A unification of the hypercontractivity and its exponential variant of the Ornstein-Uhlenbeck semigroup

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Abstract

Let \( \gamma_d \) be the \( d \)-dimensional standard Gaussian measure and \( \{Q_t\}_{t \geq 0} \) the Ornstein-Uhlenbeck semigroup acting on \( L^1(\gamma_d) \). We show that the hypercontractivity of \( \{Q_t\}_{t \geq 0} \) is equivalent to the property that

\[
\left\{ \int_{\mathbb{R}^d} \exp\left(e^{2t}Q_t f\right) \, d\gamma_d \right\}^{1/e^{2t}} \leq \int_{\mathbb{R}^d} e^f \, d\gamma_d,
\]

which holds for any \( f \in L^1(\gamma_d) \) with \( e^f \in L^1(\gamma_d) \) and for any \( t \geq 0 \). We then derive a family of inequalities that unifies this exponential variant and the original hypercontractivity; a generalization of the Gaussian logarithmic Sobolev inequality is obtained as a corollary. A unification of the reverse hypercontractivity and the exponential variant is also provided.

1 Introduction

For a given positive integer \( d \), we denote by \( \gamma_d \) the standard Gaussian measure on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) with \( \mathcal{B}(\mathbb{R}^d) \) the Borel \( \sigma \)-field on \( \mathbb{R}^d \). For every \( p > 0 \), define

\[
L^p(\gamma_d) := \left\{ f : \mathbb{R}^d \to \mathbb{R}; \text{f is measurable and satisfies} \int_{\mathbb{R}^d} |f(x)|^p \, \gamma_d(dx) < \infty \right\}
\]

and set

\[
\|f\|_p := \left\{ \int_{\mathbb{R}^d} |f(x)|^p \, \gamma_d(dx) \right\}^{1/p}, \quad f \in L^p(\gamma_d).
\]

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Although $\|\cdot\|_p$ is not a norm for $p < 1$, we abuse the common notation in the sequel.)

We denote by $Q = \{Q_t\}_{t \geq 0}$ the Ornstein-Uhlenbeck semigroup acting on $L^1(\gamma_d)$: for $f \in L^1(\gamma_d)$ and $t \geq 0$,

$$(Q_t f)(x) := \int_{\mathbb{R}^d} f \left( e^{-t}x + \sqrt{1 - e^{-2t}}y \right) \gamma_d(dy), \quad x \in \mathbb{R}^d;$$

note that $|Q_t f| < \infty \, \gamma_d$-a.e. when $f \in L^1(\gamma_d)$, since there holds the identity

$$\int_{\mathbb{R}^d} Q_t |f| \, d\gamma_d = \int_{\mathbb{R}^d} |f| \, d\gamma_d$$

for any measurable function $f$ on $\mathbb{R}^d$. It is well known that $Q$ enjoys the hypercontractivity: if $f \in L^p(\gamma_d)$ for some $p > 1$, then

$$\|Q_t f\|_{q(t)} \leq \|f\|_p \quad \text{for all} \quad t \geq 0,$$

with $q(t) = e^{2t}(p - 1) + 1$.

The hypercontractivity (1.1) was firstly observed by Nelson [8] and applied in quantum field theory; it was found later by Gross [5] to be equivalent to the (Gaussian) logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^d} |f|^2 \log |f| \, d\gamma_d \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d + \|f\|_2^2 \log \|f\|_2,$$

where $f$ is any weakly differentiable function in $L^2(\gamma_d)$ with $|\nabla f| \in L^2(\gamma_d)$. Because of their dimension-free formulation, the hypercontractivity (1.1) as well as the logarithmic Sobolev inequality (1.2) have importance in the Malliavin calculus; see, e.g., the monograph [10] by Shigekawa. We also remark that the Gaussian logarithmic Sobolev inequality (1.2) goes back to Stam [11], on which we refer the reader to [7, Section 8.13].

In this paper, we show the equivalence between the hypercontractivity (1.1) and the following property of $Q$: for any $f \in L^1(\gamma_d)$ with $e^f \in L^1(\gamma_d)$, it holds that

$$\|\exp (Q_t f)\|_{e^{2t}} \leq \|e^f\|_1 \quad \text{for all} \quad t \geq 0.$$ 

**Proposition 1.1.** *The hypercontractivity (1.1) and the property (1.3) are equivalent.*

In fact, by using Jensen’s inequality, it is easily seen that (1.1) implies (1.3); on the other hand, it can also be seen that (1.3) implies the logarithmic Sobolev inequality (1.2), and hence implies (1.1) thanks to the above-mentioned equivalence between (1.1) and (1.2).

We then show that the above two properties (1.1) and (1.3) of $Q$ are unified into

**Theorem 1.1.** *Let a positive function $c : (0, \infty) \to (0, \infty)$ be in $C^1((0, \infty))$ and satisfy*

$$c' > 0 \quad \text{and} \quad \frac{c}{c'} \quad \text{is concave on} \quad (0, \infty).$$

**C**
We set
\[ u(t, x) = \int_0^x c(y) e^{yt} \, dy, \quad t \geq 0, \ x > 0. \] (1.4)

Then for any nonnegative, measurable function \( f \) on \( \mathbb{R}^d \) such that
\[ u(0, f) \in L^1(\gamma_d), \] (1.5)
we have
\[ v(t, \|u(t, Q_t f)\|_1) \leq v(0, \|u(0, f)\|_1) \quad \text{for all } t \geq 0. \] (1.6)

Here for every \( t \geq 0 \), the function \( v(t, \cdot) \) is the inverse function of \( u(t, x), \ x > 0 \).

It is easily checked that two functions \( x^{p-1} \) with \( p > 1 \) and \( e^x \) fulfill the condition \( (\mathcal{C}) \); in fact, they both satisfy \( (c/c')'' = 0 \). These two choices of \( c \) in Theorem 1.1 lead to (1.1) and (1.3), respectively; see Remark 4.1 for details. For other examples of \( c \) satisfying \( (\mathcal{C}) \), see Example 4.1. We also show that differentiating the left-hand side of (1.6) at \( t = 0 \) gives us a generalization of the logarithmic Sobolev inequality (1.2); see Corollary 1.1.

Remark 1.1. (1) The condition \( (\mathcal{C}) \) imposed on a nonnegative \( f \) implies \( f \in L^1(\gamma_d) \), hence \( Q_t f \) in (1.6) is well-defined. To see this, note that the positivity and concavity of \( c/c' \) entail that there exist positive constants \( \kappa_1, \kappa_2 \) such that
\[ \frac{c}{c'}(x) \leq \kappa_1 x + \kappa_2 \quad \text{for all } x > 0, \]
from which it follows that for all \( x, y > 0 \) with \( x > y \),
\[ \frac{c(x)}{c(y)} \geq \left( \frac{\kappa_1 x + \kappa_2}{\kappa_1 y + \kappa_2} \right)^{1/\kappa_1}. \]
Therefore if (1.5) is fulfilled, then we have \( f \in L^{1+1/\kappa_1}(\gamma_d) \) for some \( \kappa_1 > 0 \).

(2) The left-hand side of (1.6) is nonincreasing in \( t \):
\[ v(t + s, \|u(t + s, Q_{t+s} f)\|_1) \leq v(s, \|u(s, Q_s f)\|_1) \] (1.7)
for any \( s, t \geq 0 \). To see (1.7), fix \( s \geq 0 \) and set for \( t \geq 0 \) and \( x > 0 \),
\[ c^s(x) = \{c(x)\}^{e^{xs}} \quad \text{and} \quad u^s(t, x) = \int_0^x \{c^s(y)\}^{e^{yt}} \, dy \equiv u(t + s, x). \]

We also write \( v^s(t, \cdot) \) for the inverse function of \( u^s(t, \cdot) \). Since \( c^s/(c^s)' = e^{-2s}c/c' \), the function \( c^s/(c^s)' \) is also concave on \((0, \infty)\). Therefore we may apply Theorem 1.1 to \( Q_s f \) with replacing \( c, u \) and \( v \) therein by \( c^s, u^s \) and \( v^s \), respectively, to get
\[ v^s(t, \|u^s(t, Q_t(Q_s f))\|_1) \leq v^s(0, \|u^s(0, Q_s f)\|_1) , \]
which is (1.7) thanks to the identities \( u^s(t, \cdot) = u(t + s, \cdot) \), \( v^s(t, \cdot) = v(t + s, \cdot) \) and the semigroup property \( Q_t(Q_s f) = Q_{t+s} f \). The monotonicity (1.7) may also be seen directly from the proof of the theorem given in Section 4.
It will also be shown that if we replace \((C)\) by the condition that \(c' < 0\) and \(c/c'\) is convex on \((0, \infty)\), then the concluding inequality \((1.6)\) is reversed, yielding in particular the reverse hypercontractivity of \(Q\): if a \(\gamma_d\)-a.e. positive \(f \in L^1(\gamma_d)\) satisfies \(1/f \in L^\alpha(\gamma_d)\) for some \(\alpha > 0\), then it holds that

\[
\|1/Q_t f\|_{e^{2t(\alpha+1)-1}} \leq \|1/f\|_\alpha \quad \text{for all } t \geq 0. \tag{1.8}
\]

See Section 5.

