DEFINABLY AMENABLE GROUPS IN CONTINUOUS LOGIC

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Abstract. We generalize the notions of definable amenability and extreme definable amenability to continuous structures and show that the stable and ultracompact groups are definable amenable. In addition, we characterize both notions in terms of fixed-point properties. We prove that, for dependent theories, definable amenability is equivalent to the existence of a good $S_1$ ideal. Finally, we show the randomizations of first-order definable amenable groups are extreme definably amenable.

1. Introduction

The notion of amenability has been an important tool in model theory. Starting with the seminal work [HPP08] and continued in [CS18] among others, amenability proved to be a key notion in model theory of dependent theories, with important contributions towards, for example, isolating the role of the non forking relation (outside the context of stability where it had been crucial) and of finitely satisfiable generics, understanding the quotient of a group $G$ definable in a (dependent) theory by its smallest type-definable subgroup of bounded index $G^{00}$, and the relation with flows and the Ellis group of definable groups. We believe that it may have a similar role in the context of continuous logic; this paper suggests a notion of amenability in this context, and studies some of its basic properties.

Amenability was introduced first in connection with the Banach-Tarski paradox and paradoxical decompositions. This notion was generalized to locally compact topological groups (the discrete case coinciding with the original definition) and it is defined by any of the following equivalent conditions (see Definition 4.2, Theorem 4.19, Theorem 5.4 and Proposition 4.23 in [Pie84]).

Fact 1.1. Let $G$ be a topological locally compact group. Then the following conditions are equivalent.

1. There is a finitely additive probability left invariant measure $\mu$ on $\mathcal{B}(G)$, the set of Borel subsets of $G$ which is absolutely continuous with respect to the Haar measure.
2. There is a left invariant mean $m$ on $C_B(G)$, the set of real-valued bounded continuous functions on $G$.
3. Any separately continuous and affine action of $G$ on a convex compact set $X$ has a fixed point.
(4) Any separately continuous action of $G$ on a compact set $X$ admits an invariant probability measure $\mu$ on $X$.

(5) There is a left invariant mean $m$ on $\text{RUCB}(G)$, the set of real-valued bounded uniformly continuous functions on $G$.

Here we say that a mean $m$ is invariant if for any function $f$ in the domain of $m$ we have $m(f) = m(gf)$ where $gf(x) = f(g^{-1}x)$, and a finitely additive measure $\mu$ is left invariant if $\mu(A) = \mu(gA)$ for any $A$ in the domain of $\mu$.

Conditions (3) and (4) are always equivalent, condition (3) is known as the Fixed Point Property, and are key when studying amenable locally compact topological groups. Condition (3) gives rise to the following definition:

**Definition 1.2.** A topological group $G$ is extremely amenable if every separately continuous action of $G$ on a compact Hausdorff space $X$ has a fixed point.

In the context of non locally-compact groups, these conditions are not equivalent any more: The group $U(\ell^2(\mathbb{R}))$, the unitary group of the space $\ell^2(\mathbb{R})$ with the strong operator topology is extremely amenable \cite{GM83} but it does not satisfy condition (2). It does however, satisfy condition (5) (also (3) and (4) since they are direct consequences of extreme amenability). These and other examples indicate that conditions (3), (4), and (5) provide the correct generalization of amenability for non-locally compact topological groups. Thus, in the literature, one has the following definitions for general topological groups (see for example \cite{GdH17}):

**Definition 1.3.** Let $G$ be a topological group. Then $G$ is amenable if there is a left invariant mean $m$ on $\text{RUCB}(G)$, the set of right uniformly continuous bounded functions on $G$. $G$ is $B$-amenable if there is a left invariant mean $m$ on $\text{C}_B(G)$, the set of real-valued bounded continuous functions on $G$.

Although in Section 2 we will give a more detailed account of the definitions and notions of continuous logic (based mostly in \cite{YBH08}), we will describe some of its basic ideas to make it easier to follow the rest of the introduction. The universe $M$ of a structure in continuous logic is a metric space, and the predicates are bounded uniformly continuous functions from the space to the real numbers. The interesting examples of models of continuous logic are usually non-locally compact and many of the definitions are based on bounded uniformly continuous functions from the universe of the structures to the real numbers. It therefore makes sense to suggest the following definitions:

**Definition 1.4.** Let $G$ be a structure in continuous logic with a definable group structure (so that the group operation is bi-continuous). We say that $G$ is definably amenable if there is a mean $m$ on the set $\mathcal{P}_M^e|_G$ of unary predicates contained in $G$ (i.e. bounded right-uniformly continuous functions from $G$ to the real numbers) and definable with parameters from $M$, such that $m(P) = m(g \cdot P)$ for any $P \in \mathcal{P}_M^e|_G$. 
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This generalizes a notion that appears naturally in the continuous logic setting: Recall that, in this setting, a global type over a structure $M$ is a complete set of predicates (over $M$) that is realized by a tuple $\bar{a}$ in an elementary extension $N$ of $M$. Therefore, any type $p$ is associated with the mean $m_p(P) = P^N(\bar{a})$. If a group $G$ admits a type $p$ is $G$-invariant, then $m_p$ witnesses that $G$ is definably amenable and $\ker m_p$ is a complete type. The converse also holds by Theorem 2.4 below. We will therefore say that $G$ is extremely definable amenable if it has a $G$-invariant global type. Although not with this name, these groups were studied in [BM21] in the context of randomizations of first order amenable dependent groups.

The structure of the paper is as follows. In Section 2 we recall the basic definitions of continuous logic and show that the notions of definable amenability and extreme definable amenability are logically sound (in the sense that they are preserved under elementary equivalence) and that any amenable group will be definably amenable when interpreted as a continuous logic structure (same for extremely amenable groups). In section 3 we recall some important properties about the space of means in continuous logic and give characterizations of (extreme) definable amenability in terms of fixed-point properties. This generalizes the authors work from [CDOZ21] to the continuous setting.

Section 4 starts the study of the relation between amenable groups and the structure of $G/G^{00}$. This is inspired by first order equivalences proved in [HPP08] and [CS18], we characterize definably amenable dependent groups in terms of the existence of an $f$-generic type, modulo the existence of a wide-type, (whose existence in the classical case is trivial, but we ignore if it always exists in the continuous setting); the extremely definable case, however, is very well understood.

In Section 5 we prove that any bounded stable group is definably amenable.

Finally, in Section 3 we prove that any randomization of a first order amenable group is extremely definably amenable as a continuous logic structure, generalizing Proposition 4.18 in [BM21] outside the context of dependent continuous theories, thus providing a body of examples of extremely definably amenable groups outside dependent theories.

We hope this work starts interesting directions for future work. For example, about the existence of wide-types given an ideal $S$: The existence of such types will prove the existence of $f$-generic types in definably amenable dependent groups, which would complete (via Theorem 4.6) a characterization of definable amenability in the continuous dependent logic context very much in line with what is known in the first order case.

2. Preliminaries, notation and first results

First, we summarize the notation we will use throughout the paper and recall some key theorems from continuous logic. For a more detailed exposition the reader may check [YBHU08].
Note: in this paper we are not assuming that the truth values are in the interval $[0, 1]$; instead, every formula may have its own (bounded) range in the reals. This does not make any difference in the theory since we can always normalize the formulas and their semantic remains intact, but it allows us to treat the set of formulae and predicates as a vector space, which facilitates the translation between logic and functional analysis.

A language $L$ is a triplet $(S, F, R)$ which contains the following data:

1. $S$ is the set of sorts; for each sort $S \in S$, there is a symbol $d_S$ meant to be interpreted as a metric bounded by a positive number $M_S$.
2. $F$ is the set of function symbols; for each function symbol $f \in F$, we formally specify $\text{dom}(f)$ as a sequence $(S_1, ..., S_n)$ from $S$ and $\text{rng}(f) = S$ for some $S \in S$. We will want $f$ to be interpreted as a uniformly continuous function. To this effect we additionally specify, as part of the language, functions $\delta_f^i : \mathbb{R}^+ \to \mathbb{R}^+$, for $i \leq n$. These functions are called uniform continuity moduli.
   
