Stability study of digital multi-connected automatic control system

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Abstract. A study of the stability of a multiply connected automatic control system containing digital links by the frequency method is proposed. The stability criterion is based on the system description of the dynamic properties of the system at the level of characteristics of subsystems and characteristics of communication.

1. Introduction
In various fields of technology, multiply connected automatic control systems (MCACS) are widespread. In addition, in connection with the massive transition of control technology to a digital base, it becomes possible to implement complex control algorithms based on the achievements of modern control theory. Therefore, the study of the stability of multiply connected control systems with digital elements is relevant.

The paper considers multiply connected automatic control systems and uses a system description of the dynamic properties of the system at the level of characteristics of subsystems and characteristics of communication [1, 2].

The stability criterion proposed in [1] allows one to graphically investigate the stability of a continuous multi-connected system with subsystems of the same type.

For the stability of linear MCACS of the same type, it is necessary and sufficient that the hodograph of the amplitude-phase characteristic (APC) of the subsystems \( \Phi(j\omega) \), \( \omega \in (0, +\infty) \), built on the plane of the roots of the constraint equation, does not cover any of its roots [2, 3].

The article discusses the possibility of using the frequency stability criterion of MCACS [2, 3] to study the stability of digital and hybrid MCACS containing a continuous control object and a discrete controller.

2. Investigation of the stability of digital MCACS.
Let the transfer function of digital subsystems of the system (Fig. 1) with a quantization period \( T_0 \) have the form:

\[
C(z) = \frac{Y(z)}{R(z)} = \frac{G(z)D(z)}{1 + G(z)D(z)}, \quad C(z) = C_1(z) = \ldots = C_n(z).
\]  

(1)
Characteristics of a multidimensional relationship is as follows [2]:

\[
H_w = \sum_{i,j=1}^{n} H_{ij} = \frac{\det[G(z)\gamma_{ij}]}{\det[G(z)\delta_{ij}]}, \quad k = 2, n,
\]

\[
\gamma_{ij} = \begin{cases} 
1, & i \neq j; \\
0, & i = j; 
\end{cases} \quad \delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j, 
\end{cases} \quad i, j = 1, n.
\]

The characteristic equation of a system with holonomic constraints, i.e. \(H_m(s) = h_m\) has the form

\[
A(h,s) = 1 + h_sC^2(s) + h_sC^3(s) + \ldots + h_sC^n(s) = 0. 
\]

The coupling equations with respect to the variable \(x\) for a digital system is:

\[
A(x) = 1 + h_x x^2 + h_x x^3 + \ldots + h_x x^n = 0
\]

and is obtained from (2) by substitution \(C(z) = x\).

Let us evaluate the validity of the criterion proposed in [2] for evaluating the stability of digital multi-connected control systems.

Let us describe the procedure for its application.

1. Find the transfer function of the separate subsystems \(C(z)\) and replace \(z = e^{sT_0}\).

2. Construct the amplitude-phase characteristic (APC) of separate subsystems \(C(x^{T_0})\). For digital control systems, the corresponding frequency domain analysis involves replacing \(s = j\omega\).

\[
z = e^{j\omega T_0} = e^{j\overline{\omega}k} = \cos(\overline{\omega}k) + j \sin(\overline{\omega}k), \quad \overline{\omega} = \omega T_0, \quad (-\pi < \omega < \pi),
\]

\[
C(e^{j\overline{\omega}}) = \frac{G(e^{j\overline{\omega}})D(e^{j\overline{\omega}})}{1 + G(e^{j\overline{\omega}})D(e^{j\overline{\omega}})}.
\]

3. Find the roots of the characteristic coupling equation \(x_i (3)\) and place them on the same complex plane.

4. **Stability standard.** For the stability of digital MCACS, it is necessary and sufficient that the hodograph of the amplitude-phase characteristics of the subsystems \(C(e^{j\overline{\omega}})\), \(\overline{\omega} \in (0, +\pi)\), built on the plane of the roots of the constraint equation (3), does not cover any of its roots \(x_i\).

