TWO S-UNIT EQUATIONS WITH MANY SOLUTIONS

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1. Introduction

In this note we consider two $S$-unit equations for which we will exhibit many solutions. Our first problem concerns solutions to the equation $a+b=c$ where $a$, $b$, and $c$ are coprime integers such that all prime factors of $abc$ lie in a given set $S$ of $s$ primes. In [8] J.-H. Evertse showed that this $S$-unit equation has at most $\exp(4s+6)$ solutions. On the other hand, in [6] P. Erdős, C. Stewart, and R. Tijdeman showed that there exist arbitrarily large sets $S$ such that the $S$-unit equation $a+b=c$ has more than $\exp((4-\epsilon)\sqrt{s/\log s})$ solutions (see also [9] for a refinement of their result). The set $S$ that they exhibited is rather special, and they conjectured that if $S$ were the set of the first $s$ prime numbers then there should be $\gg \exp(s^{\frac{2}{3}-\epsilon})$ solutions to the $S$-unit equation. Moreover, for any set $S$ they conjectured that there are $\ll \exp(s^{\frac{2}{3}+\epsilon})$ solutions. We remark that recently J. Lagarias and K. Soundararajan [12] have shown that if $S$ is the set of the first $s$ prime numbers and the Generalized Riemann Hypothesis is true then the $S$-unit equation has $\gg \exp(s^{\frac{1}{12}-\epsilon})$ solutions. Our first result improves the construction of Erdős, Stewart, and Tijdeman and shows the existence of arbitrarily large sets $S$ with more than $\exp(s^{2-\sqrt{2}-\epsilon})$ solutions.

**Theorem 1.** Let $\beta$ be any positive number with $\beta < 2 - \sqrt{2}$. There exist arbitrarily large sets $S$ of $s$ prime numbers such that the $S$-unit equation $a+b=c$ has at least $\exp(s^\beta)$ solutions in coprime integers $a$, $b$ and $c$ having all their prime factors from $S$.

The second $S$-unit equation that we will consider is a special case of the first: namely, the equation $a+1=c$ with all prime factors of $ac$ lying in the set $S$. Although this is a much more restrictive equation than our first, we are able to find arbitrarily large sets $S$ with many solutions to this equation.

**Theorem 2.** There exist arbitrarily large sets $S$ of $s$ prime numbers such that the equation $a+1=c$ has at least $\exp(s^{\frac{1}{16}})$ solutions where all prime factors of $ac$ lie in $S$. In fact, there exist arbitrarily large integers $N$ such that

$$\#\{d: \ d(d+1)|N\} \geq \exp((\log N)^{\frac{1}{16}}).$$

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The second, stronger, conclusion of Theorem 2 advances a line of inquiry initiated by Erdős and R.R. Hall [5]. They showed the existence of arbitrarily large numbers \( N \) with \( \# \{ d : \ d(d+1)|N \} \gg (\log N)^{\frac{\beta}{2}-\epsilon} \). From the work of A. Hildebrand [10] on consecutive smooth numbers it follows that there are large \( N \) with \( \# \{ d : \ d(d+1)|N \} \gg (\log N)^{\frac{\beta}{2}} \) for any given positive number \( A \). In [1] A. Balog, Erdős, and G. Tenenbaum quantified this and obtained large \( N \) with \( \# \{ d : \ d(d+1)|N \} \gg (\log N)^{\log_3 N/9 \log_4 N} \) where \( \log_3 \) and \( \log_4 \) denote the third and fourth iterated logarithms. For upper bounds on the quantity \( \# \{ d : \ d(d+1)|N \} \) we refer the reader to [3], [4], and [7].

