ADAPTIVE OPTIMAL SIGNAL (CARDIOGRAM) PROCESSING, 
with boundary values and energy precisely measurement.

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ABSTRACT.

We construct an adaptive asymptotically optimal in order in the weight Hilbert space norms signal denoising on the background noise and its energy measurement, with high precision near the boundary of the signal.

An offered method used the Fourier-Riesz expansion on the orthonormal polynomials, for instance, Jacobi’s polynomials, relative unbounded near the boundary weight function.

An applications: technical and medical, in particular, cardiac diagnosis.

Key words and phrases: Adaptive estimations, signal processing, weight, functional, non-parametrical statistics, loss function, slowly and regular varying functions, minimax sense, Fourier-Riesz series, weight Hilbert spaces, orthonormal polynomial and system, Jacobi’s polynomials, recursion, confidence region, speed of convergence, statistical tests, energy of high order, diagnosis, cardiogram.

1 Introduction. Statement of problem

A mathematical model: the cardiogram \( f = f(x), x \in [-1, 1] \) is Riemann integrable unknown function observed in the points \( \{x(i) = x(i, n), i = 1, 2, \ldots, n\} \) on the background additive white noise \( \varepsilon(i) \), on the other terms: REGRESSION PROBLEM (R):

\[
\xi(i) = f(x(i)) + \sigma \cdot \varepsilon(i),
\]

(0)

where \( \{\varepsilon(i)\} \) is an additive noise; \( \varepsilon(i) \) are independent centered: \( \mathbf{E}\varepsilon(i) = 0 \) normed random variables (r.v.), \( \text{Var}[\varepsilon(i)] = 1 \), with unknown distribution; in particular, r.v. \( \varepsilon(i) \) not necessary to be Gaussian distributed; so that

\[
\sigma^2 = \text{Var}[\xi(i)] = \text{const} \in (0, \infty),
\]

\[
x(i) = x(i, n) = (2i - n - 1)/(n - 1), \quad (n \geq 15),
\]

\( f = f(x) \) be a Riemann integrable unknown function (on the other terms, signal, for instance, cardiogram).

Our aim is to offer the so-called adaptive optimal in order as \( n \to \infty \) in the modular Hilbert space sense estimation of a function (signal) \( f = f(x) \) and its energy of arbitrary order with a great precision especially near the boundaries \( x = \pm 1 \).
In the ordinary Hilbert spaces this problem is considered in many publications; see, for instance, [1], [2], [3], [4], [7], [8], [11], [15] and reference therein.

2 Notations and definitions

1. Modular Hilbert spaces. Let $X = [-1, 1], x \in X, \Phi = \{\phi_k(x)\}, k = 1, 2, \ldots$ be some fixed complete bounded:

$$\sup_k \sup_{x \in [-1,1]} |\phi_k(x)| < \infty,$$

uniform Lipschitz continuous:

$$|\phi_k(x) - \phi_k(y)| \leq C \cdot k \cdot |x - y|$$

orthonormal polynomial system relative some non-trivial non-negative integrable weight signal function $\gamma = \gamma(x)$:

$$(\phi_k, \phi_l)_\gamma := \int_{-1}^1 \phi_k(x)\phi_l(x)\gamma(x)dx = \delta(k, l),$$

$\delta(k, l)$ is the Kroneker’s symbol;

$$\deg \phi_k = k - 1, \ k = 1, 2, \ldots.$$  

We impose henceforth the following conditions on the weight function $\gamma(x)$: $\gamma(x) \geq 0,$

$$K(\gamma) \overset{df}{=} (2\pi)^{-1} \int_{-1}^1 \frac{\gamma(x)}{\sqrt{1 - x^2}} dx = (2\pi)^{-1} \int_{-\pi}^{\pi} \gamma(\cos \phi) d\phi < \infty; \quad (1.a)$$

$$\int_{-1}^1 \log(\gamma(x)) \frac{dx}{\sqrt{1 - x^2}} > -\infty; \quad (1.b)$$

$$\exists \kappa = \text{const} < 1/2, \ \forall x \in (-1, 1) \ \Rightarrow |\gamma'(x)| \leq C \left[(x + 1)^{-\kappa - 1} + (x - 1)^{-\kappa - 1}\right]. \quad (1.c)$$

