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AN EXPLICIT UPPER BOUND FOR THE LEAST PRIME IDEAL IN THE CHEBOTAREV DENSITY THEOREM

by Jeoung-Hwan AHN & Soun-Hi KWON (*)

Abstract. — Lagarias, Montgomery, and Odlyzko proved that there exists an effectively computable absolute constant $A_1$ such that for every finite extension $K$ of $\mathbb{Q}$, every finite Galois extension $L$ of $K$ with Galois group $G$ and every conjugacy class $C$ of $G$, there exists a prime ideal $p$ of $K$ which is unramified in $L$, for which $[L/K] = C$, for which $N_{K/\mathbb{Q}}p$ is a rational prime, and which satisfies $N_{K/\mathbb{Q}}p \leq 2d_L A_1$. In this paper we show without any restriction that $N_{K/\mathbb{Q}}p \leq d_L^{12577}$ if $L \neq \mathbb{Q}$, using the approach developed by Lagarias, Montgomery, and Odlyzko.

Résumé. — Lagarias, Montgomery, et Odlyzko ont démontré qu’il existe une constante absolue effectivement calculable $A_1$ telle que pour chaque extension finie $K$ de $\mathbb{Q}$, chaque extension galoisienne finie $L$ de $K$ à groupe de Galois $G$, et chaque classe de conjugaison $C$ de $G$, il existe un idéal premier $p$ de $K$ qui est nonramifié dans $L$, pour lequel $[L/K] = C$, pour lequel $N_{K/\mathbb{Q}}p$ est un nombre premier rationnel, et qui satisfait $N_{K/\mathbb{Q}}p \leq 2d_L A_1$. Dans cet article nous démontrons sans aucune restriction que $N_{K/\mathbb{Q}}p \leq d_L^{12577}$ si $L \neq \mathbb{Q}$, en suivant la méthode développée par Lagarias, Montgomery, et Odlyzko.

1. Introduction

Let $K$ be a finite algebraic extension of $\mathbb{Q}$, and $L$ a finite Galois extension of $K$ with Galois group $G$. Let $d_L$ and $d_K$ denote the absolute values of discriminants of $L$ and $K$, respectively, and let $n_L = [L : \mathbb{Q}]$, $n_K = [K : \mathbb{Q}]$. To each prime ideal $p$ of $K$ unramified in $L$ there corresponds a certain

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conjugacy class $C$ of $G$ consisting of the set of Frobenius automorphisms attached to the prime ideals $\mathfrak{P}$ of $L$ which lie over $p$. Denote this conjugacy class by the Artin symbol $\left[ \frac{L/K}{\mathfrak{p}} \right]$. For a conjugacy class $C$ of $G$ let

$$\pi_C(x) = \left| \{ \mathfrak{p} \mid \mathfrak{p} \text{ a prime ideal of } K, \text{ unramified in } L, \left[ \frac{L/K}{\mathfrak{p}} \right] = C, \text{ and } N_{K/Q} \mathfrak{p} \leq x \} \right|.$$ 

The Chebotarev density theorem states that

$$\pi_C(x) \sim \frac{|C|}{|G|} Li(x)$$

as $x \to \infty$. (See [15], [53], [28], [39], and [50]. See also [47] for some extensions of Chebotarev’s theorem and applications.) The error term of this theorem was estimated in [24], [41], and [59]. Lagarias, Montgomery, and Odlyzko estimated upper bound for the least prime ideal $\mathfrak{p}$ with $\left[ \frac{L/K}{\mathfrak{p}} \right] = C$ under the Generalized Riemann Hypothesis (GRH), and unconditionally, in [24] and [23], respectively.

**Theorem A** (Lagarias and Odlyzko [24]). — There exists an effectively computable positive absolute constant $A_0$ such that if the GRH holds for Dedekind zeta function of $L \neq \mathbb{Q}$, then for every conjugacy class $C$ of $G$ there exists an unramified prime ideal $\mathfrak{p}$ in $K$ such that $\left[ \frac{L/K}{\mathfrak{p}} \right] = C$ and

$$N_{K/Q} \mathfrak{p} \leq A_0 (\log d_L)^2.$$ 

Oesterlé ([41]) has stated that if GRH holds, then one may have $A_0 = 70$. Bach and Sorenson ([4]) has improved this result in two ways: If GRH holds, then for any class $C$ of $G$ there is a prime $\mathfrak{p}$ in $K$ of degree 1 over $\mathbb{Q}$ with $\left[ \frac{L/K}{\mathfrak{p}} \right] = C$ and $N_{K/Q} \mathfrak{p} \leq (4 \log d_L + 2.5 n_L + 5)^2$. (See also [3], [38], and [22].) Let

$$P(C) = \left\{ \mathfrak{p} \mid \mathfrak{p} \text{ a prime ideal of } K, \text{ unramified in } L, \text{ of degree one over } \mathbb{Q} \text{ and } \left[ \frac{L/K}{\mathfrak{p}} \right] = C \right\}.$$

**Theorem B** (Lagarias, Montgomery, and Odlyzko [23]). — There is an absolute, effectively computable constant $A_1$ such that for every finite extension $K$ of $\mathbb{Q}$, every finite Galois extension $L$ of $K$, and every conjugacy class $C$ of $G$, there exists a prime $\mathfrak{p}$ in $P(C)$ which satisfies

$$N_{K/Q} \mathfrak{p} \leq 2d_L A_1.$$
See also [57]. When $K = \mathbb{Q}$ and $L = \mathbb{Q}(e^{2\pi i/q})$, the conjugacy classes of $G$ correspond to the residues classes modulo $q$ and Theorem B gives an upper bound for the least prime in an arithmetic progression ([24] and [23]). In this case Theorem B is weaker than Linnik's theorem ([29], [30], [5]). For the least prime in an arithmetic progression, see for example [7], [8], [13], [14], [17], [18], [42], [43], [55], [56], and [61]. If $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{D})$, and $\rho$ is the non identity in $Gal(L/\mathbb{Q})$, Theorem B gives an upper bound for the least quadratic nonresidue module $D$. For this case no upper bound better than Theorem B is known ([54], [6], [24], [23], [2], [25], [26]). In this paper we compute the constant $A_1$.

**Theorem 1.1.** — For every finite extension $K$ of $\mathbb{Q}$, every finite Galois extension $L(\neq \mathbb{Q})$ of $K$ with Galois group $G$, and every conjugacy class $C$ of $G$, there exists a prime ideal $\mathfrak{p}$ in $P(C)$ which satisfies

$$N_{K/\mathbb{Q}} \mathfrak{p} \leq d_L A_1$$

with $A_1 = 12577$.

To compute the constant $A_1$ we follow the method developed by [23]. In particular, we express zero-free regions for Dedekind zeta functions, density of zeros of Dedekind zeta functions, and Deuring–Heilbronn phenomenon with explicit constants in Sections 5-7 below. Zaman showed in [63] that $N_{K/\mathbb{Q}} \mathfrak{p} \ll d_L^{40}$ for sufficiently large $d_L$. See also [51]. Winckler proved $A_1 = 27175010$ without any restriction in [60].

### 2. Outline of Lagarias–Montgomery–Odlyzko’s method

Let $\Re z$ and $\Im z$ denote the real part and imaginary one of $z \in \mathbb{C}$, respectively. We review the procedure for the proof of Theorem B in [23]. Let $g \in C$ and

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\psi} \overline{\psi}(g) \frac{L'}{L}(s, \psi, L/K),$$

where $\psi$ runs over the irreducible characters of $G$ and $L(s, \psi, L/K)$ is the Artin L-function attached to $\psi$. The main parts of [23] consist of estimates of inverse Mellin transforms

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s) k(s) \, ds$$

where $k(s)$ is a kernel function. The main steps of the proof of Theorem B in [23] are as follows:
(i) From the orthogonality relations for the characters \( \psi \) it follows that for \( \Re s > 1 \)

\[
F_C(s) = \sum_p \sum_{m=1}^{\infty} \theta(p^m)(\log N_{K/Q} p)(N_{K/Q} p)^{-ms}
\]

where for prime ideals \( p \) of \( K \) unramified in \( L \)

\[
\theta(p^m) = \begin{cases} 1 & \text{if } \left[ \frac{L/K}{p} \right]^m = C, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( |\theta(p^m)| \leq 1 \) if \( p \) ramifies in \( L \). So we can separate the \( p^m \) with \( \left[ \frac{L/K}{p} \right]^m = C \) from the others. (See [24, Section 3].)

(ii) Using a method due to Deuring ([10] and [35]) \( F_C(s) \) can be written as a linear combination of logarithmic derivatives of Hecke \( L \)-functions instead of Artin \( L \)-functions. Let \( H = \langle g \rangle \) be the cyclic subgroup generated by \( g \), \( E \) the fixed field of \( H \). Then

\[
(2.1) \quad F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \chi(g) \frac{L'}{L}(s, \chi, E),
\]

where \( \chi \) runs over the irreducible characters of \( H \), and \( L(s, \chi, E) \) is a Hecke \( L \)-function attached to field \( E \) with \( \chi(p) = \chi \left( \left[ \frac{L/E}{p} \right] \right) \) for all prime ideals \( p \) of \( E \) unramified in \( L \). (See [24, Section 4].) So, all the singularities of \( F_C(s) \) appear at the zeros and the pole of \( \zeta_L(s) \).

(iii) The kernel functions which weight prime ideals of small norm very heavily are used. Set

\[
k_0(s; x, y) = \left( \frac{y^{s-1} - x^{s-1}}{s-1} \right)^2 \quad \text{for } y > x > 1,
\]

\[
k_1(s) = k_0(s; x, x^2) \quad \text{for } x \geq 2,
\]

and

\[
k_2(s) = k_2(s; x) = x^{s^2 + s} \quad \text{for } x \geq 2.
\]

In the case that \( \zeta_L(s) \) has a real zero very close to 1 we use the kernel \( k_2(s) \). Otherwise we use the kernel \( k_1(s) \). The use of the kernel functions is the main innovation of [23].
(iv) For \( u > 0 \) we denote by \( \hat{k}(u) \) the inverse Mellin transform of the kernel function \( k(s) \). Then, for \( \Re s > 1 \),

\[
I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s)k(s)\,ds
\]

\[
= \sum_p \sum_{m=1}^{\infty} \theta(p^m)(\log N_{K/Q}p)\hat{k}(N_{K/Q}p^m),
\]

where the outer sum is over all prime ideals of \( K \). An upper bound \( E(\log d_L) \) for (2.2) was estimated in [23, (3.15) and (3.16)].

(v) The integral \( I \) is evaluated by contour integration:

\[
I = \left| \frac{|C|}{|G|} k(1) - \sum_{\rho \chi} \chi(g) \sum_{\rho \chi} k(\rho \chi) \right| + \mathcal{O} \left( \frac{|C|}{|G|} n_Lk(0) + \frac{|C|}{|G|} k \left( -\frac{1}{2} \right) \log d_L \right),
\]

where \( \rho \chi \) runs over the zeros of \( L(s, \chi, E) \) in the critical strip. (See [23, Section 3].) So we get

\[
|C| \left| k(1) - \sum_{\rho} |k(\rho)| \right| \geq k(1) - \sum_{\rho} |k(\rho)| - c_6 \left\{ n_Lk(0) + k \left( -\frac{1}{2} \right) \log d_L \right\},
\]

where \( \rho \) runs over the zeros of \( \zeta_L(s) \) in the critical strip and \( c_6 \) is some constant. Note that \( \zeta_L(s) = \prod \chi L(s, \chi, E) \), where \( \chi \) runs over the irreducible characters of \( H = \text{Gal}(L/E) \). From (2.2) and (2.3) it follows that

\[
\sum_{\rho \in P(C)} (\log N_{K/Q}p)\hat{k}(N_{K/Q}p) \geq \frac{|C|}{|G|} k(1) - \frac{|C|}{|G|} \sum_{\rho} |k(\rho)|
- c_6 \frac{|C|}{|G|} \left\{ n_Lk(0) + k \left( -\frac{1}{2} \right) \log d_L \right\} - E(\log d_L).
\]

(vi) The sum

\[
k(1) - \sum_{\rho} |k(\rho)|
\]

is estimated from below. To do this we need to know the location and the density of the zeros of \( \zeta_L(s) \). If the possible exceptional zero exists, say \( \beta_0 \), then \( k(\beta_0) \) is large. The term \( k(1) - |k(\beta_0)| \)
must be controlled compared to \( \sum_{\rho \neq \beta_0} |k(\rho)| \). We need an enlarged zero-free region which makes possible \( \sum_{\rho \neq \beta_0} |k(\rho)| \) to be small. The Deuring–Heilbronn phenomenon guarantees that the other zeros of \( \zeta_L(s) \) cannot be very close to 1.

(vii) We choose \( x \) of the kernel \( k(s) \) in terms of \( d_L \) so that the right side of (2.4) is positive.

Then Theorem B follows. In the remaining sections of this paper we will make explicit numerically the constants intervening in the zero free regions, the density of zeros, and Deuring–Heilbronn phenomenon of \( \zeta_L(s) \), and ultimately \( A_1 \).

### 3. Prime ideals in \( P(C) \)

In this section we will estimate from above

\[
I - \sum_{p \in P(C)} (\log N_K/Qp) \tilde{k}(N_K/Qp) \left| \vphantom{\sum} \right.
\]

We will treat carefully their bounds in [23, Section 3]. We begin by recalling the inverse Mellin transform of the kernel functions. They can be easily computed. For \( x \geq 2 \) and \( u > 0 \) we have

\[
\tilde{k}_1(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left\{ \frac{x^{2(s-1)} - x^{s-1}}{s-1} \right\}^2 u^{-s} ds
\]

\[
= \begin{cases} 
    u^{-1} \log \frac{x^4}{u} & \text{if } x^3 \leq u \leq x^4, \\
    u^{-1} \log \frac{u}{x^2} & \text{if } x^2 \leq u \leq x^3, \\
    0 & \text{otherwise},
\end{cases}
\]

and

\[
\tilde{k}_2(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{s^2 + s} u^{-s} ds = (4\pi \log x)^{-\frac{1}{2}} \exp \left\{ -\left( \frac{\log \frac{u}{x}}{4 \log x} \right)^2 \right\},
\]

where \( a > -\frac{1}{2} \).

**Lemma 3.1.** — Let \( \sum^R \) denote summation over the prime ideals \( p \) of \( K \) that ramify in \( L \). For \( x \geq 2 \) we have then

\[
(1) \sum^R \sum_{m=1}^{\infty} \theta(p^m)(\log N_K/Qp) \tilde{k}_1(N_K/Qp^m) \leq \frac{2 \log x}{x^2} \log d_L;
\]
(2) \[ \sum_{R} \sum_{m \geq 1} \sum_{\substack{N_K/Q \mathfrak{p}^m \leq x^5 \mathfrak{p}}} \theta(p^m) (\log N_K/Q \mathfrak{p}) \bar{k}_2(N_K/Q \mathfrak{p}^m) \leq \frac{5}{2 \sqrt{\pi} \log 3} (\log x)^{\frac{1}{2}} \log d_L. \]

Proof.
(1) — Let \( \mathfrak{p} \) be a prime ideal of \( K \) that is ramified in \( L \). Note that \( N_K/Q \mathfrak{p} \geq 2 \) and \( \sum R \log N_K/Q \mathfrak{p} \leq \log d_L \). We have
\[
\sum_{R} \sum_{m=1}^{\infty} \theta(p^m) (\log N_K/Q \mathfrak{p}) \bar{k}_1(N_K/Q \mathfrak{p}^m)
\leq \log x \sum R \log N_K/Q \mathfrak{p} \sum_{m \geq 1} (N_K/Q \mathfrak{p}^m)^{-1}
\leq \log x \sum R \log N_K/Q \mathfrak{p} \left( \frac{1}{N_K/Q \mathfrak{p}^m} \right)^{-1}
\leq 2 \log x \sum R \log N_K/Q \mathfrak{p} \left( \frac{1}{1 - \frac{N_K/Q \mathfrak{p}^m}{N_K/Q \mathfrak{p}^m}} \right)
\leq 2 \log x \sum R \log N_K/Q \mathfrak{p} \left( \frac{1}{1 - \frac{N_K/Q \mathfrak{p}^m}{N_K/Q \mathfrak{p}^m}} \right)
\leq \frac{5}{2 \sqrt{\pi} \log 3} (\log x)^{\frac{1}{2}} \log d_L.
\]

where \( m_p = \left\lceil \frac{\log(x^2)}{\log N_K/Q \mathfrak{p}} \right\rceil \).

