Linear Bounds between Contraction Coefficients for $f$-Divergences

Anuran Makur and Lizhong Zheng

Abstract

Data processing inequalities for $f$-divergences can be sharpened using constants called “contraction coefficients” to produce strong data processing inequalities. For any discrete source-channel pair, the contraction coefficients for $f$-divergences are lower bounded by the contraction coefficient for $\chi^2$-divergence. In this paper, we elucidate that this lower bound can be achieved by driving the input $f$-divergences of the contraction coefficients to zero. Then, we establish a linear upper bound on the contraction coefficients for a certain class of $f$-divergences using the contraction coefficient for $\chi^2$-divergence, and refine this upper bound for the salient special case of Kullback-Leibler (KL) divergence. Furthermore, we present an alternative proof of the fact that the contraction coefficients for KL and $\chi^2$-divergences are equal for a Gaussian source with an additive Gaussian noise channel (where the former coefficient can be power constrained). Finally, we generalize the well-known result that contraction coefficients of channels (after extremizing over all possible sources) for all $f$-divergences with non-linear operator convex $f$ are equal. In particular, we prove that the so called “less noisy” preorder over channels can be equivalently characterized by any non-linear operator convex $f$-divergence.

CONTENTS

I Introduction .......................... 2
I-A Outline ................................ 2

II Overview of Contraction Coefficients ........................................... 2
II-A $f$-Divergence .................................. 2
II-B Contraction Coefficients of Source-Channel Pairs .................. 5
II-C Coefficients of Ergodicity ............................................. 9
II-D Contraction Coefficients of Channels ................................. 10

III Main Results and Discussion ............................................. 12
III-A Local Approximation of Contraction Coefficients .............. 13
III-B Linear Bounds between Contraction Coefficients ............... 14
III-C Contraction Coefficients of Gaussian Random Variables .... 16
III-D Less Noisy Preorder and Operator Convexity ..................... 17

IV Proofs of Linear Bounds between Contraction Coefficients ........ 18
IV-A Bounds on $f$-Divergences using $\chi^2$-Divergence ................ 18
IV-B Proofs of Theorems 8 and 10 ....................................... 22
IV-C Ergodicity of Markov Chains ...................................... 23
IV-D Tensorization of Bounds between Contraction Coefficients .. 24

V Proof of Equivalence between Gaussian Contraction Coefficients . 25

VI Proof of Equivalent Characterizations of the Less Noisy Preorder 28
VI-A Operator Convex Functions ....................................... 28
VI-B Proof of Theorem 12 ............................................. 29

VII Conclusion .......................................... 30

Appendix A: Proof of Proposition 2 ........................................ 30
Appendix B: Proof of Proposition 3 ........................................ 31

* A. Makur and L. Zheng are with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, USA (e-mail: a_makur@mit.edu; lizhong@mit.edu).
This research was supported in part by the National Science Foundation under Award 1216476 and in part by the Hewlett-Packard Fellowship. This work was presented in part at the 2015 53rd Annual Allerton Conference on Communication, Control, and Computing [1].
I. INTRODUCTION

Strong data processing inequalities for Kullback-Leibler divergence (KL divergence or relative entropy) and mutual information [2]–[6], and more generally (Csiszár) \( f \)-divergences [7]–[9], have been studied extensively in various contexts in information theory. They are obtained by tightening traditional data processing inequalities using distribution dependent constants known as “contraction coefficients.” Contraction coefficients for \( f \)-divergences come in two flavors: those pertaining to source-channel pairs, and those pertaining only to channels. The broad goal of this paper is to study various inequalities and equalities between contraction coefficients in both settings. In the source-channel pair setting, we will primarily establish general bounds on contraction coefficients for certain classes of \( f \)-divergences, as well as specific bounds on the contraction coefficient for KL divergence, in terms of the contraction coefficient for \( \chi^2 \)-divergence (or squared Hirschfeld-Gebelein-Rényi maximal correlation). On the other hand, in the channel only setting, we will prove an appropriate generalization of the well-known result that the contraction coefficient for KL divergence is equal to the contraction coefficient for any \( f \)-divergence with non-linear operator convex \( f \) [10].

A. Outline

We briefly delineate the remainder of our discussion. We will first provide an overview of the burgeoning literature on contraction coefficients in section II. This section will compile formal definitions and key properties of both the aforementioned variants of contraction coefficients, and briefly outline their genesis in the study of ergodicity. Then, we will state and explain our main results, and discuss related literature in section III. In section IV, we will present some useful bounds between \( f \)-divergences and \( \chi^2 \)-divergence, and use them to prove linear upper bounds on contraction coefficients of source-channel pairs for a certain class of \( f \)-divergences and KL divergence. Following this, we will prove the equivalence between certain contraction coefficients of Gaussian sources with additive Gaussian noise channels in section V. In section VI, we will prove equivalent characterizations of the less noisy preorder over channels using non-linear operator convex \( f \)-divergences by generalizing the main result of [10]. Finally, we will conclude our discussion and propose future research directions in section VII.

II. OVERVIEW OF CONTRACTION COEFFICIENTS

In this section, we will define contraction coefficients for \( f \)-divergences and present some well-known facts about them. We begin by introducing some preliminary definitions and notation pertaining to \( f \)-divergences in subsection II-A, and then give a brief prelude on contraction coefficients and strong data processing inequalities in the ensuing subsections.

A. \( f \)-Divergence

Consider a discrete sample space \( \mathcal{X} \triangleq \{1, \ldots, |\mathcal{X}|\} \) with \( 2 \leq |\mathcal{X}| < +\infty \), where we let singletons in \( \mathcal{X} \) be natural numbers without loss of generality. Let \( \mathcal{P}_\mathcal{X} \subseteq (\mathbb{R}^{|\mathcal{X}|})^\ast \) denote the probability simplex in \((\mathbb{R}^{|\mathcal{X}|})^\ast\) of all probability mass functions (pmfs) on \( \mathcal{X} \), where \((\mathbb{R}^{|\mathcal{X}|})^\ast\) is the dual vector space of \( \mathbb{R}^{|\mathcal{X}|} \) consisting of all row vectors of length \(|\mathcal{X}|\). We perceive \( \mathcal{P}_\mathcal{X} \) as the set of all possible probability distributions of a random variable \( X \) with range \( \mathcal{X} \), and construe each pmf \( P_X \in \mathcal{P}_\mathcal{X} \) as a row vector \( P_X = (P_X(1), \ldots, P_X(|\mathcal{X}|)) \in (\mathbb{R}^{|\mathcal{X}|})^\ast \). We also let \( \mathcal{P}_\mathcal{X}^\circ \triangleq \{P_X \in \mathcal{P}_\mathcal{X} : \forall x \in \mathcal{X}, P_X(x) > 0\} \) denote the relative interior of \( \mathcal{P}_\mathcal{X} \). A popular notion of “distance” between pmfs in information theory is the \( f \)-divergence, which was independently introduced by Csiszár in [11], [12] and by Ali and Silvey in [13].

Definition 1 (\( f \)-Divergence [11]–[13]). Given a convex function \( f : (0, \infty) \rightarrow \mathbb{R} \) that satisfies \( f(1) = 0 \), we define the \( f \)-divergence of a pmf \( P_X \in \mathcal{P}_\mathcal{X} \) from a pmf \( R_X \in \mathcal{P}_\mathcal{X} \) as:

\[
D_f(R_X||P_X) \triangleq \sum_{x \in \mathcal{X}} P_X(x) f \left( \frac{R_X(x)}{P_X(x)} \right) \tag{1}
\]

\[
= \mathbb{E}_{P_X} \left[ f \left( \frac{R_X(X)}{P_X(X)} \right) \right] \tag{2}
\]
where $\mathbb{E}_{P_X}[\cdot]$ denotes the expectation with respect to $P_X$, and we assume that $f(0) = \lim_{t \to 0^+} f(t)$, $0f(0/0) = 0$, and for all $r > 0$, $0f(r/0) = \lim_{p \to 0^+} pf(r/p) = r \lim_{p \to 0^+} pf(1/p)$, based on continuity and other considerations (see [14, Section 3] for details).

The $f$-divergences generalize several well-known divergence measures that are used in information theory, statistics, and probability theory. We present some examples below:

1) **Total variation (TV) distance:** When $f(t) = \frac{1}{2}|t - 1|$, the corresponding $f$-divergence is the TV distance:

$$
\|R_X - P_X\|_\text{tv} \triangleq \max_{\lambda \in X} |R_X(A) - P_X(A)|
\text{ (3)}
= \frac{1}{2} \|R_X - P_X\|_1
\text{ (4)}
$$

where $P_X(A) = \sum_{x \in A} P_X(x)$ for any $A \subseteq X$, $\|\|_p$ denotes the $\ell^p$-norm for $p \in [1, \infty]$, and the second equality as well as several other characterizations of TV distance are proved in [15, Chapter 4].

2) **KL divergence:** When $f(t) = t \log(t)$ (where $\log(.)$ is the natural logarithm with base $e$ throughout this paper), the corresponding $f$-divergence is the KL divergence:

$$
D(R_X||P_X) \triangleq \sum_{x \in X} R_X(x) \log \left( \frac{R_X(x)}{P_X(x)} \right).
\text{ (5)}
$$

3) **$\chi^2$-divergence:** When $f(t) = (t - 1)^2$ or $f(t) = t^2 - 1$, the corresponding $f$-divergence is the $\chi^2$-divergence:

$$
\chi^2(R_X||P_X) \triangleq \sum_{x \in X} \frac{(R_X(x) - P_X(x))^2}{P_X(x)}.
\text{ (6)}
$$

4) **Hellinger divergence of order $\alpha 
\in \langle 0, \infty\rangle \backslash \{1\} \text{ [16, Definition 2.10]: When } f(t) = \frac{t^\alpha - 1}{\alpha}, \text{ the corresponding } f \text{-divergence is the Hellinger divergence (or Tsallis divergence) of order } \alpha:

$$
\mathcal{H}_\alpha(R_X||P_X) \triangleq \frac{1}{\alpha - 1} \left( \sum_{x \in X} R_X(x)^\alpha P_X(x)^{1-\alpha} - 1 \right).
\text{ (7)}
$$

where $\frac{1}{2}\mathcal{H}^2(R_X||P_X)$ is the squared Hellinger distance, $\mathcal{H}_2(R_X||P_X) = \chi^2(R_X||P_X)$ is the $\chi^2$-divergence, and $\alpha = 1$ corresponds to KL divergence, $\mathcal{H}_1(R_X||P_X) = D(R_X||P_X)$, by analytic extension, cf. [17, Section II].

5) **Vince-Le Cam divergence of order $\lambda 
\in \langle 0, 1\rangle \text{ [18]-[20]: When } f(t) = \lambda \frac{(t-1)}{\lambda t + (1-\lambda)}, \text{ where } \lambda = 1 - \lambda, \text{ the corresponding } f \text{-divergence is the Vince-Le Cam divergence of order } \lambda:

$$
\mathcal{ LC}_\lambda(R_X||P_X) \triangleq \frac{\lambda \chi^2(R_X||\lambda R_X + \bar{P}_X)}{\lambda R_X(x) + \bar{P}_X(x)}
\text{ (8)}
$$

where the special case of $\lambda = \frac{1}{2}$ is known as the Vince-Le Cam distance or triangular discrimination.

Although $f$-divergences are not valid metrics in general,1 they satisfy several useful properties. To present some of these properties, we let $\mathcal{Y} \triangleq \{1, \ldots, |\mathcal{Y}|\}$ denote another discrete alphabet with $2 \leq |\mathcal{Y}| < +\infty$, and corresponding probability simplex $P_\mathcal{Y}$ of possible pmfs of a random variable $Y$ with range $\mathcal{Y}$. Furthermore, we let $P_{\mathcal{Y}|X}$ denote the set of $|X| \times |\mathcal{Y}|$ row stochastic matrices in $\mathbb{R}^{|X| \times |\mathcal{Y}|}$. Throughout our discussion, the discrete channel of conditional pmfs $\{P_{Y|x=x} \in P_\mathcal{Y}: x \in X\}$ will correspond to a transition probability matrix $W \in P_{\mathcal{Y}|X}$ (where the $x$th row of $W$ is $P_{Y|x=x}$). We interpret $W: P_X \rightarrow P_\mathcal{Y}$ as a map that takes input pmfs $P_X \in P_X$ to output pmfs $P_Y = P_XW \in P_\mathcal{Y}$. Some well-known properties of $f$-divergences are presented next, cf. [11], [12]:

1) **Non-negativity and reflexivity:** For every $R_X, P_X \in P_X$, $D_f(R_X||P_X) \geq 0$ (by Jensen’s inequality) with equality if $R_X = P_X$. Furthermore, if $f$ is strictly convex at unity,2 then equality holds if and only if $R_X = P_X$.

2) **Affine invariance:** Consider any affine function $\alpha(t) = a(t-1)$ with $a \in \mathbb{R}$. Clearly, $D_{\alpha}(R_X||P_X) = 0$ for every $R_X, P_X \in P_X$. Hence, $f$ and $f + \alpha$ define the same $f$-divergence, i.e. $D_{f+\alpha}(R_X||P_X) = D_f(R_X||P_X)$ for every $R_X, P_X \in P_X$.3

1We often distinguish $f$-divergences that are metrics by dubbing them as “distances” (e.g. TV distance, Hellinger distance), while the term “divergence” is reserved for $f$-divergences that are not metrics (e.g. KL divergence, $\chi^2$-divergence).

2Strict convexity of $f: (0, \infty) \rightarrow \mathbb{R}$ at unity implies that for every $x, y \in (0, \infty)$ and $\lambda \in (0, 1)$ such that $\lambda x + (1-\lambda)y = 1$, $\lambda f(x) + (1-\lambda)f(y) > f(1)$. The aforementioned examples of $f$-divergences have this property.

3Note that $f + \alpha: (0, \infty) \rightarrow \mathbb{R}$ is the function $(f + \alpha)(t) = f(t) + a(t-1)$.
3) **Csizárs duality**: Let the Csizárs conjugate function of \( f \) be \( f^* : (0, \infty) \to \mathbb{R}, f^*(t) = tf\left(\frac{1}{t}\right) \), which is also convex and satisfies \( f^{**} = f \). Then, \( D_{f^*}(P_X || R_X) = D_f(R_X || P_X) \) for every \( R_X, P_X \in \mathcal{P}_X \).

4) **Joint convexity**: The map \((R_X, P_X) \rightarrow D_f(R_X || P_X)\) is convex in the pair of input pmfs.

5) **Data processing inequality (DPI)**: For every \( W \in \mathcal{P}_{Y|X} \), and every \( R_X, P_X \in \mathcal{P}_X \), we have (by the convexity of perspective functions):

\[
D_f(R_X W || P_X W) \leq D_f(R_X || P_X)
\]  

where equality holds if and only if \( Y \) is a sufficient statistic of \( X \) for performing inference about the pair \((R_X, P_X)\) (see e.g. [21, Section 3.1]).

While [12] and [22, Section 2] contain the original presentation of these properties, we also refer readers to [21, Section 6] for a more didactic presentation. Note that due to property 2, we only consider \( f \)-divergences with non-linear \( f \).

We next define a notion of “information” between random variables corresponding to any \( f \)-divergence that also exhibits a DPI. For random variables \( X \) and \( Y \) with joint pmf \( P_{X,Y} \) (consisting of \( (P_X, W) \)), the mutual \( f \)-information between \( X \) and \( Y \) is defined as [23]:

\[
I_f(X;Y) \equiv D_f(P_{X,Y} || P_X P_Y) = \sum_{x \in X} P_X(x) D_f(P_{Y|X=x} || P_Y)
\]

where \( P_X P_Y \) denotes the product distribution specified by the marginal pmfs \( P_X \) and \( P_Y \) (also see [9, Equation (V.8)], [24, Equation (11)]). When \( f(t) = t \log(t) \), mutual \( f \)-information corresponds to standard mutual information. Moreover, mutual \( f \)-information possesses certain natural properties of information measures. For example, if \( X \) and \( Y \) are independent, then \( I_f(X;Y) = 0 \), and the converse holds when \( f \) is strictly convex at unity.

Now suppose \( U \) is another random variable with discrete alphabet \( \mathcal{U} \equiv \{1, \ldots, |\mathcal{U}|\} \) such that \( 2 \leq |\mathcal{U}| < +\infty \). If \((U, X, Y)\) are jointly distributed and form a Markov chain \( U \rightarrow X \rightarrow Y \), then they satisfy the DPI [22]:

\[
I_f(U;Y) \leq I_f(U;X)
\]

where equality holds if and only if \( Y \) is a sufficient statistic of \( X \) for performing inference about \( U \) (i.e. \( U \rightarrow Y \rightarrow X \) also forms a Markov chain). Needless to say, the DPIs (10) and (13) are generalizations of the better known DPIs for KL divergence and mutual information that can be found in standard texts on information theory, e.g. [25]. Finally, note that although we cite [11], [12] and [22] for the DPIs (10) and (13) respectively, both DPIs were also independently proved in [26], [27].

We end this subsection with a brief exposition of the “local quadratic behavior” of \( f \)-divergences. Local approximations of \( f \)-divergences are geometrically appealing because they transform neighborhoods of stochastic manifolds, with certain \( f \)-divergences as the distance measures, into inner product spaces with the Fisher information metric [28]–[30]. Consider any reference pmf \( P_X \in \mathcal{P}_X \) (which forms the “center of the local neighborhood” of pmfs that we will be concerned with), and any other pmf \( R_X \in \mathcal{P}_X \). Let us define the spherical perturbation vector of \( R_X \) from \( P_X \) as:

\[
K_X \equiv (R_X - P_X) \text{diag}\left(\sqrt{P_X}\right)^{-1}
\]

where \( \sqrt{\cdot} \) denotes the entry-wise square root when applied to a vector, and diag(\(\cdot\)) denotes a diagonal matrix with its input vector on the principal diagonal. Using \( K_X \), we can construct a trajectory of spherically perturbed pmfs:

\[
R_X^{(\epsilon)} = P_X + \epsilon K_X \text{diag}(\sqrt{P_X})
\]

which is parametrized by \( \epsilon \in (0, 1) \), and corresponds to the convex combinations of \( R_X \) and \( P_X \). Note that \( K_X \) provides the direction of the trajectory (15), and \( \epsilon \) controls how close \( R_X^{(\epsilon)} \) and \( P_X \) are. Furthermore, (15) clarifies why \( K_X \) is called a “spherical perturbation” vector; \( K_X \) is proportional to the first order perturbation term as \( \epsilon \to 0 \) of \( \sqrt{R_X^{(\epsilon)}} \) from \( \sqrt{P_X} \), which are embeddings of the pmfs \( R_X^{(\epsilon)} \) and \( P_X \) as vectors on the unit sphere in \((\mathbb{R}^{[X]})^\mathbb{R}\).

4Note also that \( f \) is strictly convex at unity if and only if \( f^* \) is strictly convex at unity.

5Although Csizárs studies a different notion known as \( f \)-informativity in [22], (13) can be distilled from the proof of [22, Proposition 2.1].

6In particular, Ziv and Zakai studied generalized information functionals in [27], and a specialization of [27, Theorem 5.1] yields \( D_f(P_U, P_Y || P_{U,Y}) \leq D_f(P_U, P_X || P_{U,X}) \) for any Markov chain \( U \rightarrow X \rightarrow Y \). By the Csizárs duality property of \( f \)-divergences, this implies (13).
Now suppose the function \( f : (0, \infty) \to \mathbb{R} \) that defines our \( f \)-divergence is twice differentiable at unity with \( f''(1) > 0 \). Then, Taylor’s theorem can be used to show that this \( f \)-divergence is locally proportional to \( \chi^2 \)-divergence, cf. [31, Section 4] (or [25] for the KL divergence case):

\[
D_f(R_X^{(\epsilon)} \| P_X) = \frac{f''(1)}{2} \epsilon^2 \chi^2(R_X \| P_X) + o(\epsilon^2) \tag{17}
\]

\[
= \frac{f''(1)}{2} \epsilon^2 \|K_X\|_2^2 + o(\epsilon^2) \tag{18}
\]

where we use the Bachmann-Landau asymptotic little-\( o \) notation. The local approximation in (18) is somewhat more flexible than the version in (17). Indeed, we can construct a trajectory (15) using a spherical perturbation vector \( K_X \in (\mathbb{R}^{|X|})^* \) that satisfies the orthogonality constraint \( \sqrt{T_X}K_X^T = 0 \), but is not of the form (14). For sufficiently small \( \epsilon \neq 0 \) (depending on \( P_X \) and \( K_X \)), the vectors \( R_X^{(\epsilon)} \) defined by (15) are in fact valid pmfs in \( \mathcal{P}_X \). So, the approximation in (18) continues to hold because it is concerned with the regime where \( \epsilon \to 0 \).

It is also straightforward to verify that \( f \)-divergences with \( f''(1) > 0 \) are locally symmetric, i.e. \( D_f(R_X^{(\epsilon)} \| P_X) = D_f(P_X \| R_X^{(\epsilon)}) + o(\epsilon^2) \). Hence, they resemble the standard Euclidean metric within a “neighborhood” of pmfs around a reference pmf in \( \mathcal{P}_X \). Note that the advantage of using spherical perturbations \( \{K_X \in (\mathbb{R}^{|X|})^* : \sqrt{T_X}K_X^T = 0 \} \) over additive perturbations (e.g. \( R_X - P_X \) is that they form an inner product space equipped with the standard Euclidean inner product. This allows us to recast (17) using the \( \ell^2 \)-norm of \( K_X \) instead of a weighted \( \ell^2 \)-norm of the additive perturbation \( K_X \text{diag}(\sqrt{T_X}) \). Consequently, we benefit from more polished notation and simpler algebra later on—see our proof of Theorem 7. Finally, we remark that perturbation ideas like (15) have also been exploited in various other contexts in information theory, and we refer readers to [28], [29], [32], and [33] for a few examples.

**B. Contraction Coefficients of Source-Channel Pairs**

The DPIs, (10) and (13), can be maximally tightened into so called strong data processing inequalities (SDPIs) by inserting in pertinent constants known as contraction coefficients. There are two variants of contraction coefficients: the first depends on a source-channel pair, and the second depends solely on a channel. We introduce the former kind of coefficient in this subsection, and defer a discussion of the latter kind to the next subsection.

**Definition 2** (Contraction Coefficient of Source-Channel Pair). For any input pmf \( P_X \in \mathcal{P}_X \) and any discrete channel \( W \in \mathcal{P}_Y|X \) corresponding to a conditional distribution \( P_{Y|X} \), the contraction coefficient for a particular \( f \)-divergence is:

\[
\eta_f(P_X, P_{Y|X}) \triangleq \sup_{0 < D_f(R_X \| P_X) < +\infty} \frac{D_f(R_X W \| P_X W)}{D_f(R_X \| P_X)}
\]

where the supremum is taken over all pmfs \( R_X \) such that \( 0 < D_f(R_X \| P_X) < +\infty \). Furthermore, if \( X \) or \( Y \) is a constant almost surely (a.s.), we define \( \eta_f(P_X, P_{Y|X}) = 0 \).

