Giuseppe Veronese and Ernst Witt – Neighbours in PG(5, 3)

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Dedicated to János Aczél on the occasion of his 75th birthday

Summary. Let P be a point of the Veronese surface V in PG(5, 3). Then there are four conics of V through P. We show that the internal points of those conics form a 12–cap which is a point model for Witt’s 5–(12, 6, 1) design. In fact, this construction is “dual” to a similar construction that has been established in [6] recently. We give an explicit parametrization of the cap K; the domain is a dual affine plane which arises from PG(2, 3) by removing one point. Thus, as a by–product, we obtain an easy approach to the extended ternary Golay code G_{12}. Finally, we discuss some other procedures that yield 12–sets of points from the Veronese surface.

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1 Introduction

A construction of a cap K, in PG(5, 3), which is a point model for Witt’s 5–(12, 6, 1) design W_{12} (see, among others, [3, Chapter IV]) has been found by H.S.M. Coxeter [5] and, independently, by G. Pellegrino [9]. The cap K has exactly twelve points and any five distinct points of K span a prime (hyperplane) of PG(5, 3) which contains exactly six points of K. Such a K is projectively unique. The group of collineations fixing K, as a set, is the automorphism group of W_{12}, i.e. the Mathieu group M_{12}. Also, J.A. Todd [10] has shown that there are exactly twelve primes of PG(5, 3) carrying no point of K. Those primes gives rise to a point model K∗ of W_{12} in the dual space of PG(5, 3).

The Veronese surface V in PG(5, 3) is a set of thirteen points; cf. e.g. [8, Chapter 25]. It determines uniquely its dual Veronese surface V∗ in the dual space of PG(5, 3). As has been pointed out in [6], the following holds true: If one conic of the Veronese surface V is replaced with the set formed by the internal points of that conic, then a point model K of W_{12} is obtained. Figure 1 illustrates a conic
in a projective plane of order three: The four points of the conic, its three internal points, and its six external points are drawn as squares, triangles, and hexagons, respectively.

Figure 1: A conic in PG(2,3)

Clearly, now the question about the connection between the dual Veronese surface $V^*$ and the dual point model $K^*$ arises. For our purposes it will be convenient to adopt a dual point of view from the very beginning: So we start with $V$, then go over to its dual $V^*$, next apply the procedure of [6] to obtain a $K^*$ from $V^*$, and finally go back to $K^{**} =: K$.

In Theorem 1 we give a direct description (avoiding the dual space) of that construction. It turns out that the present construction is different from the one given in [6], since now we do not have a distinguished conic of $V$, but a distinguished point $P \in V$. In contrast to [6], $V$ and $K$ do not have common points, but there is a bijection $\rho : V \setminus \{P\} \rightarrow K$ such that $Y \in V \setminus \{P\}, Y^\rho \in K$, and $P$ are always collinear.

The Veronese mapping $\varphi$ is a bijection of PG(2,3) onto $V$. By combining it with the above-mentioned bijection $\rho$, we find a bijection of a dual affine plane (a PG(2,3) with one point removed) onto $K$. Using homogeneous coordinates in PG(2,3) and PG(5,3) then gives an explicit parametric representation $\psi$ of $K$ described in Theorem 2. It is a peculiar feature that one of the coordinate functions of $\psi$ contains an inhomogeneous term, something which usually does not make sense. But here it is meaningful, since 1 is the only non-zero square in GF(3).

In Section 4 we discuss procedures, similar to that of Theorem 1, which yield 12-sets of points from the Veronese surface $V$. Some of them give point models of $W_{12}$, others do not. However, one should keep in mind that each point model of $W_{12}$ arises in some way or another from a fixed Veronese surface; so our discussion is far from being complete. In fact, all our 12-sets are “close neighbours” of the
Veronese surface $V$, as they belong to the algebraic hypersurface (of order three) which is formed by all points of $\text{PG}(5,3)$ that are on a chord of $V$.

2 The Veronese Surface and its Dual

Let us recall some properties of the Veronese surface $V$ in $\text{PG}(5,3)$ [4, Kapitel V], [7], [8, Chapter 25]: The term conic plane is used for a plane which meets $V$ in a conic. Any two distinct conic planes have one and only one point in common. This point belongs to $V$. Let $C$ be the set of all conics of $V$. Then $(V, C, \in)$ is a $\text{PG}(2,3)$.

For each $m \in C$ there is a unique prime which meets $V$ exactly in $m$; this osculating prime (or contact prime) of $V$ along $m$ will be denoted by $H_m V$. A line $l$ is called a tangent of $V$, if $l$ is a tangent of a conic $m \subset V$. Given $P \in V$ the tangent plane of $V$ at $P$ is the union of all tangents of $V$ which are running through $P$. It is written as $T_P V$. Another description of osculating primes and tangent planes is given by

$$H_m V = \text{span} \left( \bigcup_{X \in m} T_X V \right) (m \in C), \quad (1)$$

$$T_P V = \bigcap_{P \in m \in C} H_m V \quad (P \in V). \quad (2)$$

Any two distinct tangent planes have a unique common point; this point is not on $V$.

