Conic sector analysis using integral quadratic constraints

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Funding information
Engineering and Physical Sciences Research Council, Grant/Award Number: EP/N00924X/1

Summary
This paper interprets the Conic Sector stability results of Zames, and the multivariable generalizations provided by Safonov and Athans, in an integral quadratic constraint context. The ideas are formulated for the case where one element of a feedback interconnection is linear and the other satisfies a conic sector condition. This scenario allows the main stability results to be formulated using integral quadratic constraints, with all finite-sector cases being captured using this framework. The stability results are expressed using frequency domain inequalities or linear-matrix inequalities. Some examples show how the main technical result can, in the multivariable case, be used to provide a reduction in conservatism over standard results.

KEYWORDS
absolute stability, conic sector, integral quadratic constraints

1 | INTRODUCTION

One of the fundamental problems in nonlinear control engineering is the analysis of the stability properties of feedback interconnections. The 1960's was particularly fruitful and the inception of many well-known results can be traced back to this period. Notable achievements are the small gain theorem, which underpins the robust control literature1,2 and the passivity theorem and related results.3,4 In addition, other researchers have considered the case when one of the systems in the feedback interconnection was linear and this has yielded a number of absolute stability results such as the popular Circle and Popov criteria, and also the theory associated with Zames-Falb multipliers.5-8

The circle criterion originally arose in the two-part paper of Zames,9,10 where the stability of systems of the form depicted in Figure 1 was studied. In most applications of the Circle Criterion, the troublesome nonlinearity, \( \Delta \), is considered to be static with a graph belonging to a particular sector. However, the Circle Criterion is a special case of the result proved in the work of Zames,9 which treated the more general case where \( \Delta \) belonged to a conic sector; this can be considered a dynamic generalization of the static sector. In fact, the conic sector notion introduced by Zames generalizes a number of different stability notions, most notably, passivity and small gain.

Since the work of Zames, numerous papers have discussed and applied the circle criterion to various different problems, but the original result of Zames, the conic sector theorem, has been somewhat overlooked. An early and comprehensive generalization of Zames’ result was provided by Safonov and Athans11 where results were provided for conic sectors described by linear operators. However, like Zames’ original results, Safonov and Athans’ methods were somewhat difficult to apply and analysis was chiefly based on singular value inequalities. Recent papers by Bridgeman and Forbes12,13 have interpreted the conic sector theorem in notation more familiar to the control theorist of today, and have devised approaches for synthesizing controllers which meet the conditions of the conic sector theorem. Some of this work was inspired by the earlier work of Gupta and Joshi.14 In addition, Bridgeman and Forbes15 extended Zames’ original theorem to cases not covered in the work of Zames,9 so there are a number of modern design and analysis tools available.

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FIGURE 1  System under consideration: $G(s)$ is a stable LTI system; $\Delta(\cdot)$ is a bounded causal operator.

The main idea of this paper is to approach conic sector stability analysis from the viewpoint of integral quadratic constraints (I-QCs). When one of the elements of Figure 1 is linear, IQCs provide a simple framework for stability analysis and enable one to perform a combined analysis of systems containing various different uncertainty types. The results derived here enable one to cast multivariable conic sector stability analysis problems either as a frequency domain inequality, or as a set of linear matrix inequalities (LMIs). The results improve upon those in the work of Bridgeman and Forbes due to their multivariable nature and they improve upon the work of Safonov and Athans because the singular value inequality is replaced by more convenient LMIs. In addition, the inverse of certain operators, which are assumed in the work of the aforementioned authors, is not required in the IQC formulation.

The results in this paper can be considered an input-output counterpart to some of the (Q,S,R) dissipativity results developed elsewhere in the literature; see the works of Hill and Moylan for early work and the works of Xia et al and Kottenstette et al for recent contributions. The appealing feature of the approach considered here is that the results are developed from a purely input-output perspective and do not require many of the complicated notions from (Q,S,R) passivity, eg, input/output/state strict passivity is not required and even the notion of a state is coincidental. On the other hand, if required, internal (state-space) stability can be inferred from the results here under mild conditions.

This paper is organized as follows. The next section introduces conic sector definitions and the connections between them. Following this, the IQC machinery of the works of Megretski and Rantzer and Seiler is introduced and comments are made on its application. The section after this introduces the main stability result. A penultimate section provides examples, which demonstrate the superiority of the results here over those already known.

1.1 | Notation

The notation is standard and mainly follows the works of Zhou et al and Bridgeman and Forbes. The $L_2$ norm of a vector $x(t) \in \mathbb{R}^m$ is defined as

$$\|x\|_2 := \left( \int_0^\infty x(t)^t x(t) dt \right)^{\frac{1}{2}}.$$  

By Parseval’s identity, this is equal to the $L_2$ norm of the vector $\hat{x}(j\omega) \in \mathbb{C}^m$, where $\hat{x}(j\omega)$ is the Fourier transform of $x(t)$. A vector $x(t) \in L_2^m$ if $\|x\|_2 < \infty$.

The finite horizon $L_2$ norm of a vector $x(t) \in L_2^m$ is defined as

$$\|x\|_{2T} := \left( \int_0^T x(t)^t x(t) dt \right)^{\frac{1}{2}}.$$  

A vector $x(t) \in L_{2T}^m$ if $x(t) \in \mathbb{R}^m$ is such that $\|x\|_{2T} < \infty$ for any $T \in \mathbb{R}_+$.

The inner product of vectors $x(t), y(t) \in L_{2T}^m$ is defined as

$$\langle x, y \rangle_T := \int_0^T x(t)^t y(t) dt = \int_0^T y(t)^t x(t) dt.$$  

An operator $H : L_{2T}^m \rightarrow L_{2T}^p$ is said to be bounded if there exists a scalar $\gamma > 0$ such that

$$\|H(x)\|_2 \leq \gamma \|x\|_2 \quad \forall x \in L_{2T}^m.$$  


The truncation operator $P_T$ is such that, for a signal $u(t)$ and a positive scalar, $T > 0$,

$$P_T[u(t)] = \begin{cases} u(t) & t \leq T \\ 0 & t > T. \end{cases}$$

The space of real rational $m \times p$ transfer function matrices, bounded on the imaginary axis is denoted $\mathbb{R}H_{\infty}^{m \times p}$; the subspace, which is, in addition, analytically continuous in the right-half complex plane is denoted $\mathbb{R}H_{\infty}^{m \times p}$, or for short-hand $\mathbb{RH}_{\infty}$.

