Whitham Deformations and Tau Functions in 
$N = 2$ Supersymmetric Gauge Theories *

Kanehisa Takasaki
Department of Fundamental Sciences, Kyoto University
Yoshida, Sakyo-ku, Kyoto 606-8501, Japan
E-mail: takasaki@yukawa.kyoto-u.ac.jp

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Abstract
We review new aspects of integrable systems discovered recently in
$N = 2$ supersymmetric gauge theories and their topologically twisted
versions. The main topics are (i) an explicit construction of Whitham
deformations of the Seiberg-Witten curves for classical gauge groups, (ii)
its application to contact terms in the $u$-plane integral of topologically
twisted theories, and (iii) a connection between the tau functions and the
blowup formula in topologically twisted theories.

1 Introduction

The Seiberg-Witten low energy effective action of four-dimensional
$N = 2$ supersymmetric gauge theories (with and without matters) is described by the
geometry of a family of complex algebraic curves (the Seiberg-Witten curves)
$C$ fibered over the Coulomb moduli space $U$. Each curve is equipped with a
meromorphic differential (the Seiberg-Witten differential) $dS_{SW}$. This differential
induces a “special geometry” on $U$ with special coordinates $a_j$ and their
duals $a_j^D$ defined by the period integrals of $dS_{SW}$ along suitable cycles $\alpha_j$ and
$\beta_j$ on $C$. The prepotential $F$ of this special geometry determines the effective
action.

Shortly after the original $SU(2)$ theories of Seiberg and Witten were general-
ized to other gauge groups [1, 2, 3, 4, 5, 6, 7], a close connection with integrable
systems was discovered [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] as a guiding principle for the study of
$N = 2$ supersymmetric gauge theories. This connection appears in two different
aspects. The first aspect is a direct relation with integrable systems such

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as the periodic Toda systems and the elliptic Calogero-Moser systems. In this connection, the Seiberg-Witten curves are interpreted to be the spectral curves of a finite-dimensional integrable system, the special coordinates are nothing but the action variables, and the electro-magnetic duality of the abelianized theory is encoded into the structure of an associated polarized Abelian variety. Thus the building blocks of the Seiberg-Witten effective theory fit well to the general setup of integrable systems, in particular, of “algebraically integrable systems” [30, 31]. The second aspect is the notion of Whitham deformations of the spectral curve. This leads to a new interpretation of recursion relations of instanton expansion [32] and the so called RG equation [33, 34].

In this paper, we review recent developments in these lines [35, 36, 37, 38, 39, 40, 41, 42, 43], mostly focussed on the connection with topologically twisted $N = 2$ theories [44, 45, 46]. Section 2 is a short summary of the fundamental geometric stuff (complex algebraic curves, differentials, special coordinates and prepotentials). Section 3 is concerned with an explicit construction of Whitham deformations, which has been applied to instanton expansion and soft breaking of $N = 2$ supersymmetry, as well as the topological theories. Section 4 deals with the relation to the topological theories in more detail. Here a central role is played by the notion of tau functions.

2 Seiberg-Witten Geometry for Classical Gauge Groups

2.1 Seiberg-Witten geometry for $SU(\ell + 1)$

The Seiberg-Witten curve $C$ for the gauge group $SU(\ell + 1)$ is a complex algebraic curve of the form

$$z + \frac{\mu^2}{z} = P(z) = x^{\ell+1} - \sum_{j=2}^{\ell+1} u_j x^{\ell+1-j}, \quad (2.1)$$

where $\mu$ denotes the power $\Lambda^{\ell+1}$ of the renormalization group parameter $\Lambda$, and $u_j$’s are the Coulomb moduli. This is a hyperelliptic curve of genus $\ell$. By the simple transformation

$$z = \frac{P(x) + y}{2}, \quad (2.2)$$

the above equation can be indeed converted to the usual hyperelliptic equation

$$y^2 = P(z)^2 - 4\mu^2. \quad (2.3)$$

The prepotential of the Seiberg-Witten effective theory is defined by use of special coordinates $a_j, a_j^D (j = 1, \cdots, \ell)$ on the Coulomb moduli space $U$. These
special coordinates are given by the period integrals

\[ a_j = \frac{1}{2\pi i} \oint_{\alpha_j} dS_{SW}, \quad a_j^D = \frac{1}{2\pi i} \oint_{\beta_j} dS_{SW}, \quad (2.4) \]

where \( \alpha_j \) and \( \beta_j \) are a symplectic basis of cycles on the curve \( C \), and \( dS_{SW} \) the Seiberg-Witten differential

\[ dS_{SW} = \frac{dz}{z}. \quad (2.5) \]

The integrals \( a_j \) and \( a_j^D \) give, respectively, a set of local coordinates on the Coulomb moduli space. If one interpret \( a_j \) as such coordinates and the duals \( a_j^D \) as a function of the former, one can define the prepotential \( F = \mathcal{F}(a_1, \ldots, a_\ell) \) as a solution of the equations

\[ \frac{\partial F}{\partial a_j} = a_j^D. \quad (2.6) \]

Furthermore, the second derivatives of \( F \) reproduce the period matrix \( \mathcal{T} = (T_{jk})_{j,k=1,\ldots,\ell} \):

\[ \frac{\partial^2 F}{\partial a_j \partial a_k} = \mathcal{T}_{jk} = \frac{1}{2\pi i} \oint_{\beta_j} d\omega_k. \quad (2.7) \]

Here \( d\omega_j \ (j = 1, \ldots, \ell) \) are a basis of holomorphic differentials on \( C \) normalized as

\[ \frac{1}{2\pi i} \oint_{\alpha_j} d\omega_k = \delta_{jk}. \quad (2.8) \]

This matrix \( \mathcal{T} \) appears in the analytic expression

\[ \text{Jac}(C) \simeq \mathbb{C}^\ell/(\mathbb{Z}^\ell + \mathcal{T} \mathbb{Z}^\ell) \quad (2.9) \]

of the Jacobi variety \( \text{Jac}(C) \) of the curve \( C \).

In the context of integrable systems, the curve \( C \) is nothing but the spectral curve \( \det(L(z) - xI) = 0 \) of a Lax representation of the \( \ell \)-periodic Toda system, i.e., the affine Toda system of the \( A_{\ell+1}^{(1)} \) type; the Lax pair is constructed in the vector representation of \( sl(\ell + 1) \). It is well known \[17, 18, 19, 20\] that the dynamics of this Toda system is mapped to linear flows on the Jacobi variety \( \text{Jac}(C) \). In a more abstract language, the \( 2\ell \)-dimensional total space of the fiber bundle of the Jacobi varieties over the Coulomb moduli space \( \mathcal{U} \) becomes an algebraically integrable system \[30, 31\].

\[ ^1 \text{As opposed to my previous papers} \ [36, 39], \text{I have inserted the factor} \ "1/(2\pi i)" , \text{which is rather standard in the physical literature. Similar changes of convention have been done throughout this paper.} \]
2.2 Seiberg-Witten curves for other classical gauge groups

The Seiberg-Witten curves for other classical gauge groups (i.e., orthogonal and symplectic groups) can be written

\[ z + \frac{\mu^2}{z} = W(x), \]  

where \( W(x) \) is a polynomial or a Laurent polynomial of the form

\[
\text{SO}(2\ell + 1) : W(x) = x^{-1} \left( x^{2\ell} - \sum_{j=1}^{\ell} u_j x^{2\ell - 2j} \right),
\]

\[
\text{Sp}(2\ell) : W(x) = x^{2\ell} - \sum_{j=1}^{\ell} u_j x^{2\ell - 2j} + 2\mu.
\]

\[
\text{SO}(2\ell) : W(x) = x^{-2} \left( x^{2\ell} - \sum_{j=1}^{\ell} u_j x^{2\ell - 2j} \right).
\]  

In the case of \( SU(\ell + 1) \) and \( SO(2\ell) \), \( W(x) \) coincide with the superpotential of the topological Landau-Ginzburg models (or \( d < 1 \) strings) for the \( A_\ell \) and \( D_\ell \) isolated singularities [51, 52]. Guided by this analogy, basic notions in the topological Landau-Ginzburg models, such as the Gauss-Manin system and the WDVV equations, have been generalized to the Seiberg-Witten effective theories [53, 54, 55, 56, 57, 58].

