Probing the inflationary background of gravitational waves from large to small scales

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The detection of Primordial Gravitational Waves (PGWs) is one of the most important goals of modern cosmology since PGWs can both provide substantial evidence for primordial inflation and shed light on its physical nature. Small scale experiments on gravitational waves such as Ligo-Virgo and, in future, LISA and Einstein Telescope (ET), are sensitive to the stochastic background, $\Omega_{GW}$. So they can be used to constrain the inflationary parameters. In performing this kind of analysis, as recently done also by the Planck collaboration, the primordial spectrum of gravitational waves is usually parametrized with a power law that includes only the amplitude and the spectral tilt. In this paper, we investigate the robustness of such constraints showing that the higher-order terms in the power law approximation (i.e. the runnings of the tensor tilt), even small on the CMB scales, can lead to non-negligible corrections on small scales. We start investigating these corrections in the simplest scenario of slow-roll inflation that predicts an almost scale-invariant, slightly red tilted primordial spectrum of gravitational waves. Nevertheless also in this case, fixing the tensor to scalar ratio to $r = 0.064$, we estimate a correction on $\Omega_{GW}$ of about 39% on the Ligo-Virgo scales. Then we focus on blue models of inflation analyzing the robustness of the constraints that can be obtained using small scale measurements of the stochastic background. We prove that it is effectively possible to constrain the inflationary parameters on the CMB scales with the small scale data, but also that the higher-order corrections are nonnegligible on small scales. We provide an example of this, constraining with the Ligo Virgo data a physical model of blue inflation that employs a pseudo scalar axion naturally coupled to gauge fields, both including and neglecting the spectral runnings. For the same model, we also study the future constraints from the next generation of gravitational waves experiments such as Lisa and ET. We prove that, once the higher-order corrections are considered in the analysis, future experiments can tightly constraints the spectral tilt $n_t$ improving at least of an order of magnitude the constraints.

Keywords: Inflation, primordial gravitational waves.

I. INTRODUCTION

The search for primordial gravitational waves (PGWs) is one of the main goals of modern cosmology since they can both provide a substantial evidence for primordial inflation and shed light on its physical nature[1–3, 5]. In fact long-wavelength gravitational waves are predicted by primordial cosmic inflation [6–9] and, at least in the simplest models, the scale at which inflation occurs is itself related to the amount of PGWs [4, 5, 10–13, 15–17]. The fraction of the energy density of the universe due to PGWs at the present time and at a given scale $k = 2 \pi f$ is given by [18–22]

$$\Omega_{GW}(k) \equiv \frac{1}{\rho_c} \frac{d\rho_{GW}}{d \log k} = \frac{\mathcal{P}_t(k)}{24 z_{eq}}$$

(1)

where $\mathcal{P}_t$ is the primordial tensor spectrum and $z_{eq}$ is the redshift at the matter-radiation equivalence [18]

$$1 + z_{eq} = \frac{\Omega_m}{\Omega_r} \simeq 3400$$

(2)

The primordial tensor spectrum $\mathcal{P}_t$ is usually parameterized with a power law that includes only the amplitude and the spectral index (or tilt) $n_t$:

$$\mathcal{P}_t(k) = r \mathcal{P}_s(k_* \left( \frac{k}{k_*} \right)^{n_t}$$

(3)
where $k_*$ is the pivot scale that we fix to $k_* = 0.05 \text{ Mpc}^{-1}$, $P_s$ is the primordial scalar spectrum measured to be $P_s(k_*) \simeq 2 \times 10^{-9}$ [18] and $r = P_t(k_*)/P_s(k_*)$ is the tensor to scalar ratio. Recently, analyses carried out by the Planck collaboration [18] have shown that, using the most recent CMB data together with the Ligo-Virgo data [23], it is possible to constrain the spectral index $n_t$ on the CMB scales ($k \sim 0.05 \text{ Mpc}^{-1}$) for the blue models of inflation (i.e. models that allow a positive tensor tilt) being $n_t < 0.53$ at 95% C.L. In performing this kind of analysis, it is assumed that the tensor fluctuations follow the power law (3) and so $n_t$ is assumed as scale-independent. This is clearly an approximation since $n_t$ is scale dependent and further parametrization that includes its runnings should be considered [24] above all if one wants to constrain physics at the CMB scales using small scale data such as the Ligo-Virgo ones ($k_{LV} \simeq 10^{16} \text{ Mpc}^{-1}$).

In this paper we want to study the small scale corrections on the total amount of the PGWs due to the higher order terms in the power law approximation. The paper is organized as follows.

In section II we modify the power law relation (3) introducing the higher order corrections and the tensor runnings. We first provide an estimation of the effects induced by these corrections on the simplest slow roll models of inflation. In this paper we want to study the small scale corrections on the total amount of the PGWs due to the higher order terms in the power law approximation since $n_t$ is scale dependent and further parametrization that includes its runnings should be considered [24] above all if one wants to constrain physics at the CMB scales using small scale data such as the Ligo-Virgo ones ($k_{LV} \simeq 10^{16} \text{ Mpc}^{-1}$).

In section III we provide a working example of this, studying a model of inflation that employs a pseudo scalar axion naturally coupled to gauge fields. First of all we prove that it is effectively possible to constrain the inflationary parameters at the CMB scales using the Ligo Virgo constraints on $\Omega_{GW}$; nevertheless strong corrections may arise from the higher order terms in the power law approximation, even for small values of the runnings. Then we focus on blue tilted models of inflation that instead can be probed by small scale data such as the Ligo Virgo ones and so that are largely studied as well [18, 20–23, 25]. We first perform a phenomenological analysis in order to show that the Planck and Ligo-Virgo constraints on $n_t$ are very sensitive to the assumption of considering it constant.

