On the $N=1$ super Liouville four-point functions

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Abstract: We construct the four-point correlation functions containing the top component of the supermultiplet in the Neveu–Schwarz sector of the $N=1$ SUSY Liouville field theory. The construction is based on the recursive representation for the NS conformal blocks. We test our results in the case where one of the fields is degenerate with a singular vector on the level $3/2$. In this case, the correlation function satisfies a third-order ordinary differential equation, which we derive. We numerically verify the crossing symmetry relations for the constructed correlation functions in the nondegenerate case.

1. Introduction

The $N=1$ supersymmetric extension of the Liouville field theory (SLFT) [1] plays an important role in a world-sheet description of the noncritical fermionic string theory. It has at least two remarkable properties. First, it is one of the simplest supersymmetric conformal field theories (CFT); hence, investigating it might be useful for possible more complicated generalizations. Second, it, in addition to the bosonic Liouville theory, is another example of a “noncompact” CFT, having a continuous spectrum; this is a rather new object in the area of integrable models, worth investigating in its own right. One of the main problems in any quantum field theory is calculating correlation functions. Recent results obtained in [2] open the way for constructing the basic four-point correlation functions in the NS sector of the SLFT. But there are still a few subtleties in realizing this program. One is to classify the main correlation functions clearly in terms of the basic superconformal block functions. Another concerns the problem of finding the so-called elliptic uniformizing representation for the four-point superconformal block functions. In this paper, we partially tackle these problems.
We briefly recall the main features of the SLFT (see, e.g., [3–7] for more details). Local properties of the SLFT are encoded in the Lagrangian density

\[ \mathcal{L}_{\text{SLFT}} = \frac{1}{8\pi} (\partial_a \phi)^2 + \frac{1}{2\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) + 2i\mu b^2 \bar{\psi} \psi e^{b\phi} + 2\pi b^2 \mu^2 e^{2b\phi}. \] (1.1)

The symmetry is generated by the holomorphic and antiholomorphic components of the supercurrent \( S \) and the stress tensor \( T \). In terms of the Laurent components of \( S \) and \( T \), the algebra takes the conventional form of the Neveu–Schwarz–Ramond [8,9] algebra \( SVir \):

\[
\begin{align*}
[L_n; L_m] &= (n - m) L_{n+m} + \frac{\hat{c}}{8} (n^2 - n) \delta_{n,-m}, \\
\{G_r; G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2} \left( r^2 - \frac{1}{4} \right) \delta_{r,-s}, \\
[L_n; G_r] &= \left( \frac{1}{2} n - r \right) G_{n+r},
\end{align*}
\] (1.2)

where

\[ r, s \in \mathbb{Z} + \frac{1}{2} \] for the NS sector,

\[ r, s \in \mathbb{Z} \] for the R sector.

The central charge of the superconformal algebra is related to the parameter \( b \) in (1.1) via the “background charge” \( Q \),

\[ \hat{c} = 1 + 2Q^2, \quad \text{where } Q = b + \frac{1}{b}. \] (1.3)

Local fields in the SLFT belong to highest-weight representations of \( SVir \otimes \overline{SVir} \) algebra. Each representation, a so-called Verma module, consists of a primary field \( V_a \) with the conformal dimension \( \Delta = a(Q-a)/2 \) (we sometimes use another convenient parameterization of the conformal dimension \( \Delta = Q^2/8 - \lambda^2/2 \) and all its superconformal descendants. In this paper, we consider only the spinless primary fields with \( \bar{\Delta} = \Delta \). The general form of the descendant operator is

\[ \mathcal{L}_{\vec{k}} V_a = L_{-k_1} \cdots L_{-k_n} G_{-r_1} \cdots G_{-r_m} V_a, \] (1.4)

where \( \vec{k} \) denotes \( \{k_i, r_j\} \), which is an ordered set of positive integers and half-integers correspondingly. The relation \( \sum_i k_i + \sum_j r_j = N \) fixes the particular level in the Verma module. It is useful to introduce a special notation for the other components of the primary supermultiplet:

\[
\begin{align*}
\Lambda_a &= G_{-1/2} V_a, \\
\bar{\Lambda}_a &= \bar{G}_{-1/2} V_a, \\
W_a &= G_{-1/2} \bar{G}_{-1/2} V_a.
\end{align*}
\] (1.5)

The space of fields in the SLFT forms an operator algebra closed under the operator product expansion (OPE), which is continuous and involves integration over the internal
conformal dimension. The basic OPE is that of two primary fields (for the sake of brevity below, we set $\Delta = \Delta_{Q/2+iP}$ and $\Delta_i = \Delta_{a_i}$):

$$V_{a_1}(x)V_{a_2}(0) = \int \frac{dP}{4\pi} (x\bar{x})^{\Delta_1 - \Delta_2} \left( \mathcal{C}_{a_1,a_2}^{Q/2+iP}[V_{Q/2+iP}]_{ee} + \mathcal{C}_{a_1,a_2}^{Q/2+iP}[V_{Q/2+iP}]_{oo} \right).$$ \tag{1.6}

The integration contour is basically along the real axis but should sometimes be deformed (see [10]) under analytic continuation in the parameters $a_1$ and $a_2$. In (1.6), the “chains” $[V_{Q/2+iP}]_{ee}$ and $[V_{Q/2+iP}]_{oo}$ denote the respective contributions of integer and half-integer descendents (the second subscript denotes the antiholomorphic part; see Sec. 4). These contributions are unambiguously defined by the superconformal invariance, but unlike the standard conformal symmetry, the integer and half-integer descendents here enter independently. Hence, we obtain two different structure constants $C^p_{a_1,a_2}$ and $\tilde{C}^p_{a_1,a_2}$. All other OPEs of two arbitrary local fields can be derived from (1.6). In particular,

$$W_{a_1}(x)V_{a_2}(0) = \int \frac{dP}{4\pi} (x\bar{x})^{\Delta_1 - \Delta_2 - 1/2} \left( \mathcal{C}_{a_1,a_2}^{Q/2+iP}[V_{Q/2+iP}]_{ee} + \mathcal{C}_{a_1,a_2}^{Q/2+iP}[V_{Q/2+iP}]_{oo} \right).$$ \tag{1.7}

We note that the structure constants are the same in both OPE (1.6) and (1.7).

In this paper, we deal with the NS sector, which is closed under the OPE. In the NS sector, the field $V_{m,n}$ with $m$ and $n$ being either both even or both odd positive integers corresponds to the “degenerate” primary field with the conformal dimension $\Delta = \Delta(\lambda_{m,n})$,

$$\lambda_{m,n} = \frac{mb^{-1} + nb}{2}. \tag{1.8}$$

The “degenerate” primary field $V_{m,n}$ has a singular vector at the level $N = mn/2$ [11]. We introduce a “singular-vector creation operator” $D_{m,n}$ [12] such that the singular vector appears when $D_{m,n}$ is applied to $V_{m,n}$. The normalization is fixed by taking the coefficient of the leading term to be unity, $D_{m,n} = G_{m,n}$. The first nontrivial null vector in the NS sector occurs on the level $N = 3/2$:

$$D_{13}V_{-b} = (G_{-1/2}^3 + b^2 G_{-3/2})V_{-b} = 0. \tag{1.9}$$

This paper is organized as follows. In Secs. 2 and 3, we consider the four-point correlation functions in terms of the NS superconformal blocks and analyze the special case where one field is the degenerate field $W_{-b}$. We derive the corresponding differential equation, which becomes an important tool in investigating the four-point correlation function in the subsequent sections. In Sec. 4, we recapitulate the so-called recursive “c-representation” and use it to construct the four-point correlation function $\langle WVVV \rangle$. In Sec. 5, we suggest an elliptic representation for the correlation function under consideration, making an additional assumption about the asymptotic behavior of the superconformal blocks considered as functions of the internal conformal dimension, and numerically verify the crossing symmetry for the constructed four-point correlation function. The last section contains a brief summary and some discussion.
2. Four-point correlation function and conformal blocks