We remark that since there are not involved any constants dependent on the dimension \(d\), every result mentioned above can be extended to the framework of abstract Wiener space through finite-dimensionalization.

We give an outline of the paper. In Section 2 we provide preliminary lemmas. In Section 3 we prove Proposition 1.1. In Section 4, we give a proof of Theorem 1.1 as well as examples of functions \(c\) satisfying the assumption of the theorem. As a corollary to Theorem 1.1, we also derive a family of inequalities that includes the logarithmic Sobolev inequality \((1.2)\) as a particular case. In the final section, we show a unification of the reverse hypercontractivity \((1.8)\) and the exponential variant \((1.3)\) of the hypercontractivity; some related inequalities are also presented.

In the sequel, we denote by \(x \cdot y\) the inner product of \(x\) and \(y\) in \(\mathbb{R}^d\) and by \(|x|\) the Euclidean norm of \(x\): \(|x| = \sqrt{x \cdot x}\). Given a positive integer \(m\), the symbol \(C^1_b(\mathbb{R}^m)\) stands for the set of bounded \(C^1\)-functions on \(\mathbb{R}^m\) with bounded derivatives. We denote by \(C^1_{b,0}(\mathbb{R}^m)\) the set of functions \(f \in C^1_b(\mathbb{R}^m)\) bounded away from 0: \(\inf_{x \in \mathbb{R}^m} f(x) > 0\).

For a given multivariate function \(g(t, x)\), its subscripts denote partial differentiations: 

\[g_x(t, x) = (\partial g/\partial x)(t, x), \quad g_{tx}(t, x) = (\partial^2 g/\partial x \partial t)(t, x), \quad \text{and so on.}\]

For two functions \(h_1(z), h_2(z)\) in a variable \(z\), we often write \((h_1/h_2)(z)\) to denote \(h_1(z)/h_2(z)\). Other notation will be introduced as needed.

## 2 Preliminaries

In this section, we state and prove preliminary lemmas. For this purpose, we prepare a probability space \((\Omega, \mathcal{F}, P)\) on which a \(d\)-dimensional standard Brownian motion \(W = \{W_t\}_{0 \leq t \leq 1}\) is defined. We denote by \(\{\mathcal{F}_t\}_{0 \leq t \leq 1}\) the (augmentation of) the natural filtration of \(W\). Pick \(f \in L^1(\gamma_d)\) and set

\[M_t \equiv M_t(f) := \mathbb{E}[f(W_1)|\mathcal{F}_t], \quad 0 \leq t \leq 1.\]

Then by the Markov property of \(W\),

\[M_t = \mathbb{E}[f(W_{1-t} + x)]|_{x=W_t} \quad \text{a.s.}\]

for any \(0 \leq t \leq 1\), which leads to the identity in law:

\[(Q_t f, \gamma_d) \overset{(d)}{=} (M_{e^{-ut}}, \mathbb{P}) \quad \text{for every } t \geq 0. \tag{2.1}\]
Our proof of Proposition 1.1 and Theorem 1.1 utilizes this identity.

For given \(-\infty \leq l < r \leq \infty\), let \(u(t,x), (t,x) \in (0,1] \times (l,r)\), be a nonnegative \(C^{1,2}\)-function such that \(u_x\) does not vanish. In the remainder of this section, we let \(f\) be in \(C^1_b(\mathbb{R}^d)\) and suppose that \(f\) fulfills

\[
l < \inf_{x \in \mathbb{R}^d} f(x) \leq \sup_{x \in \mathbb{R}^d} f(x) < r.
\]

In the subsequent sections, we take either \(-\infty\) or 0 for \(l\) and \(\infty\) for \(r\). In order to develop the process \(\{u(t,M_t)\}_{0 \leq t \leq 1}\) by applying Itô's formula, we use the martingale representation for \(\{M_t\}_{0 \leq t \leq 1}\). Set a \(d\)-dimensional process \(\theta = \{\theta_t\}_{0 \leq t \leq 1}\) by

\[
\theta_t = \mathbb{E} \left[ \nabla f(W_{1-t} + x) \right] |_{x=W_t}.
\]

Lemma 2.1. We have \(\mathbb{P}\)-a.s.,

\[
M_t = \mathbb{E} \left[ f(W_1) \right] + \int_0^t \theta_s \cdot dW_s \quad \text{for all } 0 \leq t \leq 1.
\]

The above lemma is an immediate consequence of the Clark-Ocone formula. Because that formula will be used again, we provide its rough formulation as introducing the necessary notation; we do this in a slightly general situation although what we use repeatedly is the simplest case with \(m = 1\): let \(F(W)\) be a functional of \(W\) of the form

\[
F(W) = \phi(W_{t_1}, \ldots, W_{t_m})
\]

for some positive integer \(m\) and \(0 \leq t_1, \ldots, t_m \leq 1\), and for some \(\phi \in C^1_b(\mathbb{R}^{d \times m})\). We denote by \(DF(W)\) the Malliavin derivative of \(F(W)\), which is expressed as

\[
D_s F(W) = \sum_{i=1}^m 1_{[0,t_i]}(s) \nabla_{x_i} \phi(W_{t_1}, \ldots, W_{t_m}), \quad 0 \leq s \leq 1.
\]

Then the Clark-Ocone formula states that \(\mathbb{P}\)-a.s.,

\[
\mathbb{E} \left[ F(W) | \mathcal{F}_t \right] = \mathbb{E} [F(W)] + \int_0^t \mathbb{E} \left[ D_s F(W) | \mathcal{F}_s \right] \cdot dW_s
\]

for all \(0 \leq t \leq 1\). For more detailed accounts of the formula, see, e.g., [6, Appendix E], [9, Section 1.3].

Proof of Lemma 2.1. Applying (2.5) to \(f(W_1)\), we have

\[
M_t = \mathbb{E} \left[ f(W_1) \right] + \int_0^t \mathbb{E} \left[ \nabla f(W_1) | \mathcal{F}_s \right] \cdot dW_s.
\]

By the Markov property of \(W\),

\[
\mathbb{E} \left[ \nabla f(W_1) | \mathcal{F}_s \right] = \mathbb{E} \left[ \nabla f(W_{1-s} + x) \right] |_{x=W_s} \quad \text{a.s.,}
\]

which ends the proof due to the definition (2.3) of \(\theta\).
By (2.4) and by Itô’s formula, we have \( P \)-a.s.,
\[
 u(t, M_t) - u(s, M_s) = \int_s^t u_t(\tau, M_\tau) \, d\tau + \int_s^t u_x(\tau, M_\tau) \theta_\tau \cdot dW_\tau + \frac{1}{2} \int_s^t u_{xx}(\tau, M_\tau) |\theta_\tau|^2 \, d\tau \tag{2.6}
\]
for all \( 0 < s \leq t \leq 1 \). As \( f \) is assumed to satisfy (2.2), the stochastic integral above gives rise to a true martingale. Therefore, taking the expectation on both sides of (2.6) and differentiating both sides with respect to \( t \), we obtain the relation
\[
\frac{d}{dt} \mathbb{E}[N_t] = \mathbb{E}[u_t(t, M_t)] + \frac{1}{2} \mathbb{E}[u_{xx}(t, M_t)|\theta|_t^2]. \tag{2.7}
\]
Here and in what follows, we write \( N_t = u(t, M_t), \quad 0 < t \leq 1 \).

