Naked Singularity formation in scalar field collapse

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We construct here a class of collapsing scalar field models with a non-zero potential, which result in a naked singularity as collapse end state. The weak energy condition is satisfied by the collapsing configuration. It is shown that physically it is the rate of collapse that governs either the black hole or naked singularity formation as the final state for the dynamical evolution. It is seen that the cosmic censorship is violated in dynamical scalar field collapse.

There has been considerable interest and discussion recently in the string theory context in the phenomena of possible naked singularity formation, and validity or otherwise of the cosmic censorship conjecture [1]. These considerations are mainly centered around models of four dimensional gravity coupled to a scalar field with potential $V(\phi)$, and various modifications and numerical studies of such a scenario. Such models satisfy the positive energy theorem, however, in general may violate the energy conditions, within an asymptotically anti de Sitter framework. While these considerations have provided us with some insights, the basic question of cosmic censorship in scalar field collapse still remains very much open to further analysis.

The issue of naked singularity formation in gravitational collapse is, in fact, of great interest in gravity physics and has been investigated already in considerable detail (see e.g. [2] and references therein) within the framework of Einstein’s gravity. This is because the occurrence of naked singularities will offer us the possibility to observe the quantum gravity effects taking place in such visible ultra-strong gravity regions. The generic conclusion here is, depending on the nature of the regular initial data (in terms of the initial distribution of matter fields and other collapse parameters) from which the collapse evolves, the final outcome is either a black hole or a naked singularity. Such studies have involved various forms of matter such as dust, perfect fluids, radiation collapse, and general (type I) matter fields.

Special importance however, is attached at times to the investigation of collapse for a scalar field. This is because one would like to know if cosmic censorship is necessarily preserved or violated in gravitational collapse for fundamental matter fields, which are derived from a suitable Lagrangian.

Our purpose here is to construct a class of continual collapse models of scalar field with potential, such that no trapped surfaces form in the spacetime as the collapse evolves in time, and the singularity that develops as collapse endstate is necessarily naked. We require that the weak energy condition is preserved throughout the collapse, though the pressures would be negative closer to the singularity. The interior collapsing sphere is matched with a generalized Vaidya exterior spacetime to complete the model. We thus see that naked singularities are created in scalar field collapse from generic initial conditions, thus violating the cosmic censorship hypothesis.

In order to present a transparent consideration, let us examine a spherically symmetric homogeneous scalar field, $\Phi = \Phi(t)$, with a potential $V(\Phi)$. This ensures that the interior spacetime must have a Friedmann-Robertson-Walker (FRW) metric. Further, let us choose the marginally bound $(k = 0)$ case. Then the interior metric is of the form,

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + r^2 d\Omega^2\right]$$

where $d\Omega^2$ is the line element on a two-sphere. In this comoving frame, the energy-momentum tensor of the scalar field is given as,

$$T^t_t = -\rho(t) = -\left[\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\right]$$

and

$$T^r_r = T^\theta_\theta = T^\phi_\phi = p(t) = \left[\frac{1}{2}\ddot{\Phi}^2 - V(\Phi)\right]$$

with all other off-diagonal terms being zero.

It may be noted that the comoving coordinate system we have chosen has a particular physical significance as compared to an arbitrary system, and the quantities $\rho$ and $p$ are interpreted as the density and pressure respectively of the scalar field. It is then easily seen that the scalar field behaves like a perfect fluid, as the radial and tangential pressures are equal. We take the scalar field to satisfy the weak energy condition, that is, the energy density as measured by any local observer be non-negative, and for any timelike vector $V^i$, we have,

$$T_{ik} V^i V^k \geq 0$$

This amounts to,

$$\rho \geq 0; \quad \rho + p \geq 0;$$

The dynamic evolution of the system is now determined by the Einstein equations, which for the metric \( \text{I} \) become (in the units $8\pi G = c = 1$),

$$\rho = \frac{F'}{R^2 R'}; \quad p = -\frac{F}{R^2 R'}$$

where

| $X$ | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $\rho$ | $p$ | $\rho + p$ | $\rho - p$ | $\rho$ |

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| $X$ | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $\rho$ | $p$ | $\rho + p$ | $\rho - p$ | $\rho$ |

