TEMPERATURE ON RODS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. We consider solutions $u_f$ to the one-dimensional Robin problem with the heat source $f \in L^1[-\pi,\pi]$ and Robin parameter $\alpha > 0$. For given $m$, $M$, and $s$, $0 \leq m < s < M$, we identify the heat sources $f_0$, such that $u_{f_0}$ maximizes the temperature gap $\max_{[-\pi,\pi]} u_f - \min_{[-\pi,\pi]} u_f$ over all heat sources $f$ such that $m \leq f \leq M$ and $\|f\|_{L^1} = 2\pi s$. In particular, this answers a question raised by J. J. Langford and P. McDonald in [5]. We also identify heat sources, which maximize/minimize $u_f$ at a given point $x_0 \in [-\pi,\pi]$ over the same class of heat sources as above and discuss a few related questions.

1. Heating with Robin boundary conditions.

Recently, J. J. Langford and P. McDonald [5] studied the one-dimensional Poisson equation with Robin boundary conditions. They considered the following physical setup: Suppose that a metal rod of length $2\pi$ is located along the interval $[-\pi,\pi]$. Suppose that to half of the locations of the rod, heat is generated uniformly; call this set $E$. On the remaining half, heat is neither generated nor absorbed. The ends of the rod interact with the cooler environment so that there is a heat flux from the rod which is proportional to the temperature at each end (Newton’s law of cooling).

Let $u$ be the steady-state temperature function; it satisfies the Poisson equation

$$-u''(x) = \chi_E(x), \quad x \in [-\pi,\pi]$$

with Robin boundary conditions

$$-u'(-\pi) + \alpha u(-\pi) = u'(\pi) + \alpha u(\pi) = 0,$$

where $\alpha > 0$ and $\chi_E$ stands for the characteristic function of the set $E$.

Langford and McDonald [5] studied the problem of where one should locate the heat sources to maximize the hottest steady-state temperature. In other words, for which set $E$, for the solution $u$ of the boundary value problem (1.1)-(1.2), $\max_{[-\pi,\pi]} u$ is maximal. They showed that this quantity is maximal when $E$ is an interval located symmetrically in the middle of the rod; namely $E = [-\pi/2, \pi/2]$. The authors of [5] actually studied a more general problem and obtained much stronger comparison results. They observed, however, that the symmetric interval is not extremal for another problem.
They considered the temperature gap over \([-\pi, \pi]\), i.e. the quantity
\[
\text{osc}(u) := \max_{[-\pi, \pi]} u - \min_{[-\pi, \pi]} u
\]
and showed that osc\(u\) is not maximized for \(E = [-\pi/2, \pi/2]\). So they raised the following question:

**Problem 1.** Where should we place the heat sources to maximize the temperature gap?

They suggested that the extremal set \(E\) is again an interval which, however, is not symmetrically located on \([-\pi, \pi]\). In the present note, we will study this conjecture. As in [5], we will consider a more general setting.

The Robin problem for \(f \in L^1[-\pi, \pi]\) and \(\alpha > 0\) is to find \(u \in C^1[-\pi, \pi]\) such that

1. \(u'\) is absolutely continuous on \([-\pi, \pi]\),
2. \(-u'' = f\) a.e. on \((-\pi, \pi)\),
3. \(-u'(-\pi) + \alpha u(-\pi) = u'(\pi) + \alpha u(\pi) = 0\).

It was shown in Proposition 2.1 in [5] that the Robin problem has a unique solution given by the equation

\[
u_f(x) = \int_{-\pi}^{\pi} G(x, y) f(y) \, dy. \tag{1.3}
\]

Here, \(G(x, y)\) stands for the Green’s function for Robin problem, which is

\[
G(x, y) = -\frac{1}{2}c_{\alpha} xy - \frac{1}{2}|x - y| + \frac{1}{2c_{\alpha}}, \quad x, y \in [-\pi, \pi], \tag{1.4}
\]

where
\[
c_{\alpha} = \frac{\alpha}{1 + \alpha \pi}. \tag{1.5}
\]

To recall a few basic facts about solutions of the Robin problem, we note first that, as simple Calculus shows, \(G(x, y) > 0\) for all \(x, y \in [-\pi, \pi]\). Another simple but important conclusion from (1.3) is that the solution to the Robin problem is an additive function of the heat source; i.e. if \(m_1, m_2\) are constants and \(f_1, f_2 \in L^1\) (here and below \(L^1\) stands for \(L^1[-\pi, \pi]\)), then

\[
u_f = m_1 u f_1 + m_2 u f_2, \quad \text{where} \quad f = m_1 f_1 + m_2 f_2. \tag{1.6}
\]

Furthermore, if \(f \geq 0\), then it is immediate from property 2 above that \(u_f\) is concave on \([-\pi, \pi]\) and therefore it takes its minimal value at one of the end points \(x = \pm \pi\) and it takes its maximal value either at a single point or on some closed subinterval of \([-\pi, \pi]\). If \(u_f\) takes its minimal value at \(-\pi\), it follows from property 3 that \(u_f(x) \geq u_f(-\pi) = \alpha^{-1} u_f'(-\pi) > 0\) and therefore, in this case, \(u_f(x) > 0\) for \(x \in [-\pi, \pi]\) unless \(f \equiv 0\). The same conclusion follows if \(u_f\) takes its minimal value at \(\pi\).

To put the question in Problem 1 in a more general setting, we consider, for given \(m\), \(M\), and \(s\) such that \(0 \leq m < s < M\), the class \(\mathcal{F} = \mathcal{F}(m, M, s)\) of heat sources \(f \in L^1\) such that \(m \leq f(x) \leq M\) for \(x \in [-\pi, \pi]\) and \(\|f\|_{L^1} = 2\pi s\). The parameters \(m\), \(M\), and \(s\) can be interpreted as the ground heat, the top heat, and the average heat over the rod.
Our main goal in this note is to prove the following theorem, which solves the maximal temperature gap problem for the class \( F(m,M,s) \) and therefore, as a special case, it provides a solution to Problem 1. In this theorem and below, \( I(a,l) \subset \mathbb{R} \) denotes the closed interval of length \( 2l > 0 \) centered at \( a \).

**Theorem 1.** Let \( u_f \) solves the Robin problem for \( f \in F(m,M,s) \) and \( \alpha > 0 \). Then

\[
\text{osc}(u_f) \leq (M - m)\Theta_{\alpha}(l,\delta),
\]

where \( l = \pi(s - m)/(M - m) \), \( \delta = m/(M - m) \) and \( \Theta_{\alpha}(l,\delta) \) is defined by equations (4.11), (4.12) and (4.13) in Section 4.

Equality holds in (1.7) if and only if \( f(x) = f_0(x) \) or \( f(x) = f_0(-x) \) a.e. on \( [-\pi,\pi] \), where \( f_0 = m + (M-m)\chi_{I(a_g,l)} \) with \( l = \pi(s-m)/(M-m) \) and \( a_g \) defined by equation (4.10) in Section 4.

For \( m = 1, M = 3, \) and \( s = \frac{7}{5} \), the graph of the maximal temperature gap \( (M - m)\Theta_{\alpha}(l,\delta) \) considered as a function of \( \alpha \) is shown in Figure 1(a). Figure 1(b) displays the graph of the extremal function \( u_{f_0}(x) \), where \( f_0 = m + (M-m)\chi_{I(a_g,l)} \) with \( m = 1, M = 3, s = \frac{7}{5} \) and \( \alpha = 1/2 \).

