SOME SHARP ESTIMATES FOR CONVEX HYPERSURFACES OF PINCHED NORMAL CURVATURES

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Abstract. For a convex domain $D$ bounded by the hypersurface $\partial D$ in a space of constant curvature we give sharp bounds on the width $R - r$ of a spherical shell with radii $R$ and $r$ that can enclose $\partial D$, provided that normal curvatures of $\partial D$ are pinched by two positive constants. Furthermore, in the Euclidean case we also present sharp estimates for the quotient $R/r$.

1. Preliminaries and the main results

In [1] A. Borisenko and V. Miquel proved that a closed hypersurface with normal curvatures $k_n$ satisfying the inequality $k_n \geq 1$ in the Lobachevsky space $\mathbb{H}^m(-1)$ can be put into a spherical shell between two concentric spheres of radii $R$ and $r$ such that the width $R - r$ of the shell satisfies $R - r \leq \ln 2$. A similar estimate holds in Hadamard manifolds (see [2]). In [3] these results were extended for Riemannian manifolds of constant-signed sectional curvatures and hypersurfaces with normal curvatures bounded below.

In the present paper we refine some results from [3]. For this purpose we consider hypersurfaces with normal curvatures at any point and in any direction pinched by two positive constants. Such restriction allows us to obtain sharper estimates for the width $R - r$ than in [3]. Furthermore, for such surfaces we are able to derive an upper bound on the quotient $R/r$, which can be arbitrarily large for a hypersurface with normal curvatures just bounded below.

Let us denote by $\mathbb{M}^m(c)$ a complete simply connected $m$-dimensional Riemannian manifold of constant sectional curvatures equal to $c$. In order to state the main results, we need the following definition.

Definition 1.1. A hypersurface $F \subset \mathbb{M}^m(c)$ is said to be $\kappa_1, \kappa_2$-convex (with $\kappa_2 \geq \kappa_1$, and for $c = 0$ we assume that $\kappa_1 > 0$, for $c > 0$ we assume that $\kappa_1 \geq 0$ and for $c < 0$ we assume that $\kappa_1 > \sqrt{-c}$), if for any point $P \in F$ there exist two nested geodesic spheres $S_2 \subset S_1 \subset \mathbb{M}^m(c)$ of constant normal curvatures equal to, respectively, $\kappa_1$ and $\kappa_2$, and passing through $P$ such that locally near $P$ the hypersurface $F$ lies inside $S_1$ and outside $S_2$.

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Observe that $\kappa_1, \kappa_2$-convex hypersurfaces are, in particular, $\kappa_1$-convex (see [2] and [3]).

We should note that for $C^r$-smooth hypersurfaces with $r \geq 2$ the property of being $\kappa_1, \kappa_2$-convex is equivalent to that all its normal curvatures $k_n$ are $\kappa_1, \kappa_2$-pinched, that is $\kappa_1 \leq k_n \leq \kappa_2$. In general, since some small neighborhood of any point $P$ on a $\kappa_1, \kappa_2$-convex hypersurface $F$ lies between two tangent at $P$ geodesic spheres, we have that $F$ is $C^{1,1}$-smooth. Therefore, by Rademacher’s theorem at almost all points a $\kappa_1, \kappa_2$-convex hypersurfaces has well-defined normal curvature satisfying the inequality shown above.

A closed domain $D \subset M^m(c)$ is called $\kappa_1, \kappa_2$-convex domain if its boundary $\partial D$ is a $\kappa_1, \kappa_2$-convex hypersurface. Such domains are homeomorphic to geodesic balls of the corresponding spaces.

