Stochastic strong zero modes and their dynamical manifestations

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(Dated: May 4, 2023)

Strong zero modes (SZMs) are conserved operators localised at the edges of certain quantum spin chains, which give rise to long coherence times of edge spins. Here we define and analyse analogous operators in one-dimensional classical stochastic systems. For concreteness, we focus on chains with single occupancy and nearest-neighbour transitions, in particular particle hopping and pair creation and annihilation. For integrable choices of parameters we find the exact form of the SZM operators. Being in general non-diagonal in the classical basis, the dynamical consequences of stochastic SZMs are very different from those of their quantum counterparts. We show that the presence of a stochastic SZM is manifested through a class of exact relations between time-correlation functions, absent in the same system with periodic boundaries.

Recent successes have transformed our understanding of how long relaxation times—and potential non-ergodicity—emerge in quantum many-body systems (for reviews see e.g. [1–6]). One simple mechanism in some systems with open boundaries is that of a strong zero mode (SZM) [7–18]. An SZM is an operator localised at the boundary that commutes with the Hamiltonian, up to exponentially small corrections. Its presence affects the structure of the whole spectrum of the Hamiltonian, resulting for example in boundary degrees of freedom having very long coherence times [11–15]. For certain integrable spin chains, SZMs can be constructed exactly and explicitly [9].

In a classical stochastic system, continuous-time Markov dynamics are defined by a stochastic generator, just like a Hamiltonian generates unitary dynamics in a quantum system. While being in general non-Hermitian, stochastic generators often share many properties with Hamiltonians, thus connecting classical stochastic and quantum problems at the technical level. An example of such a connection is between the simple exclusion process and the XXZ quantum chain, see e.g. [19–22]. A natural question to ask, therefore, is whether SZMs exist in classical stochastic systems, and if they do, what consequences they have for the dynamics.

Here we answer this question. For simplicity we focus on systems of particles on a one-dimensional chain with at most single occupancy per site. We consider transitions between neighbouring sites, including hopping and pair creation or annihilation. Detailed balance need not be obeyed. For certain choices of the transition rates the generators are integrable, and for these we find the explicit form of boundary localised operators that commute with the generator (either exactly or up to corrections that are exponentially small in the system size). These stochastic SZMs are non-diagonal in the classical basis, and as such do not correspond to classical observables. They represent “hidden” conservation laws which, as we show below, manifest themselves in the dynamics through a class of exact relations among time-correlation functions observable at finite times.

We study a system of particles stochastically hopping on a one-dimensional chain of length $L$, while obeying an exclusion constraint so that each site can be occupied by at most one particle. A particle can hop to a neighbouring site (either left or right) if it is empty, two particles positioned on consecutive sites can evaporate from the lattice, and two particles can condense on a pair of empty sites. As illustrated in Fig. 1 the left- and right-hopping transitions have rates $D(1 + \delta)$ and $D(1 - \delta)$ respectively, while evaporation and condensation occur with rates $\gamma(1 + \kappa)$, and $\gamma(1 - \kappa)$. At the edges we typically assume open boundary conditions, where the first and last site each have only one nearest neighbour [23].

At each time the configuration of the system can be expressed in terms of a $L$-tuple $\mathbf{n} = (n_1, n_2, \ldots, n_L) \in \mathbb{Z}_2^L$, where $n_j = 1$ if there is a particle on site $j$ and $n_j = 0$ when empty. To describe dynamics of macroscopic states (i.e. probability distributions) we use bra-ket notation, $|p\rangle = [p_0, p_1, \ldots, p_{2^L - 1}] \in \mathbb{R}^{2L}$, where each component $p_n \geq 0$ represents a probability of the configuration given by the binary representation of the subscript $n$, and the sum of all components is one, $\sum_n p_n = 1$. Diagonal...
operators represent observables, i.e. quantities that can be measured. Their expectation values are by definition given by the sum \( \langle a \rangle_p = \sum_n a_n p_n = -\langle a | p \rangle \), where we introduced the flat state \( \langle - | = [1 \ 1] \otimes^L \). The normalization condition for \( | p \rangle \) then can be equivalently expressed as \( \langle - | p \rangle = 1 \). Encoding the stochastic transitions with a generator \( W \) means that an initial state \( | p \rangle \) evolves in time as \( | p(t) \rangle = e^{tW} | p \rangle \). Conservation of probability under time evolution requires \( \langle - | e^{tW} = \langle - | \). This notation gives convenient expressions for more complicated objects, such as the expectation value at time \( t \) after starting from some non-stationary initial state \( \langle - | e^{tW} | p \rangle \), or correlation functions between multiple observables at different times, \( \langle | p \rangle \).

We restrict the discussion to two different integrable limits of the generator \( W \) (see e.g. [24]), for which the Hamiltonian counterparts are known to exhibit conserved edge modes [3] [4] [25].

(i) In the regime \( \gamma = D \) the generator is quadratic in fermionic operators (see Sec. [A]), so we refer to it as the free-fermion model. The stochastic generator with open boundaries has the form

\[
W^{(FF)} = \sum_{j=1}^{L-1} \left[ X_j X_{j+1} + \kappa Z_j + i \frac{\delta}{2} X_j Y_{j+1} + i \frac{\kappa - \delta}{2} Y_j X_{j+1} - 1 \right] + \frac{\kappa + \delta}{2} (Z_L - Z_1)
\]

(1)

where \( X_j, Y_j \), and \( Z_j \) denote Pauli matrices acting on the site \( j \). Without loss of generality, we rescaled the unit of time so that \( D = 1 \). Note that when \( \delta \neq 0 \), hopping is asymmetric and \( W \) does not obey detailed balance.

(ii) The second integrable regime arises when \( \kappa = \delta = 0 \), i.e. there is no asymmetry between the left and right hopping, and the rates for condensation and evaporation are the same. This model was studied with periodic boundaries in Ref. [26], and, more recently, solutions to the boundary-driven setup have been found [27]. In this case the generator takes a form of a rotated anisotropic Heisenberg XXZ Hamiltonian,

\[
W^{(XZZ)} = \sum_{j=1}^{L-1} \left[ \frac{1 - \gamma}{2} (Y_j Y_{j+1} + Z_j Z_{j+1}) + \frac{1 + \gamma}{2} (X_j X_{j+1} - 1) \right],
\]

(2)

so we refer to it as XZZ model. We again chose \( D = 1 \).

In analogy to the quantum setting, a conserved edge mode \( \Psi \) is an operator that commutes with the stochastic generator, \( [\Psi, W] = 0 \), squares into identity, \( \Psi^2 = 1 \), and is localised at an edge—its local densities that involve sites far from the edge are exponentially suppressed.

