Assessment of Bending Reinforced Concrete Beams Crack Resistance

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Abstract. The problem of determining the stress state in a section of bending reinforced concrete beams with an initial and operational cracks has been solved analytically. To this end, a system of two nonlinear algebraic equations has been obtained from the equilibrium conditions of the part of the beam cut along the crack line. From this system, for a beam with an initial crack, the height of the compression zone and the nominal stress at the crack tip are determined, and for an operational crack, the height of the compression zone and the length of the crack are determined. The remaining parameters of the stress state are expressed in terms of these values. There is also determined the value of the external moment above which an increase in the initial length of the crack occurs.

Determining the stress intensity factor (SIF) is based on the assumptions that the longitudinal forces are equal at the crack tip, with and without stress concentration. The size of the stress concentration zone is determined from the condition that the local stress is equal to the nominal stress. On this basis, the formula for determining the SIF is obtained. The paper analyzes the SIF dependence on the crack length and the bending moment.

The method of calculation is valid for beams of arbitrary cross-section, but explicit dependences are given for beams of rectangular cross-section most frequently encountered in production.

The results obtained allow estimating the bearing capacity of beams with a crack, as well as their crack resistance according to the force criterion of fracture mechanics.

1. Introduction
Cracks are a very common type of defect in reinforced concrete structures. They occur at the manufacturing stage and at the operation stage. The appearance of cracks in bending elements does not mean exhaustion of its bearing capacity. It leads to increasing the internal efforts in the sections with a crack, which reduces the element strength. Due to the opening of the crack width, corrosion of reinforcement increases, which reduces the durability of structures. A lot of books [1, 2] and articles [3-7] are dealing with calculating the stress state of reinforced concrete structures with cracks. In these works, the length of the crack is considered known. However, during bending the length of the crack depends on the bending moment, and the length of the existing crack may increase. The crack opening width is one of the criteria for the ultimate state of reinforced concrete structures with cracks. Works
[8–10] are dealing with its definition. The issues of crack formation are also considered in works [11, 12].

Under certain conditions, unstable crack development is possible, which is estimated in terms of the parameters of fracture mechanics [13, 14]. The issues of applying fracture mechanics criteria are still under discussion [15, 16]. The most frequently used stress intensity factor (SIF) is the main parameter of linear fracture mechanics. In the general case, the SIF is determined by the finite element method in the linear formulation. However, the relationship between stress and strain in concrete is not linear. This introduces inaccuracies in the SIF definition in a reinforced concrete element. During bending, the development of a crack is limited to reinforcement in the tension zone and is blocked by the presence of a compression zone, which introduces its own characteristics in the SIF definition.

In this paper, the authors propose an approximate analytical method for determining the SIF in reinforced concrete beams. For this, the stress state in a beam with a crack is first determined. The authors assume that during bending the crack starts from the contour point of the tension zone that is the farthest from the neutral axis and spreads across the cross section.

2. Determining stress state

Let’s consider the case of bending an I-beam with a pre-stressed lower reinforcement and a crack (see Figure 1,a). When deriving the Formulas, we assume that the law of flat sections is satisfied. Two cases must be distinguished here: a) the nominal stress at the crack tip is lower than or equal to the ultimate tensile strength of concrete b) exceeds which leads to increasing the initial crack length l.

When calculating the stress state, we consider the law of flat sections to be fair. Here it is necessary to distinguish two cases: a) the nominal stress at the crack tip \( \sigma_m \) is lower than or equal to the tensile strength of concrete \( R_{bt} \); b) this stress exceeds \( R_{bt} \) which leads to increasing the initial crack length \( l \).

Let’s consider an I-section with the vertical axis of symmetry (Fig. 1, a). Let’s introduce designations:

- \( A_s \), \( A_t \) is the area of the shelves overhang in the tension and compression zones;
- \( h_s, h_t \) is the thickness of the shelves in the tension and compression zones;
- \( a_s, a_t \) is the thickness of the protective layer of concrete in the tension and compression zones;
- \( A_s, A_t \) is the area of reinforcement in the tension and compression zones;
- \( N_{sa}, N_{ta} \) are internal forces in the reinforcement in the tension and compression zones.

Let’s cut the beam by the section passing through the crack and show the curve of the stress distribution in it.

For the tension zone the relationship between stress and strain has the form of the exponential law [17].

\[
\sigma = 1.1R_{lt} \left[ 1 - \exp(-0.9\varepsilon_E / R_{lt}) \right].
\]
where \( E_b \) and \( R_{bt} \) are the elasticity modulus and the concrete ultimate strength.

Due to the low stress level (\( \sigma_m < R_{bt} \)), the distribution of compressive stresses is assumed to be linear. Then the stress profile in the cross section will take the form shown in Figure 1, b.

