A Bernstein type result of translating solitons

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Abstract
We prove a Bernstein type theorem for complete translating solitons of the mean curvature flow, whose images of their Gauss maps are contained in an appropriate neighborhood of the Grassmannian manifold.

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1 Introduction
Let $X : M^n \rightarrow \mathbb{R}^{m+n}$ be an isometric immersion from an $n$-dimensional oriented Riemannian manifold $M$ to the Euclidean space $\mathbb{R}^{m+n}$. The mean curvature flow (MCF) in Euclidean space is a one-parameter family of immersions $X_t = X(\cdot, t) : M^m \rightarrow \mathbb{R}^{m+n}$ with the corresponding images $M_t = X_t(M)$ such that

$$\frac{\partial}{\partial t} X(x, t) = H(x, t), \quad x \in M,$$

$$X(x, 0) = X(x),$$

is satisfied, where $H(x, t)$ is the mean curvature vector of $M_t$ at $X(x, t)$ in $\mathbb{R}^{m+n}$.

We call $M^n$ a translating soliton in $\mathbb{R}^{m+n}$ if it satisfies

$$H = V_0^N,$$

where $V_0$ is a fixed vector in $\mathbb{R}^{m+n}$ with unit length and $V_0^N$ denotes the orthogonal projection of $V_0$ onto the normal bundle of $M^n$. The translating solitons give rise to eternal solutions $X_t = X + tV_0$ to (1.1). They are not only special solutions to the mean curvature flow, but also they often occur as the Type II singularities of the mean curvature flow (see [2, 3, 21, 24, 25, 45, 46]). And the geometry of the translating soliton has been paid much attention during the past two decades, see the references (not exhaustive): [12, 13, 22, 27, 32, 35–39, 44, 48], etc.
Translating solitons can be regarded as minimal submanifolds if we make a conformal change of the metric of the ambient space. This important observation is due to Ilmanen [26]. So we firstly recall the related theory of minimal submanifolds. The Bernstein problem has been a central problem in the study of minimal submanifolds. The classical Bernstein theorem states that the entire minimal graph in the Euclidean space \( \mathbb{R}^3 \) is a plane [6]. Many efforts had been made to generalize the Bernstein theorem to higher dimensions. Eventually, Simons [42] proved that entire minimal graphic hypersurfaces in \( \mathbb{R}^{n+1} \) must be hyperplanes for \( n \leq 7 \) (see De Giorgi [15] for \( n = 3 \) and Almgren [1] for \( n = 4 \)), while Bombieri–De Giorgi–Giusti [7] constructed a counterexample for \( n \geq 8 \). However, Moser [34] had earlier showed that, under the additional assumption that the gradient of the graph function is uniformly bounded, the entire minimal graphic hypersurface has to be planar in arbitrary dimension (see also [23]). Later, Ecker–Huisken [17] improved Moser’s theorem by using the curvature estimate technique. There are plenty of works on the study of the Bernstein type problems on minimal hypersurfaces in Euclidean spaces (see [14, 16, 19, 20, 29, 40, 41, 43, 47]).

The Bernstein problem in higher codimensions becomes more complicated, for example, the Moser’s theorem can be generalized to higher codimensional complete minimal graphs for dimension 2 and 3 (see [5, 11, 18]), but the counterexample of a nontrivial minimal graph with bounded slope, constructed by Lawson–Osserman [33], sets a limit for how far one can go. They also raised a question for finding the “best” constant possible in the same paper. Afterward, the Bernstein type theorem can be achieved that any minimal graph \( M^n \) in \( \mathbb{R}^{m+n} (n \geq 3, m \geq 2) \) must be an affine \( n \)-plane provided the slope \( \leq 3 \) (see [23, 28, 30, 31, 49]). Recently, Assimos–Jost [4] extended Moser’s theorem to codimension 2 by using Sampson’s maximum principle.

