A WEIGHTED RELATIVE ISOPERIMETRIC INEQUALITY IN CONVEX CONES

EMANUEL INDREI

ABSTRACT. A weighted relative isoperimetric inequality in convex cones is obtained via the Monge-Ampere equation. The method improves several inequalities in the literature, e.g. constants in a theorem of Cabre–Ros–Oton–Serra. Applications are given in the context of a generalization of the log-convex density conjecture due to Brakke and resolved by Chambers: in the case of $\alpha$–homogeneous ($\alpha > 0$), concave densities, (mod translations) balls centered at the origin and intersected with the cone are proved to uniquely minimize the weighted perimeter with a weighted mass constraint. In particular, if the cone is taken to be $\{ x_n > 0 \}$, reflecting the density, balls intersected with $\{ x_n > 0 \}$ remain (mod translations) unique minimizers in the $\mathbb{R}^n$ analog in the case when the density vanishes on $\{ x_n = 0 \}$.

Suppose $\mathcal{C} \subset \mathbb{R}^n$ is an open, convex cone ($n \geq 2$). If $|E| < \infty$ is a set of finite perimeter with reduced boundary $\mathcal{F}E$, $K = B_1 \cap \mathcal{C}$, then

$$n|E|^\frac{n-1}{n}|K|^{\frac{1}{n}} \leq \mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{C})$$

where equality holds if and only if $E = \text{mod translations}$ (if the cone contains lines) $sK$ for $s > 0$. If $|E| = |K|$, the inequality states

$$n|E| \leq \mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{C})$$

and this via scaling is equivalent; a sharp stability result for this inequality was proved in Figalli-Indrei [FI13]: the simple version states that if $\mathcal{C}$ contains no line

$$|E \Delta K| \lesssim \left( \frac{\mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{C})}{n|E|} - 1 \right)^{\frac{1}{2}}$$

($E \Delta K$ is the symmetric difference). In the theorem below, a weighted version of the relative isoperimetric inequality in convex cones is shown. The proof is based on the Monge-Ampere equation [Caf96, Caf92, Urb97, TW08]

$$\det D^2 \phi = f.$$
Theorem 0.1. If $C \subset \mathbb{R}^n_+$ is an open convex cone, $E \subset C$ is a set of finite perimeter with $|E| = |K|$, $h(x) \geq 0$, $\inf_{a \in K} \nabla h \cdot a \geq 0$, then

$$\int_E nh(x)dx \leq \int_{FE \cap C} h(x)d\mathcal{H}^{n-1},$$

moreover, if $C \subset \mathbb{R}^n_+$, $h = h(x_n) > 0$ for $x_n > 0$, & equality holds, then (up to sets of measure zero) $E = K$ (if $E = K$, then

$$\int_K nhdx + \int_K \nabla h \cdot xdx = \int_{\partial K \cap C} h d\mathcal{H}^{n-1}).$$

Proof. Let $d\mu^+ = \chi_E dx$, $d\mu^- = \chi_K dy$ and $T\# \mu^+ = d\mu^-$ denote the optimal map (i.e. $T = \nabla \phi$, $\phi$ convex) then for a.e. $x \in E$

$$\det DT(x) = 1.$$ 

Thus,

$$\int_E nhdx = \int_E n(\det DT)^{\frac{1}{n}} hdx$$

$$\leq \int_E (\text{div} T) hdx = \int_E \text{div} Thdx - \int_E \nabla h \cdot Tdx$$

$$\leq \int_E \text{div} Thdx$$

$$\leq \int_{E^{(1)}} \text{div} Thdx + (\text{Div} Th)_{s}(E^{(1)})$$

$$= \text{Div} Th(E^{(1)})$$

$$= \int_{FE} \text{tr}_E(Th) \cdot \nu_E(x) d\mathcal{H}^{n-1}$$

$$\leq \int_{FE \cap C} h d\mathcal{H}^{n-1}$$

($E^{(1)}$ is the set of density 1, $(\text{Div} Th)_{s}$ is the singular part of the measure $\text{Div} Th$, and $\text{tr}_E$ is the trace, see [FI13]). Let

