Time and Time Functions
in Parametrized Non-Relativistic Quantum Mechanics

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Abstract

The “evolving constants” method of defining the quantum dynamics of time-reparametrization-invariant theories is investigated for a particular implementation of parametrized non-relativistic quantum mechanics (PNRQM). The wide range of time functions that are available to define evolving constants raises issues of interpretation, consistency, and the degree to which the resulting quantum theory coincides with, or generalizes, the usual non-relativistic theory. The allowed time functions must be restricted for the predictions of PNRQM to coincide with those of usual quantum theory. They must be restricted to have a notion of quantum evolution in a time-parameter connected to spacetime geometry. They must be restricted to prevent the theory from making inconsistent predictions for the probabilities of histories. Suitable restrictions can be introduced in PNRQM but these seem unlikely to apply to a reparametrization invariant theory like general relativity.
I. INTRODUCTION

Relativistic theories of gravity such as general relativity or string theory are invariant under reparametrizations of time. The quantization of such theories presents a number of problems of principle loosely known as “the problem of time”\(^*\). Beyond any issues of finiteness or consistency such theories may exhibit, the usual quantum mechanical framework for prediction must be formulated anew. That is because a fixed notion of time is central to these usual frameworks — but no such fixed notion is supplied by a theory which does not distinguish between two time variables differing by a reparametrization.

The method of “evolving constants” is a concrete proposal for defining quantum dynamical predictions in the context of Dirac quantization of theories with a single time reparametrization invariance. Features of this idea may be found in the work of DeWitt [4], Page and Wootters [5,6], and Carlip [7,8]. However, the most complete development is to be found in the work of Rovelli [9,10] which we shall rely on. We shall review the method of “evolving constants” in some detail in Section III, but the essential features are as follows:

Classically a reparametrization invariance of the action implies a constraint on the canonical coordinates \(q^i\) and their conjugate momenta \(p_i\) which may be written

\[
H(p_i, q^i) = 0.
\] (1.1)

In a time-reparametrization theory \(H\) is the superhamiltonian and may be chosen to generate reparametrizations.

In the Dirac approach to the quantization of constrained systems, the \(p\)'s, \(q\)'s, and \(H\) become operators acting on a linear space. The operator \(\hat{H}\) annihilates physical states and commutes with observables

\[
[\hat{F}, \hat{H}] = 0.
\] (1.2)

(We use a caret to denote operators while quantities without carets are classical functions.) Observables are thus “constants” in the sense of commuting with the superhamiltonian.

Eq. (1.2) would seem to prohibit any reparametrization-invariant quantum evolution. However, the proposal of [9,10] is to define evolution using families of operators labeled by a time parameter \(\tau\). Each operator in a family satisfies (1.2) and represents a physical quantity at a value of \(\tau\). Each one-parameter family is generated from a time function \(T(p_i, q^i)\). Given a classical function on phase space \(F(p_i, q^i)\), its value may be found at an intersection of a classical phase space trajectory and the surface \(T(p_i, q^i) = \tau\). There are different trajectories for different initial conditions \((p_i, q^i)_0\), at say \(\tau = 0\), and thus the values \(F\) at \(\tau\) become functions of \((p_i, q^i)_0\). When, with suitable ordering, these functions of \(p\)'s and \(q\)'s are turned into operators, there results a one parameter family of operators \(\hat{F}(\tau)\), each satisfying (1.2), in which \(\hat{F}(\tau)\) may be said to represent the classical quantity \(F(p_i, q^i)\) at the time parameter \(\tau\). This family of “evolving constants” \(\hat{F}(\tau)\) may then be employed, as in the usual Heisenberg picture, to define evolution as a function of a time parameter \(\tau\). For example, one can discuss the variation in an expected value \(\langle \hat{F}(\tau) \rangle\) as a function of \(\tau\).

\(^*\)For recent and, by now, classic reviews of the problem of time see Refs [1–3]
time parameter $\tau$. Different time functions may be used to construct different families of “evolving constants”. Any time function in a large set yields a family of evolving constants for each classical quantity $F(p_i, q_i)$. The quantum mechanical probabilities that result from all families are the predictions for quantum evolution.

The severe difficulties that would arise in actually implementing such a program for realistic systems have been discussed by Hájíček [11], Unruh [3], and Kuchař [2]. Hájíček and Kuchař stressed the ambiguities that arise in finding a consistent operator ordering when deriving quantum evolution from classical evolution. Unruh emphasized the serious problem with even exhibiting the classical evolution of a chaotic system. However, it is not the purpose of this paper to reexamine these problems.† Rather we shall assume that the difficulties raised by Hájíček, Kuchař, and Unruh can be solved and investigate a series of complementary questions bearing on the consistency of the “evolving constant” proposal and the extent to which it reproduces the predictions of familiar quantum mechanics in a fixed flat background spacetime.

Time reparametrization invariance is not just a feature of geometric theories of gravity. As stressed by Dirac [13] and Kuchař [14], any classical theory may be made time reparametrization invariant by parametrizing the time and elevating it to the status of a canonical variable. For example, suppose the dynamics of a single non-relativistic particle moving in one dimension is summarized by an action

$$S[x(t)] = \int^{t''}_{t'} dt \ell(dx/dt, x), \quad (1.3)$$

for some Lagrangian $\ell$. Simply by writing $t = t(\lambda)$ and $x = x(\lambda)$ (“parametrizing” them) this action may be rewritten as

$$S[x(\lambda), t(\lambda)] = \int d\lambda \dot{t} \ell(\dot{x}/\dot{t}, x), \quad (1.4)$$

where a dot denotes a derivative with respect to $\lambda$. This is an equally good summary of classical dynamics, for the equations of motion of (1.4) imply those of (1.3). The action (1.4) is evidently reparametrization invariant because the way in which $x$ and $t$ were parametrized was not specified in its construction. This parametrized mechanics is fully equivalent to usual mechanics.

