Homotopy groups of $E_{C}^{hG_{24}} \wedge A(1)$

Viet-Cuong Pham

Abstract

Let $A(1)$ be any of the four finite spectra whose cohomology is isomorphic to the subalgebra $A(1)$ of the Steenrod algebra. Let $E_{C}$ be the second Morava-E theory associated to a universal deformation of the formal completion of the supersingular elliptic curve $(C) : y^2 + y = x^3$ defined over $\mathbb{F}_4$ and $G_{24}$ a maximal finite subgroup of automorphism groups $S_C$ of the formal completion $F_C$. In this paper, we will compute the homotopy groups of $E_{C}^{hG_{24}} \wedge A(1)$ by means of the homotopy fixed point spectral sequence.

Keywords: Davis-Mahowald spectral sequence; Higher real $K$-theory; Topological modular forms; Homotopy fixed point spectral sequence

Introduction

One of the central problems in stable homotopy theory is to understand the homotopy groups of the sphere spectrum localized at each prime $p$, $\pi_*(S^0_{(p)})$. The chromatic point of view offers a promising tool to analyze $\pi_*(S^0_{(p)})$ in a systematic way by decomposing it into smaller pieces. More precisely, the chromatic convergence theorem shows that every $p$-local finite spectrum $X$ is the homotopy limit of the tower

$$\ldots \to L_nX \to L_{n-1}X \to \ldots$$

where $L_n$ is Bousfield localization functor with respect to an $n$-th Morava $E$-theory, which is usually chosen to be the Lubin-Tate spectrum associated to the universal deformation of the height $n$ Honda formal group law via Landweber’s exact functor theorem [Rez98]. Furthermore, the chromatic fracture square asserts that $L_n$ can be determined from $L_{n-1}$ and $L_{K(n)}$ as a homotopy pull-back

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for any finite spectrum $X$. Here, $L_{K(n)}$ denotes Bousfield localization with respect to the Morava $K$-theory $K(n)$. See [Rav92].

Therefore, in the chromatic approach to stable homotopy theory, it is crucial to understand the $K(n)$-local homotopy category at all primes and all natural number $n$. For this purpose, a general strategy is to analyze the homotopy type of the $K(n)$-localization of various finite spectra. The turning point of the theory is the work of Devinatz and Hopkins [DH04] which expresses the $K(n)$-localization as the continuous homotopy fixed point spectrum

$$L_{K(n)}X = E_n^{h\mathbb{G}_n} \wedge X$$

where $\mathbb{G}_n \cong S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, called extended Morava stabilizer group, is the semi-product of the automorphism groups $S_n$ of the Honda formal group law $F_n$ and the Galois group $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Moreover, for any closed subgroup $F$ of $\mathbb{G}_n$, there is a $K(n)$-local $E_n$-based spectral sequence or homotopy fixed point spectral sequence converging to $\pi_*(E_n^{hF} \wedge X)$ with $E_2$-term being the continuous cohomology of $F$ with coefficient in $E_n^*(X)$:

$$H_c^*(F, E_n^*(X)) \Rightarrow \pi_*(E_n^{hF} \wedge X) \quad (1)$$

If $F$ is a finite subgroup of $\mathbb{G}_n$, then this spectral sequence coincides with the usual homotopy fixed point spectral sequence.

Chromatic level one is very well understood at all primes: the homotopy groups of $L_{K(1)}S^0$ has been completely computed. Chromatic level two has been also thoroughly investigated at odd primes. It started with Shimomura and his collaborators computations of $L_2$ localization of various finite spectra (see [SY95], [Shi97], [Shi00], [SW02]). Later Goerss et al. in [GHMR05] proposed a conceptual framework to organize the $K(2)$-local homotopy category at the prime 3. See also [GHM04], [HKM13], [GH16] for further investigations at $n = 2$ and $p = 3$ and [Beh12] for an exposition of $L_2S^0$ at $p \geq 5$.

We are now leaning on the edge of our current knowledge. Considerable effort has being made to understand the $K(2)$-local homotopy category at the prime 2. The reason why the latter is hard to deal with lies largely in the fact that the cohomological properties of the group $\mathbb{G}_2$ are much more complicated at the prime 2.
2. At chromatic level 2 and at the prime 2, we can replace the height 2 Honda formal group law by the formal completion \( F_C \) of the supersingular elliptic curve \((C) : y^2 + y = x^3 \) defined over \( \mathbb{F}_4 \). Let \( S_C \) be the group of automorphisms of \( F_C \). Let \( E_C \) the Lubin-Tate spectrum associated to \( F_C \). Because \( F_2 \) and \( F_C \) are isomorphic over the algebraic closure of \( \mathbb{F}_2 \) [Laz55], Bousfield localization with respect to \( K(2) \) can also be described as

\[
L_{K(2)} X \cong (E_C^{hS_C})^{h\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \wedge X
\]

For computational purposes that we will explain in more detail later, we will work with \( F_C \) for the rest of the paper. Let us consider \( S^1_C \) defined to be the kernel of the reduced determinant

\[
det : S_C \to \mathbb{Z}_2 \tag{2}
\]

We have a cofiber sequence

\[
E_C^{hS_C} \to E_C^{hS^1_C} \xrightarrow{\pi^{-1}} E_C^{hS^1_C}
\]

where \( \pi \in S_C \) be a lift of a topological generator of the determinant homomorphism in \( (2) \). Therefore, understanding \( E_C^{hS^1_C} \) is a major step towards \( L_{K(2)} S^0 \). In [Bea15], Beaudry established a finite resolution, known as the algebraic duality resolution, of the trivial \( S^1_C \) module \( \mathbb{Z}_2 \). This algebraic spectral sequence has a topological counterpart, the topological duality resolution. Let \( G_{24} \) be a maximal finite subgroup of \( S^1_C \), \( G'_{24} = \pi G_{24} \pi^{-1} \) and \( C_6 \) a cyclic subgroup of order 6 of \( G_{24} \). The following theorem had been first announced by Henn in [Hen07] and has been recently completed and proved by Bobkova and Goerss in [BG16].

\[ \text{Theorem 0.0.1. There exists a sequence of maps in the } K(2)\text{-local homotopy category at the prime 2} \]

\[
E_C^{hS^1_C} \to E_C^{hG_{24}} \to E_C^{hC_6} \to \Sigma^{48} E_C^{hC_6} \to \Sigma^{48} E_C^{hG'_{24}}
\]

such that all compositions and Toda bracket are trivial modulo indeterminacy.

This theorem offers a useful instrument to organize the \( K(2) \)-local homotopy theory at the prime 2. In particular, it produces a small spectral sequence converging to \( \pi_*(E_C^{hS^1_C} \wedge X) \) with \( E_1 \)-term being identified from \( \pi_*(E_C^{hG_{24}} \wedge X) \) and \( \pi_*(E_C^{hC_6} \wedge X) \). To obtain the inputs for the duality spectral sequence, we start analyzing \( \pi_*(E_C^{hG_{24}} \wedge X) \) be means of the homotopy fixed point spectral sequence

\[ H^*(G_{24}, E_{C_4}(X)) \Rightarrow \pi_*(E_C^{hG_{24}} \wedge X) \tag{3} \]

for various finite spectra \( X \).
Let us consider $A(1)$ any of the four type 2 finite spectra whose cohomology is isomorphic to the subalgebra $A(1) = \langle Sq^1, Sq^2 \rangle$ of the Steenrod algebra $A$. These spectra were first considered by Davis and Mahowald in [DM82].

The aim of this paper is to give a detailed computation of the homotopy fixed point spectral sequence for $E_{hG}^{hG_{24}} \wedge A(1)$. We could only determine some differentials with prior information about the homotopy groups of $E_{hG}^{hG_{24}} \wedge A(1)$. To this end, we resort to the spectrum $tmf$ of topological modular forms which is closely related to the homotopy fixed point spectrum $E_{hG}^{hG_{24}}$. In fact, we prove that there is a homotopy equivalence (Theorem 4.1.3):

$$(\Delta^8)^{-1}tmf \wedge A(1) \cong (E_{hG_{24}}^{hG_{24}})^{hGal(F_4/F_2)} \wedge A(1)$$

where $\Delta^8$ is the periodicity generator of $\pi_*tmf$. Thus we can try to understand the homotopy groups of $tmf \wedge A(1)$, then invert $\Delta^8$ to get information about the homotopy groups of $E_{hG_{24}}^{hG_{24}} \wedge A(1)$. The homotopy groups of $tmf \wedge A(1)$ are accessible through the classical Adams spectral sequence

$$\text{Ext}^s_{A(2)_*}(F_2, \pi_s(A(1))) \Rightarrow \pi_{s-t}(tmf \wedge A(1))$$

We notice that in [BEM17] Batacharya, Egger, Maholwald briefly discussed this Adams spectral sequence. Our approach is however different and contains more details. For example, we give an explicit description of the $E_2$-term of the Adams spectral sequence using the Davis-Mahowald spectral sequence and determine some differentials without appealing to Bruner’s software. Finally, the calculation of $\pi_* (E_{hG_{24}}^{hG_{24}} \wedge A(1))$ presented here will be an essential input in our future work to analyze $\pi_* (E_{hG_{24}}^{hG_{24}} \wedge A(1))$.

The paper is organized as follows. Section 1 is devoted to a slight generalization of the Davis-Mahowald spectral sequence. In Section 2, we recollect certain information of the Davis-Mahowald spectral sequence for the $A(2)$-comodule $F_2$. Then we come to discuss the Davis-Mahowald spectral sequence for $A(1)$ and obtain the $E_2$-term of the Adams spectral sequence. Finally, we compute two products one of which is exotic and can not be seen by the Davis-Mahowald spectral sequence. These products allow us to determine some differentials in the Adams spectral sequence for $A(1)$. In Section 3, we discuss some differentials in the later and then extract some suitable information about $\pi_* (tmf \wedge A(1))$ for later use. Finally in Section 4, we study the homotopy fixed point spectral sequence for $E_{hG_{24}}^{hG_{24}} \wedge A(1)$.

Convention Without otherwise stated, all spectra are localized at the prime 2. $H^*(X)$ and $H_*(X)$ denote the mod-2 (co)homology of the spectrum X. Given
a Hopf algebra $A$ over a field $k$ and $M$ a $A$-comodule, we will often abbreviate $\Ext^*_A(k, M)$ by $\Ext^*_A(M)$. In general, we will write $C_f$ for the cofiber of a map $f : X \to Y$ except that we will write $V(0)$ for the Moore spectrum which is the cofiber of the multiplication by $p$ on the sphere. We reserve the notation $C_p$ for the cyclic group of order $p$.

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Contents

1 Davis-Mahowald spectral sequence

2 On the Davis-Mahowald spectral sequence for the $A(2)_*$-comodules
   $A(1)$
   2.1 Recollections on the Davis-Mahowald spectral sequence for the
       $A(2)_*$-comodule $\mathbb{F}_2$ ................................. 12
   2.2 Davis-Mahowald spectral sequence for $A(1)$ ............................. 26
   2.3 Two products ......................................................... 33

3 Partial study of the Adams spectral sequence for $tmf \wedge A(1)$ 36

4 Homotopy fixed point spectral sequence $E^{hG_{24}}_2 \wedge A(1)$
   4.1 Preliminaries and recollection on cohomology of $G_{24}$ .... 41
   4.2 The $E_2$-term : $H^*(G_{24}, (E_C)_*(A(1)))$ .................. 44
   4.3 Differentials of the homotopy fixed point spectral sequence for
       $E^{hG_{24}}_2 \wedge A(1)$ ........................................... 48

1 Davis-Mahowald spectral sequence

Construction of the spectral sequence. Let $k$ be a field of characteristic 2. We will later specialize to the case $k = \mathbb{F}_2$, the field of two elements. Let $(A, \Delta, \mu, \epsilon, \eta, \chi)$ be a commutative Hopf algebra over $k$ with $\Delta, \mu, \epsilon, \eta, \chi$ being coproduct, product, counit, unit, the conjugation, respectively.
Definition 1.0.1. Let $E$ be the graded exterior algebra of a finite dimensional $k$-vector space $V$ with all elements of $V$ having degree 1. An $A$-comodule algebra structure on $E$ is called almost graded if the natural embedding $k \oplus V \to E$ is a map of $A$-comodules.

Let $E$ be an almost graded $A$-comodule exterior algebra (of a finite dimensional $k$-vector space $V$). We will construct a $A$-comodule polynomial algebra, called the Koszul dual of $E$ as follows. Let $P$ be the graded polynomial algebra of $V$ with all element of $V$ having degree 1. Let us denote by $E_i$ and $P_i$ the subspace of elements of homogeneous degree $i$ for $i \geq 0$ of $E$ and $P$, respectively. Let us also denote by $E_{\leq i}$ the the direct sum $\bigoplus_{j=0}^i E_j$. Notice that $P_1$ sits in a short exact sequence:

$$0 \to k \to k \oplus E_1 \xrightarrow{\mu} P_1 \to 0 \quad (4)$$

The embedding $k \to k \oplus E_1$ is clearly a map of left $A$-comodules. Thus $P_1$ admits a (unique) structure of left $A$-comodule such that $p : k \oplus E_1 \to P_1$ is a map of $A$-comodules.

Lemma 1.0.2. There is a unique $A$-comodule structure on $P_n$ such that the tau-tological multiplication $P_1^\otimes n \to P_n$ is a map of $A$-comodules. Here, $P_1^\otimes n$ is equipped with the usual diagonal $A$-comodule structure.

Proof. This map is surjective and its kernel is spanned by elements of the form $y_1 \otimes \cdots \otimes y_n - y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}$ where $\sigma$ belongs to the permutation group on the set of $n$ elements. Since $A$ is commutative, this kernel is stable under coaction of $A$. Thus, $P_n$ inherits an $A$-comodule structure from $P_1^\otimes n$. \(\square\)

This lemma shows that $P = \bigoplus_{i \geq 0} P_i$ admits a left $A$-comodule algebra structure.

Now, let us define a cochain complex, called the Koszul complex,

$$(E \otimes P, d) \quad (5)$$

with

d) \quad (E \otimes P)_{-1} = k$

ii) $d) (E \otimes P)_m = E \otimes P_m$ for $m \geq 0$

iii) $d : k = (E \otimes P)_{-1} \to E = (E \otimes P)_0$ being the unit of $E$

iv) $d(\prod_{j=1}^n x_{i_j} \otimes z) = \sum_{t=1}^n \prod_{j \neq t} x_{i_j} \otimes p(x_{i_t})z$ where $x_{i_j} \in E_1$, $z \in P_m$ and $p$ is the projection of (4).
Remark 1.0.3. In other words, \( d : E_{\leq n} \otimes P_m \to E_{\leq n-1} \otimes P_{m+1} \) is the unique homomorphism making the following diagram commute

\[
\begin{array}{c}
E_{\leq 1} \otimes P_{\leq m} & \xrightarrow{(\sum (Id \otimes (n-1) \otimes p) \circ \sigma) \otimes Id} & E_{\leq 1} \otimes (n-1) \otimes P_{m+1} \\
\downarrow \mu \otimes Id & & \downarrow \mu \otimes \mu \\
E_{\leq n} \otimes P_m & \xrightarrow{d} & E_{\leq n-1} \otimes P_{m+1}
\end{array}
\]

(6)

where in the upper horizontal map, the sum is taken over all cyclic permutations on \( n \) factors of \( E_1 \) in the tensor product \( E_{\leq n} \otimes P_{m} \) and \( p \) is the restriction on \( E_1 \) of the map of (4).

Proposition 1.0.4. The complex \((E \otimes P, d)\) is an exact sequence of \( A \)-comodules. Furthermore, \((E \otimes P, d)\) has a structure of differential graded algebra induced from the algebra structure of \( E \) and \( P \).

Proof. Let \( x_1, \ldots, x_n \) be a basis of \( E_1 \). As a cochain complex over \( k \), \((E \otimes P, d)\) is isomorphic to the tensor product of \((E(x_i) \otimes k[y_i], d_i)\) where \( y_i = p(x_i) \) for \( 1 \leq i \leq n \). Here, each \((E(x_i) \otimes k[y_i], d_i)\) is defined in the same manner as \((E \otimes P, d)\) is. It is not hard to see that the cochain complex \((E(x_i) \otimes k[y_i], d_i)\) is exact. Hence, \((E \otimes P, d)\) is exact by the Künneth theorem. This proves the first part.

Let us check that \( d \) is a map of \( A \)-comodules. In the diagram (6), the two vertical maps are ones of \( A \)-comodule algebras because \( E \) and \( P \) are \( A \)-comodule algebras. In addition, they are surjective. It remains to check that the upper horizontal map is a map of \( A \)-comodules. Or equivalently, each map \( E_{\leq 1} \otimes (Id \otimes (n-1) \otimes p) \circ \sigma \otimes Id \) is a map of \( A \)-comodules where \( \sigma \) is a cyclic permutation on \( n \) elements. This is true because \( \sigma \) is a map of \( A \)-comodules as \( A \) is commutative and \( p \) is a map of \( A \)-comodules by definition. The second part follows.

Finally, It is straightforward from the formula of \( d \) in (5.iv) that \( d \) is a map of algebras.

This lemma allows us to construct a spectral sequence of algebras converging to \( \text{Ext}^s_A(k, k) \) see ([Rav86], Thm A1.3.2).

Proposition 1.0.5. (1) There is a spectral sequence of algebras converging to \( \text{Ext}^s_A(k, k) \)

\[
\text{Ext}^s_A(k, E \otimes P_t) \Rightarrow \text{Ext}^{s+t}_A(k, k)
\]

(7)
(2) If $M$ is an $A$-comodule, then there is a spectral sequence converging to $\Ext^*_A(k, M)$

$$\Ext^*_A(k, E \otimes P_t \otimes M) \Longrightarrow \Ext^{s+t}_A(k, M)$$

Furthermore, this spectral sequence is a spectral sequence of modules over that of (7).

**Terminology.** We will call these spectral sequences the Davis-Mahowald spectral sequences or DMSS for short.

In the perspective of carrying out explicit computations of products in $\Ext^*_A(k)$ and the action $\Ext^*_A(k)$ on $\Ext^*_A(M)$, we recall a double complex from which the above spectral sequence is derived.

For each $t \geq 0$, let $(C^s(A, E \otimes P_t), d_v)_{s \geq 0}$ be the cobar complex whose cohomology is $\Ext^*_A(k, E \otimes P_t)$, i.e.,

$$C^s(A, E \otimes P_t) = A^\otimes s \otimes E \otimes P_t$$

and $d_v : A^\otimes s \otimes E \otimes P_t \to A^\otimes s+1 \otimes E \otimes P_t$ is given by

$$d_v(a_1 \otimes \ldots \otimes a_n \otimes m) = 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes m + \sum_{i=1}^n a_1 \otimes \ldots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \ldots \otimes m + a_1 \otimes \ldots \otimes a_n \otimes \Delta(m)$$

We will abbreviate $a_1 \otimes \ldots \otimes a_n \otimes m$ by $[a_1|\ldots|a_n|m]$ . By an abuse of notation, we will denote by $d_v$ the differentials in the cobar complexes associated to $E \otimes P_t$ for different $t$. The fact that $d : E \otimes P_t \to E \otimes P_{t+1}$ is a map of $A$-comodules implies that maps $d_h = Id^\otimes s \otimes d : C^s(A, E \otimes P_t) \to C^s(A, E \otimes P_{t+1})$ assemble to give a map of cochain complexes $d_h : (C^s(A, E \otimes P_t), d_v)_{s \geq 0} \to (C^s(A, E \otimes P_{t+1}), d_v)_{s \geq 0}$. Finally, it is easily seen that the maps of cochain complexes assemble to form a double complex $(C^s(A, E \otimes P_t), d_v, d_h)_{s,t \geq 0}$.
We can see that the spectral sequence associated to the horizontal filtration has the $E_1$-term isomorphic to $(A^s \otimes k, d_1)_{s \geq 0}$ which identifies with the cobar complex of the trivial $A$-comodule $k$. Thus this spectral sequence is degenerate at the $E_2$-term and the $E_\infty = E_2$-term identifies with $\text{Ext}^s_A(k)$. Since there is no possible extension problem, the cohomology of the total complex is isomorphic to $\text{Ext}^s_A(k)$. This spectral sequence converging to the cohomology of the total complex $\text{Ext}^{s+t}_A(k)$ is exactly the one appearing in Proposition 1.0.5.

Remark 1.0.6. The differential $d_1 : \text{Ext}^0_A(E \otimes P_t) \to \text{Ext}^0_A(E \otimes P_{t+1})$ is nothing but the restriction of the derivation $d$ in (5) on the $A$-primitives of $E \otimes P_t$.

Naturality of the Davis-Mahowald spectral sequence. We notice that the above construction is natural in pairs $(A, E)$ where $A$ is a commutative Hopf algebra and $E$ is an almost graded left $A$-comodule exterior algebra. This allows us to compare Davis-Mahowald spectral sequences associated to different pairs $(A, E)$. We will make use of this property to reduce computations in a crucial way. Let us first define morphisms between such pairs.

Definition 1.0.7. Let $(A, E)$ and $(B, F)$ be such that $A$ and $B$ are commutative Hopf algebras, $E$ and $F$ are almost graded exterior comodule algebras over $A$ and $B$, respectively. A morphism between $(A, E)$ and $(B, F)$ consists of $f_1 : A \to B$ and $f_2 : E \to F$ where $f_1$ is a map of Hopf algebras and $f_2$ is a map of $B$-comodule graded algebras with the $B$-comodule structure on $E$ being induced from $f_1$.

Proposition 1.0.8. A morphism between $(A, E)$ and $(B, F)$ induces a map between the associated Davis-Mahowald spectral sequences.