2.1. **Example**

Consider as an example a three-connected control system, which is described by transfer functions,

\[
D(s)G(s) = \frac{1}{s(s+1)} \begin{bmatrix} 
1 & 2 & 0 \\
0.02 & 1 & 1 \\
0.04 & 0 & 1 
\end{bmatrix}, \quad \text{the transfer function of subsystems } \Phi_i(s) = \frac{1}{s^2 + s + 1}.
\]

The transients of a continuous system are shown in Figure 2.
The transfer function of a subsystem of the same digitized system with a quantization period $T_0=0.025$ sec has the form $C_i(z) = \frac{0.3679z + 0.2642}{z^2 - z + 0.6321}$.

The transients corresponding to the digital system are shown in Figure 3. As you can see, the system is stable.

Fig. 2. Transient processes in a continuous system.  
Fig. 3. Transients in a digital system with a quantization period $T_0=0.025$ sec.

The transfer function of subsystems of the continuous system (1) digitized with a quantization period $T_0=1$ sec is as follows:

$$C_i(z) = \frac{1.135z + 0.594}{z^2 + 0.7293}.$$ 

Transient processes correspond to an unstable system (Fig. 4).

Fig. 4. Transients in a digital system with a quantization period $T_0=1$ sec.

1. Let us construct the amplitude-phase characteristic for separate subsystems $C(z)$ using the expression $z = e^{j\omega T_0} = e^{j\bar{\omega}}$, ($-\pi < \bar{\omega} < \pi$), on the s-plane.

2. Find the roots $x_i$ of the characteristic constraint equation.

$$A(x) = 1 + 0.04x^2 - 0.08x^3 = 0.$$ 

The roots are equal to $x_{1,2} = -1.0 \pm j2.0$, $x_3 = 2.5$. We place them on the same complex plane as the amplitude-phase characteristic.

The stability of the system is affected by the sampling period $T_0$. For example, hodographs of a continuous system and the same digitized system with quantization periods $T_0 = 0.025$ sec and $T_0 = 1$ sec are presented in Fig. 5. Hodographs of the continuous system and the digital one with $T_0 = 0.025$ sec correspond to stable systems according to the criterion, and with $T_0 = 1$ sec – to unstable ones, which is also confirmed by transient processes.
3. Determination of the critical value of the quantization period $T_0$

The critical value of the quantization period $T_0$ can be determined when the system is at the stability boundary. In this case, the hodograph of the subsystem intersects with the root of the characteristic constraint equation:

$$C(e^{j\omega T_0}) = \frac{G(e^{j\omega T_0})D(e^{j\omega T_0})}{1 + G(e^{j\omega T_0})D(e^{j\omega T_0})} = x_i.$$  

From the equality we obtain a system of two equations for $T_0$ and $\omega$:

$$\begin{cases} 
|C(e^{j\omega T_0})| = |x_i| \\
\arg C(e^{j\omega T_0}) = \arg x_i
\end{cases} \quad \text{or} \quad \begin{cases} 
\Re(C(e^{j\omega T_0})) = \Re(x_i) \\
\Im(C(e^{j\omega T_0})) = \Im(x_i)
\end{cases}.$$  

Having solved the system of two equations, one can find the critical value of the quantization period $T_0$.

4. Investigation of stability of hybrid MCACS

The considered stability criterion for digital MCACS allows us to study the stability of hybrid systems. The study revealed that for the stability of hybrid multi-connected systems, it is necessary and sufficient that the hodograph of stable closed subsystems does not cover the roots of the coupling equation. The article accepts the following systems as hybrid systems.

1. A digital system has elements with different quantization periods of $T_0$:

$$C(e^{j\omega T_0}) = \frac{G(e^{j\omega T_{01}})D(e^{j\omega T_{02}})}{1 + G(e^{j\omega T_{01}})D(e^{j\omega T_{02}})}.$$  

For the subsystems to be stable, each digital element in the subsystem must be stable, and the sampling rate of each digital element must not exceed the critical sampling rate of the entire system. This condition for a digital system consisting of two digital elements is shown in Fig. 6.

Fig. 5. Hodographs of a continuous system and digital systems with a quantization period of 1 sec and 0.025 sec.

