The dual foliation of some singular Riemannian foliations

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Abstract: In this paper, we use methods of subriemannian geometry to study the dual foliation of the singular Riemannian foliation induced by isometric Lie group actions on a complete Riemannian manifold $M$. We show that under some conditions, the dual foliation has only one leaf.

Key Words: dual foliation, torus actions, subriemannian geometry

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1 Introduction

We recall some basic notions about singular Riemannian foliations, for further details, we refer readers to [1, 8, 13, 16]. A singular Riemannian foliation $F$ on a Riemannian manifold $M$ is a decomposition of $M$ into smooth injectively immersed submanfolds $L(x)$, called leaves, such that it is a singular foliation and any geodesic starting orthogonally to a leaf remains orthogonal to all leaves it intersects. Such a geodesic is called a horizontal geodesic. For all $x \in M$, we denote by $H_x$ the orthogonal complement to the tangent space $T_x(L(x))$, and call it the horizontal space at $x$. If all the leaves have the same dimension, then $F$ is called a regular Riemannian foliation.

A curve is called horizontal if it meets the leaves of $F$ perpendicularly. In [16], Wilking associates to a given singular Riemannian foliation $F$ the so-called dual foliation $F^\#$. The dual leaf through a point $p \in M$ is defined as all points $q \in M$ such that there is a piecewise smooth, horizontal curve from $p$ to $q$. Let $L^\#_p$ be the dual leaf through $p$. Wilking proved that when $M$ has positive curvature, the horizontal connectivity holds on $M$, i.e. the dual foliation has only one leaf. Wilking also used the theory of dual foliation to show that the Sharafudinov projection in the soul theorem is smooth. For more applications of dual foliation, the reader is referred to [7, 9, 10, 11, 12, 16].

Let $G$ be a compact connected Lie group. An isometric action of $G$ induces a singular Riemannian foliation $F$ on a Riemannian manifold $M$, throughout we assume that the action of $G$ is effectively. Applying Wilking's theorem, we can get the horizontal connectivity by assuming positive curvature of $M$. However, in this paper we are interested in getting the horizontal connectivity by methods of subriemannian geometry. We may refer readers to [4, 5, 14] for further knowledge about subriemannian geometry.
Let $H$ be the horizontal distribution on $M$, the Lie brackets of vector fields in $H$ generate the flag

$$H = H^1 \subset H^2 \subset \cdots \subset H^r \subset \cdots \subset TM$$

with

$$H^{r+1} = H^r + [H, H^r] \text{ for } r \geq 1$$

where

$$[H, H^r] = \text{span}\{[X, Y] : X \in H, Y \in H^r\}$$

At a point $p \in M$, this flag gives a flag of subspaces of $T_pM$:

$$H_p = H^1_p \subset H^2_p \subset \cdots \subset H^r_p \subset \cdots \subset T_pM$$

**Definition** We say that $H$ is bracket generating at $p$ if there is an $r \in \mathbb{Z}^+$ such that $H^r_p = T_p M$, and $H$ is bracket generating if for all $x \in M$ there is an $r(x) \in \mathbb{Z}^+$ such that $H^r(x) = T_x M$.

The smallest integer $r = r(p)$ such that $H^r_p = T_p M$ is called the step of the distribution at $p$. Set $n_i(p) = \dim H^i_p$, the integer list $(n_1(p), n_2(p), \cdots, n_r(p))$ of dimensions is called the growth vector of $H$ at $p$ (note that $n_1 = k$ is the rank of the $H$ and $n_r = n$ is the dimension of $M$).

**Chow's theorem** ([6]). Let $M$ be a connected manifold and $H \in TM$ be a bracket generating distribution, then the set of points that can be connected to $p \in M$ by a horizontal path coincides with $M$.

When Chow’s condition fails on some subset of $M$, sometimes the horizontal connectivity also fails ([5], P. 82), sometimes the horizontal connectivity still holds ([14], P. 25).

However, to get the horizontal connectivity in a regular Riemannian foliation induced by isometric action of a compact connected Lie group, we only need to assume the horizontal distribution $H$ is bracket-generating at one point:

**Lemma 1.1.** Let a compact connected Lie group $G$ acts isometrically on a complete Riemannian manifold $M$, suppose all of the orbits have the same dimension. If the horizontal distribution $H$ is bracket-generating at some point $p \in M$, then the dual foliation has only one leaf. If $M$ is compact, then there exists a constant $C = C(M, G)$ such that any two points of $M$ can be connected by a horizontal curve of length $\leq C$.

