A Time-Symmetric Soliton Dynamics à la de Broglie

Aurélien Drezet

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Abstract
In this work we develop a time-symmetric soliton theory for quantum particles inspired from works by de Broglie and Bohm. We consider explicitly a non-linear Klein–Gordon theory leading to monopolar oscillating solitons. We show that the theory is able to reproduce the main results of the pilot-wave interpretation for non-interacting particles in an external electromagnetic field. In this regime, using the time symmetry of the theory, we are also able to explain quantum entanglement between several solitons and we reproduce the famous pilot-wave nonlocality associated with the de Broglie-Bohm theory.

Keywords De Broglie double solution · Soliton · Time symmetry · Bohmian mechanics

1 Introduction: General Description of the Model
Seventy years ago Bohm [6], rediscovering some older results made by de Broglie [1, 15], published his deterministic hidden-variables theory showing that quantum mechanics can be reproduced by a dynamics where particles follow trajectories guided by a \( \psi \)-wave solution of the Schrödinger equation. This pilot wave theory (PWI) is clearly a counter example against the complacency of the previous period when even the possible existence of hidden variable was contested (e.g., by the von Neumann theorem). Moreover, the PWI is counter-intuitive: It involves a nonlocal ‘spooky’ quantum potential in tension with the theory of relativity, and there is no back-reaction of the particle on the guiding wave (which nature is by the way unclear). For these reasons the proposal made by Bohm is often rejected or criticized. That was the case of Louis de Broglie who actually invented the PWI already in 1926 [1, 15] but favored a different approach namely the double solution program (DSP) where a particle is a kind of singularity or localized wave (i.e., a soliton)
in an oscillating field guiding its motion \cite{15, 16} (for reviews see \cite{10, 19, 21, 25, 26}). The DSP was motivated by classical works made by Poincaré, Abraham, Mie \cite{37}, and Einstein to understand particles and waves as merging objects in a deeper (local) field theory. However, due to the constraints imposed by quantum mechanics and Bell’s theorem the DSP was never successfully developed. Interestingly the first version of the DSP proposed by de Broglie (1925b) is mostly unknown. This model was based on a time symmetric field involving retarded and advanced waves focused on the particle and conspiring in order to reproduce the phase coherence associated with standard quantum waves \cite{13, 14} (for a review see \cite{21}).

In the present work we develop and extend such a theory for a scalar $u$-field solution of a non-linear Klein–Gordon (NLKG) equation involving moving solitons in an external electromagnetic field. As we will see by choosing a specific non-linear term in the NLKG equation (the so called Lane-Emden nonlinearity), and by using ‘a phase harmony’ condition reminiscent of de Broglie DSP, a self-consistent model can be developed where the soliton core is obeying a dynamics recovering the usual PWI, i.e. standard quantum mechanics. In our approach the core of the soliton is guided by the phase of the linear Klein–Gordon (LKG) equation. In turn, since the model is completely local, we show that time symmetry associated with waves propagating forward and backward in time is needed in order to reproduce the nonlocal properties of the PWI for several entangled solitons. In the model the nonlocality is thus not fundamental but just effective and results from watching the particle trajectories while ignoring the underlying time symmetric $u$-field that propagates in the 4D space–time and guides the solitons. The present work modifies an earlier analysis \cite{22} in which a nonlocal soliton theory was developed in order to recover the PWI. Moreover, many of the mathematical results derived in \cite{22} are still valid and used in the present work.

In order to describe our model we start with the Lagrangian density

\[ L = Du(x)D^* u^*(x) - U(u^*(x)u(x)) \]

for a scalar complex field $u(x) \in \mathbb{C}$ with $x := [t, \mathbf{x}] \in \mathbb{R}^4$, and with $D = \partial + ieA(x)$, where $A(x)$ is the electromagnetic potential four-vector and $e$ an electric charge). The nonlinear function $U(y)$ leads to the (Euler–Lagrange) NLKG equation:

\[ D^2 u(x) = -N(u^*(x)u(x))u(x) \]  \hspace{1cm} (1)

with $N(y) := \frac{dU(y)}{dy}$. In the present work we need to use the so called Lane–Emden nonlinearity

\[ N(uu^*) = -\frac{3r^2_0}{4g} (uu^*)^2, \]  \hspace{1cm} (2)

\footnote{We use the Minkowski metric $\eta_{\mu\nu}$ with signature $+, -, -, -$ and the convention $\hbar = 1, c = 1.$}
where $g$ is a (nondimensional) coupling constant and $r_0$ a length that will define the typical radius of our soliton. As we will show this nonlinearity allows us to define explicit analytical solitons (at least in the near-field of the particle).

Equation 1 is different from the standard LKG equation

$$D^2\Psi(x) = -\omega_0^2 \Psi(x),$$

where $\Psi(x)$ is the usual quantum (relativistic) wavefunction for a particle of mass $\omega_0$. In the present theory the wavefunction $\Psi(x)$ plays a role to guide the soliton core. Moreover, here $\Psi(x)$ is not an external physical field. Rather, it represents a mathematical method to integrate the nonlinear $u$-field equation 1. This is similar to the role played by the Hamilton–Jacobi function in classical mechanics and to the role played by the phase of the wavefunction in ‘Bohmian mechanics’, i.e., the minimalistic version of the PWI advocated by Bell and others, where $\Psi(x)$ has a nomological role for guiding the (assumed) point-like particle but no ontological (material) status. However, here the localized solitonic $u$-field replaces the idealized point-like object of the PWI. In other words, in our theory there is only one physical field: the $u$-field solution of Eq. 1.

We stress that our approach differs from the most known version of the DSP advocated by de Broglie and Vigier in the 1950s [16] where the $u$-field is usually supposed to be the sum $u(x) = v(x) + u_0(x)$ of a base wave $v(x) \propto \Psi(x)$ solution of the linear equation 3 and of a localised solution $u_0(x)$ guided by the phase of $v(x)$. In the usual approach of de Broglie and Vigier [16] the amplitude of the soliton $u_0$ decays rapidly with the distance to the core and at large distances $r$ only the guiding base wave $v(x)$ survives. In our approach, which takes its inspiration from the first time-symmetric version of the DSP advocated by de Broglie [13, 14], there is no need for a physical base wave $v(x)$ guiding the soliton core. Instead, the soliton amplitude decays progressively to zero as $\sim 1/r$ when the distance $r$ to the particle center grows. Moreover, as we will show, the far-field of the soliton can be decomposed into a sum $u \equiv u_{\text{sym}} = \frac{1}{2} [u_{\text{adv}} + u_{\text{ret.}}]$ where a retarded and advanced contributions are needed to satisfy boundary conditions on the particle and in order to recover the famous guidance condition or guidance law postulated by de Broglie [15].

Remarkably, the present model unifies the first DSP of de Broglie developed in 1925 [13, 14] with two classes of methods that were used by de Broglie collaborators in the 1950s. First, in Sect. 2 in order to define the near-field of the soliton (i.e., for distances to the soliton center $r$ smaller than the typical Compton wavelength $\omega_0^{-1}$) we will use a collective coordinate method developed by Gérard Petiau with de Broglie [41]. This is a mathematical development of the ‘phase harmony condition’ postulated by de Broglie in the DSP [16]. The method was only developed in the classical regime for soliton driven by a classical dynamics. Here, we extend and modify this method in order to deal with quantum particles reproducing the PWI. We show that with the Lane–Emden nonlinearity given by Eq. 2 the soliton has a dilation invariance that allows us to define collective coordinates evolving adiabatically during the time evolution along the trajectory of the soliton center $z(\tau)$. These collective coordinates geometrically correspond
to internal deformations that force the soliton to keep its motion guided by the phase $S(x)$ of the wavefunction $\Psi(x)$. In turn, we show that this adiabatic evolution of the collective coordinates accounts naturally for the existence of a quantum potential that is acting along the trajectory $z(\tau)$ of the soliton and allows us to recover the Bohmian dynamics of the PWI.

Second, in order to define the far-field of the soliton and to connect it with the near-field we use in Sect. 3 a Green function approach inspired by works made by Francis Fer under de Broglie supervision [27]. The justification of our approach comes from the Lane–Emden nonlinearity that actually vanishes asymptotically when the field amplitude $|u|$ decays. In the far-field where the soliton amplitude approaches zero Eq. 1 thus reduces to the linear equation $D^2 u(x) \simeq 0$. The far-field associated with the soliton can thus be described by a linear equation with a source term associated with a singularity. Remarkably, Fer assumed that the soliton (singularity) far-field is described only by a (usual) retarded potential associated with an oscillating moving point source. This implied complicated conditions on the phase of the $u$-field that are in general impossible to fulfill to recover the guidance formula of the PWI. However, here we show that in order to recover the guidance conditions and the phase harmony defined in the near-field we must use the time symmetric decomposition $u := u_{\text{sym}} = \frac{1}{2}(u_{\text{adv}} + u_{\text{ret}})$. This fully agrees with the first intuitions of de Broglie [13, 14] that clearly emphasized the importance of such a time symmetric structure to justify the existence of the quantum phase wave and thus of the famous wave-particle duality.

It is obvious that having a time symmetric $u$-field will have strong implications in relation with the problem of non-locality, the EPR paradox [24] and Bell’s theorem [4]. Indeed, Bell’s theorem shows that a strong tension exists between quantum mechanics and special relativity. If we are assuming an ontological description of the world in term of hidden variables these variables must be nonlocal (like in the PWI) or we must relax some assumptions concerning causality. This is clearly what is suggested in ‘exotic’ approaches involving retrocausality (causality acting from the future to the past) [11, 12] or superdeterminism (see for examples the remarkable papers by Bohm [7] and also Gerard ’t Hooft [48]). Superdeterminism is often seen as a conspiratorial approach and is associated with the end of physics. Moreover, it is clear as the present author claimed elsewhere [20], that a retrocausal or time symmetric theory is just a particular example of superdeterministic theory where the conspiracy is deciphered and justified. Moreover, a time-symmetric approach of classical electrodynamics, the so-called absorber theory of Wheeler and Feynman [52], exists and cannot be said to be non-scientific. In Sect. 4 of this work we actually extend our DSP to the case involving several particles. We show that based on our $u$-field theory which is fundamentally local we can still develop a model able to reproduce the predictions of the PWI for many entangled particles. Moreover, since the PWI is explicitly nonlocal this actually means that we explain and interpret physically the existence of nonlocality as an effective description of the problem obtained when ones consider only the ‘Bohmian’ particle trajectories and ignore the complex time symmetric $u$-field surrounding the particles. In our model the particles are seen as moving clocks synchronized by the $u$-field coming from the past.
and future and that focus at the right time and right place to be in phase harmony with the soliton core field. In that way we explain and interpret nonlocality as an effective description and reproduce precisely the PWI predictions violating Bell’s inequalities. We believe that this new development of de Broglie’s intuition is the most remarkable consequence of a model able to explain wave-particle duality, and quantum entanglement in one unified deterministic, non linear but definitely local picture.

