Effect of Thermal Fluctuations on a Charged Dilatonic Black Saturn

Behnam Pourhassan\textsuperscript{a}\textsuperscript{*} and Mir Faizal\textsuperscript{b}\textsuperscript{†}

\textsuperscript{a}School of Physics, Damghan University, Damghan, Iran
\textsuperscript{b}Department of Physics and Astronomy, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada

Abstract

In this paper, we will analyze the effect of thermal fluctuations on the thermodynamics of a charged dilatonic black Saturn. These thermal fluctuations will correct the thermodynamics of the charged dilatonic black Saturn. We will analyze the corrections to the thermodynamics of this system by first relating the fluctuations in the entropy to the fluctuations in the energy. Then, we will use the relation between entropy and a conformal field theory to analyze the fluctuations in the entropy. We will demonstrate that similar physical results are obtained from both these approaches. We will also study the effect of thermal fluctuations on the phase transition in this charged dilatonic black Saturn.

1 Introduction

Black Saturn is an interesting black object in higher dimensions, where a black hole is surrounded by a black ring \cite{1, 2}. The black ring and the black hole are in thermodynamic equilibrium with each other \cite{3}. This thermodynamic equilibrium is obtained because of the rotation of the black ring. However, it is also possible to obtain thermodynamically stable black Saturn solutions with a static black ring \cite{4, 5}. In this case, the black hole remains in thermodynamic equilibrium with the static black ring because of an external magnetic field. It may be noted that conditions for meta-stability of a black Saturn have also been studied \cite{6}. It has been demonstrated that the black Saturn is causal stably on the closure of the domain of outer communications \cite{7}. It has been possible to obtain a relation between the black Saturn and Myers-Perry black holes \cite{8}. Furthermore, the thermodynamics of charged black rings and a dilatonic black Saturn has been analyzed using Einstein-Maxwell-dilaton theory in five dimensions. \cite{9}. In this paper, we will analyze the effects of thermal fluctuations on the thermodynamics of this charged dilatonic black Saturn.

\textsuperscript{*}Email: b.pourhassan@du.ac.ir
\textsuperscript{†}Email: f2mir@uwaterloo.ca
It is possible to study the thermodynamics of black Saturn as an entropy is associated with all black objects. This is needed to prevent the violation of the second law of thermodynamics \cite{10, 11}. This is because the entropy of the universe would spontaneous reduce when an object would crosses the horizon of any black object, if we do not associate an entropy with that black object. So, in order to prevent the violation of second law of thermodynamics for all black objects, an entropy is associated with them. In fact, black holes have more entropy than any other object with the same volume \cite{12, 13}. This maximum entropy which is associated with a black holes is proportional to the area of the horizon \cite{14}. The fact that this entropy scales with the area and not the volume of the black holes has lead to the development of the holographic principle \cite{15, 16}. According to the holographic principle the degrees of freedom in a region of space are equal to the degrees of freedom on the boundary surrounding that region of space. It is expected that this holographic principle will get modified near Planck scale \cite{17, 18}.

It is generally expected that any thermodynamical system will undergo thermal fluctuations. These thermal fluctuations will lead to corrections in the thermodynamics of that system. As black holes are also thermodynamic systems, they will also be effected by thermal fluctuations. In fact, the effect of such thermal fluctuations on the entropy of the black holes has been analyzed \cite{19, 20}. These thermal fluctuations modify the entropy-area law of the black holes. This is an interesting result as in Jacobson formalism, that the Einstein’s equation can be derived from the first law of thermodynamics \cite{21, 22}. This is done by requiring that the Clausius relation holds for all the local Rindler causal horizons through each space-time point. As there exists a relation between the geometry of black holes and thermodynamics, we expect that thermal fluctuations will also give rise to the fluctuations of the metric in Jacobson thermodynamic formalism. Hence, we can expect that these thermal fluctuations in the thermodynamic of black holes to occur because of quantum fluctuations in the geometry of space-time. We can neglect such fluctuations for large black holes. However, as the black holes reduce in size due to Hawking radiation, the effect of quantum fluctuations on the geometry of the black holes cannot be neglected. Thus, at this stage, the effect of thermal fluctuations on the thermodynamics of black holes also cannot be neglected.

It is interesting to note that the thermal fluctuations correct the entropy of the black holes by a logarithmic terms \cite{19, 20}. This is because such logarithmic corrections also occur in various different approaches to quantum gravity. In fact, non-perturbative quantum general relativity has been used for calculating the corrections to the thermodynamics of black holes \cite{23}. In this approach, the leading order corrections to the entropy of a black hole are logarithmic corrections. These logarithmic corrections have been calculated from the density of states of a black hole. This density of states are calculated using the conformal blocks of a well defined conformal field theory. Such logarithmic correction have also been calculated using the Cardy formula \cite{24}. The logarithmic correction for a BTZ have been obtained using an exact partition function \cite{25}. Such corrections terms have also been obtained by analyze the effect of matter fields surrounding a black hole \cite{26, 27, 28}. 

