Restricted Admissible Limit for Domains of Finite Type

Steven G. Krantz and Baili Min

Abstract: We investigate the boundary behavior of holomorphic functions with respect to a family of curves in a domain of finite type. This work is a generalization of Čirka’s classical result on the unit ball and it supplements the result by Cima and Krantz on the Lindelöf principle for general domains. See [KRA2] for some recent developments in this subject.

Our discussion is carried out in \( \mathbb{C}^2 \).

1. Background

The classical Lindelöf principle says that, in the unit disk, if a bounded holomorphic function has a boundary limit \( L \) at \( e^{i\theta} \) along the radial approach \( re^{i\theta}, r \to 1^- \), then it has non-tangential limit \( L \) at \( e^{i\theta} \). When it comes to several variables, the story is much more subtle. Much has been investigated for the unit ball \( B \subset \mathbb{C}^n \) and Čirka’s generalization in [CIR] is of outstanding significance. His version is to consider the function’s boundary behavior along two types of curves: special and restricted ones. Basically speaking, a special curve in \( B \) is one that is contained in a paraboloid, and a restricted curve is a special one that also has a non-tangential projection in the complex normal direction. These definitions can also be found in Rudin’s book [RUD], as well as the result given by Čirka which supplemented the classical Lindelöf principle as follows.

**Theorem 1.1** (Čirka). Let \( S \), the unit sphere, be the boundary of the ball \( B \). Suppose that \( f \in H^\infty(B) \), \( \zeta \in S \), and \( \Gamma_0(t) \) is a special curve with \( \lim_{t \to 1^-} \Gamma_0(t) = \zeta \), and

\[
\lim_{t \to 1^-} f(\Gamma_0(t)) = L.
\]

Then, for any restricted curve \( \Gamma(t) \) with \( \lim_{t \to 1^-} \Gamma(t) = \zeta \), we have

\[
\lim_{t \to 1^-} f(\Gamma(t)) = L.
\]

One of the important parts of this theorem is that it is linked to the admissible convergence of bounded holomorphic functions on the unit ball because of the fact that any restricted curve will eventually be contained in an admissible approach region based at \( \zeta \), as discussed in [RUD]. Readers are referred to [KRA1] and [RUD] for more about admissible approach regions.

There are still many unknowns and developments about this principle for more general domains in \( \mathbb{C}^n \), among which Cima and Krantz formulated another version of the Lindelöf principle from the point of view of normal functions in their work [CIK].
In this paper, we discuss the generalization to a domain Ω ⊂ C² that is of finite type. We first define the generalized special and restricted curves.

Suppose that Γ = Γ(t) is a curve in Ω approaching to ζ ∈ ∂Ω and let γ = γ(t) be its orthogonal projection to the complex line through a transversal vector to ∂Ω at ζ.

**Definition 1.2.** The curve Γ is special if |Γ(t) − γ(t)|^m = o(|ζ − γ(t)|), where the number m is the type of the point ζ.

**Definition 1.3.** The curve Γ is restricted if it is special, and γ is non-tangential.

Careful discussion of these matters will be carried out in the next section.

The Lindelöf principle in terms of these curves is then as follows. This is the main result of the present paper. We shall explain the concept of finite type in the next section.

**Theorem 1.4.** Suppose that Ω ⊂ C² is a domain of finite type and ζ ∈ ∂Ω. If f ∈ H^∞(Ω), ζ ∈ ∂Ω, and Γ₀ is a restricted curve that approaches to ζ along which f converges:

$$\lim_{t \to 1} f(\Gamma₀(t)) = L,$$

then f has restricted admissible limit L at ζ, which means, for any restricted curve Γ(t) in Ω with \(\lim_{t \to 1−} Γ(t) = ζ\), we have the limit

$$\lim_{t \to 1} f(Γ(t)) = L.$$

2. Special and Restricted Curves

Let Ω be a bounded domain in C², and suppose that ζ is a boundary point in ∂Ω with ν the unit outward normal vector.

The complex tangent space to ∂Ω at ζ is the largest complex subspace of the real tangent space Tζ. The complex normal space is just Cν.

Suppose that Γ is a curve in Ω approaching ζ, Γ : [0, 1) → Ω, and

(2.1) \(\lim_{t \to 1−} Γ(t) = ζ\).

Then let γ be its orthogonal projection to Cν

(2.2) γ = γ(t) = ⟨Γ(t), ν⟩ν.

The definition of the type of ζ is subtle. Geometrically speaking, the type is the order of contact: if l is a nonsingular complex curve tangent to ∂Ω at ζ, and for \(z ∈ l\),

(2.3) \(\text{dist}(z, ∂Ω) = O(|z − ζ|^τ)\),
and if the number \(\tau\) cannot be improved, then we say \(\zeta\) has the type \(\tau\). See [KRA1] for more on finite type. Readers are also referred to [JJK], [BLG], [KRA1], and [NSW] for the discussion of the concept of type.

