Abstract. The paper deals with the problem posed by Katz and Morris whether the free product with amalgamation of any Hausdorff topological groups is Hausdorff, the negative solution of which (even for the particular case of a closed amalgamated subgroup) easily follows from the relevant result by Uspenskij. The topology of such a product is characterized by proving that it coincides with the so-called $X_0$-topology in the sense of Mal’tsev for the corresponding pushout $X$ in the category of Hausdorff topological spaces. Applying this characterization, it is proved that the canonical mappings of Hausdorff groups into their amalgamated free product are open homeomorphic embeddings if an amalgamated subgroup is open. This immediately implies that in that case this product is Hausdorff.

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Key words and phrases: Free product of topological groups; Hausdorff topological group determined by a Hausdorff topological space and a set of relations; $X_0$-topology.

1. Introduction

Under the free product of topological groups $A$ and $C$ (being or not being Hausdorff) with a common topological subgroup $B$ amalgamated we mean the topological group $D$ given by the pushout

$$
\begin{array}{ccc}
B & \xrightarrow{\gamma} & C \\
\downarrow{\alpha} & & \downarrow{\lambda} \\
A & \xrightarrow{\kappa} & D
\end{array}
$$

(1.1)
in the category of all (!) topological groups; here $\alpha$ and $\gamma$ are obvious embeddings. Recall that the fact that (1.1) is a pushout means that for every pair $\mu$ and $\nu$ of continuous homomorphisms of resp. $A$ and $C$ into any topological group $E$, which agree on $B$, there exists a unique continuous homomorphism $\phi$ of $D$ into $E$ with $\phi\kappa = \mu$ and $\phi\lambda = \nu$.

In [5] Katz and Morris posed the problem whether (i) the free product with amalgamation of any Hausdorff topological groups is Hausdorff, (ii)
its underlying group is the amalgamated free product of the underlying groups and (iii) the canonical mappings of the given groups into the above-mentioned product are homeomorphic embeddings. Note that the negative answer to (i) easily follows from the result by Uspenskij [12], [13] asserting that not every epimorphism in the category of Hausdorff topological groups has the dense range, and Remark 2.1 below. Indeed, if the free product of a Hausdorff topological group $A$ by itself with an amalgamated subgroup $B$ is Hausdorff, then $B$ is an equalizer of some pair of continuous mappings into a Hausdorff space, and hence $B$ is closed. But even in that case the free product of Hausdorff groups with amalgamation is not necessarily Hausdorff since otherwise every closed embedding would be an equalizer of some pair of continuous homomorphisms which contradicts the above-mentioned result by Uspenskij.

Though (i) has the negative answer, the range of particular cases where the problem has the positive answer is known. For instance, Ordman settled the problem for some locally invariant Hausdorff groups [11]. Khan and Morris solved the problem in the case where the amalgamated subgroup is central [7], [8]. The important case where the given Hausdorff groups are $k_\omega$–groups is dealt with in three papers by Katz and Morris. Namely, in [4] they obtained the desired result for the case where the amalgamated subgroup is closed and normal, while in [5] the authors proved the statement for the case where the amalgamated subgroup is compact. In [6] they weakened the latter condition, replacing it by the so-called beseder condition. Later, Nickolas generalized these result by proving that the free product of any $k_\omega$–groups with a closed subgroup amalgamated is a $k_\omega$–group [10] and the canonical mappings of these groups into this product are closed homeomorphic embeddings.

Let us also distinguish the particular case of the trivial amalgamated subgroup. In that case the answer to the above-posed problem is given by the following theorems applied in our discussion.

**Theorem 1.1** (Graev [3]). The free product of Hausdorff topological groups is Hausdorff.

**Theorem 1.2** (Mal’tsev [9]). The canonical mappings of Hausdorff topological groups into their free product are closed homeomorphic embeddings.

