Symplectic connections, Noncommutative Yang Mills theory and Supermembranes

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Abstract

In built noncommutativity of supermembranes with central charges in eleven dimensions is disclosed. This result is used to construct an action for a noncommutative supermembrane where interesting topological terms appear. In order to do so, we first set up a global formulation for noncommutative Yang Mills theory over general symplectic manifolds. We make the above constructions following a pure geometrical procedure using the concept of connections over Weyl algebra bundles on symplectic manifolds. The relation between noncommutative and ordinary supermembrane actions is discussed.

1 Introduction

The appearance of noncommutative geometry in the study of string theory was first seen in [1] where a noncommutative product was defined as overlap of half strings. This fact and subsequent work, including that of mathematicians using string techniques see [2] and references therein, led to the present believe that noncommutative geometry is necessarily relevant in the understanding of string physics. The best
understood context in which noncommutative geometry arises is in the interaction of open strings with background fields, where a constant background antisymmetric field is much larger than the background metric \[3\].

In \[4\] the description of \(Dp\)-branes in terms of fields on a noncommutative space was analysed. In there open strings were quantized in the presence of a constant \(B\) background field. The precise limits, in which the noncommutativity appeared, were obtained when it was observed that the \(\alpha' \sim 0\) limit allows to express the product of vertex operators as a noncommutative star product.

It is also important to understand from the point of view of the eleven dimensional \(M\) theory how the noncommutativity arises. There has been several articles on the matter \[2\] mainly through M(atrix) theories, where the noncommutativity is related to constant components of the three-antisymmetric tensor potential \(C\). One would expect that the noncommutative description of \(D\)-branes could be obtained from a corresponding geometrical analysis of classical supermembranes without any need for a coupling to an external background field. After all, the supermembrane in the light cone gauge over a flat background shows as a gauge symmetry area preserving diffeomorphisms which correspond to symplectomorphisms in two dimensions. This result suggests that there should be a formulation of the supermembrane in terms of symplectic noncommutative gauge theories constructed from an intrinsic symplectic structure. Also, such noncommutative formulation of the supermembrane should correspond to the noncommutative construction drawn from the quantum analysis of the open strings by analogy with the case of the Born Infeld action. This action, describing \(D\)-branes for slow varying fields, may be obtained either from the one-loop analysis of strings or from a direct study of classical supermembranes \[5\].

It is important to emphasize that the analysis on a constant \(B\) field, which directly defines a symplectic two-form in the construction of the star product, may only be considered in a local approach since any symplectic two-form is locally diffeomorphic to a canonical form with constant coefficients (Darboux’s Theorem). However, the corresponding global study (with general symplectic two-forms not necessarily constant) is needed to provide insight into the structure of the action for noncommutative gauge field theories.

In this article we will address the problem of noncommutativity arising intrinsically in the context of supermembranes in eleven dimensions as well as a more general framework for noncommutative Yang- Mills theories where global aspects
are taken into account.

Essentially, to see any noncommutativity one has to look for a Poisson structure intrinsically defined in the supermembrane theory. In our work, for the sake of simplicity we will restrict the study to the case of symplectic manifolds which is enough for the discussion of the supermembrane and for most of the applications of noncommutative Yang-Mills, the more general formulation on Poisson manifolds will be treated elsewhere.

A non-degenerate symplectic two-form emerges naturally in supermembranes when non-trivial central charges are considered in the supersymmetric algebra. These charges imply the existence of a closed two-form with integral periods over the two dimensional spatial worldvolume. We may reinterpret this two-form as a Poisson structure over the worldvolume. In the case of the supermembrane with a non trivial central charge, monopole type solutions were found for the minima energy levels of the Hamiltonian which provide proper non degenerate closed two-forms, i.e. symplectic structures over the spatial worldvolume. In this picture, the symplectic structure may be constant only on a Darboux chart. Hence, in order to have a correct and general formulation it is relevant to have a global construction of noncommutative gauge theories over symplectic manifolds with general symplectic two-forms.

One way to achieve this goal is to follow the original ideas of deformation quantization in and . In the latter, the idea of glueing charts with Moyal type star products (constant $B$) was used. Later a more geometrical approach was introduced particularly in , see also . We will take results from to build in a geometrical formulation on a Weyl bundle, the noncommutative Yang-Mills connections and their corresponding action using a different point of view from . Its projection to central terms of the bundle defined over the symplectic manifold (the world volume in the case of the supermembrane) provide the global construction of noncommutative Yang Mills theory. We will obtain explicit formulae where the presence of a symplectic connection appears manifestly together with the Yang-Mills potential. Having then this geometrical construction we may then apply it to the description of the $D = 11$ supermembrane as a noncommutative gauge theory as shown in section 5. In section 2, we introduce needed geometrical concepts, definitions and notation in Weyl algebra bundles seeking to be self-contained. In section 3, we construct the noncommutative Yang Mills theory for the symplectic ‘flat’ case. In section 4, previous results are extended to general symplectic manifolds, there
new terms in the noncommutative Yang Mills action appear naturally associated to the symplectic curvature. In section 5, the noncommutative supermembrane action is constructed and its relation to the ordinary known supermembrane action with central charges is considered.

2 Connections over the Weyl algebra bundle

The purpose of this section is to introduce notation and definitions we will use in subsequent sections, for more details see [3]. We will consider here a symplectic manifold \((\Sigma, \omega)\) of dimension \(2n\). The two-form \(\omega\) defines a symplectic structure on each tangent space \(T_x \Sigma\). The corresponding tangent bundle is \(T \Sigma\). We will denote by \(y^\mu\) the components of \(y \in T_x \Sigma, \mu = 1, \ldots, 2n\), and \(\omega_{\mu\nu}\) the nondegenerate antisymmetric tensor defining the symplectic structure over the fibers of \(T \Sigma\). Also, we denote \(x^\mu\) the local coordinates over \(\Sigma, \mu = 1, \ldots, 2n\).

