Numerical Solutions of Free Convection Boundary Layer Flow on a Solid Sphere with Convective Boundary Conditions

H T Alkasasbeh, M Z Salleh, R M Tahar and R Nazar

1 Faculty of Industrial Science and Technology, University Malaysia Pahang, 26300 UMP Kuantan, Pahang, Malaysia
2 Faculty of Technology, Universiti Malaysia Pahang, 26300 UMP Kuantan, Pahang, Malaysia
3 School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia

Email: zukikuj@yahoo.com

Abstract. The free convection boundary layer flow on a solid sphere with convective boundary conditions has been investigated. The basic equations of boundary layer are transformed into a non-dimensional form and reduced to nonlinear systems of partial differential equations are solved numerically using an implicit finite difference scheme known as the Keller-box method. Numerical results are obtained for the wall temperature, the local heat transfer coefficient and the local skin friction coefficient, as well as the velocity and temperature profiles of the fluid. The features of the flow and heat transfer characteristics for Prandtl number, Pr = 0.7, 7 and 100, the conjugate parameter \( \gamma = 0.05, 0.1, 0.2 \) and the coordinate running along the surface of the sphere, \( 0^\circ \leq x \leq 120^\circ \) are analyzed and discussed.

1. Introduction

Free convection boundary layer flow about a solid sphere represents also an important problem, which is related to numerous engineering applications. This returns to the fact that a sphere presents an important geometry for the study of the convection flow in many engineering applications (spherical storage tanks, packed beds of spherical bodies, etc.). The above mentioned investigations have been reviewed by Jafarpur and Yovanovich [1]. The boundary layer theory was first introduced by a Ludwig Prandtl, in his lecture on “Fluid Motion with Very small Friction” at the Heidelberg Mathematical Congress in August 1904 (Schlichting [2]). Using theoretical considerations together with some simple experiments, Prandtl showed that the flow past a body can be divided into two major parts. The larger part concerns a free stream of fluid, far from any solid surface, which is considered to be in viscid. The smaller part is a thin layer adjacent to the solid surface in which the effects of viscosity are felt. This thin layer where friction effects cannot be ignored is called the boundary layer (see Burmeister [3], Acheson [4]).

1 To whom any correspondence should be addressed.
The analysis of free, forced and mixed convection about a sphere studied by Chen and Mucoglu [5]. Nazar et al. [6-8] considered the free and mixed convection boundary layer flows on a sphere in micropolar and viscous fluids, respectively. The natural convection heat and mass transfer from a sphere in micropolar fluids with constant wall temperature and concentration was presented by Cheng [9]. All the papers above considered the boundary condition of either constant wall temperature or constant heat flux.

It is worth mentioning that Newtonian heating (NH), in which the heat transfer from the surface is proportional to the local surface temperature has been first used by Merkin [1]. Salleh et al. [11-14] considered the free and mixed convection boundary layer flow on a sphere with Newtonian heating in viscous and micropolar fluids, respectively. The free convection boundary layer flow on solid sphere in viscoelastic fluid with Newtonian heating studied by Kasim et al. [15].

On the other hand, the convective boundary conditions in which the heat is supplied through a bounding surface of finite thickness and finite heat capacity recently used by Aziz [16] who obtained the similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition. Subsequently, Makinde and Aziz [17] studied the magnetohydrodynamic (MHD) mixed convection flow from a vertical plate embedded in a porous medium with a convective boundary condition. Ishak et al. [18, 19] obtained the similarity solutions for flow and heat transfer on the thermal boundary layer flow over a moving plate with convective boundary conditions, respectively. The mixed convection boundary-layer flow past a horizontal circular cylinder embedded in a porous medium filled with a nanofluid with convective boundary condition presented by Rashad et al. [20]. The numerical solutions of the steady magnetohydrodynamic two dimensional stagnation point flow of an incompressible nanofluid towards a stretching cylinder with convective boundary condition using fourth-order Runge-Kutta-Fehlberg method with a shooting technique has been investigated by Akbar et al. [21]. Recently, the numerical solutions of stagnation point flow over a stretching surface with convective boundary conditions using the shooting method has been studied by Mohamed et al. [22].

Motivated by the above mentioned studies, therefore, the aim of the present paper is to study the free convection boundary layer flow on a solid sphere with convective boundary conditions. The governing boundary layer equations are first transformed into a system of non-dimensional equations via the non-dimensional variables, and then into non-similar equations before they are solved numerically by the Keller-box method, as described in the books by Cebeci and Bradshaw [23] and Na [24].

