CURVATURE ESTIMATES AND GAP THEOREMS FOR EXPANDING RICCI SOLITONS

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ABSTRACT. We derive a sharp lower bound for the scalar curvature of non-flat and non-compact expanding gradient Ricci soliton provided that the scalar curvature is non-negative and the potential function is proper. Upper bound for the scalar curvature of expander with nonpositive Ricci curvature will also be given. Furthermore, we provide a sufficient condition for the scalar curvature of expanding soliton being nonnegative. Curvature estimates of expanding solitons in dimensions three and four will also be established. As an application, we prove a gap theorem on three dimensional gradient expander.

1. INTRODUCTION

Let $(M^n, g)$ be an $n$ dimensional smooth connected Riemannian manifold and $X$ be a smooth vector field on $M$. The triple $(M, g, X)$ is said to be a Ricci soliton if there is a constant $\lambda$ such that the following equation is satisfied

\[ \text{Ric} + \frac{1}{2} L_X g = \lambda g, \]

where Ric and $L_X$ denote the Ricci curvature and Lie derivative with respect to $X$ respectively. A Ricci soliton is called expanding (steady, shrinking) if $\lambda < 0$ ($= 0$, $> 0$). Upon scaling the metric by a constant, we assume $\lambda \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. The soliton is called complete if $(M, g)$ is complete as a Riemannian manifold. It is said to be gradient if $X$ can be chosen such that $X = \nabla f$ for some smooth function $f$ on $M$. In this case, $f$ is called a potential function and (1) can be rewritten as

\[ \text{Ric}_f := \text{Ric} + \nabla^2 f = \lambda g. \]

We shall abbreviate shrinking, steady and expanding solitons as shrinker, steadier and expander respectively.

Hereinafter, $S$ denotes the scalar curvature of the Riemannian manifold. Fix a point $p_0$ in $M$, for any $x$ in $M$, $r$, $r(x)$ and $d(x, p_0)$ will be used interchangeably and refer to the distance between $x$ and $p_0$. $B_R(p)$ is the geodesic ball centered at $p$ with radius $R$. For any smooth function $\omega$, we define the weighted Laplacian w.r.t. $\omega$ by $\Delta_\omega := \Delta - \nabla \omega \cdot \nabla$, where $\Delta$ is the usual Laplacian.

Ricci flow was introduced by Hamilton in his seminal work [25] to study closed three manifolds with positive Ricci curvature:

\[ \frac{\partial g(t)}{\partial t} = -2 \text{Ric}(g(t)). \]

Ricci soliton is of great importance since it is a self similar solution to the Ricci flow and usually arises as a rescaled limit of the flow near its singularities (see [27] and [4]). In particular, expanding Ricci soliton is related to the limit solution
of Type III singularities of the Ricci flow (see [3], [13] and [32]). Ricci soliton is also a natural generalization of the Einstein metric. The corresponding curvature quantity $\text{Ric}_f := \text{Ric} + \nabla^2 f$ plays a significant role in the theory of smooth metric measure spaces (see [31], [48], [37], [38] and [36]).

Chen [10] showed that any complete ancient solutions to the Ricci flow have non-negative scalar curvature. Consequently, any complete shrinking and steady gradient Ricci solitons have non-negative scalar curvature (see also [50] for a different proof). By the strong minimum principle, the scalar curvature of gradient shrinker and steadier must be positive unless the manifolds are Ricci flat (and hence is flat in the former case). Chow-Lu-Yang [16] applied an estimate of the potential function for gradient shrinker by Cao-Zhou [6] and obtained a sharp positive lower bound for the scalar curvature of noncompact and nonflat shrinker.

**Theorem 1.** [16] Let $(M^n, g, f)$ be an $n$ dimensional complete noncompact and nonflat shrinking gradient Ricci soliton. Then there exists a positive constant $c$ such that

$$S \geq \frac{c}{r^2}$$

outside a compact set of $M$.

**Remark 1.** As pointed out in [16], the estimate (4) is sharp on the shrinker constructed by Feldman-Ilmanen-Knopf [22].

For the case of steady soliton, Chow-Lu-Yang [16] also provided a sharp lower estimate for the scalar curvature under some conditions on the potential function (see also [36]).

**Theorem 2.** [16] Let $(M^n, g, f)$ be an $n$ dimensional complete noncompact and non Ricci flat steady gradient Ricci soliton with the scaling convention $|\nabla f|^2 + S = 1$. Suppose $f \leq 0$ and $\lim_{x \to \infty} f(x) = -\infty$. Then

$$S \geq \frac{1}{\sqrt{2^2 + 2e^{f}}}$$

on $M$.

**Remark 2.** Using some localization and cut off function arguments, Munteanu-Sung-Wang [36] showed $S \geq Ce^{f}$ without assuming $\lim_{r \to \infty} f = -\infty$. However the constant $C$ depends not only on the dimension $n$, but also the metric $g$. The decay rate in (5) is sharp on cigar soliton (see [16], [26] and [14]).

Unlike shrinker and steadier, expander may have negative scalar curvature. Pigola-Rimoldi-Setti [44] and S. J. Zhang [49] independently improved an estimate in [50] and showed the scalar curvature $S$ of an $n$ dimensional complete expanding gradient Ricci soliton satisfies

$$S \geq -\frac{n}{2},$$

with equality holds somewhere if and only if the expander is Einstein with scalar curvature $-\frac{n}{2}$ (see [14] and [39]).

Roughly speaking, the inequalities (1) and (5) in the above theorems provide some quantitative estimates to measure the gap between the solitons and the flat or Ricci-flat spaces. It is natural to ask whether such a gap exists or not if the scalar curvature of the expander is nonnegative. We provide an affirmative answer to the above question if the potential function $f$ is proper. No upper boundedness of the scalar curvature is assumed. Here is the main result of this paper.
Theorem 3. Let \((M^n, g, f)\) be an \(n\) dimensional complete non-compact gradient expanding Ricci soliton with \(n \geq 2\). Assume that \(\lim_{r \to \infty} f = -\infty\) and \((M, g)\) is non-flat and has non-negative scalar curvature \(S\). Then there exists a positive constant \(C\) such that
\[
S \geq Cv^{1 - \frac{n}{2}}e^{-v} \quad \text{on } M,
\]
where \(v := \frac{n}{2} - f\).

Remark 3. The estimate (7) is sharp when \(n = 2\) on 2 dimensional positively curved gradient expander and when \(n = 2m, m \geq 2\) on the Kähler gradient expander constructed by Feldman-Ilmanen-Knopf (see [22] and [47]). By (6) and (12), \(f \leq \frac{n}{2}\), hence \(f\) is proper (preimages of compact subsets are compact) if and only if \(\lim_{r \to \infty} f = -\infty\). One sufficient condition for the properness of \(f\) is that for some \(\varepsilon > 0\),
\[
\text{Ric} \geq (\varepsilon - \frac{1}{2})g
\]
outside a compact subset of \(M\).

Remark 4. Under the additional assumption that \(\lim_{r \to \infty} r^2|\text{Ric}| = 0\), Deruelle [21] also gave a lower bound (possibly zero) of \(v^{\frac{n}{2}} - 1e^vS\) which depends on \(\lim \inf_{r \to \infty} v^{\frac{n}{2}} - 1e^vS\). The limit \(\lim_{r \to \infty} v^{\frac{n}{2}} - 1e^vS\) is also related to the notion of the scalar curvature at infinity in [20].

For expander with nonpositive Ricci curvature and proper potential function, we prove an upper bound for the scalar curvature.

Theorem 4. Let \((M^n, g, f)\) be an \(n\) dimensional complete non-compact gradient expanding Ricci soliton with \(n \geq 2\). Suppose that \(\lim_{r \to \infty} f = -\infty\) and \((M, g)\) is non-flat and has non-positive Ricci curvature. Then there is a positive constant \(C\) such that
\[
S \leq -Cv^{1 - \frac{n}{2}}e^{-v},
\]
where \(v := \frac{n}{2} - f\).

Remark 5. The upper estimate is sharp on two dimensional negatively curved gradient expanders with curvature decaying to zero at infinity.

In view of Theorem 3, it is interesting to see when the scalar curvature of an expander is nonnegative. Sufficient condition for \(S \geq 0\) on Yamabe soliton was studied by Ma-Miquel [34]. The lower bound of the scalar curvature is also important since it is related to the connectedness at infinity of the expander (see [37], [12] and [40]). It is known that there doesn’t exist any compact expander with nonnegative scalar curvature. Indeed, any compact expander must be Einstein with scalar curvature \(-\frac{n}{2}\) (see [14]). Motivated by various studies of solitons with integrable curvature (see [44], [19], [7], [36] and [9]), we prove an analog in the noncompact case. Here is the second main result of the paper.

Theorem 5. Let \((M^n, g, f)\) be a complete noncompact gradient expanding Ricci soliton with dimension \(n \geq 3\). We denote the negative part of the scalar curvature \(S\) by \(S_-\), i.e. \(S_- := (-S)_+ := \max\{-S, 0\}\). If \(S_-\) is integrable, that is,
\[
\int_M S_- dv_g < \infty,
\]
then either $S \geq 0$ everywhere or $M$ is isometric to an Einstein manifold with scalar curvature $-\frac{n^2}{2}$ and finite volume, where $dv_g$ is the volume element with respect to the metric $g$.

Remark 6. The curvature of negatively curved Cao’s Kähler expander in complex dimension one decays exponentially in $r^2$ (see [3] and [17]). Hence Theorem 5 doesn’t hold in real dimension two.

One immediate consequence of the above theorem is that any complete noncompact gradient expander with nonnegative scalar curvature outside compact subset must have nonnegative scalar curvature everywhere. Using the asymptotic curvature estimates in [21], we also have:

**Corollary 1.** Let $(M^n, g, f)$ be a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. Suppose in addition that

$$
\lim_{x \to \infty} r^2 |\text{Ric}| = 0,
$$

where $r$ is the distance function from a fixed point. Then $M$ has nonnegative scalar curvature. In particular, $M$ is connected at infinity.

It was proven by Pigola-Rimoldi-Setti [44] that any gradient expander with $S \geq 0$ and $S \in L^1(M, e^{-f} dv_g)$ must be flat. Using $f \leq \frac{n^2}{2}$ on $M$, their result and Theorem 5, we get a slightly more general classification for $n \geq 3$.

**Corollary 2.** Given $(M^n, g, f)$ a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. If the scalar curvature $S$ is in $L^1(M, e^{-f} dv_g)$, then $M$ is isometric either to $\mathbb{R}^n$ or to an Einstein manifold with scalar curvature $-\frac{n^2}{2}$ and finite volume.

Using Theorem 5, we give another sufficient condition for nonnegativity of the scalar curvature.

**Theorem 6.** Let $(M^n, g, f)$ be a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$ and bounded scalar curvature $S$. Suppose the following conditions are satisfied:

1. $\lim_{x \to \infty} f = -\infty$;
2. $\liminf_{x \to \infty} r^2 S \geq 0$.

Then the scalar curvature $S \geq 0$ on $M$. In particular, $M$ is connected at infinity.

Remark 7. It can be seen from the proof of Theorem 5 that $S \geq 0$ holds under a weaker assumption on $S$, namely $\limsup_{x \to \infty} v^{-1}S < 1$ instead of $S$ being bounded, where $v = \frac{n^2}{2} - f$. Under the conditions of Corollary 1 or Theorem 5, it follows from Theorem 5 that the scalar curvature $S$ satisfies (17) provided that $M$ is not flat. In view of the negatively curved Bryant’s expander and Cao’s Kähler expander, we know that the quadratic factor $r^2$ in Corollary 1 and Theorem 6 is sharp (see [3], [2], [14] and [17]). We also see from the F.I.K. expander [22] that analogous result is not true if one replaces the scalar curvature $S$ by the Ricci curvature $\text{Ric}$.

We then study the curvature estimates for expanders of low dimensions, namely dimensions 3 and 4. Motivated by the estimates of the curvature tensor of shrinker and steadier in [39], [3], [42] and [41], we prove some analogs for 3 dimensional gradient expander.
Theorem 7. Let \((M^3, g, f)\) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Then the curvature tensor \(Rm\) is bounded if the scalar curvature \(S\) is bounded. Moreover, the following hold:

(a) If \(S\) is nonnegative and bounded, then
\[
|Rm| \leq c\sqrt{S} \text{ on } M,
\]
for some positive constant \(c\);
(b) If \(S \to 0\) as \(x \to \infty\), then \(\lim_{x \to \infty} |Rm| = 0\).

As an application, we show that \(f\) is proper if the scalar curvature \(S\) decays at infinity in dimension three.

Corollary 3. Let \((M^3, g, f)\) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Suppose that \(\lim_{x \to \infty} S = 0\). Then \(\lim_{x \to \infty} f = -\infty\), furthermore
\[
\lim_{x \to \infty} \frac{4f(x)}{r^2(x)} = -1.
\]

Remark 8. In view of Theorem 12 and Proposition 1, we see that (10) is also true in higher dimensions if in addition \(|Rm|\) is bounded on \(M\).

