Central Values of $GL(2) \times GL(3)$ Rankin-Selberg $L$-functions with Applications\footnote{This work is supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2014AQ002) and Innovative Research Team in University (Grant No. IRT16R443).}

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Abstract Let $f$ be a normalized holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight $k$ with $k \equiv 0 \pmod{4}$. By the Kuznetsov trace formula for $GL_3(\mathbb{R})$, we obtain the first moment of central values of $L(s, f \otimes \phi)$, where $\phi$ varies over Hecke-Maass cusp forms for $SL_3(\mathbb{Z})$. As an application, we obtain a non-vanishing result for $L(1/2, f \otimes \phi)$ and show that such $f$ is determined by $\{L(1/2, f \otimes \phi)\}$ as $\phi$ varies.

Keywords: central values, the Rankin-Selberg $L$-function, the Kunznetsov trace formula

MSC 11F11, 11F67

1. Introduction

Special values of $L$-functions are expected to carry important information on relevant arithmetic and geometric objects. In 1997, Luo and Ramakrishnan [LR1997] asked the question that to what extent modular forms are actually characterized by their special $L$-values. In the same paper, they considered the moment of $\chi_d(p)L(1/2, f \otimes \chi_d)$ as $d$ varies, and showed that a cuspidal normalized holomorphic Hecke newform $f$ is uniquely determined by the family $\{L(1/2, f \otimes \chi_d)\}$ for all quadratic characters $\chi_d$. Since then, this problem has been studied by many authors ([La1999], [CD2005], [Li2007], [Li2009], [GHS2009], [Mu2010], [Li2010], [Pi2010], [Pi2011], [Zh2011], [Ma2014], [Pi2014], [Su2014],[MS2015]).

Let $f$ be a normalized holomorphic Hecke-cusp form for $SL_2(\mathbb{Z})$ of fixed weight $k$ with $k \equiv 0 \pmod{4}$. Let $\{\phi\}$ be a Hecke basis of the space of Maass cusp forms for $SL_3(\mathbb{Z})$. In this paper, we consider central values of Rankin-Selberg $L$-functions $L(s, f \otimes \phi)$. By calculating the twisted moment of $A_\phi(p, p)L(1/2, f \otimes \phi)$ where $A_\phi(p, p)$ is the Hecke eigenvalue of $\phi$ at $(p, p)$, we show that $f$ is uniquely determined by the family $\{L(1/2, f \otimes \phi)\}$ as $\phi$ varies over a Hecke basis of the space of Maass cusp forms for $SL_3(\mathbb{Z})$.

To state our result, we give the following notations.

- For a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$, let $\mu_\phi = (\mu_1, \mu_2, \mu_3)$ be the Langlands parameter of $\phi$. We know that $\mu_\phi$ is a point in the region

$$\Lambda_{1/2}' := \left\{(\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad |\text{Re}(\mu_j)| \leq \frac{1}{2}, \quad \mu_1 + \mu_2 + \mu_3 = 0, \quad \{\mu_1, \mu_2, \mu_3\} = \{-\overline{\mu}_1, -\overline{\mu}_2, -\overline{\mu}_3\} \right\}$$

in the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Let

$$\nu_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2 - \mu_3), \quad \nu_3 = -\nu_1 - \nu_2$$

which are known as the spectral coordinates.

- Fix a point $\mu^0 \in \Lambda_{1/2}'$ such that

$$|\mu^0_j| \times \|\mu^0\| := T, \quad 1 \leq j \leq 3.$$
As in [BB2015] (or see [HLZ2017]), we choose the test function $h(\mu)$ to localize at a ball of radius $M = T^\theta$ with $0 < \theta < 1$ about $w(\mu^0)$, where $w$ are elements in the Weyl group $W$. For a precise definition of $h(\mu)$, we refer to section 2.3.

- Let $d\mu = d\mu_1 d\mu_2$ and $d_{\text{spec}} \mu = \text{spec} (\mu) d\mu$ with

$$\text{spec} (\mu) = \prod_{j=1}^3 (3\nu_j \tan \left(\frac{3\pi \nu_j}{2}\right)).$$

Our main result is in the following.

**Theorem 1.1.** Let $f$ be a normalized holomorphic Hecke cusp form for $SL_2(\mathbb{Z})$ of weight $k$ with $k \equiv 0 \mod 4$. Let $\{\phi\}$ be a Hecke basis of the space of cusp forms for $SL_3(\mathbb{Z})$ and $A_{\phi}(p, p)$ be the Hecke eigenvalue of $\phi$ at $(p, p)$. One has

$$\sum_{\phi} \frac{h(\mu_0)}{N_0} A_{\phi}(p, p) L(1/2, f \otimes \phi) = \frac{\lambda_f (p)}{p^{3/2}} M_k (h) + O_{k, \epsilon} (p^{7/16 + \epsilon} T^{5/2 + \epsilon} M^2)$$

for $T \gg_k p^{3 + \frac{7}{16} + \epsilon}$. Here $N_0$ is the normalized factor defined in (2.5) and

$$M_k (h) = \frac{1}{192 \pi^5} \int_{\Re(\mu) = 0} h (\mu) \left( 1 + \prod_{j=1}^3 \frac{\Gamma \left( \frac{k}{2} + \mu_j \right)}{\Gamma \left( \frac{k}{2} - \mu_j \right)} \right) d_{\text{spec}} \mu.$$

Note that $M_k (h) \asymp_k T^3 M^2$. On taking $p = 1$, the above theorem implies the existence of non-vanishing of $L(1/2, f \otimes \phi)$ as $\phi$ varies. Moreover, by the strong multiplicity one theorem (see [PS1979]), we have the following corollary.

**Corollary 1.** Let $f$ and $f'$ be two normalized holomorphic cusp forms for $SL_2(\mathbb{Z})$ of fixed weight $k$ with $k \equiv 0 \mod 4$. If $L(1/2, f \otimes \phi) = L(1/2, f' \otimes \phi)$ for all Hecke-Maass cusp forms $\phi$ for $SL_3(\mathbb{Z})$, then $f = f'$.

We remark that central values of $L(s, f \otimes \phi)$ vanish for $k \equiv 2 \mod 4$. In this case we can consider $\frac{\partial}{\partial s} L(1/2, f \otimes \phi)$ instead of $L(1/2, f \otimes g)$ as in [Zhi2011]. But we do not address this here.

This paper is arranged as follows. In section 2, we review the Kuznetsov trace formula in the version of [Bu2014], choose the test function and give the approximate functional equation of the Rankin-Selberg $L$-function. Theorem 1.1 will be proved in section 3, where we apply the approximate functional equation and the Kuznetsov trace formula, and give estimations on each terms. The main term in (1.1) comes from the geometric term associated to the trivial Weyl’s element, and the error term comes from the maximal Eisenstein series in the continuous spectrum.

### 2. Preliminaries

In this section, we review the definition of automorphic forms on $SL(3, \mathbb{Z})$ in [Bi2013] (or see [Go2006]), the Kuznetsov’s trace formula in [Bu2014] (or see [BB2015]) and the approximate functional equation of Rankin-Selberg $L$-functions.