Proof. Let $P$ and $Q$ be the Koszul dual of $E$ and $F$, respectively. The map of $B$-comodule algebras $f_2 : E \to F$ induces a map of graded $B$-comodule algebras $P \to Q$ such that the following diagram is commutative

$$
\begin{array}{ccc}
k \oplus E_1 & \xrightarrow{p} & P_1 \\
\downarrow f_2 & & \downarrow \\
k \oplus F_1 & \xrightarrow{p} & Q_1
\end{array}
$$

Then one can check that the induced map $E \otimes P \to F \otimes Q$ is a map of Koszul complexes. Therefore one obtains a map of double complexes $(A^{\otimes s} \otimes E \otimes P_t) \to (B^{\otimes s} \otimes F \otimes Q_t)$, hence a map of spectral sequences. 

Remark 1.0.9. Although we only treat ungraded situations so far, the construction carries verbatim to graded ones in which an extra grading is added to cohomology groups and differentials preserve them.
Now, we will examine some examples which are our main objects of interest.

**Example 1.** Let $A_\ast$ be the Hopf algebra dual to the Steenrod algebra $A$ at the prime 2 [Mil58]. As a graded algebra, $A_\ast = \mathbb{F}_2[\xi_i | i \geq 1]$ where $\xi_i$ is in degree $|\xi_i| = 2^i - 1$. The coproduct is given by the formula

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_i^{2^k-i} \otimes \xi_{k-i}$$

where $\xi_0 = 1$. Let us denote by $\overline{\xi_i}$ the conjugate of $\xi_i$. As we will mainly work with conjugates, we record the coproduct formula for them

$$\Delta(\overline{\xi_k}) = \sum_{i+j=k} \overline{\xi_i} \otimes \overline{\xi_j}^{2^k} \quad (8)$$

We recall that a Hopf ideal of a Hopf algebra $A$ is an ideal $I$ such that $\Delta(I) \subset I \otimes A + A \otimes I$. If $I$ is a Hopf ideal of $A$, then $A/I$ inherits a structure of Hopf algebra from $A$. Let $A(n)_\ast$ be the quotient of $A_\ast$ by the Hopf ideal $I_n$ generated by $(\overline{\xi_1}^{2^n+1} ; \overline{\xi_2}^{2^n} ; \ldots ; \overline{\xi_{n+1}^2} ; \overline{\xi_{n+2}} ; \ldots)$. As an algebra,

$$A(n)_\ast = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_{n+1}]/(\overline{\xi_1}^{2^n+1} ; \overline{\xi_2}^{2^n} ; \ldots ; \overline{\xi_{n+1}}^2)$$

It is dual to the subalgebra $A(n) = \langle Sq^1, Sq^2, \ldots, Sq^{2^n} \rangle$ of the Steenrod algebra $A$. The canonical projection $\pi : A(n)_\ast \rightarrow A(n-1)_\ast$ induced by the inclusion $I_n \subset I_{n-1}$ of Hopf ideals is a map of Hopf algebras, hence induces on $A(n)_\ast$ a structure of right $A(n-1)_\ast$-comodule algebra:

$$(id \otimes \pi) \Delta : A(n)_\ast \rightarrow A(n)_\ast \otimes A(n)_\ast \rightarrow A(n)_\ast \otimes A(n-1)_\ast$$

The following well-known lemma describes the structure of the primitives $A \square_B k$ of this comodule algebra.

**Lemma 1.0.10.** Let $A$ be a Hopf algebra and $B$ be a Hopf quotient of $A$. Give $A$ the usual right $B$-comodule structure. Then $A \square_B k$ has a left $A$-comodule algebra structure such that the inclusion $A \square_B k \rightarrow A$ is a map of $A$-comodule algebras.

An easy computation shows that

$$A(n)_\ast \square A(n-1)_\ast \mathbb{F}_2 = E(\overline{\xi_1}^{2^n} ; \overline{\xi_2}^{2^{n-1}} ; \ldots ; \overline{\xi_{n+1}})$$

which is abstractly isomorphic to $E_n = E(x_1, \ldots, x_{n+1})$ where $x_i$ stands for $\overline{\xi_i}^{2^{n+1-i}}$. Here and elsewhere in this paper, $E(X)$ denotes the exterior algebra.
Similarly to Example 1, the projection $x \to x_{n+1}$ converging to a sequence of algebras $I_n$ inherits a Hopf ideal $J$ in the corresponding Hopf algebra $A(n)_s$-comodule. Let $R_n$ denote the Koszul dual of $E_n$. In particular, it follows from Proposition 1.0.5 that for any left $A(n)_s$-comodule $M$, there is a spectral sequence converging to $\text{Ext}_{A(n)_s}^*(\mathbb{F}_2, M)$ with the $E_1$-term isomorphic to $\text{Ext}_{A(n)}^*(\mathbb{F}_2, E_n \otimes R_n^* \otimes M)$. The change-of-ring isomorphism tells that

$$\text{Ext}_{A(n)_s}^*(\mathbb{F}_2, E_n \otimes R_n^* \otimes M) \cong \text{Ext}_{A(n-1)_s}^*(\mathbb{F}_2, R_n^* \otimes M)$$

That means that the problem of computing $\text{Ext}_{A(n)_s}^*(\mathbb{F}_2, -)$ can be reduced to two steps: first computing $\text{Ext}_{A(n-1)_s}^*(\mathbb{F}_2, -)$, then studying the corresponding Davis-Mahowald spectral sequence. We will demonstrate the efficiency of this method by carrying out explicit computation in the case $n = 2$ and some relevant $M$.

**Example 2.** Let $B(n)_s$ be the quotient of $A_s$ by the Hopf ideal $J_n$ generated by $(\xi_1^{2^n}, \xi_2^{2^n}, \xi_3^{2^n-1}, ..., \xi_{n+1}^{2^n}, \xi_{n+2}, ...)$. So

$$B(n)_s = \mathbb{F}_2[\xi_1, \xi_2, ..., \xi_{n+1}]/(\xi_1^{2^n}, \xi_2^{2^n}, \xi_3^{2^n-1}, ..., \xi_{n+1}^{2^n})$$

Similarly to Example 1, the projection $B(n)_s \to A(n-1)_s$ resulted from the inclusion of Hopf ideals $J_n \subset I_{n-1}$ defines a structure of right $A(n-1)_s$-comodule algebra on $B(n)_s$. A calculation shows that

$$B(n)_s \otimes_{A(n-1)_s} \mathbb{F}_2 = E(\xi_2^{2^n-1}, \xi_3^{2^n-2}, ..., \xi_{n+1})$$

which is abstractly isomorphic to $F_n := E(x_2, ..., x_{n+1})$. The notation is chosen in coherence with Example 1. Lemma 1.0.10 implies that $F_n$ inherits a structure of left $B(n)_s$-comodule algebra from that of $B(n)_s$. In particular, the coaction is given by the formulae

$$\Delta(x_k) = \sum_{i=0, i \neq 1}^k \xi_i^{2^{n+1-k}} \otimes x_{k-i}, \quad 2 \leq k \leq n + 1$$

where $x_0 = 1$. Let $S_n$ denote the Koszul dual of $F_n$. There arises then a spectral sequence of algebras

$$\text{Ext}_{B(n)_s}^{*,*}(\mathbb{F}_2 \otimes S_n^\sigma) \Rightarrow \text{Ext}_{B(n)_s}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$
By the change-of-ring theorem, the $E_1$-term is isomorphic to $\text{Ext}^{s,t}_{A(n-1)*}(S^*_n)$ because $F_n = B(n) \otimes_{A(n-1)*} \mathbb{F}_2$. Moreover, for any left $B(n)*$-comodule $M$, there is a spectral sequence of modules over the above spectral sequence
\[
\Ext^{s,t}_{B(n)*}(F_n \otimes S^*_n \otimes M) \cong \Ext^{s,t}_{A(n-1)*}(S^*_n \otimes M) \Rightarrow \Ext^{s+\sigma,t}_{B(n)*}(\mathbb{F}_2, \mathbb{F}_2)
\]

**Comparison of DMSS** There is a morphism between $(A(n)*, E_n)$ and $(B(n)*, F_n)$ given by two projections

\[
A(n)* \to B(n)*; \xi_i \mapsto \overline{\xi}_i
\]

\[
E_n \to F_n; x_1 \mapsto 0, x_i \mapsto x_i \text{ for } i \geq 2
\]

and so it induces a map of spectral sequences
\[
\begin{align*}
\Ext^{s,t}_{A(n)*}(E_n \otimes R^*_n \otimes M) & \Rightarrow \Ext^{s,t}_{B(n)*}(F_n \otimes S^*_n \otimes M) \\
\Ext^{s+\sigma,t}_{A(n)*}(\mathbb{F}_2, M) & \Rightarrow \Ext^{s+\sigma,t}_{B(n)*}(\mathbb{F}_2, \mathbb{F}_2)
\end{align*}
\]

As was mentioned earlier, this comparison allows us to transfer some computations in the former SS to the latter SS which are in practice simpler because apparently, all modules involved in the latter are smaller. This observation will be made concrete in the next section.

## 2 On the Davis-Mahowald spectral sequence for the $A(2)*$-comodules $A(1)$

The goal of this section is to describe the structure of $\text{Ext}^{*,*}_{A(2)*}(\mathbb{F}_2, A(1))$ as a module over $\text{Ext}^{*,*}_{A(2)*}(\mathbb{F}_2, \mathbb{F}_2)$ for different $A(2)*$-comodules $A(1)$ that will be recalled in Subsection 2.2. To achieve a part of this goal, we will study the DMSS
\[
\Ext^{s,t}_{A(2)*}(\mathbb{F}_2, E_2 \otimes S^*_2 \otimes A(1)) \Rightarrow \Ext^{s+\sigma,t}_{A(2)*}(\mathbb{F}_2, A(1))
\]

as a spectral sequence of modules over the spectral sequence of algebras
\[
\Ext^{s,t}_{A(2)*}(\mathbb{F}_2, E_2 \otimes S^*_2) \Rightarrow \Ext^{s+\sigma,t}_{A(2)*}(\mathbb{F}_2, \mathbb{F}_2)
\]

We obtain then the structure of $\text{Ext}^{*,*}_{A(2)*}(\mathbb{F}_2, A(1))$ as a graded abelian group and a partial action of $\text{Ext}^{*,*}_{A(2)*}(\mathbb{F}_2, \mathbb{F}_2)$ on it. However, there is an important action
of an element of $\text{Ext}_{A(2)_*}^*(\mathbb{F}_2, \mathbb{F}_2)$ on some elements of $\text{Ext}_{A(2)_*}^*(\mathbb{F}_2, A(1))$ that cannot be seen at the $E_1$-term of the DMSS. One way of understanding these exotic products is to carry out computations at the level of double complexes: find representatives of the cohomological classes in question in the double complexes from which the DMSS is derived and carry out products at that level. It turns out that a brute-force attack is messy. Instead, computations are simplified drastically by comparing the DMSS associated to $(A(2)_*, E_2)$ to that of $(B(2)_*, F_2)$:

\[
\begin{array}{c}
\text{Ext}_{A(2)_*}^{s,t}(E_n \otimes R_{\sigma}^2 \otimes A(1)) \quad \text{Ext}_{B(2)_*}^{s,t}(F_n \otimes S_{\sigma}^2 \otimes (1)) \\
\downarrow \quad \downarrow \\
\text{Ext}_{A(2)_*}^{s+\sigma,t}(\mathbb{F}_2, A(1)) \quad \text{Ext}_{B(2)_*}^{s+\sigma,t}(\mathbb{F}_2, A(1))
\end{array}
\]

### 2.1 Recollections on the Davis-Mahowald spectral sequence for the $A(2)_*$-comodule $\mathbb{F}_2$

To set up notations, we recollect some information relevant for our purposes. This material was originally treated in [DM82] and reviewed in an unpublished course note of Rognes. As we will specialize to the case $n = 2$, we will simplify the notation by writing $R, R_{\sigma}, S, S_{\sigma}$ for $R_2, R_{\sigma}^2, S_2, S_{\sigma}^2$, respectively.

Recall that $R$ is a homogenous graded polynomial algebra on three generators, say $y_1, y_2, y_3$ and $R_{\sigma}$ is its subspace of homogeneous elements of degree $\sigma$ for $\sigma \geq 0$. Let us first explicitly give the coaction of $A(2)_*$ on $R = \mathbb{F}_2[y_1, y_2, y_3]$ with $|y_1| = 4, |y_2| = 6, |y_3| = 7$. From Example 1 of Section 2, we have

\[
\begin{align*}
\Delta(y_1) &= 1 \otimes y_1 \\
\Delta(y_2) &= \xi_1 \otimes y_1 + 1 \otimes y_2 \\
\Delta(y_3) &= \xi_2 \otimes y_1 + \xi_1 \otimes y_2 + 1 \otimes y_3
\end{align*}
\]

By the change-of-ring theorem, the $E_1$-term of the DMSS converging to $\text{Ext}_{A(2)_*}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to $\text{Ext}_{A(1)_*}^{*,t}(\mathbb{F}_2, R_{\sigma})$. The coaction of $A(1)_*$ on $R_1$ is induced from that of $A(2)_*$ and hence is given by

\[
\begin{align*}
\Delta(y_1) &= 1 \otimes y_1 \\
\Delta(y_2) &= \xi_1 \otimes y_1 + 1 \otimes y_2 \\
\Delta(y_3) &= \xi_2 \otimes y_1 + \xi_1 \otimes y_2 + 1 \otimes y_3
\end{align*}
\]

In particular, $y_1, y_2^2, y_3^3$ are $A(1)_*$-primitives of $R$. Let $R_{\sigma}'$ denote the $A(1)_*$-subcomodule \{\(y_1^i y_2^j y_3^k \in R_{\sigma} | k \leq 3\)\} of $R_{\sigma}$.
Lemma 2.1.1. As an \( A(1)_* \)-comodule, \( R_{\sigma} \) can be decomposed as

\[
R_{\sigma} \cong \bigoplus_{i \equiv \sigma \pmod{4}, i \leq \sigma} R_i' \otimes \mathbb{F}_2\{y_3^{\sigma-i}\}
\]

Therefore,

\[
\bigoplus_{\sigma \geq 0} R_{\sigma} = (\bigoplus_{\sigma \geq 0} R_{\sigma}') \otimes \mathbb{F}_2[y_3]
\]

Proof. If one views \( \mathbb{F}_2\{y_3^{\sigma-i}\} \) as a subvector space of \( R_{\sigma-i} \), then the product of \( R \) produces an isomorphism of vector spaces

\[
\bigoplus_{i \equiv \sigma \pmod{4}, i \leq \sigma} R_i' \otimes \mathbb{F}_2\{y_3^{\sigma-i}\} \cong R_{\sigma}
\]

Since \( y_3^i \) is a \( A(1)_* \)-primitive of \( R_{\sigma} \), this map is also a map of \( A(1)_* \)-comodules. The lemma follows.

Let us denote \( \text{Ext}_{A(1)_*}^{*,*} (R_{\sigma}) \) by \( G_{\sigma} \), so that

\[
\text{Ext}_{A(1)_*}^{*,*} (R) \cong (\bigoplus_{\sigma \geq 0} G_{\sigma}) \otimes \mathbb{F}_2[v_2^4]
\]

Determining the full multiplicative structure of \( \text{Ext}_{A(1)_*}^{*,*} (R) \) is quite involving. Instead, we will be content to work module \( (v_2^4) \). This will suffice for us to obtain a set of algebra generators of \( \text{Ext}_{A(1)_*}^{*,*} (R) \). More precisely, since the product \( R_i' \otimes R_i' \to R_{\sigma+i} \) factorises through \( R_i' \oplus (R_i' - 4 \otimes \mathbb{F}_2\{y_3^i\}) \), we obtain a map

\[
G_{\sigma} \otimes G_{\tau} \to G_{\sigma+i} \oplus (G_{\sigma+i-4} \otimes \mathbb{F}_2\{v_2^i\})
\]

We will analyze the map \( G_{\sigma} \otimes G_{\tau} \to G_{\sigma+i} \) which is the composite

\[
G_{\sigma} \otimes G_{\tau} \to G_{\sigma+i} \oplus (G_{\sigma+i-4} \otimes \mathbb{F}_2\{v_2^i\}) \to G_{\sigma+i}
\]

where the second map is the projection on the first factor.

In what follows, we compute \( G_i \) for \( i \geq 0 \) as modules over \( G_0 \). For this, we decompose \( R_i' \) into smaller pieces, compute the Ext groups over \( A(1)_* \) of these pieces, then determine \( G_i \) via long exact sequences. Next, we study the pairings

\[
G_{\sigma} \otimes G_{\tau} \to G_{\sigma+i}
\]

which allows us to determine a set of algebra generators of the \( E_1 \)-term. Finally, we compute \( d_1 \)-differentials on this set of algebra generators. We do not intend
to describe completely the Ext_{A(2), \text{Ext}^*}^*, (\mathbb{F}_2, \mathbb{F}_2)$ but only a subalgebra which we are interested in.

Since $y_1$ is primitive, multiplication by $y_1$ induces injections of $A(1)_{*}$-comodules

$\Sigma^4 R'_{\sigma} \rightarrow R'_{\sigma+1}$

**Lemma 2.1.2.** There are short exact sequences of $A(1)_{*}$-comodules

(a) $0 \rightarrow H_*(\Sigma^{12} C_{\eta}) \rightarrow R'_2 \rightarrow \Sigma^8 (A(1)_{*} \square A(0)_{}, \mathbb{F}_2) \rightarrow 0$

where $\eta : S^1 \rightarrow S^0$ is the Hopf map and the map $H_*(\Sigma^{12} C_{\eta}) \rightarrow R'_2$ sends the generators of $H_{12}(\Sigma^{12} C_{\eta})$ and $H_{14}(\Sigma^{12} C_{\eta})$ to $y^2_2$ and $y^3_3$, respectively.

(b) $0 \rightarrow \Sigma^4 R'_1 \rightarrow R'_2 \rightarrow \Sigma^{12} V_3 \rightarrow 0$

where $V_3 = H_*(S^0 \cup_2 e^1 \cup_\eta e^2)$

**Proof.** For part (a), the map $\Sigma^{12} H_*(C_{\eta}) \rightarrow R'_2$ described in the statement of the Lemma 2.1.2 is a map of the $A(1)_{*}$-comodules. Its quotients is isomorphic to $\mathbb{F}_2 \{y^2_1, y_1 y^2_2, y_1 y^3_3, y^2_3\}$ with the $A(1)_{*}$-comodule structure given by

$\Delta(y^2_2) = 1 \otimes y^2_2 + \xi^2_1 \otimes y^2_3$

$\Delta(y_1 y^2_3) = 1 \otimes y_1 y^3_3 + \xi^2_2 \otimes y^2_3$

$\Delta(y^3_3) = 1 \otimes y^3_3 + \xi^2_1 \otimes y^2_3$

$\Delta(y^3_3) = 1 \otimes y^3_3 + \xi^2_2 \otimes y^2_3$

We can check that this module is isomorphic to $\Sigma^8 (A(1)_{*} \square A(0)_{}, \mathbb{F}_2)$ as $A(1)_{*}$-comodules.

For part (b), the quotient of $R'_2$ by $\Sigma^4 R'_1$ is isomorphic to $\mathbb{F}_2 \{y^2_2, y_2 y^3_3, y^2_3\}$ with $A(1)_{*}$-comodule structure given by

$\Delta(y^2_2) = 1 \otimes y^2_2$

$\Delta(y_2 y^3_3) = \xi^2_1 \otimes y^2_3 + 1 \otimes y_2 y^3_3$

$\Delta(y^3_3) = \xi^2_1 \otimes y^3_3 + 1 \otimes y^2_3$

On can check that this quotient is isomorphic to $\Sigma^{12} V_3$.

**Lemma 2.1.3.** For every $\sigma \geq 3$, there is a short exact sequence of $A(1)_{*}$-comodules

$0 \rightarrow \Sigma^4 R'_{\sigma-1} \rightarrow R'_\sigma \rightarrow \Sigma^6 V_4 \rightarrow 0$

where $V_4$ is $H_*(V(0) \wedge C_{\eta})$.

15
Lemma 2.1.5. Let us denote $C_2 \wedge C_\eta$ by $Y$.

Proof. The quotient of $R'_\sigma$ by $\Sigma^4 R'_{\sigma-1}$ is isomorphic to $\mathbb{F}_2 \{y_2^\sigma, y_2^{\sigma-1}y_3, y_2^{\sigma-2}y_3^2, y_2^{\sigma-3}y_3^3\}$ with $A(1)_\ast$-comodule structure given by

\[
\begin{align*}
\Delta(y_2^\sigma) &= 1 \otimes y_2^\sigma \\
\Delta(y_2^{\sigma-1}y_3) &= \xi_1 \otimes y_2^\sigma + 1 \otimes y_2^{\sigma-1}y_3 \\
\Delta(y_2^{\sigma-2}y_3^2) &= \xi_1^2 \otimes y_2^\sigma + 1 \otimes y_2^{\sigma-2}y_3^2 \\
\Delta(y_2^{\sigma-3}y_3^3) &= \xi_1^3 \otimes y_2^\sigma + \xi_1^2 \otimes y_2^{\sigma-1}y_3 + \xi_1 \otimes y_2^{\sigma-2}y_3^2 + 1 \otimes y_2^{\sigma-3}y_3^3
\end{align*}
\]

It can be easily seen that this quotient is isomorphic to $\Sigma^6 \sigma V_4$. \qed

Next we describe the Ext groups of some $A(1)_\ast$-comodules as basis steps towards computing $G_\sigma$. These calculations are elementary and classical. See for example ([Rav86], Chapter 3).

Proposition 2.1.4. There is an isomorphism of algebras

\[
G_0 := \text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, h_1, v, v_1^4]/(h_1^3, h_0h_1, h_1v, v^2 - h_0^2v_1^4)
\]

![Figure 1 – Ext_{A(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) in the range 0 \leq t - s \leq 8](image)

Lemma 2.1.5. As a module over $\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$,

1. $\text{Ext}_A^{s,t}(\mathbb{F}_2, H_*(V(0)))$ is generated by $h^0 \in \text{Ext}^{0,0}, h^1 \in \text{Ext}^{1,3}$ with the following relations $h_0h^0 = vh^0 = vh^1 = 0$ and $h_1^2h^0 = h_0h^1$.

