SM Stability for Time-Dependent Problems

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Abstract. Various classes of stable finite difference schemes can be constructed to obtain a numerical solution. It is important to select among all stable schemes such a scheme that is optimal in terms of certain additional criteria. In this study, we use a simple boundary value problem for a one-dimensional parabolic equation to discuss the selection of an approximation with respect to time. We consider the pure diffusion equation, the pure convective transport equation and combined convection-diffusion phenomena. Requirements for the unconditionally stable finite difference schemes are formulated that are related to retaining the main features of the differential problem. The concept of SM stable finite difference scheme is introduced. The starting point are difference schemes constructed on the basis of the various Padé approximations.

1 Introduction

When time-dependent problems of mathematical physics are solved numerically, much emphasis is placed on computational algorithms of higher orders of accuracy (e.g., see [1, 2]). Along with improving the approximation accuracy with respect to space, improving the approximation accuracy with respect to time is also of interest. In this respect, the results concerning the numerical methods for ordinary differential equations (ODEs) [3, 4] provide an example. Taking into account the specific features of time-dependent problems for PDEs, we are interested in numerical methods for solving the Cauchy problem in the case of stiff equations [3, 7].

When time-dependent problems are solved approximately, the accuracy can be improved in various ways. In the case of two-level schemes (the solution at two adjacent time levels is involved), polynomial approximations of the scheme operators on the solutions are used explicitly or implicitly. The most popular representatives of such schemes are Runge-Kutta methods [7, 8], which are widely used in modern computations. The main feature of the multilevel schemes (multistep methods) manifests itself in the approximation of time derivatives with a higher accuracy on a multipoint stencil. A characteristic example is provided by multistep methods based on backward numerical differentiation [9].

Various classes of stable finite difference schemes can be constructed to obtain a numerical solution [10, 11]. It is important to select among all stable schemes such a scheme that is optimal in terms of certain additional criteria. In the theory
of finite difference schemes, there is the class of asymptotically stable schemes (see \[12, 13\]) that ensure the correct long-time behavior of the approximate solution. In the theory of numerical methods for ODEs (see \[7, 9\]), the concept of \(L\)-stability is used, which reflects the long-time asymptotic behavior of the approximate solution from a different point of view.

In \[14\] the properties of two-level difference schemes of high order approximation for the approximate solution of the Cauchy problem for evolutionary equations with self-adjoint operators are considered. The simplest boundary value problem for the one-dimensional parabolic equation serves as a basic problem. The concept of SM stability (Spectral Mimetic stability) of a difference scheme is introduced. This property is connected with the behavior of individual harmonics of the approximate solutions.

In this paper, we continue to study the SM properties of difference schemes for the approximate solutions of unsteady problems of mathematical physics. On the model boundary value problem for one-dimensional parabolic equation, the spectral characteristics of the approximations in space and in time are considered. In particular, good approximation properties (third order approximation in space) are observed for the convection operator. Two-level schemes of higher order of approximation in time, based on the Padé approximation, are considered for solving problems of mathematical physics with symmetric and skew-symmetric operators.

2 Problem Formulation

We consider finite-dimensional real Hilbert space \(H\), where the scalar product and the norm are \((\cdot, \cdot)\) and \(\| \cdot \|\), respectively. Let \(u(t) (0 \leq t \leq T > 0)\) be defined as the solution of the Cauchy problem for evolutionary equation of first order:

\[
\frac{du}{dt} + Au = f(t), \quad 0 < t \leq T, \tag{1}
\]

\[
u(0) = u_0. \tag{2}
\]

The right-hand side \(f(t) \in H\) of equation (1) is given and \(A\), depending on \(t\) (\(A = A(t) \geq 0\)), is a linear non-negative, in generally, not self-adjoint operator from \(H\) to \(H\).

For problem (1), (2) the estimate of stability is easily established. Taking into account the skew-symmetric property of operator \(A\), we have the equality

\[
\|u\| \frac{d\|u\|}{dt} = (f, u).
\]

By using

\[
(f, u) \leq \|u\| \|f\|
\]

we obtain a simple estimate of stability for the solution of (1), (2) with respect to the initial data and the right-hand side:

\[
\|u(t)\| \leq \|u_0\| + \int_0^t \|f(\theta)\| d\theta. \tag{3}
\]