Let

$$h^3 = \left\{ \begin{array}{c} z = \begin{pmatrix} 1 & x_2 & x_3 \\ 1 & x_1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & \ast & \ast \\ \ast & y_1 & \ast \\ \ast & \ast & 1 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}, y_1, y_2 \in \mathbb{R}^+ \end{array} \right\}$$

$$\simeq GL_3(\mathbb{R})/O_3(\mathbb{R})Z(\mathbb{R})$$
be the generalized Poincare upper half plane. Given a spectral parameter \((\nu_1, \nu_2) \in \mathbb{C}^2\), the function \(I_{\nu_1, \nu_2}\) on \(\mathfrak{h}^3\) is defined by
\[
I_{\nu_1, \nu_2}(z) = y_1^{1+2\nu_1+\nu_2} y_2^{1+\nu_1+2\nu_2}
\]
and the Jacquet-Whittaker function is defined by
\[
\mathcal{W}_{\nu_1, \nu_2}(z) := \int_{\mathbb{R}^3} I_{\nu_1, \nu_2} \left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix} \right) e(-u_1 \pm u_2) du_1 du_2 du_3
\]
where \(e(x) = \exp(2\pi ix)\).

Let \(\nu_3 = -\nu_1 - \nu_2\) and
\[
\mu_1 = \nu_1 + 2\nu_2, \quad \mu_2 = \nu_1 - \nu_2, \quad \mu_3 = -2\nu_1 - \nu_2.
\] (2.1)
We will simultaneously use \(\mu = (\mu_1, \mu_2, \mu_3)\) and \((\nu_1, \nu_2, \nu_3)\) coordinates,
\[
\nu_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2 - \mu_3), \quad \nu_3 = -\nu_1 - \nu_2.
\]

2.1. Automorphic forms for \(SL(3, \mathbb{Z})\).

2.1.1. Hecke-Maass cusp forms. A Hecke-Maass cusp form for \(\Gamma = SL_3(\mathbb{Z})\) of type \((\frac{1}{3} + \nu_1, \frac{1}{3} + \nu_2)\) is a function \(\phi : \Gamma \backslash \mathfrak{h}^3 \to \mathbb{C}\) which has the Fourier expansion
\[
\phi(z) = \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A_\phi(m_1, m_2)}{m_1|m_2|} \sum_{\gamma \in U_2 \backslash SL_2(\mathbb{Z})} \mathcal{W}_{\nu_1, \nu_2}(\gamma) \left( \begin{pmatrix} \gamma & m_2 \\ m_1 & 1 \end{pmatrix} \right) c_{\nu_1, \nu_2}.
\]

Here \(A_\phi(m_1, m_2)\) are eigenvalues of \(\phi\) at \((m_1, m_2)\) satisfying
\[
A_\phi(m_1, m_2) \ll m_1 m_2,
\]
\(\mathcal{W}_{\nu_1, \nu_2}(z)\) is the Jacquet-Whittaker function and \(c_{\nu_1, \nu_2}\) is a constant depending only on \(\nu_1\) and \(\nu_2\) (see formula (2.13) in [B2013]).

Let \(\mu_\phi = (\mu_1, \mu_2, \mu_3)\) be the Langlands parameter of \(\phi\) where \(\mu_j\) are given by (2.1). The \(L\)-function associated to \(\phi\) is defined by
\[
L(s, \phi) := \sum_{m \geq 1} A_\phi(1, m) \frac{m^s}{m^s}
\]
for \(\text{Re}(s) > 2\). It has analytic continuation for \(s \in \mathbb{C}\) and satisfies the functional equation
\[
\Lambda(s, \phi) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}}(s + \mu_j) L(s, \phi) = \Lambda(1 - s, \phi^\vee).
\]
Here \(\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(s/2)\) and \(\phi^\vee\) is the dual Hecke-Maass cusp form of \(\phi\) with
\[
A_{\phi^\vee}(m_1, m_2) = A_\phi(m_2, m_1), \quad \mu_{\phi^\vee} = (-\mu_1, -\mu_2, -\mu_3).
\]
2.1.2. *The minimal Eisenstein series.* Let $P_{1,1,1}$ be the standard minimal parabolic subgroup of $GL_3$ and $U_3$ be the unipotent radical of $P_{1,1,1}$. Given a spectral parameter $(\nu_1, \nu_2) \in \mathbb{C}^2$, let $\mu = (\mu_1, \mu_2, \mu_3)$ be the Langlands parameter given by (2.1). The minimal Eisenstein series

$$E_{\nu_1,\nu_2}^{\min}(z) := \sum_{\gamma \in U_3(\mathbb{Z}) \setminus \Gamma} I_{\nu_1,\nu_2}(\gamma z)$$

is defined for $\text{Re}(\nu_1)$ and $\text{Re}(\nu_2)$ sufficiently large and has meromorphic continuation to all $(\nu_1, \nu_2) \in \mathbb{C}^2$. The Hecke eigenvalues $A_{\nu_1,\nu_2}^{\min}(m, n)$ of $E_{\nu_1,\nu_2}^{\min}(z)$ at $(m, n)$ are defined by

$$A_{\nu_1,\nu_2}^{\min}(1, n) = \sum_{d_1d_2d_3=n} d_1^{-\mu_1} d_2^{-\mu_2} d_3^{-\mu_3}$$

and by Hecke relations

$$A_{\nu_1,\nu_2}^{\min}(m, 1) = \overline{A_{\nu_1,\nu_2}^{\min}(1, m)},$$

$$A_{\nu_1,\nu_2}^{\min}(m_1, m_2) = \sum_{d \mid (m_1, m_2)} \mu(d) A_{\nu_1,\nu_2}^{\min}(\frac{m_1}{d}, 1) A_{\nu_1,\nu_2}^{\min}(1, \frac{m_2}{d}).$$

The $L$-function associated to $E_{\nu_1,\nu_2}^{\min}(z)$ is

$$L(s, E_{\nu_1,\nu_2}^{\min}) := \sum_{m \geq 1} \frac{A_{\nu_1,\nu_2}^{\min}(1, m)}{m^s} = \zeta(s + \mu_1) \zeta(s + \mu_2) \zeta(s + \mu_3)$$

where $\mu_i$ are given by (2.1).

2.1.3. *The Maximal Eisenstein series.* Let $g : SL_2(\mathbb{Z}) \backslash \mathbb{H}^2 \to \mathbb{C}$ be a Hecke-Maass cusp form with the spectral parameter $it_g \in i\mathbb{R}$ and Hecke eigenvalues $\lambda_g(m)$. We assume that $g$ is normalized by $\|g\| = 1$. Let

$$P_{2,1} = \left[ \begin{array}{ccc} * & * & * \\ * & * & * \\ & & * \end{array} \right]$$

be the standard maximal parabolic subgroup of $GL_3$. For $u \in \mathbb{C}$, the maximal Eisenstein series

$$E_{g, u}^{\max}(z) := \sum_{\gamma \in P_{2,1}(\mathbb{Z}) \setminus \Gamma} \det(\gamma z)^{\frac{1}{2} + u} g(m_{P_{2,1}}(\gamma z))$$

is defined for $\text{Re}(u)$ sufficiently large. Here $m_{P_{2,1}}$ is the restriction to the upper left corner,

$$m_{P_{2,1}} : \mathfrak{h}^3 \to \mathfrak{h}^2, \quad \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ y_1 & x_1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} y_2 & x_2 \\ 1 \end{pmatrix}.$$ 