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II. BEYOND THE POWER LAW APPROXIMATION: HIGHER ORDER CORRECTIONS ON SMALL SCALES

Let us start by generalizing the power law parametrization to the following expansions:

$$P_t(k) = r P_s(k_*) \left( \frac{k}{k_*} \right)^{n_t(k_*) + \sum_{n=1}^{\infty} \frac{\alpha^s_n(k_*)}{n!} \left[ \log \left( \frac{k}{k_*} \right) \right]^n}$$

(4)

where

$$\alpha^s_n(k_*) \equiv \left( \frac{d}{d \log k} \right)^n n_t(k) \bigg|_{k=k_*}$$

(5)

is the $n$-order tensor running\(^1\). Our aim is to instigate the corrections induced by the runnings of $n_t$ on the stochastic background $\Omega_{GW}$ on the small scales probed by experiments such as Ligo-Virgo: $k_{LV} \sim 10^{16} \text{ Mpc}^{-1}$ [23]. In fact we see that the quantity

$$\log \left( \frac{k_{LV}}{k_*} \right) = \log \left( \frac{1.3 \times 10^{16}}{0.05} \right) \simeq 40$$

(6)

and so the exponent of the power law approximation that involves the runnings, at the Ligo-Virgo scale $k_{LV}$, can be non negligible:

$$\sum_{n=1}^{\infty} \frac{\alpha^s_n}{(n+1)!} \left[ \log \left( \frac{k}{k_*} \right) \right]^n \bigg|_{k=k_{LV}} \simeq \sum_{n=1}^{\infty} \frac{40^n}{(n+1)!} \alpha^s_n$$

(7)

Clearly in order to exactly compute the sum we have to know both the value and the sign of all the runnings $\{ \alpha^s_n \}$ and these depend from the specific model of inflation that one considers. However from the sum we can extract

\(^1\) In what follows we will usually avoid to specify that the spectral tilt and the runnings are computed on the pivot scale $k_*$ and, to simplify the notation, we will only write $n_t$ and $\alpha^s_n$.
information about the size of the runnings in order to give a negligible contribution at the Ligo Virgo scales. In particular

\[ |\alpha_n| \ll \frac{(n+1)!}{\log \left( \frac{n+1}{4}\right)} |n_t| \quad \forall n \tag{8} \]

and this is a non trivial requirement since \( \left[\log \left( \frac{n+1}{4}\right)\right]^n \) dominates over \((n+1)!\), above all for small \(n\). Moreover, since we have a power law, also relatively small contributions arising from the runnings can translate into appreciable differences in the fraction of PGWs. In the next subsection we plan to estimate the sum (7) in the simplest single field slow roll scenario that predicts a total amount of PGWs too small to be probed by the present and future small scale GWs experiments. Nevertheless we will show that the runnings can give appreciable contribution also in these almost scale independent models. Moreover the discussion below will be useful in the next section when we will study a model that can be effectively probed by the Ligo Virgo data.

A. A toy model for the Slow Roll inflation

In the simplest scenario of single field slow roll inflation one can compute the primordial spectra both for scalar and tensor perturbations to obtain [14, 15, 26–30]:

\[ P_s = \left(\frac{1}{8\pi^2 M_p^2}\right) \left(\frac{H^2}{\epsilon_1}\right) \tag{9} \]

\[ P_t = \left(\frac{2}{\pi^2 M_p^2}\right) H^2 \tag{10} \]

where \(M_p\) is the reduced Planck mass (\(= 2.435 \times 10^{18}\) GeV), \(H\) is the Hubble parameter, \(\epsilon_1 = -\frac{1}{2} \frac{d \log H^2}{d \log k}\) is the first of the slow roll parameters \(\{\epsilon_1, \ldots, \epsilon_n\}\) defined as \(\epsilon_n = \frac{d \log \epsilon_{n-1}}{d \log k}\). All these relations are to be considered calculated at the horizon crossing. We also introduce the slow roll relations for the tensor to scalar ratio \(r\), the tensor tilt \(n_t\) and the scalar tilt \(n_s\) [15, 24]:

\[ r = \frac{P_t(k_*)}{P_s(k_*)} = 16\epsilon_1 \tag{11} \]

\[ n_t = \frac{d \log P_t}{d \log k} = -2\epsilon_1 \tag{12} \]

\[ n_s - 1 = \frac{d \log P_s}{d \log k} = -2\epsilon_1 - \epsilon_2. \tag{13} \]

We see that the single field slow roll inflation predicts the consistency relation

\[ n_t = -\frac{r}{8} \tag{14} \]

In order to estimate the contribution coming from the runnings, we build a simple model when each running is \(\lambda\) times smaller than the previous:

\[ \alpha_n = \lambda \alpha_{n-1} = \lambda^2 \alpha_{n-2} = \cdots = \lambda^{n-1} \alpha_1 = \lambda^n n_t \tag{15} \]

with \(0 < \lambda \ll 1\). This toy model is motivated by the fact that in the slow roll paradigm each running is expected to be one order smaller in the slow roll parameters so that \(\alpha_n = O(\epsilon^{n+1}) \approx \epsilon^n n_t\). In other words if we consider the relation \(n_t = -2\epsilon_1\), we can easily see that if \(\epsilon_{n>2} \ll \epsilon_2\) we can assume

\[ \epsilon_2 \approx \text{const.} \tag{16} \]
and we have $\alpha_n^i \approx (\epsilon_2)^n n_t$ that is noting else that the assumption (15) with $\lambda \equiv \epsilon_2$. Therefore we can fix the parameter $\lambda$ simply using the relations (12) and (13) obtaining:

$$\lambda = 1 + (n_t - n_s)$$

(17)

Note that in this way, since the scalar tilt $n_s$ is measured with a great precision to be $n_s \simeq 0.96$, the sign of $\lambda$ (and so of all the runnings) depends only on $n_t$ (i.e. $r$). For $n_t > n_s - 1$ (i.e. $r \gtrsim 0.32$) $\lambda > 0$ and all the runnings are negative. If instead $n_t < n_s - 1$ (i.e. $r \lesssim 0.32$) so $\lambda < 0$ and all the runnings are positive. Because of the Planck bounds on $r$ [18], we expect the runnings to be negative. Under the assumption (15), one can easily compute the series (7):

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{(n+1)!} \left[ \log \left( \frac{k}{k_*} \right) \right]^n = n_t \sum_{n=1}^{\infty} \frac{\lambda \log \left( \frac{k}{k_*} \right)}{(n+1)!} = n_t \left[ -1 + \frac{\left( \frac{k}{k_*} \right)^\lambda - 1}{\lambda \log \left( \frac{k}{k_*} \right)} \right]$$

(18)

obtaining for $\Omega_{GW}(k)$:

$$\Omega_{GW}(k) = \frac{r \mathcal{P}_s(k_*)}{24 z_{eq}} \left( \frac{k}{k_*} \right)^{n_t} \left[ \frac{\left( \frac{k}{k_*} \right)^\lambda - 1}{\lambda \log \left( \frac{k}{k_*} \right)} \right]$$

(19)

where $\lambda$ is given by eq. (17). Note that, since $n_t$ has to respect the consistency relation (14) and being $\mathcal{P}_s$ and $n_s$ fixed by observations, $\Omega_{GW}(k)$ depends only on $r$.

