Time as a dynamical variable in quantum decay

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Abstract

We present a theoretical analysis of quantum decay in which the survival probability is replaced by a decay rate that is equal to the absolute value squared of the wave function in the time representation. The wave function in the time representation is simply the Fourier transform of the wave function in the energy representation, and it is also the probability amplitude generated by the Positive Operator Valued Measure of a time operator. The present analysis endows time with a dynamical character in quantum decay, and it is applicable only when the unstable system is monitored continuously while it decays. When the analysis is applied to the Gamow state, one recovers the exponential decay law. The analysis allows us to interpret the oscillations in the decay rate of the GSI anomaly, of neutral mesons, and of fluorescence quantum beats as the result of the interference of two resonances in the time representation. In addition, the analysis allows us to show that the time of flight of a resonance coincides with its lifetime.

Keywords: Gamow states; resonances; time operators; time of flight; continuous measurements; Zeno effect

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1 Introduction

In quantum mechanics, time plays the role of an external parameter, and therefore it is apparently not a dynamical variable, as made clear by Pauli’s theorem [1]. However, there are many experimental situations such as the time of flight or the decay of an unstable particle in which time seems to play a dynamical role. For example, the lifetime of a particle seems to be an intrinsic dynamical property of the particle, not just a mere parameter.
Many authors have constructed time operators that endow time with a dynamical character, see for example Refs. [2–15] and references therein. Such time operators are usually [2–11] associated with Positive Operator Valued Measures (POVMs) and therefore circumvent Pauli’s theorem. POVMs not only provide a natural setting for time operators, but also for phase operators and for the momentum operator of a one-dimensional particle on the half line. Rather than being uncommon, POVMs are standard tools in the quantum theory of open systems [16] and in quantum information and computation [17,18].

Although the mathematical aspects of the POVMs associated with time operators are well established, their phenomenological signatures have remained elusive [19]. The purpose of this paper is to propose a theoretical analysis of quantum decay in which the decay rate is given by the probability distribution associated with the POVM of a time operator. In such analysis, time appears explicitly as a dynamical variable (or, more precisely, as a random variable). We will show that the probability distribution associated with the POVM of the time operator is different from the survival probability. We will also show that the time representation of the Gamow states describes the exponential region of quantum decay while explicitly displaying the dynamical character of time.

As we will stress along the paper, describing the decay on an unstable system in the time representation is necessary only in experiments that monitor the system’s decay continuously. One such experiment is the so-called GSI anomaly [20], where Litvinov et al. observed that K-shell electron capture decay rates of Hydrogen-like $^{140}$Pr$^{58+}$ and $^{142}$Pm$^{60+}$ ions show an oscillatory modulation superimposed on the exponential decay. Because Litvinov et al. monitored individual ions continuously, we will interpret the GSI anomaly as the result of the interference of two resonances in the time representation. We will also see that such interpretation could be applied to the decay of $K$ and $B$ mesons and to fluorescence quantum beats if the decay of these systems were monitored continuously.

In Sec. 2, we recall the basic phenomenological features of exponential decay. In Sec. 3 we recall the standard theoretical analysis of quantum decay. In Sec. 4, we construct the time representation and use dimensional analysis to identify the decay rate with the absolute value squared of the wave function in the time representation. In Secs. 5 and 6 we obtain the time representation of a Gamow state and show that such time representation accounts for the phenomenology of exponential decay. In Sec. 7 we compare the survival probability $p_s(\tau)$ with the non-decay probability $P(t)$ associated with the time representation, and we point out that $P(t)$ does not exhibit the Zeno effect. In Secs. 8 and 9 we show that the interference of two resonances in the time representation can account for the GSI anomaly, for fluorescence quantum beats, and for the decay of neutral mesons. In Sec. 10 we compare the pulsed and the continuous measurements of the survival probability $p_s(\tau)$ with the measurement of the non-decay probability $P(t)$, and we argue that the measurement of $P(t)$ is inherently continuous. In Sec. 11 we use the time representation to derive an expression for the time of flight.
of a particle, and we show that the time of flight of a resonance is equal to its lifetime, as it is usually assumed. Section 12 contains our conclusions.

2 Phenomenology of radioactive decay

The standard phenomenological treatment of the decay of a radioactive sample is as follows. When a sample of radioactive nuclei contains \( N(t) \) radioactive nuclei at time \( t \), the rate at which nuclei decay is proportional to \( N(t) \),

\[
\frac{dN(t)}{dt} = -\lambda N(t),
\]

where \( \lambda \) is the decay constant. Straightforward integration yields

\[
N(t) = N_0 e^{-\lambda t},
\]

where \( N_0 \) is the number of radioactive nuclei at \( t = 0 \). The non-decay and the decay probabilities are

\[
\mathcal{P}(t) = \frac{N(t)}{N_0} = e^{-\lambda t},
\]

\[
\mathcal{P_d}(t) = \frac{N_d(t)}{N_0} = \frac{N_0 - N(t)}{N_0} = 1 - \mathcal{P}(t) = 1 - e^{-\lambda t},
\]

where \( N_d(t) \) is the number of atoms that have decayed at time \( t \), that is, the number of detector clicks that result from observing the decay products of a radioactive reaction.

Quite often, as for example in Ref. [20], we are interested in the decay rate. The decay rate is defined as

\[
R(t) \equiv \frac{dN_d(t)}{dt} = -\frac{dN(t)}{dt},
\]

where the minus sign in Eq. (2.5) comes from the fact that the rate at which the mother nuclei have decayed is the opposite to the rate at which such nuclei have not decayed. The decay rate also follows the exponential law,

\[
R(t) = R_0 e^{-\lambda t},
\]

where \( R_0 = \lambda N_0 \). The decay rate has dimensions of probability/time (i.e., counts/time):

\[
[R(t)] = \frac{1}{T}.
\]

When we measure \( R(t) \), we can obtain \( N(t) \) from \( R(t) \) by integration:

\[
N(t) = N_0 - \int_0^t R(t')dt'.
\]
When we measure the decay of a single radioactive nucleus (as it is done in Ref. [20]), we need to repeat the experiment \(N_0\) times, and the above analysis carries through, except that the number of initial radioactive nuclei \(N_0\) is replaced by the number of times that we repeat the experiment.

The output data of a decay experiment are usually expressed by plotting either the number of decaying events (i.e., the number of detector “clicks”) as a function of time, or the decay rate as a function of time. When the system is monitored continuously, such output data can also be viewed as a temporal probability distribution of decay events, in very much the same way that the output data of experiments that measure quantities such as arrival times, times of flight or tunneling times can be viewed as temporal probability distributions of arrival, flight or tunneling events.

Because an unstable quantum system decays at a random time, the measurement of quantum decay requires that we monitor the system continuously, or else we may miss the moment when it decays.

