Pseudo-Riemannian Symmetries on Heisenberg group $\mathbb{H}_3$

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Abstract

The notion of $\Gamma$-symmetric space is a natural generalization of the classical notion of symmetric space based on $\mathbb{Z}_2$-grading of Lie algebras. In our case, we consider homogeneous spaces $G/H$ such that the Lie algebra $\mathfrak{g}$ of $G$ admits a $\Gamma$-grading where $\Gamma$ is a finite abelian group. In this work we study Riemannian metrics and Lorentzian metrics on the Heisenberg group $\mathbb{H}_3$ adapted to the symmetries of a $\Gamma$-symmetric structure on $\mathbb{H}_3$. We prove that the classification of $\mathbb{Z}_2^2$-symmetric Riemannian and Lorentzian metrics on $\mathbb{H}_3$ corresponds to the classification of left invariant Riemannian and Lorentzian metrics, up to isometries. This gives examples of non-symmetric Lorentzian homogeneous spaces.

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1 $\Gamma$-symmetric spaces

Let $\Gamma$ be a finite abelian group. A $\Gamma$-symmetric space is an homogeneous space $G/H$ such that there exists an injective homomorphism

$$\rho : \Gamma \to Aut(G)$$

where $Aut(G)$ is the group of automorphisms of the Lie group $G$, the subgroup $H$ satisfying $G^\rho_e \subset H \subset G^\rho$ where $G^\rho = \{ x \in G / \rho(\gamma)(x) = x, \forall \gamma \in \Gamma \}$ and $G^\rho_e$ is the connected identity component of $G^\rho$ of $G$.

The notion of $\Gamma$-symmetric space is a generalization of the classical notion of symmetric space by considering a general finite abelian group of symmetries $\Gamma$ instead of $\mathbb{Z}_2$. The case $\Gamma = \mathbb{Z}_k$, the cyclic group of order $k$, was considered by A.J. Ledger, M. Obata [13], A. Gray, J. A. Wolf, [8] and O. Kowalski [11] in terms of $k$-symmetric spaces. The general notion of $\Gamma$-symmetric spaces was

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introduced by R. Lutz [12] and was algebraically reconsidered by Y. Bahturin and M. Goze [1]. In this last work the authors proved, in particular, that a $\Gamma$-symmetric space $M = G/H$ is reductive and the Lie algebra $\mathfrak{g}$ of $G$ is $\Gamma$-graded, that is,

$$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$$

with

$$[\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}] \subset \mathfrak{g}_{\gamma\gamma'} \quad \forall \gamma, \gamma' \in \Gamma.$$

Examples.

1. If $\Gamma = \mathbb{Z}_2$ and $\mathfrak{g}$ a complex or real Lie algebra, a $\Gamma$-grading of $\mathfrak{g}$ corresponds to the classical symmetric decomposition of $\mathfrak{g}$.

2. If $\mathfrak{g}$ is a simple complex Lie algebra and $\Gamma = \mathbb{Z}_k$, $k \geq 3$, we have the notion of generalized symmetric spaces and the classification of $\Gamma$-gradings are described by V. Kac in [10].

3. Let $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ be a Lie algebra $\Gamma$-graded. For any commutative associative algebra $\mathcal{A}$, the current algebra $\mathcal{A} \otimes \mathfrak{g}$ (see [18]) also admits a $\Gamma$-grading.

4. In [1], the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading on classical simple complex Lie algebras are classified.

One proves also in [1] that the structure of $\Gamma$-symmetric space on $G/H$ is, when $G$ is connected, completely determinate by the $\Gamma$-grading of $\mathfrak{g}$. Thus, if $G$ is connected, the classification of the $\Gamma$-symmetric spaces is equivalent to the classification of the $\Gamma$-graded Lie algebras. Many results of this last problem concern more particularly the simple Lie algebras. For solvable or nilpotent Lie algebras, it is an open problem. A first approach is to study induced grading on Borel or parabolic subalgebras of simple Lie algebras. In this work we describe $\Gamma$-grading of the Heisenberg algebra $\mathfrak{h}_3$. Two reasons for this study

- Heisenberg algebras are nilradical of some Borel subalgebras.
- The Riemannian and Lorentzian geometries on the 3-dimensional Heisenberg group have been studied recently by many authors. Thus it is interesting to study the Riemannian and Lorentzian symmetries with the natural symmetries associated with a $\Gamma$-symmetric structure on the Heisenberg group. In this paper we prove that these geometries are entirely determined by Riemannian and Lorentzian structures adapted to $(\mathbb{Z}_2^2 \times \mathbb{Z}_2^2)$-symmetric structures.

Recall that the Heisenberg algebra $\mathfrak{h}_3$ is the real Lie algebra whose elements are matrices

$$\begin{pmatrix}
0 & x & z \\
0 & 0 & y \\
0 & 0 & 0
\end{pmatrix} \quad \text{with} \quad x, y, z \in \mathbb{R}$$

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The elements of $h_3$, $X_1$, $X_2$, $X_3$, corresponding to $(x, y, z) = (1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ form a basis of $h_3$ and the Lie brackets are given in this basis by

$$\begin{align*}
[X_1, X_2] &= X_3 \\
[X_1, X_3] &= [X_2, X_3] = 0.
\end{align*}$$

The Heisenberg group is the real Lie group of dimension 3 consisting of matrices

$$\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}$$

$\alpha, b, c \in \mathbb{R}$

and its Lie algebra is $h_3$.

## 2 Finite abelian subgroups of $Aut(h_3)$

Let us denote by $Aut(h_3)$ the group of automorphisms of the Heisenberg algebra $h_3$. Every $\tau \in Aut(h_3)$ admits, with regards to the basis $\{X_1, X_2, X_3\}$, the following matricial representation:

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & 0 \\
\alpha_3 & \alpha_4 & 0 \\
\alpha_5 & \alpha_6 & \Delta
\end{pmatrix}$$

with $\Delta = \alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$.

Let $\Gamma$ be a finite abelian subgroup of $Aut(h_3)$. It admits a cyclic decomposition. If $\Gamma$ contains a cyclical component isomorphic to $\mathbb{Z}_k$, then there exists an automorphism $\tau$ satisfying $\tau^k = Id$. The aim of this section is to determinate the cyclic decomposition of any finite abelian subgroup $\Gamma$.

