YONEDA EXT-ALGEBRAS OF TAKEUCHI SMASH PRODUCTS
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ABSTRACT. We prove that the Yoneda Ext-algebra of a Takeuchi smash product is the graded Takeuchi smash product of the Yoneda Ext-algebras of the two algebras or modules involved. As an application, we prove that graded Takeuchi smash products preserve Artin-Schelter regularity, and describe the Nakayama automorphism of the product.

INTRODUCTION

Let $H$ be a Hopf algebra with bijective antipode, $A$ be a left $H$-module algebra and $B$ be a left $H$-comodule algebra. There is a canonical algebra construction on the space $A \otimes B$ called the Takeuchi smash product of $A$ and $B$, denoted by $A \# B$ (see Definition 1.1). Usual smash products and Ore extensions are examples of Takeuchi smash products (see Example 1.2).

If $A$ is a positively graded algebra and $G$ is a finite group acting gradely on $A$, Martinez-Villa [Mar, Theorem 10] proved that the Yoneda Ext-algebra of the skew group algebra $A \# G$ is isomorphic to the skew group algebra $\text{Ext}_A^\bullet(A_0, A_0) \# G$ under some mild conditions. Bergh and Oppermann proved that the Yoneda Ext-algebra of a class of Takeuchi smash product (see Example 1.3) is isomorphic to the Takeuchi smash product of the Yoneda Ext-algebras [BO, Theorem 3.7]. The main purpose of this paper is to give a precise description of the Yoneda Ext-algebras of the general Takeuchi smash products.

Let $M$ be a left $A \# H$-module and $X$ be a left $(H, B)$-Hopf module. Then there is a natural left $A \# B$-module structure on $M \otimes X$ denoted by $M \# X$, which is called the Takeuchi smash product module of $M$ and $X$. If $P_\bullet \to M \to 0$ is a projective $A \# H$-module resolution of $M$, then $\text{Hom}_A^\bullet(P_\bullet, P_\bullet)$ is a differential graded left $H$-module algebra and $\text{Ext}_A^\bullet(M, M) = \bigoplus_{n \geq 0} \text{Ext}_A^n(M, M)$ is a graded left $H$-module algebra. If $X$ has a resolution $Q_\bullet \to X \to 0$ in the category of left $(H, B)$-Hopf modules such that each $Q_i$ is finitely generated as $B$-module, then $\text{Hom}_B^\bullet(Q_\bullet, Q_\bullet)$ is a differential graded left $H$-comodule algebra and $\text{Ext}_B^\bullet(X, X)$ is a graded left $H$-comodule algebra. Therefore, we can form the graded Takeuchi smash product algebra $\text{Ext}_A^\bullet(M, M) \# \text{Ext}_B^\bullet(X, X)$.

The following is one of the main results in this paper, which generalizes the result in the usual tensor product case, and [Mar, Theorem 10], [BO, Theorem 3.7]. Theorem 4.2 is the right-sided version of it.

**Theorem 0.1.** Suppose that $M$ is a left $A \# H$-module which is pseudo-coherent as $A$-module, and $X$ is a left $(H, B)$-Hopf module which is pseudo-coherent as $B$-module. Then, as graded algebras,

$$\text{Ext}_{A \# B}^\bullet(M \# X, M \# X) \cong \text{Ext}_A^\bullet(M, M) \# \text{Ext}_B^\bullet(X, X)$$

where the last $\#$ means graded Takeuchi smash product.

A module is called pseudo-coherent if it has a finitely generated projective resolution.

The other result in this paper is the graded Takeuchi smash product of AS-regular algebras is AS-regular (see Theorem 4.3). In the case that $H$ is trivial or the actions of $H$ is trivial, the result is well known.

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Theorem 0.2. Let $A$ and $B$ be AS-regular algebras of dimension $d_1$ and $d_2$ respectively. If $H$ is a positively graded Hopf algebra, $A$ is an $H$-module graded algebra and $B$ is an $H$-comodule graded algebra, then $A\#B$ is an AS-regular algebra of dimension $d_1 + d_2$.

The Nakayama automorphism of $A\#B$ is described as an application of Theorems 0.1 and 0.2.

The paper is organized as follows. In section 1, we define Takeuchi smash products of algebras or modules, and give some basic facts. In section 2, we study homological properties of Takeuchi smash products, in particular, the Tor-groups and Ext-groups. In section 3, we introduce Takeuchi smash products of differential graded algebras, and prove Theorem 3.2 that is, the Yoneda Ext-algebra of the Takeuchi smash product of modules is the graded Takeuchi smash product of the Yoneda Ext-algebras of the modules involved. In section 4, we prove that Takeuchi smash product preserves the AS-regularity, and describe the Nakayama automorphism of the product.

1. Takeuchi smash products

Throughout, let $k$ be a field. All algebras and modules are over $k$, and the unmarked tensor product $\otimes$ (resp. Hom) means $\otimes_k$ (resp. Hom$_k$). For the basic concept and theory concerning Hopf algebras we refer to $\text{Sw}$ and $\text{Mo}$ as basic references.

1.1. Takeuchi smash products of algebras. Let $H$ be a Hopf algebra, $A$ be a left $H$-module algebra and $B$ be a left $H$-comodule algebra $\text{[Mo, Definitions 4.1.1, 4.1.2]}$. The following construction (see $\text{Tak, section 8}$ and the references therein) is a generalization of smash products, which is now called Takeuchi smash product in literature.

Definition 1.1. The Takeuchi smash product $A\#B$ of $A$ and $B$ is the associative algebra with underlying $k$-space $A \otimes B$, and multiplication defined by

$$(a \# b)(a' \# b') := \sum_{(b)} a(b_{-1} \rightarrow a') \# b_0 b'$$

for any $a, a' \in A$ and $b, b' \in B$, where $\sum_{(b)} b_{-1} \otimes b_0 = \rho(b)$ is given by the comodule structure map of $B$.

If $B = H$, the Takeuchi smash product $A\#B$ is the usual smash product $A\#H$ $\text{[Mo, Definition 4.1.3]}$.

Example 1.2. Any Ore extension $A[x, \sigma, \delta]$ can be viewed as a Takeuchi smash product, where $A$ is a $k$-algebra, $\sigma$ is an algebra automorphism of $A$ and $\delta$ is a $\sigma$-derivation of $A$.

In fact, let $H = k\langle g^\pm 1, X \rangle$ be the free algebra with the Hopf algebra structure given by

$\Delta(g) = g \otimes g, \Delta(X) = g \otimes X + X \otimes 1, \varepsilon(g) = 1, \varepsilon(X) = 0, S(g) = g^{-1}, S(X) = -g^{-1}X.$

Then $A$ has a left $H$-module structure, given by $g \rightarrow a = \sigma(a)$ and $X \rightarrow a = \delta(a)$. It is easy to see that

$g \rightarrow (ab) = (g \rightarrow a)(g \rightarrow b)$ and $X \rightarrow (ab) = (g \rightarrow a)(X \rightarrow b) + (X \rightarrow a)b$

for all $a, b \in A$. So $m_A$ and $u_A$ are $H$-module morphisms, that is, with the action defined above $A$ is a left $H$-module algebra.

Let $B = k[X]$ be the subalgebra of $H$ generated by $X$. In fact, $B$ is a left coideal subalgebra of $H$. The Ore extension $A[x; \sigma, \delta]$ is isomorphic to the Takeuchi smash product $A\#B$, which is a subalgebra of $A\#H$. 

Example 1.3. A tensor product of graded algebras twisted by a bicharacter can also be viewed as a Takeuchi smash product. Let $A$ and $B$ be two $k$-algebras that are graded by abelian groups $G$ and $L$ respectively: $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{l \in L} B_l$.

Let $t : G \times L \to k^\times$ be a bicharacter, that is,

$$t(g + g', l) = t(g, l)t(g', l) \text{ and } t(g, l + l') = t(g, l)t(g, l')$$

for any $g, g' \in G$ and $l, l' \in L$.

With this data, the twisted tensor product $A \otimes^t B$ (see \cite{BO} Definition/Construction 2.2) is by definition the vector space $A \otimes B$, with multiplication given by

$$(a \otimes b)(a' \otimes b') = t(g, l) a a' \otimes b b',$$

where $a \in A$, $a' \in A_g$, $b \in B_l$ and $b' \in B$.

In fact, $B$ is a $kL$-comodule algebra, and $A$ is a $kL$-module algebra via $l \to a := t(g, l)a$ for any $g \in G$, $l \in L$ and $a \in A_g$. The twisted tensor product $A \otimes^t B$ is the Takeuchi smash product $A \#_B$ over the Hopf algebra $kL$.

