On the Mahler measure of a family of genus 2 curves

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Abstract We establish a general identity between the Mahler measures \( m(Q_k(x, y)) \) and \( m(P_k(x, y)) \) of two polynomial families, where \( Q_k(x, y) = 0 \) and \( P_k(x, y) = 0 \) are generically hyperelliptic and elliptic curves, respectively.

Keywords Mahler measure · L-value · Elliptic curve · Hyperelliptic curve · Elliptic integral

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1 Introduction

In his 1998 Experimental Mathematics paper [6] Boyd gave many conjectures about (logarithmic) Mahler measures of two-variable polynomials,

\[
m(P(x, y)) = \frac{1}{(2\pi i)^2} \iint_{|x|=|y|=1} \log |P(x, y)| \frac{dx}{x} \frac{dy}{y},
\]

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the conjectures that started the whole area of research on the border of number theory, $K$-theory, algebraic geometry and analysis. One of the exemplar families include evaluations of $m(P_k(x, y))$ for

$$P_k(x, y) = (x + 1)y^2 + (x^2 + kx + 1)y + (x^2 + x), \ k \in \mathbb{Z},$$

which can be characterised by $m(P_k)/L'(E, 0) \in \mathbb{Q}^\times$, where $E : P_k(x, y) = 0$ is (generically) an elliptic curve and $L'(E, 0)$ is the derivative of its $L$-function $L(E, s)$ at $s = 0$. Very few of these are proven so far: $k = 0, 6$ (conductor 36) by Rodriguez-Villegas [9], $k = 1, 10, -5$ (conductor 14) by Mellit [8], and $k = 4, -2$ (conductor 20) by Rogers and the second author [10]. Note that these particular cases (as well as all other proven cases of elliptic type, for both CM and non-CM curves) accidentally fall under application of the Mellit–Brunault formula [12].

Boyd also indicated in [6] some families of Mahler measures related to genus 2 curves. One of them was studied in great detail in Bosman’s thesis [5]: namely, he considered the family $m(Q_k(X, Y))$, where

$$Q_k(X, Y) = Y^2 + (X^4 + kX^3 + 2kX^2 + kX + 1)Y + X^4,$$

and showed that the curve $C : Q_k(X, Y) = 0$ has genus 2 for $k \in \mathbb{C} \setminus \{-1, 0, 4, 8\}$; genus 1 for $k = -1$ and 4; genus 0 for $k = 8$; and reducible to two irreducible components of genus 0 for $k = 0$. The principal result of [5] includes the following three evaluations:

$$m(Q_2) = L'(E_{36}, 0), \quad m(Q_{-1}) = 2L'_{(3-1)}, \quad m(Q_8) = 4L'_{(3-4), -1}. \quad (1)$$

In fact, the other genus 1 case $k = 4$ admits a parametrisation by modular units (of level $N = 20$), so that Theorem 1 from [12] applies to produce the evaluation

$$m(Q_4) = 4L'(E_{20}, 0), \quad m(Q_{-1}) = 2L'(3-1), \quad m(Q_8) = 4L'_{(3-4), -1}. \quad (2)$$

From the parametrisation in [5], one can recognise a clear link between the families $Q_k(X, Y)$ and $P_{2-k}(x, y)$. In fact, the Jacobian of $C : Q_k(X, Y) = 0$ generically splits into two elliptic curves, the first being birationally equivalent to $E : P_{2-k}(x, y) = 0$. This gives birth to a relation between the corresponding Mahler measures.

**Theorem 1** The following is true for real values of $k$:

$$m(Q_k) = \begin{cases} 2m(P_{2-k}) & \text{for } 0 \leq k \leq 4, \\ m(P_{2-k}) & \text{for } k \leq -1. \end{cases}$$

Bosman comments in [5] on his proof of evaluations of $m(Q_2)$, $m(Q_{-1})$ and $m(Q_8)$:

“Although I succeeded in proving these identities, I still don’t understand why such identities must be valid. These proofs give a deduction which one can verify step-by-step but they seem to exist coincidentally. It is not at all clear how the arithmetic structure of the polynomials is related to the arithmetic structure of the $L$-series. We will rewrite the integral to so-called dilogarithm sums and these we will rewrite in terms of $L$-series. It is an open problem whether this is always possible. In the elliptic curve case it is not clear at all how to prove a relation between the dilogarithm sums and the $L$-series, except for a few instances.”
We believe that Theorem 1 and its proof below provides us with at least a partial understanding of the magic behind the $L$-series evaluations of Mahler measures. The results in [8–10] and the theorem also lead to different proofs of the formulae for $m(Q_2)$ and $m(Q_{-1})$, as well as to four more equalities conjectured by Boyd in [6].