These curves coincide with the spectral curve \( \det(L(z) - xI) = 0 \) of a Lax pair (in minimal dimensions) of an affine Toda system, namely, the affine Toda system associated with the dual Lie algebra \( \mathfrak{g}(1)^{\vee} \) of the affine algebra \( \mathfrak{g}(1) \) of the gauge group \( G \) [10]. Note that the dual Lie algebra \( \mathfrak{g}(1)^{\vee} \) for the non-simply laced gauge groups \( SO(2\ell + 1) \) and \( Sp(2\ell) \) is a twisted affine algebra.

All these curves are hyperelliptic. One can find an equivalent expression in the usual expression \( y^2 = R(x) \) of hyperelliptic curves:

\[
\text{SO}(2\ell + 1) : y^2 = Q(x^2)^2 - 4\mu^2 x^2, \quad z = (Q(x^2) + y)/2x.
\]

\[
\text{Sp}(2\ell) : y^2 = Q(x^2)(x^2 Q(x^2) + 4\mu), \quad z = (2\mu + x^2 Q(x^2) + xy)/2.
\]

\[
\text{SO}(2\ell) : y^2 = Q(x^2)^2 - 4\mu^2 x^4, \quad z = (Q(x^2) + y)/2.
\]

Actually, it is in this form (or the quotient curve \( C' \) discussed later on) that the relevant complex algebraic curves for classical gauge groups were first derived [59, 60, 61, 62, 63, 64].

2.3 Involutions and Prym varieties

The curves \( C \) for the orthogonal and symplectic gauge groups have two involutions:

\[
\sigma_1 : (x, y) \mapsto (x, -y), \quad (x, z) \mapsto (x, \mu^2/z),
\]

\[
\sigma_2 : (x, y) \mapsto (-x, y), \quad (x, z) \mapsto (-x, z).
\]
The first involution is the hyperelliptic involution, which also exists on the curve for \(SU(\ell + 1)\); the second involution is a characteristic of the other cases. The quotient \(C_2 = C/\sigma_2\) by the second involution is again a hyperelliptic curve. Some fundamental properties of these curves are summarized in the following table.

| \(G\)          | genus(\(C\)) | genus(\(C_2\)) | covering \(C \to C_2\) |
|-----------------|--------------|-----------------|-----------------------|
| \(Sp(2\ell)\)   | \(2\ell\)   | \(\ell\)       | unramified            |
| \(SO(2\ell + 1), SO(2\ell)\) | \(2\ell - 1\) | \(\ell - 1\)   | ramified              |

The double covering \(C \to C_2\) determines the Prym variety \(\text{Prym}(C/C_2)\). This is an \(\ell\)-dimensional polarized Abelian variety, which plays the role of the Jacobi variety \(\text{Jac}(C)\) for the \(SU(\ell + 1)\) gauge theory. Let us specify the structure of this Prym variety in some detail [59].

In the complex analytic language, \(\text{Prym}(C/C_2)\) is a complex torus of the form

\[
\text{Prym}(C/C_2) \simeq \mathbb{C}^\ell / (\Delta \mathbb{Z}^\ell + 2\mathcal{P} \mathbb{Z}^\ell),
\]

where \(\Delta\) is a diagonal matrix \(\Delta = \text{diag}(d_1, \cdots, d_N)\) with positive integers on the diagonal line, and \(\mathcal{P}\) is a complex symmetric matrix \((\mathcal{P}_{jk})\) with positive definite imaginary part. The diagonal elements of \(\Delta\) represent the polarization:

\[
\begin{align*}
Sp(2\ell) & : (d_1, \cdots, d_\ell) = (2, \cdots, 2, 2), \\
SO(2\ell + 1), SO(2\ell) & : (d_1, \cdots, d_\ell) = (2, \cdots, 2, 1).
\end{align*}
\]

The matrix elements of \(\mathcal{P} = (\mathcal{P}_{jk})\) are given by the period integrals

\[
\mathcal{P}_{jk} = \frac{d_j}{4\pi i} \oint_{\beta_j} d\omega_k,
\]

where \(d\omega_j (j = 1, \cdots, \ell)\) are holomorphic differentials on \(C\) that are “odd” under the action of \(\sigma_2\), i.e.,

\[
\sigma_2^* d\omega_j = -d\omega_j,
\]

and uniquely determined by the normalization condition

\[
\frac{1}{2\pi i} \oint_{\alpha_j} d\omega_k = \delta_{jk}.
\]

The cycles \(\alpha_j, \beta_j (j = 1, \cdots, \ell)\) in these period integrals have to be chosen as follows:

- For \(Sp(2\ell)\): The \(4\ell\) cycles \(\alpha_j, -\sigma_2(\alpha_j), \beta_j, -\sigma_2(\beta_j) (j = 1, \cdots, \ell)\) form a symplectic basis of cycles of \(C\).
- For \(SO(2\ell + 1)\) and \(SO(2\ell)\): The homology classes \([\alpha_\ell]\) and \([\beta_\ell]\) are “odd” under the action of \(\sigma_2\), i.e., \(\sigma_2([\alpha_\ell]) = -[\alpha_\ell]\) and \(\sigma_2([\beta_\ell]) = -[\beta_\ell]\). The \(4\ell - 2\) cycles \(\alpha_j, -\sigma_2(\alpha_j), \beta_j, -\sigma_2(\beta_j) (j = 1, \cdots, \ell - 1)\) and \(\alpha_\ell, \beta_\ell\) altogether form a symplectic basis of cycles of \(C\).

