ON LARGE $F$-DIOPHANTINE SETS

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Abstract. Let $F \in \mathbb{Z}[x, y]$ and $m \geq 2$ be an integer. A set $A \subset \mathbb{Z}$ is called an $(F, m)$-Diophantine set if $F(a, b)$ is a perfect $m$-power for any $a, b \in A$ where $a \neq b$. If $F$ is a bivariate polynomial for which there exist infinite $(F, m)$-Diophantine sets, then there is a complete qualitative characterization of all such polynomials $F$. Otherwise, various finiteness results are known. We prove that given a finite set of distinct integers $S$ of size $n$, there are infinitely many bivariate polynomials $F$ such that $S$ is an $(F, 2)$-Diophantine set. In addition, we show that the degree of $F$ can be as small as $4\lfloor n/3 \rfloor$.

1. Introduction

The polynomial $F(x, y) = xy + 1$ is a bivariate polynomial of degree 2 that satisfies the following property: there are infinitely many sets of integers of size 4, $\{a_1, a_2, a_3, a_4\}$, such that $F(a_i, a_j)$ is a perfect square for any $1 \leq i, j \leq 4$ where $i \neq j$. While it is known that there are no such sets of size 6, it was proved that there are at most finitely many such sets of size 5, see [2]. Recently, a proof of the non-existence of such sets of integers of size 5 was presented, [3]. In fact, a set of integers $S$ of size $m$ enjoying the property that $F(a_i, a_j)$, $a_i, a_j \in S$, $i \neq j$, is a perfect square is said to be a Diophantine $m$-tuple. So the previous facts can be restated as follows: there are infinitely many Diophantine quadruples while there are no Diophantine $m$-tuples when $m \geq 5$.

One may replace the polynomial $F(x, y) = xy + 1$ with any bivariate polynomial in $\mathbb{Z}[x, y]$ and then look for sets of distinct integers $S$ such that $F(a_i, a_j)$ is a perfect $m$-power, $m \geq 2$, for any $a_i, a_j \in S$, $i \neq j$. The set $S$ is then called an $(F, m)$-Diophantine set. One remarks that this definition allows patterns that were not seen when tackling the Diophantine $m$-tuple problem. For example, there is an infinite $(F, m)$-Diophantine set when $F(x, y) = xy$ for any $m \geq 2$.

Many questions may be posed now. Fixing a bivariate polynomial $F(x, y) \in \mathbb{Z}[x, y]$, is it possible that there is an infinite $(F, m)$-Diophantine set? If not, then what is the size of the largest $(F, m)$-Diophantine set? Or at least, can one find an upper bound on the size of such set? In addition, given a set of integers $S$, can we construct a bivariate polynomial $F$ such that $S$ is an $(F, m)$-Diophantine set? If so, then what is the smallest possible degree of $F$?

$(F, m)$-Diophantine sets were introduced in [1]. The polynomials $F(x, y)$ for which there are infinite $(F, m)$-Diophantine sets $S$ are completely characterized. They are of the form

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$f(x)^sg(y)^t$, for some $s, t \geq 0$, where $0 \leq \deg f, \deg g \leq 2$, see [1, Theorem 2.2]. Otherwise, bounds on the size of $S$ are given. Those bounds are not explicit and they depend on $F$ and $m$. Finally, the authors gave explicit bounds on the size of $(F, 2)$-Diophantine sets when $F$ is a quartic bivariate polynomial of certain form.

In this note, given a set of distinct integers $S = \{x_1, \ldots, x_n\}$, we prove that there are infinitely many polynomials $f(x) \in \mathbb{Z}[x]$ such that $S$ is an $(F, 2)$-Diophantine set, where $F_f(x, y) = f(x)f(y)$. Moreover, the degree of $f(x)$ is $n - 2$. In order to find such $f$, we use ideas of Yamagishi, [4, 5, 6]. We associate a hypersurface in the projective space to the variety defined by the latter equations will be called $(1)$.

