ON THE NUMBER AND SUMS OF EIGENVALUES OF SCHRÖDINGER-TYPE OPERATORS WITH DEGENERATE KINETIC ENERGY

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Dedicated to the memory of Sergey N. Naboko

Abstract. We estimate sums of functions of negative eigenvalues of Schrödinger-type operators whose kinetic energy vanishes on a codimension one submanifold. Our main technical tool is the Stein–Tomas theorem and some of its generalizations.

1. Introduction

For \( d \geq 1 \) we consider Schrödinger-type operators of the form

\[
H = T(-i\nabla) - V \quad \text{in } L^2(X^d), \quad X \in \{\mathbb{R}, \mathbb{Z}\}
\]

where the kinetic energy \( T(\xi) \) vanishes on a codimension-one submanifold. A prime example is \( T = |\Delta + 1| \), which naturally appears, e.g., in the BCS theory of superconductivity and superfluidity, see, e.g., Frank, Hainzl, Naboko, and Seiringer [20], Hainzl, Hamza, Seiringer, and Solovej [23], and Hainzl and Seiringer [28, 29], as well as the Hartree–Fock theory of the electron gas (jellium), see, e.g., Gontier, Hainzl, and Lewin [26]. The potential \( V \) is assumed to be real-valued and sufficiently regular, so that \( H \) can be realized as self-adjoint operator. In this note we are interested in estimates for sums of functions of negative eigenvalues of \( H \) when \( V \in L^q \) for some \( q < \infty \). We now state our assumptions on \( T \).

Assumption 1.1. Assume that \( T(\xi) \geq 0 \) attains its minimum on a smooth compact codimension one submanifold \( S = \{\xi \in \mathbb{R}^d : T(\xi) = 0\} \). Assume that there exists an open, precompact neighborhood \( \Omega \subseteq \mathbb{R}^d \) of \( S \) such that the following holds.

1. There exists \( P \in C^\infty(\Omega) \) such that \( T(\xi) = |P(\xi)| \). Let \( \tau := \max_{\xi \in \Omega} T(\xi) \).
2. There exist \( c_P > 0 \) such that \( |\nabla P(\xi)| \geq c_P \) for all \( \xi \in \Omega \).
3. There exist constants \( C_1, C_2 > 0 \) and \( s \in (0, d) \) such that \( T(\xi) \geq C_1|\xi|^s + C_2 \) for \( \xi \in \mathbb{R}^d \setminus \Omega \).

For \( t > 0 \), consider the level set \( S_t := \{\xi \in \mathbb{R}^d : |P(\xi)| = t\} \) which is a smooth compact codimension one submanifold embedded in \( \mathbb{R}^d \) with corresponding surface measure \( d\Sigma_{S_t} \). We set \( d\sigma_{S_t}(\xi) := d\Sigma_{S_t}(\xi)/|\nabla P(\xi)| \) and assume that

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there is $r > 0$ such that $\sup_{x \in (0,\tau)} |(d\sigma_S)^\vee(x)| \lesssim \tau (1 + |x|)^{-r}$, where $(d\sigma_S)^\vee(x) = \int_{S_t} e^{2\pi i x \cdot \xi} d\sigma_S(\xi)$ denotes the Fourier transform of $d\sigma_S_t$.

Assumptions (1)-(3) also appear in the work of Hainzl and Seiringer \[31\], where it is assumed that $V \in L^1 \cap L^{d/s}$. These assumptions imply that the quadratic form $(u, (T(-i\nabla) - V)u)$ is bounded from below, whenever $u \in C_c^\infty(\mathbb{R}^d)$. The Friedrichs extension then provides us with a self-adjoint operator $H = T(-i\nabla) - V$. Note that the constants $\tau, c_p, C_1, C_2$ in Assumption \[1.1\] are fixed $O(1)$-quantities.

Assumption (4) is related to the curvature of $S$, and is crucial since it allows us to consider $V \in L^q \cap L^{d/s}_{\text{loc}}$ with $q > 1$. Littman \[43\] showed that if $S$ has $2r \in \{0, 1, \ldots, d-1\}$ non-vanishing principles curvatures, then one has the decay $|(d\sigma_S)^\vee(x)| = O(|x|^{-r})$. In particular, Assumption (4) holds for $T = |\Delta + 1|$ with $r = (d - 1)/2$. Note also that this assumption is always guaranteed in the nonzero curvature case, whenever one has the decay $(d\sigma_S)^\vee(x) = O(|x|^{-(d-1)/2})$ for $t = 0$. (See, e.g., \[9\] Proposition 4.1.)

For $V \in L^q$ with $q \in [d/s, \infty)$ the essential spectrum $\sigma_{\text{ess}}(H) = [0, \infty)$ coincides with that of $T(-i\nabla)$. The discrete spectrum of the operator $H_\lambda := T(-i\nabla) - \lambda V$ for $0 < \lambda \ll 1$ has recently received considerable interest. For $V \in L^1 \cap L^{d/s}(\mathbb{R}^d)$ it has been shown, e.g., by Frank, Hainzl, Naboko, and Seiringer \[20\] and Hainzl and Seiringer \[29, 31\] that for any eigenvalue $a_S^j > 0$ of the operator

$$L^2(S, d\sigma_S) \to L^2(S, d\sigma_S), \quad u \mapsto \int_S \check{V}(\xi - \eta)u(\eta)d\sigma_S(\eta), \quad u \in L^2(S, d\sigma_S),$$

(1.2)

there is a corresponding eigenvalue $-e_j(\lambda) < 0$ of $T - \lambda V$ which satisfies

$$e_j(\lambda) = \exp \left(-\frac{1}{2\lambda a_S^j}(1 + o(1))\right) \quad \text{as } \lambda \to 0. \quad (1.3)$$

Here, $\check{V}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi}V(x) \, dx$ denotes the Fourier transform of $V$ in $\mathbb{R}^d$. Recently, the authors \[9\] extended this result to a substantially larger class of potentials, such as $V \in L^q(\mathbb{R}^d)$ with $q \in [d/s, r + 1]$, whenever $T(-i\nabla)$ satisfies also the curvature assumption (4) with $r + 1 \geq d/s$ in Assumption \[1.1\]. This is clearly the case for $T = |\Delta + 1|$ with $r = (d - 1)/2$.

On the other hand, Laptev, Safronov, and Weidl \[37\] studied the asymptotic behavior of the eigenvalues $-e_j < 0$ of $T - V$ as $j \to \infty$, when $V$ is of the form $V(x) = v(x)(1 + |x|)^{-1-\varepsilon}$, where $v \in L^\infty(\mathbb{R}^d)$ satisfies $v(x) = w(x/|x|)(1 + o(1))$ as $|x| \to \infty$ with $w \in C^\infty(S^{d-1})$. Similarly as in \[13\], the eigenvalue asymptotics is determined by that of the eigenvalues $a_S^j > 0$ of the operator in \[1.2\]. Their main result \[37\] Theorem 4.4] essentially relied on an abstract theorem (Theorem 3.4 there) which connected the spectral asymptotics of $H$ and \[1.2\] with each other. In turn, the limit $\lim_{j \to \infty} a_S^j$ is well understood thanks to the works \[8\] of Birman and Solomjak on singular values of (asymptotically) homogeneous pseudodifferential operators with symbol $h_1(x)a(x, \xi)h_2(x)$. Here $h_1, h_2 \in C_c^\infty$, and $a(x, t\xi) = t^{-\beta}a(x, \xi)$ for all $|\xi| \geq 1$.
and $t > 1$. By a change of coordinates, the operator in $(1.2)$ can be transformed into this operator modulo “error operators” which do not change the leading order of the spectral asymptotics of $(1.2)$. We refer to Birman and Yafaev [5] for a detailed exposition and for the explicit expression for $\lim_{j \to \infty} a_j$. For $V$ merely in $L^q(\mathbb{R}^d)$, the results of Birman and Solomjak are not applicable. It would be interesting to study the asymptotics $\lim_{j \to \infty} e_j$ in this case.

The purpose of this note is to prove estimates for sums of functions $f(x)$ on $\mathbb{R}^d$ of the absolute values of the negative eigenvalues of $T - V$ when $V \in L^q$. For $f(x) = x^\gamma$ this will lead to modifications of the celebrated Lieb–Thirring inequality [41, 42, 38] for the pseudorelativistic Chandrasekhar operator, are important in the analysis of many-particle quantum systems. In fact, using the techniques of [38], Daubechies extended (1.5) to a larger class of operators $T(-i \nabla)$. However, these results are not applicable in the present situation, since they require $T(\xi)$ to be a spherically symmetric and strictly increasing function with $T(0) = 0$. This condition is not satisfied by the operators $T$ we consider here, such as $T = |\Delta + 1|$. Further examples of eigenvalue estimates involving a sum of two terms were proved, e.g., by Lieb, Solovej, and Yngvason [40] in the context of the Pauli operator and by Exner and Weidl [15] in the context of Schrödinger operators in wave guides $\omega \times \mathbb{R}$ with $\omega \subset \mathbb{R}^{d-1}$. For two-term estimates for eigenvalue sums of Schrödinger operators on metric trees, we refer to Frank and Kovařík [21, Theorem 6.1], see also Ekholm, Frank, and Kovařík [13], Molchanov and Vainberg [45], and the references therein for further results. Finally, we refer to Frank, Lewin, Lieb, and Seiringer [22] for two-term estimates for eigenvalue sums of Schrödinger operators in presence of a constant positive background density.