2. Mathematical Analyses
Consider a heated sphere of radius $a$, which is immersed in a viscous and incompressible fluid of ambient temperature $T_\infty$. The surface of the sphere is subjected to a convective boundary conditions (CBC), as shown in figure 1.

![Figure 1. Physical model and coordinate system.](image_url)
Under the Boussinesq and boundary layer approximations, the basic dimensional equations of the flow are

\[ \frac{\partial}{\partial x} (\bar{\rho} \bar{u}) + \frac{\partial}{\partial y} (\bar{\rho} \bar{v}) = 0 \]  

(1)

\[ \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} + g \beta (T - T_\infty) \sin \left( \frac{x}{a} \right) \]  

(2)

\[ \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} = \alpha \frac{\partial^2 \bar{T}}{\partial y^2} \]  

(3)

subject to the boundary conditions

\[ \bar{u} = \bar{v} = 0, \quad -k \frac{\partial \bar{T}}{\partial y} = h_f (T_f - T) \text{ at } \bar{y} = 0 \]

\[ \pi \to 0, \ T \to T_\infty \text{ as } \bar{y} \to \infty, \]  

(4)

where \( \bar{u}(x) = a \sin(\bar{x}/a) \), \( \bar{u} \) and \( \bar{v} \) are the velocity components along the \( \bar{x} \) and \( \bar{y} \) directions, respectively. \( T \) is the local temperature, \( g \) is the gravity acceleration, \( \beta \) is the thermal expansion coefficient, \( T_f \) is the temperature of the hot fluid, \( \nu \) is the kinematic viscosity, \( k \) is the thermal conductivity, \( \alpha \) is the thermal diffusivity and \( h_f \) is the heat transfer coefficient.

We introduce now the following non-dimensional variables (Salleh et al. [11], Aziz [16]):

\[ x = \bar{x}, \ y = Gr^{1/4} \left( \frac{\bar{y}}{a} \right), \ r = \bar{r}/a, \]

\[ u = \left( \frac{a}{\nu} \right) Gr^{-1/2} \bar{u}, \ v = \left( \frac{a}{\nu} \right) Gr^{-1/4} \bar{v}, \ \theta = \frac{T - T_\infty}{T_f - T_\infty} \]  

(5)

where \( Gr = g \beta (T_f - T_\infty) \frac{a^3}{\nu^3} \) is the Grashof number. Substituting variables (5) into (1) to (3) then become

\[ \frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial y} = 0 \]  

(6)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + \theta \sin x \]  

(7)

\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} \]  

(8)

The boundary conditions (4) become

\[ u = v = 0, \ \frac{\partial \theta}{\partial y} = -\gamma (1 - \theta) \text{ on } y = 0 \]

\[ u \to 0, \ \theta \to 0 \text{ as } y \to \infty, \]  

(9)
where $Pr$ is the Prandtl number, $\gamma = ah_f Gr^{-1/4} / k$ is the conjugate parameter for the convective boundary conditions, respectively. It is noticed that when $\gamma = 0$ is for the insulated plate and $\gamma \to \infty$ is when the surface temperature is constant. To solve equations (6) to (8), subjected to the boundary conditions (9), we assume the following variables:

$$\psi = x r(x, y), \quad \theta = \theta(x, y),$$

which satisfies the continuity equation (6). Thus, equations (7) and (8) become

$$\frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} + (1 + x \cot x) \frac{\partial \theta}{\partial y} = x \left( \frac{\partial f}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \theta}{\partial y} \right)$$

subject to the boundary conditions

$$f = \frac{\partial f}{\partial y} = 0, \quad \frac{\partial \theta}{\partial y} = -\gamma (1 - \theta) \quad \text{at} \quad y = 0$$

$$\frac{\partial f}{\partial y} \to 0, \quad \theta \to 0 \quad \text{as} \quad y \to \infty$$

It can be seen that at the lower stagnation point of the sphere, $x \approx 0$, equations (12) and (13) reduce to the following ordinary differential equations:

$$f'' + 2f'f' - (f')^2 + \theta = 0 \quad (15)$$

$$\frac{1}{Pr} \theta'' + 2f \theta' = 0 \quad (16)$$

and the boundary conditions (14) become

$$f(0) = f'(0) = 0, \quad \theta'(0) = -\gamma (1 - \theta(0)) \quad \text{(CBC)}$$

$$f' \to 0, \quad \theta \to 0 \quad \text{as} \quad y \to \infty$$

along with $\theta'(0) = -\gamma (1 + \theta(0))$ (NH). Where primes denote differentiation with respect to $y$. The physical quantities of interest in this problem are the local skin friction coefficient, $C_f$, and the local heat transfer coefficient, $Q_u(x)$ which are given by