We redefine the weight function $\gamma = \gamma(x)$ at the boundary points $x = \pm 1$ as follows:

$$\gamma(-1) = \gamma(1) = 0.$$  

The conditions (1.a), (1.b) and (1.c) are satisfied, e.g. for the weight function

$$\gamma_{\alpha, \beta}(x) = (1 - x)^\alpha(1 + x)^\beta, \ \alpha, \beta > -1/2, \ x \in (-1, 1); \gamma_{\alpha, \beta}(\pm 1) = 0,$$

for the classical (normed!) Jacobi’s polynomial, which we will denote

$$\phi_k^{(\alpha, \beta)}(x) : \deg(\phi_k^{(\alpha, \beta)}(x)) = k - 1,$$

$$\int_{-1}^1 \phi_k^{(\alpha, \beta)}(x)\phi_l^{(\alpha, \beta)}(x)\gamma_{\alpha, \beta}(x)dx = \delta(k, l).$$

As long as we need to estimate with great precision the behavior of unknown function $f(x)$ near the points $x = \pm 1$, we will suppose further that
\[ \alpha, \beta \in (-1/2, 0]. \] (2)

Note that
\[
K(\gamma_{\alpha, \beta}) = 2^{\alpha+\beta} \pi^{-1} B(\alpha + 1/2, \beta + 1/2),
\]
obviously when \( \alpha, \beta > -1/2; \) \( B(\alpha, \beta) \) denotes ordinary Beta- function.

Many used for us information about orthogonal polynomials see in [5], [13], [14]. For instance, there are described the recurrence relations of a view
\[
\phi_{k+2}(x) = (A(k; \Phi) x + B(k; \Phi))\phi_{k+1}(x) + C(k; \Phi)\phi_k(x), k = 0, 1, 2, \ldots,
\]
with initial conditions
\[
\phi_0(x) = \left[ \int_{-1}^{1} \gamma(z) \, dz \right]^{-1/2} =: C_0 = C_0(\gamma),
\]
\[
\phi_1(x) = \lambda(x - C_1),
\]
where
\[
C_1 = C_1(\gamma) = \frac{\int_{-1}^{1} x\gamma(x) \, dx}{\int_{-1}^{1} \gamma(x) \, dx},
\]
\[
\lambda^{-2} = \lambda^{-2}(\gamma) = \int_{-1}^{1} (x - C_1)^2 \gamma(x) \, dx.
\]
For instance,
\[
\int_{-1}^{1} \gamma_{\alpha, \beta}(x) \, dx = 2^{\alpha+\beta+1} B(\alpha + 1, \beta + 1),
\]
\[
C_1(\gamma_{\alpha, \beta}) = \frac{\beta - \alpha}{\alpha + \beta + 2},
\]
\[
\lambda^{-2}(\gamma_{\alpha, \beta}) = 2^{\alpha+\beta+1}B(\alpha + 1, \beta + 1) \times \frac{3\alpha^2\beta + 3\alpha\beta^2 + 3\alpha^2 + 3\beta^2 + 16\alpha\beta + 14\alpha + 14\beta + 12}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2}.
\]

These recurrent relations with correspondent initial conditions may be used for the programming of offered algorithms.

**Remark 1.** By our opinion, it is convenient for the practical using to choose the interval (0,1) instead (-1,1) and the weight function
\[
\gamma(x) = \frac{x^{\alpha-1}(1 - x)^{\beta-1}}{B(\alpha, \beta)}, \quad x \in (0, 1);
\]
\[
\gamma(0) = \gamma(1) = 0; \quad \alpha, \beta > 0; \quad \text{i.e. } \gamma(x) \text{ is the density of the well-known Beta distribution.}
\]
Then \( \phi_0(x) = 1, \)
\[
\phi_1(x) = \frac{(\alpha + \beta)x - \alpha}{\sqrt{\alpha\beta/(\alpha + \beta + 1)}}.
\]
The useful values of the parameters $\alpha, \beta$ are following:

$$\alpha, \beta \in (1/2, 1].$$

For arbitrary integrable function $f : [-1, 1] \to R$ we put

$$f(x) = \sum_{k=1}^{\infty} c(k) \phi_k(x);$$

here

$$c(k) = \int_{-1}^{1} f(x) \phi_k(x) \gamma(x) \, dx$$

be the Fourier-Riesz coefficients of a function $f(\cdot)$ over the system $\Phi$.

Let $w = w(k), k = 1, 2, \ldots$ be some positive function (sequence), for example, $w(k) = k^m, m \geq 0$. The value of $m$ is said to be the order of energy; not necessary to be integer.

We will called the function $w = w(k)$ the energy weight, in contradiction to the notion signal weight $\gamma = \gamma(x)$.

