State-independent preparation uncertainty relations

Hubert de Guise,1 Lorenzo Maccone,2 Barry C. Sanders,3,4,* and Namrata Shukla3

1Department of Physics, Lakehead University, Thunder Bay, Ontario P7B 5E1, Canada
2Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy
3Institute for Quantum Science and Technology, University of Calgary, Calgary, Alberta, T2N 1N4, Canada
4Program in Quantum Information Science, Canadian Institute for Advanced Research, Toronto, Ontario M5G 1M1, Canada

Preparation uncertainty relations establish a trade-off in the statistical spread of incompatible observables. However, the Heisenberg-Robertson (or Schrödinger’s) uncertainty relations are expressed in terms of the product of variances, which is null whenever the system is in an eigenstate of one of the observables. So, in this case the relation becomes trivial and in the other cases it must be expressed in terms of a state-dependent bound. Uncertainty relations based on the sum of variances do not suffer from this drawback, as the sum cannot be null if the observables are incompatible, and hence they can capture fully the concept of quantum incompatibility. General procedures to construct generic sum-uncertainty relations are not known. Here we present one such procedure, based on Lie algebraic properties of observables that produces state-independent bounds. We illustrate our result for the cases of the Weyl-Heisenberg algebra, special unitary algebras up to rank 4, and any semisimple compact algebra.

For \( \Delta w^2 \) signifying the variance of measurement outcomes for the observable \( w \), Heisenberg’s uncertainty relation for position \( x \) and momentum \( p \) is

\[
[x, p] = i \mathbb{1}, [x, \mathbb{1}] = 0 = [p, \mathbb{1}] \implies \Delta x^2 \Delta p^2 \geq 1/4, \quad (1)
\]

for \( \mathbb{1} \) the identity operator, and fortuitously has a constant lower bound due to the appealing algebraic properties of observables \( x \) and \( p \). Unfortunately, Robertson’s generalization to \( \Delta A^2 \Delta B^2 \geq |\langle [A, B] \rangle |^2/4 \) for arbitrary observables \( A \) and \( B \) does have a state-dependent lower bound \( \mathbb{1} \), which fails to capture the intrinsic incompatibility of non-commuting observables \( \mathbb{2}, \mathbb{3} \). This cannot be amended as the underlying product of uncertainties is null whenever one of the uncertainties is null, an observation that provided impetus for the emergence of entropic uncertainty relations \( \mathbb{4, 5, 6} \) that eschew entropy in favor of variance.

Properly assessing uncertainty is important for foundational quantum mechanics \( \mathbb{10, 12} \) and for quantum information and communication \( \mathbb{13, 14, 15} \); variance is closer than entropy for practical quantum mechanics, a driving motivation behind research into sum-uncertainty relations (SURs), which deliver state-independent lower bounds \( \mathbb{16, 24} \). Here we provide a general framework to build SURs using algebraic properties of the observables involved, with examples of the Weyl-Heisenberg \( \mathfrak{w} \), special unitary \( \mathfrak{su}(n) \) and \( \mathfrak{su}(1,1) \) and generally semi-simple compact algebras. Our method (which is the main result of the paper) recovers some known uncertainty relations, enclosing them in a general framework and, most importantly, allows one to derive new ones, some of which we detail here.

We strongly emphasize that our results refer to the “preparation uncertainty” \( \mathbb{11, 12, 25} \) and not to the “measurement uncertainty”. The former refers to the variance of the outcomes of measurements of different observables performed on different systems prepared in identical states. The latter, which has been the subject of a lively debate recently \( \mathbb{13, 23, 31} \), refers to the relation between errors and post-measurement disturbance in an apparatus. We underline that they are very different notions \( \mathbb{13, 26, 28, 31} \), but this difference is often obscured in the literature.

