ON THE HILBERT FUNCTION OF FAT POINTS ON A RATIONAL NORMAL CUBIC

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Abstract: In this paper we find an algorithm which computes the Hilbert function of schemes $Z$ of "fat points" in $\mathbb{P}^3$ whose support lies on a rational normal cubic curve $C$. The algorithm shows that the maximality of the Hilbert function in degree $t$ is related to the existence of fixed curves (either $C$ itself or one of its secant lines) for the linear system of surfaces of degree $t$ containing $Z$.

Introduction

The aim of this paper is to consider linear systems $J_t$ defined by particular schemes of fat points, where with "fat points" we mean 0-dimensional schemes defined by homogeneous ideals of type

(*) \[ J = \bigoplus_{t \geq 0} J_t = p_1^{m_1} \cap \ldots \cap p_s^{m_s} \]

where each $p_i$ is the homogeneous ideal in $R = k[x_0, \ldots, x_r]$ of a point $P_i \in \mathbb{P}^r = \mathbb{P}_k^r$ ($k$ being an algebraically closed field of characteristic 0), and the $m_i$'s are non negative integers. We will denote a scheme of fat points by $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s)$.

In [3], a bound for the regularity of the linear systems of type $J_t$ is given when the points $P_i \in \mathbb{P}^r$ are in (linear) generic position (i.e. no three on a line, no four on a plane, etc.). It turns out that the "worst" case for $J_t$ is when the points $P_i$ lie on a rational normal curve (see also [7]).

This leads to the following conjecture:

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**Conjecture:** Let \( J \subseteq \mathbb{R} \) be an ideal of fat points in \( \mathbb{P}^r \) (i.e. \( J \) is as in (*)), and let \( H(\mathbb{R}/J,t) \) denote the Hilbert function of \( \mathbb{R}/J \). Then, if the points \( P_i \) are in linear generic position, \( \forall t \in \mathbb{N} \) we have that \( H(\mathbb{R}/J,t) \geq H(\mathbb{R}/I,t) \), where \( I \) is an ideal of type (*), with the same multiplicities \( m_i \) as \( J \) and whose support is given by points on a rational normal curve \( C_r \subseteq \mathbb{P}^r \). Moreover the value \( H(\mathbb{R}/I,t) \) does not depend on the choice of the \( s \) points on \( C_r \).

In this paper we analyze the case \( r = 3 \), and we show (via Theorem 2.2) that there is an algorithm which computes \( H(\mathbb{R}/I,t) \) for ideals \( I \) as above (i.e. for fat points on a cubic curve \( C \)). The algorithm will only depend on the data \( s, m_1, ..., m_s \), thus showing that the Hilbert function does not depend on the position of the (distinct) points on the curve.

It will also turn out that \( H(\mathbb{R}/I,t) \) has its maximal value (i.e. the fat points impose independent conditions to surfaces of degree \( t \)), if and only if for every \( P_i \) with \( m_i > 0 \) the linear system \( (I : p_i)_t \) has neither \( C \) nor any line \( P_i P_j \) as fixed locus (Corollary 2.3).

The paper is divided as follows: the first section is devoted to studying the following question: which numerical ("Bezout-like") conditions imply that a multiple of a curve (a line \( P_i P_j \) or the curve \( C \) in our case) is a fixed locus for the linear system \( I_t \)? In this section we also consider whether the numerical conditions that we find are necessary and we compute the Hilbert function of all the multiples of \( C \), i.e. of the ideals \( I^n_C \).

In §2 we state the main result and describe it, while §3 is dedicated to several lemmata which will be used in §4 to prove Theorem 2.2.

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1. A "Bezout-like" condition for multiples of \( C \) and of lines.

From now on we assume that \( I \) is an ideal of type (*) in \( k[x_0, ..., x_3] \), i.e. that the points \( P_i \) are in \( \mathbb{P}^3 \). Let \( L \) be the line \( P_i P_j \), and let \( n, t \) be natural numbers; Proposition 1.1 will give a Bezout-type condition that forces the elements of the linear system \( I_t \) to contain the line \( L \) with multiplicity at least \( n \) (it is actually just Bezout for \( n = 1 \)).

Assuming further that the \( P_i \)'s lie on a rational normal curve \( C \), Proposition 1.3 gives an analogous condition that forces the elements of \( I_t \) to contain the curve \( C \) with multiplicity at least \( n \).

Let \((x)^+ = \max\{x, 0\}\). We have
Proposition 1.1: Let \( I = p_i^{m_i} \cap \ldots \cap p_s^{m_s} \) be an ideal of type (\(*\)), and \( L \) be the line \( P_i P_j \). If \( I_L \) is the ideal of \( L \), and \( n \leq (m_i + m_j - t)^+ \), then \( I_t \subseteq (I_L^n)_t \).

Proof: The statement is obvious for \( n = 0 \) and \( n = 1 \), so let \( n > 1 \). We may assume \( P_i = (0 : 0 : 0 : 1), P_j = (0 : 0 : 1 : 0) \), hence \( I_L = (x_0, x_1) \). Let \( f \in I_t \): by Bezout’s theorem applied to the intersection of \( \{ f = 0 \} \) with the plane \( \{ x_1 - ax_0 = 0 \}, a \in k \), it is easy to prove that \( f(x_0, ax_0, x_2, x_3) \in (x_0)^n, \forall a \in k \). Hence \( f(x_0, x_1, x_2, x_3) \in (x_0, x_1)^n \).

\[ \square \]

Definition 1.2: Let \( I \) be as above, and \( t, n \in \mathbb{N} \). We will say that \( I_t \) satisfies property \( \mathcal{P}(n) \) if and only if \( \forall l, 1 \leq l \leq n : 3t + 5(1 - l) < \sum_{i=1}^{s}(m_i - l + 1)^+ \).

Proposition 1.3: Let \( I = p_i^{m_i} \cap \ldots \cap p_s^{m_s} \) be an ideal of type (\(*\)) such that the \( P_i \)'s are distinct points on a rational cubic curve \( C \) and let \( t, n \in \mathbb{N} \) be such that \( I_t \) satisfies property \( \mathcal{P}(n) \). Then, if \( I_C \) is the ideal of \( C \), we have that \( I_t \subseteq (I_C^n)_t \).

Proof: Note that \( I_C^n = I_C^{(n)} \), i.e. \( I_C^n \) is saturated and represents the \( n^{th} \) infinitesimal neighborhood of \( C \). In fact, by [5], Cor. 2.2, we have:

Corollary (Robbiano): Let \( I \) be the ideal associated to a complete intersection of codimension \( \leq 2 \) inside the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^{r-1} \) in \( \mathbb{P}^{2r-1} \).

Then \( I^n \) is primary for every \( n \).

Since the ideal of a rational normal cubic \( C \subseteq \mathbb{P}^3 \) can be obtained by intersecting the ideal of \( \mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5 \) which is given by the maximal minors of a matrix of type

\[
\begin{pmatrix}
  x_0 & x_1 & x_2 \\
  x_5 & x_4 & x_3
\end{pmatrix},
\]

with the hyperplanes \( \{ x_1 = x_5 \}, \{ x_2 = x_4 \} \), we get what we want (see also [A-S-V], 6.9). \( \square \)

Let \( X \) be the blow-up of \( \mathbb{P}^3 \) along \( C \). Then we have \( \text{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z} \), and we can choose as generators the exceptional divisor \( E \) and the divisor \( H \), corresponding to the strict transform of a generic plane of \( \mathbb{P}^3 \).

Let \( S \) be a surface in the linear system \( I_t \) and let \( S' \subset X \) be its strict transform. Then Proposition 1.3 is equivalent to:

Proposition 1.4: Let \( I, n, t \) be as in Proposition 1.3. Then \( S' \in |tH - nE| \).

