Generalized mutual informations of quantum critical chains

F. C. Alcaraz\textsuperscript{1} and M. A. Rajabpour\textsuperscript{2}
\textsuperscript{1} Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970, São Carlos, SP, Brazil
\textsuperscript{2} Instituto de Física, Universidade Federal Fluminense, Av. Gal. Milton Tavares de Souza s/n, Gragoatá, 24210-346, Niterói, RJ, Brazil

(Dated: January 14, 2015)

We study the Rényi mutual information $I_\alpha$ of the ground state of different critical quantum chains. The Rényi mutual information definition that we use is based on the well established concept of the Rényi divergence. We calculate this quantity numerically for several distinct quantum chains having either discrete Z($Q$) symmetries (Q-state Potts model with $Q = 2, 3, 4$ and Z($Q$) parafermionic models with $Q = 5, 6, 7, 8$ and also Ashkin-Teller model with different anisotropies) or the U($1$) continuous symmetries (Klein-Gordon field theory, XXZ and spin-1 Fateev-Zamolodchikov quantum chains with different anisotropies). For the spin chains these calculations were done by expressing the ground-state wavefunctions in two special basis. Our results indicate some general behavior for particular ranges of values of the parameter $n$ that defines $I_\alpha$. For a system, with total size $L$ and subsystem sizes $\ell$ and $L - \ell$, the $I_\alpha$ has a logarithmic leading behavior given by $\frac{\alpha}{4} \log (\frac{\pi}{4} \sin (\frac{\alpha}{2}))$ where the coefficient $c_\alpha$ is linearly dependent on the central charge $c$ of the underlying conformal field theory (CFT) describing the system’s critical properties.

PACS numbers: 11.25.Hf, 03.67.Bg, 89.70.Cf, 75.10.Pq

I. INTRODUCTION

The entanglement entropy, as a tool to detect and classify quantum phase transitions, has been playing an important role in the last fifteen years (see\textsuperscript{[1]} and references therein). In one dimension, where most of the critical quantum chains are conformal invariant, the entanglement entropy provides a powerful tool to detect, as well to calculate, the central charge $c$ of the underlying CFT. For example, for quantum chains, the ground-state entanglement entropy of a subsystem formed by contiguous $\ell$ sites of an infinite system, with respect to the complementary subsystem has the leading behavior $S = \frac{c}{3} \ln \ell$ if the system is critical or $S = \frac{c}{3} \log \xi$, when the system is noncritical with correlation length $\xi$\textsuperscript{[2]}. Although there are plenty of proposals to measure this quantity in the lab\textsuperscript{[3, 4]} the actual experiments were out of reach so far. Strictly speaking the central charge of quantum spin chains has never been measured experimentally. Recently other quantities, that are also dependent of the central charge has been proposed\textsuperscript{[5, 6, 7]}. Among these proposals interesting measures that, from the numerical point of view, are also efficient in detecting the phase transitions as well as the universality class of critical behavior, are the Shannon and Rényi mutual informations\textsuperscript{[8, 9]} (see also the related works\textsuperscript{[12, 13]}). The Rényi mutual information (the exact definition will be given in the next section) has a parameter $n$ that recovers the Shannon mutual information at the value $n = 1$. The results derived in\textsuperscript{[8, 11]} indicate that the Shannon and Rényi mutual informations of the ground state of quantum spin chains, when expressed in some special local basis, similarly as happens with the Shannon and Rényi entanglement entropy, show a logarithmic behaviour with the subsystem’s size whose coefficient depends on the central charge.

Recently additional new results concerning the Shannon and Rényi mutual information in quantum systems were obtained, see\textsuperscript{[16, 23]}. There are also studies of the mutual information in classical two dimensional spin systems\textsuperscript{[15, 24]}. Most of the results regarding the Shannon and the Rényi mutual information, except for the case of harmonic chains, are based on numerical analysis, especially for systems with central charge not equal to one. One of the main problems in a possible analytical derivation comes from the presence of a discontinuity at $n = 1$ of the Rényi mutual information. This discontinuity prevents the use of the replica trick, which is normally a necessary step for the analytical derivation of the Shannon mutual information.

In this paper we will consider, for many different quantum chains, another version of the Rényi mutual information, which is also parametrized by a parameter $n$ that reduces at $n = 1$ to the Shannon mutual information. The motivation for our calculations is two fold. Firstly this definition is more appropriate from the point of view of a measure of shared information among parts of a system, since it has the expected properties. This will be discussed in the Appendix.

