Near-Optimal Algorithm for Distribution-Free Junta Testing

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Abstract

We present an adaptive algorithm with one-sided error for the problem of junta testing for Boolean function under the challenging distribution-free setting, the query complexity of which is \(\tilde{O}(k/\epsilon)\). This improves the upper bound of \(\tilde{O}(k^2/\epsilon)\) by [LCS+19]. From the \(\Omega(k \log k)\) lower bound for junta testing under the uniform distribution by [Sağ18], our algorithm is nearly optimal. In the standard uniform distribution, the optimal junta testing algorithm is mainly designed by bridging between relevant variables and relevant blocks. At the heart of the analysis is the Efron-Stein orthogonal decomposition. However, it is not clear how to generalize this tool to the general setting. Surprisingly, we find that junta could be tested in a very simple and efficient way even in the distribution-free setting. It is interesting that the analysis does not rely on Fourier tools directly which are commonly used in junta testing.

1 Introduction

Property testing of Boolean functions dates back to the seminal work of [BLR93, RS96]. Various properties of Boolean functions have been investigated in the succeeding works, for example, [AKK+05, BBM12, BMPR16, BB16, BKS+10, Sağ18]. Junta is an important property that is widely used in the machine learning setting ([Gol10]). A function \(f : \{0,1\}^n \rightarrow \{0,1\}\) is referred to as a \(k\)-junta if it depends on at most \(k\) variables. The Junta tester is used as a basic building block for testing various properties of Boolean functions, such as function isomorphism ([FKR+04]), halfspaces ([MORS10]), and concise representations ([DLM+07, Ser10]). It is therefore very motivating to design efficient algorithms to distinguish between \(k\)-junta and those far from every \(k\)-junta.

The problem of junta testing was firstly introduced by [PRS02], and [FKR+04] provided an algorithm that uses \(\tilde{O}(k^2/\epsilon)\) queries. In the uniform distribution framework, an \(\Omega(\log k)\) lower bound was introduced by [FKR+04] for adaptive testing, and was later improved to \(\Omega(k)\) by [CG04]. [Sağ18] further improved the lower bound to \(\Omega(k \log k)\). [Bla08] proposed a non-adaptive algorithm with query complexity \(\tilde{O}(k^{3/2})/\epsilon\). [Bla09] further presented an adaptive algorithm that uses \(\tilde{O}(k/\epsilon)\) queries, which achieves optimal query complexity.

Distribution-free property testing is very attractive since it allows an unknown and arbitrary distribution, which is more applicable in some cases than the uniform distribution. In this setting, the Boolean function is unknown. Moreover, the distribution that measures distance is also unknown and arbitrary, making the testing problem very challenging. One might therefore conjecture that the lower bound of query complexity is exponential instead of being polynomial. Surprisingly, [LCS+19] showed that a polynomial adaptive algorithm for this setting exists, despite an \(\Omega(2^{k/3})\) lower bound for any non-adaptive algorithm. The query complexity of the adaptive algorithm with one-sided error introduced by [LCS+19] is \(\tilde{O}(k^2/\epsilon)\). [Bsh19] further proposed an adaptive algorithm with two-sided error for distribution-free junta testing, which uses \(\tilde{O}(k/\epsilon)\) queries and is near-optimal.
The interesting problem of whether there exists a one-sided distribution-free adaptive algorithm with query complexity $\tilde{O}(k)$ remains open.

In the uniform setting, [Bla09] showed that if $f$ is $\epsilon$-far from being a $k$-junta, then $f$ is $\epsilon/2$-far from $k$-part with high probability. Here far from $k$-part means far from being determined by the union of coordinates in at most $k$ parts in a random partition of the coordinates. This property plays a key role in the proposal of the optimal adaptive algorithm. In the very challenging distribution-free setting, it is not clear whether this property still holds. Instead, [LCS+19] showed that if $f$ is $\epsilon$-far from being a $k$-junta, then $f$ is $\epsilon/2$-far from $k$-coordinate with high probability. They designed an adaptive algorithm with $\tilde{O}(k^2)/\epsilon$ query complexity based on this weaker property. This advancement in distribution-free junta testing leads to a series of natural but challenging problems. Could the upper bound of $\tilde{O}(k^2)/\epsilon$ be improved to $\tilde{O}(k)/\epsilon$ to match the lower bound from the uniform distribution setting? How to prove it without relying on the Fourier tools? Is the lower bound for adaptive junta testing under the distribution-free setting significantly larger than that under the uniform distribution?

The definition of $k$-junta relies on the total number of relevant variables. In order to design an algorithm with query complexity independent of $n$, the commonly used approach is to divide $[n]$ into poly$(k)$ blocks. In the standard uniform setting, [Bla09] show the effectiveness of the reduction between $k$-variables and poly$(k)$ blocks using the Efron-Stein orthogonal decomposition, and an optimal algorithm with query complexity $\tilde{O}(k)/\epsilon$ is consequently obtained. Given the optimal query complexity of [Bla09], it seems to be a natural way to try to generalize the argument to the distribution-free setting, and prove the correctness of the argument without relying on the Efron-Stein orthogonal decomposition ([Xie18]). However, it turns out to be very challenging to conduct a similar analysis to the distribution-free setting. [BWY15] further proposed another analysis approach which relies on the sub-additivity of influence and the property of intersecting family. However, the properties required by this analysis might not hold in an arbitrary distribution.

Usually, some stronger structural properties accompanied with complicated analysis are necessary to achieve better query complexity. The analysis of junta testing relies on the Fourier tools in most prior works. By investigating this challenging distribution, we are forced to design algorithms that could be analyzed without using the commonly used Fourier tools. This pursues us to explain the problem of junta testing from another point of view. It is amazing to find that near-optimal query complexity could be achieved simply with some properties obtained by [LCS+19]. This improvement is achieved by viewing the problem of finding the part containing the literal as the problem of best block identification, reusing some samples appropriately, and designing a stopping condition that could be triggered earlier. To the best of our knowledge, this is the first algorithm that achieves near-optimal query complexity with one-sided error.

1.1 Our Main Results

We give an adaptive algorithm with one-sided error for the distribution-free junta testing. The query complexity of our proposed algorithm is nearly optimal.

**Theorem 1.** If $f$ is $\epsilon$-far from every $k$-junta under the distribution $\mathcal{D}$, then there exists an algorithm rejects with probability at least $2/3$. The query complexity of the algorithm could be upper bounded by $\tilde{O}(k)/\epsilon$.

[LCS+19] gave an adaptive algorithm for distribution-free junta testing with one-sided error, the query complexity is $\tilde{O}(k^2)/\epsilon$. [Bsh19] further improves the query complexity to $\tilde{O}(k/\epsilon)$, but with two-sided error. Our algorithm is adaptive and has one-sided error. This upper bound essentially
reaches near-optimal query complexity, compared with the adaptive uniform-distribution lower bound for junta testing (there is no better lower bound for adaptive distribution-free junta testing).

Table 1: A summary of works on junta testing

| Distribution | Algorithm | Query complexity       | Type of algorithm | Type of error |
|--------------|-----------|------------------------|-------------------|---------------|
| Product      | [Bla09]   | $O(k \log k) / \epsilon$ | adaptive          | one-sided     |
| Uniform      | [Sag18]   | $\Omega(k \log k)$     | adaptive          | one-sided     |
| Arbitrary    | [LCS+19]  | $\Omega(2^{k/3})$       | non-adaptive      | one-sided     |
| Arbitrary    | [LCS+19]  | $\tilde{O}(k^2) / \epsilon$ | adaptive          | one-sided     |
| Arbitrary    | [Bsh19]   | $\tilde{O}(k) / \epsilon$ | adaptive          | two-sided     |
| Arbitrary    | This work | $\tilde{O}(k) / \epsilon$ | adaptive          | one-sided     |

1.2 Technical Overview

We firstly introduce the development of the algorithms from uniform setting to distribution-free setting. Readers who are familiar with the previous work of [Bla09] and [LCS+19] may feel free to skip the following two paragraphs.

