The fourth-order Hermitian Toeplitz determinant for convex functions

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Abstract
The sharp bounds for the fourth-order Hermitian Toeplitz determinant over the class of convex functions are computed.

Keywords Hermitian Toeplitz determinant · Univalent functions · Convex functions · Carathéodory class

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{A}$ be its subclass normalized by $f(0) := 0$, $f'(0) := 1$, that is, functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \quad z \in \mathbb{D},$$

(1)

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions. Let $\mathcal{S}^c$ denote the subclass of $\mathcal{S}$ of convex functions, that is, univalent functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is a convex domain in $\mathbb{C}$. By the well-known result of Study [11] (see also [5, p. 42]), a function

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$f$ is in $\mathcal{S}^c$ if and only if

$$\Re \left\{ 1 + \frac{z^2 (z)}{f'(z)} \right\} > 0, \quad z \in \mathbb{D}. \quad (2)$$

Given $q, n \in \mathbb{N}$, the Hermitian Toeplitz matrix $T_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1) is defined by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \cdots & a_n \end{bmatrix},$$

where $\bar{a}_k := \bar{a}_k$. Let $|T_{q,n}(f)|$ denote the determinant of $T_{q,n}(f)$. In particular, the third Toeplitz determinant $|T_{3,1}(f)|$ is given by

$$|T_{3,1}(f)| = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 1 + 2 \Re \left( a_2^2 \bar{a}_3 \right) - 2|a_2|^2 - |a_3|^2 \quad (3)$$

and the fourth Toeplitz determinant $|T_{4,1}(f)|$ is given by

$$|T_{4,1}(f)| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \bar{a}_2 & a_1 & a_2 & a_3 \\ \bar{a}_3 & \bar{a}_2 & a_1 & a_2 \\ \bar{a}_4 & \bar{a}_3 & \bar{a}_2 & a_1 \end{vmatrix}$$

$$= 1 - 2 \Re \left( a_2^3 \bar{a}_4 \right) + 4 \Re \left( a_2^2 \bar{a}_3 \right) - 2 \Re \left( a_2^2 \bar{a}_3 a_4 \right)$$

$$+ 4 \Re (a_2 a_3 \bar{a}_4) + |a_2|^4 - 3|a_2|^2 + |a_3|^4 - 2|a_3|^2$$

$$+ |a_2|^2 |a_4|^2 - 2|a_2|^2 |a_3|^2 - |a_4|^2. \quad (4)$$

In recent years a lot of papers has been devoted to the estimation of determinants built with using coefficients of functions in the class $\mathcal{A}$ or its subclasses. Hankel matrices i.e., square matrices which have constant entries along the reverse diagonal (see e.g., [3] with further references), and the symmetric Toeplitz determinant (see [1]) are of particular interest.

For this reason looking on the interest of specialists in [4] and [7] the study of the Hermitian Toeplitz determinants on the class $\mathcal{A}$ or its subclasses has begun. Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics as well as in physics and technical sciences.

In [4] the conjecture that the sharp inequalities $0 \leq |T_{q,1}(f)| \leq 1$ for all $q \geq 2$, holds over the class $\mathcal{S}^c$ was proposed and was confirmed for $q = 2$ and $q = 3$. The purpose of this paper is to prove this conjecture for $q = 4$. 
Let $\mathcal{P}$ be the class of all $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \quad (5)$$

having a positive real part in $\mathbb{D}$.

The key to the proof of the main result is the following lemma. It contains the well-known formula for $c_2$ (see e.g., [10, p. 166]) and the formula for $c_3$ due to Libera and Zlotkiewicz [8,9].

