Analytic solutions of the 1D finite coupling delta function Bose gas

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An intensive study for both the weak coupling and strong coupling limits of the ground state properties of this classic system is presented. Detailed results for specific values of finite \( N \) are given and from them results for general \( N \) are determined. We focus on the density matrix and concomitantly its Fourier transform, the occupation numbers, along with the pair correlation function and concomitantly its Fourier transform, the structure factor. These are the signature quantities of the Bose gas. One specific result is that for weak coupling a rational polynomial structure holds despite the transcendental nature of the Bethe equations. All these new results are predicated on the Bethe ansatz and are built upon the seminal works of the past.

I. INTRODUCTION

The one dimensional delta function Bose gas is a classic in the field of exactly solvable integrable systems. Following on the realization that in the infinite coupling limit, the impenetrable limit, the system had many of the properties of the free Fermi gas \([1]\), in their seminal work, Lieb and Liniger \([2]\) solved the model exactly. They derived the Bethe ansatz and Bethe equations and went on to solve for the excitations in the thermodynamic limit \([2, 3]\).

In this paper, we present an intensive study of the finite system, in the weak coupling and the strong coupling limits. Extensive analytical solutions are given for finite values of \( N \), the particle number, and from them, the analytical solutions for general \( N \) are determined. Except for a couple of papers that give the excitations for a system of three particles \([4]\) and larger number of particles \([5, 6]\), this is the first intense study of this finite \( N \) body system for finite coupling. We concentrate on the principle quantities that are the signatures of the Bose gas.

After a preliminary Section II and Appendix A introducing the Bethe ansatz and Bethe equations and their properties, we give extensive analytical solutions, in Sections III and IV with the aide of Appendices B and C for the density matrix and concomitantly the occupation numbers. In Sections V and VI we do likewise for the pair correlation function and concomitantly the structure factor. As such we build upon the seminal work of Lenard \([7]\), the very important work of Jimbo and Miwa \([8]\), the Leningrad group \([9]\), and our recent works \([10, 11, 12]\). We conclude the paper in Section VII with some further comments following upon all these results.

The occupation numbers and structure factors for the modes are the experimentally realizable signatures of the Bose gas. The spur to the recent, over the past decade, revival of active interest in this system is due to Olshanii \([13]\), who pointed out how this system could be realized in nature and to the current experimental activity seeking to realize it \([14, 15, 16, 17, 18, 19]\). We refer to the introduction in our earlier paper \([10]\) for a discussion of the relevant physical parameters involved.

As all the new analytical solutions in this work are based upon the Bethe ansatz, it is appropriate to observe that this year celebrates the 75th anniversary of Bethe’s paper \([20]\) in which he introduced the now famous ansatz in his study of the Heisenberg spin-chain. Ever since, it has been a golden key in unlocking the solutions to many exactly solvable integrable systems. Baxter \([21]\) gave a concise review of these in his tribute to Yang, who has recently written \([22]\) on the Bethe ansatz to honor Bethe.

II. CONSTRUCTING THE WAVEFUNCTION

The purpose of this section is to present the Bethe ansatz wavefunction and concomitantly the Bethe equations for the one-dimensional delta function Bose gas in periodic boundary conditions, as derived in the seminal paper by Lieb and Liniger \([2]\). The Schrödinger equation for this system with \( \hbar = 1, 2m = 1 \) is

\[
\left( -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \right) \psi_N(x_1, x_2, \ldots, x_N) = E \psi_N(x_1, x_2, \ldots, x_N)
\]

(2.1)

where \( c \) controls the strength of the \( \delta \)-function. Throughout this paper we consider only \( c \geq 0 \), the repulsive case.
Here we display the solution for periodic boundary conditions

$$\psi_N(x_1, x_2, \ldots, x_N) = \sum_{p \in S_N} a(p)e^{i\sum_{j=1}^N k_{p(j)} x_j}$$  \hspace{1cm} (2.2)

where $S_N$ is the symmetric group on $N$ symbols, and hence the wavefunction is a sum over $N!$ quantities. The function $\epsilon(p)$ is the signature of the permutation $p$, and the function $a(p)$ is given by

$$a(p) = \prod_{i<j}^N \left[ 1 + \frac{i}{c}(k_{p(j)} - k_{p(i)}) \right].$$  \hspace{1cm} (2.3)

Note that $a(p)$ may be given in a number of forms (any change absorbed into the normalisation), we choose (2.3) from Jimbo and Miwa [8]. Throughout this paper we use the normalisation given by

$$N^2 = \int_{R_{N-1}} dx_1 \ldots dx_{N-1} |\psi_N(0, x_1, \ldots, x_{N-1})|^2$$  \hspace{1cm} (2.4)

where $R_{N-1}$ is the domain of integration specified by

$$R_{N-1} : 0 < x_1 < \ldots < x_{N-1} < L.$$  \hspace{1cm} (2.5)

In this paper we are concerned only with the ground state of the system, where $\sum_{i=1}^N k_i = 0$, which leads to

$$k_i = -k_{N+1-i} \quad \forall \; i = 1, \ldots, N.$$  \hspace{1cm} (2.6)

The $k_i$ are ordered such that $k_N > k_{N-1} > \ldots > k_2 > k_1$. The $N$ (real) numbers $k_i$ are determined as the solution of the Bethe equations. These can be given in many forms [2, 8, 23], we display that of [3]

$$(k_{j+1} - k_j)L = 2\pi + \sum_{i=1}^{N} 2 \arctan \left( \frac{k_i - k_{j+1}}{c} \right) - 2 \arctan \left( \frac{k_i - k_j}{c} \right) \quad j = 1, \ldots, N - 1.$$  \hspace{1cm} (2.7)

While these equations cannot be solved explicitly for $k_j$ as a function of $c$, they can be solved for both small and large $cL$ expansions, using the method of quadrature. We list explicit small $cL$ expansions in Appendix A and at this point we highlight that

$$k_j = \sqrt{\frac{2c}{L}} h_j^{(N)} \left( 1 - \frac{1}{24}(cL) + O(cL)^2 \right)$$  \hspace{1cm} (2.8)

where $h_j^{(N)}$ is the $j$th zero of the $N$th Hermite polynomial. The leading term in (2.8) is found in reference to this problem by Gaudin [23], for more detail refer to Szegő [24]. The coefficient $-1/24$ in the next term of the expansion is new, and appears to be universal (see Appendix A).

Here we have chosen to display the large $cL$ expansion for $k_j$ in its general $N$ form as given in [8], for its particular utility in the following sections.

$$k_j = (2j - N - 1) \frac{\pi}{L} \left[ 1 - 2N \left( \frac{1}{cL} \right) + 4N^2 \left( \frac{1}{cL} \right)^2 \right]$$  
$$+ \left\{ -8N^3 + \frac{4}{3} N \left[ 2j^2 + (N + 1)(N - 2j) \right] \pi^2 \right\} \left( \frac{1}{cL} \right)^3 + O \left( \frac{1}{cL} \right)^4.$$  \hspace{1cm} (2.9)
It is now possible to explicitly construct the wavefunction \( \psi_N(x_1, \ldots, x_N) \) using (2.2) and (2.3). We close here by exhibiting the unnormalised wavefunction \( \psi_2(x_1, x_2) \) by way of example

\[
\psi_2(x_1, x_2) = \left( 1 - \frac{2i k_2}{c} \right) e^{i k_2 (x_2 - x_1)} - \left( 1 + \frac{2i k_2}{c} \right) e^{-i k_2 (x_2 - x_1)}. \tag{2.10}
\]

which has normalisation

\[
\mathcal{N}^2 = 2L - \frac{\sin 2k_2 L}{k_2} - \frac{8 \sin^2 k_2 L}{c} + \frac{4k_2 (2k_2 L + \sin 2k_2 L)}{c^2}. \tag{2.11}
\]

In all that follows the weak coupling expansion corresponds to the d imensionless parameter \( cL \ll 1 \) and the strong coupling expansion corresponds to \( cL \gg 1 \).

### III. DENSITY MATRIX AND OCCUPATION NUMBERS

The normalised density matrix \( \rho_N(x, 0) \) is defined as

\[
\rho_N(x, 0) = \frac{1}{\mathcal{N}^2} \sum_{j=0}^{N-1} \int_{R_{N-1,j}(x)} dx_1 \ldots dx_{N-1} \psi_N(0, x_1, x_2, \ldots, x_{N-1}) \overline{\psi_N(x_1, \ldots, x_j, x, x_{j+1}, \ldots, x_{N-1})} \tag{3.1}
\]

where the overbar implies complex conjugation, the normalisation \( \mathcal{N}^2 \) is specified by (2.4), and the domain of integration is specified by

\[
R_{N,j}(x) : 0 \leq x_1 < \ldots < x_j < x < x_{j+1} < \ldots < x_N \leq L. \tag{3.2}
\]

The density matrix is normalised such that \( \rho_N(0, 0) = \rho_0 = N/L \). Hence to compute the density matrix for \( N \) particles, one must perform \( N \) lots of \( N - 1 \) dimensional integrals over \((N!)^2\) terms (e.g. 4 triple integrals over 576 terms for \( N = 4 \)). This is a computationally expensive task, and hence we were able to determine solutions only up to \( N = 4 \) using this method.

The occupation numbers \( c_n(N) \) are determined as a Fourier transform of the density matrix

\[
c_n(N) = \int_0^L \rho_N(x, 0) e^{2i \pi n x / L} dx = \int_0^L \rho_N(x, 0) \cos(2\pi n x / L) dx. \tag{3.3}
\]

They have the physical interpretation of being the expectation value of the number of particles in mode \( n \).

