Corrector estimates and homogenization errors of unsteady flow ruled by Darcy’s law

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Abstract

Focusing on Darcy’s law incorporating memory effects, this paper studies non-stationary Stokes equations on perforated domains. We establish a sharp homogenization error for both velocity and pressure in terms of the energy norm. The main challenge lies in gauging the boundary layers induced by the incompressibility condition. To address this, we construct boundary-layer correctors using Bogovskii’s operator. Also, the present work provides detailed regularity estimates for these correctors, where a significant difficulty arises from the incompatibility between initial and boundary values. The methodologies developed herein hold a great potential for tackling the same issue in other evolutionary models beyond a homogenization setting.

Key words: homogenization error; perforated domain; unsteady Stokes equation; Darcy’s law with memory; boundary layer.

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1 Introduction

1.1 Motivation and main results

We start from introducing the main object of this paper. Let the ratio of the period to the overall size of the porous medium be denoted by a parameter $\varepsilon$, which allowed to approach zero, and the porous medium is contained in a bounded domain $\Omega$. Its fluid part is represented by $\Omega_{\varepsilon}$, which is also referred to as a perforated domain. Precisely, let $\Omega \subset \mathbb{R}^d$ (with $d \geq 2$) be a bounded domain with $C^2$ boundary, and set $Y := [-\frac{1}{2}, \frac{1}{2}]^d \cong \mathbb{R}^d / \mathbb{Z}^d$ to be the elementary cell for the lattice $\mathbb{Z}^d$, made of two complementary parts: the solid part $Y_s$ and the fluid part $Y_f$. Assume that $Y_s \subset Y$ is a connected subset of $Y$ with $C^2$ boundary and a strictly positive Lebesgue measure in $\mathbb{R}^d$, and we are interest in the case where $Y_f := Y \setminus \overline{Y}_s$ is a connected set. We now define the configuration and the perforated domain, respectively.

$$\omega := \bigcup_{z \in \mathbb{Z}^d} (Y_f + z); \quad \Omega_{\varepsilon} := \Omega \cap \varepsilon \omega = \Omega \setminus \bigcup_k \varepsilon(\overline{Y}_s + z_k),$$

(1.1)

where $z_k \in \mathbb{Z}^d$ and the union is taken over those $k$'s such that $\varepsilon(\overline{Y} + z_k) \subset \Omega$ and $\text{dist}(\partial \Omega, \partial \Omega_{\varepsilon} \setminus \partial \Omega) \geq \kappa_0 \varepsilon$ with $\kappa_0 \geq 2$ (which is not essential). These assumptions will be utilized without stating in later sections.

The incompressible fluid movement in $\Omega_{\varepsilon}$ can be described by the unsteady Stokes equations: for $T > 0$,

$$\begin{cases}
\partial_t u_{\varepsilon} - \varepsilon^2 \mu \Delta u_{\varepsilon} + \nabla p_{\varepsilon} = f & \text{in } \Omega_{\varepsilon} \times (0, T);
\nabla \cdot u_{\varepsilon} = 0 & \text{in } \Omega_{\varepsilon} \times (0, T);
\n u_{\varepsilon} = 0 & \text{on } \partial \Omega_{\varepsilon} \times (0, T);
\n u_{\varepsilon}_{|t=0} = 0 & \text{on } \Omega_{\varepsilon}.
\end{cases}
$$

(1.2)

We denote by $u_{\varepsilon}$ and $p_{\varepsilon}$ the velocity and pressure of the fluid, respectively, while $f$ represents the density of forces acting on the fluid, where $u_{\varepsilon}$ and $f$ are vector-valued functions but $p_{\varepsilon}$ is a scalar. The fluid viscosity $\mu$ is a constant and we assume $\mu = 1$ throughout the paper for simplicity. In terms of the scaling $\varepsilon^2$ of the viscosity in the equations (1.2), it is not a simple change of variable as in the stationary case because the density in front of the inertial term has been scaled to 1. The present scaling in (1.2) will precisely lead to a limit problem depending on time in a nonlocal manner (i.e., memory effects), which is the critical case shown in [27].

The limit behavior of the equations (1.2) was pioneered by J.-L. Lions [17] through a formal asymptotic expansion argument, demonstrating that the effective equation is governed by Darcy’s law with memory. Then, under the geometry assumption (1.1), G. Allaire [1] gave a rigorous proof by using the two-scale convergence method, while A. Mikelić [20] developed a similar result independently. Also, G. Sandrakov [27] studied the same type models by considering different scalings on viscosity.

Since the solution $(u_{\varepsilon}, p_{\varepsilon})$ to the equations (1.2) is not defined in a fixed domain, a method of extending the solution to the whole domain $\Omega$ is required to state the homogenization result. As in the stationary case, the velocity $u_{\varepsilon}$ naturally owns a zero-extension, denoted by $\tilde{u}_{\varepsilon}$. Let $Y_{\varepsilon}^k := Y_s + z_k$ with $z_k \in \mathbb{Z}^d$. Let $\tilde{p}_{\varepsilon}$ be the extension of $p_{\varepsilon}$ to $\Omega$, given as follows:

$$\tilde{p}_{\varepsilon}(x, t) := \begin{cases}
p_{\varepsilon}(x, t) & \text{if } x \in \Omega_{\varepsilon}; \\
p_{\varepsilon}(\cdot, t) & \text{if } x \in \varepsilon Y_{\varepsilon}^k \text{ and } \varepsilon Y_{\varepsilon}^k \subset \Omega \text{ for some } z_k \in \mathbb{Z}^d.
\end{cases}$$

(1.3)

We now present the main results in the qualitative homogenization theory.

**Theorem 1.1** (homogenization theorem [1, 3, 17, 20, 27]). Let $0 < T < \infty$, and $d \geq 2$. There exists an extension $(\tilde{u}_{\varepsilon}, \tilde{p}_{\varepsilon})$ of the solution $(u_{\varepsilon}, p_{\varepsilon})$ to (1.2), which weakly converges in $L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)/\mathbb{R})$ to the unique...
solution \((u_0, p_0)\) of the homogenized problem (known as Darcy’s law with memory):

\[
\begin{align*}
\left\{ \begin{array}{ll}
u_0(x, t) &= \int_0^t dsA(t-s)(f - \nabla p_0)(x, s) & \text{in } \Omega \times (0, T); \\
\nabla \cdot u_0 &= 0 & \text{in } \Omega \times (0, T); \\
\nabla \cdot \nbar{n} &= 0 & \text{on } \partial \Omega \times (0, T), \\
u_0 \cdot \nbar{n} &= 0 \\
\end{array} \right.
\tag{1.4}
\end{align*}
\]

where \(\nbar{n}\) is the unit outward normal vector of \(\partial \Omega\). The quantity \(A = (A_{ij})\) is a symmetric, positive defined, and time-dependent permeability tensor, determined by

\[
A_{ij}(t) = \int_{Y_j} dyW_j(y, t) \cdot e_i,
\tag{1.5}
\]

where \(e_i = (0, \ldots, 1, \ldots, 0)\) with 1 in the \(i^{th}\) place, and \((W_j, \pi_j) \in L^2(0, T; H^1(\Omega_j)^d) \times H^{-1}(0, T; L^2(\Omega_j)^d))\) is the corrector associated with \(e_j\) by the following equations:

\[
\begin{align*}
\partial_t W_j - \Delta W_j + \nabla \pi_j &= 0 \quad \text{in } \omega \times (0, T); \\
\nabla \cdot W_j &= 0 \quad \text{in } \omega \times (0, T); \\
W_j &= 0 \quad \text{on } \partial \omega \times (0, T), \\
W_j|_{t=0} &= e_j \quad \text{on } \omega.
\end{align*}
\tag{1.6}
\]

Moreover, there holds the strong convergence

\[
\int_0^T dt\|\tilde{u}_\varepsilon(\cdot, t) - \int_0^t dsW(\cdot/\varepsilon, t-s)(f - \nabla p_0)(\cdot, s)\|_{L^2(\Omega)}^2 \to 0, \quad \text{as } \varepsilon \to 0.
\tag{1.7}
\]

The main purpose of this paper is to quantify the strong convergence as described in (1.7). For the reader’s convenience, we introduce the following notations:

\[\tilde{A}_{ij}(t) = \int_{Y_j} dyW_j(y, t) \cdot e_i, \quad \Omega_{\varepsilon, T} := \Omega_{\varepsilon} \times (0, T].\]

**Theorem 1.2** (error estimates). Let \(0 < T < \infty\), and \(d = 2\) or 3. Assume that \(\Omega \subset \mathbb{R}^d\) is a bounded \(C^2\) domain, and the perforated one \(\Omega_{\varepsilon}\) satisfies the geometrical hypothesis (1.1). Given \(f \in L^2(0, T; C^{1, \frac{1}{2}}(\Omega)^d)\), let \((u_\varepsilon, p_\varepsilon) \in L^2(0, T; H^1(\Omega_{\varepsilon})^d) \times L^2(0, T; H^1(\Omega_{\varepsilon})^d))\) be the weak solution of (1.2). Then, we have

\[
\|u_\varepsilon - W(\cdot/\varepsilon) * (f - \nabla p_0)\|_{L^2(\Omega_{\varepsilon}, T)} + \|\varepsilon \nabla u_\varepsilon - \nabla W(\cdot/\varepsilon) * (f - \nabla p_0)\|_{L^2(\Omega_{\varepsilon}, T)} \leq C\varepsilon^{1/2}\|f\|_{L^2(0, T; C^{1, 1/2}(\tilde{\Omega}))},
\tag{1.8}
\]

and there further holds

\[
\|\partial_t u_\varepsilon - \partial_t W(\cdot/\varepsilon) * (f - \nabla p_0) - W(0)(f - \nabla p_0)\|_{L^2(\Omega_{\varepsilon}, T)} + \inf_{c \in \mathbb{R}^d} \|p_\varepsilon - p_0 - c\|_{L^2(\Omega_{\varepsilon}, T)} \leq C\varepsilon^{1/2}\|f\|_{L^2(0, T; C^{1, 1/2}(\tilde{\Omega}))},
\tag{1.9}
\]

where the constant \(C\) depends only on \(d, T, |Y_f|\), and the characters of \(\Omega\) and \(Y_f\).

**Corollary 1.3.** Assume the same conditions as in Theorem 1.2. Let \(\tilde{p}_\varepsilon\) be the extension of \(p_\varepsilon\) as in (1.3). Then, there holds the following estimate

\[
\inf_{c \in \mathbb{R}^d} \left( \int_0^T dt \int_{\Omega} |\tilde{p}_\varepsilon(\cdot, t) - p_\varepsilon(\cdot, t) - c|^2 \right)^{1/2} \leq C\varepsilon^{1/2}\|f\|_{L^2(0, T; C^{1, 1/2}(\tilde{\Omega}))},
\tag{1.10}
\]

where the constant \(C\) depends only on \(d, T, |Y_f|\), and the characters of \(\Omega\) and \(Y_f\).

Roughly speaking, the primary contribution of this paper is the derivation of the following asymptotic expansion for the velocity:

\[
u_\varepsilon \approx W(\cdot/\varepsilon) * (f - \nabla p_0) + \xi_\varepsilon + \tilde{\eta}_\varepsilon + O(\varepsilon).
\tag{1.11}
\]
While the form of the above expansion should be well-known to experts in qualitative analysis (see e.g. [17]), how to specifically construct the boundary layer part and make a well quantitative analysis remains a significant challenge in this field. In this work, we have successfully constructed the boundary-layer correctors, denoted by \( (\hat{\xi}_\varepsilon, \hat{\eta}_\varepsilon) \), for the boundary layer part. Furthermore, we derived that

\[
\|\hat{\xi}_\varepsilon\|_{L^2(\Omega, \varepsilon)} + \varepsilon\|\nabla \hat{\xi}_\varepsilon\|_{L^2(\Omega, \varepsilon)} + \|\hat{\eta}_\varepsilon\|_{L^2(\Omega, \varepsilon)} + \varepsilon\|\nabla \hat{\eta}_\varepsilon\|_{L^2(\Omega, \varepsilon)} = O(\sqrt{\varepsilon}),
\]

which are optimal even for the stationary case. Consequently, with respect to the convergence rate, the error estimates (1.8) and (1.9) are sharp, given the \( O(\sqrt{\varepsilon}) \) loss of the boundary-layer correctors.

Beyond the challenges posed by the boundary layer in spatial variables, Theorem 1.1 has highlighted the considerable difficulties associated with the temporal variable when considering higher-order regularity estimates.

**Q1.** Given \( f \in L^2(0,T;C^{1,1/2}(\bar{\Omega})^d) \), how to establish the well-posedness of the equations (1.4)?

Within the framework of Hilbert spaces, A. Mikelić [21] and G. Sandrakov [27] developed well-posedness results using Laplace’s transform methods. However, these methods are difficult to be applied to general Bochner spaces. Inspired by J.-L. Lions [17, pp.170], this work establishes a rigorous well-posedness theory for the integral-differential equations (1.4) as follows.

**Proposition 1.4** (well-posedness for Darcy’s law with memory). Let \( 1 \leq q \leq \infty, m \geq 1 \) and \( \alpha \in (0, 1) \). Given \( 0 < T < \infty \) and \( d = 2 \) or \( 3 \), suppose that \( f \in L^q(0,T;C^{m+1,\alpha}(\bar{\Omega})) \) and \( \partial \Omega \in C^{m+1,\alpha} \). Then, there exists a unique \( p_0 \in L^q(0,T;C^{m+1,\alpha}(\bar{\Omega})) \) to the equations (1.4) with the condition \( \int_\Omega p_0(\cdot, t) = 0 \) for a.e. \( t \geq 0 \). Moreover, let the permeability tensor \( A \) be given as in (1.5). Then, for any \( 0 < \beta < (2/21) \), one can derive that

\[
|\partial_t A| \in L^{1+\beta}(0,T),
\]

As a result, there holds the regularity estimate

\[
\|p_0\|_{L^q(0,T;C^{m+1,\alpha}(\bar{\Omega}))} \leq C\|f\|_{L^q(0,T;C^{m,\alpha}(\bar{\Omega}))},
\]

where the constant \( C \) depends on \( d, |Y_f|, \Omega, \) and \( T \).

To the best of the authors’ knowledge, the most recent and relevant results (see [27, pp.127]) indicate that \( |\partial_t A| \in L^1(0,T) \); and the estimate (1.14) is only valid for \( L^2(0,T;H^1(\Omega)) \). Although the upper bound of \( \beta \) may not be optimal, the aforementioned estimate (1.13), combined with the definition of \( A \) given in (1.5), has already shown that a more refined regularity estimate for the correctors is required. This naturally leads to another key issue addressed in this article.

**Q2.** How can one derive a refined regularity estimate for \( (W, \pi) \) to (1.6) without compatibility conditions between the initial and boundary data?

In fact, the methodology employed for constructing boundary-layer correctors renders us highly dependent on obtaining more accurate estimates for the corrector \( (W, \pi) \). To the best authors’ knowledge, there are currently no established theoretical results available for reference concerning the issue of incompatible initial and boundary conditions. This article provides a more in-depth investigation into this problem, employing three key tools:

- An interior Caccioppoli-type inequality (as well as, interior higher-order regularity estimates);
- Decay estimates of Stokes semigroup;
- Interpolations.

In this regard, the paper achieves the crucial improvement\(^1\) in the semigroup estimates as follows: for any \((1/3) < \theta \leq (1/2)\), there exists a constant \( C_\theta \) such that

\[
\|\nabla W_j(\cdot, t)\|_{L^2(Y_f)} \leq C_\theta t^{-\theta}, \quad \forall t > 0.
\]

(1.15)

The time weight \( t^\alpha \), introduced below, is utilized to characterize the singularity of the velocity field and pressure near the initial moment. The intensity of this singularity is quantified by the exponent \( \alpha \) of the weight function. Leveraging the estimate (1.15), we have established

\[
\|t^\alpha \partial_t W_j\|_{L^2(0,T;L^2(Y_f))} + \|t^\alpha \pi_j\|_{L^2(0,T;L^2(Y_f))} \leq C_\alpha, \quad \forall \alpha > (1/3).
\]

(1.16)

\(^1\)Instead of establishing time-decay estimates as time goes to infinity, we are interested in the “decay” as time approaches zero. In this sense, we regard the estimate (1.15) as an improvement over original semigroup estimates.
We are uncertain whether the lower bound of (1/3) is optimal; however, it suffices for establishing the desired results in this paper, including the previously stated estimates (1.12), (1.13), and (1.14). It is important to note that the technique developed herein primarily pertains to the regularity theory of PDEs, which constitutes an area of independent interest in the current literature.

Remark 1.5. The fundamental reason for restricting the dimension to 2D or 3D in this paper is based on the technique of the corrector’s estimate. As mentioned earlier, we are going to employ the interior Caccioppoli-type estimates, which utilizes the basic properties of the curl operator in three-dimensional case (see Lemma 3.5). The high-dimensional generalization of these basic properties involves the redefinition of the curl operator. We do not intend to discuss this issue in the present work.

1.2 Relation to other works

The pioneer literature on homogenization errors was contributed by E. Marušić-Paloka, A. Mikelić, L. Paoli [19, 22], where they obtained an $O(\varepsilon^{1/6})$-error for steady Stokes problems and an $O(\varepsilon)$-error for non-stationary incompressible Euler’s equations, respectively. These results were obtained in dimension two. Recently, the quantitative estimates on homogenization of Stokes equations in porous medias have seen rapid development (see e.g. [4, 11, 14, 15, 18, 29, 30, 31]). Regarding the optimal homogenization error for Darcy’s law, the main challenge lies in accurately gauging the boundary layers created by the incompressibility condition combined with the discrepancy of boundary values between the solution and the leading term in the related asymptotic expansion (1.11). To date, apart from our approach, there are two other methods to address this difficulty, as outlined below:

1. Z. Shen [29] derived the optimal error estimate for steady Stokes systems for $d \geq 2$. The primary approach involves reducing all challenges to a boundary value problem, followed by defining boundary correctors with tangential boundary data and normal boundary ones, respectively, to achieve a sharp estimate. The analysis therein relies on some advanced concepts such as non-tangential maximal function and Rellich estimates.

2. The approach proposed by L. Balazi, G. Allaire, and P. Omnes [4] is also noteworthy. In essence, their method involves first correcting the discrepancy between the boundary values of the solution and the leading term. Subsequently, the problem is reduced to correcting the error term to satisfy the incompressibility condition. Regarding the techniques in their framework, they smartly concealed a cut-off function within the curl operator, based upon the result stated in [4, Lemma 4.10], previously developed in [7].

E. Marušić-Paloka, A. Mikelić in [19, pp.2] ever noted that “the cut-off argument does not work for the Stokes and Navier-Stokes systems because of the incompressibility condition, which creates a boundary layer in the neighbourhood of $\partial \Omega$ destroying the estimate.” By introducing the concept of radial cut-off function, this work directly addresses this issue. Unlike the method developed by L. Balazi et al. [4, 11], our strategy reduces the problem to constructing boundary-layer correctors associated with Bogovskii’s operator (see Subsection 2.1).

Regarding the geometric modeling of the porous medium, the isolated obstacles condition $\text{dist}(\partial Y, \partial Y_0) > 0$ appeared in [29], is not necessary in the present work. For a detailed description of the notion of isolated (or connected) obstacles, we refer the reader to [4, Section 2] or [6, Section 14] for the details.

In the end, we would like to highlight two points. Firstly, the work of L. Balazi et al. [4, 11] convinced us that the idea of introducing a radial cut-off function was a viable approach. This inspired us to revisit and significantly improve our previous work [34] (see Remark 4.9). Secondly, it is important to note that our work is not a continuation of the research presented in [4, 11, 29] in any sense, since our approach is independent of the most important tool mentioned above in their analysis. Additionally, the idea of the radial cut-off function has already been applied in recent studies, such as [15], demonstrating its potential for broader applications.

1.3 Structure of the paper

In Section 2, we introduce the main ideas leading to the conclusion stated in Theorem 1.2. The detailed motivation for boundary-layer correctors is presented in Subsection 2.1. To provide readers with an overview of the proof for Theorem 1.2, we outline the main steps and key estimates required in Subsection 2.2.

In Section 3, we present the main results on correctors in Proposition 3.1. To achieve the required estimates, we employ two rounds of regularity enhancement. Subsection 3.1 focuses on the first round of regularity improvement, as detailed in (1.15); Subsection 3.2 addresses the second round. Subsection 3.3 mainly provides the proof of weighted estimates (1.16).

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G. Jankowiak, A. Lozinski [11] developed a similar strategy in dimension two.
In Section 4, the estimates of boundary-layer correctors are presented in Propositions 4.1, 4.2, and 4.3. Subsection 4.1 establishes the existence of the desired radial cut-off function. Due to the truncation, we categorize boundary-layer correctors into co-layer type and layer type, with the corresponding proofs provided in Subsections 4.2 and 4.3, respectively.

In Section 5, we present the proof of Theorem 1. Section 6 serves as an appendix to this paper, providing the proof of Proposition 1.4 for the reader’s convenience.

1.4 Notations

1. Notation related to domains

We denote the co-layer part of Ω by Σε := \{x ∈ Ω, dist(x, ∂Ω) ≥ ε\}, while the region Ω \ Σε is known as the layer part of Ω. There are two type cut-off functions used throughout the paper: One is the so-called radial cut-off function ψε (defined in Lemma 4.5): The other is a general cut-off function, denoted by φε, satisfying that φε ∈ C_{0}^{1}(Ω) and φε = 1 on Σε/2 with |∇φε| ≤ 1/ε. Let supp(f) represent the support of f, and we denote supp(∇φε) by Oε.

2. Notation for special quantities

- Let F := f − ∇p_{0} be given with p_{0} satisfying (1.4), and G := S_{δ}(φ_{ε}F) with δ = \frac{ε}{2}, where S_{δ} is a smoothing operator\(^3\), and φ_{ε} is a general cut-off function defined above.
- Let φ∗(x, t) := φ(\frac{x}{ε}, t), Wε(x, t) := W(\frac{x}{ε}, t), and πε(x, t) := π(\frac{x}{ε}, t), in which correctors (W, π) and φ are defined in the equations (1.6) and (2.3), respectively; We also introduce two important quantities:

\[
J_{1} := \nabla \psi_{ε} \cdot [(Wε − A) * G];
\]

\[
J_{2} := \nabla \psi_{ε} \cdot (A * G + ε \nabla \psi_{ε} \cdot (φ∗ *_{2} \partial G) + ψ_{ε} \frac{A}{|J_{f}|} *_{2} \partial G + ε \psi_{ε} φ∗ *_{3} \partial^{2} G). \tag{1.17}
\]

(We recommend that readers initially skip the derivation of the above expression and treat it merely as a notation. After reviewing Subsection 2.1 and Remark 3.2, readers will naturally encounter the source of the expression at the beginning of Section 4.)

3. Notation for derivatives

Spatial derivatives: \nabla v = (\partial_{1} v, \cdots, \partial_{d} v) (or denoted by ∇v) is the gradient of v, where ∂_{i} v = ∂v/∂x_{i} denotes the ith derivative of v. \nabla^{2} v (or ∇^{2} v) denotes the Hessian matrix of v; \nabla \cdot \vec{u} = \sum_{i=1}^{d} \partial_{i} u_{i} denotes the divergence of \vec{u}, where \vec{u} = (u_{1}, \cdots, u_{d}) is a vector-valued function. Also, curl \vec{u} is identified with the 2-tensor ω_{ij} = ∂_{i} u_{j} − ∂_{j} u_{i} for all d ≥ 2. In particular, we have the following convention

\[
\text{curl } \vec{u} = \nabla \times \vec{u} \quad \text{as } \quad d = 3.
\]

Temporal derivatives: ∂v := ∂v/∂t represents the derivative with respect to the temporal variable. In the Appendix, the time derivative of A is also denoted as A′.

4. Notation for spaces

We mention that this paper merely involves some simple Bochner spaces whose definition is standard (see e.g. [8, Subsection 5.9.2]): Note that X_{per} represents its element being a periodic object, where X could be any Sobolev or Hölder space; Also, X^{d} indicates that its collected elements are d-dimensional, and X/K represents a quotient space (see e.g. [6, 28]).

5. Notation for estimates

≤ and ≥ stand for ≤ and ≥ up to a multiplicative constant, which may depend on some given parameters introduced in the paper, but never on ε; We write ~ when both ≤ and ≥ hold; We use ≲ instead of ≤ when the inverse of multiplicative constant is much larger than 1.

