Quantum Hamilton-Jacobi analysis of PT symmetric Hamiltonians

S. Sree Ranjani¹*, A.K. Kapoor¹†, and P. K. Panigrahi¹,²‡

¹ School of Physics, University of Hyderabad, Hyderabad-500 046, India
² Physical Research Laboratory, Navrangpura, Ahmedabad-380 009, India

We apply the quantum Hamilton-Jacobi formalism, naturally defined in the complex domain, to a number of complex Hamiltonians, characterized by discrete parity and time reversal (PT) symmetries and obtain their eigenvalues and eigenfunctions. Examples of both quasi-exactly and exactly solvable potentials are analyzed and the subtle differences, in the singularity structures of their quantum momentum functions, are pointed out. The role of the PT symmetry in the complex domain is also illustrated.

I. INTRODUCTION

Complex Hamiltonians possessing real eigenvalues have attracted considerable attention in the current literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. These quantal systems, apart from being counter intuitive, are not well understood because of their recent origin. These Hamiltonians are characterized by discrete parity and time reversal symmetries. In the case when the wave functions are also PT symmetric, the eigenvalues are real and the violation of PT symmetry by the wave function leads to eigenvalues, which occur in complex conjugate pairs. Besides identifying new Hamiltonians belonging to this class, the role of various discrete symmetries is also under current investigation.

The presence of complex potentials in these systems makes them ideal candidates to be probed using the quantum Hamilton-Jacobi (QHJ) formalism, since this approach has been formulated in the complex domain [15, 16]. In the QHJ formalism, the singularity structure of the quantum momentum function (QMF) plays a crucial role in the determination of the eigenvalues and the eigenfunctions. Recently, we have studied the structure of the QMF in the complex domain for exactly [17] and quasi-exactly solvable [18] potentials.

In this light, it is extremely interesting to investigate the properties of the QMF of the PT symmetric Hamiltonians, in order to systematically find out their differences and similarities with exactly solvable (ES) real potentials, as also with the quasi-exactly solvable (QES) ones.

* akksprs@uohyd.ernet.in
† akksp@uohyd.ernet.in
‡ prasanta@prl.ernet.in
It is worth mentioning that, as compared to the solvable potentials, the QMF of QES models reveal significant differences in their singularity structure.

The goal of this letter is to investigate the structure of the QMF of a class of PT symmetric Hamiltonians, for which the eigenfunctions and the eigenvalues are also simultaneously obtained, through the QHJ approach. The differences and similarities of these novel systems, with their ES and QES counterparts, are clearly brought out.

II. QUANTUM HAMILTON-JACOBI FORMALISM

In this section, we proceed to describe briefly the QHJ formalism and its working. More details can be found in our earlier paper [17]. With the definition of QMF:

\[ p = -i\hbar \frac{d}{dx} \ln(\psi), \]  

the Schrödinger eigenvalue equation \( H\psi = E\psi \) can be cast in the form of the Riccati equation,

\[ p^2 - i\hbar p' = 2m[E - V(x)]. \]

To find the eigenvalues, Leacock and Padgett [15, 16] suggested using the following exact quantization condition for the bound states of a real potential:

\[ \frac{1}{2\pi} \oint_C pdx = n\hbar, \]

where, the contour \( C \) encloses the \( n \) moving poles in the complex domain, corresponding to the nodes of the wave function located in the classical region. This quantization rule is an exact one and follows from the oscillation theorem in Strum-Liouville theory. Using this quantization rule, Bhalla et. al [23, 24] studied several ES models and showed that the eigenvalues could be obtained without obtaining the eigenfunctions. Briefly, the integral in Eq.(3) is evaluated using the knowledge of the singular points of \( p(x) \) outside the contour \( C \) and their corresponding residues.

The singularities of \( p \) consists of fixed and moving ones. The fixed singular points of \( p \) corresponding to the singular points of the potential, can be found by inspection. From Eq.(1), one can see that the nodes of the wave function correspond to the moving poles of the QMF. In general, the QMF may have other moving poles at locations which are not easy to determine. However, the residues can be computed by substituting a Laurent expansion in the Eq.(2). In fact, the residue at a moving pole can be easily seen to be \(-i\hbar\).

