Exact Solutions for Equations of Bose–Fermi Mixtures in One-Dimensional Optical Lattice

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Abstract. We present two new families of stationary solutions for equations of Bose–Fermi mixtures with an elliptic function potential with modulus \( k \). We also discuss particular cases when the quasiperiodic solutions become periodic ones. In the limit of a sinusoidal potential \( k \to 0 \) our solutions model a quasi-one dimensional quantum degenerate Bose–Fermi mixture trapped in optical lattice. In the limit \( k \to 1 \) the solutions are expressed by hyperbolic function solutions (vector solitons). Thus we are able to obtain in an unified way quasi-periodic and periodic waves, and solitons. The precise conditions for existence of every class of solutions are derived. There are indications that such waves and localized objects may be observed in experiments with cold quantum degenerate gases.

Key words: Bose–Fermi mixtures; one dimensional optical lattice

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1 Introduction

Over the last decade, the field of cold degenerate gases has been one of the most active areas in physics. The discovery of Bose–Einstein Condensates (BEC) in 1995 (see e.g. [1, 2]) greatly stimulated research of ultracold dilute Boson-Fermion mixtures. This interest is driven by the desire to understand strongly interacting and strongly correlated systems, with applications in solid-state physics, nuclear physics, astrophysics, quantum computing, and nanotechnologies.

An important property of Bose–Fermi mixtures wherein the fermion component is dominant is that the mixture tends to exhibit essentially three-dimensional character even in a strongly elongated trap. During the last decade, great progress has been achieved in the experimental realization of Bose–Fermi mixtures [3, 4], in particular Bose–Fermi mixtures in one-dimensional lattices. Optical lattices provide a powerful tool to manipulate matter waves, in particular solitons. The Pauli exclusion principle results in the extension of the fermion cloud in the transverse direction over distances comparable to the longitudinal dimension of the excitations. It has been shown recently, however, that the quasi-one-dimensional situation can nevertheless be realized in a Bose–Fermi mixture due to strong localization of the bosonic component [5, 6]. With account of the effectiveness of the optical lattice in managing systems of cold atoms, their effect on the dynamics of Bose–Fermi mixtures is of obvious interest. Some of the aspects of this problem have already been explored within the framework of the mean-field approximation. In particular, the dynamics of the Bose–Fermi mixtures were explored from the point of view of designing quantum dots [8]. The localized states of Bose–Fermi mixtures with attractive (repulsive) Bose–Fermi interactions are viewed as a matter-wave realization of quantum dots.
and antidots. The case of Bose–Fermi mixtures in optical lattices is investigated in detail and the existence of gap solitons is shown. In particular, in [8] it is obtained that the gap solitons can trap a number of fermionic bound-state levels inside both for repulsive and attractive boson-boson interactions. The time-dependent dynamical mean-field-hydrodynamic model to study the formation of fermionic bright solitons in a trapped degenerate Fermi gas mixed with a Bose–Einstein condensate in a quasi-one-dimensional cigar-shaped geometry is proposed in [9]. Similar model is used to study mixing-demixing in a degenerate fermion-fermion mixture in [10].

Modulational instability, solitons and periodic waves in a model of quantum degenerate boson-fermion mixtures are obtained in [11].

Our aim is to derive two new classes of quasi-periodic exact solutions of the time dependent mean field equations of Bose–Fermi mixture in one-dimensional lattice. We also study some limiting cases of these solutions. The paper is organized as follows. In Section 2 we give the basic equations. Section 3 is devoted to derivation of the first class quasi-periodic solutions with non-trivial phases. A system of $N_f + 1$ equations, which reduce quasi-periodic solutions to periodic are derived. In Section 4 we present second class (type B) nontrivial phase solutions. In Section 5 we obtain 14 classes of elliptic solutions. Section 6 is devoted to two special limits, to hyperbolic and trigonometric functions. In Section 7 preliminary results about the linear stability of solutions are given. Section 8 summarizes the main conclusions of the paper.

## 2 Basic equations

At mean field approximation we consider the following $N_f + 1$ coupled equations [7, 8, 12, 11]

$$i\hbar \frac{\partial \Psi^b}{\partial t} + \frac{1}{2m_B} \frac{\partial^2 \Psi^b}{\partial x^2} - V \Psi^b - g_{BB} |\Psi^b|^2 \Psi^b - g_{BF} \rho_f \Psi^b = 0,$$

(2.1)

$$i\hbar \frac{\partial \Psi^f}{\partial t} + \frac{1}{2m_F} \frac{\partial^2 \Psi^f}{\partial x^2} - V \Psi^f - g_{BF} |\Psi^b|^2 \Psi^f = 0,$$

(2.2)

where $\rho_f = \sum_{i=1}^{N_f} |\Psi^f_i|^2$ and

$$g_{BB} = \frac{2a_{BB}}{a_s}, \quad g_{BF} = \frac{2a_{BF}}{a_s \alpha}, \quad \alpha = \frac{m_B}{m_F}, \quad a_s = \sqrt{\frac{\hbar}{m_B \omega_\perp}},$$

$a_{BB}$ and $a_{BF}$ are the scattering lengths for $s$-wave collisions for boson-boson and boson-fermion interactions, respectively. In recent experiments [13, 14] the quantum degenerate mixtures of $^{40}$K and $^{87}$Rb are studied where $m_B = 87 m_p$, $m_B = 40 m_p$ and $\omega_\perp = 215$ Hz. Equations (2.1), (2.2) have been studied numerically in [7]. The formation of localized structures containing bosons and fermions has been reported in the particular case in which the interspecies scattering length $a_{BF}$ is negative, which is the case of the $^{40}$K-$^{87}$Rb mixture. An appropriate class of periodic potentials to model the quasi-1D confinement produced by a standing light wave is given by [15]

$$V = V_0 \text{sn}^2(\alpha x, k),$$

where $\text{sn}(\alpha x, k)$ denotes the Jacobian elliptic sine function with elliptic modulus $0 \leq k \leq 1$.

