Energy momentum tensor in the nonsymmetric gravity

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(Dated:)

General relativity is the theory with unclear energy momentum tensor. The proposed approach allows to construct the energy momentum tensor for relativity with non-symmetric metric. A consequence of the approach is confirmed in the nuclear physics and may be applicable to astrophysics

PACS numbers: 04.20.Cv, 04.20.Ex, 04.50.Kd, 04.40.Nr

INTRODUCTION

The theory of relativity allows the possibility of a non-symmetric metric tensor. A. Einstein, in his attempts to build out an Unified Field Theory, associated the anti-symmetric part of this tensor with electromagnetism [1]. Afterwards it was found that the antisymmetric metric tensor may represent a generalized gravity [2] with a new forces and properties [3],[4].

Presently, Einstein’s interpretation is not popular among the physical community. This scepticism is expressed in the phrase: "Research in this direction ultimately proved fruitless; the desired unified field theory was not found”. One of the problems is the transition to classical electrodynamics. Another problem is the energy momentum tensor.

However, the potential of Einstein’s interpretation is not exhausted.

The paper presents an approach, which on the one hand considers Einstein’s interpretation, on the other hand is applicable to the generalized Moffat gravity. The distinguish feature of the approach is the construction of the energy momentum tensor so that the field equation are fulfilled due to Maxwell’s equation. Such a construction may consist of the inclusion of missing terms as well as the compensation of redundant terms. All inserts in the tensor should be material, that is, they should disappear when the electric current goes to a zero.

In a sense, the structure of Maxwell’s equation is optimal for any field described by an antisymmetric tensor of the second rank. Therefore, for simplicity, we use here the terminology of electrodynamics, in spite of the fact that this approach is applicable also to Moffat’s interpretation.

In our approach we come back to the initial Einstein’s works. We postulate that the covariant derivative of the metric tensor equals zero, and equate the Einstein tensor to energy tensor.

Generally the symmetric $g(\mu\nu)$ and antisymmetric $g<\mu\nu>$ part of the metric tensor describes the gravitational and electromagnetic field. However, for a better understanding of the role of the electromagnetic field and the principle of constructing the energy tensor, preferably to consider the case of absence of the gravitational field. In this case $g(\mu\nu) = \delta_{\mu\nu}$ and the interval has the Euclidean form. However all manipulations with such a form of the metric tensor allows only the orthogonal coordinate transformations. Therefore we assume a very weak gravitational field, $g_{\mu\nu} = g(\mu\nu) - \delta_{\mu\nu}, |g_{\mu\nu}| \ll |g<\mu\nu|$, which allows more general transformations. In addition, we assume also $|g<\mu\nu| \ll 1$.

Using the assumptions, we expand parameters of the field in power series in the electromagnetic tensor. This leads to arising symmetric terms in the energy tensor. The terms may be sources of the gravitational field. The field has the order of the electromagnetic field squared.

For simplicity, we use the expansion including the second order. It is enough to demonstrate main features of the approach.

The approach provides a simple algorithm of matching the field equation and Maxwell’s equation for every order of the approximation.
The metric tensor and Christoffel’s symbol

We use the normalized (dimensionless) metric tensor, electric current density and coordinates with \( x_k = i t \).

The covariant derivative of vectors \( V_\mu \) and \( V^\mu \) is defined as

\[
V_{\mu,\nu} = V_{\mu,\nu} - \nabla_\nu G^\sigma_{\mu,\sigma}, \quad V^\mu = V^\mu + \nabla^\sigma G^\nu_{\mu,\nu},
\]

(1)

the semicolon and comma denotes the covariant and partial differentiation, respectively.

We assume that

\[
g_{\mu\nu} = \delta_{\mu\nu} + \varphi_{\mu\nu} + \gamma_{\mu\nu},
\]

(2)

where \( \delta_{\mu\nu} \) is the Kronecker delta, \( \varphi_{\mu\nu} \) is the electromagnetic tensor.

The covariant derivative of the metric tensor obeys the equation

\[
g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - g_{\kappa\nu} \Gamma^\kappa_{\mu,\sigma} - g_{\mu\kappa} \Gamma^\kappa_{\nu,\sigma} = 0,
\]

(3)

Both the tensors \( g^{\mu\nu} \) and \( g_{\mu\nu} \) are connected by the relation

\[
g^{\mu\sigma} g_{\nu\sigma} = \delta^\mu_\nu = g^{\mu\sigma} g_{\nu\sigma}
\]

(4)

From this relation \( g^{\mu\nu} \) is defined as the corresponding minor divided by the determinant \( g = |g_{\mu\nu}| \).

