BALL PACKINGS WITH HIGH CHROMATIC NUMBERS
FROM STRONGLY REGULAR GRAPHS

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ABSTRACT. Inspired by Bondarenko’s counter-example to Borsuk’s conjecture, we notice some strongly regular graphs that provide examples of ball packings whose chromatic numbers is significantly higher than the dimension. In particular, we obtain from generalized quadrangles the first non-constant lower bound for the difference between the chromatic number and the dimension.

1. THE PROBLEM AND PREVIOUS WORKS

A ball packing in $d$-dimensional Euclidean space is a collection of balls with disjoint interiors. The tangency graph of a ball packing takes the balls as vertices and the tangent pairs as edges. The chromatic number of a ball packing is defined as the chromatic number of its tangency graph.

The Koebe–Andreev–Thurston disk packing theorem says that every planar graph is the tangency graph of a 2-dimensional ball packing. The following question is asked by Bagchi and Datta in [BD13] as a higher dimensional analogue of the four-colour theorem:

Problem. What is the maximum chromatic number $\chi(d)$ over all the ball packings in dimension $d$?

The authors gave $d + 2 \leq \chi(d)$ as a lower bound since it is easy to construct $d + 2$ mutually tangent balls. By ordering the balls by size, the authors also argued that $\kappa(d) + 1$ is an upper bound, where $\kappa(d)$ is the kissing number for dimension $d$.

However, the case of $d = 3$ has already been investigated by Maehara [Mae07], who proved that $6 \leq \chi(3) \leq 13$. His construction for the lower bound uses a variation of Moser’s spindle, which is the tangency graph of an unit disk packing in dimension 2 with chromatic number 4, and the following lemma:

Lemma. If there is a unit ball packing in dimension $d$ with chromatic number $\chi$, then there is a ball packing in dimension $d + 1$ with chromatic number $\chi + 2$.

The technique of Maehara [Mae07] can be easily generalized to higher dimensions and gives $d + 3 \leq \chi(d)$.

Another progress is made by Cantwell in an answer on MathOverflow [Can]. He proved that the graph of the halved 5-cube (also called the Clebsch graph) is the tangency graph of a 5-dimensional unit ball packing with chromatic number 8. Then the lemma above implies that $10 \leq \chi(6)$. This argument can be generalized to higher dimensions using a result of Linial, Mechulam and Tarsi [LMT88, Theorem 4.1], and gave $d + 4 \leq \chi(d)$ for $d = 2^k - 2$.

As we have seen, both constructions study the chromatic number of unit ball packings and conclude by the lemma. We will do the same. A unit ball packing can be regarded as a set of points such that the minimum distance between pairs of points is at least 1, then the tangency graph of the packing is the unit-distance graph for these points. The finite version of the Borsuk conjecture can be formulated as follows: the chromatic number of the unit-distance graph for a set of points with maximum distance 1 is at most $d + 1$. So the chromatic number problem for unit ball packings is the opposite of the Borsuk conjecture. By ordering the unit balls by height, we see that the chromatic number of a unit ball packing is at most one plus the one-side kissing number.

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The Borsuk conjecture was first disproved by Kahn and Kalai [KK93]. Recently, Bondarenko [Bon14] found a counter-example for Borsuk conjecture in dimension 65. His construction was then slightly improved by Jenrich [JB14] to dimension 64, which is the current record for the smallest counter-example. Their construction is based on geometric representations of strongly regular graphs.

In this note, we use the technique of Bondarenko to find unit ball packings whose chromatic numbers is significantly higher than their dimensions. In particular, the graphs of generalized quadrangles of parameter $(q, q^2)$ correspond to unit ball packings in dimension $q^3 - q^2 + q$ with chromatic number $q^3 + 1$. This is the first non-constant lower bound for the difference $\chi(d) - d$.

In view of [KK93], we propose the following conjecture, and hope that examples in this note may help further improvement of the lower bound.

**Conjecture.** There is a constant $c$ such that $\chi(d) \geq c\sqrt{d}$.

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## 2. Strongly regular graphs

We use [CvL91] for general references of strongly regular graphs.

