Relativistic spin operator and Dirac equation

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We give a direct link between description of Dirac particles in the abstract framework of unitary representation of the Poincaré group and description with the help of the Dirac equation. In this context we discuss in detail the spin operator for a relativistic Dirac particle. We show also that the spin operator used in quantum field theory for spin $s = 1/2$ corresponds to the Foldy-Wouthuysen mean-spin operator.

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I. INTRODUCTION

The field of relativistic quantum information theory has emerged several years ago [1]. Since then a lot of papers have been published (see e.g. [2, 32]) and different aspects of relativistic quantum information have been studied (including, for example, correlations in vector boson systems [11], correlations of massive particles in helicity formalism [12, 13, 14, 15, 16, 17, 18]).

However, the spin-1/2 massive particles are often considered as the best objects to study relativistic effects on entanglement and violation of Bell-type inequalities. They have been considered in many papers [1, 4, 5, 10, 16, 20, 21, 22]. The most recent ones [23, 24, 25, 26, 27] also discuss relativistic effects in a system of spin-1/2 massive particles.

In some papers such particles are described in the framework of unitary representations of the Poincaré group (e.g. [1, 26]) while other authors use Dirac equation (e.g. [24, 27]).

In the present paper we give a direct link between these two approaches. To this end we formulate Dirac formalism in an abstract Hilbert space which is a carrier space of an unitary representation of the Poincaré group. To include negative energy solutions of the Dirac equation we consider as a Hilbert space the direct sum of carrier spaces of positive and negative energy unitary representations of the Poincaré group for a massive spin-1/2 particle. In this Hilbert space there exists the standard basis (we call it spin basis) defined in the context of unitary representation of the Poincaré group (e.g. [1, 26, 27]).

II. RELATIVISTIC DESCRIPTION OF A DIRAC PARTICLE

A. Dirac equation

Relativistic spin-1/2 particle is described by Dirac equation which has the following form

\[ (i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \]

where bispinor $\psi(x)$ is a four-component column and we have used the standard notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$. For the conventions concerning Dirac matrices and metric tensor see Appendix C. We use natural units with $\hbar = c = 1$.

Dirac equation (1) is covariant under Lorentz transformations $\Lambda$

\[ x'^\mu = \Lambda^\mu_\nu x^\nu. \]

On the level of Dirac bispinors Lorentz transformations are realized as

\[ \psi'(x') = S(\Lambda)\psi(x), \]
where $S(\Lambda)$ is a 4-dimensional, bispinor representation of the Lorentz group and $x' = \Lambda x$. Bispinor representation is generated by $\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$:

$$S(\Lambda) = \exp(i\frac{\omega_{\mu\nu}}{2}\Sigma^{\mu\nu}),$$

therefore

$$S^{-1}(\Lambda) = \gamma^0 S(\Lambda) \gamma^0.$$  (4)

Covariance of the Dirac equation gives the following condition:

$$S^{-1}(\Lambda) \gamma^\mu S(\Lambda) = \Lambda^\mu_{\nu} \gamma^\nu.$$  (6)

We define the invariant scalar product of two Dirac spinors $\psi(x)$ and $\phi(x)$ at time $x_0$ by

$$(\psi, \phi) = \int_{x' = \text{const}} d^3 x \bar{\psi}(x) \gamma^0 \phi(x),$$

where $\bar{\psi}(x) = \psi^\dagger(x) \gamma^0$. This scalar product can be written in a manifestly invariant form

$$(\psi, \phi) = \int d\sigma_\mu(x) \bar{\psi}(x) \gamma^\mu \phi(x),$$

where integration is performed over a space-like surface $\sigma$ and $d\sigma_\mu(x) = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] dx^\nu \wedge dx^\sigma \wedge dx^\lambda$.

**B. Unitary representations of the Poincaré group for a Dirac particle**

As is well known, the Dirac equation possesses negative energy solutions as well as positive energy ones. Therefore, to describe in a consistent manner both types of solutions as a space of states for a spin-1/2 relativistic particle we will take the direct sum of carrier spaces of two unitary, irreducible representations of the Poincaré group, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, corresponding to positive and negative energy, respectively.

The carrier space of a unitary irreducible representation corresponding to the massive, spin-1/2 particle, $\mathcal{H}_\epsilon$ ($\epsilon = +1$ corresponds to positive energy while $\epsilon = -1$ corresponds to negative energy), is spanned by the eigenvectors of the four-momentum operators $P^\mu$

$$P^\mu|e\sigma, \sigma\rangle = \epsilon P^\mu|e\sigma, \sigma\rangle,$$

where $\sigma = \pm 1/2$. The above choice of basis is explained in Appendix A.

We denote by $U(\Lambda)$ an unitary operator representing a Lorentz transformation $\Lambda$. It holds

$$U(\Lambda) = \exp \left(i\frac{\omega_{\mu\nu}}{2}\tilde{J}^{\mu\nu}\right),$$

where $\tilde{J}^{\mu\nu}$ are generators of the Lorentz group.

Action of $U(\Lambda)$ on the basis vectors reads [for derivation see Appendix A]:

$$U(\Lambda)|e\sigma, \sigma\rangle = \mathcal{D}(R(\Lambda, p))_{\lambda\sigma}|e\Lambda p, \lambda\rangle,$$

where $\mathcal{D}$ is the standard unitary spin-1/2 representation of the rotation group, Wigner rotation $R(\Lambda, p) = L_{\Lambda p}^\dagger L_p$ and $L_p$ denotes standard Lorentz transformation which is defined by the conditions: $L_p q = p, L_q = I$ with $q = (m, 0)$.