Lemma 2.2. It holds that for any \( 0 < t \leq 1 \),
\[
 2u_x(t, \nu(t, \mathbb{E}[N_t])) \frac{d}{dt} \nu(t, \mathbb{E}[N_t]) = -u_t(t, \nu(t, \mathbb{E}[N_t])) + \mathbb{E}[u_t(t, M_t)] + \frac{1}{2} \mathbb{E}[u_{xx}(t, M_t)|\theta|_t^2]. \tag{2.9}
\]
Proof. Observe that due to the relation \( x = u(t, \nu(t, x)) \),
\[
 \nu_t(t, x) = -\frac{u_t}{u_x}(t, \nu(t, x)),
\]
from which we see that
\[
\frac{d}{dt} \nu(t, \mathbb{E}[N_t]) = \nu_t(t, \mathbb{E}[N_t]) + \nu_x(t, \mathbb{E}[N_t]) \frac{d}{dt} \mathbb{E}[N_t]
\]
\[
= -\frac{u_t}{u_x}(t, \nu(t, \mathbb{E}[N_t])) + \frac{1}{u_x(t, \nu(t, \mathbb{E}[N_t]))} \frac{d}{dt} \mathbb{E}[N_t].
\]
Combining this expression with (2.7), we obtain the lemma. \( \square \)

In the next lemma, we assume further that for every \( 0 < t \leq 1 \), \( u_t \) is twice continuously differentiable with respect to the spatial variable \( x \). Set
\[
U(t, x) := \left\{ \left( \frac{u_x}{u_x}, \frac{1}{u_x} \right) \right\}(t, x), \quad (t, x) \in (0, 1) \times (l, r). \tag{2.10}
\]

Lemma 2.3. It holds that for any \( 0 < t \leq 1 \),
\[
2u_x(t, \nu(t, \mathbb{E}[N_t])) \frac{d}{dt} \nu(t, \mathbb{E}[N_t])
\]
\[
= \int_0^1 \mathbb{E}[U(t, \nu(t, \mathbb{E}[N_t] F_s))] |\mathbb{E}[D_s N_t | F_s]|^2 \, ds + \mathbb{E}[u_{xx}(t, M_t)|\theta|_t^2]. \tag{2.11}
\]
Proof. By the definitions of \( v \) and \( N_t \), we may rewrite the integrand of the first term on the right-hand side of (2.7) as

\[
u_t(t, M_t) = u_t(t, v(t, \mathbb{E}[N_t|\mathcal{F}_t])).
\]

We apply Itô’s formula to the process \( u_t(t, v(t, \mathbb{E}[N_t|\mathcal{F}_t])) \), \( 0 \leq \tau \leq 1 \), noting the Clark-Ocone formula (see (2.5)) for \( \mathbb{E}[N_t|\mathcal{F}_\tau] \):

\[
\mathbb{E}[N_t|\mathcal{F}_\tau] = \mathbb{E}[N_t] + \int_0^\tau \mathbb{E}[D_s N_t|\mathcal{F}_s] \cdot dW_s, \quad 0 \leq \tau \leq 1, \ \mathbb{P}\text{-a.s.}
\]

Then it holds that \( \mathbb{P}\text{-a.s.} \)

\[
u_t(t, v(t, \mathbb{E}[N_t|\mathcal{F}_\tau]))
\]

\[
= u_t(t, v(t, \mathbb{E}[N_t])) + \int_0^\tau \frac{u_{tx}(t, v(t, \mathbb{E}[N_t]))}{u_x} \mathbb{E}[D_s N_t|\mathcal{F}_s] \cdot dW_s
\]

\[
+ \frac{1}{2} \int_0^\tau U(t, v(t, \mathbb{E}[N_t|\mathcal{F}_s])) \mathbb{E}[D_s N_t|\mathcal{F}_s]^2 ds
\]

for all \( 0 \leq \tau \leq 1 \). Here we used the fact that \( v_x(t, x) = 1/u_x(t, v(t, x)) \). Taking the expectation on both sides (again the stochastic integral is a true martingale thanks to the boundedness (2.2) of \( f \)) and putting \( \tau = 1 \), we have

\[
\mathbb{E}[u_t(t, M_t)]
\]

\[
= u_t(t, v(t, \mathbb{E}[N_t])) + \frac{1}{2} \int_0^1 \mathbb{E}[U(t, v(t, \mathbb{E}[N_t|\mathcal{F}_s])) \mathbb{E}[D_s N_t|\mathcal{F}_s]^2] ds,
\]

(2.12)

where we used Fubini’s theorem for the last term. Plugging (2.12) into (2.9), we arrive at the conclusion.

3 Proof of Proposition 1.1

This section is devoted to the proof of Proposition 1.1. We start with the proof of the fact that the property that (1.3) holds for any \( f \in L^1(\gamma_d) \) with \( e^f \in L^1(\gamma_d) \), is necessary for (1.1) to hold for any \( p > 1 \) and \( f \in L^p(\gamma_d) \).

Lemma 3.1. (1.1) implies (1.3).

Proof. Fix \( t \geq 0 \) and let \( f \in L^1(\gamma_d) \) be such that \( e^f \in L^1(\gamma_d) \). Fix \( p > 1 \) arbitrarily and set \( g = e^{f/p} \). By Jensen’s inequality, \( Q_t g \geq \exp \{(1/p)Q_t f\} \ \gamma_d\text{-a.e.} \) The hypercontractivity (1.1) applied to \( g \) yields \( \|Q_t g\|_{q(t)} \leq \|e^f\|_1^{1/p} \). Combining these two inequalities, we have

\[
\|\exp \{(1/p)Q_t f\}\|_{q(t)}^p \leq \|e^f\|_1
\]

for any \( p > 1 \). Noting that \( q(t)/p \to e^{2t} \) as \( p \to \infty \), we let \( p \to \infty \) on the left-hand side of the above inequality to conclude (1.3).
We turn to the sufficiency. As mentioned in Section 1, we use the fact (1.3) that the hypercontractivity (1.1) is equivalent to the logarithmic Sobolev inequality (1.2). Thanks to the equivalence, Proposition 1.1 immediately follows once we show the following lemma:

**Lemma 3.2.** (1.3) implies (1.2).

*Proof of Proposition 1.1.* Lemma 3.2 indicates that (1.3) is sufficient for (1.1) to hold. Combining this fact with Lemma 3.1, we have the proposition. □

It remains to prove Lemma 3.2. We recall the well-known fact that taking the derivative of the left-hand side of (1.1) at \( t = 0 \) leads to the logarithmic Sobolev inequality (1.2); the same argument works for (1.3) as well.

*Proof of Lemma 3.2.* By density arguments, it suffices to prove (1.2) for any \( f \in C^{1}_{b}(\mathbb{R}^{d}) \) (for the notation, see the end of Section 1). Pick such an \( f \) arbitrarily and set \( g = 2 \log f \in C^{1}_{b}(\mathbb{R}^{d}) \). In view of (2.1) with \( g \) replacing \( f \) therein, (1.3) is restated as

\[
 t \log \mathbb{E} \left[ \exp \left\{ (1/t)M_{t}(g) \right\} \right] \leq \log \mathbb{E} \left[ \exp \left\{ M_{1}(g) \right\} \right] \quad \text{for all} \ 0 < t \leq 1,
\]

which in particular entails that

\[
 \frac{d}{dt} \mathbb{V}(t, \mathbb{E}[N_{t}]) \bigg|_{t=1} \geq 0
\]

with \( N_{t} = u(t, M_{t}(g)) \). Here we set \( u(t, x) = \exp(x/t) \) for \((t, x) \in (0, 1] \times \mathbb{R} \) with \( \mathbb{V}(t, x) = t \log x \) the inverse function of \( u(t, \cdot) \) as in the notation of Section 2. Observe that by choosing \( l = -\infty \) and \( r = \infty \), the lemmas in the previous section are applicable to \( g \) and \( u \); in particular, we may apply Lemma 2.2 to see that the last inequality is rewritten as

\[
 -u_{t}(1, \mathbb{V}(1, \mathbb{E}[N_{1}])) + \mathbb{E} \left[ u_{t}(1, M_{1}(g)) \right] + \frac{1}{2} \mathbb{E} \left[ u_{xx}(1, M_{1}(g)) |\nabla g(W_{1})|^{2} \right] \geq 0
\]

by the positivity of \( u_{x} \) and by the definition (2.3) of \( \theta \). Therefore by the definition of \( u \), we obtain

\[
 \mathbb{E} \left[ e^{\theta(W_{1})} \right] \log \mathbb{E} \left[ e^{\theta(W_{1})} \right] - \mathbb{E} \left[ g(W_{1}) e^{\theta(W_{1})} \right] + \frac{1}{2} \mathbb{E} \left[ e^{\theta(W_{1})} |\nabla g(W_{1})|^{2} \right] \geq 0.
\]

Substituting \( g = 2 \log f \) leads to (1.2) and ends the proof. □

### 4 Proof of Theorem 1.1

In this section we prove Theorem 1.1. On account of the identity (2.1) in law, the theorem follows once we show the
**Proposition 4.1.** For a function $c$ on $(0, \infty)$ satisfying the assumptions in Theorem 1.1, set

$$u(t,x) = \int_0^x c(y)^{1/t} \, dy, \quad 0 < t \leq 1, \ x > 0. \quad (4.1)$$

Then for any nonnegative, measurable function $f$ on $\mathbb{R}^d$ satisfying (1.5), we have

$$v(t, E[u(t,M_t(f))]) \leq v(1, E[u(1,M_1(f))]) \quad \text{for all} \ 0 < t \leq 1. \quad (4.2)$$

Here for every $0 < t \leq 1$, we denote by $v(t, \cdot)$ the inverse function of $u(t,x)$, $x > 0$, as in preceding sections.