We recall that for $\kappa_1, \kappa_2$-convex domains well-known Blaschke’s rolling theorem holds (see [4], [5], and [6] for a smooth case, and [7], [8] for a general case). More precisely, it states the following. Suppose $D \subset M^m(c)$ is a $\kappa_1, \kappa_2$-convex domain; for any point $P \in \partial D$ let $S_1$ and $S_2$ be two spheres of normal curvature equal to, respectively, $\kappa_1$ and $\kappa_2$, and that are tangent to $\partial D$ at $P$; then $B_2 \subseteq D \subseteq B_1$, where $B_i$ is the closed geodesic ball bounded by $S_i$, $i \in \{1, 2\}$.

The main result of the present paper consists of the following three theorems.

**Theorem 1.** If $D \subset M^m(c)$ is a $\kappa_1, \kappa_2$-convex domain, then the hypersurface $\partial D$ can be put into a spherical shell between two concentric spheres of radii $R$ and $r$ (with $R \geq r$) such that

(1) for $c = 0$,

$$R - r \leq \left(\sqrt{2} - 1\right) \left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right);$$

(2) for $c = k^2$ with $k > 0$,

$$R - r \leq \frac{2}{k} \arccos \sqrt{\cos(k(R_1 - R_2))} - (R_1 - R_2);$$

(3) for $c = -k^2$ with $k > 0$,

$$R - r \leq \frac{2}{k} \arccosh \sqrt{\cosh(k(R_1 - R_2))} - (R_1 - R_2),$$

where $R_1$ and $R_2$ are the radii of circles with geodesic curvatures equal to, respectively, $\kappa_1$ and $\kappa_2$, and lying in the corresponding 2-planes $M^2(c)$.

Moreover, these estimates are sharp.

**Remark 1.1.** As $\kappa_2 \to \infty$, estimates (1.1) – (1.3) tend to the corresponding estimates in spaces of constant curvature from [3].

**Remark 1.2.** By sharpness of the inequalities above and below we mean that the shown bounds are attended by so-called rounded $\kappa_1, \kappa_2$-convex spindle-shaped surfaces (see Fig. 1), which we describe in details in the next section.
In the Euclidean case we can give even more interesting estimate for the quotient \(R/r\). This estimate is related to the result of A.V. Pogorelov about convex three-dimensional surfaces with almost umbilical points (see \[9\], p. 493).

**Theorem 2.** If \(D \subset \mathbb{E}^m\) is a \(\kappa_1, \kappa_2\)-convex domain in the Euclidean space, then the hypersurface \(\partial D\) can be put into a spherical shell between two concentric spheres of radii \(R\) and \(r\) (with \(R \geq r\)) such that

\[
\frac{R}{r} \leq \sqrt{\frac{\kappa_2}{\kappa_1} + \sqrt{2}} \sqrt{\frac{\kappa_1}{\kappa_2} + \sqrt{2}}.
\]

Moreover, this estimate is sharp.

**Corollary 1.1.** In the condition of Theorem 2 it is true that

\[
\frac{R}{r} \leq \frac{\kappa_2}{\kappa_1}.
\]

**Remark 1.3.** We note that if in the above \(\kappa_1 = \kappa_2\), then from all estimates (1.4) – (1.7) it follows that \(R = r\), and thus the domain \(D\) is a geodesic ball of the corresponding space.

Theorems 1 and 2 are based on the following result, which is useful by itself. It gives the sharp upper bound on the outer radius \(R\) of the spherical shell in terms of the inner radius \(r\) of that shell.

**Theorem 3.** If \(D \subset \mathbb{M}^m(c)\) is a \(\kappa_1, \kappa_2\)-convex domain, then the hypersurface \(\partial D\) can be put into a spherical shell between two concentric spheres of radii \(R\) and \(r\) (with \(R \geq r\)) such that

1. for \(c = 0\),

\[
R \leq \sqrt{(R_1 - R_2)^2 - (R_1 - r)^2} + R_2;
\]

2. for \(c = k^2\) with \(k > 0\),

\[
R \leq \frac{1}{k} \arccos \left( \frac{\cos (k(R_1 - R_2))}{\cos (k(R_1 - r))} \right) + R_2;
\]

3. for \(c = -k^2\) with \(k > 0\),

\[
R \leq \frac{1}{k} \arccosh \left( \frac{\cosh (k(R_1 - R_2))}{\cosh (k(R_1 - r))} \right) + R_2,
\]

where \(R_1\) and \(R_2\) are the radii of circles with geodesic curvature equal to, respectively, \(\kappa_1\) and \(\kappa_2\), and lying in the corresponding 2-planes \(\mathbb{M}^2(c)\).

