Eigenvalues and degree deviation in graphs

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March 29, 2022

Abstract
Let $G$ be a graph with $n$ vertices and $m$ edges and let $\mu(G) = \mu_1(G) \geq ... \geq \mu_n(G)$ be the eigenvalues of its adjacency matrix. Set $s(G) = \sum_{u \in V(G)} |d(u) - 2m/n|$. We prove that
\[
\frac{s^2(G)}{2n^2 \sqrt{2m}} \leq \mu(G) - \frac{2m}{n} \leq \sqrt{s(G)}.
\]
In addition we derive similar inequalities for bipartite $G$.

We also prove that the inequality
\[
\mu_k(G) + \mu_{n-k+1}(\overline{G}) \geq -1 - 2\sqrt{2s(G)}
\]
holds for every $k = 1, ..., n - 1$.
We show that these inequalities are tight up to a constant factor.
Finally we prove that for every graph $G$ of order $n$,
\[
\mu_n(G) + \mu_n(\overline{G}) \leq -1 - \frac{s^2(G)}{n^3}.
\]

AMS classification: 15A42, 05C50
Keywords: graph eigenvalues, degree sequence, measure of irregularity, semiregular graph

1 Introduction
Our notation is standard (e.g., see [3], [5], and [10]); in particular, all graphs are defined on the vertex set $\{1, 2, ..., n\} = [n]$ and $G(n, m)$ stands for a graph with $n$ vertices and $m$ edges. We write $\Gamma(u)$ for the set of neighbors of the vertex $u$ and set $d(u) = |\Gamma(u)|$.
Given a graph $G$ of order $n$, we assume that the eigenvalues of the adjacency matrix of $G$ are ordered as $\mu(G) = \mu_1(G) \geq ... \geq \mu_n(G)$. As usual, $\overline{G}$ denotes the complement of a graph $G$.

Collatz and Sinogowitz [6] showed that $\mu(G) \geq 2m/n$ for every graph $G = G(n, m)$. Since equality holds if and only if $G$ is regular, they proposed the value $\epsilon(G) = \mu(G) - \frac{2m}{n}$.
2m/n as a relevant measure of irregularity of $G$. Two other closely related measures of graph irregularity are the functions

$$\text{var}(G) = \frac{1}{n} \sum_{u \in V(G)} \left( d(u) - \frac{2m}{n} \right)^2,$$

$$s(G) = \sum_{u \in V(G)} \left| d(u) - \frac{2m}{n} \right|.$$

Bell [1] compared $\epsilon(G)$ to $\text{var}(G)$ and showed that none of them could be preferred to the other one as a measure of irregularity. He did not, however, give explicit inequalities between $\epsilon(G)$ and $\text{var}(G)$. In this note we prove that for every graph $G$ with $n$ vertices and $m$ edges,

$$\frac{\text{var}(G)}{2\sqrt{2m}} \leq \mu(G) - \frac{2m}{n} \leq \sqrt{s(G)} \quad (1)$$

Thus, in view of

$$\frac{s^2(G)}{n^2} \leq \text{var}(G) \leq s(G),$$

we also have

$$\frac{s^2(G)}{2n^2\sqrt{2m}} \leq \mu(G) - \frac{2m}{n} \leq \sqrt{\text{var}(G)}.$$

In addition we derive similar inequalities specifically for bipartite graphs.

Another well-known inequality involving graph eigenvalues is

$$\mu_k(G) + \mu_{n-k+1}(\overline{G}) \leq -1, \quad (2)$$

holding for every graph $G$ of order $n$ and every $k = 1, \ldots, n-1$. Note that if $G$ is regular, equality holds in (2) but the converse is not always true (e.g., $G = K_{a,b}$, $b > a > 2$, $k = 2$). A natural problem is to find a lower bound on $\mu_k(G) + \mu_{n-k+1}(\overline{G})$ implying explicit equality in (2) for regular $G$. In this note we show that for every $k = 1, \ldots, n-1$,

$$\mu_k(G) + \mu_{n-k+1}(\overline{G}) \geq -1 - 2\sqrt{2s(G)}. \quad (3)$$

We show that inequalities (1) and (3) are tight up to a constant factor.