Application of the QHJ to PT symmetric potentials requires an approach different from the one used earlier. In the absence of a generalization of the oscillation theorem [3], for this class of potential, it is not clear whether the quantization rule Eq.(3), is valid. Even
in the case of the violation of this quantization rule it is not clear which contour should be used. Earlier studies indicate that, for all ES and QES models, after a suitable change of variables, the quantum momentum function has a finite number of moving poles and the point at infinity is at most a pole. In this case the quantization condition still holds for a contour, which encloses all the moving poles. We shall assume this to be the case for the PT symmetric models to be taken up in this paper.

When one attempts to compute the residue at a pole using the QHJ, one gets two answers. A boundary condition, in the limit $\hbar \to 0$, has been suggested by Leacock and Padgett to select the right residue. Several other conditions, such as square integrability, have also been utilized in earlier papers for this purpose [17, 18]. In some cases [27, 28], in the absence of any criterion to select a residue, one must consider all the values which are consistent with the other equations of the theory.

In studying the complex potentials, we will first perform a suitable change of variable and bring the resultant equation to the QHJ form so that the potential is replaced by a rational function. We analyze the potential introduced by Khare and Mandal in section 3 and the complex Scarf potential $V(x) = -A \text{sech}^2 x - iB \tanh x \text{sech} x$ with $A > 0$ in section 4. After comparing the structure of QMF for PT symmetric, ES and QES systems, we conclude in the final section, with the remarks about problems and future directions of work.

III. KHARE-MANDAL MODEL

The potential expression, for the Khare-Mandal model is given by, $V(x) = -(\zeta \cosh 2x - iM)^2$. This potential has complex or real eigenvalues depending on whether $M$ is odd or even [7, 9]. It is worth noting that, parity operation in this case is given by $x \to i\pi/2 - x$, whereas the time reversal remains same as the conventional $i \to -i$. Using the QHJ formalism, we first obtain the QES condition, for the odd and even values of $M$. Subsequently, the explicit expressions for the eigenvalues and eigenfunctions for the cases, $M = 3$ and $M = 2$ are also obtained. The QHJ equation in terms of $q \equiv ip$, setting $\hbar = 2m = 1$, can be written as

$$q^2 + \frac{dq}{dx} + E + (\zeta \cosh 2x - iM)^2 = 0. \quad (4)$$

This above form has the advantage that, the residue at each moving pole is one. To bring the potential to the rational form we do a change of variable $t \equiv \cosh 2x$. Substitution of this in Eq.(4) gives

$$q^2 + 2\sqrt{t^2 - 1} \frac{dq}{dt} + E + (\zeta t - iM)^2 = 0. \quad (5)$$
One observes that the coefficient of \( \frac{dq}{dt} \) is not one. Hence, in order to bring the above equation to the form of Eq.(4), we define

\[
q = 2(\sqrt{t^2 - 1})\phi, \quad \phi = \chi - \frac{t}{2(t^2 - 1)},
\]

which transforms (5) to

\[
\chi^2 + \frac{d\chi}{dt} + \frac{t^2 + 2}{4(t^2 - 1)^2} + \frac{E + (\zeta t - iM)^2}{4(t^2 - 1)} = 0,
\]

which has the convenient form wherein the residue at a moving pole is one. From here on, \( \chi \) will be called as the QMF and Eq.(7), the QHJ equation.

**Singularity structure of \( \chi \):** As already explained in the previous section, we shall assume that \( \chi \) has a finite number of moving poles in the complex \( t \) plane and that the point at infinity is at most a pole. Besides the moving poles, \( \chi \) has fixed poles at \( t = \pm 1 \). It is seen from Eq.(7) that the function \( \chi \) is bounded at \( t = \infty \). Assuming that \( \chi \) has only these above mentioned singularities, we separate the singular part of \( \chi \) and write it in the following form:

\[
\chi = \frac{b_1}{t - 1} + \frac{b'_1}{t + 1} + \frac{P_n}{P_n} + C.
\]

Here \( b_1 \) and \( b'_1 \) are the residues at fixed poles \( t = \pm 1 \); \( P_n \) is a polynomial of degree \( n \) and equals \( \prod_{k=1}^{n} (t - t_k) \), where \( t_k \)'s are the locations of the moving poles of the QMF. \( C \) gives the analytic part of \( \chi \) and is a constant due to the Liouville’s theorem. From Eq.(7), one can see that for large \( t \), \( \chi \) goes as \( \pm \frac{\zeta}{2} \), which are the values of \( C \).

