Multiparameter Quantum Function Algebra
at Roots of 1

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Introduction

In this paper we consider a multiparameter deformation $F^\varphi_q[G]$ of the quantum function algebra associated to a simple algebraic group $G$. This deformation has been introduced by Reshetikhin ([R], cf. also [D-K-P1]) and is constructed from a skew endomorphism $\varphi$ of the weight lattice of $G$. When $\varphi$ is zero we get the standard quantum group, that is the algebra studied by [H-L1-2-3], [Jo] and, in the compact case, by [L-S2]. In the case ($\varphi = 0$) and when the quantum parameter $q$ is a root of unity, important results are contained in [D-L] and [D-P2]. A general $\varphi$ has been considered in [L-S2] for $G$ compact and $q$ generic. Here we study the representation theory at roots of one for a non trivial $\varphi$.

Our arguments are similar to those used by De Concini and Lyubashe nko. Nevertheless there is a substantial difference: when $\varphi = 0$ the major tool is to understand in detail the $SL(2)$-case which allows to construct representations. Unfortunately there are not multiparameter deformations for $SL(2)$. Moreover the usual right and left actions of the braid group on $F^\varphi_q[G]$ are not so powerful as in the case $\varphi = 0$.

We first (sections 1.,2.,3.) give some properties of the multiparameter quantum function algebra $F^\varphi_e[G]$ at $\varepsilon$, $l$-th root of 1. To do this we principally use a duality, given in [C-V], between some Borel type subHopf algebras of $F^\varphi_q[G]$. In section 4. we compute the dimension of the symplectic leaves of $G$ for the Poisson structure determined by $\varphi$. Our main result (cor. 5.7) is the link between this dimension and the dimension of the representations of $F^\varphi_e[G]$, for "good" $l$. More precisely, we can see $F^\varphi_e[G]$ as a bundle of algebras on $G$. Its theory of representations is constant over the T-biinvariant Poisson submanifold of $G$ (T being the Cartan torus of $G$) and we have

**Theorem** Let $l$ be a "good" integer (see 5.5 ) and let $p$ be a point in the the symplectic leaf $\Theta$ of $G$. Then the dimension of any representation of $F^\varphi_e[G]$ lying over $p$ is divisible by $l^{\frac{1}{2}\dim \Theta}$.

Finally, using the results in the first three sections, we describe explicitely (5.8) a class of representations of $F^\varphi_e[G]$.

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Notations. For the comultiplication in a coalgebra we use the notation \( \Delta x = x_{(1)} \otimes x_{(2)} \). If \( H \) is a Hopf algebra, we denote by \( H^{op} \) the same coalgebra with the opposite multiplication and by \( \Delta_{op} \), the same algebra with the opposite comultiplication.

Let \( F \) be a field and let \((H_i, \eta_i, \Delta_i, \varepsilon_i, S_i), \ i = 1, 2, \) be Hopf algebras. Then an \( F \)-linear pairing \( \pi : H_1 \otimes H_2 \rightarrow F \) is called an Hopf algebra pairing \([T]\) if:

\[
\pi(uv \otimes h) = \pi(u \otimes h_{(1)})\pi(v \otimes h_{(2)}), \quad \pi(u \otimes hl) = \pi(u_{(1)} \otimes h)\pi(u_{(2)} \otimes l)
\]

\[
\pi(\eta_1 1 \otimes h) = \varepsilon_2 h, \quad \pi(u \otimes \eta_2 1) = \varepsilon_1 u
\]

\[
\pi(S_1 u \otimes h) = \pi(u \otimes S_2 h),
\]

for \( u, v \in H_1, \ h, l \in H_2 \). Moreover \( \pi \) is said perfect if it is not degenerate.

We denote by \( R \) the ring \( \mathbb{Q}[q, q^{-1}] \) and by \( K \) its quotient field \( \mathbb{Q}(q) \). Take a positive integer \( l \) and let \( p_l(q) \) be the \( l \)th cyclotomic polynomial. We define \( \mathbb{Q}(\varepsilon) = \mathbb{Q}(q)/(p_l(q)) \) (\( \varepsilon \) being a primitive \( l \)th root of unity). Finally we recall the definition of the \( q \)-numbers:

\[
(n)_q = \frac{q^n - 1}{q - 1}, \quad (n)_q! = \prod_{m=1}^{n} (m)_q, \quad \left( \begin{array}{c} n \\ m \end{array} \right)_q = \frac{(n)_q!}{(m)_q!(n-m)_q!},
\]

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{m=1}^{n} [m]_q, \quad \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{[n]_q!}{[m]_q![n-m]_q!}.
\]

1. The Multiparameter Quantum Group

1.1. Let \( A = (a_{ij}) \) be an indecomposable \( n \times n \) Cartan matrix; that is let \( a_{ij} \) be integers with \( a_{ii} = 2 \) and \( a_{ij} \leq 0 \) for \( i \neq j \) and let \( (d_1, \ldots, d_n) \) be a fixed \( n \)-uple of relatively prime positive integers \( d_i \) such that the matrix \( TA \) is symmetric and positive definite. Here \( T \) is the diagonal matrix with entries \( d_i \).

Consider the free abelian group \( P = \sum_{i=1}^{n} \mathbb{Z} \omega_i \) with basis \( \{ \omega_i | i = 1, \ldots, n \} \) and define

\[
\alpha_i = \sum_{j=1}^{n} a_{ij} \omega_j \quad (i = 1, \ldots, n), \quad Q = \sum_{i=1}^{n} \mathbb{Z} \alpha_i, \quad P_+ = \sum_{i=1}^{n} \mathbb{Z}_+ \omega_i;
\]

\( P \) and \( Q \) are called respectively the weight and the root lattice, the elements of \( P_+ \) are the dominant weights.

Define a bilinear \( \mathbb{Z} \)-valued pairing on \( P \times Q \) by the rule \( (\omega_i, \alpha_j) = d_i \delta_{ij} \) (\( \delta_{ij} \) is the Kronecker symbol); it can be extended to symmetric pairings

\[
P \times P \rightarrow \mathbb{Z}\left[ \frac{1}{\det(A)} \right], \quad Q \times Q \rightarrow \mathbb{Q}
\]

where \( QP = \sum_{i=1}^{n} \mathbb{Q} \omega_i \).

To this setting is associated a complex simple finite dimensional Lie algebra \( g \) and a complex connected simply connected simple algebraic group \( G \).
1.2. Fix an endomorphism \( \varphi \) of the \( \mathbb{Q} \)-vector space \( \mathbb{Q} P \) which satisfies the following conditions:

\[
\begin{align*}
(1.1) \quad (\varphi x, y) &= -(x, \varphi y) \quad \forall x, y \in \mathbb{Q} P, \\
\varphi \alpha_i &= \alpha_i + 2\tau_i, \quad \tau_i \in \mathbb{Q}, i = 1, \ldots, n, \\
\frac{1}{2}(\varphi \lambda, \mu) &\in \mathbb{Z} \quad \forall \lambda, \mu \in P
\end{align*}
\]

We will see later the motivation of the third assumption, now observe that it implies \( \varphi P \subseteq P \).

If \( \tau_i = \sum_{j=1}^{n} x_{ij} \omega_j = \sum_{j=1}^{n} y_{ij} \alpha_j \), let put \( X = (x_{ij}), \ Y = (y_{ij}) \). Then \( TX \) is an antisymmetric matrix and the last two conditions in (1.1) are equivalent to the following:

\[
Y \in M_n(\mathbb{Z}) \cap T^{-1}A_n(\mathbb{Z})A
\]

where \( A_n(\mathbb{Z}) \) denotes the submodule of \( M_n(\mathbb{Z}) \) given by the antisymmetric matrices.

The maps

\[
1 \pm \varphi : \mathbb{Q} P \rightarrow \mathbb{Q} P, \quad \alpha_i \mapsto \alpha_i \pm \delta_i
\]

are \( \mathbb{Q} \)-isomorphisms (cf. [C-V]); moreover we have

\[
((1 + \varphi)\pm^1 \lambda, \mu) = (\lambda, (1 - \varphi)\pm^1 \mu), \quad \forall \lambda, \mu \in P
\]

and so \( (1 + \varphi)^\pm (1 - \varphi)^\mp \) are isometries of \( \mathbb{Q} P \). Let us put \( r = (1 + \varphi)^{-1}, T = (1 - \varphi)^{-1} \).

We like to stress that if we want to enlarge the results of this paper to the semisimple case it is enough to ask that \( 2AYA^{-1} \in M_n(\mathbb{Z}) \), which guarantees \( \varphi P \subseteq P \).

1.3. The multiparameter simply connected quantum group \( U_q^\varphi(\mathfrak{g}) \) associated to \( \varphi \) ([R],[D-K-P1],cf. [C-V]) is the \( K \)-algebra on generators \( E_i, F_i, K_{\omega_i \sigma_i}^\pm \), \( i = 1, \ldots, n \), with the same relations of the Drinfel'd-Jimbo quantum group \( U_q(\mathfrak{g}) = U_q^\varphi(\mathfrak{g}) \) and with an Hopf algebra structure given by the following comultiplication \( \Delta_\varphi \), counity \( \varepsilon_\varphi \) and antipode \( S_\varphi \) defined on generators \( (i = 1, \ldots, n; \lambda \in P) \)

\[
\begin{align*}
\Delta_\varphi E_i &= E_i \otimes K_{\alpha_i + \tau_i} + K_{-\alpha_i + \tau_i} \otimes E_i, \\
\Delta_\varphi F_i &= F_i \otimes K_{\alpha_i + \tau_i} + K_{-\alpha_i + \tau_i} \otimes F_i, \\
\Delta_\varphi K_\lambda &= K_\lambda \otimes K_\lambda, \quad \varepsilon_\varphi E_i = 0, \quad \varepsilon_\varphi F_i = 0, \quad \varepsilon_\varphi K_\lambda = 1, \quad S_\varphi E_i = -K_{\alpha_i} E_i, \quad S_\varphi F_i = -F_i K_{-\alpha_i}, \quad S_\varphi K_\lambda = K_{-\lambda}
\end{align*}
\]

where for \( \lambda = \sum_{i=1}^{n} m_i \omega_i \in P \) we use the notation \( K_\lambda = \prod_{i=1}^{n} K_{\omega_i}^{m_i} \).

Put \( K_i = K_{\alpha_i} , \ q_i = q^{\alpha_i} \); we recall the relations in the algebra \( U_q^\varphi(\mathfrak{g}) \) ([D1],[J]) :

\[
(1.2) \quad K_{\omega_i} K_{\omega_i}^{-1} = 1 = K_{-\omega_i}^{-1} K_{\omega_i}, \quad K_{\omega_i} K_{\omega_j} = K_{\omega_j} K_{\omega_i},
\]

\[
(1.3) \quad K_{\omega_i} E_j K_{\omega_i}^{-1} = q_i^{\delta_{ij}} E_j, \quad K_{\omega_i} F_j K_{\omega_i}^{-1} = q_i^{-\delta_{ij}} F_j,
\]

\[
(1.4) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.
\]

\[
(1.5) \quad \sum_{m=0}^{\frac{1-a_{ij}}{2}} (-1)^m \left[ \begin{array}{c}
1 - a_{ij} \\
1 - a_{ij}
\end{array} \right] \begin{array}{c}
m \end{array} \frac{G_i^{1-a_{ij}-m} G_j^{m} G_i^{m}}{q_i}, \quad (i \neq j),
\]

3
in the two cases $G_i = E_i, F_i$.