A formal Weyl algebra \(W_x\) corresponding to the symplectic space \(T_x \Sigma\) is an associative algebra over the complex space \(\mathbb{C}\) with a unit, its elements being a formal series

\[
a(y, h) = \sum_{k,p \geq 0} h^k a_{k,\mu_1 \ldots \mu_p} y^\mu_1 \ldots y^\mu_p
\]

where \(h\) is a formal parameter, \(\mu_0 = 0\) and \(\mu_p\) runs from 1 to \(2n\) when \(p \neq 0\). To order terms in the summation, we give the following degrees to variables: \(\deg y^\mu = 1\), \(\deg h = 2\) and we order by increasing degrees \(2k + p\). The Weyl product of elements \(a, b \in W_x\) is defined as

\[
a \circ b = \sum_{k=0}^{\infty} \left(-\frac{i}{2}\right)^k \frac{1}{k!} \omega^{\mu_1 \nu_1} \ldots \omega^{\mu_k \nu_k} \frac{\partial^k a}{\partial y^{\mu_1} \ldots \partial y^{\mu_k}} \frac{\partial^k b}{\partial y^{\nu_1} \ldots \partial y^{\nu_k}}
\]

This product is associative and independent of the basis in \(T_x \Sigma\). The union of \(W_x\) defines the Weyl algebra bundle \(W\). Sections of the Weyl bundle are functions

\[
a(x, y, h) = \sum_{k,p \geq 0} h^k a_{k,\mu_1 \ldots \mu_p}(x) y^\mu_1 \ldots y^\mu_p
\]

where \(a_{k,\mu_1 \ldots \mu_p}(x)\) are symmetric covariant tensor fields on \(\Sigma\). We will also consider \(q\)-forms on \(\Sigma\) with values in \(W\),

\[
a(x, y, h) = \sum h^k a_{k,\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q}(x) y^\mu_1 \ldots y^\mu_p dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_q}
\]
where coefficients are covariant tensor fields symmetric with respect to $\mu_1 \ldots \mu_p$ and antisymmetric in $\nu_1 \ldots \nu_q$. The differential forms constitute an algebra with multiplication defined by means of the exterior product of differentials $dx^\nu$ and the Weyl product of polynomials in $y^\mu$. The commutator of two forms $a \in W \otimes \Lambda^{q_1}$, $b \in W \otimes \Lambda^{q_2}$ is

$$[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a. \quad (5)$$

A central form $a$ is such that for any $b \in W \otimes \Lambda$, $[a, b] = 0$

We now introduce connections on the bundle $W \otimes \Lambda$. They are differential operators

$$\mathcal{D} : W \otimes \Lambda^q \mapsto W \otimes \Lambda^{q+1} \quad (6)$$

such that for any scalar form $\phi \in \Lambda^q$ and section $a$ of $W \otimes \Lambda$

$$\mathcal{D}(\phi \wedge a) = d\phi \wedge a + (-1)^q \phi \wedge \mathcal{D}a \quad (7)$$

To construct connections on the bundle $W \otimes \Lambda$, we will use the concept of symplectic connections on $\Sigma$. There always exist a symplectic connection on any symplectic manifold. It is a torsion free connection that preserves the covariant tensor $\omega_{\mu \nu}$, i.e.

$$D_\rho \omega_{\mu \nu} = 0, \quad (8)$$

$D_\rho$ being a covariant derivative with respect to the basis $\{ \frac{\partial}{\partial x^\rho} \}$

The connection symbols $\omega_{\mu \lambda} \Theta^\lambda_{\rho \nu}$ associated with the symplectic connection $D_\rho$ are completely determined by the defining equation (8) only up to an arbitrary completely symmetric tensor.

$$\Theta_{\mu \rho \nu} = \frac{1}{3} \zeta_{\mu \rho \nu} + \frac{1}{3} \frac{\partial \omega_{\mu \nu}}{\partial x^\rho} + \frac{1}{3} \frac{\partial \omega_{\mu \rho}}{\partial x^\nu} \quad (9)$$

where $\zeta_{\mu \rho \nu}$ is completely symmetric and arbitrary.

We now lift the symplectic connections to the Weyl bundle, defining connections $\mathcal{D}_S$ on sections of $W \otimes \Lambda$. They are defined as

$$\mathcal{D}_S a = dx^\rho \wedge D_\rho a \quad (10)$$

The properties of the symplectic connection $D_\rho$ imply the following property of the connection $\mathcal{D}_S$ on the Weyl product of the bundle $W \otimes \Lambda$

$$\mathcal{D}_S(a \circ b) = \mathcal{D}_S a \circ b + (-1)^{q_1} a \circ \mathcal{D}_S b \quad (11)$$
for \( a \in W \otimes \Lambda^q \) and \( b \in W \otimes \Lambda^q \). Consequently they satisfy

\[
\mathcal{D}_S[a, b] = [\mathcal{D}_S a, b] + (-1)^q [a, \mathcal{D}_S b].
\] (12)

In Darboux local coordinates (coordinates where the components of the symplectic two-form \( \omega \) are constants), the connection \( \mathcal{D}_S \) can be written as

\[
\mathcal{D}_S a = da + \frac{i}{\hbar} [\Theta, a]
\] (13)

where \( \Theta = \frac{1}{2} \Theta_{\mu\nu\rho} y^\mu y^\nu dx^\rho \).

