Construction of a series of topologically distinct $\nu = 2/5$ fractional quantum Hall wave functions by conformal field theory

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(Dated: July 7, 2020)

In this paper, a series of $\nu = 2/5$ fractional quantum Hall wave functions are constructed from conformal field theory(CFT). They share the same topological properties with states constructed by Jain’s composite fermion approach. Upon exact lowest Landau level(LL) projection, some of Jain’s composite fermion states would not survive if constraints on Landau level indices given in appendices of this paper are not satisfied. By contrast, states constructed from CFT always stay in LLL. These states are characterized by different multi-body relative angular momenta and topological shifts, thus belong to different topological sectors. As a by-product, in appendices we prove the necessary conditions for general $\nu = p/(2p+1)$ composite fermion states to have non-vanishing LLL-projection.

I. INTRODUCTION

Strongly correlated electron system has long been a topic of focus in condensed matter physics, especially the fractional quantum Hall system of electrons on a 2D surface, at temperature as low as several Kelvins and in a high magnetic field of several Teslas. This system exhibits fractional quantum Hall effect, in which Hall resistance $R_H$ is quantized as $\frac{h}{e^2}$ in certain range of magnetic field, with Planck constant $h$, electron charge $e$ and fractional filling factor $\nu$. Fractional quantum Hall effect, as well as integer quantum Hall effect in which filling factor $\nu$ is an integer, can be explained by the notion of Landau level(LL). Under the above experimental circumstances, energy spectra for a single electron will form discrete energy levels known as LLs. Completely filled LLs account for integer quantum Hall effect while partially filled LLs with certain fractional filling factor $\nu$ lead to fractional quantum Hall effect, where $\nu = N_e/N_F$. Here $N_e$ is electron number and $N_F$ is LL degeneracy, which is equal to the number of flux quanta piercing the system. The most well known fractional quantum Hall effect is the one with $\nu = 1/3$ where one third of the lowest Landau level(LL) is filled. In the regime of fractional quantum Hall effect, if we project the full Hamiltonian of 2D electron system in a high magnetic field onto LLL, the single-particle kinetic energy term is quenched while interaction terms remain. Still this Hamiltonian cannot be easily solved, so physicists resort to trial wave functions, such as the Laughlin wave function[1] for the filling factor 1/3 and Jain’s composite fermion wave function for the series with filling factors $p/(2p+1)$. Model Hamiltonians can be derived from these trial wave functions, which are easier to deal with on analytical grounds than most realistic Hamiltonians. When projected onto specific LLs of interest, these Hamiltonians usually assume forms of 1D frustration-free lattice Hamiltonians which, in their second-quantized forms, have been studied thoroughly and many properties of their zero modes(zero energy ground states) have been discovered[2-5]. On the other hand, trial wave functions for fractional quantum Hall effect have been connected to conformal field theory(CFT). Laughlin wave function, as an example, can be cast as the conformal correlator of massless free boson in CFT[6,7]. Later on, new wave functions have been proposed from conformal correlators, such as the famous Moore-Read Pfaffian and Read-Rezayi wave functions, argued to possess quasiparticle/quasihole excitation obeying non-Abelian anyonic statistics[8,9]. The justification for the construction of trial wave functions from CFT is that the boundary theory of Chern-Simons theory which characterizes the quantum Hall effect is a CFT[10], and the ground state wave function can be viewed as the amplitude of particle configuration in a time slice of such a 2+1 D system.

It took a few more years for people to realize that Jain’s composite fermion wave functions, such as that for $\nu = 2/5$, once projected to LLL, is also a conformal correlator[11-14]. Since Jain’s $\nu = 2/5$ composite fermion wave function (which is our primary example for Jain states) has two degrees of freedom associated with two lowest LLs for the composite fermion (more details will be presented in Sec. [11]), its corresponding conformal correlator is constructed from two independent massless free bosons. However unlike the Read-Rezayi sequence which includes the Laughlin and Moore-Read states, constructing the Jain states involves not only the primary, but also descendant fields of the corresponding CFT. Since there is a large degree of freedom in the choice of the latter even when $\nu$ is fixed, there should be a corresponding family of Jain states. Motivated by the developments and consideration mentioned above, we have generalized Jain’s $\nu = 2/5$ composite fermion wave function to cases in which the composite fermions occupy higher LLs, while the lower one(s) may be empty. Since LLL projection is performed, they represent distinct yet legitimate LLL states at the same filling factor. As we are going to show, they correspond to the different choice of descendant fields in the CFT construction. Upon LLL projection, some of them will vanish. Nonetheless, the
corresponding conformal correlators, which are holomorphic and thus reside in the LLL by construction, are non-vanishing. Most importantly, we find while these family of Jain states share the same K matrix with the original one constructed by Jain, they have different shifts and other topological properties: as a result we have constructed a family of topologically distinct Jain states for a given filling factor.

The structure of this paper is as follows, in Sec. II we introduce Jain’s composite fermion approach and trial wave functions in this approach, known as composite fermion wave functions. In Sec. III we introduce the conformal field theoretical construction of composite fermion wave functions. In Secs. IV and V we propose a series of fractional quantum Hall trial wave functions of filling factor 2/5 constructed from CFT correlators and discuss their connections with general LLL-projected composite fermion wave functions. In Sects. III and IV we characterize their topological properties by two criteria, one is shift and the other is multi-body relative angular momentum. In appendices, we give the necessary conditions for a general composite fermion state to have non-vanishing LLL-projection.

II. COMPOSITE FERMION APPROACH TO FRACTIONAL QUANTUM HALL EFFECT

\[ P_{LLL} \prod_{i<j} (z_i - z_j)^2 \Psi_2, \]

where \( z = x + iy \) is the complex coordinate on disk, \( P_{LLL} \) projects the wave function onto the lowest Landau level, \( \Psi_2 \) is the \( N \)-particle wave function at filling factor \( \nu = 2 \), i.e., two Landau level(LL)s are filled. The Laughlin-Jastrow factor \( \prod_{i<j} (z_i - z_j)^2 \) has the effect of attaching two flux quanta to each electron. The explicit form of \( \Psi_2 \) is

\[
\Psi_2 = \begin{vmatrix}
\eta_{1,m_1}(z_1) & \cdots & \eta_{1,m_N}(z_N) \\
\vdots & \ddots & \vdots \\
\eta_{2,m_N}(z_1) & \cdots & \eta_{2,m_N}(z_N)
\end{vmatrix},
\]

in which \( \eta_{n,m}(z) \) is the single-particle wave function of angular momentum \( m \hbar \) in the \( n \)-th Landau level on disk by choosing a symmetric gauge for the magnetic field \( B = -B \hat{z} \). The expression of \( \eta_{n,m}(z) \) is

\[
\eta_{n,m}(r) = \frac{(-1)^n \sqrt{n!}}{\sqrt{2\pi 2^m (n + m)!}} z^m L_n^m \left( \frac{\bar{z} z}{2} \right) \ e^{-\frac{z^2}{4}},
\]

in which the magnetic length \( l_B = \sqrt{\hbar/eB} \) has been set to 1 and \( L_n^m(x) \) is the generalized Laguerre polynomial,

\[
L_n^m(x) = \sum_{i=0}^{n} \frac{(-1)^i (n + m) x^i}{i!}.
\]

Note that the maximum power of \( \bar{z} \) in \( \eta_{n,m}(r) \) is \( n \), which will be used in the following sections. For example, single-particle wave functions in 0LL, 1LL(the first excited Landau level) and 2LL(the second excited Landau level) are

\[
\eta_{0,m}(z) = \frac{z^m e^{-|z|^2/4}}{\sqrt{2\pi 2^m m!}},
\]

\[
\eta_{1,m}(z) = \frac{(\bar{z} z^{m+1} - 2(m + 1) z^m) e^{-|z|^2/4}}{\sqrt{2\pi 2^{m+2} (m + 1)!}}
\]

and

\[
\eta_{2,m}(z) = e^{-|z|^2/4} \times \frac{(\bar{z}^2 z^{m+2} - 4(m + 2) \bar{z} z^{m+1} + 4(m + 2)(m + 1) z^m)}{\sqrt{2\pi 2^{m+5} (m + 2)!}}.
\]

Besides the Gaussian factor, the single-particle wave function in 0LL is analytic in \( z \), while that in 1LL and 2LL has \( \bar{z} \) and \( \bar{z}^2 \), respectively.