1.4. Let $U_q^\phi(b_+)$ and $\overline{U}_q^\phi(b_+)$ be the sub-Hopf algebras of $U_q^\phi(g)$ generated by the $E_i's$ ($i = 1, \ldots, n$) and respectively by the sets $\{K_\lambda | \lambda \in \mathbb{P}\}$, $\{K_\lambda | \lambda \in \mathbb{Q}\}$. Similarly let $U_q^\phi(b_-)$ and $\overline{U}_q^\phi(b_-)$ be the sub-Hopf algebras of $U_q^\phi(g)$ generated by the $F_i's$ ($i = 1, \ldots, n$) and respectively by $P$ and $Q$ in the multiplicatively notation of the $K_\lambda's$.

Take an element $u$ in the algebraic closure of $K$ such that $u^{\det(A + D)} = q$. Then we know (cf. [C-V]) that the following bilinear map is a perfect Hopf algebra pairing:

$$\pi_\phi : U_q^\phi(b_-)_{op} \otimes U_q^\phi(b_+), Q(u), \left\{ \begin{array}{l} \pi_\phi(K_\lambda, K_\mu) = q^{\langle r(\lambda), \mu \rangle} \\ \pi_\phi(E_i, K_\lambda) = \pi_\phi(F_i, K_\lambda) = 0 \\ \pi_\phi(E_i, F_j) = \frac{\delta_{ij}}{q^{r(i) - q^{r(i), r_j}}} q^{\langle r(i), r_j \rangle} \end{array} \right.$$ for $\lambda, \mu \in \mathbb{P}$, $i = 1, \ldots, n$. Consider now the antisymorphism $\zeta_\phi$ of Hopf algebras relative to a $\mathbb{Q}$-algebra antisymorphism $\zeta : K \rightarrow K$, $q \mapsto q^{-1}$ of the basic ring, namely

$$\zeta_\phi : U_q^\phi(g) \rightarrow U_q^{-\phi}(g), E_i \mapsto F_i, F_i \mapsto E_i, K_\lambda \mapsto K_{-\lambda},$$

which send $U_q^\phi(b_+)$ into $U_q^{-\phi}(b_-)$ and vice versa. Then $\overline{\pi}_\phi = \zeta \circ \pi_\phi \circ (\zeta_\phi \otimes \zeta_\phi)$ is a perfect Hopf algebra pairing:

$$\overline{\pi}_\phi : U_q^\phi(b_+)_{op} \otimes U_q^\phi(b_-), Q(u), \left\{ \begin{array}{l} \overline{\pi}_\phi(K_\lambda, K_\mu) = q^{-\langle r(\lambda), r_\mu \rangle} \\ \overline{\pi}_\phi(E_i, K_\lambda) = \overline{\pi}_\phi(K_\lambda, F_i) = 0 \\ \overline{\pi}_\phi(E_i, F_j) = \frac{\delta_{ij}}{q^{r(i) - q^{r(i), r_j}}} q^{-\langle r(i), r_j \rangle} \end{array} \right.$$ for a monomial $E_\underline{x} = E_{i_1} \cdots E_{i_n}$ in the $E_i's$ and a monomial $F_\underline{y} = F_{j_1} \cdots F_{j_n}$ in the $F_i's$ we define the weight $p(E_\underline{x})$ and $p(F_\underline{y})$ in the following way:

$$(1.6) \quad p(E_r) = p(F_r) = \alpha_r, \quad p(E_\underline{x}) = p(F_\underline{y}) = \sum_{j=1}^n \alpha_{i_j}.$$ If $v$ is a monomial in the $E_i's$ or in the $F_i's$ with $p(v) = \varepsilon$ we will write $s(v)$, $r(v)$, $\tau(v)$, instead of $\frac{1}{2} \varphi(\varepsilon)$, $\frac{1}{2} r(\varepsilon)$, $\frac{1}{2} \tau(\varepsilon)$.

The next lemma is proved in [C-V].

1.5. Lemma For $x$ and $y$ homogeneous polynomials in the $E_i's$ and the $F_i's$ respectively and for $\lambda, \mu \in \mathbb{P}$ it holds:

$$(i) \quad \pi_\phi(yK_\lambda, xK_\mu) = \pi_\phi(y, x)q^{\langle r(\lambda), \mu - s(x) \rangle} - \langle r(y), \mu \rangle = \pi_0(y, x)q^{\langle r(\lambda), \mu - s(x) \rangle},$$

$$(ii) \quad \overline{\pi}_\phi(K_\lambda x, K_\mu y) = \overline{\pi}_\phi(x, y)q^{\langle r(\lambda), s(x) - \mu \rangle} - \langle \tau(y), \mu \rangle = \overline{\pi}_0(x, y)q^{\langle r(\lambda), \mu - s(x) \rangle}.$$
1.6. Consider the following $R$-subalgebra of $U_q^\varphi(g)$:

$$\Gamma(t) = \{ f \in K[Q] \mid \pi_\varphi(f, K_{(1-\varphi)\lambda}) = \pi_0(f, K_\lambda) \in R \forall \lambda \in P \}.$$ 

In [D-L] is given a $K$-basis $\{ \xi_t \mid t = (t_1, \ldots, t_n) \in \mathbb{Z}^n_+ \}$ of $K[Q]$ which is an $R$-basis of $\Gamma(t)$, namely

$$\xi_t = \prod_{i=1}^n \left( \frac{K_{i_1}}{q^0_{t_1}} \right) \cdots \left( \frac{K_{i_t}}{q^0_{t_t}} \right) = \prod_{s=1}^t \frac{K_{\delta_{is} - s + 1}}{q^s_{t_t} - 1} \left( \frac{K_{\delta_{is}}}{q^0_{t_t}} \right).$$

(for a positive integer $s$, $[s]$ denote the integer part). Note that

(1.7) $\{ f \in K[(1 + \varphi)P] \mid \pi_\varphi(f \otimes \Gamma(t)) \subseteq R \} = R[(1 + \varphi)P],$

(1.8) $\{ f \in K[(1 - \varphi)P] \mid \pi_\varphi(\Gamma(t) \otimes f) \subseteq R \} = R[(1 - \varphi)P].$

1.7. Let $W$ be the Weyl group associated to the Cartan matrix $A$, that is let $W$ be the finite subgroup of $GL(P)$ generated by the automorphisms $s_i$ of $P$ given by $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$. If $\Omega = \{ \alpha_1, \ldots, \alpha_n \}$, the root system corresponding to $A$ is $\Phi = W\Omega$ while the set of positive roots is $\Phi_+ = \Phi \cap \sum_{i=1}^n \mathbb{Z}^+ \alpha_i$. Fix a reduced expression for the longest element $\omega_0$ of $W$, say $\omega_0 = s_{i_1} \cdots s_{i_N}$ and consider the usual total ordering on the set $\Phi_+$ induced by this choice:

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \ldots, \beta_N = s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N}.$$ 

Introduced, for $k = 1, \ldots, N$, the corresponding root vectors:

$$G_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(G_{i_k}), \ G_i = E_i, F_i,$$

where the $T_i$'s are the algebra automorphisms of $U_q^\varphi(g)$ (and so of $U_q^\varphi(g)$) introduced by Lustig up to change $q \leftrightarrow q^{-1}, K_\lambda \leftrightarrow K_{-\lambda}$ (see[L2]).

For a positive integer $s$ define

$$G_{\beta_k}^{(s)} = \frac{G_{\beta_k}}{[s]q^s_{t_t}!}, \ G_{\beta_k}^{(s)} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(G_{i_k}^{(s)}),$$

always in the two cases $G_i = E_i, F_i$.

For $\alpha \in \Phi_+$ let put

$$q_\alpha = q^{\frac{\langle \alpha, \omega \rangle}{2}}, \ \tau_\alpha = \frac{1}{2} \varphi \alpha;$$

$$e_{\tau_\alpha} = (q_{\alpha}^{-1} - q_{\alpha})E_\alpha K_{\tau_\alpha}, \ f_{\tau_\alpha} = (q_{\alpha} - q_{\alpha}^{-1})F_\alpha K_{\tau_\alpha};$$

$$e_{\alpha} = e_{\tau_\alpha}, \ f_{\alpha} = f_{\tau_\alpha}.$$ 

Note that $K_{\tau_\alpha}$ commutes with every monomial of weight $\alpha$. 
1.8. Define $R_q^\varphi[B_+]'$ and $R_q^\varphi[B_-]'''$ as the $R$-subalgebras of $U_q^\varphi(b_+)^{op}$ and $U_q^\varphi(b_-)^{op}$ respectively generated by the elements $e_i^\varphi$, $K_{(1-\varphi)\omega_i}$ $(i = 1, \ldots, n, \alpha \in \Phi_+)$.

Similarly denote by $R_q^\varphi[B_+]''$ and $R_q^\varphi[B_-]''''$ the $R$-subalgebras of $U_q^\varphi(b_+)^{op}$ and $U_q^\varphi(b_-)^{op}$ respectively generated by the elements $f_i^\varphi$, $K_{(1+\varphi)\omega_i}$ $(i = 1, \ldots, n, \alpha \in \Phi_+)$.

Then, by restriction from $\pi_\varphi$, we obtain the following two pairings:

$$\pi'_\varphi : U_q^\varphi(b_-) \otimes_R R_q^\varphi[B_-]' \longrightarrow K \quad \pi''_\varphi : R_q^\varphi[B_+]'' \otimes_R U_q^\varphi(b_+) \longrightarrow K$$

while by restriction from $\pi'_\varphi$ we get the other two:

$$\pi'_\varphi : U_q^\varphi(b_+) \otimes_R R_q^\varphi[B_+]' \longrightarrow K \quad \pi''_\varphi : R_q^\varphi[B_-]'' \otimes_R U_q^\varphi(b_-) \longrightarrow K.$$

We get:

$$\begin{cases}
\pi'_\varphi(F_j, e_i^\varphi) = -\delta_{ij} \\
\pi'_\varphi(K_\lambda, K_{(1-\varphi)\mu}) = q^{(\lambda, \mu)}
\end{cases}
\quad
\begin{cases}
\pi''_\varphi(f_j^\varphi, E_j) = \delta_{ij} \\
\pi''_\varphi(K_{(1+\varphi)\lambda}, K_\mu) = q^{(\mu, \lambda)}
\end{cases}
\quad
\begin{cases}
\pi'_\varphi(e_i^\varphi, F_j) = -\delta_{ij} \\
\pi''_\varphi(K_{(1+\varphi)\lambda}, K_\mu) = q^{-(\lambda, \mu)}.
\end{cases}$$

We can choose as bases of $U_q^\varphi(b_+)$ and $U_q^\varphi(b_-)$ the elements (see [L2],[D-L]):

$$\xi_{m,t} = \prod_{j=N}^1 E_{\beta_j}^{(m_j)} \prod_{i=1}^n \left( K_i; 0 \right) K_i^{-[\frac{1}{2}]}$$

$$\eta_{m,t} = \prod_{j=N}^1 F_{\beta_j}^{(m_j)} \prod_{i=1}^n \left( K_i; 0 \right) K_i^{-[\frac{1}{2}]}$$

1.9. Proposition

$$q^{-\sum_{i<j}(n_i, n_j, \beta_i)} \pi'_\varphi(\eta_{m,t}, \prod_{j=N}^1 (e_i^\varphi)^{m_j} K_{(1-\varphi)\lambda}) = q^{\sum_{i<j}(n_i, n_j, \beta_i)} \pi''_\varphi(\prod_{j=N}^1 (f_i^\varphi)^{m_j} K_{(1+\varphi)\lambda}, \xi_{m,t}) =$$

$$\prod_{j=1}^N \frac{\delta_{n_j, m_j} q_{\beta_j}}{N} \prod_{i=1}^N \left( \frac{(\alpha_i, \lambda)}{t_i} \right) q^{-(\alpha_i, \lambda)[\frac{1}{2}]} = q^{-\sum_{i=1}^n (n_i, \beta_i, \lambda)}.$$

Similar formulas hold for $\pi'_\varphi$ and $\pi''_\varphi$.

