INHOMOGENEOUS DIRICHLET–BOUNDARY VALUE PROBLEM FOR TWO-DIMENSIONAL QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS

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(Received 2 October 2019 and revised 11 December 2019)

Abstract. We consider the inhomogeneous Dirichlet–boundary value problem for the quadratic nonlinear Schrödinger equations, which is considered as a critical case for the large-time asymptotics of solutions. We present sufficient conditions on the initial and boundary data which ensure asymptotic behavior of small solutions to the equations by using the classical energy method and factorization techniques of the free Schrödinger group.

1. Introduction

We consider the initial–boundary value problem for the nonlinear Schrödinger equations on the upper half-line with a critical nonlinearity:

\[
\begin{aligned}
&i \partial_t u + \frac{1}{2} (\partial_{x_1}^2 + \partial_{x_2}^2) u = \lambda |u|^2 u, \quad (x_1, x_2) \in \mathbb{D}, \quad t > 0, \\
&u(0, x_1, x_2) = u_0(x_1, x_2), \quad (x_1, x_2) \in \mathbb{D}, \\
&u(t, x_1, 0) = h(t, x_1),
\end{aligned}
\]

(1.1)

where \( \mathbb{D} = \mathbb{R} \times \mathbb{R}^+ \), \( \mathbb{R}^+ = (0, \infty) \). The main purpose of this paper is to show asymptotic behavior of small solutions for a large time.

In the case of homogeneous data \( h(t, x_1) = 0 \), there are a lot of papers; see e.g. [6–9, 23, 24, 27] for more general domains and nonlinearities. There are many works devoted to the study of one-dimensional initial–boundary value problems with inhomogeneous boundary conditions. One of them can be seen in [10], in which a general approach was proposed to study the well-posedness and qualitative properties of solutions of pseudodifferential nonlinear equations on a half-line by using the Laplace transform and contraction mapping principle.

Also there are many results concerned with the one-dimensional nonlinear Schrödinger equation on a half-line. In [26], the authors proved the existence of global solutions in the energy space \( \mathbf{H}^1 \) with the conditions such that the boundary data are in \( \mathbf{C}^2 \) and have compact support and satisfy the compatibility conditions. This solution was obtained as the limit of a sequence of solutions of the approximate problems for which an energy method

2010 Mathematics Subject Classification: Primary 35Q35.

Keywords: nonlinear Schrödinger equation; large-time asymptotics; inhomogeneous initial–boundary value problem; upper half-plane.
provides suitable \textit{a priori} bounds. In [13] and [3], by the boundary integral method, the local well-posedness was obtained if the initial data belong to Sobolev spaces $H^s(\mathbb{R}^+)$ for $s > 0$. Also globally well-posedness was established for $s \geq 1$. In particular, it was shown that the relevant Dirichlet–boundary value problem is globally well-posed in $H^1(\mathbb{R}^+)$ if the initial data lie in $H^1(\mathbb{R}^+)$ and the boundary data are selected from $H^{3/4}(\mathbb{R}^+)$. In paper [4], the well-posedness for the cubic nonlinear Schrödinger equation (CNLS) on the half-line with data $(u(0, x), u(t, 0))$ in Sobolev spaces $(H^s_x(\mathbb{R}^+)) \times (H^{(2s+1)/4}_t(0, T), s > \frac{1}{2})$, was established via the formula obtained through the unified transform method and a contraction mapping approach. This unified transform method also provides an approach for obtaining the large-time asymptotics of the CNLS equation for certain particular boundary conditions called linearizable (see [5]).

Another method for analyzing one-dimensional initial boundary value problems, based on the Riemann–Hilbert approach, was introduced in [14]. By this method, in [15], it was shown that long-range scattering occurs in the CNLS equation. The advantage of the Riemann–Hilbert approach is that it can also be applied to non-integrable equations with general inhomogeneous boundary data, but some technical problems have to be overcome (see [16, 18–20]). There are also some papers discussing the inhomogeneous initial–boundary value problem for multidimensional nonlinear Schrödinger equations. Among them, we refer to [1, 2, 21, 22, 25]. These papers are devoted to fundamental questions of the local existence and uniqueness of solutions in Sobolev space $H^s, s > 0$, by applying classical tools such as Strichartz and energy estimates. The global well-posedness was also discussed in $H^1$ space. The main idea of the proof for the local well-posedness was to derive a boundary integral operator for the corresponding inhomogeneous boundary condition and obtain the Strichartz estimates for this operator.

We now introduce some results concerning asymptotics of solutions for large time which is of interest in the present paper. Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet–boundary control were studied in [22]. It was shown that the decay rate of the solutions up to an exponential one holds, and some regularity and stabilization properties were obtained for the strong solutions. The proof was based on the direct multiplier method combined with monotonicity and compactness techniques. In [17], the asymptotic behavior of solutions of the inhomogeneous Dirichlet initial–boundary value problem for nonlinear Schrödinger equations on an upper-right quarter-plane was studied by using the method based on the Riemann–Hilbert approach and the theory of the Cauchy-type integral equations.

In the present paper, we study (1.1) with a critical power nonlinearity on the upper half-plane. We focus on optimal time decay estimates of solutions in $L^\infty$ space. As far as we know, there are no results for the time decay of solutions to the inhomogeneous boundary value problem of nonlinear Schrödinger equations on the upper half-plane with a critical power nonlinearity. We note here that our method is based on the energy method and factorization techniques of the Schrödinger evolution operator with homogeneous data and is different from the one used in [17].
1.1. Notation and main results

Weighted Sobolev space is defined by

$$H^{m,s}(\Omega) = \left\{ \phi \in L^2(\Omega) \left| \|\phi\|_{H^{m,s}(\Omega)} = \sum_{|\alpha| \leq s, |\beta| \leq m} \|x^\alpha \partial^\beta \phi\|_{L^2(\Omega)} < \infty \right. \right\},$$

where

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}, \quad |\alpha| = \alpha_1 + \alpha_2, \quad \partial^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}, \quad |\beta| = \beta_1 + \beta_2,$$

and $$\Omega = \mathbb{D}$$ or $$\Omega = \mathbb{R}$$. We also let $$C_b([0, \infty); X)$$ be the function space of $$X$$-valued bounded continuous functions and

$$C^l_b([0, \infty); X) = \left\{ \|h\|_{C^l_b([0, \infty); X)} = \sum_{1 \leq n \leq l} \sup_{t \in [0, \infty)} \|\partial_t^n \phi(t)\|_X < \infty \right\}.$$

For simplicity we write $$H^m(\Omega) = H^{m,0}(\Omega)$$.

In [11], we considered the same problem as in the present paper with a super-critical nonlinearity:

$$\begin{align*}
&i \partial_t u + \frac{1}{2} (\partial^2_{x_1} + \partial^2_{x_2}) u = \lambda |u|^{p-1} u, \quad (x_1, x_2) \in \mathbb{D}, \quad t > 0, \\
u(0, x_1, x_2) = u_0(x_1, x_2), \quad (x_1, x_2) \in \mathbb{D}, \\
u(t, x_1, 0) = h(t, x_1),
\end{align*}$$

and showed the following.

**Proposition 1.** We assume that $$p > 2$$, and the compatibility condition $$u_0(x_1, 0) = h(0, x_1)$$,

$$u_0 \in X_0 = H^3(\mathbb{D}) \cap H^{2,1}(\mathbb{D}) \cap H^{0,2}(\mathbb{D})$$

and

$$h \in X,$$

where

$$X = C^2_b([0, \infty); L^2(\mathbb{R})) \cap C^1_b([0, \infty); H^2(\mathbb{R}) \cap C_b([0, \infty); H^4(\mathbb{R}))$$

$$\cap C^1_b([0, \infty); H^{0,2}(\mathbb{R})) \cap C_b([0, \infty); H^{2,2}(\mathbb{R})) \cap C_b([0, \infty); H^{0,4}(\mathbb{R})).$$

Furthermore we assume that

$$\|u_0\|_{X_0} \leq \rho,$$

$$\langle t \rangle^{9/2 + \gamma} (\|\partial^2_t h(t)\|_{L^2(\mathbb{R})} + \|\partial_t h(t)\|_{H^2(\mathbb{R})} + \|h(t)\|_{H^4(\mathbb{R})})$$

$$+ \langle t \rangle^{5/2 + \gamma} (\|\partial_t h(t)\|_{H^{0,2}(\mathbb{R})} + \|h(t)\|_{H^{2,2}(\mathbb{R})}) + \langle t \rangle^{1/2 + \gamma} \|h(t)\|_{H^{0,4}(\mathbb{R})} \leq \rho^p,$$

where $$\gamma > 0$$ is small. Then there exist $$\rho, \gamma > 0$$ such that (1.2) has a unique global in time solution

$$u \in L^\infty(0, \infty; X_0)$$

satisfying the time decay estimate

$$\|u(t)\|_{L^\infty(\mathbb{D})} \leq C(t)^{-1}.$$
Our purpose in this paper is to consider (1.5) with the critical nonlinearity \( p = 2 \). We now state our main result in the paper.

**Theorem 1.1.** We assume that
\[
  u_0 \in \tilde{X}_0 = H^2(\mathbb{D}) \cap H^{0.2}(\mathbb{D})
\]
and
\[
  h \in \tilde{X},
\]
where
\[
  \tilde{X} = C^2_b([0, \infty); L^2(\mathbb{R})) \cap C^1_b([0, \infty); H^2(\mathbb{R})) \cap C_b([0, \infty); H^4(\mathbb{R})) \cap C^1_b([0, \infty); H^{0.1}(\mathbb{R})) \cap C_b([0, \infty); H^{2.1}(\mathbb{R})) \cap C_b([0, \infty); H^{0.2}(\mathbb{R})).
\]

Furthermore we assume that \( u_0(x_1, 0) = h(0, x_1) = 0 \) holds and that
\[
  \|u_0\|_{\tilde{X}_0} \leq \rho,
\]
\[
  (t)^{5/2+\gamma} \left( \|\partial_t^2 h(t)\|_{L^2(\mathbb{R})} + \|\partial_t h(t)\|_{H^2(\mathbb{R})} + \|h(t)\|_{H^4(\mathbb{R})} \right) + (t)^{1+\gamma} \left( \|\partial_t h(t)\|_{H^{0.1}(\mathbb{R})} + \|h(t)\|_{H^{2.1}(\mathbb{R})} \right) + (t)^{1/2+\gamma} \|h(t)\|_{H^{0.2}(\mathbb{R})} \leq \rho^2,
\]
where \( \gamma > 0 \). Then there exist \( \rho, \gamma > 0 \) such that (1.1) has a unique global in time solution
\[
  u \in L^\infty(\mathbb{R}^+; \tilde{X}_0)
\]
satisfying the time decay estimate
\[
  \|u(t)\|_{L^\infty(\mathbb{D})} \leq C \rho(t)^{-1}.
\]
Moreover, for any small \( (u_0, h) \in \tilde{X}_0 \times \tilde{X} \), there exists a unique \( \Psi_+ \in L^\infty(\mathbb{D}) \) such that the asymptotics of solutions
\[
  \left\| u(t) - \frac{1}{it} \exp \left( \frac{i|x|^2}{2t} \right) \Psi_+ \left( \frac{x}{t} \right) \exp \left( -i\lambda|\Psi_+ \left( \frac{x}{t} \right) | \log t \right) \right\|_{L^\infty(\mathbb{D})} = o(t^{-1})
\]
holds.