\(^3\)Fix a nonnegative function ζ ∈ C_{0}^{∞}(B(0,1/2)) with \int dζ(x) = 1. For any f ∈ L^{p}(\mathbb{R}^{d}) with 1 ≤ p < ∞, we define the smoothing operator as S_{δ}(f)(x) := \int dγf(x − y)ζ_{δ}(y), where ζ_{δ}(y) = δ^{−d}ζ(y/δ).
6. Notation for convolutions

Let \( a, b \) be vectors, \( A \) be a matrix, and \( C, D \) be tensors (higher than second order); For the ease of the statement, we introduce the following notations:

\[
a \ast_1 b(t) := \int_0^t ds(a(t - s) \cdot b(s)); \quad A \ast b(t) := \int_0^t dsA(t - s)b(s); \\
C \ast_2 A(t) := \int_0^t dsC(t - s) : A(s); \quad C \ast_3 D(t) := \int_0^t dsC_{ijk}(t - s)D_{ijk}(s),
\]

where the notation "\( : \)" represents the tensor’s inner product of second order, and Einstein’s summation convention for repeated indices is used throughout. Together with the convention on derivatives\(^4\), we list the following quantities frequently appeared in this paper:

\[
(W^\varepsilon \ast G)(x, t) := \int_0^t dsW(x/\varepsilon, t - s)G(x, s); \quad \text{(vector)} \\
(\pi^\varepsilon \ast_1 G)(x, t) := \int_0^t ds\pi_j(x/\varepsilon, t - s)G_j(x, s); \quad \text{(scalar)} \\
(\phi^\varepsilon \ast_2 \partial G)(x, t) := \int_0^t ds\phi_{k,j}(x/\varepsilon, t - s)\partial_k G_j(x, s); \quad \text{(vector)} \\
(\phi^\varepsilon \ast_3 \partial^2 G)(x, t) := \int_0^t ds\phi_{ik,j}(x/\varepsilon, t - s)\partial^2_{ik} G_j(x, s); \quad \text{(scalar)} \\
(\phi^\varepsilon \ast_2 \nabla \partial G)(x, t) := \int_0^t ds\phi_{k,j}(x/\varepsilon, t - s) \otimes \nabla \partial_k G_j(x, s), \quad \text{(matrix)}
\]

where the notation "\( \otimes \)" represents the tensor product.

2 Sketch of the proof

2.1 On the error term (main ideas)

This subsection clarifies the rationale for the form of the expansion in (1.11), particularly from an error analysis perspective, which is fundamental to the paper. In particular, in the following statements we simply treat the time variable as a parameter.

Firstly, the qualitative result (1.7) suggests that the error term should be the form of

\[
w^{(1)}_{\varepsilon} = u_{\varepsilon} - W^\varepsilon \ast F.
\]

(See the notations \( F \) and \( G \) in Subsection 1.4.) However, this kind of error term leads to the following inhomogeneous conditions:

\[
\nabla \cdot w^{(1)}_{\varepsilon} = -W^\varepsilon \ast_2 \partial F \quad \text{in} \quad \Omega_{\varepsilon} \times (0, T); \\
w^{(1)}_{\varepsilon} = -W^\varepsilon \ast F \quad \text{on} \quad \partial \Omega \times (0, T), \quad T > 0.
\]

Secondly, to correct the inhomogeneous conditions above, a more precise formulation of (1.11) is as follows:

\[
w_{\varepsilon} = \underbrace{u_{\varepsilon} - \psi_{\varepsilon} W^\varepsilon \ast G + \xi_{\varepsilon} + \eta_{\varepsilon} - \varepsilon \psi_{\varepsilon} \phi^\varepsilon \ast_2 \partial G.}_{\text{zero-order expansion term}} \quad \underbrace{\phi^\varepsilon \ast_2 \partial G}_{\text{first-order expansion term}} \quad \underbrace{\text{boundary layer terms}}_{\text{boundary layer terms}}
\]

The analogous pattern was first identified by J.-L. Lions [17, pp. 147] for stationary Stokes equations in a specific domain. We will elucidate this error expansion through a structured "questions & answers" format.

\(^4\)If the components of the gradient are involved in the inner product of a tensor and the convolution operation at the same time, we will use \( \partial \) instead of \( \nabla \) to stress this point, such as shown in (1.19).
• Why do we introduce the radial cut-off function \( \psi_\varepsilon \)?
  - To homogenize the inhomogeneous boundary condition induced by \( W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G \) on \( \partial \Omega \);
  - By virtue of the property: \( \nabla \psi_\varepsilon = -|\nabla \psi_\varepsilon| \hat{n} \) near \( \partial \Omega \), we can derive that
    \[
    \| \nabla \psi_\varepsilon \cdot u_0 \|_{L^2(\Omega)} = O(1)
    \]
  from the boundary condition \( \hat{n} \cdot u_0 = 0 \) on \( \partial \Omega \). The advantage of introducing \( \psi_\varepsilon \) becomes evident when compared to a general cut-off function \( \varphi_\varepsilon \):
    \[
    \| \nabla \varphi_\varepsilon \cdot u_0 \|_{L^2(\Omega)} = O(\varepsilon^{-1/2}) \implies \| \nabla \psi_\varepsilon \cdot u_0 \|_{L^2(\Omega)} = O(1)
    \]
  This structural benefit has been crucial in boundary-layer corrector estimates (see Lemma 4.7).

• Why do we introduce the corrector of Bogovskii’s operator \( \phi \)?
  - Roughly speaking, it was used to correct inhomogeneous divergence part of \( w_\varepsilon^{(1)} \). Using the divergence-free condition \( A * 2 \partial F = \nabla \cdot u_0 = 0 \) in \( \Omega \), we have
    \[
    \nabla \cdot w_\varepsilon^{(1)} = -W^\varepsilon * 2 \partial F = (|Y_f|^{-1}A - W^\varepsilon) * 2 \partial F.
    \]
  This leads to the following equation:
    \[
    \begin{cases}
    \nabla \cdot \phi = -W + |Y_f|^{-1}A & \text{in } \omega \times (0, T); \\
    \phi = 0 & \text{on } \partial \omega \times (0, T).
    \end{cases}
    \]
  (2.3)
  Then, one can substitute the right-hand side of (2.2) for \( \varepsilon(\nabla \cdot \phi)^\varepsilon * 2 \partial F \), further deriving that
    \[
    \nabla \cdot (w_\varepsilon^{(1)} - \varepsilon \phi^\varepsilon * 2 \partial F) = -\varepsilon \phi^\varepsilon * 2 \partial^2 F.
    \]
  Theoretically, this expansion can provide a first-order convergence rate. But for the sake of simplicity, the above formula does not take the cut-off function into account. A more detailed discussion on this is scheduled for Remark 3.2, where we will further touch on the concept of the flux corrector.

• Why do we introduce boundary-layer correctors associated with Bogovskii’s operator?
  - To see this, let \( w_\varepsilon^{(2)} = u_\varepsilon - \psi_\varepsilon (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G) \). There holds
    \[
    \begin{cases}
    -\nabla \cdot w_\varepsilon^{(2)} = J_1 + J_2 & \text{in } \Omega_\varepsilon \times (0, T); \\
    w_\varepsilon^{(2)} = 0 & \text{on } \partial \Omega_\varepsilon \times (0, T),
    \end{cases}
    \]
  (2.4)
  where \( J_1 \) and \( J_2 \) are defined in (1.17);
  - The crucial idea is to introduce a magical quantity
    \[
    \sum_i (\int_{O_\varepsilon^i} J_1) 1_{O_\varepsilon^i},
    \]
  where \( 1_{O_\varepsilon^i} \) is the indicator function of \( O_\varepsilon^i \), and \( \{ O_\varepsilon^i \} \) is a family of the non-overlap subsets of \( O_\varepsilon = \text{supp}(J_1) \), such that
    \[
    O_\varepsilon = \bigcup_i O_\varepsilon^i \quad \text{and} \quad |O_\varepsilon^i| \sim \varepsilon^d,
    \]
  (2.5)
in which \( O_\varepsilon^i \) is approximately an \( d \)-dimensional cube obtained by cutting the \( O_\varepsilon \) in the normal direction \( \hat{n} \) associated with \( \partial \Omega \) (see Remark 4.6). Then, we have
    \[
    J_1 + J_2 = J_1 - \sum_i (\int_{O_\varepsilon^i} J_1) 1_{O_\varepsilon^i} + \sum_i (\int_{\Omega_\varepsilon} J_1) 1_{O_\varepsilon} + J_2.
    \]
  Thereupon, it is possible to split the equation (2.4) to have some meaningful estimates, since \( \Pi \) and \( \mathcal{H} \) satisfy the compatibility conditions: \( \int_{\Omega_\varepsilon} \Pi = 0 \) and \( \int_{\Omega_\varepsilon} \mathcal{H} = 0 \), respectively;
Taking into account of temporal variable, for any $t \geq 0$, we first construct solutions $(\xi_\varepsilon, \eta_\varepsilon)$ to
\begin{align}
(1) \begin{cases}
\nabla \cdot \xi_\varepsilon(\cdot, t) = \partial_t H(\cdot, t) & \text{in } \Omega_\varepsilon; \\
\xi_\varepsilon = 0 & \text{in } \partial \Omega_\varepsilon,
\end{cases}
\text{ and (2) } \begin{cases}
\nabla \cdot \eta_\varepsilon(\cdot, t) = \partial_t \Pi(\cdot, t) & \text{in } O_\varepsilon; \\
\eta_\varepsilon = 0 & \text{in } \partial O_\varepsilon.
\end{cases}
\end{align}
(2.6)
Then, let $(\hat{\xi}_\varepsilon, \hat{\eta}_\varepsilon)$ be the primitive function of $(\xi_\varepsilon, \eta_\varepsilon)$, defined as follows:
\begin{align}
\hat{\xi}_\varepsilon(\cdot, t) = \int_0^t \xi_\varepsilon(\cdot, s)ds; \quad \hat{\eta}_\varepsilon(\cdot, t) = \int_0^t \eta_\varepsilon(\cdot, s)ds.
\end{align}
(2.7)
In this regard, one can verify that $-(\hat{\xi}_\varepsilon + \hat{\eta}_\varepsilon)$ satisfies the equation (2.4). Therefore, we have the error term precisely in the form of $w_\varepsilon = w_\varepsilon^{(2)} + \hat{\xi}_\varepsilon + \hat{\eta}_\varepsilon$.

**Remark 2.1.** In terms of the equation (2.4), if $w_\varepsilon^{(2)}$ is regarded as an unknown vector field, the problem "$\nabla \cdot v = g$" has no uniqueness of the solution (see e.g. [9, Theorem III.3.1]). Thus, there is no contradiction with that the corrector and (2) in (2.6) are different (see Section 4).

**2.2 Outline the proof of Theorem 1.2**

We start from introducing some terminologies used throughout the paper to distinguish the different types of correctors:

1. Corrector, denoted by $(W, \pi)$, defined in (1.6);
2. Corrector of Bogovskii’s operator\(^5\), denoted by $\phi$, defined in (2.3);
3. Boundary-layer correctors, denoted by $(\hat{\xi}_\varepsilon, \hat{\eta}_\varepsilon)$, defined in (2.7), respectively.

- **Main steps:**
  
  - **Step 1.** Find a good formula on the first-order expansion. Precisely speaking, we derive the following error term associated with velocity and pressure, respectively.

\begin{align}
\begin{cases}
w_\varepsilon = w_\varepsilon - \psi_\varepsilon(W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast_2 \partial G) + \hat{\xi}_\varepsilon + \hat{\eta}_\varepsilon; \\
q_\varepsilon = p_\varepsilon - p_0 - \varepsilon \psi_\varepsilon \pi^\varepsilon \ast_1 G,
\end{cases}
\end{align}
(2.8)

in which $(\hat{\xi}_\varepsilon, \hat{\eta}_\varepsilon)$ are boundary-layer correctors, and $\psi_\varepsilon$ is known as the radial cut-off function (see Lemma 4.5 for the existence). Also, the notations $W^\varepsilon$, $G$, $\ast_2$, etc have been defined in Subsection 1.4. A heuristic explanation on (2.8) has been presented in Subsection 2.1.

- **Step 2.** Derive the equations that error term $(w_\varepsilon, q_\varepsilon)$ satisfies. Plugging the error term (2.8) into unsteady Stokes operators in (1.2), there holds

\begin{align}
\begin{cases}
\partial_t w_\varepsilon - \varepsilon^2 \Delta w_\varepsilon + \nabla q_\varepsilon = I_1 + \varepsilon I_2 + \varepsilon^2 I_3 + \varepsilon^3 I_4, & \text{in } \Omega_\varepsilon \times (0, T); \\
\nabla \cdot w_\varepsilon = 0, & \text{in } \Omega_\varepsilon \times (0, T),
\end{cases}
\end{align}
(2.9)

with zero initial-boundary data and $T > 0$, where $I_1$, $I_2$, $I_3$ and $I_4$ have been formulated by

\begin{align*}
I_1 := & \nabla \cdot \left( - f - \psi_\varepsilon W^\varepsilon(\cdot, 0)G + \xi_\varepsilon + \eta_\varepsilon \right);
I_2 := & - \psi_\varepsilon \left[ \partial_t \phi^\varepsilon + \phi^\varepsilon(0) : \partial G \right] + 2 \psi_\varepsilon(\partial W^\varepsilon) \ast_2 \partial G - \nabla \psi_\varepsilon \pi^\varepsilon \ast_1 G - \psi_\varepsilon \nabla G \ast \pi^\varepsilon;
I_3 := & \psi_\varepsilon \nabla \cdot \left[ (\nabla \phi)^\varepsilon \ast_2 \partial G \right] + \nabla \cdot \left[ \nabla \psi_\varepsilon \ast (W^\varepsilon \ast G) \right] + \nabla (W^\varepsilon \ast G) \nabla \psi_\varepsilon + \psi_\varepsilon W^\varepsilon \ast \Delta G - \Delta \hat{\xi}_\varepsilon - \Delta \hat{\eta}_\varepsilon;
I_4 := & \nabla (\partial \phi^\varepsilon \ast_2 \partial G) \nabla \psi_\varepsilon + \nabla \cdot \left[ \nabla \psi_\varepsilon \ast (\partial \phi^\varepsilon \ast_2 \partial G) \right] + \psi_\varepsilon \nabla \cdot \left[ \partial \phi^\varepsilon \ast_2 \partial G \right].
\end{align*}

(See the details in the proof Lemma 5.1, which can be temporarily skipped for an initial reading.)

\(^5\)In [4, 29], the same type corrector was constructed by using Stokes equations. To the authors’ best knowledge, this type corrector was originally introduced by E. Marušić-Paloka, A. Mikelić [19].
– **Step 3.** *Rewriting the right-hand side of (2.9) as the form of*

\[
\Theta + \varepsilon \nabla \cdot \Lambda + \varepsilon \psi \frac{\nabla \cdot \Xi}{\varepsilon},
\]

*it follows from the energy estimate (see e.g. [23], or Lemma 5.2) that*

\[
\|w_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla w_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq \|(\Theta, \Lambda, \psi_\varepsilon \Xi)\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.
\]

*Then, reduce the error estimates to show:*

\[
\|(\Theta, \Lambda, \psi_\varepsilon \Xi)\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = O(\varepsilon^{1/2}).
\]  

(2.10)

*(See the details in Subsection 5.2.)*

– **Step 4.** *The desired estimate (2.10) consequently relies on the following three type estimates:*

1. **Smoothness of the correctors, i.e.,** for any \(1 < q < \infty,

\[
\|(\partial x W_j, \nabla \pi_j)\|_{L^1(0,T;L^q(Y_j))} + \|(\partial t \Phi, \partial r \phi)\|_{L^1(0,T;W^{1,q}(Y_j))} \leq 1,
\]  

(2.11)

*(See Propositions 3.1 and 3.3.)*

2. **Regularity estimates on boundary-layer correctors, i.e.,**

\[
\|(\xi_j, \eta_j)\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \varepsilon \|(\nabla \xi_j, \nabla \eta_j)\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = O(\varepsilon^{1/2}).
\]

(2.12)

*(See Propositions 4.1, 4.2 and 4.3.)*

3. **Well-posedness of the homogenized system (1.4), i.e.,**

\[
\|p_0\|_{L^q(0,T;C^{m,\alpha}(\Omega_\varepsilon))} \lesssim \|f\|_{L^q(0,T;C^{m,\alpha}(\Omega_\varepsilon))}, \quad \alpha \in (0,1) \text{ and } m \geq 1.
\]  

(2.13)

*(See Proposition 1.4.)*

– **Step 5.** *Show the error estimates on the pressure term.*

We first establish the error estimates on the inertial term as follows:

\[
\|\partial_t w_\varepsilon\|_{L^2(\Omega_\varepsilon,T)} = O(\varepsilon^{1/2}).
\]  

(2.14)

Then, base upon the estimates (2.10) and (2.14), a duality argument consequently leads to the error on the pressure term, i.e.,

\[
\inf_{c \in \mathbb{R}} \|q_\varepsilon - c\|_{L^2(\Omega_\varepsilon,T)} = O(\varepsilon^{1/2}).
\]

(See the details in Subsection 5.3.)

### 3 Correctors

**Proposition 3.1** (corrector & flux corrector). *Let \(0 < T < \infty, \ d = 2 \text{ or } 3, \text{ and } 1 < q < \infty. \ Suppose that \((W_j, \pi_j)\)*

is a weak solution of (1.6) with \(j = 1, \cdots, d. \) Then there holds the energy estimate

\[
\|W_j\|_{L^\infty(0,T;L^2(Y_j))} + \|W_j\|_{L^2(0,T;H^1(Y_j))} \leq 1,
\]

(3.1)

*where the multiplicative constant depends only on \(d \text{ and } T. \) Also, for any \(\alpha > (1/3), \) the weak solution possesses higher regularity estimates:

\[
\|W_j\|_{L^2(0,T;W^{1,q}(Y_j))} + \|\partial \alpha W_j\|_{L^2(0,T;L^2(Y_j))} + \|\partial \pi_j\|_{L^2(0,T;L^2(Y_j))} \leq 1 \quad \text{and} \quad \|\nabla \pi_j\|_{L^1(0,T;L^2(Y_j))} \leq 1,
\]

(3.2a)

\[
\|\partial_t W_j\|_{L^1(0,T;L^2(Y_j))} + \|\nabla \pi_j\|_{L^1(0,T;L^2(Y_j))} \leq 1,
\]

(3.2b)

*in which the up to constant relies on \(q, \ d, \text{ and the character of } Y_f. \) Moreover, let \(\tilde{W}_j\) be the zero-extension of \(W_j\) on the solid part \(Y_s. \) For each \(t \in (0,T], \) we can define \(b_{ij}(\cdot,t) := \tilde{W}_j(\cdot,t) \cdot e_i - A_{ij}(t). \) Then, there exists \(\Phi = \{\Phi_{ij,k}\}_{1 \leq i,j,k \leq d} \text{ with } \Phi_{ij,k}(\cdot,t) \in H^1_{loc}(\mathbb{R}^d), \) which is also 1-periodic and satisfies

\[
\nabla_k \Phi_{ij,k} = b_{ij} \quad \text{and} \quad \Phi_{ij,k} = -\Phi_{ik,j},
\]

(3.3)
as well as, the following regularity estimates:
\[
\|\partial_t \Phi\|_{L^1(0,T;H^1(\Omega))} + \|\Phi\|_{L^1(0,T;H^1(\Omega))} + \|\Phi(\cdot,0)\|_{H^1(\Omega)} \lesssim 1; \\
\|\partial_t \Phi\|_{L^1(0,T;W^{1,q}(\Omega))} + \|\Phi\|_{L^1(0,T;W^{1,q}(\Omega))} + \|\Phi(\cdot,0)\|_{L^\infty(\Omega)} \lesssim 1.
\] (3.4a)

(3.4b)

Consequently, we also have $\Phi_{ki,j} \in C([0,T];C_{per}(\Omega))$.

The estimate (3.1) directly follows from a traditional argument while the pressure is merely estimated by $H^{-1}$-norm with respect to the temporal variable (see e.g. [23]). Although the incompatibility will bring the negative influence to the solution’s regularity, we infer that this effect only occurs in the region where the solution has just begun to evolve from the initial value. The main idea on the estimates (3.2a) and (3.2b) is to employ interior regularity estimates to improve the associated semigroup estimates in $L^2$-norm (see Subsections 3.1 and 3.2), and then use interpolation arguments to get the desired results, which, to the authors’ best knowledge, seems to be new even for the general evolutionary Stokes equations without compatibility conditions.

**Remark 3.2.** We refer to $\Phi$ as the flux corrector, and its estimates are grounded in correctors’. To elucidate its role, we observe that

\[\nabla \cdot (u_\varepsilon - \psi_\varepsilon W^\varepsilon * F) = \underbrace{\nabla \psi_\varepsilon \cdot (A - W^\varepsilon) * F - \nabla \psi_\varepsilon \cdot A * F}_\text{layer part} + \underbrace{\psi_\varepsilon (Y_f^{-1} A - W^\varepsilon) *_2 \partial F}_\text{co-layer part}.\] (3.5)

By definition, the flux corrector plays a crucial role in addressing the boundary layer effects. Specifically, it collaborates with a corrector of Bogovskii’s operator, introduced below, to correct the inhomogeneous divergence component in the error expansion. Notably, unlike the corrector of Bogovskii’s operator, the flux corrector does not appear explicitly in the final form of the error expansion. This feature is consistent with the flux corrector introduced in the context of elliptic homogenization problems.

**Proposition 3.3** (corrector of Bogovskii’s operator). Let $0 < T < \infty$, $d = 2$ or $3$, and $1 < q < \infty$. Suppose that the corrector $W$ and the permeability tensor $A$ are given as in Theorem 1.1. Then, for any $t \geq 0$, there exists at least one weak solution $\phi$ associated with $W$ and $A$ by the following equations:

\[
\begin{aligned}
\nabla \cdot \phi_{i,j}(\cdot, t) &= -W_{ij}(\cdot, t) + |Y_f|^{-1} A_{ij}(t) \quad \text{in } \omega, \\
\phi_{i,j}(\cdot, t) &= 0 \quad \text{on } \partial \omega,
\end{aligned}
\] (3.6)

which is merely the component form of (2.3), with $1 \leq i,j,k \leq d$, whose solution is $1$-periodic and satisfies $\phi_{ki,j}(\cdot, t) \in H_{per}(Y_f)$. Moreover, there holds refined regularity estimate

\[
\|\phi\|_{L^1(0,T; W^{1,q}(\Omega))} + \|\phi(\cdot,0)\|_{W^{1,q}(\Omega)} + \|\partial_t \phi\|_{L^1(0,T; W^{1,q}(\Omega))} \lesssim 1, 
\] (3.7)

and we concludes that $\partial_t \phi_{ki,j} \in L^1(0,T; C_{per}(Y_f))$ and $\phi_{ki,j} \in C([0,T]; W^{1,q}_{per}(Y_f))$.

**Remark 3.4.** The corrector $(W_j, \pi_j)$ defined in (1.6) is taken from A. Mikelić [20], which slightly differs from that given by G. Allaire [1], i.e.,

\[
\begin{aligned}
\partial_t w_j - \Delta w_j + \nabla \pi_j &= \epsilon_j \quad \text{in } \omega \times (0,T); \\
\nabla \cdot w_j &= 0 \quad \text{in } \omega \times (0,T); \\
w_j &= 0 \quad \text{on } \partial \omega \times (0,T); \\
w_j|_{\nu=0} &= 0 \quad \text{in } \omega, \quad T > 0.
\end{aligned}
\] (3.8)

We observe that there holds the equality: $\partial_t w_j(\cdot, t) = W_j(\cdot, t)$ for any $t \geq 0$, in the sense of Stokes semigroup representation, up to a projection\(^6\). However, this relationship does not alleviate the complexity arising from the incompatibility among the given data in the equations (1.6) or (3.8). By modifying the initial conditions, G. Sandrakov introduced a revised version of the corrector equations [27, Theorem 2]. However, this modification did not resolve this incompatibility between the initial and boundary data.

\(^6\)Let $A := P(-\Delta)$ and $P$ is the Helmholtz projection (see e.g. [24]). By functional analytic approach, we have $w_j(\cdot, t) = \int_0^t ds \exp(-s A) \epsilon_j$, and then using a spectral representation, one can derive $\partial_t w_j = \exp(-t A) \epsilon_j$. On the other hand, by noting that $P(\nabla \kappa) = 0$, there holds $W_i(\cdot, t) = \exp(-t A) \epsilon_i = \exp(-t A) P(\epsilon_i - \nabla \kappa)$. Thus, combining the previous two formulas, the equality $\partial_t w_j = W_j$ holds up to the projection $P$. 