**Proof of Theorem 1.1.** On noting the identity

$$u(t,x) = u(e^{-2t},x) \quad \text{for all} \ t \geq 0 \ \text{and} \ x > 0,$$

with a common function $c$, the assertion of Theorem 1.1 is immediate from that of Proposition 4.1 and the identity (2.1).

It remains to prove Proposition 4.1. To this end, we assume first that $f$ is in $C^1_{b,0}(\mathbb{R}^d)$. This assumption will be removed later by density arguments. Note that the assumption on $f$ and the definition of $u$ allow us to apply the lemmas in Section 2 by choosing $l = 0$ and $r = \infty$; in particular, the identity (2.11) holds true for the above pair of $f$ and $u$, from which we start the proof of the proposition. Set

$$\varphi(t,x) := -\frac{1}{U(t,v(t,x))}, \quad 0 < t \leq 1, \ x > 0. \quad (4.3)$$

**Lemma 4.1.** For every $0 < t \leq 1$, the function $(0, \infty) \ni x \mapsto \varphi(t,x)$ is positive and concave.

**Proof.** Noting $(u_x/u_x)(t,x) = -(1/t^2) \log c(x)$, we see that $U$ is expressed as

$$U(t,x) = -\frac{1}{t^2} \frac{c'(x)}{c(x)} \frac{1}{\{c(x)\}^{1/t}}$$

by the definition (2.10) of $U$. Therefore we have the expression

$$\varphi(t,x) = t^2 \frac{c(v(t,x))}{c'(v(t,x))} \{c(v(t,x))\}^{1/t}. \quad (4.4)$$

The positivity is obvious because $c$ and $c'$ are positive. To check the concavity, note that $\varphi(t,x)$ is both right- and left-differentiable with respect to $x$ since $c/c'$ is concave and $v(t,x)$ is strictly increasing and differentiable with respect to $x$; in fact, if we denote by $(c/c')_+$ (resp. $(c/c')_-$) its right-(resp. left-)derivative, then

$$\frac{1}{t^2} \lim_{h \to 0^+} \frac{\varphi(t,x+h) - \varphi(t,x)}{h} = \left(\frac{c}{c'}\right)_+ (v(t,x)) v_x(t,x) c(v(t,x))^{1/t}$$

$$+ \frac{c}{c'}(v(t,x)) \times \frac{1}{t} c(v(t,x))^{1/t-1} c'(v(t,x)) v_x(t,x)$$

$$= \left(\frac{c}{c'}\right)_+ (v(t,x)) + \frac{1}{t}, \quad (4.5)$$
where the second equality follows from the fact that \( v_x(t, x) = c(v(t, x))^{-1/t} \); in the same way,

\[
\frac{1}{t^2} \lim_{h \to 0^-} \frac{\varphi(t, x + h) - \varphi(t, x)}{h} = \left( \frac{c}{c'} \right)'(v(t, x)) + \frac{1}{t^2}. \tag{4.6}
\]

From these identities, the concavity follows because their right-hand sides are nonincreasing in \( x \) by the concavity assumption on \( c/c' \).

Thanks to the above lemma, we have the following lower bound for the expectation in the first term on the right-hand side of (2.11):

**Lemma 4.2.** It holds that for every \( 0 < t \leq 1 \) and \( 0 \leq s \leq 1 \),

\[
\mathbb{E} \left[ U(t, \varphi(t, \mathbb{E}[N_t|\mathcal{F}_s])) \right] \leq \mathbb{E} \left[ |D_s N_t|^2 \right]. \tag{4.7}
\]

**Proof.** By the definition (4.3) of \( \varphi \), the left-hand side of (4.7) is written as

\[
-\mathbb{E} \left[ \frac{\mathbb{E}[D_s N_t | \mathcal{F}_s]^2}{\varphi(t, \mathbb{E}[N_t|\mathcal{F}_s])} \right]. \tag{4.8}
\]

Observe the identity

\[
\mathbb{E} \left[ \varphi(t, N_t) \left| \frac{D_s N_t}{\varphi(t, N_t)} \right| \mathbb{E}[D_s N_t | \mathcal{F}_s] \right]^2 \mid \mathcal{F}_s = \mathbb{E} \left[ \frac{|D_s N_t|^2}{\varphi(t, N_t)} \mid \mathcal{F}_s \right] - 2 \mathbb{E} \left[ \frac{|D_s N_t| |F_s|^2}{\varphi(t, \mathbb{E}[N_t|\mathcal{F}_s])} \right] + \mathbb{E} \left[ \varphi(t, N_t) | \mathcal{F}_s \right] \mathbb{E} \left[ \frac{|D_s N_t|^2}{\varphi(t, \mathbb{E}[N_t|\mathcal{F}_s])} \right]^2 \quad \text{a.s.} \tag{4.9}
\]

Note that by Lemma 4.1 and by the conditional Jensen inequality,

\[
\mathbb{E} \left[ \varphi(t, N_t) | \mathcal{F}_s \right] \leq \varphi (t, \mathbb{E}[N_t | \mathcal{F}_s]) \quad \text{a.s.}
\]

Plugging this estimate into the third term on the right-hand side of the above identity and using the positivity of \( \varphi \), we have

\[
0 \leq \mathbb{E} \left[ \frac{|D_s N_t|^2}{\varphi(t, N_t)} \mid \mathcal{F}_s \right] - \mathbb{E} \left[ \frac{|D_s N_t|^2}{\varphi(t, \mathbb{E}[N_t|\mathcal{F}_s])} \right]^2 \quad \text{a.s.} \tag{4.10}
\]

Taking the expectation, we see that (4.8) is bounded from below by the right-hand side of (4.7), which ends the proof.

We are in a position to prove Proposition 4.1.

**Proof of Proposition 4.1.** First let \( f \) be as above, that is, suppose \( f \in C^1_{b,0}(\mathbb{R}^d) \). By the definition (2.8) of \( N_t \) and by the chain rule for the Malliavin derivative \( D \),

\[
D_s N_t = u_x(t, M_t)D_s M_t.
\]
Since $M_t$ is written as $M_t = \mathbb{E}[f(W_{1-t} + x)]|_{x=W_t}$, we see that

$$D_s M_t = 1_{[0,t]}(s) \mathbb{E}[\nabla f(W_{1-t} + x)]|_{x=W_t}$$

(recall $\nabla f$ is also assumed to be bounded), hence

$$= 1_{[0,t]}(s) \theta_t$$

by the definition (2.3) of $\theta_t$. By combining these and by the definition of $N_t$, the right-hand side of (1.7) is expressed as

$$-1_{[0,t]}(s) \mathbb{E} \left[ \frac{(u_x(t, M_t))^2}{\varphi(t, u(t, M_t))} |\theta_t|^2 \right].$$

By the last expression and by Lemmas 2.3 and 4.2, we have for any $0 < t \leq 1$,

$$2u_x(t, v(t, \mathbb{E}[N_t])) \frac{d}{dt} v(t, \mathbb{E}[N_t]) \geq \mathbb{E} \left[ \left\{ -t \frac{(u_x(t, M_t))^2}{\varphi(t, u(t, M_t))} + u_{xx}(t, M_t) \right\} |\theta_t|^2 \right]. \quad (4.11)$$

Recall (4.3) to note that

$$\varphi(t, u(t, x)) = t^2 \frac{c(x)}{c'(x)} \{c(x)\}^{1/t}.$$ 

We also note the expressions of $u_x$ and $u_{xx}$ in terms of $c$:

$$u_x(t, x) = c(x)^{1/t}, \quad u_{xx}(t, x) = \frac{1}{t} \frac{c'(x)}{c(x)} \{c(x)\}^{1/t}.$$ 

From these expressions, it follows that for all $0 < t \leq 1$ and $x > 0$,

$$-t \frac{(u_x(t, x))^2}{\varphi(t, u(t, x))} + u_{xx}(t, x) = \left( -t \times \frac{1}{t^2} + \frac{1}{t} \right) \frac{c'(x)}{c(x)} \{c(x)\}^{1/t}$$

$$= 0,$$

and hence by (4.11),

$$\frac{d}{dt} v(t, \mathbb{E}[N_t]) \geq 0 \quad \text{for any} \ 0 < t \leq 1,$$

because $u_x(t, x)$ is positive for all $0 < t \leq 1$ and $x > 0$. Consequently, we have proven (4.2) when $f \in C_{b,0}^1(\mathbb{R}^d)$.