Besides sums of powers of eigenvalues, we also prove estimates for sums of powers of logarithms (i.e., $f(x) = (\log(2 + 1/x))^{-\gamma}$) of eigenvalues of $T - V$. This is natural, as (1.3) indicates that the eigenvalues of $T - V$ cluster with an exponential rate at zero. In particular, the proofs of these results yield estimates on the eigenvalues $e_j$ and show
how fast they cluster at zero as \( j \to \infty \), see (3.16). However, we do not investigate the asymptotics for \( \lim_{j \to \infty} e_j \) here. The idea of deriving estimates for logarithms of eigenvalues is not new and has already been considered by Kovařík, Vugalter, and Weidl [39] in the context of two-dimensional Schrödinger operators \(-\Delta - V\), whose eigenvalues also cluster exponentially fast at the bottom of the essential spectrum, see Simon [52].

If \( T \) degenerates sublinearly, we are able to prove Cwikel–Lieb–Rosenbljum-type estimates [11, 38, 47] for the number of negative eigenvalues. We illustrate this using \( T = |\Delta + 1|^{1/s} \) with \( s > 1 \).

Finally, we generalize our results to lattice Schrödinger-type operators on \( \ell^2(\mathbb{Z}^d) \). Under the same curvature assumption we obtain better estimates than in \( L^2(\mathbb{R}^d) \) due to the absence of high energies.

**Organization and notation.** In Section 2 we collect facts about Schatten spaces and Fourier restriction theory that are used in the subsequent sections. In Section 3 we prove estimates for the number of eigenvalues of \( T - V \) in \( L^2(\mathbb{R}^d) \) below a fixed threshold \(-e < 0\) (Theorem 3.1). Then we prove inequalities for sums of powers of eigenvalues (Theorem 3.4), and for sums of powers of logarithms of eigenvalues of \( T - V \) (Theorem 3.6). We conclude with a Cwikel–Lieb–Rosenbljum bound for \( |\Delta + 1|^{1/\sigma} - V \) with \( \sigma > 1 \) (Theorem 3.8). In Section 4 we consider the corresponding problems for Schrödinger operators on \( \ell^2(\mathbb{Z}^d) \). We first recall two versions of a discrete Laplace operator and a modification of the “BCS operator” \(|\Delta + 1| - V\) to \( \ell^2(\mathbb{Z}^d) \). In Section 4.2 we prove estimates on the number of negative eigenvalues of \( T - V \) in \( \ell^2(\mathbb{Z}^d) \) below a threshold \(-e < 0\) (Theorem 4.1), ordinary and logarithmic Lieb–Thirring-type inequalities (Theorems 4.2 and 4.4), and a Cwikel–Lieb–Rosenbljum bound for powers of the modified BCS operator in \( \ell^2(\mathbb{Z}^d) \) (Theorem 4.7).

We write \( A \lesssim B \) for two non-negative quantities \( A, B \geq 0 \) to indicate that there is a constant \( C > 0 \) such that \( A \leq CB \). If \( C = C_\tau \) depends on a parameter \( \tau \), we write \( A \lesssim_\tau B \). The dependence on fixed parameters like \( d \) and \( s \) is sometimes omitted. Constants are allowed to change from line to line. The notation \( A \sim B \) means \( A \lesssim B \lesssim A \). All constants are denoted by \( c \) or \( C \) and are allowed to change from line to line. We abbreviate \( A \wedge B := \min\{A, B\} \) and \( A \vee B := \max\{A, B\} \). The Heaviside function is denoted by \( \theta(x) \). We use the convention \( \theta(0) = 1 \). The indicator function and the Lebesgue measure of a set \( \Omega \subseteq \mathbb{R}^d \) are denoted by \( 1_\Omega \) and \( |\Omega| \), respectively. For \( x \in \mathbb{R}^d \) we write \( \langle x \rangle := (2 + x^2)^{1/2} \).

2. Preliminaries

2.1. Trace ideals. We collect some facts on trace ideals that are used in this note, see also, e.g., Birman–Solomjak [4, Chapter 11] or Simon [53].

Let \((\mathcal{B}, \| \cdot \|)\) denote the Banach space of all linear, bounded operators on a Hilbert space \( \mathcal{H} \). The \( p \)-th Schatten space of all compact operators \( T \in \mathcal{S}^\infty(\mathcal{H}) \) whose singular values \( \{s_n(T)\}_{n \in \mathbb{N}} \) (in non-increasing order, appearing according to their multiplicities)
satisfy \( \|T\|_{S_p(H)}^p := \sum_{n \geq 1} s_n(T)^p < \infty \) for \( p > 0 \) is denoted by \( S^p(H) \). We denote the \( p \)-th weak Schatten space over \( H \) by
\[
S^{p,\infty}(H) := \{ T \in S^\infty(H) : \|T\|_{S^{p,\infty}(H)}^p := \sup_{\lambda > 0} \lambda^p n(\lambda, T) < \infty \} \supseteq S^p(H), \tag{2.1}
\]
where
\[
n(\lambda, T) := \#\{ n : s_n(T) > \lambda \}, \quad \lambda > 0. \tag{2.2}
\]
Note that
\[
\|T\|_{S^{p,\infty}(H)} = \sup_m s_m(T)m^{\frac{1}{p}}, \tag{2.3}
\]
which together with (2.1) implies in particular
\[
s_m(T) \leq \|T\|_{S^{p,\infty}(H)} m^{-\frac{1}{p}} \text{ and } n(\lambda, T) \leq \|T\|_{S^{p,\infty}(H)}^{-p}. \tag{2.4}
\]
If \( T : H \to H' \) is a linear operator between two Hilbert spaces \( H \) and \( H' \) we denote its \( p \)-th Schatten norm by \( \|T\|_{S^p(H,H')} \). If \( H = H' \), we either write \( \|T\|_{S^p(H)} \), \( \|T\|_{S^p} \), or \( \|T\|_p \), and abbreviate \( S^p(H) = S^p \). Analogous notation is used for \( S^{p,\infty} \).

2.2. Fourier restriction and extension. Let \( X \in \{ \mathbb{R}, \mathbb{Z} \} \), \( \hat{X} = \mathbb{R} \) when \( X = \mathbb{R} \), and \( \hat{X} = \mathbb{T} \) when \( X = \mathbb{Z} \), where \( \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d \) denotes the \( d \)-dimensional torus with Brillouin zone \([-1/2,1/2)^d \). If \( X = \mathbb{Z} \), then the \( L^q(X^d) \)-spaces are equipped with counting measure so that \( L^q(\mathbb{Z}^d) \equiv l^q(\mathbb{Z}^d) \) for any \( q > 0 \).

Let \( S \) be a smooth, compact codimension one submanifold embedded in \( \hat{X}^d \) with induced Lebesgue surface measure \( d\sigma_S \). If \( S \) is the level set of a smooth real-valued function \( P \in C^\infty(\hat{X}^d) \), i.e., \( S = \{ \xi \in \hat{X}^d : P(\xi) = 0 \} \), then the Leray measure \([25]\) is \( d\sigma_S(\xi) = |\nabla P(\xi)|^{-1}d\Sigma_S(\xi) \). We introduce the Fourier restriction operator
\[
F_S : S(X^d) \to L^2(S,d\sigma_S), \quad \varphi \mapsto (F_S\varphi)(\xi) = \widehat{\varphi}(\xi) \big|_S = \int_{X^d} e^{-2\pi i x \cdot \xi} \varphi(x) \, dx \big|_S \tag{2.5}
\]
and its adjoint, the Fourier extension operator
\[
F^*_S : L^2(S,d\sigma_S) \to S'(X^d), \quad u \mapsto (F^*_S u)(x) = \int_S u(\xi)e^{2\pi i x \cdot \xi} \, d\sigma_S(\xi). \tag{2.6}
\]
Under the additional assumption that the Gaussian curvature of \( S \) is non-zero everywhere, the Stein–Tomas theorem \([56, 54, 7]\) asserts that \( F_S : L^p(X^d) \to L^2(S) \) is bounded for all \( p \in [1, 2(d+1)/(d+3)] \). Its proof relies on the bound \( |(d\sigma_S)^\vee(x)| \lesssim \langle x \rangle^{-\frac{d+1}{2}} \). By duality, the Stein–Tomas theorem is equivalent to the operator norm bound
\[
\|W_1 F^*_S F_S W_2\|_{L^2(X^d) \to L^2(X^d)} \lesssim \|W_1\|_{L^2n(X^d)}\|W_2\|_{L^2n(X^d)} \text{ for all } W_1, W_2 \in L^{2q},
\]
whenever \( 1/q = 1/p - 1/p' \) and \( p \in [1, 2(d+1)/(d+3)] \), i.e., \( q \in [1, (d+1)/2] \). Frank and Sabin \([24]\, \text{Theorem 2}\) upgraded this to a Schatten norm estimate. For smooth compact hypersurfaces \( S \subseteq \hat{X}^d \) with everywhere non-vanishing Gaussian curvature and
\[
\sigma(q) := \frac{(d-1)q}{d-q}, \quad q \in [1,d), \tag{2.7}
\]
Frank and Sabin proved

$$\|W_1 F_S^* F_S W_2\|_{S^{s(q)}(L^q(X^d))} \lesssim_{d,S,q} \|W_1\|_{L^{2q}(X^d)} \|W_2\|_{L^{2q}(X^d)}, \quad q \in \left[1, \frac{d+1}{2}\right]. \quad (2.8)$$

Note that $\sigma(1) = 1$, $\sigma((d+1)/2) = d+1$, and $\sigma(q) \geq q$.