In general, any Takeuchi smash product is a twisted tensor product \cite{CSV}. Let $A$ and $B$ be two algebras, and $\tau : B \otimes A \to A \otimes B$ be a linear map. Then there is a multiplication map on the vector space $A \otimes B$ given by

$$m_{A \otimes^\tau B} := (m_A \otimes m_B) \circ (\text{id}_A \otimes \tau \otimes \text{id}_B)$$

where $m_A$ and $m_B$ are the multiplication maps of $A$ and $B$ respectively. If, with this multiplication, $A \otimes B$ is an associative algebra with unit $1_A \otimes 1_B$, then it is called a twisted tensor product determined by $\tau$ and is denoted by $A \otimes^\tau B$. If $A \# B$ is a Takeuchi smash product, then it is the twisted tensor product $A \otimes^\tau B$ given by $\tau : B \otimes A \to A \otimes B$, $b \otimes a \mapsto \sum(b_1 \to a) \otimes b$.

1.2. Takeuchi smash product of modules. Let $A\mathcal{M}$ (resp. $\mathcal{M}_A$) denote the left (resp. right) $A$-module category for an algebra $A$. Let $A$ be a left $H$-module algebra and $B$ be a left $H$-comodule algebra. Recall that a left $B$-module $X$ with a left $H$-coaction $\rho$ is called a left $(H, B)$-Hopf module \cite{Ma} Definition 8.5.1 if $\rho(bx) = \rho(b)\rho(x)$ for all $b \in B$ and $x \in X$. Let $A \mathcal{M}_B$ denote the left $(H, B)$-Hopf module category. The $(H, B)$-Hopf module category $H\mathcal{M}_B$ is defined analogously.

Note that $A \# B \to (A \# H) \otimes B$, $a \# b \mapsto \sum(b)(a \# b_1) \otimes b_0$ is an algebra map.

For any $M \in A \# H \mathcal{M}$ and $X \in B \mathcal{M}$, there is a natural left $A \# B$-module structure on $M \otimes X$, given by

$$a \# b \mapsto m \otimes x = \sum(b)(a(b_1 \to m) \otimes b_0 x = \sum(b)(a \# b_1)m \otimes b_0 x$$

for any $a \in A$, $b \in B$, $m \in M$ and $x \in X$. This module, denoted by $M \# X$, is called the Takeuchi smash product of $A \# H M$ and $B X$, where the element $m \otimes x$ in $M \# X$ is denoted by $m \# x$.

For any $M \in A \# H \mathcal{M}$ and $X \in B \mathcal{M}$, there is also a right $A \# B$-module structure on $M \otimes X$ given by

$$(m \otimes x)(a \# b) = \sum(b)m(a \# b_1) \otimes xb_0.$$

This module, denoted also by $M \# X$, is called the Takeuchi smash product of $M_{A \# H}$ and $X_B$.

For any $N \in A \mathcal{M}$ and $Y \in H \mathcal{M}_B$, there is a natural right $A \# B$-module structure on $N \otimes Y$, given by

$$n \otimes y \leftarrow a \# b = \sum(y)n(y_1 \to a) \otimes y_0 b$$

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for any $a \in A$, $b \in B$, $n \in N$ and $y \in Y$. This module, denoted by $N\# Y$, is called the Takeuchi smash product of $N_A$ and $Y \in H\mathcal{M}_B$, where the element $n \otimes y$ in $N\# Y$ is denoted by $n\#y$.

For any $N \in \mathcal{A}\mathcal{M}$ and $Y \in H\mathcal{M}$, there is a left $A\# B$-module structure on $N \otimes Y$ which is given by

$$(a\# b)(n \otimes y) = \sum_{(b)} (S^{-1}(b_{-1}y_{-1}) \to a)n \otimes b_0y_0.$$  

This module, denoted also by $N\# Y$, is called the Takeuchi smash product of $AN$ and $Y \in H\mathcal{M}$.

**Remark 1.4.** A right-sided version of Takeuchi smash product $B\# A$ is defined similarly for a right $H$-comodule algebra $B$ and a right $H$-module algebra $A$. There are also four types of Takeuchi smash product of modules similar to the module structures defined by (1.1), (1.2), (1.3) and (1.4). The module structures given by (1.3) and (1.4) seem less natural than the two given by (1.1) and (1.2), but they are dual to the two natural ones in the right-sided version in some sense (in particular when $H$ is finite-dimensional).

### 1.3. Adjointness

The following lemma is modified from the usual smash product case, the proof is routine.

**Lemma 1.5.** Let $M \in A\# H\mathcal{M}$, $X \in B\mathcal{M}$ and $T \in A\# B\mathcal{M}$.

1. There is a natural left $B$-module structure on $\text{Hom}_A(M, T)$, given by

$$(b \to f)(m) = \sum_{(b)} b_0f(S^{-1}b_{-1} \to m).$$

2. The Hom-Tensor adjoint isomorphism $\text{Hom}(X, \text{Hom}(M, T)) \xrightarrow{\cong} \text{Hom}(M \otimes X, T)$ restricts to a natural isomorphism $\text{Hom}_B(X, \text{Hom}_A(M, T)) \xrightarrow{\cong} \text{Hom}_{A\# B}(M\# X, T)$, that is, $\text{Hom}_A(M, -) : A\# B\mathcal{M} \to B\mathcal{M}$ and $M\#: B\mathcal{M} \to A\# B\mathcal{M}$ is a pair of adjoint functors.

Then, for any $M \in A\# H\mathcal{M}$ and $T \in A\# B\mathcal{M}$, there exists a natural $B$-module structure on the Ext-groups $\text{Ext}^n_{A\# B}(M, T)$. In fact, the $H$-module structure on the Ext-groups can be induced by any resolution of $M$ in $A\# H\mathcal{M}$ such that each term is projective $A$-module, or any resolution of $T$ in $A\# B\mathcal{M}$ of which all terms are injective $A$-modules. And these two induced $B$-module structures on $\text{Ext}^n_{A\# B}(M, T)$ coincide, as shown in the following lemma.

**Lemma 1.6.** Let $M \in A\# H\mathcal{M}$ and $T \in A\# B\mathcal{M}$. Let $P_\bullet \to M \to 0$ and $P'_\bullet \to M \to 0$ be exact sequences in $A\# H\mathcal{M}$, where all $P_n$ and $P'_n$ are projective $A$-modules, and let $0 \to T \to I^\bullet$ and $0 \to T \to I'^\bullet$ be exact sequences in $A\# B\mathcal{M}$, where all $I^n$ and $I'^n$ are injective $A$-modules. Then, for any $n \in \mathbb{N}$, as $B$-modules,

$H^n(\text{Hom}_A(P_\bullet, T)) \cong H^n(\text{Hom}_A(M, I'^\bullet)) \cong H^n(\text{Hom}_A(P'_\bullet, T)) \cong H^n(\text{Hom}_A(M, I^\bullet)).$

**Proof.** Consider the double complex $\text{Hom}_A(P_\bullet, I^\bullet)$ of $B$-modules. Then

$H^*(\text{Hom}_A(P_\bullet, T)) \cong H^*(\text{Hom}_A(M, I^\bullet))$

as $B$-modules.

For any $M_{A\# H}$, $X_B$ and $T_{A\# B}$, there is a right module version of Lemma 1.5 and Lemma 1.6.

There is also a version of Lemma 1.5 for any $N \in \mathcal{M}_A$, $Y \in H\mathcal{M}_B$ and $T \in \mathcal{M}_{A\# B}$, which can be viewed as a dual version.

**Lemma 1.7.** Let $N \in \mathcal{M}_A$, $Y \in H\mathcal{M}_B$ and $T \in \mathcal{M}_{A\# B}$.

1. There is a right $A$-module structure on $\text{Hom}_{B\#}(Y, T)$ given by

$$(fa)(y) = \sum_{(y)} f(y_0)(S^{-1}y_{-1} \to a).$$
Lemma 1.8. Let $X \in {}^B_B \mathcal{M}$ (resp. $^H_B \mathcal{M}$).

(1) Let $X$ viewed as left (resp. right) $B$-module is finitely generated if and only if there is a finite-dimensional left $H$-comodule $V$ and a surjective morphism $\pi : B \otimes V \to X$ in $^H_B \mathcal{M}$ (resp. $^H_B \mathcal{M}$).

