Simultaneously solving the $H_0$ and $\sigma_8$ tensions with late dark energy

Lavinia Heisenberg, Hector Villarrubia-Rojo, Jann Zosso
Institute for Theoretical Physics, ETH Zürich, Wolfgang-Pauli-Strasse 27, 8093, Zürich, Switzerland
(Dated: January 28, 2022)

In a model independent approach, we derive generic conditions that any late time modification of the ΛCDM expansion history must satisfy in order to consistently solve both the $H_0$ and the $\sigma_8$ tensions. Our results are fully analytical and the method is merely based on the assumption that the late-time deviations from ΛCDM remain small. For the concrete case of a dark energy fluid with deviations encoded in the expansion history and the gravitational coupling constant, we present necessary conditions on its equation of state. Solving both the $H_0$ and $\sigma_8$ tensions requires that $w(z)$ must cross the phantom divide if $G_{\text{eff}} = G$. On the other hand, for $G_{\text{eff}} = G + \delta G(z)$ and $w(z) \leq -1$, it is required that $\frac{\delta G(z)}{G} < \alpha(z)\frac{\delta H(z)}{H(z)} < 0$ at some redshift $z$.

INTRODUCTION

Today’s era of precision cosmology poses a serious challenge to the theoretical description of our universe providing a guideline on the search for new physics. While the dark sector of the current cosmological standard model, that is a cosmological constant $\Lambda$ together with cold dark matter (ΛCDM), remains poorly understood, the data is beginning to require a departure from the model itself with increasing statistical significance. This is mainly due to the mismatch or tensions between the values of cosmological observables inferred from the cosmic microwave background (CMB) on the one hand and several independent local measurements on the other.

The two most prominent tensions are discrepancies in the several independent local measurements on the other. Simultaneously solving the CMB acoustic scale and a minimal set of assumptions alone, allows us to derive necessary conditions which severely constrain late time dark energy (DE) models characterized by an equation of state $w(z)$ which deviate from ΛCDM through the expansion history $\delta H(z)$ and a modification of the gravitational constant $\delta G(z)$. For any such theory we show the following:

Solving both the $H_0$ and $\sigma_8$ tensions requires:

i) If $G_{\text{eff}} = G$, $w(z)$ must cross the phantom divide.

ii) If $G_{\text{eff}} = G + \delta G(z)$ and $w(z) \leq -1$,

\[
\frac{\delta G(z)}{G} < \alpha(z)\frac{\delta H(z)}{H(z)} < 0 \quad \text{at some redshift } z,
\]

where in the second case, due to the condition on the equation of state, $\delta H(z) \neq 0$ and $\alpha(z)$, which we define in (15), is strictly positive.

In the following we will describe the method which allows us to arrive at the above results, referring to the companion paper [39] for details. We want to stress that our method is applicable in a much broader context and it is a priori not tied to dark energy models nor any specific cosmological tension.

METHOD

Setup. The starting point is a ΛCDM cosmology, which at late times can effectively be described by two free parameters, the Hubble constant $H_0$ and the matter abundance $\Omega_m$ through

\[
H_{\Lambda\text{CDM}}^2 = H_0^2 \left( \Omega_m (1+z)^3 + \Omega_\Lambda \right). \tag{1}
\]

In the following, we will define $\omega_m = \Omega_m h^2$ and $\omega_\Lambda = \Omega_\Lambda h^2$, where $H_0 \equiv 100 \text{ km s}^{-1} \text{Mpc}^{-1}$, such that

\[
\omega_\Lambda = h^2 - \omega_m. \tag{2}
\]

Alternative models can then be characterized by variations of the expansion history $\delta H(z)$

\[
H(H_0, \omega_m) = H_{\Lambda\text{CDM}}(H_0, \omega_m) + \delta H(z), \tag{3}
\]

as well as variations in other quantities, such as the gravitational constant $G_{\text{eff}} = G + \delta G(z)$. For brevity, however, we will focus on $\delta H(z)$ during the exposition of the
method and refer to the last paragraph of this section for the general case. At this stage, $\delta H(z)$ is an arbitrary function which captures deviations from $\Lambda$CDM for fixed $H_0$ and $\omega_m$. Restricting ourselves to late-time modifications, we will assume that $\delta H(z) = 0$ for $z > 300$.

