Maximal 4-degenerated subgraph of a planar graph*

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Abstract

A graph $G$ is $k$-degenerated if it can be deleted by subsequent removals of vertices of degree $k$ or less. We prove that every planar graph of order $n$ has a 4-degenerated induced subgraph of order at least $8/9 \cdot n$. We also consider a local variation of this problem and show that in every planar graph with at least 7 vertices, deleting a suitable vertex allows us to subsequently remove at least 6 more vertices of degree four or less.

1 Degeneracy and choosability

A graph $G$ is $k$-degenerated if every subgraph of $G$ has a vertex of degree $k$ or less. Equivalently, a graph is $k$-degenerated if we can delete the whole graph by subsequently removing vertices of degree at most $k$. The reverse of this sequence of removed vertices can be used to colour (or even list-colour) $G$ with $k + 1$ colours in a greedy fashion. Graph degeneracy is therefore a natural bound on both chromatic number and list chromatic number. In fact, for some problems graph degeneracy provides the best known bounds on the choice number \[3\].

Every subgraph of a planar graph has a vertex of degree at most 5 because it is also planar, therefore, every planar graph is 5-degenerated. If $k < 5$, we can still choose at least some subset of the vertices of $G$, and if these vertices induce a $k$-degenerated graph, then they can be greedily coloured by $k + 1$ colours. Thus an interesting question is how large $k$-degenerated subgraph can be guaranteed in a planar graph $G$. We discuss this question for particular values of $k$ in the following paragraphs.

Let $G$ be a planar graph. To find a maximal induced 0-degenerated subgraph, we want to find a maximal independent set. By employing 4-colour theorem, we can find an independent set of order at least $1/4 \cdot |V(G)|$. This bound is tight since $K_4$ has no two independent vertices.

To find a maximal induced 1-degenerated subgraph, we need a large induced forest in $G$. Borodin \[4\] proved that every planar graph $G$ is acyclically 5-colourable, that is, we can partition the vertices of $G$ into five classes such that the union of any two classes is acyclic (hence, a forest). By taking two largest classes we can guarantee an induced forest of order at least $2/5 \cdot |V(G)|$ in $G$. The Albertson-Berman conjecture \[1\] asserts that every planar graph has an induced forest with at least half of the vertices. This conjecture is tight as $K_4$ has no induced forest.

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of order greater than 2. Borodin and Glebov proved that the Albertson-Berman conjecture is true for planar graphs of girth at least 5.

Let \( G \) be a plane graph. The vertices that belong to the unbounded face induce an outerplanar graph. Let us delete those vertices. Then again, the vertices that belong to the unbounded face induce an outerplanar graph and we can repeat the process. In this way we can create a sequence of outerplanar layers such that only vertices in neighbouring layers can be adjacent. If we take every second layer, the vertices from these layers induce a disjoint union of outerplanar graphs. Since outerplanar graphs are 2-degenerated (every outerplanar graph contains a vertex of degree at most 2, see [6]), we can partition the vertices of \( G \) into two sets such that each set induces a 2-degenerated graph. The larger of this two sets has clearly at least \( \frac{1}{2} \cdot V(G) \) vertices. On the other hand, in the octahedron we can take at most 4 vertices into an induced 2-degenerated subgraph, so the maximum 2-degenerated subgraph has order \( \frac{2}{3} \cdot V(G) \) vertices.

Degeneracy 3 was studied by Oum and Zhu who were interested in the order of maximal 4-choosable induced subgraph of a planar graph. They showed that every planar graph has an induced 3-degenerated subgraph of order at least \( \frac{5}{7} \cdot V(G) \). For the upper bound, the best we are aware of is that both octahedron and icosahedron contain no induced 3-degenerated subgraph of order at least \( \frac{5}{6} \cdot V(G) \).

To the authors’ knowledge, there are no published results on maximum 4-degenerated induced subgraphs of planar graphs. A likely reason is that such bounds are not interesting for list-colouring applications: Thomassen proved that every planar graph is 5-choosable.

The problem of maximal degenerated subgraphs was also studied for general graphs by Alon, Kahn and Seymour. They determined exactly how large \( k \)-degenerated induced subgraph one can guarantee depending only on the degree sequence of \( G \).

## 2 Degeneracy 4

The rest of this paper focuses on degeneracy 4. We define two operations for vertex removal: deletion and collection. To delete a vertex \( v \), we remove \( v \) and its incident edges from the graph. To collect a vertex \( v \) is the same as to delete \( v \), but to be able to collect \( v \) we require \( v \) to be of degree at most 4. Though the definitions are very similar, for our purpose there is a clear difference: we want to collect as many vertices as possible and to delete as few as possible. The collected vertices induce a 4-degenerated subgraph whose order we are trying to maximize. We say we can collect a set \( S \) of vertices if there exists a sequence in which the vertices of \( S \) can be collected. Vertices that are deleted or collected are collectively called removed.

Note that a graph \( G \) is 4-degenerated if and only if we can collect all its vertices.

The main results of this paper are the following two theorems.

**Theorem 1.** In each planar graph \( G \) we can delete at most \( \frac{1}{9} \) of its vertices in such a way that we can collect all the remaining ones.

**Theorem 2.** In each planar graph with at least 7 vertices we can delete a vertex in such a way that we can subsequently collect at least 6 vertices.

Note that the statement of Theorem 1 can be equivalently reformulated as “Every planar graph \( G \) has a 4-degenerated induced subgraph of order at least \( \frac{8}{9} \cdot V(G) \).”

These results are probably not the best possible. The worst known example known to us is the icosahedron from which we need to delete one vertex out of twelve to be able to collect the remaining eleven. We believe that this is the worst case possible.
**Conjecture 2.1.** In each planar graph $G$ we can delete at most $1/12$ of its vertices in such a way that we can collect all the remaining ones.

**Conjecture 2.2.** In each planar graph with at least 12 vertices we can delete a vertex in such a way that we can subsequently collect at least 11 vertices.

### 3 Induction invariants

To prove Theorem 2, we only need to find vertex, whose deleting allows us to collect 6 vertices in the neighbourhood. To prove Theorem 1 in this straightforward manner we would need to collect 8 vertices per one deleted vertex. We cannot guarantee this immediately in all cases. However, even if we collect only 6 vertices we do also something other that helps us. We create a large face, which means that we decrease the average degree in the graph. Let $G$ be a planar graph. Let

$$
\Phi(G) = \sum_{v \in V(G)} (\deg(v) - 5). \tag{1}
$$

This is the actual theorem we are going to prove. The function $tc(G)$ stands for the number of tree components of $G$.

**Theorem 3.** Suppose that $G$ is a planar graph. Then

1. Either we can collect all vertices from $G$ or we can delete a vertex and collect 6 more vertices.

2. There exists a positive real number $\lambda \leq 1/36$ such that the following statement holds: There is a set $S \subseteq V(G)$ with at most $\Gamma(G) = v/12 + \lambda \Phi(G) + 2\lambda tc(G)$ vertices such that if we delete $S$ we can subsequently collect all the vertices from $G$.

We prove Theorem 3 for $\lambda = 1/36$. However, certain other constraints are imposed on $\lambda$ throughout the proof. One of these constraints is an upper bound $\lambda \leq 1/36$. All the other constraints are lower bounds. Indeed, lowering the value of $\lambda$ improves the result, therefore the lower bounds hinder an improvement of Theorem 3. This approach demonstrates which configurations have to be treated more finely in order to strengthen Theorem 3.

**Lemma 4.** Any smallest counterexample to Theorem 3 is connected.

**Proof.** Let $G$ be a smallest counterexample to Theorem 3 which is not connected. Let $G_1$ be a component of $G$ and let $G_2 = G - G_1$.

Suppose that statement 3 of Theorem 3 does not hold for $G$. Then $G$ cannot be collected. Therefore either $G_1$ or $G_2$ cannot be collected. Since $G$ is the smallest counterexample to Theorem 3, the graph $G_1$ or $G_2$ contains a vertex whose deleting allows us to collect 6 vertices, a contradiction.

