Maximally extended, explicit and regular coverings of the Schwarzschild–de Sitter vacua in arbitrary dimension

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Abstract
Maximally extended, explicit and regular coverings of the Schwarzschild–de Sitter family of vacua are given, first in spacetime (generalizing a result due to Israel) and then for all dimensions $D$ (assuming a $D-2$ sphere). It is shown that these coordinates offer important advantages over the well-known Kruskal–Szekeres procedure.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A maximally extended and regular covering of the Schwarzschild vacuum is now a fundamental part of any introduction to general relativity. Almost always the covering is given by way of the implicit Kruskal–Szekeres procedure [1–3]. However, as emphasized long ago by Ehlers [4], an explicit maximally extended and regular covering of the Schwarzschild vacuum is known and was first given by Israel [5]. Unfortunately, despite the fact that these coordinates offer many advantages over the Kruskal–Szekeres coordinates, they are almost never used. In this paper, I extend Israel’s procedure to the Schwarzschild–de Sitter class of vacua in four dimensions and then to arbitrary dimensions $D$ assuming a $D-2$ sphere as in the Tangherlini generalization of the Schwarzschild vacuum. Moreover, it is made clear that these coordinates offer important advantages over the Kruskal–Szekeres procedure. These advantages include: an explicit representation of the line element that can be extended to arbitrary dimension, a

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1 Although usually obtained by way of coordinate transformations, the Kruskal–Szekeres coordinates can be obtained in an algorithmic way without guessing transformations. See, for example, [3].
simultaneous covering of both the black-hole and cosmological horizons and derivation by direct integration of Einstein’s equations\(^2\) without recourse to coordinate transformations.

2. Coordinate construction

We start with a hyper-spherically symmetric spacetime where \(k^\alpha = \delta^\alpha_w\) is a null vector so that the line element takes the form \(^3\)

\[
ds^2 = f(u, w) \, du^2 + 2h(u, w) \, du \, dw + r(u, w)^2 \, d\Omega^2_{D-2},
\]

where \(d\Omega^2_{D-2}\) is the metric of a unit \(D-2\) sphere. Further, setting \(k^\beta \nabla_\beta k^\alpha = 0\) (so that trajectories of constant \(u\) and angular coordinates are radial null geodesics affinely parameterized by \(w\)) it follows that \(\partial h/\partial w = 0\). Defining \(U \equiv \int h(u) \, du\) and rewriting \(U\) as \(u\) we set \(h = 1\) in (1) which remains a completely general spacetime subject to the stated symmetries. The expansion of \(k^\alpha\) reduces to

\[
\nabla_\alpha k^\alpha = \frac{D - 2}{r} \frac{\partial r}{\partial w}.
\]

(2)

Associated with the vector field \(l^\alpha = \delta^\alpha_u\) we have \(l^\alpha l_\alpha = f\) and the non-zero components of the acceleration are

\[
l^\beta \nabla_\beta l^u = -\frac{1}{2} \frac{\partial f}{\partial w}, \quad l^\beta \nabla_\beta l^w = \frac{1}{2} \left( \frac{\partial f}{\partial u} + f \frac{\partial f}{\partial w} \right).
\]

(3)

The associated expansion is

\[
\nabla_\alpha l^\alpha = \frac{D - 2}{r} \frac{\partial r}{\partial u}.
\]

(4)

The trajectories for which all coordinates are constant except \(u\) are radial null geodesics (affinely parameterized by \(u\)) only for \(f = 0\). Finally, if \(\partial^2 f/\partial u^2 = 0\), then \(m^\alpha\) with non-zero components \(m^u = -2/f\), and \(m^w = 1\) is tangent to a radial null geodesic with \(w\) again affine and the associated geodesics immediately follow as

\[
\begin{align*}
\frac{du}{d\delta} &= -\frac{2}{f} \frac{dw}{d\delta} + \frac{1/\delta}{f}, \quad \text{where } \delta \text{ is a constant.}
\end{align*}
\]

Important examples of this special case are given below.