As discussed in the introduction, if $S$ has $2r \in \{0, 1, \ldots, d-1\}$ non-vanishing principle curvatures, then one has the weaker decay $|\langle(d\sigma_j S)^\vee(x)\rangle| \lesssim \langle x \rangle^{-r}$, which, as Greenleaf showed, implies that $F_S : L^p(X^d) \to L^2(S)$ is bounded for all $p \in [1, (2+2r)/(2+r)]$. For a given decay rate of $|\langle(d\sigma_j S)^\vee(x)\rangle|$ the first author proved the following generalization of (2.8).

**Proposition 2.1** ([8 Proposition A.5]). Let $S \subseteq \mathbb{X}^d$ be a smooth compact hypersurface with normalized defining function $P : \mathbb{X}^d \to \mathbb{R}$ and Lebesgue surface measure $d\Sigma_S$ and Leray measure $d\sigma(S) = |\nabla P(\xi)|^{-1} d\Sigma_S(\xi)$. Assume that

$$\sup_{x \in \mathbb{X}^d} (1 + |x|^r)|\langle(d\sigma_j S)^\vee(x)\rangle| < \infty \quad (2.9)$$

for some $r > 0$. Let $1 \leq q \leq 1 + r$ and define

$$\sigma(q,r) := \begin{cases} \frac{2(d-1-r)q}{d-q} & \text{if } \frac{d}{d-r} \leq q \leq 1 + r, \\ \frac{2rq+1}{2rq-d(q-1)} & \text{if } 1 \leq q < \frac{d}{d-r} \end{cases}. \quad (2.10)$$

Here, $2rq+$ means $2rq+\varepsilon$ with $\varepsilon > 0$ arbitrarily small but fixed. Then for all $W_1, W_2 \in L^{2q}(X^d)$, we have

$$\|W_1 F_S^* F_S W_2\|_{S^{\sigma(q,r)}(L^q(X^d))} \lesssim \|W_1\|_{L^{2q}(X^d)} \|W_2\|_{L^{2q}(X^d)}, \quad (2.11)$$

where the implicit constant is independent of $W_1, W_2$.

**Remarks 2.2.**

1. We have $\sigma(q,r) \geq q$ when $r \leq (d-1)/2$ and $\sigma(q, (d-1)/2) = \sigma(q)$ with $\sigma(q)$ as in (2.7).

2. The estimates (2.8) and (2.11) were proved for $\mathbb{R}^d$, but their (Fourier-analytic) proofs readily generalize to $\mathbb{Z}^d$.

3. The estimate in [8 Proposition A.5] involved the resolvent of $P(-i\nabla)$. As usual, this implies (2.11) since the imaginary part of the limiting resolvent equals the spectral measure.

4. Littman’s bound $|\langle(d\sigma_j S)^\vee(x)\rangle| \lesssim \langle x \rangle^{-r}$ is rarely optimal except when the surface is completely flat in the vanishing curvature direction. (For a more detailed discussion and references to generic results, see, e.g., [10] by Schippa and the first author.)

5. As is discussed, e.g., in Ikromov, Kempe, and Müller [33, 34, 35], sharp decay estimates do not always imply $L^2 \to L^p$ Fourier restriction bounds with optimal $p$.\footnote{This means that $S = \{P = 0\}$ and $|\nabla P| = 1$ on $S$.}
3. Bounds on number and sums of functions of eigenvalues in \( L^2(\mathbb{R}^d) \)

Suppose that the kinetic energy \( T(-i\nabla) \) satisfies (1)-(3) in Assumption \([\text{I1}])\). Let \(-e_1 \leq -e_2 \leq \cdots < 0\) denote the negative eigenvalues of \( H = T - V \) in non-decreasing order (counting multiplicities) and

\[
N_e(V) := \sum_{e_j > e} 1
\]

(3.1)
denote the number of negative eigenvalues of \( H \) below \(-e \leq 0\). Let

\[
BS(e) := |V|^\frac{d}{2}(T + e)^{-1}V^\frac{d}{2} \quad \text{on} \quad L^2(\mathbb{R}^d)
\]

(3.2)
where \( V^{1/2}(x) := |V(x)|^{1/2} \text{sgn}(V(x)) \) with \( \text{sgn}(V(x)) = 1 \) if \( V(x) = 0 \). By the Birman–Schwinger principle (cf. [53, Proposition 7.2], [11, p. 99], [39, Proposition 6]), one has

\[
N_e(V) = n(1, BS(e)) \leq \|BS(e)^m\|_{S^m,\infty} \leq \|BS(e)^m\|_{S^m} \quad \text{for all} \quad m > 0.
\]

(3.3)
As a consequence of the variational principle, i.e.,

\[
N_e(V) \leq N_{e/(2)}(V_+ - e/2) \leq N_{e/2}((V_+ - e/2)_+),
\]

(3.4)
one can estimate for any \( \gamma > 0 \),

\[
\text{Tr}(T(-i\nabla) - V)^{\gamma}_- \leq \gamma \int_0^\infty e^{\gamma - 1}N_{e/2}((V_+ - e/2)_+) \, de.
\]

(3.5)

3.1. Number of eigenvalues below a threshold. We first prove estimates for Schatten norms of \( BS(e) \).

**Theorem 3.1.** Let \( e > 0 \) and suppose \( T(\xi) \) satisfies (1)-(3) in Assumption \([\text{I1}])\).

1. Let \( m > d/s \). If \( V \in L^m(\mathbb{R}^d) \), then there exists a constant \( c_S > 0 \) (which also depends on \( d, s, m, \tau \)) such that

\[
\|BS(e)^m\|_m \leq c_S (e^{1-m}\theta(1-e) + e^{d/s-m}\theta(e-1))\|V\|^m_m.
\]

(3.6)

2. Suppose \( T \) also satisfies (4) in Assumption \([\text{I1}])\) with \( r > 0 \). Let \( q \in [1, r + 1] \) and \( m = \sigma(q, r) \) be as in (2.10). Suppose additionally \( m > d/s \) and let \( V \in L^m \cap L^q(\mathbb{R}^d) \). Then there is a constant \( c_S \) (which also depends on \( d, s, m, \tau, q, r \)) such that

\[
\|BS(e)^m\|_m \leq c_S \left[ \|V\|^m_m + \log(2 + 1/e)^m\|V\|^m_q \right] \theta(1-e) + c_S \left[ e^{d/s-m}\|V\|^m_m + e^{-m}\min\{\|V\|^m_m, \|V\|^m_q\} \right] \theta(e-1).
\]

(3.7)

3. In addition to the assumptions in (2), suppose \( q > d/s \). Then

\[
\|BS(e)^m\|_m \leq c_S\left[ \log(2 + 1/e)^m\theta(1-e) + e^{md/s-m}\theta(e-1) \right].
\]

(3.8)
If \( q = d/s \), then (3.8) holds with \( \| \cdot \|^m_m \) on the left side replaced by \( \| \cdot \|^m_{m,\infty} \).
Proof. We begin with the proof of (3.6). Hölder’s inequality yields
\[
\|W_1 F_{S}^* F_{S} W_2\|_{S^1(L^2(\mathbb{R}^d))} \leq \|W_1 F_{S}^*\|_{S^2(L^2(S,d\sigma_S),L^2(\mathbb{R}^d))} \|W_2 F_{S}^*\|_{S^2(L^2(S,d\sigma_S),L^2(\mathbb{R}^d))} = \|W_1\|_{L^2} \|W_2\|_{L^2} \sigma_S(S). \tag{3.9}
\]
For $\tau > 0$ as in Assumption 1.1 we separate high and low energies using a bump function $\chi \in C_c^\infty(\mathbb{R}_+ : [0,1])$ with supp $\chi \subseteq [0,1]$. By the Kato–Seiler–Simon inequality [33, Theorem 4.1] (with $m > d/s$), we obtain
\[
\|B S(e)\|_m^m \lesssim_m \left(\|V\|^{\frac{1}{2}}(T + e)^{-1}\chi(T/\tau)|V|^{\frac{1}{2}}m + \|V\|^{\frac{1}{2}}(T + e)^{-1}(1 - \chi(T/\tau))|V|^{1/2}m\right)^m \lesssim \|V\|^{\frac{1}{2}}(T + e)^{-1}\chi(T/\tau)|V|^{\frac{1}{2}}m + \|V\|_m^m \min\{1, e^{d/s-m}\}. \tag{3.10}
\]
To treat the low energy part we use the Lieb–Thirring trace inequality [42, Theorem 9]
\[
\|B^{1/2} A B^{1/2}\|_m^m \leq \|B^{m/2} A^m B^{m/2}\|_{S^1}, \quad m \geq 1 \tag{3.11}
\]
for linear operators $A, B \geq 0$ in a separable Hilbert space, the spectral theorem, and (3.9). We obtain
\[
\|V\|^{\frac{1}{2}}(T + e)^{-1}\chi(T/\tau)|V|^{\frac{1}{2}}m \leq \|V\|^{\frac{m}{2}}(T + e)^{-m}\chi(T/\tau)^m|V|^{\frac{m}{2}}1 \leq \int_0^\tau dt \frac{\|V\|^{m/2} F_{S}^* F_{S} |V|^{m/2}}{(t + e)^m} \leq \|V\|_m^m \int_0^\tau dt (t + e)^{-m} \sigma_{S}(S_t) \tag{3.12}
\]
\[
\leq c_{S,\tau} \|V\|_m^m \int_0^\tau \frac{dt}{(t + e)^m} \leq c_{S,\tau,m} \|V\|_m^m \min\{e^{1-m}, e^{-m}\},
\]
where we used Assumption 1.1 to estimate
\[
\sigma_{S_t}(S_t) \leq \sup_{t \in [0,\tau]} \sup_{\xi \in S_t} \frac{\Sigma_{S_t}(S_t)}{\sqrt{P(\xi)}} \leq c_{S,\tau}. \tag{3.13}
\]
Combining (3.10) and (3.12) proves (3.6). To prove (3.7), we proceed as in the proof of (3.6) but estimate the low energies using the Stein–Tomas estimate for trace ideals (2.11) instead. For $q \in [1, r + 1]$, we obtain
\[
\|V\|^{\frac{1}{2}}(T + e)^{-1}\chi(T/\tau)|V|^{\frac{1}{2}}m_{\sigma(q,r)} \leq \int_0^\tau \frac{dt}{t + e} \|V\|^{1/2} F_{S}^* F_{S_t} |V|^{1/2}m_{\sigma(q,r)} \leq c_S \min\{\log(1 + \tau/e), \tau/e\} \|V\|_q. \tag{3.14}
\]
Setting $m = \sigma(q, r)$ on the left side of (3.14) and combining it with (3.6) yields (3.7). The final estimate (3.8) follows from the proof of (3.7) by replacing the estimate for the high energies in the second and third line of (3.10) by the following estimate,
\[
\|V\|^{\frac{1}{2}}(T + e)^{-1}(1 - \chi(T/\tau))|V|^{1/2}m \leq \|V\|^{\frac{1}{2}}(T + e)^{-1}(1 - \chi(T/\tau))|V|^{1/2}m \lesssim \|V\|_q^m \min\{1, e^{m/(m-m)}\},
\]
where \( m \geq q > d/s \). (Here we used the Kato–Seiler–Simon inequality again.) This concludes the proof of (3.8). For \( q = d/s \) we use Cwikel’s bound (see [53, Theorem 4.2] or (3.37) below), which is applicable since \( q > 1 \) in this case.