More general covariant derivatives \( \mathcal{D} \) on the bundle may be considered with one-form connections \( \gamma \) globally defined on \( \Sigma \) and with values in \( W \),

\[
\mathcal{D} a = \mathcal{D}_S a + \frac{i}{\hbar} [\gamma, a]
\] (14)

The two-form \( \Omega \)

\[
\Omega = R + \mathcal{D}_S \gamma + \frac{i}{2\hbar} [\gamma, \gamma]
\] (15)

is the Weyl curvature of the connection \( \mathcal{D} \). \( R \) is the curvature of the connection \( \mathcal{D}_S \). The Weyl curvature satisfies the Bianchi identity

\[
\mathcal{D} \Omega = \mathcal{D}_S \Omega + \frac{i}{\hbar} [\gamma, \Omega] = 0
\] (16)

moreover, for any section \( a \in W \otimes \Lambda \)

\[
\mathcal{D}^2 a = \frac{i}{\hbar} [\Omega, a]
\] (17)

In general, transitions on the bundle \( T\Sigma \) will induce transitions on the algebra \( W \). The infinitesimal gauge transformations on elements of the algebra are expressed as automorphisms given by

\[
a \rightarrow a + [a, \lambda]
\] (18)

with ‘infinitesimal’ \( \lambda \in W \).

The corresponding gauge transformations for the connections \( \mathcal{D} \) are

\[
\mathcal{D} \rightarrow \mathcal{D} + \mathcal{D} \lambda.
\] (19)

Consequently

\[
\mathcal{D} a \rightarrow \mathcal{D} a + [\mathcal{D} a, \lambda].
\] (20)
Abelian connections $\mathcal{D}_A$ are connections $\mathcal{D}$ with Weyl curvature $\Omega$ being a central form of the algebra. Let us denote it $\Omega_A$. It then satisfies

$$[\Omega_A, a] = 0$$

for any section $a \in W$.

There always exist abelian connections on $W$, an example of these connections may be expressed as

$$\mathcal{D}_A = \mathcal{D}_S + \frac{i}{\hbar} [\omega_{\mu\nu} y^\mu dx^\nu + r, \cdot]$$

with $\deg r \geq 3$. Associated with $\mathcal{D}_A$ there is a subalgebra of $W$, denoted $W_A$, defined by

$$W_A = \{ a \in W : \mathcal{D}_A a = 0 \}.$$ (23)

There is a one to one correspondence between the $C^\infty$ functions $a_0(x)$ over $\Sigma$ and the elements of $W_A$. In fact, given $a \in W_A$, one defines the projection

$$\sigma a := a(x, y = 0, h) = a_0(x),$$ (24)

and given $a_0(x)$ there is a unique element $a \in W_A$ with such projection. The explicit expression of this element, obtained by solving $\mathcal{D}_A a = 0$, is

$$a(x, y, h) = a_0(x) + D_\mu a_0(x) y^\mu + \frac{1}{2} D_\mu D_\nu a_0(x) y^\mu y^\nu + \frac{1}{6} D_\mu D_\nu D_\rho a_0(x) y^\mu y^\nu y^\rho$$

$$- \frac{1}{24} R_{\mu\nu\rho\lambda} \omega^{\lambda\sigma} D_\sigma a_0 y^\mu y^\nu y^\rho + \ldots$$ (25)

where the remaining terms are of higher degree. If $a$ and $b \in W_A$, then

$$\sigma(a \circ b) = a_0 \star b_0$$ (26)

where $\star$ is the globally defined star product. In the particular case when $\omega_{\mu\nu}$ is constant and the symplectic connection is zero, the formula agrees with the Moyal product.

Finally, we may obtain the above constructions also for more general Weyl bundles where the fibre $T_x \Sigma$ is replaced by $L_x$. The latter being any symplectic vector space on $\Sigma$ of dimension $2n$ with a given symplectic structure $\omega$ and corresponding symplectic connection $D_L$. The bundle $L$ is assumed to be isomorphic to $T\Sigma$ where the bundle isomorphism is

$$\varepsilon : TM \rightarrow L$$ (27)
Introducing a local symplectic frame \((e_1 \ldots e_{2n})\) of \(L\) and a dual frame \((e^1 \ldots e^{2n})\) for the dual \(L^*\), we will have local one-forms on \(\Sigma\) \(\varepsilon^*(e^i) = \varepsilon_i^\mu dx^\mu\) corresponding to a basis \(\{dx^\rho\}\) of \(T^*\Sigma\). The form \(\omega\) in \(L\) may be transported to \(T\Sigma\) giving a nondegenerate two-form on \(\Sigma\)

\[\omega = \frac{1}{2} \omega_{ij} e^i \wedge e^j = \frac{1}{2} \tilde{\omega}_{\mu\nu} dx^\mu \wedge dx^\nu\]  \((28)\)

Once the isomorphism between \(L\) and \(T\Sigma\) is defined the corresponding concepts like symplectic and abelian connections apply as above.

3 Yang-Mills connections over the Weyl bundle

We will consider in this section one-form connections and curvatures on \(u(1)\) valued Weyl bundles over a manifold \(\Sigma\) with a symplectic two-form whose coefficients are constants, in this case we take \(\varepsilon^i_\mu = \delta^i_\mu\). We will assume also that the totally symmetric symplectic one-form connection \(\Theta\) is zero. These assumptions will be relaxed in the next section. They are valid, for example, on the symplectic vector space \(R^{2n}\) with a constant symplectic two-form \(\omega\). The extension to the \(u(N)\) nonabelian case may be achieved by considering the Weyl algebra bundle with elements \(a\) valued in the \(u(N)\) algebra. The following construction may then be extended in a straightforward way.