**Theorem 2** The evaluations (1), (2) and

\[
m(Q_1) = 2m(P_1) = 2L'(E_{14}, 0), \quad \text{where } E_{14}: y^2 + xy + y = x^3 - x,
\]
\[
m(Q_{-2}) = m(P_4) = 3L'(\hat{E}_{20}, 0), \quad \text{where } \hat{E}_{20}: y^2 = x^3 + x^2 - x,
\]
\[
m(Q_{-4}) = m(P_6) = 2L'(\hat{E}_{36}, 0), \quad \text{where } \hat{E}_{36}: y^2 = x^3 - 15x + 22,
\]
\[
m(Q_{-8}) = m(P_{10}) = 10L'(\hat{E}_{14}, 0), \quad \text{where } \hat{E}_{14}: y^2 + xy + y = x^3 - 11x + 12,
\]

are valid.

## 2 Bosman’s family of genus 2 curves

The polynomials

\[
Q_k(X, Y) = Y^2 + (X^4 + kX^3 + 2kX^2 + kX + 1)Y + X^4
\]

define generically the family of curves $Q_k(X, Y) = 0$ of genus 2. The split of its Jacobian into two families of elliptic curves correspond to a reduction of hyperelliptic integrals, a phenomenon first observed in general settings by Goursat [7]. Since $Q_k(X, Y) = X^4 \tilde{Q}_k(X, Y/X^2)$ where

\[
\tilde{Q}_k(X, Y) = Y^2 + \left(\left(X + \frac{1}{X}\right)^2 + k \left(X + \frac{1}{X}\right) + 2k - 2\right)Y + 1,
\]

we conclude that $m(Q_k(X, Y)) = m(\tilde{Q}_k(X, Y))$. We now investigate where the curve $\tilde{Q}_k(X, Y) = 0$ cuts the torus $|X| = |Y| = 1$.

On $|X| = 1$, both

\[
B(X) := B_k(X) = \left(X + \frac{1}{X}\right)^2 + k \left(X + \frac{1}{X}\right) + 2k - 2
\]

and

\[
\Delta(X) := \Delta_k(X) = B(X)^2 - 4
\]

\[
= \left(\left(X + \frac{1}{X}\right)^2 + k \left(X + \frac{1}{X}\right) + 2k\right)\left(X + \frac{1}{X} + 2\right)\left(X + \frac{1}{X} + k - 2\right)
\]

(3)

are real-valued. Then

\[
\tilde{Q}_k(X, Y) = Y^2 + B(X)Y + 1 = (Y - Y_1(X))(Y - Y_2(X)),
\]

where

\[
Y_1(X), Y_2(X) = \frac{-B(X) \pm \sqrt{\Delta(X)}}{2}.
\]
By Viète’s theorem \( Y_1(X)Y_2(X) = 1 \); therefore, we get \(|Y_1(X)| = |Y_2(X)| = 1\) for \( \Delta(X) \leq 0 \). If \( \Delta(X) > 0 \), then both zeroes \( Y_1(X) \) and \( Y_2(X) \) are real of the same sign as \(-B(X)\). In this case we numerate them in such a way that

\[ |Y_2(X)| < 1 < |Y_1(X)|; \]

in other words,

\[ Y_1(X) = \frac{-B(X) - \text{sign}(B(X))\sqrt{\Delta(X)}}{2}. \]

**Lemma 1** If \( \Delta(X) > 0 \) on the unit circle, then signs of \( B(X) \) and \( k \) coincide.

**Proof** Taking \( X = e^{i\theta} \), where \(-\pi < \theta < \pi\), so that \( X + 1/X = 2 \cos \theta \) is between \(-2\) and \(2\), we see from (3) that the sign of \( \Delta(X) \) is fully controlled by the signs of

\[ B(X) + 2 = \left( X + \frac{1}{X} \right)^2 + k \left( X + \frac{1}{X} \right) + 2k = 4 \cos^2 \theta + 2k(1 + \cos \theta) \]

and \( X + 1/X + k - 2 = 2 \cos \theta + k - 2 \).

