THE ALGEBRAIC STRUCTURES OF COMPLEX INTUITIONISTIC FUZZY SOFT SETS ASSOCIATED WITH GROUPS AND SUBGROUPS

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Abstract. In recent years, the theory of complex fuzzy sets has captured the attention of many researchers, and research in this area has intensified in the past five years. This paper focuses on developing the algebraic structures pertaining to groups and subgroups for the complex intuitionistic fuzzy soft set model. Besides examining some of the properties of these structures, the relationship between these structures and corresponding structures in fuzzy group theory is also examined.

1. Introduction

Uncertainty, imprecision and vagueness are characteristics that are pervasive in problems occurring in the real world and these features cannot be handled effectively using mathematical tools that are traditionally used to deal with uncertainties and vagueness. Some of the pioneering theories used to deal with these limitations are fuzzy set theory [1], intuitionistic fuzzy set theory [2] and soft set theory [3]. To overcome the problems that are inherent in each of these theories, researchers have chosen to combine these theories to develop new fuzzy based hybrid models. The more well-known among these include fuzzy soft sets [4], intuitionistic fuzzy soft sets [5], interval-valued fuzzy soft sets [6], interval-valued intuitionistic fuzzy soft sets [7], and vague soft sets [8]. Although all of the above-mentioned theories are able to handle the uncertainties and fuzziness that exists in the data, all of these models are not able to handle the periodicity or seasonality that exists in many real-life problems. This led to the introduction of the complex fuzzy set model in [9], and subsequently the development and extension of this theory.

The notion of complex sets stems from the concept of complex numbers which is a primary concept in solving problems especially in the field of engineering. Complex sets has the ability to solve many problems that cannot be solved using traditional mathematical concepts such as number theory, probability theory and fuzzy set theory. Examples of these instances include solving the improper integrals that are used to represent resistance in electrical engineering.
and also representing the phase or wave-like qualities in two-dimensional problems. This led to
the notion of complex fuzzy sets in [9] which is an improved and extended version of ordinary
fuzzy sets. Kumar & Bajaj [10] then proposed the notion of complex intuitionistic fuzzy soft
sets (CIFSS) which combines the characteristics and advantages of complex sets, soft sets and
intuitionistic fuzzy sets in a single set. The CIFSS is parametric in nature and characterized
by an amplitude term which is equivalent to the membership and non-membership functions
in an ordinary IFSS, and a phase term which represents the seasonality and/or periodicity of
the elements. The novelty of CIFSS is manifested in the additional dimension of membership
which is the phase of the grade of membership. This feature gives CIFSS the added advantage
of being able to represent data or information which occurs repeatedly over a period of time,
which is often the case with problems that are two-dimensional in nature.

Although research pertaining to the theory of CFSs and other complex fuzzy based models
is still at its infancy, it has been steadily gaining momentum in recent years. As of now, almost
all of the work done in this area has revolved around the study of the theoretical properties of
CFSs, complex fuzzy computing and modeling, complex fuzzy logic, complex fuzzy optimization
and decision making, and the application of these in solving time-periodic problems. The phase
term in the structure of CFSs is the key defining feature of this model, and can be used to
model the seasonality and/or periodicity of time-periodic phenomena. However, this is not
the only interpretations for the phase term. Instead, the phase term can be used to represent
different aspects of the information, depending on the context of the scope of the problem or
area that is being studied. In most of the existing literature, the phase term has been used
to represent the time factor and seasonality of the problems, and has been applied in multi-
attribute decision making problems in a myriad of areas including supplier selection, economics,
pattern recognition, engineering and artificial intelligence.

The phase term can also be used to accurately represent the cycles present in fuzzy algebraic
structures. In the study of complex fuzzy algebraic theory, the fuzzy algebraic structures are
defined in a complex fuzzy setting, and therefore the structures consists of an amplitude term
and a phase term. The amplitude term is equivalent to the membership function in ordinary
fuzzy sets, whereas the phase term can be used to aptly represent the cycles of the algebraic
structures. For example, when dealing with fuzzy alternating groups, the different cycles can be
represented aptly and accurately using the phase term if the fuzzy alternating groups are defined in terms of CFSs or any complex fuzzy based models. This would make it easier to identify the different cycles and its corresponding membership functions in a systematic manner. The desire to utilize this unique ability of the phase term present in the CFS model and other complex fuzzy based models in the study of fuzzy algebra served as the main motivation to introduce and develop the theory of complex intuitionistic fuzzy soft groups in this paper. In this regard, we introduce and develop the notion of CIFS groups and other supporting algebraic structures for CIFSGs. The lack of proper research pertaining to the algebraic theory of complex fuzzy based models in the literature served as another motivation for the study done in this paper.

The rest of this paper is organized as follows. In Section 2, some important background information pertaining to the concepts introduced here are recapitulated. In Section 3, the algebraic structures of complex intuitionistic fuzzy subgroups and complex intuitionistic fuzzy soft groups are derived and the properties and structural characteristics of these algebraic structures are proposed and subsequently verified. The relationship between the structures introduced here and corresponding concepts in fuzzy group theory and classical group theory are also examined and verified in this section. In Section 4, normal complex intuitionistic fuzzy soft groups are proposed and the properties of this structure are discussed and verified. Concluding remarks are presented in Section 5, followed by acknowledgments and the list of references.

2. Preliminaries

In this section, we recapitulate some of the important background information pertaining to the development of the algebraic structures that will be proposed here.

2.1. Intuitionistic fuzzy sets.

An intuitionistic fuzzy set (abbr. IFS) [2] is an extension of the classical fuzzy set, and is characterized by a membership function and a non-membership function, each of which describes the degree of belongingness and non-belongingness of the elements with respect to each attribute. The concept of IFS was then further extended by incorporating the concept of soft set to derive the concept of intuitionistic fuzzy soft set (abbr. IFSS) [5].

In all that follows, $U$ shall be used to denote a universal set.
Definition 2.1. [2] Let \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\} \), where both \( \mu_A \) and \( \nu_A \) are functions from \( U \) to \([0, 1]\), satisfying \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \) for all \( x \in U \). Then \( A \) is called an \textit{intuitionistic fuzzy set} on \( U \), where \( \mu_A \) is the \textit{membership function} of \( A \), and \( \nu_A \) is the \textit{non-membership function} of \( A \).

Define \( \pi_A(x) = 1 - \mu_A - \nu_A \). Then, for each \( x_0 \in U \):

(i) the value of \( \mu_A(x_0) \) is called the \textit{degree of belongingness} of \( x_0 \) to \( A \);

(ii) the value of \( \nu_A(x_0) \) is called the \textit{degree of non-belongingness} of \( x_0 \) to \( A \);

(iii) the value of \( \pi_A(x_0) \) is called the \textit{degree of uncertainty} or \textit{indeterminacy} of \( x_0 \) to \( A \).

Henceforth, \( A \) and \( B \) shall be used to denote two intuitionistic fuzzy sets on \( U \), which are as defined below:

\[
A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\}, \quad B = \{(x, \mu_B(x), \nu_B(x)) : x \in U\}.
\]

Definition 2.2. [2] The \textit{subset} and \textit{equality} of \( A \) and \( B \) are as defined below:

(a) \( A \subseteq B \), if \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) for all \( x \in U \).

(b) \( A = B \), if \( A \subseteq B \) and \( B \subseteq A \).

Definition 2.3. [2] The \textit{complement}, \textit{union} and \textit{intersection} of \( A \) and \( B \) are as defined below:

(a) \( A = \{(x, \nu_A(x), \mu_A(x)) : x \in U\} \).

(b) \( A \cup B = \{(x, \max \{\mu_A(x), \mu_B(x)\}, \min \{\nu_A(x), \nu_B(x)\}) : x \in U\} \).

(b) \( A \cap B = \{(x, \min \{\mu_A(x), \mu_B(x)\}, \max \{\nu_A(x), \nu_B(x)\}) : x \in U\} \).

Definition 2.4. [2] The set \( \{x \in U : \nu_A(x) > 0\} \) is called the \textit{support} of \( A \) and is denoted by \( \mathcal{S}_A \). Moreover,

(a) \( A \) is said to be \textit{null} if \( \mathcal{S}_A = \emptyset \), otherwise it is said to be \textit{non-null}.

(b) \( A \) is said to be \textit{absolute} if \( \mathcal{S}_A = U \).

2.2. Soft sets and intuitionistic fuzzy soft sets.

Definition 2.5. [3] Let \( E \) be a set of parameters. Denote \( \varphi(U) \) to be the power set of \( U \), and let \( \mathcal{F} \) be a function from \( E \) to \( \varphi(U) \). Then the set of ordered pairs \( \{(\varepsilon, \mathcal{F}(\varepsilon)) : \varepsilon \in E, \mathcal{F}(\varepsilon) \in \varphi(U)\} \),
denoted by \((\mathcal{F}, E)\), is called a soft set on \(U\). Moreover, for each \(\varepsilon_0 \in E\), \(\mathcal{F}(\varepsilon_0)\) is called the set of \(\varepsilon_0\)-elements of \((\mathcal{F}, E)\), or the \(\varepsilon_0\)-approximate elements of \((\mathcal{F}, E)\).

**Definition 2.6.** [11] Let \(E\) be a set of parameters. Let \((\mathcal{F}, E)\) be a soft set on \(U\). Then the set \(\{\varepsilon \in E : \mathcal{F}(\varepsilon) \neq \emptyset\}\), denoted by \(\mathcal{S}(\mathcal{F}, E)\), is called the support of \((\mathcal{F}, E)\). Moreover, \((\mathcal{F}, E)\) is said to be null if \(\mathcal{S}(\mathcal{F}, E) = \emptyset\), otherwise it is said to be non-null.

**Definition 2.7.** [5] Let \(E\) be a set of parameters. Denote \(\text{IFS}(U)\) to be collection of all intuitionistic fuzzy sets on \(U\), and let \(\mathcal{F}\) be a function from \(E\) to \(\text{IFS}(U)\). Then the set of ordered pairs \(\{(\varepsilon, \mathcal{F}(\varepsilon)) : \varepsilon \in E, \mathcal{F}(\varepsilon) \in \text{IFS}(U)\}\), denoted by \((\mathcal{F}, E)\), is called an intuitionistic fuzzy soft set on \(U\).