Let $G$ be a strongly regular graph of parameter $(n, k, \lambda, \mu)$. That is, $G$ is a $k$-regular graph of $v$ vertices, such that every pair of adjacent vertices have $\lambda$ neighbors in common and every pair of non-adjacent vertices have $\mu$ neighbors in common. We assume that

$$\lambda - \mu \geq -2k/(v - 1).$$

If this is not the case, we may replace $G$ by its complement $\bar{G}$, which is a strongly regular graph of parameter $(v, v - k - 1, v - 2k - 2 + \mu, v - 2k + \lambda)$. For any vertex of $G$, the graphs induced by its neighbors in $G$ and by its neighbors in $\bar{G}$ are respectively the first and the second subconstituent of $G$.

The adjacency matrix of $G$ has three eigenvalues $k, r, s$ with multiplicities $1, f, g$ respectively. They can be expressed in terms of the parameters as follows:

$$r, s = (\lambda - \mu \pm \delta)/2,$$

$$f, g = (v - 1 \pm \Delta)/2,$$

where $\delta = \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}$ and $\Delta = ((v - 1)(\mu - \lambda) - 2k)/\delta \leq 0$. The eigenvalues of $\bar{G}$ are $v - k - 1, -s - 1, -(r + 1) - \mu$, with multiplicities $1, g, f$ respectively. Note that $r > 0 > s + 1$ and $f \leq g$.

We may identify vertices of $G$ with an orthonormal basis of $\mathbb{R}^v$. The projection of the base vectors onto the $r$-eigenspace form a spherical 2-distance set on the sphere $S^{f-1} \subset \mathbb{R}^f$, with cosines $\alpha = r/k$ for adjacent vertices and $\beta = -(r + 1)/(v - k - 1)$ for non-adjacent vertices. After a proper scaling, we obtain a point set with minimum distance 1, whose unit-distance graph is $G$.

For strongly regular graphs, we can calculate the Lovász theta number using the formula [Lov79, Theorem 9; Hae81]

$$\theta(G) = 1 + (v - k - 1)/(1 + r).$$

One verifies that $\theta(\bar{G}) = 1 - k/s = v/\theta(G)$ [God94, Hae81]. By the Lovász sandwich theorem [Kum94], we have $\omega(G) \leq \theta(G) \leq \chi(G)$, where $\omega(G)$ is the clique number. We will use $\theta(G)$ as a lower bound for the chromatic number of $G$.

## 3. Results

From Brouwer’s online list of strongly regular graphs [Bro], we notice some graphs such that $f + 3 < \theta(G)$, therefore improve the previous lower bounds. In Table 1, we list their parameters, eigenvalues with multiplicities, and the theta number of the complement. For comparison, we highlight the dimension $f$ and the theta number $\theta(G)$.
Remark. For many graphs in the table, the conventional parameters in the literature do not satisfy our assumption (1). For these graphs, we use the parameters of their complements, as explained in the beginning of Section 2. 

Two infinite families are not included in the table: One is the complements to the C20 family from Hubaut [Hub75], recovered by Godsil [God92, Lemma 5.3], with the parameters $(q^3, (q + 1)(q^2 - 1)/2, (q + 3)(q^2 - 3)/4 + 1, (q + 1)(q^2 - 1)/4)$ where $q$ is an odd prime power. They can be represented unit ball packings in dimension $f = q^2 - q$ with chromatic number at least $\theta(G) = q^2$, which already improve the previous lower bounds. The other family is the complements to the point graphs of the generalized quadrangles of parameter $(q, q^3)$, which provides an even better lower bound. We now introduce the generalized quadrangles in detail.

A generalized quadrangle [PT09] of parameter $(s, t)$, denoted by $GQ(s,t)$, is an incidence structure $(P, L, \in)$, where $P$ is the set of points and $L$ is the set of lines, satisfying the following axioms:

- Each point is incident with $t + 1$ lines and two distinct points are incident with at most one line.
- Each line is incident with $s + 1$ points and two distinct lines are incident with at most one point.
- For a point $p$ and a line $\ell$ such that $p \notin \ell$, there is a unique pair $(p', \ell')$ such that $p \in \ell' \ni p' \in \ell$.

The point graph of a generalized quadrangle takes $P$ as the set of vertices and connect two vertices by an edge if they are incident to the same line. It is a strongly regular graph with parameters $((st + 1)(s + 1), s(t + 1), s - 1, t + 1)$.