If \( k > 0 \), the former expression is strictly positive; hence \( \Delta(X) = (B(X) - 2)(B(X) + 2) > 0 \) implies \( B(X) - 2 > 0 \), in particular, \( B(X) > 0 \).

If \( k < 0 \), then \( 2 \cos \theta + k - 2 < 0 \), so that \( \Delta(X) > 0 \) is equivalent to \( B(X) + 2 < 0 \), hence \( B(X) < 0 \). \( \square \)

Using Jensen’s formula and the symmetry \( Y_1(X) = Y_1(X^{-1}) \), we get

\[ \tilde{g}(k) = m(Q_k(X, Y)) = m(Q_k(X, Y)) \]

\[ = \frac{1}{(2\pi i)^2} \int\int_{|X|=|Y|=1} \log |Q_k(X, Y)| \frac{dX}{X} \frac{dY}{Y} \]

\[ = \frac{1}{2\pi i} \int_{|X|=1} \log |Y_1(X)| \frac{dX}{X} \]

\[ = \frac{1}{\pi i} \int_{|X|=1} \Re \log Y_1(X) \frac{dX}{X} \]

\[ = \frac{1}{\pi} \Re \int_{0}^{\pi} \log Y_1(e^{i\theta}) \ d\theta \]

\[ = \frac{1}{\pi} \Re \int_{0}^{\pi} \log \left( \frac{B(e^{i\theta}) + \text{sign}(k)\sqrt{\Delta(e^{i\theta})}}{2} \right) \ d\theta, \quad k \neq 0. \]

**Lemma 2** For \( k < -1 \) and \( k > 0 \),

\[ \frac{d\tilde{g}(k)}{dk} = \frac{\text{sign}(k)}{\pi} \Re \int_{-\infty}^{k(3-k)} \frac{dv}{\sqrt{-v + 4}(v^2 + k(k - 4)v + 4k^2)}. \]

**Proof** We have

\[ \frac{d}{dk} \log \frac{B \pm \sqrt{\Delta}}{2} = \frac{d}{dB} \log \frac{B \pm \sqrt{B^2 - 4}}{2} \cdot \frac{dB}{dk} = \mp \frac{1}{\sqrt{B^2 - 4}} \cdot \left( X + \frac{1}{X} + 2 \right). \]
Therefore,
\[
\frac{d\tilde{g}(k)}{dk} = -\frac{\text{sign}(k)}{\pi} \Re \int_0^\pi \frac{2(\cos \theta + 1)}{\sqrt{\Delta_k(e^{i\theta})}} \, d\theta.
\]

Changing to \( t = \cos \theta = (X + 1/X)/2 \) we can write the result as
\[
\frac{d\tilde{g}(k)}{dk} = \frac{\text{sign}(k)}{\pi} \Re \int_{-1}^1 \frac{t + 1}{\sqrt{(2t^2 + kt + k)(t + 1)(2t + k - 2)}} \frac{dt}{\sqrt{1 - t^2}}.
\]

For \( k < -1 \) and \( k > 0 \) we further employ the substitution \( t = (v + 2k^2 - 2k)/(v - 4k) \) to get the desired form. \( \square \)

We remark that for \( k > 4 \) the formula of Lemma 2 translates into
\[
\frac{d\tilde{g}(k)}{dk} = \frac{1}{\pi} \int_{-\infty}^{k(3-k)} \frac{dv}{\sqrt{-(v + 4)(v^2 + k(k - 4)v + 4k^2)}}.
\]

The result is an incomplete elliptic integral, and this fact creates a natural obstruction for having no relations between the two Mahler measures in Theorem 1 when \( k > 4 \).