Figure 1: we plot the $\Omega_{GW}(k = k_{LV})$ as a function of the tensor to scalar ratio $n_t = -\frac{r}{8}$ both including (black solid line) and neglecting (green dashed line) the contribution of the runnings calculated under the assumption (15). For $n_t \simeq -0.008$ (i.e. $r \simeq 0.064$, limit by Planck) the error that one commits neglecting the runnings at the Ligo Virgo scale is about 39%.

In figure 1, we plot the $\Omega_{GW}(k_{LV})$ as a function of the tensor to scalar ratio $n_t = -\frac{r}{8}$ both including and neglecting the contribution due to the runnings. Since in our toy model all the runnings are negative for the range of $r$ we have

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2 Note that the assumption (16) translates into $\alpha_n^i \approx \alpha_n^0$.

3 Note that when $\lambda \rightarrow 0$ (i.e. $\epsilon_2 \rightarrow 0$) we recover the scale independent limit:

$$\lim_{\lambda \rightarrow 0} \Omega_{GW}(k) = \frac{r \mathcal{P}_s(k_*)}{24 z_{eq}} \left( \frac{k}{k_*} \right)^{n_t}$$
Figure 2: $\Omega_{GW}(f)$ for different values of the tensor to scalar ratio $r$ both including (solid lines) and neglecting (dashed lines) the runnings.

considered, they reduce the total amount of energy in PGWs and the difference between the two cases grows with $r$. For example when $r \approx 0.064$ (that corresponds to the limit imposed by the Planck collaboration), the error that one commits ignoring the runnings at the Ligo-Virgo scale is about 39%:

$$\left(\frac{|\Delta \Omega_{GW}(k_{LV})|}{\Omega_{GW}^{\text{nt}+\text{run}}(k_{LV})}\right)_{r \approx 0.064} \approx 0.39$$

where $\Delta \Omega_{GW} = \Omega_{GW}^{\text{nt}+\text{run}} - \Omega_{GW}^{\text{only\ nt}}$. Instead in figure 2 we plot the $\Omega_{GW}(f)$ for three different values of the tensor to scalar ratio $r$ both including (solid lines) and neglecting (dashed lines) the runnings. One can see that, the contribution of the runnings is appreciable only in the case $r = 0.064$ while it is totally negligible when $r = 0.001$ ($\frac{|\Delta \Omega_{GW}(k_{LV})|}{\Omega_{GW}^{\text{nt}+\text{run}}(k_{LV})} \lesssim 0.02\%$). These results show that if $r$ is sufficiently large, also in the simplest slow roll scenario that predicts an almost scale independent spectrum, the runnings can give a satiable contribution at high frequency. As we will see in the next subsection, the importance of the runnings is instead crucial in models of inflation that admit a tensor tilt $n_t$ positive and of order one. However since this estimation depends on the assumption of considering the parameter $\epsilon_2 \approx \text{const}$, we would like to spend a few words on this approximation showing that, in order to make the series (7) converge, we have to require that

$$\epsilon_{n>2} \ll \epsilon_2$$

that justifies the validity of our approximation. In fact let us suppose that $\epsilon_{n>2} \approx \pm \epsilon_2$, it is simply to show that in this case we would have:

$$\alpha_n^t \approx n! (\pm \lambda)^n n_t$$

where $\lambda$ is always given by the eq. (17). So the sum of all the runnings (7) will be:

$$\sum_{n=1}^{\infty} \frac{\alpha_n^t}{(n+1)!} \left[ \log \left( \frac{k}{k_*} \right) \right]^n = n_t \sum_{n=1}^{\infty} \frac{\pm \lambda \log \left( \frac{k}{k_*} \right)^n}{n+1}$$

this sum coverages only when

$$\left| \pm \lambda \log \left( \frac{k}{k_*} \right) \right| < 1$$

this condition is difficult to ensure when $k \gg k_*$ since we need large values of $r$ excluded by observations. So in general this sum does not converge and we can conclude that in order to make things work, we have to require that $\epsilon_{n>2} \ll \epsilon_2$ and so that our approximation $\epsilon_2 \approx \text{const}$ makes sense.
B. Constraining blue models of inflation with the Ligo-Virgo data

In the frequency range \( f \in (20 - 85.8) \text{ Hz} \), which corresponds to the wavenumber range \( k \in (1.3 - 5.5) \times 10^{16} \text{ Mpc}^{-1} \), Ligo and Virgo set an upper bound on the stochastic background of

\[
\Omega_{GW}(k_{LV}) \leq 1.7 \times 10^{-7}
\]  