Munteanu-Wang [39] showed that any 4 dimensional complete gradient shrinker must have bounded curvature if the scalar curvature is bounded (see also [5] for the estimates in steady soliton). We prove that it is also true for gradient expanding soliton if in addition the potential function \(f \to -\infty\) as \(r \to \infty\).

Theorem 8. Let \((M^4, g, f)\) be a 4 dimensional complete non-compact gradient expanding Ricci soliton with bounded scalar curvature. Suppose that \(\lim_{r \to \infty} f = -\infty\). Then the curvature tensor \(Rm\) is bounded.

It was proved by Deng-Zhu [17] and Deruelle [21] that any gradient expander with non-negative Ricci curvature must have bounded scalar curvature (their arguments work well if \(\text{Ric} \geq 0\) outside some compact subset of \(M\)). Together with Remark 8, we have the following corollary.

Corollary 4. Let \((M^4, g, f)\) be a 4 dimensional complete noncompact gradient expanding Ricci soliton with \(\text{Ric} \geq 0\) outside some compact subset of \(M\). Then it has bounded curvature tensor.

Using Shi’s estimate [15], Theorem 12 and Lemma 8, we establish the equivalence of the properness of \(f\) and the curvature decay at infinity.

Corollary 5. Let \((M^4, g, f)\) be a 4 dimensional complete noncompact gradient expanding Ricci soliton with \(\lim_{x \to \infty} S(x) = 0\). Then \(\lim_{x \to \infty} f(x) = -\infty\) if and only if \(\lim_{x \to \infty} |Rm|(x) = 0\).

As an application of the previous curvature estimates, we study the gap theorem of expanding soliton. Gap phenomenon of manifold has long been an interesting problem in geometry. A special case of the problem is concerned with the flatness of noncompact manifold, more precisely, for a non-compact manifold with curvature decaying sufficiently fast at infinity, is it necessarily flat? The tangent cone at infinity of gradient expander with \(\lim_{x \to \infty} x^2 |Rm| = 0\) was proven to be flat by Chen [11] and Chen-Deruelle [12]. The expander itself is also flat if in addition \(\text{Ric} \geq 0\) (see [1] and [12]). Deng and Zhu [17] proved that a Kähler gradient expander with
complex dimension \( n \geq 2 \), \( \text{Ric} \geq 0 \) and \( \lim_{r \to \infty} r^2 S = 0 \) is isometric to \( \mathbb{C}^n \) (see \[35\], \[24\] and \[1\] for more results on the gap theorems of general Riemannian manifolds). In either cases, Ricci curvature is assumed to be non-negative. Using the positive mass theorem, asymptotic curvature estimates in \[21\], Theorems 5 and 7(b), we are able to prove a gap theorem for 3 dimensional gradient expander by only imposing condition on the scalar curvature.

**Theorem 9.** Let \((M^3, g, f)\) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Suppose the scalar curvature \(S\) satisfies
\[
\lim_{x \to \infty} r^2(x)S(x) = 0.
\]
Then \(M\) is isometric to \(\mathbb{R}^3\).

**Remark 9.** Analogous result is not true in higher even dimensions. The non-flat Kähler gradient expander constructed by Feldman-Ilmanen-Knopf \[22\] has non-negative scalar curvature which decays exponentially in \(r\).

**Remark 10.** Positive mass theorem was also used by Ma \[33\] to prove the flatness of 3 dimensional Ricci pinched gradient expander. Using the asymptotic curvature estimates \[21\] and volume comparison theorem, Deruelle showed that any 4 dimensional Ricci pinched gradient expander is flat (see \[33\] and \[21\] for more results in higher dimensions under additional assumptions). Again \(\text{Ric} \geq 0\) was assumed in both cases.

Recently the positive mass theorem for asymptotically flat manifold with dimension \(n \geq 9\) has been proven by Lohkamp \[29\], \[30\], and later also by Schoen-Yau \[46\]. Hence similar argument for Theorem 9 gives the following gap theorem in higher odd dimensions without sign condition on the Ricci curvature.

**Theorem 10.** Let \((M^{2m+1}, g, f)\) be a complete noncompact gradient expanding Ricci soliton with odd dimension \(2m + 1\) and \(m \geq 2\). If \(\lim_{x \to \infty} r^2|\text{Rm}| = 0\), then \(M\) is isometric to \(\mathbb{R}^{2m+1}\).

**Remark 11.** The theorem is not true in even dimensions. The non-flat Feldman-Ilmanen-Knopf Kähler expander \[22\] has flat asymptotic cone and satisfies \(\lim_{x \to \infty} r^2|\text{Rm}| = 0\). Nonetheless, if in addition either one of the following conditions holds:

1. \(M\) is smoothly asymptotic to the cone \((C(\mathbb{S}^{n-1}(1)), dt^2 + t^2g_{\mathbb{S}^{n-1}(1)})\) in sense of \[21\];
2. \(\lim \inf_{R \to \infty} \frac{\text{Vol}_{g_{\mathbb{R}^{n}}}(B_R(p_0))}{R^n} > \frac{\omega_{n-1}}{2n}\), where \(\omega_{n-1}\) is the volume of \(\mathbb{S}^{n-1}(1)\) in \(\mathbb{R}^n\) with respect to the Euclidean metric;
3. \(M\) is simply connected at infinity;
4. \(\text{Ric} \geq (\delta - \frac{1}{2})g\) on \(M\) for some \(\delta > 0\),

then \(M^n\) is isometric to \(\mathbb{R}^n\), where \(n = 2m\) and \(m \geq 2\). As mentioned in Remark 7, the quadratic factor \(r^2\) in Theorems 9 and 10 is optimal.

The paper is organized as follows. We include all the preliminaries and computations for the proof of Theorems 3 and 4 in Section 2. Theorems 5 and 6 will then be proved in Sections 3 and 4 respectively. In Section 5, We give a proof for Theorem 5 using integration by part. In Section 6, we shall show Theorems 7 and 8. We then justify Theorem 6 in Section 7. The proof of Theorem 8 will be given in the last section.
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2. Preliminaries

Let \( (M^n, g, f) \) be an \( n \) dimensional gradient expanding Ricci soliton, i.e.
\[
\text{Ric} + \nabla^2 f = -\frac{1}{2} g.
\]
It generates a self similar solution to the Ricci flow. Indeed, let \( \psi_t \) be the flow of the vector field \( \nabla f \) with \( \psi_0 \) being the identity map. We define \( g(t) := (1 + t) \psi_t^* g \), then \( g(t) \) is a solution to the Ricci flow for \( t \in (-1, \infty) \) with \( g(0) = g \). The following equations for gradient expanders are known (see [14], [43] and [21])

11. \[
S + \Delta f = -\frac{n}{2},
\]
12. \[
S + |\nabla f|^2 = -f,
\]
13. \[
\Delta f f = f - \frac{n}{2},
\]
14. \[
\Delta f S = -S - 2|\text{Ric}|^2,
\]
15. \[
R_{kij} - R_{ki,j} = R_{ijkl} f_l,
\]
16. \[
\Delta f R_{ij} = -R_{ij} - 2R_{iklj} R_{kl}
\]
and
17. \[
\Delta f R_m = -R_m + R_m * R_m,
\]
where \( S \) is the scalar curvature and \( \Delta f := \Delta - \nabla \nabla f \). We define a function \( v \) in the following way:

\[
v := \frac{n}{2} - f.
\]
From \([12]\) and \([13]\), we have

19. \[
\Delta f v = v \quad \text{and}
\]
20. \[
S + |\nabla v|^2 = v - \frac{n}{2}.
\]
By \([9]\) and \([20]\), we have \( |\nabla v|^2 \leq v \) and \( |\nabla \sqrt{v + \delta}| \leq \frac{1}{2} \) for any \( \delta > 0 \). Integrating the inequality along minimizing geodesics and letting \( \delta \to 0 \), we get

21. \[
|\nabla f| = |\nabla v| \leq \sqrt{v} \leq \frac{1}{2} r + \sqrt{v}(p_0),
\]
where \( p_0 \) is a fixed point in \( M \). Suppose that \( S \) is now non-negative. Then we have by \([12]\) that \( f \leq 0 \). Using strong minimum principle [23] and \([14]\), we conclude that \( S > 0 \) unless \( S \equiv 0 \) on \( M \) (i.e. flat, see [44]). Moreover \( v \) satisfies

22. \[
v \geq \frac{n}{2}.
\]
Lemma 3. Under the same assumption in Lemma 2, then at

This completes the proof of the lemma. □

Proof. From (19) and (20), we see that

\( R > - \frac{2}{v^2} \) somewhere on \( M \), then \( v > 0 \) on \( M \).

If \( \lim_{r \to \infty} f = - \infty \), then \( v > 0 \) somewhere and hence \( v > 0 \). The following lemmas are immediate consequences of the computations by Deruelle [21]. We include the calculations for the sake of completeness.

Lemma 2. Let \( (M^n, g, f) \) be an \( n \) dimensional complete gradient expanding Ricci soliton and \( p \in M \). If \( v(p) > 0 \), then at \( p \)

\[ \Delta f^2 + 2 \ln v(\nabla v^2) = -2v|Ric|^2 - 2|\nabla v|^2 vS. \]  

Proof. Using (14) and (19),

\[ \Delta_f (vS) = v\Delta_f S + S\Delta_f v + 2\langle \nabla v, \nabla S \rangle \]

\[ = v(-S - 2|Ric|^2 + Sv + 2\langle \nabla v, \nabla (vSv^{-1}) \rangle) \]

\[ = -2v|Ric|^2 + 2\langle \nabla \ln v, \nabla (vS) \rangle - 2|\nabla \ln v|^2 vS. \]

This completes the proof of the lemma. □

Lemma 3. Under the same assumption in Lemma 3, then at \( p \)

\[ \Delta_{f + 2 \ln v}(e^{\frac{1}{v^2}} v^2 - \frac{2}{v} e^{-v}) = e^{\frac{1}{v^2}} v^2 - \frac{2}{v} e^{-v} \left\{ \frac{1}{2\sqrt{v}} - \frac{1}{v^2} + \frac{1}{v^2} \frac{1}{v^2} \right\} \]

\[ = e^{\frac{1}{v^2}} v^2 - \frac{2}{v} e^{-v} \left\{ \frac{1}{2\sqrt{v}} - \frac{1}{v^2} + \frac{1}{v^2} \right\} \]

(24)

Consequently if \( M \) is noncompact and \( v \to \infty \) as \( r \to \infty \), then near infinity

\[ \Delta_{f + 2 \ln v}(e^{\frac{1}{v^2}} v^2 - \frac{2}{v} e^{-v}) = e^{\frac{1}{v^2}} v^2 - \frac{2}{v} e^{-v} \left\{ \frac{1}{2\sqrt{v}} - \frac{1}{v^2} + \frac{1}{v^2} \right\} \]

Proof. \( \Delta_f v^2 - \frac{2}{v} \) \( \Delta_f v + \frac{n}{2} - 2(1)v^2 - 1) v^2 |\nabla v|^2 \]

\[ = \left( 2 - \frac{n}{2} \right) v^2 - \frac{2}{v} \Delta_f v + \frac{n}{2} - 2(1)v^2 - 1) v^2 |\nabla v|^2 \]

By (20),

\[ \Delta_f e^{-v} = -e^{-v} \Delta_f v + e^{-v} |\nabla v|^2 \]

\[ = e^{-v} |\nabla v|^2 - v \]

\[ = -(S + \frac{n}{2}) e^{-v}. \]
\[ 2(\nabla(v^2 - \frac{v}{2}), \nabla e^{-v}) = 2(2 - \frac{n}{2}) v^{1 - \frac{v}{2}} |\nabla v|^2 (-e^{-v}) \]
\[ = 2\left(\frac{n}{2} - 2\right) v^{1 - \frac{v}{2}} e^{-v} - (2S + n)\left(\frac{n}{2} - 2\right) v^{1 - \frac{v}{2}} e^{-v}. \]

\[ \Delta f(v^2 - \frac{v}{2} e^{-v}) = e^{-v} \Delta f v^2 - \frac{v}{2} + v^2 - \frac{v}{2} \Delta f e^{-v} + 2(\nabla(v^2 - \frac{v}{2}), \nabla e^{-v}) \]
\[ = (2 - \frac{n}{2}) v^{2 - \frac{v}{2}} e^{-v} + (\frac{n}{2} - 2)(\frac{n}{2} - 1) v^{2 - \frac{v}{2}} e^{-v} |\nabla \ln v|^2 \]
\[ - (S + \frac{n}{2}) v^2 - \frac{v}{2} e^{-v} + 2(\frac{n}{2} - 2) v^{2 - \frac{v}{2}} e^{-v} \]
\[ - (2S + n)(\frac{n}{2} - 2) v^{1 - \frac{v}{2}} e^{-v} \]
\[ = v^2 - \frac{v}{2} e^{-v}\left[ -2 + (\frac{n}{2} - 2)(\frac{n}{2} - 1) |\nabla \ln v|^2 \right. \]
\[ \left. - (2S + n)(\frac{n}{2} - 2) v^{-1}\right]. \]

