The lattice of worker-quasi-stable matchings

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Abstract

In a many-to-one matching model, we study the set of worker-quasi-stable matchings when firms’ choice functions satisfy substitutability. Worker-quasi-stability is a relaxation of stability that allows blocking pairs involving a firm and an unemployed worker. We show that this set has a lattice structure and define a Tarski operator on this lattice that models a re-equilibration process and has the set of stable matchings as its fixed points.

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1 Introduction

In this paper, we study a many-to-one matching model in which agents in one side of the market (that we call firms) have to be assigned to subsets of agents on the other side of the market (that we call workers) and the only requirement on subsets of workers that each firm’s choice function has to satisfy is substitutability. For this model, using a partial order first studied by Blair (1988), we show that the set of worker-quasi-stable

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matchings has a lattice structure. Worker-quasi-stability is a relaxation of stability that allows blocking pairs involving a firm and an unemployed worker. The importance of these matchings is twofold. First, from a practical standpoint, worker-quasi-stability captures an interim situation when, starting from a stable matching, new workers arrive or firms downsize and laid off workers have to look for employment elsewhere. Furthermore, if our aim is to pursue stability, a re-equilibration process can be described as a layoff chain dynamic within worker-quasi-stable matchings that brings the market back to stability. Second, from a theoretical standpoint, much of the literature that studies stability through Tarski’s fixed point theorem (Tarski, 1955) carries out its analysis by means of a lattice that strictly contains the set of matchings (see Adachi, 2000; Fleiner, 2003; Echenique and Oviedo, 2004, among others). However, we show that a fixed point approach can be performed within the realm of worker-quasi-stability.

Our general framework assumes substitutability on firms’ choice functions. This condition, first introduced by Kelso and Crawford (1982), is the less restrictive requirement in firms’ choice functions in order to guarantee the existence of stable matchings. A firm has substitutable choice functions if it wants to continue hiring a worker even if other workers become unavailable. Blair (1988) defines a partial order over the set of matchings, and shows that when choice functions are substitutable, the set of stable matchings has a lattice structure. A matching Blair-dominates another matching if each firm wishes to keep the workers hired under the first one even if all the workers hired under the second one are also available, and does not wish to hire any new worker. In our paper, given two worker-quasi-stable matchings, we define a new one by means of a choice function that selects, for each firm, the best subset of workers among those that this firm is matched to in either matching. This new matching turns out to be the join (least upper bound) of the two original matchings according to Blair’s order. In this way, we extend Blair’s result to the whole set of worker-quasi-stable matchings. More specifically, we prove that the set of worker-quasi-stable matchings forms a finite join-semilattice with a minimum element, implying that it is a lattice.

Furthermore, we define a Tarski operator in the worker-quasi-stable lattice that describes a possible re-equilibration process. This process models how, starting from

1The lattice structure of the set of stable matchings is introduced by Knuth (1976) for the one-to-one matching model. This result is generalized in different directions by several papers (see for instance Blair, 1988; Martínez et al., 2001; Alkan, 2002; Wu and Roth, 2018, among others).

2In the many-to-one matching literature, stability can be thought of as the conjunction of “envy-freeness” and “non-wastefulness” (see, for example, Kamada and Kojima, 2022; Wu and Roth, 2018). The presence of blocking pairs involving an unemployed worker and an acceptant firm can be interpreted as wastefulness of the matching.

3Most of these papers rely on the notion of “pre-matching”. In a pre-matching, the fact that agent a is matched to agent b does not imply that b is matched to a.
any worker-quasi-stable matching, a decentralized sequence of offers in which unemployed workers are hired (causing new unemploymets), produces a sequence of worker-quasi-stable matchings that converges to a stable matching. As a by-product, applying Tarski’s fixed point theorem to our operator, we give an alternative proof of the fact that the set of stable matchings (the fixed point set of our operator) is non-empty and has a lattice structure as well. Finally, we present some additional results when firms’ choice functions satisfy, in addition to substitutability, the “law of aggregate demand”. This condition says that when a firm chooses from an expanded set, it hires at least as many workers as before. Under the “law of aggregate demand” we can identify the fixed point of the operator that can be obtained by iterating it starting at a worker-quasi-stable matching: it is the join of that worker-quasi-stable matching and the worker-optimal stable matching. We also show that (i) the join of a worker-quasi-stable matching and a stable matching is stable, and (ii) every worker-quasi-stable matching that weakly Blair-dominates the worker-optimal stable matching is stable.

The paper closest to ours is Wu and Roth (2018). In a many-to-one matching model in which firms have responsive preferences (a more restrictive requirement than substitutable choice functions), they obtain a lattice structure for the set of firm-quasi-stable matchings under the common preference of workers. Given two matchings, the Conway-like join for workers that they use matches all workers to their most preferred firm between their two original partners. Their paper is the first one to present a Tarski operator defined on a lattice of matchings (in their case, the set of firm-quasi-stable matchings), that can be interpreted as modeling vacancy chains, and show that the operator has the set of stable matchings as its fixed points. Concerning the set of worker-quasi-stable matchings, they show the difficulties of defining a Conway-like join for firms even when firms have responsive preferences. Instead, we are able to sidestep this problem following Blair’s insight.

Another paper that relates a Tarski operator with the notion of firm-quasi-stability is Kamada and Kojima (2022). In a school choice setting with constraints, they define a Tarski operator over a space of “cutoff profiles”, and use it to characterize the firm-quasi-stable matchings as its fixed points. Thus, their approach is tangential to ours and Wu and Roth’s.

The rest of the paper is organized as follows. In Section 2, we present the model

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4This property is first studied by Alkan (2002) under the name of “cardinal monotonicity”. See also Hatfield and Milgrom (2005).
5Firm-quasi-stability (called envy-freeness by Wu and Roth (2018)) is a relaxation of stability that allows blocking pairs involving a worker and an empty position of a firm.
6For Kamada and Kojima (2022), a firm-quasi-stable matching is an envy-free matching that fulfills a pre-specified constraint.
and preliminaries. The lattice structure of the worker-quasi-stable set is analyzed in Section 3. In Section 4, we introduce our Tarski operator, which allows us to prove that the set of stable matchings is non-empty and forms a lattice when firms’ choice functions satisfy substitutability. Moreover, we present a re-equilibration process via layoff chains based on our operator. Further results that give some insight on the behavior of the Tarski operator are gathered in Section 5, where in addition to substitutability we require firms’ choice functions to satisfy the “law of aggregate demand”. Finally, in Section 6, we present some conclusions.

2 Model and preliminaries

We consider a many-to-one matching model where there are two disjoint sets of agents: the set of firms $F$ and the set of workers $W$. Each worker $w \in W$ has a strict preference relation $\succ_w$ over the individual firms and the prospect of being unmatched, denoted by $\emptyset$. Each firm $f \in F$ has a choice function $C_f$ over the set of all subsets of $W$ that satisfies substitutability, i.e., for $S' \subseteq S \subseteq W$, we have $C_f(S) \cap S' \subseteq C_f(S')$. In addition, we assume that $C_f$ satisfies $C_f(S') = C_f(S)$ whenever $C_f(S) \subseteq S' \subseteq S \subseteq W$. This property is known in the literature as consistency. If $C_f$ satisfies substitutability and consistency, then it also satisfies

$$C_f(S \cup S') = C_f(C_f(S) \cup S') \quad (1)$$

for each pair of subsets $S$ and $S'$ of $W$.