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The string theory considerations have also lead to logarithmic corrections in the entropy of black holes [29, 30, 31, 32]. In fact, such corrections terms have been also obtained for dilatonic black holes [33]. The studies done on the partition function of a black hole have also lead to such logarithmic correction [34]. The generalized uncertainty principle has also been used for calculating the corrections to the thermodynamics of a black hole [35, 36]. The entropy of a black hole gets modified by the generalized uncertainty principle. The correction terms obtained this way can be expressed as logarithmic functions of the area of the horizon.

It may be noted that quantum fluctuations become important for all black geometries, when the size of the such geometries is sufficiently reduced. So, at such small scales the thermal fluctuations also become important for all black geometries. The effect of thermal fluctuations on the thermodynamics of an AdS charged black hole has already been analyzed [37]. This was done by relating the fluctuations in the entropy of this black hole to a conformal field theory. Furthermore, the effect of thermal fluctuations on the thermodynamics of a black Saturn has also been studied [38]. This was done by relating the fluctuations in the entropy of the black Saturn to the fluctuations in its energy. However, so far no work has been done on the effect of thermal fluctuations on the thermodynamics of a charged dilatonic black Saturn. So, in this paper, we will analyze the effects of thermal fluctuations on the thermodynamics of a charged dilatonic black Saturn. We will first analyze the thermal fluctuations of the charged dilatonic black Saturn by relating the fluctuations in the entropy of the charged dilatonic black Saturn to the fluctuations in its energy. Then we will use the relation between the fluctuations in the entropy and a conformal field theory to analyze the effect of such thermal fluctuations. We will demonstrate that both these approaches lead to the same physical effects for the charged dilatonic black Saturn.

2 Charged Dilatonic Black Saturn

In this section, we review some basic properties of charged dilatonic black Saturn [9]. The metric for black Saturn can be written as [9]

\[
\begin{align*}
 ds^2 &= -V_\beta(\rho, z)^{-\frac{2}{3}}H_y H_x \left[ dt + \left( \frac{\omega\psi}{H_y} + q \right) d\psi \right]^2 \\
 &\quad + V_\beta(\rho, z)^{-\frac{4}{3}}H_x \left[ k^2 P(d\rho^2 + dz^2) + \frac{G_y}{H_y} d\psi^2 + \frac{G_x}{H_x} d\varphi^2 \right],
\end{align*}
\]

where \( q \) and \( k \) are constants, and \( \beta \) is related to the charge of the black Saturn. Furthermore, we also have

\[
 V_\beta(\rho, z) = \cosh(\beta)^2 - \frac{H_y}{H_x} \sinh(\beta)^2,
\]
Furthermore, we also have

\[ G_x = \frac{\mu_4}{\mu_3 \mu_5} \rho^2 \]
\[ G_y = \frac{\mu_3 \mu_5}{\mu_4}. \]  \hspace{1cm} (3)

Here, we have used

\[ P = (\mu_3 \mu_4 + \rho^2)(\mu_1 \mu_5 + \rho^2)(\mu_4 \mu_5 + \rho^2), \]  \hspace{1cm} (4)

and

\[ \mu_i = \sqrt{\rho^2 + (z - a_i)^2} - (z - a_i) = R_i - (z - a_i). \]  \hspace{1cm} (5)

Here \( a_i \) \((i = 1, ..., 5)\) are real constant parameters which satisfy

\[ a_1 \leq a_5 \leq a_4 \leq a_3 \leq a_2. \]  \hspace{1cm} (6)

The non-zero components of the vector potential are given by,

\[ A_t = \frac{(H_x - H_y) \sinh(\beta) \cosh(\beta)}{H_x \cosh(\beta)^2 - H_y \sinh(\beta)^2}, \]
\[ A_\psi = \frac{(\omega_\psi + q H_y) \sinh(\beta)}{H_y \sinh(\beta)^2 - H_x \cosh(\beta)^2}. \]  \hspace{1cm} (7)

and the dilaton function is given by,

\[ \Phi = -\frac{\sqrt{6}}{3} \ln \left( \frac{\cosh(\beta)^2 - \frac{H_y}{H_x} \sinh(\beta)^2}{\cosh(\beta)} \right). \]  \hspace{1cm} (8)

Furthermore, we also have

\[ H_x = \frac{M_0 + c_1^2 M_1 + c_2^2 M_2 + c_1 c_2 M_3 + c_1^2 c_2^2 M_4}{F} \]
\[ H_y = \frac{1}{F} \frac{\mu_3}{\mu_4} \left[ \frac{\mu_1}{\mu_2} M_0 - c_1^2 M_1 \frac{\rho^2}{\mu_1 \mu_2} - c_2^2 M_2 \frac{\mu_1 \mu_2}{\rho^2} + c_1 c_2 M_3 + c_1^2 c_2^2 M_4 \frac{\mu_2}{\mu_1} \right], \]  \hspace{1cm} (9)

