Certainty Equivalent Control of LQR is Efficient

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Abstract

We study the performance of the certainty equivalent controller on the Linear Quadratic Regulator (LQR) with unknown transition dynamics. We show that the sub-optimality gap between the cost incurred by playing the certainty equivalent controller on the true system and the cost incurred by using the optimal LQR controller enjoys a fast statistical rate, scaling as the square of the parameter error. Our result improves upon recent work by Dean et al. [11], who present an algorithm achieving a sub-optimality gap linear in the parameter error. A key part of our analysis relies on perturbation bounds for discrete Riccati equations. We provide two new perturbation bounds, one that expands on an existing result from Konstantinov et al. [21], and another based on a new elementary proof strategy. Our results show that certainty equivalent control with \( \varepsilon \)-greedy exploration achieves \( \tilde{O}(\sqrt{T}) \) regret in the adaptive LQR setting, yielding the first guarantee of a computationally tractable algorithm that achieves nearly optimal regret for adaptive LQR.

1 Introduction

One of the most straightforward methods for controlling a dynamical system with unknown transitions is based on the certainty equivalence principle: a model of the system is fit by observing its time evolution, and a control policy is then designed by treating the fitted model as the truth [8]. Despite the simplicity of this method, it is challenging to guarantee its efficiency because small modeling errors may propagate to large, undesirable behaviors on long time horizons. As a result, most work on controlling systems with unknown dynamics has explicitly incorporated robustness against model uncertainty [11, 12, 20, 25, 35, 36].

In this work, we show that for the standard baseline of controlling an unknown linear dynamical system with a quadratic objective function, known as the Linear Quadratic Regulator (LQR), certainty equivalent control synthesis achieves better cost than prior methods that account for model uncertainty. In the case of offline control, where one collects some data and then designs a fixed control policy to be run on an infinite time horizon, we show that the gap between the performance of the certainty equivalent controller and the optimal control policy scales quadratically with the error in the model parameters. Our work improves upon the recent result of Dean et al. [11], who present an algorithm achieving a sub-optimality gap linear in the parameter error. In the case of online control, where one adaptively improves the control policy as new data comes in, our offline result implies that a simple, polynomial time algorithm using \( \varepsilon \)-greedy exploration suffices to achieve nearly optimal \( \tilde{O}(\sqrt{T}) \) regret. Prior to our work, existing algorithms for adaptive
LQR either require sub-routines for which efficient algorithms are not known [1, 14, 19], achieve sub-optimal regret [3, 4, 12], or apply only to scalar systems [5].

This paper is structured as follows. In Section 2 we present background concepts and discuss our main result. In particular, Section 2.2 explains why a quadratic dependence on the estimation error of the sub-optimality gap ensures that an online version of the certainty equivalent controller yields $\tilde{O}(\sqrt{T})$ regret for adaptive LQR. Our results rely on a study of the sensitivity to parameter perturbations of the Bellman equation of LQR, known as the discrete algebraic Riccati equation. In Section 3, we assume the existence of a sensitivity guarantee, and use the guarantee to prove a meta theorem which quantifies the performance of the certainty equivalent controller. Then, in Section 4 we offer two explicit and interpretable upper bounds on the sensitivity of the Riccati solution: one based on a proof strategy proposed by Konstantinov et al. [21] and one based on a direct approach that is of independent interest. We conclude and discuss future directions in Section 5.

1.1 Related Work

Because LQR is a fundamental problem in optimal control, there is a vast literature surrounding it. We focus on related work concerning LQR with unknown dynamics. In particular, we divide existing literature into two main categories: offline batch evaluation and the online adaptive setting.

For the offline batch setting, the work of Fiechter [16] was the first to consider the LQR problem with unknown dynamics. Fiechter proved that the sub-optimality gap $\hat{J} - J_*$ scales as $O(\varepsilon)$ for certainty equivalent control, where $\hat{J}$ denotes the cost achieved by the certainty equivalent controller, $J_*$ denotes the optimal cost, and $\varepsilon$ is the error of the transition parameters. A crucial assumption of his analysis, however, is that the certainty equivalent controller stabilizes the true unknown system. One of our contributions is to give bounds on when this assumption is valid. Recently, Dean et al. [11] proposed a computationally efficient robust controller synthesis procedure which takes model uncertainty into account in the design. They show that the suboptimality gap $\hat{J} - J_*$ of their procedure also scales as $O(\varepsilon)$, where $\hat{J}$ is now the cost achieved by the proposed method. Tu and Recht [33] show that the gap $\hat{J} - J_*$ of certainty equivalent control scales asymptotically as $O(\varepsilon^2)$ instead of $O(\varepsilon)$; our contribution here is to provide a non-asymptotic analogue of this result. Fazel et al. [15] and Malik et al. [23] analyze a model-free approach to policy optimization for LQR, in which the parameters of the controller are directly optimized from sampled rollouts. Malik et al. [23] showed that, after collecting $N$ rollouts, a derivative free optimization method achieves a discounted cost gap that scales as $O(1/\sqrt{N})$ or $O(1/N)$, depending on the oracle model used.

In the online adaptive setting it is well understood that using the certainty equivalence principle without adequate exploration can result in a lack of parameter convergence [7, see e.g.]. Abbasi-Yadkori and Szepesvári [1] showed that optimism in the face of uncertainty (OFU) when applied to adaptive LQR yields $O(\sqrt{T})$ regret. Later work from Faradonbeh et al. [14] removed some unnecessary assumptions of the previous analysis. Ibrahimi et al. [19] showed that when the underlying system is sparse, the dimension dependent constants in the regret bound can be improved. The main issue with OFU is that it is not known how to efficiently solve the non-convex optimization problem required for optimistic exploration. In order to deal with this, both Dean et al. [12] and Abbasi-Yadkori et al. [3] propose polynomial time algorithms for adaptive LQR based on $\varepsilon$-greedy exploration which achieve $O(T^{2/3})$ regret. Prior to this work, it was open whether or not it was possible to achieve $O(\sqrt{T})$ regret with a tractable algorithm. Partial progress towards this was recently obtained by Abeille and Lazaric [5], who show that Thompson sampling achieves $O(\sqrt{T})$ (frequentist) regret for the case when the state and inputs are both scalars. In parallel to our work,
Cohen et al. [10] also give an efficient algorithm based on semidefinite programming that achieves \( \tilde{O}(\sqrt{T}) \) regret and is based on a relaxation developed by Cohen et al. [9]. We first remark that their main result, Theorem 4, requires the initial parameter error to scale as \( \mathcal{O}(1/T^{1/4}) \) in Frobenius norm in order for the guarantee to apply. While they propose a \( \mathcal{O}(\sqrt{T}) \) length warmup period to get around this, we note that our analysis of \( \varepsilon \)-greedy control does not require \( o(T) \) accuracy of the initial parameters. Furthermore, we remark that practically speaking, there are specialized algorithms for the solution to the discrete algebraic Riccati equation, that are more efficient than general off-the-shelf semidefinite programming solvers. We leave a detailed empirical comparison of these algorithms to future work. We note that in a Bayesian setting when there is a prior distribution over the model, Ouyang et al. [26] showed that Thompson sampling achieves \( \tilde{O}(\sqrt{T}) \) expected regret.

Finally, there are several works that study online learning with LQR where the transition dynamics are assumed to be known, but the cost is unknown to the learner Abbasi-Yadkori et al. [2], Cohen et al. [9].

A key part of our analysis involves bounding the perturbation of solutions to the discrete algebraic Riccati equation. While there is a rich line of work studying perturbations of Riccati equations [21, 22, 31, 32], the results in the literature are either asymptotic in nature or difficult to use and interpret. We give two new results. First, we clarify the operator-theoretic result of Konstantinov et al. [21] and provide an explicit upper bound on the perturbation based on their proof strategy. Second, we take a new direct approach and use an extended notion of controllability to give a constructive and simpler result. While the result of Konstantinov et al. [21] applies more generally to systems that are stabilizable, we give examples of linear systems for which our new perturbation result is tighter.

## 2 Main Result

An instance of the linear quadratic regulator (LQR) is defined by four matrices: two matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times d} \) that define the linear dynamics and two positive semidefinite matrices \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{d \times d} \) that define the cost function. Given these matrices, the goal of LQR is to solve the optimization problem

\[
\min_{u_0, u_1, \ldots} \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T} x_t^\top Q x_t + u_t^\top R u_t \right]
\]

subject to \( x_{t+1} = Ax_t + Bu_t + w_t \),

where \( x_t, u_t \) and \( w_t \) denote the state, input (or action), and noise at time \( t \), respectively. The expectation is over the initial state \( x_0 \sim \mathcal{N}(0, I_n) \) and the i.i.d. noise \( w_t \sim \mathcal{N}(0, \sigma^2_w I_n) \). The state and noise vectors are \( n \) and \( d \) dimensional, respectively. The input at time \( t \) is allowed to depend on the state at time \( t \) and all the previous states and actions. Nonetheless, when the problem parameters \( (A, B, Q, R) \) are known the optimal policy is given by linear feedback, \( u_t = K_x x_t \), and can be computed efficiently [6, see e.g.]. More precisely, \( K_x = -(R + B^\top PB)^{-1}B^\top PA \) where \( P \) is the (positive definite) solution to the discrete Riccati equation

\[
P = A^\top PA - A^\top PB (R + B^\top PB)^{-1}B^\top PA + Q
\]

and can be computed efficiently. Problem (1) considers an average cost over an infinite horizon. The optimal controller for the finite horizon variant is also static and linear, but time-varying. The LQR solution in this case can be computed efficiently via dynamic programming.
In this work we are interested in the control of a linear dynamical system with unknown transition parameters \((A_*, B_*)\) based on estimates \((\hat{A}, \hat{B})\). The cost matrices \(Q\) and \(R\) are assumed known. We analyze the certainty equivalence approach: use the estimates \((\hat{A}, \hat{B})\) to solve the optimization problem (1) while disregarding the modeling error, and use the resulting controller on the true system \((A_*, B_*)\). We interchangeably refer to the resulting policy as the certainty equivalent controller or, following Dean et al. [11], the nominal controller. We denote by \(\hat{P}\) the solution to the Riccati equation (2) associated with the parameters \((\hat{A}, \hat{B})\) and let \(\hat{K}\) be the corresponding controller. We denote by \(J(A, B, K)\) the cost (1) obtained by using the actions \(u_t = K\hat{x}_t\) on the system \((A, B)\), and we use \(J\) and \(J_*\) to denote \(J(A_*, B_*, \hat{K})\) and \(J(A_*, B_*, K_*)\), respectively.

