2–dimensional Regge gravity
in the conformal gauge

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Abstract

By restricting the functional integration to the Regge geometries, we give the discretized version of the well known path integral formulation of 2–dimensional quantum gravity in the conformal gauge. We analyze the role played by diffeomorphisms in the Regge framework and we give an exact expression for the Faddeev–Popov determinant related to a Regge surface; such an expression in the smooth limit goes over to the correct continuum result.

While Regge discretized formulation of gravity is well understood at the classical level, the same cannot be said at the quantum level. Here the discussion centers mainly about the integration measure to be adopted in the functional integral and the role played by diffeomorphisms.

In lattice QCD there exists at the discretized level a well defined invariance group, i.e. the local gauge group which dictates the unique integration measure; the volume of the gauge group is finite and in principle no gauge fixing is necessary as we can integrate over the whole group. With regards to gravity if one wants to maintain the classical definition of diffeomorphism one must describe the space–time as a continuous manifold. The Regge approach can be understood as a description of continuous geometries in which the geometries are restricted to the ones which are piecewise flat [1, 2]. From this point of view one can consider diffeomorphisms in the same way as in the usual continuous formulation. In the following we shall adhere to a Regge theory understood as exactly invariant under the full group of diffeomorphisms.

With regard to the functional integration in the usual gauge theories it is performed on the gauge fields and not on the invariants of the theory; also in gravity we shall stick to the integration over the metrics and not over some invariants of the metric. In order to do that a distance among different metrics has to be given and we shall adopt the usual De-Witt super-metric, which is the unique ultra–local distance invariant under diffeomorphisms

\[
(\delta g^{(1)}, \delta g^{(2)}) = \int \sqrt{g} \delta g_{\mu\nu}^{(1)} \left( g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + C g^{\mu\nu} g^{\alpha\beta} \right) \delta g_{\alpha\beta}^{(2)}.
\]

Due to the infinite volume of the diffeomorphism group, it is necessary to introduce a gauge–fixing and the related Faddeev–Popov determinant has to be taken into account. Such an approach is the one which has proven successful in the continuous formulation [3, 4] and has been suggested by Jevicki and Ninomiya [5] in the Regge approach. In the usual parameterization, the Regge surface in two dimensions (as in all dimensions) is described by a number of bone lengths \(l_i\) which have the meaning of geodesic distances among points of singular curvature. An infinite family of \(g_{\mu\nu}(x, l_i)\) will describe such a Regge surface and they will be related among each other by a diffeomorphism. We remark that to choose an element of such equivalence class is not sufficient to provide a \(g_{\mu\nu}\) for each triangular face (in \(D = 2\)) which correctly describes the metric properties of such a face, e.g. by stating that the metric in the triangle \(T_1\) in the figure is given by

\[
g_1 = \left( \frac{1}{2} l_1^2 + \frac{1}{2} l_2^2 - l_3^2 \right)^{-1} \left( l_1^2 l_2^2 - l_3^2 \right) \left( l_1^2 + l_2^2 - l_3^2 \right).
\]
In order to keep the usual meaning of manifold and hence of diffeomorphism, we must cover our space by open charts (e.g. \( C_1 \) and \( C_2 \)) which on the overlap regions must be connected by transition functions which do not depend on the \( l_i \) i.e. on the metric \( \hat{g} \). Under this requirement it is impossible to extend continuously the metric \( \hat{g} \) on the chart \( C_1 \) and the analogous metric for the triangle \( T_2 \) on the chart \( C_2 \). Requiring compatibility of the metric on the overlapping regions severely restricts the consistent choice of the \( g_{\mu\nu} \) on each chart, as we will see below in the case of the sphere topology.

