Critical behavior of magnetic systems with extended impurities in general dimensions

V. Blavats'kuk
Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine, 79011 Lviv, Ukraine and
Theoretical Polymer Physics, Hermann-Herder-Str. 3, 79104 Freiburg University, Germany

C. von Ferbec

Yu. Holovatch
Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine, 79011 Lviv, Ukraine and
Ivan Franko National University of Lviv, 79005 Lviv, Ukraine

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We investigate the critical properties of \(d\)-dimensional magnetic systems with quenched extended defects, correlated in \(\varepsilon_d\) dimensions (which can be considered as the dimensionality of the defects) and randomly distributed in the remaining \(d - \varepsilon_d\) dimensions; both in the case of fixed dimension \(d = 3\) and when the space dimension continuously changes from the lower critical dimension to the upper one. The renormalization group calculations are performed in the minimal subtraction scheme. We analyze the two-loop renormalization group functions for different fixed values of the parameters \(d, \varepsilon_d\). To this end, we apply the Chisholm-Borel resummation technique and report the numerical values of the critical exponents for the universality class of this system.

I. INTRODUCTION

The study of critical properties of the various kinds of disordered systems remains one of the central problems in condensed matter physics. We focus our attention here to the case of \(d\)-dimensional magnetic systems with quenched nonmagnetic impurities. The first question that arises here is: how does quenched disorder influence criticality, i.e., will the universal scaling laws governing the critical behavior of the “pure” system be altered by quenched disorder introduced into the system in a quenched manner?

The effect of weak quenched uncorrelated point-like disorder on the critical behavior of magnetic systems is predicted by the Harris criterion. Disorder changes the critical exponents only if the critical exponent \(\alpha_p\) of the pure (undiluted) system is positive:

\[
\alpha_p = 2 - d\nu_p > 0, \quad (1)
\]

\(\nu_p\) being the correlation length critical exponent of the pure system. Of the \(d\)-dimensional spin systems that are described by the \(m\)-vector model, only the pure Ising model is characterized by a value of \(\alpha_p > 0\) and thus is affected by point-like weak disorder at criticality.

Real magnetic crystals often contain defects in the form of linear dislocations, planar grain boundaries, 3-dimensional cavities or regions of different phases, embedded in the matrix of the original crystal, as well as various complexes (clusters) of point-like non-magnetic impurities. Systems with such “extended” (macroscopic) defects have attracted much interest\(^{3,4,5,6,7,8,9,10,11,12,13,14}\).

Dorogovtsev\(^{15}\) proposed the model of a \(d\)-dimensional \(m\)-component spin system with quenched random nonmagnetic impurities, that are strongly correlated in \(\varepsilon_d\) dimensions and randomly distributed over the remaining \(d - \varepsilon_d\) dimensions. Such a system is no longer isotropic; the idea of two different correlation lengths naturally arises since the system is expected to behave differently along the directions “parallel” to the \(\varepsilon_d\)-dimensional impurity and along the “perpendicular” directions. The case \(\varepsilon_d = 0\) is associated with point-like defects, and extended parallel linear (planar) defects are related to the cases \(\varepsilon_d = 1(2)\). Generalizing \(\varepsilon_d\) to non-negative real numbers it may be interpreted as an effective fractal dimension of a complex random defect system.\(^{16}\) The critical behavior of such a model was examined by means of the renormalization group (RG) method\(^{17,18}\). A double expansion in both \(\varepsilon = 4 - d, \varepsilon_d\) was suggested and RG functions were calculated\(^{17}\) to order \(\varepsilon, \varepsilon_d\); qualitatively, the crossover to a new universality class in the presence of extended defects was found. These calculations were extended to the second order in Ref.\(^{18}\). Here, it was argued, that the Harris criterion is modified in the presence of extended impurities: the randomness is relevant, if

\[
\varepsilon_d > d - \frac{2}{\nu_p}, \quad (2)
\]

For point-like disorder \(\varepsilon_d = 0\) the Harris criterion\(^{11}\) is restored from \(^{17}\). In particular, \(^{17}\) defines for each value of \(m\) a lower marginal defect dimension \(\varepsilon_d^{\text{marg}}\), above which the critical exponents are influenced by disorder. Note that for a negative value of \(\varepsilon_d^{\text{marg}}\) any amount of quenched impurities will induce new critical behavior as far as \(\varepsilon_d\) is always positive.

Taking the best known estimates from a 6-loop RG expansion for the exponent \(\nu_p\) of the different \(m\)-component systems\(^{11}\), one finds that the disorder with extended defects is relevant for \(d = 3\) over a wider range of \(m\) than the point defect disorder. We show the lower marginal value \(\varepsilon_d^{\text{marg}}\) for these systems in Fig. \(^{11}\). Its asymptotic value for
large \( m \) is \( \varepsilon_d^{\text{marg}}(m = \infty) = 1 \). This estimate is easily obtained using the value of the correlation length critical exponent for \( d = 3 \) spherical model \( \nu(d = 3, m = \infty) = 1 \) in Ref. 2.

![Figure 1: The marginal value of \( \varepsilon_d^{\text{marg}} \) for 3-dimensional \( m \)-component magnetic systems as function of \( m \).

The solid line is obtained by substituting into (2) the 6-loop results for the critical exponents of pure \( m \)-vector magnet. The dashed line shows our present results, obtained in 2-loop approximation.](image)

Although the RG functions of Dorogovtsev’s model were obtained in Ref. 2 within the two loop accuracy, a numerical analysis was provided only to order \( \varepsilon, \varepsilon_d \). For the class of extended-defect systems with cubic anisotropy, the second order \( \varepsilon, \varepsilon_d \) RG analysis has been performed.

