MULTILINEAR FRACTIONAL CALDERÓN-ZYGMUND OPERATORS ON WEIGHTED HARDY SPACES

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ABSTRACT. We prove norm estimates for multilinear fractional integrals acting on weighted and variable Hardy spaces. In the weighted case we develop ideas we used for multilinear singular integrals [7]. For the variable exponent case, a key element of our proof is a new multilinear, off-diagonal version of the Rubio de Francia extrapolation theorem.

1. Introduction

The purpose of this paper is to continue the study of multilinear operators on Hardy spaces begun in [7, 10]. In those papers we considered multilinear Calderón-Zygmund operators and multipliers. Here we consider the multilinear fractional Calderón-Zygmund operators introduced by Lin and Lu [15]. In the linear case, fractional Calderón-Zygmund operators have been studied by a number of authors. See, for instance, the recent papers [17, 18].

Given positive integers $m, n$ and a real number $0 < \gamma < mn$, let $K_\gamma$ be a function defined in $\mathbb{R}^{(m+1)n}$ away from the diagonal $x = y_1 = \cdots = y_m$ that satisfies the size condition

$$
|K_\gamma(x, y_1, \ldots, y_m)| \lesssim (|x - y_1| + \cdots + |x - y_m|)^{\gamma - mn},
$$

and the smoothness condition

$$
\sum_{i=1}^m \sum_{|\beta| = N} |\partial_\beta^i K_\gamma(x, y_1, \ldots, y_m)| \lesssim (|x - y_1| + \cdots + |x - y_m|)^{\gamma - mn - N}
$$

for some sufficiently large integer $N$. We define the multilinear fractional Calderón-Zygmund operator $T_\gamma$ by

$$
T_\gamma(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} K_\gamma(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) \, d\vec{y}.
$$
The simplest example of such an operator is the multilinear fractional integral introduced by Kenig and Stein [14]:

\[ I_\gamma(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} \frac{f_1(y_1) \cdots f_m(y_m)}{|x - y_1| + \cdots + |x - y_m|^{mn-\gamma}} \, dy. \]

They proved that for \(1 < p_1, \ldots, p_m \leq \infty\) and \(q\) such that \(\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\gamma}{n} > 0\),

\[ \|I_\gamma(f_1, \ldots, f_m)\|_{L^q} \lesssim C \|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}}. \]

Moreover, if \(p_i = 1\) for some \(i\), then the above inequality is replaced by the corresponding weak-type estimate.

Lin and Lu [15] proved Hardy space estimates for multilinear fractional Calderón-Zygmund operators, generalizing the results of Grafakos and Kalton [13] for multilinear singular integrals and the results in the linear case for fractional integrals due to Strömberg and Wheeden [20] and Gatto, et al. [12]. More precisely, they proved that if \(0 < p_1, \ldots, p_m, q \leq 1\), then

\[ \|T_\gamma(f_1, \ldots, f_m)\|_{L^q} \lesssim C \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}. \]

However, they had to make the restrictive assumption that \(0 < \gamma < n\).

Our first theorem is a generalization of the result of Lin and Lu to weighted Hardy spaces. To state it, we first recall some basic definitions from the theory of Muckenhoupt weights. By a weight we mean a non-negative, locally integrable function. Given a weight \(w\) and \(1 < p < \infty\), we say \(w\) is in the Muckenhoupt class \(A_p\), denoted by \(w \in A_p\), if for every cube \(Q\),

\[ \left( \int_Q w \, dx \right) \left( \int_Q w^{1-p'} \, dx \right)^{p-1} \leq C < \infty. \]

The smallest such constant \(C\) is denoted by \([w]_{A_p}\). The \(A_p\) classes are nested: \(A_p \subset A_q\) if \(p < q\). Hence we can define \(A_\infty\) as the union of all the \(A_p\) classes, and define \(r_w = \inf \{ p : w \in A_p \}\). For \(s > 1\), we say that \(w\) satisfies a reverse Hölder inequality with exponent \(s\), denoted by \(w \in RH_s\), if for every cube \(Q\),

\[ \left( \int_Q w^s \, dx \right) \leq C \int_Q w \, dx. \]

The infimum of all the constants for which this is true is denoted by \([w]_{RH_s}\). A weight is in \(A_p\) for some \(p > 1\) if and only if it is in \(RH_s\) for some \(s > 1\).

**Theorem 1.1.** Let \(0 < \gamma < mn\). Given \(0 < p_1, \ldots, p_m < \infty\), define \(0 < p < \infty\) by

\[ \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} > \frac{\gamma}{n}, \tag{1.3} \]

and define \(0 < q < \infty\) by

\[ \frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}. \tag{1.4} \]
Suppose that \((w_1, \ldots, w_m)\) is a vector of weights satisfying \(w_i \in RH_{p_i}^q\). If \(K_{\gamma}\) satisfies (1.1) and (1.2) for some positive integer \(N > \max\{mn^{(\frac{r_{w_i}}{p_i}} - 1), 1 \leq i \leq m\}\), then

\[
\|T_{\gamma}(f_1, \ldots, f_m)\|_{L^q(w)} \lesssim m \prod_{i=1}^m \|f_i\|_{H^{p_i}(w_i)},
\]

where

\[
w = \prod_{i=1}^m w_i^{\frac{1}{p_i}}.
\]

**Remark 1.2.** Even in the unweighted case Theorem 1.1 is a more general result than that of Lin and Lu, since we extend the values of \(\gamma\) to the full range \(0 < \gamma < mn\).

Below, we will prove Theorem 1.1 as a special case of a more general result, Theorem 3.1. It is more complicated to state, since it requires the existence of certain \(q_i > p_i\) such that \(w_i \in RH_{p_i}^{q_i}\). However, this result has the advantage that it respects the product structure in the multilinear setting, in that we do not have to assume an identical condition on each weight \(w_i\). This phenomenon does not appear in the diagonal case for multilinear singular integrals considered in [7], but it does play a role in the conditions for multilinear multipliers given in [10].

