Massive relativistic particle model with spin from free two-twistor dynamics 
and its quantization

Dedicated to the memory of our collaborator Andreas Bette

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We consider a relativistic particle model in an enlarged relativistic phase space, \( M^{18} = (X_{\mu}, p_{\mu}, \eta_{\alpha}, \bar{\eta}_{\beta}, \sigma_{\alpha}, \bar{\sigma}_{\beta}, e, \phi) \), which is derived from the free two-twistor dynamics. The spin sector variables \((\eta_{\alpha}, \bar{\eta}_{\beta}, \sigma_{\alpha}, \bar{\sigma}_{\beta})\) satisfy two second class constraints and account for the relativistic spin structure, and the pair \((e, \phi)\) describes the electric charge sector. After introducing the Liouville one-form on \( M^{18} \), derived by a non-linear transformation of the canonical Liouville one-form on the two-twistor space, we analyze the dynamics described by the first and second class constraints. We use a composite orthogonal basis in four-momentum space to obtain the scalars defining the invariant spin projections. The first-quantized theory provides a consistent set of wave equations, determining the mass, spin, invariant spin projection and electric charge of the relativistic particle. The wavefunction provides a generating functional for free, massive higher spin fields.

I. INTRODUCTION

The choice of the basic geometric variables that describe the most fundamental dynamics is an important and open question. In particular, by analogy with the ‘hidden’ quark structure of hadronic particles, one may assume that the spacetime coordinates are composite variables that can be expressed in terms of more basic, fundamental geometric spinorial variables. This is the twistor theory approach, proposed by Penrose in \( D = 4 \) (see e.g. [1,2]) which has been generalized to higher dimensions (see e.g. [3,4]) as well as supersymmetrized (see e.g. [5,6]).

Most of the studies of twistor dynamics have been restricted to the one-twistor geometry, based on four complex spinorial coordinates \( Z_A = (Z_1, Z_2, Z_3, Z_4) \) supporting a fundamental representation of the spinorial covering \( SU(2, 2) \supset SO(4, 2) \) of the \( D = 4 \) conformal group. A twistor is described by a pair of \( D = 4 \) Lorentz spinors, \( Z^A = (\omega^\alpha, \bar{\omega}^\beta) \), \( A = 1, 2, 3, 4; \alpha, \beta = 1, 2 \).

In the one-twistor framework the four-momentum is introduced as follows

\[
P_{\alpha\beta} = \pi_\alpha \bar{\pi}_\beta \quad (P_\mu = \frac{1}{\sqrt{2}} (\sigma_\mu)^{\alpha\beta} P_{\alpha\beta}) \]  

If we recall the Penrose basic relations

\[
\omega^\alpha = i z^{\alpha\beta} \pi_\beta \quad , \quad \bar{\omega}^\alpha = -i \bar{\pi}_\beta \bar{\pi}^{\beta\alpha} \]

and assume that the \( SU(2, 2) \) twistor norm is zero,

\[
\langle Z, \bar{Z} \rangle = \omega^\alpha \pi_\alpha + \bar{\omega}^\beta \bar{\pi}_\beta = 0 \]

the Minkowski space coordinates \( z^{\alpha\beta} = x^{\alpha\beta} + iy^{\alpha\beta} = \frac{1}{\sqrt{2}} (\sigma_\mu)^{\alpha\beta} z^\mu \) in formula (1.5) become real (i.e. \( y^{\alpha\beta} = 0 \)). Using \( (1.2) \) and \( y^{\alpha\beta} = 0 \) one can show the equivalence of the following three Liouville one-forms:

\[
\Theta^{(1)} = P_{\alpha\beta} dx^{\alpha\beta} = \pi_\alpha \bar{\pi}_\beta dx^{\alpha\beta} = \frac{i}{2} (\omega^\alpha d\pi_\alpha + \bar{\omega}^\beta d\bar{\pi}_\beta) + c.c. \]

If we observe that the vanishing mass condition \( (P^2 = 0) \) follows from \( (1.2) \) and take into account eqs. \( (1.4) \) and \( (1.5) \), the following three models describing a free massless scalar relativistic particle are seen to be equivalent:

\[
L_1 = \int d\tau \left( P_{\mu} \dot{x}^{\mu} - \lambda P^2 \right) \quad \text{(relat. phase space description)} ; 
\]
\[
L_2 = \int d\tau \pi_\alpha \bar{\pi}_\beta \dot{x}^{\alpha\beta} \quad \text{(mixed space-twistor description)} ; 
\]
\[
L_3 = \int d\tau \left( Z_A \dot{Z}^A - \lambda \langle Z, \bar{Z} \rangle \right) \quad \text{(pure twistor description)} .
\]
To describe particles with mass and spin in a twistor formalism, it is necessary to use a pair of twistors \[Z^A_{\alpha} = (\omega^{\alpha}, \bar{\pi}^\beta), \quad Z^A_{\beta} = (\lambda^\alpha, \pi^\beta)\] (1.7)

In comparison with the one-twistor description of spacetime, the use of two twistors produces important changes:

i) the spacetime coordinates \(x_\mu\) can be considered as two-twistor composites;

ii) the appearance of non-vanishing spin and internal (electric) charge leads to the complexification of the spacetime coordinates.

The composite complex Minkowski coordinates \(z_\mu = x_\mu + iy_\mu\) \((z_\mu = \frac{1}{\sqrt{2}}(\sigma_\mu)_{\alpha\beta}z^{\alpha\beta})\) are described by the well known Penrose formula:

\[z^{\alpha\beta} = \frac{i}{f}(\omega^{\alpha}\pi^\beta - \lambda^\alpha\pi^\beta)\] (1.8)

where \(f = \pi^\alpha\eta_\beta\). (1.9)

From the free two-twistor Liouville one-form \(\Theta^{(2)} = \frac{i}{2}(Z^A_{\alpha}d\bar{Z}^A_{\beta} - \bar{Z}^A_{\beta}dZ^A_{\alpha})\) one obtains the following twistorial Poisson brackets (TPB)

\[\{\pi_\alpha, \omega^\beta\} = \frac{i}{2}\delta^{\beta}_{\alpha}, \quad \{\eta_\alpha, \lambda^\beta\} = \frac{i}{2}\delta^\beta_{\alpha}, \quad \{\pi_\alpha, \lambda^\beta\} = -\frac{i}{2}\delta^{\beta}_{\alpha}, \quad \{\eta_\alpha, \omega^\beta\} = \frac{i}{2}\delta^{\beta}_{\alpha}\] (1.11a)

