Embeddings of $k$-complexes in $2k$-manifolds and minimum rank of partial symmetric matrices

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Abstract

Let $K$ be a $k$-dimensional simplicial complex having $n$ faces of dimension $k$, and $M$ a closed orientable $(k - 1)$-connected PL $2k$-dimensional manifold. We prove that for $k \geq 3$ odd $K$ embeds into $M$ if and only if there are

- a skew-symmetric $n \times n$-matrix $A$ with $\mathbb{Z}$-entries whose rank over $\mathbb{Q}$ does not exceed $\text{rk} H_k(M; \mathbb{Z})$,
- a general position PL map $f : K \to \mathbb{R}^{2k}$, and
- orientations on $k$-faces of $K$

such that for any nonadjacent $k$-faces $\sigma, \tau$ of $K$ the element $A_{\sigma, \tau}$ equals the algebraic intersection of $f\sigma$ and $f\tau$.

We prove some analogues of this result including those for $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability. Our results generalize the Bikeev-Fulek-Kynčl criteria for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of graphs to surfaces, and are related to the Harris-Krushkal-Johnson-Paták-Tancer criteria for the embeddability of $k$-complexes into $2k$-manifolds.

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1 Introduction

1.1 Main results

The study of graph drawings on 2-dimensional surfaces is an active area of mathematical research. Higher-dimensional generalization is classical, and has attracted some attention recently, see Remark 1.1.7.bcf. Just like for $k = 1$, it is known and simple that any $k$-dimensional (simplicial) complex embeds into some $2k$-dimensional manifold. A major problem asks if there exists an algorithm for recognizing the embeddability of $k$-dimensional complexes into a given $2k$-dimensional manifold. A quick algorithm based on a beautiful mathematical result is (as always) preferable. If $k = 1$, or if the manifold is a sphere (or a ball or $\mathbb{R}^{2k}$), then there is a classical algorithm, see e.g. survey [MTW, §1]. For $k > 1$ and a closed manifold different from $S^{2k}$ no algorithm is known.

We shorten ‘$k$-dimensional (face, manifold, etc)’ to ‘$k$- (face, manifold, etc)’.

Our main results are criteria for embeddability (and for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability defined below) of $k$-complexes to $2k$-manifolds (Theorems 1.1.1, 1.1.2 and 1.1.5). We reduce embeddability to finding minimal rank of a matrix, of whose entries some are fixed, and the other can be changed. This ‘low rank matrix completion problem’ is related to the Netflix problem from machine learning, and is extensively studied in computer science, see e.g. [Ko21], surveys [MC, NKS], and the introduction [DGN+] accessible to students.

These criteria allow to prove new interesting Corollary 1.1.4.b, and [KS21e, Theorem 2.4.2] due to E. Kogan [Ko]. We also present Corollary 1.1.4.ac proved using an essentially known criterion.

![Figure 1: Two curves intersecting at an even number of points the sum of whose signs is zero (left) or non-zero (right).](image)

In this paper $k \geq 1$ is any integer, $K$ is any $k$-complex, and $n = n_K$ is the number of $k$-faces of $K$. See Definitions 1.1.6.ab of a general position map, and of the algebraic intersection number $f_\sigma \cdot f_\tau$ (Figure 1).

The complex $K$ is called compatible to an $n \times n$-matrix $A$ with $\mathbb{Z}$-entries if there is a general position map $f : K \to \mathbb{R}^{2k}$ such that for some orientations on $k$-faces (for $k = 1$ edges) of $K$ we have

$$(C_{f,\mathbb{Z}}) \quad A_{\sigma,\tau} = f_\sigma \cdot f_\tau \quad \text{for any non-adjacent $k$-faces $\sigma, \tau$ of $K$.}$$

It is not clear whether compatibility is algorithmically decidable (although a general position map $f : K \to \mathbb{R}^{2k}$ can be ‘algorithmized’ using the case $M = \mathbb{R}^{2k}$ of Lemma 2.3.4).

In this paper we shorten ‘compact $2k$-manifold possibly, with boundary’ to ‘manifold’. Unless otherwise indicated, we consider only piecewise linear (PL) manifolds and maps (thus we mostly omit ‘PL’). The analogues of Theorems 1.1.1 and 1.1.2.a are correct for topological embeddings (by the PL approximation theorem [Br72, Theorem 1], cf. [DS22, Remark 1.3.b]), as well as for almost embeddings and $\mathbb{Z}$-embeddings (defined and discussed after Theorem 1.1.2 and in §1.2).
Let $\Delta^j_N$ be the union of the $j$-faces of the $N$-simplex. A manifold $M$ is $s$-connected if for any $j = 0, 1, \ldots, s$ any continuous map $\Delta^j_{j+1} \to M$ extends to a continuous map $\Delta^j_{j+1} \to M$.

Let $R$ be either $\mathbb{Z}_2$ or $\mathbb{Z}$, and $M$ a manifold. For simple accessible to non-specialists in topology definitions of the homology group $H_k(M; R)$ and intersection product $\cap_{R,M} : H_k(M; R) \times H_{n-k}(M; R) \to R$ see [HG], [IF, §2], [Sk20, §6, §10]. We omit $\mathbb{Z}_2$-coefficients from the notations of homology groups and intersection products. By the rank of a matrix with integer entries, or of a bilinear form on a $\mathbb{Z}$-module, we mean its rank over $\mathbb{Q}$. We abbreviate $\text{rk}_M$ to $\text{rk} M$, and $\text{rk}_M \cap Z$ to $\text{rk}_Z M$.\footnote{E.g., if $M$ is the connected sum of $s$ copies of $S^k \times S^k$, then $\text{rk} M = \text{rk}_Z M = 2s$; also $\text{rk} \mathbb{C} P^2 = \text{rk}_Z \mathbb{C} P^2 = \text{rk} \mathbb{H} P^2 = \text{rk}_Z \mathbb{H} P^2 = 1.$}

**Theorem 1.1.1.** Assume that $k \geq 3$ is odd, and $M$ is a closed orientable $(k-1)$-connected $2k$-manifold. The complex $K$ embeds into $M$ if and only if $K$ is compatible to a skew-symmetric matrix whose rank does not exceed $\text{rk} Z M$.

This follows by Theorem 1.3.1, Lemma 2.1.6, and the fact that any unimodular skew-symmetric bilinear form over $\mathbb{Z}$ is isomorphic to the symplectic form [IF, Proposition 6.2].

Theorems 1.1.1 and 1.1.2.a could be useful for algorithmic questions [PT19, Problem 26].

**Theorem 1.1.2.** (a) A 4-complex embeds into the quaternionic projective space $\mathbb{H} P^2$ if and only if the complex is compatible to a rank 1 symmetric $n \times n$-matrix with $\mathbb{Z}$-entries, whose any diagonal element is the square of an integer.

(b) A 2-complex has a $\mathbb{Z}$-embedding (defined below) to the complex projective space $\mathbb{C} P^2$ if and only if the complex is compatible to a matrix as in (a).

For an orientable $2k$-manifold $M$, a general position map $h : K \to M$ is called a $\mathbb{Z}$-embedding if $h \sigma \cdot h \tau = 0$ for any pair $\sigma, \tau$ of non-adjacent $k$-faces. (The sign of $h \sigma \cdot h \tau$ depends on an arbitrary choice of orientations of $M, \sigma, \tau$, and on the order of $\sigma, \tau$, but the condition $h \sigma \cdot h \tau = 0$ does not.)

Theorem 1.1.2 follows by Theorem 1.3.1 and a simple Lemma 2.1.7 due to E. Kogan [KS21e, Lemma 2.1.7], [Ko] (for (a) we also need Theorem 1.2.1.a).

For a $2k$-manifold $M$, a general position map $h : K \to M$ is called a $\mathbb{Z}_2$-embedding if $|h \sigma \cap h \tau|$ is even for any pair $\sigma, \tau$ of non-adjacent $k$-faces. Clearly, any $\mathbb{Z}$-embedding is a $\mathbb{Z}_2$-embedding. An example of a $\mathbb{Z}_2$-embedding which is not a $\mathbb{Z}$-embedding is shown in Fig. 1, right. For motivations and discussion of $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings see §1.2.

The rank $P(M)$ is the minimal number $\text{rk}_Z M$, where $M$ is a $2k$-manifold, and $K$ has an embedding to $M$. The $Z_2$-rank $R_{Z_2}(K)$ is the minimal number $\text{rk}_Z M$, where $M$ is a $2k$-manifold, and $K$ has a $\mathbb{Z}_2$-embedding to $M$. The $\mathbb{Z}$-rank $R_{\mathbb{Z}}(K)$ is the minimal number $\text{rk}_Z M$, where $M$ is an orientable $2k$-manifold, and $K$ has a $\mathbb{Z}$-embedding to $M$.

**Proposition 1.1.3** (additivity). For any $k$-complexes $X, Y$ we have

(a) $P(X \sqcup Y) = P(X) + P(Y)$;

(b) $R_{Z_2}(X \sqcup Y) = R_{Z_2}(X) + R_{Z_2}(Y)$;

(c) $R_{\mathbb{Z}}(X \sqcup Y) = R_{\mathbb{Z}}(X) + R_{\mathbb{Z}}(Y)$.

This is proved in §2.4 independently of any embeddability criteria; the proof appeared in a discussion with T. Garaev.

Define $\rho(K), r_{Z_2}(K), r_{\mathbb{Z}}(K)$ analogously to $P(K), R_{Z_2}(K), R_{\mathbb{Z}}(K)$ but adding ‘$(k-1)$-connected’ before ‘$2k$-manifold’.
Corollary 1.1.4 (additivity). For any $k$-complexes $X, Y$ we have

(a) $\rho(X \sqcup Y) = \rho(X) + \rho(Y)$ for $k \geq 3$;
(b) $r_{\mathbb{Z}_2}(X \sqcup Y) = r_{\mathbb{Z}_2}(X) + r_{\mathbb{Z}_2}(Y)$;
(c) $r_{\mathbb{Z}}(X \sqcup Y) = r_{\mathbb{Z}}(X) + r_{\mathbb{Z}}(Y)$.

By Theorem 1.2.1.a $\rho(K) = r(K)$ for $k \geq 3$. So (a) follows by (c). Parts (b,c) are proved in §2.4 using Theorems 1.1.5 and 1.3.1 below.

Denote by $|S|_2 \in \mathbb{Z}_2$ the number of elements modulo 2 in a finite set $S$.

