Canonical structure of classical field theory in the polymomentum phase space

Igor V. Kanatchikov

Laboratory of Analytical Mechanics and Field Theory
Institute of Fundamental Technological Research
Polish Academy of Sciences
Świętokrzyska 21, Warszawa PL-00-049, Poland

(Submitted Sept. 1996 —— Accepted July 1997)

Canonical structure of classical field theory in \( n \) dimensions is studied within the covariant polymomentum Hamiltonian formulation of De Donder–Weyl (DW). The bi-vertical \((n+1)\)-form, called polysymplectic, is put forward as a generalization of the symplectic form in mechanics. Although not given in intrinsic geometric terms differently than a certain coset it gives rise to an invariantly defined map between horizontal forms playing the role of dynamical variables and the so-called vertical multivectors generalizing Hamiltonian vector fields. The analogue of the Poisson bracket on forms is defined which leads to the structure of \( \mathbb{Z} \)-graded Lie algebra on the so-called Hamiltonian forms for which the map above exists. A generalized Poisson structure appears in the form of what we call a “higher-order” and a right Gerstenhaber algebra. The equations of motion of forms are formulated in terms of the Poisson bracket with the DW Hamiltonian \( n \)-form \( H \) (\( \text{vol} \) is the space-time volume form, \( H \) is the DW Hamiltonian function) which is found to be related to the operation of the total exterior differentiation of forms. A few applications and a relation to the standard Hamiltonian formalism in field theory are briefly discussed.

1 Introduction

The Hamiltonian formalism is based on the representation of the equations of motion in the first order form and the Legendre transform. The mathematical structures emerging from such a formulation of dynamics are known to be of fundamental importance in wide area of applications from the study of integrable systems to quantization.

---

\*PACS classification: 03.50, 02.40
\†AMS classification: 70 G 50, 58 F 05, 53 C 80, 81 S 10
\‡Keywords: classical field theory, Poincaré–Cartan form, De Donder–Weyl theory, Hamiltonian formalism, polysymplectic form, multivector fields, differential forms, Schouten–Nijenhuis bracket, Poisson bracket, Gerstenhaber algebra.

\&e-mail: kai@fuw.edu.pl
The generalization of Hamiltonian formalism to field theory is well-known which is based on the functional derivative equations of first order in partial time derivative. This formulation requires an explicit singling out of the time, or evolution variable and leads to the idea of a field as a dynamical system with a continually infinite number of degrees of freedom. The canonical quantization in field theory is known to be based on this approach. It can be given a covariant form a version of which is discussed for instance in [1]. Other discussions of the Hamiltonian formalism in field theory and further details can be found for example in [2, 3, 4, 5]. Within this approach the geometrical constructions of classical mechanics can in principle be extended to field theory using the functional analytic framework of infinite-dimensional geometry, but the applicability of such constructions is often more restricted as, for example, known difficulties in geometric quantization of field theory demonstrate. It is also not clear whether the framework of the canonical quantization based on the present Hamiltonian formalism; which requires the space-time to be, topologically, a direct product of space and time, is adequate for theories like General Relativity.

However, another formulation of field equations in the form of first order partial differential equations exists, the construction of which is also similar to the way the Hamiltonian formulation is constructed in mechanics, but which keeps the symmetry between space and time explicit. This formulation seems to be much less commonly known in theoretical physics in spite of the fact that its essential elements appeared more than sixty years ago in the papers by De Donder [6], Carathéodory [8], Weyl [7] and others on the multiple integral variational calculus (see for example [9, 10, 11, 12] for a review and further references). In this approach (see Sect. 2 for more details) the generalized coordinates are the field variables \( y^a \) (not the field configurations \( y^a(\mathbf{x}) \)) to which a set of \( n \) momentum-like variables, called polymomenta, \( p^i_a := \partial L / \partial (\partial_i y^a) \), is associated (here \( i = 1, ..., n \) is the space-time index). Similar to mechanics, the covariant Legendre transform: \( \partial_i y^a \rightarrow p^i_a, L(y^a, \partial_i y^a, x^i) \rightarrow H_{DW}(y^a, p^i_a, x^i) := p^i_a \partial_i y^a - L \) is performed, where the latter expression defines the covariant field theoretical analogue of Hamilton’s canonical function which we will call the De Donder–Weyl (DW) Hamiltonian function. Unlike the Hamiltonian density in the standard (instantaneous) Hamiltonian formalism, which is the time component of the energy-momentum tensor, the DW Hamiltonian function is a scalar quantity a direct physical interpretation of which is not evident. What is interesting is that in terms of the variables above the Euler-Lagrange field equations take the form of the system of first order partial differential equations (see eqs. (2.3) below) which naturally generalizes the Hamilton canonical equations of motion to field theory. This form of field equations is entirely space-time symmetric and, in addition, is formulated in the finite dimensional covariant analogue of the phase space, the space of variables \( y^a \) and \( p^i_a \) which we call here the polymomentum phase space (in our earlier papers the term “DW phase space” was used). By this means a field theory appears as a kind of generalized Hamiltonian system with many “times” the role of which is played both by space and time variables treated in a completely symmetric manner. In doing so note that the DW Hamiltonian function does not generate a time evolution of a field from a given Cauchy data, as the standard Hamiltonian does, but rather controls a space-time variation, or development,
of a field.

Because of the fundamental features which the formulation outlined above shares with the Hamiltonian formulation of mechanics we refer to it as the DW Hamiltonian formulation. Surprisingly enough, basic elements of a possible canonical formalism based on this formulation, and even the existence of some of these, still are rather poorly understood. In fact, at present it is not even evident whether or not the DW formulation of field equations does indeed provide us with a starting point for a certain canonical formalism in field theory, with an appropriate covariant analogue of the symplectic or Poisson structure, Poisson brackets and the related geometrical constructions, as well as the starting point for a quantization.

Recall that there exists also the Hamilton-Jacobi theory inherently related to the DW Hamiltonian formulation of field equations (see e.g. [9, 10] for a review). This theory is formulated in terms of the certain covariant partial differential equation for \( n \) Hamilton-Jacobi functions \( S^i(y^a, x^i) : \partial_i S^i = H_{DW}(y^a, p^a_i := \partial_a S^i, x^i) \) which obviously reduces to the familiar Hamilton-Jacobi equation in mechanics when \( n = 1 \). However, while the connection between the Hamilton-Jacobi equation and the Schrödinger equation in quantum mechanics is a well known fact which underlies both the quasi-classical approximation and the De Broglie-Bohm interpretation of quantum mechanics, a similar possible connection between the DW Hamilton-Jacobi equation and quantum field theory still remains completely unexplored. In this context it is worth remembering that historically the arguments based on the Hamilton-Jacobi theory led Schrödinger to his famous equation, and Wheeler to the Wheeler–DeWitt equation in quantum gravity. It is quite natural to ask, therefore, whether the DW Hamilton-Jacobi equation also may help us to reveal a certain new aspect of quantum field theory.

It should be noted that the DW canonical theory is, in fact, only the simplest representative of the whole variety of covariant canonical formulations of field theories in a polymomentum analogue of the phase space which are based on different choices of the "Lepagean equivalents" of the so-called Poincaré-Cartan form (see e.g. [10, 13, 14, 17]) and different definitions of polymomenta. The famous canonical theory of Carathéodory [8, 9, 11, 17] is another interesting example of such a formulation. A discussion of the corresponding more general "polymomentum canonical theories" for fields, as we suggest to name them, is left beyond the scope of the present paper.

Despite all of the attractive features of polymomentum canonical theories, such as finite dimensionality and a manifest covariance, which seem especially relevant in the context of the canonical analysis and quantization of general relativity and string theory (note, that no restriction to the globally hyperbolic underlying space-time manifold is in principle implied here), there are a surprisingly small number of their applications in the literature. The following papers contain some applications to classical field theory [20, 21, 22], gauge fields [23, 24], classical bosonic string [28, 29, 30, 31, 32], general relativity [33, 34, 35], and integrable systems [36, 37]. Several interesting examples are also considered in the book [16].

For possible applications in field theory the understanding of the interrelations between the polymomentum canonical theories and the standard instantaneous Hamiltonian formalism is important. A recent discussion of this body of questions can be
found in the papers by Gotay [15] (see also the preprint of the book [16] and the paper by Śniatycki [18]). Let us note also that the standard functional Hamiltonian form of field equations can in principle be derived directly from the DW covariant Hamiltonian equations [19].

Among the questions concerning the applications of the polymomentum canonical theories one of the most interesting ones is whether it is possible, or has any sense, to develop a field quantization starting from the DW Hamiltonian formulation or a more general polymomentum canonical theory. Indeed, it might be questioned if it so necessary to at first split the space-time in order to obtain the Hamiltonian formulation, and then to quantize a field according to the standard prescriptions of quantum theory and to prove finally the procedure to be consistent with the relativistic symmetries. Or, perhaps, it is possible to develop an inherently covariant field quantization based on the polymomentum covariant Hamiltonian framework, without altering the space-time symmetry, and then to obtain the results referring to a particular reference frame, if necessary, from the manifestly covariant formulation of quantum field dynamics. Another related question is whether a sort of quasi-classical transition from some formulation of quantum field theory to the Hamilton-Jacobi equations corresponding to various polymomentum canonical formulations of classical fields (see [9, 10]) exists. Obviously, to approach these questions one has to gain a deeper insight into those geometric and algebraic structures of classical polymomentum canonical theories whose analogues in mechanics form a classical basis of the quantization procedures. This paper may be considered as a step in this direction.

Recall that the problem of field quantization based on the DW Hamiltonian formalism was briefly discussed in the middle of the thirties by Born [38] and Weyl [39]. Then, in the early seventies considerable progress was made in understanding the differential geometric structures underlying the De Donder–Weyl theory [10, 11, 12, 13] (see also earlier papers by Dedecker [14] who studied more general canonical theories, and the recent paper by Gotay [13] for a subsequent development) which led to the so-called multisymplectic formalism [12]. However, the known attempts [16, 17, 12, 13] to approach a field quantization from this viewpoint essentially concentrated on only establishing links with the conventional formulation based on the instantaneous Hamiltonian formalism and have not led to any new formulation. More recently, an attempt to construct a quantization of field theory based entirely on the polymomentum canonical framework was reported by Günther in [18] who used his own geometrical version of De Donder-Weyl theory – the so-called ”polysymplectic Hamiltonian formalism” [18]. Unfortunately, the ideas of his brief report [18] were not developed to the point where a comparison with results of the conventional quantum field theory would be possible. A few other related discussions may be also found in recent papers [25, 26, 27].

The main obstacle to the development of field quantization based on a polymomentum Hamiltonian formulation seems to be the lack of an appropriate generalization of the Poisson bracket. Within the multisymplectic formalism [14, 15, 16, 20, 12] which is closely related to the DW canonical theory a Poisson bracket was proposed in [11, 12, 13] which is defined on forms of degree \((n-1)\) corresponding to observables in field theory (see also [16] and the recent paper [13]). However, the related construction
proved to be too restrictive to reproduce the algebra of observables, or currents, in the theo-
ries of a sufficiently general type \[1\], and it was not appropriate for representing the DW Hamiltonian field equations in Poisson bracket formulation. Moreover, the Jacobi identity for this bracket was found to be fulfilled modulo the exact terms only (see e.g. \[11, 13\]), although this fact may seem interesting from the point of view of homotopy Lie algebras. Other approaches, by Good \[50\], Edelen \[51\] and Günther \[47\], enable in principle writing of DW Hamiltonian field equations in a certain bracket formulation, but the algebraic properties of the brackets introduced by these authors and, therefore, their usefulness for quantization are rather obscure. Another relevant discussion of a covariant Poisson bracket in field theory may be also found in \[52\].

The purpose of the present study is to develop those elements of the DW canonical theory whose analogues in the Hamiltonian formalism of mechanics are important for canonical or geometric quantization. These in particular include the symplectic form, the Poisson bracket, the notion of canonically conjugate variables, and the representation of the equations of motion in Poisson bracket formulation. There is probably no more reliable basis for attacking this problem than to start from the most fundamental object of any canonical theory – the Poincaré-Cartan (PC) \(n\)-form – and to try to develop the subsequent elements of the formalism by searching for the proper generalizations to the DW formulation of field theory of the corresponding elements of the canonical formalism of mechanics (as they are presented, for example, in classical texts \[2\] and \[53\]).

The structure of the paper is as follows. In Sect. 2 we demonstrate how the DW Hamiltonian field equations readily follow from the PC form. This consideration indicates a suitable generalization to field theory of the notion of the canonical Hamiltonian vector field. This is the multivector field of degree \(n\) whose integral \(n\)-surfaces in the extended polymomentum phase space of the DW theory represent the extremals of the variational problem describing a field, that is the solutions of field equations. We point out that the vertical components of this multivector field essentially contain all information about the equations of motion.

The latter fact leads us to the notion of the polysymplectic form in Sect. 3. This is a closed "bi-vertical" form of degree \((n+1)\), eq. (10), which is proposed as a generalization of the symplectic form to the DW Hamiltonian formulation of field theory. The function of the polysymplectic form is that it provides us with the map between vertical multivector fields (generalizing Hamiltonian vector fields in mechanics) and the so-called Hamiltonian horizontal forms (which play the role of dynamical variables). We construct the bracket operations on Hamiltonian multivector fields and Hamiltonian forms and show that they endow the corresponding spaces with the structure of \(\mathbb{Z}\)-graded Lie algebra. We next discuss two generalizations of the derivation property of the usual Poisson bracket to the bracket operation on differential forms (which is called the graded Poisson bracket) introduced here. This leads to what we call respectively a higher-order and a right Gerstenhaber algebra as generalizations of a Poisson algebra to the DW Hamiltonian formulation of field theory.

The graded Poisson bracket of forms is used in Sect. 4 for obtaining the equations
of motion of Hamiltonian forms of degree \((n-1)\) in Poisson bracket formulation. As a by-product, the proper generalization of the notions of an integral of the motion and of canonically conjugate variables to the DW formulation is given.

In Sect. 5 the bracket representation of the equations of motion of arbitrary forms is considered. For this purpose the bracket with forms of degree \(n\) is defined which requires the enlargement of the space of Hamiltonian multivector fields by vertical-vector valued one-forms. We show that the equations of motion may be written in terms of the bracket with the DW \(n\)-form \(H_{\text{vol}}\) and shortly discuss the problems related to the algebraic closure of the enlargement above.

In Sect. 6, a few examples of applications of our graded Poisson bracket on forms to the system of interacting scalar fields, electrodynamics and the Nambu-Goto string are considered. A general discussion of our results, which also includes observations concerning a relation of the graded Poisson bracket on forms to the standard Poisson bracket of functionals, is presented in Sect. 7. In App. 1 some details of the calculation related to the higher-order graded Leibniz rule can be found.

2 De Donder–Weyl Hamiltonian field equations and the Poincaré–Cartan form

Let us consider a field theory given by the first order variational problem

\[
\delta \int L(y^a, \partial_i y^a, x^i) \, \tilde{\text{vol}} = 0, \tag{2.1}
\]

where \(\{y^a\}, 1 \leq a \leq m\) are field variables, \(\{x^i\}, 1 \leq i \leq n\) are space-time variables, and

\[
\tilde{\text{vol}} := dx^i \wedge ... \wedge dx^n,
\]

is the volume form on the space-time manifold. In order to simplify formulae, in the definition of the \(n\)-volume form \(\text{vol}\) and the following discussion it is implied that the coordinates on the \(x\)-space are chosen so that the metric determinant \(|g| = 1\).

The standard way of studying the solutions of the variational problem is solving the Euler-Lagrange equations. A more geometrical approach is formulated in terms of the so-called Poincaré–Cartan form which is usually written in terms of the Lagrangian coordinates \((y^a, \partial_i y^a, x^i)\). For our purposes, however, we introduce a new set of Hamiltonian like variables:

\[
p^i_a := \frac{\partial L}{\partial (\partial_i y^a)}
\]

and

\[
H_{\text{DW}}(y^a, p^i_a, x^i) := p^i_a \partial_i y^a - L \tag{2.2}
\]

which are defined in a completely space-time symmetric manner. The set of variables above is central to the so-called De Donder–Weyl (DW) canonical theory for fields which appeared for the first time in the papers on the calculus of variations in the middle of the 1930s [3, 4]. We shall call these variables the polymomenta and the DW Hamiltonian function respectively. It is a rather straightforward calculation to
demonstrate that in terms of the variables above the Euler-Lagrange equations may be written in the following first order form (see e.g. [9, 10, 11, 14, 16]):

$$\partial_i p^i_a = -\frac{\partial H}{\partial y^a}, \quad \partial_i y^a = \frac{\partial H}{\partial p^i_a}. \quad (2.3)$$

This is the canonical form of field equations within the DW theory. In view of the evident similarity of eqs. (2.3) to Hamilton’s equations in mechanics (whose multidimensional, or rather “multi-time” generalization they provide us with) we will call them the DW Hamiltonian field equations.

