Model independent determination of the gluon condensate in four-dimensional SU(3) gauge theory

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(Dated: March 27, 2014)

We determine the non-perturbative gluon condensate of four-dimensional SU(3) gauge theory in a model independent way. This is achieved by carefully subtracting high order perturbation theory results from non-perturbative lattice QCD determinations of the average plaquette. No indications of dimension two condensates are found. The value of the gluon condensate turns out to be of a similar size as the intrinsic ambiguity inherent to its definition.

PACS numbers: 12.38.Gc, 12.38.Bx, 11.55.Hx, 12.38.Cy, 11.15.Bt

The operator product expansion (OPE) [1] is a fundamental tool for theoretical analyses in quantum field theories. Its validity is only proven rigorously within perturbation theory, to arbitrary finite orders [2]. The use of the OPE in a non-perturbative framework was initiated by the ITEP group [3] (see also the discussion in [4]), which postulated that the OPE of a correlator could be approximated by the following series:

\[
\text{correlator}(Q) \simeq \sum_{d} \frac{1}{Q^d} C_d(\alpha) \langle O_d \rangle, \tag{1}
\]

where the expectation values of local operators \(O_d\) are suppressed by inverse powers of a large external momentum \(Q \gg \Lambda_{\text{QCD}}\), according to their dimensionality \(d\). The Wilson coefficients \(C_d(\alpha)\) encode the physics at momenta scales larger than \(Q\). These are well approximated by perturbative expansions in the strong coupling parameter \(\alpha\). The large-distance physics is described by the matrix elements \(\langle O_d \rangle\) that usually have to be determined non-perturbatively.

Almost all QCD predictions of relevance to particle physics phenomenology are based on factorizations that are generalizations of the above generic OPE.

For correlators where \(O_0 = 1\), the first term of the OPE expansion is a perturbative series in \(\alpha\). In pure gluodynamics the first non-trivial gauge invariant local operator has dimension four. Its expectation value is the so-called non-perturbative gluon condensate

\[
\langle O_G \rangle = -\frac{2}{\beta_0} \left\langle \frac{\beta(\alpha)}{\alpha} G_{\mu\nu} G^{\mu\nu} \right\rangle \tag{2}
\]

\[
= \left\langle \Omega \left[ 1 + \mathcal{O}(\alpha) \right] \frac{\alpha}{\pi} G_{\mu\nu} G^{\mu\nu} \right\rangle. \tag{3}
\]

This condensate plays a fundamental role in phenomenology, in particular in sum rule analyses, as for many observables it is the first non-perturbative OPE correction to the purely perturbative result. In this Letter we will compute (and define) this object. For this purpose we use the expectation value of the plaquette calculated in Monte-Carlo (MC) simulations in lattice regularization with the standard Wilson gauge action [5]

\[
\langle P \rangle_{\text{MC}} = \frac{1}{N^4} \sum_{x \in \Lambda_E} \langle P_x \rangle, \tag{4}
\]

where \(\Lambda_E\) is a Euclidean spacetime lattice and

\[
P_{x,\mu\nu} = 1 - \frac{1}{6} \text{Tr} \left( U_{x,\mu\nu} U^\dagger_{x,\mu\nu} \right). \tag{5}
\]

For details on the notation see [6]. The corresponding OPE reads

\[
\langle P \rangle_{\text{MC}} = \sum_{n=0}^{\infty} p_n a^{n+1} + \frac{\alpha^2}{36} C_G(\alpha) a^4 \langle O_G \rangle + \mathcal{O} \left( a^6 \right), \tag{6}
\]

where \(a\) denotes the lattice spacing.

The perturbative series is divergent due to renormalons [7] and other, subleading, instabilities. This makes any determination of \(\langle O_G \rangle\) ambiguous, unless we define how to truncate or how to approximate the perturbative series. A reasonable definition that is consistent with \(\langle O_G \rangle \sim \Lambda_{\text{QCD}}^4\) can only be given if the asymptotic behaviour of the perturbative series is under control. This has only been achieved recently [6], where the perturbative expansion of the plaquette was computed up to \(\mathcal{O}(\alpha^{35})\). The observed asymptotic behaviour was in full compliance with renormalon expectations, with successive contributions starting to diverge for orders around \(\alpha^{27} - \alpha^{30}\) with the range of couplings \(\alpha\) typically employed in present-day lattice simulations.

Extracting the gluon condensate from the average plaquette was pioneered in [6–11] and many attempts followed during the next decades, see, e.g. [12–19]. These suffered from insufficiently high perturbative orders and, in some cases, also finite volume effects. The failure to make contact to the asymptotic regime prevented a reliable lattice determination of \(\langle O_G \rangle\). We solve this problem in this paper.