Suppose that statement 3 of Theorem 3 does not hold for $G$ but the statement holds for $G_1$ and $G_2$. We obtain sets $S_1$ and $S_2$ satisfying the conditions of statement 3 of Theorem 3. Then the choice of $S = S_1 \cup S_2$ satisfies statement 3 of Theorem 3, a contradiction.

**Lemma 5.** Any smallest counterexample to Theorem 3 does not contain a vertex that can be collected.
Proof. Collecting a vertex \( v \) of degree \( d \), \( d \leq 4 \), impacts the value of \( \Gamma \) as follows. First, it decreases \( \Gamma \) by \( 1/12 \) because the number of vertices decreases by one. Second, it increases \( \Gamma \) by \( \lambda(5-d) \) because we remove \( v \) from the sum defining \( \Phi \) (equation (1)). Third, it decreases \( \Gamma \) by \( \lambda d \) because neighbours of \( v \) are of smaller degree after deleting \( v \). And finally, it increases \( \Gamma \) by at most \( 2\lambda(d-1) \) because the number of tree components of \( G \) can increase by at most \( d-1 \). Since \( \lambda \leq 1/36 \) collecting a vertex does not decrease \( \Gamma \).

Suppose that \( G \) is a smallest counterexample (with respect to order) to Theorem 3. We have proved that \( G \) is connected and has minimum degree 5. We fix this graph \( G \) through the rest of the paper to show that this graph cannot exist by deriving a contradiction via discharging.

### 4 Discharging procedure

In this section, we assume that \( G \) is embedded into the plane. The actual embedding of \( G \) which we will use is described in the next section; the idea of the embedding is to avoid small cut-sets and consequent complications of case analysis.

For a positive integer \( k \), let \( V_k \) be the set of vertices of \( G \) of degree \( k \), \( V_k^+ \) the set of the vertices of \( G \) of degree at least \( k \) and \( F_k \) the set of the faces of length at least \( k \). The sets of all vertices, edges and faces of \( G \) are denoted by \( V \), \( E \), and \( F \), respectively.

We will also use the symbol \( k \) to denote a vertex of degree \( k \) in figures. (See Figure 1 for an example.) If the number \( k \) in the figure has an upper index “+”, then the corresponding vertex has degree at least \( k \). If no number is given, then the vertex has degree 5 or 6.

Each vertex of \( V_6^+ \) is assigned a certain type according to Table 1. If \( w \) is of degree \( d \), it is contained on at least \( n5 \) non-triangular faces, and has at most \( n5 \) neighbours from \( V_5 \), then \( w \) can have type \( c \). The type of \( w \) is the type that occurs first in the table among all the types \( w \) can have. The symbol \( n5 = 3c \) means that \( w \) has exactly three neighbours of degree five and all of them are consecutive in the embedding of \( G \).

Let \( vw \) be an edge such that \( v \in V_5 \) and \( w \in V_6^+ \). For every such edge we define the maximal charge \( mc(v,w) \) that \( v \) can send to \( w \). This maximal charge is given in the last column of Table 1. If \( w \) is of type \( 9c \) or \( 8c \), then \( mc(v,w) \) depends on the position of \( v \) with respect to \( w \): the value of \( mc(v,w) \) is 1 when \( v \) is the central one of the three consecutive neighbours of \( w \) of degree 5 and 9/10 otherwise.

First, we assign certain initial charges to the vertices and faces of \( G \). Each vertex \( v \) of degree \( d \) receives charge \( 6-d \) and each face of length \( \ell \) receives charge \( 2(3-\ell) \). In the following discharging procedure, we redistribute the charges between vertices and faces in a certain way such that no charge is created or lost. The charge of a vertex or a face \( x \) is denoted by \( ch(x) \). For a set \( S \subset V \cup F \), the expression \( ch(S) \) denotes the total charge of the set \( S \), that is, the sum of charges of the elements of \( S \). To distinguish the states between and after discharging we will use \( ch_0 \) instead of \( ch \) for the initial charges of the elements of \( G \).

According to Euler’s theorem, the initial total charge \( ch_0(V \cup F) \) is equal to

\[
\sum_{v \in V} (6 - \deg(v)) + \sum_{f \in F} 2(3 - \ell) = 6|V| - 2|E| + 6|F| - 4|E| = 12.
\]

Our aim is to move charge from vertices to faces. Note that only the vertices of degree
| Type | Degree | Min. number of non-tr. faces | Max. number of $V_5$ neigh. | Maximal charge |
|------|--------|-----------------------------|----------------------------|----------------|
| $(t)$ | $(d)$  | $(n_{11})$                  | $(n_5)$                    | (mc)           |
| 10a  | 10+    | 0                           | 3                         | 1              |
| 10b  | 10+    | 0                           | $\infty$                  | $1/2$          |
| 9a   | 9      | 1                           | 3                         | 1              |
| 9b   | 9      | 0                           | 2                         | 1              |
| 9c   | 9      | 0                           | 3                         | 9/10, 1, 9/10  |
| 9d   | 9      | 0                           | 9                         | $1/2$          |
| 8a   | 8      | 0                           | 1                         | 1              |
| 8b   | 8      | 1                           | 2                         | 1              |
| 8c   | 8      | 2                           | 3                         | 9/10, 1, 9/10  |
| 8d   | 8      | 0                           | 2                         | 9/10           |
| 8e   | 8      | 0                           | 8                         | $1/2$          |
| 7a   | 7      | 0                           | 1                         | 4/5            |
| 7b   | 7      | 1                           | 2                         | 13/20          |
| 7c   | 7      | 0                           | 2                         | 2/5            |
| 7d   | 7      | 0                           | 7                         | 1/3            |
| 6a   | 6      | 1                           | 1                         | 2/5            |
| 6b   | 6      | 0                           | 6                         | 0              |

Table 1: Maximal charges that can be send to a vertex.

5 have positive charge in a smallest counterexample. The discharging procedure consists of the following three steps.

**Step 1: Discharging to faces.** For each vertex $v$ and for every face $f \in F_{4+}$ that contains $v$ do the following:

1. If $v$ is of degree 6, then send $2/5$ from $v$ to $f$.
2. If $v$ has not degree 6, but both its neighbours on $f$ have degree 6, then send $3/5$ from $v$ to $f$.
3. If $v$ has not degree 6 and one of its neighbours has not degree 6, then send $1/2$ from $v$ to $f$.

**Step 2: Distance discharging.** In every subgraph of $G$ isomorphic to the configuration in Figure 1 send $1/5$ from vertex $v$ to vertex $w$ (vertices are denoted as in Figure 1; the depicted vertices are pairwise distinct).

**Step 3: Final discharging of the vertices of degree five.** For each vertex

![Figure 1: Distance discharging.](image)
$v \in V_5$ carry out the following procedure. Order the neighbours $w \in V_{6+}$ of $v$ according to the value of $mc(v, w)$ starting with the largest value; let $w_1, w_2, \ldots$ be the resulting ordering. If the value of $mc(v, w)$ is the same for two neighbours of $v$, then we order them arbitrarily. For $i = 1, 2, \ldots$, send $\max\{mc(v, w_i), ch_a(v)\}$ from $v$ to $w_i$, where $ch_a(v)$ denotes the current charge of $v$. If $mc(v, w_i) \geq ch_a(v)$, then we say that $v$ discharges completely into $w_i$.

The discharging procedure we have carried out has not changed the total charge.

**Lemma 6.** After the discharging procedure, no face has positive charge. Consequently, $ch(V) \geq 12$.

**Proof.** The only step where a face can obtain positive charge is Step 1. Let $f$ be a non-triangular face of $G$. Let $l$ be the length of $f$. Note that if a vertex sends $3/5$ to $f$, then both neighbours of $v$ on $f$ send $2/5$. Therefore the number of vertices that send $3/5$ to $f$ is less than or equal to the number of vertices that send $2/5$ to $f$. This shows that the face $f$ receives charge at most $l/2$. This is not enough to make the charge of $f$ positive.