Secondly, this quantity does not show any discontinuity at $n = 1$, so it might be a good starting point for the analytical calculation of the Shannon mutual information with some sort of analytical continuation of the parameter $n$.

Having the above motivations in mind we firstly calculated numerically (using exact diagonalization) the Rényi mutual information for several critical quantum spin chains. We considered models with $Z(Q)$ symmetries like the $Q$-state Potts modes for $Q = 2, 3$ and $4$, the $Z(4)$ Ashkin-Teller model and the $Z(Q)$ parafermionic models.
with $Q = 5 - 8$. We then calculated the Rényi mutual information for quantum critical harmonic chains (discrete version of Klein-Gordon field theory) and also for quantum spin chains with $U(1)$ symmetry like the XXZ and the spin-1 Fateev-Zamolodchikov quantum chains.

The structure of the paper is as follows: in the next section we will present the essential definitions of the Shannon and Rényi mutual informations. In section three we will present the numerical results of the Rényi mutual information for many different critical quantum spin chains. Finally in the last section we present our conclusions.

II. THE RÉNYI MUTUAL INFORMATIONS: DEFINITIONS

Consider the normalized ground state eigenfunction of a quantum spin chain Hamiltonian $|\psi_G\rangle = \sum_i a_i |I\rangle$, expressed in a particular local basis $|I\rangle = |i_1, i_2, \ldots\rangle$, where $i_1, i_2, \ldots$ are the eigenvalues of some local operators defined on the lattice sites. The Rényi entropy is defined as

$$S_{hn}(\mathcal{X}) = \frac{1}{1-n} \ln \sum_I p^n_I, \quad (1)$$

where $p_I = |a_I|^2$ is the probability of finding the system in the particular configuration given by $|I\rangle$. The limit $n \to 1$ gives us the Shannon entropy $S_h = -\sum_I p_I \ln p_I$. Since we are considering only local basis it is always possible to decompose the configurations as a combination of the configurations inside and outside of the subregions as $|I\rangle = |I_A \bar{A}\rangle$. One can define the marginal probabilities as $p_{IA} = \sum_{I_A} p_{IA} I_A$ and $p_I = \sum_{I_A} p_{IA} I_A$.

In a previous paper [11] we studied the naive definition of the Rényi mutual information:

$$I_n(A, \bar{A}) = S_{hn}(A) + S_{hn}(\bar{A}) - S_{hn}(A \cup \bar{A}). \quad (2)$$

From now on instead of using $p_{IA} I_A$ we will use just $p_I$. The known results of the Rényi mutual informations of quantum critical chains are obtained by using the definition (2). For special basis, usually the ones where part of the Hamiltonian is diagonal (see [11]), the definition (2) for the Rényi mutual information gives us a logarithmic behavior with the subsystem size, for arbitrary values of $n$. However, as observed numerically for several quantum chains (see [10, 11, 13]), it shows a discontinuity at $n = 1$, that forbids the use of large-$n$ analysis to obtain the most interesting case where $n = 1$, namely the standard Shannon mutual information. Although the definition (2) has its own uses it is not the one which normally has been considered in information sciences. For example $I_n$ for $n \neq 1$ is not necessarily a positive function, a property that we naturally expect to hold for the mutual informations. In this paper we consider a definition that is common in information sciences [29].

The Rényi mutual information with the desired properties, as a measure of shared information (see Appendix), is defined as [29]:

$$\hat{I}_n(A, \bar{A}) = \frac{1}{n-1} \ln \sum_I \frac{p^n_I}{p^{n-1}_{I_A} p^{n-1}_{I_{A'}}}, \quad (3)$$

where $p_{IA}$ and $p_{I_{A'}}$, as before, are the probabilities that the subsystems are independently in the configurations $|I_A\rangle$ and $|I_{A'}\rangle$ that forms the configuration $|I\rangle$ that occurs with probability $p_I$.

Hereafter $L$ will represent the size of the whole system and $\ell - \ell'$ the sizes of the subsystems. With this new notation one can write $\hat{I}_n(A, \bar{A})$ as $\hat{I}_n(\ell, L-\ell)$. This definition of the Rényi mutual information comes from the natural extension of the relative entropy to the Rényi case and measures the distance of the full distribution from the product of two independent distributions. In the limit $n \to 1$ one easily recovers the Shannon mutual information $I(\ell, L-\ell) = \text{Sh}(\ell) + \text{Sh}(L-\ell) - \text{Sh}(L)$, where $\text{Sh} = -\sum_I p_I \ln p_I$ is the standard Shannon entropy. One of the important properties of $\hat{I}_n$, that is not shared by $I_n$, is its nondecreasing behavior as a function of $n$ (see Appendix). Our calculations for a set of distinct quantum spin chains will be done numerically, since up to our knowledge an analytical method to consider these quantum chains is still missing.