The layout of the article goes as follows: in Sect. 2 we describe the physics of the near-field of our soliton model and connect it with results concerning the ‘guidance formula’ obtained by de Broglie and Petiau [15, 16, 43]. We then solve our NLKG equation after using properties (such as a dilation invariance) of the Lane–Emden nonlinearity and introducing collective coordinates evolving adiabatically during the soliton motion. We also connect our findings with results obtained with a version of Ehrenfest’s theorem derived in [22] and show how the weakness of our soliton confinement (with field decaying as \( \propto \frac{1}{r} \) at large distance but still in the near field) allows us to circumvent some constraints imposed by the Ehrenfest theorem. In Sect. 3 we describe the far-field of our soliton. More precisely, by using the fact that the Lane–Emden nonlinearity can be neglected when the \( u \)-field amplitude is small in the far-field we show that a Green function approach (based on early works [27, 28]) can be applied to define the soliton far-field. Importantly, we show that in order to connect the far-field with the near-field we must use a time symmetric Green function reminiscent of old works made by de Broglie [13, 14] and Wheeler–Feynman [52]. This time symmetry is also mandatory in order to recover the result deduced from the guidance theorem and telling that the particle is guided by the phase of the wavefunction \( \Psi \). Finally, in Sect. 4 we summarize our findings and discuss some extensions of the model. The most important extension concerns the many-body problem and the case of several solitons moving in external fields. Using the time symmetry of our model we show that we can recover the PWI for many entangled particles. Moreover in this new theory the configuration space is not fundamental and the motion of solitons is described in the 4D space–time. In particular, we emphasize the importance of our model to discuss the issue of nonlocality in relation with the PWI. Indeed, since the PWI (i.e., Bohmian mechanics) is nonlocal and violate Bell’s inequalities our model shows that a fundamentally local but non-linear wave equation can reproduce, justify and interpret the apparent nonlocality of the particles in the PWI. The PWI is seen only as an effective nonlocal theory obtained when one is ignoring the complicated but local time symmetric motion of the \( u \)-field and consider only the point-like particle motions.
2 The Soliton Near-Field and the Phase Harmony Condition of de Broglie

The goal of this section is to define the structure of the soliton near-field. For this purpose the precise form of the Lane–Emden nonlinearity \( N(uu^*) = -\frac{3r_0^2}{(u^*)^4} \) is central since it is one of the few known nonlinearities admitting analytical solitonic solution in 3D space. Moreover, in order to apply this property to the DSP we need to connect the solution of the nonlinear wave equation to the so called guidance formula of de Broglie \([15, 16]\) in which the phase of the soliton defined along the trajectory of the soliton center (its ‘core’) is defined and compared to the phase of the standard LKG wavefunction \( \Psi(x) \). Therefore, before describing the near-field of the soliton we must first remind the methodology proposed by de Broglie to justify his guidance theorem. Here we will use results derived by us in a recent and longer article \([22]\). Due to the technicality of the proofs we will only sketch the major steps of the deductions. We emphasize that the methodology is inspired by old and mostly unnoticed results obtained by Petiau in the 1950s for charactering the soliton motion \( z(\tau) \) using collective coordinates, i.e., parameters defining the soliton shape and strength that evolve adiabatically (slowly) along the particle path in order to keep its main structure invariant with time \([41–43]\) (some of these important results were later rediscovered by others \([5, 46]\)). We also stress that the results derived in this work are fully relativistic and covariant. In the nonrelativistic regime it recovers the standard PWI advocated by Bohm (see \([22]\)).

We first introduce the Madelung/de Broglie representation \( u(x) = f(x)e^{i\varphi(x)} \), with \( f(x), \varphi(x) \in \mathbb{R} \), in Eq. 1 and obtain a pair of coupled hydrodynamic equations:

\[
(\partial \varphi(x) + eA(x))^2 = N(f^2(x)) + \frac{\Box f(x)}{f(x)} := M_\varphi^2(x), \tag{4a}
\]

\[
\partial [f^2(x)(\partial \varphi(x) + eA(x))] = 0. \tag{4b}
\]

At that stage the precise form of the nonlinearity is irrelevant and the following results will be general. Moreover, for the LKG equation we write similarly \( \Psi(x) = a(x)e^{iS(x)} \), with \( a(x), S(x) \in \mathbb{R} \), yielding the pair of coupled hydrodynamic equations:

\[
(\partial S(x) + eA(x))^2 = \omega_0^2 + Q_\psi(x) := M_\psi^2(x), \tag{5a}
\]

\[
\partial [a^2(x)(\partial S(x) + eA(x))] = 0, \tag{5b}
\]

with \( Q_\psi(x) = \frac{\Box a(x)}{a(x)} \) the so called quantum potential \([6, 16]\). Here we consider only the cases \( M_\psi^2(x) > 0 \) avoiding tachyonic trajectories. In the following we show how these equations are solved in the vicinity of the soliton core associated with the localized particle.
As explained in Sect. 1 we don’t have to suppose the wave guiding the soliton. The guiding
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More precisely, assuming that such a soliton exists as a solution of Eq. 1 or 4 we write
z(τ) the trajectory of the soliton center labeled by the proper time τ. Associated with this particle motion we define a local Lorentz (proper) rest-frame \( R_\tau \) and an hyperplane \( \Sigma(\tau) \) with normal direction given by the velocity \( \dot{z}(\tau) \). From geometrical considerations a point \( x \) belonging to \( \Sigma(\tau) \) satisfies the constraint
with \( \xi = x - z(\tau) \) (in \( R_\tau \) we have \( \xi := [0, \xi] \)) (see Fig. 1). As we showed in [22] for points \( x \in \Sigma(\tau) \) near \( z(\tau) \) we can define univocally the structure of the soliton using the variable \( \xi \) in \( R_\tau \) if \( 1 \gg |a| \cdot |\xi| \) where \( a \) is the instantaneous acceleration of the center in the rest-frame \( R_\tau \) (i.e., \( \xi \ddot{\xi} \ll 1 \)). This is interpreted as a condition for defining the notion of quasi-rigidity of the soliton in a relativistic context.
Assuming this condition of rigidity is fulfilled we now introduce the so-called ‘phase-harmony condition’ inspired from de Broglie’s DSP [15, 16]:

\[
\text{To every regular solution } \Psi(x) = a(x)e^{iS(x)} \text{ of Eq. 3 corresponds a localized solution } u(x) = f(x)e^{i\varphi(x)} \text{ of Eq. 1 having locally the same phase } \varphi(x) \simeq S(x), \text{ but with an amplitude } f(x) \text{ involving a generally moving soliton centered on the path } z(\tau) \text{ and which is representing the particle.}
\]

As explained in Sect. 1 we don’t have to suppose the \( \Psi \)-field to be a physical external wave guiding the soliton. The guiding \( \Psi \)-wave and in particular its phase is in our work interpreted as a way to mathematically integrate and solve the nonlinear wave equation for the \( u \)-field. As in [22] (inspired by some earlier non-relativistic results by Petiau [41–43] and others [5, 46]) we here write for points \( x \in \Sigma(\tau) \) near \( z(\tau) \):

\[
\varphi(x) \simeq S(z(\tau)) - eA(z(\tau))\xi + B(z(\tau))\frac{\xi^2}{2} + O(\xi^3)
\]  

(7)

which defines the phase-harmony condition up to the second-order approximation in power of \( \xi \). The scalar function \( B(z(\tau)) := B(\tau) \) is a new collective coordinate required in order to consider the deformation of the soliton with time \( \tau \). We mention
that Eq. 7 is not gauge invariant and presupposes the Coulomb–Gauge constraint $\nabla \cdot \mathbf{A} = 0$ in the local rest frame $R_r$. Moreover, the full theory is naturally gauge invariant.

With these properties we can extract from Eq. 4 several important results derived in [22]. First, if we define $v_u(x)$, and $v_\Psi(x)$ the velocity of the NLKG and LKG equations respectively

$$v_u(x) = -\frac{\partial \varphi(x) + ea(x)}{M_u(x)},$$

$$v_\Psi(x) = -\frac{\partial S(x) + ea(x)}{M_\Psi(x)},$$

we have along $\Sigma(\tau)^2$

$$v_u(x) \simeq v_\Psi(z(\tau)) + O(\xi),$$

$$M_u(x) \simeq M_\Psi(z(\tau)) + O(\xi).$$

Moreover, as shown in [22], we deduce $v_u(z(\tau)) = \dot{z}(\tau)$ and therefore we obtain here a guidance condition

$$v_u(z(\tau)) = \dot{z}(\tau) = v_\Psi(z(\tau))$$

as postulated in the original DSP of de Broglie. The phase-harmony condition that we postulate is thus imposing $\partial \varphi(z(\tau)) = \partial S(z(\tau))$. The two phase waves $\varphi$ and $S$ are thus connected along the curve $z(\tau)$. Yet, we emphasize that we don’t here impose the second-order matching $\partial_{\mu,\nu}^2 \varphi(z(\tau)) = \partial_{\mu,\nu}^2 S(z(\tau))$ but only a first-order contact [22] meaning that $\partial_{\mu,\nu}^2 \varphi(z(\tau))$ and $\partial_{\mu,\nu}^2 S(z(\tau))$ are in general different. The relation $\dot{z}(\tau) = v_\Psi(z(\tau))$ is actually the definition given to the particle velocity in the PWI fixing a first order dynamical law. In this PWI we directly deduce a second order dynamical law [22]:

$$\frac{d}{d\tau} [M_\Psi(z(\tau))\dot{z}_\nu(\tau)] = \partial^\mu [M_\Psi(z(\tau))] + eF^\mu\nu(z(\tau))\dot{z}_\nu(\tau)$$

with $F^\mu\nu(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ the Maxwell tensor field at point $x : = z$.

Moreover, we also deduce in the vicinity of $z(\tau)$:

$$v_u(x) \partial \ln [f^2(x)] + \frac{d}{d\tau} \ln [M_\Psi(\tau)] = \frac{3B(\tau)}{M_\Psi(\tau)} + O(\xi).$$

In [22] we obtained the expression for $M_u(x)$ (Eq. 66) without expliciting the first order correction $O(\xi)$. The details of the calculations show that we have $M_u(x) \simeq M_\Psi(z(\tau)) + \xi \cdot \nabla M_\Psi(z(\tau)) + O(\xi^2)$. A careful analysis shows that we have $\partial_{\mu}M_u(z(\tau)) = \partial_{\mu}M_\Psi(z(\tau))$. 

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where we have used the Lagrangian derivative \( \frac{d}{d\tau} \equiv v_\Psi(z) \partial_z \) for the \( \Psi \)-field along the particle trajectory \( z(\tau) \). From Eq. 12 we deduce

\[
\frac{d}{d\tau} \ln [f^2(z(\tau))M_\Psi(\tau)] = \frac{3B(\tau)}{M_\Psi(\tau)}. \tag{13}
\]

To physically interpret Eq. 13 it is interesting to note that from Eq. 4b we have \( v_u \partial \ln (f^2 M_u) := \frac{d}{d\tau} \ln (f^2 M_u) = -\partial v_u \) and that from relativistic hydrodynamics we can define an elementary comoving 3D fluid volume \( \delta^3 \sigma_0 \) (defined in \( \mathcal{R}_r \)) driven by the fluid motion and such that \( v_u \partial \ln (\delta^3 \sigma_0) := \frac{d}{d\tau} \ln (\delta^3 \sigma_0) = +\partial v_u \). Regrouping all these conditions and using \( M_\Psi(\tau) = M_u(\tau) \) we obtain

\[
-\partial v_u(z(\tau)) = -\frac{d}{d\tau} \ln [\delta^3 \sigma_0(z(\tau))] = \frac{3B(\tau)}{M_\Psi(\tau)} = \frac{d}{d\tau} \ln [f^2(z(\tau))M_\Psi(\tau)] \tag{14}
\]

which shows that a non-vanishing value for \( B(\tau) \) involves a compressibility and ‘deformability’ of the soliton droplet.

A final important equation can be derived in the limit \( \xi \ll 1 \) where we have \( |\partial_x f| \ll |\nabla^2 f| \) in the rest-frame \( \mathcal{R}_r \). We obtain [22] the partial differential equation for the soliton profile for points \( x \) belonging to \( \Sigma(\tau) \) and localized near \( z(\tau) \):

\[
[\omega_0^2 + Q_\Psi(z(\tau))]f(x) + \nabla^2 f(x) \simeq N(f^2(x))f(x) \tag{15}
\]

with \( \nabla : = \frac{\partial}{\partial x} \). Moreover, we suppose the soliton core size \( r_0 \) to be much smaller than the Compton wavelength \( \omega_0^{-1} \) or even \( M_\Psi^{-1} \) and therefore in the near-field we have

\[
\nabla^2 f(x) \simeq N(f^2(x))f(x). \tag{16}
\]

This presentation summarizes the main mathematical results discussed at length in our recent work [22].

At that stage we must now apply these results to the Lane–Emden NLKG equations 13. Moreover, in order to further motivate the choice of such nonlinear function we also remind a negative result derived in [22] and concerning strongly localized and undeformable solitons like ‘Gaussons’ considered for instance in [5]. Indeed, in [22] we showed that it is not in general possible to satisfy simultaneously Eqs. 16 and 14 because the condition of existence of an undeformable localized wave contradicts Eq. 14. More precisely, assuming a undeformable solution of Eq. 16 we have \( f(z(\tau)) = f_0 = \text{Const. \forall \tau} \), i.e., from Eq. 14 \( B(\tau) = \frac{1}{3} \frac{d}{d\tau} M_\Psi(\tau) \). But since the soliton is supposed undeformable we have also \( -\partial v_u(z(\tau)) = -\frac{d}{d\tau} \ln [\delta^3 \sigma_0(z(\tau))] = 0 \) and therefore \( B(\tau) = 0 \). This implies \( M_\Psi(\tau) = \text{Const.} \) along the trajectory \( z(\tau) \) and contradicts the PWI (where the quantum potential is in general not a constant of motion).