For our purposes, it makes sense to use the superfield formalism. Then the four components of the supermultiplet can be joint in the primary superfield $\hat{V} = V + \theta_i \Lambda_i + \bar{\theta}_i \bar{\Lambda}_i + \bar{\theta}_i \theta_j W$. In the supersymmetric case, in addition to the standard anharmonic ratio $z$ for the four-point correlation functions, there are two more independent superprojective invariants that should be taken into account [3]. We let $\tau_1$ and $\tau_2$ denote them. The solution of the superprojective constraints or, equivalently, the transformation law for the primary superfields under superprojective transformations leads to the general form of the the four-point correlation function

$$\langle \hat{V}_{a_1}(z_1) \hat{V}_{a_2}(z_2) \hat{V}_{a_3}(z_3) \hat{V}_{a_4}(z_4) \rangle = |z_{41}|^{-4\Delta_1} |z_{24}|^{2\Delta_1+3-2-4} |z_{34}|^{2\Delta_1+2-3-4} |z_{23}|^{2\Delta_1+1-2-3} \hat{G}\left(\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}; z, \bar{z}, \tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2 \right), \quad (2.1)$$

where $z_i = \{x_i, \theta_i\}$ is the holomorphic superspace coordinate and $z_{ij} = x_i - x_j + \theta_i \theta_j$. Here, the reduced four-point function $\hat{G}$ depends only on superprojective invariants. Our choice of the superprojective invariants is

$$z = \frac{z_{12} z_{34}}{z_{23} z_{41}},$$

$$\tau_1 = \theta_{324},$$

$$\tau_2 = z^{-1/2}(1 - z)^{-1/2} \theta_{321},$$

(2.2)

where the “odd” three-point superprojective invariant is

$$\theta_{ijk} = \frac{z_{jk} \theta_i + z_{ki} \theta_j + z_{ij} \theta_k - \theta_i \theta_j \theta_k}{(z_{ij} z_{ik} z_{jk})^{1/2}}. \quad (2.3)$$

For considering the correlation functions of the spinless fields, it suffices to introduce four auxiliary functions $G_0$, $G_1$, $G_2$, and $G_3$:

$$\hat{G}\left(\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}; z, \tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2 \right) = G_0(z, \bar{z}) + G_1(z, \bar{z}) \bar{\tau}_1 \bar{\tau}_2 + G_2(z, \bar{z}) \bar{\tau}_2 \tau_1 + G_3(z, \bar{z}) \bar{\tau}_1 \tau_1 \bar{\tau}_2 \tau_2. \quad (2.4)$$

For the given choice of the superprojective invariants, the functions $G_i$ are uniquely related to the basic four-point correlation functions of the components of primary supermultiplets. Our main focus in this paper is the four-point correlation function involving one “top” component $W$ of a primary supermultiplet, while the others are “bottom.” For our choice of the superprojective invariants,

$$\langle W_{a_1}(x) V_{a_2}(0) V_{a_3}(1) V_{a_4}(\infty) \rangle = G_2\left(\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}; x, \bar{x} \right). \quad (2.5)$$


It can be evaluated via the integral representation based on OPEs (1.6) and (1.7).

\[
\langle W_{a_1}(x)V_{a_2}(0)V_{a_3}(1)V_{a_4}(\infty) \rangle = \int \frac{dP}{4\pi} C^{Q_{a_1 a_2}} Q_{a_3 a_4} \mathcal{F}_e \left( \begin{array}{cc} \hat{a}_1 & a_3 \\ a_2 & a_4 \end{array} \middle| \Delta, x \right) \mathcal{F}_o \left( \begin{array}{cc} \hat{a}_1 & a_3 \\ a_2 & a_4 \end{array} \middle| \Delta, \bar{x} \right) + \int \frac{dP}{4\pi} C^{Q_{a_1 a_2}} Q_{a_3 a_4} \mathcal{F}_e \left( \begin{array}{cc} \hat{a}_1 & a_3 \\ a_2 & a_4 \end{array} \middle| \Delta, x \right) \mathcal{F}_o \left( \begin{array}{cc} \hat{a}_1 & a_3 \\ a_2 & a_4 \end{array} \middle| \Delta, \bar{x} \right), \tag{2.6}
\]

where \( \mathcal{F}_{e,o} \) are the corresponding four-point superconformal blocks with the intermediate dimension \( \Delta = Q^2/4 + \mathcal{P}^2 \) (the hat over \( a_1 \) highlights that the field is the “top” component of the supermultiplet). These functions were initially introduced in [14]. In Secs. 4 and 5, we present some details and explicit constructions concerning this object. The basic NS structure constants in (2.6) were evaluated in [7, 15] and were recently rederived without referring to the Ramond sector [2]:

\[
C^{Q_{a_1 a_2}} = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) b^{1-b^2} \right)^{(Q-a)/b} \frac{\Upsilon'_\text{NS}(0)\Upsilon_{\text{NS}}(2a_1)\Upsilon_{\text{NS}}(2a_2)\Upsilon_{\text{NS}}(2a_3)}{\Upsilon_{\text{NS}}(a_1+2+3 - Q)\Upsilon_{\text{NS}}(a_1+2-3)\Upsilon_{\text{NS}}(a_2+3-1)\Upsilon_{\text{NS}}(a_3+1-2)}, \tag{2.7}
\]

where \( a \) here denotes \( a_1 + a_2 + a_3 \) and we use the convenient notation in [13] for the special functions

\[
\Upsilon_{\text{NS}}(x) = \Upsilon_b \left( \frac{x}{2} \right) \Upsilon_b \left( \frac{x + Q}{2} \right), \tag{2.8}
\]

\[
\Upsilon_{\text{R}}(x) = \Upsilon_b \left( \frac{x + b}{2} \right) \Upsilon_b \left( \frac{x + b^{-1}}{2} \right)
\]

expressed in terms of the “upsilon” function \( \Upsilon_b \) that is standard in the Liouville field theory (see [10, 16]).

3. Differential equations corresponding to the null vector (1,3)

We now turn to a four-point function with one singular field \( W_{-b} \). It is natural to renumber the operators, setting \( W_{a_1} \) to be \( W_{-b} \) and respectively setting \( V_{a_2}, V_{a_3}, \) and \( V_{a_4} \) to be \( V_1, V_2, \) and \( V_3 \). As described in Appendix A, the decoupling of the singular vector (1,3) leads to the partial differential equation for the four-point correlation function

\[
\left\{ -b^{-2}\partial_i D_i + \sum_{i=1}^{3} \left( \frac{2\Delta}{z_{4i}} \theta_{4i}^{2} + \frac{1}{z_{4i}} \left[ 2\theta_{4i} \partial_i - D_i \right] \right) \right\} (\hat{W}_{-b}(z_4) \hat{V}_1(z_1) \hat{V}_2(z_2) \hat{V}_3(z_3)) = 0, \tag{3.1}
\]
which can be reduced to two systems of ordinary differential equations for the holomorphic functions \( g_0, g_1, g_2, \) and \( g_3 \), which are just the conformal blocks contributing to the functions \( G_i \) introduced in (2.4), with the changes of the arguments described above. For example,

\[
g_2(x) \Rightarrow G_2 \left( \begin{array}{c} -b \\ a_1 \\ a_3 \\ \end{array} \begin{array}{c} a_2 \\ x, \bar{x} \end{array} \right). \tag{3.2}
\]

The first system of equations is for \( g_0 \) and \( g_3 \),

\[
-b^{-2}x g_0'' + \frac{3x - 2}{x - 1} g_0' + b^{-2} g_3' + \left[ \frac{\gamma_{13}}{x} + \frac{\gamma_{23}}{x - 1} \right] g_0 + \frac{1 - 2x}{x(x - 1)} g_3 = 0,
\tag{3.2}
\]

\[
b^{-2} g_0'' + \frac{1 - 3x}{x(x - 1)} g_0' + \left[ \frac{2 \Delta_1}{x^2} + \frac{2 \Delta_2}{(x - 1)^2} + \frac{2 \gamma_{12}}{x(x - 1)} \right] g_0 + \frac{1}{x(x - 1)} g_3 = 0.
\tag{3.3}
\]

The second is for \( g_1 \) and \( g_2 \),

\[
-b^{-2} g_1'' - b^{-2} x g_2'' + \frac{3x - 1}{x(x - 1)} g_1' + \left[ b^{-2} + \frac{3x - 2}{x - 1} \right] g_1' - \left[ \frac{2 \Delta_1}{x^2} + \frac{2 \Delta_2}{(x - 1)^2} + \frac{2 \gamma_{12}}{x(x - 1)} \right] g_2 + \frac{1 - 2x}{x(x - 1)} g_1 = 0,
\tag{3.4}
\]

\[
-b^{-2} g_1'' + \frac{2x - 1}{x(x - 1)} g_2' + \frac{1}{x(x - 1)} g_1 = 0.
\tag{3.5}
\]

which leads to a third-order linear differential equation for \( g_2 \),

\[
g_2''' + \frac{2(1 - b^2)(1 - 2x)}{x(1 - x)} g_2''
+ \left( \frac{b^4 - b^2 + 2b^2(\Delta_1(1 - x) + \Delta_2 x)}{x^2(1 - x)^2} - \frac{5b^4 + b^2(2\Delta_3 - 7) + 2}{x(1 - x)} \right) g_2'
+ b^4 \left( \frac{3(\Delta_1 - \Delta_2) + (b^2 + \Delta_3 - 1)(1 - 2x)}{x^2(1 - x)^2} + \frac{2\Delta_2 x - 2\Delta_1 (1 - x)}{x^3(1 - x)^3} \right) g_2 = 0.
\tag{3.5}
\]