In particular, these cycles have the intersection numbers \(\alpha_j \cdot \beta_k = \delta_{jk}, \alpha_j \cdot \alpha_k = \beta_j \cdot \beta_k = 0\).
2.4 Special coordinates and prepotential

The Seiberg-Witten differential for the orthogonal and symplectic gauge groups is given by

\[ dS_{SW} = \frac{x}{z} \frac{dS_{SW}}{z} = \frac{xW'(x)dx}{\sqrt{W(x)^2 - 4\mu^2}}. \]  

(2.18)

This differential, like \( d\omega_j \), is odd under the action of \( \sigma_2 \):

\[ \sigma_2^*dS_{SW} = -dS_{SW}. \]  

(2.19)

Given a set of cycles \( \alpha_j, \beta_j \) as mentioned above, one can define the special coordinates \( a_j \) and their duals \( a^D_j \) on the Coulomb moduli space \( \mathcal{U} \) by the period integrals

\[ a_j = \frac{1}{2\pi i} \oint_{\alpha_j} dS_{SW}, \quad a^D_j = \frac{1}{2\pi i} \oint_{\beta_j} dS_{SW}. \]  

(2.20)

The prepotential \( \mathcal{F} = \mathcal{F}(a_1, \ldots, a_\ell) \) is again characterized by the differential equation

\[ \frac{\partial \mathcal{F}}{\partial a_j} = a^D_j. \]  

(2.21)

The matrix elements \( P_{jk} \) of \( \mathcal{P} \) can be expressed as second derivatives of the prepotential:

\[ \frac{\partial^2 \mathcal{F}}{\partial a_j \partial a_k} = P_{jk}. \]  

(2.22)

2.5 Quotient curve of genus \( \ell \)

The Prym variety \( \text{Prym}(C/C_2) \) can be identified, up to isogeny, with the Jacobi variety \( \text{Jac}(C') \) of the quotient curve \( C' = C/\sigma' \) obtained by the following involution:

\[ \begin{align*}
\text{Sp}(2\ell) & : \quad \sigma' = \sigma_2, \\
\text{SO}(2\ell + 1), \text{SO}(2\ell) & : \quad \sigma' = \sigma_1\sigma_2.
\end{align*} \]  

(2.23)

The quotient curve \( C' \) is also hyperelliptic and has genus \( \ell \). The holomorphic differentials \( d\omega_j \), as well as \( dS_{SW} \), are “even” (i.e., invariant) under the action of \( \sigma' \), so that they are the pull-back of differentials on \( C' \). The matrix \( \mathcal{P} \) is actually the period matrix of \( C' \):

\[ \text{Jac}(C') \cong \mathcal{C}/(\mathbb{Z}^{\ell} + \mathcal{P}\mathbb{Z}^{\ell}). \]  

(2.24)

To find an explicit expression of \( C' \), one can use the following invariants \( \xi \) and \( \eta \) of \( \sigma' \):

\[ \begin{align*}
\text{Sp}(2\ell) & : \quad \xi = x^2, \quad \eta = y, \\
\text{SO}(2\ell + 1), \text{SO}(2\ell) & : \quad \xi = x^2, \quad \eta = xy.
\end{align*} \]  

(2.25)
In terms of these coordinates, \( C' \) can be written as follows:

\[
SO(2\ell + 1) : \eta^2 = \xi \left( Q(\xi^2) - 4\mu^2\xi \right).
\]

\[
Sp(2\ell) : \eta^2 = Q(\xi) \left( \xi Q(\xi) + 4\mu \right).
\]

\[
SO(2\ell) : \eta^2 = \xi \left( Q(\xi)^2 - 4\mu^2\xi^2 \right).
\]

Let us compare the two curves \( C \) and \( C' \). The curve \( C \) is a double covering of the \( x \)-sphere and has two points \( P^+_{\infty} \) at infinity above \( x = \infty \). These two points correspond to \( z = \infty \) and \( z = 0 \), and mapped to each other by the hyperelliptic involution \( \sigma_1 \). This is a characteristic of the spectral curves of affine Toda systems \([47, 48, 49, 50]\). The curve \( C' \), in contrast, has a single point at infinity above the \( \xi \)-sphere, so that \( C' \) is branched over \( \xi = \infty \). Hyperelliptic curves of this type arise in the KdV hierarchy \([47, 60, 61]\). It is well known that the KdV hierarchy is a specialization the KP hierarchy \([63, 64, 65]\), in which only “odd” time variables \( t_{2n-1} \) remain non-trivial among all possible flows \( t_1, t_2, t_3, \cdots \) of the KP hierarchy. We shall see a similar structure later on in Whitham deformations.

### 3 Construction of Whitham Deformations

#### 3.1 What are Whitham deformations?

In all the aforementioned cases, the Seiberg-Witten differential \( dS_{SW} \) plays the role of a “generating differential”, namely, differentiating against the moduli \( u_j \) give a basis of (odd) holomorphic differentials:

\[
\left. \frac{\partial}{\partial u_j} dS \right|_{z = \text{const.}} = dv_j.
\]

Here “\( \cdot \)\( _{z = \text{const.}} \)” means differentiating while leaving \( z \) constant. For instance, in the case of \( SU(\ell + 1) \), the following holomorphic differentials \( dv_j \) are thus reproduced:

\[
dv_j = \frac{x^{\ell+1-j}dx}{y} \quad (j = 2, \cdots, \ell + 1).
\]

If we use \( a_j \) as independent variables, the outcome are the normalized holomorphic differentials:

\[
\left. \frac{\partial}{\partial a_j} dS \right|_{z = \text{const.}} = d\omega_j.
\]

Note that the moduli \( u_j \) in this situation are a function \( u_j = u_j(\vec{a}) \) of \( \vec{a} = (a_1, \cdots, a_\ell) \), which gives an inverse of the period map \( \vec{u} \mapsto \vec{a}, \vec{u} = (u_2, \cdots, u_{\ell+1}) \). The curve \( C \) is now deformed as \( \vec{a} \) varies, \( C = C(\vec{a}) \).
“Whitham deformations” are an extension of these deformations with new “time variables” $T_n (n = 1, 2, \cdots)$. More precisely, we consider the following setup.

- The moduli $u_j$ of the curve $C$ are deformed as a function $u_j = u_j(\vec{a}, \vec{T})$ of $\vec{a}$ and $\vec{T} = (T_1, T_2, \cdots)$. This induces a family of deformations $C \to C(\vec{a}, \vec{T})$ of the curve $C$. At $\vec{T} = (1, 0, 0, \cdots)$, they are required to reduce to the Seiberg-Witten family. It should be noted that, apart from this “Seiberg-Witten point”, the $a_j$’s are no longer identical to the special coordinates for the deformed curve $C(\vec{a}, \vec{T})$ defined by the period integrals of the Seiberg-Witten differential on this curve.

- The following equations are satisfied:
  \[
  \frac{\partial}{\partial a_j} dS \bigg|_{z=\text{const.}} = d\omega_j, \quad \frac{\partial}{\partial T_n} dS \bigg|_{z=\text{const.}} = d\Omega_n. \quad (3.4)
  \]
  Here $d\Omega_n (n = 1, 2, \cdots)$ are meromorphic differentials of the second kind with poles at $P^\pm_\infty$ only (with some more conditions on the singular part, see below) and vanishing $\alpha$-periods
  \[
  \oint_{\alpha_j} d\Omega_n = 0 \quad (j = 1, \cdots, \ell), \quad (3.5)
  \]
  and $dS$ is a linear combination of these meromorphic differentials and the normalized holomorphic differentials of the form
  \[
  dS = \sum_{n \geq 1} T_n d\Omega_n + \sum_{j=1}^{N} a_j d\omega_j. \quad (3.6)
  \]

- $dS$ reduces to $dS_{SW}$ at $\vec{T} = (1, 0, 0, \cdots)$.

- For the prepotential $F$ to be extended to this family of deformations, the singular behavior of $d\Omega_n$ at $P^\pm_\infty$ (i.e., $z = \infty$ and $z = 0$) should be of the form
  \[
  d\Omega_n = df^\pm_n (z) + \text{non-singular}, \quad (3.7)
  \]
  where $f^\pm_n (z)$ is a polynomial in $z^\pm$ with constant coefficients.

### 3.2 Construction of Whitham deformations for $SU(\ell + 1)$

The solution of Gorsky et al. [35] for the $SU(\ell + 1)$ gauge theory is constructed by the following steps.