To find the residues at the fixed poles, say \( t = 1 \), one needs to expand \( \chi \) in a Laurent series:

\[
\chi = \frac{b_1}{t - 1} + a_0 + a_1(t - 1) + \cdots.
\]

Substituting this in Eq.(7) and comparing coefficients of different powers of \( t \), one obtains the following two values for \( b_1 \):

\[
b_1 = \frac{3}{4}, \quad \frac{1}{4}.
\]

Similarly the two values of residues at \( t = -1 \) turn out to be,

\[
b'_1 = \frac{3}{4}, \quad \frac{1}{4}.
\]

**Behaviour at infinity:** It has been assumed that the point at infinity is an isolated singularity. In order to find leading behaviour of \( \chi \) at infinity, one expands \( \chi \) as,

\[
\chi = a_0 + \frac{\lambda}{t} + \frac{\lambda_1}{t^2} + \cdots.
\]
Substitution of Eq. (12) which in Eq. (7), gives $\lambda = \frac{iM\zeta}{4a_0}$ along with, $a_0 = \pm \frac{i\zeta}{2}$, which is equal to $C$. Due to this $\lambda$ takes the following two values:

$$\lambda = \frac{M}{2}, -\frac{M}{2}$$

(13)

This should match with the leading behaviour of $\chi$ coming from Eq. (8), which is $\frac{b_1 + b'_1 + n}{t}$, for large $t$. Hence equating the two equations, one obtains

$$b_1 + b'_1 + n = \lambda.$$  

(14)

From Eq. (10) and (11), we see that the right hand side of Eq. (14) is positive. Hence for Eq. (14) to be true, we choose only the positive value of $\lambda$ i.e, $\lambda = \frac{M}{2}$, which means we choose $a_0 = C = \pm \frac{i\zeta}{2}$. It should be noted that, there is no way of choosing a particular value of residue at a fixed pole, since one does not have information regarding the square integrability of the solutions. Hence, one needs to consider both the values of $b_1$ and $b'_1$. Thus taking all possible combinations of $b_1$ and $b'_1$ in Eq. (13), one obtains the QES condition for each combination along with a constraint on $M$, as given in Table 1. From Table 1, we see that sets 1 and 2 are valid only when $M$ is odd and sets 3 and 4 are valid only when $M$ is even.

**Forms of the wavefunction:** From Eq. (11) one obtains $\psi(x)$ is terms of $p$ with $\hbar = 2m = 1$, as

$$\psi(x) = \exp(i \int p dx).$$  

(15)

Doing the change of variable and writing $p$ in terms of $\chi$, one gets,

$$\psi(t) = \exp \left( \frac{b_1}{t-1} + \frac{b'_1}{t+1} + \frac{P''_n}{P_n} + \frac{i\zeta}{2} - \frac{t}{2(t^2-1)} \right) dt.$$  

(16)

Hence, one can substitute sets 1 and 2 in Eq. (16) if $M$ is odd and sets 3 and 4 if $M$ is even to obtain the form of the wavefunction. The expression for the wave function is in terms of the unknown polynomial $P_n$, where $n$ gives the number of zeros of $P_n$. In order to calculate the polynomial, we substitute $\chi$ from Eq. (8) in (11), to get

$$\frac{P''_n}{P_n} + \frac{2P'_n}{P_n} \left( \frac{b_1}{t-1} + \frac{b'_1}{t+1} + \frac{i\zeta}{2} \right) + \frac{b_1^2 - b_1}{(t-1)^2} + \frac{(b'_1)^2 - b'_1}{(t+1)^2} + \frac{t^2 + 2}{4(t^2-1)^2} + \frac{E + (\zeta t - iM)^2 + 8b_1b'_1 - 4\zeta^2(t^2-1)}{4(t^2-1)} + i\zeta \left( \frac{b_1}{t-1} + \frac{b'_1}{t+1} \right) = 0.$$  

(17)

This leads to $n$ linear homogeneous equations, for the coefficients of different powers of $t$ in $P_n$. The energy eigenvalues are obtained by setting the corresponding determinant equal to zero. The explicit eigenvalues and eigenfunctions are obtained for $M = 3$ and $M = 2$ cases.

**Case 1:** $M = 3$,
Here $M$ is odd so we can use sets 1 and 2 from table 1 and get the required results as below.  

*Set 1*: $b_1 = \frac{1}{4}$, $b'_1 = \frac{1}{4}$ and $n = 1$.  