As an application of our weighted estimates we extend our results to the variable exponent setting. Variable exponent spaces are generalizations of the classical \(L^p\) and \(H^p\) spaces where the constant exponent \(p\) is replaced by an exponent function \(p(\cdot)\). Intuitively, \(L^p(\cdot)\) consists of all functions such that

\[
\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx < \infty.
\]

Harmonic analysis has been extensively studied on these spaces: see [1] for the history and detailed references. The theory of variable exponent Hardy spaces \(H^{p(\cdot)}\) was introduced in [11]. Our second main result is Theorem 1.3. For brevity, we defer some technical definitions to Section 4.

**Theorem 1.3.** Given \(0 < \gamma < mn\), let \(p_i(\cdot) \in P_0\), \(1 \leq i \leq m\), be log-Hölder continuous exponent functions such that \(0 < [p_i(\cdot)]_- \leq [p_i(\cdot)]_+ < \infty\) and

\[
\frac{1}{[p_1(\cdot)]_+} + \cdots + \frac{1}{[p_m(\cdot)]_+} > \frac{\gamma}{n}.
\]

Define \(q(\cdot)\) by

\[
\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \cdots + \frac{1}{p_m(\cdot)} - \frac{\gamma}{n};
\]

then

\[
\|T_{\gamma}(f_1, \ldots, f_m)\|_{L^{q(\cdot)}} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}(\cdot)}.
\]
The key tool in the proof Theorem 1.3 is a multilinear, off-diagonal version of Rubio de Francia extrapolation, Theorem 4.1, which is of interest in its own right. This result generalizes earlier multilinear extrapolation theorems into the scale of variable exponent spaces \([7, 8]\) and also the multilinear extrapolation theory in [3].

The remainder of this paper is organized as follows. In Section 2 we gather some definitions and preliminary results needed in the proof of Theorem 1.1. In Section 3 we prove Theorem 1.1. Our proof draws upon ideas from [7, 10], but significant modifications were required to handle the fractional nature of the kernel. Finally, in Section 4 we give the necessary definitions and prove Theorems 4.1 and 1.3.

Throughout this paper, we will use \(n\) to denote the dimension of the underlying space, \(\mathbb{R}^n\), and will use \(m\) to denote the “dimension” of our multilinear operators. By a cube \(Q\) we will always mean a cube whose sides are parallel to the coordinate axes, and for \(\tau > 1\) let \(\tau Q\) denote the cube with same center such that \(\ell(\tau Q) = \tau \ell(Q)\). In particular, let \(Q^* = 2\sqrt{n}Q\) and \(Q^{**} = (Q^*)^*\). By \(C, c, \) etc. we will mean constants that may depend on the underlying parameters in the problem. The values of these constants may change from line to line. If we write \(A \lesssim B\), we mean that \(A \leq cB\) for some constant \(c\).

2. Preliminary results

For \(0 \leq \gamma < n\), the fractional maximal function \(M_\gamma\) is defined by

\[
M_\gamma f(x) = \sup_Q \ell(Q)^\gamma \left( \frac{1}{Q} \int_Q |f(y)| dy \right) \chi_Q(x).
\]

When \(\gamma = 0\) we get the Hardy-Littlewood maximal operator and write \(Mf\) instead of \(M_0 f\).

For \(1 < p < \infty\) and \(1 < r < \infty\), given \(w \in A_p\) we have the Fefferman-Stein inequality:

\[
\left\| \left( \sum_k M(f_k)^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \lesssim \left\| \left( \sum_k |f_k|^{r'} \right)^{\frac{1}{r'}} \right\|_{L^p(w)}.
\]

A similar result holds for \(M_\gamma\). Given \(0 < \gamma < n\), \(1 < p < \frac{n}{\gamma}\) and \(\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}\), we say \(w \in A_{p,q}\) if for all cubes \(Q\),

\[
\left( \int_Q w^q \, dx \right)^{\frac{1}{q}} \left( \int_Q w^{-p'} \, dx \right)^{\frac{1}{p'}} \leq C < \infty.
\]

Muckenhoupt and Wheeden [16] showed that if \(w \in A_{p,q}\), then

\[
\|M_\gamma f\|_{L^q(w^q)} \lesssim \|f\|_{L^p(w^p)}.
\]

As a consequence of the off-diagonal Rubio de Francia extrapolation [4, Theorem 3.23], we have that for \(1 < r < \infty\) and \(w \in A_{p,q}\),

\[
\left( \sum_k M_\gamma(f_k)^r \right)^{\frac{1}{r}} \|_{L^q(w^q)} \lesssim \left( \sum_k |f_k|^{r'} \right)^{\frac{1}{r'}} \|_{L^p(w^p)}.
\]
For $\gamma > 0$, we have that
\begin{equation}
\ell(Q)^\gamma \chi_{Q^*} \lesssim M_{\gamma \delta}(\chi_Q)^\frac{1}{\delta}
\end{equation}
for all $0 < \delta \leq 1$. If we combine this estimate with (2.2) we get the following vector-valued estimate.

**Lemma 2.1.** Given $0 < \gamma < \infty$ and $0 < p < \frac{n}{\gamma}$, define $q > 0$ by $\frac{1}{q} = \frac{1}{p} - \frac{2}{n}$. Then for any $w \in RH_{\frac{q}{p}}$,
\[ \left\| \sum_j \lambda_j \ell(Q_j)^\gamma \chi_{Q_j}^* \right\|_{L^q(w^{\frac{q}{p}})} \lesssim \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^p(w)}, \]
where $\lambda_j > 0$ and $\{Q_j\}_j$ is any sequence of cubes.

**Remark 2.2.** Lemma 2.1 was first proved in [20] when $1 < p < n/\gamma$ in a two weight setting. Our proof is much simpler. For another proof that also uses extrapolation but avoids the vector-valued inequality see [6, Lemma 4.9].

