UNIMODALITY OF THE BETTI NUMBERS FOR HAMILTONIAN CIRCLE ACTION WITH ISOLATED FIXED POINTS

YUNHYUNG CHO AND MIN KYU KIM*

ABSTRACT. Let $(M, \omega)$ be an eight-dimensional closed symplectic manifold equipped with a Hamiltonian circle action with only isolated fixed points. In this article, we will show that the Betti numbers of $M$ are unimodal, i.e. $b_0(M) \leq b_2(M) \leq b_4(M)$.

1. INTRODUCTION

Let $(M, \omega, J)$ be an $n$-dimensional closed Kähler manifold. Then $(M, \omega, J)$ satisfies the hard Lefschetz property so that the Betti numbers are unimodal, i.e.

$$b_i(M) \leq b_{i+2}(M)$$

for all $i < n$. In symplectic case, the unimodality of the Betti numbers is obviously not clear in general. In this paper, we will consider the following conjectural question which is addressed in [JHKLM].

Conjecture 1.1. Let $(M, \omega)$ be a closed symplectic manifold with a Hamiltonian circle action. Assume that all fixed points are isolated. Then the Betti numbers are unimodal.

The reason why we put the condition “isolated fixed points” is that, as far as the authors know, all known examples of Hamiltonian circle action with only isolated fixed points admit a Kähler structure. In particular, Y. Karshon [Ka] proved that every symplectic 4-manifold with Hamiltonian circle action with only isolated fixed points is symplectomorphic to some smooth projective toric variety. In this paper, we will show that

Theorem 1.2. Let $(M, \omega)$ be an 8-dimensional closed symplectic manifold equipped with a Hamiltonian circle action with only isolated fixed points. Then the Betti numbers of $M$ are unimodal, i.e. $b_0(M) \leq b_2(M) \leq b_4(M)$.

We would like to give a remark that our method to approach this problem is purely “topological” in the sense that we do not use any geometric structure, like an almost complex structure or metric. Moreover, we cannot be sure whether our method does work in higher dimensional cases.

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2. Equivariant cohomology

In this section, we briefly review an elementary equivariant cohomology theory and the localization theorem for a circle action which will be used in Section 3. Let $S^1$ be a unit circle group and let $M$ be an $S^1$-manifold. Then the equivariant cohomology $H^*_S(M)$ is defined by

$$H^*_S(M) := H^*(M \times S^1, ES^1)$$

where $ES^1$ is a contractible space on which $S^1$ acts freely. Since $M \times S^1, ES^1$ has a natural $M$-bundle structure over the classifying space $BS^1 := ES^1/S^1$, the equivariant cohomology $H^*_S(M)$ admits a $H^*(BS^1)$-module structure. Note that $H^*(BS^1; \mathbb{R})$ is isomorphic to the polynomial ring $\mathbb{R}[u]$ where $u$ is of degree two. For the fixed point set $M^{S^1}$, the inclusion map $i : M^{S^1} \hookrightarrow M$ induces a $H^*(BS^1)$-algebra homomorphism

$$i^* : H^*_S(M) \rightarrow H^*_S(M^{S^1}) \cong \bigoplus_{F \in M^{S^1}} H^*(F) \otimes H^*(BS^1)$$

and we call $i^*$ a restriction map to the fixed point set. Note that for an inclusion $i_F : F \hookrightarrow M^{S^1}$, it induces a natural projection $i_F^* : H^*_S(M^{S^1}) \rightarrow H^*_S(F) \cong H^*(F) \otimes H^*(BS^1)$. For every $\alpha \in H^*_S(M)$, we will denote by $\alpha|_F$ an image $i_F^*(\alpha)$. The main technique for proving Theorem 2.3 is the following, which is called Atiyah-Bott-Berlin-Vergne localization theorem.

**Theorem 2.1** (A-B-B-V localization theorem). Let $M$ be a compact manifold with a circle action with isolated fixed points. Let $\alpha \in H^*_S(M)$. Then as an element of $\mathbb{Q}(u)$, we have

$$\int_M \alpha = \sum_{F \in M^{S^1}} \frac{\alpha|_F}{e_F}$$

where the sum is taken over all fixed points, and $e_F$ is the equivariant Euler class of the normal bundle to $F$.