Let \(\varepsilon \geq 0\) such that \(\|A_* - \hat{A}\| \leq \varepsilon\) and \(\|B_* - \hat{B}\| \leq \varepsilon\). (Here and throughout this work we use \(\|\cdot\|\) to denote the Euclidean norm for vectors as well as the spectral (operator) norm for matrices.) Dean et al. [11] introduced a robust controller that achieves \(\hat{J} - J_* \leq C_1(A_*, B_*, Q, R)\varepsilon\) for some complexity term \(C_1(A_*, B_*, Q, R)\) that depends on the problem parameters. We show that that the nominal controller \(u_t = \hat{K}\hat{x}_t\) achieves \(\hat{J} - J_* \leq C_2(A_*, B_*, Q, R)\varepsilon^2\). Both results require \(\varepsilon\) to be sufficiently small (as a function of the problem parameters) and it is important to note that \(\varepsilon\) must be much smaller for the nominal controller to be guaranteed to stabilize the system than for the robust controller proposed by Dean et al. [11]. Nonetheless, our result shows that once the estimation error \(\varepsilon\) is small enough, the nominal controller performs better: the sub-optimality gap scales as \(O(\varepsilon^2)\) versus \(O(\varepsilon)\). Both the more stringent requirement on \(\varepsilon\) and better performance of nominal control compared to robust control, when the estimation error is sufficiently small, have been observed empirically Dean et al. [11].

Before we can formally state our result we need to introduce a few more concepts and assumptions. We note that it is common to assume that the cost matrices \(Q\) and \(R\) are positive definite. Under an additional observability assumption, this condition can be relaxed to \(Q\) being positive semidefinite. We leave extending our results to this setting as future work.

**Assumption 1.** The cost matrices \(Q\) and \(R\) are positive definite. Since scaling both \(Q\) and \(R\) does not change the optimal controller \(K_*\), we can assume without loss of generality that \(\varepsilon(R) \geq 1\).

A square matrix \(M\) is stable if its spectral radius \(\rho(M)\) is (strictly) smaller than one. Recall that the spectral radius is defined as \(\rho(M) = \max\{|\lambda| : \lambda\) is an eigenvalue of \(M\)\). A linear dynamical system \((A, B)\) in feedback with \(K\) is fully described by the closed loop matrix \(A + BK\). More precisely, in this case \(x_{t+1} = (A + BK)x_t + w_t\). For a static linear controller \(u_t = Kx_t\) to achieve finite LQR cost it is necessary and sufficient that the closed loop matrix is stable.

In order to quantify the growth or decay of powers of a square matrix \(M\), we define

\[
\tau(M, \rho) := \sup \left\{ \|M^k\|\rho^{-k} : k \geq 0 \right\}.
\]

In other words, \(\tau(M, \rho)\) is the smallest value such that \(\|M^k\| \leq \tau(M, \rho)\rho^k\) for all \(k \geq 0\). We note that \(\tau(M, \rho)\) might be infinite, depending on the value of \(\rho\), and it is always greater or equal than one. If \(\rho\) is larger than \(\rho(M)\), we are guaranteed to have a finite \(\tau(M, \rho)\) (this is a consequence of Gelfand’s formula). In particular, if \(M\) is a stable matrix, we can choose \(\rho < 1\) such that \(\tau(M, \rho)\) is finite. Also, we note that \(\tau(M, \rho)\) is a decreasing function of \(\rho\); if \(\rho \geq \|M\|\), we have \(\tau(M, \rho) = 1\).

At a high level, the quantity \(\tau(M, \rho)\) measures the degree of transient response of the linear system \(x_{t+1} = Mx_t + w_t\). In particular, when \(M\) is stable, \(\tau(M, \rho)\) can be upper bounded by the \(\mathcal{H}_\infty\)-norm of the system defined by \(M\), which is the \(\ell_2\) to \(\ell_2\) operator norm of the system and a fundamental quantity in robust control [see 34, for more details].
Throughout this work we use the quantities $\Gamma_\star := 1 + \max\left\{\|A_\star\|, \|B_\star\|, \|P_\star\|, \|K_\star\|\right\}$ and $L_\star := A_\star + B_\star K_\star$. We use $\Gamma_\star$ as a uniform upper bound on the spectral norms of the relevant matrices for the sake of algebraic simplicity. We are ready to state our meta theorem.

**Theorem 1.** Suppose that $d \leq n$. Let $\gamma$ be a real value such that $\rho(L_\star) \leq \gamma < 1$. Also, let $\varepsilon > 0$ such that $\|\hat{A} - A_\star\| \leq \varepsilon$ and $\|\hat{B} - B_\star\| \leq \varepsilon$ and assume $\|\hat{P} - P_\star\| \leq f(\varepsilon)$ for some function $f$ such that $f(\varepsilon) \geq \varepsilon$. Then, under Assumption 1 the certainty equivalent controller $u_t = \hat{K} x_t$ satisfies the suboptimality gap

$$\hat{J} - J_\star \leq 200 \sigma_w^2 d \Gamma^9 \frac{\tau(L_\star, \gamma)^2}{1 - \gamma^2} f(\varepsilon)^2,$$

as long as $f(\varepsilon)$ is small enough so that the right hand side is smaller than $\sigma_w^2$.

In Section 4 we present two upper bounds $f(\varepsilon)$ on $\|\hat{P} - P_\star\|$: one based on a proof technique proposed by Konstantinov et al. [21] and one based on our direct approach. Both of these upper bounds satisfy $f(\varepsilon) = O(\varepsilon)$ for $\varepsilon$ sufficiently small. For simplicity, in this section we only specialize our meta-theorem (Theorem 1) using the perturbation result from our direct approach.

In order to state the specialization of Theorem 1 using our direct upper bound $f(\varepsilon)$ we need a few more concepts. A linear system $(A, B)$ is called **controllable** when the **controllability matrix**

$$[B \ AB \ A^2 B \ldots \ A^{n-1} B]$$

has full row rank. Controllability is a fundamental concept in control theory; it states that there exists a sequence of inputs to the system $(A, B)$ that moves it from any starting state to any final state in at most $n$ steps. In this work we quantify how controllable a linear system is. We denote, for any integer $\ell \geq 1$, the matrix $C_\ell := [B \ AB \ldots \ A^{\ell-1} B]$ and call the system $(\ell, \nu)$-controllable if the $n$-th singular value of $C_\ell$ is greater or equal than $\nu$, i.e. $\sigma(C_\ell) = \sqrt{\lambda_{\min}(C_\ell C_\ell^T)} \geq \nu$. Intuitively, the larger $\nu$ is, the less control effort is needed to move the system between two different states.

**Assumption 2.** We assume the unknown system $(A_\star, B_\star)$ is $(\ell, \nu)$-controllable, with $\nu > 0$.

We note that this assumption was used in a different context by Cohen et al. [9]. For any controllable system $(A, B)$ and any $\ell \geq n$ there exists $\nu > 0$ such that the system is $(\ell, \nu)$-controllable. Therefore, $(\ell, \nu)$-controllability is really not much stronger of an assumption than controllability. As $\ell$ grows minimum singular value $\sigma(C_\ell)$ also grows and therefore a larger $\nu$ can be chosen so that the system is still $(\ell, \nu)$ controllable.

We remark that controllability is not a necessary condition for LQR to have a well-defined solution: the weaker requirement is that of stabilizability, in which there exists a feedback matrix $K$ such that $A_\star + B_\star K$ is stable. In fact, the result of Dean et al. [11] only requires stabilizability. While our upper bound on $\|\hat{P} - P_\star\|$ requires a controllable system $(A_\star, B_\star)$, the result of Konstantinov et al. [21] only requires a stabilizable system. However, our upper bound $f(\varepsilon)$ on $\|\hat{P} - P_\star\|$ is sharper for some classes of systems, as discussed in Section 4. Together with Theorem 1, our perturbation result, presented in Section 4, yields the following guarantee.

**Theorem 2.** Suppose that $d \leq n$. Let $\rho$ and $\gamma$ be two real values such that $\rho(A_\star) \leq \rho$ and $\rho(L_\star) \leq \gamma < 1$. Also, let $\varepsilon > 0$ such that $\|\hat{A} - A_\star\| \leq \varepsilon$ and $\|\hat{B} - B_\star\| \leq \varepsilon$ and define $\beta = \max\{1, \varepsilon \tau(A_\star, \rho) + \rho\}$. 


