Abstract

We study a higher derivative (HD) field theory with an arbitrary order of derivative for a real scalar field. The degree of freedom for the HD field can be converted to multiple fields with canonical kinetic terms up to the overall sign. The Lagrangian describing the dynamics of the multiple fields is known as the Lee-Wick (LW) form. The first step to obtain the LW form for a given HD Lagrangian is to find an auxiliary field (AF) Lagrangian which is equivalent to the original HD Lagrangian up to the quantum level. Till now, the AF Lagrangian has been studied only for $N = 2$ and 3 cases, where $N$ is the number of poles of the two-point function of the HD scalar field. We construct the AF Lagrangian for arbitrary $N$. By the linear combinations of AF fields, we also obtain the corresponding LW form. We find the explicit mapping matrices among the HD fields, the AF fields, and the LW fields. As an exercise of our construction, we calculate the relations among parameters and mapping matrices for $N = 2, 3,$ and 4 cases.
1 Introduction

Lee and Wick (LW) constructed quantum electrodynamics with the higher derivative (HD) propagator for a photon [1]. The HD term in the denominator of the propagator improves the ultraviolet convergence of the Feynman diagram since the propagator falls off more quickly as momentum grows. A minimal set of the HD term leads to an additional physical pole in the propagator, corresponding to a massive LW-photon with the wrong-sign residue. It seems that the wrong sign gives rise to the instability or the violation of unitarity in the theory. Lee and Wick proposed a deformation of integration contours in the Feynman diagram so that the theory can be free from the instability and the violation of unitarity.

Based on the idea of Lee and Wick [1], Lee-Wick Standard Model (LWSM) has been proposed as a candidate to solve the hierarchy problem in the Standard Model [2]. Every field in the LWSM has a higher derivative kinetic term. Using the auxiliary field (AF) method, the HD term can be converted into the degree of freedom of a massive field with wrong-sign kinetic term, referred as the LW partner. Since the LW partner can decay into ordinary fields, the ‘wrong sign’ does not cause the violation of unitarity at macroscopic scales [1,3,4]. Due to the presence of the LW partner, the radiative quantum correction to the Higgs-boson mass-squared is free from the quadratic divergence, which can be a resolution of the hierarchy problem in the Standard Model. The simple mechanism solving the hierarchy problem has motivated phenomenological studies [5–18]. The LWSM was also extended to studying cosmology [19].

In LWSM [2], a minimal set of the HD term was considered. The number of poles in the two-point function of each field in LWSM is two, representing the number of physical degrees of freedom which correspond to the ordinary Standard Model particle and its LW partner. From now on, we refer the number of poles in the two-point function of the HD theory as \(N\). That is, LWSM [2] is an \(N = 2\) HD theory. In general we can include the HD terms beyond the minimal set. In this direction, Carone and Lebed in Ref. [17] constructed the \(N = 3\) HD theory by providing the mapping among the HD Lagrangian, the AF Lagrangian, and the LW form. Here the LW form represents the Lagrangian having canonical quadratic terms aside from the overall sign. While the \(N = 2\) HD theory has a single LW partner with wrong-sign quadratic terms in the LW form, the \(N = 3\) HD theory has two LW partners one of which has the correct sign (corresponding the ordinary particle) while the other has the wrong sign. That is, there is an additional ordinary LW partner in the LW form of the \(N = 3\) HD theory.

Due to the existence of the additional ordinary LW partner, the \(N = 3\) HD theory is qualitatively different from the \(N = 2\) HD theory. For instance, the gauge coupling unification in the Standard Model, which is difficult to be realized in the \(N = 2\) HD theory, can be achieved at the one-loop level in the \(N = 3\) HD theory without introducing additional fields [18]. Therefore, the
investigation of the HD theory beyond the minimal set is an interesting subject.

With this motivation, we generalize the Carone and Lebed’s construction \[17\] for a scalar field. We consider a general HD Lagrangian of a self-interacting real scalar field:

\[
\mathcal{L}^{(N)}_{\text{HD}} = \frac{1}{2} \sum_{n=1}^{N} (-1)^{n+1} a_n \phi \Box^n \phi - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_{\text{int}}(\phi)
\]  

(1.1)

with \( a_1 = 1 \), where \( N \) indicates the number of physical poles in the propagator of \( \phi \), \( a_n \) is the coefficient of mass dimension \([a_n] = 2 - 2n\), and the last term represents interactions. The equation of motion for \( \phi \) is given by

\[
\sum_{n=1}^{N} (-1)^{n+1} a_n \Box^n \phi - m^2 \phi + \mathcal{L}'_{\text{int}} = 0,
\]

(1.2)

where \( \mathcal{L}'_{\text{int}} = d\mathcal{L}_{\text{int}}/d\phi \). We construct the AF Lagrangians for \( N = \) even and \( N = \) odd cases separately for the HD Lagrangian. By integrating out the auxiliary fields, one can restore the original HD Lagrangian (1.1) and the equation of motion (1.2). Since the auxiliary fields are linear or quadratic in the AF Lagrangian (see section 2), the HD Lagrangian and the corresponding AF Lagrangian are equivalent at the quantum level also. We obtain the explicit transformation from the HD field to the AF field.\(^2\)

For general \( N \), we construct the transformation from the HD field to the LW field. We also show that this LW field can be transformed to the AF field. Therefore, the AF Lagrangian can be written in the LW form by linear combinations of the AF fields, which indicates that the HD Lagrangian and the LW form are also equivalent at the quantum level.\(^3\)

Studying the HD field theory has a long history. In 1950, Pais and Uhlenbeck \[20\] investigated the HD theories in terms of the low-order theories (so, equivalently the LW form) in order to study purely quantum-mechanical theories. In the modern point of view, we can also find the origin of the HD field theory from string theory, for example, the tachyon effective action of the truncated open string field theory, the \( p \)-adic string theory, and etc. One of the interesting features of these theories is the presence of the infinite order of derivatives. Recently there has been much interest in applying these nonlocal theories with infinite order of derivatives to cosmological models \[21–31\].

\(^1\)We can rewrite the kinetic term as the canonical form, such as \( \frac{1}{2} \phi \Box \phi = -\frac{1}{2} \nabla \mu \phi \nabla^\mu \phi \), by neglecting the total derivative term. However, mostly we use D’Alembertian \( \Box \equiv \sqrt{-g} \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \right) \), instead of \( \nabla \mu \) for simplicity.

\(^2\)In this work, the “AF field” denotes the auxiliary field plus the ordinary field in the AF Lagrangian.

\(^3\)For a pure scalar-field theory, one can show directly that the HD lagrangian is equivalent to the LW form at the quantum level, without the aid of the AF lagrangian. However, introducing the AF lagrangian shall be useful for future investigation involving gauge fields.
and to analyzing the mathematical structure of those theories [32–35]. The supersymmetric extension of the HD field theories is also an interesting subject [36]. Our result can be accommodated to this type of interesting work.

