TOTAL-POSITIVITY PRESERVERS

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ABSTRACT. We prove that the only entrywise transforms of rectangular matrices which preserve total positivity or total non-negativity are either constant or linear. This follows from an extended classification of preservers of these two properties for matrices of fixed dimension. We also prove that the same assertions hold upon working only with symmetric matrices; for total-positivity preservers our proofs proceed through solving two totally positive completion problems.

1. Introduction

A rectangular matrix with real entries is called 
totally positive if each of its minors is positive, and 
totally non-negative if each of its minors is non-negative. (The monographs [31, 37] refer to strict total positivity and total positivity instead.) For almost a century, these classes of matrices surfaced in the most unexpected circumstances, and this trend continues in full force today. Although this chapter of matrix analysis remains somewhat recondite, it has reached maturity due to the dedicated efforts of several generations of mathematicians. The foundational work [21], the survey [2], the early monograph [22], and the more recent publications [31, 23, 37, 13] offer ample references to the fascinating history of total positivity, as well as accounts of its many surprising applications. Total positivity continues to make impacts in areas such as representation theory [35, 36, 39], network analysis [38], cluster algebras [9, 18, 19], and combinatorics [11, 12]. A fascinating link between positive Grassmannians, seen as the geometric impersonation of total positivity, and integrable systems [33, 34] is also currently developing at a fast pace.

The aim of the present note is to classify entrywise transforms of matrices which preserve total positivity. The question of why entrywise operations should be considered, rather than the standard matrix functional calculus, has a history in itself. By reversing the chronology, in modern applications, statisticians often apply some form of thresholding or shrinkage to the entries of estimated covariance matrices [41]. Similarly, in probability theory, entrywise operations provide a natural way to construct more realistic models of common physical phenomena that display some levels of sparsity – see [4], for example. But such operations on matrices are much older; recall the Schur product theorem [44]. It was, however, Schönberg who proved in the 1940s the first rigidity theorem for entrywise transforms preserving positivity (that is, positive semi-definiteness) of matrices of all sizes: they must be given by convergent power series with non-negative coefficients [43]. Schönberg’s discovery was part of a larger
project of classifying the invariant distances on homogeneous spaces which are isometrically equivalent to a Hilbert-space distance; see Bochner’s very informative article [10] for more details. This circle of ideas was further extended by the next generation of analysts, to operations which preserve Fourier coefficients of measures [26].

The analogous result to Schöenberg’s theorem for matrices of a fixed size is more subtle and not accessible in closed form. Roger Horn’s doctoral dissertation contains the fundamental observation, attributed by Horn to Löwner, that the size of the positive matrices preserved by a smooth transform imposes non-negativity constraints on roughly the same number of its derivatives [27]. This observation left a significant mark on probability theory [28].

More historical comments and details about the evolution and applications of fixed-size matrix positivity transforms are contained in our recent articles [7, 8]. This study and the study of entrywise transforms preserving total positivity have also recently revealed novel connections to type-A representation theory and to combinatorics. We refer the reader to the recent works [7, 24] and the recent preprint [32] by Khare and Tao for more details.

The aim of the present note is to prove the following theorem.

**Theorem 1.1.** An entrywise transform $F$, acting on rectangular matrices with real entries, preserves total non-negativity if and only if it is constant, so that $F(x) = c$, or linear, so that $F(x) = cx$, with $c \geq 0$. The same holds if “non-negativity” is replaced by “positivity”, and $c > 0$.

In fact, we prove more; we provide a full characterization of entrywise transforms that preserve total non-negativity or total positivity on $m \times n$ matrices, for any fixed values of $m$ and $n$. We also prove the analogous classifications for symmetric square matrices of each size.

The proof strategy is broadly as follows. For preservers of total non-negativity, note that totally non-negative matrices of smaller size can be embedded into larger ones; this allows us to use, at each stage, properties of preservers for lower dimensions. Thus, we show the class of preservers to be increasingly restrictive as the dimension grows, and already for $5 \times 5$ matrices we obtain the main result.

For total positivity, the problem is more subtle: as zero entries are not allowed, one can no longer use the previous technique. Instead, the key observation is that totally positive matrices are dense in totally non-negative matrices, which reduces the problem for continuous functions to the previous case. The next step then is to prove the continuity of all total-positivity preservers; we achieve this by solving two totally positive matrix-completion problems.

**Acknowledgments.** We thank Percy Deift for raising the question of classifying total positivity preservers and pointing out the relevance of such a result for current studies in mathematical physics. We also thank Alan Sokal for his comments on a preliminary version of the paper. D.G. is partially supported by a University of Delaware Research Foundation grant, by a Simons Foundation collaboration grant for mathematicians, and by a University of Delaware strategic initiative grant. A.K. is partially supported by a Young Investigator Award from the Infosys Foundation.
2. Preservers of total non-negativity

We begin by setting notation. Throughout this note, the abbreviation “TN” stands either for the class of totally non-negative matrices, or the total-non-negativity property for a matrix, and similarly for “TP”. The convention $0^0 := 0$ is adopted throughout. Given a domain $I \subset \mathbb{R}$ and a function $F : I \to \mathbb{R}$, the function $F$ acts entrywise on a matrix $A = (a_{ij}) \in I^{m \times n}$ via the prescription $F[A] := (F(a_{ij}))$. We denote the Hadamard powers of $A$ by $A^\alpha := (a_{ij}^\alpha)$.

The main result in this section is ramified according to the size of the matrices in question.

**Theorem 2.1.** Let $F : [0, \infty) \to \mathbb{R}$ be a function and let $d := \min(m, n)$, where $m$ and $n$ are positive integers. The following are equivalent.

1. $F$ preserves TN entrywise on $m \times n$ matrices.
2. $F$ preserves TN entrywise on $d \times d$ matrices.
3. $F$ is either a non-negative constant or
   (a) $(d = 1)$ $F(x) \geq 0$;
   (b) $(d = 2)$ $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 0$;
   (c) $(d = 3)$ $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;
   (d) $(d \geq 4)$ $F(x) = cx$ for some $c > 0$.

**Proof.** That (1) $\iff$ (2) is obvious, since the minors of a $m \times n$ matrix have dimension at most $d$. We will now prove that (2) $\iff$ (3) for each value of $d$.

The result is obvious when $d = 1$, since in this case a matrix is TN if and only if its entry is non-negative.