## 2 Conic Sector Notions

This section introduces the various notions of a conic sector which will be used in this paper. The first is a small modification in the seminal work of Zames.\(^9\)

**Definition 1.** An operator $H : \mathcal{L}_2^m \mapsto \mathcal{L}_2^m$ is said to belong to the Cone$[\alpha, \beta]$, for $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$ if

$$\langle H(x) - ax, H(x) - \beta x \rangle_T \leq 0 \quad \forall x \in \mathcal{L}_2^m, \quad \forall T \in \mathbb{R}_+.$$  \hspace{1cm} (1)

If the inequality holds strictly for all $0 \neq x \in \mathcal{L}_2^m$, $H \in \text{Cone}(\alpha, \beta)$.

This can be equivalently expressed as the following definition.

**Definition 2.** An operator $H : \mathcal{L}_2^m \mapsto \mathcal{L}_2^m$ is said to belong to the Cone$[\alpha, \beta]$, for $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$ if

$$\|H(x)\|_{2T}^2 + \alpha \|x\|_{2T}^2 - \langle (\alpha + \beta)x, H(x) \rangle_T \leq 0 \quad \forall x \in \mathcal{L}_2^m, \quad \forall T \in \mathbb{R}_+.$$  \hspace{1cm} (2)

If the inequality holds strictly for all $0 \neq x \in \mathcal{L}_2^m$, $H \in \text{Cone}(\alpha, \beta)$.

This last definition is useful because it enables one to connect conicity to other concepts. For example, from Definition 2, if $H \in \text{Cone}[\beta, \beta]$, for some $\beta > 0$, inequality (2) becomes

$$\|H(x)\|_{2T}^2 - \beta^2 \|x\|_{2T}^2 \leq 0 \quad \forall x \in \mathcal{L}_2^m, \quad \forall T \in \mathbb{R}_+.$$  \hspace{1cm} (3)

Taking the limit as $T \to \infty$ then implies that $H$ has a finite $L_2$ gain of $\beta$. Similar observations can be made to connect conicity to passivity.

Noting the first two definitions, the following matrix generalization of a scalar conic sector appears natural.

**Definition 3.** An operator $H : \mathcal{L}_2^m \mapsto \mathcal{L}_2^p$ is said to belong to the Cone$[A, B]$, for matrices $A, B \in \mathbb{R}^{p \times m}$ if

$$\langle H(x) - Ax, H(x) - Bx \rangle_T \leq 0 \quad \forall x \in \mathcal{L}_2^m, \quad \forall T \in \mathbb{R}_+.$$  \hspace{1cm} (4)

If the inequality holds strictly for all $0 \neq x \in \mathcal{L}_2^m$, $H \in \text{Cone}(A, B)$.

**Definition 4.** An operator $H : \mathcal{L}_2^m \mapsto \mathcal{L}_2^p$ is said to belong to the Cone$[A, B]$, for matrices $A, B \in \mathbb{R}^{p \times m}$ if

$$\|H(x)\|_{2T}^2 + \langle Ax, Bx \rangle_T - \langle (A + B)x, H(x) \rangle_T \leq 0 \quad \forall x \in \mathcal{L}_2^m, \quad \forall T \in \mathbb{R}_+.$$  \hspace{1cm} (5)

If the inequality holds strictly for all $0 \neq x \in \mathcal{L}_2^m$, $H \in \text{Cone}(A, B)$.

There is some abuse of notation here. The notation for the matrix cone, $H \in \text{Cone}[A, B]$, implies that the scalar cone should be written as $H \in \text{Cone}[aI_m, bI_m]$, where $I_m$ is the identity matrix. The identity matrix is often suppressed for succinctness.

Safonov and Athans\(^{11}\) provided a more general definition of a conic sector, which is interpreted as Definition 5 as follows.
Definition 5. Consider the linear operators $S : L_2^m \rightarrow L_2^m$, $C : L_2^m \rightarrow L_2^m$, and $R : L_2^m \rightarrow L_2^m$. An operator $H : L_2^m \rightarrow L_2^m$ is said to belong to the Cone$(S, R, C)$ if
\[
\|S(H(x) - Cx)\|_{L_2^m}^2 - \|Rx\|_{L_2^m}^2 \leq 0 \quad \forall x \in L_2^m, \quad \forall T \in \mathbb{R}_+.
\]

If the inequality holds strictly for all $0 \neq x \in L_2^m$, $H \in$ Cone$(S, R, C)$.

Again, notation is being slightly abused. Note also the arguments of the cone are, in general, dynamic systems, not constant scalars as in the work of Zames. The operator form of $(Q, S, R)$ dissipativity, and its approximation, was considered in the work of Forbes and Damaren, which is similar to the idea of allowing $S, R, C$ to be operators, as in Definition 5. However, if $S, R, C$ are chosen as constant matrices, it is easy to see that Definition 5 coincides with Definition 3 with $C = (A + B)/2$, $S = I$ and $R = S(A - B)/2$. For the remainder of the work described here, we work with Definitions 3 or 4, i.e., we work with constant matrices rather than linear operators. In principle, the same IQC approach proposed here can be used to treat the more general cones in Definition 5.

3 INTEGRAL QUADRATIC CONSTRAINTS AND CONICITY

3.1 Stability using IQCs

This section briefly reviews the IQC stability theory developed in the works of Megretski and Rantzer and Seiler since this will form the cornerstone of the stability results, which will be developed. Consider Figure 1, which shows a feedback interconnection of a stable linear operator $G(s) \in \mathbb{R}^{m \times n}_{\infty}$ and a causal bounded nonlinear operator $\Delta(f) : L_2^m \rightarrow L_2^p$. In keeping with the robust control framework and the paper of Megretski and Rantzer, a positive feedback convention is used; this will provide results in a slightly unusual form when compared to their classical counterparts. The interconnection, depicted in Figure 1, is said to be stable if $r_1 \in L_2^p, r_2 \in L_2^m$ implies that the signals $w \in L_2^p, y \in L_2^m$.

