Instanton representation of Plebanski gravity: XVI.
Hamiltonian and Hamilton–Jacobi dynamics on
superspace

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Abstract

In this paper we focus on the Hamiltonian dynamics on the kinematic phase space of the instanton representation, in the full theory. The general solutions for pure gravity are reduced to quadratures and fixed point iteration both for vanishing and nonvanishing cosmological constant, with convenient physical interpretations. A Hamilton–Jacobi analysis is performed of the semiclassical orbits and their relation to the quantum theory motivated. We have constructed Hamilton–Jacobi functionals mimicking the classical dynamics. A main result is emergence of a natural time variable on the configuration space with respect to which the remaining variables evolve.
1 Introduction

The purpose of this paper is to provide an analysis of the classical and the semiclassical dynamics implied by the instanton representation of Plebanski gravity. The basic phase space variables for this representation are the CDJ matrix $\Psi_{ae}$ and the Ashtekar self-dual $SU(2)$ connection $A^a_i$. Some of the background for the instanton representation can be found in Paper II and references therein. To unobscure the dynamics of the theory, we will treat the case where $\Psi_{ae}$ is diagonal, which is regarded as the kinematic momentum space $P_{Kin}$.1

A canonical structure where the CDJ matrix $\Psi_{ae}$ is diagonal presents at least three main advantages for treatment of the classical and the quantum theory of gravity. First, the canonical structure can be chosen such that all terms containing spatial gradients automatically cancel out without requiring these terms to be zero.2 This affords one the benefits of the simplicity of minisuperspace while still treating the full theory. Secondly, this accomplishes a natural projection from the full unconstrained phase space $\Omega_{Inst}$ to the physical degrees of freedom, wherein the kinematic constraints (Gauss’ law constraint $G_a$ and diffeomorphism constraint $H_i$) can be eliminated from the starting action. The third point is that the remaining variables, which are regarded as the physical degrees of freedom, reside entirely within the Hamiltonian constraint which in the instanton representation affords one the luxury of focusing on the true dynamics of gravity both from the classical and from the quantum standpoint, free of kinematic effects.

We will be treating the full theory in densitized momentum space variables $\tilde{\Psi}_{ae} = \Psi_{ae} (\text{det} A)$, where $(\text{det} A) \neq 0$ is the determinant of $A^a_i$, since as it is shown in Paper XIII that these variables admit a well-defined canonical structure with globally holonomic configuration space variables. Hence in this paper we will analyse the Hamiltonian dynamics on this phase space, and the ensuing Hamilton–Jacobi formalism.

The full theory of the instanton representation admits configurations based on specific combinations of $A^a_i$, such that all terms containing spatial gradients cancel out from the canonical one form $\theta$. Unlike for the kinematic constraints, the Hamiltonian constraint remains intact at the level of the Lagrangian since it contains explicitly the essential degrees of freedom encoding the dynamics. The constraint becomes implemented only subse-

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1 It is shown in Paper IV that the $SO(3, C)$ rotation angles which make $\Psi_{ae}$ nondiagonal are ignorable coordinates in the canonical structure. Hence there is no loss of generality in using a diagonal CDJ matrix, since this corresponds to the intrinsic $SO(3, C)$ frame.

2 This comprises the quantizable configurations of the instanton representation, as shown in Paper XIII.
quent to computing the Hamilton’s equations of motion, and as part of this paper we verify consistency of the the constraint with the ensuing dynamics.

The organization of this paper is as follows. In section 2 we set up the canonical structure and put in place the notation, definitions and variables which will be used. Having expressed the Hamiltonian explicitly in terms of the densitized momentum space variables, we proceed in section 3 to derive the Hamiltonian equations of motion. There are two case to consider which divide the dynamics into two regimes, namely for $\Lambda = 0$ and for $\Lambda \neq 0$. The $\Lambda = 0$ case is analyzed in section 4, where it is found that the dynamics mimic the classical dynamics of a free particle on a two dimensional configuration space. The gravitational degrees of freedom evolve linearly with respect to $T$, seen as a time variable on configuration space. Section 5 treats the $\Lambda \neq 0$ case where the evolution dynamical evolution is nonlinear and more complicated. Nevertheless we are able to make certain inferences about the form of the solution. In this section we provide a fixed point procedure whereby solutions may in principle be constructed. In section 6 we provide a precursory analysis of the Hamilton–Jacobi aspects of the theory. We have constructed a Hamiltonian Jacobi functional which mimics the classical dynamics of the previous sections, and provides a glance into the quantum theory which is treated in a separate paper. One main feature is the stationary of the gravitational degrees of freedom with respect to evolution in $T$, which defines the functional on the initial and the final spatial hypersurfaces. It appears for $\Lambda = 0$ that the functional is truly stationary in $T$ and for $\Lambda \neq 0$ that the functional evolves in $T$. This seems to be congruous with the classical dynamics thus computed, and forms a good basis for comparison with the quantum theory.
2 Setting the stage

We will start without loss of generality from the following canonical structure for the instanton representation, where the momentum space variables $\Psi_{ae} \in SO(3,C) \otimes SO(3,C)$ are diagonal.

$$\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \Psi_{ae} B_i^e \delta A^a_i$$

$$= \frac{1}{G} \int_{\Sigma} d^3x \left( \Psi_{11} A_2^3 A_3^3 \delta A_1^1 + \Psi_{22} A_3^3 A_1^1 \delta A_2^2 + \Psi_{33} A_1^1 A_2^2 \delta A_3^3 \right). \quad (1)$$

In (1) $A^a_i$ is the left handed $SO(3,C)$ Ashtekar connection and $B_i^e = B_i^e[A]$ is the Ashtekar magnetic field. The subscript $Kin$ in (1) refers to the kinematical level, namely to the level subsequent to implementation of the Gauss’ law and diffeomorphism constraints and prior to implementation of the Hamiltonian constraint. To obtain coordinates which are globally holonomic on configuration space $\Gamma$ we define the densitized momentum space variables

$$\tilde{\Psi}_{11} = \Psi_{11}(A_1^1 A_2^2 A_3^3); \quad \tilde{\Psi}_{22} = \Psi_{22}(A_1^1 A_2^2 A_3^3); \quad \tilde{\Psi}_{33} = \Psi_{33}(A_1^1 A_2^2 A_3^3), \quad (2)$$

where $(A_1^1 A_2^2 A_3^3) \neq 0$. Hence the ranges of the coordinates as $0 < |A_f^i| < \infty$.

Using (2) for the fundamental momentum space variables, the canonical one form (1) reduces to

$$\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \left( \tilde{\Psi}_{11} \frac{\delta A_1^1}{A_1^1} + \tilde{\Psi}_{22} \frac{\delta A_2^2}{A_2^2} + \tilde{\Psi}_{33} \frac{\delta A_3^3}{A_3^3} \right). \quad (3)$$

Having identified the variables of interest, next rewrite (1) in the form

$$\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \left( (\tilde{\Psi}_{11} - \tilde{\Psi}_{33}) \frac{\delta A_1^1}{A_1^1} + (\tilde{\Psi}_{22} - \tilde{\Psi}_{33}) \frac{\delta A_2^2}{A_2^2} + \tilde{\Psi}_{33} \left( \frac{\delta A_1^1}{A_1^1} + \frac{\delta A_2^2}{A_2^2} + \frac{\delta A_3^3}{A_3^3} \right) \right) \quad (4)$$

and make the following definitions for the momentum space variables

$$\tilde{\Psi}_{11} - \tilde{\Psi}_{33} = \alpha; \quad \tilde{\Psi}_{22} - \tilde{\Psi}_{33} = \beta; \quad \tilde{\Psi}_{33} = \lambda \quad (5)$$

and correspondingly for variations in the tangent space $T_X(\Gamma)$ to configuration space $\Gamma$.