Three independent solutions for \( g_2 \) with a diagonal monodromy near \( x = 0 \) are just the s-channel conformal blocks corresponding to \( G_2 \). Using the definitions of the conformal blocks (see (4.2) below) to compare the exponents for \( x \to 0 \), we find

\[
g_2^{(+)} = x^{(1 + b^2 + 2b\lambda_1)/2} \sum_{n=0}^{\infty} A_n^{(+)} x^n = F_o \left( \begin{array}{c} \hat{\Delta}_{13} \\ \hat{\Delta}_1 \\ \Delta_4 \end{array} \begin{array}{c} \Delta_3 \\ \Delta_1 + \Delta_4 \\ (\lambda_1 \mp b) \end{array} \right) x, \tag{3.6}
\]

\[
g_2^{(0)} = x^{b^2} \sum_{n=0}^{\infty} A_n^{(0)} x^n = F_e \left( \begin{array}{c} \hat{\Delta}_{13} \\ \hat{\Delta}_1 \\ \Delta_4 \end{array} \begin{array}{c} \Delta_3 \\ \Delta_1 \\ \Delta(\lambda_1) \end{array} \right) x,
\tag{3.7}
\]

corresponding to the overall normalization

\[
A_0^{(+)} = 1, \\
A_0^{(0)} = -\frac{\Delta(\lambda_1 \mp b) + \Delta_{13} - \Delta(\lambda_1)}{2\Delta(\lambda_1 \mp b)}.
\]
The first terms in the series expansion can be easily found by substituting these expansions in differential equation (3.5) and solving the recursive relations for the coefficients order by order. For example, for the solution \( g_2^{(0)} \) and \( g_2^{(+)} \), we have

\[
A_0^{(0)} = 1, \quad A_1^{(0)} = \left( b^2(-1 - 2b^2 - b^4 + 4b^2\lambda_1^2 + 4b^2\lambda_2^2 - 4b^2\lambda_3^2) \right) \times (2(1 + b^2 - 2b\lambda_1)(1 + b^2 + 2b\lambda_1))^{-1},
\]

\[
A_2^{(0)} = \left( b^2(-22 - 47b^2 - 24b^4 + 6b^6 + b^{10} + 96b^2\lambda_1^2 - 24b^4\lambda_1^2 - 16b^6\lambda_1^2 - 8b^8\lambda_1^2 - 32b^4\lambda_1^4 + 16b^6\lambda_1^4 - 80b^2\lambda_2^2 - 8b^6\lambda_1^2 - 32b^6\lambda_2^2 - 8b^2\lambda_3^2 - 8b^4\lambda_3^2 + 32b^6\lambda_2^2 \lambda_3^2 - 64b^4\lambda_2^2\lambda_3^2 + 32b^6\lambda_2^2\lambda_3^2 \right) \times (1 + b^2 - 2b\lambda_1)(3 + b^2 - 2b\lambda_1)^{-1},
\]

\[
A_0^{(+)} = 2b^2(-1 - b^2 - 2b\lambda_1)^{-1}, \quad A_1^{(+)} = b^2(3 - 10b^2 + 3b^4 + 8b\lambda_1 - 8b^3\lambda_1 + 4b^2\lambda_1^2 + 4b^2\lambda_2^2 - 4b^2\lambda_3^2) \times (2(1 - b^2 + 2b\lambda_1)(-3 + b^2 - 2b\lambda_1))^{-1}.
\]

These expressions coincide with the corresponding terms (4.29), (4.33) of the conformal block series expansions calculated for the same parameter choices and thus confirm the recursive relations for the conformal blocks discussed in the next sections.

To conclude this section, we analyze the solutions of differential equation (3.5) and their monodromy properties. We show that these properties lead to functional relations for the structure constants that are consistent with (2.7). The decoupling equation restricts OPEs (1.6) and (1.7) to the “discrete” form (cf. (3.6))

\[
V_{-b}(x)V_{a}(0) =
(x\bar{x})^{ab}C_-(a)[V_{a-b}]_{ee} + (x\bar{x})^{1/2+b^2}\tilde{C}_0(a)[V_{a}]_{oo} + C_+(a)(x\bar{x})^{1-ba+b^2}[V_{a+b}]_{ee};
\]

\[
W_{-b}(x)V_{a}(0) =
(x\bar{x})^{ab}C_-(a)[V_{a-b}]_{oo} + (x\bar{x})^{b^2}\tilde{C}_0(a)[V_{a}]_{ee} + C_+(a)(x\bar{x})^{1-ba+b^2}[V_{a+b}]_{oo},
\]

where the special structure constants are

\[
C_-(a) = \text{res}_{\epsilon=0} C_{a,-b+\epsilon}^{a-b-\epsilon} = 1,
\]

\[
C_0(a) = \text{res}_{\epsilon=0} \frac{C_{a,-b+\epsilon}^{a-b-\epsilon}}{\gamma(-b^2)\gamma(ba_1)\gamma(1+b^2-ba_1)} = \frac{2\pi i\mu}{\gamma(-b^2)\gamma(ba_1)\gamma(1+b^2-ba_1)}.
\]

\[
C_+(a) = \text{res}_{\epsilon=0} C_{a,+b-\epsilon}^{a+b-\epsilon} = \left( \gamma \left( \frac{Qb}{2} \right) \right)^2 \frac{(\pi \mu)^2 b^4 \gamma(a_1 b - 1/2 - b^2/2)}{\gamma(1/2 + b^2/2 + a_1 b)}.
\]

These expressions can be found (see, e.g., [2]) directly from SLFT Lagrangian (1.1) by using perturbative calculations with the screening operator, similar to that developed in [17]. With
general expression \( (2.6) \) and OPE \( (3.13) \) taken into account, the correlation function \( G_2 \) is combined as
\[
G_2 \left( \begin{array}{ccc}
\frac{-b}{a_3} & a_2 \\
0 & a_1
\end{array} \right) x, \bar{x} = C_-(a_1) \tilde{C}_{a_1-b,a_2,a_3} F_o(x) F_o(\bar{x}) + C_+(a_1) \tilde{C}_{a_1+b,a_2,a_3} F_o^+(x) F_o^+(\bar{x}) + \tilde{C}_0(a_1) C_{a_1,a_2,a_3} F_e(0) F_e(0) (x, \bar{x}),
\]
(3.15)

On the other hand, substituting
\[
g_2(x) = x^{a_1 b}(1 - x)^{a_2 b} F(x)
\]
(3.16)
converts \( (3.5) \) to the form
\[
x^2(1 - x)^2 F'''' - x(1 - x)(K_1 x - K_2(1 - x)) F'''' + (L_1 x^2 + L_2(1 - x)^2 - L_3 x(1 - x)) F' + (M_1 x - M_2(1 - x)) F = 0.
\]
(3.17)

It turns out that this differential equation (as well as the one recently considered for the correlation function \( g_0 [2] \)) is of a special type \([17]\) and can be solved in terms of two-fold contour integrals of the form
\[
\int_{C_\alpha} \int_{C_\beta} dt_1 dt_2 |A|(1 - t_1)(1 - t_2)|B|(x - t_1)(x - t_2)|C|t_1 - t_2|^{2g}.
\]
(3.18)

For ansatz \( (3.18) \) to be the solution of differential equation \( (3.17) \), the exponents \( A, B, C, \) and \( g \) must satisfy the system of constraints
\[
\begin{align*}
K_1 &= -2g - 3B - 3C, & K_2 &= -2g - 3A - 3C, \\
L_1 &= (B + C)(2B + 2C + 2g + 1), & L_2 &= (A + C)(2A + 2C + 2g + 1), \\
L_3 &= 4AB + 4(2A + 2B + 2C + 1)C + 4(A + B + 3C)g + 4g^2 + 2g, \\
M_1 &= -2C(A + B + C + g + 1)(2B + 2C + 2g + 1), \\
M_2 &= -2C(A + B + C + g + 1)(2A + 2C + 2g + 1),
\end{align*}
\]
(3.19)