In the case of \( W^{(FF)} \) we take advantage of the free-fermionic form to straightforwardly find the expression for \( \Psi^{(FF)} \) (see Sec. [A] for the derivation),

\[
\Psi^{(FF)} = \sum_{j=1}^{L} \lambda^{j-1} \mu_{j-1} (X_j + i \lambda Y_j),
\]

(3)

where the disorder operator \( \mu_j = \prod_{k=1}^{j} Z_k \) is a string of \( Z_k \)s originating at the left edge, and the parameter \( \lambda \) is expressed in terms of \( \kappa \) and \( \delta \) as

\[
\lambda = \frac{1 - \sqrt{1 + \delta^2 - \kappa^2}}{\delta + \kappa}
\]

(4)

with \( |\lambda| \leq 1 \). This edge mode is exactly conserved, i.e. \( \langle \Psi^{(FF)} | \Psi^{(FF)} \rangle = 0 \) with no corrections. For simplicity we neglect exponentially small corrections to the normalization: \( \Psi^{(FF)}^2 = 1 + O(\lambda^L) \).

The XZZ generator (2) is Hermitian and has exactly the same form as the XXZ Hamiltonian with appropriately chosen couplings, therefore we can directly adapt the exact form of Ref. [9] to obtain

\[
\Psi^{(XZZ)} = \sum_{S=0}^{\infty} \sum_{1 \leq a_1 < \cdots < a_2 \leq L} \lambda^{S(b-1)} (1 - \lambda^2) \left( 1 - \frac{1}{\lambda^2} \right)^S \lambda^{-\sum_{j=1}^{2S} (-1)^{a_j} X_0 \prod_{j=1}^{S} (Y_{a_{2j-1}} Y_{a_{2j-1}} + Z_{a_{2j-1}} Z_{a_{2j}})},
\]

(5)

where the value of \( \lambda, |\lambda| \leq 1 \), is now given by

\[
\lambda = \frac{1 - \gamma}{1 + \gamma}
\]

(6)

Unlike the free-fermion case, the edge mode now no longer exactly commutes with the stochastic generator, but rather does so up to corrections of the order \( O(\lambda^L) \). In both cases (3) and (5), \( |\lambda| \leq 1 \) implies the exponential suppression of local densities on sites far from the edge of the lattice, making the SZM localised at the boundary.

Since neither SZM is diagonal, they cannot be directly interpreted as classical observables. Their effect on the dynamics therefore is not immediately obvious. A key observation is that the expectation value of an off-diagonal operator \( \hat{A} \) always can be interpreted as an expectation value of a corresponding diagonal operator \( \hat{A} \) defined by

\[
\langle - | A = \langle - | \hat{A} \quad \implies \quad \langle - | \hat{A} | p(t) \rangle = \langle - | A | p(t) \rangle.
\]

(7)

Pauli operators obey the two simple identities

\[
[1 \ 1] X_j = [1 \ 1] \quad \text{and} \quad [1 \ 1] Y_j = i [1 \ 1] Z_j,
\]

(8)

which can be linearly extended to provide the diagonal operator \( \hat{A} \) for an arbitrary \( A \). Therefore, the existence of a non-diagonal operator \( \Psi \) commuting with \( W \) implies the existence of a classical observable \( \Phi \) whose expectation value does not change with time,

\[
\langle - | \Psi e^{W} | p \rangle = \langle - | \Psi e^{W} | p \rangle = \langle - | e^{W} \Psi | p \rangle = \langle - | \Psi | p \rangle = \langle - | \hat{A} | p \rangle,
\]

(9)
where we utilised the defining property (7) and the conservation of probability.

In our cases, a little more work is needed. Indeed, the identities (8) imply that $\langle \cdot |$ is (up to terms exponentially small in $L$) a left eigenvector of both $\hat{\Psi}^{(FF)}$, and $\hat{\Psi}^{(XZZ)}$,

$$\langle \cdot | \hat{\Psi} = \langle \cdot |,$$  

(10)

and therefore the conservation of $\langle \cdot | \hat{\Psi} | \psi(t) \rangle$ gives us no meaningful restriction on the dynamics.

Nonetheless, it is possible to define a dynamical protocol, under which the existence of the boundary mode gives nontrivial effects. We require the initial state $| \alpha \rangle$ to be an eigenvector of $\hat{\Psi}$ with eigenvalue $1$: $\hat{\Psi} | \alpha \rangle = | \alpha \rangle$ (28). The conservation of $\hat{\Psi}$ implies the existence of observables whose expectation value remains constant after starting from these states. A general expectation value of an observable $a$ at time $t$ obeys

$$\langle a e^{\delta W} | \alpha \rangle = \langle a e^{\delta W} \hat{\Psi}^t | \alpha \rangle = \langle a \hat{\Psi} e^{\delta W} | \alpha \rangle = -\langle \hat{\Psi} a e^{\delta W} | \alpha \rangle + \langle a, \hat{\Psi} e^{\delta W} | \alpha \rangle,$$  

(11)

which follows from the normalization $\hat{\Psi}^2 = 1$, the definition of $| \alpha \rangle$, and the conservation of the edge mode. Because $\langle \cdot |$ is the left eigenvector of $\hat{\Psi}$ (cf. Eq. (10)), we obtain a connection between the expectation value of $a$ at any time $t$ and that of its anticommutator $\{a, \hat{\Psi}\}$:

$$\langle a e^{\delta W} | \alpha \rangle = -\langle \{a, \hat{\Psi}\} e^{\delta W} | \alpha \rangle.$$  

(12)

This general identity can now be used to obtain some nontrivial constraints on dynamics.

Let us start with the free-fermionic model, and consider $a = Z_j$. After a series of elementary manipulations similar to the ones of Eq. (10), one obtains

$$\begin{align*}
\frac{1}{2} \langle \{ Z_j, \hat{\Psi}^{(FF)} \} | \alpha \rangle &= \langle \{ Z_j - \lambda^j - 1 | \mu_j - \lambda (\cdot | \mu_{j-1} \rangle, \\
\langle | Z_j e^{\delta W} | \alpha \rangle &= \lambda \langle | e^{\delta W} | \alpha \rangle = \lambda, \\
\langle | \mu_k e^{\delta W} | \alpha \rangle &= \lambda \langle | e^{\delta W} | \alpha \rangle = \lambda^k.
\end{align*}$$  

(13)

The second equality in both rows follows from the conservation of probabilities, $\langle \cdot | e^{\delta W} = \langle \cdot |$ and the normalization of the initial state, $\langle | \alpha \rangle = 1$. The expectation values of $\mu_k = \prod_{j=1}^k Z_j$ are therefore constant in time, even though the initial state is not stationary and the system must undergo nontrivial dynamics before relaxing. For $t = 0$ the relation (13) is the property of the initial state and does not depend on whether or not $\hat{\Psi}^{(FF)}$ is conserved: the surprising consequence of the existence of the edge mode is that it holds also when $t > 0$.