Two signals $v(t) \in L_2^m$ and $w \in L_2^p$ are said to satisfy the soft IQC defined by the Hermitian matrix $\Pi(j\omega) \in \mathbb{R}^{(m+p) \times (m+p)}$ if the following inequality holds:
\[
\int_{-\infty}^{\infty} \left| \begin{bmatrix} v(j\omega) \\ w(j\omega) \end{bmatrix} \right|^* \Pi \left| \begin{bmatrix} v(j\omega) \\ w(j\omega) \end{bmatrix} \right| dt \geq 0.
\]

Furthermore, if $w(t)$ is generated by the operator $\Delta(\cdot) : L_2^m \rightarrow L_2^p$, i.e., $w(t) = \Delta(v(t))$ and is such that inequality (7) is satisfied for all $v(t) \in L_2^m$, we say that $\Delta$ satisfies the IQC defined by $\Pi(j\omega)$, or for simplicity, $\Delta(\cdot) \in$ IQC$(\Pi(j\omega))$. It is often more convenient to express IQCs in the time domain and, in the case that $\Pi(j\omega)$ is simply a constant matrix (see the work of Seiler for the more general case), the signals $v(t) \in L_2^m$ and $w \in L_2^p$ are said to satisfy the hard IQC defined by the symmetric matrix $\Pi$ if the following inequality holds:
\[
\int_0^T \left| \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \right|^* \Pi \left| \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \right| dt \geq 0,
\]
for all $T \in \mathbb{R}_+$. This latter IQC characterization will be exploited later since, in our analysis, it transpires that $\Pi(j\omega)$ is indeed a symmetric matrix, independent of $\omega$.

The great appeal of the IQC perspective on robust control is that it allows the “uncertainties,” $\Delta(\cdot)$, to be treated in a unified and systematic manner. The first IQC stability result, which will be used in this paper, is taken from the work of Megretski and Rantzer.

Theorem 1. Let $G(s) \in \mathbb{R}^{m \times n}_{\infty}$ and let $\Delta(\cdot) : L_2^m \rightarrow L_2^p$ be a bounded causal operator. Assume that

(i) for every $\tau \in [0, 1]$ the interconnection of $G(s)$ and $\tau \Delta$ is well-posed;
(ii) for every $\tau \in [0, 1]$, the IQC defined by $\Pi(j\omega)$ in inequality (7) is satisfied by $\tau \Delta$;
(iii) there exists an $\epsilon > 0$ such that
\[
\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
\]

Then, the feedback interconnection of $G(s)$ and $\Delta(\cdot)$, shown in Figure 1, is $L_2$ stable.
Under the assumption of well-posedness, one can see that the IQC theorem requires two conditions to be satisfied, i.e., item (ii), which requires the IQC defined by $\Pi(j\omega)$ to hold for all $\tau \Delta, \tau \in [0,1]$, rather than just $\Delta$; and a frequency domain stability condition. Often, item (ii) holds regardless, but, for the conicity results discussed here, this may not be the case. However, as noted in the work of Megretski and Rantzer, item (ii) will hold if the following condition is satisfied.

**Fact 1.** Assume $\Pi(j\omega) \in \mathbb{R}^{\mathbb{H}(m+p) \times (m+p)}$ is partitioned as

$$
\Pi(j\omega) = \begin{bmatrix}
\Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\
\Pi_{21}(j\omega) & \Pi_{22}(j\omega)
\end{bmatrix}.
$$

Then, condition (ii) in Theorem 1 holds if

$$
\Pi_{11}(j\omega) \geq 0 \quad \text{and} \quad \Pi_{22}(j\omega) \leq 0 \quad \forall \omega \in \mathbb{R}.
$$

Motivated by the restrictive homotopy conditions (i) and (ii) in Theorem 1, Seiler developed an alternative dissipation-based IQC result, which requires far milder assumptions on the system. The following result is a combination of Theorem 1 and Lemma 1 in the work of Seiler for the special case that $\Pi$ is a constant matrix.

**Theorem 2.** Let $G(s) \in \mathbb{R}^{\mathbb{H}_2m} \rightarrow \mathbb{L}_2^n$ be a bounded causal operator. Assume that

(i) The interconnection of $G(s)$ and $\Delta$ is well-posed;

(ii) The IQC defined by $\Pi = \Pi' \in \mathbb{R}^{(m+p) \times (m+p)}$ in inequality (9) is satisfied;

(iii) There exists a matrix $P = P' \geq 0$, which satisfies the following matrix inequality:

$$
\begin{bmatrix}
A_p'P + PA_p & PB_p \\
B_p'P & 0
\end{bmatrix} + \begin{bmatrix}
C_p' & 0 & M_p \\
D_p' & I &
\end{bmatrix} \Pi \begin{bmatrix}
C_p & D_p & 0 \\
0 & 0 & I
\end{bmatrix} < 0,
$$

where $(A_p, B_p, C_p, D_p)$ is a state-space realization for $G(s)$.

Then, the feedback interconnection of $G(s)$ and $\Delta(t)$, shown in Figure 1, is $\mathbb{L}_2$ stable.

### 3.2 Integral quadratic constraint interpretation of conicity conditions

Conicity conditions can be interpreted as *hard* IQCs, which then imply the *soft* frequency domain IQC used in Theorem 1. To see this, consider a (nonlinear) operator $\Delta(\cdot) \in \text{Cone}[A,B]$, where $A, B \in \mathbb{R}^{p \times m}$. Define $w(t) = \Delta(v(t))$. Then, from Definition 3, it follows that

$$
\langle w(t) - Av(t), w(t) - Bv(t) \rangle_T \leq 0 \quad \forall v \in \mathbb{L}_2^m \quad \forall T \in \mathbb{R}_+.
$$

This can be written as the hard IQC in the time domain

$$
\Leftrightarrow \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' \begin{bmatrix}
-(A' + B'A) & (A + B)' \\
A + B & -2I
\end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall v \in L_2^{m} \quad \forall T \in \mathbb{R}_+.
$$

Taking the limit as $T \to \infty$ then implies

$$
\int_0^\infty \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' \begin{bmatrix}
-(A' + B'A) & (A + B)' \\
A + B & -2I
\end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall v \in L_2^{m}.
$$

Taking Fourier transforms then implies the IQC in inequality (7), where $\Pi(j\omega) = \Pi_C(A, B)$ is a static symmetric matrix. Mathematically,

$$
\Delta(\cdot) \in \text{Cone}[A,B] \Rightarrow \Delta \in \text{IQC}(\Pi_C(A, B)).
$$

In fact, in certain cases, it can be shown that the converse is also true, i.e., $\Delta(\cdot) \in \text{IQC}(\Pi_C(A, B)) \Rightarrow \Delta(\cdot) \in \text{Cone}[A,B]$; this is discussed later in this paper.
4 | STABILITY RESULTS USING IQCS

The aim of this section is to obtain stability results for systems satisfying matrix conic sector conditions introduced earlier, using Theorems 1 and 2. The results are considered novel since, to the authors’ knowledge, IQCs have not been used in conic sector stability analysis and, moreover, they result in appealing frequency domain and LMI-based algorithms for stability verification.