Let us consider the parity operation $\mathcal{P}$: $\mathcal{P}(p^0, p) = (p^0, -p) \equiv p^\tau$. Action of parity on basis vectors reads

$$U(\mathcal{P})|e\sigma, \sigma\rangle = P^\sigma_{e\lambda}|e\sigma\tau, \lambda\rangle.$$  (12)

Consistency of Eqs. (11) and (12) leads to

$$P^\sigma_{e\lambda} = \xi^\sigma \delta_{\sigma\lambda},$$

where $|\xi^\sigma| = 1$. Therefore, finally

$$U(\mathcal{P})|e\sigma, \sigma\rangle = \xi^\sigma|e\sigma\tau, \sigma\rangle.$$  (13)

We adopt the following, Lorentz-covariant normalization of the basis vectors:

$$\langle e'\sigma', \sigma'|e\sigma, \sigma\rangle = 2\omega(p)\delta_{e', e}\delta_{\sigma', \sigma},$$

where $\omega(p) = \sqrt{p^2 + m^2}$.

We can define in a natural way a hermitian operator $\hat{\mathcal{E}}$ with eigen-values $\epsilon = \pm 1$ and corresponding eigen-vectors $|e\sigma, \sigma\rangle$:

$$\hat{\mathcal{E}} = \frac{\tilde{p}^0}{\omega(p)}.$$  (16)

Indeed, Eq. (10) imply

$$\hat{\mathcal{E}}|e\sigma, \sigma\rangle = \epsilon|e\sigma, \sigma\rangle.$$  (17)

Let us notice that operator corresponding to absolute value of energy can be written as

$$|\tilde{p}^0| = \sqrt{m^2 + \hat{\mathcal{E}}^2} = \hat{\mathcal{E}}\tilde{p}^0.$$  (18)

We will use this observation later on.

One can also check that the following resolution of unity holds

$$\mathbb{1} = \sum \sum \int \frac{d^3 p}{2\omega(p)}|e\sigma, \sigma\rangle\langle e\sigma, \sigma|,$$

where the measure $d^3 p/(2\omega(p))$ is Lorentz-invariant.

**C. Manifestly covariant basis**

The spin basis $\{|e\sigma, \sigma\rangle\}$, although naturally defined in the framework of unitary representation of Poincaré group, is not manifestly covariant [Eq. (11)].

Therefore, we define another basis in the space $\mathcal{H}$

$$|e\sigma, \sigma\rangle = \sum |q_{\alpha\sigma}(p)|e\sigma, \sigma\rangle,$$  (19)

where $\sigma = \pm 1/2$. The above choice of basis is explained in Appendix A.
where we demand
\[ U(\Lambda)|\alpha, \epsilon p\rangle = S^{-1}(\Lambda)_{\alpha\beta}|\beta, \epsilon \Lambda p\rangle. \]  
(21)

In Eq. (21) \(S(\Lambda)\) denotes the bispinor representation of the Lorentz group [compare Eq. (3)]. Therefore, \(\alpha\) is a bispinor index.

Inserting \(\Lambda = \mathbb{P}\) in Eq. (6), we can uniquely determine the bispinor representation of the parity operator
\[ S(\mathbb{P}) = \xi^0, \]  
(22)

where \(|\xi| = 1\). Therefore
\[ U(\mathbb{P})|\alpha, \epsilon p\rangle = \xi^* \sum_\beta \gamma^0_{\alpha\beta}|\beta, \epsilon p^\tau\rangle. \]  
(23)

Of course, phase factors \(\xi\) [Eq. (22)] and \(\xi^*\) [Eq. (14)] are related. Indeed, by virtue of Eqs. (14, 20, 23) we obtain
\[ \xi^* v'(p) = \xi^* \gamma^0 v'(p^\tau), \]  
(24)

where \(v'(p) = [v^\alpha_\sigma(p)]\).

Eqs. (11, 21) imply the following Weinberg consistency condition:
\[ S(\Lambda)v'(p)D^T(R(\Lambda, p)) = v'(\Lambda p). \]  
(25)

By virtue of the above equation we have
\[ v'(p) = S(L_p) v'(q), \]  
(26)

where \(q = (m, 0)\) is the four-momentum in the rest frame of a particle. The Weinberg condition (25) and Eq. (5) imply
\[ D^*(R(\Lambda, p)) \bar{v}'(p) v'(p)D^T(R(\Lambda, p)) = \bar{v}'(\Lambda p)v'(\Lambda p), \]  
(27)

\[ S(\Lambda)v'(p) \bar{v}'(p) S^{-1}(\Lambda) = v'(\Lambda p) \bar{v}'(\Lambda p), \]  
(28)

where \(\bar{v}'(p) \equiv v'^\dagger(p) \gamma^0\).

Now, Eqs. (22, 26, 27, 28, 5) and Schur’s Lemma lead to
\[ v'(p) \bar{v}'(p) = \epsilon \Lambda_\epsilon(p), \]  
(29)

where we have introduced the standard projectors
\[ \Lambda_\epsilon(p) = \frac{mI + \epsilon p \cdot \gamma}{2m}. \]  
(30)

and
\[ \bar{v}'(p)v'(p) = \epsilon \delta_{\epsilon \epsilon'} I_2. \]  
(31)

Eq. (29) imply the following condition:
\[ \sum_\epsilon \epsilon v'(p) \bar{v}'(p) = I. \]  
(32)

The explicit form of amplitudes \(v'(p)\) can be easily determined with help of Eqs. (20, 29, 30, 31) and is given in Appendix D (Eq. (D1)).

