An Analytical Method for Plane Elasticity Problems Involving Circular Boundaries

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Abstract. Complex structure with circular boundaries is commonly used in engineering practice, and it is essential to conduct a detailed analysis of the interior stress field of the structure. The Michell stress function is a well-known general solution for plane elasticity problems in the polar coordinates, particularly when circular boundaries are involved. This work presents an analytical method with the assistance of the Michell solution, which could be seamlessly combined with Fourier analysis. Using the expansion of Fourier series, problems with arbitrarily distributed loads can be handled via a standard procedure. A complete analytical solution is elaborated for an arbitrarily loaded circular ring, and a classical elastic solution is provided for verification.

1. Introduction
The structures with circular boundaries are common in real world applications, such as sleeve, multi-hole plate, and steel tube to name a few. To prevent fatigue cracks or brittle fractures near the holes of the structure, it is necessary to conduct a stress analysis, especially in a localized high stress zone. Furthermore, some important characters may be obtained from such a stress analysis, e.g. the birefringence direction of circular-core side-hole fiber may be affected by the stresses [1].

Analytical and numerical methods are frequently used to obtain the stress distribution. The analytical solution using complex function can be effective [2,3], but related knowledge of the complex analysis is indispensable, and complicated derivations are often encountered. The finite element method [4] is widely applied for structural analyses. However, for a complex pore structure, it is a nontrivial task to generate high-quality mesh. To get the stress of circular structure, the commonly used analytical solutions are applicable for cases of simple boundary conditions [5-8]. However, boundary loading is usually complex in practice. A fundamental analytical solution for the complex structure under any type of loading would be of significance.

In this paper, the arbitrary boundary load is approximated by Fourier series, and the analytical expressions of the stresses of a circular ring are obtained by using Michell's general solution [9]. The cases of a disk and an infinite plate with a circular hole are considered. Furthermore, a classical analytical solution is provided to verify the present work.

2. The General Solution of Michell's Stress Function
In Michell’s paper published in 1898 [10], an elastic general solution of stress function is presented in plane polar coordinates. By employing Michell’s stress function, complicated mathematic physics equations are circumvented. The solution may be obtained by determining the coefficients of terms...
from the boundary conditions.

In the polar coordinates of the elastic plane, the general solution of stress function [11] in the form of Fourier series is given as

$$\phi(r, \theta) = \sum_{n=0}^{\infty} f_n(r) \cos(n\theta) + \sum_{n=1}^{\infty} g_n(r) \sin(n\theta)$$

(1)

According to the theory of elasticity, equation (1) should satisfy the biharmonic equation. By applying the theory of partial differential equation and considering the additional terms for multivalued problems, Michell's stress function [11] could be written as

$$\phi = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \theta + a_0' \theta + \frac{a_1}{r} \cos \theta + (b_1 + b_0' \ln r) \cos \theta + \frac{c_1}{2} r \cos \theta + \frac{d_1}{r} \sin \theta$$

$$+ \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + \frac{c_n}{r} + \frac{d_n}{r^{n-2}}) \cos n\theta + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + \frac{c_n'}{r} + \frac{d_n'}{r^{n-2}}) \sin n\theta$$

(2)

where the three separate lines of equation (2) corresponds to \(n = 0\), \(n = 1\), and \(n \geq 2\) respectively, and the physical meaning of each term can be found in the textbook [12]. In view of Michell's stress function, the stresses of the structure with circular boundaries can be obtained.

3. The Stress Distribution of a Circular Ring under Arbitrary Boundary Loads

As shown in figure 1, an isotropic circular ring is studied. The center of the ring is placed at the origin \(o\). The inner radius is \(a\), and the outer radius is \(b\). Arbitrarily distributed normal and shear forces are acted on the inner and outer boundaries. By using Michell's stress function, the problem could be considered in three cases: \(n = 0\), \(n = 1\) and \(n \geq 2\). The arbitrary loads on the boundaries can be approximated by Fourier series as

$$\begin{cases}
    (\sigma_r)_{r=a} &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta), \\
    (\tau_{r\theta})_{r=a} &= \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\theta) + \sum_{n=1}^{\infty} D_n \sin(n\theta) \\
    (\sigma_r)_{r=b} &= \frac{A_0'}{2} + \sum_{n=1}^{\infty} A'_n \cos(n\theta) + \sum_{n=1}^{\infty} B'_n \sin(n\theta), \\
    (\tau_{r\theta})_{r=b} &= \frac{C_0'}{2} + \sum_{n=1}^{\infty} C'_n \cos(n\theta) + \sum_{n=1}^{\infty} D'_n \sin(n\theta)
\end{cases}$$