### 3 The standard theoretical treatment of quantum decay

Quantum mechanics describes the evolution of a system through wave functions \(\varphi(x; \tau)\) that satisfy the time-dependent Schrödinger equation

\[
\text{i}\hbar \frac{d\varphi(x; \tau)}{d\tau} = H\varphi(x; \tau),
\]  

where \(\tau\) is a time parameter that labels the evolution of the system and has no dynamical character [21]. In the position representation, the position operator acts as multiplication by \(x\), and Born’s rule says that the probability density to find a particle at position \(x\) is \(|\varphi(x)|^2\). When the wave functions are normalized to 1, \(\int dx |\varphi(x)|^2 = 1\), both \(|\varphi(x)|^2\) and \(|\varphi(x; \tau)|^2\) have dimensions of 1/length. In general, any given operator \(A\) acts as multiplication by \(a\) in the \(a\)-representation (where \(a\) runs over the spectrum of \(A\)), and for any normalized wave function \(\varphi\), \(|\varphi(a)|^2\) has dimensions of 1/\([a]\). By Born’s rule, \(|\varphi(a)|^2\) is interpreted as a probability density.

It is customary to assume that the number of unstable particles that have not decayed at time \(\tau\) is given by \(N(\tau) = |\varphi(\tau)|^2\). The probability that the particle has not decayed is then given by

\[
P(\tau) = \frac{N(\tau)}{N(0)} = \frac{|\varphi(\tau)|^2}{|\varphi(0)|^2}.
\]  

(3.2)

For a Gamow state of width \(\Gamma_R\), it can be easily shown that

\[
P(\tau) = e^{-\Gamma_R \tau/\hbar}.
\]  

(3.3)
However, because definition (3.2) assumes that time is just a parameter, we are going to construct a wave-function description of quantum decay that utilizes the time representation.

4 The Time Representation

In the remainder of this paper, we are going to work with a Hamiltonian $H = H_0 + V$, where $H_0$ is the free Hamiltonian and $V$ is a smooth, spherically symmetric potential. We will assume that $V(r)$ is not too singular at the origin and that it falls at infinity faster than exponentials (Appendix A of Ref. [22] contains the detailed mathematical characterization of the class of potentials we will use). We will restrict ourselves to the s partial wave, since the generalization to higher-order waves is straightforward. We will also restrict ourselves to the continuous part of the spectrum, which will be assumed to be $[0, \infty)$.

In order to construct the time representation, we first need to construct the energy representation. We will use the energy representation associated with the “out” Lippmann-Schwinger eigenfunctions $\chi^-(r; E) = \langle r | E^- \rangle$. The “out” energy representation of a wave function $\varphi(r)$ is given by [23]

$$\varphi(E) = \int_0^\infty dr \varphi(r)\overline{\chi^-(r; E)}, \quad (4.1)$$

which in Dirac’s bra-ket notation reads

$$\langle -E | \varphi \rangle = \int_0^\infty dr \langle -E | r \rangle \langle r | \varphi \rangle. \quad (4.2)$$

In the energy representation, the Hamiltonian $H$ acts as multiplication by $E$. Thus, by analogy to the position representation, where the position operator $Q$ acts as multiplication by $x$ and the momentum operator acts as $P = -i\hbar d/dx$, the time operator is usually defined as

$$T\varphi(E) = -i\hbar \frac{d\varphi(E)}{dE}. \quad (4.3)$$

Clearly, $T$ and $H$ satisfy the Heisenberg commutation relation $[T, H] = -i\hbar I$. However, although $T$ is a Hermitian operator (more precisely, $T$ is a symmetric operator), it is not self-adjoint, and therefore $T$ does not contradict Pauli’s theorem. In particular, the Hilbert-space spectrum of $T$ is the whole lower half of the complex plane, including the whole real line. Because its Hilbert-space spectrum is complex, it seems that $T$ should be discarded. However, it has been realized that such operators can be justified if they are understood as POVMs [2–10].

In quantum mechanics, POVMs usually arise whenever the system is embedded in a bath, and we trace out the bath degrees of freedom [16]. By contrast, in the case of the time operator, the POVM arises because there is a lower bound in the energy,
not because we are assuming that the system is embedded in a bath. From the point of view of the quantum theory of measurement, the POVMs of time operators arise because we are performing a continuous measurement on the system [6].

For real values of \( t \), i.e., for values of \( t \) that have zero imaginary part, the normalized eigenfunctions of \( T \), 
\[
\langle -E | t \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{iEt/\hbar},
\]
are not delta-normalized but rather they satisfy
\[
\langle t' | t \rangle = \int_0^\infty dE \langle t' | E^- \rangle \langle -E | t \rangle = \frac{1}{2} \delta(t-t') + \frac{i}{2\pi} \text{P} \frac{1}{t-t'}.
\]
(4.4)

From a calculational point of view, the non-orthogonality of the eigenfunctions of \( T \) for real \( t \) is the main difference with respect to the case of a self-adjoint operator. Everything else, including the use of Dirac’s bra-ket notation, is very similar.

We can use the eigenfunctions \( \langle -E | t \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{iEt/\hbar} \) to construct the time representation of a wave function by way of the Fourier transform [23],
\[
\varphi(t) = \int_0^\infty dE \varphi(E) \frac{1}{\sqrt{2\pi \hbar}} e^{-iEt/\hbar},
\]
(4.5)
which in Dirac’s bra-ket notation reads
\[
\langle t | \varphi \rangle = \int_0^\infty dE \langle t | E^- \rangle \langle -E | \varphi \rangle.
\]
(4.6)

Because for real \( t \) the eigenfunctions of \( T \) form a resolution of the identity,
\[
\int dt \langle t | t \rangle = I,
\]
(4.7)
the probability density of a wave function is normalized to 1, as it should be,
\[
\int dt |\varphi(t)|^2 = 1.
\]
(4.8)

It is important to realize that \( \varphi(t) \) is not the same as the (unmonitored) time evolved state \( \varphi(\tau) = e^{-iH\tau/\hbar} \varphi \) [24]. It is also important to realize that, although the Hilbert-space spectrum of \( T \) contains complex numbers, only real values of \( t \) are used to calculate probabilities. Thus, similar to the case of a Hermitian Hamiltonian that produces resonances [25], the time operator’s Hilbert-space spectrum (which is the lower half of the complex plane) does not coincide with its physical spectrum (which is just the real line).

In the time representation, the operator (4.3) acts as multiplication by \( t \) (at least when it acts on wave functions that satisfy \( \varphi(E = 0) = 0 \)). Because any operator \( A \) acts as multiplication by \( a \) in the \( a \)-representation and \( |\varphi(a)|^2 \) is interpreted as the probability density that the measurement of \( A \) (or, equivalently, the measurement of \( |a\rangle\langle a| \)) on the state \( \varphi \) yields the value \( a \), we can interpret the operator (4.3) as the time operator and \( |\varphi(t)|^2 \) as the probability density that the measurement of \( T \)
(or, equivalently, the measurement of $|t\rangle\langle t|$) on the state $\varphi$ yields the value $t$. Thus, the time representation is just like any representation associated with a self-adjoint operator, except that in the time representation one deals with POVMs (rather than with projective measurements) and with probability densities $|\varphi(t)|^2$ that represent temporal probability distributions of events.