4.1 | A preliminary lemma

In the work of Zames,9 several useful lemmas were obtained, which allowed conic information of two systems to imply conic information about a composite system. In particular, the so-called “sum rule” was established, which states If \( H_1 \in \text{Cone}[a_1, b_1] \) and \( H_2 \in \text{Cone}[a_2, b_2] \); then, \( H_1 + H_2 \in \text{Cone}[a_1 + a_2, b_1 + b_2] \), where \( a_1, b_1, a_2, b_2 \in \mathbb{R} \). Unfortunately, in the matrix case, a result of the same elegance does not appear to be true. However, a general sum rule is not required, i.e., a special case where one element is conic and the other is linear is sufficient.

**Lemma 1.** Let \( H \in \text{Cone}[A_1, B_1] \), where \( A_1, B_1 \in \mathbb{R}^{p \times m} \), and let \( K \in \mathbb{R}^{p \times m} \) be another real matrix. Then,

\[
H - K \in \text{Cone}[A_1 - K, B_1 - K].
\]

**Proof.** First, from Definition 4, \( H \in \text{Cone}[A_1, B_1] \) implies that, for all \( y \in L_2^m \),

\[
\|H(y)\|_2^2 \leq \langle (A_1 + B_1)y, H(y) \rangle_T - \langle A_1 y, B_1 y \rangle_T.
\]  

(15)

Now, consider

\[
\|H(y) - Ky\|_2^2 = \|H(y)\|_2^2 - \langle 2Ky, H(y) \rangle_T + \langle Ky, Ky \rangle_T
\]

(16)

\[
\leq \langle (A_1 + B_1)y, H(y) \rangle_T - \langle A_1 y, B_1 y \rangle_T - \langle 2Ky, H(y) \rangle_T + \langle Ky, Ky \rangle_T
\]

(17)

\[
= \langle (A_1 - K + B_1 - K)y, H(y) \rangle_T - \langle A_1 y, B_1 y \rangle_T + \langle Ky, Ky \rangle_T.
\]

(18)

This last equality requires \( K \) to be linear. Adding and subtracting \( \langle (A_1 - K + B_1 - K)y, Ky \rangle_T \) then gives

\[
\|H(y) - Ky\|_2^2 \leq \langle (A_1 - K + B_1 - K)y, H(y) \rangle_T - \langle A_1 y, B_1 y \rangle_T + \langle Ky, Ky \rangle_T
\]

\[
+ \langle (A_1 - K + B_1 - K)y, Ky \rangle_T - \langle (A_1 - K + B_1 - K)y, Ky \rangle_T
\]

(19)

\[
= \langle (A_1 - K + B_1 - K)y, H(y) - Ky \rangle_T + \langle (A_1 - K + B_1 - K)y, Ky \rangle_T
\]

\[
- \langle A_1 y, B_1 y \rangle_T + \langle Ky, Ky \rangle_T
\]

(20)

\[
= \langle (A_1 - K + B_1 - K)y, H(y) - Ky \rangle_T + \langle A_1 y, Ky \rangle_T + \langle B_1 y, Ky \rangle_T
\]

\[
- 2\langle Ky, Ky \rangle_T - \langle A_1 y, B_1 y \rangle_T + \langle Ky, Ky \rangle_T
\]

(21)

\[
= \langle (A_1 - K + B_1 - K)y, H(y) - Ky \rangle_T + \langle A_1 y, Ky \rangle_T + \langle B_1 y, Ky \rangle_T
\]

\[
- \langle Ky, Ky \rangle_T - \langle A_1 y, B_1 y \rangle_T.
\]

(22)

Now,

\[
\langle (A_1 - K)y, (B_1 - K)y \rangle_T = \langle A_1 y, B_1 y \rangle_T + \langle Ky, Ky \rangle_T - \langle A_1 y, Ky \rangle_T - \langle B_1 y, Ky \rangle_T.
\]

(23)

Then, it follows from inequality (22) that

\[
\|H(y) - Ky\|_2^2 \leq \langle (A_1 - K + B_1 - K)y, H(y) - Ky \rangle_T - \langle (A_1 - K)y, (B_1 - K)y \rangle_T.
\]

(24)
Comparing inequality (24) to that in Definition 4 then implies $H - K \in \text{Cone}[A_1 - K, B_1 - K]$ as required.

Remark 1. The only property used in the proof of Lemma 1 is that $K$ is linear. Thus, the “weak” sum rule would also hold with $K$ replaced by a linear operator $K$.

4.2 Main results

It has been observed that, if $\Delta(\cdot) \in \text{Cone}[A, B]$, then it satisfies the hard IQC defined by the matrix $\Pi_C(A, B)$; that is, $\Delta(\cdot) \in \text{IQC}(\Pi_C(A, B))$. Moreover, this hard IQC implies the soft IQC with the same matrix. The following theorem provides both frequency domain and LMI conditions for stability of the interconnection in Figure 1 under certain conditions. The LMI condition can be obtained using a straightforward application of Theorem 2, but the frequency domain condition requires a little more work. This is because condition (ii) of Theorem 1 requires $\tau\Delta$ to satisfy the IQC defined by $\Pi_C(A, B)$ for all $\tau \in [0, 1]$. Referring to the expression for $\Pi_C(A, B)$ in Equation (13) and noting Fact 1, it can be seen that this will be the case, providing that

$$A' B + B' A \leq 0,$$

but otherwise, the application of Theorem 1 is not straightforward. The following is the main result of this paper.

**Theorem 3.** Consider Figure 1, where $G(s) \in \mathbb{R}^{m \times p}$ and $\Delta : \mathcal{L}_2^m \mapsto \mathcal{L}_p^p$ is a bounded causal operator such that $\Delta \in \text{Cone}[A, B]$ for matrices $A, B \in \mathbb{R}^{p \times m}$. Assume that the interconnection is well-posed for all $\Delta(\cdot)$ in this class. Then, the system is $\mathcal{L}_2$ stable for all $\Delta$ described above if

(i) there exists a symmetric matrix $P \succeq 0$ such that the following matrix inequality is satisfied:

$$
\begin{bmatrix}
A_p' P + P A_p & C_p'$
\end{bmatrix} -
\begin{bmatrix}
P B_p & -D_p & C_p (A + B)' + (A + B) D_p
\end{bmatrix} < 0,
$$

where $(A_p, B_p, C_p, D_p)$ is a state-space realization of $G(s)$.

