Effective Range in Doublet S-wave Neutron-Deuteron Scattering

F. Gabbiani

Department of Physics and Astronomy, Iowa State University, Ames, IA 50014, USA
Email: fg@eft.physics.iastate.edu

Abstract

The effective field theory approach is applied to the three-nucleon process of $S = 1/2$ neutron-deuteron scattering in the S-wave, including the effective range parameters summed at all orders. This is achieved through a modification of the power counting, following a recent suggestion in the literature. It is shown that with this procedure, while the convergence of the loop integrals improves, one cannot meaningfully include a three-body effective term to describe low-energy data affected by the presence of the triton bound state.

I. Introduction

Recently, three-body interactions in nucleon systems have attracted considerable attention, both within more classical approaches based on potential models \cite{1} and in the context of effective field theories (EFT's) \cite{2,3,4,5,6}.

The three-nucleon system is a natural test-ground for the understanding of nuclear forces that has been reached in the two-nucleon system. A convenient example is given by neutron-deuteron scattering \cite{7,8,9}, since in this case Coulomb interactions are negligible. Neutron-deuteron scattering involves two S-wave channels, corresponding to total spin $S = 1/2$ and $S = 3/2$. In the $S = 1/2$ channel all spins are aligned and the two-nucleon interactions are only in the $3S_1$ partial wave. The two-body interaction is attractive but the Pauli exclusion principle forbids the three nucleons to be at the same point in space. Therefore this channel is insensitive to short-distance physics and disallows a three-body bound state. Thus one is able to obtain precise ($\sim$ 4%) predictions in a straightforward way \cite{10,11,12} in the $S = 1/2$ channel two-nucleon interaction can take place either in the $3S_1$ or in the $1S_0$ partial waves. This leads to an attractive interaction which implies a three-body bound state, the triton.
The $S = 1/2$ channel also shows a strong sensitivity to short-distance physics as the Pauli principle does not apply. In this note EFT is applied to the latter channel, but including the effective range parameters for the $^3S_1$ and $^1S_0$ $NN$ scattering partial waves in the calculation, following a suggestion proposed in [1]. In that paper the authors argued that taking both the scattering length $a$ and the effective range $r$ in the $NN$ scattering effective range expansion of order $Q^{-1}$, and summing range corrections to all orders, improves convergence and may solve problems encountered in the pionful theory [1] with KSW power counting [3]. The authors then successfully applied this modified power counting to a few two-body processes. It is therefore important to test this new scheme on a three-body system. It will be shown that the new power counting improves the convergence of the loop integrals necessary for the $nd$ scattering calculation, so that the cutoff introduced in [4, 8] is no longer needed for the system of integral equations without a three-body interaction term. However, even this improved ultraviolet behavior of the kernels in the integral equations is not enough, since adding a three-body interaction introduces new divergences which make the equations insensitive to the three-body term in the Lagrangian, as it will be shown below.

This independence of the amplitudes with respect to an additional parameter prevents us from applying the fitting procedure of ref. [5] and therefore from predicting the energy dependence of the $nd$ phase shifts.

II. Formalism

It is convenient to use a Lagrangian [12, 9] expressed in terms of two auxiliary fields $d^i$ and $t^A$ with the quantum numbers of the deuteron and of a dibaryon field in the $^1S_0$ channel of $NN$ scattering respectively:

$$\mathcal{L}_{Nd} = N^\dagger (i\partial_0 + \frac{\vec{\partial}^2}{4M}) N +$$
$$+ d^i \left[-(i\partial_0 + \frac{\nabla^2}{4M}) - \Delta_i^{(-1)} - \Delta_i^{(0)} \right] d^i + y_d \left[d^i (N^T P_d^i N) + h.c. \right] +$$

$$+ t^A \left[-(i\partial_0 + \frac{\nabla^2}{4M}) - \Delta_i^{(-1)} - \Delta_i^{(0)} \right] t^A + y_t \left[t^A (N^T P_t^A N) + h.c. \right] + \ldots$$

(1)

where $N = (p_n)$ is the nucleon doublet of two-component spinors, the subscripts $d$ and $t$ denote the $^3S_1$ and $^1S_0$ channel of $NN$ scattering, and $M$ is the average nucleon mass. $P_d^i$ and $P_t^A$ are the projectors onto the $^3S_1$ and $^1S_0$ channels, respectively

$$\begin{align*}
(P_d^i)_{ab}^{\beta\alpha} &= \frac{1}{\sqrt{8}} (\sigma_2 \sigma \sigma^i)^{\beta\alpha} (\tau_2)_{a}^{b}, & (P_t^A)_{ab}^{\beta\alpha} &= \frac{1}{\sqrt{8}} (\sigma_2)^{\beta\alpha} (\tau_2 \tau^A)_{a}^{b},
\end{align*}$$