which coincides with the one that appeared in \([2]\) but with different values of \( K_i, L_i, \) and \( M_i \). This system is overdetermined for an arbitrary choice of \( K_i, L_i, \) and \( M_i \). But it can be solved in our case and yields the following expressions for \( A, B, C, \) and \( g \) in terms of the basic parameters \( a_1, a_2, a_3, \) and \( b \):
\[
\begin{align*}
A &= -1 + \frac{b a_{0+3-1}}{2}, & B &= -1 + \frac{b a_{1-2+3}}{2}, \\
C &= -\frac{b a_{1+2+3}}{2} + b^2, & g &= \frac{1}{2} - \frac{b^2}{2}.
\end{align*}
\]
(3.20)
Hence, the analogous consideration in [2] is also applicable in this case. In particular, the single-valued solution of Eq. (3.25) (and of the same equation with respect to \( \bar{x} \)) can be constructed as
\[
g_2(x, \bar{x}) = X_1 I_1(x) I_1(\bar{x}) + X_2 I_2(x) I_2(\bar{x}) + X_3 I_3(x) I_3(\bar{x}). \tag{3.21}
\]
Here, \( I_i(x) \) are the three independent solutions with a diagonal monodromy around \( x = 0 \)
\[
I_1(x) = I_1^{(0)}(1 + \ldots),
I_2(x) = x^{1+A+C} I_2^{(0)}(1 + \ldots),
I_3(x) = x^{2+2A+2C+2g} I_3^{(0)}(1 + \ldots),
\tag{3.22}
\]
where the dots denote a regular series in \( x \) and
\[
I_1^{(0)} = \frac{\Gamma(2g) \Gamma(1+B) \Gamma(1+B+g) \Gamma(-1-2g-A-B-C) \Gamma(-1-g-A-B-C)}{\Gamma(g) \Gamma(-g-A-C) \Gamma(-A-C)},
I_2^{(0)} = \frac{\Gamma(1+A) \Gamma(1+B) \Gamma(1+C) \Gamma(-1-2g-A-B-C)}{\Gamma(2+A+C) \Gamma(-2g-A-C)},
I_3^{(0)} = \frac{\Gamma(2g) \Gamma(1+A) \Gamma(1+A+g) \Gamma(1+C) \Gamma(1+C+g)}{\Gamma(2+A+C+g) \Gamma(2+A+C+2g)}. \tag{3.23}
\]
The function \( g_2(x, \bar{x}) \) coincides with correlation function (3.15) up to overall normalization. It was established in [18] that
\[
\begin{align*}
X_3 &= \sin \pi A \sin \pi C \sin \pi (A+C) \sin \pi (A+g) \sin \pi (C+g), \\
X_1 &= \sin \pi B \sin \pi (B+g) \sin \pi (A+B+C+g) \sin \pi (A+C+2g) \sin \pi (A+B+C+2g), \\
X_2 &= \sin \pi (A+C+g) \sin \pi A \sin \pi C, \\
X_1 &= 2 \cos \pi g \sin \pi (B+g) \sin \pi (A+B+C+g) \sin \pi (A+C+2g).
\end{align*} \tag{3.24}
\]
Comparing expressions (3.15) and (3.21) (with the normalization of the conformal blocks encoded in (1.6) and (1.7) and resulting in (3.7) taken into account) results in the relations for the structure functions
\[
\frac{C_{-(a_1)} \tilde{C}_{a_1-b,a_2,a_3}}{C_0(a_1) C_{a_1,b,a_2,a_3}} = \frac{(a_1-b)^2 X_1 I_1^{(0)} X_2 I_2^{(0)}}{b^2 X_2 I_2^{(0)}}, \tag{3.25}
\]
\[
\tilde{C}_{a_1-b,a_2,a_3}^{a_1,a_2,a_3} = 2 \pi i \mu \gamma(1+b^2) \gamma(\frac{a_1+2a_2+3}{2} - b^2) \gamma(\frac{1+b_{a_1-2a_2+3-b^2}}{2}) \gamma(1+b_{a_1+2a_2-3-b^2}) \gamma(\frac{1+b_{a_1-2a_2+3-b^2}}{2}) \tag{3.26}
\]
\[
\gamma(ba_1 - b^2) \gamma(\frac{ba_1+2a_2+3}{2}) \gamma(\frac{1+b_{a_1+2a_2-3-b^2}}{2}).
\]
Similarly,

\[
\frac{C_+(a_1) \tilde{C}_{a_1+b,a_2,a_3}}{C_-(a_1) \tilde{C}_{a_1-b,a_2,a_3}} = \frac{(a_1 - b^{-1})^2 X_3 x_3^{(0)} (a_1 - b) X_1 x_1^{(0)} 2}{(a_1 - b) X_1 x_1^{(0)} 2},
\]

\[
\tilde{C}_{a_1+b,a_2,a_3} \tilde{C}_{a_1-b,a_2,a_3} = \gamma(ba_1) \gamma(ba_1 - b^2) \gamma(1+2a_1-b^2) \gamma(1+2ba_1+b^2) \pi^2 m^2 b^4 \gamma(1+b^2) \gamma(ba_1+b+2+3) \gamma(ba_1+2+3-b^2) \gamma(ba_1+2+3+b^2) \gamma(1+ba_1+2+3-b^2) \gamma(1+ba_1+2+3+b^2).
\]

It can be verified directly that structure constants (2.7) satisfy functional relations (3.26) and (3.28).

4. Analytic properties of the conformal blocks and the \( \hat{c} \)-recursion

Taking the superprojective invariance into account, we can see that there are only four independent spinless four-point correlation functions (see (2.4)). The other twelve spinless correlation functions are related to these four by superprojective transformations. We take the correlators \( \langle VVVV \rangle, \langle WVVV \rangle, \langle VVWW \rangle, \) and \( \langle WVVV \rangle \) as our basis four-point functions. The correlation function \( \langle VVVV \rangle \) of the “bottom” components of the supermultiplet was considered recently (see [19, 20] for the details). In this paper, we are interested in the correlation function \( \langle WVVV \rangle \). The idea for evaluating the conformal blocks contributing to (2.6) is similar to that in the previous case. The necessary s-channel superconformal blocks are defined via the expansions

\[
F_e \left( \hat{a}_1 \ a_2 \ a_3 \ \Delta \ \bigg| \ x \right) = x^{\Delta - \Delta_1 - \Delta_2 - 1/2} \sum_{N \text{ integer}} x^N \langle \tilde{N}|N\rangle_{34},
\]

\[
F_o \left( \hat{a}_1 \ a_2 \ a_3 \ \Delta \ \bigg| \ x \right) = x^{\Delta - \Delta_1 - \Delta_2 - 1/2} \sum_{N \text{ half-integer}} x^N \langle \tilde{N}|N\rangle_{34},
\]

where the “chain” vectors \( |N\rangle \) and \( \tilde{N} \) are the respective \( N \)-th level descendents of the intermediate state with the conformal dimension \( \Delta \) appearing in OPEs (1.6) and (1.7) for \( V(x)V(0) \) and \( W(x)V(0) \),

\[
[V_\Delta]_{ee,oo} = \sum_{N \in \mathbb{Z}, \mathbb{Z}/2} x^N |N\rangle,
\]

\[
[V_{\tilde{\Delta}}]_{ee,oo} = \sum_{N \in \mathbb{Z}, \mathbb{Z}/2} x^N |\tilde{N}\rangle.
\]
Here, we suppress the dependence of the chain operators on the external dimensions. We introduce the operators \( \mathcal{C}(N, \Delta) \) creating the chain vector \(|N\rangle = \mathcal{C}(N, \Delta)V_{\Delta}\) and the similar operator \( \tilde{\mathcal{C}}(N, \Delta) \) for the chain vector \(|\tilde{N}\rangle\). The vectors \(|N\rangle\) and \(|\tilde{N}\rangle\) are completely determined by the superconformal symmetry leading to the following relations for the vectors of the chain that grows from the vacuum vector \(V_{\Delta}\):

\[
G_k \mathcal{C}(N, \Delta)V_{\Delta} = \tilde{\mathcal{C}}(N-k, \Delta)V_{\Delta}
\]

\[
G_k \tilde{\mathcal{C}}(N, \Delta)V_{\Delta} = (\Delta + 2\Delta_1 k - \Delta_2 + N - k)\mathcal{C}(N-k, \Delta)V_{\Delta}
\]

(4.4)

for \(0 < k \leq N\).

We note that the conformal blocks contributing to the other two basis correlation functions are also expressed in terms of the chain vectors \(|N\rangle\) and \(|\tilde{N}\rangle\). Namely, the superconformal blocks have the forms

\[
F_e \left( \begin{array}{c}
a_1 \\
a_2 \\
a_4 \\
\end{array} \right | \Delta | x \right) = x^{\Delta-\Delta_1-\Delta_2} \sum_{N \geq 0}^{N \text{ integer}} x^{N_{12}} \langle N | \tilde{N} \rangle_{34},
\]

(4.5)

\[
F_o \left( \begin{array}{c}
a_1 \\
a_2 \\
a_4 \\
\end{array} \right | \Delta | x \right) = x^{\Delta-\Delta_1-\Delta_2} \sum_{N > 0}^{N \text{ half-integer}} x^{N_{12}} \langle N | \tilde{N} \rangle_{34}
\]

(4.6)

for the correlation function \(\langle VVWW \rangle\) and

\[
F_e \left( \begin{array}{c}
\hat{a}_1 \\
\hat{a}_2 \\
\hat{a}_4 \\
\end{array} \right | \Delta | x \right) = x^{\Delta-\Delta_1-\Delta_2-1/2} \sum_{N \geq 0}^{N \text{ integer}} x^{N_{12}} \langle \tilde{N} | \tilde{N} \rangle_{34},
\]

(4.7)

\[
F_o \left( \begin{array}{c}
\hat{a}_1 \\
\hat{a}_2 \\
\hat{a}_4 \\
\end{array} \right | \Delta | x \right) = x^{\Delta-\Delta_1-\Delta_2-1/2} \sum_{N > 0}^{N \text{ half-integer}} x^{N_{12}} \langle \tilde{N} | \tilde{N} \rangle_{34}
\]

(4.8)

for the correlation function \(\langle WVWV \rangle\).