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A further refinement on the degree of $f$ is displayed. We associate a determinantal variety consisting of the intersection of diagonal quadrics to $S$. By observing that this variety has several rational points, we prove that it is rational. This implies the existence of infinitely many polynomials of degree $2[\#S/3]$ such that $S$ is an $(F, 2)$-Diophantine set, where $\lfloor \cdot \rfloor$ is the greatest integer function.

2. The polynomial $F(x, y)$

Given a polynomial $f(x) \in \mathbb{Z}[x]$, we will write $F_f(x, y)$ for $f(x)f(y)$. In this section, we will show that for any positive integer $k$ there exists a polynomial $f(x) \in \mathbb{Z}[x]$ with $\deg f(x) = k - 2$ such that there is an $(F, 2)$-Diophantine set whose size is $k$. More precisely, given a set of integers $S$ containing $k$ distinct elements, we show the existence of a polynomial $f(x)$ with integer coefficients and degree $k - 2$ for which $S$ is an $(F, 2)$-Diophantine set.

Let $f(x) = f_0 + f_1x + \ldots + f_kx^k$. We choose $x_i \in \mathbb{Z}$, $i = 0, \ldots, n$, where $x_i \neq x_j$ if $i \neq j$ and $n > k$. We consider the following system of Diophantine equations

\[ z_i^2 = f(x_0)f(x_i), \quad i = 1, \ldots, n. \]

The variety defined by the latter equations will be called $V_n^k$. In fact, $V_n^k$ lies in the projective space $\mathbb{P}^{n+k}$ with coordinates $(f_0, f_1, \ldots, f_k, z_1, z_2, \ldots, z_n)$.

Lemma 2.1. If the system (1) has a solution $(f_0, f_1, \ldots, f_k, z_1, z_2, \ldots, z_n)$ with $f_i \in \mathbb{Z}$, $z_i \in \mathbb{Z}$, then $A = \{x_0, x_1, \ldots, x_n\}$ is a Diophantine-$(F, 2)$ set, where $f(x) = f_0 + f_1x + \ldots + f_kx^k$.

Proof: If such a solution exists, then it is clear that $f(x_0)f(x_i)$, $i = 1, \ldots, n$, is a perfect square. It follows that the product of $f(x_0)f(x_i)$ and $f(x_0)f(x_j)$ is also a perfect square for any $1 \leq i, j \leq n$. Thus, $F(x_i, x_j) = f(x_i)f(x_j) = (z_i z_j/f(x_0))^2$, hence a perfect square for any choice of $i$ and $j$. \[ \square \]
3. The variety $W^k_n$, $n > k$

In what follows we are going to write

$$
\begin{vmatrix}
0 & 1 & 2 & \ldots & k & i \\
Z_0 & Z_1 & Z_2 & \ldots & Z_k & Z_i
\end{vmatrix}, \ i \geq k + 1,
$$

for the $(k+2) \times (k+2)$ determinant

$$
\begin{vmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
x_0 & x_1 & x_2 & x_3 & \ldots & x_k & x_i \\
x_0^2 & x_1^2 & x_2^2 & x_3^2 & \ldots & x_k^2 & x_i^2 \\
x_0^3 & x_1^3 & x_2^3 & x_3^3 & \ldots & x_k^3 & x_i^3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_0^k & x_1^k & x_2^k & x_3^k & \ldots & x_k^k & x_i^k \\
Z_0 & Z_1 & Z_2 & Z_3 & \ldots & Z_k & Z_i
\end{vmatrix}
$$

Given $x_i \in \mathbb{Q}$, $i = 0, 1, \ldots, n$, we define the variety $W^k_n$ in $\mathbb{P}^n$ by the following system of determinantal equations:

$$
\begin{vmatrix}
0 & 1 & 2 & \ldots & k & i \\
Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_k^2 & Y_i^2
\end{vmatrix} = 0, \ i = k + 1, \ldots, n.
$$

In fact, the variety $W^k_n$ is an intersection of $n - k$ quadrics in $\mathbb{P}^n$.

