Magnetic dipole moment of a moving electric dipole

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The relativistic transformations of the polarization (electric moment density) \( \mathbf{P} \) and magnetization (magnetic moment density) \( \mathbf{M} \) of macroscopic electrodynamics imply corresponding transformations of the electric and magnetic dipole moments \( \mathbf{p} \) and \( \mathbf{m} \), respectively, of a particle. Thus, to first order in \( v/c^2 \)

\[
\mathbf{p} = \mathbf{p}_0 + \frac{1}{c} \mathbf{v} \times \mathbf{m}_0, \tag{1}
\]

\[
\mathbf{m} = \mathbf{m}_0 - \frac{1}{c} \mathbf{v} \times \mathbf{p}_0, \tag{2}
\]

Here, the subscripts 0 denote quantities in the particle’s rest frame and \( \mathbf{v} \) is the particle’s velocity. According to Eq. (1), a moving rest-frame magnetic dipole \( \mathbf{m}_0 \) develops an electric dipole moment \( \mathbf{p} = \mathbf{v} \times \mathbf{m}_0/c \). While this fact is well known and understood, the complementary effect that a moving electric dipole acquires a magnetic moment does not seem to be understood equally well. There does not appear to be a consensus in the literature as to the transformation (2). For example, Barut and Vekstein agree to first order in \( v/c \) with (2), but according to Fisher

\[
\mathbf{m} = \mathbf{m}_0 - \frac{1}{2c} \mathbf{v} \times \mathbf{p}_0, \tag{3}
\]

implying that a moving rest-frame electric dipole \( \mathbf{p}_0 \) acquires a magnetic dipole moment \( \mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/2c \), which differs by a factor of \( 1/2 \) from that of Eq. (2). The tasks of some problems involving a moving electric dipole in the authoritative text of Jackson seem at first sight to be consistent with Fisher’s transformation (3).

The aim of this note is to clear up the inconsistency of the differing transformations (2) and (3). To this end, we shall first express the electric current created by a moving electric dipole as the sum of polarization and magnetization currents, and then calculate the magnetic field of the moving dipole as the sum of the magnetic fields due to these currents.

A rest-frame electric dipole \( \mathbf{p}_0 \), located at \( \mathbf{r} = \mathbf{r}_0(t) \) and assumed for simplicity to be a point-like particle, produces a polarization

\[
\mathbf{P}(\mathbf{r}, t) = \mathbf{p}_0 \delta(\mathbf{r} - \mathbf{r}_0(t)). \tag{4}
\]

Although its net charge vanishes, the dipole is a carrier of a bound charge distribution

\[
\rho_b(\mathbf{r}, t) = -\mathbf{v} \cdot \mathbf{P}(\mathbf{r}, t)
= -\mathbf{p}_0 \cdot \mathbf{v} \delta(\mathbf{r} - \mathbf{r}_0(t)). \tag{5}
\]

When the dipole is moving with a velocity \( \mathbf{v} = d\mathbf{r}_0(t)/dt \), it creates a current density

\[
\mathbf{J}(\mathbf{r}, t) = \mathbf{v} \rho_b(\mathbf{r}, t)
= -\mathbf{v}(\mathbf{p}_0 \cdot \nabla) \delta(\mathbf{r} - \mathbf{r}_0(t)). \tag{6}
\]

The same current density can obviously be obtained by modeling the moving electric dipole as two equal and opposite point charges, separated by an infinitesimal displacement and moving with the same velocity \( \mathbf{v} \).

On the other hand, the time-dependent polarization (4) produces a polarization current density \( \mathbf{J}_p(\mathbf{r}, t) \) according to

\[
\mathbf{J}_p(\mathbf{r}, t) = \frac{\partial \mathbf{P}(\mathbf{r}, t)}{\partial t}
= -\mathbf{p}_0(\mathbf{v} \cdot \nabla) \delta(\mathbf{r} - \mathbf{r}_0(t)), \tag{7}
\]

assuming that the dipole moment \( \mathbf{p}_0 \) itself does not depend on time. If the moving electric dipole develops a magnetic dipole moment \( \mathbf{m} \), then, in addition to the polarization (4), there is also a magnetization

\[
\mathbf{M}(\mathbf{r}, t) = \mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0(t)), \tag{8}
\]

which produces a magnetization current \( \mathbf{J}_m(\mathbf{r}, t) \) according to

\[
\mathbf{J}_m(\mathbf{r}, t) = c \mathbf{v} \times \mathbf{M}
= c \mathbf{v} \times [\mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0(t))]. \tag{9}
\]

If now the magnetic dipole moment \( \mathbf{m} \) generated by the motion of the electric dipole is given by

\[
\mathbf{m} = -\frac{1}{c} \mathbf{v} \times \mathbf{p}_0, \tag{10}
\]

then the sum of the bound currents (7) and (9) equals the “convection” current (6). Indeed, we then have

\[
\mathbf{J}_p(\mathbf{r}, t) + \mathbf{J}_m(\mathbf{r}, t)
= -\mathbf{p}_0(\mathbf{v} \cdot \nabla) \delta(\mathbf{r} - \mathbf{r}_0(t)) - \mathbf{v} \times [(\mathbf{v} \times \mathbf{p}_0) \delta(\mathbf{r} - \mathbf{r}_0(t))]
= -\mathbf{v}(\mathbf{p}_0 \cdot \nabla) \delta(\mathbf{r} - \mathbf{r}_0(t)), \tag{11}
\]

where the last line was obtained using standard vector calculus identities. The circulating magnetization current due to the magnetic moment (10) is modified by the currents arising from the polarization current density (4) so that the resulting net current is directed along the dipole’s velocity \( \mathbf{v} \).