(2) — Let \( N_R \) be the number prime ideals of \( K \) that are ramified in \( L/K \). Note that \( d_L \geq 3^N \). (See [46, Chapters III and IV]). We have
\[
\sum_{R} \sum_{m \geq 1} \theta(p^m) (\log N_K/Q \mathfrak{p}) \bar{k}_2(N_K/Q \mathfrak{p}^m)
\leq (4 \pi \log x)^{-\frac{1}{2}} \sum_{R} \log N_K/Q \mathfrak{p} \sum_{m \geq 1} 1
\leq (4 \pi \log x)^{-\frac{1}{2}} \sum_{R} 5 \log x
\leq \frac{5}{2 \sqrt{\pi} \log 3} (\log x)^{\frac{1}{2}} \log d_L. \quad \Box
\]

Lemma 3.2.

(1) (Rosser and Schoenfeld [44]) For \( x > 1 \),
\[ \pi(x) < \alpha_0 \frac{x}{\log x} \]
with \( \alpha_0 = 1.25506 \), where \( \pi(x) \) is the number of primes \( p \) with \( p \leq x \).
For $x > 1$,  
\[ S(x) \leq \frac{2\alpha_0}{\log 2} \sqrt{x}, \]
where $S(x)$ is the number of prime powers $p^h$ with $h \geq 2$ and $p^h \leq x$.

For $x \geq 101$  
\[ \sum_{p \text{ prime}} p^{-h} \leq \frac{4.02\alpha_0}{x \log x}. \]

Proof.

(1). — See [44, Corollary 1].

(2). — We have
\[ S(x) \leq \pi(\sqrt{x}) \frac{\log x}{\log 2} \leq \frac{2\alpha_0}{\log 2} \sqrt{x} \]
by (1).

(3). — We have
\[ \sum_{p \text{ prime}} p^{-h} = \sum_{p \text{ prime}} \frac{p^{-h_p}}{1 - p^{-1}}, \]
where $h_p = \max \left( \left\lceil \frac{\log(x^2)}{\log p} \right\rceil, 2 \right)$ for each prime $p$. We observe that
\[ \sum_{p \leq x} \frac{p^{-h_p}}{1 - p^{-1}} \leq \frac{2}{x^2} \pi(x) \leq \frac{2\alpha_0}{x \log x}. \]
For $x \geq 101$
\[ \sum_{p > x} \frac{p^{-h_p}}{1 - p^{-1}} \leq \sum_{p > x} \frac{p^{-2}}{1 - p^{-1}} \leq \frac{x}{x - 1} \sum_{p > x} p^{-2} \leq 1.01 \sum_{p > x} p^{-2}. \]

By using the integration by parts and (1) we estimate $\sum_{p > x} p^{-2}$ from above. Namely,
\[ \sum_{p > x} p^{-2} \leq \int_{x}^{\infty} \frac{1}{t^2} d\pi(t) \leq \int_{x}^{\infty} \frac{2\pi(t)}{t^3} dt \leq \int_{x}^{\infty} \frac{2\alpha_0}{t^2 \log t} dt \leq \frac{2\alpha_0}{x \log x}. \]
Hence,
\[ \sum_{p \text{ prime}} \frac{p^{-h_p}}{1 - p^{-1}} \leq \frac{4.02\alpha_0}{x \log x}, \]
which yields (3). \qed
Lemma 3.3. — For \( y \leq \infty \), let \( \sum_P^y \) denote summation over those \((p, m)\) for which \( N_{K/Q}p^m \) is not a rational prime and \( N_{K/Q}p^m \leq y \). Then

(1) for \( x \geq 101 \)

\[
\sum_{\infty}^P \theta(p^m)(\log N_{K/Q}p)\hat{k}_1(N_{K/Q}p^m) \leq 16.08 \alpha_0 n_K \frac{\log x}{x};
\]

(2) for \( x \geq 10^{10} \)

\[
\sum_{\infty}^P \theta(p^m)(\log N_{K/Q}p)\hat{k}_2(N_{K/Q}p^m) \leq \alpha_1 n_K x^\frac{3}{4} (\log x)^\frac{3}{2}
\]

with

\[
\alpha_1 = \frac{\alpha_0}{3\sqrt{\pi}} \log 2 \left( \frac{15}{10^{\frac{4}{7}}} \log 10 + 7 + \frac{37}{10^{\frac{7}{2}}} \right) = 2.4234 \ldots.
\]

Proof.

(1). — Since for a positive integer \( q \) there are at most \( n_K \) distinct prime power ideals \( p^m \) with \( N_{K/Q}p^m = q \), it follows that

\[
\sum_{\infty}^P \theta(p^m)(\log N_{K/Q}p)\hat{k}_1(N_{K/Q}p^m) \leq \log x \sum_{\infty}^P (\log N_{K/Q}p)(N_{K/Q}p^m)^{-1}
\]

\[
\leq 4(\log x)^2 n_K \sum_{p \text{ prime}} \frac{p^{-h}}{x^2 \leq p^h \leq x^4, h \geq 2}.
\]

Hence, by Lemma 3.2(3) we obtain (1).

(2). — We have

\[
\sum_{\infty}^P \theta(p^m)(\log N_{K/Q}p)\hat{k}_2(N_{K/Q}p^m) \leq n_K \sum_{p \text{ prime}} (\log p^h)\hat{k}_2(p^h)
\]

\[
\leq n_K \int_{4}^{x^5} (\log u) \hat{k}_2(u) dS(u),
\]

where \( S(u) \) is as Lemma 3.2(2). According to Lemma 3.2(2), we have

\[
S(u) \leq \frac{2\alpha_0}{\log 2} \sqrt{u}.
\]
Hence,
\[
\int_4^{x^5} (\log u) \hat{k}_2(u) \, dS(u) \\
\leq (\log x^5)\hat{k}_2(x^5)S(x^5) + \int_4^{x^5} \hat{k}_2(u) \left( \frac{\log u \log \frac{u}{x}}{2 \log x} - 1 \right) S(u) \, du \\
\leq \frac{5\alpha_0}{\sqrt{\pi \log 2}} x^{-\frac{3}{2}} (\log x)^{\frac{1}{2}} + \int_{\log \frac{4}{x}}^{4 \log x} \hat{k}_2(xe^t) \left\{ \frac{(t + \log x)t}{2 \log x} \right\} S(xe^t) \, dt \\
\leq \frac{\alpha_0}{3\sqrt{\pi \log 2}} \left( \frac{15}{x^\frac{1}{2} \log x} + 7 + \frac{37}{x^\frac{1}{2}} \right) x^{\frac{3}{2}} (\log x)^{\frac{3}{2}}. \quad \square
\]

**Lemma 3.4.** — For \( x \geq 2 \), we have
\[
\sum_p \sum_{m \geq 1, N_{K/Q}p^m > x^5} \theta(p^m) (\log N_{K/Q}p) \hat{k}_2(N_{K/Q}p^m) \leq \alpha_2 n_K x (\log x)^{\frac{1}{2}}
\]
with \( \alpha_2 = \frac{5}{\sqrt{\pi}} \).

**Proof.** — We have
\[
\sum_p \sum_{m \geq 1, N_{K/Q}p^m > x^5} \theta(p^m) (\log N_{K/Q}p) \hat{k}_2(N_{K/Q}p^m) \\
\leq n_K \int_{x^5}^{\infty} (\log u) \hat{k}_2(u) \, dT(u),
\]
where \( T(u) \) is the number of prime powers \( p^h \) with \( h \geq 1 \) and \( p^h \leq u \). Since \( T(u) \leq u \) for \( u > 0 \), we have
\[
\int_{x^5}^{\infty} (\log u) \hat{k}_2(u) \, dT(u) \\
\leq \int_{x^5}^{\infty} \hat{k}_2(u) \left( \frac{\log u \log \frac{u}{x}}{2 \log x} - 1 \right) T(u) \, du \\
\leq \int_{\log x}^{\infty} \hat{k}_2(xe^t) \left\{ \frac{(t + \log x)t}{2 \log x} - 1 \right\} T(xe^t) \, dt \\
\leq \alpha_2 x (\log x)^{\frac{1}{2}}. \quad \square
\]

From Lemmas 3.1, 3.3, and 3.4 we deduce an upper bound for
\[
\left| I_j - \sum_{p \in P(C)} (\log N_{K/Q}p) \hat{k}_j(N_{K/Q}p) \right|
\]
for \( j = 1, 2 \) as follows.
Proposition 3.5. — Let \( k_j(s) \) be as above. Let
\[
I_j = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s)k_j(s)\, ds.
\]
Assume that \( L \neq \mathbb{Q} \). Then

(1) for \( x \geq 101 \)
\[
\left| I_1 - \sum_{p \in \mathcal{P}(C)} (\log N_{K/Q}p)\hat{k}_1(N_{K/Q}p) \right| \\
\leq \frac{2\log x}{x^2} \log d_L + 16.08\alpha_0 n_K \log \frac{x}{x} \\
\leq \frac{\log x}{x} \log d_L
\]
with
\[
\alpha_3 = \frac{2}{101} + \frac{32.16\alpha_0}{\log 3} = 36.759 \ldots
\]

(2) for \( x \geq 10^{10} \)
\[
\left| I_2 - \sum_{\substack{p \in \mathcal{P}(C) \\ N_{K/Q}p \leq x^5}} (\log N_{K/Q}p)\hat{k}_2(N_{K/Q}p) \right| \\
\leq \frac{5}{2\sqrt{\pi}\log 3}(\log x)^{\frac{3}{2}} \log d_L + \alpha_1 n_K x^{\frac{3}{2}}(\log x)^{\frac{3}{2}} + \alpha_2 n_K x(\log x)^{\frac{3}{2}} \\
\leq \alpha_4 x(\log x)^{\frac{3}{2}} \log d_L
\]
with
\[
\alpha_4 = \frac{1}{\log 3} \left( \frac{10^{-9}}{4\sqrt{\pi}} + \frac{\alpha_1 \log 10}{5\sqrt{10}} + 2\alpha_2 \right) = 5.4567 \ldots
\]

Note that \( d_L \geq 3^{n_L/2} \) for \( n_L \geq 2 \). It follows from the Hermite–Minkowski’s inequality \( d_L > \frac{\pi}{3} \left( \frac{3\pi}{4} \right)^{n_L-1} \) for \( n_L > 1 \). For \( n_L = 2 \), \( d_L \geq 3 \), and for \( n_L \geq 3 \), \( \frac{\pi}{3} \left( \frac{3\pi}{4} \right)^{n_L-1} = \frac{4}{9} \left( \frac{3\pi}{4} \right)^{n_L} > 3^{n_L/2} \). (See also [48, p. 140] and [23, p. 291].)

4. The Contour integral

In this section we will evaluate the integrals \( I_1 \) and \( I_2 \) by contour integration. We will use \( L(s, \chi) \) to denote \( L(s, \chi, E) \). Let \( \mathcal{F}(\chi) \) be the conductor.
of $\chi$ and $A(\chi) = d_{E}N_{E/Q}F(\chi)$. Let
\[
\delta(\chi) = \begin{cases} 
1 & \text{if } \chi \text{ is the principal character}, \\
0 & \text{otherwise}.
\end{cases}
\]
We recall that for each $\chi$ there exist non-negative integers $a(\chi), b(\chi)$ such that
\[
a(\chi) + b(\chi) = [E : Q] = n_{E},
\]
and such that if we define
\[
\gamma_{\chi}(s) = \left\{ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \right\}^{a(\chi)} \left\{ \pi^{-\frac{s+1}{2}} \Gamma \left( \frac{s+1}{2} \right) \right\}^{b(\chi)},
\]
and
\[
\xi(s,\chi) = \{s(s-1)\}^{\delta(\chi)} A(\chi)^{s/2} \gamma_{\chi}(s)L(s,\chi),
\]
then $\xi(s,\chi)$ satisfies the functional equation
\[
\xi(1-s,\chi) = W(\chi)\xi(s,\chi),
\]
where $W(\chi)$ is a certain constant of absolute value 1. Furthermore, $\xi(s,\chi)$ is an entire function of order 1 and does not vanish at $s = 0$. By Hadamard product theorem we have for every $s \in \mathbb{C}$
\[
-\frac{L'}{L}(s,\chi) = \frac{1}{2} \log A(\chi) + \delta(\chi) \left( \frac{1}{s} + \frac{1}{s-1} \right) + \frac{\gamma'_{\chi}(s)}{\gamma_{\chi}(s)} - B(\chi) - \sum_{\rho_{\chi} \in Z(\chi)} \left( \frac{1}{s - \rho_{\chi}} + \frac{1}{\rho_{\chi}} \right),
\]
where $B(\chi)$ is some constant and $Z(\chi)$ denotes the set of nontrivial zeros of $L(s,\chi)$. (See [48] and [24].) According to [40, (2.8)]
\[
\Re B(\chi) = - \sum_{\rho_{\chi} \in Z(\chi)} \Re \frac{1}{\rho_{\chi}}.
\]
Hence, for every $s \in \mathbb{C}$
\[
(4.1) \quad \Re \left\{ -\frac{L'}{L}(s,\chi) \right\} = \frac{1}{2} \log A(\chi) + \delta(\chi) \Re \left( \frac{1}{s} + \frac{1}{s-1} \right) + \Re \frac{\gamma'_{\chi}(s)}{\gamma_{\chi}(s)} - \sum_{\rho_{\chi} \in Z(\chi)} \Re \frac{1}{s - \rho_{\chi}}.
\]
For $j = 1, 2$ we have
\[
I_{j} = \frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) J_{j}(\chi) \quad \text{by (2.1),}
\]
where
\[ J_j(\chi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi)k_j(s) \, ds. \]

Assume that \( T \geq 2 \) does not equal the ordinate of any of the zeros of \( L(s, \chi). \) Consider
\[ J_j(\chi, T) = \frac{1}{2\pi i} \int_{B(T)} -\frac{L'}{L}(s, \chi)k_j(s) \, ds \]
for \( j = 1, 2, \) where \( B(T) \) is the positively oriented rectangle with vertices \( 2 - iT, 2 + iT, -\frac{1}{2} + iT, \) and \(-\frac{1}{2} - iT. \) By Cauchy’s theorem
\[ (4.2) \quad J_j(\chi, T) = \delta(\chi)k_j(1) - \{a(\chi) - \delta(\chi)\} k_j(0) - \sum_{\rho_\chi \in Z(\chi)} k_j(\rho_\chi) \]
for \( j = 1, 2. \)

**Lemma 4.1.** — Let
\[ V_j(\chi) = \frac{1}{2\pi i} \int_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} -\frac{L'}{L}(s, \chi)k_j(s) \, ds \]
for \( j = 1, 2. \) Then

1. for \( x \geq 101 \)
\[ |V_1(\chi)| \leq k_1 \left( -\frac{1}{2} \right) \{\mu_1 \log A(\chi) + n_E \nu_1\}, \]
where \( \mu_1 = 0.75296 \ldots \) and \( \nu_1 = 19.405 \ldots ; \)

2. for \( x \geq 10^{10} \)
\[ |V_2(\chi)| \leq k_2 \left( -\frac{1}{2} \right) \{\mu_2 \log A(\chi) + n_E \nu_2\}, \]
where \( \mu_2 = 0.058787 \ldots \) and \( \nu_2 = 1.4793 \ldots . \)

