Modified Chaplygin Gas Cosmology from Geometrothermodynamics

Hachemi B. Benaoum and Hernando Quevedo

1 Department of Applied Physics and Astronomy, University of Sharjah, United Arab Emirates
2 Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, Ciudad de México 04510, Mexico
3 Dipartimento di Fisica and ICRANet, Università di Roma “La Sapienza”, I-00185 Roma, Italy
4 Institute of Experimental and Theoretical Physics, Al-Farabi Kazakh National University, Almaty 050040, Kazakhstan

Abstract

The modified Chaplygin gas (MCG) cosmological model is derived by using the geometrothermodynamics (GTD) formalism. We show that the MCG corresponds to a system with internal thermodynamic interaction and describes the current accelerated expansion of the universe. Moreover, we also show that the MCG can be interpreted as a unified model for dark energy and dark matter that is perfectly consistent with the current SNe observations and CMB anisotropy measurements.

*Electronic address: hbenoum@sharjah.ac.ae, quevedo@nucleares.unam.mx
I. INTRODUCTION

A growing number of observational data indicate that the observable universe is undergoing a phase of accelerated expansion [1]-[10]. The most popular explanation for these unexpected observations states that the source of this cosmic acceleration is due to an unknown dark energy component with a negative pressure, which dominates the universe at recent cosmological time. Several mechanisms have been proposed to describe the physical nature of this dark energy component; among them, the single-component fluid known as Chaplygin gas has attracted a lot of interest in recent times [11]-[27]. Many variants of the Chaplygin gas model have been proposed in the literature. A further general model named modified Chaplygin gas (MCG) has been introduced by Benaoum [28] and obeys the following equation of state:

\[ p = A\rho - B \rho^\alpha \]  

where \( A, B \) and \( \alpha \) are universal positive constants.

The equation of state with \( A = 0 \) and \( \alpha = 1 \) known as Chaplygin gas (CG) was first introduced by Chaplygin to study the lifting forces on a plane wing in aerodynamics. Its generalized form with \( A = 0 \) and \( \alpha > 0 \) is known as the generalized Chaplygin gas (GCG) was first introduced by Kamenshchik et al. [29] and Bento et al. [30].

The MCG has the remarkable property of describing the dark sector of the universe (i.e. dark energy and dark matter) as a single component that acts as both dark energy and dark matter. It interpolates from a matter-dominated era to a cosmological constant-dominated era. It is well known that the CG, MCG and theirs variants have been extensively studied in the literature [31]-[33]. Several papers discussing various aspects of the behavior of MCG to reconcile the standard cosmological model with observations have been considered [36]-[39].

Geometrothermodynamics (GTD) is a formalism that has been developed during the past few years to describe ordinary thermodynamics by using differential geometry [40]. This formalism has found diverse applications in black hole thermodynamics [41], relativistic cosmology [42], mathematical chemistry [43], and others.
In this work we derive the MCG cosmological model by means of GTD. In particular, we combine thermodynamic and geometric considerations in the GTD framework, and find out the equation of state of the MCG which can explain the recent cosmological observations. This work is organized as follows. In Section 2, we briefly review the fundamentals of GTD. Then, in Section 3, we present the MCG cosmological model that follows from a GTD system with thermodynamic interaction. In Section 4, we study the MCG cosmological model by using Type Ia supernovae observational data from Union 2.1 compilation. Finally, in Section 5, we discuss our results.

II. GEOMETROTHERMODYNAMICS

Geometrothermodynamics is a differential geometric formalism for thermodynamics. The starting point of this formalism is a \((2n + 1)\)-dimensional contact manifold \(T\), called phase space, which is endowed with a Riemannian metric \(G\) and a contact 1-form \(\Theta\). A set of coordinates \(\{Z^A\}_{A=1,\ldots,2n+1} = \{\Phi, E^a, I^a\}_{a=1,\ldots,n}\) are introduced where \(\Phi\) represents the thermodynamic potential and \(E^a\) and \(I^a\) represent extensive and intensive thermodynamic variables, respectively.