This means that after the discharging procedure, the sum of charges on vertices is positive.

## 5 Technical lemma

Recall that $G$ is the smallest counterexample to Theorem 3, thus it is connected and has minimum degree 5. In this section we construct an embedding of $G$ that will we used in discharging. Moreover, we show that if we apply the discharging procedure on this embedding of $G$, then there will be a vertex with positive charge outside certain types of edge-cuts. This allows us to significantly reduce the number of cases examined later.

We need to avoid the following structures in the graph $G$. A **triangle-cut** $C$ is a subgraph of $G$ isomorphic to $C_3$ such that $V(C)$ is a cut-set of $G$. A **chordless quadrilateral-cut** $C$ is a subgraph of $G$ isomorphic to $C_4$ such that $V(C)$ is a cut-set of $G$ and $C_4$ has no chord in $G$. A **quadrilateral-cut with chord** $C$ is a subgraph of $G$ isomorphic to $C_4$ such that $V(C)$ is a cut-set of $G$ and $C_4$ has a chord in $G$. A **bad cut-set** is a triangle-cut, a chordless quadrilateral-cut, or a quadrilateral-cut with chord.

A **good subgraph** of $G$ is either $G$ itself if $G$ contains no bad cut-set or a proper subgraph $H$ which satisfies all the following conditions:

1. $H$ contains a cut-set $C$ of $G$ is a triangle-cut or a chordless quadrilateral-cut.
2. $H$ can be embedded in the plane in such a way that $C$ is on the outer face of $H$.
3. This embedding can be extended to the embedding of $G$ in such a way that all vertices of $G - V(H)$ are in the outer face of $H$. If this condition is satisfied, then $C$ cuts $H$ from $G$.
4. $H - C$ contains no vertex $v$ that is in a bad cut-set of $G$ except for the case when $C$ is triangle cut-set and two vertices of $C$ together with $v$ form a triangular face in our embedding of $H$. (Note that in the latter case, $C \cup \{v\}$ is a quadrilateral-cut with a chord.)

**Lemma 7.** Graph $G$ has a good subgraph $H$. 


Proof. Assume that $G$ has a triangle-cut $C_A^1$. We embed $C_A^1$ into the plane. This embedding has two faces, the outer face and the inner face. We embed the remaining vertices of $G$ so that both the inner and outer face contain at least one component of $G - C_A^1$. Let $X_A^1$ be the subgraph of $G$ induced by the vertices embedded into the inner face of $C_A^1$. Let $H_A^1$ be the subgraph of $G$ induced by $V(C_A^1) \cup V(X_A^1)$. This construction guarantees that $C_A^1$ cuts $H_A^1$ from $G$.

Suppose now that $X_A^1$ contains a vertex that is in a triangle-cut $C_A^2$ of $G$. Note that the vertices of $C_A^1$ all belong to the same component $K$ of $H_A^1 - V(C_A^2)$. We embed $C_A^2$ into the plane in such a way that $K$ lies in the outer face. We embed the remaining vertices of $G$ in such a way that some vertex will be embedded into the inner face of $C_A^2$. Let $X_A^2$ be the subgraph of $G$ induced by the vertices of $G$ embedded inside $C_A^2$ and let $H_A^2$ be the subgraph of $G$ induced by $V(C_A^2) \cup V(X_A^2)$. This construction guarantees that $H_A^2$ is a subgraph of $H_A^1$ and that $C_A^2$ cuts $H_A^2$ from the rest of $G$.

We continue in this way until $H_A^i - V(C_A^i)$ contains no vertices lying in triangle-cuts of $G$. (The number of vertices of $H_A^i$ decreases in each step, therefore the process is finite.) Let $H_A = H_A^i$ and $C_A = C_A^i$. If $G$ has no triangle-cut, we take $H_A = G$ and $V(C_A) = \emptyset$. Note that $H_A - C_A$ contains no vertex in a triangle cut; the same holds for any graph $H_B^i - C_B^i$ constructed later.

Suppose that $H_A - V(C_A)$ contains a vertex that is in a chordless quadrilateral-cut $C_B^1$ of $G$. We embed $C_B^1$ into the plane in such a way that the component of $G - C_B^1$ containing $C_A$ goes into the outer face of $C_B^1$ (if $V(C_A) = \emptyset$, then we choose an arbitrary component). We embed the remaining components arbitrarily; however, at least one component must be embedded into the inner face of $C_B^1$. We denote $X_B^1$ the subgraph of $G$ induced by the vertices embedded into the inner face of $C_B^1$. Let $H_B^1$ be the subgraph of $G$ induced by $V(C_B^1) \cup V(X_B^1)$. Note that $H_B^1$ is a subgraph of $H_A$.

Suppose now that $H_B^i - V(C_B^i)$ contains a vertex $v$ that is in a chordless quadrilateral-cut $C_B^2$ of $G$. We show that we can choose a chordless quadrilateral-cut $C_B^2$ of $G$ so that only one component of $G - C_B^2$ contains vertices from $C_B^1$. Assume that $C_B^2$ does not have the desired property and let $C_B^1 = v_1v_2v_3v_4v_1$. Only non-adjacent vertices can belong to different components of $G - C_B^2$, hence we may assume without a loss of generality that $v_1$ and $v_3$ are in different components of $G - C_B^2$. Consequently, vertices $v_2$ and $v_4$ must be in $C_B^2$. Vertex $w_1$ has degree at least 5 therefore it has a neighbour $x$. Then $x$ is either inside $w_1v_2v_1v_4$ or $w_1v_2v_3v_4$. Without loss of generality let it be $w_1v_2v_1v_4$. Let $C_B^2 = w_1v_2w_3w_4w_1$. The circuit $C_B^2$ of $G$ without chord: $v_2$ is inside $C_B^2$, vertex $v_3$ is outside $C_B^2$, $v_2v_4$ is not a chord since $C_B^2$ is without chord, and finally $w_1w_4$ is not chord because if it was, then either $v_1w_1v_2$ or $v_1v_1w_4$ is a triangle-cut of $G$ (triangle-cuts were removed from $H_B^i - V(C_B^1)$ before).

We embed $C_B^2$ into plane. Only one component of $G - C_B^2$ contains vertices from $C_B^1$. We embed this component into the outer face of $C_B^2$. We embed the remaining vertices so that some vertex is embedded into the inner face of $C_B^2$. Let $H_B^3$ be a subgraph of $G$ induced by $V(C_B^2)$ and the vertices embedded inside of $C_B^2$. This guarantees that $H_B^3$ is a subgraph of $H_B^i$ and that $C_B^2$ cuts $H_B^3$ from $G$.

We can continue in this way until step $i$ when $H_B^i - V(C_B^i)$ does not contain a vertex that is in a chordless quadrilateral-cut of $G$. Note that the number of vertices of $H_B^i$ decreases in each step, therefore the process is finite. Let $H_B = H_B^i$ and $C_B = C_B^i$ If $G$ has no chordless quadrilateral-cut, then let $H_B = H_A$ and $C_B = C_A$.

Suppose now that a vertex $v$ from $H_B - V(C_B)$ is in a quadrilateral-cut with chord of $G$. This implies the existence of a triangle-cut which may not exist except for the case when $C_B$ is a triangle-cut and $v$ is neighbour of two vertices from $C_B$. This is, however, allowed. Therefore $H = H_B$ with choice $C = C_B$ satisfies the
conditions of the lemma.

For the rest of the paper, we will consider that $G$ is embedded according to the proof of this lemma. The vertices from $V(C)$ will be called cut vertices. The vertices from $H - V(C)$ that are in a quadrilateral-cut with chord of $G$ will be called extraordinary. The other vertices from $H - V(C)$ will be called ordinary.

**Lemma 8.** If no extraordinary vertex $v$ in $H$ has the final charge $ch(v) \geq 2$, then the sum of the charges of ordinary vertices in $H$ is positive after the discharging procedure.