The Christoffel symbol \( \Gamma^\sigma_{\mu,\nu} \) can be expressed as a function of the metric tensor \( g_{\mu\nu} \), using the algebraic equation (3) relative \( \Gamma^\kappa_{\mu,\sigma} \) and \( \Gamma^\kappa_{\sigma,\nu} \). However, in contrast to the symmetric case, this expression is too sophisticated.

Therefore we use successive approximations expanding \( \Gamma^\sigma_{\mu,\nu} \) in power series in \( \varphi_{\mu\nu} \) and taken into account that \( \gamma_{\mu\nu} \) can be presented as a sum of even power in \( \varphi \).

With help of Eq. (2) express \( \varphi_{\mu\nu,\sigma} = (g_{\mu\nu} - g_{\mu\nu})_{,\sigma} \) and \( (g_{\mu\nu} + g_{\mu\nu})_{,\sigma} \) in terms of the metric tensor and Christoffel’s symbols. Then substitute the expression (2) and make the cyclic interchange \( \mu, \nu, \sigma \), we obtain the expansion, including the terms of the second order

\[
\Gamma^\sigma_{\mu,\nu} = \varphi_{\mu\nu,\sigma} + \varphi_{\kappa\nu} \varphi_{\mu\sigma,\kappa} + \varphi_{\mu\kappa} \varphi_{\sigma\nu,\kappa} + \\
+ \frac{1}{2} (\gamma_{\sigma,\mu,\nu} + \gamma_{\nu,\sigma,\mu} - \gamma_{\mu,\nu,\sigma}) + \ldots,
\]

(5)

The Maxwell equation

In the Euclidean space the equation can be written as follows

\[
\varphi_{\{\mu,\nu,\sigma\}} = 0, \quad \varphi_{\mu,\nu,\rho} = j_\mu,
\]

(6)

where the bracers mean the cyclic interchange \( \mu, \nu, \sigma \). \( j_\mu \) is the current density

\[
\rho = \frac{dx^\mu}{ds},
\]

(7)

\( \rho \) is the charge density. In the Euclidean space co- and -contravariant vectors coincides.

It is well know from quantum theory that the potential is a more fundamental characteristic than components of the electromagnetic field. In the Euclidean space \( \varphi_{\mu\nu} \) can be expressed in terms of the potential as follows

\[
\varphi_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}
\]

(8)

Due to the expression (8) the first equation in (6) is fulfilled identically. But the Einstein tensor contain \( \varphi_{\mu\nu} \) with two derivatives. Keeping it in mind, differentiate the first equation of (5) in respect to \( x_\sigma \). Then, using the second equation, obtain

\[
\varphi_{\mu\nu,\sigma} - j_\mu,\nu + j_\nu,\mu = 0.
\]

(9)

This equation integrates both the field \( \varphi_{\mu\nu} \) and matter \( j_\mu \). We should expect appearance of this equation in the first approximation of the field equation (6).

The Einstein tensor and field equation

Einstein’s tensor is equivalent to that in the general relatively

\[
G_{\mu,\nu} = R_{\mu,\nu} - \frac{1}{2} g_{\mu,\nu} R.
\]

(10)

where \( R_{\mu,\nu} \) is the Ricci tensor.

\[
R_{\mu,\nu} = G^\kappa_{\mu,\rho,\nu} - G^\kappa_{\mu,\nu,\rho} + \Gamma^\kappa_{\mu,\nu} G^\rho_{\kappa,\rho} - \Gamma^\kappa_{\mu,\rho} G^\rho_{\kappa,\nu}.
\]

(11)

The covariant derivative of \( G_{\mu,\nu} \) equals zero. Equating \( G_{\mu,\nu} \) to the energy tensor \( G_{\mu,\nu} = T_{\mu,\nu} \) we obtain the field equation similarly to the symmetric theory. For convenience, the modified energy tensor \( T^*_\mu,\nu = T_{\mu,\nu} - \frac{1}{2} g_{\mu,\nu} T \), where \( T = g^{\mu\nu} T_{\mu,\nu} \) is in common use. Thus the field equation is as follows

\[
R_{\mu,\nu} = T^*_\mu,\nu.
\]

The first approximation

In the first approximation

\[
\Gamma^\sigma_{\mu,\nu} = \varphi_{\mu,\nu,\sigma}, \quad R_{\mu,\nu} = \varphi_{\mu,\nu,\sigma} - \varphi_{\mu,\nu,\sigma},
\]