**Proof.** — Let \( s = -\frac{1}{2} + it. \) By [59, Lemme 5.1]
\[ \left| -\frac{L'}{L} \left( -\frac{1}{2} + it, \chi \right) \right| \leq \log A(\chi) + n_E v(t), \]
where
\[ v(t) = \log \left( \sqrt{\frac{1}{4} + t^2 + 2} \right) + \frac{19683}{812}. \]
Moreover, for \( x \geq 101 \)
\[
\left| k_1 \left( \frac{1}{2} + it \right) \right| \leq \frac{x^{-3}(1 + x^{-\frac{3}{2}})^2}{\frac{9}{4} + t^2} \\
= k_1 \left( \frac{1}{2} \right) \left( \frac{1 + x^{-\frac{3}{2}}}{1 - x^{-\frac{3}{2}}} \right)^2 \left( \frac{9}{9 + 4t^2} \right) \\
\leq k_1 \left( \frac{1}{2} \right) v_1(t)
\]
with \( v_1(t) = \left( \frac{1 + 101 - \frac{3}{2}}{1 - 101 - \frac{3}{2}} \right)^2 \left( \frac{9}{9 + 4t^2} \right) \) and for \( x \geq 10^{10} \)
\[
\left| k_2 \left( \frac{1}{2} + it \right) \right| = x^{-\frac{1}{4} - t^2} = k_2 \left( \frac{1}{2} \right) x^{-t^2} \leq k_2 \left( \frac{1}{2} \right) v_2(t)
\]
with \( v_2(t) = 10^{-10t^2} \). Hence,
\[
\left| \frac{1}{2\pi i} \int_{-\frac{1}{2} + iT}^{-\frac{1}{2} - iT} - \frac{L'}{L}(s, \chi) k_j(s) \, ds \right| \\
\leq \frac{1}{\pi} k_j \left( \frac{1}{2} \right) \int_0^T \{ \log A(\chi) + n_E v(t) \} v_j(t) \, dt.
\]
Set
\[
\mu_j = \frac{1}{\pi} \int_0^\infty v_j(t) \, dt \quad \text{and} \quad \nu_j = \frac{1}{\pi} \int_0^\infty v(t) v_j(t) \, dt.
\]
The result follows. \( \square \)

On the two segments from \( 2 \pm iT \) to \( -\frac{1}{2} \pm iT \) we proceed with the same way as [24, Section 6]. (See [23, Section 3], [59, Section 5], and [27].) Let
\[
\mathcal{H}_j(T) = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{-\frac{1}{2} + iT} \left\{ \frac{L'}{L}(\sigma + iT, \chi) k_j(\sigma + iT) - \frac{L'}{L}(\sigma - iT, \chi) k_j(\sigma - iT) \right\} \, d\sigma
\]
and
\[
\mathcal{H}^*_j(T) = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{-\frac{1}{2} + iT} \left\{ \frac{L'}{L}(\sigma + iT, \chi) k_j(\sigma + iT) - \frac{L'}{L}(\sigma - iT, \chi) k_j(\sigma - iT) \right\} \, d\sigma.
\]
Then
\[
\mathcal{H}_j(T) + \mathcal{H}^*_j(T) = \frac{1}{2\pi i} \left\{ \int_{2 + iT}^{-\frac{1}{2} + iT} - \frac{L'}{L}(s, \chi) k_j(s) \, ds + \int_{-\frac{1}{2} - iT}^{2 - iT} - \frac{L'}{L}(s, \chi) k_j(s) \, ds \right\}.
\]

**Lemma 4.2.** — For \( j = 1, 2 \) we have
\[
\mathcal{H}_j(T) \ll |k_j(iT)| (\log A(\chi) + n_E \log T).
\]
Proof. — Let $s = \sigma \pm iT$ with $-\frac{1}{2} \leq \sigma \leq -\frac{1}{4}$. Then

$$\frac{L'}{L}(s, \chi) \ll \log A(\chi) + n_E \log T$$

by [24, Lemma 6.2] and $k_j(s) \ll |k_j(iT)|$. The result follows. \( \square \)

**Lemma 4.3.** — Let $-\frac{1}{4} \leq \sigma \leq 2$. Then, we have

$$\frac{L'}{L}(\sigma \pm iT, \chi) - \sum_{\rho \chi \in \mathbb{Z}(\chi)} \frac{1}{\sigma \pm iT - \rho \chi} \ll \log A(\chi) + n_E \log T.$$  

Proof. — See [24, Lemma 5.6]. (See also [59, Lemma 4.8].) \( \square \)

Therefore, for $j = 1, 2$

$$H_j^*(T) = \frac{1}{2\pi i} \int_{-\frac{1}{4}}^{\frac{1}{2}} \left\{ k_j(\sigma + iT) \sum_{\rho \chi \in \mathbb{Z}(\chi)} \frac{1}{\sigma + iT - \rho \chi} - k_j(\sigma - iT) \sum_{\rho \chi \in \mathbb{Z}(\chi)} \frac{1}{\sigma - iT - \rho \chi} \right\} d\sigma \ll |k_j(iT)|(\log A(\chi) + n_E \log T)$$

since $k_j(\sigma \pm iT) \ll |k_j(iT)|$ for $-\frac{1}{4} \leq \sigma \leq 2$.

**Lemma 4.4.** — Let $\rho \chi \in \mathbb{Z}(\chi)$ with $t \neq \Im \rho \chi$. If $|t| \geq 2$, then

$$\int_{-\frac{1}{4}}^{\frac{1}{2}} \frac{k_j(\sigma + it)}{\sigma + it - \rho \chi} d\sigma \ll |k_j(it)|$$

for $j = 1, 2$.

**Proof.** — Suppose first that $\Im \rho \chi > t$. Let $B_t$ be the positive oriented rectangle with vertices $2 + i(t - 1)$, $2 + it$, $-\frac{1}{4} + it$, and $-\frac{1}{4} + i(t - 1)$. By Cauchy’s theorem,

$$\int_{B_t} \frac{k_j(s)}{s - \rho \chi} ds = 0$$

for $j = 1, 2$. However, on the three sides of the rectangle other than the segment from $-\frac{1}{4} + it$ to $2 + it$, the integrand is majorized by

$$\alpha_5 |k_j(it)|$$

for some positive constant $\alpha_5$ depending on $x$, which proves the result for $\Im \rho \chi > t$. A similar proof for $\Im \rho \chi < t$ uses the rectangle with vertices $2 + it$, $2 + i(t + 1)$, $-\frac{1}{4} + i(t + 1)$, and $-\frac{1}{4} + it$. \( \square \)
For $j = 1, 2$ we have

\[
\frac{1}{2\pi i} \int_{-\frac{1}{2} + iT}^{\frac{1}{2} + iT} \left\{ k_j(\sigma + iT) \sum_{\substack{\rho \chi \in \mathbb{Z}(\chi) \\ |\Im \rho \chi - T| \leq 1}} \frac{1}{\sigma + iT - \rho \chi} \right. \\
\left. - k_j(\sigma - iT) \sum_{\substack{\rho \chi \in \mathbb{Z}(\chi) \\ |\Im \rho \chi + T| \leq 1}} \frac{1}{\sigma - iT - \rho \chi} \right\} d\sigma
\]

\[
\ll |k_j(iT)| \{ n_\chi(T) + n_\chi(-T) \}
\]

\[
\ll |k_j(iT)|(\log A(\chi) + n_E \log T) \text{ by [24, Lemma 5.4]},
\]

where $n_\chi(T)$ denotes the number of zeros $\rho \chi \in \mathbb{Z}(\chi)$ with $|\Im \rho \chi - T| \leq 1$. We may then conclude as follows.

**Lemma 4.5.** For $j = 1, 2$ we have

\[
\mathcal{H}_j^*(T) \ll |k_j(iT)|(\log A(\chi) + n_E \log T).
\]

**Lemma 4.6.** For $j = 1, 2$ we have

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \left\{ \int_{2+ iT}^{2+ iT} - \frac{L'}{L}(s, \chi)k_j(s) \, ds + \int_{-\frac{1}{h} - iT}^{-\frac{1}{h} + iT} - \frac{L'}{L}(s, \chi)k_j(s) \, ds \right\} = 0.
\]

**Proof.** By Lemmas 4.2 and 4.5

\[
\mathcal{H}_j(T) + \mathcal{H}_j^*(T) \ll |k_j(iT)|(\log A(\chi) + n_E \log T).
\]

Since

\[
|k_j(iT)| \leq \begin{cases} 
\frac{9}{4x^2(1+T^2)} & \text{if } j = 1, \\
x^{-T^2} & \text{if } j = 2,
\end{cases}
\]

the result follows. \hfill \square

Letting $T \to \infty$ in (4.2) and combining this and Lemmas 4.6 yield

\[
J_j(\chi) + V_j(\chi) = \delta(\chi)k_j(1) - \{ a(\chi) - \delta(\chi) \} k_j(0) - \sum_{\rho \chi \in \mathbb{Z}(\chi)} k_j(\rho \chi)
\]
for \( j = 1, 2 \). Hence, we have
\[
\frac{|G|}{|C|} I_j = \sum_{\chi} \overline{\chi}(g) J_j(\chi)
\]
\[
= k_j(1) - k_j(0) \sum_{\chi} \overline{\chi}(g) \{ a(\chi) - \delta(\chi) \} - \sum_{\chi} \overline{\chi}(g) \left( \sum_{\rho \in Z(\chi)} k_j(\rho) \right)
\]
\[
- \sum_{\chi} \overline{\chi}(g) V_j(\chi)
\]
for \( j = 1, 2 \). Note that by the conductor-discriminant formula ([46, Chapter VI, Section 3])
\[
\sum_{\chi} \log A(\chi) = \log d_L.
\]
We therefore conclude as follows.

**Proposition 4.7.** — For \( j = 1, 2 \) we have
\[
(4.3) \quad \frac{|G|}{|C|} I_j \geq k_j(1) - \sum_{\rho \in Z(\zeta_L)} |k_j(\rho)| - \mu_j k_j \left( -\frac{1}{2} \right) \log d_L
\]
\[
- n_L \left( k_j(0) + \nu_j k_j \left( -\frac{1}{2} \right) \right)
\]
where \( Z(\zeta_L) \) denotes the set of all nontrivial zeros of \( \zeta_L(s) \), \( \mu_j \) and \( \nu_j \) are as in Lemma 4.1.

### 5. Density of zeros of Dedekind zeta functions

To begin with, we recall that for every \( s \in \mathbb{C} \) we have
\[
(5.1) \quad \Re \left\{ -\zeta_L'(s) \right\} = \frac{1}{2} \log d_L + \Re \left( \frac{1}{s} + \frac{1}{s - 1} \right)
\]
\[
+ \Re \frac{\gamma_L'}{\gamma_L}(s) - \sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{s - \rho},
\]
where
\[
\gamma_L(s) = \left\{ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \right\}^{r_1+r_2} \left\{ \pi^{-\frac{s+1}{2}} \Gamma \left( \frac{s+1}{2} \right) \right\}^{r_2},
\]
r_1 and 2r_2 are the numbers of real and complex embeddings of \( L \). (See [24, Lemma 5.1] or [48].)

For any real number \( t \) we let
\[
n_L(t) = |\{ \rho = \beta + i\gamma \mid \zeta_L(\rho) = 0 \text{ with } 0 < \beta < 1 \text{ and } |\gamma - t| \leq 1 \}|.
\]
For any complex number $s$ and positive real number $r > 0$ we let
\[ n(r; s) = \left| \{ \rho \in \mathbb{Z}(\zeta_L) \mid |\rho - s| \leq r \} \right|. \]

From (4.1) Lagarias and Odlyzko deduced that
\[ n_\chi(t) \ll \log A(\chi) + n_E \log(|t| + 2) \]
for all $t$. (See [24, Lemma 5.4].) In this section we will bound $n_L(t)$ and $n(r; s)$ from above using (4.1). To do this we need some lemmas.

**Lemma 5.1.** — Let $s = \sigma + it$ with $\sigma > 1$. We have
\[ \sum_{\rho \in \mathbb{Z}(\zeta_L)} \Re \frac{1}{s - \rho} \geq f_0(\sigma)n_L(t), \]
where
\[ f_0(\sigma) = \frac{1}{2} \min \left\{ \frac{\sigma - 1}{(\sigma - 1)^2 + 1}, \frac{\sigma - 1}{2} \right\} \]
\[ + \frac{1}{2} \min \left\{ \frac{\sigma - 1}{(\sigma - 1)^2 + 1}, \frac{1}{\sigma^2 + 1} \right\}. \]

**Proof.** — We have
\[ \sum_{\rho \in \mathbb{Z}(\zeta_L)} \Re \frac{1}{s - \rho} \geq \frac{1}{2} \sum_{\beta + i\gamma \in \mathbb{Z}(\zeta_L)} \left\{ \frac{\sigma - \beta}{(\sigma - \beta)^2 + 1} + \frac{\sigma + \beta - 1}{(\sigma + \beta - 1)^2 + 1} \right\} \]
\[ \geq f_0(\sigma)n_L(t). \qed \]

**Lemma 5.2.** — If $\Re s = \sigma > 1$, then
\[ \Re \frac{\zeta'_L}{\zeta_L}(s) \leq n_L f_1(\sigma), \]
where
\[ f_1(\sigma) = -\frac{\zeta'_Q}{\zeta_Q}(\sigma). \]

**Proof.** — For $\Re s > 1$,
\[ -\frac{\zeta'_L}{\zeta_L}(s) = \sum_{\mathfrak{p}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^s - 1} = \sum_{\mathfrak{p}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^m} \sum_{m=1}^{\infty} N\mathfrak{p}^{-ms}, \]
where $\mathfrak{p}$ runs over all prime ideals of $L$. Comparing $-\frac{\zeta'_L}{\zeta_L}(\sigma)$ with $-\frac{\zeta'_Q}{\zeta_Q}(\sigma)$ yields
\[ \Re \frac{\zeta'_L}{\zeta_L}(s) \leq \left| -\frac{\zeta'_L}{\zeta_L}(s) \right| \leq -\frac{\zeta'_L}{\zeta_L}(\sigma) \leq n_L \left\{ -\frac{\zeta'_Q}{\zeta_Q}(\sigma) \right\}. \]
(See [24, Lemma 3.2].) \qed
See also [9], [31, Lemma (a)], [59, Lemma 3.2], [11, p. 184], and [33, Proposition 2].

**Lemma 5.3.** — Assume that $\Re s > \frac{1}{2}$. We have

1. $\Re \frac{\Gamma'}{\Gamma}(s) \leq \log |s| + \frac{1}{3} \leq \alpha_6 \log(|s| + 2)$
   with $\alpha_6 = 1.08$;

2. $\Re \frac{\Gamma'}{\Gamma}(s) \geq \log |s| - \frac{4}{3} \geq \log(|s| + 2) - \alpha_7$
   with $\alpha_7 = \frac{4}{3} + \log 5 = 2.9427 \ldots$.

**Proof.** — For $\Re s > 0$,

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - 2 \int_0^\infty \frac{v}{(s^2 + v^2)(e^{2\pi v} - 1)} \, dv.$$  

(See [58, p. 251].) Since $|s^2 + v^2| \geq (\Re s)^2$, we have

$$\left| \int_0^\infty \frac{v}{(s^2 + v^2)(e^{2\pi v} - 1)} \, dv \right| \leq \frac{1}{(\Re s)^2} \int_0^\infty \frac{v}{e^{2\pi v} - 1} \, dv = \frac{1}{24 (\Re s)^2}.$$  

If $\Re s > \frac{1}{2}$, then

$$\Re \frac{\Gamma'}{\Gamma}(s) \leq \log |s| + \frac{1}{12 (\Re s)^2} \leq \log |s| + \frac{1}{3}$$

and

$$\Re \frac{\Gamma'}{\Gamma}(s) \geq \log |s| - \frac{1}{2|s|} - \frac{1}{12 (\Re s)^2} \geq \log |s| - \frac{4}{3}.$$  

Set $\varphi_1(v) = \alpha_6 \log(v + 2) - \log v - \frac{1}{3}$ for $v > \frac{1}{2}$. Then,

$$\varphi'_1(v) = \frac{(\alpha_6 - 1)v - 2}{v(v + 2)}$$

and $\varphi_1(v) > \varphi_1 \left( \frac{2}{\alpha_6 - 1} \right) > 0$.