Under Assumptions 1 and 2, the certainty equivalent controller \( u_t = \hat{K}x_t \) satisfies the suboptimality gap

\[
\hat{J} - J_* \leq \mathcal{O}(1) \sigma_w^2 d \ell^5 \Gamma_*^{15} \tau(A_*, \rho)^6 \beta(\ell-1) \frac{\tau(L_*, \rho)^2}{1 - \gamma^2} \frac{\max\{\|Q\|^2, \|R\|^2\}}{\min\{\sigma(Q)^2, \sigma(R)^2\}} \left(1 + \frac{1}{\nu}\right)^2 \varepsilon^2, \tag{5}
\]

as long as \( \varepsilon \) is small enough so that the right hand side is smaller than \( \sigma_w^2 \). Here, \( \mathcal{O}(1) \) denotes a universal constant.

The exact form of Equation 5, such as the polynomial dependence on \( \ell, \Gamma_* \), etc., can be improved at the expense of conciseness of the expression. In our proof we optimized for the latter. The factor \( \max\{\|Q\|^2, \|R\|^2\}/\min\{\sigma(Q)^2, \sigma(R)^2\} \) is the squared condition number of the cost function, a natural quantity in the context of the optimization problem (1), which can be seen as an infinite dimensional quadratic program with a linear constraint. The term \( \frac{\tau(L_*, \rho)^2}{1 - \gamma^2} \) quantifies the rate at which the optimal controller drives the state towards zero. Generally speaking, the less stable the optimal closed loop system is, the larger this term becomes.

An interesting trade-off arises between the factor \( \ell^5 \beta(\ell-1) \) (which arises from upper bounding perturbations of powers of \( A_* \) on a time interval of length \( \ell \)) and the factor \( \nu \) (the lower bound on \( \sigma(G_\ell) \)), which is increasing in \( \ell \). Hence, the parameter \( \ell \) should be seen as a free-parameter that can be tuned to minimize the right hand side of (5). Now, we specialize Theorem 2 to a few cases.

Case: \( A_* \) is contractive, i.e. \( \|A_*\| < 1 \). In this case, we can choose \( \rho = \|A_*\| \) and \( \varepsilon \) small enough so that \( \varepsilon \leq 1 - \|A_*\| \). Then, (5) simplifies to:

\[
\hat{J} - J_* \leq \mathcal{O}(1) d \sigma_w^2 \ell^5 \Gamma_*^{15} \tau(A_*, \rho)^6 \frac{\tau(L_*, \rho)^2}{1 - \gamma^2} \frac{\max\{\|Q\|^2, \|R\|^2\}}{\min\{\sigma(Q)^2, \sigma(R)^2\}} \left(1 + \frac{1}{\nu}\right)^2 \varepsilon^2.
\]

Case: \( B_* \) has rank \( n \). In this case, we can choose \( \ell = 1 \). Then, (5) simplifies to:

\[
\hat{J} - J_* \leq \mathcal{O}(1) d \sigma_w^2 \Gamma_*^{15} \tau(A_*, \rho)^6 \frac{\tau(L_*, \rho)^2}{1 - \gamma^2} \frac{\max\{\|Q\|^2, \|R\|^2\}}{\min\{\sigma(Q)^2, \sigma(R)^2\}} \left(1 + \frac{1}{\nu}\right)^2 \varepsilon^2.
\]

2.1 Comparison to Theorem 4.1 of Dean et al. [11].

Theorem 4.1 of Dean et al. [11] states that when their robust synthesis procedure is run with estimates \( (\hat{A}, \hat{B}) \) satisfying \( \|\hat{A} - A_*\| \leq \varepsilon \) and \( \|\hat{B} - B_*\| \leq \varepsilon \), then, as long as \( \varepsilon \leq \frac{1}{2} \sqrt{1 + \|K_*\|} L_* \), the resulting controller satisfies:

\[
\sqrt{\hat{J}} - \sqrt{J_*} \leq 5(1 + \|K_*\|) \Psi_* \sqrt{J_*} \varepsilon. \tag{6}
\]

Here, the quantity \( \Psi_* := \sup_{z \in \mathbb{T}} \|(zI_n - L_*)^{-1}\| \) is the \( \mathcal{H}_\infty \)-norm of the optimal closed loop system \( L_* \). Equation 6 implies that:

\[
\hat{J} - J_* \leq 10(1 + \|K_*\|) \Psi_* J_* \varepsilon + \mathcal{O}(\varepsilon^2). \tag{7}
\]

In order to compare Equation 7 to Equation 5, we upper bound the quantity \( \Psi_* \) in terms of \( \tau(L_*, \gamma) \) and \( \gamma \). In particular, by an infinite series expansion of the inverse:
We can also upper bound $J_\star = \sigma_w^2 \text{Tr}(P_\star) \leq \sigma_w^2 n \Gamma_\star$. Therefore, Equation 7 gives us that:

$$\hat{J} - J_\star \leq O(1) n \sigma_w^2 \Gamma_\star \frac{\tau(L_\star, \gamma)}{1 - \gamma} \varepsilon + O(\varepsilon^2).$$

We see that the dependence on the parameters $\Gamma_\star$ and $\tau(L_\star, \gamma)$ is significantly milder compared to Equation 5. Furthermore, this upper bound is valid for larger $\varepsilon$ than the upper bound given in Theorem 2. Comparing these upper bound suggests that there is a price to pay for obtaining a fast rate, and that in regimes of moderate uncertainty (moderate size of $\varepsilon$), being robust to model uncertainty is important. This observation is supported by the empirical results of Dean et al. [11].

A similar trade-off between slow and fast rates arises in the setting of first-order convex stochastic optimization. The convergence rate $O(1/\sqrt{T})$ of the stochastic gradient descent method can be improved to $O(1/T)$ under a strong convexity assumption. However, the performance of stochastic gradient descent, which can achieve a $O(1/T)$ rate, is sensitive to poorly estimated problem parameters [24]. Similarly, in the case of LQR, the nominal controller achieves a fast rate, but it is much more sensitive to estimation error than the robust controller of Dean et al. [11].

End-to-end guarantees. Theorem 2 can be combined with finite sample learning guarantees (e.g. [11, 13, 28, 29]) to obtain an end-to-end guarantee similar to Proposition 1.2 of Dean et al. [11]. In general, estimating the transition parameters from $N$ samples yields an estimation error that scales as $O(1/\sqrt{N})$. Therefore, Theorem 2 implies that $\hat{J} - J_\star \leq O(1/N)$ instead of the $\hat{J} - J_\star \leq O(1/\sqrt{N})$ rate from Proposition 1.2 of Dean et al. [11]. This is similar to the case of linear regression, where $O(1/\sqrt{N})$ estimation error for the parameters translates to a $O(1/N)$ fast rate for prediction error. Furthermore, Seichowitz et al. [29] and Sarkar and Rakhlin [28] showed that faster estimation rates are possible for some linear dynamical systems. Theorem 2 translates such rates into control suboptimality guarantees in a transparent way.

Finally, we remark that our result explains the behavior observed in Figure 4 of Dean et al. [11]. The authors propose two procedures for synthesizing robust controllers for LQR with unknown transitions: one which guarantees robustness of the performance gap $\hat{J} - J_\star$, and one which only guarantees the stability of the closed loop system. Dean et al. [11] observed that the latter performs better in the small estimation error regime, which happens because the robustness constraint of the synthesis procedure becomes inactive when the estimation error is small enough. Then, the second robust synthesis procedure effectively outputs the certainty equivalent controller, which we now know to achieve a fast rate.

2.2 Nearly optimal $\tilde{O}(\sqrt{T})$ regret in the adaptive setting

The regret formulation of adaptive LQR was first proposed by Abbasi-Yadkori and Szepesvári [1]. The task is to design an adaptive algorithm $\{u_t\}_{t \geq 0}$ to minimize regret, as defined by:

$$\text{Regret}(T) := \sum_{t=1}^{T} x_t^\top Q x_t + u_t^\top R u_t - TJ_\star.$$  

(8)

Abbasi-Yadkori and Szepesvári [1] study the performance of optimism in the face of uncertainty (OFU) and show that it has $\tilde{O}(\sqrt{T})$ regret, which is nearly optimal for this problem formulation. However, the OFU algorithm repeatedly requires solutions to the following optimization problem:

$$(\bar{A}, \bar{B}) = \arg \min_{(A, B) \in \mathcal{S}} J(A, B).$$

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\[ J(A, B) \text{ is shorthand for the cost of the optimal controller on the LQR instance } (A, B, Q, R) \text{ and } S \text{ denotes a confidence set. Even if } S \text{ is a convex set, this optimization problem is non-convex and no provably efficient is currently known.} \]

To deal with the computational issues of OFU, Dean et al. [12] propose to analyze the behavior of \( \varepsilon \)-greedy exploration using the suboptimality gap results of Dean et al. [11]. In the context of continuous control, \( \varepsilon \)-greedy exploration refers to the application of the control law

\[
u_t = \pi(x_t, x_{t-1}, \ldots, x_0) + \eta_t \sim \mathcal{N}(0, \sigma^2_{\eta,t} I_d),\tag{9}\]

where \( \pi \) is the policy, updated in epochs, and \( \sigma^2_{\eta,t} \) is the variance of the exploration noise.

Dean et al. [12] set the variance of the exploration noise as \( \sigma^2_{\eta,t} \sim t^{-1/3} \), and show that their method achieves \( \mathcal{O}(T^{2/3}) \). To see why this is the case, we note that by using epochs of size \( 2^i \), the regret \( (8) \) can be decomposed roughly as \( \text{Regret}(T) = \mathcal{O} \left( T(\hat{J} - J^*) + T\sigma^2_{\eta,T} \right) \). Since the estimation error of the model parameters scales as \( \mathcal{O}((\sigma_{\eta,T} \sqrt{T})^{-1}) \), and since the suboptimality gap \( \hat{J} - J^* \) of the robust controller is linear in the estimation error, we have \( \text{Regret}(T) = \mathcal{O} \left( \frac{1}{\sigma_{\eta,T}} + T\sigma^2_{\eta,T} \right) \).