The deviation from $\Lambda$CDM will modify the observationally preferred values of $H_0$ and $\omega_m$ such that working at first order in deviations the Hubble parameter in the alternative cosmology takes the general form

$$H(H_0 + \delta H_0, \omega_m + \delta \omega_m) = H_{\Lambda CDM}(H_0, \omega_m) + \Delta H.$$  \hspace{1cm} (4)

For late-time modifications, the deviations in the matter abundances are generally negligible, see \cite{39}, so for simplicity we impose $\delta \omega_m = 0$ in the reminder of this letter. In this case, the total variation in the Hubble parameter reads

$$\frac{\Delta H(z)}{H(z)} = \frac{H_0^2}{H(z)^2} \frac{\delta H_0}{H_0} + \frac{\delta H(z)}{H(z)},$$  \hspace{1cm} (5)

where, since we are working to first order, we are denoting $H_{\Lambda CDM}$ simply as $H$. This allows us to express the variation of any cosmological observable $O$ as

$$\frac{\Delta O(z)}{O(z)} = I_O(z) \frac{\delta H_0}{H_0} + \int_0^\infty \frac{dz}{1 + z} R_O(x,z,2) \frac{\delta H(x)}{H(x)},$$  \hspace{1cm} (6)

**Relation to observations.** It is now enough to choose a single very well measured observable, $Q_8$, whose value should not change in the alternative model with the aim of remaining compatible with observations, and hence impose its variation to vanish, $\Delta O_8 = 0$, in order to relate the modified expansion history $\delta H(z)$ to the variation in the inferred Hubble constant $\delta H_0$ through a response function

$$\frac{\delta H_0}{H_0} = - \int \frac{dz}{1 + z} R_{Q_8}(x,z) \frac{\delta H(x)}{H(x)} \hspace{1cm} \equiv \int \frac{dz}{1 + z} R_{H_0}(x,z) \frac{\delta H(x)}{H(x)}.$$  \hspace{1cm} (7)

Of course, agreeing with just one observable is not enough for a model to be viable. However, this simple method will allow us to derive conditions that a model must at least satisfy in order not to be directly excluded. And these necessary conditions will pose stringent analytic constraints on the allowed modifications in the expansion history.

**Deriving necessary conditions.** Combining (6) and (7) allows to compute the response function of arbitrary observables

$$\frac{\Delta O(z)}{O(z)} = \int_0^\infty \frac{dz}{1 + z} R_O(x,z,2) \frac{\delta H(x)}{H(x)}.$$  \hspace{1cm} (8)

It is now possible to formulate necessary conditions on the functional form of $\delta H$ in order to achieve the desired modifications in the inferred values of the quantities that could alleviate the tensions, e.g. decreasing $\sigma_8$. These conditions crucially depend on the shape of the response functions.

**Generalizations.** As shown in the companion paper \cite{39}, various generalizations to the above method are possible. First of all, everything goes through without imposing $\delta \omega_m = 0$ which is only a valid approximation for low redshift modifications of $\Lambda$CDM. In fact, the method is a priori not even tight to choosing $\Lambda$CDM as a starting point such that deformations of generic Hubble parameter functions can be considered. Moreover, as already mentioned, it is possible to allow for deviations of additional quantities $\delta Q_i(z)$ in form of generic functions. Typically, alternative theories to $\Lambda$CDM involve such additional deviations at the level of perturbations. Considering all this, (6) generalizes to

$$\frac{\Delta O}{O} = I_O \frac{\delta H_0}{H_0} + J_O \frac{\delta \omega_m}{\omega_m} + \int_0^\infty \frac{dz}{1 + z} R_O(x,z) \frac{\delta H(x)}{H(x)}$$

$$+ \sum_i \int_0^\infty \frac{dz}{1 + z} Q_i(z) \frac{\delta Q_i(x)}{Q_i(x)}.$$  \hspace{1cm} (9)

As generality increases, however, constraining $\delta H$ as well as $\delta Q_i$ will require additional conditions such as imposing multiple observational constraints.

**RESULTS**

We will now present our main results, which follow from applying the previous method to the specific context of the $H_0$ and $\sigma_8$ tensions and derive general requirements on the functional form of deviations from $\Lambda$CDM in order to achieve desired variations in the Hubble constant and the clustering amplitude. For computations and various details we refer the reader to the companion paper \cite{39}.

The first task is to obtain an analytic expression for $\sigma_8$, the amplitude of the density fluctuation power spectrum evaluated on spheres of radius $R = 8h^{-1}$ Mpc \cite{40}, in order to compute its variation in the form of (9). This is possible by adopting the Eisenstein-Hu fitting formula \cite{41} that takes into account the baryonic suppression at small scales which proves important for an accurate computation of the clustering amplitude. The resulting expression is proportional to the growing mode of the growth factor which for sub horizon modes and neglecting radiation as well as neutrino masses can be expressed analytically as well. See \cite{39} for the final expression.