Suppose $F[-]$ preserves TN on $2 \times 2$ matrices and note that $F(x) \geq 0$ for all $x \geq 0$. Next, consider the following totally non-negative matrices:

$$A(x, y) := \begin{pmatrix} x & xy \\ 1 & y \end{pmatrix} \quad \text{and} \quad B(x, y) := \begin{pmatrix} xy & x \\ y & 1 \end{pmatrix} \quad (x, y \geq 0). \quad (2.1)$$

Considering the determinants of $F[A(x, y)]$ and $F[B(x, y)]$ gives that

$$F(xy)F(1) = F(x)F(y) \quad \text{for all } x, y \geq 0. \quad (2.2)$$

If $F(1) = 0$ then $F(x)F(y) = 0$, so $F(x) = 0$ for all $x \geq 0$. We will therefore assume that $F(1) > 0$. If $F(x) = 0$ for any $x > 0$ then Equation (2.2) implies that $F \equiv 0$, so we assume that $F(x) > 0$ for all $x > 0$. Applying $F$ to the TN matrix

$$\begin{pmatrix} x & \sqrt{xy} \\ \sqrt{xy} & y \end{pmatrix} \quad (x, y \geq 0), \quad (2.3)$$

we conclude that $F(\sqrt{xy})^2 \leq F(x)F(y)$. As a result, the function $G(x) = \ln F(e^x)$ is mid-point convex on $\mathbb{R}$. Also, applying $F$ to the TN matrix

$$\begin{pmatrix} y & x \\ x & y \end{pmatrix} \quad (y \geq x \geq 0)$$

implies that $F$, so $G$, is non-decreasing. By [40, Theorem 71.C], we conclude that $G$ is continuous on $\mathbb{R}$, and so $F$ is continuous on $(0, \infty)$. Moreover, since $F(1) \neq 0$, Equation (2.2) implies

$$\frac{F(xy)}{F(1)} = \frac{F(x)F(y)}{F(1)F(1)},$$
i.e., the function \( F/F(1) \) is multiplicative. From these facts, there exists \( \alpha \geq 0 \) such that \( F(x) = F(1)x^\alpha \) for all \( x > 0 \). Finally, setting \( y = 0 \) in Equation \((2.2)\), we see that

\[
F(0)F(1) = F(x)F(0)
\]

for all \( x \geq 0 \).

Thus either \( F(0) = 0 \) or \( F \equiv F(1) \); in either case, the function \( F \) has the required form. The converse is immediate, and this proves the result in the case \( d = 2 \).

Next, suppose \( F \) preserves \( T \text{N} \) on \( 3 \times 3 \) matrices and is non-constant. Since the matrix \( A \oplus 0_{1 \times 1} \) is totally non-negative if the \( 2 \times 2 \) matrix \( A \) is, we conclude by part (b) that \( F(x) = cx^\alpha \) for some \( c > 0 \) and \( \alpha \geq 0 \). The matrix

\[
C := \begin{pmatrix}
1 & 1/\sqrt{2} & 0 \\
1/\sqrt{2} & 1 & 1/\sqrt{2} \\
0 & 1/\sqrt{2} & 1
\end{pmatrix}
\]

(2.4)

is totally non-negative, and \( \det F[C] = c^3(1-2^{1-\alpha}) \). It follows that \( F \) does not preserve \( T \text{N} \) on \( 3 \times 3 \) matrices when \( \alpha < 1 \). For higher powers, we use the following result \([30, \text{Theorem 4.2}]\); see \([15, \text{Theorem 5.2}]\) for a shorter proof.

\[
\alpha \geq 1 \implies x^\alpha \text{ preserves } T \text{N and } T \text{P on } 3 \times 3 \text{ matrices.} \tag{2.5}
\]

This concludes the proof of the case \( d = 3 \).

Finally, suppose \( F \) is non-constant and preserves \( T \text{N} \) on \( 4 \times 4 \) matrices. Similarly to the above, considering matrices of the form \( A \oplus 0_{1 \times 1} \) gives, by part (c), that \( F(x) = cx^\alpha \) for some \( c > 0 \) and some \( \alpha \geq 1 \). We now appeal to \([15, \text{Example 5.8}]\), which examines Hadamard powers of the family of matrices \( N(\epsilon, x) := 1_{4 \times 4} + xM(\epsilon) \), where

\[
1_{4 \times 4} := \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\quad \text{and} \quad
M(\epsilon) := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 \\
0 & 2 & 4 + \epsilon & 6 + 5\epsilon \\
0 & 3 & 8 & 14 + \epsilon
\end{pmatrix}.
\]

(2.6)

As shown therein, the matrix \( N(\epsilon, x) \) is \( T \text{N} \) for all \( \epsilon \in (0, 1) \) and \( x > 0 \). Moreover, for small \( x \) and any \( \alpha > 1 \), the determinant of the Hadamard power

\[
\det N(\epsilon, x)^{\alpha x} = \epsilon^2 x^3 + \frac{1}{4}(8 - 70c - 59\epsilon^2 - 4\epsilon^3)(\alpha^3 - \alpha^4)x^4 + O(x^5).
\]

Thus \( \det F[N(\epsilon, x)] < 0 \) for sufficiently small \( \epsilon = \epsilon(\alpha) > 0 \) and \( x > 0 \). We conclude that \( F(x) = cx \) if \( d = 4 \). More generally, if \( F \) preserves \( T \text{N} \) on \( d \times d \) matrices, where \( d \geq 4 \), then \( F \) also preserves \( T \text{N} \) on \( 4 \times 4 \) matrices, and so \( F(x) = cx \) for some \( c > 0 \), as desired. The converse is immediate. \( \square \)

Remark 2.2. Given positive integers \( m, n \) and \( r \), where \( r \leq \min(m, n) \), let \( T_{N_{r}}(m, n) \) denote the set of \( m \times n \) matrices that have all \( k \times k \) minors non-negative, for \( k = 1, \ldots, r \). Such matrices are said to be totally non-negative of order \( r \). Theorem [2.1] immediately classifies all entrywise functions preserving this set, since the entrywise transform \( F[-] \) is an endomorphism of \( T_{N_{r}}(m, n) \) if and only if \( F \) preserves \( T \text{N} \) entrywise on the set of \( r \times r \) \( T \text{N} \) matrices.
2.1. Symmetric totally non-negative matrices. Theorem 2.1 has a natural analogue for totally non-negative matrices which are symmetric. Note that any such matrix has non-negative principal minors and is therefore positive semidefinite.

Theorem 2.3. Let $F : [0, \infty) \to \mathbb{R}$ and let $d$ be a positive integer. The following are equivalent.

1. $F$ preserves TN entrywise on symmetric $d \times d$ matrices.
2. $F$ is either a non-negative constant or
   (a) $(d = 1)$ $F \geq 0$;
   (b) $(d = 2)$ $F$ is non-negative, non-decreasing, and multiplicatively mid-convex, i.e., $F(\sqrt{xy})^2 \leq F(x)F(y)$ for all $x, y \in [0, \infty)$, so continuous;
   (c) $(d = 3)$ $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;
   (d) $(d = 4)$ $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \in \{1\} \cup [2, \infty)$;
   (e) $(d \geq 5)$ $F(x) = cx$ for some $c > 0$.

Proof. The result is trivial when $d = 1$. When $d = 2$, a symmetric matrix is TN if and only if it is positive semidefinite, so part (b) follows immediately from [25, Theorem 2.5].