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3 We have the fluid conservation: \( v_u \partial \ln (f^2 M_u \delta^3 \sigma_0) := \frac{d}{d\tau} \ln (f^2 M_u \delta^3 \sigma_0) = 0. \)
Furthermore, in [22] we also showed from a version of the Ehrenfest theorem adapted to our nonlinear wave equation that the existence of a very small soliton with fastly decaying amplitude with the distance $|\xi|$ generally contradicts the PWI unless $M_{\Psi}(\tau) = \text{Const.}$ along the trajectory $z(\tau)$. In other words, a strongly localized soliton obeys a classical dynamics characterized by a constant mass $M_{\Psi}(\tau) = \text{Const.}$ and don’t reproduce quantum mechanics.

In order to circumvent these objections against the DSP we need here to relax our constraints and we must define a nonlinear function $N(uu^*)$ such that the NLKG equation (i) admit a soliton deformable along the trajectory $z(\tau)$, and (ii) that the soliton is not too strongly localized in order to avoid the conclusion of Ehrenfest theorem.

For this purpose we here consider a power-law non-linearity (also called Lane–Emden non linearity in the context of astrophysics for modeling stellar structures [9]) $N(y) = -\gamma y^p$ with $\gamma > 0$ and $p \in \mathbb{R}$ an index. Here, we use specifically $p = 2$ which leads to a nontrivial but simple analytical solution that was also obtained by G. Mie in his nonlinear electrodynamics involving solitons [37] (see also [44, 47]). The choice $p = 2$ has many remarkable properties that can be exploited in the context of the DSP. Writing

$$U(f^2) = -\frac{r_0^2}{\left(\frac{g}{4\pi}\right)^4} f^6, \quad (17a)$$

$$N(f^2) = -\frac{3r_0^2}{\left(\frac{g}{4\pi}\right)^4} f^4, \quad (17b)$$

we have in the near field [i.e., Eq. 16 with $M_{\Psi}(z(\tau))r_0(\tau) \ll 1$]

$$\nabla^2 f(x) = -\frac{3r_0^2}{\left(\frac{g}{4\pi}\right)^4} f^5(x), \quad (18)$$

which admits the radial (non topological) soliton

$$f(x) := F(r) = \frac{g}{4\pi} \frac{1}{\sqrt{r^2 + r_0^2}}, \quad (19)$$

---

4 Ehrenfest theorem has been also applied by Bialynicki-Birula and Mycielski [5] in the context of classical Gaussion dynamics driven by external electromagnetic forces. The theorem has also been used by Durt and coworkers [10, 33] in the particular context of the nonrelativistic Schrödinger–Newton equations. Our own independent results are based on the relativistic NLKG an is valid for a large range of equations.
(with \( r = |\mathbf{x}| \)) as it can be checked by direct substitution. Equation 19 has the asymptotic monopolar limit \( F(r) \approx \frac{g}{4\pi r} \) if \( r \gg r_0 \) and \( F(0) = \frac{g}{4\pi r_0} \). \( g \) is thus interpreted as a soliton charge (satisfying the integral condition \( -\int d^3x N(f^2)f = g \) as shown in “Appendix 1”) while \( r_0 \) acts as a typical radius for the soliton structure. The static energy of the soliton is (see “Appendix 1”) \( E_s = \frac{g^2}{32r_0} \).

The non-linearity equation 17b is particularly interesting in the context of the DSP since with Eq. 19 it actually vanishes asymptotically for \( r \gg r_0 \), i.e., \( N(f^2)f = -\frac{3g}{4\pi (r^2 + r_0^2)^2} \rightarrow -\frac{3g}{4\pi r^2} \). Far-away of the soliton core the monopolar approximation is thus very good and in this limit it is justified to use instead Poisson’s equation \( \nabla^2 f(x) = -g\delta^3(x) \) for a point-like source. More generally, it is visible that with Eq. 17 the NLKG equation reduces to \( \Box u(x) \approx 0 \) if the amplitude of \( u \to 0 \). It is important to observe that the asymptotic field \( u \propto \frac{1}{r} \) decays too slowly for applying Ehrenfest theorem (we proved this results in [22]). Therefore we evade the conclusions discussed before for a strongly localized droplet.

Equation 18 possesses an interesting dilation invariance. Indeed, it can directly checked that if \( f(x) \) is a solution of Eq. 18 so is the function \( \tilde{f}(x) = \sqrt{\alpha} f(\sqrt{\alpha} x) \) where \( \alpha \in \mathbb{R} \). In other words, from Eq. 19:

\[
\tilde{f}(x) = \tilde{F}(r) = \frac{\sqrt{\alpha} g}{4\pi} \frac{1}{\sqrt{\alpha^2 r^2 + r_0^2}} = \frac{g}{4\pi \sqrt{\alpha}} \frac{1}{\sqrt{r^2 + \frac{r_0^2}{\alpha^2}}}.
\]  

(20)

The second expression shows that the new soliton corresponds to a particle with a new characteristic radius \( \tilde{r}_0 = \frac{r_0}{\alpha} \) and a new charge \( \tilde{g} = \frac{g}{\sqrt{\alpha}} \left(-\int d^3x N(\tilde{f})\tilde{f} = \tilde{g} \right) \), and Eq. 18 can alternatively be written as

\[
\frac{d^2}{dr^2}\tilde{F}(r) + \frac{2}{r} \frac{d}{dr}\tilde{F}(r) + \frac{3\tilde{r}_0^2}{4g^4} \tilde{F}^5(r) = 0
\]  

(21)

with \( \frac{\tilde{r}_0^2}{g^4} = \frac{r_0^2}{g^4} \). Furthermore, observe that the quasi-static energy is invariant, i.e. \( \tilde{E}_s = \frac{\tilde{g}^2}{32\tilde{r}_0} = E_s \) during this transformation. Moreover, if \( r \gg \tilde{r}_0 \) we have the asymptotic field

\[
\tilde{F}(r) \approx \frac{g}{4\pi \sqrt{\alpha}} \frac{1}{r} = \frac{\tilde{g}}{4\pi} \frac{1}{r}
\]  

(22)

which again confirms that we have a monopolar quasi-static term corresponding to a charge \( \tilde{g} \).

\[ \text{This invariance allows us to circumvent the conclusions of the Hobart–Derrick theorem [17, 31, 34] which usually precludes the existence of static and stable solitons in 3D space. In “Appendix 2” we give an elementary proof of this result.} \]
Physically, the parameter $\alpha$ can be interpreted as a new collective coordinate for the soliton. More precisely, we now assume that during its motion the soliton typical extension $r_0(\tau)$ changes with time $\tau$. We thus write

$$r_0(\tau) := \tilde{r}_0 = r_0(0)/\alpha(\tau), \quad g(\tau) := \tilde{g} = \frac{g(0)}{\sqrt{\alpha(\tau)}}, \quad \quad (23)$$

where $\alpha(\tau)$ defines the dynamics concerning the radius. Therefore, for points located not too far from the soliton center equation 22 generally describes the field along the hyperplane $\Sigma(\tau)$ of Fig. 1. In particular, in the near-field the description is supposed to be very robust because of the condition $\mathcal{M}_\Psi(z(\tau))r_0(\tau) \ll 1$. We thus write in the near-field defined in the proper rest frame $\mathcal{R}_c$:

$$F_c(r) := \tilde{F}(r) = \sqrt{\frac{\alpha(\tau)g(0)}{4\pi}} \frac{1}{\sqrt{\alpha(\tau)^2r^2 + r_0(0)^2}}, \quad \quad (24)$$

where the parameter $\alpha(\tau)$ now describes the compressibility or ‘deformability’ of the moving soliton droplet defined along the hyperplane $\Sigma(\tau)$.

The picture obtained is thus the one of a deformable or compressing moving soliton. It is important to see that $B(\tau)$ and $\alpha(\tau)$ can easily be connected. More precisely, writing the local conservation law $\frac{d}{dt} \log [f^2(z(\tau))\mathcal{M}_\Psi(\tau)\delta^3\sigma_0(\tau)] = 0$ for a fluid element located at the soliton center we have by integration

$$f^2(z(\tau))\mathcal{M}_\Psi(\tau)\delta^3\sigma_0(\tau) = f^2(z(0))\mathcal{M}_\Psi(0)\delta^3\sigma_0(0). \quad \quad (25)$$

Furthermore, from Eq. 24 $f(z(\tau)) = F_c(0) = \sqrt{\alpha(\tau)}F_0(0) = \sqrt{\alpha(\tau)}f(z(0))$ and $\delta^3\sigma_0(\tau) = \frac{1}{\alpha^3(\tau)}\delta^3\sigma_0(\tau)$ (a more rigorous justification is given in “Appendix 3”). Therefore, Eq. 25 leads to

$$\alpha(\tau) = \sqrt{\frac{\mathcal{M}_\Psi(\tau)}{\mathcal{M}_\Psi(0)}}, \quad \quad (26)$$

Moreover, we also have $\frac{\partial\nu}{\partial t}(z(\tau)) = \frac{d}{dt} \log (\delta^3\sigma_0(\tau)) = \frac{d}{dt} \log (\alpha^{-3}(\tau)) = -\frac{3}{2} \frac{d}{dt} \log (\mathcal{M}_\Psi(\tau))$ yielding:

$$B(\tau) = \frac{1}{2} \frac{d}{dt} \mathcal{M}_\Psi(\tau). \quad \quad (27)$$

Together Eqs. 26 and 27 define the complete deformation/compression of the soliton near-field. In particular, in the non-relativistic regime where the mass $\mathcal{M}_\Psi(\tau)$ is approximately constant we have $\alpha(\tau) \simeq 1$ et $B(\tau) \simeq 0$, i.e., $\partial\nu_u(z(\tau)) \simeq 0$. We thus recover the picture of an incompressible soliton. In the general relativistic case we get by integration of Eq. 13 and the value of $B(\tau)$ the relation
In the end we thus succeeded in obtaining a description of a moving soliton with near-field $x \in \Sigma(r)$ and where $F(r)$ is given by Eq. 24, $\varphi$ by Eq. 7 (phase harmony condition), and the constraints for $a(\tau)$ and $B(\tau)$ are given by Eqs. 26 and 27. The dynamics of the soliton core $z(\tau)$ is piloted by the guidance formula equation 10 and recovers the PWI (e.g., Eq. 11).

\section{The Time-Symmetric de Broglie Double Solution in the Far-Field}

The previous theory developed for the near-field can be used to define the mide-field and far-field of the soliton. We go back to Eq. 4a written as Eq. 15 and don’t neglect the mass term $M(\varphi(z(\tau)))$. We consider first the case of an uniform motion where $M(\varphi(z(\tau))) = \text{Const.} = \omega_0$ and search for a spherical solution of

$$f(z(\tau)) = \left( \frac{M(\varphi(z(\tau)))}{M(\varphi(0))} \right)^{1/4} f(z(0))$$

in agreement with $f(z(\tau)) = \sqrt{\alpha(\tau)} f(z(0))$ and Eq. 26.

Fig. 2 Soliton profile $F(r)$ as a function of the radius $r$. The black curve shows the solution $F(r)$ of the equation $\Delta F = -3F^5 - AF$ with $A = 0.1$. The red dashed curve corresponds to the case $A = 0$ (i.e., the Lane–Emden soliton $F(r) = \frac{1}{\sqrt{r+1}}$). The solutions are obtained numerically by imposing $F(0) = 1$ and $\frac{\partial}{\partial \tau} F(0) = 0$. The blue dotted curve corresponds to the asymptotic stationary monopole field $F(r) = \frac{\text{const}(\varphi)}{r}$. The inset in the right upper corner of the figure is a zoom of the three curves near the soliton center (Color figure online).
\[
\begin{aligned}
\frac{d^2}{dr^2} F(r) + \frac{2}{r} \frac{d}{dr} F(r) + \frac{3r^2}{4\pi} F^5(r) + M_\Psi F(r) = 0.
\end{aligned}
\] (30)

In the near-field (i.e., \( M_\Psi r \ll 1 \)) we have Eq. 19 with asymptotic monopolar limit \( F(r) \approx \frac{g}{4\pi} \frac{1}{r} \). In the far-field (i.e., \( M_\Psi r \gg 1 \)) we have \( \frac{d^2}{dr^2} F(r) + M_\Psi F(r) \approx 0 \) which admits the monopole solution: \( F(r) \approx \frac{g}{4\pi} \frac{\cos(M_\Psi(z(r)t))}{r} \). We can easily interpolate these two solutions by writing a solution of Eq. 30 as quasi-static solutions (see Fig. 2 for a numerical calculation). We check that which is indeed an interpolation between the monopolar and the Lane–Emden solution of the inhomogeneous d’Alembert equation

\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0.
\]

This

\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = g(x) e^{-i\omega_0 t},
\]

and merges with the far-field of our soliton equation 32 and \( u = F(r)e^{-i\omega_0 t} \) if \( \omega_0 := M_\Psi \).

