On realizations of the Virasoro algebra\textsuperscript{*}

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Abstract

We obtain complete classification of inequivalent realizations of the Virasoro algebra by Lie vector fields over the three-dimensional field of real numbers. As an application we construct new classes of nonlinear second-order partial differential equations possessing infinite-dimensional Lie symmetries.

1 Introduction

Since its introduction in 19th century Lie group analysis has become a very popular and powerful tool for solving nonlinear partial differential equations (PDEs). Given a PDE that possesses nontrivial symmetry, we can utilize symmetry reduction procedure to construct its exact solutions \textsuperscript{[9, 10]}. Not surprisingly, the wider symmetry of an equation under study is, the better off we are when applying Lie approach to solve it. This is especially the case when symmetry group is infinite-parameter. If a nonlinear differential equation admits infinite Lie symmetry then it is often possible either to linearize it or construct its general solution \textsuperscript{[9]}. The classical example is the hyperbolic type Liouville equation

\begin{equation}
    u_{tx} = \exp(u). \tag{1}
\end{equation}

It admits the infinite-parameter Lie group

\begin{equation}
    t’ = t + f(t), \quad x’ = x + g(x), \quad u’ = u - \dot{f}(t) - \dot{g}(x), \tag{2}
\end{equation}

where \( f \) and \( g \) are arbitrary smooth functions. General solution of (1) can be obtained by the action of transformation group (2) on its particular traveling wave

\textsuperscript{*}Supported by the NSF of China (Grant Nos. 11101332, 11201371), the Foundation of Shaanxi Educational Committee, China (Grant No. 11JK0482) and the NSF of Shaanxi Province, China (Grant No. 2012JQ1013).
solution of the form $u(t, x) = \varphi(x + t)$ (see, e.g., [9]). An alternative way to solve the Liouville equation is to linearize it [9].

Note that the Lie algebra of Lie group [2] is the direct sum of two infinite-dimensional Witt algebras, which are subalgebras of the Virasoro algebra.

Unlike the finite-dimensional algebras, infinite-dimensional ones have not been systematically studied within the context of classical Lie group analysis of nonlinear PDEs. The situation is however drastically different for the case of generalized (higher) Lie symmetries which played critical role in success of the theory of integrable systems in $(1 + 1)$ and $(1 + 2)$ dimensions (see, e.g., [16]).

The breakthrough in the analysis of integrable systems was nicely complemented by development of the theory of infinite-dimensional Lie algebras such as loop algebras [28], Kac-Moody algebras [19] and Virasoro algebras [17].

Virasoro algebra plays an increasingly important role in mathematical physics in general [4, 13] and in the theory of integrable systems in particular. Study of nonlinear evolution equations in (1+2)-dimensions arising in different areas of modern physics shows that in many cases the Virasoro algebra is their symmetry algebra. Let us mention among others the Kadomtsev-Petvishilvi (KP) [7, 8, 14], modified KP, cylindrical KP [22], the Davey-Stewartson [6, 15], Nizhnik-Novikov-Veselov, stimulated Raman scattering, (1+2)-dimensional Sine-Gordon [30] and the KP hierarchy [26] equations.

Note that there exist integrable equations which admit infinite-dimensional symmetry algebras that are not of Virasoro type. For instance, the breaking soliton and Zakharov-Strachan equations do not possess Virasoro type symmetry while being integrable [30].

It is a common belief that nonlinear PDEs admitting symmetry algebras of Virasoro type are prime candidates for the roles of integrable systems. Consequently, systematic classification of inequivalent realizations of the Virasoro algebra is a crucial step of symmetry approach to constructing integrable systems (see, e.g., [23, 24]).

Classification of Lie algebras of vector fields of differential operators within action of local diffeomorphism group has been pioneered by Sophus Lie himself. It remains a very powerful method for group analysis of nonlinear differential equations. Some of more recent applications of this approach include geometric control theory [18], theory of systems of nonlinear ordinary differential equations possessing superposition principle [31], algebraic approach to molecular dynamics [2, 29] to mention only a few. Still the biggest bulk of results has been obtained in the area of classification of nonlinear PDEs possessing point and higher Lie symmetries (see [8] and references therein). Analysis of realizations of Lie algebras by first-order differential operators is in the core of almost every approach to group classification of partial differential equations (see, e.g., [1, 3, 10, 12, 20, 21]).

In this paper we concentrate on study of realizations of Virasoro algebras by first-order differential operators (Lie vector fields) in the space $\mathbb{R}^n$ with $n \leq 3$. One of our motivations was to utilize these realizations for constructing $(1 + 1)$-
dimensional PDEs that possess infinite-dimensional Lie symmetry and in this sense are integrable.

The paper is organized as follows. In Section 2 we give a brief account of necessary facts and definitions. In addition, the algorithmic procedure for classifying realizations of infinite-dimensional Virasoro and Witt algebras is described in detail. We construct all inequivalent realizations of the Witt algebra (a.k.a. centerless Virasoro algebra) in Section 3. The next section is devoted to analysis of realizations of the full Virasoro algebra. We prove that there are no central extensions of the Witt algebra in the space $\mathbb{R}^3$ that possess nonzero central element. In Section 5 we construct broad classes of nonlinear PDEs admitting infinite dimensional symmetry algebras which are realizations of the Witt algebra. The last section contains discussion of the obtained results and the outline of further work.

2 Notations and definitions

The Virasoro algebra, $\mathfrak{W}$, is the infinite-dimensional Lie algebra with basis elements $C, L_n, n = 0, \pm 1, \pm 2, \ldots$ which satisfy the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m,-n}C, \quad [L_m, C] = 0, \quad m, n \in \mathbb{Z}. \quad (3)$$

Hereafter we denote the commutator of two Lie vector fields $P$ and $Q$ as $[Q, P]$, i.e., $[Q, P] = QP - PQ$. The symbol $\delta_{a,b}$ stands for the Kronecker delta

$$\delta_{a,b} = \begin{cases} 1, & a = b, \\ 0, & \text{otherwise}. \end{cases}$$

The operator $C$ commuting with all other elements is called the central element of algebra $\mathfrak{W}$. In the case when $C$ equals to zero the algebra $\mathfrak{W}$ is called the centerless Virasoro algebra or Witt algebra $\mathfrak{W}$. Consequently, the full Virasoro algebra is the universal central extension of the Witt algebra.

