Non-degeneracy of cohomological traces for general Landau-Ginzburg models

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Abstract: We prove non-degeneracy of the cohomological bulk and boundary traces for general open-closed Landau-Ginzburg models associated to a pair \((X,W)\), where \(X\) is a non-compact complex manifold with trivial canonical line bundle and \(W\) is a complex-valued holomorphic function defined on \(X\), assuming only that the critical locus of \(W\) is compact (but may not consist of isolated points). These results can be viewed as certain “deformed” versions of Serre duality. The first amounts to a duality property for the hypercohomology of the sheaf Koszul complex of \(W\), while the second is equivalent with the statement that a certain power of the shift functor is a Serre functor on the even subcategory of the \(\mathbb{Z}_2\)-graded category of topological D-branes of such models.

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Introduction

In references [1,2], we discussed a general framework for Landau-Ginzburg models inspired by the physics arguments of [3,4] and conjectured that certain objects associated to a Kählerian Landau-Ginzburg pair obey the defining axioms of an open-closed TFT datum (see [5]). In the present paper, we establish non-degeneracy of the cohomological bulk and boundary traces constructed in [1], thus proving part of that conjecture. Since the bulk and boundary traces can be defined without the Kählerianness requirement, our proof does not assume that condition.

By definition, a holomorphic Landau-Ginzburg pair (LG pair) is a doublet \((X,W)\) such that\(^1\):

\(^1\) A holomorphic LG pair is called Kählerian if \(X\) admits at least one Kähler metric.
1. $X$ is a non-compact complex manifold (which we assume to be connected and paracompact) with holomorphically trivial canonical line bundle.

2. $W : X \to \mathbb{C}$ is a non-constant holomorphic function.

Let $(X, W)$ be a holomorphic Landau-Ginzburg pair such that $\dim_\mathbb{C} X = d$. Let:

$$\mu \overset{\text{def}}{=} \hat{d} \in \mathbb{Z}_2$$

be the modulo 2 reduction of $d$, which is known as the signature of the LG pair $(X, W)$. Let:

$$Z_W \overset{\text{def}}{=} \{ x \in X \mid (\partial W)(x) = 0 \}$$

be the critical set of $W$. Let $O(X)$ be the commutative ring of complex-valued holomorphic functions defined on $X$.

The cohomological bulk algebra of $(X, W)$ (see [1, Section 3]) is the $\mathbb{Z}$-graded associative and supercommutative algebra $\text{HPV}(X, W)$ defined as the total cohomology algebra of the $O(X)$-linear dg-algebra $(\text{PV}(X), \delta_W)$, where the $O(X)$-module:

$$\text{PV}(X) \overset{\text{def}}{=} \bigoplus_{i=0}^{d} \bigoplus_{j=0}^{d} A^i(X, \wedge^j T_X)$$

is endowed with the multiplication given by the wedge product and with the $\mathbb{Z}$-grading which places $A^i(X, \wedge^j T_X) \overset{\text{def}}{=} \Omega^0,i(X, \wedge^j T_X)$ in degree $i-j$. The twisted differential $\delta_W$ is defined through:

$$\delta_W \overset{\text{def}}{=} \overline{\partial_{TX}} + \iota_W,$$  \hspace{1cm} (0.3)

where $\overline{\partial_{TX}}$ is the Dolbeault differential of the holomorphic vector bundle $\wedge^j T_X \overset{\text{def}}{=} \oplus_{j=0}^{d} \wedge^j T_X$ and $\iota_W \overset{\text{def}}{=} -i\partial W$, where $i$ denotes the imaginary unit.

Suppose that the critical set $W$ is compact. In this case, it was shown in [1] that $\text{HPV}(X, W)$ is finite-dimensional over $\mathbb{C}$ and that any holomorphic volume form $\Omega$ on $X$ induces a natural $\mathbb{C}$-linear map $\text{Tr}_\Omega : \text{HPV}(X, W) \to \mathbb{C}$ of degree 0 (known as the bulk cohomological trace, see [1, Section 5]) which is graded cyclic in the sense that it satisfies:

$$\text{Tr}_\Omega(u_1 u_2) = (-1)^{\deg u_1 \deg u_2} \text{Tr}_\Omega(u_2 u_1)$$

for any elements $u_1, u_2 \in \text{HPV}(X, W)$ which are homogeneous with respect to the grading on $\text{HPV}(X, W)$. In this paper, we prove:

**Theorem A.** Suppose that the critical set $Z_W$ is compact. Then the graded-symmetric pairing $\langle \cdot, \cdot \rangle_\Omega : \text{HPV}(X, W) \times \text{HPV}(X, W) \to \mathbb{C}$ given by:

$$\langle u_1, u_2 \rangle_\Omega \overset{\text{def}}{=} \text{Tr}_\Omega(u_1 u_2), \quad \forall u_1, u_2 \in \text{HPV}(X, W)$$

is non-degenerate. Hence $(\text{HPV}(X, W), \text{Tr}_\Omega)$ is a finite-dimensional $\mathbb{Z}$-graded supercommutative Frobenius algebra over $\mathbb{C}$.

Let $K_W$ denote the sheaf Koszul complex defined by $\iota_W$:

$$(K_W) : \quad 0 \longrightarrow \wedge^d T_X \overset{\iota_W}{\longrightarrow} \wedge^{d-1} T_X \overset{\iota_W}{\longrightarrow} \cdots \overset{\iota_W}{\longrightarrow} T_X \overset{\iota_W}{\longrightarrow} O_X \longrightarrow 0.$$  \hspace{1cm} (0.4)
When $Z_W$ is compact, we have isomorphisms $\HPV^k(X,W) \simeq \mathbb{H}^k(K_W)$, where $\mathbb{H}(K_W)$ is the hypercohomology of the bounded complex of analytic sheaves $K_W$. Thus Theorem A. has the equivalent formulation:

**Theorem A'.** Suppose that the critical set $Z_W$ is compact. Then for every $k \in \{-d, \ldots, d\}$ we have isomorphisms of vector spaces:

$$\mathbb{H}^k(K_W) \simeq \mathbb{H}^{-k}(K_W)^\vee,$$

which depend on the choice of a holomorphic volume form $\Omega$.

Another fundamental datum defined by the holomorphic LG pair $(X, W)$ is the *cohomological twisted Dolbeault category of holomorphic factorizations* $\HDF(X,W)$ (see [1, Section 4]). This is a $\mathbb{Z}_2$-graded $\mathcal{O}(X)$-linear category defined as the total cohomology category of a $\mathbb{Z}_2$-graded $\mathcal{O}(X)$-linear dg-category $\DF(X,W)$ whose objects are the holomorphic factorizations of $W$. By definition, a *holomorphic factorization* of $W$ is a pair $(E,D)$, where $E = E^0 \oplus E^1$ is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on $X$ and $D$ is a holomorphic section of the bundle $\End^\hat{1}(E) \defeq \Hom(E^0,E^1) \oplus \Hom(E^1,E^0)$ which satisfies the condition $D^2 = \text{Wid}_D$. Given two holomorphic factorizations $a_1 = (E_1,D_1)$ and $a_2 = (E_2,D_2)$ of $W$, the space of morphisms from $a_1$ to $a_2$ in the category $\DF(X,W)$ is the $\mathcal{O}(X)$-module:

$$\Hom_{\DF(X,W)}(a_1,a_2) \defeq \mathcal{A}(X,\Hom(E_1,E_2)) = \bigoplus_{i=0}^{d} \mathcal{A}^i(X,\Hom(E_1,E_2)),$$

endowed with the $\mathbb{Z}_2$-grading which places the submodule $\oplus_{i+\kappa=\tau} \mathcal{A}^i(X,\Hom^\kappa(E_1,E_2))$ (where $i \in \{1, \ldots, d\}$ and $\kappa \in \mathbb{Z}_2$) in degree $\tau \in \mathbb{Z}_2$. This module is endowed with the differential:

$$\delta_{a_1,a_2} \defeq \overline{\partial}_{\Hom(E_1,E_2)} + \partial_{a_1,a_2},$$

where $\overline{\partial}_{\Hom(E_1,E_2)}$ is the Dolbeault differential of the holomorphic vector bundle $\Hom(E_1,E_2)$ and $\partial_{a_1,a_2}$ is the *defect differential*, which is uniquely determined by the condition:

$$\partial_{a_1,a_2}(\omega \otimes f) = (\Delta^f)\omega \otimes (D_2 \circ f) - (\Delta^f)\omega \otimes (f \circ D_1),$$

for all $\omega \in \mathcal{A}^i(X)$ and $f \in \Gamma(X,\Hom^\kappa(E_1,E_2))$. The composition of morphisms in $\DF(X,W)$ is induced in the obvious manner by the wedge product and by the fiberwise composition of linear maps.

Suppose that the critical locus of $W$ is compact. In this case, it was shown in [1] that $\DF(X,W)$ is $\Hom$-finite as a $\mathbb{C}$-linear category and that any holomorphic volume form $\Omega$ on $X$ naturally induces a $\mathbb{C}$-linear map $\tr^\Omega : \Hom_{\DF(X,W)}(a,a) \rightarrow \mathbb{C}$ of $\mathbb{Z}_2$-degree $d$ (called the *cohomological boundary trace* [1, Section 6]) for any holomorphic factorization $a$ of $W$, such that the following graded cyclicity condition is satisfied for any two holomorphic factorizations $a_1, a_2$ of $W$:

$$\tr^\Omega_{a_1}(t_2 t_1) = (-1)^{\kappa_1 \kappa_2} \tr^\Omega_{a_2}(t_1 t_2), \quad \forall t_1 \in \Hom_{\DF(X,W)}^{\kappa_1}(a_1,a_2), \quad \forall t_2 \in \Hom_{\DF(X,W)}^{\kappa_2}(a_2,a_1),$$

where $\kappa_1, \kappa_2 \in \mathbb{Z}_2$. Let $\tr^\Omega$ denote the family $(\tr^\Omega_a)_{a \in \Ob\DF(X,W)}$. The second main result of this paper is:
Theorem B. Suppose that the critical set $Z_W$ is compact. Then the bilinear pairing
\[ \langle \cdot, \cdot \rangle_{a_1,a_2}^\Omega : \text{Hom}_{\text{HDF}(X,W)}(a_1,a_2) \times \text{Hom}_{\text{HDF}(X,W)}(a_2,a_1) \to \mathbb{C} \]
defined through:
\[ \langle t_1,t_2 \rangle_{a_1,a_2}^\Omega \overset{\text{def}}{=} \text{tr}^\Omega_{a_2}(t_1 t_2) , \quad \forall t_1 \in \text{Hom}_{\text{HDF}(X,W)}(a_1,a_2) , \quad \forall t_2 \in \text{Hom}_{\text{HDF}(X,W)}(a_2,a_1) \]
is non-degenerate for any two holomorphic factorizations $a_1,a_2$ of $W$. Hence $(\text{HDF}(X,W), \text{tr}^\Omega)$ is a Calabi-Yau supercategory of signature $\mu = d$ in the sense of [1, Section 2].

The $\mathbb{Z}_2$-graded category $\text{HDF}(X,W)$ admits an automorphism $\Sigma$ which squares to the identity and comes with natural isomorphisms:
\[ \text{Hom}_{\text{HDF}(X,W)}(a_1,\Sigma(a_2)) \simeq \text{Hom}_{\text{HDF}(X,W)}(\Sigma(a_1),a_2) \simeq \Pi \text{Hom}_{\text{HDF}(X,W)}(a_1,a_2) , \]
where $\Pi$ is the parity change functor of the category of $\mathbb{Z}_2$-graded vector spaces. As a consequence, $\text{HDF}(X,W)$ can be reconstructed from its even subcategory $\text{HDF}^0(X,W)$ (which is obtained from $\text{HDF}(X,W)$ by keeping only morphisms of $\mathbb{Z}_2$-degree equal to 0). Then Theorem B. can be reformulated as follows:

Theorem B'. Suppose that the critical set $Z_W$ is compact. Then $\Sigma^d$ is a Serre functor for the category $\text{HDF}^0(X,W)$, where:
\[ \Sigma^d = \begin{cases} \Sigma & \text{if } d \text{ is odd} \\ \text{id}_{\text{HDF}^0(X,W)} & \text{if } d \text{ is even} \end{cases} . \]

Notice that $\Sigma^d$ depends only on the signature $\mu = d$.

The differentials $\delta_W = \overline{\partial} \wedge TX + \iota_W$ and $\delta_{a_1,a_2} = \overline{\partial}_{\text{Hom}(E_1,E_2)} + \partial_{a_1,a_2}$ can be viewed as deformations of the Dolbeault operators $\overline{\partial} \wedge TX$ and $\overline{\partial}_{\text{Hom}(E_1,E_2)}$ respectively. When $W = 0$, the differential $\delta_W$ reduces to $\overline{\partial} \wedge TX$ and one can check that Theorem A. reduces to Serre duality (on the non-compact complex manifold $X$) for the holomorphic vector bundle $\wedge TX$. In this case, a particular class of holomorphic factorizations is given by pairs of the form $(E,D) = (E,0)$, for which Theorem B. again reduces to Serre duality. Accordingly, both theorems can be viewed as “deformed” versions of ordinary Serre duality [6,7,8] on (non-compact) complex manifolds. We will prove them by reduction to the latter by using certain spectral sequences which relate the cohomology of the differentials $\delta_W$ (respectively $\delta_{a_1,a_2}$) to the Dolbeault cohomology of the holomorphic vector bundles $\wedge TX$ (respectively $\text{Hom}(E_1,E_2)$).

The paper is organized as follows. Section 1 recalls some well-known facts regarding duality for complexes of Fréchet-Schwartz (FS) and dual of Fréchet-Schwartz (DFS) spaces. Section 2 proves some results regarding spectral sequences which will be used later on. In Section 3, we discuss an extension of the notion of Serre pairing to the case of graded holomorphic vector bundles. Section 4 gives our proof of Theorems A and A', while Section 5 proves Theorems B and B'. Appendix A collects some properties of linear categories and supercategories with involutive shift functor.
**Notations and conventions.** We use the same notations and conventions as in reference [1]. In particular, the symbol \( \mathbb{Z}_2 \) stands for the field \( \mathbb{Z}/2\mathbb{Z} \), whose elements we denote by \( 0 \) and \( 1 \). Given an integer \( n \in \mathbb{Z} \), we denote its reduction modulo 2 by \( \overline{n} \in \mathbb{Z}_2 \). The symbol \( i \) denotes the imaginary unit, while \( j \) denotes the contraction between differential forms and polyvector fields.

Throughout the paper, \( X \) is a connected and non-compact complex manifold with holomorphically trivial canonical line bundle and \( W \) is a non-constant holomorphic complex-valued function defined on \( X \). Since \( X \) is connected, the following conditions are equivalent:

(a) \( X \) is paracompact
(b) \( X \) is second countable
(c) \( X \) is \( \sigma \)-compact
(d) \( X \) is countable at infinity.