Obviously, this formulation of field equations is explicitly space-time symmetric, and, unlike the instantaneous approach, it involves the finite dimensional analogue of the phase space, the \((m + mn + n)\)-dimensional space of variables

$$z^M := (y^a, p^j_b, x^i), \quad 1 \leq M \leq m + mn + n$$

which we call the extended polymomentum phase space.

The present paper is devoted to the investigation of the mathematical structures underlying the above formulation of field equations. The question of particular interest for us is whether a generalization of the Poisson brackets to the DW Hamiltonian formulation exists and whether the DW Hamiltonian field equations can be written in terms of these brackets. In mechanics the answers to these questions are well known and they lead to the geometrical description of mechanics in terms of the symplectic or the Poisson structure (see e.g. [2, 5, 53]). This description is known to have its roots in the geometrical formulation of the one dimensional calculus of variations in terms of the Poincaré–Cartan (PC) form. In a certain sense, all the elements of the canonical formalism in mechanics can be obtained step by step proceeding directly from the PC form. In field theory considered from the point of view of the DW canonical theory the analogues of the corresponding constructions are either unknown or rather poorly understood while the notion of the PC form is well defined, at least for first order theories. In this paper we try to recover the structures underlying the DW formulation of field theory by proceeding from the corresponding PC form and developing the elements of the canonical formalism based on the analogy with the corresponding constructions in mechanics.

As the first step of realization of this idea one has to find an appropriate description of the DW Hamiltonian field equations in terms of the PC form. In mechanics the equations of motion appear as the equations for the integral curves of the total canonical Hamiltonian vector field which annihilates the exterior differential of the PC one-form. A field theoretical generalization of this fact is presented below.

The PC form in terms of the DW Hamiltonian variables is known to be given by (see e.g. Refs. [14, 16, 11])

$$\Theta_{DW} = p_i^a \wedge dy^a \wedge \partial_i \tilde{vol} - H_{DW} \tilde{vol}.$$  

Its exterior differential, the canonical \((n + 1)\)-form, is

$$\Omega_{DW} = dp_i^a \wedge dy^a \wedge \partial_i \tilde{vol} - dH_{DW} \wedge \tilde{vol}. \quad (2.4)$$
The symbol $\mathcal{J}$ denotes the interior product of a (multi)vector to its left and a form to its right. In the following, we will omit the subscript DW in $H_{DW}$, but the scalar quantity to be denoted as $H$ and termed the (DW) Hamiltonian function should not be confused with the usual Hamiltonian or its density which will not appear in this paper.

The canonical $(n+1)$-form $\Omega_{DW}$ contains all the information about a field’s dynamics, because it is constructed immediately from the Lagrangian density. In particular, the DW Hamiltonian form of field equations can be derived directly from $\Omega_{DW}$. Recall that from the calculus of variations it is known that the extremals of the variational problem are isotropic subspaces of the PC form spanned by the (vertical) vector fields annihilating $\Omega_{DW}$ (cf. e.g. [10, 14, 16, 20, 41, 46, 49]). For our purposes another formulation of this fact proved to be useful. Namely, the solutions of the variational problem (2.1) may be viewed as $n$-dimensional surfaces in the extended polymomentum phase space. Let us describe these surfaces by the multivector field of degree $n$ (or $n$-vector, for brevity) denoted $\mathcal{X}$

$$\mathcal{X} := \frac{1}{n!} \mathcal{X}^{M_1...M_n}(z) \partial_{M_1...M_n}$$

(2.5)

which represents their tangent $n$-planes. Here the notation

$$\partial_{M_1...M_n} := \partial_{M_1} \wedge \ldots \wedge \partial_{M_n}$$

is introduced. The $n$-vector field $\mathcal{X}$ naturally generalizes the velocity field of the canonical Hamiltonian flow in classical mechanics to the multidimensional, $n > 1$, case of field theory.

The condition on $\mathcal{X}$ to determine classical extremals is that the form $\Omega_{DW}$ should vanish on $\mathcal{X}$, that is

$$\mathcal{X} \mathcal{J} \Omega_{DW} = 0,$$

(2.6)

or in terms of the components of $\mathcal{X}$

$$\mathcal{X} \mathcal{J} \Omega = \frac{1}{n!}[\mathcal{X}^i_1...i_n \partial_{i_1...i_n}$$

$$+ \mathcal{X}^a_{i_1...i_{n-1}} \partial_{a i_1...i_{n-1}} + \mathcal{X}^a_{i_1...i_{n-1}} \partial^a_{i_1...i_{n-1}}$$

$$+ \mathcal{X}^a_{b i_1...i_{n-2}} \partial_a \wedge \partial^b \wedge \partial_{i_1...i_{n-2}} + \ldots ] \mathcal{J} \Omega_{DW} = 0.$$  

This gives rise to the following expressions

$$nX^{ai_1...i_{n-1}} \epsilon_{ii_1...i_{n-1}} = \partial^a_i H X^{i_1...i_n} \epsilon_{i_1...i_n},$$

$$nX^a_{i_1...i_{n-1}} \epsilon_{ii_1...i_{n-1}} = \partial_a H X^{i_1...i_n} \epsilon_{i_1...i_n},$$

$$(-1)^n(n-1)X^a_{b i_1...i_{n-2}} \delta^a_b \epsilon_{i_1...i_{n-2}ij} = \partial_b H X^{vi_1...i_{n-1}} \epsilon_{i_1...i_{n-1}j}.$$  

(2.7)

Further, the integral $n$-surfaces of the multivector field $\mathcal{X}$ can be described by the differential equation written in terms of the Jacobian

$$\mathcal{X}^{M_1...M_n}(z) = \mathcal{N} \left( \frac{\partial(z^{M_1}, \ldots, z^{M_n})}{\partial(x^1, \ldots, x^n)} \right),$$

(2.8)
where \( \mathcal{N} \) depends on the choice of the parameterization of an \( n \)-surface in the extended polymomentum phase space which represents a solution of field equations. Obviously, the equation above is a natural generalization of the o.d.e. for the integral curves of a vector field. Substituting the expression for the components of \( \vec{X} \), eq. (2.7), into (2.8) we obtain the DW Hamiltonian field equations, eqs. (2.3), from the “vertical” components \( X^{ai_1...i_{n-1}}_a \) and \( X^{ai_1...i_{n-1}}_{a_1} \) of \( \vec{X} \). The “bi-vertical” components, \( \bar{X}^{ai_1...i_{n-2}}_a \), lead to a consequence of the DW field equations. Thus, the information about the classical dynamics of fields is essentially encoded in the ”vertical” components of the multivector field annihilating the canonical \((n+1)\)-form (2.4). This observation underlies our construction in the following section. Note that the issue of the integrability of the multivector field \( \vec{X} \) essentially does not arise here because the condition (2.6) specifies in fact only a small fraction of the components of \( \vec{X} \) (cf. eq. (2.7)), so that the remaining components are only restricted by the consistency of eq. (2.8).

3 Polysymplectic form, Hamiltonian multivector fields and forms, and graded Poisson brackets

In this section a generalization of the basic structures of classical Hamiltonian mechanics, such as the symplectic form, Hamiltonian vector fields and functions, and the Poisson bracket, to the DW Hamiltonian formulation of field theory is presented.

Our starting point is the observation that the well-known formula \( X_H \mathcal{J}_\omega = dH \) relating the canonical Hamiltonian vector field \( X_H \) to Hamilton’s function \( H \) by means of the map given by the symplectic form \( \omega \) has its natural counterpart, eq. (3.2), in the DW theory.

For this purpose let us note that the extended polymomentum phase space can be viewed as a bundle \( \pi: Z \rightarrow M \) over the space-time manifold \( M \). The local coordinates on \( Z \) are \( \{z^M\} \) introduced above. The fiber coordinates

\[
z^v = (y^a, p_a^i), \quad v = 1, ..., m, m + 1, ..., m + mn
\]

will be referred to as vertical, as well as the corresponding subspace and the objects related to it. Correspondingly, the space-time coordinates \( x^i \) are to be referred to as horizontal.

Now, let us introduce several objects which will appear in the subsequent discussion. The definitions will be given in local coordinate terms used throughout the present paper. Although we will also make some remarks concerning the invariant meaning of the introduced objects, the issue of the intrinsic geometric formulation of our approach is left beyond the scope of this paper.

(i) A vertical vector \( X^V \) is an element of the vertical tangent bundle of \( Z \), \( VTZ := \ker T\pi \). In local coordinates

\[
X^V := X^a \partial_a + X^i_a \partial^a_i.
\]

(ii) A form of degree \( p \), \( \bar{F} \), is called horizontal if in local coordinates it takes the
form
\[
\tilde{F} := \frac{1}{p!} F_{i_1...i_p}(z) \, dx^{i_1} \wedge ... \wedge dx^{i_p}.
\]
Intrinsically this means that a contraction of a horizontal form with a vertical vector always vanishes. More generally, a form of degree \(p\) is called \((p - q)\)-horizontal if its contraction with any \((q + 1)\) vertical vectors vanishes. This means that it has \((p - q)\) or more horizontal components. The space of \((p - q)\)-horizontal \(p\)-forms is to be denoted \(\wedge_{q}^{p}\).

(iii) A multivector field of degree \(p\) is called *vertical* and denoted \(\tilde{X}^V\) if in a local coordinate system it can be written in the form
\[
\tilde{X}^V := \frac{1}{p!} \tilde{X}^{vi_1...i_{p-1}}(z) \, \partial_v \wedge \partial_{i_1} \wedge ... \wedge \partial_{i_{p-1}}.
\]
This definition is not invariant, as in a different coordinate system higher vertical components may appear. The intrinsic definition is that the inner product of the vertical \(p\)-multivector with any horizontal \(p\)-form vanishes. This means that \(\tilde{X}^V\) has at least one vertical component.

(iv) For an arbitrary \(p\) form \(\Phi = \frac{1}{p!} \Phi^{M_1...M_p} dz^{M_1} \wedge ... \wedge dz^{M_p}\), where \(\{dz^M\} := \{dz^v, dx^i\}\) is a holonomic basis of \(T^*Z\), the operation of the vertical exterior differentiation, \(d^V\), in local coordinates is given by
\[
d^V \Phi = \frac{1}{p!} \partial_v \Phi^{M_1...M_p} dz^v \wedge dz^{M_1} \wedge ... \wedge dz^{M_p}.
\]
In particular,
\[
d^V H = \partial_a H dy^a + \partial_i^a H dp^i_a.
\]
As under the coordinate transformation \(dz^v\) transform as \(dz'^v = \frac{\partial z'^v}{\partial z^w} dz^w + \frac{\partial z'^w}{\partial z^v} dx^i\) the expressions above are not invariant. However, if \(\Phi \in \wedge_{q}^{p}\) and \(d\) denotes the exterior differential on the extended polymomentum phase space \(\tilde{Z}\), the expression \(d^V \Phi\) can be understood as the coset \(d^V \Phi := [d\Phi \mod \wedge_{q+1}^{p}]\), so that the highest horizontal part is factored off.

Further, let us separate out the non-horizontal part of the PC-form in Sect. 2:
\[
\Theta^V := p!^a dy^a \wedge \partial_i \tilde{\text{vol}}.
\]
This can be done in the simplified, yet nonetheless interesting, case of the trivial bundle structure of the extended polymomentum phase space. In a more general situation the expression above can be understood as the coset: \(\Theta^V := [\Theta \mod \wedge_{q+1}^{p}]\).

By taking the vertical exterior differential of \(\Theta^V\) we obtain \(\Omega^V := d^V \Theta^V\), which in local coordinates can be written as follows
\[
\Omega^V := -dy^a \wedge dp^i_a \wedge \partial_i \tilde{\text{vol}}. \quad (3.1)
\]
This object is not a form unless the extended polymomentum phase space is a trivial bundle over the space-time. In a more general case it can be treated as the coset $\Omega^V := [\Omega_{DW} \mod \Lambda^n]$.  

Now, the geometrical condition (2.6) which leads to the DW field equations can be written in the form

$$\nabla_X^V \Omega^V = (-1)^n d^V H,$$

(3.2)

if the parameterization in (2.8) is chosen in such a way that

$$\frac{1}{n!} X_{i_1...i_n} \partial_{i_1...i_n} \tilde{vol} = 1.$$

This observation justifies a significance of the objects introduced above and inspires the construction below.

We shall call the object $\Omega^V$ in (3.1) the polysymplectic form adopting the term earlier introduced by Guenther [47] for a similar object – the horizontal-vector valued vertical two-form $dy^a \wedge dp^i \otimes \partial_i$ – in the context of his polysymplectic Hamiltonian formalism. The polysymplectic form defined here has a potential advantage that it is related to the multidimensional PC form of the calculus of variations in exactly the same way as the symplectic form in mechanics is related to the one dimensional PC form. In what follows, the polysymplectic form $\Omega^V$ will be denoted simply as $\Omega$, and the superscripts $^V$ labeling the vertical multivectors will also be dropped, since all the multivectors in the subsequent discussion are vertical. Note also that the subsequent relations involving $d^V$ and $\Omega$ have to be understood as the relations between the corresponding equivalence classes (modulo terms of the highest horizontal degree).

Let us now recall (see e.g. [4, 53] for details) that the structures of classical Hamiltonian mechanics are incorporated essentially in a single statement, that Lie derivative of a symplectic form $\omega$ with respect to the vertical vector fields $X$ which generate the infinitesimal canonical transformations vanishes: $\mathcal{L}_X \omega = 0$. This is the well-known canonical symmetry of mechanics. Since $\omega$ is closed, it implies locally that $X_F \cdot \omega = dF$ for some function $F$ of the phase space variables. If the latter equality holds globally, the vector field $X_F$ is said to be a (globally) Hamiltonian vector field associated with the Hamiltonian function $F$. When $F$ is taken to be the canonical Hamiltonian function $H$, the equations for the integral curves of $X_H$, the canonical Hamiltonian vector field, reproduce Hamilton’s canonical equations of motion. Further, the canonical symmetry and the map above allow to construct appropriate brackets on vector fields and functions. Our intention here is to find an analogue of this scheme for the DW Hamiltonian formulation in field theory.

From the above considerations, it can already be concluded that the vertical $n$-vector field $\tilde{X}$ associated with the DW Hamiltonian function according to eq. (3.2) is similar to the canonical Hamiltonian vector field in mechanics, and that the polysymplectic form $\Omega$ is analogous to the symplectic 2-form. To pursue this parallel further, let us introduce the operation of generalized Lie derivative with respect to a vertical multivector field of degree $p$, $\tilde{X}$, and postulate, as a fundamental symmetry principle
extending the canonical symmetry of mechanics, that

\[ \mathcal{L}_\mathring{X} \Omega = 0. \tag{3.3} \]

A generalized Lie derivative of any form \( \Phi \) with respect to the vertical multivector field \( \mathring{X} \) of degree \( p \) is given by

\[ \mathcal{L}_\mathring{X} \Phi := \mathring{X} \bigwedge d^V \Phi - (-1)^p d^V (\mathring{X} \bigwedge \Phi) \tag{3.4} \]

which is the simplest generalization of the Cartan formula relating the Lie derivative of a form along the vector field to the exterior derivative and the inner product with a vector. However, unlike the \( p = 1 \) case the operation \( \mathcal{L}_\mathring{X} \) does not preserve the degree of a form it acts on. Instead, it maps \( q \)-forms to \( (q - p + 1) \)-forms. Note, that a similar definition of generalized Lie derivative with respect to a multivector field was considered earlier in [54].