An important property of GTD is that all its geometric objects are constructed such that they are invariant with respect to Legendre transformations. In classical thermodynamics, this is equivalent to saying that the physical properties of a given thermodynamic system do not depend on the choice of thermodynamic potential used for its description. A particular GTD metric \(G\) which is invariant with respect to partial and total Legendre transformations can be written as (summation over all repeated indices is implied)

\[
G = \Theta^2 + \Lambda (E_\alpha dE^\alpha dI^\alpha)
\]  

where the fundamental Gibbs 1-form is \(\Theta = d\Phi - \delta_{ab}I^a dE^b\) and \(\Lambda\) is a real constant, which can be set equal to one without loss of generality.

The equilibrium \(n\)-dimensional submanifold \(E \in \mathcal{T}\) is defined by the smooth map,

\[
\varphi : \mathcal{E} \longrightarrow \mathcal{T}
\]

\[
\{E^a\} \longrightarrow \{\Phi(E^a), E^a, I^a(E^a)\}
\]

such that the condition \(\varphi^*(\Theta) = 0\) is satisfied, implying that the first law of thermodynamics is satisfied on \(\mathcal{E}\). Applying the pullback \(\varphi^*\) to the metric \(G\), we get the induced
thermodynamic metric $g$ given by:

$$g = \varphi^*(G) = \left( E_a \frac{\partial \Phi}{\partial E^a} \right) \frac{\partial^2 \Phi}{\partial E_b \partial E_c} \delta^{ab} \delta^{de} dE^a dE^c$$

(4)

According to the GTD prescription, one only needs to specify the fundamental equation $\Phi = \Phi(E^a)$ in order to find explicitly the metric $g$ of the equilibrium submanifold $\mathcal{E}$.

III. MODIFIED CHAPLYGIN GAS FROM GTD

We choose the entropy $S = S(U,V)$ to be the thermodynamic potential and $U, V$ to be the extensive variables. Then, the corresponding thermodynamic phase space $\mathcal{T}$ is a five dimensional space endowed with the set of independent coordinates $Z^A = \{ S, U, V, \frac{1}{T}, \frac{P}{T} \}$.

The Gibbs’ fundamental 1-form $\Theta_S$ is given by:

$$\Theta_S = dS - \frac{1}{T} dU - \frac{P}{T} dV .$$

(5)

By defining the space of equilibrium states $\mathcal{E}$ by $\varphi^*_S(\Theta_S) = 0$, we obtain both the first law of thermodynamics,

$$dS = \frac{1}{T} dU + \frac{P}{T} dV ,$$

(6)

and the equilibrium conditions,

$$\frac{\partial S}{\partial U} = \frac{1}{T} , \quad \frac{\partial S}{\partial V} = \frac{P}{T} .$$

(7)

To construct the modified Chaplygin gas (MCG) model by means of GTD, we consider the fundamental equation

$$S = c_0 \ln V + \frac{c_1}{1 + \beta} \ln \left( U^{\alpha+1} + c_2 V^{\beta+1} \right) ,$$

(8)

where $c_0, c_1, c_2, \alpha$ and $\beta$ are real constants. This function is a solution of the Nambu-Goto system of differential equations

$$\Box Z^A = \frac{1}{\sqrt{\det(g)}} \left( \sqrt{\det(g)} g^{ab} Z^A_{,a} \right)_{,b} + \Gamma^A_{BC} Z^B_{,b} Z^C_{,c} g^{bc} = 0$$

(9)

where $\Box$ is the d’Alembert operator and $\Gamma^A_{BC}$ are the Christoffel symbols associated with the metric $G_{AB}$ of the phase space. The above system of differential equations follow from
the variational principle \( \delta \int \sqrt{\text{det}(g)} d^n E = 0 \), which implies that the equilibrium space \( \mathcal{E} \) with metric \( g \) constitutes an extremal subspace of the phase space \( T \).

