KNOTS
From combinatorics of knot diagrams to combinatorial topology based on knots

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Introduction

This book is about classical Knot Theory, that is, about the position of a circle (a knot) or of a number of disjoint circles (a link) in the space $R^3$ or in the sphere $S^3$. We also venture into Knot Theory in general 3-dimensional manifolds.

Lecture Notes on Knot Theory, published in Polish in 1995 [P-18], is the predecessor of this book\(^1\). A rough translation of the Notes (by J. Wiśniewski) was ready by the summer of 1995. It differed from the Polish edition with the addition of the full proof of Reidemeister’s theorem. While I could not find time to refine the translation and prepare the final manuscript, I did add new material and rewrote existing chapters. In this way I created a new book based on the Polish Lecture Notes but expanded three-fold. Only the first part of Chapter III (formerly Chapter II), on Conway’s algebras is essentially unchanged from the Polish book and is based on preprints [P-1].

As to the origin of the Lecture Notes, I taught an advanced course on the theory of 3-manifolds and Knot Theory at Warsaw University and it was only natural to write down my talks (see Introduction to (Polish) Lecture Notes).

\(^1\)The Polish edition was prepared for the “Knot Theory” mini-semester at the Stefan Banach Center, Warsaw, Poland, July-August, 1995.
We describe in this chapter the idea of building an algebraic topology based on knots (or more generally on the position of embedded objects). That is, our basic building blocks are considered up to ambient isotopy (not homotopy or homology). For example, one should start from knots in 3-manifolds, surfaces in 4-manifolds, etc. However our theory is, until now, developed only in the case of links in 3-manifolds, with only a glance towards 4-manifolds. The main object of the theory is a skein module and we devote this chapter mostly to the description of skein modules in 3-dimensional manifolds. In this book we outline the theory of skein modules often giving only ideas and outlines of proofs. The author is preparing a monograph devoted exclusively to skein modules and their ramifications [P-30].

IX.1 History of skein modules

H. Poincaré, in his paper “Analysis situs” (1895), abstractly defined homology groups starting from formal linear combinations of simplices, choosing cycles and dividing them by relations coming from boundaries [Po].

The idea behind skein modules is to use links instead of cycles (in the case of a 3-manifold). More precisely we consider the free module generated

\footnote{Before Poincaré the only similar construction was the formation of “divisors” on an algebraic curve by Dedekind and Weber [D-W], that is the idea of considering formal linear combinations of points on an algebraic curve, modulo relations yielded by rational functions on the curve.}
by links modulo properly chosen (local) skein relations.

Skein modules have their origin in the observation made by J. W. Alexander ([Al-3], 1928)\(^2\) that his polynomials of three links \(L_+, L_-\) and \(L_0\) in \(S^3\) are linearly related (here \(L_+, L_-\) and \(L_0\) denote three links which are identical except in a small ball as shown in Figure 1.1). J. H. Conway rediscovered the Alexander observation and normalized the Alexander polynomial so that it satisfies the skein relation

\[
\Delta_{L_+}(z) - \Delta_{L_-}(z) = z \Delta_{L_0}(z)
\]

([Co-1], 1969). In the late seventies, Conway advocated the idea of considering the free \(\mathbb{Z}[z]\)-module over oriented links in an oriented 3-manifold and dividing it by the submodule generated by his skein relation \([Co-2]\) (cited in \([Gi]\)) and \([Co-3]\) (cited in \([Ka-1]\)). However, there is no published account of the content of Conway’s talks except when \(S^3\) or its submanifolds are analyzed. The original name Conway used for this object was “linear skein”.

Conway’s idea was then pursued by Giller \([Gi]\) (who computed the linear skein of a tangle), and Kauffman \([Ka-1, Ka-8, Ka-3]\), as well as Lickorish and Millett \([L-M-1]\) (for subspaces of \(S^3\)).

In graph theory, the idea of forming a ring of graphs and dividing it by an ideal generated by local relations was developed by W. Tutte in his 1946 PhD thesis \([Tut-1]\), but the relation to knot theory was observed much later.

The theory of Hecke algebras, as introduced by N. Iwahori ([Iw], 1964), is closely connected to the theory of skein modules, however the relation of Hecke algebras to knot theory was noticed by V. Jones in 1984, 20 years after Iwahori’s and Conway’s work (it was crucial for Jones-Ocneanu construction of Markov traces).

\(^2\)We can argue further that Alexander was motivated by the chromatic polynomial introduced in 1912 by George David Birkhoff ([Birk-1]). Compare also the letter of Alexander to O. Veblen ([A-V], 1919) discussed in Chapter II.
IX.2. GOAL OF SKEIN MODULE THEORY

The Temperley-Lieb algebra ([TL], 1971) is related to the Kauffman bracket skein module of the tangle, but any relation to knot theory was again observed first by Jones in 1984.

At the time when I introduced skein modules, in April of 1987, I knew the fundamental paper of Conway [Co-1], and [Gl, Ka-8, Li-10] as well as [Li-10]. However, the most stimulating paper for me was one by J. Hoste and M. Kidwell [Ho-K] about invariants of colored links, which I read in March of 1987. The goal of Hoste and Kidwell was to find the common generalization of the multivariable Alexander polynomial and the Jones-Conway (Homflypt) polynomial, and they succeeded, in the limited manner, requiring one color to represent a trivial component\(^3\). I quickly realized that what Hoste and Kidwell actually computed were invariants of links in the solid torus, \(S^1 \times D^2\), up to the Jones-Conway skein relation. I wrote the introductory paper on skein modules in May of 1987 [P-5]. Skein modules were also introduced independently by V. Turaev [Tu-2].

IX.2 Goal of skein module theory

Our goal is to build an algebraic topology based on knots. We call the main object used in the theory a skein module and we associate it to any 3-dimensional manifold. The essence is that skein modules are quotients of free modules over ambient isotopy classes of links in 3-manifolds by properly chosen local (skein) relations.

These new objects are, after a slow start, intensively studied and their properties seem to be topologically very significant. In particular, one should look for their features that are similar to the Seifert-Van Kampen and Mayer-Vietoris theorems. Another interesting question concerns the relation between the skein modules of the base and the skein modules of the covering space, for coverings and branched coverings. At present we can say something about the above question only in a very special situation and then results concern symmetries of links and 3-manifolds. As in the case of homology, one should try to understand the free and torsion part of the module. In particular the torsion part of the module seems to reflect the geometry of the manifold (i.e. their incompressible surfaces).

There is an ingenious argument that skein modules have, in general, \(^3\)At the first Cascade Mountain Seminar, in January of 1987, Jim Hoste explained to me his (then unfinished) work with Mark Kidwell and we realized that an analogous result for the Kauffman bracket is much easier to prove. We did not present, initially, our work in the language of skein modules [P-5].
a different nature than homotopy invariants. This was first observed by F. Jaeger [Ja-1] and explored in [J-V-W]. Namely, computation of Jones type invariants is usually NP hard (and so, up to the famous conjecture cannot be performed in polynomial time [G-J]) while computing, say, the Alexander polynomial or homology groups, can be accomplished in polynomial time.

The most promising aspect of theory of skein modules is their interpretation as deformation of coordinate rings of character varieties. In the case of Kauffman bracket skein module it is an $SL(2,\mathbb{C})$ character variety $\text{[Bu-4, Bu-5, Bu-6, P-S-1, P-S-2]}$ and for Homflypt skein module it is an $SL(n,\mathbb{C})$ character variety $\text{[Si-4, Si-5]}$.

**IX.3 Skein modules of 3-manifolds; ideas and examples**

Skein modules are quotients of free modules over ambient isotopy classes of links in a 3-manifold by properly chosen local (skein) relations. The choice of relations is a delicate task as we should take into account several factors:

(i) Is the module we obtain accessible (computable)?

(ii) How precise are our modules in distinguishing 3-manifolds and links in them?

(iii) Does the module reflect topology/geometry of a 3-manifold (e.g., surfaces in a manifold, geometric decomposition of a manifold)?

(iv) Does the module admit some additional structure (e.g., filtration, gradation, multiplication, Hopf algebra structure)? Is it leading to a Topological Quantum Field Theory (TQFT) by taking a finite dimensional quotient?

From a practical point of view there is yet a fifth important factor

(v) The “historical factor” in the choice of (skein) relations: the relations of links which were already studied (possibly for totally different reasons) will be compared with the new structures, just to see how they work in the new setup. For example, if we consider the Jones skein relation we can be sure that even for $S^3$ we get a nontrivial result.

The idea of the skein module should become more apparent after we consider some examples.
Example IX.3.1 ([P-5]) Let $M$ be an arbitrary 3-dimensional manifold and let $R$ be a commutative ring with unit. Moreover, let $\mathcal{L}$ denote the set of ambient isotopy classes of oriented links in $M$ and let $R\mathcal{L}$ denote the free $R$-module generated by $\mathcal{L}$. In $R\mathcal{L}$ we consider the submodule $S$ generated by skein expressions of type $L_+ - L_-$, where $L_+$ and $L_-$ are two oriented links in $M$ which are different in a small ball where they are depicted in Fig. 3.1.

![Fig. 3.1](image)

Let us note that, in order to define $S$, we do not assume that the manifold $M$ is oriented and there is no need to distinguish between $L_+$ and $L_-$ (sometimes it is not even possible to distinguish between them). Now we define the skein module $S(M; R, L_+ - L_-)$, which we denote simply by $S_+(M)$, as the quotient $R\mathcal{L}/S$.

It is not hard to see that $S_+(M)$ is a free module over the homotopy classes of closed curves (links) in the manifold. Let us have a closer look at this example. The skein module $S_+(M)$ admits a natural multiplication, defined as $L_1 \cdot L_2 = L_1 \sqcup L_2$ (i.e. the multiplication comes from the disjoint sum of links). Because of the skein relations we considered, the multiplication does not depend on the position of $L_2$ with respect to $L_1$. The multiplication is associative and commutative but there is no multiplicative identity. However, if we extend the set $\mathcal{L}$ by an empty link $\emptyset$ (and we call the resulting set $\mathcal{L}^{alg}$) then $S(M)$ extends to the quotient $S_+^{alg}(M)$ which is then a commutative algebra with an identity (moreover the ring $R$ embeds into $S_+^{alg}(M)$ by the mapping $r \mapsto r \cdot \emptyset$). As an algebra, $S_+^{alg}(M)$ is isomorphic to the polynomial algebra with coefficients in $R$ and variables which are homotopy classes of knots in $M$; in other words, the set of variables, $\pi$, is equal to the set of conjugacy classes of the fundamental group $\pi_1(M)$. Equivalently, the skein module $S_+^{alg}(M)$ is an $R$-algebra isomorphic to symmetric tensor algebra over $R\pi$ (which we usually denote by $SR\pi$). Let us recall that the tensor algebra $TR\pi$ is the graded sum $\bigoplus T^i R\pi$ where $T^0 R\pi = R, T^1 R\pi = R\pi, T^{i+1} R\pi = T^i R\pi \otimes R\pi$, and the symmetric algebra is the quotient $SR\pi = TR\pi/(a \otimes b - b \otimes a)$. 


We consider this example in such detail since it turns out to be the Vassiliev-Gusarov skein module of degree 0. The Vassiliev-Gusarov skein modules (which in fact are Hopf algebras) give a good framework in the work on Vassiliev invariants of links (in any 3-manifold); see Section 9 and [P.9].

Now, the above simple example can be generalized (or “quantized”) to the framed links, that is to the classes of annuli embedded into a manifold (the central curve of such an annulus determines an unframed link), see Fig. 3.2.

Example IX.3.2. Let \( \mathcal{L}^{fr} \) denote the set of ambient isotopy classes of framed oriented links in an oriented 3-manifold \( M \). Let \( R = \mathbb{Z}[q^{\pm 1}] \). Let us consider the submodule \( S^{fr} \) of \( R\mathcal{L}^{fr} \) which is generated by skein expressions presented in Fig. 3.3, that is, \( S^{fr} \) is generated by expressions of type \( L_+ - q^2 L_- \) and \( L^{(1)} - q L \), where \( L^{(1)} \) denotes a link obtained from \( L \) by one positive twist of the framing of \( L \).

Now we define the skein module as the quotient \( S^{fr}_\pm (M) = R\mathcal{L}^{fr} / S^{fr} \).

The computation of \( S^{fr}_\pm (M) \) in general was an open problem for a long time. The hard part is dealing with manifolds which contain a non-separating torus (because such a torus is a source of a torsion in the module). Initially I was only able to prove:
Theorem IX.3.3 ([P-19])  
(a) If $M$ is an oriented 3-manifold without non-separating 2-spheres and tori then $S^{fr}_{\pm}(M) = S(M, Z) \otimes \mathbb{Z}[q^{\pm 1}]$.

(b) If $M$ contains a non-separating 2-sphere or torus then $S^{fr}_{\pm}(M)$ has a torsion element:

(i) If $L$ is a link in $M$ with the algebraic crossing number with some 2-sphere in $M$ equal to $k$, $k \neq 0$, then $(q^{2k} - 1)L = 0$ in $S^{fr}_{\pm}(M)$.

(ii) Let $L'$ be a link in $M$ with the algebraic crossing number with some torus in $M$ equal to $k$, $k \neq 0$. Let $L$ be a link obtained by adding to $L'$ a noncontractible curve on the torus. Then $(q^{2k} - 1)L = 0$ in $S^{fr}_{\pm}(M)$.

The link $L$ from (i) and (ii) is not equal to zero in the skein module because it is homotopy nontrivial and $S^{fr}_{\pm}(M)$ reduces to $S_{\pm}(M)$ when we set $q = 1$.