(12)

we find the field equation

\[
\varphi_{\mu,\nu,\sigma} - \varphi_{\mu,\nu,\sigma} = T^*_\mu,\nu.
\]

(13)

A comparison with (12), in accordance with our approach, allows to construct the energy tensor in the first approximation

\[
T^*_\mu,\nu = - j_\mu,\nu.
\]

(14)
This is an example of the inserted term. Eq. (13) easily can be rearranged as follows

\[(\varphi_{\mu\nu,\sigma} + \varphi_{\sigma\mu,\nu} + \varphi_{\nu\sigma,\mu}),\sigma - \varphi_{\nu\sigma,\mu\sigma} = -j_{\nu,\mu}\]  

Provided the condition (5), we obtain \(\varphi_{\nu\sigma,\sigma} = j_{\nu}\) and the Maxwell equation (6).

\(j_{\nu,\mu}\) is a 'strange' term since corresponding energy density changes the sign together with electromagnetic field. Moreover its contribution in energy density changes the sign together with electromagnetic field. This is an example of the compensating term.

\[\text{The second approximation}\]

Using Eqs. (5), (6), it can be shown that the antisymmetric part of the Ricci tensor equals

\[R_{<\mu\nu>} = \varphi_{\mu\nu,\sigma}j_{\sigma}.\]  

(16)

Obviously, the contribution in energy from this term also vanishes by the integration over all the spatial volume, in both the conditions: \(j_{\mu} = 0\) at the boundary of integration and the current conservation \(j_{\mu,\mu} = 0\).

The symmetric part of the Ricci tensor in the second approximation is as follows

\[R_{\mu\nu} = j_{\mu}j_{\nu} + (\varphi_{\mu\nu,\sigma}j_{\sigma} - \frac{1}{4}\varphi_{\sigma\kappa,\mu\nu} + \frac{1}{2}(\gamma_{\sigma,\mu\nu} + \gamma_{\nu,\sigma,\mu} - \gamma_{\mu,\sigma,\nu} - \gamma_{\sigma,\mu\nu}).\]  

(18)

The term

\[j_{\mu}j_{\nu} = \rho^{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\]  

(19)

is of special interest. Its shape is similar to the mass term in the general relativity with symmetric metric, but the role of \(\rho^{2}\) plays the mass density. Since we use the dimensionless units the mass density should be proportional to the gravitational constant, which is determined by comparing the field equation with the Newtonian theory of gravity.

From Eq. (19) it follows that the charge density squared \(\rho^{2}\) corresponds to a mass density [1]. This parameter, analogously to the mass density, is proportional to an electrodynamic constant. For this constant only the dimension is known.

The term \(j_{\nu,\mu}\) must be inserted in the energy tensor. Conveniently to divide the inserts into two parts. The first part is the compensating term [19], which inserted uniquely. The second part is inserted as the same term but with a constant \(C\) on the same basis as in the general theory of relativity.

With this definitions \(T_{\mu\nu}^{*}\) takes the form

\[T_{\mu\nu}^{*} = -j_{\nu,\mu} + 2j^\alpha\varphi_{\mu\nu,\alpha} + (1 + C)j_{\mu}j_{\nu}.\]  

(20)

The gravitational field obeys field equation

\[\frac{1}{2}(\gamma_{\sigma,\mu\nu} + \gamma_{\nu,\sigma,\mu} - \gamma_{\mu,\sigma,\nu} - \gamma_{\sigma,\mu\nu}) + (\varphi_{\mu\kappa,\nu\sigma}\kappa - \frac{1}{4}\varphi_{\sigma\kappa,\mu\nu}) = Cj_{\mu}j_{\nu}.\]  

(21)

In the general relativity a supplementary condition is imposed for simplification of the field equation. The purpose of the simplification is to cancel the first three terms in [21]

\[\gamma_{\sigma,\mu\nu} + \gamma_{\nu,\sigma,\mu} - \gamma_{\mu,\sigma,\nu} = 0.\]  

(23)

This condition consists of 16 equations connecting 10 functions \(\gamma_{\mu\nu}\), some of the equations coincide. However the number of equations may be reduced by symmetry. The same result can be obtained using the four conditions

\[\Xi_{\mu\nu,\nu} = 0,\]  

where

\[\Xi_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\gamma_{\sigma\sigma}.\]  

(24)

The four conditions are interpreted as an "analog of the Lorentz condition" in electrodynamics.

Under this condition the equation for the gravitational field is simplified

\[-\frac{1}{2}\gamma_{\sigma,\mu\nu} + (\varphi_{\mu\kappa,\nu\sigma}\kappa - \frac{1}{4}\varphi_{\sigma\kappa,\mu\nu}) = C\rho^{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}.\]  

(25)

Up to now we considered the gravitational and electromagnetic field as originated from one source.

