THE QUANTUM WAVE PACKET AND THE FEYNMAN-DE BROGLIE-BOHM PROPAGATOR OF THE LINEARIZED KOSTIN EQUATION ALONG A CLASSICAL TRAJECTORY

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Abstract: In this paper we study the quantum wave packet and the Feynman-de Broglie-Bohm propagator of the linearized Kostin equation along a classical trajetory.

PACS 03.65 - Quantum Mechanics

1. Introduction

In the present work we investigate the quantum wave packet and the Feynman-de Broglie-Bohm propagator of the linearized Kostin equation along a classical trajetory by using the

2. The Kostin Equation

Em 1972, [1] M. D. Kostin proposed a non-linear Schrödinger to represent time dependent physical systems, given by:
\[ i \hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \]
\[ + [V(x, t) + \frac{\hbar \nu}{2m} \ln \left( \frac{\psi(x, t)}{\psi^*(x, t)} \right)] \psi(x, t) \]  
(2.1)

where \( \psi(x, t) \) and \( V(x, t) \) are, respectively, the wavefunction and the time dependent potential of the physical system in study, and \( \nu \) is a constant.

Writing the wavefunction \( \psi(x, t) \) in the polar form, defined by the Madelung-Bohm [2, 3]:

\[ \psi(x, t) = \phi(x, t) \exp \left[ i S(x, t) \right] \]  
(2.2)

where \( S(x, t) \) is the classical action and \( \phi(x, t) \) will be defined in what follows, and using eq. (2.2) in eq. (2.1), we get: [4]

\[ i \hbar \left( i \frac{\partial S}{\partial t} + \frac{1}{\phi} \frac{\partial \phi}{\partial t} \right) \psi = \]
\[ = -\frac{\hbar^2}{2m} \left[ i \frac{\partial^2 S}{\partial x^2} + \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} - \left( \frac{\partial S}{\partial x} \right)^2 + 2 \frac{i}{\phi} \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} \right] \psi + \]
\[ + [V(x, t) + \frac{\hbar \nu}{2m} \ln \left( \frac{\phi e^{iS}}{\phi e^{-iS}} \right)] \psi \rightarrow \]
\[ i \hbar \left( i \frac{\partial S}{\partial t} + \frac{1}{\phi} \frac{\partial \phi}{\partial t} \right) \psi = \]
\[ = -\frac{\hbar^2}{2m} \left[ i \frac{\partial^2 S}{\partial x^2} + \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} - \left( \frac{\partial S}{\partial x} \right)^2 + 2 \frac{i}{\phi} \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} \right] \psi + \]
\[ + [V(x, t) + \hbar \nu S] \psi \]  
(2.3)

Taking the real and imaginary parts of eq. (2.3), we obtain:

**a) imaginary part**

\[ \frac{\hbar}{\phi} \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 S}{\partial x^2} + \frac{2}{\phi} \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} \right) \rightarrow \]
\[ \frac{\partial \phi}{\partial t} = -\frac{\hbar}{2m} \left( \phi \frac{\partial^2 S}{\partial x^2} + 2 \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} \right) \]  
(2.4)

**b) real part**

\[ -\hbar \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} \left[ \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} - \left( \frac{\partial S}{\partial x} \right)^2 \right] + \]
\[ + [V(x, t) + \hbar \nu S] \rightarrow \]
\[- \frac{\hbar}{m} \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2m^2} \frac{1}{\phi} [\frac{\partial^2 \phi}{\partial x^2} - \phi (\frac{\partial S}{\partial x})^2] + \]
\[+ \frac{1}{m} [V(x, t) + \hbar \nu S] . \quad (2.5)\]

2.1 Dynamics of the Kostin Equation

Now, let us see the correlation between eqs. (2.4,5) and the traditional equations of the Real Fluid Dynamics: a) continuity equation and b) Navier-Stokes’s equation. To do is let us perform the following correspondences:

\[\sqrt{\rho(x, t)} = \phi(x, t) , \quad (2.6) \quad \text{(quantum mass density)}\]

\[v_{\text{qu}}(x, t) = \frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} , \quad (2.7) \quad \text{(quantum velocity)}\]

\[V_{\text{qu}}(x, t) = - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} = - \frac{\hbar^2}{2m} \phi \frac{\partial^2 \phi}{\partial x^2} . \quad (2.8a,b) \quad \text{(Bohm quantum potential)}\]