   (When interpreted, the function $f$ satisfies that $d_S(x_i, y_i) < \delta_f^i(\epsilon)$ for $i \leq n \Rightarrow d_S(f(\bar{x}), f(\bar{y})) < \epsilon$.)
3. $R$ is the set of relation symbols; for each relation symbol $R \in R$ we formally specify $\text{dom}(R)$ as a sequence $(S_1, ..., S_n)$ from $S$ and $\text{rng}(R) = K_R$ for some bounded interval $K_R$ in $\mathbb{R}$. As with function symbols, we additionally specify, as part of the language, functions $\delta_R^i : \mathbb{R}^+ \to \mathbb{R}^+$, for $i \leq n$, called uniform continuity moduli.

For each sort $S \in S$, we have infinitely many variables $x^S_i$.

Terms are defined inductively in the same way as in the first order logic:

1. A variable $x^S_i$ is a term with domain and range $S$
2. If $f \in F$, $\text{dom}(f) = (S_1, ..., S_n)$ and $\tau_1, ..., \tau_n$ are terms with $\text{rng}(\tau_i) = S_i$ then $f(\tau_1, ..., \tau_n)$ is a term with range the same as $f$ and domain determined by the $\tau_i$’s.

Formulas are also defined inductively, but the definition of connectives and quantifiers require more attention:

1. If $R$ is a relation symbol (possibly a metric symbol) with domain $(S_1, ..., S_n)$ and $\tau_1, ..., \tau_n$ are terms with ranges $S_1, ..., S_n$ respectively then $R(\tau_1, ..., \tau_n)$ is a formula. Both the domain and uniform continuity moduli of $R(\tau_1, ..., \tau_n)$ can be determined naturally from $R$ and $\tau_1, ..., \tau_n$. These are the atomic formulas.
2. (Connectives) If $f : \mathbb{R}^n \to \mathbb{R}$ is a uniformly continuous function and $\varphi_1, ..., \varphi_n$ are formulas then $f(\varphi_1, ..., \varphi_n)$ is a formula. The domain and uniform continuity moduli are determined from $f$ and $\varphi_1, ..., \varphi_n$.
3. (Quantifiers) If $\varphi$ is a formula and $x$ is a variable of sort $S$, then both $\inf_{x \in S} \varphi$ and $\sup_{x \in S} \varphi$ are formulas.

Given a language $L = (S, F, R)$, a metric structure will be a collection of metric spaces, one for each sort $S$, with a metric $d^M_S$ of diameter at most $M_S$. 
Together with a collection of functions $F^M_i$ and relations $R^M_i$ interpreted in the natural way.

The set of formulas with free variables in $\bar{x}$ in the language $\mathcal{L}$ is denoted by $\mathfrak{F}_x^\mathcal{L}$. The set of sentences $\mathfrak{S}_x^\mathcal{L}$ is denoted by $\text{Sent}_\mathcal{L}$.

When interpreted in a metric structure $M$, formulas become bounded uniformly continuous functions $\varphi^M(\bar{x}) : M^n \to \mathbb{R}$. If the formula $\varphi$ is a sentence, then $\varphi^M$ is just a real number.

A theory $T$ is a set of sentences. The theory of a metric structure $M$ is the set of sentences $\varphi$ such that $\varphi^M = 0$. We say that $M \models T$ if $T \subset \text{Th}(M)$.

Given a theory $T$, we may endow $\mathfrak{S}_x^\mathcal{L}$ with the seminorm $|\varphi(\bar{x})| := \sup\{|\varphi(\bar{a})| : \bar{a} \in M^n, M \models T\}$.

The quotient of $\mathfrak{S}_x^\mathcal{L}$ by the predicates of seminorm 0 is a normed vector space whose completion is denoted as $\mathcal{P}_x^\mathcal{L}$, the set of definable predicates. In other words, a function $P(\bar{x})$ is a definable predicate if there is a sequence of formulas $\varphi(\bar{x})$ that converges uniformly to $P(\bar{x})$ in every model $M \models T$.

We will denote as $\mathcal{P}_x^\mathcal{L}(\bar{A})$ the set of predicates with parameters in the set $\bar{A}$. If the theory $T$ is clear from the context, we simply write $\mathcal{P}_x^\mathcal{L}$.

**Definition 2.1.** A closed set $D$ of $M^n$ is a definable set if the relation $\text{dist}(\bar{x},D)$ is a definable predicate (which we will denote simply by $D(\bar{x})$).

**Definition 2.2.** Let $D_1 \subset M^n$ and $D_2 \subset M^k$ be definable sets. A function $f : D_1 \to D_2$ is definable if there is a definable predicate $P : M^{n+k} \to [0,1]$ such that, for $(x,y) \in D_1 \times D_2$, we have that $P(x,y) = d(f(x),y)$.

**Remark 2.3** (See [FHL+16] Lemma 3.2.5 and Theorem 3.2.6). Let $D \subset M$ be a closed set. The following are equivalent:

- $D$ is a definable set.
- There exists a definable predicate $P(x)$ such that $D$ is the zero-set of $P(x)$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $P(x) < \delta$ then $d(x,D) < \epsilon$.
- For every formula $\psi(x,y)$, the functions $\sup_{x \in D} \psi(x,y)$ and $\inf_{x \in D} \psi(x,y)$ are definable predicates.
- There exists a definable predicate $P(x)$ such that $D$ is the zero-set of $P(x)$ and for every ultraproduct $M_U = \prod_U M_i$ of models of $\text{Th}(M)$ (in the sense of continuous logic), we have that $P(M_U) = \prod_U P(M_i)$.

A partial type $\pi$ (over a set $A$) is any consistent set of definable predicates. If the set is maximal, we say that the type is complete. In this case, $p$ is the kernel of a functional $\hat{p} : \mathcal{P}_x^\mathcal{L} \to \mathbb{R}$ such that, for some model $M$, there exists a tuple $\bar{a}$ in $M^{|\pi|}$ such that $\hat{p}(P) = P(a)$. 


The following theorem probably is too obvious to have been stated anywhere else, so we write down the proof here:

**Theorem 2.4.** A functional \( \hat{p} : \mathcal{P}_M^\bar{x} \to \mathbb{R} \) is Banach-algebra homomorphism if and only if its kernel \( p \) is a complete type.

**Proof.** (\( \Rightarrow \)). Clear.

(\( \Leftarrow \)). We need to show that the kernel of \( \hat{p} \) is finitely consistent. First of all, notice that if a predicate \( \varphi \geq 0 \) satisfies \( \varphi(x) > 0 \) in every model \( M \), then \( \varphi(x) \geq \epsilon \) for some \( \epsilon > 0 \). Whence \( \hat{p}(\varphi) \geq \epsilon \). Therefore, if \( \varphi \geq 0 \) and \( \hat{p}(\varphi) = 0 \), then there exists \( a \) such that \( \varphi(a) = 0 \). Now, take \( \varphi_1, \ldots, \varphi_n \in \ker(\hat{p}) \). Since \( \hat{p} \) is an homomorphism, then

\[
(\hat{p})(\sum_{i=1}^n \varphi_i^2) = (\sum_{i=1}^n \hat{p}(\varphi))^2 - 2 \sum_{i \neq j} \hat{p}(\varphi_i)\hat{p}(\varphi_j) = 0.
\]

Therefore, \( \sum_{i=1}^n \varphi_i^2 \) has a realization \( a \). In particular, \( \varphi_i(a) = 0 \) for all \( i \). \( \square \)

**Theorem 2.5** (Compactness [YBH08] Proposition 8.6). The space of types \( S_M^\bar{x} \) is compact with the weak* topology induced from the dual of \( \mathcal{P}_M^\bar{x} \).

Analogous notions of amenability and extremely amenability for definable groups and definable theories in first-order logic have been studied widely in [HPP08] and [HKP19]. In this paper, we propose such notions for the metric case.