III. THE RÉNYI MUTUAL INFORMATION IN QUANTUM CHAINS

In this section we will numerically calculate the ground-state Rényi mutual information of two series of critical quantum spin chains with slightly different structure. In the first part we will calculate the Rényi mutual information for systems with discrete symmetries such as the Q-state Potts models with $Q = 2, 3$ and $4$, the Ashkin-Teller model and the parafermionic $Z(Q)$-quantum spin chain [32] for the values of $Q = 4, 5, 6, 7$ and $8$. In the second part we will calculate the Rényi mutual information for systems with $U(1)$ symmetry such as the Klein-Gordon field theory, the XXZ model and the Fateev-Zamolodchikov model with different values of their anisotropy parameters.

A. The Rényi mutual information in quantum chains with discrete symmetries

In this subsection we will study the Rényi mutual information of the ground state of different critical spin chains with $Z(Q)$ discrete symmetries. The results we present were obtained by expressing the ground-state wavefunction in two specific basis where the systems show some universal properties.
1. The Rényi mutual information of the quantum $Q$-state Potts model and the quantum Ashkin-Teller model

Our results show that the $Q$-state Potts model and the Ashkin-Teller model share a similar behaviour. For this reason we discuss them together. The critical $Q$-state Potts model in a periodic lattice is defined by the Hamiltonian \[ H_Q = -\sum_{i=1}^{L} \sum_{k=1}^{Q-1} (S_i^k S_{i+1}^{Q-k} + R_i^k), \]

where $S_i$ and $R_i$ are $Q \times Q$ matrices satisfying the following $Z(Q)$ algebra: $[R_i, R_j] = [S_i, S_j] = [S_i, R_j] = 0$ for $i \neq j$ and $S_i R_j = e^{i\pi\ell} R_j S_i$ and $R_i^Q = S_i^Q = 1$. The model has its critical behavior governed by a CFT with central charge $c = 1 - 6 / (m(m+1))$ where $\sqrt{Q} = 2 \cos(\frac{\pi}{m+1})$. The $Q = 2$ Potts chain is just the standard Ising quantum chain. The Ashkin-Teller model has a $Z(2) \otimes Z(2)$ symmetry and a Hamiltonian given by:

\[ H = -\sum_{i=1}^{L} \left( S_i^3 S_{i+1}^3 + S_i^3 S_{i+1}^3 + \Delta S_i^3 S_{i+1}^3 + R_i + R_i^3 + \Delta R_i^3 \right), \]

where $S_i$ and $R_i$ are the same matrices introduced in the $Q = 4$ Potts model. The model is critical and conformal invariant for $-1 < \Delta \leq 1$ with the central charge $c = 1$.

It is worth mentioning that at $\Delta = 1$ we recover the $Q = 4$ Potts model and at $\Delta = 0$ the model is equivalent to two decoupled Ising models.

In the paper \[11\] we already showed that the Shannon and Rényi mutual informations, as defined in \[24\], are basis dependent. In other words one can get quite distinct different finite-size scaling behaviors by considering different basis. Surprisingly in some particular basis, that we called conformal basis, the results shows some universality. For example, the results for the $Q$-state Potts model and for the Ashkin-Teller model in the basis where the matrices $R_i$ or the matrices $S_i$ are diagonal are the same, and follow the asymptotic behavior

\[ I_n(\ell, L - \ell) = \frac{c_n}{4} \ln \left( \frac{L}{\pi \sin(\frac{\pi \ell}{L})} \right) + \ldots, \]

with

\[ c_n = c \left\{ \begin{array}{ll} 1, & n = 1 \\ \frac{n-1}{n-1}, & n > 1.5 \end{array} \right. \]

We should mention that in \[10\], based on numerical results, it was claimed that for $n = 1$ the coefficient $c_1$ might not be exactly equal to the central charge. As it was discussed in \[10\] \[11\] it is quite likely that $I_n$ is not a continuous function around $n = 1$ and so any attempt to do the replica trick using this definition of Rényi mutual information will be useless. This makes the analytical calculation a challenge. This is an additional reason to examine the behavior of $I_n$, besides being the correct extension, from the point of view of a measure of shared information. Having this in mind we calculated the $\tilde{I}_n$ for $Q = 2, 3$ and $Q = 4$ Potts chains and for the Ashkin-Teller model in the $R$ and the $S$ basis. We found that in some regimes of variation of the parameter $n$ one can fit the data nicely to

\[ \tilde{I}_n(\ell, L - \ell) = \frac{c_n}{4} \log \left( \frac{L}{\pi \sin(\frac{\pi \ell}{L})} \right) + \ldots, \]

being $c_n$ a monotonically nondecreasing function of $n$, consistent with what we expect for the Renyi mutual information, since it is a good measure of shared information (see the Appendix).