Because $|\varphi(t)|^2$ has dimensions of $1$/time, Eq. (2.7) suggests that when $\varphi$ describes an unstable state, it is not $N(t) = |\varphi(t)|^2$ but rather

$$R(t) \equiv |\varphi(t)|^2 \quad \text{(for one unstable particle)}; \quad (4.9)$$

that is, the decay rate of a single unstable particle is given by the absolute value squared of its wave function in the time representation. If initially we have $N_0$ unstable particles, the decay rate is

$$R(t) \equiv N_0|\varphi(t)|^2 \quad \text{(for $N_0$ unstable particles).} \quad (4.10)$$

Equations (4.9) and (4.10) are simply a rule to calculate probabilities in the time representation. Such rule is essentially the same as the Born rule that we use to calculate probabilities in the representation associated with a self-adjoint operator. In fact, by combining Eqs. (2.3), (2.4), (2.5) and (4.10), we can write the decay and non-decay probabilities in terms of the wave function in the time representation as follows:

$$\frac{dP_d(t)}{dt} = |\varphi(t)|^2 \quad \text{(for one particle)}; \quad (4.11)$$

$$\frac{dP(t)}{dt} = -|\varphi(t)|^2 \quad \text{(for one particle).} \quad (4.12)$$

In the remainder of this paper, we will apply the rule (4.9)-(4.12) to exponential decay, to the interference of two resonances, and to the time of flight of an unstable particle.

To finish this section, we would like to comment on the important issue of non-uniqueness of the time operator. Indeed, if $A$ is an invariant of the motion, $[H, A] = 0$, then the operator $T' \equiv T + \alpha A$ also canonically commutes with $H$, where $\alpha$ is a dimensionful constant that makes $\alpha A$ have dimensions of time. In addition, a given Hamiltonian has several energy representations that, although unitarily equivalent, are physically nonequivalent [10, 26]. One can use the energy representation associated with the “in” Lippmann-Schwinger eigenfunctions $\langle r|E^+ \rangle$ [10, 26], the one associated with the “out” Lippmann-Schwinger eigenfunctions $\langle r|E^- \rangle$ [10, 26], the one associated with the regular solution $\langle r|E \rangle$ of the Schrödinger equation [26, 27], or the one associated with the eigenfunctions $\langle r|E_\alpha \rangle$ that are time-reversal invariant [10]. Thus, we may associate many time operators with a given Hamiltonian, and one must select the most appropriate time operator for the situation at hand. Some selection criteria can be found in Ref. [10]. The reason why we have selected the energy representation associated with the “out” Lippmann-Schwinger eigenfunctions is that, as explained in Ref. [26], such “out” energy representation incorporates the final (or detection) boundary conditions of a scattering experiment, by contrast to the initial (or preparation) conditions of $\langle r|E^+ \rangle$. 

7
5 The Time Representation of a Gamow state

Although the rule (4.9)-(4.12) does not rely on the Gamow states, it is nevertheless enlightening to see what such states tell us about such rule. As shown in Ref. [22], the Gamow state
\[ u(r; z_R) = \langle r | z_R \rangle \]
associated with a resonant energy \( z_R = E_R - i\Gamma_R/2 \) has the following expression in the “out” energy representation:
\[ \langle -E | z_R \rangle = i\sqrt{2\pi} N_R \delta(E - z_R), \]  
where \( \delta(E - z_R) \) is the complex delta function and \( N_R^2 = i \text{res}[S(z)]_{z=z_R} \) is Zeldovich’s normalization factor. The time representation of the Gamow state is easily obtained by Fourier transforming Eq. (5.1):
\[ u(t; z_R) = \langle t | z_R \rangle = \int_0^\infty dE \frac{1}{\sqrt{2\pi\hbar}} e^{-iEt/\hbar} i\sqrt{2\pi} N_R \delta(E - z_R) = \frac{iN_R}{\sqrt{\hbar}} e^{-iz_Rt/\hbar}. \]  

Hence,
\[ R(t) = |u(t; z_R)|^2 = \frac{|N_R|^2}{\hbar} e^{-\Gamma_Rt/\hbar} \]  
(5.3)

Thus, the time representation of the Gamow states yields the exponential law and therefore describes the exponential region of quantum decay. It should be noted that this decay rate is not exactly the same as the decay rate associated with the probability of Eq. (3.3):
\[ R_P(\tau) = -\frac{dP(\tau)}{d\tau} = \frac{\Gamma_R}{\hbar} e^{-\Gamma_R\tau/\hbar}. \]  
(5.4)

6 A single-resonance system

As is well known, quantum mechanics predicts deviations from exponential decay. Such deviations occur because the Gamow state cannot be prepared experimentally: All that can be prepared is a square-integrable wave function \( \varphi(t) \). When one resonance is dominant and we can approximate \( \varphi(t) \) by the Gamow state of the resonance, then one can say that for all practical purposes the decay is purely exponential and the Gamow state is the wave function describing quantum decay. It would be therefore interesting to see what is the exact expression of the decay rate (4.9) when one, and only one, resonance needs to be taken into account. In such a case, we can assume that the \( S \) matrix has one, and only one, pole at the resonant energy \( z_R = E_R - i\Gamma_R/2 \). Because \( S(E) \) has only one pole and because it is unitary, the residue of \( S(E) \) at \( E = z_R \) is given by
\[ \text{res}[S(z)]_{z=z_R} = -i\Gamma_R \]  
(one resonance only).
Hence, $|N_R|^2 = \Gamma_R$ and Eq. (5.3) becomes

$$R(t) = |u(t; z_R)|^2 = \frac{\Gamma_R}{\hbar} e^{-\Gamma_R t/\hbar} = \frac{1}{\tau_R} e^{-t/\tau_R} \quad \text{(a single-resonance system).} \quad (6.2)$$

If initially there are $N_0$ resonances of the same energy $z_R = E_R - i\Gamma_R/2$, the decay rate is

$$R(t) = N_0 |u(t; z_R)|^2 = \frac{N_0}{\tau_R} e^{-t/\tau_R} \quad \text{($N_0$ copies of a single-resonance system).} \quad (6.3)$$

Comparison of Eqs. (6.3) and (2.6) shows that, when only one resonance needs to be taken into account, the Gamow state in the time representation yields the exponential law and the correct initial decay rate: $R_0 = N_0 \lambda = \frac{N_0}{\tau_R}$. If we now plug Eq. (6.3) into Eq. (2.8), we obtain Eq. (2.2). Thus, when a quantum system can be approximated by a lone resonance and its wave function can be approximated by a lone Gamow state, the time representation of the Gamow state provides a complete quantum-mechanical description of the phenomenology of exponential decay. In addition, the probability density associated with the Gamow state is automatically normalized to 1,

$$\int_0^\infty dt |u(t; z_R)|^2 = \int_0^\infty dt \frac{1}{\tau_R} e^{-t/\tau_R} = 1. \quad (6.4)$$

7 The survival probability vs. the non-decay probability

Quantum decay is usually analyzed by way of the survival probability, see for example Refs. [3, 4, 28, 29] and references therein. The survival probability is given by $p_s(\tau) = |a_s(\tau)|^2$, where $a_s(\tau)$ is the survival amplitude [30],

$$a_s(\tau) = \langle \varphi | e^{-iH\tau/\hbar} | \varphi \rangle = \int_0^\infty e^{-iE\tau/\hbar} |\varphi(E)|^2 dE. \quad (7.1)$$