(1) $$\int_E nh(x_n)dx = \int_{FE \cap C} h(x_n) d\mathcal{H}^{n-1};$$

in particular, there is a.e. equality in the arithmetic-geometric mean inequality which yields $DT = \text{Id}$ a.e. and

$$\int_{FE} \text{tr}_E(Th) \cdot \nu_E(x) d\mathcal{H}^{n-1} = \int_{FE \cap C} h d\mathcal{H}^{n-1}$$
implies $T(x) = x + x_0$ for $x_0 \in \mathbb{R}^n$. Therefore $E$ is connected and since
\[
\int_E T_n h'(x_n) dx = 0,
\]
h' = 0 a.e. on $E$; in particular $h$ is constant on $E$ and (I) implies $x_0 = 0$. (If $E = K$, then $T(x) = x$.)

**Remark 0.2.** In the case when $C$ contains a line, equality holds up to translations along the line (a general convex cone splits: $C = C_k \times \mathbb{R}^{n-k}$, where $C_k \subset \mathbb{R}^k$ is a convex cone containing no line [FI13, RV15]).

**Remark 0.3.** In the function space \( \{ f : f = h \chi_E, \inf_{a \in K} \nabla h \cdot a \geq 0, |E| = |K| \} \), the inequality is sharp: equality is attained for $f = \alpha \chi_K$, $\alpha \in \mathbb{R}$.

**Remark 0.4.** Weighted isoperimetric and/or Sobolev inequalities were studied in [CROS16, CnR14, BGK, MR14, DHHT12, ABC+19b, BC20, ABC+19a, BCM12, CFR20, CJQW08, MP13, CRO13, DDNT10, CGP+20, Bob96, CK01, BH97]. Furthermore, I recently proved the following result which answers a long-standing open problem due to Almgren (the anisotropic isoperimetric problem is considered in a convex background) [Ind]: let
\[
f : \mathbb{R}^2 \to [0, \infty)
\]
be a surface tension (a convex positively 1-homogeneous function),
\[
\mathcal{P}(E) = \int_{\partial E} f(\nu_E) d\mathcal{H}^{n-1};
\]
\[
\mathcal{G}(E) = \int_E g(x) dx;
\]
\[
\mathcal{E}(E) = \mathcal{P}(E) + \mathcal{G}(E).
\]
Suppose $g : \mathbb{R}^2 \to [0, \infty)$ is coercive and convex. If $m \in (0, \infty)$, there exists an – up to translations & sets of measure zero – unique convex set which minimizes the free energy $\mathcal{E}$ among sets with measure $m$ (inter-alia I proved non-quantitative stability). Moreover, there exists a convex $g$ such that there are no solutions for any $m > 0$.

**Corollary 0.5.** If $C \subset \mathbb{R}^n_+$ is an open convex cone, $E_\epsilon \subset C$ is a set of finite perimeter with $|E_\epsilon| = |K|$, $h(x) \geq 0$,
\[
\int_K \nabla h \cdot x dx > 0,
\]
\[
E_\epsilon \to K
\]
in $L^1_{\text{loc}}$, then there exists $a(\epsilon) > 0$, such that $a(\epsilon) \to 0$ as $\epsilon \to 0^+$ (mod a subsequence), &

$$-a(\epsilon) \int_{\partial K \cap C} h d\mathcal{H}^{n-1} \leq \int_{\partial E \cap C} h d\mathcal{H}^{n-1} - \int_{\partial K \cap C} h d\mathcal{H}^{n-1}. $$

Hence, if $\frac{a(\epsilon)}{\alpha_\epsilon} \to 0^+$,

$$\lim \inf_{\epsilon \to 0^+} \frac{1}{\alpha_\epsilon} \left( \int_{\partial E \cap C} h d\mathcal{H}^{n-1} - \int_{\partial K \cap C} h d\mathcal{H}^{n-1} \right) \geq 0. $$