Proof. First of all observe that

$$\Delta_\varphi e_i^\varphi = e_i^\varphi \otimes 1 + K_{(1-\varphi)\alpha_i} \otimes e_i^\varphi, \quad \Delta_\varphi f_i^\varphi = f_i^\varphi \otimes K_{(1+\varphi)\alpha_i} + 1 \otimes f_i^\varphi,$$

and, for $\alpha \in \Phi_+$,

$$\Delta_\varphi e_\alpha^\varphi = e_\alpha^\varphi \otimes 1 + K_{(1-\varphi)\alpha_i} \otimes e_\alpha^\varphi + e, \quad \Delta_\varphi f_\alpha^\varphi = f_\alpha^\varphi \otimes K_{(1+\varphi)\alpha} + 1 \otimes f_\alpha^\varphi + f,$$

where $e (f)$ is a sum of terms $u_i \otimes v_i$, $u_i$ and $v_i$ being linear combination of monomials in the $e_i^\varphi$ ($f_i^\varphi$) and $K_\lambda$ and $ht(\beta) < ht(\alpha)$. Moreover

$$\pi'_\varphi(F_\alpha, e_\alpha^\varphi) = \pi'_\varphi(F_\alpha, e_\alpha K_{\tau_\alpha}) = \pi''_\varphi(F_\alpha, e_\alpha) \quad \forall \alpha \in \Phi_+.$$
Put now \( F = \eta_{n,0}, \) \( M = \eta_{0,t}, \) \( e^\varphi = \prod_{j=1}^N (e^\varphi_{\beta_j})^{m_j}, \) then

\[
\pi'_\varphi(F, e^\varphi K_{(1-\varphi)\lambda}) = \pi'_\varphi(F \otimes M, \Delta_{\varphi} e^\varphi K_{(1-\varphi)\lambda}) = \pi'_\varphi(F, e^\varphi K_{(1-\varphi)\lambda}) \pi'_\varphi(M, K_{(1-\varphi)\lambda}) = \\
= \pi'_\varphi(\Delta_{\varphi} F, K_{(1-\varphi)\lambda} \otimes e^\varphi) \pi'_\varphi(M, K_{\lambda}) = q^{-(e(F), (1-\varphi)\lambda)} \pi'_\varphi(F, e^\varphi) \pi'_0(M, K_{\lambda}) = \\
= q^{-(e(F), \lambda)} \pi'_\varphi(F, e^\varphi) \pi'_0(M, K_{\lambda}).
\]

Now, if \( e^0 = \prod_{j=1}^N (e^0_{\beta_j})^{m_j}, \) we have

\[
\pi'_\varphi(F, e^\varphi) = q \sum_{i < j} (m_{i\tau_i, m_j\tau_j}) \pi'_\varphi(F, \xi^0 \sum_{m_i \tau_i} \beta_i) = q \sum_{i < j} (m_{i\tau_i, m_j\tau_j}) \pi'_0(F, e^0),
\]

where the powers of \( q \) arises from the commutation of \( K_{\tau_i} \) and the last equality from 1.5. Since the value of \( \pi'_0(F, e^0) \) is calculated in [D-L] (formula (3.2)) we are done.

For the other equality as well as for the case of \( \pi'_\varphi \) we proceed in the same way. \( \square \)

1.10. Define the following \( R \)-submodules of \( U_q^\varphi(g) \):

\[
\Gamma^\varphi(b_+) = \{ x \in \overline{U}_q^\varphi(b_+) \mid \pi'_\varphi(R_q^\varphi[B+]_\text{op} \otimes x) \subset R \} \\
\Gamma^\varphi(b_-) = \{ x \in \overline{U}_q^\varphi(b_-) \mid \pi'_\varphi(x \otimes R_q^\varphi[B-]_{\text{op}}) \subset R \}.
\]

It is clear from prop.1.2. that the \( \xi_{m,t} \)'s and the \( \eta_{m,t} \)'s are \( R \)-bases of \( \Gamma^\varphi(b_+) \) and \( \Gamma^\varphi(b_-) \) respectively and so first of all they are algebras (cf. [L]) and secondly as algebras they are isomorphic to \( \Gamma^0(b_+) \) and \( \Gamma^0(b_-) \) respectively. They are also sub-coalgebras of \( U_q^\varphi(g) \), namely

\[
\begin{align*}
\Delta_{\varphi} F_i^{(p)} &= \sum_{r+s=n} q_i^{-rs} E_i^{(p)} K_{s(\tau_i, -\alpha_i)} \otimes E_i^{(s)} K_{-r\tau_i}, \\
\Delta_{\varphi} F_i^{(p)} &= \sum_{r+s=n} q_i^{-rs} F_i^{(r)} K_{s\tau_i} \otimes F_i^{(s)} K_{r(\alpha_i, +\tau_i)}, \\
\Delta_{\varphi} (K_i; 0) &= \sum_{r+s=t} q_i^{-rs} \left( K_i; 0 \right) \otimes \left( K_i; 0 \right)
\end{align*}
\]

(1.9)

(the first two equalities are proved in [C-V], the last in [D-L]).

1.11. As a consequence of 1.9, we get, by restriction, two pairings

\[
\pi'_\varphi : \Gamma^\varphi(b_-) \otimes_R R_q^\varphi[B_-] \rightarrow R, \quad \pi''_\varphi : \pi'_\varphi : \Gamma^\varphi(b_+) \otimes_R \Gamma^\varphi(b_-) \rightarrow R.
\]

Moreover the same formulas in 1.9. and (1.7), (1.8) give

\[
\{ f \in U_q^\varphi(b_+) \mid \pi'_\varphi(\Gamma^\varphi(b_-)_{\text{op}} \otimes f) \subset R \} = R_q^\varphi[B_+], \\
\{ f \in U_q^\varphi(b_-) \mid \pi''_\varphi(f \otimes \Gamma^\varphi(b_+)) \subset R \} = R_q^\varphi[B_-]^\text{op}.
\]

Clearly analogous results hold for \( \pi'_\varphi, \pi''_\varphi \) and so we have the two perfect pairings

\[
\pi'_\varphi : \Gamma^\varphi(b_+) \otimes_R R_q^\varphi[B_-] \rightarrow R, \quad \pi''_\varphi : \Gamma^\varphi(b_-) \otimes_R \Gamma^\varphi(b_-) \rightarrow R.
\]

7
Most of the definitions and notations introduced up to now are generalisations to the multiparameter case of the ones given in [D-L]. In extending De Concini-Lyubashenko results we shall only write the parts of the proofs which differ from theirs.

1.12. Lemma The algebras $R_q^\varphi[B_-]'$, $R_q^\varphi[B_+]'$, $R_q^\varphi[B_+]''$, $R_q^\varphi[B_-]''$ have an Hopf algebra structure for which $\pi_\varphi'$, $\pi_\varphi''$, $\pi_\varphi'''$ become perfect Hopf algebra pairings.

Proof. Consider for example $R_q^\varphi[B_-]'$ and let $U_+$ be the sub-K-algebra of $U_q^\varphi(b_+)^\text{op}$ generated by $\{e_\alpha^\varphi, K_{1-\varphi}\alpha \mid \alpha \in \Phi_+, \lambda \in P\}$. We know that (see [L]) the set $\{e_\alpha^\varphi, K_{1-\varphi}\alpha \mid i = 1, \ldots, n, \lambda \in P\}$ is a generating set for $U_+$. Moreover since

$$\Delta_\varphi e_i^\varphi = e_i^\varphi \otimes 1 + K_{1-\varphi}\alpha_i \otimes e_i^\varphi, \quad S_\varphi e_i^\varphi = -K_{1-\varphi}\alpha_i e_i^\varphi, \quad \varepsilon_\varphi e_i^\varphi = 0,$$

$U_+$ is an Hopf algebra. So $\Delta_\varphi e \in U_+ \otimes U_+$ for every $e \in R_q^\varphi[B_-]'$. In order to see that indeed $\Delta_\varphi e \in R_q^\varphi[B_-'] \otimes R_q^\varphi[B_-]$ and to conclude the proof we can proceed as in [D-L](Lemma 3.4). \qed

2. The Multiparameter Quantum Function Algebra

2.1. Consider the full subcategory $C_\varphi$ in $U_q^\varphi(g) - \text{mod}$ consisting of all finite dimensional modules on which the $K_i$'s act as powers of $q$. If $V$ and $W$ are objects of $C_\varphi$ the tensor product $V \otimes W$ and the dual $V^*$ are still in $C_\varphi$, namely one can define

$$a(v \otimes w) = \Delta_\varphi(a(v) \otimes w), \quad (af)v = f((Sa)v), \quad a \in U_q^\varphi(g), \quad v \in V, \quad w \in W, \quad f \in V^*.$$

Given $V \in C_\varphi$, for a vector $v \in V$ and a linear form $f \in V^*$ we define the matrix coefficient $c_{f,v}$ as follows:

$$c_{f,v} : U_q^\varphi(g) \longrightarrow K, \quad x \mapsto f(xv).$$

The $K$-module $F_q^\varphi[G]$ spanned by all the matrix coefficients is equipped with the usual structure of dual Hopf algebra. The comultiplication $\Delta$ (which doesn’t depend on $\varphi$) is given by:

$$(\Delta c_{f,v})(x \otimes y) = c_{f,v}(xy),$$

while the multiplication $m_{\varphi}$ is given by:

$$m_{\varphi}(c_{f,v} \otimes c_{g,w}) = c_{f \otimes g, v \otimes w},$$

where $V, W \in C_\varphi$, $v \in V, w \in W$, $f \in V^*, g \in W^*$, $x, y \in U_q^\varphi(g)$.

