Kummer’s Original Type Congruence Relation
for the Universal Bernoulli Numbers

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Preface.
The aim of this paper is to give a congruence on universal Bernoulli numbers which
congruence is the same type of Kummer’s original paper [K]. The remarkable thing
is the index of prime power of the modulus of the congruence is the half of the
original one. We mention in this paper that this estimate is best possible. It is
surprising fact for the author that the critical index is not less than half of the
original.

The motivation of this work is the investigation on generalized Bernoulli-Hurwitz
numbers by the author himself in [Ô]. Kummer’s original type congruences in [Ô] hold modulo the same power as the original one. When the author was working to
get a proof of such Kummer type congruences in [Ô], he knew several researches
on universal Bernoulli numbers, especially Adelberg’s remarkable papers [A1] and
[A2].

There are three key lemmas for our proof of the main result, namely Lemma
3.2.1, Lemma 3.2.8, and Lemma 3.3.1. Lemma 3.2.8 is a very natural extension of
already proved Lemma 3.2.1. Regrettably, Lemma 3.2.8 is not yet proved. The
Lemma 3.2.8 might be true by several reason. The author hope that 3.2.8 would
be proved in the near future.

Professor Adelberg gave several crucial comments on the first version of this
paper. This version is much improved by his comments. The number theorists in
Japan did not know the universal Bernoulli numbers. The author expects that this
research would be useful for the researchers who are interested in Bernoulli numbers
and Hurwitz numbers.
Combinatorics

1.1. Properties of factorial
1.2. Lagrange inversion formula

Universal Bernoulli numbers and their properties

2.1. Definition of universal Bernoulli numbers
2.2. Schur function type expression of universal Bernoulli numbers
2.3. Clarke’s theorem

Kummer type congruence for universal Bernoulli numbers

3.1. Main results
3.2. Preparation for the proof of main theorem (1)
3.3. Preparation for the proof of main theorem (2)
3.4. Proof of the main theorem
3.5. Kummer-Adelberg congruence

References

Convention.

(1) For a rational number $\alpha$, we denote by $\lfloor \alpha \rfloor$ the greatest integer not exceed $\alpha$, and denote by $\lceil \alpha \rceil$ the least integer not less than $\alpha$.

(2) We use the notation

$$(n)_r = n(n-1) \cdots (n-r+1).$$

(3) The congruence relations on polynomials in several variables means the congruence on the coefficients of each the similar terms.

(4) Let $R$ be a commutative ring and $z$ be an indeterminate. We denote

$$R\langle\langle z \rangle\rangle = \left\{ \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \mid a_n \in R \text{ for all } n \right\}.$$ 

For two power series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad \text{and} \quad \psi(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}$$

in $R\langle\langle z \rangle\rangle$ and for a subset $S \subset R$, the congruence

$$\varphi(z) \equiv \psi(z) \mod S$$

means that $a_n - b_n \in S$ for all $n$. Especially, if $p$ is a prime element in $R$, and $S = p^d R$ for a positive integer $d$, then we write simply that $\varphi(z) \equiv \psi(z) \mod p^d$. 
1. Combinatorics.

1.1. Properties of factorial. We recall the following two properties about factorial. Let \( n \) and \( k \) be non-negative integers, and \( p \) be a rational prime. If \( n = kp + a \ (0 \leq a < p) \), then

\[
\text{ord}_p(n!) = \text{ord}_p((kp)!) = \text{ord}_p(k!) + k.
\]

We denote by \( S_p(n) \) the sum of \( p \)-adic digits of \( n \). It is well known that

\[
\text{ord}_p(n!) = n - S_p(n)/p - 1.
\]

1.2. Lagrange inversion formula. For a power series \( F(z) \), we denote by \([z^n] F(z)\) the coefficient of the term \( z^n \). The following is called Lagrange inversion formula. This is a very strong tool.

Proposition 1.2.1. Let \( \varphi(u) = u + \cdots \) is a power series with only positive terms in \( u \). The coefficient of degree 1 term is supposed to be 1. Let \( \psi(t) = \varphi^{-1}(t) \) be the inverse power series of \( \varphi(u) \), namely \( \varphi(\psi(t)) = t \). Then

\[
[u^n] \left( \frac{u}{\varphi(u)} \right) = \frac{\ell}{\ell - n} [t^n] \left( \frac{\psi(t)}{t} \right)^{\ell - n}.
\]

About the proof of this, see the reference in [A1] Cp.123CProposition 2.1.