Experimental realization of two-component Bose–Einstein condensates have stimulated considerable attention in general [16] and in particular in the quasi-1D regime [17, 18] when the Gross–Pitaevskii equations for two interacting Bose–Einstein condensates reduce to coupled nonlinear Schrödinger (CNLS) equations with an external potential. In specific cases the two component CNLS equations can be reduced to the Manakov system [19] with an external potential.
Important role in analyzing these effects was played by the elliptic and periodic solutions of the above-mentioned equations. Such solutions for the one-component nonlinear Schrödinger equation are well known, see [20] and the numerous references therein. Elliptic solutions for the CNLS and Manakov system were derived in [21, 22, 23].

In the presence of external elliptic potential explicit stationary solutions for NLS were derived in [15, 24, 25]. These results were generalized to the n-component CNLS in [18]. For 2-component CNLS explicit stationary solutions are derived in [26].

3 Stationary solutions with non-trivial phases

We restrict our attention to stationary solutions of these CNLS

\[ \Psi^b(x, t) = g_0(x) \exp \left( -i \frac{\omega_0}{\hbar} t + i \Theta_0(x) + i \kappa_0 \right), \]

\[ \Psi^f_j(x, t) = q_j(x) \exp \left( -i \frac{\omega_j}{\hbar} t + i \Theta_j(x) + i \kappa_{0,j} \right), \]

where \( j = 1, \ldots, N_f, \kappa_0, \kappa_{0,j} \), are constant phases, \( q_j \) and \( \Theta_0, \Theta_j(x) \) are real-valued functions connected by the relation

\[ \Theta_0(x) = C_0 \int_0^x \frac{dx'}{q_0^2(x')}, \quad \Theta_j(x) = C_j \int_0^x \frac{dx'}{q_j^2(x')}, \]

\( C_0, C_j, j = 1, \ldots, N_f \) being constants of integration. Substituting the ansatz (3.1), (3.2) in equations (2.1) and separating the real and imaginary part we get

\[ \frac{1}{2m_B} q_0^3 q_{0xx} - g_{BB} q_0^6 - V q_0^4 - g_B \left( \sum_{i=1}^{N_f} q_i^2 \right) q_0^4 + \omega_0 q_0^4 = \frac{1}{2m_B} C_0^2, \]

\[ \frac{1}{2m_F} q_j^3 q_{jxx} - g_{BF} q_0^2 q_j^4 - V q_j^4 + \omega_j q_j^4 = \frac{1}{2m_F} C_j^2. \]

We seek solutions for \( q_0^2 \) and \( q_j^2 \), \( j = 1, \ldots, N_f \) as a quadratic function of \( \text{sn} (\alpha x, k) \):

\[ q_0^2 = A_0 \text{sn}^2 (\alpha x, k) + B_0, \quad q_j^2 = A_j \text{sn}^2 (\alpha x, k) + B_j. \]

Inserting (3.5) in (3.3) and equating the coefficients of equal powers of \( \text{sn} (\alpha x, k) \) results in the following relations among the solution parameters \( \omega_j, C_j, A_j \) and \( B_j \) and the characteristic of the optical lattice \( V_0, \alpha \) and \( k \):

\[ A_0 = \frac{\alpha^2 k^2 - m_F V_0}{m_F g_B}, \quad \sum_{j=1}^{N_f} A_j = \frac{\alpha^2 k^2}{g_B} \left( \frac{1}{m_B} - \frac{g_B}{m_F g_B} \right) - \frac{V_0}{g_B} \left( 1 - \frac{g_B}{g_B} \right), \]

\[ \omega_0 = \frac{\alpha^2 (k^2 + 1)}{2m_B} + g_B B_0 + g_B \sum_{i=1}^{N_f} B_i + \frac{\alpha^2 k^2 B_0}{2m_B A_0}, \]

\[ \omega_j = \frac{\alpha^2 (k^2 + 1)}{2m_B} + g_B B_0 + \frac{\alpha^2 k^2 B_j}{2m_B A_j}, \]

\[ C_j^2 = \frac{A_0}{A_j} (A_0 + B_0)(A_0 + B_0k^2), \quad C_j^2 = \frac{A_0}{A_j} (A_j + B_j)(A_j + B_jk^2), \]

where \( j = 1, \ldots, N_f \). Next for convenience we introduce

\[ B_0 = -\beta_0 A_0, \quad B_j = -\beta_j A_j, \quad j = 1, \ldots, N_f, \]
In order for our results (3.5) to be consistent with the parametrization (3.1)–(3.3) we must ensure

\[ j \leq \beta \leq 1/k^2 \]

In order for our results (3.5) to be consistent with the parametrization (3.1)–(3.3) we must ensure that both \( q_0(x) \) and \( \Theta_0(x) \) are real-valued, and also \( q_j(x) \) and \( \Theta_j(x) \) are real-valued; this means that \( C_0^2 \geq 0 \) and \( q_2^2(x) \geq 0 \) and also \( C_j^2 \geq 0 \) and \( q_j^2(x) \geq 0 \) (see Table 1). \( W_B = (\alpha^2 k^2 - m_B V_0) \), \( W_T = (\alpha^2 k^2 - m_T V_0) \). An elementary analysis shows that with \( l = 0, \ldots, N_f \) one of the following conditions must hold

a) \( A_l \geq 0, \quad \beta_l \leq 0, \quad 1 \leq \beta_l \leq \frac{1}{k^2} \).