**Lemma 1** If $p \in \mathcal{P}$ is of the form (5) with $c_1 \geq 0$, then

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta \quad (6)$$

and

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \quad (7)$$

for some $\zeta, \eta \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$

### 2 Main result

In [4] the sharp bounds for the Hermitian-Toeplitz determinants of the second and third-order for the class of convex functions of order $\alpha$ were computed. In particular for $\alpha = 0$, the results obtained are reduced for the class of convex functions, namely, $0 \leq |T_{2,1}(f)| \leq 1$ and $0 \leq |T_{3,1}(f)| \leq 1$ when $f \in \mathcal{S}^c$. Both results suggested the conjecture that $0 \leq |T_{q,1}(f)| \leq 1$ for every $q \geq 2$. In this paper, this conjecture was confirmed for $q = 4$. For the sake of consistency, we also provide a short proof of the case $q = 3$.

**Theorem 1** If $f \in \mathcal{S}^c$, then

$$0 \leq |T_{3,1}(f)| \leq 1. \quad (8)$$

Both inequalities are sharp with equalities attained by

$$f(z) = \frac{z}{1 - z}, \quad z \in \mathbb{D}, \quad (9)$$

and by the identity, respectively.

**Proof** Let $f \in \mathcal{S}^c$ be the form (1). Then by (2),

$$f'(z) + zf''(z) = p(z)f'(z), \quad z \in \mathbb{D}, \quad (10)$$
for some \( p \in P \) of the form (5). Substituting the series (1) and (5) into (10) by equating the coefficients we get

\[
a_2 = \frac{1}{2} c_1, \quad a_3 = \frac{1}{6} (c_1^2 + c_2), \quad a_4 = \frac{1}{24} (c_1^3 + 3c_1c_2 + 2c_3). \tag{11}
\]

Noting that the class \( S_c \) and \(|T_{3,1}(f)|\) are rotationally invariant, we may assume that 
\( c := c_1 \) with \( c \in [0, 2] \) ([2], see also [6, Vol. I, page 80, Theorem 3]), i.e., by (11), \( a_2 \in [0, 1] \). Then by (3) and (11) we have

\[
|T_{3,1}(f)| = 1 + 2a_2^2 \Re a_3 - 2a_2^2 - |a_3|^2
\]

\[
= \frac{1}{72} \left( 72 - 36c_1^2 + 4c_4^2 + 2c_2^2 \Re c_2 - 2|c_2|^2 \right).
\]

Now by using (6) and (7) we get

\[
0 \leq |T_{3,1}(f)| = \frac{1}{144} (4 - c_2^2)(9 - |\xi|^2) \leq 1,
\]

which shows (8).

It is clear that the function (9) and the identity make the results sharp. \( \square \)

We will now estimate the fourth-order Toeplitz determinant \(|T_{3,1}(f)|\) for \( f \in \mathcal{S}_c \).

**Theorem 2** If \( f \in \mathcal{S}_c \), then

\[
0 \leq |T_{4,1}(f)| \leq 1. \tag{12}
\]

Both inequalities are sharp with equalities attained by the function (9) and the identity, respectively.

**Proof** Assuming as in the proof of Theorem 1 that \( c := c_1 \) with \( c \in [0, 2] \), so \( a_2 \in [0, 1] \), from (4) and (11) we obtain

\[
|T_{4,1}(f)| = 1 + a_2^4 - 3a_2^2 + 4a_2^2 \Re a_3 - 2a_2^2|a_3|^2 - 2a_3^2 \Re a_4 + a_2^2|a_4|^2
\]

\[
+ 4a_2 \Re (a_3 \overline{a}_4) - 2a_2 \Re (a_3^2 \overline{a}_4) - 2|a_3|^2 + |a_3|^4 - |a_4|^2
\]

\[
= \frac{1}{20736} \left( 20736 - 15552c_1^2 + 3600c_4^2 - 252c_6 + c_8 + 1152c_2^2 \Re c_2
\]

\[
- 288c_4^2 \Re c_2 + 64c_4^2(\Re c_2)^2 - 2c_6 \Re c_2 + 252c_2^2|c_2|^2
\]

\[
- 31c_4^2|c_2|^2 + 16|c_2|^4 - 1152|c_2|^2 - 24c_4^2 \Re c_2^2
\]