Using Definition 2, we may write the following SDPI from the DPI for \( f \)-divergences in (10):

\[
D_f(R_X W \| P_X W) \leq \eta_f(P_X, P_{Y|X}) D_f(R_X \| P_X) \tag{19}
\]

which holds for every \( R_X \in \mathcal{P}_X \), with fixed \( P_X \in \mathcal{P}_X \) and \( W \in \mathcal{P}_Y|X \). The next proposition illustrates that the DPI for mutual \( f \)-information can be improved in a similar fashion.

**Proposition 1** (Mutual \( f \)-Information Contraction Coefficient [9, Theorem V.2]). For any input pmf \( P_X \in \mathcal{P}_X \), any discrete channel \( P_{Y|X} \in \mathcal{P}_Y|X \), and any convex function \( f : (0, \infty) \to \mathbb{R} \) that is differentiable, has uniformly bounded derivative in some neighborhood of unity, and satisfies \( f(1) = 0 \), we have:

\[
\eta_f(P_X, P_{Y|X}) = \sup_{U \in \mathcal{U}|X} \frac{I_f(U; Y)}{I_f(U; X)}
\]

where the supremum is taken over all conditional distributions \( P_{U|X} \in \mathcal{P}_U|X \) and finite alphabets \( \mathcal{U} \) of \( U \) such that \( U \to X \to Y \) form a Markov chain.\footnote{Given two functions \( f(\epsilon) \) and \( g(\epsilon) \) such that \( g(\epsilon) \) is non-zero, we write \( f(\epsilon) = o(g(\epsilon)) \) if and only if \( \lim_{\epsilon \to 0} f(\epsilon)/g(\epsilon) = 0 \).}\footnote{For larger values of \( \epsilon \) (in magnitude), although \( R_X^{(\epsilon)} \) always sums to 1 since \( \sqrt{T_X}K_X^T = 0 \), it may not be entry-wise non-negative.}\footnote{It suffices to let \(|\mathcal{U}| = 2 \) in the extremization [9, Theorem V.2].}
Proposition 1 is proved in [9, Theorem V.2]. The special case of this result for KL divergence was proved in [4] (which tackled the finite alphabet case) and [7] (which derived the general alphabet case). Intuitively, the variational problem in Proposition 1 determines the probability model that makes $Y$ as close to a sufficient statistic of $X$ for $U$ as possible (see the comment after (13)). Furthermore, the result illustrates that under regularity conditions, the contraction coefficient for any $f$-divergence gracefully unifies the DPIs for the $f$-divergence and the corresponding mutual $f$-information as the tightest factor that can be inserted into either one of them. Indeed, when the random variables $U \rightarrow X \rightarrow Y$ form a Markov chain, we can write the SDPI version of (13): 
\[
I_f(U;Y) \leq \eta_f(P_X, P_{Y|X}) I_f(U;X)
\]
which holds for every conditional distribution $P_{U|X}$, with fixed $P_X \in \mathcal{P}_X$ and $P_{Y|X} \in \mathcal{P}_{Y|X}$. Note that even if the conditions of Proposition 1 do not hold, (20) is still true (but $\eta_f(P_X, P_{Y|X})$ may not be the tightest possible constant that can be inserted into (13)).

There are two contraction coefficients that will be particularly important to our study. The first is the contraction coefficient for KL divergence:
\[
\eta_{KL}(P_X, P_{Y|X}) = \sup_{0 < D(R_X \| P_X) < +\infty} \frac{D(R_X W \| P_X W)}{D(R_X \| P_X)}. \tag{21}
\]
This quantity is related to the fundamental notion of hypercontractivity in statistics [2]. In fact, the authors of [2] and [5] illustrate how $\eta_{KL}(P_X, P_{Y|X})$ can be defined as the chordal slope of the lower boundary of the hypercontractivity ribbon at infinity in the discrete and finite setting.

The contraction coefficient for KL divergence elucidates a striking dichotomy between the extremizations in Definition 2 and Proposition 1. To delineate this contrast, we first specialize Proposition 1 for KL divergence [4, 7]:
\[
\eta_{KL}(P_X, P_{Y|X}) = \sup_{I(U;Y)} \sup_{I(U;X) > 0} \frac{I(U;Y)}{I(U;X)} \tag{22}
\]
where the optimization is (equivalently) over all $P_U \in \mathcal{P}_U$ with $U = \{0, 1\}$ (without loss of generality, cf. [7, Appendix B]) and all $P_{X|U} \in \mathcal{P}_{X|U}$ such that marginalizing yields $P_X$. We next recall an example from [4] where $U = \{0, 1\}$, $X \sim \text{Bernoulli}(\frac{1}{2})$, and $P_{Y|X}$ is an asymmetric erasure channel. In this numerical example, the supremum in (22) is achieved by the sequences of pmfs \{\(P_{X|U = k}^{(k)} \in \mathcal{P}_X : k \in \mathbb{N}\\}, \{P_{X|U = 1}^{(k)} \in \mathcal{P}_X : k \in \mathbb{N}\}, and \{P_{U|1}^{(k)} \in \mathcal{P}_U : k \in \mathbb{N}\} (where $\mathbb{N} \equiv \{0, 1, 2, \ldots\}$) satisfying the following conditions:
\[
\lim_{k \to \infty} P_{X|U = 0}^{(k)}(1) = 0, \tag{23}
\lim_{k \to \infty} D(P_{X|U = 0}^{(k)} \| P_X) = 0, \tag{24}
\lim_{k \to \infty} D(P_{X|U = 1}^{(k)} \| P_X) > 0, \tag{25}
\limsup_{k \to \infty} \frac{D(P_{X|U = 0}^{(k)} \| P_{Y|X})}{D(P_{X|U = 0}^{(k)} \| P_X)} < \eta_{KL}(P_X, P_{Y|X}), \tag{26}
\limsup_{k \to \infty} \frac{D(P_{X|U = 1}^{(k)} \| P_{Y|X})}{D(P_{X|U = 1}^{(k)} \| P_X)} = \eta_{KL}(P_X, P_{Y|X}). \tag{27}
\]
This example conveys that in general, although (22) is maximized when $I(U;X) \to 0$ [3], (21) is often achieved by a sequence of pmfs \{\(R_{X|U = k}^{(k)} \in \mathcal{P}_X \setminus \{P_X\} : k \in \mathbb{N}\\) that does not tend to $P_X$ (due to the non-concave nature of this extremal problem). At first glance, this is counter-intuitive because the DPI (10) is tight when $R_X = P_X$. However, Theorem 7 (presented in subsection III-A) will portray that maximizing the ratio of KL divergences with the constraint that $D(R_X \| P_X) \to 0$ actually achieves $\eta_{KL}(P_X, P_{Y|X})$, which is often strictly less than $\eta_{KL}(P_X, P_{Y|X})$ [4]. Therefore, there is a stark contrast between the behaviors of the optimization problems in (21) and (22).

\[6\]

\[10\]Hypercontractivity refers to the phenomenon that some conditional expectation operators are contractive even when their input functional space has a (probabilistic) $L^q$-norm while their output functional space has a (probabilistic) $L^p$-norm with $1 < q < p$ (see e.g. [5]). This notion has found applications in information theory because hypercontractive quantities are often imparted with tensorization properties which permit single letterization.
The second important contraction coefficient is the contraction coefficient for $\chi^2$-divergence:

$$
\eta_{\chi^2}(P_X, P_{Y|X}) = \sup_{0 < \chi^2(R_X || P_X) < +\infty} \frac{\chi^2(R_X W || P_X W)}{\chi^2(R_X || P_X)}
$$

(28)

which is closely related to a generalization of the Pearson correlation coefficient between $X$ and $Y$ known as the \textit{Hirschfeld-Gebelein-Rényi maximal correlation}, or simply maximal correlation [34]–[36]. We next define maximal correlation, which was proven to be a measure of statistical dependence satisfying seven natural axioms (some of which will be given in Proposition 3 later) that such measures should exhibit [36].

**Definition 3** (Maximal Correlation [34]–[36]). For two jointly distributed random variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, the maximal correlation between $X$ and $Y$ is given by:

$$
\rho(X; Y) = \sup_{f: \mathcal{X} \rightarrow \mathbb{R}, \, g: \mathcal{Y} \rightarrow \mathbb{R}: \, \mathbb{E}[f(X)] = \mathbb{E}[g(Y)] = 0} \mathbb{E}[f(X)g(Y)]
$$

(29)

where the supremum is taken over all Borel measurable functions $f$ and $g$ satisfying the constraints. Furthermore, if $X$ or $Y$ is a constant a.s., there exist no functions $f$ and $g$ that satisfy the constraints, and we define $\rho(X; Y) = 0$.

It can be shown that the contraction coefficient for $\chi^2$-divergence is precisely the squared maximal correlation [37]:

$$
\eta_{\chi^2}(P_X, P_{Y|X}) = \rho^2(X; Y).
$$

(30)

Furthermore, the next proposition portrays that maximal correlation can be represented as a singular value; this was originally observed in [36] in a slightly different form (also see [4], [24], [38] and [39, Theorem 3.2.4]).

**Proposition 2** (Singular Value Characterization of Maximal Correlation [36]). Given the random variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with joint pmf $P_{X,Y}$ (consisting of $(P_X, W)$), we may define a divergence transition matrix (DTM):

$$
B \triangleq \text{diag}(\sqrt{P_X}) W \text{diag}(\sqrt{P_Y})
$$

(31)

where $\dagger$ denotes the Moore-Penrose pseudoinverse. Then, the maximal correlation $\rho(X; Y)$ is the second largest singular value of $B$.

**Proof.** See Appendix A. \hfill \blacksquare

From Proposition 2 and (29), we see that the contraction coefficient for $\chi^2$-divergence is in fact the squared second largest singular value of the DTM $B$. We can write this using the Courant-Fischer variational characterization of eigenvalues or singular values (cf. [40, Theorems 4.2.6 and 7.3.8]) as:

$$
\eta_{\chi^2}(P_X, P_{Y|X}) = \max_{\|x\|_2 = 0, \sqrt{P_X}x = x} \frac{\|B^T x\|_2^2}{\|x\|_2^2}
$$

(32)

where $\mathbf{0}$ denotes the zero vector of appropriate dimension, and $\sqrt{P_X}^T$ is the right singular vector of $B^T$ corresponding to its maximum singular value of unity.

Singular value decompositions (SVDs) of DTMs and their relation to $\chi^2$-divergence have been well-studied in statistics. For instance, one direction of work concerns the analysis and identification of so called Lancaster distributions [41], [42]. In particular, given a joint distribution $P_{X,Y}$ over a product measurable space $\mathcal{X} \times \mathcal{Y}$ such that $\chi^2(P_{X,Y} || P_X \times P_Y) < \infty$, [12] Lancaster proved in [41] that there exist orthonormal bases, $\{f_j \in L^2(\mathcal{X}, P_X) : 0 \leq j < \|\mathcal{X}\|\}$ and $\{g_k \in L^2(\mathcal{Y}, P_Y) : 0 \leq k < \|\mathcal{Y}\|\}$, and some sequence $\{\sigma_k \geq 0 : 0 < k < \min\{\|\mathcal{X}\|, \|\mathcal{Y}\|\}\}$ of non-negative correlations, such that $P_{X,Y}$ is a \textit{Lancaster distribution} exhibiting the decomposition:

$$
dP_{X,Y}(x, y) = \sum_{k=0}^{\min\{\|\mathcal{X}\|, \|\mathcal{Y}\|\}} \sigma_k f_k(x) g_k(y)
$$

(33)

\footnote{Note that for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $P_X(x) > 0$ and $P_Y(y) > 0$, $[B]_{x,y} = P_{X,Y}(x, y)/\sqrt{P_X(x)P_Y(y)}$, and $[B]_{x,y} = 0$ otherwise.}

\footnote{Note that $P_X$ and $P_Y$ are marginal distributions of $P_{X,Y}$, and $P_X \times P_Y$ denotes their product distribution. Furthermore, $\chi^2(P_{X,Y} || P_X \times P_Y) < \infty$ implies that $P_{X,Y}$ is absolutely continuous with respect to $P_X \times P_Y$.}

\footnote{Here, $L^2(\mathcal{X}, P_X)$ (respectively $L^2(\mathcal{Y}, P_Y)$) is the Hilbert space of square integrable functions from $\mathcal{X}$ (respectively $\mathcal{Y}$) to $\mathbb{R}$ with inner product defined by $P_X$ (respectively $P_Y$).}
where \( d\mathbb{P}_{X,Y}/d(\mathbb{P}_X \times \mathbb{P}_Y) \) is the Radon-Nikodym derivative of \( \mathbb{P}_{X,Y} \) with respect to \( \mathbb{P}_X \times \mathbb{P}_Y \). When \( X \) and \( Y \) are finite, the decomposition in (32) precisely captures the SVD structure of \( B \) corresponding to \( P_{X,Y} \). We refer readers to [43, Section II-D] for further references.

While Lancaster decompositions are usually studied in infinite-dimensional settings, the related area of applied statistics known as correspondence analysis deals with understanding the dependence between categorical variables. In particular, simple correspondence analysis views a bivariate pmf \( P_{X,Y} \) as a contingency table, and decomposes the dependence between \( X \) and \( Y \) into so called principal inertia components using the SVD of \( B \), cf. [44], [45, Section 2], and the references therein. More recently, the authors of [24] have studied principal inertia components (which are eigenvalues of the Gramian matrix \( B^T B \)) in the context of information and estimation theory. They generalize the first principal inertia component (i.e. squared maximal correlation) into a quantity known as \( k \)-correlation for \( k \in \{1, \ldots, \min\{|X|,|Y|\} - 1\} \) (which is the Ky Fan \((k+1)\)-norm of \( B^T B \) minus 1), prove some properties of \( k \)-correlation such as convexity and DPI [24, Section II], and demonstrate several applications.

Yet another line of research has focused on the computational aspects of decomposing DTMs. A well-known method of computing SVDs of DTMs is the alternating conditional expectations (ACE) algorithm—see [46] for the original algorithm in the context of non-linear regression, and [47] for a variant of the algorithm in the context of feature selection. At its heart, the ACE algorithm employs a power iteration method to estimate singular vectors of the DTM. It turns out that such singular vectors corresponding to larger singular values can be identified as “more informative” score functions. This insight has been exploited to perform inference on hidden Markov models in an image processing setting in [48], and has been framed as a means of performing universal feature selection in [49].

Having introduced the pertinent contraction coefficients, we now present some well-known properties of contraction coefficients for \( f \)-divergences.

**Proposition 3** (Properties of Contraction Coefficients of Source-Channel Pairs). The contraction coefficient for an \( f \)-divergence satisfies the following properties:

1) (Normalization) For any joint pmf \( P_{X,Y} \), we have that \( 0 \leq \eta_f(P_{X,Y}) \leq 1 \).
2) (Independence) Given random variables \( X \) and \( Y \) with joint pmf \( P_{X,Y} \), if \( X \) and \( Y \) are independent, then \( \eta_f(P_{X,Y}) = 0 \). Conversely, if \( f \) is strictly convex at unity and \( \eta_f(P_{X,Y}) = 0 \), then \( X \) and \( Y \) are independent.
3) (Convexity [9, Proposition III.3]) For fixed \( P_X \in \mathcal{P}_N^X \), the function \( \mathcal{P}_{Y|X} : \mathcal{P}_{Y|X} 
\rightarrow \eta_f(P_X, P_{Y|X}) \) is convex in the channel \( P_{Y|X} \).
4) (Tensorization [9, Theorem III.9]) If the convex function \( f : (0, \infty) \rightarrow \mathbb{R} \) that defines our \( f \)-divergence also induces a sub-additive and homogeneous \( f \)-entropy,\(^\text{14}\) and \( \{P_{X_i,Y_i} : P_{X_i} \in \mathcal{P}_N^X \) and \( P_{Y_i} \in \mathcal{P}_N^Y \) for \( i \in \{1, \ldots, n\} \) \} are independent joint pmfs, then we have:
\[
\eta_f(P_{X_i,Y_i}) = \max_{1 \leq i \leq n} \eta_f(P_{X_i,Y_i})
\]
where \( X_i^n = (X_1, \ldots, X_n) \) and \( Y_i^n = (Y_1, \ldots, Y_n) \).
5) (Sub-multiplicativity) If \( U \rightarrow X \rightarrow Y \) are discrete random variables with finite ranges that form a Markov chain, then we have:
\[
\eta_f(P_{U,X,Y}) \leq \eta_f(P_{U,Y}) \eta_f(P_{X,Y})
\]
Furthermore, for any fixed joint pmf \( P_{X,Y} \) such that \( X \) is not a constant a.s., we have:
\[
\eta_f(P_{X,Y}) = \sup_{P_{U|X}:U \rightarrow X \rightarrow Y} \frac{\eta_f(P_{U,X,Y})}{\eta_f(P_{U,Y})}
\]
where the supremum is over all arbitrary finite ranges \( U \) of \( U \), and over all conditional distributions \( P_{U|X} \in \mathcal{P}_{U|X} \) such that \( U \rightarrow X \rightarrow Y \) form a Markov chain.
6) (Maximal Correlation Lower Bound [9, Theorem III.3], [8, Theorem 2]) Suppose we have a joint pmf \( P_{X,Y} \) such that the marginal pmfs satisfy \( P_X \in \mathcal{P}_N^X \) and \( P_Y \in \mathcal{P}_N^Y \). If the function \( f : (0, \infty) \rightarrow \mathbb{R} \) that defines our \( f \)-divergence is twice differentiable at unity with \( f''(1) > 0 \), then we have:
\[
\eta_f(P_{X,Y}) = \rho^2(X; Y) \leq \eta_f(P_{X,Y})
\]

**Proof.** See Appendix B for certain proofs, as well as relevant references for specializations of the results.

\(^{14}\)For a convex function \( f : (0, \infty) \rightarrow \mathbb{R} \), the \( f \)-entropy of a non-negative random variable \( Z \) is defined as \( \text{Ent}_f(Z) \equiv \mathbb{E}[f(Z)] - f(\mathbb{E}[Z]) \), where it is assumed that \( \mathbb{E}[f(Z)] < \infty \) (see [9, Section II] and the references therein).
We now make some relevant remarks. Firstly, since parts 1 and 2 of Proposition 3 illustrate that contraction coefficients are normalized measures of statistical dependence between random variables, we can perceive the sub-multiplicativity property in part 5 as an SDPI for contraction coefficients in analogy with (20). In fact, part 5 also portrays that the contraction coefficient of the SDPI for \( \eta_f \) is given by \( \eta_f \) itself.

Secondly, the version of the DPI for \( \eta_{kl} \) presented in [2] (also see [5, Section II-A]) holds for general \( \eta_f \). Indeed, if \( U \to X \to Y \to V \) are discrete random variables with finite ranges that form a Markov chain, then a straightforward consequence of parts 1 and 5 of Proposition 3 is the following monotonicity property:

\[
\eta_f(P_U, P_{V|U}) \leq \eta_f(P_X, P_{Y|X}).
\]  

(33)

Thirdly, the maximal correlation lower bound in part 6 of Proposition 3 can be achieved with equality. For instance, let \( f(t) = t \log(t) \) and consider a doubly symmetric binary source (DSBS) with parameter \( \alpha \in [0, 1] \). A DSBS describes a joint distribution of two uniform Bernoulli random variables \( (X, Y) \), where \( X \) is passed through a binary symmetric channel (BSC) with crossover probability \( \alpha \) to produce \( Y \). It is proven in [2] that for \( (X, Y) \sim \text{DSBS}(\alpha) \), the maximal correlation lower bound holds with equality:

\[
\eta_{\text{kl}}(P_X, P_{Y|X}) = \eta_{\text{kl}}^2(P_X, P_{Y|X}) = (1 - 2\alpha)^2
\]  

(34)

where \( \eta_{\text{kl}}(P_X, P_{Y|X}) = (1 - 2\alpha)^2 \) can be readily computed using the singular value characterization of maximal correlation presented in Proposition 2. As another example, consider \( P_{Y|X} \) defined by an \( |\mathcal{X}| \)-ary erasure channel \( E_\beta \in P_{Y|X} \) with erasure probability \( \beta \in [0, 1] \), which has input alphabet \( \mathcal{X} \) and output alphabet \( \mathcal{Y} = \mathcal{X} \cup \{ \epsilon \} \), where \( \epsilon \) is the erasure symbol. Recall that given an input \( x \in \mathcal{X} \), \( E_\beta \) erases \( x \) and outputs \( \epsilon \) with probability \( \beta \), and copies its input \( x \) with probability \( 1 - \beta \). It is straightforward to verify that \( D_f(R_X E_\beta || P_X E_\beta) = (1 - \beta) D_f(R_X || P_X) \) for every \( R_X, P_X \in \mathcal{P}_X \). Therefore, for every input pmf \( P_X \in \mathcal{P}_X \) and every \( f \)-divergence, \( \eta_f(P_X, P_{Y|X}) = 1 - \beta \).

Finally, we note that although we independently proved part 6 of Proposition 3 using the local approximation of \( f \)-divergence idea from our conference paper [1, Theorem 5], the same idea is used by [9, Theorem III.3] and [8, Theorem 2] to prove this result. In fact, this idea turns out to stem from the proof of [50, Theorem 5.4] (which is presented later in part 5 of Proposition 5).

C. Coefficients of Ergodicity

Before discussing contraction coefficients that depend solely on channels, we briefly introduce the broader notion of coefficients of ergodicity. Coefficients of ergodicity were first studied in the context of understanding ergodicity and convergence rates of finite state-space (time) inhomogeneous Markov chains, cf. [51, Section 1]. We present their definition below.

Definition 4 (Coefficient of Ergodicity [52, Definition 4.6]). A coefficient of ergodicity is a continuous scalar function \( \eta : \mathcal{P}_{Y|X} \to [0, 1] \) from \( \mathcal{P}_{Y|X} \) (with fixed dimension) to \([0, 1]\). Such a coefficient is proper if for any \( W \in \mathcal{P}_{Y|X} \), \( \eta(W) = 0 \) if and only if \( W = 1_p P_Y \) for some pmf \( P_Y \in \mathcal{P}_Y \) (i.e. \( W \) is unit rank), where \( 1 \in \mathbb{R}^{[X]} \) denotes a column vector with all entries equal to 1.

One useful property of proper coefficients of ergodicity is their connection to weak ergodicity. Consider a sequence of row stochastic matrices \( \{ W_k \in \mathcal{P}_{X|X} : k \in \mathbb{N} \} \) that define an inhomogeneous Markov chain on the state space \( \mathcal{X} \). Let the forward product of \( r \geq 1 \) of these consecutive matrices starting at index \( p \in \mathbb{N} \) be:

\[
T_{(p,r)} \triangleq \prod_{i=0}^{r-1} W_{p+i}.
\]  

(35)

The Markov chain \( \{ W_k \in \mathcal{P}_{X|X} : k \in \mathbb{N} \} \) is said to be weakly ergodic (in the Kolmogorov sense) if for all \( x_1, x_2, x_3 \in \mathcal{X} \) and all \( p \in \mathbb{N} \) [52, Definition 4.4]:

\[
\lim_{r \to \infty} [T_{(p,r)}]_{x_1, x_3} - [T_{(p,r)}]_{x_2, x_3} = 0.
\]  

(36)

This definition captures the intuition that the rows of a forward product should equalize when \( r \to \infty \) for an ergodic Markov chain. The next proposition conveys that weak ergodicity can be equivalently defined using proper coefficients of ergodicity.

