Domain Wall Nonlinear Quantization

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Abstract—Nonlinear quantization of the domain wall (relativistic membrane of codimension 1) is considered. The membrane dust equation is considered as an analogue of the Hamilton–Jacobi equation, which allows us to construct its quantum analogue. The resulting equation has the form of a nonlinear Klein–Fock–Gordon equation. It can be interpreted as the mean field approximation for a quantum domain wall. Dispersion relations are obtained for small perturbations (in a linear approximation). The group speed of perturbations does not exceed the speed of light. For perturbations propagating along the domain wall, in addition to the massless mode (as in the classical case), a massive one appears. The result may be interesting in condensed matter theory and in membrane quantization in superstring and supergravity theories.

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1. INTRODUCTION

Quantization of extended objects is of interest in the program of geometrization of physics. The most significant advances in this field are considered to be the quantization of the boson string and superstring. String theory has naturally led to consideration of membranes of other dimensions, for which the quantization problem is also of great interest [1].

Strings and membranes are degenerate continuous media, so quantization can be interesting for condensed matter physics too [2].

We develop an approach based on the Hamilton–Jacobi equation. Partially similar approaches were used in a series of earlier papers (see [4] and references therein). The main difference is the way the world surface is parametrized in order to quantize. The paper [4] used scalar fields with gradients tangent to the world surface of the membrane. The resulting quantized equations were essentially nonlinear, having no linear term. We use a scalar field with a gradient orthogonal to the world surface of the membrane. In the case of codimension 1 (domain wall), the resulting equation is a nonlinear Klein–Fock–Gordon equation. It is nonlinear but includes standard linear terms.

In this paper, we develop the following scheme for the membrane quantization.

1. Transition from a single membrane to a continuous distribution of membranes (membrane dust).
2. Description of continuous membrane dust using the Hamilton–Jacobi type equation.
3. Reconstruction of the classical generalized Hamiltonian from the Hamilton–Jacobi type equation.
4. Replacing the classical generalized Hamiltonian with a quantum one.

This scheme can be considered as a generalization of canonical quantization. In particular, if you start the procedure with the standard Hamilton–Jacobi equation, you can reproduce the canonical quantization scheme.

The scheme was implemented for the case of a codimension one membrane (domain wall), since in this case the membrane dust is described using a single real scalar function \( \varphi \), for which the membranes are level surfaces \( \varphi = \text{const} \). This scalar function corresponds to the action variable in the Hamilton–Jacobi equation.

This scheme of quantization of the membrane wall naturally gives the nonlinear Klein–Fock–Gordon equation

\[
(|\psi|^4 + \Box) \psi = 0.
\]

This approach to membrane quantization differs from the standard approach adopted in string theory. Therefore, quantization of the same systems can provide different results. In these cases, the discrepancy with the generally accepted results is not a disadvantage since we consider a different physical model.
2. CLASSICAL DOMAIN WALL

2.1. Action of a Single Domain Wall

The action of a single domain wall is a standard Dirac action for a relativistic membrane (for 2-dimensional world-sheet it is known as the Nambu–Goto action). It is a measure of the world surface defined by an induced metric $h_{\alpha\beta}$. The fields $X^M(\xi^\alpha)$ are space-time coordinates $X^M$ defined as functions of coordinates on the world surface $\xi^\alpha$:

$$S_0[X^M(\xi^\alpha)] = -T \int \sqrt{-\det h_{\alpha\beta}} \, d^{D-1}\xi, \quad M = 0, \ldots, D - 1, \quad \alpha, \beta = 1, \ldots, D - 1,$$

where $g_{MN}(X)$ is the space-time metric with signature $(-, +, +, \ldots, +)$.

2.2. Domain Wall Dust Action 1

Let there be a family of non-interacting domain walls numbered by the continuous parameter $\phi$, then the corresponding action differs from (1) by integrating over the parameter $\phi$,

$$S_1[X^M(\xi^\alpha, \phi)] = -\int \sqrt{-\det h_{\alpha\beta}} \, d^{D-1}\xi \, d\phi, \quad M = 0, \ldots, D - 1, \quad \alpha, \beta = 1, \ldots, D - 1,$$

where $X^M$ are the Euler space-time coordinates.