Uwe Kaiser completed description of $S^{fr}_{\pm}(M)$ by proving that the only method of producing torsion is in fact described in (b)(i) and (b)(ii) [Kai-4].

In the example described above we used skein relations involving a crossing change. In the next example a smoothing of the crossing will be explored.

Example IX.3.4 ([H-P-1]) Suppose that $M$ is an arbitrary manifold and $R$ is a commutative ring with unit. Let $\mathcal{L}$ denote the set of isotopy classes of oriented links in $M$ and $R\mathcal{L}$ denote a free $R$ module over $\mathcal{L}$. Let $S_2$ be the submodule of $R\mathcal{L}$ generated by skein expressions: $\includegraphics{link}$.

We define the second skein module $S_2(M; R)$ to be the quotient $R\mathcal{L}/S_2$.

Lemma IX.3.5 ([H-P-1]) Let $\phi: R\mathcal{L} \rightarrow RH_1(M, Z)$ be the epimorphism of $R$ modules such that $\phi(L) = |L|$ where $|L|$ is the class of $L$ in $H_1(M, Z)$. Then $\phi$ can be factored through an isomorphism

$$\hat{\phi}: S_2(M; R) \rightarrow RH_1(M, Z)$$

The idea of the proof is to show that:

(i) If two links, $L_1$ and $L_2$, represent the same element in $H_1(M, Z)$ then $L_1 \sqcup -L_2$ is the boundary of an oriented surface contained in $M$,

(ii) If $L_1 \sqcup -L_2$ is the boundary of an oriented surface in $M$ then the link $L_2$ can be reached starting from $L_1$ via a sequence of elementary operations which are either modifications of $\includegraphics{link}$ to $\includegraphics{link}$ or their inversions.
There is also a framed version of the second skein module and it can be called a \( q \)-deformation of the first homology group. This version of the skein module contains torsion related to closed surfaces in \( M \) which do not separate \( M \).

**Example IX.3.6** Let \( M \) be an oriented 3-manifold and let \( R = \mathbb{Z}[q^{\pm 1}] \). Let us denote by \( S^{fr} \) the submodule of \( RL^{fr} \) generated by skein expressions pictured in Fig. 3.4, i.e. \( L_+ - qL_0 \) and \( L^{(1)} - qL \), where \( L^{(1)} \) denote a link obtained from \( L \) by twisting the framing of \( L \) once in the positive direction.

\[
\begin{align*}
&\begin{array}{c}
\text{Fig. 3.4}
\end{array}
\end{align*}
\]

The quotient \( RL^{fr}/S^{fr} \) is called the second skein module and it is denoted by \( S_2(M,q) \).

It is not hard to prove that \( S_2(M,q) \) is a free \( R \) module if \( M \) is a rational homology sphere or its compact submanifold. In this case \( S_2(M,q) = S_2(M,Z) \otimes \mathbb{Z}[q^{\pm 1}] \). On the other hand, non-separating closed surfaces in \( M \) yield torsion in the skein module. We can compute the module \( S_2(M,q) \) for any 3-manifold.

**Theorem IX.3.7** [P-12]

\[
S_2(M,q) = \mathbb{Z}[q^{\pm 1}]T(H_1(M,Z)) \oplus \bigoplus_{\alpha \in H_1(M,Z) - T(H_1(M,Z))} \mathbb{Z}[q^{\pm 1}]/(q^{2\text{mul}(\alpha)} - 1),
\]

where \( T(H_1(M,Z)) \) and the multiplicity of \( \alpha \), \( \text{mul}(\alpha) \), are defined as follows:

Let \( \phi : H_1(M,Z) \times H_2(M,Z) \to \mathbb{Z} \) be the bilinear form of the intersection of 1-cycles with 2-cycles on \( M \). Then \( \alpha \in T(H_1(M,Z)) \) if and only if \( \phi_\alpha(\beta) = 0 \) for any \( \beta \). Otherwise \( \text{mul}(\alpha) \) is defined as the positive generator of \( \text{im}\phi_\alpha(H_2(M,Z)) \).

One should notice that for a closed 3-manifold \( M \), \( T(H_1(M,Z)) \) is the torsion part of \( H_1(M,Z) \).

The second skein module should provide “experimental data” that is useful when computing harder skein modules.
We can consider even simpler skein module, without a skein relation, but keeping the framing relation. We call such a skein module a framing skein module.

**Example IX.3.8** Let $M$ be an oriented 3-manifold and let $R = \mathbb{Z}[q^{\pm1}]$. Let us denote by $S_{fr}^f$ the submodule of $RK_{fr}$ generated by framing expressions $K^{(1)} - qK$, for any framed knot in $K_{fr}$. The quotient $S_0(M, q) = RK_{fr}/S_{fr}^f$ is called the framing skein module.

We proved (with J. Hoste) that if $M$ has no non-separating $S^2$ then the module is free (with basis $K$) \cite{H-P-2} and we were referred by D. McCullough to his paper \cite{Mc} from which it follows that

**Theorem IX.3.9**

For a 3-manifold $M$, $S_0(M, q) = \mathbb{Z}[q^{\pm1}]K^f \oplus \bigoplus_{K \in K_{fr}} \mathbb{Z}[q^{\pm1}]/(q^2 - 1)$, where $K^f$ is composed of knots which do not cut any 2-sphere in $M$ in exactly one point.

Theorem 3.9 was also proved, by different method, by Chernov \cite{Cher}.

To summarize our examples from the geometric point of view, we stress that a nonseparating $S^2$ in $M$ is detected by torsion in the skein module $S_0^{fr}(M)$, a nonseparating $T^2$ in $M$ is detected by torsion in the skein module $S_{\pm}^{fr}(M)$, and a nonseparating oriented closed surface in $M$ is detected by torsion in the skein module $S_2^{fr}(M)$. More complicated skein modules detect, to various degree, separating surfaces as well (compare Section 8).

**IX.4 Skein module based on Homflypt skein relation**

The Jones-Conway (Homflypt) skein relation $(v^{-1}L_+ - vL_- = zL_0$, where $L_+, L_-, L_0$ are oriented links as in Fig. 4.1) is based on a relation which was hinted at by Alexander and Conway and can be thought of as a deformation of the crossing change (or, of a 2-move; see Chapter VI).

**Definition IX.4.1** Let $M$ be an oriented 3-manifold, $R = \mathbb{Z}[v^{\pm1}, z^{\pm1}]$, $\mathcal{L}$ the set of all oriented links in $M$ up to ambient isotopy of $M$ and $S_3$ the submodule of $RL$ generated by the skein expressions $v^{-1}L_+ - vL_- - zL_0$. For convenience we allow the empty knot, $\emptyset$, and add the relation $v^{-1}\emptyset - v\emptyset - zT_1$, where $T_1$ denotes the trivial knot.
Then the third skein module of $M$ is defined to be:

$$S_3(M) = S_3(M; \mathbb{Z}[v^{\pm 1}, z^{\pm 1}], v^{-1}L_+ - vL_+ - zL_0) = RL/S_3.$$  

![Diagram](image)

The third skein module of links in $S^3$ can be formulated as follows:

**Theorem IX.4.2** ([FYHLMO] [P-T-1]) $S_3(S^3) = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, and the empty link is a generator of the module. In particular $L = P_L(v, z)T_1^2$, where $P_L(v, z)$ is the Jones-Conway (Homflypt) polynomial of $L$.

The computations for $S^3$ and $S^1 \times D^2$ are special cases of the general computation for the product of a surface and an interval, $F \times I$. The third skein module of $F \times I$ has a structure of algebra ($L_1 \cdot L_2$ denotes the link obtained by placing $L_1$ above $L_2$) and the result is reminiscent of the Poincaré-Birkhoff-Witt theorem on the universal enveloping algebra of a Lie algebra.

**Theorem IX.4.3** ([P-14]) $S_3(F \times I)$ is an algebra which, as an $R$ module, is a free module isomorphic to the symmetric tensor algebra, $SR\hat{\pi}^0$, where $\hat{\pi}^0$ denotes the conjugacy classes of nontrivial elements of $\pi_1(F)$.

**Theorem IX.4.4** ([Tu-3] [P-15]) $S_3(F \times I)$ is an involutory Hopf algebra.

$S_3(F \times I)$ can be interpreted as a quantization ([Ho-K] [Tu-3] [P-5] [Tu-2] [P-15], and $S_3(M)$ is related to the algebraic set of $SL(n, \mathbb{C})$ representations of the fundamental group of the manifold $M$, ([St-4] [St-5].

If $F$ is a non-orientable surface, then the twisted $I$-bundle over $F$ is an oriented 3-manifold ($M = F \times I$) and we have, as in the product case, the concept of a diagram of a link. This gives hope that the Homflypt skein
module could also be computed in this case. M. Mroczkowski achieved this in the case of $F = \mathbb{R}P^2$ proving, in particular, that the module is free \cite{Mro}.

**Theorem IX.4.5** (\cite{Mro})

$S_3(\mathbb{R}P^2 \times I)$ is freely generated by standard oriented unlinks\footnote{The term unlink can be a little misleading as e.g. $\text{lk}(L_2) = \frac{1}{2}$ in $\mathbb{R}P^3$ (or $(\mathbb{R}P^2 \times I) = \mathbb{R}P^3 \# D^3$), but we follow the terminology in \cite{Mro}.} $\tilde{L}_n$. $\tilde{L}_n$ is the link composed of $n$ copies of noncontractible simple closed curves on $\mathbb{R}P^2$ as presented in Fig. 4.2 ($\mathbb{R}P^2$ is presented as a disk with antipodal points identified). $L_0$ is the empty link (if we allow it; otherwise, as in \cite{Mro}, one can take the trivial knot as the basic unlink $L_0$).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{unlink.png}
\caption{The standard oriented unlink $\tilde{L}_5$}
\end{figure}

Theorems 4.3 and 4.5 support the following conjecture

**Conjecture IX.4.6**

If $M$ is a submanifold of a rational homology sphere and it does not contain a closed, oriented incompressible surface then its Homflypt skein module $S_3(M)$ is free and isomorphic to the symmetric tensor algebra over module spanned by conjugacy classes of nontrivial elements of the fundamental group, $S_3(M) = \mathbb{S}R\tilde{\pi}^0$.

The idea of proving Conjecture 4.6 for lens spaces $L(n, 1)$ is sketched in \cite{L-P}, but the paper is not finished yet.

Skein modules can be defined for relative links (properly embedded 1-dimensional manifolds) in the same manner as for absolute links (that is links composed only of closed curves). We assume usually that relative links are considered modulo ambient isotopy constant on the boundary. To define
a relative skein module we should choose boundary points (or, at least, if the boundary is connected, the number of points\(^5\)). As before, the accessible case is the product of a surface with boundary, \(F\), and the interval. The case \(S_3(F \times I, 2n)\) for \(F\) a disk is a skein module of tangles, closely related to an \(A\)-type Hecke algebra. It is known to be free of \(n!\) generators. For \(F\) being an annulus, we demonstrate after [L-P] that the Homflypt skein module is free and related to a \(B\)-type Hecke algebra.

**Theorem IX.4.7** Let \(F = F_{0,2}\) be an annulus with \(2n\) points, \(p_1,...,p_{2n}\) on one of its boundary components. We assume \(p_1,...,p_n\) are inputs and \(p_{n+1},...,p_{2n}\) are outputs. \(S_3(F \times I, 2n)\) denotes the relative Homflypt skein module with the chosen \(2n\) points.

(i) The module is an infinitely generated free \(\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]\)-module.

(ii) Let \(R_0 = S_3(F \times I)\) denote a ring which is the skein module of the product of an annulus and the interval. Then \(S_3(F \times I, 2n)\) is a free \(R_0\) module with basis, \(B(F_{0,2}, 2n)\) described in subsequent definitions ([L-P]).

Sketch of the proof.
Our main tool is the Hoste-Kidwell-Turaev theorem (Theorem 4.3 for an annulus). First we need several definitions including the notion of “half-way” Hecke algebra of type \(B\), \(H_n(p, q; \infty)\). In what follows we think about \(F_{0,2} \times I\) as the exterior of the (thickened) \(z\)-axis in \(D^2 \times I\). The first, special string of a braid is represented by the \(z\)-axis

**Definition IX.4.8**

(i) The \(B\)-type Artin braid group \(B_{1,n} = \{(t, \sigma_1, ..., \sigma_{n-1} \mid t\sigma_1t\sigma_1 = \sigma_1t\sigma_1, t\sigma_i = \sigma_it \text{ for } i > 1, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |j - i| > 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}. It is a normal subgroup of a (type \(A\)) braid group \(B_{n+1}\). The braid relation \(t\sigma_1\sigma_1 = \sigma_1t\sigma_1t\) is illustrated in Fig. 4.3.