Let $\succ_W$ be the preference profile for all workers, and let $C_F$ be the profile of choice functions for all firms. A many-to-one matching market is denoted by $(W, F, \succ_W, C_F)$.

**Definition 1** A matching $\mu$ is a function from set $F \cup W$ into $2^{FUW}$ such that, for each $w \in W$ and each $f \in F$:

(i) $\mu(w) \subseteq F$ with $|\mu(w)| \leq 1$.

(ii) $\mu(f) \subseteq W$.

(iii) $w \in \mu(f)$ if and only if $\mu(w) = \{f\}$.

Usually, we will omit the curly brackets. For instance, instead of condition (iii) we will write: “$w \in \mu(f)$ if and only if $\mu(w) = f$”.

\footnote{Substitutability is equivalent to the following: for each $w \in W$ and each $S \subseteq W$ such that $w \in S$, $w \in C_f(S)$ implies that $w \in C_f(S' \cup \{w\})$ for each $S' \subseteq S$.}

\footnote{This property is known in the literature as path independence (see Alkan, 2002).}
Agent $a \in F \cup W$ is matched if $\mu(a) \neq \emptyset$, otherwise $a$ is unmatched. A matching $\mu$ is blocked by a worker $w$ if $\emptyset \succ_w \mu(w)$; that is, worker $w$ prefers being unemployed rather than working for firm $\mu(w)$. Similarly, $\mu$ is blocked by a firm $f$ if $\mu(f) \neq C_f(\mu(f))$; that is, firm $f$ wants to fire some workers in $\mu(f)$. A matching is individually rational if it is not blocked by any individual agent.

A matching $\mu$ is blocked by a firm-worker pair $(f, w)$ if $w \in C_f(\mu(f) \cup \{w\})$, and $f \succ_w \mu(w)$; that is, if they are not matched through $\mu$, firm $f$ wants to hire $w$, and worker $w$ prefers firm $f$ rather than $\mu(w)$. A matching $\mu$ is stable if it is individually rational and it is not blocked by any firm-worker pair. The set of stable matchings for market $(W, F, \succ_w, C_F)$ is denoted by $\mathcal{S}$. A matching is worker-quasi-stable if it is individually rational and each firm-worker blocking pair $(f, w)$ satisfies that $\mu(w) = \emptyset$. Let $\mathcal{Q}$ denote the set of worker-quasi-stable matchings for market $(W, F, \succ_w, C_F)$. Notice that, for each market $(W, F, \succ_w, C_F)$, the set $\mathcal{Q}$ is always non-empty since the empty matching in which each agent is unmatched belongs to this set.

Blair (1988) defines a partial order over matchings in which a matching dominates another matching if each firm wishes to keep the workers hired under the first one, even if all the workers hired under the second one are also available, and do not wish to hire any new worker. Formally, given two sets of workers $S, T \in 2^W$, we write $S \succeq^B T$ when $S = C_f(S \cup T)$. We also write: $S \succ^B f T$ when $S \succeq^B f T$ and $S \neq T$. Furthermore, given two matchings $\mu$ and $\mu'$, we say that $\mu$ weakly Blair-dominates $\mu'$, and write $\mu \succeq^B \mu'$, when $\mu(f) \succeq^B f \mu'(f)$ for each $f \in F$. If $\mu \succeq^B \mu'$ and $\mu \neq \mu'$, we say that $\mu$ Blair-dominates $\mu'$ and write $\mu \succ^B \mu'$.

### 3 Lattice structure

In this section, we prove that the set of worker-quasi-stable matchings forms a lattice under the partial order $\succeq^B$. Formally,

**Theorem 1** The set of worker-quasi-stable matchings is a lattice under the partial order $\succeq^B$.

The rest of the section is devoted to proving this theorem. In order to do so, we need to construct the join of two worker-quasi-stable matchings. Given two worker-quasi-stable matchings $\mu$ and $\mu'$, we define a function $\lambda_{\mu, \mu'} : F \cup W \to 2^{F \cup W}$ as follows:

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9The notion of worker-quasi-stable matching in many-to-one models generalizes the notion of "simple" matching in one-to-one models studied by Sotomayor (1996). A one-to-one matching is simple if, in the case of a blocking pair $(f, w)$ exists, $\mu(w) = \emptyset$.

10Given a partially ordered set $(\mathcal{L}, \succeq)$, and two elements $x, y \in \mathcal{L}$, an element $z \in \mathcal{L}$ is an upper bound of $x$ and $y$ if $z \succeq x$ and $z \succeq y$. An element $w \in \mathcal{L}$ is the join of $x$ and $y$ if and only if (i) $w$ is an upper bound of $x$ and $y$, and (ii) $t \succeq w$ for each upper bound $t$ of $x$ and $y$. The definitions of lower bound and meet of $x$ and $y$ are dual and we omit them.
(i) for each \( f \in F \), \( \lambda_{\mu, \mu'}(f) = C_f (\mu(f) \cup \mu'(f)) \),

(ii) for each \( w \in W \), \( \lambda_{\mu, \mu'}(w) = \{ f \in F : w \in \lambda_{\mu, \mu'}(f) \} \).

Notice that by item (i), if \( \mu(f) = \mu'(f) = \emptyset \), then \( \lambda_{\mu, \mu'}(f) = \emptyset \). Under \( \lambda_{\mu, \mu'} \), (i) firms want to hire the best subset of workers among those hired by them in either matching, and (ii) workers agree with the firms that want to employ them. The following lemma shows that \( \lambda_{\mu, \mu'} \) is well-defined (i.e., it is a matching) and, furthermore, that it is worker-quasi-stable.

**Lemma 1** If \( \mu \) and \( \mu' \) are two worker-quasi-stable matchings, then \( \lambda_{\mu, \mu'} \) is a worker-quasi-stable matching.

**Proof.** Let \( \mu, \mu' \in \mathcal{Q} \). First, we show \( \lambda_{\mu, \mu'} \) is a matching. By definition of \( \lambda_{\mu, \mu'} \) we have \( \lambda_{\mu, \mu'}(w) \subseteq F \) for each \( w \in W \) and \( \lambda_{\mu, \mu'}(f) \subseteq W \) for each \( f \in F \). For \( \lambda_{\mu, \mu'} \) to be matching, it is necessary to show that \( \lambda_{\mu, \mu'}(w) \leq 1 \) for each \( w \in W \). Assume that there is \( w \in W \) such that \( |\lambda_{\mu, \mu'}(w)| > 1 \). Thus, there are \( f, f' \in F \) with \( f \neq f' \) such that \( w \in \lambda_{\mu, \mu'}(f) \) and \( w \in \lambda_{\mu, \mu'}(f') \). Given that \( \mu \) and \( \mu' \) are matchings, w.l.o.g., assume that \( w \in \mu(f) \) and \( w \in \mu'(f') \). Thus, \( \mu(w) = f \) and \( \mu'(w) = f' \). Since \( w \in \lambda_{\mu, \mu'}(f) = C_f (\mu(f) \cup \mu'(f)) \) then, by substitutability, \( w \in C_f (\mu(f) \cup \{w\}) \). As \( \mu' \in \mathcal{Q} \) and \( \mu'(w) = f' \), then \( (f, w) \) is not a blocking pair for \( \mu' \). Therefore,

\[
\mu'(w) = f' \succ_w f.
\]

By analogous reasoning, \( w \in \lambda_{\mu, \mu'}(f') \) implies

\[
\mu(w) = f \succ_w f'.
\]

By (2) and (3) we get a contradiction. Therefore, \( |\lambda_{\mu, \mu'}(w)| \leq 1 \) and \( \lambda_{\mu, \mu'} \) is a matching.