Putting eq. (2.6,7) into (2.4) we get:

\[\frac{\partial \phi}{\partial t} = - \frac{\hbar}{2m} \left( 2 \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} + \sqrt{\rho} \frac{\partial^2 S}{\partial x^2} \right) \rightarrow \]

\[\frac{1}{\sqrt{\rho}} \frac{\partial \rho}{\partial t} = - \frac{\hbar}{2m} \left( 2 \frac{\partial S}{\partial x} 2 \sqrt{\rho} \frac{\partial \rho}{\partial x} + \sqrt{\rho} \frac{\partial^2 S}{\partial x^2} \right) \rightarrow \]

\[\frac{1}{\rho} \frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} \left( \frac{\hbar}{m} \frac{\partial S}{\partial x} \right) - \frac{1}{\rho} \left( \frac{\hbar}{m} \frac{\partial S}{\partial x} \right) \frac{\partial \phi}{\partial x} \rightarrow \]

\[\frac{1}{\rho} \frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} \left( \frac{\hbar}{m} \frac{\partial S}{\partial x} \right) - \frac{\partial \rho}{\partial x} = 0 \rightarrow \]

\[\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_{\text{qu}}}{\partial x} + v_{\text{qu}} \frac{\partial \rho}{\partial x} = 0 , \quad (2.9)\]

which represents the continuity equation of the mass conservation law of the Fluid Dynamics. We must note that this expression also indicates descoerence of the considered physical system represented by (2.1).

Now, taking the eq. (2.5) and using the eqs. (2.7,8b), will be:

\[- \frac{\hbar}{m} \frac{\partial S}{\partial t} = - \left( \frac{\hbar^2}{2m} \phi \right) \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} m \left( \frac{\hbar}{m} \frac{\partial S}{\partial x} \right)^2 + \]

\[\frac{1}{m} [V(x, t) + \hbar \nu S] . \quad (2.5)\]
\[ + [V(x, t) + \hbar \nu S] \rightarrow \]

\[ \hbar \left( \frac{\partial S}{\partial t} + \nu S \right) + \left( \frac{1}{2} m v_{qu}^2 + V + V_{qu} \right) = 0. \quad (2.10) \]

Differentiating the eq. (2.5) with respect \(x\) and using the eqs. (2.7,8b) we have:

\[-\frac{\hbar}{m} \frac{\partial^2 S}{\partial x \partial t} = -\frac{\hbar^2}{2 m^2} \frac{\partial}{\partial x} \left[ \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right] - \frac{1}{m} \frac{\partial V}{\partial x} + \frac{\hbar}{m} \nu \frac{\partial S}{\partial x} \rightarrow \]

\[-\frac{\partial}{\partial t} \left( \frac{\hbar}{m} \frac{\partial S}{\partial x} \right) = \frac{1}{m} \frac{\partial}{\partial x} \left( -\frac{\hbar^2}{2 m} \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right) + \]

\[+ \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\hbar}{m} \frac{\partial S}{\partial x} \right)^2 + \frac{1}{m} \frac{\partial V}{\partial x} + \nu \frac{\hbar}{m} \frac{\partial S}{\partial x} \rightarrow \]

\[\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \]

\[+ \nu v_{qu} = -\frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) , \quad (2.11) \]

which is an equation similar to the Navier-Stokes’s equation which governs the motion of a real fluid.

Considering the ”substantive differentiation” (local plus convective) or ”hidrodynamic differentiation”: \(d/dt = \partial/\partial t + v_{qu} \partial/\partial x\) and that \(v_{qu} = dx_{qu}/dt\), the eq. (2.11) could be written as:[5]

\[m \frac{d^2 x}{dt^2} = -\nu v_{qu} - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) , \quad (2.12) \]

that has a form of the second Newton law.