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in which the constant \( C \) near boundary 

Figure 1: This figure provides a heuristic illustration of the decomposition of region \( Y_f \) in two dimensional case. Furthermore, it is important to note that region \( Y \) is a periodic domain.

### 3.1 Semigroup estimate I

**Lemma 3.5** (semigroup estimate I). Let \( d = 2 \) or \( 3 \), and \( 2 < p < \infty \). Suppose that \((W_j, \pi_j)\) is a weak solution of (1.6) with \( j = 1, \cdots, d \). Then, for any \( t > 0 \), there holds a decay estimate

\[
\left( \int_{Y_f} |\nabla W_j(\cdot, t)|^2 \right)^{1/2} \leq C_p t^{-\frac{1}{p-2}}, \tag{3.9}
\]

in which the constant \( C_p \) depends on \( d, p, \) and the character of \( Y_f \).

The key observation on the estimate (3.9) is that Caccioppoli-type inequality offers a good time-decay in the interior region, but produces a bad spacial scale-factor. Meanwhile, the semigroup estimate can dominate the region near boundary \( \partial Y_f \), owning a relatively bad time-decay, but creating a good spacial scale-factor. Thus, the idea is to bring in a parameter \( \rho \) to balance their advantage and disadvantage such that we can “improve” the decay power of Stokes semigroup estimates.

**Proof.** It suffices to show the estimate (3.9) for any \( t \in (0, t_0) \), where \( t_0 \in (0, 1/2) \) is usually a small number, while for the case \( t > t_0 \) it can be simply inferred from the semigroup estimate itself. Also, we merely establish the concrete proof in the case of \( d = 3 \). For \( d = 2 \), one can simply set

\[
\tilde{W}_j(y_1, y_2, y_3, t) = (W_j(y_1, y_2), 0, t); \quad \tilde{\pi}(y_1, y_2, y_3, t) = \pi(y_1, y_2, t); \quad \tilde{W}_j(y_1, y_2, y_3, 0) = e_j, \quad j = 1, 2.
\]

It is not hard to verify that \((\tilde{W}_j, \tilde{\pi})\) satisfies the corrector equation (1.6) in three dimensional case. Therefore, once three dimensional case is proved, two dimensional problem can be directly approached in this way.

Firstly, we decompose the integral domain \( Y_f \) into two parts (see Figure 1): \((Y_f)_\rho\) and \((Y_f \setminus (Y_f)_\rho)\), where \((Y_f)_\rho := \{ y \in Y_f : \text{dist}(y, \partial Y_f) \geq 2\rho \}\) for the parameter \( \rho > 0 \), which will be fixed later. In this respect, the desired estimate (3.9) is reduced to showing: (For the ease of the statement, we omit the subscript of \( W_j \) in the proof.)

\[
\left( \int_{Y_f} |\nabla W(\cdot, t)|^2 \right)^{1/2} \lesssim \left( \int_{(Y_f \setminus (Y_f)_\rho)} |\nabla W(\cdot, t)|^2 \right)^{1/2} + \left( \int_{(Y_f)_\rho} |\nabla W(\cdot, t)|^2 \right)^{1/2} =: I_1 + I_2. \tag{3.10}
\]

The easier term is \( I_1 \), and by using Hölder’s inequality and the semigroup estimates (see e.g. [24, pp.82]) in the order, we obtain

\[
I_1 \lesssim \rho^{2-\frac{2}{p}} \left( \int_{Y_f} |\nabla W(\cdot, t)|^p \right)^{1/p} \lesssim \rho^{2-\frac{2}{p}} t^{-\frac{2}{p}}. \tag{3.11}
\]

We now turn to the second term \( I_2 \), and start from giving a family of cut-off functions, denoted by \( \{\chi_i\} \), which satisfy that \( \chi_i(y) := \chi_0(y + y_i) \), and \((Y_f)_\rho \subset \bigcup_i B(y_i, \rho) \subset (Y_f \setminus (Y_f)_\rho). \) We assume that

- \( \chi_0 \in C^1_{\text{per}}(Y) \) is a cut-off function;
- \( \chi_0 = 1 \) on \( B(0, \rho/2) \) and \( \text{supp}(\chi_0) \subset B(0, \rho) \) with \( |\nabla \chi_0| \lesssim \frac{1}{\rho} \);
- For any point in the region \((Y_f)_\rho\), it is covered by, at most, five elements of the covering family \( \{B(y_i, \rho)\} \).
From the assumptions on \( \{ \chi_i \} \), it follows that \( \text{dist}(\partial Y, \text{supp}(\chi_i)) \geq \rho \). Moreover, we define a family of indicator functions associated with \( \{ \chi_i \} \) as follows:

\[
\tilde{\chi}_i = 1 \text{ in } B(y_i, \rho) \quad \text{and} \quad \tilde{\chi}_i = 0 \text{ outside } B(y_i, \rho).
\]

Therefore, the second term in the right-hand side of (3.10) can be estimated by

\[
I_2 \leq \left( \sum_i \int_{Y_f} \chi_i^2 |\nabla W(\cdot, t)|^2 \right)^{1/2},
\]

while we claim that: for each \( \{ Z \} \) one can acquire

\[
\text{Step 1. We now verify the claims (3.12a), (3.12b), and we start from dealing with the estimate (3.12b) therein.}
\]

Admitting the claims (3.12a), (3.12b) for a moment, and combining them, we will get

\[
\int_{Y_f} \chi_i^2 |\nabla W(\cdot, t)|^2 dy \leq \frac{1}{\rho^2} \left\{ \int_0^T \int_{Y_f} \tilde{\chi}_i |\nabla W|^2 + \sup_{0 \leq t \leq T} \int_{Y_f} \tilde{\chi}_i |W(\cdot, t)|^2 \right\},
\]

In view of the assumptions on \( \chi_i \) and the above estimate (3.13), we obtain that

\[
I_2 \leq \left( \sum_i \int_{Y_f} \chi_i^2 |\nabla W(\cdot, t)|^2 \right)^{1/2} \leq \frac{1}{\rho} \left( \int_0^T \int_{Y_f} |\nabla W|^2 + \sup_{0 \leq t \leq T} \int_{Y_f} |W(\cdot, t)|^2 \right)^{1/2} \lesssim (\text{3.1}) \frac{1}{\rho}.
\]

As a result, plugging (3.11) and (3.14) back into (3.10), and then taking \( \rho = t^{\frac{\rho}{\frac{3}{2}}} \) (which requires \( t \) to be small), one can acquire

\[
\left( \int_{Y_f} |\nabla W(\cdot, t)|^2 \right)^{1/2} \lesssim \rho^{\frac{1}{2}} t^{-\frac{1}{2}} + \rho^{-1} \lesssim t^{-\frac{3}{2}},
\]

which is the desired estimate (3.9).

We plan to use two steps to complete the whole proof, which is originally inspired by [13].

**Step 1.** We now verify the claims (3.12a), (3.12b), and we start from dealing with the estimate (3.12b) therein. Let \( \nabla \times (\chi_i^2 \nabla \times W) \) be the test function and act on both sides of (1.6), and by noting the facts: \( \nabla \cdot (\nabla \times (\chi_i^2 \nabla \times W)) = 0 \), \( \partial Y \cap \text{supp}(\chi_i) = \emptyset \) and the periodicity of \( \chi_i \), integration by parts gives us

\[
0 = \int_{Y_f} (\partial_t W - \Delta W + \nabla \pi) \cdot \nabla \times (\chi_i^2 \nabla \times W)
= \int_{Y_f} \partial_t W \cdot \nabla \times (\chi_i^2 \nabla \times W) - \int_{Y_f} \Delta W \cdot \nabla \times (\chi_i^2 \nabla \times W).
\]

By the basic formula:

\[
\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}),
\]

we can further derive that

\[
\int_{Y_f} \partial_t W \cdot \nabla \times (\chi_i^2 \nabla \times W) \overset{(3.16)}{=} -\int_{Y_f} \nabla \cdot [\partial_t W \times (\chi_i^2 \nabla \times W)] + \int_{Y_f} (\nabla \times \partial_t W) \cdot (\chi_i^2 \nabla \times W)
= -\int_{\partial Y_f} \vec{n} \cdot [\partial_t W \times (\chi_i^2 \nabla \times W)] + \int_{Y_f} (\nabla \times \partial_t W) \cdot (\chi_i^2 \nabla \times W)
= \frac{1}{2} \frac{d}{dt} \int_{Y_f} \chi_i^2 |\nabla \times W|^2,
\]

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where we merely take \( \vec{a} = \partial_i W \) and \( \vec{b} = \chi_i^2 \nabla \times W \). This together with (3.15) leads to
\[
0 = \frac{d}{dt} \int_{Y_j} \chi_i^2 |\nabla \times W|^2 - \int_{Y_j} \Delta W \cdot \nabla \times (\chi_i^2 \nabla \times W)
= \frac{d}{dt} \int_{Y_j} \chi_i^2 |\nabla \times W|^2 + \int_{Y_j} \chi_i^2 |\Delta W|^2 - \int_{Y_j} \Delta W \cdot \nabla \chi_i^2 \times (\nabla \times W),
\] (3.17)
where the second equality is due to the following computation:
\[
\nabla \times (\chi_i^2 \nabla \times W) = \nabla \chi_i^2 \times (\nabla \times W) + \chi_i^2 \nabla \times \nabla \times W
\nabla \times \nabla \times W = -\Delta W + \nabla (\nabla \cdot W) = -\Delta W.
\]

Rewriting the above equality (3.17), there holds
\[
\frac{d}{dt} \int_{Y_j} \chi_i^2 |\nabla \times W|^2 + \int_{Y_j} \chi_i^2 |\Delta W|^2 = \int_{Y_j} \Delta W \cdot \nabla \chi_i^2 \times (\nabla \times W)
\leq 2 \int_{Y_j} |\chi_i||\Delta W||\nabla \chi_i||\nabla \times W|.
\] (3.18)
(Note that the repeated index \( i \) does not represent a sum throughout the proofs of Lemmas 3.5, 3.6.) Applying Young’s inequality to the right-hand side of (3.18), it is not hard to see that
\[
\frac{d}{dt} \int_{Y_j} \chi_i^2 |\nabla \times W|^2 + \int_{Y_j} \chi_i^2 |\Delta W|^2 \lesssim \int_{Y_j} |\nabla \chi_i|^2 |\nabla \times W|^2,
\]
and for any \( t > 0 \) we consequently derive that
\[
\int_{Y_j} \chi_i^2 |\nabla \times W(\cdot, t)|^2 \leq \int_{Y_j} |\nabla \chi_i|^2 |\nabla \times W|^2 + \int_{Y_j} \chi_i^2 |\nabla \times W(\cdot, 0)|^2 \leq \frac{1}{\rho^2} \int_{Y_j} \int_{0}^{T} \chi_i |\nabla \chi_i|^2 |\nabla \times W|^2,
\]
which gives the claim (3.12b).

**Step 2.** We turn to study the claim (3.12a). Recalling the formula \( \nabla \times (\chi_i^2 \nabla \times W) = \nabla \chi_i^2 \times (\nabla \times W) - \chi_i^2 \Delta W \)
in \( Y_j \), it leads us to the following integral equality
\[
-\int_{Y_j} \chi_i^2 \Delta W \cdot W = \int_{Y_j} [\nabla \times (\chi_i^2 \nabla \times W)] \cdot W - 2 \int_{Y_j} \chi_i (\nabla \chi_i \times \nabla \times W) \cdot W.
\] (3.19)

Integrating by parts, the left-hand side of (3.19) is equal to
\[
\int_{Y_j} \chi_i^2 |\nabla W|^2 + 2 \int_{Y_j} \chi_i \nabla W : \nabla \chi_i \otimes W,
\]
and moving the second term above to the right-hand side of (3.19) we then derive that
\[
\int_{Y_j} \chi_i^2 |\nabla W|^2 \lesssim \int_{Y_j} |\chi_i||\nabla W||\nabla \chi_i||W| + \int_{Y_j} \nabla \times (\chi_i^2 \nabla \times W) \cdot W.
\] (3.20)

Continue the computation as follows:
\[
\int_{Y_j} \nabla \times (\chi_i^2 \nabla \times W) \cdot W \overset{(3.16)}{=} \int_{Y_j} \nabla \cdot [(\chi_i^2 \nabla \times W) \times W] + \int_{Y_j} (\chi_i^2 \nabla \times W) \cdot (\nabla \times W)
= \int_{\partial Y_j} \vec{n} \cdot [(\chi_i^2 \nabla \times W) \times W] + \int_{Y_j} (\chi_i^2 \nabla \times W) \cdot (\nabla \times W) = \int_{Y_j} \chi_i^2 |\nabla \times W|^2.
\]

Inserting the above equality back into (3.20) and using Young’s inequality again, there holds
\[
\int_{Y_j} \chi_i^2 |\nabla W|^2 \leq \theta \int_{Y_j} \chi_i^2 |\nabla W|^2 + C_\theta \int_{Y_j} |\nabla \chi_i|^2 |W|^2 + \int_{Y_j} \chi_i^2 |\nabla \times W|^2, \quad \theta \in (0, 1),
\]
which immediately implies the stated result (3.12a). We have completed all the proof.
3.2 Semigroup estimate II

In fact, we repeat the same philosophy used in Lemma 3.5 to show the estimate (3.21) in $L^2$-norm, and then appeal to an interpolation, where we adopt the stream function method to get a higher-order interior estimate.

**Lemma 3.6** (semigroup estimate II). Let $d \geq 2$ and $p \geq 2$ be sufficiently large, and $\gamma := \frac{p(19p-14)}{(3p-2)(5p-2)}$. Suppose that $(W_j, \pi_j)$ is a weak solution of (1.6) with $j = 1, \cdots, d$. Then, for any $r \in (1, \infty)$, there exists a constant $q \in (1, \infty)$, such that $\lambda := \frac{2(q-r)}{r(q-2)}$ satisfying $0 \leq \lambda \leq 1$, and there holds the decay estimate:

$$\|\partial_t W_j(\cdot, t)\|_{L^r(Y_f)} \lesssim t^{1+\lambda(1-\gamma)}$$

(3.21)

for any $t > 0$, where the multiplicative constant depends on $d, p, q$, and the character of $Y_f$.

**Proof.** In general, we adopt a similar strategy and notation as presented in Lemma 3.5, with the additional use of an interpolation argument. The entire proof is structured into four steps.

**Step 1. One step reduction.** The main idea is that we firstly improve the decay estimate in $L^2$-spatial norm, and then appeal to the interpolation argument to get a weaker improvement in terms of $L^r$-spatial norms with $r \neq 2$. Thus, the key step is to establish the following estimate: (Here we omit the subscript of $W_j$ throughout the proof.)

$$\left( \int_{Y_f} |\Delta W(y, t)|^2 \right)^{1/2} \lesssim t^{-\gamma}$$

(3.22)

for any $t > 0$. Admitting the above estimate for a while, we introduce the infinitesimal generator $A := P(-\Delta)$, where the operator $P$ is known as the Helmholtz projection, and then it follows from the boundedness of $P$ and (3.22) that

$$\|AW(\cdot, t)\|_{L^r(Y_f)} = \|P(-\Delta)W(\cdot, t)\|_{L^r(Y_f)} \lesssim \|\Delta W(\cdot, t)\|_{L^r(Y_f)} \lesssim t^{-\gamma}.$$  

Recalling Stokes semigroup estimates (see e.g. [24, pp.81]): for $1 < q < \infty$, we have

$$\|AW(\cdot, t)\|_{L^q(Y_f)} \lesssim t^{-1}.$$  

Then, by preferring a suitable $q \in (1, \infty)$ such that $q < r \leq 2$ or $2 \leq r < q$, one can employ the interpolation inequality to get that

$$\|AW(\cdot, t)\|_{L^r(Y_f)} \lesssim \|AW(\cdot, t)\|_{L^2(Y_f)}^{1-\lambda} \|AW(\cdot, t)\|_{L^q(Y_f)}^{\lambda},$$

where $\lambda := \frac{2(q-r)}{r(q-2)}$. Using the representation of the solution of (1.6) by the semigroup theory, we have $\partial_t W = AW$ in $Y_f \times (0, \infty)$, which implies the stated estimate (3.21).

**Step 2. A further reduction.** The remainder part of the proof is devoted to establishing the crucial estimate (3.22) from the two important ingredients. Introducing the parameter $\rho > 0$, we decompose the following integral into two parts: the inner part and the near-boundary part, i.e.,

$$\left( \int_{Y_f} |\Delta W(\cdot, t)|^2 \right)^{1/2} \leq \left( \int_{(Y_f)_i} |\Delta W(\cdot, t)|^2 \right)^{1/2} + \left( \int_{(Y_f)_{e}} |\Delta W(\cdot, t)|^2 \right)^{1/2}. $$

Let $p \geq 2$, and we claim that there hold

$$\left( \int_{(Y_f)_i} |\nabla^2 W(\cdot, t)|^2 \right)^{1/2} \lesssim \rho^{-3} t^{-\frac{3}{p+2}},$$

(3.24a)

$$\left( \int_{(Y_f)_{e}} |\Delta W(\cdot, t)|^2 \right)^{1/2} \lesssim \rho^{\frac{1}{q} - \frac{1}{r}} t^{-1}.$$  

(3.24b)

Admitting the above two estimates temporarily, we can proceed with our analysis. Inserting (3.24a) and (3.24b) back into (3.23), we have

$$\left( \int_{Y_f} |\Delta W(\cdot, t)|^2 \right)^{1/2} \lesssim \rho^{\frac{1}{q} - \frac{1}{r}} t^{-1} + \rho^{\frac{1}{q} - \frac{1}{r}} t^{-1},$$

which completes the proof.
and minimizing the right-side above by taking \( \rho > 0 \) such that \( \rho^{-3} t^{-\frac{p}{2}} = \rho^{1 - \frac{1}{2}} t^{-1} \), there holds
\[
\left( \int_{Y_f} |\Delta W(\cdot, t)|^2 \right)^{1/2} \lesssim t^{\frac{-3p(19p-14)}{-3p(19p-14)+2p(19p-14)}}.
\]
which gives us the core estimate (3.22)

\[ \text{for any } j. \]

It is well known from the energy estimate (3.1) that \( v \) holds a new parabolic system:
\[
\begin{cases}
\partial_t v - \Delta v = f & \text{in } Y_f \times (0, T] \\
v|_{\partial Y_f} = \text{curl } W & \text{for } 0 < t \leq T \\
v|_{t=0} = \chi v|_{t=0} = 0.
\end{cases}
\]

For we are interested in the interior estimates, by setting \( u = \chi v \), from the above parabolic system we get
\[
\begin{cases}
(\partial_t - \Delta) u = -\Delta \chi v - 2 \nabla \chi \cdot \nabla v := g & \text{in } Y_f \times (0, T] \\
u|_{\partial Y_f} = 0, & \text{for } 0 < t \leq T \\
u|_{t=0} = \chi v|_{t=0} = 0.
\end{cases}
\]

Step 3. Show the estimate (3.24a) by setting \( \gamma := \frac{p(19p-14)}{3p-2(19p-2)} \).

To see this, it is known that \( \partial_k W \) with \( k = 1, \ldots, d \) satisfies the equations in \( Y_f \times (0, T] \) as the same as \( W \) does, and therefore it follows from the estimate (3.13) that
\[
\int_{Y_f} \chi_i^2 |\nabla^2 W(\cdot, t)|^2 \lesssim \|
abla \chi_i\|_{L^2(Y)}^2 \left\{ \int_0^T \int_{Y_f} \chi_i |\nabla^2 W|^2 + \int_{Y_f} \chi_i |\nabla W(\cdot, t)|^2 \right\},
\]
which immediately gives the desired estimate (3.25). The desired estimate is reduced to estimating the first term in the bracket above. We also note that the interior estimate is translation-invariant within \( Y_f \), and thus the center position of the estimated region can be disregarded. Therefore, we only need to establish the following estimate:
\[
\int_0^T dt \|
abla^2 W(\cdot, t)\|_{L^2(B/2)}^2 \lesssim \|
abla \chi\|_{L^\infty(Y)}^2 \int_0^T dt \|
abla W(\cdot, t)\|_{L^2(B)}^2,
\]
where \( B := B(y, 2\rho) \) with \( 2B \subset \omega \) is arbitrary, and \( \chi \in C^\infty_0(Y) \) is a cut-off function satisfying \( \chi = 1 \) in \( B \) and \( \operatorname{supp}(\chi) \subset 2B \) with \( |\nabla \chi| \lesssim \frac{1}{\rho} \). Consequently, plugging the estimate (3.26) back into (3.25), we have
\[
\left( \int_{Y_f} \chi_i^2 |\nabla^2 W(\cdot, t)|^2 \right)^{1/2} \lesssim \rho^{-3} + \rho^{-1} t^{-\frac{n}{2p-2}} \lesssim \rho^{-3} t^{-\frac{n}{2p-2}},
\]
which is the stated estimate (3.24a).

Then, we turn to study the estimate (3.26). Set \( v = \text{curl } W \) in \( \omega \times [0, T] \). By recalling the equations (1.6), there holds a new parabolic system:
\[
\begin{cases}
(\partial_t - \Delta) v = 0 & \text{in } Y_f \times (0, T] \\
v|_{\partial Y_f} = \text{curl } W & \text{for } 0 < t \leq T \\
v|_{t=0} = 0.
\end{cases}
\]

It is well known from the energy estimate (3.1) that \( v_j \in L^2(0, T; L^2_{per}(Y_f)^d) \), (where \( v_j = \text{curl } W_j \) represents the \( j \)-th component of \( v \)) whereupon we can verify that each of the components of \( g \) belongs to \( L^2(0, T; H^{-1}(Y_f)) \).

Moreover, there holds
\[
\|g(\cdot, t)\|_{H^{-1}(Y_f)} \lesssim \|\nabla \chi \cdot \nabla v(\cdot, t)\|_{H^{-1}(\operatorname{supp}(\chi))} + \|\Delta \chi\|_{L^\infty(Y)} \|v(\cdot, t)\|_{L^2(\operatorname{supp}(\chi))} \]
\[
\lesssim \|\nabla^2 \chi\|_{L^\infty(Y)} \|v(\cdot, t)\|_{L^2(\operatorname{supp}(\chi))}
\]
for any \( t > 0 \), and we get
\[
\int_0^T dt \|g(\cdot, t)\|_{H^{-1}(Y_f)}^2 \lesssim \|\nabla^2 \chi\|_{L^\infty(Y)}^2 \int_0^T dt \int_{2B} |v(\cdot, t)|^2.
\]

This together with the energy estimates for (3.27) gives us
\[
\|\nabla u\|_{L^2(0, T; L^2(Y_f))} \lesssim \|\nabla^2 \chi\|_{L^\infty(Y)} \left( \int_0^T dt \int_{2B} |v|^2 \right)^{1/2}.
\]
Recalling the definition of $u$ and $v$, it is not hard to see that
\[
\left( \int_0^T \int_{Y_f} \chi^2 |\nabla v|^2 \right)^{1/2} \lesssim \left( \int_0^T \int_{Y_f} |\nabla u|^2 \right)^{1/2} + \|\nabla \chi\|_{L^\infty(Y)} \left( \int_0^T \int_{2B} |v|^2 \right)^{1/2},
\]
\[(3.28)\]
and therefore, we have
\[
\left( \int_0^T \int_B |\nabla v|^2 \right)^{1/2} \lesssim \|\nabla \chi\|_{L^\infty(Y)} \left( \int_0^T \int_{2B} |v|^2 \right)^{1/2}.
\]
\[(3.29)\]
By noting that $v = \text{curl } W$, we have $-\Delta W = \text{curl } v$ in $Y_f \times [0,T]$, where we employ the vector identity $-\Delta W = \text{curl curl } W - \nabla(\nabla \cdot W)$ (see e.g. [24, Chapter 5]). Thus, from the interior $H^2$ estimates for elliptic equation (see e.g. [10, Chapter 4]), it follows that
\[
\|\nabla^2 W(\cdot,t)\|_{L^2(B/2)} \lesssim \|\nabla \chi\|_{L^\infty(Y)} \|\nabla W(\cdot,t)\|_{L^2(B)} + \|\text{curl } v(\cdot,t)\|_{L^2(B)}
\]
holds for any $t > 0$. Hence, integrating both sides above with respect to $t$ from $0$ to $T$, we can derive that
\[
\int_0^T dt \|\nabla^2 W(\cdot,t)\|_{L^2(B/2)}^2 \lesssim \|\nabla \chi\|_{L^\infty(Y)} \int_0^T dt \|\nabla W(\cdot,t)\|_{L^2(B)}^2 + \int_0^T dt \|\nabla v(\cdot,t)\|_{L^2(B)}^2.
\]
\[\text{(3.29)}\]
which is the stated estimate (3.26).