\[
- 8c_2^2|c_2|^2 \Re c_2 - 12c_5 \Re c_3 + 144c_2 \Re (c_2 \overline{c}_3)
\]

\[
+ 12c_3 \Re (c_2 \overline{c}_3) - 48c_2 \Re (c_2^2 \overline{c}_3) - 144|c_3|^2 + 36c_2^2|c_3|^2 \right). \tag{13}
\]
Hence by using (6) and (7) after tedious computation we get

\[
|T_{4,1}(f)| = \frac{1}{82944} (4 - c^2)^3 \left[ 1296 - 9(c^2 + 32)|\xi|^2 \\
+ 6c^2|\xi|^2 \text{Re} \, \xi + (16 - c^2)|\xi|^4 - 36c(1 - |\xi|^2) \text{Re}(\xi \bar{\eta}) \\
+ 12c(1 - |\xi|^2) \text{Re}(\xi^2 \bar{\eta}) - 36(1 - |\xi|^2)^2 |\eta|^2 \right].
\]

(14)

A. Suppose that \(\xi = 0\). Then

\[0 \leq |T_{4,1}(f)| = \frac{1}{64} (4 - c^2)^3 \leq 1.\]

Suppose that \(\eta = 0\). Then

\[
|T_{4,1}(f)| = \frac{1}{82944} (4 - c^2)^3 \left[ 1296 - 9(c^2 + 32)|\xi|^2 \\
+ 6c^2|\xi|^2 \text{Re} \, \xi + (16 - c^2)|\xi|^4 \right].
\]

(15)

We have

\[
1296 - 9(c^2 + 32)|\xi|^2 + 6c^2|\xi|^2 \text{Re} \, \xi + (16 - c^2)|\xi|^4 \\
\geq 1296 - 9(c^2 + 32)|\xi|^2 - 6c^2|\xi|^3 + (16 - c^2)|\xi|^4 \\
= 1296 - 288|\xi|^2 + 16|\xi|^4 - |\xi|^2(3 + |\xi|)^2 c^2 \\
\geq 1296 - 288|\xi|^2 + 16|\xi|^4 - 4|\xi|^2(3 + |\xi|)^2 \\
= 1296 - 324|\xi|^2 - 24|\xi|^3 + 12|\xi|^4 \geq 960, \quad 0 \leq |\xi| \leq 1.
\]

Hence and from (15) it follows that \(|T_{4,1}(f)| \geq 0\).

We have

\[
1296 - 9(c^2 + 32)|\xi|^2 + 6c^2|\xi|^2 \text{Re} \, \xi + (16 - c^2)|\xi|^4 \\
\leq 1296 - 9(c^2 + 32)|\xi|^2 + 6c^2|\xi|^3 + (16 - c^2)|\xi|^4 \\
= 1296 - 288|\xi|^2 + 16|\xi|^4 - |\xi|^2(3 - |\xi|)^2 c^2 \\
\leq 1296 - 288|\xi|^2 + 16|\xi|^4 \leq 1296, \quad 0 \leq |\xi| \leq 1.
\]

Hence and from (15) it follows that \(|T_{4,1}(f)| \leq 1\).

B. Let now \(\xi, \eta \in \overline{D} \setminus \{0\}\). By setting \(\xi := xe^{i\theta}, \eta := ye^{i\psi}, x, y \in (0, 1], \theta, \psi \in [0, 2\pi)\), from (14) we get

\[
|T_{4,1}(f)| = \frac{1}{82944} (4 - c^2)^3 F(c, x, y, \theta, \psi),
\]

(16)
where

\[ F(c, x, y, \theta, \psi) := 1296 - 9(c^2 + 32)x^2 + 6c^2x^3 \cos \theta + (16 - c^2)x^4 \]
\[ - 36c(1 - x^2)xy \cos(\theta - \psi) + 12c(1 - x^2)x^2y \cos(2\theta - \psi) \]
\[ - 36(1 - x^2)^2y^2. \]