3. The Quantum Wave Packet of the Linearized Kostin Equation along a Classical Trajectory

In order to find the quantum wave packet of the non-linear Kostin equation, let us considerer the following ansatz:[6]

\[\rho (x, t) = \left[2\pi a^2(t)\right]^{-1/2} \exp \left( - \frac{|x - q(t)|^2}{2 a^2(t)} \right) , \quad (3.1) \]

where \(a(t)\) and \(q(t)\) are auxiliary functions of time, to be determined in what follows; they represent the width and center of mass of wave packet, respectively.

Substituting (3.1) into (2.3) and integrated the result, we obtain:[4]

\[v_{qu} (x, t) = \frac{a(t)}{a(t)} [x - q(t)] + \dot{q}(t) , \quad (3.2) \]
where the integration constant must be equal to zero since $\rho$ and $\rho \, \partial S/\partial x$ vanish for $|x| \to \infty$. In fact, any well-behaved function of $(x - X)$ multiplied by $\rho$ clearly vanishes as $|x| \to \infty$.

To obtain the quantum wave packet of the linear Kostin equation along a classical trajectory given by (2.1), let us expand the functions $S(X, T), V(x, t)$ and $V_{qu}(x, t)$ around of $q(t)$ up to second Taylor order. In this way we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] [x - q(t)] + \frac{S''[q(t), t]}{2} [x - q(t)]^2 , \quad (3.3)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{V''[q(t), t]}{2} [x - q(t)]^2 . \quad (3.4)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] [x - q(t)] + \frac{V''_{qu}[q(t), t]}{2} [x - q(t)]^2 . \quad (3.5)$$

Differentiating (3.3) in the variable $x$, multiplying the result by $\frac{\hbar}{m}$, using the eqs. (2.7) and (3.2), taking into account the polynomial identity property and also considering the second Taylor order, we obtain:

$$\frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} = \frac{\hbar}{m} \left( S'[q(t), t] + S''[q(t), t] [x - q(t)] \right) = v_{qu}(x, t) = \left[ \frac{\dot{q}(t)}{\dot{a}(t)} \right] [x_{qu} - q(t)] + \dot{q}(t) \to S'[q(t), t] = \frac{m \dot{q}(t)}{\hbar} , \quad S''[q(t), t] = \frac{m}{\hbar} \frac{\ddot{a}(t)}{a(t)} , \quad (3.6a,b)$$

Substituting (3.6a,b) into (3.3), results:

$$S(x, t) = S_o(t) + \frac{m \dot{q}(t)}{\hbar} [x - q(t)] + \frac{m}{2 \hbar} \frac{\ddot{a}(t)}{a(t)} [x - q(t)]^2 , \quad (3.7)$$

where:

$$S_o(t) \equiv S[q(t), t] , \quad (3.8)$$

are the classical actions.

Differentiating the (3.7) with respect to $t$, we obtain (remembering that $\frac{\partial x}{\partial t} = 0$):

$$\frac{\partial S}{\partial t} = \dot{S}_o(t) + \frac{\dot{q}(t)}{\hbar} \left( \frac{m}{\hbar} [x - q(t)] \right) + \frac{m}{2 \hbar} \left[ \frac{\ddot{a}(t)}{a(t)} \right] [x - q(t)]^2 \to \frac{\partial S}{\partial t} = \dot{S}_o(t) + \frac{m \dot{q}(t)}{\hbar} [x - q(t)] - \frac{m \dot{q}(t)^2}{\hbar} +$$

$$+ \frac{m}{2 \hbar} \left[ \frac{\ddot{a}(t)}{a(t)} - \frac{a^2(t)}{a^2(t)} \right] [x - q(t)]^2 - \frac{m}{\hbar} \frac{\ddot{a}(t)}{a(t)} [x - q(t)] . \quad (3.9)$$
Considering the eqs. (2.6) and (3.1), let us write \( V_{qu} \) given by (2.8a,b) in terms of potencies of \([x - q(t)]\). Initially using (2.5) and (3.1), we calculate the following derivations:

\[
\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \left( [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \right) = [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \frac{\partial}{\partial x} \left( - \frac{\mathbf{x} - \mathbf{q}(t)^2}{4 a^2(t)} \right) \rightarrow
\]

\[
\frac{\partial \phi}{\partial x} = - [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \frac{\partial}{\partial x} \left( \frac{\mathbf{x} - \mathbf{q}(t)}{2 a(t)} \right),
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x} \left( - [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \frac{\partial}{\partial x} \left( - \frac{\mathbf{x} - \mathbf{q}(t)}{4 a^2(t)} \right) \right) =
\]

\[
- [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \frac{\partial}{\partial x} \left( - \frac{\mathbf{x} - \mathbf{q}(t)}{4 a^2(t)} \right) \rightarrow
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = - [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \frac{1}{2 \pi a^2(t)} + [2 \pi a^2(t)]^{-1/4} e^{-i(\mathbf{x} - \mathbf{q}(t))^2 / 4 a^2(t)} \left( \frac{\mathbf{x} - \mathbf{q}(t)}{4 a^2(t)} \right) =
\]

\[
= - \phi \frac{1}{2 a^2(t)} + \phi \left[ \frac{\mathbf{x} - \mathbf{q}(t)}{4 a^2(t)} \right] \rightarrow \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = \frac{\mathbf{x} - \mathbf{q}(t)}{4 a^2(t)} - \frac{1}{2 a^2(t)}. \quad (3.10)
\]

Substituting (3.10) into (2.8a) and taking into account (3.5), results:

\[
V_{qu}(x, t) = V_{qu}[q(t), t] + V_{qu}'[q(t), t] [x - q(t)] + \frac{V_{qu}''[q(t), t]}{2} [x - q(t)]^2 \rightarrow
\]

\[
V_{qu}(x, t) = \frac{\hbar^2}{4 m a^2(t)} - \frac{\hbar^2}{8 m a(t)} [x - q(t)]^2. \quad (3.11)
\]

Inserting the eqs. (2.7), (3.2,3,4), (3.7,8,9) and (3.11), into (2.11), we obtain [remembering that \( S_0(t), a(t) \) and \( q(t) \nabla \nabla \nabla \nabla \nabla \nabla \nabla ]:

\[
\hbar \left[ \frac{\partial S}{\partial t} + \nu S + \left( \frac{1}{2} m v_{qu}^2 + V + V_{qu} \right) \right] = 0. \quad (3.12)
\]

\[
\hbar \left( \dot{S}_0 + \frac{m \ddot{q}(t)}{\hbar} [x - q(t)] - \frac{m \dot{q}^2(t)}{\hbar} + \frac{m}{2 \hbar} \left[ \frac{\dot{a}(t)}{a(t)} - \frac{\dot{a}^2}{a^2(t)} \right] [x - q(t)]^2 - \frac{m \dot{q}(t)}{\hbar} \frac{\dot{a}(t)}{a(t)} [x - q(t)] \right) + \frac{m}{2} \left( \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t) \right)^2 +
\]

\[
+ \hbar \nu \left( S_0(t) + \frac{m \ddot{q}(t)}{\hbar} [x - q(t)] + \frac{m}{2 \hbar} \frac{\dot{a}(t)}{a(t)} [x - q(t)]^2 \right) +
\]
\[ + V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{m}{2} V''[q(t), t] [x - q(t)]^2 + \]
\[ + \frac{\hbar^2}{4 m a^2(t)} - \frac{\hbar^2}{8 m a^4(t)} [x - q(t)]^2 = 0 . \] (3.13)