at 95 % C.L. \([18, 23]\). As stated in the introduction, recently, analyses carried by the Planck collaboration have put constraints on blue models of inflation using the Ligo Virgo limit \((25)\). Fixing \(k_{LV} \approx 1.3 \times 10^{16} \text{ Mpc}^{-1}\), from the Ligo Virgo limit \((25)\) and the Planck data, they constrained \(n_t < 0.53\) at 95% CL. This result is obtained considering the spectral index \(n_t\) as constant over a range of about eighteen order of magnitude , \(k \in [0.05, 1.3 \times 10^{16}]\), and neglecting the possible contribution coming from the runnings. In this section we want to investigate the role that the runnings may play in such kind of analyses. In figure 3 we fix the tensor to scalar ratio to be \(r \approx 0.01\) that corresponds to the maximum value we can choose in order to cross the Ligo Virgo limit on \(\Omega_t\) at \(n_t = 0.5\) (see fig. 3). As we can see always in figure 3, if one includes a small running \(|\alpha_1| \approx 0.01 n_t\), the Ligo Virgo limit is crossed for smaller or larger values of \(n_t\). The fact that we really want to stress is that a small negative running \(\alpha_1 = -0.05 \times n_t\) exactly compensates the spectral index \(n_t\) on the Ligo Virgo scales and so the Planck bounds on the positive values of \(n_t\) obtained with the Ligo Virgo limit will be canceled out. Of course also the other higher order runnings may play an important role since, as explained in the previous section, on the Ligo-Virgo scale the running of order \(n\) is amplified by a factor \(\sim 40^n\). Therefore we can conclude that these constraints are very sensitive to the assumption of a constant \(n_t\) and that also the higher order corrections in the power law approximation should be considered if one wants to constrain the inflationary parameters correctly on the CMB scales using the small scale bounds on \(\Omega_{GW}\). In the next section we want to provide a physical example effectively using the Ligo Virgo limit \((25)\) in order to probe physics at the CMB scales. At the same time, we show that the higher order corrections in the power law approximation cannot be neglected also in this physical model of blue inflation.

![Figure 3: We plot \(\Omega_{GW}(k_{LV})\) versus \(n_t > 0\). In the black solid line the power law relation (3) is assumed and the tensor to scalar ratio is fixed to \(r \approx 0.013\) that is the value required in order to cross the Ligo Virgo limit (red line) for \(n_t \approx 0.5\). The blue (green) line shows how the situation changes if one considers a positive (negative) running \(\alpha_1\). As one can see a small negative running \(\alpha_1 = -0.05 \times n_t\) (green solid line) exactly compensates the spectral index \(n_t\) on the Ligo Virgo scales](image-url)
III. PARTICLE PRODUCTION DURING INFLATION

In this section we want to provide a working example for our phenomenological discussion showing that we can actually use small scales data in order to constrain physics at the CMB scales, even if we have to take into account all the warnings discussed so far. In particular we choose a non trivial model of inflation that can predict a blue tensor tilt and that can be constrained using the limits on the stochastic background. We show that this model predicts relatively small tensor runnings but that, once included in the analysis, they can effectively provide strong corrections on small scales. The inflationary model we want to consider employs a pseudo scalar axion naturally coupled to gauge fields. In such model a mechanism of particle production takes place during the rolling inflation and this can be translated into a blue gravitational wave spectrum with a blue tilt of the order we need. We give a brief description of the model, more details can be found in [31–36]. We consider a simple theory of a Pseudo Nambo Goldstone Boson inflation. In this model the inflaton field $\phi$ and the axion $\psi$ are minimally coupled to gravity and the axion is also coupled with a $U(1)$ gauge field in a way consistent with symmetries\(^4\). The action of the theory is

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) - \frac{1}{2} (\partial \psi)^2 - U(\psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\psi^4}{4f} F_{\mu\nu} \tilde{F}^{\mu\nu} \right]$$

(26)

$F_{\mu\nu}$ and $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ are the field-strength tensor of the gauge field and its dual, respectively; $f$ is the axion decay constant while $V(\phi)$ and $U(\psi)$ are the inflation and axion potential. We also assume a flat FRW metric and that both the inflaton and the axion take a homogeneous vacuum expectation value (vev) while the gauge field carries no vev. Under this assumption the equations of motion for the inflaton and the axion are

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0$$

(27)

$$\ddot{\psi} + 3H \dot{\psi} + U'(\psi) = 0$$

(28)

where the prime denotes the derivatives with respect to the argument and the over-dots denotes the derivatives with respect to time. We also assume that the contribution of the axion on the background evolution is negligible compared to that of the inflaton i.e $|U| \ll V$ and $\psi^2 \ll \dot{\phi}^2$. We introduce the parameter

$$\xi \equiv \frac{\dot{\psi}}{2Hf}$$

(29)

that will play a crucial role in our future discussion. We assume $\xi$ to be nearly but not exactly constant so that its logarithm derivative

$$\xi_1 \equiv \frac{d \log \xi}{d \log k} = \frac{\dot{\xi}}{\xi H} \ll 1$$

(30)

we also assume $\xi_1 \approx \text{const.}$ i.e. $\xi_1 > \frac{d \log \xi}{d \log k} \approx 0$. In our future discussion we restrict our attention to the case $\xi > 1$ that, as we are going to see, is the case that allows a blue tensor tilt. We are not going to discuss in details the peculiarities of this model such as the gauge quanta production [31] that are reviewed also in [33, 36] and the references within, but for our purpose is sufficient to observe that, in order to avoid a significant back-reaction of the produced gauge quanta to the background dynamics, we have to require that

$$\frac{e^{4\pi \xi}}{\xi^{5/2}} \ll \frac{13.5}{\sqrt{\epsilon_1 P_0} M_p}$$

(31)

where $P_0 = \left( \frac{1}{8\pi M_p^2} \right) \left( \frac{M_p^2}{v^2} \right)$ is primordial scalar spectrum without source (i.e. as predicted by the slow roll inflation). The scalar and tensor spectra for this model are [37, 38]:

$$P_s \simeq P_0 \left( 1 + \epsilon_2 \epsilon_1^4 P_0 \frac{e^{4\pi \xi}}{\xi^{5/2}} \right)$$

(32)

\(^4\) Note that the axion is not the inflaton itself but another distinct field.
\[ \mathcal{P}_s \simeq 16 \epsilon_1 \mathcal{P}_0 \left( 1 + c_t \epsilon_1 \mathcal{P}_0 \frac{e^{4\pi \xi}}{\xi^6} \right) \]  

(33)

where \( c_s = 2.5 \cdot 10^{-6} \) and \( c_t = 3.4 \cdot 10^{-5} \) are constants. In this subsection we compute the spectral tilts from the relation (32) and (33). Let us start from the equations (32) and (33), taking the logarithm derivatives we have:

\[ n_s - 1 = \frac{d \log \mathcal{P}_s}{d \log k} = \frac{d \log \mathcal{P}_0}{d \log k} + \frac{c_s}{1 + c_s \epsilon_1} \frac{d}{d \log k} \left( \epsilon_1^2 \mathcal{P}_0 \frac{e^{4\pi \xi}}{\xi^6} \right) \]