Moreover, since the algebras $U_q^\varphi(g)$ and $U_q(g)$ are equal, in order to obtain $F_q^\varphi[G]$ (that as coalgebra is equal to $F_q^0[G]$) it is enough to consider the subcategory of $C_\varphi$ given by the highest weight simple modules $L(\Lambda), \Lambda \in P_+.$

We recall that for these modules we have:

$$L(\Lambda) = \bigoplus_{\lambda \in \Omega(\Lambda) \subseteq P} L(\Lambda)_{\lambda}, \quad L(\Lambda)^* \simeq L(-\omega_0 \Lambda), \quad L(\Lambda)^*_{\mu} = (L(\Lambda)_{\mu})^*.$$
and that
\[ F_q(G) = \bigoplus_{\Lambda \in P_+} L(\Lambda) \otimes L(\Lambda)^*. \]

2.2. We want now to link the comultiplication \( \Delta \varphi \) in \( U^\varphi_q(\mathfrak{g}) \) and the multiplication \( m_\varphi \) in \( F^\varphi_q(G) \) with a bivector \( u \in \Lambda^2(\mathfrak{h}) \), \( \mathfrak{h} \) being the Cartan subalgebra of \( \mathfrak{g} \) and to do this we firstly give the Drinfel’d definition of quantized universal enveloping algebra \( U^\varphi_q(G) \).

Let \( \mathbb{Q}[[\hbar]] \) be the ring of formal series in \( \hbar \), then \( U^\varphi_q(\mathfrak{h}) \) is the \( \mathbb{Q}[[\hbar]] \)-algebra generated, as an algebra complete in the \( \hbar \)-adic topology, by the elements \( E_i, F_i, H_i \) \( (i = 1, \ldots, n) \) and defining relations :

\[ [H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j \]

added to relations that we can deduce from (1.5) by replacing \( q \) with \( \exp(\frac{\hbar}{2}) \) and \( K_i \) with \( \exp(\frac{\hbar}{2}d_i H_i) \).

Put now
\[ u = \sum_{i,j=1}^n d_{ij} \hbar H_i \otimes H_j \in \Lambda^2(\mathfrak{h}), \]

where the matrix \( TU = (d_{ij}u_{ij}) \) is antysimmetric.

Then for all \( x \in U^\varphi_q(\mathfrak{g}) \) using the identity \([R]\)
\[ \exp(-u)(\Delta_0 x)\exp(u) = \Delta_\varphi x \]

we can compute the \( \psi_{ij} \)'s, namely
\[ U = A^{-1} X A^{-1}. \]

Moreover we get the following useful equality (see \([L-S2]\)):

\[ m_\varphi(c_{f_1,v_1} \otimes c_{f_2,v_2}) = q^{\delta((\varphi\mu_1, \mu_2) - (\varphi\lambda_1, \lambda_2))} m_0(c_{f_1,v_1} \otimes c_{f_2,v_2}), \]

for \( \Lambda_i \in P_+ \), \( v_i \in L(\Lambda_i)_{\mu_i} \), \( f_i \in L(\Lambda_i)^*_{\lambda_i} \), \( i = 1, 2 \).

Observe that (2.1) justifies the condition \( \frac{1}{2}(\varphi\lambda, \mu) \in \mathbb{Z} \), \( \forall \lambda, \mu \in \mathbb{P} \), required for \( \varphi \) (see (1.1)).

2.3. Since we are interested in the study at roots of 1 we need an integer form \( R^\varphi_q[G] \) of the multiparameter quantum function algebra. For this purpose define \( \Gamma^\varphi(\mathfrak{g}) \) to be the \( R \)-subHopf algebra of \( U^\varphi_q(\mathfrak{g}) \) generated by \( \Gamma^\varphi(\mathfrak{b}_+) \) and \( \Gamma^\varphi(\mathfrak{b}_-) \) and consider the subcategory \( \mathcal{D}_\varphi \) of \( \Gamma^\varphi(\mathfrak{g}) - \text{mod} \)

given by the free \( R \)-modules of finite rank in which \( K_i, \left( \begin{array}{c} K_i; 0 \\ t \end{array} \right) \) act by diagonal matrices with eingevalues \( q_i^m \), \( \left( \begin{array}{c} m \\ t \end{array} \right) \). Define \( R^\varphi_q[G] \) as the submodule generated by the matrix coefficients constructed with the objects of \( \mathcal{D}_\varphi \). Similarly define \( R^\varphi_q[B_+] \) and \( R^\varphi_q[B_-] \) starting with opportune subcategories of \( \Gamma^\varphi(\mathfrak{b}_+) - \text{mod} \) and \( \Gamma^\varphi(\mathfrak{b}_-) - \text{mod} \) respectively.
In completely analogy with the case \( \varphi = 0 \) and essentially in the same way (cf. prop.4.2 in [D-L]) we can prove that the pairings \( \varphi', \varphi'', \varphi'' \) induce the Hopf algebra isomorphisms

\[
R_q^\sigma[B_+]' \simeq R_q^\sigma[B_-] \simeq R_q^\sigma[B_-]''
\]

and in fact these isomorphisms are the motivations for having introduced the pairings.

2.4. Consider now the maps

\[
\Gamma_q^\sigma(b_-) \otimes_R \Gamma_q^\sigma(b_+) \overset{\iota_-}{\longrightarrow} \Gamma_q^\sigma(g) \otimes_R \Gamma_q^\sigma(g) \overset{m}{\longrightarrow} \Gamma_q^\sigma(g)
\]

where \( \iota_\pm \) are the natural embedding and \( m \) is the moltiplication map. The corresponding dual maps composed with the isomorphisms (2.2) give the injections:

\[
\mu'_\varphi: R_q^\sigma[G] \overset{\Delta}{\longrightarrow} R_q^\sigma[G] \otimes_R R_q^\sigma[G] \overset{r}{\longrightarrow} R_q^\sigma[B_-] \otimes_R R_q^\sigma[B_+] \simeq R_q^\sigma[B_-]' \otimes_R R_q^\sigma[B_+]'
\]

\[
\mu''_\varphi: R_q^\sigma[G] \overset{\Delta}{\longrightarrow} R_q^\sigma[G] \otimes_R R_q^\sigma[G] \overset{r}{\longrightarrow} R_q^\sigma[B_+] \otimes_R R_q^\sigma[B_-] \simeq R_q^\sigma[B_+]'' \otimes_R R_q^\sigma[B_-]''.
\]

Let put, for \( M \) in \( \Gamma(\iota), \lambda \) in \( P \),

\[
M(\lambda) = \pi'_\varphi(M, K_{(1-\varphi)\lambda}) = \pi'_\varphi(M, K_{-(1+\varphi)\lambda}) = \pi''_\varphi(K_{(1+\varphi)\lambda}, M) = \pi''_\varphi(K_{-(1-\varphi)\lambda}, M) = \pi_0(M, K_\lambda).
\]

It is now easy to prove the following (see Lemma 4.3.in [D-L])

2.5. Lemma

(i) The image of \( \mu'_\varphi \) is contained in the \( R \)-subalgebra \( A'_\varphi \) generated by the elements

\[
e_\alpha^\varphi \otimes 1, \ 1 \otimes f_\alpha^\varphi, \ K_{(1-\varphi)\lambda} \otimes K_{-(1+\varphi)\lambda}, \ \lambda \in P, \ \alpha \in \Phi_+.
\]

(ii) The image of \( \mu''_\varphi \) is contained in the \( R \)-subalgebra \( A''_\varphi \) generated by the elements

\[
1 \otimes e_\alpha^\varphi, \ f_\alpha^\varphi \otimes 1, \ K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}, \ \lambda \in P, \ \alpha \in \Phi_+.
\]

2.6. Define as in [D-L] the matrix coefficients \( \psi_{\pm \lambda}^\alpha \), that is for each \( \lambda \in P_+ \) call \( v_\lambda \) (resp. \( v_{-\lambda} \)) a choosen highest (resp. lowest) weight vector of \( L(\lambda) \) (resp. of \( L(-\lambda) \), the irreducible module of lowest weight \( -\lambda \)). Let \( \phi_{\pm \lambda} \) the unique linear form on \( L(\pm \lambda) \) such that \( \phi_{\pm \lambda} v_{\pm \lambda} = 1 \) and \( \phi_{\pm \lambda} \) vanishes on the unique \( \Gamma(\iota)-\)invariant complement of \( K v_{\pm \lambda} \subset L(\pm \lambda) \).

For \( \rho = \sum_{i=1}^n \omega_i \), put \( \psi_{\pm \rho} = c_{\phi_{\pm \rho}, v_{\pm \rho}} \), and for \( \alpha \in \Phi_+ \) define

\[
\psi_\lambda^{\alpha}(x) = \phi_\lambda(E_\alpha x v_\lambda), \ \psi_\lambda^{-\alpha}(x) = \phi_\lambda(x F_\alpha v_\lambda),
\]

10
\( \psi_{-\lambda}^\varphi(x) = \phi_{-\lambda}(xE_{\alpha}v_{-\lambda}), \ psi_{-\lambda}^{-\varphi}(x) = \phi_{-\lambda}(F_{\alpha}xv_{-\lambda}) \).

2.7. Proposition  The maps \( \mu_\varphi', \mu_\varphi'' \) induce algebra isomorphisms

\[
R_q^\varphi[G]\{\psi^{-1}\} \simeq A_\varphi', \quad R_q^\varphi[G]\{\psi^{-1}\} \simeq A_\varphi''.
\]

Proof. First of all we specify that what we want to prove is that the subalgebra generated by \( \text{Im}(\mu_\varphi') \) and \( \mu_\varphi'((\psi^{-1})) \) is indeed \( A_\varphi' \) and similarly for \( A_\varphi'' \). Consider the case of \( \mu_\varphi' \). First of all we have

\[
\mu_\varphi'(\psi_p) = K_{(1-\varphi)p} \otimes K_{-(1+\varphi)p}.
\]

Moreover an easy calculation gives

\[
\mu_\varphi'(\psi_p^{\alpha_i}) = -q^{-\frac{1}{2}(\varphi_{\alpha_i}, \omega_i)} e_i^{\varphi} K_{(1-\varphi)\omega_i} \otimes K_{-(1+\varphi)\omega_i},
\]

from which we get \( e_i^{\varphi} \otimes 1 \in < \text{Im}(\mu_\varphi'), \mu_\varphi'((\psi^{-1})) > \).

To see that \( e_i^{\varphi} \otimes 1 \in < \text{Im}(\mu_\varphi'), \mu_\varphi'((\psi^{-1})) > \) we proceed as in [D-L], by induction on \( ht(\alpha) \), namely

\[
\mu_\varphi'((\psi_p^{\alpha_i})) = (-q^{-(\tau_{\alpha, \lambda})} x(\alpha, \lambda)e_i^{\varphi} + d) K_{(1-\varphi)\lambda} \otimes K_{-(1+\varphi)\lambda}
\]

where \( d \) is a \( R \)-linear combination of monomials of degree \( \alpha \) in \( e_i^{\varphi} \) with \( ht(\beta) < ht(\alpha) \) and

\[
x(\alpha, \lambda) = \frac{q^{(\alpha, \lambda)} - q^{-(\alpha, \lambda)}}{q_\alpha - q_\alpha^{-1}}.
\]

Similar arguments hold for \( 1 \otimes f_i^{\varphi} \).

\[\square\]

3. Roots of one

3.1. Consider a primitive \( l \)-th root of unity \( \varepsilon \) with \( l \) a positive odd integer prime to 3 if \( g \) is of type \( G_2 \) and define \( \Gamma_\varepsilon^\varphi(g) = \Gamma^\varphi(g) \otimes_R Q(\varepsilon), \ F_\varepsilon^\varphi[G] = R_q^\varphi[G] \otimes_R Q(\varepsilon), \ \psi : R_q^\varphi[G] \rightarrow F_\varepsilon^\varphi[G], \ \psi(c_{f,v}) = \tau_{f,v}, \) the canonical projection. By abuse of notations, the image in \( \Gamma_\varepsilon^\varphi(g) \) of an element of \( \Gamma^\varphi(g) \) will be indicated with the same symbol.