We stress that Eq. 33 reads \( gG^{(0)}_{\text{sym},\omega_0}(r)e^{-i\omega_0 t} \) where

\[
G^{(0)}_{\text{sym},\omega_0}(r) = \frac{\cos(\omega_0 R)}{4\pi R} = \frac{1}{2} \left[ \frac{e^{i\omega_0 R}}{4\pi R} + \frac{e^{-i\omega_0 R}}{4\pi R} \right]
\] (34)

We stress that de Broglie rejected the non singular wave \( u(t, r) = e^{-i\omega_0 t, \text{sin}(\omega_0 r)} \) solution of the homogeneous wave equation. This alternative solution was later rediscovered by many authors including Mackinnon [36] and Barut [3]. At the experimental level such non dispersive de Broglie waves have been recently created [32] using optical methods inspired from diffraction-free beams works [23]. This is also related to the work of Fink concerning time reversal mirrors in acoustics and optics [2, 29] that plays with causality. This could perhaps play a role in order to develop a physical wave analog of our time symmetric soliton.
$G(R) = \frac{1}{4\pi} \frac{\delta^3(x - x')}{R}$ is the time-symmetric Green function of the Helmholtz equation: $[\alpha^2 + \nabla^2]G^{(0)}_{\alpha}(R) = -\delta^3(x - x')$. In other words, $u(t, r)$ in Eq. 33 is a time-symmetric solution $u = \frac{1}{2}(u_{\text{ret}} + u_{\text{adv}})$ of $\Box u(t, x) = g\delta^3(x)e^{-i\omega t}$. This is fundamental because it leads to the stability of the micro-object: The energy radiation losses associated with the retarded wave are exactly compensated by the energy flow associated with the converging advanced wave. Furthermore, it implies a time-symmetric causality which is reminiscent of early ideas by Tetrode and Page [39, 49] for explaining the stability of atomic orbits. Such ideas were later resurrected by Fokker [30], Feynman and Wheeler in their absorber theory [52], and by Hoyle and Narlikar for cosmological models involving a time-symmetric creation-field [35]. Interestingly, this idea involving time-symmetry was also discussed in 1925 by de Broglie [13, 14] but was soon abandoned by him and he never came back to this suggestion (even after

Footnote 7: We have $G^{(0)}_{\text{sym,ret}}(R) = \frac{1}{2}[G^{(0)}_{\text{ret}}(R) + G^{(0)}_{\text{adv}}(R)]$ and $G^{(0)}_{\text{ret/adv,ret}}(R) = \frac{\cos \theta}{4\pi R}$ are the retarded and advanced Green functions respectively.
his collaborator Costa de Beauregard developed a retrocausal interpretation of the EPR paradox [11].

To appreciate the time-symmetric nature of our soliton we represent in Fig. 3\(\Re[u(x, y, z = 0, t = 0)]\) in the laboratory frame where the soliton moves at the velocity \(v_x\) along the +x direction. Using Eq. 32 for the interpolated solution we have

\[
\Re[u(x, t)] = \frac{g}{4\pi} \frac{\cos (\omega_0 R) \cos[\omega_0 \gamma (t - v_x x)]}{\sqrt{r_0^2 + R^2}}
\]

with \(R = \sqrt{(x - v_x t)^2 + y^2 + z^2}\) and \(\gamma = (1 - v_x^2)^{-\frac{1}{2}}\). Moreover, writing \(u := u_{\text{sym}} = \frac{1}{2}[u_{\text{adv}} + u_{\text{ret}}]\) and still using the interpolated field we can define

\[
\Re[u_{\text{ret./adv.}}(x, t)] = \frac{g}{4\pi} \frac{\cos [\omega_0 \gamma (t - v_x x) \mp \omega_0 R]}{\sqrt{r_0^2 + R^2}}
\]

associated with propagating diverging/converging waves. Comparing Eqs. 36 and 35 shows that the particle core is surfing a wave front reminiscent of what is occurring with an airplane in the subsonic regime (with here the velocity of light replacing the velocity of sound). The wave front precedes the particle in the retarded case and follows the particle in the advanced case. The superposition \(\Re[u_{\text{sym}}(x, y, z = 0, t = 0)]\) induces a phase wave \(\cos [\omega_0 \gamma (t - v_x x)]\) associated with de Broglie’s guiding field (i.e., the guiding wave involved in the PWI).

The previous theory for the far-field can be generalized. For this purpose start from Eq. 1 written as \(D^2 u(x) = -\hat{N}(u^*(x)u(x))u(x) := J(x)\). Using Green’s theorem a formal solution reads

\[
u(x) = \int K(x, y)J(y)d^4y + u_{\text{free}}(x),
\]

where \(u_{\text{free}}(x)\) is a solution of the homogeneous equation \(D^2 u(x) = 0\), and the propagator \(K(x, x')\) satisfies \(D^2 K_{\text{sym}}(x, x') = \delta^4(x - x')\). Consider first the case \(A(x) = 0\)
(i.e., absence of external field). We see that a natural choice corresponds to \( u_{\text{free}}(x) = 0 \) and \( K^{(0)}_{\text{sym}}(x, x') = \frac{K_{\text{ret}}^{(0)}(x, x') + K_{\text{adv}}^{(0)}(x, x')}{2} \) where

\[
K^{(0)}_{\text{sym}}(x, x') = \frac{\delta[(x - x')^2]}{4\pi} = \frac{1}{2} \left[ \frac{\delta(t - t' - R)}{4\pi R} + \frac{\delta(t - t' + R)}{4\pi R} \right]
\]

(with \( R = |x - x'|^2 \)) is the time-symmetric propagator.\(^8\)

As shown in Fig. 4a the evaluation of the far-field at point \( x \) requires the knowledge of the source term \( J(x) \) in the vicinity of the trajectory \( z(\tau) \) along two hyperplanes \( \Sigma(\tau_{\text{ret}}), \Sigma(\tau_{\text{adv}}) \) associated with retarded and advanced emissions by the particle. These planes are obtained by finding intersections of the trajectory \( z(\tau) \) with the backward and forward light cones with common apex located at point \( x \). We have approximately:

\[
u(x) \simeq \frac{1}{2} \int \int_{\Sigma(\tau_{\text{ret}})} K^{(0)}_{\text{ret}}(x, z(\tau_{\text{ret}}))J(\tau_{\text{ret}}, \xi) d^3 \xi d\tau_{\text{ret}}.
\]

\[
+ \frac{1}{2} \int \int_{\Sigma(\tau_{\text{adv}})} K^{(0)}_{\text{adv}}(x, z(\tau_{\text{adv}}))J(\tau_{\text{adv}}, \xi) d^3 \xi d\tau_{\text{adv}},
\]

where the integrations are done in the local rest frame \( \mathcal{R}_{\text{ret}} \) and \( \mathcal{R}_{\text{adv}} \). After spatial integration over the hyperplanes \( \Sigma(\tau_{\text{ret}}), \Sigma(\tau_{\text{adv}}) \) and using the lowest order approximation

\[
\int_{\Sigma(\tau)} J(\tau, \xi) d^3 \xi = - \int d^3 x N(F_2^2) F_3 e^{iS(z(\tau))} \simeq g(\tau)e^{iS(z(\tau))} = g(0)\frac{e^{iS(z(\tau))}}{\sqrt{a(\tau)}}
\]

(see Eq. 29 with \( \varphi \simeq S \)) we obtain for the far-field:

\[
u(x) = g(0) \int_{(C)} K^{(0)}_{\text{sym}}(x, z(\tau)) \frac{e^{iS(z(\tau))}}{\sqrt{a(\tau)}} d\tau
\]

with \( a(\tau) = \sqrt{\frac{M_{\varphi}(\tau)}{M_{\varphi}(0)}} \) and the integration is over the whole trajectory \( C \). This field is a solution of

\[
\Box u(x) = g(0) \int_{(C)} \delta^4(x - z(\tau)) \frac{e^{iS(z(\tau))}}{\sqrt{a(\tau)}} d\tau
\]

\[
= g(0)\delta^3(x - z(t)) \frac{e^{iS(z(t))}}{\sqrt{a(\tau)}} \sqrt{1 - v^2(t)}
\]

which shows that the coupling vanishes when the velocity of the particle \( v \) approaches the celerity of light.

We stress that using Eq. 29 in \( \int_{\Sigma(\tau)} J(\tau, \xi) d^3 \xi \) looks like a physical ansatz. To justify the self-consistency of the ansatz we now explicit Eq. 41 using Eq. 38:

\(^8\) We have also \( K^{(0)}_{\text{sym}}(x, x') = \int_{-\infty}^{+\infty} G^{(0)}_{\text{sym},\omega}(R)e^{-i\omega(t - t')} \frac{d\omega}{2\pi} \) with \( G^{(0)}_{\text{sym},\omega}(R) \) given by Eq. 34.
with \( \rho(\tau) = |(x - z(\tau)) \cdot \dot{z}(\tau)| \). The derivation of this formula is clearly reminiscent from the Lienard–Wiechert potentials in classical electrodynamics.\(^9\) To justify the ansatz used above the goal is to compute \( u(x) \) with Eq. 42 in the vicinity of the trajectory \( z(\tau) \) in order to recover the asymptotic near-field \( u \propto 1/r \) obtained in Sect. 2. The method has been already developed by Dirac [18] for the classical electron and requires lengthy calculations that will not be shown here for questions of space. We here summarize the main steps of the methods. As shown in Fig. 4b the field is evaluated in the hyperplane \( \Sigma(\tau) \) at a distance \( r = \sqrt{-\xi^2} \) that requires the retarded and advanced fields in Eq. 42 at (proper) times \( \tau_{\text{ref}} = \tau - \sigma_- \) and \( \tau_{\text{ref}} = \tau + \sigma_+ \) with \( \sigma_\pm \sim r \). Using the conditions \( (x - z(\tau \mp \sigma_\pm))^2 = 0 \) for the points on a light cone we deduce after lengthy calculations \( [18] \) the values \( \rho_\pm = \rho(\tau \mp \sigma_\pm) \), i.e.:

\[
\frac{1}{\rho_\pm} = \frac{1}{r} \left[ 1 + \frac{1}{2} \xi \ddot{\xi} + \frac{3}{8} (\dot{\xi}^2)^2 + \frac{5}{24} (r_0^2)^2 + O(r^3) \right],
\]

where the derivatives and \( \xi \) are calculated at time \( \tau \). The field is thus \( u(x) = \frac{u_++u_-}{2} \) with \( u_\pm = \frac{g(\tau \mp \sigma_\pm) e^{i S(\tau \mp \sigma_\pm)}}{8 \pi \rho_\pm} \) that leads to:

\[
u(x) = \frac{g(\tau) e^{i S(\tau)}}{4 \pi r} \left[ 1 + \frac{\xi \ddot{\xi}}{2} + \frac{r^2}{2} \left( i \ddot{\xi} - \left( \dddot{\xi} - i \ddot{\xi} \right) \right)^2 \right] + \frac{r^2}{2} \frac{\partial^2}{\partial \xi^2} \ln(g) + \frac{3}{8} (\dot{\xi}^2)^2 + \frac{5}{24} (r_0^2)^2 + O(r^3)
\]

for points \( x \in \Sigma(\tau) \) at a distance \( r \) from \( z(\tau) \).

From Eq. 44 we deduce that at the lowest order we have indeed \( u(x) \approx \frac{g(\tau) e^{i S(\tau)}}{4 \pi r} \) and we recover the asymptotic soliton near-field discussed in Sect. 2. This shows that our ansatz concerning \( \int_{\Sigma(\tau)} J(\tau, \xi) d^3 \xi \) is indeed justified. Moreover, in the case of an uniform motion with \( \ddot{\xi} = 0, \dot{\xi} = 0, \dot{\sigma} = -\omega_0, \ddot{\sigma} = 0, \dddot{\sigma} = 0 \) we have \( u(x) = \frac{e^{i \omega_0 r}}{4 \pi r} \left[ 1 - \frac{\alpha_0 r^2}{2} + O(r^3) \right] \approx \frac{e^{i \omega_0 r} \cos(\omega_0 r)}{4 \pi r} \) that is the field associated with the monopole discussed above.