---

15The set \( \mathcal{P}_{Y|X} \) is endowed with the standard topology induced by the Frobenius norm. Furthermore, \( \mathcal{X} \) is typically the same as \( \mathcal{Y} \) in Markov chain settings.

16Note that if the limiting row stochastic \( \lim_{r \to \infty} T_{(p,r)} \) exists for all \( p \in \mathbb{N} \), then the Markov chain is strongly ergodic [52, Definition 4.5].
Proposition 4 (Weak Ergodicity [52, Lemma 4.1]). Let \( \eta : \mathcal{P}_{X|X} \to [0,1] \) be a proper coefficient of ergodicity. Then, the inhomogeneous Markov chain \( \{W_k \in \mathcal{P}_{X|X} : k \in \mathbb{N} \} \) is weakly ergodic if and only if:
\[
\forall p \in \mathbb{N}, \ \lim_{r \to \infty} \eta(T_{(p,r)}) = 0.
\]

To intuitively understand this result, notice that \( T_{(p,r)} \) becomes (approximately) unit rank as \( r \to \infty \) for a weakly ergodic Markov chain. So, we also expect \( \lim_{r \to \infty} \eta(T_{(p,r)}) = 0 \), since a proper coefficient of ergodicity is continuous, and equals zero when its input is unit rank. We refer readers to [52, Lemma 4.1] for a formal proof of Proposition 4.

One of the earliest and most notable examples of proper coefficients of ergodicity is the Dobrushin contraction coefficient. Given a row stochastic matrix \( W \), we also suggest [51], [52, Chapters 3 and 4], [53], [54, Chapter 3], and the references therein for further expositions and equals zero when its input is unit rank. We refer readers to [52, Lemma 4.1] for a formal proof of Proposition 4.

**Proposition 4** (Weak Ergodicity [52, Lemma 4.1])

An ergodic Markov chain. So, we also expect \( \{ \eta \) and \( (41) \) can be found in (or easily deduced from) [52, Chapter 4.3]. The characterization in (41) illustrates that \( \eta \) or presented as the coefficient may also be attributed (at least partly) to both Doeblin and Markov. In fact, the coefficient has been called the Doeblin contraction coefficient or presented as the Markov contraction lemma in the literature (see e.g. [56, p.619]).

In addition to the properties of proper coefficients of ergodicity, \( \eta \) also exhibits the following properties:

1) **Lipschitz continuity** [53, Theorem 3.4, Remark 3.5]: For every \( V, W \in \mathcal{P}_{Y|X} \), \( |\eta_{TV}(V) - \eta_{TV}(W)| \leq \|V - W\|_{TV} \), where \( \|\cdot\|_{\infty} \) denotes the induced \( \ell^{\infty} \)-norm, or maximum absolute row sum, when applied to a matrix.

2) **Sub-multiplicativity** [52, Lemma 4.3]: For every \( V \in \mathcal{P}_{X|X} \) and \( W \in \mathcal{P}_{Y|X} \), \( \eta_{TV}(VW) \leq \eta_{TV}(V)\eta_{TV}(W) \).

3) **Sub-dominant eigenvalue bound** [51, p.584, (9)]: For every \( V \in \mathcal{P}_{X|X} \), \( \eta_{TV}(V) \geq |\lambda| \) for every sub-dominant eigenvalue \( \lambda \neq 1 \) of \( W \).

The last two properties make \( \eta_{TV} \) a convenient tool for analyzing inhomogeneous Markov chains. As explained in [53, Section 1], for a homogeneous Markov chain \( W \in \mathcal{P}_{X|X} \) with stationary pmf \( \pi \), it is well-known that the second largest eigenvalue modulus (SLEM) of \( W \), denoted \( \mu(W) \), controls the rate of convergence to stationarity.

Indeed, if \( \mu(W) < 1 \), then \( \mu(W^n) = \mu(W)^n, \) and \( \lim_{n \to \infty} W^n = \pi \) with rate determined by \( \mu(W) \). However, for an inhomogeneous Markov chain \( \{W_k \in \mathcal{P}_{X|X} : k \in \mathbb{N} \}, \mu(T_{(0,n)}) \neq \prod_{i=0}^{n-1} \mu(W_i) \) in general because SLEMs are not multiplicative. The last two properties of \( \eta_{TV} \) illustrate that it is a viable replacement for SLEMs in the study of inhomogeneous Markov chains.

**D. Contraction Coefficients of Channels**

Contraction coefficients of channels form a broad class of coefficients of ergodicity. They are defined similarly to (37), but using \( f \)-divergences in place of TV distance.

---

17 Based on the bibliographic discussion in [52, pp.144-147], the Dobrushin contraction coefficient (or equivalently, the Dobrushin ergodicity coefficient) may also be attributed (at least partly) to both Doeblin and Markov. In fact, the coefficient has been called the Doeblin contraction coefficient or presented as the Markov contraction lemma in the literature (see e.g. [56, p.619]).

18 Thus, \( \eta_{TV}(W) < 1 \) if and only if the zero error capacity of \( W \) is 0 [57].
Definition 5 (Contraction Coefficient of Channel). For any discrete channel $W \in \mathcal{P}_{Y|X}$ corresponding to a conditional distribution $P_{Y|X}$, the contraction coefficient for a particular $f$-divergence is:

$$
\eta_f(P_{Y|X}) \triangleq \sup_{P_X \in \mathcal{P}_X} \eta_f(P_X, P_{Y|X}) 
= \sup_{R_X, P_X \in \mathcal{P}_X; \ 0 < D_f(R_X || P_X) < +\infty} \frac{D_f(R_X W || P_X W)}{D_f(R_X || P_X)}
$$

where the supremum is taken over all pmfs $R_X$ and $P_X$ such that $0 < D_f(R_X || P_X) < +\infty$. Furthermore, if $Y$ is a constant a.s., we define $\eta_f(P_{Y|X}) = 0$.

This definition transparently yields SDPIs analogous to (19) and (20) for contraction coefficients of channels. Furthermore, a version of Proposition 1 also holds for contraction coefficients of channels. Indeed, using Definition 5 and Proposition 1, we observe that for any discrete channel $P_{Y|X} \in \mathcal{P}_{Y|X}$, and any convex function $f : (0, \infty) \rightarrow \mathbb{R}$ that is differentiable, has uniformly bounded derivative in some neighborhood of unity, and satisfies $f(1) = 0$, we have:

$$
\eta_f(P_{Y|X}) = \sup_{P_{U,X} : U \rightarrow X \rightarrow Y} \frac{I_f(U; Y)}{I_f(U; X)}
$$

(42)

where the supremum is taken over all joint pmfs $P_{U,X}$ and finite alphabets $\mathcal{U}$ of $U$ such that $U \rightarrow X \rightarrow Y$ form a Markov chain. The specialization of this result for KL divergence can be found in [58, p.345, Problem 15.12] (finite alphabet case) and [8] (general alphabet case).

There are two important examples of contraction coefficients of channels: the Dobrushin contraction coefficient for TV distance (defined in (37)), and the contraction coefficient for KL divergence. As seen earlier, given a channel $P_{Y|X}$, we use the notation $\eta_{\alpha}(P_{Y|X})$, $\eta_{\alpha}(P_{Y|X})$, and $\eta_{\chi^2}(P_{Y|X})$ to represent the contraction coefficient of $P_{Y|X}$ for TV distance, KL divergence, and $\chi^2$-divergence, respectively. It is proved in [2] that for any channel $P_{Y|X}$, we have:

$$
\eta_{\alpha}(P_{Y|X}) = \eta_{\chi^2}(P_{Y|X})
$$

(43)

Therefore, we do not need to consider $\eta_{\alpha}$ and $\eta_{\chi^2}$ separately when studying contraction coefficients of channels. We remark that an alternative proof of (43) (which holds for general measurable spaces) is given in [8, Theorem 3].

Furthermore, a perhaps lesser known observation is that the proof technique of [59, Lemma 1, Theorem 1] (which analytically computes $\eta_{\alpha}(P_{Y|X})$ for any binary channel $P_{Y|X}$ with $|X| = |Y| = 2$, when appropriately generalized for arbitrary finite alphabet sizes, also yields a proof of (43). It is worth mentioning that the main contribution of Evans and Schulman in [59] is an inductive approach to upper bound $\eta_{\alpha}$ in directed acyclic graphs. We refer readers to [8] for an insightful distillation of this approach, as well as for proofs of its generalization to TV distance and its connection to site percolation.

We next present some well-known properties of contraction coefficients of channels.

Proposition 5 (Properties of Contraction Coefficients of Channels). The contraction coefficient for an $f$-divergence satisfies the following properties:

1) (Normalization) For any discrete channel $P_{Y|X} \in \mathcal{P}_{Y|X}$, we have that $0 \leq \eta_f(P_{Y|X}) \leq 1$.

2) (Independence [50, Section 4]) Given a channel $P_{Y|X} \in \mathcal{P}_{Y|X}$, if $X$ and $Y$ are independent, then $\eta_f(P_{Y|X}) = 0$. Conversely, if $f$ is strictly convex at unity and $\eta_f(P_{Y|X}) = 0$, then $X$ and $Y$ are independent.

3) (Convexity [50, Section 4], [9, Proposition III.3]) The function $\mathcal{P}_{Y|X} \ni P_{Y|X} \mapsto \eta_f(P_{Y|X})$ is convex.

4) (Sub-multiplicativity [50, Section 4]) If $U \rightarrow X \rightarrow Y$ are discrete random variables with finite ranges that form a Markov chain, then we have:

$$
\eta_f(P_{Y|U}) \leq \eta_f(P_{X|U}) \eta_f(P_{Y|X})
$$

5) ($\chi^2$-Divergence Contraction Lower Bound [50, Theorem 5.4], [23, Proposition II.6.15]) Given a channel $P_{Y|X} \in \mathcal{P}_{Y|X}$, if the function $f : (0, \infty) \rightarrow \mathbb{R}$ that defines our $f$-divergence is twice differentiable at unity with $f''(1) > 0$, then we have:

$$
\eta_{\chi^2}(P_{Y|X}) \leq \eta_f(P_{Y|X})
$$

6) (TV Distance Contraction Upper Bound [50, Theorem 4.1], [23, Proposition II.4.10]) For any channel $P_{Y|X} \in \mathcal{P}_{Y|X}$, we have:

$$
\eta_f(P_{Y|X}) \leq \eta_{\alpha}(P_{Y|X})
$$
We omit proofs of these results, because the proofs are either analogous to the corresponding proofs in Proposition 3, or are given in the associated references. Parts 1, 2, and 3 of Proposition 5 portray that contraction coefficients of channels are often valid proper coefficients of ergodicity.\(^{19}\) We also remark that an extremization result analogous to part 5 of Proposition 3, albeit less meaningful, can be derived in part 4 of Proposition 5.

While (43) shows that part 5 of Proposition 5 can be easily achieved with equality, the inequality in part 6 is often strict. For example, when \(P_{Y|X}\) is a binary channel with parameters \(a, b \in [0, 1]\) and row stochastic transition probability matrix:

\[
W = \begin{pmatrix}
1-a & a \\
b & 1-b
\end{pmatrix}
\]

it is straightforward to verify that \(\eta_\alpha(P_{Y|X}) \leq \eta_\nu(P_{Y|X})\), with the inequality usually strict, since we have:

\[
\eta_\alpha(P_{Y|X}) = 1 - \left( \sqrt{a(1-b)} + \sqrt{b(1-a)} \right)^2
\]

\[
\eta_\nu(P_{Y|X}) = |1-a-b|
\]

where (45) is proved in [59, Theorem 1], and (46) is easily computed via (40). Moreover, in the special case where \(P_{Y|X}\) is a BSC with crossover probability \(\alpha \in [0, 1]\), we get [2]:

\[
\eta_\alpha(P_{Y|X}) = (1-2\alpha)^2 \leq |1-2\alpha| = \eta_\nu(P_{Y|X}).
\]

On the other hand, as shown towards the end of subsection II-B, \(\eta_f(P_{Y|X}) = 1 - \beta\) for every \(f\)-divergence when \(P_{Y|X}\) is an \(|X|\)-ary erasure channel with erasure probability \(\beta \in [0, 1]\).

In view of part 5 and (43), it is natural to wonder whether there are other \(f\)-divergences whose contraction coefficients (for channels) also collapse to \(\eta_\nu\). The following result from [10, Theorem 1] generalizes (43) and addresses this question.

**Proposition 6** (Contraction Coefficients for Operator Convex \(f\)-Divergences [10, Theorem 1], [23]). For every non-linear operator convex function \(f : (0, \infty) \rightarrow \mathbb{R}\) such that \(f(1) = 0\), and every channel \(P_{Y|X}\), we have:

\[
\eta_f(P_{Y|X}) = \eta_\nu(P_{Y|X}).
\]

The proof of [10, Theorem 1] relies on an elegant integral representation of operator convex functions. Such representations are powerful tools for proving inequalities between contraction coefficients, and we will use them to generalize Proposition 6 in subsection III-D. In fact, part 6 of Proposition 5 can also be proved using an integral representation argument, cf. [9, Theorem III.1].

In closing this overview, we also refer readers to [8, Section 2] for a complementary and comprehensive survey of contraction coefficients, and for references to various applications of these ideas in the literature.

### III. MAIN RESULTS AND DISCUSSION

We will primarily derive bounds between various contraction coefficients in this paper. In particular, we will address the following leading questions:

1) **Can we achieve the maximal correlation lower bound in Proposition 3 by adding constraints to the extremal problem that defines contraction coefficients of source-channel pairs?**

   Yes, we can constrain the input \(f\)-divergence to be small as shown in Theorem 7 in subsection III-A.

2) **While we typically lower bound \(\eta_\alpha(P_X, P_{Y|X})\) using \(\eta_\alpha(P_X, P_{Y|X})\) (Proposition 3 part 6), we typically upper bound it using \(\eta_\nu(P_{Y|X})\) (Proposition 5 part 6). Is there a simple upper bound on \(\eta_\alpha(P_X, P_{Y|X})\) in terms of \(\eta_\alpha(P_X, P_{Y|X})\)?**

   Yes, two such bounds are given in Corollary 9 and Theorem 10 in subsection III-B.

3) **Can we extend this upper bound for KL divergence to other \(f\)-divergences?**

   Yes, a more general bound is presented in Theorem 8 in subsection III-B.

4) When \(X\) and \(Y\) are jointly Gaussian, the mutual information characterization in (22) can be used to establish that \(\eta_\alpha(P_X, P_{Y|X}) = \eta_\nu(P_X, P_{Y|X})\) [3, Theorem 7]. Is there a simple proof of this result that directly uses the definition of \(\eta_\alpha\)? Does this equality hold when we add a power constraint to the extremization in \(\eta_\alpha\)?

   Yes, we discuss the Gaussian case in subsection III-C, and prove this equality for \(\eta_\alpha\) with a power constraint in Theorem 11. Our proof also establishes the known equality using the KL divergence definition of \(\eta_\alpha\).

---

\(^{19}\)The convexity of \(P_{Y|X} \mapsto \eta_f(P_{Y|X})\) in part 3 of Proposition 5 implies that this map is continuous on the interior of \(\mathcal{P}_{Y|X}\). So, only \(\eta_f\) that are also continuous on the boundary of \(\mathcal{P}_{Y|X}\) are valid coefficients of ergodicity.
5) Contraction coefficients of channels are closely related to the less noisy preorder over channels [8, Section 6]. Can we generalize the result in Proposition 6 to say something more about the less noisy preorder?

Yes, we introduce the less noisy preorder in subsection III-D, and derive a class of equivalent characterizations for it in Theorem 12.

The bounds we will derive in response to questions 2, 3, and 4 have the form of the upper bound in:

\[ \eta_\chi^2(P_X, P_{Y|X}) \leq \eta_f(P_X, P_{Y|X}) \leq C \eta_\chi^2(P_X, P_{Y|X}) \] (48)

where the first inequality is simply the maximal correlation lower bound from Proposition 3, and the constant \(C\) depends on \(P_{X,Y}\) and \(f\); note that \(C = 1\) in the setting of question 4. We refer to such bounds as linear bounds between contraction coefficients of source-channel pairs. We state our main results in the next few subsections.

A. Local Approximation of Contraction Coefficients

We assume in this subsection and in subsection III-B that we are given the random variables \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\) with joint pmf \(P_{X,Y}\) such that the marginal pmfs satisfy \(P_X \in \mathcal{P}_X\) and \(P_Y \in \mathcal{P}_Y\). Our first result portrays that forcing the input \(f\)-divergence to be small translates general contraction coefficients into the contraction coefficient for \(\chi^2\)-divergence.

**Theorem 7** (Local Approximation of Contraction Coefficients). Suppose we are given a convex function \(f : (0, \infty) \to \mathbb{R}\) that is strictly convex and twice differentiable at unity with \(f(1) = 0\) and \(f''(1) > 0\). Then, we have:

\[ \eta_\chi^2(P_X, P_{Y|X}) = \lim_{\delta \to 0^+} \inf_{R_X \in \mathcal{P}_X} \sup_{0 < D_f(R_X || P_X) \leq \delta} \frac{D_f(R_X W || P_X W)}{D_f(R_X || P_X)} \]

where \(W \in \mathcal{P}_{Y|X}\) is the row stochastic transition probability matrix representing the channel \(P_{Y|X}\).

We refer readers to Appendix C for the proof, and note that the specialization of Theorem 7 for KL divergence already captures the intuition that performing the optimization of \(\eta_f(P_X, P_{Y|X})\) over local perturbations of \(P_X\) yields \(\eta_\chi^2(P_X, P_{Y|X})\) due to (18) and (31). However, this proof (with minor modifications) only demonstrates that \(\eta_\chi^2(P_X, P_{Y|X})\) is upper bounded by the right hand side of Theorem 7. Although it is intuitively clear that this upper bound is met with equality, the formal proof contains a few technical details as shown in Appendix C.

Secondly, Theorem 7 transparently portrays that the maximal correlation lower bound in part 6 of Proposition 3 can be achieved when the optimization of \(\eta_f(P_X, P_{Y|X})\) imposes an additional constraint that the input \(f\)-divergence is small. (Hence, Theorem 7 implies the maximal correlation bound.) This insight has proved useful in comparing \(\eta_\chi^2(P_X, P_{Y|X})\) and \(\eta_{\delta_0}(P_X, P_{Y|X})\) in statistical contexts [60, p.5].

Thirdly, Theorem 7 can be construed as a minimax characterization of \(\eta_\chi^2(P_X, P_{Y|X})\) since the supremum of the ratio of \(f\)-divergences is a non-increasing function of \(\delta\) and the limit (as \(\delta \to 0^+\)) can therefore be replaced by an infimum (over all \(\delta > 0\)).

Fourthly, when the conditions of Proposition 1 and Theorem 7 hold, it is straightforward to verify that:

\[ \eta_f(P_X, P_{Y|X}) = \lim_{\delta \to 0^+} \inf_{P_U \in \mathcal{P}_{U|X}} \sup_{0 < I_f(U; X) \leq \delta} \frac{I_f(U; Y)}{I_f(U; X)} \] (49)

where the supremum is taken over all conditional distributions \(P_U \in \mathcal{P}_{U|X}\) such that \(U = \{0, 1\}, U \sim \text{Bernoulli}(\frac{1}{2})\), and \(U \to X \to Y\) form a Markov chain. Thus, the small input \(f\)-divergence constraint in the \(f\)-divergence formulation of \(\eta_f(P_X, P_{Y|X})\) corresponds to the small \(I_f(U; X)\) and \(U \sim \text{Bernoulli}(\frac{1}{2})\) constraints in (49).

Lastly, consider the trajectory of input pmfs \(R_{X|\epsilon}^\epsilon = P_X + \epsilon K_X^* \text{diag}(\sqrt{P_X})\), where \(\epsilon > 0\) is sufficiently small, and \(K_X^* \in (\mathbb{R}^{|\mathcal{X}|})^*\) is the left singular vector corresponding to the second largest singular value of the DTM \(B\) (see (31)). As the proof in Appendix C illustrates, this trajectory satisfies \(\lim_{\epsilon \to 0} D_f(R_{X|\epsilon}^\epsilon || P_X) = 0\) and achieves \(\eta_\chi^2(P_X, P_{Y|X})\) in Theorem 7:

\[ \lim_{\epsilon \to 0} \frac{D_f(R_{X|\epsilon}^\epsilon W || P_X W)}{D_f(R_{X|\epsilon}^\epsilon || P_X)} = \eta_\chi^2(P_X, P_{Y|X}). \] (50)

13
The corresponding trajectory of conditional distributions for (49) is \( \{P_{X|U=u} = P_X + (2u - 1) \epsilon K_X^* \text{diag}(\sqrt{P_X}) : u \in \{0, 1\} \} \), where \( \epsilon > 0 \) is sufficient small. This trajectory satisfies \( \lim_{\epsilon \to 0} I_f(P_U, P_{X|U}^{(\epsilon)}) = 0 \), produces \( P_X \) after \( (P_U, P_{X|U}^{(\epsilon)}) \) is marginalized, and achieves \( \eta^2(P_X, P_{Y|X}) \) in (49):

\[
\lim_{\epsilon \to 0} \frac{I_f(P_U, P_{X|U}^{(\epsilon)})}{I_f(P_U, P_{X|U})} = \eta^2(P_X, P_{Y|X})
\]

(51)

where \( U \sim \text{Bernoulli}(\frac{1}{2}) \), and \( P_{X|U}^{(\epsilon)} = P_{X|U}^{(\epsilon)} P_{Y|X} \) as row stochastic matrices.

**B. Linear Bounds between Contraction Coefficients**

For any joint pmf \( P_{X,Y} \) with \( P_X \in \mathcal{P}_2^\R \) and \( P_Y \in \mathcal{P}_2^\R \), our next result provides a linear upper bound on \( \eta_f(P_X, P_{Y|X}) \) using \( \eta^2(P_X, P_{Y|X}) \) for a certain class of \( f \)-divergences.

**Theorem 8** (Contraction Coefficient Bound). Suppose we are given a continuous convex function \( f : [0, \infty) \to \R \) that is thrice differentiable at unity with \( f'(1) = 0 \) and \( f''(1) > 0 \), and satisfies (69) for every \( t \in (0, \infty) \) (see subsection IV-A). Suppose further that the difference quotient \( g : (0, \infty) \to \R \), defined as \( g(x) = \frac{f(x) - f(0)}{x} \), is concave. Then, we have:

\[
\eta_f(P_X, P_{Y|X}) \leq \frac{f'(1) + f(0)}{f''(1)} \eta^2(P_X, P_{Y|X})
\]

Theorem 8 is proved in subsection IV-B. The conditions on \( f \) ensure that the resulting \( f \)-divergence exhibits the properties of KL divergence required by the proof of Theorem 10 (see below). So, a similar proof technique also works for Theorem 8. A straightforward specialization of this theorem for KL divergence (which we first proved in the conference version of this paper [1, Theorem 10]) is presented next.

**Corollary 9** (KL Contraction Coefficient Bound).

\[
\eta_{\text{KL}}(P_X, P_{Y|X}) \leq \frac{\eta^2(P_X, P_{Y|X})}{\min_{x \in \mathcal{X}} P_X(x)}.
\]

**Proof.** This can be recovered from Theorem 8 by verifying that \( f(t) = t \log(t) \) satisfies the conditions of Theorem 8, cf. [61]. See Appendix D for details.