3. Vacua with \(\Lambda\) \((D = 4)\)

For the spacetime (1) with \(h = 1\),

\[
f = \frac{w}{3urC^2} (-2C(C - r)^2 + uw(2C + r)),
\]

(5)

and

\[
r = \frac{uw(3M - C)}{C^2} + C,
\]

(6)

where \(C (\neq 0)\) and \(M (\geq 0)\) are constants, it follows that

\[
R^\beta_u = \Lambda \delta^\beta_u.
\]

(7)

\(^2\) Israel [5] (in the published form of his work) introduced the coordinates by way of a transformation which, in his notation reads as \(t = r - 2m \ln(u/(8m^2C))\). In this paper Einstein’s equations for vacua have been solved directly without prior coordinates. This is not possible for Kruskal–Szekeres coordinates wherein even at \(D = 4\) with \(\Lambda = 0\) the vacuum field equations are difficult to solve. See, for example, [3].

\(^3\) We use Einstein’s equations (with cosmological constant), geometrical units and a signature of \(D - 2\). The sign conventions for \(w\) and \(u\) are that \(w\) can decrease and \(u\) increase only in spacelike directions. For convenience, explicit functional dependence is usually shown only on the first appearance of a function.
and
\[ R = 4\Lambda, \quad (8) \]

where
\[ R_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = \frac{48 M^2}{r^6}, \quad (9) \]

\[ \Lambda = \frac{3(C - 2M)}{C^3}, \quad (10) \]

\( R_{\alpha\beta} \) is the Ricci tensor, \( R \) is the Ricci scalar and \( C_{\alpha\beta\gamma\delta} \) is the Weyl tensor. Viewing \( \Lambda_1 \) as a constant of nature\(^4\) and \( M \) a property of the vacuum, it is clear from (10) that the specification of \( C \) does not determine the physical situation uniquely. Since, by (6), the axes are centred on \( r = C \), it is important to distinguish ranges in \( C \). Writing \( C = \alpha M \) with \( \alpha \neq 0 \) and \( M \neq 0 \) it follows immediately from (10) that
\[ 9\Lambda M^2 = \frac{27(\alpha - 2)}{\alpha^3}. \quad (11) \]

It follows from (11) that all values of \( \Lambda M^2 \) are determined uniquely by \( \alpha \) using the following ranges: \(-\infty < 9\Lambda M^2 < 1 \) for \( 0 < \alpha < 3 \) and \( 9\Lambda M^2 > 1 \) for \(-6 < \alpha < 0 \). The latter range is of no interest and is not discussed here.

The one independent invariant derivable from the Riemann tensor without differentiation can be taken to be (9) and so with \( u \) and \( w \) extending over the reals, the vacua are maximally extended and regular for \( 0 < r < \infty \) (and for all finite \( r \) if \( M = 0 \)). In particular, note that
\[ g_{uu} \big|_{u=C} = \frac{w^2}{C^2}. \quad (12) \]

With (6) it follows that
\[ \nabla_\alpha k^\alpha = \frac{2u(3M - C)}{r C^2} = \frac{u}{w} \nabla_\alpha \theta^\alpha. \quad (13) \]

From (5) and (6) it follows that trajectories of constant \( w, \theta \) and \( \phi \) are radial null geodesics only for \( w = 0 \). More generally, the radial null geodesics that satisfy
\[ \frac{dw}{du} = -\frac{1}{2} f, \quad (14) \]

with \( f \) given by (5) and (6), can be written down in terms of elementary functions (as explained below). Note that for these trajectories \( dw/dw \to 0 \) as \( r \to 0 \) (\( C \neq 3M \)) and \( dw/du \to 0 \) as \( w \to 0 \). We distinguish the cases: \( M = 0 \) (de Sitter), \( C = 3M \) (Bertotti–Kasner), \( C = 2M \) (Schwarzschild), \( 2M < C < 3M \) (Schwarzschild–de Sitter), \( 0 < C < 2M \) (Schwarzschild–anti-de Sitter) and discuss them below. For the case \( C = 2M \) the coordinates were first obtained by Israel [5]. Not covered as special cases are anti-de Sitter space (see appendix A) and the degenerate Schwarzschild–de Sitter spacetime (see appendix B).