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3It is shown in Paper XIII that this this combined with any of six configurations for the connection $A^a_i$ are quantizable configurations upon densitization of $\Psi_{ae}$. 
\[
\frac{\delta A_1^1}{A_1^1} = \delta X; \quad \frac{\delta A_2^2}{A_2^2} = \delta Y; \quad \frac{\delta A_1^1}{A_1^1} + \frac{\delta A_2^2}{A_2^2} + \frac{\delta A_3^3}{A_3^3} = \delta T. \tag{6}
\]

Equation (6) implies the existence of globally holonomic coordinates \((X, Y, T)\) on the kinematic configuration space \(\Gamma_{Kin}\), such that

\[
A_1^1 = a_0 e^X; \quad A_2^2 = a_0 e^Y; \quad (\det A) = A_1^1 A_2^2 A_3^3 = a_0^3 e^T \tag{7}
\]

for some numerical constant \(a_0\) of mass dimension \([a_0] = 1\). At the kinematical level the variables (5) and (6) define a symplectic two form

\[
\Omega_{Kin} = \frac{1}{G} \int_\Sigma d^3x \left( \delta \alpha \wedge \delta X + \delta \beta \wedge \delta Y + \delta \lambda \wedge \delta T \right). \tag{8}
\]

The mass dimensions of the dynamical variables are

\[
[\alpha] = [\beta] = [\lambda] = 1; \quad [X] = [Y] = [T] = 0, \tag{9}
\]

which makes (8) dimensionless. The ranges of the new coodinates are \(-\infty < |X|, |Y|, |T| < \infty\). To construct a starting action we must define the Hamiltonian \(H_{Kin}\) on the kinematic phase space \(\Omega_{Kin}\). This is given by

\[
H_{Kin} = \frac{1}{G} \int_\Sigma d^3x N(\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr}\Psi^{-1}), \tag{10}
\]

where \(\Lambda\) is the cosmological constant and \(N\) the lapse function.\(^4\) Substitution of the CDJ Ansatz

\[
(\Psi^{-1})^{ae} = B_c^i (\tilde{\sigma}^{-1})_i^a \tag{11}
\]

into (10) yields the smeared Hamiltonian constraint from the Ashtekar formalism for general relativity, given by (see e.g. [1], [2], [3])

\[
H_{Ash} = \frac{1}{2} \int_\Sigma d^3x N(\det \tilde{\sigma})^{-1/2} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_d^i \tilde{\sigma}_b^j \left( \frac{\Lambda}{3} \tilde{\sigma}_c^k + B_c^k \right), \tag{12}
\]

where \(B_c^k\) is the Ashtekar magnetic field derived from the Ashtekar self-dual \(SU(2)\) connection \(A_c^a\) (\(i, j, k, \ldots\) refer to spatial indices and \(a, b, c, \ldots\) refer to internal left-handed \(SU(2)\) indices), and \(\tilde{\sigma}_d^i\) is the densitized triad.

\(^4\)We will from now on omit the prefactor of \(G\) multiplying the Hamiltonian, since it is not essential to the analysis
Equation (11) substituted into (12) eliminates $\tilde{\sigma}_{i}^{a}$ in favor of $\Psi_{ae}$ as a fundamental variable, while preserving the remaining variables along with their physical interpretations. The instanton representation regards (10) as the fundamental starting point. The Hamiltonians (10) and (12) are equivalent only when $(\tilde{\sigma}^{-1})^{i}_{i}$ exists, which requires that $\tilde{\sigma}_{i}^{a}$ be nondegenerate. The antiself dual part of the Weyl curvature tensor in unprimed $SL(2, C)$ indices, $\psi_{ABCD}$, can be written as a three by three matrix

$$
\psi_{ae} = \psi_{ABCD} \eta_{a}^{AB} \eta_{e}^{CD},
$$

where $\eta_{a}^{AB}$ is an isomorphism from unprimed symmetric $SL(2, C)$ index pairs $AB = (00, 01, 10)$ to single $SO(3, C)$ indices $a = (1, 2, 3)$. The CDJ matrix $\Psi_{ae}$ is directly related to $\psi_{ae}$ through the relation

$$
\Psi_{ae}^{-1} = \frac{1}{3} \delta_{ae} \text{tr} \Psi^{-1} + \psi_{ae},
$$

where $\psi_{ae} \in SO(3, C) \otimes SO(3, C)$ is symmetric and traceless in its indices. As a consequence of the relation (14), it then follows that the dynamics implied by (10) are those of the eigenvalues of the antiself-dual part of the Weyl curvature expressed in $SO(3, C)$ language.\(^5\)

We will now write (10) in terms of the densitized momentum variables of the instanton representation using the identities

$$
\tilde{\Psi}_{ae} = \Psi_{ae} (\det A); \quad \text{tr} \tilde{\Psi}^{-1} = (\det A) \text{tr} \tilde{\Psi}^{-1}; \quad \sqrt{\det \tilde{\Psi}} = \sqrt{\det \Psi} (\det A)^{-3/2},
$$

Substitution of (15) into (10) yields a Hamiltonian density

$$
H[N] = \int_{\Sigma} d^{3}x N (\det B)^{1/2} (\det A)^{-3/2} \sqrt{\det \tilde{\Psi}} \left( \Lambda + (\det A) \text{tr} \tilde{\Psi}^{-1} \right).
$$

Defining the densitized eigenvalues of $\Psi_{ae}$ by $\lambda, \lambda + \alpha$ and $\lambda + \beta$, we can write the constituents of (16) as

$$
(\det A) = a_{0}^{3} e^{T}; \quad \det \tilde{\Psi} = \lambda (\lambda + \alpha) (\lambda + \beta);
$$

$$
\text{tr} \tilde{\Psi}^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta}.
$$

\(^5\)The Weyl curvature encodes the algebraic classification of spacetime [4]. The rationale behind the instanton representation is that these properties, which are invariant under change of coordinates and tetrad frames, are ideal for describing general relativity from both the classical and from the quantum standpoint.
While all terms of the theory containing spatial gradients cancel out of the canonical one form (4) for the given choice of $A^a_i$ (see e.g. Paper XIII), they do not cancel out of the magnetic field $B^i_a$ whose determinant appears in the Hamiltonian. For a diagonal connection $A^a_i = \delta^a_i A^a_1$ the Ashtekar magnetic field is given in matrix form by

$$B^i_a = \begin{pmatrix} A^2_2 A^3_3 & -\partial_3 A^2_3 & \partial_2 A^3_3 \\ \partial_2 A^1_1 & A^3_2 A^1_1 & -\partial_1 A^3_2 \\ -\partial_1 A^2_1 & \partial_1 A^2_2 & A^1_2 A^2_1 \end{pmatrix}.$$ 