Moreover, estimates (1.4) – (1.7) are sharp.
2. Proofs of the main results

We are going to prove the above theorems by using a comparison argument. Let us introduce an object to compare with.

In $\mathbb{M}^m(c)$ let us consider a spindle-shaped $\kappa_1$-convex hypersurface that is obtained by rotating a circular arc of geodesic curvature equal to $\kappa_1$ (see [3]). Such surface have two vertexes where its normal curvatures blow up. After smoothing these vertexes using two spherical caps of normal curvature equal to $\kappa_2$ we obtain a convex $C^{1,1}$-smooth hypersurface (see Fig. 1). We will call such surfaces rounded $\kappa_1, \kappa_2$-convex spindle-shaped hypersurfaces.

![Figure 1. Rounded $\kappa_1, \kappa_2$-convex spindle-shaped hypersurface](image)

It appears that the following comparison lemma holds.

**Lemma 2.1.** Let $D \subset \mathbb{M}^m(c)$ be a closed $\kappa_1, \kappa_2$-convex domain, $r$ be the radius of the inscribe sphere for $D$ with center at a point $O$. Let $\tilde{F} \subset \mathbb{M}^m(c)$ be a rounded $\kappa_1, \kappa_2$-convex spindle-shaped hypersurface, and let $\tilde{r}, \tilde{R}$ be the radii of its inscribe and circumscribe spheres. If

$$\tilde{r} = r,$$

then

$$(2.1) \quad \max\text{ dist } (O, \partial D) \leq \tilde{R}.$$  

Moreover, this bound is sharp.

**Proof.** We will argue by contradiction. Suppose that $(2.1)$ is not true, and the inverse inequality

$$(2.2) \quad \max\text{ dist } (O, \partial D) > \tilde{R}$$

holds. For simplicity, we denote the hypersurface $\partial D$ by $F$.

Let $M \in F$ be a point such that $\max\text{ dist } (O, F) = |OM|$ (here and below $| \cdot |$ denotes the distance between two points); then from $(2.2)$ it follows that on the
geodesic segment $OM$ there exists a point $A$ such that
\begin{equation}
|OA| = \tilde{R} < |OM|.
\end{equation}

Assume that the rounded $\kappa_1, \kappa_2$-convex spindle-shaped hypersurface $\tilde{F}$ is centered at $O$, and its rotational axis coincides with the geodesic line $OA$. Then $A \in \tilde{F}$.

Since the point $M$ is a point for which the maximal distance from $O$ is attained, we have that a totally geodesic hyperplane which touches $F$ at $M$ is perpendicular to the geodesic line $OM$. Therefore, if $\omega_2 \subseteq D$ is a sphere with the center at a point $O_2$, and with normal curvature equal to $\kappa_2$ that touches from inside the hypersurface $F$ at $M$ (such a sphere exists by Blaschke’s rolling theorem), then $O_2$ lies on the segment $OM$. We note that from (2.3) it follows
\begin{equation}
|OO_2| > \tilde{R} - R_2,
\end{equation}
where $R_2$ is the radius of $\omega_2$.

Let us denote the inscribe sphere for $F$ by $\omega$. Since the hypersurface $F$ is $\kappa_1, \kappa_2$-convex, then by Blaschke’s rolling theorem $F$ lies in a ball of radius $R_1$ (we remind that $R_1$ is the radius of a sphere of normal curvature equal to $\kappa_1$). Hence, both $\omega$ and $\omega_2$ lies in this ball too. Now we show that there is a sphere of radius $R_1$ that touches externally both $\omega$ and $\omega_2$ simultaneously.