1. Consider the meromorphic differentials
  \[
  d\hat{\Omega}_n = R_n(x) \frac{dz}{z}, \quad R_n(x) = \left( P(x)^{n/(\ell + 1)} \right)_+. \quad (3.8)
  \]
Here \((\cdots)_+\) denotes the polynomial part of a Laurent series of \(x\). The fractional power of \(P(x)\) is understood to be a Laurent series of the form \(x^n + \cdots\) at \(x = \infty\). Since \(R_1(x) = x\), \(d\Omega_1\) is nothing but the Seiberg-Witten differential.

2. Consider the differential
\[
dS = \sum_{n \geq 1} T_n d\hat{\Omega}_n \tag{3.9}\]
and its period integrals
\[
a_j = \frac{1}{2\pi i} \oint_{\alpha_j} dS = \sum_{n \geq 1} \frac{T_n}{2\pi i} \oint_{\alpha_j} d\hat{\Omega}_n. \tag{3.10}\]
These period integrals are functions of the moduli \(u_j\) and the deformation parameters \(T_n\). They determine a family of deformations of the Seiberg-Witten period map \(\vec{u} \mapsto \vec{a}\) with parameter \(T_n\).

3. The period map \(\vec{u} \mapsto \vec{a}\) from the \(\vec{u}\)-space to the \(\vec{a}\)-space is invertible if \(\vec{T}\) is close to \((1, 0, 0, \cdots)\), because the Seiberg-Witten period map at this point is invertible. The inverse map \(\vec{a} \mapsto \vec{u} = (u_2(\vec{a}, \vec{T}), \cdots, u_{\ell+1}(\vec{a}, \vec{T}))\) gives deformations of the Seiberg-Witten moduli \(u_j = u_j(\vec{a})\), hence of the curve \(C\), with parameters \(T_n\).

4. The differentials
\[
d\Omega_n = d\hat{\Omega}_n - \sum_{j=1}^{N} c_j^{(n)} d\omega_j, \quad c_j^{(n)} = \frac{1}{2\pi i} \oint_{\alpha_j} d\Omega_n \tag{3.11}\]
satisfy the required normalization condition. \(dS\) thereby becomes a linear combination of \(d\Omega_n\) and \(d\omega_j\) of the required form.

The following property of \(d\Omega_n\) plays a central role in this construction:

- \((\partial/\partial u_j) d\hat{\Omega}_n |_{z=\text{const.}}\) are holomorphic differentials on \(C\).

Note that this is a generalization of the property of the Seiberg-Witten differential that we have mentioned in the beginning of this section. Once this property is established, it is rather straightforward to verify that the other requirements are indeed fulfilled.

Somewhat delicate part is the determination of the Laurent polynomials \(f_n^\pm(z)\) that represent the singular part of \(d\Omega_n\) at \(P_\infty^\pm\). This can be worked out by using the identity
\[
P(x)^{n/(\ell+1)} = \left(z + \frac{\mu^2}{z}\right)^{n/(\ell+1)}. \tag{3.12}\]
The singular part of Laurent expansion of the right hand side at \(z = \infty\) or \(z = 0\) determines the Laurent polynomials \(f_n^\pm(z)\). Obviously the singular part
is a Laurent polynomial with constant coefficients. Accordingly $f^+_n(z)$, too, turn out to have constant coefficients.

Having this solution, we can now define the prepotential $F = F(\vec{a}, \vec{T})$ by the equation

$$\frac{\partial F}{\partial a_j} = \frac{1}{2\pi i} \oint_{\beta_j} dS,$$

$$\frac{\partial F}{\partial T_n} = -\frac{1}{(2\pi i)^2} \oint_{P_+^n} f^+_n(z) dS - \frac{1}{(2\pi i)^2} \oint_{P_-^n} f^-_n(z) dS. \quad (3.13)$$

The second derivatives are also related to period integrals:

$$\frac{\partial^2 F}{\partial a_j \partial a_k} = \frac{1}{2\pi i} \oint_{\beta_j} d\omega_k,$$

$$\frac{\partial^2 F}{\partial a_j \partial T_n} = -\frac{1}{(2\pi i)^2} \oint_{P_+^n} f^+_n(z) d\omega_j - \frac{1}{(2\pi i)^2} \oint_{P_-^n} f^-_n(z) d\omega_j,$$

$$\frac{\partial^2 F}{\partial T_m \partial T_n} = -\frac{1}{(2\pi i)^2} \oint_{P_+^m} f^+_m(z) d\Omega_n - \frac{1}{(2\pi i)^2} \oint_{Q_-^m} f^-_m(z) d\Omega_n. \quad (3.14)$$

In particular, the second derivatives $\partial^2 F / \partial a_j \partial T_n$ are identical to the matrix elements of the period matrix $\mathcal{T}$ for the deformed curve $C(\vec{a}, \vec{T})$. Moreover, by Riemann’s bilinear relation, the mixed derivatives can also be written

$$\frac{\partial^2 F}{\partial a_j \partial T_n} = -\frac{1}{2\pi i} \oint_{\beta_j} d\Omega_n. \quad (3.15)$$

Gorsky et al. [35] further proceeded to calculating these period integrals explicitly. This yields a formula for the second derivatives $\partial^2 F / \partial T_m \partial T_n$ ($m, n = 1, \ldots, \ell$) in terms of a theta function, which is related to contact terms of topologically twisted $N = 2$ theories. We shall return to this issue later on.

### 3.3 Solutions for other classical gauge groups

A new feature that arises in the orthogonal and symplectic gauge groups is the “parity”: The differentials and cycles in the Seiberg-Witten geometry have to respect the parity under the action of the involutions. This is also the case for Whitham deformations. Apart from this feature, it is straightforward to generalize the method of Gorsky et al. [35] to the other classical gauge groups, which we now present below [36].

The first step of the construction is to seek for suitable meromorphic differentials of the form

$$d\Omega_n = R_n(x) \frac{dz}{z}, \quad R_n(x) = \text{polynomial}, \quad (3.16)$$

with the following properties:
\begin{itemize}
  \item $R_n(x)$ is an odd polynomial.
  \item $(\partial/\partial u_j)\hat{\Omega}_n|_{z=\text{const.}}$ are holomorphic differentials on $C$.
\end{itemize}

A solution to this problem can be found by applying the fractional power construction to $W(x)$:

\begin{align*}
SO(2\ell + 1) : \quad R_n(x) &= \left( W(x)^{(2n-1)/(2\ell-1)} \right)_+ , \\
Sp(2\ell) : \quad R_n(x) &= \left( W(x)^{(2n-1)/(2\ell+2)} \right)_+ , \\
SO(2\ell) : \quad R_n(x) &= \left( W(x)^{(2n-1)/(2\ell-2)} \right)_+ . \quad (3.17)
\end{align*}

Note that $R_n(x)$ is a polynomial of the form $x^n + \cdots$. In particular, as in the case of $SU(\ell + 1)$, $d\hat{\Omega}_1$ is nothing but the Seiberg-Witten differential $dS_{SW}$.

The other part of the construction is fully parallel to the case of $SU(\ell + 1)$. We have only to pay an extra attention to the parity. It is easy to see that the differentials $dS$, $d\hat{\Omega}_n$, $d\Omega_n$ and $d\omega_j$ are "odd" under the action of the involution of $\sigma_2$.

