ROUGH SOLUTIONS OF THE FIFTH-ORDER KDV EQUATIONS

ZIHUA GUO, CHULKWANG KWAK, AND SOONSIK KWON

ABSTRACT. We consider the Cauchy problem of the fifth-order equation arising from the Korteweg-de Vries (KdV) hierarchy

\[ \begin{cases} \partial_t u + \partial_x^5 u + c_1 \partial_x u \partial_x^3 u + c_2 u \partial_x^3 u = 0 & x, t \in \mathbb{R} \\ u(0, x) = u_0(x) & u_0 \in H^s(\mathbb{R}) \end{cases} \]

We prove a priori bound of solutions for $H^s(\mathbb{R})$ with $s \geq \frac{5}{4}$ and the local well-posedness for $s \geq 2$.

The method is a short time $X^{s,b}$ space, which is first developed by Ionescu-Kenig-Tataru [11] in the context of the KP-I equation. In addition, we use a weight on localized $X^{s,b}$ structures to reduce the contribution of high-low frequency interaction where the low frequency has large modulation.

As an immediate result from a conservation law, we have the fifth-order equation in the KdV hierarchy,

\[ \partial_t u - \partial_x^5 u - 30u^2 \partial_x u + 20 \partial_x u \partial_x^3 u + 10u \partial_x^3 u = 0 \]

is globally well-posed in the energy space $H^2$.

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1. Introduction

We consider the Cauchy problem for the fifth-order KdV equation

\[ \begin{cases} 
\partial_t u + \partial_x^5 u + c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u = 0, & x, t \in \mathbb{R} \\
0 \leq s < \frac{5}{4}, & u(0, x) = u_0(x) \in H^s(\mathbb{R}) 
\end{cases} \]

(1.1)

The fifth-order KdV type equation (1.1) generalizes the second equation appearing in the KdV hierarchy:

\[ \partial_t u - \partial_x^5 u - 30u^2 \partial_x u + 20 \partial_x u \partial_x^2 u + 10u \partial_x^3 u = 0. \]

(1.2)

Due to the theory of complete integrability, the KdV equation and the higher-order equations in the hierarchy are solved via the scattering and the inverse scattering methods at least for regular and well-decaying initial data. Moreover, they enjoy infinitely many conservation laws. However, the theory of complete integrability is rigid so that one cannot apply to non-integrable equations. For instance, if one slightly change a coefficient of the equation, then the inverse scattering transform does not work. We note that there are some other physical background of the equation (1.1) such as higher-order water wave models, a lattice of an harmonic oscillators. See [19] for further discussion.

The purpose of this article is to study the Cauchy problem for Sobolev initial data with low regularity in analytic manner. Previously, Ponce [19] showed (1.1) is locally well-posed for \( s \geq 5 / 4 \). Later, the third author [17] improved the local well-posedness for \( s > \frac{5}{2} \). Both works are based on the energy method and local and global smoothing estimates from dispersive effects. Furthermore, in [17], it is shown that the flow map fails to be uniformly continuous on a bounded set of initial data for any \( s \in \mathbb{R} \). So, the Picard iteration method cannot apply and such a less perturbative way is necessary. This is a sharp contrast to the KdV equation, which is solved via the Picard iteration [15]. The issue is a strong low-high frequencies interaction in the nonlinearity. Since the quadratic nonlinearity has too many derivatives, they cannot be overcome by the dispersive effect of linear part. As a result the equation shows a quasilinear dynamics. This type of phenomenon is observed in other equations, such as the Benjamin-Ono equation and the KP-I equation. Now we state our main results:

**Theorem 1.1.** (a) Let \( s \geq 5 / 4, R > 0 \). For any \( u_0 \in B_R = \{ f \in H^\infty : \|f\|_{H^s} \leq R \} \), there exists a time \( T = T(R) > 0 \) and a unique solution \( u = S^\infty_T(u_0) \in C([-T, T], H^\infty) \) to the fifth-order KdV equation (1.1). In addition, for any \( \sigma \geq 5 / 4 \)

\[ \sup_{|t| \leq T} \| S^\infty_T(u_0)(t) \|_{H^\sigma} \leq C(T, \sigma, \| u_0 \|_{H^\sigma}). \]

(1.3)

\[ ^1 \text{If one consider a weighted Sobolev spaces, then Picard iteration may work. See, for example, [14].} \]
(b) Let $s \geq 2$. The map $S_T^\infty : B_R \to C([-T,T], H^\infty)$ extends uniquely to continuous mapping

$$S_T : \{ f \in H^s : \| f \|_{H^s} \leq R \} \to C([-T,T], H^s).$$

We have a priori bound of solutions for $H^s(\mathbb{R})$ with $s \geq \frac{5}{4}$, while the well-posedness holds for $s \geq 2$. For the proof of the theorem, we use $X^{s,b}$ type structure in a short time interval depending on frequencies. This is first developed by Ionescu, Kenig, and Tataru [11] in the context of KP-I equation, see [4] for a similar idea and [16] in the setting of Strichartz norms. The method is a combination of modified Bourgain’s $X^{s,b}$ space and the energy method.

First, observe that the bilinear $X^{s,b}$-estimates $\| uv_{xxx} \|_{X^{s,b-1}} \lesssim \| u \|_{X^{s,b}} \| v \|_{X^{s,b}}$ fails in usual $X^{s,b}$ for any $s$, and the difficult term is the high-low interaction component with very low frequency of the following type

$$(P_{\leq 0} u) \cdot (P_{\text{high}} v_{xxx})$$

where the low frequency is very small such that these interactions have a small resonance and coherence. But if one use $X^{s,b}$ structure for a short time interval ($\approx (\text{frequency})^{-2}$), the contribution of high frequency and low modulation is reduced so that we can prove the bilinear estimate (See Remark 2.3).

To compensate this short-time estimates, we need an energy-type estimates. In fact, we could not close the energy estimate solely using Ionescu-Kenig-Tataru’s method. The enemy is the high-low interactions where the low frequency component has the largest modulation. In [9], the first author et. al. used a weight that was first used in [10] to strengthen estimates for this interaction. It turns out that we have to use the weight differentially on low and high frequencies. (See (2.1)) Intuitively, we put more modulation regularity for the low frequency component $P_{\leq 0} u$. Then this helps improving the high-low interaction and the energy estimates.

The trade-off of using such a weight is to worsen the high – high $\to$ low interactions in the nonlinear estimates

$$P_{\leq 0} (P_{\text{high}} u \cdot P_{\text{high}} v_{xxx}).$$

Fortunately, after rewriting the nonlinear term in the divergence form $c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u = c'_1 \partial_x (\partial_x u \partial_x u) + c'_2 \partial_x (u \partial_x^2 u)$, we are able to choose a weight to balance both purposes.

Around the time when we completed this work, we learned Kenig and Pilod [13] have worked on the same problem with a similar idea an obtained the same result. They used the short-time $X^{s,b}$ structure and the modified energy method. The modified energy is an energy norm with cubic correctional terms that make a cancellation and so improve the energy bound.

There are many works on similar types of higher-order dispersive equations. For instance, the Kawahara equation, which is the fifth-order equation with nonlinearity $uu_x$ [1] and the
fifth-order equation in the modified KdV hierarchy, which has the cubic nonlinearities such as \( u^2 u_{xxx} \) \cite{18}. As opposed to (1.1), the dynamics of these equations are semi-linear in the sense that the flow map is locally Lipschitz continuous and so solved via the Picard iteration, since nonlinear feedback of these equations are weaker.

Combined with the second conservation law in the KdV hierarchy,

\[
H_2(u) = \int \frac{1}{2} (\partial_x^2 u)^2 - 5u\partial_x(u^2) + \frac{5}{2} u^4 \, dx,
\]

we can obtain the global well-posedness for the equation (1.2).

**Corollary 1.2.** The Cauchy problem (1.2) is globally well-posed in \( H^2 \).

The paper is organized as follows: In Section 2, we present notations and define function spaces. In Section 3 and 4, we prove the bilinear estimates, the energy estimates. In Section 5, we sketch the proof of well-posedness. We collect the proofs for known propositions in the appendix for convenience of readers.

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## 2. Notations and Definitions

For \( x, y \in \mathbb{R}_+ \), \( x \lesssim y \) means that there exists \( C > 0 \) such that \( x \leq Cy \). And \( x \sim y \) means \( x \lesssim y \) and \( y \lesssim x \). Let \( a_1, a_2, a_3 \in \mathbb{R} \). The quantities \( a_{\max} \geq a_{\med} \geq a_{\min} \) can be conveniently defined to be the maximum, median and minimum values of \( a_1, a_2, a_3 \) respectively.

For \( f \in S' \) we denote by \( \hat{f} \) or \( \mathcal{F}(f) \) the Fourier transform of \( f \) with respect to both spatial and time variables,

\[
\hat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x, t) \, dx \, dt.
\]

Moreover, we use \( \mathcal{F}_x \) and \( \mathcal{F}_t \) to denote the Fourier transform with respect to space and time variable respectively. Let \( \mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty) \). Let \( I_{\leq 0} = \{ \xi : |\xi| < 3/2 \} \), \( \tilde{I}_{\leq 0} = \{ \xi : |\xi| \leq 2 \} \). For \( k \in \mathbb{Z}_{>0} \) let \( I_k \) and \( \tilde{I}_k \) be dyadic intervals with \( I_k \subset \tilde{I}_k \). More precisely \( I_k = \{ \xi : |\xi| \in [(3/4) \cdot 2^k, (3/2) \cdot 2^k] \} \), \( \tilde{I}_k = \{ \xi : |\xi| \in [2^{k-1}, 2^{k+1}] \} \).

Let \( \eta_0 : \mathbb{R} \to [0, 1] \) denote a smooth bump function supported in \([-2, 2]\) and equal to 1 in \([-1, 1]\). For \( k \in \mathbb{Z} \), let \( \chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1}) \), which is supported in \( \{ \xi : |\xi| \in [2^{k-1}, 2^{k+1}] \} \), and

\[
\chi_{[k_1, k_2]} = \sum_{k=k_1}^{k_2} \chi_k \quad \text{for any } k_1 \leq k_2 \in \mathbb{Z}.
\]

\textsuperscript{2} (1.2) have an additional cubic term. The nonlinear estimate and the energy estimate for this term are essentially easier. Thus, we only sketch the proof. See Remark 3.9 and 4.3.
For simplicity, let \( \eta_k = \chi_k \) if \( k \geq 1 \) and \( \eta_k \equiv 0 \) if \( k \leq -1 \). Also, for \( k_1 \leq k_2 \in \mathbb{Z} \) let

\[
\eta_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \eta_k \quad \text{and} \quad \eta_{\leq k_2} = \sum_{k=-\infty}^{k_2} \eta_k.
\]

\( \{\chi_k\}_{k \in \mathbb{Z}} \) is the homogeneous decomposition function sequence and \( \{\eta_k\}_{k \in \mathbb{Z}^+} \) is the inhomogeneous decomposition function sequence to the frequency space. For \( k \in \mathbb{Z} \) let \( P_k \) denote the operators on \( L^2(\mathbb{R}) \) defined by \( \hat{P_k} u(\xi) = 1_{I_k}(\xi) \hat{u}(\xi) \). By a slight abuse of notation we also defined the operators \( P_k \) on \( L^2(\mathbb{R} \times \mathbb{R}) \) by formulas \( F(P_k u)(\xi, \tau) = 1_{I_k}(\xi) F(u)(\xi, \tau) \). For \( l \in \mathbb{Z} \) let

\[
P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.
\]

For \( k, j \in \mathbb{Z}_+ \) let

\[
D_{k,j} = \{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in \tilde{I}_k, \tau - w(\xi) \in \tilde{I}_j \}, \quad D_{k,\leq j} = \bigcup_{l \leq j} D_{k,l}.
\]

For \( \xi \in \mathbb{R} \), let

\[
w(\xi) = -\xi^5,
\]

which is the dispersion relation associated to the equation (1.1). For \( \phi \in L^2(\mathbb{R}) \), let \( W(t)\phi \in C(\mathbb{R} : L^2) \) be the linear solution given by

\[
F_x[W(t)\phi](\xi, t) = e^{itw(\xi)} F_x(\phi)(\xi).
\]

We introduce that \( X^{s,b} \) norm associated to Eq. (1.1) which is given by

\[
\|u\|_{X^{s,b}} = \|\langle \tau - w(\xi) \rangle^b \langle \xi \rangle^s F(u)\|_{L^2(\mathbb{R}^2)}
\]

where \( \langle \cdot \rangle = (1+|\cdot|) \). The space \( X^{s,b} \) turns out to be very useful in the study of low-regularity theory for the dispersive equations. These space were used systematically to study nonlinear dispersive wave problems by Bourgain [2] and used by Kenig, Ponce and Vega [15] and Tao [20]. Klainerman and Machedon [12] used similar ideas in their study of the nonlinear wave equation. We denote the space by \( X_T \) localized to the interval \([-T,T]\).

For \( k \in \mathbb{Z}_+ \), we define the weighted \( X^{s,\frac{1}{2},1} \)-type space \( X_k \) for frequency localized functions \( f_k \),

\[
X_k = \left\{ f \in L^2(\mathbb{R}^2) : f(\xi, \tau) \text{ is supported in } \tilde{I}_k \times \mathbb{R} \text{ if } k = 0 \right\}
\]

\[
\text{and } \|f\|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - w(\xi)) \cdot f(\xi, \tau)\|_{L^2_{\xi,\tau}} < \infty
\]

where

\[
\beta_{k,j} = \begin{cases} 2^{j/2}, & k = 0, \\ 1 + 2(j-5k)/8, & k \geq 1. \end{cases}
\]

(2.1)
Remark 2.1. We choose a parameter $\frac{1}{8}$ in the weight. In fact, we can choose any parameter from 1/8 to 3/16. But this change will not affect our result since the weight helps to reduce only the low-high interactions where the low frequency part has large modulation.