Using (1.8) it follows that the spacetime coordinates \(x_\mu = \text{Re} z_\mu\) have non-vanishing TPB’s (1.11b)

\[\{x_\mu, x_\nu\} = -\frac{1}{(2|f|^2)2\epsilon_{\mu\nu\rho\tau}W^\rho P^\tau\}\] (1.12)

where \(P_\mu\) is the two-twistor composite momentum,

\[P_{\alpha\beta} = \pi_\alpha \bar{\pi}^\beta + \eta_\alpha \eta^\beta\] (1.13)

\[P^2 = P_{\alpha\beta}P^{\alpha\beta} = 2(\pi_\alpha \eta^\beta)(\bar{\pi}^\alpha \bar{\eta}^\beta) = 2|f|^2\] (1.14)

and \(W_\mu\) is the composite Pauli-Lubański spin four-vector, \(W^\mu P_\mu = 0\). If we denote \(\pi_\alpha \equiv \pi_{\alpha 1}, \eta_\alpha \equiv \pi_{\alpha 2}, \bar{\pi}^\alpha \equiv \bar{\pi}^{\alpha 1}, \bar{\eta}^\alpha \equiv \bar{\pi}^{\alpha 2}\) and introduce

\[P^{(r)}_{\alpha\beta} = (\pi_{\alpha 1}\bar{\pi}^{\beta 1})(\tau^r)^{i\beta}, \quad r = 1, 2, 3\] (1.15)

then \(W_\mu\) in (1.12) is given by the formula

\[W_{\alpha\beta} = t^r P^{(r)}_{\alpha\beta}\] (1.16)

\[t^r = (\tau^r)^{j\alpha}t^j_i = Z^A_{\alpha\beta}Z^{A\beta}_{\alpha}^{-1}\] (1.17)

where \(\tau^r \equiv \sigma^r\) are the three 2 \(\times\) 2 Pauli matrices satisfying the standard algebra \((\tau^r)^{j\alpha}_{k\beta} = \delta^{jr}\delta^\alpha_k + i\epsilon^{rst}(\tau^s)^{i\beta}_{k\alpha}\).

The three composite four-vectors \(P^{(A)}_\mu\) (eq. (1.15)) together with \(P^{(0)}_\mu\) (eq. (1.16)) describe an orthonormal basis in four-momentum space \((\eta_{AB} = (+, - , - , - ))\)

\[P^{(A)}_\mu P^{(B)}_{\nu} = P^2\eta^{AB}, \quad A, B = 0, 1, 2, 3\] (1.18)

In order to complete the set of observables we add to the spin four-vector square \(W^\mu W_\mu\) the projection of the relativistic spin vector \(W_\mu\) on any orthogonal direction to \(P_\mu\). Taking into consideration the orthogonal basis (1.18), the Lorentz invariant spin projection can be represented by the scalar product of \(S_\mu = \frac{W_\mu}{\sqrt{P^2}}\) and the four-vector \(P^{(3)}\), normalized to unity. Denoting the projection by \(S^{(3)}\) and using (1.11) one obtains

\[S^{(3)} = -\frac{1}{P^2}P^{(3)}S_\mu = -\frac{1}{P^2}P^{(3)}W_\mu = t^3\] (1.19)

In general, we find

\[S^{(r)} = -\frac{1}{P^2}P^{(r)}W_\mu = t^r, \quad r = 1, 2, 3\] (1.20)

where \(t^r\) denote the three scalars that provide the three invariant spin projections. Thus, contrarily to the standard choice of the projection on a fixed space direction (e.g., the z-axis), we project the spin four-vector in an invariant way on the three directions in momentum space defined by the composite four-vectors (1.15) orthogonal to the four-vector (1.13). We observe that using eq. (1.17) one can calculate the TPB of the \(t^r\),

\[\{t^r, t^s\} = \epsilon^{rst}t^u\] (1.21)

Thus, the \(t^r\) variables determine a \(su(2)\) algebra of invariant spin projections.

The composite four-momentum formulae (1.13) can be supplemented with the composite Lorentz generators \(M_{\alpha\beta}, M_{\hat{\alpha}\hat{\beta}}\)

\[M_{\alpha\beta} = \pi_\alpha \omega^{\beta} + \eta_\alpha \lambda^{\beta}\] (1.22a)

\[M_{\hat{\alpha}\hat{\beta}} = \bar{\pi}^\alpha \omega^{\hat{\beta}} - \bar{\eta}^\alpha \lambda^{\hat{\beta}}\] (1.22b)

As mentioned, the four-vector (1.16) can be identified with the composite Pauli-Lubański \(W_\mu\) \((W_\mu = \frac{1}{2}(\epsilon_{\mu\nu\rho\sigma}P^{\nu}M^{\rho})\) that defines the square of the relativistic spin four-vector through \(S^{(r)2} = S_\mu S^\mu = S^2 = \frac{1}{2}W^2\) (so that \(W^2 = -m^2s^2\)). From (1.16) and (1.17) we obtain

\[S^2 = -S^{(r)2}S^{(r)} = -(t_1^2 + t_2^2 + t_3^3)\] (1.23)
Therefore, the relativistic spin square is defined in any Lorentz frame as the \( su(2) \) Casimir \( t^2 \), here given by the sum of the squares of the three invariant spin projections.

In order to complete the description of spin in the two-twistor space one can show that \( (y^{α\dot{β}} = \text{Im} \, z^{α\dot{β}}) \) by performing the standard Fourier transform

\[
y^{α\dot{β}} = \frac{1}{P^2} \left( W^{α\dot{β}} - t^0 P^{α\dot{β}} \right) , \tag{1.24}
\]

where \( t^0 = Z_μ^A \tau_A^i \). \( \tag{1.25} \)

The fourth generator \( t^0 \) enlarges \( su(2) \) to \( u(2) \) due to the vanishing of the TPB relation

\[
\{t^i, t^j\} = 0 ; \tag{1.26}
\]

\( t^0 \) can be associated with an internal \( U(1) \) charge. Thus, we see from eq. \( \ref{eq:1.22} \) that the non-trivial Pauli-Lubański four-vector and the internal charge \( \ref{eq:1.22} \) generate the non-vanishing imaginary part of the composite spacetime.