**Proof.** If $V(C) = \emptyset$, then $H = G$ and all vertices are ordinary; the statement of the lemma is immediately implied by Lemma 6. We split the rest of the proof into two cases according to the type of the bad cut-set $C$—either $C$ is a triangle-cut or $C$ is a quadrilateral-cut without chord. In our fixed embedding of $G$, the inner vertices and the inner faces of $C$ will be collectively called the kernel of $H$ and will be denoted by $K$. The “outer face” of $H$ (bounded by $C$) is denoted by $f$.

**Case 1:** $C$ is a quadrilateral-cut without a chord. There are no extraordinary vertices in this case. Let $C = v_1v_2v_3v_4v_1$. Let us compute the initial charge on the kernel of $H$. Note that this charge is the same as if we have assigned initial charges in $H$ instead of $G$; hence, we do it now. The total charge in $H$ is at least $12$. The outer face has charge $-2$. The initial charge of the vertices of $C$ depends on their degree within $H$. The initial charge of the kernel is thus

$$ch_0(K) = 12 + 2 - \sum_{i=1}^{4} (6 - \deg_H(v_i)).$$

How can the charge in the kernel change during the discharging procedure applied to $G$? Since the initial charge of each vertex is at most 1, each neighbour of $C$ can discharge at most 1 outside of $K$ (that is into $C$). Other vertices can send charge only via distance discharging, but each single distance discharging implies that two neighbours have degree 6 which prevents these two vertices to discharge any charge outside $K$.

The straightforward computation of the resulting charge in the kernel is the following:

$$ch(K) \geq 12 + 2 - \sum_{i=1}^{4} (6 - \deg_H(v_i)) - \sum_{i=1}^{4} (\deg_H(v_i) - 2) = -2$$

(the terms in the last summation reflect that the number of neighbours of $v_i$ belonging to the kernel is at most $\deg_H(v_i) - 2$ since it has two neighbours in $C$). This is not enough to prove our lemma, but we can make certain small improvements.

Let us focus on the edge $v_1v_2$. This edge is on the boundary of some face $f_1$ from $K$. If $f_1$ is triangular, then a common neighbour $w$ of $v_1$ and $v_2$ exists in $K$. The vertex $w$ can discharge at most 1 outside of $K$ and we counted it twice in equation (2). If $f_1$ is not triangular, then the neighbours of $v_1$ and $v_2$ on $f_1$ both send at least $2/5$ into $f_1$ which belongs to $K$. In either case, the charge of $K$ is increased by at least $4/5$ compared to the rough calculation in (2).
Since the same reasoning applies also for the edges \( v_2v_3, v_3v_4, \) and \( v_4v_1, \) we can increase the bound in (2) by \( 16/5, \) thus \( \text{ch}(K) \geq 6/5. \) Thanks to Lemma 6 faces do not have positive charge and so the sum of charges of ordinary vertices is positive.

**Case 2: C is a triangle-cut.** Let \( v_1, v_2, \) and \( v_3 \) be the vertices of \( C. \) We compute the initial charge of the kernel:

\[
\text{ch}_0(K) = 12 - \sum_{i=1}^{3} (6 - \deg_H(v_i)).
\]

Supposing that each neighbour of \( C \) discharges 1 outside \( K \) (if non-neighbours discharge through distance discharging, then two neighbours are of degree 6 and cannot discharge 1 outside \( K \)).

\[
\text{ch}(K) \geq 12 - \sum_{i=1}^{3} (6 - \deg_H(v_i)) - \sum_{i=1}^{3} (\deg_H(v_i) - 2) = 0. \tag{3}
\]

By Lemma 6 faces do not have positive charge. Therefore sum of charges on ordinary and extraordinary vertices is non-negative.

Suppose now that \( K \) has an extraordinary vertex. Note that each extraordinary vertex \( v_e \) is counted as it would discharge 2 outside \( K. \) For each extraordinary vertex \( v_e \) we can do the following:

- If \( v_e \in V_5, \) then \( v_e \) discharges \( c_e, \) \( 0 \leq c_e \leq 1, \) outside \( K. \) Then the final charge \( \text{ch}(v_e) \leq 1 - c_e. \) In (3) \( v_e \) is considered as it would discharge 2 outside \( K. \) However it discharges only \( c_e. \) The final charge on \( v_e \) is at most \( 1 - c_e. \) Therefore \( \text{ch}(K - v_e) \geq \text{ch}(K) + (2 - c_e) - (1 - c_e) \geq 1. \)

- If \( v_e \not\in V_5 \) then \( v_e \) does not discharge outside \( K. \) This allows us to modify the result of (3) by at least 2, which is more than \( \text{ch}(v_e) \) since the lemma asserts that \( \text{ch}(v_e) < 2. \) Therefore \( \text{ch}(K - v_e) \geq \text{ch}(K) > 0. \)

We do this for all extraordinary vertices and we obtain that the final charge of \( K \) minus the final charge on extraordinary vertices is positive. Since only vertices have positive charge, this implies that there is an ordinary vertex with positive charge.

If there is no extraordinary vertex, then the edge \( v_1v_2 \) is on a non-triangular face \( f_1 \) in \( K. \) But then a neighbour \( w_1 \) of \( v_1 \) in \( f_1 \) discharges at least partially into \( f_1 \) and cannot discharge 1 outside of \( K. \) This is enough to guarantee that \( \text{ch}(K) > 0. \) This, again, guarantees that there is an ordinary vertex with positive charge.

This guarantees the existence of a vertex with positive charge outside a triangle-cut and quadrilateral-cut or the existence of an extraordinary vertex with charge at least 2.

### 6 Analysis of configurations

We have proved in Section 3 that a minimal counterexample to Theorem 3 is a connected graph \( G \) which contains no vertices that can be collected (that is, \( G \) has minimum degree 5). The discharging procedure defined in Section 4 ensures that after the redistribution of charges, there is an ordinary positive vertex or an extraordinary vertex with charge at least 2. Our aim is to prove that both cases
lead to a contradiction. In either case, we will do it by examining an exhaustive list of configurations and showing that none of those configurations can occur in a minimal counterexample $G$. Before we describe how to do it, recall that

$$
\Gamma(G) = \frac{v}{12} + \lambda \Phi(G) + 2\lambda \text{tc}(G).
$$

In each of the configurations, we obtain a graph $G'$ from $G$ by deleting a vertex $v$ and collecting several other vertices (always at least six) subsequently. This ensures that the statement $\text{(3)}$ of Theorem $\text{[3]}$ is true for $G$. The graph $G'$ has less vertices than $G$, hence Theorem $\text{[4]}$ holds for $G'$ and thus there exists a subset $S' \subseteq V(G')$ whose deletion allows us to collect the remaining vertices of $G'$. Consequently, the deletion of the set $S = S' \cup \{v\}$ allows us to collect all the vertices of $G$. If $\Gamma(G) \geq \Gamma(G') + 1$, then this contradicts the fact that $G$ is a counterexample to Theorem $\text{[4]}$. This would show that the examined configuration is not contained in $G$.

The critical part is to check whether $\Gamma(G) \geq \Gamma(G') + 1$. The computation consists of several steps; we demonstrate it in full detail in the proof of Lemma $\text{[9]}$. After that, we introduce a short notation that will allow us to skip repetitive arguments and help the reader to track all the details.

Lemma $\text{[9]}$ constitutes an important step in our proof because together with Lemma $\text{[8]}$ it guarantees that $G$ has an ordinary vertex with positive charge.

**Lemma 9.** The graph $G$ has no extraordinary vertex $v$ with $\text{ch}(v) \geq 2$.

**Proof.** Suppose, for a contradiction, that $G$ contains such a vertex $v$. Apparently any vertex of degree five has initial charge 1 and then its charge only decreases, so $v \in V_{6+}$. The initial charge of $v$ is $6 - \deg_G(v)$, so it has to receive charge at least $\deg(v) - 4$ during the discharging phase. Note that $v$ can receive at most $\deg(v)/5$ via distance discharging, so most of the charge has to come from neighbours of degree five in Step 3. Moreover, any distance discharging reduces the possible number of neighbours of degree five. A short case analysis left to the reader shows that $v$ must be of type 8e. In addition, $v$ must be surrounded by 8 vertices $v_1, \ldots, v_8$ of degree 5, and all the faces surrounding $v$ are triangles.

