Casimir Energy for a Wedge with Three Surfaces and for a Pyramidal Cavity

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Abstract

Casimir energy calculations for the conformally coupled massless scalar field for a wedge defined by three intersecting planes and for a pyramid with four triangular surfaces are presented. The group generated by reflections are employed in the formulation of the required Green functions and the wave functions.

I. Introduction

Having new geometries in hand for which we can evaluate the Casimir energies is of interest: We learn more about the phenomena itself, and hope that experimentalists may realize some of the geometries to measure the effect. At this point we like to emphasize that all experiments so far performed are of two body ones, a single cavity measurement has not been done. [1]

For geometries with planar boundaries, if the planes are parallel or perpendicular to each other, the method of images is easily applicable. The parallel plates and in general rectangular prisms of any dimensions are of that type [2]. For these geometries the groups generated by reflections are abelian. If the walls of the cavity are not perpendicularly intersecting, the reflection groups are not commutative. This was the case for a previously studied triangular region [3]. If the reflections in the geometry we consider generate a non-abelian group, one has to study the structure of this group, for it is essential in the construction of the required Green functions and wave functions. For example in the present study the octahedral group provides the basic tools for the calculations.

In coming section we first calculate the Casimir energy density for the massless scalar field for a wedge with three boundary surfaces.

Section III is devoted to the calculation of the Casimir energy in a pyramidal cavity with four triangular surfaces. We get a positive result.

The corresponding groups generated ( related to the Octahedral group ) by reflections play vital role in the construction of the Green function and

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the wave functions. The required group theoretical details are given in the appendices.

II. Pyramidal wedge

Consider the region in the first quadrant \((x_1 > 0, x_2 > 0, x_3 > 0)\) inside the following three planes (Figure)

\[
P_1: x_1 = x_3, \quad P_2: x_2 = 0, \quad P_3: x_1 = x_2.
\]

(1)

Reflection operators with respect to these planes are

\[
q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

(2)

A group \(G\) of order 48 is generated by the above reflections. Its elements are \(g_j\) and \(ig_j\; j = 0, \ldots, 23\). Here 24 elements \(g_j\) form the Octahedral group (see Appendix A) and \(i = -1\) is the generator of the inversion group.

Consider the function

\[
K(x, y) = \sum_{j=0}^{23} [G(g_j x, y) - G(ig_j x, y)],
\]

(3)

where \(G(x, y)\) is the Green function for the massless scalar field in the Minkowski space

\[
G(x, y) = \frac{1}{4\pi^2} \frac{1}{|x - y|^2}.
\]

(4)

Here \(x\) and \(y\) are four vectors with interval \(|x - y|^2 = |\vec{x} - \vec{y}|^2 - (x_0 - y_0)^2\); and, \(g_j\) act only on the spatial components. To check the boundary conditions let us consider (with \(q_1 = ig_{21}\))

\[
K(q_1 x, y) = \sum_{j=0}^{23} [G(ig_j g_{21} x, y) - G(g_j g_{21} x, y)].
\]

(5)

Since for any \(j\) we have the element \(g_k = g_j g_{21}\) in \(G\), (5) is equal to

\[
K(q_1 x, y) = \sum_{k=0}^{23} [G(ig_k x, y) - G(g_k x, y)]
\]

(6)

which implies

\[
K(q_1 x, y) = -K(x, y).
\]

(7)

In a similar fashion one can verify the antisymmetry property for the elements \(q_2\) and \(q_3\). Therefore the function \(K(x, y)\) satisfies Dirichlet boundary conditions on the planes of \(P_1, P_2, P_3\).

Note that in order the equation (with \(\eta = \text{diag}(-1, 1, 1, 1)\))

\[
\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} K(x, y) = \delta(x - y)
\]

(8)
to be satisfied by (3), every point in the region between the planes $P_1, P_2$ and $P_3$ must represent different orbits under the action of the group $G$, which is indeed the case. In other words the region we consider is the fundamental domain of the group $G$ (see Appendix B).