As in [11] at frequency $2^k$ we will use the $X^{s,\frac{1}{2},1}$ structure given by the $X_k$ norm, uniformly on the $2^{-2^k}$ time scale. For $k \in \mathbb{Z}_+$, we define function spaces

$$F_k = \left\{ f \in L^2(\mathbb{R}^2) : \hat{f}(\xi, \tau) \text{ is supported in } \tilde{I}_k \times \mathbb{R} \ (\tilde{I}_{\leq 0} \times \mathbb{R} \text{ if } k = 0) \quad \text{and} \quad \|f\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|F[f \cdot \eta_0(2^{2k}(t - t_k))]|_{X_k} < \infty \right\},$$

$$N_k = \left\{ f \in L^2(\mathbb{R}^2) : \hat{f}(\xi, \tau) \text{ is supported in } \tilde{I}_k \times \mathbb{R} \ (\tilde{I}_{\leq 0} \times \mathbb{R} \text{ if } k = 0) \quad \text{and} \quad \|f\|_{N_k} = \sup_{t_k \in \mathbb{R}} \| (\tau - \omega(\xi) + i2^{2k})^{-1} F[f \cdot \eta_0(2^{2k}(t - t_k))] |_{X_k} < \infty \right\}.$$

Since the spaces $F_k$ and $N_k$ are defined on the whole line, we define then local versions of the spaces in standard ways. For $T \in (0, 1]$ we define the normed spaces

$$F_k(T) = \{ f \in C([-T, T] : L^2) : \|f\|_{F_k(T)} = \inf_{\tilde{f} = f \text{ in } \mathbb{R} \times [-T, T]} \|\tilde{f}\|_{F_k} \},$$

$$N_k(T) = \{ f \in C([-T, T] : L^2) : \|f\|_{N_k(T)} = \inf_{\tilde{f} = f \text{ in } \mathbb{R} \times [-T, T]} \|\tilde{f}\|_{N_k} \}.$$

We assemble these dyadic spaces in a Littlewood-Paley manner. For $s \geq 0$ and $T \in (0, 1]$, we define function spaces solutions and nonlinear terms:

$$F^s(T) = \left\{ u : \|u\|^2_{F^s(T)} = \sum_{k=1}^\infty 2^{2sk}\|P_k(u)\|^2_{F_k(T)} + \|P_{\leq 0}(u)\|^2_{F_0(T)} < \infty \right\}.$$

$$N^s(T) = \left\{ u : \|u\|^2_{N^s(T)} = \sum_{k=1}^\infty 2^{2sk}\|P_k(u)\|^2_{N_k(T)} + \|P_{\leq 0}(u)\|^2_{N_0(T)} < \infty \right\}.$$

We define the dyadic energy space as follows: For $s \geq 0$ and $u \in C([-T, T] : H^\infty)$

$$\|u\|^2_{E^s(T)} = \|P_{\leq 0}(u(0))\|^2_{L^2} + \sum_{k \geq 1} \sup_{t_k \in [-T, T]} 2^{2sk}\|P_k(u(t_k))\|^2_{L^2}.$$

These $l^1$-type $X^{s,b}$ structures were first introduced in [23] and used in [10, 11, 21, 8]. The weighted $l^1$-type $X^{s,b}$ structures in here were used in [9].

Lemma 2.2 (Properties of $X_k$). Let $k, l \in \mathbb{Z}_+$ with $l \leq 5k$ and $f_k \in X_k$. Then

$$\sum_{j=l+1}^{\infty} 2^{l/2} \beta_{k,j} \left\| \eta_j(\tau - \omega(\xi)) \int_{\mathbb{R}} |f_k(\xi, \tau')| 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} + 2^{l/2} \left\| \eta_{\leq l}(\tau - \omega(\xi)) \int_{\mathbb{R}} |f_k(\xi, \tau')| 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \lesssim \|f_k\|_{X_k}.$$  (2.2)
In particular, if \( t_0 \in \mathbb{R} \) and \( \gamma \in \mathcal{S}(\mathbb{R}) \), then

\[
\| \mathcal{F}[\gamma(2^j(t - t_0)) \cdot \mathcal{F}^{-1}(f_k)] \|_{X_k} \lesssim \| f_k \|_{X_k}.
\]

(2.3)

**Proof.** It follows directly from the definition that

\[
\left\| \int_\mathbb{R} |f_k(\xi, \tau')| d\tau' \right\|_{L^2_\xi} \lesssim \| f_k \|_{X_k}.
\]

(2.4)

First, assume \( k \geq 1 \). For the second term on the left-hand side of (2.2), it follows from Cauchy-Schwarz inequality and (2.4) that

\[
2^{j/2} \left\| \eta_l(\tau - w(\xi)) \int_\mathbb{R} |f_k(\xi, \tau')|2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2}
\]

\[
\lesssim \left\| \int_\mathbb{R} |f_k(\xi, \tau')| d\tau' \right\|_{L^2_\xi} \lesssim \| f_k \|_{X_k}.
\]

It remains to control the first term on the left-hand side of (2.2), let \( f_k(\xi, \tau') = \sum_{l \geq 0} f_{k,l} \), where \( f_{k,l} = f_k(\xi, \tau')\eta_l(\tau' - w(\xi)) \). For \( l < j \leq 5k \), we have \( \beta_{k,l} \sim 1 \). Thus, we have from Cauchy-Schwarz inequality that

\[
\sum_{l<j \leq 5k} \sum_{j_1 \geq 0} 2^{j/2} \left\| \eta_l(\tau - w(\xi)) \int_\mathbb{R} f_{k,j_1}(\xi, \tau')2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2}
\]

\[
\lesssim \sum_{l<j \leq 5k} \sum_{j_1 \geq 0} 2^{j} 2^{3l-4} \max(j,j_1) \left\| \int_\mathbb{R} f_{k,j_1}(\xi, \tau') d\tau' \right\|_{L^2_\xi} \lesssim \| f_k \|_{X_k}.
\]

For the rest term \( (j > 5k) \), since \( l \leq 5k \), we get similarly as before that

\[
\sum_{5k<j \leq 5k} \sum_{j_1 \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_l(\tau - w(\xi)) \int_\mathbb{R} f_{k,j_1}(\xi, \tau')2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2}
\]

\[
\lesssim \sum_{5k<j \leq 5k} \sum_{j_1 \geq 0} 2^{j} 2^{(j-5k)/8} 2^{3l-4} \max(j,j_1) \left\| \int_\mathbb{R} f_{k,j_1}(\xi, \tau') d\tau' \right\|_{L^2_\xi} \lesssim \| f_k \|_{X_k}.
\]

Now consider \( k = 0 \). In this case, we also get the condition \( l = 0 \). So, it is relatively simpler than \( k \geq 1 \) case. Similarly as before, we have

\[
\sum_{j \geq 0} \sum_{j_1 \geq 0} 2^{j/2} \beta_{0,j} \left\| \eta_l(\tau - w(\xi)) \int_\mathbb{R} f_{0,j_1}(\xi, \tau')(1 + |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2}
\]

\[
\lesssim \sum_{j \geq 0} \sum_{j_1 \geq 0} 2^{3j/2} 2^{-4} \max(j,j_1) \left\| \int_\mathbb{R} f_{0,j_1}(\xi, \tau') d\tau' \right\|_{L^2_\xi} \lesssim \| f_0 \|_{X_0}.
\]

Thus, we complete the proof of the lemma. \( \square \)
As mentioned before, the bilinear estimates do not hold on the standard $X \ast \cdot \ast$ spaces defined above, this low-high interaction counter-example shows that for any $k \in \mathbb{Z}_+$ we define the set $S_k$ of $k$-acceptable time multiplication factors

$$S_k = \{ m_k : \mathbb{R} \to \mathbb{R} : \| m_k \|_{S_k} = \sum_{j=0}^{10} 2^{-2jk} \| \partial_x^j m_k \|_{L^\infty} < \infty \}.$$ 

Direct estimates using the definitions and \ref{2.3} show that for any $s \geq 0$ and $T \in (0, 1]$ \ref{2.5}

$$\left\{ \begin{aligned}
\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \|_{F^*(T)} &\lesssim (\sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k}) \cdot \| u \|_{F^*(T)}; \\
\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \|_{N^*(T)} &\lesssim (\sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k}) \cdot \| u \|_{N^*(T)}; \\
\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \|_{E^*(T)} &\lesssim (\sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k}) \cdot \| u \|_{E^*(T)}.
\end{aligned} \right.$$ 

Remark 2.3. As mentioned before, the bilinear estimates do not hold on the standard $X^{s,b}$ spaces:

$$\| \partial_x^2 (uv) \|_{X^{s,b-1}} \not\lesssim C \| u \|_{X^{s,b}} \| v \|_{X^{s,b}},$$

due to strong low-high interactions. Indeed, for fixed large frequency $N$, define the characteristic functions supported on the sets:

$$A = \{ (\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^5| \leq 1, N \leq |\xi| \leq N+1 \} \quad \text{and} \quad B = \{ (\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^5| \leq 1, |\xi| \leq 1 \};$$

$$F(u)(\tau, \xi) = 1_A(\tau, \xi) \quad \text{and} \quad F(v)(\tau, \xi) = 1_B(\tau, \xi).$$

By simple calculation, we have $\text{LHS} = N N^s$, while $\text{RHS} = N^s$, respectively. However, if one use the short time $X^{s,b}$ spaces defined above, this low-high interaction counter-example is resolved. Concretely, consider the following sets:

$$\tilde{A} = \{ (\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^5| \leq N^2, N \leq |\xi| \leq N + N^{-2} \}$$

and

$$\tilde{B} = \{ (\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^5| \leq 1, |\xi| \leq 1 \};$$

And define functions $\tilde{u}$ and $\tilde{v}$ which satisfy

$$F(\tilde{u})(\tau, \xi) = 1_{\tilde{A}}(\tau, \xi) \quad \text{and} \quad F(\tilde{v})(\tau, \xi) = 1_{\tilde{B}}(\tau, \xi).$$

Computing both side, we have that for any $s \in \mathbb{R}$,

$$\| \partial_x^2 (\tilde{u} \tilde{v}) \|_{N^s} \sim N^s N^3 N^{-1} N^{-2} N \sim N^s N \quad \text{and} \quad \| \tilde{u} \|_{F^s}, \| \tilde{v} \|_{F^s} \sim N^s N.$$ 

Moreover, this example explains how we choose time length ($= (\text{frequency})^{-2}$) on which we apply $X^{s,b}$ structures.
3. Bilinear estimates

In this section we show the bilinear estimates. For \( \xi_1, \xi_2 \in \mathbb{R} \), let
\[
H(\xi_1, \xi_2) = \xi_1^5 + \xi_2^5 - (\xi_1 + \xi_2)^5
\]
be the resonance function, which plays an crucial role in the bilinear \( X^{s,b} \)-type estimates. For compactly supported functions \( f, g, h \in L^2(\mathbb{R}^2) \), we define
\[
J(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \tau_1)g(\xi_2, \tau_2)h(\xi_1 + \xi_2, \tau_1 + \tau_2 + H(\xi_1, \xi_2))d\xi_1d\xi_2d\tau_1d\tau_2.
\]
By simple changes of variables in the integration, we have
\[
|J(f, g, h)| = |J(g, f, h)| = |J(f, h, g)| = |J(\tilde{f}, g, h)|,
\]
where \( \tilde{f}(\xi, \tau) = f(-\xi, -\tau) \).

Lemma 3.1. Let \( k_i \in \mathbb{Z}, j_i \in \mathbb{Z}_+, i = 1, 2, 3 \). Let \( f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) be nonnegative functions supported in \([2^{k_i-1}, 2^{k_i+1}] \times I_{j_i}\).

(a) For any \( k_1, k_2, k_3 \in \mathbb{Z} \) with \( |k_{\max} - k_{\min}| \leq 5 \) and \( j_1, j_2, j_3 \in \mathbb{Z}_+ \), then we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \lesssim 2^{j_{\min}/2}2^{j_{\med}/4}2^{-\frac{3}{2}k_{\max}}\prod_{i=1}^{3} \|f_{k_i, j_i}\|_{L^2}.
\]  
(3.1)

(b) If \( 2^{k_{\min}} \ll 2^{k_{\med}} \ll 2^{k_{\max}} \), then for all \( i = 1, 2, 3 \) we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \lesssim 2^{(j_1+j_2+j_3)/2}2^{-3k_{\max}/2}2^{-(k_i+j_i)/2}\prod_{i=1}^{3} \|f_{k_i, j_i}\|_{L^2}.
\]  
(3.2)

(c) For any \( k_1, k_2, k_3 \in \mathbb{Z} \) and \( j_1, j_2, j_3 \in \mathbb{Z}_+ \), then we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \lesssim 2^{j_{\min}/2}2^{k_{\min}/2}\prod_{i=1}^{3} \|f_{k_i, j_i}\|_{L^2}.
\]  
(3.3)

Lemma 3.1 is obtained in a similar way to Tao’s (20), Proposition 6.1) in the context of the KdV equation. For the fifth-order equation, it was first shown by Chen, Li, Miao and Wu [5]. But there was error in the high-high \( \rightarrow \) high case and was corrected in [3]. See [3] for the proof. We rewrite the lemma in the following form:

Corollary 3.2. Assume \( k_i \in \mathbb{Z}, j_i \in \mathbb{Z}_+, i = 1, 2, 3 \) and \( f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) be functions supported in \( D_{k_i, j_i}, i = 1, 2 \).

(a) For any \( k_1, k_2, k_3 \in \mathbb{Z} \) with \( |k_{\max} - k_{\min}| \leq 5 \) and \( j_1, j_2, j_3 \in \mathbb{Z}_+ \), then we have
\[
\|1_{D_{k_3, j_3}}(\xi, \tau)(f_{k_1, j_1} \ast f_{k_2, j_2})\|_{L^2} \lesssim 2^{j_{\min}/2}2^{j_{\med}/4}2^{-\frac{3}{2}k_{\max}}\prod_{i=1}^{2} \|f_{k_i, j_i}\|_{L^2}.
\]  
(3.4)
(b) If $2^{k_{\min}} \ll 2^{k_{\text{med}}} \sim 2^{k_{\max}}$, then for all $i = 1, 2, 3$ we have

$$
\|1_{D_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{(j_1+j_2+j_3)/2}2^{-(k_1+j_1)/2}2^{-(k_2+j_2)/2}2^{-(k_3+j_3)/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}.
$$

(3.5)

(c) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$
\|1_{D_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{j_{\min}/2}2^{k_{\min}/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}.
$$

(3.6)

Remark 3.3. If the assumption is replaced by $f_{k_{i,j}}$ is supported in $\tilde{T}_{k_i} \times \tilde{T}_{\leq j} (D_{k_i, \leq j})$ for $k_1, k_2, k_3 \in \mathbb{Z}_+$, then part (a) and (c) of Lemma 3.1 (and of Corollary 3.2) also hold. In addition, if $k_{\min} \neq 0$, then part (b) holds, else if $k_{\min} = 0$, part (b) holds for $i \in \{1, 2, 3\}$ with $k_i \neq 0$. See [7] for the proof.