A related model with long-range-correlated quenched disorder, has been proposed in the work of Weinrib and Halperin and was recently studied in Refs. 10,14. This model is characterized by a correlation function that has a power law decay \( g(r) \sim r^{-b} \) with distance \( r \). This type of disorder has a direct interpretation for integer values of \( b \). Namely, the case \( b = d \) corresponds to point-like defects, while \( b = d - 1(\text{or} \; d - 2) \) describes straight lines (planes) of impurities of random orientation, in contrast to the parallel extended impurities, considered above. The results of Weinrib and Halperin were confirmed by a Monte Carlo study of the 3-dimensional random Ising model with linear defects of random orientation. An extension of this model describes \( d \)-dimensional systems with quenched disorder that is completely correlated in \( \varepsilon_d \) dimensions and long-range correlated in the remaining \( d - \varepsilon_d \) dimensions. This model applies when parallel dislocations in a crystal induce an elastic stress field in the direction perpendicular to the dislocation axis. The stress field may couple to the local critical temperature and play the role of a random potential. Real linear dislocations are described by \( \varepsilon_d = 1 \). The case \( \varepsilon_d = 0 \) reproduces the model of Weinrib and Halperin. The renormalization group treatment has been performed in a one-loop expansion within triple \( (\varepsilon, \varepsilon_d, \sigma = d - b) \) and \( (\varepsilon, \varepsilon_d, \delta = 4 - b) \) expansions.

Although systems with extended quenched defects are subject to a number of studies, a complete classification of the critical behavior of magnets of this type has yet to be established. The results that have been obtained so far using second order double expansions are of qualitative character only. The series of the RG functions are known to be asymptotic at best, and to obtain reliable quantitative results, appropriate resummation techniques should be applied. Such have proved to be fruitful for the analysis of point-like quenched disorder. In our previous paper, we have already applied this analysis to the 3-dimensional magnetic systems with extended defects. Previously, resummation techniques were not exploited for these systems, apart from the special case \( d = 3 \) and \( \varepsilon_d = 1 \), corresponding to linear defects. There, critical exponents were calculated to second order in an expansion in \( \varepsilon, \varepsilon_d \) and Padé-like approximants were used to provide numerical estimations. In our present paper we explore a wider region of the phase diagram shown in Fig. 1 and apply resummation schemes to estimate numerically the critical exponents of extended-defect systems both in the case of space dimension \( d = 3 \) and in general dimensions, using the two loop RG functions of Refs. 15.

Our paper is organized as follows. In the following section II we present the model. The field-theoretical analysis of critical behavior is reviewed in section III, and expressions are given for the RG functions. Resummation is applied to the RG functions both at \( d = 3 \) and in the case of general dimensions and numerical estimates for the critical exponents are presented in section IV followed by conclusions in section V.

II. THE MODEL

We consider the model of an \( m \)-vector magnet with \( \varepsilon_d \)-dimensional impurities, each extending throughout the system along the coordinate directions symbolized as \( x_\parallel \), whereas in the remaining \( d - \varepsilon_d \) dimensions they are randomly distributed. The effective Hamiltonian of the model reads:

\[
\mathcal{H} = \int d^d x \left[ \frac{1}{2} (\mu_0^2 + V(x)) \bar{\phi}^2(x) + (\nabla_\parallel \bar{\phi}(x))^2 \right. \\
+ \left. a_0 (\nabla_\perp \bar{\phi}(x))^2 + \frac{a_0}{4!} (\bar{\phi}^2(x))^2 \right].
\] (3)

Here, \( \bar{\phi} \) is an \( m \)-component vector field: \( \bar{\phi} = \{ \phi^1 \cdots \phi^m \} \), \( \mu_0 \) and \( u_0 \) are the bare mass and the coupling of the magnetic model, \( a_0 \) is the bare anisotropy constant, \( \nabla_\parallel \) and \( \nabla_\perp \) stand for differentiation in the coordinates \( x_\parallel \) (along \( \varepsilon_d \) directions) and \( x_\perp \) (along \( d - \varepsilon_d \) directions) respectively, \( V(x) \) represents the impurity potential. The probability distribution for the impurities is defined to yield:

\[
\langle \langle V(x) \rangle \rangle = 0,
\]
\[ \langle V(x)V(y) \rangle = -v_0 \delta^{d-\varepsilon_d}(x_\perp - y_\perp). \] (4)

Here, \( \langle \ldots \rangle \) stands for the average over the potential distribution, \( -v_0 \) is a positive constant proportional to both the concentration of impurities and the strength of their potential. The extended defects make the space coordinate anisotropic; the anisotropy is parameterized by the constant \( a_0 \). It can initially be assigned a value of unity, but will be renormalized away from this value in the scaling limit.

As far as the disorder is considered to be quenched, the free energy of the system is obtained as follows:

\[ \mathcal{F} = -kT \langle \ln Z_c \rangle, \] (5)

where \( Z_c \) is the partition function for a given (quenched) configuration of impurities, and the average is performed over the impurity probability distribution. To avoid averaging the logarithm of the partition function in (5), the standard procedure is to use the replica trick. This amounts to formally implementing the identity \( \ln Z = \lim_{n \to 0} (Z^n - 1)/n \) by analytic continuation in \( n \). After performing the average the \( n \)-replicated effective Hamiltonian then reads:

\begin{align*}
\mathcal{H} &= \sum_{\alpha=1}^n \int d^d x \left\{ \frac{1}{2} \mu_0^2 \partial_\alpha^2 \bar{\phi}_\alpha(x) + (\nabla_\perp \bar{\phi}_\alpha(x))^2 \right. \\
&\quad + a_0 (\nabla_\parallel \bar{\phi}_\alpha(x))^2 \left. + \frac{v_0}{4} \partial_\alpha^2 \bar{\phi}_\alpha(x)^2 \right\} \\
&\quad + \frac{v_0}{2} \sum_{\alpha, \beta=1}^n \int d^d x \int d^d y \delta^{d-\varepsilon_d}(x_\perp - y_\perp) \partial_\alpha^2 \bar{\phi}_\alpha(x) \partial_\beta^2 \bar{\phi}_\beta(y). \tag{6}
\end{align*}

Here, Greek indices denote replicas. The last term introduces an additional coupling \( v_0 < 0 \). It is present only for non-zero dilution and is directly responsible for the effective interaction between replicas due to the presence of impurities. The replica limit \( n \to 0 \) is implied for all quantities calculated in the following.

