Ordering of multivariate probability distributions with respect to extreme portfolio losses

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Abstract

A new notion of stochastic ordering is introduced to compare multivariate stochastic risk models with respect to extreme portfolio losses. In the framework of multivariate regular variation comparison criteria are derived in terms of ordering conditions on the spectral measures, which allows for analytical or numerical verification in practical applications. Additional comparison criteria in terms of further stochastic orderings are derived. The application examples include worst case and best case scenarios, elliptically contoured distributions, and multivariate regularly varying models with Gumbel, Archimedean, and Galambos copulas.

1 Introduction

This paper is dedicated to the comparison of multivariate probability distributions with respect to extreme portfolio losses. A new notion of stochastic ordering named \( \preceq_{\text{apl}} \) is introduced. Specially designed for the ordering of stochastic risk models with respect to extreme portfolio losses, this notion allows to compare the inherent extreme portfolio risks associated with different model parameters such as correlations, other kinds of dependence coefficients, or diffusion parameters.

In a recent paper of Mainik and Rüschendorf (2010) the notion of extreme risk index has been introduced in the framework of multivariate regular variation. This index, denoted by \( \gamma_\xi \), is a functional of the vector \( \xi \) of portfolio

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weights and of the characteristics of the multivariate regular variation of $X$ given by the tail index $\alpha$ and the spectral measure $\Psi$. It measures the sensitivity of the portfolio loss to extremal events and characterizes the probability distribution of extreme losses. In particular, it serves to determine the optimal portfolio diversification with respect to extreme losses. Within the framework of multivariate regular variation the notion of asymptotic portfolio loss ordering introduced in this paper is tightly related to model comparison in terms of the extreme risk index $\gamma_{\xi}$. Thus this paper can be seen as a supplement of the previous one, allowing to order multivariate risk models with respect to their extremal portfolio loss behaviour.

In Section 2 of the present paper we introduce the asymptotic portfolio loss order $\preceq_{apl}$ and highlight some relationships to further well-known ordering notions. It turns out that even strong dependence and convexity orders do not imply the asymptotic portfolio loss order in general. We present counterexamples, based on the the inversion of diversification effects in models with infinite loss expectations. Another example of particular interest discussed here is given by the elliptical distributions. In this model family we establish a precise criterion for the asymptotic portfolio loss order, which perfectly accords with the classical results upon other well-known order relations. Section 3 is devoted to multivariate regularly varying models. We discuss the relationship between the asymptotic portfolio loss order and the comparison of the extreme risk index and characterize $\preceq_{apl}$ in terms of a suitable ordering of the canonical spectral measures. These findings allow to establish sufficient conditions for $\preceq_{apl}$ in terms of spectral measures, which can be verified by analytical or numerical methods. In particular, we characterize the dependence structures that yield the best and the worst possible diversification effects for a multivariate regularly varying risk vector $X$ in $\mathbb{R}^d_+$ with tail index $\alpha$. For $\alpha \geq 1$ the best case is given by the asymptotic independence and the worst case is the asymptotic comonotonicity. The result for $\alpha \leq 1$ is exactly the opposite (cf. Theorem 3.7 and Corollary 3.8). Restricting $X$ to $\mathbb{R}^d_+$ means that $X$ represents only the losses, whereas the gains are modelled separately. This modelling approach is particularly suitable for applications in insurance, operational risk, and credit risk. If $X$ represents both losses and gains, these results remain valid if the extremal behaviour of the gains is weaker than that of the losses, so that there is no loss-gain compensation for extremal events. In Section 4 we discuss the interconnections between $\preceq_{apl}$ or ordered canonical spectral measures and other well-known notions of stochastic ordering. Ordering of canonical spectral measures allows to conclude $\preceq_{apl}$ from the (directionally) convex or the supermodular order. It is not obvious how to obtain this implication in a general setting. Finally, in Section 5 we present a series of examples with graphics illustrating the
numerical results upon the ordering of spectral measures. The relationship to spectral measures provides a useful numerical tool to establish $\preceq_{\text{apl}}$ in practical applications.

## 2 Asymptotic portfolio loss ordering

To compare stochastic risk models with respect to extreme portfolio losses, we introduce the asymptotic portfolio loss order $\preceq_{\text{apl}}$. This order relation is designed for the analysis of the asymptotic diversification effects and the identification of models that generate portfolio risks with stronger extremal behaviour.

Before stating the definition, some basic notation is needed. Focusing on risks, let $X$ be a *random loss vector* with values in $\mathbb{R}^d$, i.e., let positive values of the components $X^{(i)}$, $i = 1, \ldots, d$, represent losses and let negative values of $X^{(i)}$ represent gains of some risky assets. Following the intuition of diversifying a unit capital over several assets, we restrict the set of portfolios to the unit simplex in $\mathbb{R}^d$:

$$\Sigma^d := \left\{ \xi \in \mathbb{R}^d_+ : \sum_{i=1}^d \xi_i = 1 \right\}.$$  

The portfolio loss resulting from a random vector $X$ and the portfolio $\xi$ is given by the scalar product of $\xi$ and $X$. In the sequel it will be denoted by $\xi \top X$.

**Definition 2.1.** Let $X$ and $Y$ be $d$-dimensional random vectors. Then $X$ is called smaller than $Y$ in asymptotic portfolio loss order, $X \preceq_{\text{apl}} Y$, if

$$\forall \xi \in \Sigma^d \limsup_{t \to \infty} \frac{P\{\xi \top X > t\}}{P\{\xi \top Y \geq t\}} \leq 1.$$  

(1)

Here, $\frac{0}{0}$ is defined to be 1.

**Remark 2.2.** (a) Although designed for random vectors, $\preceq_{\text{apl}}$ is also defined for random variables. In this case, the portfolio set has only one element, $\Sigma^1 = \{1\}$.

(b) It is obvious that $\preceq_{\text{apl}}$ is invariant under componentwise rescaling. Let $vX$ denote the componentwise product of $v, x \in \mathbb{R}^d$:

$$vX := (v^{(i)} x^{(i)}, \ldots, v^{(d)} x^{(d)}),$$  

(2)

Then it is easy to see that $X \preceq_{\text{apl}} Y$ implies $vX \preceq_{\text{apl}} vY$ for all $v \in \mathbb{R}^d_+$. Hence condition (1) can be equivalently stated for $\xi \in \mathbb{R}^d_+$.
The ordering statement \( X \preceq_{\text{apl}} Y \) means that for all portfolios \( \xi \in \Sigma^d \) the portfolio loss \( \xi^\top X \) is asymptotically smaller \( \xi^\top Y \). Thus \( \preceq_{\text{apl}} \) concerns only the extreme portfolio losses. In consequence, this order relation is weaker than the (usual) stochastic ordering \( \preceq_{\text{st}} \) of the portfolio losses:

\[
\xi^\top X \preceq_{\text{st}} \xi^\top Y \text{ for all } \xi \in \Sigma^d \implies X \preceq_{\text{apl}} Y. \tag{3}
\]

Here, for real random variables \( U, V \) the stochastic ordering \( U \preceq_{\text{st}} V \) is defined by

\[
\forall t \in \mathbb{R} \quad P\{U > t\} \leq P\{V > t\}. \tag{4}
\]

Some related, well-known stochastic orderings (cf. Müller and Stoyan, 2002; Shaked and Shanthikumar, 1997) are collected in the following list. Remind that \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is called supermodular if

\[
\forall x, y \in \mathbb{R}^d \quad f(x \wedge y) + f(x \vee y) \geq f(x) + f(y). \tag{5}
\]

**Definition 2.3.** Let \( X, Y \) be random vectors in \( \mathbb{R}^d \). Then \( X \) is said to be smaller than \( Y \) in

(a) (increasing) convex order, \( X \preceq_{\text{cx}} Y \) (\( X \preceq_{\text{icx}} Y \)), if \( Ef(X) \leq Ef(Y) \) for all (increasing) convex functions \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) such that the expectations exist;

(b) linear convex order, \( X \preceq_{\text{lcx}} Y \), if \( \xi^\top X \preceq_{\text{cx}} \xi^\top Y \) for all \( \xi \in \mathbb{R}^d \);

(c) positive linear convex order, \( X \preceq_{\text{plcx}} Y \), if \( \xi^\top X \preceq_{\text{cx}} \xi^\top Y \) for all \( \xi \in \mathbb{R}^d_+ \);

(d) supermodular order \( X \preceq_{\text{sm}} Y \), if \( Ef(X) \leq Ef(Y) \) for all supermodular functions \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) such that the expectations exist;

(e) directionally convex order, \( X \preceq_{\text{dcx}} Y \), if \( Ef(X) \leq Ef(Y) \) for all directionally convex, i.e., supermodular and componentwise convex functions \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) such that the expectations exist.

The stochastic orderings listed in Definition 2.3 are useful for describing the risk induced by larger diffusion (convex risk) as well as the risk induced by positive dependence (supermodular and directionally convex). The following implications are known to hold generally for random vectors \( X, Y \) in \( \mathbb{R}^d \):

(a) \( (X \preceq_{\text{sm}} Y) \Rightarrow (X \preceq_{\text{dcx}} Y) \Rightarrow (X \preceq_{\text{plcx}} Y) \)

(b) \( (X \preceq_{\text{cx}} Y) \Rightarrow (X \preceq_{\text{lcx}} Y) \Rightarrow (X \preceq_{\text{plcx}} Y) \)

(c) \( (X \preceq_{\text{icx}} Y) \Rightarrow (X \preceq_{\text{plcx}} Y) \)
Remark 2.4. (a) It is easy to see that the usual stochastic order \( \preceq_{st} \) implies \( \preceq_{apl} \) in the univariate case.

(b) In spite of being strong risk comparison orders, the order relations outlined in Definition 2.3 do not imply \( \preceq_{apl} \) in general. For instance, it is known that the comonotonic dependence structure is the worst case with respect to the strong supermodular ordering \( \preceq_{sm} \), whereas it is not necessarily the worst case with respect to \( \preceq_{apl} \) (cf. Examples 5.1 and 5.2).

The following proposition helps to establish sufficient criteria for \( \preceq_{apl} \) in the univariate case. To obtain multivariate results, it can be separately applied to each portfolio loss \( \xi^\top X \) for \( \xi \in \Sigma^d \).

**Proposition 2.5.** Let \( R_1, R_2 \geq 0 \) be real random variables and let \( V \) be a real random variable independent of \( R_i, i = 1, 2 \).

(a) If \( R_1 \preceq_{apl} R_2 \) and \( V < K \) for some constant \( K \), then
\[
R_1 V \preceq_{apl} R_2 V. \tag{6}
\]

(b) If \( R_1 \preceq_{st} R_2 \), then
\[
(R_1 V)_+ \preceq_{st} (R_2 V)_+ \quad \text{and} \quad (R_2 V)_- \preceq_{st} (R_1 V)_-. \tag{7}
\]

In addition, if \( V \) and \( R_i \) are integrable and \( EV \geq 0 \), then
\[
R_1 V \preceq_{icx} R_2 V. \tag{8}
\]

Moreover, if \( EV = 0 \), then \( R_1 V \preceq_{cx} R_2 V \).

**Proof.**

Part (a). Since \( R_1 V \preceq_{apl} R_2 V \) is trivial for \( V \leq 0 \), we assume that \( P\{V > 0\} > 0 \). Hence \( V \leq K \) implies for all \( t > 0 \)
\[
P\{R_1 V > t\} = \int_{(0,K)} P\{R_1 > t/v\} dP^V(v)
= \int_{(0,K)} f(t/v) P\{R_2 > t/v\} dP^V(v), \tag{9}
\]
where
\[
f(z) := \frac{P\{R_1 > z\}}{P\{R_2 > z\}}.
\]

An obvious consequence of (9) is the inequality
\[
P\{R_1 V > t\} \leq \sup \{f(z) : z > t/K\} \cdot P\{R_2 V > t\} \tag{10}
\]
Since $R_1 \preceq_{apl} R_2$ is equivalent to $\limsup_{z \to \infty} f(z) \leq 1$, we obtain
\[
\limsup_{t \to \infty} \frac{P\{R_1V > t\}}{P\{R_2V > t\}} \leq 1.
\]

Part (b). By the well-known coupling principle for the stochastic ordering $\preceq_{st}$ we may assume without loss of generality that $R_1 \leq R_2$ pointwise on the underlying probability space. This implies
\[
P\{R_1V > t\} \leq P\{R_2V > t\}, \quad t \geq 0,
\]
and, similarly,
\[
P\{R_1V \leq t\} \leq P\{R_2V \leq t\}, \quad t \leq 0.
\]
In consequence we obtain (7).