Then, setting \( \sigma^2_{\eta,t} \sim t^{-1/3} \) balances these two terms and yields \( \mathcal{O}(T^{2/3}) \) regret. However, Theorem 2, which states that the gap \( \hat{J} - J^* \) for the nominal controller depends quadratically on the estimation rate, implies that online certainty equivalent control achieves \( \text{Regret}(T) = \mathcal{O} \left( \frac{1}{\sigma_{\eta,T}} + T\sigma^2_{\eta,T} \right) \). Here, the optimal variance of the exploration noise scales as \( \sigma^2_{\eta,t} \sim t^{-1/2} \), yielding \( \mathcal{O}(\sqrt{T}) \) regret.

**Corollary 1.** (Informal) \( \varepsilon \)-greedy exploration \( (9) \) with exploration schedule \( \sigma^2_{\eta,t} \sim t^{-1/2} \) combined with certainty equivalent control synthesis yields an adaptive LQR algorithm with regret \( (8) \) bounded as \( \mathcal{O}(\sqrt{T}) \), with high probability.

### 3 Proof of Theorem 1

In this section we prove our meta theorem; we show how an upper bound \( \| \hat{P} - P_* \| \leq f(\varepsilon) \) can be used to quantify the mismatch between the performance of the the nominal controller and the optimal controller. First, we upper bound \( \| \hat{K} - K_* \| \) and offer a condition on this mismatch size so that \( A_* + B_* \hat{K} \) is a stable matrix. The next two optimization results are helpful in proving \( \| \hat{K} - K_* \| \) is small.

**Lemma 1.** Let \( f_1, f_2 \) be two \( \mu \)-strongly convex twice differentiable functions. Let \( \mathbf{x}_1 = \arg \min_{\mathbf{x}} f_1(\mathbf{x}) \) and \( \mathbf{x}_2 = \arg \min_{\mathbf{x}} f_2(\mathbf{x}) \). Suppose \( \| \nabla f_1(\mathbf{x}_2) \| \leq \varepsilon, \) then \( \| \mathbf{x}_1 - \mathbf{x}_2 \| \leq \frac{\varepsilon}{\mu}. \)

**Proof.** Taylor expanding \( \nabla f_1 \), we have:

\[
\nabla f_1(\mathbf{x}_2) = \nabla f_1(\mathbf{x}_1) + \nabla^2 f_1(\tilde{x})(\mathbf{x}_2 - \mathbf{x}_1) = \nabla^2 f_1(\tilde{x})(\mathbf{x}_2 - \mathbf{x}_1) .
\]

for \( \tilde{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2 \) with some \( t \in [0, 1] \). Therefore:

\[
\mu \| \mathbf{x}_1 - \mathbf{x}_2 \| \leq \| \nabla^2 f_1(\tilde{x})(\mathbf{x}_2 - \mathbf{x}_1) \| = \| \nabla f_1(\mathbf{x}_2) \| \leq \varepsilon .
\]

\[\Box\]
Lemma 2. Define \( f_i(u; x) = \frac{1}{2} u^T R u + \frac{1}{2} (A_i x + B_i u)^T P_i (A_i x + B_i u) \) for \( i = 1, 2 \), with \( R, P_1, \) and \( P_2 \) positive definite matrices. Let \( K_i \) be the unique matrix such that \( u_i := \arg \min_u f_i(u; x) = K_i x \) for any vector \( x \). Also, denote \( \Gamma := 1 + \max\{\|A_1\|, \|B_1\|, \|P_1\|, \|K_1\|\} \). Suppose there exists \( \varepsilon \) such that \( 0 \leq \varepsilon < 1 \) and \( \|A_1 - A_2\| \leq \varepsilon, \|B_1 - B_2\| \leq \varepsilon, \) and \( \|P_1 - P_2\| \leq \varepsilon \). Then, we have

\[
\|K_1 - K_2\| \leq \frac{7\varepsilon \Gamma^3}{\sigma(R)}.
\]

Proof. We first compute the gradient \( \nabla f_i(u; x) \) with respect to \( u \):

\[
\nabla f_i(u; x) = (B_i^T P_i B_i + R) u + B_i^T P_i A_i x.
\]

Now, we observe that:

\[
\|B_i^T P_i B_i - B_2^T P_2 B_2\| \leq 7\Gamma^2 \varepsilon \quad \text{and} \quad \|B_i^T P_i A_i - B_2^T P_2 A_2\| = 7\Gamma^2 \varepsilon.
\]

Hence, for any vector \( x \) with \( \|x\| \leq 1 \), we have

\[
\|\nabla f_1(u; x) - \nabla f_2(u; x)\| \leq 7\Gamma^2 \varepsilon (\|u\| + 1).
\]

We can bound \( \|u_1\| \leq \|K_1\| \|x\| \leq \|K_1\| \). Then, from Lemma 1 we obtain

\[
\sigma(R) \|(K_1 - K_2)x\| = \sigma(R) \|u_1 - u_2\| \leq 7\varepsilon \Gamma^3.
\]

Recall that \( \Gamma_* := 1 + \max\{\|A_*\|, \|B_*\|, \|P_*\|, \|K_*\|\} \). Now, we upper bound \( \|\hat{K} - K_*\| \).

Proposition 1. Let \( \varepsilon > 0 \) such that \( \|\hat{A} - A_*\| \leq \varepsilon \) and \( \|\hat{B} - B_*\| \leq \varepsilon \). Also, let \( \|\hat{P} - P_*\| \leq f(\varepsilon) \) for some function \( f \) such that \( f(\varepsilon) \geq \varepsilon \). Then, under Assumption 1 we have

\[
\|\hat{K} - K_*\| \leq 7\Gamma_*^3 f(\varepsilon).
\]

Let \( \gamma \) be a real number such that \( \rho(L_*) < \gamma < 1 \). Then, if \( f(\varepsilon) \) is small enough so that the right hand side of (10) is smaller than \( \frac{1 - \gamma}{2\tau(L_*, \gamma)} \), we have

\[
\tau \left( L_*, \frac{1 + \gamma}{2} \right) \leq \tau(L_*, \gamma).
\]

Proof. By our assumptions \( \|\hat{A} - A_*\|, \|\hat{B} - B_*\|, \) and \( \|\hat{P} - P_*\| \) are smaller than \( f(\varepsilon) \), and \( \sigma(R) \geq 1 \). Then, Lemma 2 ensures that

\[
\|\hat{K} - K_*\| \leq 7\Gamma_*^3 f(\varepsilon).
\]

Finally, when \( \varepsilon \) is small enough so that the right hand side of (10) is smaller or equal than \( \frac{1 - \gamma}{2\tau(A_* + B_* K_*, \gamma)} \), we can apply Lemma 5, presented in Section 4, to guarantee that \( \|(A_* + B_* K)^k\| \leq \tau(A_* + B_* K_*, \gamma) \left( \frac{1+\gamma}{2} \right)^k \) for all \( k \geq 0 \).
In order to finish the proof of Theorem 1 we need to quantify the suboptimality gap \( \hat{J} - J_* \) in terms of the controller mismatch \( \hat{K} - K_* \). For a stable matrix \( L \) and a symmetric matrix \( M \), we let \( \text{dlyap}(L, M) \) denote the solution \( X \) to the Lyapunov equation \( L^T X L - X + M = 0 \). The following lemma offers a useful second order expansion of the average LQR cost.

**Lemma 3** (Lemma 12 of Fazel et al. [15]). Let \( K \) be an arbitrary static linear controller that stabilizes \((A_*, B_*)\). Denote \( \Sigma(K) := \text{dlyap}((A_* + B_* K)^T, \sigma_w^2 I_n) \) the covariance matrix of the stationary distribution of the closed loop system \( A_* + B_* K \). We have that:

\[
J(A_*, B_*, K) - J_* = \text{Tr}(\Sigma(K)(K - K_*)^T(R + B_*^T P_* B_*) (K - K_*)).
\]

(11)

Now, we have the necessary ingredients to complete the proof of Theorem 1. Equation 11 implies:

\[
J(A_*, B_*, K) - J_* \leq \|\Sigma(K)\|\|R + B_*^T P_* B_*\|\|K - K_*\|^2_F.
\]

Proposition 1 states that \( \hat{K} \) stabilizes the system \((A_*, B_*)\) when the estimation error is small enough. More precisely, under the assumptions of Theorem 1, we have \( \tau(A_* + B_* \hat{K}, \frac{1}{2}) \leq \tau(L_*, \gamma) \). When \( \hat{L} = A_* + B_* \hat{K} \) is a stable matrix we know that \( \Sigma(K) = \sigma^2 \sum_{t \geq 0} (L_*)^t L^t \). Then, by the triangle inequality we can bound

\[
\|\Sigma(K)\| \leq \sigma_w^2 \frac{\tau(L_*, \gamma)^2}{1 - \left(\frac{\gamma + 1}{2}\right)^2} \leq \frac{4\sigma_w^2 \tau(L_*, \gamma)^2}{1 - \gamma^2}.
\]

Recalling that \( \Gamma_* := 1 + \max\{\|A_*\|, \|B_*\|, \|P_*\|, \|K_*\|\} \), we have \( \|R + B_*^T P_* B_*\| \leq \Gamma^3 \). Then,

\[
J(K) - J_* \leq 4\sigma_w^2 \Gamma^3 \frac{\tau(L_*, \gamma)^2}{1 - \gamma^2} \|K - K_*\|^2_F
\]

\[
\leq 4\sigma_w^2 \min\{n, d\} \Gamma^3 \frac{\tau(L_*, \gamma)^2}{1 - \gamma^2} \|K - K_*\|^2
\]

\[
\leq 200\sigma_w^2 \frac{\tau(L_*, \gamma)^2}{1 - \gamma^2} f(\varepsilon)^2,
\]

where we used Proposition 1 and the assumption on \( f(\varepsilon) \).