As our relations are based on the sum of variances, they easily relate more than two observables and possess a simple physical interpretation as the diagonal of the uncertainty volume, depicted in Fig. 1. The uncertainty relations that we derive from algebraic properties of observables are stated below, with the explanation and derivation to follow. Some of our relations are prior knowledge and some are new. Each algebra is defined by its commutator relation, given by Eq. \( \mathbb{11} \) for \( \mathfrak{w} \) and by

\[
[J_x, J_y] = i J_z \quad \quad \quad [J_y, J_z] = i J_x \quad [J_z, J_x] = i J_y
\]

\[
[K_x, K_y] = -i K_z \quad \quad \quad [K_y, K_z] = i K_x \quad [K_z, K_x] = i K_y
\]

for \( \mathfrak{su}(2) \) and \( \mathfrak{su}(1,1) \), respectively. For semisimple com-

\[ FIG. 1: \text{Sum of variances is a measure of total uncertainty. Given a (green) box with the uncertainties as edges, the sum of variances is the squared length of the (red) diagonal. [Here 30000 (blue) points with Gaussian distribution corresponding to } \Delta x = \Delta y = .5, \Delta z = .25 \text{ are plotted as an illustration.} \]

pact Lie algebras, we use \{e_i\} in an operators basis for
diagonal Killing form, \{\lambda_i\} as group irrep labels,
|\Lambda\rangle the dominant integral weight and |\delta\rangle the Weyl root.

We claim the following tight state-independent SURs:
\[ \mathfrak{su}(1,1) : \Delta K^2 - \Delta K^2_{\delta} \geq \kappa \]  
and the semisimple compact case
\[ \frac{1}{2} \sum_i \Delta e^2_i \geq 2\langle \Lambda | \delta \rangle, \]
which yields special cases
\[ \mathfrak{su}(2) : \Delta J_x^2 + \Delta J_y^2 + \Delta J_z^2 \geq j, \text{ e.g. in} \]  
\[ \mathfrak{su}(3) : \frac{1}{2} \sum_i \Delta e^2_i \geq 2 \langle \lambda_1 + \lambda_2 \rangle, \]
\[ \mathfrak{su}(4) : \frac{1}{2} \sum_i \Delta e^2_i \geq 3 \lambda_1 + 4 \lambda_2 + 3 \lambda_3, \]
\[ \mathfrak{su}(5) : \frac{1}{2} \sum_i \Delta e^2_i \geq 4 \lambda_1 + 6 \lambda_2 + 6 \lambda_3 + 4 \lambda_4, \]
with all these state-independent lower bounds given by
constants that depend only on the choice of irreducible representation (irrep) As discussed later, there are states
for which the equality holds. A geometric intuition for relations follows from Pythagoras’ theorem: the left-hand-side is the squared length of the diagonal of a “box” with uncertainties as edges shown in Fig.1. These edges are a measure of the “total” uncertainty and bounded below by a positive constant.

We begin by developing our algebraic approach for the
familiar SUR based on the familiar \mathfrak{su}(1,1) algebra treated by Heisenberg. Heisenberg’s uncertainty relation follows from
\[ (\Delta x - \Delta p)^2 \geq 0 \implies \Delta x^2 + \Delta p^2 \geq 2 \Delta x \Delta p \geq 1, \]
which incorporates both the sum and the Heisenberg product \Delta x \Delta p \geq 1/2 relations in the same expression. Our focus is on the sum relation, and we now review how this SUR is derived by another approach.