Furthermore, from Eq. 44 and the definition \( u = e^{i \phi} \) we compute the ratio \( \frac{u(x)}{u^*(x)} = e^{i 2 \phi(x)} \) and obtain

\(^9\) We mention that F. Fer in 1957 developed a method for analyzing the motion of singularities in the context of the DSP [27, 28]. Moreover, his approach using only retarded Green’s functions missed the time-symmetry needed to recover the wave particle duality considered here.

\(^{10}\) More precisely we have \( \sigma_\pm = r \left( 1 + \frac{\xi_0^2}{2} + \frac{r_0^2 \xi_0^2}{6} - \frac{5}{24} (\xi_0^2)^2 + \frac{3}{8} (\xi_0^2)^2 \right) + O(r^4) \) with \( \xi_0^3 := \frac{d^3 \xi(t)}{dt^3} \).
\[
\frac{u(x)}{u_s(x)} = e^{i2\varphi(x)} = e^{i2S} \left[ 1 + ir^2 \left( \frac{\hat{S} + 2\frac{\xi g}{g}}{2} \right) + O(r^3) \right].
\] (45)

Comparing this with the Taylor expansion \( e^{i2\varphi(x)} = e^{i2\varphi(z)} [1 + i2\xi \varphi(z) + O(r^2)] \) we deduce immediately that the first-order term must vanish

\[
\xi \cdot \partial \varphi(z) = 0
\] (46)

and since we have also by definition \( \xi \xi = 0 \) we have thus \( \hat{z}(\tau) \) parallel (i.e., proportional) to \( \partial \varphi(z) \). In other words, since \( \hat{z}^2 = 1 \), we recover the guidance formula equation \( 10\hat{z}(\tau) = -\frac{\partial \varphi(z(\tau))}{\sqrt{\partial \varphi(z(\tau))^2}} \) (in absence of external field \( A(x) \)) discussed in Sect. 2 in the near-field. The fact that we can recover this result from Eq. 42 associated with the far-field again confirms the self-consistency of our approach.

In presence of an external field \( A(x) \neq 0 \) the previous propagator method can be generalized. For this we first replace the partial derivative \( \partial \) by the covariant derivative \( D = \partial + i eA(x) \) in Eq. 41 leading to

\[
D^2 u(x) = g(0) \int_{(C)} \delta^4(x - z(\tau)) \frac{e^{iS(z(\tau))}}{\sqrt{a(\tau)}} d\tau
\] (47)

with solution

\[
u(x) = g(0) \int_{(C)} K_{\text{sym}}(x, z(\tau)) \frac{e^{iS(z(\tau))}}{\sqrt{a(\tau)}} d\tau
\] (48)

and \( K_{\text{sym}}(x, x') \) the time-symmetric propagator solution of \( D^2 K_{\text{sym}}(x, x') = \delta^4(x - x') \).

A formal solution for the propagator reads

\[
K_{\text{sym}}(x, x') = K_{\text{sym}}^{(0)}(x, x') + K_{\text{sym}}^{(\text{ref})}(x, x')
\] (49)

where \( K_{\text{sym}}^{(\text{ref})}(x, x') = \int d^4y K_{\text{sym}}^{(0)}(x, y) \hat{\mathcal{O}}_y K_{\text{sym}}(y, x') \) (with the operator \( \hat{\mathcal{O}}_y := e^2 A(y)^2 - ie \partial_y A(y) - 2ieA(y)\partial_y \)) defines the reflected part of the propagator resulting from the interaction of the vacuum solution \( K_{\text{sym}}^{(0)} \) with the potential \( A \).

Therefore, the \( u \)-field splits as

\[
u(x) = u^{(0)}(x) + u^{(\text{ref})}(x),
\] (50)

where \( u^{(0)}(x) \) evaluated using \( K_{\text{sym}}^{(0)}(x, x') \) leads to Eq. 42 and \( u^{(\text{ref})}(x) \) is evaluated using \( K_{\text{sym}}^{(\text{ref})}(x, x') \). As we saw the singular field \( u^{(0)}(x) \) is diverging as \( u(x) \approx \frac{g(\tau)e^{S(z(\tau))}}{4\pi r} \) near \( x \sim z(\tau) \). The reflected part is in general a much more regular and weaker field near \( x \sim z(\tau) \).
Moreover, near the point \( x \sim z(\tau) \) we can assume \( A(x) \approx A(z) = \text{const.} \) (the soliton is supposed much smaller than the variation of \( A \)) and we check directly\(^{11} \) that the function \( K_{\text{sym}}(x, x') = K_{\text{sym}}^{(0)}(x, x')e^{-ieA(z)(x-x')} \) is a solution of \( (\partial + ieA(z))K_{\text{sym}}(x, x') = \delta^4(x-x') \). Inserting this result in Eq. 40 where \( K_{\text{sym}}(x, z(\tau)) \) replaces \( K_{\text{sym}}^{(0)}(x, z(\tau)) \) we obtain once more Eq. 42 with the substitution \( e^{iS(x, \tau_{\text{ret/adv}})} \rightarrow e^{iS(z, \tau_{\text{ret/adv}})}e^{- ie(x-x, \tau_{\text{ret/adv}})} \). In the vicinity of \( z(\tau) \) in the hyperplane \( \Sigma(\tau) \) we obtain at the lowest order:

\[
\frac{u(x)}{u^*(x)} = e^{i2S}[1 - i2e^{\xi}A(z(\tau)) + O(r^2)] = e^{i2\phi(z)}[1 + i2\xi \partial \phi(z) + O(r^2)],
\]

which implies

\[
\xi \cdot (\partial \phi(z) + eA(z)) = 0.
\]

Therefore, using once more \( \xi \dot{z} = 0 \), we recover the guidance formula equation 10, i.e., \( \dot{z}(\tau) = - \frac{\partial \phi(z(\tau)) + eA(z(\tau))}{\sqrt{\partial^2 \phi(z(\tau)) + eA(z(\tau))}^2} \) derived in Sect. 2.

This analysis shows that even if in general \( |u^{(0)}(x)| > |u^{(\text{ref})}(x)| \) in the vicinity of \( x \sim z \) the phase is however globally influenced by the presence of the external field \( A(x) \) imposing the guiding formula.

### 4 Discussion: Entanglement, Generalizations, Perspectives

In order to conclude this article we would like to emphasize some general properties of our model and extend its range of application to the many-body problem involving entangled solitons violating Bell’s inequalities. First, concerning the methodology we started in Sect. 2 with a near-field approach assuming a field with a spherical symmetry \( f = F_z(r) \) (see Eq. 24). This actually neglects the contribution of the reflected field. The consistency of our model becomes more obvious if we formally write the full \( u \)-field as \( u(x) = u^{(0)}(x) + u^{(\text{ref})}(x) \) with

\[
u^{(0/\text{ref})}(x) = - \frac{3r_0^2}{(g/4\pi)^4} \int K^{(0/\text{ref})}_{\text{sym}}(x, y)(f(y))\xi e^{i\phi(y)} d^4 y
\]

\[
sim - \frac{3r_0^2}{(g/4\pi)^4} \int K^{(0/\text{ref})}_{\text{sym}}(x, y)(f^{(0)}(y))\xi e^{i\phi(y)} d^4 y,
\]

\(^{11} \) A proof is obtained by using the Fourier transform \( K(x, x') = \int \frac{d^dk}{(2\pi)^d} e^{i(kx' - kx)}G_k \). Equation 49 reads thus \( G_k = G_k^{(0)} + G_k^{(0)}(e^2A^2 - 2eAk)G_k \), i.e., \( G_k = \frac{G_k^{(0)}}{1 - e^2A^2 - 2eAkG_k} \). Using \( G_k^{(0)} = -1/k^2 \) we deduce

\[
K^{(0)}_{\text{sym}}(x, x') = K^{(0)}_{\text{sym}}(x, x')e^{-ieA(z(x-x'))}.
\]
where we used the approximation $f \sim f^{(0)}$ for evaluating the source term in the second line. This is justified since we assume $f^{(\text{ref})} \ll f^{(0)}$ in the core region of the soliton where the integral contributes. This shows that $u(0/\text{ref})(x)$ are determined by the knowledge of the soliton $f^{(0)}(x)$ in the near-field as assumed in Sect. 2. Moreover we could in principle obtain deviations to this approximation. That could occur for regimes where the reflected field is not small, e.g., in very strong (relativistic) fields leading to further non-linearities.

To give an illustration of this issue consider a particle at rest in the middle of a spherical ideal cavity of radius $R$ with a perfectly reflecting wall associated with an infinite potential wall. The far-field stationary spherical solution of the equation

$$\Box u(t, x) = g\delta^3(x)e^{-ikt}$$

reads

$$u(t, r) = \frac{ge^{-ikt}}{4\pi r} \left[ \cos(\omega r) - \cotan(\omega R) \sin(\omega r) \right]$$

and obeys the boundary condition $u(R, t) = 0$. Near the origin (where $\omega r \ll 1$) the reflected field $f^{(\text{ref})} = \cotan(\omega R) \frac{g\sin(\omega r)}{4\pi r} \sim -\cotan(\omega R) \frac{2g}{4\pi}$ is in general much smaller than $f^{(0)} = \frac{g\cos(\omega r)}{4\pi} \sim \frac{g}{4\pi}$ unless the ‘cotan’ term is diverging which occurs if $\omega = \frac{m\pi}{R}$, with $m \in \mathbb{N}$. If that happens then the field in the cavity blows up and the approximations $f^{(\text{ref})} \ll f^{(0)}$ breaks down. Moreover, don’t forget that the particle is actually guided by the LKG equation with spherical eigen-solutions

$$\Psi_n(t, r) = \frac{\sin(n\pi R)}{n\pi R} e^{-i\omega_n t}$$

with $\omega_n = \sqrt{\left(\omega_0^2 + \frac{n^2\pi^2}{R^2}\right)} \simeq \omega_0 + \frac{n^2\pi^2}{2\omega_0 R^2}$ and $n \in \mathbb{N}$. The particle is at rest in agreement with the PWI and we have $\omega = \omega_n \simeq \omega_0$ in Eq. 54. We see that problem occurs only if $\omega_0 \simeq \frac{m\pi}{R}$. But this possibility can be rejected for at least two reasons. First, this would imply a strong conspiracy or fine-tuning where the Compton wavelength of the particle $\lambda_0 = 2\pi/\omega_0$ matches $2R/m$. This
corresponds to very small cavities of the size of the particle and we enter in the QED regime where particle/antiparticle pairs could be created. This regime is not considered in our analysis.

Moreover, the second more physical reason for rejecting this implausible resonance is that in general the potential barrier is not infinite and we can show that if the potential step $|eV|$ is smaller than $\omega_0$ the field given by Eq. 54 is modified: There is no strong reflectivity at the boundary $r = R$ and $\psi^0$ is mostly unaffected,\footnote{To prove this rather general statement a qualitative argument could go like this: Considering the LKG equation $D^2\psi = -\omega_0^2 \psi$ in a electrostatic potential $V(x)$ the first Born order scattering amplitude for an incident plane wave $\psi_0(x) = e^{ikR} \cdot \hat{\Phi}$ (with $k^2 = \omega^2 - \omega_0^2$) reads $\psi_\omega \approx -2\omega_0 \int d^3x' G_0^{(0)}(x,x')eV(x')\psi_0(x') \approx -2\omega_0 \frac{eV_\hat{q}}{4\pi} \hat{V}_q$ where we neglected the quadratic term $e^2V^2$.} \(i.e., \ u \approx \psi^0\). Again all this analysis is consistent if the potential is not too strong so that particle/antiparticle pairs are not generated (pairs are potentially generated if $|eV| \approx 2\omega_0$).

An important related problem concerns energy conservation and causality for a particle moving in an external field. Consider a particle following a curved trajectory like the one shown in Fig. 5 and emitting a retarded + advanced field as given in Eqs. 40 and 42, \(i.e.,\) neglecting the reflected part for simplicity. As visible on Fig. 5a the time-symmetric field implies that waves are constantly radiated into the future and into the past directions. De Broglie \[16\] analyzed the problem in terms of retarded waves (as shown in Fig. 5b) and concluded that the basic DSP leads to a paradox known as Perrin’s objection (for a discussion see \[21\]): following this objection a particle interacting with an external field, like a beam splitter, should radiate energy in empty branches not followed by the particle. After several interactions of that kind the particle (\(i.e.,\) the $u$-wave) should have lost all its energy in contradiction with experiments showing that particles are detected with a finite energy (the same issue remains in the double-slit experiment where the potential acting on the particle is mostly of quantum origin). This problem is reminiscent of the interpretation of empty waves in the PWI where their peculiar energetic properties are often seen as a difficulty.

