Spectroscopy of the Einstein-Maxwell-Dilaton-Axion black hole

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Abstract: The entropy spectrum of a spherically symmetric black hole was derived via the Bohr-Sommerfeld quantization rule in Majhi and Vagenas’s work. Extending this work to charged and rotating black holes, we quantize the horizon area and the entropy of an Einstein-Maxwell-Dilaton-Axion (EMDA) black hole via the Bohr-Sommerfeld quantization rule and the adiabatic invariance. The result shows the area spectrum and the entropy spectrum are respectively equally spaced and independent on the parameters of the black hole.

1. Introduction

It is widely believed that black holes have discrete horizon area spectra. The horizon area was first quantized in Bekenstein’s work by adding a particle into the black hole and the discrete spectrum was derived as [1]

$$A_n = 8\pi nl_p^2,$$

where $l_p$ is the Planck length and $n = 1, 2, 3, \ldots$. This implies that the area spectrum is equally spaced and the minimal spacing is $8\pi l_p^2$.

One popular method to quantize the horizon area was first put forward by Hod [2]. In this method, quasinormal modes (QNM) were needed. Using the quasinormal mode frequency and the Bohr’s correspondence principle, he proposed that the area spacing was $\Delta A = 4l_p^2 \ln 3$ and related to the real part of the modes. The value of this area spacing is in consistence with both the Boltzmann-Einstein formula in statistics and the area-entropy thermodynamic relation for black holes. However, there are some problems needed to
be solved. First, the factor $4 \ln 3$ associated with the real part is not a universal value. In Ref. [3], the generic spin-$j$ perturbations were investigated. It showed that the leading asymptotic value of the quasinormal mode frequencies is related to spin perturbations and given by $e^{8\pi M \omega_n} = - (1 + 2 \cos \pi j)$, where $\omega_n = \omega_R + i\omega_I$. When $j = 0$, $j = 1$ and $j = 2$, it describes the cases of scalar perturbation, vector perturbation and gravitational perturbation, respectively. It can be found that the real part would vanish for certain values of $j$. The research on the asymptotic behavior of the quasinormal mode frequencies of gravitational perturbations furthermore exhibits this problem [3, 4, 5]. Moreover, he only considered the case from the ground state to a state with large $n$. The result would be changed when one considers two arbitrary states. Black holes perturbed by exterior fields can be seen as damped oscillators. The frequency of a damped oscillator is its physical frequency and is related to both the real part and the imaginary part. Therefore the physical frequency of the perturbed black hole also has this property and is

$$\omega = \sqrt{\omega_R^2 + \omega_I^2}. \quad (2)$$

This view was proved in Maggiore’s work [6]. Applying this new explanation to the quantization of the horizon area of spherically symmetric black holes, he found that the area spacing is $8\pi l_p^2$. This value is different from that derived by Hod, but in consistence with Bekenstein’s result. Subsequently people have applied this new explanation to quantize the horizon area of other black holes and all of the results show the validity of this mode [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

The entropy/area quantization via tunneling mechanism was proposed in the work of Majhi and his collaborators [20, 21, 22]. From the Hawking radiation as tunneling effect, the entropy/area spectra were obtained. The result showed the minimal interval of entropy spectrum is 1, which is different from Bekenstein and Maggiore’s results. The area spectra can be obtained from the relation between the entropy and the area, while the relation is different in different gravity theories.

Recently, the work of Majhi and Vagenas has showed that the black hole entropy could be quantized without QNMs [23]. In their work, the black hole is non-extreme and the horizon is seen as an adiabatic invariant. From the Hamilton function, they quantized the entropy of a spherically symmetric black hole via the Bohr-Sommerfeld quantization rule. The entropy spectrum was derived and equally spaced, which is full in consistence with that derived
by Maggiore and Bekenstein. One interesting point is that the QNMs didn’t appear in their work. Subsequently, this work has been extended and people have quantized the entropy via the periodicity of the particle’s wave function [24, 25].