On each surface $X^M(\xi^\alpha, \phi)$, where $\phi = \text{const}$, we obviously have the same field equations that are obtained for a separate membrane from the action (1).

Let the world surfaces of the domain walls corresponding to different values of $\phi$ not intersect, and the Jacobian $\frac{DX}{D(\xi, \phi)} \neq 0$, then $\xi^\alpha$ and $\phi$ can also be considered as space-time coordinates, which are naturally called Lagrangian coordinates.

The functions $X^M(\xi, \phi)$ (dynamic fields) represent the Euler coordinates as functions of the Lagrangian coordinates, $(\xi, \phi) \rightarrow X$.

2.3. Domain Wall Dust Action 2 (to Build the Hamilton–Jacobi Type Equation)

$X^M$ are the Euler space-time coordinates; $\xi^\alpha$ and $\phi$ are the Lagrangian space-time coordinates;

Previously (in (2)) we considered the fields to be the Euler coordinates as functions of the Lagrangian coordinates, now we will take as fields the inverse functions $[5, 6]$.

The fields $\varphi(X^M)$ and $\xi^\alpha(X^M)$ are new dynamic fields $X \rightarrow (\xi, \varphi)$. (Here, $\phi$ is the independent variable, meanwhile $\varphi$ is the same coordinate, represented as a function of $X$.)

$$S_1[X^M(\xi^\alpha, \phi)] = -\int \sqrt{-\det h_{\alpha\beta}} \, d^{D-1}\xi \, d\phi, \quad M = 0, \ldots, D - 1, \quad \alpha, \beta = 1, \ldots, D - 1,$$

where $g_{MN}(X)$ is the space-time metric with signature $(-, +, +, \ldots, +)$.

2.4. Field Equation and the Energy–Momentum Tensor

The equation for the field $\varphi$ is equivalent to the standard membrane equation of motion at all world surfaces $\varphi = \text{const}$:

$$\frac{1}{\sqrt{-g}} \frac{\delta S_2}{\delta \varphi} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial X^M} \sqrt{-g} \, \text{div} g_{MN} \frac{\partial \varphi}{\partial X^N} = 0. \quad (4)$$

The equivalence of the actions (2) and (3) (provided that the mapping $X \rightarrow (\xi, \varphi)$ is nondegenerate) is obvious from the form of the energy–momentum tensor (the field equations can be derived from the continuity equations $\nabla_M T^{MN} = 0$).

The energy–momentum tensor has the form of a scalar multiplied by the orthogonal projector $P_{MN}$ on the surface $\varphi = \text{const}$,

$$T^{MN} = \frac{2}{\sqrt{-g}} \frac{\delta S_2}{\delta g_{MN}} = -\sqrt{-g} K^L \frac{\partial \varphi}{\partial X^K} \frac{\partial \varphi}{\partial X^L} P_{MN},$$

where $g_{MN}(X)$ is the space-time metric with signature $(-, +, +, \ldots, +)$.
We will consider the \( \varphi_0 \) field as an unperturbed solution. Now let us add a small perturbation to the field:

\[
\varphi(X) = \varphi_0(X) + \varepsilon f(X), \quad \varepsilon = \text{const} \ll 1.
\]

Then,

\[
\partial_M \varphi = c \left( \frac{K_m z + K_m X^m + \varepsilon \partial_m f}{1 + K z + K_m X^m} \right) + o(X) + o(\varepsilon),
\]

\[
(\partial_M \varphi)(\partial^M \varphi) = c^2 \left( 1 + 2 K z + 2 K_m X^m \right) \times \left( 1 - K - K_m X^m - \varepsilon \partial_z f \right) + o(X) + o(\varepsilon)
\]

\[
= \left( K_m z + K_m X^m \right) + o(X) + o(\varepsilon).
\]

Thus for a linear perturbation we have a wave equation on the world surface of the membrane wall

\[
\partial_m \partial^m f = 0, \quad m = 0, 1, \ldots, D - 2.
\]

The perturbation is transferred along the membrane wall at a unit speed (i.e., the speed of light), which means that the principle of causality is valid.