(34)

\[ = -2(1 + f_s)\epsilon_1 - (1 - f_s)\epsilon_2 + f_s(4\pi \xi - 6)\xi_1 \]

(35)

\[ \approx -2\epsilon_1 - \epsilon_2 \]

(36)

and

\[ n_t = \frac{d \log \mathcal{P}_t}{d \log k} = \frac{d \log \epsilon_1 \mathcal{P}_0}{d \log k} + \frac{c_t}{1 + c_t \epsilon_1} \frac{d}{d \log k} \left( \epsilon_1 \mathcal{P}_0 \frac{e^{4\pi \xi}}{\xi^6} \right) \]

(37)

\[ = -2(1 + f_t)\epsilon_1 + f_t(4\pi \xi - 6)\xi_1 \]

(38)

where we have defined the functions

\[ f_s = \frac{c_s \mathcal{P}_0 \epsilon_1^2 \frac{e^{4\pi \xi}}{\xi^6}}{1 + c_s \mathcal{P}_0 \epsilon_1^2 \frac{e^{4\pi \xi}}{\xi^6}} \ll 1 \]

(39)

and

\[ f_t = \frac{c_t \mathcal{P}_0 \epsilon_1 \frac{e^{4\pi \xi}}{\xi^6}}{1 + c_t \mathcal{P}_0 \epsilon_1 \frac{e^{4\pi \xi}}{\xi^6}} \]

(40)

that weigh the corrections to the slow roll predictions respectively for the scalar and tensor inflationary parameters. As we will show the corrections to the scalar spectrum are completely negligible, \( f_s \approx 0 \), and the scalar parameters are essentially equal to that obtained in the simplest slow roll models. On the other hand the corrections to the tensor spectrum can be dominant for an appreciable range of the parameter space. In what follows we:

1. fix \( \mathcal{P}_s \) and \( n_s \) to the observed values of \( \mathcal{P}_s \approx 2 \cdot 10^{-9} \) and \( n_s \approx 0.96 \);
2. fix the tensor to scalar ratio \( r \) to a given value;
3. fix \( \xi_1 \) to be\(^5\) \( \xi_1 \simeq 0.007 \ll 1 \);
4. use the eqs. (32), (33) in order to find \( \epsilon_1 \) as a function of \( \xi \) for the value of \( r \) chosen at the point 2: \( \epsilon_1 = \epsilon_1(\xi) \).
   In this way, since \( \xi_1 \) has been fixed at the point 3, also \( n_t \) is only a function of \( \xi \): \( n_t = n_t(\xi) \);
5. use the relation (36) to find \( \epsilon_2 \) as a function of \( \xi \): \( \epsilon_2 = \epsilon_2(\xi) \);
6. let \( \xi \) vary in the range \( \xi \in [1, 7] \).

First of all we want to stress again that in this model the scalar spectrum (32) is existentially equal to that predicted by the single field slow roll inflation for all the range of \( \xi \) that we have considered in our simulations\(^6\). In fact if we decompose the scalar spectrum \( \mathcal{P}_s = \mathcal{P}_0 + \mathcal{P}_{s, \text{sourced}} \), the sourced term induces corrections that are extremely small compared to the vacuum contribution: \( \mathcal{P}_{s, \text{sourced}} \sim 10^{-4} \mathcal{P}_0 \). This fact can be understood noting that the corrections are suppressed by a factor \( \epsilon_1^2 \mathcal{P}_0 \) and that \( \epsilon_1 \) exponentially decreases with \( \xi \) when \( r \) is fixed, as one can see from fig. 4

where we plot the slow roll parameter \( \epsilon_1 \) as a function of the particle production parameter \( \xi \) for different values of the tensor to scalar ratio. From the same figure we see that for \( \xi \lesssim 3 \) the particle production is negligible and \( r \approx 16 \epsilon_1 \) as predicted by the slow roll paradigm. Instead when \( \xi \gtrsim 3 \) in order to keep \( r \) constant \( \epsilon_1(\xi) \) needs to decrease very quickly becoming totally negligible for \( \xi \gtrsim 4 \). The fact that the scalar spectrum is essentially indistinguishable from

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\(^5\) This value corresponds to \( 10^{-3} : \xi_{\text{max}} \) where \( \xi_{\text{max}} = 7 \) is the maximum value of \( \xi \) that we are considering in our analysis and that corresponds to \( n_{s, \text{max}} \approx 0.57 \) that is the order we need to use the Ligo-Virgo limit to constrain this model. Note that our results are of course sensitive to the value of \( \xi_1 \) but we are interested neither into a parameter analysis nor into the robustness of our constraints for this specific model: our task is to provide a working example of using the Ligo-Virgo limit to constrain quantities at the CMB scales.

\(^6\) For example the function \( f \) (that weighs the corrections to the tensor parameters), albeit very small for \( \xi \lesssim 3 \), converges to 1 for \( \xi \gtrsim 5 \) while the analog function for the scalar case, \( f_s \), converges to \( f_s(\xi \gtrsim 4) \sim 10^{-5} \) suppressing all the corrections also for large values of \( \xi \).
Figure 4: The slow roll parameter $\epsilon_1$ as a function of the particle production parameter $\xi$ for three different fixed values of the tensor to scalar ratio $r$: $r = 0.064$ (green line), $r = 0.01$ (blue line) and $r = 0.001$ (red line). As one can see when $\xi$ increases, $\epsilon_1(\xi)$ needs to decrease very quickly in order to keep $r$ constant.