\(^5\)We give more detailed discussion of this point in the case of the relative Kauffman bracket skein module; see Section 7.
(ii) The “half-way” Coxeter group $W_n(\infty) = B_{1,n}/(\sigma_i^2 = 1) = \{t, s_1, ..., s_{n-1} : \)

$t s_1 t s_1 = s_1 t s_1 t$, $t s_i = s_i t$ if $i > 1$, $s_i s_j = s_j s_i$ if $|i - j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i^2 = 1\}$. We use $s_i$ for the image of $\sigma_i$ in the quotient space. The group $W_n(\infty)$ (which is “half-way” between Coxeter group and related Artin group because we do not assume the relation $t^2 = 1$) is well understood as it can be interpreted as a group of weighted (or framed) permutations. Thus we know that $W_n(\infty) = \mathbb{Z}^n \rtimes S_n$ (semidirect product) so we have another familiar presentation coming from this semi-direct product decomposition: $W_n(\infty) = \{s_1, ..., s_{n-1}, v_0, v_1, ..., v_{n-1} \mid s_i s_j = s_j s_i$ if $|i - j| > 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i^2 = 1, v_i v_j = v_j v_i, s_i v_j s_i = v_j$ for $j \neq i - 1, j$, $s_i v_{i-1} s_i = v_i, s_i v_i s_i = v_{i-1}\}$. When going from the second presentation to the first we have $v_i = s_i s_{i-1} ... s_1 s_{i-1} s_i$ ($v_3$ is presented in Fig. 4.4). When we think of $W_n(\infty)$ as of “framed” permutations then we interpret $v_i$ as one positive twist on the framing of the $i + 1$ string.

(iii) For $R = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}]$ we define $H(p, q; \infty)$ (the “half-way” Hecke alge-
Skein modules\) as the quotient of the group ring \(RB_{1,n}\) by the quadratic relations:
\[
\sigma_i^2 = p\sigma_i + q.
\]
We can write, therefore, the presentation of \(H(p,q;\infty)\) as follows:
\[
H_n(p,q;\infty) = \{ t, g_1, \ldots, g_{n-1} \mid tg_1t g_1 = g_1tg_1t, tg_i = g_it if i > 1, g_ig_j = g_jg_i if |i-j| > 1 \text{ and } g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, g_i^2 = pg_i + q \}.
\]
We use \(g_i\) for the class of \(\sigma_i\). The “half-way” Hecke algebra \(H(p,q;\infty)\) is a deformation of \(RW_n(\infty)\) with respect to the equations \(s_i^2 = 1\), where \(R = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}]\) and \(RW_n(\infty)\) is the group algebra of \(W_n(\infty)\).

Proposition IX.4.9 (Natural generating set; basis)

(i) Every element of \(W_n(\infty)\) can be uniquely written in the normal form. We give an inductive definition. For \(n = 1\) the normal form is \(t^i, i \in \mathbb{Z}\). If normal words for \(W_n(\infty)\) are defined, then normal words for \(W_{n+1}(\infty)\) are of the form \(s_k s_{k-1} \ldots s_1 t^j s_1 \ldots s_n w, j \neq 0\) or \(s_is_{i+1} \ldots s_n w\) where \(w\) is a normal word in \(W_n(\infty)\) and \(k \leq n, 1 \leq i \leq n + 1\). In particular these words form a basis of \(RW_n(\infty)\).

(ii) The words from (i) where we take \(g_i\) in place of \(s_1\) form a generating set of \(H(p,q;\infty)\). In Figure 4.5 we illustrate \(g_2g_1 t^2 g_1 g_2 g_3 g_4 g_1 t^{-1} g_1 g_2 g_3 g_1 g_2 g_1\).

(iii) The generating set described in (ii) is a basis of \(H(p,q;\infty)\).

Sketch of the proof.

Part (i) follows easily from our “framed” permutation interpretation of \(RW_n(\infty)\). Part (ii) can be proved by induction on the number of letters in a braid (quadratic relations allows us to reduce the number of letters (crossings) or move us closer to a normal form). To prove that our generating set is in fact a basis of \(H(p,q;\infty)\) is more difficult but we can do it very
quickly if we assume the theorem of Kidwell-Hoste and Turaev about the structure of the skein module of the solid torus. It is the same trick used by Morton and Traczyk \cite{Mo-Tr-2} to find the skein module of a tangle when the existence of the Homflypt polynomial of links in $S^3$ is established.

We show that our generators from (ii) are linearly independent by showing that certain form is nondegenerated. It is more convenient here to consider a slightly adjusted version of the “half-way” Hecke algebra, $H_n(v, z; \infty)$ with the relation $v^{-1}s_i^2 = zs_i + v$, which is also a Homflypt skein module relation. Let $e_1, e_2, ...$ denote our generating set of $H_n(v, z; \infty)$ from (ii). We prove that the above elements are linearly independent. Let $\sum_i a_i(v, z)e_i = 0$ and $j$ be an index in the sum for which $z$ degree is the smallest. Consider the bilinear function:

$$\Phi : H_n(v, z; \infty) \times H_n(v, z; \infty) \to \mathcal{S}_3(S^1 \times D^2)$$

Where $\Phi(\alpha, \beta)$ is obtained by composing $\alpha$ and $\beta$ first, then closing the result and filling up the braid axis\(^6\). The result is a link in the solid torus (with given $I$-bundle structure) and we consider its value in $\mathcal{S}_3(S^1 \times D^2)$ as $\Phi(\alpha, \beta)$. Now consider $\Phi(\sum_i a_i(v, z)e_i, e_j^{-1})$, in the standard basis of $\mathcal{S}_3(S^1 \times D^2)$, composed of layered families of torus knots of type $(2, n)$. We can see easily that the unique coefficient of the basis element with the lowest power of $z$ is given by $\Phi(a_j(v, z)e_j, e_j^{-1}) = a_j(v, z)(v^{-1} - v)^nz^{-n}$. Thus from $\sum_i a_i(v, z)e_i = 0$ follows $a_j(v, z) = 0$ and $a_i(v, z) = 0$ for any $i$. These complete the proof of Proposition 4.9.

To complete a proof of Theorem 4.7 notice that in $\mathcal{S}_3(F_{0, 2} \times I, 2n)$ we start from $n$ arcs and some number of closed components in $F_{0, 2} \times I$ and we assume that arcs have orientations from the level $F_{0, 2} \times \{1\}$ to $F_{0, 2} \times \{0\}$ (we call such relative links – string links). Using Homflypt skein relations we can reduce every string link to a linear combination of elements of the generating set from Proposition 4.9(ii) (over the ring $R_0$ as we have to allow closed curves). We organize our reduction along some properly chosen complexity: for arcs, distance from the normal form, and for closed curves, how far they are from the basic curves of $\mathcal{S}_3(F_{0, 2} \times I)$. The fact that our braids from (ii) are linearly independent allows us to complete the proof of Theorem 4.7.

As mentioned at the end of Section 3, torsion in skein modules is often related to surfaces in a 3-manifold. We illustrate this for the Homflypt polynomial in one simple instance: a nonseparating 2-sphere.

\(^6\)Equivalently we consider braids up to conjugation and Markov moves.
Proposition IX.4.10 Let $S^2$ be a nonseparating 2-sphere in an oriented 3-manifold $M$ and $L$ a link cutting $S^2$ in exactly one point. Then \((\frac{v^{-1}-v}{z})^2 - 1) L = 0\) in $S_3(M)$.

Proof: Consider the link $L'$ composed of $L$ and meridian circle, $C$, around it (Hopf component), Fig. 4.6. On one hand $C$ can be taken out of $L$ using the “other side” of $S^2$ (thus $C$ is a trivial circle in $M - L$). Therefore $L' = v\frac{v^{-1}-v}{z} L$ in $S_3(M)$. On the other hand we can use skein relation to compute $L'$ and get $L' = v^2(\frac{v^{-1}-v}{z}) L + vz L$, Fig. 4.6. Therefore $\frac{v^{-1}-v}{z} L = (v^2(\frac{v^{-1}-v}{z}) + vz)L$ and $(frac{v^{-1}-v}{z} - vz(v^2 - 1) + vz)L = 0$, and finally $(\frac{v^{-1}-v}{z})^2 - 1) L = 0$. □

IX.5 Homotopy skein module

If we ignore self-crossings in the third (Homflypt) skein module we obtain its simplified version which we call the homotopy skein module $[\text{H-P-2}]$.

Definition IX.5.1 Let $M$ be an oriented 3-manifold, $\mathcal{L}$ the set of oriented links in $M$, $R$ a commutative ring with unit, and $z$ a fixed element of $R$. The homotopy skein module $\mathcal{H}S(M, R, z)$ is defined as the quotient of the free $R$ module over $\mathcal{L}$ and the submodule $\mathcal{H}$ generated by skein relations $L_+ - L_-$ for a selfcrossing and $L_+ - L_- - zL_0$ for a crossing between different components of $L_\pm$. That is, $\mathcal{H}S(M, R, z) = RL/\mathcal{H}$.

If $M = F \times [0, 1]$, where $F$ is an oriented surface, then we have a multiplication in the module: $L_1 \cdot L_2$ denotes the link obtained by placed $L_1$ above $L_2$ in $F \times [0, 1]$ with respect to “height” $[0, 1]$. If we assume that $\mathcal{L}$ contains the empty knot then $\mathcal{H}S(M; R, z)$ becomes an algebra (in $[\text{H-P-2}]$ a skein module with the empty set allowed is called a reduced skein module).

Theorem IX.5.2 ([\text{H-P-2}, \text{Tu-2}])

(a) $\mathcal{H}S(F \times [0, 1]; R, z)$ is an algebra which as an $R$ module is a free module isomorphic to the symmetric tensor algebra, $SR\tilde{\pi}$, where $\tilde{\pi}$ denotes the conjugacy classes of elements of $\pi_1(F)$. 

Fig. 4.6
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(b) \( \mathcal{HS}(F \times [0,1]; R, 1) \) is an \( R \)-algebra isomorphic to the universal enveloping algebra of the Goldman-Wolpert Lie algebra \( U(R\pi) \). Going “backwards” we can describe the Lie bracket \( [\alpha, \beta] \) on \( R\pi \) as a projection on \( F \) of \( K_\alpha \cdot K_\beta - K_\beta \cdot K_\alpha \), where \( K_\alpha \) and \( K_\beta \) are knots representing curves \( \alpha \) and \( \beta \) respectively.

(c) \( \mathcal{HS}(F \times [0,1]; R, z) \) is a Drinfeld-Turaev quantization of the Goldman-Wolpert Poisson algebra of curves on \( F \).

The following \( q \)-analogue of the homotopy skein module is of considerable interest.

**Definition IX.5.3** Let \( M \) be an oriented 3-manifold, \( \mathcal{L} \) the set of oriented links in \( M \), \( R \) a commutative ring with unit, \( z \) a fixed element of \( R \), and \( q \) an invertible element of \( R \). The \( q \) homotopy skein module \( \mathcal{HS}(M; R, q, z) \) is defined as the quotient of the free \( R \)-module over \( \mathcal{L} \) and the submodule \( \mathcal{H}^q \) generated by skein relations \( L_+ - L_- \) for a selfcrossing and \( q^{-1}L_+ - qL_- - zL_0 \) for a crossing between different components of \( L_\pm \). That is, \( \mathcal{HS}(M; R, q, z) = R\mathcal{L}/\mathcal{H}^q \).

We use the notation \( \mathcal{HS}^q(M) \) for \( \mathcal{HS}(M; Z[q^{\pm 1}, z], q, z) \). For \( M = S^3 \) the skein module is freely generated by trivial links and the value of a link in the module depends only on the number of components and linking numbers between components (linking matrix) [P-6, P-12]. The skein module can be described using the dichromatic polynomial of the associated weighted graph: vertices correspond to link components and weights of edges correspond to linking numbers. An interesting feature of the skein module is that it distinguishes some links with the same Jones-Conway (Homflypt) polynomial.

**Example IX.5.4**

The 3-component links in Fig. 5.1 have the same Jones-Conway polynomial [Bi-2] but they are different in \( \mathcal{HS}^q(M) \). Namely \( L_1 = -(1 + q^2 + q^4 + q^6)z^2T_1 + (q^{-1} + q + q^3 + q^5 - q^7)zT_2 + q^6T_3 \) and \( L_2 = -(q^4 + 2q^6 + q^8)z^2T_1 + (q + 2q^3 + 2q^5 - q^7 - q^9)zT_2 + q^6T_3 \).
We generalize, partially the case of $M = S^3$ or rather $M = D^3 = D^2 \times I$ into the case of $M = F \times I$. Let $\mathcal{L}^h$ denote the set of homotopy links in $M$, that is, $\mathcal{L}^h = \mathcal{L}/(L_+ - L_-)$ where relations are yielded by self-crossings. Let $\hat{\pi}$ denote the set of conjugacy classes in $\pi_1(M)$, or equivalently the set of homotopy knots in $M$. Choose some linear ordering, denoted by $\leq$, of elements of $\hat{\pi}$. Given a homotopy link $L = \{K_1, K_2, \ldots, K_n\}$ in $F \times I$, we shall say that $L$ is a layered homotopy link with respect to the ordering of $\hat{\pi}$ if each $K_i$ is above $K_{i+1}$ in $F \times I$ and $K_i \leq K_{i+1}$. Let $\mathcal{B}$ be the set of all layered homotopy links with respect to the ordering of $\hat{\pi}$, including the empty link.

**Theorem IX.5.5**

(i) The $q$-homotopy skein module $\mathcal{HS}^q(F \times I)$ is generated by $\mathcal{B}$.

(ii) The homotopy skein module $\mathcal{HS}(F \times I)$ is freely generated by $\mathcal{B}$: \cite{H-P-1}.

(iii) If $\pi_1(F)$ is abelian then the $q$-homotopy skein module $\mathcal{HS}^q(F \times I)$ is freely generated by $\mathcal{B}$.

For the proof we refer to \cite{P-6, P-30}.

If the Euler characteristic, $\chi(F)$, is negative then the $q$-homotopy skein module has torsion. This is described in Theorem 5.6. On the other hand we know, Theorem 5.2(a), that for $q = \pm 1$ the module is free.

