Edge-weighted Online Stochastic Matching: Beating $1 - \frac{1}{e}$

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Abstract

We study the edge-weighted online stochastic matching problem. Since Feldman, Mehta, Mirrokni, and Muthukrishnan [6] proposed the $(1 - \frac{1}{e})$-competitive Suggested Matching algorithm, there has been no improvement for the general edge-weighted online stochastic matching problem. In this paper, we introduce the first algorithm beating the $1 - \frac{1}{e}$ barrier in this setting, achieving a competitive ratio of 0.645. Under the LP proposed by Jaillet and Lu [13], we design an algorithmic preprocessing, dividing all edges into two classes. Then based on the Suggested Matching algorithm, we adjust the matching strategy to improve the performance on one class in the early stage and on another class in the late stage, while keeping the matching events of different edges highly independent. By balancing them, we finally guarantee the matched probability of every single edge.

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1 Introduction

Since the online bipartite matching problem was introduced by Karp, Vazirani, and Vazirani [15], it has been playing an important role in the field of online algorithms. One of the most important applications of online matching is online advertising. When a user searches on a search engine, appropriate advertisements should be selected to show. In this problem, advertisers are modeled as offline vertices which are known to the algorithm in the beginning, and impressions (user searches) are modeled as online vertices. Edges between them indicate whether advertisers are interested in impressions. When an online vertex arrives, the online algorithm should immediately decide how to match it.

Karp et al. [15] proposed the worst case model, which measures the online algorithm’s performance in the worst instance. They introduced the Ranking algorithm, achieving a competitive ratio of $1 - \frac{1}{e} \approx 0.632$ for unweighted matching, and proved its optimality.

To beat the $1 - \frac{1}{e}$ bound, some new models are introduced, including the online stochastic matching model proposed by Feldman, Mehta, Mirrokni, and Muthukrishnan [6]. In this model, each online vertex samples its type from a known distribution. They proposed the $(1 - \frac{1}{e})$-competitive Suggested Matching algorithm as a benchmark algorithm, and then proposed the first algorithm beating the $1 - \frac{1}{e}$ bound for unweighted matching, under an assumption of integral arrival rates. Later works removed the integral assumption [10, 12, 13, 17] and generalized the results to vertex-weighted matching [10, 12] and edge-weighted matching with free disposal [12]. On the other hand, under the integral assumption, the $1 - \frac{1}{e}$ ratio was also surpassed for edge-weighted matching (without free disposal) [4, 9]. However, for the general edge-weighted online stochastic matching problem without any further assumption, there are no positive results beyond Suggested Matching.

1.1 Our Contributions and New Techniques

In this paper, we propose a 0.645-competitive algorithm for the edge-weighted online stochastic matching, which is the first algorithm beating the $1 - \frac{1}{e}$ bound in this setting. We will introduce the algorithm in the Poisson arrival model, where instead of a fixed number of online vertices, they arrive following a Poisson process. This model has been proved asymptotically equivalent to the original online stochastic matching model [10].

Preprocessing Under the Jaillet-Lu LP Like previous works, we solve an LP to get a fractional matching which bounds the offline optimal and guides our online algorithm. We use the LP proposed by Jaillet and Lu [13]. Under this LP, we can preprocess the solution to make it satisfy some stronger constraints. We show that under the Jaillet-Lu LP, without loss of generality, an online algorithm can assume that there are only two classes of online vertex types: one matched with only one offline vertex (in the fractional matching) and the other matched with two offline vertices symmetrically.

Multistage Suggested Matching We take the Suggested Matching algorithm proposed by Feldman et al. [6] as the starting point. After dividing edges into two classes, we optimize them separately. In Suggested Matching, the match strategy of first-class edges itself cannot be further improved, while that of second-class edges can. So on the one hand, we improve the strategy of

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1 They stated Suggested Matching and its related LP in unweighted form as they only considered unweighted matching, but it’s straightforward to change it into edge-weighted form (see Section 2 for more details).
2 The expected number of vertices of each online vertex type is an integer.
3 An offline vertex can be matched multiple times, but only the heaviest edge remains in the end.
second-class edges in the late stage to avoid affecting first-class edges too much, and on the other hand, we don’t match second-class edges in a short period of time in the beginning to increase the match probabilities of first-class edges. Balancing the durations of these two operations, our algorithm achieves a competitive ratio over $1 - \frac{1}{e}$ on every single edge.