We introduce the following Hilbert modular space $B = B(\Phi, w)$, which will called modular Hilbert weight space, consisting on all the (measurable) functions $\{f\}$ with finite quadratic functional

$$W(w)[f] = W(\Phi, w)[f] \overset{df}{=} \sum_{k=1}^{\infty} w(k) |c(k)|^2. \quad (3)$$

The Hilbert’s norm in this space may be defined as usually:

$$||f||W(w) = \sqrt{W(w)[f]}.$$}

For instance, $W(w)[f]$ may be integral on the square of derivative of an arbitrary order of the signal $f(\cdot)$:

$$W(w)[f] = \int_{-1}^{1} [f^{(r)}(t)]^2 \, dt$$

or its weight modification:

$$W(w)[f] = \int_{-1}^{1} \zeta(t) [f^{(r)}(t)]^2 \, dt$$

In the case when $w(k) = w(k) = w_\theta(k) \simeq k^\theta, \theta = \text{const} \geq 0$ we will denote $W(\Phi, w_\theta) = W_\theta(\Phi) = W_\theta$.

Evidently, if $w(k) = w_0(k) = 1$, then

$$\sum_{k=1}^{\infty} c_k^2 = ||f||^2(\Phi, w_0) = (f, f)_\gamma.$$

For instance, the Sobolev’s spaces $W^m_2 = W^m_2[-1, 1], m = 0, 1, 2, \ldots$ consisting on all the periodical (mod 2) $m-$ times differentiable functions $f$ with finite norm

$$||f||^2W^m_2 = [(||f||L_2[-1, 1])^2 + (||f^{(m)}||L_2[-1, 1])^2]^{1/2}$$
is weighted Hilbert modular space relative the trigonometric system $T$.

The functional

$$W(f) = W(f; \Phi, w) = ||f||^2 W(\Phi, w)$$

will be called Energy or more exactly $w$-Energy of the signal $f$. We denote any estimation of $W(f)$ as $W(n, f)$.

2. Loss functions. We define the following loss function $L = L(g(n, \cdot), f(\cdot))$ for an arbitrary estimation of the function $f$ (more exactly, the sequence of estimations) $g(n, x) = g(n, x; \{\xi(i)\})$ as

$$L_f(g(n, \cdot), f(\cdot)) = L_f(\Phi) \overset{\text{def}}{=} E ||g(n, \cdot) - f(\cdot)||^2.$$  (4.a)

We denote as for the arbitrary energy estimation $W(n, f)$ as $M(W(n, f), W(f))$ the following loss function (functional):

$$Z(W(n, f), W(f)) \overset{\text{def}}{=} E(W(n, f) - W(f))^2.$$  (4.b)

3. Denotations. We denote:

$$S(N; m) = \sum_{k=1}^{N} w^m(k), \quad S(N) = S(N, 1) = \sum_{k=1}^{N} w(k);$$

$$\rho_w(N; f) = \rho_w(N) = \sum_{k=N+1}^{\infty} w(k) |c(k)|^2, \quad \rho(N) = \sum_{k=N+1}^{\infty} |c(k)|^2;$$  (5)

then

$$f \in W(\Phi, w) \Leftrightarrow \lim_{N \to \infty} \rho(N; f, 1) = 0.$$ 

3 Adaptive signal estimation: denoising

1. Offered estimations. Let us denote the truncated Fourier-Riesz’s sum for the function $f(x)$ as follows:

$$f_{(N)}(x) = \sum_{k=1}^{N} c(k) \phi_k(x),$$

and correspondingly

$$f_{(N,n)}(x) = \sum_{k=1}^{N} c_n(k) \phi_k(x),$$

where

$$c_n(k) = n^{-1} \sum_{i=1}^{n} f(x(i, n)) \cdot \phi_k(x(i, n)) \cdot \gamma(x(i, n)).$$
The non-random variables \( c_n(k) \) are usually consistent approximations of a Fourier (or Fourier-Riesz) coefficients \( c(k) \); they are Riemann integral sums for the function \( f(x)\phi_k(x)\gamma(x) \).