**Definition 2.8.** [5] Let \((\mathcal{F}, E)\) be an intuitionistic fuzzy soft set on \(U\). Then the set \(\{\varepsilon \in E : \mathcal{F}(\varepsilon) \neq \emptyset\}\), denoted by \(\mathcal{S}(\mathcal{F}, E)\), is called the support of \((\mathcal{F}, E)\). Moreover, \((\mathcal{F}, E)\) is said to be null if \(\mathcal{S}(\mathcal{F}, E) = \emptyset\), otherwise it is said to be non-null.

**Definition 2.9.** [5] Let \((\mathcal{F}_1, E_1)\) and \((\mathcal{F}_2, E_2)\) be two intuitionistic fuzzy soft sets on \(U\). Then \((\mathcal{F}_1, E_1)\) is an intuitionistic fuzzy soft subset of \((\mathcal{F}_2, E_2)\), denoted as \((\mathcal{F}_1, E_1) \subseteq (\mathcal{F}_2, E_2)\), if

(i) \(E_1 \subseteq E_2\).

(ii) \(\mathcal{F}_1(\varepsilon) \subseteq \mathcal{F}_2(\varepsilon)\) for all \(\varepsilon \in \mathcal{S}(\mathcal{F}_1, E_1)\).

**Remark.** For each \(\varepsilon \in \mathcal{S}(\mathcal{F}_1, E_1)\), \(\mathcal{F}_1(\varepsilon)\) is non-null. So if \((\mathcal{F}_1, E_1) \subseteq (\mathcal{F}_2, E_2)\), then \(\mathcal{F}_1(\varepsilon) \subseteq \mathcal{F}_2(\varepsilon)\), and we also have \(\mathcal{F}_2(\varepsilon)\) being non-null, which implies \(\varepsilon \in \mathcal{S}(\mathcal{F}_2, E_2)\). As a result, the condition \(\mathcal{S}(\mathcal{F}_1, E_1) \subseteq \mathcal{S}(\mathcal{F}_2, E_2)\) follows.

**Definition 2.10.** [5] Let \((\mathcal{F}_1, E_1)\) and \((\mathcal{F}_2, E_2)\) be two intuitionistic fuzzy soft sets on \(U\). Define \(R = E_1 \cup E_2\), \(S = E_1 \cap E_2\); and for all \(\varepsilon \in S\), \(\mathcal{H}(\varepsilon) = \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon)\) and \(\mathcal{K}(\varepsilon) = \mathcal{F}_1(\varepsilon) \cap \mathcal{F}_2(\varepsilon)\).

\[
\mathcal{H}(\varepsilon) = \begin{cases} 
\mathcal{F}_1(\varepsilon), & \varepsilon \in E_1 - S \\
\mathcal{F}_2(\varepsilon), & \varepsilon \in E_2 - S, \text{ and } \mathcal{K}(\varepsilon) = \begin{cases} 
\mathcal{F}_1(\varepsilon), & \varepsilon \in E_1 - S \\
\mathcal{F}_2(\varepsilon), & \varepsilon \in E_2 - S.
\end{cases}
\end{cases}
\]

Then

(i) \((\mathcal{H}, R)\) is called the union of \((\mathcal{F}_1, E_1)\) and \((\mathcal{F}_2, E_2)\), and is denoted as

\[(\mathcal{H}, R) = (\mathcal{F}_1, E_1) \sqcup (\mathcal{F}_2, E_2);\]
(ii) \((K, R)\) is called the intersection of \((F_1, E_1)\) and \((F_2, E_2)\), and is denoted as 
\[
(K, R) = (F_1, E_1) \cap (F_2, E_2);
\]
(iii) \((\overline{H}, S)\) is called the restricted union of \((F_1, E_1)\) and \((F_2, E_2)\), and is denoted as 
\[
(\overline{H}, S) = (F_1, E_1) \hat{\cup} (F_2, E_2);
\]
(iv) \((\overline{K}, S)\) is called the restricted intersection of \((F_1, E_1)\) and \((F_2, E_2)\), and is denoted as 
\[
(\overline{K}, S) = (F_1, E_1) \hat{\cap} (F_2, E_2);
\]

2.3. Complex fuzzy sets.

In this section, an overview of the concept of complex fuzzy sets (abbr. CFS) [9] and complex intuitionistic fuzzy soft sets (abbr. CIFSS) [10] are presented. Since the introduction of CFS, attempts to improve and overcome the drawbacks that are inherent in the CFS model has led to the introduction of several complex fuzzy based hybrid models. We refer the readers to [10, 12-18] for more details on these models.

**Definition 2.11.** [9] A complex fuzzy set \(A\) defined on a universe of discourse \(U\) is characterized by a membership function \(\mu_A(x)\) that assigns a complex-valued grade of membership in \(A\) to any element \(x \in U\). By definition, all values of \(\mu_A(x)\) lie within the unit circle in the complex plane and are expressed by 
\[
\mu_A(x) = r_A(x) e^{i \omega_A(x)},
\]
where \(i = \sqrt{-1}\), \(r_A(x)\) and \(\omega_A(x)\) are both real-valued, \(r_A(x) \in [0, 1]\) and \(\omega_A(x) \in (0, 2\pi]\). A complex fuzzy set \(A\) is thus of the form 
\[
A = \{(x, \mu_A(x)) : x \in U\} = \{(x, r_A(x) e^{i \omega_A(x)}) : x \in U\}.
\]

Henceforth, the symbol \(i\) is used to denote the imaginary unit \(\sqrt{-1}\), whereas the symbol \(O_1\) will be used to denote \(\{z \in \mathbb{C} : |z| \leq 1\}\). Up till Section 2.2, we have reached the concept of intuitionistic fuzzy soft sets (IFSS), which involves the relations \(\geq, \leq, \max, \min\) on the outcomes of membership and non-membership functions of the IFSS model. Such relations are inherently defined for real numbers only. On the other hand, a CFS and all its generalizations have membership and non-membership functions that can lie anywhere in \(O_1\). We must therefore generalize the concept of \(\geq, \leq, \max, \min\) for all complex numbers in \(O_1\). To achieve these, we give the following definitions and lemmas.

**Definition 2.12.** Let \(\mu = re^{i \omega}\) and \(\nu = \tau e^{i \psi}\), with \(r, \tau \in [0, 1]\) and \(\omega, \psi \in (0, 2\pi]\). We define the relations \(\geq\) and \(\leq\) as follows:
(i) \( \mu \geq \nu \), when both \( r \geq \tau \) and \( \omega \geq \psi \), or when \( \nu = 0 \).

(ii) \( \mu \leq \nu \), when both \( r \leq \tau \) and \( \omega \leq \psi \), or when \( \mu = 0 \).

**Remark.** The usual definition of \( \geq \) and \( \leq \) on the real interval \([0, 1]\) is a special case of this definition. However, there remain pairs of elements of \( O_1 \), to whom neither \( \geq \) nor \( \leq \) can be established between them, such as \( 0.1e^{3i} \) and \( 0.4e^{2i} \), because \( 0.1 < 0.4 \) but \( 3 > 2 \). Nonetheless, \( 0 \leq \mu \leq 1 \) still holds for all \( \mu \in O_1 \).

**Definition 2.13.** Let \( S = \{\mu_n : n \in V\} \subseteq O_1 \). Then max \( S \) and min \( S \) are as defined below.

(i) (a) \( \max S \geq \mu_n \) for all \( n \in V \).

(b) If \( \xi \in O_1 \) is such that \( \xi \geq \mu_n \) for all \( n \in V \), then \( \xi \geq \max S \).

(ii) (a) \( \min S \leq \mu_n \) for all \( n \in V \).

(b) If \( \zeta \in O_1 \) is such that \( \zeta \leq \mu_n \) for all \( n \in V \), then \( \zeta \leq \max S \).

**Remark.** Unlike subsets of \( \mathbb{R} \), max \( S \) and min \( S \) may not be in \( S \), even if \( S \) is finite. For example, if \( S_0 = \{0.1e^{3i}, 0.4e^{2i}\} \), then \( \max S_0 = 0.4e^{3i} \) and \( \min S_0 = 0.1e^{2i} \).

**Definition 2.14.** Let \( \mu = re^{i\omega} \), with \( r \in [0, 1] \) and \( \omega \in (0, 2\pi] \). The **complement** of \( \mu \), denoted as \( 1 \sim \mu \), is defined as \( 1 \sim \mu = (1 - r)e^{i\omega'} \), where

\[
\omega' = \begin{cases} 
2\pi - \omega, & \omega < 2\pi \\
\omega, & \omega = 2\pi
\end{cases}
\]

**Remark.** If \( \mu \in [0, 1] \), then \( 1 \sim \mu = 1 - \mu \).

**Lemma 2.15.** For all \( \mu \in O_1 \), \( 1 \sim (1 \sim \mu) = \mu \).

**Remark.** Let \( \mu = re^{i\omega} \), with \( r \in [0, 1] \) and \( \omega \in (0, 2\pi] \). Then \( |\mu| = r \).

**Lemma 2.16.** For all \( \mu \in O_1 \), \( |1 \sim \mu| = 1 - |\mu| \).

2.4. Complex intuitionistic fuzzy soft sets.

The object of study in this paper is the CIFSS model [10] which is an adaptation of the original CFS model [9]. It is a hybrid between complex fuzzy sets, intuitionistic fuzzy sets and soft sets that is characterized by a membership and non-membership function that represents the degree of belongingness and non-belongingness of the elements with respect to the attributes that are under consideration.
**Definition 2.17.** [10] Let $E$ be a set of parameters, $\text{CIFS}(U)$ denote the collection of all complex intuitionistic fuzzy sets on $U$, and $\tilde{F}$ be a function from $E$ to $\text{CIFS}(U)$. Then the set of ordered pairs $\left\{ (\varepsilon, \tilde{F}(\varepsilon)) : \varepsilon \in E, \tilde{F}(\varepsilon) \in \text{CIFS}(U) \right\}$, denoted by $\left( \tilde{F}, E \right)$, is called a **complex intuitionistic fuzzy soft set (CIFSS)** on $U$. Note that, for each $\varepsilon \in E$,

$$\tilde{F}(\varepsilon) = \left\{ (x, \mu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(x)) : x \in U \right\} = \left\{ (x, r_{\tilde{F}(\varepsilon)}(x)e^{i\omega_{\tilde{F}(\varepsilon)}(x)}, \tau_{\tilde{F}(\varepsilon)}(x)e^{i\nu_{\tilde{F}(\varepsilon)}(x)}) : x \in U \right\}$$

In all that follows, let $\text{CIFSS}(U)$ denote the collection of all complex intuitionistic fuzzy soft sets on a universe $U$. Furthermore, we write $\left( \tilde{F}, E \right) \in \text{CIFSS}(U)$ to denote that $\left( \tilde{F}, E \right)$ is a complex intuitionistic fuzzy soft set on $U$.