The single field slow roll models is crucial since in this way all the tight constraints on the scalar perturbations (e.g. their high level of gaussianity) are respected as well [38, 39]. On the other hand the sourced tensor modes could leave a sizable non-gaussianity of nearly equilateral shape on the CMB anisotropies and polarization. The amount of non-gaussianity is controlled by the parameter $f_{NL}$ estimated as [33, 40]:

$$f_{NL} \simeq 1.1 \times 10^{-14} \left( \epsilon_1 \frac{e^{2\pi \xi}}{\xi^3} \right)^3$$

Figure 5: The non-gaussianity parameter $f_{NL}$ as a function of $\xi$ for three different values of the tensor to scalar ratio $r$. As we can see the total amount of primordial tensor non gaussianity increases with $\xi$ converging to an asymptotic value for $\xi \gtrsim 5$. For $r = 0.064$ the expected tensor non gaussianity is high being $f_{NL}(\xi \gtrsim 5) \simeq 157$, but it
decreases with $r$: $f_{\text{NL}}(\xi \gtrsim 5) \simeq 10$ for $r = 0.01$ and $f_{\text{NL}}(\xi \gtrsim 5) \simeq 0.31$ for $r = 0.001$. These values are allowed by the present bounds on primordial non-gaussianity [41–44]. In figure 6 we finally plot $n_t$ as a function of $\xi$ for the same three different values of $r$. Note that when $\xi \lesssim 3$, being the function $f_t \approx 0$, from the equation (38) we recover the slow roll relation $n_t \approx -2\epsilon_1$ for $\xi \lesssim 3$ while $n_t \approx (4\pi\xi - 6)\xi$ for $\xi \gtrsim 4$.

In what follows we fix the tensor to scalar ratio to $r = 0.01$ but at the end we will provide the final results also for other values of $r$.

A. The runnings

In this subsection we want to estimate the higher order corrections to the power law approximation. We start carrying out an analytic approach performing a second order computation. We focus our attention on the tensor running $\alpha_1 \equiv \frac{dn_t}{d \log k}$ and the running of the running $\alpha_2 \equiv \frac{d\alpha_1}{d \log k}$ that can be analytically computed starting from equation (38) and taking the derivatives with respect to $\log k$:

$$\alpha_1 \equiv \frac{dn_t}{d \log k} = -2(1 + f_t)\epsilon_1\epsilon_2 - 2f_t'\epsilon_1 + f_t'(4\pi\xi - 6)\xi_1 + 4\pi f_t\xi_1^2$$

$$\alpha_2 \equiv \frac{d\alpha_1}{d \log k} = -2(1 + f_t)\left(\epsilon_1\epsilon_2^2 + \epsilon_1\epsilon_2\epsilon_3\right) - 4f_t'\epsilon_1\epsilon_2 - 2f_t''\epsilon_1 + f_t''(4\pi\xi - 6)\xi_1 + 8\pi f_t\xi_1^2 + 4\pi f_t\xi_1^3$$

where we have supposed $\xi_1 \approx \text{const.}$ and where we have defined:

$$f_t' \equiv \frac{df_t}{d \log k} = \left[\frac{-2\epsilon_1 + (4\pi\xi - 6)\xi_1}{1 + \epsilon_1 P_0 \epsilon_1 e^{4\pi\xi}}\right] f_t$$

![Figure 6](image-url): The tensor tilt as a function of $\xi$ for three different values of the tensor to scalar ratio $r$: $r = 0.064$ (green line), $r = 0.01$ (blue line) and $r = 0.001$ (red line). As one can see, $n_t \approx -2\epsilon_1$ for $\xi \lesssim 3$ while $n_t \approx (4\pi\xi - 6)\xi$ for $\xi \gtrsim 4$. Instead when $\xi \gtrsim 4$ we have $f_t \approx 1$ and $\epsilon_1(\xi \gtrsim 4) \approx 0$ (see figure 4), and the relation for $n_t$ becomes $n_t \approx (4\pi\xi - 6)\xi_1$ and so independent from the value of $r$ that we choose. In the middle region between $\xi \sim 3$ and $\xi \sim 4$, the function $f_t$ increases and it is no more approximately constant so that we have a transient region between the two cases discussed so far. We conclude that in this model the sourced gravitational waves can dominate over the vacuum tensor modes even for small values of $r$ but sufficient large value of $\xi$ resulting into a blue tensor tilt big enough to be constrained using the Ligo-Virgo limit on the stochastic background. This is exactly what we plan to do: we will estimate the runnings of $n_t$, quantifying their contribution on small scales and investigating their impact on the final constraints.
and

\[ f''_t = \frac{d f'_t}{d \log k} = \left[ \frac{-2\epsilon_1 + (4\pi \xi - 6)\xi_1}{1 + c_t P_0 \epsilon_1 \frac{d\epsilon_1}{d\xi}} \right]^2 f_t + \left[ \frac{(1 + c_t P_0 \epsilon_1 \frac{d\epsilon_1}{d\xi}) \left(\frac{d\epsilon_1}{d\xi}\right)^2}{1 + c_t P_0 \epsilon_1 \frac{d\epsilon_1}{d\xi}} \right]^2 f_t \]

As explained before, apart from the slow roll parameter \( \epsilon_3 \) whose contribute can be ignored\(^7\) being irrelevant in the economy of the equation (43), all these quantities are known functions of \( \xi \). In figure 7 we plot \( \alpha_1^t \) and \( \alpha_2^t \) obtained both with this analytic computation (black solid line) and with the numerical approach (green dashed line) described below. As one can see \( \alpha_1^t \) and \( \alpha_2^t \) are relatively small compared to \( n_t \). Nevertheless their contribution on the Ligo

\text{Figure 7: The running } \alpha_1^t \text{ (left panel) and the running of running } \alpha_2^t \text{ (right panel) as functions of } \xi \text{ both for the analytic (black solid line) and the numerical (green dashed line) approach.}

Virgo scale is relevant and the abundance of primordial gravitational waves is very sensitive to them. In figure 9 we plot \( \Omega_{GW}(k_{LV}) \) as a function of \( \xi \). The gray dashed line represents the case of a constant \( n_t \), while the green and blue dashed lines represent respectively the case in which the running \( \alpha_1^t \) and the running of running \( \alpha_2^t \) are included in the parametrization. Our analysis proves that, also in physical models of inflation, the presence of relatively small runnings becomes really important when we want to constrain these models using the small scale gravitational waves experiments such as Ligo and Virgo. Therefore to probe physics on the CMB scales using small scale experiments, albeit possible, requires some precautions. We may also ask if a second order computation is enough to ensure a good level of precision and if the higher order runnings could give further appreciable contributions. Therefore we change our approach in order to estimate as many runnings as we need to ensure the convergence of the sum (7) within a good level of precision, improving our results. We estimate all the \( n \) runnings, \( \alpha_n^t = \left( \frac{d}{d \log k} \right)^n n_t \), following a numerical approach: starting from the spectral tilt \( n_t(\xi) \) given by the equation (38) and using that

\[ \frac{d}{d \log k} = \frac{d\xi}{d \log k} \frac{d}{d\xi} = \xi_1 \frac{d}{d\xi} \]

we write an algorithm able to perform the numeric derivatives with respect to \( \xi \) iteratively. Using our algorithm, for the first and second order runnings, \( \alpha_1^t \) and \( \alpha_2^t \), we find the same results as our analytical computation within a precision better than 3\%, as one can see from figure 7. Therefore we use this approach in order to include in the