2. $\text{Ext}_A^{s,t}(H_*(C))$ is generated by $\{h^i \in \text{Ext}^{i,3i} | 0 \leq i \leq 3\}$ with $h_1h^i = 0, vh^0 = h_0h^2, vh^1 = h_0h^3$.

3. $\text{Ext}_A^{s,t}(H_*(S^0 \cup_2 e \cup_\eta e^2))$ is generated by $h^0 \in \text{Ext}^{0,0}, h^1 \in \text{Ext}^{1,3}, a^1 \in \text{Ext}^{1,3}, h^2 \in \text{Ext}^{2,6}, h^3 \in \text{Ext}^{3,9}$ with $h_0h^0 = h_1h^9 = h_1h^1 = h_0a^1 = va^1 = h_1h^2 = vh^2 = h_1h^3 = vh^3 = 0$ and $h_0h^2 = h_1^2a^1$. 

16
Remark 2.1.6. We are using the same notations $h^i$ for $i = 0, 1, 2, 3$ to denote some generators of the above groups. This abuse of notation is justified by the fact that these generators have close relationships which are described in the next lemma. Of course, the context will clarify the use of the notations.

Consider cell inclusions $V(0) \to Y$ and $S^0 \cup_2 e^1 \cup_\eta e^2 \to Y$. The induced homomorphisms in Ext over $A(1)_*$ are described as

**Lemma 2.1.7.** (i) The homomorphism $\text{Ext}^*_{A(1)_*}(H_*(V(0))) \to \text{Ext}^*_{A(1)_*}(H_*(Y))$ sends each of the classes $h^0$ and $h^1$ to the non-trivial class of the same name.

(ii) The homomorphism $\text{Ext}^*_{A(1)_*}(H_*(S^0 \cup_2 e^1 \cup_\eta e^2)) \to \text{Ext}^*_{A(1)_*}(H_*(Y))$ sends each of the classes $h^0, h^1, h^2, h^3$ to the non-trivial class of the same name.

**Proof.** For part (i), consider the short exact sequence of $A(1)_*$-comodules

$$0 \to H_*(V(0)) \to H_*(Y) \to H_*(\Sigma^2 V(0)) \to 0$$

By degree reasons, the classes $h^0$ and $h^1$ of $\text{Ext}^*_{A(1)_*}(\mathbb{F}_2, H_*(V(0)))$ do not belong to the image of the connecting homomorphism $\text{Ext}^*-1_{A(1)_*}(H_*(\Sigma^2 V(0))) \to$
Therefore, they are sent to nontrivial classes of the same name in $\text{Ext}_{A(1)_*}^*(H_*(Y))$. For part (ii), consider the short exact sequence of $A(1)_*$-comodules

$$0 \to H_*(S^0 \cup_2 e^1 \cup_3 e^2) \to H_*(Y) \to \Sigma^3 F_2 \to 0$$

and the resulting long exact sequence

$$\text{Ext}_{A(1)_*}^{s,t}(H_*(Y)) \xrightarrow{\partial} \text{Ext}_{A(1)_*}^{s,t-1}(\Sigma^3 F_2) \to \text{Ext}_{A(1)_*}^{s,t}(H_*(Y))$$

By degree reason, the classes $h_0^1$, $h_2^1$, $h_3^1$ of $\text{Ext}_{A(1)_*}^{s,t}(H_*(S^0 \cup_2 e^1 \cup_3 e^2))$ are not in the image of the connecting homomorphism, and thus are sent to $h_0^1$, $h_2^1$, $h_3^1$ in $\text{Ext}_{A(1)_*}^*(H_*(Y))$, respectively. Next, by degree reasons, the classes $h_0 h_1$ and $h_1 a^1$ are sent to $0 \in \text{Ext}_{A(1)_*}^*(H_*(Y))$. The only way for this to happen is that the connecting homomorphism sends $\Sigma^3 1 \in \text{Ext}_{A(1)_*}^0(F_2, H_*(\Sigma^3 F_2))$ to the sum $h_1^1 + a^1$. It follows that $h_1^1$ is not in the image of the connecting homomorphism, and therefore is sent to $h_1^1 \in \text{Ext}_{A(1)_*}^{1,3}(F_2, H_*(Y))$.

**Lemma 2.1.8.** $H_*(Y)$ has a structure of a $A(1)_*$-comodule algebra. The resulting structure on $\text{Ext}_{A(1)_*}^{s,t}(H_*(Y))$ is that of a polynomial algebra.

**Proof.** It is not hard to see that $H_*(Y)$ is isomorphic to $A(1)_* \boxtimes_{E(1)_*} F_2$ as $A(1)_*$-comodules. In particular, $H_*(Y)$ has a structure of a $A(1)_*$-comodule algebra according to Lemma [1.0.10]. As a consequence, $\text{Ext}_{A(1)_*}^{s,t}(H_*(Y))$ is an algebra and is furthermore isomorphic to $\text{Ext}_{E(1)_*}^{s,t}(F_2)$ by the change-of-ring isomorphism. It is well-known that the latter is a polynomial algebra on one variable.

We are now computing $G_\sigma = \text{Ext}_{A(1)_*}^{s,t}(R'_1)$. We will denote by $\alpha_{s,t-s,\sigma}$ a nontrivial class of $G_\sigma$ whenever there is a unique such one.

**Proposition 2.1.9.** As a module over $G_0$, $G_1 = \text{Ext}_{A(1)_*}^{s,t}(R'_1)$ is generated by $\alpha_{0,4,1} \in \text{Ext}_{A(1)_*}^{0,4}(R'_1)$ and $\alpha_{1,8,1} \in \text{Ext}_{A(1)_*}^{1,9}(R'_1)$ with the relations $h_1 \alpha_{0,4,1} = 0$ and $v \alpha_{0,4,1} = h_0^2 \alpha_{1,8,1}$.

**Proof.** Consider the short exact sequence of $A(1)_*$-comodules

$$0 \to \Sigma^4 F_2 \to R_1 \to \Sigma^6 H_*(V(0)) \to 0$$

The connecting homomorphism

$$\partial : \text{Ext}_{A(1)_*}^{s,t-6}(V(0)) \to \text{Ext}_{A(1)_*}^{s+1,t-4}(F_2)$$
of the resulting long exact sequence sends $h^0$ to $h_1$ and $h^1$ to $0$. The latter follows from degree reasons and the former from the following map of short exact sequences of $A(1)_*$-comodules and the naturality of the connecting homomorphism

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Sigma^4 F_2 & \longrightarrow & R_1 & \longrightarrow & \Sigma^6 H_*(C'_\eta) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \Sigma^4 F_2 & \longrightarrow & H_*(\Sigma^4 C_\eta) & \longrightarrow & \Sigma^6 F_2 & \longrightarrow & 0
\end{array}
$$

It follows that $G_1$ is $v_4^1$-periodic on the following generators (Figure 6)

Figure 6 – $G_1$ with a missing $h_0$-multiplication represented by the dashed line

What remains to be established is the multiplication by $h_0$ on the generator of bidegree $(2, 8)$. This is done by a similar consideration of the connecting homomorphism associated to the short exact sequence of $A(1)_*$-comodules

$$
0 \rightarrow \Sigma^4 C_\eta \rightarrow R_1 \rightarrow \Sigma^7 F_2 \rightarrow 0
$$

**Proposition 2.1.10.** As a module over $G_0$, $\text{Ext}_{A(1)_*}^{s,s} (R'_2) = G_2$ generated by $\alpha_{s,t,2} \in \text{Ext}_{A(1)_*}^{s,s+t} (R'_2)$ where $(s, t) \in \{(0, 8), (0, 12), (1, 14), (2, 16), (3, 18)\}$ with $h_1 \alpha_{s,t,2} = 0$, $v\alpha_{0,8,2} = h_0^3 \alpha_{0,12,2}$ and $v\alpha_{0,12,2} = h_0 \alpha_{2,16,2}$ and $v\alpha_{1,14,2} = h_0 \alpha_{3,18,2}$.

**Proof.** The short exact sequence in part (a) of Lemma 2.1.2 gives rise to the long exact sequence

$$
\rightarrow \text{Ext}_{A(1)_*}^{s,t-12} (H^*(C_\eta)) \rightarrow \text{Ext}_{A(1)_*}^{s,t} (R'_2) \rightarrow \text{Ext}_{A(0)_*}^{s,t-8} (F_2) \rightarrow \text{Ext}_{A(1)_*}^{s+1,t-12} (H^*(C_\eta)) \rightarrow
$$

Combining that $\text{Ext}_{A(0)_*}^{s,t} (F_2) \cong F_2 [h_0]$ and the description of $\text{Ext}_{A(1)_*}^{s,t} (H^*(C_\eta))$, we see that the connecting homomorphism is trivial for bidegree reasons.
What remains is to establish the $v_4^1$-multiplication on the class $\alpha_{0,8,2}$ of bidegree $(0, 8)$. Consider the long exact sequence associated to the short exact sequence in part (b) of Lemma 2.1.2

$$\text{Ext}^{s-1,t}_{A(1),s}(F_2, \Sigma^{12}V_3) \xrightarrow{\partial} \text{Ext}^{s,t}_{A(1),s}(\Sigma^4R_1') \to \text{Ext}^{s,t}_{A(1),s}(F_2, R_2')$$

One can check that the class $\Sigma^4\alpha_{0,4,1} \in \text{Ext}^{s,t}_{A(1),s}(\Sigma^4R_1')$ is not in the image of $\partial$, and so is sent to $\alpha_{0,8,2} \in \text{Ext}^{s,t}_{A(1),s}(R_2')$. By degree reasons, we see that $v_4^1\Sigma^4\alpha_{0,4,1}$ is neither in the image of $\partial$, thus $v_4^1\alpha_{0,8,2}$ is nontrivial in $G_2$. This completes the proof. Even though it is not necessary, an complete effect of the connecting homomorphism can be carried out giving rise to the chart Figure 8.

\[ \text{Figure 8} - G_2 \ - \text{The red part is the contribution of } G_1 \text{ and the black one of } \text{Ext}^{s,t}_{A(1),s}(F_2, V_3). \]

\[ \text{Lemma 2.1.11. As a module over } G_0, \text{ Ext}^{s,t*}_{A(1),s}(R_3') = G_3 \text{ generated by } \alpha_{s,t,3} \text{ where } (s, t) \in \{(0, 12), (0, 16), (0, 18), (1, 20), (2, 22), (3, 24)\} \text{ with } h_1\alpha_{s,t,3} = \]

\]
Proof. The short exact sequence in Lemma 2.1.3 gives the long exact sequence
\[ \rightarrow \text{Ext}_{A(1)}^{s,t}(\Sigma^4 R'_2) \rightarrow \text{Ext}_{A(1)}^{s,t}(R'_3) \rightarrow \text{Ext}_{A(1)}^{s,t}(\Sigma^18 V_4) \rightarrow \text{Ext}_{A(1)}^{s+1,t}(F_2, \Sigma^4 R'_2) \rightarrow \]

For degree reasons, the connecting homomorphism is trivial, hence we obtain the additive structure of $G_3$ as in Figure 11. We need to establish the non-trivial $h_0$-multiplication on the generators $\{\alpha_{s,18+2s,3} \mid s \geq 0\}$. Taking the $v^1$-periodicity into account, we reduce to show this property for the generators of $\alpha_{0,18,3}, \alpha_{1,20,3}, \alpha_{2,22,3}, \alpha_{3,24,3}$.

![Figure 9 - $G_3$ - The red part is the contribution of $G_2$ and the black one of $\text{Ext}_{A(1)}^{s,t}(Y)$](image)

For this, we can check that there are the following short exact sequences:
\[ 0 \rightarrow \Sigma^{18} H_*(C_\eta) \rightarrow R_3 \rightarrow R_3/\Sigma^{18} H_*(C_\eta) \rightarrow 0 \]

and
\[ 0 \rightarrow \Sigma^4 R_2 \rightarrow R_3/\Sigma^{18} H_*(C_\eta) \rightarrow \Sigma^{19} H_*(C_\eta) \rightarrow 0 \]

where as a sub $A(1)_*$-comodule of $R_3$, $\Sigma^{18} H_*(C_\eta)$ is equal to $\mathbb{F}_2\{y_1 y_3^2 + y_2, y_2 y_3^2\}$ and the map $\Sigma^4 R_2 \rightarrow R_3/\Sigma^{18} C_\eta$ is the composite $\Sigma^4 R_2 \xrightarrow{xy_1^3} R_3 \rightarrow R_3/\Sigma^{18} H_*(C_\eta)$. As a consequence, $\text{Ext}_{A(1)}^*(R_3/\Sigma^{18} H_*(C_\eta))$ sits in a long exact sequence
\[ \text{Ext}_{A(1)}^{s-1,t}(\mathbb{F}_2, \Sigma^{19} H_*(C_\eta)) \xrightarrow{\partial} \text{Ext}_{A(1)}^{s,t}(\Sigma^4 R_2) \rightarrow \text{Ext}_{A(1)}^{s,t}(R_3/\Sigma^{18} H_*(C_\eta)) \rightarrow \]

Since $\partial$ is $G_0$-linear, one only needs to compute $\partial$ on the two generators of $\text{Ext}_{A(1)}^{0,19}(\mathbb{F}_2, \Sigma^{19} H_*(C_\eta))$ and $\text{Ext}_{A(1)}^{1,21}(\mathbb{F}_2, \Sigma^{19} H_*(C_\eta))$. Direct computations show...
that $\partial$ act non-trivially on these classes. It follows that $\partial$ is a monomorphism and so $\text{Ext}^s_{A(1)_*}(R_3/\Sigma^{18}H_*(C_\eta))$ is $v_1$-free on the generators depicted in Figure 10. It follows immediately from the exact sequence

$$0 \to \Sigma^{18}H_*(C_\eta) \to R_3 \to R_3/\Sigma^{18}H_*(C_\eta) \to 0$$

that $\text{Ext}^s_{A(1)_*}(R_3)$ is depicted in Figure 11. In particular, missing $h_0$-extensions are established.

Figure 10 – $\text{Ext}^s_{A(1)_*}(R_3/\Sigma^{18}H_*(C_\eta))$

$$\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
\end{array}$$

Figure 11 – $G_3$ -The red part is the contribution of $\text{Ext}^s_{A(1)_*}(R_3/\Sigma^{18}H_*(C_\eta))$ and the black one of $\text{Ext}^s_{A(1)_*}(\Sigma^{18}H_*(C_\eta))$

\[\begin{array}{cccccccccccccccc}
5 & 4 & 3 & 2 & 1 & 0 \\
12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
\end{array}\]

\[\square\]

**Theorem 2.1.12.** As a module over $G_0$, we have

(a) For every $\sigma \geq 2$, $\text{Ext}^s_{A(1)_*}(R_\sigma) = G_\sigma$ generated by $\alpha_{s,t,\sigma} \in \text{Ext}^{s,t+s}_{A(1)_*}(R_\sigma)$ where $(s, t) \in \{(0, 4\sigma), (0, 2j + 4\sigma) \mid 2 \leq j \leq \sigma, (j, 6\sigma + 2j) \mid 1 \leq j \leq 3\}$ with $h_1\alpha_{s,t,\sigma} = 0$.

(b) For all pairs of triples $(s_1, t_1, \sigma_1)$ and $(s_2, t_2, \sigma_2)$ with $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$ except for $(2, 9, 1)$ and $(3, 10, 1)$, we have that

$$\alpha_{s_1,t_1,\sigma_1}\alpha_{s_2,t_2,\sigma_2} = \alpha_{s_1,s_2,t_1+t_2,\sigma_1+\sigma_2}$$
Proof. (a) The statement for $\sigma = 2$ is Lemma 2.1.10. Let us prove the claim for $\sigma \geq 3$ by induction. The basis step is Lemma 2.1.11.

Suppose the claim is true for some $\sigma \geq 3$. The long exact sequence associated to the short exact sequence in Lemma 2.1.3 reads

$$\to \text{Ext}^{s,t}_{A(1)_*}(\Sigma^4 R'_\sigma) \to \text{Ext}^{s+1,t}_{A(1)_*}(\Sigma^4 R'_\sigma) \to \text{Ext}^{s,t}_{A(1)_*}(\Sigma^6 R'_\sigma) \to$$

Combining the additive structure of $\text{Ext}^{s,t}_{A(1)_*}(\Sigma^4 R'_\sigma)$ and that

$$\text{Ext}^{s,t}_{A(1)_*}(\Sigma^6 R'_\sigma) \cong \Sigma^6 F_2[v_1]$$

we obtain the additive structure of $G_{\sigma+1}$ as described in the lemma because the connecting homomorphism vanishes for degree reasons. To establish the non-trivial $h_0$-multiplication on the generators $\{\alpha_{s,2s+6\sigma+6,s+1} \mid s \geq 0\}$, we need a priori the following identities

1. $G_{\sigma+1} \ni \alpha_{0,4,1,0,2s+6d,\sigma} \neq 0 \forall \sigma \geq 1$
2. $\alpha_{1,8,1,0,2s+6\sigma-6,\sigma-1} = \alpha_{s+1,2s+6\sigma+2,\sigma} \forall \sigma \geq 2$
3. $\alpha_{0,12,2,0,2s+6\sigma-6,\sigma-1} = \alpha_{s,2s+6\sigma+6,\sigma+1} \forall \sigma \geq 3$

These identities are content of part (b). For the sake of presentation, we postpone the proof to (b). This is legitimate because as we will see the proof of (b) only serves the additive structure of $G'_\sigma s$. Let us show how these identities allows us to conclude the proof of (a). Indeed, the classes $\sigma_{s,2s+6\sigma-6,\sigma-1}$ exist (non-trivial) for all $\sigma \geq 3$ and $s \geq 0$. Therefore, we have that for all $\sigma \geq 3$

$$h_0 \alpha_{s,2s+6\sigma+6,s+1} = h_0 \alpha_{0,12,2,0,2s+6\sigma-6,\sigma-1} \text{ (multiplying both sides of (iii) by } h_0)$$

$$= \alpha_{0,4,1,0,1,8,1,0,2s+6\sigma+6,\sigma-1} = \alpha_{s+1,2s+6\sigma+2,\sigma} \text{ (because of (i))}$$

$$= \alpha_{0,4,1,0,1,8,1,0,2s+2+6\sigma,\sigma} \text{ (because of (ii))}$$

$$\neq 0 \text{ (because of (i))}$$
(b) For every $\alpha, \tau \geq 1$, there is a commutative diagram of $A(1)_*\text{-comodules}$

\[
\begin{array}{ccc}
R'_\alpha \otimes R'_\tau \quad \xrightarrow{\mu} \quad R_{\sigma+\tau} \\
\downarrow \quad \downarrow \quad \downarrow \\
H_* (\Sigma^{6\sigma} X_\sigma) \otimes H_* (\Sigma^{6\tau} X_\tau) \quad \xrightarrow{\mu} \quad H_* (\Sigma^{6\sigma+6\tau} X_{\sigma+\tau}) \\
\downarrow \quad \downarrow \\
H_* (\Sigma^{6\sigma} Y) \otimes H_* (\Sigma^{6\tau} Y) \quad \xrightarrow{\mu} \quad H_* (\Sigma^{6\sigma+6\tau} Y)
\end{array}
\]

Let us explain maps in this diagram. The spectrum $X_\sigma$ is $V(0), S^0 \cup_1 e^1 \cup_2 e^2$ or $Y$ if $\sigma = 1, 2$ or $\sigma > 2$ respectively; and in each case the map $R'_\sigma \to H_* (X_\sigma)$ is the projection appearing in the proof of Lemma 2.1.9, Lemma 2.1.2 or Lemma 2.1.3 respectively. The other vertical arrows are inclusions of $X_\sigma$ into $Y$. The bottom horizontal arrow is multiplication on $H_* (Y)$ described in Lemma 2.1.8 and the medium one is induced by the latter. The second upper arrow is the projection on the factor $R'_{\sigma+\tau}$ of the decomposition in Lemma 2.1.1.

The induced homomorphisms in Ext over $A(1)_*$ of all vertical arrows are studied in the proof of Lemmas 2.1.9, 2.1.10, 2.1.12 and Lemma 2.1.7 according to which the classes $\alpha_{s,t,\sigma}$ are sent non-trivially in a unique way to $\text{Ext}_{A(1)_*}^s (H_* (Y))$, hence their products are non-trivial by Lemma 2.1.8. This proves (b).

**Remark 2.1.13.** Let us summarise what has been done so far. First, Lemma 2.1.1 implies that

\[ \text{Ext}_{A(1)_*}^* (R) \cong \bigoplus_{i \geq 0} G_i \otimes F_2[v^4] \]

where $v^4 \in \text{Ext}^{4,28} (F_2, R_4)$ represented by $y^4_2$. Next, Lemma 2.1.12 describes completely products between $G_i$’s module the ideal generated by $(v^4_2)$. It is then straightforward verify that the $\text{Ext}_{A(1)_*}^* (R)$ is generated by the classes of

\[ h_0, h_1, v, v^4_1, \alpha_{0,4,1}, \alpha_{1,8,1}, \alpha_{0,12,2}, \alpha_{1,14,2}, \alpha_{3,18,2}, \alpha_{0,18,3}, v^4_2 \] (9)

Let us describe the subalgebra of primitives.