\[
\text{Remarks 3.2.} \quad (1) \text{The terms proportional to } \|V\|_m^m \text{ in (3.7) and the term that scales like } e^md/m \text{ in (3.8) are due to high energies.}
\]

(2) If \( r = (d - 1)/2 \), \( d > s \geq d/q \) and \( 0 < e < 1 \), then (3.8) implies for \( q \in (1, (d + 1)/2] \) and \( m = \sigma(q) \),

\[
N_e(V) \leq c_S \log(1/e)^{\sigma(q)} \|V\|_q^{\sigma(q)}. \tag{3.15}
\]

Thus, the \( n \)-th negative eigenvalue \(-1 < -e_n < 0\) satisfies

\[
e_n \leq \exp \left( -\frac{n^{1/\sigma(q)}}{c_S^{1/\sigma(q)} \|V\|_q} \right). \tag{3.16}
\]

We close this subsection by proving a slight refinement of the bound for \( N_e(V) \) that follows immediately from (3.3) and (3.6). To that end we apply Fan’s inequality [16] (see also [53, Theorem 1.7]), which asserts

\[
s_{j+\ell+1}(A + B) \leq s_{j+1}(A) + s_{\ell+1}(B) \tag{3.17}
\]

for all \( j, \ell \in \mathbb{N}_0 \) and all \( A, B \in \mathcal{S}^\infty \).

\[
\text{Corollary 3.3.} \quad \text{Let } e, \tau > 0, m_1, m_2 \geq 1. \text{ Let } L_{\text{loc}}^1(\mathbb{R}^d) \ni T(\xi) \geq 0 \text{ and } V \in L_{\text{loc}}^1(\mathbb{R}^d) \text{ so that } BS_{<}(e) := |V|^1/2(T + e)^{-1}1_{\{T < \tau\}}V^{1/2} \text{ and } BS_{>}(e) := |V|^1/2(T + e)^{-1}1_{\{T > \tau\}}V^{1/2} \text{ are compact operators. Then}
\]

\[
N_e(V) \leq 2 \cdot [2^{m_1}\|BS_{<}(e)\|_{\mathcal{S}^\infty_1,m_1}^{m_1} + 2^{m_2}\|BS_{>}(e)\|_{\mathcal{S}^\infty_2,m_2}^{m_2}]. \tag{3.18}
\]

In particular, for \( T(\xi) \) satisfying (1)-(3) in Assumption 1.1 \( m_1 > 1 \) and \( m_2 > d/s \), there is \( c_S > 0 \) (which also depends on \( d, s, m_1, m_2, \tau \)) such that

\[
N_e(V) \leq c_S[(e^{m_1-1}\|V\|_{m_1}^{m_1} + \|V\|_{m_2}^{m_2} \theta(1 - e) + e^{d/s-m_2}\|V\|_{m_2}^{m_2} \theta(1 - e))]. \tag{3.19}
\]

\[
\text{Proof.} \quad \text{We begin with proving (3.18). By (3.17), we have } s_{n+1}(BS(e)) \leq s_{n/2+1}(BS_{<}(e)) + s_{n/2+1}(BS_{>}(e)) \text{ for even } n \text{ and } s_{n+1}(BS(e)) \leq s_{(n+1)/2+1}(BS_{<}(e)) + s_{(n-1)/2+1}(BS_{>}(e)) \text{ for odd } n. \text{ Thus,}
\]

\[
\{n \in 2\mathbb{N}_0 : s_{n+1}(BS(e)) > 1\} \subseteq \{n \in 2\mathbb{N}_0 : s_{n/2+1}(BS_{<}(e)) > 1/2\} \cup \{n \in 2\mathbb{N}_0 : s_{n/2+1}(BS_{>}(e)) > 1/2\} \tag{3.20}
\]

and a similar statement holds for odd \( n \). Combining this with

\[
\{n \in \mathbb{N}_0 : s_{n+1}(BS(e)) > 1\} = \{n \in 2\mathbb{N}_0 : s_{n+1}(BS(e)) > 1\} \cup \{n \in (2\mathbb{N}_0 + 1) : s_{n+1}(BS(e)) > 1\}
\]
and (3.3) yields (3.18), because

\[
N_e(V) = \#\{n \in \mathbb{N} : s_n(BS(e)) > 1\} \\
\leq 2(\#\{n \in \mathbb{N}_0 : s_{n+1}(BS_<(e)) > 1/2\} + \#\{n \in \mathbb{N}_0 : s_{n+1}(BS_>(e)) > 1/2\}) \\
\leq 2(2^{m_1}\|BS_<(e)\|^{m_1}_{S^1_{\infty}} + 2^{m_2}\|BS_<(e)\|^{m_2}_{S^2_{\infty}}). \tag{3.21}
\]

To prove (3.19), we first write

\[
N_e(V) = n(1, BS(e))\theta(1 - e) + n(1, BS(e))\theta(e - 1). \tag{3.22}
\]

The second summand is estimated using the Kato–Seiler–Simon inequality by

\[
n(1, BS(e))\theta(e - 1) \leq \|BS(e)\|^{m_2}_{m_2}\theta(e - 1) \lesssim \|V\|^{m_2}_{m_2}e^{d/s - m_2}\theta(e - 1). \tag{3.23}
\]

The first summand in (3.22) is estimated using (3.18) by

\[
n(1, BS(e))\theta(1 - e) \lesssim (\|BS_<(e)\|^{m_1}_{m_1} + \|BS_>(e)\|^{m_2}_{m_2})\theta(1 - e) \\
\lesssim (e^{1-m_1}\|V\|^{m_1}_{m_1} + \|V\|^{m_2}_{m_2})\theta(1 - e), \tag{3.24}
\]

where we used the steps in the proof of (3.6).

\[
\]

3.2. Sums of powers of eigenvalues. We now use (3.3)–(3.5) and Theorem 3.4 to obtain estimates for sums of powers of eigenvalues of $T - V$.

**Theorem 3.4.** Suppose $T(\xi)$ satisfies (1)-(3) in Assumption 1.1.

1. If $\gamma > 0$ and $V \in L^{\gamma+1} \cap L^{\gamma+d/s}(\mathbb{R}^d)$ then there exists a constant $c_S > 0$ (which also depends on $d, s, m, \gamma$) such that

\[
\text{Tr}_{L^2(\mathbb{R}^d)}(T(-i\nabla) - V)_\gamma^\gamma \leq c_S \int_{\mathbb{R}^d} (V_+(x)^{\gamma+1} + V_+(x)^{\gamma+d/s}) \, dx. \tag{3.25}
\]

2. Suppose $T$ also satisfies (4) in Assumption 1.1 with $r > 0$. Let $q \in [1, r+1]$ and $m = \sigma(q, r)$. Suppose additionally $m > d/s$. If $\gamma > m - d/s$ and $V \in L^q \cap L^{\gamma+d/s}(\mathbb{R}^d)$, then there is a constant $c_S$ (which also depends on $d, s, m, q, \gamma, r$) such that

\[
\text{Tr}_{L^2(\mathbb{R}^d)}(T(-i\nabla) - V)_\gamma^\gamma \leq c_S(\|V_+\|_q^m + \|V_+\|^{\gamma+d/s}_{\gamma+d/s}). \tag{3.26}
\]

**Proof.** By the variational principle we can assume $V = V_+ \geq 0$. To prove (3.25) we apply (3.19) in Corollary 3.3 for any $m_1 > 1$ and $m_2 > d/s$ and obtain

\[
N_{e/2}(V(x) - e/2)_+ \leq c_S \left[ e^{1-m_1} \int_{\mathbb{R}^d} (V(x) - e/2)_+^{m_1} + e^{d/s - m_2} \int_{\mathbb{R}^d} (V(x) - e/2)_+^{m_2} \right].
\]
Plugging this into (3.5) with \( \gamma > \max\{m_1 - 1, m_2 - d/s\} \) yields
\[
\text{Tr}(T(-i\nabla) - V)^\gamma \\
\leq c_S \int_0^\infty \, de \left[ e^{\gamma - m_1} \int_{\mathbb{R}^d} \, dx \, (V(x) - e/2)^{m_1} + e^{\gamma - 1 + d/s - m_2} \int_{\mathbb{R}^d} \, dx \, (V(x) - e/2)^{m_2} \right] \\
\leq c_S \int_{\mathbb{R}^d} \, dx \, (V(x)^{\gamma + 1} + V(x)^{\gamma + d/s}),
\]
where \( c_S \) also depends on \( d, s, m, \gamma \). This proves (3.25).