The realization space of the Virasoro algebra is the infinite-dimensional Lie algebra $\mathcal{L}_\infty$ of first-order differential operators of the form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u \quad (4)$$

over the space $\mathbb{R}^3 \ni (t, x, u)$. One can readily verify that the set of operators $\mathcal{H}$ is invariant under the transformation of variables $t, x, u$

$$t \to \tilde{t} = T(t, x, u), \quad x \to \tilde{x} = X(t, x, u), \quad u \to \tilde{u} = U(t, x, u), \quad (5)$$

provided $D(T, X, U)/D(t, x, u) \neq 0$. Indeed, applying $\mathcal{H}$ to an arbitrary element of $\mathcal{L}_\infty$ of the form $\mathcal{H}$ we get

$$\tilde{Q} = (\tau T_t + \xi T_x + \eta T_u)\partial_{\tilde{t}} + (\tau X_t + \xi X_x + \eta X_u)\partial_{\tilde{x}} + (\tau U_t + \xi U_x + \eta U_u)\partial_{\tilde{u}}.$$
Evidently, $\tilde{Q} \in \mathcal{L}_\infty$.

It is a common knowledge that correspondence $Q \sim \tilde{Q}$ is the equivalence relation and as such it splits the set of operators $4$ into equivalence classes. Any two elements within the equivalence class are related through a transformation $5$, while two elements belonging to different equivalence classes cannot be transformed one into another by a transformation of the form $5$. Hence to describe all possible realizations of the Virasoro algebra within the class of Lie vector fields $4$ one needs to construct a representative of each equivalence class. The remaining realizations are obtained by applying transformations $5$ to the representatives in question.

The procedure for constructing realizations of inequivalent algebra $3$ consists of the following three steps:

- Describe all inequivalent forms of $L_0$, $L_1$ and $L_{-1}$ such that the commutation relations of the Virasoro subalgebra,

$$
[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0,
$$

hold together with the relations $[L_i, C] = 0$, $(i = 0, 1, -1)$. Note that the algebra $6$ is isomorphic to $sl(2, \mathbb{R})$.

- Construct all inequivalent realizations of the operators $L_2$ and $L_{-2}$ which satisfy the following commutation relations:

$$
[L_0, L_2] = -2L_2, \quad [L_{-1}, L_2] = -3L_1, \quad [L_1, L_{-2}] = 3L_{-1},
$$

$$
[L_0, L_{-2}] = 2L_{-2}, \quad [L_2, L_{-2}] = 4L_0 + \frac{1}{2}C, \quad [L_i, C] = 0, \ (i = 2, -2).
$$

- Describe the remaining basis operators of the Virasoro algebra through the recursion relations

$$
L_{n+1} = (1 - n)^{-1}[L_1, L_n], \quad L_{-n-1} = (n - 1)^{-1}[L_{-1}, L_{-n}]
$$

and

$$
[L_{n+1}, L_{-n-1}] = 2(n + 1)L_0 + \frac{1}{12}n(n + 1)(n + 2)C, \quad [L_i, C] = 0
$$

with $i = n + 1, -n - 1$ and $n = 2, 3, 4, \ldots$.

In the Sections 3 and 4 we implement the above algorithm to construct all inequivalent realizations of the Witt and Virasoro algebras by operators $4$. 

4
3 Realizations of the Witt algebra

Turn now to describing realizations of the Witt algebra. We remind that the Witt algebra is obtained from the Virasoro algebra by putting $C = 0$. We begin by letting the vector field $L_0$ be of the general form (4), i.e.,

$$L_0 = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$ 

Transformation (5) maps $L_0$ into

$$\tilde{L}_0 = (\tau T_t + \xi T_x + \eta T_u)\partial_t + (\tau X_t + \xi X_x + \eta X_u)\partial_x + (\tau U_t + \xi U_x + \eta U_u)\partial_u.$$ 

We have $\tau^2 + \xi^2 + \eta^2 \neq 0$, since otherwise $L_0$ is zero. Consequently, we can choose the solutions of equations

$$\tau T_t + \xi T_x + \eta T_u = 1, \quad \tau X_t + \xi X_x + \eta X_u = 0, \quad \tau U_t + \xi U_x + \eta U_u = 0.$$ 

as $T$, $X$ and $U$ and reduce $L_0$ to the form $L_0 = \partial_t$ (hereafter we drop the tildes). Consequently, the vector field $L_0$ is equivalent to the canonical operator $\partial_t$.

With $L_0$ in hand we now proceed to constructing $L_1$ and $L_{-1}$ which obey the commutation relations (6). Let $L_1$ be of the general form (4). Inserting it into $[L_0, L_1] = -L_1$ yields

$$L_1 = e^{-t}f(x,u)\partial_t + e^{-t}g(x,u)\partial_x + e^{-t}h(x,u)\partial_u,$$ 

where $f$, $g$, $h$ are arbitrary transformation smooth functions. To further simplify vector field $L_1$ we utilize the equivalence transformation (5) preserving the form of $L_0$. Applying (5) to $L_0$ gives

$$L_0 \rightarrow \tilde{L}_0 = T_t\partial_t + X_t\partial_x + U_t\partial_u = \partial_t.$$ 

Consequently, transformation

$$\tilde{t} = t + T(x,u), \quad \tilde{x} = X(x,u), \quad \tilde{u} = U(x,u)$$ 

is the most general transformation that does not alter the form of $L_0$. It maps the Lie vector field $L_1$ into

$$\tilde{L}_1 = e^{-t}(f + g T_x + h T_u)\partial_t + e^{-t}(g X_x + h X_u)\partial_x + e^{-t}(g U_x + h U_u)\partial_u.$$ 

To further analyze the above class of realizations of $L_1$ we need to differentiate between two separate cases $g^2 + h^2 = 0$ and $g^2 + h^2 \neq 0$.

**Case 1.** If $g^2 + h^2 = 0$, then $\tilde{L}_1 = e^{-t}f(x,u)\partial_t$. Choosing $\tilde{t} = t - \ln|f(x,u)|$ we have $L_1 = e^{-t}\partial_t$. Let $L_{-1}$ be of the general form (4). Inserting $L_0$, $L_1$, $L_{-1}$ into the commutation relations $[L_0, L_{-1}] = L_{-1}$ and $[L_1, L_{-1}] = 2L_0$ yields $L_{-1} = e^t\partial_t$.

**Case 2.** If $g^2 + h^2 \neq 0$ then we choose $\tilde{t} = t + T(x,u)$ where $T(x,u)$ satisfies the equation

$$e^{-T} = f + g T_x + h T_u,$$
and take solutions of the equations
\[ gX + uX = e^{-T}, \quad gU + hU = 0 \]
as \( X \) and \( U \) thus mapping \( L_1 \) into \( e^{-t}(\partial_t + \partial_x) \). Choosing \( L_{-1} \) in the general form (4) and taking into account commutation relations \([L_0, L_{-1}] = L_{-1} \) and \([L_1, L_{-1}] = 2L_0 \) we get
\[ L_{-1} = e^t(1 - e^{-2x} f_1(u))\partial_t + e^t(-1 - e^{-2x} f_1(u) + e^{-x} g_1(u))\partial_x + e^{t-x} h_1(u) \partial_u, \]
where \( f_1, g_1, h_1 \) are arbitrary smooth functions.