Finally we prove that for every graph $G$ of order $n$,

$$\mu_n(G) + \mu_n(\overline{G}) \leq -1 - \frac{s^2(G)}{n^3}, \quad (4)$$

implying that for any highly irregular graph $G$ either $\mu_n(G)$ or $\mu_n(\overline{G})$ must be large in absolute value.

Let us note that these results are readily applicable to the study of quasirandom graph properties.

The rest of the note is organized as follows. In Section 2 we describe algorithms for regularizing graphs with few edge changes. Section 3 contains basic results about spectra of blown-up graphs. In Sections 4, 5, and 6 we prove inequalities (1), (3), and (4).
2 Efficient regularization

Consider the following natural problem: given a graph $G$, what is the minimum number of edges $\rho(G)$ that must be changed to obtain a regular graph. Writing $A(G)$ for the adjacency matrix of a graph $G$, we see that

$$\rho(G) = \frac{1}{2} \min \{ \| A(G) - A(R) \|_2 : R \text{ is regular graph of order } v(G) \}.$$

It is almost certain that the problem of estimating $\rho(G)$ has been raised and solved in the literature but, lacking a proper reference, we shall solve it from scratch.

We first show that there exists a graph $R^*$ whose degrees differ by at most one and such that

$$\| A(G) - A(R^*) \|_2 \leq s(G).$$

Next we find a regular graph $R$ such that

$$\| A(R) - A(R^*) \|_2 < 3n.$$ 

Finally we show that for every graph $G$,

$$\rho(G) \geq \frac{s(G)}{2}$$

implying that our upper bounds on $\rho(G)$ are not too far from the best possible ones.

2.1 Rough regularization

The main result in this section is the following theorem.

**Theorem 1** For every graph $G = G(n, m)$, there exists a graph $R = G(n, m)$ such that $\Delta(R) \leq \delta(R) + 1$ and $R$ differs from $G$ in at most $s(G)$ edges. In particular, if $2m/n$ is integer then $R$ is $(2m/n)$-regular.

**Proof** We shall describe a simple algorithm that produces the graph $R$ by deleting and adding edges of $G$. Set $d = \lfloor 2m/n \rfloor$.

**Step 1**

While $\delta(G) < d$ and $\Delta(G) > d + 1$ select $u, v$ with $d(u) = \delta(G)$ and $d(v) = \Delta(G)$. Since $\Gamma(v) \setminus \Gamma(u) \neq \emptyset$, there exists $w \in \Gamma(v) \setminus \Gamma(u)$; delete the edge $vw$ and add the edge $uw$.

Write $G'$ for the graph obtained upon exiting Step 1. Since Step 1 is iterated as long as $\delta(G) < d$ and $\Delta(G) > d + 1$, we have either $\delta(G') = d$ or $\Delta(G') = d + 1$; we may assume $\delta(G') = d$, since the other case is reduced to this one by considering $\overline{G'}$.

If $\Delta(G') \leq d + 1$ then terminate the procedure with $R = G'$. Otherwise write $A$ for the set of vertices of degree $d$, $B$ for the set of vertices of degree $d + 1$, and $C$ for the set of vertices of degree $d + 2$ or higher.

**Step 2**
While \( C \neq \emptyset \) select \( u \in A, v \in C \). Since \( |\Gamma (v)| > |\Gamma (u)| \), we may select \( w \in \Gamma (v) \setminus \Gamma (u) \); delete the edge \( vw \) and add the edge \( uw \).