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|-----|-----|-----|-----|-----|
| $\rho$ | $p$ | $\rho + p$ | $\rho - p$ | $\rho$ |

| $X$ | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $\rho$ | $p$ | $\rho + p$ | $\rho - p$ | $\rho$ |

| $X$ | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $\rho$ | $p$ | $\rho + p$ | $\rho - p$ | $\rho$ |
\[ R^2 = \frac{F}{R} \] (7)

Here \( F = F(t, r) \) is an arbitrary function, and in spherically symmetric spacetimes it has the interpretation of the mass function for the cloud, with \( F \geq 0 \). The quantity \( R(t, r) = ra(t) \) is the area radius for the shell labeled by the comoving coordinate \( r \). In order to preserve the regularity of the initial data, we have \( F(t_i, 0) = 0 \), that is, the mass function should vanish at the center of the cloud. From equation (6) it is evident that on any regular epoch \( F \approx r^3 \) near the center.

The \( \text{Klein-Gordon} \) equation for the scalar field is given by,
\[
\frac{d}{dt} [a^3 \Phi] = -a^3 V(\Phi) \phi. \tag{8}
\]

Since our aim here is to construct a continual collapse model, we consider the class with \( \dot{a} < 0 \), which is the collapse condition implying that the area radius of a shell at a constant value of comoving radius \( r \) decreases monotonically. As such there may be classes of solutions where a scalar field may disperse away also (see e.g. [3]). Our objective, however, is to examine here whether the singularities forming in scalar field collapse could be naked, or necessarily covered within a black hole, and if so under what conditions. The singularity resulting from continual collapse is given by \( a = 0 \), that is when the scale factor vanishes and the area radius for all the collapsing shells becomes zero. At the singularity we must have \( \rho \to \infty \).

The key factor that decides the visibility or otherwise of the singularity is the geometry of trapped surfaces which may form as the collapse evolves. These are two-surfaces in the spacetime from which both outgoing and ingoing wavefronts necessarily converge [3]. The boundary of the trapped region in a spherically symmetric spacetime is given by the equation,
\[ F = R \] (9)

The spacetime region where the mass function \( F \) satisfies \( F < R \) is not trapped, while \( F > R \) describes a trapped region (see e.g. [3]).

In terms of the scale variable \( a \), the mass function can be written as,
\[
F = \frac{1}{3} a^3 \left[ \frac{1}{2} \Phi(a)^2 \dot{a}^2 + V(\Phi(a)) \right]. \tag{10}
\]

This is because from equation (6) we can solve for the mass function as,
\[
F = \frac{1}{3} \rho(t) R^3 \] (11)

From equation (10), we then see that,
\[
\frac{F}{R} = \frac{1}{3} \rho(t) r^3 a^2 \] (12)

The above relation decides the trapping or otherwise of the spacetime as the collapse develops. We would like to construct and investigate the classes of collapse solutions for a scalar field with potential, where the trapping is avoided till the singularity formation, thereby allowing the singularity to be visible. Towards such a purpose, consider the class of models where near the singularity the divergence of the density is given by,
\[
\rho(t) \approx \frac{1}{a(t)} \] (13)

Then using equation (10), we see that the above condition [13] implies, near the singularity,
\[
\frac{1}{2} \Phi(a)^2 \dot{a}^2 + V(\Phi(a)) = \frac{1}{a} \] (14)

Now solving the equation of motion (7) we get,
\[
\dot{a} = \frac{\sqrt{a}}{\sqrt{3}} \] (15)

The negative sign above implies a collapse scenario where \( \dot{a} < 0 \). Using equations (15) and (14) in later part of equation (8) we can now solve for \( \Phi \) which is given as,
\[
\Phi(a) = -\ln a \] (16)

We note that as we approach the singularity, \( a \to 0 \) implies \( \Phi \to \infty \), that is, the scalar field blows up at the singularity. Finally, using equations (15) and (16) in equation (8) we can solve for the potential \( V \) as,
\[
V(\Phi) = \frac{5}{6} e^\Phi \] (17)

Thus we see that near the singularity,\[
\rho(t) \approx \frac{1}{a(t)} ; \ p(t) \approx -\frac{2}{3a(t)} \] (18)

It is seen that in the limit of approach to the singular epoch \( t = t_s \) we get \( F/R = 0 \) for all shells and there is no trapped surface in the spacetime.