The transformation
\[
\tilde{t} = t, \quad \tilde{x} = x + X(u), \quad \tilde{u} = U(u)
\]
evidently does not alter the form of \( L_0, L_1 \). Applying (5) to \( L_{-1} \) yields
\[
\tilde{L}_{-1} = e^t(1 - e^{-2x} f_1(u))\partial_t + e^t(-1 - e^{-2x} f_1(u) + e^{-x} g_1(u) + e^{-x} h_1(u) \dot{X}) \partial_x + e^{t-x} h_1(u) \dot{U} \partial_u.
\]
Here and after the dot over the symbol stands for derivative of the corresponding function of one variable.

To complete the analysis we need to consider separately the cases \( f_1(u) \neq 0 \) and \( f_1(u) = 0 \).

Provided \( f_1(u) \neq 0 \) we can choose
\[
X(u) = -\ln \sqrt{|f_1(u)|}, \quad \phi(u) = (g_1(u) + h_1(u) \dot{X}) \frac{\dot{X}}{\sqrt{|f_1(u)|}}.
\]
Selecting \( U \) to be a solution of \( h_1(u) \dot{U} = \sqrt{|f_1(u)|} \) if \( h_1 \neq 0 \) or an arbitrary non-constant function if \( h_1 = 0 \), we have
\[
L_{-1} = e^t(1 + \alpha e^{-2x}) \partial_t + e^t(-1 + \alpha e^{-2x} + e^{-x} \phi(u)) \partial_x + \beta e^{t-x} \partial_u,
\]
where \( \alpha = \pm 1 \) and \( \beta = 0, 1 \).

The case \( f_1(u) = 0 \) leads to the realization
\[
\tilde{L}_{-1} = e^t \partial_t + e^t(-1 + e^{-x} g_1(u) + e^{-x} h_1(u) \dot{X}) \partial_x + e^{t-x} h_1(u) \dot{U} \partial_u.
\]
Choosing \( X = 0 \) and \( U \) to be a solution of \( h_1(u) \dot{U} = 1 \) if \( h_1 \neq 0 \) or an arbitrary non-constant function if \( h_1 = 0 \), we obtain
\[
L_{-1} = e^t \partial_t + e^t(-1 + e^{-x} g_1(u)) \partial_x + \beta e^{t-x} \partial_u
\]
with \( \beta = 0, 1 \).

We summarize the results obtained above in the following lemma.
Lemma 1. There exist only two inequivalent realizations of the algebra \( \mathfrak{g} \):

1. \( L_0 = \partial_t, \quad L_1 = e^t \partial_t, \quad L_{-1} = e^t \partial_t; \) \( (9) \)
2. \( L_0 = \partial_t, \quad L_1 = e^{-t}(\partial_t + \partial_x), \quad L_{-1} = e^t(1 + \alpha e^{-2x})\partial_t \)
   \[ + e^t(-1 + \phi(u)e^{-x} + \alpha e^{-2x})\partial_x + \beta e^{(t-x)}\partial_u. \] \( (10) \)

Here \( \alpha = 0, \pm 1, \beta = 0, 1 \) and \( \phi(u) \) is an arbitrary smooth function.

To get a complete description of inequivalent realizations of the Witt algebra, we need to extend algebras (9) and (10) by the operators \( L_2 \) and \( L_{-2} \) and perform the last two steps of the classification procedure described in Section 2. We first present the final result and then give the detailed proof.

Theorem 1. There exist at most eleven inequivalent realizations of the Witt algebra \( \mathfrak{w} \) over the space \( \mathbb{R}^3 \). Below we give the list of representatives of each equivalence class \( \mathfrak{w}_i, i = 1, 2, \ldots, 11 \).

\( \mathfrak{w}_1 \):

\[ L_n = e^{-nt} \partial_t, \]

\( \mathfrak{w}_2 \):

\[ L_n = e^{-nt} \partial_t + e^{-nt} \left[ n + \frac{1}{2} n(n-1)\alpha e^{-x} \right] \partial_x, \]

\( \mathfrak{w}_3 \):

\[ L_n = e^{-nt+(n-1)x} \left[ e^{2x} - (n + 1)\gamma e^x + \frac{1}{2} n(n+1)n \right] (e^x - \gamma)^{-n-1} \partial_t \]
\[ + e^{-nt+(n-1)x} \left[ ne^x - \frac{1}{2} n(n+1)n \gamma \right] (e^x - \gamma)^{-n} \partial_x, \]

\( \mathfrak{w}_4 \):

\[ L_0 = \partial_t, \]
\[ L_1 = e^{-t}\partial_t + e^{-t}\partial_x, \]
\[ L_{-1} = e^t(1 + \gamma e^{-2x})\partial_x + e^t(-1 + \gamma e^{-2x} + e^{-x}\phi)\partial_x, \]
\[ L_2 = e^{-2t}f(x,u)\partial_x + e^{-2t}g(x,u)\partial_x, \]
\[ L_{-2} = e^{2t} \left[ 1 + 3\gamma e^{-2x} - \frac{1}{2} e^{-3x} (6\gamma \phi + \phi^3) \pm (4\gamma + \phi^2)^{3/2} \right] \partial_t \]
\[ + e^{2t} \left[ -2 + 3e^{-x}\phi + 6\gamma e^{-2x} - \frac{1}{2} e^{-3x} (6\gamma \phi + \phi^3) \pm (4\gamma + \phi^2)^{3/2} \right] \partial_x, \]
\[ L_{n+1} = (1 - n)^{-1}[L_1, L_n], \quad L_{n-1} = (n - 1)^{-1}[L_{-1}, L_{-n}], \quad n \geq 2, \]
where

\[ f(x, u) = e^x \left[ 4e^{4x} - 10e^{3x} \phi - 36\gamma e^{2x} + 2e^x \left( 31\gamma \phi + 6\phi^3 \pm 6(4\gamma + \phi^2)^{3/2} \right) \right] - 64\gamma^2 - 54\gamma \phi^2 - 9\phi^4 \mp 9\phi(4\gamma + \phi^2)^{3/2} \right] r^{-1} \]

\[ g(x, u) = e^x \left[ 8e^{4x} - 16e^{3x} \phi - 2e^{2x}(44\gamma + 5\phi^2) + 2e^x \left( 44\gamma \phi + 9\phi^3 \pm 9(4\gamma + \phi^2)^{3/2} \right) \right] - 64\gamma^2 - 54\gamma \phi^2 - 9\phi^4 \mp 9\phi(4\gamma + \phi^2)^{3/2} \right] r^{-1}, \]

\[ r = 4e^{5x} - 10e^{4x} \phi - 40\gamma e^{3x} + 10e^{2x} \left( 6\gamma \phi + \phi^3 \pm (4\gamma + \phi^2)^{3/2} \right) - 10e^x \left( 6\gamma^2 + 6\gamma \phi^2 + \phi^3 \pm (2\gamma + 3\phi^2)(4\gamma + \phi^2)^{3/2} \right), \]

\[ \mathcal{M}_5 : \]