The involution $\sigma'$ and the quotient curve $C' = C/\sigma'$ lead to an alternative view. The differentials $dS$, $d\Omega_n$, $d\Omega_n$ and $d\omega_j$ are all invariant under the involution $\sigma'$. Accordingly, they actually descend to (or, equivalently, are the pull-back of) differentials on $C' = C/\sigma'$.

The status of the meromorphic differentials $d\Omega_n$ is particularly interesting from the second point of view. These meromorphic differentials have two poles at $P^\pm_\infty$. Since these two points are mapped to the same point $Q_\infty$ ($\xi = \infty$) of $C'$, the corresponding meromorphic differentials on $C'$ have a single pole at $Q_\infty$, and by a direct calculation, one can see that this is a pole of order $2n$. As already mentioned, this is a property shared by the meromorphic differentials that arise in the KdV hierarchy. More precisely, the singular behavior of those meromorphic differentials at $Q_\infty$ is such that

\begin{equation}
\text{(3.18)}
\end{equation}

Our meromorphic differentials $d\Omega_n$ are a linear combination of those "normalized" meromorphic differentials.

4 Application to Topologically Twisted $N = 2$ Theories

4.1 Tau functions and modular transformations

The algebro-geometric tau functions of integrable hierarchies (KP, Toda, etc.) are determined by a set of algebro-geometric data (the so called Krichever data) including a non-singular complex algebraic curve (Riemann surface) $C$ of genus
Such a tau function can be generally written
\[
\tau(\vec{t}) = e^{2\pi i Q(\vec{t})} \Theta\left(\sum_n t_n V_n + c \mid \mathcal{T}\right), \quad \vec{t} = (t_1, t_2, \cdots). \tag{4.1}
\]
Here \(Q(\vec{t})\) is a quadratic form (including linear and constant terms),
\[
Q(\vec{t}) = \frac{1}{2} \sum_{m,n} q_{mn} t_m t_n + \sum_n r_n t_n + r_0, \tag{4.2}
\]
\(V_n = (V_{(n)}^j)_{j=1,\cdots,g}\) and \(c = (c_j)_{j=1,\cdots,g}\) are \(g\)-dimensional vectors, and \(\Theta(w \mid \mathcal{T})\) denotes the Riemann theta function
\[
\Theta(w \mid \mathcal{T}) = \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i n \cdot \mathcal{T} n + 2\pi i n \cdot w\right), \tag{4.3}
\]
where the “\(\cdot\)” means the inner product, e.g., \(n \cdot w = \sum_{j=1}^g n_j w_j\). The constants \(r_n, r_0\) and \(c_j\) are arbitrary, but this is not the case for \(q_{mn}\) and \(V_{(n)}^j\). As we illustrate below, they are given by some period integrals on \(C\).

**Example related to KP hierarchy**

Let us consider the algebro-geometric tau functions of the KP hierarchy in a somewhat non-standard (in the sense specified below) formulation. The algebro-geometric data are a non-singular algebraic curve \(C\) of genus \(g\) with a marked point \(Q_{\infty}\), a local coordinate \(\kappa\) in a neighborhood of \(Q_{\infty}\) with \(\kappa(Q_{\infty}) = 0\), a set of polynomials \(f_n(\kappa)\) \((n = 1, 2, \cdots)\) in \(\kappa^{-1}\) with constant coefficients, and a symplectic basis \(\alpha_j, \beta_j\) \((j = 1, \cdots, g)\) of cycles on \(C\). (This is the setup that we encounter when the Seiberg-Witten geometry for orthogonal and symplectic gauge groups is reformulated in the language of the quotient curve \(C'\).) Let \(d\omega_j\) \((j = 1, \cdots, g)\) be a basis of holomorphic differentials on \(C\) normalized by the condition
\[
\frac{1}{2\pi i} \oint_{\alpha_j} d\omega_k = \delta_{jk}. \tag{4.4}
\]
The period matrix \(\mathcal{T} = (T_{jk})_{j,k=1,\cdots,g}\) is defined by the period integrals
\[
\frac{1}{2\pi i} \oint_{\beta_j} d\omega_k = T_{jk}. \tag{4.5}
\]
Furthermore, a set of meromorphic differentials \(d\Omega_n\) \((n = 1, 2, \cdots)\) are uniquely determined by the following conditions:

- \(d\Omega_n\) is non-singular outside \(Q_{\infty}\), has a pole at \(Q_{\infty}\), and the leading part of singularity at \(Q_{\infty}\) is given by \(df_n(\kappa)\),
\[
d\Omega_n = df_n(\kappa) + \text{non-singular}. \tag{4.6}
\]
• The $\alpha$-periods of $d\Omega_n$ vanish,

$$\oint_{\alpha_j} d\Omega_n = 0 \quad (j = 1, \cdots, g). \quad (4.7)$$

Now the constants $q_{mn}$ and $V_{j}^{(m)}$ are given by contour integrals along a small circle around $Q_\infty$:

$$q_{mn} = - \frac{1}{(2\pi i)^2} \oint_{Q_\infty} f_m(\kappa) d\Omega_n, \quad (4.8)$$

$$V_{j}^{(m)} = - \frac{1}{(2\pi i)^2} \oint_{Q_\infty} f_m(\kappa) d\omega_j.$$

By Riemann’s identity, one can readily see that $q_{mn}$ is symmetric and $V_{j}^{(m)}$ can be rewritten

$$V_{j}^{(m)} = \frac{1}{2\pi i} \oint_{\beta_j} d\Omega_m. \quad (4.9)$$

Remarks

1. In the “standard” formulation of the KP hierarchy, $f_n(\kappa)$ is chosen to be $\kappa^{-n}$, so that the condition on the singular behavior of $d\Omega_n$ at $Q_\infty$ becomes

$$d\Omega_n = d\kappa^{-n} + \text{non-singular}. \quad (4.10)$$

Accordingly, the definition of $q_{mn}$ and $V_{j}^{(m)}$ has to be modified as

$$q_{mn} = - \frac{1}{(2\pi i)^2} \oint_{Q_\infty} \kappa^{-m} d\Omega_n, \quad (4.11)$$

$$V_{j}^{(m)} = - \frac{1}{(2\pi i)^2} \oint_{Q_\infty} \kappa^{-m} d\omega_j.$$

but the other part remains intact. The aforementioned formulation is simply to take an arbitrary set of directional vectors for the time variables $t_n$ in this standard formulation of the KP hierarchy.

2. In particular, suppose that if $C$ is a hyperelliptic curve of the form

$$\eta^2 = R(\xi) = \xi^{2g+1} + c_1 \xi^{2g} + \cdots + c_{2g+1}, \quad (4.12)$$

$Q_\infty$ the point at infinity $\xi = \infty$, and $\kappa$ is given by

$$\kappa = \xi^{-1/2}. \quad (4.13)$$

Then in the standard formulation as mentioned above, all even members $d\Omega_{2n}$ of the meromorphic differentials become exact,

$$d\Omega_{2n} = d\xi^n. \quad (4.14)$$
thereby the directional vectors $V_{2n}$ for the even time variables $t_{2n}$ all vanish. The coefficients $q_{mn}$ of the Gaussian factor also vanish for even indices. This means that all the “even” flows become trivial, so that the KP hierarchy reduces to the KdV hierarchy.