A quantum theory of parametrized non-relativistic mechanics (PNRM) may be constructed by restricting $\ell$ to be of usual non-relativistic form, employing the principles of Dirac quantization, and calculating quantum evolution by the method of “evolving constants”. To what extent does the resulting parametrized non-relativistic quantum mechanics (PNRQM) coincide with familiar non-relativistic quantum mechanics or differ from it? As shown in Refs [3,10], with the choice of time function

$$T(p_x, p_t, x, t) = t \quad (1.5)$$

the operators $\hat{F}(\tau)$ coincide with the usual Heisenberg picture operators. The predictions of PNRQM thus agree with usual non-relativistic quantum mechanics for this choice of time function.

†Recently A. Anderson [12] has argued that they may not be as severe as they seem.
However, the method of evolving constants permits a much wider variety of time functions than (1.3), and the predictions of all of them must be considered. In this paper we discuss the character of the predictions of PNRQM that arise from time functions other than (1.3). First we show that the general time functions yield predictions that are nothing like those of non-relativistic quantum mechanics. Next we point out that measurements of the time variables corresponding to certain time functions are very different in character from measurements of the Newtonian time of non-relativistic spacetime. Finally, we examine the predictions of the “evolving constants” method for time histories.

Usual quantum mechanics predicts probabilities for sequences of alternatives that constitute time histories. Such probabilities are necessary if the theory is to make predictions about such everyday phenomena as the orbit of the moon or the evolution of the universe. In predicting such probabilities, usual quantum mechanics employs, in addition to the usual law of unitary evolution, some variant of a “second law of evolution” sometimes called “the reduction of the state vector”.

The original work on “evolving constants” did not explicitly address the prediction of the probabilities for histories, but the usual quantum mechanical rules for these predictions can be straightforwardly included. However, we show in a parametrized relativistic field theory that allowing arbitrary time functions in this straightforward extension can lead to inconsistent predictions for the probabilities of field histories.

In Section II we review what we mean by usual non-relativistic quantum mechanics to which PNRQM is to be compared. Section III reviews the details of the “evolving constants” method for defining quantum evolution. In Section IV we compare the predictions of “good” time functions which define surfaces in phase space that a classical trajectory intersects at most once with “bad” time functions which do not have this property. Section V discusses the connection between time functions and non-relativistic spacetime. In Section VI we display the inconsistencies that may arise if arbitrary time functions are permitted in theories with “many-fingered time”. Section VII offers some brief conclusions and opinions.

II. USUAL NON-RELATIVISTIC QUANTUM MECHANICS

In this section we describe what we mean by usual non-relativistic quantum mechanics. We shall define this as narrowly and as specifically as possible, not to be contentious concerning the meaning of the term, but to highlight any contrast with PNRQM.

We work in the Heisenberg picture where states are fixed vectors in a Hilbert space \( \mathcal{H} \). An exhaustive set of “yes-no” alternatives at time \( t \) is represented by a set of projection operators \( \{ \hat{\mathcal{P}}_\alpha(t) \} \) where the discrete index \( \alpha \) labels the particular alternative. These operators evolve according to the Heisenberg equations of motion

\[
\hat{\mathcal{P}}_\alpha(t) = e^{i\hat{h}t} \hat{\mathcal{P}}_\alpha(0) e^{-i\hat{h}t},
\]

where \( \hat{h} \) is the Hamiltonian. (Here, as throughout, we employ units where \( \hbar = 1 \).)

A set of alternative time histories for the system is defined by a succession of alternatives \( \{ \hat{\mathcal{P}}_{\alpha_1}^1(t_1) \}, \{ \hat{\mathcal{P}}_{\alpha_2}^2(t_2) \}, \ldots \) at a sequence of times \( t_1 < \cdots < t_n \). An individual history corresponds to a particular sequence of alternatives \( \alpha_1, \cdots, \alpha_n \) whose probability is

\[
p(\alpha_n, \cdots, \alpha_1) = \| \hat{\mathcal{P}}_{\alpha_n}^n(t_n) \cdots \hat{\mathcal{P}}_{\alpha_1}^1(t_1) | \psi \rangle \|^2,
\]

where \( | \psi \rangle \) is the state vector of the system.
where $|\psi\rangle$ is the Heisenberg state of the system. The operators in (2.2) are time ordered. The relation (2.2) exhibits compactly the two laws of evolution in quantum mechanics — unitary evolution between alternatives [eq. (2.1)] and reduction of the state vector (by the action of the projections) at them.

For the present discussion, (2.2) may be regarded in either of two ways. It can be thought of as the probability of the outcomes of a sequence of ideal measurements on a subsystem in the approximate quantum mechanics of measured subsystems (aka the “Copenhagen” formulation). In that case $\mathcal{H}$ is the Hilbert space of the measured subsystem. More fundamentally (2.2) may be thought of as the probability of a history in a decohering set of alternative histories of a closed system, most generally the universe. The reason (2.2) may be viewed either way is that we shall only be concerned with whether PNRQM reproduces the form of (2.2), not issues of decoherence.

While not always stated explicitly, usual non-relativistic quantum mechanics assumes fixed Newtonian spacetime geometry. In particular the $t$ in (2.2) is the familiar Newtonian time. That assumption means that the time $t$ may be measured by a clock that is entirely separate from any subsystem under study. Indeed, that is how time is measured in typical experiments. It is because $t$ is a property of an assumed fixed spacetime geometry, and not a property of the quantum system itself, that it is represented in the theory as a parameter describing evolution and not as an operator in the Hilbert space of the quantum system.