In his book [RUD], Rudin discussed the relationship between restricted curve and the admissible approach regions in the unit ball. In this section, we also hope to set up a similar context. Since we are dealing with a weakly pseudoconvex domain while the unit ball is strongly pseudoconvex, the definition of the admissible approach region is very different. In our case, the admissible region is broader in the complex tangential direction and relies on the type of the boundary point. See [EMS] and [NSW] for more about this definition.

One of the results in the paper [DBF] implies that the shape of the admissible approach region \(A_{\alpha}(1, 0)\) is reflected by the order of contact such that if the dimension in the complex tangential direction is \(r\) then the dimension in the complex normal direction is \(r^m\).

Let us review the definitions of special and restricted curves:

**Definition 2.1.** A \(\zeta\)-curve \(\Gamma = \Gamma(t)\) is special if
\[
|\Gamma(t) - \gamma(t)|^m = o(|\zeta - \gamma(t)|).
\]

Since \(\gamma\) is the projection of \(\Gamma\) onto the complex normal direction \(\nu\), \(\zeta - \gamma(t)\) is exactly the complex normal component at \(\zeta\) of \(\zeta - \Gamma(t)\), while \(\Gamma(t) - \gamma(t)\) is the complex tangential component. The equation (2.4) relates the complex tangential and normal components. Recall that the type of \(\zeta\) also reflects the order of contact in the sense of complex tangential and normal components in a similar way.

However, a special curve may not be in an admissible approach region with vertex \(\zeta\), which is tangent to \(\partial\Omega\) only in the complex tangential directions.

**Definition 2.2.** The curve \(\Gamma\) is restricted if it is special, and \(\gamma\) is non-tangential to \(\partial\Omega\) at \(\zeta\).

With this extra requirement, we have a quick result:

**Lemma 2.3.** If \(\Gamma(t)\) is a restricted curve then, for \(t\) close enough to 1, \(\Gamma(t)\) lies in an admissible approach region \(A_{\alpha}(1, 0)\).

Consequently we define

**Definition 2.4.** A function \(f : \Omega \rightarrow \mathbb{C}\) has a restricted admissible limit \(L\) at \(\zeta \in \partial\Omega\) if
\[
\lim_{t \to 1} f(\Gamma(t)) = L
\]
for every restricted \(\zeta\)-curve \(\Gamma(t)\).

We observe that admissible convergence implies the restricted admissible convergence, but not vice versa.
3. Proof of the Main Theorem

We now prove our main result, Theorem 1.4, with two major steps.

First we want to see that

$$
\lim_{t \to 1} \left( f(\Gamma(t)) - f(\gamma(t)) \right) = 0.
$$

for any special $\zeta$-curve $\Gamma(t)$.

Suppose that $\Gamma(t)$ is a curve in $\Omega$ that approaches $\zeta \in \partial \Omega$ with $\gamma(t)$ as its projection onto $C\nu$, the complex normal direction to $\partial \Omega$ at $\zeta$. Fix at $t$ the point $(1 - \lambda)\gamma(t) + \lambda \Gamma(t) = \gamma(t) + \lambda(\Gamma(t) - \gamma(t))$, which has the component $\lambda(\Gamma(t) - \gamma(t))$ in the complex tangential direction from $\zeta$ and complex normal component $\zeta - \gamma(t)$.

The type of $\zeta$ is $m$, which is the greatest order of contact to the boundary along a nonsingular complex curve that is tangential to the boundary. Notice that $\Omega \subset \mathbb{C}^2$ and there is only one complex tangential direction to $\partial \Omega$ at $\zeta$, so we can always do a holomorphic change of coordinates to change the curve of best contact into the complex line passing through $\zeta$, along the complex tangential direction. Details can be found in [GUN]. Due to this geometry, there exists a constant $k$ such that, if

$$
|\lambda|^m |\Gamma(t) - \gamma(t)|^m < k |\zeta - \gamma(t)|,
$$

then the point $(1 - \lambda)\gamma(t) + \lambda \Gamma(t) = \gamma(t) + \lambda(\Gamma(t) - \gamma(t))$ is in $\Omega$.

Set

$$
R(t) = \frac{(k |\zeta - \gamma(t)|)^{1/m}}{|\Gamma(t) - \gamma(t)|}.
$$

The argument above indicates that, if $|\lambda| < R$, then $(1 - \lambda)\gamma + \lambda \Gamma \in \Omega$.

