Axial Quasi-normal modes of Neutron Stars with Exotic Matter

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Abstract We investigate the axial w quasi-normal modes of neutron stars for 18 realistic equations of state in order to study the influence of hyperons and quarks on the modes. The study has been developed with a new method based on Exterior Complex Scaling with variable angle, which allow us to generate pure outgoing quasi-normal modes. A complete study of the junction conditions has been done. We have obtained that w-modes can be used to distinguish between neutron stars with exotic matter and without exotic matter for compact enough stars.

1 Quasi-normal modes Formalism and Numerical Method

We will consider a spherical and static star. The matter inside of it is considered to be a perfect fluid. Following the original papers (see reviews [1, 2]), we make perturbations over the spherical static metric and the stress-energy tensor, taking into account only the axial perturbations. After some algebra, it can be seen that the perturbations satisfy the well-known Regge-Wheeler equation [3], both inside and outside the star. The eigen-frequency of the axial mode is a complex number $\omega = \omega_\Re + i\omega_\Im$. Inside the star an equation of state must be provided, so in general also the static configuration must be solved numerically. Outside the star the metric is known (Schwarzschild) and only the perturbation must be integrated.

We are only interested in purely outgoing waves. In general a solution of the Regge-Wheeler equation will be a composition of incoming and outgoing oscillating
waves. Because the outgoing wave diverges towards infinity, the purely outgoing quasi-normal mode condition could only be imposed as a behavior far enough from the star, but every small numerical error in the imposition of this behavior will be amplified as we approach the border of the star, resulting in a mixture of outgoing with ingoing waves. Note also that in general the exterior solution will oscillate infinitely towards infinity. We have developed the following method, based on the Colsys package [4], to deal with these difficulties. We make use of previously well known techniques and new ones.

**Exterior solution:** We study the phase function (logarithmic derivative of the Regge-Wheeler function), which does not oscillate. Hence, the differential equation outside the star is reduced to a Riccati equation and we can compactify the radial variable. The boundary condition must grant the outgoing wave behavior. In order to impose a constringent enough condition, we make use of Exterior Complex Scaling method [5] with variable angle, where the integration coordinate is considered to be a complex variable. The principal advantage with respect other methods is that in principle no assumption on the imaginary part (i.e. damping time) of the quasi-normal mode is done.

**Interior solution:** The interior part of the solution is integrated numerically. As we want to obtain realistic configurations, we implement the equations of state in two different ways: 1) A piece-wise polytrope approximation, done by Read et al [6], in which the equation of state is approximated by a polytrope in different density-pressure intervals. 2) A piece-wise monotone cubic Hermite interpolation satisfying local thermodynamic conditions.

We generate two independent solutions inside of the star for the same static configuration. These two solutions must be combined to match the exterior solution with the appropriate junction conditions. We use Darmois conditions (continuity of the fundamental forms of the matching hypersurface). This formulation allow us to introduce surface layers of energy density on the border of the star, that allow us to approximate the exterior crust as a thin layer enveloping the core.

**Determinant method:** The junction conditions can be used to construct what we call the \textit{determinant method}: We construct a matrix in terms of the derivatives of the Regge-Wheeler function whose determinant must be zero only if the matching conditions are fulfilled, i.e., when \( \omega \) corresponds to a quasi-normal mode for the static configuration integrated. The matrix is calculated using both independent solutions in the interior of the star together with the exterior phase function.

This method has been successfully extended to study polar modes of realistic neutron stars. These results will be presented elsewhere.

2 Numerical Results

We have made several tests on our method successfully reproducing data from previous works for axial modes. As an example, we reproduce the results from [7] with a precision of \( 10^{-7} \). In this section we will present our results for new realis-
tic EOS. Using the parametrization presented by Read et al. [6], we can study the 34 equations of state they considered. We have used, following their notation, SLy, APR4, BGN1H1, GNH3, H1, H4, ALF2, ALF4. After the recent measurement of the $1.97 M_\odot$ for the pulsar PSR J164-2230 [8], several exotic matter EOS have been proposed satisfying this condition. We have considered the following ones using the cubic Hermite interpolation: two EOS presented by Weissenborn et al with hyperons in [9], we call them WCS1 and WCS2; three EOS presented by Weissenborn et al with quark matter in [10], we call them WSPHS1, WSPHS2, WSPHS3; four EOS presented by L. Bonanno and A. Sedrakian in [11]; we call them BS1, BS2, BS3, BS4; and one EOS presented by Bednarek et al in [12], we call it BHZBM.

Empirical relations between the frequency and damping time of quasi-normal modes and the compactness of the star can be useful in order to use future observations of gravitational waves to estimate the mass and the radius of the neutron star, as well as to discriminate between different families of equations of state. In top of Figure 1 we present the frequency of the fundamental mode scaled to the radius of each configuration. The softest equations of state that include hyperon matter, H1 and BGN1H1, present a quite different behavior than the rest of EOS considered.
Nevertheless, as the detection of the recent $2M_{\odot}$ pulsar suggest, these two particular EOS cannot be realized in nature.

Another exception is found in pure quark matter stars (WSPH51-2 EOS). Their behavior is clearly differentiated from the rest because of the different layer structure found at the exterior of the star.

In general, for hyperon matter EOS and hybrid stars, we obtain linear relations between the scaled frequency and the compactness. These relations could be used, applying the technique from [13], to measure the radius of the neutron star and constrain the equation of state.

We plot at the bottom left of Fig. 1 a new phenomenological relation between the real part and the imaginary part of the frequency of the w quasi-normal modes valid for all the EOS. We plot $\bar{\omega}_R = 2\pi \frac{1}{\sqrt{p_c (cm^{-2})}} \frac{10^6}{c} \omega (KHz)$ and $\bar{\omega}_I = \frac{1}{\sqrt{p_c (cm^{-2})}} \frac{10^6}{c} \frac{1}{\tau (\mu s)}$.

Although the empirical relation between $\bar{\omega}_R$ and $\bar{\omega}_I$ is quite independent of the EOS, the parametrization of the curve is EOS dependent. So a possible application of this empirical relation is the following. If the frequency $\omega (KHz)$ and the damping time $\tau (\mu s)$ are known, we can parametrize a line defining $\bar{\omega}_R$ and $\bar{\omega}_I$ with parameter $p_c$ using the observed frequency and damping time. The crossing point of this line with the empirical relation presented in the bottom left of Fig. 1 gives us an estimation of the central pressure $p_c$ independent of the EOS. Now, we can check which EOS is compatible with this $p_c$, i.e., which one have the measured wI0 mode near the crossing point for the estimated central pressure. Hence, this method could be used to constrain the equation of state. Note that if mass and radius are already measured, we would have another filter to impose to the EOS.

Also, the precision of our algorithm allows us to construct explicitly the universal low compactness limiting configuration for fundamental wII modes (bottom right of Fig. 1) around $M/R = 0.106$ for which the fundamental wII mode vanishes [14].

We also study the impact of the core-crust transition pressure on the quasi-normal mode spectrum. We obtain that variations of the transition pressure from $10^{32} dyn/cm^2$ to $10^{33} dyn/cm^2$ affect the frequency and damping time order 0.1%.

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