$$C_f = x \frac{\partial^2 f}{\partial y^2} (x, 0), \quad \text{and} \quad Q_u(x) = \gamma (1 - \theta(x, 0))$$

$$\psi = x r(x, y), \quad \theta = \theta(x, y),$$
where $C_f = \tau_v / (\rho U_w^2)$ is the skin friction coefficient and $\tau_v = \mu (\partial u / \partial y)_{\tau=0}$ is the wall shear stress. At the lower stagnation point of the sphere, $x \approx 0$, the skin friction coefficients and the heat transfer coefficient are measured by $f'(0)$ and $-\theta'(0)$ respectively.

3. Results and Discussion

Equations (12) and (13) subject to the boundary conditions (14) were solved numerically using the Keller-box method for both cases of boundary conditions, Newtonian heating (NH) and convective boundary conditions (CBC) with several parameters considered, namely the Prandtl number $Pr$, the conjugate parameter $\gamma$ and the coordinate running along the surface of the sphere, $x$. The numerical solutions start at the lower stagnation point of the sphere, $x \approx 0$, with initial profiles as given by equations (15) and (16) and proceed round the sphere up to $120^\circ$ (see Nazar et al. [6, 7]). Values of $Pr$ considered are $Pr = 0.7, 7$ and 100. It is worth mentioning that small values of $Pr (\ll 1)$ physically correspond to liquid metals, which have high thermal conductivity but low viscosity, while large values of $Pr (>1)$ correspond to high-viscosity oils. It is also worth to be pointed out that specifically, the Prandtl number considered in this study, namely $Pr = 0.7, 7$ and 100, correspond to air, water and engine oil, respectively.

Tables 1 show the values of the wall temperature $\theta(0)$ for the case of Newtonian heating (NH). Some results reported by Salleh et al. [11] are included in these tables. It is seen that the agreement between the previously published results with the present ones is very good.

For the case of convective boundary conditions (CBC), Table 2 presents the values of the wall temperature $\theta(0)$ and the skin friction coefficient $f'(0)$ at the lower stagnation point of the sphere, $x \approx 0$, when $Pr = 0.7, 7, 100$ and $\gamma = 0.1$. It is found that, when $Pr$ increases, the values of $\theta(0)$ and $f'(0)$ decrease.

Tables 3, 4 and Figures 6, 7 shows the values of the local heat transfer coefficient $Q_{w}(x)$ and the skin friction coefficient, $C_{f}$ for various values of $x$ when $Pr = 0.7, 7, 100$ and $\gamma = 0.1$, respectively. It is found that as $Pr$ increases, the values of $Q_{w}(x)$ increases and $C_{f}$ decrease. On the other hand, for a fixed $Pr$, as $x$ increases, i.e. between the interval $0' \leq x \leq 120^\circ$, the values of $Q_{w}(x)$ decrease and $C_{f}$ increase. From Table 3 and Figures 6, it is seen that the value of $Q_{w}(x)$ is significantly higher at $x = 0^\circ$ than those at $0' < x \leq 120^\circ$, because the sphere temperature is almost equal to fluid temperature at $x = 0^\circ$, and has a different value when $0' < x \leq 120^\circ$. From Table 4 and Figures 7, it is found that the value of $C_{f} = 0$ at $x = 0^\circ$, because at this point the value of the wall shear stress $\tau_{f}$ is very small. On the other hand, the maximum value of $C_{f}$ appears when $x = 120^\circ$, because in this case the value of $\tau_{f}$ is very high.

Figure 2 illustrates the variation of the wall temperature $\theta(x,0)$ with conjugate parameter $\gamma$ when $Pr = 0.7, 7$ and 100. To get a physically acceptable solution, $\gamma$ must be less than or equals to some critical value, say $\gamma_{c}$, i.e. $\gamma \leq \gamma_{c}$, depending on $Pr$. It is found that, the critical value of $\gamma_{c}$ is 0.3766 when $Pr = 0.7$, 0.5971 when $Pr = 7$, and 0.6892 when $Pr = 100$.