On the other hand, by [20]

\[ -2(\nabla \ln v, \nabla(v^2 - \frac{v}{2} e^{-v})) = 2\left(\frac{n}{2} - 2\right) v^{1 - \frac{v}{2}} e^{-v} (\frac{\nabla v}{v}, \nabla v) \]
\[ + 2v^2 - \frac{v}{2} e^{-v} (\frac{\nabla v}{v}, \nabla v) \]
\[ = 2\left(\frac{n}{2} - 2\right) v^{2 - \frac{v}{2}} e^{-v} |\nabla \ln v|^2 \]
\[ + 2v^2 - \frac{v}{2} e^{-v} (v - S + \frac{n}{2}) \]
\[ = v^2 - \frac{v}{2} e^{-v}\left[ 2 - (2S + n)v^{-1} \right. \]
\[ \left. + 2\left(\frac{n}{2} - 2\right) |\nabla \ln v|^2 \right]\]

Hence

\[ \Delta f_{+2\ln v}(v^2 - \frac{v}{2} e^{-v}) = v^2 - \frac{v}{2} e^{-v}\left[ -S + (\frac{n}{2} - 2)(\frac{n}{2} + 1) |\nabla \ln v|^2 \right. \]
\[ \left. - (2S + n)(\frac{n}{2} - 1) v^{-1}\right]. \]

\[ \Delta f e^{-\frac{v}{2}} = -\frac{1}{2} v^{1 - \frac{v}{2}} e^{-\frac{v}{2}} \Delta f v + e^{-\frac{v}{2}} \left[ \frac{3}{4} v^{1 - \frac{v}{2}} + \frac{1}{4} v^{-3} \right] |\nabla v|^2 \]
\[ = -\frac{1}{2} v^{1 - \frac{v}{2}} e^{-\frac{v}{2}} + e^{-\frac{v}{2}} \left[ \frac{3}{4} v^{1 - \frac{v}{2}} + \frac{1}{4} v^{-3} \right] (v - S + \frac{n}{2}) \]
\[ = -\frac{1}{2} v^{1 - \frac{v}{2}} e^{-\frac{v}{2}} + v^{1 - \frac{v}{2}} e^{-\frac{v}{2}} \left[ \frac{3}{4} + \frac{1}{4} v^{-3} \right] \]
\[ - v^{-\frac{v}{2}} e^{-\frac{v}{2}} (S + \frac{n}{2}) \left[ \frac{3}{4} + \frac{1}{4} v^{-3} \right]. \]
\[
\Delta_{f+2\ln v} e^{\sqrt{v}} = \Delta f e^{\sqrt{v}} - 2\left(\frac{\nabla v}{v} , \nabla e^{\sqrt{v}}\right)
= \Delta f e^{\sqrt{v}} + \left(\frac{v - S - \frac{5}{2}}{2}\right) e^{\sqrt{v}}
= -\frac{1}{2} v^{-\frac{3}{2}} e^{\sqrt{v}} + v^{-\frac{3}{2}} e^{\sqrt{v}} \left[\frac{7}{4} + \frac{1}{4} v^{-\frac{3}{2}}\right]
= -v^{-\frac{3}{2}} e^{\sqrt{v}} \left(S + \frac{n}{2} \right) \left[\frac{7}{4} + \frac{1}{4} v^{-\frac{3}{2}}\right].
\]

\[
2 \langle \nabla e^{\sqrt{v}} , \nabla (v^{2 - \frac{3}{2}} e^{-v}) \rangle = -v^{-\frac{3}{2}} e^{\sqrt{v}} \langle \nabla v , \nabla (2 - \frac{n}{2}) e^{-v}\rangle - v^{-\frac{3}{2}} e^{\sqrt{v}} \langle \nabla v , \nabla (v^{2 - \frac{3}{2}} (-e^{-v}))\rangle
= v^{-\frac{3}{2}} e^{\sqrt{v}} (v - S - \frac{n}{2}) \left(\frac{n}{2} - 2\right) v^{1 - \frac{3}{2}} e^{-v}
+ v^{-\frac{3}{2}} e^{\sqrt{v}} (v - S - \frac{n}{2}) v^{2 - \frac{3}{2}} e^{-v}
= v^{-\frac{3}{2}} e^{\sqrt{v}} v^{2 - \frac{3}{2}} e^{-v} - (S + \frac{n}{2}) v^{-\frac{3}{2}} e^{\sqrt{v}} v^{2 - \frac{3}{2}} e^{-v}
+ v^{-\frac{3}{2}} e^{\sqrt{v}} \left(\frac{n}{2} - 2\right) v^{2 - \frac{3}{2}} e^{-v}
-(S + \frac{n}{2}) v^{-\frac{3}{2}} e^{\sqrt{v}} \left(\frac{n}{2} - 2\right) v^{2 - \frac{3}{2}} e^{-v}.
\]

Using (24), we have
\[
\Delta_{f+2\ln v} (e^{\sqrt{v}} v^{2 - \frac{3}{2}} e^{-v}) = e^{\sqrt{v}} \Delta_{f+2\ln v} (v^{2 - \frac{3}{2}} e^{-v}) + v^{2 - \frac{3}{2}} e^{-v} \Delta_{f+2\ln v} (e^{\sqrt{v}})
+ 2 \langle \nabla e^{\sqrt{v}} , \nabla (v^{2 - \frac{3}{2}} e^{-v}) \rangle
= e^{\sqrt{v}} v^{2 - \frac{3}{2}} e^{-v} \left\{\frac{1}{2\sqrt{v}} - (S + \frac{n}{2}) v^{-\frac{3}{2}} + v^{-\frac{3}{2}} \left(\frac{n}{2} - 2\right)\right\}
-(S + \frac{n}{2}) v^{-\frac{3}{2}} \left[\frac{n}{2} - 2\right] - S + \frac{n}{2} v^{-\frac{3}{2}} \left[\frac{7}{4} + \frac{1}{4} v^{-\frac{3}{2}}\right]
- (2S + n) \left(\frac{n}{2} - 1\right) v^{-1} + v^{-\frac{3}{2}} \left[\frac{7}{4} + \frac{1}{4} v^{-\frac{3}{2}}\right]
- v^{-\frac{3}{2}} \left(S + \frac{n}{2} \right) \left[\frac{7}{4} + \frac{1}{4} v^{-\frac{3}{2}}\right].
\]

We showed (24). We can then separate the terms with S from those without S in R.H.S of (24) to get (25). \[\Box\]

3. Proof of Theorem \[\Box\]

With all the computations in the previous section, we are going to prove Theorem \[\Box\]
Theorem. Let \((M^n, g, f)\) be an n dimensional complete non-compact gradient expanding Ricci soliton with \(n \geq 2\). Suppose that \(\lim_{r \to \infty} f = -\infty\) and \((M, g)\) is not flat and has non-negative scalar curvature \(S\). Then there exists a positive constant \(C\) such that

\[
S \geq Cv^{1 - \frac{n}{2}}e^{-v} \text{ on } M,
\]

where \(v := \frac{n}{2} - f\).

Proof. Since the scalar curvature \(S \geq 0\) and \(v \geq \frac{n}{2}\), we have by (23)

\[
\Delta f + 2 \ln v (vS) \leq 0.
\]

Moreover by \(\lim_{x \to \infty} v(x) = \infty\) and (25), we see that outside a compact subset of \(M\),

\[
\Delta f + 2 \ln v (e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)}) = e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)} \left\{ \left( \frac{1}{2} + o(1) \right) \frac{1}{\sqrt{v}} + S \left( -1 + o(1) \right) \right\}.
\]

From the above equation,

\[
\Delta f + 2 \ln v (e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)}) \geq e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)} \left( S - 1 + o(1) \right).
\]

Since \(v = \frac{n}{2} - f \to \infty\) as \(r \to \infty\), by taking a larger compact set if necessary, we may assume that

\[
\frac{1}{8} > e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)}
\]

near infinity. Let \(R_0\) be a large positive number such that (27), (28) and (29) hold on \(M \setminus B_{R_0}(p_0)\). Hence by \(S > 0\) and (22), there exists a constant \(b \in (0, 1)\) such that

\[
vS > be^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)} \text{ on } \partial B_{R_0}(p_0).
\]

Let \(Q := vS - be^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)}\). It is not difficult to see that

\[
\liminf_{r \to \infty} Q \geq 0.
\]

Fix any \(y\) in \(M \setminus B_{R_0}(p_0)\) and any \(\varepsilon > 0\), there exists a large positive \(T > R_0\) such that \(y \in B_T(p_0)\) and

\[
Q \geq -\varepsilon \text{ on } \partial B_T(p_0).
\]

Let \(\Omega := B_T(p_0) \setminus B_{R_0}(p_0)\) and \(z \in \Omega\) such that \(Q\) attains its minimum over \(\overline{\Omega}\) at \(z\), i.e.

\[
Q(z) = \min_{\overline{\Omega}} Q.
\]

If \(z \in \partial \Omega\), then by (30) and (31), we have

\[
Q(y) \geq Q(z) \geq -\varepsilon.
\]

If \(z \in \Omega\), we have by (27) and (28) that at \(z\)

\[
0 \leq \Delta f + 2 \ln v Q = \Delta f + 2 \ln v (vS) - b\Delta f + 2 \ln v (e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)}) \leq -b\Delta f + 2 \ln v (e^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)}) \leq 2be^{\frac{1}{\sqrt{v}} \left( \frac{v}{2} \frac{v}{2} - \frac{n}{2} e^{-v} \right)} (S - \frac{1}{8\sqrt{v}}).\]
Together with (29), we know that at
\[ vS \geq \sqrt{v} \]
\[ \geq e^{\frac{1}{2}v^{2-n}e^{-v}} \]
\[ \geq be^{\frac{1}{2}v^{2-n}e^{-v}}, \]
i.e. \( 0 \leq Q(z) \leq Q(y) \). Hence in any cases, \( Q(y) \geq -\varepsilon \). Result then follows by letting \( \varepsilon \to 0 \) and choosing a smaller \( b \) to make \( Q \geq 0 \) on the entire \( M \).

4. Proof of Theorem 4

To prepare for the maximum principle argument, we first show that the scalar curvature is negative in a gradient expander with nonpositive Ricci curvature.

**Lemma 4.** Let \((M^n, g, f)\) be a complete noncompact and nonflat expanding gradient Ricci soliton with \( \text{Ric} \leq 0 \). Then \( S < 0 \) on \( M \).

**Proof.** Using \( \text{Ric} \leq 0 \), we have \(|\text{Ric}|^2 \leq S^2\) and by (14)
\[
\Delta fS = -S - 2|\text{Ric}|^2 \\
\geq -(1 + 2S)S.
\]
We argue by contradiction. If \( S(z) = 0 \) for some \( z \) in \( M \), then \( S \) attains its interior maximum at \( z \). By the strong maximum principle [23], \( S \equiv 0 \) and hence by (14) \( \text{Ric} \equiv 0 \). From (2), \( -2\nabla^2f = g \). \( g \) is flat by a result in [44], which is absurd. \( \square \)

With Lemma 4, we can finish the proof of Theorem 4

**Theorem.** Let \((M^n, g, f)\) be an \( n \) dimensional complete non-compact gradient expanding Ricci soliton with \( n \geq 2 \). Suppose that \( \lim_{r \to \infty} f = -\infty \) and \( (M, g) \) is not flat and has non-positive Ricci curvature. Then there is a positive constant \( C \) such that
\[ S \leq -Cv^{1-\frac{n}{2}}e^{-v}, \]
where \( v := \frac{n}{2} - f \).