**Remark 1.1.** The regularity of solutions such that \( u \in L^\infty(\mathbb{R}^+; H^2(\mathbb{D})) \) implies the derivative \( u_{x_2}(t, x) \) is continuous at the origin. Therefore the compatibility condition \( u_0(x_1, 0) = h(0, x_1) \) is necessary. However the conditions \( u_0(x_1, 0) = 0 \) and \( h(0, x_1) = 0 \) are not natural for the problem (1.1). By using these conditions, we treat (1.1) by separating the problem into two problems (1.5) and (1.6) (see below). Furthermore the conditions \( u_0(x_1, 0) = h(0, x_1) = 0 \) make boundary and data conditions weaker than those used in the previous paper [11]. We think that these zero conditions could be relaxed by the method of paper [17] with some additional conditions on the boundary and data conditions.

Denote by
\[
  U_D(t)\phi = \frac{1}{2\pi it} \int_0^\infty \left( \exp \left( -\frac{(x_2 - y_2)^2}{2it} \right) - \exp \left( -\frac{(x_2 + y_2)^2}{2it} \right) \right) \times \int_\mathbb{R} \exp \left( -\frac{|x_1 - y_1|^2}{2it} \right) \phi(x_1 - y_1, x_2 - y_2) dy_1 dy_2 \quad (1.3)
\]
and
\[ U_1(t)\phi = \frac{1}{(2\pi t)^{1/2}} \int_{\mathbb{R}} \exp \left( i \frac{(x_1 - y_1)^2}{2t} \right) \phi(y_1) \, dy_1. \tag{1.4} \]

In [11], it was proved that the solution of (1.1) has the following integral representation:
\[ u(t) = U_D(t)u_0 + z(t) - i \int_0^t U_D(t - \tau) f(\tau) \, d\tau, \]
where
\[ z(t, x_1, x_2) = \frac{1}{\sqrt{2\pi t}} \int_0^t \frac{x_2}{\tau^{3/2}} \exp \left( i \frac{x_2^2}{2\tau} \right) U_1(\tau) h(t - \tau, x_1) \, d\tau. \]

By the additional assumptions such that \( u_0(x_1, 0) = h(0, x_1) = 0 \), we find that the solution of (1.1) can be reduced to the combination of the solution to the homogeneous boundary value problem
\[
\begin{aligned}
\begin{cases}
  i \partial_t w + \frac{1}{2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 \right) w = \lambda |u|^2 u, & (x_1, x_2) \in \mathbb{D}, \ t > 0, \\
  w(0, x_1, x_2) = u_0(x_1, x_2), & (x_1, x_2) \in \mathbb{D}, \\
  w(t, x_1, 0) = 0,
\end{cases}
\end{aligned}
\tag{1.5}
\]
and the solution to the inhomogeneous boundary value problem with zero initial data
\[
\begin{aligned}
\begin{cases}
  i \partial_t z + \frac{1}{2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 \right) z = 0, & (x_1, x_2) \in \mathbb{D}, \ t > 0, \\
  z(0, x_1, x_2) = 0, & (x_1, x_2) \in \mathbb{D}, \\
  z(t, x_1, 0) = h(t, x_1).
\end{cases}
\end{aligned}
\tag{1.6}
\]

Equation (1.5) can be written as the integral form
\[ w(t, x_1, x_2) = U_D(t)u_0 - i \int_0^t U_D(t - \tau) \lambda |u|^2 u(\tau) \, d\tau. \tag{1.7} \]

Then the solution of (1.6) can be written explicitly as
\[ z(t, x_1, x_2) = \frac{1}{\sqrt{2\pi t}} \int_0^t \frac{x_2}{\tau^{3/2}} \exp \left( i \frac{x_2^2}{2\tau} \right) U_1(\tau) h(t - \tau, x_1) \, d\tau \tag{1.8} \]
(see [11]).

We close the section to summarize the Sobolev-type inequality and the estimates of the nonlinear terms through the operator
\[ J_{x_j} = x_j + it \partial_{x_j} = it e^{i|x|^2/2t} \partial_{x_j} e^{-i|x|^2/2t}, \]
which were shown in [11].

**Lemma 1.1.** Let \( \phi, J_{x_1}\phi, J_{x_2}\phi, J_{x_1}J_{x_2}\phi \in L^2(\mathbb{D}). \)

The following estimate
\[ \|\phi\|_{L^\infty(\mathbb{D})} \leq Ct^{-1} \|\phi\|_{L^2(\mathbb{D})}^{1/2} \|J_{x_1}\phi\|_{L^2(\mathbb{D})}^{1/2} + Ct^{-1} \|J_{x_1}\phi\|_{L^2(\mathbb{D})}^{1/2} \|J_{x_2}\phi\|_{L^2(\mathbb{D})}^{1/2}, \]
holds.
Lemma 1.2. Let
\[ \phi, J_{x_1} \phi, J_{x_2} \phi, J_{x_1}^2 \phi, J_{x_2}^2 \phi, J_{x_1} J_{x_2} \phi \in L^2(\mathbb{D}) \]
and
\[ \phi(t, x_1, 0) = h(t, x_1) \in L^\infty(\mathbb{R}). \]
Then the following estimates
\[ \| J_{x_1} J_{x_2} |\phi| \|_{L^2(\mathbb{D})} \leq C \|\phi\|_{L^\infty(\mathbb{D})} \| J_{x_1} J_{x_2} \phi \|_{L^2(\mathbb{D})} + C(\| h(t) \|_{L^\infty(\mathbb{R})} + \|\phi\|_{L^\infty(\mathbb{D})}) \| J_{x_2}^2 \phi \|_{L^2(\mathbb{D})}, \]
are valid.

In Section 2, some preliminary estimates are obtained by the energy method. Section 3 is devoted to the proof of Theorem 1.1.

2. Energy estimate

In order to get the main result, we apply the classical energy estimate to (1.5) and (1.6). We have the following results.

Lemma 2.1. Let \( u_0 \in L^2(\mathbb{D}) \), where \( \mathbb{D} = \mathbb{R} \times \mathbb{R}^+ \), and
\[ \| h(t) \|_{L^2(\mathbb{R})} + \langle t \rangle^{1/2} \| \partial_\tau h(t) \|_{L^2(\mathbb{R})} + \langle t \rangle^{1/2} \| \partial_{x_1}^2 h(t) \|_{L^2(\mathbb{R})} \leq C \rho^2 (t)^{-1/2-\gamma}, \]
where \( 0 < \gamma \). Then the solutions of (1.5) and (1.6) satisfy
\[ \| w(t) \|_{L^2(\mathbb{D})} \leq \| u_0 \|_{L^2(\mathbb{D})} + C \int_0^t \| u(t) \|_{L^2(\mathbb{D})} \, d\tau \]
and
\[ \| z(t) \|_{L^2(\mathbb{D})} \leq C \rho^2. \]

Proof. the first estimate follows from the energy method. To get the last one, we apply the energy estimate to (1.6) to get
\[ i \partial_\tau \| z(t) \|_{L^2(\mathbb{D})}^2 = \frac{1}{2} \int_{\mathbb{R}} \left( h(t, x_1) \partial_{x_2} z(t, x_1, 0) - h(t, x_1) \partial_{x_2} \bar{z}(t, x_1, 0) \right) \, dx_1, \]
which implies
\[ \| z(t) \|_{L^2(\mathbb{D})}^2 \leq C \int_0^t \| \partial_{x_2} z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} \| h(\tau) \|_{L^2(\mathbb{R})} \, d\tau. \]
we need a priori estimates of \( z(t, x_1, 0) \) to get the desired estimates of \( z(t, x_1, x_2) \). We put \( y = x_2 \tau^{-1/2} \) in (1.8), then

\[
z(t, x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_0^t x_2 \tau^{-3/2} e^{i x_2^2 / 2\tau} U_1(\tau) h(t - \tau, x_1) d\tau \\
= -\frac{2}{\sqrt{2\pi}} \int_{x_2/\sqrt{\tau}}^\infty e^{-y^2 / 2\tau} U_1 \left( \frac{x_2^2}{y^2} \right) h \left( t - \frac{x_2^2}{y^2}, x_1 \right) dy.
\]

Hence,

\[
z(t, x_1, 0) = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-y^2 / 2\tau} dy h(t, x_1) = h(t, x_1)
\]

and

\[
\partial_\tau z(t, x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_0^t x_2 \tau^{-3/2} e^{i x_2^2 / 2\tau} U_1(\tau) \partial_\tau h(t - \tau, x_1) d\tau
\]

since \( h(0, x_1) = 0 \). We also have with \( y = x_2 \tau^{-1/2} \) that

\[
\partial_{x_2} z(t, x_1, x_2) = -\frac{2}{\sqrt{2\pi}} \int_{x_2/\sqrt{\tau}}^\infty e^{-y^2 / 2\tau} 2 \frac{x_2}{y^2} \partial_{x_2^2} \left( U_1 \left( \frac{x_2^2}{y^2} \right) h \left( t - \frac{x_2^2}{y^2}, x_1 \right) \right) dy
\]

\[
= -\frac{2}{\sqrt{2\pi}} \int_0^t e^{-x_2^2 / 2\tau} \frac{1}{\sqrt{x_2}} \partial_\tau \left( U_1(\tau) h(t - \tau, x_1) \right) d\tau
\]

\[
= 2i \int_0^t e^{-x_2^2 / 2\tau} \frac{1}{\sqrt{x_2}} \partial_\tau \left( U_1(\tau) \left( i \partial_\tau + \frac{1}{2} \partial_{x_1}^2 \right) h(t - \tau, x_1) \right) d\tau,
\]

which implies

\[
\partial_{x_2} z(t, x_1, 0) = \frac{2i}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{\tau}} U_1(\tau) \left( i \partial_\tau + \frac{1}{2} \partial_{x_1}^2 \right) h(t - \tau, x_1) d\tau. \tag{2.5}
\]

So, we have

\[
\| \partial_{x_2} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \int_0^t \frac{1}{\sqrt{\tau}} \left( \| \partial_\tau h(t - \tau, \cdot) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 h(t - \tau, \cdot) \|_{L^2(\mathbb{R})} \right) d\tau.
\]