Write \( R \) for the graph obtained after executing Step 2. Let \( G', A, B, C \) be as defined prior to Step 2; set \( |A| = k, |C| = s \). Each iteration in Step 1 changes two edges and decreases \( s(G) \) by 2; therefore, after the execution of Step 1, at most \( s(G) - s(G') \) edges of \( G \) are changed. Set

\[
l = \sum_{u \in C} (d(u) - d - 1).
\]

Each iteration in Step 2 changes two edges and decreases \( l \) by 1; therefore, there are \( l \) iterations in Step 2 and at most 2\( l \) edges are changed. To complete the proof we have to show that \( S(G') \geq 2l \).

\[
2m/n = \frac{1}{n} \sum_{u \in V(G)} d(u) = \frac{1}{n} \sum_{u \in A} d(u) + \frac{1}{n} \sum_{u \in B} d(u) + \frac{1}{n} \sum_{u \in C} d(u) = kd + (n - k - s)(d + 1) + s(d + 1) + l = d + \frac{n - k + l}{n}
\]

it follows that \( k > l \). Furthermore,

\[
s(G') - 2l = \sum_{u \in V} |d_{G'}(u) - 2m/n| - 2l
\]

\[
= \sum_{u \in A} |d_{G'}(u) - 2m/n| + \sum_{u \in B} |d_{G'}(u) - 2m/n| + \sum_{u \in C} |d_{G'}(u) - 2m/n| - 2l
\]

\[
= k \frac{n - k + l}{n} + (n - k - s) \frac{k - l}{n} + s \frac{k - l}{n} - l = 2 \frac{(k - l)(n - k)}{n} > 0,
\]

completing the proof. \( \square \)

### 2.1.1 Rough regularization of bipartite graphs

Call a bipartite graph \emph{semiregular} if vertices belonging to the same vertex class have equal degrees.

Let \( G \) be a bipartite graph and \( A, B \) be its vertex classes, \( |A| = a, |B| = b \). Define the function

\[
s_2(G) = \sum_{u \in A} \left| d(u) - \frac{m}{a} \right| + \sum_{u \in B} \left| d(u) - \frac{m}{b} \right| ;
\]

\( s_2(G) \) is the equivalent to \( s(G) \) for bipartite graphs. Clearly, \( s_2(G) = 0 \) if and only if \( G \) is semiregular.

Modifying slightly the proof of Theorem \( \square \) we obtain the following special case for bipartite graphs.

\textbf{Theorem 2} For every bipartite graph \( G = G(n, m) \) with vertex classes \( A, B \), there exists a bipartite graph \( R = G(n, m) \) with the same vertex classes such that:
(i) \(|d_R(u) - d_R(v)| \leq 1\) for every \(u, v\) belonging to the same vertex class;
(ii) \(R\) differs from \(G\) in at most \(s_2(G)\) edges.
In particular, if \(m/|A|\) and \(m/|B|\) are integer then \(R\) is semiregular.

### 2.2 Fine regularization

If we allow \(m\) to change, we may further regularize the graph \(R\) obtained in Theorem 1.

**Theorem 3** Let the degrees of a graph \(G = G(n, m)\) be either \(d\) or \(d + 1\). There exists an \(r\)-regular graph \(R\) such that either \(r = d\) or \(r = d + 1\), and \(R\) differs from \(G\) in at most \(3n/2\) edges.

**Proof** Write \(A\) for the set of vertices of degree \(d + 1\) and \(B = V(G) \setminus A\). Clearly either \(|A|\) or \(|B|\) is even. We shall assume that \(|A|\) is even, otherwise we may apply the argument to the complementary graph. Set \(a = |A|\). Our goal is to construct a \(d\)-regular graph by changing at most \(3a/2\) edges. We shall describe a procedure constructing \(R\).

**Step 1**
While \(E(A) \neq \emptyset\), select \(uv \in E(A)\) and remove it.

**Step 2.**
While \(A \neq \emptyset\), select two distinct \(u, v \in A\) and two disjoint vertices \(t \in \Gamma(v), w \in \Gamma(u)\). Delete the edges \(uw\) and \(vt\); add the edge \(wt\).