If $S$ is a subspace of $\text{PG}(5,3)$, then let $[S]^*$ be the star of primes through $S$. All osculating primes of $V$ form the dual Veronese surface $V^*$, i.e. a Veronese surface in the dual space, since Char GF(3) $\neq 2$. There is a one–one correspondence between the tangent and conic planes of $V$ with the “conic” and “tangent planes” of $V^*$, respectively:

Each tangent plane $T_P V$ yields the “conic plane” $[T_P V]^*$ of $V^*$. The osculating primes of $V$ that are passing through $P$ comprise the corresponding “conic” $c^* \subset V^*$. If we choose one “point” of the “conic” $c^*$, say $H_m V$, then its “tangent” is given by the pencil $[T_P V \lor \text{span } m]^*$. An “internal point” of $c^*$ is a prime $I \in [T_P V]^*$ which is on no “tangent” of $c^*$, i.e.

$$\text{span } m \cap I = \text{span } m \cap T_P V \text{ for all } m \in C \text{ with } P \in m. \quad (3)$$

Alternatively, an “internal point” of $c^*$ may be characterized as a prime $I$ of $\text{PG}(5,3)$ satisfying

$$I \cap V = \{P\}, \quad (4)$$

since (1) implies that $I$ corresponds to a quadric in $\text{PG}(3,2)$ consisting of one double point only, so that $T_P V \subset I$ (cf. [4] p. 168, Satz 1).

Likewise, each conic plane $\text{span } m$ ($m \in C$) yields the “tangent plane” $[\text{span } m]^*$ of $V^*$ at the “point” $H_m V \in V^*$.
3 Point Models of $W_{12}$

Let $K$ be a set of twelve points in a $\text{PG}(5,3)$. Define a block of $K$ as hyperplane section of $K$ which contains exactly six points. Write $B$ for the set of all such blocks. If $(K, B, \in)$ is Witt’s $5-(12, 6, 1)$ design $W_{12}$, then $K$ is called a point model of $W_{12}$ in $\text{PG}(5,3)$.

In what follows we put $\text{GF}(3) = \{0, 1, 2\} =: F$.

**Theorem 1** Let $P$ be a point of the Veronese surface $V$ in $\text{PG}(5,3)$. The four conics of $V$ through $P$ are denoted by $m_k$ ($k \in F \cup \{\infty\}$). The set of internal points of each $m_k$ is written as $\Delta_k$. Also let $c^*$ be the set of osculating primes of $V$ through $P$ and $\Delta^*$ the set of all primes that meet $V$ in $P$ only. Then the following holds true:

1. The set
   \[ K := \bigcup_{k \in F \cup \{\infty\}} \Delta_k \]  
   is a point model of the Witt design $W_{12}$.

2. No point of $K$ is incident with a prime belonging to
   \[ K^* := (V^* \setminus c^*) \cup \Delta^*. \]  

**Proof.** According to Section 2, $c^*$ is a “conic” of the dual Veronese surface and $\Delta^*$ is the set of its “internal points”. Hence $K^*$ is a point model of $W_{12}$ in the dual space of $\text{PG}(5,3)$ [8, Remark 3]. By a result of J.A. Todd [10, p. 408], applied to $K^*$, there are exactly twelve points of $\text{PG}(5,3)$ which are not in any prime belonging to $K^*$. Moreover, those points form a point model of $W_{12}$.

Each conic $m_k$ has exactly three internal points. The planes of two distinct conics of the Veronese surface do not have common internal points. Thus $\#K = 12$ and the theorem follows, if we can show that no point of $K$ lies in a prime belonging to $K^*$.

Let $I$ be one of the three “internal points” of $c^*$. By (3), the prime $I$ meets the plane of each $m_k$ ($k \in F \cup \{\infty\}$) in the tangent of $m_k$ at $P$, so that

\[ I \cap K = \emptyset. \]  

Any of the remaining nine primes in $K^*$ is an osculating prime $H_c \gamma$ along a conic $c \in C \setminus \{m_0, m_1, m_2, m_\infty\}$. Put $\{S_k\} := c \cap m_k$. By (2), $T_{S_k} \gamma \subset H_c \gamma$, whence $H_c \gamma \cap \text{span} m_k$ is the tangent of $m_k$ at $S_k$. It follows that

\[ H_c \gamma \cap K = \emptyset, \]  
which completes the proof. \qed
Now we introduce coordinates in order to obtain a parametric representation of $K$. Assume that $\text{PG}(2,3)$ and $\text{PG}(5,3)$ are projective spaces $\mathcal{P}(F^3)$ and $\mathcal{P}(F^6)$, respectively. The Veronese mapping is given as

$$\varphi : \mathcal{P}(F^3) \rightarrow \mathcal{P}(F^6), \quad F(x_0, x_1, x_2) \mapsto F(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$  \hspace{1cm} (9)