The determinant of $B^i_a$ is given by

$$\text{det}B = (A^1_1 A^2_2 A^3_3)^2 + r,$$

where $r$ is a remainder which contains the spatial gradients, given by

$$r = (\partial_2 A^3_3)(\partial_3 A^1_1)(\partial_1 A^2_2) - (\partial_3 A^2_2)(\partial_1 A^3_3)(\partial_2 A^1_1) + \frac{1}{4} \left[ \partial_1 (A^2_2)^2 \partial_1 (A^3_3)^2 + \partial_2 (A^3_3)^2 \partial_2 (A^1_1)^2 + \partial_3 (A^1_1)^2 \partial_3 (A^2_2)^2 \right]. \quad (19)$$

It is convenient to define a dimensionless quantity $U$ such that

$$\text{det}B = (U \text{det}A)^2 \rightarrow U = 1 + r(\text{det}A)^{-2}, \quad (20)$$

which factors out the leading order behavior of $\text{det}A = A^1_1 A^2_2 A^3_3$ from $\text{det}B$. Putting (17) and (20) into (16), we have

$$H[N] = \int_{\Sigma} d^3x NU(a_0^{-3/2} e^{-T/2} \sqrt{\text{det}\bar{\Psi}} \left( \frac{1}{\lambda + \alpha + \beta} \right)^{-1})$$

$$= \int_{\Sigma} d^3xNU a_0^{-3/2} e^{-T/2} \sqrt{\frac{1}{\lambda + \alpha + \beta}} \left[ \Lambda + a_0^3 e^T \left( \frac{1}{\lambda + \alpha + \beta} \right) \right]. \quad (21)$$

Having expressed the Hamiltonian explicitly in terms of the physical variables we can now construct the action, given by

$$\int dt \int_{\Sigma} d^3x \left( \alpha \dot{X} + \beta \dot{Y} + \lambda \dot{T} \right) - H[N], \quad (22)$$

where the Hamiltonian is given by (21).
3 Hamilton’s equations of motion

Having set up the canonical structure and the Hamiltonian in terms of the physical variables, we are ready to compute the classical dynamics. We will compute the Hamilton’s equations of motion using the Hamiltonian

\[ H[N] = \int \Sigma d^3 x N a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} \left[ \Lambda + a_0^3 e^T \left( \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} \right) \right]. \tag{23} \]

As a note prior to proceeding, the Hamiltonian constraint can be reduced to the vanishing of the rightmost term in square brackets, given by

\[ h = \Lambda + a_0^3 e^T \left( \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} \right) \sim 0 \tag{24} \]

on the constraint shell, for configurations where the quantities to the left of the square brackets in (23) are nonvanishing. In this paper we will restrict attention to these nondegenerate configurations, which are based on the nondegeneracy of \( B_i \) and \( \Psi_{ae} \) in the CDJ Ansatz (11).

The Hamilton’s equations of motion for the configuration space variables \((X, Y, T) \in \Gamma_{Kin}\) are given by

\[ \dot{X} = \frac{\delta H}{\delta \alpha} \sim -N U a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} \left( \frac{1}{\lambda} \right)^2; \]
\[ \dot{Y} = \frac{\delta H}{\delta \beta} \sim -N U a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} \left( \frac{1}{\lambda + \beta} \right)^2; \]
\[ \dot{T} = \frac{\delta H}{\delta \lambda} \sim -N U a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} \left[ \left( \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda + \alpha} \right)^2 + \left( \frac{1}{\lambda + \beta} \right)^2 \right]. \tag{25} \]

We have used a tilde to remind the reader that we have used (24) in deriving (25), hence the equations of motion hold only as long as the Hamiltonian constraint is satisfied. For here onward we will treat these as strong equalities since the equations of motion must always be consistent with the initial value constraints as a matter of Dirac consistency [5].

Equations (25) can be written in the form

\[ \dot{X} = p_{\alpha, \beta}(\lambda) \dot{T}; \quad \dot{Y} = q_{\alpha, \beta}(\lambda) \dot{T}, \tag{26} \]

where we have defined

\[ p_{\alpha, \beta}(\lambda) = \left( \frac{1}{\lambda + \alpha} \right)^2 \left[ \left( \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda + \alpha} \right)^2 + \left( \frac{1}{\lambda + \beta} \right)^2 \right]^{-1}; \]
\[ q_{\alpha, \beta}(\lambda) = \left( \frac{1}{\lambda + \beta} \right)^2 \left[ \left( \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda + \alpha} \right)^2 + \left( \frac{1}{\lambda + \beta} \right)^2 \right]^{-1}. \tag{27} \]
Moving on to the momentum space variables we have

\[\dot{\alpha} = -\frac{\delta H}{\delta X} = -\int_{\Sigma} d^3x N \left( \frac{\delta U}{\delta X} \right) a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} h;\]

\[\dot{\beta} = -\frac{\delta H}{\delta Y} = -\int_{\Sigma} d^3x N \left( \frac{\delta U}{\delta Y} \right) a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} h;\]

\[\dot{\lambda} = -\frac{\delta H}{\delta T} = -NU a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} \left[ a_0^{-3} e^T \left( \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} \right) \right].\] (28)

where \(h\) is given by (24). Note that the first two lines of (28) are directly proportional to the Hamiltonian constraint since according to (24), \(h\) vanishes. There will be spatial gradients upon integration by parts of the \(\delta U/\delta T\), \(\delta U/\delta X\) and \(\delta U/\delta Y\) terms in (28), which act on \(h\). We will assume that such terms \(\partial_i h\) vanish on the constraint surface since \(h\) vanishes. We are still in compliance with the requirement of Dirac that the constraints cannot be used prior to evaluating derivatives [5]. Our interpretation is that the derivatives referred to by Dirac are functional derivatives with respect to the phase space variables or time derivatives, but not spatial gradients since the latter are not dynamical. Another way to see this is that the functional form of the constraints

\[h(\alpha(x), \beta(x), \lambda(x); T(x)) = 0 \forall x \in \Sigma\] (29)

must be preserved on each spatial slice \(\Sigma\) for each time \(t\), due to Dirac consistency. To evaluate the spatial gradient discretize 3-space into a lattice of spacing \(\epsilon\). Then the spatial gradient is approximated by

\[\partial h \sim \frac{1}{2\epsilon} \left[ h(\alpha(x_{n+1}), \beta(x_{n+1}), \lambda(x_{n+1}); T(x_{n+1})
- h(\alpha(x_{n-1}), \beta(x_{n-1}), \lambda(x_{n-1}); T(x_{n-1}) \right] = 0 - 0.\] (30)

We have used (29) to argue that for any \(\epsilon > 0\), each term of (30) vanishes on the constraint shell, regardless of the lattice spacing. Hence we conclude in the limit \(\epsilon \to 0\) the spatial gradients must still be zero since they are proportional to the constraint.

