Notes on the complexity of 3-valent graphs in 3-manifolds

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Abstract

A theory of complexity for pairs \((M, G)\) with \(M\) an arbitrary closed 3-manifold and \(G \subset M\) a 3-valent graph was introduced by the first two named authors, extending the original notion due to Matveev. The complexity \(c\) is known to be always additive under connected sum away from the graphs, but not always under connected sum along (unknotted) arcs of the graphs. In this article we prove the slightly surprising fact that if in \(M\) there is a sphere intersecting \(G\) transversely at one point, and this point belongs to an edge \(e\) of \(G\), then \(e\) can be canceled from \(G\) without affecting the complexity. Using this fact we completely characterize the circumstances under which complexity is additive under connected sum along graphs. For the set of pairs \((M, K)\) with \(K \subset M\) a knot, we also prove that any function that is fully additive under connected sum along knots is actually a function of the ambient manifold only.

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The definition of complexity for a (compact) 3-manifold was originally given by Matveev in [5] and proved extremely fruitful over the time, leading to an excellent topological and geometric understanding [4] of a large number of the “least complicated” such manifolds. The main technical idea of Matveev was that to use simple spines (defined below) rather than triangulations, which allowed him to show in particular that complexity is additive under connected sum, and finite-to-one on irreducible manifolds.

The first extension of the notion of complexity for pairs \((M, G)\) where \(M\) is a closed 3-manifolds and \(G \subset M\) is a 3-valent graph (possibly with knot
components) was suggested in [8] and then thoroughly investigated in [6]. (In
the latter paper the main emphasis and most statements were given for $G$’s
without vertices, i.e., for links, but all the results and constructions extend
to the graph context — see below.)

It was shown already in [8] that the complexity function $c$ is additive
under connected sum away from the graphs, and then it was proved in [6]
that additivity under connected sum along the graphs holds under some
restriction (explained below), that cannot be avoided. In this paper:

1. We show that if an edge $e$ of a graph $G \subset M$ intersects transversely
   at a point a sphere that meets $G$ at that point only, then $e$ can be
canceled without affecting the complexity of $(M, G)$;

2. Using this fact we provide a very explicit characterization of the cir-
cumstances under which complexity is additive under connected sum
along graphs;

3. Along the same lines of reasoning we show that if a function $\psi$ on
   the set of pairs $(M, K)$, with $M$ a 3-manifold and $K \subset M$ a knot,
is fully additive under connected sum along knots, then $\psi$ is actually
insensitive to knots, namely $\psi(M, K)$ depends on $M$ only.

Result (1) was first established in [9] in the context of an extended theory
of complexity for unitrivalent graphs in 3-manifolds. The extension itself
proved only mildly interesting, because one easily sees that for a unitriva-
lent $G$ one has $c(M, G) = c(M, G')$ with $G'$ obtained from $G$ by collapsing as
long as possible and removing isolated points. But it was only within this ex-
tended context that the “edge-cancelation theorem” was first guessed. With
the statement at hand, we later worked out a proof fully contained in the
original trivalent context, and we draw result (2) as a conclusion. Result (3)
also follows from the same machinery, and is actually deduced from a more
abstract theorem of categorical flavour concerning the Grothendieck group
associated to the semigroup of knot-pairs.

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1 Definition of complexity and review of known facts

In this section we will rapidly recall the definitions and facts from [8] and [6] that we will need below. For the sake of simplicity we will only consider orientable (but unoriented) 3-manifolds, even if our results hold for non-orientable manifolds as well.

If $M$ is a closed 3-manifold and $G \subset M$ is a trivalent graph, we call $X = (M, G)$ a graph-pair, and we set $M(X) := M$ and $G(X) := G$. We allow $G$ to be disconnected and to have knot components, and we regard as vertices of $G$ only the trivalent ones (so a knot component of $G$ does not contain vertices). We say that $X$ is a link-pair if $G(X)$ is a link, i.e., if it does not have vertices.

