Wavelet Nyström fast numerical algorithm for the integral equation in scattering problems

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Abstract. We present a fast Nyström algorithm to solving the inhomogeneous Lippmann–Schwinger equation for scattering wave field problems. This algorithm transforms the discrete matrix of the linear integral operator into a sparse matrix utilizing the compression property of the wavelet transform. The sparse linear equations are solved by the double conjugate gradient method. In the numerical part, we verified the feasibility and effectiveness of our method by modeling the wave field for a synthetic model.

1. Introduction

The numerical solution of a scattering integral equation is generally achieved by the numerical integral formula or the moment method [1,2]. The traditional moment method has a complex calculation process, and the inner product integral must be calculated for each element of the coefficient matrix, which requires a large amount of calculation, resulting in low efficiency [3]. In comparison, the Nyström method forms the discrete matrix from the integral operator directly, and the coefficient matrix can be calculated by a simple evaluation of the integral kernel [4]. Elements close to the singularity can be accurately calculated using the high-order interpolation basis function, local correction, high-precision integration, and other techniques. This method requires less computational work and produces high computational efficiency. The pretreatment speed is greatly improved.

Wavelet analysis is a very important mathematical tool and has been widely used in applied mathematics, time-series analysis, signal and image processing, geophysics, approximation theory, and other engineering and scientific endeavors. Because of the orthogonality, symmetry, regularity, vanishing moment, compactness, and attenuation of the wavelet base, the wavelet functions constructed by stretching and shifting orthogonal polynomials are widely used for integral equations. The combination of wavelet analysis and traditional numerical solutions has derived lots of methods to solve numerical modeling problems like integral equations problem [5].

To obtain the acoustic scattering wave field, we use the Nyström method to solve the acoustic scattering Lippmann–Schwinger equation. First, the integral equation is discretized using the Nyström method, and the discrete matrix of the linear integral operator is then transformed into a sparse matrix via wavelet transform. Finally, the sparse linear equations are solved using the double conjugate gradient method. By comparing the numerical results with the finite difference method, we demonstrate the feasibility and effectiveness of the proposed method.
2. Algorithm

2.1. Integral equation for the scattering problem

The scattering problem can be established using a displacement field, \( u(x, x', \omega) \), in the frequency domain, where \( x \) is the observation position, \( x' \) is the source location, and \( \omega \) is the frequency. In the homogeneous background medium \( \Omega \) with velocity \( c(x) \), there is an anomaly \( \Omega' \), with a velocity distribution \( v(x) \), and the displacement field satisfies the scalar Helmholtz equation:

\[
Lu = [\nabla^2 + k^2]u(x, x', \omega) = -s(x, x', \omega),
\]

where the Helmholtz operator \( L = [\nabla^2 + k^2] \), the wavenumber \( k = \omega / v(x) \), and source signal, \( s(x, x', \omega) = s(\omega)\delta(x - x') \). Assuming that the wave velocity \( v(x) \) can be represented as a disturbance in the background profile, \( v(x) = v_0(x) + \alpha(x) \), the equivalent Helmholtz equation can be derived as:

\[
\frac{1}{V(x)} = \frac{1}{c(x)}[1 + \alpha(x)]
\]

where the equivalent Helmholtz operator \( L_0 = [\nabla^2 + k_0^2] \) and the background wavenumber \( k_0 = \omega / c(x) \). According to Green's function theorem, we can get the Lippmann–Schwinger equation as follows:

\[
u(x, x', \omega) = s(\omega)G(x, x', \omega) + k_0^2\int_{\Omega'}\alpha(x')u(x', x', \omega)G(x, x', \omega)dx'.
\]

The Green’s function \( G(x, x', \omega) \) in the homogeneous background medium has the form of

\[
G(x, x', \omega) = \frac{i}{4}H_0^{(1)}(k_0|x - x'|),
\]

where \( H_0^{(1)} \) is the first kind zero-order Hankel function. The incident wave field is

\[
S(x, x', \omega) = s(\omega)G(x, x', \omega),
\]

the scattered field is

\[
u_s(x, x', \omega) = u(x, x', \omega) - S(x, x', \omega),
\]

and

\[
F(x, x', \omega) = \alpha(x)u(x, x', \omega).
\]

Then, by substituting Eqs. (5), (6), and (7) into Eq. (3), we get the scattered field:

\[
u_s(x, x', \omega) = k_0^2\int_{\Omega'}F(x', x', \omega)G(x, x', \omega)dx'.
\]

2.2. Wavelet Nyström algorithm for acoustic scattering

The Nyström method is a mechanical quadrature method based on the quadrature formula, which can avoid large numbers of integral calculations to generate a discrete matrix. The scattering integral, Eq. (8), under the full space hypothesis is the second kind weak singular Fredholm integral equation which
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has a singularity along the diagonal. Singularities can be eliminated by the singularity subtraction. Thus, Eq. (8) can be written as

\[
F(x, x', \omega) + \left( k_0^2 - k^2 \right) \int_{\Omega'} [F(x', x', \omega) - F(x, x, \omega)] G(x, x', \omega) dx' + \left( k_0^2 - k^2 \right) F(x, x, \omega) \int_{\Omega'} G(x, x', \omega) dx' = \alpha(x) S(x, x, \omega), \quad x \in \Omega'.
\]  