**Theorem IX.5.6** Let $F$ be a surface (not necessary compact) which contains a disc with 2 holes or a torus with a hole embedded $\pi_1$-injectively; equivalently, $\pi_1(F_0)$ is not abelian for a connected component $F_0$ of $F$ (in the compact connected case this means that $\chi(F) < 0$). Then
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(a) $\mathcal{HS}^q(F \times I)$ has torsion.

(b) Let $\alpha : R\mathcal{B} \to \mathcal{HS}^q(F \times I)$ be an $R$-homomorphism given by $\alpha(L) = L$. Then $\ker \alpha \neq \{0\}$.

U. Kaiser generalized Theorem 5.6, fully characterizing oriented 3-manifolds for which $q$-homotopy skein module has torsion $[\text{Kai}^2]$.

IX.6 The Kauffman bracket skein module

The skein module based on the Kauffman bracket relation is, so far, the most extensively studied object in algebraic topology based on knots. We describe in this section the basic properties of the Kauffman Bracket Skein Module (KBSM) and list manifolds for which the structure of the module is known. In the seventh section, we give the detailed proof of the structure of KBSM of a 3-manifold which is an interval bundle over a surface. In the ninth section we analyze the structure of algebra in the case of a surface times an interval, we introduce the notion of the skein algebra of a group and investigate the relation with representations of the group to $\text{SL}(2, \mathbb{C})$.

Definition IX.6.1 ($[\text{P-5}, \text{H-P-3}]$)

Let $M$ be an oriented 3-manifold, $\mathcal{L}_{fr}$ the set of unoriented framed links in $M$ (including the empty knot, $\emptyset$), $R$ any commutative ring with identity and $A$ its invertible element. Let $S_{2,\infty}$ be the submodule of $R\mathcal{L}_{fr}$ generated by skein expressions $L_+ - AL_0 - A^{-1}L_\infty$, where the triple $L_+, L_0, L_\infty$ is presented in Fig.6.1, and $L \sqcup T_1 + (A^2 + A^{-2})L$, where $T_1$ denotes the trivial framed knot. We define the Kauffman bracket skein module, $S_{2,\infty}(M; R, A)$, as the quotient $S_{2,\infty}(M; R, A) = R\mathcal{L}_{fr}/S_{2,\infty}$.

![Fig. 6.1.](image-url)
Notice that $L^{(1)} = -A^3L$ in $S_{2,\infty}(M; R, A)$; we call this the framing relation. We use the simplified notation $S_{2,\infty}(M)$ for $S_{2,\infty}(M; Z[A^\pm 1], A)$.

We list below several elementary properties of KBSM including description of the KBSM of any compact 3-manifold using generators and relators.

**Proposition IX.6.2**

1. An orientation preserving embedding of 3-manifolds $i : M \to N$ yields the homomorphism of skein modules $i^* : S_{2,\infty}(M; R, A) \to S_{2,\infty}(N; R, A)$. The above correspondence leads to a functor from the category of 3-manifolds and orientation preserving embeddings (up to ambient isotopy) to the category of $R$-modules (with a specified invertible element $A \in R$).

2. (i) If $N$ is obtained from $M$ by adding a 3-handle to it (i.e. capping off a hole), and $i : M \to N$ is the associated embedding, then $i^* : S_{2,\infty}(M; R, A) \to S_{2,\infty}(N; R, A)$ is an isomorphism.

(ii) If $N$ is obtained from $M$ by adding a 2-handle to it, and $i : M \to N$ is the associated embedding, then $i^* : S_{2,\infty}(M; R, A) \to S_{2,\infty}(N; R, A)$ is an epimorphism.

3. If $M_1 \sqcup M_2$ is the disjoint sum of 3-manifolds $M_1$ and $M_2$ then

$$S_{2,\infty}(M_1 \sqcup M_2; R, A) = S_{2,\infty}(M_1; R, A) \otimes S_{2,\infty}(M_2; R, A).$$

4. (Universal Coefficient Property)

Let $r : R \to R'$ be a homomorphism of rings (commutative with 1). We can think of $R'$ as an $R$ module. Then the identity map on $L_{fr}$ induces the isomorphism of $R'$ (and $R$) modules:

$$\overline{r} : S_{2,\infty}(M; R, A) \otimes_R R' \to S_{2,\infty}(M; R', r(A)).$$

5. Let $(M, \partial M)$ be a 3-manifold with the boundary $\partial M$, and let $\gamma$ be a simple closed curve on the boundary. Let $N = M_\gamma$ be the 3-manifold obtained from $M$ by adding a 2-handle along $\gamma$. Furthermore let $L_{fr}^{gen}$ be a set of framed links in $M$ generating $S_{2,\infty}(M; R, A)$. Then $S_{2,\infty}(N; R, A) = S_{2,\infty}(M; R, A)/J$, where $J$ is the submodule of $S_{2,\infty}(M; R, A)$ generated by expressions $L - sl_\gamma(L)$, where $L \in L_{fr}^{gen}$ and $sl_\gamma(L)$ is obtained from $L$ by sliding it along $\gamma$ (i.e. handle sliding).
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(6) Let $M$ be an oriented compact manifold and consider its Heegaard decomposition (that is $M$ is obtained from the handlebody $H_n$ by adding 2- and 3-handles to it, then $M$ has a presentation as follows: generators of $S_{2,\infty}(M; R, A)$ are generators of $S_{2,\infty}(H_n; R, A)$ and relators are yielded by 2-handle slidings.

Proof:

(1) $i_\ast$ is well defined because if framed links $L_1$ and $L_2$ are ambient isotopic in $M$ then $i(L_1)$ and $i(L_2)$ are ambient isotopic in $N$. Furthermore any skein triple $L_+, L_0, L_\infty$ in $M$, is sent by $i$ to a skein triple in $N$. Finally $i(T_1)$ is a trivial framed knot in $N$. Notice that if $i_\ast: M \to N$ is an orientation reversing embedding then $i_\ast$ is a $\mathbb{Z}$-homomorphism and $i(Aw) = A^{-1}i(w)$.

(2) (i) It holds because the cocore of a 3-handle is 0-dimensional.\footnote{A manifold $N$ is obtained from an $n$-dimensional manifold $M$ by attaching a $p$-handle, $D^p \times D^{n-p}$, to $M$, if $N = M \cup_f D^p \times D^{n-p}$ where $f: \partial D^p \times D^{n-p} \to M$ is an embedding. $D^p \times \{0\}$ is a core of the handle and $\{0\} \times D^{n-p}$ is a cocore of the handle [R-S].}
(ii) It holds because the cocore of a 2-handle is 1-dimensional.

(3) This is a consequence of the well known property of short exact sequences, [Bl]:
If $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ are short exact sequences of $R$-modules then $0 \to A' \otimes B + A \otimes B' \to A \otimes B \to A'' \otimes B'' \to 0$ is a short exact sequence.

(4) The exact sequence of $R$ modules

$$S_{2,\infty}(R, A) \to RL_{fr} \to S_{2,\infty}(M; R, A) \to 0$$
leads to the exact sequence of $R'$ modules ([C-E], Proposition 4.5):

$$S_{2,\infty}(R, A) \otimes_R R' \to RL_{fr} \otimes_R R' \to S_{2,\infty}(M; R, A) \otimes_R R' \to 0.$$
(5) It follows from (2)(ii) because any skein relation can be performed in \(M\), and the only difference between KBSM of \(M\) and \(N\) lies in the fact that some nonequivalent links in \(M\) can be equivalent in \(N\); the difference lies exactly in the possibility of sliding a link in \(M\) along the added 2-handle (that is \(L\) is moving from one side of the cocore of the 2-handle to another).

(6) It follows from (5) and (2)(i).

\(\square\)

**Remark IX.6.3** The Universal Coefficient Property holds for all skein modules considered in this work. It is the case because we always have the isomorphism \(R\mathcal{L} \otimes_R R' \rightarrow R'\mathcal{L}\) and the epimorphism of submodules of skein relations \(S(R) \otimes_R R' \rightarrow S(R')\), thus the proof described in (5), based on “five lemma,” works for our skein modules.

In the next theorem we list manifolds for which the exact structure of the Kauffman bracket skein module has been computed.

**Theorem IX.6.4** ([P-5], [H-P-4], [H-P-5], [H-P-6], [Bu-2])

(a) \(S_{2,\infty}(S^3) = \mathbb{Z}[A^{\pm 1}]\), more precisely: \(\emptyset\) is the generator of the module and \(L = \langle L \rangle \) is the Kauffman bracket polynomial of a framed link \(L\).

(b) \(S_{2,\infty}(F \times [0,1])\) is a free module generated by links (simple closed curves) on \(F\) with no trivial component (but including the empty knot). Here \(F\) denotes an oriented surface (see also Theorems 5.1 and 5.9). This applies in particular to a handlebody, because \(H_n = P_n \times I\), where \(H_n\) is a handlebody of genus \(n\) and \(P_n\) is a disc with \(n\) holes.

(c) \(S_{2,\infty}(L(p,q))\) is a free \(R\) module and it has \([p/2]+1\) generators, where \([x]\) denotes the integer part of \(x\).

(d) \(S_{2,\infty}(S^1 \times S^2) = \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}[A^{\pm 1}]/(1 - A^{2i+4})\)

(e) The skein module of the complement of the torus knot of type \((k,2)\) is free.

(f) Let \(W\) be the classical Whitehead manifold, then \(S_{2,\infty}(W)\) is infinitely generated torsion free but not free.
In [H-P-5], (f) is proved for a large class of genus one Whitehead type manifolds, and the paper ends with an optimistic note: *for the classical Whitehead manifold it seems feasible to find the exact structure of $S_{2,\infty}(W)$, and we plan to address that in a future paper.*

We prove (b), with its generalizations, in the next section. The part (e) was generalized recently by Thang Le ([Le] April 2004) to the Kauffman bracket skein modules of the exteriors of 2-bridge knots.

**Theorem IX.6.5 (T.Le)** For a 2-bridge (rational) knot $K_{\frac{p}{m}}$ the skein module is the free $\mathbb{Z}[A^{\pm 1}]$ module with the basis $\{x^iy^j\}$, $0 \leq i, 0 \leq j \leq \frac{p-1}{2}$, where $x^iy^j$ denotes the element of the skein module represented by the link composed of $i$ parallel copies of the meridian curve $x$ and $j$ parallel copies of the curve $y$; see Fig. 6.2. Le’s theorem generalizes results in [Bu-3] and [B-L].

![Fig. 6.2](image_url)

One should compare here Le’s Theorem with the part (c) of computation of Hoste-Przytycki [H-P-4]. These two results are related by the fact that the lens space $L(p,m)$ is the double branched cover of $S^3$ branched along the 2-bridge knot $K_{\frac{p}{m}}$. We will discuss this in the monograph [P-30].
IX.7 KBSM and relative KBSM of $F \times I$ and $\hat{F} \times I$

The understanding of the Kauffman bracket skein module of the product of a surface and the interval is the first step to understanding KBSM of a general 3-manifold. Furthermore the case of $F \times I$ is relatively easy to understand because we can project links onto the surface and work with diagrams of links. This can be generalized to twisted $I$-bundles over $F$ and one can have reasonable hopes that the method can work for other 3-manifolds by projecting links to spines of 3-manifolds. The relative case is described in Theorem 7.10.

**Theorem IX.7.1** Let $M$ be an oriented 3-manifold which is either equal to $F \times I$, where $F$ is an oriented surface, or it equal to a twisted $I$ bundle over $F$ ($F\hat{\times}I$), where $F$ is an unorientable surface. Then the KBSM, $S_{2,\infty}(M; R, A)$, is a free $R$-module with a basis $B(F)$ consisting of links in $F$ without contractible components (but including the empty knot).

**Proof:** We will give here the proof of Theorem 5.1 which is based on the original proof of Kauffman on the existence of his bracket polynomial. Let $M$ be an oriented 3-manifold which is an $I$-bundle over a surface $F$. Let $B(F)$ consist of all links in $F$ which have no trivial components (including $\emptyset$). Furthermore each link is equipped with an arbitrary, but specific framing (to be concrete we can assume that if a knot in $F$ preserves the orientation of $F$ then we choose as its framing the regular neighborhood of $K$ in $F$ ("blackboard" framing), if $K$ is changing the orientation on $F$ then its regular neighborhood is a Möbius band so to get a framing we perform a positive half twist on it). Now one can quickly see that $B(F)$ is a generating set of $S_{2,\infty}(M; R, A)$. Namely every link in $M$ has a regular projection on $F$ and any link can be reduced by skein relations so that a projection has no crossings. Then another relation allows us to eliminate trivial components and finally the framing relations allow us to adjust framing. We will prove that $B(F)$ is a basis for $S_{2,\infty}(M; R, A)$. First we need however to consider the space of link diagrams (for a nonorientable surface $F$ the proof is still easy but requires great care).

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8Because $M$ is oriented therefore for $\gamma$ in $F$ changing orientation of $F$, the restriction of the $I$-bundle to $\gamma$ is a nontrivial bundle (Möbius band). For $\gamma$ preserving orientation of $F$, the bundle is trivial (an annulus).
Definition IX.7.2

(a) A link diagram on $F$ is a 4-valent graph in $F$ (allowing loops without vertices) such that one corner of each vertex is marked. $F$ does not need to be oriented for this definition.

