Monotonicity of the first Dirichlet eigenvalue of the Laplacian on manifolds of non-positive curvature

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Abstract

Let $(M, g)$ be a complete Riemannian manifold with nonpositive scalar curvature, let $\Omega \subset M$ be a suitable domain, and let $\lambda(\Omega)$ be the first Dirichlet eigenvalue of the Laplace-Beltrami operator on $\Omega$. We prove several bounds for the rate of decrease of $\lambda(\Omega)$ as $\Omega$ increases, and a result comparing the rate of decrease of $\lambda$ before and after a conformal diffeomorphism. Along the way, we prove a reverse-Hölder inequality for the first eigenfunction, which generalizes results of Chiti to the manifold setting and maybe be of independent interest.

1 Introduction

Let $(M, g)$ be a complete Riemannian manifold, and let $\Omega \subset M$ be a domain with $\overline{\Omega}$ compact and $\partial \Omega \in C^\infty$. The first Dirichlet eigenvalue of the Laplace-Beltrami operator on $\Omega$ is a natural and important object. It controls the slowest rate of heat dissipation from $\Omega$, the largest value of the expected exit time of a Brownian particle from $\Omega$, and the fundamental frequency of vibration of $\Omega$ (when considered as a vibrating membrane with stationary boundary). Many results have linked the first eigenvalue and its associated eigenfunction to the geometry of $\Omega$, and also to that of the ambient space $(M, g)$.

The first Dirichlet eigenvalue $\lambda(\Omega)$ of the Laplace-Beltrami $\Delta_g$ operator on $\Omega$ is

$$\lambda(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 \, dm}{\int_{\Omega} u^2 \, dm} : u \in W^{1,2}_0(\Omega) \right\},$$

where $dm$ represents the volume element on $M$. This infimum is realized by a nontrivial function $\phi$, which satisfies

$$\Delta_g \phi + \lambda \phi = 0, \quad \phi|_{\partial \Omega} = 0, \quad \phi > 0 \text{ inside } \Omega.$$

If $\overline{\Omega}$ is compact then $\lambda$ is a positive, simple eigenvalue. In the case that $(M, g)$ is Euclidean space, a fundamental result is the Faber-Krahn inequality:

$$\lambda(\Omega) \geq \lambda(B_1) \left( \frac{\omega_n}{|\Omega|} \right)^{2/n},$$

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and equality can only occur if $\Omega$ is a round ball. Here $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega_0|$ is the volume of $\Omega$. Similar inequalities hold if the ambient space is hyperbolic space, or the ambient space is a sphere and $\Omega$ is convex. One can find an excellent introduction to all this and more in Chapter 1 of [5].

Our main goal is to explore the rate of change of $\lambda(\Omega_t)$, where $\Omega_t$ is an evolving family of domains. More precisely, we let $\zeta(t, p)$ be a one-parameter family of diffeomorphisms, and let $\Omega_t = \zeta(t, \Omega)$. Suppose that $\langle \eta, \frac{\partial \zeta}{\partial t} \rangle > 0$, where $\eta$ is the outward unit normal of $\Omega_t$. Then $\Omega_s \subset \Omega_t$ for $s < t$, so domain monotonicity implies $\lambda$ is a decreasing function of $t$, and we would like to estimate its rate of decrease.

The isoperimetric inequality is a key tool we use, so we will need some standing hypotheses on the ambient space $(M, g)$ and the domain $\Omega$. We always assume one of the following holds:

- either $(M, g)$ is compact, and $S_g \leq -n(n-1)\kappa^2$ for some fixed $\kappa \in \mathbb{R}$, and $|\Omega|_g$ is sufficiently small,
- or $(M, g)$ is complete, and $S_g \leq -n(n-1)\kappa^2$ for some fixed $\kappa \in \mathbb{R}$, and $\Omega$ is contained in a small geodesic ball $B_r$, whose radius might depend on its center,
- or $\dim(M) = 2$ and $S_g \leq 0$ is nonpositive.

Here $S_g$ is the scalar curvature of the metric $g$. One should be able to prove results similar to ours in the presence of a weaker scalar curvature bound, such as $S_g \leq n(n-1)\kappa^2$, using straightforward adaptations of our proofs below.

Three examples of what we are able to prove are the following theorems.

**Theorem 1.** Let $(M, g)$ be a complete Riemannian manifold, and let $\Omega \subset M$ be a domain with $\bar{\Omega}$ compact and $\partial \Omega \in C^\infty$. Suppose that $\partial \Omega$ evolves with velocity vector $\eta$, the unit length outward normal to $\Omega$, and let $\lambda(t) = \lambda(\Omega_t)$. If $\dim(M) = 2$ and $(M, g)$ has nonpositive Gauss curvature then

$$\frac{d}{dt} \log(\lambda) \leq -\frac{4\pi}{|\partial \Omega|_g}.$$ 

If $\dim(M) = n \geq 3$, and $(M, g)$ has nonpositive scalar curvature and $\Omega$ is sufficiently small (see Section 2.2 below) then

$$\frac{d}{dt} \left( \lambda^{\frac{n-2}{2}} \right) \leq -\left( \frac{n-2}{2} \right) K \frac{1}{|\partial \Omega|_g},$$

where $K$ is a constant depending only on $n$. Equality in either case can only occur if $\Omega$ is isometric to a round ball in the appropriate dimensional Euclidean space.

**Theorem 2.** Let the hypotheses of Theorem 1 hold, with $n = \dim(M) = 2$, and suppose additionally that $\Omega$ is strictly convex. Let $\partial \Omega$ evolve with velocity $k_g \eta$, where $k_g$ is the geodesic curvature of $\partial \Omega$. Then

$$\frac{d}{dt} \log(\lambda) \leq -\frac{4\pi}{\int_{\partial \Omega} k_g^{-1} d\sigma}.$$ 

Equality can only occur if $\Omega$ is isometric to a round disk in the Euclidean plane.
Theorem 3. Let the hypotheses of Theorem 1 hold, with \( n = \dim(M) \geq 3 \), and suppose additionally that \( \Omega \) is strictly mean convex. Let \( \partial \Omega \) evolve with velocity \( H \eta \), where \( H \) is the mean curvature of \( \partial \Omega \). Then

\[
\frac{d}{dt} \left( \lambda^{n-2} \right) \leq - \left( \frac{n-2}{2} \right) \frac{K}{\int_{\partial \Omega} H^{-1} d\sigma},
\]

where \( K \) is a constant depending only on \( n = \dim(M) \). Equality can only occur if \((M,g)\) is Euclidean space and \( \Omega \) is a round ball.