We note the normalisation property \( \sum_{n=-\infty}^{\infty} c_n(N) = N \); we have confirmed this result for all occupation number formulas that follow. It is sometimes useful to discuss occupation number per particle; we introduce the notation \( c^*_n(N) = c_n(N)/N \). We shall now display some explicit solutions for \( N = 2, 3, 4 \) and general \( N \).

| \( N \) | \( cL \) | \( kN L \) |
|---|---|---|
| 2 | \( \pi \) | \( \pi/2 \) |
| 3 | \( \pi/2 \) | \( \pi(\sqrt{17} - 3)/4 \) |
| 3 | \( \pi \) | \( \sqrt{2\pi} \) |
| 3 | \( 3\pi/2 \) | \( 3\pi(\sqrt{17} + 3)/4 \) |

TABLE I: A sample of exact solutions to (2.7)
A. $N = 2$

Within Section III this subsection heralds the most complete set of results, with results for $N = 3$ and 4 becoming increasingly exigous as the intricacies of the equations develop. For example, it is only possible to display the complete density matrix for $N = 2$, as for $N = 3$ already the equation would take many pages to display. We have utilised for $N = 2$ (3.7) to obtain

$$c = 2k_2 \tan \left( \frac{k_2 L}{2} \right)$$

hence producing a concise form of the density matrix using (2.2), (2.3) and (3.1)

$$\rho_2(x,0) = \frac{2}{L} k_2 x \cos(k_2 (L-x)) + k_2 (L-x) \cos(k_2 x) + \sin(k_2 (L-x)) + \sin(k_2 x) \quad 0 \leq k_2 \leq \pi$$

with corresponding occupation numbers from (3.3)

$$c_n(2) = \frac{4(k_2 L)^3 (1-\cos(k_2 L))}{(4n^2 \pi^2 - k_2^2 L^2)^2 (k_2 L + \sin(k_2 L))} \quad 0 \leq k_2 \leq \pi.$$  

The expansion of the density matrix for small $cL$ is given using (A1)

$$\rho_2(t,0) = \frac{2}{L} \left[ 1 - \frac{t^2(\pi - t)^2}{24 \pi^4} (cL)^2 + \frac{t^2(\pi - t)^2(t^2 - \pi t + 2\pi^2)}{360 \pi^6} (cL)^3 \right.$$ 

$$+ \frac{t^2(\pi - t)^2(16\pi^4 - 24\pi^3 t + 27\pi^2 t^2 - 6\pi t^3 + 3t^4)}{40320 \pi^8} (cL)^4 + O(cL)^5 \left].ight.$$  

Note that we have introduced here $t = \pi x / L$, henceforth we switch between $t$ and $x$ as appropriate. The corresponding occupation numbers for (3.7) are given by

$$c_n(2) = 2 \left\{ \frac{1}{16n^2 \pi^4} (cL)^2 + \frac{1}{1200n^3 \pi^4} (cL)^3 - \frac{1}{3840n^4 \pi^4} (cL)^4 + O(cL)^5 \right\} \quad \text{when } n = 0$$

$$+ \frac{1}{3072n^2 \pi^4} (cL)^2 + \frac{1}{2880n^3 \pi^4} (cL)^3 + \frac{4n^4 \pi^4 - 30n^2 \pi^2 + 15}{3840n^4 \pi^4} (cL)^4 + O(cL)^5 \quad \text{when } n \neq 0.$$  

Utilising (2.3), we obtain a large $cL$ expansion for the density matrix

$$\rho_2(t,0) = \frac{2}{L} \left\{ \frac{t (\pi - 2t) \cos t + 2 \sin t}{\pi} + \frac{8(\pi - t)t \sin t}{\pi} \left( \frac{1}{cL} \right) \right.$$ 

$$+ \frac{8 [(\pi^2 - 3\pi t + 2t^2) \cos t - (\pi^2 + 6\pi t - 6t^2) \sin t]}{\pi} \left( \frac{1}{cL} \right)^2 + O \left( \frac{1}{cL} \right)^3 \right\}.\right.$$  

Note that in the limit $cL \to \infty$, we recover (20) from [10]. The corresponding occupation numbers for (3.9) are given by

$$c_n(2) = 2 \left\{ \frac{8}{(4n^2 - 1)^2 \pi^2} - \frac{32(4n^2 + 1)}{(4n^2 - 1)^3 \pi^2} \left( \frac{1}{cL} \right) \right.$$ 

$$+ \frac{3072n^2 (4n^2 + 1)}{(4n^2 - 1)^4 \pi^2} \left( \frac{1}{cL} \right)^2 + \frac{64(4n^2 - 1)}{(4n^2 - 1)^2} \left\{ \left( \frac{1}{cL} \right)^2 + O \left( \frac{1}{cL} \right)^3 \right\} \right\} \quad \text{when } n \neq 0.$$  

which in the limit $cL \to \infty$ recovers (42) from [10]. We also display here the density matrix for the exact solution to (2.7) from Table I, when $cL = \pi, k_2 L = \pi / 2$

$$\rho_2(t,0) = \frac{2}{L} \left( -t + \pi + 2 \right) \cos \frac{t}{\pi} + (t + 2) \sin \frac{t}{\pi}$$

$$\pi / 2$$
and the corresponding occupation numbers are given by

\[ c_n(2) = \frac{16}{(16n^2 - 1)^2} \pi (\pi + 2). \] (3.12)

\[ c_n(3) = 3 \begin{cases} 
1 - \frac{1}{180}(cL)^2 + \frac{1}{180}(cL)^3 - \frac{163}{18\pi} (cL)^4 + O(cL)^5 \quad \text{when } n \geq 1 \\
\frac{1}{8\pi^2} (cL)^2 - \frac{1}{48\pi^2} (cL)^3 - \frac{9n^3 + 17n^2 + 1515}{120\pi^2} (cL)^4 + O(cL)^5 \quad \text{when } n \neq 0.
\end{cases} \] (3.14)

\[ \rho_3(t, 0) = \frac{3}{L} \left\{ \frac{(\pi - 2t)^2 + 12(\pi - 2t) \sin 2t + 4(2t - \pi + 2)(2t - \pi - 2) \cos 2t + 4t + 15}{6\pi^2} \right. \\
+ \frac{4 \sin t [(\pi - 2t) (1 + 8t - 8t^2) \cos t - (\pi - 2t) \cos 2t + 8 (\pi - t) \sin t]}{\pi^2} \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 \} \] (3.15)

which in the limit \( cL \to \infty \) recovers (21) of \([11]\).

The occupation numbers corresponding to (3.15) are given by

\[ c_n(3) = 3 \begin{cases} 
\frac{35}{1680\pi^2} - \frac{385}{36\pi^2} (\frac{1}{cL})^2 \quad \text{when } n \geq 3.
\end{cases} \] (3.16)

\[ \frac{2(3n^2 - 1)\pi^2}{3n^2} = \frac{4(9n^6 - 28n^4 - 6n^2 + 8)}{n(n^2 - 1)^3(n^2 - 4)\pi^2} \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 \] (3.17)

The density matrix and occupation numbers for the other exact values listed in Table I for \( N = 3 \), although calculated, are too lengthy to display here.
C. $N = 4$

The intricacy of the density matrix at this value of $N$ is already so great that we go on to calculate the occupation numbers without explicitly exhibiting it, and as such we give only $c_0(4)$. We utilise (3.20) and (A3) to produce the result

$$c_0(4) = 4 \left[1 - \frac{1}{240}(cL)^2 + \frac{13}{10080}(cL)^3 - \frac{383}{1209600}(cL)^4 + O(cL)^5\right].$$  \hspace{1cm} (3.19)

D. General $N$

It is interesting to observe that with the presence of the irrational numbers in the Bethe equations for small $cL$ (Appendix A), that when the final results for the occupation numbers appear they contain purely rational numbers. Encouraged by this remarkable observation and other indications in the preceding subsections, we looked for a pattern (Appendix A), that when the final results for the occupation numbers appear they contain purely rational numbers.

To quantify this last point, note that in the thermodynamic limit the dimensionless parameter is $c/\rho_0 = \gamma$ and went on to show that for asymptotically large $N$, we have written (3.20) as given to emphasise the detailed structure of the coefficients in this expansion. Note that the coefficient of $(cL)^5$ term we would need $c_0(N)$ for four specific values of $N$ in their rational form. Unfortunately, it was too computationally expensive for us to obtain any $N - 1$ polynomial for any higher order than $(cL)^4$. We conjecture that this is the pattern for all $N \geq 2$.

The fact that the coefficient of $(N - 1)(cL)^p$ appears to be a polynomial of order $p - 2$ in $N - 1$ suggests that for any finite $N$ there is always an interval $cL \in [0, D_N)$ such that the series is convergent, but with $D_N \rightarrow 0$ as $N \rightarrow \infty$. To quantify this last point, note that in the thermodynamic limit the dimensionless parameter is $c/\rho_0 = \gamma$ and the coefficient is proportional to $N^{2p-1}$, which suggests that the corresponding radius of convergence is proportional to $1/N^2$. In particular this means that no information can be gleaned as to the functional form of $c_0^*(N)$ as a function of $\gamma$ about $\gamma = 0$ except that it is not analytic.

IV. DENSITY MATRICES AND OCCUPATION NUMBERS FOR LARGE $cL$

In the previous section, we gave large $cL$ expansions for the density matrices and occupation numbers for $N = 2$ and $N = 3$. Again, it was numerically prohibitive to go beyond $N = 4$, furthermore as can be seen from the coefficients in these expansions (and as can be witnessed in the Tables III-VIII which we will refer to in what follows), that the numbers are highly irrational with no hope of finding an analogous pattern as we were fortunate enough to do in (3.20).

We therefore turn to a totally different mathematical strategy to obtain a large $cL$ expansion for these quantities. In the impenetrable limit ($cL = \infty$, arbitrary $L$), Lenard developed the theory for the density matrix for arbitrary $N$, and went on to show that for asymptotically large $N$ that $c_0(N) \sim N^{1/2}$.

In our recent work [10] we employed Lenard’s theory to obtain the results for the occupation numbers for a range of finite $N$, and from them determine the results for general $N$, which continue on to the asymptotically large $N$ limit [31].

To go beyond the impenetrable limit for these quantities, we turn to the very valuable work of Jimbo and Miwa. Building upon the work of Lenard, they developed an expansion for the density matrix in the large $cL$ limit for general $N$ in principle. We say in principle because while their expansion is superb in the form given, it is as numerically prohibitive to use as was the method we employed in the previous section.

To resolve this difficulty we have recast their theory using mathematical techniques evidenced in our recent work [12] and presented in great detail by Forrester into a new form that is readily amenable to numerical calculation.
We will find in what follows that the specific results for \( N = 2 \) and \( N = 3 \) as in Section III are useful specific checks to the theory.

In Subsection IV A along with Appendix E we present the full details of our derivation for the density matrix. Following upon that, in Subsection IV B we are now able to calculate the occupation numbers for a finite range of \( N \) values and from these results are able to determine the results for general \( N \) which again continue to asymptotically large \( N \).

