Randall-Sundrum Scenario at Nonzero Temperature

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Abstract

The effect of temperature is investigated in the Randall–Sundrum brane–world scenario. It is shown that for a spacetime ansatz motivated by similarity with AdS/CFT correspondence several features of the model, such as its $Z_2$ symmetry, are not maintained at nonzero temperatures.

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The Randall–Sundrum (RS) scenario was designed to solve the gauge hierarchy problem \[^1\]. Furthermore, the RS model involves a massless mode in the graviton fluctuation equation which is interpreted as a confined gravity on the brane \[^2\]. In view of its various promising features the RS brane–world scenario has been applied to various areas of particle physics such as cosmology \[^3–6\], the cosmological constant problem \[^7,8\], and black holes \[^9–11\]. It is interesting to check whether or not the promising features of the RS model are maintained at nonzero temperatures.

The most interesting feature of the RS scenario is its compatibility with five–dimensional Einstein theory in spite of its $Z_2$ symmetry. We first show that this remarkable feature is not maintained when the temperature is nonzero.

The first step to proceed is to examine how the bulk spacetime is modified at finite temperature. The most appropriate candidate can be constructed by gluing together the two copies of the Schwarzschild-$AdS_5$ spacetime in a $Z_2$-symmetric manner along a boundary which is interpreted as the three-brane world volume:

$$ds^2 = e^{-2k|y|} \left[ -\left( 1 - \frac{U_T^2}{k^4} e^{4k|y|} \right) dt^2 + \sum_{i=1}^3 dx_i dx_i \right] + \frac{dy^2}{1 - \frac{U_T^2}{k^4} e^{4k|y|}} \quad (1)$$

where $U_T$ is the horizon parameter and proportional to the external temperature $T$ defined by $T = U_T/\pi R_{ads}$. The temperature enters in Eq.(1) through the periodic identification of $t \rightarrow t + 1/T$ in the Euclidean time of Schwarzschild-$AdS_5$ space to make the horizon at $U = U_T$ regular \[^12\]. The reason why the spacetime (1) is the most appropriate candidate for a finite temperature RS scenario is its similarity at zero temperature with the $AdS/CFT$ correspondence \[^13,14\]. In fact, recently, much attention has been paid to the similarity of these somewhat different scenarios \[^10,15–20\]. As we will show below, however, Eq.(1) is not a solution of the 5d Einstein equation although the fine-tuning of the 5d cosmological constant $\Lambda$ and the brane tension $\nu_b$ is appropriately adopted.

We now consider the 5d Einstein equation. Although the following analysis can also be applied to the RS two brane scenario \[^1\], we will confine ourselves to the RS one brane scenario \[^2\] for simplicity.
The 5d Einstein equation in this scenario is
\[
R_{MN} - \frac{1}{2}G_{MN}R = -\frac{1}{4M^3} [\Lambda G_{MN} + v_b G_{\mu\nu} \delta_{M}^{\mu} \delta_N^{\nu} \delta(y)]
\] (2)
which is derived by taking the variation of the action
\[
S = \int d^4x \int dy \sqrt{-G} \left[ -\Lambda + 2M^3R - v_b \delta(y) \right].
\] (3)
In Eqs. (2) and (3) \(M, N, \cdots\) and \(\mu, \nu, \cdots\) are 5d spacetime and 4d worldvolume indices respectively. Motivated by Eq.(1) we make the ansatz
\[
ds^2 = e^{-2\sigma(y)} \left[ -f(y)dt^2 + dx^i dx^i \right] + \frac{dy^2}{f(y)}.
\] (4)
Of course, one could choose a different ansatz for the extension to nonzero temperatures. But it is clearly desirable to select in the first place one with a maximum of similarity between the RS scenario and AdS/CFT contexts. Then it is straightforward to show that the Einstein equation (2) yields three independent equations:
\[
6f\sigma'^2 - 3f'\sigma' = 6k^2,
\] (5)
\[
f'' = 4f'\sigma',
\]
\[
3\sigma'' = \frac{v_b}{4M^3f} \delta(y),
\]
where \(k^2 = -\Lambda/(24M^3)\) and the prime denotes differentiation with respect to \(y\). If we choose \(\sigma = k \frac{y}{|y|}\) and \(f = 1 - \xi e^{4k|y|}\) as in Eq. (1), then the first and third equations of (5) can be solved using a fine-tuning condition \(v_b = 24M^3k(1 - \xi)\). However, it is impossible to solve the second equation due to a delta-function occurring in \(f''\).