**Remark 2.2.** Sometimes, the integral $\int_M$ is called an integration along the fiber $M$. In fact, as a Cartan model, every equivariant cohomology class can be written as a sum of the form $x \otimes u^k \in H^*_S(M) \cong H^*(M) \otimes H^*(BS^1)$ and the operation $\int_M$ acts on the ordinary cohomology factor. Hence if $\alpha \in H^*_S(M)$ is of degree less than a dimension of $M$, then we have

$$\int_M \alpha = \sum_{F \in M^{S^1}} \frac{\alpha|_F}{e_F} = 0.$$

When our manifold has a symplectic structure $\omega$ and the action preserves $\omega$, then the equivariant cohomology satisfies a remarkable property as follows.

**Theorem 2.3.** [K] Let $(M, \omega)$ be a closed symplectic manifold and $S^1$ act on $(M, \omega)$ in a Hamiltonian fashion. Then the restriction map $i^*$ to the fixed point set is injective.

Theorem 2.3 enables us to study the ring structure of $H^*_S(M)$ more easily via the restriction map. For instance, assume that all fixed points are isolated. Then $H^*_S(M^{S^1})$
is nothing but \( \bigoplus_{F \in \mathbb{S}^1} H^*(BS^1) \cong \bigoplus_{F \in \mathbb{S}^1} \mathbb{R}[u] \) where \( u \) is a degree two generator of \( H^*(BS^1) \). Hence we can think of an element \( f \in H^*_S(M) \) as a function \( i^*(f) \) from the fixed point set \( M^S \) to the polynomial ring \( \mathbb{R}[u] \) with one-variable \( u \). Also, for any elements \( f \) and \( g \) of \( H^*_S(M) \), the product \( f \cdot g \) can be computed by studying \( i^*(f \cdot g) \), which is simply the product of \( i^*(f) \) and \( i^*(g) \) component-wise.

Now, consider a Hamiltonian \( S^1 \)-manifold \( (M, \omega) \) with a moment map \( H : M \to \mathbb{R} \). Then we may construct an equivariant symplectic class on the Borel construction as follows. For \([\text{McT}]\) and \( S \). Tolman found a remarkable family of equivariant cohomology classes as follows. In \( Y \) the restriction of \( H^*_S(M) \) to the polynomial ring \( \mathbb{R}[u] \) by a sub-bundle of \( \nu_F \) each connected fixed component \( F \) are isolated, let \( \mathbb{S}^1 \) be the product of \( \text{tonian circle action with a moment map } \) the product space \( \theta \) we may construct an equivariant symplectic class on the Borel construction as follows. For \([\text{Au}]\) Let \( (M, \omega) \) be a fixed component of \( M^S \) such that \( \mathbb{S}^1 \)-invariant and vanishes on the fiber \( S^1 \) over \( M \times \mathbb{S}^1 \), \( ES^1 \). So we may push-forward \( \omega_H \) to the Borel construction \( M \times \mathbb{S}^1 \), \( ES^1 \), and denote by \( \widetilde{\omega}_H \) the push-forward of \( \omega_H \). Obviously, the restriction of \( \widetilde{\omega}_H \) on each fiber \( M \) is precisely \( \omega \) and we call a class \( [\widetilde{\omega}_H] \in H^2_{S^1}(M) \) an equivariant symplectic class with respect to \( H \). By definition of \( \widetilde{\omega}_H \), we have the following proposition.

**Proposition 2.4.** \([\text{Au}]\) Let \( F \in M^S \) be an isolated fixed point of the given Hamiltonian circle action. Then we have

\[
[\widetilde{\omega}_H]|_F = H(F)u.
\]

**3. Main Theorem**

Let \( (M, \omega) \) be a closed symplectic manifold, \( S^1 \) be the unit circle group acting on \( (M, \omega) \) in a Hamiltonian fashion, and \( H : M \to \mathbb{R} \) be a moment map for the given action. For each connected fixed component \( F \subset M^S \), let \( k_F \) be the index of \( F \) with respect to \( H \). Let \( \nu_F \) be a normal bundle of \( F \) in \( M \). Then the negative normal bundle \( \nu_F^+ \) of \( F \) is defined by a sub-bundle of \( \nu_F \) whose fiber is contained in an unstable submanifold of \( M \) at \( F \) with respect to \( H \). We denote by \( e_F \in H^*_S(F) \) the equivariant Euler class of \( \nu_F^+ \). \( \text{D. McDuff and S. Tolman found a remarkable family of equivariant cohomology classes as follows.} \)**