Since \( \Omega \) is closed with respect to the vertical exterior differential, from (3.3) and (3.4) it follows that

\[ d^V (\mathring{X} \bigwedge \Omega) = 0, \tag{3.5} \]

so that locally we can write

\[ \mathring{X} \bigwedge \Omega = d^V 0^F, \tag{3.6} \]

for some 0-form \( 0^F \) which depends on the polymomentum phase space variables \( z^M \).

By analogy with mechanics, we can call \( \mathring{X} \) the (globally) Hamiltonian \( n \)-vector field associated with the Hamiltonian 0-form \( 0^F \), if such a form exists globally. Similarly, the multivector field \( \mathring{X} \) satisfying eq. (3.3) or (3.5) can be called locally Hamiltonian. From eqs. (3.2) and (3.6) we see that our symmetry postulate in eq. (3.3), together with the definition of generalized Lie derivative in eq. (3.4), is consistent with the DW Hamiltonian field equations when the 0-form \( 0^F \) in eq. (3.6) is taken to be the DW Hamiltonian function \( H \).

Now, given two locally Hamiltonian \( n \)-vector fields it might seem natural to define their bracket \([\ , \]\) by means of the equality

\[ [\mathring{X}_1 , \mathring{X}_2] \bigwedge \Omega := \mathcal{L}_{\mathring{X}_1} (\mathring{X}_2 \bigwedge \Omega), \tag{3.7} \]

which is just an extension of the definition of the Lie bracket of vector fields. From this definition it formally follows that

\[ d^V ([\mathring{X}_1 , \mathring{X}_2] \bigwedge \Omega) = 0, \tag{3.8} \]

so that \([\mathring{X}_1 , \mathring{X}_2]\) is also locally Hamiltonian. However, the bracket above does not map \( n \)-vectors to \( n \)-vectors; instead it mixes multivectors of different degrees. Moreover, the
counting of degrees in eq. (3.7) gives \( \text{deg}([X_1, X_2]) = 2n - 1 \), so that the bracket identically vanishes unless \( n = 1 \). This simple consideration indicates that multivector fields and, consequently, forms of various degrees should come into play. We are thus led to the following construction.

Given the polysymplectic \((n + 1)\)-form \( \Omega \), we define the set of \textit{locally Hamiltonian} (LH) multivector fields \( \hat{X}^p \), \( 1 \leq p \leq n \), which satisfy the condition

\[
\mathcal{L}_{\hat{X}^p} \Omega = 0. \tag{3.9}
\]

The \( p \)-vector field is then called \textit{Hamiltonian} if there exists a horizontal \((n - p)\)-form \( \hat{F} \) such that

\[
\hat{X}^p \llcorner \Omega = d^v n^{-p}, \tag{3.10}
\]

The multivector field \( \hat{X}^p_F \) is said to be the \textit{Hamiltonian multivector field} associated with the form \( n^{-p} \hat{F} \). The horizontal forms \( \hat{F} \) to which a Hamiltonian multivector field can be associated are to be referred to as \textit{Hamiltonian forms}. Hamiltonian forms of various degrees extend to field theory, within the present approach, Hamiltonian functions, or dynamical variables, in mechanics. The inclusion of forms of various degrees is motivated also by the fact that the dynamical variables of interest in a field theory in \( n \) dimensions can be represented in terms of horizontal forms of various degrees \( p \leq n \) (\( n \)-forms are to be included in Sect. 5). It should be noted, however, that, in contrast with the symplectic formulation of mechanics, the map in eq. (3.10) implies a rather strong restriction on the allowable dependence of the components of Hamiltonian forms on the polymomenta (see e.g. eq. (4.5) below for the case of \((n - 1)\)-forms). This limits the class of admissible Hamiltonian forms.

Note that the map in (3.10) is invariant. In fact, under a coordinate transformation the vertical multivector \( n^{-p} \hat{X}^p_F \) may gain higher vertical contributions (let us denote them \( n^{-p} \hat{X}^p v^v \)), the polysymplectic form may gain an \( n \)-horizontal part \( \tilde{\text{vol}} \) in which \( W \) is essentially a vertical covector, and the vertical exterior differential of \( \hat{F} \) may gain a pure \((p + 1)\)-horizontal addition \( F^{p+1} \). Thus, in another coordinate system (3.10) takes the form \( (n^{-p} \hat{X}^p + n^{-p} \hat{X}^p v^v) \llcorner (\Omega + W \wedge \tilde{\text{vol}}) = d^v \hat{F}^{p+1} + f \) whence it follows: \( n^{-p} \hat{X}^p \llcorner \Omega = d^v \hat{F}^{p+1} \) and \( n^{-p} \hat{X}^p (W \wedge \tilde{\text{vol}}) + n^{-p} \hat{X}^p v^v \llcorner \Omega = f \). The former equation coincides with (3.10), while the latter, being pure horizontal, is factored off, for (3.10) is understood as an equality of equivalence classes modulo the horizontal contributions of highest degree.

The bracket of two locally Hamiltonian fields may now be defined by

\[
[X_1^p, X_2^q] \llcorner \Omega := \mathcal{L}_{X_1^p} (X_2^q \llcorner \Omega). \tag{3.11}
\]

It is easy to show that the bracket defined above maps the pair of LH fields to a Hamiltonian field and that
(i) it generalizes the Lie bracket of vector fields,
(ii) its multivector degree is
\[
\deg([\dot{X}_1, \dot{X}_2]) = p + q - 1, \tag{3.12}
\]
(iii) it is graded antisymmetric
\[
[\dot{X}_1, \dot{X}_2] = -(\dot{X}_2, [\dot{X}_1, \dot{X}_2]), \tag{3.13}
\]
and
(iv) it fulfills the graded Jacobi identity
\[
(-1)^{g_1 g_3} [\dot{X}, [\dot{X}, \dot{X}]] + (-1)^{g_2 g_3} [\dot{X}, [\dot{X}, \dot{X}]] + (-1)^{g_2 g_3} [\dot{X}, [\dot{X}, \dot{X}]] = 0, \tag{3.14}
\]
where \(g_1 = p - 1\), \(g_2 = q - 1\) and \(g_3 = r - 1\).

All the properties above are known for the Schouten–Nijenhuis (SN) bracket of multivector fields. This is not of surprise, as the relation between the notion of Lie derivative with respect to a multivector field and the SN bracket is already evident from (54) (see also a related discussion in [56]). However, in our case there is a little difference due to the fact that the bracket is defined on vertical multivectors, so that it contains only the derivatives with respect to the vertical variables, while the multivectors themselves have both vertical and horizontal indices. Despite this difference, the bracket of locally Hamiltonian multivector fields defined in (3.11) will be referred to here as the (vertical) SN bracket. It is clear from the above equations that the set of LH multivector fields equipped with the SN bracket is a \(\mathbb{Z}\)-graded Lie algebra.

Now, taking \(\dot{X}_1\) and \(\dot{X}_2\) to be Hamiltonian multivector fields, we obtain:
\[
[\dot{X}_1, \dot{X}_2] \Leftrightarrow \Omega = \mathcal{L}_{\dot{X}_1} d^V \mathcal{F}_2
\]
\[
= (-1)^{p+1} d^V (\dot{X}_1 \Leftrightarrow d^V \mathcal{F}_2)
\]
\[
= -d^V \{F_1, \mathcal{F}_2\}, \tag{3.15}
\]
where \(r = n - p\) and \(s = n - q\). The first equality in (3.15) follows from (3.10) and (3.11). The second one essentially proves that the SN bracket of two Hamiltonian multivector fields is also a Hamiltonian multivector field. The third equality identifies the form which is associated with the SN bracket of two Hamiltonian multivector fields with the bracket of forms which are associated with these multivector fields.

We shall call the bracket operation on forms introduced in (3.15) the \textit{graded Poisson bracket}. In the following subsection it will be shown that it fulfills a certain graded generalization of the properties of the usual Poisson bracket. It is also evident from the definition that our graded Poisson bracket of forms is related to the Schouten-Nijenhuis bracket of multivector Hamiltonian fields and the polysymplectic form in just
the same way as the usual Poisson bracket of functions is related to the Lie bracket of Hamiltonian vector fields and the symplectic form.

From eq. (3.15) the following useful formulae for the graded Poisson bracket of forms can be deduced

\[
\{F_1^r, F_2^s\} = (-1)^{(n-r)} X_1^{n-r} dV^s F_2 - (-1)^{(n-r)} X_1^{n-r} X_2^{s-r} \Omega.
\] (3.16)

These relations generalize the usual definitions of the Poisson bracket in mechanics, but they are merely a consequence, as in mechanics, of the more fundamental definition, eq. (3.9). Note, that in spite of the fact that the definition in eq. (3.15) determines only the vertical exterior differential of the Poisson bracket, there is no arbitrariness "modulo an exact form" in the definition of the Poisson bracket itself (as opposed to the previous definitions of the Poisson bracket of \((n-1)\)-forms, cf. [11, 46, 12]) because the latter maps horizontal forms to horizontal forms, while the \(dV\)-exact addition would be necessarily vertical (if understood as a coset). Note also that using a consideration similar to that which demonstrated the invariance of the map (3.10) the invariance of the graded Poisson bracket given by (3.16) can be established.

3.1 Algebraic properties of graded Poisson bracket

Let us consider the algebraic properties of the graded Poisson bracket of forms. The degree counting in eq. (3.15) gives

\[
\text{deg}\{F_1^p, F_2^q\} = p + q - n + 1,
\] (3.17)

so that the bracket exists if \(p + q \geq n - 1\). From the graded anticommutativity of the SN bracket the similar property can be deduced for the graded Poisson bracket:

\[
\{F_1^p, F_2^q\} = -(-1)^{g_1 g_2} \{F_2^q, F_1^p\},
\] (3.18)

where \(g_1 := n - p - 1\) and \(g_2 := n - q - 1\) are degrees of corresponding forms with respect to the bracket operation. Furthermore, by a straightforward calculation the graded Jacobi identity can be proven:

\[
(-1)^{g_1 g_2} \{F_1^p, \{F_2^q, F_3^r\}\} + (-1)^{g_1 g_3} \{F_2^q, \{F_3^r, F_1^p\}\} + (-1)^{g_2 g_3} \{F_3^r, \{F_1^p, F_2^q\}\} = 0,
\] (3.19)

where \(g_3 := n - r - 1\).

Thus, the space of Hamiltonian forms equipped with the graded Poisson bracket operation defined in (3.15), is a \(\mathbb{Z}\)-graded Lie algebra. Now, the question naturally
arises as to whether this bracket gives rise also to some appropriate analogue of the Poisson algebra structure, as is the case in mechanics.

In order to answer this question, the analogue of the Leibniz rule has to be considered. It should be noted from the very beginning that the space of Hamiltonian forms on which the bracket \{\ddot{x}_i, \ddot{y}_j\} has so far been defined is not stable with respect to the exterior product. For example, if an \((n-1)\)-form \(F\) is Hamiltonian, then it is easy to show that its product with any function which depends on the polymomenta is not a Hamiltonian \((n-1)\)-form because Hamiltonian \((n-1)\)-forms must have a very specific (linear) dependence on the polymomenta (cf. eq. (4.5)). Therefore, there is little sense in the Leibniz rule with respect to the exterior product within the space of Hamiltonian forms. However, the dynamical variables in field theory which cannot be represented as Hamiltonian forms can be easily constructed. For example, it is quite natural to associate the \((n-1)\)-form \(P_j := T_{ij} \partial_i \tilde{\text{vol}}\) to the canonical energy-momentum tensor \(T_{ij} := p_a^i \partial_j y^a - \delta_j^i L\). It is easy to see that expressing it in terms of the DW variables so that \(T_{ij} = T_{ij}(y^a, p_a^i, x^i)\) one is led to the \((n-1)\)-form \(P_j\) which is not Hamiltonian for most field theories that are of interest. One may expect, therefore, that the algebraic structure on Hamiltonian forms is embedded in some more general structure which does also involve non-Hamiltonian forms. Investigation of the validity of the Leibniz rule with respect to the exterior product may be considered as an attempt to gain insight into this more general structure (see also the related discussion at the end of Sect. 5). An alternative tactic which is not considered here might be a construction of the multiplication law of horizontal forms with respect to which the space of Hamiltonian forms is stable.

Let us consider first the left Leibniz rule. The calculation in the Appendix yields the following result:

\[
\{\{F, \dot{F} \wedge \ddot{F}\} = \{\{F, \dot{F}\} \wedge \ddot{F} + (-1)^q(n-p-1) \dot{F} \wedge \{\{F, \ddot{F}\}\}
- (n-p)(-1)^{n-p-q-p-1} \sum_{s=1} \sum_{i_1 \ldots i_{n-p-1}} X^{v_{i_1 \ldots i_{n-p-1}}} ([(-1)^q(n-p-s-1) \partial_{i_1 \ldots i_s} J] F)
\wedge \partial_{i_1 \ldots i_s} J \partial_{i_1 \ldots i_s} J F),
\]

where \(X^p\) is the Hamiltonian multivector field associated with \(F\) and \(\partial_{i_1 \ldots i_s} J \partial_{i_1 \ldots i_s} J F\).

The first two terms in (3.20) are typical for the graded Leibniz rule. However, the supplementary terms are also present, appearing due to the fact that the multivector field of degree \((n-p)\) is not a graded derivation on the exterior algebra, but rather a kind of graded differential operator of order \((n-p)\) (see App. 1 for more details). If there were no additional higher order terms in the last two lines of eq. (3.20), a \(Z\)-graded Poisson algebra with different gradings with respect to the graded commutative

---

1See the Note at the end of Sect. 5.
exterior product and the graded anticommutative bracket operation would be arrived at. This structure is known as a Gerstenhaber algebra \cite{57}. However, the expression obtained here is essentially a higher-order analogue of the graded Leibniz rule (see App. 1), so that the algebraic structure which emerges here may be viewed as a higher-order generalization of Gerstenhaber algebra. This notion means that the bracket operation acts as a higher-order graded differential operator on a graded commutative algebra instead of being a graded derivation.

The above formulation of the higher-order Leibniz rule is not quite satisfactory as it is not given entirely in terms of the bracket with $F^p$ and refers explicitly to the components of the multivector field associated with the form $F^p$. More appropriate formulation may be given in terms of the $r$-linear maps $\Phi^r_D$ introduced by Koszul \cite{58}, which are associated with a graded higher-order differential operator $D$ on a graded commutative algebra.

On the space of horizontal forms $\Lambda^*_0$ the map $\Phi^r_D : \otimes^r \Lambda^*_0 \to \Lambda^*_0$ is defined by

$$\Phi^r_D(F_1, ..., F_r) := m \circ (D \otimes 1) \lambda^r(F_1 \otimes ... \otimes F_r),$$

for all $F_1, ..., F_r$ in $\Lambda^*_0$. Here $m$ denotes the multiplication map:

$$m(F_1 \otimes F_2) := F_1 \wedge F_2,$$

$\lambda^r$ is a linear map $\otimes^r \Lambda^*_0 \to \Lambda^*_0 \otimes \Lambda^*_0$ such that

$$\lambda^r(F_1 \otimes ... \otimes F_r) := \lambda(F_1) \wedge ... \wedge \lambda(F_r),$$

and $\lambda : \Lambda^*_0 \to \Lambda^*_0 \otimes \Lambda^*_0$ is the map given by

$$\lambda(F) := F \otimes 1 - 1 \otimes F.$$

The graded differential operator $D$ is said to be of $r$-th order iff $\Phi^r_{D} = 0$ identically. This definition may be checked to be consistent with the idea of $r$-th order partial derivative on the multiplicative algebra of functions.