**Independent Analysis for Every Single Edge** Previous analyses for vertex-weighted matching [10, 12] upperbound the unmatched probability of each offline vertex to calculate the competitive ratio. However, for edge-weighted matching, this cannot reflect the matched probability of each edge. On the other hand, the analysis for edge-weighted matching with free disposal [12] pays attention to the overall matching progress, but it cannot handle the loss when a light edge occupies an offline vertex and then blocks a heavy edge. In contrast, our algorithm independently lowerbounds the unmatched probability of each offline vertex at any time, so that we can independently calculate an edge’s matched probability given its match rate.

### 1.2 Related Work

Karp, Vazirani, and Vazirani [15] first started the research of online matching. They considered the situation that the graph can be arbitrary and the algorithm does not know anything about the online vertices, known as the **worst case model**. They introduced the Ranking algorithm, achieving a competitive ratio of $1 - \frac{1}{e} \approx 0.632$ for unweighted matching, and also proved no algorithm can do better than that. Then, Aggarwal, Goel, Karande, and Mehta [1] generalized this algorithm to vertex-weighted matching, keeping the same competitive ratio. For edge-weighted matching with free disposal, Fahrbach, Zadimoghaddam, Huang, and Tao [5] proposed a 0.5086-competitive algorithm, then Gao, He, Huang, Nie, Yuan, and Zhong [7] and Blanc and Charikar [3] independently improved the ratio to 0.519 and 0.5239 respectively.

In the **random order model**, we still consider any bipartite graph, but assume that the online vertices arrive in a random order, and measure the expected performance of an online algorithm. Goel and Mehta [8] proved that the Greedy algorithm is $(1 - \frac{1}{e})$-competitive, and gave an upper bound of $\frac{5}{6}$. Mahdian and Yan [16] showed that Ranking is 0.696-competitive for unweighted matching. Huang, Tang, Wu, and Zhang [11] generalized Ranking to vertex-weighted matching with a competitive ratio of 0.653, and Jin and Williamson [14] improved it to 0.662.

In the **online stochastic matching** model, all online vertices are sampled independently and identically from a distribution, which is known to the online algorithm at the beginning. Feldman, Mehta, Mirrokni, and Muthukrishnan [6] proposed a 0.67-competitive algorithm for unweighted matching, with a further assumption that the arrival rate of each online vertex type must be integral. They also proved an upper bound of 0.989. Bahmani and Kapralov [2] improved the ratio and the upper bound to 0.699 and 0.902 respectively. Manshadi, Gharan, and Saberi [17] removed the integral assumption and improved the ratio and the upper bound to 0.702 and 0.823 respectively. Jaillet and Lu [13] improved the ratio to 0.706. Huang and Shu [10] improved the ratio to 0.711, and generalized it to vertex-weighted matching with a competitive ratio of 0.7009. Huang, Shu, and Yan [12] improved the ratio for vertex-weighted matching to 0.716, and proposed a 0.706-competitive algorithm for edge-weighted matching with free disposal. They also proved an upper bound of 0.703 for edge-weighted matching without free disposal. Meanwhile, Tang, Wu, and Wu [18] proposed a 0.704-competitive algorithm for vertex-weighted matching, and generalized it to non i.i.d. arrivals with a competitive ratio of 0.666.

On the other hand, under the integral assumption, Manshadi et al. [17] improved the ratio for unweighted matching to 0.705. Jaillet and Lu [13] improved the ratio to 0.729, and generalized it to vertex-weighted matching with a competitive ratio of 0.725. For edge-weighted matching,
Haeupler, Mirrokni, and Zadimoghaddam [9] proposed a 0.667-competitive algorithm. Brubach, Sankararaman, Srinivasan, and Xu [4] improved the ratios to 0.7299 for vertex-weighted matching and 0.705 for edge-weighted matching.

2 Preliminaries

Online Stochastic Matching In this paper, we consider the edge-weighted case of online stochastic matching. Consider a set $I$ of online vertex types and a set $J$ of offline vertices. Let $J_i \subseteq J$ denote the set of offline vertices adjacent to online type $i \in I$. Let $E = \{(i, j) : i \in I, j \in J_i\}$ denote the set of edges, where each edge $(i, j)$ has a non-negative weight $w_{ij}$. Each online vertex type $i \in I$ has an arrival rate $\lambda_i$. $\Lambda = \sum_{i \in I} \lambda_i$ online vertices arrive one by one. Each of them draws its type $i$ with probability $\frac{\lambda_i}{\Lambda}$ independently. The objective is to maximize the expected total weight of all matched edges.