The proof of Proposition 4.1 is completed by density arguments. To this end, let a measurable function $f$ on $\mathbb{R}^d$ be such that

$$\varepsilon \leq f \leq K \quad \gamma_d\text{-a.e.} \quad (4.12)$$
for some $0 < \varepsilon \leq K < \infty$. Then we may choose a sequence $\{f_n\}_{n=1}^{\infty} \subset C_b^1(\mathbb{R}^d)$ such that
\[
\lim_{n \to \infty} \mathbb{E} [ |f_n(W_1) - f(W_1)| ] = 0
\] (4.13)
and $\varepsilon \leq f_n(x) \leq K$ for all $n \geq 1$ and $x \in \mathbb{R}^d$. To see this, we fix $n$ arbitrarily. Since any measurable function on $\mathbb{R}^d$ is approximated by continuous functions in the sense of $\gamma_d$-a.e. convergence (see, e.g., [4, Theorem V.16 (a)]), we may pick a continuous function $g$ in such a way that
\[
\|f - g\|_1 < n^{-1} \quad \text{and} \quad \varepsilon \leq \inf_{x \in \mathbb{R}^d} g(x) \leq \sup_{x \in \mathbb{R}^d} g(x) \leq K.
\]

Convoluting $g$ with a mollifier on $\mathbb{R}^d$, we find a $\tilde{g}$ in $C_b^1(\mathbb{R}^d)$ (in fact, in $C_b^\infty(\mathbb{R}^d)$) such that
\[
\|g - \tilde{g}\|_1 < n^{-1} \quad \text{and} \quad \varepsilon \leq \inf_{x \in \mathbb{R}^d} \tilde{g}(x) \leq \sup_{x \in \mathbb{R}^d} \tilde{g}(x) \leq K.
\]
Taking $f_n = \tilde{g}$, we have a desired sequence since $\|f - f_n\|_1 < 2n^{-1}$ by triangular inequality. We have already seen that (4.12) holds true for each $f_n$:
\[
\nu(t, \mathbb{E} [u(t, \mathbb{E} [f_n(W_1)|\mathcal{F}_t])] \leq \nu(1, \mathbb{E} [u(1, f_n(W_1))]).
\] (4.14)
By the definition of $u$ and by the nonnegativity of $c'$, it holds that for any $0 < t \leq 1$,
\[
|u(t, x_1) - u(t, x_2)| \leq c(K)^{1/4} |x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in [\varepsilon, K].
\] (4.15)
Therefore we have the convergence
\[
\mathbb{E} [u(t, \mathbb{E} [f_n(W_1)|\mathcal{F}_t])] - \mathbb{E} [u(t, \mathbb{E} [f(W_1)|\mathcal{F}_t])]
 \leq \mathbb{E} [ |u(t, \mathbb{E} [f_n(W_1)|\mathcal{F}_t]) - u(t, \mathbb{E} [f(W_1)|\mathcal{F}_t])|]
 \leq c(K)^{1/4} \mathbb{E} [ |f_n(W_1) - f(W_1)|]
 \xrightarrow{n \to \infty} 0
\]
by (4.13), which is true for any $0 < t \leq 1$. Here for the third line, we used (4.15) as well as the conditional Jensen inequality when $t < 1$. Letting $n \to \infty$ on both sides of (4.14), we have
\[
\nu(t, \mathbb{E} [u(t, \mathbb{E} [f(W_1)|\mathcal{F}_t])]) \leq \nu(1, \mathbb{E} [u(1, f(W_1))])
\] (4.16)
for $f$ satisfying (4.12) for some $\varepsilon$ and $K$.

For a general nonnegative and measurable $f$ satisfying (1.5), we set
\[
f_{m,n} := \min \{ \max \{ f, 1/m \} , n \}
\]
for positive integers $m, n$. Then we have (4.13) for these $f_{m,n}$’s. Appealing to the (conditional) monotone convergence theorem, we first let $m \to \infty$ and then $n \to \infty$ to conclude the proof. \qed
As noted in Section 1, two choices $x^{p-1}$ ($p > 1$) and $e^x$ for $c(x)$ both fulfill (C). In the remark below, we explain how the hypercontractivity (1.1) and its variant (1.3) are recovered from Theorem 1.1 applied to these $c$’s and reveal a specific feature of the two functions.

**Remark 4.1.** (1) For $c(x) = x^{p-1}$, we have

$$u(t,x) = \frac{1}{q(t)} e^{\theta(t)} \quad \text{and} \quad v(t,x) = \{q(t)x\}^{1/q(t)}, \quad t \geq 0, \ x > 0,$$

with $q(t) = e^{2t(p-1)} + 1$, and hence Theorem 1.1 entails that (1.1) holds for every nonnegative $f \in L^p(\gamma_d)$. If $f \in L^p(\gamma_d)$ is not necessarily nonnegative, then noting the fact that $|Q_t f| \leq Q_t |f|_{\gamma_d}$-a.e. for every $t \geq 0$ (or equivalently, $|M_t(f)| \leq M_t(|f|)$ a.s. for every $0 < t \leq 1$ in the formulation of the present section), we obtain (1.4) for any $f \in L^p(\gamma_d)$. As to the choice $c(x) = e^x$, the corresponding $u$ and $v$ are given respectively by

$$u(t,x) = e^{-2t} \{\exp(e^{2t}x) - 1\} \quad \text{and} \quad v(t,x) = e^{-2t} \log(e^{2t}x + 1)$$

for $t \geq 0$ and $x > 0$. Thus for a nonnegative, measurable $f$ such that $e^f \in L^1(\gamma_d)$, (1.6) is restated as

$$e^{-2t} \log \|\exp(e^{2t}Q_t f)\|_1 \leq \log \|e^f\|_1 \quad \text{for all } t \geq 0,$$

which is nothing but (1.3). For a general $f$ satisfying $f \in L^1(\gamma_d)$ and $e^f \in L^1(\gamma_d)$, set $f_n = \max\{f, -n\}$ for each positive integer $n$. Then Theorem 1.1 applies to $f_n + n$, yielding (1.3) with $f_n$ replacing $f$. Appealing to the monotone convergence theorem, we let $n \to \infty$ on both sides and conclude that (1.3) holds true for any $f \in L^1(\gamma_d)$ satisfying $e^f \in L^1(\gamma_d)$.

(2) In both of the above two cases of $c$, the corresponding $\varphi$ defined by (1.3) is a linear function in $x$, which fact may be deduced from the expressions (4.5) and (1.6). Therefore in those cases, the right-hand side of (4.9) and that of (4.10) coincide. In addition, the inequality (4.7) may also be seen by applying the conditional Schwarz inequality to $|\mathbb{E}[D_s N_t | \mathcal{F}_s]|^2$. We note that the Clark-Ocone formula and the conditional Schwarz inequality are both main ingredients in a simple derivation [3] of the logarithmic Sobolev inequality over Wiener space. We also remark that if $c(x)$ satisfies $(c/c')'' = 0$, then it is identical, up to affine transformation for $x$, with either $x^\alpha$ for some $\alpha \neq 0$ or $e^x$.

The next remark is on the proof of Proposition 4.1.

**Remark 4.2.** (1) Let $f$ be in $C^1_{b,0}(\mathbb{R}^d)$. In view of the identity (4.9), if we set a nonnegative function $\Phi \equiv \Phi_{c,f}$ on $(0, 1]$ by