We can delete $v$ and collect $v_1, \ldots, v_8$, obtaining a graph $G'$. All we have to show is that $\Gamma(G) \geq \Gamma(G') + 1$. The function $\Gamma$ depends on three parameters: the number of vertices, the value of the function $\Phi$, and the number of tree components. This is how the removal of vertices $v, v_1, \ldots, v_8$ affects the value of $\Gamma$ as follows:

1. The number of vertices decreases by 9.

2a. The value of $\Phi$ decreases because the removed vertices do not contribute to the sum $\text{[1]}$ anymore. In most of our configurations, we do not know the degree of every vertex precisely, but for every vertex $w$ we have a lower bound $\mindeg(w)$ on its degree. Let $b_i$ be the number of vertices with $\mindeg(w) = i$ for $i \in \{5, \ldots, 9\}$ and let $b_{10}$ be the number of vertices with $\mindeg(w) \geq 10$. Let $b = (b_5, b_6, b_7, b_8, b_{10})$; in our case, $b = (8, 0, 0, 1, 0, 0)$. The value of $\Phi$ is decreased by at least $b_6 + 2b_7 + 3b_8 + 4b_9 + 5b_{10}$ because of the removal of the vertices from the sum $\text{[1]}$. In our case, $\Phi$ decreased by at least 3.

2b. The value of $\Phi$ is also decreased because the neighbours of the removed vertices have smaller degree in $G'$. Let $\Sigma_d$ denote the sum of the degrees of the removed vertices; clearly $\Sigma_d \geq 5b_6 + 6b_7 + 7b_8 + 8b_9 + 9b_{10} + 10b_{10}$. Let $\Sigma_e$ be the number of edges of the graph induced by $v, v_1, \ldots, v_8$. Clearly, the value of $\Phi$ decreases by at least $\Sigma_d - 2\Sigma_e$ because of the neighbours’ degree reduction.
The only remaining problem is to count $\Sigma_e$. We have 8 edges between $v$ and $v_1, \ldots, v_8$. We have another 8 edges between vertices $v_1, \ldots, v_8$ on the triangular faces containing $v$. No other edge may exist in the graph induced by $v, v_1, \ldots, v_8$. Such an edge would together with $v$ create a triangle-cut of $G$, which contradicts that $v$ is extraordinary. Therefore $\Sigma_e \leq 16$. (Note that we require only an upper bound on $\Sigma_e$.) We conclude that $\Phi$ decreases by 16 because of decreased degrees of the neighbours of removed vertices. In total $\Phi$ decreases by 19.

3. We have to show that no new tree component is created. Assume that such a tree component exists; it must have a vertex $w$ of degree at most 1 in $G'$. The vertex $w$ therefore has at least four neighbours among the deleted vertices. All four these neighbours are neighbours of $v$; let $v_i$ and $v_j$ be two non-adjacent of them (note that we only need three of the four neighbours to proceed with this kind of argument). The cycle $vv_iwv_jv$ is a quadrilateral cut without a chord, and this contradicts the assumption that $v$ is extraordinary. Therefore, no tree component is created.

This argument also holds in configurations when $w$ can be a neighbour of $v$. In this case a triangle-cut containing $v$, $w$ and some other neighbour of $v$ is created.

The value of $\Gamma$ decreases by $9/12 + 19\lambda$, which is at least 1 if $\lambda \geq 1/76$. For any such $\lambda$, we obtain a contradiction with the assumption that $G$ is a counterexample to Theorem 3. In other words, the statement of Lemma 9 holds for any $\lambda \geq 1/76$. \[\Box\]

The reduction idea used in the proof of Lemma 9 will be used many times over.

We summarize the key points of the calculation in the following lemma; its statement is structured with respect to the phase of the computation.

**Lemma 10.** Suppose that we removed a vertex $v$ together with some neighbours $v_1, \ldots, v_m$ of $v$ and some vertices $v_1', \ldots, v_{m2}'$ at distance 2 from $v$. The following statements are true:

$(\Sigma_e)$: Two neighbours $v_i$ and $v_j$ are joined by an edge if and only if $vv_i v_j$ is a triangular face. The vertex $v_k'$ can be joined with at most two neighbours of $v$. If $v_k'$ is adjacent to two neighbours of $v$, say $v_i$ and $v_j$, then either $v_k'v_i v_j$ and $vv_i v_j$ are triangular faces, or $v_k'v_i v_j$ is a quadrangular face.

$(\Delta \Phi)$: $\Delta \Phi \geq 5b_5 + 7b_6 + 9b_7 + 11b_8 + 13b_9 + 15b_{10} - 2\Sigma_e$.

(tc): If $m_2 \leq 1$, then no new tree component is created.

$(\lambda)$: $\lambda \geq (12 - \Delta \nu)/(12\Phi + 24tc)$.

**Proof.** Statement $(\Sigma_e)$ is implied by the fact that $v_k'v_i v_j$ is not a quadrilateral cut (with or without chord). Statement $(\Delta \Phi)$ is implied by the definitions of $\Phi$ and $b$. Statement (tc) is implied by the argumentation from the proof of Lemma 9 (enumeration of the change of $\Gamma$, part 3). Statement $(\lambda)$ follows from the inequality $\Delta \Gamma = 1/12\Delta \nu + \lambda \Delta \Phi + 2\lambda tc \geq 1$. \[\Box\]

To make our computations easier to follow we will present the key points in a concise form (the following example captures the computation used in proving Lemma 9):

Delete($v$) Collect($v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$)

$\Delta \nu = 9$ $\ b = (8, 0, 0, 1, 0, 0)$ $\Sigma_e \leq 16$ $\Delta \Phi \geq 19$ $\ tc = 0$ $\lambda \geq 1/76$.

Whenever possible, the order of the collected vertices will match the order in which these vertices can be collected.
All the computations except for the computation of $\Sigma$ are entirely routine. If Lemma 10 cannot be used or a further clarification is needed, we signalize it by a star. The explanation will follow in square brackets.

7 Charges on vertices of degree 6 and more

In this section we show that no ordinary vertex of degree at least 6 may have positive charge. The initial charge of any such vertex is not positive, so if it has final positive charge, it must have obtained it during the discharging phase. First, we determine the maximal charge a vertex can obtain via distance discharging in Step 2 of the discharging procedure described in Section 4.

Lemma 11. If a vertex $w$ of degree $k \geq 7$ has at least $m$ neighbours of degree 5, then $w$ can receive charge at most \( \left\lfloor \frac{(k-m-1) / 3}{5} \right\rfloor \) for $m > 0$ and at most \( \left\lfloor \frac{(k-m)}{3} / 5 \right\rfloor \) for $m = 0$ via distance discharging.

Proof. If a vertex $w$ of degree $k \geq 7$ receives charge $1/5$ via distance discharging, it must have four consecutive neighbours forming a path in the following order (cf. Figure 1):

1. a vertex $v_1$ of degree 6 or more.
2. a vertex $v_2$ of degree 6 that has one neighbour of degree 5.
3. a vertex $v_3$ of degree 6 that has one neighbour of degree 5.
4. a vertex $v_4$ of degree 6 or more.

Altogether, four consecutive neighbours of $w$ of degree at least 6 are required.

Note that $v_2$ and $v_3$ cannot be contained in any other path of this type: they have only one neighbour of degree 5 and only vertices of degree 5 can send charge via distance discharging. The vertices $v_1$ and $v_4$ can be shared by two paths of this type around $v$. Consequently, $v$ must have at least $3i$ neighbours from $V_{6+}$ to get charge $i/5$ via distance discharging. For $m > 0$, $v$ must have at least $3i + 1$ such neighbours. This last two statements imply the lemma.