All three of these theorems are special cases of Theorem 11 below. Additionally, one can state similar results for other geometric flows, such as flow by Gauss curvature or the Willmore flow, under appropriate geometric hypotheses, as corollaries of Theorem 11.

Our secondary goal is a general Schwarz Lemma for the first eigenvalue. To make sense of this statement, one should think of the Schwarz Lemma geometrically, as interpreted by Pick and Ahlfors: if \( D \) is the unit disk in the complex plane \( \mathbb{C} \), then any holomorphic mapping \( F : D \to D \) is a contraction in the Poincaré metric. More recently, Burckel, Marshall, Minda, Poggi-Corradini, and Ransford [3] proves a variety of results in the spirit of the Schwarz Lemma for other geometric quantities, such as diameter, logarithmic capacity, and area. Bringing the eigenvalue into the picture, Laugesen and Morpurgo [17] prove the following result as a special case of a more general theorem: if \( F : D \to \mathbb{C} \) be conformal, then the function

\[
r \mapsto \frac{\lambda(F(rD))}{\lambda(rD)}
\]

is a strictly decreasing function, unless \( F \) is linear (in which case it is constant). Geometrically, the Laugesen-Morpurgo result states that \( \lambda(F(rD)) \) decreases more rapidly than \( \lambda(rD) \), as \( r \) increases.

At first glance, all these variations on the Schwarz Lemma seem unique to the complex plane. However, this is not the case, and one can find several versions of the classical Schwarz Lemma in higher dimensions. For instance, Yau [21] proved a version of the Schwarz Lemma for holomorphic mappings between Hermitian manifolds, and later Chen, Cheng, and Look [6] proved a different version of Yau’s result. Motivated by these papers, we prove a version of the Schwarz Lemma for the first eigenvalue under a conformal diffeomorphism in Theorem 12 below.

The rest of the paper is organized as follows. In Section 2 we collect some useful facts from Riemannian geometry. In Section 3 we prove some rearrangement and reverse-Hölder theorems, which may be of independent interest, and finally we prove Theorems 11 and 12 in Section 4.

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2 Geometric preliminaries

In this section we collect some useful results from Riemannian geometry.
2.1 Model spaces

One can find most of the material below in textbooks such as [4], or in the survey article [15].

We begin with some basic formulas regarding our model space \((M_\kappa, g_\kappa)\), the complete, simply connected space with constant sectional curvature \(-\kappa^2\). First observe that, by the Cartan-Hadamard theorem, \(M_\kappa\) is diffeomorphic to \(\mathbb{R}^n\), so we use global polar coordinates.

In these coordinates, the metric has the form

\[ g_\kappa = dr^2 + \frac{1}{\kappa^2} \sinh^2(\kappa r) d\theta^2, \tag{2.1} \]

where \(d\theta^2\) is the round metric on the unit sphere. Using the expansion

\[ \frac{\sinh(\kappa r)}{\kappa} = r + \frac{1}{3!} \kappa^2 r^3 + \frac{1}{5!} \kappa^4 r^5 + \cdots, \]

we (formally) recover the Euclidean metric in polar coordinates as \(\kappa \to 0^+\):

\[ g_0 = dr^2 + r^2 d\theta^2 = \lim_{\kappa \to 0^+} \left[ dr^2 + \frac{1}{\kappa^2} \sinh^2(\kappa r) d\theta^2 \right]. \]

It is also convenient to observe that \(\kappa^{-1} \sinh(\kappa r) > r\) for all \(\kappa > 0\); geometrically, this says geodesics spread apart more rapidly (in fact, exponentially more rapidly) in hyperbolic space than in Euclidean space. From (2.1) we see that

\[ |\partial B_r|_\kappa = n \omega_n \kappa^{1-n} (\sinh(\kappa r))^{n-1} \tag{2.2} \]

and

\[ |B_r|_\kappa = n \omega_n \kappa^{1-n} \int_0^r (\sinh(\kappa t))^{n-1} dt = v_\kappa(r), \tag{2.3} \]

where \(B_r\) is a geodesic ball of radius \(r\), and \(\omega_n\) is the volume of an \(n\)-dimensional Euclidean unit ball. Later, it will be convenient to invert the model volume function \(v_\kappa(r)\), and write its inverse as \(r_\kappa(v)\), which we call the volume radius. Again, we can recover the familiar Euclidean formulae by taking a limit as \(\kappa \to 0\):

\[ |\partial B_r|_0 = n \omega_n r^{n-1}, \quad |B_r|_0 = \omega_n r^n = v_0(r), \quad r_0(v) = \left( \frac{v}{\omega_n} \right)^{1/n}. \]

The first eigenfunction \(\psi_\kappa\) of a geodesic ball in the model space \((M_\kappa, g_\kappa)\) is radial, and so it satisfies

\[ -\lambda \psi_\kappa = \Delta_\kappa \psi_\kappa = (\sinh(\kappa r))^{1-n} (\sinh(\kappa r))^{n-1} \psi_\kappa', \tag{2.4} \]

where we use \('\) to denote differentiation with respect to \(r\). (Where it can be understood from context, we suppress the subscript \(\kappa\).) If we change variables to volume and write \(\psi^*(v) = \psi(r_\kappa(v))\) this equation becomes

\[ -\lambda \psi^*(v) = n^2 \omega_n^2 \kappa^{2-2n} \frac{d}{dv} \left( \frac{\sinh^{2n-2}(\kappa r_\kappa(v))}{d \psi^*} \right), \]
which we can integrate once to obtain

\[- (\psi^*)'(v) = n^{-2} \omega_n^{-2} \lambda \left[ \frac{\sinh(\kappa r(v))}{\kappa} \right]^{2 - 2n} \int_0^v \psi^*(t) dt. \quad (2.5)\]