**Step 4.** Show the estimate (3.24b). We start from rewriting the equations (1.6) as follows: for any $t > 0$,
\[
\begin{cases}
-\Delta W(\cdot,t) + \nabla \pi(\cdot,t) = -\partial_t W(\cdot,t) & \text{in } Y_f; \\
\nabla \cdot W(\cdot,t) = 0 & \text{in } Y_f; \\
W(\cdot,t)|_{\partial Y_f} = 0.
\end{cases}
\]

For any $1 < p < \infty$, from the $L^p$ theory for stationary Stokes system (see e.g. [9, Chapter IV]), it follows that
\[
\|\nabla^2 W(\cdot,t)\|_{L^p(Y_f)} + \|\nabla \pi(\cdot,t)\|_{L^p(Y_f)} \lesssim \|\partial_t W(\cdot,t)\|_{L^p(Y_f)},
\]
\[(3.30)\]
where the up to constant depends on $p, d$ and the character of $Y_f$.

Then, for any $p \geq 2$, by using Hölder’s inequality, the estimate (3.30), and the observation $\partial_t W = AW$, as well as, Stokes semigroup estimates (see e.g. [24, Chapter 5]), in the order, we obtain that
\[
\left( \int_{Y_f \setminus \{Y_f\}_0} |\Delta W(\cdot,t)|^2 \right)^{1/2} \lesssim p^{1 - \frac{1}{p}} \left( \int_{Y_f \setminus \{Y_f\}_0} |\nabla^2 W(\cdot,t)|^p \right)^{1/p}
\]
\[
\lesssim p^{1 - \frac{1}{p}} \left( \int_{Y_f} |\partial_t W(\cdot,t)|^p \right)^{1/p} = p^{1 - \frac{1}{p}} \left( \int_{Y_f} |AW(\cdot,t)|^p \right)^{1/p} \lesssim p^{1 - \frac{1}{p}} t^{-1},
\]
which is the desired estimate (3.24b). We have completed the whole proof.

\[\square\]

### 3.3 Proof of Proposition 3.1

**Corollary 3.7** (weighted estimates). Let $0 < T < \infty$, and $2 < p < \infty$ with $\vartheta := \frac{p}{2p - 2}$. Suppose that $(W_j, \pi_j)$ is a weak solution of (1.6) with $j = 1, \cdots, d$, and the condition $\int_{Y_f} \pi_j(\cdot,t) = 0$ for any $t > 0$. Then, for any $\alpha > 2\vartheta$, we have a refined weighted estimate
\[
\left( \int_0^T dt \int_{Y_f} |\partial_t W_j(\cdot,t)|^{2\vartheta} \right)^{1/2} + \left( \int_0^T dt \int_{Y_f} |\pi_j(\cdot,t)|^{2\vartheta} \right)^{1/2} \lesssim 1,
\]
\[(3.31)\]
where the up to constant depends on $d, \alpha, T$, and the character of $Y_f$. 

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Proof. The advantage of the present proof avoids using advanced analysis results\(^7\). We firstly establish the weighted estimate for \(\partial_t W_j\), and then translate the same type estimate to the pressure term \(\pi_j\). To do so, we take \(\partial_t W_j\) as the test function for the equations (1.6), and for any \(t > 0\) there holds:

\[
\int_{Y_j} |\partial_t W_j(\cdot,t)|^2 + \frac{1}{2} \frac{d}{dt} \int_{Y_j} |\nabla W_j(\cdot,t)|^2 = 0.
\]

Integrating both sides of the above equation with respect to temporal variable from \(\tau\) to \(T\), and then appealing to Lemma 3.5, we obtain that

\[
\int_{\tau}^{T} \int_{Y_j} |\partial_t W_j|^2 \leq \frac{1}{2} \int_{\tau}^{T} \int_{Y_j} |\nabla W_j(\cdot,\tau)|^2 \lesssim \tau^{-2\vartheta}.
\]

Multiplying the factor \(\tau^{\alpha-1}\) on the both sides above, and then integrating with respect to \(\tau\) from 0 to \(T\), one can further derive that

\[
\int_{0}^{T} d\tau \left( \tau^{\alpha-1} \int_{\tau}^{T} \int_{Y_j} |\partial_t W_j|^2 \right) \lesssim T^{\alpha-2\vartheta}.
\]

By using Fubini’s theorem in the left-hand side above, i.e.,

\[
\int_{0}^{T} dt \int_{Y_j} |\partial_t W_j(\cdot,t)|^2 t^\alpha \lesssim T^{\alpha-2\vartheta}.
\]

We now continue to study the pressure term, and begin with constructing an auxiliary function associated with Bogovskii’s operator, i.e., for any \(t > 0\) we have

\[
\begin{cases}
\nabla \cdot v(\cdot,t) = \pi(\cdot,t), & \text{in } Y_j; \\
v(\cdot,t) = 0, & \text{on } \partial Y_j, \ y \mapsto v(y,t) \text{ is 1-periodic}
\end{cases}
\]

with the estimate \(\|v(\cdot,t)\|_{H^1(Y_j)} \lesssim \|\pi(\cdot,t)\|_{L^2(Y_j)}\). By taking \(v\) as a test function acting on (1.6), it is not hard to get

\[
\int_{Y_j} |\pi_j(\cdot,t)|^2 \lesssim \int_{Y_j} |\partial_t W_j(\cdot,t)|^2.
\]

Consequently, multiplying \(t^\alpha\) on the both sides above and then integrating it from 0 to \(T\), we obtain

\[
\int_{0}^{T} dt \int_{Y_j} |\pi_j(\cdot,t)|^2 t^\alpha \lesssim \int_{0}^{T} dt \int_{Y_j} |\partial_t W_j(\cdot,t)|^2 t^\alpha \lesssim T^{\alpha-2\vartheta}.
\]

This together with (3.32) leads to the desired estimate (3.31), and ends the whole proof. \(\square\)

**Corollary 3.8.** Let \(\vartheta\) be given as in Corollary 3.7. Suppose that \((W_j,\pi_j)\) is a weak solution of (1.6) with \(j = 1,\ldots,d\). Then, for any \(1 < q < \infty\), there exists a constant \(\lambda \in (0,1]\) as the same as in Lemma 3.6 such that a refined decay estimate

\[
\left( \int_{Y_j} |\nabla W_j(\cdot,t)|^q \right)^{1/q} \lesssim t^{-\frac{1}{2} + \lambda(\frac{q}{2}-\vartheta)}
\]

holds for any \(t > 0\), which further leads to

\[
\|W_j\|_{L^2(0,T;W^{1,q}(Y_j))} \lesssim 1,
\]

where the up to constant depends on \(d, q, \lambda, T\), and the character of \(Y_j\).

\(^7\)The case \(\alpha = 1\) simply come from a testing function argument, and we pursue a consistency even in this refined estimate, although there exists a way to save effort on the pressure term.
Proof. Using the same arguments as given for Lemma 3.6, we can derive the estimate (3.33) from the interpolation between the refined estimate (3.9) and the semigroup estimates (see e.g. [24, Lemma 5.1]), while the stated estimate (3.34) directly follows from (3.33) coupled with Poincare’s inequality (by noting zero-boundary condition of $W_j$ for $t > 0$), and this completes the proof.

Lemma 3.9 (antisymmetry and regularities). For any $0 < t < T$, let $B(\cdot, t) = \{b_{ij}(\cdot, t)\}_{1 \leq i,j \leq d}$ be given as in Proposition 3.1. Then there hold the structure properties:

\begin{equation}
\begin{aligned}
& (i) \quad \nabla \cdot B(\cdot, t) = 0; \quad (ii) \quad \int_Y B(\cdot, t) = 0. \\
& \text{Moreover, there exists } \Phi(\cdot, t) = \{\Phi_{ki,j}(\cdot, t)\}_{1 \leq i,j,k \leq d} \text{ with } \Phi_{ki,j}(\cdot, t) \in H^1_{\text{loc}}(\mathbb{R}^d) \text{ being 1-periodic, and satisfying}
\end{aligned}
\end{equation}

\begin{equation}
\nabla \cdot \Phi(\cdot, t) = B(\cdot, t) \quad \text{in} \ Y
\end{equation}

under the antisymmetry condition (i.e., $\Phi_{ki,j} = -\Phi_{ik,j}$). Also, for any $1 < q < \infty$, we have the regularity estimate

\begin{equation}
\|\Phi(\cdot, t)\|_{W^{1,q}(Y)} \lesssim \|B(\cdot, t)\|_{L^q(Y)},
\end{equation}

where the up to constant is independent of $t$.

Proof. Here we merely treat temporal variable as a parameter. In this regard, once the properties (3.35) had been verified, the existence of the solution $\Phi$ of the equation (3.36) would be established as the same as the stationary case under the antisymmetry condition. As a consequence, it would similarly satisfies the regularity estimate (3.37). Thus, the main job is only left to check the equalities in (3.35). Recalling the formula of $B$ in Proposition 3.1, the equality (i) in (3.35) follows from the divergence-free condition of $W$, while the equality (ii) directly comes from the definition of the effective matrix $A$ (see (1.5)).

The main structure of the proof of Proposition 3.1 is presented by the following flow chart.

![Figure 2: The proof structure of Proposition 3.1](image)

The proof of Proposition 3.1. In terms of the equations (1.6) within a finite time, the existence of the weak solution $(W_j, \pi_j)$, as well as, the energy estimate (3.1) had been well known for a long time (see e.g. [23, Chapter 3]). Based upon the estimate (3.9) stated in Lemma 3.5, we have derived weighted estimates (3.31) in Corollary 3.7, which is in fact one part of (3.2a). The other part of (3.2a) concerns the estimate on the quantity $\|\nabla W_j\|_{L^1(0,T;L^q(Y_j))}$ for $1 < q < \infty$, which has been shown in Corollary 3.8. Then we turn to the estimate (3.2b), and it follows from the improved estimate (3.21) stated in Lemma 3.6, which gives us the estimate on the first term of (3.2b). Once we had the estimate on $\|\partial_t W_j\|_{L^1(0,T;L^q(Y_j))}$, the other two terms of (3.2b) would be derived from the estimate (3.30). In the end, we address the flux corrector. Its existence and antisymmetry properties in (3.3) have been shown in Lemma 3.9. On account of the equation (3.36) and the estimate (3.37), the desired estimates (3.4a) and (3.4b) are reduced to showing the related decay estimates of $\|B(\cdot, t)\|_{L^q(Y_j)}$ and $\|\partial_t B(\cdot, t)\|_{L^q(Y_j)}$. By the definition of $B = \{b_{ij}(\cdot, t)\}$ in Proposition 3.1, it suffices to show the associated decay estimates on $\|\nabla W(\cdot, t)\|_{L^q(Y_j)}$, $\|\nabla W(\cdot, t)\|_{L^q(Y_j)}$, which can be found in Corollary 3.8 and Lemma 3.6, respectively. The conclusion of $\Phi_{ki,j} \in C([0,T];C_{\text{per}}(Y_j))$ follows from the estimate (3.4b) for the case of $q > d$ (see e.g. [8, Theorem 2 in Subsection 5.9.2]). We have completed the whole proof.

\end{document}
4 Boundary-layer correctors

Before proceeding the concrete results of boundary-layer correctors that we have introduced in Subsection 2.1, we further explain the source of $J_1$ and $J_2$ presented in (1.17). Recalling the formula (2.4), if computed $w^{(2)}_\varepsilon$ directly, we would find the following formula:

$$\nabla \cdot w^{(2)}_\varepsilon = \nabla \psi_z \cdot (W^r \ast G + \varepsilon \phi^r \ast_2 \partial G) + \psi_z A \int_{Y_f} \ast_2 \partial G + \varepsilon \psi_z \phi^r \ast_3 \partial^2 G.$$  

By virtue of $\nabla \cdot u_0 = 0$ in $\Omega \times (0, T)$, there is no big problem on co-layer part, while the layer part is problematic. Fortunately having flux corrector, it inspires us to introduce the quantity $\nabla \psi_z \cdot A \ast G$ into the layer part (see Remark 3.2). Therefore, the present form of $J_1$ and $J_2$ in (1.17) is cogent.

We now state the results of boundary-layer correctors:

**Proposition 4.1** (Boundary-layer corrector I). Let $0 < T < \infty$. Given $f \in L^2(0, T; C^{1,1/2}(\bar{\Omega})^d)$, assume the same geometry assumptions on perforated domains as in Theorem 1.2. Let $J_1$ and $J_2$ be given as in (1.17). Then, for a.e. $t \geq 0$, there exists at least one weak solution to

$$\begin{cases}
\nabla \cdot \xi_\varepsilon(\cdot, t) = \frac{\partial J_2}{\partial t} + \sum_i \left( \int_{O^I_\varepsilon} \frac{\partial J_1}{\partial t} \right) 1_{O^I_\varepsilon}, & \text{in } \Omega_\varepsilon; \\
\xi_\varepsilon(\cdot, t) = 0, & \text{on } \partial \Omega_\varepsilon,
\end{cases}  \tag{4.1}$$

where we recall that $1_{O^I_\varepsilon}$ is the indicator function of $O^I_\varepsilon$, and $\{O^I_\varepsilon\}$ is a family of the non-overlap subsets of $O_\varepsilon$, satisfying (2.5). Also, the solution satisfies the following estimate

$$\|\xi_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \xi_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} \lesssim \varepsilon^{1/2} \|f\|_{L^2(0, T; C^{1,1/2}(\bar{\Omega}))},  \tag{4.2}$$

in which the up to constant depends on $d$, $|Y_f|$, $T$, and the characters of $Y_f$ and $\Omega$.

**Proposition 4.2** (Boundary-layer corrector II). Assume the same conditions as in Theorem 1.2. Let $J_1$ be given as in (1.17). Then, for a.e. $t \geq 0$, there exists at least one weak solution to

$$\begin{cases}
\nabla \cdot \eta_\varepsilon(\cdot, t) = \frac{\partial J_1}{\partial t} - \sum_i \left( \int_{O^I_\varepsilon} \frac{\partial J_1}{\partial t} \right) 1_{O^I_\varepsilon}, & \text{in } O_\varepsilon; \\
\eta_\varepsilon(\cdot, t) = 0, & \text{on } \partial O_\varepsilon,
\end{cases}  \tag{4.3}$$

satisfying the following estimate

$$\|\eta_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \eta_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} \lesssim \varepsilon^{1/2} \|f\|_{L^2(0, T; C^{1,1/2}(\bar{\Omega}))},  \tag{4.4}$$

where the up to constant depends on $d$, $|Y_f|$, $T$, and the characters of $Y_f$ and $\Omega$.

**Proposition 4.3** (Boundary-layer corrector III). Suppose $\xi_\varepsilon$ and $\eta_\varepsilon$ are the two solutions of (4.1) and (4.3) given in Propositions 4.1 and 4.2, respectively. Let $\hat{\xi}_\varepsilon$ and $\hat{\eta}_\varepsilon$ be defined as in (2.7)\footnote{Note that $\eta_\varepsilon$ can be trivially zero-extended to the whole region $\Omega$.}: Moreover, there also hold the regularity estimates:

$$\begin{align*}
\|\hat{\xi}_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \hat{\xi}_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} & \lesssim \varepsilon^{1/2} \|f\|_{L^2(0, T; C^{1,1/2}(\bar{\Omega}))};  \\
\|\hat{\eta}_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \hat{\eta}_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} & \lesssim \varepsilon^{1/2} \|f\|_{L^2(0, T; C^{1,1/2}(\bar{\Omega}))},
\end{align*} \tag{4.5a,b}$$

where the multiplicative constant depends on $d$, $|Y_f|$, $T$, and the characters of $Y_f$ and $\Omega$.

It is well known that for the problem: $\nabla \cdot v = g$ in $\Omega$; and $v = 0$ on $\partial \Omega$ with the compatibility condition $\int_{\Omega} f = 0$, there exists at least one solution $v \in H^{1}_0(\Omega)^d$ together with a constant $C$, depending on $\Omega$, such that $\|v\|_{H^{1}_0(\Omega)} \leq C \|f\|_{L^2(\Omega)}$ (see [9, lemma III.3.1]). Therefore, if replaced by a perforated domain, the constant $C$ will additionally depend on the size $\varepsilon$ of holes in the perforated domain, which has been stated below.
Theorem 4.4 (Bogovskii’s operator on perforated domains [6]). Assume the same geometry assumptions on the perforated domain as in Theorem 1.2. Then, for any \( g \in L^2(\Omega_\varepsilon) \) with \( \int_{\Omega_\varepsilon} g = 0 \), there exists a vector-valued function \( v_\varepsilon \in H^1_0(\Omega_\varepsilon)^d \) such that \( \nabla \cdot v_\varepsilon = g \) in \( \Omega_\varepsilon \), satisfying

\[
\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\|g\|_{L^2(\Omega_\varepsilon)},
\]

where the constant \( C \) is independent of \( \varepsilon \) and \( g \).

By virtue of Theorem 4.4, the existence of the solution \( \xi_\varepsilon \) to (4.1) is reduced to verifying compatibility condition, while the desired estimate (4.2) follows from the estimates on the quantities

\[
\left( \int_0^T dt \int_{\Omega_\varepsilon} |\partial J_i^\varepsilon|^2 \right)^{1/2} \quad \text{and} \quad \left( \int_0^T dt \int_{\Omega_\varepsilon} \sum_i \left( \int_{O_i^\varepsilon} |\partial J_i^\varepsilon|^2 1_{O_i^\varepsilon} \right)^2 \right)^{1/2},
\]

which will be addressed in Lemmas 4.7 and 4.8, separately.

The key ideas are summarized as follows:

- By introducing radial cut-off function, together with the special structure of the effective solution (1.4) on the boundary, i.e., \( \vec{n} \cdot u_0 = 0 \) on \( \partial\Omega \times (0, T) \), it is possible to produce a desired smallness near the boundary, simply by using Poincaré’s inequality (see Lemma 4.7), which is a crucial observation for Proposition 4.1.

- By decomposing the boundary layer region, the proof of Proposition 4.2 does not rely on Theorem 4.4. The solution \( \eta_\varepsilon \) to the equation (4.3) consists of “piecewise” solutions of cell problems, according to the decomposed element \( O_i^\varepsilon \). On the other hand, due to \( \Omega_i^\varepsilon \) obtained by cutting the boundary layer region \( O_\varepsilon \) in the normal direction \( \vec{n} \) associated with \( \partial\Omega \), this enables us to consistently apply the antisymmetry property of the flux corrector (see Lemma 4.8), which is also important to the proof of Proposition 4.1.

4.1 Radial cut-off functions

We first describe the important concept of this paper: radial cut-off function, and then we will discuss the decomposition of the boundary layer \( O_\varepsilon \) with details.

Lemma 4.5 (radial cut-off functions). Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^2 \) domain. Then, for any \( 0 < \varepsilon \ll 1 \), there exists a cut-off function \( \psi_\varepsilon \) satisfying the following properties:

- (a) \( \psi_\varepsilon \in C_0^\infty(\Omega) \) with \( \text{supp}(\psi_\varepsilon) = \Sigma_\varepsilon \), \( \psi_\varepsilon = 1 \) on \( \Sigma_{2\varepsilon} \), and \( 0 \leq \psi_\varepsilon \leq 1 \) (where \( \Sigma_\varepsilon \) is defined in Subsection 1.4);

- (b) For any \( x \) with \( \text{dist}(x, \partial\Omega) < 10\varepsilon \), there exists a unique element \( \hat{x} \in \partial\Omega \) such that \( \text{dist}(x, \partial\Omega) = |x - \hat{x}| \) and \( \hat{x} \in \partial\Omega \), and there also holds

\[
\nabla \psi_\varepsilon(x) = -|\nabla \psi_\varepsilon(x)|\hat{n}(\hat{x}) \quad \text{on} \ O_\varepsilon,
\]

where \( O_\varepsilon := \text{supp}(\nabla \psi_\varepsilon) \subset \Omega \setminus \Sigma_2 \varepsilon \); Also, we have \( |\nabla \psi_\varepsilon(x)| = |\nabla \psi_\varepsilon(y)| \), provided \( \text{dist}(x, \partial\Omega) = \text{dist}(y, \partial\Omega) \) for \( x, y \in O_\varepsilon \);

- (c) \( |\nabla \psi_\varepsilon| \lesssim \frac{1}{\varepsilon} \) and \( |\nabla^2 \psi_\varepsilon| \lesssim \frac{1}{\varepsilon^2} \).

Proof. We initially construct the radial function as follows:

\[
\phi_\varepsilon(r) = \begin{cases} 
0, & 0 \leq r \leq (4/3)\varepsilon; \\
3 \varepsilon (r - \frac{4}{3} \varepsilon), & (4/3)\varepsilon < r \leq (5/3)\varepsilon; \\
1, & r > (5/3)\varepsilon.
\end{cases}
\]

Let \( g_\varepsilon := g \ast \phi_\varepsilon \), where the kernel \( \varrho \in C^\infty(\mathbb{R}) \) is the 1-dimensional standard mollifier\(^9\). Thus, we have \( g_\varepsilon \in C^\infty(\mathbb{R}) \), and \( 0 \leq g_\varepsilon \leq 1 \) with \( g_\varepsilon(r) = 0 \) if \( r \leq \varepsilon \); and \( g_\varepsilon(r) = 1 \) if \( r > 2\varepsilon \). Also, we have \( |\frac{d^k g_\varepsilon}{dr^k}| \lesssim 1/\varepsilon^k \) for any positive integer \( k \). Now, we introduce the distance function \( \delta(x) := \text{dist}(x, \partial\Omega) \) and then the desired radial cut-off function can be defined by \( \psi_\varepsilon(x) := g_\varepsilon(\delta(x)) \) on \( \Omega \). It is not hard to see that \( \text{supp}(\psi_\varepsilon) = \Sigma_\varepsilon \), \( \psi_\varepsilon \equiv 1 \) on \( \Sigma_{2\varepsilon} \), and \( 0 \leq \psi_\varepsilon \leq 1 \). Also, we denote the compact support of \( \nabla \psi_\varepsilon \) by \( O_\varepsilon \), which is included in the layer type region \( \Sigma_{2\varepsilon} \setminus \Sigma_\varepsilon \).

\(^9\)The mollifier \( \varrho(r) := C \exp\left\{ -\frac{1}{1-r^2} \right\} \) for \( r < 1 \), and vanishes for \( r \geq 1 \), in which the constant \( C \) is such that \( \int \varrho = 1 \).
Since $\Omega$ is assumed to be a bounded $C^2$ domain and $0 < \varepsilon \ll 1$, for any $x \in O_\varepsilon$ there exists a unique point $\tilde{x} \in \partial \Omega$ such that $\delta(x) = |x - \tilde{x}|$. Thus, we can find that

$$\nabla \psi_\varepsilon(x) = g'_\varepsilon(\delta(x)) \nabla \delta(x) = -g'_\varepsilon(\delta(x)) \vec{n}(\tilde{x}) \tag{4.9}$$

for any $x \in O_\varepsilon$, where $g'_\varepsilon$ represents the derivative of $g_\varepsilon$. This further implies

$$|g'_\varepsilon(\delta(x))| = |\nabla \psi_\varepsilon(x)| \quad \text{and} \quad |\nabla \psi_\varepsilon(x)| = |g'_\varepsilon(\delta(x))| = |g'_\varepsilon(\delta(y))| = |\nabla \psi_\varepsilon(y)|, \tag{4.10}$$

whenever $\delta(x) = \delta(y)$. Combining the equalities (4.9) and (4.10) leads to the stated conclusions in (b). Moreover, it is clear to see $|\nabla \psi_\varepsilon| \lesssim 1/\varepsilon$, and there holds

$$\nabla^2 \psi_\varepsilon = g''_\varepsilon(\delta) \nabla \delta \otimes \nabla \delta \quad \text{in} \quad O_\varepsilon,$$

where $g''_\varepsilon$ represents the second-order derivative of $g_\varepsilon$, and this gives us $|\nabla^2 \psi_\varepsilon(x)| \leq |g''_\varepsilon(\delta(x))| \lesssim 1/\varepsilon^2$. We have completed the proof. $\square$

**Remark 4.6.** As mentioned in Proposition 2.1, $\{O^i_\varepsilon\}$ is a family of the non-overlap subsets of $O_\varepsilon$, such that

$$O_\varepsilon = \bigcup_i O^i_\varepsilon \quad \text{and} \quad |O^i_\varepsilon| \sim \varepsilon^d,$$

in which $O^i_\varepsilon$ is an approximately $d$-dimensional cube obtained by cutting $O_\varepsilon$ in the normal direction $\vec{n}$ associated with $\partial \Omega$ (see Figure 3 for an example). The boundary of $O^i_\varepsilon$ consists of two parts:

(a). $\{R^i_j\} := \partial O^i_\varepsilon \cap \partial O_\varepsilon$. Let $\vec{n}_R$ be the unit normal vector of $R^i_j$, satisfying $\vec{n}_R \perp R^i_j$. Then, there holds

$$\vec{n}_R \parallel \vec{n} \quad \text{on} \quad R^i_j.$$

(b). $\{S^i_j\} := \partial O^i_\varepsilon \cap O_\varepsilon$. Let $\vec{n}_S$ be the unit normal vector of $S^i_j$, satisfying $\vec{n}_S \perp S^i_j$. Due to $S^i_j$ obtained by the cutting in the normal direction $\vec{n}$, we have the following relationship:

$$\vec{n}_S \perp \vec{n} \quad \text{on} \quad S^i_j. \tag{4.11}$$

### 4.2 Proof of Proposition 4.1

**Lemma 4.7** (estimates for $J_2$). Let $\psi_\varepsilon$ be the radial cut-off function defined in Lemma 4.5, and $J_2$ be given as in (1.17). Assume the same conditions as in Proposition 4.1. Then, the first term in the right-hand side of (4.1)
By virtue of the equality (4.8), together with the structure of the effective solution, we can find a cancellation
Proof.
where the multiplicative constant depends on $d$, $|Y_j|$, and the character of $\Omega$, but independent of $\varepsilon$.