\[ L_n = e^{-nt+(n-1)x}(e^x \pm n)(e^x \pm 1)^{-n}\partial_t + ne^{-nt+(n-1)x}(e^x \pm 1)\partial_x. \]

\[ \mathcal{M}_6 : \]

\[ L_n = e^{-nt}\partial_t + \gamma e^{-nt}[e^{nx} - (e^x - \gamma)^n](e^x - \gamma)^{-1}\partial_x. \]

\[ \mathcal{M}_7 : \]

\[ L_0 = \partial_t, \]

\[ L_1 = e^{-t}\partial_t + e^{-t}\partial_x, \]

\[ L_{-1} = e^t(1 + \gamma e^{-2x})\partial_t + e^t(-1 + \gamma e^{-2x} + e^{-x}\phi)\partial_x, \]

\[ L_2 = e^{-2t+x}(e^x - \phi)(e^{2x} - e^x \phi - \gamma)^{-1}\partial_t + e^{-2t+x}(2e^x - \phi)(e^{2x} - e^x \phi - \gamma)^{-1}\partial_x, \]

\[ L_{-2} = e^{2t-3x}(e^{3x} + 3\gamma e^x + \phi^3 - \gamma \phi)\partial_t + e^{2t-3x}(2e^x - \phi)(-e^{2x} + e^x \phi + \gamma)\partial_x, \]

\[ L_{n+1} = (1 - n)^{-1}[L_1, L_n], \quad L_{-n-1} = (n - 1)^{-1}[L_{-1}, L_{-n}], \quad n \geq 2, \]

\[ \mathcal{M}_8 : \]

\[ L_n = e^{-nt}\partial_t + e^{-nt} \left[ n - sgn(n)\frac{\gamma}{2} \sum_{j=1}^{n-1} j(j+1)e^{-2x} \right] \partial_x, \]

\[ \mathcal{M}_9 : \]

\[ L_n = \frac{e^{-nt+(n-1)x}}{(e^x - 1)^{n+2}} \left[ -1 + \sum_{j=1}^{n-1} (2j+1)n + (2n+1)e^x - (n+2)e^{2x} + e^{3x} \right] \]

\[ + sgn(n)\frac{\phi}{2} \sum_{j=1}^{n-1} j(j+1)\partial_t + \frac{e^{-nt+(n-1)x}}{(e^x - 1)^{n+1}} \left[ -(-1 + \sum_{j=1}^{n-1} (2j+1)n \right] \]

\[ -2ne^x + ne^{2x} - sgn(n)\frac{\phi}{2} \sum_{j=1}^{n-1} j(j+1) \partial_x, \]
\[ W_0: \]
\[ L_n = e^{-nt} \partial_t + ne^{-nt} \partial_x + \frac{\text{sgn}(n)}{2} \sum_{j=1}^{\lfloor n \rfloor} j(j-1)e^{-nt-2x} \partial_u, \]

\[ W_1: \]
\[ L_n = e^{-nt} \partial_t + e^{-nt} \left[ n + \frac{\alpha n(n-1)}{2} e^{-x} \right] \partial_x + \frac{n(n-1)}{2} e^{-nt} \partial_u, \]

where \( n \in \mathbb{Z}, \alpha = 0, \pm 1, \gamma = \pm 1, \text{sgn}(\cdot) \) is the standard sign function and
\[ \phi(u) = \begin{cases} c, & c \in \mathbb{R}, \\ u. & \end{cases} \]

**Proof.** To prove the theorem it suffices to consider the case when the operators \( L_0, L_1, L_{-1} \) are of the form (9) or (10).

**Case 1.** If \( L_0, L_1, L_{-1} \) are given by (9), then it is straightforward to verify that due to (7) \( L_2 \) and \( L_{-2} \) are of the forms
\[ L_2 = e^{2t} \partial_t, \quad L_{-2} = e^{2t} \partial_t. \]

The remaining basis elements of the Witt algebra are easily obtained through recursion, which yields \( L_n = e^{-nt} \partial_t, \ n \in \mathbb{Z} \). We arrive at the realization \( W_1 \) from Theorem [1].

**Case 2.** Turn now to the realization (10). Inserting \( L_0, L_1, L_{-1} \) into the commutation relations \([L_0, L_{-2}] = 2L_{-2} \) and \([L_1, L_{-2}] = 3L_{-1} \) and solving the latter for the coefficients of the operator \( L_{-2} \) yield
\[ L_{-2} = e^{2t} (1 + 3ae^{-2x} + \psi_1(u)e^{-3x}) \partial_t + e^{2t} (-2 + 3\phi(u)e^{-x} + \psi_2(u)e^{-2x} + \psi_3(u)e^{-2x}) \partial_u, \]

where \( \psi_1, \psi_2, \psi_3 \) are arbitrary smooth functions of \( u \).

Utilizing the commutation relations \([L_0, L_2] = -2L_2 \) and \([L_{-1}, L_2] = -3L_1 \) in a similar fashion we derive the form of the basis element \( L_2 \)
\[ L_2 = e^{-2t} f(x, u) \partial_t + e^{-2t} g(x, u) \partial_x + e^{-2t} h(x, u) \partial_u, \]

where \( f, g, h \) are solutions of the system of three PDEs
\begin{align*}
-3(\alpha e^{-2x} + 1)f + 2\alpha e^{-2x}g + (\phi e^{-x} + \alpha e^{-2x} - 1)f_x + \beta e^{-x}f_u + 3 &= 0, \quad (11a) \\
(1 - \phi e^{-x} - \alpha e^{-2x})f + (\phi e^{-x} - 2g) - \phi_u e^{-x}h + (\phi e^{-x} + \alpha e^{-2x})g_x &= 0, \quad (11b) \\
\beta e^{-x}f - \beta e^{-x}g + 2(1 + \alpha e^{-2x})h - (\phi e^{-x} + \alpha e^{-2x} - 1)h_x - \beta e^{-x} h_u &= 0. \quad (11c)
\end{align*}
Inserting $L_2$ and $L_{-2}$ into the commutation relation $[L_2, L_{-2}] = 4L_0$ yields three more PDEs

\[
(4e^{-3x}\psi_1 + 12\alpha e^{-2x} + 4)f + (-6\alpha e^{-2x} - 3e^{-3x}\psi_1)g + he^{-3x}\psi_1 \\
+ (-e^{-3x}\psi_1 + 2 - 3\phi e^{-x} - e^{-2x}\psi_2)f_x + (-e^{-2x}\psi_3 - 3e^{-x}\beta)f_u - 4 = 0,
\]

\[
(2e^{-3x}\psi_1 + 6\phi e^{-x} + 2e^{-2x}\psi_2 - 4)f + (6\alpha e^{-2x} + 2 - 3\phi e^{-x} - e^{-3x}\psi_1 - 2e^{-2x}\psi_2)g \\
+ (e^{-3x}\psi_1 + 3\phi e^{-x} + e^{-2x}\psi_2)h + (-e^{-3x}\psi_1 + 2 - 3\phi e^{-x} - e^{-2x}\psi_2)g_x \\
+ (-e^{-2x}\psi_3 - 3\beta e^{-x})g_u = 0,
\]

\[
(2e^{-2x}\psi_3 + 6\beta e^{-x})f + (-3\beta e^{-x} - 2e^{-2x}\psi_3)g + (6\alpha e^{-2x} + 2e^{-3x}\psi_1 + e^{-2x}\psi_3 + 2)h \\
+ (-e^{-3x}\psi_1 + 2 - 3\phi e^{-x} - e^{-2x}\psi_2)h_x + (-e^{-2x}\psi_3 - 3\beta e^{-x})h_u = 0.
\]  