In this paper we describe the topology of the free product of any Hausdorff topological groups with amalgamation by proving that it coincides

* In [9], the author actually proved a more general statement in which the groups are replaced by algebras from any variety, assuming that they satisfy a certain condition.
with the so-called \( X_0 \)-topology in the sense of Mal’tsev [9] for the corresponding pushout

\[
\begin{array}{ccc}
B & \xrightarrow{\gamma} & C \\
\alpha \downarrow & & \downarrow \psi \\
A & \xrightarrow{\varphi} & X
\end{array}
\]  

(1.2)

in the category of Hausdorff topological spaces. The proof is based on the simple observation that a pushout of closed homeomorphic embeddings in the category of Hausdorff topological groups is precisely the Hausdorff topological group determined in the sense of [9] by \( X \) and a special set of relations. According to the found description a subset \( O \) is open in \( D \) if and only if the following conditions are satisfied:

(i) \( O \cap X \) is open in \( X \);

(ii) for any \( n \geq 2 \) and \( x_1, x_2, \ldots, x_n \in X \) with

\[
x_1 x_2 \cdots x_n \in O,
\]

there exist neighborhoods \( W_1, W_2, \ldots, W_n \) in \( X \) of resp. \( x_1, x_2, \ldots, x_n \) with

\[
W_1 W_2 \cdots W_n \subset O;
\]

(iii) for any \( x \in X \) with

\[
x^{-1} \in O,
\]

there exists a neighborhood \( W \) in \( X \) of \( x \) with

\[
W^{-1} \subset O.
\]

Applying this fact, we prove that the canonical mappings of Hausdorff topological groups into their amalgamated free product are open homeomorphic embeddings if an amalgamated subgroup is open. This immediately implies that in that case this product is Hausdorff.

Though we use the category approach to the issue, only little knowledge of category theory is needed for reading the paper. We assume that the reader is familiar with basic notions such as “category“, “pushout“, “equalizer“, “(left adjoint) functor“, and the well-known fact that a left adjoint functor preserves pushouts (and other colimits).

2. Free Products of Hausdorff Groups with Amalgamation

Let us begin with the following

Remark 2.1. Since the forgetful functor

\[
F : \text{Top(Grp)} \longrightarrow \text{Grp},
\]
where \( \textbf{Top(Grp)} \) is the category of topological groups, while \( \textbf{Grp} \) is that of discrete groups, has a right adjoint (sending a discrete group to itself, but equipped with an antidiscrete topology), it preserves the free products with amalgamation. Therefore the underlying group of the free product \( D \) of topological groups \( A \) and \( C \) with an amalgamated subgroup \( B \) is the amalgamated free product of the corresponding underlying groups. Therefore, \( \kappa(A) \cup \lambda(C) \) generates \( D \) algebraically. Moreover, for any pushout (1.1) in \( \textbf{Top(Grp)} \) with injective \( \alpha \) and \( \gamma \), so are \( \kappa \) and \( \lambda \), and we have
\[
\kappa(A) \cap \lambda(C) = \kappa \alpha(B). \tag{2.1}
\]

Below, when no confusion might arise, we will identify the elements of \( A \) and \( C \) with their images under resp. \( \kappa \) and \( \lambda \).

For the reader, with deeper knowledge of category theory, we note that the functor \( F \) is, in fact, topological (Wyler [14]). Therefore \( \textbf{Top(Grp)} \) has all (small) limits and colimits, and \( F \) preserves them (see, for example, [1]).

Let us now describe the topology of the free product of Hausdorff topological groups with amalgamation. To this end recall Mal’tsev’s notion [9] of the Hausdorff topological group determined by a Hausdorff topological space \( X \) and a set of relations
\[
t_i(x_{i_1}, x_{i_2}, \ldots, x_{i_{n_i}}) = 1, \quad i \in I,
\]
where \( t_i \) is a group term on the variables \( x_{i_1}, x_{i_2}, \ldots, x_{i_{n_i}} \) from \( X \). Such a group is defined as a Hausdorff group \( G \) equipped with a continuous mapping \( \sigma : X \to G \), for which
\begin{enumerate}
\item \( G \) is the smallest closed subgroup containing \( \sigma(X) \);
\item \( \sigma \) maps the left-hand part of (2.2) into the unit for all \( i \in I \) and, moreover, is universal among continuous mappings (into Hausdorff groups) with this property, i.e. for any such continuous mapping \( \theta : X \to G' \) with Hausdorff group \( G' \), there exists a (unique) continuous homomorphism \( \phi : G \to G' \) with \( \phi \sigma = \theta \).
\end{enumerate}

The problem of the existence (and uniqueness up to an isomorphism) of the Hausdorff topological group determined by any Hausdorff topological space \( X \) and any set of relations was positively solved in [9].