The iteration in Step 2 may always be executed since, for every two distinct \(u, v \in A\), there exist disjoint vertices \(t \in \Gamma(v)\) and \(w \in \Gamma(u)\). Indeed, if \(\Gamma(u) \neq \Gamma(v)\), select \(w \in \Gamma(u) \setminus \Gamma(v)\). Since \(d(w) = d < |\Gamma(v)|\), there exists \(t \in \Gamma(v)\) that is disjoint from \(w\) and the assertion is proved. If \(\Gamma(u) = \Gamma(v)\) then \(\Gamma(u)\) cannot induce a complete graph, since \(\Gamma(u) \subset B\) and so the vertices in \(\Gamma(u)\) have degree \(d\).

Each iteration in Step 1 removes two vertices from \(A\) and changes two edges. Each iteration in Step 2 removes two vertices from \(A\) and changes three edges. Therefore, after changing at most \(3|A|/2\) edges, we obtain a \(d\)-regular graph \(R\), as claimed. \(\square\)

### 2.3 Optimal regularization

Summarizing Theorems 1 and 3, we obtain the following corollary.

**Corollary 4** For every graph \(G\) of order \(n\),

\[
\rho(G) \leq s(G) + 3n/2.
\]

It turns out that this bound is quite close to the optimal one, no matter what the graph \(G\) is. We shall show that

\[
\rho(G) \geq s(G) /2.
\]

Let \(R\) be \(r\)-regular graph with \(V(R) = V(G)\). For every vertex \(v \in V(G)\), we have

\[
|\Gamma_G(u) \setminus \Gamma_R(u)) \cup (\Gamma_R(u) \setminus \Gamma_G(u))| \geq d(u) + r - 2 \min(d, r) \geq |d(u) - r|.
\]
Hence, summing over all vertices \( v \in V(G) \) we find that
\[
2 \rho(G) \geq \|A(G) - A(R)\|_2 \geq \sum |d(u) - r| \geq s(G),
\]
as claimed.

We note without proof that \( \rho(K_{a,b}) \geq 3s(K_{a,b})/4 \).

3 The spectra of blown-up graphs

In this section we introduce two operations on graphs and consider how they affect graph spectra.

Let \( G = G(n,m) \) and \( t > 0 \) be integer. Write \( G(t) \) for the graph obtained by replacing each vertex \( u \in V(G) \) by a set \( V_u \) of \( t \) vertices and joining \( x \in V_u \) to \( y \in V_v \) if and only if \( uv \in E(G) \). Notice that \( v(G(t)) = tn \). The following theorem holds.

**Theorem 5** The eigenvalues of \( G(t) \) are \( t\mu_1(G), \ldots, t\mu_n(G) \) together with \( n(t-1) \) additional \( 0 \)'s.

Set \( G[t] = \overline{G[t]} \), i.e., \( G[t] \) is obtained from \( G(t) \) by joining all vertices within \( V_u \) for every \( u \in V(G) \); note also that \( G(t) = \overline{G[t]} \). The following theorem holds.

**Theorem 6** The eigenvalues of \( G[t] \) are \( t\mu_1(G) + t - 1, \ldots, t\mu_n(G) + t - 1 \) together with \( n(t-1) \) additional \( (-1) \)'s.