Let us recall that a **definable group** \( (G, \cdot) \) in a structure \( M \) is a group such that the set \( G \subset M^n \) is definable and the multiplication is a definable function \( \cdot : G \times G \to G \) (so there is a definable predicate \( Q(x, y, z) \) such that \( Q(g_1, g_2, h) = d(g_1 \cdot g_2, h) \) for every \( g_1, g_2, h \in G \)).

If \( P(\bar{x}) \) is any definable predicate contained in \( G(\bar{x}) \), then we may define a new predicate \( g \cdot P(x) := \inf_y (P(y) + Q(g^{-1}, x, y)) \). Notice that if \( x \in G \), then \( g \cdot P(x) = P(g^{-1}, x) \).

If \( G \) is a definable group and \( p \) is a complete type in \( \mathcal{L}(M) \) such that \( G(x) \in p \), then, for each \( h \in G \), we define \( q = h \cdot p \) to be the type of \( h \cdot g \), where \( g \) is any realization of \( p \).

**Definition 2.6.** A functional \( m \) on \( \mathcal{P}_M^\bar{x} \) is called a **mean** if \( m(P) \geq 0 \) when \( P \geq 0 \) and \( m(1) = 1 \), where \( 1 \) is the constant function of value 1.

Notice that, for any type \( p \), the function \( \hat{p} \) is a mean.

**Definition 2.7.** A mean over \( \mathcal{P}_M^\bar{x} \) is **M-definable** if it is \( M \)-invariant and for every \( M \)-formula \( \varphi(\bar{x}, \bar{y}) \), and \( r, s \in \mathbb{R} \), the set
\[
\{ q \in S_y(M) : r < m(\varphi(\bar{x}, b)) < s \text{ for any } b \in M; b \models q \}
\]
is an open subset of \( S_y(M) \).

**Remark 2.8.** A complete type \( p \) is **definable** over \( M \) (as a mean) if for each \( M \)-formula \( \varphi(\bar{x}, \bar{y}) \) there exists a predicate \( P(\bar{y}) \) definable over \( M \) such that for all the tuples \( \bar{b} \in M \) we have that \( P(\bar{b}) = r \) if and only if \( \hat{p}(\varphi(\bar{x}, \bar{b})) = r \).
Quoting known results of continuous logic, we can now prove some easy facts of our notions of amenability and extreme amenability. This provide evidence that they are the correct ones in this context.

It is noted in [Iva21] that every group $G$ that admits a continuous structure has a bi-invariant metric. Thus, every bounded uniformly continuous function is in $RUCB(G)$. In particular, formulas are $RUCB$.

**Remark 2.9.** If a metric group $G$ is (extremely) amenable, then it is (extremely) definably amenable.

**Proof.** If $G$ is amenable, then there is a mean on $RUCB(G)$. Since the formulas are all bounded right uniformly continuous, we have that $G$ is definably amenable.

If $G$ is extremely amenable, then its action on the space of types $S(G)$ has a fixed point, this is an invariant type, hence $G$ is definably extremely amenable. $\square$

**Theorem 2.10.** Let $\{M_i\}_{i \in I}$ be a family of metric structures elementary equivalent to each other and let $G$ be a definable group. If $G(M_i)$ is (extremely) definably amenable for every $i$, then $G(\prod_U M_i)$ is (extremely) definably amenable.

**Proof.** Let $m_i$ be an invariant mean on $G(M_i)$ and $\varphi(x,a_i)$ a predicate with parameters in $a_i$. Since $M_i \equiv M_j$, then there exists a real number $r$ such that $M_i \models |\varphi(x,y)| \leq r$. In particular, $M_i \models |\varphi(x,a_i)| \leq r$; thus,

$$\{m_i(\varphi(x,a_i)) : i \in I\} \subset [-r,r].$$

Since $[-r,r]$ is compact and Hausdorff, the ultralimit $\lim_{i \in U}(m_i(\varphi(x,a_i)))$ exists. We define $\hat{m}$ in the ultraproduct as

$$\hat{m}(\varphi([x],[a_i])) = \lim_{i \in U}(m_i(\varphi(x,a_i))).$$

Now, it is routine to check that the ultralimit of positive functionals is a positive functional and that $\hat{m}(1) = 1$, then $\hat{m}$ is a mean. Moreover, since $m_i$ is $G(M_i)$ invariant, we have that $\hat{m}$ is $G(\prod_U M_i)$ invariant. Finally, if $m_i$ is a type, by Theorem 2.4 it is a Banach-homomorphism, then $\hat{m}$ is also a Banach-homomorphism, therefore it is a type. $\square$

**Corollary 2.11.** If $M \prec N$ and $G$ is a definable group, then $G(M)$ is (extremely) definably amenable if and only if $G(N)$ is.

**Proof.** If $G(M)$ is (extremely) definably amenable, then by Keisler-Shelah theorem for continuous logic, there exists $\prod_U M$ an ultrapower of $M$ such that $N \prec \prod_U M$. By the previous theorem, $G(\prod_U M)$ is (extremely) definably amenable. $\square$

### 3. The space of means and fixed point theorems

In order to prove the existence of invariant means, it is useful to look at the space of means in general and take advantage of its topological properties.
As before, $\mathcal{P}^x$ denotes the set of unary definable predicates, which is a normed space with the norm given by

$$|\varphi| = \sup_{x \in M} |\varphi(x)|.$$

Let us denote by $\Sigma(M)$ the space of means over $M$ and, as usual, $S_M(x)$ denotes the space of complete types over $M$. By the previous remark, we know that $S_M(x) \subset \Sigma(M)$.

By Alaouglu’s theorem, the unit ball of the dual set of $\mathcal{P}^x$ is compact with the weak$^*$-topology. Moreover, the $\Sigma(M)$ is closed in the unit ball: it is the preimage of 1 of the (continuous) function $\nu' \to \nu'(1)$, whence it is compact as well. Finally, notice that both the $S_M(x)$ and $\Sigma(M)$ are closed as well, therefore compact.

Clearly, the set $\Sigma(M)$ is convex, hence the closure of the convex hull of $S_M(x)$ is contained in $\Sigma(M)$. We will show that the converse is also true. This is a direct corollary of the following theorem:

**Theorem 3.1** (Phelps [Phe63]). Let $A$ and $B$ be algebras of real valued functions on the sets $X$ and $Y$ respectively and suppose that $1 \in A$. Let $K_0'(A,B)$ be the convex set of all linear operators $T$ from $A$ to $B$ which satisfy $T \geq 0$ and $T(1) \leq 1$. Then $T$ is an extreme point of $K_0'(A,B)$ if and only if $T$ is multiplicative.

**Theorem 3.2.** The closure of the convex hull of the $S_M(x)$ is $\Sigma(M)$.

**Proof.** Clearly, $\Sigma(M)$ is convex and compact. Now, the extreme points of $\Sigma(M)$ are precisely $S_M(x)$ by the previous theorem and Theorem 2.4 since $(\mathcal{P}^x_L)'$ is locally convex, then by Choquet’s theorem [CM63], the closure of the convex hull of $S_M(x)$ is $\Sigma(M)$. $\square$

**Definition 3.3.**
- Let $D$ be a definable set in a structure $M$ and $K$ be a compact Hausdorff space. We say that $f : D \to K$ is definable if, for every $C \subset U \subset K$, with $C$ closed and $U$ open, there exists a definable predicate $P(x)$ and $\epsilon > 0$ such that

$$f^{-1}(C) \subset \{d \in D : P(d) = 0\} \subset \{d \in D : P(d) < \epsilon\} \subset f^{-1}(U).$$

- A definable action of $G$ on $X$ (or definable $G$-flow) is an action $G \times K \to K$ such that
  - For every $g \in G$, $x \to gx$ is an homeomorphism.
  - For every $k \in K$, $g \to gk$ is definable.