Although our main interest is to analyze periodic solutions, note that the solutions \( \Psi^b, \Psi^f \) in (2.1), (2.2) are not always periodic in \( x \). Indeed, let us first calculate explicitly \( \Theta_0(x) \) and \( \Theta_j(x) \) by using the well known formula, see e.g. (27):

\[ \int_0^x \frac{du}{\varphi(\alpha u) - \varphi(\alpha v)} = \frac{1}{\varphi'(\alpha v)} \left[ 2x\zeta(\alpha v) + \frac{1}{\alpha} \ln \frac{\sigma(\alpha u - \alpha v)}{\sigma(\alpha u + \alpha v)} \right], \]

where \( \varphi, \zeta, \sigma \) are standard Weierstrass functions.

In the case a) we replace \( v \) by \( iv_0 \) and \( v \) by \( iv_j \), set \( \text{sn}^2(i\alpha v_0; k) = \beta_0 < 0 \), \( \text{sn}^2(i\alpha v_j; k) = \beta_j < 0 \) and

\[ e_1 = \frac{1}{3} (2 - k^2), \quad e_2 = \frac{1}{3} (2k^2 - 1), \quad e_3 = -\frac{1}{3} (1 + k^2), \]

and rewrite the l.h.s in terms of Jacobi elliptic functions:

\[ \int_0^x \frac{du \text{sn}^2(i\alpha v; k)\text{sn}^2(\alpha u; k)}{\text{sn}^2(i\alpha v; k) - \text{sn}^2(\alpha u; k)} = -\beta_0 x - \beta_0^2 \int_0^x \frac{du}{\text{sn}^2(\alpha u, k) - \beta_0}, \]

and for \( j = 1, \ldots, N_f \) we have

\[ \int_0^x \frac{du \text{sn}^2(i\alpha v; k)\text{sn}^2(\alpha u; k)}{\text{sn}^2(i\alpha v; k) - \text{sn}^2(\alpha u; k)} = -\beta_j x - \beta_j^2 \int_0^x \frac{du}{\text{sn}^2(\alpha u, k) - \beta_j}. \]

Skipping the details we find the explicit form of

\[ \Theta_0(x) = C_0 \int_0^x \frac{du}{A_0(\text{sn}^2(\alpha u; k) - \beta_0)} = -\tau_0 x + \frac{i}{2} \ln \frac{\sigma(\alpha x + i\alpha v_0)}{\sigma(\alpha x - i\alpha v_0)}, \]

\[ \tau_0 = i\alpha \zeta(\alpha v_0) + \frac{\alpha}{\beta_0} \sqrt{-\beta_0 (1 - \beta_0)(1 - k^2 \beta_0)}. \]

and for \( \Theta_j(x), j = 1, \ldots, N_f \) we have

\[ \Theta_j(x) = C_j \int_0^x \frac{du}{A_j(\text{sn}^2(\alpha u; k) - \beta_j)} = -\tau_j x + \frac{i}{2} \ln \frac{\sigma(\alpha x + i\alpha v_j)}{\sigma(\alpha x - i\alpha v_j)}, \quad (3.9) \]
\[ \tau_j = i\alpha\zeta(i\alpha v_j) + \frac{\alpha}{\beta_j} \sqrt{-\beta_j(1 - \beta_j)(1 - k^2\beta_j)}. \]

These formulae provide an explicit expression for the solutions \( \Psi^b, \Psi^f \) with nontrivial phases; note that for real values of \( v_0 \Theta_0(x), v_j \Theta_j(x) \) are also real. Now we can find the conditions under which \( Q_j(x,t) \) are periodic. Indeed, from (3.9) we can calculate the quantities \( T_0, T_j \) satisfying:

\[ \Theta_0(x + T_0) - \Theta_0(x) = 2\pi p_0, \quad \Theta_j(x + T_j) - \Theta_j(x) = 2\pi p_j, \quad j = 1, \ldots, N_f. \]

Then \( \Psi^b, \Psi^f \) will be periodic in \( x \) with periods \( T_0 = 2m_0\omega/\alpha, T_j = 2m_j\omega/\alpha \) if there exist pairs of integers \( m_0, p_0, \) and \( m_j, p_j, \) such that:

\[ \frac{m_0}{p_0} = -\pi \left[ \alpha v_0(\omega) + \omega \tau_0/\alpha \right]^{-1}, \quad \frac{m_j}{p_j} = -\pi \left[ \alpha v_j(\omega) + \omega \tau_j/\alpha \right]^{-1}, \quad j = 1, \ldots, N_f. \]

where \( \omega \) (and \( \omega' \)) are the half-periods of the Weierstrass functions.

### 4 Type B nontrivial phase solutions

For the first time solutions of this type were derived in [15, 24, 25] for the case of nonlinear Schrödinger equation and in [18] for the \( n \)-component CNLSE. For Bose–Fermi mixtures solutions of this type are possible

- when we have two lattices \( V_B \) and \( V_F \),
- when \( m_B = m_F \).

We seek the solutions in one of the following forms:

\[ q_0^2 = A_0 \text{sn} (\omega x, k) + B_0, \quad q_j^2 = A_j \text{sn} (\omega x, k) + B_j, \quad j = 1, \ldots, N_f. \]

(4.1)

\[ q_0^2 = A_0 \text{cn} (\omega x, k) + B_0, \quad q_j^2 = A_j \text{cn} (\omega x, k) + B_j, \quad j = 1, \ldots, N_f. \]

(4.2)

\[ q_0^2 = A_0 \text{dn} (\omega x, k) + B_0, \quad q_j^2 = A_j \text{dn} (\omega x, k) + B_j, \quad j = 1, \ldots, N_f. \]

(4.3)