The model described by the effective Hamiltonian has a rich scaling behavior. An introduction to its critical exponents and a derivation of their scaling relations may be found in Refs. \( ^3 \). Here, we summarize the main quantities that describe the critical behavior of the model. Due to the spatial anisotropy two correlation lengths exist, one perpendicular and one parallel to the extended impurities direction: \( \xi_{\perp} \) and \( \xi_{\parallel} \). As the critical temperature \( T_c \) is approached, their divergences are characterized by corresponding critical exponents \( \nu_{\perp}, \nu_{\parallel} \):

\[ \xi_{\perp} \sim |t|^{-\nu_{\perp}}, \quad \xi_{\parallel} \sim |t|^{-\nu_{\parallel}}, \] (7)

where \( t \) is the reduced distance to the critical temperature \( t = (T - T_c)/T_c \). The correlation of the order parameter fluctuations in two different points depends on the orientation of their distance vector. Thus, the critical exponents \( \eta_{\perp} \) and \( \eta_{\parallel} \), that characterize the behavior of the correlation function in the directions, perpendicular and parallel to the extended defects, must be distinguished.

On the other hand, as far as the interaction of all order parameter components with defects is the same, the system susceptibility is isotropic and can be expressed by the pair correlation functions:

\[ \chi(k_{\perp}, k_{\parallel}, t) = |t|^{-\gamma} g(k_{\perp}(|t|^{1/\nu_{\perp}}), \frac{k_{\parallel}}{|t|^{1/\nu_{\parallel}}}, \pm 1) \]

\[ \begin{cases} k_{\perp}^{2-\eta_{\perp}} g(1, \frac{k_{\parallel}}{|t|^{1/\nu_{\parallel}}}, \frac{|t|}{k_{\parallel}}) , \\ k_{\parallel}^{2-\eta_{\parallel}} g(1, \frac{k_{\parallel}}{|t|^{1/\nu_{\parallel}}}, \frac{|t|}{k_{\parallel}}) , \end{cases} \] (8)

In (8), \( k_{\perp}, k_{\parallel} \) are the components of the momenta along \( \varepsilon_d \) and \( d - \varepsilon_d \) directions, respectively, \( \gamma \) is the magnetic susceptibility critical exponent and \( g \) is the scaling function. For the above introduced critical exponents the following scaling relations hold:

\[ \gamma = (2 - \eta_{\perp}) \nu_{\perp} = (2 - \eta_{\parallel}) \nu_{\parallel}. \] (9)

The critical exponent \( \alpha \) of the specific heat is related to \( \nu_{\perp}, \nu_{\parallel} \) by another scaling relation that differs from the ordinary one:

\[ \alpha = 2 - (d - \varepsilon_d) \nu_{\perp} - \varepsilon_d \nu_{\parallel}. \] (10)

All the other scaling relations are of the standard form. This implies that one should calculate at least three independent exponents (e.g., \( \nu_{\parallel}, \nu_{\perp}, \gamma \)) instead of two, as in the standard case, to find the others by scaling relations.

III. RENORMALIZATION GROUP FUNCTIONS

To describe the long-distance properties of the model near the second order phase transition we use the field-theoretical renormalization group (RG) method. In this approach one observes the behavior of the system under a scaling transformation that rescales the length e.g. by a factor \( \ell \). At the critical point the system assumes scale invariance. Away from the critical point the change of couplings \( u_0, v_0 \to u, v \) and the anisotropy constant \( a_0 \to a \) under rescaling defines a flow in parametric space, expressed by RG functions:

\[ \beta_u = \frac{\partial u}{\partial \ln \ell}|_0, \quad \beta_v = \frac{\partial v}{\partial \ln \ell}|_0, \quad \zeta_a = \frac{\partial \ln a}{\partial \ln \ell}|_0. \] (11)

Here, \( \ell \) is the rescaling factor and the notation \( |_0 \) indicates differentiation at fixed bare parameters. The bare field \( \phi \) and the bare mass \( \mu_0 \) are related to the renormalized field \( \varphi \) and mass \( \mu \) by:

\[ \phi = Z_{\varphi}^{1/2} \varphi; \quad \mu_0^2 = Z_{\mu^2} \mu^2. \] (12)

The fixed points (FPs) \( u^*, v^* \) of the RG transformation are the solutions of the system of FP equations: \( \beta_u(u^*, v^*) = 0, \beta_v(u^*, v^*) = 0 \). A FP is stable if the eigenvalues of the stability matrix \( B_{ij} = \partial^2 \beta_{u_i}(u^*, v^*)/\partial u_j(u^*, v^*) \) have positive real parts. The
stability that can be reached starting from the initial values of the coupling constants (in our case, \( u > 0 \), \( v < 0 \)), corresponds to the critical point of the system.

At this point, the magnetic susceptibility and correlation length critical exponents are given by the relations:

\[
\begin{align*}
\gamma^{-1} &= 1 - \frac{\tilde{\gamma}_\phi^2}{2 - \gamma_\phi}, \\
\nu^{-1}_\perp &= 2 - \tilde{\gamma}_\phi - \gamma_\phi, \\
\nu|_\parallel &= (1 - \frac{\zeta_\alpha}{2})\nu, \\
\end{align*}
\]

while the expressions for the RG functions \( \gamma_\phi, \tilde{\gamma}_\phi^2 \) read:

\[
\gamma_\phi = \frac{\partial \ln Z_\phi}{\partial \ln \ell}|_0, \quad \tilde{\gamma}_\phi^2 = \frac{\partial \ln Z_{\phi^2}}{\partial \ln \ell}|_0 - \gamma_\phi.
\]

The other critical exponents can be obtained from the scaling relations \[9\], \[10\].