If all the types based on $G$ are definable, then the action of $G$ onto its space of types is definable. In general, this hypothesis is too strong because a theory $T$ is stable if and only if all its types over all models are definable. Since we are not assuming stability here, we propose a weaker definition of definable action:

**Definition 3.4.** A weak definable action of $G$ on a compact space $K$ (or weak definable $G$-flow) is an action $G \times K \to K$ such that
For every \( g \in G \), \( x \mapsto gx \) is an homeomorphism.

For some \( k_0 \in K \), \( g \mapsto k_0x \) is definable.

In this case, the triple \( (K,G,k_0) \) is called a weak definable ambit.

Notice that the natural action of \( G \) on its space of types makes \( (S(G),G,tp(e)) \) a weak definable ambit:

- For every \( g \in G \), \( p \mapsto gp \) is a bijection in \( S(G) \) that sends the open basic sets \( [P(x) < \epsilon] \) to the open set \( [g^{-1}P(x) < \epsilon] \). Thus it is an homeomorphism.
- Let \( C \subset U \subset S(G) \), with \( C \) closed and \( U \) open. By definition of the logic topology, there exists a definable predicate \( P(x) \) and \( \epsilon > 0 \) such that
  \[
  C \subset [P(x) = 0] \subset [P(x) < \epsilon] \subset U.
  \]

By taking the inverse image of by the function \( f(g) = gtp(e) \) we get that
  \[
  f^{-1}(C) \subset \{ g \in G : P(g) = 0 \} \subset \{ g \in G : P(g) < \epsilon \} \subset f^{-1}(U).
  \]

Moreover, by a similar reasoning it can be shown that the action of \( G \) on its space of means makes \( (\Sigma(G),G,\hat{tp}(e)) \) a weak definable ambit as well. If all the types (respectively means) are definable, then \( (S(G),G,tp(e)) \) (\( (\Sigma(G),G,\hat{tp}(e)) \) respectively) is a definable ambit.

The following result has been proved for the first-order case (when all the types are definable \([GPP14]\)). The result is also valid in this context:

**Theorem 3.5.** The weak definable ambit \( (S(G),G,tp(e)) \) is universal in the category of weak definable ambits. In particular, if all the types are definable, then \( (S(G),G,tp(e)) \) is universal in the category of definable ambits.

**Proof.** Let \( (X,G,x_0) \) a weak definable ambit and let \( f : G \rightarrow X \) given by \( f(g) = g \cdot x_0 \). We extend \( f \) to a function \( f^* : S(G) \rightarrow X \) as follows:

\[
  f^*(p) = \bigcap_{P \in p} \overline{fP(G)}
\]

First of all let us check that the function \( f^* \) is well-defined: suppose that \( a \neq b \) are \( \bigcap_{P \in p} \overline{fP(G)} \) and let \( C_1 \) and \( C_2 \) disjoint neighborhoods of \( a \) and \( b \). Since \( f \) is definable, then

\[
  f^{-1}(C_1) \subset \{ g \in G : P(g) = 0 \} \subset \{ g \in G : P(g) < \epsilon \} \subset f^{-1}(C_2^c).
\]

Thus, \( b \notin \overline{fP(G)} \), a contradiction.

Now let use check that \( f^* \) is a homomorphism of \( G \)-ambits (by construction, it is the only possible one):
• The function $f^*$ is a homomorphism of $G$-spaces:

$$gf^*(p) = g \bigcap_{P \in p} f(P(G))$$

$$= \bigcap_{P \in p} gP(G)x_0$$

$$= \bigcap_{P \in p} gP(G)x_0$$

$$= \bigcap_{P \in p} \bar{f(gP(G))}$$

$$= \bigcap_{Q \in gp} \bar{Q(G)}$$

$$= f^*(gp).$$

• The image of $tp(e)$ is $x_0$: since $f(e) = x_0$, we have that $x_0 \in \bar{f(P(G))}$ for every $P \in tp(e)$. Thus $x_0 = \bigcap_{P \in tp(e)} \bar{fP(G)} = f^*(tp(e)).$

Using Theorem 3.2 we get the following corollary:

**Corollary 3.6.** The weak definable ambit $(\Sigma(G), G, \hat{tp}(e))$ is universal in the category of weak definable ambiits $(K,G,k_0)$, where $K$ is a compact convex set of a locally convex vector space.

Finally, we can establish the main theorem of this section, whose proof follows from the discussion above:

**Theorem 3.7.** Let $G$ be a definable group. Then

1. Characterization of extreme definable amenability:
   • $G$ is extremely definably amenable if and only if every weak definable action over a compact set has a fixed point. Moreover, if all the types in $S(G)$ are definable, then:
     • $G$ is extremely definably amenable if and only if every definable action over a compact set has a fixed point.

2. Characterization of definable amenability:
   • $G$ is definably amenable if and only if every weak definable action over a compact convex subset on a locally convex space has a fixed point. Moreover, if all the means in $\Sigma(G)$ are definable, then:
     • $G$ is definably amenable if and only if every definable action over a compact convex subset on a locally convex space has a fixed point.
4. Amenability, S1 ideals, f-generic types and $G^{00}$

This section is an attempt to recover some of the results in [HPP08] and [CS18] relating (in the dependent first-order context) amenability with the structure of $G/G^{00}$ in terms of stabilizers of f-generic types.

We will start with some definitions in the metric setting:

**Definition 4.1.** A type $\pi(x; B_0)$ *divides* over $C$ if there exists an indiscernible sequence $\{B_i\}_{i<\omega}$ such that the $\bigcup \pi(x; B_i)$ is not satisfiable.

We say that $\pi(x, b_0)$ *forks* over $C$ if there exists $D \supset \{b_0\} \cup C$ such that every complete type $p$ over $D$, with $p \supset \pi(x, b_0)$, divides over $C$.

Let $T$ denote a complete continuous theory and $\bar{M}$ a $\kappa$-saturated, $\kappa$-strongly homogeneous model of $T$, for $\kappa$ large enough.

**Definition 4.2.** A formula $\varphi(x, y)$ has the independence property (IP) if there are $r<s$, sequences $\{a_i\}_{i<\omega}$ and $\{b_I\}_{I \subset \omega}$ in $\bar{M}$ such that $\varphi(a_i, b_I) < r$ for $i \in I$ and $\varphi(a_i, b_I) > s$ for $i \notin I$. A theory $T$ is *dependent* if no formula has the independence property.

Let $G$ be a definable group, by $G^{00}_A$ we denote the intersection of all type-definable subgroups over $A$ of bounded index. If $G^{00}_A$ does not depend on $A$ we say that $G^{00}$ exists and it is equal to $G^{00}_A$.

The existence of $G^{00}$ in continuous dependent theories can be proved analogously to the classic case (see [BM21], Theorem 2.19).

If the group is extremely definably amenable, we know that $G^{00} = G$, however, for the definably amenable case, things get much more complicated.

We start with the following theorem. The proof is analogous to the classic one [Sim15].

**Theorem 4.3.** Let $T$ be a dependent theory. A global type $p \in S(M)$ is $A$-invariant if and only if $p$ does not fork over $A$.

**Proof.** Since $p$ is a global type, not forking over $A$ is equivalent to not dividing over $A$. Hence every $A$-invariant type does not fork over $A$ (this direction does not require dependent). Now, assume that $p$ is not $A$-invariant. Hence $r = \hat{p}(\varphi(x, a_0)) \neq \hat{p}(\varphi(x, a_1) = s$ for some $A$-indiscernible sequence $<a_i>_{i<\omega}$. By *dependence*, there does not exist any element $b$ such that $\varphi(b, a_{2n}) = r$ for and $\varphi(b, a_{2n+1}) = s$, whence, the formula $(\varphi(x, a_0) - r)(\varphi(x, a_1) - s)$ in $p$ divides. \(\Box\)

These properties are not known outside the dependent context. It is for this reason that we will initially work under the assumption of dependence. We will also need the notion of f-generic types, defined as in [HPP08]:

**Definition 4.4.** A global type $p$ is *f-generic* if there is a small model $M_0$ such that neither $p$ nor its translates (under the group action) fork over $M_0$.