**Corollary 2.1.14.** There is the following isomorphism of graded algebras

\[ \text{Ext}_{A(1)_*}^0 (R) \cong F_2 [\alpha_{0,4,1}, \alpha_{0,12,2}, v^4_2, \alpha_{0,18,3}] / (\alpha_{0,18,3}^2 = \alpha_{0,12,2}^3 + \alpha_{0,4,1}^2 v^4_2) \]
Proof. The group $\text{Ext}_{A(1)_*}^0(\mathbb{F}_2, R)$ is naturally identified with a subalgebra of $R = \mathbb{F}_2[y_1, y_2, y_3]$. Through this identification, $\alpha_{0,4,1}, \alpha_{0,12,2}, v_4^1, \alpha_{0,18,3}$ become $y_1, y_2^2 + y_1 y_3^2$, respectively. Thus $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_4^1, \alpha_{0,18,3}] / (\alpha_{0,18,3} = \alpha_{0,12,2} + \alpha_{0,4,1} v_4^1)$ is isomorphic to the subalgebra of $\text{Ext}_{A(1)_*}^0(\mathbb{F}_2, R)$ generated by $\alpha_{0,4,1}, \alpha_{0,12,2}, v_4^1, \alpha_{0,18,3}$. On the other hand, it follows from Remark (2.1.13) that $\alpha_{0,4,1}, \alpha_{0,12,2}, v_4^1, \alpha_{0,18,3}$ generate the whole subalgebra of primitives of $\text{Ext}_{A(1)_*}^0(\mathbb{F}_2, R)$. This concludes the proof of the lemma. □

The differentials $d_1$. Since the DMSS for $\mathbb{F}_2$ is a spectral sequence of algebras, all $d_1$-differentials can be determined on the set of algebra generators of $(9)$.

Proposition 2.1.15. There are following $d_1$-differentials

1) $d_1(h_0) = 0$
2) $d_1(h_1) = 0$
3) $d_1(\alpha_{0,4,1}) = 0$
4) $d_1(\alpha_{1,14,2}) = 0$
5) $d_1(\alpha_{0,18,3}) = 0$
6) $d_1(v_4^1) = 0$
7) $d_1(\alpha_{0,12,2}) = \alpha_{0,4,1}^3$
8) $d_1(\alpha_{1,8,1}) = h_0 \alpha_{0,4,1}^2$
9) $d_1(v) = h_0^3 \alpha_{0,4,1}$
10) $d_1(\alpha_{3,18,2}) = h_0^3 \alpha_{0,18,3}$
11) $d_1(v_2^3) = \alpha_{0,4,1} \alpha_{0,12,2}$

Proof. 1), 2), 4) By degree reasons, there is no room for a non-trivial $d_1$-differential on $h_0, h_1, \alpha_{1,14,2}$

3) It is easy to see that $\text{Ext}_{A(2)_*}^{1,4}(\mathbb{F}_2, \mathbb{F}_2)$ is non-trivial and that $\alpha_{0,4,1}$ is the only class in the $E_1$-term that can contributes to it. Therefore $\alpha_{0,4,1}$ is a permanent cycle.

5) We see that $h_0 \alpha_{0,18,3} = \alpha_{0,4,1} \alpha_{1,14,2}$. By Leibniz rule, $h_0 d_1(\alpha_{0,18,3}) = 0$. As $h_0$ acts injectively on $G_3$, it follows that $d_1(\alpha_{0,18,3}) = 0$.

6) Since $h_0^3 v_4^1 = v^2$, $h_0 d_1(v_4^1) = 0$. 6) follows because $d_1(v_4^1)$ takes value in $\text{Ext}_{A(1)_*}^{4,8}(\mathbb{F}_2, R_3^1)$ on which $h_0$ acts injectively.

7) We have $\alpha_{0,12,2}$ is represented by the $A(2)$-primitive $[1|y_2^2] + [x_1|y_1^2] \in E \otimes R_2$. By Remark (1.0.6), $d_1(\alpha_{0,12,2})$ is represented by $d([1|y_2^2] + [x_1|y_1^2]) = [1|y_1^3] \in E \otimes R_3$, hence is equal to $\alpha_{0,4,1}^3$. 25
the spectral sequence converging to elements of Ext higher differentials. At least, the classes $\alpha$ It turns out that the DMSS collapses at the bidegrees. Following [DFHH14], those elements are denoted by sequence of a $d$ respectively. Furthermore, $h$ implies the relation $\beta$ Therefore, $d_1(\alpha_{3,18,2}) = h_0^3\alpha_{0,18,3}$. Therefore, $d_1(\alpha_{3,18,2}) = h_0^3\alpha_{0,18,3}$.

11) We check that $v_2^3$ is represented by the $A(2)$-primitive $[1|y_3^1] + [x_1|y_2^1]$ in $E \otimes R_4$. By Remark 1.0.6, $d_1(v_2^3)$ is represented by $[1|g_1y_2^1]$, hence is equal to $\alpha_{0,4,1}\alpha_{0,12,2}^2$.

It turns out that the DMSS collapses at the $E_2$-term because there is no room for higher differentials. At least, the classes $\alpha_{1,14,2}, \alpha_{0,4,1}, \alpha_{0,12,2}, v_2, \alpha_{0,18,3}$ survive the spectral sequence converging to elements of $\text{Ext}_{A(2)_*}^*(\mathbb{F}_2, \mathbb{F}_2)$ in appropriate bidegrees. Following [DFHH14], those elements are denoted by $\alpha, h_2, g, w_2, \beta$, respectively. Furthermore, $h_2, g, w_2, \beta$ generate a subalgebra of $\text{Ext}_{A(2)_*}^*(\mathbb{F}_2, \mathbb{F}_2)$ isomorphic to $\mathbb{F}_2[h_2, g, w_2, \beta]/(h_2^4, h_2g, \beta^3 - g^3)$. The relation $\beta^4 = g^3$ is a consequence of a $d_1$-differential. In effect, the relation $\alpha_{0,18,3}^2 = \alpha_{0,12,2}^2 + \alpha_{0,4,1}^2v_2^3$ implies the relation $\beta^4 - g^3 - h_2^4w_2 = 0$ in $\text{Ext}_{A(2)_*}^*(\mathbb{F}_2, \mathbb{F}_2)$. But $\alpha_{0,4,1}^2v_2^3$ gets hit by the differential $d_1(v_2^3\alpha_{0,4,1}\alpha_{0,12,2}) = v_2^3\alpha_{0,4,1}d_1(\alpha_{0,12,2}) = v_2^3\alpha_{0,4,1}^2$ Thus the relation $\beta^4 = g^3 + h_2^4w_2$ becomes $\beta^4 = g^3$.

2.2 Davis-Mahowald spectral sequence for A(1)

Review on $A(2)_*$-comodule structure of $A(1)$ In [DM81], Davis and Mahowald constructed four type 2 finite spectra whose mod 2 cohomology are isomorphic to a free module of rank one over the subalgebra $A(1) = \langle Sq^1, Sq^2 \rangle$ of the Steenrod algebra. Let us review the construction of these spectra and their module structure over the subalgebra $A(2) = \langle Sq^1, Sq^2, Sq^4 \rangle$ of the Steenrod algebra $A$. Recall
Figure 13 – Diagram of $H^*(Y)$: the straight lines represent $Sq^1$ and the curved lines represent $Sq^2$, the numbers represent the degree of the cell

that $Y$ is $V(0) \wedge C_n$. The $A$-module structure of $H^*(Y)$ is depicted in Figure 13. An element of $\text{Ext}^{1,3}_{A(1)}(H^*(Y), H^*(Y))$ can be represented by an $A(1)$-module $M$ sitting in a short exact sequence of $A(1)$-modules

$$0 \to H^*(\Sigma^3 Y) \to M \to H^*(Y) \to 0$$

It can be checked that $M$ must be isomorphic either to $H^*(\Sigma^3 Y) \oplus H^*(Y)$ or to $A(1)$ as an $A(1)$-module. It means that

$$\text{Ext}^{1,3}_{A(1)}(H^*(Y), H^*(Y)) \cong \mathbb{Z}/2$$ (10)

The $A(1)$-module structure of $A(1)$ is depicted in Figure 14. One can ask whether

Figure 14 – Diagram of $A(1)$: The straight lines represent $Sq^1$, the curved lines $Sq^2$, the numbers represent the degree of the class

$A(1)$ admits a structure of $A(2)$-module. If such a structure exists, then according to the Adem relations $Sq^2 Sq^1 Sq^2 = Sq^4 Sq^1 + Sq^1 Sq^4$, there must be a nontrivial action of $Sq^4$ on the nontrivial class of dimension 1. It is straightforward to verify that the latter is the only constraint to put an $A(2)$-module structure on $A(1)$. There are also possibilities for $Sq^4$ to act nontrivially on the class of dimension 0 and 2. These give in total four different $A(2)$-module structures on $A(1)$. In other words, the inclusion of Hopf algebras $A(1) \hookrightarrow A(2)$ induces a surjective homomorphism

$$\text{Ext}^{1,3}_{A(2)}(H^*(Y), H^*(Y)) \to \text{Ext}^{1,3}_{A(1)}(H^*(Y), H^*(Y))$$

whose kernel contains 4 element. Therefore,

$$\text{Ext}^{1,3}_{A(2)}(H^*(Y), H^*(Y)) \cong \mathbb{Z}/2 \oplus 3^{27}$$

27
Next, one observes that

$$\text{Ext}^{1,3}_{A}(H^*(Y), H^*(Y)) \cong \text{Ext}^{1,3}_{A(2)}(H^*(Y), H^*(Y))$$

because for any $A$-module $M$ sitting in a short exact sequence

$$0 \to H^*(\Sigma^3 Y) \to M \to H^*(Y) \to 0$$

there can not be any non-trivial $Sq^k$ for $k \geq 8$ on $M$. In fact, it is proved in [DM81] that four classes of $\text{Ext}^{1,3}_{A}(H^*(Y), H^*(Y))$ that are sent to the unique non-trivial class of $\text{Ext}^{1,3}_{A(1)}(H^*(Y), H^*(Y))$ are permanent cycles in the Adams spectral sequence and converge to four $v_1$-self-maps of $Y$, i.e., maps $\Sigma^2 Y \to Y$ that induce isomorphisms in $K(1)$-homology theory. As a consequence, the cofibers of these $v_1$-selfmaps realize the four different $A$-module structures on $A(1)$. Furthermore, these four complexes exhaust all homotopy types of finite complexes having mod 2 cohomology isomorphic to $A(1)$. Following [BEM17], we denotes by $A_1[i,j]$, $i, j \in \{0, 1\}$ the model of $A(1)$ having the non-trivial $Sq^4$ on the generator of degree 0 and 2 if and only if $i = 1$ and $j = 1$ respectively. We will use $A(1)$ if we want to refer to any of the models of $A(1)$. As a $\mathbb{F}_2$-vector spaces,

$$H_*(A_1[i,j]) \cong \mathbb{F}_2\{a_0, a_1, a_2, a_3, \overline{a_3}, a_4, a_5, a_6\}$$  \hspace{1cm} (11)

Here, $a_0, a_1, a_2, a_4, a_5, a_6$ are duals to the generators of degree 0, 1, 2, 4, 5, 6 of $H^*(A_1[i,j])$, respectively and $a_3, \overline{a_3}$ are duals to the images of the generator of degree 0 by $Sq^3, Sq^3 + Sq^3 Sq^1$, respectively. From now on, we denote by $A_1[i,j]$ the homology of $A_1[i,j]$. By taking duals to the action of $A(2)$ on $A_1[i,j]$, we obtain

**Proposition 2.2.1.** The left coaction of $A(2)_*$ on $A_1[i,j]$ is given by

i) $\Delta(a_1) = [1]a_1 + [\xi_1]a_0$

$\Delta(a_2) = [1]a_2 + [\xi_2]a_1$

$\Delta(a_3) = [1]a_3 + [\xi_1 a_2] + [\xi_2]a_1 + [\xi_1]a_0$

$\Delta(a_4) = [1]a_4 + [\xi_1 a_3] + [\xi_2]a_2 + [\xi_1]a_1 + [\xi_2]a_0$

$\Delta(a_5) = [1]a_5 + [\xi_2 a_3] + [\xi_1]a_2 + [\xi_2]a_1 + [\xi_2]a_0$

$\Delta(a_6) = [1]a_6 + [\xi_1 a_5] + [\xi_2]a_4 + [\xi_2]a_3 + [\xi_1]a_3 + [\xi_2]a_2 + [\xi_2]a_1 + [\xi_2]a_0$

$\beta_{i,j} = [\xi_2 a_2] + [\xi_2]a_2 + \beta_{i,j} [\xi_1]a_1 + \gamma_{i,j} [\xi_1 a_0] + [\xi_2 a_1] + [\xi_2]a_0 + \lambda_{i,j} [\xi_2]a_0$

ii) In the above formulas,

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } (i,j) \in \{(0,0), (0,1)\} \\ 1 & \text{if } (i,j) \in \{(1,0), (1,1)\} \end{cases}$$

$$\beta_{i,j} = \begin{cases} 0 & \text{if } (i,j) \in \{(0,0), (0,1)\} \\ 1 & \text{if } (i,j) \in \{(1,0), (1,1)\} \end{cases}$$

28
\[ \gamma_{i,j} = 1 + \alpha_{i,j} \]

and
\[ \lambda_{i,j} = \alpha_{i,j} + \beta_{i,j} \]

\( \alpha \) being 0 for \( A_1[00] \) and \( A_1[01] \); being 1 for \( A_1[10] \) and \( A_1[11] \)

\( \beta \) being 1 for \( A_1[01] \) and \( A_1[11] \); being 0 for \( A_1[00] \) and \( A_1[10] \)

\( \gamma \) being 1 for \( A_1[00] \) and \( A_1[01] \); being 0 for \( A_1[10] \) and \( A_1[11] \)

\( \lambda \) being 1 for \( A_1[10] \) and \( A_1[01] \); being 0 for \( A_1[00] \) and \( A_1[11] \)

Proof. The proof is a straightforward translation from \( A(2) \)-module structure to \( A(2) \)\(^*\)-comodule structure using the formula of the duals of the Milnor basis in \cite{Mil58}.

DMSS for \( A(1) \) In what follows, we will apply in many places the shearing homomorphism that we implicitly used in the proof of Proposition 2.1.15 to find primitives representing certain cohomology classes. It is useful to recall it here.

In general, let \( C \) be a Hopf algebra with conjugation \( \chi \) and \( B \) be Hopf-algebra quotient of \( C \). Given a \( C \)-comodule \( M \), consider the composite
\[
C \otimes M \xrightarrow{id \otimes \Delta} C \otimes C \otimes M \xrightarrow{id \otimes \chi \otimes id} C \otimes C \otimes M \xrightarrow{\psi \otimes id} C \otimes M
\]

When restricting to \( C \Box_B M \), this composite factors through \((C \Box_B k) \otimes M\) inducing the shearing isomorphism of \( C \)-comodules
\[
Sh : C \Box_B M \rightarrow (C \Box_B k) \otimes M
\]

Here \( C \) coacts on \( C \Box_B M \) via the left factor and on \((C \Box_B k) \otimes M\) diagonally. Combining with the change-of-ring isomorphism, the shearing isomorphism gives
\[
\text{Ext}^*_B(k, M) \cong \text{Ext}^*_C(k, C \Box_B M) \cong \text{Ext}^*_C(k, (C \Box_B k) \otimes M)
\]

In particular, via these isomorphism a class \( x \in \text{Ext}^0_B(k, M) \) is sent to \( Sh(1 \otimes x) \).

Proposition 2.2.2. The \( E_1 \)-term of the Davis-Mahowald spectral sequence converging to \( \text{Ext}^{s,t}_{A(2)}(A(1)) \) is given by
\[
E_1^{s,\sigma,*} \cong \begin{cases} 
0 & \text{if } s > 0 \\
R_\sigma & \text{if } s = 0
\end{cases}
\]

As a module over \( \mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2]\)(\( \subset \text{Ext}^{*,*}_{A(1)}(R) \)), \( E_1^{s,r,t} \) is free of rank 8 on the following generators of
\[
1, y_3, y_5^2, y_6^3, y_7, y_2 y_3, y_2 y_5^2, y_2 y_6^3
\]

\( (12) \)
Proof. In effect, $E^{s,t\sigma}_{2}$ is equal to $\text{Ext}^{s,t\sigma}_{A(1)_{s}}(R^\sigma \otimes A(1))$ by definition. The coaction of $A(1)_{s}$ on $R^\sigma \otimes A(1)$ is the usual diagonal coaction on tensor products. In addition, $A(1)$ is isomorphic to $A(1)_{s}$ as $A(1)_{s}$-comodules. By the change-of-ring isomorphism, we obtain that

$$\text{Ext}^{s,t\sigma}_{A(1)_{s}}(\mathbb{F}_2, R^\sigma \otimes A(1)) \cong \text{Ext}^{s,t\sigma}_{\mathbb{F}_2}(\mathbb{F}_2, R^\sigma) \cong R^\sigma$$ (13)

The first part of the proposition follows immediately. For the second part, the action of $\text{Ext}^{s,t\sigma}_{A(1)_{s}}(R)$ on $E^{s,t\sigma}_{2}$ is induced by the multiplication on $R$:

$$R \otimes (R \otimes A(1)) \to R \otimes A(1)$$

Now let $r \in \text{Ext}^{0,s\sigma}_{A(1)_{s}}(\mathbb{F}_2, R) \subset R$ and $s \in R \cong \text{Ext}^{0,s\sigma}_{A(1)_{s}}(\mathbb{F}_2, R \otimes A(1))$. By applying the shearing isomorphism, the class $s$ is represented by a unique element of the form $s \otimes a_0 + \sum s_i \otimes a_i \in R \otimes A(1)$ where $a_i$ are in positive degrees. The action of $r$ on $s$ is then represented by $rs \otimes a_0 + \sum rs_i \otimes a_i$ which represents $rs \in R \cong \text{Ext}^{s,t\sigma}_{A(1)_{s}}(\mathbb{F}_2, R \otimes A(1))$ via (13). Equivalently, the action of $\text{Ext}^{0,s\sigma}_{A(1)_{s}}(\mathbb{F}_2, R)$ on $\text{Ext}^{0,s\sigma}_{A(1)_{s}}(\mathbb{F}_2, R \otimes A(1))$ is given by the multiplication of the polynomial algebra $R$. The proof follows from the fact that $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4]$ is identified with the subalgebra of $R$ generated by $y_1, y_2^2, y_3^4$. \hfill \qed

Let us analyse the differentials in this spectral sequence. As the $d_r$-differentials decrease $s$-filtration by $r - 1$, i.e., $d_r : E^{s,t\sigma}_{r} \to E^{s-r+1,t\sigma+r}_{r}$ and $E^{s,t\sigma}_{1} = 0$ if $s > 0$, the spectral sequence collapses at the $E_2$-term and there are no extension problems. Therefore,

$$E^{s,t\sigma}_{2} \cong \text{Ext}^{s,t\sigma}_{A(2)_{s}}(\mathbb{F}_2, A(1))$$

We can now turn our attention to the $d_1$-differentials. As all elements of the $E_1$-term are in $\text{Ext}^{0,s\sigma}_{A(1)_{s}}(E \otimes R \otimes A(1))$, we can apply the remark after Proposition 1.0.5. We have already determined $d_1$-differentials on classes $\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4$ in Proposition 2.1.15. By Leibniz rule, it remains to determine $d_1$-differential on the classes of (12).

**Proposition 2.2.3.** There are following $d_1$-differentials

1) $d_1(1) = 0$
2) $d_1(y_2) = 0$
3) $d_1(y_3) = 0$
4) $d_1(y_2y_3) = 0$

30
5) \( d_1(y_2y_3^2) = 0 \)
6) \( d_1(y_2y_3^3) = 0 \)
7) \( d_1(y_3^2) = \alpha^2_{0,4,1}y_2 \)
8) \( d_1(y_3^3) = \alpha^2_{0,4,1}y_2y_3 \)

Proof. Parts 1–4 follow from the sparseness of the E₁-term.

5) The only nontrivial \( d_1 \)-differential that \( y_2y_3^2 \) can support is

\[
d_1(y_2y_3^2) = \alpha^2_{0,4,1}y_0,12,2 \quad 1
\]

However,

\[
d_1(\alpha^2_{0,4,1}y_0,12,2) = \alpha^2_{0,4,1}d_1(\alpha_0,12,2) = \alpha^5_{0,4,1}1 \neq 0
\]

This means that \( \alpha^2_{0,4,1}y_0,12,2 \) is not a \( d_1 \)-cycle, and so can not be hit by a \( d_1 \)-differential. Therefore,\( y_2y_3^2 \) is a \( d_1 \)-cycle.

6) Similarly, a nontrivial \( d_1 \)-differential on \( y_2y_3^3 \) should be

\[
d_1(y_2y_3^3) = \alpha^2_{0,4,1}y_0,12,2y_3
\]

But,

\[
d_1(\alpha^2_{0,4,1}y_0,12,2y_3) = \alpha^5_{0,4,1}y_2 \neq 0
\]

by Leibniz rule. Thus, \( y_2y_3^3 \) is a \( d_1 \)-cycle.

7–8) We can prove these two differentials by completing \([1[y_i]a_0] i \in \{2,3\}\) to \( A(2)_1 \)-comodule primitives in \( E_2 \otimes R_2 \otimes A(1) \) and then applying the differential of the complex \( (E_2 \otimes R_2, d) \). For example, \([1[y_3]a_0] + [x_2[y_i]a_0] + [x_1[y_i^2]a_2] + [1[y_3^2]a_2] \) is a \( A(2)_1 \)-primitive representing \( y_3^2 \). Then \( d_1(y_3^2) \) is represented by \([1[y_3^2]a_0] + [1[y_3]a_2] \) which represents \( \alpha^2_{0,4,1}y_2 \). An \( A(2)_1 \)-primitive representing \( y_3^3 \) contains even more terms. To avoid this tedious primitive completion procedure, it suffices to prove that \( \nu^2 y_2 = 0 \) and \( \nu^2 y_2 y_3 = 0 \) in \( \text{Ext}^*_{A(2)_1}(A(1)) \) because differentials in part 7) and 8) are the only possibilities for the latter to occur. We will proceed using juggling formulas for Massey products. In effect, the classes 1 and \( y_3 \) being permanent cycles by part 1) and part 3), they converge to classes in \( \text{Ext}^*_{A(2)_1}(A(1)) \) and \( \text{Ext}^*_{A(2)_1}(A(1)) \), respectively. By sparseness even at the level of the \( E_1 \)-term of the DMSS, \( \eta_1 = \eta y_3 = 0 \). Hence the Massey product \( \langle \nu, \eta, y_3^i \rangle \) with \( i \in \{0, 1\} \) can be formed. We have that

\[
\nu^2 y_3^i = \langle \eta, \nu, \eta \rangle y_3^i = \eta \langle \nu, \eta, y_3^i \rangle
\]

By sparseness of the DMSS, \( \alpha^2_{0,4,1}y_3 \) survives the DMSS and so \( \nu^2 y_3^i \neq 0 \). It follows that \( \langle \nu, \eta, y_3^i \rangle \) is nontrivial and must be equal to \( y_2y_3^i \). The fact that \( \nu^3 = \)}
$0 \in \text{Ext}_{A(2)}^{3,12}(F_2)$ allows us to do the following juggling

$$\nu^2 y_2 y_3^i = \nu^2 \langle \nu, \eta, y_3^i \rangle = \langle \nu^2, \nu, \eta \rangle y_3^i$$

However, the Massey product $\langle \nu^2, \nu, \eta \rangle$ lives in the group $\text{Ext}_{A(2)}^{3,14}(F_2)$ which can be checked to be zero. This concludes the proof of part 7) and 8).