Second, we show that \( \lambda_{\mu, \mu'} \) is an individually rational matching. By (1), for any \( f \in F \) and \( S \subseteq W \), \( C_f (C_f (S)) = C_f (S) \). Thus, \( C_f (\lambda_{\mu, \mu'} (f)) = C_f (\mu(f) \cup \mu'(f)) = \lambda_{\mu, \mu'} (f) \), and \( \lambda_{\mu, \mu'} \) is not blocked by any firm. By definition of \( \lambda_{\mu, \mu'} \), \( w \in \mu(f) \) or \( w \in \mu'(f) \). Thus, \( f = \mu(w) \) or \( f = \mu'(w) \). Since \( \mu \) and \( \mu' \) are individually rational matchings, \( \mu(w) \succ_w \emptyset \) and \( \mu'(w) \succ_w \emptyset \). Therefore, \( f \succ_w \emptyset \) and \( \lambda_{\mu, \mu'} \) is not blocked by any worker. This implies that \( \lambda_{\mu, \mu'} \) is an individually rational matching.

Finally, we show that \( \lambda_{\mu, \mu'} \) is worker-quasi-stable. Assume that \( \lambda_{\mu, \mu'} \) is not a worker-quasi-stable matching. Then, there is a blocking pair \((f, w)\) for \( \lambda_{\mu, \mu'} \) and \( \lambda_{\mu, \mu'}(w) \neq \emptyset \). Assume, w.l.o.g., that \( \lambda_{\mu, \mu'}(w) = \mu(w) \). Recall that, since \((f, w)\) is a blocking pair for \( \lambda_{\mu, \mu'} \), \( w \in C_f (\lambda_{\mu, \mu'}(f) \cup \{w\}) \), and \( f \succ_w \lambda_{\mu, \mu'}(w) \). By definition of \( \lambda_{\mu, \mu'} \),

\[
w \in C_f (C_f (\mu(f) \cup \mu'(f)) \cup \{w\}).
\]
By (1), \( w \in C_f(\mu(f) \cup \mu'(f) \cup \{w\}) \). Since firms have substitutable choice functions, \( w \in C_f(\mu(f) \cup \{w\}) \). Recall that \( f \succ_w \lambda_{\mu,\mu'}(w) = \mu(w) \). Therefore, \((f, w)\) is also a blocking pair for \( \mu \). Since we assumed that \( \mu(w) = \lambda_{\mu,\mu'}(w) \neq \emptyset \), we contradict the fact that \( \mu \) is, by hypothesis, a worker-quasi-stable matching. 

Next, we show that \( \lambda_{\mu,\mu'} \) is indeed the join of \( \mu \) and \( \mu' \) according to \( \succeq^B \). Let \( \mu, \mu' \in Q \) and \( f \in F \).

**Lemma 2** If \( \mu \) and \( \mu' \) are two worker-quasi-stable matchings, then \( \lambda_{\mu,\mu'} \) is the join of \( \mu \) and \( \mu' \).

**Proof.** Let \( \mu, \mu' \in Q \). We know by Lemma 1, that \( \lambda_{\mu,\mu'} \in Q \). First, we prove that \( \lambda_{\mu,\mu'} \) is an upper bound of \( \mu \) and \( \mu' \). By definition of \( \lambda_{\mu,\mu'} \) and (1), for each \( f \in F \),

\[
C_f(\lambda_{\mu,\mu'}(f) \cup \mu(f)) = C_f(C_f(\mu(f) \cup \mu'(f)) \cup \mu(f))
\]

\[
= C_f(\mu(f) \cup \mu'(f) \cup \mu(f)) = C_f(\mu(f) \cup \mu'(f)) = \lambda_{\mu,\mu'}(f).
\]

This implies that \( \lambda_{\mu,\mu'}(f) \succeq^B f \mu(f) \) for each \( f \in F \) and then \( \lambda_{\mu,\mu'} \succeq^B f \mu \). Similarly, \( \lambda_{\mu,\mu'} \succeq^B f \mu' \). Therefore, \( \lambda_{\mu,\mu'} \) is an upper bound of \( \mu \) and \( \mu' \). Second, we prove that \( \lambda_{\mu,\mu'} \) is the join of \( \mu \) and \( \mu' \). Let \( v \in Q \) be such that \( v \succeq^B f \mu \) and \( v \succeq^B f \mu' \). That is,

\[
v(f) = C_f(v(f) \cup \mu(f)) \quad \text{and} \quad v(f) = C_f(v(f) \cup \mu'(f)) \quad (4)
\]

for each \( f \in F \). We need to show that \( v \succeq^B \lambda_{\mu,\mu'} \), that is, \( v(f) = C_f(v(f) \cup \lambda_{\mu,\mu'}(f)) \) for each \( f \in F \). Thus, using repeatedly (1) and (4), and the definition of \( \lambda_{\mu,\mu'} \),

\[
v(f) = C_f(v(f) \cup \mu(f)) = C_f(C_f(v(f) \cup \mu'(f)) \cup \mu(f))
\]

\[
= C_f(v(f) \cup \mu'(f) \cup \mu(f)) = C_f(v(f) \cup C_f(\mu'(f) \cup \mu(f))) = C_f(v(f) \cup \lambda_{\mu,\mu'}(f))
\]

for each \( f \in F \). Thus, \( v \succeq^B \lambda_{\mu,\mu'} \). Therefore, \( \lambda_{\mu,\mu'} \) is the join for \( \mu \) and \( \mu' \). \( \square \)

From now on, given two worker-quasi-stable matchings \( \mu \) and \( \mu' \), we denote \( \lambda_{\mu,\mu'} \) as \( \mu \lor \mu' \).

To finish the proof of Theorem 1, we make three observations. First, by Lemma 2 the set of worker-quasi-stable matchings \( Q \) forms a join-semilattice under the partial order \( \succeq^B \). Second, the empty matching \( \mu_{\emptyset} \) in which all workers are unmatched (that is by definition a worker-quasi-stable matching) is the minimum element of \( Q \) under the partial order \( \succeq^B \). To see this, let \( \mu \in Q \). Since \( \mu(f) = C_f(\mu(f) \cup \emptyset) \) for each \( f \in F \), it follows that \( \mu = \mu \lor \mu_{\emptyset} \). Thus, \( \mu \succeq^B f \mu_{\emptyset} \) for each \( \mu \in Q \). Finally, given that the

\[\text{[11]}\] A partially ordered set \( L \) is called a join-semilattice if any two elements in \( L \) have a join. If any two elements in \( L \) also have a meet, then \( L \) is called a lattice (see Stanley, 1986, for more details).
set of worker-quasi-stable matchings is finite and is a join-semilattice with a minimum element, it follows that the set of worker-quasi-stable matchings forms a lattice under the partial order $\succeq^B$ (see Stanley, 1986, for more details). This completes the proof of Theorem 1.

The following example illustrates the lattice structure of the set of worker-quasi-stable matchings.