(2) Suppose $X$ viewed as left (resp. right) $B$-module is pseudo-coherent. Then there is an exact sequence in $^H_B \mathcal{M}$ (resp. $^H_B \mathcal{M}$), where all the $V_n$'s are finite-dimensional left $H$-comodules,

$$\cdots \to B \otimes V_n \to \cdots \to B \otimes V_0 \to X \to 0 \text{ (resp. } \cdots \to V_n \otimes B \to \cdots \to V_0 \otimes B \to X \to 0).$$

Proof. (1) Let $\{x_1, \ldots, x_m\}$ be a set generators of $B\times$. By the Finiteness Theorem [Mo, 5.1.1], there exists a finite-dimensional left $H$-comodule $V \subseteq X$ such that $\{x_1, \ldots, x_m\} \subseteq V$. Hence a surjective morphism $\pi : B \otimes V \to X$ in $^H_B \mathcal{M}$ exists.

(2) By (1), there exists a finite-dimensional left $H$-comodule $V_0$ with a surjective morphism $\pi : B \otimes V_0 \to X$ in $^H_B \mathcal{M}$. By Schanuel’s lemma, the kernel of $\pi$ is also a pseudo-coherent $B$-module. Then the conclusion follows by induction. \(\square\)

1.4. Comodule structure on the B-morphism space of (H, B)-Hopf modules. The material in this subsection is essentially modified from [Ul, especially Ul Lemma 2.2].

For any $X, Y \in ^H_B \mathcal{M}$, we try to endow $\text{Hom}_B(X, Y)$ with a left $H$-comodule structure, say, when $B\times$ is finitely generated. So, firstly we have to define a map $\text{Hom}_B(X, Y) \to H \otimes \text{Hom}_B(X, Y)$. We use the natural embedding

$$H \otimes \text{Hom}_B(X, Y) \subseteq \text{Hom}_B(X, H \otimes Y), \quad h \otimes f \mapsto (x \mapsto h \otimes f(x)).$$

Let $\rho = \rho_Y : \text{Hom}_B(X, Y) \to \text{Hom}_B(X, H \otimes Y)$ be the morphism given by

$$(\rho f)(x) = \sum_{(x), (f(x))} (Sx_{-1})f(x_{-1}) \otimes f(x_0)$$

where $f \in \text{Hom}_B(X, Y)$ and $x \in X$. Let $N \subseteq \text{Hom}_B(X, Y)$ be the $H$-comodule structure, say, $N \subseteq \text{Hom}_B(X, H \otimes Y)$. Then, $N \subseteq \text{Hom}_B(X, Y)$ is a left $H$-comodule. A proof of this is added in the following for the convenience.

Lemma 1.9. Let $X, Y \in ^H_B \mathcal{M}$.

(1) $\text{Hom}_B(X, Y)$ has a left $H$-comodule structure as given by (1.6).

(2) If $B\times$ is finitely generated, then $\text{Hom}_B(X, Y) = \text{Hom}_B(X, Y)$, and so it is a left $H$-comodule (see also [Ul, Proposition 4.2]).

Proof. (1) For any $f \in \text{Hom}_B(X, Y)$, suppose $\rho(f) = \sum_i h_i \otimes f^i \in H \otimes \text{Hom}_B(X, Y)$, where $\{h_i\} \subseteq H$ and $\{f^i\} \subseteq \text{Hom}_B(X, Y)$ are $k$-linear independent respectively. Then

$$\rho(f)(x) = \sum_i h_i \otimes f^i(x) = \sum_{(x), (f(x))} (Sx_{-1})f(x_{-1}) \otimes f(x_0),$$

and

$$\sum_{(x)} Sx_{-1} \otimes \rho(f)(x_0) = \sum_{i, (x)} Sx_{-1} \otimes h_i \otimes f^i(x_0) = \sum_{(x), (f(x))} Sx_{-2} \otimes (Sx_{-1})f(x_{-1}) \otimes f(x_0).$$
By acting by \( \rho \) on the last component of both sides in the last equality, and switching the first two components obtained, then
\[
\sum_i \sum_{(x),(f'(x_0))} h^i \otimes (Sx_{-1})f^i(x_0) \otimes f'(x_0) = \sum_{(x),(f(x_0))} \sum_i (Sx_{-1})f(x_0) \otimes (Sx_{-2})f(x_0) \otimes f(x_0).
\]
Consider the images of \( \sum_i (h^i) \otimes h^i \otimes f^i \in H \otimes H \otimes \text{Hom}_B(X,Y) \) and \( \sum_i h^i \otimes \rho(f^i) \in H \otimes \text{Hom}_B(X,H \otimes Y) \) in \( \text{Hom}_B(X, H \otimes H \otimes Y) \). By acting on \( x \in X \), then, on one hand,
\[
(\Delta \otimes 1)\rho(f)(x) = (\Delta \otimes 1)\sum_i (h^i \otimes f^i(x)) = \sum_i (h^i \otimes h^i \otimes f^i(x)).
\]
On the other hand,
\[
(\Delta \otimes 1)\rho(f)(x) = \sum_{(x),(f(x_0))} (Sx_{-1})f(x_0) \otimes (Sx_{-2})f(x_0) \otimes f(x_0) \]
\[
= \sum_i \sum_{(x),(f'(x_0))} h^i \otimes (Sx_{-1})f^i(x_0) \otimes f'(x_0) = \sum_i h^i \otimes \rho(f^i(x))
\]
\[
= (1_H \otimes \rho)(h^i \otimes f^i)(x) = (1_H \otimes \rho)\rho(f)(x).
\]
Now it is easy to see that \( f^i \in \text{HOM}_B(X,Y) \) for all \( i \), and \( \text{HOM}_B(X,Y) \) is a left \( H \)-comodule.

(2) By Lemma 1.8, there exists a finite-dimensional left \( H \)-subcomodule \( V \) of \( X \), and a surjective morphism \( \pi : B \otimes V \to X \) in \( H_B \). So, \( \text{Hom}_B(X,Y) \xrightarrow{\pi^*} \text{Hom}_B(B \otimes V,Y) \) is injective, and the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Hom}_B(X,Y) & \xrightarrow{\pi^*} & \text{Hom}_B(B \otimes V,Y) \\
\rho_X & & \rho_{B \otimes V} \\
\text{Hom}_B(X, H \otimes Y) & \xrightarrow{\pi^*} & \text{Hom}_B(B \otimes V, H \otimes Y) \\
\downarrow & & \downarrow \circ \pi^* \\
H \otimes \text{Hom}_B(X,Y) & \xrightarrow{1_H \otimes \pi^*} & H \otimes \text{Hom}_B(B \otimes V,Y)
\end{array}
\]

Now for any \( f \in \text{Hom}_B(X,Y) \), suppose \( \rho_{B \otimes V}(f) = \sum h^i \otimes g^i \), where \( \{h^i\} \in H \) and \( \{g^i\} \in \text{Hom}_B(B \otimes V,Y) \) are \( k \)-linear independent respectively. Then, for any \( x' \in \ker \pi \),
\[
\sum h^i \otimes g^i(x') = \rho_X(f)(\pi(x')) = 0.
\]

It follows that \( g^i(x') = 0 \) for any \( i \), and so \( g^i = f^i \pi \) for some \( f^i \in \text{Hom}_B(X,Y) \). Since \( \text{Hom}_B(X,Y) \xrightarrow{\pi^*} \text{Hom}_B(B \otimes V,Y) \) is injective, \( \rho_X(f) = \sum h^i \otimes f^i \in H \otimes \text{Hom}_B(X,Y) \) and \( \text{Hom}_B(X,Y) = \text{HOM}_B(X,Y) \).

If fact, \( \text{HOM}_B(-,-) : (H_B \mathcal{M})^{op} \times H_B \mathcal{M} \to H_B \mathcal{M} \) is a functor. For any \( X, X' \in H_B \mathcal{M} \) with \( B \)-\( X \) pseudo-coherent, by Lemma 1.9 there is a natural \( H \)-comodule structure on \( \text{Ext}^1_B(X, X') \). The \( H \)-comodule structure on the Ext-groups is independent of the choices of the resolutions of \( X \) in \( H_B \mathcal{M} \) such that each term is finitely generated projective as \( B \)-module and the resolutions of \( X' \) in \( H_B \mathcal{M} \) of which all terms are injective as \( B \)-modules.