4 Bounds on \( \mu(G) \)

In this section we shall prove inequalities (1). Recall first the inequality
\[
\mu^2(G) \geq \frac{1}{n} \sum_{u \in V(G)} d^2(u),
\]
due to Hofmeister [9] and observe that Stanley’s inequality [11]
\[
\mu(G) \leq -1/2 + \sqrt{2m + 1}/4
\]
implies
\[
\mu^2(G) \leq 2m.
\]

We thus find that
\[
2\sqrt{2m} (\mu(G) - 2m/n) \geq 2\mu(G)(\mu(G) - 2m/n) \geq \mu^2(G) - (2m/n)^2
\geq \frac{1}{n} \sum_{u \in V(G)} d^2(u) - (2m/n)^2 = \text{var}(G),
\]
obtaining the lower bound in (1). To prove the upper bound we need the following proposition.
Proposition 7 If $G_1$ and $G_2$ are graphs with $V(G_1) = V(G_2)$ then

$$\mu(G_1) - \mu(G_2) \leq \sqrt{2|E(G_1) \setminus E(G_2)|}.$$ 

Proof Setting $G' = (V(G_1), E(G_1) \cup E(G_2))$, $G'' = (V(G_1), E(G_1) \setminus E(G_2))$, from Weyl’s inequalities ([10], p. 181), we have

$$\mu(G_1) \leq \mu(G') \leq \mu(G_2) + \mu(G'').$$

By (5), we have, $\mu(G'') \leq \sqrt{2|E(G_1) \setminus E(G_2)|}$, completing the proof. □

We shall deduce the upper bound in (1) essentially from Theorem 1.

Theorem 8 For every graph $G = G(n, m)$,

$$\mu(G) - 2m/n \leq \sqrt{s(G)}.$$ 

Proof Theorem 1 implies that there exists a graph $R = G(n, m)$ such that $\Delta(R) \leq \delta(r) + 1$ and $R$ differs from $G$ in at most $s(G)$ edges. Since $e(R) = e(G)$ it follows that $|E(G) \setminus E(R)| = |E(R) \setminus E(G)|$ and so $2|E(G) \setminus E(R)| \leq s(G)$. Hence, by Proposition 4

$$\mu(G) - 2m/n \leq \mu(G) - \lfloor 2m/n \rfloor + 1 \leq \mu(G) - \mu(R) + 1 \leq 1 + \sqrt{s(G)}. \quad (7)$$

Notice that $v(G^{(t)}) = tn$, $e(G^{(t)}) = t^2m$, and $s(G^{(t)}) = t^2s(G)$. Applying Theorem 4 we also see that

$$\mu(G^{(t)}) = t\mu(G).$$

From (7) it follows that

$$(\mu(G) - 2m/n) t = \mu(G^{(t)}) - 2e(G^{(t)}) / v(G^{(t)}) \leq 1 + \sqrt{s(G^{(t)})} = 1 + t\sqrt{s(G)}.$$ 

Hence, dividing by $t$ and letting $t$ tend to infinity, the desired inequality follows. □

4.1 Tightness of inequalities (1)

It is natural to ask how large $c$ could be so that the inequality

$$\mu(G) - \frac{2m}{n} \geq \frac{c^2}{n^2}\sqrt{m}$$

holds for every graph $G = G(n, m)$. Taking the graph $G = K_{n,n+1}$ for $n$ large enough, we see that $c$ may be at most $1/2$.

Similarly, let $c$ be such that the inequality

$$\mu(G) - 2m/n \leq c\sqrt{s(G)}$$

holds for every graph $G = G(n, m)$. Taking $G = K_n \cup K_1$ we see that $c$ must be at least $1/\sqrt{2}$.

We venture the following conjecture.
Conjecture 9 For every graph \(G\) of sufficiently large order \(n\) and size \(m\),

\[
\frac{s^2(G)}{2n^2 \sqrt{m}} \leq \mu(G) - \frac{2m}{n} \leq \sqrt{s(G)/2}
\]

4.2 Bounds on \(\mu(G)\) when \(G\) is bipartite

It is possible to modify inequalities (1) to better suit bipartite graphs.

Let \(G\) be a bipartite graph and \(A, B\) be its vertex classes, \(|A| = a, |B| = b\). Then, by Rayleigh’s principle we have,

\[\mu(G) \geq \frac{e(G)}{\sqrt{ab}}.\]

A careful analysis shows that equality is possible if and only if \(G\) is semiregular. In fact the following theorem holds.