We assume throughout the paper that one (and hence all) of these equivalent conditions is satisfied by \( X \). The \( \mathbb{C} \)-algebra of smooth complex-valued functions defined on \( X \) is denoted by \( \mathcal{C}^\infty(X) \), while the \( \mathbb{C} \)-algebra of holomorphic complex valued functions defined on \( X \) is denoted by \( \mathcal{O}_X \). Given a holomorphic vector bundle \( V \) defined on \( X \), its \( \mathcal{O}(X) \)-module of globally-defined holomorphic sections is denoted by \( \Gamma(X,V) \) while its \( \mathcal{C}^\infty(X) \)-module of globally-defined smooth sections is denoted by \( \Gamma_{\text{sm}}(X,V) \). We sometimes tacitly identify a holomorphic vector bundle with its sheaf of local holomorphic sections.

1. **Duality for complexes of topological vector spaces**

In this section, we summarize some properties of complexes of Fréchet-Schwartz (FS) and duals of Fréchet-Schwartz (DFS) spaces, following [9,10,11,12].

Throughout this paper, a *topological vector space* (tvs) means a topological vector space over the normed field \( \mathbb{C} \) of complex numbers. Given a tvs \( F \), let \( F^* \) denote the topological dual of \( F \), endowed with the strong topology. Given a continuous linear map \( f : F_1 \to F_2 \) between two topological vector spaces, we denote its transpose by \( f^* : F_2^* \to F_1^* \); the transpose of \( f \) is continuous with respect to the strong topologies on the dual spaces. A continuous linear map \( f : F_1 \to F_2 \) is called a *topological homomorphism* if the corestriction \( f_0 : F_1 \to f(F_1) \) of \( f \) to its image is an open map when the image subspace \( f(F_1) \) is endowed with the induced topology. The linear map \( f \) is called a *topological isomorphism* if it is a homeomorphism. Given two topological vector spaces \( F_1 \) and \( F_2 \), we write \( F_1 \simeq F_2 \) if there exists at least one topological isomorphism from \( F_1 \) to \( F_2 \). This defines an equivalence relation on the collection of all topological vector spaces. Given two topological vector spaces \( F_1 \) and \( F_2 \), we endow the vector space \( F_1 \times F_2 = F_1 \oplus F_2 \) with the product topology, which makes it into a tvs.

Suppose that \( F_1 \) and \( F_2 \) are Fréchet spaces and \( f : F_1 \to F_2 \) is a continuous map. Then \( f \) is a topological homomorphism iff \( f \) has closed range. Moreover, the open mapping theorem states that any continuous and surjective linear map \( f : F_1 \to F_2 \) is open. In particular, \( f \) is a topological isomorphism iff it is continuous and bijective.

A Fréchet-Schwartz (FS) space is a Fréchet space which is also a Schwartz space (see [9,10]). Every such space is reflexive, i.e. naturally topologically isomorphic with the strong topological dual of its strong topological dual. A tvs is called a DFS space if it is the strong topological dual of an FS space; DFS spaces are also reflexive.

**Definition 1.1** Let \( F_1 \) and \( F_2 \) be topological vector spaces. A topological pairing between \( F_1 \) and \( F_2 \) is a bilinear map \( \langle \cdot, \cdot \rangle : F_1 \times F_2 \to \mathbb{C} \) which is (jointly) continuous.
**Definition 1.2** Let \( \langle \cdot, \cdot \rangle : F_1 \times F_2 \to \mathbb{C} \) be a topological pairing. The left Riesz morphism of \( \langle \cdot, \cdot \rangle \) is the linear map \( \tau_l : F_1 \to F_2^\ast \) defined through:

\[
\tau_l(u)(v) \overset{\text{def}}{=} \langle u, v \rangle, \quad \forall u \in F_1, \forall v \in F_2.
\]

The right Riesz morphism of \( \langle \cdot, \cdot \rangle \) is the linear map \( \tau_r : F_2 \to F_1^\ast \) defined through:

\[
\tau_r(v)(u) \overset{\text{def}}{=} \langle u, v \rangle, \quad \forall u \in F_1, \forall v \in F_2.
\]

We say that \( \langle \cdot, \cdot \rangle \) is a perfect topological pairing if both \( \tau_l \) and \( \tau_r \) are topological isomorphisms.

**Remark 1.1.** If \( \langle \cdot, \cdot \rangle : F_1 \times F_2 \to \mathbb{C} \) is a perfect topological pairing, then the topological vector spaces \( F_1 \) and \( F_2 \) are reflexive:

\[
(F_1^\ast)^\ast \simeq F_2^\ast \simeq F_1,
\]

\[
(F_2^\ast)^\ast \simeq F_1^\ast \simeq F_2.
\]

If we identify \( F_2 \) with \( F_1^\ast \) using the right Riesz isomorphism and \( F_2^\ast \) with \( F_1 \) using the left Riesz isomorphism, then the bidual of \( F_1 \) identifies with \( F_1 \) and the perfect pairing identifies with the duality pairing between \( F_1 \) and \( F_1^\ast \).

**Definition 1.3** A topological cochain complex is a sequence:

\[
(F^\bullet): \quad \ldots \to F^{k-1} \xrightarrow{\delta_{k-1}} F^k \xrightarrow{\delta_k} F^{k+1} \xrightarrow{\delta_{k+1}} \ldots
\]

where \( F^k \) are topological vector spaces and \( \delta_k \) are continuous linear maps which satisfy \( \delta_{k+1} \circ \delta_k = 0 \) for all \( k \in \mathbb{Z} \). The cohomology of such a complex in degree \( k \in \mathbb{Z} \) is the vector space:

\[
H^k(F^\bullet, \delta) \overset{\text{def}}{=} \ker \delta_k / \text{im} \delta_{k-1},
\]

endowed with the quotient topology. We say that the topological complex \( F^\bullet \) is bounded if there exist integers \( k_1 < k_2 \) such that \( F^k = 0 \) unless \( k_1 \leq k \leq k_2 \).

Given a bounded topological cochain complex \( F^\bullet \), we set \( F \overset{\text{def.}}{=} \oplus_{k \in \mathbb{Z}} F^k = F^{k_1} \times \ldots \times F^{k_2} \) (endowed with the direct product topology) and \( \delta = \sum_{k \in \mathbb{Z}} \delta_k = \sum_{k=k_1}^{k_2} \delta_k \). Then \( F \) is a finitely \( \mathbb{Z} \)-graded topological vector space and \( \delta \) is a continuous degree +1 endomorphism of \( F \) which satisfies \( \delta^2 = 0 \). The total cohomology:

\[
H(F^\bullet, \delta) = \ker \delta / \text{im} \delta = H(F, \delta) = \bigoplus_{k \in \mathbb{Z}} H^k(F, \delta) = H^{k_1}(F, \delta) \times \ldots \times H^{k_2}(F, \delta)
\]

is a finitely \( \mathbb{Z} \)-graded topological vector space. When \( u \in F^k \), we set \( \deg u \overset{\text{def.}}{=} k \).

**Definition 1.4** Let \( (F, \delta) \) be a bounded topological cochain complex. The topological dual of \( (F, \delta) \) is the topological cochain complex \( (F^\ast, \delta^\ast) \) defined through:

\[
(F^\ast)^k \overset{\text{def.}}{=} (F^{-k})^\ast, \quad (\delta^\ast)_k \overset{\text{def.}}{=} \delta_{-k-1}^t
\]

where \( \delta^t_j \) denotes the transpose of \( \delta_j \).
Definition 1.5 Let \((F, \delta)\) and \((\hat{F}, \hat{\delta})\) be two bounded topological cochain complexes. A \(\mathbb{C}\)-bilinear map \(\langle \cdot, \cdot \rangle : F \times \hat{F} \to \mathbb{C}\) is called a topological pairing of complexes if it satisfies the following conditions:

1. \(\langle \cdot, \cdot \rangle\) is jointly continuous.
2. \(\langle \cdot, \cdot \rangle\) has degree zero:
   \[\langle \cdot, \cdot \rangle|_{F^i \times \hat{F}^j} = 0\] if \(i + j \neq 0\).
3. \(\langle \delta u, v \rangle + (-1)^{\deg u} \langle u, \hat{\delta} v \rangle = 0\) for all homogeneous elements \(u \in F\) and \(v \in \hat{F}\).

In this case, we say that \(\langle \cdot, \cdot \rangle\) is a perfect topological pairing of complexes if its restriction to \(F^k \times \hat{F}^{-k}\) is perfect for all \(k \in \mathbb{Z}\).

A topological pairing of bounded topological cochain complexes induces a degree zero topological pairing \(\langle \cdot, \cdot \rangle^H : H(F, \delta) \times H(\hat{F}, \hat{\delta}) \to \mathbb{C}\) between total cohomologies.

Definition 1.6 A topological pairing of bounded topological cochain complexes is called cohomologically perfect if the restriction \(\langle \cdot, \cdot \rangle^H\) is a perfect topological pairing for all \(k \in \mathbb{Z}\).

When \(\langle \cdot, \cdot \rangle\) is cohomologically perfect, the vector spaces \(H^k(F, \delta)\) and \(H^k(\hat{F}, \hat{\delta})\) are reflexive for all \(k \in \mathbb{Z}\) and \(\langle \cdot, \cdot \rangle^H\) induces topological isomorphisms \(H^k(F, \delta) \cong H^{-k}(\hat{F}, \hat{\delta})^*\).

Proposition 1.7 Let \((F, \delta)\) be a bounded topological cochain complex of FS spaces such that \(H^k(F, \delta)\) is finite-dimensional for all \(k \in \mathbb{Z}\). Then:

1. \(\delta_k\) is a topological homomorphism for all \(k \in \mathbb{Z}\).
2. The dual complex \((F^*, \delta^t)\) is a finite topological cochain complex of DFS spaces whose differentials are topological homomorphisms and whose cohomology is finite-dimensional in every degree.
3. The natural linear map:
   \[H^{-k}(F^*, \delta^t) \to H^k(F, \delta)^*\]
   is bijective for all \(k \in \mathbb{Z}\).

Proof. Follows from [12, Theorem 1.5] and [12, Theorem 1.6] upon noticing that \(H^k(F, \delta)\) is separated and that its topological dual coincides with the algebraic dual since \(H^k(F, \delta)\) is finite-dimensional. \(\square\)

Corollary 1.8 Let \((F, \delta)\) and \((\hat{F}, \hat{\delta})\) be two bounded topological complexes and let \(\langle \cdot, \cdot \rangle : F \times \hat{F} \to \mathbb{C}\) be a perfect topological pairing between these complexes. Suppose that \(H^k(F, \delta)\) is finite-dimensional for all \(k \in \mathbb{Z}\). Then \(H^k(\hat{F}, \hat{\delta})\) is finite-dimensional for all \(k \in \mathbb{Z}\) and \(\langle \cdot, \cdot \rangle\) is cohomologically perfect.

Proof. Follows immediately from Proposition 1.7. \(\square\)
2. Some results on spectral sequences

In this section, we use the notations and conventions of [13]. Let \( K = \oplus_{p,q \in \mathbb{Z}} K^{p,q} \) be a double complex of \( \mathbb{C} \)-vector spaces with vertical differential \( d_1 : K^{p,q} \rightarrow K^{p,q+1} \) and horizontal differential \( d_2 : K^{p,q} \rightarrow K^{p+1,q} \). We are interested in the following special cases, which will arise in later sections:

A. \( K \) is concentrated in the first quadrant, i.e. \( K^{p,q} \) vanishes unless \( p > 0 \) and \( q > 0 \).

B. \( K \) is concentrated in a horizontal strip above the horizontal axis, i.e. there exists \( N > 0 \) such that \( K^{p,q} \) vanishes unless \( 0 \leq q \leq N \).

Let \( K = \oplus_{n \in \mathbb{Z}} K^n \) be the decomposition corresponding to the total grading of \( K \), where:

\[
K^n \overset{\text{def}}{=} \bigoplus_{p+q=n} K^{p,q}.
\]

Let \( \delta \overset{\text{def}}{=} d_1 + d \) be the total differential, where \( d \overset{\text{def}}{=} (-1)^p d_2 \). The double complex can be endowed with the standard decreasing filtration \( F \) given by:

\[
F^p K \overset{\text{def}}{=} \bigoplus_{i \geq p} \bigoplus_{q \in \mathbb{Z}} K^{i,q}.
\]

This filtration can in general be unbounded (as in case B. above). However, for any \( n \in \mathbb{Z} \), the spaces:

\[
\text{gr}_F^p (K^n) \overset{\text{def}}{=} \frac{[K^n \cap F^p K]}{[K^n \cap F^{p-1} K]} = \left[ \bigoplus_{i \geq p, q = n-i} K^{i,q} \right] \big/ \left[ \bigoplus_{i \geq p-1, q = n-i} K^{i,q} \right] \simeq_{\mathbb{C}} K^{p,n-p}
\]

vanish in both cases A. and B. except for a finite number of values of \( p \). Here and below, the symbol \( \simeq_{\mathbb{C}} \) denotes isomorphism of \( \mathbb{C} \)-vector spaces. Applying the theory of exact couples to the bigraded complex \( K \) endowed with the filtration \( F \), we obtain a spectral sequence which computes the cohomology \( H^n(K) \) of the total complex \((K, \delta)\):

**Proposition 2.1** Assume that the double complex \((K,d_1,d_2)\) satisfies either of the conditions A. or B. above. Then the filtration (2.1) defines a spectral sequence \( E = (E_r,d_r)_{r \geq 0} \) which converges to the total cohomology \( H_n(K) \overset{\text{def}}{=} \oplus_{n \in \mathbb{Z}} H^{p,n}_K(K) \). For each \( r \geq 0 \), the page \( E_r \) is endowed with a bigrading given by the decomposition \( E_r = \oplus_{p,q \in \mathbb{Z}} E^{p,q}_r \) and with a differential\(^2\) \( d_r : E^{p,q}_r \rightarrow E^{p+1,r,q+1}_r \) defined recurrently by:

\[
E^{p,q}_r \overset{\text{def}}{=} \text{H}(E^{p,q}_{r-1},d_{r-1}).
\]

For the first pages we have \( d_0 = d_1 \) and \( d_1 = d := (-1)^p d_2 \), hence:

\[
E^{p,q}_1 = \text{H}_{d_1}^\delta (\text{gr}_F^p K), \quad E^{p,q}_2 = \text{H}_d^p (E^{p,q}_1).
\]

For each \( n \in \mathbb{Z} \), the filtration (2.1) induces a decreasing filtration \((F^p H^n(K))_{p \in \mathbb{Z}}\) of the vector space \( H^n(K) \), whose associated graded \( \text{gr}^p F^n H^n(K) \overset{\text{def}}{=} \frac{F^p H^n(K)}{F^{p-1} H^n(K)} \) satisfy:

\[
\text{gr}^p F^n H^n(K) \simeq_{\mathbb{C}} E^{p,n-p}_{\infty}, \quad \forall p \in \mathbb{Z},
\]

where \( E_{\infty} = \oplus_{p,q \in \mathbb{Z}} E^{p,q}_{\infty} \) is the limit of \( E \).