We express \mathfrak{su} in terms of lowering \( a := (x + ip) / \sqrt{2} \) and raising \( a^\dagger \) ladder operators so
\[ \mathfrak{su} = \text{span} \{ a, a^\dagger, 1 \}, \ [a, a^\dagger] = 1, \]
with “weight”, or number, operator denoted \( n = a^\dagger a \) The
(Fock) eigenstates \{m\} satisfy \(|m\rangle : n \in \mathbb{N}\) such that
\[ n |m\rangle = m |m\rangle, \ a^\dagger |m\rangle = \sqrt{m + 1} |m + 1\rangle. \]
Henceforth a general arbitrary state is expressed as a sum
\[ |\psi\rangle = \sum \psi_m |m\rangle \]
over the weights \{m\} determined by diagonal operators for
the algebra being studied. If some weights are repeated, the sum extends over the orthogonal states of the same weights. The sum of variances
\[ \Delta x^2 + \Delta p^2 = 2\langle a^\dagger a \rangle + 1 - \langle x^2 \rangle - \langle p^2 \rangle. \]
is bounded by
\[ \langle x \rangle + i \langle p \rangle = \sqrt{2} \sum \nu_m, \nu_m := \sqrt{m + 1} \psi_{m+1} \psi_m^*. \]
The Cauchy-Schwarz inequality yields
\[ \langle x \rangle^2 + \langle p \rangle^2 = 2 \sum \nu_m^2 \leq 2 \sum (n + 1) |\psi_m + 1\rangle^2 \sum |\psi_{n'}|^2 = 2\langle a^\dagger a \rangle, \]
which proves Eq. (3), and the “lowest-weight state” |0\rangle saturates this bound.
Next we apply this \mathfrak{su} approach to the ubiquitous \mathfrak{su}(2) algebra, pertinent to systems of spin-\( j \) for \( 2j \in \mathbb{N} \). For \( J_\pm = J_x \pm i J_y \) the \mathfrak{su}(2) raising/lowering operators, \( \mathfrak{su}(2) = \text{span} \{ J_+, J_-, J_z \} \) such that
\[ |J_+, J_- \rangle = 2J_z, \ |J_z, J_\pm \rangle = \pm J_z, \]
The eigenstates \(|m\rangle : 0 \leq m \leq 2j \) of the weight operator \( J_z \), satisfying \( J_z |m\rangle = (m - j) |m\rangle \)
form a basis for the \( (2j + 1) \)-dimensional irrep of \mathfrak{su}(2) with
\[ C_2 = J^2_z + \frac{1}{2} (J_+ J_- + J_- J_+) = J^2_z + J^2_x + J^2_y = c_2 \mathbb{I}. \]
and eigenvalue \( c_2 = j(j + 1) \). Then
\[ \Delta J^2_x + \Delta J^2_y + \Delta J^2_z = c_2 - \sum_{i=x,y,z} \langle J_i \rangle^2. \]
This sum (20) can be bounded, analogous to the \mathfrak{su} case, by expanding for an arbitrary state \(|13\rangle\) as we now see.
For \( \mu_m := \psi_{m+1}^* \psi_m \sqrt{(m + 1)(2j - m)} \), we have
\[ \langle J_z \rangle = \sum_{m=0}^{2j} m |\psi_m|^2 - j, \langle J_x \rangle + i \langle J_y \rangle = \sum_{m=0}^{2j-1} \mu_m, \]
which leads to
\[ \langle J_x \rangle^2 + \langle J_y \rangle^2 \leq \left( \sum_{m=0}^{2j-1} |\psi_{m+1}|^2 (m + 1) \right) \left( \sum_{m' = 0}^{2j-1} |\psi_{m'}|^2 (2j - m') \right) \]
\[ \leq \left( \sum_{m=0}^{2j} |\psi_{m+1}|^2 m \right) \left( \sum_{m' = 0}^{2j} |\psi_{m'}|^2 (2j - m') \right) \]
\[ = 2j \sum_{m} |\psi_m|^2 m - \left( \sum_{m} |\psi_m|^2 m \right)^2. \]
using the Cauchy-Schwarz inequality. As

\[ (J_z)^2 = j^2 - 2j \sum_m |\psi_m|^2 m + \left( \sum_m |\psi_m|^2 m \right)^2 \]  

(24)

we obtain the desired \( su(2) \) uncertainty relation \[ \big( \mathfrak{s}\mathfrak{u}(2) \big) \]. Next we see that this approach robustly extends to the non-compact case.