### III. PARAMETRIZED NON-RELATIVISTIC QUANTUM MECHANICS

In this section we lay out in more detail than was possible in the Introduction the essential features of PNRQM [13,14]. We shall be brief because we are merely reviewing the development of [9,10]. In this and the succeeding two sections we restrict attention to a non-relativistic particle moving in one dimension with a Hamiltonian of the form

$$h(p_x, x) = \frac{p_x^2}{2m} + V(x). \quad (3.1)$$

We shall consider aspects of field theory in Section VI.

The extended phase space for the parametrized non-relativistic mechanics summarized by the action (1.4) is spanned by coördinates $(t, x)$ and momenta $(p_t, p_x)$. We write these $x^\mu$ and $p_\mu$ respectively where $\mu = 1, 2$, and as $z^A$ when we wish to speak of all four together. The $z^A$ are the coördinates of a point in phase space, $A = 1, 2, 3, 4$. The action of parametrized non-relativistic mechanics (1.4) implies a constraint of the form

$$H(p_\mu, x^\mu) = p_t + h(p_x, x) = 0, \quad (3.2)$$

which defines the “constraint surface” in the extended phase space. The constraint generates classical trajectories $\gamma$ in the constraint surface according to

$$\dot{z}^A(\lambda) = \{z^A(\lambda), N(\lambda)H\}, \quad (3.3)$$

where $\{ , \}$ is the Poisson bracket and $N(\lambda)$ is a multiplier (the “lapse”) defining the particular parametrization of these trajectories. The set of classical trajectories $\gamma$ constitute
the reduced phase space. The functions $F(z^A)$ that are constant on these curves are the classical observables and satisfy
\[ \{ F, H \} = 0 . \] (3.4)

To pass to quantum mechanics, we consider the linear space of wave functions $\psi(x^\mu)$ and represent $x^\mu$ and $p_\mu$ by operators in the usual way, e.g., $p_\mu = -i \partial / \partial x^\mu$. Physical states are annihilated by the operator form of the constraint
\[ \hat{H} \psi(x^\mu) = \left[ -i \frac{\partial}{\partial t} + \hat{h} \left( -i \frac{\partial}{\partial x}, x \right) \right] \psi(t, x) = 0 , \] (3.5)
which will be recognized as the Schrödinger equation. The inner product between physical states is
\[ (\psi, \phi) = \int_{-\infty}^{+\infty} dx \psi^*(t, x) \phi(t, x) \] (3.6)
and is independent of $t$ as a consequence of (3.5).

To define a family of “evolving constants” we choose a time function $T(z^A)$ such that every classical trajectory intersects each surface of constant $T$ at least once. (We shall return below to the question of whether classical trajectories may intersect such surfaces more than once.) Evidently $T$ cannot be an observable because then it would be constant along classical trajectories.

Consider any function $F(z^A)$ on the extended phase space. For each value of a parameter $\tau$, we may define the classical observable $F(\tau, \gamma)$ as having the value of $F(z^A)$ at the point $z^A$ where $\gamma$ intersects the surface $T(z^A) = \tau$. If more than one intersection is possible then more than one family of observables $F(\tau, \gamma)$ may be defined.

Classical trajectories may be labeled by their location $z^A_0$ in phase space at $\lambda = 0$. Thus $F(\tau, \gamma)$ becomes a function of $z^A_0$ The idea now is to replace $x^\mu_0$ and $p_\mu_0$ by their corresponding operators and carry out a suitable operator ordering to yield a one parameter family of quantum observables $\hat{F}(\tau)$ satisfying
\[ [\hat{F}(\tau), \hat{H}] = 0 . \] (3.7)

The operator $\hat{F}(\tau)$ represents the quantity $F$ at the time parameter $\tau$.

Probabilities may be calculated as in ordinary quantum mechanics treating the parameter $\tau$ as time. Suppose, for example, a sequence of ideal measurements is made on a subsystem of ranges of values of quantities $F^1, \cdots, F^n$ at a sequence of time parameters $\tau_1 < \cdots < \tau_n$. Let $\hat{P}^k_{\alpha_k}(\tau_k)$, $\alpha_k = 1, 2, \cdots$ be the projection operators on the ranges of the spectrum of the operator $\hat{F}^k(\tau_k)$ that define the possible outcomes of the measurement at time parameter $\tau_k$. Then the probability of a particular sequence of outcomes $(\alpha_1, \cdots, \alpha_n)$ is
\[ p(\alpha_n, \cdots, \alpha_1) = \left\| \hat{P}^n_{\alpha_n}(\tau_n) \cdots \hat{P}^1_{\alpha_1}(\tau_1) |\psi\rangle \right\|^2 , \] (3.8)
where the operators are ordered by the value of $\tau$. A notationally indistinguishable formula holds for the probability of a decoherent set of histories of a closed system. The probabilities of a history of alternatives at a sequence of times were not explicitly discussed in [9,10]. In
employing formulae such as (3.8) we are assuming that histories with general time functions would be treated just as they are in the Newtonian time.

With perhaps some suitable general restrictions, any choice of time function yields a family of evolving constraints for each classical quantity $F(p_i, q^i)$. For each such choice of time parameter, PNRQM predicts probabilities according to (3.8). The totality of all these probabilities for all allowed time functions are the predictions of the quantum theory.