\(^2\) We explain later (see (2.2) and (2.3)) the relation between \( d_r \), \( d_1 \) and \( d_2 \).
Proof. The proof can be found in [13, §14] (see Theorem 14.6 and Theorem 14.14). The proof is not restricted to the case when \( F^p K \) is a finite filtration, but requires only that the induced filtration \( F^p K^n \) is finite for each \( n \), which is true when condition A. or condition B. is satisfied. \( \square \)

Consider another double complex \(( \hat{K}, \hat{d}_1, \hat{d}_2 )\) with the total differential \( \hat{\delta} = \hat{d}_1 + \hat{d} \), where \( \hat{d} = (-1)^p \hat{d}_2 \). Let \( \tau : K \to \hat{K} \) be a morphism of double complexes and \( \tau_* : H_\delta(K) \to H_\delta(\hat{K}) \) denote the morphism of graded \( \mathbb{C} \)-vector spaces induced by \( \tau \) on total cohomology. Let \( F^p \hat{K} \) denote the analogue of the filtration \((2.1)\) for \( \hat{K} \) and \((\hat{F}^p H^p_\delta(\hat{K}))_{p \in \mathbb{Z}}\) denote the filtration induced by \( F^p \) on the homogeneous components of total cohomology.

**Theorem 2.2** Suppose that \( \tau \) is injective and that it induces isomorphisms of vector spaces \( H^p_\delta(K) \cong H^p_\delta(\hat{K}) \) for all \( p,q \in \mathbb{Z} \) in vertical cohomology. Assume that both double complexes \((K,d_1,d_2)\) and \((\hat{K},\hat{d}_1,\hat{d}_2)\) satisfy condition A. or that both satisfy condition B. above. Then \( \tau_* \) satisfies:

\[
\tau_* (\hat{F}^p H^p_\delta(K)) \subset \hat{F}^p H^p_\delta(\hat{K}) , \quad \forall p,n \in \mathbb{Z} .
\]

and restricts to isomorphisms of vector spaces between the associated gradeds:

\[
\tau_* : \text{gr}_F H^p_\delta(K) \cong \text{gr}_F H^p_\delta(\hat{K}) , \quad \forall p,n \in \mathbb{Z} .
\]

**Proof.** We apply Proposition 2.1 for both double complexes \((K,d_1,d_2)\) and \((\hat{K},\hat{d}_1,\hat{d}_2)\) and denote by \( E \) and \( \hat{E} \) the spectral sequences defined by the standard filtration \( F^p \). By assumption, the first pages of these two spectral sequences coincide. The statement of the theorem follows if we show that the spectral sequences coincide on each page. To show this, we have to look closer at the components \( E^p_{r,q} \), whose elements arise as cohomology classes of previous pages.

An element \( b \in K \) represents a cohomology class in \( \text{ker} d_1 \) if and only if \( b \) is a cocycle in all \( E^p_{r,q} \) for \( p,q \in \mathbb{Z} \). Let us denote by \([b]_r\) the image of \( b \) in \( E^p_r \) if defined. An explicit description of this can be found in [13, page 164], which states that \( b \in K^{p,q} \) represents an element of \( E^p_{r,q} \) if there exists a chain of elements \( c_i \in K^{p+i,q-i} \) with \( 1 \leq i \leq r - 1 \) such that:

\[
d_1 b = 0 , \quad d b = -d_1 c_1 \quad \text{and} \quad d c_{i-1} = -d_1 c_i
\]

for \( i = 2, \ldots, r - 1 \). Moreover, the differential \( d_r \) of \( E^p_r \) is given by:

\[
d_r [b]_r \overset{\text{def}}{=} [d c_{r-1}]_r .
\]

The differential \( d_r [b]_r \) is defined for \( b \) belonging to \( \ker d_1 \) and depends only on the class \([b]_1 \in H^p_\delta(K) \). Indeed, for any \( b' \in K^{p,q-1} \), the element \( \tilde{b} := b + d_1 b' \) satisfies \( d_1 \tilde{b} = 0 \), \( d \tilde{b} = d b + d(d_1 b') = -d_1 c_1 - d_1 b' = -d_1 (c_1 + b') \). We also have \( d(c_1 + b') = d c_1 \). This means that \( \tilde{b} \) satisfies the same system of equations \((2.2)\) after modifying \( c_1 := c_1 + b' \). Thus, for any \( b \) such that \([b]_1 = [b]_1 \in H^p_\delta(K) \), we have \( d_r [b]_r = [d c_{r-1}]_r = d_r [b]_r \). The definition of \( d_r \) does not depend on the choice of the elements \( c_1, \ldots, c_{r-1} \) representing classes \([c_i]_i \in E_i \) and satisfying \((2.2)\). Everything said above also holds for the differentials \( d_r \) of the spectral sequence defined by the filtration \( F \) of the complex \( \hat{K}^{p,q} \). We will also use the notation \( \tau_* \) for the map induced by \( \tau \) on any of \( E^p_{r,q} \).

In what follows we prove by induction on \( s \) that the following statements hold for any \( p,q \in \mathbb{Z} \):

1. The inclusion \( \tau \) induces an isomorphism \( \tau_* : E^p_{r,q}(K) \cong \hat{E}^p_{r,q}(\hat{K}) \),

2. \([\tau b]_s = \tau_* [b]_s\) for any \( b \in K^{p,q} \) such that \([b]_s \) is defined,
For \( s = 1 \), statement (i) is just the assumption of the theorem while the other two statements follow from the fact that \( \tau \) is an injective map of double complexes.

Let us assume that (i) – (iii) hold for all \( s \leq r - 1 \). To show (ii) for \( s = r \), consider an element \( b \in K^{p,q} \) such that \([b]_r\) is defined. Then we have \( \hat{d}_{r-1}[b]_{r-1} = 0 \) as well as \( \hat{d}_{r-1}[\tau b]_{r-1} = 0 \). We also have \( [\tau b]_{r-1} = \tau_s [b]_{r-1} \). Using (i) and (iii) for \( s = r - 1 \) gives:

\[
[\tau b]_{r-1} + \hat{d}_{r-1}\hat{E}_{r-1} = \tau_s [b]_{r-1} + \hat{d}_{r-1}\hat{E}_{r-1} = \tau_s ([b]_{r-1} + d_{r-1}E_{r-1}) ,
\]

which shows that \([\tau b]_r = \tau_s [b]_r\) as cohomology classes in \( E_r \). To show that (iii) holds for \( s = r \), consider an element \( b \in \hat{K}^{p,q} \) which represents \([b]_r\) in \( \hat{E}_r \). Since \( \tau_s : H^p_{dq}(K) \to H^p_{dq}(\hat{K}) \) is an isomorphism, there exist elements \( \hat{b} \) satisfying (2.2) for all \( 1 \leq i \leq r - 1 \). Let \( \tau b \) is defined.

To show that (i) holds for \( s = r \), we start from statement (iii) for \( s = r - 1 \), which tells us that the differentials are compatible with the isomorphism \( \tau_s \), namely we have \( \hat{d}_{r-1}(\tau_s [b]) = \tau_s (\hat{d}_{r-1}[b]) \). This implies that the cohomology taken with respect to these differentials is also compatible with \( \tau_s \). Thus:

\[
\hat{E}^{p,q}_r = H(\hat{E}^{p,q}_r, \hat{d}_{r-1}) = H(\tau_s(E^{p,q}_r), \hat{d}_{r-1}) = \tau_s (H(E^{p,q}_r, \hat{d}_{r-1})) = \tau_s (E^{p,q}_r) .
\]

Since the bigradings of both graded complexes \( K \) and \( \hat{K} \) satisfy either condition A. or condition B., it follows that for any \( n \) there exists \( N = N(n) \) such that \( d_r(K^{p,q}) = 0 = d_r(\hat{K}^{p,q}) \) for all \( r > N, p+q = n \) and for both sets of pairs \((p',q') = (p+r, q-r+1)\) and \((p',q') = (p-r, q+r-1)\). Thus for \( r > N \) and \( p+q = n \), we have \( \tau_s (E^{p,q}_r) = \tau_s (E^{p,q}_r) = E^{p,q}_r = E^{p,q}_r \). By Proposition 2.1, this is equivalent with the statement that the map \( \tau_s : gr^p F H^q(K) \to gr^p F H^q(\hat{K}) \) is an isomorphism for any \( p \) and \( n \).

\[\square\]

3. Graded Serre pairings

In this section, we discuss a version of the Serre pairing (see [6]) which exists for graded holomorphic vector bundles. This pairing will arise later on in the proof of non-degeneracy of cohomological bulk and boundary traces.

3.1. Topological complexes of differential forms valued in a holomorphic vector bundle. For any holomorphic vector bundle \( V \) on \( X \), let \( \Omega^{p,q}(X,V) = \Gamma_{sm}(X, \Lambda^p T^*X \otimes \Lambda^q T^*X \otimes V) \) denote the space of \( V \)-valued smooth forms of type \((p,q)\) defined on \( X \) and \( \Omega^{p,q}_{\text{sm}}(X,V) \) denote the subspace of \( \Omega^{p,q}(X,V) \) consisting of compactly-supported forms. Then \( \Omega^{p,q}(X,V) \) is an FS space [6,11] when endowed with the topology of uniform convergence of all derivatives on compact subsets.
The subspace $\Omega^{p,q}(X, V)$ is dense in $\Omega^{p,q}(X, V)$, being the space of “test sections” of the bundle $\bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes V$. Let:

$$\Omega(X, V) \overset{\text{def}}{=} \bigoplus_{p,q=0}^d \Omega^{p,q}(X, V), \quad \Omega_c(X, V) \overset{\text{def}}{=} \bigoplus_{p,q=0}^d \Omega_c^{p,q}(X, V)$$

and

$$\mathcal{A}(X, V) \overset{\text{def}}{=} \bigoplus_{q=0}^d \Omega^{0,q}(X, V), \quad \mathcal{A}_c(X, V) \overset{\text{def}}{=} \bigoplus_{q=0}^d \Omega_c^{0,q}(X, V).$$

Then $\Omega(X, V) = \Gamma_{\text{sm}}(X, \bigwedge^* T^* X \otimes \bigwedge^* T^* X \otimes V)$ and $\mathcal{A}(X, V) = \Gamma_{\text{sm}}(X, \bigwedge^* T^* X \otimes V)$ are FS spaces which contain $\Omega_c(X, V)$ (respectively $\mathcal{A}_c(X, V)$) as dense subspaces. Notice that $\mathcal{A}(X, V)$ is a closed subspace of $\Omega(X, V)$. Moreover, $(\mathcal{A}(X, V), \mathcal{A}_V)$ is a finite topological complex of FS spaces [6,11], where $\mathcal{A}_V$ denotes the Dolbeault differential of $V$.

### 3.2. Topological complexes of bundle-valued currents with compact support and the classical Serre pairing.

Let $\hat{\Omega}^{p,q}(X, V)$ denote the space of distributions with compact support valued in the bundle $\bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes V$. Consider the bigraded topological vector space:

$$\hat{\Omega}(X, V) \overset{\text{def}}{=} \bigoplus_{p,q=0}^d \hat{\Omega}^{p,q}(X, V)$$

of distributions with compact support valued in the vector bundle $\bigwedge^* T^* X \otimes \bigwedge^* T^* X \otimes V$. Let $V^\vee \overset{\text{def.}}{=} \text{Hom}(V, \mathcal{O}_X)$ denote the dual bundle to $V$. Then $\hat{\Omega}^{d-p,d-q}(X, V)$ is topologically isomorphic (see [6]) with the topological dual $\Omega^{p,q}(X, V^\vee)^*$, where the latter is endowed with the strong topology. The corresponding perfect duality pairing is known as the **Serre pairing** and it is given by (see [6]):

$$(\omega, T) \rightarrow \int_X \omega \wedge T, \forall \omega \in \Omega^{p,q}(X, V^\vee), \forall T \in \hat{\Omega}^{d-p,d-q}(X, V),$$

where $\int_X$ denotes integration of compactly supported currents of type $(d, d)$ on $X$ with respect to the orientation induced by the complex structure of $X$. Below, we introduce a version of this pairing adapted to the case when $V$ is replaced by a $\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded holomorphic vector bundle.

### 3.3. The graded Serre pairing of a $\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded holomorphic vector bundle.

Let $A$ be either of the Abelian groups $\mathbb{Z}$ or $\mathbb{Z}_2$. Let $Q = \bigoplus_{j \in A} Q^j$ be an $A$-graded holomorphic vector bundle\(^3\) defined on $X$. Let $Q^\vee$ denote the dual vector bundle of $Q$, which we grade by the decomposition:

$$Q = \bigoplus_{j \in A} (Q^\vee)^j,$$

where $(Q^\vee)^j \overset{\text{def}}{=} (Q^{-j})^\vee$. In this case, the bundles $\bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes Q$ and $\bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes Q^\vee$ are $\mathbb{Z}^2 \times A$-graded with homogeneous components:

$$(\bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes Q)^{p,q,i} \overset{\text{def}}{=} \bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes Q^i,$$

$$(\bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes Q^\vee)^{p,q,j} \overset{\text{def}}{=} \bigwedge^p T^* X \otimes \bigwedge^q T^* X \otimes (Q^{-j})^\vee.$$

\(^3\) For $A = \mathbb{Z}$, the grading of $Q$ is necessarily concentrated in a finite number of degrees, since $Q$ has finite rank.
Viewing $O_X$ as an $A$-graded holomorphic vector bundle concentrated in degree zero, the bundle
$\wedge T^*X \otimes \wedge T^*X \simeq \wedge T^*X \otimes \wedge T^*X \otimes O_X$ is also $\mathbb{Z}^2 \times A$-graded with the third grading concentrated in degree zero. The spaces $\Omega(X, Q) = \Gamma_{sm}(X, \wedge T^*X \otimes \wedge T^*X \otimes \wedge T^*X)$ and $\Omega(X, Q)$ are trigraded accordingly. Notice that $\Omega(X, Q)$ is an FS space, while $\Omega(X, Q)$ is a DFS space.

**Definition 3.1** The graded duality morphism of $Q$ is the morphism $ev_Q : Q \otimes Q^\vee \to O_X$ of $A$-graded holomorphic vector bundles determined uniquely by the condition:

$$ev_Q(x)(v)(w) \overset{\text{def}}{=} (-1)^i \delta_{i+j,0} w(v), \ \forall v \in Q^i, \ \forall w \in (Q^\vee)^j, \ \forall x \in X .$$

Together with the wedge product of differential forms, $ev_Q$ induces a morphism of holomorphic vector bundles:

$$S_Q : (\wedge T^*X \otimes \wedge T^*X \otimes Q) \otimes (\wedge T^*X \otimes \wedge T^*X \otimes Q^\vee) \to \wedge T^*X \otimes \wedge T^*X$$

which is determined uniquely by the condition:

$$S_Q(x)(\omega_1 \otimes v, \omega_2 \otimes w) \overset{\text{def}}{=} (-1)^{(p_2+q_2+1)} \delta_{i+j,0} w(v) \omega_1 \wedge \omega_2$$

for $\omega_1 \in \wedge^{p_1} T^*_X \otimes \wedge^{q_1} T^*_X$ and $\omega_2 \in \wedge^{p_2} T^*_X \otimes \wedge^{q_2} T^*_X$ and $v \in Q^i_x, w \in (Q^\vee)^j_x$ (where $x \in X$). The bundle morphism $S_Q$ induces a continuous bilinear map:

$$S_Q : \Omega(X, Q) \times \hat{\Omega}(X, Q^\vee) \to \hat{\Omega}(X) .$$

With respect to the trigradings described above, the maps $S_Q$ and $S_Q$ are homogeneous of tridegree $(0,0,0)$.