**Proof.** For each $\delta \in (0, p)$, set $q_{\delta} = q/\delta$, $p_{\delta} = p/\delta$ and $u_{\delta} = w^{\frac{1}{p_{\delta}}}$. Since $w \in RH_{\frac{q}{p}}$, there exists $\delta > 0$ sufficiently small so that
\[ u_{\delta}^{q_{\delta}} = w^{\frac{q}{p}} \in A_{1+\frac{q_{\delta}}{p_{\delta}}} . \]
Therefore, it follows from the definitions that $u_{\delta} \in A_{p_{\delta}, q_{\delta}}$. Then we can apply inequalities (2.3) and (2.2) to get that
\[ \left\| \sum_j \lambda_j \ell(Q_j)^\gamma \chi_{Q_j}^* \right\|_{L^q(w^{\frac{q}{p}})} \lesssim \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^p(w)} . \]
\[ \square \]

**Lemma 2.3.** Let $\gamma, p,$ and $q$ be real numbers as in Lemma 2.1 and suppose $w \in RH_{\frac{q}{p}} \cap A_r$ for some $r > 1$. Then for $\epsilon > \max(\frac{nr}{p}, n)$ and any sequence $\{Q_j\}_j$ of cubes,
\[ \left\| \sum_j \lambda_j \ell(Q_j)^\gamma \chi_{Q_j}^* \right\|_{L^q(w^{\frac{q}{p}})} \lesssim \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^p(w)}, \]
where $\lambda_j > 0$ and $c_j$ is the center of the cube $Q_j$. 

Proof. For each \( j \), we first decompose \( \mathbb{R}^n \setminus Q_j^* \) into annuli and then into non-overlapping cubes \( R_{j}^{kl} \) such that
\[
\mathbb{R}^n \setminus Q_j^* = \bigcup_{l=0}^{\infty} \bigcup_{k=1}^{3^n-1} R_{j}^{kl}
\]
and such that \( |x - c_j| \approx 3^l \ell(Q_j) \approx \ell(R_{j}^{kl}) \) for all \( x \in R_{j}^{kl} \) and all \( 1 \leq h \leq 3^n - 1 \). Consequently, for any fixed \( s > 0 \),
\[
|x - c_j|^{-s} \chi(Q_j^*)^c(x) \approx \sum_{l=0}^{\infty} \sum_{k=1}^{3^n-1} (3^l \ell(Q_j))^{-s} \chi_{R_{j}^{kl}}(x).
\]
(2.4) Then by Lemma 2.1 and the equivalence (2.4) we have that
\[
\left\| \sum_{j} \lambda_j \frac{\ell(Q_j)^s \chi(Q_j^*)^c}{|x - c_j|^t - \gamma} \right\|_{L^p(w^{\frac{t}{p}})} \lesssim \left\| \sum_{l=0}^{\infty} \sum_{k=1}^{3^n-1} \lambda_j 3^{-l} \ell(R_{j}^{kl})^s \chi_{R_{j}^{kl}} \right\|_{L^p(w^{\frac{s}{p}})}
\]
(3.1) \( \lesssim \left\| \sum_{l=0}^{\infty} \sum_{k=1}^{3^n-1} \lambda_j 3^{-l} \chi_{R_{j}^{kl}} \right\|_{L^p(w)}
\]
(3.2) \( = \left\| \sum_{j} \lambda_j \ell(Q_j)^s \sum_{l=0}^{\infty} \sum_{k=1}^{3^n-1} (3^l \ell(Q_j))^{-s} \chi_{R_{j}^{kl}} \right\|_{L^p(w)}
\]
(3.3) \( \lesssim \left\| \sum_{j} \lambda_j \ell(Q_j)^s |x - c_j|^{-s} \chi(Q_j^*)^c \right\|_{L^p(w)}
\)
(3.4) \( \lesssim \left\| \sum_{j} \lambda_j M(\chi Q_j)^{\frac{s}{\gamma}} \right\|_{L^p(w)}
\)
(3.5) \( \lesssim \left\| \left( \sum_{j} \lambda_j M(\chi Q_j)^{\frac{s}{\gamma}} \right)^{\frac{\gamma}{s}} \right\|_{L^{\frac{sp}{s+p}}(w)}
\)
(3.6) \( \lesssim \left\| \sum_{j} \lambda_j \chi Q_j \right\|_{L^p(w)}.
\)

The last inequality holds because by our choice of \( \epsilon, \frac{sp}{s+p} > r \), so \( w \in A_{\frac{sp}{s+p}} \) and we can apply inequality (2.1). \( \square \)

3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1, and as we said in the introduction, we actually prove a more general result.

Theorem 3.1. Given \( 0 < \gamma < mn \) and \( 0 < p_1, \ldots, p_m < \infty \), define \( p \) as in (1.3) and \( q \) as in (1.4). Suppose that \( q_i \) are such that \( p_i < q_i < \infty \), \( 1 \leq i \leq m \), and \( \frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q} \), and \((w_1, \ldots, w_m)\) are weights such that \( w_i \in RH_{\frac{p_i}{q_i}} \). If \( K_\gamma \) satisfies
(1.1) and (1.2) for some positive integer }N > \max\{mn\left(\frac{r_{wi}}{p_{i}} - 1\right), 1 \leq i \leq m\}, then
\|T_\gamma(f_1, \ldots, f_m)\|_{L^q(\Omega)} \lesssim \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(w_i)},
where
\bar{w} = \prod_{i=1}^{m} w_{i}^{\frac{1}{p_i}}.

**Remark 3.2.** Theorem 1.1 follows from Theorem 3.1 by taking }q_i = \frac{q}{p_i}.

**Proof.** Recall that for }w \in A_\infty, H^{p_i}(w) is defined as the set of all distributions }f such that \mathcal{M}_{N_0} f \in L^p(w); here }N_0 is some large constant whose precise value does not matter, though it will be implicit in our constants. For more information, see [19]. Let }N be the positive integer as in the hypotheses of Theorem 1.1. Define
\mathcal{O}_N = \{ f \in \mathcal{C}_0^\infty : \int_{\mathbb{R}^n} x^\beta f(x) \, dx = 0, \quad 0 \leq |\beta| \leq N \}.

Then }\mathcal{E}_i = \mathcal{O}_N \cap H^{p_i}(w_i) is dense in }H^{p_i}(w_i) for all }1 \leq i \leq m (see [7, 19]). As proved in [7, Theorem 2.6] for each }f_i \in \mathcal{E}_i, 1 \leq i \leq m, we have a finite atomic decomposition:
(3.1)
\begin{equation}
f_i = \sum_{k_i} \lambda_{i,k_i} a_{i,k_i},
\end{equation}
where }\lambda_{i,k_i} > 0, |a_i| \leq \chi_{Q_{i,k_i}} for some cube }Q_{i,k_i}, \int x^\alpha a_{i,k_i} \, dx = 0 for all }|\alpha| \leq N, and
(3.2)
\| \sum_{k_i} \lambda_{i,k_i} \chi_{Q_{i,k_i}} \|_{L^{p_i}(w_i)} \lesssim \| f_i \|_{H^{p_i}(w_i)}.