**Proof.** Using the discussion after the proof of Lemma 4, we have \( v > 0 \) on \( M \). By (20),
\[
|\nabla \ln v|^2 v = \frac{|\nabla v|^2}{v} = 1 - \frac{\frac{n}{2} + S}{v}.
\]
Using \( \text{Ric} \leq 0 \) and (23) we see that \(|\text{Ric}|^2 \leq S^2\) and
\[
\Delta f_{T+2\ln v}(vS) \geq -2\nu S^2 - 2|\nabla \ln v|^2 vS \\
= -2\nu S^2 - 2S + \frac{nS + 2S^2}{v} \\
= S\left(-2 + \frac{n + 2S}{v} - 2\nu S\right).
\]
From \( \lim_{r \to \infty} v = \infty \) and (23), we see that
\[
\Delta f_{T+2\ln v}(e^{\frac{1}{2}v^2 - \frac{n}{2}e^{-v}}) = e^{\frac{1}{2}v^2 - \frac{n}{2}e^{-v}}\left\{ \left(\frac{1}{2} + o(1)\right)\frac{1}{\sqrt{v}} + S\left(-1 + o(1)\right) \right\} \\
\geq 0
\]
near infinity. We now consider a large $R_0$ such that on $M \setminus B_{R_0}(p_0)$, $v > 1$ and (33) are true, moreover the following hold:

\begin{equation}
-1 + \frac{n}{1 - \frac{v}{v^*}} \leq -\frac{1}{2}
\end{equation}

and

\begin{equation}
\frac{1}{2} \geq e^{\frac{1}{2}v^2 - \frac{n}{2}v} e^{-v}.
\end{equation}

For such $R_0$, there exists a positive constant $\alpha \in (0, 1)$ such that

\begin{equation}
\int_{M \setminus B_{R_0}(p_0)} S_{-} \, dv < \infty,
\end{equation}

\begin{equation}
\int_{M \setminus B_{R_0}(p_0)} S_{-} \, dv < \infty.
\end{equation}

5. Proof of Theorem 5

Before moving to the proof of Theorem 5, we recall the statement of the theorem:

**Theorem.** Let $(M^n, g, f)$ be a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,

\begin{equation}
\int_M S_- \, dv < \infty,
\end{equation}

then $M$ is compact with finite Gaussian curvature and $f \equiv 0$. Thus $M^n$ is a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature $S$ by $S_-$, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If $S_-$ is integrable, that is,
then either $S \geq 0$ everywhere or $M$ is isometric to an Einstein manifold with scalar curvature $-\frac{4}{n}$ and finite volume.

**Proof.** W.L.O.G., we may assume $S < 0$ somewhere and show that $M$ is Einstein. We are going to prove that $M$ has constant scalar curvature $-\frac{4}{n}$. By ([13] and Cauchy Schwarz inequality,

$$\Delta f S = -S - 2|\text{Ric}|^2 \leq -S - \frac{2}{n}S^2.$$ 

Let $\delta \in (0, \frac{1}{4})$, $\epsilon \in (0, 1)$ and $\phi$ be any nonnegative compactly supported function, we multiply the above inequality by $(-S)_+(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2 \geq 0$ and integrate the inequality over $M$,

$$(I) + (II) := \int_M (-S)_+(S^2 + \epsilon)^{\delta - \frac{1}{2}}\Delta S \phi^2 - \int_M (-S)_+(S^2 + \epsilon)^{\delta - \frac{1}{2}}\langle \nabla f, \nabla S \rangle \phi^2$$
$$\leq \int_M (-S)_+(-S - \frac{2}{n}S^2)(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2$$
$$= \int_{\{S < 0\}} (-S)(-S - \frac{2}{n}S^2)(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2$$
$$= \int_{\{S < 0\}} \frac{2}{n}S^2(S + \frac{n}{2})(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2.$$ 

By [24], the weak derivative of $(-S)_+$ is given by

$$\nabla(-S)_+ = \chi_{\{S < 0\}}\nabla(-S) = -\chi_{\{S < 0\}}\nabla S,$$

where $\chi_{\{S < 0\}}$ is the characteristic function of the set $\{S < 0\}$ which is 1 on $\{S < 0\}$ and vanishes elsewhere. Using integration by part, we see that

$$(I) = \int_M (-S)_+(S^2 + \epsilon)^{\delta - \frac{1}{2}}\Delta S \phi^2$$
$$= -\int_M \chi_{\{S < 0\}}(-\nabla S, \nabla S)(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2 - \int_M \chi_{\{S < 0\}}(-S)\phi^2\langle \nabla(S^2 + \epsilon)^{\delta - \frac{1}{2}}, \nabla S \rangle$$
$$- \int_M \chi_{\{S < 0\}}(-S)2\phi\langle \nabla \phi, \nabla S \rangle(S^2 + \epsilon)^{\delta - \frac{1}{2}}$$
$$\geq \int_{\{S < 0\}} |\nabla S|^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2 + (2\delta - 1) \int_{\{S < 0\}} |\nabla S|^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}S^2 \phi^2$$
$$- \int_{\{S < 0\}} 2\phi|\nabla \phi||\nabla S||S|(S^2 + \epsilon)^{\delta - \frac{1}{2}}$$
$$\geq 2\delta \int_{\{S < 0\}} |\nabla S|^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2 - \delta \int_{\{S < 0\}} |\nabla S|^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2$$
$$- \frac{1}{\delta} \int_{\{S < 0\}} S^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}|\nabla \phi|^2$$
$$= \delta \int_{\{S < 0\}} |\nabla S|^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}\phi^2 - \frac{1}{\delta} \int_{\{S < 0\}} S^2(S^2 + \epsilon)^{\delta - \frac{1}{2}}|\nabla \phi|^2.$$


By (11) and integration by part again, we have

\[(II) = - \int_M (-S)^+ (S^2 + \varepsilon) \phi^2 \\nabla f, \nabla f) \phi^2 \]
\[= \frac{1}{2\delta + 1} \int_M (\nabla f, \nabla f) (-S)^+ (S^2 + \varepsilon) \phi^2 \]
\[= \frac{1}{2\delta + 1} \int_M \Delta f (-S)^+ (S^2 + \varepsilon) \phi^2 - \frac{2}{2\delta + 1} \int_M \phi \nabla \phi, \nabla f)(-S)^+ (S^2 + \varepsilon) \phi^2 \]
\[= \frac{1}{2\delta + 1} \int_M (S + \frac{n}{2})(-S)^+ (S^2 + \varepsilon) \phi^2 = \frac{2}{2\delta + 1} \int_M \phi \nabla \phi, \nabla f)(-S)^+ (S^2 + \varepsilon) \phi^2. \]

All in all, we have

\[0 \leq \frac{1}{\delta} \int_{\{S < 0\}} |\nabla S|^2 (S^2 + \varepsilon) \phi^2 \]
\[\leq \frac{1}{\delta} \int_{\{S < 0\}} S^2 (S^2 + \varepsilon) \phi^2 + \frac{2}{2\delta + 1} \int_M \phi \nabla \phi, \nabla f)(-S)^+ (S^2 + \varepsilon) \phi^2 \]
\[+ \int_{\{S < 0\}} \frac{2}{n} (S + \frac{n}{2})(S^2) \phi^2 - \frac{1}{2\delta + 1} \int_{\{S < 0\}} (S + \frac{n}{2})(S^2) \phi^2 \]
\[= \frac{1}{\delta} \int_{\{S < 0\}} (S^2) \phi^2 + \frac{2}{2\delta + 1} \int_{\{S < 0\}} \phi \nabla \phi, \nabla f)(S^2) \phi^2 \]
\[+ \frac{2}{n} \int_{\{S < 0\}} (S + \frac{n}{2})(S^2) \phi^2. \]

Since \(n \geq 3\) and \(\delta < \frac{1}{4}\), we have \(\frac{2}{n} - \frac{1}{2\delta + 1} < 0\) and by (10)

\[0 \leq \frac{1}{\delta} \int_{\{S < 0\}} (S + \frac{n}{2})(S^2) \phi^2 \]
\[\leq \frac{1}{\delta} \int_{\{S < 0\}} (S^2) \phi^2 + \frac{2}{2\delta + 1} \int_{\{S < 0\}} \phi \nabla \phi, \nabla f)(S^2) \phi^2. \]

Let \(R \geq 1\) and \(\psi : [0, \infty) \to \mathbb{R}\) be a smooth real valued function satisfying the following: \(0 \leq \psi \leq 1\), \(\psi' \leq 0\),

\[\psi(t) = \begin{cases} 
1 & 0 \leq t \leq 1 \\
0 & 2 \leq t 
\end{cases} \]

and

\[|\psi'(t)| \leq c\] for all \(t \geq 0\).

We take the cut off function \(\phi(x) := \psi(\frac{1+x}{R})\), then

\[\phi = \begin{cases} 
1 & \text{on } B_R(p_0) \\
0 & \text{on } M \setminus B_{2R}(p_0) 
\end{cases} \]
and

\[(40) \quad |\nabla \phi| = \frac{|\psi'|}{R} \leq \frac{c}{R} \chi_{B_{2R} \setminus B_R},\]

where \(\chi_{B_{2R} \setminus B_R}\) is the characteristic function of the set \(B_{2R}(p_0) \setminus B_R(p_0)\) which is 1 on \(B_{2R}(p_0) \setminus B_R(p_0)\) and vanishes elsewhere. We have

\[
\frac{1}{\delta} \int_{\{S < 0\}} (S^2)^{\delta + \frac{1}{2}} |\nabla \phi|^2 \leq \frac{c^2 n^{2\delta}}{\delta^4 R^2} \int_{\{S < 0\} \setminus B_R} |S| = \frac{c^2 n^{2\delta}}{\delta^4 R^2} \int_{M \setminus B_R} (-S)_+.\]

Since \(|\nabla f| \leq C(r + 1)\) (see (21)),

\[
\frac{2}{2\delta + 1} \int_{\{S < 0\}} \phi(\nabla \phi, \nabla f)(S^2)^{\delta + \frac{1}{2}} \leq \frac{2cn^{2\delta}}{(2\delta + 1)4^\delta} \int_{\{S < 0\} \setminus B_R} |S| = \frac{2cn^{2\delta}}{(2\delta + 1)4^\delta} \int_{M \setminus B_R} (-S)_+.
\]

By (38), we let \(R \to \infty\) and conclude by (39) that

\[
\int_{\{S < 0\}} (S + \frac{n}{2})(S^2)^{\delta + \frac{1}{2}} = 0.
\]

Hence we may apply (6) again to get

\[
(S + \frac{n}{2})(S^2)^{\delta + \frac{1}{2}} \equiv 0 \quad \text{on} \quad \{S < 0\}.
\]

From this we see that \(S = -\frac{n}{2}\) wherever \(\{S < 0\}\). Since we assume that \(\{S < 0\}\) is nonempty, we know by connectedness that \(S \equiv -\frac{n}{2}\). \(M\) is Einstein since

\[
0 = \Delta_f S = -S - 2|\text{Ric}|^2 = -\frac{2S}{n}(S + \frac{n}{2}) - 2|\text{Ric} - \frac{S}{n}g|^2.
\]

Finiteness of volume is a now consequence of (38).

\[ \square \]

6. PROOF OF THE THEOREMS 7 AND 9

In this section, we study the geometry of three dimensional gradient expander. For the convenience of reader, we recall the statement of Theorem 7(a).

**Theorem.** Let \((M, g, f)\) be a 3 dimensional complete non-compact gradient expanding Ricci soliton with bounded non-negative scalar curvature. Then the curvature tensor \(Rm\) is bounded, moreover

\[ |Rm| \leq c\sqrt{S} \quad \text{on} \quad M, \]

for some constant \(c > 0\).

**Proof.** Throughout this proof, we use \(c\) to denote an absolute constant, its value may be different from line by line. W.L.O.G, we may assume \(M\) is not flat, or else we have nothing to prove. By the strong minimum principle and (14), \(S\) is positive...
on $M$. Since the Weyl tensor is zero in dimension three, it suffices to bound the Ricci tensor. By (43)

$$\Delta_f |\text{Ric}|^2 = 2|\nabla \text{Ric}|^2 + 2\langle \text{Ric}, \Delta_f \text{Ric} \rangle$$

(41)

$$= 2|\nabla \text{Ric}|^2 - 2|\text{Ric}|^2 - 4R_{ij} R_{klij} R_{kl}$$

$$\geq 2|\nabla \text{Ric}|^2 - 2|\text{Ric}|^2 - c|\text{Ric}|^3,$$

for some absolute constant $c$. Using (44),

$$\Delta_f S^{-1} = -S^{-2} \Delta_f S + 2S^{-1} |\nabla \ln S|^2$$

(42)

$$= -S^{-2}(S - 2|\text{Ric}|^2) + 2S^{-1} |\nabla \ln S|^2$$

$$= S^{-1} + 2S^{-2}|\text{Ric}|^2 + 2S^{-1} |\nabla \ln S|^2.$$

Hence by Kato’s inequality, (41) and (42),

$$\Delta_f (S^{-1}|\text{Ric}|^2) = S^{-1} \Delta_f |\text{Ric}|^2 + 2\langle \nabla S^{-1}, \nabla |\text{Ric}|^2 \rangle + |\text{Ric}|^2 \Delta_f S^{-1}$$

(43)

$$\geq 2S^{-1} |\nabla \text{Ric}|^2 - 2S^{-1} |\text{Ric}|^2 - cS^{-1} |\text{Ric}|^3$$

$$- 4S^{-1} |\nabla \ln S||\text{Ric}||\nabla \text{Ric}| + S^{-1} |\text{Ric}|^2$$

$$+ 2S^{-2} |\text{Ric}|^4 + 2S^{-1} |\nabla \ln S|^2 |\text{Ric}|^2.$$

By completing square, we see that

$$2S^{-1} |\nabla \text{Ric}|^2 - 4S^{-1} |\nabla \ln S||\text{Ric}||\nabla \text{Ric}| = 2S^{-1} \left( |\nabla \text{Ric}| - |\nabla \ln S||\text{Ric}| \right)^2 - 2S^{-1} |\nabla \ln S|^2 |\text{Ric}|^2$$

$$\geq -2S^{-1} |\nabla \ln S|^2 |\text{Ric}|^2.$$

$$\Delta_f (S^{-1}|\text{Ric}|^2) \geq 2S^{-2} |\text{Ric}|^4 - cS^{-1} |\text{Ric}|^3 - S^{-1} |\text{Ric}|^2.$$

Let $u := S^{-1}|\text{Ric}|^2$, the above differential inequality can be rewritten as

$$\Delta_f u \geq 2u^2 - cS^{-1} u^3 - u$$

(44)

$$\geq u^2 - c(1 + S) u$$

$$\geq u^2 - c_0 u,$$

where $c_0$ is a positive constant depending on the global upper bound of the scalar curvature. Let $R \geq 1$ and $\psi : [0, \infty) \to \mathbb{R}$ be a smooth real valued function satisfying the following: $0 \leq \psi \leq 1$, $\psi' \leq 0$,

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 2 \leq t \end{cases}$$

and

$$|\psi''(t)| + |\psi'(t)| \leq c$$

for all $t \geq 0$.