**Proposition 3.1.** The variety $V^k_n$ is $\mathbb{Q}$-birationally equivalent to $W^k_n$ for any $n \geq k + 1$.

**Proof:** We will give an explicit description of the birational equivalence. We define the map $\phi : V^k_n \to W^k_n$ by

$$(f_0, f_1, \ldots, f_k, z_1, \ldots, z_n) \mapsto (f(x_0), z_1, \ldots, z_n).$$

In fact, the point $(f(x_0), z_1, \ldots, z_n)$ is a point in $W^k_n$ by using the fact that $z_i^2 = f(x_0)f(x_i)$ and the linearity of determinants.
We define the map $\psi : W^k_n \to V^k_n$ by sending the point $(Y_0, Y_1, \ldots, Y_n) \in W^k_n$ to

$$
\begin{pmatrix}
  x_0 & x_1 & x_2 & \ldots & x_k \\
  x_0^2 & x_1^2 & x_2^2 & \ldots & x_k^2 \\
  x_0^3 & x_1^3 & x_2^3 & \ldots & x_k^3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_0^k & x_1^k & x_2^k & \ldots & x_k^k \\
  Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_k^2
\end{pmatrix}
- \begin{pmatrix}
  1 & 1 & 1 & \ldots & 1 \\
  1 & 1 & 1 & \ldots & 1 \\
  1 & 1 & 1 & \ldots & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & 1 & \ldots & 1 \\
  x_0 & x_1 & x_2 & \ldots & x_k \\
  x_0^2 & x_1^2 & x_2^2 & \ldots & x_k^2 \\
  x_0^3 & x_1^3 & x_2^3 & \ldots & x_k^3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_0^k & x_1^k & x_2^k & \ldots & x_k^k \\
  Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_k^2
\end{pmatrix}

$$

where

$$
D = \begin{pmatrix}
  1 & 1 & 1 & \ldots & 1 \\
  x_0 & x_1 & x_2 & \ldots & x_k \\
  x_0^2 & x_1^2 & x_2^2 & \ldots & x_k^2 \\
  x_0^3 & x_1^3 & x_2^3 & \ldots & x_k^3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_0^k & x_1^k & x_2^k & \ldots & x_k^k \\
  Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_k^2
\end{pmatrix}
$$

This means that the coefficient $(-1)^j f_j$ is obtained by deleting the last column and the $j$-th row in the $(k+2) \times (k+2)$-determinant

$$
\begin{vmatrix}
  0 & 1 & 2 & \ldots & k & i \\
  Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_k^2 & Y_i^2
\end{vmatrix}
$$

where the latter determinant is clearly 0 if $i \leq k$, and it is 0 by the definition of $W^k_n$ if $i \geq k + 1$. Furthermore, the expansion of the determinant is $f(x_i) + (-1)^{k+1}DY_i^2 = 0$ where $0 \leq i \leq n$. It follows that $f(x_0) f(x_i) = D^2 Y_i^2 Y_i^2$, hence a point on $V^k_n$.

Checking that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps on $V^k_n$ and $W^k_n$ respectively is performed by direct calculations. \hfill \Box

**Remark 3.2.** We remark that we may assume that the coordinates $Y_i$ on the projective variety $W^k_n$ are integers. Therefore, if $x_i \in \mathbb{Z}$, $i = 0, 1, \ldots, n$, where $x_i \neq x_j$ when $i \neq j$,
then the map $\psi$ will yield a polynomial $f(x) \in \mathbb{Z}[x]$ of degree $k$. In particular, the existence of a rational point on $W^k_n$ together with our choice of integer $x_i$’s imply the existence of an $(F_f, 2)$-Diophantine set of size $n + 1$, see Lemma 2.1.