For the third line of (28), the term in square brackets can be replaced with \(-\Lambda\) on account of (24). The result is that the Hamilton’s equations of motion (26) and (28) reduce to

\[\dot{X} = p_{\alpha,\beta}(\lambda) \dot{T}; \quad \dot{Y} = q_{\alpha,\beta}(\lambda) \dot{T}\] (31)

and
\[ \dot{\alpha} = 0; \quad \dot{\beta} = 0; \quad \dot{\lambda} = \Lambda NU a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)}. \]

Immediately from (32) one reads off that

\[ \alpha(x, t) = \alpha(x); \quad \beta(x, t) = \beta(x) \]

are arbitrary functions only of spatial position, independent of time. For vanishing cosmological constant \( \Lambda = 0 \), one would also have that \( \lambda(x, t) = \lambda(x) \) is a time independent function of position. However, for \( \Lambda \neq 0 \) the variable \( \lambda \) will undergo a nontrivial time evolution while \( \alpha \) and \( \beta \) remain constant in time. Therefore we will consider each possibility separately.
4 Hamiltonian dynamics for $\Lambda = 0$

When the cosmological constant is vanishing, the Hamiltonian constraint (24) reduces to

$$h = \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} = 0. \quad (34)$$

From (34) one can solve a quadratic equation directly for $\lambda$ explicitly as a function of $\alpha$ and $\beta$, given by

$$3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta = 0 \rightarrow \lambda = \lambda_{\alpha,\beta} = -\frac{1}{3}(\alpha + \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2}). \quad (35)$$

The result is that the three momentum space degrees of freedom on the kinematic phase space $\Omega_{Kin}$ reduce to two degrees of freedom on the physical phase space $\Omega_{Phys}$ on the Hamiltonian constraint shell. We will now verify that this is consistent with the Hamilton’s equations of motion as previously indicated.\(^6\) Starting from the Hamilton’s equations of motion (26) and (32)

$$\dot{X} = p_{\alpha,\beta}(\lambda)\dot{T}; \quad \dot{Y} = q_{\alpha,\beta}(\lambda)\dot{T};$$

$$\dot{\alpha} = 0; \quad \dot{\beta} = 0; \quad \dot{\lambda} = \Lambda NU a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} = 0, \quad (36)$$

the third equation on the bottom line of (36) vanishes since $\Lambda = 0$ and we have that $\lambda(x, t) = \lambda(x)$ is an arbitrary function of position, independent of time. Hence it follows for $\Lambda = 0$ that

$$\alpha(x, t) = \alpha(x); \quad \beta(x, t) = \beta(x); \quad \lambda(x, t) = \lambda_{\alpha,\beta}(x). \quad (37)$$

with $\lambda_{\alpha,\beta}$ given by (35). Since $\lambda$ is no longer an independent degree of freedom, then (27) reduces to

$$p_{\alpha,\beta}(\lambda) \rightarrow p_{\alpha,\beta}(\lambda_{\alpha,\beta}) \equiv p_{\alpha,\beta};$$

$$q_{\alpha,\beta}(\lambda) \rightarrow q_{\alpha,\beta}(\lambda_{\alpha,\beta}) \equiv q_{\alpha,\beta}, \quad (38)$$

whence both quantities now carry two independent labels $\alpha$ and $\beta$ instead of three, and become time independent functions of position in $\Sigma$. The result is that the two equations in the top line of (36) can be directly integrated to

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\(^6\)We have excluded $\lambda_{\alpha,\beta} = 0$, which implies that either $\alpha$ or $\beta$ must vanish. This is an example of the degenerate case, which forms a set of measure zero.
\[ X(x, t) = X(x, 0) + p_{\alpha, \beta}(x) (T(x, t) - T(x, 0)); \]
\[ Y(x, t) = X(x, 0) + q_{\alpha, \beta}(x) (T(x, t) - T(x, 0)). \]  (39)

The classical dynamics for \( \Lambda = 0 \) are those of a free point particle undergoing straight line motion in a two dimensional configuration space \((X, Y)\) per spatial point, where the motion evolves linearly with respect to \( T(x, t) \) regarded as a time variable on this space.

The specific time dependence of \( T(x, t) \) can be found from its equation of motion, given by the third line of (25),
\[ \dot{T} = -NUa^3/2 \frac{e^{-T/2}}{\sqrt{Q_{\alpha, \beta} M_{\alpha, \beta}}}, \]  (40)
Making the definitions
\[ Q_{\alpha, \beta}(\lambda) = \lambda(\lambda + \alpha)(\lambda + \beta); \]
\[ M_{\alpha, \beta}(\lambda) = \left( \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda + \alpha} \right)^2 + \left( \frac{1}{\lambda + \beta} \right)^2; \]
\[ O_{\alpha, \beta}(\lambda) = 3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha \beta, \]  (41)
the following notation is useful in view of the fact that \( \lambda \) on the solution space (35) is no longer an independent degree of freedom
\[ Q_{\alpha, \beta}(\lambda_{\alpha, \beta}) \equiv Q_{\alpha, \beta}; \quad M_{\alpha, \beta}(\lambda_{\alpha, \beta}) \equiv M_{\alpha, \beta}, \]  (42)
Then (40) is given by
\[ e^{-T/2} \dot{T} = -2 \frac{d}{dt} e^{-T/2} = -NUa^3/2 \sqrt{Q_{\alpha, \beta} M_{\alpha, \beta}}, \]  (43)
which integrates directly to
\[ T(x, t) = T_{\alpha, \beta}(x, t; N) = \ln \left( e^{-T_0/2} + \frac{1}{2} a^3/2 \sqrt{Q_{\alpha, \beta} M_{\alpha, \beta}} \int_0^t N(x, t')U(x, t')dt' \right)^{-2} \]  (44)
where \( T_0 = T_{\alpha, \beta}(x, 0) \). Equation (44) can be written in the shorthand notation
\[ e^T = \left( e^{-T_0/2} + \sqrt{Q_{\alpha, \beta} M_{\alpha, \beta}} \rho(x, t) \right)^{-2}, \]  (45)
where the time dependence has been absorbed into the term
\[ \rho(x, t) = \int_0^t N(x, t')U(x, t')dt' = \rho(x; T(x, t)); \]  (46)
5 Hamiltonian dynamics for $\Lambda \neq 0$

For nonvanishing cosmological constant the Hamiltonian constraint is given by (24), now with $\Lambda \neq 0$,

$$h = \Lambda + a_0^3 e^T \left( \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} \right) = 0$$  \hspace{1cm} (47)

which can be written as

$$3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta + \Lambda a_0^{-3} e^{-T}\lambda(\lambda + \alpha)(\lambda + \beta) = O_{\alpha,\beta}(\lambda) + a_0^{-3} e^{-T}Q_{\alpha,\beta}(\lambda) = 0.$$  \hspace{1cm} (48)

Equation (48) can be solved as a cubic polynomial equation in $\lambda$, which is displayed later in this paper. The solution yields three roots labelled by $\alpha$ and $\beta$ but containing explicit dependence on time through the factor $e^T$.