Remark that for \( q = l = 1 \) the quotient of \( \Gamma^\varphi_1(g) \) by the ideal generated by the \( (K_i - 1)'s \) is isomorphic, as Hopf algebra, to the usual enveloping algebra \( U(g) \) of \( g \) over the field \( Q \); while the Hopf algebra \( F_1^\varphi[G] \) is isomorphic to the coordinate ring \( Q[G] \) of \( G \).

3.2. It is important to stress some results of Lusztig [L1] and De Concini-Lyubashenko [D-L] in the case \( \varphi = 0 \) which still hold in our case principally by virtue of formulas (1.9). More precisely :

(i) There exists an epimorphism of Hopf algebras (use (1.9)) \( \phi : \Gamma_\varepsilon^\varphi(g) \rightarrow U(g)\Q(\varepsilon) \) relative to \( R \rightarrow Q(\varepsilon) \) such that \( (i = 1, \ldots, n; \ p > 0) : \)

\[
\phi E_i^{(p)} = e_i^{(p)}, \ \phi F_i^{(p)} = f_i^{(p)}, \ \phi \left( \frac{K_i}{p} ; 0 \right) = \left( \frac{h_i}{p} \right) (\text{if } l|p, \ 0 \text{ otherwise}); \ \phi q = \varepsilon.
\]
Here \( e_i, f_i, h_i \) are Chevalley generators for \( \mathfrak{g} \). Generators for the kernel \( J \) of \( \phi \) are the elements:

\[
E_i^{(p)}, F_i^{(p)}, \left( \frac{K_i}{p}, K_i - 1, p_i(q) \right), \text{ where } p_i(q) \text{ is divisible by } p; l \neq p.
\]

Moreover if \( \Gamma_l \) is the free \( R \)-module with basis

\[
\prod_{\beta} F_{\beta}^{(m_\beta)} \xi_l \prod_{\alpha} E_{\alpha}^{(n_\alpha)}, \quad m_\beta, t_i, n_\alpha \equiv 0 \pmod{l},
\]

then \( U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)} \simeq \Gamma_l/p_l(q)\Gamma_l \).

(ii) Denote by \( I \) the ideal of \( \Gamma_{\mathbb{Q}(\varepsilon)}(\mathfrak{g}) \) generated by \( E_i, F_i, K_i - 1 \) \((i = 1, \ldots, n)\). The elements \( \prod_{\beta} F_{\beta}^{(m_\beta)}M \prod_{\alpha} E_{\alpha}^{(m_\alpha)} \), where \( M \) is in the ideal \((K_i - 1|i = 1, \ldots, n) \subset \Gamma_{\mathbb{Q}(\varepsilon)}(t) \) or one of the exponents \( n_\beta, m_\alpha \) is not divisible by \( l \), constitute an \( R \)-basis of \( I \). The epimorphism \( \phi \) induces the Hopf algebras isomorphism \( U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)} \simeq \Gamma_{\mathbb{Q}(\varepsilon)}(\mathfrak{g})/I \) and an \( R \)-basis for \( U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)} \) is given by the elements

\[
\prod_{\beta} F_{\beta}^{(m_\beta)}M \prod_{\alpha} E_{\alpha}^{(m_\alpha)}, \quad n_\beta, m_\alpha \equiv 0 \pmod{l}, \quad M \text{ polynomial in } \left( \frac{K_i}{l} \right).
\]

### 3.3. An important consequence of 3.2.

is the existence of a central Hopf subalgebra \( F_0 \) of \( F^\varepsilon_\varepsilon[G] \) which is naturally isomorphic to \( \mathbb{Q}(\varepsilon)[G] \). An element of \( F^\varepsilon_\varepsilon[G] \) belongs to \( F_0 \) if and only if it vanishes on \( I \) and we deduce from \([L1]\) that

\[
(3.1) \quad F_0 = \langle \bar{r}_{f,v} | f \in L((\Lambda)_{\mu}^\varepsilon), v \in L((\Lambda)_\mu), \mu, \mu \in P_+ \rangle,
\]

where \( <> \) denotes the \( \mathbb{Q}(\varepsilon) \)-span.

### 3.4. Lemma

Let \( \bar{r}_{f,v} \) be an element of \( F_0 \) and \( \bar{r}_{g,w} \) an element of \( F^\varepsilon_\varepsilon[G] \). Then

\[
m_\phi(\bar{r}_{f,v} \otimes \bar{r}_{g,w}) = m_0(\bar{r}_{f,v} \otimes \bar{r}_{g,w}).
\]

**Proof.** It is enough to consider identity (3.1) and to apply formula (2.1). \( \Box \)

### 3.5. Proposition

\( F^\varepsilon_\varepsilon[G] \) is a projective module over \( F_0 \) of rank \( \dim G \).

**Proof.** By 3.4. \( F^\varepsilon_\varepsilon[G] \) and \( F^0_\varepsilon[G] \) are the same \( F_0 \)-modules and so the result follows from \([D-L]\). \( \Box \)

### 3.6. Define \( A^\varepsilon_\varepsilon = A^\varepsilon_\varepsilon^0 \otimes_R \mathbb{Q}(\varepsilon) \). Let \( \mu^\varepsilon : F^\varepsilon_\varepsilon[G] \rightarrow A^\varepsilon_\varepsilon \) be the injection induced by \( \mu^\varepsilon_\varepsilon \); we get the isomorphism (see 2.7.) \( F^\varepsilon_\varepsilon[G][\psi^{-1}_{\varepsilon}] \simeq A^\varepsilon_\varepsilon \). Denote by \( A^\varepsilon_0 \) the subalgebra of \( A^\varepsilon_\varepsilon \) generated by

\[
1 \otimes (e^\varepsilon_\alpha)^I, (f^\varepsilon_\alpha)^I \otimes 1, K_{-(1+\varepsilon)(\mu,\lambda)} \otimes K_{(1-\varepsilon)(\mu,\lambda)} (\alpha \in \Phi_+, \lambda \in \Lambda),
\]

12
then \( \mu^\varphi_0 (F_0) [\psi^{-1}_\varphi] = A^\varphi_0 \) (it is a consequence of 3.2, 3.3.)

3.7. A basis for \( A^\varphi_\psi \) is the following

\[
(F_{\beta N} K_{\tau^\varphi N}) \times \cdots (F_{\beta N} K_{\tau^\varphi N}) \times (K_{-(1+\varphi)^\omega_1} \cdots K_{-(1-\varphi)^\omega_n} \otimes K_{(1+\varphi)^\omega_1} \cdots K_{(1-\varphi)^\omega_n} (E_{\beta N} K_{\tau^\varphi N}) \times \cdots (E_{\beta N} K_{\tau^\varphi N}) \times \cdots)
\]

Moreover \( A^\varphi_\psi \) is a maximal order in its quotient division algebra. We can prove this following the ideas in [D-P1], th. 6.5. (cf also [D-K-P1]).

3.8. Theorem \( F^\varphi_\psi [G] \) is a maximal order in its quotient division algebra.

Proof. In order to repeat the reasoning in th.4 of [D-L] we need elements \( x_1, \ldots, x_r \) in \( F_0 \) such that \( (x_1, \ldots, x_r) = (1) \) and \( F^\varphi_\psi [G] [x^{-1}_i] \) is finite over \( F_0 [x^{-1}_i] \). In fact, when \( \varphi \neq 0 \) we cannot use left translations (by elements of \( W \)) of \( \psi^{-1}_\varphi \). For \( g \in G \), let \( \mathcal{M}_g \) be the maximal ideal in \( \mathcal{Q}(G) \) determined by it. Then \( (F^\varphi_\psi [G])_{\mathcal{M}_g} \) is a free \( (F_0)_{\mathcal{M}_g} \)-module of finite type (by 3.5.) and there exists \( x_g \in F_0 \setminus \mathcal{M}_g \) (that is \( x_g (g) \neq 0 \)) such that \( F^\varphi_\psi [G] [x_g^{-1}] \) is a free \( F_0 [x_g^{-1}] \)-module of finite type. Now \( G = \bigcup_x D(x_g) \), where \( D(x_g) = \{ x \in G \mid x_g (x) \neq 0 \} \), and so there exist \( x_1, \ldots, x_r \in F_0 \) for which \( G = \bigcup_{i=1}^r D(x_i) \), that is the assert.

4. Poisson structure of \( G \)

4.1. To the quantization \( \Gamma^\varphi (g) \) of \( U(g)_{\mathcal{Q}(\psi)} \) is associated, in the sense of [D2], a Manin triple \((\mathfrak{d}, g, g_\varphi)\) and a Poisson Hopf algebra structure on \( F_0 = \mathcal{Q}(\psi) G \).

The Manin triple is composed of \( g_\varphi \), identified with the diagonal subalgebra of \( g = g \times g \), and of \( g_\psi = g_\varphi \oplus u \), where \( g_\varphi = \{ (x + \varphi(x), x + \varphi (x)) \mid x \in h \} \), \( u = \{ n_+ \times n_- \} \), \( n_+ \) is the nilpotent radical of a fixed Borel subalgebra \( \mathfrak{b}_\pm \) of \( g \). Here we denote, by abuse of notation, again by \( \varphi \) the endomorphism of \( h \) obtained by means of the identification \( h \leftrightarrow h^* \) with the Killing form. The bilinear form on \( g_\varphi \), for which \( g_\varphi \) and \( g_\psi \) become isotropic Lie subalgebras, is defined by

\[
\langle x, y \rangle \ast \langle x', y' \rangle = \langle x, x' \rangle - \langle y, y' \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the Killing form on \( g \).

In order to define a bracket \( \{ \cdot, \cdot \}_\varphi \) on \( F_0 \) we can proceed as in [D-L], namely lemma 8.1. still hold after substitution \( \Delta \leftrightarrow \Delta_\varphi \). We want here to give also a direct construction starting from the bracket \( \{ \cdot, \cdot \}_0 \) corresponding to \( \varphi = 0 \).