4. The variety $W^k_{k+1}$

We recall that a variety is rational if it is $\mathbb{Q}$-birational to $\mathbb{P}^m$ for some $m \geq 1$. In this section, we fix our choice of $x_i \in \mathbb{Q}$, $i = 0, 1, \ldots, k + 1$, where $x_i \neq x_j$ when $i \neq j$. We then prove that the variety $W^k_{k+1}$ is rational. Hence $V^k_{k+1}$ is also rational, see Proposition 3.1.

**Theorem 4.1.** Let $x_i \in \mathbb{Q}$, $i = 0, 1, \ldots, k + 1$, be such that $x_i \neq x_j$ when $i \neq j$. Then the variety $W^k_{k+1}$ is rational.

**Proof:** Recall that $W^k_{k+1}$ is defined by the determinant

$$\begin{vmatrix} 0 & 1 & 2 & \ldots & k & k+1 \\ Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_k^2 & Y_{k+1}^2 \end{vmatrix} = 0.$$  

Therefore, $W^k_{k+1}$ is a quadric hypersurface in $\mathbb{P}^{k+1}$ with coordinates $(Y_0, \ldots, Y_{k+1})$. Furthermore, the point $P = (1, 1, \ldots, 1)$ lies in the set of rational points $W^k_{k+1}(\mathbb{Q})$. In particular, $W^k_{k+1}$ is birationally equivalent to $\mathbb{P}^k$. More precisely, we have a rational map $\phi : \mathbb{P}^k \to W^k_{k+1}$ such that $\phi(Q)$, where $Q = (q_0, q_1, \ldots, q_k, 0)$, is the intersection of the line $L$ that joins the points $P$ and $Q$ with $W^k_{k+1}$. For ease of calculations, we assume that the line spanned by $P$ and $Q$ is given by $2\mu Q - \nu P$. Using determinantal properties, one may show that $\phi(Q)$ is described as follows:

$$(Y_0, Y_1, \ldots, Y_k, Y_{k+1}) = (2\mu q_0 - \nu, 2\mu q_1 - \nu, \ldots, 2\mu q_k - \nu, -\nu)$$

where

$$\mu = \begin{vmatrix} 0 & 1 & 2 & \ldots & k & k+1 \\ q_0 & q_1 & q_2 & \ldots & q_k & 0 \end{vmatrix}, \text{ and } \nu = \begin{vmatrix} 0 & 1 & 2 & \ldots & k & k+1 \\ q_0^2 & q_1^2 & q_2^2 & \ldots & q_k^2 & 0 \end{vmatrix}.$$  

One may easily check that the inverse rational map $W^k_{k+1} \to \mathbb{P}^k$ is described as follows:

$$(Y_0, Y_1, \ldots, Y_k, Y_{k+1}) \mapsto (Y_0 - Y_{k+1}, Y_1 - Y_{k+1}, \ldots, Y_k - Y_{k+1}).$$

\[\square\]

**Corollary 4.2.** Given $x_i \in \mathbb{Z}$, $i = 0, \ldots, k + 1$, where $x_i \neq x_j$ if $i \neq j$, there are infinitely many polynomials $f(x) \in \mathbb{Z}[x]$ with $\deg f = k$ such that the set $\{x_i : 0 \leq i \leq k + 1\}$ is an $(F_f, 2)$-Diophantine set where $F_f(x, y) = f(x)f(y)$.  


Proof: This follows from Proposition 3.1 since $V_{k+1}^k$ is birational to $W_{k+1}^k$ where the latter is rational, see Theorem 4.1. Now one concludes using Remark 3.2. 

5. Constructing $f$ of smaller degree

In Corollary 4.2, given any subset $S$ of $\mathbb{Z}$ of size $k$ we may find a polynomial $f(x)$, with $\deg f = k - 2$, such that $S$ is an $(F_f, 2)$-Diophantine set. In this section we investigate the possibility of finding $f'(x) \in \mathbb{Z}[x]$ of smaller degree than $k - 2$ such that $S$ is still an $(F_f', 2)$-Diophantine set.