Closely related to \( su(2) \) is the non-compact \( su(1,1) = \text{span} \{ K_+, K_-, K_\pm \} \) with ladder operators \( K_\pm = K_x \pm iK_y \) and commutation relations

\[ [K_+, K_-] = -2K_z, \quad [K_-, K_\pm] = \pm K_\pm, \]

(25)

where the operators \( K_{x,y,z} \) are self-adjoint bijections on Hilbert space. \( K_z \) eigenstates \( \{ m \} \), such that \( K_z | m \rangle = (m + \kappa) | m \rangle \) and \( m, \kappa \geq 0 \), form a basis for the infinite-dimensional unitary irrep \( \kappa \). We restrict our discussion to irreps of the positive discrete series, where the representation label common in physics are \( \kappa = 1/2, 1, 3/2, \ldots \). The analysis applies to the two limits of discrete series with labels \( \kappa = 1/4, 3/4 \). The eigenvalue \( m \) is discrete; continuous \( m \) \[ \text{[32]} \] is a topic for future investigation. The \( su(1,1) \) raising operators satisfies

\[ K_+ | m \rangle = \sqrt{(m+1)(2\kappa + m)} | m + 1 \rangle, \]  

(26)

and \( K_- = K_+^\dagger \). Evidently the ladder of \( | m \rangle \) states is unbounded above, but the \( K_z \) eigenstate \( | m = 0 \rangle \) with eigenvalue \( \kappa \) is annihilated by \( K_- \). The quadratic Casimir operator is

\[ C_2 f = K_x^2 - \frac{1}{2} (K_+ K_- + K_- K_+) = K_x^2 - K_y^2 - K^2 - c_2 1 \]

with \( c_2 = \kappa (\kappa - 1) \). The sum of variances is

\[ \Delta K_x^2 + \Delta K_y^2 + \Delta K_z^2 \geq \Delta K_x^2 + \Delta K_y^2 - \Delta K_z^2 \]

(27)

where \( \Delta K^2 \) is the highest-weight for the irrep \( \Lambda = (\Lambda_1, \ldots, \Lambda_r) \). The \( \ell \)-r operators are combinations of raising and lowering operators so, crucially, have null expectation value on any eigenstate of the Cartan elements, i.e. on any state of definite weight.

The Casimir operator \( C_2 \) and its state-independent eigenvalue \( c_2 \) are

\[ C_2 = \frac{1}{2} \sum_{k=1}^\ell e_k \]

(33)

with \( e_k \) a diagonal Cartan element and \( e_m \) a non-diagonal operator. For \( su(2) \) this would be the Hermitian basis \( J_{x,y} \). The \( \ell \)-r operators are combinations of raising and lowering operators so, crucially, have null expectation value on any eigenstate of the Cartan elements, i.e. on any state of definite weight.

The Casimir operator \( C_2 \) and its state-independent eigenvalue \( c_2 \) are

\[ C_2 = \frac{1}{2} \sum_{k=1}^\ell e_k^2, \quad c_2 = 2 \langle \Lambda | \delta \rangle + \langle \Lambda | \Lambda \rangle, \quad \Lambda := \sum_{i=1}^r \lambda_i | w_i \rangle \]

(34)

where, in the last equality, we assume the system state \( | \Lambda \rangle \) is an eigenstate of the \( r \) Cartan operators so that \( \langle e_m \rangle = \langle \Lambda | e_m | \Lambda \rangle = 0 \) for \( m > r \) due to the action of the raising and lowering operators. For the weight \( | \Lambda \rangle \),

\[ \frac{1}{2} \sum_{k=1}^r \langle e_k \rangle^2 = \langle \Lambda | \Lambda \rangle \leq \langle \Lambda | | \Lambda \rangle \]

(35)

where the upper bound is attained for the highest-weight state. Combining Eqs. \[ \text{[34]} \] and \[ \text{[35]} \] yields

\[ \frac{1}{2} \sum_{k=1}^\ell \Delta e_k^2 \geq c_2 - \langle \Lambda | | \Lambda \rangle = 2 \langle \Lambda | \delta \rangle, \]

(36)
which is the desired SUR \([5]\) for semisimple compact Lie algebras. Moreover, the uncertainty-sum relation is tight as the inequality is saturated by the highest-weight state \(|\Lambda\rangle\), its Weyl-reflected images and any state in the group orbit of \(|\Lambda\rangle\), i.e., any coherent state and the algebra’s coherent states \([38]\).