The constant in this upper bound on \( \eta_{\text{KL}}(P_X, P_{Y|X}) \) can be improved, and the ensuing theorem presents this improvement.

**Theorem 10** (Refined KL Contraction Coefficient Bound).

\[
\eta_{\text{KL}}(P_X, P_{Y|X}) \leq \frac{2 \eta^2(P_X, P_{Y|X})}{\phi\left(\max_{A \subseteq \mathcal{X}} \pi(A)\right) \min_{x \in \mathcal{X}} P_X(x)}
\]

where \( \pi(A) = \min\{P_X(A), 1 - P_X(A)\} \) for any \( A \subseteq \mathcal{X} \), and the function \( \phi : [0, \frac{1}{2}] \to \R \) is defined in (62) (see subsection IV-A).

Theorem 10 is also proved in subsection IV-B, and it is tighter than the bound in Corollary 9 due to (64) in subsection IV-A. We now make some pertinent remarks about Corollary 9 and Theorems 8 and 10.

Firstly, as shown in Figure 1(a), the upper bounds in these results can be strictly less than the trivial bound of unity. For example, when \( (X, Y) \sim \text{DSBS}(p) \) for some \( p \in [0, 1] \) (which is a slice along \( \mathbb{P}(X = 1) = \frac{1}{2} \) in Figure 1(a)), the upper bounds in Corollary 9 and Theorem 10 are both equal to:

\[
\frac{2 \eta^2(P_X, P_{Y|X})}{\phi\left(\max_{A \subseteq \mathcal{X}} \pi(A)\right) \min_{x \in \mathcal{X}} P_X(x)} = \frac{\eta^2(P_X, P_{Y|X})}{\min_{x \in \mathcal{X}} P_X(x)} = 2(1 - 2p)^2
\]

(52)

using (34) and the fact that \( \max_{A \subseteq \mathcal{X}} \pi(A) = \frac{1}{2} \). This upper bound is tighter than the trivial bound of unity when:

\[
2(1 - 2p)^2 < 1 \iff 2 - \sqrt{2} < p < \frac{2 + \sqrt{2}}{4}.
\]

(53)
(a) Plots of $\eta_{KL}(P_X, P_{Y|X})$ (red mesh), and linear upper bounds on $\eta_{KL}(P_X, P_{Y|X})$. The green mesh denotes the upper bound from Corollary 9, and the yellow mesh denotes the tighter upper bound from Theorem 10.

(b) Plots of upper bounds on the ratio $\eta_{KL}(P_X, P_{Y|X})/\eta_{\chi^2}(P_X, P_{Y|X})$, denoted by the red mesh. The bound $1/\min_{x \in X} P_X(x)$ from Corollary 9 is the green mesh, and the bound $2/(\phi(\max_{A \subseteq X} \pi(A)) \min_{x \in X} P_X(x))$ from Theorem 10 is the blue mesh.

Fig. 1. Plots of the contraction coefficient bounds in Corollary 9 and Theorem 10 for a BSC, $P_{Y|X}$, with crossover probability $p \in [0, 1]$, and input random variable $X \sim \text{Bernoulli}(P(X = 1))$.

We also note that this upper bound is not achieved with equality in this scenario since $\eta_{KL}(P_X, P_{Y|X}) = \eta_{\chi^2}(P_X, P_{Y|X}) = (1 - 2p)^2$, as shown in (34).

Secondly, our proofs of Theorems 8 and 10 will rely on extensions of the well-known Pinsker’s inequality (or Csiszár-Kemperman-Kullback-Pinsker inequality, cf. [62, Section V]) which upper bound TV distance using KL and other $f$-divergences. So, it is natural to ask: Are these bounds tighter than the TV distance contraction bound in part 6 of Proposition 5? As the ensuing example illustrates, our bounds are tighter in certain regimes. Let $(X, Y) \sim \text{DSBS}(p)$ for some $p \in [0, 1]$. Then, (52) presents the upper bounds in Corollary 9 and Theorem 10, and the TV distance contraction bound is:

$$\eta_{KL}(P_X, P_{Y|X}) \leq \eta_{KL}(P_{Y|X}) \leq \eta_{\text{tv}}(P_{Y|X}) = |1 - 2p|$$

(54)
using Definition 5, part 6 of Proposition 5, and (47). Hence, our bound in (52) is tighter than the \( \eta_{TV} \) bound when:

\[
2(1 - 2p)^2 < |1 - 2p| \iff \frac{1}{4} < p < \frac{1}{2} \text{ or } \frac{1}{2} < p < \frac{3}{4}.
\]

(55)

Since our upper bounds can be greater than 1 (see (53)), we cannot hope to beat the \( \eta_{TV} \) bound in all regimes. On the other hand, one advantage of our upper bounds is that they "match" the \( \eta_{\chi^2} \) lower bound in part 6 of Proposition 3; we will illustrate a useful application of this in subsection IV-C.

Thirdly, we intuitively expect a bound between contraction coefficients to depend on the cardinalities \(|\mathcal{X}|\) or \(|\mathcal{Y}|\). Since the minimum probability term in all our upper bounds satisfies \(1/\min_{x \in \mathcal{X}} P_X(x) \geq |\mathcal{X}|\), we can superficially construe it as "modeling" \(|\mathcal{X}|\). Unfortunately, this intuition is quite misleading. Simulations for the binary case, depicted in Figure 1(b), illustrate that the ratio \( \eta_{\chi^2}(P_X, P_{Y|X})/\eta_{\chi^2}(P_X, P_{Y|X}) \) increases significantly near the boundary of \( P_X \) when any of the probability masses of \( P_X \) is close to 0. This effect, while unsurprising given the skewed nature of probability simplices at their boundaries with respect to KL divergence as the distance measure, is correctly captured by the upper bounds in Corollary 9 and Theorem 10 because \(1/\min_{x \in \mathcal{X}} P_X(x) \) increases when any of the input probability masses tends to 0 (see Figure 1(b)). Clearly, linear upper bounds on \( \eta_{KL}(P_X, P_{Y|X}) \) that are purely in terms of \(|\mathcal{X}|\) or \(|\mathcal{Y}|\) cannot capture this effect. This gives credence to the existence of the minimum probability term in our linear bounds.

Finally, we note that the inequality \(1/\min_{x \in \mathcal{X}} P_X(x) \geq |\mathcal{X}|\) does not preclude the possibility of \(1/\min_{x \in \mathcal{X}} P_X(x) \) being much larger than \(|\mathcal{X}|\). So, our bounds can become loose when \(|\mathcal{X}|\) is large (see the example in subsection IV-D). As a result, the bounds in Theorem 8, Corollary 9, and Theorem 10 are usually of interest in the following settings:

1) \(|\mathcal{X}|\) and \(|\mathcal{Y}|\) are small: Figure 1 portrays that our bounds can be quite tight when \(|\mathcal{X}| = |\mathcal{Y}| = 2\).
2) Weak dependence i.e. \( I(X;Y) \) is small: This situation naturally arises in the analysis of ergodicity of Markov chains—see subsection IV-C.
3) Product Distributions: If the underlying joint pmf is a product pmf, we can exploit tensorization of contraction coefficients (Proposition 3 part 4)—see subsection IV-D.

C. Contraction Coefficients of Gaussian Random Variables

In this subsection, we consider contraction coefficients for KL and \( \chi^2 \)-divergences corresponding to Gaussian source-channel pairs. Suppose \( X \) and \( Y \) are jointly Gaussian random variables. Their joint distribution has three possible forms:

1) \( X \) or \( Y \) are constants a.s., and we define the contraction coefficients to be \( \eta_{KL}(P_X, P_{Y|X}) = \eta_{\chi^2}(P_X, P_{Y|X}) = 0 \).
2) \( aX + bY = c \) a.s. for some constants \( a, b, c \in \mathbb{R} \) such that \( a \neq 0 \) and \( b \neq 0 \). Here, it is straightforward to verify that \( \rho(X;Y) = 1 \), which implies that \( \eta_{KL}(P_X, P_{Y|X}) = \eta_{\chi^2}(P_X, P_{Y|X}) = 1 \).
3) The joint probability density function (pdf) \( P_{X,Y} \) exists with respect to the Lebesgue measure.

The final non-degenerate case is our regime of interest. For simplicity, we will assume that \( X \) and \( Y \) are zero-mean, and analyze the classical additive white Gaussian noise (AWGN) channel model [25, Chapter 9]:

\[
Y = X + W, \quad X \perp \!
\perp W
\]

(56)

where the input is \( X \sim \mathcal{N}(0, \sigma_X^2) \) with \( \sigma_X^2 > 0 \) (i.e. \( X \) has a Gaussian pdf with mean 0 and variance \( \sigma_X^2 \)), the Gaussian noise is \( W \sim \mathcal{N}(0, \sigma_W^2) \) with \( \sigma_W^2 > 0 \), and \( X \) is independent of \( W \). This relation also defines the channel conditional pdfs \( \{P_{Y|X=x} = \mathcal{N}(x, \sigma_W^2) : x \in \mathbb{R} \} \).

For the jointly Gaussian pdf \( P_{X,Y} \) define above, the contraction coefficients for KL and \( \chi^2 \)-divergences are given by (cf. (21) and (28)):

\[
\begin{align*}
\eta_{KL}(P_X, P_{Y|X}) &= \sup_{R_X: 0 < D(R_X||P_X) < +\infty} \frac{D(R_Y||P_Y)}{D(R_X||P_X)} \\
\eta_{\chi^2}(P_X, P_{Y|X}) &= \sup_{R_X: 0 < \chi^2(R_X||P_X) < +\infty} \frac{\chi^2(R_Y||P_Y)}{\chi^2(R_X||P_X)}
\end{align*}
\]

(57)

(58)

Note that Definition 3 holds for general random variables, and (29) and part 6 of Proposition 3 (which also hold generally—see [8, Equations (9) and (13)]) can be used to conclude \( \eta_{KL}(P_X, P_{Y|X}) = \eta_{\chi^2}(P_X, P_{Y|X}) = 1 \).
where the suprema are over pdfs $R_X$ (which differ from $P_X$ on a set with non-zero Lebesgue measure),\(^{21}\) and $R_Y$ denotes the marginal pdf of $Y$ after passing $R_X$ through the AWGN channel $P_{Y|X}$. In particular, $R_Y = R_X \star \mathcal{N}(0, \sigma^2_W)$, where $\star$ denotes the convolution operation. Furthermore, we define the contraction coefficient for KL divergence with average power constraint $p \geq \sigma^2_X$ as:

$$
\eta_{KL}^{(p)}(P_X, P_{Y|X}) \triangleq \sup_{X: R_X \in P_X \text{ s.t. } \mathbb{E}[X^2] \leq p} \frac{D(R_Y \| P_Y)}{D(R_X \| P_X)}
$$

(59)

where the supremum is over pdfs $R_X$ satisfying the average power constraint $\mathbb{E}[X^2] \leq p$. Note that setting $p = +\infty$ yields the standard contraction coefficient in (57).

It is well-known in the literature that $\eta_{KL}(P_X, P_{Y|X}) = \eta_{KL}(P_X, P_{Y|X})$ for the jointly Gaussian pdf $P_{X,Y}$ in (56). For example, [3, Theorem 7] proves this result in the context of investment portfolio theory, [63, p.2] proves a generalization of it in the context of Gaussian hypercontractivity, and [60, Section 5.2, part 5] proves it in an effort to axiomatize $\eta_{KL}$. While the proofs in [3, Theorems 6 and 7] and [60, Section 5.2, part 5] use the mutual information characterization of $\eta_{KL}$ in (22) (cf. [8, Theorem 4]), we provide an alternative proof of this result in section V that directly uses the KL divergence definition of $\eta_{KL}$ in (57). Furthermore, our proof also establishes that $\eta_{KL}^{(p)}(P_X, P_{Y|X})$ equals $\eta_{KL}(P_X, P_{Y|X})$ for every $p \in [\sigma^2_X, +\infty)$. Although this latter result follows easily from our proof, it has not explicitly appeared in the literature to our knowledge. The ensuing theorem states these results formally.

**Theorem 11 (Gaussian Contraction Coefficients).** Given the jointly Gaussian pdf $P_{X,Y}$, defined via (56) with source $P_X = \mathcal{N}(0, \sigma^2_X)$ and channel $\{P_{Y|X} = \mathcal{N}(x, \sigma^2_W) : x \in \mathbb{R}\}$ such that $\sigma^2_X, \sigma^2_W > 0$, the following quantities are equivalent:

$$
\eta_{KL}(P_X, P_{Y|X}) = \eta_{KL}^{(p)}(P_X, P_{Y|X}) = \eta_{KL}(P_X, P_{Y|X}) = \frac{\sigma^2_X}{\sigma^2_X + \sigma^2_W}
$$

where the average power constraint $p \geq \sigma^2_X$.

As mentioned earlier, we prove this in section V. In contrast to Theorem 11, where $\eta_{KL}(P_X, P_{Y|X})$ and $\eta_{KL}^{(p)}(P_X, P_{Y|X})$ can both be strictly less than 1, we note that the contraction coefficients for KL divergence of channels (i.e. the setting of Definition 5) are equal to 1 regardless of whether we impose power constraints, cf. [7, Section 1.2] and [64, Section 1].

**D. Less Noisy Preorder and Operator Convexity**

Our last main result presents an equivalent characterization of the less noisy preorder over channels that generalizes the result in Proposition 6. We begin by defining the less noisy preorder. Within the finite alphabet setting of subsection II-A, consider an input random variable $X \in \mathcal{X}$, and two output random variables $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$, where $\mathcal{Z} \triangleq \{1, \ldots, |\mathcal{Z}|\}$ such that $2 \leq |\mathcal{Z}| < +\infty$. Let $P_{Y|X}$ and $P_{Z|X}$ be any two channels with the same input alphabet $\mathcal{X}$, and corresponding row stochastic transition probability matrices $W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}$ and $V \in \mathcal{P}_{\mathcal{Z}|\mathcal{X}}$, respectively. We say that $P_{Y|X}$ is less noisy than $P_{Z|X}$, denoted $P_{Y|X} \succeq_{n} P_{Z|X}$, if and only if:

$$
D(R_X W \| P_X W) \geq D(R_X V \| P_X V)
$$

(60)

for every pair of pmfs $R_X, P_X \in \mathcal{P}_{\mathcal{X}}$. It is straightforward to verify that (60) defines a preorder over channel matrices. Moreover, this definition conveys that the pair of pmfs $R_X W$ and $P_X W$ is “more distinguishable” than the pair $R_X V$ and $P_X V$, which indeed intuitively corresponds to $W$ being “less noisy” than $V$. There are several other equivalent characterizations of $\succeq_{\ell}$; for example, via channel coding [65, Definition B, Proposition 2], mutual information [65, Proposition 2], and the van Dijk functional [66, Theorem 2]. We refer readers to [67, Sections I-B, I-D, II-A, IV] and the references therein for further details on the less noisy preorder.

The authors of [8, Section 6] illustrate that less noisy domination of a given channel by an erasure channel is closely related to the contraction coefficient for KL divergence of the channel. Recall that $E_{1-\beta}$ denotes an $|\mathcal{X}|$-ary erasure

\(^{21}\)When $P_X$ is a general probability measure and $P_{Y|X}$ is a Markov kernel between two measurable spaces, the contraction coefficients for KL and $\chi^2$-divergences are defined exactly as in (21) and (28) using the measure theoretic definitions of KL and $\chi^2$-divergences [8, Section 2]. In (57), when we optimize over all probability measures $R_X$ on $\mathbb{R}$ (with its Borel $\sigma$-algebra), the constraint $D(R_X \| P_X) < +\infty$ implies that $R_X$ must be absolutely continuous with respect to the Gaussian distribution $P_X$, cf. [21, Section 1.6]. Hence, the supremum in (57) can be taken over all pdfs $R_X$ such that $0 < D(R_X \| P_X) < +\infty$. A similar argument applies for (58). (Note that KL and $\chi^2$-divergences for pdfs are defined just as in (5) and (6) with Lebesgue integrals replacing summations.)
channel with erasure probability $1 - \beta \in [0,1]$, input alphabet $\mathcal{X}$, and output alphabet $\mathcal{X} \cup \{e\}$ (as defined towards the end of subsection II-B). It can be deduced from [8, Proposition 15] that for any channel $P_{Y|X}$:

$$\eta_{kl}(P_{Y|X}) = \inf \{ \beta \in [0,1] : E_{1-\beta} \geq_{n} P_{Y|X} \}.$$  \hfill (61)

In [67, Section IV-A], the authors note that while (61) conveys that $\eta_{kl}$ characterizes less noisy domination by erasure channels, (43) portrays that $\eta_{n,2}$ also characterizes this domination. This begs the question: Does $\chi^2$-divergence characterize the less noisy preorder in general? To answer this question, [67, Theorem 1] (which we present later as Lemma 23) characterizes $\geq_{n}$ using $\chi^2$-divergence, thereby generalizing (43).

Inspired by these results, we consider Proposition 6, which shows that $\eta_{kl}(P_{Y|X}) = \eta_{f}(P_{Y|X})$ for all non-linear operator convex functions $f$ (defined in subsection VI-A). The ensuing theorem generalizes both Proposition 6 and [67, Theorem 1], and portrays that non-linear operator convex $f$-divergences also characterize the less noisy preorder.

**Theorem 12** (Equivalent Characterizations of $\geq_{n}$). Consider any non-linear operator convex function $f : (0, \infty) \to \mathbb{R}$ such that $f(1) = 0$. Then, for any two channels $P_{Y|X}$ and $P_{Z|X}$ on the same input alphabet $\mathcal{X}$ with row stochastic transition probability matrices $W \in \mathcal{P}_{Y|X}$ and $V \in \mathcal{P}_{Z|X}$ respectively, $P_{Y|X} \geq_{n} P_{Z|X}$ if and only if:

$$D_{f}(R_{X}W\|P_{X}W) \geq D_{f}(R_{X}V\|P_{X}V)$$

for every pair of input pmfs $R_{X}, P_{X} \in \mathcal{P}_{X}$.

Theorem 12 is proved in subsection VI-B using techniques from [10]. It is well-known that $f(t) = t \log(t)$ and $f(t) = \frac{t^{\alpha} - 1}{\alpha - 1}$ for any $\alpha \in (0, 1) \cup (1, 2]$ are operator convex functions (see [68, Theorems V.2.5 and V.2.10, Exercises V.2.11 and V.2.13], and apply the affine transformation property in subsection VI-A appropriately). Hence, one class of $f$-divergences that satisfy the conditions of the theorem are the Hellinger divergences of order $\alpha \in (0, 2]$, where the cases $\alpha = 1$ and $\alpha = 2$ correspond to KL and $\chi^2$-divergences, respectively.

### IV. PROOFS OF LINEAR BOUNDS BETWEEN CONTRACTION COEFFICIENTS

In this section, we will prove Theorems 8 and 10. The central idea to establish these results entails upper and lower bounding the $f$-divergences in the numerator and denominator of Definition 2 respectively, using $\chi^2$-divergences. To this end, we will illustrate some simple bounds between $f$-divergences and $\chi^2$-divergence in the next subsection, and prove the main results in subsection IV-B.

#### A. Bounds on $f$-Divergences using $\chi^2$-Divergence

We first present bounds between KL divergence and $\chi^2$-divergence. To derive our lower bound on KL divergence, we will require the following “distribution dependence refinement of Pinsker’s inequality” proved in [69].

**Lemma 13** (Distribution Dependent Pinsker’s Inequality [69, Theorem 2.1]). For any two pmfs $R_{X}, P_{X} \in \mathcal{P}_{X}$, we have:

$$D(R_{X}\|P_{X}) \geq \phi \left( \max_{A \subseteq X} \pi(A) \right) \|R_{X} - P_{X}\|^{2}_{TV}$$

where $\pi(A) = \min\{P_{X}(A), 1 - P_{X}(A)\}$ for any $A \subseteq X$, and the function $\phi : [0, \frac{1}{2}] \to \mathbb{R}$ is defined as:

$$\phi(p) \triangleq \begin{cases} \frac{1}{1 - 2p} \log \left( \frac{1 - p}{p} \right), & p \in [0, \frac{1}{2}] \\ 2, & p = \frac{1}{2} \end{cases}.$$  \hfill (62)

Moreover, this inequality uses the optimal distribution dependent constant in the sense that for any fixed $P_{X} \in \mathcal{P}_{X}$:

$$\inf_{R_{X} \in \mathcal{P}_{X} \backslash \{P_{X}\}} \frac{D(R_{X}\|P_{X})}{\|R_{X} - P_{X}\|^{2}_{TV}} = \phi \left( \max_{A \subseteq X} \pi(A) \right).$$

Recall that Pinsker’s inequality states that for any $R_{X}, P_{X} \in \mathcal{P}_{X}$ (see e.g. [25, Lemma 11.6.1]):

$$D(R_{X}\|P_{X}) \geq 2 \|R_{X} - P_{X}\|^{2}_{TV}.$$  \hfill (63)

Hence, Lemma 13 is tighter than Pinsker’s inequality, because $0 \leq \max_{A \subseteq X} \pi(A) \leq \frac{1}{2}$, and hence:

$$\phi \left( \max_{A \subseteq X} \pi(A) \right) \geq 2.$$  \hfill (64)

---

22Note that when $\beta = 1$, $E_{0}$ is the identity channel and $E_{0} \geq_{n} P_{Y|X}$. 

with equality if and only if \( \max_{A \subseteq X} \pi(A) = \frac{1}{2} \), cf. [69, Section III]. The ensuing lemma uses Lemma 13 to lower bound KL divergence using \( \chi^2 \)-divergence.

**Lemma 14 (KL Divergence Lower Bound).** Given any two pmfs \( R_X, P_X \in \mathcal{P}_X \), we have:

\[
D(R_X || P_X) \geq \phi \left( \max_{A \subseteq X} \pi(A) \right) \min_{x \in X} P_X(x) \cdot \chi^2(R_X || P_X)
\]

where \( \pi(\cdot) \) and \( \phi : [0, \frac{1}{2}] \to \mathbb{R} \) are defined in Lemma 13.

**Proof.** Observe that if \( R_X = P_X \) or \( \min_{x \in X} P_X(x) = 0 \), then the inequality is trivially satisfied. So, we assume without loss of generality that \( R_X \neq P_X \) and \( P_X \in \mathcal{P}_X \).