This implies $P_n$ is a first degree polynomial say $Bt + D$. Substituting these values in Eq. (17) and comparing various powers of $t$, one obtains a $2 \times 2$ matrix for $B$ and $D$ as follows

\[
\begin{pmatrix}
1 + \frac{E - 9 + \zeta^2}{4} & -i\zeta \\
-i\zeta & E - \frac{9 + \zeta^2}{4}
\end{pmatrix}
\begin{pmatrix}
B \\
D
\end{pmatrix} = 0.
\]  

(18)

Equating the determinant of this matrix to zero, one obtains the two values for energy and the polynomials as

\[
E = 7 - \zeta^2 \pm 2\sqrt{1 - 4\zeta^2}, \quad P_1 = \frac{B}{2}(2t - \frac{i}{\zeta}(1 \pm \sqrt{1 - 4\zeta^2})).
\]  

(19)

Substituting the values of $b_1$, $b'_1$ and $P_1$ in Eq. (16) gives, the two eigenfunctions corresponding to the two eigenvalues:

\[
\psi(x) = e^{i\zeta\cosh 2x}
\left(2\cosh 2x - \frac{i}{\zeta}(1 \pm \sqrt{1 - 4\zeta^2})\right).
\]  

(20)

*Set 2*: $b_1 = \frac{3}{4}$, $b'_1 = \frac{3}{4}$ and $n = 0$.  

Here we see that $n = 0$ implies $P$ is a constant. Substituting these values in Eq. (17) and proceeding in the same manner as before one obtains,

\[
E = 5 - \zeta^2, \quad \psi(x) = e^{i\zeta\cosh 2x}\sinh 2x
\]  

(21)

which are the known results [7, 9]. Below we elaborate on the case when $M$ is even.  

**Case 2: $M = 2$,**  
In this case, one makes use of sets 3 and 4 in table 1 and proceed in the same way as was done for case 1.  

*Set 3*: $b_1 = \frac{1}{4}$, $b'_1 = \frac{3}{4}$ and $n = 0$.  

the eigenvalues and eigenfunctions obtained are

\[
E = 3 - \zeta^2 + 2i\zeta, \quad \psi(x) = e^{i\zeta\cosh 2x}(\cosh 2x + 1)^{1/2}.
\]  

(22)

*Set 4*: $b_1 = \frac{3}{4}$, $b'_1 = \frac{1}{4}$ and $n = 0$.  

In this case, one obtains

\[
E = 3 - \zeta^2 - 2i\zeta, \quad \psi(x) = e^{i\zeta\cosh 2x}(\cosh 2x - 1)^{1/2}.
\]  

(23)

These match with the solutions given in [9]. Thus for any given positive value of $M$, odd or even, one can obtain the eigenvalues and eigenfunctions for the Khare-Mandal potential. In the next section, we study the complex Scarf -II potential.
IV. COMPLEX SCARF-II POTENTIAL

The expression for the complex Scarf-II potential is given by

\[ V = A \sech^2 x + iB \sech x \tanh x. \quad (24) \]

Note that, unlike the previous case, here parity operation is given by \( x \to -x \), time reversal operation remaining the same. The corresponding QHJ equation, in terms of \( q \), where \( q = \frac{d\ln \psi}{dx} \), is

\[ q^2 + \frac{dq}{dx} + E - A \sech^2 x - iB \sech x \tanh x = 0. \quad (25) \]

Carrying out the change of variable, \( y = i \sinh x \), proceeding in the same manner as before, one obtains the QHJ equation for \( \chi \): \n
\[ \chi^2 + \frac{d\chi}{dy} + \frac{2 + y^2}{4(1-y^2)^2} - \frac{E}{1-y^2} - \frac{A - By}{(1-y^2)^2} = 0, \quad (26) \]

where

\[ \chi = \left( \phi - \frac{y}{2(1-y^2)} \right), \quad q = i(\sqrt{1-y^2})\phi. \quad (27) \]

Along with the \( n \) moving poles with residue one, \( \chi \) has poles at \( y = \pm 1 \). We assume that except for these poles there are no other singularities and that the point at infinity is an isolated singularity. The residue at \( y = 1 \) and \(-1\) are respectively given by,

\[ b_1 = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1}{4} + A - B}, \quad (28) \]

and

\[ b'_1 = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1}{4} + A + B}. \quad (29) \]

As in Eq. (12), considering the behaviour of \( \chi \) at infinity one gets,

\[ \lambda = \frac{1}{2} \pm \sqrt{-E}. \quad (30) \]

It should be noted that, unlike the previous QES case, energy explicitly enters in \( \lambda \). As will be soon seen, this is the reason all the energy values can be obtained here, making this exactly solvable model.