It is known that $GQ(q, q^2)$ exists when $q$ is a prime power, and is unique for $q = 2, 3$. For a generalized quadrangle $GQ(q, q^2)$, we abuse the notation $GQ(q, q^2)$ for the complement of its point graph. It is a strongly regular graph with parameters $((q + 1)(q^2 + 1), q^4, q(q - 1)(q^2 + 1), (q - 1)q^3)$.

Up to complement, $GQ(2, 4)$ is the Schl"afli graph and $GQ(3, 9)$ is the second subconstituent of the McLaughlin graph (112 vertices). $GQ(q, q^2)$ is represented in dimension $f = q^3 - q^2 + q$ and the theta number of its complement is $q^3 + 1$. So the chromatic number of $GQ(q, q^2)$ is at least $q^3 + 1$.

A spread of a generalized quadrangle is a set of lines such that each point is incident with a unique line in the set. By [PT09, Theorem 3.4.1(ii)], a generalized quadrangle $GQ(q, q^2)$ has spreads, meaning that the vertices of the point graph can be partitioned into $q^3 + 1$ cliques of size $q + 1$. So the chromatic number of $GQ(q, q^2)$ (complement to the point graph) is at most $q^3 + 1$.

To conclude, the chromatic number of $GQ(q, q^2)$ is exactly $q^3 + 1$. By the lemma in Section 1, we have constructed a ball packing in dimension $q^3 - q^2 + q + 1$ with chromatic number $q^3 + 3$, which gives the first non-constant lower bound for the difference $\chi(d) - d$.

| name or ref       | $v$  | $k$  | $\lambda$ | $\mu$ | $r$ | $f$ | $s$ | $q$ | $\theta(G)$ |
|-------------------|------|------|-----------|-------|-----|-----|-----|-----|-------------|
| Higman-Sims       | 100  | 77   | 60        | 56    | 7   | 22  | -4  | 87  | 80/3        |
| [DC08, LR13]      | 105  | 72   | 51        | 45    | 9   | 20  | -3  | 84  | 25          |
| [DC08, LR13]      | 120  | 77   | 52        | 44    | 11  | 20  | -3  | 99  | 80/3        |
| [HK92]            | 126  | 75   | 48        | 39    | 12  | 20  | -3  | 105 | 26          |
| 1st subconst. of McL | 162  | 105  | 72        | 60    | 15  | 21  | -3  | 153 | 36          |
| [Hae81b]          | 175  | 102  | 65        | 51    | 17  | 21  | -3  | 153 | 35          |
| [Hae81b, DC08]    | 176  | 105  | 68        | 54    | 17  | 21  | -3  | 154 | 36          |
| [Hae81b]          | 176  | 85   | 48        | 34    | 17  | 22  | -3  | 153 | 88/3        |
| [Del73]           | 243  | 132  | 81        | 60    | 24  | 22  | -3  | 220 | 45          |
| [DC08, LR13]      | 253  | 140  | 87        | 65    | 25  | 22  | -3  | 230 | 143/3       |
| McLaughlin        | 275  | 162  | 105       | 81    | 27  | 22  | -3  | 252 | 55          |
| [GS75]            | 276  | 135  | 78        | 54    | 27  | 23  | -3  | 252 | 46          |
| [BR13]            | 729  | 520  | 379       | 350   | 22  | 112 | -5  | 616 | 621/5       |

Table 1. Strongly regular graphs with high chromatic numbers
Remark. Up to complement, the Clebsch graph, the Higman–Sims graph, the McLaughlin graph, the first subconstituent (162 vertices) of the McLaughlin graph, and $GQ(q, q^2)$ are the only known examples of Smith graphs [Smi75, CGS78]. They can be constructed from a rank 3 permutation group such that the stabilizer of a vertex has rank $\leq 3$ on both subconstituents. It turns out that Smith graphs tend to have high chromatic number (the Clebsch graph is noticed by Cantwell [Can]).

Remark. The Lovász theta number is not always a good bound for the chromatic number. Our author believes that there are other strongly regular graphs with high chromatic number. However, for the conjecture in Section 1, the power of strongly regular graphs might be very limited. Further improvement on the lower bound is encouraged.

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