Now, the higher-order Leibniz rule fulfilled by the (right) bracket with a $p$-form may be written compactly as follows:

$$\Phi^{n-p+1}_D(F_1, ..., F_{n-p+1}) = 0.$$  \hspace{1cm} (3.22)

For $p = (n-1)$ this reproduces the usual Leibniz rule, because the vector field associated with a form of degree $(n-1)$ is a derivation on the exterior algebra. The simplest non-trivial example is obtained when $p = (n-2)$. This leads to the second-order graded Leibniz rule

$$\Phi^2_{n-2}(F_r, F, F) = 0.$$
or, in the explicit form,
\[
\left\{ n^{-2}, q^r \right\} = \left\{ n^{-2}, q^r \right\} \wedge \hat{F} + (-1)^{q(r+s)} \left\{ n^{-2}, q^r \right\} \wedge \hat{F} + (-1)^{s(q+r)} \left\{ n^{-2}, q^r \right\} \wedge \hat{F} - \left\{ n^{-2}, q^r \right\} \wedge \hat{F} + (-1)^{s(q+r)} \left\{ n^{-2}, q^r \right\} \wedge \hat{F} + (-1)^{s(q+r)} \left\{ n^{-2}, q^r \right\} \wedge \hat{F} + (-1)^{s(q+r)} \left\{ n^{-2}, q^r \right\} \wedge \hat{F},
\]
where the expression of \( \Phi^3_D \) as found in [\text{58}] is used. One can better understand the expression above by comparing it with the “Leibniz rule for the second derivative”
\[
(abc)^{\prime\prime} = (ab)^{\prime\prime}c + (ac)^{\prime\prime}b + (bc)^{\prime\prime}a - a^{\prime\prime}bc - ab^{\prime\prime}c - abc^{\prime\prime}.
\]

The structure defined by eqs. (3.18), (3.19), (3.22) and the graded commutativity of the exterior product generalizes both the Poisson algebra structure on functions and the Gerstenhaber algebra structure on graded commutative algebra. We suggest to call this structure a higher-order Gerstenhaber algebra.

Now, let us consider the right Leibniz rule. In this case we are interested in the expression
\[
\left\{ \hat{F} \wedge \hat{F}, \hat{F} \right\} = (-1)^{n-q-r} \hat{X}^q_{\hat{F}} \wedge \hat{F} \wedge d^\Omega (F \wedge \hat{F}),
\]
where the symbol \( \hat{X}^q_{\hat{F}} \wedge \hat{F} \) denotes the object which is associated with the form \( \hat{F} \wedge \hat{F} \).

Because the exterior product of two Hamiltonian forms is not necessarily Hamiltonian this object is not generally a multivector. Still, we can attribute a certain meaning to it as follows. By definition,
\[
\hat{X}^q_{\hat{F}} \wedge \hat{F} \wedge \int \Omega = d^\Omega (F \wedge \hat{F}),
\]
where the right hand side may be written as follows
\[
d^\Omega (F \wedge \hat{F}) = (-1)^{r(q+1)} \hat{F} \wedge d^\hat{F} q^r + (-1)^q \hat{F} \wedge d^\hat{F} r^q = (-1)^{r(q+1)} \hat{F} \wedge X^q_{\hat{F}} \wedge \int \Omega + (-1)^q \hat{F} \wedge X^r_{\hat{F}} \wedge \int \Omega.
\]
Hence, we can formally take
\[
\hat{X}^q_{\hat{F}} \wedge \hat{F} := (-1)^{r(q+1)} \hat{F} \circ X^q_{\hat{F}} + (-1)^q \hat{F} \circ X^r_{\hat{F}},
\]
where the right hand side is understood as a composition of two operations, the inner product with a multivector, and the exterior product with a form.

With this definition the Poisson bracket of interest takes the form
\[
\left\{ \hat{F} \wedge \hat{F}, \hat{F} \right\} = \hat{F} \wedge \left\{ \hat{F}, \hat{F} \right\} + (-1)^{r^q} \hat{F} \wedge \left\{ \hat{F}, \hat{F} \right\},
\]
so that the right graded Leibniz rule appears to be fulfilled. The algebraic structure defined by the graded Lie algebra properties of the bracket, eqs. (3.18) and (3.19), graded commutativity of the exterior product, and the right graded Leibniz rule (3.26) is known as a right Gerstenhaber algebra (see e.g. [59] for the explicit definition).

Thus, evidence of two possible ways in which a generalization of the (graded) Poisson structure is introduced on the space of horizontal forms by the bracket operation defined in (3.15) is found. The first leads to the new notion of higher-order Gerstenhaber algebra, while the second provides us with the structure of right Gerstenhaber algebra. Note however, that both structures are based on a certain extrapolation beyond the space of Hamiltonian forms, and their adequate substantiation requires a proper definition of the bracket operation on arbitrary horizontal forms. The formal construction of the object in (3.25) associated with the exterior product of two forms suggests that non-Hamiltonian forms can be associated with multivector valued forms. Note also that the puzzling difference between left and right Leibniz rules, as well as the subsequent consideration in Sect. 5 of the bracket with \( n \)-forms which appears to be not graded antisymmetric in general, poses the question whether the graded antisymmetry of the Poisson bracket of forms can be preserved for non-Hamiltonian forms.

3.2 Pre-Hamiltonian fields

In the present formulation there exists a nontrivial set of Hamiltonian multivector fields \( \hat{X}_0 \) which annihilate the polysymplectic form:

\[
\hat{p}_X \bigwedge \Omega := 0,
\]

\( p = 1, ..., n \). We shall call them pre-Hamiltonian fields\(^2\) As a consequence of their existence the map in eq. (3.10) specifies a Hamiltonian multivector field associated with a given form only up to an addition of a pre-Hamiltonian field. This means that it actually maps Hamiltonian \((n-p)\)-forms to the equivalence classes \([\hat{p}_X]\) of Hamiltonian multivector fields of degree \( p \) modulo an addition of pre-Hamiltonian \( p \)-vector fields: \([\hat{p}_X] = [\hat{p}_X + \hat{p}_{X_0}]\). It is easy to show that pre-Hamiltonian fields form an ideal \( \mathcal{X}_0 \) in the graded Lie algebra \( \mathcal{X} \) of Hamiltonian multivector fields with respect to the SN bracket, and that the graded Poisson bracket in (3.16) does not depend on the choice of the representative of the equivalence class of multivectors associated with \( \hat{F} \) or \( \hat{F'} \). Therefore the graded quotient algebra \( \mathcal{X}/\mathcal{X}_0 \) is essentially a field theoretical analogue of a Lie algebra of Hamiltonian vector fields in mechanics. Note also that this is on the quotient space of Hamiltonian multivector fields modulo pre-Hamiltonian fields the polysymplectic form may be considered as non-degenerate.

\(^2\)In our earlier papers the term “primitive Hamiltonian fields” have been used instead.
4 The equations of motion of Hamiltonian \((n-1)\)-forms

In this section, the operation on Hamiltonian forms generated by the graded Poisson bracket with the DW Hamiltonian function is considered, and the equations of motion of Hamiltonian \((n - 1)\)-forms are written with the help of this bracket. Next, the field theoretic analogues within the polymomentum formulation of the integrals of the motion and the canonically conjugate variables are discussed.

From the degree counting in eq. (3.16) it is evident that only the forms of degree \((n - 1)\) have nonvanishing brackets with \(H\) and that the resulting brackets are forms of degree 0. Let us calculate the bracket of a Hamiltonian \((n - 1)\)-form

\[ F := F^i \partial_i \widetilde{vol} \]

with the DW Hamiltonian function \(H\):

\[ \{[F, H] \} = -X_F \partial_i \widetilde{vol}. \]  

The components of the vector field associated with \(F\),

\[ X_F := X^a \partial_a + X^i_a \partial^a_i, \]

are given by the equation

\[ X_F \partial_i \Omega = d^V F \]

or in components

\[ (-X^a dp^i_a + X^i_a dy^a) \wedge \partial_i \widetilde{vol} = (\partial_a F^i dy^a + \partial^a_i F^i dp^i_a) \wedge \partial_i \widetilde{vol}. \]  

From (4.2) we obtain

\[ X^a \delta^i_j = -\partial^a_j F^i, \]  

\[ X^i_a = \partial_a F^i. \]  

We see that no arbitrary \((n - 1)\)-forms can be Hamiltonian, that is to say, no arbitrary \((n - 1)\) forms can ensure the consistency of both sides of (4.2) and be associated with a certain Hamiltonian vector field, but only those which satisfy the condition (4.3) which restricts the admissible dependence of the components of Hamiltonian \((n - 1)\)-form on polymomenta \(p_a^i\). Namely, from the integrability condition of (4.3) we can deduce that the most general admissible Hamiltonian \((n - 1)\)-form has the components

\[ F^i = -p^i_a X^a(y, x) + g^i(y, x), \]

where \(X^a\) and \(g^i\) are arbitrary functions of field and space-time variables. For such \((n - 1)\)-forms, from (4.1) and (4.3) - (4.5) it follows

\[ \{[H, F] \} = \partial_a F^i \partial^a_i H + X^a \partial_a H. \]  

Now, let us introduce the total (i.e. evaluated on sections \(z^v = z^v(x)\)) exterior differential \(d\) of the form \(F\)

\[ dF := (\partial_a F^j \partial_i y^a + \partial^a_j F^i \partial_a p^k_a + \partial_i F^j) dx^i \wedge \partial_j \widetilde{vol}. \]
Taking into account the condition (4.3) yields
\[ dF = (\partial_a F^i \partial_i y^a - X^a \partial_p p^i_a + \partial_i F^i) \tilde{vol}. \]

Using the DW Hamiltonian field equations, eqs. (2.3), in the equation above, and comparing the result with (4.6), for an arbitrary Hamiltonian \((n - 1)\)-form \(F\) we obtain
\[ dF = \{[H, F]\} \tilde{vol} + d^{hor} F, \tag{4.7} \]
where the last term \(d^{hor} F = (\partial_i F^i) \tilde{vol}\) appears for forms which have an explicit dependence on the space-time variables. Taking the inverse Hodge dual of (4.7)\(^3\):
\[ \star^{-1} dF = \{[H, F]\} + \partial_i F^i, \tag{4.8} \]
we conclude that the Poisson bracket of a Hamiltonian \((n - 1)\)-form with the DW Hamiltonian function is related to the inverse Hodge dual of the total exterior differential of the former. Eq. (4.8) is the equation of motion of a Hamiltonian form of degree \((n - 1)\) in Poisson bracket formulation. It obviously extends the familiar Poisson bracket formulation of the equations of motion in mechanics to the DW formulation of field theory. Note, that the dual of the total exterior derivative \(\star^{-1} d\) of \((n - 1)\)-forms can be related to the generalized Lie derivative with respect to the \(n\)-vector field annihilating the canonical \((n + 1)\)-form \(\Omega_{DW}\) (see eqs. (2.6)–(2.8)) making the analogy with the equations of motion in mechanics even more transparent. In the subsequent section we discuss how the bracket representation of the equations of motion can be generalized to forms of degrees \(p \leq (n - 1)\).

### 4.1 DW canonical equations in Poisson bracket formulation and canonically conjugate variables

Eq. (4.8) contains, as a special case, the entire set of DW Hamiltonian field equations, eqs. (2.3). In fact, on account of \(d^{hor} y^a = 0\) and \(d^{hor} p^i_a = 0\), by substituting \(p_a := p^i_a \partial_i \tilde{vol}\) and then \(y^a_i := y^a \partial_i \tilde{vol}\) for the \((n - 1)\)-form \(F\) in (4.8), using (4.6) we obtain
\[ \star^{-1} dp_a = \{[H, p_a]\} = -\partial_a H, \tag{4.9} \]
\[ \star^{-1} dy^a_i = \{[H, y^a_i]\} = \partial_i^a H. \]
which is the Poisson bracket formulation of DW Hamiltonian field equations, eqs. (2.3).

The representation above of the canonical field equations in terms of the graded Poisson bracket sheds also light on the question as to which variables may be considered as canonically conjugate within the formalism under discussion. As we know from mechanics, the canonically conjugate variables have "simple" mutual Poisson brackets

\(^3Recall that \(\star \tilde{vol} = \sigma\), where \(\sigma = +1\) for Euclidean and \(\sigma = -1\) for Minkowski signature of the metric; \(\star^{-1} \star := 1\), therefore on \(n\)-forms \(\star^{-1} = \sigma \star\) or, in general, \(\star^{-1} = \sigma (-1)^{p(n-p)} \star\) on \(p\)-forms.
It is easy to see that in our approach the pair of variables
\[ y^a \quad \text{and} \quad p_a := p^i_a \partial_i \int \widetilde{\text{vol}}, \] (4.10)
one of which is a 0-form and another is an \((n - 1)\)-form, may be considered as a pair of canonically conjugate variables, for the Poisson brackets of these variables
\[ \{y^a, p_b\} = -\delta^a_b, \quad \{y^a, y^b\} = 0, \quad \{p_a, p_b\} = 0 \] (4.11)
agree with those of coordinates and canonically conjugate momenta in mechanics.

Indeed, from (3.16) we deduce
\[ \{y^a, p_b\} = -\{p_b, y^a\} = \frac{1}{X(p_b)} \int dy^a = -\delta^a_b, \]
where in the last equality we used the fact that the vector field \(\frac{1}{X(p_b)} \int \Omega\) associated with the \((n - 1)\)-form \(p_b\) is given by
\[ \frac{1}{X(p_b)} \int \Omega = dp_b, \]
whence it follows that
\[ \frac{1}{X(p_b)} = -\partial_b. \]

It should be noted, however, that the choice above of the pair of canonically conjugate variables is not unique. For example, one could also choose the pair \((y^a \partial_i \int \widetilde{\text{vol}}, p^i_b)\) or \((y^a \partial_i \int \widetilde{\text{vol}}, p_b)\) since in this case the (non-vanishing) Poisson brackets are also similar to the canonical bracket in mechanics and reduce to the latter when \(n = 1\):
\[ \{y^a \partial_i \int \widetilde{\text{vol}}, p^i_b\} = -\delta^i_j \delta^a_b, \quad \{y^a \partial_i \int \widetilde{\text{vol}}, p_b\} = -\delta^a_b \partial_i \int \widetilde{\text{vol}}. \] (4.12)
Such a freedom in specifying the pairs of canonically conjugate variables is due to the “canonical graded symmetry”, eq. (3.9), mixing forms of different degrees. It may be especially useful in field theories where the field variables themselves are forms. For example, the 1-form potential \(A_i dx^i\) in electrodynamics or the 2-form potential \(B_{ij} dx^i \wedge dx^j\) in the Kalb-Ramond field theory. In general, an \((n - p - 1)\)-form conjugate momentum may be associated to a \(p\)-form field variable so that their mutual Poisson bracket is a constant equal to one when \(n = 1\) (cf. Sect. 6.2 where the example of electrodynamics is considered).

4.2 The conserved currents

The Poisson bracket formulation of the equations of motion suggests a natural generalization to field theory of the classical notion of an integral of the motion known
in analytical mechanics. Let $\mathcal{J}$ be a Hamiltonian $(n-1)$-form which does not depend explicitly on space-time coordinates and whose Poisson bracket with the DW Hamiltonian function vanishes. Then, from eq. (4.8) the conservation law follows:

$$d\mathcal{J} = 0.$$ 

Therefore, the field theoretical analogues of integrals of the motion within the present formulation are $(n-1)$-forms corresponding to the conserved currents. Similar to the conserved quantities in mechanics, they are characterized by the condition

$$\{\mathcal{J}, H\} = 0.$$ 

Taking $\mathcal{J}_1$ and $\mathcal{J}_2$ to be the $(n-1)$-forms satisfying eq. (4.10) and using the Jacobi identity for the bracket we obtain

$$\{H, \{\mathcal{J}_1, \mathcal{J}_2\}\} = 0.$$ 

Therefore, the Poisson bracket of two conserved currents is again a conserved current. The latter statement extends to field theory the Poisson theorem [53], to the effect that the Poisson bracket of two integrals of the motion is again an integral of the motion. One can also conclude that the set of conserved $(n-1)$-form currents having a vanishing Poisson bracket with the DW Hamiltonian function is closed with respect to the Poisson bracket and therefore forms a Lie algebra which is a subalgebra of graded Lie algebra of Hamiltonian forms.

Furthermore, eq. (4.10) means that the Lie derivative of $H$ with respect to the vertical vector field $X_J$ associated with $\mathcal{J}$ vanishes, i.e. $H$ is invariant with respect to the symmetry generated by $X_J$, and $\mathcal{J}$ is the conserved current corresponding to this symmetry of the DW Hamiltonian. This is obviously a field theoretical extension (within the present framework) of the Hamiltonian Noether theorem (cf. for example Ref. [53] a §40 or Ref. [53] b §15.1). Note that this extension is proved here for the symmetries generated by vertical vector fields, that is only for internal symmetries. Another considerable limitation of the present discussion is that $(n-1)$-forms $\mathcal{J}$ are supposed to be Hamiltonian, that is restricted to have the form (4.5).