### 2.1 Subgroups of $Aut(h_3)$ isomorphic to $\mathbb{Z}_2$

Let $\tau \in Aut(h_3)$ satisfying $\tau^2 = Id$. If

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & 0 \\
\alpha_3 & \alpha_4 & 0 \\
\alpha_5 & \alpha_6 & \Delta
\end{pmatrix}$$

is its matricial representation, then the involution can be written in matrix form

$$\begin{pmatrix}
\alpha_1^2 + \alpha_2\alpha_3 & \alpha_1\alpha_2 + \alpha_2\alpha_4 & 0 \\
\alpha_1\alpha_3 + \alpha_3\alpha_4 & \alpha_2\alpha_3 + \alpha_4^2 & 0 \\
\alpha_1\alpha_5 + \alpha_3\alpha_6 + \Delta\alpha_5 & \alpha_2\alpha_5 + \alpha_4\alpha_6 + \Delta\alpha_6 & \Delta^2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

The resolution of the system can be done using formal calculation software. Here we use Mathematica.
Proposition 2.1. Any involutive automorphism $\tau$ of $\text{Aut}(\mathfrak{h}_3)$ is equal to one of the following automorphisms

$$\text{Id}, \quad \tau_1(\alpha_3,\alpha_6) = \begin{pmatrix} -1 & 0 & 0 \\ \frac{\alpha_3}{\alpha_3 \alpha_6} & 1 & 0 \\ \frac{\alpha_6}{2} & \alpha_6 & -1 \end{pmatrix}, \quad \tau_2(\alpha_3,\alpha_5) = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_3 & -1 & 0 \\ \alpha_5 & 0 & -1 \end{pmatrix},$$

$$\tau_3(\alpha_1,\alpha_2 \neq 0,\alpha_6) = \begin{pmatrix} \frac{1 - \alpha_1^2}{\alpha_2} & \alpha_2 & 0 \\ -\alpha_1 & 0 & \alpha_6 \\ (1 + \alpha_1)\alpha_6 & \alpha_2 & -1 \end{pmatrix}, \quad \tau_4(\alpha_5,\alpha_6) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \alpha_5 & \alpha_6 & 1 \end{pmatrix}.$$  

Corollary 2.2. Any subgroup of $\text{Aut}(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_2$ is equal to one of the following:

1. $\Gamma_1(\alpha_3,\alpha_6) = \{\text{Id}, \tau_1(\alpha_3,\alpha_6)\}$,
2. $\Gamma_2(\alpha_3,\alpha_5) = \{\text{Id}, \tau_2(\alpha_3,\alpha_5)\}$,
3. $\Gamma_3(\alpha_1,\alpha_2,\alpha_6) = \{\text{Id}, \tau_3(\alpha_1,\alpha_2,\alpha_6), \alpha_2 \neq 0\}$,
4. $\Gamma_4(\alpha_5,\alpha_6) = \{\text{Id}, \tau_4(\alpha_5,\alpha_6)\}$.

2.2 Subgroups of $\text{Aut}(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_3$

Let $\tau$ be an automorphism satisfying $\tau^3 = \text{Id}$. This identity is equivalent to $\tau^2 = \tau^{-1}$. If we have

$$\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ \alpha_5 & \alpha_6 & \Delta \end{pmatrix}, \quad \Delta = \alpha_1 \alpha_4 - \alpha_2 \alpha_3,$$

then

$$\tau^{-1} = \frac{1}{\Delta} \begin{pmatrix} \alpha_4 & -\alpha_2 & 0 \\ -\alpha_3 & \alpha_1 & 0 \\ \alpha_3 \alpha_6 - \alpha_4 \alpha_5 & \Delta & \alpha_2 \alpha_5 - \alpha_1 \alpha_6 \\ \alpha_5 & \alpha_6 & 1 \end{pmatrix}.$$  

The condition $\tau^2 = \tau^{-1}$ implies $\Delta^3 = 1$ and the only real solution is $\Delta = 1$. Thus $\tau^2 = \tau^{-1}$ is equivalent to

$$\begin{align*}
\alpha_1 \alpha_4 - \alpha_2 \alpha_3 &= 1, \\
\alpha_1^2 + \alpha_2 \alpha_3 &= \alpha_4, \\
\alpha_4^2 + \alpha_2 \alpha_3 &= \alpha_1, \\
\alpha_2 (1 + \alpha_1 + \alpha_4) &= 0, \\
\alpha_3 (1 + \alpha_1 + \alpha_4) &= 0, \\
\alpha_5 (1 + \alpha_1 + \alpha_4) &= 0, \\
\alpha_6 (1 + \alpha_1 + \alpha_4) &= 0.
\end{align*} \tag{1}$$
If $\alpha_1 + \alpha_4 \neq -1$, then $\alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0$ and $\alpha_1 = \alpha_4 = 1$. In this case
$$\tau = Id.$$ Let us assume $\alpha_1 + \alpha_4 = -1$. Then (I) is reduced to
$$\alpha_1^2 + \alpha_1 + \alpha_2 \alpha_3 + 1 = 0$$
If $\alpha_2 \alpha_3 > -\frac{3}{4}$, we have no solutions. Assume that $\alpha_2 \alpha_3 \leq -\frac{3}{4}$. Then
$$\alpha_1 = \frac{-1 \pm \sqrt{-3 - 4\alpha_2 \alpha_3}}{2}.$$ So we obtain
$$\tau_5 = \begin{pmatrix} \frac{-1 - \sqrt{-3 - 4\alpha_2 \alpha_3}}{2} & \alpha_2 & 0 \\ \alpha_3 & -1 + \sqrt{-3 - 4\alpha_2 \alpha_3} & 0 \\ \alpha_5 & 0 & \alpha_6 \end{pmatrix},$$
and
$$\tau_5' = \begin{pmatrix} \frac{-1 + \sqrt{-3 - 4\alpha_2 \alpha_3}}{2} & \alpha_2 & 0 \\ \alpha_3 & -1 - \sqrt{-3 - 4\alpha_2 \alpha_3} & 0 \\ \alpha_5 & 0 & \alpha_6 \end{pmatrix}.$$ Since
$$\tau_5^2(\alpha_2, \alpha_3, \alpha_5, \alpha_6) = \tau_5'(-\alpha_2, -\alpha_3, \alpha_5', \alpha_6'),$$
where
$$\alpha_5' = \frac{\alpha_5 - \sqrt{-3 - 4\alpha_2 \alpha_3 \alpha_5 - 2\alpha_3 \alpha_6}}{2}, \quad \alpha_6' = \frac{\alpha_6 + \sqrt{-3 - 4\alpha_2 \alpha_3 \alpha_6 - 2\alpha_2 \alpha_5}}{2},$$
and
$$\tau_5' \tau_5(\alpha_2, \alpha_3, \alpha_5, \alpha_6) = \tau_5(-\alpha_2, -\alpha_3, \alpha_5'', \alpha_6''),$$
where
$$\alpha_5'' = \frac{\alpha_5 + \sqrt{-3 - 4\alpha_2 \alpha_3 \alpha_5 - 2\alpha_3 \alpha_6}}{2}, \quad \alpha_6'' = \frac{\alpha_6 - \sqrt{-3 - 4\alpha_2 \alpha_3 \alpha_6 - 2\alpha_2 \alpha_5}}{2},$$
we deduce