We prove that given a set $S$ consisting of $k$ integers, there is a polynomial $f(x) \in \mathbb{Z}[x]$ of degree $2 \left\lfloor \frac{k}{3} \right\rfloor$ such that $S$ is an $(F_f, 2)$-Diophantine set, and $\left\lfloor . \right\rfloor$ is the greatest integer function.

Theorem 5.1. Let $x_i \in \mathbb{Q}$, $i = 0, 1, \ldots, 3k + 1$, be such that $x_i \neq x_j$ when $i \neq j$. Then the variety $W_{3k+1}^k$ is rational.

Proof: The variety $W_{3k+1}^k$ is defined by the intersection of the following $k + 1$ diagonal quadrics in $\mathbb{P}^{3k+1}$:

\[
\begin{vmatrix}
0 & 1 & 2 & \ldots & 2k & i \\
Y_0^2 & Y_1^2 & Y_2^2 & \ldots & Y_{2k}^2 & Y_i^2 \\
\end{vmatrix} = 0, \quad i = 2k + 1, \ldots, 3k + 1.
\]

The variety $W_{3k+1}^k$ contains the plane $\Pi$ spanned by the following $k + 1$ points

\[
T_0 = (1, 1, 1, \ldots, 1), \quad T_1 = (x_0, x_1, x_2, \ldots, x_{3k+1}), \quad T_2 = (x_0^2, x_1^2, x_2^2, \ldots, x_{3k+1}^2), \ldots,
T_k = (x_0^k, x_1^k, x_2^k, \ldots, x_{3k+1}^k).
\]

In order to prove the rationality of $W_{3k+1}^k$, we will project away from $\Pi$. More precisely, we display the following birational map $\phi : \mathbb{P}^{2k} \to W_{3k+1}^k$ defined by sending $Q = (g_0, \ldots, g_{2k}, 0, \ldots, 0)$ to the intersection of the $(k + 1)$-dimensional plane $L$ passing through $Q$ and $\Pi$ with $W_{3k+1}^k$. We may define $L$ by $\mu_0 T_0 + \mu_1 T_1 + \ldots + \mu_k T_k + \mu_{k+1} Q$. The map $\phi$ is defined explicitly once we give the values of the $\mu_i$’s. An intersection point of $L$ with $W_{3k+1}^k$ is a point which lies on $L$ and the $k + 1$ diagonal quadrics defining $W_{3k+1}^k$. Therefore, using the properties of the determinant, one may conclude that an intersection point will be
a solution for the following system of equations:

$$A \begin{pmatrix} 
\mu_0 \\
\mu_1 \\
\vdots \\
\mu_k \\
\mu_{k+1} 
\end{pmatrix} = 0$$

where $A = (A_{i,j})$ is a $(k+1) \times (k+2)$-full rank matrix whose entries are given as follows:

$$A_{m-2k,1} = 2 \begin{vmatrix} 
0 & 1 & \ldots & 2k & m \\
q_0 & q_1 & \ldots & q_{2k} & 0 
\end{vmatrix}, \quad A_{m-2k,2} = 2 \begin{vmatrix} 
0 & 1 & \ldots & 2k & m \\
q_0 x_0 & q_1 x_1 & \ldots & q_{2k} x_{2k} & 0 
\end{vmatrix}, \ldots,$$

$$A_{m-2k,k+1} = 2 \begin{vmatrix} 
0 & 1 & \ldots & 2k & m \\
q_0 x_0^k & q_1 x_1^k & \ldots & q_{2k} x_{2k}^k & 0 
\end{vmatrix}, \quad A_{m-2k,k+2} = 2 \begin{vmatrix} 
0 & 1 & \ldots & 2k & m \\
q_0^2 & q_1^2 & \ldots & q_{2k}^2 & 0 
\end{vmatrix}$$

where $m = 2k + 1, 2k + 2, \ldots, 3k + 1$.