Poisson Arrival Model In this variant of online stochastic matching, we do not assume that the number of online vertices is fixed. Instead, the online vertices of each type $i$ arrive independently following a Poisson process with time horizon $0 \leq t \leq 1$ and arrival rate $\lambda_i$. In this paper, we use Poisson arrival model instead of the original online stochastic matching model, as Huang and Shu [10] have proved they are asymptotically equivalent.

Online Algorithms When an online vertex arrives, an online algorithm should immediately and irrevocably match it to an unmatched adjacent offline vertex, or discard it. Define the competitive ratio of an online algorithm to be the infimum of the ratio of its expected objective to the expected objective of the offline optimal matching.

Suggested Matching Here we briefly restate the edge-weighted version of the Suggested Matching algorithm proposed by Feldman et al. [6]. We first calculate the optimal solution of the following LP, which is a fractional matching bounding the offline optimal.

$$\text{maximize} \quad \sum_{(i,j) \in E} w_{ij} x_{ij}$$

subject to

$$\sum_{j \in J} x_{ij} \leq \lambda_i \quad \forall i \in I$$

$$\sum_{i \in I} x_{ij} \leq 1 \quad \forall j \in J$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in E$$

Then, when an online vertex of type $i$ arrives, we match it to each unmatched adjacent offline vertex $j$ with probability $\frac{x_{ij}}{\lambda_i}$. In the end, the matched probability of each edge $(i, j)$ will be at least $(1 - \frac{1}{e})x_{ij}$. 

3
Jaillet-Lu LP  We will use the LP proposed by Jaillet and Lu [13] instead of the above LP. We restate the Jaillet-Lu LP as follows:

maximize \( \sum_{(i,j) \in E} w_{ij} x_{ij} \)
subject to \( \sum_{j \in J} x_{ij} \leq \lambda_i \quad \forall i \in I \)
\( \sum_{i \in I} x_{ij} \leq 1 \quad \forall j \in J \)
\( \sum_{i \in I} (2x_{ij} - \lambda_i)^+ \leq 1 - \ln 2 \quad \forall j \in J \) \tag{1}
\( x_{ij} \geq 0 \quad \forall (i,j) \in E \)

The only difference between these two LPs is Constraint (1), which limits the situation that an online vertex type puts most of its arrival rate on one incident edge. Although this LP was initially proposed for the original online stochastic matching model, Huang and Shu [10] have shown that it also bounds the offline optimal in Poisson arrival model. For convenience, we artificially define \( x_{ij} = 0 \) for any \((i, j) \notin E\). We further define \( x_i = \sum_{j \in J} x_{ij} \) and \( x_j = \sum_{i \in I} x_{ij} \). We will also call \( x_{ij} \) the flow of edge \((i, j)\) or the flow from \(i\) to \(j\).

3 Preprocessing Under the Jaillet-Lu LP

When we use the Jaillet-Lu LP to bound the offline optimal, we can preprocess the solution to make it satisfy some new constraints, while the objective remains unchanged and all original constraints still hold. This on the one hand facilitates designing our algorithm, and on the other hand characterizes the worst instances we need to face.

For each online vertex type \(i\) such that \(x_i < \lambda_i\), we add \(m = \max\{\lceil \lambda_i - x_i \rceil, 2 \}\) dummy\(^4\) offline vertices connecting to \(i\), each with a flow \(\frac{\lambda_i - x_i}{m}\). The new dummy vertices satisfy Constraint (1) since \(m \geq 2\). It’s easy to check that other constraints of the Jaillet-Lu LP still hold and we have:

\[ \forall i \in I, \quad x_i = \lambda_i. \] \tag{2}

Then we add two more dummy offline vertices, and one dummy online vertex type with a flow \(1 - x_j\) to every offline vertex \(j\) (including the two dummy offline vertices), which has an appropriate arrival rate satisfying Equation (2). Since the arrival rate of this new vertex is at least 2, it won’t contribute to Constraint (1) for any offline vertex. After that, all constraints of the Jaillet-Lu LP and Equation (2) still hold, and we have:

\[ \forall j \in J, \quad x_j = 1. \] \tag{3}

The final step is more complicated. We want to make the solution satisfy:

\[ \forall (i, j) \in E, \quad x_{ij} \in \left\{ 0, \frac{1}{2} \lambda_i, \lambda_i \right\}, \] \tag{4}
which means there are only two classes of online vertex types: one with an adjacent offline vertex such that \(x_{ij} = \lambda_i\), and the other with two adjacent offline vertices such that \(x_{ij1} = x_{ij2} = \frac{1}{2} \lambda_i\).