We denote by \(\Omega^0_A(\Sigma, W)\) the set of sections \(a(x, y)\) of \(W\) satisfying

\[\mathcal{D}_A a(x, y) = 0\]  \((29)\)

where \(\mathcal{D}_A\) in \((22)\) with \(r = 0\) takes the particular form

\[\mathcal{D}_A a(x, y) = da + \frac{i}{h}[\varepsilon_{ij} y^i dx^j, a],\]  \((30)\)

and \(\Omega^1_A(\Sigma, W \otimes \Lambda^1)\) the set of sections \(b = e^i b_i(x, y)\) of Weyl algebra valued one-forms in \(W \otimes \Lambda^1\) in a particular basis \((e^1 \ldots e^{2n})\) that fulfill the condition

\[\mathcal{D}_A b_i(x, y) = 0\]  \((31)\)

the coefficients \(b_i \in \Omega^0_A(\Sigma, W)\). Any element \(a \in \Omega^0_A(\Sigma, W)\) satisfy the equation

\[a = a_0 + \delta^{-1}(\mathcal{D}_A + \delta)a(x, y)\]  \((32)\)
where $\mathcal{D}_A$ is the abelian connection of section 2 and
\[
\delta = e^i \frac{\partial}{\partial y^i}, \quad (33)
\]
$\delta^{-1}$ is an operator acting on every term $c_{pq} = c_{i_1 \ldots i_p, j_1 \ldots j_q} y^{i_1} \cdots y^{i_p} e^{j_1} \cdots e^{j_q}$ of $c(x, y) \in W \otimes \Lambda$ as follows
\[
\delta^{-1} c_{pq} = \frac{1}{p+q} y^k \iota(e_k) c_{pq} \quad (34)
\]
here $\iota(e_k)$ is the contraction on one-forms operation. These operators satisfy a formula similar to the Hodge decomposition
\[
c(x, y) = \delta \delta^{-1} c + \delta^{-1} \delta c + c_{00}(x) \quad (35)
\]
where $c_{00}(x)$ is the zero-form central term of $c$. Notice that $\delta$ decreases the degree of every term of $c$ in one, while $\delta^{-1}$ increases it in one.

Given the first term $a_0(x)$ in (32), the rest of the terms in the series $a(x, y)$ can be calculated using an iterative procedure by applying successively the same equation (32).

We introduce now the lifted symplectic connections and curvatures on $W$ whose projections to central terms yield the usual noncommutative Yang-Mills field strengths.

We consider first the connection $\hat{\mathcal{D}}$ given explicitly by
\[
\hat{\mathcal{D}} = d + i \frac{\hbar}{\hbar} [\gamma, \bullet] \quad (36)
\]
locally $\gamma$ is in $\Omega^1_A(\Sigma, W \otimes \Lambda^1)$.

If $a \in \Omega^0_A(\Sigma, W)$, then
\[
\hat{\mathcal{D}} a = e^i (\partial_i a + i \hbar [\gamma_i, a]) \in \Omega^1_A, \quad (37)
\]
its projection yields
\[
\sigma \hat{\mathcal{D}} a = e^i (\partial_i a_0(x) + i \hbar \{A_i, a_0 \}_{Moyal}) \quad (38)
\]
where $\sigma \gamma_i = A_i$.

Condition $\gamma \in \Omega^1_A(\Sigma, W \otimes \Lambda^1)$ is necessary to get the desired projection over $\Lambda^1(\Sigma)$ in (38). The more general condition $\mathcal{D} \gamma = 0$ is not enough to achieve the required projection.

We define the infinitesimal gauge variations $\Delta_g$ of sections $a \in \Omega^0_A(\Sigma, W)$ as
\[
\Delta_g a(x, y) = [a, \lambda] \quad (39)
\]
where the infinitesimal parameter $\lambda \in \Omega^0_A(\Sigma, W)$. We then define
\[ \Delta_g \gamma(x, y) = \hat{D}\lambda, \]
(40)

implying
\[ \Delta_g \hat{D}a = [\hat{D}a, \lambda]. \]
(41)

In particular, we obtain from equation (40)
\[ \Delta_g A_i = \partial_i \lambda_0 + \frac{i}{\hbar} \{A_i(x), \lambda_0(x)\}_\text{Moyal} \]
(42)

where $\sigma\lambda(x, y) = \lambda_0(x)$.

We remark that $\Delta_g \gamma \in \Omega^1_A(\Sigma, W \otimes \Lambda^1)$, this is a non-trivial property of the construction. In fact, if the curvature of the symplectic connection was not zero, then $e^i \partial_i \lambda$ would not belong to $\Omega^1_A$ and the above construction had to be modified so that all gauge equivalence classes could belong to $\Omega^1_A$. We notice that $\hat{D}A$ corresponds to the zero in the space of connections we are considering and it is invariant under infinitesimal gauge transformations. This result implies that $\hat{D} - \hat{D}A$ defines the same connection as $\hat{D}$ on any $\Omega^q_A(\Sigma, W \otimes \Lambda^q)$. So in considering physical connections we will take those ones normalized to zero, in this case $D = \hat{D} - \hat{D}A$ besides, this connection representation is more appropriate to study global aspects of the bundle.