5 The equations of motion of forms of arbitrary degree

In Sect. 3 it was argued that the proper field theoretical generalization of Hamiltonian functions or dynamical variables are horizontal forms of various degrees from 0 to $(n-1)$, on the subspace of which given by the Hamiltonian forms a graded analogue of the Poisson bracket operation can be defined. However, in Sect. 4 only the equations of motion of Hamiltonian forms of degree $(n-1)$ were formulated in terms of the graded Poisson bracket. It seems natural to ask whether this circumstance is due to some privileged place of $(n-1)$-forms in the present formalism (which might indeed be the case since these forms yield classical observables of the field after integrating over the space-like hypersurface) or a possibility exists to represent the equations of motion of forms of any degree in the bracket formulation. We show in this section that the second
alternative may indeed be realized. This requires, however, a slight generalization of
the construction presented in Sect. 3. In particular, an extension of the notions of a
Hamiltonian form and the associated Hamiltonian multivector field is required.

The problem faced when trying to extend the Poisson bracket formulation of the
equations of motion from \((n-1)\)-forms to forms of arbitrary degree is that the bracket
\( \{pF, H \} \) vanishes identically when \( p < (n-1) \). A way out is suggested by the observation
that for all \( p \) the bracket with the \( n \)-form \( H \widehat{vol} \) would not vanish, if properly defined, as
the formal degree counting based on eq. (3.17) indicates that its degree is to be \( (p+1) \).
In order to define such a bracket our hierarchy of equations relating Hamiltonian
forms to Hamiltonian multivector fields, eq. (3.10), has to be extended to horizontal forms
of degree \( n \) so that a certain object generalizing Hamiltonian multivector field could
be associated with the form \( H \widehat{vol} \). It is easily seen that the formal multivector degree
of such an object has to be equal to zero.

This is possible indeed, if the object \( \tilde{X}^{V} \) which is associates with the horizontal
\( n \)-form \( \tilde{F} \) by means of the map
\[
\tilde{X}^{V}_{\tilde{F}} \hookrightarrow \Omega = d^{\tilde{V}}_{n} \tilde{F}
\]
is a vertical-vector valued horizontal one-form
\[
\tilde{X}^{V} := X^{v}_{k} dx^{k} \otimes \partial_{v}
\]
which acts on the form to the right through the Frölicher-Nijenhuis (FN) inner product.
Recall that the FN inner product of a vector valued form with a form is defined as
follows (see e.g. [60, 61, 62])
\[
\tilde{X}^{V}_{\Omega} := X^{v}_{k} dx^{k} \wedge (\partial_{v} \Omega).
\]
Here we continue to use the usual symbol of the inner product of vectors and forms and
imply that the tilde over the argument to the left indicates that it is a vector valued
form, so that the sign \( \hookrightarrow \) respectively denotes the FN inner product.

By extending formulae (3.16) to the case when one of the arguments in the Poisson
bracket is an \( n \)-form the bracket of the \( p \)-form \( \tilde{F}^{p} \) with the \( n \)-form \( \tilde{F}^{n} \) can be defined as
follows
\[
\{ \tilde{F}^{n}, \tilde{F}^{p} \}^{'} := \tilde{X}^{V}_{\tilde{F}^{n}} \hookrightarrow d^{\tilde{V}}_{p} \tilde{F}^{p}.
\]
This expression may be substantiated by considerations similar to those which led from
eq. (3.9) to eq. (3.16). To this end, the hierarchy of symmetries in eq. (3.9) has to be
supplemented with the additional equation
\[
\mathcal{L}_{\tilde{X}^{V}} \Omega = 0
\]
which formally corresponds to \( p = 0 \). The generalized Lie derivative with respect to
the vertical-vector valued form \( \tilde{X}^{V} \) is naturally given by (cf. (3.4))
\[
\mathcal{L}_{\tilde{X}^{V}} \Phi := \tilde{X}^{V} \hookrightarrow d^{\tilde{V}} \Phi - d^{\tilde{V}} (\tilde{X}^{V} \hookrightarrow \Phi)
\]
for an arbitrary form $\Phi$. Obviously, $\mathcal{L}_{\tilde{X}^V}$ maps $p$-forms to $(p+1)$-forms.

It should be noted that the bracket operation with an $n$-form as defined in (5.4) is not graded antisymmetric in general. That is why it is marked with a prime. Nevertheless, this is the definition which is shown below to be appropriate for representing the equations of motion of arbitrary horizontal forms in Poisson bracket formulation.

Taking now $\bar{\mathcal{V}}^X = \bar{H} \bar{\text{vol}}$, let us calculate the components of the associated vertical-vector valued form $\tilde{X}_{\bar{H}\bar{\text{vol}}}$:

$$
\tilde{X}_a^b = \partial^a_b H, \quad \tilde{X}_{ai}^k \delta_i^k = -\partial_a H. \tag{5.7}
$$

It is evident that $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ is given up to an addition of an arbitrary (pre-Hamiltonian) vector valued form $\tilde{X}_0$ which satisfies

$$
\tilde{X}_0 \cdot \Omega = 0,
$$

so that the only non-vanishing components of $\tilde{X}_0$ obey $\tilde{X}_0^i a_k \delta_i^k = 0$. Therefore, the symbol $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ in (5.1) is actually the equivalence class of Hamiltonian vector valued forms modulo an addition of pre-Hamiltonian vector valued forms. However, the bracket

$$
\{[\bar{H}\bar{\text{vol}}, F]_{\bar{p}}\}' = \frac{1}{p!} \tilde{X}_v^k \, dx^k \wedge \partial_v F_i \ldots dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{5.8}
$$

in general cannot be understood as given in terms of the equivalence class $\tilde{X}_{\bar{H}\bar{\text{vol}}} \mod \tilde{X}_0$, as it obviously depends on the choice of a particular representative $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ (this is in particular related to the lack of the antisymmetry of the bracket). Nevertheless for some forms, at least those which belong to the center of generalized Gerstenhaber algebra of graded Poisson brackets of forms, this dependence cancels out so that the bracket can be evaluated “off-shell”. The complete characterization of forms for which the bracket (5.8) does not depend on the choice of the representative $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ and for which the bracket (5.8) is graded antisymmetric is as yet not studied.

For other forms the bracket with $\bar{H} \bar{\text{vol}}$ can only be understood “on-shell” in the sense specified below.

Namely, let us note that $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ may be regarded, along with the canonical $n$-vector field $\bar{X}$ in Sect. 2, as a suitable analogue of the canonical Hamiltonian vector field whose integral curves are known to be the trajectories of a mechanical system in the phase space. Indeed, the expression of the components of $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ in (5.7) gives rise to the DW Hamiltonian field equations (2.3) if the geometrical object associated with the vector valued form $\tilde{X}_{\bar{H}\bar{\text{vol}}}$ and generalizing the integral curves of a vector field is given by

$$
\tilde{X}_k^v = \frac{\partial z^v}{\partial x^k} \tag{5.9}
$$

Geometrically, eqs. (5.9) describe a web of $n$ integral curves associated with the field of vector valued forms. This web spans an $n$-dimensional surface in the polymomentum phase space and represents a solution of field equations. The bracket in (5.8) is said
to be evaluated “on-shell” if the value of $\tilde{X}^v_k$ in the right hand side of (5.8) is taken “on-shell”, i.e. on a specific section $z^v = z^v(x)$, on which it is given by (5.9).

Now, let us define the total exterior differential, $d$, of a $p$-form:

$$d^p F := \frac{1}{p!} \partial_v F_{i_1 \ldots i_p} (z^v, x^i) \frac{\partial z^v}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + d^{hor} F,$$

(5.10)

where the latter term

$$d^{hor} F := \frac{1}{p!} \partial_i F_{i_1 \ldots i_p} (z^v, x^i) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

By comparing (5.10) with (5.8) and (5.9) we conclude that

$$d^p F = \{ [H \tilde{vol}, F] \} + d^{hor} F,$$

(5.11)

where the bracket is understood as evaluated “off-shell” when possible, or “on-shell” otherwise. The equation above can be viewed as the equation of motion of a $p$-form dynamical variable in Poisson bracket formulation. The left hand side of (5.11) generalizes the total time derivative in the similar equation of motion of a dynamical variable in mechanics: $dF/dt = \{ H, F \} + \partial F/\partial t$; and its last term present if the dynamical variable explicitly depends on space-time coordinates generalizes the partial time derivative term. It is clear also that essentially $d = L_{\tilde{X}^{tot}}$, where $\tilde{X}^{tot} = \tilde{X}^{vol} + \tilde{X}^{hor}$ and $\tilde{X}^{hor} := \delta^i_k dx^k \otimes \partial_i$. Thus, the equation of motion in the form (5.11) has a similar meaning to that of the equation of motion in mechanics: it determines a (generalized) Lie-dragging of a dynamical variable along the (generalized) Hamiltonian flow. In the present context the “Lie-dragging” with respect to the field of vector valued forms is formally given sense by the expression for a generalized Lie derivative, eq. (5.6); and the analogue of the “Hamiltonian flow” is given by eqs. (5.7), (5.9).

It should be noted that the enlarging of the set of Hamiltonian multivector fields by including the vector valued one-forms associated with horizontal $n$-forms implies a certain extension of graded Lie algebras of Hamiltonian and locally Hamiltonian multivector fields. Let us outline some of the problems related to this extension. At first, the bracket of two vector valued forms should be specified. The natural choice would be the (vertical) Frölicher-Nijenhuis (FN) bracket [60, 61, 62] which maps the pair of vector valued one-forms to a vector valued two-form. The result is therefore always vanishing when acting on the polysymplectic form. Similarly, vector valued forms of higher degrees which in principle appear when closing the algebra with respect to the FN bracket will also vanish on the polysymplectic form, thus contributing only to the space of pre-Hamiltonian fields.

Next, a bracket of vector valued one-forms with Hamiltonian multivectors of arbitrary degree has to be defined. The most naive extension of (3.7), which we essentially have used when defining the bracket in (5.4), is

$$[\tilde{X}, \tilde{X}] \cup \Omega := L_{\tilde{X}}(\tilde{X} \cup \Omega),$$

(5.12)
whence it follows that \([\hat{X}, \hat{X}'] \in D_{1}^{p}\), where \(D_{q}^{p}\) denotes the space of vertical-\(p\)-vector valued horizontal \(q\)-forms. In such a manner the multivector valued one-forms come into play. Note, that the bracket defined in (5.12) is marked with the prime because it is not graded antisymmetric in general. Further closing of the algebra requires a proper definition of the bracket of multivector valued one-forms appearing above. To do this would require a proper definition of a generalized inner product, to be also denoted \(\langle \cdot, \cdot \rangle\), by which a multivector valued form acts on forms. Assuming according to [33] for \(\hat{X} \in D_{q}^{p}\)

\[
\hat{X} \langle \cdot, \cdot \rangle \Omega := \frac{1}{q!p!} \hat{X}^{v_{i_{1}} \ldots i_{p-1}}_{k_{1} \ldots k_{q}} dx^{k_{1}} \wedge \ldots \wedge dx^{k_{q}} \wedge \partial_{v_{i_{1}} \ldots i_{p-1}} \Omega,
\]

it may be concluded that the non-symmetric bracket of two multivector valued one-forms, defined similarly to (5.12), leads to multivector valued two-forms and, therefore, the algebraic closure of the algebra with respect to this kind of bracket may involve multivector valued forms \(\hat{X} \in D_{q}^{p}\) of all possible degrees \(p\) and \(q\).

Thus, essentially, we are led to the problem of the construction of the bracket operation on arbitrary multivector valued forms, which would properly generalize the SN bracket on multivector fields and the FN bracket on vector valued forms. This is related to the problem of embedding the SN and FN graded Lie algebras into a larger algebraic structure on multivector valued forms which was recently considered by A.M. Vinogradov [33]. His “unification theorem” (see also [34] for a related discussion) states that the SN and FN algebras may be embedded in a certain \(\mathbb{Z}\)-graded quotient algebra of the algebra of “super-differential operators”, or graded endomorphisms, on the exterior algebra, which are actually represented by multivector valued forms.

Note that multivector valued forms essentially have already appeared in Sect. 3.1 when constructing the analogue of the multivector field associated with the exterior product of two Hamiltonian forms (see eq. 3.25). This suggests that multivector valued forms can be associated with non-Hamiltonian horizontal forms and in this way the graded Poisson bracket can be extended to the latter. An extension of this sort is postponed to a future publication (see also [73] for a preliminary discussion).

Note added in the final revision (July 1997). After the paper was completed a further progress has been made in extending the bracket to non-Hamiltonian forms. This extension leads to a non-commutative (in the sense of Loday) graded Lie algebra structure (see [74, 75] and the corresponding non-commutative generalization of higher-order and right Gerstenhaber algebras mentioned in Sect. 3. Furthermore, the product operation preserving the space of Hamiltonian forms is presented in [76] which turns the latter into a Gerstenhaber algebra.

6 Several simple applications

6.1 Interacting scalar fields

As the simplest example of how the formalism constructed in the previous sections works, consider a system of interacting real scalar fields \(\{\phi^{a}\}\) which is described by the Lagrangian density

\[
L = \frac{1}{2} \partial_{i} \phi^{a} \partial^{i} \phi_{a} - V(\phi^{a}).
\]

(6.1)
Henceforth the space-time is assumed to be Minkowskian. The polymoments derived from (6.1) are

\[ p_i^a := \frac{\partial L}{\partial (\partial_i \phi^a)} = \partial^i \phi_a \]  

(6.2)

and for the DW Hamiltonian function we easily obtain

\[ H = \frac{1}{2} p_i^a p_i^a + V(\phi). \]  

(6.3)

In terms of the canonically conjugate (in the sense of Sect. 4.1) variables \( \phi^a \) and \( \pi_a := p_i^a \partial_i \omega \)

which have the following non-vanishing Poisson bracket

\[ \{ \phi^a, \pi_b \} = -\delta^a_b, \]  

(6.4)

we can also write

\[ H \omega = \frac{1}{2} (\ast \pi^a) \wedge \pi_a + V(\phi) \omega. \]  

(6.5)

where \( \ast \pi_a := p_i^a dx_i \). Now, the canonical DW field equations may be written in terms of the Poisson brackets on forms:

\[ d\pi_a = \{ [H \omega, \pi_a] \} = \{ \hat{X}_{H \omega} \omega, d\pi_a \} \]

\[ = \hat{X}_{ak} \omega d\pi_a \]

\[ = -\partial_a H \omega, \]  

(6.6)

\[ d\phi^a = \{ [H \omega, \phi^a] \} = \{ \hat{X}_{H \omega} \omega, dy^a \} \]

\[ = \hat{X}_{ak} \omega dx_k \]

\[ = \partial_k H dx_k \]

\[ = p_k^a dx_k \]

\[ = \ast \pi^a, \]  

(6.7)

It is easy to show that eqs. (6.6), (6.7) are equivalent to the field equations following from the Lagrangian (6.1):

\[ \square \phi_a = -\partial_a V. \]

6.2 The electromagnetic field

Let us start from the conventional Lagrangian density

\[ L = -\frac{1}{4} F_{ij} F^{ij} - j_i A^i, \]  

(6.8)
where \( F_{ij} := \partial_i A_j - \partial_j A_i \). For the polymomenta we obtain

\[
\pi^i_m := \frac{\partial L}{\partial (\partial_i A^m)} = -F^i_m, \tag{6.9}
\]

whence the "primary constraints"

\[
\pi^{im} + \pi^{mi} = 0
\]

follow. The DW Legendre transformation \( \partial_i A^m \rightarrow \pi^i_m \) is singular, so that the use of the naive DW Hamiltonian function

\[
H = -\frac{1}{4} \pi^{im} \pi^{i} m + j_m A^m \tag{6.10}
\]

leads to the incorrect equation of motion \( \partial_i A^m = \partial H / \partial \pi^i_m = 1/2 \pi^{im} \). The problems of this sort are usually handled by adding the constraints with some Lagrange multipliers to the canonical Hamiltonian function and then applying the well known Dirac’s procedure for constrained systems. This approach could in principle be extended to singular, from the point of view of DW theory, Lagrangean theories. However, our formalism provides us with another possibility which is based on the freedom in choosing the canonically conjugate field and momentum variables which was mentioned in Sect. 4.1.

