD}| = |\vec{BC}| = |\vec{CD}| = a\), we compute

\[
|\vec{AC}| = 2a \sin \frac{\psi}{2} , \quad |\vec{BD}| = 2a \sin \frac{\phi}{2} . \quad (20)
\]

Thus all sides of the tetrahedron are expressed in terms of \( a, \phi, \psi \). Then

\[
T_1 \vec{n} = \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|} .
\]

Taking into account \((15)\) and \( \vec{n} \perp \vec{AD} \equiv a\Delta_2 \vec{r} \) we get

\[
a^2 \Delta_1 \vec{n} \cdot \Delta_2 \vec{r} = T_1 \vec{n} \cdot \vec{AD} .
\]

Similarly (because the triangles \( ABC \) and \( ADC \) are homothetic) we have

\[
a^2 \Delta_2 \vec{n} \cdot \Delta_1 \vec{r} = a^2 \Delta_1 \vec{n} \cdot \Delta_2 \vec{r} = T_1 \vec{n} \cdot \vec{AD} .
\]

Finally,

\[
\Delta_2 \vec{n} \cdot \Delta_1 \vec{r} = \Delta_1 \vec{n} \cdot \Delta_2 \vec{r} = \frac{(\vec{AB} \times \vec{AC}) \cdot \vec{AD}}{a^2 |\vec{AB} \times \vec{AC}|} = \frac{\det(\vec{AB}, \vec{AC}, \vec{AD})}{a^2 |\vec{AB} \times \vec{AC}|} . \quad (21)
\]

We denote by \( H \) the height of the tetrahedron \( ABCD \), perpendicular to the plane \( ABC \) (i.e., perpendicular to \( \pi(T_1 \vec{r}) \)). The volume of the tetrahedron \( ABCD \) is given by \( V_{ABCD} = \frac{1}{3} HP_{ABC} \), and

\[
P_{ABC} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} a^2 \sin \psi , \quad (22)
\]
\[ V_{ABCD} = \frac{1}{6} \det(A\vec{B}, A\vec{C}, A\vec{D}) = \frac{1}{6} \begin{vmatrix} \vec{A}\vec{B} \cdot \vec{A}\vec{B} & \vec{A}\vec{B} \cdot \vec{A}\vec{C} & \vec{A}\vec{B} \cdot \vec{A}\vec{D} \\ \vec{A}\vec{C} \cdot \vec{A}\vec{B} & \vec{A}\vec{C} \cdot \vec{A}\vec{C} & \vec{A}\vec{C} \cdot \vec{A}\vec{D} \\ \vec{A}\vec{D} \cdot \vec{A}\vec{B} & \vec{A}\vec{D} \cdot \vec{A}\vec{C} & \vec{A}\vec{D} \cdot \vec{A}\vec{D} \end{vmatrix}. \]

All entries of the determinant can be expressed by \( a, \phi, \psi \) using (20) and the cosine rule. We get

\[ \det(A\vec{B}, A\vec{C}, A\vec{D}) = 4a^3 \sin \frac{\phi}{2} \sin \frac{\psi}{2} \sqrt{\cos \phi + \cos \psi}. \quad (23) \]

Therefore,

\[ H = \frac{\det(A\vec{B}, A\vec{C}, A\vec{D})}{a^2 \sin \psi} = \frac{4a \sin \frac{\phi}{2} \sin \frac{\psi}{2} \sqrt{\cos \phi + \cos \psi}}{\sin \psi}. \quad (24) \]

Then, from (21) and (24), we have

\[ a^2 \Delta_1 \vec{n} \cdot \Delta_2 \vec{r} = a^2 \Delta_1 \vec{n} \cdot \Delta_1 \vec{r} = H. \quad (25) \]