Following our earlier papers \( \ref{12,13} \) we modify now the standard Penrose formula \( \ref{1.5} \) by introducing the real composite spacetime coordinates \( X_μ \) firstly proposed by Bette and Zakrzewski \( \ref{14} \).

\[
X_μ = x_μ + \Delta x_μ ; \tag{1.27}
\]

they differ from \( x_μ \) by the shift \( \Delta x_μ \) (see eq. \( \ref{2.3a} \) below). In contrast with the non-commuting variables \( x_μ \), eq. \( \ref{1.12} \), the variables \( X_μ \) satisfy the conventional TPB as composites of two-twistors, namely

\[
\{X_μ, X_ν\} = \{P_μ, P_ν\} = 0 , \tag{1.28a}
\]

\[
\{P_μ, X_ν\} = δ_μ^ν \tag{1.28b}
\]

It follows from \( \ref{1.28a} \) that after quantizing the PB we will obtain a quantized theory with commuting spacetime coordinates.

In the rest of the paper we proceed as follows. In Sec. \( \text{II} \) we discuss the explicit form of the modification \( \ref{1.27} \). Subsequently we define the two-twistor counterpart of \( \ref{1.6a} \) by introducing the Liouville one-form on the \( 18 \)-dimensional enlarged relativistic phase space \( M^{18} \) \( \ref{12} \).

\[
M^{18} = (X_μ, P_μ, η_α, η_α, σ_α, σ_α, τ_α, τ_α, e, φ) . \tag{1.29}
\]

In order to identify the manifold \( M^{18} \) with the sixteen-dimensional two-twistor space (see \( \ref{14} \)) we have to introduce two constraints \( \ref{31} \). By expressing the Liouville one-form \( \ref{110} \) in terms of the variables of \( M^{18} \) we find two second class constraints \( R_2, R_3 \) which encode the composite two-twistor structure of the variables \( \ref{129} \). These constraints are taken into consideration in the quantization procedure by introducing suitable Dirac brackets.

In Sec. \( \text{II} \) we show to what extent the Dirac brackets in the space \( \ref{129} \) can be correctly quantized i.e., without the violation of associativity. It turns out that the quantized Dirac brackets can be only consistently defined in the subspace \( M^{14} = (P_μ, η_α, η_α, σ_α, σ_α, e, φ) \) of \( M^{18} \). We introduce further a differential realization of the internal momenta \( (σ_α, τ_α, e) \) in the generalized momentum space \( (P_μ, η_α, η_α, e, φ) \). In such a generalized Schrödinger representation the dynamics will be described by four \( \text{‘physical’} \) first class constraints determining the mass, spin, invariant spin projection and electric charge. They take the form of a mass shell condition and three differential wave equations. We shall look for the explicit solution of the wave equations describing spin in the form of a power series

\[
Ψ(P_μ, η_α, η_α, e, φ) = \sum_{n,m=0} η_α \cdots η_α \eta_β \cdots η_β \Psi^{(α\cdots α_n)(β\cdots β_m)}(P_μ, φ) , \tag{1.30}
\]

where the values of the four-momentum \( P_μ \) are restricted to the mass-shell \( P^2 = m^2 \) and the dependence on the gauge variable \( φ \) may be simply factorized out as a suitable \( U(1) \) phase factor. We shall express the higher spin fields in Minkowski space by performing the standard Fourier transform

\[
Ψ^{(α\cdots α_n)(β\cdots β_m)}(x_μ) = \frac{1}{(2π)^4} \int d^4 P_μ e^{iP_μ x_μ} Ψ^{(α\cdots α_n)(β\cdots β_m)}(P_μ, φ) . \tag{1.31}
\]

We stress here that the spacetime coordinates \( x_μ \) appearing in the formula \( \ref{1.30} \) cannot be identified with the commuting composite spacetime coordinates \( X_μ \) in \( M^{18} \).

We conclude in Sec. \( \text{IV} \) with some remarks and open problems.

### II. FROM THE TWO-TWISTOR FREE MODEL TO A PARTICLE MODEL IN ENLARGED SPACETIME

Let us modify the Penrose formula \( \ref{1.5} \) in order to introduce new complex composite spacetime coordinates

\[
Z^{α\dot{β}} \text{ defined as follows}
\]

\[
z^{α\dot{β}} \longrightarrow Z^{α\dot{β}} = z^{α\dot{β}} + Δz^{α\dot{β}} , \tag{2.1}
\]
where
\[ \Delta \xi^{\alpha\beta} = \Delta x^{\alpha\beta} + i \Delta y^{\alpha\beta} = \frac{i \rho}{|f|^2} \sigma^{\alpha\beta} . \] (2.2)
and \( \rho = \frac{1}{2} (t_1 - t_2) \). A calculation shows that
\[ \Delta x_{\alpha\beta} = \frac{1}{2|f|^2} \left( t^1 P^2_{\alpha\beta} + t^2 P^1_{\alpha\beta} \right) , \] (2.3a)
\[ \Delta y_{\alpha\beta} = \frac{1}{2|f|^2} \left( t^1 P^1_{\alpha\beta} - t^2 P^2_{\alpha\beta} \right) . \] (2.3b)

We use now the non-linear change of twistor variables \( Z_A , \bar{Z}^\dagger_A \) into the eighteen coordinates of \( \mathcal{M}^{18} \) given by the formula (1.13) for \( P^2 \) and the following ones (1.14)
\[ X_{\alpha\beta} = \Re Z_{\alpha\beta} = \frac{i}{2f} \left( \omega^{\alpha\beta} \bar{\eta}^3 - \lambda^\alpha \eta^\beta \right) - \frac{i}{2f} \left( \eta^\beta \bar{\omega}^\alpha - \bar{\pi}^\alpha \chi^\beta \right) + \frac{i}{2|f|^2} \rho \pi^{\alpha \beta} \eta^3 - \frac{i}{2|f|^2} \bar{\rho} \bar{\pi}^\alpha \eta^\beta , \] (2.4a)
\[ \sigma^\alpha = - \frac{1}{f} (\rho \eta^\alpha + t_3 \bar{\pi}^\alpha) , \] (2.4b)
\[ \bar{\pi}^\alpha = - \frac{1}{f} (\rho \bar{\pi}^\alpha + t_3 \pi^\alpha) , \] (2.4c)
\[ e = t_0 + t_3 , \] (2.4d)
\[ \phi = \frac{i}{2} \ln \frac{1}{f} . \] (2.4e)