where \( c_1 \) and \( c_2 \) are real constants, and

\[ M_0 = \mu_2 \mu_5^2 (\mu_1 - \mu_3)^2 (\mu_2 - \mu_4)^2 (\rho^2 + \mu_1 \mu_2)^2 (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_2 \mu_3)^2, \]
\[ M_1 = \mu_2^2 \mu_3 \mu_4 \mu_5 \rho^2 (\mu_1 - \mu_2)^2 (\mu_2 - \mu_4)^2 (\mu_1 - \mu_5)^2 (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_2 \mu_3)^2; \]
\[ M_2 = \mu_2 \mu_3 \mu_4 \mu_5 \rho^2 (\mu_1 - \mu_2)^2 (\mu_1 - \mu_3)^2 (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_2 \mu_5)^2; \]
\[ M_3 = 2 \mu_1 \mu_2 \mu_3 \mu_4 \mu_5 (\mu_1 - \mu_3)(\mu_1 - \mu_5)(\mu_2 - \mu_4)(\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2) \times (\rho^2 + \mu_1 \mu_4)(\rho^2 + \mu_2 \mu_5)(\rho^2 + \mu_2 \mu_5), \]
\[ M_4 = \mu_1^2 \mu_2 \mu_3 \mu_4 \mu_5^2 (\mu_1 - \mu_5)^2 (\rho^2 + \mu_1 \mu_2)^2 (\rho^2 + \mu_2 \mu_5)^2, \]  \hspace{1cm} (10)
with

\[ F = \mu_1\mu_5(\mu_1 - \mu_3)^2(\mu_2 - \mu_4)^2(\rho^2 + \mu_1\mu_3) \times(\rho^2 + \mu_2\mu_3)(\rho^2 + \mu_1\mu_4)(\rho^2 + \mu_2\mu_4)(\rho^2 + \mu_2\mu_5) \times(\rho^2 + \mu_3\mu_5)(\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2)(\rho^2 + \mu_3^2)(\rho^2 + \mu_5^2). \]  

(11)

Here \( \omega_\psi \) is expressed as

\[ \omega_\psi = \frac{2}{F\sqrt{G_x}} \left[ c_1R_1\sqrt{M_0M_1} - c_2R_2\sqrt{M_0M_2} + c_2^2c_2R_2\sqrt{M_1M_4} - c_1c_2^2R_1\sqrt{M_2M_4} \right]. \]

(12)

where \( R_1 \) and \( R_2 \) given by the relation [5]. Free parameters of this model are fixed as [2],

\[ L^2 = a_2 - a_1, \]

(13)

and

\[ c_1 = \pm \sqrt{\frac{2(a_3-a_1)(a_4-a_1)}{a_5-a_1}}. \]

(14)

We also have

\[ c_2 = \sqrt{2}(a_4 - a_2)\sqrt{(a_1-a_3)(a_4-a_2)(a_2-a_5)(a_3-a_5)} \pm (a_2-a_1)(a_3-a_4)\sqrt{(a_1-a_4)(a_2-a_4)(a_1-a_5)(a_2-a_5)(a_3-a_5)} \]

(15)

and

\[ k = \frac{2(a_1-a_3)(a_2-a_4)}{2(a_1-a_3)(a_2-a_4) + (a_1-a_5)c_1c_2} = \frac{2k_1\hat{k}_2}{2k_1\hat{k}_2 + c_1c_2k_3}, \]

(16)

where,

\[ \hat{k}_i = 1 - k_i = 1 - \frac{a_i+2 - a_1}{L^2}, \]

(17)

with \( i = 1, 2, 3 \). Here we have

\[ 0 \leq k_3 \leq k_2 \leq k_1 \leq 1. \]

(18)

The variable \( q \) can be written as

\[ q = \frac{2k_1c_2}{2k_1 - 2k_1k_2 + c_1c_2k_3}. \]

(19)

Thus, all the thermodynamics quantities can be written in terms of \( a_i \) with \( i = 1, 2, 3, 4, 5 \). The Hawking temperatures for the charged dilatonic black Saturn is given by [2],

\[ T = \frac{1}{2\pi L \cosh(\beta)} \sqrt{\frac{k_2\hat{k}_3}{k_1}} \left( \frac{(1+k_2c)^2}{1 + \frac{k_1k_2k_3}{k_1c^2}} \right) \]

\[ + \frac{1}{2\pi L \cosh(\beta)} \sqrt{\frac{k_1\hat{k}_3(k_1-k_3)}{2k_2(k_2-k_3)}} \left( \frac{(1+k_2c)^2}{1 - (k_1-k_2)c + \frac{k_1k_2k_3}{k_1c^2}} \right). \]