By (2.6) and (2.1), we have

\[
\| \partial_{x_2} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \int_0^t \frac{1}{\sqrt{\tau}} \left( \| \partial_\tau h(t - \tau, \cdot) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 h(t - \tau, \cdot) \|_{L^2(\mathbb{R})} \right) d\tau
\]

\[
\leq C \rho^2 \int_0^t \frac{1}{\sqrt{\tau}} \frac{1}{(t - \tau)^{1/\gamma}} d\tau \leq C \rho^2 (t)^{-1/2}. \tag{2.7}
\]

This completes the proof of the lemma.

\[\square\]

\textbf{Lemma 2.2.} Let \( u_0 \in H^1(D) \), (2.1) and

\[
(t)^{1/2} \| \partial_\tau \partial_{x_1} h(t) \|_{L^2(\mathbb{R})} + (t)^{1/2} \| \partial_{x_1}^3 h(t) \|_{L^2(\mathbb{R})} \leq C \rho^2 (t)^{-1/2 - \gamma}, \tag{2.8}
\]
where $0 < \gamma$. Then the solutions of (1.5) and (1.6) satisfy

$$
\|w(t)\|_{H^1(D)} \leq \|u_0\|_{H^1(D)} + C \int_0^t \|u\|_{H^1(D)} d\tau
$$

(2.9)

and

$$
\|z(t)\|_{H^1(D)} \leq C\rho^2.
$$

(2.10)

**Proof.** We differentiate (1.5) with respect to $x_2$ to get

$$
i \partial_t \partial_{x_2} w + \frac{1}{2} (\partial_{x_1}^2 + \partial_{x_2}^2) \partial_{x_2} w = \lambda \partial_{x_2} |u| u.
$$

We multiply both sides of the above by $\overline{\partial_{x_2} w}$, integrate in space and take the imaginary part to get

$$
\|\partial_{x_2} w(t)\|_{L^2(D)} \leq \|\partial_{x_2} u_0\|_{L^2(D)} + C \int_0^t \|\partial_{x_2} |u| u(\tau)\|_{L^2(D)} d\tau,
$$

(2.11)

where we have used the fact that $\partial_{x_2}^2 w(t, x_1, 0) = 0$ from the integral equation (1.7). Similarly, we also have (2.11), when we replace $\partial_{x_2}$ by $\partial_{x_1}$. Therefore we have (2.9). In the same way as in the proof of (2.4), we have

$$
\|\partial_{x_1} z(t)\|_{L^2(D)}^2 \leq C \int_0^t \|\partial_{x_2} \partial_{x_1} z(\tau, \cdot, 0)\|_{L^2(\mathbb{R})} \|\partial_{x_1} h(\tau)\|_{L^2(\mathbb{R})} d\tau.
$$

By (2.6) and (2.8)

$$
\|\partial_{x_2} \partial_{x_1} z(t, \cdot, 0)\|_{L^2(\mathbb{R})} \leq C \int_0^t \frac{1}{\sqrt{\tau}} \|\partial_{x_1} h(t - \tau, \cdot)\|_{L^2(\mathbb{R})} + \|\partial_{x_1}^3 h(t - \tau, \cdot)\|_{L^2(\mathbb{R})} d\tau
$$

$$
\leq C\rho^2 \int_0^t \frac{1}{\sqrt{\tau}} \frac{1}{(t - \tau)^{1+\gamma}} d\tau \leq C\rho^2 (t)^{-1/2}.
$$

(2.12)

Hence

$$
\|\partial_{x_1} z(t)\|_{L^2(D)} \leq C\rho^2.
$$

(2.13)

We apply the energy method to (1.6) to get

$$
\|\partial_{x_2} z(t)\|_{L^2(D)} \leq C \left| \int_{\mathbb{R}^+} \int_{\mathbb{R}} \partial_{x_1} (\partial_{x_1} \partial_{x_2} z \cdot \overline{\partial_{x_2} z}) \, dx_1 \, dx_2 \right|
$$

$$
+ C \left| \int_{\mathbb{R}^+} \int_{\mathbb{R}} \partial_{x_2} (\partial_{x_2}^2 z \cdot \overline{\partial_{x_2} z}) \, dx_1 \, dx_2 \right|
$$

$$
\leq C \int_0^t \|\partial_{x_2} z(\tau, \cdot, 0)\|_{L^2(\mathbb{R})} \|\partial_{x_2}^2 z(\tau, \cdot, 0)\|_{L^2(\mathbb{R})} d\tau.
$$
We get by (1.6)
\[ \partial_{x_2}^2 z(t, x_1, x_2) = -\partial_{x_1}^2 z(t, x_1, x_2) - 2i \partial_t z(t, x_1, x_2) \]
\[ = -\frac{1}{\sqrt{2\pi}} \int_0^t x_2 e^{ix_2^2/2t} U_1(\tau) \partial_{x_1}^2 h(t - \tau, x_1) d\tau \]
\[ - 2i \frac{1}{\sqrt{2\pi}} \int_0^t x_2 e^{ix_2^2/2t} U_1(\tau) \partial_t h(t - \tau, x_1) d\tau \]
\[ = \frac{2}{\sqrt{2\pi}} \int_{x_2/\sqrt{t}}^{\infty} e^{-y^2/2t} U_1 \left( \frac{x_2}{y^2} \right) \partial_{x_1}^2 h \left( t - \frac{x_2}{y^2}, x_1 \right) dy \]
\[ + 2i \frac{2}{\sqrt{2\pi}} \int_{x_2/\sqrt{t}}^{\infty} e^{-y^2/2t} U_1 \left( \frac{x_2}{y^2} \right) \partial_t h \left( t - \frac{x_2}{y^2}, x_1 \right) dy, \]
from which it follows that
\[ \partial_{x_2}^2 z(t, x_1, 0) = -\partial_{x_1}^2 h(t, x_1) - 2i \partial_t h(t, x_1). \]
Therefore, by the assumptions on \( h \), we obtain
\[ |\partial_{x_2}^2 z(t, x_1, 0)| \leq |\partial_{x_1}^2 h(t, x_1)| + 2|\partial_t h(t, x_1)|, \]
which implies
\[ \| \partial_{x_2}^2 z(t, \cdot, 0) \|_{L^p(\mathbb{R})} \leq C \| \partial_{x_1}^2 h(t, \cdot) \|_{L^p(\mathbb{R})} + C \| \partial_t h(t, \cdot) \|_{L^p(\mathbb{R})} \leq C \rho^2 (t)^{-1-\gamma} \] (2.14)
for any \( p \geq 2 \). By (2.7) and (2.14) we have
\[ \| \partial_{x_2} z(t) \|_{L^2(\mathbb{D})} \leq C \rho^2, \]
from which, with (2.13), the estimate (2.10) follows. This completes the proof of the lemma.

**Lemma 2.3.** Let \( u_0 \in H^2(\mathbb{D}) \), (2.1), (2.8) and
\[ (t)^{1/2} \| \partial_{x_1}^2 h(t) \|_{L^2(\mathbb{R})} + (t)^{1/2} \| \partial_{x_1}^4 h(t) \|_{L^2(\mathbb{R})} \leq C \rho^2 (t)^{-1/2-\gamma}, \] (2.15)
where \( \gamma > 0 \). Then the solutions of (1.5) and (1.6) satisfy
\[ \| w(t) \|_{H^2(\mathbb{D})} \leq \| u_0 \|_{H^2(\mathbb{D})} + C \int_0^t \| u(u(\tau)) \|_{H^2(\mathbb{D})} d\tau \]
and
\[ \| z(t) \|_{H^2(\mathbb{D})} \leq C \rho^2. \]

**Proof.** In the same way as in the proof of \( H^1(\mathbb{D}) \) estimate, we have
\[ \| \partial_{x_1}^2 w(t) \|_{L^2(\mathbb{D})} \leq \| \partial_{x_1}^2 u_0 \|_{L^2(\mathbb{D})} + C \int_0^t \| \partial_{x_1}^2 |u(u(\tau))| \|_{L^2(\mathbb{D})} d\tau, \]
\[ \| \partial_{x_1} \partial_{x_2} w(t) \|_{L^2(\mathbb{D})} \leq \| \partial_{x_1} \partial_{x_2} u_0 \|_{L^2(\mathbb{D})} + C \int_0^t \| \partial_{x_1} \partial_{x_2} |u(u(\tau))| \|_{L^2(\mathbb{D})} d\tau, \]
and
\[ \| \partial_{x_2}^2 w(t) \|_{L^2(\mathbb{D})} \leq \| \partial_{x_2}^2 u_0 \|_{L^2(\mathbb{D})} + C \int_0^t \| \partial_{x_2}^2 |u|u(\tau) \|_{L^2(\mathbb{D})} d\tau, \]
where we have used the fact that \( \partial_{x_2}^2 w(t, x_1, 0) = 0 \). We also have by the energy method, (2.7) and (2.1)–(2.15)
\[ \| \partial_{x_1}^2 z(t) \|_{L^2(\mathbb{D})}^2 \leq C \int_0^t \| \partial_{x_2} \partial_{x_1}^2 z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} \| \partial_{x_1}^2 h(\tau) \|_{L^2(\mathbb{R})} d\tau. \quad (2.16) \]
In the same way as in the proof of (2.7) we have by (2.15)
\[ \| \partial_{x_2} \partial_{x_1}^2 z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \rho^2 (t)^{-1/2}. \quad (2.17) \]
Therefore from (2.16) it follows that
\[ \| \partial_{x_1}^2 z(t) \|_{L^2(\mathbb{D})}^2 \leq C \rho^2. \]
We also have
\[ \| \partial_{x_1} \partial_{x_2} z(t) \|_{L^2(\mathbb{D})} \leq C \int_0^t \| \partial_{x_2} \partial_{x_1} z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} \| \partial_{x_1}^3 h(\tau) \|_{L^2(\mathbb{R})} + \| \partial_{x_1} \partial_{x_2} h(\tau) \|_{L^2(\mathbb{R})} d\tau. \]
In the same way as in the proof of (2.14), we get
\[ \| \partial_{x_2}^2 \partial_{x_1} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \rho^2 (t)^{-1-\gamma} \]
for \( \gamma > 0 \) and, as a consequence,
\[ \| \partial_{x_1} \partial_{x_2} z(t) \|_{L^2(\mathbb{D})} \leq C \rho^2. \]

We again use the energy method to obtain
\[ \partial_t \| \partial_{x_2}^2 z(t) \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 \]
\[ = \frac{1}{2} \text{Im} \int_\mathbb{R} \int_0^\infty \partial_{x_2} \left( \partial_{x_2}^3 z(t, x_1, x_2) \cdot \overline{\partial_{x_2}^2 z(t, x_1, x_2)} \right) dx_2 \, dx_1 \]
\[ = \frac{1}{2} \text{Im} \int_\mathbb{R} \int_0^\infty \partial_{x_2} \left( -2i \partial_{x_1} \overline{\partial_{x_2}^2 z(t, x_1, x_2)} \cdot \overline{\partial_{x_2}^2 z(t, x_1, x_2)} \right) dx_2 \, dx_1 \]
\[ = -\frac{1}{2} \text{Im} \int_\mathbb{R} \left( \partial_{x_2} \left( -2i \partial_{x_1} - \partial_{x_1}^2 \right) z(t, x_1, x_2) \cdot \overline{\partial_{x_2}^2 h(t, x_1)} \right) dx_1, \]
which implies with (2.17) that
\[ \| \partial_{x_2}^2 z(t) \|_{L^2(\mathbb{D})} \leq C \int_0^t \left( \| \partial_{x_2} \partial_{x_1}^2 z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} + \| \partial_{x_2} \partial_{x_2} z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} \right) \]
\[ \times \left( \| \partial_{x_1}^3 h(\tau) \|_{L^2(\mathbb{R})} + \| \partial_{x_1} \partial_{x_2} h(\tau) \|_{L^2(\mathbb{R})} \right) d\tau \leq C \rho^2. \]
This completes the proof of the lemma.