As before, we take a limit as \(\kappa \to 0\) to recover the Euclidean analogs of (2.4) and (2.5), which are (respectively)

\[- \lambda \psi_0 = \Delta_0 \psi_0 = r^{1-n} (r^{n-1} \psi'_0)' \]

and

\[- (\psi^*_0)'(v) = n^{-2} \omega_n^{-2/n} \lambda v^{-2+2/n} \int_0^v \psi^*_0(t) dt. \quad (2.6)\]

### 2.2 Isoperimetric inequalities

In this section we recall some isoperimetric inequalities for general Riemannian manifolds. Throughout this section, we take \((M, g)\) to be a complete Riemannian manifold, and we usually place a bound on its curvature. We also let \(\Omega \subset M\) be a domain with \(\partial \Omega \in C^\infty\) (though this much regularity is rarely necessary), and with \(\Omega\) compact. A theorem of Beckenbach and Radó [1] states that if \((M, g)\) is a complete surface with nonpositive Gauss curvature then

\[|\partial \Omega|^2_g \geq 4\pi |\Omega|_g,\]

and equality can only occur if \((M, g)\) is the Euclidean plane and \(\Omega\) is a round disk.

The next major break-through is a theorem is due to Croke [9], which states that if \(\text{Sect}(g) \leq 0\) then

\[|\partial \Omega|^g \geq c_1(n) \left| \Omega \right|_g^{\frac{n-1}{n}},\]

where \(c_1(n)\) is an explicit constant and \(\text{Sect}(g)\) is the sectional curvature of \((M, g)\). This inequality is only an equality when \(n = 4\) and \(\Omega\) is a round ball in Euclidean space.

The next result we quote is due to Kleiner [16], and states that if \(\text{Sect}(g) \leq -\kappa^2 \leq 0\) then \(|\partial \Omega|^g \geq |\partial B_r|_\kappa\), where \(B_r\) is a geodesic ball in the model space \((M_\kappa, g_\kappa)\), with \(|\Omega|_g = |B_r|_\kappa\). One only has equality in Kleiner’s result if \(\Omega\) is a model geodesic ball. It is worth remarking that Kleiner’s proof relies on the Gauss-Bonnet formula, so it can only work in dimension three, while Croke’s proof can only be sharp in dimension four. To date, these are the only general results one can find with no restriction on the size of \(\Omega\).

More recently, Morgan and Johnson [18] proved a result for compact manifolds \((M, g)\), so long as \(|\Omega|_g\) is sufficiently small. Their results state that if \(\text{Sect}(g) \leq -\kappa^2\) and \(|\Omega|_g\) is sufficiently small then the same inequality \(|\partial \Omega|_g \geq |\partial B_r|_\kappa\) holds, where again \(B_r\) is a geodesic ball in the model space \((M_\kappa, g_\kappa)\) with \(|B_r|_\kappa = |\Omega|_g\). Later, Druet [10] strengthened the Morgan-Johnson result to the point that one only needs a bound on the scalar curvature \(S_g\) of \(g\), of the form \(S_g \leq -n(n-1)\kappa^2\). He also shows that these results hold when \((M, g)\) is complete and \(\Omega\) is contained in a small geodesic ball (whose radius might depend on position). Again, one can only have equality in either of these theorems if \(\Omega\) is a geodesic ball in the model space. We can use our notation from the previous section to write these inequalities as

\[\Omega\text{ sufficiently small } \Rightarrow |\partial \Omega|_g \geq n \omega_n \kappa^{1-n} (\sinh(\kappa r(v)))^{n-1}, \quad (2.7)\]
where \( r_\kappa(v) \) is the volume radius function, which inverts (2.3). Again, we can contrast this with the Euclidean case, which states that \( |\partial \Omega|_0 \geq n \omega^n_{n/2} \frac{1}{n} |\Omega|^{1/n} = n \omega_n(r_0(v))^{n-1} \).

Using these later forms of the isoperimetric inequality for small volumes, Druet [11] and Fall [12] proved a Faber-Krahn theorem, and in fact obtained stability estimates. More precisely, they showed that if \( \text{Sect}(g) \leq -\kappa^2 \) and \( |\Omega|_g \) is small, then \( \lambda(\Omega) \geq \lambda(B_r) \), where \( B_r \) is the geodesic ball in the model space as before. Moreover, they estimate the difference \( \lambda(\Omega) - \lambda(B_r) \), again when \( |\Omega|_g \) is small.

### 3 Rearrangements and reverse-Hölder inequalities

In this section we discuss a rearrangement of the first eigenfunction \( \phi \) of \( \Omega \), and use it to prove an integro-differential inequality similar to that of Talenti [20]. Next, we obtain inequalities which generalize the results of Chiti [7, 8] to the Riemannian setting. As outlined in our introduction, our standing hypotheses will be that (2.7) holds. While the precise statements below have not yet appeared in the literature (to our knowledge), we suspect that much of this section is, in the words of A. Treibergs, “well-known to those who know it well.”

Recall that the first eigenfunction \( \phi \) satisfies

\[
\lambda(\Omega) = \frac{\int_{\Omega} |\nabla \phi|^2 \, dm}{\int_{\Omega} \phi^2 \, dm} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 \, dm}{\int_{\Omega} u^2 \, dm} : u \in W^{1,2}_0(\Omega) \right\},
\]

or, alternatively,

\[
\Delta_g \phi + \lambda(\Omega) \phi = 0, \quad \phi|_{\partial \Omega} = 0, \quad \phi > 0 \text{ inside } \Omega. \tag{3.2}
\]

Let \( m = \sup_{\Omega} \phi \), and for \( 0 \leq t \leq m \) define

\[
D_t = \{ \phi > t \}, \quad \mu(t) = |D_t|_g. \tag{3.3}
\]

By the co-area formula, we have

\[
\mu(t) = \int_t^m \int_{\partial D_r} \frac{d\sigma}{|\nabla \phi|} \, dr,
\]

so that

\[
\mu'(t) = -\int_{\partial D_t} \frac{d\sigma}{|\nabla \phi|} < 0. \tag{3.4}
\]

Here \( d\sigma \) is the \((n-1)\)-dimensional volume element induced on \( \partial D_t \) by its inclusion in \( \Omega \). Therefore, \( \mu \) is monotone, and so it has an inverse function we call \( \phi^*(v) \), defined by

\[
\phi^*(v) = \inf\{t \in [0, m] : \mu(t) < v\}.
\]

While the following is an easy adaptation of equation (34) of [20], we include its proof for the reader’s convenience.