### 4 Perturbation of the discrete algebraic Riccati equation

Given Theorem 1, the only remaining difficulty to show that \( \hat{J} - J_* = O(\varepsilon^2) \) is to prove that \( \|\hat{P} - P_*\| \leq L \varepsilon \) for some \( b \) and \( L \). In other words, we have to show that the solutions to the discrete Riccati equation are locally Lipschitz with respect to the problem parameters. However, we note that one cannot hope to find universal values \( b \) and \( L \) such that for any \( 0 < \varepsilon < b \) one has \( \|\hat{P} - P_*\| \leq L \varepsilon \) for arbitrary \((A_*, B_*)\) and \((\hat{A}, \hat{B})\) with \( \|\hat{A} - A_*\| \leq \varepsilon \) and \( \|\hat{B} - B_*\| \leq \varepsilon \). To see this, consider the one dimensional linear system \((n = 1)\) given by \( A_* = 1 \) and \( B_* = \varepsilon \) and consider the estimated system \( \hat{A} = 1 \) and \( \hat{B} = 0 \). Then, the estimated system is \( \varepsilon \) close to the optimal system, but the estimated system is not stabilizable and hence \( \hat{P} \) is not finite. Even when \( \hat{B} = \varepsilon/2 \), there is no universal \( L \) such that the desired inequality holds for all positive \( \varepsilon \). Therefore, \( b \) and \( L \) must depend on the system parameters \((A_*, B_*)\).
While there is a long line of work analyzing perturbations of Riccati equations, we are not aware of any result that offers explicit and easily interpretable $b$ and $L$ for a fixed system $(A_*, B_*)$. See the book by Konstantinov et al. [22] for an overview of this literature. In this section, we present two new results for Riccati perturbation which offer interpretable bounds. The first one expands upon the operator-theoretic proof of Konstantinov et al. [21]; its proof can be found in Section 4.1.

**Proposition 2.** Let $\gamma \geq \rho(L_*)$ and also let $\varepsilon$ such that $\|\hat{A} - A_*\| \leq \varepsilon$ and $\|\hat{B} - B_*\| \leq \varepsilon$. Let $\|\cdot\|_+ = \|\cdot\| + 1$. We assume $\sigma(Q) \geq 1$. Then, under Assumption 1 and assuming the system $(A_*, B_*)$ is stabilizable, we have

$$\|\hat{P} - P_*\| \leq O(1) \varepsilon \frac{\tau(L_*, \gamma)^2}{1 - \gamma^2} \|A_*\|^2 \|P_*\|^2 \|B_*\| \|R^{-1}\|_+, \leq \min \left\{ \|L_*\|^{-2}, \|P_*\|^{-1}\right\},$$

as long as

$$\varepsilon \leq O(1) \frac{(1 - \gamma^2)^2}{\tau(L_*, \gamma)^4} \|A_*\|^{-2} \|P_*\|^{-2} \|B_*\|^{-3} \|R^{-1}\|^{-2} \min \left\{ \|L_*\|^{-2}, \|P_*\|^{-1}\right\}. \leq \min \left\{ \|Q\|, \|R\|\right\} \min \left\{ \sigma(R), \sigma(Q)\right\}.$$

We now present our direct approach, which uses Assumption 2 to give a bound which is sharper for some systems $(A_*, B_*)$ than the one provided by Proposition 2. Recall that any controllable system is always $(\ell, \nu)$-controllable for some $\ell$ and $\nu$.

**Proposition 3.** Let $\rho \geq \rho(A_*)$ and also let $\varepsilon \geq 0$ such that $\|\hat{A} - A_*\| \leq \varepsilon$ and $\|\hat{B} - B_*\| \leq \varepsilon$. Let

$$\beta := \max\{1, \varepsilon \tau(A_*, \rho) + \rho\}. \leq \max\{1, \varepsilon \tau(A_*, \rho) + \rho\}.$$ Under Assumptions 1 and 2 we have

$$\|\hat{P} - P_*\| \leq 32 \varepsilon^{2} \tau(A_*, \rho)^{3} 2(\ell - 1) \left(1 + \frac{1}{\nu}\right) (1 + \|B_*\|)^{2} \|P_*\| \leq \max\{\|Q\|, \|R\|\} \min\{\sigma(R), \sigma(Q)\},$$

as long as $\varepsilon$ is small enough so that the right hand side is smaller or equal than one.

The proof of this proposition is included in Section 4.2. Proposition 3 requires an $(\ell, \nu)$-controllable system $(A_*, B_*)$, whereas Proposition 2 only requires a stabilizable system, which is a milder assumption. However, Proposition 3 can offer a sharper guarantee. For example, consider the linear system with two dimensional states ($n = 2$) given by $A_* = 1.01 \cdot I_2$ and $B_* = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}$. The cost functions $Q$ and $R$ are chosen to be the identity matrix $I_2$. This system $(A_*, B_*)$ is readily checked to be $(1, \beta)$-controllable. It is also straightforward to verify that as $\beta$ tends to zero, Proposition 2 gives a bound of $\|\hat{P} - P_*\| = O(\varepsilon/\beta^4)$, whereas Proposition 3 gives a sharper bound of $\|\hat{P} - P_*\| = O(\varepsilon/\beta^3)$.

### 4.1 Proof of Proposition 2

Given system parameters $(A, B)$ we denote by $F(X, A, B)$ the matrix expression

$$F(X, A, B) = X - A^T X A + A^T X B (R + B^T X B)^{-1} B^T X A - Q = X - A^T X \left( I + BR^{-1} B^T X \right)^{-1} A - Q. \leq (12)$$

Then, solving the Riccati equation associated to the system $(A, B)$ corresponds to finding the unique positive definite matrix $X$ such that $F(X, A, B) = 0$. We denote by $P_*$ the solution of the Riccati
equation corresponding to the trye system parameters \((A_*, B_*)\) and we denote by \(\hat{P}\) the solution associated with the estimated dynamics \((\hat{A}, \hat{B})\). Our goal is to upper bound \(\|\hat{P} - P_*\|\) in terms of \(\varepsilon\), where \(\varepsilon > 0\) such that \(\|\hat{A} - A_*\| \leq \varepsilon\) and \(\|\hat{B} - B_*\| \leq \varepsilon\).

We denote \(\Delta_P = \hat{P} - P_*\). The proof strategy goes as follows. Given the identities \(F(P_*, A_*, B_*) = 0\) and \(F(\hat{P}, \hat{A}, \hat{B}) = 0\) we construct an operator \(\Phi\) such that \(\Delta_P\) is its unique fixed point. Then, we show that the fixed point of \(\Phi\) must have small norm when \(\varepsilon\) is sufficiently small.

We denote \(S_* = B_* R^{-1} B_*^\top\) and \(\hat{S} = \hat{B} R^{-1} \hat{B}^\top\). For any matrix \(X\) such that \(I + S_*(P_* + X)\) is invertible we have

\[
F(P_* + X, A_*, B_*) = X - L_*^\top XL_* + L_*^\top X [I + S_*(P_* + X)]^{-1} S_* XL_*.
\]  

To check this identity one needs to add contraction over the set \(\hat{S}\). Now, we consider the set \(\hat{S}\). We denote \(\Delta_1 = \hat{A} - A_*\), \(\Delta_2 = \hat{B} - B_*\), and \(\Delta_3 = \hat{S} - S_*\). By assumption we have \(\|\Delta_1\| \leq \varepsilon\) and \(\|\Delta_2\| \leq \varepsilon\). Then, \(\|\Delta_3\| \leq \|B_*\| R^{-1} \|\varepsilon\|\) because \(\varepsilon \leq \|B_*\|\). The next lemma allows us to complete the argument.

**Lemma 4.** Suppose the matrices \(X, X_1, X_2\) belong to \(S_\nu\), with \(\nu \leq \min\{1, \|S_*\|^{-1}\}\). Furthermore, we assume that \(\|\Delta_1\| \leq \varepsilon\) and \(\|\Delta_2\| \leq \varepsilon\) with \(\varepsilon \leq \min\{1, \|B_*\|\}\). Finally, let \(\sigma(Q) \geq 1\). Then

\[
\|\Phi(X)\| \leq \frac{\lambda(L_*, \gamma)^2}{1 - \gamma^2} \left[\|L_*\|^2 \|S_*\| \nu^2 + \varepsilon \|A_*\|_+ \|P_*\|_+ \|B_*\|_+ \|R^{-1}\|_+\right],
\]

\[
\|\Phi(X_1) - \Phi(X_2)\| \leq \frac{32 \lambda(L_*, \gamma)^2}{1 - \gamma^2} \left[\|L_*\|^2 \|S_*\| \nu + \varepsilon \|A_*\|_+ \|P_*\|_+ \|B_*\|_+ \|R^{-1}\|_+ \right] \|X_1 - X_2\|.
\]

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The proof of this lemma is deferred to Appendix C. Now, we choose
\[ \nu = 6 \varepsilon \frac{\tau(L_*, \gamma)^2}{1 - \gamma^2} \| A_* \|_2^2 \| P_* \|_+^2 \| B_* \|_+ \| R^{-1} \|_+ . \]  
(15)

Since \( \varepsilon \) is assumed to be small enough, we know
\[ \nu \leq \min \left\{ \frac{1 - \gamma^2}{128 \tau(L_*, \gamma)^2 \| L_* \|_2^2 \| S_* \|}, \| S_* \|^{-1}, \frac{1}{2} \right\} . \]

Then, the operator \( \Phi \) satisfies \( \| \Phi(X_1) - \Phi(X_2) \| \leq \frac{1}{2} \| X_1 - X_2 \| \) for all \( X_1 \) and \( X_2 \) in \( S_\nu \). Moreover, we have \( \| \Phi(X) \| \leq \nu \) for all \( X \in S_\nu \). Since \( \nu \leq \sigma(Q) \) and \( P_* \geq Q \), we know that that \( P_* + \Phi(X) \geq 0 \).