Truncating the infinite sum at the order of the minimal contribution provides one definition of the perturbative
series. Varying the truncation order will result in changes of size $\Lambda_{QCD}^2$, where the dimension $d = 4$ is determined by that of the gluon condensate. We approximate the asymptotic series by the truncated sum

$$S_P(\alpha) \equiv S_{n_0}(\alpha), \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1}. \quad (6)$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0} \alpha^{n_0+1}$ is minimal. We then obtain the gluon condensate from the relation

$$\langle O_G \rangle = \frac{36C_G^{-1}(\alpha)}{\pi^2 a^3(\alpha)} \left[ (P)_{\text{MC}}(\alpha) - S_P(\alpha) \right] + \mathcal{O}(a^2 \Lambda_{QCD}^2). \quad (7)$$

For the plaquette, the inverse Wilson coefficient

$$C_G^{-1}(\alpha) = -\frac{2\pi \beta(\alpha)}{\beta_0 a^2} = 1 + \frac{\beta_1}{\beta_0} \frac{\alpha}{4\pi} + \frac{\beta_2}{\beta_0} \left( \frac{\alpha}{4\pi} \right)^2 + \frac{\beta_3}{\beta_0} \left( \frac{\alpha}{4\pi} \right)^3 + \mathcal{O}(a^4) \quad (8)$$

is proportional to the $\beta$-function of $\mathcal{L}_4$. For $j \leq 3$ the coefficients $\beta_j$ are known in the lattice scheme (see Eq. (25) of [21]). The corrections to $C_G$ are 1 small. However, the $\mathcal{O}(a^2)$ and $\mathcal{O}(a^3)$ terms are of similar sizes. We will account for this uncertainty in our error budget.

Integrating the $\beta$-function results in the following dependence of the lattice spacing $a$ on the coupling $\alpha$:

$$a = \frac{1}{\Lambda_{\text{lat}}} \exp \left[ -\frac{1}{t} - b \ln \frac{t}{2} + s_1 t - s_2 b^2 t^2 + \cdots \right], \quad (9)$$

where $t = \alpha \beta_0/(2\pi)$, $b = \beta_1/(2\beta_0^2)$, $s_1 = (\beta_1^2 - \beta_0^2 \beta_2)/(4\beta_0^4)$ and $s_2 = (\beta_1^3 - 2\beta_0 \beta_1 \beta_2 + \beta_0^2 \beta_3)/(16\beta_0^4 \beta_4)$. Eq. (9) is not accurate in the lattice scheme for typical $\beta$-values ($\beta \equiv 3/(2\pi a)$) used in present-day simulations.

Instead, we employ the phenomenological parametrization of $\mathcal{L}_4$ ($x = 6 - \beta$)

$$a = r_0 \exp \left( -1.6804 - 1.7331 x + 0.7849 x^2 - 0.4428 x^3 \right), \quad (10)$$

obtained by interpolating non-perturbative lattice simulation results. Eq. (10) was reported to be valid within an accuracy varying from 0.5% up to 1% in the range $22, 5.7 \leq \beta \leq 6.92$. We plot the ratio of the above two equations $\lambda_{\text{lat}}$ in Fig. 1, where we truncate Eq. (10) at different orders. The green error band corresponds to $r_0 = 0.0209(17)\Lambda_{\text{lat}} \simeq 0.5 \text{fm}$. For $\beta > 0.7$ the slope of the ratio starts to increase. This may indicate violations of Eq. (10) for $\beta > 6.7$. Therefore, we will restrict ourselves to the range $\beta \in [5.8, 6.65]$, where $a(\beta)$ is given by Eq. (10). This corresponds to $(a/r_0)^4 \in [3.1 \times 10^{-5}, 5.5 \times 10^{-3}]$, covering more than two orders of magnitude.

Following Eq. (11), we subtract the truncated sum $S_P(\alpha)$ calculated from the coefficients $p_0$, $p_1$, and $p_2$ from the MC data on $(P)_{\text{MC}}(\alpha)$ of [24]. Multiplying this difference by $36r_0^3/(\pi^2 C_G a^4)$, where $r_0/a$ is given by Eq. (10), gives $r_0^3 \langle O_G \rangle$ plus higher order non-perturbative terms. We show this combination in Fig. 2. The smaller error bars represent the errors of the MC data, the outer error bars include the error of $S_P(\alpha)$. The error band is our prediction for $\langle O_G \rangle$, Eq. (11).