In order to derive the quantitative characteristics of the critical behavior of magnetic systems with extended impurities, we analyze the 2-loop RG functions, obtained in the minimal subtraction scheme. Unfortunately, that paper contains some numerical errors, as pointed out in Ref.\[\text{2}\].

The appropriately corrected functions read:

\[
\begin{align*}
\beta_u/u &= -\varepsilon + \frac{(m + 8)}{6}u - 2v - \frac{(3m + 14)}{12}u^2 \\
&\quad + \frac{1}{12}\frac{1}{uv} \left[ 2\frac{(11m + 58)}{3}(m - 4)\frac{\varepsilon_d}{3(\varepsilon + \varepsilon_d)} \right] \\
&\quad - \frac{v^2}{144} \left[ 328 + 32\frac{\varepsilon_d}{\varepsilon + \varepsilon_d} \right] ; \\
\beta_v/v &= -\varepsilon - \varepsilon_d - \frac{4}{3}v + \frac{6}{3}u - \frac{7}{6}v^2 \\
&\quad + \frac{vu}{18} \left[ 11 - \frac{\varepsilon_d}{\varepsilon + \varepsilon_d} \right] - \frac{5}{12}m + \frac{m + 2}{3}u^2; \\
\gamma_\phi &= \frac{1}{36}v^2 - \frac{m + 2}{36}u + \frac{m + 2}{72}u^2; \\
\tilde{\gamma}_\phi^2 &= \frac{v}{6} - \frac{1}{3}v - \frac{2m + 2}{6}u^2 - 24v^2 \\
&\quad + \frac{m + 2}{24}v + \frac{m + 2}{6}v, \\
\zeta_\alpha &= \frac{1}{3}v - \frac{5}{36}v^2 + \frac{1}{36}(2 + m)vu.
\end{align*}
\]

In the case of \( \varepsilon_d = 0 \) the dimension enters the minimally subtracted RG \( \beta \)-functions \[15\] only in the form of the dimensionless parameter \( \varepsilon \). For fixed dimension \( \varepsilon = 1 \) the flow \[11\] with the \( \beta \)-functions \[15\] can then be directly evaluated\[24,25\]. Also for non-zero \( \varepsilon_d \) no singularities are introduced by the dimension dependence. We thus propose to extend the approach of direct evaluation to the RG functions of the present model, i.e. to treat them directly at \( d = 3 \) (\( \varepsilon = 1 \)) for different fixed values of the (fractal) defect dimensionality \( \varepsilon_d \).

The expansions of the RG functions in powers of the coupling are known for a number of models with high accuracy. However, the very idea of perturbation theory is that the result may successfully be approximated by accounting for higher order contributions. In the field-theoretical RG approach the series appear to be divergent; moreover, they are characterized by a factorial growth of the coefficients implying a zero radius of convergence\[22\]. A simple method to manage this problem is to truncate the series optimally, i.e., to account only for those first several terms that do not show the divergence. To take into account the higher order contributions, the application of special tools of resummation is required\[28\].

There remains the principal question about the Borel summability of the perturbation series for a given model; at present such a proof has been given only for the \( \phi^4 \) theory with one coupling\[29\]. The field-theoretical RG series for models with several couplings were analyzed as if they are asymptotically divergent without a proof of this property. Moreover, there exists strong evidence of possible Borel non-summability of the series obtained for disordered models\[30\].

Recently, it was demonstrated analytically, that the RG functions of the \( d = 0 \) random Ising model are Borel summable, if a special resummation technique is exploited\[31\]. the application of this technique to the \( d = 3 \) RG series of the random Ising model allows to restore the convergence of the results obtained in the massive RG scheme\[32\] and in the minimal subtraction scheme\[33\].

In our case, the question about the summability of the series \[15\] remains open. Nevertheless, we apply various kinds of resummation techniques to obtain reliable quantitative results. In the following section, we describe the procedure of the Chisholm-Borel resummation technique, which proves to be most effective in our case, and we present numerical results for the critical behavior of systems with extended defects both in the 3-dimensional case and in the case of general (non-integer) dimension.

IV. THE RESULTS

A. Resummation procedure

The two-variable Chisholm-Borel resummation technique\[34,35\] consists of several steps. To explain, how it is applied to the RG functions let us denote any of the expressions \[15\]–\[16\] by \( f = f(u, v) \). First, we construct the Borel image of the initial RG function \( f \):

\[
f = \sum_{i,j} a_{i,j} u^i v^j \rightarrow \sum_{i,j} \frac{a_{i,j}}{\Gamma(i + j + 1)}.
\]

where \( \Gamma(x) \) is Euler’s gamma function. Then, the Borel image is extrapolated by a rational Chisholm approximant \( [K/L](u, v) \). One constructs this ratio of two polynomials of order \( K \) and \( L \) such that its truncated Taylor expansion is equal to that of the Borel image of the function \( f \). The resummed function is then calculated by an
inverse Borel transform of this approximant:

\[ f^{res} = \int_0^\infty dt \exp(-t)[K/L](ut, vt). \]  

(17)

There is a lot of possibilities to choose a Chisholm approximant in two variables. The most natural way is to construct it such that, if any of \( u \) or \( v \) is equal to zero, the familiar results are obtained for reduced model. Here, for the Borel images of the \( \beta \)-functions we have chosen the following approximant with a linear denominator:

\[ \beta^{\text{chis}} = \frac{b_{0,0} + b_{1,0}ut + b_{0,1}vt + b_{1,1}utvt^2}{1 + c_{1,0}ut + c_{0,1}vt}. \]  

(18)

Note, that the polynomial in the numerator is chosen to be symmetric in the variables \( u \) and \( v \). As far as the Chisholm approximants enter the integrals (17), the expression under integration may contain poles; in this case an analytic continuation is needed, taking the principal values of the integral. Below we display results only for those situations that do not require such an analytic continuation.