The Hecke eigenvalue $A_{g, u}^{\max}(m, n)$ of $E_{g, u}^{\max}$ at $(m, n)$ is defined by

$$A_{g, u}^{\max}(1, n) = \sum_{d_1d_2=n} \lambda_g(d) d_1^{-u} d_2^{2u}$$

and by the Hecke relations

$$A_{g, u}^{\max}(m, 1) = \overline{A_{g, u}^{\max}(1, m)},$$

$$A_{g, u}^{\max}(m_1, m_2) = \sum_{d \mid (m_1, m_2)} \mu(d) A_{g, u}^{\max}(\frac{m_1}{d}, 1) A_{g, u}^{\max}(1, \frac{m_2}{d}).$$
The $L$-function associated to $E_{u,g}^{\text{max}}(z)$ is
\[
L(s, E_{u,g}^{\text{max}}) = \sum_{m \geq 1} \frac{A_{u,g}^{\text{max}}(1,m)}{m^s} = \zeta(s - 2u)L(s + u, g)
\]
and the complete $L$-function is
\[
\Lambda(s, E_{u,g}^{\text{max}}) = \prod_{i=1}^{3} \Gamma_R(s + \mu'_i) L(s, E_{u,g}^{\text{max}}) = \Lambda(1 - s, E_{-u,g}^{\text{max}}),
\]
where
\[
\mu'_1 = u + it_g, \quad \mu'_2 = u - it_g, \quad \mu'_3 = -2u.
\]

2.2. The Kuznetsov trace formula. We recall the Kuznetsov trace formula in the version of [Bu2014]. Let $d\mu = d\mu_1 d\mu_2$ be the standard measure on the Lie algebra $\Lambda_{\infty} := \{ \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \mu_1 + \mu_2 + \mu_3 = 0 \}$.

We set $d_{\text{spec}}(\mu) = \text{spec}(\mu)d\mu$ with
\[
\text{spec}(\mu) := \prod_{j=1}^{3} \left( 3\nu_j \tan \left( \frac{3\pi}{2} \nu_j \right) \right).
\]

2.2.1. Normalized factors. The normalized factors are defined as follows.

- For $\phi$ a Hecke-Maass cusp form with $\mu_\phi = (\mu_1, \mu_2, \mu_3)$, we denote by
  \[
  \mathcal{N}_\phi := \|\phi\|^2 \prod_{j=1}^{3} \cos \left( \frac{3\pi}{2} \nu_j \right).
  \]
  Note that for $\mu_\phi = (\mu_1, \mu_2, \mu_3)$ with $\mu_i \asymp T$, one has
  \[
  \mathcal{N}_\phi \asymp \text{Res}_{s=1} L(s, \phi \otimes \phi^\vee) \ll T^\epsilon.
  \]

- For $E_{\nu_1,\nu_2}^{\text{min}}(z)$ the minimal Eisenstein series with the Langlands parameter $\mu(E_{\nu_1,\nu_2}^{\text{min}}) = (\mu_1, \mu_2, \mu_3)$, the normalized factor is defined by
  \[
  \mathcal{N}_{\nu_1,\nu_2}^{\text{min}} := \frac{1}{16} \prod_{j=1}^{3} |\zeta(1 + 3\nu_j)|^2.
  \]

- For $E_{u,g}^{\text{max}}(z)$ the maximal Eisenstein series, we define
  \[
  \mathcal{N}_{u,g}^{\text{max}} := 8L(1, \text{Ad}^2 g)|L(1 + 3u, g)|^2.
  \]

2.2.2. Kloosterman Sums. Two type of Kloosterman sums are defined as follows. Assume $D_1 \mid D_2$, we have the incomplete Kloosterman sum
\[
\tilde{S}(n_1, n_2, m_1, D_1, D_2) := \sum_{C_1 \pmod{D_1}, C_2 \pmod{D_2}, (C_1, D_1) = (C_2, D_2, D_1) = 1} e \left( \frac{C_1 C_2}{D_1} n_2 + m_1 \frac{C_2}{D_2/D_1} + n_1 \frac{C_1}{D_1} \right).
\]
The complete Kloosterman sum is defined by
\[
S(n_1, n_2, m_1, m_2, D_1, D_2) := \sum_{B_1, C_1 \mod D_1, B_2, C_2 \mod D_2} e \left( \frac{n_1 B_1 + m_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right)
\]
where \( Y_j B_j + Z_j C_j \equiv 1 \mod D_j \) for \( j = 1, 2 \).

By the standard (Weil-type) bounds we have (see formulas 3.1 and 3.2 in [BB2015])
\[
\tilde{S}(n_1, n_2, m_1, m_2, D_1, D_2) \ll (|m_1, D_2/D_1|D_1^n, (n_1, n_2, D_1, D_2)) (D_1 D_2)^{\epsilon}
\]
and
\[
S(n_1, n_2, m_1, m_2, D_1, D_2) \ll (D_1 D_2)^{1/2+\epsilon} \{(D_1, D_2)(m_1 n_1, [D_1, D_2]) (m_2 n_2, [D_1, D_2]) \}^{1/2}.
\]

2.2.3. **Integral kernels.** Following Theorems 2 and 3 in [Bu2014], the integral kernels are given as follows. For \( s \in \mathbb{C} \) and \( \mu = (\mu_1, \mu_2, \mu_3) \), we let
\[
\tilde{G}^\pm(s, \mu) := \frac{\pi^{-3s}}{12288\pi^{7/2}} \left( \prod_{j=1}^3 \Gamma \left( \frac{1-s-\mu_j}{2} \right) \pm i \prod_{j=1}^3 \Gamma \left( \frac{1+s-\mu_j}{2} \right) \right).
\]
The integral kernel associated to \( w_4 \) is defined by
\[
K_{w_4}(y; \mu) = \int_{-i\infty}^{i\infty} |y|^{-s} \tilde{G}^\pm(s, \mu) \frac{ds}{2\pi i}
\]
for \( y \in \mathbb{R} - \{0\} \) with \( \epsilon = \text{sgn}(y) \).

For \((s_1, s_2) \in \mathbb{C}^2\) and \( \mu = (\mu_1, \mu_2, \mu_3) \), we let
\[
G(s_1, s_2, \mu) := \frac{1}{\Gamma(s_1 + s_2)} \prod_{j=1}^3 \Gamma(s_1 - \mu_j) \Gamma(s_2 + \mu_j).
\]

We also define the following trigonometric functions
\[
S^{++}(s_1, s_2; \mu) = \frac{1}{24\pi^2} \prod_{j=1}^3 \cos \left( \frac{3}{2} \pi \nu_j \right),
\]
\[
S^{+-}(s_1, s_2; \mu) = \frac{-1}{32\pi^2} \cos \left( \frac{3}{2} \pi \nu_1 \right) \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right),
\]
\[
S^{-+}(s_1, s_2; \mu) = \frac{-1}{32\pi^2} \cos \left( \frac{3}{2} \pi \nu_1 \right) \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right),
\]
\[
S^{--}(s_1, s_2; \mu) = \frac{1}{32\pi^2} \cos \left( \frac{3}{2} \pi \nu_1 \right) \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right).
\]
The integral kernel associated to the longest Weyl's element \( w_l \) is defined by
\[
K_{w_l}^{\epsilon_1, \epsilon_2}(y_1, y_2; \mu) = \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} |4\pi^2 y_1|^{-s_1} |4\pi^2 y_2|^{-s_2} G(s_1, s_2; \mu) S^{\epsilon_1, \epsilon_2}(s_1, s_2; \mu) \frac{ds_1 ds_2}{(2\pi)^2}
\]
for \((y_1, y_2) \in (\mathbb{R} - \{0\})^2\) with \( \epsilon_1 = \text{sgn}(y_1) \).
2.2.4. The Kuznetsov’s trace formula. Let $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and let $h(\mu)$ be a function that is holomorphic on