This paper is organized as follows. In section 2, we construct the AF Lagrangian from the HD Lagrangian for the $N$ = even and the $N$ = odd cases separately. In section 3, we construct the LW form from the HD Lagrangian, and obtain an explicit transformation from the LW field to the HD field. We also find a transformation from the LW field to the AF field. We apply our formalism for $N = 2, 3,$ and $4$ cases, and provide the results. We conclude in section 4.

2 Generalized Auxiliary Field Lagrangian

By introducing auxiliary fields we can write an AF Lagrangian. Integrating out these auxiliary fields from the Lagrangian reproduces the HD Lagrangian. The AF Lagrangian has the ordinary kinetic terms. The number of the fields including those auxiliary fields is $N$. Until now, the AF Lagrangians for $N = 2$ [2] and $N = 3$ [17] cases have been constructed. In this work, we construct an AF Lagrangian for general $N$. We shall separately treat the $N = 2\hat{N}$ and $N = 2\hat{N} + 1$ cases with positive integer $\hat{N}$.

2.1 $N = 2\hat{N}$ case

In this subsection we construct a generalized AF Lagrangian which is equivalent to the original HD Lagrangian (1.1) for arbitrary $N = 2\hat{N}$, and obtain the transformation between the AF and the HD fields. The form of the Lagrangian is given by

$$L_{\text{AF}}^{(2\hat{N})} = \frac{1}{2} \sum_{n=1}^{\hat{N}} \varphi_n(\Box - Q_n)\varphi_n + \sum_{n=1}^{\hat{N}-1} \chi_n(\Box \varphi_n - R_n \varphi_{n+1}) + \chi_{\hat{N}} \Box \varphi_{\hat{N}} + \frac{\hat{N}^2 R_{\hat{N}}^2}{2Q_{\hat{N}+1}} \chi_{\hat{N}}^2 + L_{\text{int}}(\varphi_1), \quad (2.3)$$

where $\varphi_n$ and $\chi_n$ are real scalar fields and $R_n$ and $Q_n$ are constant parameters of mass dimension $\text{dim} = 1$. Using the relations among parameters $(m, a_n)$ in (1.1) and $(Q_n, R_n)$ in (2.3), we can verify that $L_{\text{HD}}^{(2\hat{N})}$ and $L_{\text{AF}}^{(2\hat{N})}$ are equivalent.

In order to verify that the AF Lagrangian (2.3) is equivalent to the HD Lagrangian (1.1) with even number of poles, we first integrate out the auxiliary field $\chi_n$ in (2.3). In order to do this we write the equation of motion for the auxiliary field $\chi_n$, and then $\varphi_n$ ($n > 1$) is expressed in terms

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4We introduced $R_{\hat{N}}$ in (2.3) for the formal consistency, but we can treat $R_{\hat{N}}^2 / Q_{\hat{N}+1}$ as a single parameter.
of $\varphi_1$:

$$
\chi_1 : \varphi_2 = \frac{1}{R_1} \square \varphi_1 = S_1 \square \varphi_1,
$$

$$
\chi_2 : \varphi_3 = \frac{1}{R_2} \square \varphi_2 = S_2 \square^2 \varphi_1,
$$

$$
\vdots
$$

$$
\chi_{N-1} : \varphi_N = \frac{1}{R_{N-1}} \square \varphi_{N-1} = S_{N-1} \square^{N-1} \varphi_1,
$$

$$
\chi_N : \chi_N = -\frac{Q_{N+1}}{R_N^2} \square \varphi_N = -S_N Q_{N+1} \square^{N} \varphi_1,
$$

(2.4)

where we defined

$$
S_n \equiv \prod_{i=1}^{n} \frac{1}{R_i}.
$$

Therefore, the auxiliary field $\varphi_n$ is expressed by the HD field $\phi$,

$$
\varphi_n = S_{n-1} \phi_{n-1}, \quad (n = 2, ..., \hat{N}),
$$

(2.5)

where we defined $\phi_n \equiv \square^n \phi$, and set $\phi = \varphi_1$.

Since the auxiliary field $\chi_n$ is linear or quadratic in the Lagrangian (2.3), the equations of motion (2.4) obtained by the variation of $\chi_n$ are exact at the quantum level also. Plugging the equations in (2.4) into (2.3), we obtain

$$
\mathcal{L}^{(2\hat{N})}_{AF} = \frac{1}{2} \sum_{n=1}^{\hat{N}} S_{n-1}^2 \varphi_1 (\square - Q_n) \square^{2n-2} \varphi_1 - \frac{1}{2} S_N^2 Q_{N+1} \varphi_1 \square^{2N} \varphi_1 + \mathcal{L}_{\text{int}}(\varphi_1)
$$

$$
= \frac{1}{2} \sum_{n=1}^{\hat{N}} S_{n-1}^2 \phi \square^{2n-1} \phi - \frac{1}{2} \sum_{n=1}^{\hat{N}+1} S_{n-1}^2 Q_n \phi \square^{2n-2} \phi + \mathcal{L}_{\text{int}}(\phi).
$$

(2.6)

Next, we read the equation of motion for $\varphi_n$:

$$
\varphi_1 : (\square - Q_1) \varphi_1 + \square \chi_1 + \mathcal{L}'_{\text{int}}(\varphi_1) = 0,
$$

$$
\varphi_2 : (\square - Q_2) \varphi_2 + \square \chi_2 = R_1 \chi_1,
$$

$$
\varphi_3 : (\square - Q_3) \varphi_3 + \square \chi_3 = R_2 \chi_2,
$$

$$
\vdots
$$

$$
\varphi_{\hat{N}} : (\square - Q_{\hat{N}}) \varphi_{\hat{N}} + \square \chi_{\hat{N}} = R_{\hat{N}-1} \chi_{\hat{N}-1}.
$$

(2.7)
Combining equations (2.4) and (2.7), we can express $\chi_n$ in terms of $\varphi_1 ( = \phi)$,

$$
\chi_n = \sum_{l=n}^{\hat{N}-1} \frac{S^2}{S^2_{n-1}} \square^{2l-n+1} \varphi_1 - \sum_{l=n}^{\hat{N}} \frac{S^2 Q_{l+1}}{S^2_{n-1}} \square^{2l-n} \varphi_1, \quad (n = 1, \ldots, \hat{N}),
$$

(2.8)

$$
= \left[ \sum_{l=n}^{\hat{N}-1} \frac{S^2}{S^2_{n-1}} (\square - Q_{l+1}) \square^{2l-n} - \frac{S^2 \hat{N} Q \hat{N} + 1}{S^2_{n-1}} \square^{2\hat{N}} \square \right] \phi,
$$

(2.9)

where $S_0 = 1$. See Appendix A for the detail. The auxiliary field $\chi_n$ is now expressed by the HD field $\phi$ via Eq. (2.9).

