On weighted cumulative past extropy

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Abstract

In this paper, we study some properties and characterization of the weighted cumulative past extropy (WCPJ). Many results including some bounds, inequalities, and effects of linear transformations are obtained. We study the characterization of WCPJ based on the largest order statistics. Conditional WCPJ and some of its properties are discussed. Related studies in connection to reliability theory are covered.

Key Words: Entropy; Cumulative residual entropy; Extropy, Cumulative residual extropy; Weighted Cumulative past extropy; conditional weighted past extropy.

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1 Introduction

We encounter phenomena or events which are associated with uncertainty. Uncertainty emerges since we have less information than the total information required to describe a system and its environment. Uncertainty and information are closely associated. Since in an experiment the information provided is equal to the amount of uncertainty removed. Entropy which is a measure of uncertainty was first introduced by Shannon (1948) [26] in communication theory. It is useful to estimate the probabilities of rare events (large deviation theory) and in the study of likelihood-based inference principles. Shannon entropy is defined as the average amount of information that we receive per event and was the first defined entropy. For continuous case, it is given by

$$H(X) = -\int_{0}^{\infty} f(x) \log f(x) dx,$$

where X is a non-negative absolutely continuous random variable with probability density function (pdf) f, cumulative distribution function (cdf) F and

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survival function (sf) $\bar{F} = 1 - F$. Shannon entropy has various applications in communication theory, mathematical, physical, engineering, biological and social sciences as well. For further details on entropy one may refer Ash (1990)\cite{2} and Cover and Thomas (2006)\cite{5}.

Rao et al. (2004)\cite{22} introduced the notion of cumulative residual entropy (CRE) as:

$$\mathcal{E}(X) = - \int_{0}^{\infty} \bar{F}(x) \log \bar{F}(x) dx.$$ 

Some general results regarding this measure have been studied by Drissi et al (2008)\cite{8} and Rao (2005)\cite{23}. CRE finds applications in image alignment and in the measurement of similarity between images. Di Crescenzo (2009)\cite{6} proposed an entropy called cumulative past entropy (or cumulative entropy) i.e. CPE (or CE) as:

$$\mathcal{C}\mathcal{E}(X) = - \int_{0}^{\infty} F(x) \log F(x) dx.$$ 

Asadi et al. (2007)\cite{1}, Di Crescenzo et al. (2013)\cite{7}, Khorashadizadeh et al. (2013)\cite{11} and Navarro et al. (2010)\cite{17} investigated many aspects of CRE (CPE).

Lad et al. (2015)\cite{15} defined an alternative measure of the uncertainty of a random variable called extropy. For continuous non-negative random variable $X$ the extropy is defined as:

$$J(X) = - \frac{1}{2} \int_{0}^{\infty} f^2(x) dx = - \frac{1}{2} E \left( f(X) \right). \quad (1.1)$$ 

Some results and properties of the extropy of order statistics and record values are given by Qiu (2017)\cite{19}. Qiu et al. (2018)\cite{20} derived some of the results of the residual extropy of order statistics. Yang et al. (2019)\cite{27} studied the bounds on extropy with a variational distance constraint. Qiu (2019)\cite{21} examined certain extropy properties of mixed systems. To find out more about extropy, one may refer to Krishnan et al. (2020)\cite{12}, Noughabi et al. (2019)\cite{18} and Raqab et al. (2019)\cite{24}.

Jahanshahi et al. (2019)\cite{10} introduced the cumulative residual extropy (CRJ). For continuous non-negative random variable $X$ the cumulative residual extropy (CRJ) as:

$$\xi J(X) = - \frac{1}{2} \int_{0}^{\infty} \bar{F}^2_X(x) dx.$$ 

Kundu (2021)\cite{14} proposed an extropy called cumulative past entropy (CPJ). For continuous non-negative random variable $X$ the cumulative past extropy (CPJ) is defined as:

$$\bar{\xi} J(X) = - \frac{1}{2} \int_{0}^{\infty} F^2_X(x) dx.$$ 

(1.3)
The idea behind this is to replace the density function by distribution function in extropy (1.1). Kundu (2021)\cite{14} studied extreme order statistics on cumulative residual (past) extropy. Hashempour et al. (2022)\cite{9} proposed a new measure called weighted cumulative residual extropy. Mohammadi (2022)\cite{16} studied a new measure called interval weighted cumulative residual extropy. For continuous non-negative random variable $X$ the weighted cumulative residual extropy (WCRJ) is defined as:

$$\xi_{wJ}(X) = -\frac{1}{2} \int_{0}^{\infty} x\hat{F}_X^2(x)dx.$$ \hspace{1cm} (1.4)

This paper is organized in the following manner. In Section 2 we introduce the weighted cumulative past extropy (WCPJ) and study some of its properties. In Section 3 some bounds and inequalities are derived. In Section 4, we study the characterization of WCPJ based on the largest-order statistic. In Section 5 we focus on related studies on reliability analysis. Conditional WCPJ and some of its properties are discussed in Section 6.