Proposition 3.4 (high-low \Rightarrow high). Let $k_3 \geq 20$, $|k_2 - k_3| \leq 4$, $0 \leq k_1 \leq k_2 - 10$, then we have

$$
\|P_{k_3}\partial_x^3(u_{k_1}v_{k_2})\|_{N_{k_3}} + \|P_{k_3}(u_{k_1}\partial_x^3v_{k_2})\|_{N_{k_3}} \lesssim \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}}.
$$

(3.7)

Proof. First, we observe that each term of the left-hand side of (3.7) has the same bound since $2^{k_2} \sim 2^{k_3}$. From the definition, the left-hand side of (3.7) is bounded by

$$
\sup_{t_k \in \mathbb{R}} \|\tau^x w(\xi) + i2^{k_3}1_{k_4}(\xi)\mathcal{F}[u_{k_1} \eta_0(2^{k_3} - 2(t - t_k))] * \mathcal{F}[v_{k_2} \eta_0(2^{k_3} - 2(t - t_k))]\|_{X_{k_3}}.
$$

(3.8)

Set $f_{k_1} = \mathcal{F}[u_{k_1} \eta_0(2^{k_3} - 2(t - t_k))]$ and $f_{k_2} = \mathcal{F}[v_{k_2} \eta_0(2^{k_3} - 2(t - t_k))]$. We decompose $f_{k_i}$ into modulation dyadic pieces as $f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau) \eta_{j_i}(\tau - \xi_i)$, $i = 1, 2$, with usual modification $f_{k_i, \leq j}(\xi, \tau) = f_{k_i}(\xi, \tau) \eta_{\leq j}(\tau - \xi_i)$. Then (3.8) is bounded and reduced by

$$
\sup_{t_k \in \mathbb{R}} 2^{3k_3} \left( \sum_{0 \leq j_3 \leq 5k_3} + \sum_{j_3 \geq 5k_3} \right) \frac{2^{j_3/2} \beta_{k_3,j_3}}{\max(2^{j_3/2}, 2^{2k_3})} \sum_{j_1,j_2 \geq 2k_3} \|1_{D_{k_3,j_3}}(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} := I + II.
$$

(3.9)

For the term $I$, by Corollary 3.2 (b) we get

$$
I \lesssim \sup_{t_k \in \mathbb{R}} 2^{3k_3} \sum_{0 \leq j_3 \leq 5k_3} 2^{j_3/2} \max(j_3, 2k_3) \sum_{j_1,j_2 \geq 2k_3} 2^{j_1/2}2^{j_2/2}2^{-2k_3} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2}.
$$

\leq \sup_{t_k \in \mathbb{R}} \sum_{j_1,j_2 \geq 2k_3} 2^{j_1/2}2^{j_2/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2}. \leq \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}}.
$$
Thus we finish the proof.

and hence finish the proof of the proposition.

For the term \( 2 \), by the support properties, we have

\[
2^{j_{\text{max}}} \sim \max(2^{j_{\text{med}}}, |H|),
\]

where \( |H| \sim |\xi_{\text{max}}|^{4} |\xi_{\text{min}}| \) and \( |\xi_{\text{max}}| = \max(|\xi_{1}|, |\xi_{2}|, |\xi_{1} + \xi_{2}|) \), \( |\xi_{\text{min}}| = \min(|\xi_{1}|, |\xi_{2}|, |\xi_{1} + \xi_{2}|) \). Since in this case \( |H| \ll 2^{3k_{3}}, \) then max\((j_{1}, j_{2}) \geq 5k_{3} - 4. \) By Corollary 3.2 (c) we get

\[
II \lesssim \sup_{t_{k} \in \mathbb{R}} \sum_{j_{2} \geq 2k_{3}} 2^{j_{2}/2} \sum_{j_{1} \geq 2k_{3}} 2^{j_{1}/2} 2^{j_{2}/2} 2^{-j_{3}} 2^{j_{3}/2} 2^{-25k_{3}/2} 2k_{3}/2 \| f_{k_{1}, j_{1}} \|_{L^{2}} \| f_{k_{2}, j_{2}} \|_{L^{2}}
\]

\[
\lesssim \sup_{t_{k} \in \mathbb{R}} \sum_{j_{2} \geq 2k_{3}} 2^{j_{2}/2} \sum_{j_{1} \geq 2k_{3}} 2^{j_{1}/2} 2^{j_{2}/2} \| f_{k_{1}, j_{1}} \|_{L^{2}} \| f_{k_{2}, j_{2}} \|_{L^{2}} \lesssim \| u_{k_{1}} \|_{F_{k_{1}}} \| u_{k_{2}} \|_{F_{k_{2}}}.
\]

Thus we finish the proof.

**Proposition 3.5** (high-high \( \Rightarrow \) high). Let \( k_{3} \geq 20, |k_{1} - k_{2}|, |k_{2} - k_{3}| \leq 4, \) then we have

\[
\| P_{k_{3}} \partial_{x}^{3}(u_{k_{1}}v_{k_{2}}) \|_{N_{k_{3}}} + \| P_{k_{3}}(\partial_{x}^{3}u_{k_{1}}v_{k_{2}}) \|_{N_{k_{3}}} \lesssim 2^{-\frac{2}{3}k_{3}} \| u_{k_{1}} \|_{F_{k_{1}}} \| v_{k_{2}} \|_{F_{k_{2}}}.
\]

**Proof.** As in the proof of Proposition 3.4, both terms of the left-hand side of (3.11) are bounded by

\[
\sup_{t_{k} \in \mathbb{R}} 2^{3k_{3}} \sum_{j_{3} \geq 0} 2^{j_{3}/2} \beta_{k_{3}, j_{3}} \sum_{j_{1}, j_{2} \geq 0} (2^{j_{3}} + 2^{2k_{3}})^{-1} \| f_{k_{1}, j_{1}} \cdot f_{k_{2}, j_{2}} \|_{L^{2}}.
\]

We may assume \( 2^{j_{\text{max}}} \sim 2^{5k_{3}} \), otherwise it is easier to handle in view of (3.10) and \( |H| \sim 2^{5k_{3}}. \) Then by Corollary 3.2 (a) and (2.2) we get

\[
\lesssim 2^{-\frac{3}{4}k_{3}/2} \| u_{k_{1}} \|_{F_{k_{1}}} \| v_{k_{2}} \|_{F_{k_{2}}},
\]

and hence finish the proof of the proposition.

\[\square\]
Proposition 3.6 (high-high ⇒ low). (a) Let $k_2 \geq 20$, $|k_1 - k_2| \leq 4$, $1 \leq k_3 \leq k_2 - 10$, then we have

\[
\|P_{k_3} \partial_x (\partial_x^2 u_{k_1} v_{k_2})\|_{N_{k_3}} + \|P_{k_3} \partial_x (\partial_x u_{k_1} \partial_x v_{k_2})\|_{N_{k_3}} \lesssim 2^{(k_2-k_3)/2} (2^{k_2/2-2k_3} + 2^{-3k_3}) \|u_{k_1}\| F_{k_1} \|v_{k_2}\| F_{k_2}. \tag{3.13}
\]

(b) Let $k_2 \geq 20$, $|k_1 - k_2| \leq 4$, $k_3 = 0$, then we have

\[
\|P_{k_3} \partial_x (\partial_x^2 u_{k_1} v_{k_2})\|_{N_{k_3}} + \|P_{k_3} \partial_x (\partial_x u_{k_1} \partial_x v_{k_2})\|_{N_{k_3}} \lesssim 2^{2k_1} \|u_{k_1}\| F_{k_1} \|v_{k_2}\| F_{k_2}. \tag{3.14}
\]

Proof. (a) Since $k_3 \leq k_2 - 10$, we observe that the first term is dominated by other terms. By the definition of $X_{k_3}$ and $N_{k_3}$ one take $X^{0, 1}_{\iota}$-structure on time intervals of length $2^{-2k_3}$, while $\|u_{k_1}\| F_{k_1}$ and $\|v_{k_2}\| F_{k_2}$ are taken in smaller intervals of length $2^{-2k_2}$. We make a partition of intervals of length $2^{-2k_3}$. Let $\gamma : \mathbb{R} \to [0, 1]$ denote a smooth function supported in $[-1, 1]$ with $\sum_{m \in \mathbb{Z}} \gamma^2(x - m) \equiv 1$. The left-hand side of (3.13) is dominated by

\[
C \sup_{t_k \in \mathbb{R}} 2^{k_3} 2^{2k_2} \left\| (\tau - w(\xi) + i 2^{2k_3})^{-1} 1_{j_3} \right\| \cdot \sum_{|m| \leq C \cdot 2^{2k_2 - 2k_3}} \mathcal{F}[u_{k_1} \eta_0 (2^{2k_3} (t - t_k)) \gamma (2^{2k_3} (t - t_k) - m)] \tag{3.15}
\]

\[
\ast \mathcal{F}[u_{k_1} \eta_0 (2^{2k_3} (t - t_k)) \gamma (2^{2k_3} (t - t_k) - m)] \bigg|_{X_{k_3}}.
\]

As in the proof of Proposition 3.4 (3.15) dominated by

\[
2^{k_2} 2^{2k_2} \sum_{j_3 \geq 2k_2} 2^{j_3/2} \beta_{k_3,j_3} 2^{2k_2 - 2k_3} \sum_{j_1,j_2 \geq 2k_2} \left\| (\tau - w(\xi) + i 2^{2k_3})^{-1} 1_{D_{k_3,j_3}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2}) \right\| L^2. \tag{3.16}
\]

By Corollary 3.2 (b), we have

\[
2^{4k_2 - k_3} \sum_{j_3 \geq 2k_3} \sum_{j_1,j_2 \geq 2k_2} 2^{-j_3/2} \beta_{k_3,j_3} 2^{(j_1+j_2+j_3)/2} 2^{-3k_2/2} 2^{-(k_3+j_1)/2} \left\| f_{k_1,j_1} \right\| L^2 \left\| f_{k_2,j_2} \right\| L^2.
\]

If $2^{j_3} \sim 2^{j_{max}}$, take $j_1 = j_3$. Then by the support property (3.10), we get

\[
\left.<3.15> \lesssim 2^{4k_2 - k_3} \sum_{j_1,j_2 \geq 2k_2} 2^{-j_3/2} \beta_{k_3,j_3} 2^{(j_1+j_2)/2} 2^{-3k_2/2} 2^{-(k_3+j_1)/2} \left\| f_{k_1,j_1} \right\| L^2 \left\| f_{k_2,j_2} \right\| L^2
\]

\[
\lesssim 2^{k_2 + \frac{3}{2}k_3} \sum_{j_1,j_2 \geq 2k_2} 2^{(j_1+j_2)/2} \left\| f_{k_1,j_1} \right\| L^2 \left\| f_{k_2,j_2} \right\| L^2.
\]

Otherwise, since $j_3 \leq 4k_2 + k_3 - 10$, (3.16) can be rewritten

\[
2^{4k_2 - k_3} \sum_{2k_3 \leq j_3 \leq 4k_2 + k_3 - 10} \sum_{j_1,j_2 \geq 2k_2} 2^{-j_3/2} \beta_{k_3,j_3} 2^{(j_1+j_2)/2} 2^{-3k_2/2} 2^{-(k_3+j_1)/2} \left\| f_{k_1,j_1} \right\| L^2 \left\| f_{k_2,j_2} \right\| L^2.
\]
Take $j_i = j_{\text{max}}$. Then

\[(3.15) \lesssim 2^{k_2-k_3} \sum_{j_1,j_2 \geq 2k_2} 2^{(k_2-k_3)/2} 2^{(j_1+j_2)/2} 2^{-3k_2/2} 2^{-k_2/2} 2^{-j_{\text{max}}/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2} \lesssim 2^{(k_2-k_3)/2} 2^{-\frac{3}{2}k_3} \sum_{j_1,j_2 \geq 2k_2} 2^{(j_1+j_2)/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2}.
\]

(b) The left-hand side of (3.14) is dominated by

\[
C \sup_{t_k \in \mathbb{R}} 2^{2k_2} \left\| \xi(\tau - w(\xi) + i)^{-1} 1_{I_0} \cdot \mathcal{F}[u_{k_1} \eta_0(t - t_k)] \ast \mathcal{F}[v_{k_2} \eta_0(t - t_k)] \right\|_{X_0}. \tag{3.17}
\]

We decompose further the low frequency, then

\[
(3.17) \lesssim \sum_{l \leq 0} \sup_{t_k \in \mathbb{R}} 2^l \left( \sum_{j_3 \leq 4k_2 + l - 5} + \sum_{j_3 \geq 4k_2 + l + 5} + \sum_{j_3 = 4k_2 - l \leq 5} \right) 2^{k_2} 2^{-j_3} \left\| 1_{D_{l,j_3}} \mathcal{F}[u_{k_1} \eta_0(t - t_k)] \ast \mathcal{F}[v_{k_2} \eta_0(t - t_k)] \right\|_{X_0} = I + II + III.
\]

From the support property (3.16), the first case includes $j_3 = j_{\text{min}}$ and $j_3 = j_{\text{med}}$ with $2^{j_{\text{med}}} \ll 2^{j_{\text{max}} \sim |H|}$ cases, and we regard the second one as $2^{j_3} = 2^{j_{\text{max}}} \sim 2^{j_{\text{med}}} \geq |H|$ case. The last term is regarded as $j_3 = j_{\text{max}}$ with $2^{j_3} \sim |H|$ case.

For $I, II$ cases, we use same argument to (3.15):

\[
I \lesssim 2^{4k_2} \sum_{l \leq 0} 2^l \sum_{j_3 \leq 4k_2 + l - 5} 2^{-j_3/2} \beta_{0,j_3} \sum_{j_1,j_2 \geq 2k_2} 1_{D_{l,j_3}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2}) \|L^2,
\]

and

\[
II \lesssim 2^{4k_2} \sum_{l \leq 0} 2^l \sum_{j_3 \geq 4k_2 + l + 5} 2^{-j_3/2} \beta_{0,j_3} \sum_{j_1,j_2 \geq 2k_2} 1_{D_{l,j_3}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2}) \|L^2.
\]

First, we consider $I$.

From Corollary 3.2 (b), we estimate that

\[
I \lesssim \sum_{l \leq 0} 2^{l+4k_2} \sum_{j_3 \leq 4k_2 + l - 5} \sum_{j_1,j_2 \geq 2k_2} 2^{-j_3/2} \beta_{0,j_3} 2^{(j_1+j_3)/2} 2^{-3k_2/2} 2^{-(k_1+j_3)/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2}.
\]

Since $\beta_{0,j_3} = \frac{2^{j_3/2}}{2}$, by taking $j_i = j_{\text{max}}$ and performing $j_3$ summation, we have

\[
I \lesssim 2^{4k_2} \sum_{l \leq 0} 2^l \sum_{j_3 \leq 4k_2 + l - 5} \sum_{j_1,j_2 \geq 2k_2} 2^{j_3/2} 2^{(j_1+j_3)/2} 2^{-3k_2/2} 2^{-k_2/2} 2^{-j_{\text{max}}/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2} \lesssim 2^{2k_2} \sum_{l \leq 0} 2^{l-2} \sum_{j_3 \leq 4k_2 + l - 5} \sum_{j_1,j_2 \geq 2k_2} 2^{j_3/2} 2^{(j_1+j_3)/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2}
\]

\[
\lesssim 2^{2k_2} \sum_{l \leq 0} 2^l \sum_{j_1,j_2 \geq 2k_2} 2^{(j_1+j_3)/2} \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2}
\]

\[
\lesssim 2^{2k_2} \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}}.
\]
Now, we consider \( II \).