By contrast, in the time representation, the non-decay probability amplitude is

$$\mathcal{A}(t) \equiv \varphi(t) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty e^{-iEt/\hbar} \varphi(E) dE. \quad (7.2)$$

Comparison of Eqs. (7.1) and (7.2) shows several differences between $a_s(\tau)$ and $\mathcal{A}(t)$. First, the survival amplitude is dimensionless, whereas the non-decay amplitude has dimensions of $1/\sqrt{\text{time}}$. Second, the survival amplitude is the Fourier transform of the absolute value squared of the wave function in the energy representation, whereas the non-decay amplitude is the Fourier transform of the wave function in the energy representation. Third, in $a_s(\tau)$ time appears as a parameter, whereas in $\mathcal{A}(t)$ time appears as a random variable. Fourth, as will be further discussed in Section 10.
A(t) seems more suitable to situations in which the system is monitored continuously, whereas \( a_s(\tau) \) seems more suitable to situations where the system evolves freely until the instant \( \tau \), at which instant an instantaneous measurement is made. Fifth, the decay rate \( \dot{\rho}_s(\tau) \) associated with \( a_s(\tau) \) is always zero at \( \tau = 0 \) \([28]\), whereas the decay rate \( R(t) = |\phi(t)|^2 \) is not necessarily zero at \( t = 0 \), as shown in the Appendix by way of an example. Thus, contrary to \( \dot{\rho}_s(\tau) \), the decay rate \( R(t) = |\phi(t)|^2 \) does in general not exhibit the Zeno effect.

There are, however, some analogies between \( a_s(\tau) \) and \( A(t) \): Both \( a_s(\tau) \) and \( A(t) \) yield the exponential decay law when the wave function is the Gamow state, and both yield deviations from exponential decay when the wave function is a properly normalized wave function \( \varphi \). Thus, from a physical point of view, both \( a_s(\tau) \) and \( A(t) \) can describe quantum decay, and one should choose one over the other depending on whether the unstable system is monitored continuously.

8 Interference of two resonances in the time representation

8.1 The GSI anomaly

In 2008, Litvinov et al. \([20]\) observed that K-shell electron capture (EC) decay rates of Hydrogen-like \(^{140}\text{Pr}^{58+}\) and \(^{142}\text{Pm}^{60+}\) ions,

\[
\begin{align*}
^{140}\text{Pr}^{58+} & \rightarrow ^{140}\text{Ce}^{58+} + \nu_e, \\
^{142}\text{Pm}^{60+} & \rightarrow ^{142}\text{Nd}^{60+} + \nu_e,
\end{align*}
\]  

(8.1)

(8.2)

show an oscillatory modulation superimposed on the exponential decay. The decay rate of the GSI anomaly has been fitted with the following equation \([20]\):

\[
\frac{dN_{\text{EC}}(t)}{dt} = N_0 e^{-\lambda t} \lambda_{\text{EC}} \left( 1 + a \cos(\omega t + \phi) \right) ,
\]

(8.3)

where \( N_{\text{EC}} \) is the number of daughter ions \(^{140}\text{Ce}^{58+}\) and \(^{142}\text{Nd}^{60+}\), \( N_0 \) is the number of Hydrogen-like mother ions \(^{140}\text{Pr}^{58+}\) and \(^{142}\text{Pm}^{60+}\), the amplitude \( a \approx 0.20 \), and the period \( T = 2\pi/\omega \approx 7 \) seconds. The data of \([20]\) were obtained by continuously monitoring the decay of individual atoms \([31]\).

There are several theoretical proposals that attempt to explain the oscillations of the GSI anomaly: Refs. \([32, 35]\) use neutrino oscillations; Refs. \([36, 40]\) use the interference of two mass eigenstates; Ref. \([41]\) uses the neutrino spin precession in the static magnetic field of the storage ring; Ref. \([42]\) uses a truncated Breit-Wigner distribution with an energy-dependent width.

Similarly to Refs. \([36, 40]\), we are going to make the assumption that the oscillations of the GSI anomaly are the result of the interference of two mass eigenstates.
Because Litvinov et al. [20] continuously monitored the ions [31], we are going to express the decay rates of $^{140}\text{Pr}^{58+}$ and $^{142}\text{Pm}^{60+}$ in terms of the wave function in the time representation as in Eqs. (4.9) and (4.10).

Before proceeding with the time-representation description of the GSI anomaly, we would like to note that the following results will not explain why the GSI anomaly actually occurs, that is, why Hydrogen-like $^{140}\text{Pr}^{58+}$ and $^{142}\text{Pm}^{60+}$ ions must have two resonances that interfere to produce an oscillation superimposed on exponential decay. What the present paper will show is that if the oscillations of the GSI anomaly were due to the interference of two resonances, then such interference should be analyzed in the time representation.

When $\varphi(t)$ can be approximated by a Gamow state, Eqs. (4.9) and (4.10) lead to the exponential law. However, since in Eq. (8.3) we have an oscillation superimposed on the exponential decay, it seems natural to approximate the wave function $\varphi(t)$ by a superposition of two Gamow states in the time representation,

$$\varphi(t) = \langle t|\varphi \rangle \equiv b_1\langle t|z_1 \rangle + b_2\langle t|z_2 \rangle = b_1 c_1 e^{-i\frac{\pi}{2}t/\hbar} + b_2 c_2 e^{-i\frac{\pi}{2}t/\hbar},$$

where $z_i = E_i - i\Gamma_i/2$, $c_i = \frac{iN_i}{\sqrt{N}}$, $N_i^2 = \text{res}[S(z)]_{z=z_i}$, $b_i$ are the mixing coefficients, and $i = 1, 2$. The absolute value squared of (8.4) yields the following single-particle decay rate:

$$|\varphi(t)|^2 = |b_1 c_1 e^{-i\pi t/\hbar} + b_2 c_2 e^{-i\pi t/\hbar}|^2$$

$$= |b_1|^2|c_1|^2 e^{-\Gamma_1 t/\hbar} + |b_2|^2|c_2|^2 e^{-\Gamma_2 t/\hbar} + 2|b_1||c_1||b_2||c_2|e^{-i\frac{(\Gamma_1 + \Gamma_2)t}{2\hbar}} \cos \left(\frac{\Delta E t}{\hbar} + \delta\right),$$

(8.5)

where $\Delta E = E_1 - E_2$, $c_i = |c_i|e^{-i\delta_i}$, $b_i = |b_i|e^{-i\delta_i}$, and $\delta = \delta_1 - \delta_2 + \delta_1' - \delta_2'$. Equation (8.5) is the most general form for the interference of two Gamow states in the time representation. Such interference will in general produce an exponential decay coupled to an oscillation between the two modes of decay. If the lifetimes of the resonances are not the same, then the oscillation will be damped until the first resonance has decayed, and after that the decay will be essentially exponential through the longer-lived resonance.