3.1. \( M = 0 \) (de Sitter space)

With \( M = 0 \), \( \Lambda = 3/C^2 \), \( r = \sqrt{\frac{\Lambda}{3} (\frac{3}{\Lambda} - uw)} \) and we have de Sitter space. The metric simplifies to
\[ ds^2 = \frac{\Lambda}{3} w^2 du^2 + 2 du dw + \frac{\Lambda}{3} \left( \frac{3}{\Lambda} - uw \right)^2 d\Omega^2. \quad (15) \]

\(^4\) Vacuum solutions with \( \Lambda \neq 0 \) were first discussed by [6].
Trajectories with four tangents \( m^\alpha = \left(-\frac{6}{\Lambda u^2}, 1, 0, 0\right) \) are radial null geodesics (so \( w \) is affine for both radial null directions). We can write these geodesics in the form
\[
w = \frac{6}{\Lambda u} + \delta, \tag{16}\]
where \( \delta \) is a constant. The negative cosmological horizons \( \( r = -\sqrt{3}/\Lambda \) \) are then given by \( \delta = 0 \), where the expansion \( \nabla_\alpha m^\alpha = \frac{(6 - uw/\Lambda)}{w^3(-uw/\Lambda)} \) vanishes. Some details are shown in figure 1.

3.2. \( C = 3M \) (Bertotti–Kasner space)

With \( C = 3M = 1/\sqrt{\Lambda} = r \), we have Bertotti–Kasner space. The metric simplifies to
\[
ds^2 = \Lambda w^2 du^2 + 2 du dw + \frac{1}{\Lambda} d\Omega^2. \tag{17}\]

Trajectories with four tangents \( m^\alpha = \left(-\frac{2}{\Lambda u^2}, 1, 0, 0\right) \) are radial null geodesics (so again \( w \) is affine for both radial null directions) but now \( \nabla_\alpha k^\alpha = \nabla_\alpha m^\alpha = 0 \). We can write these geodesics in the form
\[
w = \frac{2}{\Lambda u} + \delta, \tag{18}\]
where \( \delta \) is a constant. The \( u - w \) plane is like that of de Sitter space now with \( uw = 2/\Lambda \) (again \( \delta = 0 \)) distinguishing the branches of the \( m \) geodesics. This space shows that the Birkhoff theorem does not extend directly to \( \Lambda > 0 \) (see the case \( 2M < C < 3M \) below).

\[5\] For a detailed discussion see [8].
3.3. $C = 2M$ (Schwarzschild vacuum)

With $C = 2M$, $\Lambda = 0$, $r = \frac{8M^2 + uw}{4M}$ and we have the Schwarzschild vacuum in Israel coordinates [5]. The metric simplifies to

$$ds^2 = \frac{2w^2}{uw + 8M^2} du^2 + 2 dw d\Omega + \left(\frac{8M^2 + uw}{4M}\right)^2 d\Omega^2.$$  \hspace{1cm} (19)

Trajectories with four tangents $m^\alpha = (-\frac{uw + 8M^2}{w}, w, 0, 0)$ are radial null geodesics so $w$ is not affine. Now $\nabla_\alpha m^\alpha = -\frac{16M^2}{uw + 8M^2} = -\frac{4M}{w}$. We write these geodesics in the form

$$uw = -8M^2 \ln(\delta w),$$  \hspace{1cm} (20)

where $\delta$ is a constant. Some details are shown in figure 2.

3.4. $2M < C < 3M$ (Schwarzschild–de Sitter)

Again writing $C = \alpha M$ it follows that for $2 < \alpha < 3$ there is another constant $E > C$ for which $\Lambda = \frac{(E - 2M)}{E^3}$. The constant is given by

$$E = \beta M$$  \hspace{1cm} (21)

with

$$\beta = \frac{(2 - \alpha + \sqrt{\alpha + 6})(\alpha - 2)}{2(\alpha - 2)}.$$  \hspace{1cm} (22)

The trajectories $r = E$ are radial null geodesics (with non-zero expansion) tangent to the ‘cosmological’ horizons. Note that $\beta \to 3$ as $\alpha \to 3$ and $\beta \to \infty$ as $\alpha \to 2$. Now define

$$A \equiv \alpha^4 M^2 + uw(\alpha - 2)(3 - \alpha)$$  \hspace{1cm} (23)
and
\[ B \equiv \alpha^3 M^2 + (3 - \alpha)uw. \]  
(24)