**Lemma 1.10.** Let \( X, X' \in H_B \mathcal{M} \). Suppose that \( P_* \to X \to 0 \), \( P'_* \to X \to 0 \to X' \to I^* \) and \( 0 \to X' \to I^* \) are exact sequences in \( H_B \mathcal{M} \), where all \( P_n \) and \( P'_n \) are finitely generated
projective as $B$-module, and all $I^n$ and $I^m$ are injective as $B$-module. Then, for any $n \in \mathbb{N}$, as $H$-comodules,

$$H^n(\text{Hom}_B(P_\bullet, X')) \cong H^n(\text{Hom}_B(X, I^*)) \cong H^n(\text{Hom}_B(P_\bullet', X') \cong H^n(\text{Hom}_B(X, I^*)).$$

Proof. Similar to the proof of Lemma 1.6 \hfill \Box

2. Some homological properties of Takeuchi smash products

In this section, we consider some (homological) properties of Takeuchi smash products of modules, especially, the decompositions of the Tor-groups and Ext-groups of Takeuchi smash products of modules.

Let $A$ be a left $H$-module algebra and $B$ be a left $H$-comodule algebra. With the natural module structure, $A(A\#B)$ and $(A\#B)B$ are obviously free modules.

Lemma 2.1. Both $(A\#B)_A$ and $B(A\#B)$ are free modules.

Proof. The conclusion follows from the $B$-$A$-bimodule isomorphism $\varphi : _B(A\#B)_A \rightarrow _B B \otimes _A A$ defined by $\varphi(a\#b) = \sum (b_0 \otimes (S^{-1}b_{-1} \rightarrow a)).$ \hfill \Box

Proposition 2.2. For any $M \in A\#H_M$, $X \in B_M$ and $T \in A\#B_M$, there is a convergent spectral sequence

$$\text{Ext}^p_M(X, \text{Ext}^q_N(M, T)) \Rightarrow \text{Ext}^{p+q}_{A\#B}(M\#X, T).$$

Proof. Let $P_\bullet$ be a projective resolution of $B_X$, and let $I^*$ be an injective resolution of $A\#B_T$. Since $A\#B$ is a flat right $A$-module by Lemma 2.1, each $I^i$ is injective as an $A$-module. Since $\text{Hom}_A(M, I^i)$ is a injective $B$-module by Lemma 1.6 then the double complex

$$\text{Hom}_{A\#B}(M\#P_\bullet, I^*) \cong \text{Hom}_B(P_\bullet, \text{Hom}_A(M, I^*))$$

yields the spectral sequence. \hfill \Box

Corollary 2.3. For any $X \in B_M$ and $M \in A\#H_M$, $\text{pd}_{A\#B}(M\#X) \leq \text{pd}_A M + \text{pd}_B X$. In particular, if both $A_M$ and $B_X$ are projective modules, then $M\#X$ is a projective $A\#B$-module.

There is also right module versions of Proposition 2.2 and Corollary 2.3.

Proposition 2.4. For any $N \in M_A$, $Y \in H_M$ and $T \in M_{A\#B}$, there is a convergent spectral sequence

$$\text{Ext}^p_A(N, \text{Ext}^q_B(Y, T)) \Rightarrow \text{Ext}^{p+q}_{(A\#B)^{op}}(N\#Y, T).$$

Corollary 2.5. For any $N \in M_A$, $Y \in H_M$, $\text{pd}_{(A\#B)^{op}}(N\#Y) \leq \text{pd}_A N + \text{pd}_B Y$. In particular, if both $A_N$ and $B_Y$ are projective modules, then $N\#Y$ is a projective right $A\#B$-module.

Proposition 2.6. Suppose $M \in A\#H_M$, $X \in B_M$, $N \in M_A$ and $Y \in H_M$. Then,

1. $(N\#Y) \otimes_{A\#B} (M\#X) \cong (N \otimes_A M) \otimes (Y \otimes_B X)$.
2. $\text{Tor}_n^B(N\#Y, M\#X) \cong \bigoplus_{p+q=n} \text{Tor}^A_p(N, M) \otimes \text{Tor}^B_q(Y, X)$.

Proof. (1) It is easy to check that the map $\phi : (N\#Y) \otimes_{A\#B} (M\#X) \rightarrow (N \otimes_A M) \otimes (Y \otimes_B X)$

$$(n\#y) \otimes_{A\#B} (m\#x) \mapsto (n \otimes_A y_{-1} \rightarrow m) \otimes (y_0 \otimes_B x)$$

is well-defined. The map $\phi$ is an isomorphism with the inverse $\phi^{-1}$ given by

$$(n \otimes_A m) \otimes (y \otimes_B x) \mapsto \sum_{(y)} (n\#y_0) \otimes_{A\#B} (S^{-1}y_{-1} \rightarrow m\#x).$$
Let $P_\bullet$ be a projective resolution of $A_{\# H} M$, and $Q_\bullet$ be a projective resolution of $B X$. By Corollary 2.3, the total complex $\text{Tot}(P_\bullet \# Q_\bullet)$ is a projective resolution of $A_{\# B}(M \# X)$. It follows from (1) that we have an isomorphism of complexes

$$(N \# Y) \otimes_{A_{\# B}} \text{Tot}(P_\bullet \# Q_\bullet) \cong \text{Tot}((N \otimes_A P_\bullet) \otimes (Y \otimes_B Q_\bullet)).$$

The conclusion follows by taking the homology. □

2.1. Ext-groups of Takeuchi smash product modules. As a special case in Lemma 1.5 (1) where $B = H$, $\text{Hom}_A(M, M')$ is a left $H$-module for $M, M' \in A_{\# H} \mathcal{M}$, with the $H$-action given by

$$(h \to f)(m) = \sum h_2 \mapsto f(S^{-1} h_1 \cdot m).$$

Lemma 2.7. Let $M, M' \in A_{\# H} \mathcal{M}$, $X \in \mathcal{H}_B \mathcal{M}$, and $X' \in B \mathcal{M}$. There is a morphism

$$\Phi : \text{Hom}_A(M, M') \otimes \text{Hom}_B(X, X') \to \text{Hom}_B(X, \text{Hom}_A(M, M') \otimes X')$$

$$f \otimes g \mapsto [f, g] : x \mapsto (x_{-1} \mapsto f) \otimes g(x_0)$$

where $\text{Hom}_A(M, M') \otimes X'$ is viewed as a left $B$-module via the algebra map $\rho : B \to H \otimes B$.

If $BX$ is finitely presented, then $\Phi$ is an isomorphism.

Proof. For any $k$-vector space $U$, the natural map $U \otimes \text{Hom}_B(X, X') \to \text{Hom}_B(X, U \otimes X')$, $u \otimes g \mapsto ([u, g], x \mapsto u \otimes g(x))$ is an isomorphism provided $B X$ is finitely presented.

If $U$ is a left $H$-module, then $\text{Hom}_B(X, U \otimes X') \cong \text{Hom}_B(X, U \otimes X')$, $F \mapsto (\bar{F} : x \mapsto \sum x_{-1} \mapsto F(x_0))$, where $U \otimes X'$ in the second $\text{Hom}$ is viewed as a left $B$-module via the algebra map $\rho : B \to H \otimes B$. The inverse map is $G \mapsto (\bar{G} : x \mapsto \sum S x_{-1} \mapsto G(x_0))$.

The composition of above two maps with $U = \text{Hom}_A(M, M')$ is the given morphism $\Phi$. □

Proposition 2.8. Let $M, M' \in A_{\# H} \mathcal{M}$, $X \in \mathcal{H}_B \mathcal{M}$, and $X' \in B \mathcal{M}$.

1. There is a natural morphism

$$\psi : \text{Hom}_A(M, M') \otimes \text{Hom}_B(X, X') \to \text{Hom}_{A_{\# B}}(M \# X, M' \# X')$$

$$f \otimes g \mapsto (m \# x \mapsto \sum_{(x)} (x_{-1} \mapsto f(m) \# g(x_0))).$$

If $M$ viewed as $A$-module and $X$ viewed as $B$-module are finitely presented, then $\psi$ is an isomorphism.

2. Suppose that $A M$ and $B X$ are pseudo-coherent modules. Then for any $n \in \mathbb{N}$

$$\text{Ext}^n_{A_{\# B}}(M \# X, M' \# X') \cong \bigoplus_{p+q=n} \text{Ext}^p_A(M, M') \otimes \text{Ext}^q_B(X, X').$$

Proof. (1) As noted before, $\text{Hom}_A(M, M') \otimes X'$ is viewed as a left $B$-module by the algebra morphism $\rho : B \to H \otimes B$, that is, $b \mapsto (f \otimes y) := \sum (b_{-1} \mapsto f) \otimes b_0 y$. By (1.5), $\text{Hom}_A(M, M' \# X')$ is a left $B$-module. Then the natural morphism $\sigma : \text{Hom}_A(M, M' \# X') \to \text{Hom}_A(M, M' \# X')$, $f \otimes y \mapsto (m \mapsto f(m) \# y)$ is a $B$-module morphism. If $A M$ is finitely presented as an $A$-module, then $\sigma$ is an isomorphism.