Fig. 6. a) both elements are unstable; b) one of the elements is stable; c) both elements are stable.
2. Digital controller and continuous object

\[ C(e^{j\omega T_0}, j\omega) = \frac{G(e^{j\omega T_0})D(j\omega)}{1 + G(e^{j\omega T_0})D(j\omega)}. \]

The process of evaluating a stable hybrid MCACS can be described by the following algorithm:

1. We construct the APC of separate subsystems \( C(z,s) \). When constructing the frequency characteristics of the continuous part in the expression for the transfer function, substitution \( s = j\omega \), \((0\leq\omega \leq \infty)\) is performed. For the digital part, the corresponding frequency domain analysis is associated with replacement \( z = e^{j\omega T_0} \).

2. We find the roots of the characteristic coupling equation \( x_1(3) \) and place them on the same complex plane.

3. We analyze the stability of the system using the criterion. For the stability of the hybrid MCACS, it is necessary and sufficient that the hodograph of the amplitude-phase characteristic of subsystems \( C(e^{j\omega T_0}, j\omega), \omega \in (0, \frac{\pi}{T_0}) \), constructed on the plane of the roots of the coupling equation (3), does not cover any of its roots \( x_i \).

**Example**

Consider as an example a three-connected control system that is described by transfer functions, where

\[
D(s)G(s) = \frac{0.00966s + 0.024}{s - 0.005s^2 + 1.012445s + 2.489},
\]

\[
\begin{bmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{bmatrix}.
\]

1. Find the roots \( x_1, x_2 \) of the characteristic coupling equation \( \lambda(x) = 1 - 0.333x^2 + 0.0741x^3 = 0 \).

The roots are \( x_{1,2} = 3, x_3 = -1.5 \), and we place them on the complex plane.

2. Let us construct the amplitude-phase characteristic of separate subsystems.
   a) a continuous system

\[
D(j\omega)G(j\omega) = \frac{0.00966j\omega + 0.024}{j\omega - 0.005\omega^2 + j1.012445\omega + 2.489},
\]

\[
\begin{bmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{bmatrix}.
\]

b) a hybrid system

\[
D(e^{j\omega T_0})G(e^{j\omega T_0}) = \frac{0.00966e^{j0.0036\omega} - 0.009574}{e^{j0.0036\omega} - 1 - 0.005\omega^2 + j1.012445\omega + 2.489},
\]

\[
\begin{bmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{bmatrix}.
\]

where the quantization period of the digital controller is 0.0036 sec.

c) the system is stable digital

\[
D(e^{j\omega T_0})G(e^{j\omega T_0}) = \frac{0.00966e^{j0.0036\omega} - 0.009564}{e^{j0.0036\omega} - 1 - 1.356e^{j0.0049\omega} + 10.07},
\]

\[
\begin{bmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{bmatrix}.
\]

where the quantization period of the digital controller is 0.0036 sec and the control object is 0.004 sec.

d) the system is unstable digital

The hodographs of stable subsystems cover the roots of the coupling equation:

\[
D(e^{j\omega T_0})G(e^{j\omega T_0}) = \frac{0.00966e^{j0.0055\omega} - 0.009564}{e^{j0.0055\omega} - 1 - 1.356e^{j0.0085\omega} + 0.3633},
\]

\[
\begin{bmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{bmatrix}.
\]

where the sampling rate of the digital controller and the control object is 0.005 sec, which is greater than the critical value.

Figure 7 shows transients in a three-connected system.
Fig. 7. Transients of three-connected systems: continuous, hybrid, digital-stable and unstable.

Figure 8 shows the hodographs of separate subsystems: continuous, hybrid (digital controller), and digital system with elements having different sampling periods.

Fig. 8. Hodographs of separate subsystems: continuous, hybrid, digital-stable and unstable.

Conclusions
The possibility of using the frequency criterion for determining the stability of digital and hybrid MCACS, proposed in [1, 2], is shown. The stability of the digital and hybrid MCACS depends on the sampling period $T_0$. A method for determining the critical value of the quantization period $T_0$ is considered. The reliability of the obtained results was confirmed using the MCACS simulation.
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