Using (1.13) - (1.14) and (2.4e) we obtain
\[ f = \sqrt{\frac{P^2}{2}} e^{i\phi} , \] (2.5a)
\[ \pi^\alpha = - \frac{1}{f} P_{\alpha\beta} \eta^\beta , \] (2.5b)
\[ \bar{\pi}^\alpha = \frac{1}{f} \bar{\eta}^\beta P_{\beta\alpha} . \] (2.5c)

It follows from eqs. (2.4a) - (2.4e) that the invariant spin components \( t_1 \), \( t_2 \) and \( t_3 \) (eqs. (1.17), (1.20)) are given in terms of the variables \( t_3, \rho, \bar{\rho} \) by the formulae
\[ t^3 = \eta^\alpha \sigma_\alpha = \bar{\eta}^\beta \bar{\sigma}_\beta , \] (2.6a)
\[ \rho = t_1 - i t_2 = \pi^\alpha \bar{\pi}^\alpha = \frac{1}{f} \eta^\alpha P_{\alpha\beta} \eta^\beta , \] (2.6b)
\[ \bar{\rho} = t_1 + i t_2 = \pi^\alpha \sigma_\alpha = \frac{1}{f} \sigma^\alpha P_{\alpha\beta} \bar{\eta}^\beta . \] (2.6c)

Further, the linear combination (2.4d) of the internal scalar charge \( t^0 \) and \( t^3 \) will be called electric charge. Indeed, if we identify \( t^0 \) with \( \frac{1}{2} \) (‘hypercharge’) and \( t^3 \) with the third isospin component, eq. (2.4d) takes the form of the Gell-Mann-Nishijima formula for the electric charge [32].

Using the formulæ (1.13), (2.4a) - (2.4e), we can now re-express the free two-twistor Liouville one-form (1.10) as
\[ \Theta^{(2)} = P_\mu dX^\mu - i (\sigma^\alpha d\eta_\alpha - \sigma^\alpha d\bar{\eta}_\alpha) + e d\phi , \] (2.7)
where the 18 coordinates of \( \mathcal{M}^{18} \) are restricted by two constraints. From definitions (1.13) and (2.4a) - (2.4e) we can obtain the following constraints:
\[ R_1 = \sigma_\alpha P_{\alpha\beta} \sigma_\beta - t^2 = 0 \quad (t^2 = \rho \bar{\rho} + (t^3)^2) , \] (2.8a)
\[ R_2 = \eta_\alpha P_{\alpha\beta} \eta_\beta - \frac{1}{2} P^2 = 0 . \] (2.8b)

The relation (2.6a) implies additionally the reality constraint
\[ R_3 = \eta^\alpha \sigma_\alpha - \bar{\eta}^\beta \bar{\sigma}_\beta = 0 . \] (2.9)

It can be shown that only two constraints out of the three \( R_1, R_2, R_3 \) are independent. Indeed, if we introduce
\[ \tilde{R}_1 \equiv R_1 + \frac{2 \sigma^\alpha P_{\alpha\beta} \bar{\pi}^\beta}{P^2} R_2 \equiv \frac{2 \sigma^\alpha P_{\alpha\beta} \bar{\pi}^\beta \eta_\beta P_{\beta \gamma} \bar{\eta}_\gamma}{P^2} - t^2 \] (2.10)
by using the definition of \( t^2 \) (see (2.6a) - (2.6c)) and then use the following Fierz identity
\[ (\sigma^\alpha \sigma_\alpha \bar{\pi}^\beta) (\eta^\beta \eta_\beta \bar{\pi}^\beta) = - (\sigma^\alpha \eta_\alpha) (\eta_\beta \eta^\beta) (\bar{\sigma}_\beta \bar{\pi}^\beta) + \bar{\eta}^\beta \bar{\sigma}^\beta \bar{\pi}^\beta (\sigma^\alpha \sigma_\alpha \bar{\pi}^\beta) \] (2.11)
one can show that \( \tilde{R}_1 = 0 \), i.e. \( R_1 = 0 \) follows from \( R_2 = 0 \).

We stress that the two algebraic constraints \( R_2 = R_3 = 0 \) encode the composite structure of the variables (1.29) as expressed in terms of primary twistor variables (1.17).

The Liouville one-form (2.7) and the constraints \( R_2 = R_3 = 0 \) provide the following action describing a relativistic particle model \( \dot{a} \equiv \frac{d a}{d \tau} \)
\[ S = \int d\tau L(\tau) = \int d\tau \left[ P_\mu \dot{X}^\mu - i (\bar{\pi}^\beta \bar{\eta}_\alpha - \sigma^\alpha \eta_\alpha) + e \dot{\phi} + \lambda_2 R_2 + \lambda_3 R_3 \right] , \] (2.12)
This action, supplemented only with the algebraic constraints on $\mathcal{M}^{18}$, can be quantized canonically. The action leads to the canonical PB (CPB) in the spacetime sector of $\mathcal{M}^{18}$ plus the following (non-vanishing) remaining ones

\begin{align}
\{\eta_\alpha, \sigma^\beta\}_C &= i\delta^\beta_\alpha, \\
\{\bar{\eta}_\alpha, \bar{\sigma}^\beta\}_C &= -i\delta^\beta_\alpha, \\
\{e, \phi\}_C &= 1,
\end{align}

where $\{,\}_C$ means canonical PB.