As seen in the earlier section one can write \( \chi \) in terms of its analytic and singular parts as

\[ \chi = \frac{b_1}{y-1} + \frac{b'_1}{y+1} + \frac{P_n}{P_n} + C, \quad (31) \]

where \( C \) is the analytic part of \( \chi \). \( C \) is a constant due to Liouville’s theorem, which turns out to be zero. The leading behaviour of \( \chi \) for large \( y \) from Eq.(31) is of the form, \( \frac{b_1 + b'_1 + n}{y} \).

This coefficient of \( \frac{1}{y} \), should match with \( \lambda \) in Eq.(30) \( i.e. \),

\[ b_1 + b'_1 + n = \lambda. \quad (32) \]
This gives the energy eigenvalues as

$$-E = (b_1 + b'_1 + n - \frac{1}{2})^2.$$  \hfill (33)

The wave function in terms of $\chi$ is

$$\psi(y) = \exp \left( \int dy \left( \frac{b_1}{y-1} + \frac{b'_1}{y+1} + \frac{P'_n}{P_n} + \frac{1}{2} \frac{y}{1-y^2} \right) \right),$$  \hfill (34)

which is equal to

$$\psi(y) = (y-1)^{-p}(y+1)^{-q}P_n(y),$$  \hfill (35)

where, $-p = b_1 - \frac{1}{4}$ and $-q = b'_1 - \frac{1}{4}$. To obtain the polynomial, one needs to substitute Eq.(31) in (26), which yields the following differential equation

$$P''_n + 2P'_n \left( \frac{b_1}{y-1} + \frac{b'_1}{y+1} \right) + G(y)P_n = 0,$$  \hfill (36)

where

$$G(y) = \frac{\left(4(b_1^2 - b_1 + b'_1) + 1 + 4E\right)y^2 + 2y\left(4(b_1^2 - b_1 - b'_1^2 + b'_1) + 2B\right)}{(y^2 - 1)^2} + \frac{4(b_1^2 - b_1 + b'_1^2 - b'_1) + 2 - 4A - 4E}{(y^2 - 1)^2}.$$  \hfill (37)

Substituting the expression for $E$ from Eq.(33) in (36), one obtains

$$(1 - y^2)P'' + P'(2(b_1 - b'_1) - 2(b_1 + b'_1)y) + n(n + 2(b_1 + b'_1 - 1) + 1) = 0,$$  \hfill (38)

which is in the form of the Jacobi differential equation and hence the polynomial $P_n(y) = P_n^{b_1-1,b'_1-1}(y)$ is the Jacobi polynomial. Thus the complete expression for the wave function can be written as,

$$\psi(x) = (i \sinh x - 1)^{b_1-\frac{1}{2}}(i \sinh x + 1)^{b'_1-\frac{1}{2}}P_n^{2b_1-1,2b'_1-1}(i \sinh x),$$  \hfill (39)

which matches with the answer in [8] if $b_1$ and $b'_1$ are written in terms of $p$ and $q$ respectively. Note that in this whole process, we had written the expression of the eigenvalues and the eigenfunctions in terms of $b_1$ and $b'_1$ which have two values. No particular value has been chosen. Hence, we need to choose one value of each residue to remove this ambiguity. For this case, the solutions are known to satisfy the property $\psi(\pm \infty) \to 0$. We make use of this condition to choose the right values of the residues. For this purpose, we consider two conditions on the potential parameter $A$ and $B$. For each condition, we see that the particular values of residues, which are chosen using the property of square integrability give physically acceptable results.
**Case 1:** \(|B| > A + \frac{1}{4}\)

With this restriction on \(A\) and \(B\), the residues at \(y = \pm 1\) becomes, \(b_1 = \frac{1}{2} \pm i \sqrt{B - A - \frac{1}{4}}\) and \(b'_1 = \frac{1}{2} \pm \frac{1}{2} \sqrt{A + B + \frac{1}{4}}\). In the limit \(y \to \infty\), the wave function in Eq.(35) goes as

\[
\psi(y) \approx y^{b_1 + b'_1 - \frac{n}{2} + n}.
\]