In the nonrelativistic (quasi-static) limit, the vector
potential due to the vector potential \( \mathbf{A}(r, t) = \frac{1}{c} \int d^3r \frac{\mathbf{J}(r', t)}{|r - r'|} \) is

\[
\mathbf{A}_p(r, t) = -\frac{p_0}{c} \int d^3r' \frac{(\mathbf{v} \cdot \nabla') \delta(r' - r_0(t))}{|r - r'|} = \frac{1}{c} \frac{\mathbf{v} \cdot (\mathbf{r} - r_0(t))}{|\mathbf{r} - r_0(t)|^3} p_0.
\]

(12)

Here, the integral in the 2nd line was performed by parts and the integral of the resulting gradient term was transformed into a vanishing surface integral. The vector potential of the magnetization current \( \mathbf{A}_m(r, t) \) that is due to the magnetic moment \( \mathbf{m}_0 \) is evaluated similarly:

\[
\mathbf{A}_m(r, t) = \frac{1}{c} \int d^3r' \frac{\mathbf{J}_m(r', t)}{|r - r'|} = -\frac{1}{c} \int d^3r' \frac{\mathbf{v} \times [\nabla' \times (\mathbf{v} \times \mathbf{p}_0)] (r' - r_0(t))}{|r - r'|} = -\frac{1}{c} \frac{(\mathbf{v} \times \mathbf{p}_0) \times (\mathbf{r} - r_0(t))}{|\mathbf{r} - r_0(t)|^3}.
\]

(13)

The magnetic field \( \mathbf{B} \) of the moving electric dipole is therefore the sum

\[
\mathbf{B} = \mathbf{B}_m + \mathbf{B}_p,
\]

(14)

where

\[
\mathbf{B}_m = \nabla \times \mathbf{A}_m(r, t)
\]

\[
= -\frac{1}{c} \frac{3(\mathbf{n} \cdot (\mathbf{v} \times \mathbf{p}_0)) |\mathbf{n} - \mathbf{v} \times \mathbf{p}_0|}{|\mathbf{r} - r_0(t)|^3},
\]

(15)

the magnetic field due to the vector potential \( \mathbf{A}_m \), is the magnetic field of the magnetic moment \( \mathbf{m}_0 \), and

\[
\mathbf{B}_p = \nabla \times \mathbf{A}_p(r, t)
\]

\[
= \frac{1}{c} \frac{3(\mathbf{n} \cdot \mathbf{v}) |\mathbf{n} - \mathbf{v}|}{|\mathbf{r} - r_0(t)|^3}
\]

(16)

is the magnetic field due to the vector potential \( \mathbf{A}_p \), created by the polarization current \( \mathbf{J}_p \). Here,

\[
\mathbf{n} = \frac{\mathbf{r} - r_0(t)}{|\mathbf{r} - r_0(t)|}
\]

(17)

is a unit vector along the direction from the dipole’s location \( r_0 \) to the field point \( r \). The sum of the expressions \( \mathbf{A}_m \) and \( \mathbf{A}_p \) reduces to

\[
\mathbf{B} = \frac{1}{c} \mathbf{v} \times \mathbf{E},
\]

(18)

where

\[
\mathbf{E} = \frac{3(\mathbf{n} \cdot \mathbf{p}_0) \mathbf{n} - \mathbf{p}_0}{|\mathbf{r} - r_0(t)|^3}
\]

(19)

is the electric field of the moving dipole in the nonrelativistic limit. We note that the magnetic field \( \mathbf{B}_m \) is the same as that obtained by Lorentz transforming to first order in \( v/c \) the dipole’s rest-frame electromagnetic field to the laboratory frame.

The dipole transformation \( \mathbf{J}_p \), implying that a moving rest-frame electric dipole \( \mathbf{p}_0 \) acquires a magnetic dipole moment \( \mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/c \), is thus correct, though the magnetic field due to the dipole moment \( \mathbf{m} \) is not the whole magnetic field of the moving electric dipole, which includes also the magnetic field created by the polarization current.\(^{12}\) The magnetic dipole moment \( \mathbf{m} = -(1/2c)\mathbf{v} \times \mathbf{p}_0 \) arises in a different decomposition of the moving dipole’s magnetic field.\(^{13}\)

\[
\mathbf{B} = \mathbf{B}_{m/2} - \frac{3}{2c} \mathbf{n} \times (\mathbf{v} \cdot \mathbf{n}) \mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{n}) \mathbf{v},
\]