The composition $\psi$ in the following commutative diagram is the morphism as claimed.

$$\begin{array}{ccc}
\text{Hom}_A(M, M') \otimes \text{Hom}_B(X, X') & \xrightarrow{\psi} & \text{Hom}_B(X, \text{Hom}_A(M, M') \# X') \\
\downarrow & & \downarrow \sigma \\
\text{Hom}_{A_{\# B}}(M \# X, M' \# X') & \xrightarrow{\cong} & \text{Hom}_B(X, \text{Hom}_A(M, M' \# X'))
\end{array}$$

If both $A M$ and $B X$ are finitely presented, then $\psi$ is an isomorphism.
(2) By Lemma \[\text{Lemma 2.8}\] (2), there is an exact sequence \( \cdots \rightarrow Q_n \rightarrow \cdots \rightarrow Q_0 \rightarrow X \rightarrow 0 \) in \( H^B \cdot \mathcal{M} \) with \( Q_n = B \otimes V_n \) where \( V_n \) is some finite-dimensional \( H \)-comodule. Let \( P_* \) be a projective resolution of \( A \#_H M \). It follows from Corollary \[\text{Corollary 2.3}\] that \( \text{Tot}(P_* \# Q_*) \) is a projective resolution of \( A \#_B(M \# X) \). Since \( A_M \) is pseudo-coherent, there is a finitely generated \( A \)-projective resolution \( \tilde{P}_* \rightarrow M \rightarrow 0 \) of \( M \). Then, \( P_* \) and \( \tilde{P}_* \) are homotopy equivalent, and the following is a commutative diagram of complexes

\[
\begin{array}{c}
\text{Hom}_A(P_*, M') \otimes X' \xrightarrow{\alpha_*} \text{Hom}_A(P_*, M' \# X') \\
\downarrow \text{homo. equ.} \quad \downarrow \text{homo. equ.} \\
\text{Hom}_A(\tilde{P}_*, M') \otimes X' \xrightarrow{\alpha_\cong} \text{Hom}_A(\tilde{P}_*, M' \# X')
\end{array}
\]

It follows that \( \text{Hom}_A(P_*, M') \otimes X' \xrightarrow{\alpha_*} \text{Hom}_A(P_*, M' \# X') \) is a quasi-isomorphism of \( B \)-module complexes. Hence

\[
\text{Tot}(\text{Hom}_B(Q_*, \text{Hom}_A(P_*, M') \otimes X')) \overset{\sim}{\rightarrow} \text{Tot}(\text{Hom}_B(Q_*, \text{Hom}_A(P_*, M' \# X')))
\]

where \( \overset{\sim}{\rightarrow} \) means quasi-isomorphism.

\[
\text{Hom}_{A \# B}(\text{Tot}(P_* \# Q_*), M' \# X') \\
\cong \text{Tot}(\text{Hom}_B(Q_*, \text{Hom}_A(P_*, M' \# X')) \text{ (by Lemma 1.5)}) \\
\overset{\sim}{\rightarrow} \text{Tot}(\text{Hom}_B(Q_*, \text{Hom}_A(P_*, M') \otimes X')) \\
\cong \text{Tot}(\text{Hom}_A(P_*, M') \otimes \text{Hom}_B(Q_*, X')) \text{ (as } Q_n = B \otimes V_n \text{ is finitely generated projective)}.
\]

The conclusion follows by taking cohomologies. \( \square \)

**Corollary 2.9.** Suppose further both \( A \) and \( B \) are semi-primary rings over a perfect field \( k \). If the Jacobson radical \( J_A \) of \( A \) is an \( H \)-submodule, and the Jacobson radical \( J_B \) of \( B \) is an \( H \)-subcomodule, then \( \text{gld } A \# B = \text{gld } A + \text{gld } B \).

**Proof.** By assumption, \( J := A \# J_B + J_A \# B \) is a nilpotent ideal of \( A \# B \). Since \( k \) is a perfect field, \( (A \# B)/J \cong A/J_A \# B/J_B \) is a semisimple ring. Hence \( A \# B \) is also a semi-primary ring. By \[\text{Aus. Corollary 12}\], \( \text{gld } A \# B = \sup \{n \mid \text{Ext}^n_{A \# B}(A \# B/J, (A \# B)/J)\} \). The conclusion follows from Proposition \[\text{Proposition 2.8}\]. \( \square \)

Also we have the right version of Proposition \[\text{Proposition 2.8}\].

For any \( M \in \mathcal{M}_{A \# H}, M \) is sometimes viewed as a left \( H \)-module via \( h \mapsto m := m(1 \# Sh) \). Thus \( h \mapsto (ma) = \sum_{(h)} (h_1 \mapsto m)(h_2 \mapsto a) \). For any \( M, N \in \mathcal{M}_{A \# H}, \text{Hom}_{A \# H}(M, N) \in H^H \mathcal{M}, \) with the \( H \)-action given by

\[
(h \mapsto f)(m) = \sum_{(h)} h_1 \mapsto f(Sh_2 \mapsto m)
\]

where \( m \in M, h \in H, f \in \text{Hom}_{A \# H}(M, N) \).

**Proposition 2.10.** Let \( M \in \mathcal{M}_{A \# H}, N \in \mathcal{M}_A, \) and \( X, Y \in H^H \mathcal{M}_B \). Suppose \( X_B \) is finitely generated. Then,

\[
(1) \text{Hom}_{B_{\# H}}(X, Y) \in H^H \mathcal{M}, \text{ with the } H\text{-coaction given by}
\]

\[
\rho(f)(x) = \sum_{(x)} f(x_0)_{-1} S^{-1} x_{-1} \otimes f(x_0)_0
\]

for any \( x \in X \) and \( f \in \text{Hom}_A(M, N) \).
(2) There exists a natural morphism
\[ \phi : \text{Hom}_{A^{op}}(M,N) \otimes \text{Hom}_{B^{op}}(X,Y) \rightarrow \text{Hom}_{(A^\#B)^{op}}(M^\#X,N^\#Y) \]
\[ f \otimes g \mapsto (m^\#x \mapsto \sum_{(g)} f(m(1^\#Sg^{-1}))^g_0(x)) \]
for any \( m \in M, x \in X \). If \( M_A \) and \( X_B \) are finitely presented modules, then \( \phi \) is an isomorphism.

(3) Suppose that \( M_A \) and \( X_B \) are pseudo-coherent modules. Then for any \( n \in \mathbb{N} \)
\[ \text{Ext}_{(A^\#B)^{op}}^n(M^\#X,N^\#Y) \cong \bigoplus_{p+q=n} \text{Ext}_{A^{op}}^p(M,N) \otimes \text{Ext}_{B^{op}}^q(X,Y). \]

3. Yoneda Ext-algebras of Takeuchi smash products

3.1. Smash products of differential graded algebras. A graded algebra \( \Lambda = \bigoplus_{n \in \mathbb{Z}} \Lambda_n \) is called a graded \( H \)-module algebra, if \( \Lambda \) is an \( H \)-module algebra, and \( h \cdot \Lambda_n \subseteq \Lambda_n \) for all \( h \in H \) and \( n \in \mathbb{Z} \). For any non-zero homogeneous element \( \lambda \in \Lambda_n, |\lambda| := n \) is called the degree of \( \lambda \).

Let \( \Lambda \) be a differential graded algebra (or dga, for short), that is, \( \Lambda = \bigoplus \Lambda_n \) is a graded algebra with a graded map \( d : \Lambda \rightarrow \Lambda \) of degree 1, such that \( d^2 = 0 \) and the Leibnitz rule holds:
\[ d(\lambda \cdot \lambda') = d(\lambda) \cdot \lambda' + (-1)^{|\lambda|} \lambda \cdot d(\lambda'), \]
for any homogeneous element \( \lambda \) and \( \lambda' \in \Lambda \).

\( \Lambda \) is called a differential graded left \( H \)-module algebra, if \( \Lambda = \bigoplus \Lambda_n \) is a graded \( H \)-module algebra and the differential \( d \) is an \( H \)-morphism, that is, \( d(h \cdot \lambda) = h \cdot d(\lambda) \) for all \( h \in H \) and \( \lambda \in \Lambda \).

Differential graded left \( H \)-comodule algebras are defined analogously.