It is well known that \( E \cong \mathbb{P}(N_C) \) (e.g. see [4]), hence, since \( N_C \cong \mathcal{O}_C(5) \oplus \mathcal{O}_C(5) \), we have that

\[ E \cong \mathbb{P}(N_C) \cong \mathbb{P}(\mathcal{O}_C(5) \oplus \mathcal{O}_C(5)) \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C) \]
is isomorphic to a quadric surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$.

Let $\pi : X \to \mathbb{P}^3$ be the canonical projection; then one of the two rulings of $E$ is given by the lines $L_P = \pi^{-1}(P)$, $P \in C$, and the other is given by the zero-loci $C'$ of sections of $\mathcal{N}_C$ ($\pi(C') = C$).

Let us use the notation $(a, b)$ for the divisor class of $aL_P + bC'$; then we have that $H \cdot E$, as divisor on $E$, is $(E \cdot H)|_E = (3, 0)$.

Of course $H \cdot H = H^2$ is the strict transform of a generic line of $\mathbb{P}^3$ (not touching $C$), so $H^2 \cdot E = 0$.

In order to determine the $E^2$, consider $E^2 \cdot H$. Since $E^2 = E|_E$, we have $E^2 \cdot H = E \cdot (E \cdot H) = (E \cdot H)|_E \cdot E^2$. Let $E^2 = (a, b)$. Since we have also $E^2 \cdot H = E|_H \cdot E|_H$, and $H \cong \{ \text{the blow-up of } \mathbb{P}^2 \text{ at three points} \}$, where $E|_H$ is the exceptional divisor of such a blow up, we get that $E^2 \cdot H = (E|_H \cdot E|_H) = -3$. Hence $(H \cdot E)|_E \cdot E^2 = (3, 0) \cdot (a, b) = 3b$, which implies $b = -1$.

In order to determine $a$, consider instead: $E^3 = E^2|_E = E|_E \cdot E|_E = -2a$ (since $E|_E = (a, -1)$ and $(a, -1) \cdot (a, -1) = -2a$).

On the other hand, $(3H - E)^3 = 27H^3 - 27H^2 \cdot E + 9H \cdot E^2 - E^3$, and so, since $H^2 \cdot E = 0$, $H^3 = 1$ and $H \cdot E^2 = -3$, we have $(3H - E)^3 = -E^3$.

It is not hard to compute $(3H - E)^3$: this is the number of intersections of the strict transforms $S'_1, S'_2, S'_3$ of three generic cubic surfaces $S_1, S_2, S_3$ containing $C$ in $\mathbb{P}^3$.

Consider $S'_1 \in |3H - E|$. The cubic surface $S_1$ in $\mathbb{P}^3$ is isomorphic to the blow-up of $\mathbb{P}^2$ in six generic points $P_1, ..., P_6$, so $\text{Pic}S_1 \cong \mathbb{Z}^7$, and we can write $(t; m_1, ..., m_6)$ for the divisor class in $S_1$ of the strict transform of a curve of degree $t$ in $\mathbb{P}^2$ which has multiplicity $m_i$ in $P_i, i = 1, ..., 6$; e.g. we have that a plane section of $S_1$ is in $(3; 1, 1, 1, 1, 1, 1)$.

With this notation, we can assume $C \in (1; 0, 0, 0, 0, 0, 0)$, and we have $S_2 \cdot S_1 \in (9, 3, 3, 3, 3, 3, 3)$ on $S_1$, with $S_2 \cdot S_1 = C \cup C'$ and $C' \in (8, 3, 3, 3, 3, 3, 3) \ (C'$ is a sextic curve of genus 3). So the integer we want is $S_3 \cdot C' - C \cdot C' = 18 - 8 = 10$.

Hence $E^3 = -10$ and $a = 5$.

Now we want to prove Proposition 1.4. Let $Z = (P_1, ..., P_s; m_1, ..., m_s)$ be the scheme defined by $I$ (hence all $P_i \in C$), and $S \subset \mathbb{P}^3$ be a surface of degree $t$ which contains $Z$. Note that for $n = 1$ the statement is trivial by Bezout Theorem.

Let us work by induction on $n$, the case $n = 1$ being done. Suppose the proposition is true for $n - 1 > 0$, and let us check that it is true for $n$.

Since $3t + 5(1 - l) < \sum_i(m_i - l + 1)^+$ for $1 \leq l \leq n - 1$, then, by induction, any surface of degree $t$ containing $Z$ is such that its strict transform is in $|tH - (n - 1)E|$.
So, let $S'$ be the strict transform of $S$ in $X$; then $S' \in |tH - (n - 1)E|$. Consider (on $E$)

$$S' \cdot E = (tH - (n - 1)E) \cdot E = (3t, 0) - (n - 1)(5, -1) = (3t - 5(n - 1), (n - 1)).$$

Let $L_i = \pi^{-1}(P_i)$; then $\pi^{-1}S = S' + (n - 1)E$ has to contain the divisors $m_iL_i \in (m_i, 0)$, $i = 1, ... , s$ of $E$, hence $S'$ has to contain $(m_i - n + 1)^+L_i$, i.e. at least a divisor $(\sum_i^s(m_i - n + 1)^+, 0)$ on $E$.

So, if $3t - 5(n - 1) < \sum_i^s(m_i - n + 1)^+$, $E$ has to be a fixed component in $tH - (n - 1)E$, i.e. $S' \in |tH - nE|$. □

It can be of some interest to give an actual computation of $\dim(I_C^n)_{t}$:

**Proposition 1.5:** Let $I_C \subseteq k[x_0, ..., x_3]$ be the ideal of a rational normal curve $C \subseteq \mathbb{P}^3$. Then we have:

$$\dim(I_C^n)_t = 0, \text{ for } t \leq 2n - 1, \text{ and}$$

$$\dim(I_C^n)_t = \left(t + \frac{3}{3}\right) - \left(\frac{n+1}{2}\right)(3t + 6) + 5(1 + ... + n^2), \text{ for } t \geq 2n - 1; \text{ moreover in this case, if } I_C \text{ is the ideal sheaf associated to } I_C, \text{ we have: } H^1(\mathbb{P}^3, I_C^n(t)) = 0.$$

**Proof:** The case $t \leq 2n - 1$ is obvious since $I_C^n$ is generated in degree $2n$, so consider $t \geq 2n$.

Let $q = 0$ be the equation of a smooth quadric $Q$ containing $C$; then multiplication by $q$ defines an injection $0 \to I_C^{n-1} \to I_C^n$, from which, sheafifing, we get:

$$0 \to I_C^{n-1}(-2) \to I_C^n \to O_Q(-nC) \to 0.$$

Since $C$ is of type $(2, 1)$ as a divisor on $Q$, twisting by $O_{\mathbb{P}^3}(t)$ we get

$$0 \to I_C^{n-1}(t-2) \to I_C^n(t) \to O_Q(t-2n, t-n) \to 0$$

which, passing to cohomology, yields:

$$0 \to H^0(\mathbb{P}^3, I_C^{n-1}(t-2)) \to H^0(\mathbb{P}^3, I_C^n(t)) \to H^0(Q, O_Q(t-2n, t-n)) \to$$

$$\to H^1(\mathbb{P}^3, I_C^{n-1}(t-2)) \to H^1(\mathbb{P}^3, I_C^n(t)) \to H^1(Q, O_Q(t-2n, t-n))$$

where $H^1(Q, O_Q(t-2n, t-n)) = 0$ since $t \geq 2n$.