For \( \Psi_2 \) in Eq. 1 Jain chose two filled LLs as LLL and 1LL. The LLL-projection \( P_{LLL} \) is technically accomplished in the following way: we bring all the anti-holomorphic coordinates \( \bar{z}_i \) to the leftmost of the wave function, and then replace them individually by \( 2\partial_{\bar{z}_i} \), where the derivative only acts on the polynomial part of the wave function. In Appendix C, we have also given a closed form for the LLL-projected composite fermion wave function using an alternative approach.
III. CFT CONSTRUCTION OF FRACTIONAL QUANTUM HALL WAVE FUNCTION

The wave function of LLL-projected Jain’s composite fermion state of even number of particles in Eq. 1 can be written as a conformal correlator in CFT[10,11]. In the framework of CFT, we introduce two independent free massless bosonic fields $\phi_1(z)$ and $\phi_2(z)$ compactified on two circles of radii $\sqrt{3}$ and $\sqrt{15}$, respectively. Their conformal correlator satisfies $\langle \phi_1(z)\phi_2(w) \rangle = -\delta_{z, w} \ln(z - w)$. Then we introduce two vertex operators,

$$V_0(z) = e^{i\sqrt{3}\phi_1(z)};$$

and

$$V_1(z) = \partial_z \left( e^{i\sqrt{3}\phi_1(z)} e^{i\sqrt{2}\phi_2(z)} \right);$$

where $::$ means normal ordering. Here $V_0(z)$ is a primary field and $V_1(z)$ is a descendant of primary field $e^{i\sqrt{3}\phi_1(z)} e^{i\sqrt{2}\phi_2(z)}$. It is easy to see that these two vertex operators represent two species of independent electron operators since $[V_0(z), V_1(w)] = 0$. The wave function prescribed by Jain’s composite Gaussian factor. It has been proved[9,10,11] that the LLL-projected Jain state can be written as a conformal correlator.

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IV. GENERAL CONSTRUCTION OF VERTEX OPERATOR FOR COMPOSITE FERMIONS

As we stated in Sec. [10,11] Jain has chosen two filled composite fermion LLs as LLL and 1LL. This raises an interesting question: can we choose composite fermions in the correlator we have neglected the back ground charge term[10,11] which accounts for the Gaussian factor. It has been proved[10,11] that the LLL-projected wave function prescribed by Jain’s composite fermion approach in Eq. 4 which is constructed from filled LLL and 1LL, is exactly equal to that given by CFT correlator in Eq. 14 up to a constant. Since the LLL single-particle wave function has no $\bar{z}$ and 1LL single-particle wave function has $\bar{z}$ to the power of 1, which is just $2\delta_{z, 0}$ in the process of LLL-projection[12], we can attribute the vertex operator $V_0$ to LLL and $V_1$ to LLL since $V_0$ contains no derivative and the power of the derivative in $V_1$ is 1.

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$$\mathcal{A}\{V_0(z_1)V_0(z_{\frac{N}{2} + 1})V_0(z_{\frac{N}{2} + 2}) \cdots V_0(z_N)\};$$

That is, half of electrons are represented by $V_0$ and remaining half are represented by $V_1$. This correlator leads to

$$\mathcal{A}\{\partial_{z_1}\partial_{z_2} \cdots \partial_{z_{\frac{N}{2}}} \prod_{1 < j \leq \frac{N}{2}} (z_i - z_j)^3 \prod_{\frac{N}{2} < k < l} (z_k - z_l)^3 \prod_{m \leq \frac{N}{2} < n} (z_m - z_n)^2 \};$$

by using the formula[10]

$$\langle e^{i\alpha_1\phi(z_1)} \cdots e^{i\alpha_{2N}\phi(z_{2N})} \rangle = \prod_{1 < j \leq 2N} (z_i - z_j)^{\alpha_i \alpha_j}.\]
Therefore we can construct a conformal correlator,
\[ A\{ (V_{n_2}(z_1)V_{n_2}(z_2) \cdots V_{n_2}(z_2) ) V_{n_1}(z_{\frac{N}{2}+1}) V_{n_1}(z_{\frac{N}{2}+2}) \cdots V_{n_1}(z_N) \} , \]
which is simplified as
\[ A(\partial_{z_1}^{n_2} \partial_{z_2}^{n_2} \cdots \partial_{z_i}^{n_1} \partial_{z_{i+1}}^{n_1} \partial_{z_{i+2}}^{n_1} \cdots \partial_{z_N}^{n_1} \prod_{i<j}^{N} (z_i - z_j)^3 \prod_{2<k<l}^{N} (z_k - z_l)^3 \prod_{m<\frac{N}{2}<n} (z_m - z_n)^2 \} . \]