As done in the 2–dimensional continuum formulation we shall adopt the conformal gauge fixing \( g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu}(\tau_i) \). In the classical papers [3, 4], it has been proved that in such a gauge the functional integral in 2–dimensional gravity takes the form

\[
Z = \int \mathcal{D}[\sigma] \, d\tau_i \, \frac{\det'(L^\dagger L)}{\det(\Psi_i, \Psi_j) \det(\Phi_a, \Phi_b)}
\]

(3)

where

\[
(L\xi)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - g_{\mu\nu} \nabla^\rho \xi_\rho
\]

\[
(L^\dagger h)_{\nu} = -4 \nabla^\mu h_{\mu\nu}
\]

(4)

being \( \xi_\mu \) a vector field and \( h_{\mu\nu} \) a symmetric traceless tensor field. \( \tau_i \) are the Teichmüller parameters and \( \mathcal{D}[\sigma] \) is the functional integration measure induced by the distance

\[
(\delta\sigma^{(1)}, \delta\sigma^{(2)}) = \int \sqrt{\hat{g}} e^{2\sigma} \delta\sigma^{(1)} \delta\sigma^{(2)}
\]

(5)

and \( \Psi_i \) and \( \Phi_a \) are respectively the zero modes of \( L \) and \( L^\dagger \). The dependence on \( \sigma \) of the integrand in (3) can be factorized in the expression \( e^{-26S_L} \) where

\[
S_L[\sigma, \hat{g}(\tau_i)] = \frac{1}{24\pi} \int d^2x \, \sqrt{\hat{g}} \, [\hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + R_{\sigma}]
\]

(6)
is the Liouville action. With the topology of a sphere (to which we shall restrict hereafter) there are no Teichmüller parameters and consequently no zero modes of \( L^\dagger \). Instead there are 6 independent solutions of \((L\Psi)_{\mu\nu} = 0\) which are called conformal Killing vectors, i.e. generators of diffeomorphisms which leave unaltered the conformal structure of the metric. Thus for a universe with spherical topology eq.(3) reduces to

\[
Z_{\text{sp.}} = \int \mathcal{D}[\sigma] \sqrt{\frac{\text{det}'(L^\dagger L)}{\text{det}(\Psi_i, \Psi_j)}} = \int \mathcal{D}[\sigma] e^{-26S_L} \tag{7}
\]

If we try to evaluate \( S_L \) for a conformal factor describing a piecewise flat geometry we obtain a divergent result. Nevertheless \( \frac{\text{det}'(L^\dagger L)}{\text{det}(\Psi_i, \Psi_j)} \) can be defined also for a Regge surface by means of the \( Z \)-function regularization \[\text{[7]}\]. Our main goal will be the computation of this functional determinant for a Regge surface.

Following a procedure developed by Aurell and Salomonson \[\text{[8]}\] for the scalar Laplace–Beltrami operator on a piecewise flat two dimensional surface, we derived the exact expression for such a determinant. The first point is to produce a description of a piecewise flat surface with the topology of a sphere in terms of a conformal factor. The manifold is described by a single chart given by the projective plane and the conformal factor describing a given geometry is unique apart for the action of the diffeomorphisms generated by the 6 conformal Killing vectors, i.e. the \( SL(2, \mathbb{C}) \) transformations of the complex plane

\[
\omega' = \frac{a\omega + b}{c\omega + d}, \quad \omega = \frac{\omega'd - b}{-c\omega' + a}, \quad ad - bc = 1 \tag{8}
\]

under which the conformal factor \( \sigma \) changes as

\[
\sigma(\omega) \to \sigma'(\omega') = \sigma(\omega(\omega')) - 2 \ln |a - c\omega'|. \tag{9}
\]

For a Regge surface whose singularities have location \( \omega_i \) in the projective plane and angular aperture \( 2\pi\alpha_i \) (\( \alpha_i = 1 \) is the plane), the conformal factor takes the form \[\text{[9]}\] \( e^{2\sigma} = e^{2\lambda_0} \prod_i |\omega - \omega_i|^{2(\alpha_i - 1)} \) which in the neighborhood of \( \omega_i \) becomes \( e^{2\lambda_i}|\omega - \omega_i|^{2(\alpha_i - 1)} \) with \( e^{2\lambda_i} = e^{2\lambda_0} \prod_{j \neq i} |\omega_i - \omega_j|^{2(\alpha_j - 1)} \). In the conformal gauge \( L \) and \( L^\dagger \) assume a very simple form

\[
L = e^{2\sigma} \frac{\partial}{\partial \omega} e^{-2\sigma}, \quad L^\dagger = -e^{-2\sigma} \frac{\partial}{\partial \omega}. \tag{10}
\]