Now, using Eqs. (31, D1) we can simplify Eq. (24). We get finally
\[ \xi^* = \epsilon \xi^*. \]  
(33)

The covariant basis vectors fulfill the following normalization condition:
\[ \langle \beta, \epsilon k | \alpha, \epsilon p \rangle = 2\epsilon \omega(p) \delta_{\epsilon \epsilon'} \delta^3(k - p) \langle \Lambda_\epsilon(p) | \beta, \epsilon p \rangle, \]  
(34)

where we have used the natural notation
\[ \langle \beta, \epsilon k \rangle = \sum_\alpha \langle \alpha, \epsilon k | \gamma^0_{\alpha\beta}. \]  
(35)

By virtue of Eq. (31) we can invert relation (20)
\[ |\epsilon p, \sigma\rangle = \sum_\alpha \epsilon \bar{v}^\epsilon_\alpha(p) |\alpha, \epsilon p\rangle. \]  
(36)

Now, using Eqs. (29, 36), we receive
\[ \sum_\beta (\pi_{\alpha\beta})_{\beta, \epsilon p} |\alpha, \epsilon p\rangle = \epsilon \omega(p) |\alpha, \epsilon p\rangle, \]  
(37)

or, in terms of the projectors (30)
\[ \sum_\beta (\Lambda_\epsilon(p))_{\alpha\beta} |\beta, \epsilon p\rangle = |\alpha, \epsilon p\rangle. \]  
(38)

Notice that the above equation is in fact the Dirac equation in momentum representation written in an abstract Hilbert space. Thus, in this approach the Dirac equation is a consequence of the demand of manifest covariance and form of the bispinor representation. Dirac equation (38) can be cast in an operator form
\[ \sum_\beta \langle \bar{P} - mI \rangle_{\alpha\beta} |\beta, \epsilon p\rangle = 0. \]  
(39)

Therefore, Hamiltonian acts on basis vectors as follows:
\[ \bar{P}^0 |\alpha, \epsilon p\rangle = \sum_\beta [\gamma^0(\epsilon p \cdot \gamma + mI)]_{\alpha\beta} |\beta, \epsilon p\rangle \]  
\[ = \sum_\beta \bar{H}_{\alpha\beta} |\beta, \epsilon p\rangle. \]  
(40)

Applying Eq. (39) to Eq. (11) we get
\[ 1 = \sum_\epsilon \sum_\alpha \int \frac{d^3 p}{2\omega(p)} \epsilon(\alpha, \epsilon p) \overline{\epsilon(\alpha, \epsilon p)}. \]  
(41)

D. Functional realization

In this section we construct a functional realization in terms of Dirac bispinors. We consider functional realization with help of covariant [Eq. (20)] as well as spin basis [Eq. (9)].
a. Covariant basis

Let us expand an arbitrary state vector in the covariant basis defined in Eq. (20):

$$|\psi\rangle = \sum_{\epsilon, \alpha} \int \frac{d^3p}{2\omega(p)} C_\psi(p)_\alpha^\epsilon |\alpha, \epsilon p\rangle.$$  

(42)

Using Eqs. (48) and (44) we find

$$\sum_\alpha C_\psi(p)_\alpha^\epsilon (\Lambda_\epsilon(p))_{\alpha\beta} = C_\psi(p)_\beta^\epsilon.$$  

(43)

and

$$C_\psi(p)_\alpha^\epsilon = (\alpha, \epsilon p) |\psi\rangle,$$  

(44)

respectively. We would like to connect the function $C_\psi(p)^\epsilon_{\alpha}$ with Dirac bispinor $\psi^\epsilon_{\alpha}(p)$. The above equations suggest the following identification:

$$C_\psi(p)_\alpha^\epsilon = \overline{\psi}^\epsilon_{\alpha}(p) = \sum_{\beta} \psi^\epsilon_{\beta}(p) \gamma_0^{\beta\alpha}.$$  

(45)

Therefore, we define a bispinor with definite energy ($\epsilon = +1$—positive, $\epsilon = -1$—negative) in momentum representation as follows:

$$\psi^\epsilon_{\alpha}(p) = (\alpha, \epsilon p).$$  

(46)

Using this definition, Eq. (42) takes the following form

$$|\psi\rangle = \sum_\epsilon \sum_\alpha \int \frac{d^3p}{2\omega(p)} \overline{\psi}^\epsilon_{\alpha}(p) |\alpha, \epsilon p\rangle.$$  

(47)

We can easily check that a bispinor (46) fulfills the Dirac equation (compare Eq. (43)),

$$\Lambda_{-\epsilon}(p) \psi^\epsilon(p) = 0,$$  

(48)

where $\psi^\epsilon(p)$ denotes four-component column $[\psi^\epsilon_{\alpha}(p)]$.

We can also check that bispinors defined in Eq. (46) transform properly under Lorentz transformations. Denoting

$$\psi^\epsilon_{\alpha}(p') = (\psi^\epsilon|\alpha, \epsilon p'),$$  

(49)

where

$$U(\Lambda)|\psi\rangle = |\psi'\rangle, \quad p' = \Lambda p,$$  

(50)

we receive the following transformation law:

$$\psi^\epsilon(p') = S(\Lambda) \psi^\epsilon(p),$$  

(51)

[compare Eq. (16)]. So, we can define the most general bispinor in momentum representation as follows

$$\psi_{\alpha}(p) = \sum_\epsilon \psi^\epsilon_{\alpha}(p).$$  

(52)

Finally, action of the Hamiltonian operator on a bispinor $\psi(pp)$ can be determined with help of Eqs. (48) and (49). We get

$$\hat{H} \psi^\epsilon_{\alpha}(p) = \sum_{\beta} \left[ \gamma^0 (\epsilon p \cdot \gamma + mI) \right]_{\alpha\beta} \psi^\epsilon_{\beta}(p).$$  

(53)

We define scalar product of bispinors in the following way:

$$(\psi, \phi) = \langle \phi | \psi \rangle = \sum_{\epsilon} \int \frac{d^3p}{2\omega(p)} \overline{\psi}(p) \phi^\epsilon(p).$$  

(54)

b. Spin basis

Of course, the expansion given in Eq. (42) can be performed in the non-covariant (spin) basis $\{ |\epsilon p, \sigma\rangle \}$, too. In analogy to Eq. (42), we denote

$$\overline{\psi}^\epsilon_{\alpha}(p) = (\psi|\epsilon p, \sigma),$$  

(55)

then

$$|\psi\rangle = \sum_{\epsilon, \sigma} \int \frac{d^3p}{2\omega(p)} \overline{\psi}^\epsilon_{\alpha}(p) |\epsilon p, \sigma\rangle.$$  