The complex $K$ is called **compatible modulo 2** to a symmetric $n \times n$-matrix $A$ with $\mathbb{Z}_2$-entries if there is a general position PL map $f : K \to \mathbb{R}^{2k}$ such that

$$(C_f) \quad A_{\sigma, \tau} = |f_{\sigma} \cap f_{\tau}|_2 \quad \text{for any non-adjacent $k$-faces (for $k = 1$ edges) $\sigma, \tau$ of $K$.}$$

Compatibility modulo 2 is algorithmically decidable by the case $M = \mathbb{R}^{2k}$ of Lemma 2.3.2.

For $R = \mathbb{Z}$ or $\mathbb{Z}_2$ a bilinear form $q : V \times V \to R$ on a $\mathbb{Z}_2$-vector space or on a $\mathbb{Z}$-module $V$ is called **even** if $q(v, v)$ is even for any $v \in V$, and is called **odd** otherwise. A symmetric matrix with $\mathbb{Z}_2$- or $\mathbb{Z}$-entries is called **even** (for the case of $\mathbb{Z}_2$ a.k.a. **alternate**) if its diagonal contains only even entries, and is called **odd** otherwise. The **type** of a bilinear form or of a symmetric matrix is its being even or odd.

**Theorem 1.1.5** (proved in §2.2, §2.3). Assume that $M$ is a $(k-1)$-connected $2k$-manifold. There is a $\mathbb{Z}_2$-embedding $K \to M$ if and only if $K$ is compatible modulo 2 to a matrix of the same type as $\cap_M$, and whose rank does not exceed $\text{rk} M$.

See more equivalent conditions in [KS21e, Proposition 2.5.1]. The connectedness assumption in Theorem 1.1.5 is essential for $k = 1$. It would be interesting to know if the connectedness assumption in Theorems 1.1.5 and 1.3.1 is essential for $k > 1$.

**Definition 1.1.6.** (a) A map $f : K \to M$ to a $2k$-manifold is **general position map** if

- any vertex-disjoint faces the sum of whose dimensions is less than $2k$ have disjoint images,
- the restriction of $f$ to any $k$-face has a finite set of self-intersection points,
- for any vertex-disjoint $k$-faces $\sigma, \tau$
  - the set $f_{\sigma} \cap f_{\tau}$ is finite and is disjoint with self-intersections of $f|_{\sigma}$ and $f|_{\tau}$,
  - for any point from $f_{\sigma} \cap f_{\tau}$, and some small $(2k-1)$-sphere $S$ centered at this point, the intersections $S \cap f_{\sigma}$ and $S \cap f_{\tau}$ are $(k-1)$-spheres having linking number $\pm 1$ in $S$.
(b) Let $h : K \to M$ be a general position map to an orientable $2k$-manifold. Then preimages $y_1, y_2 \in K$ of any double point $y \in M$ lie in the interiors of $k$-faces. By general position, $h$ is ‘linear’ on some neighborhood $U_j$ of $y_j$ for each $j = 1, 2$. Given orientations on $k$-faces, we can take two bases of oriented $hU_1, hU_2$. The **intersection sign** of $y$ is the sign $\pm 1$ of the base formed by the base of $hU_1$ followed by the base of $hU_2$. The **integer intersection number** $h_{\sigma} \cdot h_{\tau} \in \mathbb{Z}$ for non-adjacent oriented $k$-faces $\sigma, \tau$ is the sum of the intersection signs of all intersection points from $h_{\sigma} \cap h_{\tau}$.

Let us prove that. Since $K_{10}$ contains $K_5 \sqcup K_5$, by Corollary 1.1.3 we have $r_{\mathbb{Z}_2} K_{10} \geq 2$. Then $K_{10}$ has no $\mathbb{Z}_2$-embedding to the Moebius band. So $K_{10}$ has no $\mathbb{Z}_2$-embedding to the disjoint union $M$ of 1000 Moebius bands. However, $K_{10}$ has $10 \cdot 9/2 = 45$ edges, so it embeds into the disk with $45 \cdot 44/2 < 1000$ Moebius bands. Since $K_{10}$ has 45 edges, it is compatible modulo 2 to some odd matrix of rank at most $45 < 1000 = \text{rk} M$. 

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Remark 1.1.7 (relation to known results). (a) The cases $k = 1$ of Theorems 1.1.5 and 1.3.1, of Remarks 1.2.4 and 2.3.5 (and of the $\mathbb{Z}$-analogue of Theorem 1.1.1) are known [FK19, Proposition 9, Corollary 10], [Bi21, Theorems 1.1, 1.4].

The case $k = 1$ of Corollary 1.1.4 (equivalent to Proposition 1.1.3 for $k = 1$ as explained in its proof) should be considered known. This case is classical for Corollary 1.1.4.a, was stated as [SS13, Lemmas 6, 7] for Corollary 1.1.4.b, and is analogous for Corollary 1.1.4.c. The paper [SS13] presents interesting heuristic ideas, but not a reliable proof (as justified in Remark 2.5.1.ab). The case $k = 1$ of Corollary 1.1.3.b (1.1.3.c) is simply recovered (proved) in this paper using the non-trivial cases $k = 1$ of Theorems 1.1.5 and 1.3.1 proved in [Bi21].

(b) Criteria for the embeddability of $k$-complexes in given $m$-manifold for $2m \geq 3k + 3$ are given in [Ha69, Theorem 1 and Corollaries 6, 7, 8], in terms of isovariant maps or cohomology obstructions (in certain configuration space). Assume further that $m = 2k \geq 6$. Such criteria in a different equivalent form involving ‘intersection cocycle’ are given in [Jo02]; see exposition in §2.3, in [KS21e, Proposition 2.5.1 EH and Remark 2.5.6.b]; cf. [La70].

A formula for the intersection cocycle (and for its cohomology class) is given

(i) implicitly in homological terms in [Kr00, Theorem 3.2], see explanation in [KS21e, Remark 2.5.2.b];

(ii) for $k = 1$ in terms of crossing numbers in [FK19, §3.1, equality (1)];

(iii) in terms of cohomology class $\omega(\psi)$ (in certain configuration space) in [PT19, Theorem 4], see exposition in [KS21e, §2.5, Definitions required for (RH), Lemma 2.5.5].

(c) Such a formula reduces embeddability to solvability of a system of quadratic Diophantine equations, i.e. to realizability defined before Theorem 1.3.1. Such an embeddability criterion is the conjunction of [PT19, Theorems 4, 6 and 15], and is explicitly stated in Theorem 1.3.1 (see also Theorem 1.2.1.a), without using cohomology classes in configuration spaces as in [PT19]. Theorem 1.3.1 (and Lemma 2.3.1.a) are essentially results of Paták-Tancer (with a contribution by Harris-Krushkal-Johnson), although we present an explicit statement and (in §2.2, §2.3) a simple direct proof independent of [PT19]; see also (e).

(d) The main novelty of this paper is passing from the realizability condition to the compatibility condition (defined above). The statements in terms of the compatibility condition are simpler because they do not involve homology classes $y_\sigma$ in the manifold, only the rank of the intersection form of the manifold. (Although the definition of the intersection form requires homology, results on rank are stated without homology, see e.g. footnote 1.)
The notion of compatibility (of a graph to a matrix) appeared in [FK19, Bi21]. Thus our results are higher-dimensional generalizations of criteria for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of graphs to surfaces obtained in Bikeev’s paper [Bi21] using ideas of Fulek-Kynčl-Schaefer-Stefankovič [SS13, FK19]. Just as the criteria of [PT19] (in terms of realizability and cohomological obstruction), these generalizations are not very hard (modulo classical techniques of geometric topology, see details in (e,g)). Although we provide direct proofs of these generalizations, they could be deduced from Theorem 1.3.1, [PT19, Proposition 21 and Theorem 15] and relations between compatibility and realizability (algebraic Lemmas 2.1.1 and 2.1.6). Still, we hope these generalizations are interesting since they reveal relation to the ‘low rank matrix completion problem’, and give interesting corollaries (see the beginning of §1.1).

(e) The case $k = 2$ is excluded from [PT19, Theorems 6, 10.i, and Proposition 21]. However, [PT19, Theorem 10.i, and Proposition 21], and the analogue of [PT19, Theorem 6] obtained by replacing ‘embedding’ by ‘$\mathbb{Z}$-embedding’, do hold for $k = 2$. (By (a), these results are known for $k = 1$.) The proofs could be obtained by minor changes in [PT19]; simpler proofs (of the versions in terms of compatibility) are presented in this paper.

(f) For the Kühnel problem on embeddings of $k$-skeleta of $n$-simplices into $2k$-manifolds see recent paper [DS22] and the references therein.

(g) (known ideas in our proofs) Our constructions of $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings (in Theorems 1.1.5 and 1.3.1) use known construction of a map inducing given homomorphism in homology, cf. [KS21e, Lemma 2.5.3]. Our construction of matrix $A$ and vectors $y_\sigma$ from $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings (in Theorems 1.1.5 and 1.3.1) use essentially known Lemmas 2.3.2 and 2.3.4 on the Harris-Johnson cohomology obstructions (see Remark 2.3.3), and the modification of a given map as in [PT19, Lemma 12], cf. [KS21e, Lemma 2.5.5].

Geometric proofs (closer to [Bi21]) of Theorems 1.1.5 and 1.3.1 could perhaps be obtained using a handle decomposition of $M$.

### 1.2 Almost embeddings, $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings

A map $f : K \to Y$ of a complex $K$ to a subset $Y \subset \mathbb{R}^d$ is called an almost embedding if the images of non-adjacent faces are disjoint, i.e. if $f\sigma \cap f\tau = \emptyset$ for any non-adjacent faces $\sigma, \tau$.

This notion naturally appears in geometric topology (studies of embeddings), in combinatorial geometry (Helly-type results on convex sets), and in topological combinatorics (topological Radon and Tverberg theorems), see [FKT, §4], [Mat97, GPP+] and surveys [Sk16, §1], [Sk06]. The notions of $\mathbb{Z}_2$- and $\mathbb{Z}$-embedding naturally appear in studies of embeddings. The notion of a $\mathbb{Z}_2$-embedding (a.k.a. Hanani–Tutte drawing) is most actively studied for graph drawings on surfaces, see survey [Sc13] and [SS13, FK19, Bi21].

Some proofs of the non-embeddability of complexes into $\mathbb{R}^d$ actually show that these complexes are not almost embeddable to $\mathbb{R}^d$, are not $\mathbb{Z}_2$- or $\mathbb{Z}$-embeddable to $\mathbb{R}^d$ for $d = 2k$. This is so e.g. for the boundary of $(d + 1)$-simplex (the non-almost embeddability is the topological Radon theorem) as well as for the $k$-complex $\Delta_{k+2}^6$ and $d = 2k$. See e.g. surveys [Sk18, §§1.4, 2.2], [Sk14, Theorem 1.4] and [KS21, Remark 1.2.d]. On the other hand, some constructions of embeddings have constructions of almost or $\mathbb{Z}$-embeddings as a convenient intermediate step allowing to structure the proof, and to describe relation to known results and methods.