Corollary 12. Vertices of types 10a, 9a, 9b, 9c, 8a, 8b, 8c, 8d, 7a, 7b, 7c, 6a and 6b cannot have positive charge.

Proof. We sketch the proof for a vertex $w$ of type 7a; the other cases are proved very similarly (we need to employ $n \lnot$ and look also at Step 1 in some of them). The initial charge of $w$ is $6 - 7 = -1$.

If $w$ has a neighbour of degree 5, it receives charge at most $4/5$ in Step 3 of the discharging procedure because it has only one neighbour of degree 5 and that neighbour can send at most $mc(w)$. According to Lemma 11, it can receive at most $\left\lfloor \frac{5}{3} / 5 \right\rfloor = 1/5$ in Step 2. Altogether, $w$ receives at most $4/5 + 1/5 = 1$, and so its final charge is not positive.

If $w$ has no neighbour of degree 5, it receives nothing in Step 3 and at most $\left\lfloor \frac{7}{3} / 5 \right\rfloor = 2/5$ in Step 2 according to Lemma 11. Thus $w$ has negative final charge in this case.

We have to deal only with the remaining types of vertices.

Lemma 13. The graph $G$ has no ordinary vertex of degree 8 or more with positive charge.
Proof. Let \( v \) be the vertex. Vertex \( v \) can be only of type \( 10b, 9d \) or \( 8e \). If \( v \) has at least 6 neighbours of degree 5, then denote \( v_1, \ldots, v_6 \) some six of them. For the purpose of counting \( b \), the lower bound \( \mindeg(v) \) on the degree of \( v \), will be 8.

\[
\Delta v = 7 \quad b = (6, 0, 0, 1, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 19 \quad tc = 0 \quad \lambda \geq 5/228.
\]

We prove that if \( v \) is of type \( 10b \) or \( 9d \), then it must have at least 6 neighbours of degree five to have positive charge. Indeed, if \( v \) has exactly 5 neighbours of degree 5, then it receives charge at most 2.5 from them. Additional charge might have been received by distance discharging, but its amount is bounded from above by \( (\lfloor \deg(v) - 5 - 1 \rfloor/3)/5 \) due to Lemma 11. The initial charge of these vertices is \( 6 - \deg(v) \). Thus the final charge \( 6 - \deg(v) + 2.5 + (\lfloor \deg(v) - 5 - 1 \rfloor)/5 \) is negative. If \( v \) has at most 5 neighbours, then \( v \) receives charge at most 2 from the neighbours and charge at most \( \deg(v)/15 \) by distance discharging (we used Lemma 11 again). The final charge \( 6 - \deg(v) + 2 + \deg(v)/15 \) is therefore negative. Consequently, \( v \) is of type \( 8e \) and has at most 5 neighbours of degree 5.

Vertex \( v \) can have four neighbours of degree 5 and to obtain charge 1/5 in the distance discharging step. In this case let us denote \( v_1, \ldots, v_4 \) the neighbours of \( v \) of degree 5 (they have to be consecutive around \( v \)) and denote \( v_5, v_6 \) the two neighbours of degree 6 which are also neighbours of the vertex of degree 5 that distance-discharges into \( v \). Let \( w \) be some of the other two neighbours of \( v \).

\[
\Delta v = 8 \quad b = (4, 3, 0, 1, 0, 0) \quad \Sigma_e \leq 13 \quad \Delta \Phi \geq 26 \quad tc = 0 \quad \lambda \geq 1/78.
\]

Finally, \( v \) can have exactly 5 neighbours of degree 5, let them be \( v_1, \ldots, v_5 \). Only one face around \( v \) can be non-triangular and therefore we can always find a neighbour \( w \not\in \{v_1, v_2, \ldots, v_5\} \) of \( v \) that allows us to collect three vertices from \( \{v_1, v_2, \ldots, v_5\} \), which eventually allows us to collect \( v \) and the remaining vertices from \( \{v_1, v_2, \ldots, v_5\} \).

\[
\Delta v = 7 \quad b = (6, 0, 0, 1, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 19 \quad tc = 0 \quad \lambda \geq 5/228.
\]

Lemma 14. The graph \( G \) has no ordinary vertex of degree at least 6 with positive charge.

Proof. For a contradiction, assume that \( v \) is such a vertex. The vertex \( v \) must be of type \( 7d \) and have at least three neighbours of degree 5 due to Lemma 13 and Corollary 12. Let \( m \geq 3 \) be the number of neighbours of \( v \) of degree 5.

Case \( m = 3 \): The vertex \( v \) must be only on triangular faces and receives charge by distance discharging from some vertex \( x \). Let \( v_1, v_2, v_3 \) be the three neighbours of \( v \) of degree 5. Let \( v_4, v_5 \) be two vertices of degree 6 that are common neighbours of \( v \) and \( x \). Let \( w \) be some other neighbour of \( v \).

\[
\Delta v = 7 \quad b = (3, 3, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

Case \( m = 4 \): Vertex \( v \) must be only on triangular faces (note that distance discharging into \( v \) is impossible). We can easily find all four possible configurations
Figure 2: Possible configurations around a vertex of degree 7.

around $v$; they are displayed in Figure 2. The neighbours of $v$ are denoted according to the figure.

**Configuration 1:** If $v_7$ is of degree 6, then

\[
\Delta v = 7 \quad b = (4, 2, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

If $v_7$ has another neighbour $w \neq v_1$ of degree 5, then

\[
\Delta v = 7 \quad b = (5, 0, 2, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

Now we know that $mc(v_1, v_7) \geq 4/5$. If the edge $v_1v_7$ is on a non-triangular face $f$, the vertex $v_1$ would discharge at least $1/2$ into $f$ and then discharge completely into $v_7$, leaving nothing to discharge into $v$. Therefore there exist a common neighbour $w$ of $v_1$ and $v_7$. But then $v_7$ is of degree at least 8, otherwise

\[
\Delta v = 7 \quad b = (4, 2, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

This shows that $v_1$ completely discharges into $v_7$, leaving for $v_1$ nothing to discharge into $v$. Therefore $v$ does not have positive charge in this configuration.

**Configuration 2:** Argument is very similar to Configuration 1. It suffices to switch the roles of $v_4$ and $v_5$.

**Configuration 3:** The vertex $v_7$ has degree at least 8. Otherwise

\[
\Delta v = 7 \quad b = (4, 2, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

If $v_7$ has another neighbour $w$ of degree 5 besides $v_1$, then

\[
\Delta v = 7 \quad b = (5, 0, 1, 1, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 23 \quad tc = 0 \quad \lambda \geq 5/276.
\]
But then $v_1$ discharges completely into $v_7$. Therefore $v$ does not have positive charge in this configuration.

**Configuration 4:** The vertex $v_7$ has degree at least 8. Otherwise

$$\Delta v = 7 \quad b = (4, 2, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.$$ 

The vertex $v_5$ has degree at least 8. Otherwise

$$\Delta v = 7 \quad b = (4, 2, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.$$ 

But then $v_6$ discharges 0 to $v$. Therefore $v$ in this configuration does not have positive charge.

**Case $m = 5$:** There can be only one non-triangular face around $v$. Simple case analysis shows that $v$ always has three consecutive neighbours, say $x$, $y$ and $z$, such that $x$ is not of degree 5, $y$ and $z$ are of degree 5, and edges $xy$ and $yz$ exist. The other neighbours of $v$ of degree 5 will be denoted by $v_1, v_2, v_3$.

$$\Delta v = 7 \quad b = (5, 1, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 19 \quad tc = 0 \quad \lambda \geq 5/228.$$ 

**Case $m \geq 6$:** Let us denote some six neighbours of $v$ of degree 5 by $v_1, \ldots, v_6$.

$$\Delta v = 7 \quad b = (6, 0, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 17 \quad tc = 0 \quad \lambda \geq 5/204.$$ 

\[\square\]

## 8 Charges on vertices of degree 5

In this section we show that all vertices of degree 5 outside bad cuts completely discharge and none of them has positive charge.

