Green’s function retrieval and fluctuations of cross density of states in multiple-scattering media

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Abstract – In this work we derive the average and the variance of the cross-correlation of a noise wavefield. The noise cross-correlation function (NCF) is widely used to passively estimate the Green’s function between two probes and is proportional to the cross density of states (CDOS) in photonic and plasmonic systems. We first explain from the ladder approximation how the diffusion halo plays the role of secondary sources to reconstruct the mean Green’s function. We then show that fluctuations of NCF are governed by several non-Gaussian correlations. An infinite-range correlation term dominates fluctuations of NCF-CDOS and proves that NCF is not a self-averaging quantity with respect to the plurality of noise sources. The link between these correlations and the intensity ones is highlighted. These results are supported by numerical simulations and are of importance for passive imaging applications and material science.

Introduction. – A wave propagating in a multiple-scattering medium is completely scrambled and generates random intensity patterns. Nevertheless, Weaver and Lobkis [1] showed in 2001 that the time derivative of the cross-correlation of an equipartitioned field measured at two positions $r_A$ and $r_B$ is proportional to the difference of the causal and anti-causal temporal Green’s function (GF). This result originates from the fluctuation-dissipation theorem [2] and has provided a framework for passive imaging systems [3]. It has especially led to spectacular developments in seismology where images of the earth crust have been obtained at different scales with unprecedented resolutions [4,5]. GF retrieval from cross-correlations of a diffuse field has also been applied to acoustic waves [6], elastic waves [7,8] and recently to electromagnetic waves [9]. The NCF has been interpreted as the field that is back-propagated by a time reversal mirror that completely surrounds a multiple-scattering medium [10].

In the frequency domain, the NCF reduces to the imaginary part of the GF, $\Im G(r_A, r_B)$. When the positions of the probes coincide ($r_A = r_B$), the NCF linearly depends on the local density of states (LDOS) which counts the number of modes available at a given position. In optics, the LDOS determines spontaneous and stimulated emission of light. The LDOS exhibits spatial fluctuations caused by scatterers in the vicinity of the source [11–14]. The variance of the LDOS is indeed equal to the intensity correlation $C_0$ [12,14], which results from local interaction. This infinite spatial range correlation was identified by Shapiro [15]. It differs from the universal intensity correlations $C_1$, $C_2$ and $C_3$. The short-range contribution $C_1$ simply results from Gaussian statistics. The non-Gaussian contributions $C_2$ and $C_3$ are long- and infinite-range contributions, respectively, and characterize the statistics of enhanced intensity fluctuations [16].

About 10 years ago, van Tiggelen [17] showed that in a random medium, the NCF tends to be self-averaging even though the noise sources are not equally distributed. Because multiple scattering increases the spatial diversity of the field and therefore reduces Gaussian fluctuations, the NCF converges more rapidly towards the average NCF [18,19]. The same conclusion was derived from a parabolic approximation approach of scattering within the time reversal framework [20]. However, when the distance between $r_A$ and $r_B$ is larger than one elastic mean free path ($l_e$), the mean GF vanishes and the self-averaging property of the NCF seems to be in contradiction with the deterministic approach that claims that the NCF is given by $\Im G(r_A, r_B)$. Indeed, even if...
$\| \mathbf{r}_A - \mathbf{r}_B \| \gg l_e$, $\langle |S_G(r_A, r_B)|^2 \rangle > 0$. This result implies that non-Gaussian correlations should contribute to NCF fluctuations.

Here we first show that the ladder approximation helps to interpret the emergence of the average NCF in a multiple-scattering medium. Scatterers located less than a mean free path around the probes play the role of secondary sources. Then we use a diagrammatic expansion of the diffuse field to identify the significant non-Gaussian correlations that characterize NCF fluctuations. We derive the analytical expression of the variance of the NCF for one noise source or a continuous distribution over the scattering volume. We show that the same infinite-range contribution $\gamma_2$ explains why the NCF is not self-averaging and causes fluctuations of the cross density of states (CDOS). We highlight why this contribution cannot be deduced from the classical intensity correlation term $C_2$. Finally, we show that fluctuations of the NCF in the case of a single noise source are due to non-universal local terms. All those results are supported by numerical simulations.