In fact, it may be a formidable task to derive a solution of Eqs. (5) in a closed form. Here instead, we solve Eqs. (5) in the form of infinite series which is sufficient to examine the features of the RS scenario at nonzero temperature. Motivated by Eq. (1) again, we introduce a small dimensionless parameter \(\xi\) as follows:
\[
f = 1 + \xi f_1(y) + \xi^2 f_2(y) + \cdots,
\]
\[
\sigma = k \frac{y}{|y|} + \xi \sigma_1(y) + \xi^2 \sigma_2(y) + \cdots.
\]
The expansion parameter $\xi$ is related to the external temperature of our universe, and the explicit relation will be derived later (see Eq. (14)).

Inserting Eq. (6) into the first two equations of (5) one can solve these for $f_i$ and $\sigma_i$ step by step. First we note the equation

$$f''_1 = 4k\epsilon(y)f'_1$$  \hspace{1cm} (7)

which originates from $O(\xi)$ contributions in the second of equations (5). Eq. (7) implies $f'_1 \propto e^{4k|y|}$ which means that $f_1$ is $Z_2$-antisymmetric. If one chooses $f_1 = 0$ in an attempt to maintain the $Z_2$ symmetry of the original zero temperature RS scenario, the higher order equations imply that all $f_i$ and $\sigma_i$ are zero. Thus we return to the original zero-temperature RS scenario. The other possibility of choosing $\sigma = k|y|$ implies from the first of eqs. (5) that $f_i'(y)\delta(y) = 0$ and so has a similar effect. We therefore abandon the $Z_2$ symmetry at finite temperature. It is relatively straightforward to derive now $f_i$ and $\sigma_i$ perturbatively.

Here, we present the first few solutions explicitly:

$$f_1 = -\frac{1}{4}\epsilon(y)(e^{4k|y|} - 1), \quad f_2 = \frac{1}{8}(k \ | \ y \ | - \frac{1}{4})e^{4k|y|},$$  \hspace{1cm} (8)

$$f_3 = -\frac{1}{32}\epsilon(y) \left[(k^2 y^2 + \frac{1}{4}k \ | \ y \ |)e^{4k|y|} - \frac{1}{16}(e^{4k|y|} - 1)\right],$$

$$f_4 = \frac{k^2 y^2}{384}(3 + 2k \ | \ y \ |)e^{4k|y|},$$

$$\sigma_1 = -\frac{k}{8}y, \quad \sigma_2 = \frac{3k}{128} \ | y \ |,$$

$$\sigma_3 = -\frac{k}{256}y, \quad \sigma_4 = \frac{23k}{32768} \ | y \ |.$$  \hspace{1cm}

Then the third of Eqs. (5) yields an additional fine-tuning condition

$$v_b = 24M^3k \left[1 - \frac{\xi^2}{128} - \frac{\xi^4}{32768} + \cdots\right].$$  \hspace{1cm} (9)

A further interesting feature of the RS scenario is that its spacetime consists of two copies of $AdS_5$ space. Due to $Z_2$–symmetry breaking, however, the two spaces attached to the boundary are not necessarily identical at finite temperature. In order to show this explicitly, we define a variable
\[ z \equiv \frac{1}{k} e^{\sigma(y)} = \frac{1}{k} e^{y |y|} \left[ 1 - \frac{\xi}{8} ky + \frac{\xi^2}{128} (3k | y | + k^2 y^2) + \cdots \right]. \quad (10) \]

In terms of \( z \) the spacetime (4) becomes
\[ ds^2 = \frac{1}{k^2 z^2} \left[ -f dt^2 + dx^i dx_i + \frac{k^2}{f \sigma'^2} dz^2 \right]. \quad (11) \]