4.2. Proposition Let \( \Lambda_i \in P_+ \), \( v_i \in L(l(\Lambda_i))_{\mu_i} \), \( f_i \in L(l(\Lambda_i))_{-l(\Lambda_i)}, c_i = c_{f_i, v_i}, i = 1, 2 \) and define \( \chi(1, 2) = \frac{1}{2} ((\varphi \mu_1, \mu_2) - (\varphi \lambda_1, \lambda_2)) = -\chi(2, 1). \) Then :

\[
\{ \tau_1, \tau_2 \}_\varphi = \{ \tau_1, \tau_2 \}_0 + 2\chi(1, 2)m_\varphi (\tau_1 \otimes \tau_2).
\]

Proof. Let \([ \cdot, \cdot ]_\varphi \) be the commutator in the algebra \( R^\varphi_q [G] \). The using (2.1) we obtain :

\[
[c_1, c_2]_\varphi - [c_1, c_2]_0 = (q^2\chi(1, 2) - 1)m_0 (c_1 \otimes c_2) - (q^2\chi(2, 1) - 1)m_0 (c_2 \otimes c_1).
\]
Now we recall that, by construction (see [D-L]), if \([c_1,c_2]_{\varphi} = p_t(q)c\), we put

\[
\{\overline{c}_1, \overline{c}_2\}_{\varphi} = \left(\frac{p_t(q)}{l(q^l - 1)}\right)_{q=\epsilon} t,
\]
and that (by 3.4.) in \(F_0\), \(m_{\varphi}\) coincides with \(m_0\). Then, by projecting in \(F_\epsilon[G]\) and using the commutativity in \(F_0\), we get :

\[
\{\overline{c}_1, \overline{c}_2\}_{\varphi} = \{\overline{c}_1, \overline{c}_2\}_0 + (h_{12} - h_{21})m_{\varphi} (\overline{c}_1 \otimes \overline{c}_2),
\]
where

\[
h_{ij} = \left(\frac{d^2\chi(i,j)}{p_t(q)} - 1\right)_{q=\epsilon} = \left(\frac{d^2\chi(i,j)}{l(q^l - 1)}\right)_{q=\epsilon} = \left(\frac{q^{d^2\chi(i,j)} - 1}{l(q^l - 1)}\right)_{q=\epsilon}.
\]

Define

\[
p(x) = \frac{x^l\chi(1,2) - x^{-l}\chi(1,2)}{l(x-1)} = \frac{x^{-l}\chi(1,2)}{l}\left(\sum_{k=0}^{2l\chi(1,2)-1} x^k\right) \in \mathbb{Q}[x, x^{-1}],
\]
then \(h_{12} - h_{21} = p(1) = 2\chi(1,2)\) and we are done.

4.3. Corollary

(i) Any function \(\{\overline{c}_1, \overline{c}_2\}_{\varphi}, \overline{c}_i \in F_0\), vanishes on the torus \(T = \exp h \subseteq G\).

(ii) Right and left shift by an element of the torus are automorphisms of the Poisson algebra \(\mathbb{Q}(\epsilon)[G]\).

Proof. (i) In [D-L] the assert is proved for \(\{\cdot,\cdot\}_0\) then, by 4.2., we only need to prove that \(2\chi(1,2)m_{\varphi}(\overline{c}_1 \otimes \overline{c}_2)\) vanishes in the elements of torus. An easy calculation shows (here we use the identification \(h_i \leftrightarrow \left(K_i ; 0\right)\)) in agreement with 3.1.(i) that, for \(t \in T\), \((\overline{c}_1 \otimes \overline{c}_2)(\Delta_{\varphi} t) \neq 0\) if \(\lambda_i = \mu_i\), that is if \(\chi(1,2) = 0\).

(ii) The right shift by the element \(t \in T\) is defined as the element \(\overline{c} = \overline{c}_{11} \cdot \overline{c}_{12}(t)\) (similarly for the left shift) and then the claim follows from (i) and from formal properties of the bracket in a Poisson Hopf algebra.

4.4. Let \(T, C_{\varphi}, U_+, B_\pm\), be the closed connected subgroups of \(G\) associated to \(\mathfrak{h}, \mathfrak{c}_{\varphi}, n_\pm, b_\pm\) and let \(D\) be \(G \times G\). Put :

\[G_{\varphi} = C_{\varphi}(U_+ \times U_-), H = \{(x,x) | x \in T\}, \mathfrak{h} = \{(x,x) | x \in \mathfrak{h}\}\].

We have the Bruhat decomposition

\[D = \bigcup_{w \in W \times W} HG_{\varphi} wG_{\varphi}.
\]

The symplectic leaves, that is the maximal connected symplectic subvarieties of \(G\), are the connected components, all isomorphic, of \(X_{\varphi}^w = p^{-1}(G_{\varphi}\setminus \Delta_{\varphi} wH_{\varphi})\) for \(w\) running in \(W \times W\), where \(p : G \rightarrow D = G_{\varphi}\setminus D\) is the diagonal immersion followed by the canonical projection (see [L-W]). Moreover \(X_{\varphi}^w\) are the minimal \(T\)-biinvariant Poisson submanifolds of \(G\). Observe that \(\epsilon_{\varphi} + \mathfrak{h} = \epsilon_0 + \mathfrak{h}\) and so \(C_{\varphi}H = C_0H\), that is \(X_{\varphi}^w = X_{\varphi}^w = X_w = (B_+ w B_+ \cap \mathbb{B}_- w_2 \mathbb{B}_-)\) for all \(w = (w_1, w_2) \in W \times W\).
4.5. Proposition Let \( w = (w_1, w_2) \in W \times W \). The dimension of a symplectic leaf in \( X_{w_1, w_2} \) is equal to

\[
\dim \mathcal{T} = \dim \mathcal{T}_{w_1, w_2} + l(w_1) + l(w_2),
\]

where \( l(\cdot) \) is the length function on \( W \).

Proof. Since \( p \) is an unramified finite covering of its image, it is enough to calculate the dimension of the \( G_\varphi \)-orbits in \( G_\varphi \backslash D \). Moreover \( G_\varphi \subseteq B_+ \times B_- = B \), then we can consider the map \( \pi : G_\varphi \backslash D \to B \backslash D \), equivariant for the right action of \( G_\varphi \) and so preserving \( G_\varphi \)-orbits. In \( B \backslash D \) the \( G_\varphi \)-orbits coincide with the \( B \)-orbits which are equals to \( G \). Note that \( \pi \) is a principal \( T/T \)-bundle, where \( \Gamma = \{ t \in T | t^2 = 1 \} \). Let \( \Theta \) be a \( G_\varphi \)-orbit in \( D \) such that \( \pi(\Theta) = \Theta(w_1, w_2) \), then \( \pi : \Theta \to \Theta(w_1, w_2) \) is a principal \( T_{w_1, w_2} / T \)-bundle where \( T_{w_1, w_2} = \{ t \in T | t \Theta = \Theta \} \). From it follows \( \dim \Theta = \dim (T_{w_1, w_2} / \Gamma) + \dim \Theta(w_1, w_2) \), that is

\[
\dim \Theta = \dim T_{w_1, w_2} + l(w_1) + l(w_2).
\]

In order to calculate \( \dim T_{w_1, w_2} \) take \( n_1, n_2 \) representatives of \( w_1, w_2 \) in the normalizer of \( T \). We get \( t \Theta = \Theta \) if and only if there exist \((t_1, t_2), (s_1, s_2) \in C_\varphi \) and \((s_1, s_2)(t, t) = (n_1, n_2)(t_1, t_2)(n_1, n_2)^{-1} \). Let \( u, v \) be elements in \( h \) such that

\[
(s_1, s_2) = (\exp(-u + \varphi u), \exp(u + \varphi u)), \quad (t_1, t_2) = (\exp(-v + \varphi v), \exp(v + \varphi v)).
\]

We are so reduced to find \( x \in h \) for which

\[
\begin{cases}
- u + \varphi u + x = w_1(- v + \varphi v) \\
u + \varphi u + x = w_2(v + \varphi v)
\end{cases}
\]

that is

\[
\begin{cases}2x + 2\varphi u = (-w_1(1 - \varphi) + w_2(1 + \varphi))v \\
u = (w_1(1 - \varphi) + w_2(1 + \varphi))v
\end{cases}
\]

We find \( 2x = ((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi))v \) and so

\[
\dim T_{w_1, w_2} = rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi)).
\]

\( \square \)

5. Representations

5.1. In all this paragraph we shall substitute the basic field \( \mathbb{Q}(\varepsilon) \) with \( \mathbb{C} \). The fact that \( \mathcal{F}_\varepsilon[G] \) is a projective module of rank \( l^{\dim(G)} \) over \( F_0 \) allows us to define a bundle of algebras on \( G \) with fibers \( \mathcal{F}_\varepsilon[G](g) = \mathcal{F}_\varepsilon[G]/\mathcal{M}_g\mathcal{F}_\varepsilon[G] \) (for more details on this construction confront section 9 in [D-L]). From the results of previous chapter also in our case the algebras \( \mathcal{F}_\varepsilon[G](g) \) and \( \mathcal{F}_\varepsilon[G](h) \) are isomorphic for \( g, h \) in the same \( X_{w_1, w_2} \) that is, using the central character map \( \text{Spec}(\mathcal{F}_\varepsilon[G]) \to \text{Spec}(F_0) = G \), the representation theory of \( \mathcal{F}_\varepsilon[G] \) is constant on the sets \( X_{w_1, w_2} \).
5.2. Let \( w_1, w_2 \) be two elements in \( W \). Choose reduced expressions for them, namely \( w_1 = s_{i_1} \cdots s_{i_t} \), \( w_2 = s_{j_1} \cdots s_{j_m} \), and consider the corresponding ordered sets of positive roots \( \{ \beta_1, \ldots, \beta_t \} \) and \( \{ \gamma_1, \ldots, \gamma_m \} \) with \( \beta_i = \alpha_{i_i}, \beta_r = s_{i_{r-1}} \cdots s_{i_{r-s}} \alpha_{i_r} \) for \( r > 1 \) and similarly for the \( \gamma_i \)'s. Define \( A_{\phi}(w_1, w_2) \) as the subalgebra in \( A_\phi \) generated by the elements

\[
1 \otimes e_{\beta_i}, f_{\beta_i} \otimes 1, K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}, \quad (i = 1, \ldots, t), \quad (j = 1, \ldots, m, \lambda \in \mathbb{P}),
\]

and put \( A_{\phi}(w_1, w_2) = A_{\phi}(w_1, w_2) \cap A_\phi \). Note that these definitions do not depend on the reduced expressions (see [D-K-P2]). The algebra \( A_{\phi}(w_1, w_2) \) is a free module of rank \( l^l \) over its central subalgebra \( A_{\phi}(w_1, w_2) \) and so it is finite over its centre and has finite degree. We will call \( d_\varphi(w_1, w_2) \) the degree of \( A_{\phi}(w_1, w_2) \).

5.3. There is an algebra isomorphism \( A_{0,0}(w_1, w_2) \simeq A_{0,\varphi}(w_1, w_2) \) induced by the isomorphism between the algebras \( A_0 \) and \( A_0 \) given by

\[
1 \otimes e_{\alpha_i} \mapsto 1 \otimes (e_{\alpha_i'})^l, \quad f_{\alpha_i} \otimes 1 \mapsto (f_{\alpha_i'})^l \otimes 1, \quad K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda} \mapsto K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}.
\]

Therefore \( \text{Spec}(A_{\phi}(w_1, w_2)) \) is birationally isomorphic to \( X_{w_1, w_2} \cap \text{Spec}(A_\phi) \) (cf. prop. 10.4 in [D-L]). From this, and reasoning as in [D-L], it follows that the dimension of any representation of \( F_\varphi[G] \) lying over a point in \( X_{w_1, w_2} \) has dimension divisible by \( d_\varphi(w_1, w_2) \).