Now we work by induction on $n$. For $n = 1$, it is well known that

$$\dim H^0(\mathbb{P}^3, I_C^n(t)) = \left(t + \frac{3}{3}\right) - (3t + 1) \quad \text{and} \quad H^1(\mathbb{P}^3, I_C(t)) = 0.$$

So, suppose $n \geq 2$. Since $t-2 \geq 2(n-1)$, we have $H^1(\mathbb{P}^3, I_C^{n-1}(t-2)) = 0$ by induction hypothesis, hence $H^0(\mathbb{P}^3, I_C^n(t)) = H^0(\mathbb{P}^3, I_C^{n-1}(t-2)) + H^0(Q, O_Q(t-2n, t-n)) = \ldots = 0$.

-5- 3/3/122
= \binom{t+1}{3} - \binom{n}{2} (3(t - 2) + 6) + 5(1 + \ldots + (n - 1)^2) + (t - n + 1)(t - 2n + 1) = \\
= \binom{t+3}{3} - \binom{n+1}{2} (3t + 6) + 5(1 + \ldots + n^2).

We conclude this section with the following result, which gives, in case \(n = 1\), a sort of inverse with respect to Proposition 1.3.

**Proposition 1.6:** Let \(I\) be as in Proposition 1.3, and let \(I_t \neq \{0\}\). Then \(I_t \subseteq (I_C)_t\) if and only if \(3t < \sum_{i=1}^{s} m_i\).

**Proof:** By proposition 1.3, it remains only to prove that \(\{0\} \neq I_t \subseteq (I_C)_t\) implies \(3t < \sum_{i=1}^{s} m_i\). This follows from the fact that the inequality \(3t \geq \sum_{i=1}^{s} m_i\) allows to find a surface in \(I_t\) made of planes (each of them passing at most through three of the \(P_i\)'s), hence not contained in \((I_C)_t\).

**Remark 1.7:** Notice that it is not possible, instead, to do the same with Proposition 1.1, in fact the condition \(t \geq m_i + m_j\) does not guarantee that the line \(P_iP_j\) is not fixed for \(I_t\).

For instance, let \(Z = (P_1, \ldots, P_7; 3, 3, 2, 2, 2, 2, 1)\) with \(P_1, \ldots, P_7\) on \(C\); we have that the only surface of degree 4 containing \(Z\) is given by the union of the two quadric cones with vertices in \(P_1, P_2\) which contain \(C\).

Hence the lines \(P_1P_7\) and \(P_2P_7\) are fixed for \(I_4\), even if \(4 = t = m_1 + m_7 = m_2 + m_7\).

On the other hand, the two numerical conditions \((t \geq m_i + m_j\) and \(t \geq \sum_{i=1}^{s} m_i)\) together are equivalent to the two geometric conditions (see Corollary 2.3).

2. The algorithm to compute the Hilbert function of fat points on a cubic curve.

Let \(I, J\) be, respectively, the homogeneous ideals of the schemes \(Z, W \in \mathbb{P}^3\) of fat points

\[Z = (P_1, \ldots, P_s; m_1, \ldots, m_s), \quad W = (P_1, \ldots, P_s; m_1, \ldots, m_{s-1}, m_s + 1)\]

where, from now on, \(P_1, \ldots, P_s\) are on rational normal cubic \(C\), and

\[m_1 \geq m_2 \geq \ldots \geq m_s \geq 0.\]

We want to give a method that can compute \(\dim J_t\) (for every \(t \geq 0\)) from the data: \(t, m_1, \ldots, m_s\) and \(\dim I_t\). The result will not depend on the position of the points on \(C\).

This will answer to our question, since one will be able to compute \(\dim J_t\) working by recursion on \(s\) and \(m_s\). The algorithm will be given for \(s \geq 2\), since the case \(s = 1\) is quite
trivial. In fact for $s = 1$ we have $Z = (P, m), W = (P, m + 1)$, and so we get that $t \geq m$ implies:

$$\dim(I/J)_t = \dim I_t - \dim J_t = \binom{m + 3}{3} - \binom{m + 2}{3} = \binom{m + 2}{2};$$

while when $t < m$, trivially $\dim(I/J)_t = 0$.

Hence, in the sequel, we will always suppose $s \geq 2$.

We will determine $\dim(I/J)_t$ via a scheme of fat points $N \subset \mathbb{P}^2$.

**Definition 2.1.** Let $Z = (P_1, ..., P_s; m_1, ..., m_s)$ be a scheme of fat points in $\mathbb{P}^3$ with support on a rational normal curve $C$ and let $I$ be the corresponding homogeneous ideal. We say that $N = (Q, Q_1, ..., Q_{s-1}; n, n_1, ..., n_{s-1}) \subseteq \Pi \cong \mathbb{P}^2$ is the $t$-projection of $Z$ from $P_s$, if the points $Q_1, ..., Q_s-1$ are the projection from $P_s$ of $P_1, ..., P_{s-1}$ on a plane $\Pi$ not containing $P_s$, while $Q$ is the projection of $P_s$ itself along the tangent line to $C$ at $P_s$ and the numbers $n, n_1, ..., n_{s-1}$ are defined as follows:

$$n_i = (m_i + m_s - t)^+, \quad n = \min\{m_s + 1, \sup\{\nu \in \mathbb{N} | P(\nu) \text{ holds for } I_t}\}.$$

We can always suppose $P_s = (0 : 0 : 0 : 1)$, and $\Pi$ to be $\{x_3 = 0\}$.

Of course the points $Q, Q_1, ..., Q_{s-1}$ lie on the conic $\Gamma$ which is the projection of $C$ from $P_s$, so the Hilbert function of $N$ is known (see [2]).

Our result is:

**Theorem 2.2:** Let $I$, $J$ be respectively the homogeneous ideals of the schemes of fat points $Z = (P_1, ..., P_s; m_1, ..., m_s)$, $W = (P_1, ..., P_s; m_1, ..., m_{s-1}, m_s + 1)$ where the $P_i$’s are distinct points of $\mathbb{P}^3$ lying on a rational normal cubic $C$, and $m_1 \geq m_2 \geq \ldots \geq m_s \geq 0, s \geq 2$. We have, for every $t \geq 0$, that $\dim(I/J)_t$ equals the dimension, in degree $m_s$, of the ideal $I_N \subset k[x_0, x_1, x_2]$ of the $t$-projection $N$ of $Z$ from $P_s$, i.e.:

$$\dim(I/J)_t = \dim(I_N)_{m_s}.$$

Note that $L_i = \{\text{line } P_iP_s\}$ and $C$ are fixed multiple curves for the surfaces in the linear system $I_t$ with multiplicities at least $n_i, n$, respectively (see Proposition 1.1 and 1.3). The theorem shows their role in determining the Hilbert Function of $W$; in particular, when $n = n_i = 0$, i.e. when $N = \emptyset$, the difference between $\dim I_t$ and $\dim J_t$ is $\binom{m_s+2}{2} = -7- 3/3/122$
\((m_s + 3) - \left(\frac{m_s + 2}{3}\right)\), i.e. it is what it "should be", in the sense that passing to multiplicity \(m_s + 1\) on \(P_s\) imposes exactly \(\left(\frac{m_s + 2}{2}\right)\) new independent conditions to surfaces of degree \(t\).

Thus, we have:

**Corollary 2.3:** For any ideal \(I\) as in Theorem 2.2, if \(m_s > 0\), then the following are equivalent:

i) \(I_t\) is regular, (i.e. the fat points impose independent conditions to surfaces of degree \(t\));

ii) neither \(C\) nor, \(\forall i \in \{1, \ldots, s\}\), any of the lines \(P_iP_j, j \neq i\), is a fixed locus for \((I : p_i)_t = (p_i^{m_1} \cap \ldots \cap p_i^{m_i-1} \cap \ldots \cap p_s)_{m_s}\);

iii) \(3t \geq \sum_{i=1}^{s} m_i - 1\) and \(t \geq m_1 + m_2 - 1\).