Proof. Let $d\mu^+_\epsilon = \chi_{E \epsilon} dx$, $d\mu^-_\epsilon = \chi_K dy$ and $(T_\epsilon)\#d\mu^+_\epsilon = d\mu^-_\epsilon$ denote the optimal map. Since $T_\epsilon \chi_{E \epsilon} \rightharpoonup x \chi_K$ (along a subsequence)

$$\left| \int_{E \epsilon} T_\epsilon \cdot \nabla h dx - \int_K x \cdot \nabla h dx \right| \leq a(\epsilon) \int_K \nabla h \cdot x dx$$

for $a(\epsilon) \to 0$ ($\int_{E \epsilon} |T_\epsilon|^2 dx = \int_K |x|^2 dx$, $\lim \sup_{\epsilon \to 0} \int_{E \epsilon} |x|^2 dx < \infty$). Hence for $\epsilon$ sufficiently small,

$$\int_{E \epsilon} T_\epsilon \cdot \nabla h dx \geq (1 - a(\epsilon)) \int_K x \cdot \nabla h dx,$$

$$n \int_{E \epsilon} h dx \geq (1 - a(\epsilon)) n \int_K h dx,$$

and the proof of the theorem implies the inequality. \qed

Remark 0.6. If $|E| \neq |K|$, the previous arguments apply to $aE$ such that $|aE| = |K|$.

The log-convex density conjecture is stated in terms of the following formulation: if $h : \mathbb{R}^n \to [0, \infty)$ is radially log-convex with a smooth density ($h(x) = e^{\Phi(|x|)}$, $\Phi$ convex, smooth), then a ball $B$ around the origin such that $\int_B h dx = m$ solves

$$\inf_{\int_E h dx = m} \int_{\mathcal{F} E} h d\mathcal{H}^{n-1}. $$

There is a large collection of papers in the literature on this conjecture, e.g. [RCnBM08, Cha19, McG18, FM13, KZ11, Mor16] (see also Remark 0.4), and results vary in terms of the assumptions on $\Phi$: if $\Phi \in C^3(0, \infty)$, the above was proved by Chambers (McGillivray obtained this in $\mathbb{R}^2$ assuming $\Phi$ was non-decreasing, convex). The next theorem yields the analog in convex cones for $\alpha-$homogeneous, concave functions $h$ and characterizes the minimizers–up to translations–uniquely (see Corollary 0.9).
Theorem 0.7. If \( C \subset \mathbb{R}^n_+ \) is an open convex cone, \( E \subset C \) is a set of finite perimeter, \( h(x) \geq 0 \) is concave and \( \alpha \)-homogeneous, \( \alpha \geq 0 \), then

\[
(n + \alpha - 1) \left( \frac{|K|}{|E|} \right)^{\frac{1}{n}} \int_E hdx + \frac{1}{(\alpha + n + 1)^{\frac{1}{n}}} \int_K hdx \leq \int_{F \cap C} h d\mathcal{H}^{n-1}.
\]

Suppose \( h(x) > 0 \) for some \( x \in C \) and that equality holds, then \( aE = (\text{mod translations}) K \).

Proof. If \( \alpha = 0 \), \( h = \text{constant} \), and the claim follows via the relative isoperimetric inequality in convex cones; therefore without loss of generality, \( \alpha > 0 \).

Let \( |E| = |K| \), \( d\mu^+ = \chi_E dx \), \( d\mu^- = \chi_K dy \), and \( T\#\mu^+ = d\mu^- \) denote the optimal map (i.e. \( T = \nabla \phi \), \( \phi \) convex) then for a.e. \( x \in E \)

\[
\det DT(x) = 1.
\]

Note that since \( h \) is \( \alpha \)-homogeneous, \( ah(x) = \nabla h(x) \cdot x \) and concavity yields

\[
\nabla h(x) \cdot T = \nabla h(x) \cdot (T - x) + ah(x) \geq h(T) - h(x)(1 - \alpha).
\]

Thus

\[
\int_E nhdx = \int_E n \left( \det DT \right)^{\frac{1}{n}} hdx
\]

\[
\leq \int_E (\text{div}T)hdx = \int_E \text{div}Thdx - \int_E \nabla h \cdot Tdx
\]

\[
\leq \int_E \text{div}Thdx - \int_K hdx + (1 - \alpha) \int_E h(x)dx
\]

\[
\leq \int_{E^{(1)}} \text{div}Thdx + (\text{DivTh})_s(E^{(1)}) - \int_K hdx + (1 - \alpha) \int_E h(x)dx
\]

\[
= \text{DivTh}(E^{(1)}) - \int_K hdx + (1 - \alpha) \int_E h(x)dx
\]

\[
= \int_{F \cap C} \text{tr}_E(Th) \cdot \nu_E(x) d\mathcal{H}^{n-1} - \int_K hdx + (1 - \alpha) \int_E h(x)dx
\]

\[
\leq \int_{F \cap C} h d\mathcal{H}^{n-1} - \int_K hdx + (1 - \alpha) \int_E h(x)dx
\]