(56)

In terms of spinors defined in Eq. (55), the scalar product defined in Eq. (43) takes the form:

$$(\psi, \phi) = \sum_{\epsilon} \int \frac{d^3p}{2\omega(p)} \overline{\psi}^\epsilon_{\alpha}(p) \phi^\epsilon_{\alpha}(p).$$  

(57)

c. Relation between bases

Spinors $\psi^\epsilon_{\alpha}(p)$ and $\overline{\psi}^\epsilon_{\alpha}(p)$ are related via the following relation:

$$\psi^\epsilon_{\alpha}(p) = \sum_\sigma \psi^\epsilon_{\alpha}(p) \psi^\epsilon_{\sigma}(p),$$  

(58)

where we have used Eq. (20).

III. NEWTON-WIGNER POSITION OPERATOR

Problem of defining a proper position operator in the relativistic quantum mechanics has a very long history and no fully satisfactory solution (see e.g., Ref. [36]). In this section we discuss briefly the Newton–Wigner position operator [37] which, although non-covariant, seems to be the best proposition. The Newton–Wigner position operator is assumed to be hermitian, to have commuting components

$$[\hat{X}^i, \hat{X}^j] = 0,$$  

(59)

and to fulfill standard canonical commutation relations with four-momentum operators

$$[\hat{X}^i, \hat{P}^j] = i\delta^{ij}.$$  

(60)

Eqs. (48) imply the following relation

$$e^{ia \cdot \mathbf{p}} |\epsilon p, \sigma\rangle = N(p, \epsilon a) \| \epsilon p(a)\sigma\rangle,$$  

(61)

where we have denoted by $p(\epsilon a)$ a four-vector with components given below

$$p_0(\epsilon a) = \omega(p + \epsilon a) = \sqrt{m^2 + (p + \epsilon a)^2}, \quad \mathbf{p}(\epsilon a) = p + \epsilon a,$$  

(62)
and the normalization factor \( N(p, \epsilon a) \) is equal to
\[
N(p, \epsilon a) = \sqrt{\frac{\omega(p)}{\omega(p + \epsilon a)}} = \left( \frac{p^2 + m^2}{(p + \epsilon a)^2 + m^2} \right)^{1/4}. \tag{63}
\]

Eqs. (61, 68) imply the well-known relation
\[
\hat{X} \psi_\alpha(p) = i \epsilon \left( \nabla_p - \frac{1}{2} \frac{p}{p^2 + m^2} \right) \psi_\alpha(p). \tag{64}
\]

Using Eqs. (20, 58, 61) we find in a bispinor (covariant) basis
\[
e^{ia \cdot X} \psi(\alpha, \epsilon p) = \epsilon N(p, \epsilon a) \times \sum_\beta (\nabla(p^\alpha(p(\epsilon a)))_{\alpha\beta}) \psi_\beta(p(\epsilon a)). \tag{65}
\]

Therefore, for wave functions in a bispinor basis, defined in Eq. (40), we get
\[
\hat{X} \psi_\alpha(p) = -i \sum_\beta \left( (\nabla_p \psi^\alpha(p))_{\alpha\beta} \psi_\beta(p) \right)
+ i \epsilon \left( \nabla_p - \frac{1}{2} \frac{p}{p^2 + m^2} \right) \psi_\alpha(p). \tag{66}
\]

**IV. THE FOLDY-WOUTHEYSEN TRANSFORMATION**

The Foldy-Woutheysen (FW) transformation \[34\] is a canonical transformation which diagonalizes Dirac Hamiltonian given in Eq. (40) or (53). Hamiltonian \[10\] is defined in the covariant basis. Using Eqs. (61, 68) we can find Hamiltonian in the spin basis. We have
\[
\hat{P}^0 \psi_\alpha = \sum_{\alpha, \beta} \epsilon_\alpha \psi_\beta(p) H_D^{\alpha\beta} \psi_\beta. \tag{67}
\]

Therefore, by virtue of Eqs. (12, 13) we finally get
\[
\hat{P}^0 \psi_\alpha = \epsilon \psi_\alpha. \tag{68}
\]

Thus we see that Foldy-Woutheysen transformation corresponds to change of basis from manifestly covariant one to spin one. Foldy-Woutheysen spinors are simply spinors defined in terms of vectors spanning the carrier space of the unitary representation of the Poincaré group.

**V. SPIN OPERATOR**

In this section we clarify some questions concerning spin operator for a Dirac particle.

Spin is an internal degree of freedom. It means that a spin operator should commute with space-time observables like momentum and position. Therefore, choosing as a position operator the Newton–Wigner one, which fulfills the relations \[59 \ 60\], we find that the action of a spin component operator, \( \hat{S}^i \), on a spin basis vectors \(| \epsilon p, \sigma \rangle \) must have the following form:
\[
\hat{S}^i | \epsilon p, \sigma \rangle = \sum_\lambda A^i_{\sigma\lambda} | \epsilon p, \lambda \rangle, \tag{69}
\]

where \( A^i_{\sigma\lambda} \) are constant \( 2 \times 2 \) matrices. Moreover, we demand that spin components fulfill standard commutation relation
\[
[\hat{S}^i, \hat{S}^j] = i \epsilon_{ijk} \hat{S}^k. \tag{70}
\]

Thus, we are lead to
\[
\hat{S}^i | \epsilon p, \sigma \rangle = \frac{1}{2} \sum_\lambda (\sigma^i)_{\sigma\lambda} | \epsilon p, \lambda \rangle, \tag{71}
\]

where \( \sigma_i \) are standard Pauli matrices.