## 2 Weighted cumulative past extropy

Balakrishnan et al. (2020)\cite{3} and Bansal et al. (2021)\cite{4} independently introduced the weighted extropy. Mohammadi (2022)\cite{16} studied a new measure called interval weighted cumulative past extropy. Weighted cumulative past extropy (WCPJ) is an information measure, which is a generalization of cumulative past extropy. In this section, we study the properties of WCPJ.

**Definition 1** Let $X$ be a non-negative absolutely continuous random variable with cdf $F$. We define the WCPJ of $X$ by

$$\bar{\xi}_{wJ}(X) = -\frac{1}{2} \int_{0}^{\infty} x\hat{F}_X^2(x)dx.$$ \hspace{1cm} (2.1)

The integral (2.1) can be extended to the support of random variable $X$.

**Example 1** Let $X$ has $U[a,b]$ distribution. Then CRJ and WCRJ of the uniform distribution are

$$\xi_{J}(X) = -\frac{b - a}{6}, \text{ and, } \xi_{wJ}(X) = \frac{a - b}{24}(3a + b),$$

respectively. Then CPJ and WCPJ of the uniform distribution are

$$\bar{\xi}_{J}(X) = -\frac{b - a}{6}, \text{ and, } \bar{\xi}_{wJ}(X) = \frac{a - b}{24}(a + 3b),$$
respectively. Note that
\[ \bar{\xi}^w J(X) = \left( \frac{a + 3b}{4} \right) \xi J(X) = \left( \frac{E(X) + b}{2} \right) \xi J(X). \]

If \( \frac{E(X) + b}{2} > 1 \), then \( \bar{\xi}^w J(X) < \bar{\xi} J(X) \), and if \( \frac{E(X) + b}{2} < 1 \), then \( \bar{\xi}^w J(X) > \bar{\xi} J(X) \).

**Example 2** Let \( X \) follow power-law distribution with pdf \( f(x) = \lambda x^{-\lambda - 1}, x \in (0, 1), \lambda > 1 \). The CPJ and WCPJ of the distribution are
\[ \xi J(X) = -\frac{\lambda^2}{(\lambda + 1)(2\lambda + 1)}, \text{ and } \xi^w J(X) = -\frac{\lambda^2}{4(\lambda + 1)(\lambda + 2)}, \]
respectively. Note that
\[ \bar{\xi}^w J(X) = \left( \frac{2\lambda + 1}{4(\lambda + 2)} \right) \xi J(X) = \left( \frac{2(\lambda + 1)E(X) + 1}{4(\lambda + 2)} \right) \xi J(X). \]

For \( \lambda = \frac{-7}{2} \), \( \bar{\xi}^w J(X) = \bar{\xi} J(X) \). The CPJ and WCPJ of the distribution are
\[ \bar{\xi} J(X) = -\frac{1}{2(2\lambda + 1)}, \text{ and } \bar{\xi}^w J(X) = -\frac{1}{4(\lambda + 1)}. \]
We conclude that \( \bar{\xi}^w J(X) = \frac{2\lambda + 1}{4\lambda + 2} \bar{\xi} J(X) \). Also, \( \xi^w J(X) = -\frac{\lambda^2 + 2}{4\lambda + 1}E(X^2) \).

**Theorem 1** Let \( X \) be a non-negative continuous random variable with finite WCPJ, \( \bar{\xi}^w J(X) \). Then we have
\[ \bar{\xi}^w J(X) = -\frac{1}{2}E(G_F(X)), \]
where \( G_F(t) = \int_t^\infty xF(x)dx \).