For $i \in \{0, 1, 2, 3\}$ we call $i$-sphere in a graph-pair $X$ a sphere $S \subset M(X)$ intersecting $G(X)$ transversely at $i$ points. And for $i \in \{0, 2, 3\}$ we call trivial $i$-ball in $X$ a ball $B \subset M(X)$ such that $B \cap G(X)$ is empty for $i = 0$, an unknotted arc for $i = 2$, and an unknotted Y-graph for $i = 3$. We define a (simple) spine of $X$ to be a compact polyhedron $P \subset M(X)$ such that the link of each point of $P$ embeds in the complete graph with 4 vertices, $P$ intersects $G(X)$ transversely, and the complement of $P$ in $X$ is a union of trivial $i$-balls (with varying $i$). A vertex of $P$ is a point whose link in $P$ is the whole complete graph with 4 vertices, and the complexity $c(X)$ of $X$ is the minimal possible number of vertices for a spine of $X$. For closed manifolds, namely for pairs $X$ with $G(X) = \emptyset$, this definition agrees with that given by Matveev [5], who proved the following main facts:

- For all $n \in \mathbb{N}$ there exist finitely many irreducible manifolds $M$ (namely such that every sphere in $M$ bounds a ball in $M$) with $c(M) = n$; moreover if $n > 0$ then $c(M)$ is realized by spines dual to triangulations of $M$;

- If $M_1 \# M_2$ is a connected sum of $M_1$ and $M_2$ (namely $M_1 \# M_2 = (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$, with $B_j \subset M_j$ a ball and $f : \partial B_1 \to \partial B_2$ a homeomorphism) then $c(M_1 \# M_2) = c(M_1) + c(M_2)$.

We will not spell out here the notion of duality, but we note that these two results, combined with the fact that every 3-manifold can be uniquely realized as a connected sum of irreducible ones and copies of $S^2 \times S^1$, imply that the topological meaning of the notion of manifold complexity is very well
understood at a theoretical level. In addition, very satisfactory experimental results have been established [4]. Turning to graph-pairs, the following was shown in [8]:

• For all $n \in \mathbb{N}$ there exist finitely many $(0,1,2)$-irreducible graph-pairs $X$ (namely such every $i$-sphere in $X$ bounds a trivial $i$-ball in $X$ — in particular, no 1-sphere exists), with $c(X) = n$; moreover if $n > 0$ then $c(X)$ is realized by spines dual to triangulations of $M(X)$ that contain $G(X)$ in their 1-skeleton.

This fact naturally prompts for an analogue of the notion of connected sum for graph-pairs, for the quest of a unique decomposition theorem for such pairs, and for the investigation of additivity of complexity under connected sum. This notion itself is easily introduced: for $i \in \{0,2\}$ one says that a graph-pair $X_1 \#_i X_2$ is an $i$-connected sum of pairs $X_1$ and $X_2$ if it is obtained by removing trivial $i$-balls from $X_1$ and $X_2$ and gluing the resulting boundary $i$-spheres, matching the intersections with $G(X_1)$ and $G(X_2)$ for $i = 2$. Uniqueness of the expression of a graph-pair as a connected sum of $(0,2)$-irreducible ones however does not quite hold in a literal fashion, but weaker versions do, see [3, 7] and the discussion in [6]. Turning to our main concern, namely additivity under $i$-connected sum, it was shown in [6] using [3] that it holds for $i = 0$ and under some restrictions, but not in general, for $i = 2$:

**Theorem 1.1.** Let $X_1$ and $X_2$ be graph-pairs. Then:

• $c(X_1 \#_0 X_2) = c(X_1) + c(X_2)$;

• If $X_1 \#_2 X_2$ is obtained from $X_1$ and $X_2$ by removing trivial 2-balls that intersect edges (or knot components) $e_1$ of $X_1$ and $e_2$ of $X_2$, and if $e_j$ does not intersect any 1-sphere in $X_j$ for $j = 1, 2$, then $c(X_1 \#_2 X_2) = c(X_1) + c(X_2)$;

• There exist cases where $c(X_1 \#_2 X_2) < c(X_1) + c(X_2)$.

For the sake of brevity, we will describe the situation of the second item above by saying that $X_1 \#_2 X_2$ is a 2-connected sum of $X_1$ and $X_2$ along $e_1$ and $e_2$.