Now, suppose $F$ preserves TN entrywise on symmetric $3 \times 3$ matrices and is non-constant. Considering matrices of the form $A \oplus 0_{1 \times 1}$, it follows from part (b) that $F$ is continuous and non-decreasing on $(0, \infty)$. Applying $F$ entrywise to the matrix $x \text{Id}_3$ for $x > 0$, where $\text{Id}_3$ is the $3 \times 3$ identity matrix, it follows easily that $F(0) = 0$. Next, let $L := \lim_{\delta \to 0^+} F(\delta)$, which exists since $F$ is non-decreasing, and let $C$ be the TN matrix in Equation (2.4). Then $0 \leq \lim_{\delta \to 0^+} \det F[\epsilon C] = -L^2$, whence $L = 0$. Thus $F$ is continuous on $[0, \infty)$. Next, consider the symmetric totally non-negative matrices

$$A'(x, y) := \begin{pmatrix} x^2 & x & xy \\ x & 1 & y \\ xy & y & y^2 \end{pmatrix}$$

and

$$B'(x, y) := \begin{pmatrix} x^2y & xy & x \\ xy & y & 1 \\ x & 1 & 1/y \end{pmatrix}$$

$(x \geq 0, y > 0)$.

Note that $A'(x, y)$ contains the matrix $A(x, y)$ from Equation (2.4) as a submatrix, and the same is true for $B'(x, y)$ and $B(x, y)$. As in the proof of Theorem 2.1(a), it follows that

$$F(xy)F(1) = F(x)F(y) \quad \text{for all } x, y \geq 0.$$ 

Proceeding as there, and noting that the matrix $C$ from Equation (2.4) is symmetric, we obtain $c > 0$ and $\alpha \geq 1$ such that $F(x) = cx^\alpha$. Moreover, each function of this form preserves TN entrywise, by (2.5). This concludes the proof of part (c).

To prove (d), we suppose the non-constant function $F$ preserves TN on symmetric $4 \times 4$ matrices, and use part (c) with the usual embedding to obtain $c > 0$ and $\alpha \geq 1$ such that $F(x) = cx^\alpha$. To rule out $\alpha \in (1, 2)$, let $x \in (0, 1)$ and note that the infinite matrix $(1 + x^{i+j})_{i,j \geq 0}$ is the moment matrix of the two-point measure $\delta_1 + \delta_x$. Its leading principal $4 \times 4$ submatrix $D$ is TN, by classical results in the theory of moments [21, 15], but if $\alpha \in (1, 2)$ then $D^{\alpha}$ is not positive semidefinite, hence not TN, by [21, Theorem 1.1]. The converse follows from [15, Proposition 5.6]. This proves (d).

Finally, suppose $F$ is non-constant and preserves TN on $5 \times 5$ symmetric matrices, and apply part (d) to obtain $c > 0$ and $\alpha \in \{1\} \cup [2, \infty)$ such that $F(x) = cx^\alpha$. To rule out the case $\alpha \geq 2$, we appeal to [15, Example 5.10], which studies the symmetric,
totally non-negative matrices

\[ T(x) := 1_{5 \times 5} + x \begin{pmatrix} 2 & 3 & 6 & 14 & 36 \\ 3 & 6 & 14 & 36 & 98 \\ 6 & 14 & 36 & 98 & 276 \\ 14 & 36 & 98 & 284 & 842 \\ 36 & 98 & 276 & 842 & 2604 \end{pmatrix} \ (x > 0). \] (2.7)

It is shown there that, for every \( \alpha > 1 \), there exists \( \delta = \delta(\alpha) > 0 \) such that the upper right \( 4 \times 4 \) submatrix of \( T(x)^{\alpha} \) has negative determinant whenever \( x \in (0, \delta) \). It now follows that \( F(x) = cx \) if \( d = 5 \). The general case, where \( d \geq 5 \), follows by the usual embedding trick, and the converse is once again immediate.

2.2. Hankel totally non-negative matrices. For completeness, we conclude this section by discussing a refinement of the question considered above: what are the preservers of TN on Hankel matrices? As the study of such preservers, and of positivity preservers for Hankel matrices, is carried out in great detail in other papers \([8, 32]\), here we confine ourselves to making a few remarks in this restricted setting.

As explained in \([8]\), TN Hankel matrices constitute a test set that is closed under addition, multiplication by non-negative scalars, entrywise products, and pointwise limits. In particular, this test set, in each fixed dimension, is a closed convex cone. As the functions 1 and \( x \) preserve total non-negativity when applied entrywise, this implies the same holds for any absolutely monotonic function \( \sum_{k=0}^{\infty} c_k x^k \), where the Maclaurin coefficient \( c_k \geq 0 \) for all \( k \). It is natural to ask if there are any other preservers. In \([8]\), we show that, up to a possible discontinuity at the origin, there are no others.

**Theorem 2.4** (\([8]\)). Given a function \( f : [0, \infty) \to \mathbb{R} \), the following are equivalent.

1. Applied entrywise, \( f[-] \) preserves TN for Hankel matrices of all sizes.
2. Applied entrywise, \( f[-] \) preserves positivity for TN Hankel matrices of all sizes.
3. \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) on \( (0, \infty) \) with \( c_k \geq 0 \) for all \( k \), and \( 0 \leq f(0) \leq c_0 \).

Theorem 2.4 thus completely resolves the problem of characterising entrywise TN preservers on the set of Hankel matrices of all dimensions.

We conclude this section with some recent progress on this problem in the fixed-dimension context. The following result provides a necessary condition, analogous to a result of Horn \([27]\) for positivity preservers.

**Theorem 2.5** (\([8]\)). Suppose \( f : [0, \infty) \to \mathbb{R} \) is such that \( f[-] \) preserves TN on the set of \( d \times d \) Hankel matrices. Then \( f \) is \( d - 3 \)-times continuously differentiable, with \( f, f', \ldots, f^{(d-3)} \) non-negative on \( (0, \infty) \), and \( f^{(d-3)} \) is convex and non-decreasing. If, instead, \( f \) is analytic then the first \( d \) non-zero Maclaurin coefficients of \( f \) are positive.

Theorem 2.5 implies strong restrictions for the class of TN preservers of Hankel matrices. For instance, if one restricts to power functions \( x^\alpha \), the only such preservers in dimension \( d \) correspond to \( \alpha \) being a non-negative integer or greater than \( d - 2 \). The converse, that such functions preserve TN for \( d \times d \) Hankel matrices, was shown in \([15]\). This is the same as the set of entrywise powers preserving positivity on \( d \times d \) matrices, as proved by FitzGerald and Horn \([17]\).

Finally, we note that there exist power series with not all coefficients being non-negative which preserve total non-negativity on Hankel matrices of a fixed dimension.
The question of which of the coefficients can be negative was fully settled in [32]. Again, the characterization is the same as that for the class of positivity preservers, and this coincidence is explained by the following result of Khare and Tao.