Trajectories with four tangents \( m^u = (m^u, m^w, 0, 0) \), where
\[ m^u = -2 \frac{\alpha^3 M^2 B}{Aw^2} m^w \]  
(25)

and
\[ m^w = \delta A^2 \frac{w}{uw} e^{\frac{\alpha^2 uw^2}{4}}, \]  
(26)

where \( \delta \) is a constant, are radial null geodesics. Now we find that the expansion is given by
\[ \nabla_\alpha m^u = \frac{2\delta(\alpha - 3)FA^2 e^{\frac{\alpha^2 uw^2}{4}}}{w^{\frac{\alpha^2 uw^2 - 1}{4}}}, \]  
(27)

where
\[ F = \alpha^3 M^2 (2\alpha^3 M^2 - 3uw(\alpha - 2)) + u\omega(A - \alpha^2 M^2). \]  
(28)

The choice \( \alpha = 3, \delta = 1 \) reproduces the Bertotti–Kasner result and the choice \( \alpha = 2, \delta = \frac{16M^2}{r} \) reproduces the Schwarzschild vacuum result both for \( m^u \) as given above. For \( 2 < \alpha < 3 \), these geodesics can be given explicitly in terms of elementary functions and are reproduced in appendix C. Some details are shown in figure 3.

3.5. \( 0 < C < 2M \) (Schwarzschild–anti-de Sitter)

For \( 0 < \alpha < 2 \) there are no cosmological horizons but equations (23) through (28) hold as given above. Again the associated radial null geodesics can be given explicitly in terms of elementary functions and are reproduced in appendix D. Some details are shown in figure 4.
4. Hyper-spherical vacua with $\Lambda (D \geq 4)$

This section generalizes section 3 above. For dimensions $D \geq 4$ consider the spaces described by

$$ds^2 = \frac{w^2 \psi_D(u, w)}{(r - C)^2(D - 1)r^{D-3}C^3} du^2 + 2 du \, dw + r^2 d\Omega_{D-2}^2,$$

where the function $\psi_D$ is given by

$$3(D - 3)(C - 2M)r^{D-3}(r - C)^2 + 2C[3(D - 3)M - (D - 4)C]$$

$$\times [r^{D-2}(D - 3) - r^{D-3}C(D - 2) + C^{D-2}]$$

and $r$ signifies the function

$$r \equiv \frac{(D - 3)uw(3M - C)}{C^2} + C,$$

with $C (\neq 0)$ and $M (\geq 0)$ constants. It follows that

$$R^\mu_\alpha = \Lambda \delta^\mu_\alpha,$$

$$R = D\Lambda,$$

and

$$C_{\alpha \beta \gamma \delta} = \frac{4(D - 3)(D - 2)^2C^{2(D-4)}[(D - 4)C - 3(D - 3)M]^2}{(D - 1)r^{2(D-1)}},$$

where

$$\Lambda = \frac{3(D - 3)(C - 2M)}{C^3}.$$
Again, the one independent invariant derivable from the Riemmann tensor without differentiation can be taken to be (34) and so with \( u \) and \( w \) extending over the reals, the spaces described by (29) are maximally extended and regular for \( 0 < r < \infty \) (and for all \( r \) if \( C_{\alpha \beta \gamma \delta} C^{\alpha \beta \gamma \delta} = 0 \), see below). In particular, note that

\[
g_{uw} = \frac{w^2(D-3)((3M-C)D+5C-12M)}{C^3}. \tag{36}
\]

Again viewing \( \Lambda \) as a constant of nature and \( M \) a property of the vacuum, the specification of \( C \) does not determine the physical situation uniquely. This is discussed below. In view of the generality of the cases considered, the formulae given are remarkably simple.