**Definition 2.18.** Let $\left( \tilde{F}, E \right) \in \text{CIFSS}(U)$. Then the set $\left\{ \varepsilon \in E : \tilde{F}(\varepsilon) \text{ is non} - \text{null} \right\}$, denoted by $\mathcal{S}(\tilde{F}, E)$, is called the **support** of $\left( \tilde{F}, E \right)$. Moreover, $\left( \tilde{F}, E \right)$ is said to be **null** if $\mathcal{S}(\tilde{F}, E) = \emptyset$, otherwise it is said to be **non-null**.

**Definition 2.19.** [10] Let $\left( \tilde{F}_1, E_1 \right), \left( \tilde{F}_2, E_2 \right) \in \text{CIFSS}(U)$. Then $\left( \tilde{F}_1, E_1 \right)$ is an **complex intuitionistic fuzzy soft subset** of $\left( \tilde{F}_2, E_2 \right)$, denoted as $\left( \tilde{F}_1, E_1 \right) \subseteq \left( \tilde{F}_2, E_2 \right)$, if

(i) $E_1 \subseteq E_2$.

(ii) $\tilde{F}_1(\varepsilon) \subseteq \tilde{F}_2(\varepsilon)$ for all $\varepsilon \in \mathcal{S}(\tilde{F}_1, E_1)$.

**Remark.** When $\left( \tilde{F}_1, E_1 \right) \subseteq \left( \tilde{F}_2, E_2 \right)$ for each $\varepsilon \in \mathcal{S}(\tilde{F}_1, E_1)$, $\tilde{F}_1(\varepsilon)$ is non-null. Since $\tilde{F}_1(\varepsilon) \subseteq \tilde{F}_2(\varepsilon)$, we also have $\tilde{F}_2(\varepsilon)$ being non-null, which implies $\varepsilon \in \mathcal{S}(\tilde{F}_2, E_2)$. As a result, the condition $\mathcal{S}(\tilde{F}_1, E_1) \subseteq \mathcal{S}(\tilde{F}_2, E_2)$ follows.

**Definition 2.20.** [10] Let $\left( \tilde{F}, E \right) \in \text{CIFSS}(U)$. Then the **complement** of $\left( \tilde{F}, E \right)$, denoted as $\left( \tilde{F}, E \right)^c$ is defined as $\left( \tilde{F}, E \right)^c = \left( \tilde{F}^c, -E \right)$, where $\tilde{F}^c$ is a function from $-E$ to $\text{CIFS}(U)$ given by

$$\tilde{F}^c(-\varepsilon) = \left\{ (x, \nu_{\tilde{F}^c(-\varepsilon)}(x), \mu_{\tilde{F}^c(-\varepsilon)}(x)) : x \in U \right\} = \left\{ (x, \nu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(x)) : x \in U \right\},$$

for all $-\varepsilon \in -E$

**Remark.** Definition 2.20 can be restated as follows:
Let \((\tilde{F}, E)\) \(\in\) CIFSS\((U)\). Define \(\tilde{T}\) as a function from \(E\) to CIFSS\((U)\), where \(\tilde{T}(\varepsilon)\) is the complement of \(\tilde{F}(\varepsilon)\) for all \(\varepsilon \in E\). Then \((\tilde{T}, E)\) is called the complement of \((\tilde{F}, E)\) and this can be denoted as \((\tilde{T}, E) = (\tilde{F}, E)^c\).

Note that, for each \(\varepsilon \in E\), \(\tilde{T}(\varepsilon) = \{ (x, \nu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(x)) : x \in U \}\), in line with Definition 2.3

**Definition 2.21.** [10] Let \((\tilde{F}_1, E_1), (\tilde{F}_2, E_2) \in\) CIFSS\((U)\). Define \(R = E_1 \cup E_2, S = E_1 \cap E_2\); and for all \(\varepsilon \in S\), \(\mathcal{H}(\varepsilon) = \tilde{F}_1(\varepsilon) \cup \tilde{F}_2(\varepsilon)\) and \(\mathcal{K}(\varepsilon) = \tilde{F}_1(\varepsilon) \cap \tilde{F}_2(\varepsilon)\).

\[
\mathcal{H}(\varepsilon) = \begin{cases} 
\tilde{F}_1(\varepsilon), & \varepsilon \in E_1 - S \\
\tilde{F}_2(\varepsilon), & \varepsilon \in E_2 - S,
\end{cases}
\quad
\mathcal{K}(\varepsilon) = \begin{cases} 
\tilde{F}_1(\varepsilon), & \varepsilon \in E_1 - S \\
\tilde{F}_2(\varepsilon), & \varepsilon \in E_2 - S.
\end{cases}
\]

Then

(i) \((\mathcal{H}, R)\) is called the union of \((\tilde{F}_1, E_1)\) and \((\tilde{F}_2, E_2)\), and is denoted as \((\mathcal{H}, R) = (\tilde{F}_1, E_1) \cup (\tilde{F}_2, E_2)\);

(ii) \((\mathcal{K}, R)\) is called the intersection of \((\tilde{F}_1, E_1)\) and \((\tilde{F}_2, E_2)\), and is denoted as \((\mathcal{K}, R) = (\tilde{F}_1, E_1) \cap (\tilde{F}_2, E_2)\);

(iii) \((\overline{\mathcal{H}}, S)\) is called the restricted union of \((\tilde{F}_1, E_1)\) and \((\tilde{F}_2, E_2)\), and is denoted as \((\overline{\mathcal{H}}, S) = (\tilde{F}_1, E_1) \cup (\tilde{F}_2, E_2)\);

(iv) \((\overline{\mathcal{K}}, S)\) is called the restricted intersection of \((\tilde{F}_1, E_1)\) and \((\tilde{F}_2, E_2)\), and is denoted as \((\overline{\mathcal{K}}, S) = (\tilde{F}_1, E_1) \cap (\tilde{F}_2, E_2)\);

We now define two new operations for the CIFSS model, namely the \((\alpha, \beta)\)-level set and the characteristic set of a CIFSS, and provide some properties of these operations. The formal definitions of these operations and the properties of these operations are as given below.

**Definition 2.22.** Let \((\tilde{F}, E) \in\) CIFSS\((U)\), and \(\alpha, \beta \in O_1\). The \((\alpha, \beta)\)-level set of \((\tilde{F}, E)\), denoted as \((\tilde{F}, E)_{(\alpha, \beta)}\), is a soft set on \(U\) defined below:

\[
(\tilde{F}, E)_{(\alpha, \beta)} = \left\{ (\varepsilon, \tilde{F}_{(\alpha, \beta)}(\varepsilon)) : \varepsilon \in E, \tilde{F}_{(\alpha, \beta)}(\varepsilon) \in \varphi(U) \right\},
\]

where \(\tilde{F}_{(\alpha, \beta)}(\varepsilon) = \{ x \in U : \mu_{\tilde{F}(\varepsilon)}(x) \geq \alpha, \nu_{\tilde{F}(\varepsilon)}(x) \leq \beta \}\) for all \(\varepsilon \in E\).
Remark. Consider the particular case where 

\[
\tilde{F}(x) = \begin{cases} 
\varepsilon \in E, \tilde{F}_\alpha(\varepsilon) \in \wp(U) 
\end{cases},
\]

where \(\tilde{F}_\alpha(\varepsilon) = \{x \in U : \mu_{\tilde{F}}(x) \geq \alpha, \nu_{\tilde{F}}(x) \leq \alpha\}\) for all \(\varepsilon \in E\).

**Definition 2.23.** Let \((\tilde{F}, E) \in \text{CIFSS}(U)\), and \(S\) be a non-null proper subset of \(U\). If \(\{\mu_{\tilde{F}(\varepsilon)} : \varepsilon \in E\} = \mu_0\) and \(\{\nu_{\tilde{F}(\varepsilon)} : \varepsilon \in E\} = \nu_0\), in which

\[
\mu_0(x) = \begin{cases} 
re^{i\omega}, \quad x \in S \\
1 \sim re^{i\omega}, \quad x \in U - S
\end{cases}
\]

and \(\nu_0(x) = \begin{cases} 
\tau e^{i\psi}, \quad x \in S \\
1 \sim \tau e^{i\psi}, \quad x \in U - S
\end{cases}\),

where \(\omega + \psi \in \{2\pi, 4\pi\}\) and \(re^{i\omega} \geq \tau e^{i\psi}\), then \((\tilde{F}, E)\) is said to be characteristic over \(S\).

**Remark.** Consider the particular case where \(U = \{p, q\}\) and \(\mu_0(x) = \nu_0(x) = \frac{1}{2}\) for all \(x \in U\).