**Definition 3.2** The graded Serre pairing of the $A$-graded holomorphic vector bundle $Q$ is the topological pairing $S_Q : \Omega(X, Q) \times \hat{\Omega}(X, Q^\vee) \to \mathbb{C}$ defined through:

$$S_Q(\omega, T) \overset{\text{def}}{=} \int_X S_Q(\omega, T) ,$$

where $\int_X L$ is defined to equal zero unless $L \in \hat{\Omega}(X)$ has type $(d,d)$.

If we view $\mathbb{C}$ as a $\mathbb{Z}^2 \times A$-graded vector space whose grading is concentrated in degree $(0,0,0)$, then $S_Q$ has tridegree $(-d,-d,0)$. Thus:

$$S_Q(\omega, T) = \delta_{p+\mu,0} \delta_{q+\nu,0} \delta_{i+j,0} S_Q(\omega, T) ,$$

when $\omega \in \Omega^{p,q}(X, Q^i)$ and $T \in \hat{\Omega}^{\mu,\nu}(X, (Q^\vee)^j)$.

**Lemma 3.3** $S_Q$ is a perfect pairing between the topological vector spaces $\Omega(X, Q)$ and $\hat{\Omega}(X, Q^\vee)$.

**Proof.** Follows immediately from [6, Proposition 4] and [6, Proposition 5]. \qed

4. Non-degeneracy of the bulk trace

In this section, we prove non-degeneracy of the bulk trace defined in [1]. The proof uses the spectral sequence results obtained in Section 2 together with an adaptation of the argument of [6] to the case of $\mathbb{Z}$-graded holomorphic vector bundles.
4.1. The topological complex of polyvector-valued forms. Let \((X, W)\) be a holomorphic Landau-Ginzburg pair. Consider the cochain complex \((\text{PV}(X), \delta_W)\), where \(\delta_W = \overline{\partial}_{TX} + \iota_W\) is the twisted differential \((0.3)\), graded by the total \(\mathbb{Z}\)-grading:

\[
0 \longrightarrow \text{PV}^{-d}(X) \xrightarrow{\delta_W} \text{PV}^{-d+1}(X) \xrightarrow{\delta_W} \cdots \xrightarrow{\delta_W} \text{PV}^{-1}(X) \xrightarrow{\delta_W} \text{PV}^{0}(X) \longrightarrow 0 .
\]

Notice that \(\delta_W\) has total degree +1 with respect to this grading. Let \(\text{PV}_c(X)\) denote the subcomplex of \((\text{PV}(X), \delta_W)\) formed by compactly-supported forms valued in the holomorphic vector bundle \(\wedge TX\). Notice that \(\iota_W\) and \(\delta_W\) are continuous with respect to the Fréchet topology, since they are differential operators of order zero and one, respectively. In particular, \((\text{PV}(X), \delta_W)\) is a finite topological cochain complex of FS spaces, which contains \(\text{PV}_c(X)\) as a dense subcomplex.

4.2. The topological complex of compactly-supported polyvector-valued currents. For any \(i \in \{-d, \ldots, 0\}\) and any \(j \in \{0, \ldots, d\}\), let \(\hat{\text{PV}}^{i,j}(X) \overset{\text{def}}{=} \hat{\Omega}^{i,j}(X, \wedge^{|i|}TX)\). Consider the bigraded vector space:

\[
\hat{\text{PV}}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \hat{\text{PV}}^{i,j}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \hat{\Omega}^{i,j}(X, \wedge^{|i|}TX) .
\]

We endow \(\hat{\text{PV}}(X)\) with its total \(\mathbb{Z}\)-grading, which has homogeneous components:

\[
\hat{\text{PV}}^k(X) = \bigoplus_{i+j=k} \hat{\text{PV}}^{i,j}(X) .
\]

This grading is concentrated in degrees \(k \in \{-d, \ldots, d\}\). Let \(\hat{\delta}_W : \hat{\text{PV}}(X) \rightarrow \hat{\text{PV}}(X)\) be the natural extension of \(\delta_W\) to \(\hat{\text{PV}}(X)\). Then \((\hat{\text{PV}}(X), \hat{\delta}_W)\) is a topological cochain complex of DFS spaces:

\[
0 \longrightarrow \hat{\text{PV}}^{-d}(X) \xrightarrow{\hat{\delta}_W} \hat{\text{PV}}^{-d+1}(X) \xrightarrow{\hat{\delta}_W} \cdots \xrightarrow{\hat{\delta}_W} \hat{\text{PV}}^{-1}(X) \xrightarrow{\hat{\delta}_W} \hat{\text{PV}}^{0}(X) \longrightarrow 0 .
\]

Let \(\hat{\text{HPV}}^k(X, W) \overset{\text{def}}{=} \text{H}^k(\hat{\text{PV}}(X), \hat{\delta}_W)\) denote the cohomology of this cochain complex in degree \(k \in \{-d, \ldots, d\}\). Notice that \((\text{PV}_c(X), \delta_W)\) is naturally a subcomplex of \((\hat{\text{PV}}(X), \hat{\delta}_W)\) as explained in [1], the wedge product induces an associative and supercommutative multiplication on \(\text{PV}(X)\), which we denote by juxtaposition and which makes \((\text{PV}(X), \delta_W)\) into a supercommutative differential graded algebra. This multiplication operation is jointly continuous with respect to the Fréchet topology, since the wedge product is.

4.3. The canonical off-shell bulk pairing and its extension. Let \(\Omega\) be a holomorphic volume form on \(X\). The canonical off-shell bulk trace [1, Section 5] determined by \(\Omega\) is the continuous \(\mathbb{C}\)-linear map \(\text{Tr}_B : \text{PV}_c(X) \rightarrow \mathbb{C}\) defined through:

\[
\text{Tr}_B(\omega) \overset{\text{def}}{=} \int_X \Omega \wedge (\Omega, \omega) , \quad \forall \omega \in \text{PV}_c(X) ,
\]

where the integral over \(X\) of a form of type \((k, l)\) is defined to vanish unless \(k = l = d\). For simplicity, we do not indicate the dependence of \(\Omega\) in the notation \(\text{Tr}_B\). This map has bidegree \((d, -d)\) and hence is of degree zero with respect to the total \(\mathbb{Z}\)-grading on \(\text{PV}_c(X)\). Notice that \(\text{Tr}_B\) can be viewed as a distribution valued in the holomorphic vector bundle \(\wedge^d TX\). It can also be viewed as a current of type \((d, 0)\) valued in the holomorphic vector bundle \(\wedge^d TX\).
Definition 4.1 The canonical off-shell bulk pairing determined by $\Omega$ is the continuous bilinear map $\langle \cdot, \cdot \rangle_B : \text{PV}(X) \times \text{PV}_c(X) \to \mathbb{C}$ defined through:

$$\langle \omega, \eta \rangle_B \triangleq \text{Tr}_B(\omega \eta) = \int_X \Omega \wedge [\Omega,\omega] \eta, \quad \forall \omega \in \text{PV}(X), \forall \eta \in \text{PV}_c(X).$$

Notice that $\langle \omega, \eta \rangle_B$ is well-defined since $\omega \eta$ belongs to $\text{PV}_c(X)$. This pairing has degree zero when $\text{PV}(X)$ and $\text{PV}_c(X)$ are endowed with the total $\mathbb{Z}$-gradings.

Definition 4.2 The extended canonical off-shell bulk pairing is the continuous bilinear map $\langle \cdot, \cdot \rangle : \text{PV}(X) \times \hat{\text{PV}}(X) \to \mathbb{C}$ defined through:

$$\langle \omega, T \rangle = \int_X \Omega \wedge [\Omega,\omega] T, \quad \forall \omega \in \text{PV}(X), \forall T \in \hat{\text{PV}}(X).$$ (4.1)

We have $\langle \cdot, \cdot \rangle|_{\text{PV}(X) \times \text{PV}_c(X)} = \langle \cdot, \cdot \rangle_B$. The pairing $\langle \cdot, \cdot \rangle$ has degree zero when $\text{PV}(X)$ and $\hat{\text{PV}}(X)$ are endowed with the total $\mathbb{Z}$-gradings.

Proposition 4.3 The canonical off-shell bulk pairing $\langle \cdot, \cdot \rangle_B$ is a topological pairing of bounded $\mathbb{Z}$-graded complexes between $(\text{PV}(X), \delta_W)$ and $(\text{PV}_c(X), \delta_W)$, while the extended canonical off-shell bulk pairing $\langle \cdot, \cdot \rangle$ is a topological pairing of bounded $\mathbb{Z}$-graded complexes between $(\text{PV}(X), \delta_W)$ and $(\hat{\text{PV}}(X), \delta_W)$.

Proof. For $\omega \in \text{PV}^k(X)$ and any $\eta \in \text{PV}_c(X)$, we have:

$$\langle \delta_W \omega, \eta \rangle_B + (-1)^k \langle \omega, \delta_W \eta \rangle_B = \text{Tr}_B \left[ (\delta_W \omega) \eta + (-1)^k \omega (\delta_W \eta) \right] = \text{Tr}_B [\delta_W (\omega \eta)] = 0,$$

where in the last equality we used the property $\text{Tr}_B \circ \delta_W = 0$ (see [1, Section 5]). This shows that $\langle \cdot, \cdot \rangle_B$ is a paring of bounded complexes. Continuity of this pairing follows from continuity of $\text{Tr}_B$ and joint continuity of the multiplication in $\text{PV}(X)$. A formally similar argument shows that $\langle \cdot, \cdot \rangle$ is a topological pairing of bounded complexes.

Let us view $\wedge^* TX$ as a $\mathbb{Z}$-graded holomorphic vector bundle with $\wedge^k T^* X$ sitting in degree $+k$ and $\wedge TX$ as a $\mathbb{Z}$-graded holomorphic vector bundle with $\wedge^k TX$ sitting in degree $-k$. Let $\Omega_d : \wedge TX \to \wedge^* T^* X$ be the degree $d$ map of graded holomorphic vector bundles given by contraction with $\Omega$. Let $\Omega_{d0} : \wedge TX \to \wedge^* T^* X$ be the map of holomorphic vector bundles given by reduced contraction with $\Omega$ (see [1, Section 3]). Let $\text{ev} := \text{ev}_{\wedge TX} : \wedge^k TX \otimes \wedge^k T^* X \to \mathcal{O}_X$ denote the graded duality morphism of the $\mathbb{Z}$-graded holomorphic vector bundle $\wedge TX$ (see (3.1)).

Proposition 4.4 For any $x \in X$ and any $v_1, v_2 \in \wedge T_x X$, we have:

$$\Omega_{x,0}(v_1 \wedge v_2) = (-1)^{k_1 d} \text{ev}_{\wedge TX}(x)(v_1, \Omega_{x,0} v_2).$$

Proof. Suppose that $v_1 \in \wedge^{k_1} T_x X$ and $v_2 \in \wedge^{k_2} T_x X$. Then:

$$\text{ev}_{\wedge TX}(x)(v_1, \Omega_{x,0} v_2) = (-1)^{k_1} \delta_{k_1 + k_2, d}(\Omega_{x,0} v_2) v_1 = (-1)^{k_1} \delta_{k_1 + k_2, d} \Omega_{x,0} (v_2 \wedge v_1) = (-1)^{k_1 + k_2} \delta_{k_1 + k_2, d} \Omega_{x,0} (v_1 \wedge v_2) = (-1)^{k_1} \Omega_{x,0} (v_1 \wedge v_2).$$

Contraction and wedge product with $\Omega$ induce topological isomorphisms:

$$\Omega_d : \hat{\text{PV}}(X) \to \hat{\Omega}^d(X, \wedge^* T^* X),$$

$$\Omega^d : \text{PV}(X) \to \Omega^d(X, \wedge TX).$$ (4.2)

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**Proposition 4.5** For any \( \mathbb{Z} \)-homogeneous element \( \omega \in \text{PV}(X) \) and any \( T \in \hat{\text{PV}}(X) \), we have:

\[
(\omega, T) = (-1)^{d \deg \omega} S_{\Lambda TX}(\Omega \wedge \omega, \Omega \wedge T) \quad .
\]

**Proof.** It suffices to consider the case \( \omega = \alpha \otimes v \) and \( T = \beta \otimes w \) with \( \alpha \in \Omega^0(X) \), \( \beta \in \hat{\Omega}^0(X) \), \( v \in \Gamma_{\text{sm}}(X, \Lambda^i TX) \) and \( w \in \Gamma_{\text{sm}}(X, \Lambda^j TX) \). Then both sides of (4.3) vanish unless \( p + q = d \) and \( i + j = d \). When these conditions are satisfied, we have:

\[
S_{\Lambda TX}(\Omega \wedge (\alpha \otimes v), \Omega \wedge (\beta \otimes w)) = (-1)^{q\deg v} S_{\Lambda TX}(\Omega \wedge (\alpha \otimes v), \beta \otimes (\Omega \wedge w)) =
\]

\[
= (-1)^{qj} \int_X (\Omega \wedge (\alpha \otimes \beta)) \ev_{\Lambda TX}(v, \Omega \wedge w) = (-1)^{qj+di} \int_X \Omega \wedge (\alpha \otimes \beta) \wedge (\Omega \wedge w) =
\]

\[
= (-1)^{qj+d^2+di} \int_X \Omega \wedge (\alpha \wedge (\Omega \wedge w) = (-1)^{p j+qi} \int_X \Omega \wedge (\alpha \otimes (\Omega \wedge w) = (-1)^{d \deg \omega}(\omega, T)
\]

where we used Proposition 4.4 and noticed that:

\[
qj + d(d + i) \equiv_2 (q + d)j \equiv_2 pj \quad , \quad pj + qi = p(d - i) + (d - p)i \equiv_2 d(p + i) \equiv_2 d \deg \omega .
\]

**Proposition 4.6** The extended canonical off-shell bulk pairing \( \langle \cdot, \cdot \rangle \) is a perfect topological pairing between the topological complexes \((\text{PV}(X), \delta W)\) and \((\hat{\text{PV}}(X), \hat{\delta}_W)\).

