HIGHER RANK NUMERICAL RANGES AND UNITARY DILATIONS

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Abstract. Here we show that for \( k \in \mathbb{N} \), the closure of the \( k \)-rank numerical range of a contraction \( A \) acting on an infinite-dimensional Hilbert space \( \mathcal{H} \) is the intersection of the closure of the \( k \)-rank numerical ranges of all unitary dilations of \( A \) to \( \mathcal{H} \oplus \mathcal{H} \). The same is true for \( k = \infty \) provided the \( \infty \)-rank numerical range of \( A \) is non-empty. These generalize a finite dimensional result of Gau, Li and Wu. We also show that when both defect numbers of a contraction are equal and finite (\( = N \)), one may restrict the intersection to a smaller family consisting of all unitary \( N \)-dilations. A result of Bercovici and Timotin on unitary \( N \)-dilations is used to prove it. Finally, we have investigated the same problem for the \( C \)-numerical range and obtained the answer in negative.

1. Introduction

Let \( \mathcal{H} \) be a Hilbert space and \( \mathcal{B}(\mathcal{H}) \) be the algebra of all bounded linear maps acting on \( \mathcal{H} \). Suppose \( A \in \mathcal{B}(\mathcal{H}) \). Let \( \mathcal{K} \) be a Hilbert space containing \( \mathcal{H} \). An operator \( B \in \mathcal{B}(\mathcal{K}) \) is said to be a dilation of \( A \) (or, \( A \) is said to be a compression of \( B \)) if there exists a projection \( P \in \mathcal{B}(\mathcal{K}) \) on \( \mathcal{H} \) such that \( A = P_{\mathcal{H}}B|_{\mathcal{H}} \) or, equivalently, if \( B \) is unitarily similar to the operator matrix \( \begin{pmatrix} A & * \\ * & * \end{pmatrix} \). If \( \dim \mathcal{K} \ominus \mathcal{H} = r \) then \( B \) is called an \( r \)-dilation of \( A \). Moreover, if \( B \), being a dilation of \( A \), is unitary then \( B \) is said to be a unitary dilation of \( A \). Halmos [11] showed that every contraction \( A \in \mathcal{B}(\mathcal{H}) \) has a unitary dilation \( U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) of the form

\[
U = \begin{pmatrix}
A & -\sqrt{I-AA^*} \\
\sqrt{I-A^*A} & A^*
\end{pmatrix}.
\]

It generated a lot of research including far reaching Sz.-Nagy dilation theorem regarding power unitary dilations of a contraction.

The notion of the quadratic form associated with a matrix has been extended for an operator acting on a Hilbert space, which is known as the numerical range. The numerical range of \( A \in \mathcal{B}(\mathcal{H}) \), denoted by \( W(A) \), is defined as

\[
W(A) := \{ \langle Ax, x \rangle : \|x\| = 1 \}.
\]
It has been studied extensively because of its connections and applications to many different areas. The numerical range is a non-empty, convex set. Durszt \[7\] proved that the numerical range of a normal operator \(A\) is the intersection of all convex Borel set \(s\) such that \(E(s) = I\) where \(E\) is the unique spectral measure associated with \(A\). The closure of the numerical range of a normal operator is the closed convex hull of its spectrum \([10]\).

It is, in general, difficult to compute the numerical range. Halmos \([12]\) conjectured that for every contraction \(A \in B(H)\),

\[
W(A) = \bigcap \{W(U) : U \text{ is a unitary dilation of } A\}. \tag{1.1}
\]

Durszt \([7]\) settled (1.1) in negative using the description of the numerical range of a normal operator in terms of the convex Borel sets. Later, Choi and Li \([5]\) proved that for every contraction \(A \in B(H)\),

\[
\overline{W(A)} = \bigcap \left\{\overline{W(U)} : U \in B(H \oplus H) \text{ is a unitary dilation of } A\right\}. \tag{1.2}
\]

Note here that the closure sign can be omitted in finite dimensional case.

Let \(A \in B(H)\) be a contraction. Denote \(D_A = (I - A^*A)^{\frac{1}{2}}\), \(D_A = \text{ran}D_A\) and \(d_A = \dim D_A\) as the defect operator, the defect space and the defect number of \(A\), respectively. Bercovici and Timotin \([3]\) recently showed that if both defect numbers of a contraction are same and finite (=\(N\)) then one can restrict the intersection to a smaller family consisting of all unitary \(N\)-dilations. It complements Choi and Li’s theorem (1.2) and also generalizes a result of Benhida, Gorkin and Timotin \([2]\) for \(C_0(N)\) contractions. Let \(A \in B(H)\) be a contraction with \(d_A = d_A^* = N < \infty\). BT proved

\[
\overline{W(A)} = \bigcap \left\{\overline{W(U)} : U \text{ is a unitary } N\text{-dilation of } A \text{ to } H \oplus \mathbb{C}^N\right\}. \tag{1.3}
\]

Choi, Kribs and Žyczkowski have first defined the higher rank numerical range in the context of “quantum error correction” \([4]\), which is defined in the following way. Let \(A \in B(H)\) and \(1 \leq k \leq \infty\). The \(k\)-rank numerical range of \(A\), denoted by \(\Lambda_k(A)\), is defined as

\[
\Lambda_k(A) := \{\lambda \in \mathbb{C} : PAP = \lambda P, \text{ for some projection } P \text{ of rank } k\}
\]

or, equivalently, \(\lambda \in \Lambda_k(A)\) if and only if there is an orthonormal set \(\{f_j\}_{j=1}^k\) such that \(\langle Af_j, f_r \rangle = \lambda \delta_{j,r}\) for \(j, r \in \{1, 2, \cdots, k\}\). Clearly,

\[
W(A) = \Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_k(A) \supseteq \cdots.
\]

Li and Sze \([15]\) have described the higher rank numerical range of a matrix as an intersection of closed half planes. Let \(A \in M_n\) and \(1 \leq k \leq n\). Li and Sze showed

\[
\Lambda_k(A) = \bigcap_{\theta \in [0,2\pi)} \left\{\mu \in \mathbb{C} : \Re(e^{i\theta} \mu) \leq \lambda_k(\Re(e^{i\theta} A))\right\}, \tag{1.4}
\]

\[
\lambda_k(\Re(e^{i\theta} A)) = \frac{\lambda_k(e^{i\theta}A)}{1}, \quad \lambda_k(\Re(e^{i\theta} A)) = \frac{\lambda_k(e^{i\theta}A)}{1}.
\]
where $\lambda_k(H)$ denotes the $k$-th largest eigenvalue of the self-adjoint matrix $H \in M_n$. As an immediate consequence, it follows that the higher rank numerical range of a matrix is convex. It also proved that for a normal matrix $A \in M_n$,

\[ \Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \text{conv} \{ \lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}} \}. \]  

However, the convexity of the higher rank numerical range of any operator was shown by Woerdeman \[20\]. For the non-emptyness of the higher rank numerical ranges, the reader may refer to \[14\], \[17\].