As discussed in Ref. [9,10], one time function for which this procedure may be carried out explicitly is

$$T(z^A) = t.$$  \hspace{1cm} (3.9)

For the one parameter of observables representing position and momentum when $T = t = \tau$ we may take

$$\hat{x}(\tau) = e^{i\hat{t}(\tau - \check{t})} \check{x} e^{-i\hat{t}(\tau - \check{t})},$$  \hspace{1cm} (3.10a)

$$\hat{p}_x(\tau) = e^{i\hat{t}(\tau - \check{t})} \hat{p} e^{-i\hat{t}(\tau - \check{t})},$$  \hspace{1cm} (3.10b)

where the operators $\hat{x}$ and $\hat{t}$ act on wave functions $\psi(x^\mu)$ by multiplication, and $\hat{p}$ is $-i\partial/\partial x$. The operators (3.10) commute with the constraint (3.2) and classically correspond to the values of $x$ and $p_x$ when $T = t = \tau$. The value of any function $F(z^A)$ when $T = \tau$ may be similarly represented.

The quantities $\hat{x}(\tau)$ and $\hat{p}_x(\tau)$ are the usual representations of position and momentum in the Heisenberg picture. Thus, for the choice of time function $T = t$, the predictions of PNRQM coincide exactly through (3.8) with those of usual non-relativistic quantum mechanics (2.2). However, PNRQM permits many more choices of time functions, and it is to the predictions of these that we now turn.

### IV. GOOD AND BAD TIME FUNCTIONS

In this section we consider some simple examples of time functions of the form $T = T(t, x)$ and show that they lead to predictions which are not contained in usual non-relativistic quantum mechanics.

We begin with the elementary example of a free particle, $h = p^2/2m$, and investigate the time function $T = x$. Almost all classical trajectories intersect a surface $x = \tau$ once and only once. (The only exceptions are those with zero momentum that may remain at $x = \tau$.)

The operators corresponding to the values of the phase space coördinates when $x = \tau$ are

$$\hat{x}(\tau) = \tau,$$  \hspace{1cm} (4.1a)

$$\hat{p}_x(\tau) = \hat{p},$$  \hspace{1cm} (4.1b)

$$\hat{t}(\tau) = \check{t} + \frac{1}{2} \left[ \frac{m}{\hat{p}}, (\tau - \check{x}) \right]_+,$$  \hspace{1cm} (4.1c)

$$-\hat{p}_t(\tau) = \frac{p^2}{2m}.$$  \hspace{1cm} (4.1d)

where $[\ , \ ]_+$ is the anti-commutator making the operator symmetric.

Thus, we could calculate a probability for a measurement of $t$ at a given value of $x$. For typical wave functions $\psi(x^\mu)$, the expected value $\int dx \psi^* \hat{t}(\tau) \psi$ at a given value of $x = \tau$
would diverge because of the $1/\hat{p}$ factor in (4.1c) but for wave functions with zero amplitude for $p = 0$ it might be finite. Using (3.8) one could calculate the probabilities for values of $\hat{t}$ at a sequence of $x$’s. The operators in (3.8) would be ordered by the value of $x$.

None of these probabilities occur in usual non-relativistic quantum mechanics because that theory — as defined in Section II — deals only with alternatives at definite moments of the Newtonian time. To be consistent with the usual theory, the operator $\hat{t}(\tau)$ defined above cannot mean simply the time that the particle crosses $x$. Although the classical trajectories of a free particle cross a surface of constant $x$ at one time, non-relativistic quantum mechanics predicts a probability for the particle to be at $x$ at any time. Put differently, in quantum mechanics the particle may cross a surface of constant $x$ an arbitrary number of times. There is no one time of crossing for there to be a probability of. Some interpretation, beyond that of usual non-relativistic quantum mechanics, would have to be given to the probabilities generated from the choice $T = x$.

When there is any non-trivial dynamics, classical trajectories may intersect a surface of constant $x$ more than once. For example if $V = -gx$ then we can construct $\hat{t}(\tau)$ and $\hat{p}_x(\tau)$ by eliminating $\lambda$ between

$$\tau = \hat{x} + \frac{\hat{p}}{m}\lambda + \frac{1}{2}g\lambda^2,$$

$$\hat{t}(\tau) = \hat{t} + \lambda,$$

$$\hat{p}_x(\tau) = \hat{p} + g\lambda.$$  

Thus, there would be two one-parameter families of operators corresponding to the two branches in the solution to (4.2a). Again the resulting probabilities correspond to nothing predicted by non-relativistic quantum mechanics.

The class of allowed time functions must be restricted for PNRQM to coincide with usual quantum mechanics. A possible restriction on time functions of the form of $T(t, x)$ would be to require that they be “good” time functions in the sense that all classical trajectories intersect surfaces of constant $T$ once and only once. For many potentials this would restrict to $T = t$. Thus restricted, PNRQM would coincide with the usual theory.

Schön and Hájíček [15] have shown that good time functions do not exist in generic minisuperspace models whose constraints are analogous to those of general relativity. Their work suggests that good time functions do not exist in the phase space of general relativity. If that is the case, any restrictions on time functions necessary for general relativity to coincide with usual quantum mechanics in an appropriate limit would have be of a different character.

V. TIME FUNCTIONS AND SPACETIME

For many non-relativistic systems, good time functions of the form $T(t, p_x)$ exist even when time functions depending only on $t$ and $x$ are very restricted. In this section we consider a simple example of a time function $T(t, p_x)$ and discuss its relations to the usual non-relativistic notions of spacetime.