Below, instead of speaking in terms of the chain vectors \(|N\rangle\) and \(|\tilde{N}\rangle\), we use the chain operators \([2]\) defined as

\[
\mathcal{C}(\Delta, x) = \mathcal{C}_e(\Delta, x) + \mathcal{C}_o(\Delta, x),
\]

(4.9)

\[
\tilde{\mathcal{C}}(\Delta, x) = \tilde{\mathcal{C}}_e(\Delta, x) + \tilde{\mathcal{C}}_o(\Delta, x),
\]

where

\[
\mathcal{C}_{e,o}(\Delta, x) = \sum_{N \in \mathbb{Z}, \mathbb{Z}/2} x^{N} \mathcal{C}(N, \Delta), \quad \tilde{\mathcal{C}}_{e,o}(\Delta, x) = \sum_{N \in \mathbb{Z}, \mathbb{Z}/2} x^{N} \tilde{\mathcal{C}}(N, \Delta).
\]

(4.10)

The consideration in terms of chain operators \([4,9]\) allows presenting the results more transparently. The analytic properties of the chain operators were discussed previously \([2,19]\). Here, we present only the main results to make the material more independent. It follows from the properties of Eqs. \([4,4]\) that starting from the level \(mn/2\), the chain operators
have simple poles at $\Delta = \Delta_{m,n}$, and the residues at these poles are proportional to the new chain $C(\Delta_{m,-n}, x)D_{m,n}V_{m,n}$ or $\tilde{C}(\Delta_{m,-n}, x)D_{m,n}V_{m,n}$ growing from the vector $D_{m,n}V_{\Delta}$, which themselves satisfy chain equations (4.4) with $\Delta_{m,n}$ substituted for $\Delta$. We have the residue formulas

$$\text{res}_{\Delta=\Delta_{m,n}} C_e(\Delta, x) = x^{mn/2} \begin{cases} X_m n C_e(\Delta_{m,-n}, x)D_{m,n}, & m, n \text{ even,} \\ X_m n C_o(\Delta_{m,-n}, x)D_{m,n}, & m, n \text{ odd,} \end{cases}$$

for the chain $C(\Delta, x)$ and

$$\text{res}_{\Delta=\Delta_{m,n}} \tilde{C}_e(\Delta, x) = x^{mn/2} \begin{cases} X_m n C_e(\Delta_{m,-n}, x)D_{m,n}, & m, n \text{ even,} \\ X_m n C_o(\Delta_{m,-n}, x)D_{m,n}, & m, n \text{ odd,} \end{cases}$$

for the complementary chain $\tilde{C}(\Delta, x)$. Here,

$$X_{m,n}^{(e,o)}(\Delta_1, \Delta_2) = 2^{-p_{e,o}(m,n)} P_{m,n}^{(e,o)}(\lambda_1 + \lambda_2) P_{m,n}^{(e,o)}(\lambda_2 - \lambda_1),$$

where the degree $p_{e,o}(m, n) = \text{deg} P_{m,n}^{(e,o)}(x)$ of the “fusion polynomials” [19, 20] in the numerator,

$$P_{m,n}^{(e)}(x) = \prod_{k \in \{1-m,2,m-1\}, \ell \in \{1-n,2,n-1\}} (x - \lambda_{k,l}),$$

$$P_{m,n}^{(o)}(x) = \prod_{k \in \{1-m,2,m-1\}, \ell \in \{1-n,2,n-1\}} (x - \lambda_{k,l}),$$

(the notation $\{1-m,2,m-1\}$, for example, means “from $1-m$ to $m-1$ with step 2,” i.e., $1-m, 3-m, \ldots, m-1$), coincides with the number of multipliers in products (4.14),

$$p_{e,o}(m, n) = \begin{cases} mn/2 & \text{for } m, n \text{ even,} \\ mn/2 - 1/2 & \text{for } m, n \text{ odd.} \end{cases}$$

(4.15)

The denominator in (4.13) is related to the norm of the “quasingular” vector $D_{m,n}V_{\Delta}$, which was explicitly evaluated in [12],

$$r'_{m,n} = 2^{mn-1} \prod_{k=1-m, l=1-n}^{k+l \in \mathbb{Z}} \lambda_{k,l}.$$  

(4.16)
These simple analytic properties are inherited by superconformal blocks (4.11) and (4.12). The poles of $C_{e,o}(\Delta)V_\Delta$ become the poles of the blocks, the residues being evaluated similarly (we suppress the external dimensions in the arguments of the blocks for compactness)

$$\text{res}_{\Delta=\Delta_{m,n}} F_e(\Delta, x) = B_{m,n}^{(e)} \left( \frac{\Delta_1}{\Delta_2}, \frac{\Delta_3}{\Delta_4} \right) \begin{cases} F_e(\Delta_{m,-n}, x) & \text{for } m, n \text{ even}, \\ F_e(\Delta_{m,-n}, x) & \text{for } m, n \text{ odd}, \end{cases}$$

$$\text{res}_{\Delta=\Delta_{m,n}} F_o(\Delta, x) = B_{m,n}^{(o)} \left( \frac{\Delta_1}{\Delta_2}, \frac{\Delta_3}{\Delta_4} \right) \begin{cases} F_o(\Delta_{m,-n}, x) & \text{for } m, n \text{ even}, \\ F_o(\Delta_{m,-n}, x) & \text{for } m, n \text{ odd}, \end{cases}$$

where

$$B_{m,n}^{(e,o)} \left( \frac{\Delta_1}{\Delta_2}, \frac{\Delta_3}{\Delta_4} \right) = r_{m,n}^{(e,o)}(\Delta_1, \Delta_2) \frac{X_{m,n}^{(e,o)}(\Delta_3, \Delta_4)}{X_{m,n}(\Delta_1, \Delta_2)}.$$  \hspace{1cm} (4.17)

It follows from (4.17) that conformal blocks (4.1) and (4.2) as functions of the central charge $\hat{c}$ have one simple pole for each pair of positive integers $m$ and $n$ ($n > 1$) at $\hat{c} = \hat{c}_{m,n}(\Delta)$, where

$$\hat{c}_{m,n} = 5 + 2(T_{m,n} + T_{m,n}^{-1}),$$

$$T_{m,n} = \frac{1 - 4\Delta - mn + \sqrt{[mn - 1] + 4\Delta^2 - (m^2 - 1)(n^2 - 1)}}{n^2 - 1}.$$  \hspace{1cm} (4.19)

Hence, the residues at the poles of the conformal blocks as functions of the central charge $\hat{c}$ are also completely determined:

$$\text{res}_{\hat{c}=\hat{c}_{m,n}} F_e(\hat{c}, \Delta, x) = \hat{B}_{m,n}^{(e)}(\Delta) \begin{cases} F_e(\hat{c}, \Delta + mn/2, x) & \text{for } m, n \text{ even}, \\ F_e(\hat{c}, \Delta + mn/2, x) & \text{for } m, n \text{ odd}, \end{cases}$$

$$\text{res}_{\hat{c}=\hat{c}_{m,n}} F_o(\hat{c}, \Delta, x) = \hat{B}_{m,n}^{(o)}(\Delta) \begin{cases} F_o(\hat{c}, \Delta + mn/2, x) & \text{for } m, n \text{ even}, \\ F_o(\hat{c}, \Delta + mn/2, x) & \text{for } m, n \text{ odd}, \end{cases}$$

where

$$\hat{B}_{m,n}^{(e,o)}(\Delta) = B_{m,n}^{(e,o)}(\Delta) \frac{16(T_{m,n}(\Delta) - T_{m,n}^{-1}(\Delta))}{n^2 - 1}.$$  \hspace{1cm} (4.21)