\((E^{(1)}) \) is the set of density 1, \((\text{DivTh})_s\) is the singular part of the measure \(\text{DivTh}\), and \(\text{tr}_E\) is the trace). Thus

\[
(n + \alpha - 1) \int_E hdx + \int_K hdx \leq \int_{F \cap C} h d\mathcal{H}^{n-1}.
\]

In the case of equality, if \( C \) does not contain a line note that

\[
\nabla h(x) \cdot T = h(T) - h(x)(1 - \alpha)
\]
a.e., and thus in the case of strictly concave densities $h$, $T(x) = x$ a.e.; if $\mathcal{C}$ contains a line, equality holds up to translations; in the case when $h$ is concave but not strictly concave,

$$\int_E (\det DT) \frac{h}{n} \, dx = \int_E (\text{div} T) h \, dx;$$

in particular, there is a.e. equality in the arithmetic-geometric mean inequality which yields $DT = \text{Id}$ a.e. and

$$\int_{\mathcal{F}} \text{tr}_E (Th) \cdot \nu_E(x) d\mathcal{H}^{n-1} = \int_{\mathcal{F} \cap C} h d\mathcal{H}^{n-1}$$

implies $T(x) = x + x_0$ for $x_0 \in \mathbb{R}^n$. Suppose $|E| \neq |K|$ and let $|aE| = |K|$. The homogeneity implies

$$\int_{aE} h \, dx = a^{n+\alpha} \int_E h \, dx,$$

$$\int_{\mathcal{F}(aE) \cap C} h d\mathcal{H}^{n-1} = a^{n+\alpha-1} \int_{\mathcal{F} \cap C} h d\mathcal{H}^{n-1},$$

thus

$$(n + \alpha - 1) \left( \frac{|K|}{|E|} \right)^{\frac{1}{n}} \int_E h \, dx + \frac{1}{\left( \frac{|K|}{|E|} \right)^{\frac{n+\alpha-1}{n}}} \int_K h \, dx \leq \int_{\mathcal{F} \cap C} h d\mathcal{H}^{n-1}. $$

Remark 0.8. In the case of strictly concave $\alpha$--homogeneous densities $h \geq 0$, if $\mathcal{C} \subseteq \mathbb{R}^n_+$ and equality holds, then $aE = K$.

Corollary 0.9. If $\mathcal{C} \subset \mathbb{R}^n_+$ is an open convex cone, $E \subset \mathcal{C}$ is a set of finite perimeter, $h(x) \geq 0$ is concave and $\alpha$--homogeneous, $\alpha \geq 0$, then if $\int_E h \, dx = \int_K h \, dx$,

$$\frac{1}{n + \alpha} \left( n + \alpha - 1 \right) \left( \frac{|K|}{|E|} \right)^{\frac{1}{n}} + \frac{1}{\left( \frac{|K|}{|E|} \right)^{\frac{n+\alpha-1}{n}}} \int_{\partial K \cap C} h d\mathcal{H}^{n-1} \leq \int_{\mathcal{F} \cap C} h d\mathcal{H}^{n-1},$$

and equality holds if and only if $aE = (\text{mod translations}) K$, $a > 0$; in particular

$$\int_{\partial K \cap C} h d\mathcal{H}^{n-1} = \inf_{E: hdx = \int_K h \, dx, \mathcal{F} \cap C} \int_{\mathcal{F} \cap C} h d\mathcal{H}^{n-1},$$

and the infimum is attained uniquely by $E = (\text{mod translations}) K$.