Since the amplitude of the oscillation of the GSI anomaly is not damped, in order to reproduce the decay rate (8.3), we are going to assume that the decay widths of the two resonances $z_1$ and $z_2$ are the same (or, equivalently, that their lifetimes are the same):

$$\Gamma_1 = \Gamma_2 = \Gamma.$$

(8.6)

Substituting Eq. (8.6) into the single-particle decay rate (8.5) yields

$$|\varphi(t)|^2 = e^{-\Gamma t/\hbar}(|b_1|^2|c_1|^2 + |b_2|^2|c_2|^2) \left[1 + \frac{2|b_1||c_1||b_2||c_2|}{|b_1|^2|c_1|^2 + |b_2|^2|c_2|^2} \cos \left(\frac{\Delta E t}{\hbar} + \delta\right)\right].$$

(8.7)
By combining Eqs. (2.5), (4.10) and (8.7), we obtain the rate at which \( N_0 \) unstable nuclei decay through two resonances \( z_1 \) and \( z_2 \) of the same width,

\[
\frac{dN_{EC}(t)}{dt} = N_0 e^{-\Gamma t/h} \left[ \frac{b_1}{|b_1|^2} (|c_1|^2 + |c_2|^2) \left[ 1 + \frac{2|b_1||c_1||b_2||c_2|}{|b_1|^2|c_1|^2 + |b_2|^2|c_2|^2} \cos \left( \frac{\Delta E t}{\hbar} + \delta \right) \right] \right].
\]

Comparison of Eqs. (8.8) and (8.3) shows that those two equations are identical if we make the following identifications: \( \lambda = \Gamma/h \), \( \lambda_{EC} = |b_1|^2|c_1|^2 + |b_2|^2|c_2|^2 \), \( \omega = \frac{\Delta E}{\hbar} \), \( \phi = \delta \), and \( a = \frac{2|b_1||c_1||b_2||c_2|}{|b_1|^2|c_1|^2 + |b_2|^2|c_2|^2} \). Thus, the GSI anomaly can be interpreted as the interference of two resonances in the time representation.

### 8.2 Description of the GSI anomaly in terms of the survival probability

In this section, we are going to see that the decay rate of the survival probability of two interfering resonances has mathematical similarities to and physical differences from the decay rate of Eq. (8.8).

Let us consider a wave function that can be approximated by a coherent superposition of two Gamow states with amplitudes \( b_1 \) and \( b_2 \),

\[
|\varphi\rangle \equiv b_1|z_1\rangle + b_2|z_2\rangle.
\]

The survival amplitude of such state is \( [21] \)

\[
a_s(\tau) = \langle \varphi|e^{-iH\tau/h}|\varphi\rangle = |b_1|^2 e^{-iz_1\tau/h} + |b_2|^2 e^{-iz_2\tau/h}.
\]

where we have assumed that the Gamow states are normalized such that \( \langle z_i|z_j\rangle = \delta_{ij} \). The survival probability is given by

\[
p_s(\tau) = |a_s(\tau)|^2 = |b_1|^4 e^{-\Gamma_1\tau/h} + |b_2|^4 e^{-\Gamma_2\tau/h} + 2|b_1|^2|b_2|^2 e^{-\frac{(\Gamma_1+\Gamma_2)\tau}{2\hbar}} \cos \left( \frac{\Delta E \tau}{\hbar} \right).
\]

Since Litvinov et al. [20] did not measure a probability but rather a decay rate, we need to calculate the decay rate associated with \( p_s(\tau) \) when initially there are \( N_0 \) unstable ions,

\[
\frac{dN_s(\tau)}{d\tau} = -N_0 \frac{dp_s(\tau)}{d\tau} = N_0 \Gamma e^{-\Gamma \tau/h} \left( |b_1|^4 + |b_2|^4 \right) \left[ 1 + \frac{2|b_1|^2|b_2|^2}{|b_1|^4 + |b_2|^4} \left[ \cos \left( \frac{\Delta E \tau}{\hbar} \right) + \frac{\Delta E}{\Gamma} \sin \left( \frac{\Delta E \tau}{\hbar} \right) \right] \right],
\]

where we have assumed that \( \Gamma_1 = \Gamma_2 = \Gamma \). If we define \( A e^{-i\psi} \equiv 1 + i\frac{\Delta E}{\hbar} \), then Eq. (8.12) can be written as

\[
\frac{dN_s(\tau)}{d\tau} = N_0 \Gamma e^{-\Gamma \tau/h} \left( |b_1|^4 + |b_2|^4 \right) \left[ 1 + \frac{2|b_1|^2|b_2|^2}{|b_1|^4 + |b_2|^4} A \cos \left( \frac{\Delta E \tau}{\hbar} + \psi \right) \right].
\]
Comparison of Eqs. (8.13) and (8.3) shows that the survival probability can also account for the GSI anomaly if we define
\[ \lambda_{\text{EC}} = \Gamma \hbar (|b_1|^4 + |b_2|^4), \quad \lambda = \Gamma \hbar, \quad a = \frac{2|b_1|^2|b_2|^2}{|b_1|^2 + |b_2|^2} A, \]
and \( \phi = \psi \).

Mathematically, Eqs. (8.8) and (8.13) both consist of an oscillation superimposed on exponential decay. Physically, however, Eqs. (8.8) and (8.13) have several differences. First, the coefficients needed by Eqs. (8.8) and (8.13) in order to account for Eq. (8.3) are different and, in principle, experimentally distinguishable. For example, if we were able to vary the mixing coefficients \( b_1 \) and \( b_2 \) at will, Eqs. (8.8) and (8.13) would yield distinguishable decay rates. Second, in Eq. (8.8) time appears as a random variable, whereas in Eq. (8.13) time appears as a parameter. Third, as we will further discuss in Section 10, Eq. (8.8) is based on the assumption that the system is monitored continuously (as is the case of the GSI anomaly), whereas Eq. (8.13) assumes that the system evolves freely up to the instant \( \tau \), at which instant an instantaneous measurement is made.

### 8.3 The Quantum-Beat description of the GSI anomaly

Assuming that the GSI anomaly is due to the interference of two resonances in the time representation is very close to assuming that such anomaly is due to the “quantum beats” of two exponentially decaying mass eigenstates [36–40]. It seems therefore pertinent to compare the quantum-beat approach with Eq. (8.8).