By a standard density argument, it will suffice to show that inequality (1.5) holds for }f_i of the form (3.1). Define }0 < \gamma_i < \infty by
\begin{equation}
\frac{\gamma_i}{n} = \frac{1}{p_i} - \frac{1}{q_i}, \quad 1 \leq i \leq m.
\end{equation}
From the hypothesis (1.4) we get
(3.3)
\begin{equation}
\sum_{i=1}^{m} \gamma_i = \gamma, \quad 0 < \gamma_i < \frac{n}{p_i}, \quad 1 \leq i \leq m.
\end{equation}
By the multilinearity of }T_\gamma we get that
\begin{equation}
T_\gamma(f_1, \ldots, f_m)(x) = \sum_{k_1, \ldots, k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T_\gamma(a_{1,k_1}, \ldots, a_{m,k_m})(x).
\end{equation}
Given a cube }Q and }\bar{k} = (k_1, \ldots, k_m), define
\begin{equation}
E_{\bar{k}} = \cap_{i=1}^{m} Q_{i,k_i}^*, \quad F_{\bar{k}} = \mathbb{R}^n \setminus E_{\bar{k}}.
\end{equation}
Then we can decompose \( T_\gamma(f_1, \ldots, f_m) = G_1 + G_2 \) where

\[
G_1 = \sum_{k_1, \ldots, k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T_\gamma(a_{1,k_1}, \ldots, a_{m,k_m}) \chi_{E_k^*};
\]

\[
G_2 = \sum_{k_1, \ldots, k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T_\gamma(a_{1,k_1}, \ldots, a_{m,k_m}) \chi_{F_k^*}.
\]

To estimate the first term, we may assume that \( E_k^* \) is not empty. With \( \gamma_i \) as defined by (3.3) we can estimate \( T_\gamma(a_{1,k_1}, \ldots, a_{m,k_m})(x) \) for all \( x \in E_k^* \) as follows:

\[
|T_\gamma(a_{1,k_1}, \ldots, a_{m,k_m})(x)| \leq \int_{\mathbb{R}^m} \frac{\chi_{Q_{1,k_1}}(y_1) \cdots \chi_{Q_{m,k_m}}(y_m) \, dy}{(|x - y_1| + \cdots + |x - y_m|)^{mn - \gamma}}
\]

\[
\leq \prod_{i=1}^m \left( \int_{Q_{i,k_i}} |x - y_i|^\gamma \, dy_i \right) \chi_{Q_{i,k_i}^*}(x)
\]

(3.4)

\[
\lesssim \prod_{i=1}^m \ell(Q_{i,k_i})^\gamma \chi_{Q_{i,k_i}^*}(x).
\]

We can now estimate the quasi-norm of \( G_1 \) as follows:

\[
\|G_1\|_{L^q(\mathbb{R})} = \left\| \sum_{k_1, \ldots, k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T_\gamma(a_{1,k_1}, \ldots, a_{m,k_m}) \chi_{E_k^*} \right\|_{L^q(\mathbb{R})}
\]

\[
\lesssim \left\| \sum_{k_1, \ldots, k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} \prod_{i=1}^m \ell(Q_{i,k_i})^\gamma \chi_{Q_{i,k_i}^*} \right\|_{L^q(\mathbb{R})}
\]

\[
= \left\| \prod_{i=1}^m \left( \sum_{k_i} \lambda_{i,k_i} \ell(Q_{i,k_i})^\gamma \chi_{Q_{i,k_i}^*} \right) \right\|_{L^q(\mathbb{R})}
\]

(3.5)

\[
\leq \prod_{i=1}^m \left\| \sum_{k_i} \lambda_{i,k_i} \ell(Q_{i,k_i})^\gamma \chi_{Q_{i,k_i}^*} \right\|_{L^q(w_i/\ell)}
\]

(3.6)

\[
\lesssim \prod_{i=1}^m \left\| \sum_{k_i} \lambda_{i,k_i} \chi_{Q_{i,k_i}} \right\|_{L^p(w_i)};
\]

for inequality (3.5) we used Hölder’s inequality, and for (3.6) we used Lemma 2.1.

To estimate \( G_2 \), fix \( x \in F_k^* \). Then there exists a non-empty subset \( \Lambda \) of \( \{1, \ldots, m\} \) such that \( x \notin Q_{i,k_i}^* \) for all \( i \in \Lambda \) and \( x \in Q_{j,k_j}^* \) for all \( 1 \leq j \leq m, j \notin \Lambda \). Let \( Q_{i_0,k_{i_0}} \), for some \( i_0 \in \Lambda \), be the cube with smallest length among \( Q_{i,k_i}, i \in \Lambda \) and let \( c_{i_0,k_{i_0}} \) be the center of the cube \( Q_{i,k_i} \). Note that since \( x \notin Q_{i_0,k_{i_0}}^* \), \( |x - y_0| \lesssim |x - c_{i_0,k_{i_0}}| \) for all \( y_{i_0} \in Q_{i_0,k_{i_0}} \). Let

\[
P_N(x, y_1, \ldots, c_{i_0,k_{i_0}}, \ldots, y_m) = \sum_{|\beta| < N} \frac{\partial^\beta}{\beta!} K_\gamma(x, y_1, \ldots, c_{i_0,k_{i_0}}, \ldots, y_m) \left( y_{i_0} - c_{i_0,k_{i_0}} \right)^\beta
\]
be the Taylor polynomial of $K_r$. Then the cancellation conditions satisfied by the atoms $a_{i_0, k_0}$ and the smoothness of $K_r$ in (1.2) imply that

$$
|T_r(a_{1,k_1}, \ldots, a_{m,k_m})(x)|
\leq \int |K_r(x, y_1, \ldots, y_m) - P_N(x, y_1, \ldots, c_{i_0, k_0}, \ldots, y_m)| \prod_{i=1}^m |a_{i,k_i}(y_i)| \, dy
$$

