How to mode-lock an atom laser

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Abstract

We consider a possible technique for mode-locking an atom laser, based on the generation of a dark soliton in a ring-shaped Bose-Einstein condensate, with repulsive atomic interactions. The soliton is a kink, with angular momentum per particle equal to $\hbar/2$. It emerges naturally, when the condensate is stirred at the soliton velocity, and subsequently outcoupled with a periodic Raman pulse-train. The result is a replicating coherent field inside the atom laser. We give a numerical demonstration of the generation and stabilization of the soliton.

The discovery of Bose-Einstein condensation (BEC) in ultra-cold alkali-metal vapors \footnote{\cite{1}} in a magnetic trap, at temperatures of $10^{-6} K$ or lower, has raised the possibility of a coherent atom laser. Recent experimental developments \footnote{\cite{2}} show that this is indeed practical. All atom lasers to date, however, produce output pulses that are not phase-coherent. It would be desirable to have a mode-locked laser, in which the pulses are in phase with each other, for this will enable a wide range of interference experiments and phase-sensitive measurements. In addition, mode-locked lasers can have an enhanced intensity stability, relative to their non-mode-locked cousins.

One way to make a mode-locked atom laser is to create a periodic field circulating around a ring-shaped condensate, and outcouple it with a synchronized period. To do this, we must choose a periodic field that can be easily created, and that has sufficient stability to enable a steady-state outcoupling process. In this regard, we suggest a dark soliton in a condensate of atoms with repulsive interactions. This is a kink configuration, which stands apart from a continuum of possible excitations, owing to two distinctive features: (a) It has a characteristic propagating velocity $v_0$; (b) The angular momentum per particle, normal to the plane of the ring, is $\hbar/2$. 
The dark soliton can be created by stirring the condensate at the characteristic velocity $v_0$; but it has to be cleaned up, because the stirring also creates other excitations such as phonons. The cleansing can be achieved by applying a stroboscopic loss mechanism, which also serves as outcoupler for the atom laser. The soliton persists because of a dynamical stability due to the stroboscopic environment, which is particularly strong for the half-integer angular momentum dark kinks. In the following, we first describe the soliton as a solution of the nonlinear Schrödinger equation (NLSE) and then demonstrate mode-locking via numerical simulations.

Consider a condensate contained in a ring of radius $R$. Let the cross-sectional radius be $r_0$, and let the cross-sectional area be denoted by $A = \pi r_0^2$. We choose $R \gg r_0$, so that the transverse excitations are far more energetic than those along the ring. For the low excitation modes, therefore, we may regard the condensate as a one-dimensional system, and denote by $\theta$ the angle around the ring. The condensate wave function $\Psi(\theta,t)$ satisfies the NLSE

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2mR^2} \frac{\partial^2 \Psi}{\partial \theta^2} + U \Psi + \frac{4\pi \hbar^2 a}{m} |\Psi|^2 \Psi$$

(1)

where $m$ is the atomic mass, $a$ the $s$-wave scattering length, and $U$ is an external potential due to trap non-uniformity. The total number of atoms $N$ enters through the normalization $AR \int_0^{2\pi} d\theta |\Psi|^2 = N$, which is a constant of the motion, if $U$ is independent of the time. Continuity requires the boundary condition $\Psi(\theta + 2\pi, t) = \Psi(\theta, t)$.

The NLSE on a line is well-known in nonlinear optics, where it describes the envelope of an electromagnetic wave propagating along an optical fiber. Let us recount what is generally known in the linear case. For attractive interactions with $\alpha < 0$, the nonlinear term in the equation presents an attractive potential proportional to $|\Psi|^2$. In three spatial dimensions in free space, this would lead to “self-focusing” — the development of spots of infinite intensity in finite time. In one dimension, however, the kinetic energy counterbalances the attraction, leading to the formation of a stable bright soliton. The situation is the same when we join the ends of the line to form a ring.

For repulsive interactions with $\alpha > 0$, we can make a dark soliton in a linear condensate by requiring $\Psi$ to approach $\pm 1$ at opposite ends of the line. By continuity, the configuration must have a kink, i.e., a zero of the wave function. The slope at the kink determines its propagation velocity $v_0$, which will be finite and nonzero. Kink solitons in a cigar-shaped BEC were recently observed. Gray solitons also exist, in which the wave function has a minimum, but never vanishes. These have also been analyzed theoretically, and created experimentally. They propagate with a speed proportional to the wave intensity at the minimum, and hence come to rest in the dark-soliton limit.