Example related to Toda hierarchy

An immediate generalization of the KP-like setup is such that the algebraic curve has two marked points $P_{\pm\infty}$. Suppose that a local coordinate $\kappa_{\pm}$ at $P_{\pm\infty}$ (with $\kappa_{\pm}(P_{\pm\infty}) = 0$) and a symplectic basis $\alpha_j, \beta_j$ of cycles are given. This is exactly the setup for constructing an algebro-geometric tau function of the Toda hierarchy [68] (which is a hierarchy obtained from the two-dimensional $SU(\infty)$ Toda field theory). A standard formulation is to introduce two series of time variables $t_{\pm n}$ ($n = 1, 2, \ldots$) associated with the two marked points $P_{\pm\infty}$. We now consider a suitable linear combinations $t_n$ of those standard flows. (This is indeed the setup that takes place in the Seiberg-Witten geometry for $SU(\ell+1)$ gauge groups.) This linear combination is specified by a set polynomials $f_{\pm n}(\kappa_{\pm})$ in $\kappa_{\pm}^{-1}$ with constant coefficients. Given these data, one can define meromorphic differentials $d\Omega_n$ by the following conditions:

- $d\Omega_n$ is non-singular outside $P_{\pm\infty}$, has two poles at $P_{\pm\infty}^\pm$, and the leading part of singularity at $P_{\pm\infty}^\pm$ is given by $df_{\pm n}(\kappa_{\pm})$,

$$d\Omega_n = df_{\pm n}(\kappa_{\pm}) + \text{non-singular.} \quad (4.15)$$

- The $\alpha$-periods of $d\Omega_n$ vanish,

$$\oint_{\alpha_j} d\Omega_n = 0 \quad (j = 1, \ldots, g). \quad (4.16)$$

Defining $q_{mn}$ and $V_j^{(m)}$ as

$$q_{mn} = -\frac{1}{(2\pi i)^2} \oint_{P_{\pm\infty}^+} f_{m+}(\kappa_+) d\Omega_n - \frac{1}{(2\pi i)^2} \oint_{P_{\pm\infty}^-} f_{m-}(\kappa_-) d\Omega_n,$$

$$V_j^{(m)} = -\frac{1}{(2\pi i)^2} \oint_{P_{\pm\infty}^+} f_{m+}(\kappa_+) d\omega_j - \frac{1}{(2\pi i)^2} \oint_{P_{\pm\infty}^-} f_{m-}(\kappa_-) d\omega_j, \quad (4.17)$$

one obtains the tau function. Also here, Riemann’s bilinear identity implies that $q_{mn}$ is symmetric and that $V_j^{(m)}$ can be rewritten

$$V_j^{(m)} = \frac{1}{2\pi i} \oint_{\beta_j} d\Omega_m. \quad (4.18)$$

We here briefly mention how these stuff are related to the Whitham deformations. Upon turning on the Whitham deformations, $q_{mn}$ and $V_j^{(m)}$ are
deformed to $\vec{T}$-dependent quantities: $q_{mn} \mapsto q_{mn}(\vec{T})$, $V_j^{(n)} \mapsto V_j^{(n)}(\vec{T})$. Comparing the definition of $q_{mn}$ and $V_j^{(n)}$ with the period integral formulae of the second derivatives of the deformed prepotential $\mathcal{F}(\vec{T})$, one will soon find that

$$q_{mn}(\vec{T}) = \frac{\partial^2 \mathcal{F}(\vec{T})}{\partial T_m \partial T_n}, \quad V_j^{(n)}(\vec{T}) = \frac{\partial^2 \mathcal{F}(\vec{T})}{\partial \alpha_j \partial T_n}.$$ (4.19)

These relations are also crucial in understanding the role of the Whitham deformations and the tau functions in topologically twisted $N = 2$ theories.

We now turn to the modular property of these tau functions and its building blocks. Actually, this issue was studied in the eighties in the context of free fermions on a Riemann surface [69, 70]. The modular property turns out to be model-independent, i.e., apply to the aforementioned general setup without specifying the integrable system, the algebro-geometric data, etc. [36]

The first step is to determine the transformations of the building blocks of the tau functions under the symplectic transformations

$$\beta_j \mapsto A_{jk} \beta_k + B_{jk} \alpha_k, \quad \alpha_j \mapsto C_{jk} \beta_k + D_{jk} \alpha_k, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z})$$ (4.20)

of cycles. This induces the well known transformation

$$\mathcal{T} \mapsto (AT + B)(CT + D)^{-1}$$ (4.21)

of the period matrix $\mathcal{T}$, which stands for the period matrix $\mathcal{P}$ of $\text{Jac}(C')$ if we consider the case of orthogonal and symplectic gauge groups. The normalized holomorphic differentials $d\omega_j$ and the meromorphic differentials $d\Omega_n$ transform as

$$d\omega_j \mapsto [(CT + D)^{-1}]_{kj} d\omega_k, \quad d\Omega_n \mapsto d\Omega_n - \frac{1}{2\pi i} [(CT + D)^{-1} C]_{kj} \oint_{\beta_j} d\Omega_n \cdot d\omega_k.$$ (4.22)

Accordingly, $q_{mn}$ and $V_j^{(n)}$ transform as

$$V_j^{(n)} \mapsto [(CT + D)^{-1}]_{kj} V_k^{(m)}, \quad q_{mn} \mapsto q_{mn} - [(CT + D)^{-1} C]_{kj} V_j^{(n)} V_k^{(m)}.$$ (4.23)

It should be noted that these transformations already signal a possible connection with the contact terms in the $u$-plane integral of topological theories, which are known to obey substantially the same transformations.

The next steps is to combine these transformations with the modular property of theta functions. To this end, we introduce the theta functions with characteristics

$$\Theta[\gamma, \delta](w | \mathcal{T}) = \sum_{n \in \mathbb{Z}^g} \exp \left( \pi i (n + \gamma) \cdot \mathcal{T}(n + \gamma) + 2\pi i (n + \gamma) \cdot (w + \delta) \right).$$ (4.24)
These theta functions are known to obey the following modular transformations:

\[
\Theta[\gamma, \delta]\left(\left(C^T + D\right)^{-1}w \mid (AT + B)(C^T + D)^{-1}\right) = \epsilon \det(C^T + D)^{1/2} \exp(\pi i w \cdot (C^T + D)^{-1}Cw)\Theta[\gamma', \delta'](w \mid T). \tag{4.25}
\]

Here \(\epsilon\) is an eighth root of unity and \([\gamma', \delta']\) a transformed theta characteristic, both of which are determined by the symplectic matrix.

We now examine the modular property of tau functions of the form

\[
\tau_{\gamma, \delta}(\vec{t}) = e^{2\pi i Q(\bar{t})} \Theta[\gamma, \delta]\left(\sum_n t_n V_n \mid T\right). \tag{4.26}
\]

Here (and in the following), \(Q(\bar{t})\) is purely quadratic:

\[
Q(\bar{t}) = \frac{1}{2} \sum_{m,n} q_{mn} t_m t_n. \tag{4.27}
\]

As the identity

\[
\Theta[\gamma, \delta](w \mid T) = e^{2\pi i \gamma \cdot (w + \delta)} \Theta(w + T\gamma + \delta \mid T) \tag{4.28}
\]

implies, this is a special case of the aforementioned tau functions. One may consider the more general tau functions, but these tau functions \(\tau_{\gamma, \delta}(\vec{t})\) turn out to possess a better modular property. From the modular transformations of the theta functions, indeed, we readily find that these tau functions transform as

\[
\tau_{\gamma, \delta}(\vec{t}) \mapsto \epsilon \det(C^T + D)^{1/2} \tau_{\gamma', \delta'}(\vec{t}). \tag{4.29}
\]

Note that the multiplicative factor of the transformation is independent of \(\vec{t}\); this is not the case for the more general tau function. This fact, too, is a key to search for a connection with topological theories.