4.1. Comparison with curvature coordinates

Consider the hyper-spherically symmetric spacetime

\[
ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega_{D-2}^2. \tag{37}
\]

Note that \( r \) is now a coordinate. It is not difficult to show that the unique form of \( f \) which satisfies (32) is given by

\[
f = 1 - \frac{2\tilde{m}}{r^{D-3}} - \frac{\Lambda r^2}{D-1}, \tag{38}
\]

where \( \tilde{m} \) is a constant (which we take \( \geq 0 \)). This solution was apparently first discussed by Tangherlini [10]. The coordinates are, of course, defective at \( f = 0 \), but the associated roots depend on \( D \) and are not so easy to find from (38). The previous construction circumvents this unnecessary algebra as follows. For (37) with (38) we find

\[
C_{\alpha \beta \gamma \delta} C^{\alpha \beta \gamma \delta} = \frac{4(D-2)^2((D-2)^2 - 1)\tilde{m}^2}{r^{2D-12}}. \tag{39}
\]

and so from (34) and (39) (assuming, without loss in generality, the same coordinates for \( d\Omega_{D-2}^2 \)) we have the relation

\[
\tilde{m}^2 = \frac{(D-3)C^{2(D-4)}((D-4)C - 3(D-3)M)^2}{(D-1)((D-2)^2 - 1)}, \tag{40}
\]

which gives \( \tilde{m}^2 = M^2 \) only for \( D = 4 \). Now substituting for \( \tilde{m} \) from (40) into (38) and writing an associated root to \( f = 0 \) as \( r = C \) we obtain (35). In this way we never need to deal with (38) and so all dimensions \( D \geq 4 \) are equally difficult to deal with as regards the location of horizons.

4.2. Geometrical mass

The previous section raises the issue of ‘mass’. In spacetime with \( \Lambda = 0 \) the concept of mass in the spherically symmetric case is well established [11]. The geometrical mass in \( D \) dimensions for spaces admitting a \( D-2 \) sphere has been defined previously by way of the sectional curvature [12]. Whereas this is merely a formal definition, it is of interest to compare this definition with \( M \) and \( \tilde{m} \). In dimension \( D \) define

\[
\mathcal{M} = \frac{1}{2} R_{\phi \phi}^{\phi \phi} \left( \frac{D-1}{2} \right). \tag{41}
\]

In the special case \( \Lambda = 0 \) (\( C = 2M \)) it follows that

\[
\mathcal{M} = \tilde{m} = 2^{D-4} M^{D-3}. \tag{42}
\]
More generally, for $\Lambda \neq 0 (C \neq 2M)$, it follows that
\[
\mathcal{M} = \tilde{m} + \frac{\Lambda r^{D-1}}{2(D - 1)},
\]  
(43)
where $\Lambda$ is given by (35). We can, of course, replace $\tilde{m}$ by $M$ in (43) by solving for $M$ from (40). We now consider some specific cases.

4.3. de Sitter space ($D \geq 4$)

We define hyper-spherical de Sitter space by the requirement $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 0$, so that from (34) and (39) we have
\[
M = \frac{C(D - 4)}{3(D - 3)}, \quad \tilde{m} = 0
\]  
(44)
and so
\[
\Lambda = \frac{D - 1}{C^2}.
\]  
(45)
The metric follows as
\[
d\mathbf{s}^2 = \frac{\Lambda}{D - 1} w^2 d\mathbf{u}^2 + 2d\mathbf{u} d\mathbf{w} + \frac{\Lambda}{D - 1} \left( \frac{D - 1}{\Lambda} - uw \right)^2 d\Omega^2_{D-2}.
\]  
(46)
Trajectories with four tangents $m^\alpha = (\frac{-2(D-1)}{\Lambda w^2}, 1, 0, 0, \ldots)$ are radial null geodesics (so $w$ is affine for both radial null directions). We can write these geodesics in the form
\[
w = \frac{2(D - 1)}{\Lambda u + \delta},
\]  
(47)
where $\delta$ is a constant. The negative cosmological horizons ($r = -\sqrt{\frac{(D-1)}{\Lambda}}$) are then given by $\delta = 0$, where the expansion $\nabla_\alpha m^\alpha = (\frac{(D-2)(D-1)-uw\Lambda}{\psi(D-1)-uw\Lambda})$ vanishes. The associated diagram is qualitatively the same as figure 1.