$$\leq \int \frac{\ell(Q_{i_0,k_0})^N}{(|x-y_1| + \cdots + |x-y_m|)^{m\gamma}} \prod_{i=1}^m |a_{i,k_i}(y_i)| \, dy
$$

$$\leq \prod_{i \in A} \int \frac{\ell(Q_{i,k_i})^N}{|x-c_{i,k_i}|^{\gamma}} \prod_{i \notin A} \int \frac{\ell(Q_{i,k_i})^N}{|x-y_i|^{\gamma}} \, dy
$$

$$\leq \prod_{i \in A} \int \frac{\ell(Q_{i,k_i})^N}{|x-c_{i,k_i}|^{\gamma}} \prod_{i \notin A} \int \frac{\ell(Q_{i,k_i})^N}{|x-y_i|^{\gamma}} \, dy
$$

(3.7)

$$\leq \prod_{i \in A} \int \frac{\ell(Q_{i,k_i})^N}{|x-c_{i,k_i}|^{\gamma}} \prod_{i \notin A} \int \frac{\ell(Q_{i,k_i})^N}{|x-y_i|^{\gamma}} \, dy
$$

where $\epsilon_N = n + \frac{N}{m}$. (For details of this calculation, see [7, Lemma 3.6].) Since $w_i \in RH_{p_i} \subset A_{\infty}$, $1 \leq i \leq m$, for all $r_i > r_{w_i}, w_i \in A_{r_i}$. By our assumption on $N$ in the hypotheses, we can choose $r_i$ close enough to $r_{w_i}$ so that

$$\epsilon_N = n + \frac{N}{m} > \frac{r_i}{p_i}.$$

If we combine the above estimates we get

$$
\|G_2\|_{L^q(w)} \leq \left\| \sum_{k_1, \ldots, k_m} \lambda_{k_1} \cdots \lambda_{k_m} |T_r(a_{1,k_1}, \ldots, a_{m,k_m})| \chi_{F_k} \right\|_{L^q(w)}
$$

$$\leq \left\| \sum_{k_1, \ldots, k_m} \prod_{i \in A} \frac{\lambda_{i,k_i} \ell(Q_{i,k_i})^{\gamma}}{|x-c_{i,k_i}|^{\gamma}} \prod_{i \notin A} \int \frac{\ell(Q_{i,k_i})}{|x-y_i|^{\gamma}} \, dy \right\|_{L^q(w)}.
$$

By Hölder’s inequality, and Lemmas 2.1 and 2.3 we get

$$
\|G_2\|_{L^q(w)} \leq \prod_{i \in A} \left\| \sum_{k_i} \lambda_{i,k_i} \ell(Q_{i,k_i})^{\gamma} \chi(Q_{i,k_i}) \right\|_{L^q(w_i^{q_i/p_i})}
$$

$$\times \prod_{i \notin A} \left\| \lambda_{i,k_i} \ell(Q_{i,k_i})^{\gamma} \chi Q_{i,k_i} \right\|_{L^q(w_i^{q_i/p_i})}
$$

(3.8)

$$\leq \prod_{i=1}^m \left\| \sum_{k_i} \lambda_{i,k_i} \chi Q_{i,k_i} \right\|_{L^p_i(w_i)}.$$
Combining (3.6) and (3.8) we get

\[ \| T_\gamma(f_1, \ldots, f_m) \|_{L^q(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \left\| \sum_{k_i} \lambda_{i,k_i} \chi_{Q_{i,k_i}} \right\|_{L^{p_i}(w_i)}, \]

which, when combined with (3.2), gives us the desired estimate for \( T_\gamma. \)

**Remark 3.3.** If \( 0 < \gamma < (m-l)n \) for some \( 1 \leq l < m, \) then we can allow at most \( l \) exponents among the \( \{p_1, \ldots, p_m\} \) to be infinite and the conclusion of Theorem 3.1 is still true, replacing \( H^{p_i}(w_i) \) with \( L^\infty. \) To see this, first note that we may assume that \( p_{m-l+1} = \cdots = p_m = \infty. \) Then we can integrate in \( y_{m-l+1}, \ldots, y_m \) to estimate (3.7) as follows:

\[
\begin{align*}
|T_\gamma(a_{1,k_i}, \ldots, a_{m-l,k_{m-l}}, f_{m-l+1}, \ldots, f_m)(x)| & \leq \int_{\mathbb{R}^{mn}} \chi_{Q_{1,k_i}}(y_1) \cdots \chi_{Q_{m-l,k_{m-l}}}(y_{m-l}) (y_{m-l}) f_{m-l+1}(y_{m-l+1}) \cdots f_m(y_m) \, d\bar{y} \\
& \leq \int_{\mathbb{R}^{nl}} \chi_{Q_{1,k_i}}(y_1) \cdots \chi_{Q_{m-l,k_{m-l}}}(y_{m-l}) \, dy_1 \cdots dy_l \left( |x - y_1| + \cdots + |x - y_l| \right)^{n-l-\gamma} \prod_{i=l+1}^m \| f_i \|_{L^\infty} \\
& \leq \prod_{i=1}^l \left( \int_{Q_{i,k_i}} |x - y_i|^{n-l} \, dy_i \right) \chi_{Q_{i,k_i}}(x) \prod_{i=l+1}^m \| f_i \|_{L^\infty} \\
& \lesssim \prod_{i=1}^l \ell(Q_{i,k_i})^{n-l} \chi_{Q_{i,k_i}}(x) \prod_{i=l+1}^m \| f_i \|_{L^\infty}.
\end{align*}
\]

If we now repeat the argument in the proof of Theorem 1.1, we get

\[ \| T_\gamma(f_1, \ldots, f_m) \|_{L^q(\mathbb{R}^n)} \lesssim \prod_{i=1}^l \| f_i \|_{H^{p_i}(w_i)} \prod_{i=l+1}^m \| f_i \|_{L^\infty}. \]

In their work, Lin and Lu [15, Theorem 2.1] assumed an unweighted estimate similar to this one when \( l = m - 1. \) This helps to explain the restriction \( 0 < \gamma < n \) in their results.