In the uniform setting, a key property for the proposal of optimal adaptive algorithm is: if $f$ is $\epsilon$-far from being a $k$-junta, then $f$ is $\epsilon/2$-far from $k$-part with high probability. A simple algorithm is proposed based on this property, which achieves nearly-optimal query complexity. The algorithm randomly divides $[n]$ into $10^{20}k^9 / \epsilon^5$ blocks. Given $x, y \in \{0, 1\}^n$, $(x, y)$ is referred to as a distinguishing pair of $f$ for block $B$ if $f(x) \neq f(y)$, and $x_B = y_B$. Suppose the algorithm finds $m$ relevant blocks. Fixing these blocks, the algorithm could find a distinguishing pair $(x, y)$ in $O(1 / \epsilon)$ number of queries with high probability. This could be easily derived from the aforementioned property. Note that the coordinates of $x$ and $y$ are identical in the $m$ relevant blocks. The algorithm then uses binary search over blocks instead of over coordinates, to find a new relevant block. The resulting query complexity is $\tilde{O}(k) / \epsilon$.

In the distribution-free setting, it is not clear whether this property still holds. Instead, [LCS+19] designed an adaptive algorithm with $\tilde{O}(k^2)$ query complexity based on a weaker property under this challenging setting. The property is: if $f$ is $\epsilon$-far from being a $k$-junta, then $f$ is $\epsilon/2$-far from $k$-coordinate (see Defi. 3) with high probability. Suppose the algorithm has found $m (m \leq k)$ relevant blocks. If all the relevant blocks are close to a literal under the uniform distribution, then the algorithm randomly partitions each relevant block into two parts. Given a randomly sampled string $x$, the algorithm constructs $y$ by fixing the coordinates of $x$ in each block that contains the literal, fixing the coordinates of $x$ in a random part of the complement of all the relevant blocks, and flipping all the remaining coordinates. This procedure is equivalent to fixing $m (m \leq k)$ literals of $x$, and each of the remaining coordinate is equal to 0 with probability 1/2, and is equal to 1 with probability 1/2. With the aforementioned property, a distinguishing pair could be found in $O(1 / \epsilon)$ number of queries. Using binary search over blocks, a new relevant block could be found, and the total number of relevant blocks increases by 1. If one of the relevant blocks is far from every literal under the uniform distribution, then the original relevant block is divided into two new relevant blocks, and the total number of relevant blocks increases by 1.

One advantage of our algorithm is the proposal of the algorithm FindLiteral. It improves over the algorithm WhereIsTheLiteral of [LCS+19] by leveraging the requirement of closeness. The design of the subroutine FindLiteral is inspired by viewing the connection between the problem of finding the block that contains the literal and the problem of best block identification. Suppose
that $g : \{0, 1\}^B \to \{0, 1\}$ is $\gamma$-close to literal $l$ under the uniform distribution, and the block $B$ is randomly divided into two blocks. The goal of FindLiteral is to identify which block the literal lies in and return the corresponding distinguishing pair. We regard that each block is associated with an unknown expected reward. The expected reward of each block is defined as the probability of finding a distinguishing pair by flipping the coordinates in this block. The expected reward of the block that contains the literal is at least $1 - 2\gamma$, and the expected reward of another block is at most $2\gamma$. Setting $\gamma = 1/8$, the reward of one block is strictly larger than that of another block. Then the problem of locating the literal could be reduced to the problem of identifying the block with the highest expected reward. For the problem of identifying the best block between two blocks, we design an algorithm that is easy to understand intuitively and simple to analyze, which uses the standard "success amplification by majority" technique. With probability at least $1 - 1/\Theta(k)$, this algorithm identifies the best block within $O((1 - \gamma)^{-2}\log k)$ number of samples.

Recall that our goal is to design an algorithm which rejects with high constant probability when $f$ is $\epsilon$-far from every $k$-junta. The subroutine FindLiteral is required to find the correct part that contains the literal with probability at least $1 - 1/(\Theta(k))$, to further deal with the union bound argument over all $k$ blocks. A natural approach is to set $\gamma = 1/(8k)$, as was proposed by [Xie18]. By reducing the problem of locating the literal to the problem of identifying the block with the highest reward, our newly proposed subroutine FindLiteral achieves the same guarantee with a leveraged number of queries to perform literal testing. Therefore, the improvement of the $\gamma$-special block. The subroutine FindLiteral requires $O((1 - \gamma)^{-2}\log k)$ queries. Setting $\gamma$ as a constant, the number of queries required is $O(\log k)$.

The other advantage of our algorithm is achieved by the observation that we only need to locate the literal for the to-be-orientated $\gamma$-special block, instead of locating the literal for all the $\gamma$-special blocks in all repetitions. Besides, we design a new stopping condition to accommodate the strategy of reusing samples. Specifically, our algorithm iterates for a total of $k + 1$ repetitions, if the algorithm fails to increase the number of relevant blocks by 1 within $\Theta(1)/\epsilon$ iterations, it directly terminates and accepts. As a comparison, the algorithm proposed by [LCS+19] iterates for $\Theta(k^2)/\epsilon$ times, and determines whether to accept if it fails to find more than $k$ relevant blocks after all the iterations. Consequently, we improve the query complexity of the adaptive one-sided tester from $\tilde{O}(k^2)/\epsilon$ to the optimal $\tilde{O}(k)/\epsilon$.

## 2 Preliminaries

Let $f : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function. Let $[n]$ represent $\{1, 2, \ldots, n\}$. A nonempty subset of $[n]$ is also referred to as a block. Given $x, y \in \{0, 1\}^n$, $(x, y)$ is referred to as a distinguishing pair of $f$ for block $B$ if $f(x) \neq f(y)$, and $x_B = y_B$. For a subset $B \subseteq [n]$, let $\bar{B}$ denote the complement of $B$, i.e., $\bar{B} = [n] \setminus B$. Let $x(B)$ represent the string obtained from $x$ with coordinates in $B$ flipped. The string $S = x_{B_1}^{x_1} x_{B_2}^{x_2} \ldots x_{B_m}^{x_m}$ represents the string that is equal to $x^i$ over coordinates in $B_i$, for all $i = 1, 2, \ldots, m$. If there exist $x, y \in \{0, 1\}^n$, satisfying that $f(x) \neq f(y)$, $x$ differs from $y$ in one coordinate ($y = x(i)$), then $i$ is referred to as a relevant variable. If there exist $x, y \in \{0, 1\}^n$, satisfying that $f(x) \neq f(y)$, and $y = x^{(B)}$, then $B$ is referred to as a relevant block. Let $f, g : \{0, 1\}^n \to \{0, 1\}$, $\dist_D(f, g) = \Pr_{x \sim D}(f(x) \neq g(x))$ is used to measure the distance
between $f$ and $g$ under the distribution $\mathcal{D}$. A function $f$ is a $k$-junta if it has at most $k$ relevant variables. Let $\mathcal{J}_k$ denote the class of $k$-juntas. $\text{dist}_\mathcal{D}(f, \mathcal{J}_k) = \min_{g \in \mathcal{J}_k} \text{dist}_\mathcal{D}(f, g)$ is used to measure the distance between $f$ and $k$-junta functions under the distribution $\mathcal{D}$. If $\text{dist}_\mathcal{D}(f, \mathcal{J}_k) \geq \epsilon$, then $f$ is said to be $\epsilon$-far from every $k$-junta under the distribution $\mathcal{D}$; if $\text{dist}_\mathcal{D}(f, \mathcal{J}_k) \leq \epsilon$, then $f$ is said to be $\epsilon$-close to $k$-junta under the distribution $\mathcal{D}$. For a given Boolean function $f$ and string $x \in \{0,1\}^n$, we say block $B$ belongs to class of functions $\mathcal{C}$ if $g(u) = f(u_B x_B)$ belongs to class of functions $\mathcal{C}$.