Denote $|OO_2|$ by $d$. It is clear that the sphere mentioned above exists if and only if the three numbers $d, R_1 - r,$ and $R_1 - R_2$ satisfy the triangle inequality. Let us check this:

1. $(R_1 - r) + (R_1 - R_2) = 2R_1 - (r + R_2) > d$, since $\omega$ and $\omega_2$ lie in a ball of radius $R_1$;
2. $(\tilde{R_1} - R_2) + d > R_1 - r$, because $\omega$ is the inscribe sphere and thus either $\omega \equiv \omega_2$ and $r = R_1 = R_2$ that is a trivial case when $F$ is a sphere, or $\omega_2$ touches $\omega$ from inside, which is again a trivial case when $F$ is a sphere, or $\omega_2$ cannot lie entirely inside $\omega$, hence $r + d < R_2$.
3. $(\tilde{R_1} - r) + d > R_1 - R_2$, which is obviously true.

By rotational symmetry, along with a single sphere of radius $R_1$ there exists a family of spheres of the same radius that touches simultaneously $\omega$ and $\omega_2$ along small $(m - 2)$-dimensional spheres $\sigma$ and $\sigma_2$. To continue we need the following lemma.

**Lemma 2.2.** If $D \subset \mathbb{M}^m(c)$ is a closed $\kappa_1$-convex domain (where for $c = 0$ we assume that $\kappa_1 > 0$, for $c > 0$ we assume $\kappa_1 \geq 0$ and for $c < 0$ we assume that $\kappa_1 > \sqrt{-c}$), then for any two points $A, B$ from $D$ every smaller circular arc of geodesic curvature equal to $\kappa_1$ that joins $A$ and $B$ lies in the domain $D$.

Since $D$ is, in particular, a $\kappa_1$-convex domain, then by Lemma 2.2 the part $\Theta$ of the envelope of this family lying between two hyperplanes $\pi$ and $\pi_2$ (corresponding to the spheres $\sigma = \pi \cap \omega$ and $\sigma_2 = \pi_2 \cap \omega_2$), lies inside $D$ (see Fig. 2).
Let $\omega^-$ and $\omega^+_2$ be the spherical caps cut from $\omega$ and $\omega_2$ by the planes $\pi$ and $\pi_2$ in a way that $\Theta$ and $\omega^-$, $\Theta$ and $\omega^+_2$ lie on the different sides with respect to $\pi$ and $\pi_2$, correspondingly.

Consider a $C^{1,1}$ smooth hypersurface $\omega^- \cup \Theta \cup \omega^+_2$ denoted by $\Omega$. By the arguments above, $\Omega$ lie in $D$.

![Figure 2. The hypersurface $\Omega = \omega^- \cup \Theta \cup \omega^+_2$.](image)

Now, using (2.4), we are going to show that it is possible to inscribe inside $\Omega$ a sphere of radius $r'$ strictly greater then the radius $r$ of the inscribed sphere. This give us a desired contradiction, since by definition the inscribed sphere is a sphere of maximal radius lying in a domain.

By symmetry, the center $C$ of the inscribed in $\Omega$ sphere lie on the geodesic $OO_2$. Let us consider a section of $\Omega$ by a totally geodesic two-dimensional plane $\Pi$ passing through $OO_2$. Then the curve $\Omega'$ defined as $\Omega \cap \Pi$ consists of four parts: an arc $\omega'$ of a circle $\omega \cap \Pi$ of radius $r$, an arc $\omega'_2$ of a circle $\omega_2 \cap \Pi$ of radius $R_2$, and two circular arcs of radius $R_1$. Denote by $O_1$ the center of one of the corresponding circles (see Fig. [2]). Then $|OO_1| = R_1 - r$, $|O_2O_1| = R_1 - R_2$. Let $\alpha$ be the angle between the geodesic lines $OO_2$ and $OO_1$. Assume $\alpha \leq \pi/2$.