4.4. $C = 3M$ (Bertotti–Kasner space) ($D \geq 4$)

With $C = 3M = \sqrt{\frac{D-3}{\Lambda}} = r$, we have hyper-spherical Bertotti–Kasner space. The metric simplifies to
\[
d\mathbf{s}^2 = \Lambda w^2 d\mathbf{u}^2 + 2d\mathbf{u} d\mathbf{w} + \frac{D - 3}{\Lambda} d\Omega^2_{D-2}.
\]  
(48)
Trajectories with four tangents $m^\alpha = (\frac{-2}{\Lambda w^2}, 1, 0, 0, \ldots)$ are radial null geodesics (so again $w$ is affine for both radial null directions) and again $\nabla_\alpha k^\alpha = \nabla_\alpha m^\alpha = 0$. We can also write these geodesics in the form (18).

4.5. Tangherlini black holes ($D \geq 4$)

Asymptotically flat static vacuum black holes (admitting the $D - 2$ sphere) are unique [13] and given by the Tangherlini generalization of the Schwarzschild vacuum [10]. In our notation these correspond to the case $C = 2M$. Global, regular and explicit coordinates for these spaces have been given previously [14]. Some further discussion is given in appendix E. More generally, the radial null geodesics that satisfy
\[
\frac{dw}{du} = -\frac{1}{2} \frac{w^2 \psi_D(u, w)}{(r - C)^2(D - 1)r^{D-1}C^3},
\]  
(49)
with \( \psi_D \) given by (30) and \( r \) by (31), are of interest. For these trajectories, excluding the cases discussed above, it is clear that \( \frac{du}{dw} \to 0 \) as \( r \to 0 \) and \( \frac{dw}{du} \to 0 \) as \( w \to 0 \) \((u \neq 0)\). From (35), writing \( C = \alpha M \), we have \( \Lambda M^2 = 3(D - 3)(\alpha - 2)/\alpha^3 \). In the range \( 2 < \alpha < 3 \) there is another constant \( \beta > \alpha \) for which \( \Lambda M^2 = 3(D - 3)(\beta - 2)/\beta^3 \). The trajectories \( r = \beta M \) are radial null geodesics and are tangent to the ‘cosmological’ horizons. It is also clear that for \( C < 2M \) the solutions to (49) evolve in a fundamentally different way as in the case \( D = 4 \). In fact, the qualitative behaviour of all solutions to (49) can be obtained without explicit integration. The essential conclusion is that the figures given for \( D = 4 \) hold, qualitatively, for \( D > 4 \). This is discussed in detail elsewhere [15].

5. Summary

A maximally extended, explicit and regular covering of the Schwarzschild–de Sitter vacua in arbitrary dimension \((D \geq 4)\) has been given. It has been stressed that these coordinates offer important advantages over the Kruskal–Szekerès procedure that include: an explicit representation of the line element that can be extended to arbitrary dimension, a simultaneous covering of both the black-hole and cosmological horizons and derivation by direct integration of Einstein’s equations without recourse to coordinate transformations. In view of the generality of the problem solved, the resultant formulae obtained are of a remarkably simple form.

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Appendix A. Anti-de Sitter space \((D = 4)\)

Anti-de Sitter space is not contained in (5) with (6) but can be given in the form (1) with \( h = 1 \) and is included here for completeness. It is given by

\[
\frac{ds^2}{\text{d}t^2} = \left( \frac{w^2\Lambda}{3} - 1 \right) \text{d}u^2 + 2 \text{d}u \text{d}w + w^2 \text{d}\Omega^2,
\]

\((\text{A.1})\)

where the constant \( \Lambda < 0 \). Equations (7) and (8), of course, hold and the space is conformally flat. Trajectories with four tangents \( m^\alpha = \left(\frac{6}{\sqrt{3w^2 - \Lambda}}, -1, 0, 0\right) \) are radial null geodesics so \( w \) is again affine. We can write these geodesics in the form

\[
w = \sqrt{\frac{3}{-\Lambda}} \tan \left( \frac{1}{2} \sqrt{\frac{-\Lambda}{3}}(u + \delta) \right),
\]

\((\text{A.2})\)

where \( \delta \) is a constant. With \( u \) and \( w \) extending over the reals, (A.1) is maximally extended and regular.