**Modifying the expansion history**

A broad class of proposed solutions to the $H_0$ tension modify the $\Lambda$CDM background without introducing significant deviations in the perturbations, i.e. without in-
introducing new clustering species or modifying quantities like the gravitational coupling \( G_{\text{eff}} \approx G \).

In these models, the variation of \( \sigma_8 \) takes the form of (6). As discussed above, working with fixed \( \omega_m \) without loss of generality as confirmed in [39], we now merely need to impose a zero variation of a single observable in order to relate the deviation \( \delta H(z) \) from \( \Lambda \)CDM to \( \delta H_0 \) and subsequently to \( \Delta \sigma_8 \). In this work we will choose the variation of the CMB acoustic scale \( \theta_* \) [42] to vanish. The resulting response function \( R_{\sigma_8}(z) \) as defined in (7) is used to calculate the response function of \( \sigma_8 \) today as

\[
R_{\sigma_8}(z,0) = I_{\sigma_8}(0)R_{H_0}(z) + R_{\sigma_8}(z,0).
\]

The results are plotted in Fig. 1. The shape of the response functions immediately allows to draw simple conclusions with substantial impact. From (7) alone, it follows that in order solving the \( H_0 \) tension, hence a positive \( \delta H_0 \), necessarily requires that

\[
\delta H_0 > 0 \implies \delta H(z) < 0 \text{ at some } z . \tag{11}
\]

![FIG. 1. The response functions \( R_{H_0}(z) \) and \( R_{\sigma_8}(z,0) \) as defined in (7) and (13) respectively. Both responses remain strictly negative over the entire range 0 < z < 300 in which the expansion history is modified.](image)

On the other hand, relieving the \( \sigma_8 \) tension, which requires a negative variation \( \Delta \sigma_8 \), is only possible if

\[
\Delta \sigma_8 < 0 \implies \delta H(z) > 0 \text{ at some } z . \tag{12}
\]

In other words, increasing the value of the Hubble constant while simultaneously decreasing the clustering amplitude necessarily requires \( \delta H(z) \) to change sign.

This general result can readily be used to rule out specific models proposed in the literature. For example, for late dark energy with equation of state \( w(z) \) and under the assumption that DE perturbations only play a subleading role in driving the values of \( H_0 \) and \( \sigma_8 \), which is the case for many typical theories, we conclude that for such models increasing \( H_0 \) while decreasing \( \sigma_8 \) necessarily requires \( w(z) \) to cross the value \( w = -1 \), since the sign of \( \delta H(z) \) is directly connected to the sign of \( 1+w(z) \) [39].

**Beyond the expansion history: \( G_{\text{eff}} \)**

As remarked in the previous section, going beyond the background evolution induces additional small deviations from \( \Lambda \)CDM, parametrized by additional functions \( \delta Q_i(z) \). For example, dark energy clustering or modified gravity typically introduce modifications to the gravitational coupling \( G_{\text{eff}} = G + \delta G \). To first order, this will only affect the growth factor and therefore the variation of the clustering amplitude via

\[
\frac{\Delta \sigma_8}{\sigma_8} = \int_0^\infty \frac{dz}{1+z} R_{\sigma_8} \frac{\delta H(z)}{H(z)} + \int_0^\infty \frac{dz}{1+z} G_{\sigma_8} \frac{\delta G(z)}{G} . \tag{13}
\]

The result for \( G_{\sigma_8}(z,0) \) derived in [39] is again presented visually in Fig. 2. Given our minimal approach, that is only enforcing one observational anchor point given by the CMB acoustic scale, it is not possible to derive similarly strong conditions on the two free functions \( \delta H \) and \( \delta G \) independent of any quantitative analysis. However, if we restrict ourselves to cases where \( \delta H < 0 \) (i.e. \( w(z) \leq -1 \)), the following must hold

\[
\Delta \sigma_8 < 0 \implies \frac{\delta G(z)}{G} < \alpha(z) \frac{\delta H(z)}{H(z)} < 0 \text{ at some } z , \tag{14}
\]

where we have defined the strictly positive function

\[
\alpha(z) \equiv -\frac{R_{\sigma_8}(z,0)}{G_{\sigma_8}(z,0)} . \tag{15}
\]

![FIG. 2. The response function \( G_{\sigma_8}(z,0) \) and the function \( \alpha(z) \) as defined in (13) and (15) respectively. Both functions remain strictly positive over the entire range 0 < z < 300 in which the expansion history is modified.](image)
**DISCUSSION**