According to Eq.(3), the induced metric in the space of equilibrium states \( \mathcal{E} \) is given as follows

\[
g = g_{UU} dU^2 + 2g_{UV} dU dV + g_{VV} dV^2, \quad (10)
\]

where the components of the thermodynamic metric \( g \) can be expressed as

\[
g_{UU} = c_1^2 (1 + \alpha)^2 \frac{\alpha c_2 V^{1 + \beta} - U^{1 + \alpha}}{(1 + \beta)^2 (U^{1 + \alpha} + c_2 V^{1 + \beta})^2},
\]

\[
g_{VV} = -\frac{1}{V^2} \left( c_0 + c_1 c_2 \frac{V^{1 + \beta}}{U^{1 + \alpha} + c_2 V^{1 + \beta}} \right) \left( c_0 + c_1 c_2 \frac{V^{1 + \beta} - \beta U^{1 + \alpha}}{(U^{1 + \alpha} + c_2 V^{1 + \beta})^2} \right),
\]

\[
g_{UV} = -\frac{(1 + \alpha) c_1 c_2 \alpha^\beta V^\beta}{2(1 + \beta) (U^{1 + \alpha} + c_2 V^{1 + \beta})^2} \left( c_0 + \frac{c_1 (1 + \alpha) U^{1 + \alpha} + (1 + \beta) c_2 V^{1 + \beta}}{U^{1 + \alpha} + c_2 V^{1 + \beta}} \right). \quad (11)
\]

Using the induced metric, we obtain the scalar curvature for the particular case \( \beta = \alpha \) as:

\[
R = \frac{N(U, V)}{D(U, V)}, \quad (12)
\]

where

\[
N(U, V) = -8(\alpha + 1)^2 c_2 V^{\alpha + 1} \left( c_2 V^{\alpha + 1} + U^{\alpha + 1} \right)^3 (c_2^2 (c_0 + c_1) \quad \times \left[ (3\alpha - 5) c_0^2 + (9\alpha - 5) c_0 c_1 + 4\alpha c_1^2 \right] U^{\alpha + 1} V^{3(\alpha + 1)} + c_0 c_2 \left[ (\alpha - 7) c_0^2 - (\alpha - 1) c_1^2 - 2c_0 c_1 \right] U^{3(\alpha + 1)} V^{\alpha + 1} - 2c_0^2 (c_0 - c_1) U^{4(\alpha + 1)} + c_2^4 (c_0 + c_1)^2 (1 - \alpha) c_0 + 4\alpha c_1 \right] V^{4(\alpha + 1)} + 3c_0 c_2^2 (c_0 + c_1) \left[ (\alpha - 3) c_0 + (\alpha - 1) c_1 \right] U^{2(\alpha + 1)} V^{2(\alpha + 1)}
\]

\[
D(U, V) = c_1 \left( c_0^2 \left[ c_0^2 (\alpha^2 + 14\alpha - 11) - 2c_0 c_1 (\alpha^2 - 6\alpha + 5) + c_1^2 (\alpha^2 + 6\alpha + 1) \right] U^{2(\alpha + 1)} V^{2(\alpha + 1)} + c_1^2 (c_0 + c_1)^2 (\alpha^2 + 6\alpha + 1) U^{4(\alpha + 1)} + 2c_1^3 (c_0 + c_1) \left[ c_0 (\alpha^2 + 8\alpha - 1) - c_1 (\alpha - 1)^2 \right] U^{\alpha + 1} V^{3(\alpha + 1)} - 4c_0^2 U^{4(\alpha + 1)} + 4c_0 c_2 \left[ c_0 (\alpha - 3) + c_1 (\alpha - 1) \right] U^{3(\alpha + 1)} V^{\alpha + 1} \right)^2. \quad (13)
\]

The Legendre invariant scalar curvature is in general non-vanishing which indicates the presence of internal thermodynamic interaction. In Figures 1 and 2, we have shown the three-dimensional behavior of the thermodynamic scalar curvature \( R \) as a function of the
energy $U$ and the volume $V$ for different values of $\alpha$ and $\beta$. It is interesting to note that even for $\beta = \alpha = 0$ the scalar curvature does not vanish because of the presence of the parameter $c_0$.

FIG. 1: Contour plot of the curvature versus the energy $U$ and the volume $V$ for $\alpha = 1, \beta = 2, c_0 = 0.3, c_1 = 0.9$ and $c_2 = -0.5$.

FIG. 2: Contour plot of the curvature versus the energy $U$ and the volume $V$ for $\beta = \alpha = 0, c_0 = 0.3, c_1 = 1.$ and $c_2 = -0.1$.