Given positive integers \( k \leq d \) and a constant \( \rho \in (0, \infty] \), let \( P^k_d([0,\rho]) \) denote the set of positive semidefinite \( d \times d \) matrices of rank at most \( k \) and with entries in \([0,\rho])\).

**Theorem 2.6 ([32]).** Suppose \( f : [0,\rho) \to \mathbb{R} \) is such that the entrywise map \( f[-] \) preserves positivity on \( P^k_d([0,\rho]) \), where \( k \leq d \) and \( \rho \in [0,\infty) \). Then \( f[-] \) preserves total non-negativity on the set of Hankel matrices in \( P^k_d([0,\rho]) \).

3. **Total-positivity preservers**

We now turn to entrywise transformations which leave invariant the property of total positivity. There are two technical challenges one encounters once the underlying inequalities are strict. First, the embedding technique used to prove Theorem 2.1, which realises totally non-negative \( d \times d \) matrices as submatrices of totally non-negative \( (d+1) \times (d+1) \) matrices, is lost. Second, the crucial property of multiplicative midpoint convexity is no longer available, since the matrices in (2.1) and (2.3) are not always totally positive.

We indicate below how these challenges can be addressed, to once again obtain a complete characterization.

**Theorem 3.1.** Let \( F : (0,\infty) \to \mathbb{R} \) be a function and let \( d := \min(m,n) \), where \( m \) and \( n \) are positive integers. The following are equivalent.

1. \( F \) preserves total positivity entrywise on \( m \times n \) matrices.
2. \( F \) preserves total positivity entrywise on \( d \times d \) matrices.
3. The function \( F \) satisfies
   - (a) \( d = 1 \): \( F(x) > 0 \);
   - (b) \( d = 2 \): \( F(x) = cx^\alpha \) for some \( c > 0 \) and some \( \alpha > 0 \);
   - (c) \( d = 3 \): \( F(x) = cx^\alpha \) for some \( c > 0 \) and some \( \alpha \geq 1 \).
   - (d) \( d \geq 4 \): \( F(x) = cx \) for some \( c > 0 \).

In order to prove Theorem 3.1, we formulate two auxiliary results. The first is as follows.

**Lemma 3.2.** Fix integers \( m \) and \( n \geq 2 \). Every totally positive \( 2 \times 2 \) matrix occurs as the leading principal submatrix of a positive multiple of a totally positive \( m \times n \) generalized Vandermonde matrix.

Lemma 3.2 is an example of a totally positive completion problem [14]. Embedding results are known for arbitrary totally positive matrices, using, for example, the exterior-bordering technique discussed in [13, Chapter 9] or the parametrizations available in [9, 19]. Lemma 3.2 has the advantage of providing an explicit embedding into the well-known class of Vandermonde matrices.

The second result we require is a density theorem derived in 1952 by Whitney, using generalized Vandermonde matrices and the Cauchy–Binet identity.

**Theorem 3.3 ([48, Theorem 1]).** The set of totally positive \( m \times n \) matrices is dense in the set of totally non-negative \( m \times n \) matrices.

With these two observations to hand, we can now classify total-positivity preservers.
Proof of Theorem 3.1. That (1) ⇐⇒ (2) and (3) ⇒ (2) are immediate, with the latter using (2.5) when \( d = 3 \). We now prove (2) ⇒ (3). The case \( d = 1 \) is immediate, so we assume that \( d = 2 \). Considering the action of \( F[\cdot] \) on the matrices

\[
\begin{pmatrix}
y & x \\
x & y
\end{pmatrix} \quad (y > x > 0)
\]

gives that \( F \) takes positive values and is increasing on \((0, \infty)\). Thus \( F \) is Borel measurable and continuous outside a countable set. Let \( a > 0 \) be a point of continuity and consider the totally positive matrices

\[
A(x, y, \epsilon) := \begin{pmatrix}
ax & axy \\
\frac{a-\epsilon}{a} & ay
\end{pmatrix} \quad (x, y > 0, 0 < \epsilon < a).
\]

Then

\[
0 \leq \lim_{\epsilon \to 0^+} \det F[A(x, y, \epsilon)] = F(ax)F(ay) - F(ax)F(a)
\]

and

\[
0 \leq \lim_{\epsilon \to 0^+} \det F[B(x, y, \epsilon)] = F(a)F(axy) - F(ax)F(ay).
\]

Hence letting \( G(x) := F(ax)/F(a) \), we have that

\[
G(xy)G(1) = G(x)G(y) \quad \text{for all } x, y > 0.
\]

Since \( G \) is measurable, classical results of Sierpiński [46] and Banach [6] on the Cauchy functional equation imply there exists \( \alpha \in \mathbb{R} \) such that \( G(x) = G(1)x^{\alpha} \) for all \( x > 0 \). Thus if \( c := F(a)a^{-\alpha} > 0 \), then

\[
F(x) = F(a)(x/a)^{\alpha} = cx^\alpha \quad \text{for all } x > 0.
\]

As \( F \) is increasing, it holds that \( \alpha > 0 \) and (b) follows.

Finally, we suppose \( d \geq 3 \). By Lemma 3.2 and case (b), we obtain \( c, \alpha > 0 \) such that \( F(x) = cx^\alpha \). In particular, \( F \) admits a continuous extension \( \tilde{F} \) to \([0, \infty)\). By Theorem 3.3, we conclude that \( \tilde{F} \) preserves TN entrywise on \( d \times d \) matrices. Theorem 2.1 gives the form of \( \tilde{F} \), and restricting to \((0, \infty)\) shows that \( F \) is as claimed. This proves that (2) ⇒ (3), which completes the proof. \( \square \)

We now provide a proof of Lemma 3.2.

Proof of Lemma 3.2. Given positive constants \( u_1 < \cdots < u_K \) and real constants \( \alpha_1 < \cdots < \alpha_K \), the generalized Vandermonde matrix \( V = (u_i^{\alpha_j})_{i,j=1}^K \) is totally positive [20, Chapter XIII, §8, Example 1]. Applying the same permutation to both the rows and columns of \( V \) preserves TP, so the result also holds if both sets of inequalities are reversed.

Let

\[
A := \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \quad (a, b, c, d > 0, \; ad - bc > 0).
\]

It suffices to show that a positive multiple of this matrix can be embedded inside a \( K \times K \) generalized Vandermonde matrix of the form just discussed, where \( K := \max(m, n) \).

The proof goes through various cases. Suppose first that three entries of \( A \) are equal. Rescaling the matrix \( A \), there are four cases to consider:

\[
A_1 = \begin{pmatrix}
\lambda & 1 \\
1 & 1
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
1 & \mu \\
1 & 1
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
1 & 1 \\
\mu & 1
\end{pmatrix}, \quad A_4 = \begin{pmatrix}
1 & 1 \\
1 & \lambda
\end{pmatrix}.
\]
where $\lambda > 1$ and $0 < \mu < 1$. In the first case, the matrix $A_1$ embeds into any generalized Vandermonde matrix with $u_1 = \lambda$, $u_2 = 1$, $1 > v_3 > \cdots > u_K$ and $\alpha_1 = 1$, $\alpha_2 = 0$, $0 > \alpha_3 > \cdots > \alpha_K$. A similar embedding can easily be constructed for $A_2$, $A_3$, and $A_4$.