Theorem 10 For every bipartite graph \(G\) with vertex classes \(A, B\),

\[
\frac{s^2(G)}{2n^2 \sqrt{|A||B|}} \leq \mu(G) - \frac{e(G)}{\sqrt{|A||B|}} \leq \sqrt{s_2(G)/2}.
\]

Proof Let \(|A| = a, |B| = b, e(G) = m, v(G) = n\). We start with the proof of the first inequality. By the AM-QM inequality we have

\[
\sum_{u \in A} \left| d(u) - \frac{m}{a} \right| \leq \sqrt{a \sum_{u \in A} \left( d(u) - \frac{m}{a} \right)^2},
\]

\[
\sum_{u \in B} \left| d(u) - \frac{m}{b} \right| \leq \sqrt{b \sum_{u \in B} \left( d(u) - \frac{m}{b} \right)^2}.
\]

Hence, by Cauchy-Schwarz and inequality (5), we find that,

\[
s_2(G) \leq \sqrt{n} \sqrt{\sum_{u \in A} \left( d(u) - \frac{m}{a} \right)^2 + \sum_{u \in B} \left( d(u) - \frac{m}{b} \right)^2} = \sqrt{n} \sqrt{\sum_{u \in V(G)} \left( d^2(u) - \frac{m^2n}{ab} \right)}
\]

\[
\leq n \sqrt{\mu^2(G) - \frac{m^2}{ab}} \leq n \sqrt{\mu(G) - \frac{m}{\sqrt{ab}}} \left( 2\sqrt{ab} \right),
\]

proving the first inequality.

To prove the second inequality we first note the equivalent of Proposition 7 for bipartite graphs: if \(G_1\) and \(G_2\) are bipartite graphs with the same vertex classes then

\[\mu(G_1) - \mu(G_2) \leq \sqrt{|E(G_1) \setminus E(G_2)|}.\]

Note that the coefficient 2 under the square root is missing here, since \(\mu(G) \leq \sqrt{e(G)}\) for bipartite \(G\) (Cvetković [7], also [3], p. 92 Theorem 3.19).
Theorem 2 implies that there exists a graph $R = G(n, m)$ with vertex classes $A, B$ such that $|d_R(u) - d_R(v)| \leq 1$ for every $u, v$ belonging to the same vertex class and $R$ differs from $G$ in at most $s_2(G)$ edges. Since $e(R) = e(G)$ it follows that $|E(G) \setminus E(R)| = |E(R) \setminus E(G)|$ and so $2|E(G) \setminus E(R)| \leq s_2(G)$. Hence, by Proposition 7,

$$\mu(G) - \mu(R) \leq \sqrt{\frac{s_2(G)}{2}}.$$

Applying the inequality $\mu(G) \leq \max_{u \in V(G)} \sqrt{d(u)d(v)}$, due to Berman and Zhang [2], we find that

$$\mu(G) - \frac{m}{\sqrt{ab}} \leq \sqrt{\frac{s_2(G)}{2}} + \sqrt{n + 1},$$

and so

$$\mu(G) \leq \frac{m}{\sqrt{ab}} + \sqrt{n + 1}.$$

Now, applying the final argument from the proof of Theorem 8, the desired inequality follows.

5 A lower bound on $\mu_k(G) + \mu_{n-k+1}(\overline{G})$

The main goal of this section is the proof of inequality (2). By Weyl’s inequalities (10, p. 181), for every graph $G$ of order $n$, we have

$$\mu_k(G) + \mu_{n-k+1}(\overline{G}) \leq \mu_k(K_n) = -1.$$

**Theorem 11** For every $k = 1, \ldots, n - 1$

$$\mu_k(G) + \mu_{n-k+1}(\overline{G}) \geq -1 - 2\sqrt{2s(G)}.$$

**Proof** By Corollary 4 there exists a regular graph $R$ that differs from $G$ in at most $s(G) + 3n/2$ edges. Then, by Weyl’s inequalities,

$$\mu_k(A(G)) + \mu_1(A(R) - A(G)) \geq \mu_k(A(R)),$$

$$\mu_{n-k+1}(A(\overline{G})) + \mu_1(A(R) - A(\overline{G})) \geq \mu_{n-k+1}(A(R)).$$