Returning to the context of Problem 1, let us suppose that \( E \) is a measurable subset of \( [-\pi,\pi] \) of length (one-dimensional Lebesgue measure) equal to \( \pi \). We apply Theorem 1 with \( f = \chi_E, m = 0, M = 1, \delta = 0, \) and \( l = \pi/2 \). The parameter \( a_0 \) defined by equation (4.8) in Section 4 takes the value \( a_0 = \frac{2}{\sqrt{3}\pi} \). Let

\[
E^* = I(a_g,\pi/2) = [a_g - \pi/2, a_g + \pi/2] \quad \text{and} \quad f^* = \chi_{E^*},
\]

where (see formula (4.10))

\[
a_g = \begin{cases} 
\pi/2, & \text{if } 0 < \alpha \leq \frac{2}{\sqrt{3}\pi}, \\
\frac{\pi/2}{(1+\alpha\pi)(\pi\alpha/2 - \pi^2\alpha^2/4)}, & \text{if } \alpha \geq \frac{2}{\sqrt{3}\pi}.
\end{cases}
\]
It follows (cf. [3 Proposition 3.3]) that when \(0 < \alpha \leq \frac{2}{\sqrt{3}\pi}\), we have \(E^* = [0, \pi]\); when \(\alpha \geq \frac{2}{\sqrt{3}\pi}\), the location of the interval \(E^*\) depends on \(\alpha\) and as \(\alpha\) increases, \(E^*\) moves from the right end to the center. By Theorem 1 we have \(\text{osc}(u_f) \leq \text{osc}(u_{f^*})\). The solution to Problem 1 is: The temperature gap is maximized uniquely when the heat sources are placed on \(E^*\) or on \(-E^*\).

Another interesting problem on the distribution of heat on a rod is to identify heat sources \(f \in \mathcal{F}(m, M, s)\), which generate the maximal possible temperature and the minimal possible temperature at a fixed location \(x_0 \in [-\pi, \pi]\) of the rod. Notice that if \(f^-(x) = f(-x)\), then \(u_{f^*}(x) = u_f(-x)\). Thus, working with this problem, we may assume that \(x_0 \in [0, \pi]\). Its solution is given by the following theorem.

**Theorem 2.** Let \(u_f\) solves the Robin problem for \(f \in \mathcal{F}(m, M, s)\) and \(\alpha > 0\) and let \(x_0 \in [0, \pi]\) be fixed. Then

\[
\eta_a(x_0)M - (M - m)\nu_a(x_0, l^-) \leq u_f(x_0) \leq \eta_a(x_0)m + (M - m)\nu_a(x_0, l^+) \tag{1.8}
\]

where \(l = \pi(s - m)/(M - m)\), \(l^- = \pi - l\), and the functions \(\eta_a(x)\) and \(\nu_a(x, l)\) are defined in Section 2 by equations (2.2) and (2.4), respectively.

Equality holds in the right inequality in (1.8) if and only if \(f = f_0^+\) a.e. on \([-\pi, \pi]\), where \(f_0^+ = m + (M - m)\chi_{I(a_m, l)}\) with \(l\) defined above and \(a_m = x_0(1 - lc_\alpha)\) if \(x_0(1 - lc_\alpha) < \pi - l\) and \(a_m = \pi - l\) otherwise.

Equality holds in the left inequality in (1.8) if and only if \(f = f_0^-\) a.e. on \([-\pi, \pi]\), where \(f_0^- = M - (M - m)\chi_{I(a_m, l^-)}\) with \(l^- = \pi - l\) and \(a_m = x_0(1 - l^- c_\alpha)\) if \(x_0(1 - l^- c_\alpha) < \pi - l^-\) and \(a_m = \pi - l^-\) otherwise.

It would be also useful to know how warmer a fixed spot \(x_0\) could be compared to the edges of the rod. The answer to this question is the following.

**Theorem 3.** Let \(u_f\) solves the Robin problem for \(f \in \mathcal{F}(m, M, s)\) and \(\alpha > 0\) and let \(x_0 \in [-\pi, \pi]\) be fixed. Then

\[
u_a(x_0)M - (M - m)\tau_a(x_0, l) \leq u_f(x_0) \leq \nu_a(x_0)m + (M - m)\tau_a(x_0, l^+) \tag{1.9}\]

where \(l = \pi(s - m)/(M - m)\), the function \(\eta_a(x)\) is defined by equation (2.3) and the function \(\tau_a(x, l)\) is defined by equation (3.5).

Equality holds in (1.9) if and only if \(f = f_e\) a.e. on \([-\pi, \pi]\), where \(f_e = m + (M - m)\chi_{I(a_e, l)}\) with \(l\) defined above and \(a_e = a_e(x, l)\) defined in equation (3.4).

The main results of [3] stated in Theorems 1.3 and 1.5 concern the comparison principles for heating problems with the Robin and Newman boundary conditions, respectively. In these problems, the extremal distribution of heat is symmetric with respect to the center of the rod. With this symmetry, the authors of [3] were able to use symmetrization methods due to G. Talenti [9] and A. Baernstein II [2] to prove their theorems. We want to mention here that S. Abramovich in her paper [1] published in 1975 already used the symmetrization method due to G. Pólya and G. Szegö [11] to prove interesting results on the eigenvalues of the differential system \(y''(x) + \lambda p(x)y(x) = 0\), \(y(\pm 1) = 0\) for the function \(p(x) \geq 0\) defined on the string \((-1, 1)\). More
recently, the symmetrization method similar to the one used in \[5\] in combination with the polarization technique was used in \[3\] to study several problems on heat distribution in the cylindrical pipes heated along various regions on the surface area.

We stress here that the extremal distributions of heat in our Theorems 1, 2 and 3 are not symmetric with respect to the center of the rod, in general. Thus, the classical symmetrization technique cannot be applied in these problems while certain versions of polarization technique used in \[3\] still can be applied.

2. Heating a Fixed Spot by a Single Interval.

Suppose that the Robin rod \([-\pi, \pi]\) is heated with unit density along the interval \(I = I(a, l)\) centered at the point \(a \in (-\pi, \pi)\) with length \(2l\) such that \(-\pi \leq a - l < a + l \leq \pi\). Thus, we assume here that \(f = \chi_I\). Using the integral representation (1.4) for the Robin temperature with the Green’s function given by (1.5), we evaluate \(u_{\chi_I} = u_{\chi_{I(a,l)}}(x, \alpha)\) as follows:

\[
u_{\chi_I} = \int_{I} [(1/2)c_\alpha x y - (1/2)|x - y| + 1/(2c_\alpha)] dy = \begin{cases} 
  l[(1 - ac_\alpha)x + (c_\alpha^{-1} - a)], & -\pi \leq x \leq a - l, \\
  -\frac{1}{2}a^2 + a(1 - l c_\alpha)x + \frac{l}{c_\alpha} - \frac{\alpha^2 l^2}{2}, & a - l < x < a + l, \\
  l[-(1 + ac_\alpha)x + (c_\alpha^{-1} + a)], & a + l \leq x \leq \pi. 
\end{cases} \tag{2.1}
\]

In particular, if the whole rod \([-\pi, \pi]\) is heated with unit density, then \(a = 0, l = \pi, \) and \(u_{\chi_{[-\pi, \pi]}}(x) = \eta_{\alpha}(x)\), where

\[
\eta_{\alpha}(x) = \frac{1}{2}x^2 + \frac{\pi}{\alpha} + \frac{\pi^2}{2}, \quad -\pi \leq x \leq \pi. \tag{2.2}
\]

Next, we fix \(x_0 \in [0, \pi], l \in (0, \pi), \alpha > 0\) and treat \(u_{\chi_I}\) as a function \(F(a)\) of the variable \(a \in [-\pi + l, \pi - l]\).