Next, suppose two entries in a row or column of $A$ are equal. There are again 4 cases:

$$A_5 = \begin{pmatrix} 1 & 1 \\ \gamma & \delta \end{pmatrix}, \quad A_6 = \begin{pmatrix} \delta & 1 \\ 1 & \gamma \end{pmatrix}, \quad A_7 = \begin{pmatrix} \delta & 1 \\ \gamma & 1 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 1 & \gamma \\ \gamma & \delta \end{pmatrix},$$

where $\delta > \gamma > 0$ and $\delta, \gamma \neq 1$. For $A_5$, take $u_1 = 1$, $u_2 = \gamma$, and $\alpha_1 = 1$, and let $\alpha_2 = \log \delta / \log \gamma$. If $u_1 > u_2$, then $\alpha_2 < 1 = \alpha_1$; similarly, when $u_1 < u_2$, we have that $\alpha_1 < \alpha_2$. Thus $A_5$ embeds as desired. The other cases are similar.

The remaining case is when $\{a, d\} \cap \{b, c\} = \emptyset$. Set $\alpha_1 = 1$. We claim there exist scalars $\mu, u_1, u_2 > 0$ and $\alpha_2$ such that

$$\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_1 & u_2^{\alpha_2} \\ u_2 & u_2^{\alpha_2} \end{pmatrix},$$

and either $u_1 < u_2$ and $\alpha_1 < \alpha_2$, or $u_1 > u_2$ and $\alpha_1 > \alpha_2$. Applying the logarithm entrywise to both matrices and computing the determinants gives that

$$(L + A)(L + D) = (L + B)(L + C),$$

where $L = \log(\mu)$ and $A = \log(a)$, \ldots , $D = \log(d)$. This yields a linear equation in $L$, whence

$$\mu = \exp\left(\frac{\log(b) \log(c) - \log(a) \log(d)}{\log(ad/bc)}\right) > 0.$$ 

Clearly, $u_1 = \mu a$ and $u_2 = \mu c$. Solving for $\alpha_2$ explicitly, we obtain

$$\alpha_2 = \frac{\log(b/d)}{\log(a/c)}.$$

There are now two cases: if $u_1 < u_2$, then $a < c$, so $b/d < a/c < 1$ and $\alpha_2 > 1 = \alpha_1$. If, instead, $u_1 > u_2$, then $a/c > 1$ and $\alpha_2 < 1 = \alpha_1$. In either case, the matrix $\mu A$ is the leading $2 \times 2$ principal submatrix of a generalized Vandermonde matrix of the type being considered. This concludes the proof. \hfill $\Box$

**Remark 3.4.** Lemma 3.2 can be strengthened to the following TP completion result: given integers $m, n \geq 2$, a $2 \times 2$ matrix $A$ occurs as a minor in a totally positive $m \times n$ matrix at any given position (i.e., in a specified pair of rows and pair of columns) if and only if $A$ is totally positive. This is because the proof shows that $A$ may be written as

$$\mu^{-1} \begin{pmatrix} u_1^{\alpha_1} & u_2^{\alpha_2} \\ u_1^{\alpha_1} & u_2^{\alpha_2} \end{pmatrix},$$

where $\mu, u_1, u_2 > 0$ and $\alpha_1, \alpha_2$ are real numbers, with either $u_1 < u_2$ and $\alpha_1 < \alpha_2$, or $u_1 > u_2$ and $\alpha_1 > \alpha_2$. In the former case, and letting $K = \max(m, n)$, we can find $v_1 < \cdots < v_K$ and $\beta_1 < \cdots < \beta_K$ such that $v_p = u_1$, $\beta_q = \alpha_1$, $v_{p'} = u_2$, and $\beta_{q'} = \alpha_2$: a similar claim holds in the latter case. The generalised Vandermonde matrix $(v_i^{\beta_j})$ then realises the required embedding. We thank Prakhar Gupta and Pranjal Warade for this observation.
Remark 3.5. As in the totally non-negative setting, Theorem 3.1 classifies the entrywise maps preserving total positivity of order $r$, for matrices of each fixed size; see Remark 2.2. More precisely, given positive integers $m$, $n$ and $r$, where $r \leq \min(m, n)$, let $TP_r(m, n)$ denote the set of $m \times n$ matrices that have all $k \times k$ minors positive, for $k = 1, \ldots, r$. If $F[-]$ preserves entrywise the totally positive $r \times r$ matrices then it preserves the families $TP_r(m, n)$ whenever $m, n \geq r$. The converse is true for $r = 1$, as well as for $r = 2$ via Lemma 3.2 and Theorem 3.1. Now suppose $r \geq 3$ and $F[-]$ preserves the set $TP_r(m, n)$ for fixed $m, n \geq r$. By Lemma 3.2 and the $r = 2$ case, there exist constants $c, \alpha > 0$ such that $F(x) = cx^\alpha$, so $F$ extends continuously to $\tilde{F} : [0, \infty) \to \mathbb{R}$. Now we appeal to [15, Theorem 2.5], which asserts that $TP_r(m, n)$ is dense in $TN_r(m, n)$. From this it follows that $\tilde{F}$ preserves entrywise the set $TN_r(m, n)$, so, by Remark 2.2, $\tilde{F}$ preserves total non-negativity on $r \times r$ matrices. But then $F$ preserves total positivity on $r \times r$ matrices as claimed, using Theorems 2.1 and 3.1.

3.1. The symmetric case. As in the totally non-negative case, Theorem 3.1 has an analogue for symmetric matrices; compare the following result with Theorem 2.3.

Theorem 3.6. Let $F : (0, \infty) \to \mathbb{R}$ and let $d$ be a positive integer. The following are equivalent.

1. $F$ preserves total positivity entrywise on symmetric $d \times d$ matrices.
2. The function $F$ satisfies
   (a) $(d = 1)$ $F(x) > 0$;
   (b) $(d = 2)$ $F$ is positive, increasing, and multiplicatively mid-convex, i.e.,
       
       $F(\sqrt{xy})^2 \leq F(x)F(y)$ for all $x, y \in (0, \infty)$, so continuous;
   (c) $(d = 3)$ $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \geq 1$;
   (d) $(d = 4)$ $F(x) = cx^\alpha$ for some $c > 0$ and some $\alpha \in \{1\} \cup [2, \infty)$.
   (e) $(d \geq 5)$ $F(x) = cx$ for some $c > 0$.

We now outline our proof philosophy. Akin to Theorem 3.1, the idea is to derive the continuity of $F$ from the $2 \times 2$ case, without the use of multiplicative mid-convexity, and then use the density of symmetric totally positive matrices in symmetric TN matrices. For the first step, we require the solution of a symmetric totally positive completion problem.