From the proof of (7) it follows that the distribution functions of the products $R_iV$, $i = 1, 2$, satisfy the cut criterion of Karlin–Novikov (cf. Shaked and Shanthikumar, 1994, Theorem 2.A.17 and Müller and Stoyan, 2002, Theorem 1.5.17) Hence we obtain
\[
R_1V \succeq_{icx} R_2V.
\]
If $EV = 0$, then $E[R_1V] = E[R_2V]$ and therefore
\[
R_1V \succeq_{cx} R_2V.
\]

Remark 2.6. (a) Note that (7) implies (without assuming the existence of moments) that $(R_2V)_+ \preceq_{decx} (R_1V)_+$ where $\preceq_{decx}$ denotes the decreasing convex order. Similarly one obtains $(R_2V)_- \preceq_{icx} (R_1V)_-$.

(b) If $f(t) := P\{R_1 > t\}/P\{R_2 > t\} \leq C < \infty$ and $R_1 \preceq_{apl} R_2$, then $R_1V \preceq_{apl} R_2V$.

(c) A related problem is the ordering of products $RV_i$ for $R \geq 0$ with $V_1$ and $V_2$ independent of $R$. In the special case when $R$ is regularly varying with tail index $\alpha > 0$, i.e.,
\[
\lim_{t \to \infty} \frac{P\{R > tx\}}{P\{R > t\}} = x^{-\alpha}, \quad x > 0,
\]
extact criteria for $\preceq_{apl}$ can be obtained from Breiman’s Theorem (cf. Resnick, 2007, Proposition 7.5). If $E(V_i)^{\alpha+\varepsilon}_+ < \infty$ for $i = 1, 2$ and some $\varepsilon > 0$, then
\[
\lim_{t \to \infty} \frac{P\{RV_i > t\}}{P\{R > t\}} = E \left[(V_i)^{\alpha}_+\right].
\]
This yields
\[
\lim_{t \to \infty} \frac{P\{ RV_1 > t \}}{P\{ RV_2 > t \}} = \frac{E[(V_1)^\alpha]}{E[(V_2)^\alpha]}.
\]

An important class of stochastic models with various applications are elliptical distributions, which are natural generalizations of multivariate normal distributions. A random vector \( X \in \mathbb{R}^d \) is called elliptically distributed, if there exist \( \mu \in \mathbb{R}^d \) and a \( d \times d \) matrix \( A \) such that \( X \) has a representation of the form
\[
X \overset{d}{=} \mu + RAU,
\]
where \( U \) is uniformly distributed on the Euclidean unit sphere \( S^d_2 \),
\[
S^d_2 = \{ x \in \mathbb{R}^d : \|x\|_2 = 1 \},
\]
and \( R \) is a non-negative random variable independent of \( U \). By definition we have
\[
E\|X\|_2^2 < \infty \iff ER^2 < \infty,
\]
and in this case
\[
\text{Cov}(X) = \text{Var}(R)AA^\top.
\]
The matrix \( C := AA^\top \) is unique except for a constant factor and is also called the generalized covariance matrix of \( X \). We denote the elliptical distribution constructed according to (14) by \( \mathcal{E}(\mu, C, F_R) \), where \( F_R \) is the distribution of \( R \).

A classical stochastic ordering result going back to Anderson (1955) and Fefferman et al. (1972) (cf. Tong, 1980, p. 70) says that positive semidefinite ordering of the generalized covariance matrices
\[
C_1 \preceq_{\text{psd}} C_2,
\]
implies symmetric convex ordering if the location parameter \( \mu \) and the distribution \( F_R \) of the radial factor are fixed:
\[
\mathcal{E}(\mu, C_1, F_R) \preceq_{\text{symm}} \mathcal{E}(\mu, C_2, F_R).
\]
It is also known that for elliptical random vectors \( X \sim \mathcal{E}(\mu, C, F_R) \) the multivariate distribution function \( F(x) := P\{X_1 \leq x_1, \ldots, X_d \leq x_d \} \) is increasing in \( C_{i,j} \) for \( i \neq j \), where \( C = (C_{i,j}) \) (see, e.g., Joe, 1997, Theorem 2.21).

The following result is concerned with the asymptotic portfolio loss ordering \( \preceq_{\text{apl}} \) for elliptical distributions.
Theorem 2.7. Let $X \overset{d}{=} \mu_1 + R_1A_1U$, $Y \overset{d}{=} \mu_2 + R_2A_2U$ be elliptically distributed with generalized covariances $C_i := A_iA_i^\top$. If
\[ \mu_1 \leq \mu_2, \quad R_1 \preceq_{\text{apl}} R_2, \tag{19} \]
and
\[ \forall \xi \in \Sigma^d \quad \xi^\top C_1 \xi \leq \xi^\top C_2 \xi, \tag{20} \]
then
\[ X \preceq_{\text{apl}} Y. \tag{21} \]

Proof. It suffices to show that \( \xi^\top Y \preceq_{\text{apl}} \xi^\top Y \) for an arbitrary portfolio \( \xi \in \Sigma^d \). Furthermore, without loss of generality we can assume \( \mu_1 = \mu_2 = 0 \). For \( i = 1, 2 \) and \( \xi \in \Sigma^d \) denote
\[ a_i = a_i(\xi) := \left( \xi^\top C_i \xi \right)^{1/2} \]
and
\[ v_i = v_i(\xi) := \frac{\xi^\top A_i}{a_i}. \]

Then, by definition of elliptical distributions, we have
\[ \xi^\top X \overset{d}{=} R_1a_1v_1U \quad \text{and} \quad \xi^\top Y \overset{d}{=} R_2a_2v_2U. \tag{22} \]

Since the vectors \( v_i = v_i(\xi) \) have unit length by construction, the random variables \( v_iU \) are orthogonal projections of \( U \sim \text{unif}(S_d^2) \) on vectors of unit length. Symmetry arguments yield that the distribution of \( v_iU \) is independent of \( v_i \) and that \( v_iU \overset{d}{=} (1, 0, \ldots, 0)^\top U = U^{(1)} \).

Thus we have
\[ \xi^\top X \overset{d}{=} a_1R_1V \quad \text{and} \quad \xi^\top Y \overset{d}{=} a_2R_2V \]
with \( V := U^{(1)} \). By assumption we have \( a_1 \leq a_2 \) and \( R_1 \preceq_{\text{apl}} R_2 \). Applying Proposition 2.5(a) we obtain \( \xi^\top X \preceq_{\text{apl}} \xi^\top Y \).

Remark 2.8. (a) It should be noted that condition (20) is indeed weaker than (17). Let \(-1 < \rho_1 < \rho_2 < 1\) and consider covariance matrices
\[ C_i := \begin{pmatrix} 1 & \rho_i \\ \rho_i & 1 \end{pmatrix}, \quad i = 1, 2. \]

Straighforward calculations show that \( C_i \) satisfy (20), but not (17).

(b) For subexponentially distributed \( R_i \) the assumption \( \mu_1 \leq \mu_2 \) in (19) can be omitted.
3 Multivariate regular variation: $\preceq_{apl}$ in terms of spectral measures

This section is concerned with the characterization of the asymptotic portfolio loss order $\preceq_{apl}$ in the framework of multivariate regular variation. The results obtained here highlight the influence of the tail index $\alpha$ and the spectral measure $\Psi$ on $\preceq_{apl}$, with primary focus put on dependence structures captured by $\Psi$. It is shown that $\preceq_{apl}$ corresponds to a family of order relations on the set of canonical spectral measures and that these order relations are intimately related to the extreme risk index $\gamma_\xi$ introduced in Mainik and Rüschendorf (2010) and Mainik (2010).

The main result of this section is stated in Theorem 3.6, providing criteria for $X \preceq_{apl} Y$ in terms of componentwise ordering $X^{(i)} \preceq_{apl} Y^{(i)}$ for $i = 1, \ldots, d$ and ordering of canonical spectral measures. A particular consequence of these criteria is the characterization of the dependence structures that yield the best and the worst possible diversification effects for random vectors in $\mathbb{R}_+^d$ (cf. Theorem 3.7 and Corollary 3.8). Another application concerns elliptical distributions. Combining Theorem 3.6 with results on $\preceq_{apl}$ obtained in Theorem 2.7, we obtain ordering of the corresponding canonical spectral measures.

Recall the notions of regular variation. In the univariate case it can be defined separately for the lower and the upper tail of a random variable via (13). A random vector $X$ taking values in $\mathbb{R}^d$ is called multivariate regularly varying with tail index $\alpha \in (0, \infty)$ if there exist a sequence $a_n \to \infty$ and a (non-zero) Radon measure $\nu$ on the Borel $\sigma$-field $B([-\infty, \infty]^d \setminus \{0\})$ such that $\nu([-\infty, \infty]^d \setminus \mathbb{R}^d) = 0$ and, as $n \to \infty$,

$$nP_a^{-1}X \overset{\nu}{\to} \nu \text{ on } B([-\infty, \infty]^d \setminus \{0\}),$$

where $\overset{\nu}{\to}$ denotes the vague convergence of Radon measures and $P_a^{-1}X$ is the probability distribution of $a_n^{-1}X$.

It should be noted that random vectors with non-negative components yield limit measures $\nu$ that are concentrated on $[0, \infty]^d \setminus \{0\}$. Therefore multivariate regular variation in this special case can also be defined by vague convergence on $B([0, \infty]^d \setminus \{0\})$.

Many popular distribution models are multivariate regularly varying. In particular, according to Hult and Lindskog (2002), multivariate regular variation of an elliptical distribution $\mathcal{E}(\mu, C, F_R)$ is equivalent to the regular variation of the radial factor $R$ and the tail index $\alpha$ is inherited from $R$. Other popular examples are obtained by endowing regularly varying margins
with an appropriate copula Wüthrich (cf. 2003); Alink et al. (cf. 2004); Barbe et al. (cf. 2006)

For a full account of technical details related to the notion of multivariate regular variation, vague convergence, and the Borel σ-fields on the punctured spaces \([-\infty, \infty]^d \setminus \{0\}\) and \([0, \infty]^d \setminus \{0\}\) the reader is referred to Resnick (2007).

It is well known that the limit measure \(\nu\) obtained in (23) is unique except for a constant factor, has a singularity in the origin in the sense that
\[
\nu((-\varepsilon, \varepsilon)^d) = \infty \quad \text{for any } \varepsilon > 0,
\]
and exhibits the scaling property
\[
\nu(tA) = t^{-\alpha}\nu(A) \quad \text{(24)}
\]
for all sets \(A \in \mathcal{B} \left(\left[\begin{array}{c}-\infty, \infty \end{array}\right]^d \setminus \{0\}\right)\) that are bounded away from 0.