(3)

where \(A_0, A_n, A'_n, B_0...D_n, D'_n\) are known constants.

![Figure 1. A concentric ring with arbitrary loads on the boundary.](image)

The relationship between the stress function and stress/displacement components in polar coordinates can be found in Dundurs’ table [11]. Coefficients in the stress components are obtained through the boundary conditions when \(n \geq 2\). But for \(n = 0\) or \(1\), more conditions should be
considered.
When \( n = 0 \), the stress function can be obtained from equation (2) as
\[
\phi = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \theta + a'_0 \theta
\]  
(4)
According to Dundurs’ table [11], the stress expressions are
\[
\sigma_r = a_0 r^2 + 2b_0 + 2c_0 \ln r + c_0 + 2d_0 \theta, \quad \sigma_\theta = -a_0 r^2 + 2b_0 + 2c_0 \ln r + 3c_0 + 2d_0 \theta, \quad \tau_{r\theta} = a'_0 r^2 - d_0
\]  
(5)
The corresponding boundary conditions are
\[
(\sigma_r)_{r=a} = \frac{A_n}{2}, \quad (\sigma_r)_{r=b} = \frac{A'_n}{2}, \quad (\tau_{r\theta})_{r=a} = \frac{C_n}{2}, \quad (\tau_{r\theta})_{r=b} = \frac{C'_n}{2}
\]  
(6)
In the present problem, according to the single-valued conditions of stress and displacement, the terms of \( d_0 r^2 \theta \) and \( c_0 r^2 \ln r \) are removed. The stress function thus can be reduced to
\[
\phi = a_0 \ln r + b_0 r^2 + a'_0 \theta
\]  
(7)
Based on the physical meaning of each item in equation (2), \( a'_0 \) could be determined by the shear stress boundary conditions, which indicates that \( C_0 \) and \( C'_0 \) should satisfy static equilibrium conditions as
\[
\tau_{r\theta} = C_0 a^2 r^{-2} = C'_0 b^2 r^{-2}
\]  
(8)
and the other two normal stress boundary conditions can be used to get \( a_0 \) and \( b_0 \). The solutions for the case \( n = 1 \) can be obtained in a similar process.

4. The Stress Distribution of an Infinite Plate with a Hole under Arbitrary Boundary Loads
An infinite plate with a circular hole under arbitrarily distributed forces on the boundary is shown in figure 2. The circular hole with radius, \( a \), is centered at the origin, \( o \), of the Cartesian coordinate system.

![Figure 2. An infinite plate with a circular hole under arbitrary loads](image)

From Saint-Venant principle, it is obvious that the stresses vanish at the infinite point. Therefore, the multi-value term or the terms, which lead to stresses or strains approaching infinity with \( r \to \infty \), in equation (2) should be removed. A dimensionless variable \( \lambda = a/r \) is introduced, and the stress \( \sigma_r \) is accordingly shown below:
\[ \sigma_r = \frac{A_1 \lambda^2}{2} + \frac{\lambda}{2(1+\kappa)} \left[ 3 + \kappa - (1-\kappa) \lambda^2 \right] (A_i \cos \theta + B_i \sin \theta) \]
\\[ + \frac{3 + \kappa}{2(1+\kappa)} \left( \lambda - \lambda^3 \right) (C_i \sin \theta - D_i \cos \theta) \]
\\[ + \sum_{n=2}^{\infty} \left( -\frac{n}{2} \lambda^{n+2} + \frac{n+2}{2} \lambda^n \right) \left[ A_n \cos(n\theta) + B_n \sin(n\theta) \right] \]
\\[ + \sum_{n=2}^{\infty} \left( -\frac{n+2}{2} \lambda^{n+2} + \frac{n}{2} \lambda^n \right) \left[ C_n \sin(n\theta) - D_n \cos(n\theta) \right] \]
\[(9)\]

The stresses \( \sigma_\theta \) and \( \tau_{r\theta} \) could be obtained in a similar manner.