Therefore, \( \Phi \) maps \( S_\nu \) into itself and is a contraction over \( S_\nu \). Hence, \( \Phi \) has a fixed point in \( S_\nu \) since \( S_\nu \) is a closed set. However, we already argued that the unique fixed point of \( \Phi \) is \( \Delta_P \). Therefore, \( \Delta_P \in S_\nu \) and \( \| \Delta_P \| \leq \nu \). Proposition 2 is now proven.

### 4.2 Proof of Proposition 3

Since both noisy and noiseless LQR have the same associated Riccati equation and the same optimal controller, we can focus on the noiseless case in this section. Namely, noiseless LQR takes the form

\[
\min_u \sum_{t=0}^{\infty} x_t^\top Q x_t + u_t^\top R u_t, \quad \text{where} \quad x_{t+1} = A_* x_t + B_* u_t,
\]

for a given initial state \( x_0 \). Then, we know that the cost achieved by the optimal controller when the system is initialized at \( x_0 \) is equal to \( x_0^\top P_* x_0 \).

We denote by \( J(A, B, x_0, \{ u_t \}_{t \geq 0}) \) the cost achieved on a linear system \((A, B)\) initialized at \( x_0 \) by the input sequence \( \{ u_t \}_{t \geq 0} \). When the input sequence is given by a time invariant linear gain matrix \( K \) we slightly abuse notation and denote the cost by \( J(A, B, x_0, K) \). In this case, \( J(A, B, x_0, K) = x_0^\top P x_0 \), where \( P \) is the solution to the associated Riccati equation.

Now, let \( x_0 \) be an arbitrary unit state vector in \( \mathbb{R}^n \). Then,
\[
x_0^\top \hat{P} x_0 - x_0^\top P_* x_0 = J(\hat{A}, \hat{B}, x_0, \hat{K}) - J(A_*, B_*, x_0, K_*)
\]
\[
\leq J(\hat{A}, \hat{B}, x_0, \{ \hat{u}_t \}_{t \geq 0}) - J(A_*, B_*, x_0, K_*)
\]
for any sequence of inputs \( \{ \hat{u}_t \}_{t \geq 0} \). We denote by \( \hat{x}_t \) the states produced by \( \hat{u}_t \) on the system \((\hat{A}, \hat{B})\) and by \( x_t \) and \( u_t \) the states and actions obtained on the system \((A_*, B_*)\) when the optimal controller \( u_t = K_* x_t \) is used. To prove Proposition 3 we choose a sequence of actions \( \{ \hat{u}_t \}_{t \geq 0} \) such that \( J(\hat{A}, \hat{B}, x_0, \{ \hat{u}_t \}_{t \geq 0}) \approx J(A_*, B_*, x_0, K_*) \).

For any sequence of inputs \( \{ \hat{u}_t \}_{t \geq 0} \) such that the series defining the cost \( J(\hat{A}, \hat{B}, x_0, \{ \hat{u}_t \}_{t \geq 0}) \) is absolutely convergent, we can write
\[
J(\hat{A}, \hat{B}, x_0, \hat{K}) - J(A_*, B_*, x_0, K_*) = \sum_{j=0}^{\infty} \sum_{i=0}^{\ell-1} \left[ \hat{x}_{t+j+i}^\top Q \hat{x}_{t+j+i} - x_{t+j+i}^\top Q x_{t+j+i} \right] + \sum_{j=0}^{\infty} \sum_{i=0}^{\ell-1} \left[ \hat{u}_{t+j+i}^\top R \hat{u}_{t+j+i} - u_{t+j+i}^\top R u_{t+j+i} \right].
\]
Then, the key idea is to choose a sequence of inputs \( \{ \hat{u}_t \}_{t \geq 0} \) such that the system \( (\hat{A}, \hat{B}) \) tracks the system \( (A_*, B_*, K_*) \), i.e., \( \hat{x}_{t+1} = \hat{A} x_t + \hat{B} u_t \) for any \( j \geq 0 \) (\( \hat{x}_0 = x_0 \) because both systems are initialized at the same state). This can be done because \((\hat{A}, \hat{B})\) is \((\ell, \tau/2)\)-controllable when \((A_*, B_*)\) is \((\ell, \tau)\)-controllable and the estimation error is sufficiently small, as shown in Lemma 6. First, we present a result that quantifies the effect of matrix perturbations on powers of matrices.

**Lemma 5.** Let \( M \) be an arbitrary matrix in \( \mathbb{R}^{n \times n} \) and let \( \rho \geq \rho(M) \). Then, for all \( k \geq 1 \) and real matrices \( \Delta \) of appropriate dimensions we have

\[
\| (M + \Delta)^k \| \leq \tau(M, \rho)(\tau(M, \rho)\|\Delta\| + \rho)^k,
\]

\[
\| (M + \Delta)^k - M^k \| \leq k \tau(M, \rho)^2 (\tau(M, \rho)\|\Delta\| + \rho)^{k-1}\|\Delta\|.
\]

Recall that \( \tau(M, \rho) \) is defined in Equation 3.

The proof is deferred to Appendix A. Lemma 5 quantifies the effect of a perturbation \( \Delta \), applied to a matrix \( M \) on the spectral radius of \( M + \Delta \). We are interested in quantifying the sizes of these perturbations for all \( k = 1, 2, \ldots, \ell \). Depending on \( \|\Delta\|, M, \) and \( \rho \) the sum \( \tau(M, \rho)\|\Delta\| + \rho \) can either be greater than one or smaller than one. For notational simplicity, in the rest of the proof we denote \( \beta = \max\{1, \tau(A_*, \rho) + \rho\} \). Then, we have \( \| (A_* + \Delta)^k \| \leq \tau(A_*, \rho) \beta^{\ell-1} \) and \( \| (A_* + \Delta)^k - A_*^k \| \leq \ell \tau(A_*, \rho)^2 \beta^{\ell-1} \varepsilon \) for all \( k \leq \ell - 1 \) and all real matrices \( \Delta \) with \( \|\Delta\| \leq \varepsilon \).

We denote \( C_{\ell} = [B_* A_* A_* \ldots A_*^{\ell-1} B_*] \) and \( \hat{C}_{\ell} = [\hat{B} \hat{A} \hat{B} \ldots \hat{A}^{\ell-1} \hat{B}] \). Before presenting the next result we recall that for any block matrix \( M \) with blocks \( M_{i,j} \) we have \( \| M \|^2 \leq \sum_{i,j} \|M_{i,j}\|^2 \).

The next lemma gives us control over the smallest positive singular value of the controllability matrix \( \hat{C}_{\ell} \) in terms of the corresponding value for \( C_{\ell} \).

**Lemma 6.** Suppose the linear \((A_*, B_*)\) is \((\ell, \nu)\)-controllable and let \( \rho \) be a real number such that \( \rho \geq \rho(A_*) \). Then, if \( \| \hat{A} - A_* \| \leq \varepsilon \) and \( \| \hat{B} - B_* \| \leq \varepsilon \), we have

\[
\sigma(\hat{C}_{\ell}) \geq \tau - 3\varepsilon^2 \tau(A_*, \rho)^2 \max\{1, \tau(A_*, \rho)\|\Delta\| + \rho\}^{\ell-1}(\|B_*\| + 1).
\]

The proof is deferred to Appendix B. Lemma 6 tells us that by the assumption made in Proposition 3 on \( \varepsilon \), we have \( \sigma(\hat{C}_\ell) \geq \frac{\tau}{4} \). Hence, we know that for any \( x_0 \in \mathbb{R}^n \) and \( u_0, u_1, \ldots, u_{\ell-1} \in \mathbb{R}^d \), there exist \( \hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{\ell-1} \in \mathbb{R}^{d'} \) such that

\[
A_*^\ell x_0 + \sum_{i=0}^{\ell-1} A_*^i B_* u_{\ell-1-i} = \hat{A}^\ell x_0 + \sum_{i=0}^{\ell-1} \hat{A}^i \hat{B} \hat{u}_{\ell-1-i}
\]

because the system \((\hat{A}, \hat{B})\) is controllable. This equation implies that \( \hat{x}_\ell = x_\ell \).