It is also well known that (23) implies that the random variable \(\|X\|\) with an arbitrary norm \(\|\cdot\|\) on \(\mathbb{R}^d\) is univariate regularly varying with tail index \(\alpha\). Moreover, the sequence \(a_n\) can always be chosen as
\[
a_n := F^{-}_{\|X\|}(1 - 1/n), \quad \text{(25)}
\]
where \(F^{-}_{\|X\|}\) is the quantile function of \(\|X\|\). The resulting limit measure \(\nu\) is normalized on the set \(A_{\|\cdot\|} := \{x \in \mathbb{R}^d : \|x\| > 1\}\) by
\[
\nu \left( A_{\|\cdot\|} \right) = 1. \quad \text{(26)}
\]

Thus, after normalizing \(\nu\) by (26), the scaling relation (24) yields an equivalent rewriting of the multivariate regular variation condition (23) in terms of weak convergence:
\[
\mathcal{L} \left\{ t^{-1}X \mid \|X\| > t \right\} \xrightarrow{w} \nu|_{A_{\|\cdot\|}} \quad \text{on } \mathcal{B} \left( A_{\|\cdot\|} \right) \quad \text{(27)}
\]
for \(t \to \infty\), where \(\nu|_{A_{\|\cdot\|}}\) is the restriction of \(\nu\) to the set \(A_{\|\cdot\|}\).

Additionally to (23) it is assumed that the limit measure \(\nu\) is non-degenerate in the following sense:
\[
\nu \left( \{x \in \mathbb{R}^d : |x^{(i)}| > 1\} \right) > 0, \quad i = 1, \ldots, d. \quad \text{(28)}
\]

This assumption ensures that all asset losses \(X^{(i)}\) are relevant for the extremes of the portfolio loss \(\xi^\top X\). If (28) is satisfied in the upper tail region, i.e., if
\[
\nu \left( \{x \in \mathbb{R}^d : x^{(i)} > 1\} \right) > 0, \quad i = 1, \ldots, d, \quad \text{(29)}
\]
then \(\nu\) also characterizes the asymptotic distribution of the componentwise maxima \(M_n := (M^{(1)}, \ldots, M^{(d)})\) with \(M^{(i)} := \max \{X^{(i)}_1, \ldots, X^{(i)}_n\}\) by the limit relation
\[
P \left\{ a_n^{-1}M_n \in [-\infty, x] \right\} \xrightarrow{w} \exp \left( -\nu \left( [-\infty, \infty]^d \setminus [-\infty, x] \right) \right) \quad \text{(30)}
\]
for \( x \in (0, \infty]^d \). Therefore \( \nu \) is called exponent measure. For more details concerning the asymptotic distributions of maxima the reader is referred to Resnick (1987) and de Haan and Ferreira (2006).

Another consequence of the scaling property (24) is the product representation of \( \nu \) in polar coordinates

\[
(r, s) := \tau(x) := (\|x\|, \|x\|^{-1} x)
\]

with respect to an arbitrary norm \( \| \cdot \| \) on \( \mathbb{R}^d \). The induced measure \( \nu^\tau := \nu \circ \tau^{-1} \) necessarily satisfies

\[
\nu^\tau = c \cdot \rho_\alpha \otimes \Psi
\]  \hspace{1cm} (31)

with the constant factor

\[
c = \nu(A_{\| \cdot \|}) > 0,
\]

the measure \( \rho_\alpha \) on \( (0, \infty] \) defined by

\[
\rho_\alpha((x, \infty]) := x^{-\alpha}, \quad x \in (0, \infty],
\]  \hspace{1cm} (32)

and a probability measure \( \Psi \) on the unit sphere \( S_{\| \cdot \|}^d \) with respect to \( \| \cdot \| \),

\[
S_{\| \cdot \|}^d := \{ s \in \mathbb{R}^d : \|s\| = 1 \}.
\]

The measure \( \Psi \) is called spectral measure of \( \nu \) or \( X \). Since the term “spectral measure” is already used in other areas, \( \Psi \) is also referred to as angular measure. In the special case of \( \mathbb{R}^d_+ \)-valued random vectors \( X \) it may be convenient to reduce the domain of \( \Psi \) to \( S_{\| \cdot \|}^d \cap \mathbb{R}^d_+ \).

Although the domain of the spectral measure \( \Psi \) depends on the norm \( \| \cdot \| \) underlying the polar coordinates, the representation (31) is norm-independent in the following sense: if (31) holds for some norm \( \| \cdot \| \), then it also holds for any other norm \( \| \cdot \|_\circ \) that is equivalent to \( \| \cdot \| \). The tail index \( \alpha \) is the same and the spectral measure \( \Psi_\circ \) on the unit sphere \( S_\circ^d \) corresponding to \( \| \cdot \|_\circ \) is obtained from \( \Psi \) by the following transformation:

\[
\Psi_\circ = \Psi^T, \quad T(s) := \|s\|^{-1}_\circ s.
\]

Finally, it should be noted that multivariate regular variation of the loss vector \( X \) is intimately related with the univariate regular variation of portfolio losses \( \xi^\top X \). As shown in Basrak et al. (2002), multivariate regular variation of \( X \) implies existence of a portfolio vector \( \xi_0 \in \mathbb{R}^d \) such that \( \xi_0^\top X \) is regularly varying with tail index \( \alpha \) and any portfolio loss \( \xi^\top X \) satisfies

\[
\lim_{t \to \infty} \frac{\mathbb{P}\{\xi^\top X > t\}}{\mathbb{P}\{\xi_0^\top X > t\}} = c(\xi, \xi_0) \in [0, \infty).
\]  \hspace{1cm} (33)
This means that all portfolio losses $\xi^\top X$ are either regularly varying with tail index $\alpha$ or asymptotically negligible compared to $\xi_0^\top X$.

Moreover, it is also worth a remark that for $\mathbb{R}^d$-valued random vectors $X$ the converse implication is true in the sense that (33) and univariate regular variation of $\xi_0^\top X$ imply multivariate regular variation of $\xi^\top X$. This sort of Cramér-Wold theorem was established in Basrak et al. (2002) and Boman and Lindskog (2009).

Under the assumption of multivariate regular variation of $X$ the extreme risk index $\gamma_\xi = \gamma_\xi(X)$ is defined as

$$\gamma_\xi(X) = \lim_{t \to \infty} \frac{\mathbb{P}\{\xi^\top X > t\}}{\mathbb{P}\{\|X\|_1 > t\}}.\quad (34)$$

In Mainik and Rüschendorf (2010) the random vector $X$ is restricted to $\mathbb{R}^d_+$ and the portfolio vector $\xi$ is restricted to $\Sigma^d$. The general case with $X$ in $\mathbb{R}^d$ and possible negative portfolio weights, i.e., short positions, is considered in Mainik (2010). Normalizing the exponent measure $\nu$ by (26), one obtains

$$\gamma_\xi(X) = \nu \left( \{ x \in \mathbb{R}^d : \xi^\top x > 1 \} \right).\quad (35)$$

Rewriting this representation in terms of the spectral measure $\Psi$ and the tail index $\alpha$ yields

$$\gamma_\xi = \int_{\mathbb{S}^d} (\xi^\top s)^\alpha_+ \, d\Psi(s).\quad (36)$$

Denoting the integrand by $f_{\xi,\alpha}$, we will write this representation as $\gamma_\xi = \Psi f_{\xi,\alpha}$.

The extreme risk index $\gamma_\xi(X)$ allows to compare the risk of different portfolios. It is easy to see that (34) implies

$$\lim_{t \to \infty} \frac{\mathbb{P}\{\xi_1^\top X > t\}}{\mathbb{P}\{\xi_2^\top X > t\}} = \frac{\gamma_{\xi_1}(X)}{\gamma_{\xi_2}(X)}.\quad (37)$$

Thus, by construction, ordering of the extreme risk index $\gamma_\xi$ is related to the asymptotic portfolio loss order $\preceq_{apl}$.

However, designed for the comparison of different portfolio risks within one model, the extreme risk index $\gamma_\xi$ cannot be directly applied to the comparison of different models. The major problem is the standardization by $\mathbb{P}\{\|X\|_1 > t\}$ in (34). Indeed, since $\mathbb{P}\{\|X\|_1 > t\}$ also depends on the spectral measure $\Psi_X$ of $X$, criteria for $\preceq_{apl}$ in terms of $\gamma_\xi$ demand the specification of the limit

$$\lim_{t \to \infty} \frac{\mathbb{P}\{\|X\|_1 > t\}}{\mathbb{P}\{\|Y\|_1 > t\}}.$$
Another technical issue arises from the invariance of \( \preceq_{\text{apl}} \) under componentwise rescalings. Since the spectral measure \( \Psi \) does not exhibit this property, ordering of spectral measures needs additional normalization of margins that makes it consistent with \( \preceq_{\text{apl}} \). To solve these problems, we use an alternative representation of \( \gamma_\xi \) in terms of the so-called canonical spectral measure \( \Psi^* \), which has standardized marginal weights.

This representation is closely related to the asymptotic risk aggregation coefficient discussed by Barbe et al. (2006). Furthermore, the link between the canonical spectral measure and extreme value copulas allows to transfer ordering results for copulas into the \( \preceq_{\text{apl}} \) setting. These results are presented in Section 4.

To reduce the problem to the essentials, we start with the observation that \( \preceq_{\text{apl}} \) is trivial for multivariate regularly varying random vectors with different tail indices and non-degenerate portfolio losses.

**Proposition 3.1.** Let \( X \) and \( Y \) be multivariate regularly varying on \( \mathbb{R}^d \) and assume that \( \gamma_\xi(Y) > 0 \) for all \( \xi \in \Sigma^d \).

(a) If

\[
\begin{align*}
\lim_{t \to \infty} \frac{P\{\|X\|_1 > t\}}{P\{\|Y\|_1 > t\}} &= 0, \\
\end{align*}
\]

then \( X \preceq_{\text{apl}} Y \).

(b) If \( \alpha_X > \alpha_Y \), then \( X \preceq_{\text{apl}} Y \).

**Proof.**

(a) Using relation (34) we obtain

\[
\begin{align*}
\limsup_{t \to \infty} \frac{P\{\xi^\top X > t\}}{P\{\xi^\top Y > t\}} &= \limsup_{t \to \infty} \left( \frac{P\{\xi^\top X > t\}}{P\{\|X\|_1 > t\}} \cdot \frac{P\{|Y|_1 > t\}}{P\{\|Y\|_1 > t\}} \cdot \frac{P\{\|X\|_1 > t\}}{P\{\|Y\|_1 > t\}} \right) \\
&= \frac{\gamma_\xi(X)}{\gamma_\xi(Y)} \cdot \limsup_{t \to \infty} \frac{P\{\|X\|_1 > t\}}{P\{\|Y\|_1 > t\}} \\
&= 0.
\end{align*}
\]

(b) Recall that multivariate regular variation of \( X \) implies regular variation of \( \|X\|_1 \) with tail index \( \alpha_X \). Analogously, \( \|Y\|_1 \) is regularly varying with tail index \( \alpha_Y \). Finally, \( \alpha_X > \alpha_Y \) yields (38) and by (3.1) we obtain \( X \preceq_{\text{apl}} Y \). \( \square \)
Thus the primary setting for studying the influence of dependence structures on the ordering of extreme portfolio losses is the case of random variables $X$ and $Y$ with equal tail indices:

$$\alpha_X = \alpha_Y =: \alpha.$$ 

In the framework of multivariate regular variation, asymptotic dependence in the tail region is characterized by the spectral measure $\Psi$ or its canonical version $\Psi^*$. The canonical exponent measure $\nu^*$ of $X$ is obtained from the exponent measure $\nu$ as

$$\nu^* = \nu \circ T$$

with the transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$T(x) := \left( T_\alpha \left( \nu(B_1) \cdot x^{(1)} \right), \ldots, T_\alpha \left( \nu(B_d) \cdot x^{(d)} \right) \right),$$

where

$$T_\alpha(t) := \left( t_1^{1/\alpha} - t_1^{-1/\alpha} \right)$$

and

$$B_i := \{ x \in \mathbb{R}^d : |x^{(i)}| > 1 \}.$$ 

Furthermore, $\nu^*$ exhibits the scaling property

$$\nu^*(tA) = t^{-1}\nu^*(A), \quad t > 0,$$

and, analogously to (31), has a product structure in polar coordinates:

$$\nu^* \circ \tau^{-1} = \rho_1 \otimes \Psi^*,$$

The measure $\Psi^*$ is the canonical spectral measure of $X$.