As we mentioned above, translating solitons can be regarded as minimal submanifolds if we make a conformal change of the metric of the ambient space, it is natural to study the Bernstein type problem of translating solitons. For the codimension one case, Bao–Shi [8] proved a translating soliton version of Moser’s theorem, namely, if the image of the translating soliton \( M^n \) under the Gauss map is contained in a regular ball, then such a complete translating soliton in \( \mathbb{R}^{n+1} \) has to be a hyperplane. For higher codimensions, Kunikawa [32] generalized the result of [8] to the flat normal bundle case, and obtained that for any complete translating soliton \( M^n \) with flat normal bundle in \( \mathbb{R}^{m+n} \), if the \( w \)-function is positive and it satisfies the growth condition \( w^{-1} = o(R^{\frac{1}{2}}) \), then \( M^n \) must be an affine subspace. The function \( w \) measures the slope of the translator with respect to a fixed plane and the precise definition of \( w \) will be given in Sect. 2. If \( w > 0 \) then the function \( v = w^{-1} \) describes the volume element of the translator. In general, without the condition on the flat normal bundle, Xin [48] partially solved a counterpart of the Lawson–Osserman problem for translating solitons, that is, if the \( v \)-function satisfies \( v \leq v_1 < v_0 := \frac{2+3\frac{2}{3}}{1+3\frac{2}{3}} \), then any complete translating soliton \( M^n \) in \( \mathbb{R}^{m+n} (m \geq 2) \) has to be affine linear. It is natural to find the optimum constant \( v_0 \), such that the corresponding Bernstein type result still holds.

In this note, by adopting a new test function which is different from the one in [48] and using the gradient estimate technique, we have the following result.
Theorem 1 Let $M^n$ be a complete $n$-dimensional translating soliton in $\mathbb{R}^{m+n}$ with codimension $m \geq 2$ and positive $w$-function. If there exists a positive constant $v_1 > 0$ such that

$$v \leq v_1 < v_0 := \frac{2 \cdot 4^{\frac{3}{2}}}{1 + 4^{\frac{3}{2}}}$$

then $M^n$ is an affine subspace.

Remark 1 Xin [48] showed that the same conclusion holds under the condition that $v \leq v_1 < \frac{2 \cdot 3^{\frac{5}{2}}}{1 + 3^{\frac{5}{2}}}$. Clearly, $\frac{2 \cdot 3^{\frac{5}{2}}}{1 + 3^{\frac{5}{2}}} < \frac{2 \cdot 4^{\frac{3}{2}}}{1 + 4^{\frac{3}{2}}}$. Thus the condition in Theorem 1 is weaker than the one in [48].

Remark 2 By Proposition 6.1 in [49] (see also [50]), if a $v$-function is bounded from above by a positive constant, then any complete translating soliton $M^n$ is an entire graph. Let $u_\alpha : \mathbb{R}^n \to \mathbb{R}^m$ be the graph functions. Then the induced metric $(g_{ij})$ on $M^n$ is $g_{ij} = \delta_{ij} + \sum_\alpha u_i^\alpha u_j^\alpha$ and the $v$-function is just $\left( \det \left( \delta_{ij} + \sum_\alpha u_i^\alpha u_j^\alpha \right) \right)^{\frac{1}{2}}$. Theorem 1 claims that any entire graphic translating soliton $M^n$ in $\mathbb{R}^{m+n}$ ($m \geq 2$) has to be affine linear provided

$$\left( \det \left( \delta_{ij} + \sum_\alpha u_i^\alpha u_j^\alpha \right) \right)^{\frac{1}{2}} \leq v_1 < v_0 := \frac{2 \cdot 4^{\frac{3}{2}}}{1 + 4^{\frac{3}{2}}}$$

2 Preliminaries

Let $G_{n,m}$ be the Grassmann manifold consisting of the oriented linear $n$-subspaces in $\mathbb{R}^{m+n}$. The canonical Riemannian structure on $G_{n,m}$ makes it a natural generalization of the Euclidean sphere. $G_{n,m} = SO(m+n)/SO(n) \times SO(m)$ is an irreducible symmetric space of compact type.

For every $P \in G_{n,m}$, we choose an oriented basis $\{u_1, \ldots, u_n\}$ of $P$, and let

$$\psi(P) := u_1 \wedge \cdots \wedge u_n \in \Lambda^n(\mathbb{R}^{m+n}).$$

A different basis for $P$ shall give a different exterior product, but the two products differ only by a positive scalar; $\psi(P)$ is called the Plücker coordinate of $P$, which is a homogeneous coordinate.