On the other hand, we can try to define spin operator in terms of the generators of the Poincaré group. We know that spin square operator is well defined in the unitary representation of the Poincaré group and has the following form
\[
\hat{S}^2 = -\frac{1}{m^2} \hat{W}^\mu \hat{W}_\mu, \tag{72}
\]

where \( \hat{W}^\mu \) is the Pauli-Lubanski (pseudo)four-vector
\[
\hat{W}^\mu = \frac{1}{2} \epsilon^{\alpha\beta\mu} \hat{P}_\alpha \hat{J}_\beta, \tag{73}
\]

and \( \hat{J}_{\alpha\beta} \) are generators of the Lorentz group. Therefore, taking into account that spin is a pseudo-vector, it is natural to look for a spin operator which is a linear function of components of \( \hat{W}^\mu \). If we assume that a spin operator is such a function and (i) commutes with four-momentum operators, (ii) fulfills the canonical commutation relations \[70\], (iii) transforms like a vector under rotations, i.e.
\[
[\hat{J}^i, \hat{S}^j] = i \epsilon_{ijk} \hat{S}^k, \tag{74}
\]

where \( \hat{J}^i = \frac{1}{2} \epsilon_{ijk} \hat{J}^k \), we arrive at
\[
\hat{S} = \frac{1}{m} \left( \frac{[\hat{P}^0, \hat{W}] - \hat{W}^0 \hat{P}}{[\hat{P}^0] + m} \right). \tag{75}
\]

With help of Eq. (18) we finally get
\[
\hat{S} = \frac{1}{m} \left( \hat{\mathcal{E}} \hat{W} - \hat{W}^0 \frac{\hat{P}}{\mathcal{E} \hat{P}^0 + m} \right). \tag{76}
\]

Let us stress that in the case when only positive energies are allowed, the spin operator given in Eq. (76) takes the well-known form
\[
\hat{S} = \frac{1}{m} \left( \hat{W} - \hat{W}^0 \frac{\hat{P}}{\hat{P}^0 + m} \right). \tag{77}
\]
This spin operator is used in quantum field theory (see e.g. [38]). We used this form in our previous works [10, 11, 21, 28], where we considered only positive-energy particles.

We can determine action of the operator defined in Eq. (76) on basis vectors. Taking into account Eqs. (4, 6) we get in the covariant basis

\[ \hat{W}^\mu(\alpha, cp) = -\frac{1}{2} \epsilon_{\beta} \sum_{\beta} ((cm^\gamma + p^\gamma) \gamma)^5_{\alpha,\beta} |\beta, cp\rangle. \]  

(78)

Furthermore, from Eqs. (78, 79) we receive

\[ \hat{S}(\alpha, cp) = -\frac{1}{2} \left[ (\gamma + \frac{cp}{\epsilon p^0} + (I - \gamma^0)^5) \gamma \right]_{\alpha,\beta} |\beta, cp\rangle. \]  

(79)

Therefore, using Eqs. (21, 30, 78, 128) we find in the spin basis

\[ \hat{W}^\mu \epsilon \sigma, \rangle = -\frac{m}{2} \sum_{\lambda} \left[ \gamma' \gamma' - \gamma' \gamma' \right]_{\lambda} \epsilon \sigma, \rangle. \]  

(80)

Finally, by virtue of Eqs. (125, 126), we have

\[ \hat{W}^0 \epsilon \sigma, \rangle = \sum_{\lambda} \left( \frac{1}{2} \epsilon \sigma, T \right)_{\lambda} \epsilon \sigma, \rangle, \]  

(81)

\[ \hat{W}^0 \epsilon \sigma, \rangle = \sum_{\lambda} \left( \frac{1}{2} \epsilon \sigma, T \right)_{\lambda} \epsilon \sigma, \rangle. \]  

(82)

Applying Eq. (76) we get

\[ \hat{S}(\epsilon \sigma, \rangle) = \frac{1}{2} \sum_{\lambda} \left( \sigma^T \right)_{\lambda} \epsilon \sigma, \rangle, \]  

(83)

which coincides with Eq. (81).

As we have seen in Sec. [11,12] the Foldy-Wouthuysen basis is in fact the spin basis. In Ref. [31] spin operator which after Foldy-Wouthuysen transformation has a form given in Eq. (83) was named "mean-spin operator". Therefore, the spin operator defined in Eq. (83) coincides with Foldy-Wouthuysen mean-spin.

We can define spin operator in yet another way, as a difference between total angular momentum (which is defined with help of Poincaré group generators \( J_{\alpha\beta} \) as \( \hat{J}^i = \frac{1}{2} \epsilon_{ijk} \hat{j}^j \)) and orbital angular momentum \( \hat{\chi} \times \hat{P} \):

\[ \hat{S} = \hat{J} - \hat{\chi} \times \hat{P}. \]  

(84)

The Newton-Wigner position operator, discussed in Sec. [13] can be expressed in terms of the generators of the Poincaré group. In the case we consider here, i.e. when negative energies are allowed, the Newton-Wigner operator takes the following form:

\[ \hat{X} = -\frac{1}{2} \frac{1}{P^0} \hat{K} + \frac{1}{P^0} \hat{\chi}, \]  

\[ -\frac{\hat{P} \times \hat{W}}{mP^0(m + \hat{P}^0)}, \]  

(85)

where \( K^i = J^i \) (compare [39]). Notice, that action of \( \hat{J} \) and \( \hat{K} \) on basis vectors is independent of energy sign \( \epsilon \); action of \( \hat{P}^\mu \) and \( \hat{W} \) is given in Eqs. (41, 83, 84, 85). Inserting Eq. (85) into Eq. (84) we can check that spin operator defined in this way also coincides with (70).

A. Transformation properties of the spin operator

1. In this section we find transformation properties of the spin operator defined in Eq. (77). To do this, let us consider two inertial observers, \( O \) and \( O' \), connected by a Lorentz transformation (2). The spin operator in the reference frame of the observer \( O \) is given by Eq. (77).