Instead of the survival probability \( p_s(\tau) \), in the quantum-beat approach one obtains the following transition probability of electron capture at time \( \tau \) [36]:

\[ P_{\text{QB}}(\tau) = \bar{P} e^{-\Gamma \tau/\hbar} \left[ 1 + b \cos \left( \frac{\Delta E \tau}{\hbar} + \delta \right) \right], \quad (8.14) \]

where \( \bar{P} \), \( b \) and \( \delta \) are constants. The decay rate associated with \( P_{\text{QB}}(\tau) \) is given by

\[ \frac{dN_{\text{QB}}(\tau)}{d\tau} = N_0 \bar{P} e^{-\Gamma \tau/\hbar} \left[ \frac{\Gamma}{\hbar} \left( 1 + b \cos \left( \frac{\Delta E \tau}{\hbar} + \delta \right) \right) + b \frac{\Delta E}{\hbar} \sin \left( \frac{\Delta E \tau}{\hbar} + \delta \right) \right]. \quad (8.15) \]

If we define \( B e^{-i \psi} = b + i b \frac{\Delta E}{\hbar} \), Eq. (8.15) becomes

\[ \frac{dN_{\text{QB}}(\tau)}{d\tau} = N_0 \bar{P} \frac{\Gamma}{\hbar} e^{-\Gamma \tau/\hbar} \left[ 1 + B \cos \left( \frac{\Delta E \tau}{\hbar} + \delta + \psi \right) \right]. \quad (8.16) \]

Thus, the quantum-beat approach accounts for Eq. (8.3) but with different (and, in principle, experimentally distinguishable) coefficients than those of Eq. (8.8). In addition, similarly to the survival probability, the quantum-beat approach treats time as a parameter and implicitly assumes that the system is not monitored continuously.

It should be noted that molecular and atomic fluorescence quantum beats are studied using an equation similar to Eq. (8.14), see for example Ref. [43]. Therefore, when
the system is monitored continuously in an atomic or molecular quantum-beat experiment (as, for example, in Ref. [44]), the theoretical description of quantum beats may have to be done in the time representation [45].

9 Neutral-meson decay

Similarly to Hydrogen-like \(^{140}\text{Pr}^{58+}\) and \(^{142}\text{Pm}^{60+}\) ions, the decays of \(K\) and \(B\) mesons exhibit oscillations superimposed on exponential decay. In this section, we are going to see how the oscillations of neutral mesons can be described in the time representation (for a somewhat related approach, see Refs. [46,47]).

In the Lee-Oehme-Yang model of the kaon system, the mass operator has two mass eigenstates \(|K_L⟩\) and \(|K_S⟩\) with complex eigenvalues \(z_L = m_L c^2 - i\Gamma_L/(2\hbar)\) and \(z_S = m_S c^2 - i\Gamma_S/(2\hbar)\). Thus, \(|K_L⟩\) and \(|K_S⟩\) are two Gamow states. The \(|K^0⟩\) and \(|\bar{K}^0⟩\) can be written in terms of such Gamow states as follows

\[
|K^0⟩ = \frac{1}{\sqrt{2}} (|K_S⟩ + |K_L⟩),
\]

\[
|\bar{K}^0⟩ = \frac{1}{\sqrt{2}} (|K_S⟩ - |K_L⟩).
\]

The time representation of Eqs. (9.1) and (9.2) reads as

\[
\varphi|K^0⟩(t) = \frac{1}{\sqrt{2}} (c_S e^{-izLt/\hbar} + c_L e^{-izLt/\hbar}),
\]

\[
\varphi|\bar{K}^0⟩(t) = \frac{1}{\sqrt{2}} (c_S e^{-izLt/\hbar} - c_L e^{-izLt/\hbar}),
\]

where \(c_i = i\sqrt{\frac{N_i}{N}}, N_i^2 = i \text{res}[S(z)]_{z=z_i},\) and \(i = S, L\). We can derive the decay rates for the neutral kaons in complete analogy to the way we derived the decay rate (8.8) from Eq. (8.4). If we start off with a pure \(|K^0⟩\) beam at \(t = 0\), then the decay rate is given by

\[
\frac{dN_{|K^0⟩}}{dt} = \frac{N_0}{2} \left[ |c_S|^2 e^{-\Gamma_S t/\hbar} + |c_L|^2 e^{-\Gamma_L t/\hbar} + 2|c_S||c_L| e^{-\frac{(\Gamma_S + \Gamma_L)t}{2\hbar}} \cos \left( \frac{\Delta E t}{\hbar} + \delta \right) \right].
\]

If we start off with a pure \(|\bar{K}^0⟩\) beam at \(t = 0\), then the decay rate is

\[
\frac{dN_{|\bar{K}^0⟩}}{dt} = \frac{N_0}{2} \left[ |c_S|^2 e^{-\Gamma_S t/\hbar} + |c_L|^2 e^{-\Gamma_L t/\hbar} - 2|c_S||c_L| e^{-\frac{(\Gamma_S + \Gamma_L)t}{2\hbar}} \cos \left( \frac{\Delta E t}{\hbar} + \delta \right) \right].
\]

Equations (9.5) and (9.6) have the same form as those used in the literature to calculate the time dependence of the decay rates of \(|K^0⟩\) and \(|\bar{K}^0⟩\) (see, for example, Eqs. (7.56), (7.57) and (7.64) in Ref. [48], or Eq. (3) in Ref. [49]). A similar procedure can be applied to the decay of B mesons [50].
Thus, if in an experiment measuring the decay of neutral mesons the particles were monitored continuously, one could describe such decay in a time representation that is obtained by Fourier transforming the representation where the mass operator is diagonal [51].

10 Continuous measurements and the Zeno effect

In this section, we are going to compare the procedure to measure the survival probability \( p_s(\tau) \) with the procedure to measure the non-decay probability \( \mathcal{P}(t) \). From such comparison we will conclude that \( \mathcal{P}(t) \) is more suitable than \( p_s(\tau) \) to model the decay of a particle that is monitored continuously.

10.1 Measurement of the survival probability

Let a quantum system be initially prepared in the state \( \varphi \). It is usually assumed that the probability that the system remains in the state \( \varphi \) after a time \( \tau_0 \) is given by the survival probability, \( p_s(\tau_0) = |\langle \varphi | e^{-iH\tau_0/\hbar} | \varphi \rangle|^2 \). If we want to measure the survival probability at \( \tau = \tau_0 \), we need to prepare the system in the state \( \varphi \) at \( \tau = 0 \), let the system evolve unmonitored until \( \tau = \tau_0 \), and finally make an instantaneous measurement at \( \tau = \tau_0 \). We then say that we have measured the observable \( \mathcal{P} = |\varphi\rangle\langle\varphi| \) on the state \( e^{-iH\tau_0/\hbar}\varphi \).

If we want to measure the survival probability at time \( \tau = 2\tau_0 \), we prepare the system in the state \( \varphi \) at \( \tau = 0 \), let the system evolve unmonitored till \( \tau = 2\tau_0 \), and finally, without performing any measurement prior to \( \tau = 2\tau_0 \), perform an instantaneous measurement at \( \tau = 2\tau_0 \). We then say that we have measured the observable \( \mathcal{P} = |\varphi\rangle\langle\varphi| \) on the state \( e^{-iH2\tau_0/\hbar}\varphi \).

Thus, in order to measure \( p_s(\tau) \), we must perform a different experiment for each instant of time \( \tau \). All these measurements are projective measurements, since the observable we are measuring is the projection \( \mathcal{P} = |\varphi\rangle\langle\varphi| \).