Since \( \chi^2 \)-divergence resembles a weighted \( \ell^2 \)-norm, we first use Lemma 13 to get the lower bound:

\[
D(R_X || P_X) \geq \phi \left( \max_{A \subseteq X} \pi(A) \right) \frac{\| R_X - P_X \|_1^2}{4}
\]

where we use the \( \ell^1 \)-norm characterization of TV distance given in (4). We next notice using (6) that:

\[
\chi^2(R_X || P_X) = \sum_{x \in X} |R_X(x) - P_X(x)| \frac{|R_X(x) - P_X(x)|}{P_X(x)}
\leq \frac{\| R_X - P_X \|_\infty}{\min_{x \in X} P_X(x)} \| R_X - P_X \|_1.
\]

This implies that:

\[
\frac{\| R_X - P_X \|_1^2}{\min_{x \in X} P_X(x)} \geq \chi^2(R_X || P_X) \frac{\| R_X - P_X \|_1}{\| R_X - P_X \|_\infty}
\geq \chi^2(R_X || P_X) \min_{S_X, Q_X \in \mathcal{P}_X} \frac{\| S_X - Q_X \|_1}{\| S_X - Q_X \|_\infty}
\]

\[
= 2 \chi^2(R_X || P_X)
\]

where we use the fact that:

\[
\min_{S_X, Q_X \in \mathcal{P}_X} \frac{\| S_X - Q_X \|_1}{\| S_X - Q_X \|_\infty} = 2.
\]

To prove (67), note that for every \( S_X, Q_X \in \mathcal{P}_X \) (see e.g. [70, Lemma 1]):

\[
\| S_X - Q_X \|_\infty \leq \frac{1}{2} \| S_X - Q_X \|_1
\]

because \( (S_X - Q_X)1 = 0 \), and this inequality can in fact be tight. For example, choose any pmf \( Q_X \in \mathcal{P}_X \) and let \( x_0 = \arg \min_{x \in X} Q_X(x) \). Then, select \( S_X \in \mathcal{P}_X \) such that \( S_X(x_0) = Q_X(x_0) + \delta \) for some sufficiently small \( \delta > 0 \), \( S_X(x_1) = Q_X(x_1) - \delta \) for some \( x_1 \in X \setminus \{x_0\} \), and \( S_X(x) = Q_X(x) \) for every other \( x \). These choices of \( S_X \) and \( Q_X \) yield \( \| S_X - Q_X \|_\infty = \delta = \frac{1}{2} \| S_X - Q_X \|_1 \).

Finally, combining (65) and (66), we get:

\[
D(R_X || P_X) \geq \phi \left( \max_{A \subseteq X} \pi(A) \right) \min_{x \in X} P_X(x) \cdot \chi^2(R_X || P_X)
\]

which completes the proof.

We remark that if we apply (64) to Lemma 14, or equivalently, if we use the standard Pinsker’s inequality (63) instead of Lemma 13 in the proof of Lemma 14, then we obtain the well-known weaker inequality (see e.g. [17, Equation (338)]):

\[
D(R_X || P_X) \geq \min_{x \in X} P_X(x) \chi^2(R_X || P_X)
\]

for every \( R_X, P_X \in \mathcal{P}_X \).

23Throughout this paper, when \( \min_{x \in X} P_X(x) = 0 \) and \( \chi^2(R_X || P_X) = +\infty \), we assume that \( \min_{x \in X} P_X(x) \chi^2(R_X || P_X) = 0 \).
It is worth mentioning that a systematic method of deriving optimal bounds between any pair of $f$-divergences is given by the Harremoës-Vajda joint range [71]. However, we cannot use this technique to derive lower bounds on KL divergence using $\chi^2$-divergence since no such general lower bound exists (when both input distributions vary) [21, Section 7.3]. On the other hand, distribution dependent bounds can be easily found using ad hoc techniques.\textsuperscript{24} Our proof of Lemma 14 demonstrates one such ad hoc approach based on Pinsker’s inequality.

It is tempting to try and improve Lemma 14 by using better lower bounds on KL divergence in terms of TV distance. For example, the best possible lower bound on KL divergence via TV distance is the lower boundary of their Harremoës-Vajda joint range, cf. [71, Figure 1]. This lower boundary, known as Vajda’s tight lower bound, gives the minimum possible KL divergence for each value of TV distance, and is completely characterized using a parametric formula in [72, Theorem 1] (also see [21, Section 7.2.2]). Although Vajda’s tight lower bound yields a non-linear lower bound on KL divergence using $\chi^2$-divergence, this lower bound is difficult to apply in conjunction with Lemma 15 (shown below) to obtain a non-linear upper bound on a ratio of KL divergences using a ratio of $\chi^2$-divergences (see the proof of Theorem 10 in subsection IV-B). For this reason, we resort to using simple linear bounds between KL and $\chi^2$-divergence, which yields a linear bound in Theorem 10.

Another subtler reason for proving a linear lower bound on KL divergence using $\chi^2$-divergence is to exploit Lemma 13. Although Pinsker’s inequality is the best lower bound on KL divergence using squared TV distance over all pairs of input pmfs (see e.g. [72, Equation (9)]), the contraction coefficients in subsection III-B have a fixed source pmf $P_X$. Therefore, we can use the distribution dependent improvement of Pinsker’s inequality in Lemma 13 to obtain a tighter bound than (68).

We next present an upper bound on KL divergence using $\chi^2$-divergence which trivially follows from Jensen’s inequality. This bound was derived in the context of studying ergodicity of Markov chains in [73], and has been re-derived in the study of inequalities related to $f$-divergences, cf. [74], [75].

**Lemma 15** (KL Divergence Upper Bound [73]). Given any two pmfs $P_X, R_X \in \mathcal{P}_X$, we have:

$$D(R_X \| P_X) \leq \log(1 + \chi^2(R_X \| P_X)) \leq \chi^2(R_X \| P_X).$$

**Proof.** We provide a proof for completeness, cf. [74]. Assume without loss of generality that there does not exist $x \in \mathcal{X}$ such that $R_X(x) > P_X(x) = 0$. (If this is not the case, then $\chi^2(R_X \| P_X) = +\infty$ and the inequalities are trivially true.) So, restricting $\mathcal{X}$ to be the support of $P_X$, we assume that $P_X \in \mathcal{P}^c_X$ (which ensures that none of the ensuing quantities are infinity). Since $x \mapsto \log(x)$ is a concave function, using Jensen’s inequality, we have:

$$D(R_X \| P_X) = \sum_{x \in \mathcal{X}} R_X(x) \log \left( \frac{R_X(x)}{P_X(x)} \right) \leq \log \left( \sum_{x \in \mathcal{X}} \frac{R_X(x)^2}{P_X(x)} \right) = \log(1 + \chi^2(R_X \| P_X)) \leq \chi^2(R_X \| P_X)$$

where the third equality follows from (6) after some algebra, and the final inequality follows from the well-known inequality: $\log(1 + x) \leq x$ for all $x > -1$.

We remark that the first non-linear bound in Lemma 15 turns out to capture the Harremoës-Vajda joint range [21, Section 7.3]. Although it is tighter than the second linear bound, we will use the latter to prove Theorem 10 (as explained earlier).

We now present bounds between general $f$-divergences and $\chi^2$-divergence. To derive our lower bound on $f$-divergences, we first state a generalization of Pinsker’s inequality for $f$-divergences that is proved in [61].

**Lemma 16** (Generalized Pinsker’s Inequality for $f$-Divergence [61, Theorem 3]). Suppose we are given a convex function $f : (0, \infty) \to \mathbb{R}$ that is thrice differentiable at unity with $f(1) = 0$ and $f''(1) > 0$, and satisfies:

$$(f(t) - f'(1)(t - 1)) \left(1 - \frac{f''(1)}{3f''(1)}(t - 1)\right) \geq \frac{f''(1)}{2} (t - 1)^2$$

for every $t \in (0, \infty)$. Then, we have for every $R_X, P_X \in \mathcal{P}_X$:

$$D_f(R_X \| P_X) \geq 2 f''(1) \|R_X - P_X\|_{TV}^2.$$

\textsuperscript{24}A “distribution dependent” bound between two $f$-divergences is a bound that contains terms that are not either of the $f$-divergences, but depend on the input distributions.
Moreover, this inequality uses the optimal constant in the sense that:

\[
\inf_{R_X, P_X \in \mathcal{P}_X: \|R_X - P_X\|_\infty \neq 0} \frac{D_f(R_X \| P_X)}{\|R_X - P_X\|_\infty^2} = 2 f''(1).
\]

We remark that \( f(t) = t \log(t) \) satisfies the conditions of Lemma 16 with \( f''(1) = 1 \) as shown in Appendix D; this yields the standard Pinsker’s inequality presented in (63). Since (69) can be difficult to check for other \( f \)-divergences, the author of [61] provides sufficient conditions for (69) in [61, Corollary 4]. (These conditions can be verified to yield a variant of Pinsker’s inequality for Rényi divergences of order \( \alpha \in (0, 1) \) [61, Corollary 6].) The ensuing lemma uses Lemma 16 to establish a lower bound on certain \( f \)-divergences using \( \chi^2 \)-divergence which parallels Lemma 14 (or more precisely, (68), since it follows from the standard Pinsker’s inequality).

**Lemma 17** (\( f \)-Divergence Lower Bound). Suppose we are given a convex function \( f : (0, \infty) \to \mathbb{R} \) that is thrice differentiable at unity with \( f(1) = 0 \) and \( f''(1) > 0 \), and satisfies (69) for every \( t \in (0, \infty) \). Then, for any two pmfs \( R_X, P_X \in \mathcal{P}_X \) with \( R_X \neq P_X \), we have:

\[
D_f(R_X \| P_X) \geq f''(1) \min_{x \in \mathcal{X}} P_X(x) \chi^2(R_X \| P_X).
\]

**Proof.** We follow the proof of Lemma 14 mutatis mutandis. Assume without loss of generality that \( R_X \neq P_X \). The generalized Pinsker’s inequality for \( f \)-divergences in Lemma 16 yields:

\[
D_f(R_X \| P_X) \geq \frac{f''(1)}{2} \|R_X - P_X\|_1^2,
\]

using the \( \ell^1 \)-norm characterization of TV distance in (4). Applying (66) to this inequality produces the desired result. \( \square \)

Note that setting \( f(t) = t \log(t) \) in Lemma 17 gives (68).

Finally, we present an upper bound on certain \( f \)-divergences using \( \chi^2 \)-divergence which is analogous to Lemma 15. This upper bound was proved in [9, Lemma A.2] with the assumption that \( f \) is differentiable, but we only need to check differentiability at unity as seen below. (It is instructive for readers to revisit the proof of Lemma 15 to see how the ensuing proof generalizes it for \( f \)-divergences.)

**Lemma 18** (\( f \)-Divergence Upper Bound [9, Lemma A.2]). Suppose we are given a continuous convex function \( f : [0, \infty) \to \mathbb{R} \) that is differentiable at unity with \( f(1) = 0 \) such that the difference quotient \( g : (0, \infty) \to \mathbb{R} \), defined as \( g(x) = \frac{f(x) - f(0)}{x} \), is concave. Then, for any two pmfs \( R_X, P_X \in \mathcal{P}_X \), we have:

\[
D_f(R_X \| P_X) \leq (f'(1) + f(0)) \chi^2(R_X \| P_X).
\]

**Proof.** We provide the proof in [9] for completeness. As in the proof of Lemma 15, we may assume without loss of generality that \( P_X \in \mathcal{P}_X \) so that none of the ensuing quantities are infinity. We then have the following sequence of equalities and inequalities:

\[
D_f(R_X \| P_X) = \sum_{x \in \mathcal{X}} P_X(x) f \left( \frac{R_X(x)}{P_X(x)} \right)
= f(0) + \sum_{x \in \mathcal{X}} R_X(x) g \left( \frac{R_X(x)}{P_X(x)} \right)
\leq f(0) + g \left( \sum_{x \in \mathcal{X}} R_X(x)^2 \right) \frac{1}{P_X(x)}
= f(0) + g(1 + \chi^2(R_X \| P_X))
\leq f(0) + g(1) + g'(1) \chi^2(R_X \| P_X)
= (f'(1) + f(0)) \chi^2(R_X \| P_X)
\]

25Since \( f \) is convex, it is clearly continuous on \((0, \infty)\). So, the continuity assumption on \( f \) only asserts that \( f(0) = \lim_{t \to 0^+} f(t) \) (see Definition 1).
where the second equality uses the convention $0g(0) = 0$, the first inequality follows from Jensen’s inequality since $g : (0, \infty) \to \mathbb{R}$ is concave, the second inequality is also a consequence of the concavity of $g : (0, \infty) \to \mathbb{R}$ as shown in [76, Section 3.1.3], and the final equality holds because $g(1) = −f(0)$ (as $f(1) = 0$) and:

$$g'(1) = \lim_{\delta \to 0} \frac{g(1 + \delta) + f(0)}{\delta} = \lim_{\delta \to 0} \frac{f(1 + \delta) + \delta f(0)}{\delta} = \left( \lim_{\delta \to 0} \frac{1}{1 + \delta} \right) f(0) + \lim_{\delta \to 0} \frac{f(1 + \delta)}{\delta}$$

This completes the proof.

We note that (70) is the analog of the tighter (non-linear) bound in Lemma 15. Furthermore, we remark that $g(x) = \frac{f(x)}{x}$ (when it is assumed to be concave) is a valid definition for the function in Lemma 18 instead of the difference quotient. The proof carries through with a constant of $f'(1)$ instead of $f'(1) + f(0)$. However, we choose the difference quotient to prove Lemma 18 in view of the affine invariance property of $f$-divergences (cf. subsection II-A). It is easy to verify that the quantity $f'(1) + f(0)$ is invariant to appropriate affine shifts, but the quantity $f'(1)$ is not. We also remark that the constant $f'(1)$ in Lemma 17 is invariant to appropriate affine shifts.

B. Proofs of Theorems 8 and 10

Recall from the outset of subsection III-A that we are given a joint pmf $P_{X,Y}$ such that $P_X \in \mathcal{P}_X$ and $P_Y \in \mathcal{P}_Y$. Moreover, we let $W \in \mathcal{P}_{Y\mid X}$ denote the row stochastic transition probability matrix of the channel $P_{Y\mid X}$. Using Lemmata 14 and 15 from subsection IV-A, we can now prove Theorem 10.

**Proof of Theorem 10.** For every pmf $R_X \in \mathcal{P}_X$ such that $R_X \neq P_X$, we have:

$$\frac{D(R_X W||P_Y)}{D(R_X||P_X)} \leq \frac{2 \chi^2(R_X W||P_Y)}{\phi \left( \max_{A \subseteq X} \pi(A) \right) \min_{x \in X} P_X(x) \chi^2(R_X||P_X)}$$

using Lemmata 14 and 15, where $P_Y = P_X W$. Taking the supremum over all $R_X \neq P_X$ on both sides produces:

$$\eta_\alpha(P_X, P_{Y\mid X}) \leq \frac{2 \eta_{\chi^2}(P_X, P_{Y\mid X})}{\phi \left( \max_{A \subseteq X} \pi(A) \right) \min_{x \in X} P_X(x)}$$

using (21) and (28). This completes the proof.

We now make a few pertinent remarks. Firstly, applying (68) instead of Lemma 14 in the preceding proof yields Corollary 9.

Secondly, while the conference version of this paper proves Corollary 9 (see [1, Theorem 10]), it also proves the following weaker upper bound on $\eta_\alpha(P_X, P_{Y\mid X})$ [1, Theorem 9]:

$$\eta_\alpha(P_X, P_{Y\mid X}) \leq \frac{2 \eta_{\chi^2}(P_X, P_{Y\mid X})}{\min_{x \in X} P_X(x)}$$

which is independently derived in [9, Equation III.19]. Our proof of (71) in [1, Theorem 9] uses the ensuing variant of (68) that is looser by a factor of 2, cf. [1, Lemma 6]:

$$D(S_X||Q_X) \geq \frac{\min_{x \in X} Q_X(x)}{2} \chi^2(S_X||Q_X)$$

for all $S_X, Q_X \in \mathcal{P}_X$. This follows from executing the proof of Lemma 14 using the bound $\|S_X - Q_X\|_\infty \leq \|S_X - Q_X\|_1$ (which neglects the information that $(S_X - Q_X)1 = 0$ instead of (67), and then applying (64) to the resulting lower bound on KL divergence. Alternatively, we provide a proof of (72) via Bregman divergences in Appendix E for completeness, cf. [1, Lemma 6]. The improvement by a factor of 2 from (72) to (68) is also observed in [17, Remark 33], where the authors mention that our result [1, Theorem 9] (see (71)) in the conference version of this paper can be improved by a factor of 2 by using (68) instead (72). We believe the authors of [17] may have missed
our result [1, Theorem 10] (see Corollary 9) in the conference paper, which presents precisely this improvement by a factor of 2.

Lastly, we remark that [9, Section III-D] also presents upper bounds on $\eta_\alpha(P_X, P_{Y|X})$ that use the function $\phi : [0, 1] \rightarrow \mathbb{R}$, which stems from the refined Pinsker’s inequality in [69]. However, these upper bounds are not in terms of $\eta_{\chi^2}(P_X, P_{Y|X})$.

We next prove Theorem 8 by combining Lemmata 17 and 18 from subsection IV-A.

**Proof of Theorem 8.** The conditions of Theorem 8 encapsulate all the conditions of Lemmata 17 and 18. Hence, using Lemmata 17 and 18, for every pmf $R_X \in \mathcal{P}_X$ such that $R_X \neq P_X$, we have:

$$D_f(R_X W||P_Y) \leq \frac{(f'(1) + f(0))}{f''(1)} \chi^2(R_X W||P_X)$$

where $P_Y = P_X W$. Taking the supremum over all $R_X \neq P_X$ on both sides produces:

$$\eta_f(P_X, P_{Y|X}) \leq \frac{(f'(1) + f(0))}{f''(1)} \eta_{\chi^2}(P_X, P_{Y|X})$$

where we use Definition 2 and (28). This completes the proof.

We remark that [9, Theorem III.4] presents an alternative linear upper bound on $\eta_f(P_X, P_{Y|X})$ using $\eta_{\chi^2}(P_X, P_{Y|X})$. Suppose $f : [0, \infty) \rightarrow \mathbb{R}$ is a twice differentiable convex function that has $f'(1) = 0$, is strictly convex at unity, and has non-increasing second derivative. If we further assume that the difference quotient $x \mapsto \frac{f(x) - f(0)}{x}$ is concave, then the following bound holds [9, Theorem III.4]:

$$\eta_f(P_X, P_{Y|X}) \leq \frac{2(f'(1) + f(0))}{f''(1)} \eta_{\chi^2}(P_X, P_{Y|X})$$

where $p_* = \min_{x \in \mathcal{X}} P_X(x)$. Hence, when $f$ is additionally thrice differentiable at unity, has $f''(1) > 0$, and satisfies (69) for every $t \in (0, \infty)$, we can improve the upper bound in Theorem 8 to:

$$\eta_f \leq \min \left\{ \frac{f'(1) + f(0)}{f''(1)} - \frac{2(f'(1) + f(0))}{f''(1)p_*}, \frac{2f''(1)p_*}{f''(1)} \right\} \eta_{\chi^2} \eta$$

where $\eta_f = \eta_f(P_X, P_{Y|X})$ and $\eta_{\chi^2} = \eta_{\chi^2}(P_X, P_{Y|X})$.

Observe that our bound in Theorem 8 is tighter than that in (73) if and only if:

$$\frac{2(f'(1) + f(0))}{f''(1)p_*} \geq \frac{f'(1) + f(0)}{f''(1)}$$

$$\Leftrightarrow \frac{2f''(1)p_*}{f''(1)} \geq f''(1)p_*$$

(75)

One function that satisfies the conditions of Theorem 8 and (73) as well as (76) is $f(t) = t \log(t)$. This engenders the improvement that Corollary 9 (which can be recovered from Theorem 8) offers over (71) (which can be recovered from [9, Theorem III.4]).

As another example, consider the function $f(t) = \frac{t^{\alpha-1}}{\alpha-1}$ for $\alpha \in (0, 2] \setminus \{1\}$, which defines the Hellinger divergence of order $\alpha$ (see subsection II-A). It is straightforward to verify that this function satisfies the conditions of Theorem 8 and (73), cf. [61, Corollary 6], [9, Section III-B, p.3362]. In this case, our bound in Theorem 8 is tighter than (73) for all Hellinger divergences of order $\alpha$ satisfying (76), i.e. $2f''(1)p_* = 2\alpha p_* \geq \alpha(1/p_*)^{\alpha-2} = f''(1/p_*) \Leftrightarrow p_*^{\alpha-1} \geq \frac{1}{2}$, or equivalently, $0 < \alpha \leq 1 + (\log(2)/\log(1/p_*))$ (where $\alpha = 1$ corresponds to KL divergence—see subsection II-A).

C. Ergodicity of Markov Chains

In this subsection, we derive a corollary of Corollary 9 that illustrates one use of upper bounds on contraction coefficients of source-channel pairs via $\eta_{\chi^2}(P_X, P_{Y|X})$. Consider a Markov kernel $W \in \mathcal{P}_{X|Y}$ on a state space $\mathcal{X}$ that defines an irreducible and aperiodic (time homogeneous) discrete-time Markov chain with unique stationary pmf $P_X \in \mathcal{P}_X$ such that $P_X W = P_X$, cf. [15, Section 1.3]. For simplicity, suppose further that $W$ is reversible (i.e. the detailed balance equations, $P_X(x)W_{x,y} = P_X(y)W_{y,x}$ for all $x, y \in \mathcal{X}$, hold [15, Section 1.6]). This means that $W$ is self-adjoint with respect to the weighted inner product defined by $P_X$, and has all real eigenvalues $1 = \lambda_1(W) > \lambda_2(W) \geq \cdots \geq \lambda_{|\mathcal{X}|}(W) > -1$. Let $\mu(W) \triangleq \max\{|\lambda_2(W)|, |\lambda_{|\mathcal{X}|}(W)|\} \in [0, 1)$ denote the SLEM of $W$ (see subsection II-C).

26The matrix $W$ is primitive since the chain is irreducible and aperiodic.
Since this Markov chain is ergodic, \( \lim_{n \to \infty} R_X W^n = P_X \) for all \( R_X \in \mathcal{P}_X \) [15, Theorem 4.9]. This implies that \( \lim_{n \to \infty} D(R_X W^n || P_X) = 0 \) by the continuity of KL divergence [21, Proposition 3.1]. Let us estimate the rate at which this “distance to stationarity” (measured by KL divergence) vanishes. A naive approach is to apply the SDPI singular value of \( W \) and similar to (80) (Corollary 19).

\[
\lim_{n \to \infty} \eta_\alpha(P_X, W)^n \leq \eta_\alpha(P_X, W)
\]

(77) for all \( R_X \in \mathcal{P}_X \). Using (21), this implies that:

\[
\lim_{n \to \infty} \eta_\alpha(P_X, W)^n \leq \eta_\alpha(P_X, W)
\]

(78) which turns out to be a loose bound on the rate in general.

When \( n \) is large, since \( R_X W^n \) is close to \( P_X \), we intuitively expect \( D(R_X W^n || P_X) \) to resemble a \( \chi^2 \)-divergence (see (17) in subsection II-A), which suggests that \( \eta_\alpha(P_X, W^n) \) should scale like \( \eta_\alpha(P_X, W)^n \). This intuition is rigorously executed in [50, Section 6]. Indeed, when \( \mu(W) \) is strictly greater than the third largest eigenvalue modulus of \( W \), a consequence of [50, Corollary 6.2] is:

\[
\lim_{n \to \infty} \frac{D(R_X W^n || P_X)}{D(R_X W^{n-1} || P_X)} \leq \mu(W)^2
\]

(79) for all \( R_X \in \mathcal{P}_X \) such that the denominator is always positive. (This limit is either 0 or \( \mu(W)^2 \).) Hence, after employing a Cesàro convergence argument and telescoping, we get:

\[
\lim_{n \to \infty} \frac{D(R_X W^n || P_X)}{D(R_X || P_X)} \leq \mu(W)^2
\]

(80) which suggests that \( \lim_{n \to \infty} \eta_\alpha(P_X, W^n) = \frac{1}{n} = \eta_\alpha(P_X, W) = \mu(W)^2 \). The next result proves that this inequality is in fact tight.