We determine the form of this operator in the frame of the observer \( O' \) in terms of the spin operator \( \hat{S} \).

Firstly, let

\[ \Lambda(R) = \begin{pmatrix} 1 & 0^T \\ 0 & R \end{pmatrix}, \quad R \in SO(3) \]  

be a pure rotation. In this case we see immediately that

\[ \hat{S}' = R \hat{S}, \]  

(87)

i.e. \( \hat{S} \) transforms like a vector under rotations—compare Eq. (74). Now, let \( \Lambda(v) \) be a pure Lorentz boost. In the frame \( O' \) the spin operator has obviously the following form:

\[ \hat{S}' = \frac{1}{m} \left( \hat{W} - \hat{W}' \hat{P}' \hat{P}' - \hat{P} m + \hat{P} \hat{P}' m \right), \]  

(88)

where

\[ \hat{W}' = \Lambda(v)^\mu \epsilon \sigma, \hat{W} \epsilon \sigma, \hat{P}' = \Lambda(v)^\mu \epsilon \sigma, \hat{P}. \]  

(89)

The explicit form of the most general pure boost is given in Eq. (B2).

Inserting Eqs. (88) into Eq. (89) and using the following relations (yielded by Eq. (77) and \( \hat{P}^\mu \hat{W}_\mu = 0 \)):

\[ \hat{W}^0 = \hat{P} \cdot \hat{S}, \]  

(90)

\[ \hat{W} = m \hat{S} + (\hat{P} \cdot \hat{S}) \frac{\hat{P}}{m + \hat{P}^0}, \]  

(91)

we get finally

\[ \hat{S}' = \hat{S} + \left( 1 - \gamma \right) \left( \hat{P} \cdot \hat{S} \right) + \gamma (m + \hat{P}^0) \left( \hat{P} \cdot \hat{P} \right) \frac{\hat{P}}{m + \gamma \left( P^0 - \hat{P} \cdot \hat{P} \right)} \]  

\[ + \gamma \left( m + \left( P^0 - \hat{P} \cdot \hat{P} \right) \right) \left( \hat{P} \cdot \hat{P} \right) \frac{\hat{P}}{m + \gamma \left( P^0 - \hat{P} \cdot \hat{P} \right)} \]  

\[ + \frac{2 \gamma (v \cdot \hat{P}) (\hat{P} \cdot \hat{S})}{m + \gamma \left( P^0 - \hat{P} \cdot \hat{P} \right) \right) \]  

\[ + 2 \gamma (v \cdot \hat{P}) (\hat{P} \cdot \hat{S}) (1 + \gamma) \]  

\[ \frac{v \cdot \hat{P}}{m + \gamma \left( P^0 - \hat{P} \cdot \hat{P} \right) \right) \]  

\[ \frac{v \cdot \hat{P}}{m + \gamma \left( P^0 - \hat{P} \cdot \hat{P} \right) \right) \]  

\[ + 2 \gamma (v \cdot \hat{P}) (\hat{P} \cdot \hat{S}) (1 + \gamma) \]  

\[ \frac{v \cdot \hat{P}}{m + \gamma \left( P^0 - \hat{P} \cdot \hat{P} \right) \right) \]  

where \( \gamma = \frac{1 - v^2}{1 - v^2} \) is a Lorentz factor. Now, comparing Eq. (92) with Eq. (B2) we see that

\[ \hat{S}' = R(v, \hat{P}) \hat{S}, \]  

(93)

where \( R(v, \hat{P}) \) is given in Eq. (B3). Thus, taking into account Eqs. (87) and (92), spin operator transforms under Lorentz transformations according to Wigner rotation. Notice that \( R(v, \hat{P}) \) in Eq. (93) is an operator. In momentum basis it is an ordinary matrix but in other bases, like e.g. position basis, it is a non-local operator.
VI. PARTICLE IN ELECTROMAGNETIC FIELD

To make this paper self-contained we firstly remind here some results we discussed in details in our paper [10]. Then we find the transformation law for the Bloch vector describing fermion polarization.

Let us consider a Dirac particle with positive energy \( \epsilon = +1 \) and sharp momentum \( q \). The most general state of such a particle is described by the following density matrix:

\[
\hat{\rho}(q, \xi) = \frac{1}{2}(1 + \xi \cdot \sigma)_{\alpha\beta}|q, \sigma\rangle\langle q, \lambda|,
\]

where the Bloch vector \( \xi \) determines a polarization of a particle. Using Eqs. [81, 82, 83] we can find the normalized average value of the Pauli-Lubanski and spin operators in the state defined in Eq. (94). We get

\[
\langle \hat{W}^0 \rangle_\rho = \frac{q \cdot \xi}{2},
\]

\[
\langle \hat{W} \rangle_\rho = \frac{1}{2}(m\xi + \frac{q(q \cdot \xi)}{q^0 + m}),
\]

\[
\langle \hat{S} \rangle_\rho = \frac{\xi}{2}.
\]

Now, let us assume that a charged particle with sharp momentum moves in the external electromagnetic field. The momentum and polarization of such a particle can be regarded as functions of time

\[
q = q(t), \quad \xi = \xi(t).
\]

The expectation value of the operators representing the spin and the momentum will follow the same time dependence as one would obtain from the classical Lorentz-covariant equations of motion [40–43]. The slow motion limit of the equations of motion, in the case when the electric field is equal to zero, takes the form

\[
\frac{d\xi}{dt} = -\frac{e}{m} \xi \times B,
\]

\[
\frac{dq}{dt} = -\frac{e}{2m} \xi \cdot \nabla B,
\]

(99)

(100)

(we assume that the giromagnetic ratio \( g = 2 \)). Therefore we should really identify \( \xi \) with the polarization of a particle.