We fix the points

$$U := F(1, 0, 0) \text{ and } P := U^\varphi = F(1, 0, 0, 0, 0).$$  \hspace{1cm} (10)

Let $m_k \in \mathcal{C}$ be a conic through $P$. Each bisecant of $m_k$ contains exactly one internal point of $m_k$; see Figure 1. Thus the mapping

$$\rho : \mathcal{V} \setminus \{P\} \rightarrow K, \quad Y \mapsto Y^\psi$$  \hspace{1cm} (11)

is a well-defined bijection. Putting

$$W := \mathcal{P}(F^3) \setminus \{U\}$$  \hspace{1cm} (12)

yields the bijection

$$\psi : W \rightarrow K, \quad X \mapsto X^\psi$$  \hspace{1cm} (13)

whose domain is a dual affine plane.

**Theorem 2** In terms of homogeneous coordinates the mapping (13) takes the form

$$F(x_0, x_1, x_2)^\psi \mapsto F(x_0^2 + 1, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$$  \hspace{1cm} (14)

**Proof.** At first we note that the use of the inhomogeneous term $x_0^2 + 1$ is not ambiguous: In fact, if $q : F^{n+1} \rightarrow F$ is a quadratic form, then

$$(2x_0, \ldots, 2x_n)^q = 2^2 \cdot (x_0, \ldots, x_n)^q = 1 \cdot (x_0, \ldots, x_n)^q.$$  \hspace{1cm} (15)

Next choose a fixed pair $(x_1, x_2) \in F^2 \setminus \{(0, 0)\}$. The Veronese image of the line joining $U$ with the point $F(0, x_1, x_2) \in \mathcal{P}(F^3)$ is a conic $m$ through $P$. Then $m$ comprises those four points which are spanned by the vectors

$$v_u := (u^2, ux_1, ux_2, x_1^2, x_1x_2, x_2^2) \quad (u \in F), \quad v_\infty := (1, 0, 0, 0, 0, 0).$$  \hspace{1cm} (16)

The three internal points of the conic $m$ are the three diagonal points of the planar quadrangle $m$. We observe

$$v_u + v_\infty = 2v_{u+1} + 2v_{u+2} \text{ for all } u \in F, \hspace{1cm} (17)$$

whence $F(u, x_1, x_2)^\psi = (Fv_u)^\rho = F(v_u + v_\infty)$ for all $u \in F$.

As $(x_1, x_2)$ varies in $F^2 \setminus \{(0, 0)\}$, we obtain all four conics $\{m_0, m_1, m_2, m_\infty\}$, since those conics can be relabelled in such a way that $(x_1, x_2)$ yields the conic $m_k$ with $k = x_2/x_1$. \hspace{1cm} \Box
Remark 1 The dual Veronese mapping $\varphi^*$ assigns to each line $l$ of $\text{PG}(2, 3)$ with homogeneous coordinates $F(a_0, a_1, a_2)$ the prime of $\text{PG}(5, 3)$ with homogeneous coordinates
\[ F(a_0^2, 2a_0a_1, 2a_0a_2, a_1^2, 2a_1a_2, a_2^2), \] (18)
since $l^\varphi^*$ equals the osculating prime of $V$ along the conic $l^\varphi$. The image of $\varphi^*$ is the dual Veronese surface $V^\ast$. The nine lines which are not running through $U$ are characterized by $a_0 \neq 0$, whence their images under $\varphi^*$ are immediate from (18).

By \[6, \text{Remark 1}\], the remaining three primes in $K^\ast$ have coordinates
\[ F(0, 0, 0, 1, 0, 1), F(0, 0, 0, 2, 2, 1), F(0, 0, 0, 2, 1, 1). \] (19)

Note that the 01–, 02–, and 12–coordinates in \[6\] have to multiplied by 2 in order to match (18). By virtue of (14), (18), and (19) it is easy to verify in terms of coordinates that no point of $K$ is incident with a prime belonging to $K^\ast$. This gives an alternative proof of Theorem 1.

Remark 2 With the help of formula (14) one may immediately write down twelve vectors of $F^6$ representing the points of $K$. If those vectors are regarded as columns of a $6 \times 12$ matrix over $F$, then a generator matrix of the extended ternary Golay code $G_{12}$ arises. We refer to \[1\], \[2, 8.6\], and \[6\] for further details on the connections between the Witt design $W_{12}$ and coding theory.