**Lemma 4.** Let \( \Omega \subset (M, g) \) be a domain with compact closure, smooth boundary, and sufficiently small that (2.7) holds. Then the function \( \phi^* \) satisfies

\[
-(\phi^*)'(v) \leq n^{-2} \omega_n^{-2} \lambda(\Omega) \left( \frac{\sinh(\kappa r_\kappa(v))}{\kappa} \right)^{2-2n} \int_0^v \phi^*(t) \, dt. \tag{3.5}
\]

Moreover, equality can only occur if \( \Omega \) is isometric to a geodesic ball in the model space \((M_\kappa, g_\kappa)\).
Therefore, there exists \( k > 1 \) such that \( k\phi^v(v) > \psi^v(v) \) for all \( v \in [0, |B^*|_\kappa] \). Define \( k_0 = \inf\{k > 1 : k\phi^v(v) > \psi^v(v) \text{ on } [0, |B^*|_\kappa]\} \).
If $k_0 = 1$ then we’ve completed the proof, and otherwise there exists $v_0 \in (0, |B^*|)$ such that $k_0 \phi^*(v_0) = \psi^*(v_0)$. If we let

$$u^*(v) = \begin{cases} k_0 \phi^*(v) & 0 \leq v \leq v_0 \\ \psi^*(v) & v_0 < v \leq |B^*|, \end{cases}$$

then, by (3.5) and (2.5), we have

$$- (u^*)'(v) \leq n^{-2} 2^{2/n} \lambda \left[ \frac{\sinh(\kappa r(v))}{\kappa} \right]^{2 - 2/n} \int_0^v u^*(t) dt.$$  \tag{3.8}

Now define a radial test function on $B^*$ by $u(r) = u^*(v_\kappa(r))$. We use the chain rule and

$$\frac{dv_\kappa}{dr} = n \omega_2 \left( \frac{\sinh(\kappa r)}{\kappa} \right)^{n-1}$$

to see that

$$\int_{B^*} |\nabla u|^2 dm = \int_0^{|B^*|} n^2 \omega_2 \left[ \frac{\sinh(\kappa r(v))}{\kappa} \right]^{2n-2} (- (u^*)'(v))^2 dv \leq \lambda \int_0^{|B^*|} (- (u^*)'(v))^2 (u^*(v)) dv \int_0^v (u^*(\tau))^2 d\tau = \lambda \int_0^{|B^*|} (u^*(\tau))^2 d\tau = \lambda \int_{B^*} u^2 dm.$$  

However, this is impossible unless $u = \psi$, which would contradict $k_0 > 1$. \hfill \Box

We can integrate the inequality in Theorem 5 to obtain the following (scale-invariant) corollary.

**Corollary 6.** Let $\Omega \subset (M, g)$ be as above, and let $B^*$ be the geodesic ball in the model space $(M_\kappa, g_\kappa)$ with $\lambda(\Omega) = \lambda(B^*)$. Let $\psi$ be the first eigenfunction of $B^*$ and let $\phi$ be the first eigenfunction of $\Omega$. Then for all $p > 0$ we have

$$\frac{\|\phi\|_{L^p(\Omega)}}{\|\phi\|_{L^\infty(\Omega)}} \geq \frac{\|\psi\|_{L^p(B^*)}}{\|\psi\|_{L^\infty(B^*)}}.$$  

Equality can only occur if $\Omega$ is isometric to $B^*$.

One can find a version of the following theorem, which reverses the standard Hölder inequality, in the hyperbolic setting in Section 9 of [2]. Both proofs utilize Chiti’s method from [8].

**Theorem 7.** With the same $\Omega$ as above and any choice $0 < p < q < \infty$, there exists a positive, finite constant $C = C(n, p, q, \kappa, \lambda)$ such that the first eigenfunction $\phi$ of $\Omega$, with eigenvalue $\lambda$, satisfies

$$\left( \int_\Omega \phi^p dm \right)^{\frac{q}{p}} \geq C \left( \int_\Omega \phi^q dm \right)^{\frac{p}{q}}. \tag{3.9}$$

Equality can only occur if $\Omega$ is isometric to $B^*$.
In fact, it will be transparent from the proof that
\[
C = \left( \int_{B^*} \psi^p \, dm \right)^\frac{q}{p},
\]
where \(B^*\) is the geodesic ball in the model space with \(\lambda(B^*) = \lambda = \lambda(\Omega)\), and \(\psi\) is its first eigenfunction.

**Proof.** We use the same approach as in the proof of Theorem 5, but this time normalize \(\psi\) such that
\[
\int_{B^*} \psi^p \, dm = \int_{\Omega} \phi^p \, dm.
\]
Thus, by Corollary 6 above, \(\|\psi\|_{L^\infty(B^*)} \geq \|\phi\|_{L^\infty(\Omega)}\), with equality if and only if \(\Omega\) is isometric to \(B^*\). We may therefore assume \(\psi^*(0) = \|\psi\|_{L^\infty(B^*)} > \|\phi\|_{L^\infty(\Omega)} = \phi^*(0)\).

We also know, as before, that \(\psi^*(|B^*|_\kappa) = 0, \quad \phi^* > 0\) on \([0, |B^*|_\kappa]\), which combined with (3.11) tells us the graphs of \(\phi^*\) and \(\psi^*\) must cross, and not just touch, at least once on the interval \([0, |B^*|_\kappa]\). Define
\[
v_0 = \sup\{v \in (0, |B^*|_\kappa) : \phi^*(\tilde{v}) \leq \psi^*(\tilde{v})\text{ for all } \tilde{v} \in (0, v)\},
\]
so that we have
\[
0 < v_0 < |B^*|_\kappa, \quad \psi^* \geq \phi^* \text{ in } [0, v_0], \quad \phi^*(v_0) = \psi^*(v_0).
\]
Additionally, there must exist \(\delta > 0\) such that \(\phi^*(v) > \psi^*(v)\) for \(v \in (v_0, v_0 + \delta)\).