To obtain the energy momentum tensor for the conformally coupled massless scalar field we employ the well known coincidence limit formula [4]

$$T_{\mu\nu} = \lim_{x \to y} \frac{2}{3} \delta_{\mu}^{\rho} \delta_{\nu}^{\sigma} \left[ \partial^\rho \partial_\nu + \partial_\mu \partial_\nu \right] - \frac{\eta_{\mu\nu}}{6} \eta^{\sigma\rho} \partial_\sigma \partial_\nu + \frac{\eta_{\mu\nu}}{24} \eta^{\sigma\rho} \left( \partial_\sigma \partial_\rho + \partial_\sigma \partial_\rho \right) \xi(x, y)$$

(9)

The energy density $T(x) = T_{00}$ is given by:

$$T(x) = \frac{1}{12\pi^2} \sum_{j=1}^{23} [T(g_j) - T(ig_j)]$$

(10)

where

$$T(g) = \frac{1}{|\bar{\eta}|^4} - 2 \left[ \frac{|(1 + g)|\bar{\eta}|^2}{|\bar{\eta}|^6} \right]$$

(11)

and

$$\bar{\eta} = (1 - g)x$$

(12)

with $g$ standing for $g_j$ and $ig_j$.

Using the invariance of $T(g)$ under the $g \to g^{-1}$ we have

$$T(g_1) = T(g_3)$$
$$T(g_4) = T(g_6)$$
$$T(g_7) = T(g_9)$$
$$T(g_{10}) = T(g_{11})$$
$$T(g_{12}) = T(g_{13})$$
$$T(g_{14}) = T(g_{15})$$
$$T(g_{16}) = T(g_{17})$$

(13)

The same is true for elements $ig_j$, with $j$ running the same values as (13).

• For $g_1$ which is rotation by angle $\frac{\pi}{2}$ around the $x_2$-axis we have

$$T(g_1) = -\frac{1}{(x_1^2 + x_3^2)^2}$$

(14)

$$T(ig_1) = -\frac{3(x_1^2 + x_3^2) + 2x_2^2}{2(x_1^2 + x_3^2 + x_2^2)^3}$$

(15)

$g_4$ and $g_6$ are the rotations by the same angle around the $x_1$-axis and $x_3$-axis respectively. Therefore, $T(g_4), T(ig_4)$ and $T(g_6), T(ig_4)$ are obtained from (14) and (15) with the cyclic replacements of coordinates $(x_1, x_2, x_3) \to (x_3, x_1, x_2)$ and $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$ respectively.

• For the rotation $g_{12}$ by angle $\frac{2\pi}{3}$ around the line passing trough the origin and the point $(1, -1, 1)$ we have

$$T(g_{12}) = -\frac{3}{((x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_3 - x_1)^2)^2}$$

(16)
\[ T(\text{id}_{12}) = - \frac{6|\vec{x}|^2 + 2(x_1 x_3 - x_1 x_2 - x_2 x_3)}{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 + x_1)^2} \] (17)

Since \( g_{14} \) and \( g_{16} \) are rotation matrices by the same angle around the axis passing through the origin and the points \((-1,1,1)\) and \((1,1,-1)\) we conclude that \( T(g_{14}) \), \( T(\text{id}_{14}) \) and \( T(g_{16}), T(g_{16}) \) are given by (10) and (11) with the cyclic replacements of coordinates \((x_1, x_2, x_3) \rightarrow (x_3, x_1, x_2)\) and \((x_1, x_2, x_3) \rightarrow (x_2, x_3, x_1)\) respectively.

- We also have

\[ T(g_{10}) = -\frac{3}{((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2)} \] (18)
\[ T(\text{id}_{10}) = -\frac{6|\vec{x}|^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)}{(x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2} \] (19)

- For elements satisfying the condition \( g^2 = 1 \) the second expression in (11) vanishes. These are the elements \( g_j, j = 2, 5, 8 \) and 18, \ldots, 23. Since \( tr(\text{id}_j) = 1 \) we have \( T(\text{id}_j) = 0 \). Nonzero one is

\[ T(i) = -\frac{1}{4|\vec{x}|^4}. \] (20)

For rotations \( g_{20}, g_{21} \) and \( g_2 \) we get

\[ T(g_{20}) = -\frac{1}{2((x_1 - x_3)^2 + 2x_2^2)} \] (21)
\[ T(g_{21}) = -\frac{1}{2((x_1 + x_3)^2 + 2x_2^2)} \] (22)

and

\[ T(g_2) = -\frac{1}{8(x_1^2 + x_3^2)}. \] (23)

Remaining six terms \( T(g_{18}), T(g_{19}), T(g_{22}), T(g_{23}), T(g_5) \) and \( T(g_8) \) are obtained from the above three equations by the cyclic replacements of coordinates.