Here we summarize the results for the $Q$-state Potts and Ashkin-Teller quantum chains:

\[ \text{FIG. 1: The Rényi mutual information } \tilde{I}_n(\ell, L - \ell) \text{ of the } L = 28 \text{ sites periodic Ising quantum chain, as a function of } \ln(\sin(\frac{n\ell}{L})) / 4. \text{ The ground-state wavefunction is in the basis where the matrices } S_i \text{ are diagonal (S basis).} \]

\[ \text{FIG. 2: The Rényi mutual information } \tilde{I}_n(\ell, L - \ell) \text{ of the } L = 28 \text{ sites periodic Ising quantum chain, as a function of } \ln(\sin(\frac{n\ell}{L})) / 4. \text{ The ground-state wavefunction is in the basis where the matrices } R_i \text{ are diagonal (R basis).} \]
1. The results in general depend on the basis we choose to express the ground-state wavefunction.

2. The Rényi mutual information follows \( \mathcal{I} \) in the \( S \) and \( R \) basis but with different coefficients for different basis. To illustrate the logarithmic behavior we show in Fig. 1 and Fig. 2 the mutual information \( \mathcal{I}_n \) for the Ising model (\( Q = 2 \)) with \( L = 28 \) sites and ground-state eigenfunctions in the \( S \) and \( R \) basis, respectively. We see, from these figures, that for subsystem sizes \( \ell \geq 3 \) we have the logarithmic behavior given by \( \mathcal{I}_n \) up to \( n \approx 8 \) in the \( S \)-basis and \( n \approx 4 \) in the \( R \)-basis. As we can see our results does not exclude the existence of some relevant \( \ell \)-dependent terms in \( \mathcal{I}_n \) for large values of \( n \).

3. The coefficient of the logarithm \( \tilde{c}_n \) in \( \mathcal{I}_n \) is a continuous monotonically non-decreasing function of \( n \) and it follows the following formula in the \( S \)-basis:

\[
\tilde{c}_n = cf(n), \quad \text{with} \quad f(1) = 1, \tag{9}
\]

where \( c \) is the central charge and \( f(n) \) seems to be a continuous universal function independent of the model, as we can see in Fig. 3. In the case of the Ashkin-Teller model the results start to deviate around \( n = 6 \) from the ones obtained for the Potts models. As we can see in Fig. 3, the deviation point is dependent on the anisotropy parameter \( \Delta \) of the model.

4. In the case of the \( R \) basis, as we can see in Fig. 4, equation \( \mathcal{I}_n \) is still valid for values of \( n \) up to \( \sim 4 \).

However the function \( f(n) \) is distinct from the one obtained in the \( S \) basis. As shown in Fig. 4, up to \( n = 2 \) the form of the function \( f(n) \) seems to be also independent of the model. This figure also shows that the Ashkin-Teller model has stronger deviations in this basis, as compared with the results obtained in the \( S \) basis.

5. The coefficient of the logarithm in the \( S \) basis always goes to zero as \( n \to 0 \), differently from the \( R \) basis where it approaches to a non-trivial number. This simply means that probably in the continuum limit all the probabilities in the \( S \) basis are positive but in the \( R \) basis some of them are zero. For the definition of the \( n = 0 \) case see the Appendix.

Our numerical results indicate that \( \tilde{c}_n \) is a continuous function of \( n \) around \( n = 1 \). This means that \( \mathcal{I}_n \) should be a continuous function with respect to \( n \) and so it is a better candidate to be used in techniques exploring the analytical continuation of the value \( n \), as happens for example in the replica trick. However, the appropriate technique that may be used is still unclear to us.