We consider a function $\phi(x) := \psi\left(\frac{r(x)}{R}\right)$, then

$$\phi = \begin{cases} 1 & \text{on } B_R(p_0) \\ 0 & \text{on } M \setminus B_{2R}(p_0) \end{cases}$$

and

(45)

$$|\nabla \phi| = \frac{|\psi'|}{R} \leq \frac{c}{R}.$$
Using the Laplacian comparison theorem in [48] (see Theorem 3.1), there exists a positive constant $\beta$ (independent on $R$ and $x \in M \setminus B_1(p_0)$) such that on $M \setminus B_1(p_0)$

\[(46) \quad \Delta f r(x) \leq \frac{1}{2} r(x) + \beta.\]

Hence

\[(47) \quad \Delta f \phi = \frac{\phi'}{R} \Delta f r + \frac{\phi''}{R^2} |\nabla r|^2 \geq -\frac{c}{R} (R + \beta) - \frac{c}{R^2} \geq -c - \frac{c(1 + \beta)}{R}.\]

Let $G = \phi^2 u$, we have by (44), (45) and (47)

\[(48) \quad \phi^2 \Delta f G = \phi^4 \Delta f u + 4 \phi^3 \langle \nabla \phi, \nabla (\phi^2 u \phi^{-2}) \rangle + (2 \phi \Delta f \phi + 2 |\nabla \phi|^2) G \geq G^2 - c_0 G + 4 \phi \langle \nabla \phi, \nabla G \rangle + (2 \phi \Delta f \phi - 6 |\nabla \phi|^2) G \geq G^2 - \left( c_0 + c + \frac{c(1 + \beta)}{R} \right) G + 4 \phi \langle \nabla \phi, \nabla G \rangle.\]

Suppose $G$ attains its maximum at $q$. If $q \in B_1(p_0)$, then

\[G \leq G(q) \leq \sup_{B_1(p_0)} u,\]

R.H.S. is independent on $R \geq 1$. If $q \in M \setminus B_1(p_0)$, then by the maximum principle, we have by (43),

\[0 \geq G^2(q) - \left( c_0 + c + \frac{c(1 + \beta)}{R} \right) G(q).\]

In either cases, we get the following bound for $G$:

\[G \leq \sup_{B_1(p_0)} u + c_0 + c + \frac{c(1 + \beta)}{R},\]

result then follows by letting $R \to \infty$. \hfill \square

For general gradient expander, the scalar curvature may be negative. Nonetheless, $Rm$ is still bounded if the scalar curvature is bounded.

**Theorem 11.** Let $(M^3, g, f)$ be a 3 dimensional complete noncompact gradient expanding Ricci soliton. Then there exist positive constant $C_1$ and $C_2$ such that on $M$

\[(49) \quad |Rm| \leq C_1 S + C_2.\]

In particular the curvature tensor is bounded if in addition the scalar curvature $S$ is bounded.

**Proof.** Since in dimension 3, the curvature tensor is controlled by the Ricci tensor, we shall estimate the Ricci tensor. Using the computation (41) in the proof of Theorem 7(a) and Kato’s inequality, we have

\[\Delta f |\text{Ric}| \geq -|\text{Ric}| - c|\text{Ric}|^2\]
for some absolute constant $c > 0$ wherever $|\text{Ric}| > 0$. Hence by (14) and (6)
\begin{equation}
\Delta_f(|\text{Ric}| - \phi^2) \geq (2A - c)|\text{Ric}|^2 - |\text{Ric}| + \phi^2
\end{equation}
(50)
\begin{align*}
& \geq |\text{Ric}|^2 - |\text{Ric}| + \phi^2
\end{align*}
\begin{align*}
& \geq |\text{Ric}|^2 - |\text{Ric}| - \frac{3A}{2},
\end{align*}
for all large positive constant $A$. Let $u := |\text{Ric}| - \phi^2$ and $G := \phi^2 u$. For any $R \geq 1$, we consider the cutoff function $\phi(x) = \psi\left(\frac{r(x)}{R}\right)$ as in (45), (46) and (47). Using (6) and (50), we compute directly as in (48) on the set where $G$ is positive to get
\begin{align*}
\phi^2 \Delta_f G &= \phi^4 \Delta_f u + 4\phi(\nabla \phi, \nabla G) + (2\phi \Delta_f \phi - 6|\nabla \phi|^2) G \\
& \geq \phi^4 \Delta_f u - (c + \frac{c(1 + \beta)}{R})G + 4\phi(\nabla \phi, \nabla G) \\
& \geq \phi^4 \Delta_f u - (c + \frac{c(1 + \beta)}{R})^2 |\text{Ric}| - \frac{3A}{2}(1 + \frac{c(1 + \beta)}{R}) + 4\phi(\nabla \phi, \nabla G) \\
& \geq \phi^4 |\text{Ric}|^2 - (1 + c + \frac{c(1 + \beta)}{R})^2 |\text{Ric}| - \frac{3A}{2}(1 + c + \frac{c(1 + \beta)}{R}) + 4\phi(\nabla \phi, \nabla G),
\end{align*}
(51)
where $\beta$ is the positive constant in (40). Suppose $G$ attains its maximum at $q$. If $q \in B_1(p_0)$, then
\[ G \leq G(q) \leq \sup_{B_1(p_0)} (|\text{Ric}| + A|S|), \]
R.H.S. is independent on $R \geq 1$. If $q \in M \setminus B_1(p_0)$, we may assume $G(q) > 0$. Hence $|\text{Ric}|(q) > 0$ and $u$ is smooth near $q$. By the maximum principle, (6) and (51), we have
\begin{align*}
G & \leq G(q) \\
& = \phi^2(q)|\text{Ric}|(q) - \phi^2(q)AS(q) \\
& \leq \phi^2(q)|\text{Ric}|(q) + \frac{3A}{2} \\
& \leq 1 + c + \frac{c(1 + \beta)}{R} + \sqrt{\frac{3A}{2}(1 + c + \frac{c(1 + \beta)}{R})} + \frac{3A}{2}.
\end{align*}
Results follows by letting $R \to \infty$. 

To prove Theorem 5, we need to control the change in distance along the flow of $\nabla f$. Recall that it is a result of Zhang [50] that the flow of $\nabla f$ exists for all time $t \in \mathbb{R}$.

**Lemma 5.** Let $(M^n, g, f)$ be an $n$ dimensional complete gradient expanding Ricci soliton and $\phi_s$ be the flow of $\nabla f$ with $\phi_0$ being the identity map. Then for any $x \in M$ and $s \geq 0$,
\begin{equation}
r(\phi_s(x)) \leq e^{\frac{s}{2}}(r(x) + 2\sqrt{v(p_0)}),
\end{equation}
(52)
where $v = \frac{\Delta f}{2} - f$ and $p_0$ is the base point of the distance function $r$.

**Proof.** It follows from (21) that
\begin{equation}
|\nabla f| \leq \frac{1}{2}r + \sqrt{v(p_0)}.
\end{equation}
(53)
For simplicity, we denote $r(\phi_s(x))$ by $r_s$, $\sqrt{v(p_0)}$ by $c_1$. By triangle inequality and (53),

$$
\begin{align*}
    r_s &\leq r_0 + \int_0^s |\dot{\phi}_\tau(x)|d\tau \\
    &= r_0 + \int_0^s |\nabla f(\phi_\tau(x))|d\tau \\
    &\leq r_0 + \int_0^s \frac{1}{2} r_\tau + c_1d\tau.
\end{align*}
$$

Let $w(s) := r_0 + \int_0^s \frac{1}{2} r_\tau + c_1d\tau + 2c_1$, the above inequality can be rewritten as

$$
    w' \leq \frac{w}{2}.
$$

Integrating the inequality w.r.t. $s$, we have by (54)

$$
    r_s \leq w(s) \leq e^{s/2} (r_0 + 2c_1) = e^{s/2} (r(x) + 2\sqrt{v(p_0)}).
$$

□

Theorem 7(b) is now a consequence of Theorem 11 and the following theorem. The proof is motivated by the arguments in [18] and [42].

**Theorem 12.** Let $(M^n, g, f)$ be an $n$-dimensional complete noncompact gradient expanding Ricci soliton with bounded curvature. If $\lim_{x \to \infty} S(x) = 0$, then

$$
    \lim_{x \to \infty} |\text{Ric}(x)| = 0.
$$

**Proof.** We argue by contradiction. Suppose the claim is not true. Then there exist a sequence $y_k \in M \to \infty$ as $k \to \infty$ and $\varepsilon_0 \in (0,1)$ such that for all $k$

$$
    P_k := |\text{Ric}(g)|(y_k) \geq \varepsilon_0.
$$

Since $|\text{Rm}|$ is assumed to be bounded on $M$, we see that the derivatives of $\text{Rm}$ of any orders are also bounded by the Shi’s derivative estimate (see [15]). For any nonnegative integer $l$, we may let $L_l := \sup_M |\nabla^l \text{Rm}(g)| < \infty$. Let $g(t)$ be the Ricci flow generated by the soliton, i.e. $g(t) = (1 + t)\psi_t^* g$ for all $t > -1$, where $\psi_t$ is the flow of the non-autonomous vector field $\nabla^l g$ with $\psi_0$ being the identity map. We are going to rescale $g(t)$ near $y_k$. We consider the flow $g_k(t) := P_k g(\frac{t}{P_k}) = (P_k + t)\psi_{\frac{t}{P_k}}^* g$.

Then all $t \in [-\frac{\varepsilon_0}{10}, \frac{\varepsilon_0}{10}]$ and $z \in M$

$$
    |\text{Rm}(g_k(t))|_{g_k(t)}(z) = \frac{1}{P_k} |\text{Rm}((1 + \frac{t}{P_k})\psi_{\frac{t}{P_k}}^* g)|_{(1 + \frac{t}{P_k})\psi_{\frac{t}{P_k}}^* g}(z) = \frac{1}{P_k + t} |\text{Rm}(g)|_g(\psi_{\frac{t}{P_k}}(z)) \leq \frac{10}{9\varepsilon_0} L_0.
$$
Similarly for all the derivatives of the curvature \( Rm \),

\[
|\nabla^l_{g_k(t)}Rm(g_k(t))|_{g_k(t)}(z) = (P_k + t)|\nabla^l_{g_k(t)}Rm(\psi^*_k g)|_{g_k(t)}(z)
\]

\[
= (P_k + t)|\nabla^l_{\psi^*_k g}Rm(\psi^*_k g)|_{g_k(t)}(z)
\]

\[
= \frac{1}{(P_k + t)^{\frac{L_l}{2}}} |\nabla^l_{\psi^*_k g}Rm(\psi^*_k g)|_{\psi^*_k g}(z)
\]

\[
= \frac{1}{(P_k + t)^{\frac{L_l}{2}}} |\nabla^l_gRm(g)|_{g(\psi^*_k g)}(z)
\]

\[
\leq \left( \frac{10}{9} \right)^{\frac{L_l}{2}} L_l.
\]