To prove (3.26) we instead use (3.7) in Theorem 3.1 and plug the right side of \( N_{e/2}((V(x) - e/2)_+) \)
\[
\leq c_S(e^{d/s - m} \theta(e - 1) + \theta(1 - e)) \int_{\mathbb{R}^d} \, (V(x) - e/2)^m \\
+ c_S \theta(1 - e) \log(2 + 1/e) \left( \int_{\mathbb{R}^d} \, (V(x) - e/2)^q \right)^{\frac{m}{q}} \\
\leq c_S e^{d/s - m} \int_{\mathbb{R}^d} \, (V(x) - e/2)^m + c_S \theta(1 - e) \log(2 + 1/e) \left( \int_{\mathbb{R}^d} \, (V(x) - e/2)^q \right)^{\frac{m}{q}}
\]
into (3.5). For \( \gamma + d/s > m > d/s \) the first summand gives again rise to
\[
\int_0^\infty \, de \, e^{\gamma - 1 + d/s - m} \int_{\mathbb{R}^d} \, (V(x) - e/2)^m \, dx \leq c_S \int_{\mathbb{R}^d} \, V(x)^{\gamma + d/s} \, dx,
\]
whereas the second summand contributes with
\[
\int_0^1 \, de \, e^{\gamma - 1} \log(2 + 1/e) \left( \int_{\mathbb{R}^d} \, (V(x) - e/2)^q \right)^{\frac{m}{q}} \leq c_S \|V\|_q^m
\]
to the left hand side of (3.26) for all \( \gamma > 0 \). (As before, \( c_S \) also depends on \( d, s, m, q, \gamma, r \))
This concludes the proof. \( \square \)

**Remark 3.5.** (1) The term \( \|V^\gamma\|_{\gamma + d/s} \) on the right sides of (3.25)-(3.26) comes from high energies as can be seen from the proofs of (3.6)-(3.7). In Theorem 4.2 we will see that this term is absent for operators in \( L^2(\mathbb{Z}^d) \) since only low energies are present.

(2) The term \( \|V^\gamma\|_{\gamma + d/s} \) is necessary, which can be seen by repeating the arguments in the proof of Theorem 3.1 with \( |\Delta + \mu| \) and letting \( \mu \to 0 \).

(3) Estimate (3.26) also holds in case the level sets of \( T \) are not curved and can be seen as a Lieb–Thirring inequality since the right hand side is “local” in the sense that it involves only integrals over \( V \). In contrast, (3.26) requires non-vanishing Gaussian curvature of the level sets. Moreover, (3.26) is non-local in the sense that it involves powers of integrals of \( V \).

### 3.3. Sums of logarithms of eigenvalues

Suppose \( r = (d - 1)/2, s \in [2d/(d + 1), d), V \in L^{(d+1)/2}, \) and assume that \( T \) satisfies (1)-(4) in Assumption 1.1. By [9]...
Theorem 4.2], for any eigenvalue $a_j^S > 0$ of the operator $V_S = F_S V F_S^*$ in $L^2(S)$, there exists a negative eigenvalue $-e_j(\lambda) < 0$ of $H_\lambda = T - \lambda V$ with weak coupling limit

$$e_j(\lambda) = \exp \left( -\frac{1}{2\lambda a_j^S} (1 + o(1)) \right), \quad \lambda \to 0. \quad (3.27)$$

(Eigenvalues $-e_j < 0$ corresponding to zero-eigenvalues of $V_S$ obey $e_j(\lambda) = e^{-c_j^\lambda \lambda^{-2}}$ for some $c_j > 0$, cf. [9, Theorem 4.4]). On the other hand, as we have seen in (3.16) in Remark 3.2, if the $j$-th eigenvalue $-e_j(\lambda)$ is greater than $-1$, then it satisfies

$$e_j(\lambda) \leq \exp \left( -\frac{1}{2\lambda a_j^S} (1 + o(1)) \right), \quad \lambda \to 0. \quad (3.28)$$

Formulae (3.27)-(3.28) illustrate that the eigenvalues of $H_\lambda$ approach $\inf \sigma_{\text{ess}}(H_\lambda) = 0$ exponentially fast. This suggests to compute logarithmic moments of eigenvalues,

$$\sum_j \left( \frac{1}{\log((1/e_j))} \right)^\gamma, \quad \gamma > 0.$$

(Note that $1/\log(1/x) \geq x$ for $0 < x < 1/2$, say.) For those eigenvalues $e_j(\lambda)$ corresponding to the $a_j^S$ in the asymptotics (3.27), estimate (2.8) implies, for $\lambda$ in a sufficiently small open neighborhood of 0,

$$\sum_j \left( \frac{1}{\log((1/e_j(\lambda)))} \right)^{\sigma(q)} \sim \sum_j \left( 1 + \frac{1}{2\lambda a_j^S} \right)^{-\sigma(q)} \sim \lambda^{\sigma(q)} \text{Tr}(V_S)^{\sigma(q)} \quad (3.29)$$

$$\lesssim \lambda^{\sigma(q)} \|V\|^{\sigma(q)}$$

where $\sigma(q)$ is as in (2.7) and $q \in [1, (d + 1)/2]$. We now prove analogous estimates for $\lambda = 1$, in which case we cannot use the results in the weak coupling regime.

**Theorem 3.6.** Let $H = T - V$ with $T$ satisfying (1)-(4) in Assumption 1.1 with $r > 0$. Let $q \in [1, r + 1]$ and $m = \sigma(q, r)$. Suppose additionally $m > d/s$ and let $V \in L^m \cap L^q(\mathbb{R}^d)$. Then for any $\gamma > m$ there is a constant $c_S$ (which also depends on $d, s, m, q, \gamma, r$) such that

$$\sum_j \left[ \log((1/e_j)) \right]^{-\gamma} \leq c_S \|V_+\|_m^m + \|V_+\|_q^m. \quad (3.30)$$

Moreover, if $q \geq d/s$, then

$$\sum_j \left[ \log((1/e_j)) \right]^{-\gamma} \leq c_S \|V_+\|_q^m. \quad (3.31)$$

**Proof.** By the variational principle we can again assume $V = V_+$. To estimate the left side of (3.30) we use

$$\frac{1}{(\log((e^{-1})))^\gamma} = \gamma \int_0^e \left( \log((r^{-1})) \right)^{-\gamma-1} \cdot \left( \frac{1}{r} \right)^{-2} \frac{dr}{r^3} \quad (3.32)$$
for $\gamma > 0$. Thus,
\[
\sum_j \frac{1}{(\log((1/e_j)))^\gamma} = \gamma \int_0^\infty \left( \log((r^{-1})) \right)^{-\gamma - 1} \cdot \left( \frac{1}{r} \right)^{-2} \sum_j \theta(e_j - r) \frac{dr}{r^3} \quad \text{(3.33)}
\]
\[
= \gamma \int_0^\infty \left( \log((r^{-1})) \right)^{-\gamma - 1} \cdot \left( \frac{1}{r} \right)^{-2} N_r(V) \frac{dr}{r^3}.
\]
By (3.3) and (3.7) in Theorem 3.1 for $m > d/s$, we estimate
\[
N_r(V) \lesssim \|V\|_m^m + \left( \log \left( 2 + \frac{1}{r} \right) \right)^m \|V\|_q^m. \quad \text{(3.34)}
\]
Thus, the left side of (3.30) can be estimated by
\[
\sum_j \left( \frac{1}{\log((1/e_j)))^\gamma} \right) \lesssim \|V\|_m^m \int_0^\infty \frac{dr}{r^3} \left( \log((r^{-1})) \right)^{-\gamma - 1} + \|V\|_q^m \int_0^\infty \frac{dr}{r^3} \left( \log((r^{-1})) \right)^{-\gamma - 1} \cdot \left( \log \left( 2 + \frac{1}{r} \right) \right)^m \lesssim \|V\|_m^m + \|V\|_q^m. \quad \text{(3.35)}
\]
This concludes the proof of (3.30). The proof of (3.31) is completely analogous, but uses (3.8) instead of (3.7). Thus, estimate (3.34) is replaced by
\[
N_r(V) \lesssim \|V\|_m^m \left[ 1 + \left( \log \left( 2 + \frac{1}{r} \right) \right)^m \right]. \quad \text{(3.36)}
\]
Proceeding as in the proof of (3.30) concludes the proof of (3.31).

Remarks 3.7. (1) In contrast to the right side of (3.25), the powers of $V$ appearing on the right side of (3.30) are all the same.

(2) For $r = (d - 1)/2$ and $m = d + 1$ the power $d + 1$ on the right side of (3.30) is consistent with that on the right side of (3.29). However, (3.30) is slightly weaker than (3.29) due to the assumption $\gamma > d + 1$ and, if $q < d/s$, the additional $\|V\|_{d+1}$ term on the right of (3.30).