Before considering the dynamical first class constraints that fix the values of the mass, spin, spin projection and electric charge, we have to introduce Dirac brackets consistent with the algebraic second class constraints $R_2, R_3$. Since

$$\{R_2, R_3\}_C = -2i\eta_\alpha P^\alpha \bar{\eta}_\alpha - i P^2,$$

the Dirac bracket of two dynamical quantities $A, B$ takes the form

$$\{A, B\}_D = \{A, B\}_C - \frac{1}{iP^2} \{R_3, B\}_C - \{A, R_3\}_C \frac{1}{iP^2} \{R_2, B\}_C.$$

Thus, within the generalized phase space $\mathcal{M}^{18}$, the only coordinates $Y$ that possess non-vanishing Poisson brackets with the constraints $R_2$ and $R_3$ are

$$\{Y, R_2\}_C \neq 0 \quad \text{if} \quad Y = X_{a\alpha\beta}, \sigma_\alpha, \bar{\sigma}_\alpha$$

$$\{Y', R_3\}_C \neq 0 \quad \text{if} \quad Y' = \eta_\alpha, \bar{\eta}_\alpha, \sigma_\alpha, \bar{\sigma}_\alpha.$$

Hence, according to \eqref{2.15}, only the Dirac $\{Y, Y\}_D$ brackets will be different from the canonical $\{Y, Y\}_C$ ones. This leads to two sets of non-zero Dirac brackets, one involving the spacetime variables $X_{a\alpha\beta}$,

$$\{X_{a\alpha\beta}, \eta_\gamma\}_D = -\frac{i}{P^2}(P_{a\alpha\beta} - \eta_\alpha \bar{\eta}_\alpha)\eta_\gamma,$$

$$\{X_{a\alpha\beta}, \bar{\eta}_\gamma\}_D = -\frac{i}{P^2}(P_{a\alpha\beta} - \eta_\alpha \bar{\eta}_\alpha)\bar{\eta}_\gamma,$$

$$\{X_{a\alpha\beta}, \sigma_\gamma\}_D = \frac{i}{P^2}(P_{a\alpha\beta} - \eta_\alpha \bar{\eta}_\alpha)\sigma_\gamma,$$

$$\{X_{a\alpha\beta}, \bar{\sigma}_\gamma\}_D = \frac{i}{P^2}(P_{a\alpha\beta} - \eta_\alpha \bar{\eta}_\alpha)\bar{\sigma}_\gamma,$$

and another involving only those of the spin sector,

$$\{\eta_\alpha, \sigma^\beta\}_D = i\delta^\beta_\alpha - \frac{i}{P^2}\eta_\alpha P^\beta \bar{\eta}_\gamma,$$

$$\{\bar{\eta}_\alpha, \bar{\sigma}^\beta\}_D = \frac{i}{P^2}\bar{\eta}_\alpha \gamma, P^\beta \bar{\gamma}$$. 

$$\{\sigma^\alpha, \sigma^\beta\}_D = \frac{i}{P^2} P^\alpha \eta_\gamma \bar{\sigma}^\beta,$$

$$\{\bar{\sigma}^\alpha, \bar{\sigma}^\beta\}_D = -\frac{i}{P^2} P^\alpha \bar{\eta}_\gamma \sigma^\beta,$$

$$\{\eta_\alpha, \bar{\sigma}^\beta\}_D = -\frac{i}{P^2} P^\alpha \eta_\gamma \bar{\sigma}^\beta,$$

$$\{\bar{\eta}_\alpha, \sigma^\beta\}_D = \frac{i}{P^2} P^\alpha \bar{\eta}_\gamma \sigma^\beta.$$

All other Dirac brackets in $\mathcal{M}^{18}$ coincide with the canonical ones.

If we now replace the relations \eqref{2.18a} - \eqref{2.18h} by their quantum analogues,

$$a \rightarrow \hat{a}, \quad \{a, b\}_D \rightarrow \frac{1}{i} [\hat{a}, \hat{b}],$$

and provide a rule for the ordering of variables associated with the non-commuting operators, one can show that only the seven relations \eqref{2.18a} - \eqref{2.18h} above produce a consistent (i.e., satisfying the Jacobi identity) set of commutators

\begin{align}
\{\hat{\eta}_\alpha, \hat{\sigma}^\beta\} &= -\delta^\beta_\alpha + \frac{1}{P^2} \eta_\alpha \hat{P}^\alpha \hat{\gamma} \bar{\eta}_\gamma, \\
\{\hat{\bar{\eta}}^\alpha, \hat{\bar{\sigma}}^\beta\} &= -\frac{1}{P^2} \hat{\bar{\eta}}^\alpha \hat{\bar{\eta}}^\beta \hat{\gamma} \bar{\eta}_\gamma, \\
\{\hat{\sigma}^\alpha, \hat{\sigma}^\beta\} &= \frac{1}{P^2} \hat{\bar{P}}^\gamma \eta_\gamma \bar{\sigma}^\alpha \hat{\sigma}^\beta, \\
\{\hat{\bar{\sigma}}^\alpha, \hat{\bar{\sigma}}^\beta\} &= \frac{1}{P^2} \hat{\bar{P}}^\gamma \eta_\gamma \bar{\sigma}^\alpha \hat{\bar{\sigma}}^\beta, \\
\{\hat{\eta}_\alpha, \hat{\bar{\sigma}}^\beta\} &= \frac{1}{P^2} \hat{\bar{P}}^\gamma \eta_\gamma \bar{\sigma}^\alpha \hat{\bar{\sigma}}^\beta, \\
\{\hat{\bar{\eta}}^\alpha, \hat{\sigma}^\beta\} &= \frac{1}{P^2} \hat{\bar{P}}^\gamma \eta_\gamma \bar{\sigma}^\alpha \hat{\sigma}^\beta, \\
\{\hat{\bar{\eta}}^\alpha, \hat{\bar{\sigma}}^\beta\} &= \frac{1}{P^2} \hat{\bar{P}}^\gamma \eta_\gamma \bar{\sigma}^\alpha \hat{\bar{\sigma}}^\beta.
\end{align}

In particular, we note that the quantization \eqref{2.20d} of the Dirac bracket \eqref{2.18h} requires the specific order of the operators indicated by the r.h.s. of formula \eqref{2.18h}. Unfortunately, for any choice of ordering, the quantization of the Dirac brackets \eqref{2.18a} - \eqref{2.18d} involving the spacetime coordinate $X_\mu$ leads to a non-associative algebra.\[33\]
The algebra (2.20a) - (2.20g) can be realized in terms of differential operators on the following 8-dimensional enlarged momentum space

$$\mathcal{P}^8 = P_k = (P_\mu, \eta_\alpha, \pi_\alpha, \phi; R_2 = 0) \ .$$  \ (2.21)