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6Because of using this construction [PT19, §5, proof of Theorem 6, step 1] and because of using essentially known Lemmas 2.3.2 and 2.3.4 [PT19, §5, proof of Theorem 6, step 2], a part of our proof is similar to [PT19].
Theorem 1.2.1. (a) A complex of dimension $k \neq 2$ is embeddable into a simply connected 2k-manifold $M$ if and only if the complex is $\mathbb{Z}$-embeddable to $M$. (See [vK32, Sh57, Wu58] and also [Ha69, Jo02].)

(b) For each $k \geq 2$ there is a 2-complex $\mathbb{Z}_2$-embeddable but not $\mathbb{Z}$-embeddable to $\mathbb{R}^{2k}$ [Me06, Example 3.6].

(c) There is a 2-complex $\mathbb{Z}$-embeddable but not almost embeddable to $\mathbb{R}^4$. (See [FKT, §3.2, §3.3, §4], [AMS+, Theorem 1.6], [Al22].)

(d) There is a 2-complex almost embeddable but not PL embeddable to $\mathbb{R}^4$ [SSS, Example in p. 338].

Comments on the proof of (a). Clearly, an embedding is a $\mathbb{Z}$-embedding. Let us discuss the converse implication.

For $k = 1$ the converse follows by the Hanani-Tutte Theorem, see survey [Sk18, Theorem 1.5.3] (because a compact simply connected 2-manifold is a sphere).

Assume that $k \geq 3$. For $M = \mathbb{R}^{2k}$ the converse is proved in [vK32, Sh57, Wu58]; for a simple exposition see [FKT, §2], [Sk06, §4]. The general case is proved in the same way, just note that in [FKT, Lemma 4, 5 and application of the Whitney trick in the proof of Theorem 3] $\mathbb{R}^{2k}$ could be replaced by $M$. See details in [Jo02, Corollary 2 and Theorem 4].

The proofs of [Jo02] mentioned here and in Remark 2.3.3 are written for ‘smooth’ maps of complexes but work for PL maps.\(^7\)

See more in [Sk, §6.10 ‘Almost embeddings, $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings’].

Remark 1.2.2. (a) Clearly, the property of being an almost embedding is preserved under sufficiently small perturbation of the map (as opposed to the property of being an embedding). Thus by approximation of continuous maps with PL maps we observe that for complexes in manifolds

- topological embeddability implies PL almost embeddability;
- a PL or topological embedding can be approximated by a general position PL almost embedding;
- PL almost embeddability is equivalent to topological almost embeddability.

Analogous remark and results holds with ‘almost’ replaced by ‘$\mathbb{Z}_2$-’ or by ‘$\mathbb{Z}$-’.

Cf. [DS22, Remark 1.3.c]; observe that [DS22, Remarks 1.3.bc] hold for any complexes, not necessarily skeleta of simplices.

(b) Studies of embeddings of $k$-complexes into 2k-manifolds for $k > 1$ are analogous to studies of $\mathbb{Z}_2$-embeddings (not embeddings) of graphs to surfaces. Indeed, Euler formula $V - E + F = 2$ cannot be applied for $\mathbb{Z}_2$-embeddings of graphs to surfaces. Analogously, for $k > 1$ a $k$-hyperplane in $\mathbb{R}^{2k}$ does not split $\mathbb{R}^{2k}$, so an analogue of Euler formula is not available for $k$-complexes in 2k-manifolds.

Remark 1.2.3. Recall that

(a) there are two isomorphism classes of non-degenerate symmetric bilinear forms over $\mathbb{Z}_2$ of a given rank: odd forms and (for an even rank) even forms [IF, Theorem 6.1];

(b) $(k - 1)$-connected smooth 2k-manifolds with odd forms exist only for $k = 1, 2, 4$ (see a folklore proof in [KS21, footnote 1]; the PL analogue presumably holds).

\(^7\)Part (a) is stated as [PT19, Proposition 7 for $M = M'$] (in a slightly weaker form), and implicitly proved in (known) step 3 of [PT19, §5, proof of Theorem 6]. (A $\mathbb{Z}$-embedding in the notation of [PT19] is a map $f'' : K \to M$ such that $\vartheta f'' = 0$. So that step 3 is precisely the proof of Theorem 1.2.1.a.)

Part (a) also follows by [Ha69, Corollary 4 and its proof], and was essentially proved in a more general situation in [Ha69, §5], cf. [Ha69, Corollary 6 and the third paragraph after Corollary 5].
Remark 1.2.4. (a) Algorithms for recognizing the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of $k$-complexes to given $2k$-manifold are interesting. For a related problem see [KS21e, Remark 2.5.7]. If $M$ is a $(k - 1)$-connected $2k$-manifold, then there is an algorithm recognizing $\mathbb{Z}_2$-embeddability of $k$-complexes to $M$ [PT19, Theorem 10.i], see Remark 1.1.7.e. Simpler, this follows from Theorem 1.1.5 and the case $M = \mathbb{R}^{2k}$ of Lemma 2.3.2.

Is there such a polynomial (in $n$) algorithm for $k > 1$? Cf. [Ko21]. By (b) and Remark 1.2.3.a it suffices to answer this question for $M$ being a connected sum of several copies of $S^k \times S^k$ (for $\cap M$ even) or of $\mathbb{C}P^2$, of $\mathbb{H}P^2$ (for $\cap M$ odd). For $k = 1$ see [Bi21, Remark 1.2.b].

(b) For a $(k - 1)$-connected $2k$-manifold $M$, the $\mathbb{Z}_2$-embeddability of given $k$-complex to $M$ depends only on the rank and the type of $\cap M$. This follows from the criterion [PT19, Proposition 21] for $\mathbb{Z}_2$-embeddability in terms of a cohomology obstruction, see Remark 1.1.7.e. Simpler, this follows from Theorem 1.1.5 and Remark 1.2.3.a.

For a $(k - 1)$-connected orientable $2k$-manifold $M$, the $\mathbb{Z}$-embeddability of given $k$-complex to $M$ depends only on $\cap_{M; \mathbb{Z}}$. This follows by Theorem 1.3.1, and for $k \geq 3$ is essentially the same as [PT19, Proposition 7], cf. (c).

(c) It would be interesting to know if for $k \geq 3$ any two $(k - 1)$-connected PL $2k$-manifolds are PL homeomorphic if they have isomorphic intersection forms and boundaries $PL$ homeomorphic to $S^{2k-1}$ (cf. classification of ‘almost closed’ manifolds [Wa62, p. 170]; pre-H-spaces and H-spaces are defined on p. 168 and 169). This statement implies the $\mathbb{Z}$ case of (b) for $k \geq 3$ and $M$ closed (so it implies [PT19, Proposition 7]).

1.3 Criteria in terms of realizability

Let $V$ be an abelian group. The complex $K$ is called realizable by a bilinear form $I : V \times V \rightarrow \mathbb{Z}$ and elements $y_\sigma \in V$ indexed by $k$-faces $\sigma$ of $K$ if there are a general position map $f : K \rightarrow \mathbb{R}^{2k}$ and orientations on $k$-faces of $K$ such that

$$(R_{f,I,\mathbb{Z}}) \quad I(y_\sigma, y_\tau) = f_\sigma \cdot f_\tau \quad \text{for any non-adjacent } k\text{-faces } \sigma, \tau \text{ of } K.$$  

Comment. Here is the matrix form of the above definition. The complex $K$ is called realizable by an $s \times s$-matrix $I$ of integers and vectors $y_\sigma \in \mathbb{Z}^s$ indexed by $k$-faces of $K$ if there are a general position map $f : K \rightarrow \mathbb{R}^{2k}$, and orientations on $k$-faces of $K$ such that

$$(R_{f,I,\mathbb{Z}}) \quad y_\sigma^T I y_\tau = f_\sigma \cdot f_\tau \quad \text{for any non-adjacent } k\text{-faces } \sigma, \tau \text{ of } K.$$  

The system $(R_{f,I,\mathbb{Z}})$ of Diophantine equations with coefficients $I, f_\sigma \cdot f_\tau$ and variables $\{y_\sigma\}$ is quadratic. So it is unclear if its solvability is algorithmically decidable (although a general position map $f : K \rightarrow \mathbb{R}^{2k}$ can be ‘algorithmized’ using the case $M = \mathbb{R}^{2k}$ of Lemma 2.3.4).

Theorem 1.3.1 (see Remark 1.1.7.c). Let $M$ be a $(k - 1)$-connected orientable $2k$-manifold. There is a $\mathbb{Z}$-embedding $K \rightarrow M$ if and only if $K$ is realizable by $\cap_{M; \mathbb{Z}}$ and some homology classes $y_\sigma \in H_k(M; \mathbb{Z})$.

Corollary 1.3.2. Assume that $k$ is even, $M$ is a closed orientable $(k - 1)$-connected $2k$-manifold, and $\cap_{M; \mathbb{Z}}$ is odd indefinite with positive and negative ranks $r_+$ and $r_-$. There is a $\mathbb{Z}$-embedding $K \rightarrow M$ if and only if $K$ is realizable by vectors $y_\sigma \in \mathbb{Z}^{r_++r_-}$ and the square matrix of size $r_+ + r_-$ having $r_+$ units on the diagonal, $r_-$ elements ‘$-1$’ on the diagonal and zeros outside the diagonal.

This follows by Theorem 1.3.1 and classification of symmetric unimodular odd indefinite bilinear forms over $\mathbb{Z}$ [IF, Theorem 6.3.a].
Conjecture 1.3.3. (a) Assume that $k$ is odd, and $M$ is an orientable $(k-1)$-connected $2k$-manifold. There is a $\mathbb{Z}$-embedding $K \rightarrow M$ if and only if $K$ is compatible to a skew-symmetric matrix $A$ such that $\text{rk } A \leq r := \text{rk}_Z M$ and every $r \times r$ minor of $A$ is divisible by the determinant of the matrix of $\cap_{M, \mathbb{Z}}$ in some basis of $H_k(M; \mathbb{Z})$.

(This would follow from (b) and Theorem 1.3.1.)