Since $\preceq_{apl}$ and $\Psi^*$ are invariant under componentwise rescalings, the canonical spectral measure $\Psi^*$ is more suitable for the characterization of $\preceq_{apl}$. The following lemma provides a representation of the extreme risk index $\gamma_\xi$ in terms of $\Psi^*$. Note that the formulation makes use of the componentwise product notation (2).

**Proposition 3.2.** Let $X$ be multivariate regularly varying on $\mathbb{R}^d$ with tail index $\alpha \in (0, \infty)$. If $X$ satisfies the non-degeneracy condition (28), then

$$\gamma_\xi(X) = \int_{\mathbb{R}^d} g_{\xi,\alpha}(vs) \, d\Psi^*(s),$$

where $\Psi^*$ denotes the canonical spectral measure of $X$, the rescaling vector $v = (v^{(1)}, \ldots, v^{(d)})$ is defined by

$$v^{(i)} := (\gamma_{\xi_1}(X) + \gamma_{-\alpha}(X)),$$

and the function $g_{\xi,\alpha} : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$g_{\xi,\alpha}(x) := \left( \sum_{i=1}^d \xi^{(i)} \cdot (x^{(i)})_{+}^{1/\alpha} - (x^{(i)})_{-}^{1/\alpha} \right)_+^\alpha.$$
Proof. Denote $A_{\xi,1} := \{ x \in \mathbb{R}^d : \xi^\top x \geq 1 \}$. Then, by definition of $\nu^*$,

$$
\begin{align*}
\gamma_\xi(X) &= \nu(A_{\xi,1}) \\
&= \nu^*(T^{-1}(A_{\xi,1})) \\
&= \nu^* \left\{ x \in \mathbb{R}^d : T(x) \in A_{\xi,1} \right\} \\
&= \int_{S_1^d} \int_{(0,\infty)} 1 \left\{ \xi^\top T(rs) > 1 \right\} \, d\rho_1(r) \, d\Psi^*(s).
\end{align*}
$$

(45)

It is easy to see that (40) implies $T_\alpha(rt) = r^{1/\alpha}T_\alpha(t)$ for $r > 0$ and $t \in \mathbb{R}$. Consequently, (39) yields

$$
T(rx) = r^{1/\alpha}T(x)
$$

(46)

for $r > 0$ and $x \in \mathbb{R}^d$. Applying (46) to (45), one obtains

$$
\begin{align*}
\gamma_\xi(X) &= \int_{S_1^d} \int_{(0,\infty)} 1 \left\{ r^{1/\alpha}\xi^\top T(s) > 1 \right\} \, d\rho_1(r) \, d\Psi^*(s) \\
&= \int_{S_1^d} \int_{(0,\infty)} 1 \left\{ \xi^\top T(s) > 0 \right\} 1 \left\{ r > \left( \xi^\top T(s) \right)^{-\alpha} \right\} \, d\rho_1(r) \, d\Psi^*(s) \\
&= \int_{S_1^d} 1 \left\{ \xi^\top T(s) > 0 \right\} \left( \xi^\top T(s) \right)^\alpha \, d\Psi^*(s) \\
&= \int_{S_1^d} \left( \xi^\top T(s) \right)^\alpha_+ \, d\Psi^*(s).
\end{align*}
$$

(47)

Finally, consider the sets $B_i$ defined in (40). It is easy to see that

$$
\nu(B_i) = \gamma_{e_i}(X) + \gamma_{-e_i}(X) = v^i.
$$

Hence

$$
\left( \xi^\top T(s) \right)^\alpha_+ = \left( \sum_{i=1}^d \xi^{(i)} \cdot \left( T_\alpha \left( v^i s^{(i)} \right) \right) \right)^\alpha_+ = g_{\xi,\alpha}(vs).
$$

□

As already mentioned above, $\preceq_{\text{apl}}$ and $\Psi^*$ are invariant under rescaling of components. Consequently, characterization of $\preceq_{\text{apl}}$ can be reduced to the case when the marginal weights $v^i = \gamma_{e_i}(X) + \gamma_{-e_i}(X)$ in (42) are standardized by

$$
\forall i, j \in \{1, \ldots, d\} \quad \lim_{t \to \infty} \frac{P\{|X^{(i)}| > t\}}{P\{|X^{(j)}| > t\}} = 1.
$$

(48)

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This condition will be referred to as the balanced tails condition. The following result shows that this condition significantly simplifies the representation (42).

**Proposition 3.3.** Suppose that \( X \) is multivariate regularly varying on \( \mathbb{R}^d \) with tail index \( \alpha \in (0, \infty) \).

(a) If \( X \) has balanced tails in the sense of (48), then

\[
\frac{\gamma_\xi(X)}{\gamma_{e_1}(X) + \gamma_{-e_1}(X)} = \Psi^* g_{\xi,\alpha}. \tag{49}
\]

(b) The non-degeneracy condition (28) is equivalent to the existence of a vector \( w \in (0, \infty)^d \) such that \( wX \) has balanced tails.

(c) The extreme risk index \( \gamma_\xi \) of the rescaled vector \( wX \) obtained in part (b) satisfies

\[
\frac{\gamma_\xi(wX)}{\gamma_{e_1}(wX) + \gamma_{-e_1}(wX)} = \Psi^*_X g_{\xi,\alpha}. \tag{50}
\]

Proof. Part (a). Consider the integrand \( g_{\xi,\alpha}(vs) \) in the representation (42):

\[
g_{\xi,\alpha}(vs) = \left( \sum_{i=1}^d \xi^{(i)} \cdot \left( \frac{(v^{(i)}s^{(i)})_{+}^{1/\alpha} - (v^{(i)}s^{(i)})_{-}^{1/\alpha}}{\nu} \right) \right)^{\alpha}.
\]

The balanced tails condition (48) implies that \( X \) is non-degenerate in the sense of (28). Furthermore, all weights \( v^{(i)} \) in the representation (42) are equal:

\[
1 = \lim_{t \to \infty} \frac{P \{ |X^{(i)}| > t \}}{\nu} / \frac{\nu}{\|X\|_1 > t} = \frac{\gamma_{e_i}(X) + \gamma_{-e_i}(X)}{\gamma_{e_j}(X) + \gamma_{-e_j}(X)}
\]

and

\[
\nu^{(i)} \nu^{(j)}, \quad i, j \in \{1, \ldots, d\}.
\]

Hence \( g_{\xi,\alpha}(vs) \) simplifies to

\[
g_{\xi,\alpha}(vs) = v^{(i)} g_{\xi,\alpha}(s) = (\gamma_{e_i}(X) + \gamma_{-e_i}(X)) g_{\xi,\alpha}(s).
\]

Part (b). Suppose that \( X \) satisfies (28). Then the sets \( B_i \) defined in (40) satisfy \( \nu(B_i) > 0 \) for \( i = 1, \ldots, d \). Consequently, the random variables \( |X^{(i)}| \) are regularly varying with tail index \( \alpha \). Denoting

\[
w^{(i)} := (\nu(B_i))^{-1/\alpha},
\]

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one obtains

\[
\lim_{t \to \infty} \frac{\mathbb{P}\{w^{(i)}X^{(i)} > t\}}{\mathbb{P}\{|X| > t\}} = \lim_{t \to \infty} \left( \frac{\mathbb{P}\{X^{(i)} > t/w^{(i)}\}}{\mathbb{P}\{|X| > t\}} \cdot \frac{\mathbb{P}\{|X| > t\}}{\mathbb{P}\{|X| > t/w^{(i)}\}} \right) = (w^{(i)})^\alpha \cdot \nu(B_i) = 1
\]

for \(i = 1, \ldots, d\). Hence, for any \(i, j \in \{1, \ldots, d\}\),

\[
\lim_{t \to \infty} \frac{\mathbb{P}\{w^{(i)}X^{(i)} > t\}}{\mathbb{P}\{w^{(j)}X^{(j)} > t\}} = 1.
\]

To prove the inverse implication, suppose that \(Z := wX\) has balanced tails for some \(w \in (0, \infty)^d\). Then the exponent measure \(\nu\) of \(X\) satisfies

\[
\nu(B_i) = \lim_{t \to \infty} \frac{\mathbb{P}\{|X| > t\}}{\mathbb{P}\{|X| > t/w^{(i)}\}} \cdot \frac{\mathbb{P}\{|X| > t/w^{(i)}\}}{\mathbb{P}\{|Z| > w^{(i)}t\}} \cdot \frac{\mathbb{P}\{|Z| > w^{(i)}t\}}{\mathbb{P}\{|Z| > w^{(i)}t|w^{(1)}|\}} = (\frac{w^{(i)}}{w^{(1)}})^\alpha \in (0, \infty), \quad i \in \{1, \ldots, d\}.
\]

Since multivariate regular variation of \(X\) implies \(\nu(B_j) > 0\) for at least one index \(j \in \{1, \ldots, d\}\), this yields \(\nu(B_i) > 0\) for all \(i\).

Part (c). This is an immediate consequence of part (a) and the invariance of canonical spectral measures under componentwise rescaling. □

Representation (49) suggests that ordering of the normalized extreme risk indices \(\gamma_{\xi}/(\gamma_{e_1} + \gamma_{-e_1})\) in the balanced tails setting can be considered as an integral order relation for canonical spectral measures with respect to the function class

\[
\mathcal{G}_\alpha := \{g_{\xi,\alpha} : \xi \in \Sigma^d\}.
\]

(52)

This justifies the following definition.

**Definition 3.4.** Let \(\Psi^*\) and \(\Phi^*\) be canonical spectral measures on \(\mathbb{S}^d_1\) and let \(\alpha > 0\). Then the order relation \(\Psi^* \preceq_{\alpha} \Phi^*\) is defined by

\[
\forall g \in \mathcal{G}_\alpha \quad \Psi^* g \leq \Phi^* g.
\]

(53)
Remark 3.5. (a) For $\alpha = 1$ and spectral measures on $\Sigma^d$ the extreme risk index $\gamma_\xi(X)$ is linear in $\xi$ (cf. Mainik and Rüschendorf, 2010, Lemma 3.2). Consequently, $\preceq_{\mathcal{G}_\alpha}$ is indifferent in this case, i.e., any $\Psi^*$ and $\Phi^*$ on $\mathcal{B}(\Sigma^d)$ satisfy
$$\Psi^* \preceq_{\mathcal{G}_1} \Phi^* \quad \text{and} \quad \Phi^* \preceq_{\mathcal{G}_1} \Psi^*. \quad (54)$$

(b) The order relation $\preceq_{\mathcal{G}_\alpha}$ is mixing invariant in the sense that uniform ordering of two parametric families $\{\Psi^*_\theta : \theta \in \Theta\}$ and $\{\Phi^*_\theta : \theta \in \Theta\}$,
$$\forall \theta \in \Theta \quad \Psi^*_\theta \preceq_{\mathcal{G}_\alpha} \Phi^*_\theta,$$
implies
$$\int_{\Theta} \Psi^*_\theta \, d\mu(\theta) \preceq_{\mathcal{G}_\alpha} \int_{\Theta} \Phi^*_\theta \, d\mu(\theta)$$
for any probability measure $\mu$ on $\Theta$.

The following theorem states that $\preceq_{\text{apl}}$ is in a certain sense equivalent to the ordering of canonical spectral measures and allows to reduce the verification of $\preceq_{\text{apl}}$ to the verification of $\preceq_{\mathcal{G}_\alpha}$. Some exemplary applications are given in Section 5. Furthermore, given explicit representations of spectral measures or their canonical versions, this result allows to verify $\preceq_{\text{apl}}$ numerically, which is very useful in practice.

**Theorem 3.6.** Let $X$ and $Y$ be multivariate regularly varying random vectors on $\mathbb{R}^d$ with tail index $\alpha \in (0, \infty)$ and canonical spectral measures $\Psi^*_X$ and $\Psi^*_Y$. Further, suppose that $X$ and $Y$ satisfy the balanced tails condition (48).