For a given Boolean function $f$ and string $x \in \{0,1\}^n$, we say block $B$ is $\gamma$-close to a literal under the uniform distribution if $\text{dist}_\mathcal{U}(g, z) \leq \gamma$, where $g(u) = f(u_B x_B)$ and $z(u)$ only depends on a literal. If a block contains at least one relevant variable, and is $\gamma$-close to a literal under the uniform distribution, then this block is called a $\gamma$-special block, and the corresponding literal is referred to as special literal. A tester with one-sided error under the distribution $\mathcal{D}$ is a randomized algorithm that accepts if $f$ is $k$-junta, and rejects with probability at least $2/3$ if $f$ is $\epsilon$-far from every $k$-junta under the distribution $\mathcal{D}$. A tester with two-sided error under the distribution $\mathcal{D}$ is a randomized algorithm that accepts with probability at least $2/3$ if $f$ is $k$-junta, and rejects with probability at least $2/3$ if $f$ is $\epsilon$-far from every $k$-junta under the distribution $\mathcal{D}$.

**Definition 1** (One-sided distribution-free testing algorithm for $k$-junta). Given as input a distance parameter $\epsilon > 0$ and oracle access to a pair $(f, \mathcal{D})$, a randomized algorithm $A$ is referred to as a one-sided distribution-free testing algorithm for $k$-junta if it satisfies:

- If $f$ is a $k$-junta, then $A$ accepts.
- If $f$ is $\epsilon$-far from every $k$-junta with respect to $\mathcal{D}$, then $A$ rejects with probability at least $2/3$.

The query complexity of a distribution-free testing algorithm is the number of queries made on $f$ plus the number of samples drawn from $\mathcal{D}$.

**Definition 2** ($\epsilon$-far from $k$-part). Let $\mathcal{I}$ be a partition of $[n]$. $f$ is $\epsilon$-far from $k$-part with respect to $\mathcal{I}$ under distribution $\Omega$, if for every set $J$ formed by taking the union of $k$ parts in $\mathcal{I}$, $\forall_{\Omega, \Omega}(J) = \Pr_{x \sim \Omega, w \sim \Omega}[f(x) \neq f(x, Jw)] \geq \epsilon$.

**Definition 3** ($\epsilon$-far from $k$-coordinate). We say $f$ is $\epsilon$-far from $k$-coordinate with respect to $[n]$ under distribution $\Omega$, if for every set $J$ formed by taking the union of $k$ coordinates in $[n]$, $\forall_{\Omega, \Omega}(J) = \Pr_{x \sim \Omega, w \sim \Omega}[f(x) \neq f(x, Jw)] \geq \epsilon$.

## 3 A tester for distribution-free junta

### 3.1 Problem Statement

We aim at designing a one-sided distribution-free algorithm for testing the property of being $k$-juntas over Boolean functions. Specifically, if $f : \{0,1\}^n \rightarrow \{0,1\}$ is $\epsilon$-far from every $k$-junta under the distribution $\mathcal{D}$, then the algorithm rejects with probability at least $2/3$; If $f$ is a $k$-junta, then the algorithm accepts.

The algorithm is allowed to draw samples from distribution $\mathcal{D}$ and also from the uniform distribution $\mathcal{U}$. The key distinction of this problem from the commonly investigated setting is that the distance is now measured in terms of an unknown and arbitrary distribution $\mathcal{D}$. When the distance is measured under different distributions, the junta class that a Boolean function belongs to might also change. Let us take the Boolean function $f$ in Figure 1 as an example. $f$ is $1/4$-far from every 1-junta under the uniform distribution, while it is a 1-junta under the distribution $\mathcal{D}$. 


3.2 Overview of previous approaches

[Bla09] introduced an optimal algorithm under the product distribution. They focus on the functions of the form $f : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, and $\mathcal{Y}$ is an arbitrary finite set. Let $\Omega = \Omega_1 \times \Omega_2 \cdots \times \Omega_n$ be a product distribution over $\mathcal{X}$, where $\Omega_i = (\mathcal{X}_i, \mu_i)$, and $\mu_i$ is an arbitrary probability measure on $\mathcal{X}_i$. The coordinates in $[n]$ are randomly initially partitioned into a total of $m$ disjoint blocks, where $m = \text{poly}(k/\epsilon)$. The algorithm keeps on finding a new pair of strings with distinct function values by sampling from the distribution $\Omega$, with the coordinates in the relevant blocks fixed. Once such pair of strings is found, the algorithm uses binary search over blocks to find a block that contains at least one relevant variable.

The core structural lemma introduced by [Bla09] is as follows:

**Lemma 1** ([Bla09]). Let $s = 10^{20}k^9/\epsilon^5$. Let $\mathcal{I}$ be a random partition of $[n]$ into $s$ parts, which is obtained by uniformly and independently assigning each coordinate to a part. For a function $f : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, and $\mathcal{Y}$ is an arbitrary finite set. If $f$ is $\epsilon$-far from being a $k$-junta, then with probability at least $5/6$, $f$ is $\epsilon/2$-far from $k$-part with respect to $\mathcal{I}$.

This lemma shows that if a function is $\epsilon$-far from every $k$-junta under distribution $\Omega$, it is also $\epsilon/2$-far from $k$-part with high constant probability. That is, with probability at least $5/6$, $\mathbb{V}_{\Omega,\Omega}(J) \geq \epsilon/2$, where $J$ is any union of $k$ parts in $\mathcal{I}$. The analysis of this lemma is based on the Efron-Stein orthogonal decomposition. However, it is quite challenging to generalize the analysis to make it applicable in the distribution-free setting.

For the Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$, it was shown that the total number of blocks required is $s = 24k^2$. Note that $s$ is independent of $\epsilon$. Besides, the analysis of this structural lemma is much simpler. The core structural lemma is as follows:

**Lemma 2** ([BWY15]). Let $s = 24k^2$. Let $\mathcal{I}$ be a random partition of $[n]$ into $s$ parts, which is obtained by uniformly and independently assigning each coordinate to a part. For a function $f : \{0,1\}^n \rightarrow \{0,1\}$. If $f$ is $\epsilon$-far from being a $k$-junta, then with probability at least $5/6$, $f$ is $\epsilon/2$-far from $k$-part with respect to $\mathcal{I}$.
That is, for any union of $k$ parts $J$ in $I$, we have that $V_{UJ}(J) \geq \epsilon/2$. The analysis of this lemma uses the property of intersecting family. However, the analysis also relies on the property of sub-additivity of influence. Then the question becomes: does sub-additivity of $V$ still hold in distribution-free setting? We know that the sub-additivity of $V$ (i.e., $V_{UJ}(J \cup K) \leq V_{UJ}(J) + V_{UJ}(K)$) holds if $x$ and $w$ are both uniformly sampled from $\{0, 1\}^n$. But if $x$ is sampled from $\{0, 1\}^n$ according to an arbitrary and unknown distribution $\mathcal{D}$, and $w$ is sampled uniformly from $\{0, 1\}^n$, the sub-additivity of $V$ does not always hold true. For example, for the Boolean function $h : \{0, 1\}^3 \rightarrow \{0, 1\}$ as illustrated in Figure 1, let the first branch with probability 1 represent distribution $\mathcal{D}$, then we have $V_{\mathcal{D}U}(\{2\}) = V_{\mathcal{D}U}(\{3\}) = 0$, while $V_{\mathcal{D}U}(\{2, 3\}) > 0$. Therefore, the sub-additivity of $V$ under the distribution $\mathcal{D}$ does not hold in some cases. That is, $V_{\mathcal{D}U}(J \cup K) \leq V_{\mathcal{D}U}(J) + V_{\mathcal{D}U}(K)$.