$$\Lambda_{1/2+\delta} = \left\{ \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad \mu_1 + \mu_2 + \mu_3 = 0, \text{Re}(\mu_j) \leq \frac{1}{2} + \delta \right\}$$

for some $\delta > 0$, symmetric under the Weyl group $W$, rapidly decaying as $|\text{Im}\mu_j| \to \infty$ and satisfies

$$h(3\nu_1 + 1, 3\nu_2 + 1, 3\nu_3 + 1) = 0.$$ 

Then one has

$$c + \epsilon_{\text{min}} + \epsilon_{\text{max}} = \Delta + \Sigma_4 + \Sigma_5 + \Sigma_7,$$

where

$$c = \sum_{\phi} \frac{h(\mu_\phi)}{N_\phi} A_{\phi}(n_1, n_2) A_{\phi}(m_1, m_2),$$

$$\epsilon_{\text{max}} = \frac{1}{2\pi i} \sum_g \int_{\text{Re}(u)=0} \frac{h(u + it_g, u - it_g, -2u)}{N_{u,g}^{\text{max}}(n_1, n_2) A_{u,g}(m_1, m_2)} du,$$

$$\epsilon_{\text{min}} = \frac{1}{24(2\pi i)^2} \sum_{\text{Re}(\mu)=0} \frac{h(\mu)}{N_{\nu_1, \nu_2}^{\text{min}}(n_1, n_2) A_{\mu}(m_1, m_2)} d\mu,$$

and

$$\Delta = \delta_{m_1, n_1} \delta_{m_2, n_2} \frac{1}{192\pi^5} \int_{\text{Re}(\mu)=0} h(\mu) d\text{spec} \mu,$$

$$\Sigma_4 = \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_1 | D_2 \atop m_2 D_2 = m_1 D_2}} \tilde{S}(-\epsilon n_2, m_1, m_2, D_1, D_2) \Phi_{w_4} \left( \frac{\epsilon m_1 m_2 n_2}{D_1 D_2}; h \right),$$

$$\Sigma_5 = \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_1 | D_2 \atop m_2 D_2 = m_1 D_2}} \tilde{S}(-\epsilon n_1, m_1, m_2, D_1, D_2) \Phi_{w_5} \left( \frac{\epsilon n_1 m_2}{D_1 D_2}; h \right),$$

$$\Sigma_7 = \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \sum_{D_1, D_2} S(\epsilon_2 n_2, \epsilon_1 n_1; m_1, m_2; D_1, D_2) \Phi_{w_7} \left( \frac{-\epsilon_2 m_1 n_2 D_2}{D_1^2}, -\frac{\epsilon_1 m_2 n_1 D_1}{D_2^2}; h \right).$$

Here

$$\Phi_{w_4}(y; h) = \int_{\text{Re}(\mu)=0} h(\mu) K_{w_4}(y; \mu) d\text{spec} \mu,$$

$$\Phi_{w_5}(y; h) = \int_{\text{Re}(\mu)=0} h(\mu) K_{w_5}(-y; -\mu) d\text{spec} \mu,$$

$$\Phi_{w_7}(y_1, y_2; h) = \int_{\text{Re}(\mu)=0} h(\mu) K_{w_7}^{\text{sgn}(1), \text{sgn}(2)}(y_1, y_2; \mu) d\text{spec} \mu.$$
2.3. The choice of the test function. By unitarity and the Jacquet-Shalika’s bounds, the Langlands parameter $\mu_\phi$ of a Hecke-Maass cusp form $\phi$ for $SL_3(\mathbb{Z})$ is contained in

$$\Lambda'_{1/2} := \left\{ (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \begin{array}{l}
|\text{Re}(\mu_j)| \leq \frac{1}{2}, \\
\mu_1 + \mu_2 + \mu_3 = 0,
\end{array}\right\}.$$  

Let $\mu^0 = (\mu^0_1, \mu^0_2, \mu^0_3)$ be in generic position in $\Lambda'_{1/2}$, i.e.

$$|\mu^0_j| \gg \|\mu^0\| := T, \quad 1 \leq j \leq 3.$$  

Following [BB2015] (or see [HLZ2017]), we choose a test function $h(\mu)$ to localizes at a ball of radius $M = T^0$ with $0 < \theta < 1$ about $w(\mu^0)$ for each $w \in W$. It is defined by

$$h(\mu) := P(\mu)^2 \left( \sum_{w \in W} \psi \left( \frac{w(\mu) - \mu^0}{M} \right) \right)^2,$$

where $\psi(\mu) = \exp \left( \mu_1^2 + \mu_2^2 + \mu_3^2 \right)$ and

$$P(\mu) = \prod_{0 \leq n \leq A_0} \prod_{j=1}^3 \left( \frac{\nu_j - \frac{1}{3}(1 + 2n)}{|\nu_j|^2} \right) \left( \frac{\nu_j + \frac{1}{3}(1 + 2n)}{|\nu_j|^2} \right)$$

for some fixed large $A_0 > 0$. Here

$$W = \left\{ I, w_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, w_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, w_5 = \begin{pmatrix} 1 & 1 \\ 1 \end{pmatrix}, w_6 = \begin{pmatrix} 1 & 1 \\ 1 \end{pmatrix} \right\}$$

is the Weyl group for $GL_3(\mathbb{R})$.

We need the following two lemmas in [BB2015], which are used in truncating summations in geometric terms after the application of the Kuznetsov’s trace formula.

**Lemma 2.1.** Let $0 < |y| \leq T^{3-\varepsilon}$. Then for any constant $A \geq 0$ one has

$$\Phi_{w_4}(y; h) \ll_{\varepsilon, \beta} T^{-A}.$$

If $|y| > T^{3-\varepsilon}$ then

$$|y|^{-j} \Phi_{w_4}^{(j)}(y; h) \ll_{j, \varepsilon} T^{2} M^2 (T + |y|^{1/3})^j$$

for any $j \in \mathbb{N}_0$.

**Lemma 2.2.** Let $\mathcal{Y} := \min\{|y_1|^{1/3}, |y_2|^{1/6}, |y_1|^{1/6} |y_2|^{1/3}\}$. If $\mathcal{Y} \leq T^{1-\varepsilon}$, then

$$\Phi_{w_1}(y_1, y_2; h) \ll_{B, \varepsilon} T^{-A}$$

for any fixed constant $A \geq 0$. If $\mathcal{Y} \gg T^{1-\varepsilon}$, then

$$|y_1|^{j_1} |y_2|^{j_2} \frac{\partial^{j_1}}{\partial y_1^{j_1}} \frac{\partial^{j_2}}{\partial y_2^{j_2}} \Phi_{w_1}(y_1, y_2) \ll_{j_1, j_2, \varepsilon} T^{2} M^2 (T + |y_1|^{1/2} + |y_1|^{1/3} |y_2|^{1/6})^{j_1} (T + |y_2|^{1/2} + |y_2|^{1/3} |y_1|^{1/6})^{j_2}$$
for all $j_1, j_2 \in \mathbb{N}_0$.