Define $K = \frac{1}{2}(A + B)$ and assume further that $(I - KG(s))^{-1} G(s) \in \mathbb{R}^{m \times p}$, $\det(I - KG(\infty)) \neq 0$ and that the interconnection of $G(s)$ and $\tau\Delta$ is well-posed for all $\tau \in [0, 1]$, and then the system is $\mathcal{L}_2$ stable for all $\Delta$ described above if

(ii) there exists an $\epsilon > 0$ such that the following frequency domain inequality holds:

$$
-G(j \omega)^* (A' B + B' A) G(j \omega) - 2I + (A + B)' + (A + B) G(j \omega) \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
$$

Moreover, this condition is equivalent to the existence of a matrix $P = P'$ such that inequality (25) holds.

**Proof.** Item (i). This follows by direct application of Theorem 2. Recall that $\Pi_C(A, B)$ is given by inequality (13). Hence, stability follows if there exists a $P = P' \succeq 0$, satisfying the following LMI:

$$
\begin{bmatrix}
A_p' P + P A_p & P B_p \\
B_p' P & 0
\end{bmatrix} +
\begin{bmatrix}
C_p' & 0 \\
D_p' & I
\end{bmatrix} \Pi_C
\begin{bmatrix}
C_p & D_p \\
0 & I
\end{bmatrix} < 0.
$$

After some algebra, inequality (25) follows.

Item (ii). Using $K$, as defined in the theorem, and using loopshifting, Figure 1 can equivalently be redrawn as Figure 2, where

$$
\tilde{G}(s) = (I - G(s)K)^{-1} G(s).
$$

By the assumption that $\det(I - KG(\infty)) \neq 0$, it follows that $\tilde{G}(s)$ exists. Moreover, by assumption, $\tilde{G}(s) \in \mathbb{R}^{m \times p}$, and the loopshifted nonlinearity is given by

$$
\tilde{\Delta}(y) = \Delta(y) - Ky.
$$

$\tilde{\Delta}(\cdot) : \mathcal{L}_2^m \mapsto \mathcal{L}_p^p$ is bounded and causal because $\Delta(\cdot)$ and $K$ are. Furthermore, well-posedness follows because of the original well-posedness assumption, $\det(I - KG(\infty)) \neq 0$ and the expression for $\tilde{\Delta}(\cdot)$ above. Lemma 1 then implies

$$
\tilde{\Delta} \in \text{Cone}[A - K, B - K].
$$
Selecting $K = \frac{1}{2}(A + B)$ then yields

$$\tilde{\Delta} \in \text{Cone} \left[ \frac{1}{2}(A - B), \frac{1}{2}(B - A) \right],$$

which implies $\tilde{\Delta} \in \text{IQC}(\tilde{\Pi}_C(\frac{1}{2}(A - B), \frac{1}{2}(B - A)), \text{where } \tilde{\Pi}_C$ is given after some algebra (see Equation (13)) by

$$\tilde{\Pi}_C = \left[ \begin{array}{cc} \frac{1}{2}(A - B)'(A - B) & 0 \\ 0 & -2I \end{array} \right].$$

Clearly, the IQC defined by $\tilde{\Delta}$ is such that

$$\tilde{\Pi}_{C,11} = \frac{1}{2}(A - B)'(A - B) \geq 0$$

$$\tilde{\Pi}_{C,22} = -2I \leq 0.$$  

By Fact 1, this implies that item (ii) of Theorem 1 is satisfied. Therefore, stability holds if there exists an $\epsilon_1$ such that the following frequency domain inequality is satisfied:

$$\frac{1}{2} \tilde{G}^*(j\omega)(A - B)'(A - B)\tilde{G}(j\omega) - 2I \leq -\epsilon_1I \quad \forall \omega \in \mathbb{R}.$$  

(33)

This reduces to

$$\frac{1}{2} \tilde{G}^*(j\omega)(A - B)'(A - B)\tilde{G}(j\omega) - 2I \leq -\epsilon_1I \quad \forall \omega \in \mathbb{R}.$$  

(34)

Next, using the push-through rule, and dropping arguments of $j\omega$ for convenience,

$$\tilde{G} = (I - GK)^{-1}G = G(I - KG)^{-1}.$$  

Thus, inequality (33) can be written as

$$\frac{1}{2}(I - G^*K')^{-1}G^*(A - B)'(A - B)G(I - KG)^{-1} - 2I \leq -\epsilon_1I \quad \forall \omega \in \mathbb{R}.$$  

(35)

Because $(I - KG)$ is invertible, this is equivalent to the existence of an $\epsilon_1$ such that the following inequality is satisfied:

$$\frac{1}{2}G^*(A - B)'(A - B)G - 2(I - G^*K')(I - KG) \leq -\epsilon_1(I - G^*K')(I - KG) \quad \forall \omega \in \mathbb{R}.$$  

(36)
Defining \( \epsilon = 2c_1 \| I - KG \|_{\infty} \), this will hold if there exists an \( \epsilon > 0 \) such that

\[
G^*(A - B)'(A - B)G - 4[I - G^*K' - KG + G^*K'KG] \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
\] (37)

Simplifying, this holds if there exists an \( \epsilon > 0 \) such that

\[
G^*(A - B)'(A - B)G - 4[I - G^*K' - KG + G^*K'KG] \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
\] (38)

\[\Leftrightarrow \]

\[
G^*[(A - B)'(A - B) - 4K'K]G - 4I + 4G^*K' + 4KG \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
\] (39)

Using the fact that \( K = \frac{1}{2}(A + B) \), after some algebra, this is then equivalent to the existence of an \( \epsilon > 0 \) such that

\[
-G^*(A'B + B'A)G - 2I + G^*(A + B)' + (A + B)G \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
\] (40)

This is exactly the condition in (ii). To see that condition (ii) corresponds to condition (i) in this case, note that, using the expression for \( \Pi_C \), inequality (40) can be written as

\[
\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_C \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.
\] (41)

From the state-space realization of \( G(s) \), it follows that

\[
\begin{bmatrix} G \\ I \end{bmatrix} \sim \begin{bmatrix} A_p & B_p \\ C_p & D_p \\ 0 & I \end{bmatrix}.
\] (42)

Using the KYP lemma,\(^{21}\) it follows that inequality (41) is satisfied if there exists a symmetric matrix \( P \) such that the following matrix inequality is satisfied:

\[
\begin{bmatrix} A_p'P + PA_p & PB_p \\ B_p'P & 0 \\ 0 & I \end{bmatrix}^* \Pi_C \begin{bmatrix} C_p & D_p \\ 0 & I \end{bmatrix} < 0.
\] (43)

After a little algebra, this inequality becomes exactly (25).