Let $A \in M_n$ be a contraction. Observe that $d_A = d_{A^*}$. Gau, Li and Wu \[9\] proved the following which extends and refines (1.2) to the higher rank numerical ranges of matrices.

**Theorem 1.1** (Theorem 1.1, \[9\]). Let $A \in M_n$ be a contraction and $1 \leq k \leq n$. Then

\[ \Lambda_k(A) = \bigcap \{ \Lambda_k(U) : U \in M_{n+d_A} \text{ is a unitary dilation of } A \}. \]

It is to be noted that there exists a normal contraction $A$ for which the $k$-rank numerical range of all unitary dilations of $A$ contains $\Lambda_k(A)$ as a proper subset for $1 \leq k \leq \infty$. See \[6\]. The following is the first main theorem of this paper. It generalizes Theorem 1.1.

**Theorem 1.2.** Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction and $k \in \mathbb{N}$. Then

\[ \Lambda_k(A) = \bigcap \{ \Lambda_k(U) : U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } A \}. \]

Moreover,

\[ \Lambda_\infty(A) = \bigcap \{ \Lambda_\infty(U) : U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } A \} \]

provided $\Lambda_\infty(A)$ is non-empty.

An example is given that Theorem 1.2 is not true whenever the $\infty$-rank numerical range is empty (Example 3.3).

The following is the second main theorem of this paper which complements the previous theorem. It states that if both defect numbers of a contraction are same and finite then one can restrict the intersection to a smaller family of unitary dilations. It generalizes (1.3). A result of Bercovici and Timotin on unitary $\mathcal{N}$-dilations is used while proving it.

**Theorem 1.3.** Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction with $d_A = d_{A^*} = \mathcal{N} < \infty$ and $1 \leq k \leq \infty$. Then

\[ \Lambda_k(A) = \bigcap \{ \Lambda_k(U) : U \text{ is a unitary } \mathcal{N}\text{-dilation of } A \text{ to } \mathcal{H} \oplus \mathbb{C}^\mathcal{N} \}. \]

There are so many other generalizations of the numerical range like the higher rank numerical range. One of them is the $C$-numerical range. Let $A, C \in M_n$. The $C$-numerical range of $A$, denoted by $W_C(A)$, is defined as

\[ W_C(A) := \{ \text{tr}(CU^*AU) : U \in M_n \text{ is a unitary matrix} \}. \]  

(1.6)
Let \( C \) be unitarily similar to \( \mathbf{c} = \text{diag}(c_1, c_2, \cdots, c_n) \) where \( c_j \in \mathbb{C} \) for all \( 1 \leq j \leq n \). As \( W_c(A) \) is unitarily invariant, we obtain
\[
W_c(A) = \left\{ \sum_{j=1}^{n} c_j \langle Ae_j, e_j \rangle : \{e_j\}_{j=1}^{n} \text{ is an orthonormal basis of } \mathbb{C}^n \right\}. \tag{1.7}
\]
Note that if \( \mathbf{c} = \text{diag}(1, 0, \cdots, 0) \) then \( W_c(A) = W(A) \). Westwick [19] has shown that \( W_c(A) \) is convex if \( c_j \in \mathbb{R} \) for all \( 1 \leq j \leq n \). In addition, he gave an example which shows that for \( (c_1, c_2, \cdots, c_n) \in \mathbb{C}^n \) with \( n \geq 3 \), the \( c \)-numerical range may fail to be convex. The reader may refer to the survey article [13] for more details on the \( C \)-numerical range. We have investigated Theorem 1.2- and Theorem 1.3-type relations for the \( c \)-numerical range and have obtained the answer in negative, which is the final theorem of this paper.

**Theorem 1.4.** There exist \( A, c \) for which the intersection of the closure of the \( c \oplus 0 \)-numerical ranges of all unitary dilations of \( A \) contains \( W_c(A) \) as a proper subset.

Let us end this section by listing some basic properties of the higher rank numerical range:

\( \mathbf{P1} \) \( \Lambda_k(\alpha A + \beta I) = \alpha \Lambda_k(A) + \beta, \) for \( \alpha, \beta \in \mathbb{C} \).

\( \mathbf{P2} \) \( \Lambda_k(A^*) = \overline{\Lambda_k(A)}. \)

\( \mathbf{P3} \) \( \Lambda_k(A \oplus B) \supseteq \Lambda_k(A) \cup \Lambda_k(B). \)

\( \mathbf{P4} \) \( \Lambda_k(U^*AU) = \Lambda_k(A), \) for any unitary \( U \in \mathcal{B}(\mathcal{H}). \)

\( \mathbf{P5} \) If \( A_0 \) is a compression of \( A \) on a subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) such that \( \dim(\mathcal{H}_0) \geq k \) then \( \Lambda_k(A_0) \subseteq \Lambda_k(A). \)

2. Preliminaries

Throughout this paper, we denote \( \mathcal{H} \), an infinite dimensional separable Hilbert space and \( \mathcal{B}(\mathcal{H}) \), the algebra of all bounded linear maps acting on \( \mathcal{H} \). Let \( A \in \mathcal{B}(\mathcal{H}) \) be a normal operator and \( E \) be the unique spectral measure associated with \( A \) defined on the Borel \( \sigma \)-algebra in \( \mathbb{C} \) supported on \( \sigma(A) \). Suppose \( 1 \leq k \leq \infty \). Let
\[
S_k := \{ H : H \text{ is a half closed-half plane in } \mathbb{C} \text{ with } \dim \text{ ran } E(H) < k \}\).
\[
S_k := \bigcap_{H \in S_k} H^c. \]
Dey and Mukherjee [6] proved the following, which extends (1.5) for a normal operator acting on an infinite-dimensional Hilbert space.

**Theorem 2.1** (Theorem 3.2, [6]). Let \( A \in \mathcal{B}(\mathcal{H}) \) be normal and \( 1 \leq k \leq \infty \). Then
\[
\Lambda_k(A) = \bigcap_{V \in \mathcal{V}_k} W(V^*AV) = V_k(A),
\]
where \( \mathcal{V}_k \) is the set of all isometries \( V : \mathcal{H} \to \mathcal{H} \) such that codimension of \( \text{ran } V \) is less than \( k \).
Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator and $k \in \mathbb{N}$. Define
\[
\lambda_k(A) := \sup \{ \lambda_k(V^*AV) : V : \mathbb{C}^k \to \mathcal{H} \text{ is an isometry} \}.
\] (2.1)

Let us now observe the following.