The conditions of equilibrium (7) give

$$\frac{\partial S}{\partial U} = \frac{c_1 (1 + \alpha)}{1 + \beta} \frac{U^\alpha}{U^{1+\alpha} + c_2 V^{1+\beta}} = \frac{1}{T}$$

(14)

$$\frac{\partial S}{\partial V} = \frac{c_0}{V} + c_1 c_2 \frac{V^\beta}{U^{1+\alpha} + c_2 V^{1+\beta}} = \frac{P}{T},$$

(15)
which lead to an equation of state

\[ P = \frac{c_0}{c_1} \left( \frac{1 + \beta}{1 + \alpha} \right) \rho + c_2 \left( \frac{1 + \beta}{1 + \alpha} \right) \left( 1 + \frac{c_0}{c_1} \right) \frac{V^{-(\alpha - \beta)}}{\rho^\alpha}. \]  

(16)

By defining \( A = \frac{c_0}{c_1} \left( \frac{1 + \beta}{1 + \alpha} \right) \), \( B = -c_2 \left( A + \frac{1 + \beta}{1 + \alpha} \right) \), the above equation can be written as

\[ P = A \rho - \frac{BV^{-(\alpha - \beta)}}{\rho^\alpha} \]  

(17)

which for \( \beta = \alpha \) is the MCG equation of state. From this equation one can see that for \( \beta \neq \alpha \), it corresponds to the variable modified Chaplygin gas.

One of the advantages of knowing the fundamental equation (8) explicitly is that it can be used to derive the complete thermodynamic information of the system. In particular, two thermodynamic quantities are important for the cosmological model under consideration, namely, the temperature, which follows from Eq. (14),

\[ T = \frac{1}{c_1} \left( \frac{1 + \alpha}{1 + \beta} \right) \rho V \left( 1 + c_2 \rho^{-\alpha - 1} V^{\beta - \alpha} \right), \]  

(18)

and the heat capacity

\[ C_V = -\left( \frac{\partial S}{\partial U} \right)^2 = c_1 \left( \frac{1 + \alpha}{1 + \beta} \right) \frac{1}{1 - \alpha c_2 \rho^{-\alpha - 1} V^{\beta - \alpha}}. \]  

(19)

The arbitrary parameters which enter Eq. (18) must be chosen such that the temperature is always positive definite. Moreover, from the heat capacity one can infer the phase transition structure of the system. Indeed, for a phase transition to take place (\( C_V \rightarrow \infty \)), the condition

\[ \frac{c_2}{\rho^{\alpha + 1} V^{\alpha - \beta}} = \frac{1}{\alpha} \]  

(20)

must be satisfied. If we also take into account that the temperature must be positive definite to be physically meaningful, then the above condition implies that

\[ c_1 \alpha (1 + \beta) > 0. \]  

(21)

This means that only for this particular choice of the free parameters a physical phase transition can occur. Since we have three different parameters available to satisfy this condition, one could in principle generate models with and without phase transitions. However, observations should impose additional limits on the range of values of the parameters entering each model.
IV. MCG COSMOLOGY

We consider a Friedmann-Lemaître-Robertson-Walker (FLRW) universe described by the following metric:

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2d\Omega^2_{D-1} \right] . \]  

(22)

Here \( a(t) \) is the scale factor of the universe and the curvature \( k = 0, \pm 1 \) describes spatially flat, closed or open spacetimes, respectively.

The Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \]  

(23)

for an one-component perfect fluid

\[ T_{\mu\nu} = P g_{\mu\nu} + (\rho + P) u_\mu u_\nu \]  

(24)

lead to the Friedmann equations which govern the evolution of the scale factor:

\[ H^2 = \frac{2 \rho}{(D-1)(D-2)} - \frac{k}{a^2} , \]  

(25)

\[ \dot{H} = \frac{1}{D-2}(\rho + P) + \frac{k}{a^2} . \]  

(26)

Moreover, the conservation law equation reads

\[ \dot{\rho} + (D-1) H(\rho + P) = 0 \]  

(27)

with \( H = \frac{\dot{a}}{a} \) being the Hubble parameter. Here, we assume that \( c = 1 \) and \( 8\pi G = 1 \).