**Proposition 2.3.** Any abelian subgroup of $\text{Aut}(h_3)$ isomorphic to $\mathbb{Z}_3$ is equal to
$$\Gamma_5(\alpha_2, \alpha_3, \alpha_5, \alpha_6) = \{Id, \tau_5(\alpha_2, \alpha_3, \alpha_5, \alpha_6), \tau_5'(-\alpha_2, -\alpha_3, \alpha_5', \alpha_6'), 4\alpha_2 \alpha_3 \leq -3\}.$$
2.3 Subgroups of $\text{Aut}(h_3)$ isomorphic to $\mathbb{Z}_k$, $k > 3$

If $\tau \in \text{Aut}(h_3)$ satisfies $\tau^k = Id$, $k > 3$, its minimal polynomial has 3 simple roots and it is of degree 3. More precisely, it is written

$$m_\tau(x) = (x - 1)(x - \mu_k)(x - \overline{\mu_k})$$

where $\mu_k$ is a root of order $k$ of 1. As $\tau$ has to generate a cyclic subgroup of $\text{Aut}(h_3)$ isomorphic to $\mathbb{Z}_k$, the root $\mu_k$ is a primitive root of 1. There exists $m$, a prime number with $k$ such that $\mu_k = \exp\left(\frac{2m\pi}{k}\right)$. If $\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ \alpha_5 & \alpha_6 & \Delta \end{pmatrix}$ is the matricial representation of $\tau$, then $\Delta = 1$ and $\alpha_1 + \alpha_4 = 2 \cos \frac{2m\pi}{k}$. Thus

$$\begin{cases} 
\alpha_1 = \cos \frac{2m\pi}{k} - \sqrt{\cos^2 \frac{2m\pi}{k} - 1 - \alpha_2 \alpha_3}, \\
\alpha_4 = \cos \frac{2m\pi}{k} + \sqrt{\cos^2 \frac{2m\pi}{k} - 1 - \alpha_2 \alpha_3}, \\
\end{cases}$$

or

$$\begin{cases} 
\alpha_1 = \cos \frac{2m\pi}{k} + \sqrt{\cos^2 \frac{2m\pi}{k} - 1 - \alpha_2 \alpha_3}, \\
\alpha_4 = \cos \frac{2m\pi}{k} - \sqrt{\cos^2 \frac{2m\pi}{k} - 1 - \alpha_2 \alpha_3}, \\
\end{cases}$$

If $\tau'$ and $\tau''$ denote the automorphisms corresponding to these solutions, we have, for a good choice of the parameters $\alpha_i$, $\tau' \circ \tau'' = Id$ and $\tau'' = (\tau')^{k-1}$. Thus these automorphisms generate the same subgroup of $\text{Aut}(h_3)$. Moreover, with same considerations, we can choose $m = 1$. Thus we have determinate the automorphism $\tau_6(\alpha_2, \alpha_3, \alpha_5, \alpha_6)$ whose matrix is

$$\begin{pmatrix} 
\cos \frac{2\pi}{k} + \sqrt{\cos^2 \frac{2\pi}{k} - 1 - \alpha_2 \alpha_3} & \alpha_2 & 0 \\
\alpha_3 & \cos \frac{2\pi}{k} - \sqrt{\cos^2 \frac{2\pi}{k} - 1 - \alpha_2 \alpha_3} & \alpha_6 \\
\alpha_5 & \alpha_6 & 1 
\end{pmatrix}$$

Proposition 2.4. Any abelian subgroup of $\text{Aut}(h_3)$ isomorphic to $\mathbb{Z}_k$ is equal to

$$\Gamma_{6,k}(\alpha_2, \alpha_3, \alpha_5, \alpha_6) = \left\{ Id, \tau_6(\alpha_2, \alpha_3, \alpha_5, \alpha_6), \ldots, \tau_6^{k-1}, \ \alpha_2 \alpha_3 \leq -1 + \cos^2 \frac{2\pi}{k} \right\}.$$
2.4 General case

Now suppose that the cyclic decomposition of a finite abelian subgroup $\Gamma$ of $\text{Aut}(\mathfrak{h}_3)$ is isomorphic to $\mathbb{Z}_2^{k_2} \times \mathbb{Z}_3^{k_3} \times \cdots \times \mathbb{Z}_p^{k_p}$ with $k_i \geq 0$.

**Lemma 2.5.** Let $\Gamma$ be an abelian finite subgroup of $\text{Aut}(\mathfrak{h}_3)$ with a cyclic decomposition isomorphic to $\mathbb{Z}_2^{k_2} \times \mathbb{Z}_3^{k_3} \times \cdots \times \mathbb{Z}_p^{k_p}$.

Then

- If there is $i \geq 3$ such that $k_i \neq 0$, then $k_2 \leq 1$.
- If $k_2 \geq 2$, then $\Gamma$ is isomorphic to $\mathbb{Z}_2^{k_2}$.

**Proof.** Assume that there is $i \geq 3$ such that $k_i \geq 1$. If $k_2 \geq 1$, there exist two automorphisms $\tau$ and $\tau'$ satisfying $\tau^2 = \tau'^2 = \text{Id}$ and $\tau' \circ \tau = \tau \circ \tau'$. Thus $\tau'$ and $\tau$ can be reduced simultaneously in the diagonal form and admit a common basis of eigenvectors. As for any $\sigma \in \text{Aut}(\mathfrak{h}_3)$ we have $\sigma(X_3) = \Delta X_3$, $X_3$ is an eigenvector for $\tau'$ and $\tau$ associated to the eigenvalue 1 for $\tau'$ and $\pm 1$ for $\tau$. As the two other eigenvalues of $\tau'$ are complex conjugate numbers, the corresponding eigenvectors are complex conjugate. This implies that the eigenvalues of $\tau$ distinguished of $\Delta = \pm 1$ are equal and from Proposition 2.4, $\tau = \tau_4(\alpha_5, \alpha_6)$. But

$$\tau_4(\alpha_5, \alpha_6) \circ \tau_4(\alpha'_5, \alpha'_6) = \tau_4(\alpha'_5, \alpha'_6) \circ \tau_4(\alpha_5, \alpha_6) \iff \alpha_5 = \alpha'_5, \alpha_6 = \alpha'_6.$$  

Thus, we have to determine, in a first step, the subgroups $\Gamma$ of $\text{Aut}(\mathfrak{h}_3)$ isomorphic a $(\mathbb{Z}_2)^k$ with $k \geq 2$.