**Proof:** By Bezout’s Theorem, ii) implies iii). Let us show now that iii) implies the regularity of \(I_t\). Consider that one can get \(I_t\) starting from \((p_i^{m_1})_t\), which is regular, and "adding the multiplicities on the \(P_i's\) one at a time", i.e. considering the ideals associated to the schemes

\((P_1, \ldots, P_s; m_1, 0, \ldots, 0), \ (P_1, \ldots, P_s; m_1, 1, 0, \ldots, 0), \ (P_1, \ldots, P_s; m_1, 2, \ldots, 0)\)

and so on. At any step we have that the \(t\)-projection of such schemes from the "last" point is empty, so Theorem 2.2 tells us that adding one to the multiplicity of the last point imposes independent conditions, and the system remains regular. (See also [3]).

In order to prove that i) implies ii), suppose that either \(C\) or a line \(P_iP_j\) are fixed components for \((I : p_i)_t\) for some \(i = 1, \ldots, s\). Then we will show that \(\dim((I : p_i)/I)_t < e = \left(\frac{m_i + 2}{2}\right)\), hence that \(I_t\) cannot be regular.

If \(I_t\) were regular, let \(P_i = (0 : 0 : 0 : 1)\); then there would exist \(F_1, \ldots, F_e \in (I : p_i)_t\) such that, locally, they generate \((x, y, z)^{m_i}\) modulo \(I_t\), hence they are of type (in affine coordinates):

\[F_1 = \tilde{F}_1 + x^{m_i}; \ F_2 = \tilde{F}_2 + x^{m_i-1}y; \ \ldots; \ F_e = \tilde{F}_e + z^{m_i}\]

where the \(\tilde{F}_i\) have degree \(\geq m_i + 1\). Now, if \(C = \{x = t, y = t^2, z = t^3\}\) is fixed for \((I : p_i)_t\), the above equations should become identities in \(t\), but this is clearly impossible for the first one, since \(x^{m_i} = t^{m_i}\) while \(\tilde{F}_1\) has degree \(\geq m_i + 1\) in \(t\).

If the fixed component is a line \(P_iP_j\), we work in the same way, e.g. assuming that the line is given by \(\{x = t, y = 0, z = 0\}\).

\[\Box\]

3. Preliminary Lemmata.
The proof of Theorem 2.2 will be given showing first that \( \dim(I/J)_t \leq \dim(I_N)_{m_s} \) (Lemma 3.1). Then we will consider several particular cases, with which we will deal with lemmata 3.2 to 3.5. This will leave us only with cases in Remark 3.6.

Lemmas 3.7 to 3.10 describe geometric properties of the cases listed by the Remark, and they will be used in the next section for the proof of the theorem.

From now on, we will always suppose that \( I, J, I_N \) are as in Theorem 2.2 and \( s \geq 2 \).

**Lemma 3.1:** For every \( t \geq 0 \), we have \( \dim(I/J)_t \leq \dim(I_N)_{m_s} \).

*Proof:* Let \( P_s = (0 : 0 : 0 : 1) \). If \( F = F_{m_s}(x_0, x_1, x_2)x_3^{t-m_s} + F_{m_s+1}(x_0, x_1, x_2)x_3^{t-m_s-1} + \ldots + F_t(x_0, x_1, x_2) \) is a form of \( I_t \), then it is easy to prove that \( F_{m_s} \), i.e. a polynomial defining the tangent cone to \( \{ F = 0 \} \) in \( P_s \), is a form of \( (I_N)_{m_s} \). Since the application \( (I/J)_t \to (I_N)_{m_s} \) that maps the class of \( F \) to \( F_{m_s} \) is injective, we have the conclusion.

\[ \Box \]

**Lemma 3.2:** If \( n = m_s + 1 \) or \( n_1 \geq m_s + 1 \), then \( \dim(I/J)_t = \dim(I_N)_{m_s} = 0 \).

*Proof:* Trivially \( \dim(I_N)_{m_s} = 0 \), so, by Lemma 3.1, we are done. \[ \Box \]

**Lemma 3.3:** If \( n = 0 \) and \( n_1 = 0 \), then \( \dim(I/J)_t = \dim(I_N)_{m_s} = \frac{(m_s+2)}{2} \).

*Proof:* If \( m_{s-1} \geq m_s + 1 \), the result follows from [3], Proposition 5. For \( m_{s-1} = m_s \) it is easy to extend the above proposition to our case. \[ \Box \]

**Lemma 3.4:** Let \( n = 0, n_1 > 0 \) and one of the following cases occurs:

a) \( s = 2 \), and \( n_1 \leq m_2 \).

b) \( s = 3 \) or \( 4 \), and \( n_1 + n_2 \leq m_s \).

Then \( \dim(I/J)_t = \dim(I_N)_{m_s} \).

*Proof:* Note that in both cases \( \dim(I_N)_{m_s} \) is known since \( (I_N)_{m_s} \) is regular (e.g. see [2]).

In case a) suppose \( P_1 = [0 : 0 : 1 : 1] \) and \( P_2 = [0 : 0 : 1 : 1] \), so \( I_t \subseteq (x, y)^{n_1} \).

We have to show that \( \dim(I/J)_t \geq \dim(I_N)_{m_s} \), since we have seen (Lemma 3.1) that the opposite inclusion always holds.

Let us consider the monomials (in affine coordinates):

\[
\begin{align*}
x^{m_2}, x^{m_2-1}y, x^{m_2-2}y^2, \ldots, y^{m_2}; \\
z x^{m_2-1}, z x^{m_2-2}y, \ldots, zy^{m_2-1}; \\
\ldots \\
z^{m_2-n_1}x^{n_1}, z^{m_2-n_1}x^{n_1-1}y, \ldots, z^{m_2-n_1}xy^{n_1-1}, z^{m_2-n_1}y^{n_1}.
\end{align*}
\]
All these monomials have multiplicity $m_2$ at $P_2$, and at least $n_1$ at $P_1$. Since $t - m_2 = m_1 - n_1$, multiplying the above monomials by $(z - 1)^{t-m_2}$ we get linearly independent polynomials of degree $t$ with multiplicity at least $m_1$ at $P_1$ and exactly $m_2$ at $P_2$ (where they have independent initial forms), hence:

$$\dim(I/J)_t \geq (m_2 + 1) + m_2 + (m_2 - 1) + \ldots + (n_1 + 1) = \binom{m_2 + 2}{2} - \binom{n_1 + 1}{2} = \dim(I_N)_{m_2}.$$ 

So we are done in case $a$).

In case $b)$, for $s = 3$, let $P_1 = [0 : 0 : 1 : 1]$, $P_2 = [1 : 0 : 0 : 1]$ and $P_3 = [0 : 0 : 0 : 1]$. If $n_2 = 0$, we consider the same monomial as above, but with $m_3$ instead of $m_2$; multiplying them by $(x - z - 1)^{t-m_3}$ we get $\dim(I/J)_t = \dim(I_N)_{m_s}$, as we did above.

If $n_2 > 0$, we consider instead the monomials:

$$x^{m_3-n_2}y^{n_2}, \ldots, xy^{m_3-1}, y^{m_3};$$
$$zx^{m_3-n_2}y^{n_2-1}, \ldots, zy^{m_3-1};$$
$$\ldots$$
$$z^{n_2}x^{m_3-n_2}y^0, \ldots, z^{n_2}y^{m_3-n_2};$$
$$z^{n_2+1}x^{m_3-n_2-1}, \ldots, z^{n_2+1}y^{m_3-n_2-1};$$
$$\ldots$$
$$z^{m_3-n_1}x^{n_1}, \ldots, z^{m_3-n_1}y^{n_1}.$$

With the same kind of reasoning as before, we get

$$\dim(I/J)_t \geq (m_3 - n_2 + 1) n_2 + \frac{(m_3 - n_2 + 1 + n_1 + 1)(m_3 - n_2 - n_1 + 1)}{2} = \binom{m_3 + 2}{2} - \binom{n_2 + 1}{2} - \binom{n_1 + 1}{2} = \dim(I_N)_{m_3}.$$ 

The case $s = 4$ is completely analogous by taking $P_1 = [0 : 0 : 1 : 1]$, $P_2 = [1 : 0 : 0 : 1]$, $P_3 = [0 : 1 : 0 : 1]$ and $P_4 = [0 : 0 : 0 : 1]$, so we leave it to the reader.

\[ \square \]

**Lemma 3.5:** Let $n \leq 1$, $n_1 \leq 1$, $s \geq 3$, $m_1 = m_2 = \ldots = m_s = 1$, then $\dim(I/J)_t = \dim(I_N)_{m_s}$.