Example 1 Let $(W, F, \succ_w, C_F)$ be a matching market where $W = \{w_1, w_2, w_3, w_4\}$, $F = \{f_1, f_2\}$, the preference for the workers are given by:

\begin{align*}
\succ_{w_i}: & \{f_1\}, \{f_2\}, \emptyset \quad \text{for } i = 1, 2 \\
\succ_{w_3}: & \{f_2\}, \{f_1\}, \emptyset \\
\succ_{w_4}: & \{f_2\}, \emptyset,
\end{align*}

and the choice functions of the firms are given in Table 1.

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $C_{f_1}$ | 3 | 3 | 3 | 3 |
| $C_{f_2}$ | 12 | 12 | 24 | 13 |

Table 1: Choice functions of the firms

For example, $C_{f_1}(\{w_1, w_2, w_3, w_4\}) = \{w_3\}$. We denote each worker-quasi-stable matching as an ordered pair, in which the first component consists of the workers hired by $f_1$, and the second of the workers hired by $f_2$. In Figure 1, the lattice of all worker-quasi-stable matchings is presented. There are nineteen worker-quasi-stable matchings, two of which are stable: $(w_1w_2, w_3w_4)$ and $(w_3, w_1w_2)$. Now, we illustrate how to compute the join of two worker-quasi-stable matchings. Take, for instance, $\mu' = (w_3, w_2)$ and $\overline{\mu} = (w_1w_2, w_3w_4)$. Observe that

\[ C_{f_1}(\mu' \cup \overline{\mu}(f_1)) = C_{f_1}(\{w_3\} \cup \{w_1, w_2\}) = \{w_3\} \]

and

\[ C_{f_2}(\mu' \cup \overline{\mu}(f_2)) = C_{f_2}(\{w_2\} \cup \{w_3, w_4\}) = \{w_2, w_4\}. \]

Then, $\mu' \cup \overline{\mu} = (w_3, w_2w_4)$.

To finish this section, we compare our approach to construct a join with the one of Wu and Roth (2018). In their Example 5.1, for two worker-quasi-stable-matchings, the authors show the difficulties in defining a Conway-like join for firms even when firms have responsive preferences.\footnote{For a firm $f \in F$, preference $\succ_f$ is \textbf{responsive} if there is a quota $q_f$ such that, for each $S \subseteq W$:

(i) If $|S| > q_f$, we have $\emptyset \succ_f S$.

(ii) If $|S| < q_f$ and $w \notin S$, we have $S \cup \{w\} \succ_f S$ if and only if $w \succ_f \emptyset$.} In their example, there is only one firm $f$ with
quota $q_f = 3$ and the set of workers is $\{w_1, w_2, w_3, w_4\}$. All workers rank $f$ above $\emptyset$ and the ranking of single agents of $f$ is $w_1 \succ_f w_2 \succ_f w_3 \succ_f w_4 \succ_f \emptyset$. Now, take the worker-quasi-stable matchings $\mu = (w_1w_4)$ and $\mu' = (w_2w_3)$. Notice that the sets of agents assigned to $f$ in these two matchings are not comparable in a responsive manner, therefore the Conway-like join cannot be defined. Here in this paper, we are able to sidestep this problem following Blair’s insight. Now, we show how our results apply to their example. According to our approach, we can compute the join of $\mu$ and $\mu'$ as follows:

$$\mu \vee \mu'(f) = C_f(\mu(f) \cup \mu'(f)) = C_f(\{w_1, w_2\} \cup \{w_2, w_3\}) = \{w_1, w_2, w_3\}.$$ 

Therefore, the resulting worker-quasi-stable matching is $\mu \vee \mu' = (w_1w_2w_3)$. Further-

(iii) If $|S| < q_f$ and $w, w' \notin S$, we have $S \cup \{w\} \succ_f S \cup \{w'\}$ if and only if $w \succ_f w'$. 

more, this matching is stable.

4 Re-equilibration process via a Tarski operator

As we mentioned in the introduction, worker-quasi-stable matchings appear naturally in the short run when firms decide to downsize or new workers become available. In this section, we define our Tarski operator in the worker-quasi-stable lattice that describes a possible re-equilibration process. This process models how, starting from a worker-quasi-stable matching, a decentralized sequence of offers in which unemployed workers are hired and cause new unemployments, produces a sequence of worker-quasi-stable matchings that converges to a stable matching. In the first subsection, we present the operator, show some of its properties, and prove that the set of its fixed points is the set of stable matchings. In the second subsection, we discuss the re-equilibration process, based on our Tarski operator, that models a layoff chain that leads towards a stable matching.

4.1 A Tarski operator for worker-quasi-stable matchings

First, for each \( \mu \in Q \), define the following sets:

\[
B^\mu = \{(f, w) \in F \times W : (f, w) \text{ blocks } \mu\},
\]

\[
B^\mu_\star = \{(f, w) \in B^\mu : f \succ_w f' \text{ for each } (f', w) \in B^\mu \setminus \{(f, w)\}\}
\]

and, for each \( f \in F \),

\[
W_f^\mu = \{w \in W : (f, w) \in B^\mu_\star\}.
\]

The set \( B^\mu \) collects all possible blocking pairs for \( \mu \) (since \( \mu \in Q \), \( (f, w) \in B^\mu \) implies \( \mu(w) = \emptyset \)). Then, each blocking pair \( (f, w) \) is included in \( B^\mu_\star \) if \( f \) is \( w \)'s most preferred firm with which \( w \) forms a blocking pair. Lastly, for each \( f \in F \), \( W_f^\mu \) gathers all workers that partner with \( f \) in \( B^\mu_\star \). Notice that, if a firm \( f \) is not involved in any blocking pair in \( B^\mu_\star \), then \( W_f^\mu = \emptyset \). Now, for each \( \mu \in Q \), our Tarski operator \( T \) maps \( Q \) into the set of matchings and is defined as follows:

(i) for each \( f \in F \),

\[ T(\mu)(f) = C_f \left( \mu(f) \cup W_f^\mu \right) \]

(ii) for each \( w \in W \), if there is \( f \in F \) such that \( w \in T(\mu)(f) \) then \( T(\mu)(w) = f \). Otherwise, \( T(\mu)(w) = \emptyset \).

Remark 1 If \( w \in W_f^\mu \), (i) \( \mu(w) = \emptyset \) and, (ii) by definition of \( B^\mu_\star \), \( w \notin W_{f'}^\mu \) for each \( f' \in F \setminus \{f\} \).
The following lemma shows that the operator $T$ is well defined: it assigns to a worker-quasi-stable matching a matching.

**Lemma 3** For any worker-quasi-stable matching $\mu$, $T(\mu)$ is a matching.

**Proof.** Let $\mu$ be a worker-quasi-stable matching. By definition of $T(\mu)$, $T(\mu)(f) \subseteq W$ for each $f \in F$ and $T(\mu)(w) \subseteq F$ for each $w \in W$. To show that $T(\mu)$ is a matching, it only remains to be seen that $|\{f \in F : T(\mu)(w) = f\}| \leq 1$ for each $w \in W$. Thus, assume that there are $w \in W$ and distinct $f$ and $f'$ in $F$ such that $w \in T(\mu)(f)$ and $w \in T(\mu)(f')$. By definition of $T(\mu)$ and choice function,

$$w \in C_f \left( \mu(f) \cup W^\mu_f \right) \subseteq \mu(f) \cup W^\mu_f. \quad (5)$$

Likewise,

$$w \in C_{f'} \left( \mu(f') \cup W^\mu_{f'} \right) \subseteq \mu(f') \cup W^\mu_{f'}. \quad (6)$$

By (5), there are two cases to consider:

1. $w \in \mu(f)$. Since $\mu$ is a matching, $w \notin \mu(f')$. By (6), $w \in W^\mu_{f'}$ and, therefore, by Remark 1 (i) $\mu(w) = \emptyset$. This contradicts that $w \in \mu(f)$.