We next consider the estimates of solutions in \( H^{0,2}(\mathbb{D}) \) space under the stronger decay conditions on the boundary data.
LEMMA 2.4. Let \( u_0 \in H^{0,2}(\mathbb{D}) \) and

\[
\|h(t)\|_{H^2(\mathbb{R})} + \langle t \rangle \|\partial_t h(t)\|_{H^2(\mathbb{R})} + \langle t \rangle \|\partial^2_{x_1} h(t)\|_{H^2(\mathbb{R})} + \langle t \rangle \|\partial^3_{x_1} h(t)\|_{L^2(\mathbb{R})} \leq C \rho^2(t)^{-3/2 - \gamma}, \tag{2.18}
\]

where \( \gamma > 0 \). Then the solutions of (1.5) and (1.6) satisfy

\[
\|Jw(t)\|_{L^2(\mathbb{D})} + \|J^2w(t)\|_{L^2(\mathbb{D})} \leq \|u_0\|_{H^{0,2}(\mathbb{D})} + C \int_0^t \left( \|J|u|u(\tau)\|_{L^2(\mathbb{D})} + \|J^2|u|u(\tau)\|_{L^2(\mathbb{D})} \right) d\tau \tag{2.20}
\]

and

\[
\|Jz(t)\|_{L^2(\mathbb{D})} + \|J^2z(t)\|_{L^2(\mathbb{D})} \leq C \rho^2(t)^{1/2 - \gamma/2}, \quad \gamma > 0. \tag{2.21}
\]

Proof. In the same way as in the proof of the estimate of \( H^2(\mathbb{D}) \) norm, we get (2.20). We focus on the proof of (2.21). We apply integration by parts to (2.5) with \( h(0, x_1) = 0 \) to get

\[
\partial_{x_2}z(t, x_1, 0) = \frac{2}{\sqrt{2i \pi}} \int_0^t \frac{1}{\sqrt{\tau}} \partial_\tau(U_1(\tau)h(t - \tau, x_1)) d\tau
\]

and

\[
\partial_{x_2} \partial_t z(t, x_1, 0) = \frac{2}{\sqrt{2i \pi}} \int_0^t \frac{1}{\sqrt{\tau}} \partial_\tau(U_1(\tau)\partial_t h(t - \tau, x_1)) d\tau.
\]

Now we derive the new estimate of \( \partial_{x_2}z(t, x_1, 0) \). We have

\[
\partial_{x_2}z(t, x_1, 0) = \frac{2}{\sqrt{2i \pi}} \int_0^{t/2} \frac{1}{\sqrt{\tau}} \partial_\tau(U_1(\tau)h(t - \tau, x_1)) d\tau
\]

\[
+ \frac{2}{\sqrt{2i \pi}} \int_{t/2}^t \frac{1}{\sqrt{\tau}} \partial_\tau(U_1(\tau)h(t - \tau, x_1)) d\tau
\]

\[
+ \frac{1}{\sqrt{2i \pi}} \int_{t/2}^t \frac{1}{\sqrt{\tau}} (U_1(\tau)h(t - \tau, x_1)) d\tau.
\]

Integration by parts in time, and taking \( L^2(\mathbb{R}) \) norm in the resulting function, we obtain

\[
\|\partial_{x_2}z(t, \cdot, 0)\|_{L^2(\mathbb{R})} \leq C \int_0^{t/2} \frac{1}{\sqrt{\tau}} \left( \|\partial^2_{x_1} h(t - \tau)\|_{L^2(\mathbb{R})} + \|\partial_\tau h(t - \tau)\|_{L^2(\mathbb{R})} \right) d\tau
\]

\[
+ C \frac{1}{\sqrt{t}} \left\| h \left( \frac{t}{2} \right) \right\|_{L^2(\mathbb{R})} + C \frac{1}{t^{1/2}} \int_{t/2}^t \|h(t - \tau)\|_{L^2(\mathbb{R})} d\tau
\]

\[
\leq C \int_{t/2}^t \frac{1}{\sqrt{\tau}} \left( \|\partial^2_{x_1} h(\tau)\|_{L^2(\mathbb{R})} + \|\partial_\tau h(\tau)\|_{L^2(\mathbb{R})} \right) d\tau
\]

\[
+ C \frac{1}{\sqrt{t}} \left\| h \left( \frac{t}{2} \right) \right\|_{L^2(\mathbb{R})} + C \frac{1}{t^{1/2}} \int_0^{t/2} \|h(\tau)\|_{L^2(\mathbb{R})} d\tau
\]

\[
\leq C \rho^2(t)^{-3/2}.
\]
We apply the energy method to (1.6) to find that
\[ \| \partial_{x_2} \partial_t z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \int_{t/2}^t \frac{1}{\sqrt{t-\tau}} (\| \partial_{x_1}^2 \partial_t h(\tau) \|_{L^2(\mathbb{R})} + \| \partial_t^2 h(\tau) \|_{L^2(\mathbb{R})}) d\tau \]
\[ + C \frac{1}{\sqrt{t}} \left\| \partial_t h \left( \frac{t}{2} \right) \right\|_{L^2(\mathbb{R})} + C \frac{1}{t^{3/4}} \int_0^{t/2} \| \partial_t h(\tau) \|_{L^2(\mathbb{R})} d\tau \]
\[ \leq \rho^2(t)^{-3/2}. \]

In the same way we obtain
\[ \sum_{j=0}^2 \| \partial_{x_2} \partial_t^{j} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq \rho^2(t)^{-3/2}. \quad (2.22) \]

We apply the energy method to (1.6) to find that
\[ \| J_{x_1} z(t) \|_{L^2(\mathbb{R} \times \mathbb{R}^+) \cap \mathcal{D}}^2 \leq C \int_0^t \tau \| \partial_{x_2} \partial_t z(\tau) \|_{L^2(\mathbb{R})} \| J_{x_1} h(\tau) \|_{L^2(\mathbb{R})} d\tau \]
\[ \leq C \rho^4 \int_0^t (\tau)^{-3/2-\gamma} \leq C \rho^4. \]

We next consider the estimate of \( \| J_{x_2} z(t) \|_{L^2(\mathbb{D})}^2 \). We have by the energy method that
\[ i \partial_t \| J_{x_2} z(t) \|_{L^2(\mathbb{D})}^2 = i \operatorname{Im} \int_{\mathbb{R}} \int_0^\infty \partial_{x_2} \left( \partial_{x_2} J_{x_2} z(t) \cdot J_{x_2} z(t) \right) dx_2 \, dx_1 \]
\[ = i \operatorname{Im} \int_{\mathbb{R}} \int_0^\infty \partial_{x_2} \left( (1 + it \partial_{x_2}^2) z(t) \cdot \overline{it \partial_{x_2} z(t)} \right) dx_2 \, dx_1 \]
\[ = i \operatorname{Im} \int_{\mathbb{R}} \int_0^\infty \partial_{x_2} \left( (1 - it (2 \partial_t + \partial_{x_1}^2)) z(t) \cdot \overline{it \partial_{x_2} z(t)} \right) dx_2 \, dx_1 \]
\[ = -i \operatorname{Im} \int_{\mathbb{R}} (1 - it (2 \partial_t + \partial_{x_1}^2)) h(t) \cdot \overline{it \partial_{x_2} z(t)} \, dx_1, \]
from which it follows with (2.18) that
\[ \| J_{x_2} z(t) \|_{L^2(\mathbb{D})}^2 \leq C \int_0^t \tau^3 \| \partial_{x_2} z(\tau) \|_{L^2(\mathbb{R})} \]
\[ \times (\| \partial_t h(\tau) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 h(\tau) \|_{L^2(\mathbb{R})} + \tau^{-1} \| h(\tau) \|_{L^2(\mathbb{R})}) \, d\tau \]
\[ \leq C \rho^4 \int_0^t \tau^{-3/2} (\tau)^{-2-\gamma} \, d\tau \]
\[ \leq C \rho^4. \]

Therefore the estimate (2.21) is proved. We consider the estimates of solutions in the space \( H^{0,2}(\mathbb{D}) \). We have by (2.19) and (2.22)
\[ \| J_{x_1} z(t) \|_{L^2(\mathbb{D})} \leq C \int_0^t \tau^2 \| \partial_{x_2} \partial_{x_1} z(\tau) \|_{L^2(\mathbb{R})} \| J_{x_1} h(\tau) \|_{L^2(\mathbb{R})} \, d\tau \]
\[ \leq C \rho^4 \int_0^t \tau^2 (\tau)^{-2-2\gamma} \, d\tau \leq C \rho^4 (t)^{1-2\gamma}. \]
We also have
\[
\frac{d}{dt} \| J_{x_2} J_{x_1} z(t) \|_{L^2(\mathbb{D})}^2 = i \int_{\mathbb{R}} \int_{0}^{\infty} \partial_{x_2} \left( \partial_{x_2} J_{x_2} J_{x_1} z(t) \cdot \overline{J_{x_2} J_{x_1} z(t)} \right) \, dx_2 \, dx_1
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \int_{0}^{\infty} \partial_{x_2} \left( (1 + i t \partial_{x_2}^2) J_{x_1} z(t) \cdot i t \partial_{x_2} J_{x_1} z(t) \right) \, dx_2 \, dx_1,
\]
from which it follows that
\[
\| J_{x_1} J_{x_2} z(t) \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 \leq C \int_{0}^{t} \tau^2 \| \partial_{x_2} J_{x_1} z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})}^2 \times \left( \| \partial_{\tau} J_{x_1} h(\tau) \|_{L^2(\mathbb{R})} + \| \partial_{x_1} J_{x_1} h(\tau) \|_{L^2(\mathbb{R})} + \tau^{-1} \| J_{x_1} h(\tau) \|_{L^2(\mathbb{R})} \right) \, d\tau,
\]
since
\[
\partial_{x_2}^2 J_{x_1} z(t, x_1, 0) = -\partial_{x_1}^2 J_{x_1} h(t, x_1) - 2i \partial_{t} J_{x_1} h(t, x_1).
\]
By the assumption
\[
t \| \partial_{t} J_{x_1} h(t) \|_{L^2(\mathbb{R})} + t \| \partial_{x_1} J_{x_1} h(t) \|_{L^2(\mathbb{R})} + \| J_{x_1} h(t) \|_{L^2(\mathbb{R})} \leq C \rho^2(t)^{-\nu}
\]
and
\[
\| \partial_{x_2} J_{x_1} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \left( \| \partial_{x_2} x_1 z(t, \cdot, 0) \|_{L^2(\mathbb{R})} + t \| \partial_{x_2} \partial_{x_1} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \right)
\]
\[
\leq C \left( \| x_1 h(t) \|_{H^2(\mathbb{R})} + \| \partial_{x_1} h(t) \|_{L^2(\mathbb{R})} + \| \partial_{x_1} h(t) \|_{L^2(\mathbb{R})} \right)
\]
we get that
\[
\| J_{x_1} J_{x_2} z(t) \|_{L^2(\mathbb{D})}^2 \leq C \rho^2(t)^{-1-\nu}.
\]
We finally have
\[
\frac{d}{dt} \| J_{x_2} z(t) \|_{L^2(\mathbb{D})}^2
\]
\[
= \text{Im} \int_{\mathbb{R}} \int_{0}^{\infty} \partial_{x_2} \left( \partial_{x_2} J_{x_2}^2 z(t) \cdot \overline{J_{x_2}^2 z(t)} \right) \, dx_2 \, dx_1
\]
\[
= \text{Im} \int_{\mathbb{R}} \int_{0}^{\infty} \partial_{x_2} \left( (3it \partial_{x_2} - t^2 \partial_{x_2}^3) z(t) \cdot \overline{(it - t^2 \partial_{x_2}^2) z(t)} \right) \, dx_2 \, dx_1
\]
\[
= \text{Im} \int_{\mathbb{R}} \int_{0}^{\infty} \partial_{x_2} \left( (3it + t^2(2\partial_t + \partial_{x_1}^2)) \partial_{x_2} z(t) \cdot \overline{(it + t^2(2\partial_t + \partial_{x_1}^2)) z(t)} \right) \, dx_2 \, dx_1,
\]