### 4.2 \(u\)-plane integrals and contact terms in topologically twisted \(N = 2\) theories

The topological twisting of an \(N = 2\) supersymmetric gauge theory gives a topological field theory that detects the topology of the (compactified) four-dimensional space-time \(X\). In the following, we consider the case where \(X\) is simply connected. The topological information is encoded in the correlators of observables, which are obtained by successively applying a descent operator \(G\) to the Casimirs of the scalar field \(\phi\) in the \(N = 2\) vector multiplet, then integrating over a homology cycle of \(X\).

In the simply connected case, the relevant observables are supported by zero- and two-dimensional cycles. The correlators of those zero- and two-cycle observables can be collected into the generating function

\[
Z = \exp\left(\sum_j f_j O_j + \sum_n g_n \bar{I}_n(S)\right), \tag{4.30}
\]
where $\mathcal{O}_k$ and $I_n(S_n)$ are observables supported by a zero-cycle (point) $Q_k$ and a two-cycle (surface) $S_n$, and $f_k$ and $g_n$ are their coupling constants. Thus $\mathcal{O}_k$ is just the value of a Casimir of $\phi$ at $Q_k$ (whose position itself is irrelevant in the correlators), and $I_n(S_n)$ is an integral of the form

$$I_n(S_n) = \int_{S_n} G^2 \mathcal{P}_n,$$

where $\mathcal{P}_n$ is yet another Casimir of $\phi$. In the case of the $SU(2)$ gauge group, these correlators give the Donaldson invariants of $X$ \cite{73, 74}.

If, however, $X$ is a manifold with $b^+_2(X) = 1$ (e.g., complex rational surfaces), those observables loose topological invariance, and exhibit the so called “chamber structure” or “wall crossing phenomena”. Moore and Witten \cite{44} proposed a field theoretical interpretation of this phenomena, postulating that $Z$ is the sum

$$Z = Z_D + Z_U$$

of contributions of the strong coupling singularities and the bulk of the Coulomb moduli space (“$u$-plane”) $U$. They determined an explicit form of $Z_U$ by examining the modular property of the (low energy effective) theory under the $Sp(2\ell, \mathbb{Z})$ duality group ($\ell$ being the rank of the gauge group). This is nothing but the group of symplectic transformations cycles in $C$ that we have discussed. The outcome is an integral of the form

$$Z_U = \int_U \left[ |da\bar{a}| A^\chi B^\sigma \exp(U + \sum_{m,n} g_m g_n S_m \cdot S_n T_{mn}) \right] \Psi,$$

where $A$ and $B$ are given by the formulae

$$A = \alpha (\det \partial u_k / \partial a_j)^{1/2}, \quad B = \beta \Delta^{1/8},$$

$\alpha$ and $\beta$ are some constants, and $\Delta$ is essentially the discriminant of $W(x)^2 - 4\mu^2$; $\chi$ and $\sigma$ are the Euler characteristic and the signature of $X$; $U$ is a collection of the effects of the zero-cycle observables; $S_m \cdot S_n$ is the intersection number of $S_m$ and $S_n$; $T_{mn}$ are the “contact terms” of the two-cycle observables; $\Psi$ is a lattice sum evaluating the photon partition function in the effective $U(1)^\ell$ theory.

Our main concern lies in the contact terms. According to Moore and Witten \cite{44} (for $SU(2)$) and Mariño and Moore \cite{46} (for other gauge groups), the contact terms are uniquely determined by the following properties:

- Under the $Sp(2\ell, \mathbb{Z})$ duality group, they transform as

$$T_{mn} \rightarrow T_{mn} - \frac{1}{4\pi i} \left[ (C T + D)^{-1} C \right]_{jk} \frac{\partial \mathcal{P}_m}{\partial a_j} \frac{\partial \mathcal{P}_n}{\partial a_k}. \quad (4.35)$$

- In the semi-classical (i.e., weak coupling) limit as $\Lambda / a_j \rightarrow 0$,

$$T_{mn} \rightarrow 0. \quad (4.36)$$
Note that \( P_n \) is understood to be a function on \( \mathcal{U} \). Also recall that the matrix \( T \) stands for the relevant \( \ell \)-dimensional Abelian variety — the period matrix \( T \) of \( \text{Jac}(C) \) in the case of \( SU(\ell + 1) \) gauge groups, and the period matrix \( P \) of \( \text{Jac}(C') \) in the case of orthogonal and symplectic gauge groups.

Remarkably, the modular transformations of the contact terms \( T_{mn} \) are essentially of the same form as the modular transformations of \( q_{mn} \). (Note that \((C^T + D)^{-1}C\) is a symmetric matrix.) One might thus naively guess that \( T_{mn} \) is identical to \( q_{mn} \) (up to a multiplicative constant). This comparison also suggests to identify \( \partial P_n/\partial a_j \) with \( V_{j(n)} \). This implies that, upon turning on the Whitham deformations, \( P_n \) are given by

\[
P_n = \text{const.} \left. \frac{\partial F(T)}{\partial T_n} \right|_{T=(1,0,0,\ldots)},
\]

because of the relation \( V_j^{(n)} = \partial^2 F(T)/\partial a_j \partial T_n \) in the Whitham deformations.

A trouble in this naive identification is that \( q_{mn} \) do not fulfill the correct semi-classical property mentioned above\(^1\). A correct identification, as pointed out by Gorsky et al.\(^{[35]}\), is achieved upon subtracting a singular part \( q_{mn}^{\text{sing}} \) from \( q_{mn} \):

\[
T_{mn} = \text{const.}(q_{mn} - q_{mn}^{\text{sing}}).
\]

Of course, the subtracted term \( q_{mn}^{\text{sing}} \) must be modular invariant in order to retain the aforementioned modular property. Gorsky et al. worked out this separation for the \( SU(\ell + 1) \) theory by evaluating the period integrals for \( q_{mn} = \partial^2 F/\partial T_m \partial T_n \), and obtained the formula

\[
q_{mn} - q_{mn}^{\text{sing}} = -\frac{2\beta^2}{(2\pi i)^2} \frac{\partial \log \Theta_E(0 \mid T) \partial \mathcal{H}_{m+1} \partial \mathcal{H}_{n+1}}{\partial a_j \partial a_k},
\]

where \( \beta = 2\ell + 2 \), \( \Theta_E \) is a theta function with an “even” half-characteristic, and \( \mathcal{H}_{n+1} \) is defined by the residue

\[
\mathcal{H}_{n+1} = \frac{\ell + 1}{n} \text{Res}_{x=\infty} P(x)^{n/(\ell + 1)} dx.
\]

Thus in order to complete the identification, we have to interpret \( P_n \) as \( \mathcal{H}_{n+1} \) (up to a multiplicative constant). The singular part \( q_{mn}^{\text{sing}} \) itself is defined by

\[
q_{mn}^{\text{sing}} = -\frac{\beta}{2\pi i} \mathcal{H}_{m+1,n+1},
\]

where

\[
\mathcal{H}_{m+1,n+1} = \frac{\ell + 1}{mn} \text{Res}_{x=\infty} P(x)^{m/(\ell + 1)} d(P(x)^{n/(\ell + 1)}). +
\]

It should be also noted that these calculations are valid for the range of \( m, n \leq \ell \) only.