Energy density (10) is given by

\[ T(x) = \frac{1}{12\pi^2}[T(g_{10}) - T(\text{id}_{10}) - T(i)] + \]
\[ +\frac{1}{12\pi^2}[\frac{17}{8}T(g_1) - 2T(\text{id}_1) + 2T(g_{12}) - 2T(\text{id}_{12}) + T(g_{20}) + T(g_{21}) + \text{c.p.}] \] (24)

where c.p. stands for cyclic permutations of coordinates. The system we consider is the intersection region of three wedges \((P_1, P_2), (P_2, P_3)\) and \((P_1, P_3)\).

For \( x_3 \gg 1 \) and \( \sqrt{x_1^2 + x_2^2} \ll x_3 \) our result should reduce to the wedge problem \((P_2, P_3)\). Recall, that energy density in the wedge between two inclined planes is [5]

\[ T_W = -\frac{1}{1440\pi^2 r^4}(\frac{\pi^4}{\alpha^4} - 1), \] (25)
where $\alpha$ is the angle between two planes and $r$ is the minimal distance to the axis which is intersection of two planes. For the system $(P_2, P_3)$ we have $\alpha = \frac{\pi}{4}$ and $r^2 = x_1^2 + x_2^2$, that is \[ T_{P_2P_3} = -\frac{255}{1440\pi^2(x_1^2 + x_2^2)^2}. \] (26)

Coming to our density \[ T(x) \simeq -\frac{17}{12\pi^2 \cdot 8} T(g_0) = -\frac{17}{96\pi^2(x_1^2 + x_2^2)^2} \] (27) which is same as \[ T(g_0). \] In a similar fashion, terms $T(g_{20})$ and $T(g_{10})$ correspond in the suitable limits to the wedge problems $(P_1P_2)$ and $(P_1P_3)$.

### III. Pyramid

We add to the planes $P_1, P_2, P_3$ the fourth one $P_4 : x_3 = a$. Reflection with respect to the $P_4$ plane is given by

$$ q_4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 2 - x_3 \end{pmatrix} $$

(28)

The group generated by $q_j, j = 1, 2, 3, 4$ is the semidirect product of the group $G$ defined in the previous section and the translation group $Z^3$. The Green function vanishing on the planes $P_a, = 1, \ldots, 4$ is given by

$$ K(x, y) = \sum_{n,m,k=-\infty}^{\infty} \sum_{j=1}^{23} [G(gx + \xi, y) - G(gx + \xi, y)] $$

(29)

where

$$ \xi = \begin{pmatrix} 0 \\ 2na \\ 2ma \\ 2ka \end{pmatrix} $$

(30)

The energy momentum density is

$$ T(x) = \frac{1}{6\pi^2} \sum_{n,m,k=-\infty}^{\infty} \sum_{j=1}^{23} [T(g_j) - T(ig_j)] $$

(31)

where $T(g)$ is given by (14) with the replacement $\vec{\eta} \rightarrow \vec{\eta} + \vec{\xi}$, where $\vec{\xi}$ is the spatial part of the four dimensional vector $\xi$. We calculate explicitly the total vacuum energy of the pyramid. For the geometry in hand it is reasonable to use another representation for the Green function related to the wave function and the spectra of the quantum mechanical system inside the pyramid. The wave function which vanish on the planes $P_1, P_2$ and $P_3$ can be obtained in a similar fashion as the Green function:

$$ \Psi(\vec{x}) = \Omega \sum_{g \in G} \sum_{j=1}^{23} [e^{i(\vec{p}_j g \vec{x})} - e^{i(\vec{p}_j i g \vec{x})}] $$

(32)
\[
\Psi_{\vec{p}}(\vec{x}) = -8i\Omega[\sin p_1 x_1 \sin p_2 x_2 \sin p_3 x_3 - \sin p_1 x_1 \sin p_2 x_2 \sin p_3 x_2 + \text{c.p.}]
\]  
(33)

where \(\Omega\) is the normalization.