These belong to a different class from the kink soliton we are considering on a ring, where periodicity demands that the ends match. Therefore, the phase of the wave function must change by $n\pi$ upon one complete revolution, where $n$ is an odd integer. This makes the angular momentum per particle normal to the ring $nh/2$, for the same mathematical reason that an electron has spin $1/2$. The lowest-energy dark soliton has $n = 1$. 
It is convenient to use dimensionless variables to reduce the NLSE to the form
\[
\frac{i}{\partial \tau} \psi = -\frac{\partial^2 \psi}{\partial \theta^2} + V \psi + \alpha |\psi|^2 \psi \tag{2}
\]
where \( \tau \equiv (\hbar/2mR^2)t, \psi \equiv \Psi \sqrt{2\pi AR/N}, \ldots \) \( V = (2mR^2/\hbar^2)U, \) and \( \alpha \equiv 4NRa/A. \) The boundary condition is \( \psi(\theta + 2\pi, \tau) = \psi(\theta, \tau), \) and the normalization condition is \( \int_0^{2\pi} d\theta \psi^* \psi = 2\pi. \) The energy per particle \( \epsilon, \) and angular momentum per particle \( \ell \) normal to the ring are given by

\[
\epsilon = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[ \frac{\partial \psi^*}{\partial \theta} \frac{\partial \psi}{\partial \theta} + V \psi^* \psi + \frac{\alpha}{2} (\psi^* \psi)^2 \right] \tag{3}
\]
\[
\ell = \frac{1}{4\pi i} \int_0^{2\pi} d\theta \left( \psi^* \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi^*}{\partial \theta} \right). \tag{3}
\]

We consider only special solutions relevant to the application at hand, and discuss more general solutions in a separate paper \[8].\]

For configurations that move around the ring at speed \( v_0, \) we denote by \( \bar{\theta} = \theta - v_0 \tau \) the angle in the co-moving frame, and put

\[
\psi(\theta, \tau) = f(\bar{\theta}) \exp \left( il \bar{\theta} - i\omega \tau \right) \tag{4}
\]
where \( f \) is real, \( \omega \) is a constant to be determined by solving the equation, and \( l \) is an integer or half-integer. The boundary condition requires \( f(\bar{\theta} + 2\pi) = (-1)^l f(\bar{\theta}). \) Substituting (4) into (3), we find that the angular momentum per particle is \( l: \)

\[
l = 0, 1, \frac{1}{2}, 1, \frac{3}{2}, \ldots \tag{5}
\]

Substituting \( f \) into (3), we find the propagation velocity

\[
v_0 = 2\ell. \tag{6}
\]

The equation for \( f \) is

\[
\frac{d^2 f}{d\bar{\theta}^2} = \alpha f^3 - \beta f \tag{7}
\]
where \( \beta \equiv \omega + \ell^2. \) Mathematically, this equation describes the Newtonian motion of a particle of unit mass, in a potential \( \Phi(f) = -\frac{\alpha}{4} f^4 + \frac{\beta}{2} f^2. \) From the conserved “energy” \( E = \frac{1}{2} f'^2 + \Phi(f), \) we can find \( f \) in terms of elliptic functions. Gray solitons correspond to complex \( f, \) and are not considered here.

From now on we assume \( \alpha > 0. \) In the absence of external potential, a uniform state must have \( \psi^* \psi = 1 \) by normalization. These are the vortex states:

\[
\psi_{\text{vortex}}(\theta, \tau) = \exp \left[ il \theta - i(\alpha + \ell) \tau \right] \tag{8}
\]
\[
\epsilon_{\text{vortex}} = \frac{\alpha}{2} + \ell^2 \quad (\ell = 0, 1, 2, 3, \ldots). \tag{8}
\]
In contrast, the solutions with half-integer $\ell$ must vanish somewhere, and thus correspond to dark solitons. The width of the soliton should be proportional to $N^{-1/2}$, for the correlation length of the condensate is $(8\pi a\rho)^{-1/2}$, where $\rho$ is the particle density. In the large $N$ limit, we obtain the simple form:

$$f_{\text{soliton}}(\bar{\theta}) \approx \tanh\left(\sqrt{\frac{\alpha}{2}}(\bar{\theta} - \pi)\right)$$

$$\epsilon_{\text{soliton}} \approx \frac{\alpha}{2} + \ell^2 + \frac{\sqrt{8\alpha}}{3\pi} \quad \left(\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right).$$

(9)

where $\alpha \propto N \gg 1$. In this limit we have $\beta \approx \alpha + (2\pi)^{-1}\sqrt{2\alpha}$. These limiting solitons all have the same shape, but propagate at different velocities $2\ell$. Derivation of these and more general results will be given in a separate paper [8].

In experimental situations, the condensate is in equilibrium with a thermal cloud of uncondensed atoms. The equilibrium number of condensate atoms, relative to that in the thermal cloud, is determined by the temperature, and it determines the width of the soliton. During each soliton round-trip, a loss of particles from the condensate due to output coupling will be countered by a gain from the thermal cloud. The mean atom number in the entire BEC is self-stabilizing because the output-coupling efficiency is a nonlinear function of the dark-soliton width, which in turn is inverse to the total atom number. Thus, a fluctuation from equilibrium that increases the atom number, will also increase the efficiency of the output coupler, which therefore acts as a nonlinear absorber. Even with linear gain and loss mechanisms, we can create a self-maintained soliton and a steady stream of coherent output pulses. The feasibility of the scheme can be demonstrated by numerical simulations, as we now describe.