### 2.6. The Hamilton–Jacobi Type Equation

The field equation for domain wall dust is somewhat similar to the Hamilton–Jacobi equation

\[
\text{div} \frac{\text{grad} \varphi}{||\text{grad} \varphi||} = 0.
\]

It describes a set of non-overlapping non-interacting domain walls with different initial conditions.

However, it is a second-order equation, whereas the Hamilton–Jacobi equation is always first-order.

Let us introduce an additional field

\[
\rho = \frac{1}{||\text{grad} \varphi||}.
\]

The equation linking \( \rho \) and \( \varphi \) is a first-order differential equation for \( \varphi \), and it can be considered as
a Hamilton–Jacobi type equation (the Hamilton–Jacobi equation depending on the functional parameter \( \rho \)).

The original equation for the field (4), rewritten through the field \( \rho \), takes the form of a continuity equation

\[
\text{div}(\rho \text{grad}\varphi) = 0. \quad (11)
\]

The equations (9), (10) are nontrivial.

First, the continuity equation (10) looks tachyonic (grad\( \varphi \) is space-like), but the perturbations are not tachyonic, this is evident from the absence of tachyons for the domain wall (perturbations of an elastic medium similarly described via the Lagrangian coordinate defined as functions of the Euler coordinates are considered in [2]).

\[
\text{div}(\rho \text{grad}\varphi) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial X^M} \sqrt{g} \rho g^{MN} \frac{\partial\varphi}{\partial X^N} = 0.
\]

Second, the Hamilton–Jacobi type equation (9) depends on the density \( \rho \):

\[
||\text{grad}\varphi||^2 = g^{KL} \frac{\partial\varphi}{\partial X^K} \frac{\partial\varphi}{\partial X^L} = \frac{1}{\rho^2}.
\]

2.7. Domain Wall Dust Action 3 (for Quantization)

It is easy to find an action for \( \varphi \) and \( \rho \) as independent fields that would reproduce Eqs. (10), (11) as the Euler–Lagrange equations:

\[
S_3[\rho(X, \varphi(X))] = -\frac{1}{2} \int \left( \rho ||\text{grad}\varphi||^2 + \frac{1}{\rho^2} \right) \times \sqrt{-g} d^D X. \quad (12)
\]

The Euler–Lagrange equations reproduce (9), (10).

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_3}{\delta \rho} = \frac{1}{2} \left( \frac{1}{\rho^2} - ||\text{grad}\varphi||^2 \right) = 0,
\]

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_3}{\delta \varphi} = \frac{1}{2} \text{div}(\rho \text{grad}\varphi) = 0,
\]

\[
T_{MN} = \rho \frac{\partial\varphi}{\partial X^M} \frac{\partial\varphi}{\partial X^N} - \frac{1}{2} g^{MN} \rho \left((||\text{grad}\varphi||^2 + \frac{1}{\rho^2})\right).
\]

On solutions of the field equations (if we impose the constraint (9) between the \( \varphi \) and \( \rho \) fields), the energy-momentum tensor coincides with the previously obtained (5),

\[
T_{MN} \big|_{\delta S_3 = 0} = -||\text{grad}\varphi|| P_{MN},
\]

3. QUANTIZATION

3.1. Preparing for Quantization

The Hamilton–Jacobi type equation

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_3}{\delta \rho} = \frac{1}{2} \left( \frac{1}{\rho^2} - ||\text{grad}\varphi||^2 \right) = 0
\]

allows one to find the “extended Hamiltonian” (see, for example, [10]) by the substitution \( \frac{\partial\varphi}{\partial X^M} \rightarrow P_M \).

\[
H(X^M, P_M) = \frac{1}{2} \left( \frac{1}{\rho^2(X)} - P_M P^M \right).
\]

The extended Hamiltonian includes a time component of the relativistic momentum. On the energy surface it vanishes.

The extended Hamiltonian depends on \( \rho(X) \).