When the set of relations is empty, \( G \) is called the free Hausdorff topological group over \( X \) [9]. Note that such \( G \) is precisely the image of \( X \) under the left adjoint for the forgetful functor
\[
\text{Haus(Grp)} \to \text{Haus},
\]

** Mal’tsev introduced this notion and that of the \( X_0 \)-topology to be mentioned below, in a more general context of any variety of universal algebras.
where $\text{Haus(Grp)}$ is the category of Hausdorff topological groups, while $\text{Haus}$ is that of Hausdorff topological spaces. Moreover, the Hausdorff group determined by the Hausdorff space $X$ and any set of relations is merely the quotient-group of the free Hausdorff group over $X$ by a suitable normal closed subgroup.

Our discussion is based on the observation that for any Hausdorff topological groups $A$, $C$, their common closed subgroup $B$ and the pushout

$$
\begin{array}{ccc}
B & \xrightarrow{\gamma} & C \\
\alpha \downarrow & & \downarrow \lambda \\
A & \xrightarrow{\kappa} & D
\end{array}
$$

in the category of Hausdorff topological groups, $D$ in (2.3) is precisely the Hausdorff topological group determined by the Hausdorff space $X$ given by the pushout

$$
\begin{array}{ccc}
B & \xrightarrow{\gamma} & C \\
\alpha \downarrow & & \downarrow \psi \\
A & \xrightarrow{\varphi} & X
\end{array}
$$

in the category of Hausdorff topological spaces and the set of relations

$$
1 = 1, \\
a_1 a_2 a_3^{-1} = 1
$$

(2.5)

and

$$
c_1 c_2 c_3^{-1} = 1
$$

(2.6)

for all $a_1, a_2, a_3 \in A$ and $c_1, c_2, c_3 \in C$, for which equalities (2.5) and (2.6) are valid in resp. $A$ and $C$. Recall that the underlying set of the space $X$ is the free union of the sets $A \setminus B$ and $C$, while the open sets in it are sets $W$ such that both $\varphi^{-1}(W)$ and $\psi^{-1}(W)$ are open.

**Remark 2.2.** We have restricted our consideration to the case of a closed common subgroup $B$ since otherwise the space $X$ need not be Hausdorff. It is obvious that our observation remains valid for arbitrary $B$ if the word ”Hausdorff” is omitted everywhere in it and in the corresponding Mal’tsev’s definition, too.

Further, let us recall yet another notion by Mal’tsev [9]. Let $X$ be a topological space, and let $G$ be a (discrete) group containing $X$ as a subset. One has the so-called $X_0$-topology on $G$. The open sets in it are subsets $O$ such that for any group term $t(x_1, x_2, \ldots, x_n)$ on the variables $x_1, x_2, \ldots, x_n$ from $X$ with

$$
t(x_1, x_2, \ldots, x_n) \in O,
$$
there exist neighborhoods $W_1, W_2, \ldots, W_n$ in $X$ of resp. $x_1, x_2, \ldots, x_n$ such that
\[ t(W_1, W_2, \ldots, W_n) \subset O. \]

If $X$ is closed under inverse elements, then the latter condition is obviously equivalent to the following ones:

(i) $O \cap X$ is open in $X$;

(ii) for any $n \geq 2$ and $x_1, x_2, \ldots, x_n \in X$ with
\[ x_1 x_2 \cdots x_n \in O, \]
there exist neighborhoods $W_1, W_2, \ldots, W_n$ in $X$ of resp. $x_1, x_2, \ldots, x_n$ with
\[ W_1 W_2 \cdots W_n \subset O; \]

(iii) for any $x \in X$ with
\[ x^{-1} \in O, \]
there exists a neighborhood $W$ in $X$ of $x$ with
\[ W^{-1} \subset O. \]

It is clear that if, moreover, $X$ is a subgroup of $G$, then the condition (ii) is equivalent to its weak version, formulated only for $n = 2$. If, in addition, $X$ is a topological group with respect to the available topology, then (i) implies both (ii) and (iii).