## 4. Boundedness on Variable Hardy Spaces

In this section, we state and prove the analogue of Theorem 1.1 on the variable exponent Hardy spaces, \( H^{p(\cdot)}(\cdot). \) To do so, we first recall some basic facts about the variable exponent Lebesgue and Hardy spaces. For complete background information we refer the reader to [1].

Let \( P_0(\mathbb{R}^n) \) be the set of all measurable functions \( p(\cdot) : \mathbb{R}^n \to (0, \infty). \) Define

\[
[p(\cdot)]_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x), \quad [p(\cdot)]_+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x).
\]
Given $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ define $L^{p(\cdot)} = L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of all measurable functions $f$ such that for some $\lambda > 0$,  

$$
\rho(f/\lambda) = \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty.
$$

This becomes a quasi-Banach space with the quasi-norm $$
\|f\|_{L^{p(\cdot)}} = \inf \{ \lambda > 0 : \rho(f/\lambda) \leq 1 \}.
$$

If $[p(\cdot)]_- \geq 1$, then $\|\cdot\|_{L^{p(\cdot)}}$ is a norm and $L^{p(\cdot)}$ is a Banach space. For all $p > 0$, if $p(\cdot) = p$ a constant, then $L^{p(\cdot)} = L^p$ with equality of norms, so the variable exponent Lebesgue spaces are a generalization of the classical $L^p$ spaces.

Let $\mathcal{B}$ denote the collection of exponents $p(\cdot)$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$. A sufficient (but not necessary) condition for $p(\cdot) \in \mathcal{B}$ is that $1 < [p(\cdot)]_- \leq [p(\cdot)]_+ < \infty$ and $p(\cdot)$ is log-Hölder continuous: there exist constants $C_0, C_\infty$ and $p_\infty$ such that  

$$
|p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad 0 < |x - y| < \frac{1}{2},
$$

and  

$$
|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.
$$

Given $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, the variable Hardy space $H^{p(\cdot)}$ is defined to be the set of all distributions $f$ such that $\mathcal{M}_{N_0}f \in L^{p(\cdot)}$. Again, we here assume $N_0 > 0$ is a sufficiently large integer so that all the standard definitions of the classical Hardy spaces are equivalent in $H^{p(\cdot)}$. For further details, see [7, 11].

We will prove norm inequalities in the variable exponent Lebesgue and Hardy spaces from the corresponding weighted norm inequalities by applying the theory of Rubio de Francia extrapolation. In the linear case this approach was introduced in [1, 2, 5], and multilinear versions of extrapolation into the variable Lebesgue spaces were proved in [7, 8]. Here we need a generalization of these results similar to the multilinear extrapolation theorem proved in [3]. We state our extrapolation results in terms of extrapolation $(m+1)$-tuples; for more on this approach, see [5, 8].

**Theorem 4.1.** Let $\mathcal{F} = \{(f_1, \ldots, f_m, F)\}$ be a family of $(m+1)$-tuples of non-negative, measurable functions on $\mathbb{R}^n$. Given $0 < \gamma < \alpha n$ and exponents $0 < p_i < \infty$, $1 \leq i \leq m$, such that (1.3) holds, define $q > 0$ by (1.4). Suppose that for all exponents $p_i < q_i < \infty$ such that  

$$
\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m},
$$

and for all weights $w_i \in RH_{q_i/p_i}$, with $\overline{w} = \prod_{i=1}^m w_i^{q_i/p_i}$, we have that  

$$
\|F\|_{L^q(\overline{w})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}
$$

for all $(f_1, \ldots, f_m, F) \in \mathcal{F}$ such that $F \in L^q(\overline{w})$, where the implicit constant depends only on $n$, $p_i$, and $[w_i]_{RH_{q_i/p_i}}$. 


Let $p_i(\cdot) \in \mathcal{P}_0$, $1 \leq i \leq m$, be such that each $p_i(\cdot)$ is log-Hölder continuous, $p_i < [p_i(\cdot)]_- \leq [p_i(\cdot)]_+ < \infty$, and

\begin{equation}
\frac{1}{[p_1(\cdot)]_+} + \cdots + \frac{1}{[p_m(\cdot)]_+} > \frac{\gamma}{n}.
\end{equation}

Define $q(\cdot)$ by

\begin{equation}
\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \cdots + \frac{1}{p_m(\cdot)} - \frac{\gamma}{n}.
\end{equation}

Then for all $(f_1, \ldots, f_m, F) \in \mathcal{F}$ such that $\|F\|_{L^q(\cdot)} < \infty$,

\begin{equation}
\|F\|_{L^q(\cdot)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}}.
\end{equation}

The implicit constant only depends on $n$, $[p_i(\cdot)]_-$, $[p_i(\cdot)]_+$, and the log-Hölder constants of $p_i(\cdot)$.

**Remark 4.2.** It will be clear from the proof that we can weaken the hypothesis that each $p_i(\cdot)$ is log-Hölder continuous, and instead assume that the maximal operator is bounded on a certain family of variable exponent Lebesgue spaces. Details are left to the interested reader.

**Remark 4.3.** Theorem 4.1 is stated so that its hypotheses coincide with the weighted results in Theorem 3.1. We can also prove an extrapolation theorem starting from the weaker conclusion given in Theorem 1.1. The proof below can be modified, but we need to assume that the exponents $p_i(\cdot)$ have bounded oscillation: more precisely, that

\[ p_i < [p_i(\cdot)]_- \leq [p_i(\cdot)]_+ < \frac{q p_i}{q - p}. \]

Details of this result are left to the interested reader. The fact that we can remove this upper bound on $[p_i(\cdot)]_+$ is another reason for proving the stronger result in Theorem 3.1.