The graphs of $\theta(x,0)$ for some values of the Prandtl number $Pr$ when $\gamma = 0.05, 0.1$ and 0.2 are plotted in Figure3. It is found that, as the Prandtl number $Pr$ increases, the wall temperature $\theta(x,0)$ decrease, and $\theta(x,0)$ increase as $\gamma$ increases. For small values of $Pr \ll 1$, the difference value changing is higher than for large values of $Pr \gg 1$, and it is seen that the surface temperature is very sensitive to the Prandtl number variations.
Figures 4 and 5 display the temperature and velocity profiles, respectively, at \( x = 0^\circ, 60^\circ, 90^\circ \) when \( \text{Pr} = 0.7, 7, 100 \) and \( \gamma = 0.1 \), respectively. It is found that as when \( x \) is fixed and \( \text{Pr} \) increases, the temperature and velocity profiles decrease and also the thermal boundary layer thickness decrease. This is because for small values of the Prandtl number \( \text{Pr} \ll 1 \), the fluid is highly conductive. Physically, if \( \text{Pr} \) increases, the thermal diffusivity decreases and this phenomenon leads to the decreasing manner of the energy transfer ability that reduces the thermal boundary layer. On the other hand, in the same figure it has been found that when \( \text{Pr} \) is fixed and \( x \) increases, the temperature, velocity profiles and the thermal boundary layer thickness increase.

**Table 1.** Values of the wall temperature \( \theta(0) \) at the lower stagnation point of the sphere, \( x \approx 0^\circ \), when \( \text{Pr} = 0.7, 1, 7 \) and \( \gamma = 1 \) (NH)

| \( \text{Pr} \) | \( \theta(0) \) | Present | Salleh et al. [11] |
|---|---|---|---|
| 0.7 | 26.4590 | 26.459113 |
| 1 | 17.2876 | 17.287591 |
| 7 | 3.3635 | 3.3634765 |

**Table 2.** Values of the wall temperature \( \theta(0) \) and the skin friction coefficient \( f^*(0) \) at the lower stagnation point of the sphere, \( x \approx 0^\circ \), when \( \text{Pr} = 0.7, 7, 100 \) and \( \gamma = 0.1 \) (CBC)

| \( \text{Pr} \) | \( \theta(0) \) | Present | \( f^*(0) \) | Present |
|---|---|---|---|
| 0.7 | 0.236634 | 0.261164 |
| 7 | 0.148764 | 0.117331 |
| 100 | 0.090019 | 0.042963 |

**Table 3.** Values of the local heat transfer coefficient \( Q_w(x) \) for various values of \( x \) when \( \text{Pr} = 0.7, 7, 100 \) and \( \gamma = 0.1 \) (CBC)

| \( \text{Pr} \) | 0.7 | 7 | 100 |
|---|---|---|---|
| \( x \) | Present | Present | Present |
| 0° | 0.083615 | 0.089018 | 0.092998 |
| 10° | 0.083420 | 0.088868 | 0.092982 |
| 20° | 0.083366 | 0.088832 | 0.092960 |
| 30° | 0.083256 | 0.088769 | 0.092920 |
| 40° | 0.083098 | 0.088673 | 0.092863 |
| 50° | 0.082892 | 0.088551 | 0.092786 |
| 60° | 0.082638 | 0.088400 | 0.092678 |
| 70° | 0.082311 | 0.088213 | 0.092549 |
| 80° | 0.081906 | 0.088005 | 0.092392 |
| 90° | 0.081409 | 0.087735 | 0.092204 |
| 100° | 0.080795 | 0.087446 | 0.091969 |
| 110° | 0.080060 | 0.087112 | 0.091704 |
| 120° | 0.079085 | 0.086677 | 0.091356 |
Table 4. Values of the local skin friction coefficient, $C_f$, for various values of $x$ when $Pr = 0.7, 7, 100$ and $\gamma = 0.1$(CBC)

| $Pr$ | $0^\circ$ | $10^\circ$ | $20^\circ$ | $30^\circ$ | $40^\circ$ | $50^\circ$ | $60^\circ$ | $70^\circ$ | $80^\circ$ | $90^\circ$ | $100^\circ$ | $110^\circ$ | $120^\circ$ |
|------|-----------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 0.7  | 0.000000  | 0.024339   | 0.047659   | 0.071418   | 0.094013   | 0.115443   | 0.135174   | 0.153510   | 0.169912   | 0.184092   | 0.195770   | 0.204355   | 0.209970   |
| 7    | 0.000000  | 0.010401   | 0.020366   | 0.030349   | 0.040072   | 0.049131   | 0.057400   | 0.064763   | 0.071107   | 0.076146   | 0.080327   | 0.083037   | 0.084369   |
| 100  | 0.000000  | 0.003838   | 0.007516   | 0.011241   | 0.014760   | 0.018063   | 0.021113   | 0.023958   | 0.026411   | 0.028611   | 0.030327   | 0.031600   | 0.032472   |