In this case, since \( 2^{j_{\text{max}}} \sim 2^{j_{\text{med}}} \), we estimate from Corollary 3.2 (c) that

\[
II \lesssim 2^{4k_2} \sum_{l_0 \leq 0} 2^{l_0} \sum_{j_3 > 4k_2 + l_0 + 1} \sum_{j_1, j_2 \geq 2k_2} 2^{-j_3/2} \beta_{0,j_3} 2^{j_{\text{min}}/2} 2^{l_0/2} 2^{j_{\text{med}}/2} 2^{-j_{\text{med}}/2} \| f_{k_1,j_1} \| L^2 \| f_{k_2,j_2} \| L^2 \\
\lesssim 2^{4k_2} \sum_{l_0 \leq 0} 2^{l_0} \sum_{j_3 > 4k_2 + l_0 + 1} \sum_{j_1, j_2 \geq 2k_2} 2^{j_{\text{min}}/2} 2^{l_0/2} 2^{j_{\text{med}}/2} 2^{-j_{\text{med}}/2} \| f_{k_1,j_1} \| L^2 \| f_{k_2,j_2} \| L^2 \\
\lesssim 2^{4k_2} \sum_{l_0 \leq 0} 2^{l_0} \sum_{j_3 > 4k_2 + l_0 + 1} \sum_{j_1, j_2 \geq 2k_2} 2^{-j_3/2} 2^{j_{\text{min}}/2} 2^{l_0/2} 2^{j_{\text{med}}/2} 2^{-j_{\text{med}}/2} \| f_{k_1,j_1} \| L^2 \| f_{k_2,j_2} \| L^2 \\
\lesssim 2^{4k_2} \sum_{l_0 \leq 0} 2^{l_0} \sum_{j_3 > 4k_2 + l_0 + 1} \sum_{j_1, j_2 \geq 2k_2} 2^{(j_1 + j_2)/2} \| f_{k_1,j_1} \| L^2 \| f_{k_2,j_2} \| L^2 \\
\lesssim 2^{2k_2} \| u_{k_1} \| F_{k_1} \| v_{k_2} \| F_{k_2}.
\]

For the rest term \( III \), we get from Plancherel’s identity and \( X_k \) embedding that

\[
(3.17) \lesssim \sum_{l_0 \leq 0} \sup_{t \in \mathbb{R}} 2^{l_0} \left\| 1_{|\xi| \sim 2^l} \mathcal{F}[u_{k_1,\eta_0}(t - t_k)] \ast \mathcal{F}[v_{k_2,\eta_0}(t - t_k)] \right\|_{L^2} \\
\lesssim \sup_{t \in \mathbb{R}} 2^{2k_2} \left\| u_{k_1,\eta_0}(t - t_k) \cdot v_{k_2,\eta_0}(t - t_k) \right\|_{L^2_t L^2_x} \lesssim 2^{2k_2} \| u_{k_1} \| L^\infty_t L^2_x \| v_{k_2} \| L^\infty_t L^2_x \\
\lesssim \sup_{t_1, t_2 \in \mathbb{R}} 2^{2k_2} \| u_{k_1,\eta_0}(2^{k_1}(t - t_1)) \|_{L^\infty_t L^2_x} \| v_{k_2,\eta_0}(2^{k_2}(t - t_2)) \|_{L^\infty_t L^2_x} \\
\lesssim 2^{2k_2} \| u_{k_1} \| F_{k_1} \| v_{k_2} \| F_{k_2}.
\]

Proposition 3.7 (low-low \( \Rightarrow \) low). Let \( 0 \leq k_1, k_2, k_3 \leq 200 \), then we have

\[
\| P_{k_3} \partial_x^3 (u_{k_1} v_{k_2}) \|_{N_{k_3}} + \| P_{k_3} (\partial_x^3 u_{k_1} v_{k_2}) \|_{N_{k_3}} \lesssim \| u_{k_1} \| F_{k_1} \| v_{k_2} \| F_{k_2} \tag{3.18}
\]

Proof. As in the proof of Proposition 3.5 use (3.4), then we can get (3.18).

As a conclusion to this section we prove the bilinear estimates, using the dyadic bilinear estimates obtained above.

Proposition 3.8. (a) If \( s \geq 1 \), \( T \in (0,1] \), and \( u, v \in F^s(T) \) then

\[
\| \partial_x (\partial_x u \partial_x v) \|_{N_s(T)} + \| \partial_x (\partial_x^2 uv) \|_{N_s(T)} + \| \partial_x (u \partial_x^2 v) \|_{N_s(T)} \\
\lesssim \| u \|_{F^1(T)} \| v \|_{F^1(T)} + \| u \|_{F^s(T)} \| v \|_{F^s(T)}. \tag{3.19}
\]

(b) If \( T \in (0,1] \), \( u \in F^0(T) \) and \( v \in F^2(T) \), then

\[
\| \partial_x (\partial_x u \partial_x v) \|_{N^0(T)} + \| \partial_x (\partial_x^2 uv) \|_{N^0(T)} + \| \partial_x (u \partial_x^2 v) \|_{N^0(T)} \lesssim \| u \|_{F^0(T)} \| v \|_{F^2(T)}. \tag{3.20}
\]

Proof. The proof follows from the dyadic bilinear estimates and Young’s inequality. See [7] for a similar proof.
Remark 3.9. The equation \((1.12)\) has an additional term \(u^2u_x\). To prove Corollary \(4.2\), we need to show the nonlinear estimate for this cubic term. In fact, this term is much easier to handle. In view of the proof of each Lemma, one need to use Lemma 4.1.

\[
\sum_{j \geq 2k} 2^j \| I_{D_{k,j}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2}) \|_{L^2}.
\]

Indeed, since there is no derivative, we get from the block estimates that

\[
\sum_{j \geq 2k} 2^j \| I_{D_{k,j}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2}) \|_{L^2} \lesssim 2^{2k_1 + \frac{k}{2}} \| u_1 \|_{F_{k_1}} \| u_2 \|_{F_{k_2}},
\]

which implies

\[
\| \partial_x (u_1u_2u_3) \|_{N^2} \lesssim \| u_1 \|_{F^2} \| u_2 \|_{F^2} \| u_3 \|_{F^2}.
\]

Moreover, we also get

\[
\| \partial_x (u^2v) \|_{N^0} \leq \| u \|_{F^2} \| v \|_{N^2}.
\]

4. Energy estimates

In this section we prove the energy estimates, following the idea in \([11]\). We introduce a new Littlewood-Paley decomposition with smooth cut-offs. With

\[
\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1}), \quad k \in \mathbb{Z}.
\]

Let \(\tilde{P}_k\) denote the operator on \(L^2(\mathbb{R})\) defined by the Fourier multiplier \(\chi_k(\xi)\). Assume that \(u, v \in C([-T,T]; L^2)\) and

\[
\begin{cases}
\partial_t u + \partial_x^2 u = v, & (x,t) \in \mathbb{R} \times (-T,T) \\
u(0,x) = u_0(x)
\end{cases}
\]

Then we multiply by \(u\) and integrate to conclude that

\[
\sup_{|t_k| \leq T} \| u(t_k) \|_{L^2}^2 \leq \| u_0 \|_{L^2}^2 + \sup_{|t_k| \leq T} \left| \int_{\mathbb{R} \times [0,t_k]} u \cdot v dx dt \right|.
\]

Lemma 4.1. Let \(T \in (0,1]\) and \(k_1, k_2, k_3 \in \mathbb{Z}_+\).

(a) Assume \(|k_{\max} - k_{\min}| \leq 5\) and \(u_i \in F_{k_i}(T), i = 1, 2, 3\). Then

\[
\left| \int_{\mathbb{R} \times [0,T]} u_1 u_2 u_3 dx dt \right| \lesssim 2^{-\frac{7}{2}k_{\max}} \prod_{i=1}^3 \| u_i \|_{F_{k_i}(T)}
\]

(b) Assume \(k_{\max} \geq 10, 2^{k_{\min}} \ll 2^{k_{\med}} \sim 2^{k_{\max}}\) and \(u_i \in F_{k_i}(T), i = 1, 2, 3\). Then

\[
\left| \int_{\mathbb{R} \times [0,T]} u_1 u_2 u_3 dx dt \right| \lesssim 2^{-2k_{\max} - \frac{3}{2}k_{\min}} \prod_{i=1}^3 \| u_i \|_{F_{k_i}(T)}
\]
(c) Assume \( k_1 \leq k - 10 \). Then

\[
\left| \int_{\mathbb{R} \times [0,T]} \tilde{P}_k(u) \tilde{P}_k(\partial_x^3 u \cdot \tilde{P}_{k_1} v) dx dt \right| \lesssim 2^{2k_1} \| \tilde{P}_{k_1} v \|_{F_{k_1}(T)} \sum_{|k' - k| \leq 10} \| \tilde{P}_{k'}(u) \|_{F_{k'}(T)}^2 \quad (4.5)
\]

(d) Under the same condition as in (c), we have

\[
\left| \int_{\mathbb{R} \times [0,T]} \tilde{P}_k(u) \tilde{P}_k(\partial_x^2 u \cdot \tilde{P}_{k_1} \partial_x v) dx dt \right| \lesssim 2^{2k_1} \| \tilde{P}_{k_1} v \|_{F_{k_1}(T)} \sum_{|k' - k| \leq 10} \| \tilde{P}_{k'}(u) \|_{F_{k'}(T)}^2 \quad (4.6)
\]

**Proof.** For (a) and (b), we may assume that \( k_1 \leq k_2 \leq k_3 \) by symmetry. We fix extensions \( \tilde{u}_i \in F_{k_i} \) so that \( \| \tilde{u}_i \|_{F_{k_i}} \leq 2 \| u_i \|_{F_{k_i}(T)}, i = 1, 2, 3 \). Let \( \gamma : \mathbb{R} \to [0, 1] \) be a smooth partition of unity function (i.e. \( \text{supp } \gamma \subset [-1, 1] \)) and \( \sum_{n \in \mathbb{Z}} \gamma^3(x - n) \equiv 1, x \in \mathbb{R} \). The left-hand side of (4.3) and (4.4) is bounded by

\[
C \sum_{|n| \leq 2^{2k_3}} \left| \int_{\mathbb{R} \times \mathbb{R}} (\gamma(2^{2k_3} t - n)1_{[0,T]}(t) \tilde{u}_1) \cdot (\gamma(2^{2k_3} t - n) \tilde{u}_2) \cdot (\gamma(2^{2k_3} t - n) \tilde{u}_3) dx dt \right| \quad (4.7)
\]

Set \( A = \{ n : \gamma(2^{2k_3} t - n)1_{[0,T]}(t) \text{ non-zero and } \neq \gamma(2^{2k_3} t - n) \} \). Then one observe \( |A| \leq 4 \).

(a) First consider the summation over \( n \in A^c \). Let \( f_{k_i} = \mathcal{F}(\gamma(2^{2k_3} t - n) \tilde{u}_i) \) and \( f_{k_i,j_i} = \eta_{j_i}(\tau - \xi^5) f_{k_i}, i = 1, 2, 3 \). By Parseval’s formula and (2.2), we have

\[
|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3})| \lesssim \sup_{n \in A^c} 2^{2k_3} \sum_{j_1,j_2,j_3 \geq 0} |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3})|. \quad (4.8)
\]

By (2.2) and the support properties (3.10) we may assume \( j_1, j_2, j_3 \geq 2k_3, 2^{j_{\max}} \sim 2^{5k_3} \). Then using Lemma 3.1 (a), we get that

\[
(4.8) \lesssim \sum_{j_1,j_2,j_3 \geq 2k_3} 2^{2k_3} 2^{j_{\min}} 2^{j_{\text{med}}}/4 2^{-3k_{\max}/4} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2} \leq 2^{-\frac{7}{4} k_{\max}} \prod_{i=1}^{3} \| u_i \|_{F_{k_i}(T)}.
\]

For the summation over \( n \in A \), it is easy to handle since \( |A| \leq 4 \). Indeed, we observe that if \( I \subset \mathbb{R} \) is an interval, \( k \in \mathbb{Z}^+, f_k \in X_k \), and \( f_k = \mathcal{F}(1_I(t) \cdot F^{-1}(f_k)) \) then

\[
\sup_{j \in \mathbb{Z}^+} 2^{j/2} \| \eta_j(\tau - \omega(\xi)) \cdot f_k \|_{L^2} \lesssim \| f_k \|_{X_k}.
\]

See [7] for the proof.
(b) This part is a little trickier to prove. Thus, to overcome a trouble, we will use following form, which is different from (4.7), and it suffices to prove that

$$\left| \sum_{n \in A^c} \int_{\mathbb{R}^2} (\eta_0(2^{2k_1}t - 2^{2k_1} - 2^{2k_3}n)\tilde{u}_1)(\gamma^2(2^{2k_3}t - n)\tilde{u}_2)(\gamma(2^{2k_3}t - n)\tilde{u}_3)dxdt \right| \lesssim 2^{-2k_2} \prod_{i=1}^{3} \|u_i\|_{F_{k_i}}. $$  

(4.9)

Let \( f_1 = \eta_0(t - 2^{-2k_3}n)\tilde{u}_1 \) and \( f_i = \gamma(2^{2k_3}t - n)\tilde{u}_i, i = 2, 3. \)

First we consider the case \( k_1 \geq 1. \) Similarly, we get (4.8) and only need to consider the sum over \( n \in A^c. \) By Lemma 3.1 (b), the left-hand side of (4.9) is dominated by

$$\sum_{j_1 \geq 2k_1} \sum_{j_2,j_3 \geq 2k_3} 2^{2k_3}2^{(j_1+j_2+j_3)/2}2^{-3k_{max}/2}2^{-(k_i+j_1)/2} \prod_{i=1}^{3} \|f_{k_{i,j_i}}\|_{L^2}. $$  

(4.10)

If \( j_1 \neq j_{max}, \) take \( j_i = j_{max}. \) Then since \( 2^{k_2} \sim 2^{k_3}, \) (3.10) yields

$$2^{-3k_{max}-k_{min}} \prod_{i=1}^{3} \sum_{j_i \geq 2k_i} 2^{j_i/2} \|f_{k_{i,j_i}}\|_{L^2} \lesssim \prod_{i=1}^{3} \|u_i\|_{F_{k_i}}. $$

If \( j_1 = j_{max}, \) take \( j_i = j_1. \) Then from a similar argument, we get

$$2^{-3k_{max}-k_{min}} \prod_{i=1}^{3} \sum_{j_i \geq 4k_3+k_1-5} 2^{j_1/2} \|f_{k_{1,j_1}}\|_{L^2} \lesssim \prod_{i=1}^{3} \|u_i\|_{F_{k_i}}, $$

where we used \( \beta_{k_1,j_1} \geq 2^{(k_3-k_1)/2}. \)

Next we assume \( k_1 = 0. \) In this case, we only need to consider the sum over \( n \in A^c. \) We decompose further the low frequency component \( f_1 = \sum_{l \leq 0} f_{1,l} \) with \( f_{1,l} = F^{-1}_{\xi \sim 2^{-l-30}}F f_1. \)

Then LHS of (4.9) \( \lesssim \sum \left| \sum_{n \in A^c} \int_{\mathbb{R} \times \mathbb{R}} f_{1,l} \cdot f_2 \cdot f_3 dxdt \right| \)

\( \leq \sum_{l \leq 0} \left| \sum_{n \in A^c} \int_{\mathbb{R} \times \mathbb{R}} f_{1,l}^H \cdot f_2 \cdot f_3 dxdt \right| + \sum_{l \leq 0} \left| \sum_{n \in A^c} \int_{\mathbb{R} \times \mathbb{R}} f_{1,l}^L \cdot f_2 \cdot f_3 dxdt \right| = I + II, \)

where \( F(f_{1,l}^L) = \eta_{\leq 4k_2 + l - 30}(\tau - \xi^5)F f_{1,l}, \) and \( f_{1,l}^L = f_{1,l} - f_{1,l}^H. \) The term II is easy to handle (high frequency with large modulation). Indeed,

\( II \lesssim 2^{2k_3} \sum_{l \leq 0} 2^l \sum_{j_1,j_2,j_3 \geq 0} |J(f_{k_{1,j_1}}, f_{k_{2,j_2}}, f_{k_{3,j_3}})| \)

\( \lesssim 2^{2k_3} \sum_{l \leq 0} 2^l \sum_{j_1,j_2,j_3 \geq 0} 2^{(j_1+j_2+j_3)/2}2^{-2k_2}2^{-\max(j_2,j_3)/2} \prod_{i=1}^{3} \|f_{k_{i,j_i}}\|_{L^2} \leq 2^{-2k_2} \prod_{i=1}^{3} \|u_i\|_{F_{k_i}}, \)

(4.10)
since \( \max(j_2, j_3) \geq l + 4k_2 - 5 \) by the support property.