**Proof** Using equation (2.1) and Fubini’s theorem, we have
\[
\bar{\xi}^w J(X) = \frac{-1}{2} \int_0^\infty xF^2(x)dx = \frac{-1}{2} \int_0^\infty xF(x) \left( \int_0^x f(t)dt \right) dx = \frac{-1}{2} \int_0^\infty f(t) \left( \int_t^\infty xF(x)dx \right) dt = \frac{-1}{2} \int_0^\infty f(t)G_F(t)dt = \frac{-1}{2}E(G_F(X)).
\]

Now we see the effect of linear transformation on WCPJ in the following proposition
Proposition 1 Let $X$ be a non-negative random variable. If $Y = aX + b$, $a > 0$, $b \geq 0$, then

$$\bar{\xi}^w J(Y) = a^2 \bar{\xi}^w J(X) + ab \bar{\xi} J(X)$$

**Proof** The proof holds using (2.1) and noting that $F_Y(y) = F_X \left( \frac{y-b}{a} \right)$, $y > b$.

Here we provide an upper bound for WCPJ in terms of extropy.

**Theorem 2** Let $X$ be a random variable with pdf $f(\cdot)$ and extropy $J(X)$, then

$$\bar{\xi}^w J(X) \leq C^* \exp \{2J(X)\}, \quad (2.2)$$

where $C^* = \frac{1}{2} \exp \{E \log (XF^2(X))\}$

**Proof** Using the log-sum inequality, we have

$$\int_0^\infty f(x) \log \left( \frac{f(x)}{xF^2(x)} \right) dx \geq - \log \left( \int_0^\infty xF^2(x) dx \right).$$

Then it follows that

$$\int_0^\infty f(x) \log f(x) dx - \int_0^\infty f(x) \log (xF^2(x)) dx \geq - \log \left( \int_0^\infty xF^2(x) dx \right).$$

Note that $\log f < f$, hence

$$- \int_0^\infty f^2(x) dx + \int_0^\infty f(x) \log (xF^2(x)) dx = 2J(X) + E \log (XF^2(X)) \leq \log (-2\bar{\xi}^w J(X)) \quad (2.3)$$

Exponentiating both sides of (2.3), we have

$$\bar{\xi}^w J(X) \leq \frac{1}{2} \exp \{2J(X) + E \log (XF^2(X))\}$$

Hence the result.

**Theorem 3** $X$ is degenerate, if and only if, $\bar{\xi}^w J(X) = 0$.

**Proof** Suppose $X$ be degenerate at point $c$, then by using the definition of degenerate function and $\bar{\xi}^w J(X)$, we have $\bar{\xi}^w J(X) = 0$. Now consider $\bar{\xi}^w J(X) = 0$, i.e., $\int_0^\infty xF^2(x) dx = 0$. Noting that the integrand in the above integral is non-negative, we have $F(x) = 0$, for almost all $x \in S$, where $S$ denotes the support of random variable $X$, i.e., it is 0 in $\inf S$ and then 1.
3 Some inequalities

This section deals with obtaining the lower and upper bounds for WCPJ.

Remark 1 Consider $X$ be a non-negative random variable. then
\[
\bar{\xi}^w J(X) \geq -\frac{1}{2} \int_{0}^{\infty} x F(x) dx. \tag{3.1}
\]

Proposition 2 Consider a non-negative continuous random variable $X$ having cdf $F_X(\cdot)$ and support $[a, \infty), \ a > 0$. Then
\[
\bar{\xi}^w J(X) \leq \frac{a}{2} \bar{\xi} J(X). \tag{3.2}
\]

Proof Note that
\[
\int_{a}^{\infty} x F^2(x) dx \geq \int_{a}^{\infty} F^2(x) dx
\]
\[
-\frac{1}{2} \int_{a}^{\infty} x F^2(x) dx \leq -\frac{a}{2} \int_{a}^{\infty} F^2(x) dx
\]
\[
\bar{\xi}^w J(X) \leq \frac{a}{2} \bar{\xi} J(X).
\]

\[\square\]

Corollary 1 Let $X$ be a continuous random variable with cdf $F$ that takes values on $[0, b]$ where $b$ is finite. Then,
1. $\bar{\xi}^w J(X) \leq \frac{1}{4} \left( b^2 - E(X^2) \right) \left[ \log \left( \frac{b^2 - E(X^2)}{b^2} \right) - 1 \right]$, 
2. $\bar{\xi}^w J(X) \geq b \bar{\xi} J(X)$.