We see in the model above that the weak energy condition is satisfied as \( \rho > 0 \) and \( \rho + p > 0 \), though the pressure would be negative. Also, it should be noted that the pressure does not have to be negative from the initial epoch, because we have required the specific behavior of \( \rho \) in equation (13), only near the singularity. We can always choose a \( V(\Phi) \) such that at the initial epoch \( \frac{1}{2} \Phi^2 > V(\Phi) \), and then pressure would be positive. But near the singularity \( V(\Phi) \) should behave according to equation (17) and hence pressure should decrease monotonically from the initial epoch and tend to \(-\infty\) at the singularity.

If from an epoch \( t = t^* \) (or equivalently for some \( a = a^* \)) the density starts growing as \( a^{-1} \), then integrating equation (16) we get the singular epoch as,
\[
t_s = t^* + 2\sqrt{3} a^* \] (19)
Thus the collapse reaches the singularity in a finite collapsing time, where the matter energy density as well as the Kretschman scalar \( \kappa = R_{ijkl} R^{ijkl} \) diverges. We note that from equation of motion \( 7 \) it follows that the metric function \( a \) is given by,

\[
a(t) = \left[ \sqrt{a^2} - \frac{1}{2\sqrt{3}}(t - t^*) \right]^2 \quad (20)
\]

This completes the interior solution within the collapsing cloud, providing us with the required construction.

We can see from the above considerations that the absence or otherwise of trapped surfaces, and the behaviour of pressure, crucially depend on the rate of divergence of the density \( \rho \) near the singularity. To examine this more carefully, let us write near the singularity,

\[
\rho = a^{-n}; \quad n > 0 \quad (21)
\]

as we know \( \rho(t) \) must diverge as \( a(t) \) goes to zero in the limit of approach to the singularity. In that case, solving the Einstein equations gives,

\[
p = \frac{n - 3}{3} a^{-n} \quad (22)
\]

The corresponding values of \( \Phi \) and \( V(\Phi) \) are,

\[
\Phi = -\sqrt{n} \ln(a); \quad V(\Phi) = \left( 1 - \frac{a}{6} \right) e^{\sqrt{n} \Phi} \quad (23)
\]

Again calculating \( F/R \) in this general case, we have,

\[
\frac{F}{R} = \frac{1}{3} a^{2-n} \quad (24)
\]

Thus we see that for low enough divergences \((0 < n < 2)\) we have no trapped surfaces forming and there are negative pressures near the singularity. For \(2 \leq n < 3\), trapped surfaces do form, however pressure still remains negative at the singularity. For \(n \geq 3\) we have \( p \geq 0 \) and there are trapped surfaces in the spacetime as collapse advances. Conversely, we can say that non-negative pressure always ensures trapped surfaces in homogeneous scalar field collapse. Thus, the role of the potential \( V(\Phi) \) chosen is to control the divergence of density near the singularity, which in turn governs the development or otherwise of trapped surfaces. Fig.1 shows the behaviour of the functions \( V(\Phi) \) with respect to \( \Phi \), for different values of \( n \). It is seen that the naked singularity arises from a non-zero measure open set of initial conditions \((n < 2)\), whereas rest of the initial data set produces black hole as final collapse end state.

To complete the model, we now need to match this interior to a suitable exterior spacetime. In the following, we match this spherical ball of collapsing scalar field to a generalized Vaidya exterior geometry \( 3 \) at the boundary hypersurface \( \Sigma \) given by \( r = r_b \). Then the metric just inside \( \Sigma \) is,

\[
ds^2_- = -dt^2 + a^2(t) \left[ dr^2 + r_b^2 d\Omega^2 \right] \quad (25)
\]

while the metric in the exterior of \( \Sigma \) is,

\[
ds^2_+ = - \left( 1 - \frac{2M(r_v, v)}{r_v} \right) dv^2 - 2dvdr_v + r_v^2d\Omega^2 \quad (26)
\]

where \( v \) is the retarded (exploding) null co-ordinate and \( r_v \) is the Vaidya radius. Matching the area radius at the boundary we get,

\[
r_v a(t) = r_v(v) \quad (27)
\]