### A. Toeplitz Determinants

The fact that \(|\psi_N|^2\) consists of \((N!)^2\) terms means any method based on term-by-term integration must necessarily be restricted to small \( N \). To overcome this one must seek out structure in the form of \( \psi_N \), and this structure must be used to reduce the computational expense required to compute \( \rho_N(x,0) \). Certainly in the limit \( cL \to \infty \) there is structure in (2.2) for then (2.9) gives $k_j = (2j - N - 1)\pi/L$ while (2.3) gives \( a(p) = 1 \), and so

\[
\psi_N(x_1, \ldots, x_N)_{cL \to \infty} = \sum_{p \in S_N} \epsilon(p) \prod_{j=1}^{N} e^{i\pi(2j-N-1)x_{p(j)}/L} \tag{4.1}
\]

We will find in what follows that the specific results for \( N = 2 \) and \( N = 3 \) as in Section III are useful specific checks to the theory.

\[
\rho_N^{(0)}(x,0) = -\frac{1}{2} \Delta_1 \begin{pmatrix} x & 0 \\ 0 & -2 \end{pmatrix} \tag{4.3}
\]

where

\[
\Delta_1 \begin{pmatrix} x & \lambda \\ 0 & 1 \end{pmatrix} = \lambda e^{-i\pi(N-1)x/L} \det [A_1(j-k)]_{j,k=1,\ldots,N-1} \tag{4.4}
\]

\[
A_1(j-k) = \frac{1}{L} \left[ \int_0^L + \lambda \int_0^x \right] du (e^{2\pi i u/L} - e^{2\pi i x/L})(e^{-2\pi i u/L} - 1)e^{2\pi i(j-k)/L} \tag{4.5}
\]

\[
= \begin{cases} 
\frac{\lambda}{2\pi} \sin 2t - \frac{\lambda}{\pi} \sin t - 1 & \text{when } j-k = -1 \\
2 e^{i t} \left[ \left( \frac{\pi}{x} + 1 \right) \cos t - \frac{\lambda}{x} \sin t \right] & \text{when } j-k = 0 \\
e^{2i t} \left[ \frac{\lambda}{x} \sin 2t - \frac{\lambda}{x} - 1 \right] & \text{when } j-k = 1 \\
\frac{2 \lambda e^{i(j-k+1)}}{(j-k)(j-k)^2 - 1} \left[ (j-k) \cos((j-k)t) \sin t - \cos t \sin((j-k)t) \right] & \text{when } |j-k| \geq 2
\end{cases} \tag{4.6}
\]

and thus also can be computed in \( O(N^3) \) operations. In II this determinant formula was used to compute the density matrix and the corresponding ground state occupation numbers up to \( N = 7 \). It has been shown by Jimbo and Miwa [8] (see also Section IV B below) that determinant structures persist if \( \psi_N \) is expanded in large \( cL \), and that this implies special structures for the expansion of the density matrix. In particular, writing

\[
\rho_N(x,0) = \rho_N^{(0)}(x,0) + \left( \frac{1}{cL} \right) \rho_N^{(1)}(x,0) + O \left( \frac{1}{cL} \right)^2 \tag{4.7}
\]

it was shown in [8] that

\[
\rho_N^{(1)}(x,0) = -2 \rho_{0x} \frac{\partial}{\partial x} \rho_N^{(0)}(x,0) + F_N(x) \tag{4.8}
\]
where $\rho_N^{(0)}(x,0)$ is specified by and

$$F_N(x) = \frac{1}{\Delta(x; -2)} \left[ \frac{\partial}{\partial x} \Delta_1 \left( \frac{x}{0}; \lambda \right) \frac{\partial}{\partial \lambda} \Delta(x; \lambda) + \frac{\partial}{\partial \lambda} \Delta_1 \left( \frac{x}{0}; \lambda \right) \frac{\partial}{\partial x} \Delta(x; \lambda) - \Delta_1 \left( \frac{x}{0}; \lambda \right) \frac{\partial^2}{\partial x \partial \lambda} \Delta(x; \lambda) \right]_{\lambda = -2}.$$ (4.9)

Introducing the kernel function

$$K_{NL}(x,y) = \frac{1}{L} \sum_{j=1}^{N} e^{-i\pi(2j-N-1)(x-y)/L}$$ (4.10)

and

$$\sin \left[ \frac{N\pi(x-y)/L}{L \sin \pi(x-y)/L} \right]$$ (4.11)

the quantity $\Delta_1 \left( \frac{x}{y}; \lambda \right)$ is defined in [8] as the Fredholm minor

$$\Delta_1 \left( \frac{x}{y}; \lambda \right) = \sum_{l=0}^{\infty} \frac{\lambda^{l+1}}{l!} \int_{0}^{x} du_1 \ldots \int_{0}^{x} du_l \times \det \left[ \begin{array}{c} K_{NL}(x,y) \\ K_{NL}(u_j,y) \end{array} \right]_{j=1,\ldots,l}$$ (4.12)

while $\Delta(x; \lambda)$ is specified as the Fredholm determinant

$$\Delta(x; \lambda) = \sum_{l=0}^{\infty} \frac{\lambda^{l}}{l!} \int_{0}^{x} du_1 \ldots \int_{0}^{x} du_l \det \left[ K_{NL}(u_j,u_k) \right]_{j,k=1,\ldots,l}.$$ (4.13)

Neither (4.12) or (4.13) are suitable for computation. However for $\Delta_1$ we have the determinant formula (4.4), and $\Delta$ too can be expressed as a determinant. This can be seen by developing Lenard’s [7] derivation of (4.3), which proceeds by writing the summation of determinants of (4.12) and (4.13) as multidimensional integrals of a type which can be recognised as Toeplitz determinants (see Appendix B). In addition to (4.4) - (4.6), one deduces that

$$\Delta(x; \lambda) = \det \left[ A_0(j-k) \right]_{j,k=1,\ldots,N}$$ (4.14)

where

$$A_0(j-k) = \frac{1}{L} \left( \int_{0}^{L} + \lambda \int_{0}^{x} \right) du \exp \left[ 2\pi i u(j-k)/L \right]$$ (4.15)

$$= \begin{cases} 1 + \frac{\lambda x}{2\pi i(j-k)} & \text{when } j-k = 0 \\ \frac{\lambda x}{2\pi i(j-k)}(e^{2\pi i(j-k)x/L} - 1) & \text{when } j-k \neq 0. \end{cases}$$ (4.16)

With these determinant formulas $F_N(x)$ is expressed in a computable form. This concludes the method necessary to construct the density matrix expanded in large $cL$. We now examine the occupation numbers.

B. Occupation Numbers for Large $cL$

The following notation for the occupation number $c_n(N)$ expanded in large $cL$ is given as

$$c_n(N) = c_n^{(0)}(N) + \left( \frac{1}{cL} \right) c_n^{(1)}(N) + O \left( \frac{1}{cL} \right)^2.$$ (4.17)
We list some exact values of $c^{(0)}_n(N)$ and $c^{(1)}_n(N)$ for $n = 0$ ($N = 2, \ldots, 7$) and $n = 1, 2$ ($N = 2, \ldots, 6$) in Tables III and IV. The $c^{(1)}_n(N)$ term can be expanded further as

$$c^{(1)}_n(N) = \int_0^L \left( -2Nx \frac{\partial \rho^{(0)}_n(x, 0)}{\partial x} + F_N(x) \right) \cos \left( \frac{2\pi nx}{L} \right) \, dx$$

(4.18)

$$\quad = c^{(1,1)}_n(N) + c^{(1,2)}_n(N)$$

(4.19)

where $F_N(x)$ may be computed using the expressions in Subsection VIII above. The $c^{(1,1)}_n(N)$ term can be simplified using integration by parts to

$$c^{(1,1)}_n(N) = -2N^2 + 2Nc^{(0)}_n(N) - 4\pi nN \int_0^L \frac{x}{L} \sin(2\pi nx/L) \rho^{(0)}_n(x) \, dx.$$  

(4.20)

We list some exact values of $c^{(1,1)}_n(N)$ and $c^{(1,2)}_n(N)$ for $n = 0, 1, 2$ with $N = 2, \ldots, 6$ in Tables VII, VIII. We present all the numerical values that we computed for $c^{(0)}_n(N)$, $c^{(1,1)}_n(N)$, $c^{(1,2)}_n(N)$, and $c^{(1)}_n(N)$ obtained for $n = 0, 1, 2$ in Appendix E. In this table, the data is given to 6 significant figures for economy of presentation, while in point of fact, accuracy to 10 significant figures was needed for the analysis that now follows. We were able to achieve this numerical accuracy up to $N = 36$ for the $n = 0$ mode, and up to $N = 26$ for the modes $n = 1$ and $n = 2$. Note that $c^{(0)}_n(N)$ for $n = 0, 1, 2$ reproduces the results of [10].

We now introduce a generalised version of the ansatz first introduced in [10], expanded to $O(1/eL)$

$$c_n(N) = A_{\infty,n} \left( 1 + \frac{\alpha_n N}{eL} \right) N^{\frac{1}{2}} \sqrt{\pi N} + C_{\infty,n} \left( 1 + \frac{\gamma_n N}{eL} \right)$$

(4.21)

$$\quad = (A_{\infty,n} \sqrt{N} + C_{\infty,n}) + N(A_{\infty,n} \alpha_n \sqrt{N} + A_{\infty,n} \beta_n \sqrt{N} \ln N + C_{\infty,n} \gamma_n) \left( 1 - \frac{1}{eL} \right) + O \left( \frac{1}{eL} \right)^2. \quad (4.22)$$

It is now pertinent to determine bounds on $\alpha_n$ and $\beta_n$, and a value for $\gamma_n$. Consider the set of three linear equations describing the $1/eL$ term of (4.22) for three consecutive values of $N$

$$\begin{pmatrix}
\sqrt{N-1} & \sqrt{N-1} \ln(N-1) & 1 \\
\sqrt{N} & \sqrt{N} \ln(N) & 1 \\
\sqrt{N+1} & \sqrt{N+1} \ln(N+1) & 1
\end{pmatrix}
\begin{pmatrix}
A_{\infty,n} \alpha_n \\
A_{\infty,n} \beta_n \\
C_{\infty,n} \gamma_n
\end{pmatrix}
= \begin{pmatrix}
c^{(1)}_n(N-1) \\
c^{(1)}_n(N) \\
c^{(1)}_n(N+1)
\end{pmatrix}$$

(4.23)

and the set of two linear equations describing the $1/eL$ term of (4.22) with $C_{\infty,n} = 0$ for two consecutive values of $N$

$$\begin{pmatrix}
\sqrt{N} & \sqrt{N} \ln(N) \\
\sqrt{N+1} & \sqrt{N+1} \ln(N+1)
\end{pmatrix}
\begin{pmatrix}
A_{\infty,n} \alpha_n \\
A_{\infty,n} \beta_n
\end{pmatrix}
= \begin{pmatrix}
c^{(1)}_n(N) \\
c^{(1)}_n(N+1)
\end{pmatrix}.$$  

(4.24)

Values of $A_{\infty,n}$ and $C_{\infty,n}$ from [10] are displayed in Table I. The solution for $\alpha_n$, $\beta_n$, $\gamma_n$, of (4.23) and the solution for $\alpha_n$, $\beta_n$ for (4.24) for various values of $N$ establish bounds on $\alpha_n$ and $\beta_n$, and a value for $\gamma_n$. We establish numerical stability for these bounds by calculating them for $N = 2$ up to $N = 35$. A fractional accuracy of $\approx 10^{-10}$ is necessary, and hence the parameters are calculated at $N = 35$ for $n = 0$, and $N = 25$ for $n = 1$ and $n = 2$. We present them in Table I.