The following fact is well known in the first-order setting, but the proof can be adapted almost verbatim to the continuous setting:
Fact 4.5 (Corollary 8.20 in [Sim15]). If $G$ admits an $f$-generic type over some model $M$, then it admits an $f$-generic type over any model $M_0$.

4.1. Stabilizers and amenability in dependent theories. In this section, we will assume that $G$ is definable in a structure $M$ and $Th(M)$ is dependent. We will fix a small model $M_0$ and a saturated extension $M$ (where types are realized).

Given a type $p$ over $M$ we define 
\[ \text{stab}(p) := \{ h \in G \text{ such that } hp = p. \} \]

By saturation, if $h \in \text{stab}(p)$ then $\exists a, b \in M$ such that $a, b \models p\mid_{M_0}$ and $ha = b$.

Theorem 4.6. If $G$ is a definable group with an $f$-generic type $p$, then:

1. $\text{stab}(p) = G^{00}$.
2. $G$ is definably amenable.

Proof. (1) Let $p$ be the $f$-generic type over $M_0$.

Claim 1: The group $\text{stab}(p)$ is a type-definable group. More precisely,
\[ \text{stab}(p) = \{ g_1^{-1}g_2 | tp(g_1/M_0) = tp(g_2/M_0) \} \]

Proof of Claim 1. Since $gp$ is $Aut_{M_0}(M)$ invariant (by $f$-genericity), then, for $g_1 \equiv_{M_0} g_2$, we have that $g_1p = f(g_1p) = f(g_1)p = g_2p$, for any $f \in Aut_{M_0}(M)$. Whence $g_1^{-1}g_2p = p$.

On the other hand, if $h \in \text{stab}(p)$, then $ha = b$ for some realizations of $p$ outside $M$. This implies that $ha' = b'$ for some realizations of $p\mid_{M_0}$ inside $M$. By naming $g_1 = b^{-1}$ and $g_2 = a^{-1}$ we get the desired result. \(\square\)

Now, $\text{stab}(p)$ is a type-definable group and its index is bounded by 
\[ |S_{M_0}(G)| < 2^{|M_0|} < \kappa. \]

This implies that 
\[ G^{00} \subset \text{stab}(p). \]

On the other hand, since $G^{00}$ is an $M_0$-invariant group of bounded index (because it is $\emptyset$-invariant), then $g_1 \equiv_{G^{00}} g_2$ is an $M_0$-invariant bounded equivalence relation. Therefore, if 
\[ Lstp(g_1/M_0) = tp(g_1/M_0) = tp(g_2/M_0) = Lstp(g_2/M_0) \]

then by definition of the Lascar Type, we have that $g_1 \equiv_R g_2$ for any bounded $M_0$-invariant equivalence relation $R$. In particular, $g_1^{-1}g_2 \in G^{00}$. Thus,
\[ \text{stab}(p) = \{ g_1^{-1}g_2 | tp(g_1/M_0) = tp(g_2/M_0) \} \subset G^{00}. \]

Whence 
\[ \text{stab}(p) = G^{00}. \]

(2) In order to prove that $G$ is definably amenable, let us assume first that $T$ is a countable theory over a countable language and let $\varphi : M \rightarrow \mathbb{R}$ be any
formula defined over a model $M_0$ of countable density character. Assume that $p$ is $f$-generic over $M_0$ as well.

Let us define $f_\varphi : G/G^{00} \to \mathbb{R}$ as follows:

$$f_\varphi([g]) = \hat{p}(g\varphi).$$

Notice that $f_\varphi$ is well-defined since $p$ is $G^{00}$ invariant.

**Claim 2:** $f_\varphi$ is measurable:

*Proof of Claim 2.* We want to check that $\{ [g] \in G/G^{00} : r < \hat{p}(g\varphi(x)) < s \}$ is Borel in $G/G^{00}$. It suffices to show that $\{ g \in G : \hat{p}(g\varphi(x)) \in (r,s) \}$ is a countable union of closed sets of $G$ (because the projection of $S(G)$ onto $G/G^{00}$ is closed).

If $(a_i)_{i<\omega} \models p^{00}_{M_0}$, then $\lim_{i \to \infty} \psi(a_i) = \hat{p}(\psi(x))$. Thus $\hat{p}(\psi(x)) \in (r,s)$ if and only if

(*) There exists a rational number $q \in (r,s)$ such that, for

$$\epsilon_q = \frac{1}{2} \min\{|q-r|,|q-s|\}$$

there exists $N < \omega$ such that $|\psi(a_N) - q| < \epsilon_q$ and $|\psi(a_i) - \psi(a_j)| < \epsilon_q$ for $i,j > N$.

Fix an element $g \in G$. By dependence, for every $\epsilon$ there is a maximal $N$ such that $(a_i)_{i < N} \models p^N_{M_0}$ and $|g\varphi(a_i) - g\varphi(a_{i+1})| \geq \epsilon$ for $i < N$.

Let

$$\Phi^q_N(g) = \left\{ \inf_{x_1, \ldots, x_N} |\varphi_i(x_1, \ldots, x_n)| \cdot \left[ \prod_{i < N} (\epsilon_q - |g\varphi(x_i) - g\varphi(x_{i+1})|) \right] \cdot |g\varphi(x_N) - q| + \epsilon_q : \varphi_i \in p \right\}. $$

(This type says that there are $N$ realizations of the type $p^N_{M_0}$ satisfying (*).)

Hence,

$$f_\varphi(g) = \hat{p}(g\varphi) \in (r,s) \iff g \in \bigcup_{q \in \mathbb{Q} \cap (r,s)} \left( \bigcup_{N < \omega} (\Phi^q_N(g) \cap \neg(\Phi^q_{N+1}(g))) \right).$$

Finally, since $L$ is countable and $M_0$ has a countable dense set, we may assume that $(\Phi^q_{N+1}(g))$ is countable. Thus, $\{ g \in G : \hat{p}(g\varphi(x)) \in (r,s) \}$ is a countable union of closed sets and $\{ [g] \in G/G^{00} : r < \hat{p}(g\varphi(x)) < s \}$ is Borel in $G/G^{00}$. Therefore $f_\varphi$ is measurable.

\[ \square \]

**Claim 3:** $f_{g_0\varphi} = g_0 \cdot f_\varphi$.

*Proof of Claim 3.* $f_{g_0\varphi}([g]) = p(gg_0\varphi) = f_\varphi([g]g_0) = g_0 \cdot f_\varphi([g])$. \[ \square \]

**Claim 4:** $f_{\varphi + \psi} = f_\varphi + f_\psi$. 

Proof of Claim 4. \[ f_{\varphi + \psi}[g] = p(g(\varphi + \psi)) = p(g\varphi + g\psi) = p(g(\varphi + p(g\psi)) = (f_\varphi + f_\psi)[g]. \]

Let us define \( m \) as
\[ m(\varphi) = \int f_\varphi. \]

Claim 5: \( m \) is a \( G \)-invariant functional \( m : Form \to \mathbb{R} \) and \( m(1) = 1 \).

Proof of Claim 5.
- \( m \) is \( G \)-invariant: \( m(g \varphi) = \int f_{g\varphi} = \int g \cdot f_\varphi = \int f_\varphi. \)
- \( m \) is a functional: \( m(\varphi + \psi) = \int (f_{\varphi + \psi}) = \int f_\varphi + f_\psi = \int f_\varphi + \int f_\psi. \)
- \( m(1) = 1: f_1[g] = p(g \cdot 1) = p(1) = 1 \), hence \( m(1) = \int f_1 = \int 1 = 1. \)

So if \( G \) has an \( f \)-generic type, then it is definably amenable.

4.2. Towards proving the existence of an \( f \)-generic types in definably amenable groups. In this section we prove a partial converse to Theorem 4.6. So we assume throughout this subsection that \( G \) is a definably amenable group with mean \( m \) and all the predicates live in \( G \). For the results we do manage to prove, we don’t need any further hypothesis on the theory.