For \( c \in [0, 2] \) and \( x, y \in (0, 1] \) we have

\[ G(c, x, y) \leq F(c, x, y, \theta, \psi) \leq H(c, x, y), \tag{17} \]

where for \( x, y \in [0, 1] \),

\[ G(c, x, y) := F(c, x, y, \pi, \pi) \]
\[ = 1296 - 9(c^2 + 32)x^2 - 6c^2x^3 + (16 - c^2)x^4 \]
\[ - 12c(1 - x^2)(3 + x)xy - 36(1 - x^2)^2y^2 \]

and

\[ H(c, x, y) := 1296 - 9(c^2 + 32)x^2 + 6c^2x^3 + (16 - c^2)x^4 \]
\[ + 12c(1 - x^2)(3 + x)xy - 36(1 - x^2)^2y^2. \]

C. Let us observe that

\[ G(c, x, y) = 1296 - 9(c^2 + 32)x^2 - 6c^2x^3 + (16 - c^2)x^4 \]
\[ - 12c(1 - x^2)(3 + x)xy - 36(1 - x^2)^2y^2 \]
\[ \geq 1296 - 324x^2 - 24x^3 - 24(1 - x^2)(3 + x)x - 36(1 - x^2)^2 \]
\[ \geq 1296 - 480 = 816, \quad c \in [0, 2], \; x, y \in (0, 1]. \]

Hence and from (16) with (17) it follows that

\[ |T_{4,1}(f)| = \frac{1}{82944}(4 - c^2)^3 F(c, x, y, \theta, \psi) \]
\[ \geq \frac{1}{82944}(4 - c^2)^3 G(c, x, y) \geq \frac{17}{1728} (4 - c^2)^3 \geq 0, \quad c \in [0, 2], \]

which together with part A shows the lower bound in (12).

D. Now will discuss the upper bound of \( |T_{4,1}(f)| \).

Let first \( x = 1 \). Then for \( c \in [0, 2] \) and \( y \in (0, 1] \),

\[ H(c, 1, 1) = H(c, 1, y) = 1024 - 4c^2 \leq 1024, \quad c \in [0, 2]. \]

Let now \( x \in (0, 1) \). Then

\[ yw := \frac{cx(x + 3)}{6(1 - x^2)} \geq 0, \quad -36(1 - x^2)^2 < 0. \]
Therefore we consider two cases.

**D1.** Assume that $y_w < 1$, i.e., equivalently that $x \in (0, x_1(c))$, where

$$x_1(c) := \frac{-3c + \sqrt{9c^2 + 24c + 144}}{2(c + 6)}, \quad c \in [0, 2].$$

Note that $x_1(c) \leq 1$ for all $c \in [0, 2]$ and $x_1(0) = 1$. Let $\Delta_1 := \{(c, x) : 0 \leq c \leq 2, \ 0 \leq x \leq x_1(c)\}$. We have

$$H(c, x, y) \leq H(c, x, y_w) = h(c, x), \quad (c, x) \in \Delta_1, \ y \in (0, 1],$$

where

$$h(c, x) := 1296 - 288x^2 + 12c^2x^3 + 16x^4, \quad (c, x) \in \Delta_1.$$

(i) On the vertices of $\Delta_1$,

$$h(0, 0) = 1296, \quad h(0, x_1(0)) = h(0, 1) = 1296, \quad h(2, 0) = 1296,$$

$$h(2, x_1(2)) = h\left(2, \frac{1}{8} \left(\sqrt{57} - 3\right)\right) = \frac{9}{32} \left(3461 + 113\sqrt{57}\right) \approx 1213.349.$$

(ii) On the side $x = 0$,

$$h(c, 0) = 1296, \quad c \in (0, 2).$$

(iii) On the side $x = x_1(c)$ for $c \in (0, 2)$,

$$h(c, x_1(c))$$

$$= \frac{18}{(c + 6)^4} \left(-9c^6 - 72c^5 - 180c^4 + 216c^3 + 11840c^2 + 52224c + 73728ight.$$  

$$+(3c^5 + 20c^4 + 36c^3 + 344c^2 + 768c)\sqrt{9c^2 + 24c + 144}\right) =: \gamma(c).$$