**Lemma 2.2.** Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $k \in \mathbb{N}$. Suppose $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is an increasing sequence of closed subspaces of $\mathcal{H}$ with $\mathcal{H} = \bigcup_{n \geq 1} \mathcal{H}_n$ and $A_n = P_{\mathcal{H}_n}A|_{\mathcal{H}_n}$. Then
\[
\lim_{n \to \infty} \lambda_k(A_n) = \lambda_k(A).
\]

**Proof.** By interlacing theorem, we have, $\lambda_k(A_n) \leq \lambda_k(A_{n+1})$ for every $n \geq k$. Observe that $\lambda_k(A_n) \leq \lambda_k(A)$ for all $n \geq k$. Indeed, if not then there exists $n_o \geq k$ such that $\lambda_k(A) < \lambda_k(A_{n_o})$. Choose $A_{n_o} = \text{diag}(\lambda_1(A_{n_o}), \ldots, \lambda_k(A_{n_o}))$. Then $A_{n_o}$ is a $k$-by-$k$ compression of $A$ with $\lambda_k(A) < \lambda_k(A_{n_o})$. It contradicts the definition of $\lambda_k(A)$. So, $\lambda_k(A_n) \leq \lambda_k(A)$ for $n \geq k$. Therefore, $\{\lambda_k(A_n)\}_{n \geq k}$ is a monotonically increasing sequence and bounded above. Hence $\lim_{n \to \infty} \lambda_k(A_n) = \sup_{n \geq k} \lambda_k(A_n)$. We claim that $\sup \lambda_k(A_n) = \lambda_k(A)$.

Given $\epsilon > 0$, by the definition of $\lambda_k(A)$, there exists a $k$-by-$k$ compression $A'$ of $A$ such that $\lambda_k(A) - \frac{\epsilon}{2} < \lambda_k(A')$. Note that there exists $n_1 \geq k$ and a $k$-by-$k$ compression $A''$ of $A_{n_1}$ such that $||A' - A''|| < \frac{\epsilon}{2}$. It implies that $|\lambda_k(A') - \lambda_k(A'')| \leq \frac{\epsilon}{2}$. So, $\lambda_k(A) - \epsilon < \lambda_k(A'')$. Again, by interlacing theorem, $\lambda_k(A'') \leq \lambda_k(A_{n_1})$. Therefore, $\lambda_k(A) - \epsilon < \lambda_k(A_{n_1})$ for some $n_1 \geq k$. Hence $\lim_{n \to \infty} \lambda_k(A_n) = \lambda_k(A)$.

Let $A \in \mathcal{B}(\mathcal{H})$ and $k \in \mathbb{N}$. Define
\[
\Omega_k(A) := \bigcap_{\xi \in [0,2\pi]} \{ \mu \in \mathbb{C} : \mathfrak{R}(e^{i\xi}\mu) \leq \lambda_k(\mathfrak{R}(e^{i\xi}A)) \}.
\]

Let $\text{Int}(S)$ denote the relative interior of $S$ for $S \subseteq \mathbb{C}$. Li, Poon and Sze [16] proved the following, which extends (1.4) for an operator.

**Theorem 2.3** (Theorem 2.1, [16]). Let $A \in \mathcal{B}(\mathcal{H})$ and $k \in \mathbb{N}$. Then
\[
\text{Int}(\Omega_k(A)) \subseteq \Lambda_k(A) \subseteq \Omega_k(A) = \overline{\Lambda_k(A)}.
\]

Define $\Omega_\infty(A) = \bigcap_{k \geq 1} \Omega_k(A)$.

**Theorem 2.4** (Theorem 5.2, [16]). Let $A \in \mathcal{B}(\mathcal{H})$. Then
\[
\text{Int}(\Omega_\infty(A)) \subseteq \Lambda_\infty(A) \subseteq \Omega_\infty(A).
\]

Moreover, $\Lambda_\infty(A) = \Omega_\infty(A)$ if and only if $\Lambda_\infty(A) \neq \emptyset$.

Let $r > 0$. Denote $D(0,r) := \{ z \in \mathbb{C} : |z| < r \}$, the disc centred at 0 with radius $r$ and $\mathbb{D} = D(0,1)$, the open unit disc. Let us end this section with the following lemmas and corollary.
Lemma 2.5 (Theorem 2.1, [8]). Suppose \( n \geq 2 \) and \( 1 \leq k \leq n \). Let \( S_n \) be the \( n \)-dimensional unilateral shift on \( \mathbb{C}^n \). Then

\[
\Lambda_k(S_n) = \begin{cases} 
D(0, \cos \frac{k\pi}{n+1}), & \text{if } 1 \leq k \leq \left[ \frac{n+1}{2} \right] \\
\emptyset, & \text{if } \left[ \frac{n+1}{2} \right] < k \leq n.
\end{cases}
\]

Lemma 2.6. Let \( S \) be a shift operator with any multiplicity acting on an infinite dimensional Hilbert space. Then \( \Lambda_k(S) = \mathbb{D} \) for all \( k \in \mathbb{N} \).

Proof. Without loss of generality, we may consider \( S : l^2(\mathcal{H}) \to l^2(\mathcal{H}) \) such that

\[
S((x_0, x_1, \ldots)) = (0, x_0, x_1, \ldots) \quad \text{and} \quad (x_0, x_1, \ldots) \in l^2(\mathcal{H}),
\]

where \( \mathcal{H} \) is a Hilbert space of dimension same as the multiplicity of \( S \). So, \( S = S_0 \otimes I_{\mathcal{H}} \), where \( S_0 \) is the unilateral shift on \( l^2(\mathbb{N}) \). Let \( S_n \) be the unilateral shift on \( \mathbb{C}^n \). As \( S_n \) is a compression of \( S_0 \), by Lemma 2.5 and (P5), we obtain \( D(0, \cos \frac{k\pi}{n+1}) = \Lambda_k(S_n) \subseteq \Lambda_k(S_0) \) for \( 1 \leq k \leq \left[ \frac{n+1}{2} \right] \). Taking \( n \to \infty \), we get \( \mathbb{D} \subseteq \Lambda_k(S_0) \). Let us now show that \( \Lambda_k(S_0) \subseteq \Lambda_k(S) \).