We will first show the existence of an \( S_1 \) ideal of small positive predicates. We then show that if a type contains only formulas outside the ideal, then it is \( f \)-generic. So the only missing piece is to show that in any definably amenable group, there are types which avoid all small positive predicates.

Definition 4.7 (Classical First-Order). An ideal \( I \) of formulas is \( S_1 \) if for any formula \( \varphi(x, y) \) and indiscernible \( (a_i)_{i \in \mathbb{N}} \), if \( \varphi(x, a_i) \cap \varphi(x, a_j) \in I \) for \( i \neq j \), then some \( \varphi(x, a_i) \in I \).

In the classical first-order setting, it can be shown that if \( G \) is definably amenable, then the set of small formulas is \( S_1 \). In this context, by a small formula we mean a formula of measure 0.

We will prove an analogue of this fact in the continuous setting, however, the notion of small needs to be adjusted: intuitively, we want to define a predicate \( P(x) \) to be small if its set of zeros is small. Let us keep this assumption for now and assume that \( P(x) \geq 0 \), since, for almost all \( x \), \( P(x) > 0 \), we may find some \( \lambda \) big enough such that \( \lambda P(x) > 1 \) almost everywhere. Hence, \( m(1 - \lambda P(x)) \to 0 \) as \( \lambda \to \infty \). Since in practice we do not have any measure on the subsets of the model, we adopt this as a definition.

From now on, we will be working almost exclusively with positive predicates with parameters.

Definition 4.8. Let \( P(x) \) be a positive predicate. We define \( \overline{P}(x) \) as \( 1 - (1 - P(x)) \). This is, \( \overline{P}(x) = P(x) \) if \( P(x) < 1 \), otherwise \( \overline{P}(x) = 1 \).

Remark 4.9. If \( P \) and \( Q \) are positive predicates, then \( \overline{PQ} \leq \overline{PQ} \) and
\[
\frac{1}{2}(\overline{P} + \overline{Q}) \leq \overline{P + Q} \leq \overline{P} + \overline{Q}.
\]
Proof. Since $P \leq P$ and $Q \leq Q$, then $PQ \leq PQ$, so $PQ \leq PQ$. The other property is also easy to check. □

**Definition 4.10.** Let us denote $\lim_{\lambda \to \infty} m(\lambda P)$ as $m_\infty(P)$. We say that $P$ is **small** if it is a positive predicate and $m_\infty(P) = 1$.

**Lemma 4.11.** If $P$ and $Q$ are small, then $P + Q$ and $PQ$ are small.

Proof. By linearity of $m$ and Remark 4.9, we have that $P + Q$ is small. For $PQ$, let $\epsilon > 0$, take $\lambda_1$ such that $1 - m(\lambda_1 P) < \epsilon$ and $1 - m(\lambda_2 Q) < \epsilon$. Notice that $\lambda_1 P \lambda_2 Q \leq \lambda_1 \lambda_2 PQ$, therefore:

$$1 - m(\lambda_1 \lambda_2 PQ) = 1 + m(-\lambda_1 \lambda_2 PQ)$$

$$= 1 + m(\lambda_1 P - \lambda_1 P - \lambda_1 \lambda_2 PQ)$$

$$\leq 1 + m(\lambda_1 P - \lambda_1 P - \lambda_1 P \lambda_2 Q)$$

$$= 1 - m(\lambda_1 P) + m(\lambda_1 P(1 - \lambda_2 Q))$$

$$\leq 1 - m(\lambda_1 P) + m(1 - \lambda_2 Q)$$

$$\leq \epsilon + \epsilon$$

□

**Lemma 4.12.** For every positive predicates $P$ and $Q$ we have

$$PQ \leq 1 + P + Q - P + Q - 1/2(P + Q).$$

Proof. If $P \geq 1$ then the inequality would be

$$Q \leq 1 + 1 + Q - 1 - 1/2(P + Q)$$

which is trivially true. The same holds if $Q \geq 1$, then we may assume that $P$ and $Q$ are both less than 1.

Notice that $P = \overline{P}$, $Q = \overline{Q}$ and $1/2(P + Q) = 1/2(P + Q)$. Now, since $P$ and $Q$ are both in $[0, 1]$, then $PQ \leq \sqrt{PQ}$, therefore

$$PQ \leq 1/2(P + Q).$$

Now, since $P + Q \leq 1$, then

$$PQ \leq (1 - P + Q) + 1/2(P + Q) = 1 + P + Q - P + Q - 1/2(P + Q)$$

which is what we wanted to prove. □

**Definition 4.13.** Let $I$ be a set of positive predicates. We say that $I$ is an **ideal**, if:

1. Whenever $P \in I$ and $Q \geq P$, then $Q \in I$.
2. If $P$ and $Q$ are in $I$, then $PQ$ is in $I$.
3. $\lambda \in I$ for $\lambda > 0$. 

The ideal is *invariant over* \( A \) (\( A \)-invariant), if for every \( P(\vec{x}, \vec{a}) \in I \) and \( \vec{a} \equiv_A \vec{a}' \), we have that \( P(\vec{x}, \vec{a}') \in I \).

We say that an \( A \)-invariant ideal \( I \) is *\( S_1 \)-over* \( A \) if, for every \( A \)-indiscernible sequence of predicates \( (P_i)_{i<\omega} \), if \( P_i + P_j \in I \) for every \( i \neq j \), then \( P_i \in I \) for every \( i \).

**Theorem 4.14.** If \( G \) is definably amenable and the measure is \( M \)-invariant, then the set \( I \) of small positive predicates is an \( \lambda \)-ideal over \( M \).

**Proof.** It is clear that \( I \) is an ideal and \( M \)-invariant. Let us prove that it is \( S_1 \):

Let \((P_i)_{i<\omega} \) be a \( M \)-indiscernible sequence of predicates, such that \( P_i + P_j \) is in \( I \) for \( i \neq j \). Let us show that \( P_i \) is in \( I \). Since \( P_i \) is in \( I \) if and only if \( \overline{P_i} \) is, we are going to assume that \( P_i \leq 1 \) for every \( i \).

Suppose that \( P_i \notin I \), then there exists some \( \epsilon > 0 \) such that \( m_\infty(P_i) = 1 - \epsilon \) for every \( i \). We will prove that:

1. \((P_1P_2)_{i,j<\omega} \) is an indiscernible sequence (with the lexicographic order).
2. The predicates \( P_1P_j + P_kP_i \) are in \( I \).
3. \( m_\infty(P_1P_2) \leq 1 - 2\epsilon \).

**Proof.**

1. Clear.
2. We need to show that \( P_1P_j + P_kP_1 \) is small. Since \( (P_i + P_k), (P_i + P_j), (P_j + P_k) \) and \( (P_1 + P_k)(P_i + P_i)(P_j + P_k) \) are small, then \( (P_1 + P_k)(P_i + P_i)(P_j + P_k) \) is. Notice that this multiplication has 16 terms, 8 of those have as a factor \( P_iP_j \) and the other 8 has \( P_kP_1 \) as a factor, therefore \( (P_1 + P_k)(P_i + P_i)(P_j + P_k)(P_j + P_k) \leq 8(P_1P_j + P_kP_1) \); hence, \( 8(P_1P_j + P_kP_1) \) is in \( I \). In particular, \( (P_1P_2 + P_kP_1) \in I \).
3. Using the previous lemma, we have that
   \[
   \lambda P Q \leq 1 + \sqrt{\lambda P} + \sqrt{\lambda Q} - \sqrt{\lambda P} - \sqrt{\lambda Q} - 1/2(\sqrt{\lambda P} + \sqrt{\lambda Q})
   \]
   therefore:
   \[
   m(\lambda P Q) \leq m(\lambda I) + m(\sqrt{\lambda P}) + m(\sqrt{\lambda Q}) - m(\sqrt{\lambda P Q}) - m(\sqrt{\lambda Q}) - m(\sqrt{\lambda Q}) - m(\sqrt{\lambda P}) - m(\sqrt{\lambda P Q})
   \]
   By making \( \lambda \to \infty \), we get:
   \[
   m_\infty(P Q) \leq 1 + (1 - \epsilon) + (1 - \epsilon) - 1 - 1 = 1 - 2\epsilon
   \]

If we proceed inductively, we can show that \( m_\infty(P_1...P_n) \leq 1 - 2n\epsilon \) for every \( n \). This is a contradiction.