**Proof.** The fact that \( \langle \cdot, \cdot \rangle \) is a pairing of complexes follows from Proposition 4.3. The fact that it is a topological pairing follows from Proposition 4.5 and Lemma 3.3 applied to the graded holomorphic vector bundle \( Q := \wedge TX \), using the fact that operations (4.2) are topological isomorphisms and the existence of the natural isomorphism of holomorphic vector bundles \( Q^\vee \simeq \wedge T^* X \).

### 4.4. Non-degeneracy of the cohomological bulk pairing.

**Lemma 4.7** Suppose that the critical set \( Z_W \) is compact. Then \( \text{HPV}^k(X, W) \) and \( \hat{\text{HPV}}^k(X, W) \) are finite-dimensional for every \( k \in \{-d, \ldots , d\} \) and the extended canonical off-shell bulk pairing \( \langle \cdot, \cdot \rangle \) determined by any holomorphic volume form \( \Omega \) is cohomologically perfect. In particular, we have natural isomorphisms of \( \mathbb{C} \)-vector spaces:

\[
\hat{\text{HPV}}^k(X, W) \simeq_\mathbb{C} \text{HPV}^{-k}(X, W)^\vee \quad , \quad \forall k \in \{-d, \ldots , d\}
\]

which depend only on \( \Omega \).

**Proof.** Since \( Z_W \) is compact, the cohomology \( \text{HPV}^k(X, W) = H^k(\text{PV}(X), \delta W) \) is finite-dimensional for every \( k \) as shown in [1, Proposition 3.4]. The remaining statements follow from Proposition 4.6 and Proposition 1.7.

**Remark 4.1.** Any two holomorphic volume forms \( \Omega \) and \( \Omega' \) on \( X \) are related by:

\[
\Omega' = f \Omega ,
\]

where \( f : X \to \mathbb{C}^\times \) is a nowhere-vanishing holomorphic function, i.e. an element of the group of units \( \text{O}(X)^\times \) of the commutative ring \( \text{O}(X) \). If \( \text{Tr}^f_B \) denotes the canonical off-shell bulk trace determined by \( \Omega' \), this relation gives:

\[
\text{Tr}^f_B(\omega) = \int_X f^2 \Omega \wedge (\Omega \wedge \omega) = \text{Tr}_B(f^2 \omega) \quad , \forall \omega \in \text{PV}(X) .
\]

\(^4\) We use the notation \( \equiv_2 \) for congruence modulo 2.

\(^5\) Notice that such a function need not be constant since \( X \) is non-compact.
Since $\delta_W$ is O(X)-linear, the space $\text{HPV}(X,W)$ is a $\mathbb{Z}$-graded O(X)-module (which is finite-dimensional when $Z_W$ is compact). Since $f$ is holomorphic, multiplication with $f^2$ commutes with $\delta_W$ and hence descends to an O(X)-linear endomorphism of the graded O(X)-module $\text{HPV}(X,W)$. We thus have the \textit{squaring actions}:

$$\text{sq}_k : O(X)^k \to \text{Aut}_{O(X)}(\text{HPV}^k(X,W)), \quad \forall k \in \{-d, \ldots, d\}$$

defined by $\text{sq}_k(f)([\omega]) \overset{\text{def}}{=} [f^2 \omega]$ (where $[\omega]$ denotes the $\delta_W$-cohomology class of a $\delta_W$-closed element of $PV^k(X,W)$) and the cohomological bulk traces determined by $\Omega$ and $\Omega'$ are related through:

$$\text{Tr}' = \text{Tr} \circ \text{sq}_0(f^2).$$

Let $s : PV_c(X) \to \widehat{PV}(X)$ be the inclusion map and $s_* : \text{HPV}_c(X,W) \to \widehat{\text{HPV}}(X,W)$ be the linear map induced by $s$ on cohomology. Recall the sheaf Koszul complex (0.4).

**Proposition 4.8** Suppose that the critical set $Z_W$ is compact. Then $\text{HPV}_c(X,W)$ and $\widehat{\text{HPV}}(X,W)$ are finite-dimensional over $\mathbb{C}$ and $s_*$ is an isomorphism of $\mathbb{C}$-vector spaces. Moreover, for any $k \in \{-d, \ldots, d\}$, we have a natural isomorphism of vector spaces:

$$\widehat{\text{HPV}}^k(X,W) \simeq_{\mathbb{C}} \mathbb{H}^k_c(\mathcal{K}_W),$$

where $\mathbb{H}^k_c(\mathcal{K}_W)$ denotes compactly-supported hypercohomology of $\mathcal{K}_W$.

**Proof.** Since the category of all sheaves is Abelian with enough injectives and $\mathcal{K}_W$ is a finite complex, we can define the hypercohomology $\mathbb{H}^k_c(\mathcal{K}_W)$ (see [14, Proposition 8.6]). In fact, hypercohomology can be computed by using $\Gamma$-acyclic resolutions (see [14, Proposition 8.12]). Both $(PV_c(X), \delta_W)$ and $(\widehat{PV}(X), \hat{\delta}_W)$ are complexes of fine (and thus $\Gamma$-acyclic) sheaves (see [14, Proposition 4.36]). Since they both give $\Gamma$-acyclic resolutions of $\mathcal{K}_W$, we have isomorphisms of vector spaces:

$$\text{HPV}_c^k(X,W) \simeq_{\mathbb{C}} \mathbb{H}^k_c(\mathcal{K}_W) \simeq_{\mathbb{C}} \widehat{\text{HPV}}^k(X,W),$$

which gives the conclusion. \text{□}

**Remark 4.2.** The natural isomorphism $\text{HPV}^k_c(X,W) \simeq_{\mathbb{C}} \widehat{\text{HPV}}^k(X,W)$ also follows directly by applying Theorem 2.2 to the inclusion map $s$. Indeed, the double complexes $(\text{HPV}_c(X,W), \delta, \delta_W)$ and $(\widehat{\text{HPV}}(X,W), \hat{\delta}, \hat{\delta}_W)$ satisfy condition A. of Section 2. The columns of the zeroth page of the associated spectral sequence compute the Dolbeault cohomology of the nodes of $\mathcal{K}_W$, so the conditions $H^r_{\text{d}^1}(PV_c(X)) \simeq H^r_{\text{d}^1}(\widehat{PV}(X))$ are satisfied for all $r,s$. Now Theorem 2.2 implies:

$$\text{gr}_p^\mathbb{P} \text{HPV}^k_c(X,W) \simeq \text{gr}_p^\mathbb{P} \widehat{\text{HPV}}^k(X,W), \quad \forall p,k \in \mathbb{Z}.$$  \hspace{1cm} (4.4)

This gives the desired isomorphism $\text{HPV}^k_c(X,W) \simeq_{\mathbb{C}} \widehat{\text{HPV}}^k(X,W)$ since any extension of vector spaces splits.

### 4.5. Proof of Theorem A.

**Proof.** By Proposition 4.8, the inclusion $s : PV_c(X) \hookrightarrow \widehat{PV}(X)$ is a morphism of complexes which induces an isomorphism $s_* : \text{HPV}_c(X,W) \to \widehat{\text{HPV}}(X,W)$. Since $\langle \omega, \eta \rangle_B = \langle \omega, s(\eta) \rangle$ for all $\omega \in \text{PV}(X)$ and all $\eta \in \text{PV}_c(X)$, we have $\langle u_1, u_2 \rangle_B^H = \langle u_1, s_*(u_2) \rangle^H$ for all $u_1 \in \text{HPV}(X,W)$ and all $u_2 \in \text{HPV}_c(X,W)$. Since $\langle \cdot, \cdot \rangle^H$ is non-degenerate by Lemma 4.7 and $s_*$ is injective,
the pairing $\langle \cdot, \cdot \rangle_B^H : HPV(X, W) \times HPV_c(X, W) \to \mathbb{C}$ induced by $\langle \cdot, \cdot \rangle_B$ is also non-degenerate. On the other hand, we have $\langle u_1, u_2 \rangle_c = \langle i_*(u_1), u_2 \rangle_B^H$ for all $u_1, u_2 \in HPV_c(X, W)$, where $i_* : HPV_c(X, W) \to HPV(X, W)$ is the map induced by the inclusion $i : PV_c(X) \to PV(X)$ and $\langle \cdot, \cdot \rangle_c$ is the pairing induced by $\text{Tr}_B$ on cohomology (see [1, Proposition 5.4]). Since $Z_W$ is compact, the map $i_*$ is an isomorphism by [1, Proposition 3.7]. This shows that the pairing $\langle \cdot, \cdot \rangle_c : HPV_c(X, W) \times HPV_c(X, W) \to \mathbb{C}$ is non-degenerate. By the results of [1], we also have $\langle u_1, u_2 \rangle_c = \langle i_*(u_1), i_*(u_2) \rangle_B$ for all $u_1, u_2 \in HPV_c(X, W)$, which shows that $\langle \cdot, \cdot \rangle_B$ is non-degenerate since $i_*$ is bijective. 

4.6. Proof of Theorem A’.

Proof. The isomorphism $HPV^k(X, W) \simeq \mathbb{H}^k(K_W)$ was proved in [2, Proposition 4.1]. Since $Z_W$ is compact, we also have an isomorphism $HPV^k(X, W) \simeq HPV^k(X, W)$ which follows from [1, Proposition 3.7]. The isomorphism $HPV^k(X, W) \simeq HPV^{-k}(X, W)$ follows from Lemma 4.7. 

5. Non-degeneracy of the boundary traces

Let $\Omega$ be a holomorphic volume form on $X$. For any holomorphic factorization $a = (E, D)$ of $W$, the canonical boundary trace $\text{tr}^B_a : A_c(X, \text{End}(E)) \to \mathbb{C}$ is defined by (see [1]):

$$\text{tr}^B_a(\alpha) \overset{\text{def.}}{=} \int_X \Omega \wedge \text{str}_E(\alpha) , \ \forall \alpha \in A_c(X, \text{End}(E)) .$$

We have the canonical off-shell boundary paring:

$$\langle \alpha, \beta \rangle^{B}_{E_1, E_2} = \text{tr}_{E_2}(\alpha \beta) , \ \forall \alpha \in A(X, \text{Hom}(E_1, E_2)) , \ \forall \beta \in A_c(X, \text{Hom}(E_2, E_1)) .$$

5.1. Extended and restricted boundary pairings. Let $E$ be a holomorphic vector superbundle on $X$. Let $\text{str}_E : \text{End}(E) \to \mathcal{O}_X$ denote the morphism of holomorphic vector bundles given by the fiberwise supertrace of $E$ (see [1, Section 4]). This induces a $C^\infty(X)$-linear map $\text{str}_E : \Omega(X, \text{End}(E)) \to \Omega(X)$ (called the extended supertrace) which is determined uniquely by the condition:

$$\text{str}_E(\omega \otimes f) = \text{str}_E(f)\omega , \ \forall \omega \in \Omega(X) , \ \forall f \in \Gamma_{\text{sm}}(X, \text{End}(E)) .$$

Let $E_1$ and $E_2$ be two holomorphic vector superbundles defined on $X$.

Definition 5.1 The extended boundary pairing of $(E_1, E_2)$ is the continuous bilinear map

$$\langle \cdot, \cdot \rangle_{E_1, E_2} : A(X, \text{Hom}(E_1, E_2)) \times \tilde{A}(X, \text{Hom}(E_2, E_1)) \to \mathbb{C}$$

defined through:

$$\langle \alpha, T \rangle_{E_1, E_2} \overset{\text{def.}}{=} \int_X \Omega \wedge \text{str}_{E_2}(\alpha T) , \ \forall \alpha \in A(X, \text{Hom}(E_1, E_2)) , \ \forall T \in \tilde{A}(X, \text{Hom}(E_2, E_1)) .$$

The restriction $\langle \cdot, \cdot \rangle_{c, E_1, E_2} : A_c(X, \text{Hom}(E_1, E_2)) \times A_c(X, \text{Hom}(E_2, E_1)) \to \mathbb{C}$ of $\langle \cdot, \cdot \rangle_{E_1, E_2}$ is called the restricted boundary pairing of $(E_1, E_2)$.
5.2. Relation to the Serre pairing of $\text{Hom}(E_1, E_2)$. Let $V \overset{\text{def}}{=} \text{Hom}(E_1, E_2)$ be a $\mathbb{Z}_2$-graded vector bundle whose homogeneous components are given by $V^0 = \text{Hom}^0(E_1, E_2)$ and $V^1 = \text{Hom}^1(E_1, E_2)$. Since $V \simeq E^0 \otimes E^2$, we have $V^\vee \simeq E^\vee_1 \otimes E^\vee_2 \simeq \text{Hom}(E_2, E_1)$. This allows us to relate the Serre pairing of $\text{Hom}(E_1, E_2)$ to the extended boundary pairing of $(E_1, E_2)$. For this, notice that the isomorphism $V^\vee \simeq \text{Hom}(E_2, E_1)$ can be constructed explicitly using the morphism of holomorphic vector bundles given by the fiberwise supertrace $str_2 : \text{End}(E_2) \to \mathcal{O}_X$. Indeed, this induces an isomorphism of $\mathbb{Z}_2$-graded holomorphic vector bundles $\tilde{\sigma}_{E_1, E_2} : \text{Hom}(E_2, E_1) \to \text{Hom}(E_2, E_1)^\vee$ which is defined through:

$$\tilde{\sigma}_{E_1, E_2}(x)(g_x)(f_x) \overset{\text{def}}{=} \text{str}_{E_2,x}(f_x \circ g_x) \quad \forall x \in X, \forall f_x \in \text{Hom}(E_{1,x}, E_{2,x}) , \forall g_x \in \text{Hom}(E_{2,x}, E_{1,x}) .$$

For any $f_x \in \text{Hom}(E_{1,x}, E_{2,x})$ and any $g_x \in \text{Hom}(E_{2,x}, E_{1,x})$, we have:

$$\text{ev}_{\text{Hom}(E_1, E_2)}(x)(f_x \circ g_x) = \text{str}_{E_2,x}(f_x \circ g_x) ,$$

where ev is the graded duality pairing defined in (3.1). The even map $\tilde{\sigma}_{E_1, E_2}$ induces a topological isomorphism:

$$\sigma_{E_1, E_2} : \tilde{\Omega}(X, \text{Hom}(E_2, E_1)) \tilde{\to} \tilde{\Omega}(X, \text{Hom}(E_1, E_2)^\vee)$$

which is determined uniquely by the condition:

$$\sigma_{E_1, E_2}(\omega \otimes g) = \omega \otimes \tilde{\sigma}_{E_1, E_2}(g)$$

for all $\omega \in \Omega(X)$ and all $g \in \Omega^0(X, \text{Hom}(E_1, E_2))$. This isomorphism preserves the rank bigrading as well as the bundle grading of the space $\Omega(X, \text{Hom}(E_1, E_2))$. Recall the map $S_Q$ defined in (3.4).