Note that \(re^{i\omega} = \tau e^{i\psi} = re^{i\omega} = \tau e^{i\psi} = \frac{1}{2} \in O_1\), and \(\omega + \psi = 4\pi\), \(r = \tau\) and \(\omega = \psi\). But \(S\) can be either \(\{p\}\) or \(\{q\}\). We now have an example of \((\tilde{F}, E)\) being characteristic over more than one non-null proper subsets of \(U\).

**Proposition 2.24.** Let \((\tilde{F}, E) \in \text{CIFSS}(U)\), and \((\tilde{F}, E)\) be characteristic over \(S\), in which

\[
\mu_0(x) = \begin{cases} 
re^{i\omega}, \quad x \in S \\
1 \sim re^{i\omega}, \quad x \in U - S
\end{cases}
\]

and \(\nu_0(x) = \begin{cases} 
\tau e^{i\psi}, \quad x \in S \\
1 \sim \tau e^{i\psi}, \quad x \in U - S
\end{cases}\)

are the membership and non-membership functions of \(\tilde{F}(\varepsilon)\), respectively. Then:

(i) \(r + \tau = 1\).

(ii) \(re^{i\omega} \geq 1 \sim re^{i\omega}\) and \(\tau e^{i\psi} \leq 1 \sim \tau e^{i\psi}\).

**Proof.** (i) Note that \(\mu_0\) and \(\nu_0\) are the membership and the non-membership functions of \(\tilde{F}(\varepsilon)\), which is a complex intuitionistic fuzzy set. Then the condition \(0 \leq |\mu_0(x)| + |\nu_0(x)| \leq 1\) holds for all \(x \in U\). We now have both \(0 \leq r + \tau \leq 1\) and \(0 \leq |1 \sim re^{i\omega}| + |1 \sim \tau e^{i\psi}| \leq 1\) by Definition 2.23. From Lemma 2.16, it follows that:

\[
|1 \sim re^{i\omega}| + |1 \sim \tau e^{i\psi}| = (1 - r) + (1 - \tau) = 2 - (r + \tau),
\]

cauising \(0 \leq 2 - (r + \tau) \leq 1\), and therefore \(1 \leq r + \tau \leq 2\), which implies \(r + \tau = 1\).
(ii) As $\left(\tilde{\mathcal{F}}, E\right)$ is characteristic over $S$, $\omega + \psi \in \{2\pi, 4\pi\}$ and $re^{i\omega} \geq \tau e^{i\psi}$. Since $r \geq \tau$ and $r + \tau = 1$, it follows that $r \geq \frac{1}{2}$ and $\tau \leq \frac{1}{2}$. Thus we have $1 - r \leq \frac{1}{2}$ and $1 - \tau \geq \frac{1}{2}$. These further imply that $r \geq 1 - r$ and $\tau \leq 1 - \tau$.

Now suppose $\omega + \psi = 2\pi$. Since $\omega \geq \psi$, it follows that $\omega \geq \pi$ and $\psi \leq \pi$. We now have $1 - \omega \leq \pi$ and $1 - \psi \geq \pi$, which implies that $\omega \geq 2\pi - \omega$ and $\psi \leq 2\pi - \psi$. Also note that both $\omega, \psi < 2\pi$. By Definition 2.14, we have $re^{i\omega} \geq (1 - r)e^{i(2\pi - \omega)} = 1 \sim re^{i\omega}$ and $\tau e^{i\psi} \leq (1 - \tau)e^{i(2\pi - \psi)} = 1 \sim \tau e^{i\psi}$.

On the other hand, if $\omega + \psi = 4\pi$, then $\omega = \psi = 2\pi$. Then by Definition 2.14, we have $re^{i\omega} \geq (1 - r)e^{i\omega} = 1 \sim re^{i\omega}$ and $\tau e^{i\psi} \leq (1 - \tau)e^{i\psi} = 1 \sim \tau e^{i\psi}$.

\[\Box\]

3. Complex intuitionistic fuzzy soft groups

The study of soft algebra and fuzzy soft algebra were initiated by Aktas & Cagman [19] and Aygunoglu & Aygun [20], respectively. Other researchers such as Feng et al. [11], Acar et al. [21], Inan and Ozturk [22] and Ghosh and Samanta [23] also contributed to the development of these areas. Besides this, many more advanced algebraic structures pertaining to groups, rings and hemirings of fuzzy soft sets have been introduced in literature. Some of the latest works include the introduction of soft fuzzy rough rings and ideals by Zhu [24], I-fuzzy soft groups by Vimala et al. [25], soft union set characterizations of hemirings by Zhan [26], and neutrosophic normal soft groups by Bera and Mahapatra [27]. Yamak et al. [28], Leoreanu-Fotea et al. [29], and Selvachandran and Salleh [30-34] on the other hand, were responsible for introducing the algebraic structures of soft hyperrroupoids, fuzzy soft hypergroups as well as soft hyperrings, fuzzy soft hyperrings, vague soft hypergroups and hyperrings, respectively. Khan, Farooq and Davvaz [35] proposed the notion of soft interior hyperideals of ordered semihypergroups, whereas Ma et al. [36] studied the concept of rough soft hyperrings.

Research in the area of complex fuzzy algebra is still at its infancy. The study of complex fuzzy algebraic theory was initiated by Al-Husban, Salleh and Ahmad [37, 38] through the introduction of the algebraic structures of complex fuzzy subrings and complex fuzzy rings in [37] and [38], respectively. Al-Husban and Salleh [39] then defined the notion of a complex fuzzy group which is defined in a complex fuzzy space, instead of an ordinary universe of discourse. AlSarahead and Ahmad [40, 41] then proposed the structures of complex fuzzy subgroups and
complex fuzzy soft groups in [40] and [41], respectively. To the best of our knowledge, these are the only published works in this area of research at present.

The aim of this section is to establish the novel concept of complex intuitionistic fuzzy soft groups (CIFS-groups) in Rosenfeld’s sense (i.e. using the concept of a fuzzy subgroup of a group defined by Rosenfeld [42]). The properties and structural characteristics of the proposed algebraic structures are then examined and subsequently verified.

Henceforth, the symbol $G$ will be used to denote a group.

**Definition 3.1.** [19] Let $(\mathcal{F}, E)$ be a non-null soft set on $G$. Then $(\mathcal{F}, E)$ is said to be a soft group on $G$ if $\mathcal{F}(\varepsilon) \subseteq G$ for all $\varepsilon \in \mathcal{S}(\mathcal{F}, E)$.

**Remark.** As in classical group theory, a null set cannot be a group, a null soft set on $G$ is not a soft group on $G$.

Now, we define the notion of a complex intuitionistic fuzzy subgroup of a group $G$, and then use this to define the notion of a complex intuitionistic fuzzy soft group of a group $G$.

**Definition 3.2.** Let $M = (x, \mu_M(x), \nu_M(x)) : x \in G$ be a complex intuitionistic fuzzy set on $G$. Then $M$ is said to be a complex intuitionistic fuzzy subgroup (abbr. CIF-subgroup) of $G$, if the following conditions holds for all $x, y \in G$:

(i) $\mu_M(xy) \geq \min \{\mu_M(x), \mu_M(y)\},$

(ii) $\nu_M(xy) \leq \max \{\nu_M(x), \nu_M(y)\},$

(iii) $\mu_M(x^{-1}) \geq \mu_M(x),$

(vi) $\nu_M(x^{-1}) \leq \nu_M(x),$

Moreover, let $M$ and $N$ be two complex intuitionistic fuzzy subgroups of $G$, with $M \subseteq N$. In this case, $M$ is also said to be a complex intuitionistic fuzzy subgroup of $N$.

**Definition 3.3.** Let $\left(\tilde{\mathcal{F}}, E\right) \in \text{CIFSS}(G)$. Then $\left(\tilde{\mathcal{F}}, E\right)$ is said to be a complex intuitionistic fuzzy soft group (abbr. CIFS-group) on $G$ if $\tilde{\mathcal{F}}(\varepsilon)$ is a complex intuitionistic fuzzy subgroup of $G$ for all $\varepsilon \in \mathcal{S}(\tilde{\mathcal{F}}, E)$.

In all that follows, CIFSG($G$) denotes the collection of all complex intuitionistic fuzzy soft groups on a group $G$, and $\left(\tilde{\mathcal{F}}, E\right) \in \text{CIFSG}(G)$ denotes that $\left(\tilde{\mathcal{F}}, E\right)$ is a complex intuitionistic fuzzy soft group on $G$. 
Example 3.4. Consider the case where $G$ is the symmetric group of order 3, that is $G = S_3 = \{1, (12), (23), (13), (123), (132)\}$. Next consider a set of parameters $E = \{a, b\}$. We define $\mu_1 = 0.4e^i, \mu_2 = 0.4e^{2i}, \mu_3 = 0.7e^{3i}$; and $\nu_1 = 0.2e^{4i}, \nu_2 = 0.1e^{4i}, \nu_3 = 0.1e^{3i}$ Note that $\mu_1 \leq \mu_2 \leq \mu_3$ and $\nu_1 \geq \nu_2 \geq \nu_3$. Now we consider two CIFSSs of $G$, which are as defined below.

(i) \( \tilde{F}, E \) \( = \{ \tilde{F}(a), \tilde{F}(b) \} \),

where \( \tilde{F}(a) = \left\{ (1, \mu_3, \nu_3), ((12), \mu_1, \nu_1), ((13), \mu_1, \nu_1), ((23), \mu_1, \nu_1) \right\} \) and

\[ \tilde{F}(b) = \left\{ (1, \mu_3, \nu_3), ((12), \mu_2, \nu_2), ((13), \mu_1, \nu_1), ((23), \mu_1, \nu_1) \right\} \]

(ii) \( \tilde{G}, E \) \( = \{ \tilde{G}(a), \tilde{G}(b) \} \),

where \( \tilde{G}(a) = \tilde{F}(a) \) and

\[ \tilde{G}(b) = \left\{ (1, \mu_1, \nu_1), ((12), \mu_2, \nu_2), ((13), \mu_2, \nu_2), ((23), \mu_1, \nu_1) \right\} \]

\( \tilde{G}(b) = \left\{ (123), \mu_1, \nu_1), ((132), \mu_1, \nu_1) \right\} \]

From this it can be verified that \( \left( \tilde{F}, E \right) \in \text{CIFSS}(G) \), whereas \( \left( \tilde{G}, E \right) \notin \text{CIFSS}(G) \).

