ON THE ABSOLUTE CONTINUITY OF RADIAL PROJECTIONS

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ABSTRACT. Let $d \geq 2$ and $d - 1 < s < d$. Let $\mu$ be a compactly supported Radon measure in $\mathbb{R}^d$ with finite $s$-energy. I prove that the radial projections $\pi_x \mu$ of $\mu$ are absolutely continuous with respect to $H^{d-1}$ for every centre $x \in \mathbb{R}^d \setminus \text{spt } \mu$, outside an exceptional set of dimension at most $2(d - 1) - s$. This is sharp. In fact, for $x$ outside an exceptional set as above, $\pi_x \mu \in L^p(S^{d-1})$ for some $p > 1$.

1. INTRODUCTION

The space of compactly supported Radon measures on $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$, denote by $\pi_x : \mathbb{R}^d \setminus \{x\} \to S^{d-1}$ the radial projection

$$\pi_x(y) = \frac{y - x}{|y - x|}, \quad y \in \mathbb{R}^d \setminus \{x\}. $$

This note is concerned with the question: if $\mu \in M(\mathbb{R}^d)$ has finite $s$-energy for some $d - 1 < s < d$, then how often is $\pi_x \mu$ absolutely continuous with respect to $H^{d-1}|_{S^{d-1}}$?

Write

$$S(\mu) := \{x \in \mathbb{R}^d \setminus \text{spt } \mu : \pi_x \mu \text{ is not absolutely continuous w.r.t. } H^{d-1}|_{S^{d-1}}\}. $$

Note that whenever $x \in \mathbb{R}^d \setminus \text{spt } \mu$, the projection $\pi_x$ is continuous on $\text{spt } \mu$, and $\pi_x \mu$ is well-defined. One can check that the family of projections $\{\pi_x\}_{x \in \mathbb{R}^d \setminus \text{spt } \mu}$ fits in the generalised projections framework of Peres and Schlag [5], and indeed Theorem 7.3 in [5] yields the estimate

$$\dim_H S(\mu) \leq 2d - 1 - s.$$ 

Combining this bound with standard arguments shows that if $K \subset \mathbb{R}^d$ is a Borel set with $d - 1 < \dim_H K \leq d$, then

$$\dim_H \{x \in \mathbb{R}^d : H^{d-1}(\pi_x(K)) = 0\} \leq 2d - 1 - \dim_H K. \quad (1.1)$$

In a fairly recent paper [6], which built heavily on slightly earlier collaboration [4] with P. Mattila, I showed that the bound (1.1) is not sharp, and in fact

$$\dim_H \{x \in \mathbb{R}^d : H^{d-1}(\pi_x(K)) = 0\} \leq 2(d - 1) - \dim_H K. \quad (1.2)$$

The bound (1.2) is best possible. The proofs in [4] and [6] were somewhat indirect, and did not improve on the Peres-Schlag bound (1.1) for $\dim_H S(\mu)$. This improvement is the content of the present note:

Theorem 1.3. If $\mu \in M(\mathbb{R}^d)$ and $I_s(\mu) < \infty$ for some $s > d - 1$, then $\dim_H S(\mu) \leq 2(d - 1) - s$. 

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In fact, Theorem 1.3 follows immediately from the next statement about $L^p$-densities:

**Theorem 1.4.** Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ as in Theorem 1.3. For $p > 1$, write

$$S_p(\mu) := \{ x \in \mathbb{R}^d \setminus \text{spt } \mu : \pi_{x^\perp} \mu \notin L^p(S^{d-1}) \}.$$ 

Then $\dim_H S_p(\mu) \leq 2(d - 1) - s + \delta(p)$, where $\delta(p) \to 0$ as $p \searrow 1$.