We have to consider the following cases:

1) If \(0 \leq x_0 \leq -\pi + 2l\), then

\[
F(a) = \frac{1}{2}a^2 + x_0(1 - lc_\alpha)a - \frac{1}{2}x_0^2 + \frac{l}{c_\alpha} - \frac{l^2}{2}, \quad -\pi + l \leq a \leq -\pi + l,
\]

2) If \(-\pi + 2l < x_0 < \pi - 2l\), then

\[
F(a) = \begin{cases} 
  l(1 - x_0c_\alpha)a + l(\frac{1}{c_\alpha} - x_0), & -\pi + l \leq a \leq x_0 - l, \\
  -\frac{1}{2}a^2 + x_0(1 - lc_\alpha)a - \frac{1}{2}x_0^2 + \frac{l}{c_\alpha} - \frac{l^2}{2}, & x_0 - l \leq a \leq x_0 + l, \\
  -l(1 + x_0c_\alpha)a + l(\frac{1}{c_\alpha} + x_0), & x_0 + l \leq a \leq \pi - l.
\end{cases}
\]

3) If \(\max\{-\pi + 2l, \pi - 2l\} \leq x_0 \leq \pi\), then

\[
F(a) = \begin{cases} 
  l(1 - x_0c_\alpha)a + l(\frac{1}{c_\alpha} - x_0), & -\pi + l \leq a \leq x_0 - l, \\
  -\frac{1}{2}a^2 + x_0(1 - lc_\alpha)a - \frac{1}{2}x_0^2 + \frac{l}{c_\alpha} - \frac{l^2}{2}, & x_0 - l \leq a \leq \pi - l.
\end{cases}
\]

A simple argument, left to the interested reader, shows that, in all three cases, \(F(a)\) is positive and concave on the interval \([-\pi + l, \pi - l]\) and takes
its minimal value $\mu_\alpha = \mu_\alpha(x_0, l)$ at $a = -\pi + l$. Evaluating $\mu_\alpha = F(-\pi + l)$, we find

$$\mu_\alpha = \begin{cases} l[(\pi - l)c_\alpha - 1)x_0 + \frac{1}{c_\alpha} + l - \pi], & \text{if } -\pi + 2l < x_0 \leq \pi, \\ -\frac{l}{2}x_0^2 - (\pi - l)(1 - l)c_\alpha)x_0 - l^2 + l\left(\frac{1}{c_\alpha} + \pi\right) - \frac{\pi^2}{4}, & \text{otherwise.} \end{cases}$$

Next, we find the maximum $\nu_\alpha = \nu_\alpha(x_0, l) = \max F(a)$ taken over the interval $-\pi + l \leq a \leq \pi - l$ and identify the point $a_m \in [-\pi + l, \pi - l]$, where this maximum is achieved. Let $q(a)$ denote the quadratic function as in parts 1–3). Then $q(a)$ takes its maximum at the point $a_0 = x_0(1 - l)c_\alpha$. Notice that $0 \leq a_0 \leq x_0$ with equality sign in either of these inequalities if and only if $x_0 = 0$. If $a_0 + l \leq \pi$, then the function $F(a)$ is defined at $a_0$ and takes its maximum at this point. Thus, if $a_0 + l \leq \pi$, then $a_m = x_0(1 - l)c_\alpha$. If $a_0 + l > \pi$, then $F(a)$ takes its maximum at $a_m = \pi - l$.

Combining these cases, we have the following equation for the central point $a_m = a_m(x_0, l, \alpha)$ of the heating interval of length $2l$, which generates the maximal temperature at the point $x_0$:

$$a_m = \begin{cases} x_0(1 - l)c_\alpha), & \text{if } x_0 < \frac{\pi - l}{1 - l/c_\alpha}, \\ \pi - l, & \text{otherwise}. \end{cases} \quad (2.3)$$

With these notations, we can evaluate the maximum $\nu_\alpha = \nu_\alpha(x_0, l)$ as follows:

$$\nu_\alpha = \begin{cases} \frac{l}{2}(1 - \frac{\pi}{l}) \left(1 - \frac{x_0^2}{\pi - x_0}\right), & \text{if } x_0 < \frac{\pi - l}{1 - l/c_\alpha}, \\ \frac{l}{2}(\pi - x_0)^2 + l(1 - l)c_\alpha x_0)(2\pi - l + \frac{1}{\alpha}), & \text{otherwise}. \end{cases} \quad (2.4)$$

The inequality $a_0 + l \leq \pi$ is equivalent to the inequality

$$\alpha \geq \frac{x_0 + l - \pi}{(\pi - l)(\pi - x_0)}.$$

Let us define $\alpha_m \geq 0$ as

$$\alpha_m = \begin{cases} \frac{x_0 + l - \pi}{(\pi - l)(\pi - x_0)}, & \text{if } x_0 < \pi - l, \\ 0, & \text{otherwise}. \end{cases} \quad (2.5)$$

Now, our arguments above show that if $x_0 \in [0, \pi]$, $l \in (0, \pi)$ and $\alpha > \alpha_m$, then the function $F(a)$ achieves its maximum $\nu_\alpha(x_0, l)$, given by the first line of (2.4), at the point $a_m = x_0(1 - l)c_\alpha$, and if $0 < \alpha \leq \alpha_m$, then $F(a)$ achieves its maximum $\nu_\alpha(x_0, l)$, given by the second line of (2.4), at the point $a_m = \pi - l$.

Combining our results and using the notation introduced above, we obtain the following lemma.

**Lemma 1.** 1) Let $x_0 \in [0, \pi]$, $l \in (0, \pi)$, and $\alpha > 0$ be fixed and let $a$ varies from $-\pi + l$ to $\pi - l$.

If $\alpha > \alpha_m$, then the function $F(a) = u_{\chi_l(a,l)}(x_0, \alpha)$ increases from its minimal value $\mu_\alpha(x_0, l)$ to its maximal value $\nu_\alpha(x_0, l)$ as $a$ varies from $-\pi + l$ to $a_m = x_0(1 - l)c_\alpha < \pi - l$, and $F(a)$ decreases as $a$ varies from $a_m$ to $\pi - l$.

If $0 < \alpha \leq \alpha_m$, then the function $F(a) = u_{\chi_l(a,l)}(x_0, \alpha)$ increases from $\mu_\alpha(x_0, l)$ to $\nu_\alpha(x_0, l)$ as $a$ varies from $-\pi + l$ to $a_m = \pi - l$.
2) Furthermore, the point \( a_m \), where \( F(a) \) takes its maximum, stays at \( \pi - l \) for \( 0 < \alpha \leq a_m \) and \( a_m \) decreases from \( \pi - l \) to \( \frac{\pi - l}{\pi}x_0 \), when \( \alpha \) runs from \( a_m \) to \( \infty \).

3) Moreover, if \( x_0 \in [0, \pi] \) and \( \alpha > 0 \) are fixed and \( a_m \) is considered as a function \( a_m(l) \) of \( l \), then if \( 0 < l_1 < l_2 < \pi \), then
\[
x_0 \in [a_m(l_1) - l_1, a_m(l_1) + l_1] \subset [a_m(l_2) - l_2, a_m(l_2) + l_2].
\]

3. Temperature Gap Between a Fixed Spot and the Edges of a Rod for a Single Interval.

As in the previous section, we assume that the Robin rod \([-\pi, \pi]\) is heated with unit density along the interval \( I = I(a, l) \). Let us fix \( x_0 \in [-\pi, \pi] \), \( l \in (0, \pi) \), \( \alpha > 0 \), and consider the temperature gap between the point \( x_0 \) and the left edge of the rod as a function of \( \alpha \in [-\pi + l, \pi - l] \); i.e., we consider the function
\[
E(a) = u_{\chi_l}(x_0) - u_{\chi_l}(-\pi).
\]
To find \( E(a) \), we use equation (2.1). Depending on the values of \( l \) and \( x_0 \), we have to consider the following cases:

1) If \( -\pi \leq x_0 \leq \min\{-\pi + 2l, \pi - 2l\} \), then
\[
E(a) = \begin{cases}
-\frac{1}{2}a^2 + (x_0(1 - l\alpha) + l(1 - \pi\alpha))a - \frac{1}{2}x_0^2 + \pi l - \frac{l^2}{2}, & \text{if } -\pi + l \leq a \leq x_0 + l, \\
-l\alpha l(\pi + x_0)a + l(\pi + x_0), & \text{if } x_0 + l \leq a \leq \pi - l.
\end{cases}
\]

2) If \( -\pi + 2l < x_0 < \pi - 2l \), then
\[
E(a) = \begin{cases}
\frac{1}{2} a^2 + (x_0(1 - l\alpha) + l(1 - \pi\alpha))a - \frac{1}{2}x_0^2 + \pi l - \frac{l^2}{2}, & \text{if } -\pi + l \leq a \leq x_0 - l, \\
-l\alpha l(\pi + x_0)a + l(\pi + x_0), & \text{if } x_0 - l \leq a \leq \pi - l.
\end{cases}
\]

3) If \( \max\{-\pi + 2l, \pi - 2l\} \leq x_0 \leq \pi, \) then
\[
E(a) = \begin{cases}
\frac{1}{2} a^2 + (x_0(1 - l\alpha) + l(1 - \pi\alpha))a - \frac{1}{2}x_0^2 + \pi l - \frac{l^2}{2}, & \text{if } -\pi + l \leq a \leq x_0 - l, \\
-l\alpha l(\pi + x_0)a + l(\pi + x_0), & \text{if } x_0 - l \leq a \leq \pi - l.
\end{cases}
\]

4) If \( \pi - 2l \leq x_0 \leq -\pi + 2l \), then for \( -\pi + l \leq a \leq \pi - l \),
\[
E(a) = -\frac{1}{2}a^2 + (x_0(1 - l\alpha) + l(1 - \pi\alpha))a - \frac{1}{2}x_0^2 + \pi l - \frac{l^2}{2}.
\]

Next, we show that in each of the four cases above there is a unique point \( a_c \in [-\pi + l, \pi - l] \), where the function \( E(a) \) achieves its maximum, we call it \( \tau_c = \tau_c(x_0, l) \).

On can easily check that in all cases, \( E'(-\pi + l) > 0 \) and, in the cases 1) and 2), \( E'(\pi - l) < 0 \). Since \( E(a) \) is a smooth at most quadratic function, we conclude from this that, in the cases 1) and 2), \( E(a) \) achieves its maximum \( \tau_c \) at the point \( a_c = x_0 + l - l\alpha l(\pi + x_0) \), \( -\pi + l < a_c < x_0 + l \).

In the cases 3) and 4), we find that
\[
E'(\pi - l) = -\pi + 2l + x_0 - l(\pi + x_0)\alpha.
\]

Since, in the cases 3) and 4), $-\pi + 2l + x_0 > 0$, we conclude from (3.4) and (1.5) that $E'(\pi - l) < 0$ if $\alpha > \alpha_e$ and $E'(\pi - l) \geq 0$ if $0 < \alpha \leq \alpha_e$, where $\alpha_e$ is defined as

$$\alpha_e = \begin{cases} 
-\pi + 2l + x_0 & \text{if } x_0 > \pi - 2l, \\
0 & \text{if } x_0 \leq \pi - 2l.
\end{cases} \quad (3.3)$$

In the first of these cases, $E(a)$ takes its maximum $\tau_\alpha$ at the point $a_e = x_0 + l - lc_\alpha(\pi + x_0)$. In the second case, $E(a)$ achieves its maximum at the point $a_e = \pi - l$.

Combining these cases, we conclude that $E(a)$ achieves its maximum at the point

$$a_e = \begin{cases} 
\pi - l, & \text{if } x_0 > \pi - 2l \text{ and } 0 < \alpha \leq \alpha_e, \\
x_0 + l - lc_\alpha(\pi + x_0), & \text{otherwise}. \quad (3.4)
\end{cases}$$

Calculating the values of $\tau_\alpha$ in the cases mentioned above, we obtain:

$$\tau_\alpha = \begin{cases} 
-\frac{1}{2}x_0^2 + \frac{\pi^2}{2} + (x_0(1 - lc_\alpha) + l(1 - \pi c_\alpha))(\pi - l), & \text{if } x_0 > \pi - 2l \text{ and } 0 < \alpha \leq \alpha_0, \\
\frac{1}{2}(x_0 + l - lc_\alpha(\pi + x_0))^2 - \frac{1}{2}x_0^2 + \pi l - \frac{1}{2}l^2, & \text{otherwise}. \quad (3.5)
\end{cases}$$

Combining our results, we obtain the following lemma.

**Lemma 2.** Let $x_0 \in [-\pi, \pi]$, $l \in (0, \pi)$, and $\alpha > 0$ be fixed and let $a$ vary from $-\pi + l$ to $\pi - l$.

If $x_0$ and $l$ satisfy the inequalities in parts 1) or 2) given above in this section or if $x_0$ and $l$ satisfy inequalities given in parts 3) or 4) and also $\alpha > \alpha_e$, where $\alpha_e$ is defined in (3.3), then the function $E(a) = u_{\chi_I}(x_0) - u_{\chi_I}(\pi)$ increases from $E(-\pi + l)$ to its maximal value $\tau_\alpha = \tau_\alpha(x_0, l)$ given by the second line in the equation (3.5) as $a$ varies from $-\pi + l$ to $a_e$ defined by equation (3.4), and $E(a)$ decreases as $a$ varies from $a_e$ to $\pi - l$.

If $x_0$ and $l$ satisfy the inequalities given in parts 3) or 4) and also $0 < \alpha \leq \alpha_e$, then the function $E(a)$ increases from $E(-\pi + l)$ to its maximal value $\tau_\alpha$ given by the first line in the equation (3.5) as $a$ varies from $-\pi + l$ to $\pi - l$.

4. **Maximal temperature gap for a single interval.**

The goal here is to evaluate the temperature gap $\text{osc}(u_J)$ for the heat source $J = \delta + \chi_{I(a,l)}$, $l \in [0, \pi)$, $a \in [0, \pi - l]$, $\delta \geq 0$. It follows from our calculations in Section 2 that

$$u_J(x) = \delta \eta_\alpha(x) + u_{\chi_{I(a,l)}}(x), \quad (4.1)$$

where $\eta_\alpha(x)$ is defined by (2.2) and $u_{\chi_{I(a,l)}}$ is defined by (2.1).

An elementary calculation shows that, under our assumptions, $u_J(-\pi) \leq u_J(\pi)$ with equality sign if and only if $a = 0$. Since $u_J$ is concave on $[-\pi, \pi]$ it follows that

$$\min_{[-\pi, \pi]} u_J = u_J(-\pi) = \alpha^{-1}(\pi \delta + l(1 - ac_\alpha)). \quad (4.2)$$
To find the maximum of \( u_J \), we note that \( u_J \) is a concave piece-wise at most quadratic function, which achieves its maximum at the point

\[
x_0 = \frac{a(1 - lc_\alpha)}{1 + \delta}.
\]

(4.3)

Thus, \( 0 \leq x_0 \leq a \) with equality sign if and only if \( a = 0 \). Evaluating \( u_J(x_0) \) and simplifying, we find:

\[
\max_{[-\pi, \pi]} u_J = u_J(x_0) = \frac{1}{2} \left( \frac{(1 - lc_\alpha)^2}{1 + \delta} - 1 \right) a^2 + \frac{\pi \delta}{\alpha} + \frac{\pi^2 \delta}{2} + \frac{l}{c_\alpha} - \frac{l^2}{2}.
\]

(4.4)