For all \( k \in \mathbb{N} \), let \( \iota_k : \mathbb{R}^n \to T_{y_k} M \) (with metric \( g_k(0) \)) be any linear isometry such that \( \iota_k(0) = 0 \) and \( F_k := \exp_{y_k}^{g_k(0)} \circ \iota_k \), where \( \exp_{y_k}^{g_k(0)} \) denotes the exponential map w.r.t. \( g_k(0) \) at \( y_k \). By Rauch comparison theorem, \( F_k \) is a local diffeomorphism on \( \{ x \in \mathbb{R}^n : |x| < \sqrt{\frac{9\pi^2 \varepsilon_0}{10L_0 L_l}} \} \). By Hamilton compactness theorem (see [28], [14] and the proof of Lemma 4.4 in [13]), there exist positive \( \delta \) (depending only on \( \varepsilon_0, L_0 \) and \( n \)) and subsequence \( k_j \) such that as \( j \to \infty \)

\[
(B_\delta(0), F_{k_j}^* g_{k_j}(t)) \to (B_\delta(0), h_\infty(t))
\]

in \( C^\infty \) sense on \( B_\delta(0) \times (-\frac{\varepsilon_0}{10}, \frac{\varepsilon_0}{10}) \), where \( B_\delta(0) := \{ x \in \mathbb{R}^n : |x| < \delta \} \) and \( h_\infty(t) \) is a solution to the Ricci flow. Moreover, by the local smooth convergence of the metric, we have

\[
|Ric(h_\infty(0))|_{h_\infty(0)}(0) = \lim_{j \to \infty} |Ric(g_{k_j}(0))|_{g_{k_j}(0)}(y_{k_j})
\]

\[
= \lim_{j \to \infty} \frac{1}{P_{k_j}} |Ric(g)|_{g_{k_j}}(y_{k_j})
\]

\[
= \lim_{j \to \infty} 1
\]

\[
= 1.
\]

Hence \( h_\infty(0) \) is not Ricci flat. We claim that \( h_\infty(t) \) is scalar flat for all \( t \in (-\frac{\varepsilon_0}{10}, 0] \). We first assume the claim and prove the theorem. By the evolution equation of the scalar curvature along the Ricci flow (see [15])

\[
2|Ric(h_\infty(t))|_{h_\infty(t)}^2 = \frac{\partial}{\partial t} h_\infty(t) - \Delta h_\infty(t) S_{h_\infty(t)} = 0,
\]

which is impossible. It remains to justify our claim, i.e. \( h_\infty(t) \) is scalar flat. For all \( (z, t) \in B_\delta(0) \times (-\frac{\varepsilon_0}{10}, 0] \), the scalar curvature with respect to the metric \( h_\infty(t) \) satisfies

\[
S_{h_\infty(t)}(z) = \lim_{j \to \infty} S_{F_{k_j}^* g_{k_j}(t)}(z)
\]

\[
= \lim_{j \to \infty} \frac{1}{(P_{k_j} + t)} S_{\psi^*_k g}(F_{k_j}(z))
\]

\[
= \lim_{j \to \infty} \frac{1}{(P_{k_j} + t)} S_{g(\psi^*_k \circ F_{k_j}(z))}.
\]
By the assumption \( \lim_{x \to \infty} S(x) = 0 \), we are done if \( \lim_{i \to \infty} \psi_{x_{kj}}(F_{kj}(z)) = \infty \). Since \( r_k := d(y_k, p_0) \to \infty \) as \( k \to \infty \), there is a \( N_0 \in \mathbb{N} \) such that for all \( j \geq N_0 \),

\[
(58) \quad r_{kj} > \sqrt{\frac{10}{9} \left( \sqrt{r_k} + 2 \sqrt{v(p_0)} \right) + \frac{\delta}{\sqrt{\varepsilon}}.}
\]

We are going to show that for all \( j \geq N_0, z \in B_3(0) \) and \( t \in (-\frac{1}{10}, 0] \)

\[
(59) \quad d(\psi_{x_{kj}}(F_{kj}(z)), p_0) \geq \sqrt{r_{kj}}.
\]

Assume by contradiction that it is not true. Then \( d(\psi_{x_{kj}}(F_{kj}(z)), p_0) < \sqrt{r_{kj}} \) for some \( j \geq N_0, z \in B_3(0) \) and \( t \in (-\frac{1}{10}, 0] \). We consider the flow \( \phi_{\epsilon} \) of \( \nabla f \) with \( \phi_0 \) being the identity map, it is related to \( \psi_t \) in the following way

\[
\psi_t = \phi_{\ln(1 + \epsilon)} \text{ for all } t > -1.
\]

By Lemma 5 and \( \phi_{-\ln(1 + \epsilon)} \circ \psi_{x_{kj}}(F_{kj}(z)) \),

\[
d(F_{kj}(z), p_0) \leq \frac{1}{(1 + \frac{1}{\sqrt{r_{kj}}} \epsilon)^2} \left[ d(\psi_{x_{kj}}(F_{kj}(z)), p_0) + 2 \sqrt{v(p_0)} \right]
\]

\[
(60) \leq \sqrt{\frac{10}{9} \left( \sqrt{r_k} + 2 \sqrt{v(p_0)} \right)},
\]

we used \( \frac{1}{\sqrt{r_k}} \geq -\frac{1}{10} \) in the last inequality. By the definition of \( F_k := \exp_{y_k} \circ t_k \),

\[
d_{y_k}(0)(F_{kj}(z), y_k) = \sqrt{F_{kj}} d_g(F_{kj}(z), y_k) \leq \delta.
\]

Furthermore by triangle inequality and (60),

\[
r_{kj} - \delta \sqrt{F_{kj}} \leq d_g(F_{kj}(z), p_0) \leq \sqrt{\frac{10}{9} \left( \sqrt{r_k} + 2 \sqrt{v(p_0)} \right)},
\]

which is absurd by (55) and (58). We proved that (59) holds. \( S_{h_{\infty}}(t) = 0 \) now follows from (55), (57) and (59).

\[\square\]

**Remark 12.** By choosing \( P_k = (|\nabla^l \text{Ric}|(y_k))^{\frac{2}{2l+2}} \) in (55), where \( l \) is any positive integer, we can use a similar argument to show that \( \lim_{x \to \infty} |\nabla^l \text{Ric}| = 0 \) under the same assumptions of Theorem 12.

**Proof of Theorem 7(b):** By Theorem 11, \( M \) has bounded curvature. We then apply Theorem 12 to conclude that \( \text{Ric} \to 0 \) as \( x \to \infty \). Since the Weyl tensor is zero in dimension 3, we have \( \lim_{x \to \infty} |\text{Rm}| = 0 \).

\[\square\]

Corollary 3 is a consequence of the following proposition and Theorem 7(b). The proposition is known and we give a proof for the sake of completeness.

**Proposition 1.** Let \((M^n, g, f)\) be an \( n \) dimensional complete noncompact gradient expanding Ricci soliton. If \( \liminf_{x \to \infty} \text{Ric} \geq 0 \), then

\[
\lim_{x \to \infty} \frac{4v(x)}{r^2(x)} = 1,
\]
where \( v = \frac{r}{2} - f \).

**Proof.** It can be seen from (21) that \( \lim_{x \to \infty} \frac{v(x)}{r^2(x)} \leq \frac{1}{4} \). Since \( \liminf_{x \to \infty} \text{Ric} \geq 0 \), for any small positive number \( \varepsilon \), \( \exists R_0 \gg 1 \) such that on \( M \setminus B_{R_0}(p_0) \)
\[
\nabla^2 v = - \nabla^2 f = \text{Ric} + \frac{1}{2}g \\
\geq \left( \frac{1}{2} - \varepsilon \right) g.
\]

For any \( x \) in \( M \setminus B_{R_0}(p_0) \), consider a normalized minimizing geodesic \( \gamma \) joining \( p_0 \) to \( x \). We integrate the above inequality along the geodesic to get
\[
\langle \nabla v, \dot{\gamma}(t) \rangle - \langle \nabla v, \dot{\gamma}(t) \rangle_{\gamma(R_0)} \geq \left( \frac{1}{2} - \varepsilon \right)(t - R_0),
\]
where \( t \geq R_0 \). By integrating the inequality w.r.t. \( t \), we see that
\[
v(x) - v(\gamma(R_0)) \geq \left( \frac{1}{4} - \frac{\varepsilon}{2} \right)(r(x) - R_0)^2 + \langle \nabla v, \dot{\gamma}(\gamma(R_0)) \rangle(r(x) - R_0)
\]
\[
\geq \left( \frac{1}{4} - \frac{\varepsilon}{2} \right)(r(x) - R_0)^2 - \sup_{B_{R_0}(p_0)} |\nabla v|(r(x) - R_0).
\]
Hence it is clear that \( \lim_{x \to \infty} \frac{v(x)}{r^2(x)} \geq \frac{1}{4} - \frac{\varepsilon}{2} \). We let \( \varepsilon \to 0 \) and conclude that \( \lim_{x \to \infty} \frac{4v}{r^2} = 1 \). \( \Box \)

**Lemma 6.** Let \((M^3, g, f)\) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. If \( \lim_{x \to \infty} r^2(x)S(x) = 0 \), then \( \lim_{x \to \infty} r^2(x)|\text{Rm}|(x) = 0 \).

**Proof.** By Theorem \( [4, 21] \) \( |\text{Rm}| = o(1) \). Using local Shi’s estimate (see Lemma 2.6 in \( [21] \)), there exists a positive constant \( C \) such that for all \( p \in M \) and \( R \geq 1 \)
\[
|\text{Rm}|(p) \leq C \sup_{B_{R}(p)} |\text{Rm}| \left[ 1 + \sup_{B_{R}(p)} |\text{Rm}| + \frac{\sup_{B_{R}(p) \setminus B_{\frac{R}{2}}(p)} |\nabla f|}{R} \right]^{\frac{1}{2}}.
\]
Hence \( |\text{Rm}| = o(1) \). By Corollary \( [4, 122] \) and \( [21] \),
\[
C^{-1}r \leq |\nabla f| \leq Cr
\]
near infinity for some constant \( C > 0 \). Let \( \nu := \frac{\nabla f}{|\nabla f|} \). In dimension three, the Weyl tensor vanishes and (see also \( [21] \))
\[
R_{ij} = R_{ij \nu \nu} + \frac{S}{2}(g_{ij} - g_{\nu \nu}g_{j \nu}) - R_{\nu \nu ij}g_{\nu \nu} + R_{j \nu \nu \nu}g_{i \nu} + R_{i \nu \nu \nu}g_{j \nu}.
\]
To proceed, we need the following identity for gradient Ricci solitons which follows from Ricci identity and \( [4] \):
\[
R_{ij,k} - R_{kij} = R_{kji}f_i.
\]
By \( [62] \), we see that
\[
R_{ij \nu \nu} = \frac{R_{ij \nu} - R_{kji}f_i}{|\nabla f|} = o(r^{-1}).
\]
By (63). It is now clear that $|\text{Rm}| \leq c|\text{Ric}| = o(r^{-1})$. By (61) and (65) again, we have $|\nabla \text{Rm}| = o(r^{-1})$ and $R_{ijk\nu} = o(r^{-2})$. $|\text{Rm}| = o(r^{-2})$ then follows from (63).

Before we move on, let us recall some basic definitions:

**Definition 1.** (45), (46) An $n$ dimensional complete Riemannian manifold $M$ is asymptotically flat if there exist $R_0 > 0$, compact set $K \subseteq M$ and diffeomorphism $\psi : M \setminus K \to \mathbb{R}^n \setminus \{|x| \leq R_0\}$ such that in this coordinate

$$|x|^p |g_{ij} - \delta_{ij}| + |x|^{1+p} |\partial g_{ij}| + |x|^{2+p} |\partial^2 g_{ij}| \leq C$$

for some $p > \frac{n-2}{2}$ and $C > 0$, where $\partial$ denotes the partial derivative. We also require the scalar curvature $S = O(|x|^{-q})$ for some $q > n$.

Let $X$ be any smooth $n-1$ dimensional closed manifold with Riemannian metric $g_X$, $C(X)$ is defined to be the cone over $X$, i.e. $\{(t, \omega) : t > 0, \omega \in X\}$. $g_C$ and $\nabla_C$ denote the metric $dt^2 + t^2 g_X$ on $C(X)$ and its Riemannian connection respectively. $\overline{B(o,R)} \subseteq C(X)$ is the set given by $\{(t, \omega) : R \geq t > 0, \omega \in X\}$. With the above preparations, we are going to prove Theorem 9.

**Theorem.** Let $(M,g,f)$ be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Suppose that

$$\lim_{x \to \infty} r^2(x)S(x) = 0.$$

Then $M$ is isometric to $\mathbb{R}^3$.

**Proof.** By Theorem 1.3 in [21] and Lemma 6 the curvature tensor $\text{Rm}$ satisfies

$$|\nabla^k \text{Rm}|_g = O(t^{k-1} e^{-\frac{t^2}{2}})$$

as $r \to \infty$, for any nonnegative integer $k$.

Furthermore, $M$ is smoothly asymptotic to a three dimensional cone at exponential rate [21], i.e. there exist $R > 0$, compact set $K$ in $M$, smooth closed surface $X$ and diffeomorphism $\phi : M \setminus K \to C(X) \setminus \overline{B(o,R)}$ such that for any nonnegative integer $k$

$$|\nabla^k [(\phi^{-1})^* g - g_C]|_{g_C}(t,\omega) = O(t^{k-3} e^{-\frac{t^2}{2}})$$

as $t \to \infty$;

$$\frac{n}{2} - v \circ \phi^{-1}(t,\omega) = f \circ \phi^{-1}(t,\omega) = -\frac{t^2}{4} + c_0,$$

where $c_0$ is some constant. Hence the scalar curvature $S$ is integrable and nonnegative by Theorem 5. Moreover, $(C(X), g_C)$ is Ricci flat [21] which implies that $X$ has constant Gauss curvature equal to 1. By (65) or the proof of Theorem 3.2 in [21], $X$ is diffeomorphic to the level sets $\{-f = s\}$ for all large $s$ (see also [12]). Since $S \geq 0$, using the results of Chen-Deruelle [12] and Munteanu-Wang [37], we see that $M$ is connected at infinity. Hence $X$ is connected.