(20)
where
\[
c = \frac{1}{k_2} \left( \epsilon \frac{k_1 - k_2}{\sqrt{k_1 k_2 k_3 (k_1 - k_3)}} - 1 \right). \tag{21}
\]
Here we have used \( \epsilon = \pm 1 \). It may be noted that \( \epsilon = 0 \) gives a naked singularity. The entropy of the charged dilatonic black Saturn, in absence of thermal fluctuations, is given by
\[
S_0 = \frac{\pi L^3 \cosh(\beta)}{2} \left( \frac{2k_3}{k_2 k_3} \left( \frac{1}{k_3 k_1} + \frac{k_1 k_2 k_3 c^2}{k_3 k_1} \right) \right)
+ \frac{\pi L^3 \cosh(\beta)}{2} \left( \frac{2k_2 (k_2 - k_3)}{k_1 (k_1 - k_3) k_3} \left( 1 - (k_1 - k_2) c + \frac{k_1 k_2 k_3 c^2}{k_3} \right) \right). \tag{22}
\]
The ADM mass of the charged dilatonic black Saturn is given by,
\[
M_{\text{ADM}} = \frac{3\pi L^2 (1 + \frac{2}{3} \sinh(\beta))}{4k_3 (1 + k_2 c)^2} \left( k_3 (\hat{k}_1 + \hat{k}_2) - 2k_2 k_3 (k_1 - k_2)c + k_2 [k_1 - k_2 k_3 (\hat{k}_2 + \hat{k}_1)] c^2 \right), \tag{23}
\]
which can interpreted as enthalpy \( H = M_{\text{ADM}} \). The results for the ordinary black Saturn are recovered when \( \beta = 0 \). Now if \( a_i \) are depend to each other, then one can express \( a_i \) in terms of \( a_1 \). So, all the thermodynamical relation can express in terms of \( a_1 \). It is possible to write
\[
\begin{align*}
a_2 &= 120a_1^2, \\
a_3 &= 24a_1^2, \\
a_4 &= 6a_1^2, \\
a_5 &= 2a_1^2, \tag{24}
\end{align*}
\]
such that the condition \( (1.8) \) is satisfied. Even though this is a special choice, and we can show that, for any other choice of \( a_i = \alpha_i a_1^x \) \( (i = 2, 3, 4, 5 \text{ and } x = 1, 2, 3, \cdots) \), we will get similar results. It is useful to define the following functions,
\[
f(a_1) = \frac{1}{L} \sqrt{\frac{\hat{k}_2 \hat{k}_3}{2k_1}} \left( \frac{(1 + k_2 c)^2}{1 + \frac{k_1 k_2 k_3 c^2}{k_3 k_1}} \right), \tag{25}
\]
and
\[
g(a_1) = \frac{1}{L} \sqrt{\frac{k_1 \hat{k}_3 (k_1 - k_3)}{2k_2 (k_2 - k_3)}} \left( \frac{(1 + k_2 c)^2}{1 - (k_1 - k_2) c + \frac{k_1 k_2 k_3 c^2}{k_3}} \right). \tag{26}
\]
In the Fig. 1, we give plots of \( f \), \( g \) and \( f + g \) in terms of \( a_1 \). Surprisingly, we can see that \( f + g \) behaves like \( g \), so we can write \( f + g \approx g \).
Figure 1: $f$, $g$, and $f + g$ in terms of $a_1$.

On the other hand, from the fact that $\delta \equiv \frac{4a_1^2}{a_1(24a_1 - 1)} \ll 1$ (for small $a_1$), we have $\frac{1}{f} + \frac{\delta}{g} \approx \frac{1}{f}$. So, we can write

$$T = \frac{m(a_1)}{\cosh \beta},$$

(27)

and

$$S_0 = \frac{\cosh \beta}{n(a_1)},$$

(28)

with,

$$m(a_1) = \frac{g}{2\pi \sqrt{a_1(120a_1 - 1)}},$$

$$n(a_1) = \frac{f}{\pi^2 a_1 (24a_1 - 1) \sqrt{a_1(120a_1 - 1)}},$$

(29)

where $f$ and $g$ are given by (25) and (26). So, we obtain simple expressions for the temperature and the entropy of this black Saturn. In the Fig. 2 we give plots of $m$ and $n$. We find that for the temperature to be positive, $T \geq 0$, $a_1$ should be negative. This is because from the left plot of the Fig. 2 if $a_1 > 0$, then $m < 0$, and so $T < 0$. As the temperature cannot be negative, so we obtain the first constraint on $a_1$ i.e., $a_1 < 0$. Thus, the physical results are restricted to the left side of each plots ($a_1 < 0$). In this region, $m$ and $n$ can be expressed using simple functions. As it illustrated by solid red lines of the Fig. 2 the function $m$ with negative $a_1$ behaves as $0.0005 \frac{1}{a_1^2}$, while the function $n$ with negative $a_1$ behaves as $0.000002 \frac{1}{a_1^4}$. Coefficients 0.0005 and 0.000002 are obtained using the specific choice of coefficients chosen in (24). If we change power (to any even power) or even change coefficients defined in (24), then we can fix function by changing the value of 0.0005 and 0.000002.