2.4. **Rankin-Selberg $L$-functions.** We recall holomorphic Hecke cusp forms in [Iw1997]. Let $f$ be a normalized holomorphic Hecke cusp form of weight $k$ for $SL_2(\mathbb{Z})$ such that $f$ has the Fourier expansion

$$f(z) = \sum_{m \geq 1} \lambda_f(m) m^{\frac{k-1}{2}} e(mz),$$

where $\lambda_f(m)$ are Hecke eigenvalues of the Hecke operators $T(m)$. The $L$-function associated to $f$ is

$$L(s, f) = \sum_{m \geq 1} \frac{\lambda_f(m)}{m^s}$$

which is absolutely convergent for $\text{Re}(s) > 1$ by the Ramanujan-Deligne’s bound $\lambda_f(m) \ll m^{\epsilon}$. It has analytic continuation for all $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, f) := \Gamma_R \left( s + \frac{k-1}{2} \right) \Gamma_R \left( s + \frac{k+1}{2} \right) L(s, f) = i^k \Lambda(1 - s, f).$$

Let $f$ be as above and $\phi$ be a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ with Langlands parameter $\mu_\phi = (\mu_1, \mu_2, \mu_3)$. The Rankin-Selberg $L$-function $L(s, f \otimes \phi)$ is defined by (see Section 12.2 in [Go2006])

$$L(s, f \otimes \phi) := \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) A_\phi(m_1, m_2)}{(m_1^2 m_2)^s}$$

for $\text{Re}(s)$ sufficient large. It has analytic continuation for all $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, f \otimes \phi) = \prod_{i=1}^3 \Gamma_R \left( s + \frac{k-1}{2} - \mu_i \right) \Gamma_R \left( s + \frac{k+1}{2} - \mu_i \right) L(s, f \otimes \phi)$$

$$= (i^k)^3 \Lambda(1 - s, f \otimes \phi^\vee),$$

where $\phi^\vee$ is the dual Maass cusp form associated to $\phi$.

Let $E_{\nu_1, \nu_2}^{\min}(z)$ be the minimal Eisenstein series with the Langlands parameter $\mu(E_{\nu_1, \nu_2}^{\min})$. By Euler products of $L(s, f)$ and $L(s, E_{\nu_1, \nu_2}^{\min})$, we have

$$L(s, f \otimes E_{\nu_1, \nu_2}^{\min}) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) A_{\nu_1, \nu_2}^{\min}(m_1, m_2)}{(m_1^2 m_2)^s}$$

$$= L(s - \mu_1, f) L(s - \mu_2, f) L(s - \mu_3, f).$$

It satisfies the functional equation

$$\Lambda(s, f \otimes E_{\nu_1, \nu_2}^{\min}) = \prod_{j=1}^3 \Gamma_R \left( s + \frac{k-1}{2} - \mu_j \right) \Gamma_R \left( s + \frac{k+1}{2} - \mu_j \right) L(s, f \times E_{\nu_1, \nu_2}^{\min})$$

$$= \Lambda(1 - s, f \otimes E_{-\nu_1, -\nu_2}^{\min}).$$
For $E_{u,g}^{\text{max}}(z)$ the maximal Eisenstein series with $\mu(E_{u,g}^{\text{max}}) = (\mu'_1, \mu'_2, \mu'_3)$ where $\mu'_j$ are given by (2.4), we have

$$L(s, f \otimes E_{u,g}^{\text{max}}) := \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) A_{\mu_u}(m_1, m_2)}{(m_1 m_2)^s}$$

$$= L(s + 2u, f)L(s - u, f \otimes g),$$

where $L(s, f \otimes g)$ is the Rankin-Selberg function associated to $f$ and $g$. The complete $L$-function is

$$\Lambda(s, f \otimes E_{u,g}^{\text{max}}) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}} \left(s + \frac{k - 1}{2} - \mu'_j\right) \left(s + \frac{k + 1}{2} - \mu'_j\right) L(s, f \otimes E_{u,g}^{\text{max}})$$

$$= i^k \Lambda(1 - s, f \otimes E_{u,g}^{\text{max}}).$$

2.5. The approximate functional equation. For the Rankin-Selberg $L$-function defined in the previous section, we have the following approximate functional equation (see Theorem 5.3 in [IK2004]).

Lemma 2.3. Let $G(s) = e^{\varphi}$. We have

$$L \left(\frac{1}{2}, f \otimes \phi\right) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) A_\phi(m_1, m_2)}{(m_1 m_2)^{1/2}} V_k(m_1 m_2, \mu_\phi)$$

$$+ i^k \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_1) A_\phi(m_1, m_2)}{(m_1 m_2)^{1/2}} \tilde{V}(m_1 m_2; k, \mu_\phi),$$

where

$$V_k(y, \mu) = \frac{1}{2 \pi i} \int_{(3)} y^{-s} \prod_{i=1}^{3} \Gamma_{\mathbb{R}} \left(s + \frac{1}{2} + \frac{k - 1}{2} - \mu_i\right) \Gamma_{\mathbb{R}} \left(s + \frac{1}{2} + \frac{k + 1}{2} - \mu_i\right) G(s) \frac{ds}{s}$$

$$\left(2.6\right)$$

and

$$\tilde{V}_k(y, \mu) = \frac{1}{2 \pi i} \int_{(3)} y^{-s} \prod_{i=1}^{3} \Gamma_{\mathbb{R}} \left(s + \frac{1}{2} + \frac{k - 1}{2} + \mu_i\right) \Gamma_{\mathbb{R}} \left(s + \frac{1}{2} + \frac{k + 1}{2} + \mu_i\right) G(s) \frac{ds}{s}.$$

The functions $V_k(y, \mu)$ and $\tilde{V}_k(y, \mu)$ have the following properties, which can be proved by the method in Proposition 5.4 in [IK2004].

Lemma 2.4. Assume that $\mu = (\mu_1, \mu_2, \mu_3)$ with $\mu_i \gg T$. One has

$$y^a \frac{\partial^a}{\partial y^a} V_k(y, \mu) \ll_k \left(\frac{y}{T^3}\right)^{-A}, \quad y^a \frac{\partial^a}{\partial y^a} \tilde{V}_k(y, \mu) \ll_k \left(\frac{y}{T^3}\right)^{-A}$$

for any large number $A > 0$ and any $a \in \mathbb{N}_0$. Moreover, for $y \gg T^3$,

$$V_k(y, \mu) = 1 + O_{B,k} \left(\frac{T^3}{y}\right)^{-B}$$

$$\tilde{V}_k(y, \mu) = \prod_{i=1}^{3} \frac{\Gamma \left(\frac{1}{2} + \mu_i\right)}{\Gamma \left(\frac{1}{2} - \mu_i\right)} + O_{B,k} \left(\frac{T^3}{y}\right)^{-B}$$

for any $0 < B < \frac{k - 1}{2}$.
3. Proof of Theorem 1.1

Let $k \equiv 0 \mod 4$. For $h(\mu)$ defined in section 2.3, we consider

$$A = \sum_{\phi} \frac{h(\mu_{\phi})}{N_{\phi}} A_{\phi}(p,p)L(1/2, f \otimes \phi),$$

where $\phi$ runs over a Hecke-Maass basis of the space of Maass cusp forms for $SL_3(\mathbb{Z})$. By the approximate functional equation in Lemma 2.3, one has

$$A = A_1 + A_2$$

where

$$A_1 = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}} \sum_{\phi} \frac{h(\mu_{\phi})V_k(m_1^2m_2, \mu_{\phi})}{N_j} A_{\phi}(m_2, m_1) A_{\phi}(p,p),$$

$$A_2 = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}} \sum_{\phi} \frac{h(\mu_{\phi})V_k(m_1^2m_2, \mu_{\phi})}{N_j} A_{\phi}(m_1, m_2) A_{\phi}(p,p).$$

Thus Theorem 1.1 follows from

$$A_1 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \int_{\text{Re}(\mu)=0} h(\mu) d_{\text{spec}}(\mu) + O_{k, \epsilon}(p^{7/2 + \epsilon}T^{3/2 + \epsilon}M^2), \quad (3.1)$$

$$A_2 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \int_{\text{Re}(\mu)=0} h(\mu) \prod_{j=1}^{3} \frac{\Gamma\left(\frac{7}{2} + \mu_j\right)}{\Gamma\left(\frac{5}{2} - \mu_j\right)} d_{\text{spec}}(\mu) + O_{k, \epsilon}(p^{7/2 + \epsilon}T^{3/2 + \epsilon}M^2). \quad (3.2)$$

Since the proof of (3.2) is the same as that of (3.1). We only prove (3.1).