Proof. First, the homogeneity implies (without loss $\alpha > 0$)

$$\int_K h \, dx = \frac{1}{n + \alpha} \int_{\partial K \cap C} h d\mathcal{H}^{n-1},$$
and the first result follows from the theorem. Set
\[ g(a) = ka + \frac{1}{a^k}, \]
a \geq 0, k = n + \alpha - 1; a simple calculation shows that the minimum is attained at \( a = 1 \), and this yields
\[ \int_{\partial K \cap C} h dH^{n-1} \leq \int_{\mathcal{F} \cap C} h dH^{n-1} \]
when \( \int_E h dx = \int_K h dx \). Suppose equality is attained, then
\[ \frac{1}{n + \alpha} \left( (n + \alpha - 1) \left( \frac{|K|}{|E|} \right)^{\frac{1}{n}} + \frac{1}{(\frac{|K|}{|E|})^{\frac{n+\alpha-1}{n}}} \right) = 1, \]
and this implies \( |E| = |K| \); hence \( E = \text{ (mod translations) } K \) by the first claim.

The inequality in Corollary 0.9 implies a quantitative estimate, cf. [CGP+20, Ind16].

**Corollary 0.10.** If \( C \subset \mathbb{R}_+^n \) is an open convex cone, \( E \subset C \) is a set of finite perimeter, \( h(x) \geq 0 \) is concave and \( \alpha \)-homogeneous, \( \alpha \geq 0 \), then if \( \int_E h dx = \int_K h dx \),
\[ \int_{\mathcal{F} \cap C} h dH^{n-1} - \int_{\partial K \cap C} h dH^{n-1} \geq \int_{\partial K \cap C} h dH^{n-1} \left( 1 - \frac{1}{n + \alpha} \left( (n+\alpha-1) \left( \frac{|K|}{|E|} \right)^{\frac{1}{n}} + \frac{1}{(\frac{|K|}{|E|})^{\frac{n+\alpha-1}{n}}} \right) \right), \]
and
\[ \int_{\mathcal{F} \cap C} h dH^{n-1} = \int_{\partial K \cap C} h dH^{n-1} \]
if and only if \( E = \text{ (mod translations) } K \).

**Remark 0.11.** \( \alpha \)-homogeneous functions appear as blow-up limits in free boundary problems [Ind20, Ind19b, Ind19a].

**Corollary 0.12.** If \( h(x) \geq 0 \) is concave, \( \alpha \)-homogeneous for \( x \in \{ x_n > 0 \} \) (\( \alpha > 0 \)), \( h(x',0) = 0 \), \( h(x',x_n) = h(x',-x_n) \), then
\[ \int_{\partial B_R \cap \{ x_n > 0 \}} h dH^{n-1} = \inf_{\int_E h dx = \int_{B_1} h dx, E \subset \mathbb{R}^n} \int_{\mathcal{F} \cap C} h dH^{n-1}, \]
\( R = 2^{\frac{n+\alpha}{n}} \) and the infimum is attained in the collection of sets of finite perimeter uniquely by \( E = \text{ (mod translations) } B_R \cap \{ x_n > 0 \} \).
Proof. If \( E \subset \mathbb{R}^n \) is a minimizer, let \( E^+ = E \cap \{x_n > 0\} \), \( E^- = E \cap \{x_n < 0\} \),
\[
\int_{E^+} h \, dx + \int_{E^-} h \, dx = \int_{B_1} h \, dx.
\]
The theorem implies
\[
\int_{\partial E^+ \cap \{x_n > 0\}} h \, d\mathcal{H}^{n-1} + \int_{E^+} h \, d\mathcal{H}^{n-1} \geq \int_{\partial B_{R_1} \cap \{x_n > 0\}} h \, d\mathcal{H}^{n-1} + \int_{\partial B_{R_2} \cap \{x_n < 0\}} h \, d\mathcal{H}^{n-1},
\]
\[
\int_{B_{R_1} \cap \{x_n > 0\}} h \, dx = \int_{E^+} h \, dx, \quad \int_{B_{R_2} \cap \{x_n < 0\}} h \, dx = \int_{E^-} h \, dx;
\]
in particular,
\[
\int_{\partial E} h \, d\mathcal{H}^{n-1} \geq \frac{1}{2} \left( R_1^{n+\alpha-1} + R_2^{n+\alpha-1} \right) \int_{\partial B_1} h \, d\mathcal{H}^{n-1},
\]
\[
\frac{1}{2} \left( R_1^{n+\alpha} + R_2^{n+\alpha} \right) = 1;
\]
and minimizing
\[
R_1^{n+\alpha-1} + R_2^{n+\alpha-1}
\]
subject to
\[
\frac{1}{2} \left( R_1^{n+\alpha} + R_2^{n+\alpha} \right) = 1
\]
implies \( R_1 = 2^{\frac{1}{n+\alpha}}, R_2 = 0 \) (without loss).