The appeal of Theorem 3 is that, for a given Cone \([A, B]\), stability can be determined by considering either the LMI in condition (i) or the frequency domain inequality in condition (ii). Compared to the work of Zames,\(^{9}\) the conditions remain the same regardless of the signs of the elements of \( A \) and \( B \); the conic sector conditions are replaced by a simple frequency domain inequality, capturing all cases in which the elements of \( A \) and \( B \) are finite. Compared to the work of Safonov and Athans,\(^{11}\) the LMI given in condition (i) seems easier to check and the invertibility requirement on the matrix \( R = \frac{1}{2}(A + B) \) is removed (\( R \) does not have to even be square).

**Remark 2.** If \( G(s) \) is a scalar transfer function, inequality (26) reduces to the frequency domain condition associated with the well-known Circle Criterion. For example, if \( B = 0 \) and \( A = a \) (25) simply becomes

\[\mathfrak{R}[G(j\omega)] \leq \frac{1}{a}.\]

The slightly odd form is due to the positive feedback convention; if a negative feedback convention was used, equivalent to replacing \( G(j\omega) \) with \(-G(j\omega)\), one would obtain

\[\mathfrak{R}[G(j\omega)] \geq -\frac{1}{a},\]

which is the familiar basic Circle Criterion frequency domain inequality that can be tested graphically using the Nyquist plot.
4.3 | Relation to other results

4.3.1 | Zames’ original results

In Zames’ original paper, stability of interconnections of systems, where one system belonged to a certain cone, was stated in terms of the conic conditions required on the other system. This idea was again used, in a more modern interpretation. Instead, in Theorem 3, results are given in terms of a frequency domain inequality. The relation between the IQC approach in terms of the conic conditions required on the other system. This idea was again used, in a more modern interpretation. In Zames’ original paper, stability of interconnections of systems, where one system belonged to a certain cone, was stated in some cases, the frequency domain condition obtained in the IQC approach reduces to the conic conditions originally specified in the work of Zames.

First, note that, for the special case that $\Delta(\cdot) \in \text{Cone}[aI, bI]$, inequality (26) from Theorem 3 implies that, for some $u(t) \in L^2$, \[
\int_{-\infty}^{\infty} \hat{u}(j\omega) [-2abG(j\omega)G(j\omega) - 2I + (a + b)G(j\omega)^* + G(j\omega)] \hat{u}(j\omega) d\omega < 0. \tag{44}
\]
This can be rearranged to obtain \[
\int_{-\infty}^{\infty} \left[ \begin{array}{c} \hat{u}(j\omega) \\ G(j\omega)\hat{u}(j\omega) \\ (a + b)I \\ \end{array} \right] \left[ \begin{array}{cccc} -2I & (a + b)I \\ (a + b)I & -2abI \\ \end{array} \right] \left[ \begin{array}{c} \hat{u}(j\omega) \\ G(j\omega)\hat{u}(j\omega) \\ (a + b)I \\ \end{array} \right] d\omega < 0. \tag{45}
\]
Using Parseval’s identity, this is equivalent to \[
\int_{0}^{\infty} \left[ \begin{array}{c} u(t) \\ Gu(t) \\ \end{array} \right] \left[ \begin{array}{cccc} -2I & (a + b)I \\ (a + b)I & -2abI \\ \end{array} \right] \left[ \begin{array}{c} u(t) \\ Gu(t) \\ \end{array} \right] dt < 0, \tag{46}
\]
where $Gu(t)$ means the convolution of the impulse response of $G$ and $u(t)$. Now, using the argument in the work of Pfifer and Seiler, the aforementioned inequality implies, for all $T \in \mathbb{R}_+$, \[
\int_{0}^{\infty} \left[ \begin{array}{c} P_T[u(t)] \\ GP_T[u(t)] \\ \end{array} \right] \left[ \begin{array}{cccc} -2I & (a + b)I \\ (a + b)I & -2abI \\ \end{array} \right] \left[ \begin{array}{c} P_T[u(t)] \\ GP_T[u(t)] \\ \end{array} \right] dt < 0. \tag{47}
\]
Since $P_T[u(t)] = 0 \forall t > T$, this is equivalent to \[
\int_{0}^{T} \left[ \begin{array}{c} P_T[u(t)] \\ GP_T[u(t)] \\ \end{array} \right] \left[ \begin{array}{cccc} -2I & (a + b)I \\ (a + b)I & -2abI \\ \end{array} \right] \left[ \begin{array}{c} P_T[u(t)] \\ GP_T[u(t)] \\ \end{array} \right] dt - 2ab \int_{T}^{\infty} (GP_T[u(t)])(GP_T[u(t)]) dt < 0. \tag{48}
\]
If $ab \leq 0$, this then implies that \[
\int_{0}^{T} \left[ \begin{array}{c} P_T[u(t)] \\ GP_T[u(t)] \\ \end{array} \right] \left[ \begin{array}{cccc} -2I & (a + b)I \\ (a + b)I & -2abI \\ \end{array} \right] \left[ \begin{array}{c} P_T[u(t)] \\ GP_T[u(t)] \\ \end{array} \right] dt < 0. \tag{49}
\]
Then, since $G$ is causal, and since $P_T[u(t)] = u(t)$ on the interval $t \in [0, T]$, \[
\int_{0}^{T} \left[ \begin{array}{c} u(t) \\ Gu(t) \\ \end{array} \right] \left[ \begin{array}{cccc} -2I & (a + b)I \\ (a + b)I & -2abI \\ \end{array} \right] \left[ \begin{array}{c} u(t) \\ Gu(t) \\ \end{array} \right] dt < 0. \tag{50}
\]
This can be rewritten as \[
\|u\|_{2T}^2 + ab\|Gu\|_{2T}^2 - (a + b)\langle u, Gu \rangle_T > 0, \tag{51}
\]
for all $T \in \mathbb{R}_+$. Since we have assumed that $ab < 0$, this is equivalent to \[
\|Gu\|_{2T}^2 + \frac{1}{ab}\|u\|_{2T}^2 - \left( \frac{1}{a} + \frac{1}{b} \right) \langle u, Gu \rangle_T < 0. \tag{52}
\]
which then implies $G \in \text{Cone}(1/b, 1/a)$. This is equivalent to the classical condition in the work of Zames, except for the fact that the positive feedback convention implies $G \in \text{Cone}(1/b, 1/a)$, rather than $G \in \text{Excone}(1/b, 1/a)$. 

The central idea used in relating the IQC results to the classical results is the fact that, in some cases, a soft IQC implies a hard IQC. This is the case here when \( ab \leq 0 \), ie, the “IQC” (46) implies the “IQC” (50). When \( ab > 0 \), the soft IQC (46) does not imply the hard IQC (50), so no immediate conditions on the conic properties required of \( G(s) \) can be given. This conclusion is consistent with Theorem 4 in the work of Seiler\(^19\) where sufficient conditions are given for IQCs to admit hard factorization, ie, these conditions are not satisfied when \( ab > 0 \), but are when \( ab \leq 0 \).