The condition \(\Psi_{\vec{p}}(\vec{x}) |_{P_4} = 0\) implies that the components \(p_j\) are proportional to the nonzero positive integers

\[
p_1 = \frac{\pi}{a} n, \quad p_2 = \frac{\pi}{a} m, \quad p_3 = \frac{\pi}{a} k.
\]  
(34)

The properties

\[
\Psi_{\vec{g}, \vec{p}}(\vec{x}) = \Psi_{\vec{p}}(\vec{x}) \quad \Psi_{i\vec{p}}(\vec{x}) = \Psi_{\vec{p}}(\vec{x})
\]  
(35)

imply that the spectrum takes its values in the quotient space

\[
Z^3/G = \{ \vec{n} \in Z^3 : k \geq n \geq m \geq 0 \}
\]  
(36)

which is the discrete analogue of the pyramidal region considered in the previous section. The wave function (33) vanishes on the boundary \(B\) of \(A = Z^3/G\). Boundary of \(A\) is the union of three regions: \(k \geq n \geq m = 0\), \(k \geq n = m \geq 0\) and \(k = n \geq m \geq 0\). We have to drop these values from the spectra. Physical spectra is given by (34) with \(k > n > m > 0\) or \(\vec{n} \in A/B\).

The Green function can be written as

\[
G(x, y) = \sum_{k=3}^{\infty} \sum_{n=2}^{k-1} \sum_{m=1}^{n-1} \frac{e^{i\pi |\vec{p}|(x_0 - y_0)}}{2 |\vec{p}|} \Psi_{\vec{p}}(\vec{x}) \Psi_{\vec{p}}(\vec{y})
\]  
(37)

which implies

\[
T(x) = \frac{\pi}{2a} \sum_{k=3}^{\infty} \sum_{n=2}^{k-1} \sum_{m=1}^{n-1} \sqrt{n^2 + m^2 + k^2} |\vec{p}| \Psi_{\vec{p}}(\vec{x})^2.
\]  
(38)

After integration \(\int_0^a dx_3 \int_0^{x_3} dx_1 \int_0^{x_1} dx_2\) we have (with \(\vec{n} = (n, m, k)\))

\[
E = \frac{\pi}{2a} \sum_{\vec{n} \in A/B} |\vec{n}| = \frac{\pi}{96a} \sum_{\vec{n} \in A/B} \sum_{g \in G} |g\vec{n}| = \frac{\pi}{96a} \sum_{\vec{n} \in \bigcup g(A/B)} |\vec{n}|
\]  
(39)

Since \(A\) is the quotient space \(Z^3/G\) we have

\[
\bigcup_{g \in G} g(A/B) = Z^3/C, \quad C = \bigcup_{g \in G} gB
\]  
(40)

which implies

\[
E = \frac{\pi}{96a} \left( \sum_{\vec{n} \in Z^3} |\vec{n}| - \sum_{\vec{n} \in C} |\vec{n}| \right).
\]  
(41)

\(C\) is the union of nine planes: \(m = 0, n = m, n = k\) and other six planes are obtained by cyclic permutations of \(n, m\) and \(k\):

\[
\sum_{\vec{n} \in C} |\vec{n}| = 3 \sum_{n,m \in Z} \sqrt{n^2 + m^2 + 6} \sum_{n,m \in Z} \sqrt{2n^2 + m^2}
\]  
(42)

The Casimir energy is

\[
E = \frac{1}{6} E_1 - \frac{1}{2} E_2 - \frac{6 + 4\sqrt{2}}{16} E_3 \simeq \frac{0.069}{a}
\]  
(43)
Here $E_1$, $E_2$ and $E_3$ are the Casimir energies for the cube with sides $a$, for the rectangle with sides $a$, $\sqrt{2}a$ and for the one dimensional system of length $a$ (see [2] and references therein):

$$E_1 = \frac{\pi}{2a} \sum_{n,m,k=1}^{\infty} \sqrt{n^2 + m^2 + k^2} \simeq -\frac{0.015}{a} \quad (44)$$

$$E_2 = \frac{\pi}{2a} \sum_{n,m=1}^{\infty} \sqrt{n^2 + 2m^2} \simeq \frac{0.045}{a} \quad (45)$$

$$E_3 = \frac{\pi}{2a} \sum_{n=1}^{\infty} n \simeq -\frac{0.131}{a} \quad (46)$$

The positive result of (43) is about the same magnitude as the other well known positive Casimir energy example of the spherical cavity with radius $a$ [6]

$$E_{ball} \simeq \frac{0.045}{a} \quad (47)$$

For nanometer size, that is for $a = 10^{-7}$ cm, the energy (43) is (in $\hbar = c = 1$ unit, $1eV \cong 0.5 \times 10^9$ cm$^{-1}$) $E \simeq 35$ eV which is of considerable magnitude.