We denote the concatenation of \( u_i \), for \( i \) from \( 0 \) to \( \ell - 1 \) by \( u^{(\ell)} \). We define \( \hat{u}^{(\ell)} \) analogously. Therefore, Equation (17) can be rewritten as

\[
\left( A_*^\ell - \hat{A}^\ell \right) x_0 + \left( C_{\ell} - \hat{C}_{\ell} \right) u^{(\ell)} = \hat{C}_{\ell}(\hat{u}^{(\ell)} - u^{(\ell)}).
\]

Recall that \( \beta = \max\{1, \tau(A_*, \rho)\|\Delta\| + \rho\} \). Combining Lemma 5 and the upper bound on operator norms of block matrices we find \( \| \hat{C}_{\ell} - C_{\ell} \| \leq \varepsilon \ell^2 \tau(A_*, \rho)^2 \beta^{\ell-1}(\|B_*\| + 1) \).
We are free to choose \( \hat{u}^{(\ell)} \) anyway we wish as long as Equation (18) is true. Therefore, we can choose \( \hat{u}^{(\ell)} \) such that \( \hat{u}^{(\ell)} - u^{(\ell)} \) is perpendicular to the nullspace of \( \hat{C}_\ell \). Then,
\[
\frac{\tau_\ell}{2} \| \hat{u}^{(\ell)} - u^{(\ell)} \| \leq \| \hat{C}_\ell (\hat{u}^{(\ell)} - u^{(\ell)}) \| \leq \varepsilon \ell \tau (A_*, \rho)^2 \beta^{\ell-1} \| x_0 \| + \| \hat{C}_\ell - C_\ell \| \| u^{(\ell)} \| \\
\leq \varepsilon \ell \tau (A_*, \rho)^2 \beta^{\ell-1} \| x_0 \| + \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-1} (\| B_* \| + 1) \| u^{(\ell)} \|.
\]
Hence,
\[
\| \hat{u}^{(\ell)} - u^{(\ell)} \| \leq \frac{2\varepsilon \ell^3}{\tau_\ell} \tau (A_*, \rho)^2 \beta^{\ell-1} (\| B_* \| + 1) \left( \| x_0 \| + \| u^{(\ell)} \| \right)
=: \eta \left( \| x_0 \| + \| u^{(\ell)} \| \right). 
\tag{19}
\]
Let us consider the block Toeplitz matrix
\[
T_\ell = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ B_* & 0 & 0 & \cdots & 0 \\ A_* B_* & B_* & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ A_*^{\ell-2} B_* & A_*^{\ell-3} B_* & \cdots & B_* & 0 \end{bmatrix}.
\]
From Lemma 5 and the upper bound on operator norms of block matrices we have \( \| T_\ell - \hat{T}_\ell \| \leq \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} (\| B_* \| + 1) \). Let \( x^{(\ell)} \) be the concatenation of the vectors \( x_0, x_1, \ldots, x_{\ell-1} \). Then,
\[
x^{(\ell)} = T_\ell u^{(\ell)} + \begin{bmatrix} I_n \\ A_* \\ \vdots \\ A_*^{\ell-1} \end{bmatrix} x_0.
\]
Hence,
\[
\| x^{(\ell)} - \hat{x}^{(\ell)} \| \leq \| T_\ell u^{(\ell)} - \hat{T}_\ell \hat{u}^{(\ell)} \| + \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} \| x_0 \|
\leq \| T_\ell u^{(\ell)} - \hat{T}_\ell u^{(\ell)} \| + \| \hat{T}_\ell u^{(\ell)} - \hat{T}_\ell \hat{u}^{(\ell)} \| + \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} \| x_0 \|
\leq \| T_\ell - \hat{T}_\ell \| \| u^{(\ell)} \| + \| \hat{T}_\ell \| \| u^{(\ell)} - \hat{u}^{(\ell)} \| + \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} \| x_0 \|
\leq \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} (\| B_* \| + 1) \| u^{(\ell)} \| + \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} \| x_0 \|
\leq \ell^2 (A_*, \rho) \beta^{\ell-2} (\| B_* \| + 1) \| u^{(\ell)} \| + \varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-2} \| x_0 \|
\leq 2\varepsilon \ell^2 \tau (A_*, \rho)^2 \beta^{\ell-1} (1 + \tau^{-1}) (\| B_* \| + 1)^2 \left( \| x_0 \| + \| u^{(\ell)} \| \right)
=: \mu \left( \| u^{(\ell)} \| + \| x_0 \| \right). 
\tag{20}
\]
In Equations (19) and (20) we proved that the inputs and states of the system \( (\hat{A}, \hat{B}) \) are close to the inputs and states of the system \( (A_*, B_*) \) from time 0 to \( \ell \). Since the inputs to the system \( (\hat{A}, \hat{B}) \) satisfy Equation (18), we know that \( \hat{\mathbf{x}} \beta_j = \mathbf{x}_\beta j \) for all \( j \). We can repeat the same argument as above, with \( \mathbf{x}_\beta j \) taking the place of \( \mathbf{x}_0 \), to show that the inputs and states of the two systems are close to
each other from time \( \ell j \) to \( \ell(j + 1) \). Let us denote by \( x_j(\ell) \) the concatenation of the vectors \( x_{\ell j} \), \( x_{\ell j+1} \), \ldots, \( x_{\ell j+\ell-1} \) and let \( u_j(\ell) \) be defined analogously. Then,

\[
\| \hat{u}_j(\ell) - u_j(\ell) \| \leq \eta \left( \| u_j(\ell) \| + \| x_{\ell j} \| \right), \quad \text{and} \quad \| \hat{x}_j(\ell) - x_j(\ell) \| \leq \mu \left[ \| u_j(\ell) \| + \| x_{\ell j} \| \right]. \tag{21}
\]

Now, we note that

\[
x_0^T \hat{P} x_0 - x_0^T P_* x_0 \leq \sum_{j=0}^{\infty} \sum_{i=0}^{\ell-1} \left[ \hat{x}_{\ell j+i}^T Q \hat{x}_{\ell j+i} - x_{\ell j+i}^T Q x_{\ell j+i} \right] + \sum_{j=0}^{\infty} \sum_{i=0}^{\ell-1} \left[ \hat{u}_{\ell j+i}^T R \hat{u}_{\ell j+i} - u_{\ell j+i}^T R u_{\ell j+i} \right] \leq \sum_{j=0}^{\infty} 2Q \| x_j(\ell) \| \| x_j(\ell) - \hat{x}_j(\ell) \| + \| Q \| \| x_j(\ell) - \hat{x}_j(\ell) \|^2 + \sum_{j=0}^{\infty} 2R \| u_j(\ell) \| \| u_j(\ell) - \hat{u}_j(\ell) \| + \| R \| \| u_j(\ell) - \hat{u}_j(\ell) \|^2.
\]

Now, we use the upper bounds from (21). We always have \( \eta \leq \mu \). Since Proposition 3 assumes \( \varepsilon \) is small enough, we also have \( \mu \leq 1 \). Using these upper bounds, we find

\[
\hat{J} - J_* \leq \mu \sum_{j=0}^{\infty} 2Q \| x_j(\ell) \| \left[ \| u_j(\ell) \| + \| x_{\ell j} \| \right] + \| Q \| \left[ \| u_j(\ell) \| + \| x_{\ell j} \| \right]^2 + \mu \sum_{j=0}^{\infty} 2R \| u_j(\ell) \| \left[ \| u_j(\ell) \| + \| x_{\ell j} \| \right] + \| R \| \left[ \| u_j(\ell) \| + \| x_{\ell j} \| \right]^2.
\]

Then, we get \( \hat{J} - J_* \leq 8 \mu \max\{\| Q \|, \| R \|\} \sum_{j=0}^{\infty} \| x_j(\ell) \|^2 + \| u_j(\ell) \|^2 \) after using the inequalities \( (a+b)^2 \leq 2(a^2 + b^2) \) and \( 2ab \leq a^2 + b^2 \). Now, As long as \( \| x_0 \| \leq 1 \) we have

\[
\min\{\sigma(Q), \sigma(R)\} \sum_{j=0}^{\infty} \| x_j(\ell) \|^2 + \| u_j(\ell) \|^2 \leq \sum_{j=0}^{\infty} x_j^T Q x_j + u_j^T R u_j = x_0^T P_* x_0 \leq \| P_* \|.
\]

Since the initial state is an arbitrary unit norm vector, our upper bound on \( x_0^T (\hat{P} - P_*) x_0 \) becomes

\[
\lambda_{\max} \left( \hat{P} - P_* \right) \leq 16\varepsilon \hat{\tau}^2 \tau(A_*, \rho)^3 \beta^{2(\ell - 1)}(1 + \nu^{-1})(1 + \| B_* \|)^2 \| P_* \| \max\{\| Q \|, \| R \|\} \min\{\sigma(Q), \sigma(R)\}.
\]

Now, we can reverse the roles of \( (\hat{A}, \hat{B}) \) and \( (A_*, B_*) \) and repeat the same argument and obtain an upper bound on \( \lambda_{\max} \left( P_* - \hat{P} \right) \) analogous to Equation (22), but which has \( \| P_* \| \) replaced by \( \| \hat{P} \| \) on the right hand side. However, (22) implies that \( \| \hat{P} \| \leq \| P_* \| + 1 \leq 2 \| P_* \| \) because we assumed that \( \varepsilon \) is small enough such that the right hand side of (22) is less than one, and because \( P_* \succeq I_\nu \). The conclusion follows.
5 Conclusion

Though a naive Taylor expansion of the LQR cost suggests that the fast rates we derive here must be achievable, precisely computing such rates has been open since the 80s. This paper shows that fast rates are indeed achievable. We note that all of the pieces we used here have existed in the literature for some time, and perhaps it has just required a bit of time to align the contemporary rate-analyses in learning theory with earlier operator theoretic work in optimal control.

It is moreover quite surprising that the naive epsilon-greedy strategy of adding a bit of excitation noise to nominal control achieves optimal regret. While this method is straightforward and is used pervasively in engineering practice, its analysis has remained a longstanding challenge.

There remain many possible extensions to this work. The robust control approach of Dean et al. [11] applies to many different objective functions besides quadratic costs, such as $H_\infty$ and $L_1$ control. It would be interesting to know whether fast rates for control are possible for other objective functions. Another possible direction is to extend our results for the setting of partial observability, where we observe $y_t = Cx_t$ instead of $x_t$ directly. In this setting, some results exist on the estimation side [17, 18, 27, 30], but we are currently unaware of any bounds on the cost suboptimality gap. Finally, determining the optimal minimax rate for LQR would allow us to understand the tradeoffs between nominal and robust control at a more fine grained level.