### 3.3 The Algorithm

We propose an algorithm that has near-optimal query complexity, and ensuring that both the algorithm and analysis are simple and easy to understand intuitively. The first testing approach uses similar testing structure by [LCS+19], both testers under the distribution $\mathcal{D}$ are based on literal tester under the uniform distribution. By viewing the relationship between finding the part that the literal lies in and the biased coin identification problem, we achieve major advancement towards improving the query complexity. Using the "success amplification by majority" technique, we design a simple as well as fast subroutine. Accompanied with some sophisticated techniques including reuse some samples appropriately, our tester achieves near-optimal query complexity. Our tester iterates for a total of $k+1$ repetitions, if the tester fails to increase the number of relevant blocks by 1, it directly terminates and accepts. The flowchart of our tester is illustrated in Figure 2.

![Figure 2: The framework of our tester. The flowchart begins from the subroutine IsLiteral. If IsLiteral rejects for any $B_i$, then new relevant blocks are obtained by splitting a non-literal block into two new blocks. If IsLiteral accepts for all $B_i$, and FindRelevantBlock succeeds, then a new relevant block is obtained.](image-url)

The main distinctions between our algorithm and the algorithm MainDJunta proposed in
include:

- MainDJunta accepts if the algorithm could not find more than \(k\) relevant blocks in a total of \(\tilde{O}(k^2)/\epsilon\) queries, while our algorithm accepts if the algorithm fails to find one more relevant block in \(\tilde{O}(1)/\epsilon\) queries.

- MainDJunta uses the subroutine \texttt{WhereIsTheLiteral} for identifying the part that contains the literal, while our algorithm uses a new subroutine \texttt{FindLiteral} which will be introduced in detail in section 4.2.

- Our algorithm reuses the results obtained by the subroutine \texttt{FindLiteral}, which is an important observation towards achieving nearly-optimal query complexity. The specific approach is illustrated in Figure 4.

Algorithm 1 Distribution-Free Junta Testing

1: Let \(\mathcal{A} = \{A_1, \cdots, A_v\}\) be the set of to-be-orientated \(\gamma\)-special blocks, \(\mathcal{B} = \{B_1, \cdots, B_s\}\) be the set of to-be-checked relevant blocks, and \(\mathcal{K} = \{K_1, \cdots, K_m\}\) be the set of \(\gamma\)-special blocks, \(\mathcal{G} : \mathcal{B} \to \{0,1\}^n\), where \(B \in \mathcal{B}\) represents a block, and \(s \in \{0,1\}^n\) represents a distinguishing string satisfying that \(f(s) \neq f(s(B))\).

2: Initialization: Set \(\mathcal{A} = \mathcal{B} = \mathcal{K} = \mathcal{N} = \emptyset\), and \(w = 0\).

3: for each of \(k+1\) repetitions do
4:    for \(j = 1\) to \(|\mathcal{B}|\) do
5:        //test whether block \(B_j\) is \(\gamma\)-close to a literal under the uniform distribution
6:        if \texttt{IsLiteral}(\(\mathcal{G}(B_j), B_j\)) accepts then
7:            update \(\mathcal{K} \leftarrow \mathcal{K} \cup \{B_j\}\).
8:        else
9:            obtain two relevant blocks \(Z_1\) and \(Z_2\) (\(Z_1 \cup Z_2 = B_j\), and \(Z_1 \cap Z_2 = \emptyset\)), and update \(\mathcal{G}\).
10:       set \(\mathcal{B} \leftarrow \mathcal{B} \cup \{Z_1\} \cup \{Z_2\} \setminus \{B_j\}\), \(w \leftarrow w + 1\), and return to line 8.
11:    end if
12: end for
13: Set \(\mathcal{A} \leftarrow \mathcal{B}\).
14: if \texttt{FindRelevantBlock} \((f, w, \mathcal{A}, \mathcal{K}, \mathcal{G}, \mathcal{N})\) returns false then
15:    return accept.
16: else
17:    obtain \(\mathcal{B}\) and \(\mathcal{G}\), \(w \leftarrow w + 1\), \(\mathcal{A} \leftarrow \emptyset\).
18: end if
19: end for
20: return reject.

The algorithm maintains three collections of blocks. Let \(\mathcal{A} = \{A_1, \cdots, A_v\}\) represent the set of to-be-orientated \(\gamma\)-special blocks (\(\gamma\)-special blocks that have not been identified using the subroutine \texttt{FindLiteral}), \(\mathcal{B} = \{B_1, \cdots, B_s\}\) represent the set of to-be-checked relevant blocks (relevant blocks that have not been identified using the subroutine \texttt{IsLiteral}), and \(\mathcal{K} = \{K_1, \cdots, K_m\}\) represent the set of all \(\gamma\)-special blocks, as is shown in Algorithm 1. At each repetition, the algorithm uses the \texttt{IsLiteral} algorithm to test whether each of the to-be-checked relevant block is \(\gamma\)-close to a literal under the uniform distribution. Note that each relevant block \(B\) is accompanied with a distinguishing pair \((x, y)\) for the block \(B\). That is, \(y = x(B)\), and \(f(x) \neq f(y)\). When \(f\) is restricted to the relevant block \(B\), the input outside the block \(B\) is fixed as \(x_B\). If \texttt{IsLiteral} returns false for
some to-be-checked relevant blocks, then the algorithm divides the relevant block into two parts, each part contains at least one relevant variable. In this case, one original relevant block is divided into two new relevant blocks, and the total number of relevant blocks increases by 1. If \textbf{IsLiteral} returns true for all the to-be-checked relevant blocks, the algorithm constructs \(y\) based on \(x\) which is randomly sampled from distribution \(\mathcal{D}\), ensuring that \(f(x) \neq f(y)\) with probability at least \(\epsilon/2\), then a new block could be found among the blocks that \(x\) differs from \(y\). In this way, the algorithm maintains that in the beginning of repetition \(m + 1\), there are \(m\) relevant blocks. Before introducing the approach to construct distinguishing pairs in detail, we will review a property (illustrated in the following lemma) used to guide the design of the construction approach.

**Lemma 3** (Lemma 3.2 of \cite{LCS+19}). Let \(I\) be a subset of \([n]\). If \(f\) is \(\epsilon\)-far from every \(k\)-junta under the distribution \(\mathcal{D}\), and the size of \(I\) does not exceed \(k\), then it is satisfied that

\[ \Pr_{x \sim \mathcal{D}, w \sim \mathcal{U}}[f(x) \neq f(x_I w_I)] \geq \epsilon/2. \]  

**Proof.** \cite{LCS+19} provide a simple proof for this lemma. \cite{Bsh19} further provide an extremely simpler proof for this lemma. We will illustrate this proof to help readers understand it in a more intuitive way.

Let \(I \subset [n]\) satisfying \(|I| \leq k\). For every fixed \(w \in \{0, 1\}^n\), the function \(f(x_I w_I)\) is a \(k\)-junta and therefore \(\Pr_{x \sim \mathcal{D}}[f(x) \neq f(x_I w_I)] \geq \epsilon/2\). Thus,

\[ \Pr_{x \sim \mathcal{D}, w \sim \mathcal{U}}[f(x) \neq f(x_I w_I)] \geq \epsilon/2. \]

\[ \Box \]

**Remark:** Eq. \((1)\) could be interpreted from the following point of view,

\[ \Pr_{x \sim \mathcal{D}, R \subseteq I}[f(x) \neq f(x^{(R)})] \geq \epsilon/2. \]  

The string \(y = x_I w_I\) could be interpreted as fixing the coordinates in \(I\), and each of the remaining coordinate is uniformly sampled from \(\{0, 1\}\). For each coordinate \(y_i\) in \(\tilde{I}\), \(y_i\) is equal to 1 with probability 1/2, and is equal to 0 with probability 1/2. Since \(x_i\) is either 1 or 0, this implies that each coordinate in \(\tilde{I}\) is equal to \(x_i\) with probability 1/2, and is equal to \(\overline{x_i}\) with probability 1/2.