Now we deal with the term \( I \). By orthogonality, we may assume \( f_2, f_3 \) has frequency support in a ball of size \( 2^l \):

\[
I \leq \left( \sum_{l \leq -2k_2} + \sum_{-2k_2 \leq l \leq 0} \right) \left| \sum_{n \in A^c} \int_{\mathbb{R} \times \mathbb{R}} f^H_{1,l} \cdot f_2 \cdot f_3 \, dx dt \right| = I_1 + I_2.
\]

For the term \( I_1 \), by Hölder’s inequality and using a weight, we have

\[
I_1 \lesssim 2^{2k_3} \sum_{l \leq -2k_2} 2^l \sum_{j_1, j_2, j_3 \geq 0} |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3})|
\]

\[
\lesssim 2^{2k_3} \sum_{l \leq -2k_2} 2^l \sum_{j_1, j_2, j_3 \geq 0} 2^{2j_2/2} \prod_{i=1}^3 \| \tilde{u}_i \|_{F_{k_i}} \lesssim 2^{-2k_2} \prod_{i=1}^3 \| \tilde{u}_i \|_{F_{k_i}},
\]

since by (2.2) we may assume \( j_2, j_3 \geq 2k_2 \).

(c) We denote the commutator of \( T_1, T_2 \) by \([T_1, T_2] = T_1 T_2 - T_2 T_1\). Then the left hand side of (4.5) is dominated by

\[
\left| \int_{\mathbb{R} \times [0,T]} \tilde{P}_k(u) \tilde{P}_k(\partial_x^2 u) \tilde{P}_k(v) \, dx dt \right| + \left| \int_{\mathbb{R} \times [0,T]} \tilde{P}_k(u) \tilde{P}_k(v) (\partial_x^3 u) \, dx dt \right| =: I + II.
\]

Using \( uu_{xxx} = \frac{1}{2}(u^2)_{xxx} - 3(u_x^2)_{x} \) and integration by parts, we get

\[
I \lesssim \left| \int_{\mathbb{R} \times [0,T]} \partial_x^3 ((\tilde{P}_k(u))^2) \tilde{P}_k(v) \, dx dt \right| + \left| \int_{\mathbb{R} \times [0,T]} \partial_x ((\tilde{P}_k(\partial_x u))^2) \tilde{P}_k(v) \, dx dt \right|
\]

\[
\lesssim \left| \int_{\mathbb{R} \times [0,T]} \tilde{P}_k(u) \tilde{P}_k(\partial_x^2 v) dx dt \right| + \left| \int_{\mathbb{R} \times [0,T]} \tilde{P}_k(u_x) \tilde{P}_k(u_x) \tilde{P}_k(\partial_x v) \, dx dt \right|
\]

\[
=: I_1 + I_2.
\]

Then, use (4.4) to conclude that

\[
I_1 + I_2 \lesssim 2^{-2k-4k_1} 2^{2k_1} \| \tilde{P}_k(u) \|^2_{F_{k}(T)} \| \tilde{P}_k(v) \|_{F_{k_1}(T)}
\]

\[
+ 2^{-2k-2k_1} 2^{2k_1} 2^{2k_1} \| \tilde{P}_k(u) \|^2_{F_{k}(T)} \| \tilde{P}_k(v) \|_{F_{k_1}(T)}
\]

which suffices for (4.5).
To control $II$, we use the formula
\[
\mathcal{F}([\widetilde{P}_k, \widetilde{P}_{k_1}(v)](\partial^3_x u))(\xi, \tau)
= C \int_{\mathbb{R}^2} \mathcal{F}(\widetilde{P}_{k_1}(\partial^3_x u))(\xi_1, \tau_1) \cdot \mathcal{F}(u)(\xi - \xi_1, \tau - \tau_1) \cdot m(\xi, \xi_1) d\xi_1 d\tau_1,
\]
where
\[
m(\xi, \xi_1) = \frac{|(\xi - \xi_1)^3(\chi_k(\xi) - \chi_k(\xi - \xi_1))|}{\xi_1^3} \lesssim \sum_{|k-k'| \leq 4} \chi_{k'}(\xi - \xi_1).
\]

By the Parseval’s theorem and (4.4), we estimate
\[
II \lesssim 2^{-2k-\frac{1}{4}k_1} 2^{3k_1} \| \widetilde{P}_{k_1}(v) \|_{F_{k_1}(T)} \sum_{|k-k'| \leq 4} \| \widetilde{P}_{k'}(u) \|_{F_{k'}(T)}^2 \lesssim \text{RHS of (4.3)}.
\]

(d) is proved similarly and so we omit the detail. \(\square\)

**Proposition 4.2.** Let $T \in (0, 1]$ and $u \in C([-T, T] : H^s)$ be a solution to (1.1). Let $s \geq 5/4$. Then we have
\[
\|u\|_{E^s(T)}^2 \lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^{5/4}(T)}^2 \|u\|_{H^s(T)}^2.
\]

**Proof.** From the definition we have
\[
\|u\|_{E^s(T)}^2 - \|P_{\leq 0}(u_0)\|_{L^2}^2 \leq \sum_{k \geq 1} \sup_{t_k \in [-T, T]} 2^{2sk} \|\widetilde{P}_k(u(t_k))\|_{L^2}^2.
\]

Then we get from (4.2)
\[
2^{2sk} \|\widetilde{P}_k(u(t_k))\|_{L^2}^2 - 2^{2sk} \|\widetilde{P}_k(u_0)\|_{L^2}^2 \lesssim 2^{2sk} \left| \int_{\mathbb{R} \times [0, t_k]} \widetilde{P}_k(u) \widetilde{P}_k(u \cdot \partial^3_x u) dxdt \right|
+ 2^{2sk} \left| \int_{\mathbb{R} \times [0, t_k]} \widetilde{P}_k(u) \widetilde{P}_k(u_x \cdot \partial^3_x u) dxdt \right|
=: I + II.
\]

We further decompose $I$ as follows:
\[
I \lesssim 2^{2sk} \sum_{k_1 \leq k-10} \left| \int_{\mathbb{R} \times [0, t_k]} \widetilde{P}_k(u) \widetilde{P}_{k_1}(u \cdot \partial^3_x u) dxdt \right|
+ 2^{2sk} \sum_{k_1 \geq k-9, k_2 \in \mathbb{Z}_+} \left| \int_{\mathbb{R} \times [0, t_k]} \widetilde{P}_{k_2}^2(u) \widetilde{P}_{k_1}(u \cdot \partial^3_x \widetilde{P}_{k_2}(u)) dxdt \right|
:= I_1 + I_2.
\]

Using (4.5) then we get that
\[
I_1 \lesssim 2^{2sk} \sum_{k_1 \leq k-10} 2^{xk_1} \|\widetilde{P}_{k_1}(u)\|_{F_{k_1}(T)} \sum_{|k-k'| \leq 10} \|\widetilde{P}_{k'}(u)\|_{F_{k'}(T)}^2
\]
\[
\lesssim 2^{2sk} \|\widetilde{P}_k(u)\|_{F_k(T)}^2 \|u\|_{F^{1/2}(T)}^2.
\]
Similarly as above, we obtain

\[ k \approx \] which implies that the summation on \( k \) is bounded by \( \| u \|_{F^5/4(T)} \),

For \( I_2 \), using (4.3) and (4.4) then we get

\[ I_2 \lesssim 2^{2sk} \sum_{|k_1-k_2| \leq 10} 2^{3k_2} 2^{-2k_{\text{max}} - \frac{1}{2} k_{\text{min}}} \| \tilde{P}_k(u) \|_{F_k(T)} \| \tilde{P}_{k_1}(u) \|_{F_{k_1}(T)} \| \tilde{P}_{k_2}(u) \|_{F_{k_2}(T)} \]

\[ + 2^{2sk} \sum_{|k_1-k| \leq 10} 2^{3k_2} 2^{-2k_{\text{max}} - \frac{1}{2} k_{\text{min}}} \| \tilde{P}_k(u) \|_{F_k(T)} \| \tilde{P}_{k_1}(u) \|_{F_{k_1}(T)} \| \tilde{P}_{k_2}(u) \|_{F_{k_2}(T)} \]

\[ + 2^{2sk} \sum_{|k_1-k| \leq 10} 2^{3k_2} 2^{-2k_{\text{max}} - \frac{1}{2} k_{\text{min}}} \| \tilde{P}_k(u) \|_{F_k(T)} \| \tilde{P}_{k_1}(u) \|_{F_{k_1}(T)} \| \tilde{P}_{k_2}(u) \|_{F_{k_2}(T)} \]

\[ =: I_{2,1} + I_{2,2} + I_{2,3}. \]

For \( I_{2,1} \), since \( k \leq k_1 \), by the Cauchy-Schwarz inequality

\[ I_{2,1} \lesssim 2^{sk} \sum_{|k_1-k_2| \leq 10} 2^{3k_1} 2^{2k_2} \| \tilde{P}_k(u) \|_{F_k(T)} \| \tilde{P}_{k_1}(u) \|_{F_{k_1}(T)} \| \tilde{P}_{k_2}(u) \|_{F_{k_2}(T)} \]

\[ \lesssim 2^{sk} 2^{-(1/2+\delta)k} \| u \|_{F^4(T)} \| u \|_{F^{1+}(T)} \| \tilde{P}_k(u) \|_{F_k(T)}. \]

Similarly as above, we obtain

\[ I_{2,2} \lesssim 2^{2sk} \sum_{|k' - k| \leq 10} \| \tilde{P}_{k'}(u) \|_{F^5/4(T)}^2, \]

and

\[ I_{2,3} \lesssim 2^{sk} 2^{-\frac{1}{2}k} \| u \|_{F^4(T)} \| u \|_{F^{1+}(T)} \| \tilde{P}_k(u) \|_{F_k(T)}, \]

which implies that the summation on \( k \) of \( I \) is bounded by \( \| u \|_{F^5/4(T)} \| u \|_{F^4(T)}^2 \).

For \( II \), using the same method as \( I \) and (4.6), we have

\[ \sum_{k \geq 1} II \lesssim \| u \|_{F^5/4(T)} \| u \|_{F^4(T)}^2. \]

Therefore, we complete the proof of the proposition. \( \square \)

**Remark 4.3.** To get the energy estimates for trilinear term \( u^2 \partial_x u \) in (1.2), from (4.2), we need to control

\[ \sum_{k \geq 1} 2^{4k} \left| \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(u) \tilde{P}_k(u^2 \partial_x u) dx dt \right| \lesssim \sum_{k \geq 1} 2^{4k} \sum_{k_1 \leq k-10} \left| \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(u) \tilde{P}_k(\tilde{P}_{k_1}(u^2 \partial_x u) dx dt \right| \]

\[ + \sum_{k \geq 1} 2^{4k} \sum_{k_1 \geq k-9} \sum_{k_2 \geq k-10} \left| \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(u) \tilde{P}_{k_1}(u^2) \tilde{P}_{k_2}(\partial_x u) dx dt \right| =: I + II. \]

In view of the proof of Lemma 4.4 and Proposition 4.2, it is not difficult to obtain

\[ I + II \lesssim \| u \|_{L^2}^4, \]
We now devote to derive the estimate on \( \text{(4.2)} \), we need to control
\[
\|u_0\|_{H^s} + \|v_0\|_{H^s} \leq \epsilon \ll 1
\]
Then we have
\[
\|u - v\|_{F^0(1)} \lesssim \|u_0 - v_0\|_{L^2}
\]
and
\[
\|u - v\|_{F^s(1)} \lesssim \|u_0 - v_0\|_{H^s} + \|u_0\|_{H^{2s}} \|u_0 - v_0\|_{L^2}.
\]

**Proposition 4.4.** Assume \( s \geq 2 \). Let \( u, v \in F^s(1) \) be solutions to (\ref{1.1}) with small initial data \( u_0, v_0 \in H^\infty \) in the sense of
\[
\|u_0\|_{H^s} + \|v_0\|_{H^s} \leq \epsilon \ll 1
\]
Then we have
\[
\|u - v\|_{F^0(1)} \lesssim \|u_0 - v_0\|_{L^2}
\]
\[
\|u - v\|_{F^s(1)} \lesssim \|u_0 - v_0\|_{H^s} + \|u_0\|_{H^{2s}} \|u_0 - v_0\|_{L^2}.
\]