Let us proceed with the equations of motion

$$\dot{X} = p_{\alpha,\beta}(\lambda)\dot{T}; \quad \dot{Y} = q_{\alpha,\beta}(\lambda)\dot{T};$$

$$\dot{T} = -NUa_0^{3/2} e^{T/2} \sqrt{Q_{\alpha,\beta}(\lambda)M_{\alpha,\beta}(\lambda)};$$

$$\dot{\lambda} = 0; \quad \dot{\beta} = 0; \quad \dot{\lambda} = \Lambda NUa_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} = 0.$$  \hspace{1cm} (49)

These equations still imply that on the Hamiltonian constraint shell,

$$\alpha(x, t) = \alpha(x); \quad \beta(x, t) = \beta(x)$$  \hspace{1cm} (50)

are arbitrary time-independent functions of position exactly as in the $\Lambda = 0$ case. However, we now also have the equations for evolving degrees of freedom $\lambda$ and $T$, given by

$$\dot{\lambda} = \Lambda NUa_0^{-3/2} e^{-T/2} \sqrt{Q_{\alpha,\beta}(\lambda)};$$

$$\dot{T} = -NUa_0^{3/2} e^{T/2} \sqrt{Q_{\alpha,\beta}(\lambda)M_{\alpha,\beta}(\lambda)}.$$  \hspace{1cm} (51)

To make progress we must optimize the use of the equation for $\lambda$. Let us write this out explicitly

$$\dot{\lambda} = \Lambda NUa_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)}.$$  \hspace{1cm} (52)

To obtain an equation involving only $\lambda$ we must eliminate $e^{-T/2}$ from (52), which can be accomplished using the Hamiltonian constraint (47)
Substituting (53) into (52) we obtain

\[ \dot{\lambda} = \Lambda N U \left( \frac{\pm i}{\sqrt{\Lambda}} \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta}} \right) \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} = \pm i \sqrt{\Lambda} N U O_{\alpha,\beta}(\lambda) \]  

where \( O_{\alpha,\beta}(\lambda) \) is as defined in (41). Rearranging, we have

\[ \frac{\delta \lambda}{\sqrt{3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta}} = \pm i \sqrt{\Lambda} N U \delta t. \]  

Equation (55) integrates to

\[ \frac{1}{\sqrt{3}} \ln \left( 2 \left( \sqrt{3} \sqrt{3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta + \alpha + \beta + 3\lambda} \right) \right) = k_0 + i \sqrt{\Lambda} \int_0^t N(x, t') U(x, t') dt' = k_0(x) \pm i \rho(x, t) \]

for some time independent function \( k_0 = k_0(x) \) to be determined. This can further be written as

\[ 2\sqrt{3} \sqrt{3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta + 2(\alpha + \beta) + 6\lambda} = A(\rho), \]

where

\[ A(\rho) = A_0 e^{\pm i \sqrt{\Lambda} \rho} \]

with \( A_0 = A(x, \rho) \big|_{\rho=0} \) being the initial data. Equation (57) squares to the following quadratic equation

\[ 7 \text{In this sense functional variation and time variation are indistinguishable from each other, but distinguishable from spatial variation.} \]
\[ 9\lambda^2 + 6(\alpha + \beta)\lambda + 3\alpha\beta = 9\lambda^2 + 6\left(\alpha + \beta - \frac{A}{2}\right)\lambda + \left(\alpha + \beta - \frac{A}{2}\right)^2. \] (59)

Equation (59) has been written to show that a remarkable cancellation takes place which enables us to straightforwardly solve explicitly for \( \lambda \)

\[ \lambda = \frac{A}{12} + \frac{1}{2}\left(-\alpha - \beta + \frac{1}{A} \alpha^2 - \alpha\beta + \beta^2\right). \] (60)

As a matter of consistency we must require that (57) reduce to (35) in the limit that \( \Lambda \) approaches zero. This fixes the value of \( A_0 \) at

\[ A_0 = \pm 2\sqrt{\alpha^2 - \alpha\beta + \beta^2} = A(x, 0). \] (61)

Putting (61) back into (60) and using (58), we obtain the solution

\[ \lambda_{\alpha,\beta}(\rho) = \frac{1}{3}\left(-\alpha - \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2}\cos(\sqrt{3}\Lambda\rho)\right). \] (62)

To find the specific time evolution of the clock variable \( T \) one may perform the steps analogous to (43), (44), (45), (46) and (67), except now taking into account the time dependence induced by \( \Lambda \). This is given by

\[ T(x,t) = \ln\left(e^{-T_0/2} + \frac{1}{2}a_0^{3/2} \int_{\rho_0}^{\rho} \sqrt{Q_{\alpha,\beta}(\rho')M_{\alpha,\beta}(\rho')\delta\rho}\right)^{-2}, \] (63)

where we have defined

\[ Q_{\alpha,\beta}(\rho) = \lambda_{\alpha,\beta}(\rho)(\lambda_{\alpha,\beta} + \alpha)(\lambda_{\alpha,\beta} + \beta); \]
\[ M_{\alpha,\beta}(\rho) = \left(\frac{1}{\lambda_{\alpha,\beta}(\rho)}\right)^2 + \left(\frac{1}{\lambda_{\alpha,\beta}(\rho) + \alpha}\right)^2 + \left(\frac{1}{\lambda_{\alpha,\beta}(\rho) + \beta}\right)^2; \] (64)

with \( \lambda_{\alpha,\beta}(\rho) \) given by (62). Equation (63) is the direct analogue of (44), where now the quantities in (64) can no longer be factored out of the integral on account of the time dependence induced by \( \Lambda \not= 0 \).

It is not as straightforward to write down the explicit evolution of \( X \) and \( Y \) in the time variable \( T \) as in (39) in the \( \Lambda = 0 \) case, but one can still write the relation

\[ \delta X = p_{\alpha,\beta}(\lambda)\delta T = \left[\frac{1}{\lambda_{\alpha,\beta}(\rho)}^2 + \frac{1}{\lambda_{\alpha,\beta}(\rho) + \alpha}^2 + \frac{1}{\lambda_{\alpha,\beta}(\rho) + \beta}^2\right]\delta T \] (65)
for variations of $X$, and similarly

$$
\delta Y = q_{\alpha,\beta}(\lambda) \delta T = \left[ \frac{1}{\lambda_{\alpha,\beta}(\rho)} \right]^2 + \left( \frac{1}{\lambda_{\alpha,\beta}(\rho) + \alpha} \right)^2 \right] \delta T \quad (66)
$$

for variations of $Y$. The difference in relation to the $\Lambda = 0$ case is that now $\rho$ depends implicitly on $T$ through (63), and $\lambda_{\alpha,\beta}(\rho)$ in (62) inherits this $T$ dependence. To take this into account one must invert (63) to obtain $\rho = \rho(T)$ and then substitute into (65) and (66). The result should be a more complicated $T$ dependence, which might possibly best be handled by perturbation theory methods.

### 5.1 Fixed point iteration

Equations (63) and (45) are really recursion relations due to the explicit $T$ dependence on the right hand side contained in $\rho$ through $U$. To proceed further from this point one approach is to use a Pickard-type fixed point iteration procedure. We will illustrate the general case, which may in principle be applied for both vanishing and nonvanishing $\Lambda$. For each triple $(\alpha(x), \beta(x), N(x,t))$ and initial data $(X(x,0), Y(x,0), T(x,0))$, define a sequence $T_n(x,t;\alpha,\beta)$ such that the following recursion relation holds

$$
T_{n+1}(x,t) = \ln \left( \exp \left[ -\frac{1}{2} T_n(x,0) \right] + \frac{1}{2} \int_{\rho_0}^\rho \sqrt{Q_{\alpha,\beta}(\rho_n') M_{\alpha,\beta}(\rho_n') \delta \rho_n'} \right)^{-2}, \quad (67)
$$

where $\rho_n$ is defined by

$$
\rho_n = \rho_n(x,t) = \int_0^t N(x,t') U(x,t';T_n(x,t')) dt' \quad (68)
$$

with $T_n(x,t)$ given by (63). With each iteration one acquires spatial gradients from $U$ (recall equation (19)), which act on the position dependence of the variables, which should generate a kind of nonlinear Taylor expansion. The full solution, if convergent, would be given by $T = \lim_{n \to \infty} T_n(x,t)$.