2. The universal Bernoulli numbers and their properties.

2.1. Definition of the universal Bernoulli numbers. Let \( c_1, c_2, \cdots \) are indeterminates. We consider the power series

\[
u = u(t) = t + \sum_{n=1}^{\infty} c_n \frac{t^{n+1}}{n+1}
\]

and its inverse power series

\[
t = t(u) = u - c_1 \frac{u^2}{2!} + (3c_1^2 - 2c_2) \frac{u^3}{3!} + \cdots.
\]

We define the universal Bernoulli numbers (of degree 1) \( \hat{B}_n \in \mathbb{Q}[c_1, c_2, \cdots] \) by

\[
\frac{u}{t(u)} = \sum_{n=0}^{\infty} \hat{B}_n \frac{u^n}{n!}
\]

If we specialize \( c_n \) as \( c_n = (-1)^n \), we have \( \hat{B}_n = B_n \), the usual Bernoulli numbers, because of \( \psi(t) = \log(1 + t) \) and \( \varphi(u) = e^t - 1 \).

2.2. Schur function type expression of universal Bernoulli numbers.
We introduce several notations. For a finite sequence $U = (U_1, U_2, \cdots)$ of non-negative integers, we define the weight of $U$ to be $w(U) = \sum_j jU_j$, and the degree of $U$ to be $d(U) = \sum_j U_j$. We can regard $U$ to be a partition of $w(U)$. For simplicity, we use notations

\begin{align}
U! &= U_1! U_2! \cdots \left(\frac{d}{U}\right) = \frac{d!}{U!}, \\
\Lambda^U &= 2^{U_1} 3^{U_2} 4^{U_3} \cdots, \\
c^U &= c_1^{U_1} c_2^{U_2} c_3^{U_3} \cdots.
\end{align}

Moreover we denote

\begin{align}
(2.2.2) \quad \gamma_U &= \Lambda^U U!.
\end{align}

For the function $\psi(t)$ in 1.2.1, let $h(t) = (\psi(t)/t) - 1$. Then

\begin{align}
(2.2.3) \quad (\psi(t)/t)^s &= (1 + h(t))^s = \sum_{d=0}^{\infty} \binom{s}{d} h^d(t),
\end{align}

and

\begin{align}
(2.2.4) \quad h^d(t) &= \sum_{d(U) = d} \binom{d}{U} \frac{c^U}{\Lambda^U} t^{w(U)}.
\end{align}

Therefore, by denoting

\begin{align}
(2.2.5) \quad \tau_U &= (-1)^{d(U)-1} \frac{(w(U) + d(U) - 2)!}{\gamma_U},
\end{align}

and by using Proposition 1.2.1 for $\ell = 1$, we have

\begin{center}
\textbf{Proposition 2.2.6.}
\end{center}

\begin{equation}
\frac{\hat{B}_n}{n} = \sum_{w(U) = n} \tau_U c^U D
\end{equation}

According to Haigh’s pointing out ([C], p.594, $\ell = 1$), we call this expression \textit{Schur function expression} of $\hat{B}_n$.

\section{2.3. Clarke’s theorem.}

For convenience of the reader, we mention here Clarke’s theorem that is the universal version of von Staudt-Clausen theorem jointed with von Staudt second theorem.

We denote $a|_p = a/p^{\text{ord}_p a}$ for given positive integer $a$. 

Proposition 2.3.1.

\[
\hat{B}_1 = \frac{1}{2} c_1,
\]
\[
\frac{\hat{B}_2}{2} = -\frac{1}{4} c_1^2 + \frac{1}{3} c_2,
\]
\[
\frac{\hat{B}_n}{n} \equiv \begin{cases} 
\sum_{\substack{n = a(p-1) \\
p \text{ prime}}} \frac{a |_p - 1 \mod p^{1 + \text{ord}_p a}}{p^{1 + \text{ord}_p a}} c_{p-1}^a & \text{(if } n \equiv 0 \mod 4) \\
\frac{c_1^{n-6} c_3^2}{2} - \frac{nc_1^n}{8} + \sum_{\substack{n = a(p-1) \\
p \text{ odd prime}}} \frac{a |_p - 1 \mod p^{1 + \text{ord}_p a}}{p^{1 + \text{ord}_p a}} c_{p-1}^a & \text{(if } n \not\equiv 2 \text{ and } n \equiv 2 \mod 4) \\
c_1^n + c_1^{n-3} c_3 & \text{(if } n \not\equiv 1 \text{ and } n \equiv 1, 3 \mod 4) 
\end{cases} \mod \mathbb{Z}[c_1, c_2, \ldots].
\]

The proof is given by analysing the Schur function expression 2.2.6 of \( \hat{B}_n \). See [C], Theorem 5.

3 Kummer type congruence relations for universal Bernoulli numbers.

3.1. Main result.

We prove that Kummer’s original type congruence for universal Bernoulli numbers holds modulo \( p^{[a/2]} \) as follows.

**Theorem 3.1.1.** Fix a prime number \( p \). Let \( a \) and \( n \) be positive integers such that \( n > a \) and \( n \not\equiv 0 \mod (p - 1) \). Then

\[
\sum_{r=0}^{a} \binom{a}{r} (-1)^r c_{p-1}^{a-r} \frac{\hat{B}_{n+r(p-1)}}{n+r(p-1)} \equiv 0 \mod p^{[a/2]}.
\]

**Remark 3.1.2.** (1) Suppose \( n > a = 1 \) and \( n \not\equiv 0, 1 \mod (p - 1) \). Then the above congruence holds modulo \( p \). This fact is proved by Adelberg ([A1], Theorem 3.2). For the case of \( n \equiv 1 \mod (p - 1) \), see the main theorem of [A2].