since
\[
J_{x_2}^2 = (x_2 + it \partial_{x_2})(x_2 + it \partial_{x_2}) = x_2^2 + it + 2it x_2 \partial_{x_2} - t^2 \partial_{x_2}^2.
\]
From the assumptions (2.18) such that
\[ \| \partial_t h(t) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 h(t) \|_{L^2(\mathbb{R})} \leq \rho^2(t)^{-5/2-\gamma} \]
and
\[ \| \partial_{x_1}^2 \partial_t h(t) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 h(t) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 \partial_t h(t) \|_{H^2(\mathbb{R})} \leq C \rho^2(t)^{-3/2-\gamma}, \]
it follows that
\[ t \| h(t) \|_{L^2(\mathbb{R})} + t^2 \| \partial_t h(t) \|_{L^2(\mathbb{R})} + t^2 \| \partial_{x_1}^2 h(t) \|_{L^2(\mathbb{R})} \leq C \rho^2(t)^{-1/2-\gamma} \]
and
\[ \| \partial_{x_2} z(t, \cdot, 0) \|_{L^2(\mathbb{R})} + t \| \partial_{x_2} \partial_t z(t, \cdot, 0) \|_{L^2(\mathbb{R})} + t \| \partial_{x_2} \partial_{x_1}^2 z(t, \cdot, 0) \|_{L^2(\mathbb{R})} \leq C \rho^2(t)^{-1/2-\gamma}. \]

Hence, we obtain
\[
\| J_{x_2}^2 z(t) \|_{L^2(\mathbb{D})}^2 \leq C \int_0^t \left( t \| h(\tau) \|_{L^2(\mathbb{R})} + t^2 \| \partial_t h(\tau) \|_{L^2(\mathbb{R})} + \tau^2 \| \partial_{x_1}^2 h(\tau) \|_{L^2(\mathbb{R})} \right) 
\times (\tau \| \partial_{x_2} z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} + \tau^2 \| \partial_{x_2} \partial_t z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})}) 
+ \tau^2 \| \partial_{x_2} \partial_{x_1}^2 z(\tau, \cdot, 0) \|_{L^2(\mathbb{R})} d\tau 
\leq C \rho^4 \int_0^t (\tau)^{-2\gamma} d\tau \leq C \rho^4 (t)^{1-2\gamma}. 
\]
Collecting everything, we have the $H^{0,2}(\mathbb{D})$ estimates. This completes the proof of the lemma.

3. Proof of Theorem 1.1

We introduce the function space
\[ X_T = \{ u(t) \in C([0, T]; H^2(\mathbb{D}) \cap H^{0,2}(\mathbb{D})) \| u \|_{X_T} < \infty \}, \]
where with \( \gamma = \varepsilon^{1/4} \)
\[ \| u \|_{X_T}^2 = \sup_{t \in [0, T]} \langle t \rangle^{-2\gamma} \| u(t) \|_{H^2(\mathbb{D})}^2 
+ \sup_{t \in [0, T]} \| J^2 u(t) \|_{L^2(\mathbb{D})}^2 \langle t \rangle^{-1+2\gamma} + \sup_{t \in [0, T]} \| u(t) \|_{L^\infty(\mathbb{D})}^2 (t)^2 \]
and
\[ \| J^2 u(t) \|_{L^2(\mathbb{D})}^2 = \sum_{j=0}^2 \| J_{x_1}^j u(t) \|_{L^2(\mathbb{D})}^2 + \| J_{x_2}^j u(t) \|_{L^2(\mathbb{D})}^2 + \| J_{x_1} J_{x_2} u(t) \|_{L^2(\mathbb{D})}^2. \]

By the local existence result, putting $\rho = \varepsilon$ and small $\varepsilon > 0$ in Theorem 1.1, we may assume that
\[ \| u \|_{X_T}^2 \leq 3\varepsilon^2. \]
We prove that, for any $T > 0$, the estimate
\[ \| u \|_{X_T}^2 < \varepsilon^{4/3} \]
holds. We use the contradiction argument. We suppose that there exists a time $T$ such that
\[ \|u\|_{X_T}^2 = \varepsilon^{4/3}. \] (3.1)
We represent $u$ as $u = w + z$, where $w$ and $z$ given by (1.5) and (1.6).

**Lemma 3.1.** Let $u$ be the local solution of (1.1) satisfying (3.1). Then we have
\[ \sup_{t \in [0, T)} \langle t \rangle^{-2} \|u(t)\|_{H^1(\mathbb{R})}^2 + \sup_{t \in [0, T)} \langle t \rangle^{-1/2} \|J^2 u(t)\|_{L^2(\mathbb{R})}^2 \leq C \varepsilon \]
for $T > 0$.

**Proof.** Via Lemma 2.4
\[ \|w(t)\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} + C \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R})} \|u(\tau)\|_{H^1(\mathbb{R})} d\tau \] (3.2)
and
\[ \|J^2 w(t)\|_{L^2(\mathbb{R})} \leq \|x^2 u_0\|_{L^2(\mathbb{R})} + C \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R})} \|J^2 u(\tau)\|_{L^2(\mathbb{R})} d\tau. \] (3.3)
By (3.3) we obtain
\[ \|J^2 w(t)\|_{L^2(\mathbb{R})} \leq \|x^2 u_0\|_{L^2(\mathbb{R})} + C \varepsilon^{2/3} \int_0^t \langle \tau \rangle^{-1} \|J^2 u(\tau)\|_{L^2(\mathbb{R})} d\tau, \]
from which it follows that
\[ \sup_{t \in [0, T)} \langle t \rangle^{-1/2 + \gamma} \|J^2 w(t)\|_{L^2(\mathbb{R})} \leq C \varepsilon. \] (3.4)
By (3.2) and (3.4)
\[ \sup_{t \in [0, T)} \langle t \rangle^{-\gamma} \|w(t)\|_{H^1(\mathbb{R})} + \sup_{t \in [0, T)} \langle t \rangle^{-1/2 + \gamma} \|J^2 w(t)\|_{L^2(\mathbb{R})} \leq C \varepsilon. \] (3.5)
By Lemmas 2.1–2.4, we have
\[ \sup_{t \in [0, T)} \langle t \rangle^{-\gamma} \|z(t)\|_{H^1(\mathbb{R})} + \sup_{t \in [0, T)} \langle t \rangle^{-1/2 + \gamma} \|J^2 z(t)\|_{L^2(\mathbb{R})} \leq C \varepsilon^2. \]
Therefore we have the lemma. \Box

**Lemma 3.2.** Let $u$ be the local solution of (1.1) satisfying (3.1). Then the estimate
\[ \sup_{t \in [0, T)} \|u(t)\|_{L^\infty(\mathbb{R})} \langle t \rangle \leq C \varepsilon \]
holds for $T > 0$.

**Proof.** In order to prove the a priori estimate of solutions in the uniform norm, we use the factorization technique of the evolution operator used in [12]. We define
\[
U_2(t)\psi = \frac{1}{\sqrt{2\pi i t}} \int_0^\infty \left( \exp \left( \frac{i(x_2 - y_2)^2}{2t} \right) - \exp \left( \frac{i(x_2 + y_2)^2}{2t} \right) \right) \psi(y_2) dy_2
\]
\[ = \frac{2}{\sqrt{2\pi}} e^{ix_2^2/2t} i D_{x_2} \int_0^\infty \sin(x_2 y_2) e^{iy_2^2/2t} \psi(y_2) dy_2 \]
\[ = M_2 D_{x_2} F_{x_2} M_2 \psi \]
and

\[ U_1(t)\psi = \frac{1}{\sqrt{2\pi it}} \int_{-\infty}^{\infty} \exp\left(\frac{i(x_1 - y_1)^2}{2t}\right) \psi(y_1) \, dy_1 \]

\[ = \frac{1}{\sqrt{2\pi}} e^{i\xi_1^2/2t} D_{t,1} \int_{-\infty}^{\infty} e^{-ix_1\cdot y_1} e^{i\xi_1^2/2t} \psi(y_1) \, dy_1 \]

\[ = M_1 D_{t,1} \mathcal{F}_1 M_1 \psi, \]

where

\[ M_j = e^{i\xi_j^2/2t}, \quad D_{t,j} \phi = \frac{1}{\sqrt{it}} \phi\left(\frac{x_j}{t}\right). \]

\[ (\mathcal{F}_{s,2}\phi)(\xi_2) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(x_2\xi_2) \phi(x_2) \, dx_2, \]

\[ (\mathcal{F}_{s,2}^{-1}\hat{\phi})(x_2) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(x_2\xi_2) \hat{\phi}(\xi_2) \, d\xi_2, \]

\[ (\mathcal{F}_1\phi)(\xi_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi_1x_1} \phi(x_1) \, dx_1. \]