From the construction of $\tilde{F}$ it follows that a triangle with the side lengths $R_1 - r$, $R_1 - R_2$, and $\tilde{R} - R_2$ is a right triangle. Denote $\tilde{R} - R_2$ by $\tilde{d}$. From (2.4), we have that $d > \tilde{d}$. By the law of cosines from the geodesic triangle $\triangle OO_2O_1$, we deduce:
(1) For $c = 0$,

$$
(R_1 - R_2)^2 = (R_1 - r)^2 + d^2 - 2d(R_1 - r)\cos \alpha \\
> (R_1 - r)^2 + \tilde{d}^2 - 2d(R_1 - r)\cos \alpha.
$$

Since $(R_1 - R_2)^2 = (R_1 - r)^2 + \tilde{d}^2$, then from the above computations it follows that $\cos \alpha > 0$, thus $\alpha < \pi/2$.

(2) For $c = 1$,

$$
\cos (R_1 - R_2) = \cos (R_1 - r)\cos d + \sin (R_1 - r)\sin d\cos \alpha \\
< \cos (R_1 - r)\cos \tilde{d} + \sin (R_1 - r)\sin d\cos \alpha.
$$

Recalling that $\cos (R_1 - R_2) = \cos (R_1 - r)\cos \tilde{d}$, we obtain $\cos \alpha > 0$, $\alpha < \pi/2$.

(3) For $c = -1$,

$$
cosh (R_1 - R_2) = \cosh (R_1 - r)\cosh d - \sinh (R_1 - r)\sinh d\cos \alpha \\
> \cosh (R_1 - r)\cosh \tilde{d} + \sinh (R_1 - r)\sinh d\cos \alpha.
$$

And since $\cosh (R_1 - R_2) = \cosh (R_1 - r)\cosh \tilde{d}$, we get $\alpha < \pi/2$.

Therefore, in all the cases the angle $\alpha$ is strictly less than $\pi/2$. Hence, from the right triangle $\triangle OO_1C$ we have $|OO_1| > |O_1C|$. Thus, if $r'$ is the radius of the inscribed in $\Omega'$ circle, then $r' = R_1 - |O_1C|$ and $r' > R_1 - |OO_1| = r$. And since it is true for any plane $\Pi$, we come to the contradiction which proves (2.1).

Inequality (2.1) is sharp since the equality is obviously attained for $\tilde{F}$. Lemma 2.1 is proved.

\[ \square \]

Theorem 3 is a direct consequence of Lemma 2.1 because the right sides of inequalities (1.5) – (1.7) are the values of $\tilde{R}$ in terms of the inscribed sphere’s radius $\tilde{r} = r$.

In order to prove Theorems 1 and 2, we should derive additional estimates for the spherical shell’s width and the quotient of its radii in the case of rounded $\kappa_1, \kappa_2$-convex spindle-shaped hypersurfaces. These estimates are summarized in the following lemma.

**Lemma 2.3.** Suppose $\tilde{F} \subset \mathbb{M}^m(c)$ is a rounded $\kappa_1, \kappa_2$-convex spindle-shaped hypersurface, $\tilde{r}$ and $\tilde{R}$ are the radii of the inscribe and circumscribe spheres for $\tilde{F}$; then for the width $\tilde{R} - \tilde{r}$ estimates (1.1) – (1.3) hold. Moreover, when $c = 0$ for the quotient $\tilde{R}/\tilde{r}$ estimate (1.4) holds.