Relations (4.4) are simplified for $c = \infty$ and can be solved explicitly:

$$C(N, \Delta) = \begin{cases} \frac{(\Delta + \Delta_1 - \Delta_2)N}{N!(2\Delta)_N} G_{-1/2}^{2N}, & \text{N integer,} \\ \frac{(\Delta + 1/2 + \Delta_1 - \Delta_2)N-1/2}{(N - 1/2)!(2\Delta + 1)_{N-1/2}} G_{-1/2}^{2N}, & \text{N half-integer,} \end{cases}$$  \hspace{1cm} (4.23)

$$\tilde{C}(N, \Delta) = \begin{cases} \frac{N!(2\Delta)_N}{(\Delta + \Delta_1 - \Delta_2)N} G_{-1/2}^{2N}, & \text{N integer,} \\ \frac{(\Delta + 1/2 + \Delta_1 - \Delta_2)N+1/2}{(N - 1/2)!(2\Delta)_{N+1/2}} G_{-1/2}^{2N}, & \text{N half-integer,} \end{cases}$$  \hspace{1cm} (4.24)
where \((x)_k = x(x+1)\cdots(x+k-1)\). With

\[
G_{1/2}^2 G_{-1/2}^{2N} V_\Delta = \begin{cases} 
(2\Delta)_N N! V_\Delta, & \text{N integer,} \\
(2\Delta)_{N+1/2} (N-1/2)! V_\Delta, & \text{N half-integer,}
\end{cases}
\tag{4.25}
\]

taken into account, this leads to expressions for the asymptotic values of \(F_e\) and \(F_o\) in terms of hypergeometric functions:

\[
F_e(\hat{c} = \infty, \Delta, x) = f_e(\Delta, x) = x^{\Delta-\Delta_1-\Delta_2-1/2} F_1(\Delta + 1/2 + \Delta_1 - \Delta_2, \Delta + \Delta_3 - \Delta_4, 2\Delta, x),
\]

\[
F_o(\hat{c} = \infty, \Delta, x) = f_o(\Delta, x) = \frac{\Delta + \Delta_1 - \Delta_2}{2\Delta} \cdot 2 F_1(\Delta + \Delta_1 - \Delta_2 + 1, \Delta + \Delta_3 - \Delta_4 + 1/2, 2\Delta + 1, x).
\tag{4.26}
\]

It is hence clear that we can write the following relations for the conformal blocks \(F_e\) and \(F_o\):

\[
F_{e,o}(\hat{c}, \Delta, x) = f_{e,o}(\Delta, x) + \sum_{m,n \text{ even}} \frac{\hat{B}_{m,n}^{(e,o)}(\Delta)}{\hat{c} - \hat{c}_{m,n}(\Delta)} F_{e,o}(\hat{c}_{m,n}, \Delta + mn/2, x)
\]

\[
+ \sum_{m,n \text{ odd} \atop m > 1} \frac{\hat{B}_{m,n}^{(o,e)}(\Delta)}{\hat{c} - \hat{c}_{m,n}(\Delta)} F_{o,e}(\hat{c}_{m,n}, \Delta + mn/2, x).
\tag{4.27}
\]

We can expand \(F_e\) and \(F_o\) in \(x\) by iterating Eqs. (4.27):

\[
F_e(\hat{c}, \Delta, x) = x^{\Delta-\Delta_1-\Delta_2-1/2} \sum_{k=0}^\infty F^{(k)} e x^k,
\tag{4.28}
\]

\[
F_o(\hat{c}, \Delta, x) = x^{\Delta-\Delta_1-\Delta_2} \sum_{k=0}^\infty F^{(k)} o x^k.
\]

Using recursive relations (4.27), we easily find the first few terms of the series expansions:

\[
F^{(0)}_e = 1,
\tag{4.29}
\]

\[
F^{(1)}_e = (1/2 + \Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)(2\Delta)^{-1},
\tag{4.30}
\]
\[ \mathcal{F}_c^{(2)} = (1/2 + \Delta + \Delta_1 - \Delta_2)(3/2 + \Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4) \]
\[ \times (1 + \Delta + \Delta_3 - \Delta_4)(4\Delta(1 + 2\Delta))^{-1} \]
\[ + (4\Delta^2 + \Delta(4 + 8\Delta_1 + 8\Delta_2) - 3(4\Delta_1^2 + (1 - 2\Delta_2)^2 - 4\Delta_1(1 + 2\Delta_2))) \]
\[ \times (\Delta^2 - 3(\Delta_3 - \Delta_4)^2 + 2\Delta(\Delta_3 + \Delta_4))(24\Delta(3 + 2\Delta)(-1 + c + 16\Delta/3))^{-1} \]
\[ - (\Delta_1 - \Delta_2)(3/2 + \Delta + \Delta_1 - \Delta_2) \]
\[ \times (\Delta_3 - 2\Delta_3^2 + \Delta_4 + 4\Delta_3\Delta_4 - 2\Delta_4^2 + \Delta(-1 + 2\Delta_3 + 2\Delta_4)) \]
\[ \times ((3/2 + \Delta)(1 + 2\Delta)^2(c - 2\Delta(3 - 2\Delta)(1 + 2\Delta)^{-1}))^{-1}, \] (4.31)
\[ \mathcal{F}_o^{(0)} = (\Delta + \Delta_1 - \Delta_2)(2\Delta)^{-1}, \] (4.32)
\[ \mathcal{F}_o^{(1)} = 2(\Delta_1 - 2\Delta_1^2 + \Delta_2 + 4\Delta_1\Delta_2 - 2\Delta_2^2 + \Delta(2\Delta_1 + 2\Delta_2 - 1))(\Delta_4 - \Delta_3) \]
\[ \times ((1 + 2\Delta)^2(c - 2\Delta(3 - 2\Delta)(1 + 2\Delta)^{-1}))^{-1} \]
\[ + (1 + \Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2)(1/2 + \Delta + \Delta_3 - \Delta_4) \]
\[ \times (2\Delta(1 + 2\Delta))^{-1}. \] (4.33)

Further coefficients are also accessible as a solution of (4.27). But it turns out that another recursive representation for the conformal blocks is more convenient from the practical standpoint. We consider it in the next section.

5. q-Recursion construction for the four-point correlation function

We can use the analytic properties of the superconformal blocks in \( \Delta \), described in Sec. 4, to write a more convenient “elliptic” recursive representation. For this, we must know the asymptotic behavior of the superconformal blocks in the limit \( \Delta \to \infty \). At the moment, we are not able to derive this behavior from general principles, but with differential equation (35) in hand, we can calculate the series expansions of the degenerate superconformal blocks up to a high order in \( x \). Based on this calculation, we conclude that the asymptotic behavior is similar to the asymptotic behavior in the previous case [2]:

\[ \mathcal{F}_c \left( \frac{\hat{\Delta}_1}{\Delta_2} \Delta_3 \Delta_4 \bigg| \Delta \bigg| x \right) = (16q)^{\Delta - Q^2/8 + \sum_{i=1}^{\Delta} \frac{(1 - \Delta_1/2 - \Delta_2)(1 - \Delta_1 + 1/2 - \Delta_3)}{\theta_3^{2+4\sum_{i=1}^{\Delta} \Delta_i - 3Q^2/2}(q)} \]
\[ \times H_c \left( \frac{\hat{\lambda}_1}{\lambda_2} \frac{\lambda_3}{\lambda_4} \bigg| \Delta \bigg| q \right), \] (5.1)

\[ \mathcal{F}_o \left( \frac{\hat{\Delta}_1}{\Delta_2} \Delta_3 \Delta_4 \bigg| \Delta \bigg| x \right) = (16q)^{\Delta - Q^2/8 + \sum_{i=1}^{\Delta} \frac{(1 - \Delta_1/2 - \Delta_2)(1 - \Delta_1 + 1/2 - \Delta_3)}{\theta_3^{2+4\sum_{i=1}^{\Delta} \Delta_i - 3Q^2/2}(q)} \]
\[ \times H_o \left( \frac{\hat{\lambda}_1}{\lambda_2} \frac{\lambda_3}{\lambda_4} \bigg| \Delta \bigg| q \right), \]
where we introduce the “elliptic” blocks $H_{e,o}(\Delta, q)$. The elliptic parameter is $q = \exp(i\pi\tau)$, where

$$\tau = i\frac{K(1-x)}{K(x)}.$$  \hfill (5.2)

The asymptotic behavior of $H_{e,o}$ is independent of the internal conformal dimension $\Delta$:

$$H_{e}(\Delta, q) = \theta_3(q^2) + O(\Delta^{-1}),$$
$$H_{o}(\Delta, q) = \theta_2(q^2) + O(\Delta^{-1}),$$ \hfill (5.3)

and

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}.$$ \hfill (5.4)