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### Appendix A

Octahedral group $O$ is the group of transformations which transforms cube into itself. The order of this group is 24. We denote the identity element by $g_0$. $g_1$, $g_2$ and $g_3$ are rotations on $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$ around y-axis:

$$g_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (A.1)$$

$g_4$, $g_5$ and $g_6$ are rotations by $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$ around x-axis:

$$g_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad (A.2)$$

$g_7$, $g_8$ and $g_9$ are rotations by $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$ around z-axis:

$$g_7 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (A.3)$$

$g_{10}$ and $g_{11}$ are rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around the axis passing through the origin and the point $(1,1,1)$:

$$g_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (A.4)$$
$g_{12}$ and $g_{14}$ are rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around the axis passing through the origin and the point $(1, -1, 1)$:

$$g_{12} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, g_{13} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad (A.5)$$

$g_{14}$ and $g_{15}$ are rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around the axis passing through the origin and the point $(-1, 1, 1)$:

$$g_{14} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, g_{15} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad (A.6)$$

$g_{16}$ and $g_{17}$ are rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around the axis passing through the origin and the point $(1, 1, -1)$:

$$g_{16} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, g_{17} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad (A.7)$$

$g_{18}$ and $g_{19}$ are rotations by $\pi$ around the axis passing through the origin and the points $(1, 1, 0)$ and $(1, -1, 0)$ respectively:

$$g_{18} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, g_{19} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (A.8)$$

$g_{20}$ and $g_{21}$ are rotations by $\pi$ around the axis passing through the origin and the points $(1, 0, 1)$ and $(1, 0, -1)$ respectively:

$$g_{20} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, g_{21} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (A.9)$$

$g_{22}$ and $g_{23}$ are rotations by $\pi$ around the axis passing through the origin and the points $(0, 1, 1)$ and $(0, 1, -1)$ respectively:

$$g_{22} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, g_{23} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad (A.10)$$

The tetrahedral group $T$ is the subgroup of $O$ of order 12 with elements $g_0, g_2, g_5, g_8$ and $g_{10}, \ldots, g_{17}$. For more details we refer to [7].

**Appendix B**

Let $G$ be a point group acting in the Euclidean space $\mathbb{R}^3$. A closed subset $S$ of $\mathbb{R}^3$ is called a fundamental domain of $G$ if $\mathbb{R}^3$ is the union of conjugates of $S$, i.e.,

$$\mathbb{R}^3 = \bigcup_{g \in G} gS \quad (B.1)$$

The tetrahedral group $T$ is the subgroup of $O$ of order 12 with elements $g_0, g_2, g_5, g_8$ and $g_{10}, \ldots, g_{17}$. For more details we refer to [7].
and the intersection of any two conjugates has no interior.

The fundamental domain of the group generated by the reflections $i g_2$, $i g_5$ and $i g_8$ with respect to $y = 0$, $x = 0$ and $z = 0$ planes is the first quadrant in $R^3$. This is group of order 8 and divide $R^3$ into 8 equal parts. If one adds to this group the element $g_{10}$ we arrive at the group of order 24 which is the direct product of the tetrahedral group $T$ and inversion one $I$ generated by $i$. Rotation $g_{10}$ is three fold rotation. It divides the first quadrant into three equal parts. Therefore the fundamental domain for $T \times I$ is the region in the first quadrant between three planes $P_2$, $P_3$ and $P_5 = \{x_2 = x_3\}$. In $T \times I$ there is reflection operator $i g_2$ with respect to $P_2$ plane. The Green function constructed from the group $T \times I$ will vanish on $P_2$. For $P_3$ and $P_5$ there is no reflection operators. $g_{10}$ rotate $P_3$ into $P_5$. If we add to $T \times C$ the reflection operator with respect to $P_3$ plane we arrive at the group $O \times I$ which is of order 48. The corresponding fundamental domain can be obtained from that of $T \times C$ by dividing it into two equal parts, that is the region between three planes $P_2$, $P_3$ and $P_1$. There are reflections with respect to these planes. The Green function will vanish on these planes.

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