Note that $\beta_n$ is very close to 2 for $n = 0$, and suggestive of the value 2 for $n = 1$ and 2. We postulate that $\beta_n = 2$ for all $n$. We say more about this $\beta_n$ in Section VII.
TABLE II: Parameters for the ansatz (4.22)

| $n$ | $A_{\infty,n}$ | $C_{\infty,n}$ | $\alpha_n$ | $\beta_n$ | $\gamma_n$ |
|-----|----------------|----------------|------------|------------|------------|
| 0   | 1.54273        | -0.5725        | 0.1561     | 1.998      | 0.1599     |
| 1   | 0.5143         | -0.5739        | -5.709     | 1.972      | -1.109     |
| 2   | 0.3676         | -0.5775        | -8.350     | 1.887      | -2.736     |

V. CORRELATION FUNCTIONS AND STRUCTURE FACTORS

The definition of the two-point correlation function $g_N(x, 0)$ is

$$g_N(x, 0) = \frac{1}{N^2} \sum_{j=0}^{N-2} \int_{R_{N-2,j}(x)} dx_1 \ldots dx_{N-2} |\psi_N(0, x_1, \ldots, x_j, x, x_j+1, \ldots, x_{N-2})|^2$$

(5.1)

where the normalisation $N^2$ is specified by (2.4), and the domain of integration by (3.2). Then (5.1) has the property

$$\int_0^L g_N(x, 0) dx = N - 1,$$

(5.2)

in keeping with the interpretation of $g_N(x, 0)$ as the density of particles at position $x$, given there is a particle at the origin. The structure factor is defined by

$$S_n(N) = \frac{1}{N^2} \int_{R_{N-1,i}} dx_1 \ldots dx_{N-1} \sum_{j=0}^{N-1} e^{2i\pi x_j/nL} \left| \psi_N(0, x_1, \ldots, x_{N-1}) \right|^2$$

(5.3)

where $x_0 = 0$, which can be written in terms of $g_N(x, 0)$ to read

$$S_n(N) = 1 + \int_0^L g_N(x, 0) \cos(2\pi nx/L) dx.$$

(5.4)

In view of (5.2) this gives $S_0(N) = N$.

A. $N = 2$

This subsection yields the most complete set of results within Section V with the set of results for $N = 3$ and $N = 4$ becoming increasingly limited. For $N = 4$, we display only the small $cL$ expansion of the correlation function and structure factor.

It is possible to produce a simple, exact form of the correlation function for $N = 2$ by utilising (5.1) to obtain

$$g_2(x, 0) = \frac{2k_2 \cos^2 \left[ \frac{1}{2}k_2(L - 2x) \right]}{k_2 L + \sin(k_2 L)}$$

(5.5)

which has corresponding structure factor

$$S_n(2) = \begin{cases} 
2 & \text{when } n = 0 \\
1 + \frac{k_2^2 L^2 \sin(k_2 L)}{(k_2^2 L^2 - n^2 \pi^2)(k_2 L + \sin(k_2 L))} & \text{when } n \neq 0.
\end{cases}$$

(5.6)
The correlation function, expanded about small \( cL \) using (A1), is given by

\[
g_2(x, 0) = \frac{1}{L} \left[ 1 + \left( -\frac{x^2}{L^2} + \frac{x}{L} - \frac{1}{6} \right) (cL) + \left( \frac{x^4}{3L^4} - \frac{2x^3}{3L^3} + \frac{x^2}{2L^2} - \frac{x}{6L} + \frac{1}{60} \right) (cL)^2 \right. \\
+ \left. \left( -\frac{2x^6}{45L^6} + \frac{2x^5}{15L^5} - \frac{7x^4}{36L^4} + \frac{x^3}{6L^3} - \frac{7x^2}{90L^2} + \frac{x}{60L} - \frac{1}{945} \right) (cL)^3 + O(cL)^4 \right] \tag{5.7}
\]

and hence the structure factor is given by

\[
S_n(2) = \begin{cases} 
2 & \text{when } n = 0 \\
1 - \frac{1}{2n^2\pi^2} (cL) + \frac{2\pi^2 - 6}{12n^2\pi^2} (cL)^2 - \frac{n^2\pi^4 - 15n^2\pi^2 + 60}{120n^6\pi^6} (cL)^3 + O(cL)^4 & \text{when } n \neq 0.
\end{cases} \tag{5.8}
\]

The large \( cL \) expansion of the correlation function is given using (A2)

\[
g_2(t, 0) = \frac{1}{L} \left[ 2\sin^2 t + 8\sin t(\pi - 2t) \cos t - \sin t \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 \right] \tag{5.9}
\]

taking the Fourier transform then yields the structure factor

\[
S_n(2) = \begin{cases} 
2 & \text{when } n = 0 \\
\frac{1}{2} + 3 \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 & \text{when } |n| = 1 \\
1 - \frac{1}{(n^2 - 1)(\pi + 2)} \left( \frac{1}{cL} \right)^2 & \text{when } |n| \geq 2.
\end{cases} \tag{5.10}
\]

The correlation function for the special values \( cL = \pi, k_2L = \pi/2 \), (see Table I) is

\[
g_2(t, 0) = \frac{\pi (\sin t + 1)}{L(\pi + 2)} \tag{5.11}
\]

and hence the structure factor is

\[
S_n(2) = \begin{cases} 
2 & \text{when } n = 0 \\
1 - \frac{2}{(4n^2 - 1)(\pi + 2)} & \text{when } n \neq 0.
\end{cases} \tag{5.12}
\]

B. \( N = 3 \)

Given the length of the full correlation function for \( N = 3 \), it is not possible to display the full result here, however we shall display the small and large \( cL \) expansions for both the correlation function and the structure factor.

The small \( cL \) expansion of the correlation function is given, using (A2)

\[
g_3(x, 0) = \frac{2}{L} \left[ 1 + \left( -\frac{x^2}{L^2} + \frac{x}{L} - \frac{1}{6} \right) (cL) + \left( \frac{x^4}{12L^4} - \frac{x^3}{6L^3} + \frac{x^2}{4L^2} - \frac{x}{6L} - \frac{1}{40} \right) (cL)^2 \right. \\
+ \left. \left( \frac{x^6}{5L^6} - \frac{3x^5}{5L^5} + \frac{5x^4}{8L^4} - \frac{x^3}{4L^3} + \frac{x}{40L} + \frac{1}{280} \right) (cL)^3 + O(cL)^4 \right] \tag{5.13}
\]

and hence structure factor by

\[
S_n(3) = \begin{cases} 
3 & \text{when } n = 0 \\
1 - \frac{1}{n^2\pi^2} (cL) + \frac{2\pi^2 - 3}{12n^2\pi^2} (cL)^2 - \frac{n^2\pi^4 - 15n^2\pi^2 - 180}{40n^6\pi^6} (cL)^3 + O(cL)^4 & \text{when } n \neq 0.
\end{cases} \tag{5.14}
\]
The large \(cL\) expansion for the correlation function is given using (5.9):

\[
g_3(t, 0) = \frac{2}{L} \left\{ \frac{4(2 + \cos 2t) \sin^2 t}{3} - \frac{8 \sin t [-2(\pi - 2t)(2 \cos t + \cos 3t) + 3 \sin t + 4 \sin 3t]}{3} \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 \right\}
\]

which yields the structure factor

\[
S_n(3) = \begin{cases} 
3 & \text{when } n = 0 \\
\frac{1}{3} + \frac{32}{9} \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 & \text{when } |n| = 1 \\
\frac{2}{3} \left[ 1 + \frac{16(n^2 - 2)}{(n^2 - 4)(n^2 - 17)} \left( \frac{1}{cL} \right) + O \left( \frac{1}{cL} \right)^2 \right] & \text{when } |n| = 2 \\
1 - \frac{16(n^2 - 2)}{16(2 + \sqrt{2})} & \text{when } |n| \geq 3.
\end{cases}
\]

We now display the correlation function for the special value \(k_3L = \pi\), \(cL = \sqrt{2}\pi\), (from Table I)

\[
g_3(t, 0) = -\frac{2\pi}{L} \left( \frac{2(4\sqrt{2} - \pi) \cos t - \pi \cos 2t - 4(2 + \sqrt{2}\pi) \sin t + 2(6 + \sqrt{2}\pi) \sin 2t - 9\pi - 8\sqrt{2}}{16 + 16\sqrt{2}\pi + 9\pi^2} \right)
\]

and the corresponding structure factor by

\[
S_n(3) = \begin{cases} 
3 & \text{when } n = 0 \\
1 - \frac{32 + 16\sqrt{2} - 3\pi^2}{48 + 48\sqrt{2} + 2\pi + 27\pi^2} & \text{when } |n| = 1 \\
1 - \frac{16(2 + \sqrt{2})}{(4n^2 - 1)(16 + 16\sqrt{2} + 9\pi^2)} & \text{when } |n| \geq 2.
\end{cases}
\]

Here we display the correlation function for the small \(cL\) expansion of \(N = 4\)

\[
g_4(x, 0) = \frac{3}{L} \left\{ 1 + \left( \frac{x^2}{L^2} + \frac{x}{L} - \frac{1}{6} \right) (cL) + \left( \frac{x^4}{6L^4} + \frac{x^3}{3L^3} - \frac{x}{6L} + \frac{1}{30} \right) (cL)^2 + \left( \frac{7x^6}{18L^6} - \frac{7x^5}{6L^5} + \frac{47x^4}{36L^4} - \frac{2x^3}{3L^3} + \frac{19x^2}{180L^2} + \frac{x}{30L} - \frac{1}{135} \right) (cL)^3 + O(cL)^4 \right\}
\]

and its corresponding structure factor

\[
S_n(4) = \begin{cases} 
4 & \text{when } n = 0 \\
1 - \frac{3}{2n^2\pi^2}(cL) + \frac{n^2\pi^2}{4n^2\pi^2}(cL)^2 - \frac{2n^4\pi^4 + 60n^2\pi^2 - 525}{40n^6\pi^6}(cL)^3 + O(cL)^4 & \text{when } n \neq 0.
\end{cases}
\]