For the sake of the convenience, we recall the expression of near the boundary with respect to outward normal direction, which further provides us with the desired smallness.

To handle the first term in the right-hand side above, recalling the effective equations (1.4), we have
\begin{align*}
J_2 = \nabla \psi_\varepsilon \cdot (A \ast G) + \varepsilon \nabla \psi_\varepsilon \cdot (\phi_\varepsilon \ast \partial G) &+ \psi_\varepsilon A_F + \varepsilon \psi_\varepsilon \phi_\varepsilon \ast \partial^2 G \\
&:= J_{21} + J_{22} + J_{23} + J_{24}.
\end{align*}

We complete the whole arguments by three steps, according to the similarity of the computations.

**Step 1.** We start from the term $\partial J_{21} / \partial t$, and it follows from the equality (4.8) that
\begin{align*}
\frac{\partial J_{21}}{\partial t} &= \nabla \psi_\varepsilon \cdot \frac{\partial}{\partial t} (A \ast G) = -|\nabla \psi_\varepsilon| \bar{n} \cdot \frac{\partial}{\partial t} (A \ast G) \\
&= -|\nabla \psi_\varepsilon| \bar{n} \cdot \frac{\partial}{\partial t} [A \ast (G - F)] - |\nabla \psi_\varepsilon| \bar{n} \cdot \frac{\partial}{\partial t} (A \ast F).
\end{align*}

By noting the fact that $|\nabla \psi_\varepsilon| \lesssim \varepsilon^{-1}$ (see Lemma 4.5), there holds
\begin{equation}
\left( \int_0^T dt \int_{\Omega_x} \frac{|\partial J_{21}|}{\partial t} \right)^2 \lesssim \varepsilon^{-2} \int_0^T dt \int_{\Omega_x} |\bar{n} \cdot \frac{\partial}{\partial t} (A \ast F)|^2 + \varepsilon^{-2} \int_0^T dt \int_{\Omega_x} |\bar{n} \cdot \frac{\partial}{\partial t} [A \ast (G - F)]|^2.
\end{equation}

To handle the first term in the right-hand side above, recalling the effective equations (1.4), we have $u_0 = A \ast F$ with $\bar{n} \cdot u_0 = 0$ on the boundary $\partial \Omega \times (0, T)$, whereupon for any $x \in O_\varepsilon$ and $t \geq 0$ there holds
\begin{equation}
\bar{n} \cdot \frac{\partial}{\partial t} (A \ast F)(x, t) = \bar{n} \cdot \int_0^1 ds \nabla \frac{\partial}{\partial t} (A \ast F)(\tilde{x} + s(x - \tilde{x}), t) \cdot (x - \tilde{x}),
\end{equation}

where $\tilde{x} \in \partial \Omega$ is such that $\mathrm{dist}(x, \partial \Omega) = |x - \tilde{x}|$, which further implies
\begin{equation}
\int_{\Omega_x} \left| \bar{n} \cdot \frac{\partial}{\partial t} (A \ast F)(\cdot, t) \right|^2 \lesssim \varepsilon \int_{\Omega_x} \int_0^1 ds \left| \nabla \frac{\partial}{\partial t} (A \ast F)(\tilde{x} + s(\cdot - \tilde{x}), t) \right|^2 \lesssim \varepsilon \int_{\Omega \setminus \Sigma_{2\varepsilon}} |\nabla \frac{\partial}{\partial t} (A \ast F)(\cdot, t)|^2.
\end{equation}

Plugging this back into (4.15), and using Young’s inequality, we obtain that
\begin{align*}
\int_0^T dt \int_{\Omega_x} \left| \frac{\partial J_{21}}{\partial t} \right|^2 &\lesssim \int_0^T dt \int_{\Omega} |\nabla \frac{\partial}{\partial t} (A \ast F)|^2 + \varepsilon^{-2} \int_0^T dt \int_{\Omega_x} ((\partial_t A) \ast (G - F))^2 + |A(0)(G - F)|^2 \\
&\lesssim \int_0^T dt \int_{\Omega \setminus \Sigma_{2\varepsilon}} ((\partial_t A) \ast \nabla F)^2 + |A(0)\nabla F|^2 + \varepsilon^{-2} \int_0^T dt \int_{\Omega_x} ((\partial_t A) \ast (G - F))^2 + |A(0)(G - F)|^2 \\
&\lesssim \left( \|\partial_t A\|_{L^1(0,T)} + |A(0)|^2 \right) \left\{ \varepsilon \|\nabla F\|^2_{L^2(0,T;C(\bar{\Omega}))} + \varepsilon^{-1} \|G - F\|^2_{L^2(0,T;L^2(\supp(\psi_\varepsilon)))} \right\}.
\end{align*}

This, together with the estimate (Recall the notations $G, F$ in Subsection 1.4.)
\begin{equation}
\|G(\cdot, t) - F(\cdot, t)\|_{L^\infty(\supp(\psi_\varepsilon))} \lesssim \varepsilon \|\nabla F(\cdot, t)\|_{C(\bar{\Omega})},
\end{equation}
immediately yields
\begin{equation}
\int_0^T dt \int_{\Omega_x} \left| \frac{\partial J_{21}}{\partial t} \right|^2 \lesssim \varepsilon \left( \|\partial_t A\|_{L^1(0,T)} + |A(0)|^2 \right) \|\nabla F\|^2_{L^2(0,T;C(\bar{\Omega}))}.
\end{equation}
Step 2. We turn to study the term $J_{23}$. Appealing to the fact that $\nabla \cdot u_0 = A \ast_2 \partial F = 0$ in $\Omega \times (0, T)$, there holds

$$
\int_0^T dt \int_{\Omega_\varepsilon} \left| \frac{\partial J_{23}}{\partial t} \right|^2 = \int_0^T dt \int_{\Omega_\varepsilon} \left| \psi_x \frac{\partial}{\partial t} (A \ast_2 \partial G) \right|^2 = \int_0^T dt \int_{\Omega_\varepsilon} \left| \psi_x \frac{\partial}{\partial t} [A \ast_2 \partial (G - F)] \right|^2
$$

$$
= \int_0^T dt \int_{\Omega_\varepsilon} \left| \psi_x \frac{\partial}{\partial t} \left( [\partial_t A] \ast_2 \partial (G - F) \right) + A(0) : \partial (G - F) \right|^2.
$$

By using Minkowski’s inequality and Young’s inequality, the above equality gives us

$$
\int_0^T dt \int_{\Omega_\varepsilon} \left| \frac{\partial J_{23}}{\partial t} \right|^2 \lesssim \left( \|\partial_t A\|_{L^2(0,T; L^2(\Omega_\varepsilon))}^2 + |A(0)|^2 \right) \|\nabla (G - F)\|_{L^2(0,T; L^2(\supp(\psi_\varepsilon)))}^2.
$$

(4.17)

Step 3. Due to the analogous computations, we consider the terms $J_{22}$ and $J_{24}$ together, and start from the following estimate:

$$
\int_0^T dt \int_{\Omega_\varepsilon} \left| \frac{\partial J_{22}}{\partial t} + \frac{\partial J_{24}}{\partial t} \right|^2 \lesssim \int_0^T dt \int_{\Omega_\varepsilon} \left| \varepsilon \nabla \psi_x (x) \cdot \frac{\partial}{\partial t} (\phi_x \ast_2 \partial G) \right|^2 + \int_0^T dt \int_{\Omega_\varepsilon} \left| \varepsilon \psi_x \frac{\partial}{\partial t} (\phi_x \ast_3 \partial^2 G) \right|^2
$$

$$
\lesssim \int_0^T dt \int_{\Omega_\varepsilon} \left( \|\partial_t \phi_x\|_{L^2(\Omega_\varepsilon)}^2 + |\phi_x(\cdot, 0) : \partial G|^2 \right)
$$

$$
+ \varepsilon^2 \int_0^T dt \int_{\supp(\psi_\varepsilon)} \left( \|\partial_t \phi_x\|_{L^2(\Omega_\varepsilon)}^2 + |\phi_x(\cdot, 0)|^2 \right)
$$

Using Minkowski’s inequality and Young’s inequality, we have

$$
\int_0^T dt \int_{\Omega_\varepsilon} \left( \|\partial_t \phi_x\|_{L^2(\Omega_\varepsilon)}^2 + |\phi_x(\cdot, 0) : \partial G|^2 \right)
$$

$$
\lesssim \varepsilon \left( \|\partial_t \phi\|_{L^2(0,T; L^2(\Omega_{\varepsilon}(Y)))}^2 + \|\phi(\cdot, 0)\|_{L^2(\Omega_{\varepsilon}(Y))}^2 \right) \|\nabla G\|_{L^2(0,T; L^\infty(\supp(\psi_\varepsilon)))}^2,
$$

and it similarly follows that

$$
\int_0^T dt \int_{\supp(\psi_\varepsilon)} \left( \|\partial_t \phi_x\|_{L^2(\Omega_\varepsilon)}^2 + |\phi_x(\cdot, 0)|^2 \right) \|\nabla^2 G\|_{L^2(0,T; L^\infty(\supp(\psi_\varepsilon)))}^2.
$$

Thus, combining the three estimates above, we derive that

$$
\int_0^T dt \int_{\Omega_\varepsilon} \left| \frac{\partial J_{22}}{\partial t} + \frac{\partial J_{24}}{\partial t} \right|^2
$$

$$
\lesssim \varepsilon \left( \|\partial_t \phi\|_{L^2(0,T; L^2(\Omega_{\varepsilon}(Y)))}^2 + \|\phi(\cdot, 0)\|_{L^2(\Omega_{\varepsilon}(Y))}^2 \right) \|\nabla G\|_{L^2(0,T; L^\infty(\supp(\psi_\varepsilon)))}^2.
$$

(4.18)

Consequently, the desired estimate (4.12) follows from the estimates (4.13), (4.16), (4.17), and (4.18). This ends the proof.

Lemma 4.8 (estimates for $J_1$). Let $J_1$ be given as in (1.17). Assume the same conditions as in Proposition 4.1. Then, there holds the estimate

$$
\left( \int_0^T dt \int_{\Omega_\varepsilon} \left| \sum_i \left( \int_{\Omega_\varepsilon} \frac{\partial J_i}{\partial t} \right) 1_{\Omega_\varepsilon} \right|^2 \right)^{1/2}
$$

$$
\lesssim \varepsilon^2 \left( \|\partial_t \Phi\|_{L^2(0,T; L^\infty(\Omega_{\varepsilon}(Y)))} + \|\Phi(\cdot, 0)\|_{L^\infty(\Omega_{\varepsilon})} \right) \|\nabla G\|_{L^2(0,T; L^\infty(\Omega_{\varepsilon}))},
$$

where the multiplicative constant depends on $d$ and the character of $\Omega$, but independent of $\varepsilon$.
Proof. Step 1. Reduction. For a.e. $\tau \geq 0$, we manage to establish the following uniformly bounded estimate with respect to the index $i$, i.e.,

$$\left| \int_{O^i_t} \frac{\partial J_1}{\partial t}(\cdot,t) \right| \lesssim \int_{0}^{\tau} ds \| \partial_t \Phi(\cdot,t-s) \|_{L^\infty(Y)} \| \nabla G(\cdot,s) \|_{L^\infty(O_s)} + \| \Phi(\cdot,0) \|_{L^\infty(Y)} \| \nabla G(\cdot,t) \|_{L^\infty(O_s)}. \tag{4.20}$$

Then, it is reduced to estimating the left-hand side of (4.19) below

$$\int_{0}^{\tau} dt \int_{O^i_t} \left| \sum_i \left( \int_{O^i_t} \frac{\partial J_1}{\partial t} 1_{O_i} \right)^2 \right| \lesssim \varepsilon \int_{0}^{\tau} dt |Q(t)|^2 \int_{O^i_t} \left| \sum_i 1_{O_i} \right|^2 \lesssim \varepsilon \int_{0}^{\tau} dt |Q(t)|^2 \lesssim \varepsilon \left\{ \| \partial_t \Phi \|_{L^1(0,T;L^\infty(Y))} \| \nabla G \|_{L^2(0,T;L^\infty(O_s))} + \| \Phi(\cdot,0) \|_{L^\infty(Y)} \| \nabla G \|_{L^2(0,T;L^\infty(O_s))} \right\},$$

where we employ Young’s inequality for the second line. This gives the desired estimate (4.19).

Step 2. Arguments for (4.20). On account of the antisymmetric property of flux corrector $\Phi$ in Proposition 3.1, integrating by parts we obtain that, for any $t \in (0,T)$,

$$\int_{O^i_t} \frac{\partial J_1}{\partial t}(\cdot,t) = \int_{O^i_t} \frac{\partial}{\partial t} \left[ (W^\varepsilon - A) \ast G \right](\cdot,t) \cdot \nabla \psi \varepsilon = \int_{O^i_t} \frac{\partial}{\partial t} \left[ (\nabla \cdot \Phi) \varepsilon \ast G \right](\cdot,t) \cdot \nabla \psi \varepsilon = \int_{O^i_t} \frac{\partial}{\partial t} \left[ \Phi \varepsilon \ast G \right](\cdot,t) \cdot \nabla \psi \varepsilon + \varepsilon \int_{\partial O^i_t} dS \frac{\partial}{\partial t} [\Phi \varepsilon \ast G](\cdot,t) \cdot \nabla \psi \varepsilon \varepsilon$$

As follows:

$$E_1(t) + E_2(t),$$

in which $\mathbf{n}_s$ is the unit outward normal vector of the boundary $\partial O^i_t$.

In fact, we can establish the following estimates:

$$|E_1(t)| \lesssim |O^i_t| Q(t); \tag{4.22a}$$

$$E_2(t) = 0 \quad (\text{uniformly with respect to the index } i \text{ and } t \in (0,T]). \tag{4.22b}$$

Plugging the above two estimates back into (4.21), we obtain the stated estimate (4.20).

Step 3. Arguments for $E_1(t)$ in (4.22a). Recalling the properties of the cut-off function $\psi \varepsilon$ stated in Lemma 4.5, a direct computation leads to

$$|E_1(t)| \lesssim \int_{O^i_t} \left| \frac{\partial}{\partial t} [\Phi \varepsilon \ast G](\cdot,t) \right| = \int_{O^i_t} \left| (\partial_t \Phi \varepsilon \ast G)(\cdot,t) + \Phi(\cdot,0) : \partial G(\cdot,t) \right| \lesssim \int_{O^i_t} \left| \partial_t \Phi(\cdot,t) \right| \| \partial G(\cdot,t) \|_{L^\infty(O_s)} + |O^i_t| \| \Phi(\cdot,0) \|_{L^\infty(Y)} \| \nabla G(\cdot,t) \|_{L^\infty(O_s)}.$$

This together with the estimate

$$\int_{O^i_t} \left| \partial_t \Phi \varepsilon \ast G(\cdot,t) \right| \lesssim \int_{0}^{\tau} dt \int_{O^i_t} \left| \partial_t \Phi \varepsilon(\cdot,t) : \partial G(\cdot,t-s) \right| \lesssim |O^i_t| \int_{0}^{\tau} dt \| \partial_t \Phi(\cdot,t-s) \|_{L^\infty(Y)} \| \nabla G(\cdot,s) \|_{L^\infty(O_s)}$$

leads to the stated estimate (4.22a), where we recall the definition of the notation presented in (1.18).

Step 4. Arguments for $E_2(t)$ in (4.22b). According to Remark 4.6, we rewrite $E_2(t)$ as follows:

$$E_2(t) = \frac{\partial}{\partial t} \int_{\partial O^i_t \cap \partial O_s} dS \left[ \Phi \varepsilon \ast G \ast (\varepsilon \partial \psi \varepsilon) \otimes \mathbf{n}_R \right](\cdot,t) + \frac{\partial}{\partial t} \int_{\partial O^i_t \cap \partial O_s} dS \left[ \Phi \varepsilon \ast G \ast (\varepsilon \partial \psi \varepsilon) \otimes \mathbf{n}_S \right](\cdot,t). \tag{4.23}$$

On the one hand, it follows from the property (a) presented in Remark 4.6 that

$$(\varepsilon \partial \psi \varepsilon) \otimes \mathbf{n}_R = \mathbf{n}_R \otimes (\varepsilon \partial \psi \varepsilon) \quad \text{on } \partial O^i_t \cap \partial O_s,$$
and this together with the antisymmetric property of $\Phi$ leads to

$$
\frac{\partial}{\partial t} \int_{\partial O_L \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z) \otimes \bar{n}_R \right) \right](\cdot, t) = 0.
$$

(4.24)

On the other hand, using the antisymmetric property again, we first notice that

$$
\left[ \Phi^* \ast_3 \left( G \otimes \bar{n}_S \otimes \bar{n}_S \right) \right](\cdot, t) = \int_0^t ds \Phi_{ij,k}(\cdot, t-s) \left( G_k \bar{n}^i_S \bar{n}^j_S \right)(\cdot, s) = 0 \quad \text{on} \quad \partial O_L^i \cap O_e,
$$

where we employ the fact that $\bar{n}_S \otimes \bar{n}_S$ is a symmetric matrix. Therefore, for any $N > 0$, it follows that

$$
E_2(t) \overset{(4.24)}{=} \frac{\partial}{\partial t} \int_{\partial O_L^i \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z) \otimes \bar{n}_S \right) \right](\cdot, t)
$$

$$
= \frac{\partial}{\partial t} \int_{\partial O_L^i \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S \right) \right](\cdot, t) - N \varepsilon \frac{\partial}{\partial t} \int_{\partial O_L^i \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes \bar{n}_S \otimes \bar{n}_S \right) \right](\cdot, t)
$$

$$
= \frac{\partial}{\partial t} \int_{\partial O_L^i \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S \right) \right](\cdot, t).
$$

Then, taking $N \to \infty$, if the right-hand side above vanishes, one can obtain the desired equality (4.22b). This is true and we show it in the following.

On the one hand, we can start from the following observation

$$
\lim_{N \to \infty} \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S \right) \right](\cdot, t) = 0 \quad \text{on} \quad \partial O_L^i \cap O_e,
$$

(4.25)

which comes from the fact that $\varepsilon \nabla \psi_z + N \bar{n}_S$ becomes linearly dependent on $\bar{n}_S$, i.e.,

$$
\cos \theta = \frac{(\varepsilon \nabla \psi_z + N \bar{n}_S) \cdot \bar{n}_S}{\sqrt{N^2 + c}} = \frac{N}{\sqrt{N^2 + c}} \to 1 \quad \text{as} \quad N \to \infty,
$$

where we employ the geometry fact (4.11) by noting that $\varepsilon \nabla \psi_z \sim \bar{n}$. In other words, we have

$$
(\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S - \bar{n}_S \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \to 0, \quad \text{as} \quad N \to \infty.
$$

and this together with the antisymmetric property of $\Phi$ gives (4.25).

One the other hand, for any fixed $i$ and $\varepsilon$, $E_2(t)$ is uniform bounded with respect to any $t \in [0, T]$ and $N > 0$. Therefore, we have

$$
\lim_{N \to \infty} \frac{\partial}{\partial t} \int_{\partial O_L^i \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S \right) \right](\cdot, t)
$$

$$
= \frac{\partial}{\partial t} \left( \lim_{N \to \infty} \int_{\partial O_L^i \cap \partial O_e} dS \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S \right) \right](\cdot, t) \right)
$$

$$
= \frac{\partial}{\partial t} \int_{\partial O_L^i \cap \partial O_e} dS \left( \lim_{N \to \infty} \left[ \Phi^* \ast_3 \left( G \otimes (\varepsilon \nabla \psi_z + N \bar{n}_S) \otimes \bar{n}_S \right) \right](\cdot, t) \right) \overset{(4.25)}{=} 0,
$$

where we employed Lebesgue’s dominated convergence theorem for the second equality. This gives (4.22b).

Finally, we note that the fact that (4.22b) holds uniformly with respect to $i$ and $t \in (0, T)$, is based on consistent geometric features (i.e., $\nabla \psi_z \cdot n_S = 0$ on $\partial O_L^i \cap O_e$ for each $i$ due to (4.11)) and the same algebraic principles (i.e., “the second-order inner product of an antisymmetric matrix and a symmetric matrix is equal to zero.”). This completes all the proofs.

\begin{remark}
In the previous work [34], to estimate $E_2(t)$, we mainly relied on the periodicity of the flux corrector. Therefore, we made a more detailed division for $O_e$. However, this method becomes very complicated when addressing the decomposition of the three-dimensional boundary layer. Another obvious drawback is that it can only indicate the existence of the required decomposition for specific regions in $\mathbb{R}^3$. As given in the above proof, we now turn to make more in-depth use of the antisymmetry property of the flux corrector. This improvement greatly simplifies the decomposition process for boundary layers (see Remark 4.6).
\end{remark}
Lemma 4.10 (existences). There exists at least one weak solution for (4.1) and (4.3), respectively.

Proof. The existence of weak solution for (4.1) and (4.3) follows from compatibility condition for divergence operator and it is obvious for (4.3). Recalling the definition of $J_1$ and $J_2$ in (1.17), and the fact that

$$
\nabla \cdot \left\{ \psi_\varepsilon [W^\varepsilon * G + \varepsilon \phi_{k,j} * \nabla k G_j] \right\} = J_1 + J_2,
$$

it follows that, for a.e. $t \geq 0$,

$$
\int_{\Omega_\varepsilon} (J_1 + J_2)(\cdot,t) = \int_{\Omega_\varepsilon} \nabla \cdot \left\{ \psi_\varepsilon [W^\varepsilon * G + \varepsilon \phi_{k,j} * \nabla G_j] \right\}(\cdot,t) = \int_{\partial \Omega_\varepsilon} dS \vec{n} \cdot \left\{ \psi_\varepsilon [W^\varepsilon * G + \varepsilon \phi_k * \nabla G] \right\}(\cdot,t) = 0,
$$

where, by abusing the notation, $\vec{n}$ represents the unit outward normal vector of $\partial \Omega_\varepsilon$. Therefore, due to the fact that $\text{supp}(J_1) = O_\varepsilon = \sum_i O_{i\varepsilon}$ (see Remark 4.6), there holds

$$
\int_{\Omega_\varepsilon} \left[ J_2 + \sum_i (\int_{O_{i\varepsilon}} J_1) 1_{O_{i\varepsilon}} \right](\cdot,t) = \int_{\Omega_\varepsilon} (J_1 + J_2)(\cdot,t) = 0.
$$

Consequently, we have

$$
\int_{\Omega_\varepsilon} \left[ \frac{\partial J_2}{\partial t} + \sum_i (\int_{O_{i\varepsilon}} \frac{\partial J_1}{\partial t}) 1_{O_{i\varepsilon}} \right](\cdot,t) = \int_{\Omega_\varepsilon} \frac{\partial}{\partial t}(J_1 + J_2)(\cdot,t) = 0.
$$

This is the compatibility condition for (4.1).

Consequently, the structure of the proof of Proposition 4.1 can be presented by the following flow chart.