**Remark 1.5.** Theorem 1.4 can be viewed as an extension of Falconer’s exceptional set estimate [1] from 1982. I only discuss the planar case. Falconer proved that if $I_s(\mu) < \infty$ for some $1 < s < 2$, then the orthogonal projections of $\mu$ to all 1-dimensional subspaces are in $L^2$, outside an exceptional set of dimension at most $2 - s$. Now, orthogonal projections can be viewed as radial projections from points on the line at infinity. Alternatively, if the reader prefers a more rigorous statement, Falconer’s proof shows that if $\ell \subset \mathbb{R}^2$ is any fixed line outside the support of $\mu$, then all the radial projections of $\mu$ to points on $\ell$ are in $L^2$, outside an exceptional set of dimension at most $2 - s$. In comparison, Theorem 1.4 states that the radial projections of $\mu$ to points in $\mathbb{R}^2 \setminus \text{spt } \mu$ are in $L^p$ for some $p > 1$, outside an exceptional set of dimension at most $2 - s$. So, the size of the exceptional set remains the same even if the “fixed line $\ell$” is removed from the statement. The price to pay is that the projections only belong to some $L^p$ with $p > 1$ (possibly) smaller than 2. I do not know, if the reduction in $p$ is necessary, or an artefact of the proof.

The proof of Theorem 1.4 uses ideas from [4] and [6], but is more direct than those arguments, and perhaps a little simpler.

1.1. **Acknowledgements.** It seems quite natural to consider the problem of improving the Peres-Schlag estimate for $\dim_H S(\mu)$, but the question did not occur to me at the time of writing the papers [4] and [6]. Thanks to Pablo Shmerkin for asking it explicitly, and for nice discussions at Institut Mittag-Leffler in September 2017.

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2. **Proof of the main theorem**

Fix $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \setminus \text{spt } \mu$. For a suitable constant $c_d > 0$ to be determined shortly, consider the weighted measure

$$\mu_x := c_d k_x \, d\mu,$$

where $k_x := |x - y|^{1-d}$ is the $(d - 1)$-dimensional Riesz kernel, translated by $x$. A main ingredient in the proofs of Theorems 1.3 and 1.4 is the following identity:

**Lemma 2.1.** Let $\mu \in C_0(\mathbb{R}^d)$ (that is, $\mu$ is a continuous function with compact support) and $\nu \in \mathcal{M}(\mathbb{R}^d)$. Assume that $\text{spt } \mu \cap \text{spt } \nu = \emptyset$. Then, for $p \in (0, \infty)$,

$$\int_{S^{d-1}} \|\pi_{x^\perp} \mu_x\|^p_{L^p(S^{d-1})} \, d\nu(x) = \int_{S^{d-1}} \|\pi_{x^\perp} \mu\|^p_{L^p(\pi_{x^\perp}\nu)} \, dH^{d-1}(x).$$

Here, and for the rest of the paper, $\pi_e$ stands for the orthogonal projection onto $e^\perp \in G(d, d - 1)$.

**Proof.** Start by assuming that also $\nu \in C_0(\mathbb{R}^d)$. Fix $x \in \mathbb{R}^d$. The first aim is to find an explicit expression for the density $\pi_{x^\perp} \mu_x$ on $S^{d-1}$, so fix $f \in C(S^{d-1})$ and compute as
follows, using the definition of the measure $\mu_x$, integration in polar coordinates, and choosing the constant $c_d > 0$ appropriately:

$$
\int f(e) d[\pi_x f](e) = \int f(\pi_x(y)) d\mu_x(y) = c_d \int \frac{f(\pi_x(y))}{|x - y|^{d-1}} d\mu(y)
$$

$$
= \int_{S^{d-1}} f(e) \mu(x + re) dr d\mathcal{H}^{d-1}(e)
$$

$$
= \int_{S^{d-1}} f(e) \cdot \pi_x \mu(\pi_x(x)) d\mathcal{H}^{d-1}(e).
$$

Since the equation above holds for all $f \in C(S^{d-1})$, we infer that

$$
\pi_x \mu_x = [e \mapsto \pi_x \mu(\pi_x(x))] d\mathcal{H}^{d-1}|_{S^{d-1}}.
$$

(2.2)

Now, we may prove the lemma by a straightforward computation, starting with

$$
\int \|\pi_x \mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) = \int \int_{S^{d-1}} |\pi_x \mu_x(e)|^p \, d\mathcal{H}^{d-1}(e) \, d\nu(x)
$$

$$
= \int_{S^{d-1}} \int e^1 \int_{\pi_x^{-1}(w)} |\pi_x \mu(\pi_x(x))|^p \nu(x) \, d\mathcal{H}^1(x) \, d\mathcal{H}^{d-1}(w) \, d\mathcal{H}^{d-1}(e).
$$