The methodology developed in this work has allowed us to identify a class of models that can not solve both the $H_0$ and $\sigma_8$ tensions. Focusing on late DE models, with equation of state $w(z)$ and with a modified gravitational coupling $G_{\text{eff}} = G + \delta G(z)$, we derived the necessary conditions that must be met to solve the $H_0$ and the $\sigma_8$ tensions, i.e. $\delta H_0 > 0$ and $\Delta \sigma_8 < 0$,

i) Solving the $H_0$ tension $\implies w(z) < -1$ at some $z$.

ii) If $G_{\text{eff}} = G$:

Solving the $H_0$ and $\sigma_8$ tensions $\implies w(z)$ must cross phantom limit $w = -1$ at some $z$.

iii) If $G_{\text{eff}} = G + \delta G$ and $w(z) \leq -1$ (i.e. $\delta H < 0$):

Solving the $H_0$ and $\sigma_8$ tensions $\implies \frac{\delta G}{G} < \alpha(z) \frac{\delta H}{H} < 0$ at some $z$, where $\alpha(z) > 0$.

We would like to stress that despite the generality of these conditions obtained under a minimal set of assumptions, their implications are already significant. In words, a typical late dark energy model trying to ease the Hubble tensions, their implications are already significant. In words, these conditions obtained under a minimal set of assumptions and going through the draft. LH is supported by possibly providing hints towards building successful models work, starting with the presented results, is able to give sound horizon generally fall short [28].

Early time solutions which solely reduce the cosmic turbations often worsens the situation. On the other hand, including perturbations in $\delta H(z)$ are not able to shift $H_0$ enough to fully resolve the tensions. On the other hand, including perturbations often worsens the situation.

It should be noted at this point that so-called early time solutions to the Hubble tension are by no means in a better position. For example, it was recently shown that early time solutions which solely reduce the cosmic sound horizon generally fall short [28].

In this context, the analytic method developed in this work, starting with the presented results, is able to give valuable insights into the behavior of the dark sector, possibly providing hints towards building successful models beyond ΛCDM.

**ACKNOWLEDGEMENTS**

We would like to thank Adam Riess for useful discussions and going through the draft. LH is supported by funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme grant agreement No 801781 and by the Swiss National Science Foundation grant 179740.