- Any involutive automorphism $\tau$ commuting with $\tau_1(\alpha_3, \alpha_6)$ and distinct from it is equal to one of the following automorphisms

$$\tau_2(-\alpha_3, \alpha_5), \quad \tau_4(\alpha_5, -\alpha_6)$$

Indeed, if we set $[\tau, \tau'] = \tau \circ \tau' - \tau' \circ \tau$ then

$$[\tau_1(\alpha_3, \alpha_6), \tau_1(\alpha'_3, \alpha'_6)] = 0 \quad \text{if and only if} \quad \alpha'_3 = \alpha_3 \quad \text{and} \quad \alpha'_6 = \alpha_6$$

$$[\tau_1(\alpha_3, \alpha_6), \tau_2(\alpha'_3, \alpha'_5)] = 0 \quad \text{if and only if} \quad \alpha'_3 = -\alpha_3$$

$$[\tau_1(\alpha_3, \alpha_6), \tau_1(\alpha_1, \alpha_2, \alpha'_3)] \neq 0 \quad \text{whatever they are} \quad \alpha_1, \alpha_2, \alpha'_3$$

$$[\tau_1(\alpha_3, \alpha_6), \tau_4(\alpha_5, \alpha_6)] = 0 \quad \text{if and only if} \quad \alpha'_6 = -\alpha_6$$

These results follow directly from the matrix calculation. In addition we have

$$\tau_1(\alpha_3, \alpha_6) \circ \tau_2(-\alpha_3, \alpha_5) = \tau_4 \left( -\frac{\alpha_3 \alpha_6}{2} - \alpha_5, -\alpha_6 \right)$$

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and
\[ \tau_2(-\alpha_3, \alpha_5), \tau_4 \left( -\frac{\alpha_3 \alpha_6}{2} - \alpha_5, -\alpha_6 \right) \] = 0.
Thus
\[ \Gamma_7(\alpha_3, \alpha_5, \alpha_6) = \{ Id, \tau_1(\alpha_3, \alpha_6), \tau_2(-\alpha_3, \alpha_5), \tau_4 \left( -\frac{\alpha_3 \alpha_6}{2} - \alpha_5, -\alpha_6 \right) \} \]
is a subgroup of \( \text{Aut}(\mathfrak{h}_3) \) isomorphic to \( \mathbb{Z}_2^2 \). Moreover it is the only subgroup of \( \text{Aut}(\mathfrak{h}_3) \) of type \( (\mathbb{Z}_2)^k, k \geq 2 \), containing an automorphism of type \( \tau_1(\alpha_3, \alpha_6) \).

- Let us suppose that \( \tau_2(\alpha_3, \alpha_5) \in \Gamma \) and that \( \tau_1(\alpha_3, \alpha_6) \notin \Gamma \). We have
  \[ [\tau_2(\alpha_3, \alpha_5), \tau_2(\alpha'_3, \alpha'_5)] = 0 \] if and only if \( \alpha'_3 = \alpha_3 \) and \( \alpha'_5 = \alpha_5 \)
  \[ [\tau_2(\alpha_3, \alpha_5), \tau_3(\alpha_1, \alpha_2, \alpha_6)] \neq 0 \] because by assumption \( \alpha_2 \neq 0 \)
  \[ [\tau_2(\alpha_3, \alpha_5), \tau_4(\alpha'_5, \alpha_6)] = 0 \] if and only if \( \alpha'_5 = -\alpha_5 - \frac{\alpha_3 \alpha_6}{2} \)
But
\[ \tau_2(\alpha_3, \alpha_5) \circ \tau_4(\alpha_5 - \frac{\alpha_3 \alpha_6}{2}, \alpha_6) = \tau_1(\alpha_3, \alpha_6). \]
Thus every abelian subgroup \( \Gamma \) containing \( \tau_2(\alpha_3 \alpha_5) \) are either isomorphic to \( \mathbb{Z}_2 \) or is equal to \( \Gamma_7 \).

- Assume that \( \tau_3(\alpha_1, \alpha_3, \alpha_6) \in \Gamma \). We have
  \[ [\tau_3(\alpha_1, \alpha_2, \alpha_6), \tau_3(\alpha'_1, \alpha'_2, \alpha'_6)] = 0 \] if and only if \( \alpha'_1 = -\alpha_1 \) and \( \alpha'_2 = -\alpha_2 \).
Thus
\[ [\tau_3(\alpha_1, \alpha_2, \alpha_6), \tau_3(-\alpha_1, -\alpha_2, \alpha'_6)] = 0, \]
and
\[ [\tau_3(\alpha_1, \alpha_2, \alpha_6), \tau_4(\alpha_5, \alpha'_6)] = 0 \] if and only if \( \alpha_2 \alpha_5 + 2\alpha_6 = (\alpha_1 - 1)\alpha'_6 \).
Moreover
\[ \tau_3(\alpha_1, \alpha_2, \alpha_6) \circ \tau_3(-\alpha_1, -\alpha_2, \alpha'_6) = \tau_4 \left( \frac{\alpha'_6(1 - \alpha_1) - \alpha_6(1 + \alpha_1)}{\alpha_2}, -\alpha_6 - \alpha'_6 \right) \]
because
\[ \alpha_2 \left( \frac{\alpha'_6(1 - \alpha_1) - \alpha_6(1 + \alpha_1)}{\alpha_2} \right) + 2\alpha_6 + (1 - \alpha_1)(-\alpha_6 - \alpha'_6) = 0. \]
The subgroup of \( \Gamma_8(\alpha_1, \alpha_2, \alpha_6, \alpha'_6) \) of \( \text{Aut}(\mathfrak{h}_3) \) equal to
\[ \{ \text{Id}, \tau_3(\alpha_1, \alpha_2, \alpha_6), \tau_3(-\alpha_1, -\alpha_2, \alpha'_6), \tau_4 \left( \frac{\alpha'_6(1 - \alpha_1) - \alpha_6(1 + \alpha_1)}{\alpha_2}, -\alpha_6 - \alpha'_6 \right) \} \]
is isomorphic to \( \mathbb{Z}_2^2 \).
• We suppose that \( \tau_4(\alpha_5, \alpha_6) \in \Gamma \). If \( \Gamma \) is not isomorphic to \( \mathbb{Z}_2 \), then \( \Gamma \) is one of the groups \( \Gamma_7, \Gamma_8 \).