For $A_1$, by letting

$$H_{y}(\mu) := h(\mu)V_k(y; \mu)$$

and applying the Kunzetsov’s trace formula in section 2.2, one has

$$A_1 = D_1 + R_{1,w_4} + R_{1,w_5} - \mathcal{E}_{1,\text{max}} - \mathcal{E}_{1,\text{min}},$$

where

$$D_1 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \int_{\text{Re}(\mu)=0} h(\mu) V_k(p^3, \mu) d_{\text{spec}}(\mu),$$

$$R_{1,w_4} = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}} \sum_{\epsilon \in \{\pm\}} \sum_{D_1|D_2} \frac{\tilde{S}(\epsilon m_1, p, p; D_2)}{D_1 D_2} \Phi_{w_4} \left( \frac{\epsilon m_1 p^2}{D_1 D_2}; H_{m_1^2m_2} \right),$$

$$R_{1,w_5} = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}} \sum_{\epsilon \in \{\pm\}} \sum_{D_1|D_2} \frac{\tilde{S}(\epsilon m_2, p, p; D_1)}{D_1 D_2} \Phi_{w_5} \left( \frac{\epsilon m_2 p^2}{D_1 D_2}; H_{m_1^2m_2} \right),$$

$$R_{1,w_1} = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}} \sum_{\epsilon_1, \epsilon_2 \in \{\pm\}} \frac{\tilde{S}(\epsilon_1 m_1, \epsilon_2 m_2, p, p; D_1, D_2)}{D_1 D_2} \Phi_{w_1} \left( \frac{-\epsilon_2 p m_1 D_2}{D_1^2}, -\frac{\epsilon_1 p m_2 D_1}{D_2^2}; H_{m_1^2m_2} \right),$$

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and

\[ E_{1, \text{max}} = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \lambda_f(m_2) \frac{1}{(m_1^2 m_2)^{1/2}} \sum_{g} \frac{1}{2\pi i} \int_{\Re(u)=0} H_{m_1^2 m_2}(u + it, u - it, -2u) \frac{A_{u,g}^{\text{max}}(m_2, m_1) A_{u,g}^{\text{max}}(p, p)}{N_{u,g}^{\text{max}}} du, \]

\[ E_{1, \text{min}} = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \lambda_f(m_2) \frac{1}{(m_1^2 m_2)^{1/2}} 24(2\pi i)^2 \int_{\Re(\mu)=0} \frac{H_{m_1^2 m_2}(\mu)}{N_{\nu_1, \nu_2}^{\text{min}}} A_{\nu_1, \nu_2}^{\text{min}}(m_2, m_1) A_{\nu_1, \nu_2}^{\text{min}}(p, p) d\mu. \]

The main term in (3.1) comes from the estimation on \( D_1 \) in (3.5), and the error term comes from the contribution of \( E_{1, \text{max}} \) in (3.4). For \( E_{1, \text{min}} \) and \( R_{1, w_4}, R_{1, w_5}, R_{1, w_6} \), we will show that their contribution is negligible under the condition in Theorem 1.1.

### 3.1. Estimation on the continuous spectrum.

We consider \( E_{1, \text{min}} \) firstly. Note that \( H_y(\mu) = h(\mu)V_y(y, \mu) \). By the integral expression of \( V_y(y, \mu) \) in (2.6) and the fact that

\[ \sum_{m_1, m_2 \geq 1} \frac{\lambda_f(m_2) A_{\nu_1, \nu_2}^{\text{min}}(m_2, m_1)}{(m_1^2 m_2)^{s + \frac{1}{2}}} = L \left( \frac{1}{2} + s - \mu_1, f \right) L \left( \frac{1}{2} + s - \mu_2, f \right) L \left( \frac{1}{2} + s - \mu_3, f \right) \]

for \( \Re(s) = 3 \), one has

\[ E_{1, \text{min}} = \frac{1}{24(2\pi i)^2} \int_{\Re(\mu)=0} A_{\nu_1, \nu_2}^{\text{min}}(p, p) \frac{h(\mu)}{N_{\nu_1, \nu_2}^{\text{min}}} I_k(\mu) d\mu, \]

where

\[ I_k(\mu) = \frac{1}{2\pi i} \int_{\Re(s)=0} G(s) \prod_{i=1}^{3} \Gamma_{\Re} \left( s + \frac{1}{2} + \frac{k+1}{2} - \mu_i \right) \Gamma_{\Re} \left( s + \frac{1}{2} + \frac{k-1}{2} - \mu_i \right) L \left( \frac{1}{2} + s - \mu_i, f \right) \frac{ds}{s} \]

For \( I_k(\mu) \), moving the line of integration to \( \Re(s) = \epsilon \) and applying the subconvexity bound (see [Go1982])

\[ L(1/2 + it, f) \ll_k (1 + |t|)^{1/3 + \epsilon}, \]

one has

\[ I_k(\mu) \ll_{\epsilon,k} \prod_{j=1}^{3} (1 + |\Im(\mu_j)|)^{\frac{1}{4} + \epsilon}. \]

It gives that

\[ E_{1, \text{min}} \ll_{k, \epsilon} \int_{\Re(\mu)=0} A_{\nu_1, \nu_2}^{\text{min}}(p, p) \frac{h(\mu)}{N_{\nu_1, \nu_2}^{\text{min}}} \prod_{j=1}^{3} (1 + |\Im(\mu_j)|)^{\frac{1}{4} + \epsilon} d\mu. \]

Note that \( A_{\nu_1, \nu_2}^{\text{min}}(p, p) = O(1) \) and

\[ N_{\nu_1, \nu_2}^{\text{min}} = \frac{1}{16} \prod_{j=1}^{3} |\zeta(1 + 3\nu_{\pi,j})|^2 \gg \prod_{j=1}^{3} \left( \frac{1}{\log(1 + 3\Im(\nu_{\pi,j}))} \right)^2. \]
One has
\[ E_{1,\min} \ll_{k,\epsilon} T^{1+\epsilon} M^2. \] (3.3)

Next we consider \( E_{1,\max} \). By similar argument as above one has

\[ E_{1,\max} = \sum_g \frac{1}{2\pi i} \int_{\text{Re}(u)=0} A_{u,g}^{\max}(p,p) \frac{h(u + it_g, u - it_g, -2u)}{N_{u,g}^{\max}} \mathcal{I}_k^{\max}(u + it_g, u - it_g, -2u)du, \]

where

\[ \mathcal{I}_k^{\max}(\mu) = \frac{1}{2\pi i} \int_{(3)} \prod_{i=1}^{3} \frac{\Gamma_R\left(s + \frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_R\left(s + \frac{1}{2} + \frac{k+1}{2} - \mu_i\right)}{\Gamma_R\left(\frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_R\left(\frac{1}{2} + \frac{k+1}{2} - \mu_i\right)} L \left(\frac{1}{2} + s + 2u, f\right) L \left(\frac{1}{2} + s - u, f \otimes g\right) G(s) \frac{ds}{s}. \]