**Corollary 19 (Rate of Convergence).** For every irreducible, aperiodic, and reversible Markov chain with transition kernel \( W \in \mathcal{P}_X \) and stationary pmf \( P_X \in \mathcal{P}_X \), we have:

\[
\lim_{n \to \infty} \eta_\alpha(P_X, W)^n = \eta_\alpha(P_X, W) = \mu(W)^2.
\]

**Proof.** Since \( W \) is reversible and \( P_X \) is its stationary pmf, the DTM \( B = \text{diag}(\sqrt{P_X}) W \text{diag}(\sqrt{P_X})^{-1} \) is symmetric and similar to \( W \) (see definition (30)). Hence, \( W \) and \( B \) share the same eigenvalues, and \( \mu(W) \) is the second largest singular value of \( B \). Using Proposition 2 and (29), we have \( \eta_\alpha(P_X, W) = \mu(W)^2 \), which proves the second equality.

Likewise, \( \eta_\alpha(P_X, W^n) = \mu(W)^2 \) since \( W^n \) is reversible for any \( n \geq 1 \). This yields:

\[
\eta_\alpha(P_X, W^n) = \mu(W)^2 = \mu(W)^{2n} = \eta_\alpha(P_X, W)^n
\]

(81) where the second equality holds because eigenvalues of \( W^n \) are \( n \)th powers of eigenvalues of \( W \). Using (81), part 6 of Proposition 3, and Corollary 9, we get:

\[
\eta_\alpha(P_X, W)^n \leq \eta_\alpha(P_X, W^n) \leq \frac{\eta_\alpha(P_X, W)^n}{\min_{x \in X} P_X(x)}.
\]

Taking \( n \)th roots and letting \( n \to \infty \) yields the desired result.

**Corollary 19** portrays the well-understood phenomenon that \( D(R_X W^n || P_X) \) vanishes with rate determined by \( \mu(W)^2 = \eta_\alpha(P_X, W) \). More generally, it illustrates that the bounds in Corollary 9 and Theorems 8 and 10 are useful in the regime where the random variables \( X \) and \( Y \) are weakly dependent (e.g. \( X \) is the initial state of an ergodic reversible Markov chain, and \( Y \) is the state after a large number of time steps). In this regime, these bounds are quite tight, and beat the \( \eta_\alpha \) bound in part 6 of Proposition 5.

**D. Tensorization of Bounds between Contraction Coefficients**

In the absence of weak dependence, the upper bounds in Corollary 9 and Theorems 8 and 10 can be loose. In fact, they can be rendered arbitrarily loose because the constants in these bounds do not tensorize, while contraction coefficients do (as shown in part 4 of Proposition 3). For instance, if we are given \( P_{X,Y} \) with \( X \sim \text{Bernoulli}(\frac{1}{2}) \), then the constant in the upper bound of Corollary 9 is \( 1/\min_{x \in \{0,1\}} P_X(x) = 2 \). If we instead consider a sequence of pairs \((X_1, Y_1), \ldots, (X_n, Y_n)\) that are independent and identically distributed (i.i.d.) according to \( P_{X,Y} \), then the constant in the upper bound of Corollary 9 is \( 1/\min_{x \in \{0,1\}^n} P_X(x)^n = 2^n \). However, since \( \eta_\alpha(P_{X^n, Y^n} \mid X^n) = \eta_\alpha(P_X, P_Y | X) \) and \( \eta_\alpha(P_{X^n, Y^n} \mid X^n) = \eta_\alpha(P_X, P_Y | X) \) by the tensorization property in part 4 of Proposition 3, the
constant $2^n$ becomes arbitrarily loose as $n$ grows. The next corollary presents a partial remedy for this i.i.d. slackening attack for Corollary 9.

**Corollary 20 (Tensorized KL Contraction Coefficient Bound).** If $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d. with joint pmf $P_{X,Y}$ such that $P_X \in \mathcal{P}_X$ and $P_Y \in \mathcal{P}_Y$, then:

$$\eta_{\alpha}(P_{X^n}, P_{Y^n}|X^n) \leq \frac{\eta_{\alpha}(P_{X^n}P_{Y^n}|X^n)}{\min_{x \in X} P_X(x)}.$$  

**Proof.** This follows trivially from Corollary 9 and the tensorization property in part 4 of Proposition 3.

In the product distribution context, this corollary permits us to use the tighter factor $1/\min_{x \in X} P_X(x)$ in the upper bound of Corollary 9 instead of $1/\min_{x \in X} P_X(x)$ in the upper bound of Corollary 9. Thus, tensorization can improve the upper bounds in Corollary 9 and Theorems 8 and 10 in this context as well. Thus, tensorization can improve the upper bounds in Corollary 9 and Theorems 8 and 10.

V. **Proof of Equivalence Between Gaussian Contraction Coefficients**

We prove Theorem 11 in this section. Recall from subsection III-C that we are given the jointly Gaussian pdf $P_{X,Y}$ defined via (56), with source pdf $P_X = \mathcal{N}(0, \sigma_X^2)$ and channel conditional pdfs $\{P_{Y|X=x} = \mathcal{N}(x, \sigma_Y^2) : x \in \mathbb{R}\}$ such that $\sigma_X^2, \sigma_Y^2 > 0$. Let $T$ be the set of all pdfs with bounded support. Thus, a pdf $R_X \in T$ if and only if there exists $C \in \mathbb{R}$ such that $R_X = R_X \mathbb{1}_{[-C,C]}$ almost everywhere with respect to the Lebesgue measure. We first derive the following useful lemma.

**Lemma 21 (Bounded Support Characterization of $\eta_{\alpha}$).** The supremum in (57) can be restricted to pdfs in $T$:

$$\eta_{\alpha}(P_X, P_Y|X) = \sup_{R_X \in T, D(R_X||P_X) < +\infty} \left( \frac{D(R_Y||P_Y)}{D(R_X||P_X)} \right)$$

where $R_Y = R_X \ast P_W$ for each $R_X$, and $P_W = \mathcal{N}(0, \sigma_W^2)$.

**Proof.** Consider any pdf $R_X$ such that $0 < D(R_X||P_X) < +\infty$, and define a corresponding the sequence of pdfs $R_X^{(n)} = R_X \mathbb{1}_{[-n,n]} / C_n \in T$, where $C_n = \mathbb{E}_R_X \mathbb{1}_{[-n,n]}(X)$, the indices $n \in \mathbb{N}$ are sufficiently large so that $C_n > 0$, and $\lim_{n \to +\infty} C_n = 1$. Observe that:

$$D(R_X^{(n)}||P_X) = \frac{1}{C_n} \mathbb{E}_R_X \left( \mathbb{1}_{[-n,n]}(X) \log \left( \frac{R_X(X)}{P_X(X)} \right) \right) - \log(C_n).$$

Clearly, $\lim_{n \to +\infty} \mathbb{1}_{[-n,n]} \log(R_X/P_X) = \log(R_X/P_X)$ pointwise $R_X-a.s.$ and $\mathbb{1}_{[-n,n]} \log(R_X/P_X) \leq \log(R_X/P_X)$ pointwise $R_X-a.s.$ such that $\mathbb{E}_R_X \log(R_X/P_X) < +\infty$ (where the finiteness follows from $D(R_X||P_X) < +\infty$). Hence, the dominated convergence theorem (DCT) yields:

$$\lim_{n \to +\infty} D(R_X^{(n)}||P_X) = D(R_X||P_X).$$  

(82)

Furthermore, let $R_Y^{(n)} = R_X^{(n)} \ast P_W$ so that for every $y \in \mathbb{R}$:

$$R_Y(y) - C_n R_Y^{(n)}(y) = \mathbb{E}_R_X \left( \mathbb{1}_{R_X)[-n,n](X) P_W(y-X) \right).$$

Since for all $x, y \in \mathbb{R}$, $\lim_{n \to +\infty} \mathbb{1}_{R_X[-n,n]}(x) P_W(y-x) = 0$ and $0 \leq \mathbb{1}_{R_X[-n,n]}(x) P_W(y-x) \leq P_W(y-x)$ such that $\mathbb{E}_{R_X}[P_W(y-X)] = R_Y(y) < +\infty$, applying the DCT shows the pointwise convergence of the pdfs $\{R_Y^{(n)}\}$:

$$\forall y \in \mathbb{R}, \lim_{n \to +\infty} C_n R_Y^{(n)}(y) = \lim_{n \to +\infty} R_Y^{(n)}(y) = R_Y(y).$$

This implies that $R_Y^{(n)}$ converges weakly to $R_Y$ as $n \to +\infty$ by Scheffé’s lemma. Hence, by the weak lower semi-continuity of KL divergence [21, Theorem 3.6, Section 3.5]:

$$\liminf_{n \to +\infty} D(R_Y^{(n)}||P_Y) \geq D(R_Y||P_Y).$$  

(83)

Note that $\mathbb{1}_{[-C,C]} : \mathbb{R} \to \{0,1\}$ denotes the indicator function on $[-C,C]$, i.e. $\mathbb{1}_{[-C,C]}(x) = 1$ if $x \in [C,C]$, and $\mathbb{1}_{[C,C]}(x) = 0$ otherwise.
Combining (82) and (83), we get:
\[
\liminf_{n \to \infty} \frac{D(R^{(n)}_Y || P_Y)}{D(R^{(n)}_X || P_X)} \geq \frac{D(R_Y || P_Y)}{D(R_X || P_X)}. \tag{84}
\]

To complete the proof, we use a “diagonalization argument.” Suppose \( \{R_{X,m} : m \in \mathbb{N}\} \) is a sequence of pdfs that satisfies \( 0 < D(R_{X,m} || P_X) < +\infty \) for all \( m \in \mathbb{N} \) and achieves the supremum in (57):
\[
\lim_{m \to \infty} \frac{D(R_{Y,m} || P_Y)}{D(R_{X,m} || P_X)} = \eta_a(P_X, P_{Y|X})
\]
where \( R_{Y,m} = R_{X,m} \ast P_W \). Then, since (84) is true, we can construct a sequence \( \{R^{(n(m))}_{X,m}\} \in \mathcal{T} : m \in \mathbb{N}\}, \) where each \( n(m) \) is chosen such that for every \( m \in \mathbb{N}\):
\[
\frac{D(R^{(n(m))}_{Y,m} || P_Y)}{D(R^{(n(m))}_{X,m} || P_X)} \geq \frac{D(R_{Y,m} || P_Y)}{D(R_{X,m} || P_X)} = \frac{1}{2^m}
\]
where \( R^{(n(m))}_{Y,m} = R^{(n(m))}_{X,m} \ast P_W \). Letting \( m \to \infty \), we have:
\[
\liminf_{m \to \infty} \frac{D(R^{(n(m))}_{Y,m} || P_Y)}{D(R^{(n(m))}_{X,m} || P_X)} \geq \eta_a(P_X, P_{Y|X})
\]
Since the supremum in (57) is over all pdfs (which certainly includes all pdfs in \( \mathcal{T} \)), this inequality is actually an equality. This completes the proof. (Also note that for any \( R_X \in \mathcal{T} \), the constraint \( D(R_X || P_X) > 0 \) is automatically true since \( P_X = \mathcal{N}(0, \sigma_X^2) \). So, the supremum in the lemma statement does not include this constraint.)

We next prove Theorem 11 using Lemma 21, which ensures that all differential entropy terms in the ensuing argument are well-defined and finite.

**Proof of Theorem 11.** First note that:
\[
\eta_a^X(P_X, P_{Y|X}) = \rho^2(X; Y) = \frac{\mathbb{C} \mathbb{O} \mathbb{V}(X,Y)^2}{\mathbb{V} \mathbb{A} \mathbb{R}(X) \mathbb{V} \mathbb{A} \mathbb{R}(Y)} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}
\]
where the first equality is precisely (29) (which holds for general random variables [37]), the second equality follows from Rényi’s seventh axiom that \( \rho(X; Y) \) is the absolute value of the Pearson correlation coefficient of \( X \) and \( Y \) [36], and the final equality follows from direct computation.

We next prove that for any \( p \geq \sigma_X^2 \):
\[
\eta_a(P_X, P_{Y|X}) \geq \eta_a^{(p)}(P_X, P_{Y|X}) \geq \eta_a^X(P_X, P_{Y|X})
\]
The first inequality is obvious from (57) and (59). For the second inequality, let \( R_X = \mathcal{N}(\sqrt{\delta}, \sigma_X^2 + \sigma_Y^2 - \delta) \) and \( R_Y = R_X \ast P_W = \mathcal{N}(\sqrt{\delta}, \sigma_X^2 + \sigma_Y^2 - \delta) \) for any \( \delta > 0 \). Then, we get:
\[
\lim_{\delta \to 0^+} \frac{D(R_Y || P_Y)}{D(R_X || P_X)} = \lim_{\delta \to 0^+} \log \left( \frac{\sigma_Y^2 + \sigma_X^2}{\sigma_X^2 - \delta} \right) = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}
\]
where the second equality follows from l’Hôpital’s rule. Since \( \mathbb{E}_{R_X} [X^2] = \sigma_X^2 \) for every \( \delta > 0 \), we have \( \eta_a^{(p)}(P_X, P_{Y|X}) \geq \sigma_X^2/\sigma_Y^2 + \sigma_Y^2 \) for any \( p \geq \sigma_X^2 \).

Therefore, it suffices to prove that \( \eta_a(P_X, P_{Y|X}) \leq \sigma_X^2/\sigma_Y^2 + \sigma_Y^2 \). Using Lemma 21, we can equivalently show that:
\[
\frac{D(R_Y || P_Y)}{D(R_X || P_X)} \leq \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}
\]
for every pdf \( R_X \in \mathcal{T} \) with \( D(R_X || P_X) < +\infty \).
For any pdf $R_X$, we define the differential entropy of $R_X$ as $h(R_X) \triangleq -\mathbb{E}_{R_X} \log(R_X(X))$. To check that such differential entropy terms are well-defined and finite for $R_X \in \mathcal{T}$, we employ the argument in [77, Lemma 8.3.1, Theorem 8.3.3]. Observe that for all $x \in \text{ess supp}(R_X)$:\[^{28}\]

$$
\log(R_X(x)) = \log\left(\frac{R_X(x)}{P_X(x)}\right) - \frac{1}{2} \log\left(2\pi e\right) - \frac{x^2}{2\sigma_X^2}.
$$

Since $D(R_X || P_X)$ must be finite in (57) and $X^2 \geq 0$, we can take expectations with respect to $R_X$ to get:

$$
-h(R_X) = D(R_X || P_X) - \frac{1}{2} \log\left(2\pi e\right) - \frac{\mathbb{E}_{R_X} [X^2]}{2\sigma_X^2},
$$

which shows that $h(R_X)$ always exists, $h(R_X)$ is finite when $\mathbb{E}_{R_X} [X^2] < +\infty$, and $h(R_X) = +\infty$ when $\mathbb{E}_{R_X} [X^2] = +\infty$. Furthermore, if the pdf $R_X \in \mathcal{T}$ has bounded support, $\mathbb{E}_{R_X} [X^2] < +\infty$ and $h(R_X)$ is well-defined and finite.

Let $R_X \in \mathcal{T}$ and $R_Y = R_X * P_W$ have second moments $\mathbb{E}_{R_X} [X^2] = \sigma_X^2 + q > 0$ and $\mathbb{E}_{R_Y} [Y^2] = \sigma_X^2 + \sigma_W^2 + q > 0$ for some $q > -\sigma_X^2$. Using (86), we have:

$$
D(R_X || P_X) = \frac{1}{2} \log\left(2\pi \sigma_X^2\right) + \frac{\sigma_X^2 + q}{2\sigma_X^2} - h(R_X)
$$

$$
= h(P_X) - h(R_X) + \frac{q}{2\sigma_X^2},
$$

$$
D(R_Y || P_Y) = h(P_Y) - h(R_Y) + \frac{q}{2(\sigma_X^2 + \sigma_W^2)}
$$

where $h(R_Y)$ exists and is finite because $\mathbb{E}_{R_Y} [Y^2]$ is finite (as argued earlier using (86)). Hence, it suffices to prove that:

$$
h(P_Y) - h(R_Y) \leq \frac{\sigma_X^2 + \sigma_W^2}{\sigma_X^2} (h(P_X) - h(R_X))
$$

which is equivalent to (85). We can recast (87) as:

$$
(\mathbb{e}^{2h(P_Y) - 2h(R_Y)})^{\sigma_X^2 + \sigma_W^2} \leq (\mathbb{e}^{2h(P_X) - 2h(R_X)})^{\sigma_X^2}
$$

$$
\left(\frac{1}{2\pi e^{2h(P_Y)}}\right)^{\sigma_X^2 + \sigma_W^2} \leq \left(\frac{1}{2\pi e^{2h(R_X)}}\right)^{\sigma_X^2}
$$

$$
\left(\frac{N(P_Y)}{N(R_X)}\right)^{\sigma_X^2 + \sigma_W^2} \leq \left(\frac{N(P_X)}{N(R_X)}\right)^{\sigma_X^2}
$$

where for any pdf $Q_X$ such that $h(Q_X)$ exists, we define the entropy power of $Q_X$ as $N(Q_X) \triangleq e^{2h(Q_X)/(2\pi e)}$ [78, Section III-A]. For $P_X = N(0, \sigma_X^2)$, $P_W = N(0, \sigma_W^2)$, and $P_Y = P_X * P_W = N(0, \sigma_X^2 + \sigma_W^2)$, the entropy powers are $N(P_X) = \sigma_X^2$, $N(P_W) = \sigma_W^2$, and $N(P_Y) = \sigma_X^2 + \sigma_W^2$, respectively. Applying the entropy power inequality to $R_X, P_W$, and $R_Y = R_X * P_W$ [78, Theorem 4], we have:

$$
N(R_Y) \geq N(R_X) + N(P_W) = N(R_X) + \sigma_W^2.
$$

Hence, it is sufficient to prove that:

$$
\left(\frac{\sigma_X^2 + \sigma_W^2}{N(R_X) + \sigma_W^2}\right)^{\sigma_X^2 + \sigma_W^2} \leq \left(\frac{\sigma_X^2}{N(R_X)}\right)^{\sigma_X^2}
$$

Let $a = \sigma_X^2 + \sigma_W^2$, $b = \sigma_X^2 - N(R_X)$, and $c = \sigma_X^2$. Then, we have $a > c > 0$ and $c > b$ (which follows from the finiteness of $h(R_X)$), and it is sufficient to prove that:

$$
\left(\frac{a}{a-b}\right)^a \leq \left(\frac{c}{c-b}\right)^c
$$

which is equivalent to proving:

$$
a > c > 0 \text{ and } c > b \implies \left(1 - \frac{b}{c}\right)^c \leq \left(1 - \frac{b}{a}\right)^a.
$$

This statement is a variant of Bernoulli’s inequality proved in [79, Theorem 3.1, parts $(r_1')$ and $(r_2')$]. This completes the proof. \[\square\]

[^28]: For any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, $\text{ess supp}(f)$ denotes the essential support of $f$ (with respect to the Lebesgue measure).
VI. PROOF OF EQUIVALENT CHARACTERIZATIONS OF THE LESS NOISY PREORDER

We finally turn to deriving the equivalent characterizations of \( \succeq_w \) using operator convexity. The next subsection presents some preliminaries on operator convex function, and subsection VI-B proves Theorem 12.

A. Operator Convex Functions

For any interval \( I \subseteq \mathbb{R} \), let \( \mathbb{C}^{n \times n}_{\text{Her}}(I) \) denote the set of all \( n \times n \) (complex) Hermitian matrices with all eigenvalues in \( I \), where \( \mathbb{C}^{n \times n} = \mathbb{C}^{n \times n}(\mathbb{R}) \) is the set of all Hermitian matrices. Given a function \( f : I \rightarrow \mathbb{R} \), we can extend it to a function \( f : \mathbb{C}^{n \times n}_{\text{Her}}(I) \rightarrow \mathbb{C}^{n \times n}_{\text{Her}} \) as follows [68, Chapter V.1]:

\[
\forall A \in \mathbb{C}^{n \times n}_{\text{Her}}(I), \quad f(A) = U \text{diag}(f(\lambda_1), \ldots, f(\lambda_n)) U^H
\]

where \( A = U \text{diag}(\lambda_1, \ldots, \lambda_n) U^H \) is the spectral decomposition of \( A \) with real eigenvalues \( \lambda_1, \ldots, \lambda_n \in I \), \( U \in \mathbb{C}^{n \times n} \) is a unitary matrix, and \( U^H \) is its Hermitian transpose. We say that \( f \) is operator convex if for every \( n \geq 1 \), every pair of matrices \( A, B \in \mathbb{C}^{n \times n}_{\text{Her}}(I) \), and every \( \lambda \in [0, 1] \):

\[
\lambda f(A) + (1 - \lambda) f(B) \succeq_{\text{PSD}} f(\lambda A + (1 - \lambda) B)
\]

where \( \succeq_{\text{PSD}} \) denotes the Löwner partial order (i.e. for any two matrices \( A, B \in \mathbb{C}^{n \times n} \), \( A \succeq_{\text{PSD}} B \) if and only if \( A - B \) is positive semidefinite) [68, Chapter V.1]. Note that an operator convex function \( f : I \rightarrow \mathbb{R} \) is clearly convex, and its translated affine transformations \( g : c + I \rightarrow \mathbb{R} \), \( g(t) = af(t) + b \) are also operator convex for every \( a \geq 0, b \in \mathbb{R} \), and \( c \in \mathbb{R} \).

A more striking property of operator convex functions is that they are characterized by certain integral representations—see Löwner’s theorems in [68, Chapter V.4, Problem V.5.5]. In particular, for every operator convex function \( f : (0, \infty) \rightarrow \mathbb{R} \) with \( f(1) = 0 \), there exist constants \( a \in \mathbb{R} \) and \( b \geq 0 \), and a finite positive measure \( \mu \) on \((1, \infty)\) (with its Borel \( \sigma \)-algebra) such that:

\[
f(t) = a(t-1) + b(t-1)^2 + \int_{(1,\infty)} \frac{(t-1)(\omega t - \omega - 1)}{t + \omega - 1} \, d\mu(\omega)
\]

which follows from [10, Equation (7)] and the associated references (note that our \( f \) is related to \( g \) in [10] by \( f(t) = g(t - 1) \)). As noted in [10], the converse also holds, i.e. functions of the form (90) are operator convex. The ensuing lemma is a direct consequence of (90), which we distill from [10] and present in a more transparent form.