D. General \(N\)

Through close examination of the small \(cL\) expansions of the correlation function for \(N = 2\) (5.7), \(N = 3\) (5.13) and \(N = 4\) (5.19) we find the following polynomial structure

\[
g_N(x, 0) = \frac{N - 1}{L} \left\{ 1 - f(x)(cL) + \left[ \left( - \frac{N}{4} + \frac{5}{6} \right) f(x)^2 + \left( \frac{N}{12} - \frac{1}{9} \right) f(x) + \left( \frac{N}{720} - \frac{1}{216} \right) \right] (cL)^2 + \left[ \left( - \frac{N^2}{36} + \frac{23N}{60} - \frac{7}{10} \right) f(x)^3 + \left( \frac{N^2}{36} - \frac{7N}{40} + \frac{1}{5} \right) f(x)^2 + \left( - \frac{N^2}{144} + \frac{13N}{720} - \frac{1}{120} \right) f(x) + \left( - \frac{N^2}{6804} + \frac{79N}{90720} - \frac{1}{1080} \right) \right] (cL)^3 + O(cL)^4 \right\}
\]
where \( f(x) = x^2/L^2 - x/L + 1/6 \). We conjecture that this structure continues for all \( N \). This has corresponding structure factor

\[
S_n(N) = \begin{cases} 
N & \text{when } n = 0 \\
1 - \frac{(N-1)}{2n^2\pi^2} (cL) + \frac{(N-1)(2n^2\pi^2+9N-30)}{24n^4\pi^4} (cL)^2 \\
- \frac{(N-1)(Nn^2\pi^2-180n^2\pi^2+75N^2-1035N+1890)}{240n^6\pi^8} (cL)^3 + O(cL)^4 & \text{when } n \neq 0
\end{cases}
\]

(5.22)

We have not proceeded further than \( N = 3 \) for the large \( cL \) expansion for the same lack of utility as we encountered in the case of the density matrix in Section III. There is now, once again, a powerful way to proceed to develop such structure factor \( N \) and \( \rightarrow \) Bose result. Deviations from this begin at order \( y \). To derive this, we follow the working in \[8\], and begin by noting that the O(1) correction to the two-point correlation function (5.1), yielding in fact a closed form expression. To derive this, we follow the working in \[8\], and begin by noting that the O(1) expansion of (2.2) is

\[
\psi(x_1, x_2, \ldots, x_N) = \left( 1 + \frac{N}{cL} \left( \sum_{l=1}^{N} (-N + 2l - 1) \frac{\partial}{\partial x_l} - 2\rho_0 \sum_{l=1}^{N} x_l \frac{\partial}{\partial x_l} \right) + \ldots \right) \text{det} [e^{ik_j x_l}]
\]

(6.1)

Now put \( N \to N + 2 \) (for convenience) and label the particles as

\[
0 < x_1 < \ldots < x_j < x < x_{j+1} < \ldots < x_N < L.
\]

(6.2)

In the definition of \( g_{N+2}(x, 0) \), \( x \) will be the variable as in (6.2) and we will take \( y \to 0 \). With these labels and \( N \to N + 2 \) the operator in (6.1) reads

\[
-(N+1) \frac{\partial}{\partial y} + \sum_{l=1}^{j} (-N + 1 + 2l) \frac{\partial}{\partial x_l} + (-N + 1 + 2j) \frac{\partial}{\partial x_j} + \sum_{l=j+1}^{N} (-N + 1 + 2l) \frac{\partial}{\partial x_l} - 2\rho_0 \left( \sum_{l=1}^{N} x_l \frac{\partial}{\partial x_l} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right).
\]

(6.3)

The determinant in (6.1) has the translation invariance property

\[
\text{det} \left[ \begin{array}{c} e^{ik_j y} \\ e^{ik_j x_k} \end{array} \right] = \text{det} \left[ \begin{array}{c} 1 \\ e^{i(k_j x_k - y)} \end{array} \right]
\]

(6.4)

since \( \sum k_j = 0 \). This means that we can write

\[
\frac{\partial}{\partial y} = -\frac{\partial}{\partial x} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j}
\]

(6.5)

and using too the fact that we want \( y \to 0 \), the operator (6.3) reads

\[
\sum_{l=1}^{j} 2l \frac{\partial}{\partial x_l} + 2(1 + j) \frac{\partial}{\partial x} + \sum_{l=j+1}^{N} 2(1 + l) \frac{\partial}{\partial x_l} - 2\rho_0 \left( \sum_{l=1}^{N} x_l \frac{\partial}{\partial x_l} + x \frac{\partial}{\partial x} \right).
\]

(6.6)

VI. CORRELATION FUNCTIONS AND STRUCTURE FACTORS FOR LARGE \( cL \)

In Section IV.A results from Jimbo and Miwa \[8\] were used to express the O(1/cL) correction to the density matrix in the form of Toeplitz determinants, which could then be numerically analysed. The method of \[8\] can also be applied to the calculation of the O(1/cL) correction to the two-point correlation function (5.1), yielding in fact a closed form analytic expression. To derive this, we follow the working in \[8\], and begin by noting that the O(1/cL) expansion of (2.2) is
In keeping with (5.1), by definition

\[
g_{N+2}(x,0) = \frac{1}{\mathcal{N}^2} \lim_{y \to 0} \sum_{j=0}^{N} \int_{R_{N,j}(y,x)} dx_1 \ldots dx_N |\psi_{N+2}(y, x_1, \ldots, x_j, x, x_{j+1}, \ldots, x_N)|^2
\] (6.7)

where \( \mathcal{N}^2 \) is the normalisation, defined by (2.4), and \( R_{N,j}(y,x) \) is the region of integration specified by

\[
R_{N,j}(y,x) : 0 \leq y \leq x_1 \leq \ldots \leq x_j \leq x \leq x_{j+1} \leq \ldots \leq x_N \leq L.
\] (6.8)

With

\[
(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N+2}) = (0, x_1, \ldots, x_j, x, x_{j+1}, \ldots, x_N)
\] (6.9)

and

\[
A = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} + x \frac{\partial}{\partial x}
\] (6.10)

\[
B_j = \sum_{i=1}^{j} 2i \frac{\partial}{\partial x_i} + 2(1 + j) \frac{\partial}{\partial x} + \sum_{l=j+1}^{N} 2(1 + l) \frac{\partial}{\partial x_l}
\] (6.11)

we have

\[
\lim_{y \to 0} |\psi_{N+2}(y, x_1, \ldots, x_j, x, x_{j+1}, \ldots, x_N)|^2 = |\det[e^{ik\tilde{x}_l}]|^2 - \frac{2\rho_0}{c} A (|\det[e^{ik\tilde{x}_l}]|^2) \\
+ \frac{1}{c} B_j (|\det[e^{ik\tilde{x}_l}]|^2) + O \left( \frac{1}{cL} \right)^2.
\] (6.12)

The normalisation (2.4) was first expanded in large \( cL \) by [8], we display here the first two orders

\[
\mathcal{N}^2 = (\mathcal{N}^{(\infty)})^2 \left[ 1 + \frac{2\rho_0}{c} (N + 1) + O \left( \frac{1}{cL} \right)^2 \right].
\] (6.13)

where \( (\mathcal{N}^{(\infty)})^2 \) is the normalisation (2.4) in the limit \( cL \to \infty \). Substituting (6.12) and (6.13) in (6.7) shows that to \( O(1/cL) \)

\[
g_{N+2}(x,0) = g^{(\infty)}(x,0) - \frac{2\rho_0}{c} (N + 1) g^{(\infty)}_{N+2}(x,0)
\] (6.14)

\[
- \frac{2\rho_0}{c} \left( \frac{1}{(\mathcal{N}^{(\infty)})^2} \right) \sum_{j=0}^{N} \int_{R_{N,j}(x)} dx_1 \ldots dx_N A (|\det[e^{ik\tilde{x}_l}]|^2)
\] (6.15)

\[
+ \frac{1}{c} \left( \frac{1}{(\mathcal{N}^{(\infty)})^2} \right) \sum_{j=0}^{N} \int_{R_{N,j}(x)} dx_1 \ldots dx_N B_j (|\det[e^{ik\tilde{x}_l}]|^2).
\] (6.16)

To proceed further consider \( |\det[e^{ik\tilde{x}_l}]|^2 \) as a function of \( x_l \), \( l = 1, \ldots, N \). This function vanishes at the three points \( x_l = 0, x_j, L \) (\( j \neq l \)).

It follows that

\[
\int_{R_{N,j}(x)} dx_1 x_l \frac{\partial}{\partial x_l} |\det[e^{ik\tilde{x}_l}]|^2 = - \int_{R_{N,j}(x)} dx_1 |\det[e^{ik\tilde{x}_l}]|^2
\] (6.17)
\[
\int_{R_{N,j}(x)} \frac{dx_i}{\partial x_i} | \det [e^{ik_jx_i}] |^2 = 0. \quad (6.18)
\]

Hence
\[
g_{N+2}(x,0) = g_{N+2}^{(\infty)}(x,0) - \frac{2\rho_0}{c} (N + 1) g_{N+2}^{(\infty)}(x,0) - \frac{2\rho_0}{c} \left( x \frac{\partial}{\partial x} - N \right) g_{N+2}^{(\infty)}(x,0)
+ \frac{1}{c} \left( N^{(\infty)} \right)^2 \sum_{j=0}^{N} 2(j + 1) \frac{\partial}{\partial x} \int_{R_{N,j}(x)} dx_1 \ldots dx_N | \det [e^{ik_jx_i}] |^2 \quad (6.19)
\]
\[
= g_{N+2}^{(\infty)}(x,0) - \frac{2\rho_0}{c} \left( x \frac{\partial}{\partial x} + 1 \right) g_{N+2}^{(\infty)}(x,0)
+ \frac{1}{c} \left( N^{(\infty)} \right)^2 \frac{\partial}{\partial x} \sum_{j=0}^{N} \frac{(j + 1)}{j!(N - j)!} \int_{R_{N,j}(x)} dx_1 \ldots dx_N | \det [e^{ik_jx_i}] |^2 \quad (6.20)
\]

where (6.22) can be written
\[
\frac{2}{c} \left( N^{(\infty)} \right)^2 \frac{1}{N!} \frac{\partial}{\partial x} \xi \sum_{j=0}^{N} \xi^{j+1} \left( \begin{array}{c} N \\ j \end{array} \right) \int dx_1 \ldots \int dx_N | \det [e^{ik_jx_i}] |^2 \bigg|_{\xi=1} \quad (6.23)
\]
\[
= \frac{2}{c} \left( N^{(\infty)} \right)^2 \frac{1}{N!} \frac{\partial}{\partial x} \xi \sum_{j=0}^{N} \xi^{j+1} \left( \begin{array}{c} N \\ j \end{array} \right) \int_{0}^{L} \int_{0}^{x} | dx_1 | \det [e^{ik_jx_i}] |^2 \bigg|_{\xi=1} \quad (6.24)
\]
\[
= \frac{2}{c} \left( N^{(\infty)} \right)^2 \frac{1}{N!} \frac{\partial}{\partial x} \xi \sum_{j=0}^{N} \xi^{j+1} \left( \begin{array}{c} N \\ j \end{array} \right) \int_{0}^{L} \int_{0}^{x} | dx_1 | \det [e^{ik_jx_i}] |^2 \bigg|_{\xi=1} \quad (6.25)
\]
\[
+ \frac{2}{c} \left( N^{(\infty)} \right)^2 \frac{1}{N!} \frac{\partial}{\partial x} \xi \sum_{j=0}^{N} \xi^{j+1} \left( \begin{array}{c} N \\ j \end{array} \right) \int_{0}^{L} \int_{0}^{x} | dx_1 | \det [e^{ik_jx_i}] |^2 \bigg|_{\xi=1} \quad (6.26)
\]

we observe that
\[
\frac{1}{N!} \left( N^{(\infty)} \right)^2 \int_{0}^{L} dx_1 \ldots dx_N | \det [e^{ik_jx_i}] |^2 = g_{N+2}^{(\infty)}(x,0) = \rho_0 (1 - y(x)^2) \quad (6.27)
\]

where
\[
y(x) = \sin \left( \frac{(N + 2)\pi x}{L} \right) \left( \frac{N + 2}{N + 2} \right) = \frac{1}{\rho_0} K_{N+2,L}(x,0) \quad (6.28)
\]

where we note that (6.24) and (6.26) also appear in the CUE (Circular Unitary Ensemble), given by Dyson 26.