For \( \mathcal{I}_k^{\max}(\mu) \), by moving the line of integration to \( \text{Re}(s) = \frac{1}{2} + \epsilon \) and applying the fact that

\[ L(1 + \epsilon + 2u, f) \ll 1, \quad L(1 + \epsilon - u, f \otimes g) \ll 1, \]

which follow from the Ramanujar-Deligue’s bound and the property of Rankin-Selberg \( L \)-functions (see [RS1996]), one has

\[ \mathcal{I}_k^{\max}(\mu) \ll \int_{(3/2)+\epsilon} \prod_{j=1}^{3} \frac{\Gamma_R\left(s + \frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_R\left(s + \frac{1}{2} + \frac{k+1}{2} - \mu_i\right)}{\Gamma_R\left(\frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_R\left(\frac{1}{2} + \frac{k+1}{2} - \mu_i\right)} G(s) \frac{ds}{s} \ll_{k,\epsilon} \prod_{j=1}^{3} (1 + |\text{Im} \mu_j|)^{1/2+\epsilon}. \]

Moreover, by the definition of \( A_{u,g}^{\max}(m,n) \) in (2.2) and (2.3), and the bound \( \lambda_g(p) \ll p^{7/2+\epsilon} \) in [KS2003], one has \( A_{u,g}^{\max}(p,p) \ll p^{7/2+\epsilon} \). These together with

\[ N_{u,g}^{\max} = 8L(1, Ad^2 g)|L(1 + 3u, g)|^2 \gg \left(\frac{1}{1 + \log |u|}\right) \]

give that

\[ E_{1,\max} \ll_{k,\epsilon} p^{7/2+\epsilon} \sum_g \int_{\text{Re}(u)=0} \frac{h(u + it_g, u - it_g, -2u)}{N_{u,g}} (1 + |\text{Im} u + t_g|)^{1/2+\epsilon} \]
\[ \ll_{k,\epsilon} p^{7/2+\epsilon} T^{3/2+\epsilon} M \sum_g (1 + |2\text{Im} u|)^{1/2+\epsilon} d\mu \]
\[ \ll_{k,\epsilon} p^{7/2+\epsilon} T^{3/2+\epsilon} M^{2}, \] (3.4)

where we have used the Weyl’s law for Hecke-Mass cusp forms for \( SL_2(\mathbb{Z}) \) (see [Iw2002]).
3.2. Estimation on the diagonal term $D_1$. For the diagonal term $D_1$, by Lemma 2.4, we have
\[
D_1 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \left( 1 + O_B \left( \frac{p}{T} \right)^{3B} \right) \int_{\Re(\mu) = 0} h(\mu) d_{\text{spec}}(\mu)
\]
for $0 < B < \frac{k-1}{2}$. The choice of $h(\mu)$ in section 2.3 gives
\[
\int_{\Re(\mu) = 0} h(\mu) d_{\text{spec}}(\mu) \asymp T^3 M^2.
\]
Recall that $k \geq 12$. By (3.3) and (3.4), $D_1$ gives the main term in (3.1) if
\[
T \gg_k p^{3+\frac{7}{16}+\epsilon}.
\]

3.3. Estimation on other geometric terms. In this subsection, we show that the contribution from other geometric terms are negligible. For $\mathcal{R}_{1, w_4}$ and $\mathcal{R}_{1, w_1}$, it follows immediately from the application of the truncation Lemmas 2.1 and 2.2, respectively. To show that $\mathcal{R}_{1, w_5}$ is negligible, one needs to open the incomplete Kloosterman sum, rearrange the summation and apply the Voronoi formula for $GL_2$.

3.3.1. The term $\mathcal{R}_{1, w_4}$. Consider $\mathcal{R}_{1, w_4}$ firstly. By the property of $V_k(y; \mu)$ in lemma 2.4, the terms in summations over $m_1$ and $m_2$ are negligible for those $m_1^2 m_2 > T^{3+\epsilon}$. By Lemma 2.1, the contribution of terms in summations over $D_1$ and $D_2$ is negligible if
\[
p^2 m_1 / D_1 D_2 = p^{3/2} m_1 \sqrt{m_2} / D_1^{3/2} \leq T^{-\epsilon}.
\]
Thus one needs only to consider
\[
\sum_{m_1, m_2 \geq 1 \atop m_1^2 m_2 \leq T^{3+\epsilon}} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon \in \{\pm 1\}} \sum_{D_2 \mid D_1 \atop p D_1 = m_2 D_2^{1/3} \atop 1 \leq D_1 \leq p^{(m_1^2 m_2)^{1/3}}} \tilde{S}(-\epsilon m_1, p, D_2, D_1) \Phi_{w_4} \left( \frac{ep^2 m_1}{D_1 D_2}; H_{m_1^2 m_2} \right).
\]

Note that $m_1^2 m_2 \leq T^{3+\epsilon}$ and $1 \leq D_1 \leq p^{(m_1^2 m_2)^{1/3}}$ give $p \geq T^{1-\epsilon}$, which contradicts with (3.6). Thus these terms vanish and $\mathcal{R}_{1, w_4}$ is negligible.

3.3.2. The term $\mathcal{R}_{1, w_1}$. For $\mathcal{R}_{1, w_1}$, by the property of $V_k(y, \mu)$ in lemma 2.4, the terms in summations over $m_1$ and $m_2$ are negligible for those $m_2^2 m_1 \leq T^{3+\epsilon}$. Let $\mathcal{Y} := p^{1/2} \min \left\{ \frac{m_1^{1/3} m_2^{1/6}}{D_1^{1/2}}, \frac{m_2^{1/3} m_1^{1/6}}{D_2^{1/2}} \right\}$.

By lemma 2.2, the contribution is negligible for those terms in summations over $D_1$ and $D_2$ satisfying $\mathcal{Y} \leq T^{1-\epsilon}$. Thus we need only to estimate
\[
\sum_{m_1, m_2 \geq 1 \atop m_1^2 m_2 \leq T^{3+\epsilon}} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \sum_{D_1, D_2 \atop \mathcal{Y} > T^{1-\epsilon}} \frac{S(\epsilon_2 m_1, \epsilon_1 m_2, p, D_1, D_2)}{D_1 D_2} \Phi_{w_1} \left( -\frac{\epsilon_2 p m_1 D_2}{D_1^2}, -\frac{\epsilon_1 p m_2 D_1}{D_2^2}; H_{m_1^2 m_2} \right).
\]
Note that \( m_2^2 m_1 \leq T^{3+\epsilon} \) and \( \mathcal{V} > T^{1-\epsilon} \) give \( p \geq T^{1-\epsilon} \), which contradicts with (3.6). Thus these terms vanish and \( R_{1,w_1} \) is negligible.

3.3.3. The term \( R_{w_5} \). Consider \( R_{w_5} \). By the similar argument in previous sections, one needs only to consider the contribution of

\[
R^* : = \sum_{m_1,m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon \in \{ \pm 1 \}} \sum_{T^{3/2} \leq m_1^2 m_2 \leq T^{3+\epsilon}} \frac{\tilde{S}(\epsilon m_1,p,p;D_1,D_2)}{D_1 D_2} \Phi_{w_4} \left( \frac{\epsilon m_1^2 m_2}{D_1 D_2} ; H_{m_1^2 m_2} \right),
\]

since other terms either vanish or are negligible.