Notice that the transformation law for the spin operator, Eq. (92), is consistent with our identification. Indeed, the density matrix \( \hat{\rho}(q, \xi) \) given in Eq. (94) as seen by the observer \( \hat{O'} \) has the form

\[
\hat{\rho}' = U(\Lambda)\hat{\rho}(q, \xi)U^\dagger(\Lambda) = \hat{\rho}(\Lambda q, \xi'),
\]

where

\[
\xi' = R(\Lambda)q\xi.
\]

To obtain the above relation we used the standard homomorphism of the SU(2) group onto SO(3) group according to which

\[
D(R(\Lambda, q))(\xi \cdot \sigma)D^\dagger(R(\Lambda, q)) = (R(\Lambda, q)\xi) \cdot \sigma.
\]

(103)

Eq. (102) means that \( \xi \) transforms according to the Wigner rotation. This is consistent with the transformation law for the spin operator.

VII. POSITION REPRESENTATION

Now, we want to define bispinors in a position representation and vectors corresponding to them in an abstract Hilbert space. We are interested in a covariant picture therefore we use in our construction the covariant basis [Eq. (20)]. Thus we define

\[
|x, \alpha, \epsilon\rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega(p)} e^{-i\epsilon px}|x, \alpha, \epsilon\rangle,
\]

(104)

and

\[
|x, \alpha\rangle = \sum_\epsilon |x, \alpha, \epsilon\rangle.
\]

(105)

Inverse transformation reads

\[
|\alpha, \epsilon\rangle = \frac{2m\epsilon}{(2\pi)^{3/2}} \int d\sigma_\mu(x)e^{i\epsilon px}(\Lambda_\epsilon(p)\gamma^\mu)_{\alpha\beta}|x, \beta, \epsilon\rangle
\]

\[
= \frac{2m\epsilon}{(2\pi)^{3/2}} \int_{x^0=\text{const}} d^3xe^{i\epsilon px}(\Lambda_\epsilon(p)\gamma^0)_{\alpha\beta}|x, \beta, \epsilon\rangle,
\]

(106)

or, in terms of vectors

\[
|\alpha, \epsilon\rangle = \frac{2m\epsilon}{(2\pi)^{3/2}} \int d\sigma_\mu(x)e^{i\epsilon px}(\Lambda_\epsilon(p)\gamma^\mu)_{\alpha\beta}|x, \beta\rangle
\]

\[
= \frac{2m}{(2\pi)^{3/2}} \int_{x^0=\text{const}} d^3xe^{i\epsilon px}(\Lambda_\epsilon(p)\gamma^0)_{\alpha\beta}|x, \beta\rangle.
\]

(107)

We can check, that

\[
\langle \alpha, \epsilon | x, \beta, \epsilon \rangle = \frac{\delta_{\alpha\beta}}{(2\pi)^{3/2}}e^{-i\epsilon px}(\Lambda_\epsilon(p))_{\beta\alpha}.
\]

(108)

Now, bispinor in a position representation we define as follows:

\[
\Psi_\alpha(x) = \langle \psi|\alpha, x\rangle.
\]

(109)

Let us notice that, by virtue of Eqs. [46, 105], the above bispinor is related with bispinors in momentum representation via the standard relation

\[
\Psi_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \sum_\epsilon \int \frac{d^3p}{2\omega(p)} \psi_\epsilon^*(p)e^{-i\epsilon px}.
\]

(110)

One can also show that the scalar product [54] in terms of \( \Psi_\alpha(x) \) reads

\[
\langle \psi, \phi \rangle = \int d^3x \overline{\Psi}(x)\gamma^0\Phi(x),
\]

(111)

which is consistent with Eq. (7).
It should be stressed here that vectors \(|x, \alpha, \epsilon|\) defined in Eq. (104) are not eigenvectors of the Newton–Wigner position operator.

It is possible to define Foldy–Woutheysen transformations on the level of bispinors in position representation. However, the connection between bispinors before and after the Foldy–Woutheysen transformation is non-local.

VIII. CONCLUSIONS

In conclusion, we have formulated the Dirac formalism in an abstract Hilbert space which is a carrier space of an unitary representation of the Poincaré group. To include negative energy solutions of the Dirac equation we have considered the direct sum of carrier spaces of positive and negative energy unitary representations of the Poincaré group for a massive spin-1/2 particle. We have introduced basis which under Lorentz transformations transforms in a manifestly covariant manner according to the bispinor representation of the Lorentz group. Vectors of the covariant basis in a natural way fulfill the Dirac equation. We have also shown that the Foldy-Woutheysen transformation which diagonalizes the Dirac Hamiltonian, corresponds to the transformation between covariant basis and the standard basis in the carrier space of the Poincaré group representation.

Moreover, we have discussed in detail the relativistic spin operator for massive particle. We have shown that in the case of Dirac particles the spin operator used in quantum field theory is equal to the Foldy-Woutheysen mean-spin operator. We have also shown that this spin operator under Lorentz group action transforms according to the Wigner rotation matrix in which momentum is replaced by momentum operator. Such a “Wigner rotation” is a highly non-local operator.

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Appendix A: Basis in an abstract Hilbert space

We discuss here in detail the choice of basis in an abstract Hilbert space \(H = H_+ \oplus H_-\). Basis vectors should be labeled by four-momentum \(p\), spin \(\sigma = \pm 1/2\), and index \(\epsilon = \pm 1\) identifying sign of energy. Let us denote for a moment basis vectors as \(|p_\epsilon, \sigma\rangle\). By definition we have

\[
|p_\epsilon, \sigma\rangle = \frac{e\omega(p_\epsilon)}{\sqrt{m^2 + p_\epsilon^2}}
\]

Furthermore, let \(q_\epsilon = (cm, 0)\) denote four-momentum of a particle in its rest frame. We assume that vectors \(|p_\epsilon, \sigma\rangle\) are generated from \(|q_\epsilon, \sigma\rangle\) in the following way:

\[
|p_\epsilon, \sigma\rangle = U(L^{\epsilon\sigma}_p)|q_\epsilon, \sigma\rangle,
\]

where \(L^{\epsilon\sigma}_p\) is a standard Lorentz boost which fulfills

\[
p_\epsilon = L^{\epsilon\sigma}_p q_\epsilon, \quad L^{\epsilon\sigma}_q = I.
\]