Now a straightforward linear algebra exercise shows that $\mu_j = (-1)^j \det(A_j)$, $j = 0, 1, \ldots, k+1$, where $A_j$ is the matrix $A$ with the $(j+1)$-th column being removed. Therefore, $W_{3k+1}^k$ is birational to $\mathbb{P}^{2k}$ via the map $\phi : \mathbb{P}^{2k} \to W_{3k+1}^k$ defined by sending $(q_0 : q_1 : \ldots : q_{2k})$ to

$$(\mu_0 + \mu_1 x_0 + \mu_2 x_0^2 + \ldots + \mu_k x_0^k + \mu_{k+1} q_0, \mu_0 + \mu_1 x_1 + \mu_2 x_1^2 + \ldots + \mu_k x_1^k + \mu_{k+1} q_1, \ldots, \mu_0 + \mu_1 x_{2k} + \mu_2 x_{2k}^2 + \ldots + \mu_k x_{2k}^k + \mu_{k+1} q_{2k}, \mu_0 + \mu_1 x_{2k+1} + \mu_2 x_{2k+1}^2 + \ldots + \mu_k x_{2k+1}^k, \ldots, \mu_0 + \mu_1 x_{3k+1} + \mu_2 x_{3k+1}^2 + \ldots + \mu_k x_{3k+1}^k).$$

Using Theorem 5.1, one obtains the following result.

**Corollary 5.2.** Let $x_i \in \mathbb{Z}$, $i = 0, 1, \ldots, 3k+1$, be such that $x_i \neq x_j$ when $i \neq j$. Then there are infinitely many polynomials $f(x) \in \mathbb{Z}[x]$ of degree $2k$ such that $\{x_i : 0 \leq i \leq 3k + 1\}$ is an $(F_j, 2)$-Diophantine set.

**Proof:** The proof is a direct consequence of Theorem 5.1 together with the fact that $W_{3k+1}^k$ and $V_{3k+1}^k$ are birational, see Proposition 3.1. Now we conclude using Remark 3.2. □

The variety $V_n^k$ can possibly give rise to hyperelliptic curves whose Jacobian varieties are of high rank. More precisely, if $(f_0, \ldots, f_k, z_1, \ldots, z_n) \in V_n^k(K)$ where $K = \mathbb{Q}(x_0, \ldots, x_n)$,
then the quadratic twist \( C^T : f(x_0)y^2 = f(x) = f_0 + f_1x + \ldots + f_kx^k \) of the hyperelliptic curve \( C : y^2 = f(x) \) by \( f(x_0) \) contains the following rational points in \( C^T(K) \)

\[
P_0 = (x_0, 1), \quad P_1 = \left(x_1, \frac{z_1}{f(x_0)}\right), \quad P_2 = \left(x_2, \frac{z_2}{f(x_0)}\right), \quad \ldots, \quad P_n = \left(x_n, \frac{z_n}{f(x_0)}\right).
\]

The point \( P_0 \) is an obvious point in \( C^T \). For the points \( P_i, \ i \geq 1 \), one observes that \( z_i^2 = f(x_0)f(x_i) \), hence \( f(x_0)\left(z_i/f(x_0)\right)^2 = f(x_i) \).

We recall that the hyperelliptic curve \( C^T \) may be embedded in its Jacobian variety \( J^T \) via the Albanese map \( j : C^T \hookrightarrow J^T \) defined by \( j(P) = [(P) - (P_0)] \), where \([ \ . \ ]\) denotes the linear equivalence class of the divisor on \( C^T \). The point \( P_0 \) is called the base point of the embedding \( j \). Therefore, the points \( j(P_i), \ 1 \leq i \leq n \), generate a subgroup \( \sum \) in \( J^T(K) \) which is generically of positive rank. A natural question would be how large the rank of \( \sum \) might be. In fact, infinitely many elliptic curves with rank 7 over \( \mathbb{Q} \) were obtained by using the variety \( V_7^4 \), see [6]. The authors intend to investigate this question further in future work.

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