**Remark 1.2.** The statement of Theorem 4.8 in [6] is given for link-pairs $X_1, X_2$ and asserts that $c(X_1 \#_2 X_2) = c(X_1) + c(X_2)$ when neither $X_1$ nor $X_2$ contains 1-spheres. However one can easily check that:
• The absence of trivalent vertices in $G(X_j)$, namely the fact that $G(X_j)$ is a link rather than a graph, is never used in the proof or in the results on which the proof is based;

• For the additivity of $c$ under a 2-connected sum $X_1 \#_2 X_2$ performed along edges $e_1$ of $G(X_1)$ and $e_2$ of $G(X_2)$, the fact that $e_j$ does not meet 1-spheres in $X_j$ is sufficient: in the proof one needs to show that the 2-sphere $S$ giving the $\#_2$ is essential, namely non-trivial and unsplittable; the former condition is obvious, and for the latter one only needs the fact that $e_j$ does not intersect 1-spheres in $X_j$.

The main aim of this paper is to describe exactly in which cases one has $c(X_1 \#_2 X_2) = c(X_1) + c(X_2)$. This description will be derived from the fact that edges intersecting 1-spheres are immaterial to complexity. Using 1-spheres we will then also prove that for knot-pairs any invariant behaving in a fully additive way under 2-connected sum is actually insensitive to knots.

2 Edges intersecting 1-spheres do not contribute to the complexity

If $X$ is a graph-pair and $e$ is an edge or a knot component of $G = G(X)$, we denote by $X^{(e)}$ the pair $(M(X), G^{(e)})$ where $G^{(e)}$ is obtained from $G$ by canceling $e$, namely:

(A) If $e$ is an edge of $G$ ending at distinct trivalent vertices of $G$, then $G^{(e)}$ is $G$ minus the interior of $e$ (an open arc);

(B) If $e$ is a knot component of $G$, then $G^{(e)}$ is $G$ minus $e$ (a circle);

(C) If $e$ ends on both sides at one vertex $V$ of $G$, and $e'$ is the other edge of $G$ incident to $V$, then $G^{(e)}$ is $G$ minus $V$ and the interiors of $e$ and $e'$.

**Theorem 2.1.** Let $X$ be a graph-pair, suppose that $X$ contains a 1-sphere $S$ and let $S \cap G(X)$ be a point of an edge or knot component $e$ of $G(X)$. Then $c(X) = c(X^{(e)})$.

**Proof.** Of course any spine of $X$ is also a spine of $X^{(e)}$, whence $c(X^{(e)}) \leq c(X)$, regardless of the existence of $S$. 

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Figure 1: A spine of one of the two pairs of which $X^{(e)}$ is the 0-connected sum along $S$, and position of the ball along which the 0-connected sum is performed.

Turning to the opposite inequality, suppose first that $S$ is a separating sphere in $M$, so in particular cancelation of $e$ takes place as in case (A) above. Cutting $X^{(e)}$ along $S$ and capping with two 0-balls $B_1$ and $B_2$ we see that $X^{(e)}$ is realized as some $X_1 \#_0 X_2$, where $S$ is the image in $X^{(e)}$ of the two spheres glued to perform the 0-connected sum, and $G^{(e)}$ is the disjoint union of $G(X_1)$ and $G(X_2)$. Additivity of complexity under 0-connected sum \cite{6} shows that $c(X^{(e)}) = c(X_1) + c(X_2)$. We now claim that $c(X) \leq c(X_1) + c(X_2)$, which implies that $c(X) \leq c(X^{(e)})$ and hence the conclusion that $c(X) = c(X^{(e)})$.

To prove that $c(X) \leq c(X_1) + c(X_2)$ we start from spines $P_1$ and $P_2$ of $X_1$ and $X_2$ having $c(X_1)$ and $c(X_2)$ vertices, and we construct from $P_1$ and $P_2$ a spine of $X$ without adding vertices, which of course implies the conclusion. To do so, let us denote by $V_1$ and $V_2$ the points of $G(X_1)$ and $G(X_2)$ at which $e$ had its ends (before getting removed), and by $e_j$ the edge of $G(X_j)$ on which $V_j$ lies; recall that $B_j$ is the 0-ball in $X_j$ used to cap $X^{(e)}$ after cutting along $S$. Then:

- We can assume that $e_j$ intersects $P_j$ at a surface point of $P_j$, and that $V_j$ lies “near” this intersection point, as in Fig. 1 left;
- We can assume that $B_j$ lies near $V_j$ in the ball component of $X_j \setminus P_j$ that contains $V_j$, as in Fig. 1 right;
- Up to adding a bubble to $P_j$ we can assume that the ball component of $X_j \setminus P_j$ containing $V_j$ and $B_j$ is a trivial 2-ball, as in Fig. 2 left;
Now we add to $G(X_j)$ an arc going from $V_j$ to a point $A_j$ of $\partial B_j$, and we add to $P_j$ an arc going from $P_j$ to a different point $C_j$ of $\partial B_j$, as in Fig. 2-right; we then remove the interiors of $B_1$ and $B_2$, we glue $\partial B_1$ to $\partial B_2$ by a homeomorphism isotopic to that giving $X^{(e)}$ as $X_1 \#_0 X_2$ and mapping $A_1$ to $A_2$ and $C_1$ to $C_2$, and we call $P$ the union of $P_1$ and $P_2$, of $\partial B_1 = \partial B_2$, and of the two added arcs glued along $C_1 = C_2$.

This construction gives a spine $P$ of $X$ with $c(X_1) + c(X_2)$ vertices, and the proof for the case where $S$ is separating is complete.

We now have to deal with the case where $S$ is non-separating. Suppose first that again the cancelation of $e$ occurs as in situation (A) above. Then cutting $X^{(e)}$ along $S$ and capping with two balls $B_1$ and $B_2$ we get some $Y$ such that $X^{(e)} = Y \#_0 Z$ with $Z = (S^2 \times S^1, \emptyset)$. Since $c(Z) = 0$, additivity of complexity under $\#_0$ implies that $c(X^{(e)}) = c(Y)$, therefore we get the desired conclusion as soon as we show that $c(X) \leq c(Y)$. The construction of a spine of $X$ starting from a spine $P$ of $Y$ having $c(Y)$ vertices and without addition of vertices is actually identical to that shown above, except that one may initially have in the first place that $V_1$ and $V_2$ belong to the same arc of $G(Y) \setminus (P \cup V(G(Y)))$, as in Fig. 3-left, in which case one performs the construction described in Fig. 3-right, after which one can proceed precisely as before.

Turning to a non-separating $S$ in situation (B), we again cut $X^{(e)}$ along $S$ and cap using balls $B_1$ and $B_2$, getting some $Y$ with $X^{(e)} = Y \#_0 Z$ and $Z = (S^2 \times S^1, \emptyset)$ as above. To show that $c(X) \leq c(Y)$ and conclude we pick a point $V$ that used to belong to $e$ before it got removed, and we assume
that $V, B_1, B_2$ lie near a spine $P$ of $Y$ with $c(Y)$ vertices, as in Fig. 4-left. We then construct a spine of $X$ as suggested in Fig. 4-right, with the two visible spheres suitably glued together.

We are left to a non-separating $S$ as in (C). Once again $X^{(e)} = Y \#_0 Z$ and it is enough to show that $c(X) \leq c(Y)$. We denote by $V$ the vertex of $G(X)$ at which $e$ ends on both sides, and by $W$ the end different from $V$ of the edge $e'$ of $G(X)$ different from $e$ and incident to $V$. We can assume as in Fig. 5-left that $V, B_1, B_2$ lie near $W$ in a component of $Y \setminus P$, where $P$ is a spine of $Y$ having $c(Y)$ vertices. Fig. 5-right now suggests how to construct a spine of $X$ without adding vertices, and the proof is complete.

**Remark 2.2.** A small subtlety in the above proof is perhaps worth pointing
Figure 5: Construction of a spine of $X$ from one of $Y$ in case (C) out. One could imagine that, after cutting $X$ along $S$ and capping, the two halves in which $e$ has been cut may physically appear as “long and knotted” arcs, as in Fig. 5 left, but an arc with a free end is always “short and straight,” as in Fig. 5 right, therefore our construction always applies.

Remark 2.3. The proof of additivity of $c$ under $0$-connected sum is carried out in [6], just as the original proof in [5], in two steps. One first shows by hand that $c$ is subadditive, namely that $c(X \#_0 Y) \leq c(X) + c(Y)$ for all $X$ and $Y$, and then one shows the opposite inequality using normal surfaces

Figure 6: An arc is always short and straight: the two configurations shown are the same
and a suitable decomposition theory: for manifolds [5], the Haken-Kneser-Milnor decomposition of manifolds into prime ones, and, for graph-pairs [6], the \((0, 2)\)-reduction of Matveev’s root theory [3]. Analyzing the above proof one sees that subadditivity of \(c\) under \(\#_0\) would not be sufficient to carry out the argument. In other words, it is the hard part of additivity, depending on root theory, that one needs in order to show that edges \(e\) as in the statement can be canceled without affecting complexity.