We claim that actually \(\phi^* > \psi^*\) in the interval \((v_0, |B^*|_\kappa]\). Indeed, if this were not the case then there would exist \(v_1\) such that
\[
v_0 < v_1 < |B^*|_\kappa, \quad \psi^*(v_1) = \phi^*(v_1), \quad \phi^*(v) > \psi^*(v) \text{ for } v_0 < v < v_1.
\]
This allows us to define a test function for \(B^*\) as
\[
u^*(v) = \begin{cases} 
\psi^*(v) & 0 \leq v \leq v_0 \\
\phi^*(v) & v_0 \leq v \leq v_1 \\
\psi^*(v) & v_1 \leq v \leq |B^*|_\kappa.
\end{cases}
\]
As before, our test function satisfies
\[
-(u^*)'(v) \leq n^{-2} \omega_n^{-2/n} \lambda \left[ \frac{\sinh(\kappa \gamma(v))}{\kappa} \right]^{2-2n} \int_0^v u^*(t) \, dt,
\]
and we can define a radial test function \( u \) on \( B^* \) by \( u(r) = u^*(v_\kappa(r)) \), which in turn satisfies
\[
\int_{B^*} |\nabla u|^2 \, dm = \int_0^{\|B^*\|_r} n^2 \omega_n \left[ \frac{\sinh(\kappa r(v))}{\kappa} \right]^{2n-2} (-u^*(v))' \, dv
\]
\[
\leq \lambda \int_0^{\|B^*\|_r} (-u^*(v))' \, dv \leq \lambda \int_0^{\|B^*\|_r} u^*(\tau) \, d\tau = \lambda \int_{\|B^*\|_r} (u^*(\tau))^2 \, d\tau
\]
\[
= \lambda \int_{B^*} v^2 \, dm.
\]
As before, this is only possible if \( u = \psi \), which contradicts our assumption \( \psi^*(0) > \phi^*(0) \).

So far, we have shown there exists \( v_0 \in (0, \|B^*\|_r) \) such that \( \psi^* \geq \phi^* \) on \((0, v_0)\) and \( \phi^* > \psi^* \) on \((v_0, \|B^*\|_r)\). We extend \( \psi^* \) to be zero on the interval \([\|B^*\|_r, \|\Omega\|_g]\), and claim that
\[
v \in [0, \|\Omega\|_g] \Rightarrow \int_0^v (\psi^*(\tau))^p \, d\tau \geq \int_0^v (\phi^*(\tau))^p \, d\tau.
\] (3.12)

To prove this claim, we let
\[
I(v) = \int_0^v (\psi^*(\tau))^p \, d\tau - \int_0^v (\phi^*(\tau))^p \, d\tau
\]
and observe
\[
I(0) = I([\|\Omega\|_g]) = 0, \quad I'(v) = (\psi^*(v))^p - (\phi^*(v))^p.
\]
Thus \( I \) is increasing on the interval \([0, v_0]\) and decreasing on the interval \((v_0, \|\Omega\|_g]\). It follows immediately that \( I(v) > 0 \) for \( 0 \leq v \leq v_0 \). If we had \( I(v_1) < 0 \) for some \( v_1 \in (v_0, \|\Omega\|_g) \) then, because \( I \) is decreasing in this interval, we would also have \( I([\|\Omega\|_g]) < 0 \), which is a contradiction. We conclude \( (3.12) \). It follows from an inequality of Hardy, Littlewood, and Polya \cite{14} that for all \( q > p \) we have
\[
\left( \int \phi^q \, dm \right)^{1/q} \leq \left( \int_{B^*} \psi^q \, dm \right)^{1/q} = \frac{\left( \int_{B^*} \psi^q \, dm \right)^{1/q}}{\left( \int_{B^*} \phi^p \, dm \right)^{1/p}} \cdot \left( \int_{B^*} \phi^p \, dm \right)^{1/p},
\]
which we can rearrange to read
\[
\frac{\left( \int \phi^p \, dm \right)^{1/p}}{\left( \int \phi^q \, dm \right)^{1/q}} \geq \frac{\left( \int_{B^*} \psi^p \, dm \right)^{1/p}}{\left( \int_{B^*} \psi^q \, dm \right)^{1/q}} = \tilde{C}.
\]
Raising this inequality to the power \( pq \), we then obtain
\[
\left( \int \phi^p \, dm \right)^q \geq \tilde{C}^{pq} \left( \int \phi^q \, dm \right)^p.
\]

\[ \square \]

In the case \( S_g \leq 0 \) we can extract the explicit dependence of the constant \( C \) in \( (3.9) \) on the eigenvalue \( \lambda \). The dependence on the eigenvalue in the hyperbolic case is more challenging to understand, because the eigenfunctions on geodesic balls do not scale in a curved setting (see, for instance, Section 3 of \cite{2}).
Corollary 8. Suppose $\Omega$ is a domain in $(M,g)$, where $S_g \leq 0$, which is sufficiently small so that (2.7) applies. Let $\phi$ be its first eigenfunction, with eigenvalue $\lambda$. Then there is a constant $K = K(n,p,q)$ such that
\[
\left( \int_\Omega \phi^p \, dm \right)^q \geq K \lambda^{n(p-q)/2} \left( \int_\Omega \phi^q \, dm \right)^p.
\] (3.13)