(2) Let \( p \geq 7 \) be a prime. Let \( U \) is the partition with \( U_1 = p, U_{2p-1} = (p-3)/2, \) and \( U_j = 0 \) for the others. Then \( w(U) = p + (p-5)(2p-1)/2 \equiv -1 \mod (p - 1) \). Then we can prove

\[
\text{ord}_p(\tau_U) = (p-5)/2 \quad (\equiv \lfloor (p-4)/2 \rfloor).
\]

Therefore we see that the above is a best possible estimate by the equation (3.4.1) below for \( a = p - 4 \) and \( n = w(U) \).

(3) In the example in (2) above, if we set \( p = 5 \) then we see \( \text{ord}_5(\tau_U) = 0 \). Here \( n = w(U) = 5 \equiv 1 \mod (5 - 1) \). So this is a case removed from Theorem 3.2 in
[A1]. Keeping this example in mind, we can slightly improve Lemma 3.3.1 below. Then we can prove the case of $a = 1$ in 3.1.1.

We show in 3.5 that 3.1.1 implies directly Adelberg’s congruence ([A2], part (i) of Theorem).

**Corollary 3.1.3.** (Adelberg’s congruence relation) Let $n$ and $a$ be positive integers. If $n \not\equiv 0, 1 \pmod{p-1}$ and $n > a$, then

$$c_{p-1}p^{a-1} \frac{\hat{B}_n}{n} \equiv \frac{\hat{B}_{n+p^{a-1}(p-1)}}{n+p^{a-1}(p-1)} \pmod{p^a}.$$  

3.2. Preparation for the proof of main theorem (1).

From now on, we denote by $k$ the least non-negative integer that is congruent to $k$ modulo $p$. In this section we prove the following estimate.

**Lemma 3.2.1.** We fix an odd prime $p$. Let $a$, $q$, and $n$ be non-negative integers. Then

$$\sum_{r=0}^{a} \frac{(r+q)p+n)!}{(r+q)!p^{r+q}} \binom{a}{r} \equiv 0 \pmod{p^M},$$

where

$$M = \left\{ \begin{array}{ll} \ord_p(n!) & \text{(if } n \geq ap), \\ a - \lfloor n/p \rfloor + \ord_p(n!) - \lfloor (a - \lfloor n/p \rfloor)/p \rfloor & \text{(if } n < ap). \end{array} \right.$$  

**Proof.** We give a proof by using a generating function$^1$. We consider the function

$$(3.2.2) \quad F(v) = \exp(v^p/p) = \sum_{r=0}^{\infty} \frac{(v^p/p)^r}{r!} = \sum_{r=0}^{\infty} \frac{(r)^!}{r!p^{r} (rp)!}.$$  

Obviously this series belongs to $Z(\langle v \rangle)$. We investigate

$$(3.2.3) \quad \left( \left( \frac{d}{dv} \right)^p + 1 \right)^a \left(v^n F(v)\right).$$

This is a polynomial of $v$ times $F(v)$. The coefficients of the polynomial are as follows. The highest term is $v^{n+ap(p-1)}$ with the coefficient 1, and

(a) if $n \geq ap$, then the lowest term is $v^{n-ap}$, and the whole polynomial is a polynomial of $v^p$ times $v^{n-ap}$;

(b) if $n < ap$, then the lowest term is $v^a$, the whole polynomial is a polynomial of $v^p$ times $v^a$.

We divide these terms into several groups as follows. The highest term $v^{n+ap(p-1)}$ is itself consists one of the groups. The following higher $p$ terms, including the terms with zero coefficient, consist the next one of the groups. We continue similar

$^1$The author does not know any proofs of 3.2.1 in combinatorics.
grouping with $p$ terms each. Although the number of the finally remaining terms is possibly less than $p$, we regard the terms to be a group. If $n \geq ap$, then we devide 

$$\frac{(n + ap(p-1)) - (n - ap)}{p} + 1 = ap + 1$$

terms in to the groups. Hence we get $a + 1$ groups. If $n > ap$, then we devide

$$\{ (n + ap(p-1)) - p \} / p = \{ (n + ap(p-1)) - (n - \lfloor n/p \rfloor p) \} / p = a(p-1) + \lfloor n/p \rfloor p$$
terms into the groups. Hence we get

$$\lceil (a(p-1) + \lfloor n/p \rfloor p) / p \rceil + 1 = a - \lfloor (a - \lfloor n/p \rfloor) / p \rfloor + 1$$
groups. We denote by $w_0(v), w_1(v), \cdots$ the sums of the terms in the each group, according to the order from lower to higher. Then we have

$$\begin{align*}
(3.2.4) \quad \left( \left( \frac{d}{dv} \right)^p + 1 \right) (v^n F(v)) = \left\{ \begin{array}{ll}
\sum_{j=0}^{a} w_j(v) & \text{(if } n \geq ap), \\
\sum_{j=0}^{a-\lfloor (a-\lfloor n/p \rfloor) / p \rfloor} w_j(v) & \text{(if } n < ap). \\
\end{array} \right.
\end{align*}$$