Hence, we have a very simple expression for the temperature and entropy of this black Saturn,

$$T = \frac{\delta_1}{a_1^2 \cosh \beta},$$

(30)
Figure 2: $m$ and $n$ in terms of $a_1$. Blue dashed lines represent $m$ and $n$ given by the equation \[
(29)\] while red solid lines represent fitted functions.

\[
S_0 = \frac{a_1^4 \cosh \beta}{\delta_2},
\]

Here $\delta_1 = 0.0005$ and $\delta_2 = 0.000002$, and these values are obtained using the choice of coefficient defined in \[24\]. However, these solutions are general, and they hold for any choice of $a_i = \alpha_i a_1^{2x}$. The only thing which will change for a different choice of $a_i$, is the value of $\delta_1$ and $\delta_2$. Hence, we can consider general solution with arbitrary $\delta_1$ and $\delta_2$. It is clear that for the temperature to be positive, $\delta_1$ has to be positive. In the Fig. 3, we can see general behavior of $T$ and $S_0$. It is illustrated that, temperature and entropy are increasing function of $a_1$.

Figure 3: Typical behavior of $T$ and $S_0$ in terms of $a_1$ for arbitrary $\beta$, $\delta_1 = 0.0005$ and $\delta_2 = 0.000002$. 
3 Energy Fluctuations

The thermal fluctuations correct the thermodynamics of all black objects. This happens because they correct the partition function for these black objects. Thus, various different thermodynamical quantities get corrected because of these thermal fluctuations. It is interesting to note that the entropy of these black objects get corrected by a logarithmic term. So, if $\beta^{-1} = T$ is the temperature for the system close to equilibrium, and $\beta_0^{-1} = T_0$ is the equilibrium temperature of the system, then we have $S = S_0 - (\ln S_0')/2$ where $S_0'' = (\partial^2 S/\partial \beta^2)_{\beta = \beta_0}$. By using the fact that this second derivative of the entropy can be expressed in terms of the fluctuation of energy near the equilibrium, this expression for the total entropy can be written as [20, 38],

$$S = S_0 - \alpha \frac{1}{2} \ln |C_0 T^2| + \cdots,$$  \hspace{1cm} (32)

where $\alpha = 0$ or $\alpha = 1$. Furthermore, almost all different approaches to quantum gravity generate the logarithmic correction term, but the coefficient of this term varies between different approaches, it is useful to keep this analysis general and introduce a general parameter $\alpha$. Here $\alpha = 1$ indicates that we have taken the thermal fluctuations into account, and this hold for a very small black object. The value $\alpha = 0$ indicates that we have not taken thermal fluctuations into account, and this holds for large black objects. Here $S_0$ is the original entropy of the charged dilatonic black Saturn given by the equation (22), and a variable $\alpha$ is introduced to parameterize the effect of thermal fluctuations on the thermodynamics of charged dilatonic black Saturn. Furthermore, we have also defined

$$C_0 = T \frac{\partial S_0}{\partial T}. \hspace{1cm} (33)$$

Using the equations (30), (31) and (33), we can obtain,

$$C_0 = -2a_1^4 \frac{\cosh \beta}{\delta_2}, \hspace{1cm} (34)$$

which means that $\delta_2$ should be negative to have thermodynamical stability. It is possible if we choose some negative coefficients in the relation (24). For example, by choosing $a_2 = 120a_1^2$, $a_3 = -24a_1^2$, $a_4 = 6a_1^2$, $a_5 = -2a_1^2$, one can obtain $\delta_2 \approx -0.001$. However, the positive and the negative values $\delta_2$ do not effect the form of the logarithmic correction (32).

Now using equation (32), one can write the internal energy as

$$E = \int TdS = E_0 - \frac{\alpha}{2} T - \frac{\alpha}{4} T^2, \hspace{1cm} (35)$$

where

$$E_0 = \int TdS_0. \hspace{1cm} (36)$$

This internal energy can be expressed in terms of $a_1$,

$$E = \frac{\delta \left( 8a_1^6 \cosh^2 \beta - 2\alpha \delta_2 a_1^2 \cosh \beta - \alpha \delta_1 \delta_2 \right)}{4\delta_2 a_1^4 \cosh^2 \beta}. \hspace{1cm} (37)$$
We find that for any values of parameter, energy is decreasing function of $\alpha$ when $\delta_1$ is positive, and it is an increasing function of $\alpha$ when $\delta_1$ is negative. However, as $\delta_1$ cannot be negative because $T \geq 0$, so the inner energy cannot be an increasing function of $\alpha$. In the Fig. 4 we can observe the variation of $E$ for various values $\beta$. For the values of $\beta$, such as $10\pi$, the value of energy is $E = -0.25$. However, for the larger values of $\beta$, the energy becomes a constant.

Figure 4: $E$ in terms of $\alpha$ (a) and $a_1$ (b), with $\delta_1 = 0.0005$, $\delta_2 = -0.001$. (a) $a_1 = -0.5$, $\beta = 0$ (blue solid), $\beta = \frac{\pi}{10}$ (cyan dashed), $\beta = \frac{\pi}{5}$ (green dash dotted), $\beta = \frac{\pi}{2}$ (orange dotted), $\beta = 2\pi$ (red long dashed). (b) $\beta = 1$, $\alpha = 0$ (blue dotted), $\alpha = 1$ (red solid).