**Proof.** We prove first (\ref{4.13}). Since \( \|u_0\|_{H^s} + \|v_0\|_{H^s} \leq \epsilon \ll 1 \), we assume from (\ref{5.10}) that
\[
\|u\|_{F^s(1)} \ll 1, \|u\|_{F^s(1)} \ll 1
\]
Let \( w = u - v \), then \( w \) solves the equation
\[
\begin{align*}
\partial_t w + \partial_x^5 w + c_1^2 \partial_x (\partial_x w \partial_x (u + v)) + c_2^2 \partial_x (v \partial_x^2 w + w \partial_x^2 u) \\
w(0, x) = w_0(x) = u_0(x) - v_0(x)
\end{align*}
\]
From the linear and bilinear estimates, we obtain
\[
\begin{align*}
\|w\|_{F^0(1)} & \lesssim \|w\|_{F^0(1)} + \|\partial_x (\partial_x w \partial_x (u + v))\|_{N^0(1)} + \|\partial_x (v \partial_x^2 w + w \partial_x^2 u)\|_{N^0(1)} \\
\|\partial_x (\partial_x w \partial_x (u + v))\|_{N^0(1)} + \|\partial_x (v \partial_x^2 w + w \partial_x^2 u)\|_{N^0(1)} & \lesssim \|w\|_{F^0(1)} \|u\|_{F^s(1)} + \|v\|_{F^s(1)}.
\end{align*}
\]
We now devote to derive the estimate on \( \|w\|_{F^0(1)} \). From
\[
\|w\|_{F^0(1)}^2 - \|w_0\|_{L^2}^2 \lesssim \sum_{k \geq 1} \sup_{t_k} \|w(t_k)\|_{L^2}^2
\]
and (\ref{4.22}), we need to control
\[
\sum_{k \geq 1} \|w(t_k)\|_{L^2}^2 \lesssim \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(w) \tilde{P}_k(w \partial_x^3 u) dx dt \right|
\]
\[
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(w) \tilde{P}_k(v \partial_x^2 w) dx dt \right|
\]
\[
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(w) \tilde{P}_k(\partial_x w \partial_x^2 u) dx dt \right|
\]
\[
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(w) \tilde{P}_k(\partial_x v \partial_x^2 u) dx dt \right|
\]
\[
=: I + II + III + IV.
\]
Using (\ref{3.21}).
Using (4.3) and (4.4), $I$ is bounded by

$$
\sum_{k \geq 1} \sum_{|k-k_1| \leq 5} \sum_{k_2 \leq k-10} 2^{-2k_{max}-\frac{1}{2}k_{min}} 2^{3k_2} \|w\| F_k(1) \|w\| F_{k_1}(1) \|u\| F_{k_2}(1) \\
+ \sum_{k \geq 1} \sum_{|k-k_2| \leq 5} \sum_{k_1 \leq k-10} 2^{-2k_{max}-\frac{1}{2}k_{min}} 2^{3k_2} \|w\| F_k(1) \|w\| F_{k_2}(1) \|u\| F_{k_1}(1) \\
+ \sum_{k \geq 1} \sum_{|k-k_2| \leq 5} \sum_{k \leq k-10} 2^{-\frac{5}{4}k_{max}} 2^{3k_2} \|w\| F_{k_1}(1) \|w\| F_{k_2}(1) \|u\| F_{k_1}(1) \\
+ \sum_{k \geq 1} \sum_{k \leq k-10} \sum_{k_2 \leq k-5} 2^{-2k_{max}-\frac{1}{2}k_{min}} 2^{3k_2} \|w\| F_{k_1}(1) \|w\| F_{k_2}(1) \|u\| F_{k_2}(1)
$$

For the first part of (4.19), since $2^{k_{max}} \gtrsim 2^{k_2}$, using the Cauchy-Schwarz inequality, we have

$$
\sum_{k \geq 1} \left( \sum_{k_2 \geq 0} 2^{-2\delta k_2} \right)^{1/2} \lesssim \|w\| F_{0}^{2} \|u\| F^{2}(1).
$$

For the last term of (4.19), since $k + 10 \leq k_1, k_2$, by the Cauchy-Schwarz inequality we have

$$
\|u\| F^{2}(1) \sum_{k_1 \geq 1} 2^{-\frac{1}{4}k} \|w\| F_{k_1}(1) 2^{-\delta k_1} \|w\| F_{k_1}(1) \lesssim \|w\| F_{0}^{2} \|u\| F^{2}(1).
$$

For the second and the third parts of (4.19), similarly as above we get

$$
\sum_{k_1 \geq 1} 2^{-\delta k_1} \|w\| F_{k_1}(1) \|w\| F_{0}^{2} \|u\| F^{2}(1) \lesssim \|w\| F_{0}^{2} \|u\| F^{2}(1),
$$

and

$$
\sum_{k \geq 1} \sum_{k-5 \leq k_1, k_2 \leq k+5} 2^{\frac{5}{2}k_2} \|w\| F_{k_1}(1) \|w\| F_{k_2}(1) \|u\| F_{k_2}(1) \lesssim \|w\| F_{0}^{2} \|u\| F^{2}(1).
$$

For $II$, using Lemma 4.11 again, $II$ is dominated by

$$
\sum_{k \geq 1} \sum_{k_1 \leq k-10} 2^{k_1/2} \|v\| F_{k_1}(1) \|w\| F_{k_1}(1) \\
+ \sum_{k \geq 1} \sum_{|k-k_1| \leq 5} \sum_{k_2 \leq k-10} 2^{3k_2} 2^{-2k_{max}-\frac{1}{2}k_{min}} \|w\| F_{k}(1) \|v\| F_{k_1}(1) \|w\| F_{k_2}(1) \\
+ \sum_{k \geq 1} \sum_{|k-k_2| \leq 5} \sum_{k \leq k-10} 2^{3k_2} 2^{-2k_{max}-\frac{1}{2}k_{min}} \|w\| F_{k}(1) \|v\| F_{k_1}(1) \|w\| F_{k_2}(1) \\
+ \sum_{k \geq 1} \sum_{|k-k_2| \leq 5} \sum_{|k-k_2| \leq 5} 2^{3k_2} 2^{-2k_{max}-\frac{1}{2}k_{min}} \|w\| F_{k}(1) \|v\| F_{k_1}(1) \|w\| F_{k_2}(1).
$$

From the Cauchy-Schwarz inequality, the bound of the first term of (4.21) is easily obtained. For the second and third terms of (4.21), since $2^{k_2} \lesssim 2^{k_1}$ and $2^{k_1} \sim 2^{k_{max}} \sim 2^{k_{med}}$, it is
bounded by
\[ \sum_{k \geq 1} \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} 2^{-\delta k_1} 2^{(1+\delta)k_1} 2^{-\frac{1}{2}k_{min}} \|v\|_{F^1(1)} \|w\|_{F^2(1)} \|w\|_{F^k(1)} \]
\[ \lesssim \|v\|_{F^1(1)} \sum_{k \geq 1} \sum_{k_1 \geq 0} 2^{-\delta k} 2^{-\delta k_2} \|w\|_{F^k(1)} \|w\|_{F^k(1)} \]
\[ \lesssim \|w\|^2_{F^0(1)} \|v\|_{F^s(1)}. \]

The estimate of the rest term is similar to (4.20).

Similarly to \( I, II \), we can get
\[ III + IV \lesssim \|w\|^2_{F^0(1)} (\|u\|_{F^s(1)} + \|v\|_{F^s(1)}). \]

Therefore, we obtain the following estimate
\[ \|w\|^2_{E^0(1)} \lesssim \|w_0\|^2_{L^2} + \|w\|^2_{F^0(1)} (\|u\|_{F^s(1)} + \|v\|_{F^s(1)}), \]

hence, combined with (4.17) and (4.15) we obtain (4.13).

Now we prove (4.14). From the linear and bilinear estimates, we obtain
\[ \begin{cases} 
\|w\|_{F^s(1)} \lesssim \|w\|_{E^s(1)} + \|\partial_x (\partial_x w \partial_x (u + v))\|_{N^s(1)} + \|\partial_x (v \partial^2_x w + w \partial^2_x u)\|_{N^s(1)} \\
\|\partial_x (\partial_x w \partial_x (u + v))\|_{N^s(1)} + \|\partial_x (v \partial^2_x w + w \partial^2_x u)\|_{N^0(1)} \lesssim \|w\|_{F^s(1)} (\|u\|_{F^s(1)} + \|v\|_{F^s(1)}). 
\end{cases} \]

(4.22)

Since \( \|P_{\leq 0}(w)\|_{E^s(1)} = \|P_{\leq 0}(w_0)\|_{L^2} \), it follows from (4.22) and (4.15) that
\[ \|w\|_{F^s(1)} \lesssim \|w_0\|_{H^s} + \|P_{\geq 1}(w)\|_{E^s(1)}. \]

(4.23)

To bound \( \|P_{\geq 1}(w)\|_{E^s(1)} \), we observe that
\[ \|P_{\geq 1}(w)\|_{E^s(1)} = \|P_{\geq 1}(\Lambda^s w)\|_{E^0(1)}, \]

where \( \Lambda^s \) is the Fourier multiplier operator with the symbol \( |\xi|^s \). Thus we apply the operator \( \Lambda^s \) on both side of the (4.16) and get
\[ \partial_t \Lambda^s w + \partial^5_x \Lambda^s w = -c_1 \Lambda^s w \partial^3_x u + -c_1 \Lambda^s v \partial^3_x w \\
- c_2 \Lambda^s \partial_x w \partial^2_x u + -c_2 \Lambda^s \partial_x v \partial^2_x w. \]

We rewrite the nonlinearity in the following way:
\[ c_1 [\Lambda^s, w] \partial^2_x u + c_1 w \Lambda^s \partial^2_x u + c_1 [\Lambda^s, v] \partial^2_x w + c_1 v \Lambda^s \partial^3_x w \\
c_2 [\Lambda^s, \partial_x w] \partial^2_x u + c_2 \partial_x w \Lambda^s \partial^2_x u + c_2 [\Lambda^s, v \partial_x v] \partial^2_x w + c_2 \partial_x v \Lambda^s \partial^2_x w. \]

We write the equation for \( U = P_{\geq -10}(\Lambda^s w) \) in the form
\[ \begin{cases} 
\partial_t U + \partial^5_x U = P_{\geq -10}(-c_1 v \partial^3_x) + P_{\geq -10}(-c_2 \partial_x v \partial^2_x U) + P_{\geq -10}(G) + P_{\geq -10}(H) \\
U(0) = P_{\geq -10}(\Lambda^s w_0) 
\end{cases} \]

(4.24)
where

\[
G = -c_1 P_{\geq -10}(v) \Lambda^s \partial_x^3 P_{\leq -11}(w) - c_1 P_{\leq -11}(v) \Lambda^s \partial_x^3 P_{\leq -11}(w) \\
- c_1 [\Lambda^s, w] \partial_x^3 u - c_1 [\Lambda^s, v] \partial_x^3 w - c_1 w \Lambda^s \partial_x^3 u,
\]

and

\[
H = -c_2 P_{\geq -10}(\partial_x v) \Lambda^s \partial_x^2 P_{\leq -11}(w) - c_2 P_{\leq -11}(\partial_x v) \Lambda^s \partial_x^2 P_{\leq -11}(w) \\
- c_2 [\Lambda^s, \partial_x w] \partial_x^2 u - c_2 [\Lambda^s, \partial_x v] \partial_x^2 w - c_2 \partial_x w \Lambda^s \partial_x^2 u.
\]

It follows from (4.2) and (4.24) that

\[
\|U\|_{E^0(1)}^2 - \|w_0\|_{H^s}^2 \lesssim \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k(w \partial_x^3 U) dx dt \right] \\
+ \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k(G) dx dt \right] \\
+ \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k(\partial_x v \partial_x^2 U) dx dt \right] \\
+ \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k(H) dx dt \right] \\
= : I + II + III + IV.
\]

First, consider I and III. We can bound I, III as in (4.18) and get that

\[
I + III \lesssim \|U\|_{E^0(1)}^2 \|v\|_{F^s(1)}.
\]

For II, we estimate

\[
II \lesssim \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k(P_{\geq -10}(v) \Lambda^s \partial_x^3 P_{\leq -11}(w)) dx dt \right] \\
+ \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k([\Lambda^s, w] \partial_x^3 u) dx dt \right] \\
+ \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k([\Lambda^s, v] \partial_x^2 w) dx dt \right] \\
+ \sum_{k \geq 1} \left[ \int_{\mathbb{R} \times [0,t_k]} \tilde{P}_k(U) \tilde{P}_k(w \Lambda^s \partial_x^3 u) dx dt \right] \\
= : II_1 + II_2 + II_3 + II_4.
\]
For $II_1$, since the derivatives fall on the law frequency, we get

$$II_1 \lesssim \sum_{|k-k_1|\leq 5} \sum_{k_2\leq k-10} 2^{-2k_{\max} - \frac{1}{2}k_{\min}} \|U\|_{F_0(1)} \|v\|_{F_0(1)} \|A^s w\|_{F_2(1)}$$

$$+ \sum_{|k-k_2|\leq 5} \sum_{k_1\leq k-10} 2^{-2k_{\max} - \frac{1}{2}k_{\min}} \|U\|_{F_0(1)} \|v\|_{F_0(1)} \|A^s w\|_{F_2(1)}$$

$$+ \sum_{|k_1-k_2|\leq 5} \sum_{|k-k_2|\leq 10} 2^{-2k_{\max} - \frac{1}{2}k_{\min}} \|U\|_{F_0(1)} \|v\|_{F_0(1)} \|A^s w\|_{F_2(1)}$$

$$+ \sum_{|k-k_1|\leq 5} \sum_{|k-k_2|\leq 5} 2^{-\frac{2}{5}k_{\max}} \|U\|_{F_0(1)} \|v\|_{F_0(1)} \|A^s w\|_{F_2(1)}$$

$$\lesssim \|U\|_{F_0(1)}^2 \|v\|_{F^s(1)},$$

which comes from Lemma 4.1 and the Cauchy-Schwarz inequality.