**Proof.** Estimates (1.1) – (1.3) are obtained similarly to [3]. Let us show (1.1). In the rest of the cases computations are similar.
We can also rewrite

\[ \tilde{\kappa} = \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2 + R_2}. \]

Let us introduce a function

\[ w(\tilde{r}) = \tilde{R} - \tilde{r} = \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2 + R_2 - \tilde{r}} \]
defined for \( \tilde{r} \in [R_1, R_2] \). By construction, \( w \geq 0 \), and \( w(R_1) = w(R_2) = 0 \). Hence, this function attains the global maximum on \( (R_1, R_2) \). Solving the equation for the derivative \( dw/d\tilde{r} = 0 \), which is a linear equation with respect to \( \tilde{r} \), and substituting its solution in \( w(\tilde{r}) \), we will get (1.1).

Now let us prove estimate (1.4).

Similarly to the above, we introduce the function

\[ q(\tilde{r}) = \frac{\tilde{R}}{\tilde{r}} = \frac{1}{\tilde{r}} \left( \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2 + R_2} \right) \]
defined for \( \tilde{r} \in [R_1, R_2] \). Moreover, by construction, \( q \geq 1 \), and \( q(R_1) = q(R_2) = 1 \). Hence, this function attains the global maximum on \( (R_1, R_2) \) unless the trivial case \( R_1 = R_2 \). Let us find this maximal value.

Solving the equation \( dq/d\tilde{r} = 0 \) for \( \tilde{r} \), which simplifies to a quadratic equation, we get the root

\[ \tilde{r}_0 = \frac{1}{R_1^2 + R_2^2} \left( 2R_1^2R_2 - R_2 (R_1 - R_2) \sqrt{2R_1R_2} \right). \]

The function \( q \) attains its maximum at \( \tilde{r}_0 \). Therefore

\[ q(\tilde{r}) \leq q(\tilde{r}_0) \quad \text{for all} \quad \tilde{r} \in [R_1, R_2]. \]

Now we proceed with the computing \( q(\tilde{r}_0) \). It is straightforward to check that

\[ (R_1 - R_2)^2 - (R_1 - \tilde{r}_0)^2 = \frac{(R_1 - R_2)^2}{(R_1^2 + R_2^2)^2} \left( R_2 (R_1 - R_2) - R_1 \sqrt{2R_1R_2} \right)^2. \]

Thus, since \( R_1 \sqrt{2R_1R_2} > R_2 (R_1 - R_2) \),

\[ \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r}_0)^2} + R_2 = \frac{\sqrt{2R_1R_2}}{R_1^2 + R_2^2} \left( R_1 + \sqrt{2R_1R_2} \right) \left( R_1 - R_2 - \sqrt{2R_1R_2} \right). \]

We can also rewrite \( \tilde{r}_0 \) in the same manner:

\[ \tilde{r}_0 = \frac{\sqrt{2R_1R_2}}{R_1^2 + R_2^2} \left( R_2 + \sqrt{2R_1R_2} \right) \left( R_1 - R_2 - \sqrt{2R_1R_2} \right). \]
Combining (2.7) and (2.8), and recalling that $R_i = 1/\kappa_i$, $i \in \{1, 2\}$, we get

\[(2.9)\]

\[q(\tilde{r}_0) = \frac{R_1 + \sqrt{2R_1R_2}}{R_2 + \sqrt{2R_1R_2}} = \frac{\sqrt{\kappa_1} + \sqrt{2}}{\sqrt{\kappa_2} + \sqrt{2}}\]

Thereby, from (2.5), (2.6), and (2.9), we finally obtain

\[\tilde{R}/\tilde{r} \leq \frac{\sqrt{\frac{\kappa_1}{\kappa_2}} + \sqrt{2}}{\sqrt{\frac{\kappa_2}{\kappa_1}} + \sqrt{2}},\]

as desired.

The above bound is sharp and is attained for the rounded $\kappa_1, \kappa_2$-convex hypersurface with the radius of the inscribe sphere equal to $\tilde{r}_0$. Lemma 2.3 is proved.

Now, Theorems 1 and 2 are the direct consequences of Lemma 2.1 and Lemma 2.3.

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