\(^7\) We study the impact of the contribute arising from the slow roll parameter \( \epsilon_3 \) parametrizing \( \epsilon_3 = \gamma \epsilon_2 \) and let \( \gamma \) vary in a range \( \gamma \in [-1, 1] \). No significant changes in \( \alpha_2^t \) are observed and so the parameter \( \epsilon_3 \) has been fixed to zero and neglected in the analytic approach.
analysis also other higher order runnings $\alpha_{n>2}$ until the changes in the sum (7) become smaller and smaller. In particular we estimate that the fifth order is enough to ensure a high level of precision (about 4%) and so we extended the previous analytical analysis including in the parametrization also the third ($n = 3$), fourth ($n = 4$) and fifth ($n = 5$) order runnings, plotted in figure 8. In the next two subsections we present our results both for Ligo Virgo

![Figure 8: The third order running $\alpha_3^t$ (left panel), the fourth order running $\alpha_4^t$ (middle panel) and the fifth order running $\alpha_5^t$ (right panel) as functions of $\xi$.](image)

and for the upcoming gravitational waves experiments such as Lisa and Enstein Telescope.

### B. Constraints from Ligo Virgo

In this model, the constraints that one can obtain using the Ligo Virgo limit are themselves very sensitive to the runnings as suggested by our model independent analysis. In fact, as one can see from fig. 9, assuming $n_t$ as constant, the Ligo Virgo limit constrains $n_t < 0.51$ (i.e. $\xi < 6.25$), result that is in agreement with the model independent analysis carried out by the Planck collaboration [18]. The black solid line in the figure 9 represents $\Omega_{GW}(k_{LV})$ as a function of $n_t$ (left panel) and $\xi$ (right panel) obtained with this approach once that all the runnings up to the fifth order are included in the parameterization. Once that the runnings are considered, $n_t$ is constrained to be $n_t < 0.042$ or $0.097 < n_t < 0.43$ and, equivalently, $\xi$ to be $\xi < 3.48$ or $3.60 < \xi < 5.42$. The differences with respect to the previous analytical second order computation (blue dashed line) are small albeit observable. We recall that in this analysis the tensor to scalar ratio is fixed to $r = 0.01$, nevertheless the results are not very sensitive to the choice of $r$ as one can see from tab I where we summarize also the results obtained for different values of $r$. The fact we really

| Parametrization | $r = 0.064$ | $r = 0.01$ | $r = 0.001$ |
|-----------------|-------------|-------------|-------------|
| $n_t$ + Runnings | $n_t < 0.037$ or $0.086 < n_t < 0.39$ | $n_t < 0.042$ or $0.097 < n_t < 0.43$ | $n_t < 0.039$ or $0.11 < n_t < 0.49$ | $\xi < 3.32$ or $3.33 < \xi < 4.97$ | $\xi < 3.48$ or $3.60 < \xi < 5.42$ | $\xi < 3.67$ or $3.82 < \xi < 5.99$ |
| Only $n_t$      | $n_t < 0.46$ | $n_t < 0.51$ | $n_t < 0.56$ | $\xi < 5.73$ | $\xi < 6.25$ | $\xi < 6.91$ |

Table I: Ligo Virgo constraints on $n_t$ and $\xi$ for different values of $r$.

would like to stress is that our physical example leads us to the same conclusion of our phenomenological discussion:
Figure 9: $\Omega_{GW}(k = k_{LV})$ as a function of $n_t$ (right panel) and $\xi$ (left panel). The black solid line represents the results obtained by our computational approach including the runnings up to the fifth order while the blue (green) dashed line represents the results obtained analytically including only the second (first) order runnings.

the results of our analyses prove that the contribution of small runnings on the CMB scales can be much amplified on small scales. In particular, positive (negative) runnings, may increase (reduce) a lot the signals on small scales, also in physical models of blue inflation as those studied in this paper. In conclusion, one can effectively use small scale experimental data on $\Omega_{GW}$ to constrain the physics of inflation on the CMB scales. Nevertheless non negligible corrections can arise from the higher order terms in the power law approximation and they need to be considered in the analysis in order to correctly constrain the inflationary parameters. In particular, in this specific physical model of blue inflation, the higher order corrections let us to exclude some regions that instead will be allowed neglecting them. This because the Ligo-Virgo sensitivity on the stochastic background stays above the local minimum of the curve of $\Omega_{GW}(\xi)$. In the next subsection we are going to discuss the constraints for this model from the next generation of Gravitational Waves experiments such as Laser interferometer Space Antenna (Lisa) and the Einstein Telescope (ET).

C. Constraints from future gravitational waves experiments

So far we have discussed the constraints on $n_t$ from the Ligo Virgo limit on the stochastic background. Now we want to study the possible constraints from the next generation of gravitational waves experiments such as Lisa and ET. In fact we expect Lisa to have a sensitivity to the stochastic background $\Omega_{GW}(k_{Lisa}) \approx 1 \times 10^{-12}$ on scales $k_{Lisa} \approx 1 \times 10^{13}$ [19] while for the Einstein Telescope we expect a sensitivity of $\Omega_{GW}(k_{ET}) \approx 3 \times 10^{-13}$ on scales $k_{ET} \approx 5 \times 10^{15}$ [45] and this means that our discussion on the importance of the runnings on small scales can be extended also to these cases\(^8\). Moreover the large increase in sensitivity, combined with the effects of the higher order corrections, can be translated into tight constraints on the tensor tilt, at least in this specific model of inflation. In figure 10 we plot $\Omega_{GW}$ as a function of $n_t$ (and $\xi$) on Lisa and ET scales, both neglecting and including the runnings up to the fifth order\(^9\). Once that the runnings are included in the analysis, we can strongly improve the constraints, above all on $n_t$, gaining at least an order of magnitude. We summarize our results in tab II. We conclude that, as our

\(^8\) On the Lisa scales $\log \left( \frac{k_{Lisa}}{k_{\ast}} \right) \approx 30.9$ while on the ET scales $\log \left( \frac{k_{ET}}{k_{\ast}} \right) \approx 39.1$. Therefore the runnings are amplified by a factor similar to that on the Ligo-Virgo scales, and our previous considerations are valid, as well.