(2)

with $\sigma$ ($\tau$) the Pauli matrices acting in spin (isospin) space. The coefficients $y_d$, $\Delta_i^{(-1)}$ and $\Delta_i^{(0)}$ encode all short distance physics – like pion and $\omega$ exchanges, quarks and gluons, and resonances like the $\Delta$. Here it is necessary to split both $\Delta$’s into leading ($\Delta^{(-1)}$) and subleading pieces ($\Delta^{(0)}$).

Since the theory is nonrelativistic, all particles propagate forward in time, the nucleon tadpoles vanish, and the propagator for the nucleon fields is

$$iS(p) = \frac{i}{p_0 - p^2/2M + i\epsilon}.$$  

(3)
The deuteron and dibaryon propagators are more complicated because of the coupling to two-nucleon states. For instance, the bare deuteron propagator is simply a constant, $-i/\Delta_d^{-1}$, but the full propagator gets dressed by nucleon loops to all orders as illustrated in Fig. 1.

![Figure 1: The deuteron propagator at LO from the Lagrangian (1). The thick solid line denotes the bare propagator $-i/\Delta_d^{-1}$, the double line its dressed counterpart.](image)

The nucleon-loop integral has a linear ultraviolet (UV) divergence which can be absorbed in $y_d^2/\Delta_d^{-1}$, and a finite piece determined by the unitarity cut. Summing the resulting geometric series leads to the deuteron (and analogously the dibaryon) propagators in terms of physical quantities at LO:

$$i\Delta_d^{ij}(p) = i\delta^{ij} \Delta_d(p) = \frac{4\pi i}{M y_d^2} \frac{\delta^{ij}}{\gamma_d - \sqrt{\frac{p^2}{4} - M p_0 - i\varepsilon}},$$

$$i\Delta_t^{AB}(p) = i\delta^{AB} \Delta_t(p) = \frac{4\pi i}{M y_t^2} \frac{\delta^{AB}}{\gamma_t - \sqrt{\frac{p^2}{4} - M p_0 - i\varepsilon}}. \quad (4)$$

$\gamma_d = \sqrt{MB} = 45.7066$ MeV and the deuteron binding energy is $B = 2.225$ MeV. The typical momentum $\gamma_t = 1/a_t$ of the virtual bound state is extracted from the scattering length in this channel, $a_t = -23.714$ fm.

### III. $S = 1/2$ nd Scattering

The Lagrangian (1) can now be used to describe the $\text{nd}$-scattering in the $S = 1/2$ channel. The treatment involves coupled channel equations both in the $^3S_1$ and $^1S_0$ partial waves. The spin zero dibaryon field $t$ also contributes in intermediate states of $\text{nd}$ amplitudes. It is possible to obtain a system of two coupled integral equations (previously derived using a different method [13]) for the $d + N \to d + N$ amplitude $it_d^{ij}(\vec{k}, \vec{p}, \epsilon)$ and for the $d + N \to t + N$ amplitude $it_t^{AB}(\vec{k}, \vec{p}, \epsilon)$, pictorially represented in Fig. 2.

A momentum cutoff $\Lambda$ has to be introduced in the integral equations. In the limit $\Lambda \to \infty$ these equations do not have a unique solution because the phase of the asymptotic solution is undetermined [14]. For a finite $\Lambda$ this phase is fixed and the solution is unique. However, the equations with a cutoff display a strong cutoff dependence that does not appear in any order in perturbation theory. The amplitude $t_d(p, k = \text{const.})$ shows a strongly oscillating behavior [5]. This cutoff dependence is not created by divergent Feynman diagrams. It is a nonperturbative effect and appears although all individual diagrams are superficially UV finite. For an extended discussion on this effect, see [4].