**Definition 4.15.** A type \( p \) is *\( I \)-wide* if \( p \cap I = \emptyset \).

Following the notation here \([\text{Hru15}]\), we say that an ideal is *good* if, for every \( \varphi \notin I \), there exists a complete type \( p \supseteq \varphi \) such that \( p \) is \( I \)-wide.
the ideal is a collection of type-definable sets, but the definition is the same).

In first-order, it is easy to check that any ideal of formulas is good. However, even for the case of this precise ideal, the existence of wide types is open:

**Question 4.16.** Assume that $G$ is definably amenable and $m$ is $M$-invariant. Is the set of small formulas a good ideal?

We can now relate all to $f$-generic types.

**Lemma 4.17.** If $I$ is an ideal and $P(x, a) > 0$ in $G$, then $P \in I$.

**Proof.** By saturation, $P(x, a) \geq \lambda$ for some $\lambda > 0$, thus $P$ is in $I$. □

**Theorem 4.18.** If $I$ is a $S_1$-ideal over $M$ and $p$ is $I$-wide, then it is $f$-generic.

**Proof.** Clearly, if $p$ is $I$-wide, all its conjugates are, so it suffices to show that $p$ does not fork (divide) over $M$: otherwise, there exists a formula $\varphi(x, a_0) \in p$ and an indiscernible sequence $(a_i)_{i<\omega}$ such that $\{\varphi(x, a_i)\}_{i<\omega}$ is $k$-inconsistent for some $k$. This implies that $\sum_{i=0}^{k-1} |\varphi(x, a_i)| > 0$, thus, by the previous lemma, it is small. Therefore, by the $S_1$ property, each $|\varphi(x, a_i)|$ is in $S_1$, this is a contradiction. □

**Corollary 4.19.** If $G$ is definably amenable with an $M$-invariant mean and $I$ is the ideal of small formulas. Then any $I$-wide global type is $f$-generic.

It follows of course that a positive answer to Question 4.16 would imply that a definable group is definably amenable if and only if it has an $f$-generic type, with all the consequences this would have through Theorem 4.6. We have not been able to prove this yet.

5. **Stable Groups**

In this section, we show that stable groups are definably amenable. The proof is rather straightforward and it relies on several results proved by Ben-Yaacov in [Yaa10], but as far we know this result has not been stated explicitly before. Let us recall the definitions and results we need:

**Definition 5.1.** A formula $\varphi(x, y)$ is stable if for every $(a_i)_{i<\omega}$ and $(b_j)_{j<\omega}$, we have

$$\lim_{i \to \infty} \lim_{j \to \infty} \varphi(a_i, b_j) = \lim_{j \to \infty} \lim_{i \to \infty} \varphi(a_i, b_j).$$

A theory $T$ is stable if all its formulas are stable in every model $M \models T$.

A group $G$ is stable if $G$ is definable in some stable theory $T$.

If $G$ is a definable group, then $G$ acts continuously on $S(G)$: $gp(x) = p(g^{-1}x)$. This will help us define generic types in continuous logic:

**Definition 5.2** (Ben Yaacov). Let $S$ be a $G$-space. A set $X \subset S$ is generic if finitely many $G$-translates of $X$ cover $S$. A type $p$ is generic if every open set of $S(G)$ containing $p$ is generic.
Example 5.3. The group $S_1$ is compact (hence stable). The formulas $|x| < \epsilon$ are clearly generics, but the set $|x| = 0$ is not generic.

A group is definably connected if $G = G^{00}$. It is known that $\text{stab}(p) \subset G^{00}$ for any type $p$ (see, for example, [Wag00], Lemma 4.1.23). Thus, if $G$ is EDA, then it is definably connected.

Theorem 5.4 (Ben-Yaacov). Let $G$ be a stable group, then:

- The set of generic types $\text{Gen}$ of $G$ is non-empty.
- The group $G$ has a connected component $G^{00}$, this is, there exists the smallest type-definable group of bounded index.
- The group $G$ acts in $\text{Gen}$ and this action is homeomorphic to the action on $G$ on $G/G^{00}$.

The proof of the following fact is analogue to the first-order case (see, for example, Lemma 8.10 in [Sim15]).

Lemma 5.5. The group $G/G^{00}$ is a compact group.

Theorem 5.6. Stable groups are definably amenable.

Proof. Since $G/G^{00}$ is a compact group, it has an invariant mean $m^*$ on all its bounded uniformly continuous functions. Let $\psi : G/G^{00} \rightarrow \text{Gen}$ the isomorphism (as $G$-sets), let us define a function $m$ on all the formulas $m(\varphi)$ as $m^*(\psi \circ \varphi^*)$, where $\varphi^* : \text{Gen} \rightarrow \mathbb{R} : \varphi^*(p) = p(\varphi)$. It is easy to check that $m^*$ is an invariant mean on all the formulas. \hfill \Box

Example 5.7. Let $(\Omega, \mu)$ be a probability space. We can define a metric group $(B, *, d)$, where $B$ is the set of measurable sets of $\Omega$, the operation group is given by $X * Y = X \triangle Y$ and the distance is given by $d(X, Y) = \mu(X \triangle Y)$. This is an $\omega$-stable group, hence definably amenable.

We call this group the probability group of $(\Omega, \mu)$.

We will see in the following section that this group is, in fact, extremely definably amenable.

6. Randomizations

In this section, we prove that the randomization or first-order definably amenable groups are extremely definably amenable. This generalizes the results from Berenstein and Muñoz [BM21] that was restricted to dependent groups.

Fix $(\Omega, B, \mu)$ a probability space and $L$ a first-order language.

We define a new language $L^R$ as $L^R = \{K, B, \top, \bot, \cap, \cup, \neg, [\varphi_i(x)]\}$, where $K$ and $B$ are sorts, $\top, \bot, \cap, \cup, \neq$ are symbols for the boolean operations and $[\varphi_i(x)]$ are functions $[\varphi_i(x)] : K^n \rightarrow B$, where $n$ is the number of free variables of $\varphi_i$. 
Given a first order $L$-structure $M$, let $(\Omega, B, \mu)$ be an atomless finitely additive probability space and $\mathcal{K}$ a set of functions

$$\{f_i : \Omega \to M | i \in I\}.$$  

Both $K$ and $B$ are pre-metric spaces with

$$d_K(X, Y) = \mu(\{m \in M | X(m) \neq Y(m)\})$$

and

$$d_B(A, B) = \mu(A \Delta B).$$

The pair $(B, K)$, is called a randomization of $M$ if, as a premetric structure in $L^R$, satisfies the following:

1. For each $L$-formula $\varphi(x)$ and $f$ in $K^n$, we have
   $$[\varphi_i(X)] = \{m \in M : M \models \varphi(X(m))\}$$

2. For every $B \in B$ and $\epsilon > 0$ there are $f, g \in K$ such that
   $$\mu([f = g] \Delta B) < \epsilon$$

3. For every $L$-formula $\varphi(x; y)$, $g$ in $K^n$ and $\epsilon > 0$ there is $f \in K$ such that
   $$\mu([\varphi(f, g)] \Delta [\exists \varphi(x; g)]) < \epsilon$$

The interpretation of $\top, \bot, \sqcap, \sqcup, \neq$ are the usual boolean operations on $B$.

The completion of $K$ and $B$ are continuous structures.

One associates an $L^R$ continuous theory $T^R$ to every first order $L$ theory $T$, see [BYJK09]. This theory $T^R$ is the common theory of all the randomizations of models of $T$.

**Fact 6.1** ([BYJK09]). $T$ is complete/stable/dependent if and only if $T^R$ is.

**Definition 6.2.** Let $(\Omega, B, \mu)$ be an atomless finitely additive probability space and let $M$ be a $L$-structure.