Combining (4.2) and (4.4) and simplifying, we find that \( \text{osc}(u_J) = H_\alpha \), where \( H_\alpha = H_\alpha(a, l, \delta) \) is given by

\[
H_\alpha = \frac{1}{2} \left( \frac{(1 - lc_\alpha)^2}{1 + \delta} - 1 \right) a^2 + \frac{l}{1 + \alpha \pi} a + \pi l + \frac{\pi^2 \delta}{2} - \frac{l^2}{2}.
\]

(4.5)

Next we fix \( l \in (0, \pi) \), \( \alpha > 0 \), \( \delta \geq 0 \) and treat \( H_\alpha \) as a function \( H_\alpha(a) \) of \( a \in [0, \pi - l] \). Since \( (1 - lc_\alpha)^2 / (1 + \delta) < 1 \), it follows from (4.5) that \( H_\alpha(a) \) is concave and takes its maximum at the point

\[
a_0 = \frac{(1 + \delta)}{(1 + \alpha \pi)(\delta + 2lc_\alpha - l^2c_\alpha^2)}.
\]

(4.6)

Now, the question is whether or not the point \( a_0 \) given in (4.6) belongs to the interval \([0, \pi - l] \). Notice that the function \( \delta + 2lc_\alpha - l^2c_\alpha^2 \) in the denominator of (4.6) increases when \( \alpha \) varies from 0 to \( \infty \). This implies that, if \( \delta \) and \( l \) are fixed, then \( a_0 \), defined by (4.6) and considered as a function \( a_0 = a_0(\alpha) \) of \( \alpha \), decreases from \((1 + \delta)l/\delta \) to 0 if \( \delta > 0 \) or from \( \infty \) to 0 if \( \delta = 0 \), when \( \alpha \) varies from 0 to \( \infty \). Therefore if \((1 + \delta)l \leq \delta(\pi - l) \), then \( a_0 \in [0, \pi - l] \) for all \( \alpha > 0 \) and if \((1 + \delta) > \delta(\pi - l) \), then there is a unique \( a_0 > 0 \) such that \( a_0 \in [0, \pi - l] \) for \( \alpha \geq a_0 \) and \( a_0 > \pi - l \) for \( 0 < \alpha < a_0 \). The value of \( a_0 \) is the positive root of the quadratic equation

\[
(\pi^2 \delta + 2\pi l - l^2)\alpha^2 + \frac{2(\pi \delta + l)(\pi - l) - \pi l(1 + \delta)}{\pi - l} \alpha + \delta - \frac{(1 + \delta)l}{\pi - l} = 0,
\]

which is

\[
a_0 = \frac{-2\pi^2 \delta + 3\pi \delta l - \pi l + 2l^2 + l\sqrt{(1 + \delta)(\pi^2 \delta + 9\pi^2 - 16\pi l + 8l^2)}}{2(\pi^3 \delta - \pi^2 \delta l + 2\pi^2 l - 3\pi l^2 + l^3)}.
\]

(4.7)

(4.8)

When \( \delta = 0 \) and \( l = \pi/2 \), we obtain \( a_0 = \frac{2}{\sqrt{3}\pi} \) that is the transitional value of \( \alpha \) found in Proposition 3.3 in [5].

Thus, we can define the following parameters:

\[
 a_\varphi = \left\{\begin{array}{ll}
a_0 & \text{if } (1 + \delta)l > \delta(\pi - l), \\
0 & \text{if } (1 + \delta)l \leq \delta(\pi - l),
\end{array}\right.
\]

(4.9)

and

\[
a_g = \left\{\begin{array}{ll}
\pi - l & \text{if } (1 + \delta)l > \delta(\pi - l), \\
a_0 & \text{if } (1 + \delta)l \leq \delta(\pi - l).
\end{array}\right.
\]

(4.10)

Figure 2, which illustrates possible situations, contains graphs of \( \text{osc}(u_J) = H_\alpha(a) \) considered as a function of the parameter \( a \) for some fixed values of \( l, \delta \) and \( \alpha \). These graphs show, in particular, that when \( \alpha \) varies from 0 to \( \infty \), the central point of the segment providing the maximal oscillation for \( u_J \)
Fig 2. Graphs of $\text{osc}(u_J) = H_\alpha(a)$ as a function of $a$ for different choices of $l$, $\delta$, and $\alpha$.

moves toward the center of the rod from the right. For a special case when $\delta = 0$, $l = \pi/2$, this behavior had been already observed in Proposition 3.3 in [5].

Summarizing the results of this section and evaluating the value of the oscillation $\text{osc}(u_J) = H_\alpha(a, l, \delta)$ at the point $a = a_g$ defined in (4.10), we obtain the following.

Lemma 3. Let $l \in (0, \pi)$, $a \in [0, \pi - l]$, $\alpha > 0$, and $\delta \geq 0$ be fixed and let $u_J$ be defined as in (4.1). Then

$$\max_{a \in [0, \pi]} \text{osc}(u_J) = \Theta_\alpha(l, \delta),$$

where

$$\Theta_\alpha = \frac{1}{2} \left( \frac{1 + \delta}{\pi^2 \delta + 2\pi l - l^2} + \frac{2(\pi \delta + l) + \delta}{\pi l + \frac{\pi^2 \delta}{2} - \frac{l^2}{2}} \right),$$

when $(1 + \delta)l < \delta(\pi - l)$ and $\alpha > 0$ or $(1 + \delta)l \geq \delta(\pi - l)$ and $\alpha > \alpha_g$, and

$$\Theta_\alpha = \frac{1}{2} \left( \frac{(\pi^2 \alpha - 2\pi \alpha - \delta(\pi - l)^2}{1 + \delta} + \frac{l(\pi - l)}{\pi l + \frac{\pi^2 \delta}{2} - \frac{l^2}{2}} \right),$$

when $(1 + \delta)l \geq \delta(\pi - l)$ and $0 < \alpha \leq \alpha_g$.

Furthermore, if $l \in (0, \pi)$, $\alpha > \alpha_g$, and $\delta \geq 0$ are fixed, then $\text{osc}(u_J)$ considered as a function of $a$ increases when it varies from 0 to $a_0$ defined by (4.6) and $\text{osc}(u_J)$ decreases when it varies from $a_0$ to $\pi - l$. If $0 < \alpha \leq \alpha_g$, then $\text{osc}(u_J)$ increases when it varies from 0 to $\pi - l$.

5. APPROXIMATION BY STEP FUNCTIONS AND CONTINUITY.

In this section, we collect auxiliary results on convergence of sequences of solutions $u_{f_n}$ of the Robin problem, on approximation of $u_J$ by sequences $u_{f_n}$ with piece-wise constant functions $f_n$ and on continuity of $u_{f_n}$ as a function of the parameters of approximants. These results will be used to prove our main theorems. We start with the following convergence lemma.

Lemma 4. If $f_n \in \mathcal{F}(m, M, s)$, $n = 1, 2, \ldots$, and $f_n \to f$ ($n \to \infty$) a.e. on $[-\pi, \pi]$, then $u_{f_n} \to u_J$ uniformly on $[-\pi, \pi]$ and for some subsequence,

$$\lim_{k \to \infty} \text{osc}(u_{f_{n_k}}) = \text{osc}(u_J).$$
Lemma 5. Let the intervals $I_n$ be the subintervals of equal length. We need the following approximation result.

So, $u_{f_n}$ converges uniformly to $u_f$ on $[-\pi, \pi]$.