Let $G \times M \rightarrow M$ denote an isometric action by principal orbits of a compact Lie group $G$ on a complete Riemannian manifold $M$. The orbit space $B := M/G$ is also a manifold, and inherits the quotient metric from $M$ for which $\pi : M \rightarrow B$ becomes a Riemannian submersion. $\pi$ is then said to be a homogeneous submersion or fibration.

The following corollary of lemma 1.1 give a generalization of theorem 3.1 in [10].

**Corollary 1.2.** Let $\pi : M \rightarrow B$ be a homogeneous fibration with compact connected Lie group $G$, if the horizontal distribution $H$ is bracket-generating at some point $p \in M$, then $\pi : M \rightarrow B$ is a principal $G$-bundle.
Next we use lemma 1.1 to study the dual foliation of the singular Riemannian foliation induced by isometric torus actions on a complete Riemannian manifold $M$.

**Theorem 1.3.** Let circle $T^1$ acts isometrically on a complete manifold $M$, if $M$ is simple connected, then the dual foliation has only one leaf. If $M$ is compact, then there exists a constant $C = C(M,T^1)$ such that any two points of $M$ can be connected by a horizontal curve of length $\leq C$.

For complete nonnegatively curved manifold with torus actions, we have the following:

**Theorem 1.4.** Let torus acts isometrically on a complete nonnegatively curved Riemannian manifold $M$, then either the dual foliation has only one leaf, or else $M$ locally splits.

The following corollary give a generalization of theorem 4 in [3].

**Corollary 1.5.** Let torus $T^k$ acts isometrically on a complete, simple connected Riemannian manifold $M$, then the dual foliation has only one leaf if one of the following holds:

(a) the action of $T^k$ is free.

(b) $M$ has nonnegative curvature.

In section 2, we shall recall some fundamental facts and give the proof of lemma 1.1. The main results are proved in section 3.

## 2 Preliminaries

The following lemma is proved in [10].

**Lemma 2.1.** Let $\pi : M \to M/G$ be a homogeneous fibration on a manifold. If $\gamma : [0,1] \to M$ is a horizontal geodesic, then the isotropy groups of the action coincide for every point of $\gamma$.

In order to introduce the ball-box theorem, we shall recall some definitions and notations given in [14].

Let $H$ be a horizontal distribution on manifold $M$, denote the dimension of $M$ by $n$, the rank of $H$ by $k$. Assume $H$ is bracket-generating at some point $p \in M$ with step $r$, i.e., $H^r_p = T_p M$. Fix the base point $p$, choose a local orthonormal frame $X_i$, $i = 1, \cdots, k$, for distribution $H$.

For multi-indices $I = (i_1, i_2, \cdots, i_m)$, $1 \leq i_j \leq k$, define vector fields $X_I$ inductively by $X_I = [X_{i_1}, X_{i_2}]$, where $J = (i_2, i_3, \cdots, i_m)$. We’ll denote the length of a multi-indices $I$ by $|I|$, so $|J| = m - 1$.

By the bracket-generating assumption at $p$, we can select a local frame from amongst the $X_I$. We choose such a frame and relabel it $Y_i$, $i = 1, \cdots, n$, to respect the canonical filtration: $\{Y_1(p) = X_1(p), \cdots, Y_k(p) = X_k(p)\}$ span $H^1_p$; $\{Y_1(p), \cdots, Y_n(p)\}$ span $H^2_p$; $\{Y_1(p), \cdots, Y_n(p)\}$ span $H^3_p$; and so on, where $(k, n_2, n_3, \cdots, n_r)$ is the growth vector of $H$ at $p$. 
For each chosen $Y_i$ of the form $X_I$, let $\omega_i$ be the length $|I|$. Thus $\omega_i = m$ if and only if $Y_i(p) \in H^m_p$ and $Y_i(p) \notin H^{m+1}_p$. The assignment $i \mapsto \omega_i$ is called the weighting associated to the growth vector.