Namely, let us take as the canonical field variable the one-form

\[
\alpha := A_m dx^m
\]

per se, instead of the set of its components \( \{ A_m \} \) considered usually. Then, the momentum variable canonically conjugate to \( \alpha \) may be chosen to be the \((n-2)\)-form

\[
\pi := -F^{im} \partial_i \mathbf{J}^m \mathbf{vol} = \pi^{im} \partial_i \mathbf{J}^m \mathbf{vol}
\]

in which essentially the dual of the Faraday 2-form \( 1/2F_{ij} dx^i \wedge dx^j \) can be recognized. This is evident from the following calculation of the Poisson bracket of the 1-form \( \alpha \) with the \((n-2)\)-form \( \pi \).

The components of the bivector field \( X_\pi := X^v_i \partial_v \wedge \partial_i \) associated with \( \pi \) are given by

\[
X_\pi \mathbf{J} \Omega = d\pi = d\pi^{im} \wedge \partial_i \mathbf{J}^m \mathbf{vol}, \tag{6.11}
\]

where the polysymplectic form \( \Omega \) is

\[
\Omega = -dA^m \wedge d\pi^i_m \wedge \partial_i \mathbf{vol}.
\]

The only non-vanishing components of \( X_\pi \) are obviously those along the field directions \( \partial_{A_m} \), so that \( X_\pi = X^{A_m} i \partial_{A_m} \wedge \partial_i \). Substituting this expression to the left hand side of (6.11) we obtain

\[
X_\pi \mathbf{J} \Omega = -X^{A_m} i [\partial_{A_m} \otimes \partial_i - \partial_i \otimes \partial_{A_m}] \mathbf{J} dA^n \wedge d\pi^j_n \wedge (\partial_j \mathbf{vol})
\]

\[
= 2X^{A_m} i [\partial_i \otimes \partial_{A_m}] \mathbf{J} dA^n \wedge d\pi^j_n \wedge (\partial_j \mathbf{vol})
\]

\[
= -2X^{A_m} i d\pi^j_m \wedge \partial_i \mathbf{J}^j \mathbf{vol}.
\]
A comparison with the right hand side of (6.11) gives

$$X^{Am} = -\frac{1}{2}g^{mi},$$

(6.12)

where $g^{ik}$ is the metric tensor on the space-time. Thus,

$$\{[\pi, \alpha] = \int_x d\alpha = -2X^{Am}(\partial_l \otimes \partial_{Am})(dA_n \wedge dx^n) = n.$$ (6.13)

Eq. (6.13) justifies our choice of the $(n-2)$-form $\pi$ as the canonically conjugate variable to the one-form potential $\alpha$, as the bracket reduces to the canonical bracket when $n = 1$. Note that the form $\pi$ is a Hamiltonian form, in contrast to its dual, the Faraday 2-form, which we might naively try to associate with $\alpha$ as its conjugate momentum.

In terms of the variables $\alpha$ and $\pi$ the DW Hamiltonian $n$-form can be written as follows

$$H_{vol} = \frac{1}{2} \star \pi \wedge \pi + \alpha \wedge j,$$

(6.14)

where $j := j^i \partial_i \star vol$ is the electric current density $(n-1)$-form. The Maxwell equations may now be written in DW Hamiltonian form in terms of variables $(\alpha, \pi)$ and the graded Poisson bracket of forms:

$$d\alpha = \{[H_{vol}, \alpha] = \star^{-1} \pi,$$

$$d\pi = \{[H_{vol}, \pi] = j.$$ (6.15)

Thus, the DW Hamiltonian formulation of Maxwell’s electrodynamics is obtained without any recourse to the formalism of the fields with constraints. The constraints, however, both gauge and initial data, did not disappear, of course, but they can be taken into account after the covariant DW Hamiltonian formulation is constructed. Note that in general the systems which have Hamiltonian constraints in the sense of the usual instantaneous Hamiltonian formalism have a totally different structure of constraints (understood as the obstacles to the covariant Legendre transform) when viewed from the perspective of the DW Hamiltonian formulation, or even can appear to be constraint free within the latter, as the following example illustrates.

6.3 The Nambu-Goto string

The classical dynamics of a string sweeping in space-time the world-sheer $x^a = x^a(\sigma, \tau)$ is given by the Nambu-Goto Lagrangian

$$L = -T \sqrt{(\ddot{x} \cdot \dddot{x})^2 - \dot{x}^2 \dot{x}} = -T \sqrt{-\det[\partial_i x^a \partial_j x^a]},$$

(6.16)

where $\ddot{x}^a := \partial_\tau x^a$, $x^a := \partial_\tau x^a$, $T$ is the string rest tension, and the following notation for the world-sheer parameters $(\sigma, \tau) = (\tau^0, \tau^1) := (\tau^i)$; $i = 0, 1$ is used.
Let us define the polymomenta:

\[ p^0_a := \frac{\partial L}{\partial x^a} = T^2 \left( x' \cdot \dot{x}' - x'^2 \right) \frac{\dot{x}_a}{L}, \]

\[ p^1_a := \frac{\partial L}{\partial x'^a} = T^2 \left( x' \cdot \dot{x} - x'^2 \right) \frac{\dot{x}_a}{L}. \]  

(6.17)

From eqs. (6.17) the following identities follow

\[ p^0_a x'^a = 0, \quad (p^0)^2 + T^2 x'^2 = 0, \]  

\[ p^1_a \dot{x}^a = 0, \quad (p^1)^2 + T^2 \dot{x}^2 = 0. \]  

(6.18)

However, these identities do not have the meaning of Hamiltonian constraints within the DW Hamiltonian formalism, as they do not imply any relations between the coordinates \( x^a \) and the polymomenta \( p^i_a \). In fact, eqs. (6.17) can be easily solved (if \( L \neq 0 \)) yielding expressions for the generalized velocities \( (\dot{x}, x') \) in terms of the polymomenta; this proves \textit{de facto} that the DW Legendre transform \( (\dot{x}, x') \rightarrow (p^0_a, p^1_a) \) for the Nambu-Goto Lagrangian is regular.

In terms of the polymomenta the DW Hamiltonian function takes the form

\[ H = -\frac{1}{T} \sqrt{-\det \| p^i_a p^i_j \|}, \]

(6.19)

and can also be expressed in terms of the one-form momentum variables

\[ \pi_a := p^i_a \varepsilon_{ij} d\tau^j, \]

which are canonically conjugate (in the sense of Sect. 4.1) to \( x^a \). In fact,

\[ \det \| p^i_a p^i_j \| = \frac{1}{2} (\varepsilon_{ij} p^i_a p^j_b) (\varepsilon_{ij} p^a_i p^b_j), \]

and

\[ \varepsilon_{ij} p^i_a p^j_b = \ast^{-1} (\pi_a \wedge \pi_b). \]

The string equations of motion in terms of the graded Poisson brackets of forms can now be written as follows

\[ d x^a = \{ H \ast \text{vol}, x^a \} = \frac{\partial H}{\partial p^i_a} d\tau^i, \]

(6.20)

\[ d \pi_a = \{ H \ast \text{vol}, \pi_a \} = 0. \]

As yet another application let us show that the Poincaré algebra corresponding to the internal symmetry of the \( x \)-space in string theory can be realized with the help of our brackets on forms. In the \( x \)-space the translations and the Lorentz rotations
are generated by the vector field \( X_a := \partial_a \) and the bivector field \( X_{ab} := x_a \partial_b - x_b \partial_a \) respectively. The corresponding conserved current densities are given by one-forms:

\[
\pi_a \quad \text{and} \quad \mu_{ab} := x_a \pi_b - x_b \pi_a.
\]

(6.21)

One can easily check that the conservation laws

\[
d\pi_a = 0, \quad d\mu_{ab} = 0
\]

(6.22)

follow from the string equations of motion. Now, a straightforward calculation of the Poisson brackets of one-forms above yields:

\[
\{[\pi_a, \pi_b]\} = 0,
\]

\[
\{[\mu_{ab}, \pi_c]\} = g_{ac} \pi_b - g_{bc} \pi_a,
\]

\[
\{[\mu_{ab}, \mu_{cd}]\} = C^{ef}_{abcd} \mu_{ef},
\]

(6.23)

where \( g_{ab} \) is the \( x \)-space metric and

\[
C^{ef}_{abcd} = -g^{e}_{c}g^{f}_{a}g_{bd} + g^{e}_{c}g^{f}_{b}g_{ad} - g^{e}_{a}g^{f}_{d}g_{bc} + g^{e}_{a}g^{f}_{d}g_{ac}
\]

are the Lorentz group structure constants. Thus the internal Poincaré symmetry of a string is realized in terms of the Poisson brackets of one-forms which correspond to the conserved currents related to this symmetry.

7 Discussion

The subject of the present paper is an extension of basic structures of the mathematical formalism of classical Hamiltonian mechanics to field theory within the De Donder-Weyl polymomentum Hamiltonian formulation. As a subsequent application to the problem of quantization of field theories is implied, emphasis is given to the analogues of the structures and constructions which are important for different quantization procedures such as canonical, geometric or deformation.

Our starting point is the Poincaré–Cartan \( n \)-form corresponding to the De Donder-Weyl theory of multiple integral variational problems. We show that the De Donder-Weyl Hamiltonian field equations can be formulated in terms of the multivector field of degree \( n \), eq. (2.5), which annihilates the exterior differential of the Poincaré-Cartan form, and whose integral \( n \)-surfaces represent solutions of field equations. This observation leads us to the notion of the polysymplectic form of degree \((n+1)\). The polysymplectic form is defined in local coordinate terms as the vertical exterior differential of the non-horizontal part, \( \Theta^V \), of the Poincaré-Cartan form, see eq. (3.1). This definition, however, essentially implies a triviality of the extended polymomentum phase space as a bundle over the space-time manifold unless the potential of the polysymplectic form, \( \Theta^V \), the vertical exterior differential, \( d^V \), and the polysymplectic form itself are understood as appropriate cosets (see also \[74\], cf. \[13\]). The same concerns the equations including these objects.
Obviously, it would be highly desirable to reveal an intrinsically geometric formulation which could reproduce the essential features of the construction of the present paper. This problem, however, requires more involved intrinsic geometric techniques and the jet bundle language (see e.g. [24, 49, 65, 66, 67, 68]) using of which we avoided here.

Within the present approach, the polysymplectic form plays the role similar to that of the symplectic form in mechanics to which it reduces when \( n = 1 \). Unlike the latter the polysymplectic form is not purely vertical - it has two vertical and \((n - 1)\) horizontal components - and it is also not non-degenerate in the sense that a certain class of multivector fields called pre-Hamiltonian exists on which it vanishes. Note that the construction of the present paper could be also carried out with only minor modifications for possible alternatives of the polysymplectic form: \( dy^a \wedge dp^i_a \otimes \partial_i \) \( \text{vol} \) (see [20]), \( dy^a \wedge dp^i_a \otimes \partial_i \) (see [17]) and \( dy^a \wedge dp^i_a \wedge \text{vol} \otimes \partial_i \) (see [24]). Our main reason for preferring the polysymplectic form (3.1) is that it comes into being immediately from the Poincaré-Cartan form. This relationship may allow us to guess the analogues of the polysymplectic form in more general Lepagean canonical theories for fields (see e.g. [10, 12]), such as the Carathéodory theory [8, 9], for instance.

The basic symmetry of the theory, that of the polysymplectic form, is formulated as a statement of the vanishing of the generalized Lie derivatives of the polysymplectic form with respect to the vertical multivector fields of degrees \( 1 \leq p \leq n \), called locally Hamiltonian; see eq. (3.9). This graded symmetry may be viewed as a field-theoretical extension of the canonical symmetry known in mechanics. We show that the set of locally Hamiltonian multivector fields is a graded Lie algebra with respect to the (vertical) Schouten-Nijenhuis bracket which is defined with the help of the generalized Lie derivative with respect to a vertical multivector field.

Furthermore, the polysymplectic form gives rise to the map between vertical multivector fields and horizontal forms, see eq. (3.10). Horizontal forms play the role of dynamical variables in the present formalism. The map (3.10) generalizes the map between dynamical variables and Hamiltonian vector fields which is given in mechanics by the symplectic form. Thus we are led to the notions of a Hamiltonian multivector field, to which the horizontal form can be associated, and a Hamiltonian form, to which the vertical multivector field can be associated. Hamiltonian multivector fields also form a graded Lie algebra with respect to the vertical Schouten-Nijenhuis bracket. This algebra is in fact an ideal in the graded Lie algebra of locally Hamiltonian fields.

The existence of the multivector field associated with a horizontal form imposes certain restrictions on the dependence of the latter on the polymomentum variables. These restrictions specify the class of admissible horizontal forms called Hamiltonian. For instance, in the case of forms of degree \((n - 1)\) only the specific linear dependence on polymomenta is admissible, see eq. (4.5). As the result, the space of Hamiltonian forms is not stable under the exterior product. This poses the question as to how (or whether) the construction of the present paper can be extended to more general horizontal forms, or whether there exists a proper product operation on the space of Hamiltonian forms (cf. [75]).

It should be noted that the multivector field associated with a Hamiltonian form
is not generally uniquely specified. The arbitrariness is related to the existence of pre-Hamiltonian fields which annihilate the polysymplectic form. However, as the pre-Hamiltonian fields form an ideal in the graded polysymplectic algebra of Hamiltonian multivector fields, the map in eq. (3.10) from forms to multivector fields is essentially a map from forms to the equivalence classes of Hamiltonian multivector fields modulo an addition of pre-Hamiltonian fields.

Further, the vertical Schouten-Nijenhuis bracket of Hamiltonian multivector fields induces a bracket operation on Hamiltonian forms. This is our graded Poisson bracket operation on forms introduced in Sect. 3. It equips the space of Hamiltonian forms with the structure of \(\mathbb{Z}\)-graded Lie algebra. There are also two ways in which this bracket operation generalizes the known derivation property of the usual Poisson bracket. The first leads to the notion of a higher-order Gerstenhaber algebra with respect to the operations of the exterior product and the Poisson bracket on forms. This means that the graded Leibniz rule in the definition of a Gerstenhaber algebra is replaced by a more involved expression which has a natural interpretation of a higher-order Leibniz rule. The formulation in terms of the \(\Phi\)-maps introduced by Koszul in his discussion of graded higher-order differential operators on graded commutative algebra proved to be useful here. The second way, which is based on the fulfillment of the right graded Leibniz rule, eq. (3.26), leads to the right Gerstenhaber algebra structure. It should be noted, however, that the considerations of the analogues of graded Leibniz rules, both left and right, with respect to the exterior product of forms, are not sufficiently well-grounded because the graded Poisson bracket is defined here only on the subspace of Hamiltonian forms which is not stable under the exterior product. Correct consideration would require an extension of the definition of graded Poisson bracket to arbitrary horizontal forms which we hope to discuss in a forthcoming paper (see also \[73\]); cf. \[74, 75\] for a further progress.

Note in passing, that our graded Poisson bracket on exterior forms is different from other bracket operations which can be defined on forms by means of the Poisson bivector (see e.g. \[58, 63, 69, 70, 71\]) in the context more close to that of mechanics than to field theory. An interesting graded extension of the Poisson bracket is also constructed in \[72\].

As an application of our graded Poisson bracket of forms, the Poisson bracket formulation of the equations of motion of dynamical variables represented by horizontal forms is considered. This form of the equations of motion is given by the Poisson bracket with the DW Hamiltonian function, as it may be anticipated by analogy with the canonical formalism in mechanics. More precisely, the Poisson bracket of a Hamiltonian \((n-1)\)-form with the DW Hamiltonian function \(H\) generates the inverse Hodge dual of the total (i.e. taken on sections) exterior differential of the form, see eq. (4.8). The DW Hamiltonian form of field equations results from this more general statement, when the suitable \((n-1)\)-forms linearly constructed from the polymomenta or the field variables are substituted for a Hamiltonian \((n-1)\)-form into the bracket, see eq. (4.9).

A generalization of the Poisson bracket formulation of the equations of motion to forms of arbitrary degree requires a certain extension of the construction outlined above. Namely, the space of Hamiltonian forms has to be enlarged by adding horizontal
forms of degree \( n \) and, correspondingly, the space of Hamiltonian multivector fields by adding objects of formal degree zero, the vertical-vector valued horizontal one-forms which are associated with \( n \)-forms. This allows us to define the (left) bracket operation with \( n \)-forms and to show that the bracket of a form with the DW Hamiltonian \( n \)-form, \( \tilde{H}_{\text{vol}} \), generates the total exterior differential (see the definition in Sect. 5) of the former, i.e. \( d \cdot = \{[\tilde{H}_{\text{vol}}, \cdot]\}' \).