Proof Using log-sum inequality, we have
\[
\int_{0}^{b} F(t) t \log (F(t)) dt \geq \int_{0}^{b} F(t) t dt \log \left( \frac{\int_{0}^{b} F(t) t dt}{\int_{0}^{b} t dt} \right)
= \left( \frac{b^2 - E(X^2)}{2} \right) \log \left( \frac{b^2 - E(X^2)}{b^2} \right)
\]
Also note that $\log F(t) \leq F(t) - 1$, then
\[
\int_{0}^{b} F(t) t \log F(t) dt \leq -2 \bar{\xi}^w J(X) - \int_{0}^{b} t F(t) dt
= -2 \bar{\xi}^w J(X) - \left( \frac{b^2 - E(X^2)}{2} \right)
\]
Now using the above two inequalities, the first part follows. The second part can be verified easily.

Consider two random variables $X$ and $Y$ having cdfs $F$ and $G$, respectively. Then $X \leq_{st} Y$ whenever $F(x) \geq G(x)$, $\forall x \in \mathbb{R}$; where the notation $X \leq_{st} Y$ means that $X$ is less than or equal to $Y$ in usual stochastic order. One may refer Shaked and Shanthikumar (2007)\cite{25} for detail of stochastic ordering. In the following proposition, we show the ordering of WCPJ is implied by the usual stochastic order.

**Proposition 3** Let $X_1$ and $X_2$ be non-negative continuous random variables. If $X_1 \leq_{st} X_2$, then $\tilde{\xi}^w J(X_1) \leq \tilde{\xi}^w J(X_2)$.

**Proof** Using $X_1 \leq_{st} X_2$ and (2.1), the result follows.

Qiu et al. (2019)\cite{21} showed that the extropy of the sum of two independent random variables is larger than that of either, $J(X + Y) \geq \max\{J(X), J(Y)\}$. Hashempour et al. (2022)\cite{9} obtained the following result for weighted cumulative residual extropy (WCRJ) as

$$\xi^w J(X + Y) \geq \max\{\xi J(X) E(Y) + \xi^w J(X), \xi J(Y) E(X) + \xi^w J(Y)\}.$$  

In the following theorem, we obtain a similar result for WCPJ.

**Theorem 4** For two non-negative and independent random variables $X$ and $Y$ with finite means

$$\tilde{\xi}^w J(X + Y) \geq \max\{\tilde{\xi} J(X) E(Y) + \tilde{\xi}^w J(X), \tilde{\xi} J(Y) E(X) + \tilde{\xi}^w J(Y)\}. \quad (3.3)$$

**Proof** By supposing $X$ and $Y$ are independent, we have

$$P(X + Y \leq t) = \int_0^t F(t - y)dF(y). \quad (3.4)$$

Using Jensen’s inequality, we obtain

$$P^2(X + Y \leq t) \leq \int_0^t F^2(t - y)dF(y). \quad (3.5)$$

Multiplying both sides of (2.5) by $\frac{1}{2}$ and integrating with respect to $t$ from 0 to $\infty$, we have

$$-\frac{1}{2} \int_0^\infty tP^2(X + Y \leq t) \geq -\frac{1}{2} \int_0^\infty \int_0^t tF^2(t - y)dF(y)dt$$

$$= -\frac{1}{2} \int_0^\infty \int_y^\infty tF^2(t - y)dtdF(y)$$

$$= -\frac{1}{2} \int_0^\infty \int_0^\infty (y + v)F^2(v)dvdF(y),$$

where $\tilde{\xi} = \xi E(X)$ and $\tilde{\xi}^w = \xi^w E(X)$.
where we used a change of variable $v = t - y$. So, we have

$$\bar{\xi}^w J(X + Y) \geq \bar{\xi}J(X)E(Y) + \bar{\xi}^w J(X)$$

Using the same arguments for the random variable $X$, the proof is completed. \hfill ■

### 4 WCPJ based on largest-order statistic

Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous cdf $F_X(x)$ and pdf $f_X(x)$. Then $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the ordered statistics to random sample $X_1, \ldots, X_n$. In the following, we obtain the WCPJ of the largest-order statistic. The WCPJ of the $n$th-order statistic is

$$\bar{\xi}^w J(X_{n:n}) = -\frac{1}{2} \int_0^\infty x F^2_{X_{n:n}}(x) dx,$$  \hspace{1cm} (4.1)

where $F^2_{X_{n:n}}(x) = F^2_{X}(x)$. Using transformation $u = F(x)$ in (3.1),

$$\bar{\xi}^w J(X_{n:n}) = -\frac{1}{2} \int_0^{F^{-1}} u^{2n} F^{-1}(u) \frac{du}{f(F^{-1}(u))},$$  \hspace{1cm} (4.2)

where $F^{-1}(x)$ is the inverse function of $F(x)$.