There are at least \( x^{\frac{1}{2}+\epsilon+o(1)} = x^{\frac{1}{2}+\beta+o(1)} \) square-free numbers below \( x \) all of whose prime factors lie below \( (\log x)^{2+\delta} \). If these numbers were randomly distributed then we would expect to find about \( x^{\frac{1}{2}+\beta+o(1)} \) pairs of such consecutive numbers. This suggests that there should be arbitrarily large \( N \) with \( \# \{ d : \ d(d+1)|N \} \geq \exp((\log N)^{\frac{1}{2}+\epsilon}) \). We venture the guess that for any set \( S \), the \( S \)-unit equation \( a+1 = c \) has no more than \( \exp(s^{\frac{1}{2}+\epsilon}) \) solutions, but nothing substantially better than Evertse’s bound appears to be known.

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2. PROOF OF THEOREM 1

Let \( y \) be a large real number and let \( \beta \) and \( \gamma \) be real numbers in \((0,1)\). Consider the set \( \mathcal{L} \) which consists of square-free numbers \( \ell \) having exactly \([y^\beta]\) prime factors each from the interval \([y/2, y]\). Consider also the set \( \mathcal{M} \) which contains square-free numbers \( m \) having exactly \([\gamma y^\beta]\) prime factors each from the interval \([y/4, y/2]\). Note that the elements of \( \mathcal{L} \) are coprime to elements of \( \mathcal{M} \). Further note that

\[
|\mathcal{L}| = \left( \pi(y) - \pi\left(\frac{y}{2}\right) \right)\left[\frac{1}{y^\beta}\right] = L^{1-\beta+o(1)},
\]

where \( L = y^{[y^\beta]} \), and similarly

\[
|\mathcal{M}| = L^{\gamma(1-\beta)+o(1)}.
\]

Pick a number \( m \in \mathcal{M} \) and let \( r(\mathcal{L}; a, m) \) denote the number of elements of \( \mathcal{L} \) lying in the residue class \( a \pmod{m} \). By Cauchy-Schwarz we know that

\[
\sum_{a=1}^{m} r(\mathcal{L}; a, m)^2 \geq \frac{1}{m} \left( \sum_{a=1}^{m} r(\mathcal{L}; a, m) \right)^2 = \frac{|\mathcal{L}|^2}{m}.
\]

The left hand side counts the pairs \( (\ell_1, \ell_2) \) with \( \ell_1 \equiv \ell_2 \pmod{m} \). This congruence has \( |\mathcal{L}| \) trivial solutions, and if \( m < |\mathcal{L}|/2 \) then we are guaranteed \( \gg |\mathcal{L}|^2/m \) non-trivial solutions. Since each element of \( \mathcal{M} \) is below \( y^{[\gamma y^\beta]} \leq yL^\gamma \) we conclude that if \( \gamma < 1-\beta \) then there exist \( \gg L^{2(1-\beta)-\gamma+o(1)} \) non-trivial pairs \( (\ell_1, \ell_2) \) with \( \ell_1 \equiv \ell_2 \pmod{m} \). Therefore, if \( \gamma < 1 - \beta \)
there exist \( L^{2(1-\beta)-\beta\gamma+o(1)} \) triples \((m, \ell_1, \ell_2)\) with \(m \in \mathcal{M}\), \(\ell_1 \neq \ell_2 \in \mathcal{L}\) and \(\ell_1 \equiv \ell_2 \pmod{m}\).

Suppose below that \(\gamma < 1 - \beta\) and consider the ratios \((\ell_1 - \ell_2)/m\) arising from the triples produced above. Restricting to positive ratios, we have produced \( L^{2(1-\beta)-\beta\gamma+o(1)} \) such ratios, all below \(L^{1-\gamma+o(1)}\). Therefore if \(2(1-\beta)-\beta\gamma > 1 - \gamma\) then we can find a popular number \(u \leq L^{1-\gamma+o(1)}\) which occurs as a ratio more than \(L^{2(1-\beta)-\beta\gamma+\gamma-1+o(1)}\) times.