(a) If $\left|X^{(1)}\right| \preceq_{\text{apl}} \left|Y^{(1)}\right|$, then $\Psi^*_X \preceq_{\mathcal{G}_\alpha} \Psi^*_Y$ implies $X \preceq_{\text{apl}} Y$.

(b) If $\left|X^{(1)}\right| \preceq_{\text{apl}} \left|Y^{(1)}\right|$ and $\left|Y^{(1)}\right| \preceq_{\text{apl}} \left|X^{(1)}\right|$, then $\Psi^*_X \preceq_{\mathcal{G}_\alpha} \Psi^*_Y$ is equivalent to $X \preceq_{\text{apl}} Y$.

**Proof.** (a) Since $X$ has balanced tails, Proposition 3.3(a) yields
$$\lim_{t \to \infty} \frac{P\left\{\xi^T X > t\right\}}{P\left\{\left|X^{(1)}\right| > t\right\}} = \lim_{t \to \infty} \left(\frac{P\left\{\xi^T X > t\right\}}{P\left\{\left|X^{(1)}\right| > t\right\}} \cdot \frac{P\left\{\left|X\right| > t\right\}}{P\left\{\left|X^{(1)}\right| > t\right\}}\right) = \frac{\gamma_\xi(X)}{\gamma_{e_1}(X) + \gamma_{-e_1}(X)} = \Psi^*_X g_{\xi,\alpha}.$$

Analogously one obtains
$$\lim_{t \to \infty} \frac{P\left\{\xi^T Y > t\right\}}{P\left\{\left|Y^{(1)}\right| > t\right\}} = \Psi^*_Y g_{\xi,\alpha}.$$
Moreover, $\Psi^*_X \preceq^{g_\alpha} \Psi^*_Y$ implies
\[
\frac{\Psi^*_X g_{\xi,\alpha}}{\Psi^*_Y g_{\xi,\alpha}} \leq 1. \tag{55}
\]
Consequently,
\[
\limsup_{t \to \infty} \frac{\Pr\{\xi^\top X > t\}}{\Pr\{\xi^\top Y > t\}} = \limsup_{t \to \infty} \left( \frac{\Pr\{\xi^\top X > t\}}{\Pr\{|X| > t\}} \cdot \frac{\Pr\{|Y| > t\}}{\Pr\{|Y| > t\}} \right)
\frac{\Psi^*_X g_{\xi,\alpha}}{\Psi^*_Y g_{\xi,\alpha}} \cdot \limsup_{t \to \infty} \frac{\Pr\{|X| > t\}}{\Pr\{|Y| > t\}} \leq 1
\tag{56}
\]
due to (55) and $|X(i)| \preceq_{\text{apl}} |Y(i)|$.

(b) By part (a), it suffices to show that $X \preceq_{\text{apl}} Y$ implies $\Psi^*_X \preceq^{g_\alpha} \Psi^*_Y$. By assumption $|X(i)|$ and $|Y(i)|$ have asymptotically equivalent tails,
\[
\lim_{t \to \infty} \frac{\Pr\{|X(i)| > t\}}{\Pr\{|Y(i)| > t\}} = 1.
\]
Thus (56) yields
\[
\frac{\Psi^*_X g_{\xi,\alpha}}{\Psi^*_Y g_{\xi,\alpha}} = \limsup_{t \to \infty} \frac{\Pr\{\xi^\top X > t\}}{\Pr\{\xi^\top Y > t\}}
\]
and $X \preceq_{\text{apl}} Y$ implies $\Psi^*_X \preceq^{g_\alpha} \Psi^*_Y$. \hfill \Box

The following result answers the question for dependence structures corresponding to the best and the worst possible diversification effects for multivariate regularly varying random vectors in $\mathbb{R}^d_+$. According to Theorem 3.6, it suffices to find the upper and the lower elements with respect to $\preceq_{g_\alpha}$ in the set of all canonical spectral measures on $\Sigma^d$. It turns out that for $\alpha > 1$ the best diversification effects are obtained in case of asymptotic independence, i.e., the $\preceq_{g_\alpha}$-maximal element is given by
\[
\Psi^*_0 := \sum_{i=1}^d \delta_{e_i}, \tag{57}
\]
whereas the worst diversification effects are obtained in case of the asymptotic comonotonicity, represented by
\[
\Psi^*_1 := d \cdot \delta_{(1/d, \ldots, 1/d)}. \tag{58}
\]
For $\alpha < 1$ the situation is inverse.
Theorem 3.7. Let $\Psi^*$ be an arbitrary canonical spectral measure on $\Sigma^d$ and let $\Psi^*_0$ and $\Psi^*_1$ be defined according to (57) and (58). Then

(a) $\Psi^*_0 \preceq_{G^*} \Psi^* \preceq_{G^*} \Psi^*_1$ for $\alpha \geq 1$.

(b) $\Psi^*_1 \preceq_{G^*} \Psi^* \preceq_{G^*} \Psi^*_0$ for $\alpha \in (0, 1]$.

Proof. Let $X$ be multivariate regularly varying on $\mathbb{R}^d_+$ with canonical spectral measure $\Psi^*$. Without loss of generality we can assume that $X$ satisfies the balanced tails condition (48). Then, according to (49), we have

$$
\Psi^* g_{\xi, \alpha} = \frac{\gamma_\xi(X)}{\gamma_{e_1}(X)}. \tag{59}
$$

Furthermore, we have $\Psi^* g_{e_i, \alpha} = 1$ for $i = 1, \ldots, d$. Recall that the mapping $\xi \mapsto \gamma_\xi$ is convex for $\alpha \geq 1$ (cf. Mainik and Rüschendorf, 2010, Lemma 3.2). Due to (59) this behaviour is inherited by the mapping $\xi \mapsto \Psi^* g_{\xi, \alpha}$. Thus for $\alpha \geq 1$ we have $\Psi^* g_{\xi, \alpha} \leq 1 = \Psi^*_1 g_{\xi, \alpha}$ for all $\xi \in \Sigma^d$, which exactly means $\Psi^* \preceq_{G^*} \Psi^*_1$ for $\alpha \geq 1$.

To complete the proof of part (a), note that the normalization of canonical spectral measures yields

$$
\forall \xi \in \Sigma^d \quad \Psi^*_0 g_{\xi, \alpha} = \sum_{i=1}^d (\xi^{(i)})^\alpha = \int_{\Sigma^d} \sum_{i=1}^d (\xi^{(i)})^\alpha s^{(i)} \Psi^*(ds) \tag{60}
$$

Comparing the integrand on the right side of (60) with the function $g_{\xi, \alpha}(s) = (\xi^\top s^{1/\alpha})^\alpha$, we see that

$$
\sum_{i=1}^d (\xi^{(i)})^\alpha s^{(i)} = g_{\xi, \alpha}(s) \cdot \sum_{i=1}^d z_i^\alpha
$$

with

$$
z_i := \frac{\xi^{(i)} \cdot (s^{(i)})^{1/\alpha}}{\xi^\top s^{1/\alpha}}.
$$

Thus it suffices to demonstrate that $\sum_{i=1}^d z_i^\alpha \leq 1$, which follows from $z_i \in [0, 1]$, $z_i^\alpha \leq z_i$ for $\alpha \geq 1$, and $\sum_{i=1}^d z_i = 1$.

The inverse result for $\alpha \in (0, 1]$ stated in (b) follows from the concavity of the mapping $\xi \mapsto \Psi^* g_{\xi, \alpha}$ and the inequality $z_i^\alpha \geq z_i$. \qed

Due to Theorem 3.6, an analogue of the foregoing result for $\preceq_{spl}$ is straightforward.
**Corollary 3.8.** Let $X$ be multivariate regularly varying in $\mathbb{R}^d_+$ with tail index $\alpha \in (0, \infty)$ and identically distributed margins $X^{(i)} \sim F$, $i = 1, \ldots, d$. Further, let $Y$ be a random vector with independent margins $Y^{(i)} \sim F$, and let $Z$ be a random vector with totally dependent margins $Z^{(i)} = Z^{(1)}$ P-a.s. and $Z^{(1)} \sim F$. Then

(a) $Y \preceq_{\text{apl}} X \preceq_{\text{apl}} Z$ for $\alpha \geq 1$

(b) $Z \preceq_{\text{apl}} X \preceq_{\text{apl}} Y$ for $\alpha \in (0, 1]$. 

**Remark 3.9.** The strict assumptions of Corollary 3.8 are chosen for clearness and simplicity. The independence of $Y^{(i)}$ and the total dependence of $Z^{(i)}$ are needed only in the tail region, i.e., it suffices for $Y$ and $Z$ to be multivariate regularly varying with canonical spectral measures $\Psi^*_0$ and $\Psi^*_1$, respectively. Furthermore, the assumption of identically distributed margins can be replaced by equivalent tails:

$$1 = \lim_{t \to \infty} \frac{P\{Y^{(i)} > t\}}{P\{X^{(i)} > t\}} = \lim_{t \to \infty} \frac{P\{Z^{(i)} > t\}}{P\{X^{(i)} > t\}}, \quad i = 1, \ldots, d.$$ 

Finally, the non-negativity of $X^{(i)}$, $Y^{(i)}$, and $Z^{(i)}$ is needed only in the asymptotic sense. The ordering results remain true if the spectral measure of $X$ is restricted to the unit simplex $\Sigma^d$.

Combining Theorem 3.6 with Theorem 2.7, one obtains an ordering result for the canonical spectral measures of multivariate regularly varying elliptical distributions. The notation $\Psi^* = \Psi^*(\alpha, C)$ is justified by the fact that spectral measures of elliptical distributions depend only on the tail index $\alpha$ and the generalized covariance matrix $C$. An explicit representation of spectral densities for bivariate elliptical distributions was obtained by Hult and Lindskog (2002). Alternative representations that are valid for all dimensions $d \geq 2$ are given in Mainik (2010), Lemma 2.8.

**Proposition 3.10.** Let $C$ and $D$ be $d$-dimensional covariance matrices satisfying

$$C_{i,i} = D_{i,i} > 0, \quad i = 1, \ldots, d, \quad (61)$$

and

$$\forall \xi \in \Sigma^d \quad \xi^T C \xi \leq \xi^T D \xi. \quad (62)$$

Then

$$\forall \alpha > 0 \quad \Psi^*(\alpha, C) \preceq_{g_\alpha} \Psi^*(\alpha, D).$$
Proof. Fix $\alpha \in (0, \infty)$ and consider random vectors
\[ X \overset{d}{=} RAU, \quad Y \overset{d}{=} RBU, \]
where $A$ and $B$ are square roots of the matrices $C$ and $D$ in (62), i.e.,
\[ C = AA^\top, \quad D = BB^\top, \]
and $R$ is an arbitrary regularly varying non-negative random variable with tail index $\alpha$.

As a consequence of Theorem 2.7 one obtains $X \preceq_{\text{apl}} Y$. Furthermore, invariance of $\preceq_{\text{apl}}$ under componentwise rescaling yields $wX \preceq_{\text{apl}} wY$ for $w = (w^{(1)}, \ldots, w^{(d)})$ with
\[ w^{(i)} := C_{i,i}^{-1/2} = D_{i,i}^{-1/2}, \quad i = 1, \ldots, d. \]
Moreover, as a particular consequence of arguments underlying (22), one obtains
\[ w^{(i)} X^{(i)} \overset{d}{=} w^{(j)} Y^{(j)}, \quad i, j \in \{1, \ldots, d\}. \]

Hence the random vectors $wX$ and $wY$ satisfy the balanced tails condition (48), whereas their components are mutually ordered with respect to $\preceq_{\text{apl}}$. Finally, Theorem 3.6(b) and invariance of canonical spectral measures under componentwise rescalings yield
\[ \Psi^*(\alpha, C) = \Psi^*_{wX} \preceq_{\mathbb{G}_\alpha} \Psi^*_{wY} = \Psi^*(\alpha, D). \]

The subsequent result extends Theorem 3.6 to random vectors that do not have balanced tails.