Definition 3.5. Let \( \left( \tilde{F}_1, E_1 \right), \left( \tilde{F}_2, E_2 \right) \in \text{CIFSS}(G) \). Then \( \left( \tilde{F}_1, E_1 \right) \) is said to be a complex intuitionistic fuzzy soft subgroup (abbr. CIFSS-subgroup) of \( \left( \tilde{F}_2, E_2 \right) \) if the following conditions are satisfied:

(i) \( E_1 \subseteq E_2 \),

(ii) For all \( \epsilon \in E_1 \), \( \tilde{F}_1(\epsilon) \) is a complex intuitionistic fuzzy subgroup of \( \tilde{F}_2(\epsilon) \).

Proposition 3.6. Let \( \left( \tilde{F}, E \right) \in \text{CIFSS}(G) \), and \( 1_G \) be the identity element of \( G \). Then the following results hold for all \( \epsilon \in E \) and for all \( x \in G \)

(i) \( \mu_{\tilde{F}(\epsilon)}(x^{-1}) = \mu_{\tilde{F}(\epsilon)}(x) \) and \( \nu_{\tilde{F}(\epsilon)}(x^{-1}) = \nu_{\tilde{F}(\epsilon)}(x) \),

(ii) \( \mu_{\tilde{F}(\epsilon)}(1_G) \geq \mu_{\tilde{F}(\epsilon)}(x) \) and \( \nu_{\tilde{F}(\epsilon)}(1_G) \leq \nu_{\tilde{F}(\epsilon)}(x) \).

Proof. Let \( \epsilon \in E \), and let \( x \in G \). By Definition 3.3, \( \tilde{F}(\epsilon) \) is a CIFS-subgroup of \( G \), which enables us to utilize Definition 3.2 for proving both (i) and (ii).

(i) Both \( \mu_{\tilde{F}(\epsilon)}(x^{-1}) \geq \mu_{\tilde{F}(\epsilon)}(x) \) and \( \nu_{\tilde{F}(\epsilon)}(x^{-1}) \leq \nu_{\tilde{F}(\epsilon)}(x) \) follow directly from Definition 3.2. Since \( x \in G \), we also have \( x^{-1} \in G \). Thus it follows that

\[ \mu_{\tilde{F}(\epsilon)}(x) = \mu_{\tilde{F}(\epsilon)}(x^{-1})^{-1} \geq \mu_{\tilde{F}(\epsilon)}(x^{-1}) \]

and

\[ \nu_{\tilde{F}(\epsilon)}(x) = \nu_{\tilde{F}(\epsilon)}(x^{-1})^{-1} \leq \nu_{\tilde{F}(\epsilon)}(x^{-1}) \]
due to Definition 3.2.

(ii) Note that \(1_G = xx^{-1}\), so the conditions \(\mu_{\tilde{F}(e)}(1_G) \geq \min \{ \mu_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(x^{-1}) \}\), and 
\(\nu_{\tilde{F}(e)}(1_G) \leq \max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(x^{-1}) \}\), follow from Definition 3.2. By (i), we have 
\[
\min \{ \mu_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(x^{-1}) \} = \min \{ \mu_{\tilde{F}(e)}(y), \mu_{\tilde{F}(e)}(y) \} = \mu_{\tilde{F}(e)}(x)
\]
and 
\[
\max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(x^{-1}) \} = \max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(x) \} = \nu_{\tilde{F}(e)}(x).
\]

This completes the proof. \(\square\)

**Proposition 3.7.** Let \((\tilde{F}, E) \in \text{CIFSS}(G)\). Then \((\tilde{F}, E) \in \text{CIFSG}(G)\), if and only if the following conditions are satisfied for all \(\varepsilon \in E\) and for all \(x, y \in G\):

(i) \(\mu_{\tilde{F}(e)}(xy^{-1}) \geq \min \{ \mu_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(y) \}\)

(ii) \(\nu_{\tilde{F}(e)}(xy^{-1}) \leq \max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(y) \}\)

**Proof.** (\(\Rightarrow\)) Suppose \((\tilde{F}, E) \in \text{CIFSG}(G)\).

Let \(\varepsilon \in E\), and \(x, y \in G\). Then \(y^{-1} \in G\) too, and from Definition 3.3, \(\tilde{F}(e)\) is a CIF-subgroup of \(G\). The conditions \(\mu_{\tilde{F}(e)}(xy^{-1}) \geq \min \{ \mu_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(y^{-1}) \}\) and \(\nu_{\tilde{F}(e)}(xy^{-1}) \leq \max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(y^{-1}) \}\) follow directly from Definition 3.2. Similarly, we have \(\mu_{\tilde{F}(e)}(y^{-1}) \geq \mu_{\tilde{F}(e)}(y)\) and \(\nu_{\tilde{F}(e)}(y^{-1}) \leq \nu_{\tilde{F}(e)}(y)\), which implies that:

\[
\min \{ \mu_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(y^{-1}) \} \geq \min \{ \mu_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(y) \}
\]

and

\[
\max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(y^{-1}) \} \leq \max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(y) \}.
\]

Thus, conditions (i) and (ii) now follow.

(\(\Leftarrow\)) Suppose conditions (i) and (ii) are satisfied for all \(\varepsilon \in E\) and for all \(x, y \in G\).

By considering the case \(x = y\), we have 
\[
\mu_{\tilde{F}(e)}(1_G) = \mu_{\tilde{F}(e)}(yy^{-1}) \geq \min \{ \mu_{\tilde{F}(e)}(y), \mu_{\tilde{F}(e)}(y) \} = \mu_{\tilde{F}(e)}(y)
\]
and
\[
\nu_{\tilde{F}(e)}(1_G) = \nu_{\tilde{F}(e)}(xx^{-1}) \leq \max \{ \nu_{\tilde{F}(e)}(x), \nu_{\tilde{F}(e)}(x) \} = \nu_{\tilde{F}(e)}(x).
\]

These imply that:
\[
\mu_{\tilde{F}(e)}(y^{-1}) = \mu_{\tilde{F}(e)}(1_Gy^{-1}) \geq \min \{ \mu_{\tilde{F}(e)}(1_G), \mu_{\tilde{F}(e)}(y) \} = \mu_{\tilde{F}(e)}(y)
\]
Thus, \( \tilde{F}(\varepsilon) \) is proved to be a CIF-subgroup of \( G \) for all \( \varepsilon \in E \), and hence, it follows that \( (\tilde{F}, E) \in \text{CIFSG}(G) \).

**Proposition 3.8.** Let \( (\tilde{F}, E) \in \text{CIFSG}(G) \), and \( \alpha, \beta \in O_1 \). If \( (\tilde{F}, E)_{(\alpha, \beta)} \) is non-null, then it is a soft group on \( G \).

*Proof.* The proof is straightforward. \( \square \)

**Proposition 3.9.** Let \( S \) be a non-null subset of \( G \) and \( (\tilde{F}, E) \in \text{CIFSS}(G) \), where \( (\tilde{F}, E) \) is characteristic over \( S \). If \( (\tilde{F}, E) \in \text{CIFSG}(G) \), then \( S \) is a classical subgroup of \( G \).

*Proof.* Let \( x, y \in S \). Then by Definition 2.23, \( \{ \mu_{\tilde{F}(\varepsilon)} : \varepsilon \in E \} = \mu_0 \) and \( \{ \nu_{\tilde{F}(\varepsilon)} : \varepsilon \in E \} = \nu_0 \), in which there exist \( \alpha, \beta \in O_1 \), with \( \alpha \geq \beta \), such that \( \mu_0(x) = \mu_0(y) = \alpha \) and \( \nu_0(x) = \nu_0(y) = \beta \). Thus it follows that \( x, y \in \tilde{F}_{(\alpha, \beta)}(\varepsilon) \) for all \( \varepsilon \in E \), which further implies that \( \tilde{F}_{(\alpha, \beta)}(\varepsilon) \) is not empty for all \( \varepsilon \in E \). Therefore, we have \( \mathcal{F}(\tilde{F}, E)_{(\alpha, \beta)} = E \). Since \( \mathcal{F}(\tilde{F}, E)_{(\alpha, \beta)} \) is not empty, \( (\tilde{F}, E)_{(\alpha, \beta)} \) is non-null, and \( (\tilde{F}, E)_{(\alpha, \beta)} \) is therefore a soft group in \( G \).

Take \( \varepsilon_0 \in E \). From Definition 3.1, it follows that \( \tilde{F}_{(\alpha, \beta)}(\varepsilon_0) \) is a subgroup of \( G \), and therefore we have \( xy^{-1} \in \tilde{F}_{(\alpha, \beta)}(\varepsilon_0) \). Then by Definition 2.22, \( \mu_{\tilde{F}(\varepsilon_0)}(xy^{-1}) \geq \alpha \) and \( \nu_{\tilde{F}(\varepsilon_0)}(xy^{-1}) \leq \beta \). Recall that \( \{ \mu_{\tilde{F}(\varepsilon)} : \varepsilon \in E \} = \mu_0 \) and \( \{ \nu_{\tilde{F}(\varepsilon)} : \varepsilon \in E \} = \nu_0 \). As a result, \( \mu_0(xy^{-1}) \geq \alpha \) and \( \nu_0(xy^{-1}) \leq \beta \).

We now show that both \( \mu_0(xy^{-1}) = \alpha \) and \( \nu_0(xy^{-1}) = \beta \), which in turn implies that \( xy^{-1} \in S \).

We write \( \alpha = re^{i\omega} \) and \( \beta = re^{i\psi} \), for some \( r, \tau \in [0, 1] \) and \( \omega, \psi \in (0, 2\pi] \). Then \( \omega + \psi \in \{2\pi, 4\pi\} \) follows because of Definition 2.23. Furthermore, recall that \( r + \tau = 1 \) (and thus \( |\alpha| + |\beta| = 1 \)) due to Proposition 2.24.