Since \((x - q)^o = 1\), expanding (3.13) in potencies of \((x - q)\), we obtain:
\[
\left( \hbar \dot{S}_o(t) - m \ddot{q}(t) + \frac{1}{2} m \ddot{q}^2(t) + V[q(t), t] + \hbar \nu [S_0(t)] + \frac{\hbar^2}{4 m a^2(t)} \right) [x - q(t)]^o + \\
+ \left( m \dddot{q}(t) - m \dot{q}(t) \frac{\dddot{a}(t)}{a(t)} + m \dddot{q}(t) \frac{\dddot{a}(t)}{a(t)} + \nu m \dddot{q}(t) + V'[q(t), t] \right) [x - q(t)] + \\
+ \left( \frac{m}{2} [\frac{\dddot{a}(t)}{a(t)} - \frac{\dddot{a}(t)}{a(t)^2}] + \frac{m}{2} \frac{\dddot{a}^2(t)}{a^3(t)} + m \nu \frac{\dddot{a}(t)}{a(t)} + \\
+ \frac{1}{2} V''[q(t), t] - \frac{\hbar^2}{4 m a^2(t)} \right) [x - q(t)]^2 = 0 . \] (3.14)

As (3.14) is an identically null polynomial, all coefficients of the potencies must be all equal to zero, that is:
\[
\dot{S}_o(t) = \frac{1}{\hbar} \left( \frac{1}{2} m \ddot{q}(t)^2 - V[q(t), t] - \frac{\hbar^2}{4 m a^2(t)} - \hbar \nu S_0(t) \right) , \] (3.15)
\[
\dddot{q}(t) + \nu \dot{q}(t) + \frac{1}{m} V'[q(t), t] = 0 , \] (3.16)
\[
\dddot{a}(t) + \nu \dddot{a}(t)] + \frac{1}{m} V''[q(t), t] = \frac{\hbar^2}{4 m a^2(t)} . \] (3.17)

Assuming that the following initial conditions are obeyed:
\[
q(0) = x_o , \quad \dot{q}(0) = v_o , \quad a(0) = a_o , \quad \dddot{a}(0) = b_o , \] (3.18a-d)

and that:
\[
S_o(0) = \frac{m v_o x_o}{\hbar} , \] (3.19)

the integration of (3.15) gives:
\[
S_o(t) = \frac{1}{\hbar} \int_0^t dt' \left( \frac{1}{2} m \ddot{q}(t') + \nu [S_0(t')] - \\
- V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \right) + \frac{m v_o x_o}{\hbar} . \] (3.20)
Taking the eq. (3.20) in the eq. (3.7) results:

\[
S(x, t) = \frac{1}{\hbar} \int_o^t dt' \left( \frac{1}{2} m \dot{q}^2(t') + \hbar \nu [S_0(t')] - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \right) + \\
+ \frac{m v_0 x_0}{\hbar} + \frac{m \dot{q}(t)}{\hbar} [x - q(t)] + \frac{m}{2 \hbar} \left[ \dot{a}(t) [x - q(t)]^2 \right].
\] (3.21)

The above result permit us, finally, to obtain the wave packet for the linearized Kostin equation along a classical trajectory. Indeed, considering (2.2), (2.6), (3.1) and (3.21), we get: [6]

\[
\psi(x, t) = [2 \pi a^2(t)]^{-1/4} \exp \left[ \frac{i m}{2 \hbar} \left( \frac{\ddot{a}(t)}{a(t)} - \frac{1}{4 a^2(t)} \right) [x - q(t)]^2 \right] \times \\
\times \exp \left[ \frac{i m \dot{q}(t)}{\hbar} [x - q(t)] + \frac{i m v_0 x_0}{\hbar} \right] \times \\
\times \exp \left[ i \frac{1}{\hbar} \int_o^t dt' \left( \frac{1}{2} m \dot{q}^2(t') + \hbar \nu [S_0(t') - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \right) \right].
\] (3.22)

Note that putting \( \nu = 0 \) into (3.22) we obtain the quantum wave packet of the Schrödinger equation with the potential V(x, t). [7]

4. The Feynman-de Broglie-Bohm Propagator of the Linearized Kostin Equation along a Classical Trajectory

4.1. Introduction

In 1948, [8] Feynman formulated the following principle of minimum action for the Quantum Mechanics:

The transition amplitude between the states \( |a> \) and \( |b> \) of a quantum-mechanical system is given by the sum of the elementary contributions, one for each trajectory passing by \( |a> \) at the time \( t_a \) and by \( |b> \) at the time \( t_b \). Each one of these contributions have the same modulus, but its phase is the classical action \( S_{cl} \) for each trajectory.