(b) Let $D$ be a set of link diagrams on $F$ (up to isotopy of $F$), and $RD$ the free module over $D$. The skein space of diagrams, $SD$ is defined as a quotient:

$$SD(F; R, A) = RD / \left( \bigotimes - A \bigotimes - A^{-1} \right) (\bigotimes D \sqcup T_1 + (A^2 + A^{-2})D).$$

Lemma IX.7.3

Let $B^d(F)$ denote the set of link diagrams in $F$ without vertices and without trivial components (but allowing $\emptyset$). We can identify $B^d(F)$ with the set $B(F)$ with framing ignored. $B^d(F)$ is a subset of the set of link diagrams, so we have a homomorphism $\phi : RB^d(F) \rightarrow SD(F; R, A)$ defined by associating to a link, $\gamma \in B^d(F)$, in $F$ its class in $SD(F; R, A)$.

Then $\phi$ is an isomorphism.

Proof: For any $D \in D$ we can use the first relation to eliminate all crossings, and the second to eliminate trivial components of $D$. Thus $\phi$ is an epimorphism.

To show that it is a monomorphism we will construct the inverse map, $\psi$.

First we define a map $\hat{\psi} : RD \rightarrow RB^d(F)$. Let $D \in D$. We define $\hat{\psi}(D)$ as follows:

Choose any ordering $p_1, ..., p_n$ of crossings of $D$, and use the formula $D_{\bigotimes}^{p_i} = AD_{\bigotimes}^{p_i} + A^{-1}D_{\bigotimes}^{p_i}$, for each crossing, until all crossings are eliminated. The upper index denotes the crossing at which we perform a smoothing (crossing elimination). The result does not depend on the order of the crossings since we can make any transposition of adjacent (with respect to ordering) pairs and get the same result:

$$(D_{\bigotimes}^{p_i})_{\bigotimes}^q = A(D_{\bigotimes}^{p_i})_{\bigotimes}^q + A^{-1}(D_{\bigotimes}^{p_i})_{\bigotimes}^q =$$

$$A^2(D_{\bigotimes}^{p_i})_{\bigotimes}^q + (D_{\bigotimes}^{p_i})_{\bigotimes}^q + (D_{\bigotimes}^{p_i})_{\bigotimes}^q + A^{-2}(D_{\bigotimes}^{p_i})_{\bigotimes}^q$$

and

$$(D_{\bigotimes}^{q})_{\bigotimes}^{p_i} = A(D_{\bigotimes}^{q})_{\bigotimes}^{p_i} + A^{-1}(D_{\bigotimes}^{q})_{\bigotimes}^{p_i} =$$
\[ A^2(D_2^q)^p + (D_2^q)^p + (D_q^3)^p + A^{-2}(D_q^3)^p \]

After smoothing all crossings we eliminate trivial components by the relation 
\( D ∪ T_1 = (-A^2 - A^{-2})D \) (there is no ambiguity in the reduction). Thus \( D \) is uniquely expressed as a linear combination of elements of \( B^d(F) \), and we define \( \hat{\psi}(D) \) as this linear combination (which lies in \( RB^d(F) \)). Therefore \( \hat{\psi} \) is well defined. Now \( \hat{\psi} \) descends to \( \psi : SD(F; R, A) → RB^d(F) \) because \( \hat{\psi}(\bigtimes - A \bigtimes - A^{-1}) = 0 \) and \( \hat{\psi}(D ∪ T_1 + (A^2 + A^{-2})D) = 0 \). Now, obviously, \( \psiφ = Id \), thus \( φ \) is a monomorphism. □

Our goal is to prove that \( B(F) \) is a basis of the Kauffman bracket skein module \( S_{2,∞}(M; R, A) \), where \( M \) is an oriented 3-manifold which is an \( I \)-bundle over a surface \( F \). Because we would like to consider the case of orientable and unorientable surface simultaneously, it is convenient to consider half-integer framings of links, that is, to allow embedded Möbius bands. This suggests the following definition.

**Definition IX.7.4** Let \( M \) be any oriented 3-manifold, \( \mathcal{L}_{fr} \) the set of embeddings of annuli and Möbius bands in \( M \) (up to an ambient isotopy of \( M \)) and \( R \) a commutative ring with identity with a chosen invertible element \( \bar{A} \) (we define \( A = -\bar{A}^2 \) and we will often write \( \sqrt{-\bar{A}} \) for \( A \)). Let \( R\mathcal{L}_{fr} \) denote a free \( R \) module over \( \mathcal{L}_{fr} \) and let \( \bar{S}_{2,∞} \) denote the submodule of \( R\mathcal{L}_{fr} \) generated by expressions \( L_+ - AL_- - A^{-1}L_∞ \), and \( L^{1/2} - (\sqrt{-\bar{A}})^3L \), where \( L^{1/2} \) denotes \( L \) with its framing twisted by a half twist in a positive direction. As before, for convenience, we allow the empty knot, \( ∅ \), and add the relation 
\( T_1 = (-A^2 - A^{-2})∅ \).
Then we define \( \bar{S}_{2,∞}(M, R, \bar{A}) = R\mathcal{L}_{fr}/\bar{S}_{2,∞} \).

Consider the \( R \)-homomorphism \( g : SD(F; R, A) → \bar{S}_{2,∞}(M, R, A) \) defined on the basic elements \( γ \in B^d(F) \) by \( g(γ) = γ^{fr} \) where \( γ^{fr} \) is a framed link obtained from \( γ \) by giving it the blackboard framing (it may be an annulus or a Möbius band). Using our skein relations, in a similar manner as before, we see that \( g \) is an epimorphism. If \( D \) is any marked diagram we can describe the framed link \( g(D) \) as follows: we resolve every crossing of \( D \) according to the rule given in Fig. 7.1 and giving the link \( g(D) \) the blackboard framing (the orientation of \( M \) in a neighborhood of the crossing should agree with that of \( R^3 \) from Fig. 7.1).
Consider now the following lemma concerning Reidemeister moves on diagrams.

**Lemma IX.7.5** Consider the following moves $\bar{R}_1, \bar{R}_2, \bar{R}_3$ on marked diagrams. In $SD(F; \bar{R}, A)$ they satisfy:

1. $(\bar{R}_1) \quad \boxtimes = -A^3 \Rightarrow A \Uparrow O + A^{-1} \left( A(-A^2 - A^{-2}) + A^{-1} \right) = -A^3$.
2. $(\bar{R}_2) \quad \bar{R}_2(D) = D$, where $\bar{R}_2(\boxtimes) = \boxtimes$.
3. $(\bar{R}_3) \quad \bar{R}_3(D) = D$, where $\bar{R}_3(\boxtimes) = \boxtimes$.

**Proof:**

1. $(\bar{R}_1)$
   
   \[
   \begin{align*}
   \boxtimes &= A \Uparrow O + A^{-1} \left( A(-A^2 - A^{-2}) + A^{-1} \right) = -A^3, \\
   \boxtimes &= A^{-1} \Uparrow OA \left( A + A^{-1}(-A^2 - A^{-2}) = -A^3 \right).
   \end{align*}
   \]

2. $(\bar{R}_2)$
   
   \[
   \begin{align*}
   \boxtimes &= A \otimes + A^{-1} \boxtimes = A(\otimes + A^{-1} \boxtimes) + A^{-1}(A \otimes + A^{-1} \boxtimes) \\
   &= (A^2 + AA^{-1}(-A^2 - A^{-2}) + A^{-2}) \otimes + (\otimes + A^{-1} \boxtimes).
   \end{align*}
   \]

3. $(\bar{R}_3)$
   
   \[
   \begin{align*}
   \boxtimes &= A \otimes + A^{-1} \boxtimes = A \otimes + A^{-1} \boxtimes = \boxtimes. \quad \text{We use here the invariance under $\bar{R}_2$ moves.}
   \end{align*}
   \]

$\square$

To use the lemma in the proof that $g$ is a monomorphism, we need a variant of Reidemeister’s theorem for marked diagrams:

**Proposition IX.7.6** Let $\hat{g} : D \rightarrow \mathcal{L}_{fr}$ be a map given by Fig. 5.1. Then two marked diagrams, $D_1$ and $D_2$, represent the same framed link, $\hat{g}(D_1) = \hat{g}(D_2)$. 

![Diagram](image-url)
if and only if one can go from $D_1$ to $D_2$ using Reidemeister moves $R_i^{±1}$ and an isotopy of $F$, and additionally, for corresponding link components of $D_1$ and $D_2$, their Tait numbers are the same. One should notice here that for a knot diagram the Tait number is independent on orientation of the knot. Precisely for a knot diagram $D$ we define $\text{Tait}(D)=\sum_p \text{sgn}(p)$, where $\text{sgn}(\bigotimes) = 1$ and $\text{sgn}(\bigotimes) = -1$.

Proof: The proposition can be deduced from the classical Reidemeister theorem and the result from the PL topology; Theorem 6.2 in [Hud]9 □

Our goal is to show that the epimorphism $g : SD(F; \bar{R}, A) \to \mathcal{S}_{2,\infty}(M, \bar{R}, \bar{A})$ is an isomorphism. We use Lemma 7.5 and Proposition 7.6 to construct the map inverse to $g$. Let $\hat{h} : RL_f \to SD(F; \bar{R}, A)$ be a homomorphism defined as follows: choose a representative of a link $L \in L_f$ which has a regular projection on $F$. Let $D_L$ be a marked diagram on $F$ constructed as in Fig. 5.1, and let $t(L)$ be the number (possibly half-integer) of positive twists which should be performed on the blackboard framing of $D_L$ to get the framing of $L$. Then we define $\hat{h}(L) = (-A^3)^{t(L)} D_L$. $\hat{h}(L)$ is well defined by Lemma 7.5 and Proposition 7.6. Furthermore, $\hat{h}(L_+ - AL_+ - A^{-1}L_\infty) = 0$, $\hat{h}(L \sqcup T_1 + (A^2 + A^{-2})L) = 0$ and $\hat{h}(L^{1/2} - \sqrt{-A^2}L) = 0$ so $\hat{h}$ descends to $\hat{h} : \mathcal{S}_{2,\infty}(M; R, A) \to SD(F; \bar{R}, A)$. Of course $\hat{h} g = Id$ so $g$ is a monomorphism, as required. □

We can now complete the proof of Theorem 5.1. Because $g : SD(F; \bar{R}, A) \to \mathcal{S}_{2,\infty}(M, \bar{R}, \bar{A})$ is an isomorphism, therefore $g(B^d(F))$ is a basis of $\mathcal{S}_{2,\infty}(M, \bar{R}, \bar{A})$. On the other hand $B(F)$, whose elements may differ from elements of $g(B^d(F))$ only by framing, also forms a basis of $\mathcal{S}_{2,\infty}(M, \bar{R}, \bar{A})$. Thus they are linearly independent in $\mathcal{S}_{2,\infty}(M; R, A)$. Because $B(F)$ generates $\mathcal{S}_{2,\infty}(M; R, A)$ it is a basis of this module. The proof of Theorem 7.1 is completed.

As an immediate corollary of Theorem 7.1 we obtain the structure of KBSM of the projective space $RP^3$. This result was also obtained independently by J. Drobotukhina [Dr].

Corollary IX.7.7 $\mathcal{S}_{2,\infty}(RP^3; R, A) = R \oplus R$. As a basis of KBSM we can take $\emptyset$ and a generator of the fundamental group of $RP^3$.

9It follows from the theorem that if $C$ is a compact subset of a manifold $M$ and $F : M \times \{t\} \to M$ is the isotopy of $M$ then there is another isotopy $\hat{F} : M \times \{t\} \to M$ such that $F_0 = \hat{F}_0$, $F_1/C = \hat{F}_1/C$ and there exists a number $N$ such that the set $\{x \in M \mid F/\{x\} \times (k/N, (k+1)/N) \text{ is not constant} \}$ sits in a ball embedded in $M$.  


Proof: By Proposition 6.2(i) $S_{2,\infty}(RP^3; R, A) = S_{2,\infty}(RP^3 - \text{int}(D^3); R, A)$ and $RP^3 - \text{int}(D^3)$ is equal to the twisted $I$-bundle over a projective plane $(RP^2 I)$. By Theorem 5.1, $S_{2,\infty}(RP^2 I; R, A)$ is a free $R$-module with basis $B(RP^2)$, which has two elements: the empty knot and the noncontractible curve on $RP^2$. □

One can generalize Theorem 7.1 to relative skein modules, as long as boundary points of relative links are on the same boundary component of a manifold.

**Definition IX.7.8 (Relative Kauffman Bracket Skein Module)**

Let $x_1, x_2, \ldots, x_{2n}$ be a set of $2n$ (framed) points in $\partial M$, where $M$ is an oriented $3$-manifold. Let $L_{fr}(n)$ be a family of relative framed links in $(M, \partial M)$ such that $\partial M = \partial L = \{x_i\}$, considered up to an ambient isotopy fixing $\partial M$. Let $R$ be a commutative ring with identity and $A$ its invertible element. Let $S_{2,\infty}(n)$ be the submodule of $RL_{fr}(n)$ generated by the Kauffman bracket skein relations. We define the Relative Kauffman Bracket Skein Module (RKBSM) as the quotient:

$$S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}; R, A) = RL_{fr}(n)/S_{2,\infty}(n)$$

We list below a few useful properties of relative skein modules:

**Proposition IX.7.9**  
(a) There is a functor from the category of oriented $3$-manifolds with $2n$ framed points on the boundary and orientation preserving embeddings (up to ambient isotopy fixed on the boundary) to the category of $R$-modules (with a specified invertible element $A \in R$). The functor sends an embedding $i : (M, \{x_i\}_{i=1}^{2n}) \to (N, \{y_i\}_{i=1}^{2n})$ into $R$-modules morphism $S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}; R, A) \to S_{2,\infty}(N, \{y_i\}_{i=1}^{2n}; R, A)$.