The most important things are as follows: namely

(a) if $n \geq ap$, then

$$\begin{align*}
(3.2.5) \quad w_j(v) &= \begin{cases}
p^\text{ord}_p(n) - \lfloor \frac{n}{p} \rfloor p^a - \lfloor (a - \lfloor \frac{n}{p} \rfloor) / p \rfloor \sum_{i=0}^{p-a} F_{0i} v^{ip} & \text{(if } j = 0) \\
p^\text{ord}_p(n) - \lfloor \frac{n}{p} \rfloor - j p^a - \lfloor (a - \lfloor \frac{n}{p} \rfloor) / p \rfloor - j v^{\pi + j p^2 + (p-a)p} \sum_{i=0}^{p-1} F_{ji} v^{ip} & \text{(if } 1 \leq j < \text{ord}_p(n! - \lfloor \frac{n}{p} \rfloor)} \\
v^{n+ap(p-1)} & \text{(if } j = a - \lfloor (a - \lfloor \frac{n}{p} \rfloor) / p \rfloor). \\
\end{cases}
\end{align*}$$

(b) if $n < ap$, then

$$\begin{align*}
(3.2.6) \quad w_j(v) &= \begin{cases}
p^\text{ord}_p(n! - \lfloor n/p \rfloor - \lfloor (a - \lfloor n/p \rfloor) / p \rfloor) v^{\pi} \sum_{i=0}^{p-1} F_{0i} v^{ip} & \text{(if } j = 0) \\
p^\text{ord}_p(n!) - \lfloor n/p \rfloor - j p^a - \lfloor (a - \lfloor n/p \rfloor) / p \rfloor - j v^{\pi + j p^2 + (p-a)p} \sum_{i=0}^{p-1} F_{ji} v^{ip} & \text{(if } 1 \leq j < \text{ord}_p(n! - \lfloor n/p \rfloor)} \\
v^{n+ap(p-1)} & \text{(if } j = a - \lfloor (a - \lfloor n/p \rfloor) / p \rfloor). \\
\end{cases}
\end{align*}$$

While these facts can be proved by induction on $a$, we minimally explain on the process. In the first two cases in (3.2.6), the $p$-power of the head is appeared when
we operate \((\frac{d}{dv})^p\) to \(v^n\). Besides each operation of \((\frac{d}{dv})^p\) gives multiplication of one
\(p\), this factor is spoiled by the contribution by the similar terms obtained by the
operation by \((\frac{d}{dv})^p\) to \(F(v)\). Now we assume that \(\ell\) is a positive integer such that
there exists a positive integer \(k\) with satisfying \(p^2|k\) and \(k \leq \ell \leq k + (p - 1)\). Let us
consider the following situation; namely, after operating \((\frac{d}{dv})^p\) several times, we are
going to operating \((\frac{d}{dv})^p\) to \(v^\ell\). If this operation is done the coefficients is multiplied
by at least \(p^2\). In this situation, the similar terms come from the group of the next
higher level. So the order of \(p\) is increased by at least one. We just finish to explain
the first \(p\)-factors in the first two cases in (3.2.6).

The other \(p\)-factors come by the following reason. When \((\frac{d}{dv})^p + 1\) operates to
\(F(v)\), we have

\[
(\frac{d}{dv})^p + 1 \quad F(v) = \left(\frac{d}{dv}\right)^{p-1} v^{p-1} F(v) + F(v)
\]

\[
= (p-1)! F(v) + \cdots + F(v).
\]

Since \((p-1)! + 1 \equiv 0 \mod p\), all the coefficients of the terms in (3.2.7) are divisible
by \(p\). Only this mechanism gives rise to the other \(p\)-factors.

If we regard the right hand side of (3.2.4) to be a linear combination of terms
\(\{v^m/m!\}\), the term whose coefficient has the least \(p\)-factor is just the first terms
in (3.2.5) and in (3.2.6). In other words, if we regard the right hand side to be a
linear combination of \(\{v^j F(v)\}\), the term whose coefficient has the least \(p\)-factor is
the term \(v^j F(v)\) with the least \(j\). Because the coefficient of \(v^{qp+n}/(qp+r)!\) in

\[
\left(\left(\frac{d}{dv}\right)^p + 1\right)^a v^n F(v) = \sum_{r=0}^a \binom{a}{r} \left(\frac{d}{dv}\right)^{pr} \left(\sum_{j=0}^{\infty} \frac{(jp+n)!}{j!p^r (jp+n)!}\right)
\]

is just the left hand side of our claim, the proof has completed. \(\square\)

We need a variant with replacing \(q\) in 3.2.1 by negative \(r_0\) as follows.