In this paper, we extend this work to the charged and rotating black holes. We quantize the horizon area and the entropy of an Einstein-Maxwell-Dilaton-Axion (EMDA) black hole by the Bohr-Sommerfeld quantization rule and the adiabatic invariance. The result shows both the area spectrum and the entropy spectrum are equally spaced.

The rest is organized as follows. In Sect. 2, the EMDA spacetime is reviewed. In Sect. 3, we first derive the Hamiltonian of the EMDA black hole system, and then quantize the horizon area and the entropy via the Bohr-Sommerfeld quantization rule. In Sect. 4 we offer a summary and discussion.

2. The EMDA spacetime

The metric of the EMDA black hole is given by [26]

$$\begin{align*}
    ds^2 &= -\frac{\Delta - a^2 \sin^2 \theta}{\sum} dt^2 + \frac{\sum}{\Delta} dr^2 + \sum d\theta^2 + \frac{2a \sin^2 \theta (r^2 + 2br + a^2 - \Delta)}{\sum} dtd\phi \\
    &\quad + \frac{(r^2 + 2br + a^2)^2 - \Delta a^2 \sin^2 \theta}{\sum} \sin^2 \theta d\phi^2, \\
\end{align*}$$

(3)

with the electromagnetic vector potential

$$A'_\mu = A'_t dt + A'_\phi d\phi = \frac{Qr}{\sum} dt - \frac{Qra \sin^2 \theta}{\sum} d\phi,$$

where

$$\begin{align*}
    r_{\pm} &= M \pm \sqrt{M^2 - a^2}, \\
    \sum &= r^2 + 2br + a^2 \cos^2 \theta, \\
    \Delta &= r^2 - 2Mr + a^2 = (r - r_+) (r - r_-). \\
\end{align*}$$

$r_+ (r_-)$ is the out (inner) horizon, $M$ and $a$ are the physical mass and the angular momentum per unit mass respectively, and $b$ is defined as $b = Q^2/2M$. The ADM mass and the angular momentum are given by $M_A = M + b$, etc.
\[ J = (M + b) a. \] The thermodynamic properties of the EMDA black hole have been deeply discussed. The horizon area, entropy, angular velocity and Hawking temperature are respectively

\[
A = 4\pi \left( r_+^2 + 2br_+ + a^2 \right), \\
S = \pi \left( r_+^2 + 2br_+ + a^2 \right), \\
T = \frac{r_+^2 - a^2}{4\pi r_+ \left( r_+^2 + 2br_+ + a^2 \right)}, \\
\Omega_+ = \frac{a}{r_+^2 + 2br_+ + a^2}. 
\] (4)

In the extreme case, the out horizon and the inner horizon coincide with each other. The surface gravity vanishes in this situation and the black hole entropy is \( S = 2\pi (M + b) r_+ = 2\pi J. \)

### 3. Spectroscopy of the EMDA black hole

In this paper, we quantize the horizon area of the EMDA black hole by combining the Bohr-Sommerfeld quantization rule and the adiabatic invariance. There is a frame-dragging effect of the coordinate system in the EMDA spacetime and the matter field in the ergosphere near the horizon must be dragged. It is convenient to investigate the black hole’s properties in the dragging coordinate system. Thus we perform the dragging coordinate transformation \[ 27 \]

\[ d\phi = -\frac{g_{03}}{g_{33}} dt = \frac{a \left( r^2 + 2br + a^2 - \Delta \right)}{(r^2 + 2br + a^2)^2 - \Delta a^2 \sin^2 \theta} dt. \] (5)

Inserting Eq. (5) into the metric (3) yields

\[ ds^2 = -\frac{\Delta \sum dt^2}{(r^2 + 2br + a^2)^2 - \Delta a^2 \sin^2 \theta} + \sum dr^2 + \sum d\theta^2. \] (6)