We may now write the curvature of the connection $D$,
\[ \Omega = d\gamma + \frac{i}{2\hbar} [\gamma, \gamma] - \omega, \]
(43)

it satisfies the Bianchi identity
\[ D\Omega = 0, \]
(44)

under infinitesimal gauge transformations, we obtain
\[ \Delta_g \Omega = [\Omega, \lambda]. \]
(45)

The projection of $\Omega$ becomes the usual noncommutative Yang-Mills field strength
\[ \sigma \Omega = \frac{1}{2} e^i \wedge e^j (\partial_i A_j - \partial_j A_i) + \frac{i}{\hbar} \{A_i, A_j\}_\text{Moyal} - \omega = F - \omega \]
(46)
equation (46) corresponds to $u(1)$ noncommutative Yang-Mills theory. We may now define the $u(1)$ Yang-Mills action over the Weyl algebra bundle
\[ S_{YM} = \int_{\Sigma} \sigma(\Omega \circ *\Omega) \]
(47)
where $\ast \Omega \in W \otimes \Lambda^{2n-2}$ is the Hodge dual to $\Omega$ constructed using the induced metric $g(\cdot, \cdot)$ determined by a compatible complex structure $J$ and the symplectic structure $\omega(\cdot, \cdot)$ on the vector bundle $L$.

\[ g(u, v) = \omega(u, Jv) \text{ for any two vectors } u, v \in L, \quad (48) \]

complex structures $J$ always exist for any symplectic vector bundle. Equation (47) may be written as

\[ S_{YM} = \int_{\Sigma} F \wedge \ast F - 2\int_{\Sigma} F \wedge \ast \omega + \text{Vol}_{\Sigma} \quad (49) \]

## 4 Global construction of noncommutative gauge theories on the Weyl algebra bundle

Let $\Sigma$ be a symplectic manifold with a symplectic two-form $\omega_{\mu \nu} dx^\mu \wedge dx^\nu$. In this section, we assume $\omega$ to be an arbitrary non-degenerate closed two-form over $\Sigma$. A set of multi-beins is defined by

\[ \omega_{\mu \nu} = \varepsilon^i_\mu \varepsilon^j_\nu \epsilon_{ij}, \quad (50) \]

where $\epsilon_{ij}$ is the canonical symplectic tensor. Because of Darboux theorem, locally we always have

\[ \varepsilon^i_\mu = \partial_\mu g^i. \quad (51) \]

We may consider an atlas where on each chart we have (51). The transitions on $g^i$ between different charts preserve the symplectic structure (50). The multi-bein $\varepsilon^i_\mu$ will then have transitions over $\Sigma$, otherwise, one would have a set of $2n$ non-singular vector fields globally defined over $\Sigma$, but this is not true in general.

Let us discuss the transitions on intersection of charts in more detail. Consider two open sets $U$ and $\hat{U}$, $U \cap \hat{U} = \emptyset$ in which

\[ \hat{\varepsilon}^i_\mu = \partial_\mu \hat{g}^i, \quad (52) \]

respectively. In $U \cap \hat{U}$ we then have

\[ \omega_{\mu \nu} = \varepsilon^i_\mu \varepsilon^j_\nu \epsilon_{ij} = \hat{\varepsilon}^i_\mu \hat{\varepsilon}^j_\nu \epsilon_{ij}, \quad (53) \]
from which we obtain

\[ \hat{\varepsilon}^i_{\mu} = S^i_j \varepsilon^j_{\mu} \]

where

\[ S^i_j = \epsilon^k_{\mu} \varepsilon^l_{\mu} \epsilon_{ij}, \]

we define the inverse of \( \varepsilon \) by \( \varepsilon^{ij} \varepsilon_{jk} = \delta^i_k \). One may verify that \( S \) preserves the canonical symplectic tensor and hence \( S \in Sp(2n) \). Consequently in order to have a global construction over \( \Sigma \), one must begin by introducing a symplectic \( Sp(2n) \) connection on the tangent bundle. We first consider the following symplectic connection over \( \Sigma \),

\[ \Theta^\lambda_{\mu \nu} = \zeta^\rho_{\mu \nu \rho \lambda} + \frac{1}{3} \frac{\partial \omega^\rho_{\mu \nu}}{\partial x^\lambda} + \frac{1}{3} \frac{\partial \omega^\rho_{\mu \lambda}}{\partial x^\nu} \]

where \( \zeta^\rho_{\mu \nu \rho \lambda} \) is a totally symmetric tensor. This is the most general expression for a connection satisfying

\[ \left( \frac{\partial}{\partial x^\mu} + \Theta_\mu \right) \omega^\rho_\lambda = 0 \]

here \( \Theta^\lambda_{\mu \nu} \) is expressed in terms of \( \omega^\rho_{\mu \nu} \) and its derivatives. It is invariant under the \( Sp(2n) \) transition of the multi-bein. We now consider the following torsion free connection on the tangent space

\[ \Gamma^j_{\mu i} = \varepsilon^i_{\nu} \left( \frac{\partial \varepsilon^j_{\nu}}{\partial x^\mu} - \Theta^\lambda_{\mu \nu} \varepsilon^j_{\lambda} \right) \]

it transforms as a \( Sp(2n) \) connection under \( Sp(2n) \) transformations on the tangent space. In fact,

\[ \hat{\Gamma}^j_{\mu i} = \left( S^{-1} \right)^j_k \Gamma^k_{\mu i} S^j_i - \left( S^{-1} \right)^j_k \frac{\partial S^j_i}{\partial x^\mu} \]