### 3 Boyd’s family of elliptic curves

Let us now turn our attention to an “easier” object—the family \( P_k(x, y) = 0 \) of elliptic curves (generically), where
\[
P_k(x, y) = (x + 1)y^2 + (x^2 + kx + 1)y + (x^2 + x) = (x + 1)(y + 1)(x + y) - (2 - k)xy.
\]

Denote \( g(k) = m(P_{2-k}(x, y)) \). Since \( P_k(x^2, y^2) = x^2y^2\tilde{P}_{2-k}(x, y) \) where
\[
\tilde{P}_k(x, y) = \left( x + \frac{1}{x} \right) \left( y + \frac{1}{y} \right) \left( \frac{x}{y} + \frac{y}{x} \right) - k,
\]
we also have \( g(k) = m(\tilde{P}_k(x, y)) \). Then
\[
\tilde{P}_{-1}(x, y) = \left( x + \frac{1}{x} \right) \left( y + \frac{1}{y} \right) \left( \frac{x}{y} + \frac{y}{x} \right) + 1
\]
is real-valued and ranges between \(-7\) and \(9\) on the torus \(|x| = |y| = 1\); therefore, for \(|k| > 9\) we can write the Mahler measure
\[
m(\tilde{P}_k(x, y)) = \frac{1}{(2\pi i)^2} \int_{|x|=|y|=1} \log |k + 1 - \tilde{P}_{-1}(x, y)| \frac{dx \, dy}{x \, y}
\]
\[
= \log |k + 1| + \frac{1}{(2\pi i)^2} \int_{|x|=|y|=1} \Re \log \left( 1 - \frac{\tilde{P}_{-1}(x, y)}{k + 1} \right) \frac{dx \, dy}{x \, y}
\]
where the constant term

\[ \text{CT}(\tilde{P}_{-1}(x, y)^n) = \sum_{j, \ell, m \geq 0} \left( \frac{n!}{j! \ell! m!} \right)^2 = \sum_{j=0}^{n} \left( \frac{n}{j} \right)^2 \left( \begin{array}{c} 2j \\ j \end{array} \right). \]

Differentiating the resulting expression for \(|k| > 9\), we obtain

\[ \frac{dg(k)}{dk} = \text{Re} \sum_{n=0}^{\infty} (k + 1)^{-n+1} \sum_{j=0}^{n} \left( \frac{n}{j} \right)^2 \left( \begin{array}{c} 2j \\ j \end{array} \right). \]

The series

\[ f(z) = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \left( \frac{n}{j} \right)^2 \left( \begin{array}{c} 2j \\ j \end{array} \right) \]

satisfies the Picard–Fuchs differential equation [2,11]

\[ z(z-1)(9z-1) \frac{d^2 f}{dz^2} + (27z^2 - 20z + 1) \frac{df}{dz} + 3(3z-1) f = 0, \]

whose singularities are at \(z = 0, 1/9, 1\) and \(\infty\). Therefore, our result for \(dg(k)/dk\) can be extended on the real line by continuity to the intervals \(k < -1\) and \(k > 8\) using

\[ f(z) = \sum_{m=0}^{\infty} \frac{(3m)!}{m!^3} \frac{z^{2n}(1-z)^n}{(1-z)^{3n+1}} = \frac{1}{1-3z} F_1 \left( \frac{1}{3}, \frac{2}{3} \mid 1 \right) \left( \frac{27z^2(1-z)}{(1-3z)^3} \right) \quad \text{for } z < 0 \]

and

\[ f(z) = \sum_{m=0}^{\infty} \frac{(3m)!}{m!^3} \frac{z^n(1-z)^{2n}}{(1+3z)^{3n+1}} = \frac{1}{1+3z} F_1 \left( \frac{1}{3}, \frac{2}{3} \mid 1 \right) \left( \frac{27z(1-z)^2}{(1+3z)^3} \right) \quad \text{for } 0 < z < \frac{1}{9}, \]

respectively (see, e.g., [4, Section 3]). Namely, we have

\[ \frac{dg(k)}{dk} = \frac{1}{k-2} F_1 \left( \frac{1}{3}, \frac{2}{3} \mid 1 \right) \left( \frac{27k}{(k-2)^3} \right) \quad \text{for } k < -1 \]

(4)

and

\[ \frac{dg(k)}{dk} = \frac{1}{k+4} F_1 \left( \frac{1}{3}, \frac{2}{3} \mid 1 \right) \left( \frac{27k^2}{(k+4)^3} \right) \quad \text{for } k > 8. \]

(5)

In fact, the latter formula (and slightly more) are shown in [10, Section 4.2] to be true for \(2 < k < 8\).