**Theorem 3.11.** Let $X$ and $Y$ be multivariate regularly varying random vectors on $\mathbb{R}^d$ with tail index $\alpha \in (0, \infty)$ and canonical spectral measures $\Psi^*_X$ and $\Psi^*_Y$. Further, assume that $|X^{(i)}| \preceq_{\text{apl}} |Y^{(i)}|$ with
\[ \lambda_i := \lim_{t \to \infty} \frac{\mathbb{P}\{|X^{(i)}| > t\}}{\mathbb{P}\{|Y^{(i)}| > t\}} \in (0, 1] \] (63)
for $i = 1, \ldots, d$ and that the vector $v = (v^{(1)}, \ldots, v^{(d)})$ defined by
\[ v^{(i)} := \lambda_i^{-1/\alpha} \] (64)
satisfies
\[ X \preceq_{\text{apl}} vX \quad \text{or} \quad v^{-1} Y \preceq_{\text{apl}} Y. \] (65)
Then $\Psi^*_X \preceq_{\mathbb{G}_\alpha} \Psi^*_Y$ implies $X \preceq_{\text{apl}} Y$. 22
Proof. According to Proposition 3.3(b), there exists \( w \in \mathbb{R}^d_+ \) such that \( wY \) satisfies the balanced tails condition (48). Furthermore, the tails of the random vector

\[
vwX := (v^{(1)}w^{(1)}X^{(1)}, \ldots, v^{(d)}w^{(d)}X^{(d)})
\]

with \( v \) defined in (63) are also balanced. Indeed, it is easy to see that

\[
\lim_{t \to \infty} \frac{P\{|w^{(i)}Y^{(i)}| > t\}}{P\{|Y^{(i)}| > t\}} = \lim_{t \to \infty} \frac{P\{|v^{(i)}w^{(i)}X^{(i)}| > t\}}{P\{|v^{(i)}X^{(i)}| > t\}} = (w^{(i)})^\alpha
\]

for \( i = 1, \ldots, d \). Analogously one obtains

\[
\lim_{t \to \infty} \frac{P\{|v^{(i)}X^{(i)}| > t\}}{P\{|X^{(i)}| > t\}} = (v^{(i)})^\alpha = \lambda_i^{-1}
\]

and, as a result,

\[
\lim_{t \to \infty} \frac{P\{|v^{(i)}w^{(i)}X^{(i)}| > t\}}{P\{|w^{(i)}Y^{(i)}| > t\}} = \lambda_i^{-1} \cdot \lim_{t \to \infty} \frac{P\{|X^{(i)}| > t\}}{P\{|Y^{(i)}| > t\}} = 1
\]

for \( i = 1, \ldots, d \). Hence the balanced tails condition for \( wY \) implies that the tails of \( vwX \) are also balanced.

Furthermore, invariance of canonical spectral measures under component-wise rescaling yields

\[
\Psi^*_{vwX} = \Psi^*_{vX} \prec G_\alpha, \Psi^*_{wY} = \Psi^*_{wY}.
\]

Thus, applying Theorem 3.6(a), one obtains

\[
vwX \preceq_{\text{apl}} wY. \tag{66}
\]

Since \( v^{(i)} = \lambda_i^{-1/\alpha} > 0 \) for \( i = 1, \ldots, d \), condition (66) is equivalent to

\[
wX \preceq_{\text{apl}} v^{-1}wY. \tag{67}
\]
Moreover, assumption (65) implies
\[ wX \preceq_{\text{apl}} vwX \text{ or } v^{-1}wY \preceq_{\text{apl}} wY. \] (68)
Combining this ordering statement with (66) and (67), one obtains
\[ wX \preceq_{\text{apl}} wY. \]
Finally, invariance of \( \preceq_{\text{apl}} \) with respect to componentwise rescaling yields
\[ X \preceq_{\text{apl}} Y. \] \( \square \)
In the special case of random vectors in \( \mathbb{R}^d_+ \) Theorem 3.11 can be simplified to the following result.

**Corollary 3.12.** Let \( X \) and \( Y \) be multivariate regularly varying random vectors on \( \mathbb{R}^d_+ \) with tail index \( \alpha \in (0, \infty) \) and canonical spectral measures \( \Psi^*_X \) and \( \Psi^*_Y \). Further, suppose that
\[ \lambda_i := \limsup_{t \to \infty} \frac{\mathbb{P}\{ |X(i)| > t \}}{\mathbb{P}\{ |Y(i)| > t \}} \in (0, 1], \quad i = 1, \ldots, d. \] (69)
Then \( \Psi^*_X \preceq_{\mathcal{G}_\alpha} \Psi^*_Y \) implies \( X \preceq_{\text{apl}} Y \).

**Proof.** Assumption (69) yields that the rescaling vector \( v \) defined in (64) is an element of \( [1, \infty)^d \). Thus \( v - (1, \ldots, 1) \in \mathbb{R}^d_+ \) and, since \( X \) takes values in \( \mathbb{R}^d_+ \), we have
\[ X \preceq_{\text{apl}} X + (v - (1, \ldots, 1))X = vX. \]
Similar arguments yield \( v^{-1}Y \preceq_{\text{apl}} Y \). Hence condition (65) of Theorem 3.11 is satisfied. \( \square \)

The final result of this section is due to the indifference of \( \preceq_{\mathcal{G}_\alpha} \) for \( \alpha = 1 \) mentioned in Remark 3.5(a). This special property of spectral measures on \( \Sigma^d_1 \) allows to reduce \( \preceq_{\text{apl}} \) to the ordering of components. It should be noted that this result cannot be extended to the general case of spectral measures on \( \Sigma^d \).

**Lemma 3.13.** Let \( X \) and \( Y \) be multivariate regularly varying on \( \mathbb{R}^d_+ \) with tail index \( \alpha = 1 \). Further, suppose that \( Y \) satisfies the non-degeneracy condition (28) and that \( X(i) \preceq_{\text{apl}} Y(i) \) for \( i = 1, \ldots, d \). Then \( X \preceq_{\text{apl}} Y \).

**Proof.** According to Proposition 3.3(b), there exists \( w \in (0, \infty)^d \) such that \( wY \) satisfies the balanced tails condition (48). Furthermore, due to the invariance of \( \preceq_{\text{apl}} \) under componentwise rescaling, \( X \preceq_{\text{apl}} Y \) is equivalent to \( wX \preceq_{\text{apl}} wY \).
Thus it can be assumed without loss of generality that $Y$ has balanced tails. This yields
\[
\lambda_i := \limsup_{t \to \infty} \frac{\mathbb{P}\{X^{(i)} > t\}}{\mathbb{P}\{Y^{(i)} > t\}} = \limsup_{t \to \infty} \frac{\mathbb{P}\{X^{(i)} > t\}}{\mathbb{P}\{Y^{(1)} > t\}}, \quad i = 1, \ldots, d.
\]
Hence the assumption $X^{(i)} \preceq_{ap} Y^{(i)}$ for $i = 1, \ldots, d$ implies $\lambda_i \in [0, 1]$ for all $i$. Moreover, the balanced tails condition for $Y$ yields
\[
\gamma_{e_1}(Y) = \ldots = \gamma_{e_d}(Y). \quad (70)
\]
Now consider the random vector $X$ and denote
\[
j := \arg \max_{i \in \{1, \ldots, d\}} \gamma_{e_i}(X).
\]
Recall that $\gamma_{e_i}(X) = \nu_X(\{x \in \mathbb{R}_+^d : x^{(i)} > 1\})$ with $\nu_X$ denoting the exponent measure of $X$ and that $\nu_X$ is non-zero. This yields $\gamma_{e_j}(X) > 0$ even if $X$ does not satisfy the non-degeneracy condition (28). Moreover, for $\alpha = 1$, the mapping $\xi \mapsto \gamma_\xi(X)$ is linear. This implies
\[
\gamma_\xi(X) = \sum_{i=1}^d \xi^{(i)} \cdot \gamma_{e_i}(X) \leq \gamma_{e_j}(X), \quad \xi \in \Sigma^d \quad (71)
\]
and (70) yields
\[
\gamma_\xi(Y) = \sum_{i=1}^d \xi^{(i)} \cdot \gamma_{e_i}(Y) = \gamma_{e_1}(Y), \quad \xi \in \Sigma^d. \quad (72)
\]
Hence
\[
\limsup_{t \to \infty} \frac{\mathbb{P}\{\xi^\top X > t\}}{\mathbb{P}\{\xi^\top Y > t\}} = \left(\frac{\mathbb{P}\{\xi^\top X > t\}}{\mathbb{P}\{X^{(j)} > t\}} \cdot \frac{\mathbb{P}\{X^{(j)} > t\}}{\mathbb{P}\{Y^{(1)} > t\}} \cdot \frac{\mathbb{P}\{Y^{(1)} > t\}}{\mathbb{P}\{\xi^\top Y > t\}}\right) = \frac{\gamma_\xi(X)}{\gamma_{e_j}(X)} \cdot \lambda_j \cdot \frac{\gamma_{e_1}(Y)}{\gamma_\xi(Y)} \leq 1
\]
due to $\lambda_j \leq 1$, (71), and (72). \qed
4 Relations to convex and supermodular orders

As mentioned in Remark 2.4(b), dependence orders \( \preceq_{\text{sm}} \), \( \preceq_{\text{dcx}} \) and convexity orders \( \preceq_{\text{cx}} \), \( \preceq_{\text{icx}} \), \( \preceq_{\text{plcx}} \) do not imply \( \preceq_{\text{apl}} \) in general. However, it turns out that the relationship between \( \preceq_{\text{apl}} \) and the ordering of canonical spectral measures by \( \preceq_{G_\alpha} \) allows to draw conclusions of this type in the special case of multivariate regularly varying models. The core result of this section is stated in Theorem 4.1. It entails a collection of sufficient criteria for \( \preceq_{\text{apl}} \) in terms of convex and supermodular order relations, with particular interest paid to the inversion of diversification effects for \( \alpha < 1 \). An application to copula based models is given in Proposition 4.4.

This approach was applied by Embrechts et al. (2009b) to the ordering of risks for the portfolio vector \( \xi = (1, \ldots, 1) \) and for a specific family of multivariate regularly varying models with identically distributed, non-negative margins \( X^{(i)} \) (cf. Example 5.2 in Section 5).

The next theorem is the core element of this section. It generalizes the arguments of Embrechts et al. (2009b) to multivariate regularly varying random vectors in \( \mathbb{R}^d \) with balanced tails and tail index \( \alpha \neq 1 \). The case \( \alpha = 1 \) is not included for two reasons. First, this case is partly trivial due to the indifference of \( \preceq_{G_\alpha} \) for spectral measures on \( \Sigma^d \) (cf. Remark 3.5(a)). Second, Karamata’s theorem used in the proof of the integrable case \( \alpha > 1 \) does not yield the desired result for random variables with tail index \( \alpha = 1 \).

**Theorem 4.1.** Let \( X \) and \( Y \) be multivariate regularly varying on \( \mathbb{R}^d \) with identical tail index \( \alpha \neq 1 \). Further, assume that \( X \) and \( Y \) satisfy the balanced tails condition (48).

(a) For \( \alpha > 1 \) let

\[
\limsup_{t \to \infty} \frac{P \{ |X^{(1)}| > t \}}{P \{ |Y^{(1)}| > t \}} = 1
\]

and let there exist \( u_0 > 0 \) such that with \( h_u(t) := (t - u)_+ \)

\[
\forall u \geq u_0 \ \forall \xi \in \Sigma^d \quad E h_u (\xi^T X) \leq E h_u (\xi^T Y).
\]

Then \( \Psi_X^* \preceq_{G_\alpha} \Psi_Y^* \).