This principle is represented by the following expression known as the "Feynman propagator":

\[
K(b, a) = \int_a^b e^{i \int_x^{t_b} S_{cl}(b, a) D x(t)},
\] (4.1)
\[ S_{\text{cf}}(b, a) = \int_{t_a}^{t_b} L(x, \dot{x}, t) \, dt, \quad (4.2) \]

where \( L(x, \dot{x}, t) \) is the Lagrangean and \( D x(t) \) is the Feynman’s Measurement. It indicates that we must perform the integration taking into account all the ways connecting the states \( |a> \) and \( |b> \).

Note that the integral which defines \( K(b, a) \) is called ”path integral” or ”Feynman integral” and that the Schrödinger wavefunction \( \psi(x, t) \) of any physical system is given by (we indicate the initial position and initial time by \( x_o \) and \( t_o \), respectively): [9]

\[ \psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t, t_o) \psi(x_o, t_o) \, dx_o, \quad (4.3) \]

with the quantum causality condition:

\[ \lim_{t, t_o \to 0} K(x, x_o, t, t_o) = \delta(x - x_o). \quad (4.4) \]

### 4.2. Calculation of the Feynman-de Broglie-Bohm Propagator for the Linearized Kostin Equation along a Classical Trajectory

According to Section 3, the wavefunction \( \psi(x, t) \) that was named wave packet of the of the linearized Kostin equation along a classical trajectory, can be written as [see (3.22)]:

\[ \psi(x, t) = \left[ 2 \pi a^2(t) \right]^{-1/4} \exp \left[ \frac{i m}{\hbar} \left( \frac{\dot{q}(t)}{a(t)} - \frac{1}{4 a^2(t)} \right) [x - q(t)]^2 \right] \times \]

\[ \times \exp \left[ \frac{i m}{\hbar} \dot{q}(t) [x - q(t)] + \frac{i m v_o x_o}{\hbar} \right] \times \]

\[ \times \exp \left[ \frac{i}{\hbar} \int_{t_o}^{t} dt' \left( \frac{1}{2} m \dot{q}^2(t') + \hbar \nu [S_0(t') - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \right) \right]. \quad (4.5) \]

where [see (3.16,17)]:

\[ \ddot{q}(t) + \nu \dot{q}(t) + \frac{1}{m} V''[q(t), t] = 0, \quad (4.6) \]

\[ \ddot{a}(t) + \nu \dot{a}(t) + \frac{1}{m} V''[q(t), t] = \frac{\hbar^2}{4 m^2 a(t)^4}. \quad (4.7) \]

where the following initial conditions were obeyed [see (3.18a-d)]:

\[ q(0) = x_o, \quad \dot{q}(0) = v_o, \quad a(0) = a_o, \quad \dot{a}(0) = b_o. \quad (4.8a-d) \]

Therefore, considering (4.3), the Feynman-de Broglie-Bohm propagator will be calculated using (4.5), in which we will put with no loss of generality, \( t_o = 0 \). Thus:
\[ \psi(x, t) = f^\pm_\infty K(x, x_o, t) \psi(x_o, 0) \, dx_o. \quad (4.9) \]

Let us initially define the normalized quantity:

\[ \Phi(v_o, x, t) = (2 \, \pi \, a^2_o)^{1/4} \psi(v_o, x, t), \quad (4.10) \]

which satisfies the following completeness relation: [10]

\[ \int_{-\infty}^{+\infty} dv_o \, \Phi^*(v_o, x, t) \Phi(v_o, x', t) = \left( \frac{2 \, \pi \, \hbar}{m} \right) \delta(x - x'). \quad (4.11) \]