5. The Stress Distribution of a Homogeneous Disk under Arbitrary Boundary Loads

A homogeneous disk with radius \( b \), whose center is at the origin \( o \), of the Cartesian coordinate system, is subjected to arbitrarily distributed forces as shown in figure 3.

![Figure 3](image)

**Figure 3.** A homogeneous disk with arbitrary loads on the boundary.

Besides the boundary conditions and the single value, it should be noted that the stress and displacement are finite at the origin \( o \). A normalized variable \( \rho = r/b \) is introduced, and the stress \( \sigma_r \) is shown below.

\[ \sigma_r = \frac{A'_1}{2} + \rho A'_1 \cos \theta + \rho B'_1 \sin \theta + \sum_{n=2}^{\infty} \left( \frac{2-n}{2} \rho^n + \frac{n}{2} \rho^{n-2} \right) \left[ A'_n \cos(n\theta) + B'_n \sin(n\theta) \right] \]
\\[ + \sum_{n=2}^{\infty} \left( \frac{n-2}{2} \rho^n - \frac{n-2}{2} \rho^{n-2} \right) \left[ C'_n \sin(n\theta) - D'_n \cos(n\theta) \right] \]
\[(10)\]

The stresses \( \sigma_\theta \) and \( \tau_{r\theta} \) could also be obtained in a similar process.

6. Results and Discussions

To facilitate a systematic discussion, a classical analytical solution for verification is proposed. A plate with a circular hole, which is subjected to biaxial unequal uniform tension and compression, is shown in figure 4.
Figure 4. A plate with a circular hole subjected to biaxial forces.

The stresses in the vicinity of the small hole are [13]

\[
\sigma_r = \left(1 - \frac{a^2}{r^2}\right) \frac{q_1 + q_2}{2} + \frac{q_1 - q_2}{2} \left(1 - \frac{a^2}{r^2}\right) \cos 2\theta,
\]

\[
\sigma_\theta = \left(1 + \frac{a^2}{r^2}\right) \frac{q_1 + q_2}{2} - \frac{q_1 - q_2}{2} \left(1 + \frac{3a^2}{r^2}\right) \cos 2\theta,
\]

\[
\tau_{r\theta} = -\frac{q_1 - q_2}{2} \left(1 - \frac{a^2}{r^2}\right) \left(1 + \frac{3a^2}{r^2}\right) \sin 2\theta
\]

(11)

Based on the superposition principle, the stress components above could also be obtained through our present work as follows

\[
\sigma_r = \frac{-q_1 - q_2}{2} \lambda^2 + \left(-3\lambda^4 + 4\lambda^2 - 1\right) \frac{q_1 - q_2}{2} \cos 2\theta + \frac{q_1 + q_2}{2},
\]

\[
\sigma_\theta = \frac{q_1 + q_2}{2} \lambda^2 + \left(3\lambda^4 + 1\right) \frac{q_1 - q_2}{2} \cos 2\theta + \frac{q_1 + q_2}{2},
\]

\[
\tau_{r\theta} = \left(3\lambda^4 - 2\lambda^2 - 1\right) \frac{q_1 - q_2}{2} \sin 2\theta
\]

(12)

From the comparison between equation (11) and equation (12), it can be found that the analytical expressions are identical.

7. Conclusion

In the present work, three models, i.e. an infinite plate with a circular hole, a concentric ring, and a solid disk, under arbitrary boundary loads are analyzed. To obtain the stress results, Michell’s stress function is applied, and the boundary loads are approximated by Fourier series. The present work provides fundamental analytical solutions for other complicated problems. It is shown that Michell’s stress function is applicable for the analysis of the three classical models mentioned above, demonstrating the validity of the present work is verified.

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