We show that \( R^* \) is also negligible. Recall the smooth partition of unity

\[
1 = \sum_{\alpha \geq 0} \omega \left( \frac{m_1^2 m_2}{N_\alpha} \right),
\]

where \( \omega \) is a function which is smooth and compactly supported on \( [\frac{1}{2}, \frac{5}{2}] \) and \( N_\alpha = 2^\alpha \). One has

\[
R^* \ll \sum_{\alpha \geq 0} \sum_{m_1,m_2 \geq 1} \omega \left( \frac{m_1^2 m_2}{N_\alpha} \right) \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon \in \{ \pm 1 \}} \sum_{T^{4/3} \ll N_\alpha \ll T^{3+\epsilon}} \frac{\tilde{S}(\epsilon m_2,p,p;D_1,D_2)}{D_1 D_2} \Phi_{w_5} \left( \frac{\epsilon m_2^2}{D_1 D_2} ; H_{m_1^2 m_2} \right).
\]

Let \( D_2 = D_1 \delta \). We open the incomplete Kloosterman sum, rearrange the summation and then obtain

\[
R^* \ll \sum_{T^{8/3} \ll N_\alpha \ll T^{3+\epsilon}} \sum_{m_1 \geq 1} \frac{1}{m_1} \sum_{\epsilon \in \{ \pm 1 \}} \sum_{\frac{D_1 \geq 1}{p} \frac{D_1 \geq 1}{m_1 D_1}} \frac{1}{D_1^3 \delta} \sum_{C_1 \bmod{D_1},C_2 \bmod{(D_1 \delta)}} \sum_{(C_1,D_1)=(C_2,2)} e \left( \frac{p C_1 C_2}{D_1} + \frac{C_2}{\delta} \right) \sum_{m_2 \geq 1} \omega \left( \frac{m_1^2 m_2}{N_\alpha} \right) \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \Phi_{w_5} \left( \frac{\epsilon m_2^2}{D_1^2 \delta} ; H_{m_1^2 m_2} \right) e \left( \frac{\epsilon m_2 C_1}{D_1} \right). \tag{3.7}
\]

Thus one can apply the following \( GL(2) \) Voronoi formula (see formula (4.71) in [IK2004]).

**Lemma 3.1.** Let \( c \geq 1 \) and \((a,c) = 1\). Let \( F \) be a smooth, compactly supported function on \( \mathbb{R}^+ \). One has

\[
\sum_{m \geq 1} \lambda_f(m) e \left( \frac{am}{c} \right) F(m) = \frac{1}{c} \sum_{n \geq 1} \lambda_f(n) e \left( -\frac{\overline{a}n}{c} \right) G(n),
\]

where \( G(y) = 2\pi^k \int_0^\infty F(x) J_{k-1} \left( \frac{4\pi \sqrt{xy}}{c} \right) dx \). Here \( J_k(y) \) is the \( J \)-Bessel function.
For the summation over \( m_2 \) in (3.7), we apply the Voronoi formula in the above lemma and obtain

\[
\sum_{m_2 \geq 1} \omega \left( \frac{m_2^2}{N_\alpha} \right) \frac{\lambda_f(m_2)}{\sqrt{m_2}} \Phi \left( \frac{\sigma^2 m_2}{D_1^2 \delta}, H_{m_2^2} \right) e \left( cm_2 \frac{C_1}{D_1} \right) = \frac{1}{D_1} \sum_{m_2 \geq 1} \lambda_f(m_2)e \left( -\frac{eC_1 m_2}{D_1} \right) G(m_2),
\]

where

\[
G(m_2) = 2\pi i^k \int_0^\infty \omega \left( \frac{m_2^2 x}{N_\alpha} \right) \frac{1}{x^{1/2}} \Phi \left( \frac{\sigma^2 x}{D_1^2 \delta}, H_{m_2^2} \right) J_{k-1} \left( 4\pi \sqrt{xm_2} \right) dx.
\]

**Lemma 3.2.** We have

\[
G(m_2) \ll_{j,k,\epsilon} \frac{\sqrt{N_\alpha}}{m_1} \left( \frac{p^{1+\epsilon}}{N_\alpha^{\frac{5}{2}} m_2} \right)^j
\]

for any \( j \in \mathbb{N}_0 \).

**Proof.** For \( G(m_2) \), we change the variable \( t = \frac{m_2^2 x}{N_\alpha} \) to obtain

\[
G(m_2) = 2\pi i^k \frac{\sqrt{N_\alpha}}{m_1} \int_0^\infty \omega(t) \Phi \left( \frac{\sigma^2 N_\alpha}{D_1^2 \delta m_1^2}, H_{m_2^2} \right) J_{k-1} \left( 4\pi \frac{\sqrt{N_\alpha m_2}}{m_1 D_1} \right) dt.
\]

Let \( R = \frac{4\pi \sqrt{N_\alpha m_2}}{m_1 D_1} \). By applying the recurrence formula of the \( J \)-Bessel function

\[
\frac{d}{dy} ((R\sqrt{y})^{s+1} J_{s+1}(R\sqrt{y})) = \frac{R^2}{2} (R\sqrt{y})^s J_s(R\sqrt{y})
\]

one has

\[
G(m_2) = 2\pi i^k \frac{\sqrt{N_\alpha}}{m_1} \frac{1}{R^{k-1}} \frac{2}{R^2} \int_0^\infty \left( t^{-\frac{k}{2}} \omega(t) \Phi \left( \frac{\sigma^2 N_\alpha}{D_1^2 \delta m_1^2}, H_{m_2^2} \right) \right)^j \left( R\sqrt{t} \right)^k J_{k-1}(R\sqrt{t}) dt
\]

for any \( j \in \mathbb{N}_0 \). Note that \( \Phi \) also satisfies Lemma 2.1 and one has

\[
\left( \Phi \left( \frac{\sigma^2 N_\alpha}{D_1^2 \delta m_1^2}, H_{m_2^2} \right) \right)^j \ll T^3 M^2 \left( \frac{p^2 N_\alpha}{D_1^2 \delta m_1^2} \right)^j \left( \frac{1}{R} \right)^j.
\]

It gives that

\[
G(m_2) \ll_{k,j,\epsilon} \frac{\sqrt{N_\alpha}}{m_1} \left( \frac{\sigma^2 N}{D_1^2 \delta m_1^2} \right)^{j \left( \frac{1}{2} + \epsilon \right)} \left( \frac{1}{R} \right)^j
\]

\[
\ll_{k,j,\epsilon} \frac{\sqrt{N_\alpha}}{m_1} \left( \frac{p^{\frac{5}{3} + \epsilon}}{N_\alpha^{\frac{2}{3}} m_2^{\frac{1}{3}} \delta} \right)^j.
\]
since \( R = \frac{4\pi \sqrt{N_\alpha m_2}}{m_1 D_1} \). The lemma follows immediately from the fact that \( m_1 D_1 = p\delta \).

By lemma 3.2, the contribution is negligible for those terms in \( R^* \) satisfying
\[
\frac{p^{1+\varepsilon}}{N_\alpha \frac{1}{2} - \varepsilon} \ll_{k,\varepsilon} T^{-\varepsilon}.
\]

Note that \( N_\alpha \gg T^\frac{3}{4} \). Thus one needs only to consider terms in \( R^* \) satisfying the condition
\[
p^{1+\varepsilon} \gg_{k,\varepsilon} N_\alpha^{\frac{1}{2} - \varepsilon} T^{-\varepsilon} \gg T^{\frac{3}{4} - \varepsilon},
\]
which contradicts with (3.6). Thus the contribution of \( R^* \) is negligible.

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