**Lemma 3.2.8.** (This is a Conjecture at present.) Let \(p\) be an odd prime, and
\(a\) be a positive integer. Suppose \(r_0\) is an integer with \(0 < r_0 \leq a\). Let \(n \geq r_0 p\)
be an integer. Then

\[
\sum_{r=r_0}^a \frac{(r-r_0)p+n)!}{(r-r_0)!p^{r-r_0}} \binom{a}{r} \equiv 0 \mod p^M,
\]

where

\[
M = \begin{cases} \text{ord}_p(n!) & \text{if } n \geq ap, \\ a - \lfloor n/p \rfloor + \text{ord}_p(n!) - \lfloor (a - \lfloor n/p \rfloor)/p \rfloor & \text{if } n < ap. \end{cases}
\]

Although this lemma is not yet proved, many numerical examples suggest this would
be true and it seems natural if we replace the factorials by the function \(\Gamma\) with
comparing 3.2.1. So we can strongly expect the truth of this lemma.

**3.3. Preparation for the proof of main theorem (2).**
We show the following Lemma in this subsection.

**Lemma 3.3.1.** Let $p$ be an odd prime, and $U$ be a partition with $U_{p-1} = 0$. Assume $d(U) > 0$. Then for $\tau_U$ defined in (2.2.5) we have

$$\ord_p(\tau_U) \geq \left\lfloor \frac{w(U) + d(U) - 2}{2p} \right\rfloor.$$ 

**Remark 3.3.2.** As is mentioned in 3.1.2(2), if $U$ is the partition with $U_1 = p$, $U_{2p-1} = (p - 5)/2$, and $U_j = 0$ for the others, then we have $w(U) = p + \frac{(2p-1)(p-5)}{2}$, $d(U) = p + \frac{p-5}{2}$, and $\ord_p(\tau_U) = (p - 5)/2$. On the other hand

$$\left\lfloor \frac{w(U) + d(U) - 2}{2p} \right\rfloor = \left\lfloor \frac{p^2 - 3p - 2}{2p} \right\rfloor = \left\lfloor \frac{p - 5}{2} + \frac{p - 1}{p} \right\rfloor = \frac{p - 5}{2}.$$ 

Hence the above estimate is also best possible.

**Proof of 3.3.1.** For simplicity we write $w(U) = n$ and $d(U) = d$. Since $d(U) > 0$, we have $n + d - 2 > 0$. Firstly we suppose $U_{2p-1} \neq 0$. Then we have

$$\ord_p(\tau_U) = \ord_p((n + d - 2)! - \ord_p(\gamma_U)$$

$$= \ord_p\left(\left(-2 + \sum_{j \not\equiv p-1} (j + 1)U_j\right)! - \sum_{(\epsilon, k) \neq (1, 1)} kU_{\epsilon p} - 1 - \sum_{j \not\equiv p-1} \ord_p(U_j)\right)$$

(\text{where } \epsilon \text{ runs through the positive integers coprime to } p.)

$$\geq \ord_p\left(\left(-2 + \sum_{j \not\equiv p-1, 2p-1} jU_j + 2pU_{2p-1}\right)! - \sum_{(\epsilon, k) \neq (1, 1)} kU_{\epsilon p} - 1 - \ord_p(U_{2p-1})\right)$$

$$= \ord_p\left(\left(-2 + \sum_{p \not\equiv j+1} jU_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} (\epsilon p^k - 1)U_{\epsilon p} - 1 + 2pU_{2p-1}\right)!\right)$$

$$- \sum_{(\epsilon, k) \neq (1, 1)} kU_{\epsilon p} - 1 - \ord_p(U_{2p-1})$$

$$\geq \sum_{\nu = 1}^{\infty} \left|\frac{1}{p^\nu}\left(-2 + \sum_{p \not\equiv j+1} jU_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} (\epsilon p^k - 1)U_{\epsilon p} - 1 + 2pU_{2p-1}\right)\right|$$

$$- \sum_{(\epsilon, k) \neq (1, 1)} kU_{\epsilon p} - 1 - \ord_p(U_{2p-1})$$

$$= \left[\frac{1}{p}\left(-2 + \sum_{p \not\equiv j+1} jU_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} (\epsilon p^k - 1)U_{\epsilon p} - 1 + 2pU_{2p-1}\right)\right]$$

$$+ \sum_{\nu = 2}^{\infty} \left|\frac{1}{p^\nu}\left(-2 + \sum_{p \not\equiv j+1} jU_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} (\epsilon p^k - 1)U_{\epsilon p} - 1 + 2pU_{2p-1}\right)\right|$$

$$- \sum_{(\epsilon, k) \neq (1, 1)} kU_{\epsilon p} - 1 - \ord_p(U_{2p-1})D$$

$$\geq \left[\frac{1}{p}\left(-2 + \sum_{p \not\equiv j+1} jU_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} (\epsilon p^k - 1)U_{\epsilon p} - 1 + 2pU_{2p-1}\right)\right]$$

$$= \left[\frac{1}{p}\left(-2 + \sum_{p \not\equiv j+1} jU_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} (\epsilon p^k - 1)U_{\epsilon p} - 1 + 2pU_{2p-1}\right)\right].
3.4. Proof of the main theorem.