The XZZ regime can be treated analogously, with only the precise relations changing due to the different form of the edge mode. The left-action of the anticommutator on the flat state obeys

$$\frac{1}{2} \langle \{ Z_j, \hat{\Psi}^{(XZZ)} \} | \alpha \rangle = \langle | \mu_j - \lambda^j - 1 | \mu_j - \lambda^j \rangle \langle \cdot | \chi.$$  

(14)

where $\chi$ is a sum over the $Z_j$ with coefficients decaying exponentially away from the edge:

$$\chi = \sum_{j=1}^L \lambda^j Z_j.$$  

(15)

Inserting (14) into (12) immediately gives us the dynamical restriction for the XZZ case: the expectation value of $\chi$ is forced to be zero at all times, i.e.

$$\langle \cdot | \chi e^{\delta W} | \alpha \rangle = 0.$$  

(16)

Equations (13) and (16) provide nontrivial dynamical constraints holding in the presence of the edge mode. A few remarks are in order. First, corrections exponentially small in the system size have been ignored. Therefore one might expect that these constraints only hold up to times of the order of magnitude $1/\lambda$.

To demonstrate explicitly that relations (13) and (16) represent nontrivial constraints on the time evolution, we simulate this dynamical protocol using Monte Carlo sampling of trajectories. For clarity, we restrict the discussion to the case of symmetric hopping — i.e. we assume $\delta = 0$ in both regimes. The stationary state is then the same for both periodic and open boundary conditions, while the edge mode is only conserved in the latter case. Changing boundary conditions therefore gives a direct probe of the validity of the dynamical constraints arising from the edge mode. The initial state in the free fermionic case is

$$| \alpha^{(FF)} \rangle = \frac{1 + \hat{\Psi}^{(FF)} - 1}{2} \otimes L,$$  

(17)

while the interacting initial state is chosen as

$$| \alpha^{(XZZ)} \rangle = \frac{1 + \hat{\Psi}^{(XZZ)}}{2} \left[ \frac{1}{2} \right] \otimes L - 1.$$  

(18)

We note that these are just two concrete choices, and the full family of possible initial states is very large due to the high degeneracy of the spectra of $\hat{\Psi}^{(FF)}$, and $\hat{\Psi}^{(XZZ)}$. See Sec. [18] for more details.

The behaviour in the free-fermionic regime is shown in Fig. 2 where we compare the dynamics of the expectation value (13) between open and periodic boundaries. In both cases the initial value is equal to the stationary value, but the state itself is not stationary. Therefore
for periodic boundaries the expectation value shows non-trivial dynamics, while in the open case the edge mode prevents it from changing. The dynamics of quantities not restricted by Eq. (13) does not strongly depend on the boundary conditions, as demonstrated in the inset, where we compare the expectation value of $Z_j$ at two sites: one close to the edge and one in the bulk.

Analogous behaviour can be observed in the interacting XZZ regime in Fig. 3. The existence of conserved $\Psi^{(XZZ)}$ forces the expectation value of $\chi$ to stay at zero, see Eq. (16), while in the case of periodic boundaries there is no such restriction and $\chi$ exhibits nontrivial dynamics. However, as shown in the inset, the dynamics of generic observables shows no qualitative difference between the different boundary conditions.

In this paper we have generalised the concept of strong zero modes from quantum spin chains to one-dimensional classical stochastic systems. For choices of parameters that make the stochastic generators integrable we were able to obtain the SZMs exactly. In contrast to the quantum case, the conservation of a stochastic SZM cannot be observed directly in the dynamics, manifesting instead as specific constraints in time correlation functions. As far as we are aware these hidden conservation laws in systems with open boundaries were not identified before.

Relations (13) and (16) are just two examples of a large number of dynamical relation following from the existence of edge modes. For example, for the case of $W^{(XZZ)}$, a similar mechanism restricts the dynamics of a wide class of dynamical correlation functions in the stationary state. In particular, as we shown in Sec. D the equilibrium time-correlation functions

$$\langle -|\{A, \Psi\}e^{iW}B|\rangle = \langle -|Ae^{iW}\{B, \Psi\}|\rangle$$

(19)

are identical — up to times of the order of magnitude $1/\lambda^L$ — for any two observables $A$ and $B$. In Fig. 3 we plot the specific case of $A = Z_1$ and $B = Z_2$, where (19) reduces to $\langle -|(Z_2 - \lambda Z_1)e^{iW^{(XZZ)}}\chi|\rangle = 0$. 

Figure 2. Dynamics of disorder operators $\tilde{\mu}_k = \mu_k - \lambda_k = Z_1 Z_2 \cdots Z_k - \lambda^k$ in the free-fermionic model. The initial state $|\alpha\rangle$ is given in Eq. (17), while $\lambda$ is the expectation value of $Z_k$ in the stationary state.

Figure 3. Expectation values of $\chi$ (defined in Eq. (15)) in the interacting regime of the model. When the boundary conditions are open, the expectation value is constrained by the existence of the edge SZM (Eq. (16)), while the system with periodic boundaries exhibits nontrivial dynamics. For comparison, the dynamics of local magnetization $Z_k$ in the inset show no qualitative difference between the two boundary conditions. The initial state $|\alpha\rangle$ is given in Eq. (18), the annihilation/creation rate is $\gamma = 0.35$, the system size is $L = 20$, and the number of Monte-Carlo trajectories is $10^5$.

Figure 4. Dynamical correlation function between $\chi$ and $\lambda^k Z_j - \lambda^j Z_k$ in the stationary state $|s\rangle = 2^{-L}|-\rangle$ for the interacting regime of the model (with same numerical parameters as in Fig. 3). The vanishing at all times for the open boundaries case illustrates the identity (19) that follows from the stochastic zero mode.
Many questions remain. One is on the fate of SZMs away from integrability. Our results explicitly depend on the precise form of the SZMs, but typically the physics of these models shows no qualitative change when the stochastic rates are tuned to the integrable point. A related question is whether for non-integrable stochastic spin chains, e.g., those in Ref. [29], SZMs are only conserved parametrically, as occurs in non-integrable quantum systems [11, 12], and if so, how these “almost” SZMs manifest themselves in the dynamics. A more general issue is to describe the dynamical consequences of other conserved non-diagonal operators in classical stochastic models. For instance, setting the condensation and evaporation rates to 0, our model reduces to the asymmetric simple exclusion process [30–32], which can be mapped to the XXZ Heisenberg Hamiltonian by a similarity transformation. This mapping implies the existence of an infinite number of non-diagonal local conserved operators that are obtained from the corresponding transfer matrix [33, 34]. It would be very interesting to understand how they constrain the stochastic classical dynamics.

ACKNOWLEDGMENTS

This work has been supported by the EPSRC Grant no. EP/S020527/1 (KK, PF), EPSRC Grant no. EP/R04421X/1 (JPG) and the Leverhulme Trust Grant No. RPG-2018-181 (JPG).