**$E_2$-term of the Adams SS.** We describe $\text{Ext}_{A(2)}^{*,*}(F_2, A(1))$ as a module over $F_2[h_2, g, v_2^8]/(h_2^3, h_2 g) \subset \text{Ext}_{A(2)}^{*,*}(F_2, F_2)$ in terms of generators and relations. We will denote by $e[s, t]$ where $s, t \in \mathbb{N}$ the unique non-trivial class belonging to $\text{Ext}_{A(2)}^{s+t}(F_2, A(1))$.

**Proposition 2.2.4.** As a module over $F_2[h_2, g, v_2^8]/(h_2^3, h_2 g)$, $\text{Ext}_{A(2)}^{*,*}(F_2, A(1))$ is generated by the following generators

| $e[0,0]$ | $e[1,5]$ | $e[1,6]$ | $e[2,11]$ |
|----------|----------|----------|----------|
| $1$      | $y_2$    | $y_3$    | $y_2 y_3$|
| $(0)$    | $(h_2^2)$| $(0)$    | $(h_2^2)$|

| $e[3,15]$ | $e[3,17]$ | $e[4,21]$ | $e[4,23]$ |
|-----------|-----------|-----------|-----------|
| $y_2^i + y_1 y_2^j$ | $y_2 y_3^i$ | $y_1 y_3^i + y_2 y_3^j$ | $y_2 y_3^i$|
| $(h_2^2)$ | $(0)$     | $(h_2^2)$ | $(0)$     |

| $e[6,30]$ | $e[6,32]$ | $e[7,36]$ | $e[7,38]$ |
|-----------|-----------|-----------|-----------|
| $y_2^i + y_1 y_2^j y_3$ | $y_2 y_3^i + y_1 y_2 y_3^j$ | $y_2 y_3^i + y_1 y_3 y_2 y_3^j$ | $y_2 y_3^i + y_1 y_2 y_3 y_2^j$|
| $(h_2)$  | $(h_2)$  | $(h_2)$  | $(h_2)$  |

| $e[8,42]$ | $e[9,47]$ | $e[9,48]$ | $e[10,53]$ |
|-----------|-----------|-----------|-----------|
| $y_2^i y_3^j + y_1 y_2^i y_3^j + y_1 y_2 y_3^j$ | $y_2 y_3^i + y_1 y_2 y_3^j$ | $y_2 y_3^i + y_1 y_2 y_3^j + y_1 y_2 y_3$ | $y_2 y_3^i + y_1 y_2 y_3$|
| $(h_2)$  | $(h_2)$  | $(h_2)$  | $(h_2)$  |

The second row in the table indicates a representative in the DMSS and the third row the annihilator ideal of the corresponding generator.

**Proof.** As a corollary of Proposition 2.2.2, the $E_1$-term of the DMSS for $A(1)$ is isomorphic to a free module of rank 32 over $F_2[h_2, g, v_2^8]$. In particular, these 32 generators are $h_2$-free. It turns out that one can choose these 32-generators in such a way that there are exactly 16 $h_2$-free towers that truncate 16 others by $d_1$-differentials. The question is how one can identify these 16 $d_1$-cycles. For this, we can compute $d_1$ on 32 module generators of the $E_1$-term, say $\{y_2^i y_3^j|0 \leq i \leq 3, 0 \leq j \leq 7\}$. Some of them are $d_1$-cycles, for example $y_2, y_3$. Whereas, some of them are not $d_1$-cycles at first, but become one after being added by a multiple of $h_2$, for example $a_{0,12} y_2 + h_2 y_3^2 = y_2^3 + y_1 y_3^3$. This procedure is straightforward but lengthy, so we omit to present details here. It can be checked that the generators listed in the table are $d_1$-cycles. Finally, since $g$ and $v_2^8$ are $d_1$-cycles, Proposition 2.2.4 follows. □
2.3 Two products

Now we turn our attention to the product between \( \alpha \in \text{Ext}_{A(2)*}^{3,15}(F_2, F_2) \) and \( e[4,23] \in \text{Ext}_{A(2)*}^{4,27}(F_2, A(1)) \). This product is not detected in the DMSS because \( \alpha \) has \( \sigma \)-filtration 1 in the DMSS whereas all non-trivial groups in the DMSS converging to \( \text{Ext}_{A(2)*}^{*,*}(F_2, A(1)) \) concentrate in \( \sigma \)-filtration 0. Therefore, we need first to find a representative of \( \alpha \) in the total cochain complex of the double complex \( A(2)* \otimes E_2 \otimes R \) and that of \( e[4,23] \) in \( A(2)* \otimes E_2 \otimes R \otimes A(1) \), then take the product at the level of cochain complexes and finally check if this product is a coboundary. It is tedious to carry out this procedure because any representative of \( e[4,23] \) contains many terms, and so it is not easy to check if the product is a coboundary. Here, by a term of \( A(2)* \otimes E_2 \otimes R \) and \( A(2)* \otimes E_2 \otimes R \otimes A(1) \), we mean an element of the basis formed by the tensor products of a basis of \( A(2)*, E_2, R \) and \( A(2)*, E_2, R \otimes A(1) \), respectively. We will keep the same convention when working with \( B(2)*, F_2, S \) instead of \( A(2)*, E_2, R \). The following two lemmas simplify computations.

Lemma 2.3.1. The product of \( \alpha \) and \( e[4,23] \) is equal either to 0 or to \( ge[3,15] \)

Proof. This is trivial because \( ge[3,15] \) is the only non-trivial class in the appropriate bidegree. \( \square \)

We recall from Section 2 that there is a map of pair \( (A(2)*, E_2) \) and \( (B(2)*, F_2) \) given by

\[
\begin{align*}
A(2)* &= F_2[\xi_1, \xi_2, \xi_3]/(\xi_1^7, \xi_2^3, \xi_3^2) \\
B(2)* &= F_2[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3]/(\bar{\xi}_1^7, \bar{\xi}_2^3, \bar{\xi}_3^2) \\
E_2 &= E(x_1, x_2, x_3) \\
F_2 &= E(x_2, x_3) \\
x_1 &\mapsto 0, x_2 &\mapsto x_2, x_2 &\mapsto x_2
\end{align*}
\]

The induced map on their Koszul dual is

\[
R = F_2[y_1, y_2, y_3] \to S = F_2[y_2, y_3]
\]

\[
y_1 \mapsto 0, y_2 \mapsto y_2, y_3 \mapsto y_3
\]

By an abuse of notation, we will denote by \( p \) these projection maps. The context will make it clear which map is referred to.

Lemma 2.3.2. The map \( p_* = \text{Ext}_{A(2)*}^{7,42}(F_2, A(1)) \to \text{Ext}_{B(2)*}^{7,42}(F_2, A(1)) \) induced by the projection \( A(2)* \to B(2)* \) sends \( ge[3,15] \) to a non-trivial element.
Proof. The projection $A(2)_* \to B(2)_*$ induces a morphism at the level of DMSS. The morphism of the $E_1$-terms reads

$$\text{Ext}^{s,t}_{A(2)_*}(\mathbb{F}_2, E_2 \otimes R \otimes A(1)) \to \text{Ext}^{s,t}_{B(2)_*}(\mathbb{F}_2, F_2 \otimes S \otimes A(1))$$

By the change-of-ring isomorphisms, this morphism is the same as the projection $p : R \to S$ which is surjective. The class $ge[3, 15]$ is detected by $y^2_1(y^3_2 + y_1y^3_3) \in R^7$, which maps to $y^7_2 \in S^7$ via $p$. By naturality, $y^7_2$ is a permanent cycle in the target DMSS. The only class in the $E_1$-term which can support a differential hitting $y^7_2$ is $y^6_3$. $y^6_3$ admits $v^4_2y^3_3$ as a lift in the source DMSS. We have that

$$d_1(v^4_2y^3_3) = d_1(v^4_2)y^3_3 + v^4_2d_1(y^3_3) = (\alpha_{0,4,1})^2y^3_3 + v^4_2(\alpha_{0,4,1}y_2) = y_1y^2_2y^3_3 + y^3_2y^4_1y_2$$

This uses Leibniz rule, Proposition 2.1.15 part 11) Proposition 2.2.3 part 7). By naturality, the $d_1$-differential in the target DMSS is equal to $p(y^1_1y^2_2y^3_3 + y^3_2y^4_1y_2)$ which is equal to 0. Therefore, the image of $ge[3, 15]$ is non-trivial.

\[\square\]

Corollary 2.3.3. The product of $\alpha$ and $e[4, 23]$ is non-trivial, hence equal to $ge[3, 15]$ if and only if the product of $p_*(\alpha)$ and $p_*(e[4, 23])$ is non-trivial.

Proof. The map $p : A(2)_* \to B(2)_*$ induces the commutative diagram

$$\begin{array}{ccc}
\text{Ext}^{3,15}_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \text{Ext}^{4,27}_{A(2)_*}(\mathbb{F}_2, A(1)_*) & \longrightarrow & \text{Ext}^{7,42}_{A(2)_*}(\mathbb{F}_2, A(1)_*) \\
\text{Ext}^{3,15}_{B(2)_*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \text{Ext}^{4,27}_{B(2)_*}(\mathbb{F}_2, A(1)_*) & \longrightarrow & \text{Ext}^{7,42}_{B(2)_*}(\mathbb{F}_2, A(1)_*) \\
\text{Ext}^{3,15}_{B(2)_*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \text{Ext}^{4,27}_{B(2)_*}(\mathbb{F}_2, A(1)_*) & \longrightarrow & \text{Ext}^{7,42}_{B(2)_*}(\mathbb{F}_2, A(1)_*) \\
\end{array}$$

where the two horizontal maps are given by taking product. The proof follows from the fact that $p_*(ge[3, 15])$ is non-trivial by Lemma 2.3.2.

\[\square\]

Now let us compute the product of $p_*(\alpha)$ and $p_*(e[4, 23])$.

Lemma 2.3.4. There are the following representatives of $p_*(\alpha)$ and $p_*(e[4, 23])$ in the total cochain complex of $B(2)^{\otimes s} \otimes F_2 \otimes S$ and of $B(2)^{\otimes s} \otimes F_2 \otimes S \otimes A(1)$, respectively:

i) $p_*(\alpha)$ is represented by $[\xi_2][1][y^3_2] + [\xi_1][1][y^2_2] + [\xi_1][1][y^3_2] \in B(2) \otimes F_2 \otimes S^2$

ii) $p_*(e[4, 23])$ is represented by $[1][y^2_2][a_0] + [1][y^2_2][a_1] + [1][y^3_2y_3][a_2] + [1][y^2_2][a_3] \in F_2 \otimes S^4 \otimes A(1)$.

34
Lemma 2.3.5. \( P \) contains \( \lambda_{i,j}|1|x_2|y_2^6|a_0 \) as a term where \( \lambda_{i,j} \) is as in Proposition 2.2.7 i.e., \( \lambda_{1,0} = \lambda_{0,1} = 1 \) whereas \( \lambda_{0,0} = \lambda_{1,1} = 0 \).

Proof. The product \( MN \) contains the term \( [\xi_2|1|y_2^6|a_3] \). One can check that \( P \) must contain the term \( [1|y_2^6|a_6] \) so that \( d_v(P) \) contains the term \( [\xi_2|1|y_2^6|a_3] \). Using the formula for the coaction of \( A(2)_* \) on \( a_6 \), one sees that \( d_v(P) \) contains the term \( \lambda_{i,j}|\xi_2^2|1|y_2^6|a_0 \) which is not a term of \( MN \). In order to compensate this term, \( P \) must contain the term \( \lambda_{i,j}|1|x_2|y_2^6|a_0 \).

Lemma 2.3.6. An \((d_k + d_v)\)-cycle in \( F_2 \otimes S^7 \otimes A(1) \) gives rise to a non-trivial class in \( \text{Ext}^4_{B_{A(2)_*}}(\mathbb{F}_2, A(1)) \) if and only if it contains the term \( [1|y_2^7|a_0] \).
Proof. It is proved in the proof of Lemma 2.3.2 that
\[ \text{Ext}_{B(2),}^{7,42}(\mathbb{F}_2, A(1)) \cong \mathbb{F}_2 \]
and that this group gets contributed by
\[ \text{Ext}^{0,42}_{B(2),}(\mathbb{F}_2 \otimes S^7 \otimes A(1)) \cong \mathbb{F}_2 \{y_7^2\} \subset S^7 \]
Therefore, by the shearing homomorphism the only element in \( \mathbb{F}_2 \otimes S^7 \otimes A(1) \) that represents the non-trivial class of \( \text{Ext}^{7,42}_{B(2),}(\mathbb{F}_2, A(1)) \) contains the term \([1 | y_2^2 | a_0]\).

Proposition 2.3.7. The product \( \alpha e[4, 23] \) is equal to \( 0 \) for \( A_1[00] \) and \( A_1[11] \) and equal to \( ge[3, 15] \) for \( A_1[10] \) and \( A_1[01] \).

Proof. \( \alpha e[4, 23] \) is non-trivial if and only if \( d_h(P) \) represents a non-trivial class in \( \text{Ext}_{B(2),}^{7,42}(\mathbb{F}_2, A(1)) \). Lemma 2.3.5 shows that \( d_h(P) \) contains the term \( \lambda_{i,j}[1 | y_7^2 | a_0] \). Hence, lemma 2.3.6 concludes the proof.

The second product between \( \beta \in \text{Ext}_{A(2),}^{3,18}(\mathbb{F}_2, \mathbb{F}_2) \) and \( e[3, 15] \in \text{Ext}_{A(2),}^{3,18}(\mathbb{F}_2, A(1)) \) is easier because both have \( \sigma \)-filtration 0 in the Davis-Mahowald spectral sequence.

Proposition 2.3.8. \( \beta e[3, 15] = e[6, 30] \)

Proof. \( \beta \) is represented by \( y_3^2 + y_1y_3^2 \) in \( R^3 \) and \( e[3, 15] \) is represented by \( [y_3^2 + y_1y_3^2 | a_0] \) in \( R^3 \otimes A(1) \). So the product is represented by \( [y_2^6 + y_2^2y_3^2 | a_0] \). Notice that this class survives in the Davis-Mahowald spectral sequence and converges to \( e[6, 30] \).

3 Partial study of the Adams spectral sequence for \( tmf \wedge A(1) \)

In this section, we establish some differentials as well as some structures of the ASS for \( A(1) \). These are essential bits of information allowing us to run the homotopy fixed point spectral sequence in the next section.

Recall that the ASS for \( tmf \wedge A(1) \) with \( E_2 \)-term isomorphic to \( \text{Ext}^{*,*}_{A(2),}(A(1)) \) is a spectral sequence of module over that for \( tmf \) whose the \( E_2 \)-term is isomorphic to \( \text{Ext}^{*,*}_{A(2),}(\mathbb{F}_2) \). This follows from the fact that the mod 2 homology of \( tmf \) is isomorphic to \( A_4 \square_{A(2),} \mathbb{F}_2 \) (see [Mat16] for a proof). We first recollect some well-known properties of the ASS for \( tmf \).
Theorem 3.0.1. (i) The class $g \in Ext^4_{A(2)^*}(\mathbb{F}_2)$ is a permanent cycle detecting the image of $\pi \in \pi_{20}(S^0)$ via the Hurewicz map $S^0 \to tmf$. 

(ii) There is the following $d_2$-differential in the Adams spectral sequence for $tmf$

$$d_2(w_2) = g \alpha \beta$$

(iii) There is following $d_3$-differential in the Adams spectral sequence for $tmf$

$$d_3(w_2^3(v_2^6 \eta)) = g^6$$

(iv) The class $\Delta^8 := w_2^2$ survives the Adams spectral sequence.

Proposition 3.0.2. In the ASS converging to $tmf, A(1)$, there exists $\lambda \in \{0, 1\}$ such that the following statements are equivalent:

1. $d_2(w_2 e[4, 23]) = \lambda g^2 e[6, 30]$
2. $d_2(w_2 e[9, 48]) = \lambda g^6 e[3, 15]$
3. $d_2(w_2 e[10, 53]) = \lambda g^6 e[0, 0]$
4. $d_2(w_2 e[7, 38]) = \lambda g^4 e[1, 5]$

Proof. We will prove that $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow i)$. The chart of Figure 15 and 16 will make the proof easier to follow. First, we observe that all of the classes $e[4, 23], e[7, 38], e[9, 48], e[10, 53]$ are permanent cycles by sparseness.

2. Suppose $d_2(w_2 e[4, 23]) = g^2 e[6, 30]$. Then $d_2(g^2 w_2 e[4, 23]) = g^4 e[6, 30]$ by $g$-linearity. It follows that there is no room for a non-trivial differential on $w_2^2 e[3, 15]$. In order words, $w_2^2 e[3, 15]$ is a permanent cycle. Because of part $iii)$ of Theorem 3.0.1, $g^k$-multiple of $w_2^2 e[3, 15]$ must be hit by a differential for some $k$ less than 7. One can check that the only possibility is that $d_2(w_2^2 e[9, 48]) = g^4 w_2^2 e[3, 15]$. Since $w_2^2$ is a $d_2$-cycle in the ASS for $tmf$, the last differential implies that $d_2(w_2 e[9, 48]) = g^4 e[3, 15]$.

3. Suppose $d_2(w_2 e[9, 48]) = g^4 e[3, 15]$. Then the class $w_2^2 e[0, 0]$ is a permanent cycle by sparseness. Again, $g^k$-multiple of $w_2^2 e[0, 0]$ for some $k$ smaller than 7 must be hit by a differential. Inspection shows that the classes $w_2^2 e[10, 53]$ and $w_2^2 e[1, 5]$ are the only ones that have the appropriate bidegree to support such differentials. However, $w_2^2 e[1, 5]$ is a permanent cycle because $w_2^2$ and $e[1, 5]$ are permanent cycles in their respective ASS. Therefore, $d_2(w_2 e[10, 53]) = g^5 e[0, 0]$.

4. Suppose $d_2(w_2 e[10, 53]) = g^5 e[0, 0]$. Then the class $w_2^2 e[1, 5]$ is a permanent cycle as there is no room for non-trivial differential on it. Then $g^k w_2^2 e[1, 5]$ must be hit by a differential for some $k$ less than 7. Inspection shows that the only possibility is that $d_2(w_2^2 e[7, 38]) = g^4 w_2^2 e[1, 5]$. As $w_2^2$ is a $d_2$-cycle, it follows that $d_2(w_2 e[7, 38]) = g^4 e[1, 5]$.

i) Suppose $d_2(w_2 e[7, 38]) = g^4 e[1, 5]$. By $g$-linearity, we get that $d_3(gw_2 e[7, 38]) = g^5 e[1, 5]$. It follows by sparseness that $w_2^2 e[6, 30]$ is a permanent cycle. Then the class $g^k w_2^2 e[6, 30]$ is hit by a differential for some $k$ less than
7. Inspection shows that the only possibility is that \( d_2(w_2^3e[4, 23]) = g^2w_2^2e[6, 30] \). Therefore, \( d_2(w_2e[4, 23]) = g^2e[6, 30] \) by \( w_2^3 \)-linearity.

\[ \square \]

**Theorem 3.0.3.** We have:

i) In the Adams spectral sequence for \( \text{tmf} \wedge A_1[00] \) and \( \text{tmf} \wedge A_1[11] \), the coefficient \( \lambda \) of Proposition 3.0.2 is equal to 0.

ii) In the Adams spectral sequence for \( \text{tmf} \wedge A[10] \) and \( \text{tmf} \wedge A[01] \), the coefficient \( \lambda \) of Proposition 3.0.2 is equal to 1.

**Proof.** By Leibniz rule and part (ii) of Theorem 3.0.1,

\[ d_2(w_2e[4, 23]) = d_2(w_2)e[4, 23] = g\beta\alpha e[4, 23] \]

The proposition now follows from Proposition 2.3.7 and Proposition 2.3.8. \( \square \)

38
Figure 15 – Adams spectral sequence for A(1) in the range $0 \leq t - s \leq 48$

Figure 16 – Adams spectral sequence for A(1) in the range $48 \leq t - s \leq 101$. The arrows in bold are differentials for the models $A_1[10]$ and $A_1[01]$ and the dashed arrows for the models $A_1[00]$ and $A_1[11]$
Proposition 3.0.4. There are the following $d_3$-differentials in the Adams spectral sequence for $tmf \wedge A(1)$

\[
d_3(w_2^2e[10,53]) = g^5e[9,48] \\
d_3(w_2^3e[1,5]) = g^5w_2e[0,0]
\]

\[
\begin{array}{c|c|c|c}
148 & 149 & 150 & 151 & 152 \\
25 & 26 & 27 & 28 & 29 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
30 & & & & \\
\end{array}
\]

Figure 17 – The Adams spectral sequence in the range $148 \leq t - s \leq 152$

\textbf{Proof.} We can check from the chart that $e[9,48]$ and $we[0,0]$ are permanent cycles. Then $g^l e[9,48]$ and $g^k we[0,0]$ must be targets of some differentials for some $l$ and $k$ less than 7. Inspection of the $E_2$-term shows that either

\[
d_2(w_2^2e[10,53]) = g^5we[0,0] \text{ and } d_4(w_2^3e[1,5]) = g^5e[9,48]
\]

or

\[
d_3(w_2^2e[10,53]) = g^5e[9,48] \text{ and } d_3(w_2^3e[1,5]) = g^5w_2e[0,0]
\]

However, the former possibility is ruled out because of Leibniz rule:

\[
d_2(w_2^2e[10,53]) = d_2(w_2^2)e[10,53] = 2w_2d_2(w_2)e[10,53] = 0
\]

The first equality follows from that $e[10,53]$ is a permanent cycle by spareness.