In order to get the relations between the HD parameter $a_n$ and the AF parameters $Q_n$ and $S_n$, let us compare $L^{(2\hat{N})}_{\text{HD}}$ in (1.1) and $L^{2\hat{N}}_{\text{AF}}$ in (2.3). We decompose $L^{(2\hat{N})}_{\text{HD}}$ as

$$
L^{(2\hat{N})}_{\text{HD}} = \frac{1}{2} \sum_{n=1}^{\hat{N}} a_{2n-1} \phi \square^{2n-1} \phi - \frac{1}{2} \sum_{n=1}^{\hat{N}+1} a_{2n-2} \phi \square^{2n-2} \phi + L_{\text{int}}(\phi).
$$

(2.10)

Comparing this equation and (2.6), we obtain the relations,

$$
Q_1 = m^2, \quad Q_{n+1} = \frac{a_n}{a_{2n+1}}, \quad \frac{Q_{\hat{N}+1}}{R^2_{\hat{N}}} S^2_{\hat{N}-1} = a_{2\hat{N}},
$$

$$
S^2_0 = 1, \quad S^2_n = a_{2n+1}, \quad (n = 1, \ldots, \hat{N} - 1).
$$

(2.11)

Then $L^{2\hat{N}}_{\text{AF}}$ and $L^{2\hat{N}}_{\text{HD}}$ are equivalent. Since $Q_{\hat{N}+1}/R^2_{\hat{N}}$ is a single parameter as we discussed before, we can set $R_{\hat{N}} = 1$ for computational convenience. In this case, the parameter $Q_{\hat{N}+1}$ has mass dimension $-2$.

Now we can show that the HD field equation is reduced from the AF field equation. Inserting the relation for $\chi_1$ in (2.8) into the first equation in (2.7), we can get the field equation for $\varphi_1$,

$$
(\square - Q_1) \varphi_1 + \sum_{n=1}^{\hat{N}-1} S^2_{n}(\square - Q_{n+1}) \square^{2n} \varphi_1 - S^2 \hat{N} Q_{\hat{N}+1} \square^{2\hat{N}} \varphi_1 + L'_{\text{int}}(\varphi_1) = 0,
$$

(2.12)

where $a_0 = m^2$. Using the relation (2.11) this equation becomes the field equation (1.2) for the HD field $\phi ( = \varphi_1)$. This completes the proof that the AF Lagrangian (2.3) is equivalent to the HD Lagrangian (1.1) with even number of physical poles.

Now we obtain the explicit transformation matrix between the HD field and AF fields. Substituting the relations in (2.11) into (2.5) and (2.9), we can express the AF fields ($\varphi_n, \chi_n$) as the
HD field $\Box^a \phi$ in terms of the coefficient $a_n$,
\[
\varphi_n = \sqrt{a_{2n-1}} \phi_{n-1},
\]
\[
\chi_n = \sum_{l=n}^{\hat{N}-1} \frac{a_{2l+1}}{\sqrt{a_{2n-1}}} \phi_{2l-n+1} - \sum_{l=n}^{\hat{N}} \frac{a_{2l}}{\sqrt{a_{2n-1}}} \phi_{2l-n}
\]
\[
= \sum_{m=n}^{2\hat{N}-n} (-1)^{m+n-1} \frac{a_{m+n}}{\sqrt{a_{2n-1}}} \phi_m \quad (n = 1, \cdots, \hat{N}).
\] (2.13)

Here, we assume that $S_n > 0$. The negative $S_n$ can also be chosen. (However, the resulting AF Lagrangian has no difference from the positive $S_n$ case since the signature change in $S_n$ means the overall-signature change in the fields $\varphi_n$ and $\chi_n$.) From the results in (2.13), we can read the transformation matrix $U^{(e)}_{ij}$ satisfying
\[
\xi_i = \sum_{j=1}^{2\hat{N}} U^{(e)}_{ij} \varphi_{j-1}, \quad (i, j = 1, \cdots, 2\hat{N}),
\] (2.14)

where we combine $(\varphi_n, \chi_n)$ to a column matrix $\xi$ with $2\hat{N}$ components,
\[
\xi_n \equiv \varphi_n, \quad \xi_{\hat{N}+n} \equiv \chi_n, \quad (n = 1, \cdots, \hat{N}).
\] (2.15)

The components of the transformation matrix $U^{(e)}_{ij}$ are given by
\[
U^{(e)}_{ij} = \begin{cases} 
\sqrt{a_{2j-1}} \delta_{ij} & (1 \leq j \leq \hat{N}) \\
0 & \text{(the others)} \\
(-1)^{i+j-\hat{N}} \frac{a_{i+j-\hat{N}+1}}{\sqrt{a_{2i-2\hat{N}-1}}} & (i - \hat{N} + 1 \leq j \leq 3\hat{N} - i + 1) \\
0 & \text{(the others)} \\
\end{cases} \quad (1 \leq i \leq \hat{N}).
\] (2.16)

We can also represent the HD fields $\phi_n$ in terms of the AF fields by using inverse transformation of $U^{(e)}_{ij}$,
\[
\phi_{i-1} = \sum_{j=1}^{2\hat{N}} U^{(e)}_{ij}^{-1} \xi_j, \quad (i = 1, \cdots, \hat{N}).
\] (2.17)

where $U^{(e)}_{ij}^{-1}(a_n)$ can be expressed as $U^{(e)}_{ij}^{-1}(Q_n, S_n)$ using Eq. (2.11).

As an example, we apply the AF Lagrangian (2.3) for the $N = 2$ case. Then the AF Lagrangian (2.3) is written as
\[
\mathcal{L}^{(2)}_{AF} = \frac{1}{2} \varphi_1 (\Box - Q_1) + \chi_1 \Box \varphi_1 + \frac{1}{2Q_2} \chi_1^2 + \mathcal{L}_{\text{int}}(\varphi_1),
\] (2.18)

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where we set $R_2 = 1$ for simplicity. With the identifications given in (2.11),

$$\varphi_1 = \phi, \quad Q_1 = m^2, \quad Q_2 = \frac{a_2}{a_1} = \frac{1}{M^2},$$

one can easily check that (2.18) is the well-known Lagrangian for the $N = 2$ HD Lagrangian [2].

### 2.2 $N = 2\hat{N} + 1$ case

In this subsection we construct the AF Lagrangian with the odd number of poles, and obtain the transformation between the AF and the HD fields. The Lagrangian can be written in the following way,

$$\mathcal{L}_{AF}^{(2\hat{N}+1)} = \frac{1}{2} \sum_{n=1}^{\hat{N}+1} \varphi_n(\Box - Q_n)\varphi_n + \sum_{n=1}^{\hat{N}} \chi_n(\Box \varphi_n - R_n\varphi_{n+1}) + \mathcal{L}_{int}(\varphi_1).$$

Using the similar method to the $N = 2\hat{N}$ case in the previous subsection, we verify that (2.20) is the AF Lagrangian of $\mathcal{L}_{HD}^{(N)}$ in (1.1) with the odd number of poles. The equation of motion for the auxiliary field $\chi_n$ is given by

$$\chi_1 : \varphi_2 = \frac{1}{R_1} \Box \varphi_1 = S_1 \Box \varphi_1,$$

$$\chi_2 : \varphi_3 = \frac{1}{R_2} \Box \varphi_2 = S_2 \Box^2 \varphi_1,$$

$$\vdots$$

$$\chi_{\hat{N}} : \varphi_{\hat{N}+1} = \frac{1}{R_{\hat{N}}} \Box \varphi_{\hat{N}} = S_{\hat{N}} \Box^{\hat{N}} \varphi_1.$$}