The appearance of vector valued one-forms associated with \( n \)-forms enlarges the space of Hamiltonian multivector fields and implies a certain extension of the corresponding graded Lie algebra. The algebraic closure of this extension seems to involve multivector valued forms of higher degrees and may require an appropriate definition of a bracket operation on these objects which would generalize the Lie, Schouten-Nijenhuis (SN) and Frölicher-Nijenhuis (FN) brackets. The latter problem is related to the construction of an algebraic structure on multivector valued forms which unifies the SN and FN graded Lie algebras of, respectively, multivector fields and vector valued forms. In this connection the results of A.M. Vinogradov [63] are of great interest.

It is natural to expect that the consideration of all the elements of the hypothetical bracket algebra of multivector valued forms may allow us to avoid the restrictive conditions on the admissible dependence of Hamiltonian forms on polymomenta. In other words, the expectation is that objects of more general nature than multivectors namely, the multivector valued forms, can be associated with non-Hamiltonian horizontal forms, thus opening a possibility of extending the construction of the present paper to arbitrary horizontal forms. A preliminary discussion of this possibility can be found in [73] (see also [74, 75] for a subsequent development).

Note, that there are several reasons mentioned in the text which make an extension of the construction of graded Poisson bracket beyond the space of Hamiltonian forms desirable and necessary. First is that only in this case the generalized graded Poisson properties of the bracket, eqs. (3.22) and (3.27), may be substantiated. Second is that there exist dynamical variables in field theory which cannot be naturally related to Hamiltonian forms. The energy-momentum tensor mentioned in Sect. 3.1 is an example. Besides, the examples considered in Sect. 6 show that the Hodge duals of Hamiltonian forms (as, for instance, the one-form \( \pi^{a} \) in Sect. 6.1) naturally appear in the DW formulation of particular models, whereas it is not difficult to see that if a Hamiltonian form depends on polymomenta then its Hodge dual is not a Hamiltonian form.

Let us now sketch the connection of the formalism constructed in this paper with the conventional instantaneous Hamiltonian formalism for fields (cf. [18, 13, 10]). Let us choose a space-like surface \( \Sigma \) in the \( x \)-space (here we will assume the latter to be pseudo-euclidean with the signature \( + + \ldots + - \)). The restrictions of the polymomentum phase space variables to \( \Sigma \) will be functions of the \( x \)'s. In particular, if \( \Sigma \) is given by the equation \( x^n = t \) (\( n \) denotes the number of the time-like component of \( \{x^i\} = \{x^1, \ldots, x^{n-1}, x^n\} \), and is not an index), we have \( y^a|_{\Sigma} = y^a(x, t) \) and \( p^a|_{\Sigma} = p^a(x, t) \), where \( x \) denotes the space-like components of \( \{x^i\} \). Moreover, the restriction of forms to \( \Sigma \) implies that we must set \( dx^n = 0 \), so that for \( p_a := p^i_a \partial_i \tilde{\text{vol}} \)
we obtain $p_a|\Sigma = p_a^\mu(x,t)\partial_{\mu} \tilde{vol}$, where $\partial_{\mu} \tilde{vol}$ is clearly the $(n - 1)$-volume form on $\Sigma$, which we shall denote as $d\mathbf{x}$. The functional symplectic 2-form $\omega$ on the phase space of the instantaneous formalism may now be related to the restriction of the polysymplectic form $\Omega$ to $\Sigma$ in the following way (cf. [15, 16]):

$$\omega = \int_{\Sigma}(\Omega|_{\Sigma}) = -\int_{\Sigma} dy^a(x) \wedge dp_n^a(x)dx. \quad (7.1)$$

In addition, the equal-time Poisson bracket of $y^a(x)$ with the canonically conjugate momentum $p_n^a(x)$:

$$\{y^a(x), p_n^b(y)\}_{PB} = -\delta^a_0 \delta(x - y) \quad (7.2)$$

may be related to the Poisson bracket of the canonically conjugate variables $y^a$ and $p_n^a$ of the DW theory (see Sect. 4.3) as follows:

$$\int_{\Sigma_x} \int_{\Sigma_y} \{y^a(x), p_n^b(y)\}_{PB} f(x)g(y)dxdy = \int_{\Sigma} \{y^a, p_b\} f(x)g(x)dx, \quad (7.3)$$

where $f(x)$ and $g(x)$ are test functions. In general, one might assume that the following relationship between the generalized Poisson bracket of Hamiltonian forms and the equal-time Poisson bracket of their restrictions to the space-like surface $\Sigma$ will hold

$$\int_{\Sigma_x} \int_{\Sigma_y} \{(\phi_1 \wedge \tilde{F}_1)|_{\Sigma_x}(x), (\tilde{F}_2 \wedge \phi_2)|_{\Sigma_y}(y)\}_{PB} \sim \int_{\Sigma} \phi_1 \wedge \{[\tilde{F}_1, \tilde{F}_2]\} \wedge \phi_2, \quad (7.4)$$

where $\phi_1$ and $\phi_2$ denote horizontal ”test forms” of degree $(n - p - 1)$ and $(n - q - 1)$ respectively with components depending on the space-time coordinates only, and where the standard Poisson bracket $\{\ , \ \}_{PB}$ of forms is defined via the standard Poisson brackets of their components which survive after the restriction to a Cauchy surface. The above formula reproduces, in particular, the canonical equal-time Poisson brackets from the Poisson brackets of forms corresponding to the pairs of canonically conjugate variables (see Sect. 4.3). However, it does not enable us to reproduce all Poisson brackets of interest in field theory, because some of such brackets involve quantities, an example being the energy-momentum tensor, which cannot naturally be related to the forms belonging to the restricted class of Hamiltonian forms. Furthermore, the brackets of interest in field theory may involve higher (space-like) derivatives of field variables and it is not clear how such dynamical variables can be accounted for in the context of DW theory. Both of these are additional indications that the approach of this paper should be extended to include more general objects than Hamiltonian forms.

We would also like to pay attention to the fact that the algebraic structures that arose in the present formulation of classical field theory are cognate with those which appeared in the BRST approach to field theory, in particular, in the antibracket formalism (see, for example, [76]). The latter are, of course, established within the framework which is conceptually different from the spirit of this paper. Nevertheless, a deeper relationship than a simple algebraic analogy may not be unexpected in view of the above connection between the usual Poisson bracket and the graded Poisson bracket.
of forms put forward here; in principle, this might shed light on the geometrical origin of the structures of the BRST and BV formalism. For a related discussion of interest in this context see also [77]. It is also worth noting that the structure of Gerstenhaber algebra whose generalizations arose in our construction of the graded Poisson bracket on forms has appeared recently in the context of the BRST-algebraic structure of string theory [78].

Note in conclusion that the polysymplectic form and its analogues in more general Lepagean canonical theories than the De Donder-Weyl theory considered here can be used to define the n-ary operations of the Nambu bracket type in field theory (see [75]). This opens yet another possibility of generalizing the canonical formalism to field theory using an n-ary bracket operation of scalar quantities instead of the binary bracket operation on differential forms introduced here.

The most intriguing question for further research is whether an appropriate quantization of graded Poisson brackets of forms can lead to a new, inherently covariant formulation of the quantum theory of fields based on the polymomentum formalism. This poses, in particular, an interesting mathematical problem of quantization, or deformation, of a generalized Gerstenhaber algebra of graded Poisson brackets of forms (or its appropriate subalgebra, for the limitations of an analogue of the van Howe–Groenewold no-go theorem (see e.g. [79]) will certainly take place here). A subsequent problem would be the physical interpretation of the resulting quantum theory and the clarification of its possible relation to the well-established part of modern quantum field theory.

Appendix: Higher-order graded Leibniz rule

Here we consider the analogue of the left Leibniz rule for the graded Poisson bracket. The expression

$$\{ [n-p, q] \uparrow \downarrow F, \uparrow \downarrow F \} = (-1)^p \tilde{X} \downarrow (d^V F \wedge \tilde{F})$$

has to be calculated. Here the vertical multivector field $\tilde{X}$ acting via the inner product on the exterior forms may be viewed as a graded differential operator on exterior forms which is composed of one vertical derivation of degree $-1$ which is represented by the operation $\partial_v \mathcal{J}$, and $(p-1)$ horizontal derivations of degree $-1$ represented by operations $\partial_i \mathcal{J}$. In fact,

$$\tilde{X} \downarrow = \frac{1}{p!} \Delta^{v_1, \ldots, v_{p-1}} \partial_v \wedge \partial_{i_1} \wedge \ldots \wedge \partial_{i_{p-1}} \mathcal{J}$$

$$= (-1)^{p-1} \Delta^v \partial_v^{v_1, \ldots, v_{p-1}} \partial_i \mathcal{J} \otimes \ldots \otimes \partial_{p-1} \otimes \partial_{p-1} \mathcal{J}. \quad (A.6)$$

It can be said that the differential operator on exterior algebra is of order $p$ if it can be represented as a composition of $p$ "elementary" graded derivations. The latter
are classified by the Frölicher-Nijenhuis theorem and can be of the inner product \((i_s)\) or the differential \((d_s)\) type. The derivations of the inner product type are given by the inner products with vector fields - graded derivations of degree \(-1\) - or by the Frölicher-Nijenhuis inner product with vector valued forms - graded derivations of degree \(q \geq 0\). Note that the present notion of the higher-order graded differential operators is consistent with the more general definition given by Koszul. It is clear now, that one will not obtain the graded Leibniz rule in \((A.5)\) since the inner product with a multivector field is not a graded derivation on the exterior algebra but a composition of derivations. Nevertheless, the property following from \((A.5)\) may be considered as a higher-order analogue of the Leibniz rule.

For example, for \(p = 2\) one has
\[
\hat{X}d^V (\hat{F} \wedge \hat{F}) = \begin{array}{c} 2 \\
\end{array} \hat{X}d^V (\hat{F} \wedge \hat{F}) = (\hat{X}d^V \hat{F}) \wedge \hat{F} + (-1)^q \hat{F} \wedge (\hat{X}d^V \hat{F})
\]
\[
-2(-1)^q \hat{X}v_i (\hat{d}_v d^V \hat{F}) \wedge (\hat{d}_v \hat{F}) - 2\hat{X}v_i (\hat{d}_v d^V \hat{F}) \wedge (\hat{d}_v \hat{F}). \quad (A.7)
\]
This formula may be interpreted as the Leibniz rule for the second order graded differential operator \(\hat{X}\). It is similar to the Leibniz rule for the second derivative in the usual analysis:
\[
(fg)'' = f''g + fg'' + 2fg'.
\]
For an arbitrary \(p\) by a straightforward calculation we obtain
\[
\hat{X}d^V (\hat{F} \wedge \hat{F}) = \hat{X}d^V (d^V \hat{F} \wedge \hat{F} + (-1)^q \hat{F} \wedge d^V \hat{F}))
\]
\[
= (\hat{X}d^V \hat{F}) \wedge \hat{F} + (-1)^q \hat{F} \wedge (\hat{X}d^V \hat{F})
\]
\[
+ p(-1)^p-1 \sum_{s=1}^{p-1} \hat{X}v_{i_1i_2\ldots i_{s+1}} (\hat{F}) \wedge (\hat{d}^\otimes_{i_1\ldots i_s} \hat{F}) \wedge \hat{d}^\otimes_{i_{s+1}\ldots i_{p-1}} \hat{F} \wedge \hat{d}^\otimes_{i_1\ldots i_s} \hat{F}
\]
\[
+ (-1)^{s(q-p-s-1)} (\hat{d}^\otimes_{i_{s+1}\ldots i_{p-1}} \hat{F}) \wedge \hat{d}^\otimes_{i_1\ldots i_s} \hat{F}
\]
where the notation \(\hat{d}^\otimes_{i_1\ldots i_s} := \hat{d}_{i_1} \otimes \ldots \otimes \hat{d}_{i_s}\) is introduced. The expression above may be viewed as the analogue of the Leibniz rule for the graded differential operator of degree \(-(p-1)\) and of order \(p\).

In terms of the graded Poisson brackets we, therefore, have
\[
\{\hat{F}, \hat{F} \wedge \hat{F}\} = \{\hat{F}, \hat{F}\} \wedge \hat{F} + (-1)^q \hat{F} \wedge \{\hat{F}, \hat{F}\}
\]
\[
- (n-p)(-1)^{n-p} \hat{X}v_{i_1i_2\ldots i_{n-p}} (\hat{F}) \wedge (\hat{d}^\otimes_{i_1\ldots i_s} \hat{F}) \wedge \hat{d}^\otimes_{i_{n-p+1}\ldots i_n} \hat{F}
\]
\[
+ (-1)^{s(q-p-s-1)} (\hat{d}^\otimes_{i_{n-p+1}\ldots i_n} \hat{F}) \wedge \hat{d}^\otimes_{i_1\ldots i_s} \hat{F}
\]

This formula can be interpreted as the Leibniz rule for the second order graded differential operator \(\hat{X}\).
This is the higher-order analogue of the graded Leibniz rule for our graded Poisson bracket of forms, eq. (25) in Sect. 3.1. A more appropriate formulation in terms of Koszul’s Φ maps is given in Sect. 3.1.

Acknowledgements. I gratefully acknowledge the discussions on the earlier version of the paper with E. Binz, F. Cantrijn, M. Czachor, M. Gotay, J. Kijowski, M. Modugno, Z. Oziewicz, C. Roger, G. Rudolph, G. Sardanashvily, W. Sarlet, J. Śniatycki, I. Tyutin and I. Volovich and the letters with useful comments by P. Michor and M. Crampin. I thank L.A. Dickey for bringing the author’s attention to his paper. Thanks are due to J. Nester for reading one of the earlier versions of the paper and his helpful comments. I also thank J. Hill (Canterbury) for his remarks concerning the English of an earlier version of the paper. It is my pleasure to thank J. Śniatycki for making possible the continuation of this research project at his Laboratory and for his interest, support and benevolence. I take the opportunity to express my gratitude to André Jakob (Aachen) for the help offered to me during a year, without which this work would not be finished. I am also indebted to F. Kim (Hünxe) for his friendly support and aid in 1995. My cordial thanks are due to M. Pietrzyk (Warsaw) for her understanding and invaluable support.