Note that whenever $x \in \pi_x^{-1}(w)$, then $\pi_x(x) = w$, so the expression $[\ldots]^p$ above is independent of $x$. Hence,

$$
\int \|\pi_x \mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) = \int_{S^{d-1}} \int e^1 \int_{\pi_x^{-1}(w)} |\pi_x \mu(\pi_x(x))|^p \nu(x) \, d\mathcal{H}^1(x) \, d\mathcal{H}^{d-1}(w) \, d\mathcal{H}^{d-1}(e)
$$

$$
= \int_{S^{d-1}} \int e^1 \pi_x \mu(w)^p \pi_x \nu(w) \, d\mathcal{H}^{d-1}(w) \, d\mathcal{H}^{d-1}(e)
$$

$$
= \int_{S^{d-1}} \|\pi_x \mu\|_{L^p(\pi_x \nu)}^p \, d\mathcal{H}^{d-1}(e),
$$

as claimed.

Finally, if $\nu \in \mathcal{M}(\mathbb{R}^d)$ is arbitrary, not necessarily smooth, note that

$$
x \mapsto \|\pi_x \mu_x\|_{L^p(S^{d-1})}^p
$$

is continuous, assuming that $\mu \in C_0(\mathbb{R}^d)$, as we do (to check the details, it is helpful to infer from (2.2) that $\pi_x \mu_x \in L^\infty(S^{d-1})$ uniformly in $x$, since the projections $\pi_x \mu$ clearly have bounded density, uniformly in $e \in S^{d-1}$). Thus, if $(\psi_n)_{n \in \mathbb{N}}$ is a standard approximate identity on $\mathbb{R}^d$, we have

$$
\int \|\pi_x \mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) = \lim_{n \to \infty} \int_{S^{d-1}} |\pi_x \mu(\pi_x \psi_n)|^p \, d\mathcal{H}^{d-1}(e),
$$

(2.3)

with $\nu_n = \nu * \psi_n$. Since $\pi_x \nu_n$ converges weakly to $\pi_x \nu$ for any fixed $e \in S^{d-1}$, and $\pi_x \mu \in C_0(e^1)$, it is easy to see that the right hand side of (2.3) equals

$$
\int_{S^{d-1}} \|\pi_x \mu\|_{L^p(\pi_x \nu)}^p \, d\mathcal{H}^{d-1}(e).
$$

This completes the proof of the lemma.

\[\square\]

We can now prove Theorem 1.4, which implies Theorem 1.3.
Proof of Theorem 1.3. Fix 2(d − 1) − s < t < d − 1. It suffices to prove that if ν ∈ M(\(\mathbb{R}^d\)) is a fixed measure with \(I_t(\nu) < \infty\), and spt \(\mu\) ∩ spt \(\nu\) = ∅, then
\[
\pi_{x^T}\mu_x \in L^p(S^{d-1}) \quad \text{for } \nu \text{ a.e. } x \in \mathbb{R}^d,
\]
whenever
\[
1 < p \leq \min \left\{ 2 - \frac{t}{(d-1)}, \frac{t}{2(d-1)-s} \right\}.
\]
(2.4)

We will treat the numbers \(d, p, s, t\) as "fixed" from now on, and in particular the implicit constants in the \(\lesssim\) notation may depend on \(d, p, s, t\). Note that the right hand side of (2.4) lies in (1, 2), so this is a non-trivial range of \(p\)'s. Fix \(p\) as in (2.4). The plan is to show that
\[
\int \|\pi_{x^T}\mu_x\|_{L^p}^p \, d\nu(x) < \infty.
\]
(2.5)

This will be done via Lemma 2.1, but we first need to reduce to the case \(\mu \in C_0(\mathbb{R}^d)\). Let \((\psi_n)_{n \in \mathbb{N}}\) be a standard approximate identity on \(\mathbb{R}^d\), and write \(\mu_n = \mu * \psi_n\). Then \(\pi_{x^T}(\mu_n)_x\) converges weakly to \(\pi_{x^T}\mu_x\) for any fixed \(x \in \text{spt } \nu \subset \mathbb{R}^d \setminus \text{spt } \mu\):
\[
\int f(e) \, d[\pi_{x^T}\mu_x(e)] = \lim_{n \to \infty} \int f(e) \, d\pi_{x^T}(\mu_n)_x(e), \quad f \in C(S^{d-1}).
\]