An interesting question can be raised that whether the state in Eq. \(19\) of general choices of \(n_1\) and \(n_2\) is equal to the LLL-projected Jain’s composite fermion state constructed from filled \(n_1\)-th and \(n_2\)-th LLs. The answer is negative unless \(n_1 = 0, n_2 = 1\). Explicitly, the LLL-projected Jain’s composite fermion state on disk constructed from filled \(n_1\)-th and \(n_2\)-th LLs has the wave function
\[ P_{\text{LLL}} \prod_{1 \leq i < j \leq N} (z_i - z_j)^2 \begin{vmatrix} \eta_{n_1, -n_1}(z_1) & \cdots & \eta_{n_1, -n_1}(z_N) \\ \vdots & \ddots & \vdots \\ \eta_{n_1, -n_1-1}(z_1) & \cdots & \eta_{n_1, -n_1-1}(z_N) \\ \eta_{n_2, -n_2-1}(z_1) & \cdots & \eta_{n_2, -n_2-1}(z_N) \end{vmatrix} , \tag{20} \]
as in Eq. \(C7\). On the other hand, the wave function constructed from CFT in Eq. \(19\) can be shown to be equal to
\[ P_{\text{LLL}} \Psi_{\text{CFT}} , \tag{21} \]
where
\[ \Psi_{\text{CFT}} = \prod_{1 \leq i < j \leq N} (z_i - z_j)^2 \begin{vmatrix} z_{n_1} & \cdots & z_{n_1} \\ \bar{z}_{1} & \cdots & \bar{z}_{1} \\ \vdots & \ddots & \vdots \\ z_{n_1-1} & \cdots & z_{n_1-1} \\ \bar{z}_{1} & \cdots & \bar{z}_{1} \\ \vdots & \ddots & \vdots \\ z_{n_1} & \cdots & z_{n_1} \\ \bar{z}_{1} & \cdots & \bar{z}_{1} \end{vmatrix} , \tag{22} \]
up to a constant. We term this \(\Psi_{\text{CFT}}\) the unprojected CFT wave functions. The only difference between the two wave functions in Eqs. \(20\) \(21\) is that in contrast to the former, the latter only keeps \(z^nz^{n+m}\) in each \(\eta_{n,m}(z)\), where we omit the Gaussian factor for simplicity. In Appendix \(C\) we have proved that there are some choices of \(n_1\) and \(n_2\) for which Jain’s \(\nu = 2/5\) composite fermion state has vanishing LLL-projection if the constraints in Eq. \(C7\) are not satisfied. By contrast, CFT wave function in Eq. \(19\) is non-vanishing for general \(n_1\) and \(n_2\). Therefore, although some of LLL-projected composite fermion states in Eq. \(20\) vanish when the particle number is finite, we can still construct non-vanishing wave functions by CFT from the same filled composite fermion LLs \(n_1\) and \(n_2\). Moreover, it is easy to see that before LLL-projection, unprojected Jain’s composite fermion state in Eq. \(20\) has the same root patterns as those corresponding unprojected CFT state in Eq. \(22\) (see Appendix \(A\) for the definition of root pattern). This strongly suggests that they have the same topological feature. As will be seen in Sec. \(V\) they have the same topological shift when they have the same \(n_1\) and \(n_2\), since the topological shift is dictated by root pattern. We thus term wave function given by CFT in Eq. \(19\) as Jain \(n_1n_2\) wave function. We can safely do so since wave functions constructed from CFT correlators are described by the same Chern-Simons field theory which characterizes the \(\nu = 2/5\) composite fermion states. The justification for this follows Ref. \(10\).

From the CFT Lagrangian characterizing wave functions in Eq. \(19\) we can change the basis from massless free boson fields \(\phi_1\) and \(\phi_2\) to \(\chi_1\) and \(\chi_2\) such that the two independent quasihole operators of charge \(1/5\) are \(e^{i\chi_1}\) and \(e^{i\chi_2}\). After this change of basis, we arrive at the Chern-Simons Lagrangian characterizing \(\nu = 2/5\) state with \(K\) matrix \((\begin{smallmatrix} 3 & 2 \\ 3 & 3 \end{smallmatrix})\) and charge vector \((1, 1)^T\). Note that Jain’s \(2/5\) composite fermion states in Chern-Simons field theory, regardless of which two composite fermion LLs are filled, are characterized by the same \(K\) matrix \((\begin{smallmatrix} 3 & 2 \\ 3 & 3 \end{smallmatrix})\) and charge vector \((1, 1)^T\). Now we can safely argue that although the LLL-projected Jain states might vanish, we still have wave functions completely in LLL constructed from CFT, which capture the same important features such as filling factor \(2/5\), fractional charge \(1/5\) and Abelian exchange statistics of corresponding Jain states since these features are dictated by Chern-Simons field theory. We summarize the relation between wave functions obtained from these two approaches in Fig. \(1\).

V. TOPOLOGICAL SHIFT

We have obtained a series of Jain \(n_1n_2\) wave functions in Eq. \(19\). Now the question is how to distinguish one from another among them. We will resort to a number called topological shift which can differentiate one topological state from another. We can place the quantum Hall system on any 2D surface such as a disk or the surface of a sphere. On the surface of a sphere, the number of magnetic flux quanta piercing the sphere \(N_b\) is related
For states constructed from CFT, the boundary terms in their root patterns are more and more complicated as \( n_1 \) and \( n_2 \) increase. Their shifts are derived in the following way. Since Jain’s composite fermion state constructed from Landau levels \( n_1 \) and \( n_2 \) has the same root pattern as that of corresponding CFT state, (This fact can be easily deduced from Eq. (20) and Eq. (21)) the topological shift is the same for both states as long as they have the same \( n_1 \) and \( n_2 \). So we can use the explicit form of Jain’s composite fermion wave function to calculate the topological shift for state constructed from CFT. It is known that the monopole charge \( Q \) of the \( \nu = 2/5 \) composite fermion state is related to the monopole charge \( Q^* \) of the \( \nu = 2 \) integer quantum Hall state by the identity \( Q = Q^* + N - 1 \) where \( N \) is the particle number (see also Eq. (13)). Since we have \( n_1 \)-th and \( n_2 \)-th LLs filled, the particle number is
\[
N = 2(Q^* + n_1) + 1 + 2(Q^* + n_2) + 1.
\]
The number of magnetic flux quanta is
\[
N_\phi = 2Q = 2(Q^* + N - 1) = \frac{5}{2}N - (n_1 + n_2 + 3).
\]
Hence the topological shift is \( n_1 + n_2 + 3 \) for \( \nu = 2/5 \) Jain \( n_1n_2 \) state constructed from CFT. Now a confusion arises that two Jain \( n_1n_2 \) states of the same \( n_1 + n_2 \) have the same topological shift. A single topological number such as shift is insufficient to differentiate two topologically distinct states, we thus must resort to other topological numbers as well.

VI. MULTI-BODY RELATIVE ANGULAR MOMENTUM

Quantum Hall states are also characterized by a set of numbers \( \{S_n\} \) named pattern of zeros (PZ). These \( S_n \) are the minimum \( n \)-body relative angular momentum (unit of angular momentum is chosen as \( \hbar \)) of a specific quantum Hall state. To obtain \( S_n \), we let \( z_i \), with \( i = 2, \ldots, n \) approach \( z_1 \) in the polynomial part of wave function \( \Psi(z_1, z_2, \ldots, z_N) \) of the concerned quantum Hall state and then collect the power of \( \lambda \) in the leading term,
\[
\Psi(z_1, z_2 = z_1 + \lambda \eta_2, \ldots, z_n = z_1 + \lambda \eta_n, z_{n+1}, \ldots, z_N) = \lambda^{S_n} f(z_1, \eta_2, \ldots, \eta_n, z_{n+1}, \ldots, z_N) + O(\lambda^{S_n+1}).
\]
\( S_n \) of various Jain states are tabulated in Table I. 

As seen from Table I, Jain 12 and Jain 03 states have different PZ, thus are topologically distinct states. Generally, all Jain \( n_1n_2 \) states of the same \( n_1 + n_2 \) can be distinguished by PZ.