**Noise cross-correlation function.** – We assume a set of uncorrelated wide-band sources of noise represented by the power spectrum function $S_V(r)$ distributed over a volume $V$ (or equivalently a surface in 2D). In the frequency domain, the noise cross-correlation $\zeta_V$ between two probes at locations $r_A$ and $r_B$ is given by

$$\zeta_V(r_A, r_B) = \int_V G^*(r_B, r)G(r_A, r)S_V(r)d^d r, \quad (1)$$

where $d$ is the dimensionality of the space (here 2 or 3). The frequency dependence is kept implicit. When the noise sources are uniformly distributed ($S_V(r) = S_v$), the correlation of the fields is integrated over the entire scattering volume and the NCF is proportional to the imaginary part of the GF [21],

$$\zeta_\infty(r_A, r_B) = -\frac{l_{\text{a}}}{k_0}S_G(r_A, r_B)S_\infty. \quad (2)$$

Here $l_{\text{a}}$ is the absorption (inelastic) mean free path.

**Average of NCF.** – The average value $\langle \zeta_V \rangle$ is governed by $\langle G^*(r_2, r)G(r_1, r) \rangle$. From the Bethe-Salpeter equation and the ladder approximation, $\langle \zeta_V \rangle$ is given by

$$\langle \zeta_V(r_A, r_B) \rangle = \int \langle G^*(r_B, r) \rangle \langle G(r_A, r) \rangle S_V(r)d^d r$$

$$+ \int \langle G^*(r_B, r') \rangle \langle G(r_A, r') \rangle F(r')d^d r'. \quad (3)$$

The halo function $F(r')$ is equal to $\int |G(r, r')|^2 S_V(r)\beta(r', r')d^d r'$. The first term in eq. (3) is the coherent contribution of the field. The second term can be interpreted using eq. (1). The expressions are indeed similar but the power spectrum function is replaced by $F(r')$ and the GF are replaced by the mean ones. The halo that diffuses from the noise sources illuminates the scattering medium. The part of the halo that is closer than an elastic mean free path from points A and B contributes to the mean NCF. In other words, these last scattering events play the role of secondary sources to build up the mean NCF.

To confirm this result, we carry out 2D numerical simulations in the time domain with a finite-difference time domain (FDTD) code. The scatterers are uniformly distributed inside a ring with an inner radius of $5\lambda_0$ ($\lambda_0$ is the central frequency wavelength) and an outer radius of $20\lambda_0$. The mean free path is $l_e = 1.5\lambda_0$ and the noise is emitted from a single source outside the multiple-scattering medium. The time-dependent NCF $\zeta_V(r_A, r_B, t)$ which is the inverse Fourier transform of eq. (3) is recorded and averaged over 270 disorder realizations. In fig. 1 the maps of the $\langle \zeta_V(r_A, r_B, t) \rangle$ at different times are
displayed vs. $r_B$ for a fixed position $r_A$. We clearly observe an almost circular wavefront predicted by the coherent term in (3) that diverges from the noise source position. A second and smaller circular wavefront focuses on point A at negative times and is followed by a diverging one at positive times. As expected this contribution due to the halo appears with a skin depth of about one mean free path. The result is even more spectacular on an animation [see the movie movie4.avi of the cross-correlated mean field given as supplementary material]. Because the halo is almost uniformly distributed over at least one mean free path around $r_A$, $\langle \zeta_V(r_A,r_B) \rangle$ is proportional to $\langle G^*(r_B,r_A) \rangle - \langle G(r_B,r_A) \rangle$. Indeed, $\int \langle G(r,r_A) \rangle G^*(r,r_B) \, dV = -l_e/k_0 \Im \langle G(r_B,r_A) \rangle$ (see footnote 1). The mean GF (respectively conjugate mean GF) represents the diverging (respectively converging) coherent wave.

**Variance of NCF.** — In laboratory experiments, the NCF can easily be averaged over realizations of the disorder. For instance, in optics the scatterers randomly move as a consequence of the Brownian motion. In a microwave experiment, the beads can be mixed in a tuner. However, in seismology the NCF can only be measured in a single realization of the disorder. The NCF is expected to be self-averaging [18,20] as a result of Gaussian statistics. But, in the context of time reversal [22], or phase-conjugation focusing [23], long-range correlations that cannot be predicted by Gaussian fluctuations have been observed. Those correlations arise because of coherent interferences between wave fields of the diffusive modes within the medium and have been successfully characterized by a diagrammatic approach. We show in the following how these interferences enhance the fluctuations of the NCF. To this end, we estimate the variance $\gamma$ of $\zeta_V(r_A,r_B)$. For simplicity, we have replaced $S_\nu$ by an integration over a finite volume $V$ in eq. (1). We first consider an equipartitioned noise field, $V \to \infty$ and $\|r_A - r_B\| \gg l_e$. Equation (2) gives

$$\gamma = \frac{l_e^2}{2k_0} \langle G(r_B,r_A)G^*(r_B,r_A) \rangle .$$

(4)

This simple result shows that the NCF is not self-averaging in the sense that it does not converge towards the mean GF but towards the exact GF. The NCF is therefore sensitive to scatterings that occur at a distance larger than a mean free path from the probes. This fundamental result is crucial in monitoring applications. It shows that it is possible to follow the evolution of a scatterer hidden behind a multiple-scattering media [24].