Let us consider a so-called projection, or Tchentzov’s estimation \( f(N, n, x) \) of the function \( f(\cdot) \) of a view

\[
f(N, n, x) = \sum_{k=1}^{N} c(k, n)\phi_k(x).
\]  

(6a)

where random variables \( c(k, n) \) are usually consistent estimations of a Fourier coefficients \( c(k) \) based on the data \( \xi(i) \), \( i = 1, 2, \ldots, n \), namely, for the considered regression problem \( \mathbf{R} \)

\[
c(k, n) = n^{-1} \sum_{i=1}^{n} \xi(i) \cdot \phi_k(x(i, n)) \cdot \gamma(x(i, n)).
\]  

(6b)

We denote also

\[
A(N, n) = \rho(N) + \frac{N}{n}.
\]

Notice that

\[
A(N, n) \asymp \mathbf{E}[|f(N, n, \cdot) - f(\cdot)|^2]_{\gamma}, \ N = 1, 2, \ldots, \lfloor n/3 \rfloor;
\]

\[
A^*(n) := \min_{N \in [1, n/3]} A(N, n); \ N_0 := \arg\min_{N \in [1, n/3]} A(N, n).
\]  

(7)

We assume in addition that

\[
\lim_{N \to \infty} \frac{N_0}{\sqrt{n}} = 0.
\]  

(7a)

This condition guarantee us that

\[
\lim_{n \to \infty} \frac{|f_{N_0,n} - f_{N_0}(x)|^2}{A^*(n)} = 0.
\]  

(7b)

For example, if \( \gamma(x) = 1 \) and \( \rho(N) \asymp N^{-2\Delta}, N \to \infty \), then the last condition implies the well-known inequality \( \Delta > 1/2 \), see, e.g. [6], [8].

Analogous conclusion is true when \( \rho(N) \asymp N^{-2\Delta}L(N) \), where \( L(N) \) is slowly varying as \( N \to \infty \) function.

The function \( N \to N^{-2\Delta}L(N) \) is called regular varying as \( N \to \infty \) function. Used for us facts about slowly and regular varying function see in the classical monograph of E.Seneta [12].

The condition (7a) holds also when \( \rho(N) \asymp N^{-2\Delta}L(N) \), \( \Delta > 1/2 \), and if the signal weight function \( \gamma(x) \) satisfied the conditions (1a),(1b) and (1c).

Further, we introduce the following important (random) statistics (functionals):

\[
\tau(N, n) = \sum_{k=N+1}^{2N} |c(k, n)|^2; \ M = M(n) = \arg\min_{N \in [1, n/3]} \tau(N, n),
\]

(8)

\[
\tau^*(n) = \min_{N \in [1, n/3]} \tau(N, n) = \tau(M(n), n)
\]

(9)
and we introduce our estimation \( \hat{f}(n, x) \) of a function \( f(x) \) as follows:

\[
\hat{f}(n, x) = f(M(n), n, x) = \sum_{k=1}^{M(n)} c(k, n) \phi_k(x).
\] (10)

Note that \( M(n) \) is a random sequence which depended only on the source sample \( \{\xi(i)\}, i = 1, 2, \ldots, n \) (adaptivity).

2. We suppose also

\[
0 < \lim_{N \to \infty} \rho(2N)/\rho(N) \leq \lim_{N \to \infty} \rho(2N)/\rho(N) < 1. \tag{11a}
\]

Another case, i.e. when, e.g.,

\[
\rho(2N)/\rho(N) > 0, \quad \lim_{N \to \infty} \rho(2N)/\rho(N) = 0 \tag{12}
\]

may be investigated analogously.

3. Obviously, in the considered here regression problem \( R \) we must impose some conditions on the errors of measurements \( \epsilon(i) \). We suppose in addition that the centered and normed variables \( \{\epsilon(i)\} \) satisfy the following condition: there exist a finite positive constants \( q, Q \) such that for all positive values \( u \)

\[
\max [P(\epsilon > u), P(\epsilon < -u)] \leq \exp \left(-\frac{u}{Q}\right). \tag{13a}
\]

It may be assumed in many practical cases that the r.v. \( \{\epsilon(i)\} \) are Gaussian distributed; in this case the inequality (13a) holds with parameters \( q = 2 \) and \( Q = 1 \).

4 Main result

If we choose \( N = N_0 \), then we conclude that the optimal speed of convergence for the non-adaptive estimation is asymptotically \( A^*(n) \) : as \( n \to \infty \)

\[
E||f(n, N_0, \cdot) - f(\cdot)||^2 \asymp A^*(n). \tag{14}
\]

At the same result is true for our adaptive estimation.

Recall that we suppose the conditions (1a), (1b), (1c), (7a), (11a) and (13a) to be satisfied.