Inverting Eq.(10) one can show straightforwardly that
\[ f = (1 + \frac{\xi}{4}) \left[ 1 - \left( \frac{\xi}{4} - \frac{\xi^2}{32} + \cdots \right)(kz)^4 \right], \quad (12) \]
\[ f \sigma'^2 = k^2 \left[ 1 + O(\xi^3) \right] \left[ 1 - \left( \frac{\xi}{4} - \frac{\xi^2}{32} + \cdots \right)(kz)^4 \right] \]
for the case of positive \( y \). Expansion in the case of negative \( y \) results in the same Eq.(12) with \( \xi \) changed into \(-\xi\). If one defines a new time variable \( t_\pm \equiv \sqrt{1 \pm \frac{\xi}{4}} t \) for \( y > 0 \) and \( y < 0 \) respectively, the final form of spacetime becomes
\[ ds^2 = \frac{1}{k^2 z^2} \left[ -(1 - U_T^{(\pm)} z^4) dt_\pm^2 + dx^i dx_i + \frac{dz^2}{1 - U_T^{(\pm)} z^4} \right] \quad (13) \]
where
\[ U_T^{(\pm)} = \pm k^4 \left( \frac{\xi}{4} \mp \frac{\xi^2}{32} + \cdots \right). \quad (14) \]

If we choose \( \xi(y > 0) = -\xi(y < 0) \), Eq.(13) represents two copies of Schwarzschild–AdS\(_5\) space. In this case, however, Eq.(8) is no longer a solution of Einstein’s equation. If, on the other hand, we choose \( \xi \) to be constant, say \( \xi > 0 \), the space in the \( y > 0 \) region is a usual Schwarzschild–AdS\(_5\) space, but the space in the region \( y < 0 \) is not, due to \( U_T^{(-)} < 0 \).

Although this is also obtained from a decoupling limit of the black three-brane solution [21] by interchanging the inner and outer horizons, it is unclear why the RS scenario requires this apparent asymmetry. Eq.(14) allows one to express \( \xi \) in terms of the external temperature.

Finally, we examine features of the RS scenario at the level of the four-dimensional effective action. In the original RS scenario the fine–tuning condition \( \Lambda = -24 M^3 k^2 \) and \( v_b = 24 M^3 k \) enables one to derive a four–dimensional effective action with a vanishing cosmological constant from a wide range of five–dimensional metrics. One can show this explicitly by computing a 4\( d \) effective action from a 5\( d \) line element.
\[
 ds^2 = e^{-2k|y|} g_{\mu\nu}(x) dx^\mu dx^\nu + dy^2. \tag{15}
\]

Of course, Eq.(15) agrees with the RS solution when \( g_{\mu\nu} = \eta_{\mu\nu} \).

At a finite temperature, however, this kind of feature seems to be severely restricted. In order to see this more explicitly we introduce a line element

\[
 ds^2 = e^{-2\sigma(y)} g_{\mu\nu}(x,y) dx^\mu dx^\nu + \frac{dy^2}{f(y)} \tag{16}
\]

where

\[
 g_{\mu\nu}(x,y) = \bar{g}_{\mu\nu}(x) + (1 - f(y))\bar{g}^0_{\mu}\bar{g}^0_{\nu}. \tag{17}
\]

Here, \( \bar{g}_{\mu\nu}(x) \) represents a physical graviton in the effective theory. The curvature scalar \( R \) computed from the metric (16) can be shown to be

\[
 R = e^{2\sigma} \bar{R} + \Delta R, \tag{18}
\]

where \( \bar{R} \) is the 4d curvature scalar derived from \( \bar{g}_{\mu\nu} \) and

\[
 \Delta R = \frac{2v_b}{3M^3} \delta(y) - 20k^2 - (1 + \bar{g}^{00}) \left[ \frac{f'\sigma'}{1 + (1 - f)\bar{g}^{00}} - \frac{\bar{g}^{00}f'^2}{2[1 + (1 - f)\bar{g}^{00}]^2} \right]. \tag{19}
\]

One should note the following. In Eq.(18) we have dropped several terms which are irrelevant for the following discussion. In fact, these terms become zero for the constant metric of the worldvolume which is what we are interested in. Also we have changed \( \Delta R \) into a more convenient form for the following discussion using Eq.(3).