Let \( \lambda \in \Lambda_k(S_0) \). Then there exists an isometry \( X : \mathbb{C}^k \to l^2(\mathbb{N}) \) such that \( X^*S_0X = \lambda I_k \). Let \( h_0 \in \mathcal{H} \) with \( \|h_0\| = 1 \). Define \( Y : \mathbb{C}^k \to l^2(\mathbb{N}) \otimes \mathcal{H} \approx l^2(\mathcal{H}) \) such that \( Yh = Xh \otimes h_0 \), \( h \in \mathbb{C}^k \). Let \( h \in \mathbb{C}^k \) with \( \|h\| = 1 \). Now,

\[
\|Yh\|^2 = \langle Xh \otimes h_0, Xh \otimes h_0 \rangle = \langle Xh, Xh \rangle \langle h_0, h_0 \rangle = \|Xh\|^2 \|h_0\|^2 = 1.
\]

So, \( Y \) is an isometry. We claim that \( Y^*SY = \lambda I_k \). Let \( h \in \mathbb{C}^k \) with \( \|h\| = 1 \). Observe,

\[
\langle Y^*SYh, h \rangle = \langle (S_0 \otimes I_K)Xh \otimes h_0, Xh \otimes h_0 \rangle = \langle X^*S_0Xh, h \rangle = \lambda.
\]

So, \( Y^*SY = \lambda I_k \). Therefore, \( \lambda \in \Lambda_k(S) \). Hence \( \mathbb{D} \subseteq \Lambda_k(S_0) \subseteq \Lambda_k(S) \).

Let \( \lambda \in \Lambda_k(S) \) with \( |\lambda| = 1 \). Then there exists an orthonormal set \( \{f_j\}_{j=1}^k \) such that \( \langle Sf_j, f_r \rangle = \lambda \delta_{j,r} \) for \( 1 \leq j, r \leq k \). Note,

\[
1 = |\lambda| = |\langle Sf_j, f_j \rangle| \leq \|Sf_j\| \|f_j\| = 1
\]

for all \( 1 \leq j \leq k \). As Cauchy-Schwarz inequality is being attained for all \( 1 \leq j \leq k \), \( \lambda \) is an eigenvalue of \( S \) with multiplicity \( k \). It contradicts that \( \sigma_p(S) = \emptyset \). Hence \( \Lambda_k(S) = \mathbb{D} \). \( \square \)

Corollary 2.7. Suppose \( V \) is a proper isometry. Then \( \overline{\Lambda_k(V)} = \mathbb{D} \).

Proof. As \( V \) is a proper isometry, by Wold decomposition (c.f. [18]), we can write \( V = V_0 \oplus S \), where \( V_0 \) is unitary and \( S \) is a shift operator. So, by (P3) and Lemma 2.6, we have

\[
\mathbb{D} \supseteq W(V) \supseteq \cdots \supseteq \Lambda_k(V) = \Lambda_k(V_0 \oplus S) \supseteq \Lambda_k(V_0) \cup \Lambda_k(S) \supseteq \Lambda_k(S) = \mathbb{D}.
\]

Taking closure in both sides of (2.2), we get \( \overline{\Lambda_k(V)} = \mathbb{D} \). \( \square \)
3. Proof of Theorem 1.2

Let $A \in M_n$ and $U = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in M_{2n}$ be a unitary dilation of $A$. By polar decomposition of a matrix and using the fact that $U$ is unitary, we obtain $B = U_1 \sqrt{I - A^*A}$, $C = -\sqrt{I - AA^*}U_2$ and $D = U_1 A^* U_2$, where $U_1, U_2 \in M_n$ are unitary. Then

$$(I \oplus U_1^*) U (I \oplus U_1) = \begin{pmatrix} A & -\sqrt{I - A^*A} U_o \\ \sqrt{I - A^*A} & A^* U_o \end{pmatrix},$$

where $U_o = U_2 U_1$. Hence we may take any unitary dilation $U$ of $A$ in the form

$$U = \begin{pmatrix} A & * \\ \sqrt{I - A^*A} & * \end{pmatrix}. \quad (3.1)$$

Let us begin with the following proposition.

**Proposition 3.1.** Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A^* A + B^* B \leq I_{\mathcal{H}}$ and $k \in \mathbb{N}$. Then there exist $C, D \in \mathcal{B}(\mathcal{H})$ such that $Z = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is a contractive dilation of $A$ with $\lambda_k(Z + Z^*) = \lambda_k(A + A^*)$.

**Proof.** Let us first assume that $\dim(\mathcal{H}) = n < \infty$. Then, by Theorem 1.1, [9], there exists a unitary dilation $U_0 \in M_{n+d_A}$ of $A$ such that $\lambda_k(U_0 + U_0^*) = \lambda_k(A + A^*)$. Take $U = U_0 \oplus (-I) \in M_{2n}$. Then $U$ is a unitary dilation of $A$ with $\lambda_k(U + U^*) = \lambda_k(U_0 + U_0^*) = \lambda_k(A + A^*)$. In view of (3.1), we may take $U$ in the form

$$U = \begin{pmatrix} A & * \\ \sqrt{I - A^*A} & * \end{pmatrix}.$$  

Since $A^* A + B^* B \leq I_{\mathcal{H}}$, we have $B^* B \leq C^* C$, where $C = \sqrt{I - A^*A}$. So, $B = J \sqrt{I - A^*A}$ for some contraction $J \in M_n$. Let

$$V = \begin{pmatrix} I_n & 0_n \\ 0_n & J^* \end{pmatrix}.$$  

Then $V^* V = I_{2n}$. So, $V$ is an isometry. Let $\bar{U} = U \oplus (-I_n)$. Take $Z = V^* \bar{U} V \in M_{2n}$. Then $Z$ is a contractive dilation of $A$ with the desired form. By Cauchy’s interlacing theorem (Corollary III.1.5, [1]), we have $\lambda_k(Z + Z^*) \geq \lambda_k(A + A^*)$ and $\lambda_k(\bar{U} + \bar{U}^*) \geq \lambda_k(Z + Z^*)$ as
$Z$ is a dilation of $A$ and $\tilde{U}$ is a dilation of $Z$. So,

\[
\lambda_k(Z + Z^*) \leq \lambda_k(\tilde{U} + \tilde{U}^*) \\
= \lambda_k(U + U^*) \\
= \lambda_k(A + A^*) \\
\leq \lambda_k(Z + Z^*).
\]

It implies that $\lambda_k(Z + Z^*) = \lambda_k(A + A^*)$. This completes the proof of the proposition whenever $\mathcal{H}$ is of dimension $n < \infty$.