3.2. Canonical Quantization

Let us replace the momenta in the extended Hamiltonian with the corresponding operators and express the density \( \rho \) in terms of \( \psi \):

\[
P_M \rightarrow \hat{P}_M = -i \frac{\partial}{\partial X^M}, \quad \rho(X) = |\psi(X)|^2,
\]

\[
H(X^M, P_M) = \frac{1}{2} \left( \frac{1}{\rho^2(X)} - P_M P^M \right)
\]

\[
\rightarrow \hat{H}[\psi] = \frac{1}{2} \left( \frac{1}{|\psi(X)|^4} - \hat{P}_M \hat{P}^M \right).
\]

The corresponding equation is the Klein–Fock–Gordon nonlinear equation

\[
\hat{H}[\psi] \psi = 0,
\]

\[
\frac{1}{2} \left( \frac{1}{|\psi(X)|^4} + \square \right) \psi(X) = 0,
\]

\[
\square = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial X^M} \sqrt{-g} g^{MN} \frac{\partial}{\partial X^N}.
\]

3.3. Quantum Action

We can reproduce the nonlinear Klein–Fock–Gordon equation using the following action functional (\( \psi \) and \( i \psi^* \) are the canonical variables)

\[
\frac{1}{2} \left( \frac{1}{|\psi(X)|^4} + \square \right) \psi(X) = 0,
\]
By variational derivatives we find

\[
S_q[\psi(X), i\psi^*(X)] = -\frac{1}{2} \int \left( \frac{1}{\psi^* \psi} + g^{MN} \frac{\partial \psi^*}{\partial X^M} \frac{\partial \psi}{\partial X^N} \right) \sqrt{-g} d^D X,
\]

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_q}{\delta \psi^*} = \frac{1}{2} \left( \frac{1}{|\psi(X)|^2} + \Box \right) \psi(X) = 0,
\]

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_q}{\delta \psi} = \frac{1}{2} \left( \frac{1}{|\psi(X)|^2} + \Box \right) \psi^*(X) = 0.
\]

4. QUANTUM CORRECTIONS

To compare the resulting field equations with the classical case, we rewrite the action using the real fields \( \rho \) and \( \varphi \):

\[
\psi(X) = \sqrt{\rho(X)} e^{i\varphi(X)}.
\]

\[
S_q[\psi(X), i\psi^*(X)] = -\frac{1}{2} \int \left( \frac{1}{\psi^* \psi} + g^{MN} \frac{\partial \psi^*}{\partial X^M} \frac{\partial \psi}{\partial X^N} \right) \sqrt{-g} d^D X
\]

\[
= S_q[\rho(X), \varphi(X)] = -\frac{1}{2} \int \left( \frac{1}{\rho} + \rho ||\text{grad} \varphi||^2 + \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \right) \sqrt{-g} d^D X.
\]

By variational derivatives we find

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_q}{\delta \rho} = \frac{1}{2} \left( \frac{1}{\rho^2} - ||\text{grad} \varphi||^2 + \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \right) = 0,
\]

\[
\frac{1}{\sqrt{-g}} \frac{\delta S_q}{\delta \varphi} = \frac{1}{2} \text{div}(\rho \text{grad} \varphi) = 0,
\]

\[
T_{MN} = \rho \frac{\partial \varphi}{\partial X^M} \frac{\partial \varphi}{\partial X^N}
\]

\[
- \frac{1}{2} g_{MN} \rho \left( ||\text{grad} \varphi||^2 + \frac{1}{\rho^2} \right)
\]

\[
+ \frac{\partial \sqrt{\rho}}{\partial X^M} \frac{\partial \sqrt{\rho}}{\partial X^N} - \frac{1}{2} g_{MN} ||\text{grad} \sqrt{\rho}||^2.
\]

Under the field equation \( \delta S_q/\delta \rho = 0 \), the energy-momentum tensor has the form

\[
T_{MN} \bigg|_{\delta \rho = 0} = -||\text{grad} \varphi|| P_{MN} + \frac{1}{2} g_{MN} \sqrt{\rho} \Box \sqrt{\rho}
\]

\[
+ \frac{\partial \sqrt{\rho}}{\partial X^M} \frac{\partial \sqrt{\rho}}{\partial X^N} - \frac{1}{2} g_{MN} ||\text{grad} \sqrt{\rho}||^2
\]

\[
= -||\text{grad} \varphi|| P_{MN}.
\]