Hence (6.25) is equal to
\[
\frac{2}{c} \frac{\partial}{\partial x} g_{N+2}^{(\infty)}(x,0). \quad (6.29)
\]

Regarding (6.26), note that
where can be written in the simpler form correct to $O(1/cL)$, which is valid for all $N$ structure factor, are readily computed for any system of 2 particles does not make physical sense, But for the free Fermi system the 3-point correlation function is specified by (6.28). Note that this definition is only valid for $N \geq 3$. We have confirmed that this result recovers (5.15). Now (6.40), and concomitantly its structure factor, are readily computed for any $N \geq 3$. The structure factor in the limit $cL \to \infty$, is given by

\[ \frac{\partial}{\partial \xi} \prod_{i=1}^{N} \left( \int_{x}^{L} + \xi \int_{0}^{x} \right) dx_i \left| \det[e^{ik_j \xi}] \right|^{1} = \left( \prod_{i=1}^{N} \int_{0}^{L} dx_i \left| \det[e^{ik_j \xi}] \right|^{2} \right) \left( \prod_{j=1, j \neq l}^{N} \int_{0}^{L} dx_j \left| \det[e^{ik_j \xi}] \right|^{2} \right) \left( \prod_{j=1, j \neq l}^{N} \int_{0}^{L} dx_j \left| \det[e^{ik_j \xi}] \right|^{2} \right) \]

Therefore, (6.26) is equal to

\[ \frac{2}{c} \frac{1}{N (N-1) (N(\infty))^{2}} \frac{1}{\frac{\partial}{\partial \xi} \int_{0}^{x} dx_2 \int_{0}^{L} dx_N} \left| \det[e^{ik_j \xi}] \right|^{2} \]

But for the free Fermi system the 3-point correlation function is specified by

\[ g_{3,(N+2)}^{(\infty)}(x,0,x_1) = \frac{1}{(N-1)! (N(\infty))^{2}} \int_{0}^{L} dx_2 \int_{0}^{L} dx_N \left| \det[e^{ik_j \xi}] \right|^{2} \]

\[ = \rho_0 \det \begin{pmatrix} 1 & y(x) & y(x_1) \\ y(x) & 1 & y(x-x_1) \\ y(x_1) & y(x-x_1) & 1 \end{pmatrix} \]

\[ = \rho_0 [1 - y(x)^2 - y(x_1)^2 - y(x-x_1)^2 + 2y(x)y(x_1)y(x-x_1)] \]

where $y(x)$ is specified by (6.25). Note that this definition is only valid for $N \geq 1$, a 3-point correlation function for a system of 2 particles does not make physical sense, $g_{3,2}^{(\infty)}(x,0,x_1) = 0$. Now (6.84) reduces to

\[ \frac{2}{c} \frac{\partial}{\partial x} \int_{0}^{x} dx_1 g_{3,(N+2)}^{(\infty)}(x,0,x_1) \]

Finally, adding up all contributions gives the sought closed form expression (here we revert to $N + 2 \to N$)

\[ g_{N}(x,0) = g_{N}^{(\infty)}(x,0) - \frac{2\rho_0}{c} \left( x \frac{\partial}{\partial x} + 1 \right) g_{N}^{(\infty)}(x,0) + \frac{2}{c} \frac{\partial}{\partial x} g_{N}^{(\infty)}(x,0) + \frac{2}{c} \frac{\partial}{\partial x} \int_{0}^{x} dx_1 g_{3,N}^{(\infty)}(x,0,x_1) \]

correct to $O(1/cL)$, valid for all $N \geq 2$. This has been checked with (6.14) and (6.15). Using (6.27) and (6.30) this can be written in the simpler form

\[ g_{N}(x,0) = g_{N}^{(\infty)}(x,0) + 4N \left( \frac{1}{cL} \right) \left\{ - y(x) [\rho_0 y(x) + y'(x)] \right\} + \rho_0 \frac{\partial}{\partial x} \left[ y(x) \int_{0}^{x} y(x_1) y(x-x_1) dx_1 \right] \]

\[ + O \left( \frac{1}{cL} \right)^{2} \]

which is valid for all $N \geq 3$. We have confirmed that this result recovers (6.14). Now (6.40), and concomitantly its structure factor, are readily computed for any $N \geq 3$. The structure factor in the limit $cL \to \infty$, is given by

\[ S_n(N) = \begin{cases} N & \text{when } n = 0 \\ |n|/N & \text{when } 0 < |n| < N \\ 1 & \text{when } |n| \geq N. \end{cases} \]
In the thermodynamic limit, $y(x)$ becomes

$$y(x) = \frac{\sin(\rho_0 \pi x)}{\rho_0 \pi x} \tag{6.42}$$

and now the correlation function follows from (6.40) and is (where we use the appropriate scaled variable $\rho_0/c$)

$$g_{\infty}(x, 0) = \rho_0 \left(1 - \frac{\sin^2 \bar{x}}{\bar{x}^2}\right) - 4\rho_0 \left(\frac{\sin \bar{x}}{\bar{x}} [\pi \bar{x} \cos \bar{x} + (\bar{x} - \pi) \sin \bar{x}]\right)$$

$$- \frac{\partial}{\partial x} \left[y(x) \int_0^x y(x_1) y(x - x_1) dx_1\right] \left(\frac{\rho_0}{c}\right) + O \left(\frac{\rho_0}{c}\right)^2 \tag{6.43}$$

where $\bar{x} = \rho_0 \pi x$. This equation was first given by Korepin [27]. Recently in [28], using the Random Phase Approximation (RPA), which they indicated is valid to the $\rho_0/c$ correction, the structure factor was calculated in the thermodynamic limit from which (6.43) is recovered.

Here we explicitly evaluate the integral in (6.43) with (6.42)

$$\int_0^x y(x_1) y(x - x_1) dx_1 = \frac{\sin(\bar{x}) \text{Si}(2\bar{x}) + \cos(\bar{x}) [\text{Ci}(2\bar{x}) - \log(2\bar{x}) - \gamma]}{\rho_0 \pi \bar{x}} \tag{6.44}$$

where $\gamma$ is Euler’s constant, and Si and Ci are the sine integral and cosine integral functions, respectively.

In closing this section we note that $g_N(0, 0)$ in (6.40) and similarly (6.43) is zero to leading order, the free Fermi result. Deviations from zero do not begin until higher order than $1/cL$ as can be seen from these equations.

### VII. CONCLUDING REMARKS

The occupation numbers for the large $cL$ limit given in (4.21) can be continued to the asymptotically large $N$ limit giving

$$c_n(N) \sim N^{\frac{1}{2} + \frac{2m}{d}} \tag{7.1}$$

Very strong evidence for the exponent $\beta_n$ having the integer value 2 was presented.

In constructing the ansatz for the occupation numbers, we chose to scale $cL$ by $N$ from the outset. This was done because the $cL$ large limit is now compatible with the thermodynamic limit (unlike the small $cL$ limit as discussed after (3.20)) and further to this we found that the numerical analysis, for any finite $N$, in constructing the ansatz, failed without this scaling.

The question then is, how does (7.1) compare with the thermodynamic limit. In a very nice work, at a time coincident with the seminal work of the Japanese group [8, 29], the density matrix first given by Lenard in the impenetrable limit [7], [31], was extended to the $1/c$ correction [30]. They used the quantum inverse-scattering method in concert with the very important work [29] on the impenetrable Bose gas in terms of Painlevé V theory, and obtained

$$\rho(x) \sim \frac{1}{|x|^{\frac{d}{2} - \frac{2m}{d}}} \tag{7.2}$$

The Fourier transform of (7.2) gives the momentum distribution, with the Fermi momentum, $k_F = \pi \rho_0$,

$$c(k) \sim \frac{1}{|k|^{\frac{d}{2} + \frac{2m}{d}}} \tag{7.3}$$
Following the presentation given in (10), we can readily see that (7.1) is one-to-one with (7.3). An elementary way to see this immediately is to observe that $|k| \sim 1/L$ and thus in the thermodynamic limit $|k| \sim 1/N$. Note the integer value of 2 for the coefficient of $k_F/\pi c$ in the exponent (7.2) and (7.3).

In our work on the impenetrable Bose gas (10, 11), we studied the system in all boundary conditions, periodic, Dirichlet, Neumann, as well as for the harmonically trapped system. In all cases, we found that in the asymptotically large $N$ limit that (7.1) held in the limit $c \rightarrow \infty$. This firmly suggests that the exponent obtained, in this large $N$ limit, is universal, being the same for all boundary conditions and (low lying) $n$ modes. Therefore, we anticipate the same is true now for our (7.1).

The use of periodic boundary conditions, while facilitating the mathematics, is, nonetheless, a powerful preemptor of the analytical properties of the Bose gas, a system of both continuing and immense attraction to the theoretician and the experimentalist, in concert.