**Proof:** If $3t < s$, we have $n = 1$ and $P(2)$ does not hold for $I_t$. It follows that $3t - 5 \geq 0$, so $t \geq 2$, $s \geq 7$, $N = (Q; 1)$. Hence $\dim I_t = 3$, $\dim J_t = 1$, $\dim(I_N)_1 = 2$ and we are done.

Let $3t \geq s$, so $n = 0$. For $t = 1$ we have $s = 3$, $N = (Q_1, Q_2; 1, 1)$ so $\dim I_1 = 1$, $\dim J_1 = 0$, and $\dim(I_N)_1 = 1$. For $t > 1$, we have $n_1 = 0$, and the conclusion follows by Lemma 3.3.
In the following remark we list the cases not covered by the previous lemmata (notice that for $2 \leq s \leq 5$ we only have $n = 0$ or $n = m_s + 1$):

**Remark 3.6:** In order to complete the proof of Theorem 2.2 the following cases remain to be considered, where we always have $s \geq 3$, $m_1 \geq 2$, $n \leq m_s$, $n + n_1 > 0$ (hence $t \geq m_1$ and $m_s > 0$):

1) $n_1 + n_2 \geq m_s + 1$;
2) $s \geq 6$, $n + n_1 \geq m_s + 1$, $m_1 > m_s$;
3) $s \geq 6$, $n + n_1 \geq m_s + 1$, $m_1 = m_s > 1$;
4) $s \geq 5$, $m_s \geq n + n_1$, $m_s \geq n_1 + n_2$.

**Lemma 3.7:** In case 1) we have:

a) The line $Q_1Q_2$ is a fixed component for $(I_N)_{m_s}$;

b) The plane $P_1P_2P_s$ is a fixed component for $I_t$, $J_t$.

**Proof:** Point a) is obvious. For b), since $n_2 \geq 1$, notice that the surfaces of $I_t$ contain the line $P_1P_2$ with multiplicity $m_1 + m_2 - t$, the line $P_1P_s$ with multiplicity $m_1 + m_s - t = n_1$ and the line $P_2P_s$ with multiplicity $m_2 + m_s - t = n_2$, hence the degree of intersection of the plane $P_1P_2P_s$ with those surfaces is:

$$m_1 + m_2 - t + n_1 + n_2 = n_1 + n_2 - 2m_s + 2t - t + n_1 + n_2 = 2(n_1 + n_2 - m_s) + t \geq t + 2$$

So, by Bezout, the plane has to be a fixed component. \qed

**Lemma 3.8:** In cases 2) and 3) we have:

a) The line $Q_1Q$ is a fixed component for $(I_N)_{m_s}$;

b) The quadric cone on $\Gamma$ with vertex in $P_1$ is a fixed component for $I_t$, $J_t$.

**Proof:** As before, a) is trivial, while b) follows by Bezout, considering the fact that the surfaces in $I_t$ contain the curve $C$ with multiplicity at least $n$ and the lines $P_1P_i$, $i = 2, \ldots, s$, with multiplicity at least $m_1 + m_i - t$. Since $n_1 \geq 1$, and $P(n)$ holds for $I_t$, one gets that their multiplicity of intersection with the cone is $\geq 3n + (s-2)m_1 + \sum_{i=1}^{s} m_i - (s-1)t \geq 3n + (s-2)m_1 + (s-5)(n-1) + 3t - (s-1)t = (s-2)(n + m_1 - t - 1) + 2t + 3 = (s-2)(n + n_1 - m_s - 1) + 2t + 3 \geq 2t + 3$. \qed

**Lemma 3.9:** In case 3) we have:

a) The conic $\Gamma$ is a fixed component for $(I_N)_{m_s}$;
b) The quadric cone on $\Gamma$ with vertex in $P_s$ is a fixed component for $I_t$, $J_t$.

**Proof:** Since $m_1 = m_s$, all the cones on $\Gamma$ with vertex in a $P_i$ are fixed for $I_t$, by the previous Lemma, so b) is done.

To show a) we will check, if $m_1 = ... = m_s = m$, that

(1) \((s-1)n_1 + n-1 - 2m \geq 0.\)

Since $n_1 \geq 1$ and $P(n)$ holds for $I_t$, we have:

(2) \(m < (s-2)n_1.\)

On the other hand, since $P(n+1)$ doesn’t hold for $I_t$, we have:

(3) \((s-5)n \geq sm-3t+3m = sm-3(2m-n_1) + s-5 \leq (s-5)(n+n_1-1) + 3n_1 + s-5-m = (s-5)n + (s-2)n_1 - m.\) Thus,

(4) \((s-5)n + (s-2)n_1 - m \geq 0, \) i.e.

(5) \((s-5)(s-1)n_1 + n-1 - 2m \geq 0.\)

To prove (1), let us multiply it by $(s-5)$. We get:

(6) \((s-4)(s-2)n_1 - m - 1 \geq (s-4)(m+1 - m - 1) + 1 = 1.\) \(\square\)

**Lemma 3.10:** In case 4) we have $2m_s \geq \sum_{i=1}^{s-1} n_i + n$.

**Proof:** For $s = 5$, we have $n = 0$ and $2m_s \geq 2(n_1 + n_2) \geq \sum_{i=1}^{s-1} n_i$.

Let $s \geq 6$. If $n_4 = 0$ the result is obvious since $2m_s \geq n + n_1 + n_1 + n_2 \geq n + n_1 + n_2 + n_3$.

Now let $n_4 > 0$. The case $n = 0$ is not possible; in fact since $t = m_1 + m_s - n_1 = m_2 + m_s - n_2 = m_3 + m_s - n_3$ and $s \geq 6$, we get $3t = 3m_s + m_1 + m_2 + m_3 - n_1 - n_2 - n_3 < \sum_{i=1}^{s} m_i$.

So, $n_4 > 0$, $n > 0$. Let $r = \max\{i|n_i > 0\}$, $4 \leq r \leq s - 1$; then (since $t = m_1 + m_s - n_1 = ... = m_r + m_s - n_r$), we have:

(4) \(rt = \sum_{i=1}^{r} m_i + rm_s - \sum_{i=1}^{r} n_i.\)

Since $P(n+1)$ doesn’t hold for $I_t$, we know that $3t - \sum_{i=1}^{s} m_i + (s-5)n \geq 0$, so:

(5) \((s-5)n \geq \sum_{i=1}^{s} m_i - 3t \geq \sum_{i=1}^{r} m_i + (s-r)m_s - 3t = rt - rm_s - \sum_{i=1}^{r} n_i + (s-r)m_s - 3t = (r-3)(m_1 + m_s - n_1) + (s-2r)m_s + \sum_{i=1}^{r} n_i \geq (r-3)2m_s - (r-3)n_1 + (s-2r)m_s + \sum_{i=1}^{r} n_i = (s-6)m_s - (r-3)n_1 + \sum_{i=1}^{r} n_i \geq (s-6)m_s - (s-4)n_1 + \sum_{i=1}^{r} n_i \geq (s-4)(n + n_1) - (s-4)n_1 + \sum_{i=1}^{r} n_i - 2m_s.\)

From this we have: \(-n \geq \sum_{i=1}^{r} n_i - 2m_s, \) hence the conclusion. \(\square\)

4. Proof of Theorem 2.2.

The proof of the theorem works by induction on $\sum_{i=1}^{s} m_i$. The first steps of the induction are covered by the lemmata 3.2 to 3.5 in §3; notice that also the trivial case $m_1 = 0$, i.e. $Z$ empty, is covered (by Lemma 3.3).
Now let us consider the cases left open in Remark 3.6.