7 This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the world-wide-web from the address http://grtensor.org GRTensorIII software is in development.
Appendix B. Degenerate Schwarzschild–de Sitter space ($D = 4$)

The degenerate Schwarzschild–de Sitter black hole has $3M = \frac{1}{\sqrt{\Lambda}}$ like the Bertotti–Kasner space but it is not contained in (5) with (6) but can be given in the form (1) with $h = 1$ and is included for completeness. Coordinates for degenerate black holes are seldom discussed [7]. Now

$$f' = \frac{-w(12ur^2(u^2 + 1) + w(w + 3u^2 + 3)(w + u^2 + 1))}{3r^2(u^2 + 1)^2}$$  \hspace{1cm} (B.1)$$

with

$$r = \frac{w + u^2 + 1}{\sqrt{\Lambda}(u^2 + 1)}.$$  \hspace{1cm} (B.2)$$

In addition to the radial null geodesics given by constant $u$ and $w = 0$, the other radial null geodesics can be given by

$$w = \frac{3(u^2 + 1)}{2W(k, \pm x) - 1},$$  \hspace{1cm} (B.3)$$

where $W$ is the Lambert W function$^8$ with $k = (0(+), -1(-))$ and

$$x = \frac{1}{4} \Lambda u(u^2 + 3) + \delta,$$  \hspace{1cm} (B.4)$$

where $\delta$ is a constant. Some details are shown in figure B1.$^8$

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$^8$ This is defined by $W(x)e^{W(x)} = x$. See, for example, [9].
Appendix C. Schwarzschild–de Sitter geodesics ($D = 4$)

For $2 < \alpha < 3$, the radial null geodesics with tangents $m^\alpha$ can be given in a simple form: $GH$ is a constant of motion, where

$$G = \left( \frac{M^2\alpha^2(2\alpha + \beta) + uw(3 - \alpha)}{uw(\alpha - 3) + M^2\alpha^2(-\alpha + \beta)} \right)^{(6-\alpha)\alpha} \quad \text{(C.1)}$$

and

$$H = (M^2\alpha^3(2\alpha^3 - 3uw(-2 + \alpha)) + u^2w^2(-2 + \alpha)(\alpha - 3) + w^2)^{(2-\alpha)(\alpha + 2\beta)}.$$ \quad \text{(C.2)}

Appendix D. Schwarzschild–anti-de Sitter geodesics ($D = 4$)

Writing $C = M/\gamma$ with $\gamma > 1/2$, the radial null geodesics with tangents $m^\alpha$ can be given in the form

$$2 \arctan \left( \frac{(2\gamma - 1)(2\gamma^2(3\gamma - 1)w + 3M^2)}{\sqrt{(6\gamma^2 + 1)(2\gamma - 1)M^2}} \right) \frac{(6\gamma - 1)}{w}$$

$$= \ln \left( \frac{M^2(2M^2 + 3\gamma^2uw(2\gamma - 1)) + u^2\gamma^4w^2(3\gamma - 1)(2\gamma - 1)}{w} \right)$$

$$\times \sqrt{(6\gamma + 1)(2\gamma - 1)} + \delta,$$ \quad \text{(D.1)}

where $\delta$ is a constant.

Appendix E. Schwarzschild–Tangherlini black holes ($D \geq 4$)

In the present notation we set $C = 2M$ and so have (1) with $h = 1$ and

$$f = \frac{w^2[(D - 3)r + 2M(2 - D) + (2M)^{D-2}r^{D-3}]}{2M(r - 2M)^2},$$ \quad \text{(E.1)}$$

where

$$r = \frac{(D - 3)uw}{4M} + 2M. \quad \text{(E.2)}$$

With $D = 4$ we recover (19). Even implicit forms of $r$ in the Kruskal–Szekeres procedure are not known for all $D$ (e.g., $D = 8, 10$ [14]). However here we need only substitute for $D$ and insert (E.2) into (E.1) to give a maximally extended, explicit and regular coverings of the space. For example, with $D = 5$ we have

$$f = \frac{2w^2(6M^2 + uw)}{(4M^2 + uw)^2}. \quad \text{(E.3)}$$

Solutions to (14) with (E.1) and (E.2) can be given explicitly in terms of elementary functions in some cases. For example, with $D = 5$ we have

$$uw = -2M^2(V(\delta w) + 4),$$ \quad \text{(E.4)}$$

where $\delta$ is a constant.
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