Hence

$$\Re \frac{\Gamma'}{\Gamma}(s) \leq \alpha_6 \log(|s| + 2).$$  

Set $\varphi_2(v) = \log v - \frac{4}{3} - \log(v + 2) + \alpha_7$ for $v > \frac{1}{2}$. Then

$$\varphi'_2(v) > 0$$

and $\varphi_2(v) > \varphi_2 \left( \frac{1}{2} \right) = 0$.

Hence

$$\Re \frac{\Gamma'}{\Gamma}(s) \geq \log(|s| + 2) - \alpha_7.$$  

□
Lemma 5.4. — Let \( s = \sigma + it \). If \( \sigma > 1 \), then
\[
\Re \frac{\gamma'}{\gamma_L}(s) \leq n_L \left\{ f_2(\sigma) \log(\|t\| + 2) - \frac{1}{2} \log \pi \right\},
\]
where
\[
f_2(\sigma) = \frac{\alpha_6}{2} \left\{ \frac{\log(\sigma + 5)}{\log 2} - 1 \right\}.
\]

Proof. — By definition and Lemma 5.3(1) we have
\[
\Re \frac{\gamma'}{\gamma_L}(s) = \left( \frac{r_1 + r_2}{2} \right) \Re \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} \right) + \frac{r_2}{2} \Re \frac{\Gamma'}{\Gamma} \left( \frac{s + 1}{2} \right) - \frac{n_L}{2} \log \pi
\]
\[
\leq \frac{\alpha_6}{2} \left( \frac{r_1 + r_2}{2} \right) \log \left( \frac{\|s\|}{2} + 2 \right) + \frac{\alpha_6}{2} \log \left( \frac{\|s\| + 1}{2} + 2 \right) - \frac{n_L}{2} \log \pi
\]
\[
\leq \frac{n_L}{2} \left\{ \alpha_6 \log \left( \frac{\|s\| + 1}{2} + 2 \right) - \log \pi \right\}.
\]

It is sufficient to verify that
\[
\log \left( \frac{\|s\| + 1}{2} + 2 \right) \leq \left( \frac{\log(\sigma + 5)}{\log 2} - 1 \right) \log(\|t\| + 2).
\]
Note that \( \|s + 1\| \geq 2\|t\| \) if and only if \( \|t\| \leq (\sigma + 1)/\sqrt{3} \). If \( \|t\| \geq (\sigma + 1)/\sqrt{3} \), then (5.2) holds. We suppose now that \( \|t\| < (\sigma + 1)/\sqrt{3} \). Set \( \varphi_3(v) = \varphi_5(v)/\varphi_4(v) \) with \( \varphi_4(v) = v + 2 \) and \( \varphi_5(v) = 2 + (\sigma + 1)^2 + v^2/2 \). Then \( \varphi_3'(v) \leq 0 \) and \( \varphi_5(v) \leq \left( \frac{\varphi_5(0)}{\varphi_4(0)} \right) \varphi_4(v) \) for \( 0 \leq v < (\sigma + 1)/\sqrt{3} \). For \( 0 \leq v < (\sigma + 1)/\sqrt{3} \) we have then
\[
\log \frac{\varphi_5(v)}{\varphi_4(v)} \leq \frac{\log \varphi_4(v) + \log \varphi_5(0) - \log \varphi_4(0)}{\log \varphi_4(v)}
\]
\[
\leq \log \frac{\varphi_5(0)}{\varphi_4(0)} = \frac{\log(\sigma + 5)}{\log 2} - 1,
\]
which yields (5.2). \( \square \)

We are now ready to bound \( n_L(t) \).

Proposition 5.5. — For all \( t \) we have
\[
n_L(t) \leq 1.1 \log d_L + 2.09 \log \{ (\|t\| + 2)^{n_L} \} + 0.56 n_L + 4.05.
\]
In particular, if \( L \neq \mathbb{Q} \), then
\[
n_L(t) \leq 2.72 \log \{ d_L(\|t\| + 2)^{n_L} \}.
\]
Proof. — Combining (4.1), Lemmas 5.1, 5.2, 5.3, and 5.4 yields
\[
f_0(\sigma)n_L(t) \leq \frac{1}{2} \log d_L + \frac{1}{\sigma} + \frac{1}{\sigma - 1} + n_L \left\{ f_2(\sigma) \log(|t| + 2) - \frac{1}{2} \log \pi + f_1(\sigma) \right\}
\]
for \( \sigma > 1 \). We write
\[
n_L(t) \leq a_1(\sigma) \log d_L + a_2(\sigma) \log \{|t| + 2\}^{n_L} + a_3(\sigma)n_L + a_4(\sigma)
\]
for \( \sigma > 1 \), where
\[
a_1(\sigma) = \frac{1}{2f_0(\sigma)}, \quad a_2(\sigma) = \frac{f_2(\sigma)}{f_0(\sigma)}, \quad a_3(\sigma) = \frac{1}{f_0(\sigma)} \left\{ f_1(\sigma) - \frac{1}{2} \log \pi \right\},
\]
and
\[
a_4(\sigma) = \frac{1}{f_0(\sigma)} \left( \frac{1}{\sigma} + \frac{1}{\sigma - 1} \right).
\]
We choose now appropriate \( \sigma \). If \( \sigma = (3 + \sqrt{17})/4 \), then (5.6) yields (5.4).
For the proof of (5.5), we choose \( \sigma = 2.45 \). In this case, \( a_3(\sigma) < 0 \) and \( 2a_3(\sigma) + a_4(\sigma) > 0 \). Since \( n_L \geq 2 \), it follows from (5.6) that
\[
n_L(t) \leq a_1(\sigma) \log d_L + a_2(\sigma) \log \{|t| + 2\}^{n_L} + 2a_3(\sigma) + a_4(\sigma)
\]
\[
\leq B_1 \log d_L + B_2 \log \{|t| + 2\}^{n_L},
\]
where \( B_1 = a_1(\sigma) + \frac{1}{\log 3} \{ 2a_3(\sigma) + a_4(\sigma) \} = 2.6885 \cdots \) and \( B_2 = a_2(\sigma) = 2.7106 \cdots \). So, we obtain (5.5). \( \square \)

See also [21], [52], and [59, Lemme 4.6].

**Proposition 5.6.** — Let \( r \) be a positive real number.

1. Assume that
\[
n_L(t) \leq \alpha_8 \log \{ d_L(|t| + 2)^{n_L} \}
\]
for some \( \alpha_8 > 0 \). Then we have
\[
n(r; \sigma + it) \leq \alpha_8(1 + r) \log \{ d_L(|t| + r + 2)^{n_L} \}.
\]
2. Assume that \( L \neq \mathbb{Q} \). If \( \sigma \geq 1 \) and \( 0 < r \leq 1 \), then
\[
n(r; \sigma + it) \leq 10 \left[ 1 + \frac{2f_2(2)}{5}r \log \{ d_L(|t| + 2)^{n_L} \} \right].
\]

Proof. — Set
\[
Z(r; s) = \{ \rho \in Z(\zeta_L) \mid |\rho - s| \leq r \}
\]
and
\[
Z(t) = \{ \beta + i\gamma \in Z(\zeta_L) \mid |\gamma - t| \leq 1 \}.
\]
Note that \( n(r; s) = |Z(r; s)| \) and \( n_L(t) = |Z(t)| \).
Let $t_1, t_2, \ldots, t_{1+r}$ be real numbers such that $t - r \leq t_1 < \ldots < t_{1+r} \leq t + r$ and

$$Z(r; s) \subseteq \bigcup_{i=1}^{1+r} Z(t_i).$$

Then

$$n(r; \sigma + it) \leq \sum_{i=1}^{1+r} n_L(t_i) \leq \alpha_8 \sum_{i=1}^{1+r} \{\log d_L + n_L \log(|t_i| + 2)\}$$

$$\leq \alpha_8 (1 + r) \{\log d_L + n_L \log(|t| + r + 2)\}.$$ 

(2). Write $z = 1 + r + it$. By (4.1),

$$\sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{z - \rho} = \frac{1}{2} \log d_L + \Re \frac{\gamma_L'}{\gamma_L} (z) + \Re \frac{\zeta_L'}{\zeta_L} (z) + \Re \left( \frac{1}{z} + \frac{1}{z - 1} \right).$$

We now estimate $\Re \frac{\gamma_L'}{\gamma_L} (z)$ and $\Re \frac{\zeta_L'}{\zeta_L} (z)$ from above. By Lemma 5.4

$$\Re \frac{\gamma_L'}{\gamma_L} (z) \leq n_L \left\{ f_2 (1 + r) \log(|t| + 2) - \frac{1}{2} \log \pi \right\}$$

$$\leq f_2 (1 + r) \log \{|t| + 2 \}.$$ 

It follows from [33, Proposition 2] that

$$\Re \frac{\zeta_L'}{\zeta_L} (z) \leq \left| \frac{\zeta_L'}{\zeta_L} (z) \right| \leq - \frac{\zeta_L'}{\zeta_L} (1 + r) \leq \left( \frac{1}{2} - \frac{1}{2\sqrt{5}} \right) \log d_L + \frac{1}{r}.$$ 

Therefore,

$$\sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{z - \rho} \leq \left( 1 - \frac{1}{2\sqrt{5}} \right) \log d_L + f_2 (1 + r) \log \{|t| + 2 \} + \frac{2}{r} + \frac{1}{1 + r}.$$ 

Moreover,

$$\sum_{\rho \in Z(\zeta_L)} \Re \frac{1}{z - \rho} \geq \sum_{\rho \in Z(2r; z)} \Re \frac{1}{z - \rho} \geq \frac{1}{4r} n(2r; z).$$

Since $Z(r; \sigma + it) \subseteq Z(r; 1 + it) \subseteq Z(2r; z)$ and $1 - \frac{1}{2\sqrt{5}} < f_2 (2)$, we have

$$n(r; \sigma + it) \leq n(2r; z)$$

$$\leq 4r \left[ \left( 1 - \frac{1}{2\sqrt{5}} \right) \log d_L + f_2 (1 + r) \log \{|t| + 2 \} + \frac{2}{r} + \frac{1}{1 + r} \right]$$

$$\leq 10 \left[ 1 + \frac{2f_2 (2)}{5} r \log \{d_L \{|t| + 2 \} \} \right].$$ 

\[\Box\]
6. Zero-free regions for Dedekind zeta functions

We abbreviate $N_{L/Q}$ to $N$. The classical argument to obtain a zero-free region for $\zeta_L(s)$ starts from (4.1) and for $\sigma > 1$

$$\Re \left[ \sum_{m=0}^{d} b_m \left\{ -\frac{\zeta_L'}{\zeta_L} (\sigma + imt) \right\} \right] = \Re \sum_{m=0}^{d} b_m \left\{ \frac{\wedge(a)}{N a^{\sigma + imt}} \right\} \geq 0$$

where $b_m \geq 0$, $Q(\phi) = \sum_{m=0}^{d} b_m \cos(m\phi) \geq 0$, $\wedge(a)$ is the generalized Von Mangoldt function, and $a$ runs over all nonzero ideals of $L$.

Using Stechkin’s work one can reduce the constant $1/2$ of the term $\frac{1}{2} \log A(\chi)$ in (4.1) to $\frac{1}{2} \left( 1 - \frac{1}{\sqrt{5}} \right)$, which yields larger zero-free regions for $\zeta_L(s)$. (See [49], [45], [12], [36], [14], [19], [20], [37], [34], [32], [33], and [1].)

It is known that if $L \neq Q$, then $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ with

$$\beta > 1 - \frac{1}{2 \log d_L} \quad \text{and} \quad |\gamma| < \frac{1}{2 \log d_L}.$$ 

If this zero exists then it must be real and simple. (See [48, Lemma 3], [16, Lemma 2], and [1].) This possible zero is called the exceptional zero and denoted by $\rho_0$. In this section we will show the following:

**PROPOSITION 6.1.** — Assume that $L \neq Q$. Let $\rho = \beta + i\gamma$ be a nontrivial zero of $\zeta_L(s)$ with $\rho \neq \rho_0$ and $\tau = |\gamma| + 2$. Then

$$1 - \beta > (29.57 \log d_L \tau^{1/2})^{-1}.$$ 

For the zero-free regions of $\zeta_L(s)$ see also [20, Theorem 1.1], [59, Lemma 7.1], and [62].

We use the Stechkin’s work ([49]) as [36] and [20] and use the same notations as [36] and [20]. Set

$$s = \sigma + it, \quad \sigma_1 = \frac{1 + \sqrt{1 + 4\sigma^2}}{2}, \quad s_1 = \sigma_1 + it, \quad \kappa = \frac{1}{\sqrt{5}},$$

and

$$\Re(s, z) = \Re \left\{ \frac{1}{s - z} + \frac{1}{s - (1 - \tau)} \right\}.$$ 

For $\sigma > 1$

$$\Re \left\{ -\frac{\zeta_L'}{\zeta_L} (s) + \kappa \frac{\zeta_L'}{\zeta_L} (s_1) \right\} = \sum_{a} \frac{\wedge(a)}{N a} \left( 1 - \frac{\kappa}{N a^{\sigma_1 - \sigma}} \right) \Re(N a^{-it}),$$
where \( a \) runs over all nonzero ideals of \( L \). Moreover, by (4.1)

\[
\Re \left\{ -\frac{\zeta_L'}{\zeta_L} (s) + \kappa \frac{\zeta_L'}{\zeta_L} (s_1) \right\} = \frac{1 - \kappa}{2} \log d_L + \Re \left\{ \frac{\gamma_L'}{\gamma_L} (s) - \kappa \frac{\gamma_L'}{\gamma_L} (s_1) \right\} \\
+ \left\{ \mathcal{F}(s, 1) - \kappa \mathcal{F}(s_1, 1) \right\} - \sum_{\Re \rho \geq \frac{1}{2}} \left\{ \mathcal{F}(s, \rho) - \kappa \mathcal{F}(s_1, \rho) \right\},
\]

where

\[
\sum_{\Re \rho > \frac{1}{2}} = \frac{1}{2} \sum_{\rho \in Z(\zeta_L)} + \sum_{\rho \in Z(\zeta_L)} \frac{1}{2 < \Re \rho \leq 1}.
\]

Assume that \( b_m \geq 0 \) and \( Q(\phi) = \sum_{m=0}^{d} b_m \cos(m\phi) \geq 0 \). Then, for \( \sigma > 1 \)

\[
\sum_{m=0}^{d} b_m \Re \left\{ -\frac{\zeta_L'}{\zeta_L} (\sigma + im\gamma) + \kappa \frac{\zeta_L'}{\zeta_L} (\sigma_1 + im\gamma) \right\} = \sum_{a} \left\{ \mathcal{F}(\sigma + im\gamma, a) - \kappa \mathcal{F}(\sigma_1 + im\gamma, a) \right\} Q(\gamma \log N a) \geq 0.
\]

So,

\[
S_1(\sigma, \gamma) = \sum_{m=0}^{d} b_m \sum_{\Re \rho \geq \frac{1}{2}} \left\{ \mathcal{F}(\sigma + im\gamma, \rho) - \kappa \mathcal{F}(\sigma_1 + im\gamma, \rho) \right\},
\]

\[
S_2 = \frac{1 - \kappa}{2} Q(0) \log d_L,
\]

\[
S_3(\sigma, \gamma) = \sum_{m=0}^{d} b_m \left\{ \mathcal{F}(\sigma + im\gamma, 1) - \kappa \mathcal{F}(\sigma_1 + im\gamma, 1) \right\},
\]

and

\[
S_4(\sigma, \gamma) = \sum_{m=0}^{d} b_m \Re \left\{ \frac{\gamma_L'}{\gamma_L} (\sigma + im\gamma) - \kappa \frac{\gamma_L'}{\gamma_L} (\sigma_1 + im\gamma) \right\}.
\]

Our proof of Proposition 6.1 consists of three parts: We estimate \( S_1(\sigma, \gamma) \) from below, \( S_3(\sigma, \gamma) \) and \( S_4(\sigma, \gamma) \) from above. Note that if \( \rho \) is a nontrivial zero with \( |\gamma| < (2 \log d_L)^{-1} \), then (6.2) is satisfied. So, we may assume that

\[
\rho \in Z(\zeta_L) \text{ and } |\gamma| \geq (2 \log d_L)^{-1}.
\]

Assume that

\[
1 - \beta \leq (b \log d_L \tau^{-n_L})^{-1},
\]

where \( \tau \) runs over all nonzero ideals of \( L \). Moreover, by (4.1)
where \( b \geq 4 \) is a constant that will be specified later. Let \( \epsilon = (b \log 12)^{-1} \) and 
\( \sigma - 1 = (b \log d_L \tau^{n_L})^{-1} \). That is, \( 1 - \beta \leq \epsilon \) and \( \sigma - 1 \leq \epsilon \) with 
\( \epsilon \leq (4 \log 12)^{-1} \).