**Case 1).** Since $n_2 \geq 1$, we may consider the following subscheme of $N$:

$$N' = (Q, Q_1, ..., Q_{s-1}; n, n_1 - 1, n_2 - 1, n_3, ..., n_{s-1}).$$

By Lemma 3.7, $(I_N)_t$ has the line $Q_1 Q_2$ as a fixed component, hence we have that

$$\dim(I_N)_{m_s} = \dim(I_{N'})_{m_{s-1}}.$$

For a similar reason (the plane $P_1 P_2 P_s$ is a fixed component), we have also:

$$\dim \left( \frac{I}{J} \right)_t = \dim \left( \frac{I'}{J'} \right)_{t-1}$$

where $I'$, $J'$ are the ideals associated respectively to the schemes

$$Z' = (P_1, ..., P_s; m_1 - 1, m_2 - 1, m_3, ..., m_{s-1}, m_s - 1),$$

$$W' = (P_1, ..., P_s; m_1 - 1, m_2 - 1, m_3, ..., m_{s-1}, m_s).$$

Thus we are done, since it is quite easy to check that $N'$ is the $(t - 1)$-projection from $P_s$ of the scheme associated to $I'$ and so, by induction hypothesis, we have:

$$\dim \left( \frac{I'}{J'} \right)_{t-1} = \dim(I_{N'})_{m_{s-1}}.$$

□

**Case 2).** Note that $n_1 > 0$, $n > 0$. In this case we proceed as above, but the fixed components we ”take away” are the quadric cone $\Lambda$ on $\Gamma$ with vertex in $P_1$ and the line $Q_1 Q$ which are fixed for $I_t, J_t$ and $(I_N)_{m_s}$ respectively (by Lemma 3.8).

So, let $I', J'$ be the ideal associated to the schemes

$$Z' = (P_1, ..., P_s; m_1 - 2, m_2 - 1, ..., m_{s-1} - 1, m_s - 1) = (P_1, ..., P_s; m_1', ..., m_s'),$$

$$W' = (P_1, ..., P_s; m_1 - 2, m_2 - 1, ..., m_{s-1} - 1, m_s)$$

respectively, and let

$$N' = (Q, Q_1, ..., Q_{s-1}; n - 1, n_1 - 1, n_2, ..., n_{s-1}).$$

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Then we have:
\[
\dim \left( \frac{I}{J} \right)_t = \dim \left( \frac{I'}{J'} \right)_{t-2} ; \quad \dim (I_N)_{m_s} = \dim (I'_N)_{m_{s-1}}.
\]

In order to conclude, by induction hypothesis, we have to show that \(N'\) is the \((t-2)\)-projection of \(Z'\) from \(P_s\), i.e. that the coefficients \(n, n_1\) are the right ones.

In fact, \((m_1 - 2) + (m_s - 1) - (t - 2) = n_1 - 1\); so, if we let \(n'\) be the coefficient for \(Q\) in the \((t - 2)\)-projection of \(Z'\), it only remains to check that \(n - 1 = n'\).

Since \(\sum_{i=1}^s m'_i = \sum_{i=1}^s m_i - s - 1\), then for \(1 \leq l \leq m_s - 1\), we have \(3t + 5(1 - l) - \sum_{i=1}^s (m_i - l + 1)^+ = 3(t - 2) + 5(1 - (l - 1)) - \sum_{i=1}^s (m'_i - (l - 1) + 1)^+\). It follows that \(n' = n - 1\). \(\Box\)

**Case 3.** Here \(m_1 = \ldots = m_s \geq 2\). By Lemma 3.9, the conic \(\Gamma\) is a fixed component for \((I_N)_{m_s}\), and the quadric cone on \(\Gamma\) with vertex in \(P_s\) is fixed for \(I_t, J_t\); so, let \(I', J'\) be the ideal associated to the schemes

\[
Z' = (P_1, \ldots, P_s; m_1 - 1, m_2 - 1, \ldots, m_{s-1} - 1, m_s - 2),
\]

\[
W' = (P_1, \ldots, P_s; m_1 - 1, m_2 - 1, \ldots, m_{s-1} - 1, m_s - 1)
\]

respectively, and let

\[
N' = (Q, Q_1, \ldots, Q_{s-1}; n - 1, n_1 - 1, \ldots, n_{s-1} - 1).
\]

We have \(\dim (I/J)_t = \dim (I'/J')_{t-2}\), and \(\dim (I_N)_{m_s} = \dim (I'_N)_{m_{s-2}}\), so if we show that \(N'\) is the \((t-2)\)-projection of \(Z'\) we are done (by induction).

The kind of computations that are required are similar to the ones shown in the previous cases, so we left them to the reader. \(\Box\)

Note that in order to apply the inductive hypothesis to \(W'\) and \(Z'\) we had to use Lemma 3.9, and not Lemma 3.8 (which also applies to this case).

In order to deal with the remaining case, we will show the following lemmata:

**Lemma 4.1:** Let \(\{h = 0\}\) be the plane \(P_1P_2P_s\) and let \(I', J'\) be the homogeneous ideals associated respectively to the schemes:

\[
Z' = (P_1, \ldots, P_s; m_1 - 1, m_2 - 1, m_3, \ldots, m_{s-1}, m_s - 1),
\]

\[
W' = (P_1, \ldots, P_s; m_1 - 1, m_2 - 1, m_3, \ldots, m_{s-1}, m_s).
\]
Then the following sequence (defined via the multiplication by \( h \)) is exact:

\[
0 \rightarrow \left( \frac{I'}{J'} \right)_{t-1} \rightarrow \left( \frac{I}{J} \right)_{t} \rightarrow \left( \frac{I + (h)}{J + (h)} \right)_{t} \rightarrow 0.
\]

**Proof:** Consider the exact sequence (defined by multiplication by \( h \)):

\[
0 \rightarrow I'(-1) \rightarrow I \rightarrow \frac{I}{mI'}.
\]

We have:

\[
\text{Im} \left( \frac{I'}{J'} \right) = \frac{hI'}{hJ'} = \frac{(h) \cap I}{(h) \cap J} = \frac{(h) \cap I}{(h) \cap J \cap I} = \frac{(h) \cap I}{(h) \cap J \cap I} = \frac{(h) \cap I}{(h) \cap J \cap I}.
\]

Hence:

\[
\frac{I}{mI'} = \frac{I}{((h) \cap J \cap I)} = \frac{I}{((h) \cap J + J)} = \frac{(h) \cap J}{(h) \cap J + J}.
\]

\( \square \)

**Lemma 4.2:** In case 4) of Remark 3.6, let

\[
N' = (Q, Q_1, ..., Q_{s-1}; n, (n_1 - 1)^+, (n_2 - 1)^+, n_3, ..., n_{s-1}).
\]

Then \((I_N)_{m_s}\) and \((I'_{N'})_{m_s-1}\) are regular.