Note that, in general, the $X_0$-topology is not compatible with the algebraic structure of $G$. However, sometimes this is the case.

**Theorem 2.3 (Burgin [2])**\(^{***}\). Let $X$ be a completely regular topological space, and let $G$ be the Hausdorff topological group determined by $X$ and any set of relations. If the canonical mapping $\sigma : X \rightarrow G$ is a homeomorphic embedding, then $G$ has the $X_0$-topology.

We have

**Lemma 2.4.** Let $G$ be a topological group, and let $X$ be its subspace. Let $N$ be a normal subgroup of $G$, and $X/N$ be the quotient-set determined by the equivalence relation induced by $N$, equipped with the quotient-topology. Let $G$ have the $X_0$-topology. Then the quotient-group $G/N$ has the $(X/N)_0$-topology.

**Proof.** Let $O$ be open in the $(X/N)_0$-topology. Consider the union $O'$ of equivalence classes from $O$ and show that it is open in the $X_0$-topology. For the condition (i), we note that $O' \cap X$ is precisely the union of equivalence classes from the set $O \cap (X/N)$.

\(^{***}\) In [2] Burgin considered the general case of an arbitrary Mal’tsev variety.
To verify (ii), let us consider \(x_1, x_2, \ldots, x_n \in X\) with
\[x_1x_2 \cdots x_n \in O'.\]
Then
\[[x_1][x_2] \cdots [x_n] \in O,\]
for the equivalence classes \([x_1], [x_2], \ldots, [x_n]\) containing resp. \(x_1, x_2, \ldots, x_n\). Hence there are open neighborhoods \(W_1, W_2, \ldots, W_n\) in \(X/N\) of resp. \([x_1], [x_2], \ldots, [x_n]\) with
\[W_1W_2 \cdots W_n \subset O.\]
The unions \(W'_1W'_2 \cdots W'_n\) of equivalence classes from resp. \(W_1, W_2, \ldots, W_n\) are open in \(X\) and contain resp. \(x_1, x_2, \ldots, x_n\). Moreover,
\[W'_1W'_2 \cdots W'_n \subset O'.\]

For the condition (iii), let \(x \in X\) and \(x^{-1} \in O'\). Then \([x]^{-1} \in O\). Consequently, one has an open neighborhood \(W\) in \(X/N\) of the class \([x]\) with \(W^{-1} \subset O\). The union \(W'\) of equivalence classes from \(W\) is open in \(X\) and \(W'^{-1} \subset O'\).

Thus \(O\) is open in the quotient–topology.

The converse follows from the fact that the operations of \(G/N\) are continuous in the quotient-topology. \(\Box\)

Lemma 2.4 implies

**Theorem 2.5.** Let \(A, C\) be Hausdorff topological groups, and let the internal rectangle in the diagram

\[
\begin{array}{ccc}
B & \overset{\gamma}{\longrightarrow} & C \\
\alpha \downarrow & & \downarrow \psi \\
A & \overset{\varphi}{\longrightarrow} & X \\
\lambda \downarrow & & \downarrow \omega \\
& \overset{\kappa}{\longrightarrow} & D
\end{array}
\]

be a pushout in the category \(\text{Top}\) of topological spaces, while the external one be a pushout in that \(\text{Top(Grp)}\) of topological groups. Let \(\alpha\) and \(\gamma\) be injective mappings. Then so is \(\omega\), and \(D\) has the \(X_0\)-topology.
Proof. The injectivity of \( \omega \) follows immediately from Remark 2.1. To determine the topology of the group \( D \), consider the diagram

\[
\begin{array}{c}
\{1\} \rightarrow C \\
\downarrow \quad \downarrow \\
A \rightarrow X' \rightarrow D'
\end{array}
\]

where, again, the internal rectangle is a pushout in \( \text{Top} \), while the external one is a pushout in \( \text{Top(Grp)} \). In other words, \( X' \) is the coproduct of \((A,1)\) and \((C,1)\) in the category of topological spaces with a fixed point, while \( D' \) is the coproduct (i.e. the free product) of \( A \) and \( C \) in \( \text{Top(Grp)} \).