In the first case we have

\[ V_B = \frac{3\alpha^2 k^2}{8m_B}, \quad V_F = \frac{3\alpha^2 k^2}{8m_F}, \]

\[ A_0 = -\frac{\alpha^2 k^2}{4m_B g_B A_j} B_j, \quad B_1 = \cdots = B_N, \quad A_j = -\frac{\alpha^2 k^2}{4m_B g_B A_j} B_0 - \frac{A_0 g_B}{g_B}, \]

\[ \omega_0 = \frac{\alpha^2 (k^2 + 1)}{8m_B} + g_B B_0 + g_B A_0 - \frac{\alpha^2 k^2 B_0^2}{8m_B A_0}, \]

\[ \omega_j = \frac{\alpha^2 (k^2 + 1)}{8m_F} + g_B B_0 - \frac{\alpha^2 k^2 B_j^2}{8m_F A_j}, \]

\[ C_0 = \frac{\alpha^2}{4A_0} (B_0^2 - A_0^2)/(A_0^2 - B_0^2 k^2), \quad C_j = \frac{\alpha^2}{4A_j^2} (B_j^2 - A_j^2)/(A_j^2 - B_j^2 k^2). \]

We remark that due to relations \( \frac{B_0}{A_0} = \cdots = \frac{B_N}{A_N} \) we have that all \( q_j \) of the fermion fields are proportional to \( q_1 \).
5 Examples of elliptic solutions

Using the general solution equations (5.3)–(5.8) we have the following special cases: (these solutions are possible only when we have some restrictions on \( g_{BB}, g_{BF}, \) and \( V_0 \) see the Table [1])

**Example 1.** Suppose that \( B_0 = B_j = 0 \). Therefore we have

\[
q_0(x) = \sqrt{A_0}\text{sn} (\alpha x, k), \quad q_j(x) = \frac{\sqrt{A_j}}{g_{FB}} \text{sn} (\alpha x, k),
\]

\[
A_0 = \frac{\alpha^2 k^2 - m_F V_0}{m_F g_{BF}}, \quad \sum_j A_j = \frac{\alpha^2 k^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_B g_{FB}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right). \tag{5.2}
\]

For the frequencies \( \omega_0 \) and \( \omega_j \) we have

\[
\omega_0 = \frac{\alpha^2 (1 + k^2)}{2 m_B}, \quad \omega_j = \frac{\alpha^2 (1 + k^2)}{2 m_F},
\]

as well as \( C_0 = C_j = 0 \).

**Example 2.** Let \( B_0 = -A_0 \) and \( B_j = -A_j \) hold true. Then we have

\[
q_0(x) = \sqrt{-A_0}\text{cn} (\alpha x, k), \quad q_j(x) = \sqrt{-A_j}\text{cn} (\alpha x, k). \tag{5.3}
\]

The coefficients \( A_0 \) and \( A_j \) have the same form as (5.2). The frequencies \( \omega_0 \) and \( \omega_j \) now look as follows

\[
\omega_0 = \frac{\alpha^2 (1 - 2k^2)}{2 m_B} + V_0, \quad \omega_j = \frac{\alpha^2 (1 - 2k^2)}{2 m_F} + V_0.
\]

The constants \( C_0 \) and \( C_j \) are equal to zero again.

**Example 3.** \( B_0 = -A_0/k^2 \) and \( B_j = -A_j/k^2 \). In this case we obtain

\[
q_0(x) = \frac{\sqrt{-A_0}}{k} \text{dn} (\alpha x, k), \quad q_j(x) = \frac{\sqrt{-A_j}}{k} \text{dn} (\alpha x, k),
\]

\[
\omega_0 = \frac{\alpha^2 (k^2 - 2)}{2 m_B} + \frac{V_0}{k^2}, \quad \omega_j = \frac{\alpha^2 (k^2 - 2)}{2 m_F} + \frac{V_0}{k^2}. \tag{5.4}
\]

As before \( C_0 = C_j = 0 \).

**Example 4.** \( B_0 = 0 \) and \( B_j = -A_j \). The result reads

\[
q_0(x) = \sqrt{A_0}\text{sn} (\alpha x, k), \quad q_j(x) = \sqrt{-A_j}\text{cn} (\alpha x, k),
\]

\[
\omega_0 = \frac{\alpha^2 (1 - k^2)}{2 m_B} + V_0 + A_0 g_{BB}, \quad \omega_j = \frac{\alpha^2}{2 m_F}. \tag{5.5}
\]

By analogy with the previous examples the constants \( A_0, A_j, C_0 \) and \( C_j \) are given by formulae (5.2) and \( C_0, C_j \) are all zero.

**Example 5.** \( B_0 = 0 \) and \( B_j = -A_j/k^2 \). Thus one gets

\[
q_0(x) = \sqrt{A_0}\text{sn} (\alpha x, k), \quad q_j(x) = \frac{\sqrt{-A_j}}{k} \text{dn} (\alpha x, k),
\]

\[
\omega_0 = \frac{\alpha^2 (k^2 - 1)}{2 m_B} + \frac{V_0}{k^2} + A_0 g_{BB}, \quad \omega_j = \frac{\alpha^2 k^2}{2 m_F}. \tag{5.6}
\]
Table 2. Trivial phase solutions in the generic case. We use the quantity $W = g_{BF}m_FW_B/(m_BW_F)$.

| $q_0$ | $g_{BF} \geq 0$ | $g_{BB} \leq W$ | $V_0 \leq \alpha^2k^2/m_F$ |
|-------|----------------|----------------|--------------------------|
| $q_j = \sqrt{-A_0}sn(\alpha x, k)$ | $g_{BF} \geq 0$ | $g_{BB} \leq W$ | $V_0 \geq \alpha^2k^2/m_F$ |
| $q_j = \sqrt{A_j}sn(\alpha x, k)$ |

Example 6. Let $B_0 = -A_0$ and $B_j = 0$. Hence we have

$$q_0(x) = \sqrt{-A_0}cn(\alpha x, k), \quad q_j(x) = \sqrt{A_j}sn(\alpha x, k),$$

$$\omega_0 = \frac{\alpha^2}{2m_B} - g_{BB}A_0, \quad \omega_j = \frac{\alpha^2(1 - k^2)}{2m_F} + V_0.$$