**Proposition 5.2** For any $\alpha \in \Omega(X, \text{Hom}(E_1, E_2))$ and any $T \in \tilde{\Omega}(X, \text{Hom}(E_2, E_1))$, we have:

$$S_{\text{Hom}(E_1, E_2)}(\alpha, \sigma_{E_1, E_2}(T)) = \text{str}_{E_2}(\alpha T) .$$

In particular, the Serre pairing of the $\mathbb{Z}_2$-graded vector bundle $\text{Hom}(E_1, E_2)$ satisfies:

$$S_{\text{Hom}(E_1, E_2)}(\alpha, \sigma_{E_1, E_2}(T)) = \int_X \text{str}_{E_2}(\alpha T) .$$

**Proof.** Follows immediately from (5.1), (3.3) and from the definition of the extended supertrace. □

The wedge product with $\Omega$ induces a topological isomorphism:

$$\Omega \wedge : \check{\mathcal{A}}(X, \text{Hom}(E_1, E_2)) \tilde{\to} \check{\Omega}^{\bullet}(X, \text{Hom}(E_1, E_2)) .$$

**Corollary 5.3** For any $\alpha \in \mathcal{A}(X, \text{Hom}(E_1, E_2))$ and any $T \in \check{\mathcal{A}}(X, \text{Hom}(E_2, E_1))$, we have:

$$S_{\text{Hom}(E_1, E_2)}(\alpha \wedge, \sigma_{E_1, E_2}(T)) = \alpha \wedge \text{str}_{E_2}(\alpha T) .$$

In particular, the Serre pairing of the $\mathbb{Z}_2$-graded vector bundle $\text{Hom}(E_1, E_2)$ satisfies:

$$S_{\text{Hom}(E_1, E_2)}(\alpha \wedge, \sigma_{E_1, E_2}(T)) = \int_X \alpha \wedge \text{str}_{E_2}(\alpha T) = (\alpha, T)_{E_1, E_2} .$$

**Proof.** Follows immediately from Proposition 5.2 and from the definition of the extended supertrace. □

**Proposition 5.4** The extended boundary pairing $\langle \cdot, \cdot \rangle_{E_1, E_2}$ is a perfect topological pairing between the topological vector spaces $\mathcal{A}(X, \text{Hom}(E_1, E_2))$ and $\check{\mathcal{A}}(X, \text{Hom}(E_2, E_1))$.

**Proof.** Follows from Proposition 5.2 and Lemma 3.3, using the fact that (5.2) and (5.3) are topological isomorphisms. □
5.3. Holomorphic factorizations. Recall that a holomorphic factorization of $W$ is a pair $(E, D)$, where $E$ is a holomorphic vector superbundle and $D \in \Gamma(X, \text{End}^1(E))$ is an odd holomorphic section of $E$ such that $D^2 = \text{Wid}_E$ (see [1]). As explained in op. cit., holomorphic factorizations of $W$ form an $O(X)$-linear and $\mathbb{Z}_2$-graded dg-category $\text{DF}(X, W)$ known as the twisted Dolbeault category of holomorphic factorizations, whose morphism spaces are the spaces (0.5) of bundle-valued forms, endowed with the twisted Dolbeault differentials (0.6).

5.4. The topological complexes $(\mathcal{A}(X, \text{Hom}(E_1, E_2)), \delta_{a_1,a_2})$ and $(\mathcal{A}_c(X, \text{Hom}(E_1, E_2)), \delta_{a_1,a_2})$. Let $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ be two holomorphic factorizations of $W$. We endow the space $\mathcal{A}(X, \text{Hom}(E_1, E_2))$ with the twisted Dolbeault differential $\delta := \delta_{a_1,a_2} = \overline{\partial} + \partial$, where $\overline{\partial} := \partial_{\text{Hom}(E_1,E_2)}$ and $\partial := \partial_{a_1,a_2}$ is the defect differential (see (0.6) and (0.7)).

**Proposition 5.5** $(\mathcal{A}(X, \text{Hom}(E_1, E_2)), \delta_{a_1,a_2})$ is a $\mathbb{Z}_2$-graded topological complex of FS spaces.

**Proof.** It is clear that $\mathcal{A}(X, \text{Hom}(E_1, E_2))$ is an FS space while $\overline{\partial}_{\text{Hom}(E_1,E_2)}$ is a continuous differential. The conclusion now follows by noticing that $\delta_{a_1,a_2}$ is continuous. $\square$

Notice that $\mathcal{A}_c(X, \text{Hom}(E_1, E_2))$ is a closed subspace of $\mathcal{A}(X, \text{Hom}(E_1, E_2))$ as well as a subcomplex of $(\mathcal{A}(X, \text{Hom}(E_1, E_2)), \delta_{a_1,a_2})$. Let $\text{HDF}_{X,W}(a_1, a_2)$ denote the cohomology of the topological complex $(\mathcal{A}(X, \text{Hom}(E_1, E_2)), \delta_{a_1,a_2})$ and $\text{HDF}_{c,X,W}(a_1, a_2)$ denote the cohomology of $(\mathcal{A}_c(X, \text{Hom}(E_1, E_2)), \delta_{a_1,a_2})$.

5.5. The topological complex $(\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2)), \hat{\delta}_{a_1,a_2})$. Let $\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2)) = \hat{\Omega}^0(X, \text{Hom}(E_1, E_2))$ denote the DFS space of $\text{Hom}(E_1, E_2)$-valued compactly-supported currents of type $(0, \bullet)$ defined on $X$, which contains $\mathcal{A}_c(X, \text{Hom}(E_1, E_2))$ as a closed subspace. Let $\hat{\delta}_{a_1,a_2}$ be the canonical extension of $\delta_{a_1,a_2}$ to $\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2))$.

**Proposition 5.6** $(\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2)), \hat{\delta}_{a_1,a_2})$ is a $\mathbb{Z}_2$-graded topological complex of DFS spaces.

**Proof.** The fact that $\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2))$ is a DFS space follows from Proposition (5.4), while continuity of $\hat{\delta}_{a_1,a_2}$ follows from the fact that $\delta_{a_1,a_2}$ is continuous. $\square$

Let $\text{HDF}_{X,W}(a_1, a_2)$ denote the cohomology of the complex $(\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2)), \hat{\delta}_{a_1,a_2})$.

**Proposition 5.7** The extended boundary pairing is a perfect topological pairing of complexes between $\mathcal{A}(X, \text{Hom}(E_1, E_2))$ and $\hat{\mathcal{A}}(X, \text{Hom}(E_1, E_2))$.

**Proof.** The fact that $\left< \cdot, \cdot \right>_{E_1,E_2}$ is a pairing of complexes follows by direct computation as in [1]. The fact that this is a perfect pairing follows from Proposition 5.4. $\square$

5.6. Non-degeneracy of the cohomological boundary pairings.

**Lemma 5.8** Suppose that the critical set $Z_W$ is compact. Then the vector spaces $\text{HDF}_{X,W}(a_1, a_2)$ and $\text{HDF}_{X,W}(a_1, a_2)$ are finite-dimensional and the extended canonical off-shell boundary pairing $\left< \cdot, \cdot \right>_{E_1,E_2}$ is cohomologically perfect. In particular, we have induced isomorphisms of $\mathbb{Z}_2$-graded vector spaces:

$$\text{HDF}_{X,W}(a_1, a_2) \simeq \text{HDF}_{X,W}(a_1, a_2)$$
Proof. Since $Z_W$ is compact, the vector space $HDF_{X,W}(a_1, a_2) = H(A(X, Hom(E_1, E_2)), \delta_{a_1, a_2})$ is finite-dimensional, as shown in [1]. The remaining statements follow from Proposition 1.7.

Let $s : A_c(X, Hom(E_1, E_2)) \to \hat{A}(X, Hom(E_1, E_2))$ be the inclusion map and $s_* : HDF_{c,X,W}(a_1, a_2) \to \widehat{HDF}_{X,W}(a_1, a_2)$ be the linear map induced by $s$ on cohomology.

**Proposition 5.9** Suppose that $Z_W$ is compact. Then $HDF_{X,W}(a_1, a_2)$ and $\widehat{HDF}_{X,W}(a_1, a_2)$ are finite-dimensional and the map $s_*$ is an isomorphism of vector spaces.

**Proof.** As in [2, Subsection 5.1], we define a horizontally 2-periodic double complex $K = \oplus_{i,j} K^{i,j}$ which is an unwinding of $A_c(X, Hom(E_1, E_2))$:

$$K^{i,j} \overset{\text{def}}{=} A^j(X, Hom^i(E_1, E_2)), \quad \forall i, j \in \mathbb{Z},$$

with vertical differentials given by $\bar{\partial} := \bar{\partial}_{a_1,a_2}$ and horizontal differentials given by $(-1)^j \delta$. Then the total complex $\oplus_n K^n = \oplus_{i+j=n} K^{i,j}$ has the differential $\delta_{a_1, a_2} = \bar{\partial} + \delta$ and is 2-periodic. We similarly define a double complex $\hat{K}$ as the unwinding of $\hat{A}(X, Hom(E_1, E_2))$. The inclusion of single complexes $s : A_c(X, Hom(E_1, E_2)) \to \hat{A}(X, Hom(E_1, E_2))$ naturally defines an inclusion of double complexes $s : (K, \bar{\partial}, (-1)^j \delta) \to (\hat{K}, \hat{\partial}, (-1)^j \hat{\partial})$. The standard filtration $F$ on the double complexes defined as in (2.1) is 2-periodic and induces filtrations $\mathcal{F}$ and $\hat{\mathcal{F}}$ on $K$ and $\hat{K}$ respectively. We are going to use Theorem 2.2 for the inclusion $s$. Consider the 2-periodic complex of locally-free sheaves:

$$(\mathcal{E}_{a_1, a_2}) : \: \ldots \to Hom^1(E_1, E_2) \overset{\partial}{\to} Hom^0(E_1, E_2) \overset{\partial}{\to} Hom^1(E_1, E_2) \to \ldots \quad (5.4)$$

with $Hom^\partial(E_1, E_2)$ sitting in even degrees. The columns of both double complexes form Dolbeault resolutions of the nodes of $\mathcal{E}_{a_1, a_2}$. Thus, the zero pages of the associated spectral sequences coincide. In our case this reads:

$$E_0^{p,q} = H^p_\partial(K) = H^p_\partial(\hat{K}) = \hat{E}_0^{p,q}, \quad \forall p, q \in \mathbb{Z}. $$

Since both induced filtrations on total complexes satisfy condition B. of Section 2, the associated spectral sequences converge to the total cohomologies $H^n_\partial(K)$ and $H^n(\hat{K})$ by Proposition 2.1. Now Theorem 2.2 implies that for the graded pieces of the induced filtration on the total cohomology, the following isomorphism holds for every $p$:

$$gr_F^p H^n_\partial(K) \simeq gr_F^p H^n_\partial(\hat{K}), \quad \forall n, p \in \mathbb{Z}. \quad (5.5)$$

Returning to $HDF_{c,X,W}(a_1, a_2)$, we can express this in terms of the cohomology of the 2-periodic total complex:

$$H^n_\partial(K) = H^n(A(X, Hom(E_1, E_2)), \delta) = Hom^\partial_{HDF_{c,X,W}}(a_1, a_2), \quad \forall n \in \mathbb{Z}$$

and similarly for the complex $\hat{K}$. This give an isomorphism:

$$gr_F^p \widehat{HDF}^i_{X,W}(a_1, a_2) \simeq gr_F^p HDF^i_{c,X,W}(a_1, a_2), \quad \forall p \text{ for } i = 0, 1. \quad (5.6)$$

Since the groups $\text{Ext}^i(\cdot, \cdot)$ vanish for $i > 0$ in the category of vector spaces over $\mathbb{C}$, we obtain the desired isomorphism $s_* : \widehat{HDF}_{X,W}(a_1, a_2) \to HDF_{c,X,W}(a_1, a_2)$. □
Proposition 5.10 Suppose that $Z_W$ is compact. Then the restricted boundary pairing

$$\langle \cdot, \cdot \rangle_{c, E_1, E_2} : A_c(X, Hom(E_1, E_2)) \times A_c(X, Hom(E_2, E_1)) \to \mathbb{C}$$

is cohomologically non-degenerate.

Proof. Consider the inclusion $j : A_c(X, Hom(E_1, E_2)) \hookrightarrow A(X, Hom(E_1, E_2))$. Then it was shown in [1] that $j$ is a quasi-isomorphism of complexes from $(A_c(X, Hom(E_1, E_2)), \delta_{a_1, a_2})$ to $(A(X, Hom(E_1, E_2)), \delta_{a_1, a_2})$. Let $j_* : HDF_{c, X, W}(a_1, a_2) \overset{\sim}{\to} HDF_{X, W}(a_1, a_2)$ be the isomorphism induced by this map on cohomology. By Proposition 5.9, the inclusion $s : A_c(X, Hom(E_1, E_2)) \hookrightarrow \hat{A}(X, Hom(E_1, E_2))$ proves to be also a quasi-isomorphism since it induces an isomorphism $s_* : HDF_{c, X, W}(a_1, a_2) \overset{\sim}{\to} HDF_{X, W}(a_1, a_2)$.