**Lemma 6.2** (Stechkin [49]). — Let \( s = \sigma + it \) with \( \sigma > 1 \).

1. If \( 0 < \Re z < 1 \), then
   \[
   F(s, z) - \kappa F(s_1, z) \geq 0.
   \]
2. If \( \Im z = t \) and \( \frac{1}{2} \leq \Re z < \sigma \), then
   \[
   \Re \left( \frac{1}{s - 1 + \frac{1}{z}} \right) - \kappa F(s_1, z) \geq 0.
   \]

**Lemma 6.3.** — Keeping the above notation we have

\[
S_1(\sigma, \gamma) \geq \frac{b_1}{\sigma - \beta} - \{Q(0) - b_1\} \alpha_10 + \sum_{m \neq 1} \frac{b_m(\sigma - \beta)}{(\sigma - \beta)^2 + ((m - 1)\gamma)^2}
\]

where

\[
\alpha_9 = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad \alpha_{10} = \kappa \left\{ \frac{2\epsilon}{\alpha_9^2} + \frac{\epsilon}{(\alpha_9^{-1} - \epsilon)^2} \right\} + \frac{\epsilon}{(1 - \epsilon)^2}.
\]

**Proof.** — By Lemma 6.2(1)

\[
S_1(\sigma, \gamma) \geq \sum_{m=0}^{d} b_m \{ F(\sigma + im\gamma, \beta + i\gamma) - \kappa F(\sigma_1 + im\gamma, \beta + i\gamma) \}.
\]

When \( m = 1 \), we have

\[
F(\sigma + i\gamma, \beta + i\gamma) - \kappa F(\sigma_1 + i\gamma, \beta + i\gamma) \geq \frac{1}{\sigma - \beta}
\]

by Lemma 6.2(2). When \( m \neq 1 \), we have

\[
F(\sigma + im\gamma, \beta + i\gamma) - \kappa F(\sigma_1 + im\gamma, \beta + i\gamma) = \frac{\sigma - \beta}{(\sigma - \beta)^2 + ((m - 1)\gamma)^2}
\]

\[
- G(\sigma_1 - \beta, \sigma_1 - 1 + \beta, \sigma - 1 + \beta; (m - 1)\gamma),
\]

where

\[
G(\omega_1, \omega_2, \omega_3; \nu) = \kappa \left( \frac{\omega_1}{\omega_1^2 + \nu^2} + \frac{\omega_2}{\omega_2^2 + \nu^2} - \frac{\omega_3}{\omega_3^2 + \nu^2} \right).
\]

Note that

\[
0 < \sigma_1 - \beta - \alpha_9 \leq 2\epsilon, \quad -\epsilon \leq \sigma_1 - 1 + \beta - \alpha_9^{-1} \leq \epsilon,
\]

and

\[
-\epsilon \leq \sigma - 1 + \beta - 1 \leq \epsilon.
\]
For $u > 0$ and $u_0 > 0$

\[(6.13)\quad \left| \frac{u}{u^2 + v^2} - \frac{u_0}{u_0^2 + v^2} \right| \leq \frac{|u - u_0|}{\min(u, u_0)^2}.\]

(See the proof of [20, Lemma 2.2] or that of [21, Lemma 5].) Using (6.12), (6.13), and the fact that $\mathcal{G}(\alpha_9, \alpha_9^{-1}; 1; v) \leq 0$ for all $v \in \mathbb{R}$ ([20, Lemma 2.2(i)] or [21, Lemma 5(i)]) we get

\[(6.14)\quad \mathcal{G}(\sigma_1 - \beta, \sigma_1 - 1 + \beta, \sigma - 1 + \beta; (m - 1)\gamma) \leq \alpha_{10}.

Substituting (6.10), (6.11), and (6.14) into (6.9) yields (6.8). □

**Lemma 6.4.** — Keeping the above notation we have

\[(6.15)\quad S_3(\sigma, \gamma) \leq \frac{b_0}{\sigma - 1} + b_0f_3(1 + \epsilon) - \{\mathcal{Q}(0) - b_0\}G_0 - \alpha_{11} + \sum_{m \neq 0} \frac{b_m(\sigma - 1)}{(\sigma - 1)^2 + (m\gamma)^2},\]

where

\[f_3(\sigma) = \frac{1}{\sigma} - \kappa \left( \frac{1}{\sigma_1 - 1} + \frac{1}{\sigma_1} \right), \quad \alpha_{11} = \kappa \left( \frac{\epsilon}{\alpha_9} + \frac{\epsilon}{\alpha_{9}^{-1}} \right) + \epsilon = (3\kappa + 1)\epsilon,\]

and $G_0 = -0.121585107$.

**Proof.** — When $m = 0$, we have

\[(6.16)\quad \mathbb{F}(\sigma, 1) - \kappa \mathbb{F}(\sigma_1, 1) = \frac{1}{\sigma - 1} + f_3(\sigma) \leq \frac{1}{\sigma - 1} + f_3(1 + \epsilon)\]

since $f_3(\sigma)$ is increasing for $1 < \sigma < 1.75$. When $m \neq 0$, we have

\[(6.17)\quad \mathbb{F}(\sigma + im\gamma, 1) - \kappa \mathbb{F}(\sigma_1 + im\gamma, 1) = \frac{\sigma - 1}{(\sigma - 1)^2 + (m\gamma)^2} - \mathcal{G}(\sigma_1 - 1, \sigma_1, \sigma; m\gamma).\]

Note that $0 < \sigma_1 - 1 - \alpha_9 = \sigma_1 - \alpha_9^{-1} \leq \epsilon$ and $0 < \sigma - 1 \leq \epsilon$. Using [20, Lemma 2.2] we get

\[(6.18)\quad \mathcal{G}(\sigma_1 - 1, \sigma_1, \sigma; m\gamma) \geq G_0 - \alpha_{11}.

On feeding (6.16), (6.17), and (6.18) into (6.6) we get (6.15). □

Let

\[D(m) = \begin{cases} \frac{1}{4} \{\Gamma_1(1 + \epsilon) + \Gamma_0(1 + \epsilon)\} - \frac{1-\kappa}{2} \log \pi & \text{if } m = 0, \\ f_4(1 + \epsilon) \log m + \alpha_{12} & \text{if } m \neq 0, \end{cases}\]
where
\[ \Gamma_a(s) = \frac{\Gamma'}{\Gamma} \left( \frac{s + a}{2} \right) - \kappa \frac{\Gamma'}{\Gamma} \left( \frac{s_1 + a}{2} \right), \quad f_4(\sigma) = \frac{\alpha_6 - \kappa}{2} \left\{ \frac{\log(\sigma + 5)}{\log 2} - 1 \right\}, \]
and
\[ \alpha_{12} = \frac{\kappa \alpha_7 - (1 - \kappa) \log \pi}{2} = 0.34162 \ldots. \]

**Lemma 6.5.** — Keeping the above notation we have
\[ S_4(\sigma, \gamma) \leq \alpha_{13} \log \tau^{nL} + \alpha_{14} nL, \]
where \( \alpha_{13} = \{\mathbb{Q}(0) - b_0\} f_4(1 + \epsilon) \) and \( \alpha_{14} = \sum_{m=0}^{d} b_m D(m) \).

**Proof.** — Since \( \Gamma_0(v) \) and \( \Gamma_1(v) \) are monotonically increasing and \( \Gamma_1(v) > \Gamma_0(v) \) for \( 1 < v < 2 \),
\[ \Re \left\{ \frac{\gamma'_L(s)}{\gamma_L(s)} - \kappa \frac{\gamma'_L(s_1)}{\gamma_L(s_1)} \right\} = \frac{nL}{2} \Gamma_0(\sigma) + \frac{r_2}{2} \{\Gamma_1(\sigma) - \Gamma_0(\sigma)\} - \frac{1 - \kappa}{2} nL \log \pi \]
\[ \leq nL \left\{ \frac{1}{4} \Gamma_1(\sigma) + \frac{1}{4} \Gamma_0(\sigma) - \frac{1 - \kappa}{2} \log \pi \right\} \leq nL D(0). \]

Set \( s = \sigma + im\gamma \) and \( s_1 = \sigma_1 + im\gamma \). For \( m \geq 1 \)
\[ \Re \left\{ \frac{\gamma'_L(s)}{\gamma_L(s)} - \kappa \frac{\gamma'_L(s_1)}{\gamma_L(s_1)} \right\} \]
\[ \leq \frac{r_1 + r_2}{2} \left\{ \alpha_6 \log \left( \frac{|s|}{2} + 2 \right) - \kappa \log \left( \frac{|s_1|}{2} + 2 \right) + \kappa \alpha_7 \right\} \]
\[ + \frac{r_2}{2} \left\{ \alpha_6 \log \left( \frac{|s + 1|}{2} + 2 \right) - \kappa \log \left( \frac{|s_1 + 1|}{2} + 2 \right) + \kappa \alpha_7 \right\} \]
\[ - \frac{1 - \kappa}{2} nL \log \pi \quad \text{by Lemma 5.3} \]
\[ \leq \frac{nL}{2} \left\{ (\alpha_6 - \kappa) \log \left( \frac{|s + 1|}{2} + 2 \right) + \kappa \alpha_7 - (1 - \kappa) \log \pi \right\} \]
\[ \leq nL \{ f_4(\sigma) \log(|m\gamma| + 2) + \alpha_{12} \} \quad \text{by (5.2)} \]
\[ \leq nL \{ f_4(1 + \epsilon) \log(|\gamma| + 2) + D(m) \}. \]

Hence
\[ S_4(\sigma, \gamma) \leq b_0 nL D(0) + nL \sum_{m=1}^{d} b_m \{ f_4(1 + \epsilon) \log \tau + D(m) \} \]
\[ = \alpha_{13} \log \tau^{nL} + \alpha_{14} nL. \]
Now, Proposition 6.1 is ready to be proven. Combining (6.3), (6.5), Lemmas 6.3, 6.4, and 6.5 yields

\[ 0 \leq \frac{1 - \kappa}{2} Q(0) \log d_L + \alpha_{13} \log \tau^{n_L} + \alpha_{14} n_L + \alpha_{15} + \frac{b_0}{\sigma - 1} \]

\[ - \frac{b_1}{\sigma - \beta} + \frac{b_1(\sigma - 1)}{(\sigma - 1)^2 + \gamma^2} - \frac{b_0(\sigma - \beta)}{(\sigma - \beta)^2 + \gamma^2} \]

\[ + \sum_{m=2}^d b_m \left\{ \frac{(\sigma - 1)}{(\sigma - 1)^2 + (m\gamma)^2} - \frac{(\sigma - \beta)}{(\sigma - \beta)^2 + ((m - 1)\gamma)^2} \right\}, \]

where \( \alpha_{15} = b_0 f_3(1 + \epsilon) - \{ Q(0) - b_0 \} \{ G_0 - \alpha_{11} \} + \{ Q(0) - b_1 \} \alpha_{10} \). Since

\[ \frac{b_1(\sigma - 1)}{(\sigma - 1)^2 + \gamma^2} - \frac{b_0(\sigma - \beta)}{(\sigma - \beta)^2 + \gamma^2} \]

\[ \leq \frac{(b_1 - b_0)(\sigma - 1)}{(\sigma - 1)^2 + \gamma^2} \]

\[ \leq (b_1 - b_0) \left( \frac{4b}{4 + b^2} \right) \log d_L \]

and for \( m \geq 2 \)

\[ \frac{(\sigma - 1)}{(\sigma - 1)^2 + (m\gamma)^2} - \frac{(\sigma - \beta)}{(\sigma - \beta)^2 + ((m - 1)\gamma)^2} \leq 0, \]

it follows that

(6.19) \[ 0 \leq \alpha_{16} \log d_L + \alpha_{13} \log \tau^{n_L} + \alpha_{14} n_L + \alpha_{15} + \frac{b_0}{\sigma - 1} - \frac{b_1}{\sigma - \beta} \]

with

\[ \alpha_{16} = \frac{1 - \kappa}{2} Q(0) + (b_1 - b_0) \left( \frac{4b}{4 + b^2} \right). \]

Let \( 0 \leq \delta \leq 1 \) and \( 0 \leq \eta \leq 1 \). Note that \( d_L \geq 3^{n_L/2} \). Set

\[ B_{11} = \alpha_{16} + \frac{2\alpha_{14}}{\log 3} \delta + \frac{\alpha_{15}}{\log 3} \eta, \]

\[ B_{12} = \alpha_{13} + \frac{\alpha_{14}}{\log 2} (1 - \delta) + \frac{\alpha_{15}}{2 \log 2} (1 - \eta), \]

and

\[ B_{13} = \max(B_{11}, B_{12}). \]

The inequality (6.19) is replaced by

(6.20) \[ 0 \leq B_{13} \log d_L \tau^{n_L} + \frac{b_0}{\sigma - 1} - \frac{b_1}{\sigma - \beta}. \]

From (6.20) it follows that

\[ 1 - \beta \geq \left( \frac{b_1}{b_0 b + B_{13}} - \frac{1}{b} \right) (\log d_L \tau^{n_L})^{-1}. \]
We choose $Q(\phi)$ with $b_0 < b_1$, $b$, $\delta$, and $\eta$ as follows:

$$Q(\phi) = 4(1 + \cos \phi)(0.51 + \cos \phi)^2, \quad b = 8.7, \; \delta = 0.66, \; \text{and} \; \eta = 0.26,$$

and obtain (6.2).

7. The Deuring–Heilbronn phenomenon

The Deuring–Heilbronn phenomenon means that if the exceptional zero of $\zeta_L(s)$ exists then the other zeros of $\zeta_L(s)$ can not be very close to $s = 1$. In [23] Lagarias, Montgomery, and Odlyzko proved more precisely the following.

**Theorem C** (Lagarias, Montgomery, Odlyzko [23]). — There are positive, absolute, effectively computable constants $c_7$ and $c_8$ such that if $\zeta_L(s)$ has a real zero $\omega_0 > 0$ then $\zeta_L(\sigma + it) \neq 0$ for

$$\sigma \geq 1 - c_8 \frac{\log \left( \frac{\sqrt[6]{c_7}}{(1-\omega_0) \log \{d_L(|t| + 2)^{nL} \}} \right)}{\log \{d_L(|t| + 2)^{nL} \}}$$

with the single exception $\sigma + it = \omega_0$.

See also [30]. In this section we will estimate the values of $c_7$ and $c_8$ explicitly. We will use a power sum inequality as [23]. We begin by recalling the fact that $(s - 1)\zeta_L(s)$ is an entire function of order one. The Hadamard product theorem says that

$$(s - 1)\zeta_L(s) = s^{r_1 + r_2 - 1}e^{a + bs} \prod_{\omega} \left(1 - \frac{s}{\omega} \right) e^{s/\omega}$$

for some constants $a$ and $b$, where $\omega$ runs through all the zeros of $\zeta_L(s)$, $\omega \neq 0$, including the trivial ones, counted with multiplicity. ([48]) The Euler product for $\zeta_L(s)$ gives

$$-\frac{\zeta'}{\zeta_L}(s) = \sum_{\mathfrak{P}} \sum_{m=1}^{\infty} (\log N\mathfrak{P}) (N\mathfrak{P})^{-ms}$$

for $\Re s > 1$, where $\mathfrak{P}$ runs over all prime ideals of $L$. This series is absolutely convergent for $\Re s > 1$.