**Proof:** The conclusion follows by Lemma 3.10 applying a result by B. Segre (see [6], [2]) which says that the linear system of curves of degree \( d \) in \( \mathbb{P}^2 \), with multiplicities \( \alpha_1 \geq ... \geq \alpha_s \) at points \( P_1, ..., P_s \) lying on a non singular conic is regular if and only if

\[
d \geq \max \{ \alpha_1 + \alpha_2 - 1, \left[ \sum_{i=1}^{s} \alpha_i \right] \}.
\]

\( \square \)

Now let \( J', I', h \) be as in Lemma 4.1, and \( N' \) as in the Lemma 4.2. Consider the following diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \left( \frac{I'}{J'} \right)_{t-1} & \rightarrow & \left( \frac{I}{J} \right)_{t} & \rightarrow & \left( \frac{I + (h)}{J + (h)} \right)_{t} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (I'_{N'})_{m_s-1} & \rightarrow & (I_N)_{m_s} & \rightarrow & K & \rightarrow & 0 \\
\downarrow & & \downarrow \phi & & \downarrow \psi & & \downarrow & & \\
0 & \rightarrow & \text{coker} \phi & \rightarrow & \text{coker} \psi & & & & \\
\end{array}
\]
The first exact sequence comes from Lemma 4.1 and the second is defined via multiplication by a linear form defining the line \( Q_1Q_2 \) (\( K \) being its cokernel). The first vertical sequence is exact by induction hypothesis, the map \( \phi \) (if we take \( P_s \) to be the origin) comes from the map which associates to each \( F \in I \) the tangent cone to \( \{ F = 0 \} \) (whose equation lies in \( (I_N)_{m_s} \)). We know, by Lemma 3.1, that \( \phi \) is injective, while the map \( \psi \) is injective by the Snake’s Lemma.

From Lemma 4.2 it follows (notations as above):

**Corollary 4.3:** In case 4) of Remark 3.6 we have \( \dim K = m_s + 1 - n_1 - n_2 \).

**Proof:** Since \((I_N)_{m_s}\) and \((I_N')_{m_s-1}\) are regular, we have

\[
\dim K = \dim (I_N)_{m_s} - (I_N')_{m_s-1} = (m_s+2) - \sum_{i=1}^{s-1} \binom{n_i+1}{2} - \binom{n+1}{2} - \binom{m_s+1}{2} + \sum_{i=1}^{s-1} \binom{n_i+1}{2} - n_1 - n_2 + \binom{n+1}{2} = m_s + 1 - n_1 - n_2.
\]

\( \Box \)

Thus if we prove that, in case 4), \( \dim \left( \frac{I + (h)}{J + (h)} \right)_t \geq m_s + 1 - n_1 - n_2 \), then the map \( \psi \) will be surjective, and we will be done (\( \phi \) will be surjective too, hence an isomorphism).

First we deal with a particular case of 4), namely \( n_1 = 0 \). We will prove

**Lemma 4.4:** Let \( s \geq 5, n \leq m_s, m_s > 0 \) and \( n_1 = 0 \) (i.e. \( t \leq m_1 + m_s \)). Then

\[
\dim \left( \frac{I + (h)}{J + (h)} \right)_t \geq m_s + 1.
\]

**Proof:** We have to find \( m_s + 1 \) linearly independent forms in \( I \) such that their classes remain independent modulo \((J + (h)) \cap I \) (see proof of Lemma 4.1).

Let \( G_1, G_2 \) be the quadric forms defining the cones on \( \Gamma \) with vertices in \( P_1, P_2 \) respectively.

Let \( m_1 = \ldots = m_s \). Then it is easy to check that the forms \( G_1^x G_2^y H^{t-2m_s} \) give what we want, when \( H \) is a plane not containing any of the \( P_i \)’s and \( x + y = m_s \).

If \( m_s = 1, m_1 > m_s \), we consider a form \( S \in (p_1^{m_1-2} \cap p_2^{m_2-1} \cap \ldots \cap p_s^{m_{s-1}-1})_{t-2} \) which is not zero at \( P_s \). In order to prove that such \( S \) exists, the following Lemma (from [3], Lemma 4) will be useful:

**Lemma 4.5:** Let \( P_1, \ldots, P_{\lambda}, P \in \mathbb{P}^r \) be distinct points in general (linear) position and let \( Y = p_1^{m_1} \cap \ldots \cap p_{\lambda}^{m_{\lambda}}, \) with \( m_1 \geq \ldots \geq m_{\lambda} \geq 0 \). If \( t \in \mathbb{N} \) is such that \( rt \geq \sum_{i=1}^{\lambda} m_i \) and \( t \geq m_1 \), then we can find a form \( F \in Y_t \) avoiding \( P \).
So, by this lemma, to check that $S$ exists, it suffices to show that
\[ 3(t - 2) \geq \sum_{i=1}^{s} m_i - (s + 1), \quad t - 2 \geq m_1 - 2, \quad \text{and} \quad t - 2 \geq m_2 - 1. \]

The first inequality is verified, since $n \leq m_s = 1$ and $s \geq 5$ imply that $P(2)$ does not hold for $I_t$, i.e. $3t - 5 \geq \sum_{i=1}^{s} m_i - s$. The other two because $n_1 = 0$, hence $t \geq m_1 + m_s = m_1 + 1$.

In the same way one can find a form $S' \in (p_1^{m_1-1} \cap p_2^{(m_2-2)+} \cap \ldots \cap p_{s-1}^{m_{s-1}-1})_{t-2}$ which is not zero at $P_s$. We only notice that for $m_2 = 1$, we have to check that $3(t - 2) \geq \sum_{i=1}^{s} m_i - s = m_1 - 1$, and this follows from $t \geq m_1 + 1$.

Then the forms $G_1S$, and $G_2S'$ will give what we want.

Now let $m_s > 1$ and $m_1 > m_s$. We can work by induction on $\sum_{i=1}^{s} m_i$, using the previous cases as initial steps).

If $m_1 > m_2$, consider the homogeneous ideals $I^*,J^*$ associated to the schemes

\[ Z^* = (P_1,\ldots,P_s; m_1 - 2, m_2 - 1, m_3 - 1,\ldots,m_{s-1}, m_s - 1) = (P_1,\ldots,P_s; m_1^*, m_2^*,\ldots, m_s^*), \]

\[ W^* = (P_1,\ldots,P_s; m_1 - 2, m_2 - 1,\ldots,m_{s-1} - 1, m_s) = (P_1,\ldots,P_s; m_1^*, m_2^*,\ldots, m_s^* + 1). \]

By induction we get that there exist forms $F_1,\ldots,F_{m_s} \in I^*+(h)$ such that their classes in $\frac{I^*+(h)}{J^*+(h)}$ are linearly independent. Let us check that $t - 2, I^*, J^*$ verify the hypotheses of the Lemma; since $P(m_s + 1)$ doesn’t hold for $I_t$, then

\[ 3t - 5m_s \geq \sum_{i=1}^{s} m_i - sm_s. \]

It follows $3(t - 2) + 5(1 - m_s) \geq \sum_{i=1}^{s} m_i - sm_s - 1 = \sum_{i=1}^{s} m_i^* + s - sm_s$, so (with obvious notation) $n^* \leq m_s - 1 = m_s^*$. Moreover $n_1^* = 0$, in fact $t - 2 \geq m_1 + m_s - 2 \geq m_1^* + m_s^*$, and $m_1^* \geq m_2^* \geq \ldots \geq m_s^* > 0$, since $m_1 > m_2 \geq \ldots \geq m_s > 1$.

Hence we can consider the forms $G_1F_1,\ldots,G_1F_{m_s}$ which are independent modulo $(J+(h)) \cap I$, as required.

We have to find another one, so consider the number: $A = \sum_{i=1}^{s} m_i - m_1 - (s - 1)m_s$.