**Case 1** $M$ is orientable.

The level sets of $f$ at infinity are orientable closed surfaces in $M$. Hence $X$ is also orientable and isometric to $S^2(1)$. As a result, $C(X)$ is isometric to $\mathbb{R}^3 \setminus \{0\}$. By (66) and (67), $(M,g)$ is asymptotically flat and $\partial g_{ij}$ decay exponentially in $t = |x|$ with $x \in \mathbb{R}^3$, in the standard coordinate of $\mathbb{R}^3 \setminus \{0\}$, where $\partial$ denotes the partial derivative. The A.D.M. mass $m$ vanishes since

$$m := \frac{1}{32\pi} \lim_{t \to \infty} \int_{|x|=t} \sum_{i,j=1}^3 \frac{\partial g_{ij}}{\partial x_i} \frac{\partial g_{ij}}{\partial x_j} x_i x_j t^2 \sigma_t = 0,$$
where $d\sigma_t$ is the volume element of the Euclidean sphere $\{x| = t\}$ w.r.t. the Euclidean metric. By the rigidity case of the positive mass theorem, $M$ is isometric to $\mathbb{R}^3$ (see [15], [29], [30], [46] and ref. therein).

**Case 2** $M$ is non-orientable.

We prove that it is impossible. Suppose on the contrary that $M$ is non-orientable. Then we consider $\pi : N \to M$, the orientable double cover of $M$. With $\pi^*g$ and $\pi^*f$, $N$ is endowed with the structure of a complete expanding Ricci soliton. Moreover, $S_N \geq 0$ and $\lim_{y \to \infty} r_N^2(y)S_N(y) = 0$. By Case 1, $N$ and thus $M$ are flat. $\nabla^2 g f = -\frac{1}{2} g$ on $M$ and $M$ is diffeomorphic to $\mathbb{R}^3$, contradicting to the non-orientability of $M$. □

7. Proof of Theorem 6

We start with an estimate on the potential function $f$ under some growth conditions on $f$ and $S$.

**Lemma 7.** Let $(M^n, g, f)$ be an $n$ dimensional complete noncompact gradient expanding Ricci soliton. If $f$ is proper, i.e. $\lim_{x \to \infty} f = -\infty$, and

$$\alpha := \limsup_{x \to \infty} \frac{S}{v} < 1,$$

where $v = \frac{n}{2} - f$, then for any $\delta \in (0, 1 - \alpha)$ there exists a positive constant $C$ such that

$$\frac{\delta r^2}{4} - Cr - C \leq -f \leq \frac{r^2}{4} + Cr + C \text{ on } M.$$

**Remark 13.** It can be seen from (6) and (20) that when $f$ is proper, $\alpha$ is a real number in $[0, 1]$.

**Proof.** The upper bound of (71) follows from (21) without any conditions on $f$ and $S$. For the lower bound, fix any $\delta \in (0, 1 - \alpha)$. We consider the flow $G_t$ of the vector field $\nabla v$ with $G_0$ be the identity map, where $v = \frac{n}{2} - f$. The flow makes sense since from (20) and (70),

$$|\nabla v|^2 = v - \frac{n}{2} - S$$

$$= v \left(1 - \frac{n}{2v} - \frac{S}{v}\right)$$

$$\geq \delta v$$

$$> 1,$$

near infinity. Let $\rho$ be a large constant such that the above inequality holds on $\{v \geq \rho\}$ and $\{v = \rho\}$ is nonempty. It is not difficult to see that for all $q \in \{v \geq \rho\}$, there are $t \geq 0$ and $z \in \{v = \rho\}$ such that $G_t(z) = q$ and

$$v(q) - v(z) = \int_0^t \left\langle \nabla v, \frac{\nabla v}{|\nabla v|^2}\right\rangle (G_{\tau}(z))d\tau$$

$$= t.$$
Moreover, we have
\[
d(q, z) = d(G_t(z), z) \\
\leq \int_0^t \frac{1}{|\nabla v|(G_\tau(z))} d\tau \\
= \int_0^t \frac{1}{\sqrt{v(G_\tau(z)) - \frac{n}{2} - S}} d\tau \\
\leq \int_0^t \frac{1}{\sqrt{\delta(\tau + \rho)}} d\tau \\
\leq \frac{2}{\sqrt{\delta}} \sqrt{v(q)}.
\]
We used (72) in the second last inequality. By triangular inequality,
\[
r(q) \leq d(q, z) + \sup_{\{v=\rho\}} d(\cdot, p_0) \\
\leq \frac{2}{\sqrt{\delta}} \sqrt{v(q)} + K_0,
\]
where \(K_0 = \sup_{\{v=\rho\}} d(\cdot, p_0)\). Hence near infinity,
\[
v(q) \geq \delta r^2(q) \frac{4}{4} - \frac{\delta K_0^2(q)}{2} + \frac{\delta K_0^2}{4}.
\]

Using the above lemmas, we prove a lower bound for the scalar curvature.

**Proposition 2.** Let \((M^n, g, f)\) be a complete noncompact expanding gradient Ricci soliton with proper potential function \(f\) and dimension \(n \geq 2\). If
\[
\lim_{x \to \infty} vS \geq 0,
\]
then there exists a positive constant \(C\) such that
\[
0 \leq S + Cv^{\frac{n}{2}}e^{-v} \text{ on } M,
\]
where \(v = \frac{n}{2} - f\).

**Proof.** Let \(u = -S\). By (74),
\[
\lim_{x \to \infty} \sup_{x} v u \leq 0.
\]

From (14), we see that
\[
\Delta f u \geq -u.
\]
(75) then follows from (76), (77) and the proof of Lemma 2.9 in [21]. We include the details for the convenience of readers. The calculations were essentially done by Deruelle [21]. By the properness of \(f\), \(\lim_{x \to \infty} v = \infty\) and \(v \geq 1\) outside some compact subset of \(M\). We claim that for any constant \(a \geq 4\), there exist positive constants \(R_0\) and \(\delta\) such that
\[
\Delta f + 2\ln v - 2uv - \frac{n}{2}e^{-v} \geq 0 \text{ on } \{u \geq 0\} \setminus B_{R_0}(p_0) \text{ and}
\]
\[
\Delta f + 2\ln v - 2uv - \frac{n}{2}e^{-v} < 0 \text{ on } \{u \geq 0\} \setminus B_{R_0}(p_0).
\]
We first assume the above inequalities and prove (75). There exists a positive constant \( \beta \) such that

\[
e^{-\frac{2}{v}uv} - \beta e^{-\frac{4}{v^2}v^2 - \frac{2}{v}e^{-v}} < 0 \text{ on } \partial B_{R_0}(p_0).
\]

Let \( Q := e^{-\frac{2}{v}uv} - \beta e^{-\frac{4}{v^2}v^2 - \frac{2}{v}e^{-v}} \). From \( \lim_{x \to \infty} v = \infty \) and (76), it is evident that \( \limsup_{x \to \infty} Q \leq 0 \). For any \( y \in M \setminus \overline{B_{R_0}(p_0)} \) and \( \varepsilon > 0 \), there exists \( T > R_0 \) such that \( y \in B_T(p_0) \) and

\[
Q < \varepsilon \text{ on } \partial B_T(p_0).
\]

We consider the open set \( \Omega := B_T(p_0) \setminus \overline{B_{R_0}(p_0)} \). Suppose \( Q \) attains its maximum over \( \overline{\Omega} \) at \( z \in \Omega \). If \( z \in \partial \Omega \), then by (80) and (81), \( Q(y) \leq Q(z) \leq \varepsilon \). If \( z \in \Omega \), we show that \( Q(z) \leq 0 \). Suppose on the contrary that \( Q(z) > 0 \). Then we know that \( u(z) \geq 0 \) and by (78) and (79), we have

\[
0 \geq \Delta f + 2\ln v - 2a_v \Omega.
\]

We then obtain (75) by letting \( \varepsilon \to 0 \). It remains to prove (78) and (79). For (78), by (19), we have

\[
\Delta f (uv) = v\Delta f + u\Delta f + 2\langle \nabla u, \nabla v \rangle \\
\geq -uv + uv + 2\langle \nabla \ln v, \nabla (uv) \rangle - 2|\nabla \ln v|^2 uv;
\]

(82)

\[
\Delta f + 2\ln v (uv) \geq -2|\nabla \ln v|^2 uv.
\]

For any \( a > 0 \), we compute directly using (19) as in Lemma 2.9 of [21] to get

\[
\Delta f + 2\ln e^{-\frac{2}{v}uv} = a \frac{v}{v^3} e^{-\frac{2}{v}uv} + ( -\frac{4a}{v^3} + \frac{a^2}{v^4} ) |\nabla v|^2 e^{-\frac{2}{v}uv}.
\]

A straightforward calculation using (82) and (83) yields

\[
\Delta f + 2\ln e^{-\frac{2}{v}uv} = e^{-\frac{2}{v}uv} - \beta e^{-\frac{4}{v^2}v^2 - \frac{2}{v}e^{-v}} \\
\geq -2|\nabla \ln v|^2 e^{-\frac{2}{v}uv} + \frac{a}{v} e^{-\frac{2}{v}uv} + ( -\frac{4a}{v^3} + \frac{a^2}{v^4} ) |\nabla v|^2 e^{-\frac{2}{v}uv} \\
+2\langle e^{-\frac{2}{v}uv}, \nabla (uv) \rangle \\
= \left( \frac{a}{v} - 2|\nabla \ln v|^2 \right) e^{-\frac{2}{v}uv} + ( -\frac{4a}{v^3} + \frac{a^2}{v^4} ) |\nabla v|^2 e^{-\frac{2}{v}uv} \\
-2\langle \nabla v^{-1}, \nabla (e^{-\frac{2}{v}uv}) \rangle - \frac{2a^2}{v^4} |\nabla v|^2 e^{-\frac{2}{v}uv} \\
= \left( \frac{a}{v} - 2|\nabla \ln v|^2 \right) e^{-\frac{2}{v}uv} + ( -\frac{4a}{v^3} + \frac{a^2}{v^4} ) |\nabla v|^2 e^{-\frac{2}{v}uv} \\
-2\langle \nabla v^{-1}, \nabla (e^{-\frac{2}{v}uv}) \rangle.
\]

Hence we have

\[
\Delta f + 2\ln e^{-\frac{2}{v}uv} \geq \left( \frac{a}{v} - 2|\nabla \ln v|^2 \right) e^{-\frac{2}{v}uv} + ( -\frac{4a}{v^3} + \frac{a^2}{v^4} ) |\nabla v|^2 e^{-\frac{2}{v}uv}.
\]
On the set where \( u \) is nonnegative and for \( a \geq 4 \), we may simplify the above inequality by \( |\nabla v|^2 \leq v \)

\[
\Delta f + 2\ln v - 2a v^{-1}(e^{-\frac{n}{2}uv}) \geq \frac{a - 2}{v} e^{-\frac{n}{2}uv} - \left(\frac{4a}{v^2} + \frac{a^2}{v^3}\right) e^{-\frac{n}{2}uv} \\
\geq \frac{1}{v} e^{-\frac{n}{2}uv} \\
\geq 0
\]

near infinity. We proved the inequality (78). For (79), by \(|\nabla v|^2 \leq v\), (20), (6) and previous computation (26),

\[
\Delta f + 2\ln v(v^2 - \frac{2}{n}e^{-v}) = e^v \left[-S + \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} + 1\right)|\nabla \ln v|^2 \right. \\
- (2S + n)\left(\frac{n}{2} - 1\right)v^{-1} \bigg].
\]

\[
\leq e^v \left[-S + \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} + 1\right)v^{-1}\right].
\]

\[
2a\langle \nabla v^{-1}, \nabla (v^2 - \frac{2}{n}e^{-v}) \rangle = a(n - 4)\left|\frac{\nabla \ln v}{v}\right|^2 v^2 - \frac{2}{n}e^{-v} + 2a|\nabla \ln v|^2 v^2 - \frac{2}{n}e^{-v}
\]

\[
\leq v^2 - \frac{2}{n}e^{-v}\left[\frac{a(n - 4)}{v^2} + \frac{2a}{v}\right].
\]