Suppose that \( \Lambda \) is a differential graded left \( H \)-module algebra, and \( \Gamma \) is a differential graded left \( H \)-comodule algebra. Then \( \Lambda^\# \Gamma \) is also a (differential) graded algebra with the multiplication defined by
\[ (\lambda \# \gamma)(\lambda' \# \gamma') = \sum_{(\gamma)} (-1)^{|\lambda||\lambda'|} \lambda(\gamma_{-1} \rightarrow \lambda') \# \gamma_0 \gamma' \]
and the differential \( d_{\Lambda^\# \Gamma} \) defined by
\[ d_{\Lambda^\# \Gamma}(\lambda \# \gamma) = d_{\Lambda}(\lambda) \# \gamma + (-1)^{|\lambda|} \lambda \# d_{\Gamma}(\gamma) \]
for any homogeneous elements \( \lambda, \lambda' \in \Lambda \), and \( \gamma, \gamma' \in \Gamma \).

Let \( R \) be a ring, \( P_* \) and \( Q_* \) be two \( R \)-module complexes. Consider the graded space \( \text{Hom}^*_R(P_*,Q_*) \) with
\[ \text{Hom}^*_R(P_*,Q_*) = \prod_i \text{Hom}_R(P_i,Q_{i-n}). \]
The differentials on \( P_* \) and \( Q_* \) which are of degree \(-1\) give rise to a differential \( d_{\text{Hom}} \) of degree \( 1 \) on \( \text{Hom}^*_R(P_*,Q_*) \), where
\[ d_{\text{Hom}}(f) = d^3 \circ f - (-1)^{|f|} f \circ d^p = (d^3_{-1} \circ f_i - (-1)^{|f|} f_{i-1} \circ d^p_i) \]
for \( f = (f_i)_i \), and \( f_i \in \text{Hom}_R(P_i,Q_{i-1}). \)

In fact, \( \text{End}_R^*(P_*) = \text{Hom}_R^*(P_*,P_*) \) is a dga, as \( d_{\text{Hom}}(fg) = d_{\text{Hom}}(f)g + (-1)^{|f|} f d_{\text{Hom}}(g) \) for any \( f, g \in \text{End}_R^*(P_*) \). So, the cohomology group \( H^*(\text{End}_R^*(P_*)) \) has a graded algebra structure.

**Proposition 3.1.** Let \( M_* \) be a complex in \( \mathcal{M}_A^{H \# \ast} \), and \( X_* \) be a complex in \( ^H \mathcal{M}_B \). Suppose that each term \( X_i \) viewed as a \( B^{op} \)-module is finitely generated. Then

1. \( \text{End}_{A^{op}}^*(M_*) \) is a differential graded left \( H \)-module algebra, with the module structure given in \( (2.1) \).
2. \( \text{End}_{B^{op}}^*(X_*) \) is a differential graded left \( H \)-comodule algebra, with the comodule structure given in \( (2.2) \).
(3) There is a differential graded algebra morphism
\[ \varphi : \text{End}_A^*(M_\bullet) \# \text{End}_B^*(X_\bullet) \to \text{End}_{(A \# B)^{op}}(\text{Tot}(M_\bullet \# X_\bullet)) \]
\[
f \# g \mapsto \left( m \# x \mapsto (-1)^{|g||m|} \sum_{(g)} f(g_{-1} \to m) \# g_0(x) \right),
\]
which is induced by Proposition 2.10 (2).

Proof. (1) For any \( h \in H, f, f' \in \text{End}_A^*(M_\bullet) \) and \( m \in M_i \),
\[
((h \to (ff'))(m) = \sum_{(h)} h_1 \to (ff')(Sh_2 \to m))
\]
\[
= \sum_{(h)} h_1 \to f((Sh_2)h_3 \to f'(Sh_4 \to m)) = \sum_{(h)} h_1 \to f(Sh_2 \to ((h_3 \to f')(m)))
\]
\[
= \sum_{(h)} (h_1 \to f)(h_2 \to f')(m).
\]
It follows that \( h \to (ff') = \sum_{(h)} (h_1 \to f)(h_2 \to f') \).

Since \( (h \to \text{id}_{M_i})(m) = \sum_{(h)} h_1 \to \text{id}_{M_i}(Sh_2 \to m) = \varepsilon(h)(m) \), \( (h \to \text{id}_{M_i}) = \varepsilon(h) \text{id}_{M_i} \).

Hence \( \text{End}_A^*(M_\bullet) \) is a graded \( H \)-module algebra. In fact, the differential \( d_{\text{Hom}} \) is an \( H \)-morphism.
\[
(d_{\text{Hom}}(h \to f))(m) = d^M \circ (h \to f)(m) - (-1)^{|f|} (h \to f) \circ d^M(m)
\]
\[
= \sum_{(h)} d^M(h_1 \to f(Sh_2 \to m)) - (-1)^{|f|} h_1 \to f(Sh_2 \to d^M(m))
\]
\[
= \sum_{(h)} h_1 \to (d^M f(Sh_2 \to m)) - (-1)^{|f|} h_1 \to (fd^M(Sh_2 \to m))
\]
\[
= (h \to d_{\text{Hom}}(f))(m).
\]
It follows that \( \text{End}_A^*(M_\bullet) \) is a differential graded left \( H \)-module algebra.

(2) For any \( g, g' \in \text{End}_B^*(X_\bullet) \) and \( x \in X_i \), by (2.2),
\[
\rho(g')(x) = \sum_{(x)} g'_{-1} \otimes g_0(x) = \sum_{(x)} g'(x_{-1})^{-1} x_{-1} \otimes g'(x_0),
\]
and
\[
\rho(g)(g'(x_0)_{0}) = \sum_{(g)(x_0)} g_{-1} \otimes g_0(g'(x_0)_{0}) = \sum_{(g)(x_0)_{0}} g(g'(x_0)_{0})_{-1} (S^{-1} g'(x_{-1})_{-1} \otimes g'(x_0)_{0}).
\]
Then
\[
\sum_{(g), (g')} g_{-1} g'_{-1} \otimes g_0 g'_0(x) = \sum_{(g), (x), (g')(x_0)} g_{-1} g'(x_{0})_{-1} S^{-1} x_{-1} \otimes g_0(g'(x_0)_{0})
\]
\[
= \sum_{(x), (g')(x_0), (g'(x_0)_{0})} g(g'(x_0)_{0})_{-1} (S^{-1} g'(x_{-1})_{-1} g'(x_0)_{0}) - S^{-1} x_{-1} \otimes g(g'(x_0)_{0})
\]
\[
= \sum_{(x), (g'(x_0))} g(g'(x_0))_{-1} S^{-1} x_{-1} \otimes g(g'(x_0))
\]
\[
= \sum_{(g' g')} (g g')_{-1} \otimes (g g')_{0}(x).
\]
It follows that \( \rho(g g') = \rho(g) \rho(g') \).
Obviously, \((\varepsilon \otimes 1)\rho(g) = g\). Hence \(\text{End}^\bullet_{B^op}(X_\bullet)\) is a graded left \(H\)-comodule algebra. The differential \(d_{\text{Hom}}\) is also an \(H\)-comodule morphism as shown next. Since

\[
\rho(d^X \circ g)(x) = \sum_{(x)} (d^X \circ g)(x_0)_{-1} S^{-1} x_{-1} \otimes (d^X \circ g)(x_0)_0
\]

\[
= \sum_{(x)} g(x_0)_{-1} S^{-1} x_{-1} \otimes d^X(g(x_0)_0)
\]

\[
= \sum_{(x)} g_{-1} \otimes d^X(g_0(x)),
\]

it follows that

\[
\rho(d_{\text{Hom}}(g))(x) = \sum_{(x)} d_{\text{Hom}}(g)(x_0)_{-1} S^{-1} x_{-1} \otimes d_{\text{Hom}}(g)(x_0)_0
\]

\[
= \sum_{(x)} g(x_0)_{-1} S^{-1} x_{-1} \otimes d_{\text{Hom}}(g(x_0)_0)
\]

\[
= \sum_{(x)} g_{-1} \otimes d_{\text{Hom}}(g_0(x)).
\]

Hence \(\rho(d_{\text{Hom}}(g)) = \sum (g) g_{-1} \otimes d_{\text{Hom}}(g_0)\) and \(\text{End}^\bullet_{B^op}(X_\bullet)\) is a differential graded left \(H\)-comodule algebra.