To calculate the values of the critical exponents \( \nu_{\perp,\lvert\rvert}, \gamma \) we substitute (10) into (13), apply the resummation procedure to the resulting series, and, finally, substitute the obtained values of stable fixed point coordinates in the above expressions for the critical exponents. Note, that while we use the Chisholm approximant in the form (13) to analyze the series for \( \nu_{\perp,\lvert\rvert} \), the symmetry of the series for \( \nu_{\parallel} \) allows us to use the Chisholm approximant in the form:

\[ \nu_{\parallel}^{\text{chis}} = \frac{b_{0,0} + b_{1,0}ut + b_{0,1}vt}{1 + c_{1,0}ut + c_{0,1}vt}. \]  

(19)

In the following we apply the above resummation technique to analyze the critical behavior of the model (10) both at space dimensionality \( d = 3 \) and in general dimensions.

**B. 3-dimensional systems with extended impurities**

Let us first consider the model Eq. (10) for the three-dimensional case. Fixing the value \( \varepsilon = 1 \) (i.e. \( d = 3 \)) and treating \( \varepsilon_d \) as a variable parameter, we look for the common zeros of the resummed functions \( \beta_u/u \) and \( \beta_v/v \). Let us note that there are two possible scenarios for the change of FP stability under disorder introduced to the system. This is due to the fact that the universality class of the weakly diluted m-vector model with point-like defects (\( \varepsilon_d = 0 \)) depends in a crucial way on the order parameter dimensionality \( m \). There exists a marginal value \( m_c \) such that for \( m > m_c \) the critical exponents of the \( m \)-vector model remain unchanged by point-like defects whereas for \( m < m_c \) they split to new values. Discussing the Harris criterion we mentioned, that for integer \( m \) point-like defects lead to a new universality class only for the Ising model (\( m = 1 \)). The present estimates for \( m_c \) based on the resummation of six-loop RG functions definitely imply \( m_c < 2 \): \( m_c = 1.942 \pm 0.026 \) and \( m_c = 1.912 \pm 0.004 \).

For \( m < m_c \), at \( \varepsilon_d = 0 \) the random FP (\( u \neq 0, v \neq 0 \)) is unstable and physically accessible. Any small increase of the parameter \( \varepsilon_d \) leads to a shift of the stable FP value and therefore to new critical exponents, compared to the pure ones (see Fig. 2 a). For \( m > m_c \) the situation is different. Here, the point-like (\( \varepsilon_d = 0 \)) disorder is not relevant, and the pure fixed point (\( u^* \neq 0, v^* = 0 \)) is stable, whereas the random fixed point lies in the unphysical region and is unstable. Switching on the parameter \( \varepsilon_d \) leads to a shift of the random FP, and at some value of this parameter the random FP passes to the physically accessible region, at the same time it interchanges stability with the pure FP, i.e. it becomes stable (see Fig. 2 b). This corresponds to a crossover to another universality class with different critical exponents.

FIG. 2: The two qualitative scenarios for the change of fixed point stability. Circles mark unstable fixed points, squares are the stable ones. The polymer FP (\( u^* = 0, v^* \neq 0 \)) is stable but unphysical. a) The spin dimensionality of the magnetic system is below the marginal one. The random FP (\( u^* \neq 0, v^* \neq 0 \)) is stable at \( \varepsilon_d = 0 \). As the parameter \( \varepsilon_d \) increases, the mixed fixed point remains stable and moves within the physically accessible region. b) The spin dimensionality of the magnetic system is above the marginal one. The pure FP (\( u^* \neq 0, v^* = 0 \)) is stable at \( \varepsilon_d = 0 \), whereas the random FP is unstable and lies in the unphysical region. As \( \varepsilon_d \) increases, the mixed fixed point moves towards the physical region and at \( \varepsilon_d = \varepsilon_d^{\text{marg}} \) becomes stable.
it reproduces the well known results in this case there are no longitudinal components of the critical exponents. For \(\varepsilon_d \neq 0\), the relation \(\nu_{\parallel} > \nu_\perp\) holds for every \(\varepsilon_d\) and \(m\). This may be explained in the following way: the extended defects cut interacting paths of spins perpendicular to the extended-defect direction, so in the parallel direction the fluctuations are stronger and the correlation length more sharply diverges.

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This may be explained in the following way: the extended defects cut interacting paths of spins perpendicular to the extended-defect direction, so in the parallel direction the fluctuations are stronger and the correlation length more sharply diverges.

Let us return to the estimates of the marginal value \(\varepsilon_d^{\text{marg}}\), at which the crossover to a new universality class occurs. In Fig. 1 we plot the estimates for \(\varepsilon_d^{\text{marg}}\) as obtained from six-loop results for the correlation length critical exponent \(\nu_d(m)\) of the pure \(m\)-vector magnet: \(\varepsilon_d^{\text{marg}}(m = 1) = -0.173; \varepsilon_d^{\text{marg}}(m = 2) = 0.016; \varepsilon_d^{\text{marg}}(m = 3) = 0.172; \varepsilon_d^{\text{marg}}(m = 4) = 0.300\). The RG functions of the \(m\)-vector magnet with extended impurities exploited in this paper (expressions 15–16) are currently known with two-loop accuracy. In this approximation we get for the marginal value \(\varepsilon_d^{\text{marg}}(m = 1) = -0.105; \varepsilon_d^{\text{marg}}(m = 2) = 0.077; \varepsilon_d^{\text{marg}}(m = 3) = 0.222; \varepsilon_d^{\text{marg}}(m = 4) = 0.337\). The phase boundary of the critical behavior of \(3\)-dimensional \(m\)-component magnetic systems in the \(m, \varepsilon_d\)-plane is shown in Fig. 1 by the dashed curve for the two-loop approximation. It is remarkable that the resummed two-loop and six-loop results for \(\varepsilon_d^{\text{marg}}\) lead to a very similar phase diagram. In contrast to this the first order \(\varepsilon, \varepsilon_d\)-expansion predicts a qualitatively different behavior. The random FP is in the physical region for \(\varepsilon > \varepsilon_d\), and randomness is relevant for \(m < m_c = 4(\varepsilon + 2\varepsilon_d)/(\varepsilon - \varepsilon_d)\), while for \(m > m_c\) the random FP lies in the unphysical region and the pure FP is stable, so for \(\varepsilon > \varepsilon_d\) the stability changes at \(m_c\). For \(\varepsilon < \varepsilon_d\) the only physical fixed point is random and is always stable. From these first order results one would conclude, that the disorder is relevant for every \(m < 4\) and positive \(\varepsilon_d\).