For every $\alpha \in A_c(X, Hom(E_1, E_2))$ and every $\beta \in A_c(X, Hom(E_2, E_1))$, we have $\langle \alpha, \beta \rangle_{c, E_1, E_2} = \langle j(\alpha), s(\beta) \rangle_{E_1, E_2}$. On the cohomological level, this implies $\langle u, v \rangle_{c, H, a_1, a_2} = \langle j_*(u), s_*(v) \rangle_{H, a_1, a_2}$ for all $u \in HDF_{c, X, W}(a_1, a_2)$ and all $v \in HDF_{c, X, W}(a_2, a_1)$. On the other hand, $HDF_{c, X, W}(a_1, a_2)$ and $HDF_{c, X, W}(a_2, a_1)$ are finite-dimensional and $\langle \cdot, \cdot \rangle_{H, a_1, a_2}$ is non-degenerate by Lemma 5.8. This shows that $\langle \cdot, \cdot \rangle_{c, H, a_1, a_2}$ is also non-degenerate since $j_*$ and $s_*$ are bijective. \hfill \Box

5.7. Proof of Theorem B.

Proof. By Proposition 5.9, the inclusion map $s : A_c(X, Hom(E_1, E_2)) \to \hat{A}(X, Hom(E_1, E_2))$ induces an isomorphism $s_* : HDF_{c, X, W}(a_1, a_2) \overset{\sim}{\to} HDF_{X, W}(a_1, a_2)$. Since $\langle \alpha_1, \alpha_2 \rangle_{E_1, E_2} = (\alpha_1, s(\alpha_2))_{E_1, E_2}$ for all $\alpha_1 \in A(X, Hom(E_1, E_2))$ and all $\alpha_2 \in A_c(X, Hom(E_2, E_1))$, we have $(t_1, t_2)_B^{BH}_{E_1, E_2} = (t_1, s_*(t_2))_B^{H}_{E_1, E_2}$ for all $t_1 \in HDF_{X, W}(a_1, a_2)$ and all $t_2 \in HDF_{c, X, W}(a_2, a_1)$. Since $\langle \cdot, \cdot \rangle_{E_1, E_2}^{BH}$ is non-degenerate by Lemma 5.8 and $s_*$ is injective, it follows that the pairing $\langle \cdot, \cdot \rangle_{a_1, a_2}^{BH} : HDF_{X, W}(a_1, a_2) \times HDF_{c, X, W}(a_2, a_1) \to \mathbb{C}$ induced by $\langle \cdot, \cdot \rangle_{E_1, E_2}^{BH}$ is also non-degenerate. On the other hand, we have $\langle t_1, t_2 \rangle_{a_1, a_2}^{c_1, c_2} = (j_*(t_1), t_2)_B^{BH}_{E_1, E_2}$ for all $t_1 \in HDF_{c, X, W}(a_1, a_2)$ and all $t_2 \in HDF_{c, X, W}(a_2, a_1)$, where $j_* : HDF_{c, X, W}(a_1, a_2) \to HDF_{X, W}(a_1, a_2)$ is the map induced by the inclusion $j : A_c(X, Hom(E_1, E_2)) \to A(X, Hom(E_1, E_2))$ and $\langle \cdot, \cdot \rangle_{a_1, a_2}^c$ is the pairing induced by $tr_{a_2}^{a_1}$ on cohomology (see [1, Proposition 6.3]). Since $Z_W$ is compact, the map $j_*$ is an isomorphism by [1, Proposition 4.11]. This shows that the pairing $\langle \cdot, \cdot \rangle_{a_1, a_2}^c : HDF_{c, X, W}(a_1, a_2) \times HDF_{c, X, W}(a_2, a_1) \to \mathbb{C}$ is non-degenerate. By the results of [1], we also have $\langle t_1, t_2 \rangle_{a_1, a_2}^c = (j_*(t_1), j_*(t_2))_{a_1, a_2}^c$ for all $t_1 \in HDF_{c, X, W}(a_1, a_2)$ and all $t_2 \in HDF_{c, X, W}(a_2, a_1)$, which shows that $\langle \cdot, \cdot \rangle_{a_1, a_2}^c$ is non-degenerate since $j_*$ is bijective. \hfill \Box

As mentioned in the introduction, Theorem B. can be reformulated using Serre functors. For this, we first discuss the shift functor of the category $HDF(X, W)$. We refer the reader to Appendix A for some notions and properties used in the next subsections.

5.8. The shift functor of the category $VB_{sm}(X)$. Let $VB_{sm}(X)$ be the $Z_2$-graded $C^\infty(X)$-linear category of smooth vector bundles and smooth sections defined in [1]. This category admits a shift functor $\Pi$ defined as follows:

1. For any $Z_2$-graded vector bundle $E$ defined on $X$, let $\Pi(E)$ denote the $Z_2$-graded vector bundle with homogeneous components:

$$\Pi(E)^0 \overset{\text{def.}}{=} E^1, \quad \Pi(E)^1 \overset{\text{def.}}{=} E^0.$$
2. For any morphism $s : E_1 \to E_2$ in $\text{VB}_{\text{sm}}(X)$ (i.e. for any smooth section $s \in \Gamma_{\text{sm}}(X, \text{Hom}(E_1, E_2))$ of the vector bundle $\text{Hom}(E_1, E_2) = E_1^{c} \otimes E_2$), let:

$$\Pi(s) \overset{\text{def.}}{=} s,$$

where the right hand side is viewed as a section of the bundle $\text{Hom}(\Pi(E_1), \Pi(E_2))$.

**Remark 5.1.** For any $\mathbb{Z}_2$-graded vector bundle $E$, let $\sigma_E : E \to \Pi(E)$ be the suspension morphism of $E$, i.e. the identity endomorphism of $E$ viewed as an odd morphism for $\mathbb{Z}_2$-graded vector bundles from $E$ to $\Pi(E)$. This can be viewed as a section of the $\mathbb{Z}_2$-graded vector bundle $\text{Hom}(E, E)$ and hence as an odd invertible morphism from $E$ to $\Pi(E)$ in the category $\text{VB}_{\text{sm}}(X)$.

The fact that $\Pi$ is a functor follows by noticing the relation:

$$\Pi(s) = \sigma_{E_2} \circ s \circ \sigma_{E_1}^{-1}, \quad \forall s \in \Gamma_{\text{sm}}(X, \text{Hom}(E_1, E_2)),$$

where $\circ$ denotes the composition of $\text{VB}_{\text{sm}}(X)$. Writing $s$ as a block matrix $s = \begin{bmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{bmatrix}$ (where $s_{\kappa_1 \kappa_2} \in \Gamma_{\text{sm}}(X, \text{Hom}(E_1^{\kappa_2}, E_2^{\kappa_1}))$ for all $\kappa_1, \kappa_2 \in \mathbb{Z}_2$), we have:

$$\Pi(s)_{\kappa_1 \kappa_2} = s_{\kappa_1 + 1, \kappa_2 + 1},$$

i.e.:

$$\Pi(s) = \begin{bmatrix} s_{11} & s_{10} \\ s_{01} & s_{00} \end{bmatrix},$$

and the suspension morphism of $E$ corresponds to the matrix:

$$\sigma_E = \begin{bmatrix} 0 & \text{id}_{E_1} \\ \text{id}_{E_0} & 0 \end{bmatrix}.$$

**5.9. The shift functors of $\text{DF}(X, W)$ and $\text{HDF}(X, W)$.** Let $\text{DF}(X, W)$ be the twisted Dolbeault category of the holomorphic LG pair $(X, W)$ (which is a $\mathbb{Z}_2$-graded $\text{O}(X)$-linear category). Let $\Sigma$ be the automorphism of the underlying $\mathbb{Z}_2$-graded $\text{O}(X)$-linear category which is defined as follows:

1. For any holomorphic factorization $(E, D)$ of $W$, let:

$$\Sigma(E, D) \overset{\text{def.}}{=} (\Pi E, \Pi D).$$

2. Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ of $W$ and a morphism $\alpha \in \text{Hom}_{\text{DF}(X, W)}(a_1, a_2) = \mathcal{A}(X, \text{Hom}(E_1, E_2)) = \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_{\text{sm}}(X, \text{Hom}(E_1, E_2))$, let:

$$\Sigma(\alpha) \overset{\text{def.}}{=} (\text{id}_{\mathcal{A}(X)} \otimes \Pi)(\alpha).$$

**Remark 5.2.** Writing $D = \begin{bmatrix} 0 & G \\ F & 0 \end{bmatrix}$ (with $F \in \Gamma(X, \text{Hom}(E_0^1, E_1^1))$ and $G \in \Gamma(X, \text{Hom}(E_0^1, E_0^0))$), we have $\Pi D \overset{\text{def.}}{=} \begin{bmatrix} 0 & F \\ G & 0 \end{bmatrix}$.
Proposition 5.11 $\Sigma$ is a differential shift functor for the $\mathbb{Z}_2$-graded $O(X)$-linear dg-category $DF(X, W)$.

Proof. A simple computation shows that condition (A.3) is satisfied. □

Since $\Sigma$ is a differential shift functor on $DF(X, W)$, it induces a shift functor on the $\mathbb{Z}_2$-graded $O(X)$-linear category $HDF(X, W) = H(DF(X, W))$, which we again denote by $\Sigma$.

Proposition 5.12 Suppose that $Z_W$ is compact. Then $(HDF(X, W), tr, \Sigma)$ is a Calabi-Yau supercategory of parity $\mu = \hat{d} \in \mathbb{Z}_2$ with compatible shift functor in the sense of Definition A.15 (see Appendix A).

Proof. Follows immediately from Theorem B and the definition of $\Sigma$. □

5.10. Proof of Theorem $B'$.

Proof. Follows immediately from Proposition 5.12 and Proposition A.19 of Appendix A. □

A. Linear categories and supercategories with involutive shift functor

In this Appendix, we collect some facts regarding linear categories with involutive shift functors and Serre functors. We are particularly interested in the case of $\mu$-Calabi-Yau categories in the sense of [1]. For simplicity, we assume shift functors and Serre functors to be automorphisms (rather than autoequivalences), since this case suffices for the purpose of the present paper.

A.1. The $\mathbb{Z}_2$-graded category $Mod^s_R$ and its shift functor. Let $R$ be a unital commutative ring and $Mod_R$ be the category of $R$-modules. Recall that an $R$-supermodule is a $\mathbb{Z}_2$-graded $R$-module, i.e. an $R$-module $M$ endowed with a direct sum decomposition $M = M^0 \oplus M^1$ into two submodules $M^0$ and $M^1$. Let $Mod^s_{\mathbb{Z}_2}$ denote the ordinary category of $\mathbb{Z}_2$-graded $R$-modules, whose set of morphisms from an $R$-supermodule $M$ to an $R$-supermodule $N$ is the (ungraded) $R$-module:

$$\text{Hom}(M, N) \overset{\text{def}}{=} \text{Hom}_R(M^0, N^0) \oplus \text{Hom}_R(M^1, N^1) \ .$$

Let $Mod^s_R$ be the category whose objects are the $R$-supermodules and whose set of morphisms from an object $M$ to an object $N$ is the inner Hom $R$-supermodule $\underline{\text{Hom}}(M, N)$, whose homogeneous components are defined through:

$$\text{Hom}^0(M, N) \overset{\text{def}}{=} \text{Hom}_R(M^0, N^0) \oplus \text{Hom}_R(M^1, N^1) \ ,$$

$$\text{Hom}^1(M, N) \overset{\text{def}}{=} \text{Hom}_R(M^0, N^1) \oplus \text{Hom}_R(M^1, N^0)$$

and whose composition of morphisms is induced from $Mod_R$. We have $\text{Hom}^0(M, N) = \text{Hom}(M, N)$.

Definition A.1 The parity change functor $\Pi$ of $Mod^s_R$ is the automorphism of $Mod^s_R$ defined as follows:
1. For any $R$-supermodule $M = M^0 \oplus M^1$, the $R$-supermodule $\Pi(M)$ has homogeneous components:

$$\Pi(M)^0 \overset{\text{def}}{=} M^1, \quad \Pi(M)^1 \overset{\text{def}}{=} M^0.$$ 

2. For any morphism $f \in \text{Hom}(M, N)$ of $\text{Mod}^s_R$, the morphism $\Pi(f) \in \text{Hom}(\Pi(M), \Pi(N))$ has homogeneous components:

$$\Pi(f)^0 \overset{\text{def}}{=} f^0 \in \text{Hom}_R(M^0, N^0) \oplus \text{Hom}_R(M^1, N^1) = \text{Hom}^0(\Pi(M), \Pi(N)),$$

$$\Pi(f)^1 \overset{\text{def}}{=} f^1 \in \text{Hom}_R(M^0, N^1) \oplus \text{Hom}_R(M^1, N^0) = \text{Hom}^1(\Pi(M), \Pi(N)).$$

It is clear that $\Pi$ is involutive, i.e. we have $\Pi^2 = \text{id}_{\text{Mod}^s_R}$, where $\text{id}_{\text{Mod}^s_R}$ denotes the identity functor of $\text{Mod}^s_R$. For any $R$-supermodules $M$ and $N$, we have:

$$\text{Hom}(M, \Pi(N)) \simeq \Pi\text{Hom}(M, N) = \text{Hom}(\Pi(M), N),$$

where the second equality results from the first upon replacing $M$ and $N$ with $\Pi(M)$ and $\Pi(N)$ respectively.

**A.2. Shift functors on linear supercategories.** Let $\mathcal{T}$ be an $R$-linear supercategory, i.e. a category enriched over $\text{Mod}^s_R$. A linear functor $F : \mathcal{T} \to \mathcal{T}$ is called *even* if the following condition holds for any two objects $a$ and $b$ of $\mathcal{T}$:

$$F(\text{Hom}_\mathcal{T}^\kappa(a, b)) \subset \text{Hom}_\mathcal{T}^\kappa(a, b), \quad \forall \kappa \in \mathbb{Z}_2.$$

The *even subcategory* $\mathcal{T}$ is the subcategory obtained from $\mathcal{T}$ taking the same objects but keeping only those morphisms which have degree $0 \in \mathbb{Z}_2$ (without changing the composition of morphisms). We denote this subcategory by $\text{Ev}(\mathcal{T})$ or by $\mathcal{T}^0$.

**Definition A.2** A shift functor for $\mathcal{T}$ is an even automorphism $\Sigma$ of $\mathcal{T}$ which satisfies the following properties:

1. We have $\Sigma^2 = \text{id}_\mathcal{T}$.

2. For any two objects $a$ and $b$ of $\mathcal{T}$, there exist isomorphisms of $\mathbb{Z}_2$-graded $R$-modules:

$$\text{Hom}_\mathcal{T}(a, \Sigma(b)) \overset{\rho}{\longrightarrow} \Pi\text{Hom}_\mathcal{T}(a, b) \quad (A.1)$$

which are natural in both $a$ and $b$. More precisely, there exists an isomorphism:

$$\rho : \text{Hom}_\mathcal{T} \circ (\text{id}_\mathcal{T} \times \Sigma) \xrightarrow{\sim} \Pi \circ \text{Hom}_\mathcal{T}$$

in the category of functors from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$ and natural transformations between such.

In this case, the pair $(\mathcal{T}, \Sigma)$ is called an $R$-linear supercategory with shift.

**Remark A.1.** Let $(\mathcal{T}, \Sigma)$ be an $R$-linear supercategory with shift. The replacement in $(A.1)$ of $a$ and $b$ by $\Sigma(a)$ and $\Sigma(b)$ respectively gives isomorphisms:

$$\text{Hom}_\mathcal{T}(\Sigma(a), b) = \text{Hom}_\mathcal{T}(\Sigma(a), \Sigma^2(b)) \overset{\rho_{\Sigma(a), \Sigma^2(b)}}{\longrightarrow} \Pi\text{Hom}_\mathcal{T}(\Sigma(a), \Sigma(b)) \overset{\Sigma}{\longrightarrow} \Pi\text{Hom}_\mathcal{T}(a, b),$$

where we used the relation $\Sigma^2 = \text{id}_\mathcal{T}$. We thus have a composite isomorphism:

$$\text{Hom}_\mathcal{T}(\Sigma(a), b) \overset{\Sigma \circ \rho_{\Sigma(a), \Sigma^2(b)}}{\longrightarrow} \Pi\text{Hom}_\mathcal{T}(a, b),$$

which is natural in both $a$ and $b$. 
Definition A.3 Let \((T_1, \Sigma_1)\) and \((T_2, \Sigma_2)\) be two \(R\)-linear supercategories with shifts. A morphism of \(R\)-linear supercategories with shifts from \((T_1, \Sigma_1)\) to \((T_2, \Sigma_2)\) is a linear functor \(F : T_1 \to T_2\) such that \(F \circ \Sigma_1 = \Sigma_2 \circ F\).

With this definition of morphisms, \(R\)-linear supercategories with shifts form a category denoted \(\text{RSCat}^s\).

A.3. \(R\)-linear categories with involution. Let \(C\) be an \(R\)-linear category, i.e. a category enriched over \(\text{Mod}_R\).