Then on the hypersurface \( \Sigma \), the interior and exterior metric are given by,

\[
ds^2_{\Sigma-} = dt^2 + a^2(t)r_b^2d\Omega^2 \quad (28)
\]

and

\[
ds^2_{\Sigma+} = - \left( 1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv} \right) dv^2 + r_v^2d\Omega^2 \quad (29)
\]

Matching the first fundamental form on this hypersurface we get,

\[
\left( \frac{dv}{dt} \right)_{\Sigma} = \frac{1}{\sqrt{1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}}}; \quad (r_v)_{\Sigma} = r_v a(t) \quad (30)
\]

To match the second fundamental form (extrinsic curvature) for interior and exterior metrics, we note that the normal to the hypersurface \( \Sigma \), as calculated from the interior metric, is given as,

\[
n_{-}^a = [0, a(t)^{-1}, 0, 0] \quad (31)
\]

and the non-vanishing components of normal derived from the generalized Vaidya metric are,

\[
n_{-}^t = - \frac{1}{\sqrt{1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}}} \quad (32)
\]

\[
n_{+}^r = \frac{1}{\sqrt{1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}}} \quad (33)
\]

\[
\frac{dv}{dt}_{\Sigma} = \frac{1}{\sqrt{1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}}}
\]

\[
\frac{dr}{dt}_{\Sigma} = \frac{1}{\sqrt{1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}}}
\]

\[
\frac{d\Omega}{dt}_{\Sigma} = \frac{1}{\sqrt{1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}}}
\]
Here the extrinsic curvature is defined as,

\[ K_{ab} = \frac{1}{2} \mathcal{L}_n g_{ab} \]  

(34)

That is, the second fundamental form is the Lie derivative of the metric with respect to the normal vector \( n \). The above equation is equivalent to,

\[ K_{ab} = \frac{1}{2} \left[ g_{ab,c} n^c + g_{cb} n^c_{,a} + g_{ac} n^c_{,b} \right] \]  

(35)

Setting \([K^-_{\theta \theta} - K^+_{\theta \theta}]_\Sigma = 0\) on the hypersurface \( \Sigma \) we get,

\[ r_b a(t) = r_v \frac{1 - \frac{2M(r_v,v)}{r_v}}{\sqrt{1 - \frac{2M(r_v,v)}{r_v} + 2 \frac{dr_v}{dt}}} \]  

(36)

Simplifying the above equation using equation (30) and (7) we get, on the boundary,

\[ F(t, r_v) = 2M(r_v, v) \]  

(37)

Equations (37), (38), (39) along with (27) completely specify the matching conditions at the boundary of the collapsing scalar field, as at the boundary we know the value and the derivatives of the generalized Vaidya mass function \( M(v, r_v) \), which is free otherwise in the exterior.

Using the above equation and (30) we now get,

\[ \left( \frac{dv}{dt} \right)_\Sigma = \frac{1 + r_b \dot{a}}{1 - \frac{F(t, r_v)}{r_v a(t)}} \]  

(38)

Finally, setting \([K^-_{\tau \tau} - K^+_{\tau \tau}]_\Sigma = 0\), where \( \tau \) is the proper time on \( \Sigma \), we get,

\[ M(r_v, v, r_v) = \frac{F}{2r_v a} + r_v^2 \dot{a} \ddot{a} \]  

(39)

Thus the exterior metric along with the singularity smoothly transform to,

\[ ds = -dv^2 - 2dvdr_v + r_v^2 d\Omega^2 \]  

(41)

The above metric describes a Minkowski spacetime in retarded null coordinate. Hence we see that the exterior generalized Vaidya metric, together with the singular point at \((t_s, 0)\), can be smoothly extended to the Minkowski spacetime as the collapse completes.

From equation (38) it is seen that the generalized Vaidya geodesic, which emerges from the singularity before it evaporates into free space, is null. It follows that non-spacelike trajectories can come out from the singularity that develops as the collapse end point. Hence naked singularity is produced in the collapse of scalar field with potential as considered here, for a non-zero measure set of initial conditions, and that the occurrence of trapped surfaces in the spacetime is avoided. While we have considered here for clarity a homogeneous field, the work on inhomogeneous generalization will be reported elsewhere.

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