Mobility edges in $\mathcal{PT}$-symmetric cross-stitch flat band lattices

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(Dated: June 1, 2021)

We study the cross-stitch flat band lattice with a $\mathcal{PT}$-symmetric on-site potential and uncover mobility edges with exact solutions. Furthermore, we study the relationship between the $\mathcal{PT}$ symmetry broken point and the localization-delocalization transition point, and verify that mobility edges in this non-Hermitian model is available to signal the $\mathcal{PT}$ symmetry breaking.

I. INTRODUCTION

The unavoidable exchange of the particles, energy and quantum information with surrounding environment forms the so-called open quantum systems.$^{1-5}$ For some insightful considerations, the quantum phenomena of these systems can be well discussed by the effective non-Hermitian Hamiltonians.$^{6-10}$ Due to the non-Hermiticity, the eigenvalues of systems become complex, which is the consequence of the non-conservation of possibility. Nevertheless, if systems possess the parity-time ($\mathcal{PT}$) symmetry, they may still have purely real energy spectra, and the presence of the real spectra implies that the gain and loss of systems are balanced$^{11}$ Except the particular circumstance,$^{12,13}$ that the real spectra appear in systems without $\mathcal{PT}$-symmetry, we generally name the real-complex transition as the $\mathcal{PT}$-symmetry transition.

Since the seminal work by Bender and Boettcher$^{11}$, the $\mathcal{PT}$-symmetry has been much active in the fields of quantum field theories and mathematical physics$^{14-17}$, condensed matter physics$^{18,19}$, and optical systems$^{20-25}$. Thanks to the progress of the experimental technology, the gain and loss can be engineered controllably, which is conducive to the observation of the $\mathcal{PT}$-symmetry transition$^{26-34}$.

Similar to the $\mathcal{PT}$-symmetry, the study of Anderson localization is also a hot field, which is initially uncovered by P. W. Anderson$^{45}$. Anderson localization refers to the breakdown of the diffusion of wave packets due to the disorder. One-dimensional non-interacting systems provide good platforms to study the localization transition. A representative example is the Aubry-André (AA) model with quasiperiodic on-site potential, which presents the feature of the correlated disorder$^{36}$. The AA model undergoes a delocalization-localization transition with the increasing strength of the quasiperiodic potential, and the phase transition point can be extracted by the self-dual condition. This localization transition has been observed in the bichromatic optical lattice of ultracold-atom experiments$^{37}$. During the decades, the AA model has drawn many theoretical and experimental researches$^{38-44}$.

Recently, the localization transition is explored in some non-Hermitian systems, such as the non-Hermitian Hatano-Nelson model with asymmetric hoppings$^{45-47}$ and the generalized non-Hermitian AA models$^{48-52}$. Gong et.al. presented an intriguing topological explanation about the presence of the localization transition in a non-Hermitian Hatano-Nelson model$^{53}$. And Schiff et.al. investigated a generalized AA model with $\mathcal{PT}$-symmetry and uncovered the $\mathcal{PT}$-symmetry protected localization phase$^{54}$.

Nowadays, the combination of non-Hermiticity and the quasiperiodic potential has gradually intrigued interest in the aspect of non-Hermitian effect on the mobility edge. The physical concept of mobility edges was first proposed by Mott, based on the 3D Anderson model$^{55}$. The mobility edge refers to a critical energy-level which separating localized from extended states. In the following study, various AA-like models containing mobility edges are discussed, such as slow-varying potentials$^{56,57}$, off-diagonal disorder$^{58,59}$, long-range hoppings$^{60}$, and other generalized quasiperiodic potentials$^{61-63}$. Y. Liu et.al.$^{64}$ numerically found the simultaneous occurrence of the localization transition and the $\mathcal{PT}$-symmetry breaking. Zeng et.al.$^{65}$ demonstrated the correspondence between the winding number and the localization transition, and numerically uncovered mobility edges in the spectrum with or without $\mathcal{PT}$-symmetry. T. Liu et.al.$^{66}$ uncovered the existence of the generalized Aubry-André self-dual symmetry and obtained the exactly analytical mobility edges in non-Hermitian quasicrystals.

However, as far as we know, the influence of non-Hermitian perturbations on the flat bands has not been studied. Flat band lattices$^{67-71}$ are translationally invariant tight-binding lattices which support at least one dispersionless band in the energy spectrum. Flat band systems have usually been considered as an ideal playground to explore the strong correlation phenomena as a result of the complete quenching of the kinetic energy of electrons. Such as, a nearly flat band with non-trivial topology was proposed to simulate fractional Chern insulators$^{72}$. The classification$^{67}$ through compact localized states (CLS) gives a good framework of the properties of flat bands, i.e., the number $U$ of unit cells occupied by a CLS. For the $U = 1$ class, the CLSs form a set of orthogonal and complete bases$^{67}$, indicating that a single CLS is disentangled from the rest of unit cells, such as the sawtooth network$^{68}$. For exact solvability, in this work, we focus on the study of cross-stitch lattices...
under non-Hermitian quasiperiodic perturbations.

II. MODEL AND MOBILITY EDGES

We consider a non-Hermitian cross-stitch lattice with the complex on-site potential,

\[ \hat{\epsilon}_n \Psi_n = -\hat{V} \Psi_n - \hat{T}(\Psi_{n-1} + \Psi_{n+1}) = E \Psi_n , \]

with

\[ \hat{\epsilon}_n = \begin{pmatrix} \epsilon_n & 0 \\ 0 & -\epsilon_n \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \]

where the inter-cell hopping strength is set to be 1 and \( J \) is the intra-cell hopping strength. In the absence of the potential, \( \epsilon_n = 0 \), there is exactly one flat band \( E_{f_{tb}} = J \), associated with compact localized states \( \Psi_n \equiv (a_n, b_n)^T = (-1, 1)^T \delta_{n,n_0}/\sqrt{2} \), and one dispersive band \( E(k) = -4 \cos(k) - J \). \( \epsilon_n = V e^{i2\pi \alpha n} \) is the complex on-site potential.