It is possible to check validity of the first law of thermodynamics,

$$dE = TdS + Ada_1 + Bd\beta,$$

where $A$ and $B$ are the thermodynamic variables dual to $a_1$ and $\beta$, respectively. We will check validity of this equation with the geometro-thermodynamics method at end of this paper. In the case of $A = 0$ and $B = 0$, we have ordinary object, where $T = \frac{dE}{dS}$. The corrected entropy can be written as,

$$S = \frac{a_1^4 \cosh \beta}{\delta_2} - \frac{1}{2}\alpha \ln \left( \frac{\delta_1^2}{\delta_2 \cosh \beta} \right).$$

(39)

Also, the Helmholtz free energy is given by

$$F = E - TS = \frac{\delta_1}{a_1^2 \cosh \beta} \left( \frac{a_1^4 \cosh \beta}{\delta_2} - \frac{\alpha}{2} \ln \left( -2 \frac{\delta_1^2}{\delta_2 \cosh \beta} \right) \right)$$

$$+ \frac{2\delta_1}{\delta_2 a_1^2} - \frac{\alpha \delta_1}{2a_1^2 \cosh \beta} - \frac{\alpha \delta_1^2}{4a_1^4 \cosh^2 \beta}.$$ (40)

After some calculations, we obtain

$$F = F_0 - \frac{\alpha}{2} T(1 - \ln C_0 T^2) - \frac{\alpha}{4} T^2,$$ (41)
where

\[ F_0 = E_0 - T S_0. \]  

(42)

Finally, the specific heat at constant volume,

\[ C = T \frac{\partial S}{\partial T}, \]  

(43)

can be expressed as

\[ C = C_0 - \frac{\alpha}{2} (1 + T), \]  

(44)

where \( C_0 \) given by the equation (33). Using the equation (43), it is easy to find that,

\[ C = C_0 = -\frac{2a^4_1 \cosh \beta}{\delta_2}. \]  

(45)

So, the logarithmic corrections do not have any effect on the specific heat. Furthermore, with negative \( \delta_2 \), the specific heat is always positive, and so we have thermodynamical stability. In fact, our approximations lead to \( S_0 T^2 = C_0 T^2 = \text{constant} \). In order to analyse the effect of logarithmic corrections, we can use equations (30) and (34), to find the specific heat in terms of temperature,

\[ C = -\frac{2a^2_1}{T \delta_2} - \frac{\alpha}{2} (1 + T). \]  

(46)

For the negative \( \delta_2 \), the first term is positive while the second term is negative, so thermodynamical stability needs,

\[ T^2 + T - t \leq 0, \]  

(47)

where \( c = |4 \frac{\delta_1 a^2_1}{\delta_2 \alpha}| \) is a positive quantity. So, we can find the critical temperature,

\[ T_c = \frac{1}{2} [\sqrt{1 + 4t} - 1]. \]  

(48)

Now for \( T \leq T_c \), we have thermodynamical stability. If \( t \leq 1 \), then \( T_c \approx t \).

It is important to note that when \( C_0 < 0 \) in (32), the unperturbed black Saturn is unstable, as \( C_0 \) is the specific heat of the unperturbed solution.

### 3.1 Conformal Field Theory

It is possible to relate the microscopic degrees of freedom of a black object with a conformal field theory [19, 37]. So, using this relation for a charged dilatonic black Saturn, the modular invariance of the partition function of the conformal field theory would constraint the entropy of the charged dilatonic black Saturn \( S(\beta_\kappa) \) [24] to have the form \( S(\beta_\kappa) = a \beta_\kappa^l + b \beta_\kappa^{-j} \), where \( l, j, a, b > 0 \). The extremum of this function defines the equilibrium temperature
as $\beta_0 = (jb/la)^{1/(l+j)} = T^{-1}$. Expanding the entropy of the charged dilatonic black Saturn around this extremum, we obtain

$$S(\beta, \kappa) = [\left(\frac{j}{l}\right)^{1/(l+j)} + \left(\frac{l}{j}\right)^{j/(l+j)}](a^j b^l)^{1/(l+j)}$$
$$+ \frac{1}{2} \left[\left(\frac{l}{l} + j\right)^{j+2/(l+j)} j^{(l-2)/(l+j)} \right] (a^{j} b^{l-2})^{1/(l+j)}$$
$$\times (\beta - \beta_0)^2, \quad (49)$$

Now, we can write

$$S_0 = \left(\frac{j}{l}\right)^{1/(l+j)} + \left(\frac{l}{j}\right)^{j/(l+j)}(a^j b^l)^{1/(l+j)},$$
$$\left(\frac{\partial^2 S(\beta, \kappa)}{\partial \beta^2}\right)_{\beta, \kappa = \beta_0} = (l + j)^{j+2/(l+j)} j^{(l-2)/(l+j)} \left(a^j b^{l-2}\right)^{1/(l+j)}. \quad (50)$$