**Example 3** Let $X$ have the uniform distribution on $(0,1)$ with pdf $f(x) = 1$, $x \in (0,1)$. Then $F^{-1}(u) = u$, $u \in (0,1)$ and $f(F^{-1}(u)) = 1$, $u \in (0,1)$, hence $\bar{\xi}^w J(X_{n:n}) = -\frac{1}{4(n+1)}$.

**Example 4** let $X$ follow power-law distribution with pdf $f(x) = \lambda x^{\lambda-1}, \lambda > 1, x \in (0,1)$. Then $F^{-1}(u) = u^{\frac{1}{\lambda}}$, $u \in (0,1)$ and $f(F^{-1}(u)) = \lambda u^{\frac{1}{\lambda} - 1}, u \in (0,1)$, hence $\bar{\xi}^w J(X_{n:n}) = -\frac{1}{4(n\lambda + 1)}$.

**Remark 2** Consider $\Lambda = \bar{\xi}^w J(X_{n:n}) - \bar{\xi}^w J(X)$. Since $F^{2n}(x) \leq F^2(x)$, hence $\Lambda \geq 0$.

For the proof of Theorem 5, we need the following lemma.

**Lemma 1** [Lemma 4.1 of Hashempour et al. (2022)[9]] Let $g$ be a continuous function with support $[0,1]$, such that $\int_0^1 g(y)y^m dy = 0$, for $m \geq 0$, then $g(y) = 0$, $\forall y \in [0,1]$
Theorem 5 Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two non-negative random samples from continuous cdfs $F(x)$ and $G(x)$, respectively. Then $F(x) = G(x)$ if and only if $\bar{\xi}^w J(X_{n:n}) = \bar{\xi}^w J(Y_{n:n})$, for all $n$.

Proof The necessary condition is trivial. Hence, it remains to prove the sufficient part. If $\bar{\xi}^w J(X_{n:n}) = \bar{\xi}^w J(Y_{n:n})$, then we have

$$-\frac{1}{2} \int_0^1 u^{2n} \left( \frac{F^{-1}(u)}{f(F^{-1}(u))} - \frac{G^{-1}(u)}{g(G^{-1}(u))} \right) du = 0$$

By using Lemma 1, it follows that

$$\frac{F^{-1}(u)}{f(F^{-1}(u))} = \frac{G^{-1}(u)}{g(G^{-1}(u))} \Rightarrow F^{-1}(u) \frac{dF^{-1}(u)}{du} = G^{-1}(u) \frac{dG^{-1}(u)}{du}, \quad u \in [0,1],$$

since $\frac{dF^{-1}(u)}{du} = \frac{1}{f(F^{-1}(u))}$. Hence it follows $F^{-1}(u) = G^{-1}(u), \quad u \in [0,1]$. Thus the proof is completed. $\blacksquare$

5 Connection to reliability theory

Consider a non-negative continuous random variable $X$ with cdf $F$, such that $E(X) < \infty$. The mean inactivity time (MIT) of $X$ is defined as

$$MIT(t) = \int_0^t F(x) \frac{dx}{F(t)}, \quad t \geq 0. \quad (5.1)$$

The MIT function finds many applications in reliability, forensic science, and so on. In the following theorem, we show the relationship between WCPJ and the second moment of inactivity time (SMIT) function. For detail about SMIT one may refer Kundu et al. (2010) [13].