**Proof.** For our proof we need a family of Rubio de Francia iteration algorithms. To construct them, we will define some exponent functions and show that the maximal operator is bounded on the associated variable Lebesgue space. By (4.2), for each $i$, $1 \leq i \leq m$, we can fix $\gamma_i > 0$ such that

\[ \gamma = \sum_{i=1}^n \gamma_i, \]

and $[p_i(\cdot)]_+ < n/\gamma_i$. Define $q_i > 0$ by

\[ \frac{1}{p_i} - \frac{1}{q_i} = \frac{\gamma_i}{n} \]

and define the variable exponents $q_i(\cdot)$ by

\[ \frac{1}{p_i(\cdot)} - \frac{1}{q_i(\cdot)} = \frac{\gamma_i}{n}. \]
But then by (4.3) we have that

\[
\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)},
\]

and so

\[
\frac{1}{[q(\cdot)]_+} \leq \sum_{i=1}^{m} \frac{1}{[p_i(\cdot)]_+} - \frac{\gamma}{n} < \sum_{i=1}^{m} \frac{1}{p_i} - \frac{\gamma}{n} = \frac{1}{q}.
\]

Therefore, if we define \(\overline{q}(\cdot) = q(\cdot)/q\), then \([\overline{q}(\cdot)]_+ > 1\). Similarly, if we define \(\overline{p}_i(\cdot) = p_i(\cdot)/p_i\), then \([\overline{p}_i(\cdot)]_+ > 1\).

Now let \(\sigma_i(\cdot) = \frac{\overline{q}_i}{\overline{p}_i}(\cdot)\). We claim that \([\sigma_i(\cdot)]_+ > 1\). However, this inequality follows from some standard estimates for dual exponents in the variable exponent Lebesgue spaces [1, p. 14]: this inequality is equivalent to \([\overline{p}_i(\cdot)]_+ > \frac{\overline{q}_i}{\overline{p}_i}\), which in turn is equivalent to \([\overline{p}_i(\cdot)]_+ > \frac{\overline{q}_i}{\overline{p}_i}\), and this in turn is equivalent to

\[
[p_i(\cdot)]_+ < p_i \left(\frac{q_i}{p_i}\right) = \frac{n}{\gamma_i},
\]

which we know to hold.

We also have that each \(\sigma_i(\cdot)\) is log-Hölder continuous since each \(p_i(\cdot)\) is. Therefore, the Hardy-Littlewood maximal operator is bounded on \(L^{\sigma_i(\cdot)}\). Hence, we can define the iteration operator \(\mathcal{R}_i\), acting on non-negative functions \(h\), by

\[
\mathcal{R}_i h(x) = \sum_{j=0}^{\infty} \frac{M^j h(x)}{2^j \|M\|^{2}_{L^{\sigma_i(\cdot)}}},
\]

where \(M^j = M \circ \cdots \circ M\) is \(j\) iterates of the maximal operator, and \(M^0 h = h\). Then by a standard argument [1, p. 210] and a rescaling property of \(A_1 \cap RH\) weights [9], we have the following:

1. \(h(x) \leq \mathcal{R}_i h(x)\);
2. \(\|\mathcal{R}_i h\|_{L^{\sigma_i(\cdot)}} \leq 2 \|h\|_{L^{\sigma_i(\cdot)}}\);
3. \(\mathcal{R}_i h \in A_1\), and \(\|\mathcal{R}_i h\|_{A_1} \leq 2 \|h\|_{L^{\sigma_i(\cdot)}}\);
4. \((\mathcal{R}_i h)^{p_i/q_i} \in A_1 \cap RH_{q_i/p_i}\), and \([(\mathcal{R}_i h)^{p_i/q_i}]_{RH_{q_i/p_i}}\) depends only on \([\mathcal{R}_i h]_{A_1}\).

Define a family of auxiliary exponents \(\theta_i\), \(1 \leq i \leq m\), by

\[
\theta_i(\cdot) = \frac{q(\cdot)}{p_i \overline{p}_i(\cdot)}.
\]

Then

\[
\sum_{i=1}^{m} \theta_i(\cdot) = q \overline{q}(\cdot) \sum_{i=1}^{m} \frac{1}{p_i \overline{p}_i(\cdot)} = q \overline{q}(\cdot) \sum_{i=1}^{m} \frac{1}{p_i} \left(1 - \frac{p_i}{\overline{p}_i(\cdot)}\right)
\]

\[
= q \overline{q}(\cdot) \sum_{i=1}^{m} \left(\frac{1}{p_i} - \frac{1}{p_i \overline{p}_i(\cdot)}\right) = \overline{q}(\cdot) - \frac{\overline{q}(\cdot)}{\overline{q}(\cdot)} = 1.
\]

We can now prove the desired inequality. Fix \((f_1, \ldots, f_m, F) \in \mathcal{F}\) such that \(F \in L^{\theta(\cdot)}\). Since \(\overline{q}(\cdot) > 1\), by rescaling and the associate norm in variable exponent
Lebesgue spaces [1, Prop. 2.18, Thm. 2.34], there exists $h \in L^\varphi(\cdot)$, $\|h\|_{L^\varphi(\cdot)} = 1$, such that

\begin{equation}
(4.6) \quad \|F\|_{L^q(\cdot)}^q = \|F\|^q_{L^\varphi(\cdot)} \lesssim \int_{\mathbb{R}^n} F^q h \, dx
\end{equation}

\[ = \int_{\mathbb{R}^n} F^q \prod_{i=1}^m h^{\theta_i(\cdot)} \, dx \lesssim \int_{\mathbb{R}^n} F^q \prod_{i=1}^m \left[ \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \right] \frac{q}{p_i} \, dx. \]

By construction, we have that for each $i$

\[ \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \in A_1 \cap RH_{q_i/p_i}, \]

Assume for the moment that the last term in the above inequality is finite. If it is, then we can apply our hypothesis (4.1) and the generalized Hölder’s inequality in the scale of variable Lebesgue spaces [1, Theorem 2.26] to get

\[ \int_{\mathbb{R}^n} F^q \prod_{i=1}^m \left[ \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \right] \frac{q}{p_i} \, dx \lesssim \prod_{i=1}^m \left( \int_{\mathbb{R}^n} f_i^{p_i} \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \frac{q}{p_i} \, dx \right) \lesssim \prod_{i=1}^m \| f_i \|_{L^{p_i}(\cdot)}^{p_i} \| h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \|_{L^{q_i}(\cdot)}^{q_i}. \]