Now we can replace \(-\frac{2}{p}\) in the front by \(-\frac{1}{p}\). The reason is as follows: if the sum of the other terms in the parentheses just in \(\lfloor \frac{1}{p} \rfloor\) is divisible by \(p\), then, after operating \(\lfloor \ \rfloor\), both of \(-\frac{2}{p}\) and \(-\frac{1}{p}\) give \(-1\); on the other hand, if such the sum of the terms is not divisible by \(p\), the remainder obtained by dividing it by \(p\) is at least \(\frac{1}{p}\). About the term \(j U_j\) in the second sum, we see \(j \geq (j + 1)/2\), and about the terms in the third sum, we see \(\epsilon p^k - kp - 1 > \epsilon p^k/2\) as far as \((\epsilon, k) \neq (1, 1), (2, 1)\). Therefore

\[
\geq \left[\frac{1}{2p} \left( -2 + \sum_{p \not\mid j + 1} (j + 1) U_j + \sum_{(\epsilon, k) \neq (1, 1), (2, 1)} \epsilon p^k U_{\epsilon p^k - 1} + 2p U_{2p - 1} \right) + U_{2p - 1} \right] - U_{2p - 1}
+ \operatorname{ord}_p((-2 + 2p U_{2p - 1}!) - \left\lfloor \frac{-2 + 2p U_{2p - 1}}{p} \right\rfloor - \operatorname{ord}_p(U_{2p - 1}!)
= \left[\frac{-2 + w(U) + d(U)}{2p}\right]
+ \operatorname{ord}_p(2U_{2p - 1}!) + 2U_{2p - 1} - \operatorname{ord}_p(2p U_{2p - 1}) - \left\lfloor \frac{-2 + 2p U_{2p - 1}}{p} \right\rfloor - \operatorname{ord}_p(U_{2p - 1}!)
= \left[\frac{-2 + w(U) + d(U)}{2p}\right]
+ \operatorname{ord}_p(2U_{2p - 1}!) + 2U_{2p - 1} - \operatorname{ord}_p(2U_{2p - 1}) - 1 - (-1 + 2U_{2p - 1}) - \operatorname{ord}_p(U_{2p - 1}!)
\geq \left[\frac{-2 + w(U) + d(U)}{2p}\right] + \operatorname{ord}_p\left(\frac{(2U_{2p - 1} - 1)!}{U_{2p - 1}!}\right)D
\]

Hence we proved our statement if \(U_{2p - 1} \neq 0\). If \(U_{2p - 1} = 0\), we can get the same estimate by the similar argument by substituting \(U_{2p - 1} = 0\) at the very beginning.  

3.4. Proof of the main theorem.
We start now to prove the main result 3.1.1. By 2.2.6 (or [C], p.594, Proposition 4), and substituting Schur function type expression of $\hat{B}_n$, we have

\[ (3.4.1) \quad \sum_{r=0}^{a} \binom{a}{r} (-1)^r c_{p-1}^{-a-r} \frac{\hat{B}_{n+r(p-1)}}{n+r(p-1)} = \sum_{r=0}^{a} \binom{a}{r} (-1)^r c_{p-1}^{-a-r} \sum_{w(U)=n+r(p-1)} \tau_U c^U, \]

where

\[ \tau_U = (-1)^{d(U)-1} \frac{(w(U) + d(U) - 2)!}{\gamma_U}. \]

By enclosing as many as possible but less than or equal to $r$ factors $c_{p-1}$ from $c^U$, we have

\[ (3.4.2) \quad \sum_{w(U)=n} \tau_{U[r]} c^U c_{p-1}^{r} + \sum_{r_0=1}^{r} \sum_{\gamma_U} \tau_{U[r-r_0]} c^U c_{p-1}^{r-r_0}, \]

where $U[r]$ denotes the partition getting from $U$ by adding $r$ to the $(p-1)$-st entry. After exchanging the sum on $r$ and the sum on $U$, by writing down the terms $\tau_{U[r]}$ and $\tau_{U[r-r_0]}$, we see that

\[ (3.4.3) \quad = \sum_{w(U)=n} \frac{c^U c_{p-1}^{-a}}{\gamma_U} \sum_{r=0}^{a} \binom{a}{r} (-1)^r (-1)^{d(U[r])+r-1} \left\{ \frac{w(U[r]) + d(U[r]) - 2)!}{p^{r+U_{p-1}}(r + U_{p-1})!} \right. \]