The proof of Proposition 4.1. The existence of the solution of (4.1) has been established in Lemma 4.10, while the main job is to show the estimate (4.2). To do so, applying the result presented in Theorem 4.4 to the solution of the equations (4.1), we have

$$
\| \xi_\varepsilon \|_{L^2(\Omega_{\nu}, T)} \lesssim \varepsilon \| \nabla \xi_\varepsilon \|_{L^2(\Omega_{\nu}, T)} \quad \text{(4.6)} \lesssim \left\| \frac{\partial}{\partial t}(J_2 + \sum_i (\int_{O_{i\varepsilon}} J_1) 1_{O_{i\varepsilon}}) \right\|_{L^2(\Omega_{\nu}, T)}
$$

$$
\leq \left\| \frac{\partial}{\partial t} J_2 \right\|_{L^2(\Omega_{\nu}, T)} + \left\| \sum_i (\int_{O_{i\varepsilon}} \frac{\partial}{\partial t} J_1) 1_{O_{i\varepsilon}} \right\|_{L^2(\Omega_{\nu}, T)}.
$$

In view of Lemmas 4.7 and 4.8, the right-hand side above can be controlled by

$$
\varepsilon \frac{1}{2} \left\{ \| \partial_0 \phi \|_{L^1(0,T;L^2(Y))} + \| \phi(\cdot, 0) \|_{L^2(Y)} \right\} \| (\nabla G, \varepsilon \frac{1}{2} \nabla^2 G) \|_{L^2(0,T;L^\infty(\Omega_\nu))}
$$

$$
+ \varepsilon \frac{1}{2} \left\{ \| \partial_0 \Phi \|_{L^1(0,T;L^\infty(Y))} + \| \Phi(\cdot, 0) \|_{L^\infty(Y)} \right\} \| \nabla G \|_{L^2(0,T;L^\infty(\Omega_\nu))}
$$

$$
+ \left\{ \| \partial_0 A \|_{L^1(0,T)} + \| A(0) \| \right\} \left\{ \varepsilon \frac{1}{2} \| \nabla F \|_{L^2(0,T;L^2(\Omega_\nu))} + \| \nabla (G - F) \|_{L^2(0,T;L^2(\text{supp}(\psi_\varepsilon)))} \right\}.
$$

Applying Propositions 3.1, 3.3, and 1.4 to the above expression, we can further derive that

$$
\varepsilon \| \nabla \xi_\varepsilon \|_{L^2(\Omega_{\nu}, T)} \lesssim \varepsilon \frac{1}{2} \left\{ \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} + \| G \|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} \right\}
$$

$$
+ \varepsilon \frac{1}{2} \left\{ \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} + \varepsilon \frac{1}{2} \| \nabla^2 G \|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} \right\}
$$

$$
+ \varepsilon \frac{1}{2} \| \nabla F \|_{L^2(0,T;L^2(\Omega_\nu))} + \| \nabla (G - F) \|_{L^2(0,T;L^2(\text{supp}(\psi_\varepsilon)))}.
$$

By using Lemma 5.3, the right-hand side of (4.26) can be governed by $\varepsilon \frac{1}{2} \| f \|_{L^2(0,T;C^1,1/2(\Omega_\nu))}$, and the desired estimate (4.2) follows. This ends the whole proof. \hfill \Box
We construct $\eta_\varepsilon$ according to the decomposition introduced in Subsection 4.1. By virtue of Remark 4.6, we have the decomposition of $O_\varepsilon$. For each $O_\varepsilon^i$ and any $t \geq 0$ fixed, we can get a $\eta_\varepsilon^i$ which satisfies the following equation (4.27). The desired solution $\eta_\varepsilon$ follows from sticking these $\eta_\varepsilon^i$ together piece by piece.

\[ \begin{cases} \nabla \cdot \eta_\varepsilon^i(\cdot, t) = \frac{\partial J_1}{\partial t} - \left( \int_{O_\varepsilon^i} \frac{\partial J_1}{\partial t} \right) 1_{O_\varepsilon^i}, & \text{in } O_\varepsilon^i; \\ \eta_\varepsilon^i(\cdot, t) = 0, & \text{on } \partial O_\varepsilon^i, \end{cases} \tag{4.27} \]

Moreover, we have the following estimate

\[ \| \nabla \eta_\varepsilon^i(\cdot, t) \|_{L^2(O_\varepsilon)} \leq C \| \frac{\partial J_1}{\partial t} - \left( \int_{O_\varepsilon} \frac{\partial J_1}{\partial t} \right) 1_{O_\varepsilon} \|_{L^2(O_\varepsilon)} \leq C \| \partial_t J_1 \|_{L^2(O_\varepsilon)}, \tag{4.28} \]

where the constant $C$ does not depend on $\varepsilon$ and $i$ (see e.g. [9, Chapter III.3]).

Let $\eta_\varepsilon := \sum_i \eta_\varepsilon^i$, and it is not hard to observe that

\[ \| \nabla \eta_\varepsilon(\cdot, t) \|_{L^2(\Omega_\varepsilon)}^2 = \sum_i \| \nabla \eta_\varepsilon^i(\cdot, t) \|_{L^2(O_\varepsilon)}^2. \]

This together with (4.28) leads to

\[ \| \nabla \eta_\varepsilon(\cdot, t) \|_{L^2(\Omega_\varepsilon)}^2 \lesssim \| \partial_t J_1 \|_{L^2(\Omega_\varepsilon)}^2. \tag{4.29} \]

Now, dealing with the term in the right-hand side of (4.29), it follows from the definition of $J_1$ in (1.17) and Minkowski’s inequality that

\[ \int_0^T \int_{\Omega_\varepsilon} \left[ \frac{\partial J_1}{\partial t} \right]^2 \lesssim \varepsilon^{-2} \int_0^T \int_{\Omega_\varepsilon} \left[ (W^{\varepsilon} - A) * G \right]^2 \leq \varepsilon^{-2} \| \partial_t W^{\varepsilon} - \partial_t A \|_{L^1(0,T;L^2(\Omega_\varepsilon))}^2 + \varepsilon^{-2} \| W^{\varepsilon}(\cdot, 0) - A(0) \|_{L^2(\Omega_\varepsilon)}^2 \| G \|_{L^2(0,T;L^\infty(\Omega_\varepsilon))}^2. \tag{4.30} \]

By a rescaling argument used for $\partial_t W^{\varepsilon}$ and using its periodicity, we note that

\[ \| \partial_t W^{\varepsilon} - \partial_t A \|_{L^1(0,T;L^2(\Omega_\varepsilon))} = \int_0^T \int_{\Omega_\varepsilon} \left| \partial_t W(x/\varepsilon, t) - \partial_t A(t) \right|^2 \frac{1}{\varepsilon} \leq \varepsilon^{1/2} \left( \int_0^T \int_{\Omega_\varepsilon} \left| \partial_t A(t) \right|^2 \frac{1}{\varepsilon} \right)^{1/2} \]

\[ = \varepsilon^{1/2} \left\{ \| \partial_t A \|_{L^1(0,T)} + \| \partial_t W \|_{L^1(0,T;L^2(\Omega_\varepsilon))} \right\}; \]

By the same token, we have

\[ \| W^{\varepsilon}(\cdot, 0) - A(0) \|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{1/2}. \]

Inserting the above two estimates back into (4.30), and then together with (4.29), we obtain that

\[ \varepsilon^2 \int_0^T \int_{\Omega_\varepsilon} \| \nabla \eta_\varepsilon(\cdot, t) \|^2 \lesssim \varepsilon \left\{ \| \partial_t A \|_{L^1(0,T)} + \| \partial_t W \|_{L^1(0,T;L^2(\Omega_\varepsilon))} + 1 \right\} \| G \|_{L^2(0,T;L^\infty(\Omega_\varepsilon))}^2. \]

Consequently, appealing to Proposition 3.1 and Lemma 6.2, we have derived the main part of the stated estimate (4.4), and the remainder of the proof follows from Poincaré’s inequality. This completes the whole proof. \]

\[ \square \]

## 5 Proof of Theorem 1.2

**Lemma 5.1.** Let $p_0$ and $f$ be associated by (1.4). Suppose that $(u_\varepsilon, p_\varepsilon)$ satisfies (1.2). Let $(W, \pi)$ be the corrector given in (1.6), while the correctors $(\phi, \xi, \eta_\varepsilon)$ are related to (2.3) and (2.7). Define the error term $(w_\varepsilon, q_\varepsilon)$ as in (2.8). Then, the pair $(w_\varepsilon, q_\varepsilon)$ satisfies the following equations:

\[ \begin{cases} \partial_t w_\varepsilon - \varepsilon^2 \Delta w_\varepsilon + \nabla q_\varepsilon = I_1 + \varepsilon I_2 + \varepsilon^2 I_3 + \varepsilon^3 I_4, & \text{in } \Omega_\varepsilon \times (0, T]; \\ \nabla \cdot w_\varepsilon = 0, & \text{in } \Omega_\varepsilon \times (0, T]; \\ w_\varepsilon = 0, & \text{on } \partial \Omega_\varepsilon \times (0, T]; \\ w_\varepsilon|_{t=0} = 0, & \text{on } \Omega_\varepsilon, \end{cases} \tag{5.1} \]
in which we adopt the convention presented in (1.19) to have the expressions of $I_1, I_2, I_3,$ and $I_4$ by

$$I_1 := -\nabla p_0 - \psi_\varepsilon W^\varepsilon (\cdot, 0) G + \xi_\varepsilon + \eta_\varepsilon;$$

$$I_2 := -\psi_\varepsilon \left\{ \frac{\partial \phi^\varepsilon * 2 G}{\partial G} + 2 \psi_\varepsilon (\partial W)^\varepsilon * 2 \partial G - \nabla \psi_\varepsilon \pi^\varepsilon * 1 G - \psi_\varepsilon \nabla G * \pi^\varepsilon;$$

$$I_3 := \psi_\varepsilon \nabla \cdot \left\{ (\nabla \phi)^\varepsilon * 2 \partial G \right\} + \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (W^\varepsilon * G) \right\} + \nabla (W^\varepsilon * G) \nabla \psi_\varepsilon + \psi_\varepsilon W^\varepsilon * \Delta G - \Delta \hat{\xi}_\varepsilon - \Delta \hat{\eta}_\varepsilon;$$

$$I_4 := \nabla (\phi^\varepsilon * 2 \partial G) \nabla \psi_\varepsilon + \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (\phi^\varepsilon * 2 \partial G) \right\} + \psi_\varepsilon \nabla \cdot \left\{ \phi^\varepsilon * 2 \nabla \partial G \right\},$$

where $G$ is associated with the quantity $f - \nabla p_0$, defined in Subsection 1.4.

Proof. To obtain the equations (5.1), we merely insert the expression (2.8) into the left-hand sides of (5.1), and compute it term by term, directly. By noticing the boundary conditions of $u_\varepsilon, \psi_\varepsilon W^\varepsilon, \psi_\varepsilon \phi^\varepsilon, \xi_\varepsilon$ and $\eta_\varepsilon$, it is not hard to verify that $w_\varepsilon(\cdot, t)$ vanishes on the boundary of $\Omega_\varepsilon$ for $t \geq 0$, which satisfies the third line of (5.1). From the initial value of $u_\varepsilon$, the definition of the convolution with respect to the temporal variable, as well as $\xi_\varepsilon$ and $\eta_\varepsilon$, the last line of (5.1) can be simply checked. Therefore, the main job is devoted to deriving the first and second line of (5.1).

Part 1. We firstly address the first line of the equations (5.1), and start from dealing with the term $\partial_t w_\varepsilon$, by noting the notation presented in the first and third line of (1.19),

$$\frac{\partial w_\varepsilon}{\partial t} = \frac{\partial u_\varepsilon}{\partial t} - \psi_\varepsilon \frac{\partial}{\partial t} (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G)(\cdot, t) + \xi_\varepsilon(\cdot, t) + \eta_\varepsilon(\cdot, t).$$

Then, we turn to the term $\varepsilon^2 \Delta w_\varepsilon$,

$$- \varepsilon^2 \Delta w_\varepsilon = -\varepsilon^2 \Delta u_\varepsilon + \varepsilon^2 \Delta \left\{ \psi_\varepsilon (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G) \right\} - \varepsilon^2 \Delta \hat{\xi}_\varepsilon - \varepsilon^2 \Delta \hat{\eta}_\varepsilon$$

$$= -\varepsilon^2 \Delta u_\varepsilon - \varepsilon^2 \Delta \hat{\xi}_\varepsilon - \varepsilon^2 \Delta \hat{\eta}_\varepsilon + \varepsilon^2 \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G) + \psi_\varepsilon \nabla (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G) \right\},$$

while the last term above leads to three terms:

$$\varepsilon^2 \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G) \right\} + \varepsilon^2 \nabla (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G) \nabla \psi_\varepsilon + \varepsilon^2 \Delta (W^\varepsilon * G + \varepsilon \phi^\varepsilon * 2 \partial G).$$

The first term $R_1$ and the second term $R_2$ are quite easy, and we merely rearrange them in terms of the power of $\varepsilon$, and it follows that

$$R_1 = \varepsilon^2 \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (W^\varepsilon * G) \right\} + \varepsilon^3 \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (\phi^\varepsilon * 2 \partial G) \right\};$$

$$R_2 = \varepsilon^2 \nabla (W^\varepsilon * G) \nabla \psi_\varepsilon + \varepsilon^3 \nabla (\phi^\varepsilon * 2 \partial G) \nabla \psi_\varepsilon.$$

We continue to handle $R_3$. According to the power of $\varepsilon$, there holds

$$R_3 = \varepsilon^2 \psi_\varepsilon \left\{ \varepsilon^{-2} (\Delta W)^\varepsilon * G + 2 \varepsilon^{-1} (\partial W)^\varepsilon * 2 \partial G + W^\varepsilon * \Delta G \right\} + \varepsilon^2 \psi_\varepsilon \nabla \cdot \left\{ (\nabla \phi)^\varepsilon * 2 \partial G + \varepsilon \phi^\varepsilon * 2 \nabla \partial G \right\}$$

$$= \psi_\varepsilon (\Delta W)^\varepsilon * G + 2 \varepsilon \psi_\varepsilon \left[ (\partial W)^\varepsilon * 2 \partial G \right] + \varepsilon^2 \psi_\varepsilon W^\varepsilon * \Delta G + \varepsilon^2 \psi_\varepsilon \nabla \cdot \left\{ (\nabla \phi)^\varepsilon * 2 \partial G \right\} + \varepsilon^3 \psi_\varepsilon \nabla \cdot \left\{ \phi^\varepsilon * 2 \nabla \partial G \right\},$$

where we substituted $\nabla$ with $\partial$ at the corresponding positions to highlight that the components of the gradient are involved in the inner product of tensors (see the convention presented in the part of “notation for convolution” in Subsection 1.4). Therefore, plugging the terms $R_1, R_2$ and $R_3$ above back into (5.4) we obtain that

$$-\varepsilon^2 \Delta w_\varepsilon = \varepsilon^2 \psi_\varepsilon (\Delta W)^\varepsilon * G + 2 \varepsilon \psi_\varepsilon \left[ (\partial W)^\varepsilon * 2 \partial G \right] - \varepsilon^2 \Delta u_\varepsilon - \varepsilon^2 \Delta \hat{\xi}_\varepsilon - \varepsilon^2 \Delta \hat{\eta}_\varepsilon + \varepsilon^2 \psi_\varepsilon \nabla \cdot \left( (\nabla \phi)^\varepsilon * 2 \partial G \right)$$

$$+ \varepsilon^2 \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (W^\varepsilon * G) \right\} + \varepsilon^2 \nabla (W^\varepsilon * G) \nabla \psi_\varepsilon + \varepsilon^2 \psi_\varepsilon W^\varepsilon * \Delta G$$

$$+ \varepsilon^3 \nabla \cdot \left\{ \nabla \psi_\varepsilon \otimes (\phi^\varepsilon * 2 \partial G) \right\} + \varepsilon^3 \nabla (\phi^\varepsilon * 2 \partial G) \nabla \psi_\varepsilon + \varepsilon^3 \psi_\varepsilon \nabla \cdot \left( \phi^\varepsilon * 2 \nabla \partial G \right),$$

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Now, we turn to the pressure term
\[ \nabla q_\varepsilon = \nabla p_\varepsilon - \nabla p_0 - \varepsilon \nabla \psi_\varepsilon \pi^\varepsilon \ast_1 G - \psi_\varepsilon (\nabla \pi)^\varepsilon \ast G - \varepsilon \psi_\varepsilon \nabla G \ast \pi^\varepsilon, \tag{5.6} \]
where we refer the reader to the convention on "\ast_1" and "\ast" presented in (1.18).

Combining the equalities (5.3), (5.5) and (5.6), we have
\[
\frac{\partial u_\varepsilon}{\partial t} - \varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon = \frac{\partial u_\varepsilon}{\partial t} + \xi_\varepsilon + \eta_\varepsilon - \psi_\varepsilon \left[ \partial_t W^\varepsilon \ast G + W^\varepsilon (\cdot, 0) \partial G + \varepsilon \left( \partial_t \phi^\varepsilon \ast_2 \partial G \right) + \varepsilon \left( \partial_t \phi^\varepsilon (\cdot, 0) : \partial G \right) \right] \\
+ \psi_\varepsilon (\Delta W)^\varepsilon \ast G + 2 \varepsilon \psi_\varepsilon \left[ (\partial W)^\varepsilon \ast_2 \partial G \right] - \varepsilon^2 \Delta u_\varepsilon - \varepsilon^2 \Delta \xi_\varepsilon - \varepsilon^2 \Delta \eta_\varepsilon + \varepsilon^2 \psi_\varepsilon \nabla \cdot \left( (\nabla \phi)^\varepsilon \ast_2 \partial G \right) \\
+ \varepsilon^2 \nabla \cdot \left( \nabla \psi_\varepsilon \otimes (W^\varepsilon \ast G) \right) + \varepsilon^2 \nabla (W^\varepsilon \ast \nabla \psi_\varepsilon) + \varepsilon^2 \psi_\varepsilon W^\varepsilon \ast \triangle G \\
+ \nabla p_\varepsilon - \nabla p_0 - \varepsilon \nabla \psi_\varepsilon \pi^\varepsilon \ast_1 G - \psi_\varepsilon (\nabla \pi)^\varepsilon \ast G - \varepsilon \psi_\varepsilon \nabla G \ast \pi^\varepsilon, \\
\]
whose right-hand side further equals to
\[
\frac{\partial u_\varepsilon}{\partial t} - \varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon - \nabla p_0 - \psi_\varepsilon W^\varepsilon (\cdot, 0) \partial G + \xi_\varepsilon + \eta_\varepsilon - \psi_\varepsilon \partial_t W^\varepsilon \ast G + \psi_\varepsilon (\Delta W)^\varepsilon \ast G - \psi_\varepsilon (\nabla \pi)^\varepsilon \ast G \\
- \varepsilon \psi_\varepsilon \left[ \partial_t \phi^\varepsilon \ast_2 \partial G + \phi^\varepsilon (\cdot, 0) : \partial G \right] + 2 \varepsilon \psi_\varepsilon \left[ (\partial W)^\varepsilon \ast_2 \partial G \right] - \varepsilon \nabla \psi_\varepsilon \pi^\varepsilon \ast_1 G - \varepsilon \psi_\varepsilon \nabla G \ast \pi^\varepsilon \\
+ \varepsilon^2 \psi_\varepsilon \nabla \cdot \left( (\nabla \phi)^\varepsilon \ast_2 \partial G \right) + \varepsilon^2 \nabla \cdot \left( \nabla \psi_\varepsilon \otimes (W^\varepsilon \ast G) \right) + \varepsilon^2 \nabla (W^\varepsilon \ast \nabla \psi_\varepsilon) + \varepsilon^2 \psi_\varepsilon W^\varepsilon \ast \Delta G - \varepsilon^2 \Delta \xi_\varepsilon - \varepsilon^2 \Delta \eta_\varepsilon \\
+ \varepsilon^3 \nabla (\phi^\varepsilon \ast_2 \partial G) \nabla \psi_\varepsilon + \varepsilon^3 \nabla \cdot \left( \nabla \psi_\varepsilon \otimes (\phi^\varepsilon \ast_2 \partial G) \right) + \varepsilon^3 \psi_\varepsilon \nabla \cdot \left( \phi^\varepsilon \ast_2 \nabla \partial G \right). \\
\]

On account of the equations (1.2) and (1.6), respectively, the above expression can be rewritten as, in terms of the order of the power of \( \varepsilon \),
\[
f - \nabla p_0 - \psi_\varepsilon W^\varepsilon (\cdot, 0) \partial G + \xi_\varepsilon + \eta_\varepsilon \\
- \varepsilon \psi_\varepsilon \left[ \partial_t \phi^\varepsilon \ast_2 \partial G + \phi^\varepsilon (\cdot, 0) : \partial G \right] + 2 \varepsilon \psi_\varepsilon \left[ (\partial W)^\varepsilon \ast_2 \partial G \right] - \varepsilon \nabla \psi_\varepsilon \pi^\varepsilon \ast_1 G - \varepsilon \psi_\varepsilon \nabla G \ast \pi^\varepsilon \\
+ \varepsilon^2 \psi_\varepsilon \nabla \cdot \left( (\nabla \phi)^\varepsilon \ast_2 \partial G \right) + \varepsilon^2 \nabla \cdot \left( \nabla \psi_\varepsilon \otimes (W^\varepsilon \ast G) \right) + \varepsilon^2 \nabla (W^\varepsilon \ast \nabla \psi_\varepsilon) + \varepsilon^2 \psi_\varepsilon W^\varepsilon \ast \Delta G - \varepsilon^2 \Delta \xi_\varepsilon - \varepsilon^2 \Delta \eta_\varepsilon \\
+ \varepsilon^3 \nabla (\phi^\varepsilon \ast_2 \partial G) \nabla \psi_\varepsilon + \varepsilon^3 \nabla \cdot \left( \nabla \psi_\varepsilon \otimes (\phi^\varepsilon \ast_2 \partial G) \right) + \varepsilon^3 \psi_\varepsilon \nabla \cdot \left( \phi^\varepsilon \ast_2 \nabla \partial G \right) := I_1 + \varepsilon I_2 + \varepsilon^2 I_3 + \varepsilon^3 I_4, \\
\]
which have proved the first line of the equations (5.1).

**Part 2.** We now check the divergence-free condition of \( w_\varepsilon \). It follows that
\[ \nabla \cdot w_\varepsilon = \nabla \cdot u_\varepsilon - \nabla \cdot \left\{ \psi_\varepsilon \left[ W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast_2 \partial G \right] \right\} + \nabla \cdot \xi_\varepsilon + \nabla \cdot \eta_\varepsilon, \]
and it suffices to show
\[ \nabla \cdot \left\{ \psi_\varepsilon \left[ W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast_2 \partial G \right] \right\} = \nabla \cdot \xi_\varepsilon + \nabla \cdot \eta_\varepsilon. \tag{5.7} \]
By divergence-free condition of (1.6) and the equations (2.3) that \( \phi \) satisfies, we obtain that
\[
\nabla \cdot \left\{ \psi_\varepsilon [W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast 2 \partial G] \right\} \\
= \nabla \psi_\varepsilon \cdot [W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast 2 \partial G] + \psi_\varepsilon \nabla \cdot [W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast 2 \partial G] \\
= \nabla \psi_\varepsilon \cdot [W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast 2 \partial G] + \psi_\varepsilon \left[ \varepsilon^{-1} (\nabla \cdot W)^\ast 1 G + W^\varepsilon \ast 2 \partial G + (\nabla \cdot \phi)^\ast 2 \partial G + \varepsilon \phi^\varepsilon \ast 3 \partial^2 G \right] \\
= \nabla \psi_\varepsilon \cdot [W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast 2 \partial G] + \psi_\varepsilon \left[ A \left[ \frac{1}{|Y_f|} \right] - W^\varepsilon \right] \ast 2 \partial G + \varepsilon \psi_\varepsilon \phi^\varepsilon \ast 3 \partial^2 G,
\]
and this further implies
\[
\nabla \cdot \left\{ \psi_\varepsilon [W^\varepsilon \ast G + \varepsilon \phi^\varepsilon \ast 2 \partial G] \right\} \\
= \nabla \psi_\varepsilon \cdot [(W^\varepsilon - A) \ast G] + \nabla \psi_\varepsilon \cdot (A \ast G) + \psi_\varepsilon A \left[ \frac{1}{|Y_f|} \right] \ast 2 \partial G + \varepsilon \nabla \psi_\varepsilon \cdot (\phi^\varepsilon \ast 2 \partial G) + \varepsilon \psi_\varepsilon \phi^\varepsilon \ast 3 \partial^2 G.
\]
(5.8)

Consequently, combining (1.17), (4.1), (4.3), (2.7) and (5.8), we have proved the equality (5.7), which gives us the divergence-free condition of (5.1), and ends the whole proof. 