**Theorem 1.** Let \( K \) be arbitrary regular function space with a norm \( ||\cdot||_K \) compactly embedded in the space \( B = B(\Phi, w) \). We denote the minimax value of the loss function as \( A^*(n) = A^*(n, K, Z) \), \( Z = \text{const} \in (0, \infty) \) :

\[
A^*(n, K, Z) \overset{\text{def}}{=} \inf_{g(n, \cdot)} \sup_{f \in K, ||f||_K \leq Z} L(g(n, \cdot), f(\cdot)), \tag{15}
\]

where ” inf ” is calculated over all, including even non adaptive estimations of \( f(\cdot) \).

We assert that under conditions for adaptive estimation our estimation \( f(n, \cdot) \) is asymptotically as \( n \to \infty \) optimal up to finite multiplicative constant in the minimax sense on the arbitrary regular function classes \( K \) compactly embedded in the modular space \( W = W(\Phi, w) \):
\[ L(\hat{f}(n, \cdot), f(\cdot)) \leq C(\gamma, \Phi, K, Z) \cdot A^*(n, K, Z), \quad C(\gamma, \Phi, K, Z) < \infty. \quad (16) \]

5. **Variance estimation.** Note that the consistent \( 1/\sqrt{n} \) estimation \( \sigma^2(n) \) of the value \( \sigma^2 \) may be constructed by formulae \([8], [11]\)

\[ \sigma^2(n) = (n - M(n) - 1)^{-1} \sum_{i=1}^{n} \left( \hat{f}(n, x(i, n)) - \xi(i) \right)^2. \quad (17) \]

6. **Confidence region.** For the rough building of the confidence domain, also adaptive, in the \( B(\Phi, \gamma) \) norm we proved the following result.

**Theorem 2.** We assert under at the same conditions as in theorem 1 that there exist positive finite non-random constructive and estimative constants \( Y_j = Y_j(\gamma, \Phi, K, Z) \), \( j = 1, 2 \) and for the problem \( (R) \) \( Y_j = Y_j(\gamma, \Phi, K, Z; q, Q) \) such that

\[ \lim_{n \to \infty} \tau^*(n)/A^*(n, K, Z) = Y_1 \quad (18) \]

and correspondingly

\[ \lim_{n \to \infty} ||\hat{f}(n, \cdot) - f(\cdot)||_\gamma^2/\tau^*(n) = Y_2. \quad (19) \]

Therefore, we conclude: with probability tending to one as \( n \to \infty \)

\[ ||\hat{f}(n, \cdot) - f(\cdot)||_\gamma^2 \leq 1.05 \cdot Y_2 \cdot \tau^*(n). \quad (20) \]

More exact computation show us that

\[ \mathbb{E}||\hat{f}(n, \cdot) - f(\cdot)||_\gamma^2 \sim \mathbb{E}\tau^*(n), \quad (21) \]

\[ \text{Var}||\hat{f}(n, \cdot) - f(\cdot)||_\gamma^2 \sim 4\sigma^2 K^2(\gamma)\rho(M(n))/n + 3\sigma^4(n)K^4(\gamma)M(n)/n^2 \overset{def}{=} \delta^2(n), \quad (22) \]

where

\[ \tau^*(n) \sim \rho(M(n)) + \sigma^2(n)K^2(\gamma)M(n)/n. \quad (23) \]

Therefore, with probability tending to 0.95

\[ ||\hat{f}(n, \cdot) - f(\cdot)||_\gamma^2 \leq \tau^*(n) + 2.54 \cdot \delta(n). \quad (24) \]

## 5 Energy estimation

In order to detect a distortion in the signal, for instance, infarction, heart attack, apoplexy in the cardiodiagnostics; leakage in the pipelines, in particular, in the nuclear reactors we offer to estimate its energy and comprise with it standard value up to confidence domain.

1. **Ordinary energy.**

Let us define for simplicity the so-called ordinary energy functional
\[ G = G(f) = \sum_{k=1}^{\infty} c^2(k) = ||f||_2^2. \]  
\[ (25) \]

More exactly, we intend to consistent estimate (consistently) the truncated energy functional of a view

\[ G = G(f) = \sum_{k=1}^{N_0(n)} c^2(k) = ||P_{N_0(n)} f||_2^2, \]
\[ (26) \]

where \( P_{N_0(n)} f \) denotes the projection of the function \( f(\cdot) \) onto the subspace generated by the first \( N_0(n) \) polynomials \( \phi_k(\cdot) \), \( k = 1, 2, \ldots \).