\[ \left. + \sum_{r_0=1}^{r} \sum_{\gamma_U} \frac{c^U c_{p-1}^{-a-r_0}}{\gamma_U} \sum_{r=r_0}^{a} \binom{a}{r} (-1)^r (-1)^{d(U[r-r_0])+r-1} \right. \]

\[ \left. \cdot \left\{ \frac{w(U[r-r_0]) + d(U[r-r_0]) - 2)!}{p^{r-r_0}(r - r_0)!} \right\} \right\}, \]

where the symbol $U|_{p-1}$ is $U$ with out $(p-1)$-entry; so that $\gamma_{U|_{p-1}}$ written in the first sum means $\gamma_U$ with neglected the factors $p^{U_{p-1}}U_{p-1}!$ coming from its $(p-1)$-st entry. We remark here that

\[ (3.4.4) \quad \gamma_{U[r]|_{p-1}} = \gamma_{U[r-1]}, \]

\[ \gamma_{U[r]|_{p-1}} p^{r+U_{p-1}}(r + U_{p-1})! = \gamma_{U[r]}. \]

Note that

\[ (3.4.5) \quad \gamma_U = (2^{U_1} \cdots (p-1)^{U_{p-2}}(p+1)^{U_p} \cdots) \cdot (U_1! \cdots U_{p-2}!U_p! \cdots) \]

in the later sum does not contain the factors coming from $(p-1)$-st entry. We denote the two sum in (3.4.3) by $\sum_1$ and $\sum_2$, respectively, and denote as

\[ (3.4.6) \quad \sum_1 + \sum_2 = \sum_{w(U)=n} S_1(U) + \sum_{r_0=1}^{a} \sum_{\gamma_U} \sum_{\gamma_U} S_2(U). \]
About $S_1(U)$ for $U$ such that $w(U) = n$, we have

\[ w(U[r]) + d(U[r]) - 2 = n + (p - 1)r + d(U) + r - 2 \]

(3.4.7)

\[ = n + pr + d - 2 \]

\[ = (r + U_{p-1})p + n - pU_{p-1} + d(U) - 2. \]

Note that

\[ n - pU_{p-1} + d(U) - 2 = (n - (p - 1)U_{p-1}) + (d(U) - U_{p-1}) - 2 \]

(3.4.8)

\[ = w(U|_{p-1}) + d(U|_{p-1}) - 2. \]

About $S_2(U)$ for $U$ such that $w(U) = n + r_0(p - 1)$, we have

\[ w(U[r - r_0]) + d(U[r - r_0]) - 2 \]

(3.4.9)

\[ = n + r_0(p - 1) + (p - 1)(r - r_0) + d(U) + (r - r_0) - 2 \]

\[ = (r - r_0)p + n + r_0p + d(U) - r_0 - 2. \]

According to $n - pU_{p-1} + d - 2$ (resp. $n + r_0p + d(U) - r_0 - 2$) is $\geq ap$ or $< ap$, we divide the sum $\sum_1$ (resp. $\sum_2$) into two kinds of sums, and we say $\sum_1 = \sum_1' + \sum_1''$ (resp. $\sum_2 = \sum_2' + \sum_2''$). Here we note that $n - pU_{p-1} + d(U) - 2 > 0$.

(a) About $\sum_1'$, since $n - pU_{p-1} + d - 2 \geq ap$, 3.2.1 and 3.3.1 yield that

\[ \text{ord}_p(S_1(U)) \geq \text{ord}_p(\gamma_{U|p}) + \text{ord}_p((n - pU_{p-1} + d - 2)!) \]

(3.4.10)

\[ \geq \left\lfloor \frac{n - pU_{p-1} + d - 2}{2p} \right\rfloor \geq \left\lfloor \frac{ap}{2p} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor. \]

(b) About $\sum_1''$, we see $n - pU_{p-1} + d - 2 < ap$. We denote $N = n - pU_{p-1} + d - 2$, and $N = pb + e$ ($0 \leq e < p$). Lemma 3.3.1 shows that

\[ \text{ord}_p(N!) - \text{ord}_p(\gamma_{U|p-1}) = \text{ord}_p((w(U|_{p-1}) + d(U|_{p-1}) - 2)! - \text{ord}_p(\gamma_{U|p-1}) \]

(3.4.11)

\[ \geq \left\lfloor \frac{N}{2p} \right\rfloor. \]

Since $b < a$, by 3.2.1 we have

\[ \text{ord}_p(S_1(U)) \geq a - \left\lfloor \frac{N}{2p} \right\rfloor + \text{ord}_p(N!) - \left\lfloor \frac{a - \lfloor N/p \rfloor}{p} \right\rfloor - \text{ord}_p(\gamma_{U|p-1}) \]