It follows that
\[
\|\pi_{x^T}\mu_x\|_{L^p(S^{d-1})} \leq \liminf_{n \to \infty} \|\pi_{x^T}(\mu_n)_x\|_{L^p(S^{d-1})}^p, \quad x \in \text{spt } \nu,
\]
and consequently
\[
\int \|\pi_{x^T}\mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) \leq \liminf_{n \to \infty} \int \|\pi_{x^T}(\mu_n)_x\|_{L^p(S^{d-1})}^p \, d\nu(x)
\]
by Fatou’s lemma. Now, it remains to find a uniform upper bound for the terms on the right hand side; the only information about \(\mu_n\), which we will use, is that \(I_{s}(\mu_n) \lesssim I_{s}(\mu)\). With this in mind, we simplify notation by denoting \(\mu_n := \mu\). For the remainder of the proof, one should keep in mind that \(\pi_{x^T}\mu \in C_0^\infty(e^+)\) for \(e \in S^{d-1}\), so the integral of \(\pi_{x^T}\mu\) with respect to various Radon measures on \(e^+\) is well-defined, and the Fourier transform of \(\pi_{x^T}\mu\) on \(e^+\) (identified with \(\mathbb{R}^{d-1}\)) is a rapidly decreasing function.

We start by appealing to Lemma 2.1:
\[
\int \|\pi_{x^T}\mu_x\|_{L^p(S^{d-1})} \, d\nu(x) = \int_{S^{d-1}} \|\pi_{x^T}\mu\|_{L^p(\pi_{x^T}\nu)} \, d\mathcal{H}^{d-1}(e).
\]
(2.6)

Next, we estimate the \(L^p(\pi_{x^T}\nu)\)-norms of \(\pi_{x^T}\mu\) individually, for \(e \in S^{d-1}\) fixed. We start by recording the standard fact that \(I_t(\pi_{x^T}\nu) < \infty\) for \(\mathcal{H}^{d-1}\) almost every \(e \in S^{d-1}\), and we will only consider those \(e \in S^{d-1}\) satisfying this condition. Recall that \(1 < p \leq t/[2(d-1)-s]\). Fix \(f \in L^q(\pi_{x^T}\nu)\), with \(q = p'\) and \(\|f\|_{L^q(\pi_{x^T}\nu)} = 1\), and note that
\[
I_{t/2(d-1)-s}(f \, d\pi_{x^T}\nu) = \int f(x)f(y) \, d\pi_{x^T}\nu(x) \, d\pi_{x^T}\nu(y) \|x-y\|^t/2(d-1)-s \lesssim I_t(\pi_{x^T}\nu)^{1/p}
\]
by Hölder’s inequality. It now follows from Theorem 17.3 in [3] that
\[
\int |f| \, d\mu \leq \sqrt{\int (f^2 + |f|^2) \, d\mu} \leq \sqrt{2} \|f\|_{L^2}\|\mu\|_{L^1}
\]
Since the function \( f \in L^q(S^{d-1}) \) with \( \|f\|_{L^q(S^{d-1})} = 1 \) was arbitrary, we may infer by duality that
\[
\|\mu\|_{L^p(S^{d-1})} \leq \|f\|_{L^p(S^{d-1})} \leq \|\mu\|_{L^1}
\]
We can finally estimate (2.6). We use duality once more, so fix \( f \in L^q(S^{d-1}) \) with \( \|f\|_{L^q(S^{d-1})} = 1 \). Then, write
\[
\int_{S^{d-1}} |f| \, d\mu \leq \left( \int_{S^{d-1}} |f| \, d\mu \right)^{1/2} \|f\|_{L^q(S^{d-1})}^{1/2}
\]
Thus, \( \|\mu\|_{L^p} \) is bounded by \( \|f\|_{L^q} \). We use Hölder’s inequality again:
\[
\int_{S^{d-1}} |f| \, d\mu \leq \left( \int_{S^{d-1}} |f| \, d\mu \right)^{1/2} \|f\|_{L^q(S^{d-1})}^{1/2}
\]
Finally, we use Hölder’s inequality again:
\[
I \leq \left( \int_{S^{d-1}} |f| \, d\mu \right)^{1/2} \|f\|_{L^q(S^{d-1})}^{1/2}
\]
The second factor is bounded by \( \|f\|_{L^q} \). Thus, the result follows.
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