From PZ we may obtain projectors for which a specific Jain \( n_1n_2 \) state is a zero energy ground state, although not the unique zero energy ground state in its own sector. For example, the minimum three-body relative angular momentum \( S_3 \) of Jain 01 state is 5, so Jain
01 state is a zero energy ground state of a three-body operator $P_3^{(3)}$ which projects on three-body antisymmetric state of relative angular momentum $3$ \textsuperscript{22} However, Jain 01 state is not the unique zero energy ground state of $P_3^{(3)}$ in its own sector. In fact, the densest zero energy ground state of $P_3^{(3)}$ is Pfaffian state,\textsuperscript{22} with filling factor $\nu = 1/2$, which has root pattern 110011001100... The minimum four-body relative angular momentum $S_4$ of Jain 01 state is 12, so Jain 01 state is also a zero energy ground state of four-body projection operators $P_4^{(6)}$, $P_4^{(8)}$, $P_4^{(9)}$, $P_4^{(10)}$ and $P_4^{(11)}$ which project on four-body antisymmetric states of relative angular momentum 6, 8, 9, 10 and 11, respectively.\textsuperscript{22} If we choose the Hamiltonian as a linear combination of $P_3^{(3)}$, $P_4^{(10)}$ and $P_4^{(11)}$ with positive coefficients, (This results from the fact that Pfaffian state, being the densest zero energy ground state of $P_3^{(3)}$, has minimum four-body relative angular momentum 10. Thus Pfaffian state is automatically annihilated by $P_4^{(6)}$, $P_4^{(8)}$ and $P_4^{(9)}$.) again Jain 01 state is not the unique zero energy ground state in its sector. As a by-product, we have found the densest zero energy ground state of the above Hamiltonian to have filling factor $\nu = 3/7$, which is larger than 2/5. We have diagonalized this Hamiltonian up to 7 particles on disk and found one densest zero energy ground state at particle number 5 and 6. The root pattern of this state is 1100110011001100..., which has repetitions of 1100100. For particle number 4 and 7, there is an extra independent zero energy ground state in each case, with root pattern 1100011 and 11001001100011, respectively. For the above mentioned $\nu = 3/7$ state with root pattern 11001001100100100..., we have found its wave function on disk to have the following form (Gaussian factor omitted),
\begin{equation}
\psi_{\nu=3/7} = \psi_0 \prod_{i<j}(z_i - z_j),
\end{equation}
where $\psi_0$ is a bosonic wave function in which every three particles form a cluster. Its explicit form is given in the following way. First we divide particles into clusters, with each cluster having three particles. We then choose any two clusters, whose particle coordinates are $z_{3i+1}$, $z_{3i+2}$, $z_{3i+3}$ and $z_{3j+1}$, $z_{3j+2}$, $z_{3j+3}$, respectively. We assign to these two clusters an inter-cluster wave function
\begin{align}
(z_{3i+1} - z_{3j+1})^2(z_{3i+1} - z_{3j+2})(z_{3i+1} - z_{3j+3}) \\
(z_{3i+2} - z_{3j+1})(z_{3i+2} - z_{3j+2})^2(z_{3i+2} - z_{3j+3}) \\
(z_{3i+3} - z_{3j+1})(z_{3i+3} - z_{3j+2})(z_{3i+3} - z_{3j+3})^2.
\end{align}

To the cluster of particle coordinates $z_{3i+1}$, $z_{3i+2}$ and $z_{3i+3}$, we assign an intra-cluster wave function $(z_{3i+2} - z_{3i+3})^2$. Finally, we symmetricize the product of all inter-cluster and intra-cluster wave functions to obtain the bosonic wave function $\psi_0$. The pairings of intra-cluster and inter-cluster part of wave function for two general clusters are shown in Fig.\textsuperscript{22} It is easy to verify that $\psi_{\nu=3/7}$ has minimum 3-body relative angular momentum $S_3 = 5$ and minimum 4-body relative angular momentum $S_4 = 12$, thus is indeed a zero energy ground state of $P_3^{(3)}$, $P_4^{(10)}$ and $P_4^{(11)}$.

Similarly, Jain 02 state is a zero energy ground state of $P_4^{(6)}$ and $P_4^{(8)}$, although not the unique zero energy ground state in its sector. We have chosen a linear combination of $P_4^{(6)}$ and $P_4^{(8)}$ with positive coefficients as our parent Hamiltonian and diagonalized it for up to 10 particles on disk. We find a unique densest zero energy ground state for 4, 5, 6, 8 and 10 particles, respectively. For 7 particles, there are two densest zero energy ground states with the same angular momentum. For 9 particles, there are three densest zero energy ground states with the same angular momentum. Second-quantized form for the unique densest zero energy ground state mentioned above is shown in Supplemental Material. This state has root pattern 111000110100110011, whose filling factor is very close to 1/2 for finite number of particles as seen from the root pattern. This state has minimum 4-body relative angular momentum $S_4 = 9$, thus is indeed a zero energy ground state of $P_4^{(6)}$ and $P_4^{(8)}$. Note that Pfaffian state is a candidate for the zero energy ground state of this parent Hamiltonian. If the state with root pattern 111000110100110011 remains gapped and possesses $\nu = 1/2$ in the thermodynamic limit, it would be of interest to study its properties, obtain the close form for its first-quantized wave function and compare it with other $\nu = 1/2$ state such as Pfaffian, anti-Pfaffian,\textsuperscript{31,32} and PH-Pfaffian.\textsuperscript{33,35} Jain 12 state is a zero energy ground state of $P_3^{(3)}$, although not the unique zero energy ground state in its

| $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|------|------|------|------|------|------|------|
| Jain 01 | 1    | 5    | 12   | 21   | 33   | 47   | 64   |
| Jain 02 | 1    | 3    | 9    | 18   | 29   | 43   | 59   |
| Jain 12 | 1    | 3    | 8    | 16   | 27   | 40   | 56   |
| Jain 03 | 1    | 3    | 6    | 14   | 25   | 38   | 54   |
| Jain 13 | 1    | 3    | 6    | 13   | 23   | 36   | 51   |

Table I. The PZ of various Jain states. When constructing these states, we let the number of composite fermions in each of two composite fermion LLs to be equal. Here we list $S_n$ from $n = 2$ to 8 as these are sufficient to distinguish one state from another.
sector. In fact, the unique densest zero energy ground state of \( P_{1}^{(6)} \) is Read-Rezayi \( \mathbb{Z}_{3} \) state \(^{2} \) with \( \nu = 3/5 \).

Similar arguments can be made about other Jain \( n_{1}n_{2} \) states as well. We conjecture it is impossible to find parent Hamiltonians for which a Jain \( n_{1}n_{2} \) state is the densest zero energy ground state. This has been discussed in Refs. \(^{3} \) and \(^{36} \) for Jain 01 state, which is exactly equal to LLL-projected Jain CF state constructed from CF 0LL and 1LL. In Ref. \(^{3} \) it is argued that an exact parent Hamiltonian which can give correct edge mode counting for Jain 01 state is nonexistent due to reduced degree of freedom in Landau levels after LLL-projection. It has been conjectured in Ref. \(^{36} \) that the impossibility of finding exact parent Hamiltonian could be related to descendant fields in the CFT correlator. Indeed, introducing descendant fields as in Eqs. \(^{13} \) \(^{17} \) consequently introduces derivatives to the wave function, thus the wave function does not have property of heredity when the particle number of the system increases by 1. Below we introduce the notion of heredity. In our previous work on Laughlin state and unprojected Jain’s \( \nu = 2/5 \) states \(^{37} \), \(^{38} \), each of which is the unique densest zero energy ground state of a parent Hamiltonian, it is found that recursive formula in particle number \( N \) for the densest zero energy ground state of the parent Hamiltonian is of the following form,

\[
|\psi_{N+1}\rangle \propto \sum_{m} c_{n,m}^{\dagger} G_{r_{\text{max}}-m} |\psi_{N}\rangle. \tag{32}
\]

Here \( r_{\text{max}} \) is the maximum occupied orbital in \( |\psi_{N+1}\rangle \)

and \( G_{r_{\text{max}}-m} \) is some zero mode generator which gives a new zero mode when acting on an existing zero mode if \( r_{\text{max}} > m \). If \( r_{\text{max}} = m \), \( G_{0} \) is defined as the identity operator. \( G_{r_{\text{max}}-m} \) is automatically set to 0 if \( r_{\text{max}} < m \). Therefore, \( |\psi_{N+1}\rangle \) contains a term proportional to \( c_{n,m}^{\dagger} |\psi_{N}\rangle \). We term this property of wave function the heredity when the particle number of the system increases. Due to the existence of derivatives in their wave function, all Jain \( n_{1}n_{2} \) states do not possess the property of heredity. Using method of contradiction, the parent Hamiltonians for these states do not exist. Otherwise, they would have recursive formula following the same logic in Refs. \(^{37} \) and \(^{38} \) and thus would have the property of heredity.