### 4.3.2 Generalized scattering transformations

Recently, Polushin and Usova et al introduced a generalization of the scattering transformation\(^{23,24}\) and showed how this could be applied to the stability of interconnections of conic systems. This approach was carried out in a QRS-dissipativity framework, but the work is related to the results presented here. From an input-output perspective, for the case where \( \Delta(\cdot) : \mathcal{L}_2^m \mapsto \mathcal{L}_2^m \), the generalized scattering transformation of\(^{23}\) is equivalent to the following hard IQC:

\[
\int_{t_0}^{t_1} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' W(\phi_c, \phi_r) \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall t_1 \geq t_0, \tag{53}
\]

where

\[
W(\phi_c, \phi_r) = \lambda \begin{bmatrix} \cos 2\phi_c - \cos 2\phi_r & \sin 2\phi_c \\ \sin 2\phi_c & -(\cos 2\phi_c + \cos 2\phi_r) \end{bmatrix} \otimes I_m, \tag{54}
\]

where \( \lambda > 0 \), \( \phi_c \in \mathbb{R} \) and \( \phi_r \in (0, \pi/2) \). The hard IQC (53) implies the soft IQC

\[
\int_{0}^{\infty} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' W(\phi_c, \phi_r) \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall v \in \mathcal{L}_2^m. \tag{55}
\]

Note further that \( W(\phi_c, \phi_r) \) can be rewritten as

\[
W(\phi_c, \phi_r) = \lambda \begin{bmatrix} w_{11}I_m & w_{21}I_m \\ w_{12}I_m & -w_{22}I_m \end{bmatrix}, \tag{56}
\]

where

\[
w_{11} := \cos 2\phi_c - \cos 2\phi_r \tag{57}
\]

\[
w_{12} := \sin 2\phi_c \tag{58}
\]

\[
w_{22} := \cos 2\phi_c + \cos 2\phi_r. \tag{59}
\]

Generally, \( w_{22} \) is not positive, so slightly different transformations will be required to convert the IQC (55) to an IQC of the form (13), depending on the sign of \( w_{22} \). In the case that \( w_{22} > 0 \), we have that the IQC (55) is equivalent to

\[
\int_{0}^{\infty} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} 2w_{11}I_m & 2w_{12}I_m \\ 2w_{12}I_m & -2w_{22}I_m \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall v \in \mathcal{L}_2^m. \tag{60}
\]

Comparing this to the IQC (13), this implies that

\[
-(A'B + B'A) = 2 \frac{w_{11}}{w_{22}} I_m \tag{61}
\]

\[
A + B = 2 \frac{w_{12}}{w_{22}} I_m. \tag{62}
\]

From this, a little algebra shows that the generalized scattering IQC (55) is equivalent to the standard soft IQC (13) with \( \Delta(.) \in IQC(\Pi_C(aI_m, bI_m)) \), where

\[
a = \frac{\cos 2\phi_r - \cos 2\phi_c}{\sin 2\phi_r + \sin 2\phi_c} \tag{63}
\]

\[
b = \frac{\sin 2\phi_r + \sin 2\phi_c}{\cos 2\phi_r + \cos 2\phi_c}. \tag{64}
\]
Thus, if \( \Delta(\cdot) \) satisfies a generalized scattering IQC, an alternative to the analysis proposed in the work of Polushin\(^{23} \) is Theorem 3, with \( A = aI_n \) and \( B = bI_m \), provided of course that the other system in the interconnection is linear.

### 4.3.3 Exterior conic sector conditions

A notion closely related to that of the conic sector is the so-called exterior conic sector, again introduced in the work of Zames,\(^9 \) and used extensively in that paper. Similar to Definition 4, the matrix generalization of Zames’ exterior conic sector is given as follows.

**Definition 6.** An operator \( H : \mathcal{L}_{2e}^p \mapsto \mathcal{L}_{2e}^m \) is said to belong to the Excone\( [A, B] \), for matrices \( A, B \in \mathbb{R}^{m \times p} \) if

\[
\|H(x)\|_{2e}^2 + \langle Ax, Bx \rangle_T - \langle (A + B)x, H(x) \rangle_T \geq 0 \quad \forall x \in \mathcal{L}_{2e}^m, \quad \forall T \in \mathbb{R}_+.
\]

If the inequality holds strictly for all \( 0 \neq x \in \mathcal{L}_{2e}^m, H \in \text{Excone}(A, B) \).

It is easy to see that the exterior conic sector condition can be viewed as the hard IQC

\[
\Leftrightarrow \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \begin{bmatrix} \langle A'B + B'A \rangle - \langle A + B \rangle' \\ -(A + B) \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \quad \forall v \in \mathcal{L}_{2e}^m, \quad \forall T \in \mathbb{R}_+.
\]

(65)

It is clear that \( H \in \text{Excone}(A, B) \Rightarrow H \in \text{IQC}(\Pi_p(A, B)) = \text{IQC}(-\Pi_c(A, B)) \). With this in mind, a straightforward application of Theorem 2 then yields the following corollary.

**Corollary 1.** Consider Figure 1, where \( G(s) \in \mathbb{R}^{m \times p} \) and \( \Delta(\cdot) : \mathcal{L}_{2e}^m \mapsto \mathcal{L}_{2e}^p \) is a bounded causal operator such that \( \Delta \in \text{Excone}(A, B) \) for matrices \( A, B \in \mathbb{R}^{p \times m} \). Assume that the interconnection is well-posed for all \( \Delta(\cdot) \) in this class.

Then, the system is \( \mathcal{L}_2 \) stable for all \( \Delta \) described above if there exists a symmetric matrix \( P \geq 0 \) such that the following matrix inequality is satisfied:

\[
\begin{bmatrix}
A_p'P + PA_p + C_p'(A'B + BA')C_p & PB_p + C_p'(A'B + BA')D_p - C_p'(A + B)' \\
* & 2I + D_p'(A'B + BA')D_p - D_p'(A + B)' - (A + B)D_p
\end{bmatrix} < 0.
\]

(66)

Interestingly, this corollary reveals that, when \( \Delta(\cdot) \in \text{Excone}(A, B) \), a conclusion of stability is only possible, from the IQC perspective, if the plant has a direct feedthrough term ie, \( D_p \neq 0 \).