Suppose that $\zeta_L(s)$ has a real zero $\omega_0 > 0$. Differenciating $(2j - 1)$ times the equality

$$\sum_{\mathfrak{P}} \sum_{m=1}^{\infty} (\log N\mathfrak{P}) (N\mathfrak{P})^{-ms} = \frac{1}{s - 1} - b - \sum_{\omega} \left( \frac{1}{s - \omega} + \frac{1}{\omega} \right) - \frac{r_1 + r_2 - 1}{s}$$
yields that for $\Re s > 1$ and $j \geq 1$

\[
\frac{1}{(2j-1)!} \sum_{\mathfrak{P}} \sum_{m=1}^{\infty} (\log N\mathfrak{P})(\log N\mathfrak{P}^m)^{2j-1}(N\mathfrak{P})^{-ms} = \frac{1}{(s-1)^{2j}} - \frac{1}{(s-\omega_0)^{2j}} - \sum_{\omega \in \mathbb{Z}(\zeta_L) \setminus \omega_0} \frac{1}{(s-\omega)^{2j}} - \sum_{\tilde{m}=0}^{\infty} \frac{\ell_{\tilde{m}}}{(s+\tilde{m})^{2j}},
\]

where

\[
\ell_{\tilde{m}} = \begin{cases} 
  r_1 + r_2 - 1 & \text{if } \tilde{m} = 0, \\
  r_1 + r_2 & \text{if } \tilde{m} \neq 0 \text{ is even,} \\
  r_2 & \text{if } \tilde{m} \text{ is odd.}
\end{cases}
\]

If $s_0 = \sigma_0 + it_0$ with $\sigma_0 > 1$, then

\[
(7.1) \quad \frac{1}{(2j-1)!} \sum_{\mathfrak{P}} \sum_{m=1}^{\infty} (\log N\mathfrak{P})(\log N\mathfrak{P}^m)^{2j-1}(N\mathfrak{P})^{-\sigma_0} \{1+(N\mathfrak{P}^m)^{-it_0}\}
\]

\[
+ \sum_{\tilde{m}=2}^{\infty} \left\{ \frac{\ell_{\tilde{m}}}{(\sigma_0+\tilde{m})^{2j}} + \frac{\ell_{\tilde{m}}}{(s_0+\tilde{m})^{2j}} \right\}
\]

\[
= \frac{1}{(\sigma_0 - 1)^{2j}} - \frac{1}{(\sigma_0 - \omega_0)^{2j}} + \frac{1}{(s_0 - 1)^{2j}} - \frac{1}{(s_0 - \omega_0)^{2j}} - \sum_{n=1}^{\infty} z_n^j,
\]

where $z_n$ is the series of the terms $(\sigma_0 - \omega)^{-2}$ and $(s_0 - \omega)^{-2}$ for all $\omega \in \{0, -1\} \cup (\mathbb{Z}(\zeta_L) \setminus \{\omega_0\})$ such that $\omega$ is counted according to its multiplicity and $|z_n|$ is decreasing for $n \geq 1$. Since the real part of the left side of (7.1) is nonnegative,

\[
(7.2) \quad \Re \sum_{n=1}^{\infty} z_n^j \leq \frac{1}{(\sigma_0 - 1)^{2j}} - \frac{1}{(\sigma_0 - \omega_0)^{2j}}
\]

\[
+ \Re \left[ \frac{1}{((\sigma_0 - 1) + it_0)^{2j}} - \frac{1}{((\sigma_0 - \omega_0) + it_0)^{2j}} \right].
\]

To evaluate the constants $c_7$ and $c_8$, first, we estimate the right side of (7.2) from above.

**Lemma 7.1.** — For $\sigma_0 > 1$, $j \geq 1$, and $0 < v \leq 1$ we let

\[
f_5(\sigma_0 + it_0, j; v) = \Re \left[ \frac{1}{((\sigma_0 - 1) + it_0)^{2j}} - \frac{1}{((\sigma_0 - v) + it_0)^{2j}} \right].
\]

Then

\[
f_5(\sigma_0, j; \omega_0) + f_5(\sigma_0 + it_0, j; \omega_0) \leq \frac{4j(1 - \omega_0)}{(\sigma_0 - 1)^{2j+1}}.
\]
Proof. — We have
\[ f_5(\sigma_0 + it_0; j; v) = 2j \int_{\sigma_0 - 1}^{\sigma_0 - v} \Re \left\{ \frac{1}{(y + it_0)^{2j+1}} \right\} dy \leq 2j \frac{1 - v}{(\sigma_0 - 1)^{2j+1}}. \]
(See [60, (2.43)].) The result follows.

Second, we estimate \( \Re \sum_{n=1}^{\infty} z_n^j \) from below using [23, Theorem 4.2]. (See also [63, Theorem 2.3]). Set
\[ \mathcal{L} = \mathcal{L}(s_0) = |z_1|^{-1} \sum_{n=1}^{\infty} |z_n|. \]
According to [23, Theorem 4.2] (see also [63, Theorem 2.3]) for any \( \tilde{c} > 12 \), there exists \( j_0 \) with \( 1 \leq j_0 \leq \tilde{c} \mathcal{L} \) such that
\[ (7.3) \quad \Re \sum_{n=1}^{\infty} z_n^j \geq \left( \frac{\tilde{c} - 12}{4 \tilde{c}} \right) |z_1|^j. \]

Now we estimate \( \sum_{n=1}^{\infty} |z_n| \) from above.

Lemma 7.2. — Let \( s_0 = \sigma_0 + it_0 \), \( z_n \) and \( \omega_0 \) be as above. Then we have
\[ (7.4) \quad \sum_{n=1}^{\infty} |z_n| \leq B_{17}(\sigma_0) \log d_L + B_{18}(\sigma_0) \log \{(|t_0| + 2)^n \}
+ B_{19}(\sigma_0)n_L + B_{20}(\sigma_0), \]
where \( B_{17}(\sigma_0) = 2a_1(\sigma_0) \), \( B_{18}(\sigma_0) = a_2(\sigma_0) \), \( B_{19}(\sigma_0) = a_2(\sigma_0) \log 2 + 2a_3(\sigma_0) + \frac{2}{\sigma_0} \), and \( B_{20}(\sigma_0) = 2a_4(\sigma_0) - \frac{2}{\sigma_0} \) with
\[ a_1(\sigma_0) = \frac{1}{2(\sigma_0 - 1)}, \quad a_2(\sigma_0) = \frac{f_2(\sigma_0)}{\sigma_0 - 1}, \quad a_3(\sigma_0) = -\frac{\log \pi}{2(\sigma_0 - 1)}, \]
and
\[ a_4(\sigma_0) = \frac{1}{\sigma_0 - 1} \left( \frac{1}{\sigma_0} + \frac{1}{\sigma_0 - 1} \right). \]
(Here, \( f_2(\sigma_0) \) is as in Section 5.)

Proof. — Note that
\[ \sum_{n=1}^{\infty} |z_n| = \sum_{\omega \in \mathbb{Z}(\zeta_L)} \frac{1}{|\sigma_0 - \omega|^2} + \sum_{\omega \in \mathbb{Z}(\zeta_L) \setminus \omega_0} \frac{1}{|s_0 - \omega|^2} \]
\[ + \frac{\ell_0}{|\sigma_0|^2} + \frac{\ell_0}{|s_0|^2} + \frac{\ell_1}{|\sigma_0 + 1|^2} + \frac{\ell_1}{|s_0 + 1|^2}. \]
As
\[ \Re \frac{s - 1}{|s - \omega|^2} \leq \Re \frac{1}{s - \omega}, \]

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for $s \in \mathbb{C}$ and $\omega \in Z(\zeta_L)$ we have
\[
\sum_{\omega \in Z(\zeta_L)} \frac{\Re s - 1}{|s - \omega|^2} \leq \sum_{\omega \in Z(\zeta_L)} \frac{1}{|s - \omega|} = \frac{1}{2} \log d_L + \Re \left( \frac{1}{s} + \frac{1}{s - 1} \right) + \Re \frac{\gamma'}{\gamma_L}(s) + \Re \frac{\zeta'}{\zeta_L}(s).
\]
(7.5)

Gathering together the bound in Lemma 5.4, the fact that
\[
\Re \left\{ \frac{\zeta'}{\zeta_L}(\sigma_0) + \frac{\zeta'}{\zeta_L}(\sigma_0 + it_0) \right\} \leq 0,
\]
and (7.5) we get
\[
\sum_{\omega \in Z(\zeta_L)} \frac{1}{|\sigma_0 - \omega|^2} + \sum_{\omega \in Z(\zeta_L)} \frac{1}{|s_0 - \omega|^2}
\]
\[
\leq 2a_1(\sigma_0) \log d_L + a_2(\sigma_0) \log \{(|t_0| + 2)^n \}
+ \{a_2(\sigma_0) \log 2 + 2a_3(\sigma_0)\} n_L + 2a_4(\sigma_0).
\]

Moreover,
\[
\frac{\ell_0}{|\sigma_0|^2} + \frac{\ell_0}{|s_0|^2} + \frac{\ell_1}{|\sigma_0 + 1|^2} + \frac{\ell_1}{|s_0 + 1|^2} \leq \frac{2(r_1 + r_2 - 1)}{\sigma_0^2} + \frac{2r_2}{(\sigma_0 + 1)^2} \leq \frac{2}{\sigma_0^2} n_L - \frac{2}{\sigma_0^2}.
\]

The result follows. □

We are now ready to prove the following.

THEOREM 7.3. — Suppose that $L \neq \mathbb{Q}$ and $\zeta_L(s)$ has a real zero $\omega_0 > 0$. Let $\rho = \beta + i\gamma$ be a zero of $\zeta_L(s)$ with $\rho \neq \omega_0$.

(1) If $L$ is not an imaginary quadratic number field, then
\[
1 - \beta \geq c_8 \frac{\log \left\{ \frac{c_7}{(1 - \omega_0) \log d_L \tau^n L} \right\}}{\log d_L \tau^n L},
\]
where $\tau = |\gamma| + 2$, $c_7 = 6.7934 \cdots \times 10^{-4}$, and $c_8 = 16c_7 = \frac{1}{52}$. When $L$ is an imaginary quadratic number field, then (7.6) holds with $c_7 = 5.5803 \cdots \times 10^{-4}$ and $c_8 = 16c_7 = \frac{1}{112}$.

(2) If $\rho$ is a nontrivial zero of $\zeta_L(s)$, then (7.6) holds with $c_7 = 8.1168 \cdots \times 10^{-4}$ and $c_8 = 16c_7 = \frac{1}{77}$. 

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Proof.

(1). — If $L$ is not an imaginary quadratic number field, then $\zeta_L(s)$ has a zero at $s = 0$ and $|z_1|^{-1} \leq \sigma_0^2$. Setting $t_0 = \gamma$ in (7.4) yields
\[
\mathcal{L} \leq \sigma_0^2 \left\{ B_{17}(\sigma_0) \log d_L + B_{18}(\sigma_0) \log \tau^{n_L} + B_{19}(\sigma_0)n_L + B_{20}(\sigma_0) \right\}.
\]
Note that $B_{19}(\sigma_0) \geq 0$ for $\sigma_0 \geq 1.74$. For $\sigma_0 \geq 1.74$ and $0 \leq \delta, \eta \leq 1$, we let
\[
B_{22}(\sigma_0, \delta, \eta) = B_{17}(\sigma_0) + \frac{2B_{19}(\sigma_0)}{\log 3} \delta + \frac{B_{20}(\sigma_0)}{\log 3} \eta,
\]
\[
B_{23}(\sigma_0, \delta, \eta) = B_{18}(\sigma_0) + \frac{B_{19}(\sigma_0)}{\log 2} (1 - \delta) + \frac{B_{20}(\sigma_0)}{2 \log 2} (1 - \eta),
\]
and
\[
B_{24}(\sigma_0, \delta, \eta) = \max \{ B_{22}(\sigma_0, \delta, \eta), B_{23}(\sigma_0, \delta, \eta) \}.
\]
Then we have
\[
\mathcal{L} \leq \sigma_0^2 B_{24}(\sigma_0, \delta, \eta) \log d_L \tau^{n_L}
\]
since $d_L \geq 3^{n_L/2}$ and $n_L \geq 2$. Note that if $\rho \in Z(\zeta_L)$, then $|z_1| \geq |\sigma_0 + i\gamma - \rho|^{-2} = |\sigma_0 - \beta|^{-2}$ and if $\rho \not\in Z(\zeta_L)$, then $\rho = \beta \leq 0$ and $|z_1| \geq |\sigma_0|^{-2} \geq |\sigma_0 - \beta|^{-2}$. Thus
\[
|z_1| \geq \frac{1}{(\sigma_0 - 1)^2} \exp \left\{ -2 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\}
\]
and the bound (7.3) yields
\[
\Re \sum_{n=1}^{\infty} z_n^{j_0} \geq \left( \frac{\tilde{c} - 12}{4 \tilde{c}} \right) \frac{1}{(\sigma_0 - 1)^{2j_0}} \exp \left\{ -2j_0 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\}.
\]
Combining this with (7.2) and the bound in Lemma 7.1 we have
\[
(7.7) \quad \left( \frac{\tilde{c} - 12}{4 \tilde{c}} \right) \frac{1}{(\sigma_0 - 1)^{2j_0}} \exp \left\{ -2j_0 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\} \leq \frac{4j_0(1 - \omega_0)}{(\sigma_0 - 1)^{2j_0 + 1}}.
\]
From $j_0 \leq \mathcal{L} \leq \tilde{c}\sigma_0^2 B_{24}(\sigma_0, \delta, \eta) \log d_L \tau^{n_L}$ it follows that
\[
(7.8) \quad 1 - \beta \geq c_8(\tilde{c}, \sigma_0, \delta, \eta) \frac{\log \left\{ \frac{c_7(\tilde{c}, \sigma_0, \delta, \eta)}{(1 - \omega_0) \log d_L \tau^{n_L}} \right\}}{\log d_L \tau^{n_L}},
\]
where $c_7(\tilde{c}, \sigma_0, \delta, \eta) = (\tilde{c} - 12) c_8(\tilde{c}, \sigma_0, \delta, \eta) \frac{\sigma_0 - 1}{2 \tilde{c}\sigma_0^2 B_{24}(\sigma_0, \delta, \eta)}$. Choosing $\tilde{c} = 24$, $\sigma_0 = 7.79$, $\delta = 1$, and $\eta = 1$ we get (7.6). If $L$ is an imaginary quadratic number field, then $\zeta_L(s)$ has a zero at $s = -1$ and $|z_1|^{-1} \leq (\sigma_0 + 1)^2$. We have then
\[
\mathcal{L} \leq (\sigma_0 + 1)^2 B_{24}(\sigma_0, \delta, \eta) \log d_L \tau^{n_L}.
\]
and \( j_0 \leq \hat{c} \mathcal{L} \leq \hat{c}(\sigma_0 + 1)^2 B_{24}(\sigma_0, \delta, \eta) \log d_L r^{n_L} \). Moreover,

\[
|z_1| \geq |\sigma_0 - \beta|^{-2} \geq \frac{1}{(\sigma_0 - 1)^2} \exp \left\{ -2 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\}
\]

since \( \zeta_L(s) \) does not have a zero at \( s = 0 \). From (7.7) we get

\[
c_8(\hat{c}, \sigma_0, \delta, \eta) = \frac{\sigma_0 - 1}{2\hat{c}(\sigma_0 + 1)^2 B_{24}(\sigma_0, \delta, \eta)}.
\]

Choosing \( \hat{c} = 24 \), \( \sigma_0 = 12.21 \), \( \delta = 1 \), and \( \eta = 1 \) we get the result.