\textbf{Corollary 3.0.5.} The Toda bracket $\langle g^5, e[9,48], \nu \rangle$ can be formed and contains only elements which are divisible by $g$. 

40
Proof. In the $E_2$-term of the ASS, the Massey product $\langle g^5, e[9, 48], \nu \rangle$ has the cohomological filtration 27 and is equal to zero with zero indeterminacy. On the other hand the corresponding Toda bracket can be formed with indeterminacy containing only multiples of $g$. We can check that all conditions of the Moss’s convergence theorem [Mos70] are met. This implies that the Toda bracket $\langle g^5, e[9, 48], \nu \rangle$ contains an element detected in filtration 27 by 0, thus is a multiple of $g$. Therefore, this Toda bracket contains only multiples of $g$.

Finally, we need to have a control on the action of the class $\Delta^8 \in \text{Ext}_{A(2), A(1)}^{32,224}(F_2, F_2)$ on the $E_\infty$-term of the ASS for $tmf \wedge A(1)$. This will allow us to compare $\pi_s(tmf \wedge A(1))$ with $\pi_s(EhG_{24} \wedge A(1))$ (see Corollary 4.1.6) and hence to discuss higher differentials in the HFPSS for $EhG_{24} \wedge A(1)$.

**Proposition 3.0.6.** The class $w_2^4$ acts freely on the $E_\infty$-term of the ASS for $tmf \wedge A(1)$. As a consequence, the element $\Delta^8 \in \pi_{192}(tmf)$ acts freely on the homotopy groups of $tmf \wedge A(1)$.

**Proof.** Using the description of the $E_2$-term of the ASS for $tmf \wedge A(1)$ in Theorem 2.2.4 and an elementary bidegree inspection, we can see that if a class $y$ is in an appropriate bidegree to support a differential hitting a class of the form $w_2^4x$ for some class $x$, then $y$ is divisible by $w_2^4$. Knowing that $w_2^4$ is a permanent cycle in the ASS for $tmf$, we conclude that if a class $x$ survives the $E_r$-term, then the multiple of $x$ by all powers of $w_2^4$ also survive that term. Therefore, the Proposition follows by induction.

**Proposition 3.0.7.** For every element $x \in \pi_s(tmf \wedge A(1))$, the element $\Delta^8x$ is divisible by $\overline{\pi}$ (resp. $\nu$) if and only if $x$ is divisible by $\overline{\pi}$ (resp. $\nu$).

**Proof.** The argument is similar to that of Proposition 3.0.6. A bidegree inspection shows that if a class $y \in \text{Ext}_{A(2), A(1)}^{*,*}(F_2, F_2)$ is in an appropriate bidegree whose (exotic) product with $g$ (resp. $\nu$) might detect $\Delta^8x$, then $y$ is divisible by $w_2^4$. We can conclude proof by using that the class $w_2^4$ acts freely on the ASS for $tmf \wedge A(1)$.

## 4 Homotopy fixed point spectral sequence $E_2^{hG_{24}} \wedge A(1)$

### 4.1 Preliminaries and recollection on cohomology of $G_{24}$

Let $C$ denote the supersingular elliptic curve defined by the Weierstrass equation $y^2 + y = x^3$ over $\mathbb{F}_4$. The formal completion $\hat{F}_C$ of $C$ at the origin is a formal
The automorphism groups of $F_C$ over $\mathbb{F}_4$ is denoted by $S_C$. Let $\mathcal{W}(\mathbb{F}_4)$ denote the Witt vector over $\mathbb{F}_4$. As a ring, $\mathcal{W}(\mathbb{F}_4)$ is isomorphic to $\mathbb{Z}_2[\zeta]/(\zeta^2 + \zeta + 1)$. The Lubin-Tate theory asserts that the deformation of $F_C$ over $\mathbb{F}_4$ is classified by a universal formal group law, say $\tilde{F}_C$ over the power series ring $\mathcal{W}(\mathbb{F}_4)[[u]]$ \cite{LT66}. Let $E_C$ be the Morava E-theory associated to $\tilde{F}_C$ via the Landweber’s exact functor theorem. The coefficient ring of $E_C$ is isomorphic to $\mathcal{W}(\mathbb{F}_4)[u_1][u_\pm 1]$ with $u_1$ in degree 0 and $u$ in degree $-2$. The Goerss-Hopkins-Miller theorem shows that $E_C$ admits an essentially unique structure of $E_\infty$-ring spectrum. Furthermore, the group $S_C$ acts on $E_C$ via $E_\infty$ ring maps. See \cite{Rez98} for an exposition.

$S_C$ contains a unique maximal finite subgroup $G_{24}$ of order 24 up to conjugacy. It is known that the group of automorphisms of $C$ is of order 24 which is isomorphic to the semi-direct product $Q_8 \rtimes C_3$ of the quaternion group $Q_8$ and the cyclic group $C_3$ of order 3 (\cite{Sil09}, Appendix A). The group $Q_8$ can be represented with generators and relations as $Q_8 = \langle i, j | i^4 = j^4 = 1, ij = j^3 \rangle$ and $C_3$ acts on it by permuting $i, j, k = ij$. Explicitly, if $C_3 = \langle \omega | \omega^3 \rangle$ then

\[ \omega i \omega^{-1} = j, \omega j \omega^{-1} = k, \omega k \omega^{-1} = i \]

The group of automorphisms of $C$ is sent injectively to $S_C$, giving an interesting choice of $G_{24}$. We will work with this choice of $G_{24}$. The geometric origin of this group allows one to determine explicitly the action of $G_{24}$ on the coefficient ring $(E_C)_*$ (see (\cite{Bea17}, Section 2). We record it here for the convenience of the reader.

**Proposition 4.1.1.** The action of $G_{24}$ on $\mathcal{W}(\mathbb{F}_4)[[u_1]][u_\pm 1]$ is given by

\[
\begin{align*}
\omega(u^{-1}) &= \zeta^2 u^{-1} & \omega(v_1) &= v_1 \\
i(u^{-1}) &= \frac{-u^{-1} + v_1}{\zeta^2 - \zeta} & i(v_1) &= \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta} \\
j(u^{-1}) &= \frac{-u^{-1} + \zeta^2 v_1}{\zeta^2 - \zeta} & j(v_1) &= \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta} \\
k(u^{-1}) &= \frac{-u^{-1} + \zeta v_1}{\zeta^2 - \zeta} & k(v_1) &= \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}
\end{align*}
\]

Another reason for us to work with this choice of height two formal group law is the following folklore theorem

**Theorem 4.1.2.** There is a homotopy equivalence

\[ L_{K(2)} T MF \cong (E_{C}^{hG_{24}})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \]
We refer to [Beh06] and [BL10] for a discussion. As a consequence, it gives us a way to get access to homotopy groups of $E_{C \mathcal{G}^{24}} \land A(1)$

**Theorem 4.1.3.** There is a homotopy equivalence

$$tmf \land A(1)[(\Delta^8)^{-1}] \cong (E_{C \mathcal{G}^{24}})^{h\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \land A(1)$$

Therefore,

$$\pi_*(E_{C \mathcal{G}^{24}} \land A(1)) \cong \mathbb{V}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} (\Delta^8)^{-1}(\pi_*(tmf \land A(1)))$$

**Proof.** We know that the spectrum $TMF$ is homotopy equivalent to spectrum $tmf$ with the periodicity generator $\Delta^8$ inverted (see [DFHH14]), i.e.,

$$tmf[(\Delta^8)^{-1}] \cong TMF$$

Therefore,

$$tmf \land A(1)[(\Delta^8)^{-1}] \cong TMF \land A(1)$$

$$\cong L_2(TMFC) \land A(1)(TMF \text{ is } E_2\text{-local})$$

$$\cong L_2(TMFC) \land A(1)(L_2 \text{ is smashing})$$

$$\cong L_{K(2)}(TMF) \land A(1)(A(1) \text{ is of type } 2 \text{ ([HS99], Lem 7.2)})$$

$$\cong (E_{C \mathcal{G}^{24}})^{h\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \land A(1)(\text{Theorem 4.1.2})$$

We continue to recall some information about the HFPSS converging to $\pi_*(E_{C \mathcal{G}^{24}})$:

$$H^*(G_{24}, (E_C)_{t}) \rightarrow \pi_{t-s}(E_{C \mathcal{G}^{24}})$$

(14)

The elements $\eta \in \pi_1(S^0)$, $\nu \in \pi_{23}(S^0)$, $\overline{\pi} \in \pi_{20}(S^0)$ are sent non-trivially to elements of the same name in $\pi_*(E_{C \mathcal{G}^{24}})$ via the Hurewicz map $S^0 \rightarrow E_{C \mathcal{G}^{24}}$. As the latter factors through the unit map of $tmf$, the element $\overline{\pi}^6 = 0$ in $\pi_*(E_{C \mathcal{G}^{24}})$ because $\overline{\pi}^6 = 0$ in $\pi_*(tmf)$ (see [Bau08]). These elements are detected by $\eta \in H^1(G_{24}, (E_C)_4)$, $\nu \in H^1(G_{24}, (E_C)_{24})$, $\overline{\pi} \in H^4(G_{24}, (E_C)_{24})$, respectively. Furthermore, there is a class $\Delta \in H^0(G_{24}, (E_C)_{24})$ such that $\Delta^8$ is a permanent cycle making $\pi_*(E_{C \mathcal{G}^{24}})$ 192 periodic.

The HFPSS for $E_{C \mathcal{G}^{24}} \land A(1)$ is a spectral sequence of module over that of (14):

$$H^*(G_{24}, (E_C)_t(A(1))) \rightarrow \pi_{t-s}(E_{C \mathcal{G}^{24}} \land A(1))$$

(15)

In Subsection 4.2, we will compute $H^*(G_{24}, (E_C)_s(A(1))$ as a module over certain subalgebra of $H^*(G_{24}, (E_C)_s)$. Let $\pi : (E_C)_s \rightarrow \mathbb{F}_4[u^\pm 1]$ be the quotient of $(E_C)_s$ by the ideal $(2, u_1)$. As the ideal $(2, u_1)$ is preserved by the action of $S_C$, the ring $\mathbb{F}_4[u^\pm 1]$ inherits an action of $S_C$. We need the computation of the ring structure of $H^*(G_{24}, \mathbb{F}_4[u^\pm 1])$ (see [Bea17], Appendix A)
Proposition 4.1.4. There are classes \( z \in H^4(G_{24}, F_4[u^{\pm 1}]) \), \( a \in H^1(G_{24}, F_4[u^{\pm 1}]) \), \( b \in H^4(G_{24}, F_4[u^{\pm 1}]) \), \( v_2 \in H^0(G_{24}, (F_4[u^{\pm 1}]) \) such that there is an isomorphism of graded algebras

\[
H^*(G_{24}, F_4[u^{\pm 1}]) \cong F_4[v_2^{\pm 1}, z, a, b]/(ab, b^3 = v_2 a^3)
\]

Proposition 4.1.5. The homomorphism of graded algebras

\[
H^*(G_{24}, E_{C_\ast}) \to H^*(G_{24}, F_4[u^{\pm 1}])
\]

induced by the projection \((E_{C})_\ast \to F_4[u^{\pm 1}]\) sends \( \eta \) to \( a \), \( \nu \) to \( b \), \( \kappa \) to \( v_4^2 z \), and \( \Delta \) to \( v_4^2 \).

We end this Subsection with the following Corollary which recapitulates the relationship between \( \pi_\ast (tmf \wedge A(1)) \) and \( \pi_\ast (E_{hG_{24}} \wedge A(1)) \). Let us denote by

\[
\Theta : \mathbb{W}(F_4) \otimes_{\mathbb{Z}_2} \pi_\ast (tmf \wedge A(1)) \to \pi_\ast (E_{hG_{24}} \wedge A(1))
\]

the induced homomorphism in homotopy of the composite

\[
tmf \wedge A(1) \to tmf \wedge A(1)[(\Delta^8)^{\pm 1}] \xrightarrow{\sim} (E_{hG_{24}})^{hGal(F_4/F_2)} \wedge A(1)
\]

where the second map is the equivalence of Theorem 4.1.3.

Corollary 4.1.6. The homomorphism \( \Theta \) is injective. Moreover, it remains injective after passing to quotient by the ideal of \( \pi_\ast (S^0) \) generated by \( (\pi, \nu) \).

Proof. This follows from Theorem 4.1.3, Proposition 3.0.6 and Proposition 3.0.7.

4.2 The \( E_2 \)-term: \( H^*(G_{24}, (E_{C})_\ast (A(1))) \)

We first determine \((E_{C})_\ast (A(1))\) using the cofiber sequences through which \( A(1) \) are defined. The cofiber sequence \( \Sigma S^0 \xrightarrow{\eta} S^0 \to C_\eta \) gives rise to a short exact sequence of \( E_{C} \)-homology

\[
0 \to (E_{C})_\ast \to (E_{C})_\ast (C_\eta) \to (E_{C})_\ast (S^2) \to 0
\]

since \((E_{C})_\ast \) concentrates in even degrees. Hence, as an \((E_{C})_\ast \)-module

\[
(E_{C})_\ast (C_\eta) \cong \mathbb{W}(F_4)[[u_1]][u^{\pm 1}]{\{e_0, e_2\}}
\]

44
where \( e_0 \) is the image of 1 ∈ \((E_C)_0\) and \( e_2 \) is a lift of \( \Sigma^2 1 \in (E_C)_2(S^2) \). Next, the long exact sequence in \( E_C \)-homology associated to \( C_\eta \overset{2}{\to} C_\eta \to Y \) is the short exact sequence

\[
0 \to (E_C)_*(C_\eta) \xrightarrow{x^2} (E_C)_*(C_\eta) \to (E_C)_*(Y) \to 0
\]
since the multiplication by 2 on \((E_C)_*(C_\eta) \cong \mathcal{W}(\mathbb{F}_4)[[u_1]][u^{\pm 1}]\{e_0, e_2\}\) is injective. Therefore

\[
(E_C)_*(Y) \cong \mathbb{F}_4[[u_1]][u^{\pm 1}]\{e_0, e_2\}
\]
Now \( A(1) \) is the cofiber of some \( v_1 \)-self map of \( Y: \Sigma^2 Y \overset{v_1}{\to} Y \to A(1) \). The following lemma describes the induced homomorphism in \( E_C \)-homology of these \( v_1 \)-self maps.

**Lemma 4.2.1.** The homomorphism \((E_C)_*(v_1)\) is given by the multiplication by \( u_1 u^{-1} \) up to a unit of \((E_C)_0\). Therefore,

\[
(E_C)_*(A(1)) \cong \mathbb{F}_4[u^{\pm 1}]\{e_0, e_2\}
\]

**Proof.** Let \( K(1) \) be the first Morava K-theory at the prime 2 such that \( K(1)_* \cong \mathbb{F}_2[v_1^{\pm 1}] \) where \(|v_1| = 2\) and \( BP \) be the Brown-Peterson spectrum at the prime 2. There is a map of ring spectra \( BP \to K(1) \) that classifies the complex orientation of \( K(1) \). The induced homomorphism of coefficient rings sends \( v_1 \) to \( v_1 \). By definition, a \( v_1 \)-self-map of \( Y \) induces in \( K(1)_* \)-homology the multiplication by \( v_1 \). The map \( BP \to K(1) \) gives rise to the commutative diagram

\[
\begin{array}{ccc}
BP_*(\Sigma^2 Y) & \xrightarrow{BP_*(v_1)} & BP_*(Y) \\
\downarrow & & \downarrow \\
K(1)_*(\Sigma^2 Y) & \xrightarrow{K(1)_*(v_1)} & K(1)_*(Y)
\end{array}
\]

By a property of \( v_1 \) ([DMST], Theorem 1.2), \( K(1)_*(v_1) \) is given by the multiplication by \( v_1 \in K(1)_2 \). The above diagram forces by degree reasons that \( BP_*(v_1) \) is given by the multiplication by \( v_1 \in BP_2 \). Now, let \( c : BP \to E_C \) be the map of ring spectra that classifies the 2-typification of the formal group law of \( E_C \). One can show that the 2-series of the latter has the leading term \( u_1 u^{-1} x^2 \) modulo \((2, u_1)\). This implies that the induced homomorphism \( c_* : BP_* \to (E_C)_* \) sends \( v_1 \) to \( u_1 u^{-1} \) up to a unit of \((E_C)_0\). By naturality, \((E_C)_*(v_1)\) is also given by the multiplication by \( u_1 u^{-1} \) up to an invertible element of \((E_C)_0\).

We are now describing the action of \( G_{24} \) on \((E_C)_*(A(1))\). Recall \( c : BP \to E_C \) is the map of ring spectra classifying the complex orientation of \( E_C \). Then for any
2-local finite spectrum $X$, the map $c$ induces a map of ANSS
\[
\begin{array}{ccc}
\text{Ext}^{s,t}_{BP,BP}(BP_*, BP_* X) & \longrightarrow & \text{Ext}^{s,t}_{(E_C)_*, (E_C)_* X} \\
\pi_{t-s}(X) & \longrightarrow & \pi_{t-s}(L_{K(2)}X)
\end{array}
\]
where $(E_C)_*, E_C$ stands for $\pi_*(L_{K(2)}(E_C \wedge E_C))$. By the Morava’s change-of-ring theorem (see [Dev95]),
\[
\text{Ext}^{s,t}_{(E_C)_*, (E_C)_*}((E_C)_*, (E_C)_*) \cong H^s(G_C, (E_C)_*)
\]
Now the map $c$ induces a map of short exact sequences
\[
\begin{array}{ccc}
0 & \longrightarrow & BP_* \xrightarrow{\times 2} BP_* \longrightarrow BP_*/(2) \longrightarrow 0 \\
0 & \longrightarrow & E_{C_*} \xrightarrow{\times 2} E_{C_*} \longrightarrow E_{C_*/(2)} \longrightarrow 0
\end{array}
\]
Therefore, we obtain the commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^{0,*}_{BP_*, BP_*}(BP_*, BP_*/2) & \xrightarrow{\delta_{BP_*}} & \text{Ext}^{1,*}_{BP_*, BP_*}(BP_*, BP_*) \\
H^0_*(G_C, E_{C_*/(2)}) & \xrightarrow{\delta_{E_C}} & H^1_*(G_C, E_{C_*})
\end{array}
\]
It is well known that $\text{Ext}^{0,2}_{BP_* BP_*}(BP_*, BP_*/2) = \mathbb{Z}_2\{v_1\}$ and $\delta_{BP_*}(v_1) = \eta \in \text{Ext}^{1,2}_{BP_* BP_*}(BP_*, BP_*)$ where $\eta$ is a permanent cycle representing the Hopf element $\eta \in \pi_1(S^0)$. By naturality, $\delta_{E_C}(v_1) = c_*(\eta)$. Therefore, as a cocycle in $\text{Map}_c(G_C, (E_C)_2)$, $c_*(\eta)$ is given by
\[
G_C \rightarrow (E_C)_2, \ g \mapsto \frac{g(v_1) - v_1}{2}
\]
On the other hand, let us consider the short exact sequence
\[
0 \rightarrow E_{C_*} \rightarrow E_{C_*}(C_\eta) \rightarrow E_{C_*}(S^2) \rightarrow 0
\]
representing the class $c_*(\eta)$, so that the connecting homomorphism sends $\Sigma^21$ to $c_*(\eta)$. Thus if $e_2$ is a lift of $\Sigma^21$ in $E_{C_*}(C_\eta)$, then $c_*(\eta)$ is represented by the cocycle
\[
G_C \rightarrow (E_C)_2, \ g \mapsto g(e_2) - e_2
\]
This implies that one can modify \( e_2 \) such that

\[
g(v_1) - v_1 = g(e_2) - e_2 \forall g \in \mathbb{G}_C
\]

With this choice of \( e_2 \), we see that \( E_{C*}(C_{\eta}) = E_{C*}\{e_0, e_2\} \) and the action of \( \mathbb{G}_C \) on \( e_2 \) is given by the formula

\[
g(e_2) = e_2 + \frac{g(v_1) - v_1}{2} e_0
\]

(16)

Note that when determining (\( E_C \)\( _* \))(\( A(1) \)), we did not specify any lift \( e_2 \) of \( \Sigma^2 1 \). From now on, we will fix \( e_2 \) such that the formula of (16) holds.

**Proposition 4.2.2.** As an \( E_{C*} \)-module, \( E_{C*}(A(1)) \) is isomorphic to \( \mathbb{F}_4[u^{\pm 1}]\{e_0, e_2\} \) and the action of \( G_{24} \) on \( e_2 \) is given by

\[
\omega(u^{-1}) = \zeta^2 u^{-1}, \quad \omega(e_0) = e_0, \quad \omega(e_2) = e_2 \\
i(u^{-1}) = u^{-1}, \quad i(e_0) = e_0, \quad i(e_2) = e_2 + e_0 \\
j(u^{-1}) = u^{-1}, \quad j(e_0) = e_0, \quad j(e_2) = e_2 + \zeta^2 e_0 \\
k(u^{-1}) = u^{-1}, \quad k(e_0) = e_0, \quad k(e_2) = e_2 + \zeta e_0
\]

**Proof.** The first part of the statement is the content of Lemma 4.2.1. The second part follows from the action of \( G_{24} \) on \( v_1 \) given in Proposition 4.1.1 and the formula (16).