Thus, we obtain the values of $a, b$, and write

$$\left(\frac{\partial^2 S(\beta, \kappa)}{\partial \beta^2}\right)_{\beta, \kappa = \beta_0} = \mathcal{Y} S_0 T^2, \quad (51)$$

where

$$\mathcal{Y} = \left[\left(\frac{l}{l} + j\right)^{(l+2)/(l+j)} j^{(l-2)/(l+j)} \frac{j}{l}\right]^{2/(l+j)}. \quad (52)$$

The factors $\mathcal{Y}$ which is independent of the parameters in the charged dilatonic black Saturn can be absorbed using some redefinition, just as it was done for other black objects $[19, 37]$. So, using the relation between the corrections to the entropy and a conformal field theory, we obtain $[19, 37]$,

$$S = S_0 - \frac{\alpha}{2} \ln |S_0 T^2| + \cdots, \quad (53)$$

where the variable $\alpha$ is again introduced to parameterize the effect of thermal fluctuations on the thermodynamics of charged dilatonic black Saturn. In this paper, we will study corrected thermodynamics using both relations given by (32) and (53). We will demonstrate that both these expressions lead to the same physical results. Hence, the effect of corrections to the entropy obtained from the fluctuation of energy are the same as the effects of corrections to the entropy obtained using a conformal field theory. Now using equation (53), we can write the internal energy as follow,

$$E = \int T dS = E_0 - \frac{\alpha}{2} \ln S_0 T^2, \quad (54)$$

where $E_0$ given by the equation (36). Also we can also write

$$F = F_0 - \frac{\alpha}{2} (1 - T) \ln S_0 T^2, \quad (55)$$

where $F_0$ given by the equation (42). So, we can express the specific heat as

$$C = C_0 - \alpha \left(1 + \frac{C_0}{2S_0}\right). \quad (56)$$
The specific heat obtained here is similar to the specific heat obtained before in (45). This also means that conformal field theory can be used to analyse the thermodynamical stability of this system. This is because of the similarity between the equation (56) and (45). We can also show that both (32) and (53) yield to the similar result. Now we can write corrected entropy as

\[
S = \frac{a_1^4 \cosh \beta}{\delta_2} - \frac{1}{2} \alpha \ln \left( \frac{-2\delta_1^2}{\delta_2 \cosh \beta} \right),
\]

(57)

where the values of \(\delta_1\) and \(\delta_2\) are fixed. Now we can varies the values of \(a_1\) and \(\beta\). We can see that the only difference with the equation (39) is that \(-\frac{\alpha}{2} \ln 2\), and this does not generate any important physical effect. So, we have demonstrated that the corrections obtained from the fluctuations in the energy are similar to the corrections obtained using a conformal field theory.

It may be noted that it is possible to study the phase transition of this thermodynamical system [41]. In the geometro-thermodynamic formalism [42, 43], the thermodynamic metric given by,

\[
g = \left( E^a \frac{\partial S}{\partial E^a} \right) \left( \eta_b^c \frac{\partial^2 S}{\partial E^c \partial E^d} dE^b dE^d \right),
\]

(58)

where \(\eta_b^c = (-1, 1, 1, \ldots, 1)\) at the equilibrium. Here \(E^a\) are the relevant extensive parameters of the system. For the charged dilatonic black Saturn, these are the black hole charges \(\beta\) and \(a_1\). Therefore the thermodynamic metric reduced to,

\[
g = g_{11} da_1^2 + g_{22} d\beta^2 = \left( a_1 \frac{\partial S}{\partial a_1} + \beta \frac{\partial S}{\partial \beta} \right) \left( -\frac{\partial^2 S}{\partial a_1^2} da_1^2 + \frac{\partial^2 S}{\partial \beta^2} d\beta^2 \right).
\]

(59)

Using the equation (57), we can obtain,

\[
\delta_2^2 g_{11} = -6 \left( 2a_1^4 \beta \cosh \beta \sinh \beta + 8a_1^4 \cosh^2 \beta + \alpha \beta \delta_2 \sinh \beta \right) a_1^2,
\]

\[
4\delta_2^2 \cosh^3 \beta g_{22} = 4a_1^8 \beta \cosh^3 \beta \sinh \beta + 16a_1^8 \cosh^5 \beta + 2a_1^4 \alpha \beta \delta_2 \cosh^3 \beta \sinh \beta + 2a_1^4 \alpha \beta \delta_2 \cosh \beta \sinh \beta + 8a_1^4 \alpha \delta_2 \cosh^2 \beta + \beta \alpha^2 \delta_2^2 \sinh \beta.
\]

(60)

So, the second law of thermodynamics [44], can be expressed as

\[
\frac{\partial^2 S}{\partial a_1^2} + \frac{\partial^2 S}{\partial \beta^2} \geq 0.
\]

(61)

Using the equations (57) and (61), we obtain,

\[
\frac{2a_1^2(12 + a_1^2) \cosh^3 \beta + \alpha \delta_2}{2\delta_2 \cosh^2} \geq 0.
\]

(62)

The second law of thermodynamics is satisfied by taking negative and infinitesimal values of \(a_1\) and \(\delta_2\) along with an appropriate choice of \(\beta\) (for example, \(\alpha = 1\), \(a_1 \geq -0.00650\) and \(\delta_2 = -0.001\)). It is clear that with negative \(\delta_2\), presence of logarithmic corrections are necessary to verify the second law of thermodynamics. For instance, the following values
verify the equation (62), \( \alpha = 1, a_1 = -0.001, -2 \leq \beta \leq 2 \) and \( \delta_2 = -0.001 \).