Another interesting question concerns the existence of an upper marginal value for the defect dimensionality \(\varepsilon_d\). This question has not been raised in previous works, where the double \(\varepsilon, \varepsilon_d\)-expansion has been exploited. In our analysis, we observe the disappearance of a stable reachable FP for \(\varepsilon_d\) slightly above 1. This, in principle, may serve as evidence of an upper boundary for \(\varepsilon_d\) above which the second order phase transition ceases to exist. However, this behavior was observed in the region of couplings \(u, v\) where the Chisholm approximants (17) that enter the expressions for the resummed \(\beta\)-functions (17) have poles on the positive real axis. This excludes a definite answer about the presence and stability of the fixed points for high \(\varepsilon_d\). On the other hand, simple reasons support the idea of an upper boundary for \(\varepsilon_d\). Interpreting \(\varepsilon_d\) as the fractal dimensionality of the defects, it is clear, that it cannot exceed the dimensionality of the embedding space, \(d = 3\). Furthermore, one expects physically, that extended defects of large dimension (e.g. parallel planar defects with \(\varepsilon_d = 2\)), that extend throughout the system, divide the system into non-interacting...
regions and thus inhibit ferromagnetic order.

C. Systems with extended defects in general dimensions

Let us now consider the case of non-integer space dimension $d$. The concept of non-integer space dimension is common in the theory of critical phenomena. Considering $d$ as a continuous variable is not purely formal; for the critical behavior of spin systems placed on the sites of self-similar fractal lattices\textsuperscript{35} it has a direct geometrical interpretation.

Considerable effort has been made to calculate the critical exponents for the second order phase transition for general (non-integer) values of $d$ (see Refs.\textsuperscript{36,37,38} for review). Most papers study the Ising model. The critical exponents of the pure $m$-vector model in general dimensions have been calculated in the field-theoretical RG scheme for the case $m=1$ (Ising-like systems)\textsuperscript{36,38} and for $m=2, 3, 4$\textsuperscript{35}. Numerical results for the weakly diluted Ising model are presented in Refs.\textsuperscript{39,40}.

To analyze the critical properties of the model for general dimensions, the 3d approach, used above for the RG functions of the present model is generalized. We treat the RG functions directly at different fixed values of (non-integer) space dimension $d$ and defect dimensionality $\varepsilon_d$.

Again we are looking for the marginal value of the parameter $\varepsilon_d$ for models at different non-integer space dimensions. To make use of the generalized Harris criterion in this case, we take the known estimates from a 3-loop RG analysis of the critical exponent $\nu_p(d)$ for the pure $m$-vector model at $m=1$\textsuperscript{41} and $m=2, 3, 4$\textsuperscript{35}. In accordance with the Mermin-Wagner-Hohenberg theorem, the lower critical dimension of the $m$-vector model is $d_L=2$ for $m \geq 2$, while in the case of the Ising model ($m=1$) one has $d_L=\frac{12}{5}$ (for $d \leq d_L$, there is no spontaneous magnetic ordering at $T > 0$). In order to take into account these conditions, and also to reproduce the exact Onsager values\textsuperscript{1} for the 2d Ising model ($\nu=1$), a more sophisticated resummation procedure was exploited in Refs.\textsuperscript{37,38}, and we will use these results.

Substituting the known data\textsuperscript{37,38} for $\nu(d)$ into the generalized Harris criterion we find the marginal value of the parameter $\varepsilon_d$ for different $m$-component systems as function of space dimension, see Fig. 1. This figure serves as a phase diagram in parameter space ($d$, $\varepsilon_d$). It should be noted, that, as follows directly from the Harris criterion\textsuperscript{1}, one has $\varepsilon_d^{\text{marg}} = 0$ at $d=4$ (the upper critical dimension, where the Gaussian FP ($u^*=0, v^*=0$) is stable and $\nu=1/2$), whereas at $d=d_L$, one has $\varepsilon_d^{\text{marg}} = d_L$, independent of $m$. Note, that for $m=1, 2$ there exists a range of space dimensions $d$, at which $\varepsilon_d^{\text{marg}} < 0$, and, thus, the presence of any extended defects causes changes to the critical behavior. In the region marked as I in Fig. 1 disorder with extended defects is irrelevant for all the systems under consideration; in region II it becomes relevant for the Ising model, passing from the region II to III it gradually becomes relevant for the models with $m=2, 3, 4$. As discussed above, the concept of an upper marginal value for the parameter $\varepsilon_d$ naturally arises. The defect dimensionality $\varepsilon_d$ cannot exceed the dimension of the embedding space $d$, and thus cannot take values from the region marked as IV in Fig. 1.

To get the values of the critical exponents of the set of systems with extended defects at general dimensions $d$, we exploit the same 2-variable Chisholm-Borel resummation technique as for the 3-dimensional case above. But now we calculate the RG functions\textsuperscript{13-16} for fixed non-integer $\varepsilon = 4 - d$. We present here only the results obtained for the extended-defect Ising model ($m=1$), as far as it is most sensitive to introducing defects in a wide region of space dimensionality $d$, as can be observed from Fig. 1.