Therefore, in the same way with the previous section, we have

$$\varphi_n = S_{n-1} \varphi_{n-1}, \quad (n = 1, \ldots, \hat{N} + 1).$$

Plugging this equation into $\mathcal{L}_{AF}^{(2\hat{N}+1)}$ in (2.20), we obtain

$$\mathcal{L}_{AF}^{(2\hat{N}+1)} = \frac{1}{2} \sum_{n=1}^{\hat{N}+1} S_{n-1}^2 \varphi_1 (\Box - Q_n) \Box^{2n-2} \varphi_1 + \mathcal{L}_{int}(\varphi_1)$$

$$= \frac{1}{2} \sum_{n=1}^{\hat{N}+1} S_{n-1}^2 \phi \Box^{2n-1} \phi - \frac{1}{2} \sum_{n=1}^{\hat{N}+1} S_{n-1}^2 Q_n \phi \Box^{2n-2} \phi + \mathcal{L}_{int}(\phi).$$
Next, we write the equation of motion for \( \varphi_n \):
\[
\varphi_1 : (\Box - Q_1) \varphi_1 + \Box \chi_1 + L'_\text{int}(\varphi_1) = 0,
\]
\[
\varphi_2 : (\Box - Q_2) \varphi_2 + \Box \chi_2 = R_1 \chi_1,
\]
\[
\varphi_3 : (\Box - Q_3) \varphi_3 + \Box \chi_3 = R_2 \chi_2,
\]
\[
\vdots
\]
\[
\varphi_{\hat{N}} : (\Box - Q_{\hat{N}}) \varphi_{\hat{N}} + \Box \chi_{\hat{N}} = R_{\hat{N}-1} \chi_{\hat{N}-1},
\]
\[
\varphi_{\hat{N}+1} : (\Box - Q_{\hat{N}+1}) \varphi_{\hat{N}+1} = R_{\hat{N}} \chi_{\hat{N}}.
\]
(2.24)
Combining (2.21) and (2.24) we can express \( \chi_n \) in terms of \( \varphi_1 (= \phi) \),
\[
\chi_n = \sum_{l=n}^{\hat{N}} \frac{S_l^2}{S_{l-1}} (\Box - Q_{l+1}) \phi_{2l-n}, \quad (n = 1, \ldots, \hat{N}).
\]
(2.25)
See Appendix B for the detail.

In order to get the relations between the HD parameter \( a_n \) and the AF parameters \( Q_n \) and \( S_n \), let us compare \( L^{(2\hat{N}+1)}_{\text{HD}} \) in (1.1) and \( L^{2\hat{N}+1}_{\text{AF}} \) in (2.23). We decompose \( L^{(2\hat{N}+1)}_{\text{HD}} \) as
\[
L^{(2\hat{N}+1)}_{\text{HD}} = \frac{1}{2} \sum_{l=n}^{\hat{N}+1} a_{2n-l} \phi \Box^{2n-1} \phi - \frac{1}{2} \sum_{l=n}^{\hat{N}+1} a_{2n-l} \phi \Box^{2n-2} \phi + L'_\text{int}(\phi).
\]
(2.27)
Comparing (2.23) and (2.27), we obtain the relations,
\[
Q_1 = m^2, \quad Q_{n+1} = \frac{a_{2n+1}}{a_{2n+1}},
\]
\[
S_0^2 = 1, \quad S_n^2 = a_{2n+1}, \quad (n = 1, \ldots, \hat{N}).
\]
(2.28)
Then \( L^{2\hat{N}+1}_{\text{AF}} \) and \( L^{2\hat{N}+1}_{\text{HD}} \) become equivalent.

Similarly to the \( N = \text{even} \) case we can reproduce the field equation for \( \phi \). Inserting the relation for \( \chi_1 \) in (2.25) into the first equation in (2.24), we can get the field equation for \( \varphi_1 \),
\[
(\Box - Q_1) \varphi_1 + \sum_{n=1}^{\hat{N}} S_n^2 (\Box - Q_{n+1}) \Box^{2n} \varphi_1 + L'_\text{int}(\varphi_1) = 0.
\]
(2.29)
Using the relation (2.28) this equation reproduces the field equation (1.2) for the HD field \( \phi (= \varphi_1) \).
Plugging the relations (2.28) into (2.22) and (2.26), we can also express the AF fields \((\varphi_n, \chi_n)\) in terms of \(\phi_n = \Box^n \phi\) with the coefficient \(a_n\) in the HD field Lagrangian,

\[
\varphi_m = \sqrt{a_{2m-1}} \phi_{m-1}, \quad (m = 1, \cdots, \hat{N} + 1),
\]

\[
\chi_n = \sum_{l=n}^{\hat{N}} \left( \frac{a_{2l+1}}{\sqrt{a_{2l-1}}} \phi_{2l-1+n} - \frac{a_{2l}}{\sqrt{a_{2l-1}}} \phi_{2l-n} \right)
\]

\[
= \sum_{m=n}^{2\hat{N}-n+1} (-1)^{m+n-1} \frac{a_{m+n}}{\sqrt{a_{2n-1}}} \phi_m, \quad (n = 1, \cdots, \hat{N}).
\]

(2.30)

Now we construct the transformation matrix \(U_{ij}^{(o)}\) from the HD field to the AF field. Similarly to (2.14), we write down the transformation relation as

\[
\xi_i = \sum_{j=1}^{2\hat{N}+1} U_{ij}^{(o)} \phi_j - 1, \quad (i, j = 1, \cdots, 2\hat{N} + 1),
\]

(2.31)

where \(\xi\) is a column matrix with \(2\hat{N} + 1\) components,

\[
\xi_m \equiv \varphi_m, \quad (m = 1, \cdots, \hat{N} + 1), \quad \xi_{\hat{N}+n+1} \equiv \chi_n, \quad (n = 1, \cdots, \hat{N}).
\]

(2.32)

We read the transformation matrix \(U_{ij}^{(o)}\) from (2.31),

\[
U_{ij}^{(o)} = \begin{cases} \sqrt{a_{2j-1}} \delta_{ij} & (1 \leq j \leq \hat{N} + 1) \\ 0 & (the\ others) \\ (-1)^{i+j-\hat{N}-1} \frac{a_{i+j-2\hat{N}-2}}{\sqrt{a_{2i-2\hat{N}-3}}} & (i - \hat{N} \leq j \leq 3\hat{N} - i + 3) \\ 0 & (the\ others) \end{cases} \quad (1 \leq i \leq \hat{N} + 1)
\]

(2.33)

Again, using the inverse matrix of \(U_{ij}^{(o)}\), we can also express the HD field \(\phi_i\) in terms of the AF fields \(\xi_i\),

\[
\phi_{i-1} = \sum_{j=1}^{2\hat{N}+1} U_{ij}^{(o)-1} \xi_j.
\]

