The inverse vibration problem for fixed beam submerged in fluid

M Havlásek¹, V Habán¹, M Hudec ¹ and F Pochylý¹
¹V. Kaplan Department of Fluid Engineering, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2896/2, Brno, Czech Republic
E-mail: havlasek@fme.vutbr.cz

Abstract. Description of dynamic behaviour of the structures is based on the mass, damping and stiffness matrices. Unfortunately, this description is inapplicable in case of the fluid-structure interaction (FSI), because the matrices of system, which consists of the structure and ambient fluid, are generally not known. The matrices of system, describing the FSI, can be determined by solution of inverse vibration problem, which is an approach employing the spectral matrix and modal matrices of analysed system. The eigenvalues for creation of the spectral matrix are determined based on experimental measurement. The results of experiment can be verified by numerical simulation. The modal matrices of structure submerged in fluid can be given by experiment, which is not simple in FSI problem, or by acoustic modal analysis. Third approach for determination of the modal matrices works with the assumption, that the eigenvectors of structure are not influenced by the fluid and are identical to the eigenvectors of structure without the ambient fluid. This assumption is generally correct for majority of FSI problems. Described method is demonstrated on determination of the matrices of dynamic systems of the fixed beam submerged in water.

1. Introduction
Methods for determination of dynamic behaviour of system employ system’s mass, damping and stiffness matrices. Unfortunately, these so-called structural matrices are known only for limited number of problems. Generally, the procedure for producing of the structural matrices are not known for complex problems, e.g. the fluid-structure interaction (FSI). The structural matrices can be determined in case, when spectral matrix and modal matrices of analysed system are known. This method is called the inverse vibration problem.

O. Daněk was the first who derived formulas for determination of the structural matrices for simple matrix pencils [1] and for regular matrix pencils [2]. Generalization of the inverse vibration problem for lambda matrices was presented in [3] based on theory [4]. All of these publications were focused on derivation of inverse method to the classical modal analysis, which means determination of the system matrices from spectral matrix and modal matrices.

2. Mathematical model
Dynamic behaviour of systems are described by the general equation of motion (1), which was originally derived by Rayleigh in [5].

\[ M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t) \]  

(1)
where \( M, C \) and \( K \) are the mass, damping and stiffness matrix. The structural matrices are square matrices of order equals to number of degrees of freedom (DOF). Term \( \mathbf{u} \) represents the vector of generalized displacement and \( \mathbf{f} \) is the vector of generalized (external) forces acting on the system.

Order of differential equation can be reduced by adding identity equation \( M \ddot{\mathbf{u}}(t) - M \dot{\mathbf{u}}(t) = 0 \), which lead to (2).

\[
N \dot{\mathbf{w}}(t) + P \mathbf{w}(t) = \mathbf{g}(t)
\]

Equation (2) represents a first order matrix pencil and it uses matrices described in (3).

\[
N = \begin{bmatrix} B & M \\ M & 0 \end{bmatrix}; \quad P = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}; \quad w(t) = \begin{bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{bmatrix}; \quad g(t) = \begin{bmatrix} \mathbf{f}(t) \\ 0 \end{bmatrix}
\]

If vector \( \mathbf{g}(t) = 0 \), solution of this homogeneous equation can be expressed as the equations (4) and (5).

\[
\begin{align*}
NXS + PX &= 0 \quad (4) \\
N^*ZS^* + P^*Z &= 0 \quad (5)
\end{align*}
\]

Symbol \( * \) denotes the complex conjugate matrix. Matrix \( S \) is the spectral matrix and matrices \( X \) and \( Z \) are the modal matrix of right eigenvectors and the modal matrix of left eigenvectors, respectively. In case when all eigenvalues and eigenvectors are known, matrices \( S, X \) and \( Z \) are square matrices of order equals to \( 2 \times \text{DOF} \) and they are generally complex. The modal matrices can be written in form of equations (6).

\[
X = \begin{bmatrix} x \\ xS \end{bmatrix}; \quad Z = \begin{bmatrix} z \\ zS^* \end{bmatrix}
\]

The top half part of modal matrix of right eigenvectors \( x \) represents mode shapes of system. Modal matrices \( X \) and \( Z \), or more exactly their top halves \( x \) and \( z \) are related to each other by equation (7), which was derived in [1].

\[
\begin{align*}
xz^* &= 0 \quad (7)
\end{align*}
\]

This homogeneous equation allows to compute one modal matrix from another modal matrix. Based on structure of modal matrices decribed in (6) and equation (7), both modal matrices \( X \) and \( Z \) can be obtained from spectral matrix and first half of one of modal matrices.