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\(^1\) I overlooked this problem and wrongly identified \( T_{mn} \) with \( q_{mn} \) in my previous papers \(^{36, 39}\). I would like to take advantage of this opportunity to correct this error.
4.3 Blowup formula and tau functions

Another approach to the evaluation of contact terms is based on the blowup formula. The aforementioned theta formula of contact terms was indeed first derived by Losev et al. [45] using the technique of blowup. The blowup formula was reformulated by Moore and Witten [44] and by Mariño and Moore [46] in the language of the $u$-plane integral. This reveals a close connection to the tau functions $\tau_{\gamma,\delta}(\vec{t})$. In particular, as Marino pointed out recently [43], the theta function in the aforementioned formula of contact terms is actually the same as the theta function in the tau function. We present an outline of these observations below.

The blowup formula relates the topological correlators between $X$, which is now assumed to be a complex algebraic surface with $b_2^1(X) = 1$, and its blowup $\hat{X}$ at a point $Q$ of $X$. The inverse image of $Q$ in the blowup map $\hat{X} \to X$ is a rational curve (called an “exceptional curve”) with self-intersection $-1$. Let $B$ denote its homology class in $H_2(\hat{X}, \mathbb{Z})$. The effect of blowup on the homology is simply to add to $H_2(X, \mathbb{Z})$ a rank one module generated by $B$:

$$H_2(\hat{X}, \mathbb{Z}) = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}B. \quad (4.43)$$

One can now insert a set of observables $I_n(B)$ supported by $B$ into the generating function on $\hat{X}$:

$$Z_{\hat{X}} = \left\langle \exp\left(\sum_k f_k O_k + \sum_n g_n I_n(S) + \sum_n t_n I_n(B)\right)\right\rangle. \quad (4.44)$$

According to the formulation of Mariño and Moore [46], the effect of blowup in the $u$-plane integral is to modify the integrand for $X$ as follows:

1. In accordance with the decomposition of $H_2$ mentioned above, the photon partition function $\Psi_{\hat{X}}$ becomes a product of $\Psi_X$ and a theta function with an even half-characteristic:

$$\Psi_{\hat{X}} = \Theta[0, \delta](\sum_n t_n V_n \mid T)\Psi_X, \quad \delta = (1/2, \cdots, 1/2). \quad (4.45)$$

2. The exponential function containing contact terms undergoes a new contribution from the observables supported by $B$, which appear as the multiplicative factor

$$\exp(-\sum_{m,n} T_{mn} t_m t_n). \quad (4.46)$$

The negative sign in the exponent stems from the self-intersection $B \cdot B = -1$ of the exceptional divisor.

3. The Euler characteristic and the signature change as: $\chi(\hat{X}) = \chi(X) + 1$, $\sigma(\hat{X}) = \sigma(X) - 1$. The measure factor $A^\chi B^\sigma$ is thereby multiplied by

$$AB^{-1} = \det(\partial u_k/\partial a_j)^{1/2} \Delta^{-1/8}. \quad (4.47)$$
Let us consider the product of the first two factors (i.e., the Gaussian and the theta function), which we call \( \hat{\tau}_{0,\delta}(\vec{t}) \):
\[
\hat{\tau}_{0,\delta}(\vec{t}) = \exp\left(-\sum_{m,n} T_{mn}t_m t_n\right) \Theta[0,\delta](\sum_n t_n V_n \mid T).
\] (4.48)

Mariño and Moore \[46\] remarked that this is essentially the tau function of the Toda system. More precisely, as we have mentioned, this differs from the true tau function \( \tau_{0,\delta}(\vec{t}) \) by a Gaussian factor \( \exp(2\pi i Q_{\text{sing}}(\vec{t})) \), where \( Q_{\text{sing}}(\vec{t}) \) is the quadratic form with coefficients \( q_{\text{sing}}^{mn} \). Thus we have
\[
\hat{\tau}_{0,\delta}(\vec{t}) = \exp(-Q_{\text{sing}}(\vec{t})) \tau_{0,\delta}(\vec{t}) = \exp(2\pi i (Q(\vec{t}) - Q_{\text{sing}}(\vec{t}))) \Theta[0,\delta](\sum_n t_n V_n \mid T).
\] (4.49)

Since the subtracted term \( Q_{\text{sing}}(\vec{t}) \) is modular invariant, this modification does not spoil the modular transformation of \( \tau_{0,\delta}(\vec{t}) \); the subtraction is necessary for the correct semi-classical behavior of the integrand of the \( u \)-plane integral.

Moreover, according to Mariño \[43\], the third factor in the blowup process is nothing but the inverse of the zero-value of \( \hat{\tau}_{0,\delta}(\vec{t}) \):
\[
AB^{-1} = \frac{1}{\hat{\tau}_{0,\delta}(0)}, \quad \vec{0} = (0,0,\cdots).
\] (4.50)

The product of the three factor thus boils down to the tau quotient
\[
\frac{\hat{\tau}_{0,\delta}(\vec{t})}{\hat{\tau}_{0,\delta}(\vec{0})} = \frac{\tau_{0,\delta}(\vec{t})}{\tau_{0,\delta}(\vec{0})}.
\] (4.51)

Since \( \tau_{0,\delta}(\vec{t}) \) and \( \tau_{0,\delta}(\vec{0}) \) have the same modular property, the quotient is modular invariant — a property to be required for the consistency of the \( u \)-plane integral. Furthermore, this quotient has to be non-singular in the semi-classical region of the \( u \)-plane. On the basis of these requirements, Mariño \[43\] eventually derives the theta formula
\[
T_{mn} = 2\pi i \frac{\partial}{\partial T_{jk}} \log \Theta[0,\delta](0 \mid T) \left. \frac{\partial \Theta[0,\delta](\vec{t})}{\partial T_{jk}} \right|_{\vec{t}} V^{(m)}_{j} V^{(n)}_{k},
\] (4.52)

thus reproducing the theta formula of Gorsky et al. \[35\] from an entirely different route. This result also also shows that the theta function \( \Theta_E \) in the formula of Gorsky et al. is actually given by \( \Theta[0,\delta] \).

5 Conclusion

We have seen some new aspects of integrable systems discovered in the recent studies of \( N = 2 \) supersymmetric gauge theories and the topologically twisted versions. A particularly impressive lesson is that the combination of Whitham
deformations and tau functions can be a surprisingly powerful tool. This fact is demonstrated in Mariño’s beautiful exposition [43], part of which we have reviewed in the last section.

A number of problems still remain to be addressed. A central issue will be to extend the present approach based on integrable systems to other cases, such as the theories with matter multiplets, exceptional gauge groups, etc. Contact terms and the blowup formula for theories with matter multiplets have been studied to some extent [44, 45, 46]. Perhaps the most intriguing are the theories with an adjoint matter multiplet (equivalently, the mass deformed $N = 4$ theories). The blowup formula in the topological versions of these theories, too, will be described by some tau function (of an elliptic Calogero-Moser system?). Another interesting case can be found in toroidally compactified tensionless strings (also called $E$-strings) and related $N = 2$ or mass deformed $N = 4$ gauge theories [47, 74, 76, 77, 78, 79]. The $E_8$ theta function $\Theta_{E_8}$ arising in those models will be connected with the tau function of a yet unidentified integrable system underlying the Seiberg-Witten curves of those theories.

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