(3) We do not know whether the restriction $\gamma > m$ (especially $\gamma > d + 1$ for $r = (d - 1)/2$ and $m = d + 1$) is necessary.

3.4. CLR bounds in $L^2(\mathbb{R}^d)$. Recall that $N_0(V_+)$ equals the number of eigenvalues of $V_+^{1/2}T^{-1}V_+^{1/2}$ above one, which can be estimated by (3.3). Formula (2.4), Cwikel’s bound [11], i.e.,
\[
\|f(-i\nabla)g(x)\|_{S^{p,\infty}(L^2(\mathbb{R}^d))} \lesssim_p \|f\|_{L^p,\infty(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}, \quad p \in (2, \infty),
\]
and (3.3) yield the classical Cwikel–Lieb–Rosenbljum (CLR) bound [11, 38, 47]
\[
\|V_+^{1/2}(-\Delta)^{-1}V_+^{1/2}\|_{S^{d/2,\infty}(L^2(\mathbb{R}^d))} \lesssim_d \|
\]
for the number of negative eigenvalues $N_0(V)$ of $T - V$ when $T = -\Delta$ in $d \geq 3$. Such bounds can never hold in $d = 1, 2$, or for $T$ satisfying Assumption \ref{assumption1} in $d \geq 2$ due to the existence of weakly coupled bound states \[20, 29, 31, 32, 9\].

Interestingly, using (3.37), one does obtain a CLR bound for powers $T = |\Delta + 1|^{1/\sigma}$ of the BCS operator $|\Delta + 1|$ in $L^2(\mathbb{R}^d)$ when $\sigma > 1$. This follows from the uniform bound $\int_0^1 (t + e)^{-1/\sigma} dt \lesssim 1$ for all $e \geq 0$ and $\sigma > 1$. The following theorem generalizes this observation to powers $T^{1/\sigma}$ with $T$ satisfying (1)-(3) in Assumption \ref{assumption1} to all $d \in \mathbb{N}$. The proof is inspired by Frank [18], which, in turn, uses ideas of Rumin [48, 49].

**Theorem 3.8.** Let $d \in \mathbb{N}$, $\sigma > 1$, and suppose $T(\xi)$ satisfies (1)-(3) in Assumption \ref{assumption1} with the weaker assumption $s \leq d$. Then, for $V \in L^\sigma \cap L^\sigma_{d/s}(\mathbb{R}^d)$, one has

$$n(1, |V|^\frac{1}{2} T^{-1/\sigma} V^{\frac{1}{2}}) \lesssim_{\sigma, \sigma, s, d, T} \|V_+\|_{L^\sigma(\mathbb{R}^d)} + \|V_+\|_{L^{\sigma d/s}(\mathbb{R}^d)}.$$  

(3.38)

**Proof.** By the variational principle we can assume $V = V_+$. We first show how to prove (3.38) for $s = d \in \mathbb{N}$ using Cwikel’s estimate (3.37). For $\beta > 0$ a straightforward computation shows

$$\|T(\xi)^{-1/(2\sigma)}\|_{L^{2p} \cap L^{\infty}(\mathbb{R}^d)} = \sup_{\beta > 0} \beta^{-2p}\{\xi \in \mathbb{R}^d : T(\xi)^{-1/(2\sigma)} > 1/\beta\}$$

$$\lesssim \sup_{\beta > 0} \beta^{-2p}\left(\beta^{2\sigma} 1_{\{\beta < 1\}} + \beta^{2\sigma d/s} 1_{\{\beta \geq 1\}}\right).$$  

(3.39)

For the right side to be finite we need $p = \sigma$ and $s = d$. Thus, by (3.3) and Cwikel’s estimate (3.37), we obtain for $\sigma > 1$,

$$n(1, |V|^\frac{1}{2} T^{-1/\sigma} V^{\frac{1}{2}}) \leq \|V_+\|_{L^{\sigma}(\mathbb{R}^d)} + \|V_+\|_{L^{\sigma d/s}(\mathbb{R}^d)} \lesssim \|V\|_{L^{\sigma}(\mathbb{R}^d)},$$

which concludes the proof for $s = d$. We will now show (3.38) for $s < d$ by proceeding as in \[18\]. Let $\Gamma$ be an arbitrary operator in $L^2(\mathbb{R}^d)$ satisfying $0 \leq \Gamma \leq T^{-1/\sigma}$ and $\rho_\Gamma(x) := \Gamma(x, x)$. Let $P_E := 1_{(E, \infty)}(T^{1/\sigma})$ and $P^\perp_E = 1 - P_E$. We shall now estimate

$$\text{Tr}(\Gamma^{1/2} T^{1/\sigma} \Gamma^{1/2}) = \int_{\mathbb{R}^d} dx \int_0^\infty dE (P_E \Gamma P_E)(x, x)$$

(3.40)

from below. By a density argument it suffices to consider the case where $\Gamma$ has finite-rank and smooth eigenfunctions. For any subset $\Omega \subseteq \mathbb{R}^d$ of finite measure we have

$$\left(\int_{\Omega} \rho_\Gamma(x) dx\right)^{1/2} \leq \|\Gamma^{1/2} 1_\Omega\|_2 \leq \|\Gamma^{1/2} P_E 1_\Omega\|_2 + \|\Gamma^{1/2} P^\perp_E 1_\Omega\|_2$$

$$\leq \|\Gamma^{1/2} P_E 1_\Omega\|_2 + \|\Omega\|^{1/2} \sqrt{F(E)}$$

(3.41)
where we used $\Gamma \leq T^{-1/\sigma}$ and defined
\[
F(E) := \frac{\|T^{-1/(2\alpha)}P_E^\perp 1_{\Omega}\|^2}{|\Omega|} = \int_{\mathbb{R}^d} \frac{d\xi}{T(\xi)^{1/\sigma}} 1_{\{T(\xi)^{1/\sigma} < E\}}
\]
\[
= \int_{\mathbb{R}^d} d\xi \: 1_{\{T(\xi)^{1/\sigma} < E\}} \int_0^|\xi| dz \: 1_{\{z < T(\xi)^{-1/\sigma}\}} \tag{3.42}
\]
\[
= \int_0^\infty dz \: |\{\xi \in \mathbb{R}^d : T(\xi)^{1/\sigma} < E \wedge z^{-1}\}|.
\]
By (3.39) we have
\[
F(E) \leq \int_0^\infty dz \left[ (E \wedge z^{-1})^{\sigma} 1_{\{E \wedge z^{-1} < 1\}} + (E \wedge z^{-1})^{\sigma/s} 1_{\{E \wedge z^{-1} > 1\}} \right]
\]
\[
= E^{\sigma-1} \left[ \int_0^1 dz \: 1_{\{E < 1\}} + \int_1^\infty dz \: z^{-\sigma} 1_{\{z > E\}} \right] + E^{\sigma/s-1} \left[ \int_0^1 dz \: 1_{\{E > 1\}} + \int_1^\infty dz \: z^{-\sigma/s} 1_{\{z < E\}} \right] - \sigma^{-1} + E^{\sigma/s-1} \tag{3.43}
\]
From this and (3.41), it follows from Lebesgue’s differentiation theorem that
\[
(P_E^\perp P_E)(x, x) \geq \left( \sqrt{\rho_T(x)} - \sqrt{F(E)} \right)^2 \geq \left( \sqrt{\rho_T(x)} - c \cdot (E^{\sigma-1} + E^{\sigma/s-1}) \right)^2 \tag{3.44}
\]
for almost every $x \in \mathbb{R}^d$. Integration over $E$ shows
\[
\text{Tr}(\Gamma T^{1/\sigma}) \geq \int_{\mathbb{R}^d} dx \left( \rho_T(x)^{\sigma} 1_{\{\rho_T \leq 1\}} + \rho_T(x)^{\sigma/s} 1_{\{\rho_T \geq 1\}} \right) \tag{3.45}
\]
for all $0 \leq \Gamma \leq T^{-1/\sigma}$. By a slight generalization of the duality principle in [18, Lemma 2.4], formula (3.45) is equivalent to
\[
\text{Tr}(T^{-\frac{1}{\sigma}} VT^{-\frac{1}{\sigma}} - \mu) \geq \mu^{-\sigma+1} \int_{\mathbb{R}^d} V^{\sigma}(x) dx + \mu^{\sigma/s+1} \int_{\mathbb{R}^d} V^{\sigma/s}(x) dx \tag{3.46}
\]
for any $\mu > 0$. Noting that for any $\mu < 1$, we have
\[
n(1, V^{\frac{1}{\sigma}} T^{-1/\sigma} V^{\frac{1}{\sigma}}) = n(1, T^{-\frac{1}{\sigma}} VT^{-\frac{1}{\sigma}}) \leq (1 - \mu)^{-1} \text{Tr}(T^{-\frac{1}{\sigma}} VT^{-\frac{1}{\sigma}} - \mu), \tag{3.47}
\]
which concludes the proof of Theorem 3.8.

4. Schrödinger operators with degenerate kinetic energy in $\ell^2(\mathbb{Z}^d)$

In this section we prove analogs of the previous results for lattice Schrödinger operators. We first review two instances of the Laplacian in $\ell^2(\mathbb{Z}^d)$ and discuss an analog of the BCS operator $|\Delta + 1|$ in $\ell^2(\mathbb{Z}^d)$. Subsequently, we state and prove our results on numbers and sums of eigenvalues.