\(^4\)For a dummy vertex, the weights of its incident edges are 0, and its initial flow is 0.
We will achieve this by finding a way to split each online vertex type \( i \) to several types \( i_1, \ldots, i_k \), such that \( \sum_{n=1}^{k} x_{i_nj} = x_{ij} \) for any offline vertex \( j \). By assigning \( \lambda_{i_u} = \sum_{j u} x_{i_uj} \), any split scheme trivially satisfies Equations (2) and (3). The remaining problem is how to pair up the edges to satisfy Equation (4) while keeping Constraint (1) hold. Informally speaking, any method furthest avoiding pairing an edge with itself will work. However, to give a formal proof, here we propose a specific split scheme.

**Lemma 1.** For every online vertex type \( i \), there is a split scheme satisfying Equation (4) and Constraint (1).

**Proof.** Consider an online vertex type \( i \) with \( k \) adjacent offline vertices \( j_1, \ldots, j_k \). Define a function \( f_i : [0, \lambda_i) \to J_i \) such that \( f_i(\theta) = j_u \) if and only if \( \sum_{v=1}^{u-1} x_{ij_v} \leq \theta < \sum_{v=1}^{u} x_{ij_v} \), i.e. each edge \((j_i, j_u)\) corresponds to an interval of length \( x_{ij_u} \). Then we will pair \( f_i(\theta) \) with \( f_i(\theta + \frac{\lambda_i}{2}) \).

The interval \([0, \frac{\lambda_i}{2})\) can be divided into at most \( 2k \) subintervals, in which \( f_i(\theta) \) and \( f_i(\theta + \frac{\lambda_i}{2}) \) are both invariant. For each subinterval \([l, r)\), create an online vertex type with a flow \( r - l \) to each of \( f_i(l) \) and \( f_i(l + \frac{\lambda_i}{2}) \). If \( f_i(l) = f_i(l + \frac{\lambda_i}{2}) \), they collapse to one edge with flow \( 2(r - l) \), and only in this case it contributes to Constraint (1).

It’s easy to see these online vertex types sum up to the original type \( i \) and satisfy Equation (4). For each edge \((j_i, j_u)\) with \( x_{ij_u} \leq \frac{\lambda_i}{2} \), obviously there is no \( \theta \in [0, \frac{\lambda_i}{2}) \) such that \( f_i(\theta) = f_i(\theta + \frac{\lambda_i}{2}) = j_u \). If there is an edge \((j_i, j_v)\) with \( x_{ij_v} > \frac{\lambda_i}{2} \), there will be only one subinterval \([l, r)\) such that \( f_i(\theta) = f_i(\theta + \frac{\lambda_i}{2}) = j_u \) for \( \theta \in [l, r) \) with \( r - l = x_{ij_u} - \frac{\lambda_i}{2} \). So Constraint (1) still holds. \( \Box \)

We remark that, for an online algorithm, a split can be simulated as follows. When an online vertex of (original) type \( i \) arrives, change its type to \( i_u \) with probability \( \frac{\lambda_i}{\lambda_x} \).

In summary, for any solution of the Jaillet-Lu LP, we can preprocess it so that the constraints other than (1) can be replaced by their tighter versions, Equations (2), (3) and (4), so that the solution will have a more discrete structure. We summarize the conclusion in the following theorem.

**Theorem 2.** For any solution of the Jaillet-Lu LP, there is another equivalent fractional matching (on another equivalent instance) with the same total weight, which satisfies Constraint (1) and Equations (2), (3) and (4).