\(^9\) The runnings are estimated numerically as described in previous subsection.
Figure 10: we plot $\Omega_{GW}(k_{\text{Lisa}})$ (top panels) and $\Omega_{GW}(k_{\text{ET}})$ (bottom panels) as a functions of $n_t$ (right panels) or $\xi$ (left panels). We both include (black solid lines) and neglect (gray dashed lines) the contribution of the runnings up to the fifth order (estimated numerically). The red lines represent the Lisa and ET sensitivity. As we can see, including the runnings the constraints on $n_t$ improve significantly.

IV. CONCLUSION

The search of Primordial Gravitational Waves is one of the most important goals of modern cosmology since they can both provide a substantial evidence for primordial inflation and shed light on its physical nature. Small scale data.
scale gravitational waves experiments such as Ligo-Virgo and (in future) LISA or ET, are sensitive to the stochastic background and so can be used to constrain the physics of inflation on the CMB scales as recently done in some analyses carried out by the Planck collaboration. They have shown that using the Ligo Virgo data together with the Planck data, the tensor tilt $n_t$ can be constrained to be $n_t < 0.53$ at 95% C.L. and that the Ligo-Virgo limit on the stochastic background of PGWs improves a lot the constraints obtained using only the Planck data. In this paper we focused on the robustness of the constraints on the CMB scales obtained using small scales data (such as those of Ligo and Virgo). We have showed that when we want to probe physics on the CMB scales using the small scale bounds on the stochastic background of gravitational waves we should pay attention to the higher order terms in the power law expansion of the primordial spectrum. We have first given an estimation of the error that we commit neglecting such terms in the simplest slow roll scenario. Even if the slow roll models predict an almost scale invariant slightly red tilted primordial spectrum of GWs, we have showed that, because of the extremely small negative running, we can overestimate the PGWs production on the Ligo Virgo scale of about 39%. Clearly in the simplest slow roll models the total amount of the PGWs is too small to be probed by the Ligo Virgo data that instead can be used to constrain blue models of inflation that allow a positive tilt $n_t \sim 0.5 > 0$. We have studied the impact of the higher order corrections for these kinds of constraints and it turns out from a qualitative analysis that the constraints are extremely sensitive to the assumption of a power law primordial spectrum with a constant $n_t$ and that if one includes a small positive (negative) running of about 1% of $n_t$ the bounds can be significantly improved (reduced). In particular as shown in fig.3, a negative running of the 5% of $n_t$ will exactly compensate the growth due to the blue tilt making impossible to constrain $n_t$ using the Ligo Virgo limit. Therefore we can conclude that using the small scale experiments, such as the Ligo Virgo one, in order to probe the inflationary parameters on the CMB scales, albeit possible, is non trivial and the higher order terms may play an important role in the game. In order to better justify the validity of our qualitative analysis, we considered a physical model of inflation that can allow a positive $n_t$, of the order we need to constrain it using the Ligo Virgo data. In particular the model we have studied employs a pseudo scalar axion naturally coupled to gauge fields. A mechanism of particle production takes place during the rolling inflation and this can be translated into a blue tilted spectrum of gravitational waves. We have used the Ligo Virgo limit to probe physics on the CMB scale showing that the higher order corrections induced by the small runnings can effectively provide non negligible corrections on the Ligo Virgo scales. We have first performed a second order analytical computation focusing on the first two tensor runnings and then we have improved our results developing an algorithm able to numerically compute the higher order runnings. So we have included the first five order runnings in the analyses, enough to ensure a convergence in the expansion within a precision of about 4%. In this way we have constrained the parameters of the model only with the Ligo Virgo bounds on $\Omega_{GW}$, both neglecting and including the higher order terms. The results of our physical example, summarized in tab. 1, show how it is effectively possible to constrain blue models on inflation using the Ligo Virgo data, but also that the higher order effects can effectively provide non negligible corrections on small scales and so that they need to be included in the analysis to obtain more precise results. For the same model we analyzed the constraints we can obtain from the upcoming gravitational waves experiments such as Lisa and ET. We have shown that, including the higher order corrections, the spectral tilt can be tightly constrained and that we can gain at least an order of magnitude in the predictions that are summarized in tab. II.

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| Parametrization | Lisa constraints for $r = 0.064$ | Lisa constraints for $r = 0.01$ | Lisa constraints for $r = 0.001$ | Einstein Telescope constraints for $r = 0.064$ | Einstein Telescope constraints for $r = 0.01$ | Einstein Telescope constraints for $r = 0.001$ |
|-----------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $n_t +$ Runnings | $n_t < 0.012$                   | $n_t < 0.024$                   | $n_t < 0.030$                   | $n_t < 4 \times 10^{-4}$        | $n_t < 7.4 \times 10^{-3}$      | $n_t < 0.010$                   |
|                 | $\xi < 3.23$                   | $\xi < 3.42$                   | $\xi < 3.64$                   | $\xi < 3.13$                   | $\xi < 3.31$                   | $\xi < 3.53$                   |
| Only $n_t$      | $n_t < 0.21$                   | $n_t < 0.28$                   | $n_t < 0.35$                   | $n_t < 0.13$                   | $n_t < 0.18$                   | $n_t < 0.24$                   |
|                 | $\xi < 3.70$                   | $\xi < 4.01$                   | $\xi < 4.54$                   | $\xi < 3.52$                   | $\xi < 3.76$                   | $\xi < 4.06$                   |

Table II: Lisa and ET constraints on $n_t$ and $\xi$ for different values of $r$. 
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