Finite-Size Scaling in the $O(n) \phi^4$ Model*

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Abstract

Perturbation theory and renormalization group methods are used to derive a finite-size scaling theory for the partition function zeroes and thermodynamic functions in the $O(n) \phi^4$ model in four dimensions. The leading power–law scaling behaviour is the same as that of the mean field theory. There exist, however, multiplicative logarithmic corrections which are linked to the triviality of the theory.

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1 Introduction

The construction of the continuum limit of a quantum field theory regularized on a Euclidean space–time lattice is equivalent to the study of the critical behaviour of certain statistical physics models. While in the quantum field theory context the $O(4)$–symmetric $\phi^4$ theory (in four dimensions) appears as an essential part of the Higgs sector of the standard model, in statistical physics (below four dimensions) the $O(n)$ theory provides a model of ferromagnetism. Above the upper critical dimension $d = 4$, the scaling behaviour simplifies and the critical exponents are exactly those given by the mean field theory. It is rigorously known that the continuum limit is then trivial and described by free (Gaussian) fields $[1]$. Below four dimensions the super–renormalizable $\phi^4_d$ theory is non–trivial $[2]$.

The action for the $n$–component theory in $d$–dimensional Euclidian space–time continuum is

$$S = \int d^d x \left\{ \frac{1}{2} (\nabla \vec{\phi})^2 + \frac{m_0^2}{2} \vec{\phi}^2 + \frac{g_0}{4!} (\vec{\phi}^2)^2 \right\}, \quad (1.1)$$

where $\vec{\phi}$ is a vector in the $n$–dimensional internal space, $g_0$ the bare quartic self coupling and $m_0$ the bare bosonic mass in the quantum field theory.

Whereas in statistical physics the bare quantities have direct physical meaning, in field theory it is the renormalized quantities which are of interest. Because there is no exact solution for the $O(n)$ $\phi^4$ theory approximations such as perturbation theory (in the renormalized mass and the quartic self coupling), high and low temperature (or coupling) expansions or Monte Carlo (MC) methods have to be used. In $[3]$ the renormalization group (RG) was combined with perturbative methods to analyse the approach of the infinite volume theory to criticality. If $m_0^2$ is written as $m_0^2 + t$, where $m_0$ is the critical bare mass for which the renormalized theory is massless, then $t$ is a measure of the distance from criticality. In the statistical physics analogue the squared mass corresponds to the Boltzmann factor and $t$ is called the reduced temperature. The (perturbative) predictions of $[3]$ for the four dimensional scaling behaviour of the correlation function $\int d^d x \langle \phi(x)\phi(0) \rangle$ and the energy–energy correlation $\int d^d x \langle \phi^2(x)\phi^2(0) \rangle$ (the susceptibility and specific heat respectively in the statistical mechanics analogue) are

$$\chi_{\infty}(t) \sim c_1 |t|^{-1} \ln |t| \left| \frac{m_c}{t} \right|^{2n+8}, \quad (1.2)$$
and
\[ C_\infty(t) \sim c_2 |t|^{\frac{4-n}{n+8}} + \text{const}. \]  
(1.3)

for \( n \neq 4 \). If \( n = 4 \) (1.3) becomes
\[ C_\infty(t) \sim c_3 |\ln |t|| + \text{const.} \]  
(1.4)

In the above formulae the \( c_j \) have the form
\[ c_j = \text{const.} \left\{ 1 + O\left( \frac{\ln |\ln |t||}{\ln |t|} \right) \right\}. \]  
(1.5)

The correlation length in four dimensions is \[ \xi_\infty(t) \sim |t|^{-\frac{1}{2}} |\ln |t||^{\frac{2-n}{n+8}}. \]  
(1.6)

The leading power law scaling behaviour of these thermodynamic functions in four dimensions coincides with the predictions of mean field theory. In the formulae (1.2) to (1.6) there are multiplicative logarithmic corrections which are intimately linked to the triviality of the theory. Indeed, it has been rigorously proven that if any multiplicative logarithmic corrections to the (mean field) scaling behaviour of the susceptibility are present in a continuum limit which is used to construct a \( \phi^4 \) field theory, this limiting theory must be trivial \[ 3 \]. This has been shown to be the case (1.2) and (1.6) have been proven) for the weakly coupled version of the single component theory (i.e., for \( g_0 \) small enough) \[ 4 \]. In the strong coupling limit, \( g_0 \rightarrow \infty \), the \( O(n) \) \( \phi^4 \) model becomes the \( n \)–vector model (also called the non–linear \( \sigma \) model). It is believed that this theory remains in the same universality class as the weakly coupled version. It is therefore important to check the above formulae in a non–perturbative fashion. Such an approach is the MC method which involves the use of stochastic techniques to calculate path integrals.