**Theorem 2.1.** Any finite abelian subgroup \( \Gamma \) of \( \text{Aut}(h_3) \) isomorphic to \( (\mathbb{Z}_2)^k \) is one of the following

1. \( k = 1 \), \( \Gamma = \Gamma_1(\alpha_3, \alpha_6), \Gamma_2(\alpha_3, \alpha_5), \Gamma_3(\alpha_1, \alpha_2, \alpha_6), \alpha_2 \neq 0, \Gamma_4(\alpha_5, \alpha_6) \),
2. \( k = 2 \), \( \Gamma = \Gamma_7(\alpha_3, \alpha_5, \alpha_6), \Gamma_8(\alpha_1, \alpha_2, \alpha_6, \alpha_6') \).

Let us assume now that \( \Gamma \) is isomorphic to \( \mathbb{Z}_3^{k_3} \) with \( k_3 \geq 2 \). If \( \tau \in \Gamma_5 \), its matricial representation is

\[
\begin{pmatrix}
-1 - \sqrt{-3 - 4\alpha_2\alpha_3} \\
\alpha_3 \\
\alpha_5
\end{pmatrix} \begin{pmatrix}
\alpha_2 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
2 \\
-1 + \sqrt{-3 - 4\alpha_2\alpha_3} \\
0
\end{pmatrix}.
\]

To simplify, we put \( \lambda = \frac{-1 - \sqrt{-3 - 4\alpha_2\alpha_3}}{2} \). The eigenvalues of \( \tau \) are \( 1, j, j^2 \) and the corresponding eigenvectors \( X_3, V, V' \) with

\[
V = (1, -\frac{\lambda - j}{\alpha_2}, -\frac{\alpha_5}{1 - j} + \frac{\alpha_6(\lambda - j)}{\alpha_2(1 - j)})
\]

if \( \alpha_2 \neq 0 \). If \( \tau' \) is an automorphism of order 3 commuting with \( \tau \), then

\[
\tau'V = jV \quad \text{or} \quad j^2V.
\]

But the two first components of \( \tau'(V) \) are

\[
\lambda' - \frac{\beta_2}{\alpha_2}(\lambda - j), \beta_3 - \frac{\lambda'(\lambda - j)}{\alpha_2}
\]

where \( \beta_i \) and \( \lambda' \) are the corresponding coefficients of the matrix of \( \tau' \). This implies

\[
\alpha_2\lambda' - \beta_2(\lambda - j) = \alpha_2j \quad \text{or} \quad \alpha_2j^2.
\]

Considering the real and complex parts of this equation, we obtain

\[
\begin{cases}
\alpha_2\lambda' - \beta_2\lambda = 0, \\
\beta_2j = \alpha_2j \quad \text{or} \quad \alpha_2j^2.
\end{cases}
\]

As \( \alpha_2 \neq 0 \), we obtain \( \alpha_2 = \beta_2 \) and \( \lambda = \lambda' \). Let us compare the second component of \( \tau'(V) \). We obtain

\[
\beta_3\alpha_2 - \lambda'(\lambda - j) = -(\lambda - j)j \quad \text{or} \quad -(\lambda - j)j^2.
\]

As \( \lambda = \lambda' \), we have in the first case \( 2\lambda j = j^2 \) and in the second case \( 2\lambda j = j^3 = 1 \). In any case, this is impossible. Thus \( \alpha_2 = 0 \) and, from section 2.2, \( \tau = Id. \) This implies that \( k_3 = 1 \) or 0.
Theorem 2.2. Let $\Gamma$ be a finite abelian subgroup of $Aut(h_3)$. Thus $\Gamma$ is isomorphic to one of the following

1. $\mathbb{Z}_2 \times \mathbb{Z}_2$,

2. $\mathbb{Z}_2^{k_2} \times \mathbb{Z}_3^{k_3} \times \cdots \times \mathbb{Z}_p^{k_p}$ with $k_i = 0$ or 1 for $i = 2, \cdots, p$.

To prove the second part, we show identically to the case $i = 3$ that $k_i = 1$ as soon as $k_i \neq 0$.

Example. The group

$$\Gamma_4(0, 0) \times \Gamma_5(0, 0, 0) \times \cdots \times \Gamma_{6,k}(0, 0, 0)$$

satisfies the second property of the theorem.

Remark. We have determined the finite abelian subgroups of $Aut(h_3)$. There are non-abelian finite subgroups with elements of order at most 3. Take for example the subgroup generated by

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -1/2 & \alpha & 0 \\ 3\alpha & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha \neq 0$$

The relations on the generators are

$$\begin{cases} \sigma_1^2 = Id, \\ \sigma_2^2 = Id, \\ \sigma_1 \sigma_2 \sigma_1 = \sigma_2^2. \end{cases}$$

Thus the group generated by $\sigma_1$ and $\sigma_2$ is isomorphic to the symmetric group $\Sigma_3$ of degree 3.

3 $\Gamma$-grading of $h_3$

3.1 Description of the $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$-gradings

Let $\Gamma$ be a finite abelian subgroup of $Aut(h_3)$. We consider a $\Gamma$-grading of $h_3$

$$h_3 = \bigoplus_{\gamma \in \Gamma} h_{3, \gamma}$$

such that $h_{3,e} = \{0\}$ where $e$ is the identity of $\Gamma$. In this case, the space $\Gamma$-symmetric associated with this grading is isomorphic to the Heisenberg group $H_3$ and then $H_3$ can be studied in terms of $\Gamma$-symmetric spaces. In this section, we are particularly interested by the case $\Gamma = \mathbb{Z}_2$ or $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.

- If $\Gamma = \mathbb{Z}_2$ then the grading of $h_3$ is of the type

$$h_3 = g_0 \oplus g_1$$
with $g_0 \neq \{0\}$. In this case the corresponding symmetric homogeneous space is isomorphic to $\mathbb{H}_3/H$ where $H$ is a non trivial Lie subgroup of $\mathbb{H}_3$ whose Lie algebra is $g_0$. The group $\mathbb{H}_3$ is not provided with a symmetric space structure.