We now demonstrate the value of Eq. \([36]\) through its application to examples of compact unitary algebras, namely \(\mathfrak{su}(3)\), \(\mathfrak{su}(4)\) and \(\mathfrak{su}(5)\). For \(\mathfrak{su}(3)\), Eq. \([4]\) follows immediately from Eq. \([38]\). The Hermitian basis for the defining, i.e., 3-dimensional \((\lambda_1, \lambda_2) = (1, 0)\), irrep of \(\mathfrak{su}(3)\) is

\[
A_- = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},
\]

\[B_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad C_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[h_1 = \text{diag}(1, -1, 0), \quad h_2 = \text{diag}(1, 1, -2). \tag{37}\]

The Killing form is diagonal, and the quadratic Casimir operator

\[C_2 = \frac{1}{2} (A_-^2 + A_+^2 + B_-^2 + B_+^2 + C_-^2 + C_+^2 + h_1^2 + h_2^2)\]

with eigenvalue

\[c_2(\lambda_1, \lambda_2) = \frac{2}{3} (\lambda_1^2 + \lambda_2^2 + 3 |\lambda_1 + \lambda_2| + \lambda_1 \lambda_2) \tag{38}\]

for irrep \((\lambda_1, \lambda_2)\). For the \((1, 0)\) irrep \([37]\), \(C_2 = \frac{8}{3} \mathbb{I}\).

Now we verify inequality \([7]\) for a different \(\mathfrak{su}(3)\) irrep, namely the \((8\text{-dimensional})\) adjoint irrep \((1, 1)\). In this case

\[C_2 = 6 \mathbb{I}, \quad G_{SU(3)} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{39}\]

We obtain the lower uncertainty bound

\[
\frac{1}{2} \left[ \Delta(A_+)^2 + \Delta(A_-)^2 + \Delta(B_+)^2 + \Delta(B_-)^2 \\
+ \Delta(C_+)^2 + \Delta(C_-)^2 + \Delta(h_1)^2 + \Delta(h_2)^2 \right] \\
: = \frac{1}{2} \sum_i (\Delta \tilde{e}_i)^2 \geq 2(\lambda_1 + \lambda_2), \tag{40}\]

which confirms that the general formula \([36]\) gives the correct inequality \([7]\).

We generalize this procedure to \(\mathfrak{su}(4)\) and \(\mathfrak{su}(5)\), yielding \([5]\) and \([9]\) respectively, and confirm our procedure for irreps \((\lambda_1, \lambda_2, \lambda_3)\) and \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\), respectively. For \(\mathfrak{su}(4)\), we obtain the Gell-Mann matrices \(\Lambda_{1-15}\) in appendix following Stover’s procedure \([30]\) to obtain

\[c_2(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{4} \left[ 3 \lambda_1^2 + 2 [2 \lambda_2 + \lambda_3 + 6] \lambda_1 \\
+ 4 \lambda_2^2 + 4 \lambda_2 [\lambda_3 + 4] + 3 \lambda_3 [\lambda_3 + 4] \right]. \tag{41}\]

The lower bound \([8]\) is successfully obtained with each \(\epsilon_i\) replaced by \(\Lambda_i\) so our expression is confirmed for this \(\mathfrak{su}(4)\) irrep.

For \(\mathfrak{su}(5)\) we have the \(5 \times 5\) Gell-Mann matrices \(\Lambda_{1-24}\) given in appendix and we obtain

\[c_2(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{2}{5} [2 \lambda_1^2 + 3 \lambda_2 \lambda_1 + 2 \lambda_3 \lambda_1 + \lambda_4 \lambda_1 + 10 \lambda_1 + 3 \lambda_2^2 + 3 \lambda_3^2 + 2 \lambda_4^2 + 15 \lambda_2 + 4 \lambda_2 \lambda_3 + 15 \lambda_3 + 2 \lambda_2 \\
\lambda_4 + 3 \lambda_3 \lambda_4 + 10 \lambda_4]. \tag{42}\]

Replacing each \(\epsilon_i\) in inequality \([9]\) by \(\Lambda_i\) as detailed in appendix confirms that the SUR \([9]\) holds for this \(\mathfrak{su}(5)\) irrep. Finally, we note that the bound is the same for conjugate irreps, \(e.g., \) the \(\mathfrak{su}(3)\) irreps \((\lambda_1, \lambda_2)\) and \((\lambda_2, \lambda_1)\) have the same bound, with similar symmetry for conjugate representations holding for \(\mathfrak{su}(4)\) and \(\mathfrak{su}(5)\) irreps.