Applying the local rotations

\[ \hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} p_n \\ f_n \end{pmatrix} = \hat{U} \Psi_n. \]

Eq. (1) becomes

\[ (E + J)p_n = \epsilon_n f_n - 2(p_{n-1} + p_{n+1}), \]

\[ (E - J)f_n = \epsilon_n p_n, \]

According to the method in Ref. 73,74, by solving for the Fano coordinates \( f_n \) in Eq. (4), we obtain a new equation for the dispersive portion,

\[ (E + J)p_n = \frac{\epsilon_n^2}{E - J} p_n - 2(p_{n-1} + p_{n+1}), \]

Recalling \( \epsilon_n = V e^{i2\pi \alpha n} \), Eq. (5) becomes

\[ (E + J)p_n = \frac{V^2}{E - J} e^{i4\pi \alpha n} p_n - 2(p_{n-1} + p_{n+1}), \]

if we make

\[ \tilde{E} := E + J, \quad \tilde{V} := \frac{V^2}{E - J}. \]

Eq. (6) becomes

\[ \tilde{E}p_n = \tilde{V} e^{i4\pi \alpha n} p_n - 2(p_{n-1} + p_{n+1}), \]

Numerical and analytical results\(^{51,52,75}\) show that a metal-insulator phase transition arises at the critical point \( \tilde{V} = \pm 2 \) in Eq. (8). Thus an analytic expression is found for the mobility edge,

\[ E_m = \pm \frac{V^2}{2} + J. \]

To support the analytical result given above, we now present detailed numerical analysis of Eq. (1). In the disordered system, the localization property of wave functions can be measured by the inverse participation ratio (IPR)\(^{76}\). For any given normalized wave function, the corresponding IPR is defined as \( \text{IPR} = \sum_{n=1}^L |\psi_n|^4 \), which measures the inverse of the number of sites being occupied by particles. It is well known that the IPR of an extended state scales like \( L^{-1} \) which approaches zero in the thermodynamic limit. However, for a localized state, since only finite number of sites are occupied, the IPR is finite even in the thermodynamic limit. In Fig. 1 we show the numerical IPR diagram in the \( [\text{Re}(E), V] \) plane, different colours of the eigenvalue curves indicate different magnitudes of the IPR of the corresponding wave functions. The black eigenvalue curves denote the extended states, and the bright yellow eigenvalue curves denote the localized states. It is clearly demonstrating two mobility edges separating localized from extended states along the blue curves defined by Eq. (9).

III. REAL-COMPLEX TRANSITION

By analyzing the energy spectrum, we find that there exists the real-complex transition of spectra. In Fig. 2, we fix the size of the system \( L = 500 \) and plot the eigenvalues of Eq. (1) with various \( V \). As the figure 2(a) shows, when \( V = 1 \), the eigenvalues outside the interval [0.5, 1.5] are real and the system is in the extended phase, whereas...
the mobility edges \(E\) to the mobility edges \(V\) in Fig. 2(b), (c) and (d). Therefore, for each potential \(V\)’s, \(V = 1.5\) (b), \(V = 2\) (c) and \(V = 3\) (d), the same real-complex transitions of the spectrum occur. The total number of sites is set to be \(L = 500\). The blue solid lines represent the boundaries between the real and complex energy spectrum, which are in good agreement with the mobility edges \(E_m = \pm \frac{\sqrt{V^2}}{2} + 1\). 

those inside the interval \([0.5, 1.5]\) are complex and the system is in the localized phase. The critical energies \(E_{\text{min}} = 0.5\) and \(E_{\text{max}} = 1.5\) are exactly corresponding to the mobility edges \(E_m = \pm \frac{\sqrt{V^2}}{2} + 1\). The results of \(V = 1.3\), \(V = 2\) and \(V = 3\) are also the same, as shown in Fig. 2(b), (c) and (d). Therefore, for each potential strength \(V\), we always find the separation of real and imaginary part of the eigenvalues consistent with the exact solution from Eq. (9). The complex energy is accompanied with the localized state, whereas the real energy is accompanied with the extended state. We have also checked other combinations of parameters and get the same results as expected. Consequently, we find a perfect correspondence between the real-complex transition and analytical mobility edge energy.

IV. SUMMARY

In summary, we have studied the extended and localized phases and investigated the real-complex transition of the cross-stitch flat band lattice subject to the non-Hermitian quasiperiodic potentials. Firstly, we decouple the cross-stitch lattice and obtain the analytic form of mobility edges in the spectrum. By diagonalizing the Hamiltonian, we numerically obtain the eigenvalues and wave functions. The numerical results clearly show the existence of mobility edges and are in excellent agreement with the theoretical predictions by analysing the inverse participation ratio. Furthermore, by analysing the energy spectrum, we demonstrate that mobility edges in non-Hermitian potentials not only separate localized from extended states but also indicate the coexistence of real and complex eigenvalues. Our finding the study of the non-Hermitian mobility edges and the real-complex transition, and in the view of \(\mathcal{PT}\) symmetry, the localization transition occurs accompanied by the \(\mathcal{PT}\) symmetry breaking.

ACKNOWLEDGMENTS

T. L. acknowledges the Natural Science Foundation of Jiangsu Province (Grant No. BK20200737) and NUPTSF (Grant No. NY220090 and No. NY220208).

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