Example 7. Let $B_0 = -A_0$ and $B_j = -A_j/k^2$. We obtain

$$q_0(x) = \sqrt{-A_0}cn(\alpha x, k), \quad q_j(x) = \sqrt{-A_j}dn(\alpha x, k),$$

$$\omega_0 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_B} + \frac{1 - k^2}{k^2}A_0g_{BB}, \quad \omega_j = V_0 - \frac{\alpha^2k^2}{2m_F}.$$  

Example 8. Suppose $B_0 = -A_0/k^2$ and $B_j = 0$. Then

$$q_0(x) = \sqrt{-A_0}dn(\alpha x, k), \quad q_j(x) = \sqrt{A_j}sn(\alpha x, k),$$

$$\omega_0 = \frac{\alpha^2k^2}{2m_B} - \frac{g_{BB}A_0}{k^2}, \quad \omega_j = \frac{\alpha^2(k^2 - 1)}{2m_F} + \frac{V_0}{k^2}.$$  

Example 9. Let $B_0 = -A_0/k^2$ and $B_j = -A_j$. Thus

$$q_0(x) = \sqrt{-A_0}dn(\alpha x, k), \quad q_j(x) = \sqrt{-A_j}cn(\alpha x, k),$$

$$\omega_0 = V_0 - \frac{\alpha^2k^2}{2m_B} + \frac{k^2 - 1}{k^2}g_{BB}A_0, \quad \omega_j = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_F}.$$  

All these cases when $V_0 = 0$ and $j = 2$ are derived for the first time in [11].
5.1 Mixed trivial phase solution

Example 10. When $B_0 = 0$, $B_1 = 0$, $B_2 = -A_2$, $B_j = -A_j/k^2$, $j = 3, \ldots, N_f$ the solutions obtain the form

\[
q_0 = \sqrt{A_0} \text{sn}(\alpha x, k), \quad q_1 = \sqrt{A_1} \text{sn}(\alpha x, k),
\]

\[
q_2 = -\sqrt{A_2} \text{cn}(\alpha x, k), \quad q_j = \sqrt{-A_j} \text{dn}(\alpha x, k)/k, \quad j = 3, \ldots, N_f.
\]

Using equations (3.6)–(3.8) we have

\[
A_0 = \frac{\omega^2 k^2 - V_0 m_F}{m_F g_{BF}}, \quad \sum_{j=1}^{N_f} A_j = \omega^2 k^2 \left( \frac{1}{m_B g_{BF}} - \frac{g_{BB} g_{BF}}{m_F g_{BB}^2} \right) - V_0 \left( \frac{1}{g_{BF}} - \frac{g_{BB}}{g_{BF}^2} \right),
\]

\[
\omega_0 = \frac{\omega^2 (k^2 - 1)}{2m_B} + \frac{g_{BB}}{k^2} (A_1 + (1 - k^2) A_2) + \frac{g_{BB} A_0}{k^2} + \frac{V_0}{k^2},
\]

\[
\omega_1 = \frac{\omega^2 (1 + k^2)}{2m_F}, \quad \omega_2 = \frac{1}{2m_F} \alpha^2, \quad \omega_j = \frac{\omega^2 k^2}{2m_F}, \quad j = 3, \ldots, N_f.
\]

Example 11. Let $B_0 = B_1 = 0$ and $B_j = -A_j$ where $j = 2, \ldots, N_f$. Therefore the solutions read

\[
q_0(x) = \sqrt{A_0} \text{sn}(\alpha x, k), \quad q_1(x) = \sqrt{A_1} \text{sn}(\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{cn}(\alpha x, k).
\]

Then we obtain for frequencies the following results

\[
\omega_0 = \frac{\omega^2 (1 - k^2)}{2m_B} + V_0 + g_{BB} A_0 + g_{BF} A_1, \quad \omega_1 = \frac{\omega^2 (1 + k^2)}{2m_F}, \quad \omega_j = \frac{\omega^2 k^2}{2m_F}.
\]

Example 12. Suppose $B_0 = -A_0$, $B_1 = 0$, $B_2 = -A_2$ and $B_j = -A_j/k^2$ where $j = 3, \ldots, N_f$. The solutions have the form

\[
q_0(x) = -\sqrt{A_0} \text{cn}(\alpha x, k), \quad q_1(x) = \sqrt{A_1} \text{sn}(\alpha x, k),
\]

\[
q_2(x) = -\sqrt{A_2} \text{cn}(\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{dn}(\alpha x, k)/k.
\]

The frequencies are

\[
\omega_0 = -\frac{V_0}{k^2} - \frac{\omega^2}{2m_B} + \frac{1 - k^2}{2k^2} (g_{BB} A_0 + g_{BF} A_2) + \frac{g_{BF} A_0}{k^2} + \frac{V_0}{k^2}, \quad \omega_1 = V_0 + \frac{\omega^2 (1 - k^2)}{2m_F},
\]

\[
\omega_2 = V_0 + \frac{\omega^2 (1 - 2k^2)}{2m_F}, \quad \omega_j = V_0 - \frac{\omega^2 k^2}{2m_F}.
\]

Example 13. Let $B_0 = -A_0$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \ldots, N_f$. Then

\[
q_0(x) = -\sqrt{A_0} \text{cn}(\alpha x, k), \quad q_1(x) = -\sqrt{A_1} \text{cn}(\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{dn}(\alpha x, k)/k,
\]

\[
\omega_0 = \frac{V_0}{k^2} - \frac{\omega^2}{2m_B} + \frac{1 - k^2}{2k^2} (g_{BB} A_0 + g_{BF} A_1), \quad \omega_1 = V_0 + \frac{\omega^2 (1 - 2k^2)}{2m_F},
\]

\[
\omega_j = V_0 - \frac{\omega^2 k^2}{2m_F}.
\]