**VII. CONCLUSION AND DISCUSSION**

In this paper, we have constructed a series of \( \nu = 2/5 \) fractional quantum Hall trial wave functions from CFT, and discovered their one-to-one correspondence with Jain’s composite fermion wave functions constructed using different composite fermion Landau levels (LLs). The forms of CFT wave functions are simpler than those of composite fermion wave functions as seen in Eqs. \(^{20} \) and \(^{22} \) yet the former and the latter have the same topological properties as long as they are constructed from the same two composite fermion LLs. Among \( \nu = 2/5 \) CFT wave functions, those corresponding to different composite fermion LLs are in different topological sectors and are distinguished by topological shifts and multi-body relative angular momenta. One thing we need to pay special attention to is filling factors of all these CFT wave functions are not exactly 2/5 for finite number of particles. In fact, their filling factors deviate from 2/5 for finite number of particles and approach 2/5 only in the thermodynamic limit. Furthermore, their exact forms are difficult to deal with for large number of particles due to the action of taking derivatives and subsequent antisymmetrization. As a result, it is not easy to calculate their excitation energies over the ground state in their individual topological sector. A possible way to circumvent this difficulty is to approximate CFT wave functions from Eq. \(^{22} \) via Jain’s approximate projection \(^{39} \), \(^{40} \). Thus we leave the task of calculating their energies over the Coulomb ground state to future work.

In the procedure of trying to find parent Hamiltonians for these CFT wave functions, we discovered another two interesting states. One is a \( \nu = 3/7 \) state as the densest zero energy ground state of \( P_{3}^{(3)} \), \( P_{4}^{(10)} \) and \( P_{4}^{(11)} \). Its first-quantized wave function given in Eq. \(^{30} \) involves clusters of three particles, but the way it goes to zero when several particles come together is obviously distinct from that for Read-Rezayi \( \mathbb{Z}_{3} \) state. It will be of interest to study its CFT nature in the future. Another one is the unique densest zero energy ground state of \( P_{4}^{(6)} \) and \( P_{4}^{(8)} \), for which we only get the second-quantized wave function for up to 10 particles. This state has root pattern.
111000110100110011, with a filling factor close to 1/2 for finite number of particles. It is worth studying whether its filling factor is 1/2 in the thermodynamic limit. If so, it would be useful to study its gap and compare this state with other $\nu = 1/2$ candidate states.

ACKNOWLEDGMENTS

Part of this work is supported by US DOE, Office of BES through Grant No. [de-sc0002140], and performed at the National High Magnetic Field Laboratory, which is supported by NSF Cooperative Agreements No. DMR-1157490 and DMR-1644779, and the State of Florida. L.C. acknowledges support from NSFC Grant No. 11947027. The authors also thank Alexander Seidel and Duncan Haldane for helpful discussions.

Appendix A: Root pattern

We can always expand many-body quantum Hall wave functions in terms of Slater determinants of simultaneous single-particle eigenstates of one-body Hamiltonian and a one-body operator reflecting the symmetry of the geometry in which the quantum Hall system resides. (For example, on disk this operator is the single-particle angular momentum operator.)

$$|\psi\rangle = \sum_{\{n\}} C_{\{n\}} |\{n\}\rangle .$$  \hspace{1cm} (A1)

where $|\{n\}\rangle$ is such a Slater determinant. This Slater determinant is labeled by patterns such as $1_{n_0}01_{n_1}1_{n_2}0\ldots$, where 1 denotes occupied orbital and 0 denotes unoccupied orbital. The subscripts $n_0, n_1\ldots$ are LL indices of occupied orbitals. Of all patterns resulting from such a Slater determinant expansion, there is a special one known as root pattern in the sense that all patterns other than root pattern can be obtained from root pattern via inward squeezing. Inward squeezing involves inward pair hoppings of two particles while conserving center-of-mass of orbitals of these two particles. For example, $01_{n_1}1_{n_2}0$ can be obtained from $1_{n_1}001_{n_2}$ via inward squeezing. Note that inward squeezing can change LL indices of occupied orbitals. Another example is Slater determinant expansion of $\nu = 1/3$ Laughlin state of 3 particles on disk. Since this state resides entirely in LLL, we omit LL indices below for simplicity. Patterns of Slater determinants in its expansion are 1001001, 0101010, 0011100, and 0110001, with a filling factor close to 1.

Appendix B: Three-body and four-body relative angular momentum projectors on disk

The first-quantized three-particle states of general relative angular momentum are given in Ref. \[42\] Here we give the explicit second-quantized form of operators which projects onto a three-body state of relative angular momentum 3 in LLL on disk,

$$P_3^{(3)} = \sum_{3R-3\in\mathbb{N}} Q_R^{(3)} Q_R^{(3)} ,$$  \hspace{1cm} (B1)

with

$$Q_R^{(3)} = \sum_{i_1<i_2<i_3\atop i_1+i_2+i_3=3R} (i_1-i_2)(i_1-i_3)(i_2-i_3) \times \frac{(3R-3)!}{i_1!i_2!i_3!} c_{0,i_1} c_{0,i_2} c_{0,i_3} .$$  \hspace{1cm} (B2)

$R$ is center-of-mass angular momentum and $c_{0,i}$ is a fermionic operator which annihilates a fermion of angular momentum $i$ in LLL. Following the way in Ref. \[42\] we can also easily derive the second-quantized form of operators, each of which projects onto four-body state in LLL on disk of relative angular momentum 6, 8, 9,10 and 11, respectively.