Tensile stress at the tip of the crack \( \sigma_m \) and the height of the compression zone \( x \) are taken as the basic unknown values of the problem. We introduce designations:

\[
\sigma_m = \frac{1.1 \ln(1 - \frac{y}{c})}{y c} - \frac{R_{bt}}{R_{bt}}.
\]

Then the strain at the crack tip is determined by the expression:

\[
\varepsilon_m = \frac{1.1 y R_{bt}}{E_b}.
\]

The strain and stress at the edge of the compression zone taking into account the hypothesis of flat sections will be:

\[
\varepsilon_p = \varepsilon_m x/z_p, \quad \sigma_p = E_b \varepsilon_p = 1.1 y R_{bt} x/z_p,
\]

where \( z_p \) is the height of the tension zone.

In the zone of the crack, the adhesion of the reinforcement to the concrete is broken, the tensile forces are mainly perceived by the reinforcement. In the sections between cracks the tensile forces are perceived by reinforcement and concrete. The strains and stresses in the reinforcement and concrete, as well as the height of the compression zone between the cracks vary. This non-uniformity in the calculations is taken into account by introducing special coefficients that are equal to the ratios of the average values in the section between the cracks and the values in the section with a crack [18]

\[
\psi_s = \varepsilon_m / \varepsilon_s, \quad \psi_b = \varepsilon_m / \varepsilon_b.
\]

Taking into account the strain and stresses in the reinforcement will be:

\[
G_s = \frac{E_b \varepsilon_s}{m}, \quad \sigma_s = E_b \varepsilon_s = 1.1 y R_{bt} (h - a - x)/z_p,
\]

\[
\sigma'_p = \sigma'_p - E_b \varepsilon'_p = \sigma'_p - 1.1 y R_{bt} (x - a')/z_p,
\]

where \( \alpha \) is the ratio of the reinforcement and concrete elasticity modulus; \( \psi_b = \psi_b / \psi_s \); \( \sigma'_p \) is pre-stress in the compression zone reinforcement.

The tensile stress at the distance \( z \) from the zero line, taking into account (1) and the flat section hypothesis, has the form:

\[
\sigma = 1.1 R_{bt} [1 - \exp(-yz/z_p)].
\]

The resultant of the internal forces in the tension zone and their moment relative to the zero line are determined by integrating over the height of the tension zone:

\[
N_p = n b \sigma_m z_p, \quad M_{po} = m \sigma_m b z_p^2, \quad M_{pc} = M_{po} - N_p (h/2 - x),
\]

where \( n = 1/c - 1/y, \quad m = 0.5 + (1 - c)(1 + y)/y^2 - 1/y^2 \) / \( c \);

\( M_{pc} \) is the moment relative to the central axis.

The condition of the beam cut-off zone equilibrium as the sum of the projection of forces on the beam axis is:

\[
N_p + \sigma_p A_x - \sigma_b bx/2 + \sigma'_p A'_x - \sigma_b A_x (x - h_x/2)/x = 0.
\]

The second equilibrium equation as a sum of moments of all the forces relative to the concrete central axis has the form:

\[
M_{pc} + \sigma_p A_x (h/2 - a) + 0.5 \sigma_p bx (h/2 - x/3) + \sigma_p A_x (x - h_x/2)(h - h_x)/2x - \sigma'_p A'_x (h/2 - a') + 1.1 y R_{bt} A'_x (x - a')(h/2 - a') = M_{in},
\]

where the external moment \( M_{in} \) is equal to the bending moment \( M \) in the section with a crack for the reinforcement without pre-stress. For the pre-stressed reinforcement of the tension zone, stress \( \sigma_s \) occurs after the moment of external forces \( M \) exceeds the moment of the pretension force. Then in this equation the total external moment will be equal to

\[
M_{in} = M - \sigma'_p A_x (h/2 - a).
\]
The equilibrium equations are common for all reinforced concrete elements with and without prestressing, with various cross-sectional shapes: I-beams, T-shaped, rectangular. For a T-section \( A_t \) or \( A_{tr} \) is equal to zero. For a rectangular section the areas of both overhangs are zero. In order to obtain explicit calculated ratios, we will carry out a further solution for the most common case of the rectangular section with reinforcement in the tension zone.

Let’s introduce the following dimensionless parameters:

\[
\xi = x/h, \quad \eta = z/l, \quad \lambda = z_p/h = 1 - z - \xi, \quad \bar{h} = (h - a)/h, \quad \mu = A_c/bh.
\]

Then, taking into account the expressions for \( \sigma_b \) and \( \sigma_s \) the equilibrium equations can be written in the dimensionless form:

\[
nc\lambda^2 - 0.5y\xi^2 + y\lambda\mu\psi_{by}(\bar{h} - \xi) = 0.
\]

The solution of this system of nonlinear equations makes it possible to determine the stress at the crack tip and the height of the compression zone depending on the external moment.

This solution is valid as long as the nominal stress is lower than or equal to the tensile strength of the concrete \( R_{by} \). Otherwise, the initial crack length will increase. The value of the moment \( M_m \) at which this happens is determined from system (6), if we take \( \sigma_m = R_{by} \). At this

\[
c = 0.909, \quad y = 2.4, \quad n = 0.683, \quad m = 0.418.
\]

The new crack length corresponding to the effective moment \( M > M_m \), is also determined from system (6) at \( \sigma_m = R_{by} \). To determine stresses by formulas (3) and (4) and to calculate the SIF, it is needed to determine \( \xi \) and \( \lambda \) from this system.