Now, we express \( K \), given by (18), in terms of \( a, \phi, \psi \). By virtue of (25) we get

\[ a^2 K = -\frac{H^2}{a^2 (1 - \cos^2 \phi)} = -\frac{\cos \phi + \cos \psi}{2 \cos^2 \frac{\phi}{2} \cos^2 \frac{\psi}{2}} = \tan^2 \frac{\phi}{2} \tan^2 \frac{\psi}{2} - 1. \quad (26) \]

The angle \( \theta \), defined in Proposition 7, can be computed from the triangle \( DD'O \), where \( O \) is the foot of the height \( H \) and \( D' \) is the foot of the height of the slant \( ABD \). The area \( P_{ABD} \) is \( \frac{1}{2} \sin \phi \), therefore \( |DD'| = \sin \phi \). From Pythagoras’ theorem we get (after elementary computations)

\[ |OD'| = \sqrt{\sin^2 \phi - H^2} = 2 \sin \frac{\phi}{2} \tan \frac{\psi}{2}. \]

Then \( \cos \theta = |OD'|/|DD'| \), which yields

\[ \cos \theta = \tan \frac{\phi}{2} \tan \frac{\psi}{2}. \quad (27) \]

Comparing (26) and (27) we end the proof (\( \theta = \text{const} \) by Proposition 7).

**Remark 4.** The formula (18) can be considered as a natural discrete analogue of the Gaussian curvature (13).

**Corollary 2.** The discrete surfaces of discrete Gaussian curvature \( K = -1 \) are characterized by the condition \( a = \sin \theta \).

The same condition, \( d = \sin \sigma \), appears in the definition of the classical Bäcklund transformation for pseudospherical surfaces [10]. There \( d \) is the length of the segment joining a point of a pseudospherical surface and its Bäcklund transform, and \( \sigma \) is the angle between the corresponding tangent planes.
5 Pseudospherical surfaces on time scales

Corollary 1 shows that the assumptions of Definition 7 can be expressed completely in terms of the delta derivatives. First, given an immersion \( \vec{r} \) on a time scale, we define the normal vector

\[
\vec{n} := \frac{D_1 \vec{r} \times D_2 \vec{r}}{\sqrt{1 - (D_1 \vec{r} \cdot D_2 \vec{r})^2}}.
\]  

(28)

**Definition 8.** An immersion \( \vec{r}: \mathbb{T}_1 \times \mathbb{T}_2 \ni (u,v) \rightarrow \vec{r}(u,v) \in \mathbb{R}^3 \) such that for any \( u,v \in \mathbb{T}_1 \times \mathbb{T}_2 \)

- \( \vec{r} \) is completely delta differentiable ,
- \( \vec{n} \) is completely delta differentiable ,
- \( |D_1 \vec{r}| = |D_2 \vec{r}| = 1 \),
- \( D_1 \vec{n} \cdot D_1 \vec{r} = D_2 \vec{n} \cdot D_2 \vec{r} = 0 \),

is called a Chebyshev net on the time scale \( \mathbb{T}_1 \times \mathbb{T}_2 \) (or a pseudospherical surface on the time scale).

We conjecture that the Gaussian curvature for pseudospherical surfaces on time scales is given by the formula analogical to (18)

\[
K = -\frac{(D_1 \vec{n} \cdot D_2 \vec{r})(D_2 \vec{n} \cdot D_1 \vec{r})}{1 - (D_1 \vec{r} \cdot D_2 \vec{r})^2},
\]  

(29)

but the rigorous proof is not available yet. The formulae (15) and (18) are particular cases of (29), when \( \mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R} \) and \( \mathbb{T}_1 = \mathbb{T}_2 = a\mathbb{Z} \), respectively.

6 Conclusions

In this paper the notion of pseudospherical immersions is extended on the so called time scales, unifying the continuous and discrete cases in a single framework. In particular, the Gaussian curvature of discrete pseudospherical surfaces is defined in a way admitting a straightforward extension on time scales (Proposition 8). It would be interesting to extend other results of the integrable discrete geometry on time scales. This is especially important in the context of the numerical approximation of continuous integrable models.

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