Proof. This time our comparison domains are round balls in Euclidean space, and the dilation of an eigenfunction on a ball is an eigenfunction on the corresponding dilated ball. We have, according to (3.10)
\[
C = \left( \frac{\int_{B^*} \psi^p \, dm}{\int_{B^*} \psi^q \, dm} \right)^p.
\]
Denote the Euclidean radius of $B^*$ by $\rho$, and change variables to the unit ball by defining the function $\tilde{\psi}(r) = \psi(r\rho)$, so that
\[
C = \rho^{n(q-p)} \left( \frac{\int_{B_1} \tilde{\psi}^p \, dm}{\int_{B_1} \tilde{\psi}^q \, dm} \right)^p.
\]
Now, $\tilde{\psi}$ is the first eigenfunction on the unit ball in Euclidean space, and all that remains is to recall the scaling law for eigenvalues: $\lambda(B^*) = \rho^{-2} \lambda(B_1)$. Thus we see that
\[
\rho = \left( \frac{\lambda(B^*)}{\lambda(B_1)} \right)^{-1/2} = \left( \frac{\lambda(\Omega)}{\lambda(B_1)} \right)^{-1/2},
\]
and so
\[
C = \lambda^{-\frac{n}{2}(q-p)} \lambda(B_1)^{\frac{n}{2}(q-p)} \left( \frac{\int_{B_1} \tilde{\psi}^p \, dm}{\int_{B_1} \tilde{\psi}^q \, dm} \right)^p.
\]
We will later use the case of $p = 1$ and $q = 2$, which reads
\[
\left( \int_\Omega \phi \, dm \right)^2 \geq K \lambda^{-n/2} \int_\Omega \phi^2 \, dm.
\] (3.14)
In the case of $\text{dim}(M) = 2$ we recover an inequality of Payne and Rayner [19]:
\[
\left( \int_\Omega \phi \, dm \right)^2 \geq \frac{4\pi}{\lambda} \int_\Omega \phi^2 \, dm.
\] (3.15)
Here we have used the sharp version of the isoperimetric inequality of Beckenbach and Radó [11] for complete surfaces with nonpositive Gauss curvature. It is also important to notice that in the two-dimensional case we do not place any restriction on the size of $\Omega$.

The reverse Cauchy-Schwarz inequality (3.15) can be rewritten as a geometric isoperimetric inequality for the (singular) conformal metric $\tilde{g} = |\nabla \phi|^2 g$. We have the following corollary.
Corollary 9. Let \((M,g)\) is a surface with nonpositive Gauss curvature, and let \(\Omega\) be a domain with \(\overline{\Omega}\) compact and \(\partial \Omega \in C^\infty\). Place the (singular) conformal metric \(\tilde{g} = |\nabla \phi|^2 g\) on \(\Omega\), where \(\phi\) is the first Dirichlet eigenfunction of \(\Delta_g\) on \(\Omega\). Then, with respect to \(\tilde{g}\), we have
\[
\tilde{L}^2 \geq 4\pi \tilde{A},
\]
and equality can only occur if \(\Omega\) is isometric to a flat disk.

Proof. We begin with the left hand side of (3.15). We have
\[
\left( \int_\Omega \phi dm \right)^2 = \frac{1}{\lambda^2} \left( \int_\Omega \Delta \phi dm \right)^2 = \frac{1}{\lambda^2} \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2 = \frac{1}{\lambda^2} \left( \int_{\partial \Omega} |\nabla \phi|^2 d\sigma \right)^2 = \frac{\tilde{L}^2}{\lambda^2},
\]
where we have used the PDE satisfied by \(\phi\), the divergence theorem, and the fact that \(\phi\) is constant on \(\partial \Omega\). On the other hand, the right hand side of (3.15) is
\[
\frac{4\pi}{\lambda} \int_\Omega \phi^2 dm = \frac{4\pi}{\lambda^2} \int_\Omega |\nabla \phi|^2 dm = \frac{4\pi \tilde{A}}{\lambda^2}.
\]
The result follows.

4 Monotonicity of the first eigenvalue

In this section we study the evolution of \(\lambda\) as \(\Omega\) evolves.

A key ingredient is the reverse-Hölder inequality for the first eigenfunction we developed in Section 3. Another key ingredient is, naturally, the Hadamard variation formula for the first eigenvalue. We consider a one-parameter family of diffeomorphisms \(\zeta(t,p) : (-\epsilon, \epsilon) \times M \rightarrow M\), and let \(\Omega_t = \zeta(t, \Omega)\). The family of mappings \(\zeta\) is the flow of the time-dependent vector field \(\chi\), where
\[
\frac{\partial \zeta}{\partial t}(t,p) = \chi(t,p). \quad (4.1)
\]
In this way, if \(\Omega = \Omega_0\) satisfies our standing hypotheses, then so will \(\Omega_t\) for \(t\) sufficiently small.

We let \(\lambda(t) = \lambda(\Omega_t)\), and use a dot to denote differentiation with respect to \(t\). A classical theorem of Hadamard [13] states that
\[
\dot{\lambda}(0) = -\int_{\partial \Omega} \langle \chi, \eta \rangle \left( \frac{\partial \phi}{\partial \eta} \right)^2 d\sigma, \quad (4.2)
\]
where \(\phi\) is the first eigenfunction of \(\Omega\), normalized so that \(\int_\Omega \phi^2 dm = 1\). We include the proof for the reader’s convenience.

Proof. First we compute the time derivative of the boundary terms of the normalized first eigenfunction \(\phi\). Taking a derivative of the condition
\[
\phi(t, \zeta(t,p)) = 0, \; p \in \partial \Omega
\]
with respect to $t$ and using (4.1), we obtain
\[
\dot{\phi}(t, \zeta(t, p)) + \langle \nabla \phi(t, \zeta(t, p)), \chi(p) \rangle = 0.
\]
Here and later, the gradient refers only to the spatial derivative. Set $t = 0$ and use the fact that $\phi$ is constant along $\partial \Omega_t$ to obtain
\[
\dot{\phi}(0, p) = -\langle \nabla \phi(0, p), \chi(p) \rangle = -\left. \frac{\partial \phi}{\partial \eta} \right|_{(0, p)} \eta(p), \chi(p), \quad p \in \partial \Omega.
\]
(4.3)

Next we take the derivative of the eigenfunction equation
\[
\Delta \phi(t, \zeta(t, p)) + \lambda(t) \phi(t, \zeta(t, p)) = 0
\]
with respect to $t$, leading to
\[
0 = \Delta \left[ \dot{\phi} + \langle \nabla \phi, \chi \rangle \right] + \lambda(t) \left[ \dot{\phi} + \langle \nabla \phi, \chi \rangle \right] + \dot{\lambda}(t) \phi
\]
\[
= \Delta \dot{\phi} + \langle \nabla \Delta \phi, \chi \rangle + \lambda(t) \dot{\phi} + \lambda(t) \langle \nabla \phi, \chi \rangle + \dot{\lambda}(t) \phi
\]
\[
= \Delta \dot{\phi} + \lambda(t) \phi_t + \dot{\lambda}(t) \phi.
\]
Setting $t = 0$ and rearranging yields
\[
\Delta \left. \dot{\phi} \right|_{t=0} + \lambda(0) \left. \dot{\phi} \right|_{t=0} = -\dot{\lambda}(0) \phi |_{t=0} \quad \text{in } \Omega.
\]
(4.5)