The elliptic blocks satisfy the relations

$$H_{e}(\Delta, q) = \theta_3(q^2) + \sum_{m,n \text{ even}} \frac{q^{mn/2} R_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} H_{e}(\Delta_{m,-n}, q)$$
$$+ \sum_{m,n \text{ odd}} \frac{q^{mn/2} R_{m,n}^{(o)}}{\Delta - \Delta_{m,n}} H_{o}(\Delta_{m,-n}, q),$$ \hfill (5.5)

$$H_{o}(\Delta, q) = \theta_2(q^2) + \sum_{m,n \text{ even}} \frac{q^{mn/2} R_{m,n}^{(o)}}{\Delta - \Delta_{m,n}} H_{o}(\Delta_{m,-n}, q)$$
$$+ \sum_{m,n \text{ odd}} \frac{q^{mn/2} R_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} H_{e}(\Delta_{m,-n}, q),$$

where the residues are simply

$$R_{m,n}^{(e,o)} = \frac{P_{m,n}^{(e,o)} (\lambda_1 + \lambda_2) P_{m,n}^{(o,e)} (\lambda_1 - \lambda_2) P_{m,n}^{(e,o)} (\lambda_3 + \lambda_4) P_{m,n}^{(o,e)} (\lambda_3 - \lambda_4)}{r_{m,n}}.$$ \hfill (5.6)

Relations (5.5) allow recursively evaluating the series expansions of the elliptic blocks in $q$,

$$H_{e,o}(\Delta, q) = \sum_{N \in \mathbb{Z}, \mathbb{Z}/2} q^N h_{e,o}^{(N)}(\Delta),$$ \hfill (5.7)

where again $H_{e}(\Delta, q)$ expands in nonnegative integer powers of $q$ and $H_{o}(\Delta, q)$ expands in positive half-integer ones. Relations (5.5) give

$$h_{e,o}^{(N)}(\Delta) = \eta_{e,o}^{(N)} + \sum_{m,n \text{ even}}^{mn/2 \leq N} \frac{P_{m,n}^{(e,o)}(\Delta_{m,-n})}{\Delta - \Delta_{m,n}} + \sum_{m,n \text{ odd}}^{mn/2 \leq N} \frac{P_{m,n}^{(o,e)}(\Delta_{m,-n})}{\Delta - \Delta_{m,n}},$$ \hfill (5.8)

where $\eta_{e}^{(N)}$ and $\eta_{o}^{(N)}$ are coefficients in the $q$-expansion of the corresponding asymptotic forms (5.3). The power series in $q$ for the “elliptic” blocks converge for $|q| < 1$, i.e., on the
whole covering of the $x$ plane with three punctures. With structure constants (2.7) and the superconformal blocks known, we can now write the universal construction for four-point function (2.6):

\[
\langle W_{a_1}(x)V_{a_2}(0)V_{a_3}(1)V_{a_4}(\infty) \rangle = \frac{\hat{\theta}_3(q)\hat{\theta}_3(\bar{q})}{[\theta_3(q)\theta_3(\bar{q})]^2} \frac{\hat{\theta}_3(q)\hat{\theta}_3(\bar{q})}{[\theta_3(q)\theta_3(\bar{q})]^2+4\sum_\Delta \Delta^{-2}Q^2/2} \mathcal{H} \left( \begin{array}{c} \hat{a}_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \left| \tau, \bar{\tau} \right. \right),
\]

where

\[
\mathcal{H} \left( \begin{array}{c} \hat{a}_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \left| \tau, \bar{\tau} \right. \right) = \int \frac{dP}{4\pi} \left| 16q \right|^{P^2} \left[ \tilde{C}Q^{2+iP} CQ^{2-iP} H_e(\Delta_P, q)H_o(\Delta_P, \bar{q}) + CQ^{2+iP} \tilde{C}Q^{2-iP} H_o(\Delta_P, q)H_e(\Delta_P, \bar{q}) \right]
\]

and $\Delta_P = Q^2/8 + P^2/2$. The crossing symmetry leads to the nontrivial relation in terms of the auxiliary function $\mathcal{H}$

\[
\mathcal{H} \left( \begin{array}{c} \hat{a}_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \left| \tau, \bar{\tau} \right. \right) = (\tau \bar{\tau})^{3Q^2/4-2\sum_\Delta} \Delta^{-2} \mathcal{H} \left( \begin{array}{c} \hat{a}_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \left| -\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right. \right).
\]

Using the numerical algorithm based on the elliptic representation of the blocks, we have verified this relation for the external parameters $a_1 = a_2 = a_3 = a_4 = Q/2$ and for different values of the parameters $b$ and $\tau$. Taking terms up to the order $q^6$ in the power series for $H_{e,o}(q)$ into account, we obtain a wide range of distances where the results differ in the fifth digit, and the convergence improves as the expansion order increases.

6. Discussion

We have analyzed the NS sector of the $N=1$ supersymmetric Liouville field theory. We obtained the four-point correlation function $\langle WVVV \rangle$, involving one “top” component $W$ of a primary supermultiplet, while the others are “bottom” components. This is one of the four basic spinless four-point correlation functions in the theory. Another correlation function $\langle VVVV \rangle$, involving only the “bottom” components $V$, was recently considered in [2]. The other two basic correlation functions $\langle VVWV \rangle$ and $\langle WVWV \rangle$, are constructed in term of the corresponding superconformal blocks (4.5),(4.6) and (4.7),(4.8) in the same way as it was discussed in this paper for the correlation function $\langle WVVV \rangle$. All other spinless four-point functions involving top and bottom supermultiplet components are expressed in term of these four functions using superprojective Ward identities.

Our analysis generalizes the approach developed quite long ago in [21, 22] for the ordinary CFT. The construction involves the corresponding four-point superconformal blocks, for which an efficient recursive calculation procedure was offered. This approach is based on knowing the asymptotic properties of the conformal blocks in $\Delta$. A straightforward derivation requires a separate investigation and is the subject of a future report. But with the
additional assumption about the general structure of this asymptotic behavior, we could recover its explicit form by analyzing the degenerate case and the corresponding differential equation. We verified our assumption up to the order $q^{20}$. Remarkably, the asymptotic behavior of the superconformal blocks is the same as that considered in [2] for another correlation function. Apparently, all four basic correlation functions have the same asymptotic properties.

Of course, to complete the analysis of the four-point sector in the super Liouville field theory, the analogous research should be done for the Ramond sector of the theory. We hope that the methods discussed here will help in considering minimal supergravity, where, in contrast to the bosonic analogue for which the matrix model approach is known, it will be the only way to calculate numerically.

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A. Derivation of the differential equations for the correlation functions $\langle \hat{V}_{-b} \hat{V}_1 \hat{V}_2 \hat{V}_3 \rangle$

First, we want to derive the supersymmetric form of the differential equations that follow from decoupling condition (1.9). Temporarily ignoring the existence of the “left” subalgebra $SVir$ and completely omitting the dependence of the fields on $\bar{z}$ and $\bar{\theta}$, we write

$$\hat{V}_{-b}(z, \theta) = (1 + \theta G_{-1/2}) V_{-b}(z).$$

(A.1)

It is straightforward to verify the identity

$$(1 - \theta G_{-1/2})(G^3_{-1/2} + b^2 G_{-3/2})V_{-b} = \left( \frac{d}{dz} D + b^2 (G_{-3/2} + 2\theta L_{-2}) \right) \hat{V}_{-b} = 0,$$

(A.2)

where the supercovariant derivative $D$ is defined as

$$D = \frac{d}{d\theta} - \theta \frac{d}{dz}.$$  

(A.3)

Inside the correlation function, we can rewrite (A.2) more universally. In particular, for the four-point function, we have

$$\left\langle \left( \frac{d}{dz} D + b^2 \int_{C_z} \frac{du}{u - z} \hat{S}(u, \theta) \right) \hat{V}_{-b} \hat{V}_1 \hat{V}_2 \hat{V}_3 \right\rangle = 0,$$

(A.4)

where $\hat{S}(u, \theta) = S(u) - 2\theta T(u)$ is “super stress–energy tensor” generating superalgebra (1.2) and the contour $C_z$ encompasses the point $z$. We note that the “odd” argument of $S$ is the same as that of the field $V_{-b}$. 