From Remark 2.1 we conclude that \( \omega' \) is injective. Let us show that \( \omega' \) is a homeomorphic embedding. To this end, consider an open subset \( W \) of \( X' \) and show that for any \( x \in W \), there exists a neighborhood \( O_x \) in \( D' \) of \( x \) with \( O_x \cap X' \subset W \). Let \( U = W \cap A \) and \( V = W \cap C' \). According to Theorem 1.2, \( \kappa' \) and \( \lambda' \) are (closed) homeomorphic embeddings. Therefore \( D' \) has open subsets \( O^A \) and \( O^C \) such that \( U = O^A \cap A \) and \( V = O^C \cap C \).

If \( x = 1 \), then we can take \( O_x = O^A \cap O^C \).

If \( x \neq 1 \) and lies, say, in \( A \), then the desired neighborhood \( O_x \) of \( x \) is \( O^A \cap (D' \setminus C) \).

Since \( A \) and \( C \) are Hausdorff groups, they are also completely regular. Hence \( X' \) is also of this kind. Moreover, according to Theorem 1.2, \( D' \) is Hausdorff. Then Theorem 2.3 implies that the topology of \( D' \) is the \( X_0' \)-topology.

Let \( N \) be the smallest normal subgroup of \( D' \) containing elements of the form \( \alpha(b)(\gamma(b))^{-1} \) for all \( b \in B \). Then \( D \) is isomorphic to \( D'/N \) as a topological group. Since the mappings \( \alpha \) and \( \gamma \) are injective, Remark 1.2 implies that \( X \) is isomorphic to \( X'/N \) as a topological space and \( \omega \) is the mapping induced by \( \omega' \). Lemma 2.4 implies that \( D \) has the \( X_0' \)-topology. \( \square \)

**Lemma 2.6.** In the conditions of Theorem 2.5, the following conditions are equivalent:

(i) \( B \) is open both in \( A \) and in \( C \);

(ii) \( B \) is closed both in \( A \) and in \( C \), and the mappings \( \kappa \) and \( \lambda \) are open homeomorphic embeddings.
Proof. (i)⇒(ii): It is well-known that each open subgroup of a topological group is closed as well. Let us now show that each open subset \( W \) of \( X \) is open in \( D \), too. To verify the condition (ii) in the definition of the \( X_0 \)-topology, let us apply the principle of mathematical induction. First we consider the case where \( n = 2 \). Let \( x_1, x_2 \in X \) with \( x_1 x_2 \in W \). It is clear that both \( x_1, x_2 \) belong either to \( A \) or to \( C \) because otherwise the product \( x_1 x_2 \) would belong to \( D \setminus X \). Let \( x_1 \) and \( x_2 \) lie, say, in \( A \). Then any neighborhoods \( U_1 \) and \( U_2 \) in \( A \) of resp. \( x_1 \) and \( x_2 \) with \( U_1 U_2 \subset W \cap A \) are the desired ones.

Assume that \( n > 2 \) and the statement is valid for \( (n - 1) \). Consider \( x_1, x_2, \ldots, x_n \in X \) with
\[
x_1 x_2 \cdots x_n \in W.
\]
If no two \( x_i, x_{i+1} \) lie in one and the same \( G_j \) (here \( G_1 = A \) and \( G_2 = C \)), then
\[
x_1 x_2 \cdots x_n \in D \setminus X,
\]
which is impossible. Therefore there exist \( x_i \) and \( x_{i+1} \) from one and the same \( G_j \). Since
\[
x_1 x_2 \cdots x_{i-1} (x_i x_{i+1}) x_{i+2} \cdots x_n \in W
\]
and \( x_i x_{i+1} \in X \), by the assumption of mathematical induction, we have neighborhoods \( W_1, W_2, \ldots, W_{i-1}, W', W_{i+2}, \ldots, W_n \) in \( X \) of resp. \( x_1, x_2, \ldots, x_{i-1}, (x_i x_{i+1}), x_{i+2}, \ldots, x_n \) with
\[
W_1 W_2 \cdots W_{i-1} W' W_{i+2} \cdots W_n \subset W.
\]
By what has been proved above, there exist neighborhoods \( W_i \) and \( W_{i+1} \) in \( X \) of resp. \( x_i \) and \( x_{i+1} \) with
\[
W_i W_{i+1} \subset W'.
\]
Therefore the system of neighborhoods \( W_1, W_2, \ldots, W_{i-1}, W_i, W_{i+1}, W_{i+2}, \ldots, W_n \) is the desired one. Thus the mapping \( \omega \), and therefore \( \kappa \) and \( \lambda \), too, are open homeomorphic embeddings.