By means of the \( Z \)-function regularization we obtain

\[
- \ln(\text{det}'(L^\dagger L)) = Z'(s)|_{s=0} = \gamma_E Z(0) + \text{Finite}_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{dt}{t} \text{Tr}'(e^{-tL^\dagger L}) \tag{11}
\]

\[\text{[2]}\] We remark that the apparent singularity along an edge is only an extrinsic geometry property related to the usual representation of a Regge surface, while gravity depends only on the intrinsic geometry.
where \( \text{det}' \) and \( \text{Tr}' \) mean that the zero modes are excluded. The standard procedure is to compute the variation of \( Z'(0) \) under a variation of the conformal factor: to this end it is necessary to know the constant term in the asymptotic expansion of the heat kernel \( K(x, y, t) \) and \( H(x, y, t) \) of the operators \( L^\dagger L \) and \( LL^\dagger \). On the other hand the self adjoint extensions of \( L^\dagger L \) and \( LL^\dagger \) depend on the boundary condition one imposes on the eigenfunction at the singularities. The choice of Aurell and Salomonson for the Laplace–Beltrami operator is Dirichlet boundary conditions.

On the other hand the adoption of Dirichlet boundary condition for the operator \( L^\dagger L \) and \( LL^\dagger \) gives rise to a result which is not analytic in the angular opening of the conic singularity and does not agree with the continuum limit. This is due to the fact that Dirichlet boundary condition are equivalent to cutting off the tip of the cone. For this reason we considered a regularization which consists in replacing the tip of the cone by a segment of sphere or Poincaré pseudo–sphere and then letting the radius of curvature going to zero keeping constant the integrated curvature [10].

Such a limiting procedure can be carried through rigorously with the result that in such a limit new boundary conditions emerge, after which a well defined integral representation of the heat kernels \( K \) and \( H \) can be given for \( \frac{1}{2} < \alpha < 2 \). In particular by considering the constant terms \( c^K_0 = \sum_i c^K_0_i \) and \( c^H_0 = \sum_i c^H_0_i \) in the asymptotic expansions of the trace of \( H \) and \( K \), we obtain

\[
c^K_0_i = \frac{(\alpha_i - 1)(\alpha_i - 2)}{2\alpha_i} + \frac{1 - \alpha_i^2}{12\alpha_i} \quad (12)
\]

\[
c^H_0_i = \frac{(2\alpha_i - 1)(2\alpha_i - 2)}{2\alpha_i} + \frac{1 - \alpha_i^2}{12\alpha_i} \quad (13)
\]

where \( c^K_0_i \) and \( c^H_0_i \) are the contributions of a single singularity of opening angle \( \alpha_i \).

We notice that the \( c_0_i \) are analytic in \( \alpha_i \) and \( 2(c^K_0 - c^H_0) = 3(1 - \alpha_i) \). Thus for a generic compact surface without boundary such a relation gives

\[
2(c^K_0 - c^H_0) = 3 \sum_i (1 - \alpha_i) = 3\chi \quad (14)
\]

being \( \chi \) the Euler constant of the surface, in agreement with the Riemann–Roch theorem applied to \( L^\dagger L \) and \( LL^\dagger \) [4]. Performing a variation of the conformal factor we obtain from (11)

\[
-\delta \ln \frac{\text{det}'(L^\dagger L)}{\text{det}(\Psi_i, \Psi_j)} = \gamma_E \delta c^K_0 + \sum_i \left\{ (\delta \lambda_i - \lambda_i \frac{\delta \alpha_i}{\alpha_i})[4c^K_0_i - 2c^H_0_i] + \right. \]

\[
\left. + \text{Finite}_{\epsilon \to 0} \left[ 4 \frac{\delta \alpha_i}{\alpha_i} \int dx \ln(\alpha_i |x|) K_{\alpha_i}(x, x, \epsilon) - 2 \frac{\delta \alpha_i}{\alpha_i} \int dx \ln(\alpha_i |x|) H_{\alpha_i}(x, x, \epsilon) \right] \right\} . \quad (15)
\]