### 4 Multistage Suggested Matching

#### 4.1 Algorithm

By Theorem 2, we have a fractional matching satisfying Constraint (1) and Equations (2), (3) and (4). Naturally we ignore the edges with \( x_{ij} = 0 \), so we say \( i \) and \( j \) are neighbors if and only if \( x_{ij} > 0 \). We call edge \((i, j)\) a first-class edge if \( x_{ij} = \lambda_i \), or a second-class edge if \( x_{ij} = \lambda_i/2 \). If an online vertex/type has an incident first-class (respectively second-class) edge, we call it a first-class (respectively second-class) online vertex/type. Let \( I_1 \) (respectively \( I_2 \)) denote the set of first-class (respectively second-class) online vertex types. Define \( y_j = \sum_{i \in I_1} x_{ij} \) to be the total flow of first-class edges incident to offline vertex \( j \), then we can rewrite Constraint (1) as:

\[
\forall j \in J, \quad y_j \leq 1 - \ln 2. \tag{5}
\]

In the Suggested Matching algorithm, we match every first-class vertex to its only neighbor with probability 1, while for a second-class vertex, we only match it to a neighbor with probability \( \frac{1}{2} \) even if the other neighbor has been matched. We try to increase the match rate in this case. However, if we do it with no limitations, such as raising the probability to 1 when the other
neighbor is matched, the matched probabilities of different offline vertices and edges will become highly correlated. A lighter edge may occupy its incident offline vertex with higher probability so that a heavier edge will be more likely unmatched. In particular, it will hinder us to establish a lower bound of unmatched probability of an offline vertex at some point and then independently calculate the matched probability of each edge. Thus our algorithm only observes the status of offline vertices once, at some time point \( t_1 \). Only when a neighbor of some second-class vertex is matched at that time, we increase the match rate of the other neighbor in the remaining time.

The above method improves the performance on second-class edges, but the matched probabilities of first-class edges are affected. Since the match strategy of first-class vertices has no room to improve, we need to find another way. We sacrifice the match probabilities of second-class edges in early time to increase the unmatched probabilities of offline vertices. Before some time point \( t_0 \), our algorithm directly discards all arriving second-class vertices. Since the proportion of first-class edges is low (Eqn. (5)), their matched probability will be significantly improved. On the contrary, the previous method slightly affects first-class edges since it only happens in the late stage. Combining this two methods, our algorithm make the competitive ratio over every single edge exceed \( 1 - \frac{1}{e} \).

Therefore, our algorithm can be divided into three stages by time, and in each stage our match strategy is similar to that in Suggested Matching, so we call it Multistage Suggested Matching algorithm.

### Multistage Suggested Matching

**Input at the beginning:**
- Online vertex types \( I \), offline vertices \( J \), edges \( E \);
- Arrival rates \( (\lambda_i)_{i \in I} \);
- Fractional matching \( (x_{ij})_{(i,j) \in E} \) that satisfies Equations (2), (3), (4) and (5);
- Boundary times \( t_0, t_1 \) such that \( 0 \leq t_0 \leq t_1 \leq 1 \).

**When a first-class online vertex of type \( i \in I_1 \) arrives at time \( 0 \leq t \leq 1 \):**
- Match it to the only neighbor if the neighbor is unmatched.

**When a second-class online vertex of type \( i \in I_2 \) arrives at time \( 0 \leq t \leq t_0 \):**
- Discard it.

**When a second-class online vertex of type \( i \in I_2 \) arrives at time \( t_0 < t \leq t_1 \):**
- Match it to each unmatched neighbor with probability \( \frac{1}{2} \).

**When a second-class online vertex of type \( i \in I_2 \) arrives at time \( t_1 < t \leq 1 \):**
- If only one neighbor was unmatched at time \( t_1 \), match it to the neighbor if the neighbor is still unmatched;
- Otherwise match it to each unmatched neighbor with probability \( \frac{1}{2} \).

### 4.2 Analysis

We can see from the algorithm that the strategy of each online vertex is almost fixed. Only second-class vertices will adjust the strategy in the final stage based on an one-time observation at time \( t_1 \). This means the match events of different offline vertices are highly independent. We formalize
this by first defining for each edge \((i, j)\) the match rate \(r_{ij}(t)\) to be the rate of online vertex \(i\) trying to match offline vertex \(j\) at time \(t\). Note that for any first-class edge \((i, j)\), or for any second-class edge \((i, j)\) before time \(t_1\), \(r_{ij}(t)\) is independent of any randomness before. By the definition of the algorithm, we have:

**Fact 3.** For any first-class edge \((i, j)\), for any time \(0 \leq t \leq 1\), \(r_{ij}(t) = \lambda_i = x_{ij}\).

For any second-class edge \((i, j)\):

(1) For any time \(0 \leq t \leq t_0\), \(r_{ij}(t) = 0\);

(2) For any time \(t_0 < t \leq t_1\), \(r_{ij}(t) = \frac{1}{2}\lambda_i = x_{ij}\);