(2). — We consider \( \sum_{n=1}^{\infty} \tilde{z}_n^j \) (instead of \( \sum_{n=1}^{\infty} z_n^j \) in (7.2)), where \( \tilde{z}_n \) is the series of terms \((\sigma_0 - \omega)^{-2}\) and \((\sigma_0 + it_0 - \omega)^{-2}\) for all \( \omega \in Z(\zeta_L) \setminus \{\omega_0\} \) such that \( \omega \) is counted according to its multiplicity and \( |\tilde{z}_n| \) is decreasing for \( n \geq 1 \).

Since

\[
\Re \sum_{n=1}^{\infty} \tilde{z}_n^j + \Re \left\{ \frac{\ell_0}{\sigma_0^{2j}} + \frac{\ell_0}{(\sigma_0 + it_0)^{2j}} + \frac{\ell_1}{(\sigma_0 + 1)^{2j}} + \frac{\ell_1}{(\sigma_0 + it_0 + 1)^{2j}} \right\}
\]

\[
= \Re \sum_{n=1}^{\infty} z_n^j
\]

and

\[
\Re \left\{ \frac{1}{(\sigma_0 - \omega)^{2j}} + \frac{1}{(\sigma_0 + it_0 - \omega)^{2j}} \right\} \geq 0 \quad \text{for} \quad \omega = 0, -1,
\]

(7.9)

\[
\Re \sum_{n=1}^{\infty} \tilde{z}_n^j \leq \frac{1}{(\sigma_0 - 1)^{2j}} - \frac{1}{(\sigma_0 - \omega_0)^{2j}}
\]

\[
+ \Re \left[ \frac{1}{(\sigma_0 - 1 + it_0)^{2j}} - \frac{1}{(\sigma_0 - \omega_0)^{2j}} \right].
\]

We use the power-sum inequality in [23, Theorem 4.2] for \( \sum_{n=1}^{\infty} \tilde{z}_n^j \). Set \( \hat{\mathcal{L}} = |z_1|^{-1} \sum_{n=1}^{\infty} |\tilde{z}_n| \). For any \( \hat{c} > 12 \), there exists \( \hat{\ell}_0 \) with \( 1 \leq \hat{\ell}_0 \leq \hat{c} \hat{\mathcal{L}} \) such that

(7.10)

\[
\Re \sum_{n=1}^{\infty} \tilde{z}_n^j \geq \left( \frac{\hat{c} - 12}{4\hat{c}} \right) |z_1|^{\hat{\ell}_0}.
\]

If \( \rho \in Z(\zeta_L) \), then \( 1 - \rho \in Z(\zeta_L) \). Set \( t_0 = \gamma \). Then

\[
|z_1|^{-1} \leq \min\{(\sigma_0 - \beta)^2, (\sigma_0 - 1 + \beta)^2\} \leq \left( \sigma_0 - \frac{1}{2} \right)^2.
\]

Then we have

\[
\hat{\mathcal{L}} \leq \left( \sigma_0 - \frac{1}{2} \right)^2 \{ B_{17}(\sigma_0) \log d_L + B_{18}(\sigma_0) \log r^{n_L} + \hat{B}_{19}(\sigma_0)n_L + \hat{B}_{20}(\sigma_0) \},
\]

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where \( \hat{B}_{19}(\sigma_0) = a_2(\sigma_0) \log 2 + 2a_3(\sigma_0) \) and \( \hat{B}_{20}(\sigma_0) = 2a_4(\sigma_0) \). Note that

\[
\hat{B}_{19}(\sigma_0) \leq 0 \quad \text{and} \quad 2\hat{B}_{19}(\sigma_0) + \hat{B}_{20}(\sigma_0) \geq 0 \quad \text{for} \quad 1 < \sigma_0 \leq 11.66.
\]

So, for \( 1 < \sigma_0 \leq 11.66 \)

\[
\hat{\mathcal{L}} \leq \left( \sigma_0 - \frac{1}{2} \right)^2 \left\{ B_{17}(\sigma_0) \log d_L + B_{18}(\sigma_0) \log \tau^{n_L} + 2\hat{B}_{19}(\sigma_0) + \hat{B}_{20}(\sigma_0) \right\}.
\]

For \( 1 < \sigma_0 \leq 11.66 \) and \( 0 \leq \eta \leq 1 \), we let

\[
B_{25}(\sigma_0, \eta) = B_{17}(\sigma_0) + \frac{2\hat{B}_{19}(\sigma_0) + \hat{B}_{20}(\sigma_0)}{\log 3} \eta,
\]

\[
B_{26}(\sigma_0, \eta) = B_{18}(\sigma_0) + \frac{2\hat{B}_{19}(\sigma_0) + \hat{B}_{20}(\sigma_0)}{2 \log 2} (1 - \eta),
\]

and

\[
B_{27}(\sigma_0, \eta) = \max\{B_{25}(\sigma_0, \eta), B_{26}(\sigma_0, \eta)\}.
\]

Then we have

\[
\hat{\mathcal{L}} \leq \left( \sigma_0 - \frac{1}{2} \right)^2 B_{27}(\sigma_0, \eta) \log d_L \tau^{n_L}.
\]

Note that \( d_L \geq 3^{n_L/2} \).

Since

\[
|z_1| \geq |\sigma_0 + i\gamma - \rho|^{-2} \geq \frac{1}{(\sigma_0 - 1)^2} \exp \left\{ -2 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\},
\]

the bound (7.10) yields

\[
\Re \sum_{n=1}^{\infty} \hat{z}_n \geq \left( \frac{\hat{c} - 12}{4\hat{c}} \right) \frac{1}{(\sigma_0 - 1)^2} \exp \left\{ -2\hat{j}_0 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\}.
\]

Combining this with (7.9) and the bound in Lemma 7.1 we have

\[
\left( \frac{\hat{c} - 12}{4\hat{c}} \right) \frac{1}{(\sigma_0 - 1)^2} \exp \left\{ -2\hat{j}_0 \left( \frac{1 - \beta}{\sigma_0 - 1} \right) \right\} \leq \frac{4\hat{j}_0 (1 - \omega_0)}{(\sigma_0 - 1)^2}.
\]

From \( \hat{j}_0 \leq \hat{c} \mathcal{L} \leq \hat{c} \left( \sigma_0 - \frac{1}{2} \right)^2 B_{27}(\sigma_0, \eta) \log d_L \tau^{n_L} \) it follows that

\[
c_8(\hat{c}, \sigma_0, \eta) = \frac{\sigma_0 - 1}{2\hat{c} \left( \sigma_0 - \frac{1}{2} \right)^2 B_{27}(\sigma_0, \eta)}.
\]

Choosing \( \hat{c} = 24 \), \( \sigma_0 = 5.42 \), and \( \eta = 1 \) we get the result. \( \square \)
Remark. — To get an upper bound for $L$ the zero-density estimate for the number of zeros of $\zeta_L(s)$ was used in [23]:

$$L \ll (2 - \beta)^2 \sum_{\omega} \left( \frac{1}{|2 - \omega|^2} + \frac{1}{|2 + i\gamma - \omega|} \right) \ll \int_0^\infty \frac{1}{u^2 + 1} \, dn(u) + \int_0^\infty \frac{1}{u^2 + 1} \, dn(u + \tau) \ll \log d_L \tau^{n_L},$$

where $\omega$ runs through all the zeros of $\zeta_L(s)$ including the trivial ones. (See [23, (5.6)].) However we used

$$\sum_{\rho \in \mathbb{Z}(\zeta_L)} \sigma - \frac{1}{2} \leq \sum_{\rho \in \mathbb{Z}(\zeta_L)} \Re \frac{1}{s - \rho}$$

for $\Re s = \sigma > 1$ and (5.1). (See (7.5) above.)

**Corollary 7.4.** — Assume that $L \neq \mathbb{Q}$. Then for any real zero $\omega_0 > 0$ of $\zeta_L(s)$ we have

$$(7.11) \quad 1 - \omega_0 \geq d_L^{-c_{10}}$$

with $c_{10} = 114.72 \cdots$.

**Proof.** — When $L$ is not an imaginary quadratic number fields, we let $\bar{c} = 12.1$, $\sigma_0 = 7.79$, $\delta = 1$, and $\eta = 1$. The inequality (7.8) yields

$$(7.12) \quad 1 - \beta \geq c_8 \frac{\log c_7 + \log(1 - \omega_0)^{-1} - \log \log d_L \tau^{n_L}}{\log d_L \tau^{n_L}}$$

for any zero $\beta + i\gamma \neq \omega_0$ of $\zeta_L(s)$, where $c_7 = 2.2434 \cdots \times 10^{-5}$ and $c_8 = 2.1716 \cdots \times 10^{-2}$. Set $1 - \omega_0 = d_L^{-c}$. Since $\zeta_L(s)$ always has a trivial zero at $s = 0$ and $d_L \geq 3^n L/2$, we have

$$(7.13) \quad 1 \geq c_8 \left\{ \frac{\log c_7 + c \log d_L}{(1 + 2 \log 2 \log 3)} \log d_L - \frac{\log \log d_L 2^{n_L}}{\log d_L 2^{n_L}} \right\} \geq c_8 \left\{ \left( 1 + \frac{2 \log 2}{\log 3} \right)^{-1} \left( \frac{\log c_7}{\log d_L} + c \right) - \frac{1}{c} \right\}.$$

Note that $\frac{\log x}{x} \leq \frac{1}{e}$ for $x > 0$. Then (7.13) yields

$$c \leq \left( \frac{1}{c_8} + \frac{1}{e} \right) \left( 1 + \frac{2 \log 2}{\log 3} \right) - \frac{\log c_7}{\log 3} = 114.72 \cdots.$$

When $L$ is an imaginary quadratic number field, it is known that $\zeta_L(\sigma) \neq 0$ for $\sigma \geq 1 - \left( \frac{\pi}{d_L} \right)^{-1}$. (See [48, proof of Lemma 11].) The result follows. \[\square\]
Remarks.
(1) For the zero-free regions for $\zeta_L(s)$ see also [48].
(2) In [63], Zaman proved that, for $d_L$ sufficiently large, $1-\omega_0 \gg d_L^{-21.3}$.

8. Proof of Theorem 1.1

Theorem 1.1 is ready to be proven. We will choose appropriate kernel functions $k(s)$ and estimate

$$ k(1) - \sum_{\rho \in \mathcal{Z}(\zeta_L)} |k(\rho)| $$

from below. From now on we denote by $\beta_0$ the exceptional zero of $\zeta_L(s)$ if it exists, and $\beta_0 = 1 - (2 \log d_L)^{-1}$ otherwise. Our proof is divided into a sequence of lemmas.

**Lemma 8.1.** We have

$$ k_1(1) - k_1(\beta_0) \geq \frac{9}{10} (\log x)^2 \min\{1, (1 - \beta_0) \log x\} $$

and

$$ k_2(1) - k_2(\beta_0) \geq \frac{9}{10} x^2 \min\{1, (1 - \beta_0) \log x\}. $$

**Proof.** We have

$$ k_1(1) - k_1(\beta_0) = (\log x)^2 - \left( \frac{x^2(\beta_0 - 1) - x^2(\beta_0 - 1)}{1 - \beta_0} \right)^2 $$

$$ = (\log x)^2 \varphi_6((1 - \beta_0) \log x), $$

where

$$ \varphi_6(v) = 1 - \left( \frac{e^{-v} - e^{-2v}}{v} \right)^2. $$

It is easily verified that

$$ \varphi_6(v) \geq \begin{cases} \varphi_6(1)v & \text{for } 0 < v \leq 1, \\ \varphi_6(1) & \text{for } v \geq 1 \end{cases} $$

with $\varphi_6(1) = 0.94592 \ldots$. Hence $\varphi_6(v) \geq \varphi_6(1) \min\{1, v\}$, which yields (8.1). We have

$$ k_2(1) - k_2(\beta_0) = x^2(1 - x^2(\beta_0 - 1)(\beta_0 + 2)) \geq x^2 \varphi_7((1 - \beta_0) \log x), $$

where $\varphi_7(v) = 1 - e^{-\frac{2}{3}v}$. It is easy to see that

$$ \varphi_7(v) \geq \begin{cases} \varphi_7(1)v & \text{for } 0 < v \leq 1, \\ \varphi_7(1) & \text{for } v \geq 1 \end{cases} $$
with $\varphi_7(1) = 0.91791 \cdots$. Hence $\varphi_7(v) \geq \varphi_7(1) \min\{1, v\}$, which yields (8.2).

In the following $c_7$ and $c_8$ are as in Theorem 7.3(2).

**Lemma 8.2.** — Suppose that $\beta_0 \leq 1 - c_7^2 (\log d_L 3^n L)^{-2}$. We use the kernel function $k_1(s)$ and obtain

$$\sum_{\rho \in Z(\zeta L)} |k_1(\rho)| \leq c_{13} \log d_L + c_{14} (\log d_L)^2 \{(1 - \beta_0) \log d_L\}^{2 c_{12} \log x},$$

where $c_{12} = 6.8610 \cdots \times 10^{-4}$, $c_{13} = 124.14 \cdots$, and $c_{14} = 1.7700 \cdots \times 10^8$.

**Proof.** — Write

$$\sum_{\rho \not= \beta_0 \rho \in Z(\zeta L)} |k_1(\rho)| = \sum_{|\rho - 1| > 1} |k_1(\rho)| + \sum_{|\rho - 1| \leq 1} |k_1(\rho)|,$$

where $\sum_{|\rho - 1| > 1}$ (resp. $\sum_{|\rho - 1| \leq 1}$) denotes that we sum over $\rho = \beta + i \gamma$ such that $\rho \in Z(\zeta L)$ with $\rho \not= \beta_0$ and $|\rho - 1| > 1$ (resp. $|\rho - 1| \leq 1$). Since

$$|k_1(\rho)| = \left| \frac{x^{2(\rho - 1)} - x^\rho - 1}{\rho - 1} \right|^2 \leq \frac{4x^{-2(1 - \beta)}}{|\rho - 1|^2},$$

it follows that

$$\sum_{|\rho - 1| > 1} |k_1(\rho)| \leq 4 \int_1^\infty \frac{1}{r^2} \text{dn}(r; 1)$$

$$\leq 21.76 \int_1^\infty \frac{(1 + r)\{\log d_L + n_L \log(r + 2)\}}{r^3} \, dr$$

(by (5.5) and Proposition 5.6(1))

$$\leq c_{13} \log d_L$$

where $c_{13} = 21.76 \left(\frac{3}{2} + \frac{2 + 15 \log 3}{4 \log 3}\right) = 124.14 \cdots$. For the sum $\sum_{|\rho - 1| \leq 1} |k_1(\rho)|$ we consider two cases separately.