10.2 Pulsed and continuous measurements

Let us now assume that we perform a pulsed measurement, that is, we prepare the system in the state \( \varphi \) at \( \tau = 0 \), and then measure the probability that the system remains in the state \( \varphi \) at times \( \tau_0, 2\tau_0, 3\tau_0 \), and so on. Due to the reduction postulate, the probabilities \( p_{\text{pulsed}}(n\tau_0) \) that we will obtain at times \( n\tau_0, n = 2, 3, \ldots \), will in general be different from the survival probabilities \( p_s(n\tau_0), n = 2, 3, \ldots \). It has been found both theoretically [52–64] and experimentally [65–67] that the probabilities \( p_{\text{pulsed}}(n\tau_0) \) can be larger or smaller than \( p_s(n\tau_0) \). When they are larger, we say that the evolution (or decay, in the case of an unstable system) is hindered by the measurement, and we refer to it as the Zeno effect. When they are smaller, we say that the evolution of the system is sped up, and we refer to it as the anti-Zeno effect. The Zeno effect was observed.
experimentally for the first time in Ref. [65] for Rabi oscillations, and both the Zeno and anti-Zeno effects were first observed for a decaying system in Ref. [66].

Finally, let us consider the case in which the pulsed measurement is so frequent that it can be assumed to be a continuous measurement. In such a case, due to the reduction postulate, the measurements continuously collapse the wave function to the state \( \phi \), and the evolution slows down to a stop. This case, which was first observed for Rabi oscillations in Ref. [65], is usually referred to either as the Zeno effect, as the "watchdog effect," or as the "watched pot never boils" effect.

### 10.3 Measuring the non-decay probability vs. measuring the survival probability

We have seen that the measurement of the survival probability is a projective measurement in which the observable \( |\phi\rangle\langle\phi| \) is measured on the state \( e^{-iHt/\hbar}\phi \). A position-representation analog is the measurement of \( |x\rangle\langle x| \) on the state \( e^{-iHt/\hbar}\phi \) to obtain the probability density \( |\psi(x;\tau)|^2 \) that the state \( \phi \) is found at position \( x \) at time \( \tau \). When we measure \( |\psi(x;\tau)|^2 \) or \( p_s(\tau) \), we have complete control over the instant \( \tau \) at which the measurement is performed. The detectors are turned off prior to the instant \( \tau \), and when such instant arrives, we perform an instantaneous measurement. When we measure \( p_s(\tau) \), the question to be answered is, "what is the probability that the system has not decayed at time \( \tau ? \)" When we measure \( |\psi(x;\tau)|^2 \), the question to be answered is, "what is the probability density that the system is found at position \( x \) at time \( \tau ? \)"

By contrast, when we measure the probability density (decay rate, in the case of a resonance) \( |\psi(t)|^2 \), we measure \( |t\rangle\langle t| \) on the state \( \phi \). Because of Eq. (4.4), \( |t\rangle\langle t| \) is not a projection and therefore the measurement of \( |\psi(t)|^2 \) is not a projective measurement but rather a POVM.

When we measure \( \mathcal{P}(t) \), we have no control over the time at which the decay will occur, because the decay occurs at a random time (hence the need to promote time to a random variable). When we measure \( \mathcal{P}(t) \), we need to monitor the decay of the unstable system continuously, or else we may miss the moment when it decays. When we measure \( \mathcal{P}(t) \), the question to be answered is, “at what time does the decay event happen, and with what probability?”

Because the POVM \( |t\rangle\langle t| \) is associated with the continuous random variable \( t \), because \( t \) is an eigenvalue of a time operator, and because the POVM \( |t\rangle\langle t| \) is generated by a time operator, it seems reasonable to interpret the measurement of \( |t\rangle\langle t| \) on \( \phi \) (i.e., the measurement of \( \mathcal{P}(t) \)) as a continuous measurement.

Actually, continuous measurements are inherent to the nature of many time operators, the prototypical example being the time-of-arrival operator. When one measures the probability that a particle arrives at a given position, one needs to monitor the arrival of the particle at all times, or else one may miss the moment when the particle arrives [68].
10.4 The GSI experiment

In Ref. [20], Litvinov et al. continuously measured the decay of the ions. If Litvinov et al. were measuring the survival probability, by the “watched pot never boils” effect, the ions would not decay. However, the ions of the GSI experiment eventually do decay, and therefore the survival probability does not seem to be the quantity measured in Ref. [20]. By contrast, the non-decay probability \( P(t) \) seems a natural quantity to analyze the GSI anomaly, because \( P(t) \) can both model continuous measurements and account for oscillations superimposed on exponential decay.

There is a definitive test that would allow us to determine whether or not Litvinov et al. measured \( p_s(\tau) \). If they were able to measure the decay rate at \( t = 0 \) [69], and if such measurement yielded a non-zero initial decay rate, then we would know for sure that Litvinov et al. cannot possibly be measuring the survival probability.

10.5 Effect of the measurement on the state

When we perform a projective measurement, the effect of the measurement on the state is taken into account by the reduction postulate. In addition, one can model the measuring apparatus by way of a quantum-mechanical Hamiltonian, see for example Refs. [57–60].

It would be interesting to also account for the effect of the measurement of \( P(t) \) on the state. However, the measurement of \( P(t) \) is a POVM, for which there does not seem to exist a simple, succinct answer as to what the state is after the measurement, or as to how to model the detector with a quantum-mechanical Hamiltonian (see Ref. [5], and Chapter 3 of Ref.[17]). Thus, the results of the present paper simply provide a rule to calculate the probability for a resonance to decay when such resonance is monitored continuously, without explicitly taking the effect of the apparatus into account.

11 Time of flight

Times of flight are routinely measured in the lab, and they seem to have a dynamical character. In this section, we are going to use the time representation to construct a quantum-mechanical description of the time of flight.

Let us assume that an experimenter can measure the time of flight of an unstable particle by measuring, for example, the length of the trails left by the particle in a bubble or spark chamber. If \( v \) is the speed of the particle and \( d_i \) is the length of the trail, then the time of flight is just \( t_i = d_i/v \), where \( i \) labels the trails left by the particle in different, successive experiments. The times \( t_i \) are random, dynamical times, not parametric times over which the experimenter has complete control. The randomness of \( t_i \) arises from the seemingly irreducible randomness of quantum decay: You cannot predict when an individual particle is going to decay, all you can predict
is the probability for such a decay to occur.

Let us assume that each time of flight \( t_i \) is obtained \( N_i \) times when we repeat the same experiment \( N_0 \) times (or, equivalently, when we perform a single experiment with \( N_0 \) particles that decay independently of each other). Because the times \( t_i \) are random, the average time of flight should be given by the mean of the corresponding probability distribution,

\[
\langle t \rangle = \sum \frac{N_i}{N_0} t_i \equiv \sum p_i t_i ,
\]

(11.1)

where \( p_i \) is the probability to measure \( t_i \). In the limit that \( N_0 \) is very large, we obtain a continuous probability distribution \( p(t) \), \( p_i \) tends to \( \frac{dp(t)}{dt} \) \( dt \), and Eq. (11.1) tends to

\[
t_{\text{flight}} = \int t \frac{dp(t)}{dt} \, dt .
\]

(11.2)

By assuming that \( p(t) \) coincides with \( \mathcal{P}_d(t) \), and by using rule (4.11), we obtain the following expression:

\[
t_{\text{flight}} = \int t |\varphi(t)|^2 \, dt ;
\]

(11.3)

that is, the time of flight of a particle is just the mean (or first moment) of the probability distribution associated with the time representation of the wave function. In the case of a single-resonance system, the wave function is given by the Gamow state. By combining Eq. (6.2), Eq. (11.3), and the fact that \( \int dx \, xe^{-x} = -e^{-x}(x + 1) \), we obtain

\[
t_{\text{flight}} = \int_0^\infty t |u(t; z_R)|^2 \, dt = \frac{\hbar}{\Gamma_R} = \tau_R ;
\]

(11.4)

that is, rule (4.9)-(4.12) implies that the time of flight of a resonance is the same as its lifetime. Because we have obtained Eqs. (11.3) and (11.4) by way of the time representation, we can say that we have endowed the time of flight with a dynamical character [70].