The solution is to add one-parameter three-body force counterterm $H(\Lambda)/\Lambda^2$ that runs with the cutoff $\Lambda$ [4]. This counterterm represents a three-body force which is obtained by
Figure 2: The coupled set of Faddeev equations which need to be solved for \( t_{d,t} \) at LO in the doublet channel. The double dashed line denotes the dibaryon field \( t^i_A \).

including

\[
\mathcal{L}_3 = -\frac{2MH(\Lambda)}{\Lambda^2} \left\{ \frac{1}{2} y_d^2 N^\dagger (d^i \sigma_i)^\dagger (d^i \sigma_i) N + \frac{1}{6} y_d y_t \left[ N^\dagger (d^i \sigma_i)^\dagger (t^A \tau_A) N + h.c. \right] + \frac{1}{2} y_t^2 N^\dagger (t^A \tau_A)^\dagger (t^A \tau_A) N \right\}
\]

in the Lagrangian (1). This yields the equations:

\[
t_{d}(k, p) = 4y_d^2 M \left[ K(p, k) + \frac{2H(\Lambda)}{\Lambda^2} \right] + \frac{1}{\pi} \int_0^\Lambda dq \frac{q^2}{\sqrt{q^2 - ME - i\varepsilon - \gamma_d}} \times \left[ K(p, q) + \frac{2H(\Lambda)}{\Lambda^2} \right] - \frac{3}{\pi} \frac{y_d}{y_t} \int_0^\Lambda dq \frac{q^2}{\sqrt{q^2 - ME - i\varepsilon - \gamma_t}} \times \left[ K(p, q) + \frac{2H(\Lambda)}{\Lambda^2} \right] \quad (6)
\]

\[
t_{t}(k, p) = -12y_d y_t M \left[ K(p, k) + \frac{2H(\Lambda)}{3\Lambda^2} \right] + \frac{1}{\pi} \int_0^\Lambda dq \frac{q^2}{\sqrt{q^2 - ME - i\varepsilon - \gamma_t}} \times \left[ K(p, q) + \frac{2H(\Lambda)}{\Lambda^2} \right] - \frac{3}{\pi} \frac{y_t}{y_d} \int_0^\Lambda dq \frac{q^2}{\sqrt{q^2 - ME - i\varepsilon - \gamma_d}} \times \left[ K(p, q) + \frac{2H(\Lambda)}{\Lambda^2} \right] \quad (7)
\]

where \( k \) (\( p \)) denote the incoming (outgoing) momenta in the center-of-mass frame, \( ME = 3k^2/4 - \gamma_d^2 \) is the total energy, the kernel \( K(p, q) \) is given by

\[
K(p, q) = \frac{1}{2pq} \ln \left( \frac{q^2 + pq - p^2 - ME}{q^2 - pq - p^2 - ME} \right) \quad (8)
\]
and \( H(\Lambda) \) is defined as follows:

\[
H(\Lambda) = -\frac{\sin[s_0 \ln(\Lambda/\Lambda_*) - \arctg(1/s_0)]}{\sin[s_0 \ln(\Lambda/\Lambda_*) + \arctg(1/s_0)]}.
\]  

(9)

In eq. (9) \( s_0 \approx 1.0064 \) and \( \Lambda_* \) is a dimensionful parameter that determines the asymptotic phase of the off-shell amplitude \( \Pi \), and is fitted to reproduce the experimental value for the \( S = 1/2 \) nd scattering length, \( a_{3}^{(1/2)} = (0.65 \pm 0.04) \text{ fm} \) \cite{17}, yielding \( \Lambda_* = 0.9 \text{ fm}^{-1} \).

IV. Effective Range Expansion

The cutoff dependence is expected to be eliminated from the equations if the effective range parameters \( \rho_d \) and \( r_{0t} \) are included in the effective range expansion and are treated like the scattering length \( a \), i.e. taken of order \( Q^{-1} \), and summed to all orders. Ref. \cite{10} gave reasons to conclude that this power counting should improve the overall convergence. In the \( ^3S_1 \) channel, the expansion around the deuteron pole gives

\[
k \cot(\delta) = -\gamma_d + \frac{1}{2} \rho_d (k^2 + \gamma_d^2) + \ldots
\]  

(10)

for the nucleon-nucleon phase shifts. In the singlet channel \( ^1S_0 \), no real bound state exists, so the condition to impose is that the effective range expansion

\[
k \cot(\delta) = -\frac{1}{a_t} + \frac{1}{2} r_{0t} k^2 + \ldots
\]  