Remark 2.4. It was already shown in [6] that if \(K\) is any knot in \(S^3\) and \(D = (S^2 \times S^1, \{\ast\} \times S^1)\) then \(X = (S^3, K) \#_2 D\) has complexity 0. Our Theorem 2.1 easily implies the result just stated, because the knot \(G(X)\) encounters some sphere \(S^2 \times \{\ast\}\) at one point only, so the knot can be canceled from \(X\) without affecting the complexity, but \(X\) minus its knot is \((S^2 \times S^1, \emptyset)\), that has complexity 0.

3 Two-connected sums under which complexity is additive

Exploiting the result from the previous section and the facts already established in [6] we can now completely characterize the situations in which complexity is additive under 2-connected sum:

**Theorem 3.1.** Let \(X_1\) and \(X_2\) be graph-pairs and consider a 2-connected sum \(X_1 \#_2 X_2\) performed along edges \(e_1\) of \(X_1\) and \(e_2\) of \(X_2\). Then \(c(X_1 \#_2 X_2) = c(X_1) + c(X_2)\) if and only if one of the following holds:

1. For \(j = 1, 2\) the edge \(e_j\) does not meet 1-spheres in \(X_j\);
2. For \(j = 1, 2\) one has \(c\left(X_j^{(e_j)}\right) = c(X_j)\), namely \(e_j\) can be canceled from \(X_j\) without affecting complexity.

**Proof.** Suppose that \(c(X_1 \#_2 X_2) = c(X_1) + c(X_2)\) and that (1) does not happen. Then up to permutation we have that \(X_1\) meets a 1-sphere \(S\) at a point of \(e_1\), therefore \(c(X_1) = c\left((X_1)^{(e_1)}\right)\) by Theorem 2.1. We can of course assume that \(S\) does not meet the ball \(B_1 \subset X_1\) along which the 2-connected sum is performed.

Suppose first that \(e_1\) or \(e_2\) is a knot component of \(G(X_1)\) or \(G(X_2)\). Then the traces in \(X_1 \#_2 X_2\) of \(e_1\) and \(e_2\) give a single edge (or knot component) \(e\)
in \(X_1 \#_2 X_2\), and \(S\) is a 1-sphere in \(X_1 \#_2 X_2\) meeting \(e\) at one point, therefore Theorem 2.1 implies that \(c(X_1 \#_2 X_2) = c(X')\), where \(X' = (X_1 \#_2 X_2)^{(e)}\). Suppose next that both \(e_1\) and \(e_2\) are edges ending at vertices of \(G(X_1)\) and \(G(X_2)\), and note that we can take a parallel copy \(S'\) of \(S\) isotoped across \(B_1\), that again meets \(G(X_1)\) at a point of \(e_1\), but on the opposite side with respect to \(B_1\) to the intersection point between \(S\) and \(e_1\). The traces in \(X_1 \#_2 X_2\) of \(e_1\) and \(e_2\) now give rise in \(G(X_1 \#_2 X_2)\) to two edges \(e\) and \(e'\). By construction \(S\) and \(S'\) are 1-spheres in \(X_1 \#_2 X_2\) meeting \(G(X_1 \#_2 X_2)\) at points of \(e\) and \(e'\), therefore Theorem 2.1 implies that \(c(X_1 \#_2 X_2) = c(X')\), where \(X' = ((X_1 \#_2 X_2)^{(e)})^{(e')}\).

In both the cases just described one has that \(X' = (X_1)^{(e_1)} \#_0 (X_2)^{(e_2)}\), whence \(c(X') = c((X_1)^{(e_1)}) + c((X_2)^{(e_2)})\). But we know that \(c(X') = c(X_1 \#_2 X_2)\) and \(c(X_1) = c((X_1)^{(e_1)})\), and the standing assumption is that \(c(X_1 \#_2 X_2) = c(X_1) + c(X_2)\), whence \(c((X_2)^{(e_2)}) = c(X_2)\), as desired.