**Definition 2.2.** Coordinates $y_1, \ldots, y_n$ are said to be linearly adapted to the distribution $H$ at $p$ if $H^i_p$ is annihilated by the differentials $d\omega_{n+1}, \ldots, d\omega_n$ at $p$, where $n_i = n_i(p)$ are the coordinates of the growth vector at $p$. The $\omega$-weighted box of size $\epsilon$ is the point set

$$\Box^\omega(\epsilon) = \{y \in \mathbb{R}^n : |y_i| \leq \epsilon^{\omega_i}, i = 1, \ldots, n\}$$

Coordinates $y_i$ centered at $p$ for which $d\omega_i(p)$ are the dual basis to the $Y_i(p)$ are linearly adapted by our definition. Such coordinates will be used below.

**Lemma 2.3.** (Ball-box theorem) There exist linearly adapted coordinates $y_1, \ldots, y_n$ and positive constants $c_0$ and $\epsilon_0$ such that for all $\epsilon \leq \epsilon_0$,

$$\Box^\omega(c_0 \epsilon, p) \subset B_H(\epsilon, p)$$

here $B_H(\epsilon, p)$ denotes the subriemannian ball of radius $\epsilon$ at $p$, i.e., any $x \in B_H(\epsilon, p)$ can be connected to $p$ by a horizontal curve of length $\leq \epsilon$.

**Proof of Lemma 1.1** First, we claim that if the horizontal distribution $H$ is bracket-generating at $p \in M$ with step $r$, then $H$ is also bracket-generating at any $q \in G(p)$ with step $r$, where $G(p)$ is the orbit of $p$. Assume $q = gp$ for some $g \in G$, since the differential $g_*$ of $g$ is an isometry that preserve both horizontal and vertical distribution, for any two vector fields $X, Y \in H$, we have the identity $g_*[X, Y] = [g_*X, g_*Y]$, so if $H^i_p = T_pM$, then $H^i_q = g_*(H^i_p) = g_*(T_pM) = T_qM$, this prove our first claim.

Secondly, we claim that $G(p) \subset L^\#$. By the ball-box theorem, we know that there is positive constants $c_0$ and $\epsilon_0$ such that for all $\epsilon \leq \epsilon_0$, $\Box^\omega(c_0 \epsilon, p) \subset B_H(\epsilon, p)$, since $g_*$ is a isometric map and $H^s_{gp} = g_*(H^s_p)$ for any $s \geq 1$, we know that for any $q \in G(p)$, there is a same size ball-box as that of $p$, i.e., $\Box^\omega(c_0 \epsilon, q) \subset B_H(\epsilon, q)$ for all $\epsilon \leq \epsilon_0$. Since $G(p)$ is a compact connected, we can cover $G(p)$ by finitely many successive “box” neighborhoods for which the ball-box theorem holds. Denote the number of these covering boxes by $N_0$, then any two points $x, y \in G(p)$ can be connected by a horizontal curve with length $\leq 2N_0\epsilon_0$, this prove that $G(p) \subset L^\#_p$.

For any $x \notin G(p)$, since $G(p)$ is compact, there is a minimal geodesic $\gamma : [0, 1] \to M$ from $x$ to $G(p)$, thus $\gamma$ must be a horizontal geodesic. By $\gamma(1) \in G(p) \subset L^\#_p$, we get $x \in L^\#_p$, this prove that $L^\#_p = M$, i.e., the dual foliation has only one leaf.

If $M$ is compact, for any $x_1, x_2 \in M$, we have two horizontal minimal geodesics $\gamma_1, \gamma_2 : [0, 1] \to M$ with $\gamma_i(0) = x_i$, $\gamma_i(1) \in G(p)$ for $i = 1, 2$, we have proved that $\gamma_1(1)$ and $\gamma_2(1)$ can be connected by a horizontal curve of length $\leq 2N_0\epsilon_0$, since length($\gamma_i$) $\leq diam(M)$ for $i = 1, 2$, where $diam(M)$ is the diameter of $M$, $x_1$ and $x_2$ can be connected by a horizontal curve of length $\leq C(M, G) := 2diam(M) + 2N_0\epsilon_0$.