### 5 EXAMPLES

#### 5.1 Multiple conic uncertainties

The results can be applied to systems which are subject to a variety of perturbations. Consider the system in Figure 3. Assume that \( \Delta_1(\cdot) : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e} \) and \( \Delta_2(\cdot) : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e} \) represent two conic sector type uncertainties, viz,

\[
\Delta_1(\cdot) \in \text{Cone}[a_1, b_1]
\]

(67)

**FIGURE 3** System with multiple uncertainties. \( G(s) \) is a stable LTI system; \( \Delta_i(\cdot) \) bounded causal operators with different properties.
\[ \Delta_2(\cdot) \in \text{Cone}[a_2, b_2]. \] (68)

Assume further that the system is subject to some norm-bounded uncertainty \( \Delta_3(\cdot) : L^q_{2e} \mapsto L^q_{2e} \)

\[ \| \Delta_3(\cdot) \|_{l_2} \leq \gamma, \] (69)

where \( \| \cdot \|_{l_2} \) denotes the \( L_2 \) induced norm. This can be expressed as a conic bound

\[ \Delta_3(\cdot) \in \text{Cone}[ -\gamma I_q, \gamma I_q ]. \] (70)

Letting \( a = \min \{ a_1, a_2, -\gamma \} \) and \( b = \max \{ b_1, b_2, \gamma \} \), we have that

\[
\Delta(\cdot) := \begin{bmatrix}
\Delta_1(\cdot) \\
\Delta_2(\cdot) \\
\Delta_3(\cdot)
\end{bmatrix} \in \text{Cone}[a, b].
\] (71)

Therefore, the results of the work of Bridgeman and Forbes\(^{12}\) could be used to analyze the system's stability. Alternatively, it follows directly that

\[
\Delta(\cdot) \in \text{Cone}[A, B].
\] (72)

where \( A = \text{diag}(a_1, b_1, -\gamma I_q) \) and \( B = \text{diag}(b_1, b_2, \gamma I_q), A \leq B \), and then Theorem 3 could be used to analyze the system. One would anticipate less conservative results with the latter approach. In order to see that this is indeed the case consider the system \( G(s) \) borrowed from the work of Suykens et al\(^{25}\) with state-space matrices

\[
A_p = \text{diag}(-1, -4, -6, -2, -9, -8, -3, -10, -12)
\] (73)

\[
B_p = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\] (74)

\[
C_p = \begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1
\end{bmatrix}.
\] (75)

With the choices \( a_1 = a_2 = 0, b_1 = 0.1 \) and \( b_2 = 1 \), it transpires that the multiple-input–multiple-output conic sector analysis of Theorem 3 allows one to conclude stability for all \( \Delta(\cdot) \in \text{Cone}[A, B] \) with \( \gamma \leq 2.86 \) approximately. However, the same analysis carried out using the scalar conic sector analysis, one can only conclude stability for all \( \Delta(\cdot) \in \text{Cone}[aI_3, bI_3] \) if \( \gamma \leq 0.91 \) approximately; the matrix conic sector analysis guarantees stability for norm-bounded uncertainty of around three times the size of that for which stability is guaranteed using the scalar conic sector analysis.

5.2 \ Nonsquare conic uncertainty

This example illustrates how the results of this paper may be applied to a nonstandard robust stability problem. The problem appears to be difficult to analyze, in a straightforward manner, using other approaches at least partly because the “uncertainty” has a nonsquare form. The example is contrived, but it is hoped that it gives insight into the scope of the results.
Consider the linear system \( G_{nsq}(s) \), as depicted in Figure 4, given by the following state-space realization:

\[
G_{nsq}(s) \approx \begin{bmatrix}
A_p & B_p \\
C_p & D_p
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & -0.5 & -1 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

The uncertainty \( \Delta_{nsq}(\cdot) \) is dynamic, nonlinear, and has the form

\[
\begin{cases}
\dot{x}_3 = -\alpha x_3 + x_4 + \gamma u_2 + x_4 \text{sat}(u_1) \\
\dot{x}_4 = -x_3 - x_4^3 - x_4 \text{sat}(u_1) \\
y = x_3,
\end{cases}
\]

where \( \alpha > 2 \) and \( \gamma \in [1, 2] \) are uncertain parameters. Using the storage function

\[
V(x_3, x_4) = \frac{1}{2} (\lambda_1 x_3^2 + \lambda_2 x_4^2), \quad \lambda_1, \lambda_2 > 0,
\]

it is easy to see that \( \Delta_{nsq}(\cdot) \) satisfies the hard IQC

\[
\int_0^T y(t)(u_2(t) - y(t))dt \geq 0 \quad \forall T \geq 0
\]

for any \( \gamma \in [1, 2] \), which is equivalent to

\( \Delta_{nsq}(\cdot) \in \text{Cone}([0 \ 0], [0 \ 1]). \)

Application of Theorem 3 then allows stability to be concluded for this type of uncertainty for any \( \gamma \in [1, 2] \). Because the nonlinearity \( \Delta_{nsq} \) is nonsquare, a passivity analysis is not possible, yet it is clear that a (nonsquare) conic condition is satisfied, making Theorem 3 applicable. While it may be possible to analyze the interconnection in Figure 4 with other techniques, the conic approach advocated here appears natural.

6 | CONCLUSION

The conic sector stability results of Zames and Safonov and Athans were reported some time ago, but until recently convenient analysis tools were limited. Recently, Bridgeman and Forbes\(^\text{12}\) reported LMI-based stability analysis tools for Zames' original conic sector results, but the more general results of Safonov and Athans\(^\text{11}\) were not considered. This paper has considered both results in a unified IQC framework. Formulating the problem in this manner allows convenient LMI-based analysis tools to be derived, and it also appears that some of the invertibility conditions in Safonov and Athans' work are not required here. This paper has concentrated on the case where the sector bounds are static, but the case where the bounds are described by LTI (linear time invariant) systems should be treatable in a similar framework. An additional benefit of the IQC formulation given here is that the conic sector stability analysis can be combined with various other uncertainty analysis if required. Furthermore, because the multiplier matrices are simply static, it is possible, if tedious, to obtain linear controller synthesis procedures using this work as a basis.
ACKNOWLEDGEMENTS

This work was funded, in part, by the UK Engineering and Physical Sciences Research Council under grant EP/N00924X/1. The authors are grateful to the anonymous reviewers for their insightful and detailed comments on various versions of the manuscript; the current form of this paper owes much to these comments.

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How to cite this article: Turner MC, Forbes JR. Conic sector analysis using integral quadratic constraints. Int J Robust Nonlinear Control. 2020;30:741–755. https://doi.org/10.1002/rnc.4803