4.1. Perturbation and Causality

Let us consider the unperturbed solution of the field equation based on the unperturbed solution (6) in the classical case,

\[
\varphi_0 = cz + \frac{c}{2} (Kz^2 + 2K_m X^m z)
\]

\[
+ K_m X^m X^n + o(X^2),
\]

\[
m, n = 0, 1, \ldots, D - 2,
\]

\[
z = X^{D-1}, \quad K_m^m = 0.
\]

Here, \( K, K_m, K_{mn} = \text{const} \),

\[
\sqrt{\rho_0} = ||\text{grad} \varphi_0||^{-1/2} = \frac{1}{\sqrt{c}} \left( 1 - \frac{Kz}{2} \right) + o(X^2).
\]

Let us seek a solution in the form

\[
\varphi(X) = \varphi_0(X) + \epsilon c f(X) + o(X^2) + o(\epsilon),
\]

\( \epsilon = \text{const} \ll 1 \),

\[
\sqrt{\rho(X)} = \sqrt{\rho_0(X)} - \epsilon \frac{g(X)}{2\sqrt{c}} + o(X^2) + o(\epsilon),
\]

\[
\rho \partial_M \varphi = \left( K_m z + K_{mn} X^n + \epsilon \partial_m f \right)
\]

\[
\times (1 - Kz - K_m X^m - \epsilon g) + o(X) + o(\epsilon)
\]

\[
= \left( K_m z + K_{mn} X^n + \epsilon \partial_m f \right)
\]

\[
+ 1 + \epsilon \partial_z f - \epsilon g
\]

\[
\text{div}(\rho \text{grad} \varphi) \bigg|_{X=0} = \epsilon \left( \partial_m \partial^m f + \partial_z^2 f - \partial_z g \right)
\]

\[
+ o(\epsilon) = 0.
\]

Let us denote \( g = \partial_z f + h \), then we obtain the first equation for the perturbation

\[
\partial_m \partial^m f - \partial_z h = 0,
\]

\[
\rho^{-2} = c^2 [1 + 2Kz + 2K_m X^m + 2\epsilon (\partial_z f + h)]
\]

\[
+ o(X) + o(\epsilon),
\]

\[
||\text{grad} \varphi||^2 = c^2 (1 + 2Kz + 2K_m X^m + 2\epsilon \partial_z f)
\]

\[
+ o(X) + o(\epsilon),
\]

\[
\frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \bigg|_{X=0} = \left( 1 + \frac{Kz}{2} + \frac{K_m X^m}{2} + \frac{\epsilon}{2} (\partial_z f + h) \right)
\]

\[
\times \Box \left( -\frac{\epsilon}{2} (\partial_z f + h) \right) + o(\epsilon)
then the equations (15), (16) give
speed is less than 1 for nonzero values of
light), and when selecting the lower sign, the group
the group speed is strictly less than 1 (the speed of
It is easy to check that when selecting the upper sign,
The components of the group velocity are
\( \mu \)

The system (15), (16) takes the form
In this case, perturbations of the
domain walls
will be turned off. Thus the instability of small pertur-
bations, if any, should not deprive small perturbations
of physical meaning.

In any case, here we have studied the causality,
and the problem of stability in this context is insignif-
icant.

5. CONCLUSION
The resulting equation has the form of a nonlinear
Klein–Fock–Gordon equation. Why is it nonlinear? The most natural options are the following:

- The mean field approximation,
  - Mean field of membranes (domain walls),
  - Mean field of tachyons, \( \text{grad} \psi \) is space-
    like, but the group velocity of excitations is
    smaller than 1. So the domain wall is a
    sort of tachyonic condensate.

- Fundamental nonlinearity.

The result may be of interest in membrane quanti-
ization in superstring and supergravity theories [1] and
in condensed matter physics [2].

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