(b) Suppose that $A$ and $H$ are skew-symmetric matrices with $\mathbb{Z}$-entries, $H$ is nondegenerate, $\text{rk } A \leq \text{rk } H$, and every $\text{rk } H \times \text{rk } H$ minor of $A$ is divisible by $\text{det } H$ (the divisibility is automatic when $\text{rk } A < \text{rk } H$). Then $A = Y^T HY$ for some matrix $Y$ with $\mathbb{Z}$-entries.

(c) Both conditions of Corollary 1.3.2 are equivalent to $K$ being compatible to an odd indefinite symmetric matrix whose positive and negative ranks do not exceed $\text{rk}_Z M$.

2 Proofs

2.1 Linear algebraic lemmas

Let $V$ be a $\mathbb{Z}_2$-vector space. The complex $K$ is called **realizable modulo 2 by a bilinear form** $I: V \times V \rightarrow \mathbb{Z}_2$ and vectors $y_\sigma \in V$ **indexed by** $k$-**faces of $K** if there is a general position map $f: K \rightarrow \mathbb{R}^k$ such that

$$(R_{f,I})^k(y_\sigma, y_\tau) = |f \sigma \cap f \tau|_2$$

for any non-adjacent $k$-faces $\sigma, \tau$ of $K$.

**Lemma 2.1.1.** Let $I: V \times V \rightarrow \mathbb{Z}_2$ be an even/odd symmetric bilinear form on a $\mathbb{Z}_2$-vector space $V$. The complex $K$ is realizable modulo 2 by $I$ and some vectors $y_\sigma$ if and only if $K$ is compatible modulo 2 to an even/odd matrix $A$ such that $\text{rk } A \leq \text{rk } I$.

Lemma 2.1.1 for $\text{rk } A = \dim V$ implicitly appeared in [Bi21, §2]. Lemma 2.1.1 is deduced below from well-known Lemmas 2.1.2, 2.1.4, known Lemma 2.1.5, and simple Lemma 2.1.3 which appeared in a discussion with A. Bikeev.

**Lemma 2.1.2.** Let $F$ be $\mathbb{Z}_2$ or $\mathbb{Q}$. Let $I: V \times V \rightarrow F$ be a symmetric bilinear form on an $F$-vector space $V$. If $A$ is a Gramian matrix with respect to $I$, then $\text{rk } A \leq \text{rk } I$.

**Proof of the implication (⇒) of Lemma 2.1.1 for $I$ even.** Denote by $A$ the Gramian matrix of the vectors $y_\sigma$ with respect to $I$. Then $A$ is even and $(C_f)$ holds. By Lemma 2.1.2 we have $\text{rk } A \leq \text{rk } I$. \qed

**Lemma 2.1.3 ([Bi21, Lemma 2.2]).** Let $I: V \times V \rightarrow \mathbb{Z}_2$ be an odd symmetric bilinear form on a $\mathbb{Z}_2$-vector space $V$. If $A$ is an even Gramian matrix with respect to $I$, then $\text{rk } A \leq \text{rk } I - 1$.

**Proof of the implication (⇒) of Lemma 2.1.1 for $I$ odd.** Denote by $A$ the Gramian matrix of the vectors $y_\sigma$ with respect to $I$.

If $A$ is odd, then by Lemma 2.1.2 $\text{rk } A \leq \text{rk } I$ and $(C_f)$ holds.

If $A$ is even, then by Lemma 2.1.3 $\text{rk } A \leq \text{rk } I - 1$. Take the matrix $A'$ obtained from $A$ by replacing the entry $A_{1,1}$ by 1. Then $\text{rk } A' \leq \text{rk } I$, $A'$ is odd and $(C_f)$ holds. \qed

**Lemma 2.1.4.** Let $A, B$ be $n \times m$ and $m \times k$ matrices with entries in $\mathbb{Z}_2$, respectively. Then $\text{rk } AB \leq \text{rk } B$. In particular, $\text{rk } AB \leq m$.

Recall that the rank of an even symmetric matrix with entries in $\mathbb{Z}_2$ is even. Denote by $H_g$ the $2g \times 2g$-matrix with $\mathbb{Z}_2$-entries formed by $g$ diagonal blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and zeros elsewhere.
Lemma 2.1.5 ([AA38, Theorem 3], [MW69, Theorem 1]). Let \( A \) be a symmetric \( a \times a \)-matrix with \( \mathbb{Z}_2 \)-entries. Let \( H \) be the matrix \( H_{a \times a} \) if \( A \) is even and the \( \text{rk} A \times \text{rk} A \) identity matrix otherwise. There is a \( \text{rk} A \times a \)-matrix \( Y \) such that \( A = Y^T H Y \).

Proof of the implication \((\Leftarrow)\) of Lemma 2.1.1. (This proof appeared in a discussion with E. Kogan [KS21e, §2.1], [Ko]) Denote by \( I \) the matrix of the bilinear form \( I \) in some basis of \( V \). For \( C = A, I \) let \( H_C \) be the matrix \( H_{a \times C} \) if \( C \) is even and the \( \text{rk} C \times \text{rk} C \) identity matrix if \( C \) is odd. By Lemma 2.1.5 for each \( C = A, I \) there is a matrix \( Y_C \) such that \( C = Y_C^T H_C Y_C \). By Lemma 2.1.4 we have \( \text{rk} Y_I \geq \text{rk} H_I Y_I \geq \text{rk} I \). Hence all the \( \text{rk} I \) rows of \( Y_I \) are linearly independent. Denote \( m := \dim V \). Then there is a nondegenerate \( m \times m \)-matrix \( Y_I^0 \) whose first \( \text{rk} I \) rows are the rows of \( Y_I \). Denote by \( H_I^0 \) the \( m \times m \)-matrix \( \begin{pmatrix} H_I & 0 \\ 0 & 0 \end{pmatrix} \).

Then \( I = (Y_I^0)^T H_I^0 Y_I^0 \). Denote by \( Y \) the \( m \times \text{rk} A \)-matrix obtained from \( Y_A \) by adding \( m - \text{rk} A \) zeroes below every column of \( Y_A \). Then \( A = Y^T H_I^0 Y = ((Y_I^{0})^{-1} Y)^T I ((Y_I^{0})^{-1} Y) \). Denote by \( y_{\sigma} \in V \) the vector represented in the chosen basis of \( V \) by the corresponding column of the matrix \((Y_I^{0})^{-1} Y \). Then \((R_{f,I})\) holds.

Denote by \( H_{g,z} \) the \( 2g \times 2g \)-matrix with \( \mathbb{Z} \)-entries formed by \( g \) diagonal blocks \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and zeros elsewhere.

Lemma 2.1.6 (follows from [Bo07, §5, Theorem 1]). For every skew-symmetric \( a \times a \)-matrix \( A \) with \( \mathbb{Z} \)-entries its rank \( r \) is even and there is an \( r \times a \)-matrix \( Y \) with \( \mathbb{Z} \)-entries such that \( A = Y^T H_{r/2, a} Y \).

Lemma 2.1.7 (E. Kogan [KS21e, Lemma 2.1.7], [Ko]). If a symmetric \( n \times n \) matrix \( A \) with \( \mathbb{Z} \)-entries has rank 1, and any diagonal element of \( A \) is the square of an integer, then \( A = bb^T \) for some vector \( b \in \mathbb{Z}^n \) (standardly considered as an \( n \times 1 \)-matrix).

2.2 Construction of \( \mathbb{Z}_2 \)- or \( \mathbb{Z} \)-embeddings

Lemma 2.2.1. (a) If \( M \) is a \((k - 1)\)-connected 2k-manifold, then any homology class in \( H_k(M) \) or in \( H_k(M; \mathbb{Z}) \) is represented by a general position map \( S^k \to M \).

(b) For \( k \geq 3 \) this map can be assumed to be an embedding.

A known proof. For \( k = 1 \) the lemma is obvious, so assume that \( k \geq 2 \). Then \( M \) is simply-connected. Hence by Hurewicz theorem \( H_{k-1}(M; \mathbb{Z}) = 0 \). So by the coefficient exact sequence the reduction modulo 2 \( H_k(M; \mathbb{Z}) \to H_k(M) \) is epimorphic. Hence it suffices to prove the lemma for \( \mathbb{Z} \).

By Hurewicz theorem the map \( \pi_k(M) \to H_k(M; \mathbb{Z}) \) is epimorphic. Hence any homology class \( \alpha \in H_k(M; \mathbb{Z}) \) is represented by a continuous map \( S^k \to M \). Then by a small shift we obtain a general position map \( S^k \to M \) representing \( \alpha \).

For \( k \geq 3 \) the latter map is homotopic to an embedding by the Penrose-Whitehead-Zeeman-Irwin Theorem [Ir65], cf. [Sk06, Theorem 2.9].

Proof of the implication \((\Leftarrow)\) of Theorem 1.1.5. Since \( K \) is compatible modulo 2 to \( A \), by (the implication \((\Leftarrow)\) of) Lemma 2.1.1 \( K \) is realizable modulo 2 by \( \cap M \) and some vectors \( y_{\sigma} \in H_k(M) \). So there is a general position map \( f : K \to \mathbb{R}^{2k} \) such that \((R_{f, \cap M})\) holds.

Take a 2k-ball \( B \subset \text{Int} M \). We may assume that \( fK \subset B \). By Lemma 2.2.1.a for any \( k \)-face \( \sigma \) the class \( y_{\sigma} \) is represented by a general position map \( \tilde{\sigma} : S^k \to M \). By general position
\( \tilde{\sigma}(S^k) \cap B = \emptyset \). We may assume that the maps \( \tilde{\sigma} \) for different \( \sigma \) are in general position to each other. Since \( M \) is connected, we can take a general position map \( h : K \to M \) obtained from \( f \) by connected summation of \( f|_\sigma \) and \( \tilde{\sigma} \) along an arc \( l_\sigma \), for every \( k \)-face \( \sigma \).

(*) For \( k \geq 2 \) by general position we may assume that \( l_\sigma \cap (f_\tau \cup \tilde{\tau}S^k \cup l_\tau) = 0 \) for \( \sigma \neq \tau \). For \( k = 1 \) we may assume that for the tube \( U_\sigma \) along which we make connected summation for \( \sigma \), each of the intersections of \( U_\sigma \) with \( f_\tau \), with \( \tilde{\tau}S^k \), and with any other tube, consists of an even number of points.