Via the Plücker embedding, $G_{n,m}$ can be viewed as a submanifold of some Euclidean space. The restriction of the Euclidean inner product on $G_{n,m}$ is denoted by $w : G_{n,m} \times G_{n,m} \to \mathbb{R}$

$$w(P, Q) = \frac{\langle \psi(P) , \psi(Q) \rangle}{\langle \psi(P) , \psi(P) \rangle^\frac{1}{2} \langle \psi(Q) , \psi(Q) \rangle^\frac{1}{2}}.$$ 

If $\{e_1, \ldots, e_n\}$ is an oriented orthonormal basis of $P$ and $\{f_1, \ldots, f_n\}$ is an oriented orthonormal basis of $Q$, then

$$w(P, Q) = \langle e_1 \wedge \cdots \wedge e_n , f_1 \wedge \cdots \wedge f_n \rangle = \det W,$$

where $W = (\langle e_i , f_j \rangle)$. It is well known that

$$W^T W = O^T \wedge O$$
with $O$ an orthogonal matrix and $\Lambda = \text{diag}(\mu_1^2, \ldots, \mu_n^2)$. Here each $0 \leq \mu_i^2 \leq 1$. Putting $p := \min\{m, n\}$, then at most $p$ elements in $\{\mu_1^2, \ldots, \mu_n^2\}$ are not equal to 1. Without loss of generality, we can assume $\mu_i^2 = 1$ whenever $i > p$. We also note that the $\mu_i^2$ can be expressed as

$$\mu_i^2 = \frac{1}{1 + \lambda_i^2}$$

with $\lambda_i \in [0, +\infty)$.

The Jordan angles between $P$ and $Q$ are defined by

$$\theta_i = \arccos(\mu_i), \quad 1 \leq i \leq p.$$ 

The distance between $P$ and $Q$ is defined by

$$d(P, Q) = \sqrt{\sum \theta_i^2}.$$ 

It is a natural generalization of the canonical distance of the Euclidean sphere. Thus we have

$$\lambda_i = \tan \theta_i.$$ 

In the sequel, we shall assume $n \geq m$ without loss of generality. Let $\alpha = n + \alpha'$ and denote $\alpha$ for $\alpha'$ for simplicity.

Now we fix $P_0 \in G_{n,m}$. We represent it by the $n$-vector $\varepsilon_1 \land \cdots \land \varepsilon_i \land \cdots \varepsilon_n$. We choose $m$ vectors $\varepsilon_{n+\alpha}$, such that $\{\varepsilon_i, \varepsilon_{n+\alpha}\}$ form an orthonormal basis of $\mathbb{R}^{m+n}$. Denote

$$\mathbb{U} := \{P \in G_{n,m} : w(P, P_0) > 0\}.$$ 

The $v$-function will be

$$v(\cdot, P_0) := w^{-1}(\cdot, P_0) \quad \text{on} \quad \mathbb{U}.$$ 

For arbitrary $P \in \mathbb{U}$ determined by an $n \times m$ matrix $Z$, it is easy to see that

$$v(P, P_0) = \left(\det(I_n + ZZ^T)\right)^{\frac{1}{2}} = \prod_{\alpha=1}^{m} \sec \theta_{\alpha} = \prod_{\alpha=1}^{m} \frac{1}{\mu_{\alpha}},$$

where $\theta_1, \ldots, \theta_m$ denotes the Jordan angles between $P$ and $P_0$.

The second fundamental form $B$ of $M^n$ in $\mathbb{R}^{m+n}$ is defined by

$$B_{UW} := (\nabla_U W)^N$$

for $U, W \in \Gamma(TM^n)$. We use the notation $(\cdot)^T$ and $(\cdot)^N$ for the orthogonal projections into the tangent bundle $TM^n$ and the normal bundle $NM^n$, respectively. For $v \in \Gamma(NM^n)$ we define the shape operator $A^v : TM^n \to TM^n$ by

$$A^v(U) := -(\nabla_U v)^T.$$ 

The trace of $B$ is the mean curvature vector $H$ of $M^n$ in $\mathbb{R}^{m+n}$ and

$$H := \text{trace}(B) = \sum_{i=1}^{n} B_{e_i e_i},$$

where $\{e_i\}$ is a local orthonormal frame field of $M^n$. 
3 Proof of Theorem 1

By Corollary 6.2 in [48], the Gauss map of translating solitons is a $V^T$-harmonic map. So the study of translating solitons is naturally related to $V$-harmonic maps. Recall that a map $u$ from a Riemannian manifold $(M, g)$ to another Riemannian manifold $(N, h)$ is called a $V$-harmonic map if it solves

$$\tau(u) + du(V) = 0,$$

where $\tau(u)$ is the tension field of the map $u$, and $V$ is a vector field on $M$ (cf. [9, 10]). Clearly, it is a generalization of the usual harmonic map. Let $\Delta_V := \Delta + \langle V, \nabla \cdot \rangle$ and $V = V^T_0$.