By a direct calculation

\[ D_{t,2} \mathcal{F}_{s,2} = \mathcal{F}_{s,2}^{-1} D_{t,2}^{-1}, \]

from which it follows that

\[ U_2(t)^{-1} = M_2^{-1} \mathcal{F}_{s,2}^{-1} D_{t,2}^{-1} M_2^{-1} \]

\[ = M_2^{-1} \mathcal{F}_{s,2}^{-1} M_2^{-1} \]

\[ = M_2^{-1} D_{t,2} \mathcal{F}_{s,2} M_2^{-1} = U_2(-t). \]

We have

\[ \psi(t) = U_D(t) U_D(-t) \psi(t) \]

\[ = U_2(t) U_1(t) U_2(-t) U_1(-t) \psi(t) \]

\[ = M_2 M_1 D_{t,2} D_{t,1} \mathcal{F}_{s,2} \mathcal{F}_1 M_2 M_1 U_2(-t) U_1(-t) \psi(t) \]

\[ = M D_{t,2} \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t) \psi(t) + M D_{t,2} \mathcal{F}_{s,2} \mathcal{F}_1 (M - 1) U_D(-t) \psi(t), \]

where \( M = M_2 M_1, \) \( D_t = D_{t,2} D_{t,1}, \) \( U_D(-t) = U_2(-t) U_1(-t). \) Therefore we obtain

\[ \|\psi(t)\|_{L^\infty(\mathbb{D})} \leq C t^{-1} \|\mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t) \psi(t)\|_{L^\infty(\mathbb{D})} \]

\[ + C t^{-3/2} \|x_2 U_2(-t)x_1 U_1(-t) \psi(t)\|_{L^2(\mathbb{D})}. \quad (3.6) \]

Since

\[ x_2(\mathcal{F}_{s,2}\phi) = \sqrt{\frac{2}{\pi}} x_2 \int_0^{\infty} \sin(x_2\xi_2) \phi(\xi_2) \, d\xi_2 \]

\[ = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} (\partial_{\xi_2} \cos(x_2\xi_2)) \phi(\xi_2) \, d\xi_2 \]

\[ = \mathcal{F}_{c,2}(\partial_{\xi_2}(x_2)) + \sqrt{\frac{2}{\pi}} \phi(0), \]
with the fact that \( x_1 U_1(-t)w(t, x_1, 0) = 0 \), by a direct calculation we get
\[
x_2(\mathcal{F}_s M_2^{-1} x_1 U_1(-t)w)(t, x_1, x_2) = \mathcal{F}_c(\partial_{x_2} M_2^{-1} x_1 U_1(-t)w)(t, x_1, x_2).
\]
Hence, we obtain
\[
\|x_2 U_D(-t)x_1 U_1(-t)w(t)\|_{L^2(\mathbb{D})}
\]
\[
= \|x_2 M_2^{-1} D_{-t,2} \mathcal{F}_s M_2^{-1} x_1 U_1(-t)w(t)\|_{L^2(\mathbb{D})}
\]
\[
= \|\mathcal{F}_c \partial_{x_2} M_2^{-1} x_1 U_1(-t)w(t)\|_{L^2(\mathbb{D})}
\]
\[
\leq \|t \partial_{x_2} M_2^{-1} x_1 U_1(-t)w(t)\|_{L^2(\mathbb{D})} = \|J_{x_2} x_1 U_1(-t)w(t)\|_{L^2(\mathbb{D})}
\]
\[
= \|x_1 U_1(-t)J_{x_2} w(t)\|_{L^2(\mathbb{D})} = \|J_{x_1} J_{x_2} w(t)\|_{L^2(\mathbb{D})}. \tag{3.7}
\]
Therefore by (3.6) we have
\[
\|w(t)\|_{L^\infty(\mathbb{D})} \leq C t^{-1} \|\mathcal{F}_s \mathcal{F}_1 U_D(-t)w(t)\|_{L^\infty(\mathbb{D})}
\]
\[
+ C t^{-3/2} \|J_{x_1} J_{x_2} w(t)\|_{L^2(\mathbb{D})}. \tag{3.8}
\]
Also we get
\[
z(t) = U_D(t)U_D(-t)z(t)
\]
\[
= MD_t \mathcal{F}_s \mathcal{F} MU_D(-t)z(t)
\]
\[
= MD_t \mathcal{F}_s \mathcal{F} U_D(-t)z(t) + MD_t \mathcal{F}_s \mathcal{F}(M-1)U_D(-t)z(t).
\]
We consider the second term of the right-hand side of the above and divide it into two terms to find
\[
MD_t \mathcal{F}_s \mathcal{F}(M-1)U_D(-t)z(t)
\]
\[
= MD_t \mathcal{F}_s \mathcal{F}(M-1)U_D(-t)(z(t, x_1, x_2) - e^{-x_2 h(t, x_1)})
\]
\[
+ MD_t \mathcal{F}_s \mathcal{F}(M-1)U_D(-t)(e^{-x_2 h(t, x_1)}).
\]
Since
\[
z(t, x_1, x_2) - e^{-x_2 h(t, x_1)}|_{x_2=0} = 0,
\]
applying (3.6), we obtain
\[
\|MD_t \mathcal{F}_s \mathcal{F}(M-1)U_D(-t)(z(t, x_1, x_2) - e^{-x_2 h(t, x_1)})\|_{L^\infty(\mathbb{D})}
\]
\[
\leq C t^{-3/2} \|x_1 x_2 U_D(-t)(z(t, x_1, x_2) - e^{-x_2 h(t, x_1)})\|_{L^2(\mathbb{D})}
\]
\[
\leq C t^{-3/2} \|J_{x_1} J_{x_2} (z(t, x_1, x_2) - e^{-x_2 h(t, x_1)})\|_{L^2(\mathbb{D})}
\]
\[
\leq C t^{-3/2} \|J_{x_1} J_{x_2} z(t)\|_{L^2(\mathbb{D})} + C t^{-1/2} \|J_{x_1} h(t)\|_{L^2(\mathbb{R})}. \tag{3.9}
\]
Let us consider the term
\[
MD_t \mathcal{F}_s \mathcal{F}_1(M-1)U_D(-t)(e^{-x_2 h(t, x_1)})
\]
\[
= MD_t \mathcal{F}_s \mathcal{F}_1(M-1)U_2(-t)(e^{-x_2} U_1(-t)h(t, x_1))
\]
\[
= e^{-x_2} h(t, x_1) - MD_t \mathcal{F}_s \mathcal{F}_2 U_2(-t)(e^{-x_2} \mathcal{F}_1 U_1(-t)h(t, x_1)).
\]
We apply the identity
\[ \mathcal{F}_{x,2} U_2(-t) e^{-x_2} = e^{(i \xi_2^2/2)t} \mathcal{F}_{x,2} e^{-x_2} \]

to the right-hand side of the above equality to get
\[
\| MD_{t} \mathcal{F}_{x,2} \mathcal{F}_1 (M - 1) U_D (-t) (e^{-x_2} h(t, x_1)) \|_{L^\infty(\mathbb{D})} \\
\leq \| e^{-x_2} h(t, x_1) \|_{L^\infty(\mathbb{D})} + C t^{-1} \| \mathcal{F}_{x,2} U_2 (-t) (e^{-x_2} \mathcal{F}_1 U_1 (-t) h(t, x_1)) \|_{L^\infty(\mathbb{D})} \\
\leq C \| h(t) \|_{H^1(\mathbb{R})} + C t^{-1} \| \mathcal{F}_{x,2} e^{-x_2} \mathcal{F}_1 U_1 (-t) h(t, x_1) \|_{L^\infty(\mathbb{D})} \\
\leq C \| h(t) \|_{H^1(\mathbb{R})} + C t^{-1} \| \mathcal{F}_1 U_1 (-t) h(t) \|_{L^\infty(\mathbb{R})} \\
\leq C \| h(t) \|_{H^1(\mathbb{R})} + C t^{-1} \| J_{x_1} h(t) \|_{L^2(\mathbb{R})}.
\]

Therefore from (3.9) and (3.10) we obtain
\[
\| MD_{t} \mathcal{F}_{x,2} \mathcal{F}_1 (M - 1) U_D (-t) z(t) \|_{L^\infty(\mathbb{D})} \\
\leq C t^{-3/2} \| J_{x_1} J_{x_2} z(t) \|_{L^2(\mathbb{D})} + C t^{-1/2} \| J_{x_1} h(t) \|_{L^2(\mathbb{R})} + C \| h(t) \|_{H^1(\mathbb{R})}
\]

for \( t \geq 1 \). Thus from (3.6) it follows that
\[
\| z(t) \|_{L^\infty(\mathbb{D})} \leq C t^{-1} \| \mathcal{F}_{x,2} \mathcal{F}_1 U_D (-t) z(t) \|_{L^\infty(\mathbb{D})} + C t^{-3/2} \| J_{x_1} J_{x_2} z(t) \|_{L^2(\mathbb{D})} \\
+ C t^{-1/2} \| J_{x_1} h(t) \|_{L^2(\mathbb{R})} + C \| h(t) \|_{H^1(\mathbb{R})}.
\]

We consider the estimate of the first term on the right-hand side of (3.11). Denote
\[
U_{N,2}(t) \psi = \frac{1}{2 \sqrt{2\pi i t}} \int_0^\infty \left( \exp\left( \frac{i (x_2 - y_2)^2}{2t} \right) + \exp\left( \frac{i (x_2 + y_2)^2}{2t} \right) \right) \psi(y_2) \, dy_2 \\
= M_2 D_{t,2} \mathcal{F}_{c,2} M_2 \psi.
\]

Recalling that
\[
z(t, x_1, x_2) = \frac{1}{\sqrt{2\pi i t}} \int_0^t e^{i \xi_2^2/2t} \frac{x_2}{\tau \sqrt{\tau}} U_1(\tau) h(t - \tau) \, d\tau,
\]

we get
\[
\mathcal{F}_{x,2} U_2(-t) \mathcal{F}_1 U_1(-t) z(t, x) \\
= \mathcal{F}_{x,2} U_2(-t) \partial_{x_2} \int_0^t e^{i \xi_2^2/2t} \frac{1}{\sqrt{\tau}} \mathcal{F}_1 U_1(-(t - \tau)) h(t - \tau) \, d\tau \\
= \mathcal{F}_{x,2} \partial_{x_2} U_{N,2}(-t) \int_0^t e^{i \xi_2^2/2t} \frac{1}{\sqrt{\tau}} F(t - \tau) \, d\tau \\
= i \xi_2 \mathcal{F}_{c,2} U_{N,2}(-t) \int_0^t e^{i \xi_2^2/2t} \frac{1}{\sqrt{\tau}} F(t - \tau) \, d\tau \\
= \int_0^t i \xi_2 e^{-(i \xi_2^2/2)t} F(\tau) \, d\tau \\
= \int_0^\infty i \xi_2 e^{-(i \xi_2^2/2)t} F(\tau) \, d\tau - \int_t^\infty i \xi_2 e^{-(i \xi_2^2/2)t} F(\tau) \, d\tau.
\]
where
\[ F(t) = \mathcal{F}_1 U_1 (-t) h(t). \]