Example 14. Let $B_0 = -A_0/k^2$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \ldots, N_f$. Hence

\[
q_0(x) = \sqrt{-A_0} \text{dn}(\alpha x, k)/k, \quad q_1(x) = \sqrt{-A_1} \text{cn}(\alpha x, k), \quad q_j(x) = \sqrt{-A_j} \text{dn}(\alpha x, k)/k,
\]

\[
\omega_0 = \frac{\omega^2 (k^2 - 2)}{2m_B} + \frac{V_0}{k^2} + \frac{1 - k^2}{k^2} (g_{BB} A_0 + g_{BF} A_1), \quad \omega_1 = \frac{V_0}{k^2} - \frac{\omega^2}{2m_F} + \frac{\omega^2 (k^2 - 2)}{2m_F}.
\]

Certainly these examples do not exhaust all possible combinations of solutions and it is easy to extend this list.
6 Vector soliton solutions

6.1 Vector bright-bright soliton solutions

When $k \to 1$, $\text{sn}(\alpha x, 1) = \tanh(\alpha x)$ and $B_0 = -A_0$, $B_j = -A_j$ we obtain that the solutions read

$$ q_0 = \sqrt{-A_0} \frac{1}{\cosh(\alpha x)}, \quad q_j = \sqrt{-A_j} \frac{1}{\cosh(\alpha x)}, $$

where $A_0 \leq 0$ as well as $A_j \leq 0$. Using equations (3.6)–(3.8) we have

$$ A_0 = \frac{\alpha^2 - V_0 m_F}{m_F g_{BF}}, \quad V = V_0 \tanh^2(\alpha x), $$

$$ \sum_{j=1}^{N_j} A_j = \frac{\alpha^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_F g_{BF}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right), $$

$$ \omega_0 = V_0 - \frac{1}{2m_B} \alpha^2, \quad \omega_j = V_0 - \frac{1}{2m_F} \alpha^2. $$

As a consequence of the restrictions on $A_0$ and $A_j$ one can get the following unequalities

$$ g_{BF} > 0, \quad V_0 \geq \frac{\alpha^2}{m_F}, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, $$

$$ g_{BF} < 0, \quad V_0 \leq \frac{\alpha^2}{m_F}, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}. $$

Vector bright soliton solution when $V_0 = 0$ is derived for the first time in [11].

6.2 Vector dark-dark soliton solutions

When $k \to 1$ and $B_0 = B_j = 0$ are satisfied the solutions read

$$ q_0(x) = \sqrt{A_0} \tanh(\alpha x), \quad q_j(x) = \sqrt{A_j} \tanh(\alpha x). $$

The natural restrictions $A_0 \geq 0$ and $A_j \geq 0$ lead to

$$ g_{BF} > 0, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \leq \alpha^2/m_F, $$

$$ g_{BF} < 0, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_F V_0) m_B} g_{BF}, \quad V_0 \geq \alpha^2/m_F, $$

$$ A_0 = \frac{\alpha^2 - m_F V_0}{m_F g_{BF}}, \quad \sum_{j} A_j = \frac{\alpha^2}{g_{BF}} \left( \frac{1}{m_B} - \frac{g_{BB}}{m_F g_{BF}} \right) - \frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right). \quad (6.1) $$

For the frequencies $\omega_0$ and $\omega_j$ and the constants $C_0$ and $C_j$ we have

$$ \omega_0 = \frac{\alpha^2}{m_B}, \quad \omega_j = \frac{\alpha^2}{m_F}, \quad C_0 = C_j = 0. \quad (6.2) $$
6.3 Vector bright-dark soliton solutions

When \( k \to 1 \), \( B_0 = -A_0 \) and \( B_j = 0 \), we have

\[
q_0(x) = \sqrt{-A_0} \cosh(\alpha x), \quad q_j(x) = \sqrt{A_j} \tanh(\alpha x),
\]

\[
\omega_0 = \frac{\alpha^2}{2m_B} - g_{BB} A_0, \quad \omega_j = V_0, \quad C_0 = C_j = 0.
\]

The parameters \( A_0 \) and \( A_j \) are given by (6.1). In this case we have the following restrictions

\[
g_{BF} > 0, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_B V_0) m_B} g_{BF}, \quad V_0 \geq \alpha^2/m_F,
\]

\[
g_{BF} < 0, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_B V_0) m_B} g_{BF}, \quad V_0 \leq \alpha^2/m_F.
\]

6.4 Vector dark-bright soliton solutions

When \( k \to 1 \) and provided that \( B_0 = 0 \) and \( B_j = -A_j \) the result is

\[
q_0(x) = \sqrt{A_0} \tanh(\alpha x), \quad q_j(x) = \sqrt{-A_j} \cosh(\alpha x), \quad \omega_0 = V_0 + A_0 g_{BB}, \quad \omega_j = \frac{\alpha^2}{2m_F}.
\]

By analogy with the previous examples the constants \( A_0 \), \( A_j \), \( C_0 \) and \( C_j \) are given by formulae (6.1) and (6.2) respectively. The restrictions now are

\[
g_{BF} > 0, \quad g_{BB} \geq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_B V_0) m_B} g_{BF}, \quad V_0 \leq \alpha^2/m_F,
\]

\[
g_{BF} < 0, \quad g_{BB} \leq \frac{(\alpha^2 - m_B V_0) m_F}{(\alpha^2 - m_B V_0) m_B} g_{BF}, \quad V_0 \geq \alpha^2/m_F.
\]

6.5 Vector dark-dark-bright soliton solutions

Let \( B_0 = B_1 = 0 \) and \( B_j = -A_j \) where \( j = 2, \ldots, N_f \). Therefore the solutions read

\[
q_0(x) = \sqrt{A_0} \tanh(\alpha x), \quad q_1(x) = \sqrt{A_1} \tanh(\alpha x), \quad q_j(x) = \sqrt{-A_j} \text{sech}(\alpha x).
\]

Then we obtain for frequencies the following results

\[
\omega_0 = V_0 + g_{BB} A_0 + g_{BF} A_1, \quad \omega_1 = \frac{\alpha^2}{m_F}, \quad \omega_j = \frac{\alpha^2}{2m_F}.
\]

These examples are by no means exhaustive.