$$\Phi(t) = \int_0^1 \mathbb{E} \left[ \varphi(t, N_t) \frac{D_s N_t}{\varphi(t, N_t)} - \frac{\mathbb{E}[D_s N_t | \mathcal{F}_s]}{\varphi(t, \mathbb{E}[N_t | \mathcal{F}_s])} \right] ds,$$
then what we have in fact shown in the proof is that
\[ 2u_x(t, v(t, \mathbb{E}[N_t])) \frac{d}{dt} v(t, \mathbb{E}[N_t]) \geq \Phi(t) \]
for any \( 0 < t \leq 1 \). Here equality holds if \( \varphi \) is linear in the spatial variable, which is the case when \( c(x) \) is \( x^{p-1} \) for some \( p > 1 \), as well as when \( c(x) = e^x \) as noted in Remark 4.1(2). In the former case, by dividing both sides of the equality by the quantity
\[ v(t, \mathbb{E}[N_t])u_x(t, v(t, \mathbb{E}[N_t])) = \mathbb{E}\left[\{M_t(f)\}^{(p-1)/t+1}\right], \]
the following identity is easily deduced on account of (2.1):
\[
\|Q_tf\|_{q(t)} = \|f\|_p \exp \left\{ - \int_0^t \frac{e^{-2\tau}}{\|Q_{\tau}f\|_{q(\tau)}} \Phi(e^{-2\tau}) \, d\tau \right\} \quad \text{for all } t \geq 0;
\]
in the latter case \( c(x) = e^x \), a similar identity also holds:
\[
\|\exp (Q_tf)\|_{e^2t} = \|f\|_1 \exp \left\{ - \int_0^t \frac{e^{-2\tau}}{\|\exp (Q_{\tau}f)\|_{e^2\tau}}} \Phi(e^{-2\tau}) \, d\tau \right\} \quad \text{for all } t \geq 0.
\]
(2) Let \( u(t, x), 0 < t \leq 1, x > 0, \) be a generic, positive and smooth function with \( u_x > 0 \). The derivation of Proposition 4.1 hinges upon the fact that we are able to solve the following pair of equations in \((0, 1) \times (0, \infty)\):
\[
\begin{cases}
\psi(t, u(t, x))U(t, x) = -\frac{1}{t}, \\
\psi(t, u(t, x)) \frac{u_{xx}(t, x)}{(u_x(t, x))^2} = 1,
\end{cases}
\]
where \( U \) is defined by (2.10) and \( \psi \) is also an unknown, positive function such that \( \psi(t, \cdot) \) is required to be concave for every \( 0 < t \leq 1 \). Indeed, by these equations, \( u \) must satisfy
\[
tU(t, x) + \frac{u_{xx}(t, x)}{(u_x(t, x))^2} = 0 \quad \text{in } (0, 1) \times (0, \infty),
\]
which equation is rephrased as
\[
\frac{\partial^2}{\partial x \partial t} \{t \log u_x(t, x)\} = 0.
\]
Then \( u_x \) is expressed, up to multiple of a positive function in \( t \), as
\[
u_x(t, x) = e^{C(x)/t}
\]
with \( C \) a differentiable function in \( x \); the associated \( \psi \) is given by the product of a positive function in \( t \) and \((1/C''(\psi(t, x)) \exp\{C(\psi(t, x))/t\}, \) which is found to be concave in \( x \) when \( C' > 0 \) and \( 1/C' \) is concave. Here \( v \) is the inverse function of \( u \) in the spatial variable as in the notation of Section 2.
We give examples of functions $c$ satisfying (C) and show consequences of Theorem 1.1 corresponding to them.

**Example 4.1.** (1) For two exponents $\alpha, \beta$ satisfying $\alpha + \beta \geq 1$ and $0 < \beta \leq 1$, we take

$$c(x) = x^{\alpha + \beta - 1} \exp (x^\beta), \quad x > 0.$$ 

If we write $\rho = \alpha + \beta - 1$, then

$$c'(x) = (\rho x^{\rho - 1} + \beta x^{\rho + \beta - 1}) \exp (x^\beta),$$

which is positive for all $x > 0$ when $\rho \geq 0$ and $\beta > 0$. Noting

$$\frac{c}{c'}(x) = \frac{x}{\rho + \beta x^\beta},$$

we find that

$$\left\{ \frac{c}{c'}(x) \right\}' = \frac{\rho + (1 - \beta)y}{(\rho + y)^2} \bigg|_{y=\beta x^\beta}.$$ 

The function $\{\rho + (1 - \beta)y\}/(\rho + y)^2$ in $y > 0$ is nonincreasing when $\beta \leq 1$ and $\rho(1 + \beta) \geq 0$, and hence under the condition imposed on $\alpha$ and $\beta$, the above $c$ satisfies (C). Observe that by L’Hôpital’s rule, the corresponding $u$ admits the asymptotics

$$u(t, x) \sim \frac{e^{-2t}}{\beta} x^{2(\alpha + \beta - 1) - \beta + 1} \exp \left( e^{2t} x^\beta \right)$$

as $x \to \infty$ for every $t \geq 0$. Here and below, the notation $\sim$ indicates that the ratio of both sides in the equation converges to 1 when $x \to \infty$. As a consequence, we deduce from Theorem 1.1 that the following implication is true: for any nonnegative, measurable function $f$ on $\mathbb{R}^d$,

$$f^{\alpha} \exp \left( f^\beta \right) \in L^1(\gamma_d) \Rightarrow (Q_t f)^{\alpha} \exp \left( e^{2t} (Q_t f)^\beta \right) \in L^1(\gamma_d), \forall t \geq 0.$$ 

(2) For positive reals $\alpha$ and $\beta$, take

$$c(x) = \frac{(x + a)^\alpha}{\log^\beta (x + a)}, \quad x > 0,$$

where $a$ is a constant satisfying $a \geq e^{2+\rho}$ with $\rho := \beta/\alpha$. Then

$$c'(x) = \frac{\alpha(x + a)^{\alpha - 1}}{\log^{\beta+1} (x + a)} \{\log (x + a) - \rho\} > 0 \quad \text{for all} \ x > 0,$$

and

$$\frac{c}{c'}(x) = \frac{x + a}{\alpha} + \frac{\rho}{x + a} \cdot \frac{x + a}{\log (x + a) - \rho}.$$
is concave on $(0, \infty)$. Indeed,
\[
\left\{ \frac{x + a}{\log(x + a) - \rho} \right\}' = \frac{y - 1}{y^2} \bigg|_{y = \log(x + a) - \rho}, \quad x > 0,
\]
the function $(y - 1)/y^2$ being decreasing on $(2, \infty)$. Therefore Theorem 1.1 applies to the above choice of $c$ as well. Since the corresponding $u$ admits the asymptotics
\[
u(t, x) \sim \frac{1}{e^{2\alpha + 1}} \cdot \frac{x}{2} e^{2\alpha + 1} \log e^{2\alpha + 1} x \text{ as } x \to \infty
\]
for every $t \geq 0$, there holds the following implication: for any nonnegative, measurable function $f$ on $\mathbb{R}^d$,
\[
\frac{f^{\alpha + 1}}{log^{\alpha + 1}(f + b)} \in L^1(\gamma_d) \Rightarrow \frac{(Q_t f)^{\alpha + 1}}{log^{\alpha + 1}(Q_t f + b)} \in L^1(\gamma_d), \quad \forall t \geq 0.
\]
Here $b$ is any constant greater than 1.

We end this section by providing a generalization of the logarithmic Sobolev inequality (1.2) as a corollary to Theorem 1.1.

**Corollary 4.1.** For a function $c : (0, \infty) \to (0, \infty)$ satisfying the assumptions in Theorem 1.1 set
\[
G(x) = \int_0^x c(y) dy \quad \text{and} \quad H(x) = \int_0^x c(y) \log c(y) dy
\]
for $x > 0$. Then for any $f \in C^1_{b,0}(\mathbb{R}^d)$, we have
\[
\int_{\mathbb{R}^d} H(f) d\gamma_d \leq \frac{1}{2} \int_{\mathbb{R}^d} c'(f) |\nabla f|^2 d\gamma_d + H \circ G^{-1} (\|G(f)\|_1).
\]
Here $G^{-1}$ is the inverse function of $G$.

**Remark 4.3.** It is plausible that the inequality (4.17) would be extended to the class of functions $f$ for which every term in the inequality makes sense, however, we do not pursue it here.

The proof of Corollary 4.1 proceeds along the same lines as in the proof of Lemma 3.2.

**Proof of Corollary 4.1.** For the proof, we use Proposition 1.1, the equivalent statement of Theorem 1.1. We see from (4.2) that
\[
\frac{d}{dt} \nu(t, E[u(t, M_t(f))]) \bigg|_{t=1} \geq 0,
\]
which is rewritten, by Lemma 2.2 as
\[
-u_t(1, \nu(1, E[N_1])) + E[u_t(1, M_1(f))] + \frac{1}{2} E\left[u_{xx}(1, M_1(f)) |\nabla f(W_1)|^2\right] \geq 0
\]
due to the definition (2.3) of \( \theta \) and the positivity of \( u_x \). The last inequality is nothing but (4.17) because of the relations \( u_t(1,\cdot) = H, u(1,\cdot) = G \) and \( v(1,\cdot) = G^{-1} \) by the definitions of \( u, H \) and \( G \), as well as because of the identities \( M_1(f) = f(W_1) \) and \( N_1 = u(1,f(W_1)) = G(f(W_1)) \).

If we choose \( x^{p-1} (p > 1) \) or \( e^x \) for \( c(x) \) in (4.17), then (1.2) is recovered; details are left to the reader.

5 On the reverse hypercontractivity

This section concerns the reverse hypercontractivity of the Ornstein-Uhlenbeck semi-group \( Q \). We begin with the following proposition, which is proven by modifying slightly the proof of Theorem 4.1.