(b) Adding a 3-handle to $M$ (outside $x_i$) does not change the RKBSM, and adding a 2-handle is adding only relations to RKBSM (handle slidings yield relations); compare Proposition 6.2(2).

(c) The relative KBSM depends only on the distribution of boundary points $\{x_i\}$ among boundary components of $M$, but not on the exact position of $\{x_i\}$. In particular if $\partial M$ is connected, we can write shortly $S_{2,\infty}(M, n; R, A)$ instead of $S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}; R, A)$.

(d) The relative KBSM satisfies the Universal Coefficient Property, compare Proposition 6.2(4).
(e) For a disjoint sum of 3-manifolds we have:

\[ S_{2,\infty}(M_1 \sqcup M_2, \{x_i, y_i\}^{2n}; R, A) = S_{2,\infty}(M_1, \{x_i\}^{2n}; R, A) \otimes S_{2,\infty}(M_2, \{y_i\}^{2n}; R, A). \]

**Theorem IX.7.10** Let \( M = F \times I \) that is \( M = F \times \mathbb{I} \) or \( M = F \hat{\times} I \), then

(a) Let \( \partial F \neq \emptyset \) then \( S_{2,\infty}(M, \{x_i\}^{2n}; R, A) \) is a free \( R \)-module. Consider all \( x_i \) to lie on \( \partial F \times \{1\} \) then the basis of the module \( S_{2,\infty}(M, \{x_i\}^{2n}; R, A) \) is composed of relative links on \( F \) without trivial components.

(b) In the case of \( F_{g,0} \) closed surface of genus \( g \) \((F \neq S^2)\) the situation is more delicate so we stop on the following observation:

\[ S_{2,\infty}(F_{g,0} \hat{\times} I; \{x_i\}^{2n}; R, A) = S_{2,\infty}(F_{g,1} \hat{\times} I; \{x_i\}^{2n}; R, A)/\langle I \rangle \] where \( F_{g,1} = F_{g,0} - \text{int}(D^2) \) and assuming \( x_i \in \partial D^2 \); ideal \( \langle I \rangle \) is generated by moves in which arcs go above \( D^2 \).

**Proof:** The proof of (a) is the same as that of Theorem 7.1; as before relative link diagrams representing the same link are related by Reidemeister moves. In the case (b) it is no longer true as we need also handle sliding. \( F_{g,0} \hat{\times} I \) is obtained from \( F_{g,1} \hat{\times} I \) by adding the 2-handle along \( \partial D^2 \). Now (b) follows from Proposition 7.9(b). \( \square \)

In the case \( F \) is a closed surface the question whether \( S_{2,\infty}(F \hat{\times} I, \{x_i\}^{2n}; R, A) \) is free is open in general. If not all \( x_i \) lie on the same boundary component of \( F \times I \) the the skein module has a torsion in the case of \( F \) being a sphere or a torus; compare Section 6. We propose the following conjecture, which we are able to confirm only for \( F \) being a torus and \( n = 1 \).

**Conjecture IX.7.11** Let \( F \) be a closed surface and \( x_i \in F \times \{0\} \) for any \( i \), then the skein module \( S_{2,\infty}(F \times I, \{x_i\}^{2n}; R, A) \) is free.

**Corollary IX.7.12**

(a) \( S_{2,\infty}(D^2 \times I, \{x_i\}^{2n}; R, A) \) is a free \( R \) module of \( \frac{1}{n+1}(\binom{2n}{n}) \) free generators.

(b) \( S_{2,\infty}(\text{annulus} \times I, \{x_i\}^{2n}; R, A) \) is a free \( R[\alpha] \) module with \( \binom{2n}{n} \) free generators, where \( \alpha \) is represented by a longitude of the annulus.

**Proof:** Corollary 7.12 (a) describes, well known, module structure of the Temperley-Lieb algebra (basis having Catalan number of generators). (b) follows from work of Jones and Tom Dieck. We give here a self-sufficient proof. In lieu of Theorem 7.10, it suffices to count the crossless connections.
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of 2n points in the disc and annulus. We offer here an amazingly simple calculation for both cases simultaneously. Let \( C_n \) be the number of connections in the disc and \( D_n \) the number of connections in the annulus (all points \( x_i \) are on the “outside” circle of the annulus. Connections in the disc cut it into \( n+1 \) pieces; to get connection in the annulus we have to put a “table” (remove a disk) from \( D_n \); thus \( D_n = (n+1)C_n \). On the other hand any arc of a connection in the annulus has the first point (with respect to fixed orientation of the annulus), and any choice of \( n \) points leads to unique connection, for which given points are first. Therefore \( D_n = \binom{2n}{n} \) and thus \( C_n = \frac{1}{n+1}\binom{2n}{n} \).

We finish this section by offering the following very useful observation (compare [P-S-2]).

**Proposition IX.7.13** Consider a 3-manifold \( (M, \{x_i\}_{i=1}^{2n}) \) and let \( x_{2n+1} \) and \( x_{2n+2} \) lie on the same boundary component of \( M \). Consider the \( R \)-homomorphism of RKBSM

\[ i_\#: S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \rightarrow S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \]

generated by the identity map and with convention that \( i_\#(L) \) have \( x_{2n+1} \) connected to \( x_{2n+2} \) by a framed arc close to boundary (we push out of the boundary framed arc joining \( x_{2n+1} \) and \( x_{2n+2} \) in \( \partial M \)). Then \( i_\# \) is a monomorphism if one assumes that \( A^2 + A^{-2} \) is not an annihilator of any non-zero element of \( S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \) (i.e. \( (A^2 + A^{-2}) x = 0 \) \( \Rightarrow x = 0 \)).

**Proof:** Consider the \( R \)-homomorphism

\[ i'_\#: S_{2,\infty}(M, \{x_i\}_{i=1}^{2n+2}, R, A) \rightarrow S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \]

given by connecting \( x_{2n+1} \) and \( x_{2n+2} \) in \( \partial M \) and pushing it inside \( M \). Now clearly \( i'_\# i_\#(L) = (-A^2 - A^{-2})(L) \), thus \( i'_\# i_\#(u) = (-A^2 - A^{-2})(u) \) for any \( u \in S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \). Therefore \( i'_\# i_\# \) is a monomorphism iff \( A^2 + A^{-2} \) is not an annihilator of any non-zero element of \( S_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \). If \( i'_\# i_\# \) is a monomorphism then \( i_\# \) is a monomorphism. \( \square \)

**IX.8 Torsion in KBSM**

In all of the examples above the module is torsion free except in the case of \( S^1 \times S^2 \). In fact a non-separating \( S^2 \) in \( M \) always yields a torsion in
$S_{2,\infty}(M)$. It is enough to use the framing relation to see a torsion: Let $L$ be a framed link cutting a non-separating $S^2$ exactly in one point. We can twist $S^2$ twice, twisting also the framing of $L$ twice and then undo this by an isotopy of $M$. Thus $(A^6 - 1)L = 0$ in $S_{2,\infty}(M)$. It is less obvious that a separating $S^2$ can often yield a torsion.

**Conjecture IX.8.1** ([Kir]) If $M = M_1 \# M_2$, where $M_i$ is not equal to $S^3$, possibly with holes, then $S_{2,\infty}(M)$ has a torsion element.

We are able to prove Conjecture 8.1 only partially.

**Theorem IX.8.2** If $M_1$ and $M_2$ have first homology groups that are not 2-torsion groups, then the conjecture holds.

The proof will be divided into two lemmas, each of which is of considerable interest.

**Lemma IX.8.3** Consider the $\mathbb{Z}$-epimorphism $\phi$ from $RL^f$ onto $ZH_1(M,\mathbb{Z})$ given by $\phi(L) = (-1)^{\text{com}(L)} \sum_{L_i \in \sigma(L)} |L_i|$ where the sum is taken over all possible orientations of $L$ and $|L_i|$ denotes the homology class of the oriented link $L_i$. Furthermore $\phi(Aw) = -\phi(w)$ (i.e. $A \rightarrow -1$). Then $\phi$ descends to the map $\hat{\phi} : S_{2,\infty}(M) \rightarrow ZH_1(M,\mathbb{Z})$. More succinctly, although imprecisely, one can say that the Kauffman bracket skein module is more delicate than the homology up to orientation.

**Proof:** It suffices to show that $\phi$ sends skein expressions to zero. Because $A$ is sent to $-1$ then $\phi(L^{(1)} + A^3L) = 0$. Furthermore $\phi(L_+ - AL_0 - A^{-1}L_\infty) = \phi(L_+ + L_0 + L_\infty)$ and among three links $L_+, L_0, L_\infty$ exactly one has more components than the other two (hence twice as many orientations). Comparing these orientations, we see that all expressions reduce to zero (in $ZH_1(M,\mathbb{Z})$). □

**Lemma IX.8.4** (a) Let $(M, K) = (M_1, K_1) \# (M_2, K_2)$ and $(M, K') = (M_1, K_1)(\#, \sqcup)(M_2, K_2)$, where $(\#, \sqcup)$ means that we consider the disjoint sum of knots $K_1 \sqcup K_2$ in the connected sum $M_1 \# M_2$. Then $(A^4 - 1)(\mu K - K') = 0$ in the skein module $S_{2,\infty}(M)$, where $\mu = -A^2 - A^{-2}$.

(b) If $2|K_i| \neq 0$ in $H_1(M_i, \mathbb{Z})$, $i = 1, 2$, then $\mu K - K' \neq 0$ in $S_{2,\infty}(M)$. 

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Fig. 8.1.

\[ \text{Proof:} \]

(a) By using the “second” side of $S^2$ one gets the identity from Fig. 5.2. After reducing the right-hand-side of the equation, using skein relations, one gets in $S_{2,\infty}(M)$ the identity: $K = A^8K + (A^6 - A^2)K'$ and therefore $0 = (A^8 - 1)K + (A^6 - A^2)K' = A^2(A^4 - 1)((A^2 + A^{-2})K + K') = -A^2(A^4 - 1)(\mu K - K')$. Therefore for $f_2 = \mu K - K'$ we have $(A^4 - 1)f_2 = 0$ in $S_{2,\infty}(M)$, as required\(^{10}\).

(b) Consider the epimorphism $\hat{\phi}$ from Lemma x.8. $\hat{\phi}({\mu K - K'}) = -2\phi(K) - \phi(K')$ in $ZH_1(M, Z)$. If $K_i^\pm$ denotes two possible orientations of $K_i$ and $K_i^+ = K_i^+ \# K_2^+$, and $K_i^- = K_i^- \# K_2^-$, then $\hat{\phi}(\mu K - K') = -|K_1^+ \cup K_2^-| + |K_1^+ \cup K_2^-| = - |K_1^- \cup K_2^-| + |K_1^- \cup K_2^-|$, which is equal to zero if and only if $|K_1^+| = |K_1^-|$ or $|K_2^+| = |K_2^-|$.

\[\Box\]

**Example IX.8.5**  

(a) $S_{2,\infty}(S^1 \times S^1 \times S^1)$ has a torsion.

(b) The Kauffman bracket skein module of the double of the complement of the figure eight knot (Listing knot) has a torsion.

To prove (a) the idea of “symmetric homologies” described in Lemma 6.3 is crucial (“symmetric homology” is the image of $\hat{\phi}$ in $ZH_1(M, Z)$). For (b) the new idea is the observation by Bullock that there is an algebra homomorphism from the skein algebra ($A = -1$) to $C$ yielded by a homomorphism of the fundamental group of the manifold to $SL(2, C)$. Further we use the existence of hyperbolic structure on the complement of the Listing knot (it is probably the first connection between Jones type invariants and hyperbolic structures).

\[^{10}\] $f_2$ is related to Jones-Wenzl second idempotent in the Temperley-Lieb algebra.
IX.9 Kauffman bracket skein algebra of $F \times I$ and skein algebra of a group

We discuss in this section a possible algebra structure for a Kauffman bracket skein module. We can get the structure either by considering a special $M$ (e.g. $F \times I$ or $\mathbb{RP}^3$) or a special $A$ in $R$ (e.g. $A = \pm 1$). We allow the empty knot, $\emptyset$, in order to have a unit of an algebra.

Theorem IX.9.1 [Bullock] If $\mathcal{S}_{2,\infty}(F \times [0,1])$ is given the multiplication $L_1 \cdot L_2$ defined by placing $L_1$ above $L_2$ then the resulting algebra is finitely generated [Bu-3] and the minimal number of generators $\text{r}(\mathcal{S}_{2,\infty}(F \times [0,1]))$ is no more than $2^{\text{rank}(H_1(F))} - 1$.

More precisely we have:

Theorem IX.9.2 ([P-S-2])

(i) $\text{r}(\mathcal{S}_{2,\infty}(F \times I; Z[A^{\pm 1}], A)) = 2^{\text{rank}(H_1(F))} - 1$

(ii) If $R = Z[A^{\pm 1}, (A^2 + A^{-2})^{-1}]$ then

$$\text{r}(\mathcal{S}_{2,\infty}(F \times I; R, A)) = d + \left(\frac{d}{2}\right) + \left(\frac{d}{3}\right),$$

where $F$ is a disk with $d$ holes.

Theorem IX.9.3 $\mathcal{S}_{2,\infty}(L(2,1); R, A)$ has an algebra structure and as an algebra it is isomorphic to $R[\alpha]/(\alpha^2 - A^3 A^{-1} A^{-1})$.