Standard Wigner induction method leads to the following form of the Lorentz group action on vectors \(|p_\epsilon, \sigma\rangle\):

\[
U(\Lambda)|p_\epsilon, \sigma\rangle = D(R(\Lambda, p_\epsilon))_{\lambda\sigma}|\Lambda p_\epsilon, \lambda\rangle
\]

where \(D\) denotes spin-1/2 representation of the rotation group and \(R(\Lambda, p_\epsilon) = (L^{\epsilon\sigma}_{\Lambda p_\epsilon})^{-1} L^{\epsilon\sigma}_p\) is a Wigner rotation. We would like to have \(\Lambda p_\epsilon = (\Lambda p)_\epsilon\). Therefore we take

\[
p_\epsilon = cp = (\epsilon\omega(p), c\mathbf{p}),
\]

and, consequently, basis vectors in the following form:

\[
|p_\epsilon, \sigma\rangle = |cp, \sigma\rangle.
\]

Notice, that the choice which might seem to be the most natural, i.e. \(p_\epsilon = cp^2 = (\epsilon\omega(p), c\mathbf{p})\) is not suitable because \(\Lambda p^\pi \neq (\Lambda p)^\pi\).

Moreover, we easily see that for the standard boost defined as

\[
L_p = \begin{pmatrix} p^0 & p^\mu \gamma^\mu \frac{m}{m+p^0} \\ m & I + \frac{m}{m+p^0} \frac{p^\mu \gamma^\mu}{p^0} \end{pmatrix}
\]

we have

\[
|\Lambda p_\epsilon, \lambda\rangle = L_p|q_\epsilon, \sigma\rangle.
\]

Thus it holds

\[
R'(\Lambda, cp) = R(\Lambda, p) = L_{\Lambda p}^{-1} L_p,
\]

and we finally receive

\[
U(\Lambda)|cp, \sigma\rangle = D(R(\Lambda, p))_{\lambda\sigma}|\epsilon\Lambda p, \lambda\rangle.
\]

Appendix B: Wigner rotation

In this Appendix we give the explicit form of a Wigner rotation for a Lorentz transformation \(\Lambda(\mathbf{v})\) being a pure boost. The most general Lorentz boost \(\Lambda(\mathbf{v})\) between two inertial frames of reference, \(\mathcal{O}\) and \(\mathcal{O}'\),

\[
x'^\mu = \Lambda(\mathbf{v})^\mu_{\ \nu} x^\nu
\]

can be written in the following form

\[
\Lambda(\mathbf{v}) = \begin{pmatrix} \gamma & -\gamma v^T \\ -\gamma v & I + \frac{\gamma v^T v}{1+\gamma} \end{pmatrix},
\]
where $v$ is the velocity of a frame $O'$ with respect to a frame $O$ and $\gamma = (1 - 1/v^2)^{-1/2}$ is a Lorentz factor.

Now, using Eqs. (A8) and (B2) we can find by direct calculation

$$R(\Lambda(v), p) = L_{\Lambda(v)}^{-1} \Lambda(v) L_p = \begin{pmatrix} 1 & 0 \\ 0 & R(v, p) \end{pmatrix},$$ \hspace{1cm} (B3)

where the matrix $R(v, p) \in SO(3)$ reads

$$R(v, p) = I + \frac{1 - \gamma}{ab} p \otimes p^T + \frac{\gamma^2 (m - p^0)}{b(1 + \gamma)} v \otimes v^T + \frac{\gamma}{b} p \otimes v^T + \frac{\gamma}{b} \left( \frac{2\gamma (v \cdot p)}{a(1 + \gamma)} - 1 \right) v \otimes p^T,$$ \hspace{1cm} (B4)

where

$$a = m + \gamma \hat{p}^0,$$

$$b = m + \gamma \hat{p}^0 = m + \gamma (\hat{p}^0 - v \cdot \hat{P}).$$ \hspace{1cm} (B5)

### Appendix C: Dirac matrices

Dirac matrices fulfill the relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu}$ where the Minkowski metric tensor $g^{\mu \nu} = \text{diag}(1, -1, -1, -1)$; moreover we adopt the convention $\varepsilon^{0123} = 1$. We use the following explicit representation of gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$ \hspace{1cm} (C1)

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\sigma_i$ are standard Pauli matrices.

### Appendix D: Useful formulas

The explicit form of amplitudes $\nu^\nu(p)$ is the following:

$$\nu^\nu(p) = \frac{1}{2\sqrt{1 + \frac{p^0}{m}}} \left( I_2 + \frac{p^\mu}{m} p_\mu \sigma_\mu \right) \sigma_2,$$ \hspace{1cm} (D1)

where $\sigma_0 = I_2$. It holds

$$\bar{\nu}^\nu(p) \gamma^\mu \nu^\nu(p) = \frac{p^\mu}{m} I_2,$$ \hspace{1cm} (D2)

$$\bar{\nu}^\nu(p) \gamma^5 \nu^\nu(p) = 0.$$ \hspace{1cm} (D3)

Using Eqs. (C1) [11] we find

$$\bar{\nu}^\nu(p) \gamma^0 \nu^\nu(p) = 0,$$ \hspace{1cm} (D4)

$$\bar{\nu}^\nu(p) \gamma^0 \gamma^5 \nu^\nu(p) = -\frac{1}{m} (p \cdot \sigma^T),$$ \hspace{1cm} (D5)

$$\bar{\nu}^\nu(p) \gamma^5 \nu^\nu(p) = -\frac{1}{m} \left( m \sigma^T + \frac{p (p \cdot \sigma^T)}{m + p^0} \right).$$ \hspace{1cm} (D6)

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