2. $w \in W^\mu_{f'}$. By Remark 1 (ii) $w \notin W^\mu_{f'}$. By (6), $w \in \mu(f')$. Thus, $\mu(w) \neq \emptyset$. This contradicts Remark 1 (i).

$\square$

The following theorem states that the matching obtained by applying our operator to a worker-quasi-stable matching is: (i) worker-quasi-stable, (ii) weakly Blair-preferred by firms to the original matching, and (iii) identical to the original matching if and only if the original matching is stable.

**Theorem 2** For any worker-quasi-stable matching $\mu$, the following hold:

(i) $T(\mu)$ is a worker-quasi-stable matching,

(ii) $T(\mu) \succeq^B \mu$,

(iii) $T(\mu) = \mu$ if and only if $\mu$ is stable.

**Proof.** Let $\mu \in Q$.

(i) **$T(\mu)$ is a worker-quasi-stable matching.** Let $(f, w)$ be a blocking pair of $T(\mu)$. We want to see that $T(\mu)(w) = \emptyset$. Assume there is $f' \in F$ such that $T(\mu)(w) = f'$. Then, by definition of $T(\mu)(f')$, either $w \in \mu(f')$ or $w \in W^\mu_{f'}$. 

1. \( w \in \mu(f') \). Since \((f, w)\) blocks \(T(\mu), w \in C_f(T(\mu)(f) \cup \{w\})\). By definition of \(T(\mu)\) and (1), \( w \in C_f(\mu(f) \cup W_f^\mu \cup \{w\})\). By substitutability, \( w \in C_f(\mu(f) \cup \{w\})\). Also since \((f, w)\) blocks \(T(\mu), f \not\succeq_w T(\mu)(w) = \mu(w)\). Thus \((f, w)\) blocks \(T(\mu)(w) = \mu(w)\), contradicting that \(w \in Q\).

2. \( w \in W_f^\mu \). Thus, \((f', w) \in B^\mu_x\) and this implies that there is no firm \(f'' \in F \setminus \{f'\}\) such that \(f'' \not\succeq_w f'\) and \((f'', w)\) blocks \(T(\mu)\). In particular, when \(f'' = f\), \((f, w)\) cannot be a blocking pair of \(T(\mu)\), contradicting our assumption.

Hence, \(T(\mu)(w) = \emptyset\) and, therefore, \(T(\mu)\) is a worker-quasi-stable matching.

(ii) \(T(\mu) \succeq^B \mu\). By definition of \(T(\mu)\) and (1), for each \(f \in F\),

\[
\begin{align*}
C_f(T(\mu)(f) \cup \mu(f)) &= C_f\left( C_f\left( \mu(f) \cup W_f^\mu \cup \mu(f) \right) \right) \\
&= C_f\left( \mu(f) \cup W_f^\mu \cup \mu(f) \right) = C_f(\mu(f) \cup W_f^\mu) = T(\mu)(f).
\end{align*}
\]

Therefore, \(C_f(\mu(f) \cup W_f^\mu) = T(\mu)(f)\) for each \(f \in F\), as desired.

(iii) \(T(\mu) = \mu\) if and only if \(\mu\) is stable. Assume \(\mu \in Q \setminus S\). Thus, \(B^\mu_x \neq \emptyset\) and, therefore, there is \(f \in F\) such that \(W_f^\mu \neq \emptyset\). This implies that \(T(\mu)(f) = C_f(\mu(f) \cup W_f^\mu) \neq \mu(f)\). Hence, \(T(\mu) \neq \mu\).

Assume that \(\mu \in S\). Thus, \(B^\mu_x = B^\mu_x = \emptyset\). By definition of \(T\), \(T(\mu)(f) = \mu(f)\) for each \(f \in F\) and therefore \(T(\mu) = \mu\).

Notice that Theorem 2 implies that \(T\) is a Pareto improving operator for the firms by definition of the choice function. Now, we prove that our operator \(T\) is isotone. Recall that, for a lattice \((\mathcal{L}, \geq)\), a function \(T: \mathcal{L} \rightarrow \mathcal{L}\) is isotone if for each \(x, y \in \mathcal{L}, x \geq y\) implies \(T(x) \geq T(y)\).

**Lemma 4** If \(\mu\) and \(\mu'\) are worker-quasi-stable matchings such that \(\mu \succeq^B \mu'\), then \(T(\mu) \succeq^B T(\mu')\).

**Proof.** Let \(\mu, \mu' \in Q\) be such that \(\mu \succeq^B \mu'\) and assume that \(T(\mu) \succeq^B T(\mu')\) does not hold. This implies the existence of \(f \in F\) such that

\[
T(\mu)(f) \neq C_f(\mu(f) \cup W_f^\mu) = C_f\left( C_f\left( \mu(f) \cup W_f^\mu \right) \right).
\]

Using the definition of \(T\) and (1) twice, it follows that

\[
C_f\left( C_f\left( \mu(f) \cup W_f^\mu \right) \right) = C_f\left( C_f\left( \mu'(f) \cup W_f^\mu \right) \right).
\]
\[
\begin{align*}
= C_f \left( \mu(f) \cup W_f^\mu \cup C_f \left( \mu'(f) \cup W_f^\mu' \right) \right) &= C_f \left( (\mu(f) \cup W_f^\mu \cup \mu'(f) \cup W_f^\mu') \right) \\
&= C_f \left( (\mu(f) \cup \mu'(f) \cup W_f^\mu \cup W_f^\mu' \right).
\end{align*}
\]

Using again (1), it follows that
\[
C_f \left( (\mu(f) \cup \mu'(f)) \cup W_f^\mu \cup W_f^\mu' \right) = C_f \left( (\mu(f) \cup \mu'(f)) \cup W_f^\mu \cup W_f^\mu' \right)
\]
and, as by hypothesis \( C_f(\mu(f) \cup \mu'(f)) = \mu(f) \), using (8) and (9) we get
\[
C_f \left( T(\mu)(f) \cup T(\mu')(f) \right) = C_f \left( (\mu(f) \cup W_f^\mu \cup W_f^\mu' \right).
\]

Now, using the definition of \( T \) and (10), (7) becomes
\[
C_f \left( \mu(f) \cup W_f^\mu \right) \neq C_f \left( \mu(f) \cup W_f^\mu \cup W_f^\mu' \right).
\]

By (11), there is \( w \in W \) such that
\[
w \in C_f \left( \mu(f) \cup W_f^\mu \cup W_f^\mu' \right)
\]
and
\[
w \in W_f^\mu \setminus \left( \mu(f) \cup W_f^\mu \right).
\]

Notice that \( w \in W_f^\mu' \) implies \( \mu'(w) = \emptyset \). There are two cases to consider:

1. \( \mu(w) = \emptyset \). Since \( w \in W_f^\mu \), \( (f, w) \in B_f^\mu \) so \( f \gg_w \mu'(w) = \emptyset \). By (12) and substitutability, \( w \in C_f(\mu(f) \cup \{w\}) \), so \( (f, w) \in B^\mu \). By (13), \( w \notin W_f^\mu \) and hence there is \( f' \in F \) such that \( (f', w) \in B^\mu \) and \( f' \gg_w f \). Then,
\[
w \in C_f' \left( \mu(f') \cup \{w\} \right).
\]

By hypothesis, \( \mu(f') \succeq_{p'} \mu'(f') \) and therefore \( \mu(f') = C_{f'} (\mu(f') \cup \mu'(f')) \). Thus, using (1) we can rewrite (14) as
\[
w \in C_f' \left( \mu(f') \cup \mu'(f') \cup \{w\} \right).
\]

Hence, substitutability and (15) imply \( w \in C_{f'} (\mu'(f') \cup \{w\}) \) and therefore \( (f', w) \in B^\mu \). But \( f' \gg_w f \) contradicts the fact that \( w \in W_f^\mu' \).