(3) For any \(f, f' \in \text{End}^\bullet_{A^op}(M_\bullet), g, g' \in \text{End}^\bullet_{B^op}(X_\bullet), m \in M_i, x \in X_i,\)

\[
\varphi\left((f \# g)(f' \# g')\right)(m \# x) = (-1)^{|g||f'|} \varphi\left(\sum_{(g)} f(g_{-1} \rightarrow f') g_0 g'\right)(m \# x)
\]

\[
= (-1)^{|g||f'|} (-1)^{|m||g'|} \sum_{(g),(g')} f(g_{-1} \rightarrow f')(g_{-1} g'_{-1} \rightarrow m) g_0 g'_0(x)
\]

\[
= (-1)^{|g||f'|} (-1)^{|m||g'|} \sum_{(g),(g')} f(g_{-1} \rightarrow f')(g_{-1} g'_{-1} \rightarrow m) g_0 g'_0(x)
\]

\[
= (-1)^{|m||g'|} \sum_{(g')} \varphi(f \# g)(f'_{-1} \rightarrow m) g'_0(x)
\]

\[
= \varphi(f \# g)(f' \# g')(m \# x).
\]

Hence \(\varphi((f \# g)(f' \# g')) = \varphi(f \# g) \varphi(f' \# g').\)

Clearly \(\varphi(\text{id}_{M_\bullet} \# \text{id}_{X_\bullet}) = \text{id}_{\text{Hom}(M_\bullet \# X_\bullet)}\). So \(\varphi\) is an algebra morphism.

It is left to show that \(\varphi\) is a cochain map.

\[
(d_{\text{Hom}} \circ \varphi(f \# g))(m \# x)
\]

\[
= d^M \# X(\varphi(f \# g)(m \# x)) \rightarrow (-1)^{|f|+|g|} \varphi(f \# g)(d^M \# X(m \# x))
\]

\[
= (-1)^{|g||m|} d^M \# X\left(\sum_{(g)} f(g_{-1} \rightarrow m) g_0(x)\right) \rightarrow (-1)^{|f|+|g|} \varphi(f \# g)(d^M(m) \# x + (-1)^{|m| m} d^X(x))
\]

\[
= (-1)^{|g||m|} \left(\sum_{(g)} d^M(f_{-1} \rightarrow m) g_0(x) + (-1)^{|f|+|g|} f(g_{-1} \rightarrow m) d^X(g_0(x))\right)
\]

\[
= (-1)^{|f|+|g|} \sum_{(g)} (-1)^{|g||m|+1} f(g_{-1} \rightarrow d^M(m)) g_0(x) + (-1)^{|m|+|g|} f(g_{-1} \rightarrow m) g_0(d^X(x))
\]

\[
= \sum_{(g)} (-1)^{|g||m|} d_{\text{Hom}}(f)(g_{-1} \rightarrow m) g_0(x) + (-1)^{|f|+|g|+|m|} f(g_{-1} \rightarrow m) d_{\text{Hom}}(g_0)(x)
\]
\[\varphi(d_{\text{Hom}}(f)\#g + (-1)^{|f|}f\#d_{\text{Hom}}(g))(m\#x) = \varphi(d_{\text{Hom}}(f\#g))(m\#x).\]

Therefore, \(d_{\text{Hom}} \circ \varphi = \varphi \circ d_{\text{Hom}}\), and the conclusion follows. \(\square\)

By taking the homology, then \(\text{Ext}^\bullet_{A^\#H}(M, M) = \bigoplus_{n \geq 0} \text{Ext}_{A^\#H}^n(M, M)\) is a graded left \(H\)-module algebra, and \(\text{Ext}^\bullet_{B^\#}(X, X)\) is a graded left \(H\)-comodule algebra. Hence we can form the graded Takeuchi smash product algebra \(\text{Ext}^\bullet_{A^\#H}(M, M) \# \text{Ext}^\bullet_{B^\#}(X, X)\).

**Theorem 3.2.** Suppose that \(M \in \mathcal{M}_{A^\#H}\) is pseudo-coherent as \(A\)-module, and \(X \in H\mathcal{M}_B\) is pseudo-coherent as \(B\)-module. Then, as graded algebras,

\[
\text{Ext}^\bullet_{(A^\#B)^\#}(M \# X, M \# X) \cong \text{Ext}^\bullet_{A^\#H}(M, M) \# \text{Ext}^\bullet_{B^\#}(X, X).
\]

**Proof.** Let \(P_\bullet\) be a projective resolution of \(M_{A^\#H}\). By Lemma 1.8, there exists an exact sequence

\[
\cdots \rightarrow Q_n \rightarrow \cdots \rightarrow Q_0 \rightarrow X \rightarrow 0
\]

in \(H\mathcal{M}_B\), where all the \(Q_n\)'s are finitely generated free as \(B\)-module.

By Proposition 3.1, \(\text{End}^\bullet_{A^\#H}(P_\bullet)\) is a differential graded left \(H\)-module algebra, and \(\text{End}^\bullet_{B^\#}(Q_\bullet)\) is a differential graded left \(H\)-comodule algebra, and

\[
\varphi : \text{End}^\bullet_{A^\#H}(P_\bullet) \# \text{End}^\bullet_{B^\#}(Q_\bullet) \rightarrow \text{End}^\bullet_{(A^\#B)^\#}(\text{Tot}(P_\bullet \# Q_\bullet)),
\]

\[
f\#g \mapsto (p\#q \mapsto (-1)^{|g|p}\sum_{(g)} f(g_{-1} \rightarrow p)\#g_0(q))
\]

is a differential graded algebra morphism. By Proposition 2.10 (3), it is a quasi-isomorphism. Then the conclusion follows by taking the homology. \(\square\)

### 4. Application to Takeuchi smash products of AS-regular algebras

We recall some definitions first.

#### 4.1. Artin-Schelter regular algebras.

A graded algebra \(A = \bigoplus_{n \in \mathbb{Z}} A_n\) is called **connected**, if \(A_{<0} = 0\) and \(A_0\) is the base field \(k\). For any connected graded algebra \(A\), the left global dimension and right global dimension of \(A\) are equal, which are equal to the projective dimension of the trivial module \(k \cong A/A_{\geq 1}\) or \(k_A\).

**Definition 4.1.** A connected graded algebra \(A\) is called **Artin-Schelter regular** (for short, AS-regular) of dimension \(d\) for some integer \(d \geq 0\), if

1. \(A\) has global dimension \(d\);
2. \(\text{Ext}_A^i(k, A) \cong \begin{cases} 0 & i \neq d \\ k & i = d. \end{cases}\)

If \(A\) is a left AS-regular algebra of dimension \(d\), then it follows from the Ischebeck’s spectral sequence that the right version of the condition (2) holds.

Let \(A\) be a connected graded algebra. By [RRZ4, Lemma 1.2], \(A\) is AS-regular of dimension \(d\) if and only if it is a graded skew Calabi-Yau algebra of dimension \(d\), which is defined in the following.

**Definition 4.2.** A \(k\)-algebra \(A\) is called **skew Calabi-Yau algebra** of dimension \(d\), if

1. \(A\) is **homologically smooth**, that is, as \(A^e\)-module, \(A\) has a finitely generated projective resolution of finite length;
(ii) there exists an automorphism $\mu$ of $A$, such that
\[
\text{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A^e, & i = d \end{cases}
\]
as $A$-$A$-bimodules.

The automorphism $\mu$ is unique up to an inner automorphism, which is called a *Nakayama automorphism* of $A$, denoted sometimes by $\mu_A$.

Graded skew Calabi-Yau algebras are defined similarly in the graded module category for graded algebras.

4.2. *Takeuchi smash products of AS-regular algebras*. A Hopf algebra $H = \bigoplus_{n \in \mathbb{Z}} H_n$ is said to be a *graded Hopf algebra* if both of its algebra and coalgebra structure are graded with respect to the grading $H = \bigoplus_{n \in \mathbb{Z}} H_n$, and its antipode is also graded. Let $H$ be a graded Hopf algebra, and $A$ be a graded algebra which is also an $H$-module algebra. If $H_n \to A_n \subseteq A_{m+n}$ for all $n$ and $m$, then $A$ is called an $H$-module graded algebra. Let $B$ be a graded algebra with an $H$-comodule algebra structure $\rho$. If $\rho(B_n) \subseteq \bigoplus_i H_i \otimes B_{n-i}$ for all $n$, then $B$ is called an $H$-comodule graded algebra. In this case, it is clear that the Takeuchi smash product $A \# B$ is a graded algebra with $(A \# B)_n = \bigoplus_i A_i \otimes B_{n-i}$.

The Hopf algebra $H = k\langle g^{\pm 1}, X \rangle$ given in Example 1.2 is a graded Hopf algebra by setting $\deg(g) = 0$ and $\deg(X) = 1$. Suppose $A$ is a graded algebra, with a graded automorphism $\sigma$ of degree 0 and a graded $\sigma$-derivation $\delta$ of degree 1. Then $A$ is an $H$-module graded algebra, $B = k[X]$ is an $H$-comodule graded algebra, and the Ore extension $A[x, \sigma, \delta]$ is the graded Takeuchi smash product $A \# B$.