We consider now $II_4$. Using Lemma 4.1 again,

$$II_4 \lesssim \sum_{k_1,k_2 \geq 0} \left| \int_{\mathbb{R} \times [0,t]} \tilde{P}_k(U) \tilde{P}_{k_1}(w) \Lambda^s \partial_x^3 \tilde{P}_{k_2}(u) dx dt \right|$$

$$\lesssim \sum_{|k-k_2|\leq 5} \sum_{k_1\leq k-10} 2^{-2k_{\max} - \frac{1}{2}k_{\min} 2(3+s)k_2} \|\tilde{P}_k U\|_{F_0(1)} \|\tilde{P}_{k_1}(w)\|_{F_0(1)} \|\tilde{P}_{k_2}(u)\|_{F_2(1)}$$

$$+ \sum_{|k-k_2|\leq 5} \sum_{k_1\leq k-10} 2^{-2k_{\max} - \frac{1}{2}k_{\min} 2(3+s)k_2} \|\tilde{P}_k U\|_{F_0(1)} \|\tilde{P}_{k_1}(w)\|_{F_0(1)} \|\tilde{P}_{k_2}(u)\|_{F_2(1)}$$

$$+ \sum_{|k-k_2|\leq 5} \sum_{k_1\leq k-10} 2^{-2k_{\max} - \frac{1}{2}k_{\min} 2(3+s)k_2} \|\tilde{P}_k U\|_{F_0(1)} \|\tilde{P}_{k_1}(w)\|_{F_0(1)} \|\tilde{P}_{k_2}(u)\|_{F_2(1)}$$

$$+ \sum_{|k-k_1|\leq 5} \sum_{|k-k_2|\leq 5} 2^{-\frac{2}{5}k_{\max}} \|\tilde{P}_k U\|_{F_0(1)} \|\tilde{P}_{k_1}(w)\|_{F_0(1)} \|\tilde{P}_{k_2}(u)\|_{F_2(1)}.$$

For the first term above, since $2^{-2k_{\max} - \frac{1}{2}k_{\min} 2(3+s)k_2} \lesssim 2^{(1+s+\delta)k_2} 2^{-\delta k_2/2} 2^{-\delta k_2/2} 2^{-\frac{1}{2}k_1}$, we have the bound

$$\|U\|_{F_0(1)} \|w\|_{F_0(1)} \|u\|_{F^{2s}(1)}.$$

Otherwise, as $2^{k_2} \lesssim 2^{k_1}$ implies

$$2^{-2k_{\max} - \frac{1}{2}k_{\min} 2(3+s)k_2} \lesssim 2^{s k_1} 2^{5k_2/4} 2^{-k_{\max}/4}$$

or

$$2^{-\frac{2}{5}k_{\max} 2(3+s)k_2} \lesssim 2^{s k_1} 2^{5k_2/4},$$

the last terms above are bounded by

$$\|U\|_{F_0(1)}^2 \|u\|_{F^s(1)}.$$. 
(1.25) and (1.26) are similarly used in later inequality repeatedly. For $II_3$, we estimate

$$II_3 \lesssim \sum_{k \geq 1} \sum_{k_1 \geq k_2 - 10} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(U) \tilde{P}_k([\Lambda^s, \tilde{P}_{k_1}(v)] \partial_x^3 \tilde{P}_{k_2}(w)) dx dt \right|$$

$$+ \sum_{k \geq 1} \sum_{k_1 \geq k_2 - 9} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(U) \tilde{P}_k([\Lambda^s, \tilde{P}_{k_1}(v)] \partial_x^3 \tilde{P}_{k_2}(w)) dx dt \right|$$

$$=: II_{3,1} + II_{3,2}.$$  

We note that in the term $II_{3,2}$ the component $v$ can spare derivative and from a similar way to $II_4$ we get

$$II_{3,2} \lesssim \sum_{k \geq 1} \sum_{k_1 \geq k_2 - 9} \left| \int_{\mathbb{R} \times [0, t_k]} \Lambda^s \tilde{P}_k(U) \tilde{P}_k([\Lambda^s, \tilde{P}_{k_1}(v)] \partial_x^3 \tilde{P}_{k_2}(w)) dx dt \right|$$

$$+ \sum_{k \geq 1} \sum_{k_1 \geq k_2 - 9} \left| \int_{\mathbb{R} \times [0, t_k]} \tilde{P}_k(U) \tilde{P}_k([\Lambda^s, \tilde{P}_{k_1}(v)] \partial_x^3 \tilde{P}_{k_2}(w)) dx dt \right|$$

$$\lesssim \|U\|_{F^0(1)}^{2} \|v\|_{F^s(1)},$$

where we used $2^{k_1} \sim 2^k \sim 2^{k_{\max}}$ and a similar argument to (1.25).

For $II_{3,1}$, we need to exploit the cancellation of the commutator. By taking $\gamma$ and extending $U, v, w$ as in the proof of Lemma 4.1, then we get

$$II_{3,1} \lesssim \sum_{k \geq 1} \sum_{k_1 \leq k_2 - 10} \sum_{|n| \leq 2^{k_3}} \left| \int_{\mathbb{R} \times [0, t_k]} (\gamma(2^{k_3} t - n) 1_{[0, t_k]}(t) \tilde{P}_k(U)) \times \tilde{P}_k([\Lambda^s, \tilde{P}_{k_1}(\gamma(2^{k_3} t - n)v)] \partial_x^3 \tilde{P}_{k_2}(\gamma(2^{k_3} t - n)w)) dx dt \right|.$$  

Let $f_k = \gamma(2^{k_3} t - n) \tilde{P}_k(U)$, $g_{k_1} = \tilde{P}_{k_1}(\gamma(2^{k_3} t - n)v)$ and $h_{k_2} = \tilde{P}_{k_2}(\gamma(2^{k_3} t - n)w)$. It is easy to see from $|k_2 - k| \leq 5$ that

$$|\mathcal{F}([\Lambda^s, g_{k_1}] \partial_x^3 h_{k_2})(\xi, \tau)| \lesssim \int_{\mathbb{R}^2} |\hat{g}_{k_1}(\xi - \xi_1, \tau - \tau_1)| 2^{3k_1} 2^{k_2} |\hat{h}_{k_2}(\xi_1, \tau_1)| d\xi_1 d\tau_1,$$

which follows similarly to (4.11). Then using the same argument in the proof of Lemma 4.1 we can get that

$$II_{3,1} \lesssim \sum_{k \geq 1} \sum_{k_1 \leq k_2 - 10} 2^{3k_1} 2^{k_2} 2^{-k_{\max} - \frac{1}{2} k_{\min} -} \|\tilde{P}_k U\|_{F_k(1)} \|\tilde{P}_{k_1}(v)\|_{F_{k_1}(1)} \|\tilde{P}_{k_2}(w)\|_{F_{k_2}(1)}$$

$$\lesssim \|U\|_{F^0(1)}^{2} \|v\|_{F^s(1)}.$$  

from a similar way to (1.25).

The $II_2$ is identical to the one of $II_3$ from symmetry.

So, we now need to control $IV$, but controlling $IV$ is similar and easier than $II$ since derivatives are distributed.
Hence we have proved that
\[ \|U\|_{E^0(1)}^2 \lesssim \|w_0\|^2_{H^s} + \|U\|_{E^0(1)}^2(\|u\|_{F^s} + \|v\|_{F^s}) + \|U\|_{E^0(1)}\|w\|_{F^0(1)}\|u\|_{F^2}, \]
By (4.15) and (4.13), we get
\[ \|U\|_{E^0(1)} \lesssim \|u_0 - v_0\|_{H^s} + \|u_0 - v_0\|_{L^2}\|u_0\|_{H^2}, \]
from which combined with (4.24) we completes the proof of the proposition.

5. Proof of Theorem 1.1
In this section, we prove Theorem 1.1. The main ingredients are \( L^2 \)-convolution estimates which is proved in section 3 and energy estimates obtained in section 4. The basic idea follows the idea of Ionescu, Kenig and Tataru [11] and for the weighted \( X_k \) norm, we refer to Guo, Peng, Wang and Wang [9].

**Proposition 5.1.** Let \( s \geq 0, T \in (0, 1] \), and \( u \in F^s(T) \), then
\[ \sup_{t \in [-T, T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^s(T)} \] (5.1)

**Proposition 5.2.** Let \( T \in (0, 1] \), \( u, v \in C([-T, T] : H^\infty) \) and
\[ \partial_t u + \partial_x^2 u = v \] on \( \mathbb{R} \times (-T, T) \) (5.2)
Then we have
\[ \|u\|_{F^s(T)} \lesssim \|u\|_{E^s(T)} + \|v\|_{N^s(T)}, \] (5.3)
for any \( s \geq 0 \).

The proof of the Proposition 5.1 and 5.2 are similar to [11] and [9]. For self-containedness, we give the proof in Appendix.

Now, we show the local well-posedness of (1.1) by using the classical energy method. From Duhamel’s principle, we get that the equation (1.1) is equivalent to the following integral equation,
\[ u(t) = W(t)u_0 + \int_0^t W(t-s)v(s)ds, \] (5.4)
where \( v(t, x) = c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u. \)
We will work on the following localized version,
\[ u(t) = \eta_0(t)W(t)u_0 + \eta_0(t) \int_0^t W(t-s)v(s)ds. \] (5.5)
Then we see that if \( u \) is a solution to (5.5) on \( \mathbb{R} \), then \( u \) solves (5.4) on \([-1, 1]\).

By the scaling invariance:
\[ u_\lambda(t, x) = \lambda^{-2} u\left(\frac{t}{\lambda^5}, \frac{x}{\lambda}\right), \] (5.6)
and observing $s_c = -\frac{3}{2}$, we may assume that
\[
\|u_0\|_{H^s} \leq \epsilon \ll 1. \tag{5.7}
\]
For part (a) of Theorem 11.1, we assume that $s \geq \frac{5}{4}$.

We already know from 19 that there is a smooth solution to the (1.1) with $u_0 \in H^\infty$. So, we show a priori bound: if $T \in (0, 1]$ and $u \in C([-T, T] : H^\infty)$ is a solution of (1.1) with $\|u_0\|_{H^s} \leq \epsilon \ll 1$, then
\[
\sup_{t \in [-T, T]} \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}. \tag{5.8}
\]

It comes from the linear estimate (Proposition 5.2), the $L^2$ estimate (Proposition 3.8 (a)) and the energy estimate (Proposition 4.2). More precisely, for any $T' \in [0, T]$ we have
\[
\left\{ \begin{array}{l}
\|u\|_{F^s(T')} \lesssim \|u\|_{E^s(T')} + \|\partial_2^3(u^2)\|_{N^s(T')} + \|u\partial_2^3 u\|_{N^s(T')} \tag{5.9} \\
\|\partial_2^3(u^2)\|_{N^s(T')} + \|u\partial_2^3 u\|_{N^s(T')} \lesssim \|u\|_{F^s(T')}^2 \tag{5.10} \\
\end{array} \right.
\]

Let $X(T') = \|u\|_{E^s(T')} + \|\partial_2^3(u^2)\|_{N^s(T')} + \|u\partial_2^3 u\|_{N^s(T')}$. From similar argument as in the proof of Lemma 4.2 in 11, we know that $X(T')$ is continuous, increasing on $[-T, T]$ and satisfies
\[
\lim_{T' \to 0} X(T') \lesssim \|u_0\|_{H^s}.
\]

Moreover, we obtain from (5.9) that
\[
X(T')^2 \lesssim \|u_0\|_{H^s}^2 + X(T')^3 + X(T')^4.
\]

If $\epsilon$ is small enough, then using bootstrap (see 22), $X(T') \lesssim \|u_0\|_{H^s}$ can be obtained by (5.7). Hence we obtain
\[
\|u\|_{F^s(T')} \lesssim \|u_0\|_{H^s}, \tag{5.10}
\]

which implies (5.8) by Proposition 5.1.

For part (b), we now assume $s \geq 2$ so that we use Proposition 3.8 (b) and Proposition 4.4. In order to obtain a solution in $H^s$, we use compactness argument which follows the ideas in 11. Fix $u_0 \in H^s$. Then we can choose \(\{u_{0,n}\}_{n=1}^{\infty} \subset H^\infty\) such that $u_{0,n} \to u_0$ in $H^s$ as $n \to \infty$. Let $u_{n}(t) \in H^\infty$ is a solution to (1.1) with initial data $u_{0,n}$. Then it suffices to show that the sequence $\{u_n\}$ is a Cauchy sequence in $C([-T, T] : H^s)$. For $K \in \mathbb{Z}_+$, let $u_{n}^K = P_{\leq K}(u_{0,n})$. Then since
\[
\sup_{t \in [-T, T]} \|u_m - u_n\|_{H^s} \lesssim \sup_{t \in [-T, T]} \|u_m - u_m^K\|_{H^s} + \sup_{t \in [-T, T]} \|u_m^K - u_n^K\|_{H^s} + \sup_{t \in [-T, T]} \|u_m^K - u_n\|_{H^s},
\]

it suffices to show that for any $\epsilon > 0$ and $K$, we have
\[
\sup_{t \in [-T, T]} \|u_n - u_n^K\|_{H^s} \leq \frac{\epsilon}{3} \quad \text{and} \quad \sup_{t \in [-T, T]} \|u_m^K - u_n^K\|_{H^s} \leq \frac{\epsilon}{3}, \tag{5.11}
\]
for sufficiently large $n, m$. First, choose large $K$ such that $\|u_0 - u^K_0\|_{H^s} \leq o(1)$. Then since $u^K_{0,n} \to u^K_0$ in $H^s$ for any $K$, we get

$$\sup_{t \in [-T, T]} \|u^K_m - u^K_n\|_{H^s} \lesssim \|u^K_m - u^K_n\|_{F^s(T)}$$

$$\lesssim \|u^K_{0,m} - u^K_{0,n}\|_{H^s} + \|u^K_{0,n}\|_{H^s} + \|u^K_{0,m} - u^K_{0,n}\|_{L^2}$$

$$\lesssim \|u^K_{0,m} - u^K_0\|_{H^s} + \|u^K_{0,n} - u^K_0\|_{H^s},$$

for large $m, n$ and by Proposition 5.1 and 4.1. And this gives the second part of (5.11). From same argument to above and $\|u_{0,n} - u_0\|_{H^s} \to 0$ for large $n$, we get the first part of (5.11). Hence, we complete the existence of a solution. The uniqueness of the solution and the last part of Theorem 1.1 comes from the classical energy method, the scaling (5.5), and Proposition 5.1. We omit the detail.

**APPENDIX A.**

In appendix, we collect proofs of Proposition 5.1 and Proposition 5.2 for convenience of readers. Similar proofs are found in [11, 7, 9].

**Proof of Proposition 5.1.**

We use extended formula $\tilde{u}_k$ instead of $u$ as in proof of Proposition 3.8. First, our observation is that,

$$\sup_{t \in [-T, T]} \|u(t)\|_{H^s} \approx \sum_{k \geq 0} 2^{2sk} \sup_{t_k \in [-T, T]} \|\mathcal{F}_x[u_k(t_k)]\|_{L^2(\mathbb{R})} \quad (A.1)$$

and

$$\|u(t)\|^2_{F^s(T)} \approx \sum_{k \geq 0} 2^{2sk} \sup_{t_k \in [-T, T]} \|\mathcal{F}[^{\eta_0}(2^k(t - t_k))u_k]\|^2_{X_k} \quad (A.2)$$

From comparing (A.1) and (A.2), it is enough to prove that

$$\|\mathcal{F}_x[\tilde{u}_k(t_k)]\|_{L^2(\mathbb{R})} \lesssim \|\mathcal{F}[^{\eta_0}(2^k(t - t_k))\tilde{u}_k]\|_{X_k} \quad (A.3)$$

for $k \in \mathbb{Z}_+$, $t_k \in [-T, T]$ and $\tilde{u}_k \in F_k$, which is an extension of $u_k$. (Precise explanation of an extension will be mentioned in the proof of Proposition 5.2 later.)

Let $f_k = \mathcal{F}[^{\eta_0}(2^k(t - t_k))\tilde{u}_k]$, then we have

$$\mathcal{F}_x[\tilde{u}_k(t_k)](\xi) = c \int_{\mathbb{R}} f_k(\tau, \xi)e^{it_k\tau}d\tau.$$

From (2.4), we have

$$\|\mathcal{F}_x[\tilde{u}_k(t_k)]\|_{L^2_{\xi}} \lesssim \left\| \int_{\mathbb{R}} f_k(\tau, \xi)e^{it_k\tau}d\tau \right\|_{L^2_{\xi}} \lesssim \|f_k\|_{X_k},$$

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which implies completion of (A.3) and completes the proof of the Proposition 5.1.