For any second-class edge \((i, j)\) after time \(t_1\), \(r_{ij}(t)\) is only dependent of the status of the other neighbor of \(i\) at time \(t_1\). For simplicity, for a second-class edge \((i, j)\) where \(j\) has been matched at time \(t_1\), when online vertex \(i\) arrives after time \(t_1\), we artificially suppose it tries to match \(j\) (and of course fails) with probability \(\frac{1}{2}\) (respectively 1) if the other neighbor of \(i\) is unmatched (respectively has been matched) at time \(t_1\). Then we have:

**Fact 4.** For any second-class edge \((i, j)\), for any time \(t_1 < t \leq 1\), letting \(j'\) be the other neighbor of \(i\):

(1) If \(j'\) is unmatched at time \(t_1\), \(r_{ij}(t) = \frac{1}{2}\lambda_i = x_{ij}\);

(2) If \(j'\) has been matched at time \(t_1\), \(r_{ij}(t) = \lambda_i = 2x_{ij}\).

Using the match rate, we can bound the unmatched probability of each offline vertex at any time. For vertex-weighted matching, an upper bound of the unmatched probability of each offline vertex will be useful since it can directly contribute to the total weight. However, for edge-weighted matching, we need to compute the matched probability for each edge, so instead we lowerbound the unmatched probability of each offline vertex and then make use of the exact match rate of each edge.

For any offline vertex \(j\), let \(r_j(t) = \sum_{i \in I} r_{ij}(t)\) be the match rate of \(j\) and \(A_j(t)\) be the indicator of whether \(j\) is unmatched at time \(t\). By definition:

**Fact 5.** For any edge \((i, j)\) and any time \(0 \leq t \leq 1\), the probability that it has been matched at time \(t\) is \(\int_0^t E[r_{ij}(t')A_j(t')]dt'\).

**Fact 6.** For any edge \((i, j)\) and any time \(0 \leq t \leq 1\), \(E[A_j(t)] = 1 - \int_0^t E[r_{ij}(t')A_j(t')]dt'\).

Combining Facts 3, 4 and 6 we can get the following lemma. Recall that \(y_j = \sum_{i \in I_1} x_{ij}\) is the total flow of first-class edges incident to offline vertex \(j\).

**Lemma 7.** For any offline vertex \(j\):

(1) For any time \(0 \leq t \leq t_0\), \(E[A_j(t)] = e^{-y_j t}\);

(2) For any time \(t_0 < t \leq t_1\), \(E[A_j(t)] = e^{-y_j t_0 -(t-t_0)}\);

(3) For any time \(t_1 < t \leq 1\), \(E[A_j(t)] \geq e^{-y_j t_0 -(t_1-t_0)-(2-y_j)(t-t_1)}\).

**Proof.** (1) For \(0 \leq t \leq t_0\),

\[
r_j(t) = \sum_{i \in I_1} x_{ij}(t) + \sum_{i \in I_2} 0 = y_j
\]

independent of \(A_j(t)\), so

\[
E[A_j(t)] = 1 - y_j \int_0^t E[A_j(t')]dt' = e^{-y_j t}.
\]
(2) For \( t_0 < t \leq t_1 \),
\[
    r_j(t) = \sum_{i \in I_1} x_{ij}(t) + \sum_{i \in I_2} x_{ij}(t) = 1
\]

independent of \( A_j(t) \), so
\[
    E[A_j(t)] = E[A_j(t_0)] - \int_{t_0}^{t} E[A_j(t')]dt' = e^{-y_j(t_0 - (t-t_0)).
\]

(3) For \( t_1 < t \leq 1 \),
\[
    r_j(t) = \sum_{i \in I_1} x_{ij}(t) + \sum_{i \in I_2} x_{ij}(t) = y_j + 2(1 - y_j) = 2 - y_j
\]

for any \( A_j(t) \), so
\[
    E[A_j(t)] \geq E[A_j(t_1)] - (2 - y_j) \int_{t_1}^{t} E[A_j(t')]dt',
\]

Let
\[
    f(t) = E[A_j(t_1)] - (2 - y_j) \int_{t_1}^{t} f(t')dt',
\]

then \( E[A_j(t_1)] = f(t_1) \) and
\[
    \frac{dE[A_j(t)]}{dt} \Big/ E[A_j(t)] \geq 2 - y_j = \frac{df(t)}{dt} / f(t)
\]

for \( t_1 < t \leq 1 \), so
\[
    E[A_j(t)] \geq f(t) = e^{-y_j(t_0 - (t_1-t_0)-(2-y_j)(t-t_1)}
\]

for \( t_1 < t \leq 1 \).