First, we create a dark soliton with $\ell = 1/2$ from a uniform static condensate, by momentary stirring it at the soliton velocity $v_0 = 1$. In practice, this can be produced by a blue detuned laser beam. In the simulation, we introduce a repulsive potential $V(\theta, \tau)$ to create a moving “hole” in the condensate at $\theta = \tau - \pi$. The time origin is displaced by $\pi$, to wait for the hole to fully form. Next, we have to clean up the configuration by adding gain and loss mechanisms, for the stirring creates other excitations in addition to the dark soliton. The entire procedure is contained in the generalized equation of motion

$$i\frac{\partial \psi}{\partial \tau} = -\frac{\partial^2 \psi}{\partial \theta^2} + \alpha|\psi|^2 \psi + V(\theta, \tau)\psi + i[g - \gamma(\theta, \tau)]\psi$$

(10)

where the stirring potential is chosen as:

$$V(\theta, \tau) = V_0 \exp\left[-\left(\frac{\theta - \tau + \pi}{\sigma_1}\right)^2 - \left(\frac{\tau - \pi}{\sigma_2}\right)^4\right].$$

(11)

In the gain-loss mechanism, $g$ is a constant gain rate, representing stimulated emission from non-condensed atoms that are continuously loaded into the trap. The loss function makes localized periodic hits at the center of the soliton:
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Figure 1: Modulus of the condensate wave function as function of angle around the ring, and dimensionless time. Stirring an initially uniform condensate at the soliton velocity creates a moving hole, which develops into a dark soliton, after unwanted excitations are cleansed by the loss mechanism.

\[ \gamma(\theta, \tau) = \gamma_0 \exp \left[ - \left( \frac{\theta}{\sigma_\theta} \right)^2 - \left( \frac{\tau - \tau_1}{\sigma_\tau} \right)^2 \right] \] (12)

where \( \tau_1 = (\tau \mod 2\pi) - \pi \). This simulates a stroboscopic output coupler, realizable, for example, through a local Raman-tuned transition to a non-trapped state of the atoms. The strobe acts for the first time at \( \tau = 2\pi \), allowing time for the hole to form.

In the numerical analysis, we set \( \alpha = 7.3634 \), chosen to correspond to \( \beta = 8 \) in (7). For the stirring potential we use \( V_0 = 1.673\alpha \), \( \sigma_1 = 1.575\sqrt{2/\beta} \), and \( \sigma_2 = \pi/2 \). The gain and the loss parameters are set to: \( g = 0.01 \), \( \gamma_0 = 0.9 \), \( \sigma_\theta = 1.05\sqrt{2/\beta} \), \( \sigma_\tau = 1.05\sqrt{2/\beta} \).

Following are the results of numerical calculations, with the initial condition \( \psi = 1 \). Fig. 1 shows \( |\psi| \) as a function of \( \theta \) and \( \tau \). We see that a hole was formed, and developed into a dark soliton through the action of the gain-loss mechanisms. A more detailed view is shown in Fig. 2, which shows \( |\psi| \) and the phase of a clean soliton, as a function of \( \theta \) at a fixed time. The signature is that the phase jumps by \( \pi \) across the soliton. The modulus does not precisely vanish due to small admixtures of non-soliton excitations. Fig. 3 shows the angular momentum per particle, which rises from 0 to almost 1 as the condensate is being stirred, but settles down to \( \hbar/2 \) characteristic of a kink, under actions of the gain-loss mechanism. Fig. 4 shows the total number of atoms in the ring as a function of time. The steady-state oscillations indicate an output train of pulses coherent with each other. In this simulation, the strobscope performs the dual task of cleaning up the soliton, and acting as outcoupler for the mode-locked laser. In general, these tasks could be done by separate mechanisms.
Figure 2: Modulus and phase of the dark soliton, as functions of the angle around the ring, at a time when it has become fully stabilized. The phase jumps by \( \pi \) across the dip, indicating that the soliton is a kink.

Figure 3: The angular momentum went through transients while the condensate was being stirred, but stabilizes to a value \( \hbar/2 \) characteristic of a kink, under actions of the gain-loss mechanism.

Our numerical results are quite sensitive to the exact strength of the inter-atomic repulsive potential and to the gain and loss parameters. In actual experiments, however, one can adjust the potential, either by taking advantage of the similar couplings that exist in two-component Bose gases \([9]\), or by using tuning techniques involving Feshbach resonances\([10]\), which have been recently demonstrated experimentally\([11]\). It may also be feasible to utilize inelastic atom-atom collisions as a saturation mechanism, to further increase the mode-locked laser stability.

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Figure 4: Mode-locking — production of a coherent train of pulses — is indicated by the steady-state oscillation of the total number of atoms in the condensate.

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