- If $\Gamma = \mathbb{Z}_2^2$ then $\Gamma = \Gamma_7$ or $\Gamma = \Gamma_8$. Recall that

$$\Gamma_7 = \left\{ \text{Id}, \tau_1(\alpha_3, \alpha_6), \tau_2(-\alpha_3, \alpha_5), \tau_4 \left( -\frac{\alpha_3\alpha_6}{2} - \alpha_5, -\alpha_6 \right) \right\}$$

Denote by $L(V_1, \cdots, V_k)$ the real vector space generated by the vectors $V_1, \cdots, V_k$. Recall that each vector of the Heisenberg algebra is decomposed in the basis $\{X_1, X_2, X_3\}$. The eigenspaces associated with $\tau_1(\alpha_3, \alpha_6)$ are

$$V_1 = L \left( (0, 1, \frac{\alpha_6}{2}) \right)$$
$$V_{-1} = L \left( (1, 0, \alpha_5), (0, 0, 1) \right)$$

The eigenspaces associated to $\tau_2(-\alpha_3, \alpha_5)$ are

$$W_1 = L \left( (1, 0, \frac{\alpha_5}{2}) \right)$$
$$W_{-1} = L \left( (0, 1, 0), (0, 0, 1) \right).$$

Since $\tau_4 = \tau_1 \circ \tau_2$, the grading of $h_3$ associated with $\Gamma_7$ is

$$h_3 = V_1 \cap W_1 \oplus V_1 \cap W_{-1} \oplus V_{-1} \cap W_1 \oplus V_{-1} \cap W_{-1}$$
$$= \left\{ 0 \right\} \oplus \mathbb{R}\left\{ (0, 1, \frac{\alpha_6}{2}) \right\} \oplus \mathbb{R}\left\{ (1, 0, \frac{\alpha_5}{2}) \right\} \oplus \mathbb{R}\left\{ (0, 0, 1) \right\}.$$

Now consider the case where $\Gamma = \Gamma_8$

$$\Gamma_8 = \left\{ \text{Id}, \tau_3(\alpha_1, \alpha_2, \alpha_6), \tau_3(\alpha_1', \alpha_2', \alpha_6'), \tau_4 \left( \frac{\alpha_6'(1 - \alpha_1) - \alpha_6(1 + \alpha_1)}{\alpha_2}, -\alpha_6 - \alpha_6' \right) \right\}$$

The eigenspaces associated with $\tau_3^1$ are

$$V_1 = L \left( (1, 1 - \frac{\alpha_1}{\alpha_2}, \frac{\alpha_6}{\alpha_2}) \right)$$
$$V_{-1} = L \left( (1, 1 + \frac{\alpha_1}{\alpha_2}, 0), (0, 0, 1) \right)$$

The eigenspaces associated with $\tau_3^2$ are

$$W_1 = L \left( (1, -\frac{1 + \alpha_1}{\alpha_2}, \frac{\alpha_6'}{\alpha_2}) \right)$$
$$W_{-1} = L \left( (1, -\frac{1 - \alpha_1}{\alpha_2}, 0), (0, 0, 1) \right).$$
The grading associated with $\Gamma_8$ is therefore
\[ h_3 = V_1 \cap W_1 \oplus V_1 \cap W_{-1} \oplus V_{-1} \cap W_1 \oplus V_{-1} \cap W_{-1} = \{0\} \oplus \mathbb{R}\{(1, \frac{1-\alpha_1}{\alpha_2}, 0)\} \oplus \mathbb{R}\{(1, -\frac{1+\alpha_1}{\alpha_2}, 0)\} \oplus \mathbb{R}\{(0, 0, 1)\}. \]

**Proposition 3.1.** The $\mathbb{Z}^2_2$-grading of $h_3$ correspond to one of the following:
\[ h_3 = \{0\} \oplus \mathbb{R}\{(X_2 + \frac{\alpha_6}{2} X_3)\} \oplus \mathbb{R}\{(X_1 - \frac{\alpha_3}{2} X_2 + \frac{\alpha_5}{2} X_3)\} \oplus \mathbb{R}\{X_3\} \]
\[ h_3 = \{0\} \oplus \mathbb{R}\{(X_1 + \frac{1-\alpha_1}{\alpha_2} X_2)\} \oplus \mathbb{R}\{(X_1 - \frac{1+\alpha_1}{\alpha_2} X_2)\} \oplus \mathbb{R}\{X_3\} \]

**Remark.** If $\Gamma = \mathbb{Z}_3$, we consider the complexification $h_3, \mathbb{C}$ of the Heisenberg algebra. We still denote by $X_1, X_2, X_3$ the complex basis of $h_3, \mathbb{C}$ corresponding to the given basis of $h_3$. The grading in this case is defined by the complex eigenspaces of $\tau_5$. They are
\[ V_1 = \mathbb{C}\{(0, 0, 1)\} \]
\[ V_j = \mathbb{C}\{(1, \frac{1+2j+\sqrt{-3-4\alpha_2\alpha_3}}{\alpha_2}, 0)\} \]
\[ V_\bar{j} = \mathbb{C}\{(1, \frac{1+2\bar{j}+\sqrt{-3-4\alpha_2\alpha_3}}{\alpha_2}, 0)\} \]
We have the grading
\[ h_3, \mathbb{C} = V_1 \oplus V_j \oplus V_\bar{j} \]

### 3.2 Classification of $\mathbb{Z}^2_2$-grading up an automorphism

**Lemma 3.2.** There is an automorphism $\sigma \in Aut(h_3)$ such that
\[ \sigma^{-1}\Gamma_7 \sigma = \Gamma_8 \]

**Proof.** Denote by $(\alpha_3, \alpha_5, \alpha_6)$ the parameters of the family $\Gamma_7$ and by $(\alpha_1, \alpha_2, \alpha_6', \alpha_6'')$ those of $\Gamma_8$. If $\alpha_1^2 \neq 1$, then the automorphism
\[ \sigma = \left( \begin{array}{ccc}
\gamma & \frac{\gamma \alpha_2}{\alpha_1 - 1} & 0 \\
\delta & \frac{\alpha_2(\gamma \alpha_3 + \delta - \alpha_1 \delta)}{-1 + \alpha_2^2} & 0 \\
\rho & \mu & -\frac{\gamma \alpha_2(\gamma \alpha_3 + 2\delta)}{-1 + \alpha_2^2}
\end{array} \right) \]
with
\[ \rho = \frac{(2\gamma \alpha_5 + \gamma \alpha_3 \alpha_6 + 2\alpha_6 \delta)}{4} + \frac{(2\gamma^2 \alpha_3 a'_6 + 4\gamma \delta a'_6)(1 + \alpha_1) + (2\gamma^2 \alpha_3 a''_6 + 4\gamma \delta a''_6)(\alpha_1 - 1)}{4(\alpha_1^2 - 1)} \]
\[ \mu = \frac{2\gamma \alpha_2 \alpha_5 (1 + \alpha_1) + \alpha_2 \alpha_6 (\gamma \alpha_3 + 2\delta)(\alpha_1 - 1) + (2\gamma^2 \alpha_2 \alpha_3 + 4\gamma \alpha_2 \delta)(a'_6 + a''_6)}{4(\alpha_1^2 - 1)} \]
answers to the question.