(12)

To determine the forms of $L_2$ and $L_{-2}$ we have to solve Eqs. (11) and (12). It is straightforward to verify that the relation

\[ Q = e^{-t-4x}[\beta e^{3x} + \psi_3 e^{2x} + (\beta\psi_2 - \phi\psi_3 - 3\alpha\beta)e^x + \beta\psi_1 - \alpha\psi_3] \neq 0 \]

is the necessary and sufficient condition for the system of equations (11) and (12) to have the unique solution in terms of $f_x, f_u, g_x, g_u, h_x$ and $h_u$. By this reason we need to differentiate between the cases $Q = 0$ and $Q \neq 0$.

**Case 2.1.** Let $Q = 0$ or, equivalently, $\beta = \psi_3 = 0$. Eqs. (11) and (12) do not contain derivatives of the functions $f$, $g$, $h$ with respect to $u$. That is why the derivatives $f_x$, $g_x$, $h_x$ can be expressed in two different ways using (11) and (12). Equating the right-hand sides of the two expressions for $h_x$ yields

\[
he^x\frac{e^{4x} - 2\phi e^{3x} - \psi_2 e^{2x} - 2\psi_1 e^x + 3\alpha^2 + \phi\psi_1 - \alpha\psi_2}{(e^{2x} - \phi e^x - \alpha)(2e^{3x} - 3\phi e^{2x} - \psi_2 e^x - \psi_1)} = 0.
\]

Hence $h = 0$. Similarly the compatibility conditions for the derivatives $f_x$ and $h_x$ give two linear equations for the functions $f$ and $g$. The determinant of the obtained system of linear equations does not vanish. Hence the system in question has the unique solution $f(x), g(x)$. Computing the derivatives of the so obtained $f$ and $g$ with respect to $x$ and comparing the result with the previously obtained expressions for $f_x$ and $g_x$ yield

\[
(\psi_2 - 6\alpha)(\phi^3 + 2\psi_1 + \phi\psi_2)e^{11x} + F_{10}[x,u] = 0,
\]  

(13)

and

\[
(10\phi^3\psi_1 - 3\alpha\phi^2(3\psi_2 - 8\alpha) + 3\phi\psi_1(2\alpha + 3\psi_2) + 2(5\psi_1^2 \\
- 4\alpha(2\alpha^2 - 3\alpha_2 + \psi_2^2)))e^{10x} + F_{9}[x,u] = 0.
\]  

(14)
Hereafter $F_n[x,u]$, $n \in \mathbb{N}$ denotes a polynomial in $\exp(x)$ of the power less than or equal to $n$. To find $f$ and $g$ we need to construct the most general form of $\phi$ and $\psi_i$ $(i = 1, 2, 3)$ satisfying Eqs. $[13]$ and $[14]$. If $[13]$ holds then at least one of the following equations \( \psi_2 = 6\alpha \) and \( \psi_1 = -(\phi^3 + \phi\psi_2)/2 \) should be satisfied.

Case 2.1.1. If \( \psi_2 = 6\alpha \) then Eqs. $[13]$ and $[14]$ hold if and only if

\[
16\alpha^3 + 3\alpha^2\phi^2 - 6\alpha\phi \psi_1 - \phi^3 \psi_1 - \psi_1^2 = 0,
\]

whence \( \psi_1 = (-6\alpha \phi - \phi^3 \pm (4\alpha + \phi^2)^{\frac{3}{2}})/2 \).

Case 2.1.1.1. Suppose now that \( \psi_1 = (-6\alpha \phi - \phi^3 - (4\alpha + \phi^2)^{\frac{3}{2}})/2 \). Provided \( \alpha = 0 \) we have either \( \psi_1 = 0 \) or \( \psi_1 = -\phi^3 \). The case \( \alpha = \psi_1 = 0 \) leads to \( L_{-1} = e^t \partial_t + e^t (-1 + e^{-x}\phi) \partial_x \). Making the equivalence transformation \( \tilde{x} = x + X(u) \) we can reduce $\phi$ to one of the forms $\alpha = 0, \pm 1$ thus getting

\[
f = 1, \quad g = 2 + ae^{-x}.
\]

Utilizing the recurrence relations of the Witt algebra we arrive at the realization $\mathfrak{W}_2$.

Provided \( \alpha = 0 \) and \( \psi_1 = -\phi^3 \) we can reduce the function $\phi$ to the form $b = 0, \pm 1$ by the equivalence transformation $\tilde{x} = x + X(u)$. Note that in the case when $b = 0$ we have $\psi_1 = 0$ which leads to the realization $\mathfrak{W}_2$. The case $b \neq 0$ gives rise to the following forms of $f$ and $g$:

\[
f = \frac{e^x(e^{2x} - 3be^x + 3b^2)}{(e^x - b)^3}, \quad g = \frac{e^x(2e^x - 3b)}{(e^x - b)^2}.
\]

Hence we get the realization $\mathfrak{W}_3$.

Provided \( \alpha = \pm 1 \) we have \( \psi_1 = (-6\alpha \phi - \phi^3 - (4\alpha + \phi^2)^{\frac{3}{2}})/2 \) and the realization $\mathfrak{W}_4$ is obtained.

Case 2.1.1.2. Let the function $\psi_1$ be of the form \( \psi_1 = (-6\alpha \phi - \phi^3 + (4\alpha + \phi^2)^{\frac{3}{2}})/2 \). If \( \alpha = 0 \) then we have $\psi_1 = 0$ or $-\phi^3$. This case has already been considered when we analyzed the Case 2.1.1.1. If the relation $\alpha = \pm 1$ holds then we get the realization $\mathfrak{W}_4$.

Case 2.1.2. If \( \psi_1 = -(\phi^3 + \phi \psi_2)/2 \) then Eq. $[13]$ takes the form

\[
(4\alpha + \phi^2)(\psi_2 - (4\alpha - 5\phi^2)/4)(\psi_2 - (2\alpha - \phi^2))e^{10x} + F_3[x, u] = 0.
\]

To solve the equation above we need to analyze the following three sub-cases.