(2.34)

We apply the AF Lagrangian (2.20) for the next-to-minimal higher derivative theory \((N = 3\) case). This case was first constructed by Carone and Lebed in Ref. [17]. The \(N = 3\) HD Lagrangian with \(a_2 = 1/M_1^2\) and \(a_3 = 1/M_2^4\) is given by

\[
\mathcal{L}_{HD}^{(3)} = \frac{1}{2} \phi \Box \phi - \frac{1}{2M_1^2} \phi \Box^2 \phi + \frac{1}{2M_2^2} \phi \Box^3 \phi - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_{int}(\phi).
\]

(2.35)
We read the $N=3$ AF Lagrangian from (2.20) as
\begin{equation}
L^{(3)}_{AF} = \frac{1}{2} \phi (\Box - m^2) \phi + \frac{1}{2} \phi (\Box - \frac{M^4}{M^2}) \phi + \chi (\Box \phi - M^2 \tilde{\phi}) + L_{\text{int}}(\phi),
\end{equation}
where we redefined the fields as $\varphi_1 = \phi$, $\varphi_2 = \tilde{\phi}$, and $\chi_1 = \chi$, and used the identifications given in (2.28),
\begin{equation}
Q_1 = m^2, \quad Q_2 = \frac{a_2}{a_3} = \frac{M^4}{M^2}, \quad S_1^2 = \frac{1}{R_1^2} = a_3 = \frac{1}{M^2}.
\end{equation}

The AF Lagrangian (2.36) is slightly different from the AF Lagrangian given in Ref. [17]. There is an additional $\chi \phi$-term in Ref. [17]. Due to this term the normalization in front of the quadratic terms is different from ours. However, if we identify the parameters in (1.1) and (2.36), such as
\begin{align}
m^2 &= m_1^2 m_2^2 m_3^2 / \Lambda^4, \\
a_2 &= M_1^{-2} = (m_1^2 + m_2^2 + m_3^2) / \Lambda^4, \\
a_3 &= M_2^{-4} = 1 / \Lambda^4
\end{align}
with $\Lambda^4 = m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2$, we can see the AF Lagrangian (2.36) reproduces the same LW form given in Ref. [17], under some linear mappings. Therefore, the AF Lagrangians (2.3) and (2.20) that we constructed are not unique. However, the resulting LW form is unique, which will be studied in the next section.

3 Generalized Lee-Wick Form

In the previous section we showed that the HD Lagrangian can be recast into the AF Lagrangian by trading the higher derivatives with additional fields ($\varphi_{n>1}$) including auxiliary fields ($\chi_n$). The $N$ physical poles in the HD-field propagator are converted into $N$ degrees of freedom in the AF Lagrangian. If we construct the LW form by appropriate linear combinations of the AF fields, as done in Refs. [2,17], the resulting LW form is equivalent to the original HD Lagrangian up to the quantum level. In this section, we complete constructing the transformations among the three forms of Lagrangian, the HD Lagrangian, the AF Lagrangian, and the LW form, for arbitrary $N$.

3.1 HD Lagrangian and LW form

In this subsection, we reconstruct and extend the Pais-Uhlenbeck formalism [20] which relates the HD Lagrangian and the LW form. We explicitly express the LW field $\psi_n$ in terms of the HD field ($\Box^n \phi$, $n = 1, ..., N$) and HD parameters $m^2$ and $a_n$. 

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We consider the following LW form corresponding to the HD Lagrangian with $N$ physical poles in the HD field propagator,

$$\mathcal{L}_{\text{LW}}^{(N)} = \frac{1}{2} \sum_{n=1}^{N} \kappa_n \psi_n (\Box - \mu_n) \psi_n + \mathcal{L}_{\text{int}}(\psi),$$  \hspace{1cm} (3.39)

where $\kappa_n$ represents the overall sign ($\pm 1$), $\mu_n$ is the mass-squared parameter, and $\psi_n$ denotes the LW field. The HD field $\phi$ can be expressed as a linear combination of $\psi_n$,

$$\phi = \sum_{n=1}^{N} c_n \psi_n,$$

where $c_n$ is a real coefficient. We assume that there is no degeneracy of the mass-squared, i.e.,

$$\mu_1 < \mu_2 < \mu_3 < \ldots < \mu_N.$$  \hspace{1cm} (3.40)

The equation of motion for $\psi_n$ is given by

$$\kappa_n (\Box - \mu_n) \psi_n + c_n \mathcal{L}'_{\text{int}}(\psi) = 0,$$  \hspace{1cm} (3.41)

where $\mathcal{L}'_{\text{int}} = d\mathcal{L}_{\text{int}}/d\psi_n$.

We compare the two Lagrangians (1.1) and (3.39), and obtain the field transformation in a form,

$$\psi_i = \sum_{j=1}^{N} b_{ij} \phi_{j-1},$$  \hspace{1cm} (3.42)

where $b_{ij}$ is the component of the mapping matrix from the HD field to the LW field, and $\phi_i = \Box^i \phi$.

We can rewrite the HD Lagrangian (1.1) as

$$\mathcal{L}_{\text{HD}}^{(N)} = \frac{1}{2} \phi F(\Box) \phi + \mathcal{L}_{\text{int}}(\phi), \quad F(\Box) = - \sum_{n=0}^{N} (-1)^n a_n \Box^n;$$  \hspace{1cm} (3.43)

where $F(\Box)$ is the inverse of the HD-field propagator. Considering that the HD Lagrangian (3.43) is equivalent to the LW form (3.39), the HD-field propagator can be expressed in a form associated with $N$-physical poles,

$$F(\Box) = -a_0 \prod_{i=1}^{N} \left( 1 - \frac{\Box}{\mu_i} \right).$$  \hspace{1cm} (3.44)
By comparing the expressions of $F(\Box)$ in Eqs. (3.43) and (3.44), one can get the relations between the HD parameter $a_n$ and the LW parameter $\mu_n$,

$$a_1 = a_0 \sum_{i=1}^{N} \frac{1}{\mu_i},$$

$$a_2 = a_0 \sum_{i<j}^{N} \frac{1}{\mu_i \mu_j},$$

$$a_3 = a_0 \sum_{i<j<k}^{N} \frac{1}{\mu_i \mu_j \mu_k},$$

$$\vdots$$

(3.45)

In our parametrization, we set

$$a_1 = 1 \quad \Leftrightarrow \quad a_0 (= m^2) = \left( \sum_{i=1}^{N} \frac{1}{\mu_i} \right)^{-1}.$$  

(3.46)

Using the “partial-fraction analysis”, we can express the propagator as:

$$\frac{1}{F(\Box)} = -\frac{1}{a_0} \prod_{i=1}^{N} \left( 1 - \frac{\Box}{\mu_i} \right)^{-1} = \sum_{i=1}^{N} \frac{\eta_i}{\Box - \mu_i},$$  