If $\alpha_1 = 1$, we consider

$$\sigma = \begin{pmatrix}
0 & \beta & 0 \\
\gamma & -\beta \alpha_3 + \alpha_2 \gamma & 0 \\
\gamma \left(\frac{\alpha_6}{2} + \frac{\beta \alpha_6}{\alpha_2}\right) & \alpha_2 \gamma \alpha_6 + \frac{2\beta (\alpha_5 + \gamma \alpha_6' + \gamma \alpha_6'')}{4} & -\beta \gamma
\end{pmatrix}$$

and if $\alpha_1 = -1$, we take

$$\sigma = \begin{pmatrix}
-\frac{2\beta}{\alpha_2} & \beta & 0 \\
\beta \alpha_3 & \delta & 0 \\
-\frac{-\alpha_2 \beta \alpha_5 - (\beta^2 \alpha_3 + \gamma)^2}{\alpha_2^2} & 2 \alpha_2 \beta (\alpha_5 + \alpha_3 \alpha_6) + (2\beta^2 \alpha_3 + 4\beta \delta) (\alpha_6' + \gamma \alpha_6'') + 2 \alpha_2 \beta \alpha_6 & -\frac{\beta^2 \alpha_3 + 2 \beta \delta}{\alpha_2}
\end{pmatrix}$$

These automorphisms give an equivalence between the two subgroups.

Consequence Let $\mathfrak{h}_3 = \{0\} \oplus \mathfrak{h}_{3,a_1} \oplus \mathfrak{h}_{3,a_2} \oplus \mathfrak{h}_{3,a_3} = \{0\} \oplus \mathfrak{h}_{3,a_1}' \oplus \mathfrak{h}_{3,a_2}' \oplus \mathfrak{h}_{3,a_3}'$ be the two $\mathbb{Z}_2$-gradings of $\mathfrak{h}_3$, where $\{0, a_1, a_2, a_3\}$ are the elements of $\mathbb{Z}_2^4$. There exists $\sigma \in \text{Aut}(\mathfrak{h}_3)$ such that

$$\mathfrak{h}_{3,a_i}' = \sigma(\mathfrak{h}_{3,a_i})$$

Thus, these gradings are equivalent. (The equivalence of two grading is defined in [1]).

Lemma 3.3. There exists $\sigma \in \text{Aut}(\mathfrak{h}_3)$ such that

$$\begin{align*}
\sigma^{-1} & \tau_1(\alpha_3, \alpha_6) \sigma = \tau_1(0, 0), \\
\sigma^{-1} & \tau_2(-\alpha_3, \alpha_5) \sigma = \tau_2(0, 0).
\end{align*}$$

Proof. Indeed if

$$\sigma = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{\alpha_3}{\gamma} & 1 & 0 \\
\rho & \frac{\alpha_6}{2} & 1
\end{pmatrix}$$

then

$$\sigma^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
\frac{\alpha_3}{2} & 1 & 0 \\
-\frac{\alpha_6}{2} - \rho & -\frac{\alpha_6}{2} & 1
\end{pmatrix}$$

and

$$\sigma^{-1} \tau_1(\alpha_3, \alpha_6) \sigma = \tau_1(0, 0)$$

This automorphism satisfies

$$\sigma^{-1} \tau_2(-\alpha_3, \alpha_5) \sigma = \tau_2(0, \alpha_5 - 2\rho)$$
If $\rho = \frac{\alpha}{\alpha}$ that is

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\alpha}{\alpha} & 1 & 0 \\ \frac{\alpha}{\alpha} & \frac{\alpha}{\alpha} & 1 \end{pmatrix}$$

then we have

$$\sigma^{-1}\tau_2(-\alpha_3, \alpha_5)\sigma = \tau_2(0, 0).$$

From the previous Lemma we have

**Proposition 3.4.** Every $\mathbb{Z}_2^2$-grading on $h_3$ is equivalent to the grading defined by

$$\Gamma_7(0, 0, 0) = \{\text{Id}, \tau_1(0, 0), \tau_2(0, 0), \tau_4(0, 0)\}.$$  

This grading corresponds to

$$h_3 = \{0\} \oplus \mathbb{R}(X_2) \oplus \mathbb{R}(X_2) \oplus \mathbb{R}(X_3).$$

### 4 Riemannian structures $\mathbb{Z}_2^2$-symmetric

Let $G/H$ be an homogeneous $\Gamma$-symmetric space. We denoted by $\rho : \Gamma \rightarrow \text{Aut}(G)$ the injective homomorphism of groups. Each element $\rho(\gamma)$ for $\gamma \in \Gamma$ is called a symmetry of the $\Gamma$-symmetric space.

**Definition 1.** The $\Gamma$-symmetric homogeneous space $G/H$ is called Riemannian $\Gamma$-symmetric if there exists on $G/H$ a Riemannian metric $g$ such that

1. $g$ is $G$-invariant,
2. the symmetries $\rho(\gamma)$, $\gamma \in \Gamma$, are isometries.

According to [7], such a metric is completely determined by a bilinear form $B$ on the Lie algebra $g$ such that

1. $B$ is adjoint invariant ($B = B_e$)
2. $B(g_\gamma, g_{\gamma'}) = 0$ if $\gamma \neq \gamma' \neq e$
3. The restriction of $B$ to $\oplus_{\gamma \neq e} g_\gamma$ is positive definite.

Consider on $\mathbb{H}_3$, the Heisenberg group, a $\mathbb{Z}_2^2$-symmetric structure. It is determined, up to equivalence, by the $\mathbb{Z}_2^2$-grading of $h_3$

$$h_3 = \{0\} \oplus \mathbb{R}(X_1) \oplus \mathbb{R}(X_2) \oplus \mathbb{R}(X_3)$$

Since every automorphism of $h_3$ is an isometry of any invariant Riemannian metric on $\mathbb{H}_3$, we deduce
Theorem 4.1. Any Riemannian structure $\mathbb{Z}_2^2$-symmetric over $\mathbb{H}_3$ is isometric to the Riemannian structure associated with the grading $$h_3 = \{0\} \oplus \mathbb{R}(X_1) \oplus \mathbb{R}(X_2) \oplus \mathbb{R}(X_3)$$ and the Riemannian metric is written $$g = \omega_1^2 + \omega_2^2 + \lambda^2 \omega_3^2$$ with $\lambda \neq 0$, where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of $\{X_1, X_2, X_3\}$.

Proof. Indeed, as the components of the grading are orthogonal, the Riemannian metric $g$, which coincides with the form $B$ verifies $$g = \alpha_1 \omega_1^2 + \alpha_2 \omega_2^2 + \alpha_3 \omega_3^2$$ with $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$. According to [5], we reduce the coefficients to $\alpha_1 = \alpha_2 = 1$. \qed

Remark. According to [9] and [6], this metric is naturally reductive for any $\lambda$.

5 Lorentzian $\mathbb{Z}_2^2$-symmetric structures on $\mathbb{H}_3$

We say that a homogeneous space $(M = G/H, g)$ is Lorentzian if the canonical action of $G$ on $M$ preserves a Lorentzian metric (i.e. a smooth field of non degenerate quadratic form of signature $(n-1,1)$).