Now, let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots\}$. Take $A = (a_{ij})_{1 \leq i, j \leq \infty}$ and $B = (b_{ij})_{1 \leq i, j \leq \infty}$ with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$. Let $A_n = (a_{ij})_{1 \leq i, j \leq n}$ and $B_n = (b_{ij})_{1 \leq i, j \leq n}$ be the finite sections of $A$ and $B$ respectively. As $A^*A + B^*B \leq I_\mathcal{H}$, we have $A_n^*A_n + B_n^*B_n \leq I_\mathcal{H}$. So, by the above finite dimensional result, there exists a contractive dilation

\[
Z_n = \begin{pmatrix} A_n & C_n \\ B_n & D_n \end{pmatrix} \in M_{2n}
\]

of $A_n$ with $\lambda_k(Z_n + Z_n^*) = \lambda_k(A_n + A_n^*)$. Consider

\[
\tilde{Z}_n = \begin{pmatrix} A_n & 0 & C_n & 0 \\ 0 & 0 & 0 & 0 \\ B_n & 0 & D_n & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).
\]

Note that $\tilde{Z}_n$ converges in weak operator topology to

\[
Z = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.
\]

Now, by applying Lemma 2.2, we obtain $\lim_{n \to \infty} \lambda_k(A_n + A_n^*) = \lambda_k(A + A^*)$ and $\lambda_k(Z_n + Z_n^*) = \lambda_k(Z + Z^*)$. Finally, since $\lambda_k(Z_n + Z_n^*) = \lambda_k(A_n + A_n^*)$ for all $n \geq k$, we have $\lambda_k(Z + Z^*) = \lambda_k(A + A^*)$. This completes the proof.

The following theorem plays the key role while proving Theorem 1.2.

**Theorem 3.2.** Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction and $k \in \mathbb{N}$. Then there exists a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of $A$ such that $\lambda_k(U + U^*) = \lambda_k(A + A^*)$.

**Proof.** If $\lambda_k(A + A^*) = 2$ then the following unitary dilation

\[
U = \begin{pmatrix} A & -\sqrt{I - AA^*} \\ \sqrt{I - A^*A} & A^* \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})
\]

of $A$ will do the job. Therefore, assume that $\lambda_k(A + A^*) = \mu < 2$. 

Take $B = \sqrt{I - A^*A} \in \mathcal{B}(\mathcal{H})$. Then, by Proposition 3.1, there exists a contractive dilation

$$Z_1 = \left( \frac{A}{\sqrt{I - A^*A}} \begin{pmatrix} C \\ D \end{pmatrix} \right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

of $A$ such that $\lambda_k(Z_1 + Z_1^*) = \lambda_k(A + A^*)$ and $\|Z_1v\| = \|v\|$ for all $v \in \mathcal{H} \oplus \mathcal{O}$, where $\mathcal{O}$ is the zero subspace of $\mathcal{H}$.

Repeating the argument on $Z_1$, we get a contractive dilation

$$Z_2 = \left( \frac{Z_1}{\sqrt{I - Z_1^*Z_1}} \begin{pmatrix} \tilde{C} \\ \tilde{D} \end{pmatrix} \right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H})$$

of $Z_1$ such that $\lambda_k(Z_2 + Z_2^*) = \lambda_k(Z_1 + Z_1^*) = \lambda_k(A + A^*)$ and $\|Z_2v\| = \|v\|$ for all $v \in \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{O} \oplus \mathcal{O}$. Continuing this process, we obtain a contractive dilation $Z_{\infty}$, denoted by $U$, acting on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ such that $\lambda_k(U + U^*) = \lambda_k(A + A^*)$ and $\|Uv\| = \|v\|$ for all unit vector $v \in \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots$. Identifying $\mathcal{O} \oplus \mathcal{H}$ with $\mathcal{O} \oplus \mathcal{H} \oplus \mathcal{H} \cdots$, we may regard $U$ as an isometry acting on $\mathcal{H} \oplus \mathcal{H}$ while $A$ acts on $\mathcal{H} \oplus \mathcal{O}$. Hence we get an isometric dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of $A$ such that $\lambda_k(U + U^*) = \lambda_k(A + A^*) = \mu < 2$.

We will now show that $U$ is unitary. If possible, assume that $U$ is a proper isometry. Then, by Corollary 2.7, we get $\overline{\Lambda_k(U)} = \overline{\mathbb{D}}$. This forces $\overline{\Lambda_k(U^* + U)} = [-2, 2]$. This contradicts our assumption that $\lambda_k(U + U^*) = \mu < 2$. This completes the proof. \hfill \Box

**Proof of Theorem 1.2.** Suppose $1 \leq k \leq \infty$. Clearly,

$$\overline{\Lambda_k(A)} \subseteq \bigcap \left\{ \overline{\Lambda_k(U)} : U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } A \right\}.$$

To prove the reverse inclusion, let us first assume that $k \in \mathbb{N}$. Suppose $\xi \notin \overline{\Lambda_k(A)}$. Then, by Theorem 3.2, there exists $\theta \in [0, 2\pi)$ such that $e^{i\theta} \xi + e^{-i\theta} \xi = \lambda_k(e^{i\theta}A + e^{-i\theta}A^*)$. Now, by Theorem 2.3, there exists a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of $A$ such that $\lambda_k(e^{i\theta}U + e^{-i\theta}U^*) = \lambda_k(e^{i\theta}A + e^{-i\theta}A^*)$. Therefore, again by Theorem 2.3, we have $\xi \notin \Lambda_k(U)$.

Let $\xi \notin \overline{\Lambda_\infty(A)}$. Then, by Theorem 2.4, there exists $k_0 \in \mathbb{N}$ such that $\xi \notin \Omega_{k_0}(A)$. So, by Theorem 2.3, there exists $\theta \in [0, 2\pi)$ such that $e^{i\theta} \xi + e^{-i\theta} \xi > \lambda_{k_0}(e^{i\theta}A + e^{-i\theta}A^*)$. Now, by Theorem 3.2, there exists a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of $A$ such that $\lambda_{k_0}(e^{i\theta}U + e^{-i\theta}U^*) = \lambda_{k_0}(e^{i\theta}U + e^{-i\theta}U^*)$. So, $e^{i\theta} \xi + e^{-i\theta} \xi = \lambda_{k_0}(e^{i\theta}U + e^{-i\theta}U^*)$. Therefore, by Theorem 2.3, $\xi \notin \Omega_{k_0}(U)$. Hence, by Theorem 2.4, $\xi \notin \bigcap_{k \geq 1} \Omega_k(U) = \Omega_\infty(U) = \overline{\Lambda_\infty(U)}$ as $\emptyset \neq \Lambda_\infty(A) \subseteq \Lambda_\infty(U)$. This completes the proof. \hfill \Box