Before discussing the results obtained for general $d$ let us note, that their accuracy depends on the distance $\varepsilon = 4 - d$ from the upper critical dimension $d=4$\textsuperscript{33,34,35,39}. For larger expansion parameters $\varepsilon$ also the fixed point values of the coupling constants grow leading to expansions which are difficult to estimate, even when resummation procedures are applied. We demonstrate this for the $d=2$ and $d=3$ Ising model with point-like quenched disorder ($\varepsilon_d=0$). The Onsager solution of the $d=2$ pure Ising model\textsuperscript{1} brings about a logarithmic divergence of the specific heat, i.e. an exponent $\alpha_p = 0$. With the Harris criterion\textsuperscript{1} the well known irrelevance of this type of disorder for the $d=2$ Ising model follows. The

![FIG. 5: The phase diagram of $m$-component magnetic systems with extended defects in the $d$, $\varepsilon_d$-plane. The lower marginal value of $\varepsilon_d$ for $m=1$ (squares), 2 (circles), 3 (triangles), 4 (diamonds) is shown. In region I disorder with extended defects is irrelevant for all $m \geq 1$; region II: disorder is relevant only for the Ising model ($m=1$); region III: disorder with extended defects is relevant for systems with $m=2, 3, 4$. The solid diagonal line corresponds to $\varepsilon_d = d$: the region IV corresponds to non-physical situations.](image-url)
correlation length critical exponent in this case equals $\nu_p = 1$, leading to $\varepsilon_{d,\text{marg}} = 0$ by applying the generalized Harris criterion. However, in our 2-loop calculations refined by the Chisholm-Borel resummation technique we find for the $d = 2$ Ising model correlation length and magnetic susceptibility critical exponents $\nu_p = 1.162$ and $\gamma_p = 2.042$ correspondingly. These values differ by 16% from their exact values $\nu = 1$, $\gamma_p = 7/4$. This implies a marginal value of the extended defects dimensionality of $\varepsilon_{d,\text{marg}} = 0.278$. For $d = 3$, we reproduce the two-loop results of the random Ising model critical exponents, obtained in the minimal subtraction scheme in Ref. $\nu_\perp = 0.665$, $\gamma = 1.308$. They differ from the most accurate six-loop estimates $\nu_\perp = 0.678(10)$, $\gamma = 1.330(17)$ by only of 2%. Again, for the marginal value of the defect dimensionality we find $\varepsilon_{d,\text{marg}} = -0.105$, the six-loop value being $\varepsilon_{d,\text{marg}} = -0.173$ (see Fig. 1). To conclude, we expect the accuracy of our results for the critical exponents to be of the order of several percents for $d = 3$ and to increase to the order of $\sim 10\% - 20\%$ for $d = 2$, (recall that the approach gives the exact data for $d = 4$). For this reason we do not show any results for $d < 2$, taking that the confidence interval for them is even larger. On the other hand, the two-loop approximation restricts our investigation to dimensions $d$ smaller than 3.4; namely, for $d \geq 3.4$ the rational approximants that enter the integrals for the resummed functions have poles that exclude reliable calculations in this case.

The values of the critical exponents $\nu_\perp$ and $\gamma$ are plotted as functions of $d$ parameterized by $\varepsilon_d$ in Figs. 6, 7. The parameter $\varepsilon_d$ of the different curves changes from 0 to 1 by steps of 0.2. For values of $d$ below 2.1 the critical exponents remain equal to those of the pure model for small $\varepsilon_d$, until $\varepsilon_d$ reaches its marginal value (of about $\varepsilon_d \simeq 0.27$ in the present two-loop calculations); then, a crossover to the diluted regime takes place, and the exponents increase continuously with increasing the parameter $\varepsilon_d$ at fixed space dimension $d$. We interpret this by noting that an increase of $\varepsilon_d$ leads to an effective decrease of the space dimension implying a sharper divergence of the correlation length (characterized by a larger value of $\nu$). The results for $\nu_\parallel$, not shown here, display the same tendencies under changes of the parameters $d$, $\varepsilon_d$. Let us note however, that the relation $\nu_\parallel > \nu_\perp$ holds for all $d$, $\varepsilon_d$, as observed for $d = 3$ and explained in the previous subsection.

V. CONCLUSIONS

In our study we provide numerical estimates for the critical exponents of $m$-component magnetic systems in the presence of extended macroscopic defects. The impurities are envisaged as $\varepsilon_d$-dimensional objects, extending throughout the system, whereas in the remaining $d - \varepsilon_d$ dimensions they are randomly distributed. The presence of these impurity “lines” introduces anisotropy into the systems; as a result, the parallel and transverse correlation lengths naturally arise and are described in the vicinity of the critical point by the respective correlation length critical exponents. Such systems are not covered by the original Harris criterion; disorder with extended impurities is relevant over a wider range of $m$ and $d$ than the point defect disorder (see Figs. 6, 7).

Although the model of $m$-component magnetic system with extended quenched defects has attracted much attention and serves as a subject of study in a number of works, most results are based on the double expansion in parameters $\varepsilon, \varepsilon_d$ and are rather of a qualitative character. We analyze the RG functions of the model, obtained in the minimal subtraction scheme, treating them directly for fixed $d$ and fixed parameter $\varepsilon_d$. To evaluate the RG functions we apply appropriate resummation techniques, that have proven to be fruitful in the
analysis of point-like quenched disorder. In our case, the question about the summability of the corresponding RG functions is open; nevertheless, in order to obtain reliable quantitative results, we apply a simple two-variable Chisholm-Borel resummation technique. The values of the stable fixed point coordinates and critical exponents for 3-dimensional systems are extracted, as well as the estimate for the lower marginal value for parameter $\varepsilon_d$ is obtained. The case $\varepsilon_d = 0$ describes point-like quenched disorder and reproduces well-known results and is generally known to provide more reliable estimates as compared to the simple Padé-analysis. We note however, that our present resummation technique has shown its efficiency and accuracy in studies of models with point-like structural disorder and reproduces well-known results. In our case, $\varepsilon_d = 1$ using the Padé-analysis, they are shown in the last line of Table II. We estimate the marginal value of the parameter $\varepsilon_d$ also for magnetic systems in general (non-integer) space dimensions. In Figs. 2, 3 we plot the results for the Ising model in general dimensions, which appears to be the most “sensitive” to introducing extended defects in a rather wide range of space dimensionality $d$.