2. The Rényi mutual information in the parafermionic \( Z(Q) \)-quantum spin chains

In this subsection we consider the Rényi mutual information for some critical spin chains with discrete \( Z(Q) \) symmetry and central charge bigger than one. The quantum chains we consider are the parafermionic \( Z(Q) \)-quantum spin chains \( Z(2) \) with Hamiltonian given by...
In Figs. 6 and 7 we show the ratio \( \tilde{c}/c \) with the ground-state wavefunction expressed in the \( S \) variant with a central charge appearing in (4). This model is critical and conformal in the \( Z \) where again \( \ell = 10 \) sites periodic. To illustrate the logarithmic dependence with the subsystem size \( Q \) are shown in Figs. 5, 6 and 7. The results we obtained are very similar to the ones we already discussed in the previous case of \( - \)-parafermionic models with \( Q = 5, 6, 7 \) and 8. The ground-states are in the basis where the \( S \) matrices are diagonal. The lattice sizes of the models are shown in the figure and the coefficients \( \tilde{c}_n \) were estimated by using the subsystem sizes \( \ell = 3, 5, ..., \text{Int}[L/2] \).

### B. The Rényi mutual information of quantum chains with continuous symmetries

In this section we consider the Rényi mutual information of critical chains having a continuous \( U(1) \) symmetry. We studied a set of coupled harmonic oscillators which gives a discrete version of Klein-Gordon field theory as well as the spin-1/2 XXZ and the spin-1 Fateev-Zamolodchikov quantum chains. The last two models are interesting since, like the Ashkin-Teller model, they have an anisotropy that gives us a critical line of continuously varying critical exponents but with a fixed central charge.

#### 1. The Rényi mutual information in quantum harmonic chains

In this subsection we will first consider the Rényi mutual information of the ground state of a system of generic coupled harmonic oscillators. Then at the very end we will confine ourselves to the simple case where we have only the nonzero couplings at the next-nearest sites, that in the continuum limit gives us the Klein-Gordon field theory.

Consider the Hamiltonian of \( L \)-coupled harmonic oscillators, with coordinates \( \phi_1, \ldots, \phi_L \) and conjugated momenta \( \pi_1, \ldots, \pi_L \):

\[
\mathcal{H} = \frac{1}{2} \sum_{n=1}^{L} \pi_n^2 + \frac{1}{2} \sum_{n,n' = 1}^{L} \phi_n K_{nn'} \phi_{n'}. \tag{11}
\]

The ground state of the above Hamiltonian has the fol-
can now calculate the Rényi mutual information

\[ \tilde{I}_n = \frac{1}{2} \ln \left( \frac{\det K^{1/2}}{\det \tilde{P}_A \det \tilde{P}_A} \right) - \frac{1}{2(n-1)} \ln \left( \frac{\det \left( nK^{1/2} - (n-1) \left( \tilde{P}_A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \right)}{\det K^{1/2}} \right). \]

The following determinant formulas

\[
\det(\tilde{P}_A) \det P_A = \det K^{1/2}, \\
\det(\tilde{P}_A) \det P_A = \det K^{1/2}, \\
\det P_A \det K^{-1/2} = \det X_A, \\
\det P_A \det K^{-1/2} = \det X_{\tilde{A}},
\]

allow us to write

\[ \tilde{I}_n(\ell, L - \ell) = S_2(\ell, L - \ell) - \frac{1}{2(n-1)} \ln \det(n + (1 - n)T), \]

where

\[ T = \begin{pmatrix} X_A \tilde{P}_A & X_{\tilde{A}} \tilde{P}_A \\ X_A^T \tilde{P}_A & X_{\tilde{A}}^T \tilde{P}_A \end{pmatrix} = \begin{pmatrix} 1 & X_{\tilde{A}} \tilde{P}_A & X_A \tilde{P}_A \end{pmatrix}. \]

There is an important remark that we should mention: in principle Eq. [19] makes sense only if \( n + (1 - n)T \) is a symmetric positive definite matrix. If we start with a symmetric positive definite matrix \( K^{1/2} \) this is already warranted for \( 0 < n < 1 \) but for \( n > 1 \) one needs to check its validity. This will be an important point when we study the short-range coupled harmonic oscillators. Finally one can write

\[ \tilde{I}_n(\ell, L - \ell) = M_2(\ell, L - \ell) + \tilde{M}_n(\ell, L - \ell) = S_2(\ell, L - \ell) - \frac{1}{2(n-1)} \ln \det(1 - (1 - n)^2X_A^T \tilde{P}_A X_{\tilde{A}} \tilde{P}_A) \]

where \( \tilde{M}_n(\ell, L - \ell) \) is the only \( n \) dependent part. We notice here that by changing \( n \) to \( 2 - n \) we just change the sign of the second term, i.e., \( \tilde{M}_{2-n}(\ell, L - \ell) = -\tilde{M}_n(\ell, L - \ell) \).