Considering (2.4a), (2.9a) and (4.9), we get:

\[ \Phi^*(v_o, x, t) \psi(v_o, x, t) =
\]

\[ = (2 \, \pi \, a^2_o)^{1/4} \psi^*(v_o, x, t) \psi(v_o, x, t) = (2 \, \pi \, a^2_o)^{1/4} \rho(v_o, x, t) \rightarrow
\]

\[ \rho(v_o, x, t) = (2 \, \pi \, a^2_o)^{-1/4} \Phi^*(v_o, x, t) \psi(v_o, x, t). \quad (4.12) \]

On the other side, substituting (4.12) into (2.11), integrating the result and using (3.1) and (4.10) results [remembering that \( \psi^* \psi(\pm \infty) \rightarrow 0 \):]

\[ \frac{\partial(\Phi^* \psi)}{\partial t} + \frac{\partial(\Phi^* \psi v_{qu})}{\partial x} = 0 \rightarrow \]

\[ \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \psi + (\Phi^* \psi v_{qu})|_{- \infty}^{+ \infty} = \]

\[ = \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \psi + (2 \, \pi \, a^2_o)^{1/4} (\psi^* \psi v_{qu})|_{- \infty}^{+ \infty} = 0 \rightarrow \]

\[ \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \psi = 0. \quad (4.13) \]

Eq. (4.13) shows that the integration is time independent. Consequently:

\[ \int_{-\infty}^{+\infty} dx' \, \Phi^*(v_o, x', t) \psi(x', t) = \int_{-\infty}^{+\infty} dx_o \, \Phi^*(v_o, x_o, 0) \psi(x_o, 0). \quad (4.14) \]

Multiplying (4.14) by \( \Phi(v_o, x, t) \) and integrating over \( v_o \) and using (4.11), we obtain [remembering that \( \int_{-\infty}^{+\infty} dx' \, f(x') \, \delta(x' - x) = f(x) \):]

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o \, dx' \, \Phi(v_o, x, t) \Phi^*(v_o, x', t) \psi(x', t) = \]

\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o \, dx_o \, \Phi(v_o, x, t) \Phi^*(v_o, x, 0) \psi(x, 0) \rightarrow \]

\[ \int_{-\infty}^{+\infty} dx' \left( \frac{2 \, \pi \, \hbar}{m} \right) \delta(x' - x) \psi(x', t) = \left( \frac{2 \, \pi \, \hbar}{m} \right) \psi(x, t) = \]
\[
\psi(x, t) = \int_{-\infty}^{+\infty} \left( \frac{m}{2 \pi \hbar} \right)^{1/2} dv_0 \Phi(v_o, x, t) \Phi^*(v_o, x_0, 0) \psi(x_0, 0) \rightarrow 
\]

Comparing (4.9) and (4.15), we have:

\[
K(x, x_o, t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_0 \Phi(v_o, x, t) \Phi^*(v_o, x_0, 0). \quad (4.16)
\]

Substituting (4.5) and (4.10) into (4.16), we finally obtain the Feynman-de Broglie-Bohm Propagator of the linearized Kostin equation along a classical trajectory, remembering that \(\Phi^*(v_o, x_0, 0) = \exp(-\frac{i m v_o x_o}{\hbar})\):

\[
K(x, x_o, t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_0 \sqrt{a(t)} \times 
\]

\[
\times \exp \left[ \left( \frac{i m \dot{a}(t)}{2 \hbar a(t)} - \frac{1}{4} a^{-2}(t) \right) [x - q(t)]^2 + \frac{i m q(t)}{\hbar} [x - q(t)] \right] \times 
\]

\[
\times \exp \left[ \frac{i}{\hbar} \int_0^t dt' \left( \frac{1}{2} m \dot{q}^2(t') + 2 \hbar \nu [S_0(t')] - \frac{\hbar^2}{4 m \alpha^2(t')} \right) \right], \quad (4.17)
\]

where \(q(t)\) and \(a(t)\) are solutions of the (4.6, 7) differential equations.

Finally, it is important to note that putting \(\alpha = 0\) into (3.14), (3.15) and (4.17) we obtain the free particle Feynman propagator. [7, 9]

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