Turning to the opposite implication, we know from [6] that the desired equality \(c(X_1 \#_2 X_2) = c(X_1) + c(X_2)\) holds true under condition (1). We then assume that (2) is satisfied but (1) is not. Then again up to permutation we can assume that \(X_1\) meets a 1-sphere \(S\) at a point of \(e_1\). With precisely the same notation and arguments as above we then have \(c(X_1 \#_2 X_2) = c(X')\), and \(c(X') = c((X_1)^{(e_1)}) + c((X_2)^{(e_2)})\) because \(X' = X_1^{(e_1)} \#_0 X_2^{(e_2)}\). The standing assumption is now that \(c((X_j)^{(e_j)}) = c(X_j)\) for \(j = 1, 2\), and the desired equality \(c(X_1 \#_2 X_2) = c(X_1) + c(X_2)\) readily follows.

\[ \square \]

**Remark 3.2.** We know from Theorem 2.1 that \(c(X^{(e)}) = c(X)\) when \(e\) meets a 1-sphere in \(X\), but there are many cases where \(c(X^{(e)}) = c(X)\) and \(e\) does not meet 1-spheres in \(X\). For instance, if \(P\) is a minimal spine of a closed manifold \(M\) not containing non-separating spheres, for any knot \(K\) dual to a region of \(P\) one has \(c(M, K) = c(M)\), so \(K\) can be canceled without affecting complexity, but \(K\) does not meet 1-spheres in \((M, K)\). This shows in particular that in Theorem 3.1 cases (1) and (2) have a large overlapping.
4 The Grothendieck group of knot-pairs does not see the knots

In this section we will show a further result that can be proved with the aid of 1-spheres in graph-pairs, as our Theorem 2.1 above. We will soon be restricting our attention to knot-pairs, namely to pairs \((M, K)\) with \(K \subset M\) a knot, but we first establish the following perhaps not completely intuitive result, that basically follows from the observation made in Remark 2.2, applied to case (B) of the cancelation of an edge:

**Proposition 4.1.** Let \(X_1\) and \(X_2\) be graph-pairs such that \(M(X_1) = M(X_2)\) is connected and \(G(X_j) = G_0 \sqcup K_j\) with \(K_j\) a knot intersected by a 1-sphere \(S_j\) in \(X_j\). Then \(X_1 \cong X_2\).

**Proof.** Cutting \(X_j\) along the 1-sphere \(S_j\) and capping we get some \(N_j\) containing \(G_0\), two balls \(B_{j,1}, B_{j,2}\) and an arc \(a_j\) from \(\partial B_{j,1}\) to \(\partial B_{j,2}\) such that \(X_j\) is obtained from \((N_j, G_0 \cup a_j)\) by removing the interiors of \(B_{j,1}, B_{j,2}\) and gluing, matching the ends of \(a_j\). Now \(N_1\) and \(N_2\) can be identified to the same connected \(N\) with \(G_0 \subset N\), and the unions \(B_{j,1} \cup a_j \cup B_{j,2}\) can be isotoped to each other in \(N\) away from \(G_0\), which implies the conclusion. \(\Box\)

Turning to knot-pairs, we first refine our setting to an oriented context, namely we consider pairs \((M, K)\) with \(M\) a closed oriented 3-manifold and \(K\) an oriented knot, and we denote by \(\mathcal{X}_{\text{knot}}^{(or)}\) the set of all such pairs viewed up to homeomorphisms of pairs that preserve orientations. We next note that \(\#_2\) can now be introduced as a well-defined, binary and commutative operation on \(\mathcal{X}_{\text{knot}}^{(or)}\): to define \((M_1, K_1) \#_2 (M_2, K_2)\) one must remove trivial 2-balls \(B_j\) from \((M_j, K_j)\) and glue \(\partial B_1\) to \(\partial B_2\) insisting that the gluing homeomorphism should reverse the orientations induced on \(\partial B_j\) by \(B_j\) and on the points \((\partial B_j) \cap K_j\) by \(B_j \cap K_j\). Since \(O := (S^3, U)\), with \(U\) the unknot, is an identity element for \(\#_2\), we see that \(\mathcal{X}_{\text{knot}}^{(or)}\) has a natural structure of Abelian semigroup. We next recall that following general fact [1]:

**Proposition 4.2.** Let \(S\) be an Abelian semigroup with operation \(\oplus\) and identity element \(0\). Define \(K(S)\) as the quotient of \(S \times S\) under the equivalence relation \(\sim\), where \((a, b) \sim (c, d)\) if there exists \(u \in S\) such that \(a \oplus d \oplus u = b \oplus c \oplus u\). Then the operation \([a, b] \oplus [c, d] = [a \oplus c, b \oplus d]\) is well-defined on \(K(S)\) and turns \(K(S)\) into an Abelian group with identity
element \([o, o]\). Moreover \(\varphi : S \to K(S)\) given by \(\varphi(a) = [a, 0]\) is a homomorphism of Abelian semigroups satisfying the following universal property: if \(G\) is an Abelian group and \(\psi : S \to G\) is a homomorphism of Abelian semigroups then there exists a unique homomorphism of groups \(\gamma : K(S) \to G\) such that \(\psi = \gamma \circ \varphi\).