While for \( n \geq 1 \) computational resources limit one to relatively small lattices, the \( n = 0 \) case introduces significant simplifications. The \( n = 0 \) limit of the \( n \)–vector model is the self avoiding random walk problem. In \[ 4 \] these simplifications were used to verify (1.2) and (1.6) numerically. A numerical analysis along the lines of \[ 4 \] is not feasible for the \( n \geq 1 \) theory. Instead a finite–size extrapolation theory is needed. Finite–size scaling (FSS) was first formulated by Fisher and co–workers \[ 8, 9 \]. This allows one to extract information on the critical behaviour of systems in the thermodynamic limit.
from an analysis of the (finite) size dependency of certain thermodynamic quantities.

A finite–size scaling theory for the partition function zeroes of the single component version of the $\phi^4_4$ model has recently been derived and found to be in quantitative agreement with a numerical analysis of the four dimensional Ising model [10].

The main purpose of the present work is to extend the FSS theory of [10] to the $n$–component $\phi^4_4$ model. The strategy of [10], in which FSS is first established for the zeroes of the partition function and consequently for the thermodynamic functions, is followed here.

The layout of this paper is as follows. In sect.2 the FSS hypothesis and the concept of partition function zeroes are recalled. The (perturbative) renormalization group equation (RGE) for the free energy of a finite–size system is solved in sect.3. This is used to find the FSS of the partition function zeroes and the thermodynamic functions in sect.4 and conclusions are drawn in sect.5.

## 2 Finite–Size Scaling and Partition Function Zeroes

The usual statement of FSS for any thermodynamic quantity which exhibits an algebraic singularity in the infinite system is that near the bulk critical point $t = 0$ [11]

$$\frac{P_l(t)}{P_\infty(t)} = f\left(\frac{l}{\xi_\infty(t)}\right). \tag{2.1}$$

Here the $l$ denotes the linear extent of the system, $\xi_\infty(t)$ is the correlation length of the infinite size system and $f$ is an unknown function of its argument which is called the scaling variable.

FSS in its above form was derived from RG considerations by Brézin below four dimensions [12]. The derivation relies on two assumptions over and above the usual assumptions of RG theory. These are that the system length $l$ is not renormalized in the flow equations and that the infra–red fixed point (IRFP) $g_R^*$ is not zero. This latter assumption fails in four dimensions.

The continuum parameterization of a quantum field theory is recovered from its lattice regularized version at a phase transition of second order. In
1952 Lee and Yang provided an alternative way to understand the onset of a phase transition. The behaviour of all thermodynamic quantities is determined by the (complex) zeroes of the partition function. In the complex plane of the external odd ordering field the Lee–Yang theorem states that all of these zeroes lie on the imaginary axis (when the remaining parameters governing the system are real) \[13\]. These zeroes are called Lee–Yang zeroes to distinguish them from the zeroes in the reduced temperature plane. The study of the latter type of zeroes was first emphasized by Fisher \[9\] and they are referred to as temperature zeroes or Fisher zeroes. For a finite system, for which the partition function is an analytic function of its parameters, there is no phase transition and the zeroes are truly complex (i.e., non–real). A phase transition, which manifests itself as a point of non–analyticity can only arise if the zeroes pinch the real axis in the thermodynamic limit.

Itzykson, Pearson and Zuber managed to connect the concepts of partition function zeroes and the RG thereby deriving a FSS theory for the former below four dimensions \[14\]. A finite size scaling theory for the single component \(\phi^4_4\) theory was developed in \[10\] where it was also shown how to extract the FSS of thermodynamic quantities from the scaling of the Lee–Yang and Fisher zeroes. The primary purpose of the present work is to present the FSS of the \(O(n)\) version (in four dimensions). To this end the FSS behaviour of the partition function zeroes, the zero field magnetic susceptibility and the specific heat are calculated. All of these results are also obtainable from the modified finite–size scaling hypothesis proposed in \[10\].