(3.47)

where

$$\eta_i = \frac{\mu_i}{a_0} \prod_{j \neq i}^{N} \left( 1 - \frac{\mu_i}{\mu_j} \right)^{-1}.$$  

(3.48)

Using the relation (3.47), the HD Lagrangian (3.43) reproduces the LW form (3.39) (up to field rescaling),

$$\frac{1}{2} \phi F(\Box) \phi = \frac{1}{2} \phi \left[ F(\Box) \right]^2 \sum_{i=1}^{N} \frac{\eta_i}{\Box - \mu_i} \phi$$

$$= \frac{1}{2} \sum_{i=1}^{N} \frac{\eta_i}{\mu_i^2} \phi \left[ F(\Box) \right]^2 \frac{\mu_i^2}{\Box - \mu_i} \phi$$

$$= \frac{1}{2} \sum_{i=1}^{N} a_0 \frac{\eta_i}{\mu_i^2} \phi \left[ \prod_{j \neq i}^{N} \left( 1 - \frac{\Box}{\mu_j} \right) \right]^2 (\Box - \mu_i) \phi$$

$$= \frac{1}{2} \sum_{i=1}^{N} \kappa_i \psi_i (\Box - \mu_i) \psi_i,$$  

(3.49)

5Our parameters are related with Pais-Uhlenbeck parameters by $\mu_i = \omega_i^2$ and $\eta_i/\mu_i^2 = \eta_i^{PU}$.  

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from which we can get the transformation,

$$\psi_i = a_0 \sqrt{|\eta_i|} \prod_{j \neq i}^N \left(1 - \frac{\Box}{\mu_i}\right) \phi = \sum_{j=1}^N b_{ij} \phi_{j^{-1}}.$$  \hspace{1cm} (3.50)

The elements of this transformation matrix are given by

$$b_{i1} = \frac{a_0}{\mu_i} \sqrt{|\eta_i|},$$

$$b_{i2} = -\frac{a_0}{\mu_i} \sqrt{|\eta_i|} \sum_{j \neq i}^N (\mu_j)^{-1},$$

$$b_{i3} = \frac{a_0}{\mu_i} \sqrt{|\eta_i|} \sum_{j(\neq i) < k(\neq i)}^N (\mu_j \mu_k)^{-1},$$

$$\vdots$$  \hspace{1cm} (3.51)

We can now obtain the inverse transformation from the LW field to the HD field,

$$\phi_{j^{-1}} = \sum_{i=1}^N (b^{-1})_{ji} \psi_i.$$  \hspace{1cm} (3.52)

Using the relation (3.47), we obtain for \(j = 1\),

$$\phi = \left(\sum_{i=1}^N \frac{\eta_i}{-\Box - \mu_i}\right) F(\Box) \phi = \sum_{i=1}^N a_0 \frac{\eta_i}{\mu_i} \prod_{j \neq i}^N \left(1 - \frac{\Box}{\mu_j}\right) \phi = \sum_{i=1}^N (b^{-1})_{1i} \psi_i,$$  \hspace{1cm} (3.53)

where

$$(b^{-1})_{1i} = \kappa_i \sqrt{|\eta_i|},$$  \hspace{1cm} (3.54)

and \(\kappa_i\) is the signature of \(\eta_i\). Then

$$\phi_{j^{-1}} = \Box^{j-1} \phi = \sum_{i=1}^N \kappa_i \sqrt{|\eta_i|} \Box^{j-1} \psi_i = \sum_{i=1}^N \kappa_i \sqrt{|\eta_i|} \mu_i^{j-1} \psi_i$$

$$= \sum_{i=1}^N (b^{-1})_{ji} \psi_i,$$  \hspace{1cm} (3.55)

therefore, we have

$$(b^{-1})_{ji} = \mu_i^{j-1} (b^{-1})_{1i} = \mu_i^{j-1} \kappa_i \sqrt{|\eta_i|}.$$  \hspace{1cm} (3.56)
There are simple algebraic sum-rules for $\eta_i$:

$$\sum_{i=1}^{N} \frac{\eta_i}{\mu_i} = a_0, \quad \sum_{i=1}^{N} \eta_i \mu_i^n = 0, \quad (n = 0, 1, \ldots, N - 2).$$

(3.57)

Now we present several examples for $N = 2, 3$, and 4.

$N = 2$ case:
For simplicity, let $\mu_{ij} \equiv \mu_i - \mu_j$.

(i) The HD coefficient $a_n$ is completely determined by the LW-mass parameter from Eq. (3.45),

$$\left( \begin{array}{c} a_0 \\ a_2 \end{array} \right) = \frac{1}{\mu_1 + \mu_2} \left( \begin{array}{c} \mu_1 \mu_2 \\ 1 \end{array} \right).$$

(3.58)

(ii) The transformation matrix $b_{ij}$ is obtained by Eq. (3.51),

$$\left( b_{ij} \right) = \frac{1}{\sqrt{\mu_2(\mu_1 + \mu_2)}} \left( \begin{array}{c} \mu_2 & -1 \\ \mu_1 & -1 \end{array} \right).$$

(3.59)

$N = 3$ case:
We follow the same process above.

(i)

$$\left( \begin{array}{c} a_0 \\ a_2 \\ a_3 \end{array} \right) = \frac{1}{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1} \left( \begin{array}{c} \mu_1 \mu_2 \mu_3 \\ \mu_1 + \mu_2 + \mu_3 \\ 1 \end{array} \right)$$

(3.60)

(ii)

$$\left( b_{ij} \right) = \frac{1}{\sqrt{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1}} \left( \begin{array}{ccc} \mu_2 \mu_3 & - \mu_2 \mu_3 & 1 \\ \mu_2 \mu_3 & \mu_3 \mu_1 & \mu_3 \mu_1 \\ \mu_3 \mu_1 & \mu_1 \mu_2 & \mu_1 \mu_2 \end{array} \right).$$

(3.61)

$N = 4$ case:
For simplicity, let $\mu_A = \mu_1 \mu_2 \mu_3 + \mu_2 \mu_3 \mu_4 + \mu_3 \mu_4 \mu_1 + \mu_4 \mu_1 \mu_2$.