Summarizing, we see that if \(\gamma < 1 - \beta\) and \((2 + \gamma)(1 - \beta) > 1\) then there is a number \(u \leq L^{1-\gamma+o(1)}\) such that the equation \(\ell_1 = \ell_2 + mu\) has more than \(L^{(2+\gamma)(1-\beta)-1+o(1)}\) solutions in integers \(\ell_1 \neq \ell_2 \in \mathcal{L}\) and \(m \in \mathcal{M}\). We already know that \(\ell_1 \) and \(\ell_2\) are coprime to \(m\), so if \(\ell_1 \) and \(\ell_2\) have a common factor then it must be a divisor of \(u\). Since there are at most \(L^{o(1)}\) divisors of \(u\), after removing common factors, we find that for some divisor \(v\) of \(u\), the equation \(\ell_1 = \ell_2 + mv\) has \( L^{2+\gamma(1-\beta)-1+o(1)}\) solutions in coprime integers \(\ell_1 \neq \ell_2 \in \mathcal{L}\), and \(m \in \mathcal{M}\). Take \(S\) to be the set of all primes in \([y/4, y]\) union the prime factors of \(v\). Then \(|S| \leq \pi(y) - \pi(y/4) + \log v \leq y\), and we have exhibited more than \(\exp(y^\beta)\) solutions to this \(S\)-unit equation. If \(\beta < 2 - \sqrt{2}\) then we can find a \(\gamma\) satisfying the conditions \(\gamma < 1 - \beta\) and \((2 + \gamma)(1 - \beta) > 1\), and so Theorem 1 follows.

3. Proof of Theorem 2

Throughout we let \(y\) be a large real number. We need first the following zero-density result which may be found in [11] (see the Grand Density Theorem 10.4 on page 260).

**Lemma 3.1.** There exists a constant \(C > 0\) such that for any \(\frac{1}{2} \leq \alpha < 1\) the region

\[
\mathcal{R}(\alpha, y) := \{s : \ Re\ (s) \geq \alpha, \ |Im\ (s)| \leq y\},
\]

contains at most \((Q^2y)^{C(1-\alpha)+o(1)}\) zeros of primitive Dirichlet \(L\)-functions with conductor below \(Q\). It is permissible to take \(C = \frac{12}{5}\).

**Proposition 3.2.** Let \(\beta\) be a real number with \(0 < \beta < 1 - 3C(1-\alpha)\). Let \(K = [y^\beta]\) and put \(Z = y^K\). There exist \( \mathbb{Z}^{1-\beta+o(1)} \) square-free numbers \(q\) having exactly \(K\) prime factors each from the interval \([y/2, y]\), and such that for every non-trivial character \(\pmod{q}\) the corresponding \(L\)-function has no zeros in \(\mathcal{R}(\alpha, y)\).

**Proof.** Clearly there there are \([\pi(y) - \pi(y/2)]\) square-free integers \(q\) having exactly \(K\) prime factors each from the interval \([y/2, y]\). We must exclude those moduli for which there exists a non-trivial character whose \(L\)-function has a zero in \(\mathcal{R}(\alpha, y)\). A bad modulus \(q\) must be divisible by some number \(d\) with \(j\) prime factors \((\pmod{y/2})\) \(j\leq d \leq y^j\) and \(1 \leq j \leq K\) such that there is a primitive character \(\pmod{d}\) whose \(L\)-function has a zero in \(\mathcal{R}(\alpha, y)\). By Lemma 3.1 there are at most \(y^{(2j+1)(C(1-\alpha)+o(1))}\) possibilities for \(d\). Given a \(d\) there are at most \([\pi(y) - \pi(y/2)]\) multiples of \(d\) that must be excluded. Thus we must exclude at most

\[
\sum_{j=1}^{K} y^{(2j+1)(C(1-\alpha)+o(1))}\left(\pi(y) - \pi(y/2)\right)_{K-j}\]

moduli. Since \(\beta < 1 - 3(1-\alpha)\) this is small compared to \([\pi(y) - \pi(y/2)]\) and so we have

\[
\gg (\pi(y) - \pi(y/2)) = \mathbb{Z}^{1-\beta+o(1)}\text{ suitable moduli } q.
\]
**Proposition 3.3.** Let $X = Z^\gamma$ and suppose that $\gamma(1 - \alpha - \beta) > 1$. Let $q$ be one of the moduli produced in Proposition 3.2. Then there are $\gg Z^{(1-\beta)\gamma-1+o(1)}$ integers $\ell \leq X$ with each $\ell$ being square-free, divisible only by primes below $y$, and $\ell \equiv 1 \pmod{q}$.