3 The Renormalization Group Equations

The vacuum to vacuum transition amplitude for the \(n\)-component \(\phi^4_4\) theory in the presence of an external source \(\vec{H}(x)\) represents the evolution of the vacuum via the creation interaction and destruction of particles through the medium of the source. In Euclidean space–time continuum, and in terms of the generating functional for the connected Green’s functions it is

\[
\exp W[\vec{H}, t] \propto \int \prod_x \prod_\alpha d\phi_\alpha(x) \exp (-S), \quad (3.1)
\]
where the constant of proportionality is such that $W[\vec{0}, 0] = 0$, and $\alpha = 1, \ldots, n$ label the field components. The action is

$$S = \int d^d x \left\{ \frac{1}{2} (\nabla \vec{\phi})^2 + \frac{m_0^2}{2} \vec{\phi}^2 + \frac{g_0}{4!} (\vec{\phi}^2)^2 - \frac{t(x)}{2} \vec{\phi}^2(x) + \vec{H}(x) \vec{\phi}(x) \right\},$$

(3.2)

where $t(x)$ is a source for quadratic composite fields. Its inclusion facilitates the derivation of energy–energy correlation functions.

The function conjugate to $\vec{H}$ is

$$M_\alpha = \frac{\delta W[\vec{H}, t]}{\delta H_\alpha(x)}.$$

(3.3)

The generating functional for the one particle irreducible (1PI) vertex functions (or Schwinger functions) is written as $\Gamma[\vec{M}, t]$, and is given by a following Legendre transformation on $W[\vec{H}, t]$;

$$\Gamma[\vec{M}, t] + W[\vec{H}, t] = \int dx \vec{H}(x) \vec{M}(x).$$

(3.4)

This gives

$$H_\alpha[x, t] = \frac{\delta \Gamma[\vec{M}, t]}{\delta M_\alpha(x)}.$$

(3.5)

No difficulties beyond those already encountered in the single component $\phi^4_4$ theory studied in [10] arise in the $n$–component version. The critical theory (given by the value $m_0^2$ of $m_0^2$ for which the renormalized theory is massless) is firstly renormalized. The bulk renormalization constants are sufficient to renormalize the finite–size theory and this renormalization is performed at an arbitrary non–zero mass parameter $\mu$ in order to control infra–red divergences. In four dimensions the vertex function having no external legs and two composite fields has to be additively renormalized and this gives rise to an inhomogeneous term in the RGE. One then applies a Taylor expansion (in $t$ and $\vec{M}$) to the vertex functions to find the RGE for the massive theory (in the critical region). Following [3, 10] the solution for the free energy is (after applying dimensional analysis)

$$\Gamma_R^{(0,0)}(t, \vec{M}, g_R, \mu, l) = l^{-4} \Gamma_R^{(0,0)}(l^2 \lambda, l\vec{M}(\lambda), g_R(\lambda), l\mu(\lambda), 1) + \Pi_n(\lambda; t),$$

(3.6)
where

\[ \Pi_n(\lambda; t) = -\frac{1}{2!} \int_1^\lambda \frac{d\lambda'}{\lambda'} t(\lambda')^2 \Upsilon(g_R(\lambda')) \]  

(3.7)

Here, \( \mu(\lambda) = \lambda \mu \) is a rescaling of the arbitrary mass \( \mu \). The functions \( g_R(\lambda), M_\beta(\lambda), t(\lambda) \) and \( \Upsilon(\lambda) \) respond to this rescaling through the flow equations. To leading order in perturbation theory these flow equations for the \( O(n) \) theory are [8]

\[ \frac{d g_R(\lambda)}{d \ln \lambda} = \frac{n + 8}{6} g_R(\lambda)^2 \{ 1 + O(g_R(\lambda)) \} \ , \]  

(3.8)

\[ \frac{d \ln M_\alpha(\lambda)}{d \ln \lambda} = -\frac{n + 2}{144} g_R(\lambda)^2 \{ 1 + O(g_R(\lambda)) \} \ , \]  

(3.9)

\[ \frac{d \ln t(\lambda)}{d \ln \lambda} = \frac{n + 2}{6} g_R(\lambda) \{ 1 + O(g_R(\lambda)^2) \} \ , \]  

(3.10)

\[ \Upsilon(g_R(\lambda)) = \frac{n}{2} \{ 1 + O(g_R(\lambda)) \} . \]  

(3.11)

For \( \lambda \ll 1 \) the solutions to these flow equations are

\[ g_R(\lambda) = a_1 (-\ln \lambda)^{-1} \ , \]  

(3.12)

\[ t(\lambda) = a_2 t (-\ln \lambda)^{-\frac{n+8}{n+2}} \ , \]  