2. There is \( f' \in F \) such that \( \mu(w) = f' \). By (13), \( f' \neq f \). Notice that, as \( \mu \succeq_B \mu' \), \( w \in \mu(f') = C_{f'}(\mu(f')) = C_{f'}(\mu(f') \cup \mu'(f')) \) implies, by substitutability, that \( w \in C_{f'}(\mu'(f') \cup \{w\}) \). By (13), \( w \in W_f^\mu' \) and thus \( (f, w) \in B^\mu \). If \( f' \gg_w f \), then \( (f', w) \in B^\mu \) but this contradicts the fact that \( (f, w) \in B^\mu \). Therefore,
\[
f \gg_w f' = \mu(w) \gg_w \emptyset.
\]
By (12) and substitutability, we have that \( w \in C_f (\mu f) \cup \{ w \} \). Then, (16) implies that \((f, w) \in B^\mu \), contradicting the fact that \( \mu \) is worker-quasi-stable since \( \mu (w) = f' \neq \emptyset \).

Since in each case we reach a contradiction, it follows that

\[
T(\mu)(f) = C_f (T(\mu)(f) \cup T(\mu')(f))
\]

for each \( f \in F \), which in turn implies \( T(\mu) \succeq^B T(\mu') \).

Starting from any \( \mu \in Q \), the following example illustrates the construction of the
sets \( B^\mu, B^*_\mu \), and \( W^\mu_{f_1} \) for each \( f \in F \), in order to compute our operator \( T \). Moreover, applying our operator \( T \) (sometimes more than once) we obtain the set of fixed points of the lattice of the set of worker-quasi-stable matchings. In this particular example, the lattice of fixed points consists of the unique stable matching of the market.

**Example 2** Let \((W, F, \succ_w, C_f)\) be a matching market where \( W = \{ w_1, w_2, w_3 \} \), \( F = \{ f_1, f_2 \} \), and the preferences for the workers are is given by:

\[
\succ_w: \{ f_1 \}, \{ f_2 \}, \emptyset \text{ for } i = 1, 2, 3.
\]

Moreover, the choice functions of the firms are given in Table 2. In Figure 2, we present the

|     | 123 | 12 | 13 | 23 | 1 | 2 | 3 |
|-----|-----|----|----|----|---|---|---|
| \( C_{f_1} \) | 12 | 12 | 13 | 23 | 1 | 2 | 3 |
| \( C_{f_2} \) | 3  | \emptyset | 3 | 3 | \emptyset | \emptyset | 3 |

Table 2: Choice functions of the firms

lattice of the set of worker-quasi-stable matchings. There are eight worker-quasi-stable matchings, one of which is stable: \( \bar{\mu} = (w_1 w_2, w_3) \). Now, we show how our operator \( T \) works. Start, for instance, with \( \mu' = (w_2 w_3, \emptyset) \). First, we construct the sets \( B^{\mu'}, B^{\mu'}_* \), \( W_{f_1}^{\mu'} \) and \( W_{f_2}^{\mu'} \) and compute \( T(\mu') \):

\[
B^{\mu'} = B^{\mu'}_* = \{ (f_1, w_1) \}, \quad W_{f_1}^{\mu'} = \{ w_1 \} \text{ and } W_{f_2}^{\mu'} = \emptyset,
\]

\[
T(\mu')(f_1) = C_{f_1} (\mu'(f_1) \cup W_{f_1}^{\mu'}) = C_{f_1} (\{ w_2, w_3 \} \cup \{ w_1 \}) = \{ w_1, w_2 \},
\]

\[
T(\mu')(f_2) = C_{f_2} (\mu'(f_2) \cup W_{f_2}^{\mu'}) = C_{f_2} (\emptyset \cup \emptyset) = \emptyset.
\]

Therefore, \( T(\mu') = \bar{\mu} = (w_1 w_2, \emptyset) \). Notice that matching \( \bar{\mu} \) is not stable, since \((f_2, w_3)\) is a blocking pair. Second, we construct the sets \( B^{\bar{\mu}}, B^{\bar{\mu}}_* \), \( W_{f_1}^{\bar{\mu}} \) and \( W_{f_2}^{\bar{\mu}} \) and compute \( T^2(\mu') \equiv T(\bar{\mu}) \):

\[
B^{\bar{\mu}} = B^{\bar{\mu}}_* = \{ (f_2, w_3) \}, \quad W_{f_1}^{\bar{\mu}} = \emptyset \text{ and } W_{f_2}^{\bar{\mu}} = \{ w_3 \}.
\]
Figure 2: The lattice of Example 2.

\[ T(\tilde{\mu})(f_1) = C_{f_1} \left( (\tilde{\mu}(f_1) \cup W_{f_1}^{\tilde{\mu}}) \right) = C_{f_1} (\{w_1, w_2\} \cup \emptyset) = \{w_1, w_2\}, \]

\[ T(\tilde{\mu})(f_2) = C_{f_2} \left( (\tilde{\mu}(f_2) \cup W_{f_2}^{\tilde{\mu}}) \right) = C_{f_2} (\emptyset \cup \{w_3\}) = \{w_3\}. \]

Therefore, \( T(\tilde{\mu}) = (w_1, w_2, w_3) = \tilde{\pi}. \) It can be shown that, for each \( \mu \in Q \) such that \( \tilde{\mu} \succeq^{B} \mu, \) \( T(\mu) = \tilde{\mu} \) so \( T^2(\mu) = \tilde{\pi}. \) Furthermore, we can observe that the fixed point of our operator \( T \) is the stable matching \( \tilde{\pi}. \) \( \diamond \)

Note that in Example 2, the set of fixed points of operator \( T \) consists of the unique stable matching of the market and we cannot observe its lattice structure. However, in Example 1, it can be shown that the set of fixed points of \( T \) is the set of stable matchings \( \{(w_1, w_2, w_3, w_4), (w_3, w_1, w_2)\} \) that forms a lattice (see Figure 1). It should be clear from the previous example that an important global property of operator \( T \) is that starting from any worker-quasi-stable matching we always reach stability in a finite numbers of iterations.

Now we are in a position to give an alternative proof of the existence of stable matchings and their lattice structure. To this end, we apply Tarski’s fixed point theorem to the lattice of worker-quasi-stable matchings. Remember that Tarski’s theorem (Tarski, 1955) states that if \( (\mathcal{L}, \succeq) \) is a complete lattice and \( T : \mathcal{L} \rightarrow \mathcal{L} \) is isotone, then the set of fixed points of \( T \) is non-empty and forms a complete lattice with respect to \( \succeq. \)
**Theorem 3** The set of stable matchings is non-empty and forms a lattice with respect to $\succeq^B$.