Case 2.1.2.1. Provided \( \psi_2 = (4\alpha - 5\phi^2)/4 \), Eqs. $[13]$ and $[14]$ are satisfied if and only if \( 4\alpha + \phi^2 = 0 \).

Consequently \( \alpha \leq 0 \) and \( \phi = 2b(-\alpha)^{\frac{1}{2}} \) with \( b = \pm 1 \).

If \( \alpha = -1 \) then \( \phi = 2b, \psi_1 = 2b, \psi_2 = -6 \) and furthermore

\[
f = \frac{e^x(e^x - 2b)}{(e^x - b)^2}, \quad g = \frac{2e^x}{e^x - b}.
\]
The realization $\mathfrak{W}_5$ is obtained.

If $\alpha = 0$ and $\phi = \psi_1 = \psi_2 = 0$ then we arrive at the realization $\mathfrak{W}_2$ with $\alpha = 0$.

Case 2.1.2.2. Let $\psi_2 = 2\alpha - \phi^2$ and suppose that Eqs. (13) and (14) hold. Provided $\alpha = 0$ we can transform $\phi$ to become $b = \pm 1$ (note that the case $b = 0$ has already been considered). Then

$$f = 1, \quad g = \frac{2e^x - b}{e^x - b}$$

which yields the realization $\mathfrak{W}_6$.

Given $\alpha = \pm 1$ we have

$$f = \frac{e^x(e^x - \phi)}{e^{2x} - e^x\phi - b}, \quad g = \frac{e^x(2e^x - \phi)}{e^{2x} - e^x\phi - b},$$

where $b = \pm 1$. Since $\phi$ can be reduced to the form $\tilde{u} = \phi$ by the equivalence transformation $\tilde{u} = \psi_2$ with $\tilde{u} \neq 0$, the realization $\mathfrak{W}_7$ is obtained.

Case 2.1.2.3. If $4\alpha + \phi^2 = 0$ and Eqs. (13) and (14) holds, then we get $\alpha \leq 0$, whence $\alpha = 0, -1$.

Given the relation $\alpha = 0$ we can transform $\phi$ to become $a = 0, \pm 1$. Thus $f = 1$ and $g = 2 - ae^{-x}$ which give rise to the realization $\mathfrak{W}_8$.

In the case when $\alpha = -1$ we have

$$f = \frac{e^x(4 + 5e^x - 4e^{2x} + e^{3x} + \psi_2)}{(e^x - 1)^4}, \quad g = \frac{e^x(-4 - 4e^x + 2e^{2x} - \psi_2)}{(e^x - 1)^3}.$$ 

What is more, the function $\psi_2$ can be reduced to the form $\tilde{u} = \psi_2$ by the equivalence transformation $\tilde{u} = \psi_2$ provided $\psi_2$ is a nonconstant function. As a result we get the algebra $\mathfrak{W}_9$.

Summarizing we conclude that the case $Q = 0$ leads to the realizations $\mathfrak{W}_i$, $i = 2, 3, \cdots, 9$.

Turn now to the case $Q \neq 0$.

Case 2.2. If $Q \neq 0$ or, equivalently, $\beta^2 + \psi_3^2 \neq 0$ then Eqs. (11) and (12) can be solved with respect to $f_x, f_u, g_x, g_u, h_x$ and $h_u$. The compatibility conditions

$$f_{ux} = f_{ux}, \quad g_{ux} = g_{ux}, \quad h_{ux} = h_{ux}$$

can be rewritten in the form of the system of three linear equations for the functions $f, g, h$

$$a_1 f + a_2 g + a_3 h + d_1 = 0,$$

$$b_1 f + b_2 g + b_3 h + d_2 = 0,$$

$$c_1 f + c_2 g + c_3 h + d_3 = 0.$$ 

Here $a_i, b_i, c_i$, $d_i$, ($i = 1, 2, 3$) are functions of $t, x, \phi, \psi_1, \psi_2, \psi_3$. 

12
It is straightforward to verify that the system above has the unique solution 
\( f, g, h \) when \( \beta^2 + \psi_3^2 \neq 0 \). We do not present here the explicit formulae for these 
functions as they are very cumbersome. Inserting \( f, g, h \) into Eq. (11a) yields 
\[
\alpha \beta e^{42x} + F_{41}[x,u] = 0.
\]
Consequently we have \( \alpha = 0 \) or \( \beta = 0 \).

Case 2.2.1. If \( \beta = 0 \) then Eq. (11a) takes the form 
\[
\alpha \psi_3^6 e^{36x} + F_{35}[x,u] = 0,
\]
which gives \( \alpha = 0 \) and \( \psi_3 \neq 0 \) (since \( Q = 0 \) otherwise). Taking into account these 
relations we rewrite Eq. (12) as follows 
\[
\psi_1 \psi_3^6 e^{36x} + F_{35}[x,u] = 0,
\]
\[
(15 \phi^2 + 2 \psi_2) \psi_3^6 e^{37x} + F_{36}[x,u] = 0,
\]
\[
(57 \phi^2 - 2 \psi_2) \psi_3^7 e^{35x} + F_{34}[x,u] = 0.
\]
Hence we conclude that \( \phi = \psi_1 = \psi_2 = 0 \). Inserting these formulae into the initial 
Eqs. (11) and (12) and solving the obtained system yield 
\[
f = 1, \ g = 2, \ h = -e^{-2x} \psi_3.
\]
The function \( \psi_3 \) can be reduced to the form \(-1\) by the equivalence transformation 
\( \tilde{u} = U(u) \), where \( \tilde{U} = -1/\psi_3 \). As a result we have 
\[
f = 1, \ g = 2, \ h = e^{-2x},
\]
which leads to the realization \( \mathfrak{M}_{10} \).

Case 2.2.2. Provided \( \alpha = 0 \) Eq. (11c) takes the form 
\[
\beta^5 \left( 4 \beta \phi \psi_3 - 6 \psi_3^2 + \beta^2 \dot{\psi}_3 \right) e^{41x} + 30 \beta^5 \phi \psi_3^2 e^{40x} + F_{39}[x,u] = 0.
\]
Note that the case \( \alpha = \beta = 0 \) has been already studied while considering Case 2.2.1.
Consequently, without any loss of generality we can restrict our considerations to the cases 
\( \psi_3 = 0, \beta = 1 \) and \( \phi = 0, \beta = 1 \).

If \( \psi_3 = 0 \) then it follows from (12) and (11c) that \( \psi_1 = \psi_2 = 0 \). Taking into 
account these relations we rewrite Eqs. (11) and (12) in the form 
\[
f = 1, \ g = 2 + e^{-x} \phi, \ h = e^{-x}.
\]
What is more, the function \( \phi \) can be reduced to one the forms \( 0, \pm 1 \) by equivalence 
transformations \( \tilde{x} = x + X(u) \) and \( \tilde{u} = U(u) \). Whence we get the realization \( \mathfrak{M}_{11} \).