Assuming this Proposition for the moment we show how to deduce Theorem 2.

**Proof of Theorem 2.** Let $\alpha$, $\beta$, and $\gamma$ be as in Lemma 3.1, Propositions 3.2 and 3.3. That is

\[(3.1) \quad \frac{1}{2} \leq \alpha < 1, \quad 0 < \beta < 1 - 3C(1 - \alpha), \quad \text{and} \quad \gamma(1 - \alpha - \beta) > 1.\]

By Propositions 3.2 and 3.3 we know that there are at least $Z(1 - \beta)(1 + \gamma) - 1 + o(1)$ pairs $(\ell, q)$ satisfying the conclusions of those Propositions. Consider the ratio $(\ell - 1)/q$ which is an integer which lies below $2^K X/Z < Z^{\gamma-1+o(1)}$. If

\[(3.2) \quad (1 - \beta)(1 + \gamma) - 1 > \gamma - 1,\]

then there is a popular value $m$ which occurs as the ratio $(\ell - 1)/q$ at least $Z^{1-\beta-\beta \gamma + o(1)}$ times. Take $N = m \prod_{p \leq y} p$ and note that if $(\ell - 1)/q = m$ then $qm$ and $\ell = qm + 1$ are consecutive divisors of $N$. Therefore

$$\# \{d : d(d + 1) | N\} \geq Z^{1-\beta-\beta \gamma + o(1)} \geq \exp((\log N)^\beta),$$

since by the prime number theorem $N = e^{y+o(y)}$, and $\log Z = (1 + o(1)) y^\beta \log y$.

To complete the proof we need only find the largest $\beta$ for which (3.1) and (3.2) hold. A little calculation shows that it is best to take $\gamma$ slightly larger than $3C + \sqrt{9C^2 + 3C}$, take $\alpha = 1 - \frac{1+1/\gamma}{3C+1}$, and $\beta$ is then slightly smaller than $1 + 3C + \sqrt{9C^2 + 3C} - 1$. Since $C = \frac{12}{5}$ is permissible we conclude that $\beta = \frac{1}{16}$ is allowed.

It remains finally to prove Proposition 3.3. To this end we require the following Lemma.

**Lemma 3.4.** Let $q \leq Z$ be one of the moduli produced in Proposition 3.2 so that $L(s, \chi)$ has no zeros in the region $\mathcal{R}(\alpha, y)$, and suppose that $\beta < 1 - \alpha$. For any complex number $s$ with $\Re(s) > 0$ we define

$$F(s, \chi; y) = \sum_{\ell=1}^{\infty} \frac{\mu(\ell)^2 \chi(\ell)}{\ell^s} = \prod_{p \leq y} \left(1 + \frac{\chi(p)}{p^s}\right).$$

For any $\epsilon > 0$, if $|t| \leq y/2$ then we have

$$|F(\alpha + \epsilon + it, \chi; y)| \ll_\epsilon (qy)^\epsilon,$$

while if $|t| > y/2$ we have

$$|F(\alpha + \epsilon + it, \chi; y)| \ll \exp(y^{1-\alpha}).$$
Proof. Taking logarithms it suffices to estimate \( \sum_{p \leq y} \chi(p) p^{-\alpha - \epsilon - it} \). Since \( \alpha < 1 \) this is trivially \( \leq y^{1-\alpha} \) and the second assertion follows.