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### SUPPLEMENTAL MATERIAL

#### Appendix A: Diagonal form of the free-fermionic generator

To diagonalize the generator $\mathbb{W}^{(\text{FF})}$ (cf. (1)), we first observe that it can be put in a form that is quadratic in fermionic operators,

$$
\mathbb{W}^{(\text{FF})} = \sum_{j=1}^{L-1} \left( -i \kappa A_j B_j - i B_j A_{j+1} + \frac{\delta + \kappa}{2} B_j B_{j+1} + \frac{\delta - \kappa}{2} A_j A_{j+1} \right) + i \frac{\delta + \kappa}{2} (A_1 B_1 - A_L B_L - (L - 1)),
$$

(A1)

where we introduced Majorana operators $A_j$ and $B_j$ as

$$
A_j = Z_1 Z_2 \cdots Z_{j-1} X_j, \quad B_j = Z_1 Z_2 \cdots Z_{j-1} Y_j, \quad \{A_j, B_k\} = 0, \quad \{A_j, A_k\} = \{B_j, B_k\} = 2 \delta_{j,k}.
$$

The simplest way to find the spectrum of $\mathbb{W}^{(\text{FF})}$ is to find operators $\phi$ that obey

$$
\left[ \mathbb{W}^{(\text{FF})}, \phi \right] = 2 \varepsilon \phi.
$$

(A3)

These operators are constructed as linear combinations of Majorana operators,

$$
\phi(\alpha_1, \beta_1, \alpha_2, \ldots, \beta_L) = \sum_{\mu} (\alpha_j A_j + \beta_j B_j),
$$

(A4)

since commuting a bilinear in fermions with a linear combination always yields another linear combination. A convenient way of presenting $[\mathbb{W}^{(\text{FF})}, \phi]$ with such a $\phi = \phi(\mu)$ is to introduce the corresponding matrix $M_L$ acting on the $2L$-dimensional vector space such that

$$
\phi(\mu') = \mathbb{W}^{(\text{FF})}, \phi(\mu), \quad \mu' = 2 M_L \mu.
$$

(A5)

The possible values of $\varepsilon$ are the eigenvalues of $M_L$. In the Majorana basis the matrix is skew-symmetric, therefore its eigenvalues come in pairs, $(\varepsilon, -\varepsilon)$, and we label them as

$$
\varepsilon_{-k} = -\varepsilon_k, \quad \text{Re}(\varepsilon_k) \geq 0, \quad k = 1, 2, \ldots, L,
$$

(A6)

while the corresponding eigenvectors are labelled as $\mu_k$. The matrix has no other obvious special structure, so the eigenvalues are not guaranteed to be real, and, indeed, in general they have a non-trivial imaginary part. Furthermore, we note that if $\mu_k$ is a right eigenvector of $M_L$ corresponding to eigenvalue $\varepsilon_k$, then it is also the left eigenvector corresponding to eigenvalue $\varepsilon_{-k} = -\varepsilon_k$, which follows directly from the skew-symmetric structure of $M_L$,

$$
\mu_k M_L = - \mu_k M_L^T = -(M_L \mu_k)^T = - \varepsilon_k \mu_k.
$$

(A7)

Therefore, the eigenbasis can be chosen so that the following holds,

$$
\mu_k \cdot \mu_q = \delta_{k,-q}.
$$

(A8)
Defining now \( \phi_k = \phi(\mu_k) \), one can quickly find a diagonal form of the generator \( \mathcal{W}^{(FF)} \). First, we note that the eigenvalue condition gives us

\[
[\mathcal{W}, \phi_{\pm k}] = \pm 2\epsilon_k \phi_{\pm k}.
\]  (A9)

Second, these operators satisfy the following algebra,

\[
\{ \phi_k, \phi_q \} = 2\mu_k \cdot \mu_q = 2\delta_{k,-q},
\]  (A10)

which in particular implies also \( \phi_k^2 = 0 \). This immediately tells us that, up to a constant shift, the generator can be given in terms of \( \phi_k \phi_{-k} \),

\[
\mathcal{W}^{(FF)} = \sum_{k=1}^{L} \epsilon_k (\phi_k \phi_{-k} - 2) = \sum_{k=1}^{L} \epsilon_k \phi_{-k} \phi_k.
\]  (A11)

Here we determined the constant shift so that the eigenvalue of \( \mathcal{W}^{(FF)} \) with the largest real part is 0, which ensures the conservation of probabilities.

The diagonalization of the stochastic generator \( \mathcal{W}^{(FF)} \) thus reduces to the diagonalization of the \( 2L \times 2L \) block-three-diagonal matrix \( M_L \),

\[
M_L = \begin{pmatrix}
a_1 & b \\
-b^T & a & b \\
-b^T & a & b \\
& \ddots & \\
-b^T & a & b \\
L & -b^T & a_L
\end{pmatrix},
\]  (A12)

with the \( 2 \times 2 \) blocks given by

\[
a_1 = \frac{\kappa - \delta}{2} Y, \quad a = \kappa Y, \quad a_L = \frac{\kappa + \delta}{2} Y, \quad b = \left[ \begin{array}{cc}
-\frac{\kappa - \delta}{2} & 0 \\
-\frac{\kappa + \delta}{2} & \end{array} \right].
\]  (A13)

### 1. Conserved zero modes

In this language, conserved zero modes can be understood as eigenvectors of \( M_L \) corresponding to the eigenvalue 0. Due to the skew-symmetric structure of the matrix, they necessarily need to appear in pairs, and we assume the following homogeneous ansatz,

\[
\nu = \bigoplus_{j=1}^{L} \lambda^j \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}.
\]  (A14)

Requiring \( M_L \nu = 0 \), we obtain two linearly independent solutions,

\[
\nu_1 = \bigoplus_{j=1}^{L} \lambda^{j-1} \begin{pmatrix}
1 \\
i\lambda
\end{pmatrix}, \quad \nu_2 = \bigoplus_{j=1}^{L} \lambda^{L-j} \begin{pmatrix}
1 \\
-1
\end{pmatrix},
\]  (A15)

with the two parameters \( \lambda, \bar{\lambda} \)

\[
\lambda = \frac{1 - \sqrt{1 - \kappa^2 + \delta^2}}{\kappa + \delta}, \quad \bar{\lambda} = \frac{1 - \sqrt{1 - \kappa^2 + \delta^2}}{\kappa - \delta} = \bar{\lambda}_{\delta \leftrightarrow -\delta}.
\]  (A16)

We choose the labelling of the eigenvectors \( \mu_k \) so that the subscripts \( k = 1 \) and \( k = -1 \) correspond to \( \epsilon_{\pm 1} = 0 \), therefore \( \mu_{\pm 1} \) should be linear combinations of \( \nu_{1,2} \), which satisfy the orthogonality condition (A8). For clarity we
take into account the fact that for all sensible values of $\delta, \kappa$, the parameters $\lambda, \bar{\lambda}$ are in magnitude smaller than 1, and we require the orthogonality condition to only hold up to corrections of the order $\lambda^L, \bar{\lambda}^L$, 

$$
\mu_{\pm 1} = \frac{1}{\sqrt{2}} \prod_{j=1}^{L} \left[ \lambda^{j-1} \pm \bar{\lambda}^{L-(j-1)} \right].
$$

(A17)

At this point it is not clear that these are all the zero eigenvectors. However, as we will demonstrate later, there are exactly $2L - 2$ non-zero eigenvalues $\pm \varepsilon_k$, which means that the degeneracy of the eigenvalue 0 is exactly two.