When \( n \rightarrow 1 \) the second term vanishes and we recover the result of [8]

\[ \tilde{I}_1(\ell, L - \ell) = S_2(\ell, L - \ell). \]

For massless Klein-Gordon theory the above result in one dimension gives, as a consequence the well known result for the Rényi entanglement entropy [32],

\[ \tilde{I}_1(\ell, L - \ell) = \frac{1}{4} \ln \left( \frac{L}{\pi} \sin(\pi \ell/L) \right) + ..., \]

where the dots are the subleading terms. Our numerical analyses indicate that for short-range quantum harmonic
oscillators the matrix $n + (1 - n)T$ is symmetric positive definite up to just $n = n_c = 2\frac{10}{4}$. The numerical results show that for the values $0 < n < 2$ the equation $\tilde{c}$ is a very good approximation, as we can see for example in Fig. 8. The coefficient $\tilde{c}_n$ of the logarithmic term in $\tilde{M}$ is obtained from the fitting of the model with $L = 120$ sites is shown in Fig. 9 and in the range $0.4 < n < 1.6$ surprisingly it follows the simple formula:

$$\tilde{c}_n = f(n) = 1 + \frac{n - 1}{10}, \quad 0.4 < n < 1.6 \quad (23)$$

This is the red line in Fig. 9. At $n = 0$ we expect zero mutual information for our system, this means that based on the symmetry $n \rightarrow 2 - n$ the coefficient for $n = 2$ should be $\tilde{c}_2 = 2$. Finally one can conclude that for integer values of $n = 0, 1, 2$ the coefficient of the logarithm is

$$\tilde{c}_n = f(n) = n, \quad n = 0, 1, 2. \quad (24)$$

2. The Rényi mutual information of quantum spin chains with continuous symmetries

The Hamiltonian of the XXZ chain is defined as

$$H_{XXZ} = -\sum_{i=1}^{L} (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} + \Delta \sigma^z_j \sigma^z_{j+1}), \quad (25)$$

where $\sigma^x$, $\sigma^y$ and $\sigma^z$ are spin-$\frac{1}{2}$ Pauli matrices and $\Delta$ is an anisotropy. The model is critical and conformal invariant for $-1 \leq \Delta < 1$ with a constant central charge $c = 1$, giving us a good example to test the universality of our results with respect to the change of the anisotropy. The long-distance critical fluctuations are ruled by a CFT with central charge $c = 1$ described by a compactified boson whose action is given by

$$S = \frac{1}{8\pi} \int d^2 x (\nabla \phi)^2, \quad \phi \equiv \phi + 2\pi R, \quad (26)$$

where the compactification radius depends upon the values of $\Delta$, namely:

$$R = \sqrt{\frac{2}{\pi} \arccos \Delta}. \quad (27)$$

As it is shown in Fig. 10, in the $\sigma^z$ basis, the Rénnyi mutual information $I_n(\ell, L - \ell)$ shows the logarithmic behavior given in $\tilde{c}_n$ only for $n < 2$. This can be simply understood based on what we observed for the chain of harmonic oscillators. One can look to the Klein-Gordon field theory as a non-compactified version of the action $\tilde{c}_n$. Since we showed that in that case the Rénnyi mutual information is not defined beyond $n = 2$ we expect the same behavior also in the compactified version. Note that in our numerical calculations one can actually derive spurious big numbers for the Rénnyi mutual information even for $n > 2$, but we expect all of them go to infinity in the thermodynamic limit. This behaviour seems to be independent of the anisotropy parameter $\Delta$.

The coefficient of the logarithm in $\tilde{c}_n$ for $n < 2$ is again given by $\tilde{c}_n$, as we can see in Fig. 11, with a function $f(n)$ which fits to the results of the harmonic chain perfectly. We also considered the results in the case where the ground state wavefunction is expressed in the $\sigma^z$ basis and, except around $n = 1$, the equation $\tilde{c}_n$ is not a good approximation. The second $U(1)$-symmetric model we considered is the spin-1 Fateev-Zamolodchikov quan-
critical regimes where the charge version of the model is governed by a CFT with central $\epsilon$ for ferromagnetic ($c$ by an anisotropy $\Delta = -1/2$) symmetry or the ones where $\gamma = \pi/3, \pi/4$. The lattice sizes of the models are shown and the coefficients $\tilde{c}_n$ were estimated by using the subsystem sizes $\ell = 4, 5, ..., L/2$.