The match rate of a second-class edge after time \( t_1 \) is dependent on the status of another offline vertex at time \( t_1 \), so we also need the following lemma to bound the conditional matched probability.

**Lemma 8.** For any offline vertices \( j \neq j' \), any \( k \in \{0, 1\} \) and any time \( t_1 < t \leq 1 \),
\[
    E[A_j(t)|A_{j'}(t_1) = k] \geq e^{-y_j(t_0 - (t_1-t_0)-(2-y_j)(t-t_1)}.
\]

**Proof.** The argument is the same as that of Lemma 7. Since for \( 0 \leq t \leq t_1 \), \( r_j(t) \) is also independent of \( A_{j'}(t_1) \),
\[
    E[A_j(t_1)|A_{j'}(t_1) = k] = E[A_j(t_1)] = e^{-y_j(t_0 - (t_1-t_0)}. \]

For \( t_1 < t \leq 1 \), given \( A_{j'}(t_1) = k \), we still have \( r_j(t) \leq 2 - y_j \), so
\[
    E[A_j(t)|A_{j'}(t_1) = k] \geq E[A_j(t_1)|A_{j'}(t_1) = k] - (2 - y_j) \int_{t_1}^{t} E[A_j(t')|A_{j'}(t_1) = k]dt',
\]

so
\[
    E[A_j(t)|A_{j'}(t_1) = k] \geq e^{-y_j(t_0 - (t_1-t_0)-(2-y_j)(t-t_1)}. \]

Putting things together, we can compute the matched probability for each edge to prove our main theorem:
**Theorem 9.** Multistage Suggested Matching is 0.645-competitive for edge-weighted online stochastic matching.

**Proof.** We define the competitive ratio on each edge \((i, j)\) to be the ratio of its final matched probability to \(x_{ij}\). Since the offline optimal can be bounded by \(\sum_{(i,j) \in E} w_{ij} x_{ij}\), we only need to prove that the competitive ratio on every edge is at least 0.645. 

For any first-class edge \((i, j)\), by Facts 5 and 3, since \(r_{ij}(t) = x_{ij}\) is independent of \(A_j(t)\), the matched probability of this edge is \(\int_0^1 x_{ij} E[A_j(t)] dt\). So the competitive ratio on this edge is:

\[
\int_0^1 E[A_j(t)] dt = \int_0^{t_0} E[A_j(t)] dt + \int_{t_0}^{t_1} E[A_j(t)] dt + \int_{t_1}^1 E[A_j(t)] dt \\
\geq \int_0^{t_0} e^{-y_j t} dt + \int_{t_0}^{t_1} e^{-y_j t_0 - (t-t_0)} dt + \int_{t_1}^1 e^{-y_j t_0 - (t-t_0) - (2-y_j)(t-t_1)} dt \quad \text{(Lemma 7)} \\
= \frac{1}{y_j} (1 - e^{-y_j t_0}) + e^{-y_j t_0} (1 - e^{-(t-t_0)}) + \frac{1}{2 - y_j} e^{-y_j t_0 - (t-t_0)} (1 - e^{-(2-y_j)(1-t_1)}).
\]

Now consider any second-class edge \((i, j)\). Let \(j'\) be the other neighbor of online vertex type \(i\). By Fact 5, the matched probability of this edge is:

\[
\int_0^1 E[r_{ij}(t) A_j(t)] dt = \int_0^{t_0} E[r_{ij}(t) A_j(t)] dt + \int_{t_0}^{t_1} E[r_{ij}(t) A_j(t)] dt + \int_{t_1}^1 E[r_{ij}(t) A_j(t)] dt.
\]

By Fact 3, since \(r_{ij}(t)\) is independent of \(A_j(t)\) for \(0 \leq t \leq t_1\), we have

\[
\int_0^{t_0} E[r_{ij}(t) A_j(t)] dt = 0
\]

and

\[
\int_{t_0}^{t_1} E[r_{ij}(t) A_j(t)] dt = \int_{t_0}^{t_1} x_{ij} E[A_j(t)] dt.
\]