**Theorem 4.3.** Let $A$ and $B$ be AS-regular algebras of dimension $d_1$ and $d_2$ respectively. If $H$ is a graded Hopf algebra, $A$ is an $H$-module graded algebra and $B$ is an $H$-comodule graded algebra, such that $H \to A_{\geq 1} \subseteq A_{\geq 1}$ or $\rho(B_{\geq 1}) \subseteq H \otimes B_{\geq 1}$, then $A \# B$ is an AS-regular algebra of dimension $d_1 + d_2$.

**Proof.** Suppose $H \to A_{\geq 1} \subseteq A_{\geq 1}$ (which is automatic if $H$ is positively graded). Then the trivial module $k \cong A/A_{\geq 1}$ is an $A\#H$-module. By Corollary 2.3 $\text{pd}_{A\#B} k \leq \text{pd}_A k + \text{pd}_B k = d_1 + d_2$.

It follows from the spectral sequence in Proposition 2.2 that
\[
\text{Ext}^i_{A\#B}(k, A\#B) \cong \begin{cases} 0, & i \neq d_1 + d_2 \\ \text{Ext}^d_{B}(k, \text{Ext}^d_A(k, A\#B)), & i = d_1 + d_2. \end{cases}
\]

We claim that $\text{Ext}^d_A(k, A\#B) \cong B$ as left $B$-modules. Let $P_\bullet$ be a projective resolution of $A\#Hk$. Then there is a natural $B$-module complex morphism
\[
\text{Hom}_A(P_\bullet, A) \otimes B \longrightarrow \text{Hom}_A(P_\bullet, A\#B)
\]
where the $B$-module structure on $\text{Hom}_A(P_\bullet, A) \otimes B$ is given by $b \to (f \otimes b') = \sum_{(b)} (b_{-1} \to f) \otimes b_0 b'$, and the $B$-module structure on $\text{Hom}_A(P_\bullet, A\#B)$ is given by (1.5). Since $A_k$ is pseudo-coherent by [SZ, Proposition 3.1], it follows that $\text{Hom}_A(P_\bullet, A) \otimes B \longrightarrow \text{Hom}_A(P_\bullet, A\#B)$ is a quasi-isomorphism of $B$-module complex. Thus, with the induced $B$-module structures, $\text{Ext}^d_A(k, A\#B) \cong \text{Ext}^d_A(k, A) \otimes B$ as $B$-modules. Since $\text{Ext}^d_A(k, A)$ is an one-dimensional $k$-vector space, then there is algebra morphism $\alpha : H \to k$, such that $h \to e = \alpha(h)e$ for all $h \in H$ and $e \in \text{Ext}^d_A(k, A)$. This implies that $\text{Ext}^d_A(k, A\#B) \cong \mathbb{E}_B$, where $\mathbb{E}_B : B \to B$ is the algebra automorphism defined by $\mathbb{E}_B(b) = \sum_{(b)} \alpha(b_{-1})b_0$. Thus $\text{Ext}^d_A(k, A\#B) \cong B$ as left $B$-modules, the claim is proved.

Since $B$ is AS-regular, then $\text{Ext}^{d_1+d_2}_{A\#B}(k, A\#B) \cong \text{Ext}^d_B(k, B) \cong k$. It follows that $A\#B$ is a $(d_1 + d_2)$-dimensional AS-regular algebra.
Suppose $\rho(B_{\geq 1}) \subseteq H \otimes B_{\geq 1}$. Then $k \cong B/B_{\geq 1} \in H \mathcal{M}_B$. Then it follows Corollary \textbf{2.5} and Proposition \textbf{2.4} that $\text{pd}_{(A\#B)^{op}} k \leq \text{pd}_{A^{op}} k + \text{pd}_{B^{op}} k = d_1 + d_2$ and

$$\text{Ext}^i_{(A\#B)^{op}}(k, A\#B) \cong \begin{cases} 0, & i \neq d_1 + d_2 \\ \text{Ext}^{d_1}_{A^{op}}(k, \text{Ext}^{d_2}_{B^{op}}(k, A\#B)), & i = d_1 + d_2. \end{cases}$$

Since $A$ is AS-regular, by [SZ] Proposition 3.1, $k_B$ is pseudo-coherent. It follows from Lemma \textbf{1.8} that $k_B \cong B/B_{\geq 1}$ has a resolution $Q_n \to k \to 0$ in $H \mathcal{M}_B$ such that $Q_n = V_n \otimes B$ for some finite-dimensional left $H$-comodule $V_n$. Then, $\text{Hom}_{B^{op}}(Q_n, A\#B)$ has an $A^{op}$-module structure given by Lemma \textbf{1.7} that is, $(fa)(y) = f((y_0)(S^{-1}y_1 \to a))$, and $\text{Hom}_{B^{op}}(Q_n, B)$ has a left $H$-comodule structure given by $\rho(g)(y) = \sum(y,g(y_0))(g(y_0) - S^{-1}y_1) \otimes g(y_0)$ as defined in \textbf{2.2} (see Proposition \textbf{2.10}).

If we endow $A \otimes \text{Hom}_{B^{op}}(Q_n, B)$ with a right $A$-module structure by

$$(a \otimes g)a' = a(g_1 \to a') \otimes g_0.$$ Then, $A \otimes \text{Hom}_{B^{op}}(Q_n, B) \to \text{Hom}_{B^{op}}(Q_n, A\#B), a \otimes g \mapsto ([a, g] : Q \to A\#B, y \mapsto a\#g(y))$ is an $A^{op}$-module isomorphism. In fact.

$$[a(g_1 \to a'), y_0](y) = a(g_1 \to a')\#g_0(y) = a(g(y_0) - S^{-1}y_1 \to a')\#g(y_0), \text{ and}$$

$$([a, g]a')(y) = ([a, g](y))(S^{-1}y_1 \to a') = a(g(y_0) - S^{-1}y_1 \to a')\#g(y_0).$$

It follows that $\text{Ext}^{d_2}_{B^{op}}(k, A\#B) \cong A \otimes \text{Ext}^{d_2}_{B^{op}}(k, B) \cong (A \otimes k)_A \cong A_A$. Therefore,

$$\text{Ext}^{d_1 + d_2}_{(A\#B)^{op}}(k, A\#B) \cong \text{Ext}^{d_1}_{A^{op}}(k, \text{Ext}^{d_2}_{B^{op}}(k, A\#B)) \cong \text{Ext}^{d_1}_{A^{op}}(k, A) \cong k.$$ Hence $A\#B$ is an AS-regular algebra of dimension $d_1 + d_2$. \hfill \Box

So graded Ore extensions preserve the AS-regularity, which was proved in [AST] Proposition 2. The ungraded case was proved in [LWW] in terms of skew Calabi-Yau algebras. The Yoneda Ext-algebras recover many properties of AS-regular algebras as it is well known (see, say [LPWZ1, LPWZ2]), for instance, the Nakayama automorphisms.

**Theorem 4.4.** [RRZ2] Theorem 4.2 Let $A$ be a noetherian AS-regular algebra generated in degree 1. Let $\mu_A$ and $\mu_E$ be the Nakayama automorphism of $A$ and $\text{Ext}^\bullet_A(k, k)$ respectively. Then $\mu_A|_{A_1} = (\mu_E|_{\text{Ext}^\bullet_A(k, k)})^*$ where $A_1$ is identified with $\text{Ext}^\bullet_A(k, k)^*$. As an application of Theorems \textbf{3.2} and \textbf{4.3} we can describe the Nakayama automorphism of the Takeuchi smash product of AS-regular algebras with a similar proof to Theorem [SZL] Theorem 4.2. For the definitions of homological determinants and codeterminant we refer [KKZ] as the reference.

**Proposition 4.5.** Suppose $A$ and $B$ are noetherian AS-regular algebras generated in degree 1, and $A$ is a graded $H$-module algebra, $B$ is a graded $H$-comodule for some Hopf algebra $H$. If $A\#B$ is noetherian, then the Nakayama automorphism $\mu_{A\#B}$ of $A\#B$ is given by

$$\mu_{A\#B}(a\#b) = \sum_{(b)} \mu_A(g \to a)\#\text{hdet}(b_{-1})\mu_B(b_0),$$

where $\mu_A$ and $\mu_B$ are the Nakayama automorphisms of $A$ and $B$ respectively, $\text{hdet} \in H^\ast$ and $g \in H$ are the homological determinant of the $H$-action on $A$ and the homological codeterminant of the $H$-coaction on $B$ respectively.

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