This connection is symplectic on the tangent space. We may construct from it the most general symplectic connection on the tangent space in the following way. Let us denote

\[ \dot{D}_\mu \equiv \partial_\mu + \Gamma_\mu, \]

a symplectic connection must satisfy

\[ (\dot{D}_\mu + \Delta \Gamma_\mu) \varepsilon_{ij} = 0 \]

this equation has the general solution

\[ \Delta \Gamma^j_{\mu i} = \frac{1}{3} (\dot{D}_\mu \varepsilon_{il}) \varepsilon_{lj} + \frac{1}{3} \varepsilon^k_{\mu} \varepsilon^\nu_{i} (\dot{D}_\nu \varepsilon_{kl}) \varepsilon_{lj} + \varepsilon^k_{\mu} \tilde{\zeta}_{(ilk)} \varepsilon_{lj} \]

\( \Delta \Gamma^j_{\mu i} \) is a covariant vector on the world volume and a tensor under \( Sp(2n) \) transformations. Since the connection (57) is symplectic the first two terms of the right
hand side member in (61) are zero. We may finally construct our symplectic connection $D$, when acting on mixed indices vectors $V^i_\nu$ it yields

$$D_\mu V^i_\nu = \frac{\partial V^i_\nu}{\partial x^\mu} + (\Gamma_\mu^i + \Delta \Gamma_\mu^i)_l^j V^j_\nu - \Theta^\lambda_\mu V^i_\lambda; \quad (62)$$

it satisfies

$$D_\mu \omega^i_\rho = 0, \quad D_\mu \epsilon_{ij} = 0 \quad (63)$$

and it has the right transformation law on the world volume and in the tangent space. If we impose $\Delta \Gamma^j_\mu i$ to be zero, that is if we take the totally symmetric term in (61) zero, then the symplectic connection (62) acting on the multibein is zero. This property is valid for any totally symmetric symbol in (55). The $Sp(2n)$ symplectic connection reduces in this case to (57). In the evaluation of the final formulas we will consider this kind of connection. We will call $D$ an $Sp(2n)$ symplectic connection, when it acts on geometrical objects on the tangent space only.

We will now introduce a connection $\mathcal{D}$ on $W$ which in the flat limit when $\omega$ has constant coefficients, $\epsilon^k_\mu = \delta^k_\mu$ and $\Theta^\lambda_\mu \nu = 0$ reduces to the Yang-Mills connections of section 3. It will be a map from $\Omega^A_0 \mapsto \Omega^A_1$, which in addition to the general property of any connection (7) it satisfies the Leibnitz property for the Weyl product in $W$. All these properties should, of course, be preserved under gauge transformations. We consider

$$\mathcal{D} = \frac{i}{\hbar} [G_i e^i, \bullet] + \frac{i}{\hbar} [\gamma, \bullet] \quad (64)$$

where $G_i$ obeys the following equations

$$\mathcal{D}_A G_i = 0, \quad \sigma G_i = \epsilon_{ij} g^j(x) \quad (65)$$

where $g^j(x)$ is defined in (51) and $\gamma_i$ also obeys

$$\mathcal{D}_A \gamma_i = 0. \quad (66)$$

recalling equations (10) and (22), $\mathcal{D}_A$ is now written using the symplectic connection in (62)

$$\mathcal{D}_A = \mathcal{D}_S + \frac{i}{\hbar} [\epsilon_{ij} y^i e^j + r, \ bullet] \quad (67)$$

We will denote the projection of $\gamma_i$ as $\mathcal{A}_i(x)$, like in section 3

$$\sigma \gamma_i = \mathcal{A}_i(x) \quad (68)$$
$D$ has the following properties:

a) If $a \in \Omega^0_A$ then $Da \in \Omega^1_A$. This is so because the abelian connection $D_A$ acts directly inside the bracket, and then equations (65) and (66) assure that it annihilates all terms inside it.

b) If $a \in \Omega^q_A$ then

$$D(a \circ b) = Da \circ b + (-1)^q a \circ Db$$

(69)

because the bracket has that property.

c) If $a \in \Omega^0_A$,

$$\frac{i}{\hbar} \sigma[G, e^i, a] = e^i e^{kj} \partial_k g^j \epsilon_{ij} \partial_{\ell} a_0 + O(h) = e^i \partial_i a_0 + O(h).$$

(70)

since

$$\partial_k g^i = \epsilon^\mu_k \partial_\mu g^j = \delta^i_k$$

(71)

the terms $O(h)$ depend on the curvature of the symplectic connection (62) and becomes zero in the flat limit. Consequently, in that limit $[G, e^i, a]$ is the element of $\Omega^1_A$ with projection $e^i \partial_i a_0$. It then coincides with $D_S a$, which in the flat limit (and only there) is also an element of $\Omega^1_A$. We conclude that in the flat limit (64) is exactly the Yang-Mills connection of section 3.

d) We define the gauge transformations in a chart by

$$\Delta_\gamma = D \lambda$$

(72)

where $\lambda \in \Omega^0_A$ is the infinitesimal gauge parameter as before. It then preserves the form of (64).

e) The curvature of the connection $D$ is then given by

$$\Omega = \frac{i}{2\hbar} [G, G] + \frac{i}{\hbar} [G, \gamma] + \frac{i}{2\hbar} [\gamma, \gamma],$$