**Lemma 3** For \(0 < k < 8\),

\[ \frac{dg(k)}{dk} = \frac{1}{2\pi} \int_{0}^{1} \frac{dt}{\sqrt{t(1-t)(k^2t^2 + (4-k)kt + 4)}}. \]
Proof This integral representation is established in [10, Section 4.2] for the range $2 < k < 8$. Since the both sides are continuous on the broader interval $0 < k < 8$, the lemma follows.

Lemma 4 For $k < -1$,

$$\frac{dg(k)}{dk} = -\frac{1}{2\pi} \int_0^1 \frac{p(1+p) \, dt}{\sqrt{1-t} \left(1+2p-p^3(2+p)t\right)} \quad \text{for } k = -\frac{2}{p(1+p)},$$

where $0 < p < 1$.

Proof As in the proof of [10, Lemma 9] we use Ramanujan’s transformation [1, p. 112, Theorem 5.6]

$$\frac{1}{1+p+p^2} \binom{\frac{1}{3}, \frac{2}{3}}{\frac{1}{2}} \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} = \frac{1}{\sqrt{1+2p}} \binom{\frac{1}{2}, \frac{3}{2}}{\frac{1}{1+2p}} \frac{p^3(2+p)}{1+2p} \binom{\frac{1}{2}, \frac{3}{2}}{\frac{1}{1+2p}}

= \frac{1}{\pi} \int_0^1 \frac{dt}{\sqrt{1-t} \left(1+2p-p^3(2+p)t\right)},$$

valid for $0 \leq p < 1$. The required formula follows from this transformation by choosing $p = (\sqrt{1-8/k} - 1)/2$ in (4).

4 Comparison

Lemma 5 For the derivatives of Mahler measures defined in Sects. 2 and 3,

$$\frac{d\tilde{g}(k)}{dk} = \frac{dg(k)}{dk} \quad \text{if } k < -1, \quad \text{and} \quad \frac{d\tilde{g}(k)}{dk} = 2 \frac{dg(k)}{dk} \quad \text{if } 0 < k \leq 4.$$

Proof If $k < -1$, we let $k = -2/(p(1+p))$ where $0 < p < 1$ and write the result of Lemma 2 as

$$\frac{d\tilde{g}(k)}{dk} = -\frac{1}{\pi} \Re \int_{-\infty}^{-2/(3p^2+3p+2)/(p^2(1+p)^2)} \frac{p(p+1) \, dv}{\sqrt{-(v+4)(p^2v+4)((p+1)^2v+4)}}

= -\frac{1}{\pi} \int_{-\infty}^{-4/p^2} \frac{p(p+1) \, dv}{\sqrt{-(v+4)(p^2v+4)((p+1)^2v+4)}}.$$

On the other hand, the substitution $t = (p^2v+4)/(p^2(v+4))$ in the integral of Lemma 4 leads to

$$\frac{dg(k)}{dk} = -\frac{1}{\pi} \int_{-\infty}^{-4/p^2} \frac{p(p+1) \, dv}{\sqrt{-(v+4)(p^2v+4)((p+1)^2v+4)}},$$

so that the first equality of the lemma follows.

For $0 < k < 8$, the substitution $t = -4/v$ in the integral of Lemma 3 results in

$$\frac{dg(k)}{dk} = \frac{1}{2\pi} \int_{-\infty}^{-4} \frac{dv}{\sqrt{-(v+4)(v^2+k(k-4)v+4k^2)}}. \quad (6)$$

This coincides with the representation of Lemma 2 for the range $0 < k \leq 4$. 