One can show that the operators $\hat{\sigma}^\alpha, \hat{\sigma}^\beta, \hat{\sigma}^\gamma$ have the following generalized Schrödinger realization [34]

$$\hat{\sigma}^\alpha = \frac{\partial}{\partial \eta_\alpha} - P^\beta \eta_\beta \frac{1}{P^2} \left( \eta_\gamma \frac{\partial}{\partial \eta_\gamma} + \eta_\delta \frac{\partial}{\partial \eta_\delta} \right) \ , \ (2.22)$$

$$\hat{\sigma}^\beta = \frac{\partial}{\partial \eta_\beta} + \eta_\beta P^\gamma \frac{1}{P^2} \left( \eta_\delta \frac{\partial}{\partial \eta_\delta} + \eta_\gamma \frac{\partial}{\partial \eta_\gamma} \right) \ , \ (2.23)$$

$$\hat{\sigma}^\gamma = \frac{1}{i} \frac{\partial}{\partial \phi} \ . \ (2.24)$$

These operators satisfy (2.22a) - (2.22g), to which we may add $[\hat{\sigma}^\alpha, \hat{\sigma}^\beta] = i$. It can be checked that in the differential realization (2.22a) - (2.24), the constraint $R_3 = 0$ is identically satisfied.

In order to determine the relativistic particle states with definite mass $m$, spin $s$, invariant spin projection $s_3$ and electric charge $e_0$, we supplement the action (2.25) with the following physical constraints:

$$R_4 = t^2 - s(s + 1) = 0 \ , \ (2.25a)$$

$$R_5 = t_3 - s_3 = 0 \ , \ (2.25b)$$

$$R_6 = P^2 - m^2 = 0 \ , \ (2.25c)$$

$$R_7 = e - e_0 = 0 \ . \ (2.25d)$$

It can be shown that the CPB of the constraints $R_2, \ldots, R_7$ provide the following canonical Poisson brackets algebra besides (3.1):

$$\{R_2, R_4\}_C = -i\eta_\alpha P^\alpha \eta_\alpha R_3 \ \ (2.26a)$$

$$\{R_2, R_k\}_C = 0 \ \ k = 5, 6, 7 \ \ (2.26b)$$

$$\{R_m, R_n\}_C = 0 \ \ m, n = 3, \ldots, 7 \ \ (2.26c)$$

The set of relations (2.25a) - (2.25c) shows that the constraints $R_4, R_5, R_6, R_7$ are first class.

### III. FIRST-QUANTIZED THEORY AND WAVE EQUATIONS

The quantum form of the first class constraints (2.25a) - (2.25d) provides the following four wave equations for the function $\Psi(\mathcal{P}_k) = \Psi(P_\mu, \eta_\alpha, \pi_\alpha, \phi)$ (we set $\hbar = c = 1$)

$$\hat{t}_3 \Psi = 0 \ \ (3.1a)$$

$$\hat{t}_3 = \frac{1}{i} \left( \pi_\alpha \frac{\partial}{\partial \eta_\alpha} - \eta_\alpha \frac{\partial}{\partial \pi_\alpha} \right) \ \ (3.2a)$$

$$\hat{t}_3 \eta_\alpha = -\frac{1}{2} \eta_\alpha \ \ (3.3a)$$

$$\hat{t}_3 \eta_\alpha = -\frac{1}{2} \eta_\alpha \ \ (3.5a)$$

$$\hat{t}_3 \eta_\alpha = -\frac{1}{2} \eta_\alpha \ \ (3.7a)$$
where we recall that the Lorentz scalar $\hat{\tau}_3$ describes the projection of the relativistic spin four-vector $W_\mu$ on the direction $P_\mu^{(3)}$, orthogonal to $P_\mu$ (see $[1.15]$ and $[1.21]$). In order to identify the relations (3.8a)-(3.8c) with (3.1a)-(3.1b) we set

$$s = \frac{n}{2} \quad \text{or} \quad s = \frac{m}{2} , \quad (3.9)$$

and

$$s_3 = \frac{m - n}{2} . \quad (3.10)$$

From eqs. (3.8a)-(3.8c) we conclude that $\eta_{\alpha_1} \cdots \eta_{\alpha_n} \sim |\frac{1}{2}, \frac{1}{2} >$ and $\overline{\eta}_{\alpha_1} \cdots \overline{\eta}_{\alpha_n} \sim |\frac{1}{2}, \frac{1}{2} >$.

The general theory of $su(2)$ representations tells us that one can get all the states starting either from the lowest one, $|s, -s >$, and acting with the raising operator $\hat{\rho}$ or from the highest one, $|s, s >$, and acting with the lowering operator $\overline{\rho}$. The action of the ladder operators on a product of $\eta_\alpha$’s or $\overline{\eta}_\alpha$’s can be summarized as follows,

$$|m, \frac{n - m}{2} \rangle_{\alpha_1 \cdots \alpha_n} \sim (\hat{\rho})^m \eta_{\alpha_1} \cdots \eta_{\alpha_n} = m! \left( \frac{2}{P^2} \right)^m e^{-im\phi} \cdot$$

$$\cdot \sum_{\rho \in C_m} \sum_{\beta_1 \cdots \beta_m} P_{\alpha_1}^{\beta_1} \cdots P_{\alpha_m}^{\beta_m} \eta_{\beta_1} \cdots \eta_{\beta_m} \eta_{\alpha_{m+1}} \cdots \eta_{\alpha_n} , \quad (3.11a)$$

$$|m, -\frac{n + m}{2} \rangle_{\beta_1 \cdots \beta_n} \sim (\overline{\rho})^m \eta_{\beta_1} \cdots \eta_{\beta_n} = (-1)^m m! \left( \frac{2}{P^2} \right)^m e^{im\phi} \cdot$$

$$\cdot \sum_{\rho \in C_m} \sum_{\alpha_1 \cdots \alpha_m} P_{\alpha_1}^{\beta_1} \cdots P_{\alpha_m}^{\beta_m} \eta_{\alpha_1} \cdots \eta_{\alpha_m} \eta_{\beta_{m+1}} \cdots \eta_{\beta_n} . \quad (3.11b)$$