**Proof of Theorem 1** Let $h := \left(\frac{v^0}{2-v^0}\right)^{\frac{3}{2}}$. Clearly, $h_0 = \left(\frac{v_0}{2-v_0}\right)^{\frac{3}{2}} = 4$ and $h_1 = \left(\frac{v_1}{2-v_1}\right)^{\frac{3}{2}} < 4$ are two constants. Choose a constant $h_2$ such that $h_1 < h_2 < 4$. Since $v \geq 1$, thus we have $1 \leq h \leq h_1 < h_2 < 4$.

Fix a point $p \in M^n$ and consider an orthonormal frame $\{e_i\}$, defined on a neighborhood around $p$, such that $\nabla e_i |_p = 0$. From the translating soliton equation (1.2), we derive

$$\nabla e_j H = (\nabla e_j (V_0 - \langle V_0, e_k \rangle e_k))^N = -\langle V_0, e_k \rangle B_{e_j e_k}$$

and

$$\nabla e_i \nabla e_j H = -\langle V_0, e_k \rangle \nabla e_i B_{e_j e_k} - \langle H, B_{e_i e_k} \rangle B_{e_j e_k}.$$

Hence using the Codazzi equation, we obtain that

$$\Delta_V |H|^2 = \Delta |H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= 2\langle \nabla e_i \nabla e_j H, H \rangle + 2|\nabla H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= -2\langle H, B_{e_i e_k} \rangle^2 - 2\langle \nabla_{V^T_0} H, H \rangle + 2|\nabla H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= -2\langle H, B_{e_i e_k} \rangle^2 - |\nabla V|^2 + 2|\nabla H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= -2\langle H, B_{e_i e_k} \rangle^2 + 2|\nabla H|^2.$$

It follows that

$$\Delta_V |H|^2 \geq 2|\nabla H|^2 - 2|B|^2 |H|^2.$$

(3.1)

Denote by $X$ the position vector of the translator and set $r = |X|$. Then,

$$\nabla r^2 = 2X^T, \quad |\nabla r| \leq 1$$

$$\Delta r^2 = 2n + 2\langle H, X \rangle \leq 2n + 2r.$$

(3.2)

Let $B_a(o)$ be the closed ball of radius $a$ centered at the origin $o$ of $\mathbb{R}^{m+n}$ and $D_a(o) = M^n \cap B_a(o)$. Let $\gamma : M^n \to G_{n,m}$ be the Gauss map. Define $f : D_a(o) \to \mathbb{R}$ by

$$f = \frac{(a^2 - r^2)^2 |H|^2}{(h_2 - h \circ \gamma)^2}.$$

Since $f|_{\partial D_a(o)} = 0$, $f$ achieves an absolute maximum in the interior of $D_a(o)$, say $f \leq f(q)$, for some $q$ inside $D_a(o)$. We may also assume $|H|(q) \neq 0$. Then from

$$\nabla f(q) = 0,$$

$$\Delta_V f(q) \leq 0.$$

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we obtain the following at the point $q$:

$$
- \frac{2\nabla r^2}{a^2 - r^2} + \frac{\nabla|H|^2}{|H|^2} + \frac{2\nabla(h \circ \gamma)}{h_2 - h \circ \gamma} = 0, 
$$

(3.3)

$$
- \frac{2\Delta r^2}{a^2 - r^2} - \frac{2\nabla r^2}{(a^2 - r^2)^2} + \frac{\Delta |H|^2}{|H|^2} - \frac{|\nabla|H|^2|^2}{|H|^4} + \frac{2\Delta (h \circ \gamma)}{h_2 - h \circ \gamma} + \frac{2|\nabla(h \circ \gamma)|^2}{(h_2 - h \circ \gamma)^2} \leq 0. 
$$

(3.4)

By a direct computation we get

$$
|\nabla|H|^2|^2 = |2(\nabla H, H)|^2 \leq 4|\nabla H|^2|H|^2, 
$$

(3.5)

and

$$
|\nabla(h \circ \gamma)| \leq |dh||\nabla \gamma| \leq 3 \left( \frac{v_1}{2 - v_1} \right)^{\frac{3}{2}} |B| =: C_1|B|. 
$$