(11)

is satisfied. Resumming the effective ranges \( \rho_d \) and \( r_{0t} \) in the full deuteron and dibaryon propagators generates

\[
\begin{align*}
\text{i}\Delta^{\rho_d,ij}_d(p) &= \text{i} \delta^{ij} \Delta^{(\rho_d)}_d(p) = -\frac{4\pi \text{i}}{M y^2_d} \frac{\delta^{ij}}{-\gamma_d + \frac{1}{2} \rho_d(Mp_0 - \frac{p^2}{4} + \gamma_d^2) + \sqrt{\frac{p^2}{4} - Mp_0 - i\varepsilon}}, \\
\text{i}\Delta^{r_{0t},AB}_t(p) &= \text{i} \delta^{AB} \Delta^{(r_{0t})}_t(p) = -\frac{4\pi \text{i}}{M y^2_t} \frac{\delta^{AB}}{-\gamma_t + \frac{1}{2} r_{0t}(Mp_0 - \frac{p^2}{4}) + \sqrt{\frac{p^2}{4} - Mp_0 - i\varepsilon}}.
\end{align*}
\]

(12)

The numerical values \( \rho_d = 1.765 \text{ fm} \) and \( r_{0t} = 2.73 \text{ fm} \) \cite{16} have been used.

It is still possible to introduce in the computation a three-body term, which now is independent from the cutoff \( \Lambda \) and is in fact a constant contact interaction. The resulting integral equations are completely analogous to (3), (7), but now with the propagators (12). The integration is carried on the full real positive axis without any cutoff. The integral equations are then solved numerically using the techniques outlined in 3.

The solutions for the amplitudes coming from the equations do not show any visible dependence on the three-body interaction term \( H \). It is possible to extract the energy dependence for \( k \cot(\delta) \), where \( k \) is the incoming momentum of the particles in the center of mass frame, from the behavior of the on-shell amplitude \( t_d(k, k) \). Results are plotted on Fig. 3. Compare with the results given in ref. \cite{8} (Fig. 4).

\footnote{For an early computation including effective ranges and comparisons with potential models, see \cite{17}.}
Figure 3: Energy dependence for $S = 1/2$ nd scattering with $H = 0$ or with $H \neq 0$ and $\Lambda \to \infty$ when the effective ranges are included in the calculation.

Figure 4: Cutoff-insensitive energy dependence for $S = 1/2$ nd scattering for $\Lambda_* = 0.9$ fm$^{-1}$, as given in ref. [8], without resumming the effective ranges. Data are from the phase-shift analysis of van Oers and Seagrave [18] (crosses) and a measurement by Dilg et al. [15] (diamond).

The explanation for this outcome stems from the fact that, while the convergence of the equations has improved because of the inclusion of the effective ranges (which multiply a factor $\propto q^2$ in the denominator of the propagators), this is not yet sufficient to make the equations completely convergent. After subtracting from the amplitudes $t_{d,t}(k, p)$ the parts satisfying the (convergent) equations without the three-body term $H$, the remaining parts $t'_{d,t}(k, p)$ are described by equations containing linear divergences (plus convergent terms). Regularizing these divergences by introducing a cutoff $\Lambda$ drives $t'_{d,t}(k, p)$ to approximate quantities $\propto 1/\Lambda$, i.e. to negligibly small numbers if the cutoff is set sufficiently high. $t_{d,t}(k, p)$ are then effectively cutoff-independent, but independent from $H$ as well.

V. Conclusions

The study of three-nucleon systems using EFT methods in the $S = 1/2$ channel is more complicated than for the $S = 3/2$ channel. While for the latter $nd$ scattering accurate predictions are obtained [5], the $S = 1/2$ channel displays a strong cutoff dependence even though all individual diagrams are UV finite. This dependence can be eliminated at LO only for the equations without three-body interactions, if the effective range parameters, obtained from the effective range expansion in $NN$ scattering, are taken into account in the deuteron and dibaryon propagators. This is achieved modifying the usual power-counting scheme, assuming $\rho_d$ and $r_{0t}$ of order $Q^{-1}$ like the scattering length $a$.

Unfortunately this improvement is not sufficient to eliminate all the infinities originated in the loop diagrams from the integral equations. The remaining divergent terms have the effect of driving the three-body-dependent parts of the amplitudes to negligible values, while the amplitude parts dependent only on the finite equations (those without the three-body term) prevail.

This makes eqs. (6), (7), after the inclusion of the effective range parameters, insensitive to the three-body term necessary to describe the influence of the triton bound state and to
reproduce the experimental data for nd scattering in the S-wave doublet channel.