Now the map \( h \) is a \( \mathbb{Z}_2 \)-embedding because for any non-adjacent \( k \)-faces \( \sigma, \tau \) we have

\[
|h_\sigma \cap h_\tau|_2 \overset{(1)}{=} |f_\sigma \cap f_\tau|_2 + |\tilde{\sigma}S^k \cap \tilde{\tau}S^k|_2 \overset{(2)}{=} |f_\sigma \cap f_\tau|_2 + y_\sigma \cap_M y_\tau \overset{(3)}{=} 0.
\]

Here

- equality (1) holds because (*) holds, \( \tilde{\sigma}S^k \cap f_\tau \subset \tilde{\tau}S^k \cap B = \emptyset \) and analogously \( \tilde{\tau}S^k \cap f_\sigma = \emptyset \);
- equality (2) holds because \( \tilde{\sigma}S^k, \tilde{\tau}S^k \) represent \( y_\sigma, y_\tau \), respectively;
- equality (3) is \((R_{f, \cap_M})\).

**Proof of the implication \((\Leftarrow)\) of Theorem 1.3.1.** The proof is obtained from the proof of the implication \((\Leftarrow)\) of Theorem 1.1.5 by the following changes.

Replace the first paragraph by the following sentence. Since \( K \) is realizable by \( \cap_M \mathbb{Z} \) and some vectors \( y_\sigma \in H_k(M) \), there are a general position map \( f : K \to \mathbb{R}^{2k} \) and orientations on \( k \)-faces of \( K \) such that \((R_{f, \cap_M, \mathbb{Z}})\) holds.

Replace \( \mathbb{Z}_2 \) by \( \mathbb{Z} \), ‘an even number of points’ by ‘some points the sum of whose signs is zero’, \(|A \cap B|_2\) by \( A \cdot B \), and \( \cap_M \) by \( \cap_{M, \mathbb{Z}} \).

\[
\square
\]

### 2.3 Deduction of realizability from \( \mathbb{Z}_2 \)- or \( \mathbb{Z} \)-embeddability

**The implication \((\Rightarrow)\) of Theorem 1.1.5** follows by (the implication \((\Rightarrow)\) of) Lemma 2.1.1 and Lemma 2.3.1.a.

**Lemma 2.3.1.** (a) If \( M \) is a \( (k - 1) \)-connected \( 2k \)-manifold, and there is a \( \mathbb{Z}_2 \)-embedding \( h : K \to M \), then \( K \) is realizable modulo 2 by \( \cap_M \) and some homology classes \( y_\sigma \in H_k(M) \).

(b) Moreover, there are homology classes \( \hat{\sigma} \in H_k(K) \) for every \( k \)-face \( \sigma \) of \( K \) such that

- for any \( M, h \) as in (a) we can take \( y_\sigma = h_\sigma \hat{\sigma} \), and
- for any connected subcomplex \( K_\sigma \subset K \) containing \( \sigma \) the class \( \hat{\sigma} \) lies in the image of the inclusion-unduced homomorphism \( H_k(K_\sigma) \to H_k(K) \).

See Remark 1.1.7.c. Part (b) is only used in §2.4.

For a proof we need the following essentially known definitions and lemma.

**In this subsection \( M \) is any \( 2k \)-manifold, and \( g : K \to M \) is any general position map.**

Take any pair of non-adjacent \( k \)-faces \( \sigma, \tau \) of \( K \). By general position the intersection \( g\sigma \cap g\tau \) consists of a finite number of points. Assign to the pair \( \{ \sigma, \tau \} \) the residue

\[
\nu(g)\{\sigma, \tau\} := |g\sigma \cap g\tau|_2.
\]

Denote by \( K^* \) the set of all unordered pairs of non-adjacent \( k \)-faces of \( K \). The obtained map \( \nu(g) : K^* \to \mathbb{Z}_2 \) is called the (modulo 2) **intersection cocycle** of \( g \) (we call it ‘cocycle’ instead of ‘map’ to avoid confusion with maps to \( M \)). Maps \( K^* \to \mathbb{Z}_2 \) are identified with subsets of \( K^* \) consisting of pairs going to 1 \( \in \mathbb{Z}_2 \). (Maps \( K^* \to \mathbb{Z}_2 \) can also be regarded
as ‘partial matrices’, i.e. symmetric arrangements of zeroes and ones in those cells of the \( n \times n \)-matrix that correspond to the pairs of non-adjacent \( k \)-faces.)

Let \( \alpha \) be a \((k - 1)\)-face of \( K \) which is not contained in the boundary of a \( k \)-face \( \sigma \) of \( K \).

An **elementary coboundary** of the pair \((\alpha, \sigma)\) is the subset \( \delta_K(\alpha, \sigma) \subset K^* \) consisting of all pairs \( \{\sigma, \tau\} \) with \( \tau \supset \alpha \).

Cocycles \( \nu, \nu' : K^* \to \mathbb{Z}_2 \) (or \( \nu, \nu' \subset K^* \)) are called **cohomologous** (modulo 2) if

\[
\nu - \nu' = \delta_K(\alpha_1, \sigma_1) + \ldots + \delta_K(\alpha_s, \sigma_s)
\]

for some \((k - 1)\)-faces \( \alpha_1, \ldots, \alpha_s \) and \( k \)-faces \( \sigma_1, \ldots, \sigma_s \) (not necessarily distinct). Here \( + \) is the componentwise addition (corresponding to the sum modulo 2 of subsets of \( K^* \)).

**Lemma 2.3.2.** Let \( \nu : K^* \to \mathbb{Z}_2 \) be a cocycle. There is a general position map \( g : K \to M \) homotopic to \( g \) and such that \( \nu(g') = \nu \) if and only if \( \nu \) is cohomologous to \( \nu(g) \).

**Remark 2.3.3.** Lemmas 2.3.2 and 2.3.4 are essentially known. They were proved by van Kampen-Shapiro-Wu for \( M = \mathbb{R}^{2k} \), see a proof in [Sk18, Lemma 1.5.8 and Proposition 1.5.9] (modulo 2 version for \( k = 1 \)) and in [Sh57, Lemma 3.5], [Wu58], [FKT, §2] (integer version for any \( k \), cf. [Sk18, Remark 1.6.4]). The proof for an arbitrary \( M \) is analogous (e.g. the finger moves are done in a regular neighborhoods of a path in \( M \), which is homeomorphic to the \( 2k \)-ball). The ‘only if’ part of Lemma 2.3.4 is [Jo02, Theorem 1], [PT19, Lemma 11], the ‘if’ part of Lemma 2.3.4 is [Jo02, Theorem 6] (in which the assumptions ‘\( n \geq 3 \)’ and ‘\( M \) is 1-connected’ are superfluous).

Denote by \( K^{(k-1)} \) the union of all those faces of \( K \) whose dimension is less than \( k \).

**Proof of Lemma 2.3.1.a.** Take a \( 2k \)-ball \( B \subset \text{Int} \ M \). Since \( M \) is \((k - 1)\)-connected, the restriction \( h|_{K^{(k-1)}} \) is null-homotopic.\(^8\) Hence by the Borsuk Homotopy Extension Theorem \( h \) is homotopic to a map \( h' : K \to M - \text{Int} \ B \) such that \( h'(K^{(k-1)}) \subset \partial B \). By general position we may assume that \( h'|_{K^{(k-1)}} \) is an embedding.

Define a map \( f' : K \to B \) to be \( h' \) on \( K^{(k-1)} \), and to be the cone map over \( h|_{\partial \sigma} \) with a vertex in \( \text{Int} \ B \) on any \( k \)-face \( \sigma \). We may assume that these vertices are in general position. Then \( f' \) is a general position map.

It remains to prove that the complex \( K \) is realizable by \( \cap_M \) and the classes \( y_\sigma := [f'| \cap h'|] \). By Lemma 2.3.2 the intersection cocycles \( \nu(h) = 0 \) and \( \nu(h') \) are cohomologous. Hence by the case \( M = B \) of Lemma 2.3.2 there is a general position map \( f : K \to B \) homotopic to \( f' \), and such that

\[(*) \quad \nu(f) = \nu(f') + \nu(h').\]

Now \((R_{f} \cap M)\) follows because for any non-adjacent edges \( \sigma, \tau \) we have

\[
|f \sigma \cap f \tau|_2 = (1) \quad |f' \sigma \cap f' \tau|_2 + |h' \sigma \cap h' \tau|_2 = (2) |(f' \sigma \cup h' \sigma) \cap (f' \tau \cup h' \tau)|_2 \stackrel{(3)}{=} y_\sigma \cap_M y_\tau.
\]

Here

- equality (1) holds by \((*)\) because \( \nu(f) \{\sigma, \tau\} = |f \sigma \cap f \tau|_2 \) and the same for \( f', h' \);
- equality (2) holds because \( \sigma \cap \tau = \emptyset \) and \( h'|_{K^{(k-1)}} \) is an embedding, so that \( f' \sigma \cap h' \text{Int} \ \tau \subset B \cap h' \tau = \emptyset \), and analogously \( f' \tau \cap h' \sigma = \emptyset \);
- equality (3) holds by general position and definitions of \( y_\sigma, y_\tau \) and \( \cap_M \).

\(^8\) Instead of the \((k - 1)\)-connectedness, for (a) it suffices to assume that \( h|_{K^{(k-1)}} \) is null-homotopic.
Proof of Lemma 2.3.1.b. Recall that a (simplicial) k-cycle (modulo 2) in K is a set C of k-faces such that every (k − 1)-face is contained in an even number of k-faces from C. A maximal k-forest $T \subset K$ is an inclusion-maximal subcomplex $T \subset K$ containing no non-empty k-cycles (i.e. such that $H_k(T) = 0$). Take any maximal k-forest $T \subset K$. For any k-face $\sigma \subset T$ set $\hat{\sigma} = \hat{\sigma}_{K,T} := 0$. For any k-face $\sigma \subset K - T$ there is a unique non-empty k-cycle $\hat{\sigma} = \hat{\sigma}_{K,T}$ in $T \cup \sigma$. (For $k = 1$ the 1-cycle $\hat{\sigma}$ is the union of $\sigma$ and the simple path in $T$ joining the ends of the edge $\sigma$.)

We use the notation from the proof of (a). Since $M$ is $(k-1)$-connected, any map $T \to M$ is null-homotopic. So in the first paragraph of the proof of (a) we can make a homotopy of $\hat{\sigma}$ instead of Lemma 2.3.1. Denote by $\tilde{\kappa}$ the set of all ordered pairs of non-adjacent $k$-faces of $K$. The obtained map $\nu_{Z}(g)(\sigma, \tau) := g\sigma \cdot g\tau$.

Denote by $\tilde{K}$ the set of all ordered pairs of non-adjacent $k$-faces of $K$. The obtained map $\nu_{Z}(g) : \tilde{K} \to \mathbb{Z}$ is called the (integer) intersection cocycle of $f$. This cocycle is super-symmetric, i.e. $\nu_{Z}(g)(\sigma, \tau) = (-1)^k \nu_{Z}(g)(\tau, \sigma)$.