(i)

$$\left( \begin{array}{c} a_0 \\ a_2 \\ a_3 \\ a_4 \end{array} \right) = \frac{1}{\mu_A} \left( \begin{array}{c} \mu_1 \mu_2 \mu_3 \mu_4 \\ \mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_4 + \mu_4 \mu_1 + \mu_1 \mu_3 + \mu_2 \mu_4 \\ \mu_1 + \mu_2 + \mu_3 + \mu_4 \\ 1 \end{array} \right)$$

(3.62)
(ii)

\[
\begin{pmatrix}
\frac{\mu_2 \mu_4}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & -\frac{\mu_2 \mu_3 + \mu_2 \mu_4 + \mu_4 \mu_3}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & \mu_2 + \mu_3 + \mu_4 & -\frac{1}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} \\
\frac{\mu_2 \mu_4}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & -\frac{\mu_2 \mu_3 + \mu_2 \mu_4 + \mu_4 \mu_3}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & \mu_2 + \mu_3 + \mu_4 & -\frac{1}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} \\
\frac{\mu_4 \mu_2}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & -\frac{\mu_1 \mu_2 + \mu_4 \mu_3 + \mu_2 \mu_3}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & \mu_1 + \mu_2 + \mu_3 & -\frac{1}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} \\
\frac{\mu_4 \mu_2}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & -\frac{\mu_1 \mu_2 + \mu_4 \mu_3 + \mu_2 \mu_3}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} & \mu_1 + \mu_2 + \mu_3 & -\frac{1}{\sqrt{\mu_1 \mu_3 \mu_2 \mu_1}} \\
\end{pmatrix}
\]

(3.63)

### 3.2 AF Lagrangian and LW form

In this subsection, we construct the transformation matrix from the LW field \( \psi_n \) to the auxiliary field \( (\varphi_n, \chi_n) \). Once \( a_n \) is expressed by \( \mu_n \) from Eq. (3.45), the AF coefficients \( Q_n \) and \( R_n \) (so, \( S_n \)) can be expressed by the LW parameter \( \mu_n \) from Eqs. (2.11) and (2.28). Then the transformation matrix is completely expressed by \( \mu_n \). Using the inverse of this transformation matrix we can obtain the LW form (3.39) from the AF Lagrangians (2.3) and (2.20).

#### 3.2.1 \( N = 2 \hat{N} \) case

Plugging Eq. (3.52) into Eq. (2.14), we find the mapping,

\[
\xi_i = \sum_{j=1}^{N} V_{ij}^{(e)} \psi_j, \quad (i, j = 1, \ldots, N = 2\hat{N}),
\]

(3.64)

where \( \xi_i \) was defined in (2.15) and

\[
V_{ij}^{(e)} = \sum_{k=1}^{2\hat{N}} U_{ik}^{(e)} (b^{-1})_{kj}.
\]

(3.65)

We present the examples for \( N = 2 \) and \( N = 4 \).

#### \( N = 2 \) case:

\[
(\varphi_n) = \frac{\mu_2}{\mu_2 + \mu_1} \begin{pmatrix}
1 & -1
\end{pmatrix} \psi_m
\]

(3.66)

\[
(\chi_n) = \frac{1}{\sqrt{\mu_2 (\mu_2 + \mu_1)}} \begin{pmatrix}
\mu_2 & -\mu_1
\end{pmatrix} + \begin{pmatrix}
0 & 0
\end{pmatrix} \psi_m
\]

(3.67)

#### \( N = 4 \) case:
For simplicity, let \( \mu_B = \mu_1 + \mu_2 + \mu_3 + \mu_4 \).

\[
\begin{pmatrix}
\varphi_n \\
\chi_n
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{\mu_A}{\mu_1 \mu_3 \mu_{21}}} & -\sqrt{\frac{\mu_A}{\mu_2 \mu_3 \mu_{21}}} & \sqrt{\frac{\mu_A}{\mu_3 \mu_4 \mu_{31}}} & -\sqrt{\frac{\mu_A}{\mu_4 \mu_3 \mu_{41}}} \\
\mu_1 \sqrt{\frac{\mu_B}{\mu_4 \mu_2 \mu_{12}}} & \mu_2 \sqrt{\frac{\mu_B}{\mu_4 \mu_2 \mu_{12}}} & \mu_3 \sqrt{\frac{\mu_B}{\mu_4 \mu_3 \mu_{31}}} & \mu_4 \sqrt{\frac{\mu_B}{\mu_4 \mu_3 \mu_{41}}}
\end{pmatrix}
\begin{pmatrix}
\psi_m \\
\psi_m
\end{pmatrix}
\]

(3.68)

\[
\begin{pmatrix}
\varphi_n \\
\chi_n
\end{pmatrix} = \begin{pmatrix}
\frac{\mu_1 (\mu_3 + 2 \mu_4)}{\sqrt{\mu_1 \mu_2 \mu_3 \mu_{21}}} & \frac{\mu_2 (\mu_2 + \mu_3 + \mu_4)}{\sqrt{\mu_1 \mu_2 \mu_3 \mu_{21}}} & \frac{\mu_3 (\mu_1 + 2 \mu_4)}{\sqrt{\mu_1 \mu_2 \mu_3 \mu_{21}}} & \frac{\mu_4 (\mu_1 + 2 \mu_3 + \mu_4)}{\sqrt{\mu_1 \mu_2 \mu_3 \mu_{21}}}
\end{pmatrix}
\begin{pmatrix}
\psi_m \\
\psi_m
\end{pmatrix}
\]

(3.69)

3.2.2 \( N = 2 \hat{N} + 1 \) case

Plugging Eq. (3.52) into Eq. (2.31), we find the mapping,

\[
\xi_i = \sum_{j=1}^{N} V^{(o)}_{ij} \psi_j, \quad (i, j = 1, \ldots, N = 2 \hat{N} + 1),
\]

(3.70)

where \( \xi_i \) was defined in (2.32) and

\[
V^{(o)}_{ij} = \sum_{k=1}^{2 \hat{N} + 1} U^{(o)}_{ik} (b^{-1})_{kj}.
\]

(3.71)

\( N = 3 \) case:

\[
\begin{pmatrix}
\varphi_n \\
\chi_n
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_4}{\mu_1 \mu_2 \mu_3}} & -\sqrt{\frac{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_4}{\mu_1 \mu_2 \mu_3}} & \sqrt{\frac{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_4}{\mu_1 \mu_2 \mu_3}} & -\sqrt{\frac{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_4}{\mu_1 \mu_2 \mu_3}} \\
\mu_1 \sqrt{\frac{\mu_1 \mu_2 \mu_3}{\mu_2 \mu_3 \mu_{21}}} & \mu_2 \sqrt{\frac{\mu_1 \mu_2 \mu_3}{\mu_2 \mu_3 \mu_{21}}} & \mu_3 \sqrt{\frac{\mu_1 \mu_2 \mu_3}{\mu_2 \mu_3 \mu_{31}}} & \mu_4 \sqrt{\frac{\mu_1 \mu_2 \mu_3}{\mu_2 \mu_3 \mu_{41}}}
\end{pmatrix}
\begin{pmatrix}
\psi_m \\
\psi_m
\end{pmatrix}
\]

(3.72)

\[
\begin{pmatrix}
\varphi_n \\
\chi_n
\end{pmatrix} = \frac{1}{\sqrt{\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1}} \begin{pmatrix}
\frac{\mu_1 (\mu_2 + \mu_3)}{\sqrt{\mu_1 \mu_2 \mu_3}} & \frac{\mu_2 (\mu_2 + \mu_3)}{\sqrt{\mu_1 \mu_2 \mu_3}} & \frac{\mu_3 (\mu_2 + \mu_3)}{\sqrt{\mu_1 \mu_2 \mu_3}} & \frac{\mu_4 (\mu_2 + \mu_3)}{\sqrt{\mu_1 \mu_2 \mu_3}}
\end{pmatrix}
\begin{pmatrix}
\psi_m \\
\psi_m
\end{pmatrix}
\]

(3.73)

4 Conclusions

In this work we considered a HD field theory with \( N \) physical poles of the two-point function for a self-interacting real scalar field. We generalized the Carone and Lebed’s work \([17]\), in which the AF Lagrangian and the LW form for the \( N = 3 \) HD Lagrangian were constructed.