**Lemma 22** (Integral Representation [10, p.33]). Consider any \( f \)-divergence such that \( f : (0, \infty) \rightarrow \mathbb{R} \) is operator convex and satisfies \( f(1) = 0 \). Then, there exists a constant \( b \geq 0 \) and a positive measure \( \tau \) on \((0, 1)\) (with its Borel \( \sigma \)-algebra) such that for every \( R_X, P_X \in \mathcal{P}_X \):

\[
D_f(R_X||P_X) = b \chi^2(R_X||P_X)
\]

\[
+ \int_{(0,1)} \frac{1 + \lambda^2}{\lambda(1 - \lambda)} \mathcal{L}_\lambda(R_X||P_X) \, d\tau(\lambda).
\]

**Proof.** Fix any two pmfs \( R_X, P_X \in \mathcal{P}_X \), and suppose the random variable \( X \) has pmf \( P_X \). Then, since \( f \) exhibits the integral representation (90), we substitute \( t = R_X(X)/P_X(X) \) into (90) and take expectations to get:

\[
\mathbb{E} \left[ f \left( \frac{R_X(X)}{P_X(X)} \right) \right] = b \mathbb{E} \left[ \left( \frac{R_X(X)}{P_X(X)} \right)^2 \right]
\]

\[
+ \int_{(1,\infty)} \mathbb{E} \left[ \left( \frac{R_X(X)}{P_X(X)} - 1 \right) \left( \frac{R_X(X)}{P_X(X)} - \omega - 1 \right) \right] \, d\mu(\omega)
\]

where the first term on the right hand side of (90) vanishes after taking expectations (see the affine invariance property in subsection II-A). This implies that:

\[
D_f(R_X||P_X) = b \chi^2(R_X||P_X)
\]

\[
+ \int_{(1,\infty)} \mathbb{E} \left[ \frac{1 + \omega^2}{\omega} \left( \frac{R_X(X)}{P_X(X)} - 1 \right)^2 \right] \, d\mu(\omega)
\]

---

\[29\)The interval \( I \) can be finite or infinite, and closed or open.

\[30\)It is straightforward to verify that \( A, B \in \mathbb{C}^{n \times n}_{\text{Her}}(I) \) implies that \( \lambda A + (1 - \lambda) B \in \mathbb{C}^{n \times n}_{\text{Her}}(I) \) for all \( \lambda \in [0, 1] \).

\[31\)Note that \( c + I = \{ c + x : x \in I \} \) is the interval \( I \) translated by \( c \).
where the left hand side follows from Definition 1, the $\chi^2$-divergence term follows from the definition in subsection II-A, and the last term follows from the affine invariance property in subsection II-A and the relation:

$$\frac{(t-1)(\omega t - \omega - 1)}{t + \omega - 1} = \frac{(1 + \omega^2)(t-1)^2 - t - 1}{\omega(t+\omega-1)} - \frac{t}{\omega}$$

for any $t > 0$ and $\omega > 0$. Next, observe that the change of variables $\omega = \frac{1}{2}$ yields:

$$D_f(R_X||P_X) = b\chi^2(R_X||P_X)$$

$$+ \int_{(0,1)} \mathbb{E} \left[ \frac{(1 + \lambda^2) \left( \frac{R_X(X)}{P_X(X)} - 1 \right)^2}{\chi(\frac{R_X(X)}{P_X(X)} + 1 - \lambda)} \right] d\tau(\lambda)$$

for some positive measure $\tau$ on $(0,1)$. Finally, recognizing that the integrand on the right hand side is a scaled Vincze-Le Cam divergence (see subsection II-A), some straightforward algebra produces the desired integral representation. $\blacksquare$

Lemma 22 is used in [10, p.33] (in a slightly different form) to prove Proposition 6, cf. [10, Theorem 1]. Furthermore, [9, p.3363] also distills the key idea in [10, p.33] and presents an alternative integral representation (in terms of Vincze-Le Cam and $\chi^2$-divergences) analogous to Lemma 22. However, the representation in [9, p.3363] only holds for operator convex functions $f$ where $f(0)$ is finite, while Lemma 22 holds for infinite $f(0)$ as well.

**B. Proof of Theorem 12**

We first recall the result in [67, Theorem 1].

**Lemma 23 ($\chi^2$-Divergence Characterization of $\succeq_X$ [67, Theorem 1]).** For any two channels $P_{Y|X}$ and $P_{Z|X}$ on the same input alphabet $X$ with row stochastic transition probability matrices $W \in \mathcal{P}_{Y|X}$ and $V \in \mathcal{P}_{Z|X}$ respectively, $P_{Y|X} \succeq_n P_{Z|X}$ if and only if:

$$\chi^2(R_YW||P_XW) \geq \chi^2(R_ZV||P_XV)$$

for every pair of input pmfs $R_X, P_X \in \mathcal{P}_X$.

This lemma is proved in [67, Section IV-A]. We next use it along with Lemma 22 to prove Theorem 12.

**Proof of Theorem 12.** Fix any non-linear operator convex function $f : (0, \infty) \to \mathbb{R}$ such that $f(1) = 0$, where the non-linearity ensures that the corresponding $f$-divergence is not identically zero (see the affine invariance property in subsection II-A). We follow the proof outline in [67, Section IV-A] (also see [10] and [9, Section III-C]).

To prove the forward direction, we first use Lemma 22 and the equivalent form of Vincze-Le Cam divergences in (9) to obtain the following integral representation of our $f$-divergence in terms of $\chi^2$-divergence (cf. [10, p.33]):

$$D_f(R_X||P_X) = b\chi^2(R_X||P_X)$$

$$+ \int_{(0,1)} \mathbb{E} \left[ \frac{1 + \lambda^2}{(1 - \lambda)^2} \chi^2(R_X||\lambda R_X + \lambda P_X) \right] d\tau(\lambda)$$

(91)

for all $R_X, P_X \in \mathcal{P}_X$, where $\lambda = 1 - \lambda$. Since $P_{Y|X} \succeq_n P_{Z|X}$, Lemma 23 yields $\chi^2(R_YW||P_XW) \geq \chi^2(R_ZV||P_XV)$ and $\chi^2(R_YW||\lambda R_X + \lambda P_X) \geq \chi^2(R_ZV||\lambda R_X + \lambda P_X)$ for every $R_X, P_X \in \mathcal{P}_X$ and every $\lambda \in (0,1)$. Using these inequalities and the integral representation in (91), we get:

$$D_f(R_XW||P_XW) \geq D_f(R_XV||P_XV)$$

for every $R_X, P_X \in \mathcal{P}_X$, as desired.

To prove the converse direction, observe that the integral representation in (90) ensures that $f$ is infinitely differentiable and $f''(1) > 0$. For any $R_X, P_X \in \mathcal{P}_X$ and any $\lambda \in (0,1)$, we are given that:

$$D_f((\lambda P_X + \lambda R_X)W||P_XW) \geq D_f((\lambda P_X + \lambda R_X)V||P_XV).$$

So, when $P_X \in \mathcal{P}_X^\circ$, we can scale both sides by $\frac{2}{f''(1)|\lambda|^2} > 0$ and let $\lambda \to 0$ to obtain:

$$\chi^2(R_YW||P_XW) \geq \chi^2(R_ZV||P_XV)$$

using the local approximation of $f$-divergences in (17). Although our version of (17) requires the $P_X \in \mathcal{P}_X^\circ$ assumption, the above inequality also holds for $P_X \in \mathcal{P}_X \setminus \mathcal{P}_X^\circ$ due to the continuity of $\chi^2$-divergence in its second argument with fixed first argument. Hence, Lemma 23 yields that $P_{Y|X} \succeq_n P_{Z|X}$. This completes the proof. $\blacksquare$
We remark that even without using Lemma 23, this proof illustrates that all channel preorders that are defined using non-linear operator convex $f$-divergences (in a manner analogous to $\succeq_n$) are equivalent. Indeed, the proof shows that they are all characterized by $\chi^2$-divergence. Since $\succeq_n$ (corresponding to KL divergence) is one of these preorders, we can conclude that all the preorders are equivalent to $\succeq_n$.

Finally, we derive Proposition 6 to illustrate that it is a straightforward corollary of Theorem 12.

**Proof of Proposition 6.** Fix any non-linear operator convex function $f$ with $f(1) = 0$, and any channel $P_{Y|X}$ with row stochastic transition probability matrix $W \in \mathcal{P}_{Y|X}$. Using Theorem 12, the $|X|$-ary erasure channel $E_{1-\beta} \succeq_n P_{Y|X}$ if and only if for every pair of input pmfs $R_X, P_X \in \mathcal{P}_X$:

$$D_f(R_X W || P_X W) \leq D_f(R_X E_{1-\beta} || P_X E_{1-\beta})$$

$$= \beta D_f(R_X || P_X)$$

where the equality is shown near the end of subsection II-B. This equivalence yields the following generalization of (61):

$$\eta_f(P_{Y|X}) = \inf \{ \beta \in [0,1] : E_{1-\beta} \succeq_n P_{Y|X} \}.$$  \hspace{1cm} (92)

Therefore, the contraction coefficients $\eta_f(P_{Y|X})$ for all non-linear operator convex $f$ are equal, and in particular, they all equal $\eta_{\chi^2}(P_{Y|X})$ (since $f(t) = t^2 - 1$ is operator convex).

VII. CONCLUSION

In closing, we briefly recapitulate our main contributions and then propose some directions for future research. We first illustrated in Theorem 7 that if the optimization problem defining $\eta_f(P_X, P_{Y|X})$ is subjected to an additional “local approximation” constraint that forces the input $f$-divergence to vanish, then the resulting optimum value is $\eta_{\chi^2}(P_X, P_{Y|X})$. This transparently captures the intuition behind the maximal correlation lower bound in part 6 of Proposition 3. We then derived a linear upper bound on $\eta_f(P_X, P_{Y|X})$ in terms of $\eta_{\chi^2}(P_X, P_{Y|X})$ for a class of $f$-divergences in Theorem 8, and improved this bound for the salient special case of $\eta_{\chi^2}(P_X, P_{Y|X})$ in Theorem 10. Such bounds are useful in weak dependence regimes such as in the analysis of ergodicity of Markov chains (as shown in Corollary 19). In the spirit of comparing contraction coefficients of source-channel pairs, we also gave an alternative proof of the equivalence, $\eta_{\chi^2}(P_X, P_{Y|X}) = \eta_{\chi^2}(P_X, P_{Y|X})$, for jointly Gaussian distributions $P_{X,Y}$ defined by AWGN channels in Theorem 11 and section V. This proof showed that adding a large enough power constraint to the extremization in $\eta_{\chi^2}(P_X, P_{Y|X})$ does not change its value. Finally, in the realm of contraction coefficients of channels, we generalized Proposition 6 in Theorem 12, and established that the less noisy preorder over channels is completely characterized by any non-linear operator convex $f$-divergence.

As discussed in subsection IV-D, the constants in the linear bounds in Theorems 8 and 10 vary “blindly” with the dimension of a product distribution. While results like Corollary 20 partially remedy this tensorization issue, one compelling direction of future work is to discover linear bounds whose constants gracefully tensorize. Another, perhaps more concrete, avenue of future work is to derive the optimal distribution dependent refinement of Lemma 16 (as suggested in [61, Remark, p.5380]). Such a refinement could be used to tighten Theorem 8 so that it specializes to Theorem 10 instead of Corollary 9. However, such a refinement cannot circumvent the more critical tensorization issue that ails these bounds.

**APPENDIX A**

**Proof of Proposition 2**

**Proof.** This proof is outlined in [4], and presented in [39, Theorem 3.2.4] for the $P_X \in \mathcal{P}_X^p$ and $P_Y \in \mathcal{P}_Y^p$ case. We provide it here for completeness.

Suppose the marginal pmfs of $X$ and $Y$ satisfy $P_X \in \mathcal{P}_X^p$ and $P_Y \in \mathcal{P}_Y^p$. We first show that the largest singular value of the DTM $B$ is unity. Consider the matrix:

$$M = \text{diag} \left( \sqrt{P_Y} \right)^{-1} B^T B \text{diag} \left( \sqrt{P_Y} \right)$$

$$= \text{diag}(P_Y)^{-1} W^T \text{diag}(P_X) W$$

$$= V W$$

where $V = \text{diag}(P_Y)^{-1} W^T \text{diag}(P_X) \in \mathcal{P}_{X|Y}$ is the row stochastic reverse transition probability matrix of conditional pmfs $P_{X|Y}$. Observe that $M$ has the same set of eigenvalues as the Gramian of the DTM $B^T B$, because we are simply using a similarity transformation to define it. As $B^T B$ is positive semidefinite, the eigenvalues of $M$ and $B^T B$ are non-negative real numbers by the *spectral theorem* (see [40, Section 2.5]). Moreover, since $V$ and $W$ are both row
stochastic, their product $M = VW$ is also row stochastic. Hence, the largest eigenvalue of $M$ and $B^TB$ is unity by the **Perron-Frobenius theorem** (see [40, Chapter 8]). It follows that the largest singular value of $B$ is also unity. Notice further that $\sqrt{P_X}$ and $\sqrt{P_Y}$ are the left and right singular vectors of $B$, respectively, corresponding to the singular value of unity. Indeed, we have:

$$
\sqrt{P_X} B = \sqrt{P_X} \text{diag} \left( \sqrt{P_X} \right) W \text{diag} \left( \sqrt{P_Y} \right)^{-1} = \sqrt{P_Y},
$$

$$
B \sqrt{P_Y}^T = \text{diag} \left( \sqrt{P_X} \right) W \text{diag} \left( \sqrt{P_Y} \right)^{-1} \sqrt{P_Y} = \sqrt{P_X}^T.
$$

Next, starting from Definition 3, let $f \in \mathbb{R}^{[X]}$ and $g \in \mathbb{R}^{[Y]}$ be the column vectors representing the range of the functions $f : \mathcal{X} \to \mathbb{R}$ and $g : \mathcal{Y} \to \mathbb{R}$, respectively. Note that we can express the expectations in Definition 3 in terms of $B$, $P_X$, $P_Y$, $f$, and $g$:

$$
\mathbb{E}[f(X)g(Y)] = \left( \text{diag} \left( \sqrt{P_X} \right) f \right)^T B \left( \text{diag} \left( \sqrt{P_Y} \right) g \right),
$$

$$
\mathbb{E}[f(X)] = \sqrt{P_X} \left( \text{diag} \left( \sqrt{P_X} \right) f \right),
$$

$$
\mathbb{E}[g(Y)] = \sqrt{P_Y} \left( \text{diag} \left( \sqrt{P_Y} \right) g \right),
$$

$$
\mathbb{E}[f^2(X)] = \left\| \text{diag} \left( \sqrt{P_X} \right) f \right\|_2^2,
$$

$$
\mathbb{E}[g^2(Y)] = \left\| \text{diag} \left( \sqrt{P_Y} \right) g \right\|_2^2.
$$

Letting $a = \text{diag} \left( \sqrt{P_X} \right) f$ and $b = \text{diag} \left( \sqrt{P_Y} \right) g$, we have from Definition 3:

$$
\rho(X;Y) = \max_{a \in \mathbb{R}^{[X]}, b \in \mathbb{R}^{[Y]} : \sqrt{P_X} a = \sqrt{P_Y} b = 0, \left\| a \right\|_2 = \left\| b \right\|_2 = 1} a^T B b
$$

where the optimization is over all $a \in \mathbb{R}^{[X]}$ and $b \in \mathbb{R}^{[Y]}$ because $P_X \in \mathcal{P}_X^+$ and $P_Y \in \mathcal{P}_Y^+$. Since $a$ and $b$ are orthogonal to the left and right singular vectors corresponding to the maximum singular value of unity of $B$, respectively, this maximization produces the second largest singular value of $B$ using an alternative version (see [80, Lemma 2]) of the **Courant-Fischer min-max theorem** (see [40, Theorems 4.2.6 and 7.3.8]). This proves that $\rho(X;Y)$ is the second largest singular value of the DTM when $P_X \in \mathcal{P}_X^+$ and $P_Y \in \mathcal{P}_Y^+$.

We finally argue that one can assume $P_X \in \mathcal{P}_X^+$ and $P_Y \in \mathcal{P}_Y^+$ without loss of generality. When $P_X$ or $P_Y$ have zero entries, $X$ and $Y$ only take values in the support sets $\text{supp}(P_X) = \{ x \in \mathcal{X} : P_X(x) > 0 \} \subseteq \mathcal{X}$ and $\text{supp}(P_Y) = \{ y \in \mathcal{Y} : P_Y(y) > 0 \} \subseteq \mathcal{Y}$ respectively, which means that $P_X \in \mathcal{P}_{\text{supp}(P_X)}$ and $P_Y \in \mathcal{P}_{\text{supp}(P_Y)}$. Let $B$ denote the “true” DTM of dimension $|\mathcal{X}| \times |\mathcal{Y}|$ corresponding to the pmf $P_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$, and $B_{\text{supp}}$ denote the “support” DTM of dimension $|\text{supp}(P_X)| \times |\text{supp}(P_Y)|$ corresponding to the pmf $P_{X,Y}$ on $\text{supp}(P_X) \times \text{supp}(P_Y)$. Clearly, $B$ can be constructed from $B_{\text{supp}}$ by inserting zero vectors into the rows and columns associated with the zero probability letters in $\mathcal{X}$ and $\mathcal{Y}$, respectively. Hence, $B$ and $B_{\text{supp}}$ have the same non-zero singular values (counting multiplicity), which implies that they have the same second largest singular value. This completes the proof. \[\blacksquare\]

**APPENDIX B**

**PROOF OF PROPOSITION 3**

**Proof.**

**Part 1:** The normalization of contraction coefficients is evident from the non-negativity of $f$-divergences and their DPIs (10). We remark that in the case of $\eta_2(P_X, P_{Y|X}) = \rho^2(X;Y)$ (where we use (29)), $0 \leq \rho(X;Y) \leq 1$ is Rényi’s third axiom in defining maximal correlation [36].

**Part 2:** We provide a simple proof of this well-known property. Assume without loss of generality that $P_X \in \mathcal{P}_X^+$ by ignoring any zero probability letters of $X$. If the resulting $|X| = 1$, then $X$ is a constant a.s., and the result follows trivially. So, we may also assume that $|X| \geq 2$. Let $W \in \mathcal{P}_{Y|X}$ denote the row stochastic transition probability matrix of the channel $P_{Y|X}$. Since $W$ is unit rank (with all its columns equal to $P_Y$) if and only if $X$ and $Y$ are independent (which means $P_{Y|X=x} = P_Y$ for every $x \in \mathcal{X}$), it suffices to show that $W$ is unit rank if and only if $\eta_f(P_X, P_{Y|X}) = 0$.

To prove the forward direction, note that if $W$ is unit rank, all its rows are equal to $P_Y$ and we have $R_X = P_Y$ for all $R_X \in \mathcal{P}_X$. Hence, $\eta_f(P_X, P_{Y|X}) = 0$ using Definition 2, because $D_f(R_X W || P_X W) = 0$ for all input pmfs $R_X \in \mathcal{P}_X$. 

31
To prove the converse direction, we employ the argument in [39] that was used to prove the $\eta_{\delta_x}(P_{X,Y}|X|X)$ case. Let $R_X = \delta_x$ for any $x \in \mathcal{X}$, where $\delta_x$ is the Kronecker delta pmf such that $\delta_x(x) = 1$ and $\delta_x(i) = 0$ for $i \in X \setminus \{x\}$. (Note that such $R_X \neq P_X$ as $P_X \in \mathcal{P}_X$.) Then, for every $x \in \mathcal{X}$, the $x$th row of $W$ is $P_{Y|X=x} = \delta_x W = P_X W = P_Y$, where $\delta_x W = P_X W$ because $D_f(\delta_x W || P_X W) = 0$ and $f$ is strictly convex at unity. Hence, $W$ has unit rank.

We also note that in the $\eta_{\delta_x}(P_{X,Y}|X)$ case, this property of maximal correlation is Rényi’s fourth axiom in [36].

**Part 3:** This is proven in [9, Proposition III.3].

**Part 4:** This is proven in [9, Theorem III.9]. We also note that two proofs of the tensorization property of $\eta_{x}$ can be found in [4], and a proof of the tensorization property of $\eta_{\Delta}$ can be found in [38].

**Part 5:** To prove the first part, let $P_{U,X,Y}$ denote the joint pmf of $(U, X, Y)$, and $S \in \mathcal{P}_X$ and $W \in \mathcal{P}_Y$ denote the row stochastic transition probability matrices corresponding to the channels $P_{X|U}$ and $P_{Y|X}$, respectively. Then, $SW \in \mathcal{P}_Y$ is the row stochastic transition probability matrix corresponding to the channel $P_{Y|U}$ using the Markov property. Observe that for every pmf $P_U \in \mathcal{P}_U\setminus\{P_U\}$:

$$D_f(R_U SW || P_U SW) \leq \eta_f(P_X, P_{Y|X}) \eta_f(P_U, P_{X|U})$$

where $P_Y = P_U SW$, $P_{X} = P_U S$, and we use the SDPI (19) twice. Hence, we have:

$$\eta_f(P_U, P_{Y|U}) \leq \eta_f(P_U, P_{X|U}) \eta_f(P_X, P_{Y|X})$$

using Definition 2.

The $\eta_{\delta^2}$ specialization of this result corresponds to the sub-multiplicativity property of the second largest singular value of the DTM. Such a sub-multiplicativity property also holds for the $i$th largest singular value of the DTM, cf. [81, Theorem 2], and is useful for distributed source and channel coding applications [81]. Moreover, the result in [81, Theorem 2] is also proved in [24, Theorem 3], where the relation to principal inertia components and maximal correlation is expounded.

To prove the second part, observe that for fixed $P_{X,Y}$, and every $P_{U|X}$ such that $U \rightarrow X \rightarrow Y$ form a Markov chain and $\eta_f(P_U, P_{X|U}) > 0$ (which requires that $X$ is not a constant a.s.), we have:

$$\frac{\eta_f(P_U, P_{Y|U})}{\eta_f(P_U, P_{X|U})} \leq \eta_f(P_X, P_{Y|X})$$

using the sub-multiplicativity property established above. Let $U = X$ a.s. so that $P_{U|X} \in \mathcal{P}_X$ is the identity matrix. Then, $\eta_f(P_U, P_{X|U}) = 1$ and $\eta_f(P_U, P_{Y|U}) = \eta_f(P_X, P_{Y|X})$ using Definition 2. Therefore, equality can be achieved in (93), and the proof is complete.

We remark that the $\eta_{\delta^2}$ case of this result is presented in [82, Lemma 4], where the authors also prove that the optimal channel $P_{U|X}$ can be taken as $P_{Y|X}$ (so that $U$ is a copy of $Y$) instead of the identity matrix (where $U = X$ a.s.).

**Part 6:** Following the remark after [1, Theorem 5] in the conference version of this paper, we prove this result via the technique used to prove the $\eta_{\delta}$ case in [1, Theorem 5].