Proof of Proposition 5.2.

To prove this proposition, we see from the definitions that the square of the right-hand side of (5.3) as following,

\[ \|P_{\leq 0}(u(0))\|_{L^2}^2 + \|P_{\leq 0}(v)\|_{N_0(T)}^2 + \sum_{k \geq 1} \left( \sup_{t_k \in [-T, T]} 2^{2sk}\|P_k(u(t_k))\|_{L^2}^2 + 2^{2sk}\|P_k(v)\|_{N_k(T)}^2 \right). \]

Thus, from definitions, it is enough to prove that

\[
\begin{cases}
\|P_{\leq 0}(u)\|_{F_0(T)} \lesssim \|P_{\leq 0}(u(0))\|_{L^2} + \|P_{\leq 0}(v)\|_{N_0(T)}; \\
\|P_k(u)\|_{F_k(T)} \lesssim \sup_{t_k \in [-T, T]} \|P_k(u(t_k))\|_{L^2} + \|P_k(v)\|_{N_k(T)} \quad \text{if} \ k \geq 1.
\end{cases}
\]  

(A.4)

for \( k \in \mathbb{Z}_+ \) and \( u, v \in C([-T, T] : H^\infty) \) which solve (5.2).

Step 1: Extension of \( P_k(u) \).

Fix \( k \geq 0 \) and \( \tilde{v} \) denote an extension of \( P_k(v) \) such that \( \|\tilde{v}\|_{N_k} \leq C\|v\|_{N_k(T)} \). In view of (2.5), we may assume that \( \tilde{v} \) is supported in \( \mathbb{R} \times [-T - 2^{-2k-10}, T + 2^{-2k-10}] \). More precisely, let \( \theta(t) \) be a smooth function such that

\[ \theta(t) = 1, \quad \text{if} \quad t \geq 1; \quad \theta(t) = 0, \quad \text{if} \quad t \leq 0. \]

Let \( m_k(t) = \theta(2^{2k+10}(t + T + 2^{-2k-10})\theta(-2^{2k+10}(t - T - 2^{-2k-10})) \). Then \( m_k \in S_k \) and we see that \( m_k \) is supported in \( [-T - 2^{-2k-10}, T + 2^{-2k-10}] \) and equal to 1 in \( [-T, T] \). From (2.5), we consider \( \tilde{v} \) instead of \( m_k(t)v \). We define for \( t \geq T \),

\[ \tilde{u}(t) = \eta_0(2^{2k+5}(t - T)) \left[ W(t - T)P_k(u(T)) + \int_T^t W(t - s)P_k(\tilde{v}(s))ds \right], \]

and for \( t \leq -T \),

\[ \tilde{u}(t) = \eta_0(2^{2k+5}(t + T)) \left[ W(t + T)P_k(u(-T)) + \int_t^{-T} W(t - s)P_k(\tilde{v}(s))ds \right]. \]

For \( t \in [-T, T] \), we define \( \tilde{u}(t) = u(t) \). It is obvious that \( \tilde{u} \) is an extension of \( u \) and we get from definitions of \( F_k(T) \) and \( F_k \) that

\[ \|u\|_{F_k(T)} \lesssim \sup_{t_k \in [-T, T]} \|F[\tilde{u} \cdot \eta_0(2^{2k}(t - t_k))]|_{X_k}. \]  

(A.5)

Indeed, in view of the definition of \( F_k \), we can get (A.5) if the following holds.

\[ \sup_{t_k \in \mathbb{R}} \|F[\tilde{u} \cdot \eta_0(2^{2k}(t - t_k))]|_{X_k} \lesssim \sup_{t_k \in [-T, T]} \|F[\tilde{u} \cdot \eta_0(2^{2k}(t - t_k))]|_{X_k}. \]

(A.6)

For \( t_k > T \), since \( \tilde{u} \) is supported in \( [-T - 2^{-2k-5}, T + 2^{-2k-5}] \), we can see that

\[ \tilde{u}\eta_0(2^{2k}(t - t_k)) = \tilde{u}\eta_0(2^{2k}(t - T))\eta_0(2^{2k}(t - t_k)). \]
And we get from (2.3) that
\[
\sup_{t_k > T} \| \mathcal{F}[u \cdot \eta_0(2^{2k}(t - t_k))] \|_{X_k} \lesssim \sup_{t_k \in [-T, T]} \| \mathcal{F}[u \cdot \eta_0(2^{2k}(t - t_k))] \|_{X_k}.
\]
Using the same method for \( t_k < -T \), then we obtain (A.5).

**Step 2 : Linear estimates**

For fixed \( k \geq 0 \). In view of the definitions, (A.3) and (2.3), it is enough to prove that if \( \phi_k \in L^2 \) with \( \hat{\phi}_k \) supported in \( I_k \), and \( v_k \in N_k \) with time support in an interval \( I (|I| \leq 2^k) \), then
\[
\| \mathcal{F}[u_k \cdot \eta_0(2^{2k}t)] \|_{X_k} \lesssim \| \phi_k \|_{L^2} + \| (\tau - w(\xi)) + i 2^{2k} \|^{-1} \mathcal{F}(v_k) \|_{X_k}, \tag{A.7}
\]
where
\[
u_k(t) = W(t) \phi_k + \int_0^t W(t - s)v_k(s)ds. \tag{A.8}
\]

Then from the properties of Fourier transform, direct computations show that
\[
\mathcal{F}[u_k \cdot \eta_0(2^{2k}t)](\tau, \xi) = \mathcal{F}_{x}(\phi_k)(\xi) \mathcal{F}_{t}[e^{itw(\xi)} \eta_0(2^{2k}t)](\tau)
+ C \int_{\mathbb{R}} \mathcal{F}(v_k)(\xi, \tau') \frac{\mathcal{F}_{t}(\eta_0)(2^{-2k}(\tau - \tau')) - \mathcal{F}_{t}(\eta_0)(2^{-2k}(\tau - w(\xi)))}{i 2^{2k}(\tau' - w(\xi))} d\tau'. \tag{A.9}
\]

More precisely, for second part of (A.9), consider \( \eta_0(t) \) instead of \( \eta_0(2^{2k}t) \). Then we have
\[
\mathcal{F}[\eta_0(t) \int_0^t W(t - s)u(s)ds] = \mathcal{F}_{t}[\eta_0(t)e^{itw(\xi)} \int_0^t e^{-isw(\xi)} \mathcal{F}_{x}(u)(s)ds] = \mathcal{F}_{t}[\eta_0(t)e^{itw(\xi)} \int_{\mathbb{R}} e^{-isw(\xi)} 1_{[0,1]}(s) \mathcal{F}_{x}(u)(s)ds] = \mathcal{F}_{t}[\eta_0(t)e^{itw(\xi)} (\hat{u} * \mathcal{F}_{s}(1_{[0,1]}))(w(\xi))].
\]

Since
\[
\mathcal{F}_{s}(1_{[0,1]})(\tau) = \frac{e^{-it\tau} - 1}{-i\tau},
\]
we obtain that
\[
\mathcal{F}[\eta_0(t) \int_0^t W(t - s)u(s)ds] = \mathcal{F}_{t}[\eta_0(t)e^{itw(\xi)} \int_{\mathbb{R}} \hat{u}(\tau', \xi) e^{-it(w(\xi)-\tau')} - 1 d\tau']
= \int_{\mathbb{R}^2} e^{-it\eta_0(t)e^{itw(\xi)} \hat{u}(\tau', \xi) e^{-it(w(\xi)-\tau')} - 1 d\tau' dt
= \int_{\mathbb{R}} \hat{u}(\tau', \xi) \mathcal{F}_{t}(\eta_0)(\tau - \tau') - \mathcal{F}_{t}(\eta_0)(\tau - w(\xi)) d\tau'.
\]

Because of \( \mathcal{F}_{t}(f(\lambda t))(\tau) = \lambda^{-1} \mathcal{F}_{t}(f(t))(\lambda^{-1}\tau) \), we get the second part of (A.9).

We consider that the right-hand side of (A.9) separately.
Lemma A.1. Let $\phi_k \in L^2$ with $\phi_k$ supported in $I_k$. Then, for any $k \in \mathbb{Z}_+$, we have

$$\|\mathcal{F}[\eta_0(2^{2k}t)W(t)\phi_k]\|_{X_k} \lesssim \|\phi_k\|_{L^2}.$$  \hfill (A.10)

Proof. Since

$$\|\mathcal{F}[\eta_0(2^{2k}t)W(t)\phi_k]\|_{X_k} = \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - w(\xi))\mathcal{F}_t[e^{i\tau w(\xi)} \eta_0(2^{2k}t)]\mathcal{F}_x(\phi_k)(\xi)\|_{L^2_{\tau,\xi}}.$$ we need to show that

$$\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - w(\xi))\mathcal{F}_t[e^{i\tau w(\xi)} \eta_0(2^{2k}t)]\|_{L^2_{\tau,\xi}} \lesssim 1.$$ \hfill (A.11)

Since $\eta_0$ is a Schwartz function, which decays faster than any polynomial, we can get (A.11), which makes Lemma A.1 to be true. \hfill \Box

Lemma A.2. Let $v_k \in N_k$. Then, for any $k \in \mathbb{Z}_+$,

$$\left\|\mathcal{F}\left[\int_0^t \eta_0(2^{2k}t)W(t-s)v_k(s)ds\right]\right\|_{X_k} \lesssim \|\tau - w(\xi) + i2^{2k}\|^{-1}\mathcal{F}(v_k)\|_{X_k}.$$ \hfill (A.12)

Proof. Consider the second part of (A.9). From the oscillatory integrals for the smooth functions, we observe that

$$\left|\frac{\mathcal{F}_t[\eta_0(2^{-2k}(\tau - \tau'))]}{2^{2k}(\tau' - w(\xi))}\cdot (\tau' - w(\xi) + 2^{2k})\right| \lesssim 2^{-2k}(1 + 2^{-2k}|\tau - \tau'|)^{-4} + 2^{-2k}(1 + 2^{-2k}|\tau - w(\xi)|)^{-4}.$$ So, let $\tilde{v}_k = (\tau - w(\xi) + i2^{2k})^{-1}\mathcal{F}(v_k)$. Then we need to show that

$$\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - w(\xi))\int_{\mathbb{R}} \tilde{v}_k(\xi, \tau')2^{-2k}(1 + 2^{-2k}|\tau - \tau'|)^{-4}d\tau'\|_{L^2} \lesssim \|\tilde{v}_k\|_{X_k}$$ \hfill (A.13)

and

$$\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - w(\xi))\int_{\mathbb{R}} \tilde{v}_k(\xi, \tau')2^{-2k}(1 + 2^{-2k}|\tau - w(\xi)|)^{-4}d\tau'\|_{L^2} \lesssim \|\tilde{v}_k\|_{X_k}.$$ \hfill (A.14)

For (A.14), we observe that

$$\|\eta_j(\tau - w(\xi))\int_{\mathbb{R}} \tilde{v}_k(\xi, \tau')2^{-2k}(1 + 2^{-2k}|\tau - w(\xi)|)^{-4}d\tau'\|_{L^2} \lesssim \|\eta_j(\tau - w(\xi))2^{-2k}(1 + 2^{-2k}|\tau - w(\xi)|)^{-4}\int_{\mathbb{R}} \tilde{v}_k(\xi, \tau')d\tau'\|_{L^2_{\tau'}}.$$ Since $2^{-2k}(1 + 2^{-2k}|\tau - w(\xi)|)^{-4}$ decays faster than $\beta_k(1 + |\tau - w(\xi)|^2)^{1/4}$, we have

$$\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - w(\xi))2^{-2k}(1 + 2^{-2k}|\tau - w(\xi)|)^{-4}\|_{L^2_{\tau'}} \lesssim 1.$$
For the rest term \((j > \frac{1}{2})\), Combining (A.10) and (A.12) completes the proof of Proposition 5.2. Hence, we proved Lemma A.2 completely.

Now consider \(k < j \leq 5k\), we have \(\beta_{k,j} \sim 1\). Thus, we have from (2.4) that
\[
\sum_{2k < j \leq 5k} 2^{j/2} \beta_{k,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_k (\xi, \tau') 2^{-2k} (1 + 2^{-2k} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} 
\]
\[
\lesssim 2^{k/2} \left\| \eta_{\leq 2k} (\tau - w(\xi)) \int R \check{v}_k (\xi, \tau') 2^{-2k} (1 + 2^{-2k} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \lesssim \| \check{v}_k \|_{X_k}.
\]

For \(2k < j \leq 5k\), we also have \(\beta_{k,j} \sim 1\). Thus, we have from (2.4) that
\[
\sum_{2k < j \leq 5k} \sum_{k,j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_{k,j} (\xi, \tau') 2^{-2k} (1 + 2^{-2k} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} 
\]
\[
\lesssim \sum_{2k < j \leq 5k} \sum_{k,j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_{k,j} (\xi, \tau') 2^{-2k} (1 + 2^{-2k} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \lesssim \| \check{v}_k \|_{X_k}.
\]

For the rest term \((j > 5k)\), similarly as before, we get
\[
\sum_{5k < j \leq 1} 2^{j/2} \beta_{k,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_{k,j} (\xi, \tau') 2^{-2k} (1 + 2^{-2k} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} 
\]
\[
\lesssim \sum_{5k < j \leq 1} 2^{j/2} \beta_{k,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_{k,j} (\xi, \tau') 2^{-2k} (1 + 2^{-2k} |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \lesssim \| \check{v}_k \|_{X_k}.
\]

Now consider \(k = 0\). In this case, we can easily derive than \(k \geq 1\) case. Similarly as before, we have
\[
\sum_{j \geq 0} \sum_{j_1 \geq 0} 2^{j/2} \beta_{0,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_{0,j_1} (\xi, \tau') (1 + |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} 
\]
\[
\lesssim \sum_{j \geq 0} \sum_{j_1 \geq 0} 2^{j/2} \beta_{0,j} \left\| \eta_j (\tau - w(\xi)) \int R \check{v}_{0,j_1} (\xi, \tau') (1 + |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \lesssim \| \check{v}_0 \|_{X_0}.
\]

Hence, we proved Lemma A.2 completely. \(\square\)

Combining (A.10) and (A.12) completes the proof of Proposition 5.2. \(\square\)

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\textit{E-mail address: zihuaguo@math.pku.edu.cn}
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School of Mathematical Science, Peking University, Beijing 100871, China, Beijing International Center for Mathematical Research, Beijing 100871, China

E-mail address: ckkwak@kaist.ac.kr

Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, 291 Daehak-ro Yuseong-gu, Daejeon 305-701, South Korea

E-mail address: soonsikk@kaist.edu

Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, 291 Daehak-ro Yuseong-gu, Daejeon 305-701, South Korea