Moreover we now see that the problem disappears in our theory: the energy losses associated with radiated waves (\(i.e.,\) Fig. 5b) are compensated by the energy gain associated with the advanced waves converging on the particle (\(i.e.,\) Fig. 5c). From the point of view of usual causality this looks conspiratorial or superdeterministic. A ‘de Broglie–Bohm demon’ having access to this flow of energy and perceiving the time going from past to future would see a converging flow of energy coming from the remote space arriving precisely at the good time in a coherent way on the particle. Furthermore, a retarded wave is also emitted by the particle and the sum off both waves gives the soliton field discussed in Sects. 2 and 3 imposing the guidance

\(\psi \approx \frac{2\omega_0 eV_\hat{q}}{4\pi} \hat{V}_q\). The same calculation done for a plane wave solution of the linearized equation for $u$ $D^2u = 0$ leads to the same expression with $\omega_0^2$ replacing $k^2$. Therefore the scattered field $u_s$ is smaller than $\psi$, by a coefficient $\frac{k^2}{v^2} \approx k^2/\omega_0^2 \approx v^2$ where $v$ is the particle velocity. In general $v^2 \ll 1$ and $u_s$ is negligible.
formula. As we showed in Sect. 2 it is possible to build a stationary soliton field in the local rest frame. When merging this near-field with the far-field of Sect. 3 the time-symmetric structure is thus required for consistency. Therefore, in our model the non-linearity of the wave equation and the existence of stable stationary solitons involves a time-symmetric causality. This in turn allows us to preserve energy conservation (more on this is derived in “Appendix 4”) and reproduce the predictions of the PWI, i.e., of quantum mechanics (in the regimes considered here).

The theory discussed in this work focused on the single particle/soliton problem and we showed that a time-symmetric u-field is required. This time-symmetric causality is clearly of great importance concerning the problem of entanglement between several particles. As it is well-known in de Broglie–Bohm mechanics [8] non-locality is offered as an explanation for justifying violations of Bell’s inequality. However, the present theory is definitely local and its quantitative predictions should therefore a priori differ from the standard PWI. It is here that the time-symmetry of the model comes to the rescue.

To see how it works, we consider the many-body generalization to an ensemble of $N$ indistinguishable (bosonic) particles of the LKG equation for a single particle developed in Sect. 2. We have the wavefunction $\Psi_N(x_1, \ldots, x_N)$ solution of the set of coupled equations $D_i^2 \Psi_N = -\omega_0^2 \Psi_N$ equivalent to the set of hydrodynamic equations\(^\text{13}\):

\[
(\partial_i S_N(X) + eA(x_i)) = \omega_0^2 + \sqrt{\frac{a_N(X)}{a_N(X)}} = M_{\Psi_N, i}^2(X),
\]

\[
\partial_i[a_N^2(X)(\partial_i S_N(X) + eA(x_i))] = \partial_i[a_N^2(X)M_{\Psi_N, i}(X)v_{\Psi_N, i}(X)] = 0,
\]

where we introduced the notation $X := [x_1, \ldots, x_N]$ and the velocity $v_{\Psi_N, i}(X) := -\frac{\partial S_N(X) + eA(x_i)}{M_{\Psi_N, i}(X)}$ [we also assume $M_{\Psi_N, i}^2(X) > 0$]. In the PWI we define the velocity of the $N$ particles through the guidance relations

\[
\frac{dz_i(\lambda)}{d\tau_i} = \frac{\dot{z}_i(\lambda)}{\sqrt{\dot{z}_i(\lambda)\dot{z}_i(\lambda)}} = v_{\Psi_N, i}(Z(\lambda)).
\]

with $Z(\lambda) := [z_1(\lambda), \ldots, z_N(\lambda)]$, $\dot{z}_i(\lambda) := \frac{dz_i(\lambda)}{d\lambda}$ and $d\tau_i = \sqrt{\dot{z}_i(\lambda)\dot{z}_i(\lambda)}d\lambda$ is a proper time element along the trajectory of the $i$th particle. We stress that in this description we require a parameter $\lambda$ to synchronize the $N$ particles. Usually this is done by involving a preferred foliation $\mathcal{F}$ of Minkowski’s space-time with space-like hyperplanes. Choosing $\mathcal{F}$ in general particularizes a set of entangled trajectories defining an ensemble $M(\mathcal{F})$. Moreover, since the choice of $\mathcal{F}$ is arbitrary the PWI admits an infinite number of possible paths-ensemble $M(\mathcal{F})$. Clearly, there is an apparent

\[\text{13} \text{ We have } D_i = \partial_i + eA(x_i), \partial_i := \frac{d}{dx_i} \text{ and the polar form } \Psi_N(x_1, \ldots, x_i, \ldots, x_N) = a_N(x_1, \ldots, x_i, \ldots, x_N) e^{iS_N(x_1, \ldots, x_i, \ldots, x_N)}.\]
tension with relativity since the ensemble of trajectories is not unique and depends on a foliation sometimes identified (but not here) with a kind of ‘Bohmian-Aether’ (for a discussion on this issue see [20]). In the context of our relativistic local theory for a \( u \)-field we don’t here give any ontological content to the particular foliation used \( F_0 \) to specify the particle trajectories. Instead, it is the (infinite) ensemble of all the \( M(\mathcal{F}) \) that exhausts the set of possibilities; and the choice \( F_0 \) used in our Universe (or in the part of our Universe accessible to us and entangled with us) is associated with a particular choice on initial conditions (perhaps related to cosmological constraints [20]). We stress that the dynamics equation 56 is not in general ‘statistically transparent’, i.e., that it cannot always reproduce Born’s rule and the statistical predictions of quantum mechanics (more on this will published in a subsequent article). Moreover, in the non-relativistic regime or in finite asymptotic regions of space–time, i.e., before or after scattering or interactions with an external field, we can justify Born’s rule and recover statistical transparency.

In the present local theory for the \( u \)-field the \( N \) solitons move in the same 4D space time and not in the abstract configuration space. The far-field for the \( N \) entangled soliton is written in analogy with Eq. 48 as:
\[ u(x) = \sum_i u_i(x) = \sum_i g(0) \int_{(C_i)} K_{\text{sym}}(x, z_i(\lambda)) \frac{e^{iS_N(Z(\lambda))}}{\sqrt{\alpha_i(\lambda)}} \, d\tau_i \]  

(57)

with \( \alpha_i(\lambda) = \sqrt{\frac{\mathcal{M}_{\Psi_i(\lambda)}}{\mathcal{M}_{\Psi_i(0)}}} \). This field obeys the following local equation:

\[ D^2 u(x) = \sum_i g(0) \int_{(C_i)} \delta^4(x - z_i(\lambda)) \frac{e^{iS_N(Z(\lambda))}}{\sqrt{\alpha_i(\lambda)}} \, d\tau_i \]  

(58)

that is defined in the 4D Minkowski spacetime not in the configuration space. This wave equation and dynamics is local but the \( N \) singularities moving along the \( N \) trajectories \( C_i \) are clearly entangled through the phase \( S_N(Z(\lambda)) \) and the masses \( \mathcal{M}_{\Psi_i}(Z(\lambda)) \) defined in the PWI of Eqs. 55a and 55b. The \( N \) trajectories are synchronized using the parameter \( \lambda \) and a specific foliation \( \mathcal{F} \). We stress that each soliton are actually interacting only with their own time symmetric \( u \)-field. The coupling between solitons is neglected at a first approximation valid in the far-field (subsequent work would require to use quantum electrodynamics for interacting particles).

Consider for example a pair of particles 1 and 2 with entangled trajectories \( z_1(\lambda), z_2(\lambda) \) defined with the PWI guidance formula equation 56. The \( u \)-field at point \( x \) is the sum of the contributions \( u_1(x) \) and \( u_2(x) \) generated by the particle 1 and 2 respectively. In the example of Fig. 6a (i.e., in absence of external \( A \)-field) \( u_1(x) \) splits into a retarded contribution emitted from the particle 1 when it was at \( A_1 = [t_{A_1}, \mathbf{x}_{A_1}] \) and an advanced contribution emitted from the same particle when it was at \( D_1 = [t_{D_1}, \mathbf{x}_{D_1}] \). Moreover, when the particle 1 is at \( A_1 \) \( (D_1) \) the second particle is at point \( A_2 \) \( (D_2) \) as determined by the preferred foliation \( \mathcal{F} \) (here we consider the set of hyperplanes \( t = \text{const.} \) for such a foliation). We can use Eq. 42 to evaluate \( u_1 \):

\[ u_1(x) = \frac{g(0)}{2} \left[ \frac{e^{iS_N(z_1(\lambda), z_2(\lambda))}}{4\pi \sqrt{\alpha_1(\lambda)}} \rho_1(\lambda) \right]_{\lambda = t_{A_1}} + \left[ \frac{e^{iS_N(z_1(\lambda), z_2(\lambda))}}{4\pi \sqrt{\alpha_1(\lambda)}} \rho_1(\lambda) \right]_{\lambda = t_{D_1}} \]  

(59)

with \( \rho_1(\lambda) = \left| (x - z_1(\lambda)) \cdot \frac{\dot{z}_1(\lambda)}{\sqrt{\dot{z}_1^2(\lambda)}} \right| \) and \( \alpha_1(\lambda) = \sqrt{\frac{\mathcal{M}_{\Psi_1(z_1(\lambda), z_2(\lambda))}}{\mathcal{M}_{\Psi_1(z_1(0), z_2(0))}}} \) for \( \lambda = t_{A_1} \) or \( \lambda = t_{D_1} \). Of course, the field \( u_2(x) \) generated by particle 2 is obtained with the same method and (as shown in Fig. 6a) it will involves points \( B_2 \) (of retarded emission by particle 2), \( C_2 \) (of advanced emission by particle 2) and the correlated positions of particle 1 at points \( B_1 \) and \( C_1 \). The total field is \( u_1(x) + u_2(x) \).

More generally, the \( u \)-field obtained with Eq. 57 shows a mixture of local and non-local properties. The local part is clearly the presence of the propagator \( K_{\text{sym}}(x, z_i(\lambda)) \) associated with the field equation 58. The non-local elements are associated with the correlated phase \( S_N(Z(\lambda)) \) and masses \( \mathcal{M}_{\Psi_i}(Z(\lambda)) \) reminiscent of the PWI using the preferred foliation \( \mathcal{F} \). In this approach the \( u \)-field propagates locally in the 4D spacetime but the singularities are non-locally correlated. Moreover, in our theory this is the local \( u \)-field that is more fundamental and not the non-local (contingent) \( \Psi \)-wave. There is an other way to watch this. Indeed, the theory
is time-symmetric and as illustrated in Fig. 6b each particle is fed by an advanced $u$-wave coming from the remote past whereas it emits a retarded wave propagating into the future. This ensures energy/momentum conservation for the $u$-field and also provides an explanation for the synchronisation and entanglement of the particles. A ‘de Broglie–Bohm demon’ watching the problem from past to future (i.e., as a Cauchy problem) will explain the ‘spooky’ correlations between the particles as a superdeterministic consequence of the field preparation in the remote past. Don’t forget: From Green’s theorem the total field reads $u(x) = \frac{u_{ret}(x) + u_{adv}(x)}{2}$ where $v(x) = \frac{u_{ret}(x) - u_{adv}(x)}{2}$ is a solution of the homogeneous wave equation that can be interpreted as a free field exciting the particles in a conspiratorial looking way. But here the theory is time-symmetric as required for the solitons stability: Therefore the conspiracy is actually explained.

Furthermore, don’t forget that Eq. 58 involving Dirac distributions, and the entangled trajectories $C_i$ is just an effective wave equation for the far-field of the solitons. Fundamentally the only wave equation is $D^2 u(x) = -N(u^*(x)u(x))u(x)$ i.e., Eq. 1 that is non-linear but completely local. The separation $u(x) \approx u_i(x)$ is just a very good approximation if the solitons are not too close from each other. If we approach the $i$th soliton we get $u(x) \approx u_i(x)$ which is (an approximate) solution of Eq. 1. Now, we can apply the phase-harmony condition developed in Sect. 2 for a single soliton and here we get:

$$\varphi_i(x) \approx S_N(Z(\lambda)) - eA(z_i(\lambda))\xi_i + B_i(z_i(\lambda))\frac{\xi_i^2}{2} + O(\xi_i^3), \quad (60)$$

where $\xi_i = x - z_i(\lambda)$ is defined in the local proper hyperplane $\Sigma_i(\lambda)$ normal to the velocity $\frac{dz_i(\lambda)}{d\tau_j}$ given by Eq. 56. $B_i(z_i(\lambda))$ is a collective coordinate defined as in Sect. 2. All the deductions and theorems discussed in Sect. 2 are still valid (don’t forget we use $d\tau_j = \sqrt{\xi_i(\lambda)\xi_i(\lambda)d\lambda}$). This allows us to build the near-field of each soliton looking like the monopolar field (see Eq. 24):

$$F_{i,\lambda}(r_i) = \frac{\sqrt{\alpha_i(\lambda)g(0)}}{4\pi} \frac{1}{\sqrt{\alpha_i(\lambda)^2r_i^2 + r_0(0)^2}} \quad (61)$$

with $r_i = \sqrt{(-\xi_i^2)}$ in $\Sigma_i(\lambda)$.