If the modal matrices \( X \) and \( Z \) are square (so-called "full problem"), it is possible to restore the structural matrices of dynamic system \( M, B \) and \( K \) with equations (8)-(10). Derivation of these equations is decribed in [1].

\[
\begin{align*}
M^{-1} &= xSz^* \quad (8) \\
K^{-1} &= xS^{-1}z^* \quad (9) \\
B &= -MxS^2z^*M \quad (10)
\end{align*}
\]

If vibrating flexible body is submerged in fluid, the mathematical model of mutual interaction is described by equation (11).

\[
\left( M + \hat{M} \right) \ddot{\mathbf{u}}(t) + \left( C + \hat{C} \right) \dot{\mathbf{u}}(t) + \left( K + \hat{K} \right) \mathbf{u}(t) = \mathbf{f}(t)
\]

Matrices \( \hat{M}, \hat{C} \) and \( \hat{K} \) are the added effects of the FSI. Presented method assumes that the influence of ambient fluid on the right and left eigenvectors is negligible and only the eigenvalues
are affected by fluid, which is generally correct for majority of FSI problems. If the spectral matrix $\hat{S}$ of the FSI is known, e.g. based on measurement, the added effect matrices can be determined from equations (12)-(14).

$$\left( M + \hat{M} \right)^{-1} = x\hat{S}z^*$$ (12)

$$\left( K + \hat{K} \right)^{-1} = x\hat{S}^{-1}z^*$$ (13)

$$\left( C + \hat{C} \right) = -\left( M + \hat{M} \right)x\hat{S}^2z^*\left( M + \hat{M} \right)$$ (14)

Verification of presented method can be performed by the modal analysis with the derived structural matrices. If resultant spectral matrix and modal matrices are equal to original matrices, the derived matrices correctly represents the analysed systems.

The presented method is the same as in [6], where it was applied on analytically determined lateral vibration of beam with natural boundary conditions.

3. Description of analysed system
The method presented in the previous chapter was applied on determination of the structural matrices of fixed beam. Beam with fixed boundary condition was realized by welding on the beam on thick cylindrical steel plate with usage of the filled weld. Quality of welding process was tested by the non-destructive weld testing. The beam was positioned to the center of plate. Geometry of system is depicted in figures 1-2. Beam was made out of steel S235JR with density $\rho = 7.850 \times 10^3$ kg m$^{-3}$, Young's modulus $E = 2.1 \times 10^{11}$ Pa and Poisson's ratio 0.3.

4. Experimental modal analysis
Experimental modal analysis was performed both for the beam in air and for beam submerged in water. Figure 3 shows geometry of used tank (made out of polypropylene), location of water surface, location of the beam in water and steel props, which were used for preventing of structure movements.

The response of system was measured with the water-resistant accelerometer. Eigenvalues were evaluated from measurement of free decay. Amplitude spectrum for structure in air and in water are depicted in figure 4 and 5, respectively. Resultant eigenvalues for experiments in air and in water are presented in table 1 and 2, respectively.
Figure 3. Drawing of geometry configuration for measurement.

Figure 4. Amplitude spectrum - air.

Figure 5. Amplitude spectrum - water.

5. Computational modeling

It is possible to obtain only limited number of eigenvalues with the experimental modal analysis. And even though it is possible to experimentally obtain mode shapes of structure, Finite element method (FEM) was used for determination of top half of modal matrix of right eigenvectors \( \mathbf{x} \). This means that correct FEM model had to be created for determination of all eigenvalues and eigenvectors of the beam.

Several geometry configurations were tested and compared with results of measurements. It was found out the cylindrical plate has no influence on the eigenvalues, but the weld has to be modeled to obtain correct eigenvalues. This outcomes lead to the geometrical simplifications, which result in the geometry of computational domain depicted in figure 6.

Damping was introduced to the analysis with the concept of proportional damping. "The particular advantage of using a proportional damping model in the analysis of structures is that the modes of such a structure are almost identical to those of the undamped version of the model. Specifically, the mode shapes are identical and the natural frequencies are very similar to those of the simpler undamped system" [7]. Since analysed system is lightly damped in air, the concept of proportional damping is suitable. Based on work [8], the mass matrix multiplier and the stiffness matrix multiplier were determined for first eigenvalue. Resultant values are \( \alpha = 1.685 \) and \( \beta = 1.75 \times 10^{-7} \).
Table 1. Results for beam in air.