**Proposition 5.1.** Let a positive function \( c \) on \((0, \infty)\) be in \( C^1((0, \infty)) \) and satisfy

\[
\frac{c'}{c} < 0, \quad \text{is convex on } (0, \infty) \quad \text{and} \quad \lim_{x \to 0^+} c(x) < \infty. \tag{C'}
\]

We set the function \( u(t, x) \), \( t \geq 0, x > 0 \), by (1.4):

\[
u(t, x) = \int_0^x c(y)e^{2t} dy.
\]

Then for any \( f \in C^1_{b,0}(\mathbb{R}^d) \), we have

\[
\nu(t, \|u(t,Q_tf)\|_1) \geq \nu(0, \|u(0,f)\|_1) \quad \text{for all } t \geq 0.
\]

Here for every \( t \geq 0 \), the function \( \nu(t, \cdot) \) is the inverse function of \( u(t,x) \), \( x > 0 \).

In (C), the last condition is put to ensure the finiteness of \( u \). By the identity (2.1) in law, the assertion of the proposition is restated as

\[
\nu(t, \mathbb{E}[u(t,M_t(f))]) \geq \nu(1, \mathbb{E}[u(1,M_t(f))]) \quad \text{for all } 0 < t \leq 1,
\]

for each \( f \in C^1_{b,0}(\mathbb{R}^d) \). Here \( u \) is defined by (4.1) with \( c \) satisfying (C') and \( v \) denotes the inverse function of \( u \) in the spatial variable as before.

**Proof of Proposition 5.1.** Observe that with taking \( l = 0 \) and \( r = \infty \), the lemmas in Section 2 apply to the above choice of \( f \) and \( u \), and hence Lemma 2.3 is valid. We shall see that the right-hand side of (2.11) in that lemma does not exceed 0, which entails

\[
\frac{d}{dt} \nu(t, \mathbb{E}[N_t]) \leq 0 \quad \text{for any } 0 < t \leq 1,
\]

by the positivity of \( u_x \). To this end, recall the definition (4.3) of \( \varphi \); its expression (4.4) in terms of \( c \) is valid in the present case as well, and in particular it reveals, by (C') and
by repeating the same argument as in the proof of Lemma 4.1 that \( \varphi \) is negative and is a convex function in \( x \) for every \( 0 < t \leq 1 \). Remembering the identity (4.9) in the proof of Lemma 4.2 and noting that
\[
E \left[ \frac{D_s N_t^2}{\varphi(t, N_t)} \left| F_s \right. \right] \geq \frac{E[D_s N_t]}{\varphi(t, E[N_t]|F_s])} \text{ a.s.}
\]
in place of (4.10), due to the negativity of \( \varphi \). By the definition of \( \varphi \), the last inequality leads to (4.7) with the reversed inequality sign. The rest of the proof for (5.3) proceeds along the same lines as in the first part of the proof of Proposition 4.1. Therefore (5.2) follows and we obtain the proposition.

The next proposition shows that (5.1) unifies two properties of \( Q \), the reverse hypercontractivity (1.8) and the exponential variant (1.3) of the hypercontractivity.

**Proposition 5.2.** The property (5.1) implies (1.8) and (1.3).

In view of (2.1), in order to prove the assertion, it suffices to show that we may derive from (5.2) the following: given \( \alpha > 0 \),
\[
E \left[ \left( \frac{1}{M_t(f)} \right)^{\rho_\alpha(t)} \right]^{1/\rho_\alpha(t)} \leq E \left[ \left( \frac{1}{M_t(f)} \right)^{\alpha} \right]^{1/\alpha} \text{ for all } 0 < t \leq 1,
\]
for every a.e. nonnegative \( f \in L^1(\gamma_d) \) satisfying \( 1/f \in L^\alpha(\gamma_d) \), as well as
\[
E \left[ \exp \left\{ (1/t)M_t(f) \right\} \right] \leq E \left[ \exp \left\{ M_t(f) \right\} \right] \text{ for all } 0 < t \leq 1,
\]
for every \( f \in L^1(\gamma_d) \) satisfying \( e^f \in L^1(\gamma_d) \). Here we set \( \rho_\alpha(t) = (\alpha+1)/t - 1 \) in (5.4).

**Proof of Proposition 5.2.** We begin with the proof of (5.4). For this purpose, let \( f \) be in \( C^1_{b,0}(\mathbb{R}^d) \) first. We pick a constant \( \kappa > 0 \) in such a way that \( \kappa < \inf_{x \in \mathbb{R}^d} f(x) \) and take
\[
c(x) = \frac{1}{(x+\kappa)^{\alpha+1}}, \quad x > 0.
\]
Then the condition (C) is fulfilled; in fact, \((c/c')(x) = -(x+\kappa)/(\alpha+1)\), and hence \((c/c'') = 0\). Therefore we have (5.2) with this choice of \( c \). The corresponding \( u \) and \( v \) are given respectively by
\[
u(t, x) = \left\{ \kappa^{-\rho_\alpha(t)} - \rho_\alpha(t)x \right\}^{-1/\rho_\alpha(t)} - \kappa, \quad 0 < t \leq 1, 0 < x < \kappa^{-\rho_\alpha(t)}/\rho_\alpha(t).\]
From these expressions, we see that applying (5.2) to \( f - \kappa \) yields (5.4) for every \( f \in C^1_b(\mathbb{R}^d) \).

Next we assume that \( f \) is in \( L^1(\gamma_d) \) and satisfies
\[
\varepsilon := \operatorname{ess inf}_{x \in \mathbb{R}^d} f(x) > 0.
\]

Then we may choose a sequence \( \{f_n\}_{n=1}^{\infty} \subset C^1_b(\mathbb{R}^d) \) in such a way that
\[
\lim_{n \to \infty} \mathbb{E} \left[ |f_n(W_1) - f(W_1)| \right] = 0 \tag{5.6}
\]
and \( \inf_{x \in \mathbb{R}^d} f_n(x) \geq \varepsilon \) for all \( n \geq 1 \) (see, e.g., the middle part of the proof of Proposition 4.1). We have seen in the previous step that (5.4) holds true for each \( f_n \):
\[
\mathbb{E} \left[ \left( \frac{1}{\mathbb{E}[f_n(W_1)|\mathcal{F}_t]} \right)^{\rho_\alpha(t)^{-1}} \right]^{1/\rho_\alpha(t)} \leq \mathbb{E} \left[ \left( \frac{1}{f_n(W_1)} \right)^{\alpha} \right]^{1/\alpha} \tag{5.7}
\]
for all \( 0 < t \leq 1 \). By the inequality \(|(1/x_1)^\alpha - (1/x_2)^\alpha| \leq \alpha|x_1 - x_2|/\varepsilon^{\alpha+1}\) for \( x_1, x_2 \geq \varepsilon \), and by (5.6), we have the convergence of expectations
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{1}{f_n(W_1)} \right)^{\alpha} \right] = \mathbb{E} \left[ \left( \frac{1}{f(W_1)} \right)^{\alpha} \right]
\]
as to the right-hand side of (5.7). The same reasoning combined with the conditional Jensen inequality yields the convergence as \( n \to \infty \) of the left-hand side of (5.7) to the expression with \( f_n \) replaced by \( f \). Hence we have obtained (5.4) for \( f \in L^1(\gamma_d) \) with \( \operatorname{ess inf}_{x \in \mathbb{R}^d} f(x) > 0 \).

Finally, let \( f \) be an a.e. positive function in \( L^1(\gamma_d) \) satisfying \( 1/f \in L^\alpha(\gamma_d) \). For every positive integer \( n \), set
\[
f_n := \max \{ f, 1/n \}.
\]
Then we have (5.7) for these \( f_n \)'s as well. Since each \( f_n \) is nonnegative and dominated by the integrable function \( f + 1 \), it holds that
\[
\lim_{n \to \infty} \mathbb{E}[f_n(W_1)|\mathcal{F}_t] = \mathbb{E}[f(W_1)|\mathcal{F}_t] \quad \text{a.s.}
\]
by the conditional dominated convergence theorem. Letting \( n \to \infty \) on both sides of (5.7), we appeal to the monotone convergence theorem to conclude the validity of (5.4) for any a.e. positive \( f \in L^1(\gamma_d) \).