Note that in the main text the zero modes $\Psi, \bar{\Psi}$ are required to satisfy $\Psi^2 = \bar{\Psi}^2 = 1$, and they are not equal to $\phi_{\pm 1}$, but they are given by $\nu_1$ and $\nu_2$. In particular, the expression for $\Psi^{(FF)}$ from Eq. (3) is

$$
\Psi = \phi(\nu_1) = \sum_{j=1}^{L} \lambda^{j-1} (A_j + i\lambda B_j) = \mu_{j-1} Z_1 Z_2 \ldots Z_L (X_j + i\lambda Y_j),
$$

(A18)

while the second zero mode is

$$
\bar{\Psi} = -i \Pi \phi(\nu_2) = \Pi \sum_{j=1}^{L} \bar{\lambda}^{L-j} (\bar{\lambda} A_j + iB_j) = \sum_{j=1}^{L} \bar{\lambda}^{L-j} Z_{j+1} \ldots Z_L (X_j + i\lambda Y_j),
$$

(A19)

with $\Pi = \prod_{j=1}^{L} Z_j$. Here we take advantage of the fact that both $\Pi$ and $\phi(\nu_2)$ commute with $\mathcal{W}^{(FF)}$, and therefore also their product does. The two zero modes $\Psi$ and $\bar{\Psi}$ are exponentially localised at left and right edge respectively. Moreover, $\bar{\Psi}$ is related to $\Psi$ through the left-right reflection, since this transformation maps $\delta$ to $-\delta$, and

$$
\bar{\lambda} = \lambda \delta_{\delta^{-1}}.
$$

(A20)

2. Change of basis

To find the remaining eigenvectors it is convenient to perform a basis transformation. We first introduce fermionic creation and annihilation operators $c_j, c_j^\dagger, j = 1, \ldots, L - 1,$

$$
c_j = \frac{1}{2} (B_j + iA_{j+1}), \quad c_j^\dagger = \frac{1}{2} (B_j - iA_{j+1}).
$$

(A21)

In terms of $c_j$ and $c_j^\dagger$, the generator takes the following form,

$$
\mathcal{W}^{(FF)} = i(\delta - \kappa) A_1 c_1^\dagger + (\delta + \kappa) c_{L-1}^\dagger B_L - 2 \sum_{j=1}^{L-1} c_j^\dagger c_j + \sum_{j=1}^{L-2} \left( 2\kappa c_j^\dagger c_{j+1}^\dagger + (\delta + \kappa) c_j^\dagger c_{j+1} + (\delta - \kappa) c_j c_{j+1}^\dagger \right).
$$

(A22)

Since terms of the form $c_j c_k$, $A_1 c_1$ and $B_L c_j$ are not present, commutating with a linear combination of $c_j^\dagger$ produces another linear combination of $c_j^\dagger$, which means that $L - 1$ of the remaining eigenvectors are just linear combinations of $c_j^\dagger$. In other words, the matrix $M_L$ rewritten in the basis $\{c_1^\dagger, c_2^\dagger, \ldots, c_{L-1}^\dagger, c_1, c_2, \ldots, c_{L-1}, A_1, B_L\}$ has a non-trivial invariant subspace under right action,

$$
\tilde{M}_L = \left[ \begin{array}{ccc}
\tilde{a} & \tilde{b} & \tilde{c} \\
0 & \tilde{a}^T & 0 \\
0 & \tilde{b}^T & 0 \\
\end{array} \right],
$$

(A23)

where blocks $\tilde{a}$ and $\tilde{b}$ are $L - 1 \times L - 1$, while $\tilde{c}$ is $L - 1 \times 2$,

$$
\tilde{a} = \left[ \begin{array}{ccc}
-1 & \frac{\kappa + \delta}{2} & \frac{\kappa + \delta}{2} \\
\frac{\kappa - \delta}{2} & -1 & \frac{\kappa + \delta}{2} \\
\frac{\kappa - \delta}{2} & -1 & \frac{\kappa - \delta}{2} \\
L - 1 & \frac{\kappa - \delta}{2} & \frac{\kappa - \delta}{2} \\
\end{array} \right], \quad \tilde{b} = \left[ \begin{array}{ccc}
0 & \kappa & \kappa \\
-\kappa & 0 & \kappa \\
-\kappa & 0 & \kappa \\
L - 1 & \kappa & \kappa \\
\end{array} \right], \quad \tilde{c} = \left[ \begin{array}{ccc}
i(\kappa - \delta) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\kappa + \delta & 0 & 0 \\
\end{array} \right].
$$

(A24)
In this basis the matrix is not skew-symmetric, and therefore the sets of left and right eigenvectors, \( \{ \tilde{\mu}_k^L \}_k \) and \( \{ \tilde{\mu}_k^R \}_k \), are not the same. However, both these sets can be determined by first diagonalising \( \tilde{\alpha} \). Let \( \tilde{v}_k^R \) and \( \tilde{v}_k^L \), be the right and left eigenvectors of \( \tilde{\alpha} \) corresponding to the eigenvalue \( -\varepsilon_k \),

\[
\tilde{\alpha} \tilde{v}_k^R = -\varepsilon_k \tilde{v}_k^R, \quad \tilde{v}_k^L \tilde{\alpha} = -\varepsilon_k \tilde{v}_k^L, \quad k = 2, \ldots, L.
\]  

(A25)

These eigenvalue equations can be straightforwardly solved with the sine transform, and after some basic manipulations we obtain

\[
\tilde{v}_k^R = \sum_{j=1}^{L-1} \alpha^j \sin \left( \frac{(k-1)j\pi}{L} \right), \quad \tilde{v}_k^L = \sum_{j=1}^{L-1} \alpha^{-j} \sin \left( \frac{(k-1)j\pi}{L} \right), \quad \alpha^2 = \frac{\kappa - \delta}{\kappa + \delta} = \frac{1}{\lambda},
\]

\[
\varepsilon_k = 1 - (\kappa + \delta) \cos \left( \frac{(k-1)\pi}{L} \right).
\]  

(A26)

Note that depending on the magnitude and signs of parameters \( \kappa \) and \( \delta \), the eigenvalues \( -\varepsilon_k \) can be either complex or real, but the sign of \( \varepsilon_k \) was chosen so that \( \Re\{\varepsilon_k\} \geq 0 \) for any physically sensible values of parameters.

Using the diagonal basis of \( \tilde{\alpha} \) we are now able to immediately obtain right eigenvectors of \( \tilde{M}_L \) corresponding to eigenvalues \( \varepsilon_{-k} = -\varepsilon_k \), and left eigenvectors of \( \tilde{M}_L \) corresponding to eigenvalues \( \varepsilon_k \). Indeed, \( \tilde{\mu}_k^R \) and \( \tilde{\mu}_k^L \) defined as

\[
\tilde{\mu}_k^R = \tilde{v}_k^R \oplus \cdots \oplus \tilde{v}_k^{L-1}, \quad \tilde{\mu}_k^L = \tilde{v}_k^R \oplus \cdots \oplus \tilde{v}_k^L.
\]

(A27)

satisfy the appropriate eigenvalue equations,

\[
\tilde{M}_L \tilde{\mu}_k^R = -\varepsilon_k \tilde{\mu}_k^R, \quad \tilde{M}_L \tilde{\mu}_k^L = \varepsilon_k \tilde{\mu}_k^L.
\]  

(A28)

We note that this completely determines the spectrum of \( M_L \) (and \( \tilde{M}_L \)): two eigenvalues are zero, \( \varepsilon_{\pm 1} = 0 \), while the rest are given in pairs \( \varepsilon_{\pm k} = \pm \varepsilon_k, k \geq 2 \).