(a) By Definition 2.23, it is either \( \mu_0(xy^{-1}) = \alpha \) or \( \mu_0(xy^{-1}) = 1 \sim \alpha \). Suppose \( \mu_0(xy^{-1}) = 1 \sim \alpha \), then \( 1 \sim \alpha \geq \alpha \). Since \( 1 \sim \alpha = (1 - r)e^{i\omega} \geq re^{i\omega} = \alpha \), it follows that

\[
\nu_{\tilde{F}(\varepsilon)}(y^{-1}) = \nu_{\tilde{F}(\varepsilon)}(1gy^{-1}) \leq \max \{ \nu_{\tilde{F}(\varepsilon)}(1), \nu_{\tilde{F}(\varepsilon)}(y) \} = \nu_{\tilde{F}(\varepsilon)}(y).
\]

By considering \( y^{-1} \in G \), it follows that:

\[
\mu_{\tilde{F}(\varepsilon)}(xy) \geq \min \{ \mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y^{-1}) \} \geq \min \{ \mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y) \}
\]

\[
\nu_{\tilde{F}(\varepsilon)}(xy) \leq \max \{ \nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y^{-1}) \} \leq \max \{ \nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y) \}.
\]
\[|\beta| = 1 - |\alpha| = |1 \sim \alpha| \geq |\alpha| \geq |\beta|. \] This further implies \(|1 \sim \alpha| = |\alpha|\) and therefore \(1 - r = r\). So, we now have \(1 \sim \alpha = r e^{i\omega'}\), and there are two cases to be considered:

(i) If \(\omega = 2\pi\), then \(\omega' = \omega\) and therefore \(1 \sim \alpha = r e^{i\omega'} = r e^{i\omega} = \alpha\). Hence \(\mu_0(xy^{-1}) = \alpha\) follows.

(ii) If \(\omega < 2\pi\), we have both \(\omega' = (2\pi - \omega)\) and \(\omega + \psi = 2\pi\). Note that \(\omega \geq \psi\) because of \(\alpha \geq \beta\). As a result, \(\omega \geq \pi\) follows. On the other hand, since \(1 \sim \alpha = r e^{i(2\pi - \omega)} \geq r e^{i\omega} = \alpha\), we have \((2\pi - \omega) \geq \omega\) which implies \(\pi \geq \omega\). Thus, we now have \(\omega = \pi\), resulting in \(\omega' = \pi = \omega\) and therefore \(1 \sim \alpha = r e^{i\omega'} = r e^{i\omega} = \alpha\). Hence \(\mu_0(xy^{-1}) = \alpha\) again follows.

(b) By Definition 2.23, it is either \(\nu_0(xy^{-1}) = \beta\) or \(\nu_0(xy^{-1}) = 1 \sim \beta\). Suppose \(\nu_0(xy^{-1}) = 1 \sim \beta\), then \(1 \sim \beta \leq \beta\). Since \(1 \sim \beta = (1 - \tau)e^{i\psi} \leq \tau e^{i\psi} = \beta\), it follows that \(|\alpha| = 1 - |\beta| = |1 \sim \beta| \leq |\beta| \leq |\alpha|\). This further implies \(|1 \sim \beta| = |\beta|\) and therefore \(1 - \tau = \tau\). Thus we now have \(1 \sim \beta = \tau e^{i\psi'}\), and there are two cases to be considered:

(i) If \(\psi = 2\pi\), then \(\psi' = \psi\) and therefore \(1 \sim \beta = \tau e^{i\psi'} = \tau e^{i\psi} = \beta\). Hence \(\nu_0(xy^{-1}) = \beta\) follows.

(ii) If \(\psi < 2\pi\), we have both \(\psi' = (2\pi - \psi)\) and \(\omega + \psi = 2\pi\). Note that \(\omega \leq \psi\) because of \(\beta \leq \alpha\). As a result, \(\psi \leq \pi\) follows. On the other hand, since \(1 \sim \beta = \tau e^{i(2\pi - \psi)} \geq \tau e^{i\psi} = \beta\), so \((2\pi - \psi) \leq \psi\) which implies \(\pi \leq \psi\). Thus, we now have \(\psi = \pi\), resulting in \(\psi' = \pi = \psi\) and therefore \(1 \sim \beta = \tau e^{i\psi'} = \tau e^{i\psi} = \beta\). Hence \(\nu_0(xy^{-1}) = \beta\) again follows.

Therefore, we obtain \(xy^{-1} \in S\) whenever \(x, y \in S\). As such, it can be concluded that \(S\) is a classical subgroup of \(G\). \(\square\)

**Theorem 3.10.** Let \(S\) be a non-null subset of \(G\) and \((\tilde{F}, E) \in \text{CIFSS}(G)\), where \((\tilde{F}, E)\) is characteristic over \(S\). Then, \((\tilde{F}, E) \in \text{CIFSG}(G)\) if and only if \(S\) is a classical subgroup of \(G\).

**Proof.** In Proposition 3.9, it has already been proved that \(S\) is a classical subgroup of \(G\) whenever \((\tilde{F}, E) \in \text{CIFSG}(G)\). Therefore, it suffices to prove that \((\tilde{F}, E) \in \text{CIFSG}(G)\) whenever \(S\) is a classical subgroup of \(G\).

Since \((\tilde{F}, E) \in \text{CIFSS}(G)\) and \((\tilde{F}, E)\) is characteristic over \(S\), by Definition 2.23, we have \(\{\mu_{\tilde{F}(e)\varepsilon} : \varepsilon \in E\} = \mu_0\) and \(\{\nu_{\tilde{F}(e)\varepsilon} : \varepsilon \in E\} = \nu_0\), in which
\[ \mu_0(x) = \begin{cases} r e^{i\omega}, & x \in S \\ 1 \sim r e^{i\omega}, & x \in U - S \end{cases} \quad \text{and} \quad \nu_0(x) = \begin{cases} \tau e^{i\psi}, & x \in S \\ 1 \sim \tau e^{i\psi}, & x \in U - S \end{cases} \]

with \( \omega + \psi \in \{2\pi, 4\pi\} \) and \( r e^{i\omega} \geq \tau e^{i\psi} \).

Now, let \( \varepsilon \in E \) and \( x, y \in G \). Then both \( \{\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y), \mu_{\tilde{F}(\varepsilon)}(xy^{-1})\} \subseteq \{r e^{i\omega}, 1 \sim r e^{i\omega}\} \) and \( \{\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y), \nu_{\tilde{F}(\varepsilon)}(xy^{-1})\} \subseteq \{\tau e^{i\psi}, 1 \sim \tau e^{i\psi}\} \). Furthermore, by Proposition 2.24, \( r e^{i\omega} \geq 1 \sim r e^{i\omega} \) and \( \tau e^{i\psi} \leq 1 \sim \tau e^{i\psi} \), which imply that \( \min\{r e^{i\omega}, 1 \sim r e^{i\omega}\} = 1 \sim r e^{i\omega} \) and \( \max\{\tau e^{i\psi}, 1 \sim \tau e^{i\psi}\} = 1 \sim \tau e^{i\psi} \), respectively.

Without loss of generality, suppose \( x \in G - S \). Then \( \mu_{\tilde{F}(\varepsilon)}(x) = 1 \sim r e^{i\omega} \) and \( \nu_{\tilde{F}(\varepsilon)}(x) = 1 \sim \tau e^{i\psi} \) which causes \( \min\{\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y)\} = 1 \sim r e^{i\omega} \) and \( \max\{\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y)\} = 1 \sim \tau e^{i\psi} \), respectively. Since \( \mu_{\tilde{F}(\varepsilon)}(xy^{-1}) \in \{r e^{i\omega}, 1 \sim r e^{i\omega}\} \) and \( \nu_{\tilde{F}(\varepsilon)}(xy^{-1}) \in \{\tau e^{i\psi}, 1 \sim \tau e^{i\psi}\} \), we conclude that

\[ \mu_{\tilde{F}(\varepsilon)}(xy^{-1}) \geq \min\{r e^{i\omega}, 1 \sim r e^{i\omega}\} = 1 \sim r e^{i\omega} = \min\{\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y)\} \]

and

\[ \nu_{\tilde{F}(\varepsilon)}(xy^{-1}) \leq \max\{\tau e^{i\psi}, 1 \sim \tau e^{i\psi}\} = 1 \sim \tau e^{i\psi} = \max\{\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y)\} \]

Now let \( x, y \in S \). Since \( S \) is a classical subgroup of \( G \), \( xy^{-1} \in S \), and therefore it follows that \( \{\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y), \mu_{\tilde{F}(\varepsilon)}(xy^{-1})\} \subseteq \{r e^{i\omega}\} \) and \( \{\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y), \nu_{\tilde{F}(\varepsilon)}(xy^{-1})\} \subseteq \{\tau e^{i\psi}\} \). As a result we have \( \mu_{\tilde{F}(\varepsilon)}(xy^{-1}) = r e^{i\omega} = \min\{\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y)\} \) and \( \nu_{\tilde{F}(\varepsilon)}(xy^{-1}) = \tau e^{i\psi} = \max\{\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y)\} \).

Hence the conditions

\[ \mu_{\tilde{F}(\varepsilon)}(xy^{-1}) \geq \min\{\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y)\} \]

and

\[ \nu_{\tilde{F}(\varepsilon)}(xy^{-1}) \leq \max\{\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y)\} \]

are shown to be satisfied for all \( \varepsilon \in E \) and \( x, y \in G \). This proves that \( \tilde{F}, E \) \( \in \) CIFSG(\( G \)) \( \square \)

**Theorem 3.11.** Let \( \tilde{F}, E \) \( \in \) CIFSS(\( G \)), where \( \tilde{F}, E \) is non-null. Then the following statements are equivalent:

(i) \( \tilde{F}, E \) \( \in \) CIFSG(\( G \)).