The above equation, with the values of \(b_1\) and \(b'_1\) substituted becomes

\[
\psi(y) \approx y^{\frac{1}{2} \sqrt{B - A - \frac{1}{4}} y^\frac{1}{2} \sqrt{A + B + \frac{1}{4}} + n}
\]

For \(\psi(y)\) to go to zero for large \(y\), \(b'_1 = \frac{1}{2} + \frac{1}{2} \sqrt{A + B + \frac{1}{4}}\) is ruled out. Hence the choice of the residues for this case will be

\[
b_1 = \frac{1}{2} \pm i \sqrt{B - A - \frac{1}{4}} \quad b'_1 = \frac{1}{2} - \frac{1}{2} \sqrt{A + B + \frac{1}{4}},
\]

with the restriction on \(n\) as

\[
n < \frac{1}{2} \sqrt{A + B + \frac{1}{4} - \frac{1}{2}}.
\]

For these values of the residues at the fixed poles, \(\psi\) takes the form,

\[
\psi = (i \sinh x - 1)^{\frac{1}{2} \pm i \frac{r}{4}} (i \sinh x + 1)^{\frac{1}{2} - i \frac{r}{4}} P_n^{\mu, \nu, s}(i \sinh x),
\]

where \(r = \sqrt{B - A - \frac{1}{4}} \) and \(s = \sqrt{A + B + \frac{1}{4}}\). The expression for energy is

\[
E = - \left( n + \frac{1}{2} - \frac{1}{2} \left( \sqrt{A + B + \frac{1}{4}} \pm i \sqrt{B - A - \frac{1}{4}} \right) \right)^2,
\]

with the condition on \(n\) as

\[
n < \frac{1}{2} \sqrt{\frac{1}{4} + A - B + \frac{1}{2} \sqrt{A + B + \frac{1}{4}} - \frac{1}{2}}.
\]

**Case 2:** \(|B| \leq A + \frac{1}{4}\)

Proceeding in the same way as above, one obtains the choice of the residues as

\[
b_1 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{4} + A - B} \quad b'_1 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{4} + A + B}.
\]

The wave function is given by,

\[
\psi = (i \sinh x - 1)^{\frac{1}{2} - i \frac{r}{4}} (i \sinh x + 1)^{\frac{1}{2} + i \frac{r}{4}} P_n^{-\mu, \nu}(i \sinh x),
\]

where \(\mu = \sqrt{\frac{1}{4} + A - B} \) and \(\nu = \sqrt{\frac{1}{4} + A + B}\) with the energy

\[
E = - \left( n + \frac{1}{2} - \frac{1}{2} \left( \sqrt{\frac{1}{4} + A - B} + \sqrt{\frac{1}{4} + A + B} \right) \right)^2.
\]

Thus the answers match with those given in [8].
TABLE I: This table gives the QES condition and the number of moving poles of $\chi$ for each combination of $b_1$ and $b'_1$ for the Khare-Mandal model

| $b_1$ | $d_1$ | $n = \lambda - b_1 - b'_1$ | Condition on $M$ | QES condition |
|-------|-------|-----------------------------|-----------------|---------------|
| 1     | 1/4   | 1/4                         | $M = \text{odd}$, $M \geq 1$ | $M = 2n + 1$ |
| 2     | 3/4   | 3/4                         | $M = \text{odd}$, $M \geq 3$ | $M = 2n + 3$ |
| 3     | 1/4   | 1/4                         | $M = \text{even}$, $M \geq 2$ | $M = 2n + 1$ |
| 4     | 1/3   | 3/4                         | $M = \text{even}$, $M \geq 2$ | $M = 2n + 1$ |

V. CONCLUSIONS

In conclusion, PT symmetric potentials belonging to the QES and ES class have been investigated through the QHJ formalism. The QES solvable Khare-Mandal potential has complex or real eigenvalues, depending on whether the potential parameter $M$ is odd or even. The singularity structure of the QMF for these two cases is different. For the case when $M$ is odd, one observes from table I that the solutions fall into two groups, which consist of solutions coming from sets 1 and 2. For a solution belonging to a particular group, the number of singularities of the QMF are fixed and consist of both real and complex locations. This kind of singularity structure of the QMF has been observed in the study of periodic potentials [27]. Though the solutions, for $M$ even, fall into two groups coming from sets 3 and 4, they all have same number of singularities, which again can consist of complex and real poles. This singularity structure is same as observed in the ordinary QES models [18].