(3.6)

It follows from (3.1) and (3.5) that

$$
\frac{\Delta |H|^2}{|H|^2} \geq \frac{|\nabla|H|^2|^2}{2|h|^4} - 2|B|^2. 
$$

(3.7)

From (3.3), we obtain

$$
\frac{|\nabla|H|^2|^2}{|H|^4} \leq \frac{4|\nabla r^2|^2}{(a^2 - r^2)^2} + \frac{8|\nabla r^2||\nabla(h \circ \gamma)|}{(a^2 - r^2)(h_2 - h \circ \gamma)} + \frac{4|\nabla(h \circ \gamma)|^2}{(h_2 - h \circ \gamma)^2} 
$$

(3.8)

By (4.6) in [49] and Corollary 6.2 in [48], we obtain

$$
\Delta (h \circ \gamma) = \sum_{i=1}^n \text{Hess}(h)(d \gamma(e_i), d \gamma(e_i)) + dh(\tau(\gamma) + d \gamma(V)) 
$$

(3.9)

Substituting (3.2), (3.6), (3.7), (3.8), (3.9) into (3.4), we have

$$
\left( \frac{3h}{h_2 - h \circ \gamma} - 1 \right)|B|^2 - \frac{4C_1 r}{(a^2 - r^2)(h_2 - h \circ \gamma)}|B| - \frac{2n + 4r}{a^2 - r^2} - \frac{8r^2}{(a^2 - r^2)^2} \leq 0 
$$

Since

$$
\frac{3h}{h_2 - h \circ \gamma} - 1 = \frac{3h - (h_2 - h \circ \gamma)}{h_2 - h \circ \gamma} \geq \frac{4 - h_2}{h_2 - 1}, 
$$

Denote $C_2 := \frac{4 - h_2}{h_2 - 1}$. Obviously, $C_2$ is a positive constant. Since $h \leq h_1 < h_2$, the above two inequalities then imply

$$
C_2|B|^2 - \frac{4C_1 r}{(a^2 - r^2)(h_2 - h_1)}|B| - \frac{2n + 4r}{a^2 - r^2} - \frac{8r^2}{(a^2 - r^2)^2} \leq 0 
$$

Observe that if $ax^2 - bx - c \leq 0$ with $a, b, c$ all positive, then

$$
x \leq \max\{2b/a, 2\sqrt{c/a}\}. 
$$
Therefore, at the point $q$,

$$|B|^2 \leq \max \left\{ \frac{64C^2}{C_2^2(a^2-r^2)^2(h_2-h_1)^2} \cdot \frac{4(2n+4r)}{C_2(a^2-r^2)} + \frac{32r^2}{C_2(a^2-r^2)^2}, \right\}.$$ 

Since $|B|^2 \geq \frac{|H|^2}{n}$, thus we obtain, at the point $q$,

$$|H|^2 \leq n \max \left\{ \frac{64C^2}{C_2^2(a^2-r^2)^2(h_2-h_1)^2} \cdot \frac{4(2n+4r)}{C_2(a^2-r^2)} + \frac{32r^2}{C_2(a^2-r^2)^2}, \right\}. \quad (3.10)$$

and

$$f(q) \leq n \max \left\{ \frac{64C^2}{C_2^2(h_2-h_1)^4} \cdot \frac{4(2n+4a)^2}{C_2(h_2-h_1)^2} + \frac{32a^2}{C_2(h_2-h_1)^2}, \right\}.$$ 

Then for any point $x \in D_{a/2}(o)$, we have

$$|H|^2(x) \leq \frac{(h_2-h \circ \gamma)^2}{(a^2-r^2)^2} - f(q)$$

$$\leq \frac{16n(h_2-1)^2}{9a^4} \max \left\{ \frac{64C^2}{C_2^2(h_2-h_1)^4} \cdot \frac{4(2n+4a)^2}{C_2(h_2-h_1)^2} + \frac{32a^2}{C_2(h_2-h_1)^2}, \right\}. \quad (3.11)$$

Fixing $x$ and taking $a \to \infty$ in (3.11), we derive that $H \equiv 0$. Since $v_0 < 2$ the conclusion follows by using Jost–Xin’s results in [28].

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