To determine the rest of the right eigenvectors we take the following ansatz

\[
\tilde{\mu}_k^R = x_k^R \oplus y_k^R \oplus z_k^R, \quad x_k^R, y_k^R, z_k^R \in \mathbb{C}^{L-1}, \quad z_k^R \in \mathbb{C}^2.
\]

(A29)

Requiring that this is a right eigenvector of \( \tilde{M}_L \) corresponding to the eigenvalue \( \varepsilon_k \), we obtain the following explicit form,

\[
y_k^R = v_k^L, \quad z_k^R = -\frac{1}{2\varepsilon_k} c^T v_k^L, \quad x_k^R = (\tilde{\alpha} - \varepsilon_k)^{-1} \left( \frac{1}{2\varepsilon_k} c\tilde{c}^T - \tilde{b} \right) v_k^L.
\]  

(A30)

Note that the matrix \( \tilde{\alpha} - \varepsilon_k \) is invertible for any \( k \), since the spectrum of \( \tilde{\alpha} \) is \( \{-\varepsilon_k\}_{2 \leq k \leq L} \). The expression for \( z_k^R \) can be immediately evaluated and yields

\[
z_k^R = -\frac{(\kappa + \delta)\alpha}{2\varepsilon_k} \sin \left( \frac{(k-1)\pi}{L} \right) \left[ \frac{i}{(-1)^k \alpha^{-L}} \right].
\]

(A31)

To find a convenient form of \( x_k^R \), we express it in the basis of the eigenvectors of \( \tilde{\alpha} \) as,

\[
x_k^R = \sum_{j=2}^{L} \frac{1}{\varepsilon_k + \varepsilon_j} \left( \sum_{l=2}^{L} (v_j^L : (\tilde{b} - \frac{1}{2\varepsilon_k} c\tilde{c}^T) \cdot v_l^R) (v_j^L : v_l^R) \right) v_j^R = \sum_{j=2}^{L} \frac{1}{\varepsilon_k + \varepsilon_j} \left( \sum_{l=2}^{L} (f_{j,l} - \frac{1}{2\varepsilon_k} g_{j,l}) h_{l,k} \right) v_j^R,
\]

(A32)

where we introduced coefficients \( f_{j,l} \), \( g_{j,l} \) and \( h_{j,l} \) to encode the relevant vector overlaps. With some straightforward manipulations they can be simplified as,

\[
f_{j,l} = v_j^L \cdot \tilde{b} \cdot v_l^R = \delta_{j,l} \kappa \cos \left( \frac{(j-1)\pi}{L} \right) \left( \frac{1}{\alpha} - \alpha \right) + \frac{1}{L} \sin \left( \frac{(j-1)\pi}{L} \right) \sin \left( \frac{(l-1)\pi}{L} \right) \left( \frac{1}{\alpha} + \alpha \right),
\]

\[
g_{j,l} = v_j^L \cdot c\tilde{c}^T \cdot v_l^R = \frac{2}{L} \sin \left( \frac{(j-1)\pi}{L} \right) \sin \left( \frac{(l-1)\pi}{L} \right) \left( 1 + (-1)^{j+l} \right),
\]

\[
h_{j,l} = v_j^L \cdot v_l^R = \frac{4}{L^2} \left( \frac{(-1)^{j+l+2} \alpha^2 (1 - \alpha^2) \sin \left( \frac{(l-1)\pi}{L} \right) \sin \left( \frac{(j-1)\pi}{L} \right)}{1 - 2\alpha^2 \cos \left( \frac{(l-j+2)\pi}{L} \right) + \alpha^4} \right) \left( \frac{1 - 2\alpha^2 \cos \left( \frac{(l-j)\pi}{L} \right)}{1 - 2\alpha^2 \cos \left( \frac{(l-j)\pi}{L} \right)} \right).
\]

(A33)

The left eigenvectors \( \tilde{\mu}_{-k} \) can be determined analogously.
Appendix B: Eigenvectors of the edge mode

Since the edge mode squares into identity, $\Psi^2 = 1$, the vectors of the following form are eigenvectors corresponding to eigenvalues $+1$ and $-1$,

$$|\alpha^\pm_v\rangle = (1 \pm \Psi)|v\rangle. \quad (B1)$$

However, for an eigenvector to be used as an initial state in the protocol described in the main text, it also has to represent a valid probability distribution, i.e. all it should be normalized and all its component should be non-negative,

$$\langle -|\alpha^+_v\rangle = 1, \quad \langle \varepsilon|\alpha^+_v\rangle \geq 0. \quad (B2)$$

This requirement immediately constraints us to positive eigenvectors $|\alpha^+_v\rangle$, as $(-)$ is up to exponentially small corrections a positive left eigenvector of both the free-fermionic and the interacting edge mode, which in particular implies

$$\langle -|(1 - \Psi)|v\rangle = 0 \quad (B3)$$

for an arbitrary $|v\rangle$. To find appropriate positive eigenvectors, we need to separately treat the two cases.

1. Free-fermionic eigenvectors

a. Basis of eigenvectors

Before specialising to the case of positive eigenvectors, let us first construct a basis of eigenvectors of the form $|B1\rangle$. Choosing $|v\rangle$ in $|B1\rangle$ to be any canonical basis state $|v\rangle = |s_1s_2 \ldots s_L\rangle$, one obtains $4^L$ different vectors, which is twice the dimension of the probability space. We can explicitly show that this yields at most $2^L$ linearly independent eigenvectors, by using the fact that $|\alpha^+_v\rangle$ is an eigenvector corresponding to the eigenvalue $1$,

$$(1 + \Psi)|v\rangle = (1 + \Psi)|v\rangle \Psi|v\rangle, \quad (B4)$$

which in the example of a canonical basis vector $|v\rangle = |0s_2s_3 \ldots s_L\rangle$ yields

$$(1 + \Psi)|0s_2 \ldots s_L\rangle = (1 + \Psi)\frac{1 - \lambda}{\sqrt{1 - \lambda^2L}}|s_2 \ldots s_L\rangle + \sum_{k=2}^L (1 + \Psi)\psi_k|0s_2 \ldots s_L\rangle, \quad (B5)$$

where we use the short-hand notation

$$\psi_k = \frac{\lambda^{k-1}}{\sqrt{1 - \lambda^2L}}Z_1 \ldots Z_{k-1}(X_k + i\lambda Y_k). \quad (B6)$$

From here it follows that the eigenvectors of the form $(1 + \Psi)|s_2 \ldots s_L\rangle$ can be expressed in terms of eigenvectors obtained from basis states $|0s_2 \ldots s_L\rangle$,

$$|\alpha^+_{s_2 \ldots s_L}\rangle \in \text{Span}\left(\{|\alpha^+_0s_2 \ldots s_L\rangle\} \cup \{|\alpha^+_0s_2 \ldots 1 - s_L\rangle\}_{j=2}^L\right). \quad (B7)$$

This immediately implies that there are at most $2^{L/2}$ linearly independent eigenvectors of the form $|\alpha^+_v\rangle$. An analogous statement holds for $|\alpha^-_v\rangle$ due to the mapping $|\alpha^-_v\rangle = \prod_{j=1}^L X_j|\alpha^+_v\rangle$.