Therefore, for every $x \in [-\pi, \pi],$

$$u_{f_n}(x) = \lim_k u_{f_{n_k}}(x) \geq \lim_k u_{f_{n_k}}(x_{n_k}) = u_f(x_o).$$

So $u_f(x_o) = \min u_f$. Similarly $u_f(\tilde{x}_o) = \max u_f$. It follows that

$$\text{osc}(u_{f_{n_k}}) = u_{f_{n_k}}(\tilde{x}_{n_k}) - u_{f_{n_k}}(x_{n_k}) \rightarrow u_f(\tilde{x}_o) - u_f(x_o) = \text{osc}(u_f).$$

The proof is complete. \qed

For $n, k \in \mathbb{N}$, $1 \leq k \leq n$, let $I_{n,k} = [-\pi + 2\pi(k-1)/n, -\pi + 2\pi k/n]$. Thus, the intervals $I_{n,1}, \ldots, I_{n,n}$ constitute a partition of the interval $[-\pi, \pi]$ into $n$ subintervals of equal length. We need the following approximation result.

Lemma 5. Let $f \in F(m, M, s)$ and $\alpha > 0$. Then for every $n, k \in \mathbb{N}$, $1 \leq k \leq n$, there are constants $c_{n,k}$, $m \leq c_{n,k} \leq M$, such that $f_n = \sum_{k=1}^{n} c_{n,k} \chi_{I_{n,k}}$ satisfies the following:

1) $m \leq f_n \leq f_{n+1} \leq M$, $f_n \rightarrow f$ a.e. on $[-\pi, \pi]$, \[ ||f_n||_{L^1} \rightarrow ||f||_{L^1} = \frac{2\pi s}{\alpha} \]

2) $u_{f_n}(x) \rightarrow u_f(x)$ uniformly on $[-\pi, \pi]$.

Proof. Let $\varepsilon > 0$. By a standard approximation result (see e.g. [3] Theorem 3.14)), there exists a continuous function $g$ on $[-\pi, \pi]$ such that \[ ||f - g||_{L^1} < \varepsilon/2 \]. We can demand that $m \leq g \leq M$. For $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$, set

$$c_{n,k} = \min \{ g(x) : x \in I_{n,k} \} \quad \text{and} \quad f_n = \sum_{k=1}^{n} c_{n,k} \chi_{I_{n,k}}.$$ 

Since $f_n \leq g$ and $g$ is Riemann integrable,

$$||f_n - g||_{L^1} = \int_{[-\pi, \pi]} (g - f_n) \rightarrow 0 \quad (n \rightarrow \infty).$$

Hence $f_n \rightarrow g$ in $L^1$. So $||f_n - g||_{L^1} < \varepsilon/2$ for all sufficiently large $n$. It follows that $f_n \rightarrow f$ in $L^1$. Moreover, $(f_n)$ is an increasing sequence of functions. Therefore, $f_n \rightarrow f$ a.e. on $[-\pi, \pi]$. The other assertions come easily from Lemma 1. \qed

Let $n \in \mathbb{N}$, $m, M, s \in \mathbb{R}$, $0 \leq m \leq s \leq M$, and $\alpha > 0$ be fixed. Let $K_n = K_n(m, M, s)$ denote the compact set of points

$$(\vec{t}, \vec{c}) = (t_1, t_2, \ldots, t_n, c_1, \ldots, c_n) \in \mathbb{R}^{2n}$$
such that $-\pi = t_0 \leq t_1 \leq \ldots \leq t_n = \pi$, $0 \leq c_k \leq M - m$, and

$$\sum_{k=1}^{n} c_k(t_k - t_{k-1}) = 2\pi(s - m).$$

For $(\bar{\ell}, \bar{x}) \in K_n$ and $x \in [-\pi, \pi]$, consider the function $f_{\bar{\ell}, \bar{x}} \in \mathcal{F}(m, M, s)$ defined as

$$f_{\bar{\ell}, \bar{x}}(x) = m + \sum_{k=1}^{n} c_k \chi_{I_k}(x), \text{ where } I_k = [t_{k-1}, t_k].$$

Then the solution $u_{f_{\bar{\ell}, \bar{x}}}(x)$ of the Robin problem is a linear combination of elementary functions as in the equation (2.1). This immediately implies the following continuity lemma.

**Lemma 6.** Let $u_{f_{\bar{\ell}, \bar{x}}}(x)$ be the solution to the Robin problem considered as a function of $x \in [-\pi, \pi]$ and $(\bar{\ell}, \bar{x}) \in K_n$. Then $u_{f_{\bar{\ell}, \bar{x}}}(x)$ is continuous on the compact set $[-\pi, \pi] \times K_n$.

In particular, if $E_1$ is a compact subset of $[-\pi, \pi]$, $E_2$ is a compact subset of $K_n$, then there are points in $E_1 \times E_2$, where $u_{f_{\bar{\ell}, \bar{x}}}(x)$ achieves its minimum and maximum on $E_1 \times E_2$.

6. Main proofs.

Now we are ready to present proofs of our main results. We start with the proof of Theorem 2.

**Proof of Theorem 2.** Let $f \in \mathcal{F}(m, M, s)$. First, we prove the right inequality in (1.8). It follows from Lemma 5 that, for every $\varepsilon > 0$, there exists a piece-wise constant function $f_{n, \bar{\tau}} = m + \sum_{k=1}^{n} c_{n,k} \chi_{I_{n,k}} \in \mathcal{F}(m, M, s_1)$, where $\bar{\tau} = (c'_{n,1}, \ldots, c'_{n,n})$, such that $0 \leq c'_{n,k} \leq M - m$, $|s_1 - s| < \varepsilon$ and, for the given $x_0 \in [-\pi, \pi]$,

$$|u_{f_{n, \bar{\tau}}}(x_0) - u_f(x_0)| < \varepsilon. \quad (6.1)$$

We note here that the integer $n$ in the definition of the function $f_{n, \bar{\tau}}$ can be chosen as large as we need for our proof. Indeed, for any integer $j \geq 1$, consider the partition of $[-\pi, \pi]$ into the intervals $I_{n,j,s}$, $1 \leq s \leq n$. Then consider $\bar{\tau}'' = (c''_{n,j,1}, \ldots, c''_{n,j,n})$ such that $c''_{n,j,s} = c'_{n,k}$ if $I_{n,j,s} \subset I_{n,k}$. With these notations, we have $f_{n, \bar{\tau}''} = f_{n, \bar{\tau}}$ a.e. on $[-\pi, \pi]$.

Next, for $n$ sufficiently large, we consider $f(x_0) = m + \sum_{k=1}^{n} c_k \chi_{I_{n,k}}(x_0)$ and $u_{f}(x_0)$ as functions of $\bar{\tau} = (c_1, \ldots, c_n)$, assuming that $\bar{\tau}$ varies over the compact set defined by the following conditions:

$$0 \leq c_k \leq M - m, \quad \frac{1}{n} \sum_{k=1}^{n} c_k = s_1 - m. \quad (6.2)$$

It follows from the continuity Lemma 6, that there is $\bar{\tau}^* = (c^*_1, \ldots, c^*_n)$, satisfying conditions (6.2), such that

$$\max u_{f}(x_0) \leq u_{f_{\bar{\tau}^*}}(x_0), \quad (6.3)$$

where the maximum is taken over all $\bar{\tau}$ satisfying conditions (6.2).
Let \( a^* = a_m\left(\frac{\pi}{n}\right) \) denote the center of the interval \( I \) of length \( 2\pi/n \), which provides the maximum to the function \( u_{\chi_l}(x) \) at the point \( x = x_0 \). We recall that \( a^* = a_m\left(\frac{\pi}{n}\right) \), given by equation (2.3), is such that \( 0 \leq a^* \leq x_0 \leq a^* + \frac{\pi}{n} \).