Lemma 3.7. Fix an integer $d \geq 2$. Every symmetric totally positive $2 \times 2$ matrix occurs as the leading principal submatrix of a totally positive $d \times d$ Hankel matrix.

To prove Lemma 3.7 it would suffice to embed the matrix

$$
\begin{pmatrix}
1 & a \\
a & b
\end{pmatrix} \\
0 < a < \sqrt{b}
$$

inside the square matrix $\frac{1}{K}V^TV$, where $V = (u_i^{d-1})_{i=1, \ldots, K, j=1, \ldots, d}$ is part of a Vandermonde matrix, with $K \geq d$ and $0 < u_1 < \cdots < u_K$. Translated into probabilistic terms, this is equivalent to finding a uniform random variable supported on $\{u_1, \ldots, u_K\}$ with mean $a$ and variance $b - a^2$. In other words, one needs to solve an inverse moment problem with the size of the support bounded below. (In a sense, this has an opposite conclusion to Carathéodory’s theorem.) While it may be possible to do this using probabilistic methods (see [1], for example), we provide a constructive proof of Lemma 3.7 using a generalization of the Cauchy–Binet identity to “continuous matrices”, via an identity of Andréief [3].
Proof of Lemma 3.7. Let \( a, b > 0 \), with \( b > a^2 \). As noted above, by rescaling, it suffices to find a totally positive \( d \times d \) Hankel matrix \( A = (a_{ij})_{i,j=1}^d \) with \( a_{11} = 1, a_{12} = a \) and \( a_{22} = b \). To construct such a matrix, we first show there exists an increasing function \( h : [0, 1] \to [0, \infty) \) such that

\[
\int_0^1 f(x) \, dx = a \quad \text{and} \quad \int_0^1 f(x)^2 \, dx = b.
\]

If \( f_s(x) := a(s+1)x^s \) for \( s > 0 \), then

\[
\int_0^1 f_s(x) \, dx = a \quad \text{and} \quad \int_0^1 f_s(x)^2 \, dx = \frac{a^2(s+1)^2}{2s+1}.
\]

Since \( h(s) = (s+1)^2/(2s+1) \) is increasing and unbounded on \([0, \infty)\), and \( h(0) = 1 \), we obtain the desired function \( f = f_s \) by taking \( s = s_0 > 0 \) such that \( h(s_0) = b/a^2 \).

Now let

\[
a_{ij} := \int_0^1 f(x)^{i-1} f(x)^{j-1} \, dx = \int_0^1 f(x)^{i+j-2} \, dx. \tag{3.1}
\]

To see that \( A = (a_{ij})_{i,j=1}^d \) is totally positive, we use Andréief’s identity \([5]\) Lemma 3.1],

\[
\det \left( \int \phi_i(x) \psi_j(x) \, dx \right)_{i,j=1}^k = \frac{1}{k!} \int \cdots \int \det(\phi_i(x_j))_{i,j=1}^k \det(\psi_i(x_j))_{i,j=1}^k \, dx_1 \cdots \, dx_k,
\]

with the functions

\[
\phi_i(x) = f(x)^{\alpha_i-1} = (a(s_0+1)x^{\alpha_i})^{\alpha_i-1} \quad \text{and} \quad \psi_j(x) = f(x)^{\beta_j-1},
\]

where \( 1 \leq \alpha_1 < \cdots < \alpha_k \leq d \) and \( 1 \leq \beta_1 < \cdots < \beta_k \leq d \). The claim, hence the result, now follows from the total positivity of generalized Vandermonde matrices. \( \square \)

Remark 3.8. An alternate proof of the total positivity of the Hankel matrix \((3.1)\) may be obtained with the help of the basic composition formula, attributed by Karlin to Pólya and Szegő. See \([31]\) Chapter 1.2, Equations (2.9) and (2.10)], and take \( d\sigma(t) \) to be the differential for Lebesgue measure on \([0, 1] \).

With Lemma 3.7 at hand, we can classify the preservers of total positivity on the set of symmetric matrices.

Proof of Theorem 3.6. The result is clear when \( d = 1 \), so we assume \( d \geq 2 \) henceforth. First, suppose \((1)\) holds. By Lemma 3.7, \( F[-] \) must preserve total positivity for symmetric \( 2 \times 2 \) matrices. As shown in the proof of Theorem 3.1, it follows that \( F \) is positive and increasing on \((0, \infty)\). In particular, \( F \) has countably many discontinuities, and each of these is a jump. Let \( F^+(x) := \lim_{y \to x^+} F(y) \) for all \( x > 0 \). Then \( F^+ \) is increasing, coincides with \( F \) at every point where \( F \) is right continuous, and has the same jump as \( F \) at every point where \( F \) is not right continuous. Applying \( F[-] \) to the totally positive matrices

\[
M(x, y, \epsilon) := \left( \begin{array}{cc} x + \epsilon & \sqrt{xy} + \epsilon \\ \sqrt{xy} + \epsilon & y + \epsilon \end{array} \right), \quad (x, y, \epsilon > 0, \ x \neq y),
\]

it follows that

\[
F(\sqrt{xy} + \epsilon)^2 < F(x + \epsilon)F(y + \epsilon).
\]
Letting $\epsilon \to 0^+$, we conclude that
\[ F^+ (\sqrt{xy})^2 \leq F^+ (x) F^+ (y) \quad \text{for all } x, y > 0; \]
this inequality holds trivially when $x = y$. Thus $F^+$ is multiplicatively mid-convex on $(0, \infty)$. As in the proof of Theorem 2.1 it follows by [10, Theorem 71.C] that $F^+$ is continuous. We conclude that $F$ has no jumps and is therefore also continuous.

For $d = 2$, this completes the proof that $(1) \implies (2)$. If, instead, $d \geq 3$, note that $F$ extends to a continuous function $\tilde{F}$ on $[0, \infty)$. As observed in [15, Theorem 2.6], the set of symmetric totally positive $r \times r$ matrices is dense in the set of symmetric totally non-negative $r \times r$ matrices. By continuity, it follows that $\tilde{F}$ preserves total non-negativity entrywise, and (2) now follows immediately from Theorem 2.3.

Conversely, suppose $(2)$ holds for $d = 2$, and consider the totally positive matrix
\[
A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (a, b, c > 0, \ ac - b^2 > 0).\]
Since $F$ is increasing, we have $F(\sqrt{b^2}) < F(\sqrt{ac})$. Using the multiplicative convexity of $F$, we conclude that
\[
F(b)^2 = F(\sqrt{b^2})^2 < F(\sqrt{ac})^2 \leq F(a)F(c).
\]
Thus $F[A]$ is totally positive and (1) holds. The implications for $d = 3$ and $d = 4$ follow from [15, Theorem 5.2 and Proposition 5.6], respectively, and the case of $d = 5$ is clear.