(3.13)

\[ M_\alpha(\lambda) = b_1 M_\alpha \ , \]  

(3.14)

\[ \Pi_n(\lambda; t) \propto \begin{cases} 
  t^2 \left[ a_3 (-\ln \lambda)^{\frac{n+8}{n+2}} + \text{const.} \right] & \text{for } n \neq 4 \\
  t^2 \left[ b_2 (\ln (-\ln \lambda) + \text{const.}) \right] & \text{for } n = 4 
\end{cases} \]  

(3.15)

where the \( a_j \) have the form

\[ \text{const.} \left\{ 1 + O \left( \ln \left| \ln \lambda \right| \right) \right\} \]  

(3.16)

and the \( b_j \) are

\[ \text{const.} \left\{ 1 + O \left( \frac{1}{\ln \lambda} \right) \right\} \]  

(3.17)

Now, choosing \( \lambda = l^{-1} \) and applying perturbation theory to the homogeneous term of (3.6) as in [10], one finds

\[ \Gamma_R^{(0,0)} (t, \bar{M}, g_R, 1, l) \]

\[ = a'_1 t \bar{M}^2 (\ln l)^{-\frac{n+2}{n+8}} + a'_2 (\bar{M}^2)^2 (\ln l)^{-1} + \Pi_n(l^{-1}; t) \]  

(3.18)
where $a_1'$ and $a_2'$ are of the form (3.16) above with $\lambda$ replaced by $l^{-1}$. Applying (3.5) to this yields for the external field

$$H_\alpha(t, \vec{M}, g_{R, 1}, l) = 2a_1' t M_\alpha (\ln l)^{-\frac{n+2}{n+8}} + 4a_2' (\vec{M}^2) M_\alpha (\ln l)^{-1} .$$

(3.19)

The free energy per unit volume in the presence of an external field is

$$W_t(t, \vec{H}) = \vec{M} \vec{H}(t, \vec{M}; l) - \Gamma_R^{(0,0)}(t, \vec{M}; l) .$$

(3.20)

From (3.18) and (3.19) this is

$$W_t(t, \vec{H}) = a_1' t M_2 (\ln l)^{-\frac{n+2}{n+8}} + 3a_2' (\vec{M}^2)^2 (\ln l)^{-1} - \Pi_n (l^{-1}; t) .$$

(3.21)

From this expression the FSS relations can be derived.

From (3.19) and (3.21), if $t = 0$

$$W_t(0, \vec{H}) = c_1' (\ln l)^{\frac{4}{3}} (\vec{H}^2)^{\frac{4}{3}} ,$$

(3.22)

and if $H = 0$

$$W_t(t, \vec{0}) \propto \begin{cases} t^2 \left[ c_2' (\ln l)^{\frac{4}{n+8}} + \text{const.} \right] & \text{if } n \neq 4 \\ t^2 \left[ c_3' \ln (\ln l) + \text{const.} \right] & \text{if } n = 4 \end{cases} .$$

(3.23)

Here $c_1'$ and $c_2'$ have the form

$$\text{const.} \left\{ 1 + O \left( \frac{\ln (\ln l)}{\ln l} \right) \right\} ,$$

(3.24)

while $c_3'$ is

$$\text{const.} \left\{ 1 + O \left( \frac{1}{\ln l} \right) \right\} .$$

(3.25)

4 FSS for $O(n)$ $\phi^4_4$ Theory

Let $\vec{H}$ define the first direction of internal space such that $H_\alpha = H \delta_{1,\alpha}$. From (3.22) the total free energy at $t = 0$ is $c_1' l^4 (\ln l)^{1/3} H^{4/3}$. The partition function is therefore

$$Z_t(0, \vec{H}) = Q \left( c_1' l^4 (\ln l)^{\frac{1}{3}} H^{\frac{4}{3}} \right) ,$$

(4.1)
for some unknown function $Q$. At a Lee–Yang zero, $H_j$, this vanishes. Following [14] the FSS formula for this zero is found to be

$$H_j \sim l^{-3}(\ln l)^{-\frac{1}{4}} \left\{1 + O \left( \frac{\ln (\ln l)}{\ln l} \right) \right\}.$$  \hspace{1cm} (4.2)

Similar considerations can be applied to the Fisher zeroes. The free energy for vanishing $H$ is given by (3.23). Setting the corresponding partition function to zero and solving for $t$ gives the following FSS formula for the Fisher zeroes in four dimensions;

$$t_j^{-2} \sim \begin{cases} l^4 [c'_2 (\ln l)^{\frac{n-4}{n+8}} + \text{const.}] & \text{if } n \neq 4 \vspace{0.1cm} \\
^4 [c_3' \ln (\ln l) + \text{const.}] & \text{if } n = 4 \end{cases}.$$ \hspace{1cm} (4.3)

This completes the list of finite–size formulae for the partition function zeroes in four dimensions. The $n$–dependence of the FSS behaviour of the Lee–Yang zeroes is found in the additive corrections to (4.2).