What is of course remarkable in Eq. 60, is the presence of the nonlocal phase $S_N(Z(\lambda))$ associated with $\Psi_N(Z(\lambda))$. Even if $u(x) \approx u_i(x) = f_i(x)e^{i\varphi_i(x)}$ is a local field solution of Eq. 1 nothing prohibits us to use the non-local phase $S_N(Z(\lambda))$ obtained from $\Psi_N(X)$ and evolving in the configuration space. No violation of the conservation laws for the $u$-field will appear by doing this choice which is therefore completely legitimate. In that sense there is a gentle agreement between nonlocal and local effects in our theory. Non-locality is only an effective property allowed by the nonlinear and time-symmetric $u$-field used in our approach. This clearly defines a new paradigm where a local theory is able to reproduce the nonlocal properties of the PWI.
The implications for discussing violations of Bell’s inequalities is obvious. Consider as in Fig. 6c a pair of entangled solitons forming an EPR system. Local operations made by Alice and Bob located in remote labs on each separated particles (e.g. by applying time varying electromagnetic fields) will be interpreted non-locally using the PWI. Following the PWI the guidance equations define instantaneous (spacelike) connections associated with a special foliation $\mathcal{F}$ (see [20]). However, the nonlocal perspective is just an effective one. In reality at a deeper level the $u$-field is completely local. But the time symmetric causality involving information coming from past and future conspirates exactly in order to reproduce the predictions of quantum mechanics and in particular the strong violations of Bell’s inequalities observed by Clauser, Aspect and Zeilinger. This we think is an interesting result. We mention that other authors such as Bohm or ’Hooft [7, 48] have suspected that an explanation based on superdeterminism could explain the violation of Bell’s inequalities. But these authors didn’t apparently consider time symmetry and the full block-world picture at face value. In our approach super determinism is just a way to watch the time symmetry of the problem when looking at the experiments from past to future. An advanced wave focusing on a particle is looking magical or conspiratorial only if we don’t understand the time symmetry of the underlying dynamics. The discussion concerning time-symmetry, superdeterminism and local causality in relation with Bell’s theorem will be further developed in a subsequent paper in preparation (for a recent complete review concerning quantum mechanical retrocausal/superdeterministic models see [51]).

This is also true in the recent important experimental/theoretical work done with hydrodynamical analogs by Bush, Vervoort and coworkers [38, 40, 50]. In these experiments there is a form of superdeterminism driven by stationnary Faraday waves moving along a liquid surface. Yet, the development was for the moment limited to time-independent configurations like in the first Aspect’s experiments. Time varying settings would require a stronger conspiracy that can be generated by a time symmetric causality. The problem is thus to generate the right amount of conspiracy in order to reproduce quantum predictions and not look too magical or too contrived. This is exactly the solution offered in the present work with non-linear waves involving time symmetry.
More generally, we found remarkable that in the new paradigm all the elements are strongly connected and related together to make the theory working perfectly. Nonlinearity and time-symmetry are required for the stability of our solitons and at the same time justify the existence of a guidance formula needed for deriving the PWI. The time-symmetry modifies the usual causality from past to future and allows for emerging and effective nonlocal features (i.e., in agreement with Bell’s theorem). In that sense nonlocality emerges from local physics in a consistent way.

An other remarkable feature of our soliton model is that it circumvents the conclusions obtained in [22] with Ehrenfest’s theorem for a strongly localized soliton. Here our soliton associated with a monopole $\sim 1/r$ at large distance is not sufficiently localized to impose a classical-like dynamics. The deformation of the soliton obtained in our model allows him to follow a de Broglie–Bohm dynamics. Again, this is strongly related to the other features of the model discussed above. One interesting aspect of this weak localization must be emphasized. Indeed, consider a soliton at rest in free space with the monopolar field given by Eq. 32 reducing to de Broglie solution Eq. 33 in the far-field. The full energy of the soliton is given by

$$E = \int d^3x [U(f^2) - f^2 N(f^2) + 2\omega_0 f^2 + \nabla(f \nabla f)]$$

that can approximately written as

$$E \approx E_s + \frac{8}{4\pi} \left[ \omega_0^2 R - \frac{\cos^2(\omega_0 R)}{R} \right],$$

where $E_s = \int d^3x [U(f^2) - f^2 N(f^2)]$ has been evaluated using the near-field, i.e., $E_s = \frac{g^2}{32\omega_0}$ (see “Appendix 1”). The integration in Eq. 62 has been pushed until a large radius $R$. In the limit $R \to +\infty$ $E$ contains a diverging contribution growing linearly with $R$ the other term goes to zero as $1/R$. A diverging energy seems at first pathological. However, note that if the particle has a finite life-time $T$ the radius $R$ cannot grow indefinitely. As illustrated in Fig. 7 if the particle appears at $A$ (time $t = 0$) and disappears at $B$ (time $t = T$) the $u$-field must have a diamond like shaped structure where advanced waves coming from the past direction interfere with retarded waves emitted to the future direction and create the stationary field of Eq. 33. The diamond structure of Fig. 7 is built between the light cones coming from past and future. Integrating the total energy at times $t < 0$ or $t > T$ gives the approximate value

$$E_s \approx \frac{8}{4\pi} \omega_0^2 \frac{T}{2}$$

associated with advanced or retarded waves. During the time interval $0 \leq t \leq T$ we obtain $E = E_s + \frac{8}{4\pi} \omega_0^2 \frac{T}{2}$ the difference is attributed to the local formation of the particle at $A$ requiring an additional energy $E_s$ coming from the environment at $A$. This energy is returning to the environment at $B$ when the particle disappears.

We can naturally speculate on the scale $R$ at which the far-field energy $E_f = \frac{g^2}{4\pi} \omega_0^2 R$ in Eq. 62 becomes comparable to $E_s$. The ratio $\frac{E_f}{E_s} = 32\pi \frac{\omega_0}{\lambda_0} R$ depends on the size of the particle $r_0$ and the Compton wavelength $\lambda_0 = 2\pi/\omega_0$. In absence of

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15 The error is small in the integration since $U(f^2) - f^2 N(f^2) \sim 1/r^6$ at large distances.

16 This description made in the regime $\omega_0 T \gg 1$ is of course an approximation that neglects the transient effects associated with the discontinuities at $A$ and $B$ contributing to the energy balance.
a more precise theory fixing the value of \( r_0 \) the ratio is let undetermined. Moreover, since \( r_0 \) is supposed to be very small an effect should only be observed at astrophysical or cosmological scales. For instance consider a proton with \( \lambda_0 \sim 10^{-15} \text{ m} \) and suppose \( R_G = 10^{16} \text{ m} \) a typical size for a galaxy. If \( r_0 \) is of the order of \( r_0 \sim 10^{-48} \text{ m} \) (i.e., much smaller than the Planck length \( r_P = 10^{-35} \text{ m} \)) we obtain \( \frac{E_\text{eff}}{E_\gamma} \sim 1 \). Interestingly, \( R_G \) is the scale at which dark-matter is usually involved in order to explain the rotation curve anomaly of stars in spiral galaxy. As it is known, the density of dark-matter needed to explain the constant value of the star velocity \( v_\infty \) at large distance of the galaxy core is typically growing as \( \rho_{DM}(R) \propto 1/R^2 \) and the mass as \( M_{DM}(R) \propto R \) for \( R \sim R_G \). This is typically what we obtain in our soliton model of quantum particles where the particle-core with energy \( E_\gamma \) is surrounded by a halo of energy (mass) growing as \( E_{\text{eff}} \propto R \). With a value of \( r_0 \sim 10^{-48} \text{ m} \) our model could thus potentially explains the anomaly in the rotation curves and interpret dark-matter as the far-field gravitational contribution of the particle masses to the dynamic of galaxies. Of course this is very speculative, and in the end it is not yet very clear what is the status of the particle energy \( E \) in our theory. We point out that the conserved norm \( Q = \int d^3x 2 \omega_0 |f|^2 \) associated with the current conservation equation 4b can be computed for the same example leading to Eq. 62. We get

\[
Q(R) \simeq \frac{g^2}{4\pi} \left( \omega_0 R + \frac{\sin(2\omega_0 R)}{2} \right). \tag{63}
\]

In the limit \( R \to +\infty \) we obtain \( \frac{E}{Q} \simeq \omega_0 [1 + \frac{\lambda_0}{32\pi r_0 R}] \to \omega_0 \), i.e., \( \hbar \omega_0 \) the quantum energy formula. So perhaps it is the ratio \( \frac{E}{Q} \) that should be identified with the physical energy of the soliton. This could be important when considering coupling with the gravitational field where a definition of mass must be included.

To conclude this work, it is important to mention that several important questions are left open and unanswered. For example, in our model we ignored the self-electromagnetic field generated by the soliton. This can be a good approximation near the particle core but from Eq. 63 we see that the electric charge \( q(R) \) contained in a sphere of radius \( R \) centered on the soliton is given by \( q(R) = eQ(R) \simeq \frac{g^2}{4\pi} \omega_0 R \) which diverges linearly. From Gauss’s theorem this implies a radial electric field \( \mathbf{E}(R) = \frac{e g^2 \omega_0 R}{r_0 (4\pi)^2} \hat{R} \) different from the standard Coulomb’s field. This problem could be perhaps solved by renormalizing the electric charge or by imposing the constraint \(|q(R_U)| \ll |e| \) for a large radius \( R_U \sim 10^{26} \text{ m} \) (size of the observable Universe). This implies \( \frac{g^2 R_U}{2 \lambda_0} \ll 1 \) and for a proton we need \( g^2 \ll 10^{-41} \). If this is true we could neglect the electromagnetic coupling between solitons.\(^{17}\) New ideas should be thus inserted in the model to develop electromagnetic interactions between solitons perhaps mediated with localized solitons associated with photons. We also mention that the NLKG equation used here has some pathological features associated with the

\(^{17}\) Of course the problem is absent if we limit the present model to neutral solitons with \( e = 0 \).
tachyonic sector alluded to briefly in Sect. 2. We restricted the analysis made in this work to the case of solitons with \( \mathcal{M}_\Psi(z)^2 > 0 \) but the tachyonic sector \( \mathcal{M}_\Psi(z)^2 < 0 \) was rejected as unphysical. Perhaps this could be avoided if the model is modified to incorporate the idea of ‘fusion’ developed by de Broglie where a spin zero particle is understood as a composite object made of two solitons with spins 1/2. In the very end, similar approaches can certainly be developed for particles with integer spins like photons or gravitons, or with Dirac spinors for generating solitonic fermions with spin 1/2 (this will be discussed in subsequent articles). The regime of interacting solitons should also be considered in future works and would require to have a better mathematical understanding of the PWI applied to quantum fields and in particular quantum electrodynamics.