\( \square \)

Remark 0.13. Let \( h(x, y) = y \) if \( y \geq 0 \) and extend via even reflection. Note that
\[
\int_{B_R^+} h \, dxdy = R^3, \quad \int_{B_R^+} h \, dx = R^3, \quad \int_0^\pi \sin \theta r \, dr \, d\theta = R^3;
\]
\[
\int_{B_R} h \, dxdy = 2R^2;
\]
\[
\int_{\partial B_R \cap \{y > 0\}} h \, d\mathcal{H}^1 = R^2, \quad \int_{\partial B_1 \cap \{y > 0\}} h \, d\mathcal{H}^1 = R^2, \quad \int_0^\pi \sin \theta d\theta = 2R^2;
\]
\[
\int_{\partial B_R} h \, d\mathcal{H}^1 = 4R^2;
\]
this implies
\[
\int_{B_{R^*}} h \, dxdy = 2R^3 = \int_{B_R^+} h \, dxdy = R^3
\]
if and only if
\[ R_\ast = \frac{R}{2^\frac{\alpha}{n}}; \]
thus
\[ \int_{\partial B_{R_\ast}} hd\mathcal{H}^1 = 4R_\ast^2 = \frac{2}{2^\frac{\alpha}{n}} \int_{\partial B_R \cap \{y > 0\}} hd\mathcal{H}^1 > \int_{\partial B_R \cap \{y > 0\}} hd\mathcal{H}^1. \]

In the next corollary, the “concave–\(\alpha\) analog” of Theorem 0.7 is stated which is a strict improvement of a “concave–\(\frac{1}{\alpha}\) analog” of a theorem in [CROS16], see Remark 0.15; the equality cases have only recently appeared on arXiv [CGP+C20] after the first version of this paper was submitted.

**Corollary 0.14.** If \( C \subset \mathbb{R}_+^n \) is an open convex cone, \( E \subset C \) is a set of finite perimeter, \( h(x) \geq 0 \) is concave and \( \alpha \)-homogeneous, then
\[
\int_{\partial K \cap C} hdx \leq \frac{(n + \alpha)(\int_K hdx)(\frac{|K|}{|E|})^{\frac{n+\alpha-1}{n}}}{(\int_E hdx) + (n + \alpha - 1)(\frac{|K|}{|E|})^{\frac{n+\alpha}{n}}} \int_{\mathcal{F}_E \cap C} hd\mathcal{H}^{n-1}.
\]
Suppose \( h(x) > 0 \) for some \( |x| \neq 0 \) and that equality holds, then \( aE = (\text{mod translations}) K \).

**Proof.** The inequality is immediate via Theorem 0.7. First, assume \( |E| = |K| \) and that equality holds; then
\[
\int_E n (\det DT)^{\frac{1}{n}} hdx = \int_E (\text{div} T) hdx;
\]
in particular, there is a.e. equality in the arithmetic-geometric mean inequality which yields \( DT = \text{Id} \) a.e. and
\[
\int_{\mathcal{F}_E} \mathbf{tr}_E(T h) \cdot \nu_E(x) d\mathcal{H}^{n-1} = \int_{\mathcal{F}_E \cap C} hdx
\]
implies \( T(x) = x + x_0 \) for \( x_0 \in \mathbb{R}^n \). If \( |E| \neq |K| \), let \( |aE| = |K| \); the result follows via the previous argument applied to \( aE \). \( \square \)