(ii) For all \( \alpha, \beta \in O_1 \), either \( \tilde{F}, E \) \( \in \) CIFSG(\( G \)), or \( \tilde{F}, E \) is a soft group on \( G \).
Proof. (i) $\Rightarrow$ (ii) Take any arbitrary $\alpha, \beta \in O_1$. By Proposition 3.8, if $(\tilde{F}, E)_{(\alpha, \beta)}$ is non-null, then it is a soft group on $G$. Thus, statement (ii) is proved to be true.

(ii) $\Rightarrow$ (i) Let $\varepsilon \in E$ and $x, y \in G$, and note that $\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y), \nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y) \in O_1$. Take $\alpha = \min \left\{ \mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y) \right\}$ and $\beta = \max \left\{ \nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y) \right\}$. Then we have $\mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y) \geq \alpha$ and $\nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y) \leq \beta$, which means that $x, y \in \tilde{F}_{(\alpha, \beta)}(\varepsilon)$. This implies that $(\tilde{F}, E)_{(\alpha, \beta)}$ is not null, and therefore it is a soft group on $G$. So we now have $\tilde{F}_{(\alpha, \beta)}(\varepsilon) \subseteq G$ and therefore $xy^{-1} \in \tilde{F}_{(\alpha, \beta)}(\varepsilon)$, which in turn implies that

$$
\mu_{\tilde{F}(\varepsilon)}(xy^{-1}) \geq \alpha = \min \left\{ \mu_{\tilde{F}(\varepsilon)}(x), \mu_{\tilde{F}(\varepsilon)}(y) \right\}
$$

and

$$
\nu_{\tilde{F}(\varepsilon)}(xy^{-1}) \leq \beta = \max \left\{ \nu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(y) \right\}.
$$

Hence by Proposition 3.7, statement (i) now follows.

$\square$

Theorem 3.12. Let $(\tilde{F}_1, E_1), (\tilde{F}_2, E_2) \in \text{CIFSG}(G)$. Then $(\tilde{F}_1, E_1) \cap (\tilde{F}_2, E_2) \in \text{CIFSG}(G)$ as well.

Proof. The proof is straightforward by Definition 2.21 and is therefore omitted. $\square$

Remark. This property also holds for the restricted intersection operation between CIFSSs.

Definition 3.13. Let $U_1, U_2$ be two universal sets, $\varphi : U_1 \rightarrow U_2$ be a function, $E, B$ be two sets of parameters, $(\tilde{T}, E) \in \text{CIFSS}(U_1)$ and $(\tilde{F}, B) \in \text{CIFSS}(U_2)$. Define $(\varphi(\tilde{T}), E) \in \text{CIFSS}(U_2)$ and $(\varphi^{-1}(\tilde{F}), B) \in \text{CIFSS}(U_1)$ as follows:

(i) $(\varphi(\tilde{T}), E)$ is such that for all $y \in U_2$ and $\varepsilon \in E$,

$$
\mu_{\varphi(\tilde{T})(\varepsilon)}(y) = \max \left\{ \mu_{\tilde{T}(\varepsilon)}(u) : u \in U_1, \varphi(u) = y \right\} \cup \{0\}
$$

and

$$
\nu_{\varphi(\tilde{T})(\varepsilon)}(y) = \min \left\{ \nu_{\tilde{T}(\varepsilon)}(u) : u \in U_1, \varphi(u) = y \right\} \cup \{1\}.
$$

(ii) $(\varphi^{-1}(\tilde{F}), B)$ is such that, for all $x \in U_1$ and $s \in B$, $\mu_{\varphi^{-1}(\tilde{F})(s)}(x) = \mu_{\tilde{F}(s)}(\varphi(x))$ and $\nu_{\varphi^{-1}(\tilde{F})(s)}(x) = \nu_{\tilde{F}(s)}(\varphi(x))$.

Theorem 3.14. Let $\varphi : G \rightarrow G'$ be a surjective group homomorphism. Let $(\tilde{T}, E) \in \text{CIFSG}(G)$ and $(\tilde{F}, B) \in \text{CIFSG}(G')$. Then:
Proof. (i) Let \( \varphi(\mathcal{T}), E \) \( \in \text{CIFSG}(G') \) provided that
\[
\max \left\{ \min \left\{ \mu_{\mathcal{T}(e)}(p), \mu_{\mathcal{T}(e)}(q) \right\} : p, q \in G \quad \varphi(p) = x, \varphi(q) = y \right\} \geq \min \left\{ \mu_{\mathcal{T}(e)}(x), \mu_{\mathcal{T}(e)}(y) \right\}
\]
and
\[
\min \left\{ \max \left\{ \nu_{\mathcal{T}(e)}(p), \nu_{\mathcal{T}(e)}(q) \right\} : p, q \in G \quad \varphi(p) = x, \varphi(q) = y \right\} \leq \max \left\{ \nu_{\mathcal{T}(e)}(x), \nu_{\mathcal{T}(e)}(y) \right\}
\]
for all \( x, y \in G' \).

(ii) Let \( \varphi^{-1}(\mathcal{F}), B \) \( \in \text{CIFSG}(G) \).

Proof. (i) Let \( x, y \in G' \) and \( \varepsilon \in E \). Then by Definition 3.13, we have \( \left( \varphi(\mathcal{T}), E \right) \in \text{CIFSS}(G') \), where \( \mu_{\varphi(\mathcal{T})}(xy^{-1}) = \max \left\{ \left\{ \mu_{\mathcal{T}(e)}(u) : u \in G, \varphi(u) = xy^{-1} \right\} \cup \{0\} \right\} \). Since \( \varphi \) is surjective, we have
\[
\max \left\{ \left\{ \mu_{\mathcal{T}(e)}(u) : u \in G, \varphi(u) = xy^{-1} \right\} \cup \{0\} \right\} = \max \left\{ \mu_{\mathcal{T}(e)}(u) : u \in G, \varphi(u) = xy^{-1} \right\}.
\]
Moreover, as \( \varphi \) is also a homomorphism, so
\[
\max \left\{ \mu_{\mathcal{T}(e)}(u) : u \in G, \varphi(u) = xy^{-1} \right\} \geq \max \left\{ \mu_{\mathcal{T}(e)}(pq^{-1}) : p, q \in G, \varphi(p) = x, \varphi(q) = y \right\}.
\]
Since \( (\mathcal{T}, E) \in \text{CIFSG}(G) \), \( \mu_{\mathcal{T}(e)}(pq^{-1}) \geq \min \left\{ \mu_{\mathcal{T}(e)}(p), \mu_{\mathcal{T}(e)}(q) \right\} \) for all \( p, q \in G \). This implies that:
\[
\max \left\{ \mu_{\mathcal{T}(e)}(pq^{-1}) : p, q \in G, \varphi(p) = x, \varphi(q) = y \right\} \geq \min \left\{ \mu_{\mathcal{T}(e)}(p), \mu_{\mathcal{T}(e)}(q) \right\} : p, q \in G, \varphi(p) = x, \varphi(q) = y \}
\]
\[
\geq \min \left\{ \mu_{\mathcal{T}(e)}(x), \mu_{\mathcal{T}(e)}(y) \right\}.
\]

Similarly for the non-membership function, we have the following:
\[
\nu_{\varphi(\mathcal{T})}(xy^{-1}) = \min \left\{ \left\{ \nu_{\mathcal{T}(e)}(u) : u \in G, \varphi(u) = xy^{-1} \right\} \cup \{0\} \right\}
\]
\[
= \min \left\{ \nu_{\mathcal{T}(e)}(u) : u \in G, \varphi(u) = xy^{-1} \right\}
\]
\[
\leq \min \left\{ \nu_{\mathcal{T}(e)}(pq^{-1}) : p, q \in G, \varphi(p) = x, \varphi(q) = y \right\}
\]
\[
\leq \min \left\{ \max \left\{ \nu_{\mathcal{T}(e)}(p), \nu_{\mathcal{T}(e)}(q) \right\} : p, q \in G, \varphi(p) = x, \varphi(q) = y \right\}
\]
\[
\leq \max \left\{ \nu_{\mathcal{T}(e)}(x), \nu_{\mathcal{T}(e)}(y) \right\}.
\]

(ii) Let \( x, y \in G \) and \( s \in B \). Then \( \left( \varphi^{-1}(\mathcal{F}), B \right) \in \text{CIFSS}(G) \) by Definition 3.13. As \( (\mathcal{F}, B) \in \text{CIFSG}(G') \), by applying Proposition 3.7, alongside with Definition 3.13, we obtain
the following:

\[
\mu_{\varphi^{-1}(\tilde{F})(s)}(xy^{-1}) = \mu_{\bar{F}(s)}(\varphi(xy^{-1})) = \mu_{\bar{F}(s)}(\varphi(x)(\varphi(y))^{-1})
\geq \min \left\{ \mu_{\bar{F}(s)}(\varphi(x)), \mu_{\bar{F}(s)}(\varphi(y)) \right\}
\]

\[
\nu_{\varphi^{-1}(\tilde{F})(s)}(xy^{-1}) = \nu_{\bar{F}(s)}(\varphi(xy^{-1})) = \nu_{\bar{F}(s)}(\varphi(x)(\varphi(y))^{-1})
\leq \max \left\{ \nu_{\bar{F}(s)}(\varphi(x)), \nu_{\bar{F}(s)}(\varphi(y)) \right\}
\]

This completes the proof. \(\square\)

4. Normal complex intuitionistic fuzzy soft groups

In this section, we extend the notion of CIFS-groups by adding the normality condition to the existing conditions. Aygunoglu & Aygun [20] introduced the conditions for normality in the context of fuzzy soft sets. Here, we generalize these conditions for normality to be compatible with the CIFSS model, and subsequently use these conditions to define the notion of normal CIFS-groups.

The conditions for intuitionistic fuzzy soft normality are described in Lemma 4.1, whereas the notion of a normal CIFS-group is proposed in Definition 4.2.