When $A \geq m_s$, consider the quadric forms $G_2,\ldots,G_{s-1}$ defining the cones on $I$ with vertices in $P_2,\ldots,P_{s-1}$ respectively, and let $F = G_2^{m_2-m_s} G_3^{m_3-m_s} \ldots$, where we go on with the products until we get $m_s$ factors (i.e. deg $F = 2m_s$). Let $F \in p_1^{m_s} \cap p_2^{m_2} \cap \ldots \cap p_{s-1}^{m_{s-1}} \cap p_s^{m_s}$, then we can choose another form $S \in (p_1^{m_1-m_s} \cap p_2^{m_2-m_s} \cap \ldots \cap p_{s-1}^{m_{s-1}-m_{s-1}})_{t-2m_s}$, not passing through $P_s$ (this is possible by Lemma 4.5 since $t - 2m_s \geq m_1 - m_s$, and $3(t - 2m_s) \geq \sum_{i=1}^{s} m_i - sm_s - m_s$).

The forms $FS, G_1F_1,\ldots,G_1F_{m_s}$ are what we want.

When $A < m_s$ we consider $F = G_2^{m_2-A} G_3^{m_3-m_s} \ldots G_{s-1}^{m_{s-1}-m_s}$ and $S \in p_1^{m_1-m_s}$ not passing through $P_s$ (again, it exists since $t - 2m_s \geq m_1 - m_s$ and $3(t - 2m_s) \geq m_1 - m_s$).
Finally, when \( m_s > 1, m_1 = m_2 = \ldots = m_r > m_{r+1}, r < s \), we proceed as before, but starting with ideals \( I^*, J^* \) associated to the schemes:

\[
Z^* = (P_1, ..., P_s; m_1 - 1, ..., m_r - 1, m_r - 2, m_{r+1} - 1, ..., m_s - 1) = (P_1, ..., P_s; m_1^*, ..., m_s^*)
\]
\[
W^* = (P_1, ..., P_s; m_1 - 1, ..., m_r - 1, m_r - 2, m_{r+1} - 1, ..., m_s - 1, m_s),
\]
so that \( m_1^* \geq m_2^* \geq \ldots \geq m_s^* > 0, \{h^* = 0\} \) is the plane \( P_1P_2P_s \) again, and we use \( G_r \) instead of \( G_1 \).

\(\square\)

With the following proposition the proof of Theorem 2.2 will be complete (by Corollary 4.3).

**Proposition 4.6:** In case 4) of Remark 3.6 we have:

\[
\dim \left( \frac{I + (h)}{J + (h)} \right)_t \geq m_s + 1 - n_1 - n_2.
\]

**Hence the map** \( \psi \) **is surjective.**

**Proof:** If \( n_1 = 0 \) the conclusion follows by Lemma 4.4.

Assume \( n_1 > 0 \). Let us consider the form \( S = G_1^{n_1-n_2}G_s^{n_2} \), where, as usual, \( G_i \) defines the cone on \( \Gamma \) with vertex in \( P_i \).

The form \( S \) has degree \( 2n_1 \), multiplicity \( 2n_1 - n_2 \) at \( P_1, n_1 \) at \( P_2, ..., P_{s-1} \) and \( n_1 + n_2 \) at \( P_s \).

Now let

\[
I^* = p_1^{m_1-2n_1+n_2} \cap p_2^{m_2-n_1} \cap \ldots \cap p_{s-1}^{m_{s-1}+1-n_1} \cap p_s^{m_s-n_1-n_2} = p_1^{m_1^*} \cap \ldots \cap p_s^{m_s^*}
\]

and

\[
J^* = p_1^{m_1-2n_1+n_2} \cap p_2^{m_2-n_1} \cap \ldots \cap p_{s-1}^{m_{s-1}+1-n_1} \cap p_s^{m_s-n_1-n_2+1}.
\]

It is easy to check that \( m_1^* \geq m_2^* \geq \ldots \geq m_s^* \geq 0. \)

Since \( t \geq 2n_1 \) and any base of \( \left( \frac{I^* + (h)}{J^* + (h)} \right)_{t-2n_1} \), when multiplied by \( S \), gives independent elements in \( \left( \frac{I^* + (h)}{J^* + (h)} \right)_{t-2n_1} \), we have

\[
\dim \left( \frac{I + (h)}{J + (h)} \right)_t \geq \dim \left( \frac{I^* + (h)}{J^* + (h)} \right)_{t-2n_1}.
\]
Hence we will be finished if we show that

\[ \dim \left( \frac{I^* + (h)}{J^* + (h)} \right)_{t-2n_1} \geq m_s - n_1 - n_2 + 1. \]

Let us consider first the case when \( m_s = n_1 + n_2 \). In this situation we just have to find one form \( F \) in \( I^*_{t-2n_1} \) which is not zero at \( P_s \).

By Lemma 4.5 it suffices to check that:

a) \( 3(t - 2n_1) \geq \sum_{i=1}^{s} m_i - (s + 1)n_1 \);

b) \( t - 2n_1 \geq m_1 - 2n_1 + n_2 \).

Since \( n \leq m_s - n_1 = n_2 \leq n_1 \), then \( P(n + 1) \) does not hold for \( I_t \), i.e. \( 3t - 5n_1 \geq \sum_{i=1}^{s} m_i - sn_1 \), hence a) holds.

From \( m_s = n_1 + n_2 \) we get \( t = m_1 + m_s - n_1 = m_1 + n_2 \), which is b).

Now let \( m_s > n_1 + n_2 \). If we can apply Lemma 4.4 to \( I^*, J^*, h, t - 2n_1 \), we get exactly the inequality (5) and we will be done. Thus let us check that the hypotheses of Lemma 4.4 are satisfied.

We have \( m^* = m_s - n_1 - n_2 > 0 \), so the number of non-zero exponents in \( I^* \) is exactly \( s \geq 5 \).

We also have \( t - 2n_1 = (m_1 + m_s - n_1) - 2n_1 = m_1^* + m_s^* \).

Finally since \( n \leq m_s \), we have that \( P(n + 1) \) does not hold for \( I_t \), i.e. \( 3t - 5n \geq \sum_{i=1}^{s} m_i - sn \). This implies that \( 3(t - 2n_1) - 5(n - n_1) \geq \sum_{i=1}^{s} m_i^* - s(n - n_1) \). Hence, for \( n \geq n_1 \), we get (with obvious notation) \( n^* \leq n - n_1 \leq m_s - n_1 - n_1 \leq m_s - n_1 - n_2 = m_s^* \).

For \( n < n_1 \), since \( s \geq 5 \), we have \( n^* = 0 \leq m_s^* \).

Thus in both cases we get \( n^* \leq m_s^* \), so all the hypotheses of Lemma 4.4 are satisfied, and the proof is complete.

\[ \square \]

REFERENCES

[1]: R.Achilles, P.Schenzel, W.Vogel: Bemerkungen uber Normale Flachheit und Normale Torsionsfreiheit und Anwendungen. Per. Mat. Hung. 12, (1981), 49-75.

[2]: M.V.Catalisano: "Fat" points on a conic. Comm. Algebra 19(8), (1991), 2513-2168.

[3]: M.V.Catalisano, Trung, G.Valla: A sharp bound for the regularity index of fat points in general position. Proc. Amer. Math. Soc. 118 (1993), 717-724.

[4]: J.Harris: Algebraic Geometry. Springer-Verlag, Grad. Texts in Math. 133 (1992), Berlin, New York.
[5]: L.Robbiano: *An algebraic property of $\mathbb{P}^1 \times \mathbb{P}^N$.* Comm.in Alg. 7, (1979), 641-655.

[6]: B.Segre: *Alcune questioni su insiemi finiti di punti in Geometria Algebraica.* Atti Convegno Internazionale di Geometria Algebraica. Torino (1961), 15-33.

[7]: N.V.Trung, G.Valla: *Upper bounds for the regularity index of fat points with uniform position property.* J. Algebra, to appear.

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