But it is evident that asymptotically as \( n \to \infty \) \( G \to G(f) \).

If we define as a first approximation the following estimation for the functional \( G \) the following expression

\[ G_0(n) = \sum_{k=1}^{M(n)} c^2(k, n), \]
\[ (27) \]
then we obtain

\[ G_0(n) \sim \sum_{k=1}^{M(n)} (c(k) + \sigma K(\gamma) \eta(k)/\sqrt{n})^2, \]
\[ (28) \]

\[ \mathbb{E}G_0(n) \sim \sum_{k=1}^{N_0(n)} c^2(k) + \sigma^2 K^2(\gamma) N_0(n)/n, \]
\[ (29) \]

therefore an asymptotical non-biased estimate of a \( G = G(f) \) based on the \( G_0(n) \) has a view

\[ G(n) := \sum_{k=1}^{M(n)} c^2(k, n) - M(n) K^2(\gamma) \sigma^2(n)/n + \tau^*(n). \]
\[ (30) \]

**Theorem 3.** As \( n \to \infty \) the r.v. \( G(n) \) has Gaussian distribution with parameters

\[ G(n) \sim N(G, 4\sigma^2 K^2(\gamma)G/n). \]
\[ (31) \]

2. **Energy of general order.** Let us define

\[ W_w(f; n, N) = \sum_{k=1}^{N} w(k) \left[ c^2(k, n) \right] - n^{-1} K^2(\gamma) \sigma^2(n) + \tau^*_w(n). \]
\[ (32) \]

We consider in this pilcrow only the case \( w(k) = w_0(k) \propto k^{\theta}, \theta = \text{const} > 0, k \to \infty \); the case when \( w(k) \) is regular varying as \( k \to \infty \) may be considered analogously.

We find by direct calculation under assumption \( f \in W_{w^2} : \)

\[ Z(W_w(f; n, N), W_w(f)) \asymp n^{-1} + \rho_{w^2}(N) + n^{-2} N^{2\theta + 1}. \]
\[ (33) \]

If we choose
\[ N_w = \arg\min_N \left( \rho_w^2(f; N) + N^{2\theta+1}/n^2 \right), \]

we obtain the optimal in order non-adaptive as \( n \to \infty \) estimation of the energy \( W_w(f) \) of a view \( \hat{W}_w(f) = W_w(f; n, N_w). \)

For example, if \( \rho_w(f; N) \asymp N^{-2\Delta}, \Delta = \text{const} > 0 \), then
\[ N_w = \text{Ent}[n^{1/(2\Delta+\theta+1/2)}], \]

where \( \text{Ent}[z] \) denotes the integer part of the number \( z \), and
\[ Z(W_w(f; n, N_w), W_w(f)) \asymp \max \left( n^{-1}, n^{-4\Delta/(2\Delta+2m+1/2)} \right). \]

Notice that in the case when \( \Delta \geq 0.5\theta + 1/4 \)
\[ Z(W_w(f; n, N_w), W_w(f)) \asymp n^{-1}, \]
\[ (34) \]
i.e. optimality in order as \( n \to \infty \).

In order to construct an adaptive estimation of the variable \( W_w(f) \), we introduce the following statistics:
\[ \tau_w(N) = \tau_w(N, n) = \sum_{k=N+1}^{2N} w(k) c^2(k, n), \]
\[ \hat{N}_w = \arg\min_{N \leq n/3} \tau_w(N), \]
and introduce the following adaptive estimation (measurement) of energy \( W_w[f] \):
\[ \hat{W}_w = W_w(f; \hat{N}_w, n). \]

**Theorem 4.** Suppose \( f \in W_{2\theta} \) and
\[ \rho_w(f; N) \asymp N^{-2\Delta} L(N), \Delta = \text{const} > 0, \]
where as before \( L = L(N) \) is slowly varying as \( N \to \infty \) function. Proposition:
\[ Z \left( \hat{W}_w, W_w \right) \leq K_4(\Delta, \theta) \cdot \max \left( n^{-1}, n^{-4\Delta/(2\Delta+\theta+1/2)} \right). \]  
\[ (35) \]

Notice that if \( \Delta \geq \theta/2 + 1/2 \), then
\[ Z \left( \hat{W}_w, W_w \right) \leq K_5(\Delta, \theta) \cdot n^{-1}, \]  
\[ (36) \]
(optimality in order as \( n \to \infty \) of offered adaptive energy estimation).
6 Sketch of proofs.