Definition 2 Let $X$ be a non-negative continuous random variable. Then SMIT is

$$SMIT(t) = E \left( (t - X)^2 | X \leq t \right) = 2tMIT(t) - \int_0^t 2x \frac{F(x)}{F(t)} dx, \quad t \geq 0. \quad (5.2)$$

Theorem 6 Let $X$ be a non-negative continuous random variable with SMIT function and weighted cumulative past extropy $\bar{\xi}^w J(X)$. Thus,

$$\bar{\xi}^w J(X) \leq C^* - \frac{1}{4} E(SMIT(X)), \quad (5.3)$$

where $C^* = \frac{1}{2} \left[ E \left( XMIT(X) \right) - \int_0^\infty xF(x) dx \right]$. 
Proof Consider
\[
E(\text{SMIT}(X)) = 2E(\text{XMIT}(X)) - 2E\left( \int_0^X \frac{xF(x)}{F(X)} \, dx \right)
\]
\[
= 2E(\text{XMIT}(X)) - 2 \int_0^\infty \int_0^t x\tilde{h}(t)F(x) \, dx \, dt
\]
\[
= 2E(\text{XMIT}(X)) - 2 \int_0^\infty xF(x) \left( \int_x^\infty \tilde{h}(t) \, dt \right) \, dx
\]
\[
= 2E(\text{XMIT}(X)) - 2 \int_0^\infty xF(x) \log F(x) \, dx
\]
\[
\leq 2E(\text{XMIT}(X)) - 2 \int_0^\infty xF(x) \, dx + 2 \int_0^\infty xF^2(x) \, dx
\]
\[
= 2E(\text{XMIT}(X)) - 2 \int_0^\infty xF(x) \, dx - 4\bar{\xi}^w J(X), \tag{5.4}
\]
where \(\tilde{h}(\cdot)\) is the reversed hazard rate function. Now from (5.4), (5.3) follows.

If some information about SMIT or its behavior is given, then (5.3) may be used. Another bound for \(\bar{\xi}^w J(X)\) can be provided in terms of hazard rate function.

**Proposition 4** Let \(X\) be a non-negative continuous random variable with finite hazard rate function \(h(\cdot)\) and \(\bar{\xi}^w J(X)\). Then,
\[
\bar{\xi}^w J(X) \geq E(S(X)), \tag{5.5}
\]
where \(S(x) = -\frac{1}{2} \int_t^\infty x \left( \int_0^x h(v) \, dv \right) \, dx\), here \(h(x)\) is hazard rate function.

**Proof** Consider
\[
\bar{\xi}^w J(X) = -\frac{1}{2} \int_0^\infty xF^2(x) \, dx
\]
\[
= -\frac{1}{2} \int_0^\infty xF(x) \left( \int_0^x f(t) \, dt \right) \, dx
\]
\[
= -\frac{1}{2} \int_0^\infty f(t) \left( \int_t^\infty xF(x) \, dx \right) \, dt
\]
\[
\geq \frac{1}{2} \int_0^\infty f(t) \left( \int_t^\infty x \log F(x) \, dx \right) \, dt
\]
\[
= -\frac{1}{2} \int_0^\infty f(t) \left( \int_t^\infty x \left( \int_0^x h(v) \, dv \right) \, dx \right) \, dt
\]
\[
= E(S(X)),
\]
where \(S(x) = -\frac{1}{2} \int_t^\infty x \left( \int_0^x h(v) \, dv \right) \, dx\). Hence the result.
6 Conditional weighted past extropy

Now we consider the conditional weighted cumulative past extropy (CWCPJ). Consider a random variable $Z$ on probability space $(\Omega, \mathcal{A}, P)$ such that $E|Z| < \infty$. The conditional expectation of $Z$ given sub $\sigma$-field $\mathcal{G}$, where $\mathcal{G} \subseteq \mathcal{A}$, is denoted by $E(Z|\mathcal{G})$. For the random variable $I(Z \leq z)$, we denote $E(I(Z \leq z)|\mathcal{G})$ by $F_Z(z|\mathcal{G})$.

**Definition 3** For a non-negative random variable $X$, given $\sigma$-field $\mathcal{G}$, the CWCPJ $\bar{\xi}_w J(X|\mathcal{G})$ is defined as

$$\bar{\xi}_w J(X|\mathcal{G}) = -\frac{1}{2} \int_0^\infty x F_X^2(x|\mathcal{G}) dx. \quad (6.1)$$

Now we assume that the random variables are continuous and non-negative.

**Lemma 2** If $\mathcal{G}$ is a trivial $\sigma$-field, then $\bar{\xi}_w J(X|\mathcal{G}) = \bar{\xi}_w J(X)$.

**Proof** Since here $F_X(x|\mathcal{G}) = F_X(x)$, then the proof follows. $lacksquare$

**Proposition 5** If $X \in L^p$ for some $p > 2$, then $E[\bar{\xi}_w J(X|\mathcal{G})|\mathcal{G}^*] \leq \bar{\xi}_w J(X|\mathcal{G}^*)$, provided that $\mathcal{G}^* \subseteq \mathcal{G}$.