If the relation \( \phi = 0 \) holds Eqs. (11) and (12) are incompatible.

It is straightforward to verify that \( \mathfrak{M}_i, \ (i = 1, 2, \ldots, 11) \) cannot be transformed 
one into another with the transformation (5) and hence are inequivalent. This 
completes the proof of Theorem 1. \( \square \)

While proving the above theorem we obtained exhaustive description of realiza-
tions of the Witt algebra over the spaces \( \mathbb{R}^1 \) and \( \mathbb{R}^2 \). We present the corresponding 
results without proof.
Theorem 2. The realization $W_1$ exhausts the list of inequivalent realizations of the Witt algebra $W$ over the space $\mathbb{R}^1$.

Theorem 3. The realizations $W_1$–$W_9$ with $\phi = C$ exhaust the list of inequivalent realizations of the Witt algebra $W$ over the space $\mathbb{R}^2$.

4 Realizations of the Virasoro algebra

To construct all inequivalent realizations of the Virasoro algebra, $\mathfrak{V}$, we need to extend inequivalent realizations of the Witt algebra by all possible realizations of the nonzero central element of the Virasoro algebra. In this section we prove that there are no realizations of the Virasoro algebra with non-zero central element in the three-dimensional space $\mathbb{R}^3$.

Let us begin by constructing all possible central extensions of the subalgebra $\mathfrak{L} = \langle L_{-1}, L_0, L_1 \rangle$. In view of Lemma 1 we can restrict our considerations to realizations (9) and (10) of the algebra $\mathfrak{L}$.

Case 1. Consider first the realization (9)

$$L_0 = \partial_t, \ L_1 = e^{-t}\partial_t, \ L_{-1} = e^t\partial_t.$$

Let the basis element $C$ be of the general form (4). Inserting (4) into the commutation relations $[L_i, C] = 0$, $(i = 0, 1, -1)$ yields

$$C = \xi(x, u)\partial_x + \eta(x, u)\partial_u, \ \xi^2 + \eta^2 \neq 0.$$  

Applying the transformation

$$\tilde{t} = t, \ \tilde{x} = X(x, u), \ \tilde{u} = U(x, u),$$

which does not alter the forms of $L_0$, $L_1$ and $L_{-1}$ to the realization of $C$ above, we get

$$C \rightarrow \tilde{C} = (\xi X_x + \eta X_u)\partial_x + (\xi U_x + \eta U_u)\partial_u.$$  

Choosing solutions of the equations

$$\xi X_x + \eta X_u = 0, \ \xi U_x + \eta U_u = 1,$$

as $X$ and $U$ yields $C = \partial_u$.

Proceed now to constructing $L_2$. It easily follows from the relations $[L_0, L_2] = -2L_2$, $[L_{-1}, L_2] = -3L_1$ and $[L_2, C] = 0$ that $L_2 = e^{-2t}\partial_t$. Next, let $L_{-2}$ be of the general form (4). Then commutation relations (7) involving $L_{-2}$ yield over-determined system of PDEs for the unknown functions $\tau, \xi$ and $\eta$. This system turns out to be incompatible. Hence realization (9) cannot be extended to a realization of the Virasoro algebra with nonzero central element.
Case 2. We begin by utilizing commutation relations for the basis elements $L_0$, $L_1$ and $C$ thus getting

$$C = f(u)e^{-x} \partial_t + (g(u) + f(u)e^{-x}) \partial_x + h(u) \partial_u,$$

where $f$, $g$ and $h$ are arbitrary smooth function of $u$. Applying transformation $\tilde{g}$ that preserves the form of the basis elements $L_0$, $L_1$ to $C$ gives

$$\tilde{C} = f(u)e^{-x} \partial_t + (g(u) + f(u)e^{-x} + h(u) \dot{X}(u)) \partial_x + h(u) \dot{U}(u) \partial_u.$$

If $f(u) \neq 0$ then choosing $X(u) = -\ln|f(u)|$ we have $\tilde{C} = e^{-\tilde{x}} \partial_t + (e^{-\tilde{x}} + \beta(g + h\dot{X})) \partial_x + \beta h \dot{U} \partial_u$, where $\beta = \pm 1$. Provided $h = 0$ and $\dot{g} \neq 0$ we can make the transformation $\tilde{u} = g(u)$ and thus get $C_1 = e^{-x} \partial_t + (e^{-x} + u) \partial_x$. The case $h = \dot{g} = 0$ leads to $C_2 = e^{-x} \partial_t + (e^{-x} + \lambda) \partial_x$, where $\lambda$ is an arbitrary constant. Next, if $h \neq 0$ we choose solutions of the equations $g + h \dot{X} = 0$ and $h \dot{U} = 1/\beta$ as $X$ and $U$ getting $C_3 = e^{-x} \partial_t + e^{-x} \partial_x + \partial_u$.

Provided $f(u) = 0$ the generator $\tilde{C} = (g + h\dot{X}) \partial_x + h \dot{U} \partial_u$ is obtained. If $h \neq 0$ then it is possible to choose $X$ and $U$ so that $C_4 = \partial_u$. Given the condition $h = 0$ we have $\tilde{C} = g \partial_x$. If $g$ is nonconstant then selecting $U = g(u)$ yields $C_5 = u \partial_x$.

Finally, the case of constant $g$ leads to $C_6 = \partial_x$.

Summing up we conclude that there exist six inequivalent nonzero realizations of the central element $C$ for the case when $L_0 = \partial_t$ and $L_1 = e^{-t} \partial_t + e^{-t} \partial_x$. The next step is extending the algebras $\langle L_0, L_1, C_i \rangle$, $(i = 1, 2, \cdots, 6)$ to the realizations of the full Virasoro algebra. We present the calculation details for the case when $C_1 = e^{-x} \partial_t + e^{-x} \partial_x + \partial_u$. The remaining five cases are handled in a similar fashion.

In order to extend $\langle L_0, L_1, C_1 \rangle$ to the full Virasoro algebra, we construct all possible realizations of $L_{-1}$. Inserting $L_{-1}$ of the general form $\langle L_0, L_1, C_1 \rangle$ into the corresponding commutation relations from $\langle L_0, L_1, C_1 \rangle$ gives

$$L_{-1} = \frac{e^{t-2x}(u^2e^{2x} - 1)}{u^2} \partial_t - \frac{e^{t-2x}(ue^x + 1)^2}{u^2} \partial_x.$$

With $L_{-1}$ in hand we proceed to constructing $L_2$. Taking into account relations $\langle L_0, L_1, C_1 \rangle$ we get

$$L_2 = \frac{ue^x(ue^x + 2)}{e^{2t}(ue^x + 1)^2} \partial_t + \frac{2ue^x}{e^{2t}(ue^x + 1)} \partial_x.$$

Inserting the obtained expressions for $L_{-1}$ and $L_2$ into $\langle L_0, L_1, C_1 \rangle$ yields incompatible system of equations for the coefficients of $L_{-2}$. Whence we conclude that a realization of the algebra $\langle L_0, L_1, C_1 \rangle$ cannot be extended to a realization of the full Virasoro algebra. The same result holds for the remaining five realizations of the central elements $C_2, C_3, \ldots, C_6$.