To prove that there are no additional eigenvectors of $\Psi$ we should show that the set of vectors

$$\{|\alpha^+_0s_2s_3 \ldots s_L\rangle\}_{s_j \in \{0,1\}} \quad (B8)$$

is linearly independent. We observe that each one of vectors $|\alpha^+_0s_2s_3 \ldots s_L\rangle$ has a nonzero overlap with precisely one basis vector of the form $|s_2' \ldots s'_L\rangle$,

$$\langle 1s_2' \ldots s'_L|\alpha^+_0s_2s_3 \ldots s_L\rangle = \prod_{j=2}^L \delta_{s_j' s_j} \langle 1s_2 \ldots s_L|\psi_1|0s_2 \ldots s_L\rangle, \quad (B9)$$

which directly implies the linear independence of the set $|B8\rangle$. 

b. Positivity requirement

We start by observing that eigenvectors \( |\alpha_0\ldots\alpha_0\rangle \) and \( |\alpha_0\ldots\alpha_1\rangle \) consist of nonnegative components, therefore any (appropriately normalized) linear combination of these two eigenvectors is a valid probability distribution. To find the general condition for a positive eigenvector \( |\alpha\rangle \) to fulfil the positivity requirement, we can express it as a linear combination of basis eigenvectors,

\[
|\alpha\rangle = \sum_{s_2,s_3,\ldots,s_L} c_{0s_2s_3\ldots s_L}(1 + \Psi) |0s_2 \ldots s_L\rangle.
\]  

(B10)

The straightforward observation is that all the coefficients \( c_{0s_2\ldots s_L} \) must be non-negative, which follows from

\[
\langle 1s_2s_3\ldots s_L|\alpha\rangle = \frac{1}{\sqrt{1 - \lambda^{2L}}} c_{0s_2s_3\ldots s_L}.
\]  

(B11)

The rest of the components are expressed as a sum of \( L + 1 \) contributions,

\[
\langle 0s_2s_3\ldots s_L|\alpha\rangle = c_{0s_2s_3\ldots s_L} + \sum_{j=2}^L \lambda^{j-1} (-1)^{s_2\ldots s_{j-1}} \frac{\sqrt{1 - \lambda^{2L}}}{1 - \lambda^{2L}} c_{0s_2\ldots s_{j-1}(1-s_j)s_{j+1}\ldots s_L},
\]  

(B12)

and the positivity condition reduces to \( 2^{L-1} \) inequalities. Note that two of them are automatically satisfied, since \( \langle 00\ldots 0|\alpha\rangle \geq 0 \) and \( \langle 00\ldots 01|\alpha\rangle \geq 0 \) follow from \( c_{0s_2\ldots s_L} \geq 0 \).

In general we are unable to be more explicit. However, it is still possible to find a few more instances of allowed states. For example, the linear combination

\[
c_{0\ldots 010} |\alpha_{0\ldots 010}\rangle + c_{0\ldots 011} |\alpha_{0\ldots 011}\rangle,
\]

is positive whenever the ratio between the two coefficients is in the following range,

\[
\frac{(1 - \lambda)\lambda^{L-1}}{\sqrt{1 - \lambda^{2L}}} \leq \frac{c_{0\ldots 010}}{c_{0\ldots 011}} \leq \frac{\sqrt{1 - \lambda^{2L}}}{(1 + \lambda)\lambda^{L-1}}.
\]  

(B14)

In the numerical calculations in the main text we choose \( |\alpha\rangle \) to be proportional to the simplest basis state \( |\alpha_{00\ldots 0}\rangle \), which takes the following explicit form,

\[
|\alpha\rangle = \frac{1 + \lambda^L - \sqrt{1 - \lambda^{2L}}}{2\lambda^L} |00\ldots 0\rangle + \frac{1 - \lambda}{1 - \lambda^L + \sqrt{1 - \lambda^{2L}}} \sum_{j=1}^L \lambda^{j-1} |0\ldots 010\ldots 0\rangle
\]  

\[
\approx \frac{1}{2} |00\ldots 0\rangle + \frac{1 - \lambda}{2} \sum_{j=1}^L \lambda^{j-1} |0\ldots 010\ldots 0\rangle.
\]

(B15)

In the large system-size limit this state has a nice intuitive interpretation: with probability \( \frac{1}{2} \) the initial configuration is empty, and with probabilities decaying as \( \lambda^j \) the initial configuration consists of one particle at position \( j \).

2. Interacting eigenvectors

As in the free-fermionic case, at the moment we are unable to fully characterize the set of all positive eigenvectors with non-negative components, but we can provide a simple family of states that belong to it. In particular, we introduce the state \( |\alpha_k\rangle \) parametrized with an integer \( 1 \leq k \leq L \) and a probability parameter \( 0 \leq \alpha \leq 1 \), as

\[
|\alpha_k\rangle = \frac{1}{2} (1 + \Psi) \left( \frac{1}{2} \right)^{\otimes (k-1)} \otimes \left[ \frac{\alpha}{2} \right] \otimes \left[ \frac{1 - \alpha}{2} \right] \otimes \left( \frac{1}{2} \right)^{\otimes (L-k)}.
\]

(B16)

i.e. the vector \( |\psi\rangle \) in (B11) corresponds to the maximum-entropy state everywhere, except at position \( k \), where the probability distribution is parametrized by \( \alpha \). In the computational basis, it takes the following form,

\[
|\alpha_k\rangle = \sum_{s} \frac{1}{2^{L-1}} (\alpha + \delta_{s_{k+1}}(1 - 2\alpha) + \frac{(1 - \lambda^2)(1 - 2\alpha)}{2} \sum_{j=1}^{L-1} \lambda^{j+k-2}(-1)^s) |s\rangle,
\]  

(B17)

\begin{align}
|\alpha_k\rangle &= \sum_{s} \frac{1}{2^{L-1}} (\alpha + \delta_{s_{k+1}}(1 - 2\alpha) + \frac{(1 - \lambda^2)(1 - 2\alpha)}{2} \sum_{j=1}^{L-1} \lambda^{j+k-2}(-1)^s) |s\rangle,
\]

(B17)
and one can straightforwardly verify that each term is non-negative for any combination of parameters $-1 < \lambda < 1$, and $0 \leq \alpha \leq 1$.