The following example shows that Theorem 1.2 is not true whenever the $\infty$-rank numerical range is empty.
Example 3.3. Consider \( A = \bigoplus_{n \geq 2} \begin{pmatrix} -\frac{1}{n} & 0 \\ 0 & e^{i\pi/n} \end{pmatrix} \). Then

\[
\sigma_e(A) = \left\{ -\frac{1}{n} : n \geq 2 \right\} \cup \left\{ \frac{e^{i\pi/n}}{n} : n \geq 2 \right\},
\]

\[
\sigma(A) = \sigma_e(A) \cup \{0\}.
\]

Using Theorem 2.1, it was shown in Example 5.2, [6] that \( \Lambda_\infty(A) = \emptyset \). Let \( U \) be a unitary dilation of \( A \). If possible, assume that \( 0 \notin \Lambda_\infty(U) = \bigcap_{k \geq 1} \Lambda_k(U) \). Then there exists \( k \in \mathbb{N} \) such that \( 0 \notin \Lambda_k(U) \). By Theorem 2.1, there exists a half closed-half plane \( H_o \) at 0 such that \( \dim \text{ran}E_U(H_o) < k \). We claim that \( H_o \) cannot contain infinitely many eigenvalues of \( A \). If possible, let \( H_o \) contain infinitely many eigenvalues of \( A \), say, \( \{ \lambda_r : r \geq 1 \} \). Take \( A' = \bigoplus_{r \geq 1} (\lambda_r) \). Then

\[
\Lambda_k(A') \subseteq \Lambda_k(A) \subseteq \Lambda_k(U) \subseteq H_o^c.
\]

It is a contradiction as \( \emptyset \neq \Lambda_k(A') \subseteq \text{conv}\{\lambda_r : r \geq 1\} \subseteq H_o \). So, the only possible choice of \( H_o = \{ z \in \mathbb{C} : \Im(z) < 0 \} \cup [0, \infty) \). Let \( H_{-\frac{1}{2}} = \{ z \in \mathbb{C} : \Im(z) < 0 \} \cup [-\frac{1}{2}, \infty) \) be a half closed-half plane at \(-\frac{1}{2}\). As \( \dim \text{ran}E_U(H_o) < k \), we have \( \dim \text{ran}E_U(H_{-\frac{1}{2}}) < k \). Consider \( A'' = \bigoplus_{n \geq 2} (-\frac{1}{n}) \). Then

\[
\Lambda_k(A'') \subseteq \Lambda_k(A) \subseteq \Lambda_k(U) \subseteq H_{-\frac{1}{2}}^c.
\]

It is again a contradiction as \( \emptyset \neq \Lambda_k(A'') \subseteq \text{conv}\{-\frac{1}{n} : n \geq 2\} \subseteq H_{-\frac{1}{2}} \). Hence \( 0 \notin \Lambda_\infty(U) \) for every unitary dilation \( U \) of \( A \). This provides an example that Theorem 1.2 is not true whenever the \( \infty \)-rank numerical range is empty.

Remark 3.4. The contraction \( A \), in Example 3.3, shows that if \( W(A) \) lies in a half closed-half plane, it does not necessarily imply that there exists a unitary dilation \( U \) of \( A \) with \( W(U) \) lying in the same half closed-half plane.

4. Proof of Theorem 1.3

We begin with a few lemmas.

Lemma 4.1 (Proposition 2.2, [3]). Let \( A \in \mathcal{B}(\mathcal{H}) \) be a contraction with \( d_A = d_{A^*} = N < \infty \). Assume that \( \lambda_1, \cdots, \lambda_r \) are distinct points in \( \mathbb{T} \setminus \sigma(T) \) and \( n_1, \cdots, n_r \) are positive integers satisfying \( \sum_{j=1}^r n_j = N \). Then there exists a unitary \( N \)-dilation \( U \) of \( T \) such that \( \lambda_j \) is an eigenvalue of \( U \) with multiplicity greater than or equal to \( n_j \) for every \( j \in \{1, 2, \cdots, r\} \).

Let \( A \in \mathcal{B}(\mathcal{H}) \) be self-adjoint and \( k \in \mathbb{N} \). Define

\[
\mu_k(A) = \inf_{N \in \mathcal{H}} \sup_{x \in N, \|x\|=1} \langle Ax, x \rangle. \tag{4.1}
\]
We claim that $\lambda_k(A) \leq \mu_k(A)$. Indeed, if $\mathcal{M}, \mathcal{N}$ are two closed subspaces of $\mathcal{H}$ with $\dim \mathcal{M} = k$ and $\text{codim} \mathcal{N} < k$ then there exists a unit vector $h \in \mathcal{M} \cap \mathcal{N}$. Now,

$$\min_{x \in \mathcal{M}, \|x\|=1} \langle Ax, x \rangle \leq \langle Ah, h \rangle \leq \sup_{x \in \mathcal{N}, \|x\|=1} \langle Ax, x \rangle$$

$$\Rightarrow \lambda_k(A) = \sup_{\mathcal{M} \subseteq \mathcal{H}, \dim \mathcal{M} = k} \min_{x \in \mathcal{M}, \|x\|=1} \langle Ax, x \rangle \leq \sup_{x \in \mathcal{N}, \|x\|=1} \langle Ax, x \rangle$$

$$\Rightarrow \lambda_k(A) \leq \inf_{\mathcal{N} \subseteq \mathcal{H}} \sup_{x \in \mathcal{N}, \|x\|=1} \langle Ax, x \rangle = \mu_k(A). \quad (4.2)$$

**Lemma 4.2.** Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction with $d_A = d_{A^*} = N < \infty$ and $k \in \mathbb{N}$. Then there exists a unitary $N$-dilation $U$ of $A$ such that $\lambda_k(U + U^*) = \lambda_k(A + A^*)$.