Knowledge of the scaling behaviour of the partition function zeroes is equivalent to knowing the scaling behaviour of the partition function itself and of all derivable thermodynamic quantities. Writing the partition function as a product over its Lee–Yang zeroes,

$$Z_l(t, \vec{H}) \propto \prod_j (H - H_j),$$ \hspace{1cm} (4.4)

the magnetic susceptibility, given by the second derivative of the free energy with respect to $H$, is

$$\chi_l(t, \vec{H}) \propto \frac{1}{l^4} \sum_j \frac{1}{(H - H_j)^2}.$$ \hspace{1cm} (4.5)

The zero field susceptibility is then

$$\chi_l \left( t, 0 \right) \propto \frac{1}{l^4} \sum_j \frac{1}{H_j^2},$$ \hspace{1cm} (4.6)

which from (4.2) gives the FSS formula

$$\chi_l(0, \vec{H} = \vec{0}) \propto l^2 (\ln l)^{\frac{1}{2}} \left\{1 + O \left( \frac{\ln (\ln l)}{\ln l} \right) \right\}.$$ \hspace{1cm} (4.7)
In terms of the Fisher zeroes (in the absence of an odd external ordering field)
\[ Z_l(t, 0) \propto \prod_j (t - t_j) \quad . \quad (4.8) \]

The specific heat, given by the second derivative of the free energy with respect to \( t \), is
\[ C_l(t) = -\frac{1}{l^4} \sum_j \frac{1}{(t - t_j)^2} \quad . \quad (4.9) \]

At \( t = 0 \), then
\[ C_l(t = 0) = -\frac{1}{l^4} \sum_j \frac{1}{t_j^2} \quad . \quad (4.10) \]

In four dimensions (4.3) gives
\[ C_l(t = 0) \sim \begin{cases} c_2' (\ln l)^{\frac{1+n}{n+1}} + \text{const.} & \text{if } n \neq 4 \\ c_3' \ln (\ln l) + \text{const.} & \text{if } n = 4 \end{cases} \quad . \quad (4.11) \]

It has been known for a long time that FSS in the form (2.1) breaks down in four dimensions. A modified version of the FSS hypothesis, holding in and below four dimensions, was proposed in [10]. This is
\[ P_l(0) \sim f \left( \frac{\xi_l(0)}{\xi_\infty(t)} \right) \quad . \quad (4.12) \]

Below four dimensions where \( \xi_l \propto l \) this reduces to (2.1). In four dimensions there exist multiplicative logarithmic corrections to the finite volume correlation length too. To leading order this is [12]
\[ \xi_l(0) \sim l (\ln l)^{\frac{1}{4}} \quad . \quad (4.13) \]

This modified hypothesis succeeds in recovering the above FSS formulae for the zero field susceptibility and the specific heat in four dimensions.

5 Conclusions

Renormalization group techniques and perturbation theory have been used to derive the finite–size scaling behaviour of the \( O(n) \phi^4 \) theory in four dimensions.
In the four dimensional version of the theory there appear certain subtleties not present below four dimensions. The IRFP of the Callan–Symanzik function $B(g_R)$ moves to the origin as the dimension becomes four (the perturbative approach predicts that the theory is trivial). Secondly, in contrast to the $d < 4$ dimensional case, the fixed point is now a double zero and this is responsible for the occurrence of logarithmic corrections. A third difference comes from the inhomogeneous term in the RGE. The graph responsible for this term is not divergent in less than four dimensions where singular behaviour comes from the homogeneous term. In four dimensions the inhomogeneous term can also contribute to the leading scaling behaviour.

The multiplicative logarithmic corrections to the leading mean field FSS behaviour are, then, of primary interest. While the leading logarithmic corrections are dependent on the number of field components $n$ for the Fisher zeroes (and specific heat), they are $n$–independent in the case of the Lee–Yang zeroes (and the magnetic susceptibility). Any future numerical FSS analysis of this $n$–dependence should therefore be in terms of Fisher zeroes or even thermodynamic functions.

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