![FIG. 1. Eq. (10) over Eq. (9), truncated at different orders. The green band corresponds to $r_0\Lambda_{\text{lat}} = 0.0209(17)$ [23].](image1)

![FIG. 2. Eq. (10) evaluated using the $N = 16$ and $N = 32$ MC data of [24]. The $N = 32$ outer error bars include the error of $S_P(\alpha)$. The error band is our prediction for $\langle O_G \rangle$, Eq. (11).](image2)
ourselves to the more precise $N = 32$ data and, to keep finite size effects under control, to $\beta \leq 6.65$. We also limit ourselves to $\beta \geq 5.8$ to avoid large $O(a^2)$ corrections. At very large $\beta$-values not only the parametrization Eq. (10) breaks down but obtaining meaningful results becomes challenging numerically: the individual errors both of the individual errors both of their infinite volume limits $p_n$ that reference, coefficients $\langle \alpha \rangle$ have been obtained in [6].

The coefficients $p_n$ of $S_P(\alpha)$ have been obtained in [6]. The $p_n$ carry statistical errors and successive orders are correlated. Using the covariance matrix, also obtained in [6], the statistical error of $S_P(\alpha)$ can be calculated. In that reference, coefficients $p_n(N)$ were first computed on finite volumes of $N^4$ sites and subsequently extrapolated to their infinite volume limits $p_n$. This extrapolation is subject to parametric uncertainties that need to be estimated. We follow [6] and add the differences between determinations using $N \geq \nu$ points for $\nu = 9$ (the central values) and $\nu = 7$ as systematic errors to our statistical errors.

The data in Fig. 2 show an approximately constant behaviour. This indicates that, after subtracting $S_P(\alpha)$ from the corresponding MC values $(P)_{MC}(\alpha)$, the remainder scales like $a^4$. This can be seen more explicitly in Fig. 3, where we plot this difference in lattice units against $a^4$. The result is consistent with a linear behaviour but a small curvature seems to be present that can be parametrized as an $a^6$-correction. The right-most point ($\beta = 5.8$) corresponds to $a^{-1} \approx 1.45$ GeV while $\beta = 6.65$ corresponds to $a^{-1} \approx 5.3$ GeV. Note that $a^2$-terms are clearly ruled out.

We now determine the gluon condensate. We obtain the central value and its statistical error $\langle O_G \rangle = 3.177(36)r_0^{-1}$ from averaging the $N = 32$ data for $6.0 \leq \beta \leq 6.65$. We now estimate the systematic uncertainties. Different infinite volume extrapolations of the $p_n(N)$ data [6] result in changes of the prediction of about 6%. Another 6% error is due to including an $a^2$-term or not and varying the fit range. Next there is a scale error of about 2.5%, translating $a^4$ into units of $r_0$. The uncertainty of the perturbatively determined Wilson coefficient $C_G$ is of a similar size. This is estimated as the difference between evaluating Eq. (5) to $O(a^2)$ and to $O(a^3)$. Adding all these sources of uncertainty in quadrature and using $\langle \alpha \rangle = 0.0013r_0^{-1}$ points only.

The gluon condensate Eq. (2) is independent of the renormalization scale. However, $\langle O_G \rangle$ was obtained employing one particular prescription in terms of the observable and our choice of how to truncate the perturbative series within a given renormalization scheme. Different (reasonable) prescriptions can in principle give different results. One may for instance choose to truncate the sum at orders $n_0(\alpha) \pm \sqrt{n_0(\alpha)}$ and the result would still scale like $\langle O_G \rangle$. We estimated this intrinsic ambiguity of the definition of the gluon condensate in [6] as $\delta(\langle O_G \rangle) = 36/(\pi^2 C_G a^4)\sqrt{n_0\alpha n_0+1}$, i.e. as $\sqrt{n_0(\alpha)}$ times the contribution of the minimal term:

$$\delta(\langle O_G \rangle) = 27(11)\langle O_G \rangle.$$

Up to $1/n_0$-corrections this definition is scheme- and scale-independent and corresponds to the (ambiguous) imaginary part of the Borel integral times $\sqrt{2/\pi}$.

In QCD with sea quarks the OPE of the average plaquette or of the Adler function will receive additional contributions from the chiral condensate. For instance $\langle O_G \rangle$ needs to be redefined, adding terms $\propto \langle \gamma_m(\alpha) \psi \bar{\psi} \rangle$ [22]. Due to this and the problem of setting a physical scale in pure gluodynamics, it is difficult to assess the precise numerical impact of including sea quarks onto our estimates.

$$\langle O_G \rangle \simeq 0.077 \text{ GeV}^4, \quad \delta(\langle O_G \rangle) \simeq 0.087 \text{ GeV}^4,$$

which we obtain using $r_0 \approx 0.5$ fm [22]. While the systematics of applying Eqs. (11)–(12) to full QCD are unknown, our main observations should still extend to this case. We remark that our prediction of the gluon condensate Eq. (13) is significantly bigger than values that are typically obtained in one- and two-loop sum rule analyses, ranging from 0.01 GeV$^4$ [8, 27] up to...
0.02 GeV$^4$. However, these numbers were not extracted in the asymptotic regime, which for a $d = 4$ renormalon we expect to set in at orders $n \geq 7$ for the MS scheme. Moreover, we remark that in schemes without a hard ultraviolet cut-off, like dimensional regularization, the extraction of $O(G)$ can become obscured by the possibility of ultraviolet renormalons. Independent of these considerations, all these values are smaller than the intrinsic prescription dependence Eq. (12).