Let $\alpha$ be an oriented $(k-1)$-face of $K$ which is not contained in the boundary of a $k$-face $\sigma$ of $K$. An (integer) elementary coboundary of the pair $(\alpha, \sigma)$ is the map $\delta_{K}(\alpha, \sigma) : \tilde{K} \to \mathbb{Z}$ assigning

$$(-1)^k[\tau : \alpha] \to (\sigma, \tau), \quad [\tau : \alpha] \to (\tau, \sigma) \quad \text{and} \quad 0 \text{ to any other pair},$$

where the incidence coefficient $[\tau : \alpha]$ is defined e.g. in [HG, §3]. Cocycles $\nu, \nu' : \tilde{K} \to \mathbb{Z}$ are called (integer) cohomologous if

$$\nu - \nu' = c_1 \delta_{K}(\alpha_1, \sigma_1) + \ldots + c_s \delta_{K}(\alpha_s, \sigma_s)$$

for some integers $c_1, \ldots, c_s \in \mathbb{Z}$, oriented $(k-1)$-faces $\alpha_1, \ldots, \alpha_s$, and $k$-faces $\sigma_1, \ldots, \sigma_s$ (not necessarily distinct). Observe that change of the orientation of $\alpha$ forces change of the sign of $\delta_{K}(\alpha, \sigma)$. Hence the cohomology equivalence relation does not depend on the orientations of $(k-1)$-faces.

Lemma 2.3.4. Let $\nu : \tilde{K} \to \mathbb{Z}$ be a super-symmetric cocycle, and $M$ an oriented $2k$-manifold. There is a general position map $g' : K \to M$ homotopic to $g$, and such that $\nu_{Z}(g') = \nu$ if and only if $\nu$ is cohomologous to $\nu_{Z}(g)$.

---

9Sketch of a well-known proof. Since $T$ is a k-complex and $H_k(T) = 0$, we have $H_k(T; \mathbb{Z}) = 0$. Now the result is proved either applying the relative Hurewicz theorem to the mapping cylinder of the map, or using obstruction theory.
Remark 2.3.5. The \((k-1)\)-connectedness assumption in the ‘only if’ parts of Theorems 1.1.5, 1.3.1 can be replaced by either of the following assumptions.

(i) every map \(K^{(k-1)} \to M\) extendable to \(K\) is null-homotopic.
(ii) \(M\) is smooth \((k-1)\)-parallelizable, i.e. on the union of \((k-1)\)-faces of some triangulation of \(M\) there are \(2k\) tangent vector fields linearly independent at every point.

The first holds by footnote 8.

The second holds by the following result.

If \(Q\) is a \(k\)-subcomplex of a smooth \((k-1)\)-parallelizable \(2k\)-manifold \(M\), then \(Q\) is a subcomplex of some \((k-1)\)-connected manifold \(N\) such that \(\cap N\) is isomorphic to \(\cap M\). The same holds with \(\cap N, \cap M\) replaced by \(\cap_{N,Z}, \cap_{M,Z}\).

This result is easily proved by surgery of \(M\) below the middle dimension, not changing \(\cap M\) (or \(\cap M,Z\)) [Mi61], and outside \(X\).

2.4 Proofs of Proposition 1.1.3 and Corollaries 1.1.4.b.c

Proof of Proposition 1.1.3. (a) The inequality \(P(X \sqcup Y) \leq P(X) + P(Y)\) is clear. We have

\[
P(X \sqcup Y) \overset{(1)}{=} \mathrm{rk} \; M \overset{(2)}{=} \mathrm{rk}(X_+ \cup Y_+) \overset{(3)}{=} \mathrm{rk} \; X_+ + \mathrm{rk} \; Y_+ \geq P(X) + P(Y),
\]

where

- \(M\) is a \(2k\)-manifold such that there is an embedding \(h : X \sqcup Y \to M\) and the equality (1) holds;
- \(X_+\) and \(Y_+\) are disjoint neighborhoods of \(h(X)\) and of \(h(Y)\) in \(M\);
- equality (2) holds because \(\mathrm{rk}\) is monotone [IF, Lemma 4.1.a].

Proof of (b,c) is analogous to (a) for \(k \geq 2\). For \(k = 1\) (b,c) follow from Corollary 1.1.4.b.c because if \(K\) embeds into \(M \sqcup N\), then \(K\) embeds into \(M \# N\), so \(P(K) = \rho(K)\), and analogously \(R_G(K) = r_G(K)\) for \(G \in \{Z, Z_2\}\).

Take a \(2k\)-manifold \(M\) and a \(Z_2\)- embedding \(h : X \sqcup Y \to M\) such that the equality (1) from the proof of (a) holds. We may assume that \(h\) is in general position (see Definition 1.1.6.a). Then there are regular neighborhoods \(X_+\) and \(Y_+\) of \(h(X)\) and of \(h(Y)\) in \(M\) whose intersection is the union of some products \(B^k \times B^k\), every product intersecting \(h(X), h(Y)\), \(\partial X_+\) and \(\partial Y_+\) by \(B^k \times 0, 0 \times B^k\), \(B^k \times \partial B^k\) and \(\partial B^k \times B^k\), respectively.

We give the following argument for (b); the argument for (c) is analogous. Part (b) follows by the (in)equalities from the proof of (a). Take the homomorphisms \(i_X, i_Y\) in \(H_k\) induced by the inclusions \(X_+, Y_+ \to X_+ \cup Y_+\). Instead of equality (3) we have the inequality ‘\(\geq\)’. The latter follows because for any \(x, x' \in H_k(X_+)\) and \(y, y' \in H_k(Y_+)\) we have

\[
x \cap X_+ x' = i_X x \cap i_X x', \quad y \cap Y_+ y' = i_Y y \cap i_Y y' \quad \text{and} \quad i_X x \cap i_Y y = 0.
\]

Here \(\cap \overset{\text{def}}{=} \cap_{X_+ \cup Y_+}\) and the first two equalities are obvious. In the following paragraph we prove the latter equality.

The neighborhood \(X_+\) is homotopy equivalent to \(h(X)\). The latter is obtained from \(X\) by several identifications of pairs of points. Then the map \(h : X \to h(X)\) induces an isomorphism \(H_k(X) \to H_k(h(X))\) because \(k \geq 2\). (This is proved either by definition or by observing that \(h(X)\) is homotopy equivalent to the union \(U\) of \(X\) and several arcs, and using the exact sequence of the pair \((U, X)\).) Then \(x\) is represented by \(h\)-image of a simplicial \(k\)-cycle \(\overline{\sigma}\) in \(X\). Analogous statement holds for \(y, \overline{y}, Y\). So

\[
i_X x \cap i_Y y = \sum_{\sigma \in \overline{\sigma}, \tau \in \overline{\overline{y}}} |h\sigma \cap h\tau|_2 = 0,
\]

where the last equality holds because \(h\) is a \(Z_2\)-embedding. \(\Box\)
Comment. (a) The following example shows that the above proof of (b,c) does not work for \( k = 1 \). Let \( M \) be the Klein bottle with a hole.

There is a \( \mathbb{Z}_2 \)-embedding of the disjoint union \( K_3 \sqcup K_3 \) of two copies of \( K_3 \) to \( M \) with a certain number of holes such that the regular neighborhood of the image of each \( K_3 \) is homeomorphic to \( M \).

The surface \( M \) is homeomorphic to the union of two (‘upper’ and ‘lower’) Moebius bands which intersect by a segment in their boundaries. Take two copies of \( M \) intersecting by the union of two disks (the ‘plumbing’ intersection of two ‘upper’ Moebius bands, and the same for two ‘lower’ Moebius bands). Denote by \( M_+ \) the union of the two copies. Take a \( \mathbb{Z}_2 \)-embedding of \( K_3 \) on \( M \) having exactly one self-intersection point, which is self-intersection of an edge. These two \( \mathbb{Z}_2 \)-embeddings form together a \( \mathbb{Z}_2 \)-embedding \( h : K_3 \sqcup K_3 \to M_+ \).

Propositions 1.1.3.b,c for \( K_3 \sqcup K_3 \) are trivial. However, the above proof of (b,c) does not work for \( h \) (which is in general position). Indeed, \( M_+ \) is homeomorphic to \( M \) with a certain number of holes, so \( \text{rk} M_+ = 2 \neq 4 = 2 \text{rk} M \).

(b) For any 2k-manifold \( M \) there is a k-complex having no \( \mathbb{Z}_2 \)-embedding to \( M \) (and hence having neither embedding nor \( \mathbb{Z} \)-embedding to \( M \)).

The case \( k = 1 \) is known by [SS13, Lemmas 6, 7] or [FK19, Theorem 1]. The general case holds by a \( \mathbb{Z}_2 \)-version [PT19, Theorem 1] essentially proved in [PT19].

Alternatively, this fact follows from Proposition 1.1.3.b and non-\( \mathbb{Z}_2 \)-embeddability of \( \Delta_{2k+2}^k \) in any 2k-manifold \( M \) with trivial \( \cap M \). Such a non-\( \mathbb{Z}_2 \)-embeddability follows by Lemma 2.3.1.a together with footnote 8, and the quantitative van Kampen-Flores Theorem, see survey [Sk16, §4, (VKF\(_d^+\)) for \( d = 2k \)].

This proof shows that as an example one can take the disjoint union of \( 1 + \text{rk} M \) copies of \( \Delta_{2k+2}^k \). In other words, if the disjoint union of \( s \) copies of \( \Delta_{2k+2}^k \) (or \( \Delta_{s(2k+3)-1}^k \)) has a \( \mathbb{Z}_2 \)-embedding to \( M \), then \( \text{rk} M \geq s \). For \( \Delta_{s(2k+3)-1}^k \) this estimation is weaker than [PT19, Theorem 1(i)] but stronger (asymptotically for \( k \to \infty \)) than [GMP\(_+\), Theorem 2]. Analogously, proving even-rank additivity recovers a weaker form of [PT19, Theorem 1(ii)].

**Lemma 2.4.1.** Let \( M \) be a \((k-1)\)-connected 2k-manifold.

(a) If \( \cap M \) is odd, then there is a \((k-1)\)-connected 2k-manifold \( N_1 \) such that \( \text{rk} \cap N_1 = 1 \).

(b) Let \( A \) be the Gramian matrix of some homology classes in \( H_k(M) \) with respect to \( \cap M \). Then there is a \((k-1)\)-connected 2k-manifold \( N \) such that \( \cap N \) has the same rank and type as \( A \).