Assuming that \( \rho \) as a function of time can be represented as a function of the scale factor \( \rho(a) \), the conservation law can be integrated in a straightforward manner. To this end, we assume the GTD equation of state (17) for the particular case \( \beta = \alpha \). Then, the solution can be written as

\[ \rho = \left[ \frac{B}{A+1} + \frac{C}{a^{(D-1)(\alpha+1)(\alpha+1)}} \right]^{\frac{1}{\alpha+1}} , \]  

(28)

where \( C \) is a constant of integration.
Consider now the particular case $D = 4$. It is convenient to recast the above expression in the form

$$\rho = \rho_0 \left[ A_s + (1 - A_s)a^{-3(A+1)(\alpha+1)} \right]^{\frac{1}{\alpha+1}} \tag{29}$$

where $\rho_0$ is the present value of the dark energy density and $A_s = \frac{\rho_0}{1 + A}$.

Now, using the redshift formula $z = \frac{1}{a} - 1$, the Hubble parameter is obtained as:

$$\frac{H^2(z)}{H_0^2} = \left[ \Omega_m(1 + z)^3 + (1 - \Omega_m)(A_s + (1 - A_s)(1 + z)^3(A+1)(\alpha+1))^{\frac{1}{\alpha+1}} \right] . \tag{30}$$

Here $\Omega_m = \frac{\rho_m}{H_0^2}$ is the present value of the dimension density parameter for matter and $H_0 \sim 72 \text{km.s}^{-1}.\text{Mpc}^{-1}$.

The main evidence for the existence of dark energy was provided by Supernova Type Ia experiments. This means that the existence of dark energy is directly related to the redshift of the universe. Therefore, since 1995 two teams, the High Redshift Supernova Search and the Supernova Cosmology Project, have been working intensively and, as a result of their efforts, the have discovered several type Ia supernovae at high redshifts. The observations directly measure the distance modulus of a Supernovae and its redshift $z$. Here we will consider the recent observational data, including SNe Ia which consists of 557 data points and belongs to the Union 2.1 sample. From an observational point of view, the luminosity distance $d_L(z)$ defined as

$$d_L(z) = (1 + z)H_0 \int_0^z \frac{dz'}{H(z')} , \tag{31}$$

determines the dark energy density. Moreover, the distance modulus for Supernovae is given by:

$$\mu(z) = 5 \log_{10} \left( \frac{d_L(z)/H_0}{1 \text{Mpc}} \right) + 25 . \tag{32}$$

The best fit of the distance modulus $\mu(z)$ as a function of the redshift $z$ for the MCG cosmological model and the Supernova Type Ia Union 2.1 sample is shown in Fig.3. The best fit corresponds to $\Omega_m = 0.044, A = 0.183, A_s = 0.714$ and $\alpha = 0.613$. From the curve, we see that the MCG model is in good agreement with the Union 2.1 sample data, indicating the validity of the model.
V. CONCLUSIONS

In this work, we have used the formalism of GTD to construct a cosmological model for describing the dark sector of the universe. First, we found a particular fundamental equation that determines an equilibrium space embedded in the phase space by means of a map, satisfying the Nambu-Goto variational principle. It relates the entropy of a thermodynamic system with its internal energy and its volume. Moreover, it allows us to find all the physical properties of the system by using the standard laws and computational tools of classical thermodynamics.

The GTD fundamental equation leads to an equation of state which is then used to integrate the Friedmann equations of relativistic cosmology. By analyzing the physical properties of the resulting model, we conclude that from GTD it is possible to obtain fundamental equations for thermodynamic systems that can be used to develop physically reasonable cosmological models. In particular, we applied the formalism of GTD to construct a generalization of the MCG cosmological model. It turned out that this fluid corresponds to an equilibrium space with non-zero thermodynamic curvature, indicating the presence of internal thermodynamic interaction. We performed a detailed analysis of the behavior of the MCG and obtained that MCG, as a unified model for dark energy and dark matter, is perfectly consistent with the current SNe observations.
Acknowledgements

This work was partially supported by UNAM-DGAPA-PAPIIT, Grant No. 111617, and by the Ministry of Education and Science of RK, Grant No. BR05236322 and AP05133630.