Therefore, if \(f\) is \(\epsilon\)-far from every \(k\)-junta under the distribution \(\mathcal{D}\), and the total number of the \(\gamma\)-special blocks does not exceed \(k\), then with probability at least \(\epsilon/2\), the algorithm could find a new block which contains at least one more relevant variable.

The specific construction approach is as follows: suppose a total of \(m\) \(\gamma\)-special blocks \(K_1, K_2, \ldots, K_m\) are identified, the algorithm randomly partitions each block into two parts, and uses the \textbf{FindLiteral} algorithm (We defer to Section 4.2 detailed introduction of this algorithm) to identify which part the literal lies in. Then the algorithm flips the coordinates of \(x\) that belong to the part that does not contain the literal. For the remaining block \([n] \setminus \bigcup_{i \in [m]} K_i\), the algorithm randomly flips a subset of coordinates of \(x\) in this block. In this way, the algorithm constructs \(y\) based on \(x\). It follows from Eq. \((2)\) that with probability at least \(\epsilon/2\), \(x\) and \(y\) have distinct function values. Here, \(I\) is composed of the literals in each \(\gamma\)-special block. When the number of \(\gamma\)-special blocks does not exceed \(k\), the size of \(I\) also does not exceed \(k\). Figure 3 shows an example for constructing \(y\). If the algorithm finds a pair of strings \(x\) and \(y\) with distinct function values, then binary search over blocks is used to find a new relevant block.

**Remark:** Initially, there does not exist any relevant block. Algorithm 2 goes directly to line 15. This procedure is equivalent to the following operation: randomly select a subset \(S\) from \([n]\), and then construct \(y\) by flipping the coordinates in \(S\) of string \(x\).
Algorithm 2 FindRelevantBlock \( (f, w, A, \mathcal{K}, \mathcal{G}, \mathcal{N}) \)

1: Let \( A = \{A_1, \ldots, A_v\} \) be the set of to-be-orientated \( \gamma \)-special blocks, \( \mathcal{K} = \{K_1, \ldots, K_m\} \) be the set of \( \gamma \)-special blocks, \( \mathcal{G} : B \rightarrow \{0, 1\}^n \), where \( B \in \mathcal{B} \) represents a block, and \( s \in \{0, 1\}^n \) represents a distinguishing string satisfying that \( f(s) \neq f(s(B)) \).

2: Initialization: \( \text{FindRelevant} = 0 \), \( t = 0 \)

3: while \( t < 2(\log k + 6)/\epsilon \) do

4: for \( j = 1 \) to \( |A| \) do

5: \( z \leftarrow \mathcal{G}(A_j) \)

6: Randomly partition the to-be-orientated \( \gamma \)-special block \( A_j \) into two parts \( A_1^j \) and \( A_2^j \).

7: if \( \text{FindLiteral}(z, A_1^j, A_2^j) \) returns fail then

8: \( t \leftarrow t + 1 \), return to line 3.

9: else

10: it returns a block \( L_t^j \) and the corresponding string

11: Update \( \mathcal{N} \leftarrow \mathcal{N} \cup \{N_t^j\} \), where \( N_t^j = A_j \setminus L_t^j \).

12: end if

13: end for

14: if \( \text{FindRelevant} \) is 0 then

15: Sample \( x \) from the distribution \( \mathcal{D} \), construct \( y \) by flipping coordinates \( \cup_{i \in \mathcal{K}}\{N_t^i\} \cup \{C\} \) of \( x \), where \( C \) is a random subset of \( (\mathbb{N} \setminus \cup_{i \in \mathcal{K}}\{K_i\}) \).

16: if \( f(x) \neq f(y) \) then

17: run \( \text{BlockBinarySearch} \ (f, x, y, N_1^t, N_2^t, \ldots, N_{|\mathcal{K}|}^t, C, \mathcal{G}) \), obtain \( \mathcal{B} \) and \( \mathcal{G} \).

18: set \( \text{FindRelevant} \) as 1.

19: end if

20: end if

21: \( t \leftarrow t + 1 \).

22: end while

23: if \( \text{FindRelevant} \) is 1 then

24: return \( \mathcal{B} \) and \( \mathcal{G} \).

25: else

26: return false.

27: end if

Algorithm 3 BlockBinarySearch \( (f, x, y, N_1, N_2, \ldots, N_{|\mathcal{K}|}, C, \mathcal{G}) \)

1: **Input:** strings \( x \) and \( y \in \{0, 1\}^n \), satisfying that \( f(x) \neq f(y) \), and \( x = y(\cup_{i \in \mathcal{K}\{N_i\}} \cup C) \).

2: **Output:** the to-be-checked relevant blocks \( \mathcal{B} \), and a string \( z \) satisfying that \( f(z) \neq f(z(B_{|\mathcal{K}|+1})) \) for \( B_{|\mathcal{K}|+1} \in \{N_1, N_2, \ldots, N_{|\mathcal{K}|}, C\} \).

3: Use binary search on \( x \) and \( y \) to find a relevant block \( B_{|\mathcal{K}|+1} \) over blocks \( N_1, N_2, \ldots, N_{|\mathcal{K}|}, C \), accompanied with the distinguishing string \( z \).

4: Update \( \mathcal{G} \leftarrow \mathcal{G} \cup \{B_{|\mathcal{K}|+1}, z\} \).

5: if \( B_{|\mathcal{K}|+1} = C \) then

6: update \( \mathcal{B} \leftarrow \{B_{|\mathcal{K}|+1}\} \).

7: else if \( B_{|\mathcal{K}|+1} = N_h \) then

8: update \( \mathcal{B} \leftarrow \{L_h\} \cup \{N_h\} \).

9: end if

10: Return \( \mathcal{B} \) and \( \mathcal{G} \).
Now we illustrate the specific approach of finding a new relevant block using binary search over blocks. Suppose that $x$ differs from $y$ in coordinates in block $D$. Let $D = \bigcup_{i \in [p]} D_i$, $D_l = \bigcup_{i=1}^d D_i$ and $D_r = \bigcup_{i=d+1}^p D_i$, where $d = \lfloor p/2 \rfloor$. Since $f(x) \neq f(y)$, we have that either $f(x^{(D_l)}) \neq f(x)$ or $f(x^{(D_l)}) \neq f(y)$. Note that $f(y) = f(x^{(D_l)}) = f(x^{(D_l) \cup D_r})$. This implies that $x^{(D_l)}$ differs from $y$ in coordinates in $D_r$. If $f(x^{(D_l)}) \neq f(x)$, then the search range shrinks to $D_l$, otherwise it shrinks to $D_r$. In this way, binary search recursively shrink the search range to a half of the block set, and could finally find one block that contains at least one relevant variable. In contrast with using binary search over coordinates, the query complexity is reduced from $O(\log(n))$ to $O(\log(k))$.

![Figure 3](image.png)

Figure 3: Let the whole circle represent string $x$ over coordinates $[n]$. The algorithm finds three $\gamma$-special blocks, separately $B_1$, $B_2$ and $B_3$. Each block $B_i$ is $\gamma$-close to a literal $l_i$ under the uniform distribution, which is represented using a triangle. Then the algorithm randomly divides each block into two parts, and partitions the remaining block (complement of the union of blocks $B_1$, $B_2$ and $B_3$) into two parts. The algorithm constructs $y$ by flipping the coordinates in the blocks that are marked by the heart (the part $N_1(N_2, N_3)$ that does not contain the literal $l_1(l_2, l_3)$, and a random part of the remaining block).

It is worth noting that the subroutine FindLiteral could be reused, thereby efficiently reducing query complexity. An illustrative example is shown in Figure 4.