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### A Proof of Lemma 5

This proof is a simple modification of Lemma D.1 in [11]. We replicate the argument here for completeness.

Fix an integer $k \geq 1$. Consider the expansion of $(M + \Delta)^k$ into $2^k$ terms. Label all these terms as $T_{i,j}$ for $i = 0, \ldots, k$ and $j = 1, \ldots, \binom{k}{i}$ where $i$ denotes the degree of $\Delta$ in the term (hence there are
(k) terms with a degree of i for \(\Delta\). Using the fact that \(\|M^k\| \leq \tau(M, \rho)\rho^k\) for all \(k \geq 0\), we can bound \(\|T_{i,j}\| \leq \tau(M, \rho)^{i+1}\rho^{k-i}\|\Delta\|^i\). Hence by triangle inequality:

\[
\|(M + \Delta)^k\| \leq \sum_{i=0}^{k} \sum_j \|T_{i,j}\|
\]

\[
\leq \sum_{i=0}^{k} \binom{k}{i} \tau(M, \rho)^{i+1}\rho^{k-i}\|\Delta\|^i
\]

\[
= \tau(M, \rho) \sum_{i=0}^{k} \binom{k}{i} (\tau(M, \rho)\|\Delta\|)^i\rho^{k-i}
\]

\[
= \tau(M, \rho)(\tau(M, \rho)\|\Delta\| + \rho)^k.
\]

To prove the first part of the lemma we follow the same argument. We find

\[
\|(M + \Delta)^k - M^k\| \leq \sum_{i=0}^{k} \sum_j \|T_{i,j}\|
\]

\[
\leq \sum_{i=1}^{k} \binom{k}{i} \tau(M, \rho)^{i+1}\rho^{k-i}\|\Delta\|^i
\]

\[
= \tau(M, \rho) \sum_{i=1}^{k} \binom{k}{i} (\tau(M, \rho)\|\Delta\|)^i\rho^{k-i}
\]

\[
= \tau(M, \rho) \left[ (\tau(M, \rho)\|\Delta\| + \rho)^k - \rho^k \right]
\]

\[
\leq kC_M^2(\tau(M, \rho)\|\Delta\| + \rho)^{k-1}\|\Delta\|
\]

where the last inequality follows from the mean value theorem applied to the function \(z \mapsto z^k\).

**B Proof of Lemma 6**

We can write

\[
\sigma\left(\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{\ell-1}\hat{B} \end{bmatrix}\right) = \min_{v \in S^{d-1}} \|v^T \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{\ell-1}\hat{B} \end{bmatrix} \|.
\]

Fix an arbitrary unit vector \(v\) in \(\mathbb{R}^d\). Then,

\[
\|v^T \begin{bmatrix} B_* & A_*B_* & \ldots & A_*^{\ell-1}B_* \end{bmatrix} - v^T \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{\ell-1}\hat{B} \end{bmatrix}\|
\]

\[
\leq \|v^T \begin{bmatrix} B_* & A_*B_* & \ldots & A_*^{\ell-1}B_* \end{bmatrix} - v^T \begin{bmatrix} B_* & \hat{A}B_* & \ldots & \hat{A}^{\ell-1}B_* \end{bmatrix}\|
\]

\[
+ \|v^T \begin{bmatrix} B_* & \hat{A}B_* & \ldots & \hat{A}^{\ell-1}B_* \end{bmatrix} - v^T \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{\ell-1}\hat{B} \end{bmatrix}\|
\]

\[
\leq \varepsilon \ell^2 \tau(A_*, \rho)^2 \beta^{\ell-1}\|B_*\| + \varepsilon \sqrt{\ell} \tau(A_*, \rho, \beta)^{\ell-1}
\]

\[
\leq \varepsilon \ell^2 \tau(A_*, \rho)^2 \beta^{\ell-1} \left(\|B_*\| + 1\right).
\]

We used \(\ell \geq 1\), \(\tau(A_*, \rho) \geq 1\), Lemma 5, and the upper bound \(\|M\|^2 \leq \sum i,j \|M_{i,j}\|^2\) on the operator norm of a block matrix. The conclusion follows by the triangle inequality.
C Proof of Lemma 4

We wish to upper bound \( \| \Phi(X) \| \) and \( \| \Phi(X_1) - \Phi(X_2) \| \) for \( X, X_1, \text{ and } X_2 \) in \( S_\nu \). First, we upper bound the operator norm of the linear operator \( T^{-1} \), the inverse of \( T : X \mapsto X - L^*_X X L_* \). Since \( L_* \) is a stable matrix, the linear map \( T \) must be invertible. Moreover, when \( L_* \) is stable and \( X - L^*_X X L_* = M \) for some matrix \( M \), we know that \( X = \sum_{k=0}^{\infty} (L^*_X)^k M L^*_X \). Therefore, by the triangle inequality, the operator norm of \( T^{-1} \) can be upper bounded by \( \| T^{-1} \| \leq \frac{\tau(L_\nu, \rho)^2}{1 - \rho^2} \). Before we proceed with the rest of the proof we present a technical lemma which will be used several times.

Lemma 7. Let \( M \) and \( N \) be two positive semidefinite matrices of the same dimension. Then \( \| N(I + MN)^{-1} \| \leq \| N \| \).

Proof. We assume that \( M \) and \( N \) are invertible. If they are not, we can work with the matrices \( M + \nu I \) and \( N + \nu I \) and take the limit of \( \nu \) going to zero. Then, we have \( N(I + MN)^{-1} = N(N^{-1} + M)^{-1} \leq N \).

Next, recall that \( H(X) = L^*_X X (I + S_\nu(P_* + X))^{-1} S_* X L_* \). Then, Lemma 7 yields

\[
\| H(X) \| \leq \| L_* \| \| S_* \| \| X \|^2.
\]

We turn our attention to the difference \( F(P_* + X, A_*, B_*) - F(P_* + X, \hat{A}, \hat{B}) \). We use the notation \( P_X \) as a shorthand for \( P_* + X \). Then, by Equation 12 we find

\[
\begin{align*}
F(P_X, \hat{A}, \hat{B}) - F(P_X, A_*, B_*) &= A_1^* P_X (I + S_* P_X)^{-1} A_* - \hat{A}^* P_X (I + \hat{S} P_X)^{-1} \hat{A} \\
 &= A_1^* P_X (I + S_* P_X)^{-1} \Delta S P_X (I + \hat{S} P_X)^{-1} A_* - A_1^* P_X (I + \hat{S} P_X)^{-1} \Delta A \\
 & \quad - \Delta A^* P_X (I + \hat{S} P_X)^{-1} \Delta A_1 - \Delta A^* P_X (I + \hat{S} P_X)^{-1} \Delta A.
\end{align*}
\]

Then, \( \| F(P_* + X, \hat{A}, \hat{B}) - F(P_* + X, A_*, B_*) \| \leq \| A_* \|^2 \| P_X \|^2 \| \Delta S \| + 2 \| A_* \| \| P_X \| \| \epsilon \| + \| P_X \| \| \epsilon \|^2 \), where we used Lemma 7. Since \( X \in S_\nu \), we know \( \| X \| \leq \nu \) and hence \( \| P_X \| \leq \| P_* \| + \nu \). We assumed that \( \nu \leq 1/2 \) and so \( \| P_X \| \leq \| P_* \| + 1 \). Now, we know that \( \| \Delta S \| \leq 2 \| B_* \| \| R^{-1} \| \| \epsilon \| + \| R^{-1} \| \| \epsilon \|^2 \) and since we assumed \( \epsilon \leq \| B_* \| \), we have \( \| \Delta S \| \leq 3 \| B_* \| \| R^{-1} \| \| \epsilon \| \). Therefore,

\[
\| \Phi(X) \| \leq \frac{\tau(L_\nu, \rho)^2}{1 - \rho^2} \left[ \| L_* \|^2 \| S_* \| \| \epsilon \|^2 + 3 \| A_* \|^2 \| P_* \|^2 \| B_* \| \| R^{-1} \| \| \epsilon \| \right].
\]

We use Lemma 7, the assumption \( \epsilon \leq \| S_* \|^{-1} \), and the definition of \( H \) to upper bound

\[
\| H(X_1) - H(X_2) \| \leq \| L_* \|^2 \left[ \| S_* \|^2 \| \epsilon \|^2 + 2 \| S_* \| \| \nu \| \| X_1 - X_2 \| \right] \leq 3 \| L_* \|^2 \| S_* \| \| \nu \| \| X_1 - X_2 \|.
\]

Let us denote \( G(X) = F(P_* + X, \hat{A}, \hat{B}) - F(P_* + X, A_*, B_*) \). In order to upper bound \( \| G(X_1) - G(X_2) \| \) we first upper bound the norm of \( (I + S_* P_X)^{-1} \) and \( (I + \hat{S} P_X)^{-1} \). Since \( \| X \| \leq \nu \leq 1/2 \) and since \( P_* \geq Q \geq I \), by Lemma 7 we get

\[
\| (I + S_* P_X)^{-1} \| = \| P_X^{-1} P_X (I + S_* P_X)^{-1} \| \leq \| P_X^{-1} \| \| P_X (I + S_* P_X)^{-1} \| \leq 2 \| P_X \|.
\]

Therefore, after some algebraic manipulations, we obtain

\[
\| G(X_1) - G(X_2) \| \leq 32 \| A_* \|^2 \| P_* \|^2 \| B_* \| \| R^{-1} \|^2 \| X_1 - X_2 \|.
\]