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APPENDIX A: BETHE EQUATION SOLUTIONS

\[ N = 2 \]

\[ k_2 = \sqrt{\frac{c}{L}} \left[ 1 - \frac{1}{24}(cL) - \frac{11}{5760}(cL)^2 - \frac{17}{154828800}(cL)^3 - \frac{281}{154828800}(cL)^4 + O(cL)^5 \right] \] (A1)
\[ N = 3 \]

\[
k_3 = \sqrt{\frac{3c}{L}} \left[ 1 - \frac{1}{24}(cL) + \frac{19}{5760}(cL)^2 - \frac{299}{967680}(cL)^3 + \frac{11077}{464486400}(cL)^4 + O(cL)^5 \right] \quad (A2)
\]

\[ N = 4 \]

\[
k_4 = \sqrt{\left(3 + \sqrt{6}\right) \frac{c}{L}} \left[ 1 - \frac{1}{24}(cL) + \frac{31 - 2\sqrt{6}}{5760}(cL)^2 \\
+ \frac{-879 + 86\sqrt{6}}{967680}(cL)^3 + \frac{63381 - 5500\sqrt{6}}{464486400}(cL)^4 + O(cL)^5 \right] \quad (A3a)
\]

\[
k_3 = \sqrt{\left(3 - \sqrt{6}\right) \frac{c}{L}} \left[ 1 - \frac{1}{24}(cL) + \frac{31 + 2\sqrt{6}}{5760}(cL)^2 \\
+ \frac{-879 - 86\sqrt{6}}{967680}(cL)^3 + \frac{63381 + 5500\sqrt{6}}{464486400}(cL)^4 + O(cL)^5 \right] \quad (A3b)
\]

\[ N = 5 \]

\[
k_5 = \sqrt{\left(5 + \sqrt{10}\right) \frac{c}{L}} \left[ 1 - \frac{1}{24}(cL) + \frac{39 - 2\sqrt{10}}{5760}(cL)^2 \\
+ \frac{-1511 + 118\sqrt{10}}{967680}(cL)^3 + \frac{165589 - 13196\sqrt{10}}{464486400}(cL)^4 + O(cL)^5 \right] \quad (A4a)
\]

\[
k_4 = \sqrt{\left(5 - \sqrt{10}\right) \frac{c}{L}} \left[ 1 - \frac{1}{24}(cL) + \frac{39 + 2\sqrt{10}}{5760}(cL)^2 \\
+ \frac{-1511 - 118\sqrt{10}}{967680}(cL)^3 + \frac{165589 + 13196\sqrt{10}}{464486400}(cL)^4 + O(cL)^5 \right] \quad (A4b)
\]

We have also computed these expansions out to \( N = 10 \), due to the inordinate complexities of the numbers, the expansions for \( N \geq 6 \) are known only in decimal form.

The leading term of \( k_j \) is precisely related to the \( j \)th zero of the \( N \)th polynomial, as mentioned in Section II. Note also the universality of the coefficient of the \((cL)\) term, that is \(-1/24\) (as in (2.8)).

**APPENDIX B: FREDHOLM DETERMINANTS**

Here the Fredholm minor \((4.12)\) will be related to a multiple integral, which in turn implies the Toeplitz determinant form \((4.13)\). Consider the multiple integral

\[
A_N(x, y) = \left( \int_0^L + \lambda \int_y^x \right) dx_2 \ldots \left( \int_0^L + \lambda \int_y^x \right) dx_N \\
\times \prod_{j=2}^N \left\{ 2 \sin \left[ \pi (x - x_j)/L \right] \sin \left[ \pi (y - x_j)/L \right] \right\}^2 \prod_{2 \leq j \leq k \leq N} \left\{ 2 \sin \left[ \pi (x_k - x_j)/L \right] \right\}^2 
\]

(B1)

Setting

\[
g(u) = (1 + \lambda \chi_{[y,x]}^{(u)}) 2 \sin \left[ \pi (x - u)/L \right] 2 \sin \left[ \pi (u - y)/L \right] 
\]

(B2)

where \( \chi_{[y,x]}^{(u)} = 1 \) for \( u \in [y, x] \) and 0 otherwise, allows this to be written

\[
A_N(x, y) = \int_0^L dx_2 \ldots \int_0^L dx_N \prod_{i=2}^N g(x_i) \prod_{1 \leq j \leq k \leq N} \left| e^{2\pi x_k/L} - e^{2\pi x_j/L} \right|^2 . 
\]

(B3)
By a well known identity (see e.g. Szegő [24]) this is equal to a Toeplitz determinant,

$$A_N(x, y) = (N - 1)! \det \left[ \int_0^L du \, g(u) e^{2i\pi u (j - k)/L} \right]_{j, k = 1, \ldots, N - 1}. \quad (B4)$$

On the other hand the integral in $A_N(x, y)$ can be expanded as a power series in $\lambda$. For this define

$$\phi_N(x_1, x_2, \ldots, x_N) = \prod_{1 \leq j < k \leq N} (e^{2i\pi x_k/L} - e^{2i\pi x_j/L}) \quad (B5)$$

and introduce the free Fermi type distribution

$$\rho_{FF}^N(x, y; x_2, \ldots, x_n) = \frac{(N - 1)!}{(N - n)!} C_{N,L} \int_0^L dx_{n+1} \ldots \int_0^L dx_N \times \phi_N(x, x_2, \ldots, x_N) \phi(y, x_2, \ldots, x_N) \quad (B6)$$

where $C_{N,L}$ is such that

$$\rho_{FF}^N(x, x) = \frac{N}{L} \quad (B7)$$

and thus given by

$$NC_{N,L} = \int_0^L dx_1 \ldots \int_0^L dx_N |\phi(x_1, \ldots, x_N)|^2. \quad (B8)$$

Now, writing the integrand in (B4) in terms of (B5), expanding in $\lambda$, and making use of the definition (B6) we see

$$A_N(x, y) = C_{N,L} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_y^x dx_2 \ldots \int_y^{x_{n+1}} \rho_{FF}^N(x, y; x_2, \ldots, x_{n+1}). \quad (B9)$$

A straightforward calculation (see e.g. Forrester [27]) gives the determinant form

$$\rho_{FF}^N(x, y; x_2, \ldots, x_{n+1}) = \det \left[ K_{NL}(x, y) \quad [K_{NL}(x, u_k)]_{k=1, \ldots, n} \quad [K_{NL}(u_j, y)]_{j=1, \ldots, n} \quad [K_{NL}(u_j, u_k)]_{j,k=1, \ldots, n} \right] \quad (B10)$$

and furthermore shows from (B8) that

$$C_{N,L} = (N - 1)! L^N. \quad (B11)$$

Substituting (B11) and (B10) in (B9) and comparing with (4.12) shows

$$\frac{1}{\lambda} \Delta_1 \left( \frac{x}{0} : \lambda \right) = \frac{1}{(N - 1)! L^N} A_N(x, 0). \quad (B12)$$

With $A_N(x, y)$ in its Toeplitz determinant form (B4), this gives (4.4).
| \( N \) | \( c_0^{(0)}(N) \) | \( c_0^{(1,1)}(N) \) | \( c_0^{(1,2)}(N) \) | \( c_0^{(1)}(N) \) | \( c_1^{(0)}(N) \) | \( c_1^{(1,1)}(N) \) | \( c_1^{(1,2)}(N) \) | \( c_1^{(1)}(N) \) |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.810569 | -0.757722 | 4 | 3.24228 | 0.090633 | -0.227763 | -1.33333 | -1.56110 |
| 3 | 0.702151 | -1.78709 | 8 | 6.21291 | 0.111111 | -0.924374 | -1 | -1.92437 |
| 4 | 0.629414 | -2.96469 | 11.9525 | 8.98781 | 0.116609 | -1.85190 | -0.111842 | -1.96375 |
| 5 | 0.576137 | -4.23863 | 15.8505 | 11.6119 | 0.117046 | -2.92064 | 1.05226 | -1.86838 |
| 6 | 0.534872 | -5.58153 | 19.6958 | 14.1143 | 0.115561 | -4.08611 | 2.38235 | -1.70426 |
| 7 | 0.501065 | -6.9769 | 23.4922 | 16.5153 | 0.113315 | -5.32460 | 3.82994 | -1.50066 |
| 8 | 0.474130 | -8.41392 | 27.4840 | 18.8301 | 0.110796 | -7.61868 | 5.34000 | -1.27269 |
| 9 | 0.450832 | -9.88502 | 30.9548 | 21.0699 | 0.108225 | -7.59030 | 6.92912 | -1.02891 |
| 10 | 0.430766 | -11.3847 | 34.6282 | 23.4235 | 0.105074 | -9.34388 | 8.56031 | -0.774574 |
| 11 | 0.413239 | -12.9087 | 38.2671 | 25.3583 | 0.103278 | -10.7435 | 10.2304 | -0.513070 |
| 12 | 0.397753 | -14.4539 | 41.8741 | 27.4202 | 0.100969 | -12.1794 | 11.9327 | -0.246672 |
| 13 | 0.383935 | -16.0177 | 45.4517 | 29.4340 | 0.098779 | -13.6392 | 13.6622 | 0.023089 |
| 14 | 0.371504 | -17.5979 | 49.0018 | 31.4039 | 0.096709 | -15.1200 | 15.4150 | 0.294035 |
| 15 | 0.360239 | -19.1928 | 52.5264 | 33.3335 | 0.09476 | -16.1196 | 17.1878 | 0.568195 |
| 16 | 0.349968 | -20.8010 | 56.0269 | 35.2259 | 0.092960 | -18.1360 | 18.9782 | 0.842211 |
| 17 | 0.339505 | -22.4213 | 59.5049 | 37.0836 | 0.091157 | -19.6677 | 20.7842 | 1.11653 |
| 18 | 0.328174 | -24.0525 | 62.9616 | 38.9091 | 0.089503 | -21.2133 | 22.6041 | 1.39080 |
| 19 | 0.318444 | -25.6939 | 66.3983 | 40.7044 | 0.087934 | -22.7715 | 24.3636 | 1.66475 |
| 20 | 0.301685 | -27.3446 | 69.8160 | 42.4715 | 0.086452 | -24.3415 | 26.2977 | 1.93819 |
| 21 | 0.294941 | -29.0039 | 73.2157 | 44.2118 | 0.085029 | -25.9223 | 28.1333 | 2.19961 |
| 22 | 0.289279 | -30.6712 | 76.5983 | 45.9271 | 0.083628 | -27.5132 | 29.9961 | 2.48292 |
| 23 | 0.284626 | -32.3460 | 79.9645 | 47.6185 | 0.082390 | -29.1133 | 31.8673 | 2.75400 |
| 24 | 0.280088 | -34.0278 | 83.3150 | 49.2874 | 0.081173 | -30.7222 | 33.7464 | 3.02411 |
| 25 | 0.275677 | -35.7161 | 86.6509 | 50.9348 | 0.080030 | -32.3239 | 35.6325 | 3.29203 |
| 26 | 0.280564 | -37.4086 | 89.9724 | 52.5618 | 0.078882 | -33.9641 | 37.5254 | 3.56123 |
| 27 | 0.275723 | -39.1100 | 93.2082 | 54.1692 | 0.077809 | -35.5961 | 39.4243 | 3.82817 |
| 28 | 0.271128 | -40.8168 | 96.5749 | 55.7558 | 0.076780 | -37.2350 | 41.3290 | 4.09399 |
| 29 | 0.266761 | -42.5279 | 99.8570 | 57.3292 | 0.075791 | -38.8803 | 43.2390 | 4.35868 |
| 30 | 0.262603 | -44.2438 | 103.127 | 58.8832 | 0.074843 | -40.5318 | 45.1762 | 4.64439 |
| 31 | 0.258637 | -45.9876 | 106.385 | 60.4207 | 0.073930 | -42.1891 | 47.0737 | 4.88464 |
| 32 | 0.254849 | -47.6896 | 109.632 | 61.9426 | 0.073051 | -43.8519 | 48.9978 | 5.14590 |
| 33 | 0.251227 | -49.4190 | 112.868 | 63.4493 | 0.072203 | -45.5201 | 50.9260 | 5.40589 |
| 34 | 0.247757 | -51.1525 | 116.094 | 64.9414 | 0.071385 | -47.1931 | 52.8581 | 5.66498 |
| 35 | 0.244430 | -52.8899 | 119.309 | 66.4194 | 0.070596 | -48.8710 | 54.7939 | 5.92294 |
| 36 | 0.241237 | -54.6310 | 122.515 | 67.8839 | 0.069836 | -50.5537 | 56.7332 | 6.17953 |