By Fact 4, since \(r_{ij}(t)\) is only dependent of \(A_j(t_1)\) for \(t_1 < t \leq 1\),

\[
\int_{t_1}^1 E[r_{ij}(t) A_j(t)] dt = E[A_j(t_1)] \int_{t_1}^1 x_{ij} E[A_j(t)|A_j(t_1) = 1] dt + (1 - E[A_j(t_1)]) \int_{t_1}^1 2 x_{ij} E[A_j(t)|A_j(t_1) = 0] dt.
\]

So the competitive ratio on this edge is:

\[
\int_{t_0}^{t_1} E[A_j(t)] dt + E[A_j(t_1)] \int_{t_1}^1 E[A_j(t)|A_j(t_1) = 1] dt + 2(1 - E[A_j(t_1)]) \int_{t_1}^1 E[A_j(t)|A_j(t_1) = 0] dt \\
\geq \int_{t_0}^{t_1} e^{-y_j t_0 - (t-t_0)} dt + \left(2 - e^{-y_j t_0 - (t-t_0)}\right) \int_{t_1}^1 e^{-y_j t_0 - (t-t_0) - (2-y_j)(t-t_1)} dt \quad \text{(Lemmas 7 and 8)} \\
\geq e^{-y_j t_0} (1 - e^{-(t-t_0)}) + \left(2 - e^{-(t-t_0)}\right) \frac{1}{2 - y_j} e^{-y_j t_0 - (t-t_0)} (1 - e^{-(2-y_j)(1-t_1)}).
\]

Taking \(t_0 = 0.05\) and \(t_1 = 0.75\), under Eqn. (5), the minimum values of both ratios are achieved at \(y_j = 1 - \ln 2\). The proof is basic but tedious calculus, so we defer it to Appendix A. When \(y_j = 1 - \ln 2\), both of them are at least 0.645. \(\square\)

As the ratios are complicated, we enumerate the values of the variables to get a nearly optimal solution. Further adjustment can only improve the ratio on the 4th digit after the decimal point.
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A Missing Proof in Theorem 9

Define:
\[
a(x) = \int_{0}^{t_0} e^{-tx} \, dt = \frac{1}{x} (1 - e^{-\frac{7}{10}x}),
\]
\[
b(x) = \int_{t_0}^{t_1} e^{-t_0x-(t-t_0)} \, dt = e^{-\frac{7}{20}} (1 - e^{-\frac{7}{10}}),
\]
\[
c(x) = \int_{t_1}^{1} e^{-t_0x-(t_1-t_0)-(2-x)(t-t_1)} \, dt = \frac{1}{2-x} e^{-\frac{7}{20}} \frac{7}{10} (1 - e^{-\frac{2}{4-x}}).
\]

Then we only need to prove that \(a(x) + b(x) + c(x)\) and \(b(x) + (2 - e^{-\frac{7}{10}})c(x)\) are decreasing in \(x \in [0, 1 - \ln 2]\). Apparently, \(a(x)\) and \(b(x)\) are decreasing, so it suffices to prove:
\[
-b'(x) > (2 - e^{-\frac{7}{10}})c'(x),
\]
i.e.:
\[
\frac{1}{20} e^{-\frac{7}{10}} (1 - e^{-\frac{7}{10}}) > (2 - e^{-\frac{7}{10}}) \frac{1}{2-x} e^{-\frac{7}{20}} \frac{7}{10} (1 - e^{-\frac{2}{4-x}}) \left(\frac{1}{2-x} - \frac{1}{20}\right) - \frac{1}{4} e^{-\frac{2}{4-x}}.
\]

After rearranging terms, we get:
\[
(2-x)(e^{\frac{7}{10}} - 1) > 20(2 - e^{-\frac{7}{10}}) \left(\frac{1}{2-x} - \frac{1}{20}\right) - \frac{1}{4} e^{-\frac{2}{4-x}}.
\]
The left-hand-side achieves minimum value \(1.716\ldots\) at \(x = 1 - \ln 2\). When \(x \in [0, \frac{1-\ln 2}{2}]\), the right-hand-side is no more than:
\[
20(2 - e^{-\frac{7}{10}}) \left(\frac{1}{2-x} - \frac{1}{20}\right) - \frac{1}{4} e^{-\frac{2}{4-x}} = 1.256\ldots
\]
When \(x \in [\frac{1-\ln 2}{2}, 1 - \ln 2]\), the right-hand-side is no more than:
\[
20(2 - e^{-\frac{7}{10}}) \left(\frac{1}{2-x} - \frac{1}{20}\right) - \frac{1}{4} e^{-\frac{2}{4-x}} = 1.272\ldots
\]
So the inequality follows.