**Corollary 4.2.3.** \( E_{C*}(A(1)) \) sits in a non-split short exact sequence of \( G_{24} \)-modules

\[
0 \to \mathbb{F}_4[u^{\pm 1}]\{e_0\} \to E_{C*}(A(1)) \to \mathbb{F}_4[u^{\pm 1}]\{e_2\} \to 0
\]

(17)

**Proof.** This is immediate in view of the explicite description of the action of \( G_{24} \) on \( (E_C)_* A(1) \).

We can see that \( H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_i\}) \) \( i \in \{0, 2\} \) is free of rank one as modules over \( H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]) \). Let us choose classes \( e[0, i] \in H^0(G_{24}, (\mathbb{F}_4[u^{\pm 1}]\{e_i\})) \) \( i \in \{0, 2\} \) to be generators of \( H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_i\}) \) \( i \in \{0, 2\} \), respectively.

**Corollary 4.2.4.** The connecting homomorphism induced from the short exact sequence (17) in Lemma 4.2.3

\[
H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_2\}) \to H^{*+1}(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_0\})
\]

is \( H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]) \)-linear and sends \( e[0, 2] \) to \( ae[0, 0] \) up to a unit of \( \mathbb{F}_4 \).
Proof. That $\delta$ is $H^*(G_{24}, F_4[u^{\pm 1}])$-linear is a well-known property of the connecting homomorphism (See [Bro82], V.3). Next, since the short exact sequence in Corollary 4.2.3 does not split, the connecting homomorphism $\delta$ sends $e[2, 0]$ to a non-trivial class and hence to $ae[0, 0]$ up to a unit of $F_4$.

Using the description of $H^*(G_{24}, F_4[u^{\pm 1}])$ and the long exact sequence associated to the short exact sequence of Corollary 4.2.3, we obtain the following description of $H^*(G_{24}, (E_C)_*(A(1)))$:

**Proposition 4.2.5.** As a module over $H^*(G_{24}, F_4[u^{\pm 1}])$, there is an isomorphism

$$H^*(G_{24}, (E_C)_*(A(1))) = F_4[v_2^{\pm 1}, z, b]/(e[0, 0], e[1, 5])$$

where $e[0, 0] \in H^0(G_{24}, (E_C)_0(A(1)))$ and $e[1, 5] \in H^1(G_{24}, (E_C)_6(A(1)))$.

![Figure 18 - $H^*(G_{24}, (E_C)_t(A(1)))$ depicted in the coordinate (s, t-s)](image)

The above proposition also implies immediately the action of $H^*(G_{24}, (E_C)_*)$ on $H^*(G_{24}, (E_C)_*(A(1)))$. In effect, the action of $E_C_*$ on $(E_C)_*(A(1))$ factors through $F_4[u^{\pm 1}]$ via $E_C_* \to F_4[u^{\pm 1}]$. As a consequence the action of $H^*(G_{24}, E_C_*)$ on $H^*(G_{24}, (E_C)_*(A(1)))$ factors through the induced homomorphism in cohomology of $G_{24}$. In particular, it follows from Proposition 4.1.5 that the class $\Delta, \pi, \nu$ acts on $H^*(G_{24}, (E_C)_*(A(1)))$ the same as $v_2^2, v_2^2 z, b$ respectively do.

### 4.3 Differentials of the homotopy fixed point spectral sequence for $E_C^{bG_{24}} \wedge A(1)$

We will call the set $\{\pi^l x|l \in \mathbb{N}\}$ associated to a class $x$ in some page of the HFPSS by the $\pi$-family of that class. Proposition 4.3.3 below says that the $\pi$-families organize themself in a specific way. First, we have the following lemma:

**Lemma 4.3.1.** If there are classes $x$ and $y$ such that $\pi y$ is non-trivial and $d_r(\pi x) = \pi y$, then $d_r(x) = y$.

**Remark 4.3.2.** To avoid any confusion, we highlight that the formula $d_r(x) = y$ means two things: $x$ and $y$ survive to the $E_r$-term and $d_r(x) = y$. 

48
Proof. Suppose the converse is true: let \( r \) be the smallest integer satisfying that there are classes \( x, y \) such that (i) \( d_r(\pi x) = \pi y \) and (ii) \( d_r(x) \neq y \).

First, we claim that \( y \) survives to the \( E_r \)-term. Otherwise, there is an integer \( s < r \) and a class \( z \) such that \( d_s(y) = z \). Since \( \pi y \) survives to the \( E_r \)-term, there must be a class \( w \) and an integer \( t < s \) such that \( d_t(w) = \pi z \). Then the cohomological filtration of \( w \) is greater than 4 because

\[
\text{cf}(w) = \text{cf}(\pi z) - t = 4 + \text{cf}(z) - t > 4 + \text{cf}(z) - s \\
= 4 + \text{cf}(z) + \text{cf}(y) - \text{cf}(z) = 4 + \text{cf}(y)
\]

where \( \text{cf}(\cdot) \) stands for the cohomological filtration. Thus there is a class \( w' \) such that \( \pi w' = w \) and so \( d_t(\pi w') = \pi z \). From the minimality of \( r \), it follows that \( d_t(w') = z \), which contradicts the assumption that \( d_s(y) = z \) for \( s > t \).

Next, we claim that \( x \) also survives to the \( E_r \)-term. Suppose that this is not the case. Then there is a class \( w \) and an integer \( t < s \) such that \( d_s(w) = y \). By \( \pi \)-linearity, the differential \( d_t(x) = z \) implies that \( d_t(\pi x) = \pi z \). Because the class \( \pi x \) survives to the \( E_r \)-page by assumption, the class \( \pi z \) must be trivial in the \( E_r \)-term. Thus the class \( \pi z \) either supports a shorter differential or is hit by a shorter differential then \( d_s \). The former case cannot happen since the class \( z \) is a cycle until the \( E_r \)-term, so is the class \( \pi z \). It forces that there is a class \( w \) and an integer \( t < s \) such that \( d_t(w) = \pi z \). Then the cohomological filtration of \( w \) is greater than 4 because

\[
\text{cf}(w) = \text{cf}(\pi z) - t = 4 + \text{cf}(z) - t > 4 + \text{cf}(z) - s \\
= 4 + \text{cf}(z) + \text{cf}(x) - \text{cf}(z) = 4 + \text{cf}(x)
\]

Hence there is a class \( x' \) such that \( \pi x' = w \). By the minimality of \( s \), the relation \( d_t(\pi x') = \pi z \) implies that the class \( x' \) survives to the \( E_r \)-terms and that \( d_t(x') = z \). This contradicts the assumption that \( z \) survives to the \( E_r \)-term.

Now we know that \( x \) and \( y \) survive to the \( E_r \)-term. It follows from (i) that \( \pi(d_r(x) - y) = 0 \) in the \( E_r \)-term. Since \( \pi \) acts injectively on the classes of cohomological degree positive in the \( E_2 \)-term, condition (ii) implies that there is a class \( z \) and \( s < r \) such that \( d_s(z) = \pi(d_r(x) - y) \). On can see than that \( z \) has cohomological filtration greater than 4. So there is a class \( z' \) such that \( \pi z' = z \). By the minimality of \( r \), we obtain that \( d_s(z') = d_r(x) - y \). Consequently, \( d_r(x) - y = 0 \) in the \( E_r \)-term. This contradicts the condition (ii). This concludes the proof of Lemma 4.3.1. \( \square \)
Proposition 4.3.3. Every \( \pi \)-family is truncated by one and only one \( \pi \)-family. More explicitly, for any permanent cycle \( x \) of cohomological filtration less than 4, there exists a unique class \( y \) of cohomological filtration less than 4 and an integer \( r \) such that for all \( m \geq 0 \),

\[
d_r(\pi^n y) = \pi^{n+i} x
\]

and \( \pi^i x \), \( i = 0, ..., l-1 \) survives to the \( E_\infty \)-term. As a consequence, in any page, every class of cohomological filtration at least 4 is a multiple of \( \pi \).

Proof. This is a consequence of Lemma 4.3.1.

Remark 4.3.4. The proof of Lemma 4.3.1 uses two facts which are (i) the class \( \pi \) is a permanent cycle and (ii) the multiplication by \( \pi \) induces an isomorphism \( H^s(G_{24}, \pi_4(E_C \wedge A(1))) \to H^{s+4}(G_{24}, \pi_{4+24}(E_C \wedge A(1))) \) for \( s > 0 \). Fact (ii) follows from the fact that \( G_{24} \) is a group with periodic cohomology, so independent of the spectrum \( A(1) \), meaning that a similar result to Proposition 4.3.3 holds if \( A(1) \) is replaced by any other finite spectrum.

The following proposition gives us the horizontal vanishing line of the HFPSS for \( E_{hG_{24}} \wedge A(1) \)

Proposition 4.3.5. The HFPSS for \( E_{hG_{24}} \wedge A(1) \) has the horizontal vanishing line of height 23. As a consequence, it collapses at the \( E_{24} \)-term.

Proof. As \( \pi^6 = 0 \) in \( \pi_*(E_{hG_{24}}) \), the class \( \pi^6 \) must be hit by a differential which is of length at most 23. This is because \( \pi^6 \) has the cohomological filtration 24 and all even differentials are trivial. Hence \( \pi^6 \) is trivial in the \( E_{24} \)-term of the HFPSS for \( E_{hG_{24}} \). Next, because the \( E_{24} \)-term of the HFPSS for \( E_{hG_{24}} \wedge A(1) \) is a module over that for \( E_{hG_{24}} \), the class \( \pi^6 \) acts trivially on the \( E_{24} \)-term of the HFPSS for \( E_{hG_{24}} \wedge A(1) \). Since all classes which are not a multiple of \( \pi \) have the cohomological filtration almost 3, the HFPSS has the horizontal vanishing line of height 23.

Proposition 4.3.6. The following classes are permanent cycles

\[
e[0, 0], e[1, 5], e[0, 6], e[1, 11], e[1, 15], e[1, 17], e[1, 21], e[1, 23]
\]

Proof. Firstly, the class \( e[0, 0] \) is a permanent cycle because it detects the inclusion \( S^0 \to A(1) \) into the bottom cell of \( A(1) \). Next, we recapitulate in the following table the associated graded object with respect to the induced Adams filtration of the groups \( \pi_*(tmf \wedge A(1)) / (\pi) \) in some dimensions less than 23.

| Dim | Value | 6 | 15 | 17 | 21 | 23 |
|-----|-------|---|----|----|----|----|
|     |       | \( F_2 \oplus F_2 \) | \( F_2 \) | \( F_2 \) | \( F_2 \) | \( F_2 \oplus F_2 \) |
In the virtue of Corollary \[4.1.6\], the groups \(\pi_*(E_C^{hG_{24}} \wedge A(1))/\langle \eta \rangle\) in these dimensions must have size as big as twice of respective groups. Inspection in the \(E_2\)-term of the HFPSS through dimensions from 0 to 23 and in cohomological filtration less than 4 show that the classes \(e[0, 6], e[1, 15], e[1, 21], e[1, 23]\) are permanent cycles.

Note that the group \(\pi_0(\tmf \wedge A(1))\) and \(\pi_0(\tmf \wedge A(1))\) are annihilated by \(\eta\). It means that \(e[0, 0]\) and \(e[0, 6]\) detects two elements which are annihilated by \(\eta\). It follows that the Toda brackets \(\langle \nu, \eta, e[0, 0] \rangle\) and \(\langle \nu, \eta, e[0, 6] \rangle\) can be formed. By juggling formula,

\[\eta \langle \nu, \eta, e[0, 0] \rangle = \langle \eta, \nu, \eta \rangle e[0, 0] = \nu^2 e[0, 0]\]

and

\[\eta \langle \nu, \eta, e[0, 6] \rangle = \langle \eta, \nu, \eta \rangle e[0, 6] = \nu^2 e[0, 6]\]

Observe that \(\nu^2 e[0, 0]\) and \(\nu^2 e[0, 6]\) are nontrivial and are detected in cohomological filtration 2. Consequently, both \(\langle \nu, \eta, e[0, 0] \rangle\) and \(\langle \nu, \eta, e[0, 6] \rangle\) are nontrivial and are represented by classes in the cohomological filtration at most 1. Therefore \(e[1, 5]\) and \(e[1, 11]\) are permanent cycles.

The unique nontrivial element of \(\pi_{11}(\tmf \wedge A(1))/\langle \eta \rangle\) is annihilated by \(\nu^2\). This implies that the class \(\nu^2 e[1, 11]\) is the target of some differential. Since \(\pi_{17}(E_C^{hG_{24}} \wedge A(1))/\langle \eta \rangle\) has size at least equal to 4, the class \(e[1, 17]\) must be a permanent cycle representing the only element in dimension 17 of \(\pi_*(E_C^{hG_{24}} \wedge A(1))/\langle \eta \rangle\).

Proposition 4.3.7. As a module over \(\mathbb{F}_4[\Delta, \eta, \nu]/(\nu^3)\), the term \(E_2 = E_3\) is free on 8 generators of \((18)\)

\[e[0, 0], e[1, 5], e[0, 6], e[1, 11], e[0, 12], e[1, 17], e[0, 18], e[1, 23]\]

Proposition 4.3.8. The \(d_3\)-differential in the HFPSS for \(E_C^{hG_{24}} \wedge A(1)\) is trivial on all of the 8 generators of \((18)\) with the exception of

i) \(d_3(e[0, 12]) = \nu^2 e[1, 5]\)

ii) \(d_3(e[0, 18]) = \nu^2 e[1, 11]\)

Proof. That \(e[0, 0], e[1, 5], e[0, 6], e[1, 11], e[1, 17], e[1, 23]\) are \(d_3\)-cycles follows from Proposition \[4.3.6\]. For the two other classes, the proof of Proposition \[4.3.6\] implicitly implies that the element \(\Theta(e[1, 5])\) and \(\Theta(e[2, 11])\) are detected by \(e[1, 5]\) and \(e[1, 11]\), respectively. Moreover, the element \(e[1, 5]\) and \(e[2, 11]\) are annihilated by \(\nu^2\) in \(\pi_*(\tmf \wedge A(1))\). It follows then that in the HFPSS the classes \(\nu^2 e[1, 5]\) and \(\nu^2 e[1, 11]\) must be hit by some differentials. The only possibilities are \(d_3(e[0, 12]) = \nu^2 e[1, 5]\) and \(d_3(e[0, 18]) = \nu^2 e[1, 11]\).
Proposition 4.3.9. As a module over \( \mathbb{F}_4[\Delta, \bar{\kappa}, \nu]/(\nu^3) \), the term \( E_4 = E_5 \) is generated by 8 generators of
\[
\begin{align*}
& e[0, 0], e[1, 5], e[0, 6], e[1, 11], e[1, 15], e[1, 17], e[1, 21], e[1, 23] \\
& \text{(19)}
\end{align*}
\]
with the relations
\[
\begin{align*}
& \nu^2 e[1, 5] = \nu^2 e[1, 11] = \nu^2 e[1, 15] = \nu^2 e[1, 21] = 0 \\
& \text{(20)}
\end{align*}
\]

Proof. This is straightforward from Proposition 4.3.8 and from the fact that \( \Delta, \bar{\kappa}, \nu \) are \( d_3 \)-cycles in the HFPSS for \( E^h_{C_24} \).

\[ \square \]

\( d_5 \) - differentials We need the following \( d_5 \)-differential in the HFPSS for \( E^h_{C_24} \), that is \( d_5(\Delta) = \bar{\kappa}\nu \) (see [Bau08] for an argument).

Proposition 4.3.10. As a module over \( \mathbb{F}_4[(\Delta^8)^{\pm 1}, \bar{\kappa}, \nu]/(\bar{\kappa}\nu, \nu^3) \), \( E_6 = E_7 \) is generated by the following generators for \( i \in 0, 2, 4, 6 \) with their annihilation ideal:
Proof. Notice that if \(x\) is a class in the \(E_5\)-term, then \(d_5(\Delta^2 k x) = \Delta^2 k d_5(x)\) \(\forall k \in \mathbb{Z}\). This says in particular that the \(E_6\)-term is \(\Delta^2\)-periodic. Next, if \(x\) is a \(d_5\)-cycle and is annihilated by \(\nu^i\), then \(d_5(\Delta x) = 0\) and \(d_5(\Delta \nu^{i-1} x) = 0\). Together with the fact that all of the 8 generators of (19) are permanent cycles (Proposition 4.3.6), it is straightforward to verify that the elements given in the statement of the Proposition form a set of generators of the \(E_6\)-term as a module over \(F_4[\Delta]^{\pm 1}/(\kappa \nu, \nu^4)\). \(\square\)

Remark 4.3.11. In the above proposition, we express the \(E_6\)-term as a module over \(F_4[\Delta, \kappa, \nu]/(\kappa \nu, \nu^4)\) because only the class \(\Delta^8\) is a \(d_5\)-cycle in the HF-PSS for \(E_{hG}^{42}\). In effect, \(d_5(\Delta^4) = 4\Delta^4 \kappa \nu\) which is nontrivial. What is more, \(\Delta^8\) is a permanent cycle in the HF-PSS for \(E_{hG}^{42}\). It means that the HF-PSS for \(E_{hG}^{42} \wedge A(1)\) is a module \(F_4[\Delta]^{\pm 1}\). Note that all the \(\kappa\)-free generators in the \(E_7\)-term are of the form \((\Delta^8)^k x\) where \(k \in \mathbb{Z}\) and \(x\) is one of the generators listed in Proposition 4.3.10. By Corollary 4.3.3 these free \(\kappa\)-family pair up in the manner that each non-permanent \(\kappa\)-family truncates one and only one permanent \(\kappa\)-family. By \(\Delta^8\)-linearity, among these 64 generators, only half of them are permanent cycles and the others support differential. It reduces the problem into two steps: first identify all permanent \(\kappa\)-families, then by which \(\kappa\)-family they are truncated.

Proposition 4.3.12. The generators

\[ e[2, 30], e[2, 32], e[2, 36], e[2, 38], e[2, 42], e[3, 47], e[2, 48], e[3, 53] \]

are permanent cycles.
Proof. We give the proof for $e[2, 30]$ and the other generators are proven in a similar manner. In the $E_6$-term, the Massey product $\langle \kappa, \nu, \nu^2 e[0, 0] \rangle$ can be formed. Since $d_5(\Delta) = \kappa \nu$, we see that $\langle \kappa, \nu, \nu^2 e[0, 0] \rangle$ contains $e[2, 30]$. It is readily checked that the indeterminacy is zero. Thus

$$\langle \kappa, \nu, \nu^2 e[0, 0] \rangle = e[2, 30]$$

At the level of the homotopy groups of $\pi_*(E^G_{15} \wedge A(1))$ one can form the corresponding Toda bracket $\langle \kappa, \nu, \nu^2 e[0, 0] \rangle$ because $\nu \kappa = 0$ in $\pi_*(E^G_{15})$ and inspection in $\pi_*(tmf \wedge A(1))$ tells us that $\nu^3 e[0, 0] = 0$. Furthermore, all hypothesis of the Moss’s convergence theorem are verified. Therefore, $e[2, 30]$ is a permanent cycle representing the Toda bracket $\langle e[0, 0], \nu^3, \kappa \rangle$. For the sake of completeness, we record the Toda bracket expression for the other elements

$$\langle \kappa, \nu, e[1, 5] \rangle = e[2, 32], \langle \kappa, \nu, \nu^2 e[0, 6] \rangle = e[2, 36]$$

$$\langle \kappa, \nu, \nu e[1, 11] \rangle = e[2, 38], \langle \kappa, \nu, \nu e[1, 15] \rangle = e[2, 42]$$

$$\langle \kappa, \nu, \nu^2 e[1, 17] \rangle = e[3, 47], \langle \kappa, \nu, \nu e[1, 21] \rangle = e[2, 48]$$

$$\langle \kappa, \nu, \nu^2 e[2, 23] \rangle = e[3, 53]$$

We have already identified 16 out of 32 permanent cycles. The next 16 ones are not the same for different models of $A(1)$. The difference reflects the different behavior of the $d_2$-differential in the ASS for different models of $A(1)$ (see Proposition 3.0.3).

**Proposition 4.3.13.** In the HFPSS for all four models of $A(1)$, the following 12 generators are permanent cycles:

$$\Delta^2 e[0, 0], \Delta^2 e[1, 5], \Delta^2 e[0, 6], \Delta^2 e[1, 11], \Delta^2 e[1, 15], \Delta^2 e[1, 17]$$

$$\Delta^2 e[1, 21], \Delta^2 e[2, 30], \Delta^2 e[2, 32], \Delta^2 e[2, 36], \Delta^2 e[2, 42], \Delta^2 e[3, 47]$$

The remaining 4 permanent generators for $A_1[00]$ and $A_1[11]$ are

$$\Delta^2 e[1, 23], \Delta^2 e[2, 38], \Delta^2 e[2, 48], \Delta^2 e[3, 53]$$

whereas the remaining 4 permanent generators for $A_1[10]$ and $A_1[01]$ are

$$\Delta^4 e[1, 15], \Delta^4 e[0, 0], \Delta^4 e[1, 5], \Delta^4 e[2, 30]$$
Proof. The graded associated object of the group $\pi_*(tmf \wedge A(1))/\langle \kappa, \nu \rangle$ in certain degrees are given in the following table.

| Degree | 48 | 53 | 54 | 59 | 63 | 65 | 69 | 78 | 80 | 84 | 90 | 95 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|
| Value  | $F_2 \oplus F_2$ | $F_2 \oplus F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ | $F_2$ |

In view of Corollary 4.1.6 and Corollary 4.3.3, inspection in the $E_2$-term shows that the following 12 classes are permanent cycles in the HFPSS for all of the 4 models.

$$\Delta^2 e[0,0], \Delta^2 e[1,5], \Delta^2 e[0,6], \Delta^2 e[1,11], \Delta^2 e[1,15], \Delta^2 e[1,17]$$

$$\Delta^2 e[1,21], \Delta e[2,30], \Delta^2 e[2,32], \Delta^2 e[2,36], \Delta^2 e[2,42], \Delta^2 e[3,47]$$

Next in the ASS for $tmf \wedge A_1[00]$ and $tmf \wedge A_1[11]$, there is no differential until dimension 96. Again, inspection in the $E_2$-term shows that

$$\pi_71(tmf \wedge A_1[00])/\langle \kappa, \nu \rangle = \pi_71(tmf \wedge A_1[11])/\langle \kappa, \nu \rangle \cong F_2$$

and

$$\pi_86(tmf \wedge A_1[00])/\langle \kappa, \nu \rangle = \pi_86(tmf \wedge A_1[11])/\langle \kappa, \nu \rangle \cong F_2$$

It follows that the classes $\Delta^2 e[1,23]$ and $\Delta^2 e[2,38]$ are permanent cycles in the HFPSS for $E_{CG} \wedge A_1[00]$ and $E_{CG} \wedge A_1[11]$.