[1] N. Aghanim et al. (Planck), Astron. Astrophys. 641, A6 (2020), [Erratum: Astron.Astrophys. 652, C4 (2021)], arXiv:1807.06209 [astro-ph.CO].
[2] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri, and D. Scolnic, Astrophys. J. 876, 85 (2019), arXiv:1903.07603 [astro-ph.CO].
[3] A. G. Riess, S. Casertano, W. Yuan, J. B. Bowers, L. Macri, J. C. Zinn, and D. Scolnic, Astrophys. J. Lett. 908, L6 (2021), arXiv:2012.08534 [astro-ph.CO].
[4] D. W. Pesce et al., Astrophys. J. Lett. 891, L1 (2020), arXiv:2001.09213 [astro-ph.CO].
[5] K. C. Wong et al., Mon. Not. Roy. Astron. Soc. 498, 1420 (2020), arXiv:1907.04869 [astro-ph.CO].
[6] T. M. Abbott et al. (DES), Phys. Rev. D 98, 043526 (2018), arXiv:1708.01530 [astro-ph.CO].
[7] T. M. Abbott et al. (DES), (2021), arXiv:2105.13549 [astro-ph.CO].
[8] M. Asgari et al. (KiDS), Astron. Astrophys. 645, A104 (2021), arXiv:2007.15633 [astro-ph.CO].
[9] C. Heymans et al., Astron. Astrophys. 646, A140 (2020), arXiv:2007.15632 [astro-ph.CO].
[10] A. G. Riess, Nature Rev. Phys. 2, 10 (2019), arXiv:2001.03624 [astro-ph.CO].
[11] R. C. Nunes and S. Vagnozzi, Mon. Not. Roy. Astron. Soc. 505, 5427 (2021), arXiv:2106.01208 [astro-ph.CO].
[12] L. Knox and M. Millea, Phys. Rev. D 101, 043533 (2020), arXiv:1908.03663 [astro-ph.CO].
[13] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D. F. Mota, A. G. Riess, and J. Silk, Class. Quant. Grav. 38, 153001 (2021), arXiv:2103.01183 [astro-ph.CO].
[14] E. Di Valentino et al., Astropart. Phys. 131, 102605 (2021), arXiv:2008.11284 [astro-ph.CO].
[15] E. Di Valentino et al., Astropart. Phys. 131, 102604 (2021), arXiv:2008.11285 [astro-ph.CO].
[16] L. Perivolaropoulos and F. Skara, (2021), arXiv:2105.05208 [astro-ph.CO].
[17] V. Poulin, T. L. Smith, T. Karwal, and M. Kamionkowski, Phys. Rev. Lett. 122, 221301 (2019), arXiv:1811.04083 [astro-ph.CO].
[18] T. L. Smith, V. Poulin, and M. A. Amin, Phys. Rev. D 101, 063523 (2020), arXiv:1908.06995 [astro-ph.CO].
[19] J. Alcaniz, N. Bernal, A. Masiero, and F. S. Queiroz, Phys. Lett. B 812, 136008 (2021), arXiv:1912.03563 [astro-ph.CO].
[20] M. Zumalacarregui, Phys. Rev. D 102, 023523 (2020), arXiv:2003.06396 [astro-ph.CO].
[21] A. Gómez-Valent, V. Pettorino, and L. Amendola, Phys. Rev. D 101, 123513 (2020), arXiv:2004.00610 [astro-ph.CO].
[22] G. Ballesteros, A. Notari, and F. Rompineve, JCAP 11, 024 (2020), arXiv:2004.05049 [astro-ph.CO].
[23] J. B. Jiménez, D. Bettoni, and P. Brax, Phys. Rev. D 103, 103505 (2021), arXiv:2004.13677 [astro-ph.CO].
[24] E. Di Valentino, A. Mukherjee, and A. A. Sen, Entropy 23, 404 (2021), arXiv:2005.12587 [astro-ph.CO].
[25] G. Lambiase, S. Mohanty, A. Narang, and P. Parashari, Eur. Phys. J. C 79, 141 (2019), arXiv:1804.07154 [astro-ph.CO].
[26] R. E. Keeley, S. Joudaki, M. Kaplinghat, and D. Kirkby, JCAP 12, 035 (2019), arXiv:1905.10198 [astro-ph.CO].
[27] E. Di Valentino, A. Melchiorri, O. Mena, and S. Vagnozzi, Phys. Dark Univ. 30, 100666 (2020), arXiv:1908.04281 [astro-ph.CO].
[28] K. Jedamzik, L. Pogosian, and G.-B. Zhao, Commun. in Phys. 4, 123 (2021), arXiv:2010.04158 [astro-ph.CO].
[29] S. J. Clark, K. Vattis, J. Fan, and S. M. Koushiappas, (2021), arXiv:2110.09562 [astro-ph.CO].
[30] J. Solà Peracaula, A. Gómez-Valent, J. de Cruz Perez, and C. Moreno-Pulido, EPL 134, 19001 (2021), arXiv:2102.12758 [astro-ph.CO].
[31] G. Alestas and L. Perivolaropoulos, Mon. Not. Roy. Astron. Soc. 504, 3956 (2021), arXiv:2103.04045 [astro-ph.CO].
[32] N. Schöneberg, G. Franco Abellán, A. Pérez Sánchez, S. J. Witte, V. Poulin, and J. Lesgourgues, (2021), arXiv:2107.10291 [astro-ph.CO].
[33] G. Alestas, D. Camarena, E. Di Valentino, L. Kazantzidis, V. Marra, S. Nesseris, and L. Perivolaropoulos, (2021), arXiv:2110.04336 [astro-ph.CO].
[34] J. Renk, M. Zumalacárregui, F. Montanari, and A. Barreira, JCAP 10, 020 (2017), arXiv:1707.02263 [astro-ph.CO].
[35] N. Frusciante, S. Peirone, L. Atayde, and A. De Felice, Phys. Rev. D 101, 064001 (2020), arXiv:1912.07586 [astro-ph.CO].
[36] A. de Felice, L. Heisenberg, and S. Tsujikawa, Phys. Rev. D 95, 123540 (2017), arXiv:1703.09573 [astro-ph.CO].
[37] A. De Felice, C.-Q. Geng, M. C. Pookkillath, and L. Yin, JCAP 08, 038 (2020), arXiv:2002.06782 [astro-ph.CO].
[38] L. Heisenberg and H. Villarrubia-Rojo, JCAP 03, 032 (2021), arXiv:2010.00513 [astro-ph.CO].
[39] L. Heisenberg, H. Villarrubia-Rojo, and J. Zosso, to be submitted under preparation (2021).
[40] See e.g. [?] for a definition of $\sigma_R$.
[41] D. J. Eisenstein and W. Hu, Astrophys. J. 496, 605 (1998), arXiv:astro-ph/9709112.
[42] L. Chen, Q.-G. Huang, and K. Wang, JCAP 02, 028 (2019), arXiv:1808.05724 [astro-ph.CO].