(ii)⇒(i): If \( A = B \), then \( \kappa = \gamma \) and the assertion is proved. Let \( a \in A \setminus B \), and \( b \in B \). Since \( A \) is open in \( X \) and \( ab \in A \), we have a neighborhood \( U \) in \( A \) of \( a \) such that \( U \cap B = \emptyset \) and a neighborhood \( W \) in \( X \) of \( b \) with \( UW \subset A \). Then \( W \subset A \), and therefore \( b \) is an internal point of \( B \) in \( C \).

Lemma 2.7. In the conditions of Theorem 2.5, if \( \alpha \) and \( \gamma \) are open homeomorphic embeddings, then \( D \) is a Hausdorff topological group.

Proof. It is sufficient to show that the set \( \{1\} \) is closed in \( D \). But it is closed in \( A \) since \( A \) is Hausdorff, and, moreover, \( A \) itself is closed in \( D \) as an open subgroup (see Lemma 2.6).
Lemmas 2.6, 2.7 immediately imply

**Theorem 2.8.** Let $B$ be a common open topological subgroup of Hausdorff topological groups $A$ and $C$. Then the free product of $A$ and $C$ with the amalgamated subgroup $B$ is Hausdorff and, moreover, the canonical mappings of $A$ and $C$ into this product are open homeomorphic embeddings.

When an amalgamated subgroup is open, one can give another, more explicit, characterization of the topology of the free product of Hausdorff groups with amalgamation.

**Proposition 2.9.** In the conditions of Theorem 2.5, the $X_0$-topology on $D$ is weaker than that generated by sets of the form

\begin{align*}
U_1V_1U_2V_2\cdots U_nV_n, & \quad (2.8) \\
U_1V_1U_2V_2\cdots V_{n-1}U_n, & \quad (2.9) \\
V_1U_1V_2U_2\cdots V_nU_n, & \quad (2.10) \\
V_1U_1V_2U_2\cdots U_{n-1}V_n, & \quad (2.11)
\end{align*}

for all $n \geq 1$, open subsets $U_1, U_2, \ldots, U_n$ of $A$ and open subsets $V_1, V_2, \ldots, V_n$ of $C$. If $\alpha$ and $\gamma$ are open homeomorphic embeddings, then these two topologies coincide.

**Proof.** Let $O$ be an open set in the $X_0$-topology, and let $d \in O \setminus X$. The element $d$ can be represented as a product of elements from $A$ and $C$. Let $d$ have, say, the form

$$d = a_1c_1a_2c_2\cdots a_nc_n,$$

with $a_1, a_2, \ldots, a_n \in A$ and $c_1, c_2, \ldots, c_n \in C$. Then there exist open neighborhoods $W_1, W_1', W_2, W_2', \ldots, W_n, W_n'$ in $X$ of resp. $a_1, c_1, a_2, c_2, \ldots, a_n, c_n$ with $W_1, W_1', W_2, W_2', \ldots, W_n, W_n' \subset O$.

Let $U_i = W_i \cap A$ and $V_i = W_i' \cap C$ ($i = 1, 2, \ldots, n$). Then

$$d \in U_1V_1U_2V_2\cdots U_nV_n \subset O.$$

If $d \in O \cap A$, then similar arguments applied to the representation $d = d \cdot 1$ imply that

$$d \in U_1V_1 \subset O$$

for some open subsets $U_1$ and $V_1$ of resp. $A$ and $C$. Therefore $O$ is open in the topology induced by the sets (2.8)–(2.11).

The converse follows immediately from Lemma 2.6. \qed
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