Proposition 5.1 ([4]). Modulo an automorphism and a multiplicative constant, there exists on $h_3$ one left-invariant metric assigning a strictly positive length on the center of $h_3$.

The Lie algebra $h_3$ is generated by the central vector $X_3$ and $X_1$ and $X_2$ such that $[X_1, X_2] = X_3$. The automorphisms of the Lie algebra preserve the center and then send the element $X_3$ on $\lambda X_3$, with $\lambda \in \mathbb{R}^*$. Such an automorphism acts on the plane generated by $X_1$ and $X_2$ as an automorphism of determinant $\lambda$.

It is shown in [16] and [17] that, modulo an automorphism of $h_3$, there are three classes of invariant Lorentzian metrics on $\mathbb{H}_3$, corresponding to the cases where $||X_3||$ is negative, positive or zero.

We propose to look at the Lorentz metrics that are associated with the $\mathbb{Z}_2^2$-symmetric structure over $\mathbb{H}_3$.

Definition 2. Let $M = G/H$ be a homogeneous $\Gamma$-symmetric space. Let $g$ be a Lorentz metric on $M$. We say that the metric $g$ is $\mathbb{Z}_2^2$-symmetric Lorentzian if one of the two conditions is satisfied:

1. The homogeneous non trivial components $g_\gamma$ of the $\Gamma$-graded Lie algebra of $G$ are orthogonal and non-degenerate with respect to the induced bilinear form $B$. 

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2. One non-trivial component $\mathfrak{g}_{\lambda_0}$ is degenerate, the other components are orthogonal and non-degenerate, and there exists a component $\mathfrak{g}_{\lambda_1}$ such that the signature of the restriction to $B$ at $\mathfrak{g}_{\lambda_0} \oplus \mathfrak{g}_{\lambda_1}$ is $(1,1)$.

If $\mathfrak{g}$ is the Heisenberg algebra equipped with a $\mathbb{Z}_2^2$-grading, then by automorphism, we can reduce to the case where $\Gamma = \Gamma_7$. In this case, the grading of $\mathfrak{h}_3$ is given by:

$$\mathfrak{h}_3 = \mathfrak{g}_0 + \mathfrak{g}_{++} + \mathfrak{g}_{--}$$

with

$$\begin{cases}
\mathfrak{g}_0 = \{0\}, \\
\mathfrak{g}_{++} = \mathbb{R}(X_2 - \frac{\alpha_3}{2} X_3), \\
\mathfrak{g}_{--} = \mathbb{R}(X_1 - \frac{\alpha_3}{2} X_2 + \frac{\alpha_5}{2} X_3), \\
\mathfrak{g}_{-} = \mathbb{R}(X_3).
\end{cases}$$

Assume

$$Y_1 = X_1 - \frac{\alpha_3}{2} X_2 + \frac{\alpha_5}{2} X_3 \quad Y_2 = X_2 - \frac{\alpha_6}{2} X_3 \quad Y_3 = X_3.$$ 

The dual basis is

$$\vartheta_1 = \omega_1 \quad \vartheta_2 = \omega_2 + \frac{\alpha_3}{2} \omega_1 \quad \vartheta_3 = \omega_3 - \frac{\alpha_6}{2} \omega_2 - \left( \frac{\alpha_3 \alpha_6}{4} + \frac{\alpha_5}{2} \right) \omega_1$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of the base $\{X_1, X_2, X_3\}$.

**Case I** The components $\mathfrak{g}_{++}, \mathfrak{g}_{--}$ are non-degenerate. The quadratic form induced on $\mathfrak{h}_3$ therefore writes

$$g = \lambda_1 \omega_1^2 + \lambda_2 \left( \omega_2 + \frac{\alpha_3}{2} \omega_1 \right)^2 + \lambda_3 \left( \omega_3 - \frac{\alpha_6}{2} \omega_2 - \left( \frac{\alpha_3 \alpha_6}{4} + \frac{\alpha_5}{2} \right) \omega_1 \right)^2$$

with $\lambda_1, \lambda_2, \lambda_3 \neq 0$. The change of basis associated with the matrix

$$\begin{pmatrix}
1 & 0 \\
-\frac{\alpha_3}{2} & 1 \\
-\frac{\alpha_5}{2} \alpha_6 & -\frac{\alpha_6}{2}
\end{pmatrix}$$

is an automorphism. Thus $g$ is isometric to

$$g = \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2.$$ 

Since the signature is $(2,1)$ one of the $\lambda_i$ is negative and the two others positive.

**Proposition 5.2.** Every Lorentzian metric $\mathbb{Z}_2^2$-symmetric $g$ on $\mathbb{H}_3$ such that the components of the grading of $\mathfrak{h}_3$ are non-degenerate, is reduced to one of these two forms:

$$\begin{cases}
g = -\omega_1^2 + \omega_2^2 + \lambda^2 \omega_3^2 \\
g = \omega_1^2 + \omega_2^2 - \lambda^2 \omega_3^2
\end{cases}$$
Case II Suppose that a component is degenerate. When this component is $\mathbb{R}(x_2 + \frac{\alpha_6}{2}X_3)$ or $\mathbb{R}(X_1 - \frac{\alpha_5}{2}X_2 + \frac{\alpha_3}{2}X_3)$ then, by automorphism, we reduce to the above case.

Suppose then that the component containing the center is degenerate. Thus the quadratic form induced on $h_3$ is written

$$g = \omega_1^2 + \left[\omega_3 - \frac{\alpha_6}{2}\omega_2 - \left(\frac{\alpha_5}{2} + \frac{\alpha_3\alpha_6}{4}\right)\omega_1\right]^2 - \left[\omega_2 - \omega_3 + \frac{\alpha_6}{2}\omega_2 + \left(\frac{\alpha_5}{2} + \frac{\alpha_3\alpha_6}{4}\right)\omega_1\right]^2.$$

The change of basis associated with the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\alpha_3}{2} & 1 & 0 \\ -\frac{\alpha_5}{2} - \frac{\alpha_3\alpha_6}{4} & -\frac{\alpha_6}{2} & 1 \end{pmatrix}$$

is given by an automorphism. Thus $g$ is isomorphic to

$$g = \omega_1^2 + \omega_3^2 - (\omega_2 - \omega_3)^2.$$

**Proposition 5.3.** Every Lorentzian $\mathbb{Z}_2^2$-symmetric $g$ metric on $H_3$ such that the component of the grading of $h_3$ containing the center is degenerate, is reduced to the form

$$g = \omega_1^2 + \omega_3^2 - (\omega_2 - \omega_3)^2.$$

From [3] is the only flat Lorentzian metric, left invariant on the Heisenberg group.

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