Now the electromagnetic vector potential in three dimensional spacetime is expressed as \( A_\mu = (A_t, 0, 0), \) with \( A_t = (r^2 + 2br + a^2) Q r \left[ (r^2 + 2br + a^2)^2 - \Delta a^2 \sin^2 \theta \right]^{-1}. \) The Bohr-Sommerfeld quantization rule tells us
\[ \int p_i dq_i = nh, \tag{7} \]

where \( n = 1, 2, 3 \cdots \). To quantize the horizon area, we first Euclideanize the EMDA metric. In Ref. [28], the metric of a rotating spacetime is Euclideanized via the transformation \( t \rightarrow -i\tau \) and \( a \rightarrow ia \). In the dragging coordinate system, the Euclideanized EMDA metric is obtained by a transformation \( t \rightarrow -i\tau \) in the metric (6) and takes on the form as

\[
ds^2 = \frac{\Delta \sum (r^2 + 2br + a^2)^2 - \Delta a^2 \sin^2 \theta}{(r^2 + 2br + a^2)^2} d\tau^2 + \sum \Delta d\tau^2 + \sum d\theta^2. \tag{8}\]

Now the corresponding electromagnetic vector potential is \( A_\mu = (A_\tau, 0, 0) \).

The horizon can be seen as an adiabatic invariant when a black hole is non-extreme. Therefore, we can get

\[
\int p_i dq_i = \int \int_0^H \frac{dH'}{q_i} dq_i = \int \int_0^H dH' d\tau + \int \int_0^H \frac{dH'}{x_a} dx_a, \tag{9}\]

where \( \tau \) is the Euclidean time, which has a periodicity \( 2\pi/\kappa \) and \( \kappa \) is the surface gravity, and \( x_a \)'s denote the generalized space coordinates and \( a = 1, 2, 3 \). \( H \) is the Hamilton function of the black hole’s system and satisfies \( H = \int L d\tau \), with \( L \) being the Lagrangian. When investigate a particle tunnelling through the horizon, one has to take into account the effect of the electromagnetic field. Therefore, the gravitational system consists of the black hole and the outside electromagnetic field. The Lagrangian is composed of the part from the matter field and that from the electromagnetic field. \( L_e = -1/4F_{\mu\nu}F^{\mu\nu} \) is the Lagrangian function of the generalized coordinates \( A_\mu = (A_\tau, 0, 0) \). From the function, we find that \( A_\tau \) and \( \phi \) are cyclic coordinates. To eliminate the degrees of freedom corresponding to \( \phi \) and \( A_\tau \), the Hamilton function should be written as

\[
H = \int_{\tau_i}^{\tau_f} \left( L - P_\phi \dot{\phi} - P_A \dot{A}_\tau \right) d\tau = \int_{r_i}^{r_f} P_r dr - \int_{\phi_i}^{\phi_f} P_\phi d\phi - \int_{A_{i\tau}}^{A_{f\tau}} P_A dA_\tau.
\tag{10}\]

\( P_r, P_\phi \) and \( P_A \) are canonical momenta of \( r, \phi \) and \( A_\tau \). \( r_i \) and \( r_f \) are locations of the outer horizons before and after the emission of a particle, respectively.
To further proceed, we introduce the Hamilton canonical equations. They are expressed as

\[ \dot{r} = \left. \frac{dH}{dP} \right|_{(r;\phi,P;\phi;A,P)} , \quad dH\big|_{(r;\phi,P;\phi;A,P)} = dM' ; \]

\[ \dot{\phi} = \left. \frac{dH}{dP} \right|_{(\phi;r,P;r;A,P)} , \quad dH\big|_{(\phi;r,P;r;A,P)} = \Omega'_\tau dJ' ; \]

\[ \dot{A}_\tau = \left. \frac{dH}{dP} \right|_{(A;\phi,P;\phi;r,P)} , \quad dH\big|_{(A;\phi,P;\phi;r,P)} = A'_\tau dQ' ; \tag{11} \]