**Proof.** Let $\lambda_k(\mathcal{R}(A)) = \mu$. If $\mu = 1$ then any unitary $N$-dilation of $A$ will do the job. So, assume that $-1 \leq \mu < 1$. Let $\epsilon > 0$ be such that the line passing through $\mu + \epsilon$ and parallel to the $Y$-axis cuts the unit circle at two points, say, $\lambda_e$ and $\overline{\lambda_e}$. Then, by Lemma 4.1, there exists a unitary $N$-dilation $U_\epsilon$ (acting on $\mathcal{H} \oplus \mathbb{C}^N$) of $A$ such that $\lambda_e$ is an eigenvalue of $U_\epsilon$ with multiplicity $N$. Let $E_\epsilon$ be the spectral measure associated with $\mathcal{R}(U_\epsilon)$. We claim that $\dim \text{ran} E_\epsilon(\mu + \epsilon, \infty) < k$. If possible, let $\dim \text{ran} E_\epsilon(\mu + \epsilon, \infty) \geq k$. Suppose $f_1^{\epsilon}, \ldots, f_N^{\epsilon}$ are orthonormal eigenvectors of $\mathcal{R}(U_\epsilon)$ corresponding to the eigenvalue $\mu + \epsilon$ and $\{f_{N+1}^{\epsilon}, \ldots, f_{N+r}^{\epsilon}\}$ is an orthonormal basis of $\text{ran} E_\epsilon(\mu + \epsilon, \infty)$. Consider $\mathcal{K}_\epsilon = \text{span}\{f_1^{\epsilon}, f_2^{\epsilon}, \ldots, f_N^{\epsilon}\} \cap \mathcal{H}$. As codimension of $\mathcal{H}$ in $\mathcal{H} \oplus \mathbb{C}^N$ is $N$, we have $\dim(\mathcal{K}_\epsilon) \geq r \geq k$. Note, $P_{\mathcal{K}_\epsilon} \mathcal{R}(U_\epsilon)|_{\mathcal{K}_\epsilon} = P_{\mathcal{K}_\epsilon} \mathcal{R}(A)|_{\mathcal{K}_\epsilon}$. Let $A'$ be any $k$-by-$k$ compression of $P_{\mathcal{K}_\epsilon} \mathcal{R}(A)|_{\mathcal{K}_\epsilon}$. Then

$$[\lambda_k(A'), \lambda_1(A')] = W(A') \subseteq W(P_{\mathcal{K}_\epsilon} \mathcal{R}(A)|_{\mathcal{K}_\epsilon}) = W(P_{\mathcal{K}_\epsilon} \mathcal{R}(U_\epsilon)|_{\mathcal{K}_\epsilon}) \subseteq [\mu + \epsilon, \infty).$$

So, $\lambda_k(A') \geq \mu + \epsilon > \mu$. It contradicts $\lambda_k(\mathcal{R}(A)) = \mu$ as, by the definition of $\lambda_k(\mathcal{R}(A))$, there cannot exist any $k$-by-$k$ compression of $\mathcal{R}(A)$ whose smallest eigenvalue is strictly greater than $\lambda_k(\mathcal{R}(A)) = \mu$. Hence $\dim \text{ran} E_\epsilon(\mu + \epsilon, \infty) < k$. Now, using (4.2), we have

$$\lambda_k(\mathcal{R}(U_\epsilon)) \leq \mu_k(\mathcal{R}(U_\epsilon))$$

$$= \inf_{\mathcal{M} \subseteq \mathcal{H}, \text{codim} \mathcal{M} < k} \sup_{x \in \mathcal{M}, \|x\|=1} \langle \mathcal{R}(U_\epsilon)x, x \rangle$$

$$\leq \sup_{x \in \mathcal{N}, \|x\|=1} \langle \mathcal{R}(U_\epsilon)x, x \rangle, \text{ where } \mathcal{N} = \text{ran} E_\epsilon(-\infty, \mu + \epsilon]$$

$$\leq \mu + \epsilon. \quad (4.3)$$

Since the set of all unitary $N$-dilations of $A$ on $\mathcal{H} \oplus \mathbb{C}^N$ is compact with respect to the norm topology, $\{U_\epsilon\}_\epsilon$ has a limit point, say, $U$. Clearly, $U$ is a unitary $N$-dilation of $A$. Now, using
Corollary III.1.2, [1], we obtain

\[
\lambda_k(\Re(U)) = \sup_{M \subseteq H \oplus \mathbb{C}^N} \min_{x \in M, \|x\|=1} \langle \Re(U)x, x \rangle
\]

\[
= \lim_{\epsilon \to 0} \sup_{M \subseteq H \oplus \mathbb{C}^N, \dim M = k} \min_{x \in M, \|x\|=1} \langle \Re(U_\epsilon)x, x \rangle, \quad \text{as } U_\epsilon \to U \text{ in norm}
\]

\[
= \lim_{\epsilon \to 0} \lambda_k(\Re(U_\epsilon))
\]

\[
\leq \lim_{\epsilon \to 0} \mu + \epsilon, \quad \text{by (4.3)}
\]

\[
= \lambda_k(\Re(A)).
\]

Again by Cauchy’s interlacing theorem (Corollary III.1.5, [1]), we have \(\lambda_k(\Re(U)) \geq \lambda_k(\Re(A))\) as \(U\) is a dilation of \(A\). Hence \(\lambda_k(\Re(U)) = \lambda_k(\Re(A))\). \(\square\)

**Proof of Theorem 1.3.** We will prove it considering the following two cases.

**Case I:** Suppose \(k \in \mathbb{N}\). Clearly,

\[
\Lambda_k(A) \subseteq \bigcap \left\{ \overline{\Lambda_k(U)} : U \text{ is a unitary } N\text{-dilation of } A \text{ to } H \oplus \mathbb{C}^N \right\}.
\]

Let \(\xi \notin \Lambda_k(A)\). Then, by Theorem 2.3, there exists \(\theta \in [0, 2\pi)\) such that \(e^{i\theta}\xi + e^{-i\theta}\xi > \lambda_k(e^{i\theta}A + e^{-i\theta}A^*)\). Now, by Lemma 4.2, there exists a unitary \(N\)-dilation \(U\) of \(A\) such that \(\lambda_k(e^{i\theta}U + e^{-i\theta}U^*) = \lambda_k(e^{i\theta}A + e^{-i\theta}A^*)\). So, \(e^{i\theta}\xi + e^{-i\theta}\xi > \lambda_k(e^{i\theta}U + e^{-i\theta}U^*)\). Therefore, again by Theorem 2.3, we have \(\xi \notin \Lambda_k(U)\). This completes the proof in this case.

**Case II:** Suppose \(k = \infty\). Let us first assume that \(\Lambda_\infty(A) = \emptyset\). Suppose \(U\) is a unitary \(N\)-dilation of \(A\) to \(H \oplus \mathbb{C}^N\). If possible, let \(\lambda \in \Lambda_\infty(U)\). Then there exists an \(\infty\)-rank projection \(P \in \mathcal{B}(H \oplus \mathbb{C}^N)\) such that \(PUP = \lambda P\). Let \(K = \text{ran}P \cap H\). As codimension of \(H\) in \(H \oplus \mathbb{C}^N\) is \(N < \infty\), \(K\) is infinite dimensional. Observe,

\[
\lambda P_K = P_K(\lambda P)P_K = P_KPUPP_K = P_KUP_K = P_KAP_K.
\]

So, \(\lambda \in \Lambda_\infty(A)\), which contradicts \(\Lambda_\infty(A) = \emptyset\). Hence \(\Lambda_\infty(U) = \emptyset\) and we are done.