## 3 Proofs of Main Results

**Proof of Corollary 1.2** For $p \in M$, let $G_p$ be the isotropy group of the action of $G$ at $p$, by lemma 1.1 we know that $L^\#_p = M$, then by lemma 2.1 we get that the isotropy group
of any point must coincide with $G_p$, since the action of $G$ is effective, we have that $G_p$ is the identity, so the action of $G$ is free, and $\pi : M \to B$ is a principal $G$-bundle. \qed

**Proof of Theorem 1.3** If there is a singular orbit $x = T^1(x)$ on $M$, then the horizontal space at $x$ is $H_x = T_x M$. Since the exponential map $\exp_x : H_x \to M$ is onto, we get that $L^\#_x = M$. Thus we assume all the orbits are regular, then we get a codimension one globally horizontal distribution $H$ on $M$. If $H$ is not integrable at some point $p \in M$, then $H$ is bracket-generating at $p$, by lemma 1.1 we get that $L^\#_p = M$. So it is enough to show that there is no globally integrable horizontal distribution $H$ on $M$.

If $H$ is integrable on $M$, since $M$ is simply connected, by theorem 1.5 in [2], there are no exceptional orbits on $M$, so $T^1$ acts freely on $M$, it follows that the orbit space $B := M/T^1$ is also a manifold, and inherits the quotient metric from $M$ for which $\pi : M \to B$ becomes a Riemannian submersion with fiber $F = T^1$. By the long exact homotopy sequence of the fibration $\pi$, $B$ is simply connected since $M$ is, we claim that $M$ is diffeomorphism to $B \times T^1$.

Since $H$ is integrable, by the Frobenius theorem, for any $p \in M$, $L^\#_p$ is the integral leaf of $H$ that contains $p$. Since $\pi$ is a locally trivial bundle, $\pi |_{L^\#_p} : L^\#_p \to B$ is a covering map, and therefore a diffeomorphism, $B$ being simply connected. Hence $L^\#_p$ intersects every fiber in exactly one point.

Let $b \in B$, $F := \pi^{-1}(b)$, We consider the map $(\pi, \phi) : M \to B \times F$ defined by

$$(\pi, \phi)p = (\pi(p), L^\#_p \cap F).$$

If $(\pi, \phi)p_1 = (\pi, \phi)p_2$, then $p_2 \in L^\#_{p_1} \cap \pi^{-1}(\pi(p_1))$, since $L^\#_{p_1}$ intersects the fiber $\pi^{-1}(\pi(p_1))$ in the only one point $p_1$, we get that $p_1 = p_2$, so $(\pi, \phi)$ is a injective map.

For any $p \in M$, $\phi(p) = L^\#_p \cap F$, so $\phi |_{\pi^{-1}(\pi(p))} : \pi^{-1}(\pi(p)) \to F$ is the holonomy diffeomorphism, so $(\pi, \phi)$ is a surjective map, it follows that $(\pi, \phi)$ is a bijection map.

Since for any $p \in M$, $\pi |_{L^\#_p} : L^\#_p \to B$ is diffeomorphism, and $\phi |_{\pi^{-1}(\pi(p))} : \pi^{-1}(\pi(p)) \to F$ is the holonomy diffeomorphism, so $(\pi, \phi)$ is a diffeomorphism, this prove that $M$ is diffeomorphism to $B \times T^1$. However, this is a contradiction since $M$ is simply connected.

We have proved the horizontal connectivity on $M$, now assume that $M$ is compact, denote by $\text{diam}(M)$ the original diameter of $M$, by $\text{diam}_H(M)$ the horizontal diameter of $M$ defined by horizontal metric. If there is a singular orbit $x = T^1(x)$ on $M$, then $L^\#_x = M$, thus $\text{diam}_H(M) \leq 2\text{diam}(M)$. If all the orbits are regular, then we get a codimension one globally horizontal distribution $H$ on $M$. Since $H$ is bracket-generating at some point $p \in M$, by lemma 1.1, we get that $\text{diam}_H(M) \leq C = C(M, T^1)$. \qed

**Proof of Theorem 1.4** Suppose the dual foliation has more than one leaf, then by theorem 2 and 3 in [16], these dual leafs form a regular Riemannian foliations $\mathcal{F}^\#$. Since the torus is an ablelian group, the horizontal distribution with respect to $\mathcal{F}^\#$ is integrable, then by theorem 1.3 in [15], $M$ locally splits. \qed