In particular if $m > n$ both expressions are zero, i.e. we generate by the method described by formulas (3.11a)-(3.11b) the $(2s + 1)$-dimensional basis (we recall that $s = \frac{n}{2}$) of the irreducible representation with spin $s$. We may thus describe a wavefunction with definite spin $s$ and spin projection $s_3$ ($-s \leq s_3 \leq s$) by starting from the lowest value of $s_3$ (see (3.11a)) or from the highest one (see (3.11b)). We have the following two sequences generating the representation space basis

$$\hat{\rho} |s, s > \rightarrow |s, s - 1 > \cdots \hat{\rho} |s, -s + 1 > \rightarrow |s, -s > \quad (3.12a)$$

$$\overline{\rho} |s, -s > \rightarrow |s, -s + 1 > \cdots \overline{\rho} |s, s - 1 > \rightarrow |s, s > \quad (3.12b)$$

For example, for spin $s = 1$ we have the following three-dimensional basis, generated by the sequence (3.11a)

$$|1, -1 \rangle_{\alpha \beta} = \eta_\alpha \eta_\beta \quad (3.13a)$$

$$|1, 0 \rangle_{\alpha \beta} = \sqrt{\frac{2}{P^2}} e^{-i\phi} \left( P^{\alpha}_\alpha \right. \left. \eta_\alpha \eta_\beta + P^{\beta}_\beta \eta_\alpha \eta_\alpha \right) \quad (3.13b)$$

$$|1, 1 \rangle_{\alpha \beta} = \frac{4}{P^2} e^{-2i\phi} P^{\alpha}_\alpha P^{\beta}_\beta \eta_\alpha \eta_\beta \quad (3.13c)$$

or, alternatively, by the sequence (3.11b),

$$|1, 1 \rangle_{\alpha \beta} = \eta_\alpha \eta_\beta \quad (3.14a)$$

$$|1, 0 \rangle_{\alpha \beta} = -\frac{2}{P^2} e^{i\phi} \left( P^{\alpha}_\alpha \eta_\alpha \eta_\beta + P^{\beta}_\beta \eta_\alpha \eta_\alpha \right) \quad (3.14b)$$

$$|1, -1 \rangle_{\alpha \beta} = \frac{4}{P^2} e^{2i\phi} P^{\alpha}_\alpha P^{\beta}_\beta \eta_\alpha \eta_\beta . \quad (3.14c)$$
We add here that the particular momentum dependent coefficients (see (3.1a)–(3.1d) and (3.11a)–(3.1d)) are due to our definitions of the raising and lowering operators \(\tilde{B}, \tilde{B}\) which shift by one the invariant projections \(s_3\) given by \(\tilde{\eta}_3\).

We now derive in our framework the linear field equations for any spin (Dirac for \(s = \frac{1}{2}\) and Bargmann-Wigner for arbitrary spin). Further, we consistently assume that the relations (3.3) are fulfilled. For \(s = \frac{1}{2}\), i.e. \(n + m = 1\) in (3.30), the wavefunction is described by one complex \(\psi\) real four component Majorana spinor. The Weyl spinor equation (3.15) by introducing a new Weyl spinor \(\chi_\beta(P_\mu)\) means of the defining equation (we assume \(m \neq 0\))

\[
(\tilde{P}^2 - m^2)\psi_\alpha(P_\mu) = 0 \tag{3.15}
\]

In order to introduce the Dirac equation we linearize the equation (3.15) by introducing a new Weyl spinor \(\chi_\beta(P_\mu)\) by means of the defining equation (we assume \(m \neq 0\))

\[
\psi_\alpha(P_\mu)P_{\alpha\beta} = m\chi_\beta(P_\mu) \tag{3.16}
\]

From (3.15) and (3.16) follows also that

\[
P^{\alpha\dot{\alpha}}\chi_\alpha(P_\mu) = m\psi^{\dot{\alpha}}(P_\mu) \tag{3.17}
\]

The set of equations (3.16), (3.17) provide the standard momentum space Dirac equation in terms of two Weyl equations coupled through the particle’s mass.

In order to discuss the arbitrary spin case we introduce besides \(\Psi(P_k)\) a second function \(\Upsilon(P_k)\) satisfying the relation

\[
P_{\alpha\dot{\alpha}} \frac{\partial}{\partial \eta^{\dot{\alpha}}} \Psi(P_k) = m \frac{\partial}{\partial \eta^{\dot{\alpha}}} \Upsilon(P_k) \tag{3.18}
\]

Using (3.1a) one gets from (3.18)

\[
P^{\alpha\dot{\alpha}} \frac{\partial}{\partial \eta^{\dot{\alpha}}} \Upsilon(P_k) = m \frac{\partial}{\partial \eta^{\dot{\alpha}}} \Psi(P_k) \tag{3.19}
\]

Equations (3.18) and (3.19) describe the generalized Dirac equations in the enlarged momentum space (2.21).

Expressing (3.18), (3.19) in powers of \(\eta_\alpha\) and \(\tilde{\eta}_\ddot{\alpha}\) we obtain from the linear terms the Dirac equation (see eqs. (3.16), (3.17)) and, from the higher order terms, the Fierz-Pauli/Bargmann-Wigner equations for arbitrary spin (14)

\[
\phi^{(\alpha_1 \cdots \alpha_k)}(\beta_1 \cdots \beta_l)\psi(P_\mu)P_{\alpha_1 \beta_1} = m\chi^{(\alpha_2 \cdots \alpha_k)}(\beta_1 \cdots \beta_l)(P_\mu) \tag{3.20a}
\]

\[
P^{\alpha_1 \beta_1} \chi^{(\alpha_2 \cdots \alpha_k)}(\beta_1 \cdots \beta_l)(P_\mu) = m\phi^{(\alpha_1 \cdots \alpha_k)}(\beta_2 \cdots \beta_l)(P_\mu) \tag{3.20b}
\]