Now, let \(\Lambda_\infty(A) \neq \emptyset\). Clearly,

\[
\Lambda_\infty(A) \subseteq \bigcap \left\{ \overline{\Lambda_\infty(U)} : U \text{ is a unitary } N\text{-dilation of } A \text{ to } H \oplus \mathbb{C}^N \right\}.
\]

Let \(\xi \notin \Lambda_\infty(A)\). Then, by Theorem 2.4, there exists \(k_o \in \mathbb{N}\) such that \(\xi \notin \Omega_{k_o}(A)\). So, by Theorem 2.3, there exists \(\theta \in [0, 2\pi)\) such that \(e^{i\theta}\xi + e^{-i\theta}\xi > \lambda_{k_o}(e^{i\theta}A + e^{-i\theta}A^*)\). Now, by Lemma 4.2, there exists a unitary \(N\)-dilation \(U\) of \(A\) such that \(\lambda_{k_o}(e^{i\theta}U + e^{-i\theta}U^*) = \lambda_{k_o}(e^{i\theta}U + e^{-i\theta}U^*)\). So, \(e^{i\theta}\xi + e^{-i\theta}\xi > \lambda_{k_o}(e^{i\theta}U + e^{-i\theta}U^*)\). Therefore, by Theorem 2.3, \(\xi \notin \Omega_{k_o}(U)\). Hence, by Theorem 2.4, \(\xi \notin \bigcap_{k \geq 1} \Omega_k(U) = \Omega_\infty(U) = \overline{\Lambda_\infty(U)}\) as \(\emptyset \neq \Lambda_\infty(A) \subseteq \Lambda_\infty(U)\). \(\square\)
5. Proof of Theorem 1.4

Let \( A \in \mathcal{B}(\mathcal{H}) \) be a contraction. Suppose \( c = \text{diag}(c_1, c_2, \cdots) \) be a finite rank operator. Observe that
\[
W_c(A) \subseteq \bigcap \left\{ W_{c\oplus 0}(U) : U \text{ is a unitary dilation of } A \right\}.
\]
(5.1)

We need a few lemmas.

**Lemma 5.1** ([13]). Let \( c = \text{diag}(c_1, c_2, \cdots, c_n) \) with \( c_1 \geq c_2 \geq \cdots \geq c_n \). Suppose \( A \in M_n \) and \( \Re(A) \) has eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). If \( \alpha = \sum_{j=1}^{n} c_j \lambda_{n-j+1} \) and \( \beta = \sum_{j=1}^{n} c_j \lambda_j \) then
\[
\Re(W_c(A)) = W_c(\Re(A)) = [\alpha, \beta].
\]

Let \( A \in \mathcal{B}(\mathcal{H}) \) be self-adjoint and \( k \in \mathbb{N} \). Define
\[
\nu_k(A) = \inf_{M \subseteq \mathcal{H}, \dim M = k} \max_{x \in M, \|x\|=1} \langle Ax, x \rangle.
\]
Then \( \nu_k(A) = -\lambda_k(-A) \). Note that if \( A \in M_n \) is self-adjoint and \( 1 \leq k \leq n \) then \( \nu_k(A) = \lambda_{n-k+1}(A) \).

**Lemma 5.2.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be self-adjoint and \( c = \text{diag}(c_1, c_2, \cdots) \) with \( c_1 \geq c_2 \geq \cdots \) be a finite rank operator. Suppose \( \alpha = \sum_{j=1}^{\infty} c_j \nu_j(A) \) and \( \beta = \sum_{j=1}^{n} c_j \lambda_j(A) \). Then
\[
\overline{W_c(A)} = [\alpha, \beta].
\]

**Proof.** Let \( \{e_1, e_2, \cdots\} \) be an orthonormal basis of \( \mathcal{H} \). Suppose \( \mathcal{H}_n = \text{span}\{e_1, \cdots, e_n\} \) and \( A_n = P_{\mathcal{H}_n}A|_{\mathcal{H}_n} \). Then
\[
\overline{W_c(A)} = \bigcap_{j=1}^{\infty} \bigcup_{n \geq j} W_{c_n}(A_n), \text{ where } c_n = \text{diag}(c_1, \cdots, c_n)
\]
\[
= \bigcap_{j=1}^{\infty} \bigcup_{n \geq j} [\alpha_n, \beta_n], \text{ by Lemma 5.1, } \alpha_n = \sum_{j=1}^{\infty} c_j \nu_j(A_n), \beta_n = \sum_{j=1}^{n} c_j \lambda_j(A_n)
\]
\[
= \left[ \lim_{n \to \infty} \inf \alpha_n, \lim_{n \to \infty} \sup \beta_n \right]
\]
\[
= [\alpha, \beta], \text{ by Lemma 2.2.}
\]

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( c = I_2 \). Then \( W_c(A) = \{0\} \). If possible, let
\[
W_c(A) = \bigcap \left\{ W_{c\oplus 0}(U) : U \text{ is a unitary dilation of } A \right\}.
\]
Then
\[ \mathcal{W}_c(\mathbb{R}(A)) = \bigcap \left\{ \overline{\mathcal{W}_{c\oplus 0}(\mathbb{R}(U))} : U \text{ is a unitary dilation of } A \right\}. \]

Now, by Lemma 5.2, we obtain
\[ \overline{\mathcal{W}_{c\oplus 0}(\mathbb{R}(U))} = [\alpha_U, \beta_U] \] with \( \alpha_U = \nu_1(\mathbb{R}(U)) + \nu_2(\mathbb{R}(U)) \) and \( \beta_U = \lambda_1(\mathbb{R}(U)) + \lambda_2(\mathbb{R}(U)) \). So,
\[ \{0\} = \mathcal{W}_c(\mathbb{R}(A)) = \bigcap_U [\alpha_U, \beta_U]. \]

Then there exists a sequence of unitary operators \( \{U_n\}_{n=1}^\infty \) such that \( \beta_{U_n} \to 0 \) whenever \( n \to \infty \). Given \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \beta_{U_{n_0}} < \epsilon \), that is, \( \lambda_2(\mathbb{R}(U_{n_0})) + \lambda_1(\mathbb{R}(U_{n_0})) < \epsilon \), which implies that \( \overline{W(U_{n_0})} = \text{conv} \sigma(U_{n_0}) \) does not contain \( W(A) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\} \).

It is a contradiction as \( U_{n_0} \) is a dilation of \( A \). Hence the intersection of the closure of the \( c \oplus 0 \)-numerical ranges of all unitary dilations of \( A \) contains \( W_c(A) \) as a proper subset.

\[ \square \]

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