**Remark 0.15.** In [CROS16], the authors obtain the following: if \( C \subset \mathbb{R}_+^n \) is an open convex cone, \( E \subset C \) is a set of finite perimeter, \( h(x) \geq 0 \) is \( \alpha \)-homogeneous & \( h^{\frac{1}{\alpha}} \) is concave, then
\[
\int_{\partial K \cap C} hdx \leq \left( \frac{\int_K hdx}{\int_E hdx} \right)^{\frac{n+\alpha-1}{n+\alpha}} \int_{\mathcal{F}_E \cap C} hd\mathcal{H}^{n-1}.
\]
Define \( \bar{f}(x,y) := \frac{(n+\alpha)xy^{\frac{n+\alpha-1}{n+\alpha}}}{x+(n+\alpha-1)y} \) for \( x, y > 0 \). Then whenever the function \( h \) is concave, \( \alpha \)-homogeneous and \( h^{\frac{1}{\alpha}} \) is concave (e.g. \( 0 < \alpha \leq 1 \)) Corollary 0.14
improves the inequality if and only if
\[ f(x, y) < x^{\frac{n+\alpha-1}{n+\alpha}}. \]

Set
\[ f(x, y) = x^{-\frac{1}{n+\alpha}} - \frac{(n + \alpha)y^{\frac{n+\alpha-1}{n}}}{x + (n + \alpha - 1)y^{\frac{n+\alpha}{n}}} = x^{-\frac{1}{n+\alpha}} - \frac{(n + \alpha)}{xy^{\frac{1-n-\alpha}{n}} + (n + \alpha - 1)y^{-\alpha}}; \]
fix \( y > 0 \) and note that
\[ f(x, y) \to \infty, \quad x \to 0^+; \]
\[ f(x, y) \to 0, \quad x \to \infty; \]
if \( x > 0 \) is fixed,
\[ f(x, y) \to x^{-\frac{1}{n+\alpha}}, \quad y \to 0^+, \infty; \]
\[ \partial_y f(x, y) = \frac{(n + \alpha)}{n} \left( \frac{(n + \alpha - 1)y^{\frac{1}{n}} + (1 - n - \alpha)xy^{\frac{1-n-\alpha}{n}}}{xy^{\frac{1-n-\alpha}{n}} + (n + \alpha - 1)y^{-\alpha}} \right) \]
and thus
\[ \partial_y f(x, y) = 0 \]
when \( y(x) = x^{\frac{n}{n+\alpha}} \) and
\[ f(x, y(x)) = 0. \]
This yields that if \( x, y > 0, f(x, y) \geq 0 \) with equality if and only if \( y = x^{\frac{n}{n+\alpha}} \).
If \( |E| = |K| \),
\[ f(x, 1) = x^{-\frac{1}{n+\alpha}} - \frac{n + \alpha}{x + n + \alpha - 1} \geq 0 \]
and is equal if and only if \( x = 1 \): if \( x > 0 \)
\[ h(x) = x^{\frac{1}{n+\alpha}} = 1 + \frac{1}{(n + \alpha)}(x - 1) + h''(a_x)(x - 1)^2 \leq 1 + \frac{1}{(n + \alpha)}(x - 1); \]
in particular, when \( y(x) \neq x^{\frac{n}{n+\alpha}} \) Corollary 0.14 is a strict improvement of the result in [CROS16] and is equivalent when \( y(x) = x^{\frac{n}{n+\alpha}} \).

Remark 0.16. If \( C^{n+1} \) is a manifold with density \( h \) (a connected manifold with a Riemannian metric \( \langle \cdot, \cdot \rangle \) and smooth positive function weighing the Hausdorff measures associated to the Riemannian distance), then if \( h \) is \( \alpha \)-homogeneous,
\[ D^2 h^{\frac{1}{\alpha}} = -\frac{1}{\alpha} h^{-\frac{1}{\alpha}} \text{Ric}_h^{\alpha} = -\frac{1}{\alpha} h^{\frac{1}{\alpha}} \left( \text{Ric} - D^2 \log h - \frac{1}{\alpha} (d \log h \otimes d \log h) \right), \]
where \( \text{Ric} \) is the Ricci tensor, \( D^2 \) is the Hessian operator for the Riemannian metric, and \( \text{Ric}_h^{\alpha} \) is the \( \alpha \)-dimensional Bakry-Emery-Ricci tensor. Therefore in the case when \( \alpha > 0 \), \( h^{\frac{1}{\alpha}} \) is concave iff \( \text{Ric}_h^{\alpha} \geq 0 \) [CnR14, Lemma 3.9].
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Department of Mathematics, Purdue University, West Lafayette, Indiana, USA.