Now, thermodynamic interaction can be calculated from scalar curvature of the metric i.e., the Ricci scalar \( R \). Analytic expression of \( R \) for the metric (59) is complicated, so we will discuss it graphically. However, we can write explicit expression for some special cases.

If we set \( \alpha = \beta = 0 \) (uncorrected, uncharged) then find,

\[
R = \frac{5\delta_2^2}{8a_1^8}.
\]  

(63)

It is always positive quantity.

If we set \( \beta = 0 \) (uncharged) then find,

\[
R = \frac{(60a^8 + 36a^4\alpha\delta_2 + 7a^2\delta_2^2)\delta_2}{24(4a^8 + 4a^4\alpha\delta_2 + \alpha^2\delta_2^2)a_1^8},
\]  

(64)

which may be negative or positive depend on the values of \( \alpha, a_1 \) and \( \delta_2 \).

If we set \( \alpha = 0 \) (uncorrected) then find,

\[
R = \frac{\delta_2^2 (-\cosh^4 \beta + 8\cosh \beta \sinh \beta + 41 \cosh^2 \beta - \beta^2)}{\cosh^3 \beta a_1^8 ((12\beta^2 + 64) \cosh^3 \beta + \beta \sinh \beta (\beta^2 + 48) \cosh^2 \beta - 12\beta^2 \cosh \beta - \beta^3 \sinh \beta)}.
\]  

(65)

For the general case, we have plots of the Fig. 5. We can observe the phase transition for special values of parameters. In the simplest case, when \( \alpha = \beta = 0 \), we can see divergency of curvature around \( a_1 = 0 \) (blue solid line of the Fig. 5 (a)). Furthermore, when \( \alpha = 0 \) (uncorrected), we can see \( R \geq 0 \) everywhere.

Figure 5: \( R \) in terms of \( a_1 \) (a) and \( \beta \) (b) for \( \delta_2 = -0.001 \).  
(a) \( \alpha = 0, \beta = 0 \) (blue solid); \( \alpha = 1, \beta = 0 \) (red dash); \( \alpha = 0, \beta = 2.4 \) (orange long dash); \( \alpha = 1, \beta = 0.8 \) (cyan dot); \( \alpha = 1, \beta = 1.6 \) (green dash dot).  
(b) \( \alpha = 0, a_1 = -0.1 \) (blue solid); \( \alpha = 1, a_1 = -0.1 \) (red dash); \( \alpha = 1, a_1 = -0.09 \) (orange long dash); \( \alpha = 1, a_1 = -0.04 \) (cyan dot).
4 Conclusions

In this paper, we have analysed the corrections to the thermodynamics of a charged dilatonic black Saturn. Using numerical analysis, we found simple expressions for the temperature and entropy of this charged dilatonic black Saturn. We analysed the effects of thermal fluctuations on the thermodynamics of charged black Saturn using two different methods. Thus, we first analysed the effect of thermal fluctuations by relating the fluctuations in the entropy of this charged dilatonic black Saturn to the fluctuations in its energy. Then we analysed the fluctuations in the entropy using the relation between the entropy of this charged dilatonic black Saturn and a conformal field theory. It was demonstrated that similar physical results were obtained from both these methods. We also analysed the effect such thermal fluctuations will have on the thermodynamic stability of this charged dilatonic black Saturn. The validity of the second law of thermodynamics was investigated using this formalism.

It may be noted thermal instability and thermodynamic geometry of topological dilaton black holes coupled to nonlinear electrodynamics has been analyzed [45]. In this analysis the stability analysis was performed in both canonical and grand canonical ensembles. The phase transition and thermodynamic geometry of Einstein-Maxwell-dilaton black holes has also been discussed [46]. It was observed that this system can have three different critical behaviors near the critical points for these black holes. A thermodynamical metric was used for analyzing the thermodynamical geometry of these black holes. Magnetically charged regular black hole in a model of nonlinear electrodynamics have also been studied [47]. The heat capacity at constant charge was used for analyzing the stability of these black holes. The thermodynamics of rotating thin shells has been studied in the BTZ space-time [48]. The topological black hole solutions have been studied using a third order Lovelock Ads black holes in the presence of nonlinear electrodynamics [49]. It was demonstrated that the thermodynamic quantities of the black hole solutions satisfy the first law of thermodynamics. It would be interesting to analyze the corrections to these black holes from thermal fluctuations. We expect that the corrections to the entropy will have interesting consequences for the physics of all these black holes.

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