**Lemma 15.** The graph $G$ has no good vertex of degree 5 with positive charge.

**Proof.** For a contradiction, let $v$ be such a vertex. The vertex $v$ cannot have too many neighbours of degree more than 6. If it was so, then $v$ would completely discharge into these higher degree vertices no matter what type they are of. The vertices will be denoted according to Figure 3 in our analysis of the neighbourhood of $v$. To guarantee that we do not miss any configuration we list them in the following order:

1. Configurations where $v$ is only on triangular faces are listed before those with $v$ on a non-triangular face. The vertex $v$ cannot be in two or more non-triangular faces.

2. As the second criterion, configurations are ordered according to the number of neighbours of $v$ of degree more than 6. Vertex $v$ has at most two such neighbours, and if it has exactly two of them, then at least one of them must have degree exactly 7.

3. The third ordering criterion is the degree of the highest-degree neighbour of $v$. We may assume that this vertex is $v_1$ without a loss of generality (if there are two such candidates, we choose any of them).
Figure 3: Possible configurations around a vertex of degree 5 omitting the configuration where the vertex of degree 5 is surrounded by vertices of degree 6.

4. The fourth ordering criterion, applied if \( v \) has two neighbours of degree more than 6, is the relative position of these two neighbours. Due to the symmetry of the configuration in any such case we may suppose that the second vertex is either \( v_2 \) or \( v_3 \). Configurations where \( v_2 \) has degree more than 6 are earlier in the order.

The fourth criterion for the case that there is a non-triangular face containing \( v \) is that we order the configurations according to the position of the non-triangular face. We may without loss of generality suppose that this is the face containing either \( v_1v_v_2, v_2v_3 \), or \( v_3v_4 \). If \( v_1 \) is not of degree at most 6, then we can without loss of generality assume that \( v_1v_2 \) is in the boundary of the non-triangular face containing \( v \).

Generating the configurations in this way we get 14 possible configurations of the neighbourhood of \( v \). These configurations are displayed in Figure 3. The vertices \( v \), \( v_1,...,v_5 \) are *denoted*, the other vertices are *non-denoted*.

Suppose \( v_1 \) has a non-denoted neighbour \( w \) of degree 5. For all configurations except Configuration 10 and Configuration 14 we can do the following:

- \( \text{Delete}(v_1) \)  \( \text{Collect}(v, v_2, v_3, v_4, v_5, w) \)
- \( \Delta v = 7 \)
- \( b = (6, 0, 1, 0, 0) \)
- \( \Sigma_e \leq 12 \)
- \( \Delta \Phi \geq 15 \)
- \( tc = 0 \)
- \( \lambda \geq 1/36 \).
Therefore for these configurations we can assume that $v_1$ has no non-denoted neighbour of degree 5.

**Configuration 1, 2, 3, 4, 7:** The vertex $v$ discharges completely into $v_1$.

**Configuration 8:** If $v_1$ is on two non-triangular faces, then $v$ discharges completely into $v_1$. If $v_1$ in one non-triangular face, then $\deg(v_2) = \deg(v_5) = 5$ (otherwise $v$ discharges completely into $v_1$). Either a common non-denoted neighbour of $v_1$ and $v_2$ exists, or a common non-denoted neighbour of $v_1$ and $v_5$ exists. Say $w \neq v$ is a common non-denoted neighbour of $v_1$ and $v_2$.

$$\text{Delete}(w) \quad \text{Collect}(v_2, v, v_3, v_4, v_5, v_1)$$

$$\Delta v = 7 \quad b = (6, 0, 0, 1, 0, 0) \quad \Sigma e \leq 12 \quad \Delta \Phi \geq 17 \quad tc = 0 \quad \lambda \geq 5/204.$$ 

If $v_1$ is in no non-triangular face, then at least one of the vertices $v_2$ and $v_5$ is of degree 5 (otherwise $v$ discharges completely into $v_1$), say it is $v_2$. Let $w$ be the common non-denoted neighbour of $v_2$ and $v_1$. Then

$$\text{Delete}(w) \quad \text{Collect}(v_2, v, v_3, v_4, v_5, v_1)$$

$$\Delta v = 7 \quad b = (6, 0, 0, 1, 0, 0) \quad \Sigma e \leq 12 \quad \Delta \Phi \geq 17 \quad tc = 0 \quad \lambda \geq 5/204.$$ 

**Configuration 5:** We already showed that $v_1$ cannot have a non-denoted neighbour of degree 5. Symmetrically, $v_2$ has no non-denoted neighbour of degree 5. The vertex $v$ would completely discharge unless $\deg(v_3) = 5$ and $\deg(v_5) = 5$. There are only triangular faces around $v_1$ and $v_2$, otherwise $v$ discharges. Let $w \neq v$ be the common non-denoted neighbour of $v_5$ and $v_1$. Then

$$\text{Delete}(w) \quad \text{Collect}(v_5, v, v_4, v_3, v_2, v_1)$$

$$\Delta v = 7 \quad b = (5, 0, 2, 0, 0, 0) \quad \Sigma e \leq 12 \quad \Delta \Phi \geq 19 \quad tc = 0 \quad \lambda \geq 5/228.$$ 

**Configuration 6:** Symmetrically to $v_1$, also $v_3$ has no non-denoted neighbour of degree 5. If $\deg(v_2) = 5$ and $\deg(v_5) = 5$, then let $w$ be a non-denoted neighbour of $v_2$.

$$\text{Delete}(w) \quad \text{Collect}(v_2, v, v_5, v_4, v_3, v_1)$$

$$\Delta v = 7 \quad b = (5, 0, 2, 0, 0, 0) \quad \Sigma e \leq 12 \quad \Delta \Phi \geq 19 \quad tc = 0 \quad \lambda \geq 5/228.$$ 

The same approach works if $\deg(v_2) = 5$ and $\deg(v_4) = 5$. Now $v_1$ and $v_3$ have at most two neighbours of degree 5. This means that $v_1$ and $v_3$ are only in triangular faces, otherwise $v$ discharges fully into one of these vertices. Moreover either $v_2$ or $v_5$ is of degree 5 (otherwise $v$ discharges), say it is $v_2$. Let $w \neq v$ be the common non-denoted neighbour of $v_2$ and $v_1$.

$$\text{Delete}(w) \quad \text{Collect}(v_2, v, v_1, v_5, v_4, v_3)$$

$$\Delta v = 7 \quad b = (4, 1, 2, 0, 0, 0) \quad \Sigma e \leq 12 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.$$ 

**Configuration 9:** If one of the vertices $v_2, \ldots, v_5$ is of degree 5, say it is $v_2$, then let $w$ be a non-denoted neighbour of $v_2$.

$$\text{Delete}(w) \quad \text{Collect}(v_2, v, v_3, v_4, v_5, v_1)$$

$$\Delta v = 7 \quad b = (6, 0, 1, 0, 0, 0) \quad \Sigma e \leq 12 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.$$ 

Therefore $v_2, \ldots, v_5$ are of degree 6. If some neighbour of denoted vertices has degree 5 we proceed similarly as with $v_1$.