Classically, $p_x$ will be a single-valued function of $t$ provided there are no infinite forces, and for such systems there will be good time functions of the form $T(t, p_x)$. For example,
$t + kp_x$ is a good time function for the free particle and simple harmonic oscillator when $k$ is a constant of appropriate dimension and size. Consider this time function in the case of a free particle. Following the procedure of Section III, we may identify the operators representing $x$ and $p_x$ when $t + kp_x = \tau$. They are

\begin{align}
\hat{x}(\tau) &= \hat{x} + \frac{\hat{p}}{m}(\tau - k\hat{p} - \hat{t}) , \\
\hat{p}_x(\tau) &= \hat{p} .
\end{align}

(5.1a)  
(5.1b)

We would now like to describe a way in which measurements of quantity like $\hat{x}(\tau)$ defined by (5.1a) differ in character from those of $\hat{x}(t)$ where $t$ is the usual Newtonian time. First, note that it is not enough simply to discuss or model the measurement of an operator $\hat{x}(\tau)$ at some prescribed value of $\tau$ without also considering how intervals of $\tau$ are determined. That is because in the Heisenberg picture an operator represents some quantity at any time. Consider by way of example the operator

\[ \hat{x} + \frac{\hat{p}_x}{m} \]

(5.2)

in the usual non-relativistic quantum mechanics of a free particle. This represents $x$ at time $t = 5$. But it could also represent a particular combination of position and momentum at $t = 0$. Similarly the operator on the right hand side of (5.1a) represents $x$ when $t + kp_x = \tau$ in the evolving constant scheme, but also represents a combination of $x$ and $p_x$ when $t = \tau$, and, indeed, some other combination for any other value of $t$. In view of this, to model a measurement of $\hat{x}(\tau)$, one must not only model a determination of an eigenvalue of the operator, but also one must model a measurement of intervals of the time parameter $\tau$.

The time function $\mathcal{T} = t$ is the unique one associated with the Newtonian time defined by the geometry of non-relativistic spacetime. Intervals of $t$ may be measured by a system (a clock) that is completely independent of the free particle whose position is $x$ and momentum $p_x$. That is because usual non-relativistic quantum mechanics assumes a fixed Newtonian spacetime geometry and, in particular, the Newtonian idea of simultaneity. The readings of a clock may be simultaneous with the position of a particle without there being any interaction between the two. Indeed, that is how realistic measurements of quantities at near definite moments of time are typically carried out.

By contrast, it is difficult to see how the measurement of intervals of a time parameter like $t + kp_x$ can be carried out by a clock that is independent of the particle whose momentum is $p_x$. The $t$ in the time function may be the Newtonian time, accessible to many different

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\[ \text{Strictly speaking, the value of a time function itself cannot be measured since time functions necessarily are not observables. However, intervals of time functions can correspond to observables. For example, the integral} \]

\[ \int d\lambda (t + k\hat{p}_x) \]

\[ \text{can be reparametrization-invariant when carried out between reparametrization-invariant end points such as might be defined by positions of a clock indicator.} \]
systems, composed of different particles, but the $p_x$ refers to the particular particle being measured. A measurement of $x$ when $t + kp_x = \tau$ can certainly not be carried out by determining $x$, $p_x$ and $t$ at one point along the particles trajectory as is shown by the numerous analyses of model experiments which attempt to simultaneously define $x$ and $p_x$. Intervals of time functions such as $t + kp_x$, that depend on the variables of that subsystem, cannot be determined without intervention in the subsystem.

**VI. INCONSISTENT TIME ORDERINGS**

**A. Particles**

In this section we show that PNRQM predicts inconsistent probabilities for histories unless its allowable time functions are suitably restricted.

We begin with a simple example in the PNRQM of a single particle interpreted by the method of evolving constants, as described in Section III. First, consider the time function $T = t$ and the alternatives that the particle is localized in a region of space $\Delta$ at a value of the associated time parameter $\tau$, or not localized in that region. The alternative that the particle is in $\Delta$ when $T = \tau$ corresponds to the function on the reduced phase space which is unity on those paths which cross $\Delta$ at that time and zero otherwise. The alternative that the particle does not pass through the region corresponds to the function which is zero on the paths through $\Delta$ at $\tau$ and unity otherwise.

Evidently $\tau$ is the usual Newtonian time. As this is just the usual quantum mechanical situation, there is no difficulty with identifying the “evolving constant” corresponding to these alternatives. The alternative that the particle is localized in $\Delta$ at $\tau$ is represented by

$$\hat{P}_\Delta(\tau),$$

where $\hat{P}_\Delta(\tau)$ is the Heisenberg picture projection operator onto the range $\Delta$ of $\hat{x}(\tau)$. The alternative that the particle is not so localized is represented by

$$\hat{P}_\Delta(\tau) \equiv I - \hat{P}_\Delta(\tau).$$

Suppose measurements are carried out to determine whether the particle is in a region $\Delta_1$ when $T = \tau_1$ and in $\Delta_2$ when $T = \tau_2$ where $\tau_2 > \tau_1$. The probability that the particle is so localized is, from (3.8),

$$p(\Delta_2, \Delta_1) = \|\hat{P}_{\Delta_2}(\tau_2)\hat{P}_{\Delta_1}(\tau_1)|\psi\|^2,$$

where the operators are ordered right to left according to increasing values of the time parameter $\tau$. 