considered several integrable quantum spin chains. These quantum chains either have a $Z(Q)$ symmetry (like the $Q$-state Potts model with $Q = 2, 3$ and $4$, the Ashkin-Teller model, and the $Z(Q)$-parafermionic model with $Q = 5, 6, 7$ and $8$) or a $U(1)$ symmetry (XXZ quantum chain and the spin-1 Fateev-Zamolodchikov model). We also considered the discrete version of the Klein-Gordon field theory given by a set of coupled harmonic oscillators. In this case we have a continuum Hilbert space. We observed that by expressing the ground-state wavefunctions in general basis the obtained results are distinct. However, similarly as happens for the quantity $I_n$ given in [2] (see [11]), our results on some special basis reveal some general features. These basis are the ones where the $S$ or $R$ operators are diagonal, for the models with $Z(Q)$ symmetry or the ones where $\sigma^z$ or $S^z$ are diagonal for the models with $U(1)$ symmetry. In a continuum field theory description of these quantum chains these basis are expected to be associated to the boundaries that do not destroy the conformal invariance of the bulk underlying Euclidean conformal field theory, and for this reason we call them conformal basis [11]. Our results indicate that in these special basis the mutual information $I_n$ has the same kind of leading behavior with the subsystem size $\ell$ as we have in the Rényi entanglement entropy, namely $\tilde{I}_n(\ell, L - \ell) \sim \frac{1}{\ell} \ln(\frac{1}{\ell} \sin(\frac{n}{\ell} \pi))$, with a function $\tilde{c}_n = cf(n)$, with $f(1) = 1$. Differently from the Rényi entanglement entropy where the equivalent function $f(n)$ is universal (for any model and any basis) in the case of $\tilde{I}_n$ our results indicate that the function $f(n)$ depends on the special basis chosen to express

IV. CONCLUSIONS

In this paper we calculated the Rényi mutual information $I_n(\ell, L - \ell)$, as defined in [30], for quantum chains describing the dynamics of quantum systems with continuous or discrete degrees of freedom. Most of our analysis was purely numerical due to the absence, at the moment, of suitable analytical methods to treat this problem. We

\begin{align}
H_{FZ} & = \epsilon \sum_{i=1}^{L} \left\{ \sigma_i - (\sigma_i^z)^2 - 2(\cos \gamma - 1)(\sigma_i^+ \sigma_i^z + \sigma_i^- \sigma_i^z) \right. \\
& \left. - 2 \sin^2 \gamma (\sigma_i^- - (\sigma_i^z)^2) + 2(S_i^z)^2 \right\},
\end{align}

where $\tilde{S} = (S^x, S^y, S^z)$ are spin-1 $SU(2)$ matrices, $\sigma_i^z = S_i^z S_{i+1}^z$ and $\tilde{S}_i S_{i+1} = \sigma_i^+ + \sigma_i^-$. The model is antiferromagnetic for $\epsilon = +1$ and ferromagnetic for $\epsilon = -1$. It has a line of critical points $0 \leq \gamma \leq \frac{7}{2}$ with a quite distinct behavior in the antiferromagnetic ($\epsilon = +1$) and ferromagnetic ($\epsilon = -1$) cases. The antiferromagnetic version of the model is governed by a CFT with central charge $c = \frac{2}{5} [37]$ while the ferromagnetic one is ruled by a $c = 1$ CFT [38]. We calculated $I_n(\ell, L - \ell)$ in both critical regimes where $c = 1$ and $c = \frac{1}{2}$, and for different values of the anisotropy. We found a very similar pattern as that of the XXZ quantum chain, as can be seen in Fig. 11. The equation [5] is valid for values of $n < 2$ and the coefficient of the logarithm follows [39] with a function $f(n)$ which is quite similar to the one we found for the quantum harmonic oscillators and the XXZ chain. This shows an interesting universal pattern for critical chains with continuous $U(1)$ symmetry.

\begin{align}
\tilde{c}_n & = \tilde{c}_n(\ell, L - \ell) \sim \frac{1}{\ell} \ln(\frac{1}{\ell} \sin(\frac{n}{\ell} \pi)),
\end{align}

with a function $\tilde{c}_n = cf(n)$, with $f(1) = 1$. Differently from the Rényi entanglement entropy where the equivalent function $f(n)$ is universal (for any model and any basis) in the case of $\tilde{I}_n$ our results indicate that the function $f(n)$ depends on the special basis chosen to express

FIG. 10: The Rényi mutual information $\tilde{I}_n(\ell, L - \ell)$ of the periodic XXZ quantum chain with anisotropy $\Delta = -1/2$, as a function of $\ln(\sin(\frac{n}{\ell} \pi))/4$. The ground-state wavefunction is in the basis where the $\sigma_i^z$ matrices are diagonal ($\sigma^z$ basis). The results are for lattice sizes $L = 28$ and $L = 30$ and give an idea of the finite-size corrections.