Assume, that the source of a gravitation field (not connected with an electromagnetic field) exists. We divide the total mass of any charged object as well as nuclei or particles on the proper mass and mass connected with the charge.

In that case instead the factor \(C\) in right part of Eq. (22) would be the factor \((\gamma + C\rho^{2})\) and the tensor \(\gamma_{\mu\nu}\) is the tensor of total gravitational field originated from both the mass density \(\gamma\) and the mass density connected with the charge density \(C\rho^{2}\).

A symmetric part of the energy tensor in that case, exclusive of \((-j_{\nu,\mu})\), would be \(T_{\mu\nu} = T_{\mu\nu}^{*} - \frac{1}{2}g_{\mu\nu}T^{*}\)

\[T_{\mu\nu} = [\gamma + (1 + C)\rho^{2} + (1 + C)\rho^{2} - \frac{1}{2}g_{\mu\nu}(1 + C)\rho^{2}.\]  

(26)
In the approximation of low velocities and weak fields

\[ T_{44} \approx \gamma + \frac{1}{2} (1 + C) \rho^2 \]  

(27)
is the total mass density.

Surprisingly the last term in the expression can be found in the semi-empirical mass formula.

**The Bethe–Weizsäcker mass formula**

The semi-empirical mass formula is used to approximate the mass of an atomic nucleus

\[ m = Z m_p + N m_n - a \left( \frac{A - 2Z}{A} \right) + E_b(A, Z), \]  

(28)

where \( Z \) and \( N \) is the number of protons and neutrons, \( A = Z + N \) is the total number of nucleons, \( m_p \) and \( m_n \) are the rest mass of a proton and a neutron, respectively, \( E_b \) the binding energy of the nucleus, \( a \) and \( E_b \) are small and determined empirically.

The mass corresponding to \( \rho^2 \) is defined by the integral over the volume of nucleus. In the liquid drop model of nucleus approximately

\[ \int \rho^2 dv \approx \frac{Z^2}{A}. \]  

(29)

Eq. (28) can be changed to following form

\[ m = Z m_{pe} + N m_{ne} - 4a \frac{Z^2}{A} + E_b, \]  

(30)

where \( m_{pe} \) and \( m_{ne} \) is the effective mass of the proton and neutron in nucleus

\[ m_{pe} = m_p + 3a, \quad m_{ne} = m_n - a. \]  

(31)

The three first terms in the right part of Eq. (30) correspond to the integral of the term \( T_{44} \) from Eq. (27) over all the volume of nucleus

\[ Z m_{pe} + N m_{ne} - 4a \frac{Z^2}{A} \to \int \left[ \gamma + \frac{1}{2} (1 + C) \rho^2 \right] dv. \]  

(32)

Accordingly to Eq. (32), \( C < 0 \), moreover \((1 + C) < 0\). Nevertheless, for any atomic nucleus the integral in (32) always is positive.

However, if the charge density increases and becomes larger than nuclear, gravity may be replaced by repulsion and energy may change the sign.

**CONCLUSION**

We have considered the general relativity with non-symmetric metric and proposed the approach for the construction of the energy momentum tensor.

It was shown that the energy tensor can be build out from the inserted material terms so that the field equation turns into the equality due to Maxwell’s equation.

Distinctive feature of the approach is the start from the first approximation without the gravitational field. In the second approximation a source of gravitational field arises. This source is similar to that in the general relativity but with the mass density proportional to the charge density squared.

The term corresponding to the source finds surprising confirmation in nuclear physic. Another surprise is the negative sign of this mass density.

If such term is really applicable to nuclear physics then increasing the charge density may have far-reaching consequences from microphysics to astrophysics. However, for that we should use exact expression for the Christoffel symbols as a function of the metric tensor.

In astrophysics the charge density, which corresponds to the zeroth energy density, would determine the beginning of the Big Bang.

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[5] H. Weyl, *Space, Time, Matter*, (Methuen, 1922).

[6] In the electromagnetic field \( \frac{1}{2} \eta_{\mu\nu}\eta^{\sigma\rho} F_{\mu\nu} F^{\sigma\rho} \) is used, where \( \eta_{\mu\nu} \) is the Levi-Chivita tensor. This tensor interchanges electric and magnetic field.

[7] This result was obtained in the thesis for a master’s degree, B. V. Gisin, *Some Issues of the Last Unitary Einstein’s Theory*, 1961, and have never been published.