Also we note that the integration by parts gives
\[
\int_t^\infty i\xi_2 e^{-i\xi_2^2/2t} F(\tau) \, d\tau
= \int_t^\infty \frac{i\xi_2}{1-(i\xi_2^2/2t)} (\partial_\tau e^{-i\xi_2^2/2t} F(\tau)) \, d\tau
= \int_t^\infty \partial_\tau \left( \frac{i\xi_2}{1-(i\xi_2^2/2t)} e^{-i\xi_2^2/2t} F(\tau) \right) \, d\tau \quad \text{[using (3.12)]}
- \int_t^\infty \frac{i\xi_2 F(\tau) (1-(i\xi_2^2/2t))}{1-(i\xi_2^2/2t)} \, d\tau
- \int_t^\infty \frac{i\xi_2 F(\tau) (1-(i\xi_2^2/2t))}{1-(i\xi_2^2/2t)} \, d\tau.
\]

Hence,
\[
\left| \int_t^\infty i\xi_2 e^{-i\xi_2^2/2t} F(\tau) \, d\tau \right| \leq C \sqrt{t} \| F(t) \|_{L^\infty(\mathbb{R})} + C \int_t^\infty \frac{\| F(\tau) \|_{L^\infty(\mathbb{R})}}{\sqrt{\tau}} \, d\tau
+ C \int_t^\infty \sqrt{\tau} \| \partial_\tau F(\tau) \|_{L^\infty(\mathbb{R})} \, d\tau. \tag{3.13}
\]

Similarly,
\[
\left| \int_0^\infty i\xi_2 e^{-i\xi_2^2/2t} F(\tau) \, d\tau \right| 
\leq C \int_0^\infty \sqrt{\tau} \| \partial_\tau F(\tau) \|_{L^\infty(\mathbb{R})} \, d\tau + C \int_0^\infty \frac{\| F(\tau) \|_{L^\infty(\mathbb{R})}}{\sqrt{\tau}} \, d\tau. \tag{3.14}
\]

By \( F(t) = \mathcal{F}_1 U_1 (-t) h(t) \) and the assumptions on \( h(t) \) we have
\[ \| F(t) \|_{L^\infty(\mathbb{R})} \leq C (\| J_{x_1} h(t) \|_{L^2(\mathbb{R})} + \| h(t) \|_{H^1(\mathbb{R})}) \leq C \rho^2 (t)^{-1/2-\gamma} \]
and
\[ \| \partial_\tau F(t) \|_{L^\infty(\mathbb{R})} \leq C (\| J_{x_1} \partial_\tau h(t) \|_{L^2(\mathbb{R})} + \| \partial_\tau h(t) \|_{H^1(\mathbb{R})}) \]
\[ + C (\| J_{x_1} \partial_{x_1}^2 h(t) \|_{L^2(\mathbb{R})} + \| \partial_{x_1}^2 h(t) \|_{H^1(\mathbb{R})}) \]
\[ \leq C \rho^2 (t)^{-3/2-\gamma}. \]

By (3.12), (3.13) and (3.14) we get
\[ \| \mathcal{F}_{x,2} \mathcal{F}_1 U_D (-t) z(t) \|_{L^\infty(\mathbb{D})} \leq C \epsilon^2 \tag{3.15} \]
if we take \( \rho = \epsilon \). By (3.11), (3.15) and Lemma 2.4,
\[ \| z(t) \|_{L^\infty(\mathbb{D})} \leq C \epsilon^2 t^{-1}. \tag{3.16} \]
We next estimate \( \|w(t)\|_{L^\infty(\mathbb{D})} \). We write
\[
u = w + z = y + e^{-\lambda_2}h(t), \tag{3.17}\]
where \( y(t, x_1, 0) = 0 \). We represent the nonlinear term as
\[
|u|u = |y + e^{-\lambda_2}h(t)|(y + e^{-\lambda_2}h(t))
= |y|y + R,
\]
where
\[
|R| = ||y + e^{-\lambda_2}h(t)|(y + e^{-\lambda_2}h(t)) - |y|y| \leq C|e^{-\lambda_2}h(t)|(\|y\| + |e^{-\lambda_2}h(t)|).
\]
Applying \( \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t) \) to both sides of (1.5), we obtain by using the factorization formula of the evolution operator
\[
i\partial_t \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)w
= \lambda \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)|y|y + \lambda \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)R
= \lambda \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)|M \mathcal{D} \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y + M \mathcal{D} \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y + \lambda \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)R
= \lambda t^{-1} \mathcal{F}_{s,2} \mathcal{F}_1 M^{-1} \mathcal{F}_{s,2}^{-1} \mathcal{F}_1^{-1}|\mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y| \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y + \lambda \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)R. \tag{3.18}\]
We have
\[
\|\mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t)R\|_{L^\infty(\mathbb{D})}
\leq C \|R\|_{L^1(\mathbb{D})}
\leq C \|e^{-\lambda_2}h(t)\|_{L^1(\mathbb{D})}(\|y\|_{L^\infty(\mathbb{D})} + \|e^{-\lambda_2}h(t)\|_{L^\infty(\mathbb{D})})
\leq C \|h(t)\|_{L^1(\mathbb{R})}(\|y\|_{L^\infty(\mathbb{D})} + \|h(t)\|_{L^\infty(\mathbb{R})})
\leq C \|h(t)\|_{L^1(\mathbb{R})}(\|w(t) - e^{-\lambda_2}h(t)\|_{L^\infty(\mathbb{D})} + \|h(t)\|_{L^\infty(\mathbb{R})})
\leq C \|h(t)\|_{L^1(\mathbb{R})}(\|z(t)\|_{L^\infty(\mathbb{D})} + \|w(t)\|_{L^\infty(\mathbb{D})} + \|h(t)\|_{L^\infty(\mathbb{R})})
\leq C \|h(t)\|_{L^1(\mathbb{R})}(\|z(t)\|_{L^\infty(\mathbb{D})} + \|J_{x_1}J_{x_2}w(t)\|_{L^2(\mathbb{D})}^{1/2} \|w(t)\|_{L^2(\mathbb{D})}^{1/2} + \|h(t)\|_{L^\infty(\mathbb{R})}).
\]
We consider
\[
\lambda t^{-1} \mathcal{F}_{s,2} \mathcal{F}_1 M^{-1} \mathcal{F}_{s,2}^{-1} \mathcal{F}_1^{-1}|\mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y| \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y + \lambda t^{-1} \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y + R_1,
\]
where
\[
\|R_1(t)\|_{L^\infty(\mathbb{D})} \leq C t^{-1-1/2}(\|x_1 x_2 \mathcal{F}_{s,2}^{-1} \mathcal{F}_1^{-1} |\Psi| |\Psi|_{L^2(\mathbb{D})} + ||\Psi||_{L^2(\mathbb{D})}),
\]
with
\[
\Psi = \mathcal{F}_{s,2} \mathcal{F}_1 M U_D(-t)y.
\]
In the same way as in the proof of (3.7)

\[
\|x_1 x_2 \mathcal{F}_s^{-1} \mathcal{F}^{-1} |\Psi\|_{L^2(\Omega)} \\
\leq C \|\partial_2 \partial_1 |\Psi|\Psi\|_{L^2(\Omega)} \\
\leq C \|\Psi\|_{L^\infty(\Omega)} \|x_1 x_2 \mathcal{F}_1 U_D(-t)y\|_{L^2(\Omega)} \\
\leq C (\|\mathcal{F}_s \mathcal{F}_1 U_D(-t)y\|_{L^\infty(\Omega)} + t^{-1/2} \|J_{x_2} J_{x_1} y\|_{L^2(\Omega)}) \|J_{x_2} J_{x_1} y\|_{L^2(\Omega)}. \tag{3.19}
\]

Thus, via (3.19), we obtain

\[
\|R_1(t)\|_{L^\infty(\Omega)} \leq C t^{-3/2} (\|\mathcal{F}_s \mathcal{F}_1 U_D(-t)y\|_{L^\infty(\Omega)} + t^{-1/2} \|J_{x_2} J_{x_1} y\|_{L^2(\Omega)}) \\
\times \|J_{x_2} J_{x_1} y\|_{L^2(\Omega)}. \tag{3.20}
\]

Similarly we get

\[
\lambda t^{-1} |\mathcal{F}_s \mathcal{F}_1 MU_D(-t)y| \mathcal{F}_s \mathcal{F}_1 MU_D(-t)y \\
= \lambda t^{-1} |\mathcal{F}_s \mathcal{F}_1 U_D(-t)y| \mathcal{F}_s \mathcal{F}_1 U_D(-t)y + R_2,
\]

where

\[
\|R_2(t)\|_{L^2(\Omega)} \leq C t^{-1} (\|\mathcal{F}_s \mathcal{F}_1 MU_D(-t)y\|_{L^\infty(\Omega)} + \|\mathcal{F}_s \mathcal{F}_1 U_D(-t)y\|_{L^\infty(\Omega)}) \\
\times \|\mathcal{F}_s \mathcal{F}_1 (M - 1) U_D(-t)y\|_{L^\infty(\Omega)} \\
\leq C t^{-1/2} \|\mathcal{F}_s \mathcal{F}_1 U_D(-t)y\|_{L^\infty(\Omega)} \|J_{x_2} J_{x_1} y\|_{L^2(\Omega)} \\
+ C t^{-2} \|J_{x_2} J_{x_1} y\|_{L^2(\Omega)}^2. \tag{3.21}
\]

Thus, applying (3.20) and (3.21) to (3.18), we get

\[
i \partial_t \mathcal{F}_s \mathcal{F}_1 U_D(-t)(y + e^{-\xi_2} h(t)) - z \\
= \lambda t^{-1} |\mathcal{F}_s \mathcal{F}_1 U_D(-t)y| \mathcal{F}_s \mathcal{F}_1 U_D(-t)y + O(\varepsilon^{4/3} t^{-1-\gamma}).
\]

We rewrite this identity as

\[
i \partial_t \mathcal{F}_s \mathcal{F}_1 U_D(-t)(y + e^{-\xi_2} h(t)) \\
= \lambda t^{-1} |\mathcal{F}_s \mathcal{F}_1 U_D(-t)y| \mathcal{F}_s \mathcal{F}_1 U_D(-t)(y + e^{-\xi_2} h(t)) \\
- \lambda t^{-1} |\mathcal{F}_s \mathcal{F}_1 U_D(-t)y| \mathcal{F}_s \mathcal{F}_1 U_D(-t)e^{-\xi_2} h(t) \\
+ i \partial_t \mathcal{F}_s \mathcal{F}_1 U_D(-t)z + O(\varepsilon^{4/3} t^{-1-\gamma}). \tag{3.22}
\]