1. Denoising. We have the expansion:

\[ c(k, n) = n^{-1} \sum_{i=1}^{n} \xi(i) \phi_k(x(i, n)) \gamma(x(i, n)) = n^{-1} \sum_{i=1}^{n} \phi_k(x(i, n)) \gamma(x(i, n)) f(x(i, n)) + \]

\[ n^{-1} \epsilon(i) \gamma(x(i, n)) \phi_k(x(i, n)) \sim c(k) + n^{-1/2} \sigma \eta(k). \]  

(37)

The r.v. \{\eta(k)\} are centered, asymptotically independent and normal distributed with variance as \(n \to \infty\)

\[ \text{Var}[\eta(k)] = n^{-1} \sum_{i=1}^{n} \phi_k^2(x(i, n)) \gamma^2(x(i, n)) \to \int_{-1}^{1} \phi_k^2(x) \gamma^2(x) \, dx. \]  

Further, as \(k \to \infty\)

\[ \text{Var}[\eta(k)] \to \int_{-1}^{1} \frac{\gamma(x)}{\sqrt{1-x^2}} \, dx = 2\pi K(\gamma). \]  

(38)

On the other words, we get as a first approximation the following expression:

\[ c(k, n) \sim c(k) + C_1(\gamma) \sigma n^{-1/2} \zeta(k), \]

where \{\zeta(k)\} are independent and standard normal distributed r.v.

We find by direct calculations:

\[ A(N, n) \asymp \rho(N) + n^{-1} N; \]  

(39)

\[ \mathbb{E}[\tau(N, n)] \asymp \rho(N) + n^{-1} N = A(N, n); \]  

(40)

\[ \text{Var}[\tau(N, n)] \asymp n^{-2} N + n^{-1} \rho(N). \]  

(41)

We conclude by virtue of adaptive conditions that for some positive constant \(C = C(\gamma)\)

\[ \text{Var}[\tau(N, n)] \leq C n^{-1} \cdot A(N, n). \]

Further considerations are alike to the [1], [10], chapter 5, section 13.

Notice that with probability one

\[ \min_{N \in [1, n]} \tau(N, n) \asymp A^*(n), \quad n \to \infty. \]  

(42)

see [1], [10], chapter 5, section 13.

2. Ordinary energy estimation. We have

\[ G(n) - G = \frac{2\sigma K(\gamma)}{\sqrt{n}} \sum_{k=1}^{M(n)} c(k) \eta(k) + \frac{\sigma^2 K^2(\gamma)}{n} \sum_{k=1}^{M(n)} (\eta^2(k) - 1) \sim \]

\[ \frac{2\sigma K(\gamma)}{\sqrt{n}} + \frac{\sigma^2 K^2(\gamma)}{n} n^{-1/2} \sigma \eta(k). \]  

\[ \sim \frac{2\sigma K(\gamma)}{\sqrt{n}} \sum_{k=1}^{N_0(n)} c(k) \eta(k). \]

Therefore

\[ \text{Var}[G(n) - G] \sim 4\sigma^2 K^2(\gamma)n^{-1} \sum_{k=1}^{N_0(n)} c^2(k) \sim 4\sigma^2 K^2(\gamma)n^{-1} \sum_{k=1}^{\infty} c^2(k) = 4\sigma^2 K^2(\gamma)G/n. \quad (43) \]

The asymptotical normality follows from the classical CLT.

3. We prove here the implication \((7a) \rightarrow (7b)\).

Denote by \(F_n(x)\) the empirical function of distribution for the sequence \(x(i); \ F(x) := 0.5(x + 1), \ x \in [-1, 1]\); obviously,

\[ \sup_{x \in [-1, 1]} |F_n(x) - F(x)| \leq 2/n. \]

Further, put

\[ \sigma_{n,k} = \int_{-1}^{1} [f(x)\phi_k(x)\gamma(x)]d(F_n(x) - F(x)), \]

\[ \Sigma(n) = ||f(N) - f(N,n)||_\gamma^2; \]

then

\[ \Sigma(n) = \sum_{k=1}^{N_0} \sigma_{n,k}^2. \]

We can and will assume without loss of generality that the function \(f(\cdot)\) has a bounded derivative.