6.6 Nontrivial phase, trigonometric limit

In this section we consider a trap potential of the form \( V_{\text{trap}} = V_0 \cos(2\alpha x) \), as a model for an optical lattice. Our potential \( V \) is similar and differs only with additive constant. When \( k \to 0 \),

\[
\text{sn} \,(\alpha x, 0) = \sin(\alpha x)
\]

\[
q_0^2 = A_0 \sin^2(\alpha x) + B_0, \quad q_j^2 = A_j \sin^2(\alpha x) + B_j,
\]

\[
V = V_0 \sin^2(\alpha x) = \frac{1}{2}(V_0 - V_0 \cos(2\alpha x)),
\]

(6.3)

(6.4)
and obtain the following linearized equations

\[
A_0 = -\frac{V_0}{g_{BF}}, \quad \sum_{j=1}^{N_f} A_j = -\frac{V_0}{g_{BF}} \left( 1 - \frac{g_{BB}}{g_{BF}} \right),
\]

\[
\omega_0 = \frac{1}{2m_B} \alpha^2 + B_0 g_{BB} + g_{BF} \sum_{i=1}^{N_f} B_i, \quad \omega_j = \frac{1}{2m_F} \alpha^2 + g_{BF} B_0,
\]

\[
C^2_0 = \alpha^2 B_0 (A_0 + B_0), \quad C^2_j = \alpha^2 B_j (A_j + B_j),
\]

where

\[
\Theta_0(x) = \arctan \left( \sqrt{\frac{A_0 + B_0}{B_0}} \tan(\alpha x) \right), \quad \Theta_j(x) = \arctan \left( \sqrt{\frac{A_j + B_j}{B_j}} \tan(\alpha x) \right).
\]

This solution is the most important from the physical point of view [8].

### 7 Linear stability, preliminary results

To analyze linear stability of our initial system of equations we seek solutions in the form

\[
\psi_0(x,t) = (q_0(x) + \varepsilon \phi_0(x,t)) \exp \left( -\frac{i\omega_0}{\hbar} t + i\Theta_0(x) + i\kappa_0 \right),
\]

\[
\psi_j(x,t) = (q_1(x) + \varepsilon \phi_j(x,t)) \exp \left( -\frac{i\omega_j}{\hbar} t + i\Theta_1(x) + i\kappa_1 \right).
\]

and obtain the following linearized equations

\[
h \left( \begin{array}{c}
\Phi_0 \\
\Phi_1 \\
\vdots \\
\Phi_{N_f}
\end{array} \right) = \left( \begin{array}{cccc}
\Lambda_0 & U_1 & U_2 & \ldots & U_{N_f} \\
V_1 & \Lambda_1 & 0 & \ldots & 0 \\
V_2 & 0 & \Lambda_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V_{N_f} & 0 & 0 & \ldots & \Lambda_{N_f}
\end{array} \right) \left( \begin{array}{c}
\Phi_0 \\
\Phi_1 \\
\vdots \\
\Phi_{N_f}
\end{array} \right),
\]

\[
\Phi_0 = \left( \begin{array}{c}
\phi^R_0 \\
\phi^I_0
\end{array} \right), \quad \Phi_j = \left( \begin{array}{c}
\phi^R_j \\
\phi^I_j
\end{array} \right),
\]

where

\[
\Lambda_0 = \left( \begin{array}{cc}
S_0 & L_{0,-} \\
L_{0,+} & S_0
\end{array} \right), \quad U_j = \left( \begin{array}{cc}
0 & 0 \\
U_{0,j} & 0
\end{array} \right), \quad \Lambda_j = \left( \begin{array}{cc}
S_j & L_{j,-} \\
L_{j,+} & S_j
\end{array} \right), \quad V_j = \left( \begin{array}{cc}
0 & 0 \\
U_{1,j} & 0
\end{array} \right),
\]

\[
S_0 = -\frac{C_0}{m_{BF} q_0} \partial_x \left( \frac{1}{q_0} \right), \quad L_{0,-} = -\frac{1}{m_B} \left( \partial^2_{xx} - \frac{C^2_0}{q_0^2} \right) + V + g_{BB} q_0^2 + g_{BF} q_1^2 - \omega_0,
\]

\[
U_{0,j} = -2g_{ BF} q_0^2, \quad L_{0,+} = \frac{1}{m_B} \left( \partial^2_{xx} - \frac{C^2_0}{q_0^2} \right) - V - 3g_{BB} q_0^2 - g_{BF} q_1^2 + \omega_0,
\]

\[
S_j = -\frac{C_j}{m_F q_j} \partial_x \left( \frac{1}{q_j} \right), \quad L_{j,-} = -\frac{1}{m_F} \left( \partial^2_{xx} - \frac{C^2_j}{q_j^2} \right) + V + g_{BF} q_0^2 - \omega_j,
\]

\[
L_{j,+} = \frac{1}{m_F} \left( \partial^2_{xx} - \frac{C^2_j}{q_j^2} \right) - V - g_{BF} q_0^2.
\]
The analysis of the latter matrix system is a difficult problem and only numerical simulations are possible. Recently a great progress was achieved for analysis of linear stability of periodic solutions of type (3.1), (3.2) (see e.g. [15], [24], [25], [18], [26] and references therein). Nevertheless the stability analysis is known only for solutions of type (5.1)–(5.6) and solutions with nontrivial phase of type (6.3) and (6.4). Linear analysis of soliton solutions is well developed, but it is out of scope of the present paper.