Suppose that the centers of the intervals \( I_{n,k} \) and \( I_{n,k+1} \) lie in the interval \([-\pi, a^*]\). Under these conditions, we claim that either \( c_{k}^* = 0 \) or \( c_{k+1}^* = M - m \). Indeed, if \( c_{k}^* > 0 \) and \( c_{k+1}^* < M - m \) then, for all sufficiently small \( \delta > 0 \), \( \tilde{f} = f_{\pi^*} - \delta \chi_{I_{n,k}} + \delta \chi_{I_{n,k+1}} \in \mathcal{F}(m, M, s_1) \). Furthermore, it follows from Lemma 1 that \( u_{\chi_{I_{n,k}}}(x_0) < u_{\chi_{I_{n,k+1}}}(x_0) \). The latter together with the additivity property (1.6) imply that \( u_f(x_0) > u_{f_{\pi^*}}(x_0) \), contradicting the inequality (6.3). A similar argument shows that if the centers of the intervals \( I_{n,k} \) and \( I_{n,k+1} \) lie in the interval \([a^*, \pi]\) then either \( c_{k}^* = M - m \) or \( c_{k+1}^* = 0 \).

Using these observations, we conclude that, if the integer \( n \) is large enough, then there are integers \( k_1, k_2, 1 \leq k_1 < k_2 \leq n \), such that \( a^* \leq c_{j}^* = 0 \) for \( 1 \leq j < k_1 \) and \( k_2 < j \leq n \), \( c_{j}^* = M - m \) for \( k_1 < j < k_2 \), and \( 0 \leq c_{j}^* < M - m \), for \( j = k_1 \) and \( j = k_2 \).

Since \( f_{n, \pi^*} \in \mathcal{F}(m, M, s_1) \), it follows from (6.3) and the additivity property (1.6) that

\[
    u_{f_{n, \pi^*}}(x_0) \leq u_{f_{\pi^*}}(x_0) < u_{\tilde{f}}(x_0),
\]

where \( \tilde{f} = m + (M - m)\chi_{\tilde{I}} \) and \( \tilde{I} = [-\pi + 2\pi k_1/n, -\pi + 2\pi k_2/n] \).

Let \( l_1 = \pi(s_1 - m)/(M - m) \) and let \( \tilde{l} \) denote the half length of the interval \( \tilde{I} \). Since \( f^* \in \mathcal{F}(m, M, s_1) \), it follows from our construction of \( \tilde{I} \) that

\[
    l_1 \leq \tilde{l} \leq l_1 + \frac{2\pi}{n}.
\]

It follows from Lemma 1 and equation (6.5) that

\[
    u_{\chi_{l_1}}(x_0) \leq \nu_{\alpha}(x_0, \tilde{l}) \leq \nu_{\alpha}(x_0, l_1) + O(1/n),
\]

where \( \nu_{\alpha}(x, l) \) is defined by (2.4). The latter inequality together with equation (2.2) and the additivity property (1.6) implies the following:

\[
    u_{\tilde{f}}(x_0) \leq \eta_{\alpha}(x_0)m + (M - m)\nu_{\alpha}(x_0, l_1) + O(1/n).
\]

Combining (6.1), (6.4) and (6.7), we conclude that

\[
    u_f(x_0) \leq u_{f_{n, \pi^*}}(x_0) + \varepsilon \leq \eta_{\alpha}(x_0)m + (M - m)\nu_{\alpha}(x_0, l_1) + \delta(\varepsilon, n),
\]

where \( \delta(\varepsilon, n) \to 0 \) as \( \varepsilon \to 0 \) and \( n \to \infty \). Since \( \varepsilon > 0 \) in equation (6.1) can be chosen arbitrarily small and the integer \( n \) can be chosen arbitrarily large, (6.8) implies the second inequality in (1.8).

To prove uniqueness, we assume that \( f \) does not coincide with \( f_{0^+} = m + (M - m)\chi_{I_{0, m, l}} \) on a set of positive measure. Since \( f \in L^1 \), there are “density points” \( x_1 \) and \( x_2 \), such that \( f_{0^+}(x_1) = M \) and \( f(x) < M - \varepsilon \) on a subset \( E_1 \) of some small interval \( I_1 \) centered at \( x_1 \), \( f_{0^+}(x_2) = m \) and \( f(x) > m + \varepsilon \) on a subset \( E_2 \) of some small interval \( I_2 \) centered at \( x_2 \). Let \( E = E_1 \cap (E_2 + (x_1 - x_2)) \). Since \( x_1 \) and \( x_2 \) are density points for \( f \in L^1 \), the one-dimensional Lebesgue measure of \( E \) is strictly positive. We may assume without loss of generality that \( E \) and \( E - (x_1 - x_2) \) either both lie on the interval \([-\pi, a_m] \) or both lie on the interval \([a_m, \pi] \). Under these assumptions, it follows from the monotonicity properties of Lemma 1 that
where the maximum is taken over all $\tilde{f} \in \mathcal{F}(m, M, s)$ such that $u_f(x_0) < u_f(x_0)$. Since $\tilde{f} \in \mathcal{F}(m, M, s)$, the right inequality in (1.8) holds true for $u_f$. Therefore, for the function $u_f$ the right inequality in (1.8) holds with the sign of strict inequality.

To prove the left inequality in (1.8), we assume that $f \in \mathcal{F}(m, M, s)$ and consider the function $f^- = M + m - f$. We have $m < f^- < M$ and $\|f^-\|_{L^1} = 2\pi s^-$ with $s^- = M + m - s$, $m < s^- < M$. Thus, $f^- \in \mathcal{F}(m, M, s^-)$ and therefore, by our proof above,

$$u_f^-(x_0) \leq \eta_{\alpha}(x_0)m + (M - m)\eta_{\alpha}(x_0, 1^-),$$

where $l^- = \pi(s^- - m)/(M - m) = \pi - l$. Since $f + f^- = M + m$, we have

$$u_f(x) + u_f^-(x) = \int_{-\pi}^{\pi} G(x, y)(f(y) + f^-(y)) dy = (M + m)\eta_{\alpha}(x).$$

Combining equations (6.9) and (6.10), we obtain the left inequality in (1.8).

Furthermore, (1.8) holds with the equality sign in the left inequality if and only if (6.9) holds with the sign of equality. We proved above that the latter holds if and only if $f^- = m + (M - m)\chi_{I(a_m, l^-)}$ a.e. on $[-\pi, \pi]$, which implies that the sign of equality in the left inequality in (1.9) occurs if and only if $f = f_0^-$ a.e. on $[-\pi, \pi]$.

The structure of proofs of Theorems 1 and 3 presented below is the same as in the proof of Theorem 2. Basically, to prove these theorems, we use the same arguments as in the proof of Theorem 2 with minor changes. Below, we sketch these proofs emphasizing these minor changes.

**Proof of Theorem 3.** First, given $\varepsilon > 0$, we approximate $f \in \mathcal{F}(m, M, s)$ with $f_{n, \varepsilon} \in \mathcal{F}(m, M, s_1)$ such that $|s_1 - s| < \varepsilon$ and

$$|(u_{f_{n, \varepsilon}}(x_0) - u_{f_{n, \varepsilon}}(-\pi)) - (u_f(x_0) - u_f(-\pi))| < \varepsilon.$$  \hspace{1cm} (6.11)

Then, using the continuity Lemma 6, we find $f^* \in \mathcal{F}(m, M, s_1)$ such that

$$\max\{u_{f_{n, \varepsilon}}(x_0) - u_{f_{n, \varepsilon}}(-\pi)\} \leq u_{f^*}(x_0) - u_{f^*}(-\pi),$$

where the maximum is taken over all $\varepsilon$ satisfying conditions (6.2).

Next, performing tricks with the intervals $I_{n, k}$ and $I_{n, k+1}$ as we did in the proof of Theorem 2 and using Lemma 2, we obtain a function $\tilde{f} = m + (M - m)\chi_{\tilde{I}}$, where $\tilde{I} = [-\pi + 2\pi k_1/n, -\pi + 2\pi k_2/n]$ with appropriate $k_1$ and $k_2$, such that the following holds:

$$u_{f^*}(x_0) - u_{f^*}(-\pi) < u_f(x_0) - u_f(-\pi) \leq
\begin{align*}
&m(\eta_{\alpha}(x_0) - \eta_{\alpha}(-\pi)) + (M - m)\tau_{\alpha}(x_0, l_1) + O(1/n).
\end{align*}$$

Since $\varepsilon > 0$ in (6.11) can be taken arbitrarily small and the integer $n$ in (6.13) can be taken arbitrarily large, combining (6.11), (6.12) and (6.13), we obtain the inequality in (1.9).