If \( z \leq y \) then note that

\[
\sum_{\substack{n \leq z}} \Lambda(n) \chi(n)n^{-it} = \frac{1}{2\pi i} \int_{1+rac{1}{\log y} + i\infty}^{1+\frac{1}{\log y} + i\infty} \frac{L'(w, \chi)}{L(w, \chi)} \frac{z^w}{w} \, dw
\]

\[
= - \sum_{\rho \in \sigma} \frac{z^{\rho - it}}{\rho - it} + O(\log^2 qy),
\]

by following closely the standard argument in prime number theory leading to the ‘explicit formula’ for primes (see for example H. Davenport [2]); here \( \rho \) runs over non-trivial zeros of \( L(s, \chi) \). By assumption \( \text{Re}(\rho) \leq \alpha \) for each zero counted in our sum. Since there are \( \ll \log qy \) zeros in each interval \( k \leq |\rho - it| \leq k + 1 \) for \( 0 \leq k \leq y \) we conclude that

\[
\sum_{\substack{n \leq z}} \Lambda(n) \chi(n)n^{-it} \ll z^\alpha \log(qy) \log z + \log^2(qy) \ll z^\alpha \log(qy) \log z.
\]

Trivially we also have that this sum is bounded by \( \ll z \). Using these two estimates and partial summation we easily deduce that

\[
\sum_{2 \leq n \leq z} \frac{\Lambda(n) \chi(n)}{n^{\alpha + \epsilon + it} \log n} \ll (\log qy)^{1-\frac{\epsilon}{1-\alpha}}.
\]

This proves the Lemma.

Proof of Proposition 3.3. Using the orthogonality of characters \( (\mod q) \) we see that

(3.3)

\[
\sum_{\substack{\ell \equiv 1 \mod q \atop p|\ell \implies p \leq y}} \mu(\ell)^2 e^{-\ell/x} = \frac{1}{\phi(q)} \sum_{\substack{\ell \equiv 1 \mod q \atop p|\ell \implies p \leq y}} e^{-\ell/x} + \frac{1}{\phi(q)} \sum_{\substack{\chi \mod q \atop \chi \neq \chi_0 \atop p|\ell \implies p \leq y}} \sum_{\chi} \chi(\ell) \mu(\ell)^2 e^{-\ell/x}.
\]

We now obtain an upper bound for the contribution from non-trivial characters to (3.3). For any \( c > 0 \) we have

\[
\sum_{p|\ell \implies p \leq y} \chi(\ell) \mu(\ell)^2 e^{-\ell/x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s, \chi; y)x^s \Gamma(s) \, ds.
\]

We take \( c = \alpha + \epsilon \) and estimate the integral using Lemma 3.4. Since \( |\Gamma(c + it)| \) decays exponentially in \( |t| \) by Stirling’s formula, we obtain that the above is \( \ll x^{\alpha + \epsilon}(qy)^{c} \). Thus we conclude that

(3.4)

\[
\sum_{\substack{\ell \equiv 1 \mod q \atop p|\ell \implies p \leq y}} \mu(\ell)^2 e^{-\ell/x} = \frac{1}{\phi(q)} \sum_{\substack{\ell \equiv 1 \mod q \atop p|\ell \implies p \leq y}} e^{-\ell/x} + O(x^{\alpha + \epsilon}(qy)^{c}).
\]
We take $x = X / \log X$ in (3.4) and note that

$$\sum_{\ell \leq X \atop \ell \equiv 1 \pmod{q}, p|\ell \Rightarrow p \leq y} \mu(\ell)^2 \geq \sum_{\ell \equiv 1 \pmod{q}, p|\ell \Rightarrow p \leq y} \mu(\ell)^2 e^{-\ell/x} + O(1).$$

Now

$$\sum_{\ell \leq x \atop (\ell, q) = 1, p|\ell \Rightarrow p \leq y} \mu(\ell)^2 e^{-\ell/x} \gg \sum_{\ell \leq x \atop (\ell, q) = 1} \mu(\ell)^2 \geq \left( \frac{\pi(y) - \omega(q)}{\log x / \log y} \right) = Z^{\gamma(1 - \beta) + o(1)}.$$

Using (3.4), and recalling that $q \leq Z$ and the hypothesis that $\gamma(1 - \beta) - 1 > \gamma \alpha$, we obtain (choosing $\epsilon$ small enough) the Proposition.

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