A $B$-simple random element of $M$ is a $B$-measurable function in $M$ with finite range.

A $B$-countable random element of $M$ is a $B$-measurable function in $M^\Omega$ with countable range.

It is proven in [Kei99] that the set $K_S$ of $B$-simple random elements of $M$ is a full-randomization of $M$ and that $K_C$, the set of countable random elements is a full-randomization of $M$ that is pre complete. Moreover $K_S$ is dense in $K_C$ ([BYJK09]).

Now we prove the main theorem of this section:

**Theorem 6.3.** Let $G$ be a definably amenable group with measure $\nu$. Then $G^R$ is extremely definably amenable.
Proof. Since EDA is a property of the theory of $G^R$, we may assume that $M^R = K_C$ where $K_C$ is induced by the countable random elements. Moreover, without loss of generality, we may assume that $\Omega = [0,1]$. Let $p$ be the following type:

$$p = \{ \mu[\varphi(x,\bar{a})] = \sum_{m_i \in \text{ran}(a)} \nu(\varphi(x,m_i))\mu(\bar{a}^{-1}(m_i))|\bar{a} \in K_S}\}$$

By quantifier elimination and density of $K_S$, this determines a unique complete type.

Notice that the definition of $p$ is quite natural: if $\bar{a}$ has constant value $m$, then $\bar{p}(\varphi(x,\bar{a})) = \nu(\varphi(x,m))$. The extension to non-constant variables $\bar{a}$ is basically the integral over that function (which is equal to a finite sum since $\bar{a}$ is simple).

Let us check that $p$ is consistent: take $p_0$ a finite subset of $p$,

$$p_0 = \{ \mu[\varphi_j(x,\bar{a}_j)] = \sum_{m_{i,j} \in \text{ran}(a_j)} \nu(\varphi_j(x,m_{i,j})\mu(\bar{a}_{j}^{-1}(m_{i,j})): j \leq n\}$$

Let us define $A_{i,j} = \bar{a}_{j}^{-1}(m_{i,j})$. Notice that $\{A_{i,j}\}_{m_{i,j} \in \text{ran}(a_j)}$ is a partition of $\Omega$. Hence, the non-empty intersections of the form $\bigcap_{j \leq n} A_{i,j}$ form a partition of $\Omega$. We build $f$, a realization of $p_0$, as follows: For every

$$I = \{(i_1,1), ..., (i_n,n)\}.$$ 

We name any of such intersections as

$$A_I = A_{i_1,1} \cap ... \cap A_{i_n,n}.$$ 

Take $B_1, ..., B_l$ the atoms of the Boolean algebra generated by $\varphi_j(x,m_{i,j})$ and choose any partition of $A_I$ of the form $\{A_{I,r}\}_{r \leq l}$ such that $\mu(A_{I,r}) = \nu(B_r)\mu(A_I)$, (it exists since $\sum \nu(B_r) = 1$).

Now, for each $A_{I,r}$, we take a point $b_{I,r}$ satisfying $B_r$ and define $f : \Omega :\rightarrow G$ such that $f[A_{I,r}] = b_{I,r}$. Since $\{A_{I,r}\}$ is a measurable partition of $\Omega$, then $f \in K_S$.

Let us check that $f \models p_0$:...
\[ \mu(\varphi_j(f, \bar{a}_j)) = \mu\{\omega :|= \varphi_j(f(\omega), \bar{a}_j(\omega))\} \]
\[
= \sum_{m_{i,j} \in \text{ran}(\bar{a}_j)} \mu\{\omega \in A_{i,j} :|= \varphi_j(f(\omega), m_{i,j})\} \\
= \sum_{m_{i,j} \in \text{ran}(\bar{a}_j)} \sum_{I \in \mathcal{I}(i,j)} \mu\{\omega \in A_{I,x} :|= \varphi_j(f(\omega), m_{i,j})\} \\
= \sum_{m_{i,j} \in \text{ran}(\bar{a}_j)} \sum_{I \in \mathcal{I}(i,j)} \sum_{r \in \mathcal{B}(x, m_{i,j})} \nu(B_r) \\
= \sum_{m_{i,j} \in \text{ran}(\bar{a}_j)} \mu(A_{i,j}) \nu(\varphi_j(x, m_{i,j})) \\
= \sum_{m_{i,j} \in \text{ran}(\bar{a}_j)} \nu(\varphi_j(x, m_{i,j})) \mu(\bar{a}_j^{-1}(m_{i,j})) \\
\]

Now, let us check that \( p \) is \( G^R \)-invariant. Again, since \( K_S \) is dense, it is enough to prove that \( p \) is invariant under the simple random elements in \( G^R \):

Let \( g_R \) a simple random element in \( G^R \) and let \( h_R = g_R^{-1} \); then, in \( p \), we have that

\[
\mu(\varphi_x(\bar{a})) = \mu(\bar{a}) \\
= \sum_{h_i \in \text{ran}(h_R)} \nu(h_i) \left( \mu(h_R^{-1}(h_i)) + \mu(\bar{a}^{-1}(m_j)) \right) \\
= \sum_{m_j \in \text{ran}(\bar{a})} \nu(h_i) \sum_{h_i \in \text{ran}(h_R)} \left( \mu(h_R^{-1}(h_i)) + \mu(\bar{a}^{-1}(m_j)) \right) \\
\]

Finally, by amenability of \( G \) we have that

\[ \nu(\varphi(h_i x, m_j)) = \nu(\varphi(x, m_j)) \]

and, since \( \mu(\Omega) = 1 \)

\[ \sum_{h_i \in \text{ran}(h_R)} (\mu(h_R^{-1}(h_i))) = 1 \]

So we get that:
\[
\mu[gr\varphi(x,\bar{a})] = \sum_{m_i \in \text{ran}(\bar{a})} \nu(\varphi(x,m_i)\mu(\bar{a}^{-1}(m_i)))
\]

Which is what we wanted to prove. \qed

By definition, every extremely amenable metric group is amenable, but also extremely definably amenable. Hence, we have the following diagram:

\[
\begin{array}{ccc}
A & \subseteq & A \cap EDA \\
EA \subseteq & & A \cup EDA \subseteq DA \\
& \subseteq & \subseteq \\
EDA & & 
\end{array}
\]

All the inclusions are strict as it may be seen from First-order examples:

- \((\mathbb{R},+,<) \in A \cap EDA \setminus EA\): it is abelian and the type at \(+\infty\) is invariant, but since it is locally compact, it cannot be extremely amenable.
- The non-abelian free group is stable [Sel03] and connected [Pil08], hence it is extremely definably amenable but not amenable.
- Any non-trivial finite group is amenable but not extremely definably amenable.
- \(F_2 \times F_2\) is definably amenable, since it is stable, but it is not extremely definably amenable (because its connected component is not trivial), nor amenable.

Finally, let us recall some interesting examples from continuous logic:

- Since \(\mathbb{R}\) is definably amenable, its randomization \(L_1(\Omega,\mathbb{R})\) is extremely definably amenable. However [Pes02], it is not extremely amenable with the \(L_1\) metric. Since \(L_1(\Omega,\mathbb{R})\) is abelian, we have that \(L_1(\Omega,\mathbb{R}) \in A \cap EDA \setminus EA\).
- Let \(M_n(\mathbb{C})\) be the full matrix C*-algebra over the complex numbers. Since its unitary group is compact, it is amenable, (hence definably amenable). However, if \(U\) is a non-principal ultrafilter over \(\mathbb{N}\), the ultraproduct \(\prod_U M_n(\mathbb{C})\) is not nuclear (nor elementary equivalent to a nuclear algebra, see Proposition 7.2.5. [FHL+16]). This implies that the unitary group of \(\prod_U M_n(\mathbb{C})\) is not amenable with the weak topology (in particular, it cannot be amenable with the metric topology). However, definable amenability is preserved under ultraproducts (Theorem 2.11), so we have an example of a definably amenable group that is not purely amenable.

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