Without a proof, we state the following energy estimate.

**Lemma 5.2** (energy estimates). Let \( 0 < T < \infty \) and \( d \geq 2 \). Suppose that \( \Omega \subset \mathbb{R}^d \) is a bounded Lipschitz domain, and the perforated one \( \Omega_\varepsilon \) satisfies the geometrical hypothesis (1.1). Let \( \psi_\varepsilon \) be the radial cut-off function defined in Lemma 4.5. Given \( \Theta \in L^2(0,T;L^2(\Omega)^d) \) and \( \Lambda, \Xi \in L^2(0,T;L^2(\Omega)^{d \times d}) \), assume that \( u_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon)^d) \) is a weak solution of
\[
\begin{aligned}
\partial_t u_\varepsilon - \varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon &= \Theta + \varepsilon \nabla \cdot \Lambda + \varepsilon \psi_\varepsilon \nabla \cdot \Xi & \quad & \text{in} \quad \Omega_\varepsilon \times (0,T); \\
\nabla \cdot u_\varepsilon &= 0 & \quad & \text{in} \quad \Omega_\varepsilon \times (0,T); \\
u_\varepsilon &= 0 & \quad & \text{on} \quad \partial \Omega_\varepsilon \times (0,T); \\
u_\varepsilon &= 0 & \quad & \text{in} \quad \Omega_\varepsilon \times \{t = 0\}. 
\end{aligned}
\]
(5.9)

Then, for any \( \varepsilon > 0 \), one can derive that
\[
\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon)^d)} + \varepsilon \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \\
\lesssim \left\{ \|\Theta\|_{L^2(0,T;L^2(\Omega_\varepsilon)^d)} + \|\Lambda\|_{L^2(0,T;L^2(\Omega_\varepsilon)^{d \times d})} + \|\Xi\|_{L^2(0,T;L^2(\text{supp}(\psi_\varepsilon)))} \right\},
\]
(5.10)
where the multiplicative constant depends on \( d \) and the characters of \( \Omega \) and \( Y_\varepsilon \), but independent of \( \varepsilon \).

### 5.1 Some auxiliary results

**Lemma 5.3.** Let \( 0 < \varepsilon \ll 1 \) and \( \psi_\varepsilon \) be defined in Lemma 4.5. Given \( f \in L^2(0,T;C^{1,1/2}(\bar{\Omega})^d) \), suppose that \( p_0 \) is associated with \( f \) by the equation (1.4). Let \( F = f - \nabla p_0 \), and
\[
G(x) = S_\delta(p_\varepsilon F)(x) = \int_{\mathbb{R}^d} dy \zeta_\delta(x - y)(p_\varepsilon F)(y), \quad \zeta_\delta = \delta^{-d} \zeta(\cdot/\delta),
\]
with \( \delta = \varepsilon/4 \), where \( \zeta \in C_0^\infty(B(0,1/2)) \) satisfies \( \zeta \geq 0 \) and \( \int_{\mathbb{R}^d} \zeta = 1 \) (see also in Subsection 1.4). Then, there hold the following estimates
\[
\|G - F\|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} \lesssim \varepsilon \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}; \\
\|
\nabla \|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} \lesssim \varepsilon^{1/2} \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))},
\]
and
\[
\|G\|_{L^2(0,T;L^\infty(\Omega))} + \|\nabla G\|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} \lesssim \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}; \\
\|\nabla^2 G\|_{L^2(0,T;L^\infty(\text{supp}(\psi_\varepsilon)))} \lesssim \varepsilon^{-1/2} \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))},
\]
where the multiplicative constant depends only on \( d \) and \( \zeta \).
Remark 5.4. Once the estimate \((1.14)\) is established, a routine computation leads to the desired estimates \((5.11)\) and \((5.12)\). We mention that the cut-off function \(\varphi_\varepsilon\) in \(G\) is not so crucial as \(\psi_\varepsilon\), and the purpose of introducing \(G\) is to weaken the assumption on the smoothness of \(f\). (See e.g. \([28, 35]\).)

5.2 Error estimate on velocity

The proof of the estimate \((1.8)\). Based on Lemma 5.1, the first two steps listed in Subsection 2.2 have been completed. Now, we will show a detailed proof for Step 3 therein. The main task in this step will be reduced to estimating each term presented in \((5.2)\) according to the form of the right-hand side of the equations \((5.9)\) given in Lemma 5.2. Therefore, we mainly divide this step into four sub-steps.

Step 3-1. We start from the first line of \((5.2)\) that

\[
I_1 = f - \nabla p_0 - \psi_\varepsilon W^\varepsilon(\cdot, 0)G + \xi_\varepsilon + \eta_\varepsilon,
\]

and we need to estimate these quantities: \(\|f - \nabla p_0 - \psi_\varepsilon G\|_{L^2(\Omega, \mathbb{T})}, \|\xi_\varepsilon\|_{L^2(\Omega, \mathbb{T})}, \|\eta_\varepsilon\|_{L^2(\Omega, \mathbb{T})}\), where the last two terms had been done by Propositions 4.1 and 4.2, respectively. Note that \(W^\varepsilon(\cdot, 0)\) is the identity matrix. For the first one, we have

\[
\|f - \nabla p_0 - \psi_\varepsilon G\|_{L^2(\Omega, \mathbb{T})} \lesssim \|f - \psi_\varepsilon G + \psi_\varepsilon F - \psi_\varepsilon F\|_{L^2(\Omega, \mathbb{T})}
\]

\[
\leq \|\psi_\varepsilon(G - F)\|_{L^2(\Omega, \mathbb{T})} + \|(1 - \psi_\varepsilon)F\|_{L^2(\Omega, \mathbb{T})}
\]

\[
\lesssim \|G - F\|_{L^2(0, T; L^2(\mathbb{T}))} + \|F\|_{L^2(0, T; L^2(\mathbb{T}))}.
\]

Hence, it follows that

\[
\|I_1\|_{L^2(\Omega, \mathbb{T})} \lesssim \|G - F\|_{L^2(0, T; L^2(\mathbb{T}))} + \|F\|_{L^2(0, T; L^2(\mathbb{T}))} + \|\xi_\varepsilon\|_{L^2(\Omega, \mathbb{T})} + \|\eta_\varepsilon\|_{L^2(\Omega, \mathbb{T})}
\]

\[
\lesssim \varepsilon^{1/2} \|\| F \|_{L^2(0, T; C^{1, 1/2}(\bar{\Omega}))}. \]

Step 3-2. Recalling the second line of \((5.2)\), we have

\[
\varepsilon \xi_\varepsilon = - \psi_\varepsilon \left[ \partial_t \phi^\varepsilon \ast_2 \partial G + \phi_\varepsilon(\cdot, 0) : \partial G \right] + 2 \psi_\varepsilon(\partial W)^\varepsilon \ast_2 \partial G - \varepsilon \psi_\varepsilon \nabla \phi^\varepsilon \ast_1 G - \varepsilon \psi_\varepsilon \nabla G \ast \pi^\varepsilon
\]

\[
= I_{21} + I_{22} + I_{23} + I_{24} + I_{25}.
\]

By using Minkowski’s inequality and Young’s inequality, and then the periodicity of \(W\) and \(\phi\), we obtain that

\[
\int_{\Omega, \mathbb{T}} |I_{21} + I_{23}|^2 \lesssim \varepsilon^2 \int_0^T \int_{\Omega} \psi_\varepsilon^2 |\partial_t \phi^\varepsilon \ast_2 \partial G|^2 + \varepsilon^2 \int_0^T \int_{\Omega} \psi_\varepsilon^2 |(\partial W)^\varepsilon \ast_2 \partial G|^2
\]

\[
\lesssim \varepsilon^2 \left( \|\partial_t \phi\|_{L^2(0, T; L^2(\mathbb{T})))} + \|\nabla W\|_{L^2(0, T; L^2(\mathbb{T})))} \right) \|\nabla G\|_{L^2(0, T; L^\infty(supp(\psi_\varepsilon)))}. \]

By Hölder’s inequality and the periodicity of \(\phi\), we have

\[
\int_{\Omega, \mathbb{T}} |I_{22}|^2 = \varepsilon^2 \int_0^T dt \int_{\Omega} |\psi_\varepsilon \phi^\varepsilon (\cdot, 0) : \partial G(\cdot, 0)|^2 \lesssim \varepsilon^2 \|\phi(\cdot, 0)\|_{L^2(\mathbb{T})}^2 \|\nabla G\|_{L^2(0, T; L^\infty(supp(\psi_\varepsilon)))}^2.
\]

In view of Lemma 4.5, employing a similar computation as given for \((5.14)\), we obtain that

\[
\int_{\Omega, \mathbb{T}} |I_{24}|^2 = \varepsilon^2 \int_0^T \int_{\Omega} |\nabla \psi_\varepsilon \pi^\varepsilon \ast_1 G|^2
\]

\[
\lesssim \int_0^T \int_{\Omega} |\pi^\varepsilon \ast_1 G|^2 \lesssim \varepsilon^2 \|\pi\|_{L^1(0, T; L^2(\mathbb{T})))}^2 \|G\|_{L^2(0, T; L^\infty(supp(\psi_\varepsilon)))}^2,
\]

and

\[
\int_{\Omega, \mathbb{T}} |I_{25}|^2 = \varepsilon^2 \int_0^T \int_{\Omega} |\psi_\varepsilon \nabla G \ast \pi^\varepsilon|^2 \lesssim \varepsilon^2 \|\pi\|_{L^1(0, T; L^2(\mathbb{T})))}^2 \|\nabla G\|_{L^2(0, T; L^\infty(supp(\psi_\varepsilon)))}^2.
\]
Therefore, collecting all the estimates of $I_{21}, \ldots, I_{25}$, there holds
\[
\|\varepsilon I_2\|_{L^2(\Omega, T)} \lesssim \varepsilon \left\{ \|\partial_t \phi\|_{L^1(0, T; L^2(Y_j))} + \|\nabla W\|_{L^1(0, T; L^2(Y_j))} + \|\phi(\cdot, 0)\|_{L^2(Y_j)} \right\} \|\nabla G\|_{L^2(0, T; L^\infty(\text{supp}(\psi)))} \\
+ \varepsilon^{\frac{1}{2}} \pi_{L^1(0, T; L^2(Y_j))} \left\{ \|G\|_{L^2(0, T; L^\infty(\text{supp}(\psi)))} + \varepsilon^{\frac{3}{2}} \|\nabla G\|_{L^2(0, T; L^\infty(\text{supp}(\psi)))} \right\} 
\]
(5.15)
\[
\lesssim \varepsilon^{1/2} \|f\|_{L^2(0, T; C^{1,1/2}(\tilde{\Omega}))},
\]
where it is fine to assume $\int_{Y_j} \pi(\cdot, t) = 0$ for $t > 0$, since $\pi \in L^1(0, T; L^2(Y_j)/\mathbb{R})$ due to the estimate (3.2a).

**Step 3-3.** We now turn to the term $\varepsilon^2 I_3$. By the expression of $I_3$ in (5.2), we can rewrite it as follows:
\[
\varepsilon^2 I_3 = \varepsilon^2 \psi \cdot (W^\varepsilon * G) \nabla \psi
+ \varepsilon^2 \nabla \cdot \left\{ \psi \nabla (W^\varepsilon * G) \nabla \psi \right\}
+ \varepsilon^2 \psi \cdot \left\{ (\nabla \psi)^* \nabla (\nabla \psi)^* \right\}.
\]

Then, in view of Lemma 5.2, we need to estimate $I_{31}, I_{32}, I_{33}$ and $I_{34}$ according to the related form in the estimate (5.10). We begin with $I_{34}$, and by using Minkowski’s inequality, Young’s inequality, and the periodicity of $\phi$, there holds
\[
\int_0^T \int_{\text{supp}(\psi_\omega)} |I_{34}|^2 \lesssim \varepsilon^4 \int_0^T \int_{\text{supp}(\psi_\omega)} |(\nabla \psi)^* \nabla \psi|^2 \lesssim \varepsilon^2 \|\nabla \phi\|_{L^1(0, T; L^2(Y_j))}^2 \|\nabla G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))}. 
\]

We proceed to handle the term $I_{33}$ in the region $\Omega_{e, t}$, and a similar computation as given above leads to
\[
\int_{\Omega_{e, t}} |I_{33}|^2 \lesssim \varepsilon^2 \int_0^T \int_{\Omega_{e, t}} |(\nabla \psi \cdot W^\varepsilon * G)|^2 + |\nabla \xi|^2 \lesssim \int_0^T \int_{\Omega_{e, t}} |W^\varepsilon * G|^2 + \varepsilon^2 \int_0^T \int_{\Omega_{e, t}} |(\nabla \xi)|^2 \lesssim \varepsilon^2 \|W\|_{L^1(0, T; L^2(Y_j))} \|G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))} + \varepsilon^2 \|\xi\|^2_{L^2(\Omega_{e, t})} + \varepsilon^2 \|\nabla \xi\|^2_{L^2(\Omega_{e, t})}.
\]

By the same token, we obtain that
\[
\int_{\Omega_{e, t}} |I_{31}|^2 \lesssim \varepsilon^4 \int_0^T \int_{\text{supp}(\psi_\omega)} |W^\varepsilon \cdot \Delta G|^2 \lesssim \varepsilon^4 \|W\|_{L^1(0, T; L^2(Y_j))}^2 \|\nabla G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))},
\]
and
\[
\int_{\Omega_{e, t}} |I_{32}|^2 \lesssim \varepsilon^2 \int_0^T \int_{\Omega_{e, t}} |(\nabla (W^\varepsilon * G)|^2 \lesssim \varepsilon^2 \|\nabla W\|_{L^1(0, T; L^2(Y_j))} \|G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))} + \varepsilon^2 \|W\|_{L^1(0, T; L^2(Y_j))} \|\nabla G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))}.
\]

As a result, we have established the following estimate
\[
\|I_{31} + I_{32} + I_{33}\|_{L^2(\Omega_{e, t})} + \|I_{34}\|_{L^2(0, T; L^2(\text{supp}(\psi)))} \lesssim \varepsilon^{1/2} \left\{ \|W\|_{L^1(0, T; L^2(Y_j))} + \|\nabla W\|_{L^1(0, T; L^2(Y_j))} \right\} \|G\|_{L^2(0, T; L^\infty(\text{supp}(\psi)))} \\
+ \varepsilon \left\{ \varepsilon^{1/2} \|W\|_{L^1(0, T; L^2(Y_j))} + \|\phi\|_{L^1(0, T; L^2(Y_j))} \right\} \|\nabla G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))} \\
+ \varepsilon^2 \|W\|_{L^1(0, T; L^2(Y_j))} \|\nabla G\|^2_{L^2(0, T; L^\infty(\text{supp}(\psi)))} + \varepsilon \|\nabla \xi\|_{L^2(\Omega_{e, t})} + \varepsilon \|\nabla \xi\|_{L^2(\Omega_{e, t})} \\
\lesssim \varepsilon^{1/2} \|f\|_{L^2(0, T; C^{1,1/2}(\tilde{\Omega}))}.
\]
(5.16)
Step 3-4. Recall the last line of (5.2), and we rewrite it as follows:
\[
\varepsilon^3 I_4 = \varepsilon^3 \nabla \left( \phi \ast \partial \phi \right) \nabla \psi + \varepsilon^3 \nabla \cdot \left\{ \nabla \psi \otimes \left( \phi \ast \partial \phi \right) \right\} + \varepsilon^3 \psi \varepsilon \nabla \cdot \left( \phi \ast \partial \phi \right)
\]
\[
=: I_{41} + \varepsilon \nabla \cdot I_{42} + \varepsilon \psi \varepsilon \nabla \cdot I_{43}.
\]
By using Minkowski’s inequality and Young’s inequality, as well as, the periodicity of \( \phi \), we derive that
\[
\int_{\Omega_{\varepsilon,T}} |I_{41}|^2 \lesssim \varepsilon^4 \int_0^T \int_{\Omega_{\varepsilon}} |\nabla (\phi \ast \partial \phi)|^2 + \varepsilon^2 \int_0^T \int_{\Omega_{\varepsilon}} |(\nabla \phi) \ast \partial \phi|^2
\]
\[
\lesssim \varepsilon^3 \| \phi \|_{L^1(0,T;L^2(\Omega_{\varepsilon,T}))} \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi)))}^2.
\]
By an analogous argument employed in Step 3-3, there holds
\[
\int_{\Omega_{\varepsilon,T}} |I_{42}|^2 = \varepsilon^4 \int_0^T \int_{\Omega_{\varepsilon}} |\psi \ast \partial \phi|^2 \lesssim \varepsilon^2 \int_0^T \int_{\Omega_{\varepsilon}} |\phi \ast \partial \phi|^2
\]
\[
\lesssim \varepsilon^3 \| \phi \|_{L^1(0,T;L^2(\Omega_{\varepsilon,T}))} \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi)))}^2,
\]
and
\[
\int_0^T \int_{\text{supp}(\psi)} |I_{43}|^2 = \varepsilon^4 \int_0^T \int_{\text{supp}(\psi)} |\phi \ast \partial \phi|^2 dx dt \lesssim \varepsilon^3 \| \phi \|_{L^1(0,T;L^2(\Omega_{\varepsilon,T}))} \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi)))}^2.
\]
We now collect the above estimates and obtain
\[
\| I_{41} \|_{L^2(Q_{\varepsilon,T})} + \| I_{42} \|_{L^2(Q_{\varepsilon,T})} + \| I_{43} \|_{L^2(\Omega_{\varepsilon,T})} \lesssim \varepsilon^3 \| \phi \|_{L^1(0,T;L^2(\Omega_{\varepsilon,T}))} \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi)))}^2
\]
\[
+ \varepsilon^{3/2} \left\{ \| \phi \|_{L^1(0,T;L^2(\Omega_{\varepsilon,T}))} + \| \nabla \phi \|_{L^1(0,T;L^2(\Omega_{\varepsilon,T}))} \right\} \| \nabla G \|_{L^2(0,T;L^\infty(\text{supp}(\psi)))}.
\]
(5.17)
In the end, in view of the equations (5.1), we can rewrite its right-hand side as
\[
I_1 + \varepsilon I_2 + \varepsilon^2 I_3 + \varepsilon^3 I_4 = I_1 + \varepsilon I_2 + I_{31} + I_{32} + I_{41} + \varepsilon \nabla \cdot \left( I_{33} + I_{42} \right) + \varepsilon \psi \varepsilon \nabla \cdot \left( I_{44} + I_{43} \right),
\]
which coincides with the form of the counterpart of the equations (5.9). Thus, appealing to the estimate (5.10), together with (5.13), (5.15), (5.16) and (5.17), we obtain that
\[
\varepsilon \| \nabla w_{\varepsilon} \|_{L^2(0,T;L^2(\Omega_{\varepsilon,T}))} \lesssim \varepsilon^{1/2} \| f \|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))},
\]
(5.19)
which finally leads to the desired estimate (1.8), and complete all the proof.

5.3 Error estimate on pressure

Remark 5.5. Regarding the equation (5.18), if the right-hand side of (5.1) is considered in \( L^2(0,T;H^{-1}(\Omega_{\varepsilon,T})) \), it is generally not feasible to obtain a pressure term in \( L^2(\Omega_{\varepsilon,T}) \)-norm (see e.g. [32, Proposition 5]). However, without aiming to derive smallness from the right-hand side of (5.1), one can verify that it belongs to \( L^2(\Omega_{\varepsilon,T}) \)-space. Given the zero initial-boundary data for the equation (5.1), one can anticipate favorable regularity properties for the pressure term (see e.g. [33, Theorem 1]). Consequently, it remains reasonable to proceed with our analysis of the pressure term through meticulous computation.

The proof of the estimate (1.9). As mentioned in Step 5 in Subsection 2.2, we are required to study the inertial term \( \partial_t w_{\varepsilon} \), and then appeal to a duality argument coupled with the estimate on velocity to derive the desired estimate (1.9). Therefore, we divide the detailed proof into two sub-steps.
Step 5-1. We first show the estimate on $\partial_t w_\varepsilon$ by claiming that there holds
\[
\|\partial_t w_\varepsilon\|_{L^2(\Omega_\varepsilon, T)} \lesssim \varepsilon^{1/2}\|f\|_{L^2((0,T;C^{1,1/2}(\Omega))},
\]
(5.20)

(which have partially proved the estimate (1.9),) and then recall the estimates on the notations $\Theta, \Lambda$ and $\Xi$ presented in (5.18), i.e.,
\[
\|\Theta, \Lambda, \psi, \Xi\|_{L^2(\Omega_\varepsilon, T)} \lesssim \varepsilon^{1/2}\|f\|_{L^2((0,T;C^{1,1/2}(\Omega))}.
\]
(5.21)

Based upon the stated estimates (5.20) and (5.21), we can show the estimate (1.9) on the pressure. We start from taking any test function $H \in L^2(\Omega_\varepsilon)$ with $\int_{\Omega_\varepsilon} H = 0$, and constructing test function as follows:
\[
\begin{cases}
\nabla \cdot v_\varepsilon = H & \text{in } \Omega_\varepsilon; \\
v_\varepsilon = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}
\]
(5.22)

whose solution satisfies the estimate (4.6). In this regard, for any $\varrho \in L^2(0,T)$ and $c \in \mathbb{R}$, there holds
\[
\int_{\Omega_\varepsilon, T} \varrho(q_\varepsilon - c)H = \int_0^T \int_{\Omega_\varepsilon} \nabla q_\varepsilon \cdot v_\varepsilon
\]
(5.23)

Applying Poincaré’s inequality to $v_\varepsilon$, as well as, Hölder’s inequality, we have
\[
\int_{\Omega_\varepsilon, T} \varrho(q_\varepsilon - c)H \leq \varepsilon \varrho \|L^2(0,T)\|\nabla v_\varepsilon \|L^2(\Omega_\varepsilon)\left\{ \varepsilon \|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon, T)} + \|\partial_t w_\varepsilon\|_{L^2(\Omega_\varepsilon, T)} + \|\Theta, \Lambda, \psi, \Xi\|_{L^2(\Omega_\varepsilon, T)} \right\},
\]
and then it follows from the estimate (4.6) and the duality argument that
\[
\|q_\varepsilon - c\|_{L^2(\Omega_\varepsilon, T)} \lesssim \varepsilon \|L^2(0,T)\|\nabla w_\varepsilon \|L^2(\Omega_\varepsilon, T) + \|\partial_t w_\varepsilon\|_{L^2(\Omega_\varepsilon, T)} + \|\Theta, \Lambda, \psi, \Xi\|_{L^2(\Omega_\varepsilon, T)}
\]
(5.24)

As a result, this implies the desired estimate (1.9).