**Lemma 4.1.** Let \( (\bar{F}, E) \in \text{CIFSG}(G) \) and \( \varepsilon \in E \). Then the following statements are equivalent:

(i) \( \mu_{\bar{F}(\varepsilon)}(xyx^{-1}) \geq \mu_{\bar{F}(\varepsilon)}(y) \) and \( \nu_{\bar{F}(\varepsilon)}(xyx^{-1}) \leq \nu_{\bar{F}(\varepsilon)}(y) \), for all \( x, y \in G \).

(ii) \( \mu_{\bar{F}(\varepsilon)}(xyx^{-1}) = \mu_{\bar{F}(\varepsilon)}(y) \) and \( \nu_{\bar{F}(\varepsilon)}(xyx^{-1}) = \nu_{\bar{F}(\varepsilon)}(y) \), for all \( x, y \in G \).

(iii) \( \mu_{\bar{F}(\varepsilon)}(xy) = \mu_{\bar{F}(\varepsilon)}(yx) \) and \( \nu_{\bar{F}(\varepsilon)}(xy) = \nu_{\bar{F}(\varepsilon)}(yx) \), for all \( x, y \in G \).

**Proof.** (i) \(\Rightarrow\) (ii) Let \( x, y \in G \). Then \( y = x^{-1}(xyx^{-1})(x^{-1})^{-1} \), and both \( x^{-1}, xyx^{-1} \in G \). As a result, we have

\[
\mu_{\bar{F}(\varepsilon)}(y) = \mu_{\bar{F}(\varepsilon)}(x^{-1}(xyx^{-1})(x^{-1})^{-1}) \geq \mu_{\bar{F}(\varepsilon)}(xyx^{-1})
\]
and
\[ \nu_{\tilde{F}(e)}(y) = \nu_{\tilde{F}(e)}(x^{-1}(xyx^{-1})(x^{-1})^{-1}) \leq \nu_{\tilde{F}(e)}(xyx^{-1}). \]

Hence, statement (ii) now follows.

(ii) ⇒ (iii) Let \( x, y \in G \). Then \( xy = x(yx)x^{-1} \) and \( yx \in G \). As a result, we now have
\[ \mu_{\tilde{F}(e)}(xy) = \mu_{\tilde{F}(e)}(x(yx)x^{-1}) = \mu_{\tilde{F}(e)}(yx) \]
and
\[ \nu_{\tilde{F}(e)}(xy) = \nu_{\tilde{F}(e)}(x(yx)x^{-1}) = \nu_{\tilde{F}(e)}(yx). \]

Hence, statement (iii) now follows.

(iii) ⇒ (i) Let \( x, y \in G \). Then \( (x^{-1})(xy) = y \), where \( xy, x^{-1} \in G \). Thus, we obtain
\[ \mu_{\tilde{F}(e)}((xy)x^{-1}) = \mu_{\tilde{F}(e)}(x^{-1}(xy)) = \mu_{\tilde{F}(e)}(y) \]
and
\[ \nu_{\tilde{F}(e)}((xy)x^{-1}) = \nu_{\tilde{F}(e)}(x^{-1}(xy)) = \nu_{\tilde{F}(e)}(y). \]

Hence, statement (i) now follows. \( \square \)

**Definition 4.2.** Let \( \left( \tilde{F}, E \right) \in \text{CIFSG}(G) \). Then \( \left( \tilde{F}, E \right) \) is said to be a normal complex intuitionistic fuzzy soft group on \( G \) (abbr \( \left( \tilde{F}, E \right) \in \text{NCIFSG}(G) \)), if the conditions
\[ \mu_{\tilde{F}(e)}(xyx^{-1}) \geq \mu_{\tilde{F}(e)}(y) \text{ and } \nu_{\tilde{F}(e)}(xyx^{-1}) \leq \nu_{\tilde{F}(e)}(y) \]
are satisfied for all \( \varepsilon \in E \) and \( x, y \in G \).

**Example 4.3.** Consider the example described in Example 3.4. We define an \( \left( \tilde{H}, E \right) \in \text{CIFSS}(G) \) as \( \left( \tilde{H}, E \right) = \left\{ \tilde{H}(a), \tilde{H}(b) \right\} \), where \( \tilde{H}(a) = \tilde{F}(a) \), and \( \tilde{F}(a) \) is as defined in Example 3.4.

\[ \tilde{H}(b) = \left\{ (1, \mu_3, \nu_3), ((12), \mu_1, \nu_1), ((13), \mu_1, \nu_1), ((23), \mu_1, \nu_1), \right\}. \]

Then \( \left( \tilde{H}, E \right) \in \text{CIFSG}(G) \) and it also satisfies the conditions for normality described in Definition 4.2. Hence \( \left( \tilde{H}, E \right) \in \text{NCIFSG}(G) \).

**Theorem 4.4.** Let \( \left( \tilde{F}, E \right) \in \text{CIFSG}(G) \). Then the following statements are equivalent:

(i) \( \left( \tilde{F}, E \right) \in \text{NCIFSG}(G) \).

(ii) \( \mu_{\tilde{F}(e)}(xyx^{-1}) \geq \mu_{\tilde{F}(e)}(y) \) and \( \nu_{\tilde{F}(e)}(xyx^{-1}) \leq \nu_{\tilde{F}(e)}(y) \), for all \( \varepsilon \in E \) and \( x, y \in G \).

(iii) \( \mu_{\tilde{F}(e)}(xyx^{-1}) = \mu_{\tilde{F}(e)}(y) \) and \( \nu_{\tilde{F}(e)}(xyx^{-1}) = \nu_{\tilde{F}(e)}(y) \), for all \( \varepsilon \in E \) and \( x, y \in G \).
(iv) \( \mu_{\tilde{F}}(xy) = \mu_{\tilde{F}}(yx) \) and \( \nu_{\tilde{F}}(xy) = \nu_{\tilde{F}}(yx) \), for all \( \varepsilon \in E \) and \( x, y \in G \).

Proof. (i) \( \Rightarrow \) (ii) This follows directly from Definition 4.2.

(ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv) These follow directly from Lemma 4.1.

(iv) \( \Rightarrow \) (i) Statement (ii) follows directly from Definition 4.2, and therefore statement (i) follows by Lemma 4.1. \( \square \)

**Proposition 4.5.** Let \( (\tilde{F}, E) \in \text{NCIFSG}(G) \) and \( \emptyset \subset D \subseteq E \). Then \( (\tilde{F}, D) \in \text{NCIFSG}(G) \) as well.

Proof. Let \( x, y \in G \). By Theorem 4.4, it follows that \( \mu_{\tilde{F}}(xy) \geq \mu_{\tilde{F}}(y) \) and \( \nu_{\tilde{F}}(xy) \leq \nu_{\tilde{F}}(y) \), for all \( \varepsilon \in E \). Since \( \emptyset \subset D \subseteq E \), such statement holds for all \( \varepsilon \in D \) too. This completes the proof. \( \square \)

**Theorem 4.6.** Let \( (\tilde{F}_1, E_1), (\tilde{F}_2, E_2) \in \text{NCIFSG}(G) \). Then \( (\tilde{F}_1, E_1) \cap (\tilde{F}_2, E_2) \in \text{NCIFSG}(G) \) as well.

Proof. The proof is straightforward by Definition 2.21 and Definition 4.2. \( \square \)

**Remark.** Similar to Theorem 3.12, this property also holds for the restricted intersection operation between CIFSSs.

**Theorem 4.7.** Let \( \varphi : G \rightarrow G' \) be a surjective group homomorphism. Let \( (\tilde{F}, E) \in \text{NCIFSG}(G) \) and \( (\tilde{F}, B) \in \text{NCIFSG}(G') \). Then:

(i) \( (\varphi(\tilde{F}), E) \in \text{NCIFSG}(G') \) provided that
\[
\max \left\{ \min \left\{ \mu_{\tilde{F}}(p) : p, q \in G, \varphi(p) = x, \varphi(q) = y \right\} \right\} \geq \min \left\{ \mu_{\tilde{F}}(x), \mu_{\tilde{F}}(y) \right\}
\]
and
\[
\min \left\{ \max \left\{ \nu_{\tilde{F}}(p) : p, q \in G, \varphi(p) = x, \varphi(q) = y \right\} \right\} \leq \max \left\{ \nu_{\tilde{F}}(x), \nu_{\tilde{F}}(y) \right\}
\]
for all \( x, y \in G' \)

(ii) \( (\varphi^{-1}(\tilde{F}), B) \in \text{NCIFSG}(G) \)

Proof. The proof can be derived from Theorem 3.14, Lemma 4.1 and Definition 4.2. \( \square \)
5. Conclusion

This paper presented the initial theory of complex fuzzy algebra. We defined and developed the algebraic structures pertaining to groups and subgroups for the complex intuitionistic fuzzy soft set (CIFSS) model. The notion of CIF-subgroups, CIFS-groups and normal CIFS-groups were introduced. The fundamental properties and structural characteristics of these algebraic structures were then examined and verified. All of these were accomplished by carefully defining some important concepts pertaining to the structure of the CIFSS model, and also carefully generalizing some of the well-known operations and relations that exists between intuitionistic fuzzy soft sets to be made compatible with the structure of the CIFSS model, in which the membership and non-membership functions are defined in terms of complex numbers. Furthermore, in this paper, we have contextualized the phase term by using it to represent the different cycles of alternating groups, thereby proposing a new way of interpreting the phase term.

6. Further direction of this work

Our research in this area is still on-going. We are currently in the midst of extending the CIFSG structure introduced in this paper to introduce more advanced algebraic structures such as CIFS cyclic groups, abelian groups, dihedral groups, symmetric groups and alternating groups, using the concepts and theory that was developed in this paper. The work presented in this paper can also be used as a basis to develop other algebraic theories of complex fuzzy based models.

7. Acknowledgments

Authors Quek and Selvachandran would like to gratefully acknowledge the financial assistance received from the Ministry of Education, Malaysia under grant no. FRGS/1/2017/STG06/UCSI/03/1 and UCSI University, Kuala Lumpur, Malaysia under grant no. Proj-In-FOBIS-014.

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