in which \( A'_\tau \) and \( \Omega'_\tau \) are the electromagnetic potential and the angular velocity with the emission of particles. When the emitted particle is massless, the outgoing path is the radial null geodesic \( \dot{r} = \frac{dr}{d\tau} \) \cite{29}. The outgoing path is the phase velocity \( \dot{r} = v_p \) of the particle when the particle is charged and massive \cite{30, 31}. Inserting Eq. (11) into Eq. (10) yields

\[ H = \int_{\tau_i}^{\tau_f} \left( \frac{H}{dH'} \right)_{(r;\phi,P;\phi;A,P)} d\tau \]

\[ = \int_{\tau_i}^{\tau_f} \left( \int_0^H \frac{dH'}{(r;\phi,P;\phi;A,P)} - \int_0^H \frac{dH'}{(r;\phi,P;\phi;A,P)} - \int_0^H \frac{dH'}{(A;\phi,P;\phi;r,P)} \right) d\tau \]

\[ = \int_{\tau_i}^{\tau_f} \int_0^H dH' d\tau. \tag{12} \]

Combining Eqs.(12) and (12), we can rewrite the adiabatic invariant as

\[ \int p_i dq_i = 2 \int_0^H dH' d\tau \]

\[ = 2 \left( \int_0^H \frac{dH'}{(r;\phi,P;\phi;A,P)} - \int_0^H \frac{dH'}{(r;\phi,P;\phi;A,P)} - \int_0^H \frac{dH'}{(A;\phi,P;\phi;r,P)} \right) d\tau \]

\[ = 2 \left( \int_0^M dM' - \int_0^J \Omega'_\tau dJ' - \int_0^Q A'_\tau dQ' \right) d\tau. \tag{13} \]
Here we only consider the outgoing path, which implies the half value of periodicity of the Euclidean time is selected, namely $0 \leq \tau \leq \frac{\pi}{\kappa}$. From the first law of thermodynamics of the EMDA black hole

$$dM = TdS + \Omega dJ + \Phi dQ,$$

(14)

where $\Omega$ and $\Phi$ are the angular velocity and the electromagnetic potential at the horizon, we finish the integral and get

$$\int p_i dq_i = 2 \int_0^{\frac{\pi}{\kappa}} dH' d\tau = 2 \int_0^{\frac{\pi}{\kappa}} T dS' = \hbar S,$$

(15)

where the last equality is obtained by a relation between the surface gravity and the Hawking temperature $T = \frac{\hbar \kappa}{2 \pi}$. Introducing the Bohr-Sommerfeld quantization rule in Eq. (7), we derive the entropy spectrum as

$$S = 2\pi n.$$

(16)

Therefore the minimal interval of the entropy spectrum is $\Delta S = S_n - S_{n-1} = 2\pi$. It shows the entropy spectrum is equally spaced and independent on the parameters of the EMDA black hole. From the area-entropy law $S = \frac{A}{4l_p^2}$, the horizon area spectrum is obtained as

$$A = 8\pi nl_p^2.$$

(17)

This implies the minimal interval of the area spectrum is $\Delta A = 8\pi l_p^2$. This value is in consistence with that derived by Bekenstein and Maggiore [10]. So the area spectrum and the entropy spectrum are respectively equally spaced and independent on the parameters of the EMDA black hole.

4. Discussion and Conclusion

In this paper, extending the work of Majhi and Vagenas to the charged and rotating black holes, we quantized the horizon area and the entropy of the EMDA black hole via the Bohr-Sommerfeld quantization rule and the adiabatic invariance. The area spectrum and the entropy spectrum were derived and are respectively equally spaced, which are independent on the parameters of the EMDA black hole. This result is in consistence with that
obtained by Maggiore and that derived by Bekenstein. In the investigation, the area spectrum was derived by the area-entropy relation $S = \frac{A}{4l_p}$. However, if the relation does not satisfy $S = \frac{A}{4l_p}$, the area spectrum would be changed. This was addressed in Majhi and Vagenas’s work [23].

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