Multiplying both sides of (3.22) by

\[
\exp\left(-i \lambda \int_1^t \tau^{-1} |\mathcal{F}_s \mathcal{F}_1 U_D(-\tau)y| \, d\tau\right),
\]

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we obtain
\[
i\partial_t \left( \exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x,2}U_D(-\tau) y| \, d\tau \right) F_{x,2}U_D(-t)(y + e^{-x^2}h(t)) \right)
= -\lambda t^{-1} |F_{x,2}U_D(-t) y| F_{x,2}U_D(-t) e^{-x^2}h(t)
\times \exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x}U_D(-\tau) U_1(-\tau) y| \, d\tau \right)
+ \exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x,2}U_D(-\tau) y| \, d\tau \right) i\partial_t F_{x,2}U_D(-t) z
+ \exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x,2}U_D(-\tau) y| \, d\tau \right) \mathcal{O}(e^{4/3} t^{-1-\gamma}).
\]
Integrating the identity in time and using
\[
i\partial_t F_{x,2}U_D(-t) z(t, x) = i\xi_2 e^{-i(\xi_2^2/2)t} F_1U_1(-t) h(t),
\]
which comes from (3.12), we obtain with \(F(t) = F_1U_1(-t) h(t)\) that
\[
\exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x,2}U_D(-\tau) y| \, d\tau \right) F_{x,2}U_D(-t)(y + e^{-x^2}h(t))
= F_{x,2}U_D(-1)(y + e^{-x^2}h(1))
+ \int_1^t \exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x,2}U_D(-\tau) y| \, d\tau \right) i\xi_2 e^{-i(\xi_2^2/2)t} F(\tau) \, d\tau
+ \int_1^t \exp \left( -i\lambda \int_1^t \tau^{-1} |F_{x,2}U_D(-\tau) y| \, d\tau \right) \mathcal{O}(e^{4/3} t^{-1-\gamma}) \, d\tau.
\]
As a result, it follows that
\[
\| F_{x,2}U_D(-t)(y + e^{-x^2}h(t)) \|_{L^\infty(\mathbb{D})} \leq C \varepsilon.
\]
Therefore, via (3.17), we obtain
\[
\| F_{x,2}U_D(-t) w \|_{L^\infty(\mathbb{D})} \leq C \varepsilon. \tag{3.23}
\]
By (3.8) we have
\[
\| w(t) \|_{L^\infty(\mathbb{D})} \leq C \varepsilon(t)^{-1}. \tag{3.24}
\]
By virtue of (3.16) and (3.24), we prove the desired estimate. This completes the proof of the lemma.

By Lemmas 3.1 and 3.2, we have the desired contradiction and we have a global in time solution to (1.1).

3.1. Asymptotic behavior of solutions

We write
\[
u(t) = M D_t F_{x,2}U_D(-t) u(t) + M D_t F_{x,2}F_1(M - 1)U_D(-t) u(t). \tag{3.25}
\]
In order to get the asymptotic behavior of $F_{s, 2} F_1 U_D(-t) u(t)$ we write

$$F_{s, 2} F_1 U_D(-t) u(t) = \varphi + \psi,$$

where

$$\varphi = F_{s, 2} F_1 U_D(-t) w(t),$$

$$\psi = F_{s, 2} F_1 U_D(-t) z(t).$$

By (3.12)

$$\psi = A(\xi) + B(t, \xi),$$

$$A(\xi) = \int_0^\infty i \xi e^{-(i \xi^2/2) t} F(\tau) d\tau,$$

$$B(t, \xi) = - \int_0^\infty t i \xi e^{-(i \xi^2/2) t} F(\tau) d\tau,$$

where

$$F(t) = (F_1 U_1(-t) h)(t, \xi_1).$$

Therefore we consider $\varphi = F_{s, 2} F_1 U_D(-t) w(t)$ to get the desired estimate. We note that from (3.13) it follows that

$$|B(t, \xi)| \leq C \varepsilon \langle t \rangle^{-\gamma},$$

if $|h(t)| \leq C \varepsilon \langle t \rangle^{-3/4}$. This fact means that $B(t, \xi)$ is the remainder term. We rewrite the nonlinear term as

$$t^{-1} |\varphi + \psi|(\varphi + \psi) + O(\varepsilon^{4/3} t^{-1-\gamma})$$

$$= t^{-1} |\varphi + A + B(t)|(\varphi + A + B(t)) + O(\varepsilon^{4/3} t^{-1-\gamma})$$

$$= t^{-1} |\varphi + A|(\varphi + A) + R_3(t) + O(\varepsilon^{4/3} t^{-1-\gamma}),$$

where

$$|R_3(t)| \leq C t^{-1} |\varphi + A| |B(t)|$$

and by (3.26)

$$|R_3(t)| \leq C \varepsilon \langle t \rangle^{-1-\gamma} |\varphi + A|.$$

By the fact that $\partial_t A = 0$, we obtain

$$i \partial_t (\varphi + A) = \lambda t^{-1} |\varphi + A|(\varphi + A) + O(\varepsilon^{4/3} t^{-1-\gamma}).$$

(3.27)

Multiplying both sides of (3.27) by

$$\exp\left(i \lambda \int_1^t \tau^{-1} |\varphi(\tau) + A| d\tau\right)$$

we get

$$i \partial_t (\varphi + A) \exp\left(i \lambda \int_1^t \tau^{-1} |\varphi(\tau) + A| d\tau\right)$$

$$= O(\varepsilon^{4/3} t^{-1-\gamma}) \exp\left(i \lambda \int_1^t \tau^{-1} |\varphi(\tau) + A| d\tau\right).$$
Integrating in time, we obtain
\[ |\psi(t, \xi) + A(\xi)| \leq C\epsilon. \]

Hence, we have
\[ \|\psi(t) + A\|_{L^\infty(\mathbb{D})} = \|F_{s,2}F_1 U_D(-t)w(t) + A\|_{L^\infty(\mathbb{D})} \leq C\epsilon. \]

from which it follows that
\[ \|F_{s,2}F_1 U_D(-t)w(t)\|_{L^\infty(\mathbb{D})} \leq C\epsilon. \]

(3.28)

We begin with (3.27) and we put
\[ \varphi(t) = \exp\left(i\int_1^t \tau^{-1} |\psi(\tau) + A| \, d\tau\right), \]
then we have
\[ i\partial_t \varphi(t) = \mathcal{O}(\epsilon^{4/3} t^{-1-\gamma}). \]

Hence, integration in time gives us
\[ \|\varphi(t) - \varphi(s)\|_{L^\infty(\mathbb{D})} \leq C\epsilon, \]
which implies that there exists \( \Phi_+ \in L^\infty(\mathbb{D}) \) such that
\[ \|\Phi(t) - \Phi(t^-)\|_{L^\infty(\mathbb{D})} \leq C\epsilon, \quad \|\Phi(t) - \Phi(t^-)\|_{L^\infty(\mathbb{D})} \leq C\epsilon^{4/3} t^{-\gamma}. \]

We next consider asymptotics of the phase function. We define the function \( \Phi(t) \) as
\[ \Phi(t) = \int_1^t \tau^{-1} (|\psi(\tau)| - |\psi(t^-)|) \, d\tau + \int_1^t \tau^{-1} (|\psi(t)| - |\psi(t^-)|) \, d\tau. \]

Then
\[ \int_1^t \tau^{-1} |\psi(\tau) + A| \, d\tau = \Phi(t) + |\psi_+|^2 \log t. \]

(3.31)

From Theorem 1.1, (3.29) and (3.30) it follows that
\[ \|\Phi(t) - \Phi(s)\|_{L^\infty(\mathbb{D})} \]
\[ = C \int_s^t \tau^{-1} \|\psi(\tau)| - |\psi(t)|\|_{L^\infty(\mathbb{D})} \, d\tau + C \|\psi(t)| - |\psi_+|\|_{L^\infty(\mathbb{D})} \log \frac{t}{s} \]
\[ = C \int_s^t \tau^{-1} \|\psi(t) - \psi(t^-)\|_{L^\infty(\mathbb{D})} \, d\tau + \|\psi(t)| - |\psi_+|\|_{L^\infty(\mathbb{D})} \log \frac{t}{s} \]
\[ \leq \int_s^t C\epsilon^{4/3} \tau^{-1-\epsilon} \, d\tau + C\epsilon^{4/3} t^{-\epsilon} \log \frac{t}{s}, \quad t > s, \]

which implies that there exists a real-valued function \( \Phi_+ \in L^\infty(\mathbb{D}) \) such that
\[ \|\Phi(t) - \Phi(t^-)\|_{L^\infty(\mathbb{D})} \leq C\epsilon^{4/3} t^{-\gamma} \log t. \]

(3.32)

Hence, by (3.31) and (3.32)
\[ \left\| \int_1^t \tau^{-1} |\psi(\tau)| \, d\tau - (|\psi_+| \log t + \Phi_+) \right\|_{L^\infty(\mathbb{D})} \]
\[ \leq \|\Phi(t) - \Phi(t^-)\|_{L^\infty(\mathbb{D})} \leq C\epsilon^{4/3} t^{-\gamma} \log t. \]

(3.33)
Therefore from (3.30) we obtain
\[
\| \Psi(t) \exp \left( -i \lambda \int_1^t \tau^{-1} |\varphi(\tau) + A| d\tau \right) - \Psi_+ \exp(-i \lambda \Phi_+) \log t - i \lambda \Phi_+ \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq \| \Psi(t) - \Psi_+ \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} + C \| \Psi_+ \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \int_1^t \tau^{-1} \| \Psi(\tau) \| d\tau - (\| \Psi_+ \| \log t + \Phi_+) \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq C e^{4/3} t^{-\varepsilon} (1 + \log t),
\]
which implies
\[
\| (\varphi(t) + A) - \Psi_+ \exp(-i \lambda |\Psi_+| \log t - i \lambda \Phi_+) \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq C e^{4/3-\gamma} t^{-\gamma} (1 + \log t).
\]
We replace \( \Psi_+ \) by \( \Psi_+ e^{i\lambda \Phi_+} \) to find that
\[
\| \mathcal{F}_{s,2} \mathcal{F}_1 U_D(-t) u(t) - \Psi_+ (\xi) e^{-i |\Psi_+(\xi)| \log t} \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq C e^{4/3-\gamma} t^{-\gamma-1} (1 + \log t),
\]
from which it follows that
\[
\| u(t) - MD_t \Psi_+ (\xi) e^{-i |\Psi_+(\xi)| \log t} \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq C e^{4/3-\gamma} t^{-\gamma-1} (1 + \log t). \tag{3.34}
\]
This completes the proof of Theorem 1.1.

Acknowledgements. We would like to thank the unknown referee for his/her useful comments on the first draft. The work of N.H. is partially supported by JSPS KAKENHI Grant Numbers JP19H05597, JP15H03630. The work of E.I.K. is partially supported by CONACYT 252053-F and PAPIIT project IN100817.

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