A differential of this structure [8] can be integrated to give

\[
\ln \sqrt{\frac{\text{det}'(L^\dagger L)}{\text{det}(\Psi_i, \Psi_j)}} = 26 \left\{ \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{\alpha_i} \ln |w_i - w_j| + \lambda_0 \sum_i (\alpha_i - \frac{1}{\alpha_i}) - \sum_i F(\alpha_i) \right\} . \quad (16)
\]
where \( F(\alpha) \) is given by a well defined and convergent integral representation. Such a formula has the following appealing features

1. It is an exact result giving the value of the F.P. determinant on a two dimensional Regge surface.

2. It is invariant under the group \( SL(2, \mathbb{C}) \) which acts on \( \omega_i \) and \( \lambda_0 \) as

\[
\begin{align*}
\omega_i \rightarrow \omega'_i &= \frac{a \omega_i + b}{c \omega_i + d} \\
\lambda_0 \rightarrow \lambda'_0 &= \lambda_0 + \sum_i (\alpha_i - 1) \ln |\omega_i c + d|
\end{align*}
\]

and leaves the \( \alpha_i \) unchanged.

3. In the continuum limit, i.e. small angular deficits \( 1 - \alpha_i \) and dense set of \( \omega_i \), the first two terms of (16) go over to the well know continuum formula

\[
\frac{26}{96 \pi} \left\{ \int dx \, dy \, (\sqrt{gR})_x \left( \frac{1}{\Box} (x, y) (\sqrt{gR})_y \right) - 2 \left( \ln \frac{A}{\bar{A}} \right) \int dx \, \sqrt{gR} \right\}
\]

as can be easily checked, where \( A \) is the area \( \int dx \, \sqrt{g} \) and \( \bar{A} \) is the area evaluated for \( \lambda_0 = 0 \). The remainder \( \sum_i F(\alpha_i) \) goes over to a constant topological term.

4. While \( \alpha_i > 0 \) with \( \sum_i (1 - \alpha_i) = 2 \), the \( \omega_i \) vary without restriction in the complex plane. This as pointed out by Foerster [9] is an advantage over the equivalent parameterization of the Regge surface in term of the bone length \( l_i \) where one has to keep into account of a large number of triangular inequalities.

We come now to the last piece appearing in (3) i.e. \( D[\sigma] \). Formally one can write

\[
D[\sigma] = \sqrt{\det(J)} \, d^2 \omega_1 \ldots d^2 \omega_N \, d\alpha_1 \ldots d\alpha_{N-1} \, d\lambda_0
\]

being \( J \) the determinant of the matrix \( J_{ij} = \left( \frac{\partial \sigma}{\partial x_i}, \frac{\partial \sigma}{\partial x_j} \right) \) where \( x_i \) represent the 3N variables \( \omega_{1x}, \ldots, \omega_{Nx}, \omega_{1y}, \ldots, \omega_{Ny}, \alpha_1, \ldots, \alpha_{N-1}, \lambda_0 \). It is immediately seen that the volume element (19) is invariant under \( SL(2, \mathbb{C}) \) as it must, due to the invariance of the conformal gauge fixing under the 6 conformal Killing vectors. The matrix \( J_{ij} \) can be given a very simple geometrical meaning as follows. Doubling the number of vertices \( x_i \rightarrow x'_i, x''_i \) and defining \( A(x', x'') \) as the area of the two dimensional surface described by \( x'_i, x''_i \), \( J_{ij} \) is given by \( J_{ij} = \frac{\partial^2 A}{\partial x'_i \partial x''_j} \bigg|_{x'=x''=x} \). Diagonal elements with indexes \( \omega_{ix} \) or \( \omega_{iy} \) diverge for positive curvature (i.e. for \( 1 - \alpha_i > 0 \)) and thus the analytic continuation of \( J_{ij} \) has to be considered (as usually done in conformal theories), which does not break the \( SL(2, \mathbb{C}) \) invariance. \( A(x', x'') \) and the elements \( J_{ij} \) are given by integrals which appeared in the old time conformal theory [1] and for which up to now no explicit form is known. In addition to the ultra–locality, an interesting feature of \( \det(J) \) is that it vanishes any time one \( \alpha_i \) equals 1 (no angular deficit); this is expected from the fact that for \( \alpha_i = 1 \) the value of the position \( \omega_i \) is irrelevant in determining the geometry.

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