Coming to the case of exactly solvable PT symmetric potential, the location of the moving poles can be either real or complex. In the specific example of complex Scarf potential, it turns out that all the moving poles are off the real line. In contrast for the ordinary ES models the moving poles are always real. For both the cases, the number of moving poles of the QMF, characterize the energy eigenvalues.

Acknowledgements: We thank Dr. Z. Ahmed for illuminating discussions and Rajneesh Atre for carefully going through the manuscript.

[1] C.M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.
[2] C.M. Bender, S. Boettcher, and P.N. Meisinger, J. Math. Phys. 40 (1999) 2201.
[3] C.M. Bender, S. Boettcher and V.M. Savage, J. Math. Phys. 41 (2000) 6381.
[4] C.M. Bender, D.C. Brody and H.F. Jones, Phys. Rev. Lett. 89 (2002) 270401.
[5] C.M. Bender, M.V. Berry and A. Mandilara, J. Phys. A 35 (2002) L467.
[6] M. Znojil, J. Phys. A 33 (2000) 4561.
[7] A. Khare and B. P. Mandal, Phys. Lett. A 272 (2000) 53-56.
[8] Z. Ahmed, Phys. Lett. A. 282 (2001) 343.
[9] B. Bagchi, S. Mallik, C. Quesne and R. Roychoudhury, quant-ph/0107095
[10] G. Lévai and M. Znojil, Mod. Phys. Lett. A 16 (2001) 1973.
[11] G. Lévai and M. Znojil, J. Phys. A 35 (2002) 8793.
[12] O. Yesiltas, M. Simsek, R. Sever and C. Tezcan, Phys. Scripta. 67 (2003) 472.
[13] B. Bagchi, C. Quesne and M. Znojil, Mod. Phys. Lett. 16 (2001) 2047.
[14] B. Bagchi, F. Cannata and C. Quesne, Phys. Lett A 269 (2000) 79.
[15] R.A. Leacock and M.J. Padgett, Phys. Rev. Lett. 50 (1983) 3.
[16] R.A. Leacock and M.J. Padgett, Phys. Rev. D 28 (1983) 2491.
[17] S. Sree Ranjani, K.G. Geogo, A.K. Kapoor and P. K. Panigrahi, quant-ph/0211168
[18] K.G. Geogo, S. Sree Ranjani and A.K. Kapoor, J. Phys. A 36 (2003) 4591.
[19] V. Singh, S.N. Biswas and K. Datta, Phys. Rev. D 18 (1978) 1901;
   M. Razavy, Am. J. Phys. 48 (1980) 285, Phys. Lett. A 82, (1981) 7;
   M. Znojil, Phys. Lett. A 13 (1983) 1445;
   A.V. Turbiner and A. Ushveridze, Phys. Lett. A 126 (1987) 181;
   N. Kamran and P. J. Olver, J. Math. Anal. Appl. 145 (1990)342;
   A. González-López, N. Kamran and P.J. Olver, J. Phys. A 24 (1991) 3995.
[20] M.A. Shifman, Int. J. Mod. Phys. A 4 (1989) 2897 and references therein.
[21] A. Ushveridze, Quasi-Exactly Solvable Models in Quantum Mechanics (Inst. of Physics Publishing, Bristol, 1994).
[22] R. Atre and P.K. Panigrahi, Phys. Lett. A 317 (2003) 46.
[23] R.S. Bhalla, A.K. Kapoor and P.K. Panigrahi, Am. J. Phys. 65 (1997) 1187.
[24] R.S. Bhalla, A.K. Kapoor and P.K Panigrahi, Mod. Phys. Lett. A. 12 (1997) 295.
[25] R.S. Bhalla, A.K. Kapoor and P.K Panigrahi, Phys. Rev. A 54 (1996) 951.
[26] R.S. Bhalla, A.K. Kapoor and P.K. Panigrahi, Int. J. Mod. Phys. A. 12 (1997) 1875.
[27] S. Sree Ranjani, A.K. Kapoor and P.K. Panigrahi, quant-ph/0312041.
[28] S. Sree Ranjani, A.K. Kapoor and P.K. Panigrahi, manuscript under preparation.