We have for arbitrary values \(\epsilon \in (0, 1/2)\) :

\[ |\sigma_{n,k}| \leq \int_{-1}^{-1+\epsilon} [f(x)\phi_k(x)\gamma(x)]d(F_n(x) - F(x)) + \int_{-1+\epsilon}^{1-\epsilon} [f(x)\phi_k(x)\gamma(x)]d(F_n(x) - F(x)) + \int_{1-\epsilon}^{1} [f(x)\phi_k(x)\gamma(x)]d(F_n(x) - F(x)) = \sigma_{n,k}^{(1)} + \sigma_{n,k}^{(2)} + \sigma_{n,k}^{(3)}; \]

\[ |\sigma_{n,k}^{(1)}| \leq C_1 \int_{-1}^{-1+\epsilon} \gamma(x)dx \leq C_2 \epsilon^{1-\kappa} \]

and analogously may be estimated the value \(\sigma_{n,k}^{(3)}\).

Further,

\[ |\sigma_{n,k}^{(2)}| \leq C_3 \left[ \sigma_{n,k}^{(4)} + \sigma_{n,k}^{(5)} \right], \]
where
\[
\sigma_{n,k}^{(4)} \overset{\text{def}}{=} \int_{1-\epsilon}^{1} |F_n(x) - F(x)||\phi_k(x)||\gamma(x)| \, dx \leq C_4 k/n;
\]
\[
\sigma_{n,k}^{(5)} \overset{\text{def}}{=} \int_{1-\epsilon}^{1} |F_n(x) - F(x)||\phi_k(x)||\gamma'(x)| \, dx \leq C_5 \epsilon^{-\kappa}/n.
\]

Therefore,
\[
\sigma_{n,k} \leq C_6 \left[ \frac{k}{n} + \frac{\epsilon^{-\kappa}}{n} + \epsilon^{1-\kappa} \right].
\]

We conclude choosing \( \epsilon = 0.25/n \):
\[
\sigma_{n,k}^2 \leq C_7 \left[ \frac{k^2}{n^2} + \frac{1}{n^{2(1-\kappa)}} \right];
\]
\[
\Sigma_{N_0} \leq C_8 \left[ \frac{N_0^3}{n^2} + \frac{N_0}{n^{2(1-\kappa)}} \right]
\]

and following
\[
\lim_{n \to \infty} \frac{\Sigma_{N_0}}{A^2(n)} \leq \lim_{n \to \infty} \frac{\Sigma_{N_0}}{N_0/n} = 0.
\]

4. **The proof of theorem 4** in completely analogous; see, for example, [4] [7].

Indeed, we have as \( n \to \infty \)
\[
\hat{W}_w(n) - W_w \sim -\rho_W(N_w) + \frac{2\sigma K(\gamma)}{\sqrt{n}} \sum_{k=1}^{N_w(n)} c(k)\eta(k)w(k) + \frac{\sigma^2 K^2(\gamma)}{n} \sum_{k=1}^{N_w(n)} w(k)(\eta^2(k) - 1),
\]
therefore
\[
\mathbb{E}(\hat{W}_w(n) - W_w)^2 \sim \rho_W^2(N_w),
\]
\[
\text{Var}(\hat{W}_w(n) - W_w) \sim \frac{4\sigma^2 K^2(\gamma)}{n} \sum_{k=1}^{N_w} c^2(k)w^2(k) + \frac{3\sigma^4 K^4(\gamma)}{n^2} \sum_{k=1}^{N_w} w^2(k);
\]

and we get under formulated above conditions, in particular, \( S_w^2(f) < \infty \):
\[
Z(\hat{W}_w(n), W_w) \leq C \left[ \rho_W^2(N_w) + n^{-1} + S_w^2(f)/n^2 \right]
\]

etc.
7 Concluding remarks, applications

0. The optimality in order as $n \to \infty$ of described above signal denoising was proved by I.A.Ibragimov; see also the monograph [15], chapter 4.

1. We conclude making the so-called Fisher’s transform for the ordinary energy estimation that

$$
\sqrt{G(n)} \sim N(\sqrt{G}, 4\sigma^2 K^2(\gamma)/n).
$$

This fact allows us to built a confidence interval for the value $G$. For example, with probability $\approx 0.95$

$$
|\sqrt{G} - \sqrt{G(n)}| \leq \frac{6\sigma(n) K(\gamma)}{\sqrt{n}}.
$$

2. Let us return to the problem of diagnosis, for example, cardiodiagnostics. The offered estimations with corresponding confidence regions allow us to detect the distortions for obtained signal, for instance, in the cardiogram.

Namely, if in the current time interval the energy value does not belongs to the confidence region (or does not belongs to this region twice or three times,) we can conclude the presence of some distortion in the signal, in particular, infarction etc.

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