We multiply (4.4), with $t = 0$, by $\left. \dot{\phi} \right|_{t=0}$ and multiply (4.5) by $\phi$, subtract and obtain
\[
\dot{\lambda}(0) \phi^2(0, p) = \dot{\phi}(0, p) \Delta \phi(0, p) - \phi(0, p) \Delta \dot{\phi}(0, p), \quad p \in \Omega.
\]
(4.6)

Integrate (4.6) over $\Omega$ and use the fact that $\int_{\Omega} \phi^2 dm = 1$ to obtain
\[
\dot{\lambda}(0) = \int_{\Omega} \dot{\phi} \Delta \phi - \phi \Delta \dot{\phi} dm
\]
\[
= \int_{\partial \Omega} \left. \frac{\partial \phi}{\partial \eta} \right|_{\partial \Omega} d\sigma - \int_{\Omega} \langle \nabla \phi, \nabla \phi \rangle d\sigma + \int_{\Omega} \langle \nabla \phi, \nabla \phi \rangle d\sigma - \int_{\partial \Omega} \left. \frac{\partial \phi}{\partial \eta} \right|_{\partial \Omega} d\sigma
\]
\[
= \int_{\partial \Omega} \left. \frac{\partial \phi}{\partial \eta} \right|_{\partial \Omega} d\sigma
\]
\[
= -\int_{\partial \Omega} \left. \frac{\partial \phi}{\partial \eta} \right|_{\partial \Omega} \langle \nabla \phi, \chi \rangle d\sigma
\]
\[
= -\int_{\partial \Omega} \langle \chi, \eta \rangle \left( \frac{\partial \phi}{\partial \eta} \right)^2 d\sigma,
\]
which is equation (4.2) as claimed. In the second equality above we integrated by parts, in the next to last we used (4.3), and at the last step we used the fact that $\phi$ is constant on $\partial \Omega$ (and hence $\nabla \phi = \frac{\partial \phi}{\partial \eta} \eta$ there).

We will need to transform (3.14) and (3.15) for our use later in bounding $\dot{\lambda}$. 

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Lemma 10. Let \((M, g)\) be a complete Riemannian manifold with nonpositive scalar curvature, and let \(\Omega\) be a sufficiently small domain in \(M\) with \(\bar{\Omega}\) compact and \(\partial \Omega \in C^\infty\), so that (2.7) applies. Let \(\phi\) be the first Dirichlet eigenvalue of \(\Delta g\) on \(\Omega\), normalized so that \(\int_{\Omega} \phi^2 dm = 1\). Then

\[
K \lambda^{n/2} \leq \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2,
\]

where \(K\) is the same constant, depending only on \(n\), in (3.14). In dimension two, this inequality reads

\[
4\pi \lambda \leq \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2.
\]

Equality can only occur if \(\Omega\) is isometric to a flat ball in the appropriate dimensional Euclidean space.

Proof. By our normalization we have

\[
K \lambda^{-n/2} = K \lambda^{-n/2} \int_{\Omega} \phi^2 dm \leq \left( \int_{\Omega} \phi dm \right)^2
= \frac{1}{\lambda^2} \left( \int_{\Omega} \Delta \phi dm \right)^2 = \frac{1}{\lambda^2} \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2.
\]

The result follows. \(\square\)

Recall that we have set \(\Omega_t = \zeta(t, \Omega)\), where \(\zeta\) is a one-parameter family of diffeomorphisms on \(M\). We let \(\lambda(t) = \lambda(\Omega_t)\), and we are assuming all the hypotheses relevant to (2.7) hold.

Theorem 11. Let \(\partial \Omega\) move with velocity \(e^w \eta\), where \(\eta\) is the unit outward normal of \(\partial \Omega\) and \(w\) is a bounded and continuous function. If \(n = \dim(M) \geq 3\) then

\[
\frac{d}{dt} \left[ \lambda^{n/2} \right] \leq - \left( \frac{n - 2}{2} \right) \frac{K}{\int_{\partial \Omega} e^{-w} d\sigma},
\]

where \(K\) is the same constant in (3.14), which depends only on \(n\). If \(\dim(M) = 2\) then

\[
\frac{d}{dt} \log(\lambda) \leq - \frac{4\pi}{\int_{\partial \Omega} e^{-w} d\sigma}.
\]

In either case, equality can only occur if \(\Omega\) is isometric to a round ball in the appropriate dimensional Euclidean space.

Proof. By Cauchy-Schwarz,

\[
- \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma = \int_{\partial \Omega} \left( e^{-w/2} \left| \frac{\partial \phi}{\partial \eta} \right| \right) d\sigma \leq \left( \int_{\partial \Omega} e^{-w} d\sigma \right)^{1/2} \left( \int_{\partial \Omega} e^w \left( \frac{\partial \phi}{\partial \eta} \right)^2 d\sigma \right)^{1/2},
\]

so that

\[
\int_{\partial \Omega} e^w \left( \frac{\partial \phi}{\partial \eta} \right)^2 d\sigma \geq \frac{1}{\int_{\partial \Omega} e^{-w} d\sigma} \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2.
\]
We first prove (4.9). Using (4.2), (4.11), and (4.7), we see
\[ -\dot{\lambda} = \int_{\partial \Omega} e^w \left( \frac{\partial \phi}{\partial \eta} \right)^2 d\sigma \geq \frac{1}{\int_{\partial \Omega} e^{-w} d\sigma} \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2 \geq \frac{K\lambda^{2-\frac{2}{n}}}{\int_{\partial \Omega} e^{-w} d\sigma}, \]
which we can rearrange to read
\[ -\frac{d}{dt} \left[ \frac{2}{n-2} \lambda^{\frac{n-2}{2}} \right] = -\lambda^{\frac{n-2}{2}} \dot{\lambda} \geq \frac{K}{\int_{\partial \Omega} e^{-w} d\sigma}. \]

The proof of (4.10) is very similar. This time we replace (4.7) with (4.8) to obtain
\[ -\dot{\lambda} = \int_{\partial \Omega} e^w \left( \frac{\partial \phi}{\partial \eta} \right)^2 \geq \frac{1}{\int_{\partial \Omega} e^{-w} d\sigma} \left( \int_{\partial \Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2 \geq \frac{4\pi \lambda}{\int_{\partial \Omega} e^{-w} d\sigma}, \]
which we can rearrange to read
\[ -\frac{d}{dt} \log \lambda = -\dot{\lambda} \geq \frac{4\pi}{\int_{\partial \Omega} e^{-w} d\sigma}. \]

Now Theorem 1 follows by taking \( w = 0 \), Theorem 2 follows by taking \( w = \log k_g \), and Theorem 3 follows by taking \( w = \log H \).