Finally we discuss three special cases:

**Case I.** Let $B_0 = B_j = 0$ then for $j = 1, \ldots, N_f$ and $q_0 = \sqrt{A_0} \text{sn} (\alpha x, k), q_j = \sqrt{A_j} \text{sn} (\alpha x, k)$ we have the following linearized equations:

\[
U_{1,j} = -2gBFq_0q_j, \quad L_{j,+} = \frac{1}{2m_F} \left( \frac{\partial^2 - \frac{\omega_j^2}{q_0^2}}{q_0^2} \right) - V - gBFq_0^2 + \omega_j, \quad j = 1, \ldots, N_f.
\]

**Case II.** Let $B_0 = -A_0, B_j = -A_j$ then for $q_0 = \sqrt{-A_0} \text{cn} (\alpha x, k), q_j = \sqrt{-A_j} \text{cn} (\alpha x, k)$ we obtain the following linearized equations:

\[
\begin{align*}
\frac{d}{dt} \phi_{0,j}^R &= -\frac{1}{2m_B} \frac{d^2}{dx^2} \phi_{0,j}^R + \left( V_0 + gBBA_0 + gBF \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_{0,j}^R - \omega_0 \phi_{0,j}^R, \\
\frac{d}{dt} \phi_{0,j}^I &= \frac{1}{2m_B} \frac{d^2}{dx^2} \phi_{0,j}^I - \left( V_0 + 3gBBA_0 + gBF \sum_j A_j \right) \text{sn}^2(\alpha x, k) \phi_{0,j}^I + \omega_0 \phi_{0,j}^I, \\
\frac{d}{dt} \phi_{j,j}^R &= -\frac{1}{2m_B} \frac{d^2}{dx^2} \phi_{j,j}^R + \left( V_0 + gBF A_0 \right) \text{sn}^2(\alpha x, k) \phi_{j,j}^R + \omega_j \phi_{j,j}^R - 2gBF \sqrt{A_0 A_j} \text{sn}^2(\alpha x, k) \phi_{0,j}^R, \\
\frac{d}{dt} \phi_{j,j}^I &= \frac{1}{2m_B} \frac{d^2}{dx^2} \phi_{j,j}^I - \left( V_0 + gBF A_0 \right) \text{sn}^2(\alpha x, k) \phi_{j,j}^I + \omega_j \phi_{j,j}^I - 2gBF \sqrt{A_0 A_j} \text{sn}^2(\alpha x, k) \phi_{0,j}^I.
\end{align*}
\]

**Case III.** Let $B_0 = -A_0/k^2, B_j = -A_j/k^2$ therefore the solutions are

\[
q_0 = \sqrt{-A_0} \text{dn} (\alpha x, k)/k, \quad q_j = \sqrt{-A_j} \text{dn} (\alpha x, k)/k,
\]
and we obtain the following linearized equations

\[
\hbar \phi_{0,t}^{R} = -\frac{1}{2m_{B}} \phi_{xx}^{R} \phi_{0}^{I} + \left( V_{0} + g_{BB} A_{0} + g_{BF} \sum_{j} A_{j} \right) \text{sn}^{2}(\alpha x, k) \phi_{0}^{I} \\
- \left( g_{BB} A_{0} + g_{BF} \sum_{j} A_{j} + k^{2} \omega_{0} \right) \frac{\phi_{0}^{I}}{k^{2}},
\]

\[
\hbar \phi_{0,t}^{I} = \frac{1}{2m_{B}} \phi_{xx}^{I} \phi_{0}^{R} + \left( 3g_{BB} A_{0} + g_{BF} \sum_{j} A_{j} + k^{2} \omega_{0} \right) \frac{\phi_{0}^{R}}{k^{2}},
\]

\[
- \left( V_{0} + 3g_{BB} A_{0} + g_{BF} \sum_{j} A_{j} \right) \text{sn}^{2}(\alpha x, k) \phi_{0}^{R} + \frac{2g_{BF} A_{0}(1 - k^{2} \text{sn}^{2}(\alpha, k))}{k^{2}} \sum_{j} \phi_{j}^{R},
\]

\[
\hbar \phi_{j,t}^{R} = -\frac{1}{2m_{F}} \phi_{xx}^{R} \phi_{j}^{I} + \left( V_{0} + g_{BF} A_{0} \right) \text{sn}^{2}(\alpha x, k) \phi_{j}^{I} - \frac{g_{BF} A_{0} + k^{2} \omega_{j}}{k^{2}} \phi_{j}^{I},
\]

\[
\hbar \phi_{j,t}^{I} = \frac{1}{2m_{F}} \phi_{xx}^{I} \phi_{j}^{R} - \left( V_{0} + g_{BF} A_{0} \right) \text{sn}^{2}(\alpha x, k) \phi_{j}^{R} + \frac{g_{BF} A_{0} + k^{2} \omega_{j}}{k^{2}} \phi_{j}^{R}
\]

\[
- \frac{2g_{BF} \sqrt{A_{0} A_{j}}(1 - k^{2} \text{sn}^{2}(\alpha, k))}{k^{2}} \phi_{0}^{R}, \quad j = 1, \ldots, N_{f}.
\]

These cases are by no means exhaustive.

8 Conclusions

In conclusion, we have considered the mean field model for boson-fermion mixtures in optical lattice. Classes of quasi-periodic, periodic, elliptic solutions, and solitons have been analyzed in detail. These solutions can be used as initial states which can generate localized matter waves (solitons) through the modulational instability mechanism. This important problem is under consideration.

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