(20)

where the first term is the magnetic field due to the dipole moment \( \frac{1}{2} \mathbf{m} \) and the second term, which is symmetric in \( \mathbf{p}_0 \) and \( \mathbf{v} \), has a curl that is proportional to the displacement current of the electric quadrupole field that is created when the electric dipole’s location is off the origin, i.e. \( r_0 \neq 0 \).\(^{15}\)

The transformation \( \mathbf{J}_m \) of Fisher was arrived at using the standard definition \( \mathbf{m} = (1/2c) \int d^3r \mathbf{r} \times \mathbf{J} \). The origin of this definition is in a multipole expansion of the vector potential of a localized current distribution that is assumed to be divergenceless, i.e. \( \nabla \cdot \mathbf{J} = 0 \), so that the value of \( \mathbf{m} \) is independent of the choice of the reference point \( \mathbf{r} = 0 \).\(^{14}\) However, the net current density \( \mathbf{J} \) of a moving electric dipole is not divergenceless: by the continuity equation, \( \nabla \cdot \mathbf{J} = -\partial \mathbf{p}_0 / \partial t \), and \( \partial \mathbf{p}_0 / \partial t \neq 0 \) for the moving dipole’s charge density \( \rho_0 \) (see Eq. \( \mathbf{J} \)). But as a curl, the magnetization component \( \mathbf{J}_m \) of the net current density \( \mathbf{J} \) is divergenceless. It is only the magnetization current \( \mathbf{J}_m = -\nabla \times [\mathbf{v} \times \mathbf{p}_0] \delta(\mathbf{r} - \mathbf{r}_0) \) that, using \( \mathbf{m} = (1/2c) \int d^3r \mathbf{r} \times \mathbf{J}_m \), determines a magnetic dipole moment that is appropriate for a moving electric dipole.\(^{15}\)

The full magnetic field of a moving electric dipole can be decomposed in more than one way. Decomposition \( \mathbf{B} \) is not incorrect, but decomposition \( \mathbf{A}_m \) is singled out by the clearcut sources of its two terms: the divergenceless magnetization current due to the magnetization produced by the magnetic dipole moment \( \mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/c \) that is consistent with the transformation relations of special relativity and the non-zero-divergence polarization current due to the polarization produced by a moving electric dipole.\(^{15}\)

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1 W. K. H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd ed. (Dover, New York, 2005), Section 18-6.
2 We use Gaussian units since the relativistic aspects of relations among the dipole moments are displayed most clearly in these units.
3 Reference 1, Section 18-4.
4 D. Bedford and P. Krumm, “On the origin of magnetic dynamics,” Am. J. Phys. 54, 1036–1039 (1986).
5 D. J. Griffiths, Introduction to Electrodynamics, 3rd ed. (Prentice Hall, Upper Saddle River, 1999), Problem 12.62.
6 This situation is somewhat paradoxical since we shall see that, unlike the acquired electric dipole moment, the acquired magnetic dipole moment is not a relativistic effect.
7 A. O. Barut, Electrodynamics and Classical Theory of Fields and Particles (Dover, New York, 1980), Chapter II, Section 4.
8 G. E. Vekstein, “On the electromagnetic force on a moving dipole,” Eur. J. Phys. 18, 113–117 (1997).
9 G. P. Fisher, “The electric dipole moment of a moving magnetic dipole,” Am. J. Phys. 39, 1528–1533 (1971).
10 J. D. Jackson, Classical Electrodynamics, 3rd ed. (Wiley, New York, 1999), Problems 6.21, 6.22 and 11.27 (a).
11 When a time dependence of the electric dipole itself is allowed, the polarization current density is supplemented by a term \( \dot{p}_0 \delta(r - r_0(t)) \), where the dot denotes the time derivative. It can be shown easily that this term generates already in the dipole’s rest frame a magnetic field \( (\mathbf{p}_0 \times \mathbf{n})/c |r - r_0(t)|^3 \), where \( \mathbf{n} \) is the unit vector; this magnetic field must be then added to the moving dipole’s magnetic field.
12 The magnetic force on a moving electric dipole \( \mathbf{p}_0 \), which is the Lorentz force on the sum of the polarization and magnetization currents, can be derived in the Lagrangian formalism using an interaction \( -\mathbf{m} \cdot \mathbf{B} \), where \( \mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/c \). See V. Hnizdo, “Comment on ‘Electromagnetic force on a moving dipole’,” Eur. J. Phys. 33, L3–L6 (2012); arXiv:1108.4332.
13 Reference 10, Problem 6.22.
14 See, e.g., Reference 10, Sec. 5.6.
15 It can be shown easily by integration by parts that \( (1/2c) \int d^3r \mathbf{r} \times \mathbf{J}_m \), where \( \mathbf{J}_m = c \nabla \times [\mathbf{m} \delta(r-r_0)] \), equals the magnetic moment \( \mathbf{m} \) in the curl expression for \( \mathbf{J}_m \).
16 For additional discussion see V. Hnizdo and K. T. McDonald, “Fields and moments of a moving electric dipole,” <www.hep.princeton.edu/~mcdonald/examples/movingdipole.pdf> (2011).