We can now establish the following:

**Theorem 4.3.** If \(\varphi : X^{(or)}_{\text{knot}} \to K(X^{(or)}_{\text{knot}})\) is the universal homomorphism then \(\varphi(M, K)\) depends on \(M\) only.

**Proof.** Let \(K_1\) and \(K_2\) be any two knots in \(M\). Recall that \(\varphi(M, K_j) = [(M, K_j), O]\) with \(O = (S^3, U)\) and \(U\) the unknot. According to the definition of \(K(X^{(or)}_{\text{knot}})\) and the fact that \(O\) is the identity element of \(\#_2\) to show that \([(M, K_1), O] = [(M, K_2), O]\) we only need to exhibit an element \(Y\) of \(X^{(or)}_{\text{knot}}\) such that \((M, K_1) \#_2 Y\) is homeomorphic \((M, K_2) \#_2 Y\). For such a \(Y\) we use the pair \(D = (S^2 \times S^1, \{\ast\} \times S^1)\) already defined above, and we remark that \((M, K_j) \#_2 D = (M \#_0 (S^2 \times S^1), K_j')\) where \(K_j' \subset M \#_0 (S^2 \times S^1)\) is a knot intersecting a 1-sphere. Using Proposition 4.1 (and dealing with orientation matters) we conclude that indeed \((M, K_1) \#_2 D\) and \((M, K_2) \#_2 D\) are the same in \(X^{(or)}_{\text{knot}}\), and the conclusion follows. \(\square\)

**Corollary 4.4.** If \(\psi\) is an invariant of knot-pairs with values in an Abelian group and \(\psi\) is additive under \(\#_2\) then \(\psi(M, K)\) depends on \(M\) only.

In the context of oriented knot-pairs this corollary readily follows from the previous theorem and the universality property of \(\varphi\). But the corollary is actually true also in an unoriented context, by the same argument used to prove Theorem 4.3 for \(K_1, K_2 \subset M\) and suitably performed \(\#_2's\) one has \((M, K_1) \#_2 D \cong (M, K_2) \#_2 D\) and, using additivity of \(\psi\) and simplifying \(\psi(D)\), one concludes that \(\psi(M, K_1) = \psi(M, K_2)\). As a matter of fact one could also prove the corollary by adjusting Proposition 4.2 to the case of multivalued binary operations, using the notion of *semihypergroup*, obtained from that of *hypergroup* (due to Wall [10]) by dropping the invertibility postulate. In this context \(a \oplus b\) is a subset of \(S\). To construct \(K(S)\) one first defines an element \(x\) of \(S\) to be a *scalar* if for all \(a \in S\) the set \(a \oplus x\) consists of a single element; one then sets \(K(S) = (S \times S)/\sim\) where \((a, b) \sim (c, d)\) if there exists a scalar \(u \in S\) such that \(a \oplus d \oplus u = b \oplus c \oplus u\), and one shows that \(K(S)\) is universal with respect to maps \(\psi : S \to G\) such that
ψ is constant on \( a \oplus b \) for all \( a, b \in S \), and \( \psi \) preserves the operations and the identity elements. Since the pair \( D \) is a scalar for the semihypergroup of unoriented knot-pairs, the general machinery gives the unoriented version of Corollary 4.4.

**Remark 4.5.** We have already mentioned in Remark 2.4 the result of [6] that \( c(X) = 0 \) when \( X = (S^3, K)\#_2 D \), with \( K \subset S^3 \) any knot and \( D = (S^2 \times S^1, \{ \ast \} \times S^1) \). This was actually derived from the homeomorphism \( X \cong D \), of which the homeomorphism \( (M, K_1)\#_2 D \cong (M, K_2)\#_2 D \) used in the proof of Theorem 4.3 is an extension (take \( M = S^3 \), \( K_1 = K \) and \( K_2 = U \), the unknot).

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