Hence

\[
\Delta f + 2\ln v - 2a v^{-1}(v^2 - \frac{2}{n}e^{-v}) \leq v^2 - \frac{2}{n}e^{-v}\left[-S + \frac{C_0}{v}\right],
\]

where \( C_0 = C_0(a,n) \) is a positive constant depending on \( a \) and \( n \). Using (74), we have \( vS \geq -1 \) near infinity. We can rewrite the above inequality as

\[
(84) \quad \Delta f + 2\ln v - 2a v^{-1}(v^2 - \frac{2}{n}e^{-v}) \leq C_1 v^1 - \frac{2}{n}e^{-v},
\]

where \( C_1 = C_1(a,n) \) is a positive constant depending on \( a \) and \( n \). For any \( b > 0 \),

\[
(85) \quad 2a\langle \nabla v^{-1}, \nabla e^{-\frac{b}{v}} \rangle = -2ab|\nabla \ln v|^2 v^2 e^{-\frac{b}{v}}.
\]

By (83) and \(|\nabla v|^2 \leq v\),

\[
\Delta f + 2\ln v - 2a v^{-1}e^{-\frac{b}{v}} = e^{-\frac{b}{v}} \left[\frac{b}{v} + \left(\frac{4b}{v^3} + \frac{b^2}{v^5}\right)|\nabla v|^2 - 2ab|\nabla \ln v|^2 v^{-2}\right].
\]

\[
\leq e^{-\frac{b}{v}} \left[\frac{b}{v} + \frac{b^2}{v^3}|\nabla v|^2\right]
\]

\[
\leq e^{-\frac{b}{v}} \left[\frac{b}{v} + \frac{b^2}{v^3}\right].
\]

It follows from (20) that

\[
2\langle \nabla (v^2 - \frac{2}{n}e^{-v}), \nabla e^{-\frac{b}{v}} \rangle = 2bv^2 - \frac{2}{n}e^{-\frac{b}{v}}|\nabla v|^2 (2 - \frac{n}{2})v^1 - \frac{2}{n}e^{-v}
\]

\[
+ 2bv^2 - \frac{2}{n}e^{-\frac{b}{v}}|\nabla v|^2 (-e^{-v})v^2 - \frac{2}{n}e^{-v}
\]

\[
= 2bv^2 - \frac{2}{n}e^{-\frac{b}{v}}|\nabla v|^2 (2 - \frac{n}{2})v^1 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}
\]

\[
- \frac{2b}{v} e^{-\frac{b}{v}}v^2 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}v^2 - \frac{2}{n}e^{-v}v^2.
\]
Using (84), (86) and (87), on the set where $u$ is nonnegative (i.e. $S \leq 0$),
\[
\Delta f + 2\ln v - 2av^2 - e^{-v} = e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}}
\]
\[
+ v^2 - \frac{b}{v}v^2 e^{-v} \Delta f + 2\ln v - 2av^2 - e^{-v}
\]
\[
+ 2(\nabla^2 v - \frac{b}{v}v^2 e^{-v}), \nabla e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}}
\]
\[
\leq \frac{C_1}{v}e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}} + \left[ \frac{b}{v} + \frac{b^2}{v^3} \right] e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}}
\]
\[
- \frac{2b}{v}e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}} + \frac{b(2S + n + |n - 4|)}{v^2} e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}}
\]
\[
\leq e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}} \left[ \frac{C_1 - b}{v} + \frac{C_2}{v^2} \right],
\]
where $C_2 = C_2(b, n)$ is positive and depends on $b$ and $n$. We choose $b = C_1 + 2$, then
\[
\Delta f + 2\ln v - 2av^2 - e^{-v} \leq e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}} \leq -\frac{1}{v} e^{-\frac{b}{v}v^2 - \frac{n}{2}e^{-v}} < 0
\]
near infinity. We showed (79) and completed the proof of the proposition. □

With the above preparation, we give a proof for Theorem 6.

**Proof of Theorem 6:** From (21), we see that (74) holds. By Proposition 2, the negative part of the scalar curvature $S_-$ satisfies
\[
0 \leq S_- \leq Cv^{1-\frac{n}{2}}e^{-v}
\]
on $M$, for some constant $C > 0$. Using Lemma 7 we have $v \sim r^2$. Hence there exists $\delta > 0$ such that
\[
(88) \quad S_- \leq Ce^{-\delta r^2}.
\]
By (21) and Theorem 1.1 in [38], there exists a positive constant $C$ such that for all $R > 0$
\[
Vol_g(B_R(p_0)) \leq Ce^{\sqrt{n-1}R},
\]
where $Vol_g(B_R(p_0))$ is the volume of the geodesic ball $B_R(p_0)$. It implies that
\[
\int_M S_- < \infty.
\]
The nonnegativity of the scalar curvature is a consequence of \(\liminf_{x \to \infty} r^2 S \geq 0\) and Theorem 5. Connectedness at infinity of $M$ now follows from [12] and [37]. □

8. **Proof of Theorem 8**

It was observed by Munteanu-Wang [39] that in four dimensional gradient Ricci soliton, the Riemann curvature $Rm$ can be bounded by $\nabla Ric$ and $\nabla Ric$:

**Lemma 8.** [39] Let $(M^4, g, f)$ be a four dimensional gradient Ricci soliton. There exists a universal positive constant $A_0$ such that if $\nabla f \not= 0$ at $q \in M$, then at $q$
\[
(89) \quad |Rm| \leq A_0\left( |Ric| + \frac{\nabla Ric}{|\nabla f|} \right).
\]
Please see [39] and [8] for a proof of Lemma [8]. To prove that the curvature tensor is bounded, we first give estimate on the Ricci tensor.

**Theorem 13.** Let \((M^4, g, f)\) be a 4 dimensional complete noncompact expanding gradient Ricci soliton. Suppose that it has bounded scalar curvature and \(f \to -\infty\) as \(x \to \infty\). Then the Ricci tensor \(\text{Ric}\) is bounded.

**Proof.** Let \(L := \sup_M |S| < \infty\). If \(L = 0\), then by [14] \(M\) is Ricci flat and we are done with the proof. We may assume \(L > 0\) and introduce the function \(F(S) := (S + 3L)^{-a} > 0\), where \(a > 0\). We see that

\[
\frac{1}{4a L^a} \leq F \leq \frac{1}{2a L^a},
\]

and

\[
\nabla F = -a(S + 3L)^{-a-1}\nabla S.
\]

By [13]

\[
\Delta_f F = -a(S + 3L)^{-a-1} \Delta_f S + a(a + 1)(S + 3L)^{-a-2}|\nabla S|^2
\]

(92)

\[
= a(S + 3L)^{-a-1}(S + 2|\text{Ric}|^2) + a(a + 1)(S + 3L)^{-a-2}|\nabla S|^2.
\]

Whenever \(\nabla f \neq 0\), we compute directly using (90) and get

\[
\Delta_f |\text{Ric}|^2 = 2|\nabla \text{Ric}|^2 + 2(\text{Ric}, \Delta_f \text{Ric}) \geq 2|\nabla \text{Ric}|^2 - 2|\text{Ric}|^2 - 4|\text{Ric}|^2|\text{Rm}|\]

(93)

\[
\geq 2|\nabla \text{Ric}|^2 - 2|\text{Ric}|^2 - 4A_0|\text{Ric}|^3 - 4A_0|\text{Ric}|^2 \frac{|\nabla \text{Ric}|}{|\nabla f|}
\]

\[
\geq |\nabla \text{Ric}|^2 - 2|\text{Ric}|^2 - 4A_0|\text{Ric}|^3 - 4A_0^2|\text{Ric}|^4
\]

It is not difficult to see from Cauchy Schwarz inequality, (92), (93) and (90) that

\[
\Delta_f (F|\text{Ric}|^2) = F\Delta_f |\text{Ric}|^2 + 2(\nabla F, \nabla |\text{Ric}|^2) + |\text{Ric}|^2 \Delta_f F \geq F\Delta_f |\text{Ric}|^2 - 4F^2|\nabla S||\text{Ric}||\nabla \text{Ric}| + |\text{Ric}|^2 \Delta_f F \geq F\Delta_f |\text{Ric}|^2 - F|\nabla \text{Ric}|^2 - 4 \left(\frac{F}{F} \right) |\nabla S|^2 |\text{Ric}|^2 + |\text{Ric}|^2 \Delta_f F \geq F\Delta_f |\text{Ric}|^2 - F|\nabla \text{Ric}|^2 - 4a(S + 3L)^{-a-1}(S + 2|\text{Ric}|^2)|\text{Ric}|^2 + a(a + 1)(S + 3L)^{-a-2}|\nabla S|^2 |\text{Ric}|^2 \geq - \left(2 + \frac{a}{2}\right)F|\text{Ric}|^2 - 4A_0 F|\text{Ric}|^3 - \frac{4A_0^2 F|\text{Ric}|^4}{|\nabla f|^2} + 2a(S + 3L)^{a-1} F^2 |\text{Ric}|^4 + a(1 - 3a)(S + 3L)^{-a-2} |\nabla S|^2 |\text{Ric}|^2.
\]
We may take $a = \frac{1}{3}$ and consider the function $w := F|\text{Ric}|^2$, then the above inequalities can be rewritten as

$$\Delta f w \geq \left(2 \frac{F^2}{3} - 4A_0^2 \frac{F^2}{F|\nabla f|^2}\right)w^2 - 4A_0 F^{-\frac{1}{2}} w^\frac{3}{2} - \frac{13}{6} w.$$

By (12), $|\nabla f|^2 = -f - S \geq -f - L \to \infty$. Using (90), we have outside a compact set,

$$\Delta f w \geq \left(\frac{1}{2^\frac{1}{3} 3L^\frac{2}{3}} - 4A_0^2 \frac{L^\frac{2}{3}}{|\nabla f|^2}\right)w^2 - 4A_0 F^{-\frac{1}{2}} w^\frac{3}{2} - \frac{13}{6} w \geq \frac{1}{6L^\frac{2}{3}} w^2 - 4A_0 F^{-\frac{1}{2}} w^\frac{3}{2} - \frac{13}{6} w.$$

It is easy to see that the differential inequality of $w$ resembles the one of $u$ in (44). Hence a similar cut off function argument as in the proof of Theorem 7 (a) and the Laplacian comparison theorem (46) (see [48]) yields the boundedness of $w$. □

Theorem 8 is now a consequence of the following theorem. The proof of the theorem is essentially due to Munteanu-Wang [39] and Cao-Cui [5]. We shall only give a sketch of the proof.

**Theorem 14.** Let $(M^4, g, f)$ be a 4 dimensional complete noncompact expanding gradient Ricci soliton. Suppose that it has bounded Ricci curvature and $f \to -\infty$ as $x \to \infty$. Then the curvature tensor $R_m$ is bounded

**Proof.** Throughout this proof, we use $c_0$ to denote some constants depending on the global upper bound of $|\text{Ric}|$ and $A_0$ in Lemma 8; its value may be different from line by line. Since $|\nabla f|^2 = -f - S \to \infty$ as $x \to \infty$, we have by Lemma 8

$$|\nabla \text{Ric}|^2 \geq \frac{1}{2A_0^2} |\text{Rm}|^2 - |\text{Ric}|^2 \geq \frac{1}{2A_0^2} |\text{Rm}|^2 - c_0$$

outside a compact subset of $M$. Moreover by (93)

$$\Delta f |\text{Ric}|^2 \geq |\nabla \text{Ric}|^2 - c_0 \geq \frac{1}{2A_0^2} |\text{Rm}|^2 - c_0 \geq \frac{1}{4A_0^2} (|\text{Rm}| + \lambda |\text{Ric}|^2)^2 - \frac{\lambda^2}{2A_0^2} |\text{Ric}|^4 - c_0$$

(94)

where $\lambda$ is any non-negative constant. We apply (17) and get

$$\Delta f |\text{Rm}|^2 \geq 2|\nabla \text{Rm}|^2 + 2\langle \text{Rm, } \Delta f \text{Rm} \rangle \geq 2|\nabla \text{Rm}|^2 - 2|\text{Rm}|^2 - c|\text{Rm}|^3,$$

where $c$ is an absolute constant. By Kato’s inequality,

$$\Delta f |\text{Rm}| \geq -|\text{Rm}| - c|\text{Rm}|^2$$

on the set where $|\text{Rm}| > 0$. Using (94), we see that for all $\lambda \geq 0$,
\[ \Delta_f (\lvert Rm \rvert + \lambda \lvert Ric \rvert^2) \geq \left( \frac{\lambda}{4A_0^2} - c \right) (\lvert Rm \rvert + \lambda \lvert Ric \rvert^2)^2 - \lvert Rm \rvert - \frac{\lambda^3 c_0}{2A_0^2} - c_0 \lambda. \]

Let \( W := \lvert Rm \rvert + \lambda \lvert Ric \rvert^2 \), we may rewrite the above inequality as

\[ \Delta_f W = \left( \frac{\lambda}{4A_0^2} - c \right) W^2 - W - \frac{\lambda^3}{2A_0^2} c_0 - c_0 \lambda \]

for all sufficiently large \( \lambda \). Similar argument as in the proofs of Theorems 4(a) and 13 gives the boundedness of \( W \).

\[ \square \]

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