(c) For \( M \) orientable let \( A \) be the Gramian matrix of some \( s \) homology classes in \( H_k(M; \mathbb{Z}) \) with respect to \( \cap M; \mathbb{Z} \). Then there are an orientable \((k-1)\)-connected 2k-manifold \( N \) such that \( \text{rk}_2 N = \text{rk} A \), and homology classes \( z_1, \ldots, z_s \in H_k(N; \mathbb{Z}) \) such that \( z_i \cap N; \mathbb{Z} z_j = A_{ij} \) for every \( 1 \leq i < j \leq s \).

**Proof.** (a) If \( k = 1,2 \), then \( N_1 = \mathbb{R}P^2, \mathbb{C}P^2 \) satisfy the requirements.

Assume that \( k \geq 3 \). Since \( \cap M \) is odd, there is \( \alpha \in H_k(M) \) such that \( \alpha \cap M \alpha = 1 \). By Lemma 2.2.1.b \( \alpha \) is represented by an embedding \( f : S^k \to M \). Let \( N_1 \) be the regular neighbourhood of \( f(S^k) \). Then \( N_1 \) is \((k-1)\)-connected and \( \text{rk} \cap N_1 \leq \text{rk} H_k(N_1) = \text{rk} H_k(S^k) = 1 \). Since \( \alpha \cap M \alpha = 1 \), it follows that \( \text{rk} \cap N_1 = 1 \).

(b) If \( A \) is even, then \( \text{rk} A \) is even. So let \( N \) be the connected sum of \( \text{rk} A/2 \) copies of \( S^k \times S^k \).

If \( A \) is odd, then \( \cap M \) is odd. Hence by (a) there is a manifold \( N_1 \) such that \( \text{rk} \cap N_1 = 1 \). So let \( N \) be the connected sum of \( \text{rk} A \) copies of \( N_1 \).

\(^{10}\)Part (a) follows from the PL analogue mentioned in Remark 1.2.3.b. We present a simpler direct proof.
In both cases $\cap_N$ has the same rank and type as $A$.

(c) Take a link in $S^{2k-1}$ with unknotted components $S_i^{k-1}, i = 1, \ldots, s$, such that the linking number of $S_i^{k-1}$ and $S_j^{k-1}$ equals $A_{ij}$ for any $1 \leq i < j \leq s$. Take any framing, i.e. any extension of the inclusion $\sqcup_i S_i^{k-1} \to S^{2k-1}$ to an embedding $D^k \times \sqcup_i S_i^{k-1} \to S^{2k-1}$. Let $N$ be the result of the surgery of $B^{2k}$ along the obtained framed link. Let $z_i \in H_k(N; \mathbb{Z})$ be the class corresponding to $S_i^{k-1}$ for every $i = 1, \ldots, s$. Then $N, z_1, \ldots, z_s$ are as required. 

**Proof of Corollary 1.1.4.b**. The inequality $r_{Z_2}(X \sqcup Y) \leq r_{Z_2}(X) + r_{Z_2}(Y)$ is clear (because the connected sum of $(k - 1)$-connected $2k$-manifolds is $(k - 1)$-connected). The opposite inequality follows by

$$r_{Z_2}(X \sqcup Y) \overset{(1)}{=} \text{rk } M \overset{(2)}{\geq} \text{rk } A \overset{(3)}{=} \text{rk } A_X + \text{rk } A_Y \overset{(4)}{\geq} r_{Z_2}(X) + r_{Z_2}(Y).$$

Here

- $M$ is a $(k - 1)$-connected $2k$-manifold such that there is a $\mathbb{Z}_2$-embedding $h : X \sqcup Y \to M$ and equality (1) holds;

- $\hat{\sigma} \in H_k(X \sqcup Y)$ are classes corresponding to $k$-faces $\sigma$ of $X \sqcup Y$ given by Lemma 2.3.1.b, and $A_{\sigma, \tau} := h_\sigma \cap_M h_\tau$ is the Gramian matrix (with respect to $\cap_M$) of the classes $h_\sigma, \hat{\sigma}$;

- inequality (2) follows by Lemma 2.1.2;

- $A_X$ and $A_Y$ are the restrictions of $A$ to $k$-faces of $X$ and of $Y$, respectively;

- equality (3) follows because for any $\sigma \in X$ and $\tau \in Y$ the classes $\hat{\sigma}$ and $\hat{\tau}$ come from $H_k(X)$ and from $H_k(Y)$, respectively, so that

$$A_{\sigma, \tau} = h_\sigma \cap_M h_\tau = \sum_{\sigma', \tau' \in \hat{\sigma}} |h_{\sigma'} \cap h_{\tau'}|_2 = 0,$$

where the last equality holds because $h$ is a $\mathbb{Z}_2$-embedding;

- inequality (4) follows from the inequalities $\text{rk } A_X \geq r_{Z_2}(X)$ (proved in the following paragraph) and $\text{rk } A_Y \geq r_{Z_2}(Y)$ (proved analogously).

By Lemma 2.3.1.b $X$ is compatible modulo 2 to $A_X$ (cf. Lemma 2.1.1). By Lemma 2.4.1.b there is a $(k - 1)$-connected $2k$-manifold $N$ such that $\cap_N$ has the same rank and type as $A_X$. By Theorem 1.1.5 there is a $\mathbb{Z}_2$-embedding $X \to N$. Then $\text{rk } A_X = \text{rk } N \geq r_{Z_2}(X)$. 

**Proof of Corollary 1.1.4.c**. The proof is obtained from the proof of (b) by the changes described in §2.3, 'Proof of the implication ($\Rightarrow$) of Theorem 1.3.1'. Instead of Lemma 2.3.1 we use its integer analogue, in which $y_\sigma = -h_\sigma \hat{\sigma}$. Besides, the last paragraph of the proof of (b) is changed to the following paragraph.

11 We can construct $z_1, \ldots, z_s$ with more care so that $z_i \cap_{N, \mathbb{Z}} z_j = A_{ij}$ for any $1 \leq i \leq j \leq s$ (including $i = j$). For $k$ is odd this is automatic because both sides are zeroes. Recall that a PL framing on $S_i^{k-1} \subset S^{2k-1}$ is a PL homeomorphism from $S^{k-1} \times D^k$ onto a regular neighborhood of $S_i^{k-1}$ in $S^{2k-1}$ whose restriction to $S_i^{k-1} \times \{0\}$ is a homeomorphism onto $S_i^{k-1}$. It suffices to prove that for $k$ even there is a framing of $S_i^{k-1}$ such that $z_i \cap_{N, \mathbb{Z}} z_i = A_{ii}$ for every $i = 1, \ldots, s$. The existence of such a framing is known for $k = 2$, and is proved (by a standard argument) in the following paragraph for $k \geq 4$.

Since $k \geq 3$, by Lemma 2.2.1.b the $i$-th of the $s$ homology classes is represented by a general position embedding $y : S^k \to M$, $i = 1, \ldots, s$. Represent $S^k$ as the union of two $k$-disks $D_+^k$ and $D_-^k$ intersecting by $S^{k-1}$. Represent a regular neighborhood of $y(S^k)\cap M$ as the union of two images of embeddings

$$e_{\pm} : D_+^k \times D_+^k \to M \text{ such that } e_+(S^{k-1} \times D^k) = e_+(D_+^k \times D_+^k) \cap e_-(D_-^k \times D_-^k).$$

Take the composition of the autohomeomorphism $e_+ \circ e_-$ of $S^{k-1} \times D^k$ with a framing $S_i^{k-1} \times D^k \to S^{2k-1}$ giving $S^k \times D^k$ as the result of the surgery of $B^{2k}$ along the framed knot. This composition is the required framing of $S_i^{k-1}$.
Apply Lemma 2.4.1.c to obtain an orientable \((k - 1)\)-connected \(2k\)-manifold \(N\) such that \(\text{rk}_Z N = \text{rk} A_X\), and classes \(z_\sigma \in H_k(N; \mathbb{Z})\) for \(k\)-faces \(\sigma\) of \(X\) such that \(z_\sigma \cap_{N; \mathbb{Z}} z_\tau = y_\sigma \cap_{M; \mathbb{Z}} y_\tau = f_\sigma \cdot f_\tau\) for any non-adjacent \(\sigma, \tau\). So \(X\) is realizable by \(\cap_{N; \mathbb{Z}}\) and the classes \(z_\sigma \in H_k(N; \mathbb{Z})\). Hence by Theorem 1.3.1 there is a \(\mathbb{Z}\)-embedding \(X \to N\). Then \(\text{rk} A_X = \text{rk}_Z N \geq r_Z(X)\).

For a proof of the integer analogue of Lemma 2.3.1.b one needs the following. A maximal integer \(k\)-forest \(T \subset K\) is an inclusion-maximal subcomplex \(T \subset K\) containing no non-empty integer \(k\)-cycles (i.e. such that \(H_k(T; \mathbb{Z}) = 0\)). Take any maximal integer \(k\)-forest \(T \subset K\). For any \(k\)-face \(\sigma \subset T\) set \(\tilde{\sigma} = \tilde{\sigma}_{K,T} := 0\). For any \(k\)-face \(\sigma \subset K - T\) there is a unique integer \(k\)-cycle \(\tilde{\sigma} = \tilde{\sigma}_{K,T}\) in \(T \cup \sigma\) for which the coefficient of \(\sigma\) is 1. Then 
\[
[f'\sigma \cup (-h'\sigma)] = [f'\sigma \cup h'(\tilde{\sigma} - \sigma)] - [h'\tilde{\sigma}] = -h_s\tilde{\sigma}.
\]

Comment. An alternative proof of Corollary 1.1.4.c for \(k \geq 3\) odd is analogous to the above proofs of Corollaries 1.1.4.ab. Instead of Theorem 1.3.1 and Lemma 2.4.1.c we use Theorem 1.1.1 and the following result.

Assume that \(k\) is odd, \(M\) is an orientable \(2k\)-manifold, and \(A\) is the Gramian matrix of some homology classes in \(H_k(M; \mathbb{Z})\) with respect to \(\cap_{M; \mathbb{Z}}\). Then there is an orientable \((k - 1)\)-connected \(2k\)-manifold \(N\) such that \(\text{rk}_Z N = \text{rk} A\).

(Proof. Since \(k\) is odd, the matrix \(A\) is skew-symmetric. Then \(\text{rk} A\) is even. So let \(N\) be the connected sum of \(\text{rk} A/2\) copies of \(S^k \times S^k\).)