$$P_4^{(6)} = \sum_{4R-6\in\mathbb{N}} S_R^{(6)} S_R^{(6)} ,$$  \hspace{1cm} (B3)

with

$$S_R^{(6)} = \sum_{i_1<i_2<i_3<i_4\atop i_1+i_2+i_3+i_4=4R} (i_1-i_2)(i_1-i_3)(i_2-i_3)(i_2-i_4)(i_3-i_4) \sqrt{\frac{(4R-6)!}{i_1!i_2!i_3!i_4!}} c_{0,i_1} c_{0,i_2} c_{0,i_3} c_{0,i_4} .$$  \hspace{1cm} (B4)

$$P_4^{(8)} = \sum_{4R-8\in\mathbb{N}} S_R^{(8)} S_R^{(8)} ,$$  \hspace{1cm} (B5)
with

$$S_R^{(8)} = \sum_{i_1<i_2<i_3<i_4} f_8(i_1, i_2, i_3, i_4)(i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \quad (B6)$$

$$\times \sqrt{\frac{(4R-8)!}{i_1!i_2!i_3!i_4!}} c_{0,i_4} c_{0,i_3} c_{0,i_2} c_{0,i_1},$$

$$P_4^{(9)} = \sum_{4R-9 \in \mathbb{N}} S_R^{(9)} S_R^{(9)}, \quad (B7)$$

with

$$S_R^{(9)} = \sum_{i_1<i_2<i_3<i_4} f_9(i_1, i_2, i_3, i_4)(i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \quad (B8)$$

$$\times \sqrt{\frac{(4R-9)!}{i_1!i_2!i_3!i_4!}} c_{0,i_4} c_{0,i_3} c_{0,i_2} c_{0,i_1},$$

$$P_4^{(10)} = \sum_{4R-10 \in \mathbb{N}} S_R^{(10)} S_R^{(10)}, \quad (B9)$$

with

$$S_R^{(10)} = \sum_{i_1<i_2<i_3<i_4} f_{10}(i_1, i_2, i_3, i_4)(i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \quad (B10)$$

$$\times \sqrt{\frac{(4R-10)!}{i_1!i_2!i_3!i_4!}} c_{0,i_4} c_{0,i_3} c_{0,i_2} c_{0,i_1},$$

$$P_4^{(10)'} = \sum_{4R-10 \in \mathbb{N}} S_R^{(10)'} S_R^{(10)'}, \quad (B11)$$

with

$$S_R^{(10)'} = \sum_{i_1<i_2<i_3<i_4} f_{10}'(i_1, i_2, i_3, i_4)(i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \quad (B12)$$

$$\times \sqrt{\frac{(4R-10)!}{i_1!i_2!i_3!i_4!}} c_{0,i_4} c_{0,i_3} c_{0,i_2} c_{0,i_1},$$

and

$$P_4^{(11)} = \sum_{4R-11 \in \mathbb{N}} S_R^{(11)} S_R^{(11)}, \quad (B13)$$

with

$$S_R^{(11)} = \sum_{i_1<i_2<i_3<i_4} f_{11}(i_1, i_2, i_3, i_4)(i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \quad (B14)$$

$$\times \sqrt{\frac{(4R-11)!}{i_1!i_2!i_3!i_4!}} c_{0,i_4} c_{0,i_3} c_{0,i_2} c_{0,i_1}.$$
can be expressed in terms of elementary symmetric polynomials in \( i_1, i_2, i_3 \) and \( i_4 \).

\[
f_8 = 3e_1^4 - 8e_2 - 15e_1 + 70, \quad \text{(B15a)}
\]

\[
f_9 = e_1^3 - 4e_2e_1 + 8e_3 - 9e_2^2 + 24e_2 + 30e_1 - 120, \quad \text{(B15b)}
\]

\[
f_{10} = 3e_1^4 - 16e_2e_1^2 + 20e_2^2 + 4e_3e_1 - 16e_4
- 42e_1^3 + 128e_2e_1 - 96e_3
+ 321e_1^2 - 656e_2 - 1030e_1 + 2924, \quad \text{(B15c)}
\]

\[
f_{10}' = 3e_1^4 - 16e_2e_1^2 + 16e_2^2 + 16e_3e_1 - 64e_4
- 42e_1^3 + 144e_2e_1 - 192e_3
+ 289e_1^2 - 640e_2 - 870e_1 + 2656, \quad \text{(B15d)}
\]

\[
f_{11} = 3e_1^5 - 20e_2e_1^3 + 24e_3e_1^2 + 32e_2^2e_1 - 64e_2e_3
- 60e_1^4 + 292e_2e_1^2 - 256e_2^2 - 232e_3e_1 + 256e_4
+ 553e_1^3 - 1792e_2e_1 + 1904e_3
- 2940e_1^2 + 6160e_2 + 7980e_1 - 20944, \quad \text{(B15e)}
\]

where four elementary symmetric polynomials in \( i_1, i_2, i_3 \) and \( i_4 \) are \( e_1 = i_1 + i_2 + i_3 + i_4 \), \( e_2 = i_1i_2 + i_1i_3 + i_1i_4 + i_2i_3 + i_2i_4 + i_3i_4 \), \( e_3 = i_2i_3i_4 + i_1i_3i_4 + i_1i_3i_4 + i_2i_3i_4 \), and \( e_4 = i_1i_2i_3i_4 \).

Note that there are two independent four-particle fermionic states of relative angular momentum \( 10 \). In the above equations, we have omitted the normalization factors of all \( Q_R \), which only depend on center-of-mass angular momentum \( R \), since we are only interested in finding the common densest zero energy ground state of \( Q_R \) of all possible \( R \).

**Appendix C: The necessary conditions for general \( \nu = \frac{p}{2p+1} \) CF Jain states on disk to have non-vanishing LLL-projection**

Let us consider the simplest case in the first place, which is \( p = 2 \). We begin with \( \nu = 2/5 \) Jain state constructed from CFs filling \( n_1 \)-th and \( n_2 \)-th LLs. The number of CFs in \( n_1 \)-th and \( n_2 \)-th LL are \( N_1 \) and \( N - N_1 \), respectively. \( N_1 \) must satisfy the constraint \( 1 \leq N_1 \leq N - 1 \). The wave function of this state is

\[
\prod_{1 \leq i < j \leq N} (z_i - z_j)^2 \begin{vmatrix}
\eta_{n_1,m_1}(z_1) & \cdots & \eta_{n_1,m_1}(z_N) \\
\vdots & \ddots & \vdots \\
\eta_{n_2,m_{N_1+1}}(z_1) & \cdots & \eta_{n_2,m_{N_1+1}}(z_N)
\end{vmatrix}.
\]

We can expand the Laughlin-Jastrow factor \( \prod_{1 \leq i < j \leq N} (z_i - z_j)^2 \) as

\[
\sum_{\eta_1 + \eta_2 + \cdots + \eta_N = (N-1)N} \left| C_{i_1,i_2,\ldots,i_N} \right|^2 = \sum_{\eta_1 + \eta_2 + \cdots + \eta_N = (N-1)N} \left| C_{i_1,i_2,\ldots,i_N} \right|^2,
\]

where \( C_{i_1,i_2,\ldots,i_N} \) is the expansion coefficient. Since this Laughlin-Jastrow factor is symmetric in all variables, \( C_{i_1,i_2,\ldots,i_N} \) will be invariant under the exchange of any two indices. Then the wave function can be expanded as

\[
\sum_{\eta_1 + \eta_2 + \cdots + \eta_N = (N-1)N} \left| C_{i_1,i_2,\ldots,i_N} \right|^2 = \sum_{\eta_1 + \eta_2 + \cdots + \eta_N = (N-1)N} \left| C_{i_1,i_2,\ldots,i_N} \right|^2 \]

where we have used this symmetry of \( C_{i_1,i_2,\ldots,i_N} \). Now with the identity

\[
z^i \eta_{n,m}(z) = 2^N \sum_{k=0}^{i} \binom{i}{k} \frac{n!(n+m+i-k)!}{(n-k)!(n+m)!} \eta_{n-k,m+i}(z),
\]

the above expansion of wave function can be further simplified as
Eq. (C5) can be used to obtain composite fermion wave function projected to any LL. We can immediately see that in order for this wave function to have non-vanishing LLL-projection, each entry in the Slater determinant in Eq. (C5) must be LLL single-particle wave function. We then have \( k_1, k_2, \ldots, k_N \), \( n_1 \) and \( k_{N+1}, \ldots, k_N = n_2 \).