Furthermore, by

$$\mu_1(A(R) - A(G)) \leq \sqrt{2s(G) + 3n},$$

$$\mu_1(A(\overline{R}) - A(\overline{G})) \leq \sqrt{2s(G) + 3n},$$

and so
we find that
\[ \mu_k (G) + \mu_{n-k+1} (\overline{G}) \geq \mu_k (A(R)) + \mu_{n-k+1} (A(\overline{R})) - 2\sqrt{2s(G)} + 3n \]
\[ = -1 - 2\sqrt{2s(G)} + 3n. \]

Suppose now that \( t \) is sufficiently large and consider the graphs \( G^{(t)} \) and \( \overline{G^{(t)}} \). By Theorem 5 we have
\[ \mu_k (G^{(t)}) = t\mu_k (G). \]
Similarly in view of and \( \overline{G^{(t)}} = G^{[t]} \) and Theorem 6,
\[ \mu_{nt-k+1} (\overline{G^{(t)}}) \leq \min \{ t\mu_{n-k+1} (\overline{G}) + t - 1, -1 \} \]
Since, \( s (G^{(t)}) = t^2 s(G) \), we see that
\[ t\mu_k (G) + t\mu_{n-k+1} (\overline{G}) \geq \mu_k (G^{(t)}) + \mu_k (\overline{G^{(t)})} - t + 1 \geq -t - 2\sqrt{2s(G^{(t)})} + 3nt \]
\[ = -t - 2t\sqrt{2s(G)} + 3n/t. \]
Dividing by \( t \) and letting \( t \) tend to infinity, we obtain the desired inequality. \( \square \)

For the graph \( G = K_{1,n} \) we have \( s(G) = 2\frac{n-1}{n+1} \) and \( \mu_{n+1}(G) + \mu_2 (\overline{G}) = -1 - \sqrt{n} \). Hence,
\[ \mu_{n+1}(G) + \mu_2 (\overline{G}) = -1 - \left( \frac{1}{\sqrt{2}} + o(1) \right) \sqrt{s(G)}, \]
implying that inequality \( \text{[3]} \) is tight up to a constant factor less than 4.

6 An upper bound on \( \mu_n (G) + \mu_n (\overline{G}) \)

The main result in this section is the proof of inequality \( \text{[3]} \). We start with an auxiliary result.

Lemma 12 For every graph \( G \) of order \( n \) there exists an \( \lfloor n/2 \rfloor \)-set \( S \subset V(G) \) such that
\[ e(V(G) \setminus S) - e(S) \geq \frac{1}{2}s(G). \]

Proof Note first that for any \( a \) we have
\[ \sum_{i=1}^{n} |d_i - a| \geq \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| = s(G). \]
Let \( d(1) \leq d(2) \leq \ldots \leq d(n) \) be the degree sequence of \( G \) and set \( V = [n] \). For every \( 1 \leq k \leq n \), letting \( S = [k] \), we have
\[ \sum_{u \in V \setminus S} d(u) - \sum_{u \in S} d(u) = 2e(S) + e(S,V \setminus S) - 2e(V \setminus S) - e(S,V \setminus S) = 2e(V \setminus S) - 2e(S). \]
Assume first \( n \) even, \( n = 2k \). Letting \( a = (d(k) + d(k+1))/2 \) and \( S = [k] \), we have

\[
\sum_{u \in V \setminus S} d(u) - \sum_{u \in S} d(u) = \sum_{u \in V \setminus S} (d(u) - a) + \sum_{u \in S} (a - d(u)) = \sum_{u \in V} |d_i - a| \geq s(G),
\]

proving the assertion for even \( n \).