4 Subroutines for finding relevant blocks

In this section, we will introduce the specific approaches for finding relevant blocks. For a Boolean function $f$ that is $\epsilon$-far from every $k$-junta under the distribution $D$, if the to-be-checked relevant block is $\gamma$-far from every literal under the uniform distribution, then the total number of relevant blocks increases by 1; If the to-be-checked relevant block is $\gamma$-close to a literal under the uniform distribution, then the algorithm fixes a random subset of the $\gamma$-special block that contains the literal. According to Lemma 3, a pair of strings $(x, y)$ with distinct function values could be identified with probability at least $\epsilon/2$. The total number of relevant blocks increases by 1. Therefore, it requires to design an algorithm which could identify whether this block is $\gamma$-close to a literal under the uniform distribution, and an algorithm that could identify the part that contains the literal.
Figure 4: (a) The algorithm finds three relevant blocks $B_i = N_i \cup L_i$, $B_i$ is $\gamma$-close to literal $l_i$ under the uniform distribution, where $i = 1, 2, 3$. $L_1$, $L_2$ and $L_3$ are the blocks that contain literals $l_1$, $l_2$ and $l_3$ separately. The coordinates in $N_1, N_2, N_3$ and $R^1$ are flipped. $N_1$ is the relevant block found using binary search over $N_1, N_2, N_3$ and $R^1$. (b) $B_1$ is divided into two relevant blocks $B_4$ and $B_5$, referred to as the to-be-checked relevant block. Now the algorithm finds a total of four relevant blocks $B_i = N_i \cup L_i$, $i = 2, 3, 4, 5$. The algorithm then uses IsLiteral to test whether each of the to-be-checked relevant block $B_4$ and $B_5$ is $\gamma$-close to a literal under the uniform distribution. If IsLiteral returns true for these two blocks, then $B_4$ and $B_5$ are referred to as the to-be-orientated $\gamma$-special blocks ($A = \{B_4, B_5\}$). The algorithm then randomly partitions $B_4$ and $B_5$ to two parts, and then uses FindLiteral to find the part that contains the literal. Note that the subroutine FindLiteral over blocks $B_2$ and $B_3$ could be reused, which is an important observation for designing the algorithm with improved query complexity.

4.1 Subroutine for testing the literal

The first algorithm IsLiteral is used to test whether $g$ is $\gamma$-close to a literal under the uniform distribution. Similar to [LCS+19], this algorithm uses UniformJunta proposed by [Bla09] for junta testing under the uniform setting (the algorithm is specified in appendix). The key difference is that here $\gamma$ is set as a constant instead of a function of $k$.

- **Input:** $g: \{0, 1\}^B \rightarrow \{0, 1\}$.
- **Output:** Accept if $g$ is a literal; Reject with probability at least $1 - 1/(28k)$ if $g$ is $\gamma$-far from literal, where $\gamma = 1/8$.

Lemma 4 ([Bla09]). If $f$ is $k$-junta under the uniform distribution, then UniformJunta($f, k, \gamma$) accepts. If $f$ is $\gamma$-far from every $k$-junta under the uniform distribution, then UniformJunta($f, k, \gamma$) rejects with probability at least $2/3$. The query complexity of this algorithm is bounded by $O(k/\gamma + k \log(k))$.

Lemma 5. If $g$ is $\gamma$-far from any literal under the uniform distribution, then with probability at least $1 - 1/(28k)$, the algorithm IsLiteral rejects, where $\gamma = 1/8$. 

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4.2 Subroutine for locating the literal

In this section, we will introduce our algorithm for locating the literal. Before introducing our algorithm in detail, we will firstly compare the key results of our algorithm and the counterpart of the previous work.

4.2.1 Comparison with the previous work

\[LCS^{+19}\] used a subroutine called \texttt{WhereIsTheLiteral} to find the part the literal lies in. The property of this subroutine is illustrated in the following lemma.

**Lemma 6** \([LCS^{+19}]\). Let \(\gamma = 1/(8k)\). Assume that \(g : \{0,1\}^B \rightarrow \{0,1\}\) is \(\gamma\)-close (with respect to the uniform distribution) to a literal \(x_i\) or \(\bar{x}_i\) for some \(i \in B\). If \(i \in P\), then \texttt{WhereIsTheLiteral} \((g,P,Q)\) returns a distinguishing pair of \(g\) for \(P\) with probability at least \(1 - 4\gamma\); If \(i \in Q\), then it returns a distinguishing pair of \(g\) for \(Q\) with probability at least \(1 - 4\gamma\).

Each call of the subroutine \texttt{WhereIsTheLiteral} requires four queries. The query complexity of the algorithm \texttt{IsLiteral} is \(O(1/\gamma) = \tilde{O}(k)\) from Lemma 4. We propose a new subroutine \texttt{FindLiteral}, the result is shown in the following lemma. Compared with the above lemma, the advantage of Lemma 7 is that it improves the parameter \(\gamma\) from \(1/(8k)\) to a constant.

**Lemma 7.** If a Boolean function \(g : \{0,1\}^B \rightarrow \{0,1\}\) is \(\gamma\)-close to a literal under the uniform distribution, then we have that with probability at least \(1 - 1/(12k)\), the algorithm \texttt{FindLiteral} could return the part that contains the literal, where \(\gamma = 1/8\).

On the one hand, the query complexity of the algorithm \texttt{IsLiteral} is \(\tilde{O}(1/\gamma) = \tilde{O}(1)\) from Lemma 4. Besides, the algorithm \texttt{FindLiteral} requires \(O((1 - \gamma)^{-2} \log k) = \tilde{O}(1)\) queries. Setting \(\gamma\) as a constant, the number of queries required is \(O(\log k)\). On the other hand, the parameter \(\gamma\) controls the probability of reject if \(f\) is \(\epsilon\)-far from every \(k\)-junta. From Lemma 7, the algorithm could find the correct part that contains the literal with probability at least \(1 - 1/\Theta(k)\). By a union bound over all \(k\) blocks, the probability that the algorithm could find correctly the blocks that contain the literals is lower bounded by a constant.

4.2.2 \texttt{FindLiteral} Algorithm

Now we are ready to introduce our newly proposed algorithm. If \(g : \{0,1\}^B \rightarrow \{0,1\}\) is \(\gamma\)-close to a literal \(l\) under the uniform distribution, then block \(B\) is randomly divided into two parts \(L\) and \(N\). We want to design an algorithm to identify which part the literal lies in using \(O(\log k)\) number of queries.

- **Input:** \(g : \{0,1\}^B \rightarrow \{0,1\}\) that is \(\gamma\)-close to a literal \(l\) under the uniform distribution, block \(L\) and \(N\) (\(L \cap N = \emptyset, L \cup N = B\)), \(\gamma = 1/8\).

- **Output:** With probability at least \(1 - 1/(12k)\), return the block that contains literal \(l\).

We show that this problem could be interpreted as a variant of the problem of best block identification. Let \(L\) represent the block that contains literal \(l\). We use \(i\) to index the sample sequence. Let \(S^i_L = 1\) represent \(g(w^i) \neq g(w^i(L))\), and \(S^i_L = 0\) otherwise. We will show that \(\mathbb{E}[S^i_L] \geq 1 - 2\gamma\). Similarly, let \(S^i_N = 1\) represent \(g(x^i) \neq g(x^i(N))\), and \(S^i_N = 0\) otherwise. We will show that \(\mathbb{E}[S^i_N] \leq 2\gamma\). The expected reward of block \(L\) is at least \(1 - 2\gamma\), and that of block \(N\) is at most \(2\gamma\). Set \(\gamma = 1/8\), we know that block \(L\) has larger reward. Then, the problem of locating the literal is reduced to the problem of identifying the best block with high probability within \(O(\log k)\) number of queries.
Algorithm 4 FindLiteral \((x, L, N)\)