[1] S. J. Perlmutter et al, Nature 391, 51 (1998).
[2] S. J. Perlmutter et al, Astrophys. J. 517, 565 (1999).
[3] A. G. Riess et al., Astron. J. 116, 1009 (1998).
[4] A. G. Riess et al., Astrophys. J. 607, 665 (2004).
[5] N. A. Bachall et al, Science 284, 1481 (1999).
[6] M. Tegmark et al, Phys. Rev. D 69, 103501 (2004).
[7] D. Miller et al, Astrophys. J. 524, L1 (1999).
[8] C. Bennet et al, Phys. Rev. Lett. 85, 2236 (2000).
[9] S. Briddle et al, Science 299, 1532 (2003).
[10] D. N. Spergel et al, Astrophys. J. Suppl. 148, 175 (2003).
[11] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003).
[12] B. Ratra and P. J. E. Peebles, Phys. Rev. D37, 2406 (1998).
[13] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998).
[14] M. Sami and T. Padmanabhan, Phys. Rev. D67, 083509 (2003).
[15] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, Phys. Rev. D63, 103510 (2001).
[16] T. Chiba, Phys. Rev. D66, 063514 (2002).
[17] R. J. Scherrer, Phys. Rev. Lett. 93, 011301 (2004).
[18] A. Sen, J. High Energy Phys. 04, 048 (2002).
[19] A. Sen, J. High Energy Phys. 07, 065 (2002).
[20] G.W. Gibbons, Phys. Lett. B537, 1 (2002).
[21] R. R. Caldwell, Phys. Lett. B545, 23 (2002).
[22] E. Elizade, S. Nojiri and S. Odintsov, Phys. Rev. D70, 043539 (2004).
[23] J. M. Cline, S. Jeon and G. D. Moore, Phys. Rev. D70, 043543 (2004).
[24] A. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B511, 265 (2001).
[25] B. Feng, M. Li, Y. Piao and X. Zhang, Phys. Lett. B634, 101 (2006).
[26] P. Horava, D. Minic, Phys. Rev. Lett. 85, 1610 (2000).

[27] C. Deffayet, G. Dvali and G. Gabadadze, Phys. Rev. D65, 044023 (2002).

[28] H. B. Benaoum, “Accelerated Universe from Modified Chaplygin Gas and tachyonic Fluid”, preprint, [hep-th/0205140] (2002).

[29] A. Kamenshchik et al., Phys. Lett. B487, 7 (2000).

[30] M.C. Bento, O. Bertolami and A.A. Sen, Phys. Rev. D70, 0433507 (2004).

[31] U. Debnath, A. Banerjee, S. Chakraborty, Class. Quantum Gravity 21, 5609 (2004).

[32] H. B Benaoum, Adv. High Energy Phys. 2012, 357802 (2012).

[33] H.B. Benaoum, Int.J.Mod.Phys. D 23, 1450082 (2014).

[34] J. Lu, L. Xu, Y. Wu and M. Liu, Phys. Lett. B662, 87 (2008).

[35] J. Lu, L. Xu, Y. Wu and M. Liu, Gen. Rel. Grav. 43, 819 (2011).

[36] L. Xu, Y. Wang and H. Noh, Eur. Phys. J. C72, 1931 (2012).

[37] P. Thakur, S. Ghose and B.C. Paul, Mon. Not. R. Astron. Soc. 397, 1935 (2009).

[38] B.C. Paul, P. Thakur and A. Saha, Phys. Rev. D85, 024039 (2012).

[39] J. C. Fabris, H. E. S. Velten, C. Ogouyandjou and J. Tossa, Phys. Lett. B694, 289 (2011).

[40] H. Quevedo, J.Math.Phys. 48, 013506 (2007).

[41] J. L. Alvarez, H. Quevedo and A. Sanchez, Phys. Rev. D 77, 084004 (2008).

[42] A. Aviles, A. Bastarrachea-Almodovar, L. Campuzano, and H. Quevedo, Phys. Rev. D 86, 063508 (2012).

[43] H. Quevedo and D. Tapias, J. Math. Chem. 52, 141 (2014).