12 Conclusions

We have presented an analysis of quantum decay in which the decay of a single unstable particle is described by a single wave function in the time representation, \( \varphi(t) \). Mathematically, \( \varphi(t) \) is the Fourier transform of the wave function in the energy representation. The decay rate is given by the absolute value squared of \( \varphi(t) \). Mathematically, the decay rate is just the probability distribution generated by the POVM of a time operator. The resulting non-decay probability appears as a natural replacement for the survival probability in situations where the system is monitored continuously. When the analysis is applied to the Gamow state, one recovers all the phenomenological features of exponential decay, including the dynamical role played by time. In addition,
\( \varphi(t) \) provides a simple way to model quantum measurements that monitor the system continuously.

We have applied the analysis to the interference of two resonances, a phenomenon that occurs in a wide variety of energy ranges, from atomic and molecular fluorescence quantum beats, to neutral mesons, to (possibly) the GSI anomaly. When an unstable system can decay through two different resonances, it oscillates between them, and the ensuing decay rate is given by an oscillation superimposed on the exponential decay. We have argued that when the interfering resonances are monitored continuously, as is the case of the GSI anomaly, the decay rate should be given in terms of the wave function in the time representation, Eq. (8.8), rather than in terms of the decay rate of the survival probability, Eq. (8.13), or in terms of the decay rate of Eq. (8.16). Theoretically, Eq. (8.8) has two main advantages. First, it endows time with a dynamical character. Second, it explains why the system decays even though it is monitored continuously.

Although mathematically Eqs. (8.8), (8.13), and (8.16) are very similar, they can in principle be distinguished experimentally. In particular, if the experiment of Ref. [20] measured the decay rate around \( t = 0 \), we could find out whether such experiment should be described by Eq. (8.8) or by Eq. (8.13).

We have also introduced an expression for the time of flight of a quantum particle as the mean of the probability distribution of the wave function in the time representation. From such expression there follows that the time of flight of a resonance coincides with its lifetime.

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**Appendix**

Let \( \varphi(E) = \sqrt{\frac{2\alpha}{\hbar}} e^{-E\alpha/\hbar} \), where \( \alpha > 0 \). The time representation of \( \varphi(E) \) is [23]

\[
\varphi(t) = \int_{0}^{\infty} dE \varphi(E) \frac{1}{\sqrt{2\pi\hbar}} e^{-iEt/\hbar} = \sqrt{\frac{\alpha}{\pi}} \frac{1}{\alpha + it}.
\]

The decay rate associated with the time representation is then given by

\[
R(t) = |\varphi(t)|^2 = \frac{\alpha}{\pi(\alpha^2 + t^2)}.
\]
For the same state $\varphi(E) = \sqrt{\frac{2\alpha}{\hbar}} e^{-E\alpha/\hbar}$, the survival amplitude is

$$a_s(\tau) = \int_0^\infty e^{-iE\tau/\hbar} |\varphi(E)|^2 dE = \frac{2\alpha}{2\alpha + i\tau}.$$  \hspace{1cm} (A.3)

The survival probability and its corresponding decay rate are

$$p_s(\tau) = |a_s(\tau)|^2 = \frac{4\alpha^2}{4\alpha^2 + \tau^2},$$  \hspace{1cm} (A.4)

$$\dot{p}_s(\tau) = -\frac{8\alpha^2 \tau}{(4\alpha^2 + \tau^2)^2}.$$  \hspace{1cm} (A.5)

From the comparison of Eq. (A.2) with Eq. (A.5) it follows that $|\varphi(t)|^2 \neq -\dot{p}_s(t)$. In particular, $R(0) = 1/(\pi \alpha) \neq 0$, whereas $\dot{p}_s(0) = 0$.

Thus, the wave function $\varphi(E) = \sqrt{\frac{2\alpha}{\hbar}} e^{-E\alpha/\hbar}$ shows that in general the decay rate of the non-decay probability $\mathcal{P}(t)$ is different from the decay rate of the survival probability $p_s(\tau)$.

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[21] Whenever time has a dynamical character, we will denote it by \( t \), but when it plays the role of a parameter, we will denote it by \( \tau \).

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[23] Strictly speaking, we should attach a “minus” sign superscript to the wave function, \( \varphi \equiv \varphi^\times \), in order to specify that we are using the energy representation associated with the “out” Lippmann-Schwinger equation. In addition, we should attach a subscript to the wave function, in order to indicate whether we are dealing with the position (e.g., \( \varphi_{\text{pos}}(r) \)), the energy (e.g., \( \varphi_{\text{en}}(E) \)), or the time (e.g., \( \varphi_{\text{time}}(t) \)) representation of \( \varphi \), since in general the expression of a given wave function \( \varphi \) is different in each representation. However, for the sake of simplicity, we will drop such minus superscript, and we will simply write \( \varphi(r) \), \( \varphi(E) \) and \( \varphi(t) \) to denote the wave function in the position, energy and time representations.

[24] We can nevertheless calculate the parametric time evolution of \( \varphi(t) \), the result of which is \( e^{-iH\tau/\hbar}\varphi(t) = \varphi(t + \tau) \).
Because the Hilbert-space spectrum of a Hermitian Hamiltonian is real, it includes only the bound and the scattering spectra, thereby discarding the resonant spectrum as unphysical. However, radioactive nuclei and unstable elementary particles are physical, and they are included in the periodic table of the elements and in the particle data table. Therefore, when a Hermitian Hamiltonian produces resonances, its physical spectrum (which includes the complex resonant energies) is not the same as its Hilbert-space spectrum (which does not include the resonant energies).

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In Ref. [26], the eigenfunction \( \langle r | E \rangle \) was called the standing-wave eigenfunction, which is a misnomer.

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Although no real measurement truly monitors the system continuously, we can consider a measurement to be continuous whenever it monitors the system frequently enough. Such seems to be the case of Ref. [20], because the ions revolve in the ring with a period of the order of \( 10^{-6} \) s and are monitored nondestructively once per revolution.

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[69] In Ref. [20], the decay rate is measured only after the first few seconds. The first few seconds are needed for cooling.

[70] Equations (11.3) and (11.4) also show that there exist physically relevant quantities, such as the time of flight, that are not associated with a single eigenvalue of a time operator but rather with a mean value. See also review [12], where tunneling times are defined by way of mean values.