We conclude this section by extending the Hankel TP completion result of Lemma 3.7 to embedding in “arbitrary position”, in the spirit of Remark 3.4 for Vandermonde TP completions.

**Theorem 3.9.** Let $n$, $k$ and $N$ be integers such that $0 \leq n < n + 2k \leq 2N$. A matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ can be embedded into a TP Hankel matrix $\tilde{A}$, with first and last rows $(s_0, s_1, \ldots, s_N)$ and $(s_N, \ldots, s_{2N})$, at equally spaced points
\[
s_n = a, \quad s_{n+k} = b, \quad \text{and} \quad s_{n+2k} = c
\]
if and only if $A$ is TP.

This embedding realizes $A$ as a principal $2 \times 2$ submatrix of the matrix $\tilde{A}$, or else of its truncation $\tilde{A}^{(1)}$, where the first row and last column of $\tilde{A}$ are removed.

**Remark 3.10.** A natural path to validate Theorem 3.9 is to try to adapt the proof of Lemma 3.7 to this more general situation. However, it may be shown that such an approach cannot work uniformly. More precisely, given
\[
y \in \mathcal{Y} := \{(p, q, s, C, t) : 0 \leq p < q < \infty, \ s, C > 0, \ t > -1\} \subset \mathbb{R}^5
\]
and $\alpha \in [0, \infty)$, let
\[
m_\alpha (y) := \int_p^q f_{s,C}(x)^\alpha \, d\mu_t(x) = \int_p^q (Cx^s)^\alpha x^t \, dx
\]
be the $\alpha$th moment of the function $f_{s,C}(x) = Cx^s$ with respect to the measure with differential $d\mu_t(x) = x^t \, dx$ on $[p, q]$. There exists an absolute constant $C > 0$ such that
\begin{equation}
 m_{\alpha+\beta}(y)^2 \leq m_\alpha(y)m_{\alpha+2\beta}(y) \leq C\frac{(r+1)^2}{1 + 2r} m_{\alpha+\beta}(y)^2
\end{equation}
whenever $r \geq 0$, $\alpha > 0$ and $0 \leq \beta \leq r\alpha$. In particular, suppose $(\alpha, \beta) = (n, k)$, with $n$ and $k$ positive integers, and let $r = k/n$. Then, for no choice of $s > 0$, $t > -1$, and $[p, q] \subset [0, \infty)$ can the symmetric TP matrix \( \begin{pmatrix} 1 & 1 + C(r+1)^2 \\ 1 & 1 + \frac{C(r+1)^2}{1 + 2r} \end{pmatrix} \) be embedded into the $n$th and $(n+k)$th rows and 0th and $k$th columns of the Hankel matrix with $(i, j)$ entry
\[ \int_p^q (Cx^s)^{i+j} \, x^t \, dx, \]
where the row and column indices are taken to begin at 0.

Since the technique used in the proof of Lemma 3.7 cannot be extended to prove Theorem 3.9, we use another strategy, which requires some preliminary ingredients. The first is a well-known 1912 result of Fekete [16]: recall that a minor is contiguous if it is formed from consecutive rows and columns.

**Proposition 3.11.** Suppose $m, n \geq 1$ and let $A$ be an $m \times n$ matrix such that all its contiguous minors are positive. Then $A$ is TP.

From Proposition 3.11 we easily deduce the following corollary. Given a matrix $A$, we denote by $A^{(1)}$ the matrix obtained from $A$ by deleting its first row and last column.

**Corollary 3.12.** A square Hankel matrix $A$ is TP if and only if $A$ and $A^{(1)}$ are positive definite.

**Proof.** One implication is immediate. Conversely, suppose $A$ and $A^{(1)}$ are positive definite. As any contiguous minor of $A$ is a principal minor of either $A$ or $A^{(1)}$, it is positive, so the claim follows by Proposition 3.11. \qed

The next result solves another TP completion problem: it provides a recipe to extend TP Hankel matrices “backwards”.

**Lemma 3.13.** Suppose $A$ is a Hankel matrix, with first and last rows $(s_0, \ldots, s_N)$ and $(s_N, \ldots, s_{2N})$, respectively. Then there exist real numbers $s_{-1}$ and $s_{-2}$ such that the Hankel matrix $A''$, with first and last rows $(s_{-2}, s_{-1}, s_0, \ldots, s_{N-1})$ and $(s_{-1}, \ldots, s_{2N})$, is also TP.

**Proof.** First consider the Hankel matrix $A'$ with first row $(s_{-1}, s_0, \ldots, s_{N-1})$ and last row $(s_{N-1}, \ldots, s_{2N-1})$, where $s_{-1}$ is to be determined. Since the cofactor corresponding to $s_{-1}$ is a principal minor of $A^{(1)}$, so positive, taking sufficiently large $s_{-1} > 0$ will ensure that det $A' > 0$. The other trailing principal minors of $A'$ are also principal minors of $A^{(1)}$, hence positive. It follows that $A'$ is positive definite.

Next, we apply the same recipe to construct a square matrix $A''$ as in the statement of the lemma. The cofactor corresponding to $s_{-2}$ is precisely det $A$, so there exists $s_{-2} > 0$ such that det $A'' > 0$. By the same reasoning as above, $A''$ is positive definite. Since $A' = (A'')^{(1)}$ is also positive definite, we are done by Corollary 3.12. \qed
Equipped with these results, we may now solve the Hankel TP completion problem.

**Proof of Theorem 3.9.** If \( A \) can be embedded inside \( \tilde{A} \) as specified, then \( A \) is clearly TP. Conversely, suppose \( A \) is TP, and let \( n, k \) and \( N \) be as in the statement of the theorem. As shown in Lemma 3.7 there exists a TP Hankel matrix \( B \) with \((i, j)\) entry

\[
b_{i+j} := \int_0^1 C(x^s)^{i+j} \, dx,
\]

where column and row indices start at 0, matching the prescribed entries

\[
b_0 = a, \quad b_1 = b, \quad \text{and} \quad b_2 = c.
\]

Let the \((N + 1) \times (N + 1)\) matrix \( B' \) have \((i, j)\) entry

\[
B'_{ij} := \int_0^1 C(x^{s/k})^{i+j} \, dx.
\]

As in the proof of Lemma 3.7, the Hankel matrix \( B' \) is TP. We now apply the extension construction in Lemma 3.13 to \( B' \) exactly \( \lceil n/2 \rceil \) times, which yields a TP Hankel matrix \( M \). Depending on whether \( n \) is even or odd, the leading principal \((N + 1) \times (N + 1)\) submatrix of \( M \) or of \( M^{(1)} \) is the desired matrix \( \tilde{A} \). \( \square \)

**Remark 3.14.** Lemma 3.7 and Theorem 3.9 provide explicit embeddings of TP symmetric \( 2 \times 2 \) matrices into the class of TP Hankel matrices. Their proofs show that if the leading entry of the \( 2 \times 2 \) matrix is embedded in the leading position, then we obtain an embedding into a moment matrix of the form (3.1). While this is not guaranteed by the above construction for embeddings of the leading entry in other positions, it still happens to be the case. The reader will recognize that behind these considerations lies the truncated Stieltjes moment problem with data \( s_0, s_1, \ldots, s_{2N} \). Dealing with positive-definite forms, i.e., ruling out the degenerate cases, avoids all possible complications. For details, we refer the reader to Theorem 9.36 in the recent monograph [42].