(3.4.12)

\[ = \left\lfloor \frac{N}{2p} \right\rfloor + a - \left\lfloor \frac{N}{p} \right\rfloor - \left\lfloor \frac{a - \lfloor N/p \rfloor}{p} \right\rfloor \]

\[ \geq \left\lfloor \frac{b}{2} \right\rfloor + a - b - \left\lfloor \frac{a - b}{p} \right\rfloor \]

\[ > \left( \frac{b}{2} - 1 \right) - b + a - \frac{a - b}{p} \]

\[ = -1 + \frac{a}{2} + \frac{(a - b)(p - 2)}{2p} \]

\[ > -1 + \frac{a}{2} D \]
Here the initial side is an integer, we have shown that \( \text{ord}_p(S_1(U)) \geq |a/2| \). Hence \( \text{ord}_p(\sum_1) \geq |a/2| \). We can prove \( \text{ord}_p(\sum_2) \geq |a/2| \) by using 3.2.8 instead of 3.2.1. However, we should be careful for the case \( n + r_0p + d(U) - r_0 - 2 \leq ap \). In this case it should be \( n + r_0p + d(U) - r_0 - 2 \geq r_0p \) in order to applying 3.2.8. Therefore it must be \( n + r_0p + 1 - r_0 - 2 \geq r_0p \) because \( d(U) \geq 1 \). Namely, \( n - r_0 - 1 \geq 0 \). This condition is satisfied for \( r_0 = 1, \cdots, a \). by our assumption \( n > a \). □

**Remark 3.4.13.** If we replace the condition \( n > a \) in 3.3.1 by \( n \leq a \), we have the similar congruence modulo \( p^{a-1} \) for the generalized Bernoulli-Hurwitz numbers in [Ot]. Since those numbers are obtained by specializing the universal Bernoulli numbers, the condition \( n > a \) is crucial in 3.3.1.

### 3.5. Kummer-Adelberg congruence.

We prove Adelberg’s Theorem 3.2 in [A2], namely Corollary 3.1.3 of this paper, directly from 3.1.1.

**Proof of 3.1.3.** If \( p = 3 \), then the statement is vacuous by the assumption \( n \neq 0, 1 \mod (p - 1) \). So we may suppose \( p \geq 5 \). We prove the desired congruence by induction on \( a \). The case of \( a = 1 \) is mentioned in 3.1.2 (1). For a given \( a > 1 \), by taking \( p^{a-1} \) as \( a \) in 3.1.1, we have

\[
(3.5.1) \quad \sum_{r=0}^{p^{a-1}} (-1)^r \binom{p^{a-1}}{r} A_p p^{a-1-r} \frac{\hat{B}_{n+r(p-1)}}{n+r(p-1)} \equiv 0 \mod p^{[p^{a-1}/2]}.
\]

Because \( a \geq 2 \) and \( p \geq 5 \), we see \( [p^{a-1}/2] \geq a \). If \( r \neq 0 \), \( p^{a-1} \), then by (1.1.2)

\[
(3.5.2) \quad \text{ord}_p \left( \frac{p^{a-1}}{r} \right) = \frac{S_p(p^{a-1} - r) + S_p(r) - S_p(p^{a-1})}{p-1} = \frac{S_p(p^{a-1} - r) + S_p(r) - 1}{p-1}.
\]

Let \( \nu = \text{ord}_p(r) \). By expanding \( p^{a-1} - r = d_a - 2p^{a-2} + d_{a-3}p^{a-3} + \cdots + d_1p + d_0 \), \( (0 \leq d_j \leq p - 1) \) and \( r = h_{a-2}p^{a-2} + h_{a-3}p^{a-3} + \cdots + h_1p + h_0 \), \( (0 \leq h_j \leq p - 1) \) \( p \)-adically, we see obviously that

\[
(3.5.3) \quad d_j + h_j = \begin{cases} 
  p - 1 & (a - 2 \geq j \geq \nu + 1), \\
  p & (j = \nu), \\
  0 & (\nu - 1 \geq j \geq 0).
\end{cases}
\]

Hence \( S_p(p^{a-1} - r) + S_p(r) = (p - 1)(a - 2 - \nu) + p \). Therefore \( \text{ord}_p \left( \frac{p^{a-1}}{r} \right) = \text{ord}_p \left( \frac{p^{a-1}}{p^{a-1-r}} \right) = a - 1 - \nu \). Thanks to \( p \) is odd number, we consider the sum

\[
(3.5.4) \quad (-1)^r \binom{p^{a-1}}{r} A_p p^{a-1-r} \frac{\hat{B}_{n+r(p-1)}}{n+r(p-1)} + (-1)^{p-a-r} \binom{p^{a-1}}{p^{a-1-r}} A_p r \frac{\hat{B}_{n+(p^{a-1-r})(p-1)}}{n + (p^{a-1-r})(p-1)}
\]
for $1 \leq r \leq (p^{a-1} - 1)/2$. Then by the argument above and 2.3.1, we see that the denominator of $\frac{B_{n+r(p-1)}}{n+r(p-1)}$ for $0 < r < p^{a-1}$ is not divisible by $p$. Additionally considering the hypothesis of induction, we see the sum (3.5.4) is divisible by $p^a$. Thus we have proved 3.1.3. □

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