Using Eq.(18) and

\[
 \sqrt{-G} = \sqrt{-\bar{g}_4} e^{-4\sigma} \sqrt{\frac{1 + (1 - f)\bar{\mu}}{f}} \tag{20}
\]

where \( \bar{g}_4 = det\bar{g}_{\mu\nu}, \bar{g}_3 = det\bar{g}_{ij}(i,j = 1,2,3), \) and \( \bar{\mu} \equiv \bar{g}_3/\bar{g}_4, \) one can calculate a four-dimensional effective action whose form is

\[
 S_{eff} = \int d^4x \sqrt{-\bar{g}_4}[2M_{pl}^2\bar{R} - \Lambda_4] \tag{21}
\]

where \( M_{pl}^2 = M^3/k \) and \( \Lambda_4 \) is four-dimensional cosmological constant.
For the Ricci-flat case $\Lambda_4$ assumes the form
\[
\Lambda_4 = \int dy e^{-4\sigma} \sqrt{1 + \left(1 - \frac{f}{f'}\right)\mu} \left[\Lambda + \nu_b \delta(y) - 2M^3 \Delta R_1\right].
\] (22)

We now consider a flat space case by taking
\[
\bar{\mu} = -1,
\] (23)
\[
\Delta R_1 = \frac{2\nu_b}{3M^3} \delta(y) - 20k^2.
\]

Then it is easy to show that $\Lambda_4$ in flat space becomes
\[
\Lambda_4^{\text{flat}} = 16M^3k^2 \int dy e^{-4\sigma} - \frac{\nu_b}{3}.
\] (24)

Although not obvious at this stage, one can show as follows that the right-hand side of Eq.(24) is zero. After expanding $e^{-4\sigma}$ in terms of $\xi$, integration with respect to $y$ yields
\[
\int dy e^{-4\sigma} = \frac{1}{2k} \left[1 - \frac{\xi^2}{128} - \frac{\xi^4}{32768} + \ldots\right]
\] (25)
which indicates $\Lambda_4^{\text{flat}} = 0$ via the fine-tuning condition (9).

However, the vanishing of the cosmological constant is not maintained when the time-time component of $\bar{g}_{\mu\nu}$ deviates from Lorentz signature $-1$. To show this we choose $\bar{g}^{\mu\nu} = \eta^{\mu\nu} + \beta\eta^{0\nu}\eta_0^\mu$. Then, $\bar{\mu}$ and $\Delta R$ become
\[
\bar{\mu} = 1 - \beta, \quad \Delta R = \frac{2\nu_b}{3M^3} \delta(y) - 20k^2 - \beta \left[\frac{f'\sigma}{f + \beta(1 - f)} + \frac{(1 - \beta)f'^2}{2[f + \beta(1 - f)]^2}\right].
\] (26)

Expanding the integrand in Eq.(22) in terms of $\xi$ again, it is straightforward to show that
\[
\Lambda_4^{\text{devi}} = \Lambda^{(0)} + \Lambda^{(2)} \xi^2 + \ldots
\] (27)
where $\Lambda^{(0)} = 0$ and
\[
\Lambda^{(2)} = M^3k\beta \left[\frac{3\beta - 5}{8} - \frac{1}{2}(kL) - \frac{3}{4}(kL)^2 + \frac{3 - 2\beta}{4}e^{2kL}\right]
\] (28)
where $L$ is the length of the fifth dimension. Although $\Lambda^{(2)}$ is infinite due to the appearance of $L$, it may give a finite, but $r_c$-dependent cosmological constant if the same procedure is applied to the RS two brane scenario [1].

In summary, we have examined the RS scenario when the external temperature is nonzero by using a spacetime ansatz motivated by analogy with that of AdS/CFT correspondence. We have shown that in such a case many interesting features of the original RS model are not maintained at nonzero temperatures. It is interesting to check whether or not the finite temperature solution (3) and (5) also has a massless bound state at the fluctuation level. The more important point, however, is to reformulate the finite temperature RS scenario from the string/$M$ theory background. Then, it may be possible to find an answer why many interesting features of the RS model disappear at nonzero temperature.

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