(73)

it satisfies the Bianchi identity

$$D\Omega = 0$$

(74)

this property follows from the Jacobi identity for the bracket. The first term in (73) reduces in the flat limit to

$$\frac{i}{2\hbar} [G, G] = -\frac{1}{2} e^i \land e^j \epsilon_{ij} = -\omega$$

(75)
The projection of $\Omega$ has in general the expression
\[
\sigma \Omega = -\omega + F - \frac{h^2}{96} \left( R_{jkli}(D_j D_k D_l A_m - \frac{1}{4} R_{jk\ell p} e^{pq} D_q A_m) \right) e^{\hat{j} j} e^{\hat{k} k} e^{\hat{l} l} e^{\hat{m} m} + O(h^3) \ldots
\]
where the curvature is constructed from the $Sp(2n)$ symplectic connection (62), the remaining terms are higher order in $h$ and depend also on the derivatives of the curvature. The curvature $F$ is the Yang Mills field strength
\[
F = \frac{1}{2} e^i \wedge e^j (D_i A_j - D_j A_i + \frac{i}{\hbar} \{A_i, A_j\}_{\text{star}})
\]
constructed now with the $Sp(2n)$ covariant symplectic derivative introduced in (62), notice that the $\text{star}$ bracket in (77) is the global generalization of the Moyal bracket over the whole symplectic manifold obtained in [13], briefly presented in section 2. We notice that, because of (51) and (54), the first covariant symplectic derivative of $g^i$ is a simple derivative. We will assume the same transformation law under $Sp(2n)$ for $A$ in (68). As in the previous section the above construction may be extended to $u(N)$ valued Weyl algebras in a straightforward way.

5 Supermembranes and noncommutative gauge theories

As already said before, the supermembrane in the light cone gauge shows as a gauge symmetry symplectomorphisms in two dimensions. This result lead us to look for a formulation of the supermembrane in terms of symplectic noncommutative gauge theories constructed from an intrinsic symplectic structure.

The starting point in the construction we have discussed in previous sections is the symplectic manifold $\Sigma$. From the global symplectic two-form $\omega$ we constructed all the geometric objects which allow to formulate Yang-Mills connections over the Weyl algebra bundle. After projection we obtained a global noncommutative formulation of Yang-Mills connections. For the case of the supermembrane in the light cone gauge, the world volume $\Sigma$ is a Riemann surface. But now, how do we incorporate a symplectic structure into it?. A natural way is to consider a supermembrane with a nontrivial central charge of the SUSY algebra [17]. That is,
consider $Z^{12}$

$$Z^{12} = \int_\Sigma dX^1 \wedge dX^2 = 2\pi n$$  \hspace{1cm} (78)

which requires that when $\Sigma$ is a compact Riemann surface, $X^1$ and $X^2$ must be winded on compactified directions. By the Weil theorem, there always exists a non-degenerate closed two-form $\omega$ satisfying a condition like (78). In particular, we may consider a solution for the winded case on a compact Riemann surface $\Sigma$ where the Hodge dual of $\omega$ is an integer,

$$\ast \omega = n$$  \hspace{1cm} (79)

over all $\Sigma$, the monopole solution. The density used in $\ast \omega$ is the one introduced in the light cone gauge fixing procedure. This monopole configuration has a natural generalization for other p-branes in terms of extended self-dual connections \[9\] \[8\] \[16\].

We may thus introduce in an intrinsic way a non-degenerate closed two-form $\omega = dX^1 \wedge dX^2$ over $\Sigma$. This closed two-form is invariant under the area preserving diffeomorphism \[8\] which is the residual gauge symmetry on the supermembrane in the light cone gauge. We may now construct, using our approach of the previous sections, noncommutative Yang-Mills connections globally defined on the Weyl bundle over $\Sigma$.

We consider the seven transverse coordinates to the supermembrane as projections of corresponding elements $X^M(x, y), M = 1, \ldots, 7$ belonging to $\Omega^0_A$. The gauge transformations of $\sigma X^M$ with respect to the area preserving diffeomorphisms on the membrane correspond exactly to the gauge transformations constructed over the Weyl algebra bundle for the elements of $\Omega^0_A$. These elements will have associated conjugate momenta densities $P^M$, we will also denote $\dot{P}^M$ the associated scalar fields $\dot{P}^M = \frac{1}{\sqrt{\text{det} \omega}} P^M$ and $\dot{P}^M \in \Omega^0_A$. Also, we consider $\gamma = \gamma_i e^i, i = 1, 2$ and $\Pi = \Pi^i \omega_{ij} e^j$ the one-form gauge field and its corresponding momentum density, respectively. We denote $\dot{\Pi} = \frac{1}{\sqrt{\text{det} \omega}} \Pi$ the associated one form momentum. Notice that $\dot{\Pi}$ belong to $\Omega^1_A$. The symplectomorphisms are generated by the first class constraint

$$\mathcal{D}_i \Pi^i + [X^M, P_M] = 0$$  \hspace{1cm} (80)

All elements $X^M, \dot{P}^M$ and $\dot{\Pi}^i$ will transform homogeneously under an infinitesimal gauge transformation with parameter $\lambda$ as in the previous section

$$\Delta_{\gamma^i} = [\cdot, \lambda]$$  \hspace{1cm} (81)
also, the gauge field will transform accordingly as

\[ \Delta_\gamma = D \lambda \]  

(82)

where \( D \) is the connection introduced in section 4.