3. Determining stress intensity coefficient

Let’s determine the SIF at the tip of the crack \( K_i \). The tip of the crack is a strong stress concentrator. According to the linear theory of elasticity, stress there is determined by the

\[
\sigma_n = K_i / \sqrt{2\pi r},
\]

where \( r \) is the distance from the crack tip to the point under consideration.

The SIF determining is based on the equality of the longitudinal forces at the crack tip with and without stress concentration. Local stress is determined by formula (7), and the nominal stress by formula (5).

Let’s determine the length of the stress concentration zone “\( d \)” from the condition of equality of the rated and local stress at the end of this zone (Figure 1.c). Rated stresses at the distance “\( d \)” from the crack tip

\[
\sigma_n = 1.1R_{by}\left[1 - (1-c)^{1/\gamma}\right], \quad t = a/z_p.
\]

The longitudinal forces per unit width of the beam in the zone of concentration are

\[
I_1 = \int_0^d \sigma_n dr = K_i\sqrt{2\pi}\sqrt{\pi} = K_i / \pi\sigma_n.
\]

The longitudinal forces in the same zone without stress concentration are equal

\[
I_2 = 1.1R_{by}\left[1 - (1-c)^{1/\gamma}\right]y.
\]

By equating these forces, we obtain

\[
f(t) = t - (1-c)^{1/\gamma}(2t - 1)/y - (1-c)/y = 0.
\]

Solving this equation with the known \( c(y) \), we find the parameter \( t \).

Then from (7) we determine the SIF:

\[
K_i = \sqrt{2\pi a} \cdot \sigma_n = \sqrt{2\pi} \cdot 1.1R_{by}\left[1 - (1-c)^{1/\gamma}\right]\sqrt{\pi h}.
\]
If the external moment is larger than $M_m$, then $\sigma_m = R_y$, $c = 0.909$, $t = 0.808$ and

$$K_i = 0.914R_y \sqrt{z_p}.$$  

The calculations show that with increasing the crack length, the SIF decreases to zero. This is explained by the fact that during bending there is necessarily a minimum zone of compressive stresses. With increasing the crack length, this reduces the length of the zone of tensile stresses to zero. At $z_p = 0$, tensile forces are perceived only by reinforcement, the crack does not develop. In this case, the SIF becomes zero and the beam is destroyed due to the achievement of the yield strength in the reinforcement or crushing of the concrete of the compression zone.

The paper analyzes changing the SIF with changing the external moment. The calculations have been carried out for a rectangular cross-section beam of size (15x30) cm made of grade B25 concrete ($R_{bt}=1.6$ MPa, $R_b=14.5$ MPa) with reinforcement made of AIII steel ($R_s=370$ MPa, $\mu p_{bt}=0.15$). The beam has an initial crack $l_0=6$ cm. For this case, the crack growth moment is determined from system (6) and is equal to $M_m = 9.96$ kNm. The results of the calculations are shown in the Table 1 below.

| $M$, kNm | 2 | 6 | 10 | 16 | 26 | 34 | 43 |
|----------|---|---|----|----|----|----|----|
| $K_i$ MN/m$^{3/2}$ | 0.128 | 0.316 | 0.467 | 0.386 | 0.31 | 0.25 | 0.14 |

It can be seen in the Table that for $M<M_m$, the SIF increases with increase the moment. When $M>M_m$, the crack grows and the SIF decreases with increasing the moment.

If, with a known crack length and a given moment, the SIF is lower than the limit SIF value for a given material, then the crack will develop steadily. If $K_i \geq K_{IC}$, the crack is non-stable and rapid crack growth is possible with a slight increase in the moment. The increase in the length of the cracks continues until the SIF drops to the $K_{IC}$ value. This value of the SIF corresponds to a certain value of the crack length, which is determined from system (6) and expression (8).

Evaluation of the beam bearing capacity is performed by the value of the stress in the reinforcement (4) and the maximum compressive stress in the concrete (3). The parameters $\sigma_m$, $x$, $z_p$, found through the initial ($M<M_m$) or operational ($M>M_m$) crack length are substituted into these formulas.

### 4. Conclusions

Based on the results of the work, the following conclusions can be drawn.

1. The parameters of the stress state of bending reinforced concrete beams with an initial and operational crack are determined.
2. The value of the external moment is determined, above which the crack length increases.
3. By analyzing the stress state in the crack zone, the stress intensity factor at the crack tip has been determined.
4. The effect of the crack length and the external moment on the magnitude of the ORF is analyzed. It has been established that in a reinforced concrete beam with increasing crack length, $CIN$ decreases.

The results obtained allow estimating the bearing capacity of reinforced concrete beams with a crack, as well as estimating crack resistance of the beam according to the force criterion of fracture mechanics.

The method of calculation is valid for beams of an arbitrary cross-section, but explicit dependences are given for beams of the rectangular cross-section most frequently encountered in production.

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