Theorem IX.9.4 (i) $\mathcal{S}_{2,\infty}(F \times I; R, A)$ has no zero divisors, provided $R$ has no zero divisors.

(ii) The center of the algebra $\mathcal{S}_{2,\infty}(F \times I; R, A)$ is a subalgebra generated by the boundary components of $F$.

(iii) If $M$ is a twisted $I$ bundle over an unorientable surface $F$ (having an even number of projective planes as factors) then $\mathcal{S}_{2,\infty}(M; R, -1)$ has no zero divisors, provided $R$ has no zero divisors and $F$ is not a Klein bottle.

(iv) $\mathcal{S}_{2,\infty}(T^2 \times I; Z, -1)$ is a unique factorization domain.
Theorem IX.9.6

In what follows, use notation \( < x, y, ... > \) to denote noncommutative variables.

Theorem IX.9.7

\( S_{2,\infty}(F_{0, 4} \times I; R, A) = R[\alpha_1, \alpha_2, \alpha_3, \alpha_4] < x_1, x_2, x_3 > / (I_{0, 4}), \) where \( I_{0, 4} \) is an ideal generated by “A-bracket” relations:

\[
A^2 x_i x_{i+1} - A^{-2} x_{i+1} x_i = (A^4 - A^{-4}) x_{i+2} + (A^2 - A^{-2})(\alpha_i a_{i+1} + a_{i+2} \alpha_4),
\]

where \( i = 1, 2, 3 \) and indices are taken modulo 3, and by the “long relation”:

\[
A^4 x_1^2 + A^{-4} x_2^2 + A^4 x_3^2 - A^2 x_1 x_2 x_3 + A^2 x_1 (a_2 a_3 + a_1 a_4) + A^{-2} x_2 (a_1 a_3 + a_2 a_4) + A^{-2} x_3 (a_1 a_2 + a_3 a_4) + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_1 a_2 a_3 a_4 - A^4 - A^{-4} - 2
\]
Theorem IX.9.8

\[ S_{2,\infty}(F_{1,2} \times I; R, A) = R[a] < x, y, z, x', y', z' > /I_{1,2} \]

where \( I_{1,2} \) is the ideal generated by \( A \)-commutation relations:

\[
A xy - A^{-1}yx = (A^2 - A^{-2}) z, \quad Ayz - A^{-1}zy = (A^2 - A^{-2}) x,
\]

\[
A zx - A^{-1}xz = (A^2 - A^{-2}) y, \quad A xy' - A^{-1}y'x = (A^2 - A^{-2}) z',
\]

\[
A x' y - A^{-1}y'x' = (A^2 - A^{-2}) z' y, \quad A y' z - A^{-1}zy' = (A^2 - A^{-2}) x',
\]

\[
A z' x - A^{-1}xz' = (A^2 - A^{-2}) y', \quad xx' - x' x = yy' - y' y = z z' - z' z = 0,
\]

\[
A x' y' - A^{-1}y'x' = (A^2 - A^{-2})(z - A^{-1}(xy - x'y')), \quad A y' z' - A^{-1}z'y' = (A^2 - A^{-2}) x',
\]

\[
A z' x' - A^{-1}x'z' = (A^2 - A^{-2}) x' = A z' x - A^{-1}x'z' = (A^2 - A^{-2}) y',
\]

and the long relation:

\[
A^6 x^2 + A^2 y^2 + A^2 z^2 + A^2 x'^2 + A^2 y'^2 + A^4 z'^2 + A^2 a^2 + A^4 a x x' + a y y' + A^4 a z z' + A x y' z - A x y z - A x' y z' - A^3 x y' z' - A^3 a x y z' - A^2 (A^2 + A^{-2})^2,
\]

Remark IX.9.9 For \( A = -1 \) algebras of Proposition 5.5(2) and Theorem 5.6(1) are isomorphic. Also algebras of Theorem 6.7 and 6.8 are isomorphic. It is the case because for \( A = -1 \) the Kauffman bracket skein algebra depends only on the fundamental group (it is explored in [PSS1, PSS2] where an algebra is associated to any group).

Of considerable interest is also the \( R \) algebra monomorphism

\[
P : S_{2,\infty}(F_{1,0} \times I; R[A^\pm 1], A) \to S_{2,\infty}(F_{0,4} \times I; R[A^\pm 1], A) / J
\]

where \( J \) is the ideal generated by expressions \( (a_1 - (A + A^{-1}), a_2 - (A + A^{-1}), a_3 - (A + A^{-1}), a_4 + (A + A^{-1})) \), and \( P \) is given by \( P(A) = A^2, P(x) = x_1, P(y) = x_2 \) and \( P(z) = x_3. \)

If \( \sqrt{-1} \in R \) then \( P \) there is also another, related, \( R \) algebra monomorphism

\[
P' : S_{2,\infty}(F_{1,0} \times I; R[A^\pm 1], A) \to S_{2,\infty}(F_{0,4} \times I; R[A^\pm 1], A) / J'
\]

where \( J' \) is the ideal generated by expressions \( (a_1, a_2, a_3, a_4 - \sqrt{-1}(A^2 - A^{-2})) \), and \( P' \) is given by \( P'(A) = A^2, P'(x) = x_1, P'(y') = x_2 \) and \( P'(z') = x_3. \)

We have noticed these monomorphisms by considering the 2-fold branched covering of a torus over a 2-sphere (with the branching set of four points); [BP].

IX.10 Kauffman skein module \( S_{3,\infty}(M) \)

The deformation of the (unoriented) 2-move leads to the Kauffman skein relation and the Kauffman skein module.
Definition IX.10.1 Let $M$ be an oriented 3-manifold, and put $R = \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$. The Kauffman skein module $S_{3,\infty}(M)$ of $M$ is defined to be the $R$ module spanned by unoriented framed links in $M$ ($RL^f$) subject to the relations: the skein relation: $L_+ + L_- - x(L_0 + L_{\infty}) = 0$ (Fig. 11.1), and the framing relation: $L^{(1)} = aL$.

The Kauffman skein module has been computed for $S^3$, the solid torus, the product of a surface and the interval, and the projective space.

Theorem IX.10.2 (i) (Kauffman) $S_{3,\infty}(S^3) = R$ and $L = F_L(a, x)T_1 = \frac{a + a^{-1} - x}{x}F_L(a, x)\emptyset$.

(ii) (Hoste-Kidwell-Turaev) $S_{3,\infty}(S^1 \times D^2)$ is an infinitely generated free $R$-module.

(iii) (Lieberum) $S_{3,\infty}(F \times I)$ is a free $R$-module infinitely generated for $F$ not simply connected.

(iv) (Mroczkowski) $S_{3,\infty}(RP^3)$ is an infinitely generated free module with basis composed of standard unoriented unlinks $L_n$ (links of Fig. 4.2 with orientation ignored).

The relative Kauffman skein module has been found for a tangle, and an annular tangle in the context of type $A$ and $B$ Brauer algebra [Br].

Theorem IX.10.3 (i) (Birman-Wenzl,Murakami,Morton-Traczyk) $S_{3,\infty}(D^2 \times I; 2n)$ is a free $(2n - 1)(2n - 3) \cdots 3 \cdot 1$- dimensional $R$-algebra.

(ii) (Goodman-Hauschild) $S_{3,\infty}(F_{0,2} \times I; 2n)$ is a free $R$-algebra.

IX.11 Skein module deformation of 3- and (2,2)-moves

In Sections 3-10 we gave a description of skein modules studied extensively until now. Their skein relations can be interpreted as a deformation of 1- and 2-moves. More generally we can consider skein modules based on relations deforming n-moves: $S_n(M) = RL/(b_0L_0 + b_1L_1 + b_2L_2 + \ldots + b_{n-1}L_{n-1})$. In the unoriented case, we can add to the relation the term $b_{\infty}L_{\infty}$, to get $S_{n,\infty}(M)$, and also, possibly, a framing relation. We can also consider skein modules based on skein relations deforming more general tangle moves e.g. rational moves.
I will now describe two such examples which only recently have been considered in more detail. The first example is based on a deformation of the 3-move and the second on the deformation of the \((2,2)\)-move. The first one has been studied with my students Tsukamoto and Veve. I denote the skein module described in this example by \(S_{4,\infty}\) since it involves (in the skein relation) 4 horizontal positions and the vertical (\(\infty\)) smoothing.

**Definition IX.11.1** Let \(M\) be an oriented 3-manifold and let \(\mathcal{L}_{fr}\) be the set of unoriented framed links in \(M\) (including the empty link, \(\emptyset\)), and let \(R\) be any commutative ring with identity. Then we define the \((4,\infty)\) skein module as: 

\[
S_{4,\infty}(M; R) = R\mathcal{L}_{fr}/I_{(4,\infty)},
\]

where \(I_{(4,\infty)}\) is the submodule of \(R\mathcal{L}_{fr}\) generated by the skein relation:

\[
b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 + b_\infty L_\infty = 0
\]

and the framing relation:

\[
L^{(1)} = aL
\]

where \(a, b_0, b_3\) are invertible elements in \(R\) and \(b_1, b_2, b_\infty\) are any fixed elements of \(R\) (see Fig. 11.1).

![Fig. 11.1.](image)

It was conjectured for a while, generalizing the Montesinos-Nakanishi 3-move conjecture (Chapter I), that for \(S^3\) the fourth skein module is generated by trivial links. However the counterexamples to the Montesinos-Nakanishi 3-move conjecture, Chapter VI, can be used to show that trivial links “generically” do not generate \(S_{4,\infty}(S^3, R)\).

**Proposition IX.11.2** Assume that there is a proper ideal \(\mathcal{I} \in R\) such that \(b_1, b_2, b_\infty\) are in \(\mathcal{I}\). Then \(S_{4,\infty}(S^3, R)\) is not generated by trivial links.

**Proof:** Let \(R_{\mathcal{I}}\) be the quotient ring \(R/\mathcal{I}\). In \(R_{\mathcal{I}}, b_1 = b_2 = b_\infty = 0,\) and \(b_0, b_3\) and \(a\) are invertible. Thus the skein relation reduces to \(L_3 = (a_0a_3^{-1})L_0\). It follows from this observation that the family of links generates \(S_{4,\infty}(S^3, R_{\mathcal{I}})\) if and only if this family has a representative for every 3-move equivalence class. Thus \(S_{4,\infty}(S^3, R_{\mathcal{I}})\) is not generated by trivial links and finally \(S_{4,\infty}(S^3, R)\) is not generated by trivial links. The last conclusion follows immediately from our definitions but one can also observe that universal coefficient theorem can be used:

\[
S_{4,\infty}(S^3, R) \otimes_R R/\mathcal{I} = S_{4,\infty}(S^3, R_{\mathcal{I}}).
\]
Similarly, our fourth skein module of \( n \)-tangles needs, generically, strictly more than \( \prod_{i=1}^{n-1} (3^i + 1) \) basic tangles (with possible trivial components). The lower bound was given in Section VI using interpretation of tangles as Lagrangian in the symplectic space of Fox 3-colorings. In [P-Ts] we analyzed extensively the possibility that trivial links, \( T_n \), are linearly independent. This may happen if \( b_\infty = 0 \) and \( b_0 b_1 = b_2 b_3 \). These lead to the following conjecture:

**Conjecture IX.11.3**

1. There is a polynomial invariant of unoriented links in \( S^3 \), \( P_1(L) \in \mathbb{Z}[x,t] \), which satisfies:
   
   (i) Initial conditions: \( P_1(T_n) = t^n \), where \( T_n \) is a trivial link of \( n \) components.
   
   (ii) Skein relation: \( P_1(L_0) + xP_1(L_1) - xP_1(L_2) - P_1(L_3) = 0 \), where \( L_0, L_1, L_2, L_3 \) is a standard, unoriented skein quadruple (\( L_{i+1} \) is obtained from \( L_i \) by a right-handed half-twist on two arcs involved in \( L_i \); compare Fig.3.3).

2. There is a polynomial invariant of unoriented framed links, \( P_2(L) \in \mathbb{Z}[A^{\pm 1}, t] \) which satisfies:
   
   (i) Initial conditions: \( P_2(T_n) = t^n \),
   
   (ii) Framing relation: \( P_2(L^{(1)}) = -A^3 P_2(L) \) where \( L^{(1)} \) is obtained from a framed link \( L \) by a positive half twist on its framing.
   
   (iii) Skein relation: \( P_2(L_0) + A(A^2 + A^{-2})P_2(L_1) + (A^2 + A^{-2})P_2(L_2) + AP_2(L_3) = 0 \).