**Proof.** Let us check that operator $T$ verifies the hypothesis of Tarski’s theorem. First, notice that the lattice of worker-quasi-stable matchings is finite and therefore complete. Second, by Theorem 2 (i), $T$ maps the lattice of worker-quasi-stable matchings to itself. Finally, $T$ is isotone by Lemma 4. Then, by Tarski’s theorem, the set of fixed points of $T$ is non-empty and forms a lattice under $\succeq^B$. Moreover, by Theorem 2 (iii), the set of fixed points of our operator $T$ is the set of stable matchings. □

Another non-constructive argument for proving the existence of stable matchings can be provided following the lines of Sotomayor (1996). By using the notion of simple matchings in a one-to-one model, that paper shows the existence of stable matchings. Since a worker-quasi-stable matching is a generalization of a simple matching, our operator $T$ provides a similar proof in a many-to-one model. To see this, notice that as the set $\mathcal{Q}$ is non-empty and finite, there is a maximal element $\mu$ for the partial order $\succeq^B$. We want to see that $\mu$ is a stable matching. If $\mu$ is not stable, then there is a blocking pair $(f, w)$ for $\mu$ in which $\mu(w) = \emptyset$. Thus, $B^\mu_{\star} \neq \emptyset$ and $T(\mu) \neq \mu$. Since $T$ is a Pareto improving operator for the firms, $T(\mu) \succ^B \mu$, contradicting the maximality of $\mu$. Therefore, $\mu$ is a stable matching.

### 4.2 A re-equilibration process via layoff chains

In a labor market, sometimes new workers arrive or firms downsize and laid off workers have to look for employment elsewhere. If we start from a stable matching and some of the previous situations are considered, we can model this disruption to stability as a worker-quasi-stable matching. In order to restore stability, a re-equilibration process can be described as a layoff chain dynamic within worker-quasi-stable matchings that brings the market back to stability.\(^{13}\) Each stage of this process can be modeled by applying our operator $T$ to a worker-quasi-stable matching $\mu$, as follows:

(i) When new desirable workers become available in the market (unemployed workers in $\mu$), they propose to the most preferred firm among those they can form a blocking pair with ($B^\mu_{\star}$ in Subsection 4.1).

(ii) Then, each firm $f$ selects the most preferred subset of workers among those who just proposed to it ($W^\mu_f$ in Subsection 4.1) and its current employees.

(iii) Once the firms select their new sets of employees, a new set of unemployed workers becomes available for new proposals (unemployed workers in $T(\mu)$).

---

\(^{13}\)The notion of layoff chain is the counterpart for the workers’ side to the notion of vacancy chain for the firms’ side studied by Blum et al. (1997) and Wu and Roth (2018).
Notice that, by Theorem 2 (i) the sequence of matchings generated by this process belongs to the set of worker-quasi-stable matchings. Moreover, each matching in the sequence Pareto improves (for the firms) upon the previous matching in the sequence. Therefore, by the finiteness of the set of worker-quasi-stable matchings, this process reaches a fixed point of $T$, which is a stable matching by Theorem 2 (iii).

Thus, when a stable matching becomes a worker-quasi-stable matching due to market changes, the aforementioned process models how a decentralized sequence of offers (in which unemployed workers are hired, causing new unemployments) produces a sequence of worker-quasi-stable matchings that converges to a stable matching.

5 Further results with the Law of Aggregate Demand

In this section, by requiring an additional condition on firms’ choice functions, we can describe more accurately the re-equilibration process by means of the lattice structure of the set of worker-quasi-stable matchings. This additional condition is the “law of aggregate demand”, that says that when a firm chooses from an expanded set, it hires at least as many workers as before. Formally,

**Definition 2** Choice function $C_f$ satisfies the law of aggregate demand (LAD) if $S' \subseteq S \subseteq W$ implies $|C_f(S')| \leq |C_f(S)|$.

We know that, starting from a worker-quasi-stable matching and iterating our operator $T$, we reach a fixed point of $T$. Assuming LAD, the lattice structure can help us to identify this fixed point: it is the join of the original worker-quasi-stable matching and the worker-optimal stable matching $\mu_w$.\(^1\) To formally present this result, for $\mu \in \mathcal{Q}$, let $F(\mu)$ denote the fixed point of $T$ obtained by iterating it starting at $\mu$.

**Theorem 4** Let $\mu$ be a worker-quasi-stable matching. If firms’ choice functions satisfy LAD, then $F(\mu) = \mu \lor \mu_w$.

In order to prove Theorem 4, we first need to show that the join of a worker-quasi-stable matching and a stable matching is a stable matching (Proposition 1). However, this is not true without LAD. In Example 1, where $\mu_w = \overline{\mu}$, we can observe that for $\mu' = (w_3, w_2)$ and $\overline{\mu} = (w_1w_2, w_3w_4)$, $\mu' \lor \overline{\mu} = (w_3, w_2w_4)$ which is not stable since $(f_2, w_1)$ is a blocking pair for $\mu' \lor \overline{\mu}$. In this example firm $f_1$’s choice function, although substitutable, does not satisfy LAD. To see this, observe that $C_{f_1}(\{w_1, w_2\}) = \{w_1, w_2\}$.

\(^1\)The set of stable matchings under substitutable choice functions is very well-structured. It contains two distinctive matchings: the firm-optimal stable matching $\mu_F$ and the worker-optimal stable matching $\mu_W$. The matching $\mu_W$ is unanimously considered by all workers to be the best among all stable matchings and by all firms to be the pessimal stable matching (see Roth, 1984; Blair, 1988, for more details).
whereas $C_{f_1}\left(\{w_1, w_2, w_3\}\right) = \{w_3\}$. Nevertheless, if we restrict firms’ choice functions to satisfy substitutability and LAD, we can recover this result.

**Proposition 1** Let $\mu$ be a worker-quasi-stable matching, and $\mu'$ be a stable matching. If firms’ choice functions satisfy LAD, then $\mu \lor \mu'$ is a stable matching.

**Proof.** Let $\mu$ be a worker-quasi-stable matching and $\mu'$ be a stable matching. Assume that $\mu \lor \mu'$ is not stable. Thus there is a blocking pair $(f, w)$ such that $f \succeq_w \mu \lor \mu'(w)$ and $w \in C_f\left(\mu \lor \mu'(f) \cup \{w\}\right)$. Since $\mu \lor \mu'$ is a worker-quasi-stable matching, $\mu \lor \mu'(w) = \emptyset$. By (1), $C_f\left(\mu \lor \mu'(f) \cup \{w\}\right) = C_f\left(\mu(f) \lor \mu'(f) \cup \{w\}\right)$. Then, by substitutability, $w \in C_f\left(\mu \lor \mu'(f) \cup \{w\}\right)$ implies

$$w \in C_f\left(\mu'(f) \cup \{w\}\right).$$  \hspace{1cm} (17)