REFERENCES

[1] C. Crnkovic and E. Witten, Covariant description of canonical formalism in geometrical theories, in 300 Years of Gravitation, eds. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1987) p. 676-684; Č. Crnković, Symplectic geometry of the covariant phase space, Class. Quantum Grav. 5 (1988) 1557-1575

[2] R. Abraham and J.E. Marsden, Foundations of Mechanics, 2-nd ed., (Benjamin and Cummings, New York, 1978)

[3] P. Chernoff and J. Marsden, Properties of Infinite Dimensional Hamiltonian Systems, Lect. Notes Math. v. 425 (Springer-Verlag, Berlin etc., 1974)

[4] C. Itzykson and J.-B. Zuber, Quantum Field Theory, (Mc Graw-Hill Book Co., N.Y. 1980)

[5] P.J. Olver, Applications of Lie groups to differential equations, (Springer-Verlag, N.Y. etc., 1986)

[6] Th. De Donder, Theorie Invariantive du Calcul des Variations, Nuov. éd. (Gauthier-Villars, Paris, 1935)

[7] H. Weyl, Geodesic fields in the calculus of variations, Ann. Math. (2) 36 (1935) 607-629

[8] C. Carathéodory, Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale, Acta Sci. Math. (Szeged) 4 (1929) 193-216

[9] H. Rund, The Hamilton-Jacobi Theory in the Calculus of Variations, (D. van Nostrand Co. Ltd., Toronto, etc. 1966) [Revised and augmented reprint, Krieger Publ., New York, 1973]
[10] H. Kastrup, *Canonical theories of Lagrangian dynamical systems in physics*, Phys. Rep. 101 (1983) 1-167

[11] E. Binz, J. Śniatycki and H. Fisher, *Geometry of Classical Fields*, (North-Holland, Amsterdam, 1989)

[12] M. Giaquinta and S. Hildebrandt, *Calculus of Variations*, vols. 1,2 (Springer, Berlin, 1995/6)

[13] M.J. Gotay, *An exterior differential systems approach to the Cartan form*, in *Symplectic Geometry and Mathematical Physics*, eds. P. Donato, C. Duval, e.a. (Birkhäuser, Boston, 1991) p. 160-188

[14] M.J. Gotay, *A multisymplectic framework for classical field theory and the calculus of variations I. Covariant Hamiltonian formalism*, in *Mechanics. Analysis and Geometry: 200 Years after Lagrange*, ed. M. Francaviglia (North Holland, Amsterdam, 1991) p. 203-235

[15] M.J. Gotay, *A multisymplectic framework for classical field theory and the calculus of variations II. Space + time decomposition*, Diff. Geom. and its Appl. 1 (1991) 375-390

[16] M.J. Gotay, J. Isenberg, J. Marsden and R. Montgomery, *Momentum Maps and Classical Relativistic Fields (Lagrangian and Hamiltonian Structure of Classical Field Theories with Constraints)*, book in preparation, Berkeley, 1992 (various versions are known since 1985 as the GIMMSY paper), to appear

[17] H. Rund, *A Cartan form for the field theory of Carathéodory in the calculus of variations of multiple integrals*, in *Differential Geometry, Calculus of Variations and Their Applications*, Lect. Notes Pure and Appl. Math. vol. 100, ed. G.M. Rassias and T.M. Rassias, (Marcel Dekker etc., 1985) p. 455-469

[18] J. Śniatycki, *The Cauchy data space formulation of classical field theory*, Rep. Math. Phys. 19 (1984) 407-422

[19] J. Kijowski, private communication (March 1995)

[20] J. Kijowski and W.M. Tulczyjew, *A Symplectic Framework for Field Theories* (Springer-Verlag, Berlin etc., 1979)

[21] J. von Rieth, *The Hamilton-Jacobi theory of De Donder and Weyl applied to some relativistic field theories*, J. Math. Phys. 25 (1984) 1102-1115

[22] D.R. Grigore, *A generalized Lagrangian formalism in particle mechanics and classical field theory*, Fortschr. Phys. 41 (1993) 569-617

[23] J. Kijowski and G. Rudolph, *Canonical structure of the theory of gauge fields interacting with matter fields*, Rep. Math. Phys. 20 (1984) 385-400

[24] G. Sardanashvily and O. Zakharov, *The multimomentum Hamiltonian formalism in gauge theory*, Int. J. Theor. Phys. 31 (1992) 1477-1504; G. Sardanashvily, *Gauge Theory in Jet Manifolds*, (Hadronic Press Inc, Palm Harbor,
G. Sardanashvily, Multimomentum Hamiltonian formalism in field theory, preprint hep-th/9403172; G. Sardanashvily, Generalized Hamiltonian Formalism for Field Theory, (World Scientific, Singapore, 1995)

[25] G. Sardanashvily, Multimomentum Hamiltonian formalism in quantum field theory, preprint hep-th/9404001

[26] R.H. Good, Mass spectra from field equations I, J. Math. Phys. 35 (1994) 3333-3339; II, ibid. 36 (1995) 707-713

[27] M. Navarro, Comments on Good’s proposal for new rules of quantization, preprint Imperial-TP/94-95//25, hep-th/9503068

[28] H. Kastrup, Relativistic strings and electromagnetic flux tubes, Phys. Lett. 82B (1979) 237-238;
H. Kastrup and M. Rinke, Hamilton-Jacobi theories for strings, Phys. Lett. 105B (1981) 191-196;
M. Rinke, The relation between relativistic strings and Maxwell fields of rank 2, Comm. Math. Phys. 73 (1980) 265-271

[29] Y. Nambu, Hamilton-Jacobi formalism for strings, Phys. Lett. 92B (1980) 327-330

[30] R. Beig, On the classical geometry of bosonic string dynamics, Int. J. Theor. Phys. 30 (1991) 211-224

[31] I. Kanatchikov, Weyl spinor field from the Hamilton-Jacobi formulation of null strings, Europhys. Lett. 12 (1990) 577-581;
I. Kanatchikov, On the bivector field created by null strings and its relation to the Weyl spinor field, preprint F-52, Inst. of Physics, Eston Acad. Sci. (Tartu, 1990), 51pp. (in Russian);
I. Kanatchikov, Towards the causal interpretation of the Weyl spinor field using null string, P.N. Lebedev Physical Institute report (Moscow, March 1991) 16pp., unpublished

[32] D.R. Grigore, A derivation of the Nambu-Goto action from invariance principles, J. Phys. A 25 (1992) 3797-3811

[33] M. Ferraris and M. Francaviglia, The Lagrangian approach to conserved quantities in General Relativity, in Mechanics, Analysis and Geometry: 200 Years after Lagrange, ed. M. Francaviglia (Elsevier Sci. Publ., Amsterdam etc. 1991) pp.451-488; see also references cited therein

[34] P. Horava, On a covariant Hamilton-Jacobi framework for the Einstein–Maxwell theory, Class. Quantum Grav. 8 (1991) 2069-2084

[35] G. Esposito, G. Gionti and C. Stornaiolo, Space-time covariant form of Ashtekar’s constraints, Nuovo Cim. 110B (1995) 1137-1152; gr-qc/9506008
[36] L.A. Dickey, *Field-theoretical (multi-time) Lagrange-Hamilton formalism and integrable equations*, in *Lectures on Integrable Systems*, In Memory of Jean-Louis Verdier, eds. O. Babelon, P. Cartier and Y. Kosmann-Schwarzbach (World Scientific, 1994) pp. 103-162

[37] M. Gotay, *A multisymplectic approach to the KdV equation*, in *Differential Geometric Methods in Theoretical Physics*, eds. K. Bleuler and M. Werner (Kluwer Acad. Publ., Dordrecht etc. 1988)

[38] M. Born, *On the quantum theory of the electromagnetic field*, Proc. Roy. Soc. (London) A143 (1934) 410-437

[39] H. Weyl, *Observations on Hilbert’s independence theorem and Born’s quantization of field equations*, Phys. Rev. 46 (1934) 505-508

[40] P.L. García and A. Pérez-Rendón, *Symplectic approach to the theory of quantized fields, I.*, Comm. Math. Phys. 13 (1969) 24-44; *II.*, Arch. Rat. Mech. Anal. 43 (1971) 101-124

[41] H. Goldschmidt and S. Sternberg, *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble) 23, fasc. 1 (1973) 203-267; V. Guillemin and S. Sternberg, *Geometric Asymptotics*, Math. Surv. 14 (AMS, Providence, 1977) ch.4 §8

[42] J. Kijowski, *A finite dimensional canonical formalism in the classical field theory*, Comm. Math. Phys. 30 (1973) 99-128; J. Kijowski, *Multiphase Spaces and Gauge in the Calculus of Variations*, Bull. de l'Acad. Polon. des Sci., Sér. sci. math., astr. et Phys. XXII (1974) 1219-1225; J. Kijowski and W. Szczyrba, *A Canonical Structure for Classical Field Theories*, Comm. Math. Phys. 46 (1976) 183-206

[43] K. Gawędzki, *On the generalization of the canonical formalism in the classical field theory*, Rep. Math. Phys. 3 (1972) 307-326; K. Gawędzki and W. Kondracki, *Canonical formalism for the local-type functionals in the classical field theory*, Rep. Math. Phys. 6 (1974) 465-476

[44] P. Dedecker, *Calcul des variations, formes différentielles et champs géodésiques*, in *Géométrie Différentielle*, Colloq. Intern. du CNRS LII, Strasbourg 1953, (Publ. du CNRS, Paris, 1953) p. 17-34

[45] P. Dedecker, *On the generalization of symplectic geometry to multiple integrals in the calculus of variations*, in *Differential Geometrical Methods in Mathematical Physics*, eds. K. Bleuler and A. Reetz, Lect. Notes Math. vol. 570 (Springer-Verlag, Berlin etc., 1977) p. 395-456 (see also his papers cited there)

[46] R. Hermann, *Lie Algebras and Quantum Mechanics* (W.A. Benjamin, inc., New York, 1970)

[47] C. Günther, *The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case*, J. Diff. Geom. 25 (1987) 23-53

[48] C. Günther, *Polysymplectic quantum field theory*, in *Differential Geometric Methods in Theoretical Physics*, Proc. XV Int. Conf., eds. H.D. Doebner and J.D. Hennig (World Scientific, Singapore 1987) p. 14-27
[49] J.E. Cariñena, M. Crampin and L.A. Ibort, *On the multisymplectic formalism for first order field theories*, Diff. Geom. and its Appl. 1 (1991) 345-374

[50] R.H. Good jr., *Hamiltonian mechanics of fields*, Phys. Rev. 93 (1954) 239-243

[51] D.G.B. Edelen, *The invariance group for Hamiltonian systems of partial differential equations*, Arch. Rat. Mech. Anal. 5 (1961) 95-176

[52] J.E. Marsden, R. Montgomery, P.J. Morrison and W.P. Thompson, *Covariant Poisson bracket for classical fields*, Ann. Phys. 169 (1986) 29-47

[53] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, (Springer, New York etc., 1978);
   b) G. Marmo, E.J. Saletan, A. Simoni, B. Vitale, *Dynamical Systems. A Differential Geometric Approach to Symmetry and Reduction* (John Wiley & Sons, New York etc., 1985);
   c) P. Libermann and Ch. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987)

[54] W.M. Tulczyjew, *The graded Lie algebra of multivector fields and the generalized Lie derivative of forms*, Bull. de l’Acad. Polon. sci., Sér sci. math., astr. et phys. XXII (1974) 937-942

[55] J.A. Schouten, *Über Differentialkombinationen zweier kontravarianter Grössen*, Proc. Kon. Ned. Ak. Wet. (Amsterdam) 43 (1940) 449-452;
   J.A. Schouten, *On the differential operators of first order in tensor calculus*, in *Convegno di Geometria Differenziale 1953*, (Ed. Cremonese, Roma, 1954) p. 1-7;
   A. Nijenhuis, *Jacobi-type identities for bilinear differential concomitants of certain tensor fields*, Proc. Kon. Ned. Ak. Wet. (Amsterdam) A58 (1955) 390-403

[56] P. Dolan, *A generalization of the Lie derivative*, in *Classical General Relativity*, eds. W.B. Bonnor, J.N. Islam and M.A.H. MacCallum (Cambridge Univ. Press, Cambridge 1983) p. 53-62

[57] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. Math. 78 (1963) 267-288;
   M. Gerstenhaber and S.D. Schack, *Algebraic cohomology and deformation theory*, in *Deformation Theory of Algebras and Structures and Applications*, eds. M. Hazewinkel and M. Gerstenhaber (Kluwer Academic Publ., Dordrecht, 1988) p. 11-264

[58] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque, hors série, 1985, 257-271

[59] M. Flato, M. Gerstenhaber and A.A. Voronov, *Cohomology and deformation of Leibniz pairs*, Lett. Math. Phys. 34 (1995) 77-90

[60] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms, Part I. Derivation in the graded ring of differential forms*, Proc. Kon. Ned. Ak. Wet. (Amsterdam) A59 (1956) 338-359;
   P.W. Michor, *Note on the Frölicher-Nijenhuis bracket*, in *Differential Geometry and its
Applications, Proc. Conf. Brno 1986, ed. D. Krupka (J.E. Purkyne Univ., Brno 1988) p.197

[61] F. Mimura, T. Sakurai and T. Nôno, *Extended derivations associated with vector-valued differential forms*, Tensor, N.S. 51 (1992) 193-204

[62] I. Kolar, P.W. Michor and J. Slováek, *Natural operations in differential geometry*, (Springer-Verlag, Berlin etc. 1993)

[63] A.M. Vinogradov, *Unification of Schouten-Nijenhuis and Fröhlicher-Nijenhuis brackets, cohomology and super-differential operators*, Matem. Zametki (Math. Notices) 47 (1990) 138-140;
A. Cabras and A.M. Vinogradov, *Extensions of the Poisson bracket to differential forms and multi-vector fields*, J. Geom. Phys., 9 (1992) 75-100

[64] C. Roger, *Algèbres de Lie graduées et quantification*, in Symplectic Geometry and Mathematical Physics, eds. P. Donato, C. Duval, e.a. (Birkhäuser, Boston, 1991) p. 374-421

[65] A. Echeverría-Enríquez and M. C. Muñoz-Lecanda, *Variational calculus in several variables: a hamiltonian approach*, Ann. Inst. Henri Poincaré, 56 (1992) 27-47

[66] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda and N. Román-Roy, *Geometry of Lagrangian first-order classical field theories*, Fortschr. Phys. 44 (1996) 235-280

[67] M. de Léon, J. Marín-Solano and J. C. Marrero, *A geometrical approach to classical field theories: a constraint algorithm for singular theories*, preprint, Madrid, 1994.

[68] D. Saunders, *The Geometry of Jet Bundles*, (Cambridge Univ. Press, Cambridge, 1989)

[69] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. H. Poincaré, Phys. Theor., 53 (1990) 35-81

[70] M.V. Karasev and V.P. Maslov, *Nonlinear Poisson Brackets, Geometry and Quantization*. (Nauka, Moscow, 1991) ch. 1 §2, in Russian. English translation: (AMS Providence, Rhode Island, 1993)

[71] P. Michor, *A generalization of Hamiltonian mechanics*, J. Geom. Phys. 2 (1985) pp. 67-82;
M. Dubois-Violette and P. Michor, *A common generalization of the Frölicher–Nijenhuis bracket and the Schouten bracket for symmetric multi vector fields*, preprint ESI 70 (1994), L.P.T.H.E.-Orsay 94/05, [alg-geom/9401006](http://arxiv.org/abs/alg-geom/9401006)

[72] L.A. Ibort and J. Monterde, *A note on the existence of graded extensions of Poisson brackets*, J. Geom. Phys. 12 (1993) 29-34

[73] I.V. Kanatchikov, *From the Poincaré-Cartan form to the finite dimensional covariant canonical structure in field theory*, preprint PITHA 94/20 (April 1994);
I.V. Kanatchikov, *Basic structures of the covariant canonical formalism for fields based on the De Donder–Weyl theory*, preprint PITHA 94/47 (Oct. 1994), [hep-th/9410238](http://arxiv.org/abs/hep-th/9410238);
I.V. Kanatchikov, *On the finite dimensional covariant Hamiltonian formalism in field theory*, invited contribution to *New Frontiers in Gravitation*, ed. by G. Sardanashvily,
(Hadronic Press, Palm Harbor, 1995), unpublished due to a technical fault;
(in these papers the left Leibniz rule for the Poisson bracket should be rather replaced
with the right Leibniz rule; cf. Sect. 3.1 here)

[74] I.V. Kanatchikov, Novel algebraic structures in field theory from the polysymplectic form,
in GROUP21, Physical Applications and Mathematical Aspects of Geometry, Groups,
and Algebras, vol. 2, eds. H.-D. Doebner e.a., (World Scientific, Singapore, 1997) p.
894-899; hep-th/9612223

[75] I.V. Kanatchikov, On field theoretic generalizations of a Poisson algebra, to appear in
Rep. Math. Phys., 40 (1997) No. 2

[76] M. Henneaux and C. Teitelboim, Quantization of gauge systems, (Princeton Univ. Press,
Princeton, 1992);
E. Witten, A note on the antibracket formalism, Mod. Phys. Lett. A5 (1990) 487-494;

[77] G. Barnich and M. Henneaux, Isomorphism between the Batalin-Vilkovisky antibracket
and the Poisson bracket, preprint CGPG-96/1-7, hep-th/9601124

[78] B.H. Lian and G.J. Zuckerman, New perspectives in the BRST-algebraic structure of
string theory, Commun. Math. Phys. 154 (1993) 613-646;
M. Penkava and A. Schwarz, On some algebraic structures arising in string theory, in:
Perspectives in Mathematical Physics, eds. R. Penner and S.-T. Yau, (Internat, Press
Inc., Boston, 1994) p. 219-228; hep-th/9212072.

[79] G.G. Emch, Mathematical and Conceptual Foundations of 20th-Century Physics,
(North-Holland, Amsterdam, 1984)

[80] I.V. Kanatchikov, On the Canonical structure of the De Donder-Weyl covariant Hamilton-
ian formulation of field theory I. Graded Poisson brackets and the equations of motion,
Aachen preprint, PITHA 93/41 (Nov. 1993), 30pp. hep-th/9312162