The proof of the uniqueness statement of Theorem 3, is almost identical with the uniqueness proof of Theorem 2. Namely, given $f \in \mathcal{F}(m, M, s)$, we use our “two density points argument”, to construct a function $\tilde{f} = f - \varepsilon \chi_{E - (x_1 - x_2)} + \varepsilon \chi_{E} \in \mathcal{F}(m, M, s)$, with $\varepsilon > 0$ small enough, such that
proof of Theorem 1. It follows from Lemma 5 that, for given $f \in F(m, M, s)$ and $\varepsilon > 0$ arbitrarily small, there exists a piece-wise constant function $f_{n, \varepsilon} = m + \sum_{k=1}^{n} c'_{n,k} x_{n,k} \in F(m, M, s_1)$, where $\varepsilon = (c'_{n,1}, \ldots, c'_{n,n})$, such that $0 \leq c'_{n,k} \leq M - m$, $|s_1 - s| < \varepsilon$ and
\[|\text{osc}(u_{f_{n,\varepsilon}}) - \text{osc}(u_f)| < \varepsilon.\] (6.14)
Here, we assume once more, that the integer $n$ is chosen as large as we need for our proof.

Then, using the continuity Lemma 6, we find $f_{\varepsilon} = m + \sum_{k=1}^{n} c^*_{n,k} x_{n,k} \in F(m, M, s_1)$ such that
\[\max\{\text{osc}(u_{f_{\varepsilon}})\} \leq \text{osc}(u_{f_{\varepsilon}}),\] (6.15)
where the maximum is taken over all $\tau$ satisfying conditions (6.2). Since $u_{f_{\varepsilon}}$ is concave on $[-\pi, \pi]$, we may assume without loss of generality that $\text{osc}(u_{f_{\varepsilon}}) = u_{f_{\varepsilon}}(x_0) - u_{f_{\varepsilon}}(-\pi)$ for some $x_0 \in (-\pi, \pi]$. If this is not the case, we replace $f_{\varepsilon}(x)$ with $f_{\varepsilon}(-x)$.

Next, we consider the temperature gap $E(a) = v_{\chi_{I(a,l_n)}}(x_0) - u_{\chi_{I(a,l_n)}}(-\pi)$ for the interval $I(a, l_n)$ centered at $a$ with half length $l_n = \pi/n$. As we have shown in Section 3, there is a unique point $a^* = a_c$ with $a_c$ given by equation (3.4), where $E(a)$ achieves its maximum $\tau_a$ given by equation (3.5).

If the centers of the intervals $I_{n,k}$ and $I_{n,k+1}$ both lie in the interval $[-\pi, a^*]$ or both lie in the interval $[a^*, \pi]$ then arguing as in the proof of Theorem 2 and using the monotonicity properties of Lemma 2, we conclude that if $f_{\varepsilon}$ is maximal in the sense of equation (6.15), then either $c^*_{k} = 0$ or $c^*_{k+1} = M - m$.

The latter implies that there exists an interval $\widehat{I} = [-\pi + 2\pi k_1/n, \pi + 2\pi k_2/n]$ with half length $\widehat{l} = \pi(k_2 - k_1)/n$ such that $l_1 < \widehat{l} < l_2 + \pi/n$, where $l_1 = \pi(s_1 - m)/(M - m)$, and such that for $\widehat{f} = m + (M - m)\chi_{\widehat{l}}$ we have
\[u_{f_{\varepsilon}}(x_0) - u_{f_{\varepsilon}}(-\pi) \leq u_{\widehat{f}}(x_0) - u_{\widehat{f}}(-\pi) \leq \text{osc}(u_{\widehat{f}}).\] (6.16)

Now, it follows from equations (4.11), (4.12), (4.13) of Lemma 3 that
\[\text{osc}(u_{\widehat{f}}) \leq (M - m)\Theta_a(\widehat{l}, \delta) = (M - m)\Theta_a(l, \delta) + \beta(\varepsilon, n),\] (6.17)
where $\delta = m/(M - m)$ and $\beta(\varepsilon, n) \to 0$ when $\varepsilon \to 0$ and $n \to \infty$.

Finally, combining equations (6.14) – (6.17), we obtain the inequality in (1.7).

To prove the uniqueness statement of Theorem 1, we use once more the “two density points argument” as in the proofs of Theorems 2 and 3, to construct a function $\tilde{f} = f - \varepsilon \chi_{E_{-(x_1 - x_2)}} + \varepsilon \chi_{E} \in F(m, M, s)$, with $\varepsilon > 0$ small enough, such that $\text{osc}(u_{\tilde{f}}) < \text{osc}(u_f)$. Since $\tilde{f} \in F(m, M, s)$, the inequality in (1.7) holds true for $u_{\tilde{f}}$. Therefore, for the function $u_{f}$, (1.7) holds with the sign of strict inequality. □
7. Temperature gap in higher dimensions

One can consider a variety of higher dimensional analogs of Problem 1. Here we present two such problems on the temperature gap in pipes $P_L$, which are cylindrical domains $P_L = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, |x_3| < L\}, L > 0$. Let $S_L$ denote the cylindrical boundary of $P_L$ and let $D_L \ni (0,0,L)$ and $D_{-L} \ni (0,0,-L)$ denote the boundary disks of $P_L$. By $\frac{\partial}{\partial n}$ we denote the outward normal derivative on $\partial P_L$ (for the boundary points where it is defined).

**Problem 2.** Let $E$ be a compact subset of $P_L$ of given volume $V$, $0 < V < 2\pi L$, and let $\alpha > 0$. Suppose that $u_E$ is a bounded solution to the Poisson equation

$$-\Delta u = \chi_E \quad \text{in} \quad P_L,$$

with mixed Neumann-Robin boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad S_L,$$

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on} \quad D_{\pm L}.$$

Find $\max \text{osc}(u_E)$ over all open sets $E \subset P_L$ of volume $V$ and identify sets $E^*$ providing this maximum.

**Problem 3.** Let $\Omega$ be a compact subset of $S_L$ of given area $A$, $0 < A < 4\pi L$ and let $\alpha > 0$. Suppose that $v_\Omega$ is a bounded solution to the Laplace equation

$$\Delta v = 0 \quad \text{in} \quad P_L,$$

with mixed Dirichlet-Robin boundary conditions

$$v = \chi_\Omega \quad \text{on} \quad S_L,$$

$$\frac{\partial v}{\partial n} + \alpha v = 0 \quad \text{on} \quad D_{\pm L}.$$

Find $\max \text{osc}(v_\Omega)$ over all open sets $\Omega \subset S_L$ of area $A$ and identify sets $\Omega^*$ providing this maximum.

We assume that solutions of Problems 2 and 3 exist and are regular; for the existence and regularity of elliptic problems with Robin boundary conditions, we refer to [6] and references therein. Problems 2 and 3 look challenging. Similar problems to find $\max u_E$ or $\max v_\Omega$ on $P_L$ instead of $\text{osc}(u_E)$ in Problem 2 or $\text{osc}(v_\Omega)$ in Problem 3 can be more tractable. We expect that the symmetric configurations will provide the corresponding maxima and therefore the symmetrization methods due to Talenti and Baernstein can be applied to solve these problems.

One can also consider analogs of Problems 2 and 3 in cylindrical domains in $\mathbb{R}^n$ of any dimension $n \geq 2$.

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