(i) If an exceptional zero $\beta_0$ exists with $1 - \beta_0 \leq \left(\frac{c_7}{3}\right)^2 (\log d_L)^{-1}$, then

$$\frac{c_7}{(1 - \beta_0) \log d_L \tau^n L} \geq \frac{c_7}{3(1 - \beta_0) \log d_L} \geq \left\{(1 - \beta_0) \log d_L\right\}^{-\frac{1}{2}}.$$

Hence, by Theorem 7.3(2)

$$1 - \beta \geq c_8 \frac{\log \{(1 - \beta_0) \log d_L\}^{-\frac{1}{2}}}{\log d_L \tau^n L} \geq c_{11} \frac{\log \{(1 - \beta_0) \log d_L\}^{-1}}{\log d_L}$$

with $c_{11} = \frac{c_8}{6} = \frac{1}{462}$.
(ii) If \( 1 - \beta_0 > \left( \frac{c_7}{3} \right)^2 (\log d_L)^{-1} \), then by (6.2)
\[
1 - \beta \geq (29.57 \log d_L \tau^{n_L})^{-1} \geq (88.71 \log d_L)^{-1}.
\]
Set \( c_{12} = \left\{ 177.42 \log \left( \frac{3}{c_7} \right) \right\}^{-1} = 6.8610 \cdots \times 10^{-4} \). Then
\[
(88.71)^{-1} = 2c_{12} \log \left( \frac{3}{c_7} \right) > c_{12} \log \{(1 - \beta_0) \log d_L\}^{-1}
\]
and
\[
1 - \beta > c_{12} \frac{\log \{(1 - \beta_0) \log d_L\}^{-1}}{\log d_L}.
\]
As \( c_{11} > c_{12} \) we have
\[
1 - \beta > c_{12} \frac{\log \{(1 - \beta_0) \log d_L\}^{-1}}{\log d_L}
\]
in all cases. Let
\[
B = c_{12} \frac{\log \{(1 - \beta_0) \log d_L\}^{-1}}{\log d_L}.
\]
Then
\[
|k_1(\rho)| \leq \frac{4x^{2(\beta-1)}}{|\rho - 1|^2} \leq \frac{4x^{-2B}}{|\rho - 1|^2}.
\]
By Proposition 5.6(2),
\[
\sum_{|\rho - 1| \leq 1} |k_1(\rho)| \leq 4x^{-2B} \int_B^{1} \frac{1}{r^2} \, dn(r; 1)
\]
\[
\leq 4x^{-2B} \left\{ n(1; 1) + 20 \int_B^{1} \frac{1 + \frac{2f_2(2)}{5} \left( 1 + \frac{2\log 2}{\log 3} \right) r \log d_L}{r^3} \, dr \right\}
\]
(by Proposition 5.6(2))
\[
\leq 40x^{-2B} \left\{ B^{-2} + \frac{4f_2(2)}{5} \left( 1 + \frac{2\log 2}{\log 3} \right) B^{-1} \log d_L
\]
\[
- \frac{2f_2(2)}{5} \left( 1 + \frac{2\log 2}{\log 3} \right) \log d_L \right\}
\]
\[
\leq c_{14} (\log d_L)^2 \{(1 - \beta_0) \log d_L\}^{2c_{12} \log \log d_L}
\]
where
\[
c_{14} = \frac{40}{c_{12} \log 2} \left\{ \frac{1}{c_{12} \log 2} + \frac{4f_2(2)}{5} \left( 1 + \frac{2\log 2}{\log 3} \right) \right\} = 1.7700 \cdots \times 10^8.
\]
For the last inequality we used (6.1), which yields
\[
B = c_{12} \frac{\log \{(1 - \beta_0) \log d_L\}^{-1}}{\log d_L} \geq \frac{c_{12} \log 2}{\log d_L}.
\]
\[\square\]
We have therefore
\[(8.3) \quad k_1(1) - \sum_{\rho \in \mathbb{Z}(\zeta_L)} |k_1(\rho)| \quad \geq \quad \frac{9}{10} (\log x)^2 \min\{1, (1 - \beta_0) \log x\} \quad - \quad c_{13} \log d_L - c_{14}(\log d_L)^2 \{(1 - \beta_0) \log d_L\}^{2c_{12} \log x \log d_L}.\]

Note that for \(x \geq 101\)
\[(8.4) \quad \mu_1 k_1 \left( -\frac{1}{2} \right) \log d_L + n_L \left\{ k_1(0) + \nu_1 k_1 \left( -\frac{1}{2} \right) \right\} \quad \leq \quad \left\{ \frac{2}{\log 3} (x^{-2} - x^{-1})^2 + \frac{4}{9} \left( \mu_1 + \frac{2}{\log 3} \nu_1 \right) \left( x^{-3} - x^{-3/2} \right)^2 \right\} \log d_L \quad \leq \quad \left\{ \frac{2}{\log 3} x^{-2} + \frac{4}{9} \left( \mu_1 + \frac{2}{\log 3} \nu_1 \right) x^{-3} \right\} \log d_L \quad \leq \quad c_{15} x^{-2} \log d_L,\]

where
\[c_{15} = \frac{2}{\log 3} + \frac{4}{909} \left( \mu_1 + \frac{2}{\log 3} \nu_1 \right) = 1.9792 \ldots.\]

Gathering together the bounds (3.1), (4.3), (8.3), and (8.4) we conclude the following:

**Lemma 8.3.** — Suppose that \(\beta_0 \leq 1 - c_7^2(\log d_L 3^{n_L})^{-2}\). We have then
\[(8.5) \quad \frac{|G|}{|C|} \sum_{p \in P(C)} (\log N_{K/Q} p) \hat{k}_1(N_{K/Q} p) \quad \geq \quad \frac{9}{10} (\log x)^2 \min\{1, (1 - \beta_0) \log x\} \quad - \quad c_{13} \log d_L \quad - \quad c_{14}(\log d_L)^2 \{(1 - \beta_0) \log d_L\}^{2c_{12} \log x \log d_L} \quad - \quad c_{15} x^{-2} \log d_L \quad - \quad \alpha_3 \frac{|G|}{|C|} \frac{\log x}{x} \log d_L.\]

**Lemma 8.4.** — Suppose that \(\beta_0 \leq 1 - c_7^2(\log d_L 3^{n_L})^{-2}\). For \(\log x = c_{16} \log d_L\) with \(c_{16} = 3144.25\), we have
\[\sum_{p \in P(C)} (\log N_{K/Q} p) \hat{k}_1(N_{K/Q} p) > 0.\]

In particular, there is a prime \(p \in P(C)\) with \(N_{K/Q} p \leq x^4 = d_L^{c_{16}}\).
Proof. — Let $\log x = c_{16} \log d_L$.

(i) Suppose that $1 \leq c_{16} (1 - \beta_0) \log d_L$. (8.5) and (6.1) yield

$$
(\log d_L)^{-2} \frac{|G|}{|C|} \sum_{p \in P(C)} (\log N_{K/Q} p) \hat{k}_1(N_{K/Q} p)
$$

\[ \geq \left\{ \frac{9}{10} c_{16}^2 - c_{14} \left( \frac{1}{2} \right)^{2c_{12}c_{16}} \right\} - \epsilon_1,
\]

where

$$
\epsilon_1 = \frac{c_{13}}{\log d_L} + \frac{c_{15}}{d_{L}^{2c_{16}} \log d_L} + \frac{2\alpha_3 c_{16} \log d_L}{d_{L}^{c_{16}} \log 3}.
$$

(Note that $\frac{|G|}{|C|} \leq |G| = \frac{n_L}{n_K} \leq n_L \leq \frac{2}{\log 3} \log d_L$.) For $c_{16} = 3144.25$, we have

$$
\frac{9}{10} c_{16}^2 > c_{14} \left( \frac{1}{2} \right)^{2c_{12}c_{16}} + \epsilon_1.
$$

(ii) Suppose that $1 \geq c_{16} (1 - \beta_0) \log d_L$. Since $1 - \beta_0 \geq c_7^2 (\log d_L 3^{n_L})^{-2} \geq (\frac{c_7}{c_7})^2 (\log d_L)^{-2}$, (8.5) and (6.1) yield

$$
\{(1 - \beta_0) \log d_L\}^{-1} (\log d_L)^{-2} \frac{|G|}{|C|} \sum_{p \in P(C)} (\log N_{K/Q} p) \hat{k}_1(N_{K/Q} p)
$$

\[ \geq \frac{9}{10} c_{16}^3 - c_{14} \left\{ (1 - \beta_0) \log d_L \right\}^{2c_{12}c_{16} - 1} - \frac{c_{13}}{(1 - \beta_0)(\log d_L)^2}
\]

$$
- \frac{c_{15}}{d_{L}^{2c_{16}} (1 - \beta_0)(\log d_L)^2} - \frac{2\alpha_3 c_{16}}{d_{L}^{c_{16}} (1 - \beta_0) \log 3}
$$

\[ \geq \frac{9}{10} c_{16}^3 - c_{14} \left( \frac{1}{2} \right)^{2c_{12}c_{16} - 1} - c_{13} \left( \frac{3}{c_7} \right)^2 - \epsilon_2,
\]

where

$$
\epsilon_2 = \left( \frac{3}{c_7} \right)^2 \left\{ \frac{c_{15}}{d_{L}^{2c_{16}}} + \frac{2\alpha_3 c_{16}}{\log 3} \frac{(\log d_L)^2}{d_{L}^{c_{16}}} \right\}.
$$

For $c_{16} = 1261$, we have

$$
\frac{9}{10} c_{16}^3 > c_{14} \left( \frac{1}{2} \right)^{2c_{12}c_{16} - 1} + c_{13} \left( \frac{3}{c_7} \right)^2 + \epsilon_2.
$$

The result follows. \qed
Lemma 8.5. — Suppose that
\[ 1 - \beta_0 \leq c_7^2 (\log d_L 3^{n_L})^{-2}. \]
We have then
\[ (8.6) \quad \frac{|G|}{|C|} \sum_{p \in P(C)} (\log N_{K/Q(p)} k_2(N_{K/Q(p)}) \]
\[ \geq \frac{9}{10} x^2 \min \{1, (1 - \beta_0) \log x\} - c_{20} x \log d_L - c_{21} x^2 (1 - \beta_0)^{2c_{19} \log x} \log d_L \]
\[ - c'_{15} \log d_L - \alpha_4 \frac{|G|}{|C|} x (\log x)^{\frac{1}{2}} \log d_L, \]
where \( c_{20} = 19.16 \cdots, c_{21} = 6.1522 \cdots, c_{19} = \frac{c_8}{6} = \frac{1}{462}, \) and \( c'_{15} = 1.8291 \cdots. \)

Proof. — For \( \rho = \beta + i \gamma \in Z(\zeta_L) \) with \( |\gamma| \leq 1 \) we have by Theorem 7.3(2)
\[ 1 - \beta \geq c_8 \frac{\log \left\{ (1 - \beta_0) \log d_L 3^{n_L} \right\}}{\log d_L 3^{n_L}} \geq c_{19} \frac{\log (1 - \beta_0)^{-1}}{\log d_L} \]
with \( c_{19} = \frac{c_8}{6} = \frac{1}{462}. \) Since
\[ |k_2(\rho)| \leq x^{\beta^2 + \beta} \leq x^{2 - 2(1 - \beta)} \leq x^2 (1 - \beta_0)^{2c_{19} \log x \log d_L}, \]
\[ \sum_{|\gamma| \leq 1} |k_2(\rho)| \leq x^2 (1 - \beta_0)^{2c_{19} \log x \log d_L} \sum_{|\gamma| \leq 1} 1 \]
\[ \leq c_{21} x^2 (1 - \beta_0)^{2c_{19} \log x \log d_L} \log d_L \quad \text{by (5.5)} \]
with \( c_{21} = 2.72 \left(1 + \frac{2 \log \frac{2}{\log 3}}{\log 3}\right) = 6.1522 \cdots. \) For zeros \( \rho = \beta + i \gamma \) with \( |\gamma| > 1 \)
and \( x \geq 10^{10} \) we have
\[ \sum_{|\gamma| > 1} |k_2(\rho)| \leq x^2 \sum_{m=1}^{\infty} \left\{ n_L(2m) + n_L(-2m) \right\} x^{-(2m - 1)^2} \]
\[ \leq 5.44 x^2 \sum_{m=1}^{\infty} \left\{ \log d_L + n_L \log(2m + 2) \right\} x^{-(2m - 1)^2} \quad \text{by (5.5)} \]
\[ \leq c_{20} x \log d_L, \]
where
\[ c_{20} = 5.44 \sum_{m=1}^{\infty} \left\{ 1 + \frac{2}{\log 3} \log(2m + 2) \right\} 10^{-40m^2 + 40m} = 19.16 \cdots. \]
It follows that for $x \geq 10^{10}$

\[(8.7) \quad k_2(1) - \sum_{\rho} |k_2(\rho)| \geq \frac{9}{10} x^2 \min\{1, (1 - \beta_0) \log x\} - c_{21} x^2 (1 - \beta_0) 2c_{19}^{\frac{\log x}{\log d_L}} \log d_L - c_{20} x \log d_L.\]

Note that for $x \geq 10^{10}$

\[(8.8) \quad \mu_2 k_2 \left(-\frac{1}{2}\right) \log d_L + n_L \left\{ k_2(0) + \nu_2 k_2 \left(-\frac{1}{2}\right) \right\} \leq \left\{ \frac{2}{\log 3} + \left( \mu_2 + \frac{2}{\log 3} \nu_2 \right) x^{-\frac{3}{4}} \right\} \log d_L \leq c_{15}' \log d_L,
\]

where

\[c_{15}' = \frac{2}{\log 3} + \left( \mu_2 + \frac{2}{\log 3} \nu_2 \right) 10^{-\frac{5}{2}} = 1.8291 \ldots.\]

Combining (3.2), (4.3), (8.7), and (8.8) yields (8.6).

**Lemma 8.6.** Suppose that $1 - \beta_0 \leq c_7^2 (\log d_L 3^{n_L})^{-2}$. If $x = d_{L}^{c_{23}}$ with $c_{23} = 179$, then

\[\sum_{p \in P(C), N_{K/Q}p \leq x^5} (\log N_{K/Q}p) \hat{k}_2(N_{K/Q}p) > 0.\]

In particular, there is a prime $p \in P(C)$ with $N_{K/Q}p \leq x^5 = d_{L}^{5c_{23}}$.

**Proof.** Let $x = d_{L}^{c_{23}}$. Then (8.6) becomes

\[
\begin{align*}
\frac{|G|}{|C|} \sum_{p \in P(C), N_{K/Q}p \leq x^5} (\log N_{K/Q}p) \hat{k}_2(N_{K/Q}p) &\geq \frac{9}{10} d_{L}^{2c_{23}} \min\{1, c_{23}(1 - \beta_0) \log d_L\} - c_{20} d_{L}^{c_{23}} \log d_L \\
&\quad - c_{21} d_{L}^{2c_{23}} (1 - \beta_0)^{2c_{19}c_{23}} \log d_L - c_{15}' \log d_L - \frac{2\alpha_4 c_{23}^{\frac{1}{2}}}{\log 3} d_{L}^{c_{23}} (\log d_L)^{\frac{5}{2}}.
\end{align*}
\]
When $1 \leq c_{23}(1 - \beta_0) \log d_L$, we have
\[ d_L^{-2c_{23}} \frac{|G|}{|C|} \sum_{\substack{p \in P(C) \\ N_K/Q \leq x^5}} (\log N_K/Qp) \hat{k}_2(N_K/Qp) \geq \frac{9}{10} - c_{21} \left( \frac{c_7^2 (\log d_L)^{-2}}{d_L} \right)^{2c_{19}c_{23}} \log d_L - \epsilon_3 \]
\[ = \frac{9}{10} - c_{21} c_7^{2c_{19}c_{23}} (\log d_L)^{1 - 4c_{19}c_{23}} - \epsilon_3, \]
where
\[ \epsilon_3 = c_{20} \frac{\log d_L}{d_L^{c_{23}}} + c_{15}' \log d_L \frac{d_L^{c_{23}}}{d_L^{c_{23}} - c_{10}} + 2\alpha_4 c_{19}^{c_{23}} (\log d_L)^{\frac{3}{2}} \frac{d_L^{c_{23}}}{d_L^{c_{23}} - c_{10}}. \]
If $c_{23} = (4c_{19})^{-1} = 114.76 \cdots$, then
\[ \frac{9}{10} > c_{21} c_7 + \epsilon_3. \]

When $1 \geq c_{23}(1 - \beta_0) \log d_L$, using Corollary 7.4 we have
\[ d_L^{-2c_{23}} \left\{ (1 - \beta_0) \log d_L \right\}^{-1} \frac{|G|}{|C|} \sum_{\substack{p \in P(C) \\ N_K/Q \leq x^5}} (\log N_K/Qp) \hat{k}_2(N_K/Qp) \geq \frac{9}{10} c_{23} - \epsilon_4, \]
where
\[ \epsilon_4 = \frac{c_{20}}{d_L^{c_{23} - c_{10}}} + c_{21} c_7^{4c_{19}c_{23} - 2} (\log d_L)^{2 - 4c_{19}c_{23}} + \frac{c_{15}'}{d_L^{2c_{23} - c_{10}}} + 2\alpha_4 c_{19}^{c_{23}} (\log d_L)^{\frac{3}{2}} \frac{d_L^{c_{23} - c_{10}}}{d_L^{c_{23} - c_{10}}}. \]
If $c_{23} = 179$, then
\[ \frac{9}{10} c_{23} > \epsilon_4. \]
The result follows. \qed

Lemma 8.4 and 8.6 yield Theorem 1.1.
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