This might be suitable for constructing GR solutions for the full theory according to the following procedure. First solve (45) for $U = 1$, which amounts to setting the spatial gradients of the full theory to zero as a first approximation.\(^8\) However we are in the full theory, solving an infinite number of independent equations, one equation per spatial point whose solution

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\(^8\)This is tantamount to constructing at each spatial point a different minisuperspace theory by the way we define it, namely with all terms containing spatial gradients absent. Equation (45) constitutes an exact solution and not a recursion relation for $U = 1$, and similarly for the analogous equation in the $\Lambda \neq 0$ case.
evolves in time from the initial spatial hypersurface $\Sigma_0$. Using this solution as the zeroth order solution, with its position dependence intact based upon the initial data selected, perform the fixed point iteration procedure of (67). The choices of initial data for which this iteration procedure converges to a solution for $T(x,t)$ is a subject for further study. Irrespective of this, one can still nevertheless explicitly write the following relation for GR in the instanton representation for $\Lambda = 0$

\begin{align*}
X(x,t) &= X(x,0) + p_{\alpha,\beta}(x) + p_{\alpha,\beta}(x)T_{\alpha,\beta}(x,t; N(x,t)); \\
Y(x,t) &= Y(x,0) + q_{\alpha,\beta}(x) + q_{\alpha,\beta}(x)T_{\alpha,\beta}(x,t; N(x,t)).
\end{align*}

Equation (69) shows that the configuration space degrees of freedom $(X,Y)$ evolve linearly with respect to $T$, regarded as a time variable on a two dimensional configuration space per point $\Gamma_{Kin}$. One has complete freedom in the choice of lapse function $N$, the initial data $(X_0,Y_0,T_0)$, and the two free functions of position $(\alpha, \beta)$ which comprise the physical degrees of freedom, which is also true for $\Lambda \neq 0$. The choice of Lapse function determines the manner of evolution of the initial data, and $\alpha$ and $\beta$ remain stationary with respect to this time evolution.

The main point is that on the kinematic phase space $\Omega_{Kin}$ the dynamics can be completely reduced to evolution in $T$, with each solution labelled by $\alpha$ and $\beta$. Since we are in the full theory, we can calculate $U$ from (19) explicitly. The term on the first line vanishes for arbitrary $A_1^1 = A_1^1(T)$, $A_2^2 = A_2^2(T)$ and $A_3^3 = A_3^3(T)$, using the chain rule. This leaves behind the second line, which can be written as

\begin{align*}
A_1^1 A_2^2 (\frac{\partial A_1^1}{\partial T} \frac{\partial A_2^2}{\partial T})(\partial_3 T)^2 + A_2^2 A_3^3 (\frac{\partial A_2^2}{\partial T} \frac{\partial A_3^3}{\partial T})(\partial_1 T)^2 + A_3^3 A_1^1 (\frac{\partial A_3^3}{\partial T} \frac{\partial A_1^1}{\partial T})(\partial_2 T)^2.
\end{align*}

We will display the expression for the $\Lambda = 0$ case for simplicity. In this case we have

\begin{align*}
A_1^1 &= a_0 e^X = a_0 e^p T; \\
A_2^2 &= a_0 e^Y = e^{qT}; \\
A_3^3 &= a_0 e^{T-X-Y} = a_0 e^{(1-p-q)T},
\end{align*}

where $p = p_{\alpha,\beta}$ and $q = q_{\alpha,\beta}$. Substitution of (71) into (70) and in turn into ref (19) yields

\begin{align*}
U &= \left[1 + \left(p q e^{2(p+q-1)T}(\partial_3 T)^2 + q(1-p-q)e^{-2pT}(\partial_1 T)^2 + p(1-p-q)e^{-2qT}(\partial_2 T)^2\right)\right]^{1/2}.
\end{align*}
6 Hamilton–Jacobi analysis

We have shown that the equations of motion for general relativity on $\Omega_{Kin}$ in the instanton representation are classically integrable to a recursion relation on the variable $T$. The remaining degrees of freedom $X$ and $Y$ evolve with respect to $T$, seen as a time variable on the kinematic configuration space $\Gamma_{Kin}$. We have seen, irrespective of the value of $\Lambda$, that the functions $\alpha$ and $\beta$ are independent of time. Hence for each equivalence class of initial data, for the configuration space variables $(X,Y,T)$, $\alpha$ and $\beta$ serve as labels for the classical solution. Since time independent, these can be used as labels for the state in the quantum theory.\footnote{This interpretation is borne out in Paper XVIII.}

A possible route toward the quantum theory of the instanton representation is through the Hamilton–Jacobi formalism. Since we are dealing with the full theory, we will be generalizing the usual procedures to the infinite dimensional space of fields $\Gamma$. We assume that functional integration in this space commutes with spatial integration on spatial hypersurfaces, and that functional variation and time variation are in a sense indistinguishable. The following relation can be written for the functional variation of the Hamilton–Jacobi functional on the kinematic phase space $S_{HJ}$

$$\delta S_{HJ} = \int_{\Sigma} d^3x \left( \alpha(x)\delta X(x,t) + \beta(x)\delta Y(x,t) + \lambda(x;T)\delta T(x,t) \right), \quad (73)$$

where in (73) the functional variations are restricted to the spatial hypersurface $\Sigma$, which can be chosen arbitrarily. Also, we have allowed for the possibility that there may be explicit $T$ dependence in $\lambda$. Next, we integrate both sides of (73) in the functional space of fields $\Gamma$

$$\int \delta S_{HJ} = \int_{\Gamma} \int_{\Sigma} d^3x \left( \alpha \delta X + \beta \delta Y + \lambda_{\alpha,\beta} \delta T \right). \quad (74)$$

The left hand side of (74) is a total functional differential, so it integrates according to the usual rules for antiderivative. From here on we suppress the position dependence to avoid cluttering the notation, with the time label associated to each particular $\Sigma$. For the right hand side of (74) since $\int_{\Gamma} \int_{\Sigma} = \int_{\Sigma} \int_{\Gamma}$, we have interchanged the order of the integrations to obtain

$$S_{HJ} = \int d^3x \left( \alpha \int_{\Gamma} \delta X + \beta \int_{\Gamma} \delta Y + \int_{\Gamma} \lambda_{\alpha,\beta}[T] \delta T \right)$$

$$= \int_{\Sigma_t} d^3x (\alpha X + \beta Y) + \int_{\Sigma} d^3x \int_{\Gamma} \lambda_{\alpha,\beta}[T] \delta T. \quad (75)$$
We have used the fact that \( \alpha \) and \( \beta \) are independent of time, and therefore constants with respect to the functional integration. Note from the previous results that \( \lambda \) is constant in time only for \( \Lambda = 0 \).