(b) For \( \alpha < 1 \) suppose that \( |X^{(1)}| \) and \( |Y^{(1)}| \) are equivalent with respect to \( \preceq_{\text{apl}} \), i.e.,

\[
|X^{(1)}| \preceq_{\text{apl}} |Y^{(1)}| \quad \text{and} \quad |Y^{(1)}| \preceq_{\text{apl}} |X^{(1)}|,
\]

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and let there exist $u_0 > 0$ such that with $f_u(t) := -(t \wedge u)$,

$$\forall u \geq u_0 \quad \forall \xi \in \Sigma^d \quad E_{f_u} \left( (\xi^\top X)_+ \right) \leq E_{f_u} \left( (\xi^\top Y)_+ \right).$$

(76)

Then $\Psi^*_Y \preceq_{g_\alpha} \Psi^*_X$.

The proof will be given after some conclusions and remarks. In particular, it should be noted that the relation between $\preceq_{g_\alpha}$ and $\preceq_{apl}$ established in Theorem 3.6 immediately yields the following result.

**Corollary 4.2.** (a) If random vectors $X$ and $Y$ satisfy conditions of Theorem 4.1(a), then $X \preceq_{apl} Y$;

(b) If $X$ and $Y$ satisfy conditions of Theorem 4.1(b), then $Y \preceq_{apl} X$.

It should also be noted that conditions (74) and (76) are asymptotic forms of the increasing convex ordering $\xi^\top X \preceq_{icx} \xi^\top Y$ and the decreasing convex ordering $\xi^\top X \preceq_{dcx} \xi^\top Y$, respectively. The consequences can be outlined as follows.

**Remark 4.3.** (a) The following criteria are sufficient for (74) and (76) to hold:

(i) $(\xi^\top X)_+ \preceq_{cx} (\xi^\top Y)_+$ for all $\xi \in \Sigma^d$,

(ii) $X$ and $Y$ are restricted to $\mathbb{R}^d_+$ and $X \preceq Y$ with $\preceq$ denoting either $\preceq_{plx}$, $\preceq_{icx}$, $\preceq_{cx}$, $\preceq_{dcx}$, or $\preceq_{sm}$.

(b) Additionally, condition (74) follows from $X \preceq Y$ with $\preceq$ denoting either $\preceq_{plx}$, $\preceq_{icx}$, $\preceq_{cx}$, $\preceq_{dcx}$, or $\preceq_{sm}$.

Finally, a comment should be made upon convex ordering of non-integrable random variables and diversification for $\alpha < 1$. The so-called *phase change* at $\alpha = 1$, i.e., the inversion of diversification effects taking place when the tail index $\alpha$ crosses this critical value, demonstrates that the implications of convex ordering are essentially different for integrable and non-integrable random variables. Indeed, it is easy to see that if a random variable $Z$ on $\mathbb{R}$ satisfies $E[Z_+] = E[Z_-] = \infty$, then the only integrable convex functions of $Z$ are the constant ones. Moreover, if $Z$ is restricted to $\mathbb{R}_+$ and $EZ = \infty$, then any integrable convex function of $Z$ is necessarily non-increasing.
Proof of Theorem 4.1. (a) Consider the expectations in (74). It is easy to see that for $u > 0$
\[
\frac{1}{u} \mathrm{E} \cdot h_u (\xi^\top X) = \frac{1}{u} \int_{(u, \infty)} \mathrm{P} \{ \xi^\top X > t \} \, dt
\]
and, as a consequence,
\[
\frac{u^{-1} \mathrm{E} \cdot h_u (\xi^\top X)}{\mathrm{P} \{|X^{(1)}| > u\}} = \frac{\mathrm{P} \{ \xi^\top X > u \}}{\mathrm{P} \{|X^{(1)}| > u\}} \int_{(1, \infty)} \frac{\mathrm{P} \{ \xi^\top X > tu \}}{\mathrm{P} \{ \xi^\top X > u \}} \, dt.
\]
Moreover, Proposition 3.3(a) implies
\[
\lim_{u \to \infty} \frac{\mathrm{P} \{ \xi^\top X > u \}}{\mathrm{P} \{|X^{(1)}| > u\}} = \frac{\gamma_\xi (X)}{\gamma_{e_1} (X) + \gamma_{-e_1} (X)} = \Psi^*_X g_{\xi, \alpha}
\]
and Karamata’s theorem (cf. de Haan and Ferreira, 2006, Theorem B.1.5) yields
\[
\lim_{u \to \infty} \int_{(1, \infty)} \frac{\mathrm{P} \{ \xi^\top X > tu \}}{\mathrm{P} \{ \xi^\top X > u \}} \, dt = \int_{(1, \infty)} t^{-\alpha} \, dt = \frac{1}{\alpha - 1}.
\]
As a result one obtains
\[
\lim_{u \to \infty} \frac{u^{-1} \mathrm{E} \cdot h_u (\xi^\top X)}{\mathrm{P} \{|X^{(1)}| > u\}} = \frac{1}{\alpha - 1} \Psi^*_X g_{\xi, \alpha}
\]
and, analogously,
\[
\lim_{u \to \infty} \frac{u^{-1} \mathrm{E} \cdot h_u (\xi^\top Y)}{\mathrm{P} \{|Y^{(1)}| > u\}} = \frac{1}{\alpha - 1} \Psi^*_Y g_{\xi, \alpha}.
\]
Hence (74) and (73) yield
\[
1 \geq \limsup_{u \to \infty} \frac{u^{-1} \mathrm{E} \cdot h_u (\xi^\top X)}{u^{-1} \mathrm{E} \cdot h_u (\xi^\top Y)} = \limsup_{u \to \infty} \frac{\Psi^*_X g_{\xi, \alpha}}{\Psi^*_Y g_{\xi, \alpha}}
\]
for all $\xi \in \Sigma^d$, which exactly means $\Psi^*_X \preceq_{\gamma_o} \Psi^*_Y$.  

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(b) Note that (75) implies
\[ \lim_{t \to \infty} \frac{P \{|X(1)| > t\}}{P \{|Y(1)| > t\}} = 1 \]  \tag{78} 
and that (76) yields
\[ \forall u > u_0 \forall v \geq 0 \quad E_{f_{u+v}} (\xi^\top X) - E_{f_{u+v}} (\xi^\top Y) \leq 0. \]  \tag{79} 
Furthermore, it is easy to see that any random variable \( Z \) in \( \mathbb{R}_+ \) satisfies
\[ E[Z \land u] = \int_{(0,\infty)} (t \land u) \ dP^Z(t) \]
\[ = \int_{(0,\infty)} \int_{(0,\infty)} 1\{s < t\} \cdot 1\{s < u\} \ ds \ dP^Z(t) \]
\[ = \int_{(0,\infty)} 1\{s < u\} \int_{(0,\infty)} 1\{s < t\} \ dP^Z(t) \ ds \]
\[ = \int_{(0,u)} P\{Z > s\} \ ds. \]
This implies
\[ E_{f_{u+v}}(Z) = E_{f_u}(Z) - \int_{(u,u+v)} P\{Z > t\} \ dt. \]
Consequently, (79) yields
\[ \forall u \geq u_0 \forall v > 0 \quad E_{f_u} (\xi^\top X) - E_{f_u} (\xi^\top Y) \leq I(u,v) \]  \tag{80} 
where
\[ I(u,v) := \int_{(u,u+v)} (P\{\xi^\top X > t\} - P\{\xi^\top Y > t\}) \ dt \]
\[ = \int_{(u,u+v)} \phi(t) \cdot P\{|X(1)| > t\} \ dt \]
with
\[ \phi(t) := \frac{P\{\xi^\top X > t\} - P\{\xi^\top Y > t\}}{P\{|X(1)| > t\}}. \]
Moreover, (78), (77), and an analogue of (77) for \( Y \) yield
\[ \phi(t) = \frac{P\{\xi^\top X > t\}}{P\{|X(1)| > t\}} - \frac{P\{\xi^\top Y > t\}}{P\{|Y(1)| > t\}} \cdot \frac{P\{|Y(1)| > t\}}{P\{|X(1)| > t\}} \]
\[ \to \Psi^*_X g_{t,\alpha} - \Psi^*_Y g_{t,\alpha}, \quad t \to \infty. \]  \tag{81}
Now suppose that $\Psi^*_Y \preceq_{g_{\alpha}} \Psi^*_X$ is not satisfied, i.e., there exists $\xi \in \Sigma^d$ such that $\Psi^*_Y \xi_{\alpha} > \Psi^*_X \xi_{\alpha}$. Then (81) yields $\phi(t) \leq -\varepsilon$ for some $\varepsilon > 0$ and sufficiently large $t$. This implies

$$I(u, v) \leq -\varepsilon \int \frac{P\{|X^{(1)}| > t\}}{1} \, dt$$

(82)

for sufficiently large $u$ and all $v \geq 0$. Moreover, regular variation of $|X^{(1)}|$ with tail index $\alpha < 1$ implies $E|X^{(1)}| = \infty$. Consequently, the integral on the right side of (82) tends to infinity for $v \to \infty$:

$$\forall u > 0 \lim_{v \to \infty} \int \frac{P\{|X^{(1)}| > t\}}{1} \, dt = \infty.$$

Hence, choosing $u$ and $v$ sufficiently large, one can achieve $I(u, v) < c$ for any $c \in \mathbb{R}$. In particular, $u$ and $v$ can be chosen such that

$$I(u, v) < E f_u (\xi^T X) - E f_u (\xi^T Y),$$

which contradicts (80). Thus $\Psi^*_Y \xi_{\alpha} > \Psi^*_X \xi_{\alpha}$ cannot be true and therefore it necessarily holds that $\Psi^*_Y \preceq_{g_{\alpha}} \Psi^*_X$. \square

Now let us return to the ordering criterion in terms of the supermodular order $\preceq_{sm}$ stated in Remark 4.3. The invariance of $\preceq_{sm}$ under non-decreasing component transformations allows to transfer these criteria to copula models. Furthermore, since we are interested in the ordering of the asymptotic dependence structures represented by the canonical spectral measures, $\Psi^*_1$ and $\Psi^*_2$, we can take any copulas that yield $\Psi^*_1$ and $\Psi^*_2$ as asymptotic dependence structures.

A natural choice is given by the extreme value copulas, defined as the copulas of simple max-stable distributions corresponding to $\Psi^*_i$, i.e., the distributions

$$G^*_i(x) := \exp (-\nu^*_i (-[\infty, x]^c)),$$

(83)

where $\nu^*_i$ is the canonical exponent associated with $\Psi^*_i$ via (41). For further details on max-stable and simple max-stable distributions we refer to Resnick (1987). Since extreme value copulas and canonical spectral measures can be considered as alternative parametrizations of the same asymptotic dependence structures, we obtain the following result.

**Proposition 4.4.** Let $\Psi^*_1$ and $\Psi^*_2$ be canonical spectral measures on $\Sigma^d$. Further, for $i = 1, 2$, let $C_i$ denote the copula of the simple max-stable distribution $G^*_i$ induced by $\Psi^*_i$ according to (83) and (41). Then $C_1 \preceq_{sm} C_2$ implies...
(a) \( \Psi_1^* \preceq_{G_\alpha} \Psi_2^* \) for \( \alpha \in (1, \infty) \);

(b) \( \Psi_2^* \preceq_{G_\alpha} \Psi_1^* \) for \( \alpha \in (0, 1) \).

Proof. Let \( \nu_i^* \) denote the canonical exponent measures corresponding to \( \Psi_i^* \) and \( G_i^* \). It is easy to see that the transformed measures

\[
\nu_{\alpha,i} := \nu_i^* \circ T^{-1}, \quad i = 1, 2,
\]

with \( \alpha > 0 \) and the transformation \( T \) defined as

\[
T : x \mapsto \left( (x^{(i)})^{1/\alpha}, \ldots, (x^{(d)})^{1/\alpha} \right), \quad x \in \mathbb{R}^d_+,
\]

exhibit the scaling property with index \( -\alpha \):

\[
\nu_{\alpha,i}(tA) = t^{-\alpha} \nu_{\alpha,i}(A), \quad A \in \mathcal{B}(\mathbb{R}^d_+ \setminus \{0\}).
\]

Hence the transformed distributions

\[
G_{\alpha,i}(x) := G_i^* \circ T^{-1}(x) = \exp \left( -\nu_{\alpha,i}([0, x]^c) \right)
\]

are max-stable with exponent measures \( \nu_{\alpha,i} \).