We turn to the proof of (5.5). First we pick \( f \in C^1_b(\mathbb{R}^d) \) and let \( \kappa \geq 0 \) be a constant such that \( \inf_{x \in \mathbb{R}^d} \{-f(x)\} > -\kappa \). We may take \( c(x) = e^{-(x-\kappa)} \), \( x > 0 \), in Proposition 5.1; indeed, \( c/c' \) is identically equal to \(-1\) and thus the condition (C') is fulfilled. Then the
inequality (5.2) applied to $-f + \kappa \in C^1_{b,0}(\mathbb{R}^d)$ entails (5.5) for every $f \in C^1_b(\mathbb{R}^d)$, due to the expressions of the corresponding $u$ and $v$:

$$u(t, x) = t \left\{ e^{\kappa/t} - e^{-(x-\kappa)/t} \right\}, \quad 0 < t \leq 1, \ x > 0,$$

$$v(t, x) = -t \log \left( 1 - e^{-\kappa/t} x/t \right), \quad 0 < t \leq 1, \ 0 < x < te^{\kappa/t}.$$

Next let $f$ be in $L^1(\gamma_d)$ and satisfy $ef \in L^1(\gamma_d)$. We write $\{f_n\}_{n=1}^{\infty}$ for a sequence in $C^1_b(\mathbb{R}^d)$ that approximates $f$ in $L^1(\gamma_d)$. If we suppose that $K := \text{ess sup}_{x \in \mathbb{R}^d} f(x) < \infty$, then we may take the above sequence in such a way that

$$\sup_{x \in \mathbb{R}^d} f_n(x) \leq K$$

for all $n$ (cf. Proof of Proposition 4.1). We have observed in the previous step that for each $f_n$,

$$\mathbb{E} \left[ \exp \left\{ \left(1/t\right) \mathbb{E} \left[ f_n(W_1)|\mathcal{F}_t \right] \right\} \right] \leq \mathbb{E} \left[ e^{f_n(W_1)} \right]$$

(5.8)

holds for all $0 < t \leq 1$. By noting that $|e^{x_1} - e^{x_2}| \leq e^K |x_1 - x_2|$ for $x_1, x_2 \leq K$,

$$\left| \mathbb{E} \left[ e^{f_n(W_1)} \right] - \mathbb{E} \left[ e^{f(W_1)} \right] \right| \leq e^K \mathbb{E} \left[ |f_n(W_1) - f(W_1)| \right] \xrightarrow{n \to \infty} 0$$

since $f_n$ approximates $f$ in $L^1(\gamma_d)$. The same reasoning with $K/t$ replacing $K$, yields the convergence of the expectation in the left-hand side of (5.8), and hence we have

$$\mathbb{E} \left[ \exp \left\{ \left(1/t\right) \mathbb{E} \left[ f(W_1)|\mathcal{F}_t \right] \right\} \right] \leq \mathbb{E} \left[ e^{f(W_1)} \right]$$

when $f$ is bounded from above. For a general $f$ satisfying the assumption, we truncate $f$ from above and use the monotone convergence theorem and its conditional version (or the conditional dominated convergence theorem) to reach the conclusion. □

**Remark 5.1.** As noted in [1, p. 274], the reverse hypercontractivity (1.8) was firstly observed by Borell and Janson [2]† in the name of “converse hypercontractivity.” It is also noted in [1, Remark 5.2.4] that in the same way of the hypercontractivity (1.1) yielding the logarithmic Sobolev inequality (1.2) and vice versa, the reverse hypercontractivity is seen to be equivalent to (1.2) as well.

If we let $\alpha \downarrow 0$ in (1.8), we immediately obtain the following claim, which is of interest itself and which, to our knowledge, has not ever been stated in a clear manner.

†There seems to be some confusion in the literature. In [1], this paper is referred to as [88] in the bibliography, which is found to be Borell’s single-authored paper entitled “Positivity improving operators and hypercontractivity” (Math. Z. 180 (1982), no. 2, 225–234); it is true that in this Borell’s paper, the reverse hypercontractivity is presented with its different proof than the original one, however, the paper cited there for the original proof supposedly needs to be corrected as the reference [2] in the present paper.
Proposition 5.3. Suppose \( f : \mathbb{R}^d \to \mathbb{R} \) is positive \( \gamma_d \)-a.e. and in \( L^1(\gamma_d) \). If \( f \) satisfies
\[
\int_{\mathbb{R}^d} \log^+(1/f) \, d\gamma_d < \infty,
\]
then for every \( t > 0 \), \( 1/Q_t f \) is in \( L^{e^{2t-1}}(\gamma_d) \); in fact, it holds that
\[
\|1/Q_t f\|_{e^{2t-1}} \leq \exp \left( - \int_{\mathbb{R}^d} \log f \, d\gamma_d \right)
\]
for all \( t > 0 \). Here \( \log^+ x := \max\{\log x, 0\} \), \( x > 0 \).

Proof. By Proposition 5.2 and by the identity (2.1) in law, we may start the proof from (5.4) when \( f \in L^1(\gamma_d) \) is bounded away from 0: \( \inf_{x \in \mathbb{R}^d} f(x) > 0 \). Then by the boundedness of \( 1/f \), it is easily seen that as \( \alpha \downarrow 0 \), the left-hand side of (5.4) converges to the expression with \( \rho_\alpha(t) \) replaced by \( \rho_0(t) := 1/t - 1 \) for every \( 0 < t < 1 \). (In fact, what we actually need for the proof is a simple fact that the above-mentioned expression does not exceed the left-hand side of (5.4).) As for the right-hand side of (5.4), we rewrite it into
\[
\exp \left\{ \frac{1}{\alpha} \log \mathbb{E} \left[ (M_1(f))^{-\alpha} \right] \right\}.
\]
Recall \( M_1(f) = f(W_1) \). Observe that for any open and bounded interval \( I \subset (0, \infty) \), the random variable
\[
\sup_{\alpha \in I} (M_1(f))^{-\alpha} |\log M_1(f)|
\]
is integrable thanks to the boundedness of \( 1/f \) and the condition \( f \in L^1(\gamma_d) \). This observation entails that on \( (0, \infty) \), there holds the equality
\[
\frac{d}{d\alpha} \mathbb{E} \left[ (M_1(f))^{-\alpha} \right] = -\mathbb{E} \left[ (M_1(f))^{-\alpha} \log M_1(f) \right]
\]
whose right-hand side converges as \( \alpha \downarrow 0 \) to \(-\mathbb{E} [\log M_1(f)]\) by the dominated convergence theorem. Therefore applying L'Hôpital’s rule, we have
\[
\lim_{\alpha \downarrow 0} \frac{1}{\alpha} \log \mathbb{E} \left[ (M_1(f))^{-\alpha} \right] = -\mathbb{E} [\log M_1(f)],
\]
and hence (5.9), or the right-hand side of (5.4), converges to \( \exp \{-\mathbb{E} [\log M_1(f)]\} \) as \( \alpha \downarrow 0 \). Consequently, we obtain
\[
\mathbb{E} \left[ (M_t(f))^{-\rho_0(t)} \right]^{1/\rho_0(t)} \leq \exp \{-\mathbb{E} [\log M_1(f)]\}
\]
for any \( f \in L^1(\gamma_d) \) which is bounded away from 0.
For a general $f$ satisfying the assumption, we set $f_n = \max \{ f, 1/n \}$ for each positive integer $n$ as in the last step of the proof of [5.4]. By the same reasoning as used there,

$$\lim_{n \to \infty} E \left[ (M_t(f_n))^{-\rho_0(t)} \right] = E \left[ (M_t(f))^{-\rho_0(t)} \right].$$

We also have the convergence

$$\lim_{n \to \infty} E [\log M_1(f_n)] = E [\log M_1(f)]$$

by the monotone convergence theorem because of the domination $\log f_n \leq f_n - 1 \leq f$ for all $n$. Since [5.10] holds for any $f_n$ replacing $f$ as has already been observed, letting $n \to \infty$ on both sides leads to the desired conclusion thanks to [2.1].

We end this paper with a remark on the last proposition.

**Remark 5.2.** Proposition [5.3] may also be proven by taking

$$c(x) = \frac{1}{x + \kappa}, \quad x > 0,$$

in [5.2] with $\kappa$ a positive constant, and by repeating the same argument as in the proof of [5.4]. Moreover, if we choose

$$c(x) = \frac{1}{(x + \kappa)e^{-2s}}, \quad x > 0,$$

for a given $s > 0$, then we may deduce from [5.2] together with density arguments that for every nonnegative $f \in L^1(\gamma_d)$,

$$\|f\|_1 \geq \exp \left\{ \int_{\mathbb{R}^d} \log (Q_sf) \, d\gamma_d \right\} \geq \|Q_tf\|_{1-e^{-2(s-t)}}$$

for all $0 \leq t < s$; in particular, taking $t = 0$ leads to

$$\|f\|_1 \geq \exp \left\{ \int_{\mathbb{R}^d} \log (Q_sf) \, d\gamma_d \right\} \geq \|f\|_{1-e^{-2s}},$$

which is valid for every $s > 0$. In the last two inequalities, the upper bound $\|f\|_1$ is a consequence of Jensen’s inequality.

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