Again by rescaling we have that $\| f_i \|_{L^{p_i}(\cdot)} = \| f_i \|_{L^{p_i}(\cdot)}^{q}$. Thus to complete the proof of inequality (4.4) we will show that

\begin{equation}
(4.7) \quad \| \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \|_{L^{q_i}(\cdot)} \lesssim \| \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \|_{L^{q_i}(\cdot)} \lesssim 1.
\end{equation}

By the properties of the iteration operator and rescaling,

\[ \| \mathcal{R}_i \left( h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \right) \|_{L^{q_i}(\cdot)} \lesssim \| h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \|_{L^{q_i}(\cdot)} = \| h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \|_{L^{q_i}(\cdot)}. \]

By the relationship between the norm and the modular in variable exponent spaces [1, Prop. 2.21], since $\|h\|_{L^\varphi(\cdot)} = 1$,

\[ 1 = \int_{\mathbb{R}^n} h(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \left( h(x) \varphi^{\left(\frac{\theta_i(x)}{p_i}\right)} \right) \varphi_i(x) \, dx, \]

and this in turn implies that

\[ \| h_\varphi^{\left(\frac{\theta_i(\cdot)}{p_i}\right)} \|_{L^{q_i}(\cdot)} = 1. \]

Finally, to complete the proof we need to justify our assumption that the last term in (4.6) is finite. If we divide the second and last terms of the identity (4.5) by $\varphi(\cdot)$, we get

\[ 1 = \frac{1}{\varphi(\cdot)} + \sum_{i=1}^m \frac{q}{p_i} \frac{1}{\varphi_i(\cdot)} = \frac{1}{\varphi(\cdot)}. \]
Hence, by the multi-term generalized Hölder’s inequality in variable exponent Lebesgue spaces [1, Cor. 2.30],
\[
\int_{\mathbb{R}^n} F^q \prod_{i=1}^m \left[ \mathcal{R}_i \left( h^{(\frac{1}{p_i})} \right) \right]^{\frac{p_i}{q_i}} \, dx \lesssim \left\| F^q \right\|_{L_{p_i}^{\gamma'(\cdot)}} \prod_{i=1}^m \left[ \mathcal{R}_i \left( h^{(\frac{1}{p_i})} \right) \right]^{\frac{p_i}{q_i}} \left\| F^q \right\|_{L_{p_i}^{\gamma'(\cdot)}}.
\]
By rescaling, the first term becomes \( \| F \|_{L_{p_i}'(\cdot)} \) which is finite, and by rescaling and (4.7) we see that the remaining terms are all uniformly bounded.

Theorem 1.3 follows directly from Theorem 4.1 and a careful density argument.

**Proof of Theorem 1.3.** From condition (4.2) we can find \( p_i > 0 \) such that \( p_i < [p_i(\cdot)]_+ \) and
\[
1/p_1 + \cdots + 1/p_m > \gamma/n.
\]
Therefore, by Theorem 3.1, given \( q_i > p_i \) such that \( w_i \in RH_{q_i/p_i} \), inequality (1.5) holds. We can use this to apply Theorem 1.3 if we can define the appropriate family \( \mathcal{F} \) of extrapolation \((m+1)\)-tuples.

Since each \( p_i(\cdot) \) is log-Hölder continuous and \([p_i(\cdot)]_+ > 0 \), \( 1 \leq i \leq m \), there exists an \( N \) depending only on the \( p_i(\cdot) \) and on \( n \) such that functions of the form
\[
f = \sum_{j=1}^M \lambda_j a_j,
\]
where each \( a_j \) is an \((N, \infty)\) atom, are dense in \( HP_{\cdot}(\cdot) \) [11, Theorem. 6.3]. All such functions are also contained in \( Hp(w) \), for any \( p > 0 \) and \( w \in A_\infty \). Denote the set of such functions by \( \mathcal{A} \). Define the family of \((m+1)\)-tuples \( \mathcal{F} = \{(f_1, \ldots, f_m, F_R)\} \), where \( f_i = M_{N_0} g_i \), \( g_i \in \mathcal{A} \), \( R > 0 \), and
\[
F_R = \min \left( \{ T_{\gamma}(g_1, \ldots, g_m), R \} : \chi_{B(0,R)} \right).
\]
Since \( \| \chi_{B(0,R)} \|_{L_{p_i}(\cdot)} < \infty \) [1, Lemma 2.39], we have that \( \| F_R \|_{L_{p_i}(\cdot)} < \infty \). Further, given any weights \( w_i \in RH_{q_i/p_i} \), and \( \varpi = \prod_{i=1}^m w_i^{q_i/p_i} \), by Hölder’s inequality with exponents \( q_i/q \) we have that
\[
\| F_R \|_{L_{q_i}(\varpi)} \leq R \left( \int_{B(0,R)} \prod_{i=1}^m w_i^{q_i/p_i} \, dx \right)^{1/q} \leq R \prod_{i=1}^m \left( \int_{B(0,R)} w_i^{q_i/p_i} \, dx \right)^{1/q_i} < \infty.
\]
But then by (1.5) we have that given any \((m+1)\)-tuple in \( \mathcal{F} \),
\[
\| F_R \|_{L_{p_i}(\varpi)} \lesssim \prod_{i=1}^m \| g_i \|_{HP_i(w_i)} = \prod_{i=1}^m \| f_i \|_{L_{p_i}(\varpi)} = \prod_{i=1}^m \| g_i \|_{HP_i(i)},
\]
which gives us (4.1). Therefore, by Theorem 4.1,
\[
\| F_R \|_{L_{p_i}(\cdot)} \lesssim \prod_{i=1}^m \| f_i \|_{L_{p_i}(\cdot)} = \prod_{i=1}^m \| g_i \|_{HP_i(\cdot)}.
\]
By Fatou’s lemma in the variable exponent Lebesgue spaces [1, Theorem 2.61],
\[
\|T(g_1, \ldots, g_m)\|_{L^{q(\cdot)}} \leq \liminf_{R \to \infty} \|F_R\|_{L^{q(\cdot)}} \lesssim \prod_{i=1}^{m} \|g_i\|_{H^{q_i(\cdot)}}.
\]

This establishes the desired norm inequality of \( T \) for a dense family of functions, and Theorem 1.3 follows by a standard approximation argument. \( \square \)

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