5.4. In order to calculate the degree \( d_\varphi(w_1, w_2) \) we introduce another set of generators for \( A_{\phi}(w_1, w_2) \). Call \( \Xi \) the antimorphism of algebras \( \Xi : U_q^\varphi(\mathfrak{g}) \rightarrow U_q^\varphi(\mathfrak{g}) \) which is the identity on \( E_i, F_i, q \) and send \( K_i \) into \( K_i^{-1} \); we get \( T_i^{-1} = \Xi(T_i) \Xi \) (see [L2]). For \( \alpha \in \{ \gamma_1, \ldots, \gamma_m \} \), let \( \Xi(f_\alpha)K_{\gamma_i} = f_{\alpha'} \). Observe that, for \( r = 1, \ldots, m \), \( (F_{\gamma_i}) = T_i^{-1} \cdots T_{i-1}^{-1}(F_{\gamma_i}) \). We want now to show that the sets \( \{ f_{\beta_i} \mid i = 1, \ldots, m \} \) and \( \{ f_{\beta_i'} \mid i = 1, \ldots, m \} \) generate the same subalgebras of \( R_\varphi[B_+]^{m'} \).

Let \( H_1, H_2 \) be the subalgebras respectively generated by these sets. From \( \Xi(U_q^\varphi(n_-)) = U_q^\varphi(n_-) \) and \( T_i^{\frac{1}{2}}(R_\varphi[B_+]^{m'}) \subseteq R_\varphi[B_+]^{m'} \) for every \( i \), follow that \( f_{\gamma_i} \) belongs to the algebra generated by \( \{ f_{\gamma_i} \mid i = 1, \ldots, m \} \) and

\[
f_{\alpha'} = f_\alpha' K_{\gamma_i} = \sum c_{\phi}(f_{\gamma_i})^m \cdots (f_{\gamma_i})^n K_{\gamma_i} = \sum c_{\phi}(f_{\gamma_i})^m \cdots (f_{\gamma_i})^n \in H_1,
\]

where \( c_{\phi} = q^{r_1}c_{\phi} \) for an integer \( r_1 \). In a similar way we can show that \( f_{\alpha'} \in H_2 \).

Put now, in \( A_{\phi}(w_1, w_2) \),

\[
x_i' = 1 \otimes e_{\beta_i}, \quad (i = 1, \ldots, t), \quad y_j' = f_{\beta_i} \otimes 1, \quad (j = 1, \ldots, m), \quad z_r' = K_{-(1+\varphi)\omega_r} \otimes K_{(1-\varphi)\omega_r}, \quad (r = 1, \ldots, n).
\]

As in the case \( \varphi = 0 \), \( A_{\phi}(w_1, w_2) \) is an iterated twisted polynomial algebra and the corresponding quasipolynomial algebra is generated by elements \( x_i, y_j, z_r \) with relations which are easily found (see [L-S1] and [D-K-P2]). Namely:

\[
x_i x_j = \varepsilon(\beta_i, (1+\varphi)\beta_j) x_j x_i, \quad (1 \leq j < i \leq t), \quad y_i y_j = \varepsilon(-\gamma_i, (1+\varphi)\gamma_j) y_j y_i, \quad (1 \leq j < i \leq m),
\]
\[ z_i z_j = z_j z_i \quad (1 \leq i, j \leq n), \quad x_i y_j = y_j x_i \quad (1 \leq i \leq t, i \leq j \leq m), \]
\[ z_i x_j = \varepsilon^{((1-\varphi)\omega_i, \beta_j)} x_j z_i \quad (1 \leq i \leq n, i \leq j \leq t), \quad z_i y_j = \varepsilon^{((1+\varphi)\omega_i, \gamma_j)} y_j z_i \quad (1 \leq i \leq n, i \leq j \leq m). \]

**5.5.** Let \( \mathbb{Z}_\varphi \) be the ring \( \mathbb{Z}[(2d_1 \cdots d_n \det(1-\varphi))^{-1}] \) and denote by \( \vartheta \) the isometry \((1+\varphi)(1-\varphi)^{-1}\).
For each pair \((w_1, w_2)\) in \(W \times W\), consider the map \( e_\varphi(w_1, w_2) = 1 - w_1^{-1} \vartheta^{-1} w_2 \vartheta : P \otimes \mathbb{Z}_\varphi \leftarrow Q \otimes \mathbb{Z}_\varphi\).
Define \( l(\varphi) \) to be the least positive integer for which, for every \((w_1, w_2)\), the image of \( P \otimes \mathbb{Z}_\varphi[l(\varphi)^{-1}] \) is a split summand of \( Q \otimes \mathbb{Z}_\varphi[l(\varphi)^{-1}]\) (in special cases, namely when \( \vartheta \) fix the set of roots, one can explicitly take \( l(\varphi) = a_1 \cdots a_n \), where \( \sum_{i=1}^n a_i \alpha_i \) is the longest root, as in [D-K-P2], but in general we need a case by case analysis).

An integer \( l \) is said to be a \( \varphi \)-good integer if, besides being prime to the \( 2d_i \), it is prime to \( \det(1-\varphi) \) and \( l(\varphi) \).

**5.6. Theorem** Let \( l \) be a \( \varphi \)-good integer, \( l > 1 \). Then,
\[
\begin{align*}
d_\varphi(w_1, w_2) &= l^{\frac{1}{2}l(\varphi)} + l((w_1) + l((w_2) + \sum_{i=1}^n a_i \alpha_i) - (1+\varphi)w_2(1-\varphi)).
\end{align*}
\]

**Proof.** We work over \( S = \mathbb{Z}_\varphi[l(\varphi)^{-1}] \). Let \( w_1, w_2 \) be in \( W \). Consider free \( S \)- modules \( V_{w_1}, V_{w_2} \) with basis \( u_1, \ldots, u_t \) and \( v_1, \ldots, v_m \) respectively. Define on \( V_{w_1}, V_{w_2} \) skew symmetric bilinear forms by
\[
\langle u_i | u_j \rangle = (\beta_i, (1+\varphi)\beta_j) \quad (1 \leq j < i \leq t), \quad \langle v_i | v_j \rangle = (\gamma_i, (1+\varphi)\gamma_j) \quad (1 \leq j < i \leq m),
\]
and denote by \( C^\varphi_{w_1}, C^\varphi_{w_2} \) their matrices in the bases of the \( u_i \)’s and \( v_j \)’s respectively. Finally, let \( D^\varphi_{w_1}, D^\varphi_{w_2} \) be the \( t \times n, m \times n \) matrices whose entries are \((\beta_i, (1-\varphi)\omega_j)\) and \((\gamma_i, (1+\varphi)\omega_j)\) respectively. Put
\[
\Delta^\varphi_{w_1, w_2} = \begin{pmatrix} C^\varphi_{w_1} & 0 & D^\varphi_{w_1} \\ 0 & -C^\varphi_{w_2} & D^\varphi_{w_2} \\ -tD^\varphi_{w_1} & tD^\varphi_{w_2} & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} C^0_{w_1} & 0 & D^0_{w_1} \\ 0 & -C^0_{w_2} & D^0_{w_2} \\ -tD^0_{w_1} & tD^0_{w_2} & 0 \end{pmatrix}.
\]

We want first of all prove that \( \Delta^\varphi_{w_1, w_2} \) is equivalent to \( \Delta \), that is we want to exhibit an \( n \times t \) matrix \( M_1 \) and an \( m \times n \) matrix \( M_2 \) for which :
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & M_2 \\ 0 & 0 & 1 \end{pmatrix} \Delta^\varphi_{w_1, w_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ M_1 & 0 & 1 \end{pmatrix} = \Delta,
\]
or equivalently for which
\[
C^\varphi_{w_1} + D^\varphi_{w_1} M_1 = C^0_{w_1}, \quad C^\varphi_{w_2} + M_2(tD^\varphi_{w_2}) = C^0_{w_2}, \quad D^\varphi_{w_2} M_1 = M_2(tD^\varphi_{w_1}).
\]
First of all we need some notations. If \( f : V_1 \rightarrow V_2 \) is a linear map and \( B_1 \) (resp. \( B_2 \)) is a basis of \( V_1 \) (resp. \( V_2 \)) we will indicate by \( M(f, B_1, B_2) \) the matrix of \( f \) in these given bases. Let now
\( \alpha_i = \frac{\alpha_i}{n} \) and denote by \( \nu : SP \to (SP)^* \) the map given by \( \nu(\alpha_i) = \omega_i^* \) or, equivalently, the map which send \( \alpha_i \) to the linear form \( (\alpha_i, \cdot) \). We define the following maps:

\[
\begin{align*}
&c_i^1 : V_{w_1} \to V_{w_1}^*, \ u_j \mapsto \sum_{i>j} (\beta_j, (1+\varphi)\beta_i)u_i^* - \sum_{i<j} (\beta_i, (1+\varphi)\beta_j)u_i^*; \\
&c_i^2 : V_{w_2} \to V_{w_2}^*, \ v_j \mapsto \sum_{i>j} (\gamma_j, (1+\varphi)\gamma_i)v_i^* - \sum_{i<j} (\gamma_i, (1+\varphi)\gamma_j)v_i^*; \\
&d_i^1 : Z'P \to V_{w_1}^*, \ \omega_i \mapsto \sum_j ((1+\varphi)\beta_j, \omega_i)u_j^*; \ d_i^2 : Z'P \to V_{w_2}^*, \ \omega_i \mapsto \sum_j ((1-\varphi)\gamma_j, \omega_i)v_j^*; \\
&h_1 : V_{w_1} \to SP, \ u_j \mapsto \beta_j; \ h_2 : V_{w_2} \to SP, \ v_j \mapsto \gamma_j.
\end{align*}
\]

Then we have

\[
\begin{align*}
d_i^0 &= \nu h_1, \ d_i^0 = \nu h_2, \ d_i^1 = d_i^0(1-\varphi), \ d_i^2 = d_i^0(1+\varphi),
\end{align*}
\]

and we can easy verify that

\[
c_i^1 - c_i^0 = -d_i^0\varphi h_1, \ c_i^2 - c_i^0 = -d_i^0\varphi h_2.
\]

Moreover, if \( Z = M(\varphi, \{\omega_i\}, \{\omega_i\}) \), we get

\[
\begin{align*}
&C_{w_1}^0 = M(c_i^1, \{u_i\}, \{u_i^*\}), \ C_{w_2}^0 = M(c_i^2, \{v_i\}, \{v_i^*\}); \\
&D_{w_1}^0 = M(x_i^*, \{\omega_i\}, \{u_i^*\}) = D_{w_1}^0(1 - Z), \ D_{w_2}^0 = M(x_i^*, \{\omega_i\}, \{v_i^*\}) = D_{w_2}^0(1 + Z).
\end{align*}
\]

Let \( R = M(id, \{\alpha_i\}, \{\omega_i\}) \) and define

\[
M_1 = (1-Z)^{-1}ZR(1+Z)^{-1}, \ M_2 = D_{w_2}^0ZR(1+Z)^{-1},
\]

now it is a straightforward computation to verify that these two matrices satisfy the required properties.

Let now \( d \) be the the map corresponding to \( \Delta \) with respect to the bases \( \{u_i, v_j, \omega_r\} \) and \( \{u_i^*, v_j^*, \alpha_r\} \). We want to show that the image of \( d \) is a split direct summand, and to calculate the rank of \( \Delta \). The result will then follow from the proposition on page 34 in [D-P1], due to restrictions imposed to \( l \).