Alternatively, one can argue that the three-body force term arise not at LO but at NLO. Yet in this case it must generate a contribution to the amplitude much bigger than the $\sim 30\%$ variation one expects going from LO to NLO.

Therefore this test of the new power-counting procedure does not yield the successful results achieved in the two-body problem case. Yet it is possible that a better treatment of this problem implies the explicit inclusion of the triton propagator in the theory, analogously to what has been done so far for the deuteron and the dibaryon, and according to the spirit of ref. [10].

**Acknowledgments**

I would like to thank Paulo Bedaque for several discussions and Silas Beane for comments. This research was supported in part by the U.S. Department of Energy grant DE-FG02-01ER41155.

**References**

[1] D. Hübner et al., LA-UR-99-4996, Oct 1999, [nucl-th/9910034]. E. Epelbaum et al., *Invited talk at 17th European Conference on Few-Body Problems in Physics*, Evora, Portugal, 11-16 Sep 2000, [nucl-th/0009007]. H. Witala et al., *Phys. Rev.* C63, 024007 (2001).

[2] S. Weinberg, *Phys. Lett.* B251, 288 (1990); *Nucl. Phys.* B363, 3 (1991); C. Ordóñez and U. van Kolck, *Phys. Lett.* B291, 459 (1992); U. van Kolck, *Phys. Rev.* C49, 2932 (1994); C. Ordóñez, L. Ray, and U. van Kolck, *Phys. Rev. Lett.* 72, 1982 (1994); *Phys. Rev.* C53, 2086 (1996); *Nuclear Physics with Effective Field Theory*, edited by R. Seki, U. van Kolck, and M.J. Savage (World Scientific, Singapore, 1998); *Nuclear Physics with Effective Field Theory II*, edited by P.F. Bedaque, M.J. Savage, R. Seki, and U. van Kolck (World Scientific, Singapore, 1999).

[3] D.B. Kaplan, M.J. Savage, and M.B. Wise, *Nucl. Phys.* B534, 329 (1998); *Phys. Lett.* B424, 390 (1998); *Phys. Rev.* C59, 617 (1999).

[4] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, *Phys. Rev. Lett.* 82, 463 (1999); *Nucl. Phys.* A646, 444 (1999).

[5] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, *Phys. Rev.* C58, R641 (1998); P.F. Bedaque and U. van Kolck, *Phys. Lett.* B428, 221 (1998).

[6] H.-W. Hammer and T. Mehen, [nucl-th/0011024].

[7] J. Gegelia, *Nucl. Phys.* A680, 303 (2000); B. Blankleider and J. Gegelia, [nucl-th/0009007].

[8] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, *Nucl. Phys.* A676, 357 (2000).

[9] P.F. Bedaque and H.W. Grießhammer, *Nucl. Phys.* A671, 357 (2000); F. Gabbiani, P.F. Bedaque, and H.W. Grießhammer, *Nucl. Phys.* A675, 601 (2000).

[10] D.B. Kaplan and J.V. Steele, *Phys. Rev.* C60, 0604002 (1999); S.R. Beane and M.J. Savage, *NT@UW-00-028*, Nov 2000, [nucl-th/0011067].
[11] S. Fleming, T. Mehen, and I.W. Stewart, *Nucl. Phys.* A677, 313 (2000).

[12] D.B. Kaplan, *Nucl. Phys.* B494, 471 (1997).

[13] G.V. Skorniakov and K.A. Ter-Martirosian, *Sov. Phys. JETP* 4, 648 (1957).

[14] G.S. Danilov and V.I. Lebedev, *Sov. Phys. JETP* 17, 1015 (1963); G.S. Danilov, *Sov. Phys. JETP* 13, 349 (1961).

[15] W. Dilg, L. Koester, and W. Nistler, *Phys. Lett.* B36, 208 (1971).

[16] V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, *Phys. Rev.* C48, 792 (1993); J.J. de Swart, C.P.F. Terheggen, and V.G.J. Stoks, *Invited talk at the 3rd International Symposium “Dubna Deuteron 95”*, Dubna, Moscow Region, Russia, July 4-7, 1995, nucl-th/9509032; T.D. Cohen and J.M. Hansen, *Phys. Rev.* C59, 3047 (1999).

[17] I.V. Simenog and D.V. Shapoval, *Sov. J. Nucl. Phys.* 47, 620 (1988); *Few Body Syst.* 8, 145 (1990).

[18] W.T.H. van Oers and J.D. Seagrave, *Phys. Lett.* B24, 562 (1967).