4.1. Laplace and BCS-type operators in $\ell^2(\mathbb{Z}^d)$. 
4.1.1. Ordinary lattice Laplace. The standard lattice Laplacian is defined by

\[-\Delta u(n) = \frac{1}{2d} \sum_{\|m-n\|=1} u(m).\]  

(4.1)

Its spectrum is absolutely continuous and equal to \([-1, 1]\). The Fourier multiplier associated to (4.1) is given by \(d^{-1} \sum_{j=1}^{d} \cos(2\pi \xi_j)\). Let \(Z\) denote the set of critical values of this symbol. The level sets

\[S_t := \{ \xi \in \mathbb{T}^d : \frac{1}{d} \sum_{j=1}^{d} \cos(2\pi \xi_j) = t \}, \quad t \in [-1, 1] \setminus Z\]  

(4.2)

are strictly convex (i.e. have everywhere positive Gaussian curvature) in \(d = 2\), cf. [50, Lemma 3.3]. This implies that, for \(d = 2\), item (4) in Assumption 1.1 holds with \(r = 1/2\). In higher dimensions, \(S_t\) is not convex for \(|t| < 1 - 2/d\), cf. [51]. For \(d = 3\), Erdős–Salmhofer [14] obtained the sharp decay of the Fourier transform of the surface measure up to logarithmic factors. Recently, Schippa and the first author [10] provided a simpler proof and obtained the sharp bound

\[|\langle d\Sigma S_t \rangle \psi(x)| \lesssim (1 + |x|)^{-3/4},\]

provided \(t \neq 1\) (The level set \(S_1\) contains a flat umbilic point, see [14]). This also follows from results of Taïrâ [55], which are based on the work of Ikromov and Müller [35] and involve Newton polygon methods. We conclude that, for \(d = 3\), assumption (4) holds with \(r = 3/4\). We are not aware of any sharp estimates in dimensions \(d > 3\).

4.1.2. Molchanov–Vainberg Laplace. Molchanov and Vainberg [44] considered the following modification of \(-\Delta\), which is defined by

\[-\Delta_{\text{MV}} \psi(n) = 2^{-d} \sum_{\|m-n\|=\sqrt{d}} \psi(m).\]  

(4.3)

Again, its spectrum is absolutely continuous and equal to \([-1, 1]\). The level sets of the associated Fourier multiplier \(\prod_{j=1}^{d} \cos(2\pi \xi_j)\) are given by

\[S_t := \{ \xi \in \mathbb{T}^d : \prod_{j=1}^{d} \cos(2\pi \xi_j) = t \}, \quad t \in [-1, 1] \setminus Z.\]

The advantage over the standard Laplacian is that the level sets \(S_t\) are strictly convex for all \(t \in (-1, 1)\) as Poulin [46, Theorems 1.1 and 3.4] showed. Hence, for the Molchanov–Vainberg Laplacian, item (4) in Assumption 1.1 holds with \(r = (d-1)/2\) for all \(d \geq 2\).
4.1.3. An analog of the BCS operator in $\ell^2(\mathbb{Z}^d)$. We can define the analog of the BCS operator in $\ell^2(\mathbb{Z}^d)$ by the Fourier multiplier
\[
T(\xi) := |P(\xi) - \mu|,
\]where $P$ is the symbol of the standard Laplacian or the Molchanov–Vainberg Laplacian and $\mu \in [-1, 1] \setminus \mathbb{Z}$ is the Fermi energy (we took $\mu = 1$ in the continuum).

4.2. Number of eigenvalues below a threshold. We now generalize the results of Section 3 to $T - V$ in $\ell^2(\mathbb{Z}^d)$. Our assumptions on $T$ are the same as in Assumption 1.1 with the exception that the ellipticity assumption (3) there is not needed here. We recall that item (4) with $r = (d - 1)/2$ in Assumption 1.1 holds for the BCS operator in (4.4) in $d = 2$ and its analog where $-\Delta$ is replaced by $-\Delta_{MV}$ in all $d \geq 2$.

**Theorem 4.1.** Let $e > 0$ and suppose $T(\xi)$ satisfies (1) and (2) in Assumption 1.1.

1. Let $m \geq 1$. If $V \in \ell^m(\mathbb{Z}^d)$, then there exists a constant $c_S > 0$ (which also depends on $d, \tau, m$) such that
\[
\|BS(e)\|_{S_m(\ell^2(\mathbb{Z}^d))}^m \leq c_S \min\{e^{1-m}, e^{-m}\} \|V\|_{\ell^m(\mathbb{Z}^d)}^m.
\]
2. Suppose $T$ also satisfies (4) in Assumption 1.1 with $r \in (0, (d - 1)/2]$. Let $q \in [1, r + 1]$ and $m = \sigma(q, r)$. If $V \in \ell^q(\mathbb{Z}^d) = \ell^m \cap \ell^q(\mathbb{Z}^d)$, then there is a constant $c_S > 0$ (which also depends on $d, \tau, m, q, r$) such that
\[
\|BS(e)\|_{S_m(\ell^2(\mathbb{Z}^d))}^m \leq c_S \|V\|_{\ell^q}^m \min\{\log(2 + 1/e), 1/e\}^m.
\]

In particular,
\[
\|BS(e)\|_{S_m(\ell^2(\mathbb{Z}^d))}^m \leq c_S [\log(2 + 1/e)]^m \|V\|_{\ell^q}^m \theta(1 - e) + e^{-m} \|V\|_{\ell^q}^m \theta(e - 1).
\]

**Proof.** The proofs of (4.5) and (4.6) are exactly the same as those of (3.6) and (3.7) in the continuum case with two exceptions. Due to the absence of high energies in the estimate involving the Kato–Seiler–Simon inequality, any $e^{d/s}$-factors for $e > 1$ are absent. Secondly, by the nestedness of the $\ell^p$ spaces, we may estimate $\|V\|_{\ell^q} \lesssim \|V\|_{\ell^q}$ since $\sigma(q, r) \geq q$ (cf. (1) in Remark 2.2) to dispose of $\|V\|_{\ell^q}$-norms. Estimate (4.7) follows from (4.5)-(4.6).

4.3. Sums of powers of eigenvalues. The previous estimates allow us to prove an analog of Theorem 3.4 for the lattice Schrödinger operators considered here.

**Theorem 4.2.** Suppose $T(\xi)$ satisfies (1) and (2) in Assumption 1.1.

1. If $\gamma > 0$ and $V \in \ell^{\gamma+1}(\mathbb{Z}^d)$, then there exists a constant $c_S > 0$ (which also depends on $d, \tau, \gamma$) such that
\[
\text{Tr}_{\ell^2(\mathbb{Z}^d)}(T(-i\nabla) - V)_{\gamma}^\gamma \leq c_S \sum_{x \in \mathbb{Z}^d} V_+(x)^{\gamma+1}.
\]
(2) Suppose $T$ also satisfies (4) in Assumption 1.1 with $r \in (0, (d-1)/2]$. Let $q \in [1, r+1]$ and $m = \sigma(q, r)$. Suppose $\delta \in [0, m]$, $\gamma > \delta$, and $V \in \ell^m + \gamma - \delta \cap \ell^q(\mathbb{Z}^d)$. Then $q < m + \gamma - \delta$ and there is a constant $c_S$ (which also depends on $d, r, m, q, \gamma, \delta$) such that

$$\text{Tr}\mathcal{L}(\mathbb{Z}^d)(-i\nabla) - V)^\gamma \leq c_S(\|V_+\|_m^m + \|V_+\|_{m+\gamma-\delta}^m). \quad (4.9)$$

For $\delta = m - 1$, the bound in (4.9) restores $\|V_+\|_{\gamma+1}^{\gamma+1}$ in (4.8).

**Proof.** By the variational principle we can assume $V = V_+ \geq 0$. The proof of (4.8) is the same as that of (3.25) and we omit it. To prove (4.9) we use (4.7) in Theorem 4.1 and $m \geq q$ to bound

$$N_\xi(V) \lesssim_S \log(2 + 1/e)\theta(1 - e) \left( \sum_{x \in \mathbb{Z}^d} (V(x) - e/2)^m \right)^{m/\gamma} + e^{-m}\theta(e - 1) \sum_{x \in \mathbb{Z}^d} (V(x) - e/2)^m \lesssim \log(2 + 1/e)\theta(1 - e)\|V\|_m^m + e^{-\delta} \sum_{x \in \mathbb{Z}^d} (V(x) - e/2)^m$$

for any $0 \leq \delta \leq m$. For $\gamma > \delta$ the second term on the right contributes with

$$\int_0^\infty de e^{\gamma-1-\delta} \sum_x (V(x) - e/2)^m \lesssim_{m, \delta, \gamma} \|V\|_{m+\gamma-\delta}^m,$$

whereas the first term contributes with

$$\int_0^1 de e^{\gamma-1} \log(2 + 1/e)^m \|V\|_m^m \lesssim_{m, \gamma} \|V\|_q^m$$

to the left side of (4.9). This concludes the proof.$\square$

**Remark 4.3.** Bach, Lakaev, and Pedra [2] proved CLR bounds in $d \geq 3$ when the symbol $T \in C^2(\mathbb{T}^d)$ is a Morse function, i.e., it satisfies $T(\xi) \sim |\xi - \xi_0|^2$ near a minimum $\xi_0 \in \mathbb{T}^d$. This is needed [2, p. 21] to apply [18, Theorem 3.2] when computing $\int_{T^{-1}(0, E)} T(\xi)^{-1} d\xi$.