The above conjectures assume that \( b_\infty = 0 \) in our skein relation. Let us consider, for a moment, the possibility that \( b_\infty \) is invertible in \( R \). Using the “denominator” of our skein relation (Fig.11.2) we obtain the relation which allows us to compute the effect of adding a trivial component to a link \( L \) (we write \( t^n \) for the trivial link \( T_n \)):

\[
(\ast) \quad (a^{-3}b_3 + a^{-2}b_2 + a^{-1}b_1 + b_0 + b_\infty t)L = 0
\]

When considering the “numerator” of the relation and its mirror image (Fig. 11.2) we obtain formulas for Hopf link summands, and because the unoriented Hopf link is amphicheiral we can eliminate it from our equations to get the following formula (**):

\[
b_3(L \# H) + (ab_2 + b_1 t + a^{-1}b_0 + ab_\infty)L = 0.
\]
Skein modules

\[ b_0(L \# H) + (a^{-1}b_1 + b_2 + ab_3 + a^2b_\infty)L = 0. \]

\[ ((b_0b_1 - b_2b_3)t + (a^{-1}b_0^2 - ab_3^2) + (ab_0b_2 - a^{-1}b_1b_3) + b_\infty(ab_0 - a^2b_3))L = 0. \]

Fig. 11.2

It is possible that (*) and (**) are the only relations in the module. More precisely, we ask whether the submodule of \( S_{4,\infty}(S^3; R) \) generated by trivial link is the quotient ring \( R[t]/(I) \) where \( t \) represents the trivial link of \( i \) components and \( I \) is the ideal generated by (*) and (**) for \( L = t \). The interesting substitution which satisfies the relations is \( b_0 = b_3 = a = 1, b_1 = b_2 = x, b_\infty = y \). This may lead to a new polynomial invariant (in \( \mathbb{Z}[x, y] \)) of unoriented links in \( S^3 \) satisfying the skein relation \( L_3 + xL_2 + xL_1 + L_0 + yL_\infty = 0. \)

What about the relations to the Fox colorings? One such a relation, that was already mentioned, is the use of 3-colorings to estimate the number of basic n-tangles (by \( \prod_{i=1}^{p-1}(3^i + 1) \)) for the skein module \( S_{4,\infty} \). I am also convinced that \( S_{4,\infty}(S^3; R) \) contains full information about the space of Fox 7-colorings. It would be a generalization of the fact that the Kauffman bracket polynomial contains information about 3-colorings and the Kauffman polynomial contains information about 5-colorings. In fact, François Jaeger told me that he knew how to form a short skein relation (of the type \( (p+1, \infty) \)) involving spaces of \( p \)-colorings. Unfortunately, François died prematurely in 1997 and I do not know how to prove his statement.

\[ 11 \text{We take into account the fact that the Montesinos-Nakanishi 3-move conjecture does not hold}\]  

\[ 12 \text{If } \text{col}_p(L) = |\text{Col}_p(L)| \text{ denotes the order of the space of Fox } p \text{-colorings of the link } L, \text{ then among } p + 1 \text{ links } L_0, L_1, ..., L_{p-1}, \text{ and } L_\infty, \text{ } p \text{ of them has the same order } \text{col}_p(L) \text{ and one has its order } p \text{ times larger}\]
Finally, let us quickly describe the skein module related to the deformation of \((2,2)\)-moves. Because a \((2,2)\)-move is equivalent to the rational \(\frac{5}{2}\)-move, I will denote the skein module by \(S_{\frac{5}{2}}(M;R)\).

**Definition IX.11.4** Let \(M\) be an oriented 3-manifold. Let \(\mathcal{L}_{fr}\) be the set of unoriented framed links in \(M\) (including the empty link, \(\emptyset\)) and let \(R\) be any commutative ring with identity. We define the \(\frac{5}{2}\)-skein module as

\[
S_{\frac{5}{2}}(M;R) = R\mathcal{L}_{fr}/(I_{\frac{5}{2}})
\]

where \(I_{\frac{5}{2}}\) is the submodule of \(R\mathcal{L}_{fr}\) generated by the skein relation:

(i) \(b_2L_2 + b_1L_1 + b_0L_0 + b_\infty L_\infty + b_{-1}L_{-1} + b_{-\frac{1}{2}}L_{-\frac{1}{2}} = 0\),

its mirror image:

(ii) \(b'_2L_2 + b'_1L_1 + b'_0L_0 + b'_{\infty} L_{\infty} + b'_{-1}L_{-1} + b'_{-\frac{1}{2}}L'_{-\frac{1}{2}} = 0\)

and the framing relation:

\(L^{(1)} = aL\), where \(a, b_2, b'_2, b_{-1}, b'_{-1}\) are invertible elements in \(R\) and \(b_1, b'_1, b_0, b'_0, b_{-1}, b'_{-1}, b_\infty, b'_{\infty}\) are any fixed elements of \(R\). The links \(L_2, L_1, L_0, L_\infty, L_{-1}, L_{\frac{1}{2}}\) and \(L'_{-\frac{1}{2}}\) are illustrated in Fig. 11.3.\(^{13}\)

---

relation between Jones polynomial (or the Kauffman bracket) and \(col_3(L)\) has the form: \(col_3(L) = 3|V(e^{\pi i/3})|^2\) and the formula relating the Kauffman polynomial and \(col_5(L)\) has the form: \(col_5(L) = 5|F(1, e^{2\pi i/5} + e^{-2\pi i/5})|^2\). This seems to suggest that the formula discovered by Jaeger involved Gaussian sums.

\(^{13}\)Our notation is based on Conway’s notation for rational tangles. However, it differs from it by a sign change. The reason is that the Conway convention for a positive crossing is generally not used in the setting of skein relations.
If we rotate the figure from the relation (i) we obtain:

\[(i') \quad b_{-\frac{1}{2}}L_2 + b_{-1}L_1 + b_\infty L_0 + b_0 L_\infty + b_1 L_{-1} + b_2 L_{-\frac{1}{2}} = 0\]

One can use (i) and (i’) to eliminate \(L_{-\frac{1}{2}}\) and to get the relation:

\[
\begin{align*}
(b_2^2 - b_{-\frac{1}{2}}^2) & L_2 + (b_1 b_2 - b_{-1} b_{-\frac{1}{2}}) L_1 + ((b_0 b_2 - b_\infty b_{-\frac{1}{2}}) L_0 + (b_{-1} b_2 - b_{1} b_{-\frac{1}{2}}) L_{-1} + \\
(b_\infty b_2 - b_0 b_{-\frac{1}{2}}) & L_\infty = 0.
\end{align*}
\]

Thus, either we deal with the shorter relation (essentially the one in the fourth skein module described before) or all coefficients are equal to 0 and therefore (assuming that there are no zero divisors in \(R\)) \(b_2 = \varepsilon b_{-\frac{1}{2}}, b_1 = \varepsilon b_{-1},\) and \(b_0 = \varepsilon b_\infty.\) Similarly, we would get: \(b'_2 = \varepsilon b'_{-\frac{1}{2}}, b'_1 = \varepsilon b'_{-1},\) and \(b'_0 = \varepsilon b'_\infty,\) where \(\varepsilon = \pm 1.\) Assume, for simplicity, that \(\varepsilon = 1.\) Further relations among coefficients follow from the computation of the Hopf link component using the amphicheirality of the unoriented Hopf link. Namely, by comparing diagrams in Figure 3.6 and their mirror images we get

\[
\begin{align*}
L \# H &= -b_2^{-1}(b_1(a + a^{-1}) + a^{-2}b_2 + b_0(1 + T_1))L \\
L' \# H &= -b'_2^{-1}(b'_1(a + a^{-1}) + a^{2}b'_2 + b'_0(1 + T_1))L.
\end{align*}
\]

Possibly, the above equalities give the only other relations among coefficients (in the case of \(S^3\)). I would present below the simpler question (assuming \(a = 1, b_x = b'_x\) and writing \(t^n\) for \(T_n\)).

---

**Question IX.11.5** Is there a polynomial invariant of unoriented links in \(S^3\), \(P_\frac{x}{2}(L) \in \mathbb{Z}[b_0, b_1, t]\), which satisfies the following conditions?
(i) Initial conditions: \( P_{\frac{5}{2}}(T_n) = t^n \), where \( T_n \) is a trivial link of \( n \) components.

(ii) Skein relations

\[
P_{\frac{5}{2}}(L_2) + b_1 P_{\frac{5}{2}}(L_1) + b_0 P_{\frac{5}{2}}(L_0) + b_0 P_{\frac{5}{2}}(L_\infty) + b_1 P_{\frac{5}{2}}(L_{-1}) + P_{\frac{5}{2}}(L_{-\frac{1}{2}}) = 0.
\]

\[
P_{\frac{5}{2}}(L_{-2}) + b_1 P_{\frac{5}{2}}(L_{-1}) + b_0 P_{\frac{5}{2}}(L_0) + b_0 P_{\frac{5}{2}}(L_\infty) + b_1 P_{\frac{5}{2}}(L_1) + P_{\frac{5}{2}}(L_{\frac{1}{2}}) = 0.
\]

Notice that by taking the difference of our skein relations one gets the interesting identity:

\[
P_{\frac{5}{2}}(L_2) - P_{\frac{5}{2}}(L_{-2}) = P_{\frac{5}{2}}(L_{\frac{1}{2}}) - P_{\frac{5}{2}}(L_{-\frac{1}{2}}).
\]

Nobody has yet studied the skein module \( S_{\frac{5}{2}}(M; R) \) seriously so everything that you can find will be a new research, even a table of the polynomial \( P_{\frac{5}{2}}(L) \) for small links, \( L \).

**IX.12 Vassiliev-Gusarov skein modules**

Let \( K^{sg} \) denote the set of singular oriented knots in \( M \) where we allow only immersions of \( S^1 \) with, possibly, double points, up to ambient isotopy; additionally for any double point we choose orientation of a small ball around it (if \( M \) is oriented the chosen orientation of the ball agrees with that of \( M \)). In \( RK^{sg} \) we consider resolving singularity relations \( \sim \): \( K_{cr} = K_+ - K_- \); see Fig. 12.1.

![Figure 12.1](image)

\( RK^{sg}/\sim \) is obviously \( R \)-isomorphic to \( RK \). Let \( C_m \) be a submodule of \( RK^{sg}/\sim = RK \) generated by immersed knots with \( m \) double points. The \( m' \)th Vassiliev-Gusarov skein module \( W_m(M, R) \) is defined by \( W_m(M, R) = \)
We have the filtration: 

\[ \cdots \subset C_m \subset \cdots \subset C_1 \subset C_0 = RK \]

and therefore we have also the sequence of epimorphisms \( \{1\} \leftarrow W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow \cdots \leftarrow W_m \leftarrow \cdots \). We define the V-G skein module \( W_\infty(M; R) \) as the inverse limit 

\[ W_\infty(M; R) = \lim_{\leftarrow} W_m(M; R) \]

Equivalently the V-G skein module is the completion \( \hat{RK} \) of \( RK \) with respect to the topology yielded by the sequence of descending submodules \( C_i \). The kernel of the natural \( R \)-homomorphism \( \theta: RK \to \hat{RK} \) is equal to \( \bigcap_{i=0}^{\infty} C_i \).

\( W_\infty(S^3; R) \) is a Hopf algebra with:

- \( \mu(K_1, K_2) = K_1 \# K_2 \)
- \( \Delta(K) = K \otimes K \)
- \( \epsilon(K) = 1 \)
- \( S(K) = T_1 - (K - T_1)^2 - (K - T_1)^3 + \cdots \)

where \( K_1, K_2 \) and \( K \) are non-singular knots. \([P-9]\).

A V-G invariant of degree \( m \) of knots is defined as an element of the dual space \( V^m(M, R) \) = \( W_m^*(M, R) = \text{Hom}(W_m(M, R), R) \) (sometimes it is defined more generally, as an element of \( \text{Hom}_Z(W_m(M, Z), A) \), where \( A \) is an abelian group).

We can modify the definition of V-G skein modules by resolving the singular points differently, as suggested in \([P-9]\). The most natural resolutions are those suggested by Jones type skein relations. For example the resolution \( L_{cr} - v^{-1}L_+ - vL_+ - zL_0 \) proposed in \([P-9]\) was analyzed in more detail by Y. Rong and R. Lickorish \([Ron-2, L-R]\). The resolution of a crossing of an unoriented framed singular link \( L_{cr} - L_+ - AL_0 - A^{-1}L_\infty \) was hinted in \([B-F-K-2]\). Further work in this direction was done in \([A-T-1, A-T-2]\).

**IX.13 A glimpse into 4-dimensional skein modules**

The following approach to 4-dimensional skein modules is based on ideas of Kamada and Kawauchi \([Kaw-3]\).

**Definition IX.13.1** Let \( M \) be an oriented 4-dimensional manifold and \( F \) denote the set of immersed surfaces in \( M \) with possible double points, up to ambient isotopy. Let \( R \) be a commutative ring with unit and let \( RF \) denote the free \( R \)-module generated by \( F \). Consider the submodule \( K \) of \( RF \) generated by expressions: \( r_0F_0 + r_1F_1 + r_2F_2 \) where \( r_i \in R \) and \( F_1 \) and \( F_2 \) are obtained from \( F_0 \in F \) as follows (Fig. 13.1):

1. \( F_1 \) is obtained from \( F_0 \) by performing index 1 surgery on \( F_1 \) along \( \gamma \).
2. \( F_2 \) is obtained from \( F_0 \) by taking, instead of a 1-surgery, an immersed surface with two additional double points (one should carefully choose a convention - there are two choices).
Now in the quotient module, $\mathcal{K} = \mathcal{RF}/\mathcal{K}$, consider the filtration $\{C_i\}$, where $C_i$ is generated by surfaces from $\mathcal{F}$ with $i$ double points. The completion of the space with respect to the filtration is called the KK skein module of a four dimensional manifold $M$ and denoted by $\text{KK}_\infty(M; \mathbb{R})$. The quotient of $\mathcal{K}$ by $C_{i+1}$ are degree $i$, KK skein modules, $\text{KK}_i(M; \mathbb{R})$. Dual elements (or, more generally homomorphisms to any abelian group) are called degree $i$, KK invariants of a 4-manifold.

\begin{center}
\begin{tabular}{ccc}
\textbf{F}_0 & \textbf{F}_1 & \textbf{F}_2 \\
\includegraphics[width=0.3\textwidth]{F0.png} & \includegraphics[width=0.3\textwidth]{F1.png} & \includegraphics[width=0.3\textwidth]{F2.png}
\end{tabular}
\end{center}
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