| Experiment | Computation |
|------------|-------------|
| No | λ [rad s⁻¹] | Ω [Hz] | λ [rad s⁻¹] | Ω [Hz] | Mode shape |
| 1 | −0.87 ± 196.43i | 31.26 | −0.85 ± 198.61i | 31.61 | Bending |
| 2 | −8.03 ± 1230.20i | 195.79 | −0.98 ± 1242.77i | 197.79 | Bending |
| 3 | — | — | −1.17 ± 1923.50i | 306.13 | Bending 2 |
| 4 | −3.70 ± 2505.38i | 398.74 | −1.39 ± 2491.17i | 396.48 | Torsional |
| 5 | −17.78 ± 3459.55i | 550.60 | −1.90 ± 3478.31i | 553.59 | Bending |
| 6 | −4.24 ± 6762.22i | 1076.24 | −4.91 ± 6814.72i | 1084.60 | Bending |
| 7 | −4.17 ± 7595.45i | 1208.85 | −5.84 ± 7558.65i | 1203.00 | Torsional |
| 8 | — | — | −11.42 ± 10992.37i | 1749.49 | Bending 2 |
| 9 | −9.91 ± 11172.95i | 1778.23 | −11.93 ± 11258.27i | 1791.81 | Bending |
| 10 | −6.25 ± 12933.21i | 2058.38 | −15.35 ± 12874.38i | 2049.02 | Torsional |

Table 2. Results for beam in water.

| Experiment | Computation |
|------------|-------------|
| No | λ [rad s⁻¹] | Ω [Hz] | Ω [Hz] | Mode shape |
| 1 | −3.54 ± 141.11i | 22.46 | 22.45 | Bending |
| 2 | −11.75 ± 891.96i | 141.96 | 141.95 | Bending |
| 3 | — | — | 297.92 | Bending 2 |
| 4 | −33.31 ± 2135.00i | 339.80 | 333.63 | Torsional |
| 5 | −19.39 ± 2558.22i | 404.15 | 404.68 | Bending |
| 6 | −31.61 ± 5106.82i | 812.78 | 811.17 | Bending |
| 7 | −23.87 ± 6473.69i | 1030.32 | 1013.83 | Torsional |
| 8 | — | — | 1367.16 | Bending |
| 9 | — | — | 1707.77 | Bending 2 |
| 10 | −43.08 ± 11046.43i | 1758.09 | 1731.60 | Torsional |

Results of experiment for the beam in air and computation of the beam without ambient fluid are presented in table 1. Term λ represents the eigenvalue and Ω is the damped eigenfrequency. Several eigenvalues were not possible to analyse from measurement.

The eigenvalues of beam in water were determined experimentally but it was necessary to find out which eigenvector corresponds to each eigenvalue. The acoustics modal analysis was used for solving this problem. Results presented in table 2 show a good agreement between the
measured and computed eigenfrequencies. It is necessary to find only type of mode shape for each eigenvalue, not the exact eigenvector because the method uses the assumption that the eigenvectors remain the same.

6. Application of presented method

The full spectral matrix $S$ and full top half of modal matrix of right eigenvectors $x$ were given by the FEM model. Then the full modal matrix $X$ was created based on $S$ and $x$ from (6). The full top half of modal matrix of left eigenvectors $z$ was determined from homogeneous equation (7). Based on $S$ and $z$, the full modal matrix $Z$ was created. Full matrices $S$, $X$ and $Z$ were determined from previous steps.

It was necessary to improve the spectral matrix $S$ by changing eigenvalues from computation by eigenvalues determined experimentally for obtaining of correct model of the fixed beam in air. But it is necessary to change eigenvalues with the same eigenvector, i.e. it is necessary to find out correct eigenvalue which should be replaced and insert new eigenvalue to the same position in spectral matrix. The eigenvalues which are not determined from experiment are simply not changed.

Then the structural matrices of beam in air were determined from equations (8)-(10) and these matrices have the same spectral matrix and modal matrices as the original matrices which was verified by modal analysis.

The structural matrices of beam in water were given by formulas (12)-(14), but the spectral matrix has to be adjusted by inserting eigenvalues from the measurement in water to proper positions in spectral matrix, which was described previously. Derived structural matrices have the same eigenvalues and eigenvectors as original spectral matrix and modal matrices. But it means the new model has only few eigenvalues correct. However, only few eigenvalues and their associated eigenvectors have significant effect on the resultant dynamic response of the system and higher eigenvalues are negligible. This leads to the biggest challenge of presented method which is determination of all fundamental eigenvalues. This is similar to modal reduction method.

7. Conclusion

The derived method provides the formulas for determination of the structural matrices based on the spectral matrix and modal matrices. The presented version of the method is suitable only in the cases when the modal matrices are square. Derivation of the method for rectangular modal matrices, which is more common in engineering applications, is currently in progress.

Acknowledgments

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