FIG 1: Inconsistent probabilities for the same histories may arise from two different choices of time function when the method of “evolving constants” is used with the usual quantum mechanical framework for predicting these probabilities. The figure shows timelike separated spacelike regions $\Delta_1$ and $\Delta_2$. (The diagonal dotted lines define the domain of causal dependence of $\Delta_1$.) The regions $\Delta_1$ and $\Delta_2$ may be thought of as lying on the level surfaces of a time function $T(t,x) = t$ at constant values $\tau_1$ and $\tau_2$ respectively. Alternatively they may be thought of as lying on surfaces of a time function $T'(t,x)$ at constant values $\tau'_1$ and $\tau'_2$ shown by the solid lines above. The operators representing alternatives that refer only to $\Delta_1$ or $\Delta_2$ are independent of this choice, since the level surfaces of the two time functions coincide in these regions. However, the order “in time” of the alternatives is different in the two cases. Probabilities of histories depend on time order and will therefore the two choices of time function will result in different predictions for the probabilities of histories. Consistency in this example could be restored by restricting time functions to those whose level surfaces are spacelike — thus eliminating $T'$.

Constant values of time functions that are functions of $t$ and $x$ alone may be thought of as defining surfaces in Newtonian spacetime. A region $\Delta$ on a surface of constant $T$ may equally well be considered as a region on a surface of constant value of any other time
function $\mathcal{T}'(t, x)$ as long as the surfaces coincide inside $\Delta$. (See Figure 1.) There are many ways $\mathcal{T}'$ can differ from $\mathcal{T}$ outside $\Delta$ if no further restrictions are placed on allowable time functions. The function on phase space which is unity on the paths which pass through $\Delta$ at $\mathcal{T} = \tau$ and zero otherwise is the same as the function which is unity on the paths which pass through $\Delta$ when $\mathcal{T}' = \tau'$ and zero otherwise for an appropriate $\tau'$. The class of paths is the same in each case. Assuming that the same operator ordering is used in both cases, the corresponding “evolving constant” operators must coincide:

$$\hat{P}_{\Delta}(\tau') = \hat{P}_{\Delta}(\tau).$$

(6.4)

This is only to be expected. The operators represent, after all, the same alternative. Similarly there is an equality between $\overline{\mathcal{P}}_{\Delta}(\tau')$ and $\overline{\mathcal{P}}_{\Delta}(\tau)$ following from (6.2).

A problem of consistency arises when considering the probabilities of histories of two such regions $\Delta_1$ and $\Delta_2$ at different Newtonian times $\tau_1$ and $\tau_2$, $\tau_1 < \tau_2$. As illustrated in Figure 1, it is possible to choose another time function $\mathcal{T}'(t, x)$ which is a surface of constant Newtonian time inside $\Delta_1$ and $\Delta_2$, but such that the values of the time parameters satisfy $\tau_1' > \tau_2'$. Thus, the ordering of the same alternatives is different according to the two choices of time function. The probability $p(\Delta_1, \Delta_2)$ would be (6.3) using Newtonian time $\mathcal{T}$ but

$$p(\Delta_1, \Delta_2) = \|\hat{P}_{\Delta_1}(\tau_1')\hat{P}_{\Delta_2}(\tau_2')|\psi\rangle\|^2$$

(6.5)

using the time function $\mathcal{T}'$. Since the operators for the two alternatives agree [cf. (6.4)], but do not commute, we have an inconsistent assignment of probabilities to the same history.

A simple special case of the above ambiguity will serve to illustrate explicitly how the predictions of the “evolving constant” method can differ from those of usual quantum mechanics if arbitrary time functions are allowed. Pick a time function $\mathcal{T}''(t, x)$ corresponding to a family of surfaces in spacetime such that both region $\Delta_1$ at Newtonian time $\tau_1$ and $\Delta_2$ at Newtonian time $\tau_2$ lie on the same surface $\mathcal{T}''(t, x) = \tau''$. In the “evolving constant” construction the alternative that the particle passes through both regions would then be represented by a single projection operator, $\hat{P}_{\Delta_1\Delta_2}(\tau'')$. That projection operator would be constructed by taking the characteristic function on the region of phase space corresponding to initial conditions $(x, p)_0$ for classical trajectories that intersect both regions at the appropriate Newtonian times and turning it into a projection operator with appropriate ordering. Thus a history of alternatives which is represented by a product of projections $\hat{P}_{\Delta_2}(t_2)\hat{P}_{\Delta_1}(t_1)$ in usual quantum mechanics would be represented by a single projection in the “evolving constant” method using this choice of time function.

This example points to another type of inconsistency which can arise if the allowed time functions are not limited. Choose the time function $\mathcal{T}''(t, x)$ described above. Physically, the alternative that the particle lies in region $\Delta_1$ and $\Delta_2$ at time $\tau''$ should be the same as the history that the particle passes through $\Delta_1$ at time $\tau''$ and “then” passes through $\Delta_2$ at a time $\tau'' + \epsilon$ in the limit that $\epsilon$ tends to zero. Yet, if we are correct that the operators representing the latter two alternatives are independent of the choice of time function as in (6.4), then the operators representing these two choices would seem to be different. In the one case, the operator is a single projection. In the other, it is a product of projections which do not reduce to a single projection in the limit of vanishing $\epsilon$, because they do not do so when represented as $\hat{P}_{\Delta_2}(\tau_2)$ and $\hat{P}_{\Delta_1}(\tau_1)$. Indeed, unless the time functions are suitably
restricted, there will generally be an ambiguity in whether a history of alternatives should be represented by a single operator that is the transcription of the characteristic function of those points in the reduced phase space corresponding to the classical trajectories allowed by this sequence of alternatives, or by a sequence of operators representing a decomposition of the history into alternatives at different times, whatever time function is chosen. Those two possibilities are not the same because quantum mechanics permits non-classical trajectories.