We will show that $\gamma$ increases. Note that

$$\gamma'(c) = \frac{-36 \left(\varrho_1(c) + \varrho_2(c)\sqrt{9c^2 + 24c + 144}\right)}{(c + 6)^5\sqrt{9c^2 + 24c + 144}} > 0, \quad c \in (0, 2),$$

is equivalent to

$$\varrho_1(c) + \varrho_2(c)\sqrt{9c^2 + 24c + 144} < 0, \quad c \in (0, 2),$$
where for \( t \in \mathbb{R} \),
\[
\varrho_1(t) := -27t^7 - 630t^6 - 4224t^5 - 15084t^4 - 55800t^3 - 77472t^2 - 214272t - 331776
\]
and
\[
\varrho_2(t) := 9t^6 + 198t^5 + 1080t^4 + 2268t^3 + 9896t^2 + 7296t - 9216.
\]
Since \( \varrho_1' \) has only two real zeros at \( c \approx -12.8436 \) and \( c \approx -5.43834 \), and \( \varrho_1'(0) < 0 \), so \( \varrho_1 \) decreases on \((0, 2)\). This with \( \varrho_1(0) < 0 \) yields
\[
\varrho_1(c) < 0, \quad c \in (0, 2).
\]
Since \( \varrho_2' > 0 \) for \( c \in [0, 2] \), so \( \varrho_2 \) increases from \( \varrho_2(0) = -9216 \) to \( \varrho_2(2) = 87296 \). Thus \( \varrho_2(c) \leq 0 \) for \( c \in (0, c') \) and \( \varrho_2(c) > 0 \) for \( c \in (c', 2) \), where \( c' \approx 0.627225 \) is the unique zero of \( \varrho_2 \) in \((0, 2)\).

Hence and from (20) it follows that the inequality (19) occurs for \( c \in (0, c') \).

Note now that for \( c \in (c', 2) \) we have
\[
\varrho_1^2(c) > (9c^2 + 24c + 144)\varrho_2^2(c).
\]
Indeed, the above inequality is equivalent to
\[
48(c + 6)^5\varrho_3(c) < 0, \quad c \in (c', 2),
\]
where
\[
\varrho_3(t) := 9t^6 - 30t^5 - 4248t^4 - 31392t^3 - 70272t^2 - 208896t - 262144, \quad t \in \mathbb{R}.
\]
As easy to see \( \varrho_3'(c) < 0 \) for \( c \in (0.6, 2) \), so \( \varrho_3 \) decreases on \((0.6, 2)\), and therefore on \((c', 2)\). This and \( \varrho_3(0.6) < 0 \) yields
\[
\varrho_3(c) < 0, \quad c \in (c', 2),
\]
which confirms (21), (22), and ends the proof of (19) so of (18).

Summarizing,
\[
h(c, x_1) = \gamma(c) \leq \gamma(2) < 1296, \quad c \in (0, 2).
\]
(iv) When \( c = 0 \), then \( x_1(0) = 1 \) and
\[
h(0, x) = 1296 - 288x^2 + 16x^4 \leq 1296, \quad x \in [0, 1).
\]
(v) When \( c = 2 \), then
\[
h(2, x) = 1296 - 288x^2 + 48x^3 + 16x^4 \leq 1296, \quad x \in (0, x_1(2)).
\]
Indeed, as easy to see the function \((0, x_1(2)) \ni x \mapsto H_2(2, x)\) decreases. 