Figure 2. Variation of the wall temperature $\theta(x,0)$ with conjugate parameter $\gamma$ when $Pr = 0.7, 7, 100$

Figure 3. Variation of the wall temperature $\theta(x,0)$ with Prandtl number Pr when $\gamma = 0.05, 0.1, 0.2$
Figure 4: Temperature profiles \( \theta(x, y) \) at \( x = 0^\circ, 60^\circ, 90^\circ \) when \( Pr = 0.7, 7, 100 \) and \( \gamma = 0.1 \)

Figure 5: Velocity profiles \( f'(x, y) \) at \( x = 0^\circ, 60^\circ, 90^\circ \) when \( Pr = 0.7, 7, 100 \) and \( \gamma = 0.1 \)

Figure 6: Variation of local heat transfer coefficient with \( x \) when \( Pr = 0.7, 7, 100 \) and \( \gamma = 0.1 \)
4. Conclusions
In this paper, we have numerically studied the problem of free convection boundary layer flow on a sphere with convective boundary conditions. We are interested to see how the Prandtl number Pr and the conjugate parameter of convective boundary conditions \( \gamma = \frac{ah_f Gr^{-1/4}}{k} \) affect the flow and heat transfer characteristics. We can conclude that, when Pr increases and \( \gamma \) is fixed, the temperature and velocity profiles, the local heat transfer coefficient \( Q_w(x) \) increases and the local skin friction coefficient \( C_f \) decrease, and when Pr is fixed and the coordinate running along the surface of the sphere \( x \) increases, the temperature and velocity profiles and the thermal boundary layer thickness increase. Further, it is seen that an increase in the Prandtl number Pr, results in a decrease of the wall temperature \( \theta(x,0) \). However, increasing of conjugate parameter \( \gamma \) leads to an increase of the wall temperature \( \theta(x,0) \).

Acknowledgement
The authors gratefully acknowledge the financial supports received from the Universiti Malaysia Pahang (RDU 120390 and RDU 121302).

References
[1] Jafarpur K and Yovanovich M 1992 Int. J. Heat. Mass. Transfer 35 2195-2201
[2] Schlichting H 1979 Boundary layer theory New York:McGraw-Hill Inc
[3] Burmeister L C 1993 Convective heat transfer New York:John Wiley & Sons Inc
[4] Acheson D J 1990 Elementary fluid dynamics Oxford University Press
[5] Chen T and Mucoglu A 1977 Int. J. Heat. Mass. Transfer 20 867
[6] Nazar R, Amin N, Grosan T and Pop I 2002a Int.Comm. Heat. Mass. Transfer 29 377
[7] Nazar R, Amin N, Grosan T and Pop I 2002b Int.Comm. Heat. Mass. Transfer 29 1129
[8] Nazar R, Amin N and Pop I 2002c Arab. J. Sci. Eng 27 117
[9] Cheng C Y 2008 Int.Commm. Heat. Mass. Transfer 35 750
[10] Merkin J 1994 Int. J.Heat. Fluid. Flow 15 392
[11] Salleh M Z, Nazar R and Pop I 2010 Acta Applic Math 112 263
[12] Salleh M Z, Nazar R and Pop I 2010 ArchivesMechanics 62 283
[13] Salleh M Z, Nazar R and Pop I 2012 Meccanica 47 1261
[14] Salleh M Z, Nazar R and Pop I 2010 SRX Physics 62 283
[15] Kasim A R, Mohammad N, Aurangzaib S S and Sharidan S 2012 *World Acad. Sci Eng Tech* **64** 628
[16] Aziz A A 2009 *Commun. Nonlinear Sci. Numer. Simul.* **14** 1064
[17] Makinde O and Aziz A 2010 *Int. J. Therm. Sci.* **49** 1813
[18] Ishak A 2010 *Appl. Math. Comput.* **217** 837
[19] Ishak A, Yacob N and Bachok N 2011 *Meccanica* **46** 795
[20] Rashad A, Chamkha A and Modather M 2013 *Comp. Fluids* **86** 380
[21] Akbar N S, Nadeem S, Haq R U and Khan Z 2013 *Chinese. J Aeron* doi.org/10.1016/j.cja.2013.10.008
[22] Mohamed M K A, Salleh M Z, Nazar R and Ishak A 2013 *Boundary Value Problems* **4** 1
[23] Cebeci T and Bradshaw P 1984 *Physical and computational aspects of convective heat transfer* Springer, New York
[24] Na T Y 1979 *Computational methods in engineering boundary value problem* New York: Academic Press