4 Replacing Conics of the Veronese Surface

In what follows we shall stick to the terminology introduced in Theorem 1 as we aim at generalizing the construction given there.

Choose one conic $m_k (k \in F \cup \{\infty\})$. Let $t_k$ be the tangent of $m_k$ at $P$, $\Delta_{k,0} := m_k \setminus \{P\}$, and $\Delta_{k,1} := \Delta_k$. Denote by $\Delta_{k,2}$ the set of all external points of $m_k$ that are off the tangent $t_k$. It is easily seen from Figure 1 that there is a unique elation $\kappa_k$ of the plane span $m_k$ with centre $P$ and axis $t_k$ such that
\[ (\Delta_{k,j})^{\kappa_k} = \Delta_{k,j+1} \text{ for all } j \in F. \] (20)

Also, each $\Delta_{k,j+1}$ is the set of internal points of the conic $\Delta_{k,j} \cup \{P\}$, whence the restrictions of $\kappa_k$ and $\rho$ to the conic $m_k$ are coinciding.

All four collineations $\kappa_k$ do not simultaneously extend to a collineation of $\text{PG}(5, 3)$, since $V \setminus \{P\}$ contains four distinct coplanar points, whereas $K$ does not. However, if we choose three collineations, e.g. $\kappa_1$, $\kappa_2$, and $\kappa_\infty$, then they extend to a unique perspective collineation $\mu_0$ of $\text{PG}(5, 3)$ with centre $P$ and axis $H_{00}V$: If the numbering of the conics $m_k$ is done according to the proof of Theorem 2 then $0 = 0/1$ implies that $\mu_0$ is given by
\[ F(y_{00}, y_{01}, y_{02}, y_{11}, y_{12}, y_{22}) \mapsto F(y_{00} + y_{22}, y_{01}, y_{02}, y_{11}, y_{12}, y_{22}). \] (21)
Clearly, all points of the conic plane span $m_0 \subset H_{m_0} \mathcal{V}$ are fixed under $\mu_0$, i.e. $\mu_0$ extends $\kappa_0^0 = \text{id.}$

One may assign to each $(p, q, r, s) \in F^4$ the quadruple $(\kappa_0^p, \kappa_1^q, \kappa_2^r, \kappa_\infty^s)$ of collineations. This mapping is an isomorphism of the group $(F^4, +)$ onto the direct sum of the collineation groups generated by the $\kappa_k$’s. In addition, each $(p, q, r, s) \in F^4$ yields the point set

$$\Delta_{0,p} \cup \Delta_{1,q} \cup \Delta_{2,r} \cup \Delta_{\infty,s}$$

(22)

consisting of twelve points in $\text{PG}(5, 3)$.

Each permutation of the four conics $m_0, m_1, m_2, m_\infty$ arises from an automorphic projective collineation of the Veronese surface $\mathcal{V}$. So, if two elements of $F^4$ differ only in the arrangement of their entries, then they yield projectively equivalent 12–sets.

The subgroup $S$ of $(F^4, +)$ generated by $(1, 1, 0, 0), (1, 1, 0, 1), (1, 0, 1, 1), \text{and (0, 1, 1, 1)}$ consists of all $(p, q, r, s)$ with $p + q + r + s = 0$. Each element of $S$ yields a quadruple of planar collineations that extend to a collineation of $\text{PG}(5, 3)$ by (21). Hence all corresponding 12–sets are projectively equivalent to $\mathcal{V}$.

The elements of the coset $(1, 0, 0, 0) + S$ are characterized by $p + q + r + s = 1$ and yield 12–sets that are projectively equivalent to $\mathcal{K}$.

We mention without proof some results about the 12–sets that arise from the elements of the remaining coset $(p + q + r + s = 2)$. Each such point set, say $\mathcal{R}$, is neither projectively equivalent to $\mathcal{K}$ nor projectively equivalent to a subset of $\mathcal{V}$. Among the 364 primes of $\text{PG}(5, 3)$ there are exactly 42 which meet $\mathcal{R}$ in precisely six points. Those 42 primes have $P$ as their only common point. Thus $P$ is invariant under the group of automorphic collineations of $\mathcal{R}$. So it seems natural to project $\mathcal{R}$ through the point $P$ to a prime $\mathcal{H}$ of $\text{PG}(5, 3)$ not containing $P$. The conic planes $m_k$ ($k \in F \cup \{\infty\}$) are projected to four mutually skew lines $l_k \subset \mathcal{H}$, the tangent plane $T_P \mathcal{V}$ of the Veronese surface goes over to the only transversal line of the $l_k$’s, and the set $\mathcal{R}$ is mapped onto those twelve points of the four lines $l_k$ that are not on their common transversal line.

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