Our analysis confirms the validity of the OPE beyond perturbation theory for the case of the plaquette. Our $a^4$-scaling clearly disfavours suggestions about the existence of (non-local) dimension two condensates beyond the standard OPE framework [16, 31, 32]. In fact we can also explain why an $a^2$-contribution to the plaquette was found in [16]. In the log-log plot Fig. 4 we subtract sums $S_n$, truncated at different fixed orders $a^{n+1}$, from $(P)_{MC}$. The scaling continuously turns from $a^0$ at $O(a^0)$ to $\sim a^2$ around $O(a^3)$. Note that truncating at an $a$-independent fixed order is inconsistent, explaining why we never exactly obtain an $a^4$-slope. For $n \approx 9$ we reproduce the $a^2$-scaling reported in [16] for a fixed order truncation at $n = 7$. In view of Fig. 4 we conclude that the observation of this scaling power was accidental.

The methods used in this paper can be applied to other observables. As an example we analyse the binding energy $\Lambda = E_{MC}(\alpha) - \delta m(\alpha)$ of heavy quark effective theory. The perturbative expansion of $a\delta m(\alpha) = \sum_n a^n a^{n+1}$ was obtained in [36, 37] up to $O(a^{20})$, and its intrinsic ambiguity $\Delta \Lambda = \sum_{n=0}^{20} a^{n+1} = 0.748(42)\Lambda_{MS} = 0.450(44)r_0^{-1}$ in [35, 38]. MC data for the ground state energy $E_{MC}$ of a static-light meson with the Wilson gauge action can be found in [39, 41]. While for the gluon condensate we expected an $a^2$-scaling (see Fig. 5), for $aE_{MC}(\alpha) - a\delta m(\alpha)$ we expect a scaling linear in $a$. Comforting enough this is what we find, up to $aO(a)$ discretization corrections, see Fig. 5. Subtracting the partial sum truncated at orders $n_0(\alpha) = 6$ from the $\beta \in [5, 9, 6.4]$ data, we obtain $\Lambda = 1.55(8)r_0^{-1}$ from such a linear plus quadratic fit, where we only give the statistical uncertainty. The errors of the perturbative coefficients are all tiny, which allows us to transform the expansion $a\delta m(\alpha)$ into $\overline{MS}$-like schemes and to compute $\Lambda$ accordingly. We define the schemes $\overline{MS}_2$ and $\overline{MS}_3$ by truncating $a\delta m(a^{-1}) = a(1 + d_1 a + d_2 a^2 + \ldots)$ exactly at $O(\alpha^3)$ and $O(\alpha^4)$, respectively. The $d_j$ are known for $j \leq 3$ [37, 38]. We typically find $n_0^{\overline{MS}}(\alpha^{\overline{MS}}) = 2.3$ and obtain $\Lambda \sim 2.17(8)r_0^{-1}$ and $\Lambda \sim 1.89(8)r_0^{-1}$, respectively, see Fig. 5. We conclude that the changes due to these resummations are indeed of the size $\delta \Lambda \sim 0.5r_0^{-1}$, adding confidence that our definition of the ambiguity is neither a gross overestimate nor an underestimate. For the plaquette, where we expect $n_0^{\overline{MS}} \sim 7$, we cannot carry out a similar analysis, due to the extremely high precision that is required to resolve the differences between $S_P(\alpha)$ and $(P)_{MC}(\alpha)$, which largely cancel in Eq. (7).

In conclusion, we have obtained an accurate value of the gluon condensate in SU(3) gluodynamics after subtracting the perturbative series truncated at the order of its minimal term from the non-perturbative plaquette. The gluon condensate, Eq. (11), is found to be of a similar size as the intrinsic difference, Eq. (12), between (reasonable) subtraction prescriptions. Its scaling with the lattice spacing confirms the dimension $d = 4$ and we find no evidence of condensates of any lower dimension. Dimension $d < 4$ condensates are only obtained as artefacts of truncating the series at fixed pre-asymptotic orders, which explains earlier claims in the literature.

This work was supported by the German DFG Grant SFB/TRR-55, the Spanish Grants FPA2010-16963 and FPA2011-25948, the Catalan Grant SGR2009-00894 and the EU ITN STRONGnet 238353.