In the following, we use the diagrammatic approach to interpret eq. (4) and to address fluctuations of the NCF. For a small number of sources, i.e., a limited volume of integration $V$. In fig. 2 are shown the diagrams that contribute to $\gamma$. In the limit $\Delta r \gg \ell_e$, and $V \to \infty$, $\gamma_{2a}$ is the most significant. Since the diagram is long range both in $r - r'$ and in $r_A - r_B$, $\gamma_{2a}$ is of “infinite range” [3]. In fig. 2 the diagram of the long-range correlation $C_2$ widely used to characterize intensity fluctuations is also depicted. Even though the two diagrams involve a single Hikami box, we stress that $\gamma_{2a}$ is not equal to $C_2$ because of an exchange between positions $r_A$ and $r_B$ at the right side. $C_2$ is indeed short range in $r_A - r_B$. From the diagrammatic representation, the expression of $\gamma_{2a}$ is worked out

$$\gamma_{2a} = 2\Delta^4 \langle \Im \langle G(r,r) \rangle \rangle^2 h \left| \int_V L(r) \, d^4r \right|^2 \times \int_V \int_V \Im \langle G(r_A,r_A) \rangle \Im \langle G(r_B,r_B) \rangle \times \left| \nabla L(r_A,s) \right| \nabla L(r_B,s) \, d^4s .$$

Here $h$ is the Hikami constant and $\Delta = l_e/k_0$. The ladder $L$ is the solution of the steady-state diffusion equation with

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1. This relation [16] is similar to eq. (2) but the elastic mean free path replaces the absorption mean free path because the attenuation of the mean field is mainly due to elastic scattering.
2. The variance of a self-averaging quantity should fall towards 0 when the quantity is integrated over time or space.
absorption, i.e., \(-D\nabla^2 L(r) + L(r)c/l_\text{a} = K\delta(r)\). For 3D samples, \(h = l_0^2/48\pi k^2\), \(K = 4\pi c/l_0^2\) and \(D = l_0 c/3\) and for 2D samples, \(h = l_0^2/32k^3\), \(K = 4k_0/l_0\) and \(D = l_0 c/2\). Because the diffusion equation implies that
\[
2\int_V \nabla L(r_A, s) \nabla L(r_B, s) d^2s = \frac{K}{D} \left[ L(r_A, r_B) + L(r_B, r_A) \right],
\]
it leads to
\[
\gamma_{2a} = 2\Delta^2 \left( \frac{3}{D} \langle G(r, r) \rangle \right)^2 \frac{h}{\hbar} \left| \int_V L(r) d^2r \right|^2 \times \frac{K^2}{D^2} \langle L(r_B, r_A) \rangle \frac{3}{D} \langle G(r_A, r_A) \rangle \frac{3}{D} \langle G(r_B, r_B) \rangle.
\]