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Remark Using the evaluation (5) we can extend formula (6) to the interval \( k > 8 \) as follows:

\[
\frac{dg(k)}{dk} = \frac{1}{2\pi} \text{Re} \int_{-4}^{\infty} \frac{dv}{\sqrt{-\left(v + 4\right)\left(v^2 + k(k - 4)v + 4k^2\right)}}.
\]

As in the case \( 4 < k < 8 \), the resulting elliptic integral does not possess any clear relation with the incomplete elliptic integral obtained for \( \tilde{g}(k)/dk \) in Lemma 2.

**Proof of Theorem 1** Since both the Mahler measures \( g(k) = m(P_{2-k}(x, y)) \) and \( \tilde{g}(k) = m(Q_k(X, Y)) \) are continuous functions of real parameter \( k \), integrating the equalities of Lemma 5 imply

\[
\tilde{g}(k) = g(k) + c_- \quad \text{if} \quad k \leq -1, \quad \text{and} \quad \tilde{g}(k) = 2g(k) + c_+ \quad \text{if} \quad 0 \leq k \leq 4,
\]

where \( c_- \) and \( c_+ \) are certain constants.

In (3), \( \Delta_0(X) \leq 0 \) on \( |X| = 1 \), hence \( |Y_1(X)| = |Y_2(X)| = 1 \) on the unit circle, and \( \tilde{g}(0) = m(\tilde{Q}_0(X, Y)) = 0 \). Furthermore, \( P_2(x, y) = (x + 1)(y + 1)(x + y) \) implying \( g(0) = m(P_2(x, y)) = 0 \). Thus, \( c_+ = 0 \).

As \( k \to \infty \),

\[
\tilde{g}(k) = m(Q_k(X, Y)) = m\left(k \cdot \left((X + 1)^2Y + \frac{1}{k}(Y^2 + (X^4 + 1)Y + X^4)\right)\right)
\]

\[
= \log |k| + m((X + 1)^2Y) + O(1/k) = \log |k| + O(1/k)
\]

and, similarly,

\[
g(k) = m(P_{2-k}(x, y)) \sim \log |k| + O(1/k).
\]

In particular, \( \tilde{g}(k) - g(k) \to 0 \) as \( k \to -\infty \) implying \( c_- = 0 \). \( \Box \)

**Proof of Theorem 2** The required evaluations follow from the formulae for \( m(P_0), m(P_{-2}), m(P_1), m(P_4), m(P_6) \) and \( m(P_{10}) \) obtained in [8–10]. \( \Box \)

## 5 Conclusion

Our proof of Theorem 1 makes use of two important features of the Mahler measures of polynomial families. First, their derivatives with respect to the parameter of the family satisfy Picard–Fuchs differential equations—the property guaranteed by the fact that the Mahler measures and their derivatives are periods. Second, when a family generically corresponds to curves of genus 2 whose Jacobian splits into the product of two elliptic curves, the derivative of the Mahler measure reduces to (sometimes incomplete) elliptic integrals. Exploring this direction further we are able to establish another general equality conjectured by Boyd [6] between the Mahler measures of two hyperelliptic families and even to relate them to the Mahler measures of an elliptic family.

**Theorem 3** Define

\[
P_k(x, y) = (x^2 + x + 1)y^2 + kx(x + 1)y + x(x^2 + x + 1),
\]

\[
Q_k(x, y) = (x^2 + x + 1)y^2 + (x^4 + kx^3 + (2k - 4)x^2 + kx + 1)y + x^2(x^2 + x + 1),
\]

\[
R_k(x, y) = y^3 - y + x^3 - x + kxy,
\]
so that generically \( P_k(x, y) = 0 \) and \( Q_k(x, y) = 0 \) are families of genus 2 curves, while \( R_k(x, y) = 0 \) is a family of elliptic curves for \( k \neq 0, \pm 3 \). Then, for \( k \in \mathbb{R} \) such that \(|k| \geq 16/(3\sqrt{3}) = 3.0792\ldots\), we have \( m(P_k) = m(R_k) \). Furthermore, \( m(Q_{k+2}) = m(R_k) \) for \( k \geq 4 \).

The proof of this theorem requires some other analytical tools and is presented in our paper [3].

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