On the other hand in the ASS for $tmf \wedge A_1[10]$ and $tmf \wedge A_1[01]$, Lemma (3.0.3) and $g$-linearity induce that $d_2(g^2w_2e[4,23]) = g^4e[6,30]$ and $d_2(g^2w_2e[7,38]) = g^6e[1,5]$. Hence, $w_2^2e[3,15]$ and $w_2^2e[6,30]$ survive to the $E_\infty$-term by sparseness. It then follows that $\Delta^4 e[1,15]$ and $\Delta^4 e[2,30]$ are permanent cycles in the HFPSS for $A_1[10]$ and $A_1[01]$.

For $A_1[00]$ and $A_1[11]$, the classes $w_2e[9,48]$ and $w_2e[10,53]$ do not support differential (Lemma 3.0.3), and hence persist to the $E_\infty$-term by sparseness. They are also not divisible neither by $\kappa$ nor by $\nu$. Lastly, both $\{w_2e[48]\}$ and $\{w_2e[53]\}$ are annihilated by $\nu$. The only classes in the HFPSS that match those properties are $\Delta^2 e[2,48]$ and $\Delta^2 e[3,53]$ respectively. Thus the latter are the last 2 of the 32 permanent cycles in the HFPSS for $A_1[00]$ and $A_1[11]$.

Whereas for $A_1[10]$ and $A_1[01]$, the classes $w_2e[9,48]$ and $w_2e[10,53]$ support nontrivial $d_2$ differentials. Thus $w_2^2e[0,0]$ and $w_2^2e[1,5]$ survive to the $E_\infty$-term. By degree reasons both $\{w_2^2e[0,0]\}$ and $\{w_2^2e[1,5]\}$ are not divisible neither by $\kappa$ nor by $\nu$ and moreover their multiples by $\nu$ are not divisible by $\kappa$. Inspection shows that $\Delta^4 e[0,0]$ and $\Delta^4 e[1,5]$ are permanent cycles in the HFPSS.

\[ \square \]
After having determined all permanent $\pi$-families, we go on computing differentials. We remind that each permanent $\pi$-family is truncated by one and only one non-permanent $\pi$-family. We can proceed as follows. We take a permanent module generator, say $x$. Then we locate all non-permanent module generators that potentially support a differential that kills $\pi^nx$ for some $n \leq 6$. One of the following situations will happen:

1) There is no ambiguity, meaning that there is only one generator that supports a differential killing $\pi^nx$ for some $n \leq 6$. So this differential actually happens.

2) There are two generators that can pair up with two multiples of $x$ by powers of $\pi$. In order to decide, we inspect the $\pi$-exponent of $x$ using the ASS.

3) There are two generators that can pair up with the multiple of $x$ by a same power of $\pi$. In this case, inspection on the $\pi$-exponent of $x$ does not help. We will treat each of the particularities case by case. Some Toda brackets will be involved to solve some cases.

In order to make it easier to follow, a permanent module generator is said to be of type 1, 2, 3 respectively if its $\pi$-family is as in the situation 1, 2, 3 above respectively. Finally, the HFPSS for different models of $A(1)$ do not behave the same. To make it clear, we will treat the HFPSS for $A_1[10]$ and $A_1[01]$ in detail at the same time and then point out changes needed for $A_1[00]$ and $A_1[11]$.

Differentials continued for $A_1[01]$ and $A_1[10]$ The reader is invited to follow the discussion of higher differentials along with Figures (22) to (25) below.

The $d_9$-differentials

**Proposition 4.3.14.** There are the following $d_9$-differentials:

1) $d_9(\Delta^2e[1, 23]) = \pi^2e[2, 30]$
2) $d_9(\Delta^6e[1, 23]) = \pi^2\Delta^4e[2, 30]$

**Proof.** The classes $e[2, 30]$ and $\Delta^4e[2, 30]$ are of type 1 and the only possibilities are $d_9(\Delta^2e[1, 23]) = \pi^2e[2, 30]$ and $d_9(\Delta^6e[1, 23]) = \pi^2\Delta^4e[2, 30]$, respectively. \qed

The $d_{15}$-differentials

**Proposition 4.3.15.** There are the following $d_{15}$-differentials:

1) $d_{15}(\Delta^2e[2, 38]) = \pi^4e[1, 5]$
2) $d_{15}(\Delta^2e[2, 48]) = \pi^4e[1, 15]$
3) $d_{15}(\Delta^6e[2, 38]) = \pi^4\Delta^4e[1, 5]$
4) $d_{15}(\Delta^6e[2, 48]) = \pi^4\Delta^4e[1, 15]$

56
Proof. It is readily checked from the chart that all \(e[1,5], e[1,15], \Delta^4 e[1,5], \Delta^4 e[1,15]\) are of type 1 and their \(\pi\)-family is truncated as indicated in the proposition.

The \(d_{17}\)-differentials

**Proposition 4.3.16.** There are the following \(d_{17}\)-differentials:

1. \(d_{17}(\Delta^2 e[3,53]) = \pi^3 e[0,0]\)
2. \(d_{17}(\Delta^4 e[0,6]) = \pi^4 e[1,21]\)
3. \(d_{17}(\Delta^4 e[1,17]) = \pi^4 e[2,32]\)
4. \(d_{17}(\Delta^4 e[1,21]) = \pi^4 e[2,36]\)
5. \(d_{17}(\Delta^4 e[2,32]) = \pi^4 e[3,47]\)
6. \(d_{17}(\Delta^6 e[0,6]) = \pi^4 \Delta^2 e[1,21]\)
7. \(d_{17}(\Delta^6 e[1,17]) = \pi^4 \Delta^2 e[2,32]\)
8. \(d_{17}(\Delta^6 e[1,21]) = \pi^4 \Delta^2 e[2,36]\)
9. \(d_{17}(\Delta^6 e[2,32]) = \pi^4 \Delta^2 e[3,47]\)
10. \(d_{17}(\Delta^6 e[3,53]) = \pi^5 \Delta^4 e[0,0]\)
11. \(d_{17}(\Delta^4 e[1,23]) = \pi^4 e[2,38]\)
12. \(d_{17}(\Delta^4 e[2,38]) = \pi^4 e[3,53]\)
13. \(d_{17}(\Delta^6 e[0,0]) = \pi^4 \Delta^2 e[1,15]\)
14. \(d_{17}(\Delta^6 e[1,15]) = \pi^4 \Delta^2 e[2,30]\)

Proof. (1)-(10) All of the generators of

\[
e[0,0], e[1,21], e[2,32], e[2,36], e[3,47]
\]

\[
\Delta^2 e[1,21], \Delta^2 e[2,32], \Delta^2 e[2,36], \Delta^2 e[3,47], \Delta^4 e[0,0]
\]

are of type 1.

(11) \(e[2,38]\) is of type 2. The differentials that can truncate its \(\pi\)-family are \(d_{17}(\Delta^4 e[1,23]) = \pi^4 e[2,38]\) and \(d_{25}(\Delta^6 e[1,15]) = \pi^6 e[2,38]\). The latter can not happen because the spectral sequence collapses that the \(E_{24}\)-term.

Therefore, it must be that \(d_{17}(\Delta^4 e[1,23]) = \pi^4 e[2,38]\).

(12) \(e[3,53]\) is of type 2. Its \(\pi\)-family can be truncated by \(d_{17}(\Delta^4 e[2,38]) = \pi^4 e[3,53]\) or \(d_{25}(\Delta^6 e[2,30]) = \pi^6 e[3,53]\). There can not be any \(d_{25}\) differential in the spectral sequence because it collapses at the \(E_{24}\)-term.

Hence, we obtain that \(d_{17}(\Delta^4 e[2,38]) = \pi^4 e[3,53]\).

(13) \(\Delta^2 e[1,15]\) is of type 3. In its \(\pi\)-family only \(\pi^4 \Delta^2 e[1,15]\) can be a target of differentials which are \(d_{17}(\Delta^6 e[0,0]) = \pi^4 \Delta^2 e[1,15]\) and \(d_{15}(\Delta^4 e[2,48]) = \pi^4 \Delta^2 e[1,15]\). But if \(d_{15}(\Delta^4 e[2,48]) = \pi^4 \Delta^2 e[1,15]\) then the only class that can truncate the \(\pi\)-family of \(e[1,23]\) is \(\Delta^6 e[0,0]\) and by a \(d_{25}\)-differential: \(d_{25}(\Delta^6 e[0,0]) = \pi^6 e[1,23]\). But this contradicts the fact that the spectral sequence collapses at the \(E_{24}\)-term by Proposition 4.3.5 Thus we must have that \(d_{17}(\Delta^6 e[0,0]) = \pi^4 \Delta^2 e[1,15]\).
(14) \( \Delta^2 e[2, 30] \) is of type 2. Its \( \pi \)-family can be truncated by \( \Delta^4 e[1, 23] \) by a \( d_9 \)-differential or by \( \Delta^6 e[1, 15] \) by a \( d_{17} \)-differential. However, the former possibility cannot occur because of part (11). Therefore, \( d_{17}(\Delta^6 e[1, 15]) = \pi^4 \Delta^2 e[2, 30] \).

The \( d_{19} \)-differentials

**Proposition 4.3.17.** There are the following \( d_{19} \)-differentials:

1. \( d_{19}(\Delta^4 e[1, 11]) = \pi^3 e[0, 6] \)
2. \( d_{19}(\Delta^4 e[3, 47]) = \pi^3 e[2, 42] \)
3. \( d_{19}(\Delta^6 e[1, 11]) = \pi^5 \Delta^2 e[0, 6] \)
4. \( d_{19}(\Delta^6 e[3, 47]) = \pi^5 \Delta^2 e[2, 42] \)
5. \( d_{19}(\Delta^6 e[1, 5]) = \pi^5 \Delta^2 e[0, 0] \)
6. \( d_{19}(\Delta^4 e[3, 53]) = \pi^5 e[2, 48] \)

**Proof.** (1)-(4) All of the classes of

\[ e[0, 6], e[2, 42], \Delta^2 e[0, 6], \Delta^2 e[2, 42] \]

are of type 1.

(5) The class \( \Delta^2 e[0, 0] \) is of type 3 and its \( \pi \)-family can be truncated either by \( d_{17}(\Delta^4 e[3, 53]) = \pi^5 \Delta^2 e[0, 0] \) or by \( d_{19}(\Delta^4 e[1, 5]) = \pi^5 \Delta^2 e[0, 0] \). Suppose \( d_{17}(\Delta^4 e[3, 53]) = \pi^5 \Delta^2 e[0, 0] \). This would leave us with the differential \( d_{21}(\Delta^6 e[1, 5]) = \pi^5 \Delta^2 e[2, 48] \). It would imply the Massey product in the \( E_{22} \)-term

\[ \langle \pi^5, e[2, 48], \nu \rangle = \nu \Delta^6 e[1, 5] \]

with zero indeterminacy in the \( E_{22} \)-term. All conditions of the Moss’s convergence theorem met, the Toda bracket \( \langle \pi^5, e[2, 48], \nu \rangle \) could be formed and would contain an element represented by \( \nu \Delta^6 e[1, 5] \). This contradicts Corollary 3.0.5. This contradiction proves that

\[ d_{19}(\Delta^6 e[1, 5]) = \pi^5 \Delta^2 e[0, 0] \]

(6) The class \( e[2, 48] \) is of type 2 and its \( \pi \)-family is truncated either by \( d_{19}(\Delta^4 e[3, 53]) = \pi^5 e[2, 48] \) or by \( d_{21}(\Delta^6 e[1, 5]) = \pi^5 e[2, 48] \). However, part (5) of Proposition 4.3.17 rules out the latter because the class \( \Delta^6 e[1, 5] \) must pair up with the class \( \Delta^2 e[0, 0] \).

The \( d_{23} \)-differentials

**Proposition 4.3.18.** There are the following \( d_{23} \)-differentials:
Proof. (1)-(5) All of the classes
\[ e[1, 11], e[1, 17], e[1, 23], \Delta^2 e[1, 11], \Delta^2 e[1, 17] \]
are of type 1.

(6) The class \( \Delta^2 e[1, 5] \) is of type 2. The two possibilities are \( d_{15}(\Delta^4 e[2, 38]) = \bar{\kappa}^4 \Delta^2 e[1, 5] \) and \( d_{23}(\Delta^6 e[2, 30]) = \bar{\kappa}^6 \Delta^2 e[1, 5] \). However, part (12) of Proposition 4.3.16 rules out the former because the class \( \Delta^4 e[2, 38] \) must pair up with the class \( e[3, 38] \) by a \( d_{17} \)-differential \( d_{17}(\Delta^4 e[2, 38]) = \bar{\kappa}^4 e[3, 53] \).

\[ \square \]

Remark 4.3.19. The above differentials (from \( d_9 \) to \( d_{23} \)) together with their \( \bar{\kappa} \)-and \( \Delta^3 \)-linearity exhaust all remaining differentials.

The case of \( A_1[00] \) and \( A_1[11] \) The analysis of the HFPSS for \( A_1[00] \) and \( A_1[11] \) can be done in the same manner of that for \( A_1[10] \) and \( A_1[01] \). All of the differentials are identical except for 8 ones involving 16 of the generators of Proposition 4.3.10. We will be content to list out all the modifications.

\[ d_{17}(\Delta^4 e[1, 15]) = \bar{\kappa}^4 e[2, 30] \text{ instead of } d_0(\Delta^2 e[1, 23]) = \bar{\kappa}^2 e[2, 30] \]
\[ d_{17}(\Delta^6 e[1, 23]) = \bar{\kappa}^4 \Delta^2 e[2, 38] \text{ instead of } d_0(\Delta^6 e[1, 23]) = \bar{\kappa}^2 \Delta^2 e[2, 30] \]
\[ d_{17}(\Delta^4 e[0, 0]) = \bar{\kappa}^4 e[1, 15] \text{ instead of } d_{15}(\Delta^2 e[2, 48]) = \bar{\kappa}^4 e[1, 15] \]
\[ d_{17}(\Delta^6 e[2, 38]) = \bar{\kappa}^4 \Delta^2 e[3, 53] \text{ instead of } d_{15}(\Delta^6 e[2, 38]) = \bar{\kappa}^4 \Delta^2 e[1, 5] \]
\[ d_{19}(\Delta^4 e[1, 5]) = \bar{\kappa}^4 e[0, 0] \text{ instead of } d_{17}(\Delta^2 e[3, 53]) = \bar{\kappa}^4 e[0, 0] \]
\[ d_{19}(\Delta^6 e[3, 53]) = \bar{\kappa}^5 \Delta^2 e[2, 48] \text{ instead of } d_{17}(\Delta^6 e[3, 53]) = \bar{\kappa}^5 \Delta^2 e[0, 0] \]
\[ d_{23}(\Delta^6 e[2, 48]) = \bar{\kappa}^6 \Delta^2 e[1, 23] \text{ instead of } d_{15}(\Delta^6 e[2, 48]) = \bar{\kappa}^4 \Delta^4 e[1, 15] \]
\[ d_{23}(\Delta^4 e[2, 30]) = \bar{\kappa}^6 e[1, 5] \text{ instead of } d_{15}(\Delta^2 e[2, 38]) = \bar{\kappa}^4 e[1, 5] \]
The Figures (22) to (25) represent the HFPSS for $E_{hG^{24}} \wedge A_1[10]$ and $E_{hG^{24}} \wedge A_1[01]$ from the $E_7$-term on. Each black dot $\bullet$ represents a class generating a group $\mathbb{F}_4$ which survives to the $E_\infty$-term. Each circle $\circ$ represent a class which either is hit by a differential or supports a differential higher than $d_5$. We only represent the differentials on generators listed in Proposition 4.3.10 but not those generated by $\pi$-linearity.

Figure 22 – HFPSS for $A_1[10]$ and $A_1[01]$ from $E_7$-term with $0 \leq t - s \leq 48$
Figure 23 – HFPSS for $A_1[10]$ and $A_1[01]$ from $E_7$-term with $48 \leq t - s \leq 96$
Figure 24 – HFPSS for $A_1[10]$ and $A_1[01]$ from $E_7$-term with $96 \leq t - s \leq 144$
Figure 25 – HFPSS for $A_1[10]$ and $A_1[01]$ from $E_7$-term with $144 \leq t - s \leq 197$
References

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum tmf. In Groups, homotopy and configuration spaces, volume 13 of Geom. Topol. Monogr., pages 11–40. Geom. Topol. Publ., Coventry, 2008.

[Bea15] Agnès Beaudry. The algebraic duality resolution at $p = 2$. Algebr. Geom. Topol., 15(6):3653–3705, 2015.

[Bea17] Agnès Beaudry. Towards the homotopy of the $K(2)$-local Moore spectrum at $p = 2$. Adv. Math., 306:722–788, 2017.

[Beh06] Mark Behrens. A modular description of the $K(2)$-local sphere at the prime 3. Topology, 45(2):343–402, 2006.

[Beh12] Mark Behrens. The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited. Adv. Math., 230(2):458–492, 2012.

[BEM17] Prasit Bhattacharya, Philip Egger, and Mark Mahowald. On the periodic $v_2$-self-map of $A_1$. Algebr. Geom. Topol., 17(2):657–692, 2017.

[BG16] Irina Bobkova and Paul G. Goerss. Topological resolutions in $k(2)$-local homotopy theory at the prime 2. arXiv:1610.00158, 2016.

[BL10] Mark Behrens and Tyler Lawson. Topological automorphic forms. Mem. Amer. Math. Soc., 204(958):xxiv+141, 2010.

[Bro82] Kenneth S. Brown. Cohomology of groups, volume 87 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

[Dev95] Ethan S. Devinatz. Morava’s change of rings theorem. In The Čech centennial (Boston, MA, 1993), volume 181 of Contemp. Math., pages 83–118. Amer. Math. Soc., Providence, RI, 1995.

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill. Topological modular forms, volume 201 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2014.

[DH04] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. Topology, 43(1):1–47, 2004.

[DM81] Donald M. Davis and Mark Mahowald. $v_1$- and $v_2$-periodicity in stable homotopy theory. Amer. J. Math., 103(4):615–659, 1981.

[DM82] Donald M. Davis and Mark Mahowald. Ext over the subalgebra $A_2$ of the Steenrod algebra for stunted projective spaces. In Current trends in algebraic topology, Part 1 (London, Ont., 1981), volume 2 of CMS Conf. Proc., pages 297–342. Amer. Math. Soc., Providence, RI, 1982.
[GH16] Paul G. Goerss and Hans-Werner Henn. The Brown-Comenetz dual of the \( K(2) \)-local sphere at the prime 3. *Adv. Math.*, 288:648–678, 2016.

[GHM04] Paul Goerss, Hans-Werner Henn, and Mark Mahowald. The homotopy of \( L_2 V(1) \) for the prime 3. In *Categorical decomposition techniques in algebraic topology* (Isle of Skye, 2001), volume 215 of *Progr. Math.*, pages 125–151. Birkhäuser, Basel, 2004.

[GHMR05] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk. A resolution of the \( K(2) \)-local sphere at the prime 3. *Ann. of Math.* (2), 162(2):777–822, 2005.

[Hen07] Hans-Werner Henn. On finite resolutions of \( K(n) \)-local spheres. In *Elliptic cohomology*, volume 342 of *London Math. Soc. Lecture Note Ser.*, pages 122–169. Cambridge Univ. Press, Cambridge, 2007.

[HKM13] Hans-Werner Henn, Nasko Karamanov, and Mark Mahowald. The homotopy of the \( K(2) \)-local Moore spectrum at the prime 3 revisited. *Math. Z.*, 275(3-4):953–1004, 2013.

[HS99] Mark Hovey and Neil P. Strickland. Morava \( K \)-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.

[Laz55] Michel Lazard. Sur les groupes de Lie formels à un paramètre. *Bull. Soc. Math. France*, 83:251–274, 1955.

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal Lie groups. *Bull. Soc. Math. France*, 94:49–59, 1966.

[Mat16] Akhil Mathew. The homology of tmf. *Homology Homotopy Appl.*, 18(2):1–29, 2016.

[Mil58] John Milnor. The Steenrod algebra and its dual. *Ann. of Math.* (2), 67:150–171, 1958.

[Mos70] R. Michael F. Moss. Secondary compositions and the Adams spectral sequence. *Math. Z.*, 115:283–310, 1970.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[Rez98] Charles Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations* (Evanston, IL, 1997), volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.
[Shi97] Katsumi Shimomura. The homotopy groups of the $L_2$-localized Toda-Smith spectrum $V(1)$ at the prime 3. *Trans. Amer. Math. Soc.*, 349(5):1821–1850, 1997.

[Shi00] Katsumi Shimomura. The homotopy groups of the $L_2$-localized mod 3 Moore spectrum. *J. Math. Soc. Japan*, 52(1):65–90, 2000.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[SW02] Katsumi Shimomura and Xiangjun Wang. The homotopy groups $\pi_*(L_2S^0)$ at the prime 3. *Topology*, 41(6):1183–1198, 2002.

[SY95] Katsumi Shimomura and Atsuko Yabe. The homotopy groups $\pi_*(L_2S^0)$. *Topology*, 34(2):261–289, 1995.