Since $\Gamma$ is special, we know from equation 2.4 that

$$
\lim_{t \to 1} \frac{1}{R(t)} = 0.
$$

Now, for each $t$, we can define a holomorphic function

$$
g(\lambda) = f((1 - \lambda)\gamma(t) + \lambda \Gamma(t))
$$

and consider the disc $\mathbb{D}_{R(t)} = \{ \lambda \in \mathbb{C} : |\lambda| < R(t) \}$. Applying the Schwarz lemma to $g(\lambda) - g(0)$ in $\mathbb{D}_R$, we are able to get

$$
|g(1) - g(0)| \leq \frac{2 \|f\|_{\infty}}{R(t)}.
$$

Then it is obvious that

$$
\lim_{t \to 1} (f(\Gamma(t)) - f(\gamma(t))) = 0.
$$
Secondly, recall that a restricted curve is special, therefore we can apply the previous result to the restricted curve \( \Gamma_0(t) \) given in the hypothesis:

\[
\lim_{t \to 1} (f(\Gamma_0(t)) - f(\gamma_0(t))) = 0.
\]

By the hypothesis of the theorem,

\[
\lim_{t \to 1} f(\Gamma_0(t)) = L,
\]

we immediately know that

\[
\lim_{t \to 1} f(\gamma_0(t)) = L.
\]

Now suppose that \( \Gamma \) is an arbitrary restricted curve approaching to \( \zeta \). Notice that both \( \gamma \) and \( \gamma_0 \) are non-tangential at \( \zeta \) according to the definition of being restricted, so by the Lindelöf principle (see [HIL]) we know that

\[
\lim_{t \to 1} f(\gamma(t)) = \lim_{t \to 1} f(\gamma_0(t)) = L,
\]

and it then follows from the equation that

\[
\lim_{t \to 1} f(\Gamma(t)) = L,
\]

That concludes the proof.

4. Concluding Remarks

We have studied in this paper the detailed behavior of a holomorphic function on a special curve and a restricted curve. We have restricted attention to finite type domains in \( \mathbb{C}^2 \). There is definite interest in developing these ideas in higher dimensions, and on more general types of domains.

References

[BAS] F. Bagemihl and W. Seidel, Some boundary properties of analytic functions, *Math. Z.* 61(1954), 186–199.

[BLG] T. Bloom and I. Graham, A geometric characterization of points of type \( m \) on real submanifolds of \( \mathbb{C}^n \), *Jour. Diff. Geom.* 45(1978), 133–147.

[CIR] E.-M. Čirka, The Lindelöf and Fatou theorems in \( \mathbb{C}^n \), *Mat. U.S.S.R. Sb* 92(1973), 622–644. *Math. U.S.S.R. Sb* 21(1973), 619–641.

[CIK] J. C. Cima and S. Krantz, The Lindelöf principle and normal functions of several complex variables, *Duke Math.* 50(1983), 303–328.

[DBF] F. Di Biase and B. Fischer, Boundary behaviour of \( H^p \) functions on convex domains of finite type in \( \mathbb{C}^n \), *Pacific Jour. Math.* 183(1998), 25–38.

[DOG] F. Docquier and H. Grauert, Leisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten (German), *Math. Ann.* 140(1960), 94–123.
[GUN] R. C. Gunning, Lectures on Complex Analytic Varieties: The Local Parametrization Theorem. Princeton University Press, Princeton, NJ, 1970

[HIL] E. Hille, Analytic Function Theory. Ginn, Boston, 1959.

[JJK] J. J. Kohn, Boundary behavior of $\overline{\partial}$ on weakly pseudo-convex manifolds of dimension two, Jour. Diff. Geom. 6(1972), 523–542

[KRA1] S. G. Krantz, Function Theory of Several Complex Variables, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[KRA2] S. G. Krantz, The Lindelőf principle in several complex variables J. Math. Anal. Appl. 326(2007), 1190–1198

[NAG] A. Nagel, Smooth zero sets and interpolation sets for some algebras of holomorphic functions on strictly pseudoconvex domains, Duke Math. J. 43(1976), 323–348.

[NSW] A. Nagel, E. M. Stein, and S. Wainger, Boundary behavior of functions holomorphic on domains of finite type, Proc. Nat. Acad. Sci. 78(1981), no. 11, part 1, 6596–6599.

[RUD] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2008.

[EMS] E. M. Stein, Boundary Behavior of Holomorphic Functions of Several Complex Variables, Princeton University Press, Princeton, 1970.

S. G. Krantz
Department of Mathematics
Washington University in St. Louis
St. Louis, Missouri 63130
sk@math.wustl.edu

Baili Min
Department of Mathematics
Lafayette College
Easton, PA 18042
minbaili@gmail.com