If $f \succeq_w \mu'(w)$, the pair $(f, w)$ blocks $\mu'$, contradicting the stability of $\mu'$. Therefore, $\mu'(w) \succeq_w f$. Since $f \succeq_w \mu \lor \mu'(w) = \emptyset$, the individual rationality of $\mu'$ implies that there is $\tilde{f} \in F$ (possibly $\tilde{f} = f$) such that $\tilde{f} = \mu'(w)$. Hence, $w \in \mu'\left(\tilde{f}\right)$. Then,

$$w \in \mu'(F),$$  \hspace{1cm} (18)

where $\mu'(F) = \bigcup_{g \in F} \mu'(g)$. Also, since $\mu \lor \mu'(w) = \emptyset$, by the definition of $\mu \lor \mu'$,

$$w \notin \mu \lor \mu'(F).$$  \hspace{1cm} (19)

As $\mu'(g) \subseteq \mu(g) \lor \mu'(g)$ for each $g \in F$, LAD implies $|C_g(\mu(g) \lor \mu'(g))| \geq |C_g(\mu'(g))|$. Therefore, by definition of $\mu \lor \mu'$ and the individual rationality of $\mu'$,

$$|\mu \lor \mu'(g)| = |C_g(\mu(g) \lor \mu'(g))| \geq |C_g(\mu'(g))| = |\mu'(g)|$$

for each $g \in F$. Since $\mu \lor \mu'$ and $\mu'$ are matchings,

$$|\mu \lor \mu'(F)| = \sum_{g \in F} |\mu \lor \mu'(g)| \geq \sum_{g \in F} |\mu'(g)| = |\mu'(F)|.$$  \hspace{1cm} (20)

By (18) and (19) $w \in \mu'(F) \setminus \mu \lor \mu'(F)$. This fact together with (20) imply that there is $w' \in W$ such that $w' \in \mu \lor \mu'(F) \setminus \mu'(F)$. Then, $\mu'(w') = \emptyset$ and there is $f' \in F$ such that $w' \in C_{f'}\left(\mu(f') \lor \mu'(f')\right)$. By substitutability, $w' \in C_{f'}\left(\mu'(f') \lor \{w'\}\right)$. Furthermore, individual rationality of $\mu'$ implies that $f' \succeq_w \emptyset$. Hence, the pair $(f', w')$ blocks $\mu'$. This contradicts the stability of $\mu'$. Therefore, $\mu \lor \mu'$ is a stable matching.  \hspace{1cm} $\Box$

The next corollary, that follows easily from Proposition 1, presents an important feature of the structure of the worker-quasi-stable matching set. It states that any worker-quasi-stable matching that Blair-dominates $\mu_W$ is actually a stable matching. This happens because the join between them, that is equal to the worker-quasi-stable matching, is a stable matching by Proposition 1.
Corollary 1 Let $\mu$ be a worker-quasi-stable matching. If firms’ choice functions satisfy LAD and $\mu \succeq^B \mu_W$, then $\mu$ is a stable matching.

This corollary makes a deep connection between worker-quasi-stable matchings and stable matchings. Notice that, when choice functions satisfy substitutability and LAD, the set of stable matchings has a dual lattice structure (see for instance Alkan, 2002, for more details). Therefore, by duality, $\mu_W$ is the firm-pessimal stable matching. The implications of this are twofold: (i) a worker-quasi-stable matching that Blair-dominates any stable matching is also stable, and (ii) a matching that is worker-quasi-stable but not stable is either Blair-incomparable to or Blair-dominated by $\mu_W$.

Now we are in a position to prove the main result of this section.

Proof of Theorem 4. Let $\mu \in Q$. By Proposition 1, $\mu \vee \mu_W$ is a stable matching. By Lemma 4 and Theorem 2 (iii), $\mu \vee \mu_W \succeq^B \mu \mathcal{F}(\mu)$. Moreover, given that $T$ is a weakly Pareto improving operator by Theorem 2 (ii), $\mathcal{F}(\mu) \succeq^B \mu$ and by definition of join, $\mathcal{F}(\mu) \succeq^B \mu \vee \mu_W$. Thus, by antisymmetry, $\mathcal{F}(\mu) = \mu \vee \mu_W$. □

The following proposition provides an upper bound for the total number of workers hired in any worker-quasi-stable matching by a firm. If firms’ choice functions satisfy LAD, this number will not exceed the total number of workers matched by that firm in any stable matching.

Proposition 2 Let $\mu$ be a worker-quasi-stable matching and let $\mu'$ be a stable matching. If firms’ choice functions satisfy LAD, $|\mu(f)| \leq |\mu'(f)|$ for each $f \in F$.

Proof. Let $\mu \in Q$ and $f \in F$. By LAD and individual rationality of $\mu$, we have that $|\mathcal{T}(\mu)(f)| = |C_f(\mu(f) \cup W_f^\mu)| \geq |C_f(\mu'(f))| = |\mu'(f)|$. Iterating we obtain that $|\mathcal{F}(\mu)(f)| \geq |\mu(f)|$. By definition, $\mathcal{F}(\mu)$ is a fixed point of our Tarski operator. Thus, $\mathcal{F}(\mu)$ is stable by Theorem 2 (iii). Then, by the Rural Hospital Theorem\(^{15}\) $|\mathcal{F}(\mu)(f)| = |\mu'(f)|$ for each $\mu' \in S$ and each $f \in F$. Therefore, $|\mu(f)| \leq |\mathcal{F}(\mu)(f)| = |\mu'(f)|$ for each $\mu' \in S$ and each $f \in F$. □

Finally, even though the results that we obtain in this section under substitutability and LAD are analogous to those obtained by Wu and Roth (2018) for firm-quasi-stable matchings under responsive preferences (their Lemma 3.11, Theorem 3.12, and Corollaries 4.1 and 4.2), there are important differences. One of them is that our setting is

\(^{15}\)The Rural Hospital Theorem is proven in different contexts by many authors (see McVitie and Wilson, 1970; Roth, 1984, 1985; Martínez et al., 2000; Alkan, 2002; Kojima, 2012, among others). The version of this theorem for a many-to-many matching market where all agents have substitutable choice functions satisfying LAD, that also applies in our setting, is presented in Alkan (2002) and states that each agent is matched with the same number of partners in every stable matching.
more general, since each substitutable choice function induces a substitutable preference, and the class of substitutable preferences contains the class of responsive preferences. Another one is that their method of proof relies on the results established by Blum et al. (1997), that under responsiveness link many-to-one models with one-to-one models, while ours is independent and based on Blair’s approach under substitutability.

6 Conclusions

This paper presents the set of worker-quasi-stable matchings as an extension of the set of stable matchings for a many-to-one matching model. We show that the set of worker-quasi-stable matchings is well-structured: it forms a full lattice with respect to Blair’s partial order. Relying on the lattice structure of this set, a re-equilibration process that models a layoff chain and that converges to stability is also described. Notice that even though we explicitly construct the join between any pair of worker-quasi-stable matchings, how to compute the meet is still an open question.

The results in this paper can be generalized straightforwardly to a model of matching with contracts as in Hatfield and Milgrom (2005). This extension does not present any difficulties other than the notational ones proper to this setting.

It is usual in the literature to study many-to-one models assuming that firms’ preferences are responsive. This is due to the close relation between this model with responsive preferences and the one-to-one model (for a thorough survey on this fact, see Roth and Sotomayor, 1990). However, when firms are endowed with substitutable choice functions (a much less restrictive requirement), this relation with the one-to-one model no longer holds. Then, to extend results to the many-to-one model under the substitutable assumption, Blair’s partial order becomes crucial. We believe that to exploit Blair’s approach for extending other known results in models with responsive preferences to models with substitutable choice functions is a natural direction to pursue further research.

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