Definition A.4 An involution of \(C\) is a linear automorphism \(\Sigma\) of \(C\) such that \(\Sigma^2 = \text{id}_C\). In this case, the pair \((C, \Sigma)\) is called an \(R\)-linear category with involution.

Definition A.5 Let \((C_1, \Sigma_1)\) and \((C_2, \Sigma_2)\) be two \(R\)-linear categories with involution. A morphism of \(R\)-linear categories with involution from \((C_1, \Sigma_1)\) to \((C_2, \Sigma_2)\) is a linear functor \(F : C_1 \to C_2\) such that \(F \circ \Sigma_1 = \Sigma_2 \circ F\).

With this definition, \(R\)-linear categories with involution form a category denoted \(\text{RICat}\).

A.4. Supercompletion of an \(R\)-linear category with involution. An \(R\)-linear category with involution can be completed to a \(\mathbb{Z}_2\)-graded category as follows.

Definition A.6 Let \((C, \Sigma)\) be an \(R\)-linear category with involution. The supercompletion of \(C\) along \(\Sigma\) is the \(R\)-linear \(\mathbb{Z}_2\)-graded category \(\text{Gr}_\Sigma(C)\) defined as follows:

1. The objects of \(\text{Gr}_\Sigma(C)\) coincide with those of \(C\).
2. For any objects \(a, b\) of \(C\) and any \(\kappa \in \mathbb{Z}_2\), the \(R\)-module of morphisms from \(a\) to \(b\) in \(C\) has the \(\mathbb{Z}_2\)-grading given by the decomposition \(\text{Hom}_{\text{Gr}_\Sigma(C)}(a, b) = \text{Hom}_{\text{Gr}_\Sigma(C)}^0(a, b) \oplus \text{Hom}_{\text{Gr}_\Sigma(C)}^1(a, b)\), where:
   \[
   \text{Hom}_{\text{Gr}_\Sigma(C)}^\kappa(a, b) \overset{\text{def}}{=} \text{Hom}_C(a, \Sigma^\kappa(b)), \quad \forall \kappa \in \mathbb{Z}_2.
   \]
3. Given three objects \(a, b, c\) of \(C\), the \(R\)-bilinear composition of morphisms \(\circ : \text{Hom}_{\text{Gr}_\Sigma(C)}(b, c) \times \text{Hom}_{\text{Gr}_\Sigma(C)}(a, b) \to \text{Hom}_{\text{Gr}_\Sigma(C)}(a, c)\) of \(\text{Gr}_\Sigma(C)\) is uniquely determined by the condition:
   \[
   g \circ_{\text{Gr}_\Sigma(C)} f \overset{\text{def}}{=} \Sigma^\kappa(g) \circ f \in \text{Hom}_C(a, \Sigma^{\kappa+\nu}(c)) = \text{Hom}_{\text{Gr}_\Sigma(C)}^{\kappa+\nu}(a, c),
   \]
   for \(f \in \text{Hom}_{\text{Gr}_\Sigma(C)}^\kappa(a, b) = \text{Hom}_C(a, \Sigma^\kappa(b))\) and \(g \in \text{Hom}_{\text{Gr}_\Sigma(C)}^\nu(b, c) = \text{Hom}_C(b, \Sigma^\nu(c))\) (where \(\kappa, \nu \in \mathbb{Z}_2\)).

The proof of the following statements is elementary and left to the reader:

Proposition A.7 Let \((C, \Sigma)\) be an \(R\)-linear category with involution. Consider the functor \(\text{Gr}(\Sigma) : \text{Gr}_\Sigma(C) \to \text{Gr}_\Sigma(C)\) defined as follows:

1. For any object \(a\) of \(C\), let \(\text{Gr}(\Sigma)(a) \overset{\text{def}}{=} \Sigma(a)\).
2. For any morphism \( f = u \oplus v \in \text{Hom}_{\text{Gr}(C)}(a,b) = \text{Hom}_C(a,b) \oplus \text{Hom}_C(a,\Sigma(b)) \) (where \( u \in \text{Hom}_C(a,b) \) and \( v \in \text{Hom}_C(a,\Sigma(b)) \), let:

\[
\text{Gr}(\Sigma)(f) \overset{\text{def}}{=} \Sigma(u) \oplus \Sigma(v) \in \text{Hom}_{\text{Gr}(C)}(\Sigma(a),\Sigma(b)) = \text{Hom}_C(\Sigma(a),\Sigma(b)) \oplus \text{Hom}_C(\Sigma(a),b).
\]

Then \( \text{Gr}(\Sigma) \) is a shift functor for the supercompletion \( \text{Gr}_\Sigma(C) \).

**Proposition A.8** Let \( F : (C_1, \Sigma_1) \to (C_2, \Sigma_2) \) be a morphism of \( R \)-linear categories with involution. Consider the functor \( \text{Gr}(F) : \text{Gr}_{\Sigma_1}(C_1) \to \text{Gr}_{\Sigma_2}(C_2) \) defined through:

1. For any object \( a \) of \( C_1 \), set \( \text{Gr}(F)(a) \overset{\text{def}}{=} F(a) \).
2. For any morphism \( f = u \oplus v \in \text{Hom}_{\text{Gr}_{\Sigma_1}(C_1)}(a,b) \), where \( u \in \text{Hom}_{C_1}(a,b) \) and \( v \in \text{Hom}_{C_1}(a,\Sigma_1(b)) \), set:

\[
\text{Gr}(F)(f) \overset{\text{def}}{=} F(u) \oplus F(v) \in \text{Hom}_{\text{Gr}_{\Sigma_2}(C_2)}(F(a),F(b)) = \text{Hom}_{C_2}(F(a),F(b)) \oplus \text{Hom}_{C_2}(F(a),\Sigma_2(F(b))),
\]

where we used the relation \( F \circ \Sigma_1 = \Sigma_2 \circ F \).

Then \( \text{Gr}(F) : (\text{Gr}_{\Sigma_1}(C_1),\text{Gr}(\Sigma_1)) \to (\text{Gr}_{\Sigma_2}(C_2),\text{Gr}(\Sigma_2)) \) is a morphism of \( R \)-linear supercategories with shift.

**Proposition A.9** \( \text{Gr} \) is a functor from \( \text{RICat} \) to \( \text{RSCat}^s \).

**A.5. The even subcategory of an \( R \)-linear supercategory with shift.** A quasi-inverse of the supercompletion functor \( \text{Gr} \) can be constructed as follows, where the proof of the various statements is left to the reader.

**Proposition A.10** Let \( (\mathcal{T}, \Sigma) \) be an \( R \)-linear supercategory with shift. Then \( \Sigma \) is an involution of the even subcategory \( \mathcal{T}^0 \).

**Proposition A.11** Given a morphism of \( R \)-linear supercategories with shifts \( F : (\mathcal{T}_1, \Sigma_1) \to (\mathcal{T}_2, \Sigma_2) \), consider the functor \( \text{Ev}(f) : \text{Ev}(\mathcal{T}_1) = \mathcal{T}_1^0 \to \text{Ev}(\mathcal{T}_2) = \mathcal{T}_2^0 \) obtained by restricting \( F \) to the subcategory \( \mathcal{T}_1^0 \) of \( \mathcal{T}_1 \). Then \( \text{Ev}(f) \) is a morphism in \( \text{RICat} \) from \( (\mathcal{T}_1^0, \Sigma_1) \) to \( (\mathcal{T}_2^0, \Sigma_2) \).

**Proposition A.12** \( \text{Ev} \) is a functor from \( \text{RSCat}^s \) to \( \text{RICat} \).

Finally, one easily proves the following:

**Theorem A.13** The functors \( \text{Gr} \) and \( \text{Ev} \) are mutually quasi-inverse equivalences between \( \text{RICat} \) and \( \text{RSCat}^s \).

This shows, in particular, that \( R \)-linear supercategories with shift can be reconstructed from their even part, which is an \( R \)-linear category with involution.

**A.6. Calabi-Yau supercategories with shift.** In this subsection we consider the case \( R = \mathbb{C} \). Recall the following notion used in [1]:

**Definition A.14** A Calabi-Yau supercategory of parity \( \mu \in \mathbb{Z}_2 \) is a pair \( (\mathcal{T}, \text{tr}) \), where:
A. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category.

B. $\text{tr} = (\text{tr}_a)_{a \in \text{Ob} \mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\text{tr}_a : \text{End}_{\mathcal{T}}(a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree $\mu$ such that the following conditions are satisfied:

1. For any two objects $a, b \in \text{Ob} \mathcal{T}$, the $\mathbb{C}$-bilinear pairing $\langle \cdot, \cdot \rangle_{a,b} : \text{Hom}_{\mathcal{T}}(a, b) \times \text{Hom}_{\mathcal{T}}(b, a) \to \mathbb{C}$ defined through:
   \[
   \langle t_1, t_2 \rangle_{a,b} \overset{\text{def.}}{=} \text{tr}_b(t_1 \circ t_2), \quad \forall t_1 \in \text{Hom}_{\mathcal{T}}(a, b), \forall t_2 \in \text{Hom}_{\mathcal{T}}(b, a)
   \]
   is non-degenerate.

2. For any two objects $a, b \in \text{Ob} \mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \text{Hom}_{\mathcal{T}}(a, b)$ and $t_2 \in \text{Hom}_{\mathcal{T}}(b, a)$, we have:
   \[
   \langle t_1, t_2 \rangle_{a,b} = (-1)^{\text{deg} t_1 \cdot \text{deg} t_2} \langle t_2, t_1 \rangle_{b,a}.
   \] (A.2)

We are interested in the case of Calabi-Yau supercategories which admit a shift functor compatible with the traces $\text{tr}_a$.

**Definition A.15** A Calabi-Yau supercategory of parity $\mu \in \mathbb{Z}_2$ with compatible shift functor is a triplet $(\mathcal{T}, \text{tr}, \Sigma)$ such that:

1. $(\mathcal{T}, \text{tr})$ is a Calabi-Yau supercategory of parity $\mu$.

2. $\Sigma$ is a parity change functor on the $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category $\mathcal{T}$.

3. We have:
   \[
   \text{tr}_{\Sigma(a)}(\Sigma(t)) = \text{tr}_a(t), \quad \forall a \in \text{Ob} \mathcal{T}, \forall t \in \text{End}_{\mathcal{T}}(a).
   \]

A.7. Serre functors. Recall the notion of Serre functor introduced by Bondal and Kapranov [15]:

**Definition A.16** A Serre functor on a Hom-finite $\mathbb{C}$-linear category $\mathcal{C}$ is a linear autoequivalence $S$ of $\mathcal{C}$ such that for any two objects $a, b$ of $\mathcal{C}$, there exists a linear isomorphism:

$\text{Hom}_{\mathcal{C}}(a, S(b)) \simeq \text{Hom}_{\mathcal{C}}(b, a)^\vee$

which is natural in both $a$ and $b$. More precisely, there exists an isomorphism of functors:

$\text{Hom}_{\mathcal{C}}(\text{id}_\mathcal{C} \times S) \simeq D \circ \text{Hom}_{\mathcal{C}} \circ \tau$,

where $\tau : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is the transposition functor and $D : \text{vect}_\mathbb{C} \to \text{vect}_\mathbb{C}$ is the dualization functor on the category $\text{vect}_\mathbb{C}$ of finite-dimensional vector spaces.

One has the following equivalent description:

**Proposition A.17** Let $\mathcal{C}$ be a Hom-finite $\mathbb{C}$-linear category and let $S$ be a linear automorphism of $\mathcal{C}$. Then the following statements are equivalent:

\* Notice that we do not require $\mathcal{T}$ to be an additive category. Also notice that we do not require $\mathcal{T}$ to be triangulated.
(a) $S$ is a Serre functor for $C$.

(b) For any object $a$ of $C$, there exists a linear map $\text{tr}_a : \text{Hom}_C(a, S(a)) \to C$ such that the following conditions are satisfied for any objects $a$ and $b$ of $C$:

- $\text{tr}_a(g \circ f) = \text{tr}_b(S(f) \circ g)$, $\forall f \in \text{Hom}_C(a, b)$, $\forall g \in \text{Hom}_C(b, S(a))$.
- The bilinear map $\langle \cdot, \cdot \rangle^S_{a,b} : \text{Hom}_C(a, b) \times \text{Hom}_C(b, S(a)) \to C$ defined through:
  \[
  \langle f, g \rangle^S_{a,b} \overset{\text{def}}{=} \text{tr}_b(S(f) \circ g), \forall f \in \text{Hom}_C(a, b), \forall g \in \text{Hom}_C(b, S(a))
  \]
  is non-degenerate.

A.8. Calabi-Yau categories with involution.

Given a $\mathbb{C}$-linear category $T$ with involution and an element $\mu \in \mathbb{Z}_2$, we define:

\[
\Sigma^\mu \overset{\text{def}}{=} \begin{cases} 
\text{id}_T & \text{if } \mu = 0 \\
\Sigma & \text{if } \mu = 1
\end{cases}
\]

**Definition A.18** Let $\mu \in \mathbb{Z}_2$. Then a $\mathbb{C}$-linear category with involution $(C, \Sigma)$ is called $\mu$-Calabi-Yau if $\Sigma^\mu$ is a Serre functor for $C$.

For any $\mu \in \mathbb{Z}_2$, let $\text{ICYCat}(\mu)$ denote the full subcategory of $\text{CICat}$ consisting of all $\mu$-Calabi-Yau categories with involution and $\text{SCYCat}^s(\mu)$ denote the full subcategory of $\text{CSCat}^s$ consisting of all Calabi-Yau supercategories of signature $\mu$. The proof of the following statement is immediate:

**Proposition A.19** The restrictions of the functors $\text{Gr}$ and $\text{Ev}$ give mutually quasi-inverse equivalences between the categories $\text{ICYCat}(\mu)$ and $\text{SCYCat}^s$.

A.9. Shift functors on $\mathbb{Z}_2$-graded dg-categories.

**Definition A.20** Let $R$ be a unital commutative ring and $A$ be $\mathbb{Z}_2$-graded and $R$-linear dg-category. A differential shift functor on $A$ is a shift functor $\Sigma$ on the underlying $R$-linear supercategory of $A$ which satisfies the following condition for any objects $a$ and $b$ of $A$:

\[
\Sigma_{a,b} \circ d_{a,b} = d_{\Sigma(a),\Sigma(b)} \circ \Sigma_{a,b} : \text{Hom}_A(a, b) \to \text{Hom}_A(\Sigma(a), \Sigma(b)),
\]

where $d_{a,b}$ and $d_{\Sigma(a),\Sigma(b)}$ are the odd differentials on the $R$-supermodules $\text{Hom}_A(a, b)$ and respectively $\text{Hom}_A(\Sigma(a), \Sigma(b))$.

A differential shift functor $\Sigma$ on $A$ induces a shift functor on the total cohomology category $H(A)$, which we again denote by $\Sigma$.

Acknowledgements. This work was supported by the research grant IBS-R003-S1.
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