Let $W \in \mathcal{P}_Y$ denote the row stochastic matrix of the channel $P_{Y|X}$, and $B$ denote the DTM of the joint pmf $P_{X,Y}$. Let us define a trajectory of spherically perturbed pmfs of the form (15):

$$R_X^{(\epsilon)} = P_X + \epsilon K_X \diag\left(\sqrt{P_X}\right)$$

where $K_X \in S \triangleq \{x \in (\mathbb{R}^{|X|})^* : \sqrt{P_X}x^T = 0, \|x\|_2 = 1\}$ is a spherical perturbation vector. When these pmfs pass through the channel $W$, we get the output trajectory:

$$R_Y^{(\epsilon)}W = P_Y + \epsilon K_X B \diag\left(\sqrt{P_Y}\right)$$

where $B$ maps input spherical perturbations to output spherical perturbations [29]. Now, starting from Definition 2, we have:

$$\eta_f(P_X, P_{Y|X}) = \sup_{R_X \in \mathcal{P}_X, \delta^2 < D_f(R_X || P_X) < +\infty} \frac{D_f(R_XW || PW)}{D_f(R_X || P_X)}$$

$$\geq \liminf_{\epsilon \to 0} \sup_{K_X \in S} \frac{\|K_X B\|_2^2 + o(1)}{\|K_X\|_2^2 + o(1)}$$

$$\geq \sup_{K_X \in S} \liminf_{\epsilon \to 0} \frac{\|K_X B\|_2^2 + o(1)}{1 + o(1)}$$

32
where the second inequality follows from (18) after restricting the supremum over all pmfs of the form (15) (where \( \epsilon \neq 0 \) is some sufficiently small fixed value) and then letting \( \epsilon \to 0 \), the third inequality follows from the minimax inequality, and the final equalities follow from (31) and (29), respectively. This completes the proof.

We remark that the \( P_X \in \mathcal{P}_X \) and \( P_Y \in \mathcal{P}_Y \), assumptions, while useful for defining the aforementioned trajectories of pmfs, are not essential for this result. For the special case of \( \eta_n \), this result was first proved in [2], and then again in [6] and [1, Theorem 5]—the latter two proofs both use perturbation arguments with different flavors.

\[ \eta \]

\section*{Appendix C}

\section*{Proof of Theorem 7}

\textbf{Proof.} We begin by defining the function \( \tau : (0, \infty) \to [0, 1] \):

\[ \tau(\delta) \triangleq \sup_{0 < D_f(R_X||P_X) \leq \delta} \frac{D_f(WR_X||WP_X)}{D_f(R_X||P_X)} \]

so that what we seek to prove is:

\[ \lim_{n \to \infty} \tau(\delta_n) = \eta^2(P_X, P_{Y|X}) \]

for any decreasing sequence \( \{\delta_n \} \), such that \( \lim_{n \to \infty} \delta_n = 0 \). Note that the limit on the left hand side exists because as \( \delta_n \to 0 \), the supremum in \( \tau(\delta_n) \) is non-increasing and bounded below by 0.

We first prove that \( \lim_{n \to \infty} \tau(\delta_n) \geq \eta^2(P_X, P_{Y|X}) \). To this end, consider a trajectory of spherically perturbed pmfs of the form (15):

\[ R_X^{(n)} = P_X + \epsilon_n K_X \text{ diag} \left( \sqrt{P_X} \right) \]

where \( K_X \in \mathcal{S} = \{ x \in (\mathbb{R}^{|X|})^* : \sqrt{P_X} x^T = 0, \| x \|_2 = 1 \} \) is a spherical perturbation vector. The associated trajectory of output pmfs after passing through \( W \) is given by (94):

\[ R_Y^{(n)} W = P_Y + \epsilon_n K_X B \text{ diag} \left( \sqrt{P_Y} \right) \]

where \( B \) denotes the DTM corresponding to \( P_{X,Y} \). We ensure that the scalars \( \{\epsilon_n \} \) define our trajectory satisfy \( \lim_{n \to \infty} \epsilon_n = 0 \) and are sufficiently small such that:

\[ D_f(R_X^{(n)}||P_X) = \frac{f''(1)}{2} \epsilon_n^2 \|K_X\|^2_2 + o(\epsilon_n^2) \leq \delta_n \]

where we use (18) (and the fact that \( f''(1) \) exists and is strictly positive), and the Bachmann-Landau asymptotic little-o notation.\(^{32}\) By definition of \( \tau \), we have:

\[ \lim_{n \to \infty} \sup_{K_X \in \mathcal{S}} \frac{D_f(R_X^{(n)}||P_X)}{D_f(R_X^{(n)}||P_X)} \leq \lim_{n \to \infty} \tau(\delta_n) \]

\[ \frac{f''(1)}{2} \epsilon_n^2 \|K_X B\|^2_2 + o(\epsilon_n^2) \leq \frac{f''(1)}{2} \epsilon_n^2 \|K_X|^2_2 + o(\epsilon_n^2) \leq \lim_{n \to \infty} \tau(\delta_n) \]

\[ \lim_{n \to \infty} \sup_{K_X \in \mathcal{S}} \frac{\|K_X B\|^2_2 + o(1)}{1 + o(1)} \leq \lim_{n \to \infty} \tau(\delta_n) \]

\[ \eta^2(P_X, P_{Y|X}) \leq \lim_{n \to \infty} \tau(\delta_n) \]

where the second inequality uses (18) for both the numerator and denominator, and the final inequality uses the singular value characterization of \( \eta^2(P_X, P_{Y|X}) \) in (31).

We next prove that \( \lim_{n \to \infty} \tau(\delta_n) \leq \eta^2(P_X, P_{Y|X}) \). Observe that for each \( n \in \mathbb{N} \), there exists a pmf \( R_X^{(n)} \in \mathcal{P}_X \) satisfying two properties:

1) \( 0 < D_f(R_X^{(n)}||P_X) \leq \delta_n \)

2) \( 0 \leq \tau(\delta_n) - \frac{D_f(R_X^{(n)}||P_X)}{D_f(R_X^{(n)}||P_X)} \leq \frac{1}{2^n} \)

\(^{32}\)Given two functions \( g(n) \) and \( h(n) \) such that \( h(n) \) is non-zero, we write \( g(n) = o(h(n)) \) if and only if \( \lim_{n \to \infty} g(n)/h(n) = 0 \).
where the first property holds because $R_X \mapsto D_f(R_X||P_X)$ is a continuous map for fixed $P_X \in \mathcal{P}_X^\circledast$ (which follows from the convexity of $f$), and the second property holds because $	au(\delta_n)$ is defined as a supremum. Since $\tau(\delta_n)$ converges as $n \to \infty$, we have:

$$\lim_{n \to \infty} \frac{D_f(R_X^{(n)}W||P_XW)}{D_f(R_X^{(n)}||P_X)} = \lim_{n \to \infty} \tau(\delta_n).$$  

(95)

Using the sequential compactness of $\mathcal{P}_X$, we can assume that $R_X^{(n)}$ converges as $n \to \infty$ (in the $\ell^2$-norm sense) by passing to a subsequence if necessary. Since $\lim_{n \to \infty} D_f(R_X^{(n)}||P_X) = 0$, we have that $\lim_{n \to \infty} R_X^{(n)} = P_X$ due to the continuity of $R_X \mapsto D_f(R_X||P_X)$ for fixed $P_X \in \mathcal{P}_X^\circledast$ and the fact that an $\ell$-divergence (where $f$ is strictly convex at unity) is zero if and only if its input pmfs are equal. Let us define the spherical perturbation vectors $\{K_X^{(n)} \in \mathcal{S} : n \in \mathbb{N}\}$ using the relation:

$$R_X^{(n)} = P_X + \epsilon_n K_X^{(n)} \text{diag}(\sqrt{P_X})$$

where $\{\epsilon_n \neq 0 : n \in \mathbb{N}\}$ provide the appropriate scalings, and $\lim_{n \to \infty} \epsilon_n = 0$ (since $\lim_{n \to \infty} R_X^{(n)} = P_X$). The corresponding output pmfs are of the form (94) mutatis mutandis, and we can approximate the ratio between output and input $f$-divergences as before using (18):

$$\frac{D_f(R_X^{(n)}W||P_XW)}{D_f(R_X^{(n)}||P_X)} = \frac{\frac{f''(1)}{2} \epsilon_n^2 \|K_X^{(n)}B\|_2^2 + o(\epsilon_n^2)}{\frac{f''(1)}{2} \epsilon_n^2 \|K_X^{(n)}\|_2^2 + o(\epsilon_n^2)}$$

$$= \frac{\|K_X^{(n)}B\|_2^2 + o(1)}{1 + o(1)}.$$

Using the sequential compactness of $\mathcal{S}$, we may assume that $\lim_{n \to \infty} K_X^{(n)} = K_X \in \mathcal{S}$ by passing to a subsequence if necessary. Hence, letting $n \to \infty$, we get:

$$\lim_{n \to \infty} \tau(\delta_n) = \|K_XB\|_2^2 \leq \eta_X^2 \left(P_X, P_Y|X\right)$$

where the equality follows from (95) and the continuity of the map $(\mathbb{R}^{|X|})^* \ni x \mapsto \|xB\|_2^2$, and the inequality follows from (31). This completes the proof. 

\section*{Appendix D}

\section*{Proof of Corollary 9}

**Proof.** The convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t \log(t)$ is clearly strictly convex and thrice differentiable at unity with $f(1) = 0$, $f'(1) = 1$, $f''(1) = 1 > 0$, and $f'''(1) = -1$. Moreover, the function $g : (0, \infty) \to \mathbb{R}$, $g(x) = \frac{f(x)-f(0)}{x} = \log(x)$ is clearly concave (where $f(0) = \lim_{t \to 0^+} f(t) = 0$). So, to prove Corollary 9 using Theorem 8, it suffices to show that $f$ satisfies (69) for every $t \in (0, \infty)$ (cf. [61]):

$$(f(t) - f'(t)(t-1)) \left(1 - f'''(1) - 3f''(1)(t-1)\right) \geq \frac{f'''(1)}{2}(t-1)^2$$

which simplifies to:

$$2t(t+2) \log(t) - (5t+1)(t-1) \geq 0.$$

Define $h : (0, \infty) \to \mathbb{R}$, $h(t) = 2t(t+2) \log(t) - (5t+1)(t-1)$ and observe that:

$$h'(t) = 4(t+1) \log(t) - 8(t-1)$$

$$h''(t) = 4 \log(t) + \frac{4}{t} - 4 \geq 0$$

where the non-negativity of the second derivative follows from the well-known inequality:

$$\forall x > 0, \ x \log(x) \geq x - 1.$$

Since $h$ is convex (as its second derivative is non-negative) and $h(1) = h'(1) = 0$, $t = 1$ is a global minimizer of $h$ and $h(t) \geq 0$ for every $t \in (0, \infty)$ as required.

\footnote{We use the fact that if two sequences $\{a_n \in \mathbb{R} : n \in \mathbb{N}\}$ and $\{b_n \in \mathbb{R} : n \in \mathbb{N}\}$ satisfy $\lim_{n \to \infty} |a_n - b_n| = 0$ and $\lim_{n \to \infty} b_n = b \in \mathbb{R}$, then $\lim_{n \to \infty} a_n = b$.}
Finally, we can verify that the constant in Corollary 9 is:

\[
\frac{f''(1) + f(0)}{f''(1) \min_{x \in \mathcal{X}} P_X(x)} \geq \frac{1}{\min_{x \in \mathcal{X}} P_X(x)}
\]

which completes the proof. ■

**APPENDIX E**

**PROOF OF (72)**

**Proof.** Two proofs for (72) are provided in the conference version of this paper [1, Lemma 6]. We present the one with a convex analysis flavor. It involves recognizing that KL divergence is a Bregman divergence associated with the negative Shannon entropy function, and then exploiting the strong convexity of the negative Shannon entropy function to bound KL divergence. Let \( H_{\text{neg}} : P_X \to \mathbb{R} \) be the negative Shannon entropy function, which is defined as:

\[
\forall Q_X \in P_X, \quad H_{\text{neg}}(Q_X) \triangleq \sum_{x \in \mathcal{X}} Q_X(x) \log(Q_X(x)).
\]

Since the Bregman divergence corresponding to \( H_{\text{neg}} \) is the KL divergence, cf. [83], we have for all \( S_X \in P_X \) and \( Q_X \in P_X^\circ \):

\[
D(S_X || Q_X) = H_{\text{neg}}(S_X) - H_{\text{neg}}(Q_X) - J_X \nabla H_{\text{neg}}(Q_X)
\]

where \( J_X = S_X - Q_X \) is an additive perturbation vector, and \( \nabla H_{\text{neg}} : P_X^\circ \to \mathbb{R}^{\mathcal{X}} \) is the gradient of \( H_{\text{neg}} \). Moreover, as \( H_{\text{neg}} \) is twice continuously differentiable, we have:

\[
\forall Q_X \in P_X^\circ, \quad \nabla^2 H_{\text{neg}}(Q_X) = \text{diag}(Q_X)^{-1} \succeq_{\text{psd}} I
\]

where \( \nabla^2 H_{\text{neg}} : P_X^\circ \to \mathbb{R}^{\mathcal{X} \times \mathcal{X}} \) denotes the Hessian matrix of \( H_{\text{neg}} \), and \( I \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}} \) denotes the identity matrix. (Note that \( \text{diag}(Q_X)^{-1} - I \) is positive semidefinite because it is a diagonal matrix with non-negative diagonal entries.) Recall from [76, Chapter 9] that a twice continuously differentiable convex function \( f : S \to R \) with open domain \( S \subseteq \mathbb{R}^n \) is called strongly convex if there exists \( m > 0 \) such that for all \( x \in S \), \( \nabla^2 f(x) \succeq m I \). This means that \( H_{\text{neg}} \) is strongly convex on \( P_X^\circ \). A consequence of this strong convexity is the following quadratic lower bound [76, Chapter 9]:

\[
H_{\text{neg}}(S_X) \geq H_{\text{neg}}(Q_X) + J_X \nabla H_{\text{neg}}(Q_X) + \frac{1}{2} \| J_X \|^2
\]

\[
\iff D(S_X || Q_X) \geq \frac{1}{2} \| J_X \|^2
\]

for every \( S_X \in P_X \) and \( Q_X \in P_X^\circ \), where we allow \( S_X \in P_X \setminus P_X^\circ \) due to the continuity of \( H_{\text{neg}} \). This is precisely what we get if we loosen (65) in the proof of Lemma 14 using \( \| J_X \|_1 \geq \| J_X \|_2 \) and (64). Finally, we have for every \( S_X \in P_X \) and \( Q_X \in P_X^\circ \):

\[
D(S_X || Q_X) \geq \frac{1}{2} \| J_X \|^2 \geq \frac{\min_{x \in \mathcal{X}} Q_X(x)}{2} \chi^2(S_X || Q_X)
\]

where the second inequality follows from (6). This trivially holds for all \( Q_X \in P_X \setminus P_X^\circ \) as well. ■

**ACKNOWLEDGMENT**

A. Makur would like to thank Prof. Yury Polyanskiy for stimulating discussions related to Theorem 12, and more generally, for discussions regarding contraction coefficients.

**REFERENCES**

[1] A. Makur and L. Zheng, “Bounds between contraction coefficients,” in *Proceedings of the 53rd Annual Allerton Conference on Communication, Control, and Computing*, Allerton House, UIUC, Illinois, USA, September 29-October 2 2015, pp. 1422–1429.

[2] R. Ahlswede and P. Gács, “Spreading of sets in product spaces and hypercontraction of the Markov operator,” *The Annals of Probability*, vol. 4, no. 6, pp. 925–939, December 1976.

[3] E. Erkip and T. M. Cover, “The efficiency of investment information,” *IEEE Transactions on Information Theory*, vol. 44, no. 3, pp. 1026–1040, May 1998.

[4] V. Anantharam, A. Gohari, S. Kamath, and C. Nair, “On maximal correlation, hypercontractivity, and the data processing inequality studied by Erkip and Cover,” April 2013, arXiv:1304.6133 [cs.IT].

[5] ———, “On hypercontractivity and the mutual information between Boolean functions,” in *Proceedings of the 51st Annual Allerton Conference on Communication, Control, and Computing*, Allerton House, UIUC, Illinois, USA, October 2-4 2013, pp. 13–19.

[6] S. Kamath and V. Anantharam, “Non-interactive simulation of joint distributions: The Hirschfeld-Gebelein-Rényi maximal correlation and the hypercontractivity ribbon,” in *Proceedings of the 50th Annual Allerton Conference on Communication, Control, and Computing*, Allerton House, UIUC, Illinois, USA, October 1-5 2012, pp. 1057–1064.
[47] A. Makur, F. Kozyrski, S.-L. Huang, and L. Zheng, “An efficient algorithm for information decomposition and extraction,” in Proceedings of the 53rd Annual Allerton Conference on Communication, Control, and Computing, Allerton House, UIUC, Illinois, USA, September 29-October 2 2015, pp. 972–979.

[48] S.-L. Huang, A. Makur, F. Kozyrski, and L. Zheng, “Efficient statistics: Extracting information from iid observations,” in Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing, Allerton House, UIUC, Illinois, USA, October 1-3 2014, pp. 699–706.

[49] S.-L. Huang, A. Makur, L. Zheng, and G. W. Wornell, “An information-theoretic approach to universal feature selection in high-dimensional inference,” in Proceedings of the IEEE International Symposium on Information Theory (ISIT), Aachen, Germany, June 25-30 2017, pp. 1336–1340.

[50] J. E. Cohen, Y. Iwasa, G. Rautu, M. B. Ruskai, E. Seneta, and G. Zbaganu, “Relative entropy under mappings by stochastic matrices,” Linear Algebra and its Applications, Elsevier, vol. 179, pp. 211–235, January 1993.

[51] E. Seneta, “Coefficients of ergodicity: Structure and applications,” Advances in Applied Probability, vol. 11, no. 3, pp. 576–590, September 1979.

[52] ——. Non-negative Matrices and Markov Chains, 2nd ed., ser. Springer Series in Statistics. New York: Springer, 1981.

[53] I. C. F. Ipsen and T. M. Selee, “Ergodicity coefficients defined by vector norms,” SIAM Journal on Matrix Analysis and Applications, vol. 32, no. 1, pp. 153–200, March 2011.

[54] T. M. Selee, “Stochastic matrices: Ergodicity coefficients, and applications to ranking,” PhD Thesis in Applied Mathematics, North Carolina State University, Raleigh, North Carolina, 2009.

[55] R. L. Dobrushin, “Central limit theorem for nonstationary Markov chains. I,” Theory of Probability and Its Applications, vol. 1, no. 1, pp. 65–80, 1956.

[56] A. Kontorovich, “Obtaining measure concentration from Markov contraction,” Markov Processes and Related Fields, vol. 18, no. 4, pp. 613–638, 2012.

[57] C. E. Shannon, “The zero error capacity of a noisy channel,” IRE Transactions on Information Theory, vol. 2, no. 3, pp. 706–715, September 1956.

[58] I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems, 2nd ed. New York: Cambridge University Press, 2011.

[59] W. S. Evans and L. J. Schulman, “Signal propagation and noisy circuits,” IEEE Transactions on Information Theory, vol. 45, no. 7, pp. 2367–2373, November 1999.

[60] H. Kim, W. Gao, S. Kannan, S. Oh, and P. Viswanath, “Discovering potential correlations via hypercontractivity,” Entropy, vol. 19, no. 11, pp. 1–32, November 2017.

[61] G. L. Gilardoni, “On Pinsker’s and Vajda’s type inequalities for Csiszár’s f-divergences,” IEEE Transactions on Information Theory, vol. 56, no. 11, pp. 5377–5386, November 2010.

[62] S. Verdú, “Total variation distance and the distribution of relative information,” in Proceedings of the Information Theory and Applications Workshop (ITA), San Diego, CA, USA, February 9-14 2014, pp. 1–3.

[63] C. Nair, “An extremal inequality related to hypercontractivity of Gaussian random variables,” in Proceedings of the Information Theory and Applications Workshop (ITA), San Diego, CA, USA, February 9-14 2014, pp. 1–7.

[64] F. du Pin Calmon, Y. Polyanskiy, and Y. Wu, “Strong data processing inequalities for input constrained additive noise channels,” IEEE Transactions on Information Theory, vol. 64, no. 3, pp. 1879–1902, March 2018.

[65] I. Kontorovich and K. Marton, “Comparison of two noisy channels,” in Topics in Information Theory (Second Colloq., Keszthely, 1975), Amsterdam: North-Holland, 1977, p. 411/423.

[66] M. van Dijk, “On a special class of broadcast channels with confidential messages,” IEEE Transactions on Information Theory, vol. 43, no. 2, pp. 712–714, March 1997.

[67] A. Makur and Y. Polyanskiy, “Comparison of channels: Criteria for domination by a symmetric channel,” IEEE Transactions on Information Theory, vol. 64, no. 8, pp. 5704–5725, August 2018.

[68] R. Bhatia, Matrix Analysis, ser. Graduate Texts in Mathematics. New York: Springer, 1997, vol. 169.

[69] E. Ordentlich and M. J. Weinberger, “A distribution dependent refinement of Pinsker’s inequality,” IEEE Transactions on Information Theory, vol. 51, no. 5, pp. 1836–1840, May 2005.

[70] L. Sason, “Bounds on f-divergences and related distances,” Department of Electrical Engineering, Technion-Israel Institute of technology, Haifa, Israel, Irwin and Joan Jacobs Center for Communication and Information Technologies (CCIT) Report 859, May 2014.

[71] P. Harremoës and I. Vajda, “Ergodicity coefficients defined by vector norms,” IEEE Transactions on Information Theory, vol. 57, no. 6, pp. 3230–3235, June 2011.

[72] A. A. Fedotov, P. Harremoës, and F. Topsøe, “Refinements of Pinsker’s inequality,” IEEE Transactions on Information Theory, vol. 49, no. 6, pp. 1491–1498, June 2003.

[73] F. E. Su, “Methods for quantifying rates of convergence for random walks on groups,” PhD Thesis in Mathematics, Harvard University, Cambridge, Massachusetts, 1995.

[74] S. S. Dragomir and V. Gluščević, “Some inequalities for the Kullback-Leibler and χ²-distances in information theory and applications,” Tamkang Journal of Mathematical Sciences, vol. 17, no. 2, pp. 97–111, 2001.

[75] I. Kontorovich, “Tight bounds for symmetric divergence measures and a new inequality relating f-divergences,” in Proceedings of the IEEE Information Theory Workshop (ITW), Jerusalem, Israel, April 26-May 1 2015.

[76] S. Boyd and L. Vandenberghe, Convex Optimization. New York: Cambridge University Press, 2004.

[77] R. B. Ash, Information Theory, ser. Interscience Tracts in Pure and Applied Mathematics. New York: John Wiley & Sons, Inc., 1965, no. 19.

[78] A. Dembo, T. M. Cover, and J. A. Thomas, “Information theoretic inequalities,” IEEE Transactions on Information Theory, vol. 37, no. 6, pp. 1501–1518, November 1991.

[79] Y.-C. Li and C.-C. Yeh, “Some equivalent forms of Bernoulli’s inequality: A survey,” Applied Mathematics, vol. 4, no. 7, pp. 1070–1093, 2013.

[80] V. Rakočević and H. K. Wimmer, “A variational characterization of canonical angles between subspaces,” Journal of Geometry, vol. 78, no. 1, pp. 122–124, 2003.

[81] W. Kang and S. Ulukus, “A new data processing inequality and its applications in distributed source and channel coding,” IEEE Transactions on Information Theory, vol. 57, no. 1, pp. 56–69, January 2011.

[82] S. Asoddeh, M. Diaz, F. Alajaji, and T. Linder, “Information extraction under privacy constraints,” January 2016, arXiv:1511.02381v3 [cs.IT].

[83] A. Banerjee, S. Merugu, I. S. Dhillon, and J. Ghosh, “Clustering with Bregman divergences,” Journal of Machine Learning Research, vol. 6, pp. 1705–1749, October 2005.