**Proof of Corollary 1.5** (a) We first consider the case that the action of $T^1$ is free. We
get a sequence of principle $T^1$-bundle

$$M = M_0 \xrightarrow{\pi_1} M_1 \xrightarrow{\pi_2} \cdots M_{k-1} \xrightarrow{\pi_k} M_k = M/T^k,$$

It is easy to see that $M_i$ is simply connected for any $0 \leq i \leq k$. Notice that for any $0 \leq i \leq k-1$, the free $T^{k-i}$-action (induced by $T^k$-action) induce a Riemannian foliations $\mathcal{F}_i$ on $M_i$, a curve on $M_i$ is called $H_i$-horizontal if it meets the leaves of $\mathcal{F}_i$ perpendicularly, for any $p_i \in M_i$, denote by $L^\#_{p_i}$ the dual leaf of $\mathcal{F}_i$ that through $p_i$. We claim that if $L^\#_{p_i} = M_i$ for some $p_i \in M_i$ and $1 \leq i \leq k - 1$, then $L^\#_{p_{i-1}} = M_{i-1}$ for any $p_{i-1} \in \pi_i^{-1}(p_i)$. This in fact shows $L^\#_p = M$ for any $p \in M$.

By theorem 1.3, we get $L^\#_{p_{k-1}} = M_{k-1}$ for any $p_{k-1} \in M_{k-1}$. Now suppose $L^\#_{p_i} = M_i$ for some $p_i \in M_i$ and $1 \leq i \leq k - 1$, assume to the contrary that $L^\#_{p_{i-1}} \neq M_{i-1}$ for some $p_{i-1} \in \pi_i^{-1}(p_i)$, then by [16], $L^\#_{p_{i-1}}$ is a complete smooth immersed submanifold of $M_{i-1}$ with dimension $\dim(L^\#_{p_{i-1}}) \leq \dim(M_i)$, it is easy to see that $\pi_i | L^\#_{p_{i-1}} : L^\#_{p_{i-1}} \to L^\#_{p_i} = M_i$ is a smooth surjective map, thus $\dim(L^\#_{p_{i-1}}) = \dim(M_i)$. Notice that a $H_i$-horizontal curve can be uniquely horizontal lift to a $H_{i-1}$-horizontal curve via $\pi_i$, so we define the $H_i$-holonomy group $\text{Hol}_i(p_i)$ of $\pi_i$ at $p_i$ to be the group consisting of holonomy diffeomorphisms of $\pi_i^{-1}(p_i)$ induced by $H_i$-horizontal loops in $M_i$-based at $p_i$, since $L^\#_{p_i} = M_i$, we know that $\text{Hol}_i(p_i) = \text{Hol}_i(q_i)$ for any $p_i, q_i \in M_i$. It is easy to see that $\text{Hol}_i(p_i)$ is a subgroup of $T^1$, thus $\text{Hol}_i(p_i)$ is either $\mathbb{Z}_k(k \geq 2)$, or $\{1\}$ (the trivial subgroup of $T^1$), so $L^\#_{p_{i-1}} \cap \pi_i^{-1}(p_i) = \mathbb{Z}_k(p_{i-1})$ or $p_{i-1}$, where $\mathbb{Z}_k(p_{i-1})$ is the orbit of the action of $\mathbb{Z}_k \subset T^1$ on $p_{i-1}$. Thus $\pi_i | L^\#_{p_{i-1}} : L^\#_{p_{i-1}} \to L^\#_{p_i} = M_i$ is a locally one-to-one map, hence a covering map, by $M_i$ is simple connected, we get that $L^\#_{p_{i-1}}$ is homeomorphic to $M_i$, hence $L^\#_{p_{i-1}}$ intersects every fiber of $\pi_i$ in exactly one point. Now using the same method as that of theorem 1.3, we get that $M_i$ is homeomorphic to $M_{i-1} \times T^1$, which is a contradiction since $M_i$ is simply connected.

(b) Next, we consider the case that $M$ has nonnegative curvature. Suppose the dual foliation has more than one leaf, then by theorem 1.4 $M$ locally splits. Since $M$ is simple connected, $M$ globally splits. It is easy to see that there is an $1 \leq i \leq k$ such that $M$ is isometric to $L^\#_p \times T^1$, where $L^\#_p$ is a leaf of the dual foliations $\mathcal{F}^\#$. This is a contradiction since $M$ is simply connected.

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The dual foliation of some singular Riemannian foliations

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