APPENDIX C: NUMERICAL DATA
TABLE III: Values of $c_0^{(0)}(N)$ and $c_0^{(1)}(N)$ for $N = 2, 3, 4, 5, 6, 7$. Note that this Table extends Table II of [10].

| $N$ | $c_0^{(0)}(N)$ | $c_0^{(1)}(N)$ |
|-----|----------------|----------------|
| 2   | $\frac{16}{\pi^2}$ | $-\frac{64}{\pi^2}$ | 6.48456... |
| 3   | $\frac{1}{3} + \frac{35}{2\pi^2}$ | $8 + \frac{105}{\pi^2}$ | 18.6387... |
| 4   | $-\frac{2097152}{19845\pi^2} + \frac{320}{9\pi^2}$ | $-\frac{16777216}{19845\pi^2} + \frac{128752}{315\pi^2}$ | 35.9512... |
| 5   | $\frac{1}{5} + \frac{743649}{129600\pi^2} + \frac{4459}{216\pi^2}$ | $\frac{16}{3} + \frac{743649}{129600\pi^2} + \frac{240013}{540\pi^2}$ | 58.0593... |
| 6   | $\frac{19350784085380604199811312}{12749115781913474078125\pi^6}$ | $\frac{7749319223232214679925248}{4249386598330449159375\pi^6}$ | 84.6855... |
| 7   | $\frac{8576621135864297814}{40663643328600000\pi^6} - \frac{46891706849}{317520000\pi^6} + \frac{79679}{3000\pi^2} + 3.51155 \ldots$ | $\frac{24}{9} - \frac{57472927}{152100\pi^6} + \frac{47201}{160\pi^2}$ | 115.607... |

TABLE IV: Values of $c_1^{(0)}(N)$ and $c_1^{(1)}(N)$ for $N = 2, 3, 4, 5, 6, 7$

| $N$ | $c_1^{(0)}(N)$ | $c_1^{(1)}(N)$ |
|-----|----------------|----------------|
| 2   | $0.180127\ldots$ | $-\frac{812}{25\pi^2}$ | -3.12219... |
| 3   | $0.33333\ldots$ | $-4 + \frac{45}{2\pi^2}$ | -5.77312... |
| 4   | $\frac{466435}{2235625\pi^6} + \frac{332}{25\pi^2}$ | $\frac{5278120741968}{7032571875\pi^6} + \frac{1280912}{2025\pi^2}$ | -7.85498... |
| 5   | $\frac{1}{5} + \frac{1950989}{129600\pi^2} + \frac{3817}{216\pi^2}$ | $\frac{28}{9} - \frac{57472927}{152100\pi^6} + \frac{47201}{160\pi^2}$ | -9.34192... |
| 6   | $\frac{4458567678128563348987439874048}{4115703020232062020511144575\pi^6}$ | $\frac{3176031176261117565814042882856845312}{6061408710295087266613462634375\pi^6}$ | -10.2255... |
| 7   | $\frac{4458567678128563348987439874048}{4115703020232062020511144575\pi^6} - \frac{794}{147\pi^2}$ | $\frac{90755596276189968978208}{172210286818251875\pi^6} + \frac{8618142656}{6131125\pi^6}$ | -10.2255... |

TABLE V: Values of $c_2^{(0)}(N)$ and $c_2^{(1)}(N)$ for $N = 2, 3, 4, 5, 6$

| $N$ | $c_2^{(0)}(N)$ | $c_2^{(1)}(N)$ |
|-----|----------------|----------------|
| 2   | $0.00720506\ldots$ | $-\frac{3136}{119\pi^2}$ | -0.0941461... |
| 3   | $0.0985067\ldots$ | $-\frac{385}{12\pi^2}$ | -3.25072... |
| 4   | $\frac{74868145210948}{13268660265\pi^6} + \frac{27584}{3075\pi^2}$ | $\frac{2710310553200963316}{45865750115625\pi^6} + \frac{6735122448}{7640325\pi^6}$ | -6.28072... |
| 5   | $\frac{1}{5} + \frac{10624792}{120000\pi^2} + \frac{325}{216\pi^2}$ | $\frac{52}{9} + \frac{30350602129}{17781120\pi^6} + \frac{221099}{1808\pi^2}$ | -9.06869... |
| 6   | $\frac{16976940396817340218487564310413132}{82114358489309418643544056915\pi^6}$ | $\frac{4149702189110332829474555278533770452189184}{212455043569909107006123732109715\pi^6}$ | -11.5824... |
| 7   | $\frac{47388412779764105728}{1939504885606875\pi^6} + \frac{990928}{19845\pi^2}$ | $\frac{28872752035815184832035328}{1341756884260684346025\pi^6} + \frac{842559125}{750750\pi^6}$ | -11.5824... |
| N  | \( c_0^{(1,1)}(N) \) | \( c_0^{(1,2)}(N) \) |
|----|------------------|------------------|
| 2  | \(-8 + \frac{64}{27}\) | \(-1.51544\ldots\) | 8 | 8 |
| 3  | \(-16 + \frac{105}{256}\) | \(-5.36128\ldots\) | 24 | 24 |
| 4  | \(-32 + \frac{4677216}{19845\pi^2} + \frac{2560}{9\pi^2}\) | \(-11.8587\ldots\) | 32 + \(\frac{46384}{105\pi^2}\) | 47.8100\ldots |
| 5  | \(-48 + \frac{7436429}{12900\pi^2} + \frac{2295}{108\pi^2}\) | \(-21.9131\ldots\) | \(\frac{160}{3} + \frac{23023}{90\pi^2}\) | 79.2524\ldots |
| 6  | \(-72 + \frac{77403139222322441679923248}{424938591804449335937\pi^6}\) | \(-33.4892\ldots\) | 72 - \(\frac{7448167421360128}{517925674641375\pi^6}\) + 7061504 | 15015\pi^2 | 118.175\ldots |
|    | \(\text{TABLE VI: Values of } c_0^{(1,1)}(N) \text{ and } c_0^{(1,2)}(N) \text{ for } N = 2, 3, 4, 5, 6\) |

| N  | \( c_1^{(1,1)}(N) \) | \( c_1^{(1,2)}(N) \) |
|----|------------------|------------------|
| 2  | \(-8 - \frac{832}{27\pi^2}\) | \(-0.45527\ldots\) | \(-8\) | \(-2.66667\ldots\) |
| 3  | \(-1 - \frac{35}{2\pi^2}\) | \(-2.77312\ldots\) | \(-3\) | \(-3\ldots\) |
| 4  | \(-352 + \frac{52781507411968}{70325118756\pi^4} + \frac{65129984}{70875\pi^2}\) | \(-7.40762\ldots\) | \(352 - \frac{46728064}{70875\pi^2}\) | \(-0.447369\ldots\) |
| 5  | \(-\frac{109}{3} + \frac{57472927}{15120\pi^4} + \frac{6457}{108\pi^2}\) | \(-14.6032\ldots\) | \(\frac{355}{9} - \frac{91091}{29\pi^2}\) | \(5.26130\ldots\) |
| 6  | \(-\frac{2232}{35} + \frac{31760134172611117568514042882856845312}{303648000151866875\pi^6}\) | \(-24.5196\ldots\) | \(\frac{2232}{35} - \frac{269870354814096421376}{10938854433889093\pi^6}\) | \(14.2941\ldots\) |
|    | \(\text{TABLE VII: Values of } c_1^{(1,1)}(N) \text{ and } c_1^{(1,2)}(N) \text{ for } N = 2, 3, 4, 5, 6\) |

| N  | \( c_2^{(1,1)}(N) \) | \( c_2^{(1,2)}(N) \) |
|----|------------------|------------------|
| 2  | \(8 - \frac{3136}{125\pi^2}\) | \(0.439187\ldots\) | \(-8\) | \(-0.533333\ldots\) |
| 3  | \(4 - \frac{385}{125\pi^2}\) | \(0.749279\ldots\) | \(-4\) | \(-4\) |
| 4  | \(\frac{2272}{105} + \frac{37103105389606232}{458657305115625\pi^6} + \frac{30717772408}{38201625\pi^4}\) | \(-0.641704\ldots\) | \(-2272 + \frac{6032211968}{38201625\pi^4}\) | \(-5.63902\ldots\) |
| 5  | \(\frac{31}{3} + \frac{3035899219}{17781120\pi^6} + \frac{228405}{564\pi^2}\) | \(-4.09068\ldots\) | \(-\frac{145}{9} + \frac{830687}{756\pi^2}\) | \(-4.97802\ldots\) |
| 6  | \(\frac{3592}{105} + \frac{419470218911033289475527785337770452189184}{2212450435659351978966123732109375\pi^6}\) | \(-9.82321\ldots\) | \(\frac{3592}{105} + \frac{198359617518523964085719104}{940026976360283844375\pi^6}\) | \(-1.75923\ldots\) |
|    | \(\text{TABLE VIII: Values of } c_2^{(1,1)}(N) \text{ and } c_2^{(1,2)}(N) \text{ for } N = 2, 3, 4, 5, 6\) |