From the results in [D-K-P2] we now that the map corresponding to \( (C_{w_1}^0, D_{w_1}^\varphi) : V_{w_1} \oplus SP \to V_{w_1} \) is surjective with kernel

\[
\{(u_{(1-\varphi)\lambda}, (1+(1-\varphi)^{-1}w_{1(1-\varphi)})\lambda)| \lambda \in SP\}.
\]

Similarly the map corresponding to \( (-C_{w_2}^0, D_{w_2}^\varphi) : V_{w_2} \oplus SP \to V_{w_2} \) is surjective with kernel

\[
\{(-v_{(1+\varphi)\lambda}, (1+(1+\varphi)^{-1}w_{2(1+\varphi)})\lambda)| \lambda \in SP\}.
\]

Here, for \( \lambda = \omega_r \) \((r = 1, \ldots, n)\), we have defined

\[
I_r^1 = \{k \in \{1, \ldots, t\}| i_k = r\}, \ I_r^2 = \{k \in \{1, \ldots, m\}| j_k = r\}; \ u_\lambda = \sum_{k \in I_r^1} u_k, \ v_\lambda = \sum_{k \in I_r^2} v_k
\]

\[18\]
and the extension of the definition of \( u_\lambda, v_\lambda \) to all \( \lambda \in P \) is the only compatible with linearity. We start the study of \( d \). We consider the first row \( (C_{w_1}^0, 0, D_{w_1}^c) : V_{w_1} \oplus V_{w_2} \rightarrow V_{w_1} \); it is surjective with kernel
\[
H = \{(u_{(1-\varphi)\lambda}, v, (1+(1-\varphi)^{-1}w_1(1-\varphi))\lambda) | \lambda \in SP \ v \in V_{w_2}\}.
\]
Our aim is now to study the image of the restriction of \( d \) on \( H \). We proceed as follows. We define the composite map \( f : V_{w_2} \oplus SP \rightarrow H \rightarrow V_{w_2}^* \oplus SQ \),
\[
(v, \lambda) \mapsto (u_{(1-\varphi)\lambda}, v, (1+(1-\varphi)^{-1}w_1(1-\varphi))\lambda) \mapsto d(u_{(1-\varphi)\lambda}, v, (1+(1-\varphi)^{-1}w_1(1-\varphi))\lambda).
\]
With respect to the bases \( \{v_i, \omega_j\} \), \( \{v_i^*, \alpha_j\} \), \( f \) is represented by the matrix
\[
\begin{pmatrix}
-C_{w_2}^0 & D_{w_2}^c (1+(1+\Phi)^{-1}W_1(1-\Phi)) \\
-t^tD_{w_2}^c & -(1+\Phi)(1-W_1)(1-\Phi)
\end{pmatrix},
\]
where \( W_1, \Phi \) are the matrices representing \( w_1, \varphi \) respectively with respect to the basis \( \{\omega_j\} \). To study this matrix is equivalent to study its opposite transpose (since we are essentially interested in their elementary divisors). Let \( M = (1+(1-\Phi)^{-1}W_1(1-\Phi)), N = (1+\Phi)(1-W_1)(1-\Phi) \). We consider therefore the matrix
\[
\begin{pmatrix}
-C_{w_2}^0 & D_{w_2}^c \\
-t^tM^tD_{w_2}^c & t^tN
\end{pmatrix}.
\]
Let \( g : V_{w_2} \oplus SP \rightarrow V_{w_2}^* \oplus SQ \) be the map represented by this matrix with respect to the bases \( \{v_i, \omega_j\} \), \( \{v_i^*, \alpha_j\} \). Then \( (-C_{w_2}^0, D_{w_2}^c) : V_{w_2} \oplus SP \rightarrow V_{w_2}^* \) is surjective with kernel
\[
L = \{(u_{(1+\varphi)\lambda}, (1+(1+\varphi)^{-1}w_2(1+\varphi))\lambda) | \lambda \in SP\}
\]
and we are left to study the following composite \( e : SP \rightarrow L \rightarrow SQ \),
\[
\lambda \mapsto (-u_{(1+\varphi)\lambda}, (1+(1+\varphi)^{-1}w_2(1+\varphi))\lambda) \mapsto g((-u_{(1+\varphi)\lambda}, (1+(1+\varphi)^{-1}w_2(1+\varphi))\lambda).
\]
With respect to the bases \( \{\omega_i\} \) and \( \{\alpha_i\} \), \( e \) is represented by the matrix
\[
t^tM(1-\Phi)(1-W_2)(1+\Phi) + t^tN(1+\Phi)^{-1}W_2(1+\Phi),
\]
that is
\[
(1^t(1-\Phi)^tW_1^t(1-\Phi)^{-1})(1-\Phi)(1-W_2)(1+\Phi) + t^t(1-\Phi)(1-W_1)^t(1+\Phi)(1+(1+\Phi)^{-1}W_2(1+\Phi)).
\]
Since \( w \) is an isometry and \( \varphi \) is skew (one should use at each step appropriate bases), we get that \( e(\lambda) \) is the element
\[
((1+(1+\varphi)w_1^{-1}(1+\varphi)^{-1})(1-\varphi)(1-w_2)(1+\varphi) + (1+\varphi)(1-w_1^{-1})(1-\varphi)(1+(1+\varphi)^{-1}w_2(1+\varphi))\lambda,
\]
that is
\[
e(\lambda) = (1+(1+\varphi)w_1^{-1}(1+\varphi)^{-1})(1-\varphi)(1-w_2)(1+\varphi) + (1+\varphi)(1-w_1^{-1})(1-\varphi)(1+(1+\varphi)^{-1}w_2(1+\varphi)).
\]
It follows that
\[ e(\lambda) = 2(1 + \varphi)(1 - \varphi) - 2(1 + \varphi)w_1^{-1}(1 + \varphi)^{-1}(1 - \varphi)w_2(1 + \varphi) = 2(1 + \varphi)(1 - w_1^{-1} \vartheta^{-1} w_2 \vartheta)(1 - \varphi). \]
Since both SP and SQ are invariant under \(2(1 - \varphi)\) and \(2(1 + \varphi)\), we are left to study the map
\[ 1 - w_1^{-1} \vartheta^{-1} w_2 \vartheta : \text{SP} \rightarrow \text{SQ}. \]
The restriction imposed to \(l\) imply that the image of \(1 - w_1^{-1} \vartheta^{-1} w_2 \vartheta\)
is a split direct summand.

It is also clear at this point that the rank of \(\Delta\) is precisely \(l(w_1) + l(w_2)\). But \(rk(1 - w_1^{-1} \vartheta^{-1} w_2 \vartheta) = rk(\vartheta w_1 - w_2 \vartheta) = rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi))\) and we are done.

**5.7. Corollary** Let \(l\) be a \(\varphi\)-good integer and let \(p\) be a point of the symplectic leaf \(\Theta\) of \(G\). Then the dimension of any representation of \(F^\varphi[\mathcal{G}]\) lying over \(p\) is divisible by \(l^{\dim \Theta}\).

**5.8.** As a consequence of (2.2) we have the following isomorphisms of Hopf algebras
\[ R^\varphi_q[B_+] \cong R^0_q[B_+] \cong \Gamma^0(\mathfrak{g}-)_\text{op}. \]
Now the algebra \(\Gamma^c(\mathfrak{g}-)\) is equal to the algebra \(\Gamma^0(\mathfrak{g}-)\) and so we have the isomorphism of algebras
\[ R^\varphi_q[B_+] \cong R^0_q[B_+] \]
and similarly for the case \(B_-\). Then the representations of \(F^\varphi_c[\mathcal{G}]\) over the sets \(X_{(w,1)}\) and \(X_{(1,w)}\) are like in the case \(\varphi = 0\) (they are studied in [D-P2]). In particular there is an isomorphism between the one dimensional representations of \(F^\varphi_c[\mathcal{G}]\) and the points of the Cartan torus \(T\) (given explicitly [D-L]).

**Appendix**

In 4. we determined the dimension \(d_\varphi(w_1, w_2)\) of a symplectic leaf \(\Theta\) contained in \(X_{(w_1, w_2)}\);
\[ d_\varphi(w_1, w_2) = l(w_1) + l(w_2) + rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi)). \]
This means, of course, that \(d_\varphi(w_1, w_2)\) is an even integer. Here we give a direct proof of this fact in the more general context of finite Coxeter groups. Using the definitions from [H], let \(W = < s_1, \ldots, s_n >\) be a finite Coxeter group of rank \(n\), \(\sigma : W \leftarrow GL(V)\), the geometric representation of \(W\), \(B\) the \(W\)-invariant scalar product on \(V\), \(\Phi\) the root system of \(W\), \(l(\cdot)\) the usual length function on \(W\). We recall a fact proved in [C] for Weyl groups and which holds with the same proof for finite Coxeter groups. Each element \(w\) of \(W\) can be expressed in the form \(w = s_{r_1} \cdots s_{r_k}\), \(r_i \in \Phi\), where \(s_i\) is the reflection relative to \(v\), if \(v\) is any non zero element of \(V\). Denote by \(\bar{l}(w)\) the smallest value of \(k\) in any such expression for \(w\). We get

**Lemma** \(\bar{l}(w) = rk(1 - w).\)

**Proof.** It is Lemma 2 in [C].
We can now prove

**Proposition 1** Let $w_1, w_2$ be in $W$. Then $l(w_1) + l(w_2) + rk(w_1 - w_2)$ is even.

**Proof.** We have

$$rk(w_1 - w_2) = rk(1 - w_2w_1^{-1}) = l(w_2w_1^{-1}) \equiv l(w_2w_1^{-1}) \mod 2.$$

But $l(w_2w_1^{-1}) \equiv l(w_2) + l(w_1^{-1}) \mod 2$ and finally $l(w_1^{-1}) = l(w_1)$. Hence $l(w_1) + l(w_2) + rk(w_1 - w_2) \equiv l(w_1) + l(w_2) + l(w_1) \equiv 0 \mod 2$. $\square$

Suppose now $\varphi$ is an endomorphism of $V$ which is skew relative to $B$, and let $\vartheta$ be the isometry $(1 + \varphi)^{-1}(1 - \varphi)$. To prove the general result, we recall that, if $\eta$ is an isometry of $V$ and $r$ is the rank of $1 - \eta$, then $\eta$ can be written as a product of $r + 2$ reflections (cf. [S], where a more precise statement is given). In particular if $\eta = s_{v_1} \cdots s_{v_k}$, then $rk(1 - \eta) \equiv k \mod 2$.

**Proposition 2** Let $w_1, w_2$ be in $W$. Then $l(w_1) + l(w_2) + rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi))$ is even.

**Proof.** We have $rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi)) = rk(1 - \vartheta w_2 \vartheta^{-1}w_1^{-1})$. If we write $\vartheta, w_1, w_2$ as products of $a, a_1, a_2$ reflections respectively, we get from the previous observation that $rk(1 - \vartheta w_2 \vartheta^{-1}w_1^{-1}) \equiv a + a_2 + a + a_1 \mod 2$. Hence

$$rk(1 - \vartheta w_2 \vartheta^{-1}w_1^{-1}) \equiv rk(1 - w_2w_1^{-1}) \equiv rk(w_1 - w_2) \mod 2$$

and the result comes from prop.1. $\square$

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