The astute reader will have noticed that the time functions $T'$ and $T''$ that lead to the above ambiguities, and to orderings different from that provided by Newtonian time, are not “good” time functions in the sense defined in Section IV. While the classical trajectories that pass through $\Delta$ at $T' = \tau'$ intersect that surface once and only once, the trajectories defining the complementary alternative may intersect it many times. Thus, restricting the theory to good time functions of the form $T(t, x)$ would eliminate this particular inconsistency. However, we shall now see that this kind of inconsistency persists, even for good time functions, in a theory that possesses a “many-fingered” parametrized time.

### B. Many-Fingered Times

We consider a free massive, real relativistic field in one space and one time dimension whose dynamics is summarized by the action

$$S[\phi(t, x)] = -\frac{1}{2} \int dt dx \left[ -\left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 + m^2 \phi^2 \right].$$  \hspace{1cm} (6.6)

We foliate Minkowski spacetime by hypersurfaces $t = t(\lambda, x)$ and regard the action as a functional of both $\phi(\lambda, x)$ and $t(\lambda, x)$. The resulting parametrized field theory is invariant under independent reparametrizations of $\lambda$ for each $x$. There is therefore a constraint for each $x$ which has the form

$$\mathcal{H}(x) \equiv \pi_t(x) + h[\pi_\phi(x'), \phi(x'), t(x'); x] = 0.$$  \hspace{1cm} (6.7)

Here, $\pi_\phi(x)$ and $\pi_t(x)$ are the momenta conjugate to $\phi(x)$ and $t(x)$ respectively, and the bracket $[\cdot]$ indicates that $h$ is a functional of the argument momenta and fields while the bracket $\{\cdot\}$ indicates that it is a function of $x$.

The explicit form of $h$ is

$$h = \frac{1}{2} \left[ (1 - t'^2)^{-1} (\pi_\phi + t' \phi')^2 + (\phi')^2 + m^2 \phi^2 \right],$$  \hspace{1cm} (6.8)

where a prime indicates a partial derivative with respect to $x$. The canonical form of the action summarizing the resulting dynamics is

$$S[\pi_t, \pi_\phi, t, \phi] = \int d\lambda dx \left\{ \pi_t \dot{t} + \pi_\phi \dot{\phi} - N\mathcal{H}[\pi_t, \pi_\phi, t, \phi] \right\},$$  \hspace{1cm} (6.9)

where a dot denotes a partial derivative with respect to $\lambda$ and $N(\lambda, x)$ is a multiplier enforcing the constraint.

The extended phase space of this parametrized field theory is spanned by the coordinates $\pi_t(x), \pi_\phi(x), t(x), \phi(x)$ — four coordinates for each $x$. We denote these collectively by $z^A$.
where, in the manner of DeWitt, the index $A$ indicates $x$ as well as whether the coördinate is $\pi_t$, $\pi_\phi$, $t$ or $\phi$. Since there is one constraint for each $x$ the constraint surface has a correspondingly high co-dimension.

Classical motion in the constraint surface is generated by the constraints (6.8) according to

$$\dot{z}^A(\lambda) = \{z^A(\lambda), H[N]\},$$

(6.10)

where

$$H[N] = \int dx N(\lambda, x) \mathcal{H}[\pi_t, \pi_\phi, t, \phi; x]$$

(6.11)

for arbitrary $N(\lambda, x)$. Therefore, there is not a unique curve or trajectory describing classical evolution, but rather one function’s worth of equivalent curves

$$z^A = z^A(\lambda; N(\lambda', x')).$$

(6.12)

This function’s worth of equivalent curves defines a point in the reduced phase space.

We are now in a position to consider the generalization of our earlier discussion of “evolving constants” to theories like this with an infinite number of reparametrization invariances. In the case of a single reparametrization invariance, we defined classical observables at a point of the reduced phase space (a classical trajectory) and a value of a time parameter by the values of phase space functions at the point in the extended phase space where the classical trajectory intersected a surface $\mathcal{T}(z^A) = \tau$ specified by a time function $\mathcal{T}(z^A)$. However, the function’s worth of curves (6.12) will intersect such a surface in many points. A larger number of mutually intersecting time functions is therefore needed to define a unique point where one classical trajectory and all time functions coincide. One time function is needed for each $x$. Thus we write

$$\mathcal{T}[z^A; x] = \tau(x).$$

(6.13)

There is a time function for each $x$ and $\tau(x)$ is its value. Thus there is not just one time parameter but one for each $x$ — a freely specifiable “many-fingered time”. Classical observables are functionals of $\tau(x)$. The operator ordering problems are even more formidable in this environment, but, assuming that they can be solved, we write $\hat{F}[\tau(x)]$ for an evolving constant operator satisfying

$$[\hat{F}[\tau(x')], \hat{\mathcal{H}}(x)] = 0.$$ 

(6.14)

Similarly projections onto ranges of values of $\hat{F}$ are labeled by $\tau(x)$.

We now turn to the ordering inconsistencies that can arise in such a framework when defining the probabilities of histories. First it is clear that there is no universal notion of the ordering of functions $\tau_1(x)$ and $\tau_2(x)$ in the same way that there is for single parameters $\tau_1$ and $\tau_2$. However, one could consider restricting the notion of history to sequences of $\tau(x)$’s for which

$$\tau_n(x) > \tau_{n-1}(x) > \cdots > \tau_1(x),$$ 

(6.15)
for each \( x \). While the physical content of such a restriction is obscure in general to this author, a meaning can be given for that limited class of time functions which correspond to hypersurfaces in spacetime. We now restrict attention to this class.