### 3.2. Total-positivity preservers are continuous.

As the vigilant reader will have noticed, we have shown three similar assertions, all of the same flavor. Namely, an entrywise map \( F[-] \) preserves total non-negativity on the set of \( 2 \times 2 \) matrices, or the subset of symmetric matrices, if and only if \( F \) is non-negative, non-decreasing, and multiplicatively mid-convex, with the corresponding changes if weak inequalities are replaced by strict ones. The variation is in reducing the set of test matrices with which to work, while arriving at very similar conclusions.

Such a result was proved originally by Vasudeva [47], when classifying the entrywise preservers of positive semidefiniteness for \( 2 \times 2 \) matrices with positive entries. This remains to date the only known classification of positivity preservers in a fixed dimension greater than 1.

It is natural to seek a common strengthening of the results above, as well as of Vasudeva’s result. We conclude by recording for completeness such a characterization, which uses a small test set of totally positive \( 2 \times 2 \) matrices.
**Theorem 3.16.** Let \( F : (0, \infty) \to \mathbb{R} \) be a function. The following are equivalent.

1. The map \( F[-] \) preserves total non-negativity on the set \( \mathcal{P} \).
2. The function \( F \) is non-negative, non-decreasing, and multiplicatively mid-convex on \((0, \infty)\).
3. The map \( F[-] \) preserves positive semidefiniteness on the set \( \mathcal{P}' \cup \mathcal{P}'' \).

Moreover, every such function is continuous, and is either nowhere zero or identically zero.

The sets \( \mathcal{P}' \) and \( \mathcal{P}'' \) are in bijection with the sets \((\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R})\) and \( \mathbb{Q} \), respectively, whereas \( \mathcal{P} \) is a three-parameter family. The equivalence of (1) and (2) is Vasudeva’s result.

**Proof.** To see that (2) \( \implies \) (1), note that if \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{P} \), then \( a, b, c > 0 \) and \( 0 < b \leq \sqrt{ac} \). By (2), the matrix \( F[A] \) has non-negative entries and

\[
0 \leq F(b)^2 \leq F(\sqrt{ac})^2 \leq F(a)F(c),
\]

so \( F[A] \) is totally non-negative. Clearly, (1) \( \implies \) (3). The main challenge in the proof is to show (3) \( \implies \) (2). The first step is to observe that \( F \) is non-negative and non-decreasing on \((0, \infty)\). Let \( y > x > 0 \), choose rational \( a \) such that \( x < a < y \), and consider the matrices \( F[A(a, x)] \) and \( F[A(y, a)] \), which are both positive semidefinite. From this, it follows that \( F(y) \) is non-negative, and \( F(y)^2 \geq F(a)^2 \geq F(x)^2 \).

We now show that \( F \) is identically zero if it vanishes anywhere. Suppose \( F(x) = 0 \) for some \( x > 0 \). Then, as \( F \) is non-decreasing and non-negative, \( F \equiv 0 \) on \((0, x]\). Given \( y > x > 0 \), choose rational \( b \) and \( c \) such that \( 0 < c < x < y < b \). Considering \( F[B(1 + (b^2/c), b, c)] \) and then \( F[A(y, b)] \) shows that \( F(b) = 0 \) and then \( F(y) = 0 \). It follows that \( F \equiv 0 \).

Finally, we claim that \( F \) is multiplicatively mid-convex and continuous. Clearly this holds if \( F \equiv 0 \), so we assume that \( F \) is never zero. We first show that the function

\[
F^+ : (0, \infty) \to [0, \infty); \quad F^+(x) := \lim_{y \to x^+} F(y)
\]

is multiplicatively mid-convex and continuous. Note that \( F^+ \) is well defined because \( F \) is monotone. Given \( x, y > 0 \), we choose rational numbers \( a_n \in (x, x + 1/n) \) and \( c_n \in (y, y + 1/n) \) for each positive integer \( n \). Since \( an, c_n \) are rational, we may choose rational \( b_n \in (\sqrt{xy}, \sqrt{an,c_n}) \). The matrix \( B(a_n, b_n, c_n) \in \mathcal{P}'' \) for each \( n \), therefore

\[
0 \leq \lim_{n \to \infty} \det F[B(a_n, b_n, c_n)] = F^+(x)F^+(y) - F^+(\sqrt{xy})^2.
\]

Thus \( F^+ \) is multiplicatively mid-convex on \((0, \infty)\), and \( F^+ \) is non-decreasing since \( F \) is. Repeating the argument in the proof of Theorem 2.11 which requires the function
to take positive values, it follows that $F^+$ is continuous. Hence $F$ and $F^+$ are equal, and this gives the result. □

**Remark 3.17.** The analogous version of Theorem 3.16 holds for any bounded domain, that is, for matrices with entries in $(0, \rho)$, with $\rho > 0$. The proof is a minimal modification of that given above, except for the argument to show that either $F \equiv 0$ or $F$ vanishes nowhere. For this, see [25, Proposition 3.2(2)]. Also, clearly the set of rational numbers in the definitions of $\mathcal{P}'$ and $\mathcal{P}''$ may be replaced with any countable dense subset of the domain.

**Concluding remarks.** For the reader interested solely in the dimension-free setting of Theorem 1.1, we end with some toolkit observations. Our analysis provides a small test set of matrices for obtaining the conclusions of that theorem: if a function $F : [0, \infty) \to \mathbb{R}$ preserves TN on (a) the set of all $2 \times 2$ TN matrices, (b) the $3 \times 3$ matrix $C$ from (2.4), and (c) the two-parameter family of $4 \times 4$ TN matrices $N(\epsilon, x)$ defined above (2.6), then $F$ is either constant or linear. Specifically, by Theorem 2.1, we see using the $2 \times 2$ test set that $F$ is either a non-negative constant or $F(x) = cx\alpha$ for some $c > 0$ and $\alpha \geq 0$. Using the matrix $C$, we see that $\alpha \geq 1$. Finally, using the test set $\{N(\epsilon, x) : \epsilon \in (0, 1), x > 0\}$, we obtain $\alpha = 1$.

We also note that Theorem 1.1 for TN preservers was proved differently, in the context of Hankel positivity preservers, in [8, §4]. By comparison, the proof in the present note has clear benefits, including completing the classification in every fixed size and isolating a small set of matrices on which the invariance of the TN property can be tested. Our present approach also leads to the classification of preservers of total positivity for matrices of a prescribed size, as well as classifications of the preservers when restricted to symmetric matrices.

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