If some denoted vertex is in a non-triangular face, then $v$ discharges into $v_1$ and that vertex on a non-triangular face, even if it is of degree 6. Let $w$ be the common
non-denoted neighbour of \(v_3\) and \(v_4\). If \(\deg(w) \geq 7\), then \(v\) discharges 1/5 into \(w\) and 4/5 into \(v_1\). If \(\deg(w) \leq 6\), then

\[
\text{Delete}(v_3) \quad \text{Collect}(v, v_4, v_2, v_5, v_1, w) \\
\Delta v = 7 \quad b = (2, 4, 1, 0, 0, 0) \quad \sum_e \leq 12 \quad \Delta \Phi \geq 23 \quad tc = 0 \quad \lambda \geq 5/276.
\]

**Configuration 10:** Let \(m\) be the number of denoted vertices of degree 5. Suppose now that \(m = 6\). Assume that a neighbour \(w\) of a denoted vertex, say \(v_1\), has degree at least 7.

\[
\text{Delete}(w) \quad \text{Collect}(v_1, v, v_2, v_3, v_4, v_5) \\
\Delta v = 7 \quad b = (6, 0, 1, 0, 0, 0) \quad \sum_e \leq 12 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.
\]

If a neighbour of a denoted vertex has degree 6 and is a neighbour only to one denoted vertex, then we may without loss of generality suppose it is the vertex \(v_1\) and denote the neighbour \(w\).

\[
\text{Delete}(w) \quad \text{Collect}(v_1, v, v_2, v_3, v_4, v_5) \\
\Delta v = 7 \quad b = (6, 1, 0, 0, 0, 0) \quad \sum_e \leq 11 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.
\]

If a neighbour \(w\) of a denoted vertex has degree 6, it must be a neighbour of two neighbouring denoted vertices, say these vertices are \(v_1\) and \(v_2\). Let \(w_2\) be a neighbour of \(v_4\).

\[
\text{Delete}(w_2) \quad \text{Collect}(v_4, v, v_1, v_2, v_3, v_5, w) \\
\Delta v = 8 \quad b = (7, 1, 0, 0, 0, 0) \quad \sum_e \leq 15 \quad \Delta \Phi \geq 12 \quad tc = 0 \quad \lambda \geq 1/36.
\]

[Both vertices \(w\) and \(w_2\) may be connected to at most two denoted vertices, moreover \(w\) and \(w_2\) may be neighbours (\(v\) can be contained in a \(C_5\)-cut). Extra care is also necessary to analyse created tree components. The tree component cannot be an isolated vertex, because then it would be a neighbour of 5 removed vertices and three of them must be denoted. This is enough to guarantee that \(v\) is in a bad cut. Therefore the newly created tree component contains at least two leaves \(x_1\) and \(x_2\). Both these vertices must have four neighbours among removed vertices. Both vertices cannot be neighbours of two non-neighbouring denoted vertices, therefore at most two neighbours are among denoted vertices. This is possible. Both \(x_1\) and \(x_2\) can be neighbours of \(w, w_2\). If this happens, then there cannot be an edge between \(w\) and \(w_2\) because we would have a \(K_5\)-minor in \(G\) (containing vertices \(v, w, w_2, x_1\), and \(x_2\)). In both cases, whether we create new tree component or not, we get \(\lambda \geq 1/36\). Two new tree components cannot be created.]

Now we are left with the case where all neighbours of denoted vertices have degree 5. This is together at least 11 vertices. These vertices must have another neighbour (a planar graph that contains only 11 vertices is collectable). Removing vertices itself decreases \(\Gamma\) by at least 1. Each created tree component must have at least 5 neighbours among removed vertices. therefore \(\Delta \Phi \geq 5tc\). Therefore \(\Gamma\) decreases by at least 1.

Suppose \(m = 5\). If denoted vertices have a non-denoted neighbour of degree at least 6, then

\[
\text{Delete}(w) \quad \text{Collect}(v_1, v, v_2, v_3, v_4, v_5) \\
\Delta v = 7 \quad b = (5, 2, 0, 0, 0, 0) \quad \sum_e \leq 12 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.
\]

If denoted vertices do not have a neighbour of degree 6 we can collect at least 11 vertices after deleting \(v\). Removing vertices itself decreases \(\Gamma\) by at least 1. Each
created tree component must have at least 5 neighbours among removed vertices, therefore $\Delta \Phi \geq tc$. Therefore $\Gamma$ decreases by at least 1.

Suppose $m \leq 4$. If $v_1$, $v_2$, $v_3$, $v_4$, or $v_5$ is of degree 5 then let $w$ be a non-denoted neighbour of that vertex.

\[
\text{Delete}(w) \quad \text{Collect}(v_1, v, v_2, v_3, v_4, v_5) \quad \Delta v = 7 \quad b = (5, 2, 0, 0, 0, 0) \quad \Sigma_e \leq 12 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.
\]

If $v_1$, $v_2$, $v_3$, $v_4$, or $v_5$ has a non-denoted neighbour $w$ of degree 5, then say it is $v_1$.

\[
\text{Delete}(v_1) \quad \text{Collect}(w, v, v_2, v_3, v_4, v_5) \quad \Delta v = 7 \quad b = (2, 5, 0, 0, 0, 0) \quad \Sigma_e \leq 12 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

Let us look at the edge $v_1v_2$. If $v_1$ and $v_2$ are only on triangular faces, then they have a common neighbour $w$. The vertex $w$ has degree at least 7 otherwise:

\[
\text{Delete}(v_1) \quad \text{Collect}(v, v_2, v_3, v_4, v_5, w) \quad \Delta v = 7 \quad b = (2, 5, 0, 0, 0, 0) \quad \Sigma_e \leq 12 \quad \Delta \Phi \geq 21 \quad tc = 0 \quad \lambda \geq 5/252.
\]

In that case we can distance-discharge $1/5$ from $v$ to $w$. We repeat this argument for the edges $v_2v_3$, $v_3v_4$, $v_4v_5$, $v_5v_1$. Therefore if all vertices $v_i$, for $i \in \{1, 2, 3, 4, 5\}$, are only in triangular faces, then $v$ discharges. If some vertex $v_i$, for $i \in \{1, 2, 3, 4, 5\}$, is in a non-triangular face, then $v$ discharges $2/5$ into this vertex. This compensates for the inability to do distance discharging to neighbours of $v_i$. In all cases $v$ discharges.

**Configuration 11**: The vertex $v$ discharges to the non-triangular face and into $v_1$, which has no more than two neighbours of degree 5.

**Configurations 12 and 13**: If both $v_2$ and $v_5$ have degree 5, then let $w$ be a non-denoted neighbour of $v_2$.

\[
\text{Delete}(v_1) \quad \text{Collect}(v, v_2, v_3, v_4, v_5, w) \quad \Delta v = 7 \quad b = (6, 0, 1, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 17 \quad tc = 0 \quad \lambda \geq 5/208.
\]

Otherwise $v$ discharges to the non-triangular face and into $v_1$.

**Configuration 14**: If $\deg(v_1) = 5$, then let $w$ be a neighbour of $v_1$ that is not adjacent to any denoted vertex.

\[
\text{Delete}(w) \quad \text{Collect}(v_1, v, v_2, v_3, v_4, v_5) \quad \Delta v = 7 \quad b = (7, 0, 0, 0, 0, 0) \quad \Sigma_e \leq 10 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.
\]

Therefore $\deg(v_1) = 6$. If any other denoted vertex besides $v$ has degree 5, say it is $v_3$, then let $w$ be a neighbour of $v_3$.

\[
\text{Delete}(w) \quad \text{Collect}(v_3, v, v_2, v_1, v_4, v_5) \quad \Delta v = 7 \quad b = (6, 1, 0, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 15 \quad tc = 0 \quad \lambda \geq 1/36.
\]

If any non-denoted neighbour of a denoted vertex has degree 5, say it is $v_1$, then let $w$ be that neighbour.

\[
\text{Delete}(w) \quad \text{Collect}(v_1, v, v_2, v_3, v_4, v_5) \quad \Delta v = 7 \quad b = (2, 5, 0, 0, 0, 0) \quad \Sigma_e \leq 11 \quad \Delta \Phi \geq 23 \quad tc = 0 \quad \lambda \geq 5/276.
\]

Vertex $v$ can discharge $2/5$ into $v_1$, $2/5$ into $v_2$ and $3/5$ to non-triangular face that contains $v$. □
9 Conclusion

Now we easily prove Theorem 3.

Proof of Theorem 3. Lemma 8 and Lemma 9 show that a minimal counterexample to Theorem 3 contains an ordinary vertex with positive charge. Lemma 13, Lemma 14, and Lemma 15 say that no such vertex exists, which is a contradiction. □

The most problematic case which hinders further improvement is the case when a vertex of degree 5 is surrounded by five vertices of degree 5. One of these configurations is in Figure 4.

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