The time function most closely related to the time of spacetime geometry is

\[
T[z^A; x] = t(x).
\]

(6.16)

The values

\[
t(x) = \tau(x)
\]

(6.17)
of this time function define surfaces in spacetime. Histories are naturally restricted to sequences of \( \tau(x)'s \) which are members of a foliation of spacetime and which therefore satisfy (6.13).

The time function \( t(x) \) is a “good” time function. Fields \( \phi(\lambda, x) \) are single valued on spacetime, so for a given \( \tau(x) \) there is only one value for \( \phi, \pi_{\phi}, t \) and \( \pi_t \) on the surface \( t(x) = \tau(x) \). Yet, as we shall now show, ordering ambiguities of the kind discussed for particles in Section VIA remain if \( \tau(x) \) corresponding to all possible foliations of spacetime by hypersurfaces are permitted.

Consider the two timelike separated regions \( \Delta_1 \) and \( \Delta_2 \) as shown in Figure 1. We may consider alternative values of quantities constructed only from the values of fields in these regions. For instance, we could consider the average value of the field in one region. Such alternatives may be said to be at the time parameters \( \tau_1(x) \) and \( \tau_2(x) \) which are surfaces of constant \( t \). Alternatively they may be said to be at the time parameters \( \tau_1'(x) \) and \( \tau_2'(x) \) which are constant time surfaces inside \( \Delta_1 \) and \( \Delta_2 \) but vary outside. The operators corresponding to the alternatives are independent of which of the two sets of time parameters is used since they are the same alternatives in either case. However, the ordering of operators in the construction of the probability of a history is different depending on which of the two assignments of many-fingered time is used. Further, the two orderings are inconsistent since the operators will not commute because the regions \( \Delta_1 \) and \( \Delta_2 \) are timelike separated. If they are spacelike separated then the ordering is ambiguous, but it is also irrelevant since fields at spacelike separated points commute.

A restriction to “good” time functions will not eliminate this inconsistency as it did for the case of the particle in Section VIA. The time function \( t(x) \) is a good time function for fields as discussed above. Instead, to express usual field theory in this language, one would need to restrict, not the time function itself, but rather its values \( \tau(x) \) so that they correspond to foliations of spacetime by spacelike surfaces. This restriction eliminates the surfaces \( \tau_1'(x) \) and \( \tau_2'(x) \) since they necessarily contain some non-spacelike parts if \( \Delta_1 \) and \( \Delta_2 \) are timelike separated.

C. General Relativity

With suitable restrictions on the allowed time functions, the method of “evolving constants” can yield a consistent quantum theory of particles and fields in a fixed background spacetime. Restrict to “good” time functions. Restrict to time functions that correspond to surfaces in the background spacetime. Restrict to values of many-fingered time that define
foliations of that fixed spacetime by spacelike surfaces. The result is the usual quantum mechanics of particles and fields in fixed background spacetimes. But what of such restrictions in theories of quantum general relativity where spacetime geometry is not fixed? We have already mentioned that there may be no “good” time functions for general relativity. Certainly there cannot be a restriction to time functions that define surfaces in spacetime in a theory where spacetime geometry is not fixed. The closest analogy would be time functions that define surfaces in the superspace of three-geometries. In superspace many different curves, corresponding to different ways of foliating a classical spacetime, describe the same classical evolution. The issue of consistency of the order of operators defining histories will therefore certainly arise. It seems unlikely that it can be resolved by restricting to “spacelike” surfaces in superspace. The DeWitt metric gives a natural notion of spacelike directions in superspace, but there is unlikely to be any notion of causality in superspace similar to that in field theory. Even classical trajectories propagate outside the light cone of superspace. Further, some popular choices of time functions, such as York’s trace of the extrinsic curvature, do not even define surfaces in the superspace of three geometries. For such reasons, the nature of the restrictions on time functions and on time parameter values that would give a consistent “evolving constant” quantum dynamics of spacetime geometry coinciding with the usual fixed background theories in an appropriate limit remains an unsolved problem.

VII. CONCLUSIONS

Parametrized non-relativistic quantum mechanics (PNRQM) interpreted by the method of evolving constants does not coincide with the usual non-relativistic quantum theory unless the class of allowable time functions is severely restricted. To be sure, PNRQM reproduces the predictions of the usual theory for the choice of time function $T(t, x, p_x) = t$. However, to the extent that it allows other time functions it makes predictions that go beyond those of familiar non-relativistic theory. At best these new predictions may be generalizations of the usual framework presenting new challenges for interpretation and understanding. Certainly the central role played by a fixed Newtonian spacetime geometry in the usual theory is altered. At worst, for some non-relativistic systems, the predictions for histories may be inconsistent unless the time functions are suitably restricted or understood by a new interpretative rule.

There is no evidence of experiment or theoretical principle compelling a generalization of non-relativistic quantum mechanics such as PNRQM interpreted by the method of evolving constants represents. However, in the opinion of many workers, including the author, generalization of the familiar quantum mechanical framework, closely tied as it is to a notion of fixed background geometry, is a natural route to resolving the “problem of time” in quantum gravity. If the principles of any such generalization are to apply to the non-relativistic domain to yield a necessary generalization of the usual theory, then we are indeed in a fortunate position. For then the generalization can be analyzed for logical and experimental consistency in a well understood situation that is much more accessible to theoretical computation and experimental test than any involving quantum gravity. At present it is probably not clear whether the method of evolving constants applied to a putative quantum theory of gravity implies that a PNRQM of particles or fields should apply in the non-relativistic
domain. If so, however, progress in understanding can be made by resolving the issues of meaning and consistency raised above and demonstrating consistency with known physics. That is an obligation which is equally shared by other proposed generalizations of usual quantum mechanics.

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