In the numerical simulations in the main text we use $|\alpha\rangle \equiv |\alpha_1\rangle$, with an intermediate value of $\alpha$ ($\alpha = 0.25$).

Appendix C: Stationary states

1. Free-fermionic regime

A stationary state $|s\rangle$ is mapped to 0 under the stochastic generator $\mathbb{W}$,

$$\mathbb{W} |s\rangle = 0.$$  \hfill (C1)

In the case with $\delta = 0$, a stationary state can be found in product form,

$$|s\rangle = \begin{bmatrix} 1 + \kappa & \sqrt{1 - \kappa^2} + \kappa - 1 \\ \sqrt{1 - \kappa^2} + \kappa - 1 & 2\kappa \end{bmatrix}^\otimes L = \begin{bmatrix} 1 + \lambda^2/2 \\ 1 - \lambda^2/2 \end{bmatrix}^\otimes L,$$  \hfill (C2)

where $\lambda$ is given by the $\delta \to 0$ limit of the expression in (4). This immediately gives the stationary expectation value of an arbitrary product of $Z_j$,

$$\langle -| Z_j Z_j \cdots Z_j |s\rangle = \lambda^k,$$  \hfill (C3)

which coincides with the expectation value in the edge-mode eigenvector (cf. (13)).

We remark that the stationary state is not unique, due to the generator $\mathbb{W}$ exhibiting a $Z_2$ symmetry,

$$[\Pi, \mathbb{W}] = 0, \quad \Pi = \prod_{j=1}^L Z_j.$$  \hfill (C4)

There are two linearly independent stationary states and we can choose the basis of stationary states to consist of (normalised) eigenvectors of $\Pi$,

$$|s_+\rangle = \frac{1 + \Pi}{1 + \lambda L} |s\rangle, \quad |s_-\rangle = \frac{1 - \Pi}{1 - \lambda L} |s\rangle,$$  \hfill (C5)

so that $\Pi |s_\pm\rangle = \pm |s_\pm\rangle$. Up to exponentially small corrections the expectation values of local observables in $|s_\pm\rangle$ match the ones in $|s\rangle$, therefore the discussion in the main text holds regardless of the precise state to which the system relaxes (as long as $L - k$ is not small). Furthermore, by requiring the stationary expectation value of $\Pi$ to match the initial value $\langle -| \Pi |\alpha\rangle$, one can show that the stationary state $|s_\alpha\rangle$ to which the system relaxes after starting from $|\alpha\rangle$ matches $|s\rangle$ up to exponentially small corrections,

$$|s_\alpha\rangle = \frac{1}{2} \left( 1 + \frac{1 - \sqrt{1 - \lambda^2 L^2}}{\lambda L} \right) |s_+\rangle + \frac{1}{2} \left( 1 - \frac{1 - \sqrt{1 - \lambda^2 L^2}}{\lambda L} \right) |s_-\rangle = |s\rangle + O(\lambda L).$$  \hfill (C6)

2. Interacting generator

In the case of the interacting generator, the stationary state is the maximum-entropy state,

$$|s\rangle = \frac{1}{2^L} |\rangle, \quad \mathbb{W}^{(\text{int})} |s\rangle = 0.$$  \hfill (C7)

This can be easily understood by noticing that the left/right hopping rates are the same (therefore the state should be translationally invariant), and the rates for annihilation/creation of pairs are the same, therefore we expect the probability of finding a particle at some site the same as not finding it. Alternatively, one can check explicitly and repeat the discussion above. The expectation value of the observable $\chi$ [cf. (15)] is

$$\langle -| \chi |s\rangle = \sum_{j=1}^L \lambda^j \langle -| Z_j |s\rangle = 0,$$  \hfill (C8)

which follows from $\langle -| Z_j |\rangle = 0$ for any $j$. 
Appendix D: Stationary dynamical correlations

In the case of interacting generator the stationary state $|s\rangle = 2^{-L} |-\rangle$ is also a right eigenvector of the edge-mode,

$$\Psi |-\rangle = |-\rangle,$$

where $s$ is a right eigenvector of the edge-mode,

$$\Psi |-\rangle = |-\rangle,$$  \hspace{1cm} (D1)

which enables us to use a similar manipulation to find restrictions on dynamical correlation functions. We start with the correlation function $\langle -|Ae^{tW}B|s\rangle$ between two observables $A$ and $B$ at times 0 and $t$, and take into account (D1) and (10),

$$\langle -|Ae^{tW}B|\rangle = \langle -|Ae^{tW}\Psi B|-\rangle = \langle -|Ae^{tW}\{B,\Psi\}\rangle - \langle -|Ae^{tW}\{B,\Psi\}|-\rangle = \langle -|Ae^{tW}\{B,\Psi\}|-\rangle - \langle -|Ae^{tW}\{B,\Psi\}\rangle + \langle -|Ae^{tW}\{B,\Psi\}\rangle - \langle -|Ae^{tW}\{B,\Psi\}\rangle,$$

which immediately implies the following,

$$\langle -|\{A,\Psi\}e^{tW}B\rangle = \langle -|\{A,\Psi\}e^{tW}B\rangle - \langle -|\{B,\Psi\}e^{tW}B\rangle + \langle -|\{A,\Psi\}e^{tW}B\rangle - \langle -|\{A,\Psi\}e^{tW}B\rangle.$$  \hspace{1cm} (D2)

We note that this does not necessarily represent a relation between two correlation functions, as $\{A,\Psi\}$ and $\{B,\Psi\}$ are in general non-diagonal. However, we can extend the definition of the corresponding diagonal operator (7) for a generic operator $O$ as,

$$\langle -|\hat{O}L = \langle -|O, \hat{O}R|\rangle = O|-\rangle,$$ \hspace{1cm} (D4)

which enables us to rewrite (D3) as a relation between two genuine correlation functions,

$$\langle -|\hat{A}e^{tW}\{\hat{B},\hat{\Psi}\}L|\rangle = \langle -|\hat{A}e^{tW}\{\hat{B},\hat{\Psi}\}L|\rangle = \langle -|\hat{B}e^{tW}\{\hat{A},\hat{\Psi}\}L|\rangle.$$ \hspace{1cm} (D5)

The second equality follows from $W^T = W$.

Specializing now to the case $A = Z_j$ and $B = Z_k$ we first note that the diagonal operators appearing in (D5) read as,

$$\{Z_j, \Psi\}_L = 2Z_j - 2\lambda_j \left( \frac{1}{\lambda^2} - 1 \right) \chi, \quad \{Z_k, \Psi\}_R = 2Z_k - 2\lambda_k \left( \frac{1}{\lambda^2} - 1 \right) \chi,$$ \hspace{1cm} (D6)

with $\chi$ defined in (15). This finally gives us

$$\lambda_j \langle -|Z_k e^{tW}\chi|\rangle = \lambda_k \langle -|Z_j e^{tW}\chi|\rangle.$$ \hspace{1cm} (D7)