Finally, we apply our technique to the case that \( \Omega \) is the conformal image of a Euclidean ball. We let \( (M, g) \) be a complete Riemannian manifold of dimension \( n \) with \( S_g \leq 0 \) and let \( F : \mathbb{R}^n \to M \) be a conformal mapping. Let \( B_t \) be the ball of radius \( t \) in \( \mathbb{R}^n \), and let \( \Omega_t = F(B_t) \). Letting \( \lambda(t) = \lambda(B_t) \) and \( \tilde{\lambda}(t) = \lambda(F(B_t)) \), we wish to compare \( \lambda(t) \) to \( \tilde{\lambda}(t) \). As \( t \) increases, \( \partial B_t \) moves with velocity \( \eta = \frac{\partial}{\partial t} \), and (because \( F \) is conformal) \( \partial \Omega \) moves with velocity \( |DF|\tilde{\eta} \), where \( \tilde{\eta} \) is the outward unit normal of \( \Omega_t \). We let \( \phi \) be the first Dirichlet eigenfunction of \( \Delta \) on \( B_t \), normalized so that \( \int_{B_t} \phi^2 dm = 1 \), and let \( \tilde{\phi} \) be the first Dirichlet eigenfunction of \( \Delta_g \) on \( \Omega_t \), normalized so that \( \int_{\Omega_t} \tilde{\phi}^2 dm = 1 \). It will be convenient to define \( \psi = \tilde{\phi} \circ F \), and observe that \( |\nabla \psi| = |DF||\nabla \tilde{\phi}| \).

**Theorem 12.** Let \( F : \mathbb{R}^n \to M \) be conformal, where \( (M, g) \) is complete, with \( S_g \leq 0 \) as above. If \( n = 2 \) then
\[ -\frac{d}{dt} \log(\tilde{\lambda}/\lambda) < 0 \quad (4.12) \]
unless $F$ is an isometry when restricted to $B_t$. If $n \geq 3$, it is small enough so that (2.7) applies to $\Omega_t$, and $\int_{\partial B_t} |DF|^{n-2} d\sigma > |\partial B_t| = n \omega_n t^{n-1}$ then

$$\frac{d}{dt} \left[ \frac{2}{n-2} \tilde{\lambda}^{\frac{n-2}{2}} - \lambda^{\frac{n-2}{2}} \right] < 0. \tag{4.13}$$

Notice that we recover the (a special case of) the Laugesen-Morpurgo result in [17] in dimension two. In higher dimensions, this theorem states that if $F$ is a conformal map with a sufficiently large coformal factor then $\tilde{\lambda}^{\frac{n-2}{2}}$ decreases more rapidly than $\lambda^{\frac{n-2}{2}}$. Thus, our theorem is very much in the spirit of the results in [17] and in [3].

**Proof.** First observe that, because $\partial \Omega_t$ moves with velocity $|DF| \tilde{\eta}$, the Hadamard variation formula becomes

$$\dot{\tilde{\lambda}} = \int_{\partial \Omega_t} |DF| \left( \frac{\partial \tilde{\phi}}{\partial \tilde{\eta}} \right)^2 d\tilde{\sigma} = \int_{\partial B_t} |DF|^{n-2} \left( \frac{\partial \psi}{\partial \eta} \right)^2 d\sigma. \tag{4.14}$$

Thus, in dimension $n \geq 3$, the inequality (4.7) gives

$$K \tilde{\lambda}^{\frac{4-n}{2}} \leq \left( \int_{\partial \Omega_t} |\nabla \tilde{\phi}| d\tilde{\sigma} \right)^2 = \left( \int_{\partial B_t} |DF|^{n-2} |\nabla \psi| d\sigma \right)^2 \leq \int_{\partial B_t} |DF|^{n-2} d\sigma \cdot \int_{\partial B_t} |DF|^{n-2} |\nabla \psi|^2 d\sigma \leq -\tilde{\lambda} \int_{\partial B_t} |DF|^{n-2} d\sigma,$$

which we can rearrange to give

$$-\frac{d}{dt} \left[ \frac{2}{n-2} \tilde{\lambda}^{\frac{n-2}{2}} \right] = -\tilde{\lambda}^{\frac{n-2}{2}} \dot{\lambda} \geq \frac{K}{\int_{\partial B_t} |DF|^{n-2} d\sigma}.$$

However, the equality case in Theorem 1 tells us

$$-\frac{d}{dt} \left[ \frac{2}{n-2} \lambda^{\frac{n-2}{2}} \right] = \frac{K}{|\partial B_t|},$$

so (4.13) now follows from the inequality $\int_{\partial \Omega} |DF|^{n-2} d\sigma > |\partial B_t|$. In the two-dimensional case, we use (4.8) to see

$$4\pi \tilde{\lambda} \leq \left( \int_{\partial \Omega_t} |\nabla \tilde{\phi}| d\tilde{\sigma} \right)^2 = \left( \int_{\partial B_t} |\nabla \psi| d\sigma \right)^2 \leq |\partial B_t| \int_{\partial B_t} |\nabla \psi|^2 d\sigma = -\tilde{\lambda} |\partial B_t|,$$
which we can rearrange to give
\[
- \frac{d}{dt} \log \tilde{\lambda} = - \frac{\dot{\lambda}}{\lambda} \geq \frac{4\pi}{|\partial B_t|} = - \frac{\dot{\lambda}}{\lambda} = - \frac{d}{dt} \log \lambda,
\]
where we have again used the equality case of Theorem 1. This completes the proof of (4.12).

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