(vi) It remains to consider the interior of \(\Delta_1\). Since the system of equations

\[
\begin{cases}
  24cx^3 = 0 \\
  -576x + 36c^2x^2 + 64x^3 = 0
\end{cases}
\]

has solutions \((c_i, x_i), i = 1, 2, 3\), only when

\[
\begin{cases}
  c_1 = 0 \\
  c_2 = 0 \\
  c_3 = 0 \\
  x_1 = 0 \\
  x_2 = -3 \\
  x_3 = 3
\end{cases}
\]

so \(h\) has no critical points in the interior of \(\Delta_1\).

\[\textbf{D2.}\] Assume that \(y_w \geq 1\), i.e., equivalently that \(x \in [x_1(c), 1]\). Let \(\Delta_2 := \{(c, x) : 0 \leq c \leq 2, \ x_1(c) \leq x \leq 1\}\). Then

\[
H(c, x, y) \leq H(c, x, 1) = g(c, x), \quad (c, x) \in \Delta_2, \ y \in (0, 1],
\]

where for \((c, x) \in \Delta_2,\)

\[
g(c, x) := 1260 + 36cx - 3(3c^2 - 4c + 72)x^2 + 6(c^2 - 6c)x^3 - (c^2 + 12c + 20)x^4.
\]

(i) On the vertices of \(\Delta_2,\)

\[
g(0, x_1(0)) = g(0, 1) = 1024, \quad g(2, 1) = 1008,
\]

\[
g(2, x_1(2)) = g \left(2, \left(\sqrt{57} - 3\right)/8\right) = \frac{9}{32} \left(3461 + 113\sqrt{57}\right) \approx 1213.349.
\]

(ii) On the side \(x = x_1(c)\) we have the case B1(iii).

(iii) On the side \(x = 1,\)

\[
g(c, 1) = -4c^2 + 1024 \leq 1024, \quad c \in (0, 2).
\]

(iv) On the side \(c = 2,\)

\[
g(2, x) = -48x^4 - 48x^3 - 228x^2 + 72x + 1260, \quad x \in \left(\left(\sqrt{57} - 3\right)/8, 1\right).
\]

Since \(-192x^3 - 144x^2 - 456x + 72 = 0\) if and only if \(x \approx 0.14944\), so the function

\[
\left(\left(\sqrt{57} - 3\right)/8, 1\right) \ni x \mapsto H_3(2, x)
\]

is decreasing. Therefore

\[
g(2, x) \leq g \left(2, \left(\sqrt{57} - 3\right)/8\right) \approx 1213.349, \quad x \in \left(\left(\sqrt{57} - 3\right)/8, 1\right).
\]
(v) It remains to consider the interior of $\Delta_2$. The system of equations

\[
\begin{aligned}
36x - 6(3c - 2)x^2 + 12(c - 3)x^3 - 2(c + 6)x^4 &= 0 \\
36c - 6(3c^2 - 4c + 72)x + 18(c^2 - 6c)x^2 - 4(c^2 + 12c + 20)x^3 &= 0
\end{aligned}
\]

has the solution $c = x = 0$ evidently. Let $x \neq 0$ and $x \neq 3$. From the first equation we get

\[
c = \frac{6(1 - x^2)(x + 3)}{x(x - 3)^2}
\]

which satisfies the inequality $0 \leq c \leq 2$ only when $x \in [x', 1)$, where $x' \approx 0.81244$. Now substituting (24) into the second equation of (23) we obtain the equation

\[
4x^6 + 45x^5 - 333x^4 + 162x^3 - 486x^2 + 945x - 81 = 0
\]

which has the unique solution in $(0, 1)$, namely $x'' \approx 0.089756 < x'$. Thus $g$ has no critical point in the interior of $\Delta_2$.

Summarizing, from par D it follows that

\[
|T_{4,1}(f)| = \frac{1}{82944} (4 - c^2)^3 F(c, x, y, \theta, \psi) \leq \frac{1}{82944} \cdot 4^3 \cdot 1296 = 1. \tag{25}
\]

It is clear that equality for the upper bound in (12) holds for the identity function, and for the lower bound for the function (9).

☐

Compliance with ethical standards

Conflict of interest  The authors declare that there is no conflict of interest.

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