Step 5-2. We now show the estimate (5.20). Taking $\partial_t w_\varepsilon$ as the test function on both sides of (5.1), we have
\[
\int_{\Omega_\varepsilon} \|\partial_t w_\varepsilon\|^2 + \frac{\varepsilon^2}{2} \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 = \int_{\Omega_\varepsilon} \Theta \cdot \partial_t w_\varepsilon - \varepsilon \int_{\Omega_\varepsilon} (\Lambda + \psi \Xi) \cdot \nabla w_\varepsilon
\]
(5.25)

Then, integrating both sides above from 0 to $T$, as well as, integrating by parts with respect to the temporal variable, we can derive that
\[
\int_0^T \int_{\Omega_\varepsilon} \|\partial_t w_\varepsilon\|^2 + \frac{\varepsilon^2}{2} \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 = \int_0^T \int_{\Omega_\varepsilon} (\Theta - \varepsilon \xi \nabla \psi_\varepsilon \cdot \partial_t w_\varepsilon + \varepsilon \int_0^T \int_{\Omega_\varepsilon} (\Lambda + \psi \Xi) \cdot \nabla w_\varepsilon
\]
\[
- \varepsilon \int_0^T \int_{\Omega_\varepsilon} (\Lambda + \psi \Xi) w_\varepsilon \cdot \nabla w_\varepsilon.
\]
(5.26)

This together with Young’s inequality implies
\[
\int_{\Omega_\varepsilon, T} |\partial_t w_\varepsilon|^2 + \varepsilon \int_{\Omega_\varepsilon, T} |\nabla w_\varepsilon|^2 \lesssim \int_{\Omega_\varepsilon, T} (|\Theta|^2 + |\psi \Xi|^2 + |\varepsilon \nabla w_\varepsilon|^2) + \int_{\Omega_\varepsilon, T} (|\partial_t \Lambda|^2 + |\psi \partial_t \Xi|^2)
\]
(5.27)

\[
+ \int_{\Omega_\varepsilon, T} (|\Lambda|^2 + |\Xi|^2) =: T_1 + T_2 + T_3.
\]
(5.28)

The relatively easy term is $T_1$, and it follows from the estimates (5.21) and (5.19) that
\[
\sqrt{T_1} \lesssim \varepsilon^{1/2}\|f\|_{L^2((0,T;C^{1,1/2}(\Omega))}.
\]
(5.29)
Then, we turn to the second term $T_2$ by recalling the expression (5.18), i.e., $\partial_t \Lambda = \partial_t I_{33} + \partial_t I_{42}$ with
\[
\begin{aligned}
&\partial_t I_{33} = \varepsilon \nabla \psi \varepsilon \otimes (p_0(x,t) - \varepsilon \nabla \psi \varepsilon \otimes (W(\cdot,0)G(\cdot,t)) - \varepsilon \nabla \xi \varepsilon - \varepsilon \nabla \eta \varepsilon; \\
&\partial_t I_{42} = \varepsilon^2 \nabla \psi \varepsilon \otimes (\partial_t \phi \varepsilon) + \varepsilon \nabla \psi \varepsilon \otimes (\partial_t \phi \varepsilon : \partial G(\cdot,t));
\end{aligned}
\]
and $\partial_t \Xi = \partial_t I_{44} + \partial_t I_{13}$. A routine computation as used in Subsection 5.2 leads to
\[
\begin{aligned}
&\|\partial_t \Lambda\|_{L^2(\Omega_t, \tau)} \lesssim \varepsilon^2 \left\{ \|\partial_t W\|_{L^1(0,T;L^2(Y_t))} + 1 \right\} \|G\|_{L^2(0,T;L^\infty(\text{supp}(\psi_0)))} + \varepsilon\|\nabla \xi \varepsilon + \nabla \eta \varepsilon\|_{L^2(\Omega_t, \tau)} \\
&+ \varepsilon^3 \left\{ \|\partial_t \phi\|_{L^1(0,T;L^2(Y_t))} + \|\phi(\cdot,0)\|_{L^2(Y_t)} \right\} \|\nabla G\|_{L^2(0,T;L^\infty(\text{supp}(\psi_0)))} \\
\rangle\langle \text{5.2b),(3.7),(4.2),(4.4),(5.12)} \lesssim \varepsilon^2 \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}.
\end{aligned}
\]
By the same token, we have $\|\psi \varepsilon \partial_t \Xi\|_{L^2(\Omega_t, \tau)} \lesssim \varepsilon^2 \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}$, and we don’t reproduce the proof here. This coupled with the above estimate provides us with
\[
\sqrt{T_2} \lesssim \varepsilon^2 \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}. 
\]
Finally, we turn to the term $T_3$. According to the expression of (5.18), we merely compute the first term in $\int_{\Omega_t} |\Lambda(\cdot,T)|^2$ as an example, i.e., there holds
\[
\begin{aligned}
\varepsilon^2 \int \varepsilon^2 \nabla \psi \varepsilon \otimes G(\cdot,T)^2 \lesssim \varepsilon \left( \int ds \| W(\cdot,T-s) G(\cdot,s) \|_{L^\infty(\Omega)} \right)^2 \\
\lesssim \varepsilon \| W \|_{L^1(0,T;L^2(Y_t))} \| G \|_{L^2(0,T;L^\infty(\Omega))}^2 \lesssim \varepsilon^2 \| f \|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}^2.
\end{aligned}
\]
and this together with similar computations given for the other terms in $T_3$ provides us with
\[
\sqrt{T_3} \lesssim \varepsilon^2 \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}.
\]
Plugging the estimates (5.25), (5.26) and (5.27) back into (5.24), we have proved the stated estimate (5.20), and we have completed the whole proof.

Proof of Corollary 1.3. Treating time variable as a parameter, the idea is totally similar to that given in [29, pp.22], and we provide a proof for reader’s convenience. The key ingredient is modifying the value of the effective pressure given in (1.4) as follows
\[
\tilde{p}_0(x,t) := \begin{cases} 
p_0(x,t) & \text{if } x \in \Omega_e, \\
p_0(\cdot,t) & \text{if } x \in \varepsilon(Y_s + z_k) \text{ and } \varepsilon(Y + z_k) \subset \Omega \text{ for some } z_k \in \mathbb{Z}^d.
\end{cases}
\]

By definition, it is not hard to derive that, for any $c \in \mathbb{R}$,
\[
\int_0^T \int_{\Omega_e} \varepsilon^2 \left| \tilde{p}_e - \tilde{p}_0 - c \right|^2 \leq \frac{\| Y_s \|_d}{\| Y \|_d} \int_0^T \int_{\Omega_e} \left| p_0 - c \right|^2 \lesssim \varepsilon \| f \|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}^2. 
\]
Moreover, by the smoothness of $p_0$ and Poincérée’s inequality, we can obtain that
\[
\int_0^T \int_{\Omega_e} \left| \tilde{p}_0 - p_0 \right|^2 \lesssim \varepsilon^2 \| \nabla p_0 \|_{L^2(0,T;L^2(\Omega))}^2 
\]
Then, for any $c \in \mathbb{R}$, it is not hard to derive that
\[
\int_0^T \int_{\Omega_e} \left| \tilde{p}_e - p_0 - c \right|^2 \lesssim \int_0^T \int_{\Omega_e} \left| p_0 - p_0 - c \right|^2 + \int_0^T \int_{\Omega_e} \left| \tilde{p}_0 - p_0 \right|^2 \\
\lesssim \int_0^T \int_{\Omega_e} \left| p_0 - p_0 - c \right|^2 + \int_0^T \int_{\Omega_e} \left| \tilde{p}_0 - p_0 - c \right|^2 + \int_0^T \int_{\Omega_e} \left| \tilde{p}_0 - p_0 \right|^2 \\
\lesssim \varepsilon \| f \|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}^2.
\]
 Consequently, this implies the desired estimate (1.10) and we have completed the proof.
6 Appendix

The primary approach involves reformulating the equations (1.4) as a fixed-point problem, as presented in J.-L. Lions’s work [17, pp.170]. The key components are rooted in Schauder theory for elliptic equations and refined corrector estimates. Consequently, we can first establish the short-time existence of solutions by invoking the absolute continuity of integrals of $\partial_t A$, and then extend the solution to a finite time by induction arguments (see Lemma 6.4 and 6.5, respectively).

Remark 6.1. If replacing Hölder’s norm by Sobolev norm with respect to the spatial variable, the results similar to (1.14) would be established by the same arguments without any real difficulty, i.e.,

$$
\|p_0\|_{L^4(0,T;H^{m+1}(\Omega))} \leq C\|f\|_{L^4(0,T;L^2(\Omega))}, \quad m \geq 0,
$$

(6.1)

where we regard $H^0(\Omega)$ as $L^2(\Omega)$.

6.1 Existence of short-time solution

Lemma 6.2 (properties of $A$ [1, 17, 21, 27]). The homogenized coefficient $(A_{ij})_{1\leq i,j\leq d}$ which is defined by (1.5) is symmetric, positive defined and exponentially decay in time. Moreover, one can derive $|\partial_t A| \in L^{1+\beta}(0,T)$ with $0 < \beta < (2/21)$ and $T > 0$.

Remark 6.3. Concerned with $|\partial_t A| \in L^1(0,T)$, the proof can be found in [27, pp.127] based upon Galerkin’s methods, while the stated result relies on the refined estimate (3.21).

Lemma 6.4 (short-time solution). Assume the same conditions as in Proposition 1.4. There exists $0 < \delta_0 \ll 1$, depending on $\|\partial_t A\|_{L^1(0,T)}$, such that the equation (1.4) possesses the solution $p_0 \in L^2(0,\delta_0;C^{m+1,\alpha}(\Omega))$, satisfying the estimate

$$
\|p_0\|_{L^4(0,\delta_0;C^{m+1,\alpha}(\tilde{\Omega}))} \lesssim \|f\|_{L^4(0,\delta_0;C^{m,\alpha}(\tilde{\Omega}))},
$$

where the multiplicative constant depends on $d$, $|Y_f|$, and $\Omega$.

Proof. In view of the divergence-free and boundary conditions of (1.4), taking $t$-derivative on its both sides$^{10}$, we have a new form of the equations (1.4), i.e.,

$$
\begin{align*}
\nabla \cdot A(0) \nabla p_0(\cdot,t) + \nabla \cdot \int_0^t ds A'(t-s) \nabla p_0(\cdot,s) & = \nabla \cdot \left[ \int_0^t ds A'(t-s) f(\cdot,s) + A(0) f(\cdot,t) \right] \quad \text{in } \Omega; \\
\bar{n} \cdot A(0) \nabla p_0(\cdot,t) + \bar{n} \cdot \int_0^t ds A'(t-s) \nabla p_0(\cdot,s) & = \bar{n} \cdot \left[ \int_0^t ds A'(t-s) f(\cdot,s) + A(0) f(\cdot,t) \right] \quad \text{on } \partial \Omega.
\end{align*}
$$

(6.3)

As the argument developed by J.-L. Lions [17, pp.170], for a.e. $t > 0$, we introduce a function $\hat{p}(\cdot,t)$ as the solution of

$$
\begin{align*}
\nabla \cdot \left[ A(0) \nabla \hat{p}(\cdot,t) - \int_0^t ds A'(t-s) \nabla p(\cdot,s) \right] & = 0, \quad \text{in } \Omega; \\
\bar{n} \cdot \left[ A(0) \nabla \hat{p}(\cdot,t) - \int_0^t ds A'(t-s) \nabla p(\cdot,s) \right] & = 0, \quad \text{on } \partial \Omega;
\end{align*}
$$

(6.4)

Denote the operator $\nabla \cdot A(0) \nabla = |Y_f| \Delta$ by $L$, and

$$
K_0(p)(\cdot,t) := \int_0^t ds A'(t-s) \nabla p(\cdot,s); \quad K_1(p) := L^{-1} \nabla \cdot K_0(p).
$$

(6.5)

Then, the solution $\hat{p}$ of (6.4) can be represented by $\hat{p}(\cdot,t) = K_1(p)(\cdot,t)$ in $\Omega$. By Schauder estimates (see e.g. [10, pp.89]), there holds that

$$
\|\hat{p}(\cdot,t)\|_{C^{1,1/2}(\Omega)} \leq C_1 \left\| \int_0^t ds A'(t-s) \nabla p(\cdot,s) \right\|_{C^{1,1/2}(\tilde{\Omega})}
$$

$$
\leq C_1 \int_0^t ds |A'(t-s)||\nabla p(\cdot,s)||_{C^{1,1/2}(\tilde{\Omega})} \leq C_1 \int_0^t ds |A'(t-s)||p(\cdot,s)||_{C^{1,1/2}(\tilde{\Omega})},
$$

(6.6)

$^{10}$To shorten the formula, we use the notation $A'$ to represent $\partial_t A$ throughout the proof of Lemmas 6.4 and 6.5.
where $C_1$ depends on $|Y_f|$, $d$, and $\Omega$. In view of (6.5), we observe that $\mathcal{L} K_1(p) = \nabla \cdot \mathcal{K}_0(p)$, and therefore the equations (6.3) can be rewritten as

$$
\begin{align*}
\left\{ \begin{array}{l}
\mathcal{L} \left[ p_0(\cdot, t) + K_1(p_0)(\cdot, t) \right] = \nabla \cdot \left[ \int_0^t \! ds A'(t-s) f(\cdot, s) + A(0) f(\cdot, t) \right] \quad \text{in } \Omega; \\
\frac{\partial}{\partial \nu} \left[ p_0(\cdot, t) + K_1(p_0)(\cdot, t) \right] = \tilde{n} \cdot \left[ \int_0^t \! ds A'(t-s) f(\cdot, s) + A(0) f(\cdot, t) \right] \quad \text{on } \partial \Omega,
\end{array} \right.
\end{align*}
$$

where the conormal derivative associated with $\mathcal{L}$ is defined by $\partial/\partial \nu := \tilde{n} \cdot A(0) \nabla$. Similarly, its solution can be expressed by

$$
(p_0 + K_1(p_0))(\cdot, t) = \mathcal{L}^{-1} \nabla \cdot \left[ \int_0^t \! ds A'(t-s) f(\cdot, s) + A(0) f(\cdot, t) \right] := \tilde{f}_1(\cdot, t).
$$

Thus, for some $T_1 > 0$ (which will be fixed later), if constructing the following map:

$$
T_1(p) := \tilde{f}_1 - K_1(p) \quad \forall p \in L^q(0, T_1; C^{m+1, \alpha}(\Omega)),
$$

the unique existence of the solution of (6.3) in $L^q(0, T_1; C^{m+1, \alpha}(\Omega))$ is reduced to verifying that the map $T_1 : L^q(0, T_1; C^{m+1, \alpha}(\Omega)) \rightarrow L^q(0, T_1; C^{m+1, \alpha}(\Omega))$ is a strict contraction.

To see this, appealing to Schauder estimates again, we firstly derive that

$$
\|\tilde{f}_1(\cdot, t)\|_{C^{m+1, \alpha}(\Omega)} \lesssim \int_0^t ds |A'(t-s)||f(\cdot, s)|_{C^{m, \alpha}(\Omega)} + |A(0)||f(\cdot, t)|_{C^{m, \alpha}(\Omega)}.
$$

By integrating both sides of (6.6) and (6.9) with respect to $t$ from 0 to $T_1$, Young’s inequality, we obtain

$$
\begin{align*}
\|K_1(p)\|_{L^q(0, T_1; C^{m+1, \alpha}(\Omega))} & \lesssim \|A'\|_{L^q(0, T_1; L^p(\Omega))} \|p\|_{L^q(0, T_1; C^{m+1, \alpha}(\Omega))}; \\
\|\tilde{f}_1\|_{L^q(0, T_1; C^{m+1, \alpha}(\Omega))} & \lesssim \|A'\|_{L^q(0, T_1; L^p(\Omega))} \|f\|_{L^q(0, T_1; C^{m, \alpha}(\Omega))} + \|f(\cdot, t)\|_{C^{m, \alpha}(\Omega)}.
\end{align*}
$$

Then, it follows from the estimates (6.8) and (6.6) that

$$
\|T_1(p)(\cdot, t)\|_{C^{m+1, \alpha}(\Omega)} \lesssim \|K_1(p)(\cdot, t)\|_{C^{m+1, \alpha}(\Omega)} + \|\tilde{f}_1(\cdot, t)\|_{C^{m+1, \alpha}(\Omega)}
\lesssim \int_0^t ds |A'(t-s)||f(\cdot, s)|_{C^{m, \alpha}(\Omega)} + \|f(\cdot, t)\|_{C^{m, \alpha}(\Omega)} + \|f(\cdot, t)\|_{C^{m, \alpha}(\Omega)},
$$

and this implies

$$
\begin{align*}
\|T_1(p)\|_{L^q(0, T_1; C^{m+1, \alpha}(\Omega))} & \lesssim \|A'\|_{L^q(0, T_1; L^p(\Omega))} \|p\|_{L^q(0, T_1; C^{m+1, \alpha}(\Omega))} + \|f\|_{L^q(0, T_1; C^{m, \alpha}(\Omega))} + \|f(\cdot, t)\|_{C^{m, \alpha}(\Omega)},
\end{align*}
$$

and this coupled with (6.11) verifies the contraction property of $T_1$ in $L^q(0, \delta_0; C^{m+1, \alpha}(\Omega))$.

Therefore, it follows from Banach’s fixed-point theorem that there exists unique solution $p_0$ such that $T_1(p_0) = p_0$ in $L^q(0, \delta_0; C^{m+1, \alpha}(\Omega))$, and this together with the estimate (6.11) leads to the desired result

$$
\|p_0\|_{L^2(0, \delta_0; C^{1, 1/2}(\Omega))} \lesssim \|f\|_{L^2(0, \delta_0; C^{1, 1/2}(\Omega))}.
$$

We have completed the proof.

11By the definition, the operator $K_1$ is linear.
6.2  Extension of solution

Lemma 6.5 (inductions). Let $0 < \delta_0 \ll 1$ be given as in Lemma 6.4. Assume the same conditions as in Proposition 1.4. Let $n \geq 2$ be an arbitrary fixed large integer, and $T_k = k\delta_0$ with $k = 1, \cdots, n$. Assume that there exists a unique solution $p_0 \in L^q(0, T_{n-1}; C^{m+1,\alpha}(\bar{\Omega}))$ to the equations (1.4), satisfying the estimate

$$
\|p_0\|_{L^q(0, T_{n-1}; C^{m+1,\alpha}(\bar{\Omega}))} \leq C_{n-1}\|f\|_{L^q(0, T_{n-1}; C^{m,\alpha}(\bar{\Omega}))}.
$$

Then, there exists a unique extension of the solution $p_0 \in L^q(0, T_n; C^{m+1,\alpha}(\bar{\Omega}))$ to the equations (1.4), and satisfies the estimate

$$
\|p_0\|_{L^q(0, T_n; C^{m+1,\alpha}(\bar{\Omega}))} \leq C_n\|f\|_{L^q(0, T_n; C^{m,\alpha}(\bar{\Omega}))},
$$

where $C_n$ is monotonically ascending with respect to $n$.

Proof. We continue to adopt the notation used in Lemma 6.4. For any fixed $n \geq 2$, and for any $t \in [T_{n-1}, T_n]$, we start considering

$$
\left\{ \begin{array}{l}
\mathcal{L}[p_0(\cdot, t) + K_n(p_0)(\cdot, t)] = \nabla \cdot \left[ A' \ast f + A(0)f - \int_0^{T_{n-1}} ds A'(t-s) \nabla p_0(\cdot, s) \right] \quad \text{in } \Omega; \\
\frac{\partial}{\partial \nu}[p_0(\cdot, t) + K_n(p_0)(\cdot, t)] = \bar{n} \cdot \left[ A' \ast f + A(0)f - \int_0^{T_{n-1}} ds A'(t-s) \nabla p_0(\cdot, s) \right] \quad \text{on } \partial \Omega,
\end{array} \right.
$$

where the auxiliary function $K_n(p_0)$ is given by:

$$
\nabla \cdot \left[ A(0)\nabla K_n(p_0)(\cdot, t) - \int_{T_{n-1}}^t ds A'(t-s) \nabla p_0(\cdot, s) \right] = 0 \quad \text{in } \Omega;
$$

$$
\bar{n} \cdot \left[ A(0)\nabla K_n(p_0)(\cdot, t) - \int_{T_{n-1}}^t ds A'(t-s) \nabla p_0(\cdot, s) \right] = 0 \quad \text{on } \partial \Omega,
$$

$$
\int_\Omega K_n(p_0)(\cdot, t) = 0.
$$

In terms of Schauder estimates, we obtain that

$$
\|K_n(p_0)(\cdot, t)\|_{C^{m+1,\alpha}(\bar{\Omega})} \lesssim \left\| \int_{T_{n-1}}^t ds A'(t-s) \nabla p_0(\cdot, s) \right\|_{C^{m,\alpha}(\bar{\Omega})} \lesssim \int_{T_{n-1}}^t ds |A'(t-s)||\nabla p_0(\cdot, s)||_{C^{m,\alpha}(\bar{\Omega})}.
$$

Let $1/q + 1/q' = 1$. From Hölder’s inequality and Fubini’s theorem, it follows that

$$
\int_{T_{n-1}}^t ds |A'(t-s)||\nabla p_0(\cdot, s)||_{C^{m,\alpha}(\bar{\Omega})} \lesssim \frac{1}{2} \|p_0\|_{L^q(T_{n-1}, T_n; C^{m+1,\alpha}(\bar{\Omega}))}.
$$

This implies that

$$
\|K_n(p_0)\|_{L^q(T_{n-1}, T_n; C^{m+1,\alpha}(\bar{\Omega}))} \leq \frac{1}{2} \|p_0\|_{L^q(T_{n-1}, T_n; C^{m+1,\alpha}(\bar{\Omega}))}.
$$

Then we introduce the following notation for the ease of the statement.

$$
\tilde{f}_n(\cdot, t) := \mathcal{L}^{-1} \nabla \cdot \left[ \int_0^t ds A'(t-s) f(\cdot, s) + A(0)f(\cdot, t) - \int_0^{T_{n-1}} ds A'(t-s) \nabla p_0(\cdot, s) \right] \quad \text{in } \Omega,
$$

which would be treated as the known data in the abstract equation below. For a.e. $t \in [T_{n-1}, T_n]$, the equations (6.15) is equivalent to

$$
(p_0 + K_n(p_0))(\cdot, t) = \tilde{f}_n(\cdot, t) \quad \text{in } \Omega.
$$

(6.17)
Thus, the well-posedness of (6.15) is reduced to studying the following contraction map\(^\text{12}\):

\[ T_n(p_0) := \tilde{f}_n - K_n(p_0). \]  

(6.18)

Obviously, the strict contraction property of \( T_n \) in \( L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega})) \) is due to the estimate (6.16).

We continue to verify that the range of \( T_n \) is included in \( L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega})) \), and start from

\[ \| T_n(p_0)(\cdot, t) \|_{C^{m+1, \alpha}(\bar{\Omega})} \leq \| K_n(p_0)(\cdot, t) \|_{C^{m+1, \alpha}(\bar{\Omega})} + \| \tilde{f}_n(\cdot, t) \|_{C^{m+1, \alpha}(\bar{\Omega})}. \]

On account of the estimate (6.16), it suffices to estimate the quantity \( \| \tilde{f}_n \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} \). According to the definition of \( \tilde{f}_n \) above, by using Schauder estimates, we first have

\[ \| f_n(\cdot, t) \|_{C^{m+1, \alpha}(\bar{\Omega})} \]

\[ \lesssim \int_0^t ds |A'(t - s)||f(\cdot, s)|_{C^{m, \alpha}(\bar{\Omega})} + |A(0)||f(\cdot, t)|_{C^{m, \alpha}(\bar{\Omega})} + \int_0^{T_n} ds |A'(t - s)||\nabla p_0(\cdot, s)|_{C^{m, \alpha}(\bar{\Omega})}, \]

and then appealing to Hölder’s inequality and Fubini’s theorem, as well as, the inductive assumption, we obtain

\[ \int_0^{T_n} dt \| f_n(\cdot, t) \|_{C^{m+1, \alpha}(\bar{\Omega})} \]

\[ \lesssim \left( \int_0^{T_n} dt \| A'(t) \|_{C^{m, \alpha}(\bar{\Omega})} \right)^{\frac{1}{q} + 1} \left( \int_0^{T_n} dt \| f(\cdot, t) \|_{C^{m, \alpha}(\bar{\Omega})} \right) + \int_0^{T_n} dt \| p_0(\cdot, t) \|_{C^{m+1, \alpha}(\bar{\Omega})} \]

(6.13)

\[ \lesssim \int_0^{T_n} dt \| f(\cdot, t) \|_{C^{m, \alpha}(\bar{\Omega})}. \]

This coupled with (6.18) leads to

\[ \| T_n(p_0) \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} \leq \| K_n(p_0) \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} + \| \tilde{f}_n \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} \]

(6.16)

\[ \leq \frac{1}{2} \| p_0 \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} + C \| f \|_{L^q(0, T_n; C^{m, \alpha}(\bar{\Omega}))}. \]

Hence, by Banach’s fixed-point theorem, there exists the unique solution \( p_0 \in L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega})) \), satisfying the equation (6.17). Moreover, we have

\[ \| p_0 \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} \leq \frac{1}{2} \| p_0 \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} + C \| f \|_{L^q(0, T_n; C^{m, \alpha}(\bar{\Omega}))}, \]

which further implies

\[ \| p_0 \|_{L^q(T_{n-1}, T_n; C^{m+1, \alpha}(\bar{\Omega}))} \lesssim \| f \|_{L^q(0, T_n; C^{m, \alpha}(\bar{\Omega}))}. \]

As a result, it follows from (6.13) that

\[ \| p_0 \|_{L^q(0, T_n; C^{m+1, \alpha}(\bar{\Omega}))} \lesssim \| f \|_{L^q(0, T_n; C^{m, \alpha}(\bar{\Omega}))}, \]

and this completes the whole proof.

Proof of Proposition 1.4. Combining Lemmas 6.2, 6.4, and 6.5, we have the desired results.

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\(^{12}\)Similar to the form of \( K_1 \), the operator \( K_n \) is linear.
Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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