**Theorem 3.1.** \([\text{McT}]\) Let \( (M, \omega) \) be a closed symplectic manifold equipped with a Hamiltonian circle action with a moment map \( H : M \to \mathbb{R} \). Let \( F \) be a fixed component of the action. Then given any cohomology class \( Y \in H^i(F) \), there exists the unique class \( \bar{Y} \in H^{i+k_F}(M) \) such that

1. the restriction of \( \bar{Y} \) to \( M^{<\mathbb{S}^1(F)} \) vanishes,
2. \( \bar{Y}|_F = Y \cup e_F \), and
3. the degree of \( \bar{Y}|_F \in H^*_S(F) \) is less than the index \( k_F \) of \( F \) for all fixed components \( F' \neq F \).

We call such class \( \bar{Y} \) a canonical class with respect to \( Y \). In this case when all fixed points are isolated, let \( F \) be a fixed point of index \( k_F \) and \( 1_F \in H^0(F) \) be the identity element of \( H^0(F) \). Then Theorem 3.1 implies that there exists the unique class \( \alpha_F \in H^*_{S^1}(M) \) such that

1. \( \alpha_F|_{F'} = 0 \) for every \( F' \in M^S \) with \( H(F') < H(F) \).
\begin{align*}
(2) \quad & \alpha_F|_F = e_F = \prod_{i=1}^{k_F} w_i^{-1} u, \text{ where } w_i^{-1} \text{ is the negative weight of } S^1\text{-representation on } T_F M \\
& \text{for } i = 1, \cdots, k_F, \\
(3) \quad & \alpha_F|_{F'} = 0 \text{ for every } F' \neq F \in M^{S^1} \text{ with } k_{F'} \leq k_F.
\end{align*}

Now, we prove our main theorem.

\textbf{Proof of Theorem 1.2.} By the connectivity of \( M \), the first inequality is obvious. Now, let’s assume that \( b_2(M) > b_4(M) \). Let \( z_1, \cdots, z_k \) be the fixed points of index 2. Also, we denote by \( \alpha_i \in H^2_{S^1}(M) \) the canonical class with respect to \( z_i \) where \( k = b_2(M) \). Then our assumption \( b_2(M) > b_4(M) \) implies that there is a non-zero class \( \alpha = \sum_{i=1}^k c_i \alpha_i \) such that \( \alpha|_F = 0 \) for every fixed point \( F \) of index 4. In other words, \( \alpha \) can survive only on the fixed points of index 2, 6, or 8. Now, consider an equivariant symplectic class \( \tilde{\omega}_H \) such that the maximum of \( H \) is zero. If we let \( \beta = \alpha^2 \cdot \tilde{\omega}_H \), then for each fixed point \( F \), the restriction \( \beta|_F = (\alpha|_F)^2 \cdot H(F) u \) by Proposition 2.4. Since we took our moment map satisfying \( \max H = 0 \), we have \( H(F) < 0 \) for every fixed point \( F \) which is not maximal. Hence all coefficients of \( \beta|_F \) is non-positive. In particular, \( \beta|_F \) is non-zero for some fixed point \( F \) of index 2. Hence \( \beta \) is non-zero class of degree 6 in \( H^*_{S^1}(M) \) and survives only on the fixed points of index 2 or 6. Applying the localization theorem to \( \beta \), we have

\[
0 = \int_M \beta = \sum_{F \in M^{S^1}} \frac{\beta|_F}{e_F} = \sum_{F \in M^{S^1}, \text{isol}(F) = 2} \frac{\beta|_F}{e_F} + \sum_{F \in M^{S^1}, \text{isol}(F) = 6} \frac{\beta|_F}{e_F}.
\]

Since the coefficient of \( u^4 \) of \( e_F \) is negative for every fixed point \( F \) of index 2 or 6, it is a contradiction.

\( \square \)

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\textbf{Department of Mathematical Sciences, KAIST, 335 Gwahangno, Yu-sung Gu, Daejeon 305-701, Korea}

\textit{E-mail address: yh.cho@kaist.ac.kr}

\textbf{Department of Mathematics Education, Gyeongin National University of Education, San 59-12, Gyesan-dong, Gyeyang-gu, Incheon, 407-753, Korea}

\textit{E-mail address: mkkim@kias.re.kr}