1: Define \(g(u) = f(u_B x_B)\), where \(B = L \cup N\).
2: Set \(i \leftarrow 1\).
3: while \(i \leq 128 \log(64k)\) do
4: Query a pair of strings \((w^i, v^i)\) with coordinates in block \(L\) flipped. Specifically, \(w^i\) is sampled from \(\{0, 1\}^B\) uniformly at random, and \(v^i \leftarrow w^{i(L)}\). \(S_L^i \leftarrow 1\) if \(g(w^i) \neq g(v^i)\), and \(S_L^i \leftarrow 0\) otherwise.
5: Query a pair of strings \((w^i, v^i)\) with coordinates in block \(N\) flipped. Specifically, \(w^i\) is sampled from \(\{0, 1\}^B\) uniformly at random, and \(v^i \leftarrow w^{i(N)}\). \(S_N^i \leftarrow 1\) if \(g(w^i) \neq g(v^i)\), and \(S_N^i \leftarrow 0\) otherwise.
6: \(i \leftarrow i + 1\).
7: end while
8: \(\hat{T}_L^i \leftarrow \sum_{j=1}^i S_L^j\), \(\hat{T}_N^i \leftarrow \sum_{j=1}^i S_N^j\)
9: if \(\hat{T}_L^i > \hat{T}_N^i\) then
10: let \(w\) be any string satisfying that \(g(w) \neq g(w^{i(L)})\), \(z \leftarrow w_B x_B\).
11: return \(L\) and \(z\).
12: else if \(\hat{T}_N^i > \hat{T}_L^i\) then
13: let \(w\) be any string satisfying that \(g(w) \neq g(w^{i(N)})\), \(z \leftarrow w_B x_B\).
14: return \(N\) and \(z\).
15: else
16: return fail.
17: end if

Since \(g\) is \(\gamma\)-close to a literal \(l\) under the uniform distribution, we have that

\[
Pr_{w^i \sim \mathcal{U}} (g(w^i) = h(w^i)) \geq 1 - \gamma,
\]

(3)

where \(h(w)\) is a literal function which depends on literal \(l\).

Let \(a^i = w^{i(L)}\). Since \(h(\cdot)\) is a literal function which depends on literal \(l\), and literal \(l\) is contained in block \(L\), we have that \(h(w^i) \neq h(a^i)\). Therefore,

\[
Pr_{w^i \sim \mathcal{U}} [g(w^i) = h(a^i)] \leq Pr_{w^i \sim \mathcal{U}} [g(w^i) \neq h(w^i) \lor g(a^i) \neq h(a^i)] \leq 2\gamma.
\]

(4)

Therefore, we have

\[
Pr_{w^i \sim \mathcal{U}} [g(w^i) \neq h(a^i)] \geq 1 - 2\gamma.
\]

(5)

This implies that block \(L\) that contains literal \(r\) could be regarded as the block with expected value larger than \(1 - 2\gamma\). Therefore, \(\mathbb{E}[S_L^i] \geq 1 - 2\gamma\).

Let \(S_N^i = 1\) represent \(g(x^i) \neq g(x^{i(N)})\), and \(S_N^i = 0\) otherwise. Now we show that \(\mathbb{E}[S_N^i] \leq 2\gamma\). Let \(b = x^{i(N)}\). Since \(h(\cdot)\) is a literal function which depends on literal \(l\), and literal \(l\) is not contained in block \(N\), we have that \(h(x^i) = h(b^i)\). Therefore,

\[
Pr_{x^i \sim \mathcal{U}} [g(x^i) \neq g(b^i)] \leq Pr_{x^i \sim \mathcal{U}} [g(x^i) = h(x^i) \lor g(b^i) = h(b^i)] \leq 2\gamma.
\]

(6)
Therefore, we have
\[
\Pr_{x^i \sim U^i} [g(x^i) = g(b^i)] \geq 1 - 2\gamma. \tag{7}
\]

This implies that block \( N \) that does not contain literal \( l \) could be regarded as the block with expected value less than \( 2\gamma \). Therefore, \( \mathbb{E}[S_{N}^{i}] \leq 2\gamma \).

The core idea of the \textit{FindLiteral} algorithm is that a confidence interval is constructed for each block. If the lower bound of the confidence interval of one block is larger than the upper bound of another block, then the reward of this block is larger with high probability. When the lower bound of the reward of one block is larger than the upper bound of the reward of another block, the algorithm returns this block. The algorithm returns an arbitrary block if it fails to identify such a block, which occurs with probability at most \( \delta \).

\textbf{Lemma 8} (Restatement of Lemma 7). If a Boolean function \( g : \{0, 1\}^B \rightarrow \{0, 1\} \) is \( \gamma \)-close to a literal under the uniform distribution, then we have that with probability at least \( 1 - 1/(12k) \), the algorithm \textit{FindLiteral} could return the part that contains the literal, where \( \gamma = 1/8 \).

\textit{Proof}. Suppose that \( g \) is \( \gamma \)-close to a literal \( l \) under the uniform distribution. The algorithm then randomly divide block \( B \) into two parts. Assume that \( L \) is the part that contains literal \( l \). Set \( i = 128 \log(64k) \), \( \delta = 1/(24k) \). Let \( \hat{\mu}^{i}_{L} = \sum_{j=1}^{i} S_{L}^{j}/i = \hat{T}_{L}^{i}/i \), \( \hat{\mu}^{i}_{N} = \sum_{j=1}^{i} S_{N}^{j}/i = \hat{T}_{N}^{i}/i \), and \( r_{i} = \sqrt{\log(1/\delta)/(2i)} \).

Recall that \( g(w^i) \neq g(w^{i(L)}) \) with probability at least \( 1 - 2\gamma \), and \( g(w^i) \neq g(w^{i(N)}) \) with probability at most \( 2\gamma \). Therefore, \( \hat{\mu}^{i}_{L} = \mathbb{E}[\hat{\mu}^{i}_{L}] \geq 1 - 2\gamma \), and \( \mu^{i}_{N} = \mathbb{E}[\hat{\mu}^{i}_{N}] \leq 2\gamma \), where \( \gamma = 1/8 \).

From Hoeffding’s inequality, we have that
\[
\Pr(|\mu - \hat{\mu}| \geq r) \leq \delta. \tag{8}
\]

Therefore, with probability at least \( 1 - 2\delta \), we have
\[
(\hat{T}_{L}^{i} - \hat{T}_{N}^{i})/i \geq (\mu_{L} - \mu_{N} - 2r_{i}) > 0. \tag{9}
\]

Therefore, we have that the algorithm could return the block \( L \) that contains the literal within \( 128 \log(64k) \) iterations with probability at least \( 1 - 1/(12k) \).

\section{Analysis of the Algorithm}

In this section, we show the correctness and the query complexity of our algorithm.

\subsection{Correctness Analysis of the Algorithm}

We firstly show the correctness of our algorithm. We will present a simple argument that does not rely on the potential function.

\textbf{Theorem 2}. If \( f \) is \( \epsilon \)-far from every \( k \)-junta under the distribution \( \mathcal{D} \), then the algorithm rejects with probability at least \( 2/3 \).

\textit{Proof}. We want to prove that, if \( f \) is \( \epsilon \)-far from every \( k \)-junta under the distribution \( \mathcal{D} \), then with probability at least \( 2/3 \), the algorithm could identify at least \( k + 1 \) relevant blocks. If the total number of relevant blocks is larger than \( k \), then the algorithm returns reject. If the total number of relevant blocks does not exceed \( k \), we analyze according to the following two cases.
Case 1: If one of the to-be-checked relevant blocks is $\gamma$-far from every literal under the uniform distribution. According to Lemma 5, the algorithm IsLiteral rejects with probability at least $1 - 1/(28k)$. From Line 9 of Algorithm 1, the number of relevant blocks increases by 1.

Case 2: If each of the to-be-checked relevant block is $\gamma$-close to a literal under the uniform distribution, then we analyze according to the following two subcases.

Subcase 1: The algorithm UniformJunta rejects for one of the to-be-checked relevant blocks, therefore IsLiteral rejects. This block is then divided into two relevant blocks, and the number of relevant blocks increases by 1.