There are two cases to consider, namely \( \Lambda = 0 \) and \( \Lambda \neq 0 \). First for vanishing cosmological constant recall that

\[
\dot{\lambda} = 0 \rightarrow \lambda(x, t) = \lambda(x) = \lambda_{\alpha, \beta}(x).
\]  

(76)

For this case \( \lambda_{\alpha, \beta} \) can be factored out of the functional integral in (75) just as can \( \alpha \) and \( \beta \), and we obtain

\[
S_{\alpha, \beta}^\pm (X, Y, T) = \int_\Sigma d^3 x (\alpha X + \beta Y + \lambda_{\alpha, \beta}^\pm T),
\]  

(77)

with two solutions, one for each root of (35). One has that

\[
\alpha(x) = \frac{\delta S}{\delta X(x, t)}; \quad \beta(x) = \frac{\delta S}{\delta Y(x, t)}; \quad \lambda_{\alpha, \beta}(x) = \frac{\delta S}{\delta T(x, t)},
\]  

(78)

which define the conjugate momenta for the dynamical variables, where \( \alpha \) and \( \beta \) are constants of the motion. The variables conjugate to the constants of the motion are the initial data, which may be obtained from

\[
X(x, 0) = \frac{\delta S}{\delta \alpha(x)} = X(x, t) + \left( \frac{\partial (\lambda_{\alpha, \beta}(x) T_{\alpha, \beta}(x, t))}{\partial \alpha} \right);
\]

\[
Y(x, 0) = \frac{\delta S}{\delta \beta(x)} = Y(x, t) + \left( \frac{\partial (\lambda_{\alpha, \beta}(x) T_{\alpha, \beta}(x, t))}{\partial \beta} \right).
\]

(79)

We will not display these expressions, which should presumably imply the solution to the equations of motion which we have constructed. By exponentiation of the (77) on the physical space of solutions to the equations of motion one can obtain a wavefunctional

\[
\psi_{\alpha, \beta}^\pm [X, Y, T] = e^{(hG)^{-1}(\alpha X + \beta Y)} e^{(hG)^{-1}\lambda_{\alpha, \beta}^\pm T}.
\]

(80)

Equation (80) is labelled by two free functions of position \( (\alpha, \beta) \) and are eigenstates of the operators

\[
\hat{\Pi}_1(x, t) = (hG) \frac{\delta}{\delta X(x, t)}; \quad \hat{\Pi}_2(x, t) = (hG) \frac{\delta}{\delta Y(x, t)}; \quad \hat{\Pi}(x, t) = (hG) \frac{\delta}{\delta T(x, t)}.
\]

(81)

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10We have used an abbreviated notation \( \alpha \cdot T = \int_\Sigma \alpha(x) T(x) \), where the dot signifies an integration over 3-space \( \Sigma \).
such that if one associated operators \( \hat{X}^f \equiv (\hat{X}, \hat{Y}, \hat{T}) \) to the configuration space variables and \( \hat{\Pi}_g \equiv (\hat{\Pi}_1, \hat{\Pi}_2, \hat{\Pi}) \) to the momentum space variables, then the following commutation relations would hold

\[
[\hat{X}^f(x,t); \hat{\Pi}_g(y,t)] = \delta^{(3)}(x,y); \quad [\hat{X}^f(x,t); \hat{X}^g(y,t)] = [\hat{\Pi}_f(x,t); \hat{\Pi}_g(y,t)] = 0.
\] (82)

Equation (80) satisfies the Hamilton–Jacobi equation and also is annihilated by the quantum version of the Hamiltonian constraint obtained by making the replacements

\[\alpha \to \Pi_1; \quad \beta \to \Pi_2; \quad \lambda \to \Pi.\] (83)

For \( \Lambda \neq 0 \) the analogous manipulations may be performed, while a bit more involved albeit straightforward. In this case the \( \lambda \) now contains time dependence and cannot be factored out of the functional integral (75) as can \( \alpha \) and \( \beta \). The corresponding Hamiltonian constraint is given by

\[
r\left(\lambda^2 + \frac{2}{3}(\alpha + \beta)\lambda + \frac{1}{3}\alpha\beta\right) + \lambda(\lambda + \alpha)(\lambda + \beta) = 0,
\] (84)

where we have defined

\[
r = \left(\frac{3a^3}{\Lambda}\right)e^T.
\] (85)

Equation (84) leads to the cubic equation

\[
\lambda^3 + A\lambda^2 + B\lambda + C = 0,
\] (86)

where we have defined

\[
A = \alpha + \beta + 3r; \quad B = \alpha\beta + 2r(\alpha + \beta); \quad C = r\alpha\beta.
\] (87)

Defining the quantities

\[
p = B - \frac{1}{9}A^2; \quad q = C - \frac{1}{3}AB + \frac{2}{27}A^3,
\] (88)

then the solution to the cubic (84) is given by

\[
\lambda_{\alpha\beta}(T) = w - \frac{p}{3w} - \frac{A}{3}; \quad w^3 = \frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4}{27}p^3}\right).
\] (89)
This yields three solutions, each of which determines a Hamilton–Jacobi functional\(^{11}\)

\[
S_{HJ}(\Sigma) = S_{HJ}(\Sigma_0) + \int d^3x (\alpha X + \beta Y + \int_\Gamma \lambda_{\alpha,\beta} [T]\delta T).
\tag{90}
\]

For the special case \(\alpha = \beta = 0\), (84) reduces to

\[
\lambda^2(\lambda + r) = 0
\tag{91}
\]

with solution \(\lambda = -r\). Writing this as a functional differential equation we have

\[
(hG) \frac{\delta S}{\delta T(x, t)} = -\left(\frac{6a_0^3}{\Lambda}\right) e^T.
\tag{92}
\]

Contracting and performing an integration over 3-space \(\Sigma\), we have

\[
\delta S = \int_\Sigma d^3x \frac{\delta S}{\delta T(x, t)} \delta T(x, t) = -\left(\frac{6a_0^3}{hG\Lambda}\right) \int_\Sigma d^3x e^T \delta T
\]

\[
= -\left(\frac{6a_0^3}{hG\Lambda}\right) \int_\Sigma d^3x (\delta e^T) = \delta \left(-6(hG\Lambda)^{-1} \int_\Sigma d^3x a_0^3 e^T\right),
\tag{93}
\]

where we have used the commutativity of functional and spatial variation in the second line of (93). Taking the anti-functional derivative of both sides and exponentiating the result, we obtain

\[
\psi = e^{-6(hG\Lambda)^{-1}I_{CS}[T]},
\tag{94}
\]

which is the Kodama state \(\psi_{Kod}\) evaluated on a diagonal connection with \(\text{det} A = a_0^3 e^T\) as in (7). Note that (94) is complementary to (80) in the sense that the former depends completely on time \(T\) with \(\Lambda = 0\), and the latter has ‘spatial’ dependence \(X\) and \(Y\), spatial in the functional sense on configuration space. Equation (80), while containing \(T\) dependence, should be regarded as a nonevolving state in the sense that the Hamilton–Jacobi functional is preserved on each spatial hypersurface.\(^{12}\) If one associates a nontrivial time evolution (e.g. beyond what one would have for free particle motion) to a nonvanishing \(\Lambda\), then one sees that the Kodama state is really a time variable more so than a quantum state.