The concept of an upper marginal value of $\varepsilon_d$ naturally arises in studies of magnetic systems with extended impurities. Interpreting $\varepsilon_d$ as a fractal dimensionality of defects, it is clear, that it cannot exceed the dimensionality of embedding space, $d$. Moreover, physically one may expect, that extended defects of large dimension (e.g. parallel planar defects with $\varepsilon_d = 2$), each extending throughout the system, will divide the system into non-interacting regions and thus prevent it from ferromagnetic ordering.

Unfortunately, there is no simulational investigation of systems with parallel extended defects that we know of. However, our calculations bring about an essential change in critical exponents due to presence of extended impurities (e.g. $\nu_\perp$ for the Ising model with extended impurities increases by 13 % as the parameter $\varepsilon_d$ changes from 0 to 1). We hope that the influence of this kind of disorder on the critical behavior of three-dimensional magnetic systems may create some interest for numerical simulations or experimental measurements.

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TABLE I: The coordinates of the stable fixed points of the RG equations for 3-dimensional magnetic systems with extended εd-dimensional impurities for different m. Note, that the pure fixed point (u∗ ≠ 0, v∗ = 0) looses its stability for some marginal value εd,marg(m).

| m = 1 | m = 2 | m = 3 | m = 4 |
|-------|-------|-------|-------|
| εd   | u∗   | v∗   | u∗   | v∗   | u∗   | v∗   |
| 0.00  | 1.5772 -0.2416 1.1415 0 | 1.0016 0 0.8877 0 | 0.1  | 1.7640 -0.4187 1.1688 -0.0372 |
| 0.1   | 1.9169 -0.5635 1.2713 -0.1857 | 1.0016 0 0.8877 0 | 0.2  | 2.0478 -0.6859 1.3509 -0.3117 |
| 0.3   | 2.1633 -0.7919 1.4145 -0.4195 | 1.0467 -0.1028 0.8877 0 | 0.4  | 2.1633 -0.7919 1.4145 -0.4195 |
| 0.5   | 2.2671 -0.8853 1.4665 -0.5125 | 1.1238 -0.3188 0.9290 -0.1869 |
| 0.6   | 2.3619 -0.9688 1.5096 -0.5932 | 1.1486 -0.4053 0.9436 -0.2819 |
| 0.7   | 2.4493 -1.0445 1.5457 -0.6635 | 1.1672 -0.4799 0.9524 -0.3635 |
| 0.8   | 2.5306 -1.1131 1.5763 -0.7249 | 1.1812 -0.5440 0.9575 -0.4335 |
| 0.9   | 2.6067 -1.1762 1.6025 -0.7788 | 1.1917 -0.5991 0.9598 -0.4919 |
| 1.0   | 2.7466 -1.2896 1.6443 -0.8675 | 1.2053 -0.6858 0.9594 -0.5819 |

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| $\varepsilon_d$ | $\nu_{||}$ | $\nu_\perp$ | $\gamma$ | $\nu_{||}$ | $\nu_\perp$ | $\gamma$ | $\nu_{||}$ | $\nu_\perp$ | $\gamma$ |
|-------|-------|-------|-----|-------|-------|-----|-------|-------|-----|
| 0     | -     | 0.665 | 1.308 | -     | 0.684 | 1.344 | -     | 0.720 | 1.411 | -     | 0.751 | 1.470 |
| 0.1   | 0.714 | 0.680 | 1.338 | 0.691 | 0.688 | 1.352 | 0.720 | 0.720 | 1.411 | 0.751 | 0.751 | 1.470 |
| 0.2   | 0.741 | 0.692 | 1.362 | 0.719 | 0.705 | 1.386 | 0.720 | 0.720 | 1.411 | 0.751 | 0.751 | 1.470 |
| 0.3   | 0.765 | 0.702 | 1.384 | 0.744 | 0.718 | 1.414 | 0.740 | 0.732 | 1.438 | 0.751 | 0.751 | 1.470 |
| 0.4   | 0.786 | 0.712 | 1.402 | 0.766 | 0.730 | 1.430 | 0.763 | 0.746 | 1.467 | 0.761 | 0.760 | 1.491 |
| 0.5   | 0.805 | 0.720 | 1.419 | 0.785 | 0.739 | 1.460 | 0.784 | 0.757 | 1.493 | 0.784 | 0.772 | 1.520 |
| 0.6   | 0.822 | 0.727 | 1.434 | 0.802 | 0.747 | 1.479 | 0.801 | 0.766 | 1.515 | 0.811 | 0.782 | 1.545 |
| 0.7   | 0.838 | 0.733 | 1.448 | 0.818 | 0.754 | 1.495 | 0.817 | 0.773 | 1.533 | 0.837 | 0.789 | 1.565 |
| 0.8   | 0.853 | 0.739 | 1.460 | 0.831 | 0.760 | 1.509 | 0.831 | 0.779 | 1.549 | 0.858 | 0.795 | 1.582 |
| 0.9   | 0.867 | 0.745 | 1.472 | 0.843 | 0.765 | 1.522 | 0.842 | 0.783 | 1.562 | 0.875 | 0.799 | 1.595 |
| 1.0   | 0.880 | 0.750 | 1.483 | 0.854 | 0.769 | 1.532 | 0.852 | 0.787 | 1.573 | 0.886 | 0.801 | 1.605 |
| 1.1   | 0.892 | 0.754 | 1.493 | 0.863 | 0.773 | 1.542 | 0.860 | 0.789 | 1.581 | 0.895 | 0.803 | 1.613 |

| 1.0   | 0.84 | 0.67 | 1.34 | 0.60 | 0.56 | 1.13 | 0.66 | 0.61 | 1.24 |

TABLE II: The critical exponents for 3-dimensional magnetic systems with extended $\varepsilon$-dimensional impurities for different $m$. For comparison the last line shows the results of Ref. [5].