In conclusion, we have presented a class of state-independent tight SURs based on algebraic properties, and our scheme shows how to generalize to other algebras. Inequalities \([3]\) and \([6]\) were known previously \([18, 25, 33, 34, 40-42]\) as is inequality \([4]\) \([43]\), but bounds were not explicitly stated nor was their common algebraic origin from a similar derivation. Furthermore the state-independent nature of the tight lower bound was not investigated. Instead previous analyses focused on their connection with algebraic coherent states \([35]\). Different relations for state-independent variance-based uncertainty relations were known explicitly only for qubits \([19, 27, 44]\).

Our method exploits the relation between the SUR and the quadratic Casimir operator when the Killing form is diagonal, which is an easily generalizable notion including to infinite-dimensional irreps, but the state saturating this lower bound might not be normalizable. Some work needs to be done, such as dealing with continuous irreps, verifying SURs for various irreps and generalizing to other algebras and verifying, but an exciting path towards SURs, which are a valuable and practical alternative approach to entropic uncertainty relations, is established by our work. Some aspects of our work were known before but not in a unified, explicit, purely algebraic approach as done here.

HeDg and BCS each acknowledge NSERC support. LM acknowledges the “Blue sky” project of the University of Pavia and the FQXi foundation for financial support. NS acknowledges support from the University of Calgary Eyes High postdoctoral fellowship program.
Appendix: Gell-Mann matrices

The 4 × 4 Gell-Mann matrices are

\[
\begin{align*}
\Lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
\Lambda_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\Lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\Lambda_7 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_8 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \\
\Lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \\
\Lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{15} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_9' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{10}' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{11}' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{12}' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{13}' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{14}' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{15}' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]
\begin{align*}
\Lambda_{17} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\Lambda_{18} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\Lambda_{19} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\Lambda_{20} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
\Lambda_{21} &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\Lambda_{22} &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}, \\
\Lambda_{23} &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\Lambda_{24} &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
[32] W. Heisenberg, Z. Phys. 43, 172 (1927).

[33] R. Delbourgo, J. Phys. A: Math. Gen. 10, 1837 (1977), URL http://stacks.iop.org/0305-4470/10/i=11/a=012

[34] H. F. Hofmann and S. Takeuchi, Phys. Rev. A 68, 032103 (2003), URL https://link.aps.org/doi/10.1103/PhysRevA.68.032103

[35] G. M. D’Ariano, E. De Vito, and L. Maccone, Phys. Rev. A 64, 033805 (2001), URL https://link.aps.org/doi/10.1103/PhysRevA.64.033805

[36] J. F. Cornwell, Group Theory in Physics, vol. II (Academic, New York, 1984).

[37] R. Slansky, Phys. Rep. 79, 1 (1981), ISSN 0370-1573.

[38] A. Perelomov, Generalized Coherent States and Their Applications, Theoretical and Mathematical Physics (Springer, Berlin, 1986).

[39] C. Stover, Generalized Gell-Mann matrix, MathWorld — A Wolfram Web Resource.

[40] S. Shabbir and G. Björk, Phys. Rev. A 93, 052101 (2016), URL https://link.aps.org/doi/10.1103/PhysRevA.93.052101

[41] K. Abdelkhalek, W. Chemissany, L. Fiedler, G. Mangano, and R. Schwonnek, Phys. Rev. D 94, 123505 (2016), URL https://link.aps.org/doi/10.1103/PhysRevD.94.123505

[42] H. Fakhri and M. Sayyah-Fard, Found. Phys. 46, 1062 (2016).

[43] R. Delbourgo and J. R. Fox, J. Phys. A: Math. Gen. 10, L233 (1977), URL http://stacks.iop.org/0305-4470/10/i=12/a=004

[44] A. A. Abbott, P.-L. Alzieu, M. J. W. Hall, and C. Branciard, Mathematics 4, 8 (2016).