FIG. 11: The ratio $\tilde{c}_n/c$ of the coefficient of the logarithm in equation (8) with the central charge $c$ for the XXZ and for the spin-1 Fateev-Zamolodchikov quantum chains (F-Z). The XXZ (Fateev-Zamolodchikov) ground-state wavefunction are in the $\sigma^z$ ($S^z$) basis. The results for the XXZ are for the anisotropies $\Delta = 0, -1/2$ and in the case of the Fateev-Zamolodchikov model their are for the couplings $\gamma = \pi/3, \pi/4$. The lattice sizes of the models are shown and the coefficients $\tilde{c}_n$ were estimated by using the subsystem sizes $\ell = 4, 5, ..., L/2$.
the ground-state eigenfunction of the particular model. For the set of $Z(Q)$-symmetric models we considered the function $f(n)$, for $n < 4$, although different for the $S$ and $R$ basis are similar as the ones of the $Q$-state Potts chain ($Q = 2, 3, 4$) and the parafermionic $Z(Q)$ quantum chains ($Q = 5, 6, 7, 8$). In the case of the Ashkin-Teller model our results indicate that $f(n)$, for $n > 2$, also depends on the anisotropy $\Delta$ of the model. On the other hand our results indicate that Rényi mutual information is not defined. It is quite interesting that in these cases one can understand most of the results by just studying simple short-range coupled harmonic oscillators.

In order to conclude we should mention that an analytical approach for the Shannon entropy or the Shannon mutual information ($I_1$ or $I_\ell$ in $[2]$ and $[3]$) is a theoretical challenge. The analytical methods to treat this kind of problem normally use some sort of analytical continuation, in the parameter $n$, like the usual replica trick. The results we present showing the continuity of $\tilde{I}_n$ around $n = 1$, differently from what happens with $I_n$, indicate that $\tilde{I}_n$ is probably more appropriate for an analytical treatment.

Acknowledgments This work was supported in part by FAPESP and CNPq (Brazilian agencies). We thank J. A. Hoyos, R. Pereira and V. Pasquier for useful discussions.

V. APPENDIX: THE RELATIVE ENTROPY AND THE RÉNYI DIVERGENCE

In this appendix we review the definitions of the relative entropy and its generalization: the Rényi divergence. The relative entropy is defined as the expectation of the difference between the logarithm of the two distribution of probabilities $p$ and $q$, from the point of view of the distribution $p$, i. e.,

$$D(p \parallel q) = \sum_i p_i \ln \frac{p_i}{q_i}. \quad (29)$$

It can be considered as a measure of the difference between the two distributions $p$ and $q$. Although it is not a symmetric quantity it helps us to define the mutual information of the subsets $X$ ans $Y$ of the system as follows

$$I(X, Y) = D(p(X, Y) \parallel p(X)p(Y)). \quad (30)$$

In words, the mutual information between two parts of a system is just the relative entropy between the distribution probability for the whole system and the product of the probability distributions of the different parts. It tells how much the different parts are correlated. The natural generalization of the relative entropy is the Rényi divergence and can be defined (see $[29]$ for example), as

$$D_n(p \parallel q) = \frac{1}{n-1} \ln \sum_i p_i^n q_i^{1-n}. \quad (31)$$

It has the following properties: for $n > 0$ we have $D_n(p \parallel q) \neq 0$ and if $p = q$ then we have $D_n(p \parallel q) = 0$. The especial case $n \to 1$ gives the usual relative entropy. We also define the $n = 0$ case by:

$$D_0(p \parallel q) = -\ln q(i|p_i > 0). \quad (32)$$

It is worth mentioning that using the above definition $D_0(p \parallel q)$ is not zero except when all the $p_i$’s are positive.

Another important property is the following (see $[39]$ and references therein):

Theorem: the Rényi divergence is a continuous and nondecreasing function of the parameter $n$.

Comparing $[31]$ with $[30]$ and $[29]$ the natural definition of the Rényi mutual information is

$$\tilde{I}_n(X, Y) = D_n(p(X, Y) \parallel p(X)p(Y)). \quad (33)$$

The above definition is different from $I_n(\ell, L)$, as given by $[2]$, and has been frequently used in different areas of information science.

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