Thus we must have the following constraints on \( n_1 \) and \( n_2 \),

\[
\begin{align*}
n_1 & \leq i_1, i_2, \ldots, i_N, \\
n_2 & \leq i_{N+1}, i_{N+2}, \ldots, i_N.
\end{align*}
\]

Since \( i_1, i_2, \ldots, i_N \) are arbitrary, yet simultaneously satisfy two constraints, \( i_1 + i_2 + \cdots + i_N = (N-1)N \) and \( 0 \leq i_1, i_2, \ldots, i_N \leq 2(N-1) \), we immediately obtain equivalent constraints,

\[
N_1 n_1 + (N - N_1)n_2 \leq (N-1)N, \quad 1 \leq N_1 \leq N - 1, \\
n_1, n_2 \leq 2(N-1).
\]

We can easily generalize this analysis to \( \nu = \frac{p}{2p+1} \) CF Jain states on sphere to have non-vanishing LLL-projection.

### Appendix D: The necessary conditions for general \( \nu = \frac{p}{2p+1} \) CF Jain states on sphere to have non-vanishing LLL-projection

Let us consider a quantum Hall system on the surface of a sphere of radius \( R \), subject to a radial magnetic field \( B = \frac{kQ}{eR^2} \), where the monopole strength \( Q \) is one half of the flux quanta number \( N_0 \) piercing the sphere. The Hamiltonian of this system is

\[
H = \frac{(p_x + eA_x)^2}{2m_e} + \frac{(p_y + eA_y)^2}{2m_e} + \frac{(p_z + eA_z)^2}{2m_e},
\]

subject to the constraint imposed by the sphere surface

\[
x^2 + y^2 + z^2 = R^2.
\]

Using the gauge \( A = -\frac{Q}{eR} \cot \theta \hat{\phi} \), the Hamiltonian can be written in the sphere coordinate as\(^{13,45,46}\)

\[
H = \frac{\mathbf{A}^2}{2m_e R^2},
\]

where the square of the dynamical angular momentum \( \mathbf{A} \) is

\[
\mathbf{A}^2 = - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} - iQ \cos \theta \right)^2.
\]

The generator of rotations about the origin which commutes with the Hamiltonian is

\[
\mathbf{L} = \mathbf{A} + Q \hat{\mathbf{r}}.
\]

The eigenstates of \( H, L_z^2 \) and \( L_z \) in the \( n \)-th Landau level are monopole harmonics\(^{15,17}\)

\[
\langle \mathbf{r} | Y_{Q,l,m} \rangle = N_{Q,l,m} 2^{-m} (1 - \cos \theta)^{\frac{m-Q}{2}} (1 + \cos \theta)^{\frac{m+Q}{2}} e^{im\phi} p_{l-m}^{Q+m} (\cos \theta),
\]

where the normalization

\[
N_{Q,l,m} = \frac{\sqrt{(2l+1)(l-m)!(l+m)!}}{4\pi(l-Q)!(l+Q)!},
\]

where
$P_{l-m}^{m-Q,m+Q}$ is the Jacobi polynomial, $\theta$ and $\phi$ are polar and azimuthal angles on the sphere, respectively. The eigenvalue of $L_z^2$ is $l(l+1)\hbar^2$, with the total angular momentum $l$ being the sum of monopole charge $Q$ and LL index $n$,

$$l = Q + n.$$  

The eigenvalue of $L_z$ is $m = -l, -l+1, \ldots, l-1, l$.

The $N$-particle wave function at $\nu = 1$ can be written down in terms of spinor variables $u = \cos \frac{\theta}{2} e^{i\phi}$ and $v = \sin \frac{\theta}{2} e^{i\phi}$,

$$\Psi_1 = \prod_{1 \leq i \leq N} v_i^{N-1} \prod_{1 \leq j < k \leq N} (z_j - z_k),$$  

where $z = \frac{\pi}{\nu} = \cot \frac{\theta}{2} e^{i\phi}$.

Now let us consider $\nu = p/(2p + 1)$ Jain state constructed from CFs filling $n_1, n_2, \ldots, n_p$-th LLs. The number of CFs in $n_i$-th LL is $N_i$. The monopole charge for the $\nu = p$ integer quantum Hall effect of CFs is chosen as $Q^*$, which is different from the monopole charge $Q$ for the $\nu = p/(2p + 1)$ fractional quantum Hall effect of electrons. The relation of $Q$ to $Q^*$ will be revealed in Eq. (D13).

The wave function of this state is

$$\Psi_1^2 \times \begin{vmatrix} Y_{Q^*+n_1,m_1}(r_1) & \cdots & Y_{Q^*+n_1,m_1}(r_N) \\ \vdots & \ddots & \vdots \\ Y_{Q^*+n_2,m_{N_1+1}}(r_1) & \cdots & Y_{Q^*+n_2,m_{N_1+1}}(r_N) \\ \vdots & \ddots & \vdots \\ Y_{Q^*+n_{N_p},m_N}(r_1) & \cdots & Y_{Q^*+n_{N_p},m_N}(r_N) \end{vmatrix}$$  

In the same manner in which we expand the wave function on disk in the previous appendix, here the wave function can be expanded as

$$\sum_{i_1,i_2,\ldots,i_N} C_{i_1,i_2,\ldots,i_N} \frac{v^{2N-2} z^{1} Y_{Q^*+n_1,m_1}(r_1) \cdots v^{2N-2} z^{1} Y_{Q^*+n_1,m_1}(r_N)}{v^{2N-2} z^{1} Y_{Q^*+n_2,m_{N_1+1}}(r_1) \cdots v^{2N-2} z^{1} Y_{Q^*+n_2,m_{N_1+1}}(r_N)} \cdots \frac{v^{2N-2} z^{1} Y_{Q^*+n_{N_p},m_N}(r_1) \cdots v^{2N-2} z^{1} Y_{Q^*+n_{N_p},m_N}(r_N)}{v^{2N-2} z^{1} Y_{Q^*+n_{N_p},m_N}(r_1) \cdots v^{2N-2} z^{1} Y_{Q^*+n_{N_p},m_N}(r_N)}$$  

where $C_{i_1,i_2,\ldots,i_N}$ is the same as that in the previous appendix. Note that each entry in the above Slater determinant is of the form $v^{2N-2} z^{j} Y_{Q^*+n,m}$ with $0 \leq j \leq 2N - 2$. In order to project the wave function to LL, we need to simplify each entry as a combination of monopole harmonics. Observe that

$$v^{2N-2} z^{j} Y_{Q^*+n,m} = N_{Q^*,Q^*+n,m}^{2l-m-N} (1 - \cos \theta)^{m-Q^*+2N-2-j} (1 + \cos \theta)^{m+Q^*+j} e^{i\phi(m-N+1+j)} P_{Q^*+n-m}^{m-Q^*+j} (\cos \theta).$$  

In order to bring the above to the form of monopole harmonics, we define new monopole charge $Q$ and new $L_z$ angular momentum $m'$,

$$Q = Q^* + N - 1,$$

$$m' = m - N + 1 + j.$$  

Using this new definition, Eq. (D12) can be written as

$$v^{2N-2} z^{j} Y_{Q^*+n,m} = N_{Q^*,Q^*+n,m}^{2l-m-N} (1 - \cos \theta)^{m-Q^*+2N-2-j} (1 + \cos \theta)^{m+Q^*+j} e^{i\phi m'} (1 - \cos \theta)^{2N-2-j} P_{Q^*+n-m}^{m-Q^*+j} (\cos \theta).$$  