Finally, let us consider the question of the spacetime picture in our formalism. The equations (3.1a–3.1d) as well as (3.18), (3.19) have been derived in the enlarged momentum space (2.21). As we have shown in Sec. III the composite real spacetime coordinates (2.20) do commute with themselves (see (1.26a)), but they do not commute with the spin sector variables (see (2.18a), (2.18b)). This non-commutativity as derived from our model (2.12) is reflected at the classical level by the non-vanishing Dirac brackets. Unfortunately, we have not been able to quantize the set of Dirac brackets which contain the composite space-time coordinates in a way that leads to an associative algebra i.e., satisfying Jacobi identities. At this stage of the development of our framework we have to abandon the idea that the commuting composite spacetime coordinates of eq. (2.20) describe ‘physical’ spacetime. Instead, we may introduce another set of commuting spacetime coordinates by means of the standard Fourier transform in the physical four-momentum sector of the manifold \(P_k\),

\[
\Psi(\tilde{x}_\mu, \eta_\alpha, \bar{\eta}_\ddot{\alpha}, \phi) = \frac{1}{(2\pi)^4} \int d^4P_\mu e^{iP_\mu \tilde{x}_\mu} \Psi(P_\mu, \eta_\alpha, \bar{\eta}_\ddot{\alpha}, \phi) \tag{3.21}
\]

which leads to the formula (1.31). Due to the constraint (3.1d) the mass shell Dirac delta \(\delta(P^2 - m^2)\) is included as a factor in the momenta integral of (3.21) i.e., we obtain the standard spacetime wave function satisfying the Klein-Gordon equation (see (3.1c) and (3.15)) as well as space-time wavefunctions for arbitrary spins (see (3.21a–b)). In particular, if \(s = \frac{1}{2}\) eqs. (3.16), (3.17) lead to the standard Dirac equation in ordinary Minkowski space, with gamma matrices written in the Weyl realization.

IV. FINAL REMARKS

The aim of this paper is to exhibit the consequences of using the geometric two-twistor framework for the simplest case of the classical mechanics of free massive particles with spin. Our starting point, not present in other discussions in the twistor formalism (see e.g. [4]), is a rigorous derivation of the particle action from the free two-twistor symplectic form without introducing additional degrees of freedom [2]. In such a framework the mass, spin and electric charge appear as free parameters of our model and are determined by additional non-geometric constraints.

One point in which our paper differs from the standard Penrose framework is the definition of the spacetime coordinates. Following the usual field-theoretic description of massive particles with spin we introduce commuting, composite spacetime coordinates (see (2.20)). On the other hand, it is known that the presence of spin leads naturally to non-commutative spacetime coordinates [30] (see e.g. [18]), as reflected by the non-vanishing of the PB among the spacetime coordinates of the particles with spin; this is also the case when spin is related to the pres-
ence of the supersymmetry and the superspace extension of classical mechanics. We see therefore two options for describing a two-twistor-inspired massive spinning particle dynamics:

i) to use the standard Penrose approach with non-commutative Minkowski coordinates satisfying the PB. In such a case to complete the description of the quantized theory of particles with non-vanishing spin one is led to a field theoretic framework on non-commutative spacetime.

ii) to follow the approach used in this paper. In this case only a partial quantization of the classical phase space degrees of freedom is possible, and a complete quantization of all the Dirac brackets, including those of the composite commuting spacetime coordinates with the spin sector variables \((\eta_\alpha, \overline{\eta}_\alpha, \sigma_\alpha, \overline{\sigma}_\alpha)\), requires further study.

Nevertheless, our present geometric framework assigns a dynamical rôle to the additional twistor-motivated spinorial degrees of freedom, where the momentum is expressed in terms of the twistorial ‘constituent’ variables. Although the use of auxiliary fundamental spinor variables for the description of higher spin theories is known (see e.g. [2]), up to now most of the twistor-based applications have been concerned with massless fields with arbitrary helicity (see [24, 25, 26, 27]). We would like to observe that the Cachazo-Svrček-Witten twistor approach to maximally helicity violating vertices and tree amplitudes in gauge theories has recently been extended to tree amplitudes including massive particles [28, 29]. Nevertheless, in such a framework the description of massless four-momenta as composites of twistor coordinates has not been extended to the four-momenta and spin for massive spinning particles. We stress, however, that in the approach presented here, which considers pairs of twistors, we obtain a scheme that permits describing the four-momenta as well as the spin of massive particles as functions of twistorial ‘constituent’ variables. Also, our framework provides a twistor-motivated approach to massive, free higher spin theory.

Note added.
The case of a massive spinless particle has recently been described in terms of a single twistor by using a modified twistor-phase space transform inspired by two-time physics techniques [30].

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[31] In comparison with [12] we employ here two equivalent, but different, second class constraints.

[32] The identification of $e$ as the electric charge can be ultimately justified by coupling our model to the electromagnetic field, which is outside of the scope of this paper.

[33] The difficulties with the fulfillment of the Jacobi identity (JI) by the quantized Dirac brackets are well-known (see e.g. [13]).

[34] Notice that we do not need to check algebraically all the JI’s for (2.20a)-(2.20g); it is sufficient to show that in differential realization ($\hat{\eta}_a = \eta_a$, $\hat{\bar{\eta}}_\bar{a} = \bar{\eta}_\bar{a}$, $\hat{P}^{\alpha\dot{\alpha}} = P^{\alpha\dot{\alpha}}$ and formulae (2.22)-(2.24)) the Dirac bracket algebra (2.20a)-(2.20g) is satisfied.

[35] An example of the use of two-twistor geometry for the description of massive particles with spin which employs additional degrees of freedom (the so-called index spinor) is provided by [17]. This index spinor has been interpreted as a bosonic counterpart of the Grassmann components of the super-twistor for N=2 supersymmetry.

[36] We recall (see (1.28a)) that the non-commutativity of the spacetime coordinates follows from the standard Penrose framework. The quantization of a twistor-motivated particle model with non-commuting spacetime coordinates (i.e. within the standard Penrose definition of spacetime) will be considered in a forthcoming paper [21].

[37] We point out that non-commutative field theory has attracted considerable attention (see e.g., [22]) in the last decade, although such a non-commutativity does not follow from the non-vanishing spin.