Let now \( n \) be odd, \( n = 2k + 1 \). Letting \( a = d_{k+1} \) and \( S = [k] \), we have

\[
\sum_{u \in V \setminus S} d(u) - \sum_{u \in S} d(u) = \sum_{u \in V \setminus S} (d(u) - a) + \sum_{u \in S} (a - d(u)) = \sum_{u \in V} |d(u) - a| \geq s(G),
\]

proving the assertion for odd \( n \) as well. \( \square \)

**Theorem 13** For every graph \( G \) of order \( n \),

\[
\mu_n(G) + \mu_n(\overline{G}) \leq -1 - \frac{s^2(G)}{n^3}.
\]

**Proof** From the interlacing theorem of Haemers (see, e.g., [8], [4]), for every bipartition of \( V(G) = V_1 \cup V_2 \) we have

\[
\mu_n(G) \leq \frac{e(V_1)}{|V_1|} + \frac{e(V_2)}{|V_2|} - \sqrt{\left(\frac{e(V_1)}{|V_1|} - \frac{e(V_2)}{|V_2|}\right)^2 + \frac{e(V_1, V_2)^2}{|V_1||V_2|}}. \tag{8}
\]

Assume \( n \) even and let \( V(G) = V_1 \cup V_2 \) be a bipartition such that \( |V_1| = |V_2| = n/2 \), and \( e(V_1) - e(V_2) \geq s(G)/2 \). Letting \( e_1 = e(V_1), e_2 = e(V_2), e_3 = e(V_1, V_2), s = s(G) \), from [8], after some simple algebra, we obtain

\[
\frac{n}{2} \mu_n(G) \leq e_1 + e_2 - \sqrt{(e_1 - e_2)^2 + e_3^2} \leq e_1 + e_2 - \sqrt{\frac{s^2}{4} + e_3^2}. \tag{9}
\]

Note that \( s(G) < n^2 \) and \( e(V_1, V_2) \leq n^2/4 \); thus, we have

\[
\frac{s^4}{9n^4} + \frac{2e_3s^2}{3n^2} + e_3^2 \leq s^2\left(\frac{1}{9} + \frac{1}{6}\right) + e_3^2 \leq \frac{s^2}{4} + e_3^2,
\]

and so,

\[
\sqrt{\frac{s^2}{4} + e_3^2} \geq \frac{s^2}{3n^2} + e_3.
\]

Hence, from (9), it follows that

\[
\frac{n}{2} \mu_n(G) \leq e_1 + e_2 - e_3 - \frac{s^2}{3n^2}.
\]

Since \( s(G) = s(\overline{G}) \), we see also that

\[
\frac{n}{2} \mu_n(\overline{G}) \leq \left(\frac{n/2}{2}\right) - e_1 + \left(\frac{n/2}{2}\right) - e_2 - \frac{n^2}{4} + e_3 - \frac{s^2}{3n^2}.
\]

\[11\]
and hence,
\[ \frac{n}{2} (\mu_n (G) + \mu_n (\overline{G})) \leq 2 \left( \frac{n}{2} \right) - \frac{n^2}{4} - \frac{2s^2}{3n^2} = -\frac{n}{2} - \frac{2s^2}{3n^2}, \]
proving the assertion for even \( n \).

To prove the assertion for odd \( n \) observe that if \( t \) is even, for the graph \( G^{(t)} \) we have
\[ t\mu_n (G) + t\mu_n (\overline{G}) + t - 1 = \mu_{tn} (G^{(t)}) + \mu_{tn} (\overline{G^{(t)})} \leq -1 - \frac{t^4s^2}{t^3n^2}. \]
Dividing by \( t \) and letting \( t \) tend to infinity, the assertion follows for odd \( n \) as well. \( \Box \)

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