We may then write the following Hamiltonian density on the Weyl algebra over \( \Sigma \), subject to the above first class constraint, as a noncommutative Yang Mills coupled to the scalar fields representing the transverse coordinates to the membrane,

\[
\mathcal{H} = \frac{1}{2}(\dot{P}^M \circ \ast \dot{P}^M) + \frac{1}{2}(\dot{\Pi} \circ \ast \dot{\Pi}) - \frac{1}{2\hbar^2}([G_i e^i + \gamma, X^M] \circ \ast [G_i e^i + \gamma, X^M]) \\
- \frac{1}{4\hbar^2}([X^M, X^N] \circ \ast [X^M, X^N]) + \frac{1}{2}(\Omega \circ \ast \Omega) 
\]  

(83)

We may immediately project out the center components of this Hamiltonian yielding

\[
\sigma \mathcal{H} = \frac{1}{2}(\dot{P}^M)^2 \omega + \frac{1}{2}(\dot{\Pi} \wedge \ast \dot{\Pi}) - (D_A X^M \wedge \ast D_A X^M) \\
- \frac{1}{4\hbar^2}(X^M, X^N)^2 \omega + \frac{1}{2}(\mathcal{F} - \omega)(\ast \mathcal{F} - \ast \omega) + \text{curvature terms} 
\]  

(84)

where \( D_A = D_S X^M + \{A, X^M\}_\ast \) corresponds to the first terms in a projection of the gauge covariant derivative,

\[
\sigma D_i X^M = \frac{i}{\hbar} \{G_i, X^M\}_\ast + \frac{i}{\hbar} \{A_i, X^M\}_\ast \\
= D_S i X^M + \frac{i}{\hbar} \{A_i, X^M\}_\ast + \text{curvature terms} \ldots 
\]  

(85)

All constructions of sections 2, 3 and 4 are based on properties related to the Weyl bracket, they do not rely on the Weyl product by itself. These two basic properties are the Jacobi identity and that covariant derivatives satisfy the Leibniz condition with respect to the bracket. We may reconstruct everything if we replace the Weyl bracket by a Poisson bracket on the Weyl bundle. That is, we may change the noncommutative gauge symmetry based on a Weyl bracket in the theory for a symplectic gauge symmetry based on a Poisson bracket. The symplectic connections fulfill again the Leibniz condition with respect to it. We may then proceed to define the connections

\[ D = D_S + [\gamma, \bullet]_P \]  

(86)

where now

\[ [\cdot, \cdot]_P = \epsilon^{ij} \frac{\partial \cdot}{\partial y^i} \frac{\partial \cdot}{\partial y^j}. \]  

(87)
We then have for \( a \in W \)

\[
\mathcal{D}[a, b]_P = [\mathcal{D}a, b]_P + [a, \mathcal{D}b]_P
\]

(88)

We may construct abelian connections as before

\[
\mathcal{D}_A = \mathcal{D}_S + \frac{1}{\hbar} \omega_{ij} y^i e^j , \bullet) + [r, \bullet]_P
\]

(89)

such that

\[
\mathcal{D}_A \mathcal{D}_A a = [\Omega, a]_P = 0,
\]

(90)

\( \Omega \) being a central section of the Weyl bundle. We now deal with sections \( a \in \Omega^0_A \) of the Weyl bundle obeying

\[
\mathcal{D}_A a = 0.
\]

(91)

Its explicit expression may be obtained as before. We have again a one-to-one correspondence between \( a \in \Omega^0_A \) and its projection \( a_0 = \sigma a \). The projection of the Poisson bracket in \( W \) now yields

\[
\sigma[a, b]_P = \epsilon^{ij} \partial_i a_0 \partial_j b_0
\]

(92)

where \( \sigma a = a_0 \) and \( \sigma b = b_0 \).

We may now reconstruct Yang-Mills connections and extend them in a global way over \( \Sigma \). Everything follows exactly in the same way as before by changing the Weyl bracket by the Poisson one. The corresponding supermembrane action may now be written in terms of the Poisson bracket. It is this canonical action that describes the doubly compactified \( D = 11 \) Supermembrane as was first found in [10]. It agrees with the noncommutative action obtained from the Weyl bracket when we expand up to degree 2. It must be in this way since the supermembrane action neither depends on the parameter \( \hbar \) of the formal deformation quantization nor on the arbitrary totally symmetric symbol present in the symplectic connection. In this sense the symplectic noncommutative action of the supermembrane in [11] when formulated in terms of the star connection using a Seiberg-Witten map, provides an action for a noncommutative star theory which does not depend either on \( \hbar \) or on the totally symmetric symbol in the symplectic connection. It is natural to think that there is a one-to-one correspondence between the gauge equivalent classes of the Yang-Mills connections constructed from the Weyl bracket and with the Poisson one. We will discuss this relation and the corresponding Seiberg-Witten map in a forthcoming paper.
We notice that in both actions in addition to the standard noncommutative Yang-Mills terms, the integral of the projection of the curvature of the corresponding Yang-Mills connection on the Weyl bundle is present as well. This term characterizes the Weyl algebra bundle as a vector bundle, it is introduced from the global construction which is expressed in terms of inner derivatives only, and there is no way to avoid it from this global point of view. This term can not be eliminated in the description of the supermembrane. Fundamental properties of the theory, such as the spectrum, change dramatically if it is fixed to zero.

6 Conclusions

We have set up a global formulation for noncommutative Yang Mills theory over general symplectic manifolds using the concept of connections over Weyl algebra bundles on symplectic manifolds. This result is used to construct an action for a noncommutative supermembrane where a new topological term appears. We have found that the ordinary supermembrane action as known in the literature and the noncommutative supermembrane as constructed here may be obtained by special choices of connections in an Abelian Weyl bundle in each case. Also, that the framework of Weyl bundles suggests us to construct actions for supermembranes and Yang Mills theories straightforwardly in a natural way whenever the symplectic structure is given from the start. We notice in the case of supermembranes that this symplectic structure may be provided by central charges in general. We took as example the case when the central charge is related to winding of coordinates in the target space in eleven dimensions for compact supermembranes. Similar results may be expected for the case of charged open membranes with boundaries.

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