\(^{11}\)One would have to substitute (89), which depends on \(T\) through \(e^T\), into (90) and then carry out the functional integration to find the antiderivative. In general this procedure should produce three states, though we will not display the final expressions here.

\(^{12}\)This is our interpretation of the problem of time in quantum gravity.
6.1 Continuation

The Hamilton–Jacobi functional can be found more directly for \( \Lambda \neq 0 \) using the equations of motion. Let us rewrite the equations of motion, replacing all non-spatial differentials with functional variations as in

\[
\delta X = -\kappa e^{T/2} \left( \frac{1}{\lambda + \alpha} \right)^2 \delta t;
\]
\[
\delta Y = -\kappa e^{T/2} \left( \frac{1}{\lambda + \beta} \right)^2 \delta t;
\]
\[
\delta T = -\kappa e^{T/2} \left[ \left( \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda + \alpha} \right)^2 + \left( \frac{1}{\lambda + \beta} \right)^2 \right] \delta t,
\]

where we have defined

\[
\kappa = NUa_0^{3/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)}.
\]

Substituting (95) into (73), we obtain

\[
\delta S_{HJ} = -\int_{\Sigma} d^3x \kappa e^{T/2} \left[ \frac{\alpha}{(\lambda + \alpha)^2} + \frac{\beta}{(\lambda + \beta)^2} + \lambda \left( \left( \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda + \alpha} \right)^2 + \left( \frac{1}{\lambda + \beta} \right)^2 \right) \right] \delta t
\]
\[= -\int_{\Sigma} d^3x \kappa e^{T/2} \left( \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} \right) \delta t. \tag{97}
\]

But according the Hamiltonian constraint, which must also be satisfied, we have that

\[
\frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} = -\left( \frac{\Lambda}{a_0^3} \right) e^{-T}.
\]

Substituting (98) into (97) we have

\[
\delta S_{HJ} = \left( \frac{\Lambda}{a_0^3} \right) \int_{\Sigma} \kappa e^{-T/2} \delta t,
\]

which governs the variation of the Hamilton–Jacobi functional on time. But recall that

\[
\frac{1}{a_0^3} e^{-T/2} = NU a_0^{-3/2} e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)}
\]
\[= N(Ua_0^{3/2} e^T)e^{-3T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} = N \sqrt{\det B} \sqrt{\det \Psi}. \tag{100}
\]
We will now reverse the steps which brought us from the Ashtekar variables into the instanton representation. The following relation can be derived from the Ashtekar variables

\[ \tilde{\sigma}^i_a \tilde{\sigma}^j_a = h h^{ij} \]  

(101)

where \( h_{ij} \) is the 3-metric on the spatial slice \( \Sigma \) and \( h = \det(h_{ij}) \). The determinant of (101), under the CDJ Ansatz (11), yields

\[ \sqrt{(\det B)(\det \Psi)} = \sqrt{\det \tilde{\sigma}} = \sqrt{h}. \]  

(102)

Equation (102) can equally be read from right to left where the instanton representation holds. Substituting (102) into (99), we have

\[ \delta S_{HJ} = \Lambda \int_\Sigma d^3x \delta t \sqrt{h_{\alpha,\beta}}. \]  

(103)

Note that the 3-metric has acquired the labels \((\alpha, \beta)\). Proceeding along from (103) and using the identity \( \sqrt{-g} = N \sqrt{h} \), we have that

\[ \delta S_{HJ} = \Lambda \int_\Sigma d^3x \delta t \sqrt{-g} \]

\[ \rightarrow S_{HJ}(t) - S_{HJ}(t_0) = \Lambda \int_M d^4x \sqrt{-g} = \Lambda Vol_{\alpha,\beta}(M) \]  

(104)

which is the volume of spacetime. So we see that any time evolution of the Hamilton–Jacobi functional can be induced only by a nonvanishing cosmological constant \( \Lambda \). The exponentiation of (104) gives

\[ \psi_{\alpha,\beta} = e^{(\hbar G)^{-1} S_{HJ}(t)} = \Phi \exp \left[ \left( \frac{\Lambda}{\hbar G} \right) Vol_{\alpha,\beta}(M) \right], \]  

(105)

where the pre-factor \( \Phi \) is the value of the functional on the initial spatial hypersurface \( t_0 \). Note for \( \Lambda = 0 \) that this should reduce to (80), which provides the physical interpretation of \( \Phi \) as the state corresponding to \( \Lambda = 0 \). Equation (105) corresponds to the dominant contribution to gravitational path integrals due to gravitational instantons. It appears that (75) could be used to model the wavefunctional of the universe corresponding to the state labelled by \((\alpha, \beta)\) (See e.g. Paper XII). Note that while the classical time evolution is in general complicated for \( \Lambda = 0 \), the state has a simple mathematical form.

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13 This expresses the determinant of the spacetime metric \( g_{\mu\nu} \) in terms of its 3+1 decomposition.

14 We will now rescale the lapse function by a factor of two to return to the usual convention.
7 Conclusion

The final results of this paper are as follows. Starting from the kinematic phase space $\Omega_{\text{Kin}}$ we have provided a preview into the dynamics of the instanton representation of Plebanski gravity in the full theory. $\Omega_{\text{Kin}}$ is a suitable starting point since at this level the kinematic constraints have already been implemented, leaving remaining the Hamiltonian constraint evaluated on the physical degrees of freedom. We have treated the case of vanishing and nonvanishing $\Lambda$ for the evolution of the physical configuration space degrees of freedom relative to a clock variable $T$. For $\Lambda = 0$ the evolution is that of a free particle in a two dimensional configuration space, and for $\Lambda \neq 0$ the evolution albeit more complicated has been reduced to a quadrature. To qualify this statement in more precise terms, the time variation of explicit time evolution of the clock variable $T$ has been reduced to a fixed point iteration procedure which composes a functional integration with the evaluation of a logarithm. While the relational evolution of the dynamical variables with respect to $T$ has been determined, a remaining course of study is to test the proposed iterative procedure for various situations. For different choices of functions for the gravitational labels $\alpha$ and $\beta$, and for the lapse function $N$, one may examine the evolution from different choices of initial data for convergence.

At each stage of the process we have taken into account the Hamiltonian constraint, which implies that we are indeed considering dynamics on the reduced phase space. This was one of the aims in [6], [7] and [8], where the Hamiltonian constraint remained unresolved. Hence the implication is that the eigenvalues of the CDJ matrix are a suitable set of gauge invariant, diffeomorphism invariant degrees of freedom for gravity. Also in conformity with the aims of [6], [7] and [8], we have constructed a Hamilton–Jacobi functional for the theory. For $\Lambda = 0$ the functional has been explicitly computed, and for $\Lambda \neq 0$ it has been reduced to a quadrature in the functional space of $T$. We have also derived the Kodama state, which is directly related to the clock variable $T$. We have additionally shown that the starting action evaluated on the solution to the equations of motion exhibits the expected behavior. These latter analyses form a basis for treatment and consideration of the quantum theory in the instanton representation, a future direction of research carried out in Paper XVIII.
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