**Proof** Consider

$$E[\bar{\xi}_w J(X|\mathcal{G})|\mathcal{G}^*] = -\frac{1}{2} \int_0^\infty x E \left( |P(X \leq x|\mathcal{G})|^2 |\mathcal{G}^* \right) dx$$

$$\leq -\frac{1}{2} \int_0^\infty x E \left( (P(X \leq x|\mathcal{G})|^2 |\mathcal{G}^* \right) dx$$

$$= -\frac{1}{2} \int_0^\infty x \left[ E \left( E(I(X \leq x)|\mathcal{G}) |\mathcal{G}^* \right) \right] dx$$

$$= -\frac{1}{2} \int_0^\infty x F_X^2(x|\mathcal{G}^*) dx$$

$$= \bar{\xi}_w J(X|\mathcal{G}^*),$$

where the second step follows using the Jensen’s inequality for convex function $\phi(x) = x^2$. Hence the result. $lacksquare$

In the following theorem, we investigate the relationship between conditional extropy and $\bar{\xi}_w J(X|\mathcal{G})$.
Theorem 7 Let $\xi^w J(X|G)$ is conditional past extropy. Then we have

$$\bar{\xi}^w J(X|G) \leq B^* \exp\{2J(X|G)\}, \quad (6.2)$$

where $B^* = \frac{1}{2} \exp\{E \left( \log(XF^2(X))|G \right) \}$

Proof The proof is on the similar lines as of Theorem 2, hence omitted. ■

Theorem 8 For a random variable $X$ and $\sigma$-field $G$, we have

$$E \left( \bar{\xi}^w J(X|G) \right) \leq \bar{\xi}^w J(X), \quad (6.3)$$

and the equality holds if and only if $X$ is independent of $G$.

Proof If in Proposition 5, $G^*$ is trivial $\sigma$-field, then (6.3) can be easily obtained. Now assume that $X$ is independent of $G$, then

$$F_X(x|G) = F_X(x) \implies \bar{\xi}^w J(X|G) = \bar{\xi}^w J(X). \quad (6.4)$$

On taking expectation to both sides of (6.4), we get equality in (6.3). Conversely, assume that equality in (6.3) holds. It is sufficient to show that $F_X(x|G) = F_X(x)$, to prove independence between $X$ and $\sigma$-field $G$. Take $U = F_X(x|G)$, and since the function $\phi(u) = u^2$ is convex hence $E(U^2) \geq E^2(U) = F_X^2(x)$, and also due to equality in (6.3), we have

$$\int_0^\infty xe^{U^2}dx = \int_0^\infty xF^2_X(x)dx = \int_0^\infty xe^{2U}dx.$$

Hence $E(U^2) = E^2(U)$. Now using the Corollary 8.1 of Hashempour et al. (2022) we have $F_X(x|G) = F_X(x)$. Hence the proof.

For the Markov property for non-negative random variables $X$, $Y$ and $Z$, we have the following proposition.

Proposition 6 Let $X \to Y \to Z$ is a Markov chain, then

$$\bar{\xi}^w J(Z|X,Y) = \bar{\xi}^w J(Z|Y) \quad (6.5)$$

and

$$E \left( \bar{\xi}^w J(Z|Y) \right) \leq E \left( \bar{\xi}^w J(Z|X) \right). \quad (6.6)$$

Proof By the definition of $\xi^w J(Z|X,Y)$ and using the Markov property, (6.5) holds.

Now letting $G^* = \sigma(X), \; G = \sigma(X,Y)$ and $X = Z$ in Proposition 5 we have

$$\bar{\xi}^w J(Z|X) \geq E \left( \bar{\xi}^w J(Z|X,Y)|X \right) \quad (6.7)$$
Taking expectation on both sides of (6.7), we have

\[
E \left( \tilde{\xi}^w J(Z|X) \right) \geq E \left( E \left( \tilde{\xi}^w J(Z|X,Y)|X \right) \right) \\
= E \left( \tilde{\xi}^w J(Z|X,Y) \right) \\
= E \left( \tilde{\xi}^w J(Z|Y) \right),
\]

where the last equality holds using (6.5). Hence the result (6.6) holds. ■

Conflict of interest

The authors declare no conflict of interest.

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