(D14)
Thus we need to bring \((1 - \cos \theta)^{2N-2-j} P_m^{Q^*,m+Q^*}(\cos \theta)\) to the form of \(P_m^{Q^* - 2N+2+j,m+Q^*+j}(\cos \theta)\), that is, to lower the upper left index of \(P_m^{Q^* - 2N+2+j,m+Q^*+j}(\cos \theta)\) by \(2N - 2 - j\) and raise its upper right index by \(j\). Now with a recursive formula for Jacobi polynomials\(^{15}\)

\[
(2n + \alpha + \beta) P_n^{\alpha,\beta-1}(x) = (n + \alpha + \beta) P_n^{\alpha,\beta}(x) + (n + \alpha) P_{n-1}^{\alpha,\beta}(x),
\]

we can prove that

\[
P_m^{Q^* - 2N+2+j,m+Q^*+j}(\cos \theta) = \sum_{k=0}^{j} d_{j,k} P_m^{Q^* - 2N+2+j,m-k+Q^*+j}(\cos \theta),
\]

where \(d_{j,k}\) (which depends on \(Q^*, n\) and \(m\)) can be obtained recursively,

\[
d_{0,0} = 1,
\]

\[
d_{j,k} = \frac{Q^* + n + m - k + j}{2(Q^* + n)} \frac{d_{j-1,k} - 2k + j}{2(Q^* + n) - 2k + 2} d_{j-1,k-1} - \frac{n - k + 1}{2(Q^* + n) - 2k + 2 + j} d_{j-1,k-1} \quad \text{for} \quad j \geq 2, \quad 1 \leq k \leq j - 1,
\]

\[
d_{j,0} = \frac{Q^* + n + m + j}{2(Q^* + n) + j} d_{j-1,0},
\]

\[
d_{j,j} = \frac{n - j + 1}{2(Q^* + n) - j + 2} d_{j-1,j-1}.
\]

With another recursive formula for Jacobi polynomials\(^{15}\)

\[
(n + \alpha/2 + \beta/2 + 1)(1 - x) P_n^{\alpha+1,\beta}(x) = (n + \alpha + 1) P_n^{\alpha,\beta}(x) - (n + 1) P_{n+1}^{\alpha,\beta}(x),
\]

we obtain

\[
(1 - \cos \theta)^{2N-2-j} P_m^{Q^* - 2N+2+j,m+Q^*+j}(\cos \theta) = \sum_{k=0}^{j} \sum_{k'=0}^{2N-2-j} e_{2N-2-j,k',k} P_m^{Q^* - 2N+2+j,m-k'+Q^*+j}(\cos \theta),
\]

where \(e_{2N-2-j,k',k}\) (which not only depends on \(Q^*, n\) and \(m\), but also on \(k\)) can also be obtained recursively,

\[
e_{0,0} = 1,
\]

\[
e_{t',k'} = \frac{(n - k + k' + 1 - t') e_{t-1,k'} - (Q^* + n - m - k + k') e_{t-1,k'+1}}{Q^* + n - k + k' + 1 + j/2 - t'/2} - \frac{(Q^* + n - m - k + k') e_{t-1,k-1}}{Q^* + n - k + k' + j/2 - t'/2} \quad \text{for} \quad t' \geq 2, \quad 1 \leq k' \leq t' - 1,
\]

\[
e_{t,0} = \frac{Q^* + n - k + 1 + j/2 - t'/2}{Q^* + n - k + j/2} e_{t-1,0},
\]

\[
e_{t,t'} = \frac{Q^* + n - k + t'}{Q^* + n - k + j/2} e_{t-1,t-1}.
\]

Finally we have

\[
v^{2N-2-j} Y_{Q^*,Q^*+n,m} = N_{Q^*} Q^*+n,m\sum_{k=0}^{j} \sum_{k'=0}^{2N-2-j} d_{j,k} e_{2N-2-j,k',k} P_m^{Q^* - 2N+2+j,m-k'+Q^*+j}(\cos \theta),
\]

where \(Q\) and \(m'\) are defined in Eq. \(^{13}\). New LL index \(n'\) is related to old LL index \(n\) by

\[
n' = n - k - (2N - 2 - j - k').
\]

Since \(k \geq 0, k' \leq 2N - 2 - j\) as seen from summation indices, \(n' \leq n\) always holds, which is as expected.

In the below we will show that on sphere we recover the same constraint on LL indices as the case on disk when the thermodynamic limit is taken.

In the thermodynamic limit \(N \to \infty\), the monopole strength \(Q\) and \(Q^*\) also go to infinity. While the magnetic field on sphere \(\frac{\hbar Q}{cR^2}\) is held constant, the sphere radius \(R\) goes to infinity. Consequently, the sphere is locally equivalent to disk in this limit. It is easy to see that in the limit \(Q^* \to \infty\), the \(e_{t',k'}\) in Eq. \(^{20}\) will vanish unless \(k' = t'\). Therefore, \(k'\) only take the value \(2N - 2 - j\) in Eq.
The new LL index \( n' \) in Eq. D22 is thus \( n - k \), which is the same as that in Eq. C5 on disk. It then follows that in the thermodynamic limit, for the \( \nu = p/(2p + 1) \) Jain state on sphere to have non-vanishing LLL-projection, the constraints on indices of filled CF LLs would be the same as those in the case of disk as given in Eq. C8.

By contrast, when particle number and the monopole charge are finite, the new LL index \( n' \) is given by Eq. D22. The minimum of \( n' \) in Eq. D22 must be non-positive in order for each entry in the Slater determinant expansion of the wave function in Eq. C3 to have non-vanishing LLL-projection. In that case, we must have the following constraints on LL indices \( n_1, n_2, \ldots, n_p \),

\[
n_1, n_2, \ldots, n_p \leq 2N - 2. \tag{D23}
\]

Note that the constraint on LL indices in the case of finite particle number allows more choices than those in the thermodynamic limit.

In conclusion, the necessary conditions for general \( \nu = p/(2p + 1) \) CF Jain states on sphere constructed from \( n_1, n_2, \ldots, n_p \)-th CF LLs filled with \( N_1, N_2, \ldots, N_p \) CFs(\( N = \sum_{i=1}^{p} N_i \)) to have non-vanishing LLL-projection in the thermodynamic limit are (1) \( \sum_{i=1}^{p} N_i = (N - 1)N \) and (2) the maximum of LL indices is no greater than \( 2(N - 1) \). When the particle number \( N \) is finite, we only have the second constraint on CF LL indices.

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ically calculated zero energy ground state, \( b^\dagger_{i_1} \) is a bosonic creation operator which creates a boson of angular momentum \( i_1 \hbar \) in LLL.) we can obtain \( C_{i_1, i_2, \ldots, i_N} \). Alternatively, we can calculate \( C_{i_1, i_2, \ldots, i_N} \) by using the recursive formula for bosonic \( \nu = 1/2 \) Laughlin wave function given in Ref. \( ^{37} \).

This identity can be easily proved using the following formula for generalized Laguerre polynomial, \( L^m_n(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} L^{m-k}_n(x) \).

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