Subcase 2: The algorithm UniformJunta accepts for all the to-be-checked relevant blocks, therefore IsLiteral accepts. From Lemma 7 we have that, with probability at least $1 - 1/(12k)$, the algorithm FindLiteral could identify correctly which block the literal lies in. Let $A_t$ denote the event that the blocks containing literals could be identified correctly in iteration $t$. By a union bound,

$$\Pr(A_t) \geq 1 - k \cdot 1/(12k).$$

Conditioned on event $A_t$ holds, according to Eq. (4) of Lemma 8, with probability at least $\epsilon/2$, the algorithm could find one more relevant block. Let $B_t$ denote the event that the algorithm succeeds in finding a new relevant block in iteration $t$, we have that

$$\Pr(B_t|A_t) \geq \epsilon/2.$$ 

It follows that

$$\Pr(A_t \cap B_t) = \Pr(A_t) \cdot \Pr(B_t|A_t) \geq 11\epsilon/24.$$ 

Therefore, with probability at least $1 - 1/(3k)$, the total number of relevant blocks increases by 1 in $2(\log k + 6)/\epsilon$ iterations.

Let $E_i$ be the event that the total number of relevant blocks increases from $i - 1$ to $i$. Combining the above two cases, we have that

$$\Pr(E_i) \geq \min\{1 - 1/(28k), 1 - 1/(3k)\} \geq 1 - 1/(3k).$$

Therefore,

$$\Pr(\cap_{i \in [k]} E_i) = 1 - \Pr(\cup_{i \in [k]} E_i) \geq 1 - \sum_{i \in [k]} \Pr(E_i) \geq 2/3. $$

Therefore, with probability at least 2/3, the algorithm could find $k + 1$ relevant blocks in $\tilde{O}(k/\epsilon)$ number of queries. In conclusion, the algorithm rejects with probability at least 2/3.

Lemma 9. If $f$ is a $k$-junta, then the algorithm accepts.

Proof. If $f$ is a $k$-junta, then the algorithm could not find more than $k$ blocks each containing at least one relevant variable. According to the design of the algorithm, it accepts. \qed
5.2 Query Complexity Analysis of the Algorithm

Theorem 3. The query complexity of the algorithm could be upper bounded by $\tilde{O}(k)/\epsilon$.

Proof. If $\text{IsLiteral}$ returns reject for some to-be-checked relevant block, then the total number of relevant blocks increases by 1. If $\text{IsLiteral}$ returns accept for all the to-be-checked relevant blocks, then the algorithm iterates for $2(\log k + 6)/\epsilon$ number of times. The algorithm randomly partitions the to-be-checked relevant blocks into two parts, and uses $\text{FindLiteral}$ to identify the part that contains the literal, which requires $O(\log k)$ number of queries. Then, the algorithm queries $f(x)$ and $f(y)$ to identify whether the function value of $x$ and $y$ are distinct. If $f(x) \neq f(y)$, the algorithm further uses binary search over blocks to identify a new relevant block among all the flipped blocks.

The query complexity of algorithm $\text{IsLiteral}$ is $O(\log k \cdot \frac{1}{\gamma}) = O(\log k)$ since $\gamma = 1/8$. The query complexity of algorithm $\text{FindLiteral}$ is $O(\log k)$. Let $P =$ query complexity of algorithm $\text{IsLiteral}$, $Q =$ query complexity of algorithm $\text{FindLiteral}$, and $R =$ query complexity of binary search. The total query complexity is upper bounded by

$$c \cdot k \cdot (P + \log k/\epsilon \cdot (Q + R)) \leq O(k \cdot \log^2(k))/\epsilon = \tilde{O}(k)/\epsilon.$$  

6 Conclusions

We propose an adaptive algorithm for junta testing under distribution-free setting with one-side error, which is surprisingly simple to analyze. The query complexity of our algorithm is $\tilde{O}(k)$. Compared with the $\Omega(k \log k)$ lower bound by [Sağ18] for junta testing under the uniform distribution, our algorithm achieves nearly optimal query complexity. Junta testing was commonly solved based on Fourier analysis. In the distribution-free setting, we have no idea about how to use similar tools. This forces us to find some approaches that do not rely on Fourier analysis. It turns out that simple random algorithms and analysis suffice to achieve optimal query complexity. A natural question is whether we could use some approaches besides the commonly used Fourier analysis to make progress on some other important and open problems?

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Appendix

8.1 IsLiteral Algorithm

Algorithm 5 IsLiteral (x, B)

1: Define \( g(u) = f(u_B x_B), \gamma = 1/8 \)
2: for each of \( \log(k) + 6 \) repetitions do
3:  if UniformJunta\((g, 1, \gamma)\) rejects then
4:  return reject, and two blocks \( L \) and \( N \)
5: end if
6: Randomly partition \( B \) into \( L \) and \( N \)
7: if \( g(x^{(L)}) = g(x^{(N)}) \neq g(x) \) then
8:  return block \( L \), block \( N \), and string \( x \)
9: end if
10: \( y \leftarrow x^{(B)} \)
11: if \( g(y^{(L)}) = g(y^{(N)}) \neq g(y) \) then
12:  return block \( L \), block \( N \), and string \( y \)
13: end if
14: end for
15: return accept

8.2 Lemma 5

Lemma 5. If \( g \) is \( \gamma \)-far from any literal under the uniform distribution, then with probability at least \( 1 - 1/(28k) \), the algorithm IsLiteral rejects, where \( \gamma = 1/8 \).

Proof. If \( g \) is \( \gamma \)-far from literal under the uniform distribution, we could analyze according to the following two cases.

Case 1: \( g \) is \( \gamma \)-far from every constant function under the uniform distribution. Since \( g \) is \( \gamma \)-far from literal and constant functions, we know that \( g \) is \( \gamma \)-far from 1-junta under the uniform distribution. From the Lemma 4, the algorithm UniformJunta rejects with probability at least 2/3. Then, we have that the algorithm IsLiteral rejects in a fixed iteration with probability at least 2/3.
Case 2: $g$ is $\gamma$-close to a constant function $h$ under the uniform distribution. Without loss of generality, assume that $g(y) = h \neq g(x)$. $x^{(L)}$ could be regarded as a string uniformly sampled from distribution $U$ over $\{0,1\}^B$. Let $x^{(L)} = a$, then $x^{(N)} = \bar{a}$. The probability that the algorithm does not return reject is at most

$$\Pr_{a \in U} P(g(a) \neq h \cup g(\bar{a}) \neq h) \leq \Pr_{a \in U} P(g(a) \neq h) + \Pr_{a \in U} (g(\bar{a}) \neq h) \leq 2\gamma. \quad (16)$$

Let $\mathcal{E}_i$ be the event that $g$ is rejected by the algorithm IsLiteral at iteration $i$. The probability that the algorithm rejects during $\log(k)$ iterations is $\Pr(\mathcal{E}_i) \geq 2/3$. By a union bound over $k$ such blocks, the probability that at least one iteration of IsLiteral rejects is

$$\Pr(\bigcup_{i \in [\log(k)]} \mathcal{E}_i) = 1 - \Pr(\bigcap_{i \in [k]} \bar{\mathcal{E}_i}) \geq 1 - (1/3)^{\log(k)+6} = 1 - 1/(28k). \quad (17)$$

8.3 Inequalities

**Lemma 10 ([MR10]).** Let $X$ be the sum of $c$ i.i.d. random variables sampled from a distribution on $[0,1]$ with a mean $\mu$. For any $\delta > 0$,

$$\Pr(X - c\mu \leq -\delta \cdot c\mu) \leq \exp(-\delta^2 c\mu/2). \quad (18)$$

**Lemma 11 ([AGS10]).** Let $a > 0$. For any $t \geq (2/a)|\log(1/a) - b|^+ + at + b > \log(t)$. 

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