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The next standard step is to use supercurrent Ward identities. We deform the integration contour $C_z = - \sum_k C_k$ (permutations of the superfields do not affect the sign of the whole expression), and taking the OPE written in the supersymmetric form
\[
\tilde{S}(u, \theta') \tilde{V}_\Delta(z, \theta) = \frac{2\Delta(\theta - \theta') \tilde{V}_\Delta(z, \theta)}{(u - z)^2} + \frac{2(\theta - \theta') \partial \tilde{V}_{-\Delta}(z, \theta)}{u - z + \theta \theta'} + \frac{D \tilde{V}_{-\Delta}(z, \theta)}{u - z + \theta \theta'} + \text{reg}
\]  
(A.5)
into account, we immediately derive the supersymmetric partial differential equation for $\langle \tilde{V}_{-b} \tilde{V}_1 \tilde{V}_2 \tilde{V}_3 \rangle$:
\[
\left\{-b^{-2} \partial D_4 + \sum_{i=1}^3 \left\{ \frac{2\Delta_i}{z_{4i}^2} \theta_{4i} + \frac{1}{z_{4i}} \left[ 2\theta_{4i} \partial_i - D_i \right] \right\}\right\} \langle \tilde{V}_{-b}(z_4) \tilde{V}_1(z_1) \tilde{V}_2(z_2) \tilde{V}_3(z_3) \rangle = 0,
\]  
(A.6)
where the arguments are indexed for future purposes. In accordance with (2.1), we then introduce the function
\[
g(z, \tau_1, \tau_2) = g_0(z) + g_1(z) \tau_1 + g_2(z) \tau_2 + g_3(z) \tau_1 \tau_2
\]  
(A.7)
related to the four-point correlation function by
\[
\langle \tilde{V}_{-b}(z_4) \tilde{V}_1(z_1) \tilde{V}_2(z_2) \tilde{V}_3(z_3) \rangle = z_3^{2\Delta_4} z_1^{2\Delta_3} z_3^{2\Delta_2} z_1^{2\Delta_1} g(z, \tau_1, \tau_2).
\]  
(A.8)
Here, we again neglect the antiholomorphic dependence of the correlation function. After the field arguments are renumbered, the cross ratio is $z = z_{41}z_{23}/z_{43}z_{21}$, and the exponents are given by
\[
\begin{align*}
\gamma_{34} & = -2\Delta_4, \\
\gamma_{13} & = -\Delta_1 - \Delta_3 + \Delta_4 + \Delta_2, \\
\gamma_{23} & = -\Delta_2 - \Delta_4 + \Delta_4 + \Delta_1, \\
\gamma_{12} & = -\Delta_4 - \Delta_1 - \Delta_2 + \Delta_3.
\end{align*}
\]  
(A.9)
We derive the equation for the function $g$ from (A.6):
\[
\left[ -b^{-2} \partial_4 D_4 - b^{-2} \frac{\gamma_{34}}{z_{41}z_{42}} \left( \theta_{41} \partial_4 - D_4 \right) + \sum_{i=1}^3 \left\{ \frac{2\Delta_i}{z_{4i}^2} \theta_{4i} + \frac{1}{z_{4i}} \left[ 2\theta_{4i} \partial_i - D_i \right] \right\} \right]
\]
\[
+ \frac{\gamma_{12} \theta_{41} + \theta_{42}}{z_{41}z_{42}} + \frac{\gamma_{13} \theta_{41} + \theta_{43}}{z_{41}z_{43}} + \frac{\gamma_{23} \theta_{42} + \theta_{43}}{z_{42}z_{43}} + \frac{\gamma_{12} \theta_{43}}{z_{43}^2} \right] g = 0,
\]  
(A.10)
For the functions $g_i$, PDE (A.10) reduces to an ordinary differential equation in $z$ (of the second order) of the form
\[
A_i g''_i + B_i g'_i + C_i g_i = 0,
\]  
(A.11)
where the explicit expressions for $A_i$, $B_i$, and $C_i$ are
\[
\begin{align*}
A_0 & = -b^{-2} D_4 z \partial_4 z, \\
A_1 & = -b^{-2} D_4 z \partial_4 z \tau_1, \\
A_2 & = -b^{-2} D_4 z \partial_4 z \tau_2, \\
A_3 & = -b^{-2} D_4 z \partial_4 z \tau_1 \tau_2,
\end{align*}
\]  
(A.12)
where \( R \) is a constant. In the same manner, we write all the derivatives explicitly in terms of \( \tau \) and \( \eta \):

\[
B_0 = -b^{-2} \partial_4 D_4 z - b^{-2} \gamma_{34}^2 (\partial_4 \partial_4 z - D_4 z) + \sum_{i=1}^{3} \frac{1}{z_{4i}} (2 \partial_{4i} \partial_i z - D_i z),
\]

\[
B_1 = B_0 \tau_1,
\]

\[
B_2 = B_0 \tau_2 - b^{-2} D_4 \partial_4 \tau_2 - b^{-2} \partial_4 z D_4 \tau_2,
\]

\[
B_3 = B_0 \tau_1 \tau_2 + (b^{-2} D_4 \partial_4 \tau_2 + b^{-2} \partial_4 z D_4 \tau_2) \tau_1,
\]

\[
C_0 = \sum_{i=1}^{3} \frac{2 \Delta_i}{z_{4i}^2} \theta_{4i} + \gamma_{12} (\theta_{41} + \theta_{42}) \frac{1}{z_{41} z_{42}} + \gamma_{13} (\theta_{41} + \theta_{43}) \frac{1}{z_{41} z_{43}} + \gamma_{23} (\theta_{42} + \theta_{43}) \frac{1}{z_{42} z_{43}} + \gamma_{34} \theta_{43} \frac{1}{z_{43}^2},
\]

\[
C_1 = \sum_{i=1}^{3} \frac{1}{z_{4i}} (2 \theta_{4i} \partial_i \tau_1 - D_i \tau_1) + C_0 \tau_1,
\]

\[
C_2 = -b^{-2} \partial_4 D_4 \tau_2 - b^{-2} \gamma_{34}^2 (\theta_{43} \partial_4 \tau_2 - D_4 \tau_2) + \sum_{i=1}^{3} \frac{1}{z_{4i}} (2 \theta_{4i} \partial_i \tau_2 - D_i \tau_2) + C_0 \tau_2,
\]

\[
C_3 = b^{-2} \Delta_1 \partial_4 D_4 \tau_2 + b^{-2} \gamma_{34} \theta_{43} \partial_4 \tau_2 - D_4 (\tau_2) \tau_1
\]

\[
+ \sum_{i=1}^{3} \frac{1}{z_{4i}} [(2 \theta_{4i} \partial_i \tau_1 - D_i \tau_1) \tau_2 - (2 \theta_{4i} \partial_i \tau_2 - D_i \tau_2) \tau_1] + C_0 \tau_1 \tau_2.
\]

To simplify the calculations, we choose the “odd” superprojective invariants slightly differently here than in the main text in (2.2): \( \tau_1 = \eta \) and \( \tau_2 = \eta \).

Our next task is to find explicit expressions for each term in (A.12), (A.13), and (A.14). We again use superprojective invariance, fixing the particular choice

\[
\theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = R \eta,
\]

\[
x_1 = 0, \quad x_2 = 1, \quad x_3 = R, \quad x_4 = x,
\]

where \( R \to \infty \). We calculate the coefficients \( A_i \), \( B_i \), and \( C_i \) in terms of \( z \), \( \tau_1 \), and \( \tau_2 \). In particular,

\[
\eta = \tau_1,
\]

\[
\theta_4 = z^{1/2} (z - 1)^{1/2} \tau_2,
\]

\[
x = z [1 - z^{1/2} (z - 1)^{1/2} \tau_1 \tau_2].
\]

In the same manner, we write all the derivatives explicitly in terms of \( z \), \( \tau_1 \), and \( \tau_2 \), which results in the expressions for the coefficients \( A_i \), \( B_i \), and \( C_i \):

\[
A_0 = -b^{-2} z \tau_1 + b^{-2} z^{1/2} (z - 1)^{1/2} \tau_2,
\]

\[
A_1 = -b^{-2} z^{1/2} (z - 1)^{1/2} \tau_1 \tau_2,
\]

\[
A_2 = -b^{-2} z \tau_1 \tau_2,
\]

\[
A_3 = 0.
\]
The second set of equations, coming from the terms that are even in \( \tau_1 \) and \( \tau_2 \), involve \( g_1 \) and \( g_2 \):

\[
-b^{-2} z^{1/2}(z - 1)^{1/2} g''_1 - b^{-2} z g'_1 + \frac{3z - 1}{z(z - 1)} z^{1/2}(z - 1)^{1/2} g'_1 + (1 + b^{-2}) \frac{3z - 2}{z - 1} g_2 = 0,
\]

\[
-b^{-2} z^{1/2}(z - 1)^{1/2} g''_2 - b^{-2} z g'_2 + \frac{3z - 1}{z(z - 1)} z^{1/2}(z - 1)^{1/2} g'_2 + (1 + b^{-2}) \frac{3z - 2}{z - 1} g_1 = 0.
\]

Finally, we replace \( \tau_2 \) with \( z^{1/2}(z - 1)^{1/2} \tau_2 \). With (A.7) taken into account, this is equivalent to replacing \( g_2 \) with \( z^{-1/2}(z - 1)^{-1/2} g_2 \) and \( g_3 \) with \( z^{-1/2}(z - 1)^{-1/2} g_3 \). This substitution converts (A.20) and (A.21) to the more transparent forms (3.3) and (3.4).
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