**Theorem 4.** There are no realizations of the Virasoro algebra with nonzero central element $C$ in three-dimensional space $\mathbb{R}^n$, $n = 1, 2, 3$.
5 Some applications: PDEs admitting infinite-dimensional symmetry groups

In this section we construct several classes of second-order evolution equations in the space $\mathbb{R}^3$ of the variables $t$, $x$, $u$ that admit realization of the Witt algebra listed in Theorem 1. Given a realization of the Witt algebra, we can apply the Lie infinitesimal approach to construct the corresponding invariant equation [25, 27].

Differential equation

$$F(t, x, u, u_t, u_{tt}, u_{tx}, u_{xx}) = 0$$

is invariant with respect to the Witt algebra with basis elements $L_1, L_2, \ldots, L_n, \ldots$ if and only if the condition

$$\text{pr}^{(2)}L_n(F)|_{F=0} = 0$$

holds for any $n \in \mathbb{N}$, where $\text{pr}^{(2)}L_n$ is the second-order prolongation of the vector field $L_n$, that is

$$\text{pr}^{(2)}L_n = L_n + \eta^t \partial_{ut} + \eta^x \partial_{ux} + \eta^{tt} \partial_{utt} + \eta^{tx} \partial_{utx} + \eta^{xx} \partial_{uxx}$$

with

$$\eta^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\eta^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),$$

$$\eta^{tt} = D_t(\eta^t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),$$

$$\eta^{tx} = D_x(\eta^t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),$$

$$\eta^{xx} = D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi).$$

Here the symbols $D_t$ and $D_x$ stand for the total differentiation operators with respect to $t$ and $x$, correspondingly

$$D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{ut} + u_{tx} \partial_{ux} + \ldots,$$

$$D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{ut} + u_{xx} \partial_{ux} + \ldots.$$

As an example we consider the realization $W_1 = \langle e^{-nt} \partial_t \rangle$. Making use of the formulas above we obtain

$$\text{pr}^{(2)}L_n = e^{-nt} \partial_t + ne^{-nt} u_t \partial_{ut} + (2ne^{-nt} u_{tt} - n^2 e^{-nt} u_t) \partial_{utt} + ne^{-nt} u_{tx} \partial_{utx}. \quad (15)$$

Next step is computation of the full set of functionally-independent second-order differential invariants, $I_m(t, x, u, u_t, u_{tt}, u_{tx}, u_{xx})$ ($m = 1, 2, \ldots, 7$), associated with $L_n$. To get $I_m$ we need to solve the related system of characteristic equations

$$\frac{dt}{e^{-nt}} = \frac{dx}{0} = \frac{du}{0} = \frac{du_t}{ne^{-nt} u_t} = \frac{du_{tt}}{0} = \frac{du_{tx}}{2ne^{-nt} u_{tt} - n^2 e^{-nt} u_t} = \frac{du_{xx}}{ne^{-nt} u_{tx}} = \frac{du_{xx}}{0}.$$
Integration of the equations above yields
\[ I_1 = x, \quad I_2 = u, \quad I_3 = u_x, \quad I_4 = u_{xx}, \quad I_5 = \frac{u_{tx}}{u_t}, \quad I_6 = e^{-nt}u_t, \quad I_7 = e^{-2nt}u_{tt} - ne^{-2nt}u_t. \]

Hence the most general \( L_n \)-invariant equation is of the form
\[ F(I_1, I_2, \cdots, I_7) = 0. \]

Since this equation has to be invariant under every basis element of the infinite-dimensional Witt algebra \( \mathfrak{W}_1 \), it must be independent of \( n \). To obey this restriction function \( F \) should be independent of \( I_6 \) and \( I_7 \). Thus the final form of the most general second-order PDE invariant under \( \mathfrak{W}_1 \) reads as
\[ F(I_1, I_2, I_3, I_4, I_5) = 0, \]
or,
\[ F(x, u, u_x, u_{xx}, \frac{u_{tx}}{u_t}) = 0. \]

Below we list five more classes of second-order differential equations whose symmetry algebra is infinite-dimensional Witt algebra.

\( \mathfrak{W}_2 \) invariant PDEs
\[ F\left(u, u_x, u_{xx}, \frac{u_t u_{xx} - u_x u_{tx}}{e^x u_x}\right) = 0, \quad \text{if} \quad \alpha = 0 \]
\[ F\left(u, u_{xx} - u_x, \frac{u_t u_x - u_t u_{xx} + u_x u_{tx} + u_x^2}{e^x u_x} - 2\alpha u_x\right) = 0, \quad \text{if} \quad \alpha = \pm 1, \]

\( \mathfrak{W}_6 \) invariant PDEs
\[ F\left(u, \frac{\gamma(u_{xx} + u_{tx}) - e^x(u_x + u_{xx})}{u_x(\gamma(u_t + u_x) - e^x u_x)}\right) = 0, \]

\( \mathfrak{W}_8 \) invariant PDEs
\[ F\left(u, \frac{u_{xx} - 2u_x}{u_x^2}\right) = 0, \]

\( \mathfrak{W}_{10} \) invariant PDEs
\[ F(u_x + 2u, u_{xx} - 4u) = 0. \]

6 Concluding Remarks

The principal result of this paper is exhaustive classification of inequivalent realizations of the Virasoro algebra by Lie vector fields over the space \( \mathbb{R}^n \) with \( n = 1, 2, 3 \). These realizations are listed in Theorems 1-4. According to Theorem 1 there exist eleven inequivalent realizations of the Witt algebra in the space \( \mathbb{R}^3 \). What is more,
we proved that there exist only one realization of the Witt algebra over the space \( \mathbb{R}^1 \) and nine realizations of the Witt algebra over the space \( \mathbb{R}^2 \).

It has been established that realizations of the Virasoro algebra with nonzero central element do not exist in the space \( \mathbb{R}^n \) with \( n \leq 3 \).

As an application of our algebraic classification we construct a number of nonlinear PDEs admitting infinite-dimensional symmetry algebras, which are realizations of the Witt algebra.

An interesting application of the obtained results would be describing nonlinear PDEs whose symmetry algebras are direct sums of the Witt algebras. A nontrivial example is the Liouville equation (1). Since these equations would admit symmetry with two arbitrary functions they would automatically be classically integrable.

Since Virasoro algebra is a subalgebra of the Kac-Moody algebra, the results of this paper can be directly applied to solving the problem of classification of integrable KP type PDEs in \((1+2)\) dimensions. The starting point is a description of inequivalent realizations of the Kac-Moody algebras by differential operators over the space \( \mathbb{R}^4 \).

These problems are under study now and will be reported in our future publications.

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