5 Appendix 1

Using Eqs. 18 and 19 we define the integral

\[
G = - \int d^3 x N(f^2) f = \frac{3g}{2} \int_0^{+\infty} d\eta \frac{\sqrt{\eta}}{(\eta + 1)^{\frac{3}{2}}},
\]

where \( \eta = r^2/r_0^2 \). By definition this is related to the beta Euler function \( B\left(\frac{3}{2}, 1\right) \) by

\[
G = \frac{3g}{2} B\left(\frac{3}{2}, 1\right) = \frac{3g}{2} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} = \frac{3g}{2} \frac{\sqrt{\pi}}{\frac{3}{2} \sqrt{\pi}} = g.
\]

The static energy \( E_s \) associated with the soliton given by Eq. 19 is by definition \( E_s = \int d^3 x [U(f^2) - N(f^2)f^2] \). Inserting Eq. 18 leads after integration by part to

\[
E_s = \int d^3 x [U(f^2) - N(f^2)f^2].
\]

Using Eqs. 18 and 17 we get

\[
E_s = \frac{g^2}{4\pi r_0} \int_0^{+\infty} d\eta \frac{\sqrt{\eta}}{(\eta + 1)^{\frac{3}{2}}} = \frac{g^2}{4\pi a} B\left(\frac{3}{2}, \frac{3}{2}\right)
\]

which finally yields

\[
E_s = \frac{g^2}{4\pi r_0} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} = \frac{g^2}{4\pi r_0} \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{g^2}{32r_0}.
\]
6 Appendix 2

For a static soliton \( f(x) \in \mathbb{R} \) solution of the equation

\[
\nabla^2 f = N(f^2)f
\]

in the 3D space we can define the static energy \( E_s = \int d^3x(\nabla f)^2 + U(f^2)) \). \( E_s \) can be used to establish a variational principle \( \delta E_s = 0 \) for recovering the field equation \( \nabla^2 f = N(f^2)f \). In [17, 34] the authors consider the stretching or dilation transformation \( f(x) \rightarrow \tilde{f}(x) = \beta f(\alpha x) \) with \( \alpha > 0 \), \( \beta \in \mathbb{R} \). Here we instead consider the more general transformation

\[
f(x) \rightarrow \tilde{f}(x) = \beta f(\alpha x)
\]

with \( \beta \in \mathbb{R} \).

Under this transformation we check that the new function \( \tilde{f}(x) \) obeys Eq. 69 iff we have \( \alpha^2 N(\tilde{f}^2/\beta^2) = N(\tilde{f}^2) \). Here we consider specifically the general Lane–Emden nonlinearity \( N_p(x) = -\gamma x^p \) with \( \gamma, p \in \mathbb{R} \) (the case \( \gamma > 0, p = 2 \) is the one considered in this article). Within this family of nonlinearity functions we obtain the constraint

\[
\beta = \alpha^{1/p}
\]

which reduces to \( \beta = \sqrt{\alpha} \) used in the main text for \( p = 2 \). Moreover, by using Eq. 70 and \( N_p \) the static energy \( E_s \) for \( \tilde{f} \) becomes a function of \( \alpha \) reading

\[
E_s(\alpha) = \frac{\beta^2}{\alpha} I_k - \frac{\beta^{2(p+1)}}{\alpha^3} I_p
\]

\[= \alpha^{2/p-1}(I_k - I_p)\]

with \( I_p = \frac{\gamma}{p+1} \int d^3x f(x)^{2(p+1)} \) and \( I_k = \int d^3x (\nabla f(x))^2 \). Importantly, in passing from Eqs. 72a to 72b we used the constraint equation 71.

We now consider a first order variation \( \delta E_s = E_s(\alpha) - E_s(1) \) with \( \alpha = 1 + \epsilon \) and \( \epsilon \rightarrow 0 \). In [17, 34] the authors imposed \( \beta = 1 \) and if we use Eq. 72a the variational condition \( \delta E_s = 0 \) implies

\[
\frac{dE_s}{d\alpha} \bigg|_{\alpha=1} = -I_k + 3I_p = 0.
\]

Therefore, we deduce \( 3I_p = I_k \) which is non negative by definition of \( I_k \) and implies \( \gamma > 0 \). Similarly, we can define a second order variation \( \delta^2 E_s \) and we obtain

\[
\frac{d^2E_s}{d\alpha^2} \bigg|_{\alpha=1} = 2I_k - 12I_p = -6I_p < 0.
\]

This implies unstability of the soliton. However, a physical transformation for this soliton must rely on the constraint equation 71 in order to fulfill Eq. 72b. Therefore instead of Eq. 73 we must have:
\[
\left. \frac{dE_s}{da} \right|_{a=1} = (2/p - 1)(I_k - I_p) = 0. \tag{75}
\]

Moreover, we have \(I_k - I_p = E_s(1)\) and \(I_k = (p + 1)I_p\) (as it can be checked after integration by parts of \(I_k\) and neglecting a surface integral term) and we thus get \((2/p - 1)pI_p = 0\) which imposes the value \(p = 2\) (this result was obtained in [45]). Observe that if we insert the formula \(I_k = (p + 1)I_p\) in Eq. 73 we obtain \(-(p + 1)I_p + 3I_p = 0\) and therefore we again deduce the condition \(p = 2\) which is thus imposed by either Eq. 73 or 75. This result assumes that the field decays fast enough (i.e., at least as \(f \sim 1/r^m\) with \(m > 1/2\) for \(r\) large\(^18\)) in order to neglect the surface integral term in \(I_k\). Furthermore, with Eq. 72b we obtain

\[
\left. \frac{d^2E_s}{da^2} \right|_{a=1} = (2/p - 1)(2/p - 2)(I_k - I_p)
\]

(76)

replacing Eq. 74. Clearly, from Eq. 75 we deduce \(\left. \frac{dE_s}{da} \right|_{a=1} = 0\) which means that the soliton is not anymore unstable: it is metastable. This result evades the conclusions of the Hobart–Derrick theorem which was established without using the legitimate dilation transformation equation 71.

7 Appendix 3

We start with the local current conservation \(\partial_\mu (f^2(x)\mathcal{M}_u(x)v_\mu(x)) = 0\). Consider now the 4D volume sketched in Fig. 8 which is bound by (i) the two hyperplanes \(\Sigma(\tau)\) and \(\Sigma(\tau + \delta\tau)\) normal to respectively \(\dot{z}(\tau)\) and \(\dot{z}(\tau + \delta\tau)\) (with \(\delta\tau\) an infinitesimal delay time), and (ii) the cylindrical hypersurface \(S\) surrounding the particle trajectory. This hypersurface \(S\) is a 3D object which projects as a 2D closed surface surrounding the particle position \(z(\tau)\) in the hyperplane \(\Sigma(\tau)\).

The local height \(\delta\tau\) of the cylinder is given by [22] \(\delta\tau = \delta\tau(1 - \xi\dot{z}(\tau))\). A direct application of Gauss’s theorem applied to \(\partial_\mu (f^2(x)\mathcal{M}_u(x)v_\mu(x)) = 0\) in this 4D volume in space–time leads to

\[
\int_{\Sigma(\tau + \delta\tau)} f^2(x)\mathcal{M}_u(x)v_\mu(x) \cdot \dot{z}_\mu(\tau + \delta\tau)d^3\sigma
\]

\[
- \int_{\Sigma(\tau)} f^2(x)\mathcal{M}_u(x)v_\mu(x) \cdot \dot{z}_\mu(\tau)d^3\sigma
\]

\[
= \int_S d^2S_\mu v_\mu(x)(1 - \xi\dot{z}(\tau))\delta\tau\mathcal{M}_u(x)
\]

(77)

with \(d^2S_\mu := [0, d^2S]\) a 4-vector associated with the local surface \(d^2S\) of the 2D surface surrounding \(z(\tau)\). We have

\(^18\) Note that in order to have \(I_p < \infty\) we must have \(m > \frac{3}{2(p+1)}\) so that globally \(m > \max\left\{\frac{3}{2} \cdot \frac{3}{2(p+1)}\right\}\) [45].

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Moreover, we have near the soliton center \( v(x) \approx O(\xi) \). Therefore, writing \( d^2S = \xi^2 d^2\Omega \) (where \( d^2\Omega \) is an elementary solid angle) the surface integral in Eq. 78 is varying like \( O(\xi^3) \) which is neglected. Similarly, for the scalar products of the velocities we have \( v^\mu(x) \cdot z_{\mu}(\tau + \delta\tau) \approx 1 \) and \( v^\mu(x) \cdot z_{\mu}(\tau) \approx 1 \) and Eq. 78 reduces to:

\[
\frac{d}{d\tau} \int_{\Sigma(\tau)} f^2(x)M_u(x) d^3\sigma \approx 0 \tag{79}
\]

which leads to

\[
\int_{\Sigma(\tau)} f^2(x)M_u(x) d^3\sigma = C, \tag{80}
\]

where \( C \) is a constant (assuming the volume small).

Now, writing \( M_u(x) \approx M_{\psi}(\tau) \) and using Eq. 24 we have \( f(x) = F_r(r) = \sqrt{\alpha(\tau)F_0(\alpha(\tau)r)} \). After using the variable \( w = \alpha(\tau)\xi \) we have

\[
C = M_{\psi}(\tau) \frac{1}{\alpha^2(\tau)} \int F_0(w)d^3w \tag{81}
\]

which directly leads to Eq. 26.

8 Appendix 4

Local energy-momentum conservation for the field of Eq. 1 can be written in different equivalent ways. Here using the hydrodynamic formalism we introduce a energy–momentum tensor \( T^{\mu\nu}(x) = 2f^2(x)M^2_u(x)v^\mu(x)v^\nu(x) \) obeying the conservation law:

\[
\partial_\mu T^{\mu\nu} = 2f^2\frac{d}{d\tau_u}(M_u v^\nu) = 2f^2M_u[\partial^\nu(M_u) + eF^{\nu\mu}v_{\mu}] \tag{82}
\]
with \( \frac{d}{dx_u} \cdot = u^\mu \partial_\mu \). We used the current conservation to obtain the first equality. This leads to \( \frac{d}{dx_u} (\mathcal{M}_u\psi) = \partial^\nu (\mathcal{M}_u) + eF^{\nu\mu}v_{\mu u} \) that can be obtained directly from Eq. 4a and represents a quantum generalization of Newton’s force formula for the \( u \)-field.

A different way to write the energy–momentum conservation law is:

\[
\partial_\mu [T^{\mu\nu}_0 + \eta^{\mu\nu} V(f^2)] = 2ef^2\mathcal{M}_u F^{\nu\mu}v_{\nu u} \tag{83}
\]

with \( T^{\mu\nu}_0 = T^{\mu\nu} + 2\partial^\nu f \partial^\mu f - \eta^{\mu\nu} [(\partial f)^2 + f^2 \mathcal{M}_u^2] \). Finally, if we consider the full Maxwell’s equations we have \( \partial_\mu F^{\mu\nu} = 2ef^2\mathcal{M}_u v^\nu_u \) and therefore if we write \( T^{\mu\nu}_\text{em} \) the standard electromagnetic field energy–momentum tensor we must have \( \partial_\mu T^{\mu\nu}_\text{em} = -2ef^2\mathcal{M}_u F^{\nu\mu}v_{\nu u} \). In the end we get:

\[
\partial_\mu [T^{\mu\nu}_0 + \eta^{\mu\nu} V(f^2) + T^{\mu\nu}_\text{em}] = 0. \tag{84}
\]

In the “Appendix D” of [22] we applied Gauss’s theorem to Eq. 82 in a 4-D world tube surrounding the trajectory \( z(\tau) \) of a soliton with two ending (3D) spacelike hyper-surfaces \( \delta \Sigma_A \) and \( \delta \Sigma_B \), and obtained:

\[
\int_{\delta \Sigma_0} 2f^2(x)\mathcal{M}^2_u \psi^\nu_u - \int_{\delta \Sigma_A} 2f^2(x)\mathcal{M}^2_u \psi^\nu_u \tag{85}
\]

\[
= \int_{A}^{B} d\tau \int_{\Sigma_0(\tau)} d^3 \sigma_0 2f^2\mathcal{M}_u [\partial^\nu \mathcal{M}_u + eF^{\nu\mu}v_{\nu u}] \tag{86}
\]

In [22] we showed that for a strongly localized soliton like an undeformable Gaussian this relation leads to a form of Ehrenfest’s theorem where the quantum potential term cancels out because the \( u \)-field amplitude decays exponentially far away from \( z(\tau) \). In the present work with a weakly localized soliton with \( f \sim 1/r \) at large distance from \( z(\tau) \) we can not apply this result. Moreover, taking infinitely small cross-sections \( \delta \Sigma_\tau \) for the tube and using the fact that near the trajectory \( z(\tau) \) we have (see Footnote 2):

\[
\mathcal{M}_u(x) \simeq \mathcal{M}_\psi(z(\tau)) + \xi \cdot \nabla \mathcal{M}_\psi(z(\tau)) + O(\xi^2),
\]

\[
\partial_\mu \mathcal{M}_u(z(\tau)) = \partial_\mu \mathcal{M}_\psi(z(\tau)).
\]

This can be easily used to justify once more the dynamical law \( \frac{d}{d\tau_u} (\mathcal{M}_\psi(z)\psi^\nu) = \partial^\nu (\mathcal{M}_\psi(z)) + eF^{\nu\mu}v_{\mu u} \), associated with the PWI.

**Data Availability** No data associated in the manuscript.

**Declarations**

**Conflict of interest** Author declares no competing interest for this work.

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