### 4.4. Sums of logarithms of eigenvalues.

**Theorem 4.4.** Let $H = T - V$ in $\ell^q(\mathbb{Z}^d)$ with $T$ satisfying (1), (2), and (4) in Assumption 1.1 with $r \in (0, (d-1)/2]$. Let $q \in [1, r+1]$, $m = \sigma(q, r)$, and $V \in \ell^m \cap \ell^q(\mathbb{Z}^d) = \ell^q(\mathbb{Z}^d)$. Then for any $\gamma > m$ there is a constant $c_S$ (which also depends on $d, s, m, q, \gamma, r$) such that

$$\sum_j \left( \frac{1}{|\log(1/e_j)|} \right)^\gamma \lesssim c_S(\|V_+\|_m^m + \|V_+\|_q^m) \lesssim c_S \|V_+\|_q^m. \quad (4.10)$$

**Proof.** Without loss of generality let $V = V_+$. Using (3.4), the representation (3.32), and (4.7) in Theorem 4.1 for $\gamma > 0$ and $m \geq 1$, i.e.,

$$N_r(V) \lesssim \|V\|_m^m + \|V\|_q^m \left( \log \left( 2 + \frac{1}{r} \right) \right)^m, \quad (4.11)$$
Remark 4.5. We make an observation similar to that after Theorem \ref{thm:3.6}. The right side of (4.10) is bounded by a constant times $\|V_+\|_q^m$ which is consistent with the right side of (3.29) when $r = (d - 1)/2$, $m = d + 1$, and $q = (d + 1)/2$. However, we need to restrict ourselves again to $\gamma > d + 1$ which makes (4.10) weaker compared to (3.29). A similar question arises whether (4.10) can hold for $\gamma < d$.

4.5. A CLR bound for powers of the BCS operator in $\ell^2(\mathbb{Z}^d)$. Let $d \geq 1$. We generalize Theorem \ref{thm:3.8} to $|\Delta + \mu|^{1/s}$ with $s > 1$ and $\mu \in [-1, 1] \setminus \mathbb{Z}$ on $\ell^2(\mathbb{Z}^d)$. To that end we use that Cwikel’s estimate continues to hold in $\ell^2(\mathbb{Z}^d)$. This is a consequence of an abstract theorem by Birman, Karadzhov, and Solomyak \cite[Theorem 4.8]{3}, which also includes an extension of the Kato–Seiler–Simon inequality. Recall that the discrete unitary Fourier transform $F : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ obeys $\|F\|_{\ell^2(\mathbb{Z}^d) \to L^\infty(\mathbb{T})} \leq 1$. Adapted to our setting, their result reads as follows.

**Theorem 4.6** (\cite[Theorem 4.8]{3}). Let $q > 2$, $f \in \ell^q(\mathbb{Z}^d)$, and $g \in L^{q,\infty}(\mathbb{T}^d)$. Then

$$
\|fF^*g\|_{S_{q,\infty}(L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d))} \lesssim_q \|f\|_{\ell^q(\mathbb{Z}^d)} \|g\|_{L^{q,\infty}(\mathbb{T}^d)}.
$$

(4.12)

In combination with (3.3) (as in the proof of Theorem \ref{thm:3.8}), (4.12) yields

**Theorem 4.7.** Let $d \geq 1$, $\mu \in [-1, 1] \setminus \mathbb{Z}$, $\sigma > 1$, $p \in (1, \sigma]$, and $T_\mu(\xi)$ be defined as in \ref{eq:4.4} with the ordinary Laplace operator. Then $T_\mu(\xi)^{-1/(2\sigma)} \in L^{2p,\infty}(\mathbb{T}^d)$ (not necessarily uniformly in $\mu, \sigma, p, d$). Moreover, the number of negative eigenvalues of $(T_\mu)^{1/\sigma} - V$ is bounded by a constant (possibly depending on $\mu, \sigma, p, d$) times $\|V_+\|_{\ell^p(\mathbb{Z}^d)}^p$.

**Proof.** The proof is analogous to that of Theorem \ref{thm:3.8}. The bound for the number of negative eigenvalues follows from the variational principle, (3.3), and (4.12) (with $f(x) = |V(x)|^{1/2}$ and $g(\xi) = (T_\mu(\xi))^{-1/(2\sigma)}$ for $x \in \mathbb{Z}^d$ and $\xi \in \mathbb{T}^d$). Thus, we are left with showing $T_\mu(\xi)^{-1/(2\sigma)} \in L^{2p,\infty}(\mathbb{T}^d)$ with $p \leq \sigma$. Since $|\mathbb{T}^d| = 1$, it suffices to check

$$
|\{\xi \in \mathbb{T}^d : T_\mu(\xi) \leq \beta^{2\sigma}\}| \lesssim_{\mu,d,\sigma,p} \beta^{2p} \quad \text{for} \quad \beta \leq 1.
$$

(4.13)

Since $\mu$ is a given, fixed parameter, we may even suppose $\beta^{2\sigma} < 1 - \mu$ in the following. Then $T_\mu(\xi) \leq \beta^{2\sigma}$ is equivalent to the bounds

$$
-\beta^{2\sigma} \leq -d^{-1} \sum_{j=1}^d \cos(2\pi \xi_j) - \mu \leq \beta^{2\sigma}, \quad \xi_j \in (-\frac{1}{2}, \frac{1}{2}).
$$

(4.14)

Since $1 - x^2/2 \leq \cos x \leq 1 - x^2/(2\pi)$ for all $x \in (-\pi, \pi)$, (4.14) implies

$$
1 - \mu - \beta^{2\sigma} \leq \frac{2\pi^2}{d} |\xi|^2 \leq 1 - \mu + \beta^{2\sigma}.
$$

Thus, the left side of (4.13) is bounded from above by $\beta^{2\sigma} \leq \beta^{2p}$ since $p \leq \sigma$ and $\beta < 1$. This concludes the proof. \qed
We now give an alternative proof of Theorem 3.4 (1) for $\gamma > 0$ using an observation made by Frank [17, p. 794], together with Theorem 3.8.

Theorem A.1. Suppose $T(\xi)$ satisfies (1)-(3) in Assumption 1.1. If $\gamma > 0$ and $V \in L^{\gamma+1} \cap L^{\gamma+d/s}(\mathbb{R}^d)$ then there exists a constant $c_S > 0$ (which also depends on $d, s, \gamma$) such that

$$\text{Tr}_{L^2(\mathbb{R}^d)}(T(-i\nabla) - V)^\gamma \leq c_S \int_{\mathbb{R}^d} (V_+(x)^{\gamma+1} + V_+(x)^{\gamma+d/s}) \, dx.$$  \hspace{1cm} (A.1)

Proof. Without loss of generality we assume $V \geq 0$. For $E > 0$ and $\sigma > 1$ we record

$$T(-i\nabla) + E \geq c_\sigma \cdot (T(-i\nabla)^{1/\sigma} \cdot E^{1/\sigma'})$$  \hspace{1cm} (A.2)

for $\sigma' = (1 - 1/\sigma)^{-1}$ and some $c_\sigma > 0$. This observation and Theorem 3.8 imply that the number of eigenvalues $N(2E, T - V)$ of $T - V$ below $-2E < 0$ is bounded by

$$N(2E, T - V) = N(0, T + E - (V - E)) \leq N(0, c_\sigma E^{1/\sigma'} T^{1/\sigma} - (V - E))$$

$$= n(1, |V - E|^{1/2}(c_\sigma E^{1/\sigma'} T^{1/\sigma} - (V - E)^{1/2}))$$

$$= N(0, c_\sigma T^{1/\sigma} - E^{-1/\sigma'}(V - E))$$

$$\lesssim E^{-\frac{d}{\sigma}} \|(V - E)_+ + E_+^{d/s} \|_{L^\sigma(\mathbb{R}^d)}.$$  \hspace{1cm} (A.3)

Thus, we obtain for any $\gamma > d\sigma/(s\sigma')$,

$$\text{Tr}(T - V)^\gamma = \int_0^\infty d\xi E^{\gamma-1} \cdot N(E, T - V)$$

$$\lesssim \int_0^\infty d\xi \left[ E^{\gamma-1 - \frac{d}{\sigma'}} \int_{\mathbb{R}^d} dx \left( V(x) - E_+^\gamma + E^{\gamma-1 - \frac{d}{\sigma'}} \right) dx \left( V(x) - E_+^\gamma \right) \right]$$

$$\sim \int_{\mathbb{R}^d} (V(x)^{\gamma+1} + V(x)^{\gamma+d/s}) \, dx.$$  \hspace{1cm} (A.4)

This concludes the proof. \hfill \square

Remarks A.2. (1) We do not know whether the CLR bounds in Theorem 3.8 and an argument similar to that in the proof of Theorem A.1 can be used to prove estimates for sums of logarithms of eigenvalues as in Theorem 3.6.

(2) Theorem A.1 for $\gamma = 1$ can be proved using Rumin’s method, see also [23, Proposition 4] or [19, Section 6]. The case $\gamma > 1$ then follows from this together with the argument of Aizenman and Lieb [11] and the observation

$$\int_{\mathbb{R}^d} (T(\xi) - V(x))^\gamma \, d\xi = \int_0^{V(x)} dt \left( V(x) - t \right)^\gamma \int_{S_t} \frac{dS_t}{|\nabla P(\xi)|}$$

$$\sim \int_0^{V(x)} dt \left( V(x) - t \right)^\gamma \cdot (1 + t)^{d/s-1} \sim V(x)^{\gamma+1} + V(x)^{\gamma+d/s}.$$
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