2.5 Appendix: unreliability of some papers on embeddings

In the following Remark 2.5.1.ab I justify that the proofs of [SS13] are not reliable (as explained in [Sk21d, §1]). I suggested to the authors to make improvements upon these critical remarks, so that I could refer to an arXiv update of their paper, and omit critical remarks on the earlier version, see [Sk21d, Remark 2.3]. Thus the purpose of Remark 2.5.1.ab is (not fight for priority but) enabling people to use the results and methods of [SS13] by providing reliable proofs (preferably by the authors). From our discussions with M. Schaefer I conclude that he does not plan to produce a rigorous proof that lives up to the reliability standards pursued in Remark 2.5.1.ab. Even if only different authors would provide such a proof, I would call the block additivity stated in [SS13] ‘the result of Schaefer-Štefankovič, with some details fixed in such and such paper’, see Remark 1.1.7.a. I am not stating that it is impossible or hard to make corrections corresponding to Remark 2.5.1.ab, cf. [Sk21d, Remark 2.3.b] and Remark 2.5.1.d. Neither M. Schaefer nor D. Štefankovič nor R. Fulek nor J. Kynčl replied to my invitation of presenting their public reply to the criticism of Remark 2.5.1.abc, although I promised to present (a reference to) the reply even if I would not agree with it (see Remark 2.5.2).

Remark 2.5.1. (a) In [SS13] p. 2, the definition of \(y_e\) is not mathematically rigorous (and is unclear). The expression ‘\(e\) is pulled through the \(i\)-th crosscap an odd number of times’ has no rigorous meaning. It is not even written (and not clear) what is defined:

(i) a vector \(y_e\) for any map of the graph to the sphere with \(s\) Moebius films (‘surface with \(s\) crosscaps’ in the terminology of [SS13]);

(ii) a construction of some map of the graph to the sphere with \(s\) Moebius films, starting from vectors \(y_e\) and given map (=drawing) \(D\) from the graph to the plane (and from vectors \(x_e\), but even the case when all \(x_e\) are zeros or not mentioned has the described problem);

(iii) something else, see [Sk21d, Remark 2.1.c].
In case (i) the vector $y_e$ cannot be defined without first (N) taking specific representation the sphere with $s$ Moebius films as the union of the sphere with $s$ holes, and making the embedding ‘nice’ w.r.t. this specific representation.

Indeed,

- any non-self-intersecting curve on a surface has a neighborhood homeomorphic to the disk, so no embedded edge ‘is pulled through any crosscap’ unless we have the above specific representation;
- an edge is not a closed curve, so the number of times ‘the edge is pulled through a crosscap’ is not defined (e.g. it is not written, and it is not clear, how to define this number when one end of the edge is on the crosscap, and the other end is outside the crosscap).

But (N) is not done in [SS13, p. 2].

In case (ii) proofs of [SS13, Lemmas 3 and 4] do not work because the input there is any map of the graph to the sphere with $s$ Moebius films.

The vector $y_e$ is the main object of [SS13], so the above critical remark affects the whole paper. E.g. the statements of Lemmas 3 and 4 have no rigorous mathematical meaning (and are unclear).

(b) In [SS13, p. 2] the following result is not rigorously stated (and is unclear): ‘This definition is equivalent to the more intuitive definition given in the introduction (see, for example, Levow [5, Theorem 3]).’ Leaving aside that ‘this definition’ is not mathematically rigorous (and is unclear) as explained in (a), the ‘equivalent’ is not defined (and is unclear). The cited result [5, Theorem 3] does not have any equivalence in its statement.

After (or before) the above-quoted unclear sentence of [SS13, p. 2] it is not written that [SS13, Remark 1] contains an attempt for rigorous formulation corresponding to that sentence:

(*) ‘If $D$ is a drawing of a graph $G$ in some surface $S$, then there is a $\mathbb{Z}_2$-drawing $(D', x, y)$ of $G$ in $S$ so that $i_D(e, f) = i_{D', x, y}(e, f)$ for every pair $(e, f)$ of independent edges.’

This is not mathematically rigorous (and is unclear) because ‘a $\mathbb{Z}_2$-drawing of $G$ in $S$’ is not defined. Instead of reading a proof or a reference to a proof of (*), one reads in [SS13, Remark 1]: ‘As mentioned earlier, a result like this (with a slightly different model) was stated by Levow [5].’

Presumably ‘of $G$ in $S$’ should be replaced by ‘of $G$ in the plane’. Unravelling the definitions and getting rid of $x$ (because $x$ can be realized by change of $D'$), we obtain the following corrected rigorous version of (*).

**Algebraization Lemma.** If $D$ is a drawing of a graph $G$ in some surface $S$, then there are a drawing $D'$ of $G$ in the plane and a vector $y \in \mathbb{Z}_2^E$ such that $i_D(e, f) = i_{D'}(e, f) + y^T e y_f$ for every pair $(e, f)$ of independent edges.

This is the main basic result used in [SS13], so the above critical remark affects the whole paper.

(c) (consequences for [FK19]) The paper [FK19] uses [SS13, Lemmas 3 and 4]. Thus the proofs of [FK19] are unreliable.

Moreover, the Algebraization Lemma is stronger than (the estimation ‘$\geq$’ of) [FK19, Corollary 10 (the first sentence)]. The result [FK19, Corollary 10 (the first sentence)] uses [FK19, Proposition 9] which uses [FK19, Lemma 5] which is the same as [SS13, Lemma 4] which uses the Algebraization Lemma. This looks like a vicious circle. This is yet another

\[12\] Indeed, the Algebraization Lemma for a $\mathbb{Z}_2$-embedding $D$, and known [FK19, Lemmas 6 and 7] trivially imply that result of [FK19]. Modulo those lemmas and a choice of basis in the homology group, that result of [FK19] is Lemma 2.3.1.a for $k = 1$ (due to Bikeev-Fulek-Kynčl).
motivation for appearance of reliable proofs (although I know how to rewrite the argument to avoid vicious circle).

(d) (recovery of results) The results of [FK19] are recovered by [Bi21, Theorem 1.1] (partly attributed to Fulek-Kynčl).

The Algebraization Lemma is recovered by the implication (⇒) of [Bi21, Theorem 1.1.b] (modulo the known algebraic lemmas, and for Y = \{ye\}), or by the proof of this implication in [Bi21, §2], or alternatively by Lemma 2.3.1.a for k = 1 (whose proof is slightly different from [Bi21, §2]). Hopefully the block additivity of [SS13] can also be recovered using [Bi21, Theorem 1.1] (obtained using ideas of [FK19]), cf. Remark 1.1.7.c. Also, I have an idea of how to rewrite papers [SS13, FK19] to make the proofs reliable and closer to the initial ideas of [SS13, FK19] than to [Bi21]. Nothing of these makes the proofs of [SS13, FK19] reliable.

\textbf{Remark 2.5.2.} Here I present my letters to D. Štefankovič (analogous letter was earlier sent to M. Schaefer; Cc M. Schaefer), to R. Fulek and J. Kynčl.

(a) (Sep 2, 2024) Dear Daniel,
Hope you are fine and healthy.

The following message might seem to you way too direct and formal. If so, could you shortly contact Marcus on our latest discussion with him, of which this letter is a user-oriented result (and on our discussions with him throughout the years). Hopefully we could treat these questions in a professional way.

My purpose is to give a proper introduction to \texttt{arXiv:2112.06636}. In particular,
* to give proper credit to papers on \(\mathbb{Z}_2\)-embeddings of graphs (see Remarks 1.1.6.abcd).
* to inform a reader if some published proofs are not reliable (see \texttt{arXiv:2101.03745}).

Attached please find the planned update of \texttt{arXiv:2112.06636}. See Remark 1.1.8a and §2.5. I would be grateful if you could show that some specific sentences of Remark 2.5.1.ab are incorrect (or unclear). I am willing to make corrections corresponding to your remarks.

Could you let me know if you plan (in the forthcoming future) to make improvements upon Remark 2.5.1?

If yes, then I would be glad to refer to arXiv update of your paper, and omit critical remarks on the earlier version (see \texttt{arXiv:2101.03745}, Remark 2.3).

If no, then (possibly after working on your remarks and sending you the revision), I would be glad to publish your and Marcus’ reply to (possibly modified) Remark 2.5.1, or a reference to such a reply.

I would be glad to do that even if I disagree with your reply. Please send me your text for publication, stating that you perpetually release copyright for this text (or a reference to a text involving ‘perpetual release of copyright’ statement).

Best Regards, Arkadiy.

(b) (Sep 8, 2024) Dear Rado, Dear Jan (Honza)
Hope you are fine and healthy.

My purpose is to give a proper introduction to \texttt{arXiv:2112.06636}. In particular,
* to give proper credit to papers on \(\mathbb{Z}_2\)-embeddings of graphs (see Remarks 1.1.6.abcd).
* to inform a reader if some published proofs are not reliable (see \texttt{arXiv:2101.03745}).

Attached please find the planned update of \texttt{arXiv:2112.06636}. See Remark 1.1.8a and §2.5. Remark 2.5.1.c concerns your paper [FK19].

Could you let me know if you plan (in the forthcoming future) to update arXiv version of [FK19] to make it independent of [SS13]? Your paper [FK19] is written so nicely that it would be easy for you to do that. You implicitly prove the Algebraization Lemma. It would be easy for you to explicitly state and prove its improvement required for lemmas from [SS13] used in your paper, and to reprove the lemmas. If you are in a hurry, then you can just refer to \texttt{arXiv:2012.12070v2} for a rigorous proof of the required \(\mathbb{Z}_2\)-embeddability criterion (which is partly attributed to you there).
The main result of [FK19] is anyway not the $\mathbb{Z}_2$-embeddability criterion, but its application to the quadratic estimation for $K_{n,n}$.

If yes, then I would be glad to refer to arXiv update of your paper, and omit critical remarks on the earlier version (see arXiv:2101.03745, Remark 2.3).

If no, then I would be glad to publish your reply to Remark 2.5.1.c, or a reference to such a reply.

I would be glad to do that even if I disagree with your reply. Please send me your text for publication, stating that you perpetually release copyright for this text (or a reference to a text involving 'perpetual release of copyright' statement).

Best Regards, Arkadiy.

(c) (Sep 15, 2024) Dear Daniel, Dear Marcus, Dear Rado, Dear Jan (Honza),
Attached please find the planned update of arXiv:2112.06636. See Remark 1.1.8a and §2.5.

On one hand, the readers would be grateful to see your reply to Remark 2.5. In order to publish your reply on arXiv I need your text for publication, and your statement that you perpetually release copyright for this text (or a reference to a text involving 'perpetual release of copyright' statement).

On the other hand, I will not inform you on further development unless you express your interest.
Best Regards, Arkadiy.

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