Our work is summarized as follows:

(i) In section 2, for an HD Lagrangian with arbitrary \( N \) poles, we obtained the corresponding AF Lagrangian. We obtained the explicit transformation from the HD field to the AF field, and the inverse transformation is also possibly obtained.
(ii) In section 3.1, we found the transformation from the HD field to the LW field, and the inverse transformation was obtained explicitly.

(iii) In section 3.2, we showed that the corresponding LW form can be constructed by the redefinitions of the AF field using the transformation from the LW field to the AF field.

We presented the results of our formulation for \( N = 2, 3, \) and 4. However, in principle, one can obtain the results for \( N > 4 \) from our formulation.

There are several things to comment on the AF Lagrangian, which is one of our main results. In the construction of the AF Lagrangian, we split the HD Lagrangian into \( N = \) even and \( N = \) odd cases, since they are qualitatively different. For the \( N = \) even case, we introduced a quadratic term for the last component of auxiliary field (\( \chi^2_N \)-term), while the other auxiliary fields are linear. For the \( N = \) odd case, however, every auxiliary field is linear. Since the AF fields are linear or quadratic in the Lagrangian, the resulting equations of motion impose constraints which are exact at the quantum level. We could also obtain the LW form from the AF Lagrangian by linear mappings. Therefore, the HD Lagrangian, the AF Lagrangian, and the LW form are equivalent up to the quantum level also.

As we did in section 3.1, we can directly find the mapping matrix between the HD field and the LW field. In doing so, we could use the “partial-fraction analysis” since the numerators of the LW propagators are numbers (\( \eta_i \) in our case). By rescaling the fields we can always absorb the numerical factors. For this reason, we can actually prove that the equivalence between the HD lagrangian and the LW lagrangian up to the quantum level, without introducing the AF lagrangian.

When gauge fields are introduced in the theory, being different from the scalar-field case, the numerators of the propagators contain the gauge indices (for non-abelian gauge fields) as well as the spacetime indices. Therefore, the partial-fraction analysis is not useful for the gauge-field case. In order to prove the quantum equivalence between the HD lagrangian and the LW lagrangian, we need to introduce the AF lagrangian as an intermediate step. In Ref. [17], Carone and Lebed applied the results of the scalar field (\( m_1 = 0 \) case) to construct the AF lagrangian and the LW lagrangian from the \( N = 3 \) HD lagrangian with the gauge field. When the extension is to be made for \( N > 3 \) with gauge fields in the future, the AF lagrangian formalism which we constructed for the scalar field in this work will be very useful.

Another thing that we have to mention is that the AF Lagrangian is not unique, while the LW form is unique for a given HD Lagrangian. For example, the \( N = 3 \) AF Lagrangian constructed in Ref. [17] is different from ours (2.36). We can easily check this fact by comparing the overall normalizations of the two AF Lagrangians. However, the \( N = 3 \) LW form in Ref. [17] and ours (see section 3.1) are definitely identical.
Our \(N\)-field formulation can be applied to various interesting physical situations, for example, \(N\)-field Lee-Wick phenomenology, \(N\)-field cosmology in a relation to cosmological expansions, and etc. From the top-down aspect, once a higher derivative scalar field theory is developed, one can apply our formulation in order to see its phenomenological consequences. From the bottom-up aspect, one can deduce the theoretical set-up from the \(N\)-field interpretation of the phenomena.

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A Calculation of (2.8)

Inserting the relations in Eq. (2.4) into those in Eq. (2.7), we obtain

\[
\begin{align*}
\chi_{\hat{N}-1} &= \frac{1}{R_{\hat{N}-1}}(\Box - Q_{\hat{N}})\varphi_{\hat{N}} + \frac{1}{R_{\hat{N}-1}}\Box \chi_{\hat{N}} = \frac{S_{\hat{N}-1}}{R_{\hat{N}-1}}(\Box - Q_{\hat{N}})\Box^{\hat{N}-1} \varphi_{1} - \frac{S_{\hat{N}} Q_{\hat{N}+1}}{R_{\hat{N}-1} R_{\hat{N}}} \Box^{\hat{N}+1} \varphi_{1}, \\
\chi_{\hat{N}-2} &= \frac{1}{R_{\hat{N}-2}}(\Box - Q_{\hat{N}-1})\varphi_{\hat{N}-1} + \frac{1}{R_{\hat{N}-2}}\Box \chi_{\hat{N}-1} \\
&= \frac{S_{\hat{N}-2}}{R_{\hat{N}-2}}(\Box - Q_{\hat{N}-1})\Box^{\hat{N}-2} \varphi_{1} + \frac{S_{\hat{N}-1}}{R_{\hat{N}-2} R_{\hat{N}}} (\Box - Q_{\hat{N}})\Box^{\hat{N}} \varphi_{1} - \frac{S_{\hat{N}} Q_{\hat{N}+1}}{R_{\hat{N}-2} R_{\hat{N}-1} R_{\hat{N}}} \Box^{\hat{N}+2} \varphi_{1}, \\
&\vdots \\
\chi_{1} &= \frac{1}{R_{1}}(\Box - Q_{2})\varphi_{2} + \frac{1}{R_{1}}\Box \chi_{2} \\
&= \sum_{n=1}^{\hat{N}-1} S_{n}^{2}(\Box - Q_{n+1})\Box^{2n-1} \varphi_{1} - S_{\hat{N}}^{2} Q_{\hat{N}+1} \Box^{2\hat{N}-1} \varphi_{1},
\end{align*}
\] (A.74)
B Calculation of (2.25)

Inserting the relations in Eq. (2.21) into those in Eq. (2.24), we obtain

\begin{align*}
\chi_{\hat{N}} &= \frac{1}{R_{\hat{N}}} (\Box - Q_{\hat{N}+1})\varphi_{\hat{N}+1} = \frac{S_{\hat{N}}}{R_{\hat{N}}} (\Box - Q_{\hat{N}+1})\Box_{\hat{N}} \varphi_1, \\
\chi_{\hat{N}-1} &= \frac{S_{\hat{N}-1}}{R_{\hat{N}-1}} (\Box - Q_{\hat{N}})\Box_{\hat{N}-1} \varphi_1 + \frac{S_{\hat{N}}}{R_{\hat{N}-1} R_{\hat{N}}} (\Box - Q_{\hat{N}+1})\Box_{\hat{N}+1} \varphi_1, \\
\vdots & \\
\chi_1 &= \sum_{n=1}^{\hat{N}} S_n^2 (\Box - Q_{n+1})\Box^{2n-1} \varphi_1. \quad \text{(B.75)}
\end{align*}

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