Equation (7) is valid only for \(|r_A - r_B| > \ell_e\). In the case of coinciding probes \(r_B = r_A\), the NCF is proportional to the LDOS. In that case, NCF fluctuations are characterized by the correlation \(\gamma_{0a}\) which is proportional to the \(C_0\) intensity correlation [12,14]. This non-Gaussian term depends on the details of the local disorder around the probe and involves a non-universal vertex \(\chi_0\) [15], \(\gamma_{0a} = 2\Delta^2 \left( \frac{3}{D} \langle G(r, r) \rangle \right)^2 \frac{h}{\hbar} \left| \int_V L(r) d^3r \right|^2 \chi_0\). For a disorder obeying to white-noise Gaussian statistics, \(\chi_0 = \pi/kl_e\). Whereas the \(C_0\) and \(C_2\) correlations both characterize intensity fluctuations when a point-like source is embedded within the medium, \(\gamma_{0a}\) and \(\gamma_{2a}\) can be seen as the two asymptotic regimes \(\Delta r < \ell_e\) and \(\Delta r > \ell_e\), respectively, of the variance of the NCF in mesoscopic multiple-scattering media. We perform numerical simulations to support these derivations. The 2D multiple-scattering medium is made of \(10^4\) isotropic scatterers enclosed in a disk of diameter \(100\lambda_0\). Here, \(l_0 \sim 63\lambda_0\) and \(l_e \sim 2\lambda_0\). Those parameters ensure that the system is in the diffusive regime but with appreciable NCF fluctuations. The sample is illuminated from \(N\)-independent monochromatic noise sources embedded in the medium. The NCF is computed from a scattering matrix inversion method and statistics of the NCF are estimated from \(500\) disorder realizations. First, we estimate the average similarity (correlation coefficient) between the NCF and the imaginary part of the exact GF \(\langle G(r_A, r_B) \rangle\) when \(|r_A - r_B| > \ell_e\). Here, we define the average similarity as \(\langle \langle G(r_A, r_B) \rangle \langle \hat{G}(r_A, r_B) \rangle \rangle\) normalized by \(\sqrt{\langle \langle G(r_A, r_B) \rangle^2 \rangle \langle \langle \hat{G}(r_A, r_B) \rangle^2 \rangle}\). As expected we observe in fig. 3(a) that the NCF goes toward the exact GF for large \(N\) and is strictly proportional to \(3\langle G(r_A, r_B) \rangle\) for \(N = 10^2\). This is confirmed by simulation results in fig. 3(b) where \(\gamma\) is computed for \(N = 10^3\) with respect to the distance between the positions A and B. Indeed, the exponential decay is in very good agreement with eq. (7) in which the ladder \(L\) is the solution of the 2D diffusion equation with losses, \(L(\Delta r) = \beta K_0 (\Delta r \sqrt{c/Dl_0})/2\pi D\). The discrepancy for \(\Delta r \ll \ell_e\) results from the \(\gamma_{0a}\) contribution that enhances the fluctuations. When \(N\) is larger than \(200\), the NCF gives a very good approximation of \(3\langle G(r_A, r_B) \rangle\). As a consequence, the non-Gaussian contribution \(\gamma_{2a}\) which scales as \(N^2\) overcomes the \(\gamma_1\) Gaussian contribution (see fig. 2) which scales as \(N\) [18]. This is confirmed in fig. 4. On the other hand, \(\gamma\) is larger than \(\gamma_1\) for \(1 < N < 200\). This indicates that, in addition to \(\gamma_1\), other diagrams are contributing to \(\gamma\). Interferences indeed occur in the vicinity of the noise source location \(r_S\) and another \(C_0\)-like correlation has to be taken into account. This contribution \(\gamma_{0b}\) is shown in fig. 2. Its expression is given by
\[
2\Delta^2 \langle L(r_B, r_S) L(r_A, r_S) \rangle \langle G(r_A, r_A) \rangle \langle G(r_B, r_B) \rangle \chi_0.
\]
The volume $\delta V$ of the single source is assumed smaller than $l_s^3$. We finally note that in the case of $\Delta r \ll l_s$ and a single noise source, $\gamma$ is given by the diagram $\gamma_{00}$ shown in fig. 2 [26], $\gamma = 26V^2\Delta^2 |L(r_A, r_S)|^2 \chi_0 (3 \langle G(r_A, r_A) \rangle^2 + 3 \langle G(r_S, r_S) \rangle^2)$. Those considerations confirm the intuitive result that fluctuations of NCF caused by a source located inside a multiple-scattering medium are stronger than fluctuations caused by a source outside this disordered medium which only involves Gaussian fluctuations.

**Conclusion.** – We have used the multiple-scattering theory to demonstrate the role of scatterers in the retrieval of the GF and to interpret fluctuations of the NCF in terms of diffuse light interferences. To that end, we have introduced an original diagrammatic contribution. Those fundamental results can be applied to many different fields such as seismology, acoustics, microwave, optics or material science. In acoustics and in seismology, the estimation of the NCF is easily performed by the direct cross-correlation of recorded time-dependent fields. However, the noise sources are usually not uniformly distributed and a generalization of our approach to more complex source distributions would be a probe of the convergence of the NCF towards the GF for a single realization of disorder. This issue is of importance for imaging purposes. On the other hand, in optics, one can take benefit of the thermal noise that is uniform at thermal equilibrium. But then it is more tedious to measure the NCF. In material science, metallic nanostructures can for instance be excited with surface plasmons in disordered media. Measuring the NCF at thermal equilibrium would make possible to estimate the CDOS. We suggest the experiment consisting in the measurement of the fluctuations of the field intensity diffracted by two tips on a metallic surface at thermal equilibrium where plasmons are multiply scattered to estimate $\gamma_2$ fluctuations. This would be an extension of the thermal radiation scanning tunneling microscopy [27].

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