It is well known that max-stable distributions with identical heavy-tailed margins are multivariate regularly varying (cf. Resnick, 1987). Moreover, the limit measure \( \nu \) in the multivariate regular variation condition can be chosen equal to the exponential measure associated with the property of max-stability. Consequently, the probability distributions \( G_{\alpha,i} \) for \( i = 1, 2 \) and \( \alpha > 0 \) are multivariate regularly varying with tail index \( \alpha \) and canonical spectral measures \( \Psi_i^* \).

Furthermore, it is easy to see that \( X \sim G_{\alpha,1} \) and \( Y \sim G_{\alpha,2} \) have identical margins:

\[
X^{(i)} \overset{d}{=} Y^{(j)}, \quad i, j \in \{1, \ldots, d\}.
\]

Moreover, due to the invariance of \( \preceq_{sm} \) under non-decreasing marginal transformations, \( C_1 \preceq_{sm} C_2 \) implies

\[
G_{\alpha,1} \preceq_{sm} G_{\alpha,2}
\]

for all \( \alpha > 0 \). Thus an application of the ordering criteria from Remark 4.3 to \( X \sim G_{\alpha,1} \) and \( Y \sim G_{\alpha,2} \) completes the proof. \( \square \)
5 Examples

This section concludes the paper by a series of examples with parametric models illustrating the results from the foregoing sections. Examples 5.1 and 5.2 demonstrate application of Proposition 4.4 to copula based models and the phenomenon of phase change for random vectors in $R^d_+$. The fact that the phase change does not necessarily occur in the general case is demonstrated by multivariate Student-t distributions in Example 5.3.

Example 5.1. Recall the family of Gumbel copulas given by

$$ C_\theta(u) := \exp \left( - \left( \sum_{i=1}^d \left(- \log u^{(i)} \right)^\theta \right)^{1/\theta} \right), \quad \theta \in [1, \infty). \quad (85) $$

Gumbel copulas are extreme value copulas, i.e., they are copulas of simple max-stable distributions. According to Wei and Hu (2002), Gumbel copulas with dependence parameter $\theta \in [1, \infty)$ are ordered by $\preceq_{\text{sm}}$:

$$ \forall \theta_1, \theta_2 \in [1, \infty) \quad \theta_1 \leq \theta_2 \Rightarrow C_{\theta_1} \preceq_{\text{sm}} C_{\theta_2}. \quad (86) $$

Consequently, Proposition 4.4 applies to the family of canonical spectral measures $\Psi^*_{\theta}$ corresponding to the Gumbel copulas $C_{\theta}$. Thus $1 \leq \theta_1 \leq \theta_2 < \infty$ implies $\Psi^*_{\theta_1} \preceq_{G_{\alpha}} \Psi^*_{\theta_2}$ for $\alpha > 1$ and there is a phase change when $\alpha$ crosses the value 1, i.e., for $\alpha \in (0, 1)$ there holds $\Psi^*_{\theta_2} \preceq_{G_{\alpha}} \Psi^*_{\theta_1}$.

Applying Theorem 3.6, one obtains ordering with respect to $\preceq_{\text{apl}}$ for random vectors $X$ and $Y$ on $R^d_+$ that are multivariate regularly varying with canonical spectral measures of Gumbel type and have balanced tails ordered by $\preceq_{\text{apl}}$. In particular, this is the case if $X$ and $Y$ have identical regularly varying marginal distributions and Archimedean copulas that satisfy appropriate regularity conditions (cf. Genest and Rivest, 1989; Barbe et al., 2006).

Moreover, it is also worth a remark that multivariate regularly varying random vectors with Archimedean copulas can only induce extreme value copulas of Gumbel type (cf. Genest and Rivest, 1989).

Figure 1 illustrates the resulting diversification effects in the bivariate case, including indifference to portfolio diversification for $\alpha = 1$ and the phase change occurring when $\alpha$ crosses this critical value. The graphics show the function $\xi \mapsto \Psi^*_{\theta} g_{\xi, \alpha}$ for selected values of $\theta$ and $\alpha$. Due to $X \in R^d_+$, representation $\Psi^*_{\theta} g_{\xi, \alpha} = \gamma_{\xi}/(\gamma_{e_1} + \gamma_{-e_1})$ simplifies to $\Psi^*_{\theta} g_{\xi, \alpha} = \gamma_{\xi}/\gamma_{e_1}$ and therefore

$$ \Psi^*_{\theta} g_{e_1, \alpha} = \Psi^*_{\theta} g_{e_2, \alpha} = 1. $$

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Figure 1: Bivariate Gumbel copulas: Diversification effects represented by functions $\xi^{(1)} \mapsto \Psi^* g_{\xi,\alpha}$ for selected values of $\vartheta$ and $\alpha$. 
As already mentioned above, Theorem 4.1 generalizes some arguments from Embrechts et al. (2009b). The next example concerns Galambos copulas as addressed in that original publication.

**Example 5.2.** Another family of extreme value copulas that are ordered by \( \preceq_{\text{sm}} \) is the family of \( d \)-dimensional Galambos copulas with parameter \( \vartheta \in (0, \infty) \):

\[
C_{\vartheta}(u) := \exp \left( \sum_{I \subset \{1, \ldots, d\}} (-1)^{|I|} \left( \sum_{i \in I} (-\log u^{(i)})^{-\vartheta} \right)^{-1/\vartheta} \right). \tag{87}
\]

According to Wei and Hu (2002), \( \vartheta_1 \leq \vartheta_2 \) implies \( C_{\vartheta_1} \preceq_{\text{sm}} C_{\vartheta_2} \). Thus Proposition 4.4 yields ordering of the corresponding canonical spectral measures \( \Psi_{\vartheta}^* \) with respect to \( \preceq_{\text{G}_\alpha} \). Similarly to the case of Gumbel copulas, \( \vartheta_1 \leq \vartheta_2 \) implies \( \Psi_{\vartheta_1}^* \preceq_{\text{G}_\alpha} \Psi_{\vartheta_2}^* \) for \( \alpha > 1 \) and \( \Psi_{\vartheta_2}^* \preceq_{\text{G}_\alpha} \Psi_{\vartheta_1}^* \) for \( \alpha \in (0, 1) \).

Finally, it should be noted that Galambos copulas correspond to the canonical exponent measures of random vectors \( X \) in \( \mathbb{R}_+^d \) with identically distributed regularly varying margins \( X^{(i)} \) and dependence structure of \( -X \) given by an Archimedean copula with a regularly varying generator \( \phi(1 - 1/t) \). Models of this type were discussed in recent studies of aggregation effects for extreme risks (cf. Alink et al., 2004, 2005; Nešlehová et al., 2006; Barbe et al., 2006; Embrechts et al., 2009a,b).

The final example illustrates results established in Proposition 3.10 and Theorem 2.7. In particular, it shows that elliptical distributions do not exhibit a phase change at \( \alpha = 1 \).

**Example 5.3.** Recall multivariate Student-t distributions and consider the case with equal degrees of freedom, i.e.,

\[
X \overset{d}{=} \mu_X + RA_X U, \quad Y \overset{d}{=} \mu_Y + RA_Y U, \tag{88}
\]

where \( R \overset{d}{=} |Z| \) for a Student-t distributed random variable \( Z \) with degrees of freedom equal to \( \alpha \in (0, \infty) \). Further, let the generalized covariance matrices \( C_X = C(\rho_X) \) and \( C_Y = C(\rho_Y) \) be defined as

\[
C(\rho) := \begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix} \tag{89}
\]

and assume that \( \rho_X \leq \rho_Y \).

As already mentioned in Remark 2.8(a), \( C_X \) and \( C_Y \) satisfy condition (62) and Proposition 3.10 yields \( X \preceq_{\text{apl}} Y \). Moreover, Proposition 3.10 implies a uniform ordering of diversification effects in the sense that

\[
\Psi_{\alpha,\rho_X}^* = \Psi_{\alpha,\rho_Y}^* \preceq_{\text{G}_\alpha} \Psi_Y^*. \]
for all $\alpha \in (0, \infty)$.

Figure 2 shows functions $\xi(1) \mapsto \Psi^*_{\alpha, \rho} g_{\xi, \alpha}$ for selected parameter values $\rho$ and $\alpha$ that illustrate the ordering of asymptotic portfolio losses by $\rho$ and the missing phase change at $\alpha = 1$. The indifference to portfolio diversification for $\alpha = 1$ is also absent. Moreover, symmetry of elliptical distributions implies $\gamma_{-e_1} = \gamma_{e_1}$ and, as a result,

$$\Psi^*_{\alpha, \rho} g_{e_1, \alpha} = \Psi^*_{\alpha, \rho} g_{e_2, \alpha} = 1/2.$$ 

Thus the standardization of the plots in Figure 2 is different from that in Figure 1.

**Remark 5.4.** All examples the authors are aware of suggest that the diversification coefficient $\Psi^* g_{\xi, \alpha}$ is decreasing in $\alpha$. This means that risk diversification is stronger for lighter component tails than for heavier ones.

However, it should be noted that the influence of the tail index $\alpha$ on risk aggregation is different from that. The asymptotic risk aggregation coefficient

$$q_d := \lim_{t \to \infty} \frac{P \{ X^{(1)} + \ldots + X^{(d)} > t \}}{P \{ X^{(1)} > t \}}$$

introduced by Wüthrich (2003) is known to be increasing in $\alpha$ when the loss components $X^{(i)}$ are non-negative (cf. Barbe et al., 2006). It is easy to see that the restriction to non-negative $X^{(i)}$ implies

$$q_d = \lim_{t \to \infty} \frac{P \| X \|_1 > t}{P \{ X^{(1)} > t \}} = \frac{1}{\gamma_{e_1}}.$$ 

Moreover, denoting the uniformly diversified portfolio by $\eta$,

$$\eta := d^{-1}(1, \ldots, 1),$$

one obtains

$$q_d = \lim_{t \to \infty} \frac{P \{ \eta^T X > d^{-1} t \}}{P \{ X^{(1)} > t \}} = d^a \frac{\gamma_\eta}{\gamma_{e_1}}.$$ 

Thus $q_d$ is a product of the factor $d^a$, which is increasing in $\alpha$, and the ratio $\gamma_\eta/\gamma_{e_1}$, which is closely related to the diversification coefficient $\Psi^* g_{\xi, \alpha}$.

In particular, given equal marginal weights, i.e.,

$$\gamma_{e_1} = \ldots = \gamma_{e_d},$$

Proposition 3.3(a) yields

$$\frac{\gamma_\eta}{\gamma_{e_1}} = \Psi^* g_{\eta, \alpha}.$$
Figure 2: Bivariate elliptical distributions with generalized covariance matrices defined in (89): Diversification effects represented by functions $\xi^{(1)} \mapsto \Psi_{\alpha,\rho}^\ast g_{\xi,\alpha}$ for selected values of $\rho$ and $\alpha$. 
As already mentioned above, the coefficients $\Psi^* g_{\xi,\alpha}$ with $\xi \in \Sigma^d$ are decreasing in all examples considered here. This means that the aggregation and the diversification of risks are influenced by the tail index $\alpha$ in different, maybe even always contrary ways.

The question for the generality of this contrary influence is currently open. One can easily prove that the extreme risk index $\gamma_{\xi} = \Psi f_{\xi,\alpha}$ is decreasing in $\alpha$ for $\xi \in \Sigma^d$. However, this result cannot be extended to $\Psi^* g_{\xi,\alpha}$ directly since $\Psi^* g_{\xi,\alpha}$ is related to $\Psi f_{\xi,\alpha}$ by the normalizations (49) and (50). The question whether $\Psi^* g_{\xi,\alpha}$ with arbitrary $\xi \in \Sigma^d$ or at least $\Psi^* g_{\eta,\alpha}$ is generally decreasing in $\alpha$ is an interesting subject for further research.

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