(9)

Discretizing the object, \( \Omega' \), we obtain the linear equations,

\[
\begin{bmatrix}
1 - (k_0^2 - k^2) \sum_{j=1}^{n} w_j G(x_i, x'_j, \omega) + (k_0^2 - k^2) \int_{\Omega'} G(x, x', \omega) dx' \\
+ (k_0^2 - k^2) \sum_{j=1}^{n} w_j G(x, x'_j, \omega) F(x'_j, x, \omega) = \alpha(x) S(x, x, \omega)
\end{bmatrix} F(x, x, \omega) = \alpha(x) S(x, x, \omega),
\]  

(10)

where \( w_j, \ j = 1, \cdots, n \) is quadrature coefficient. Define \( G_j = G(x_i, x'_j, \omega), \ F_i = F(x_i, x, \omega), \) \( S_i = S(x_i, x, \omega), \) and substitute \( k^2 = k_0^2 \left( 1 + \alpha(x) \right) \) into Eq. (10), we can get the equation:

\[
[1 / k_0^2 \alpha - \sum_{j=1}^{n} \omega_j G_{i j} + M_i] F_i + \sum_{j=1}^{n} \omega_j G_{i j} F_j = S_i / k_0^2,
\]  

(11)

where \( M_i = M(x_i) = \int G(x_i, x') dx'. \)

Applying the simple trapezoidal rule to the discretization of the integral operator with step length \( h \) and defining \( F = [F_i]_{i=1}^{n}, \ S = [S_i / k_0^2]_{i=1}^{n}, \ \bar{I} = diag \left( 1 / k_0^2 \alpha_1, 1 / k_0^2 \alpha_2, \ldots, 1 / k_0^2 \alpha_N \right), \) \( M = diag(M_1, M_2, \ldots, M_n), \) \( W = diag \left( \left[ \begin{array}{cccc}
1 / 2, 1, & \ldots, & 1 / 2
\end{array} \right]^T \right), \) \( B = diag \left( \bar{G} \left[ \begin{array}{c}
1 / 2, 1, \ldots, 1 / 2
\end{array} \right]^T \right), \) and \( \bar{G}_i = \left[ \begin{array}{c}
\hbar^2 G_{i j}, i \neq j, \\
0, i = j
\end{array} \right], \) we finally get the equation

\[
\left( \bar{I} + M + B + \bar{G} W \right) F = S.
\]  

(12)

Allowing

\[
Z = \bar{I} + M + B + \bar{G} W,
\]  

(13)

and substituting Eq. (12) to (13), we get

\[
ZF = S,
\]  

(14)

where the coefficient matrix \( Z \) is a dense matrix. If the direct solution method is adopted, the calculation amount is \( O(n^3) \). By applying wavelet transform to operator \( Z \) and source vector \( S \), we get,

\[
\tilde{Z} = W \cdot Z \cdot W^T, \quad \tilde{S} = W \cdot S
\]  

(15)

where \( W \) is wavelet transform matrix. Then, we solve

\[
\tilde{Z} \cdot \tilde{F} = \tilde{S}
\]  

(16)
Finally, we apply the inverse wavelet transform to $\tilde{F}$ to obtain

$$F = W^T \cdot \tilde{F}$$

(17)

As the wavelet transform matrix $W$ is a sparse matrix, it is very cheap to get $F$ from equation 17. According to the characteristics of local support and the high-order vanishing moment of the wavelet function, it can be shown that the values of most elements in matrix $\tilde{Z}$ are very small. If the threshold value, $\epsilon$ is set, the influence of the smaller elements on the accuracy of the solution can be ignored. The new matrix is a sparse matrix with an approximate diagonal distribution, and Eq. (16) is solved by using the double conjugate gradient method, which greatly reduces the computation amount and improves the speed.

2.3. Numerical example

In this section, we test the wavelet Nyström algorithm by modeling the scattering wavefield of a synthetic model. The wavefield for this model is also simulate by the finite difference method for comparison. The model is shown in Figure 1. A ball and a single horizontal reflector are buried in the homogeneous earth. The entire modeling region has a size of $640 \times 640$ m in width and depth, and is discretized into $64 \times 64$ grid blocks which has a scale of 10 m in each direction. One source and 64 receivers are placed in the top of the model. The receivers are distributed along the source line. The central frequency of the Ricker source wavelet is designed to be 15 Hz so that the dominant wavelength is much larger than the size of a single grid block. The background velocity is 3,000 m/s.

Figure 2 shows the sparse matrix obtained by the DAUB12 wavelet transform for the discrete coefficient matrix. The coefficient matrix is a banded sparse matrix following wavelet transformation. Figs. 3 and 4 represent the seismic scattering wavefields obtained by wavelet Nyström and finite difference methods, respectively. From Figs. 3 and 4, we find that the scattering wavefield from the wavelet Nyström method matched very well with these calculated by the finite difference method.
Figure 3. The scattering wavefield using WN method. Figure 4. The scattering wavefield using FD method.

3. Conclusion
We presented a wavelet Nyström fast numerical method for solving the scattering integral equations. The sparse matrix is derived by the direct discretization of the integral operator and the wavelet matrix transform. The scattering integral equation is effectively solved using the double conjugate gradient solver. By testing out method on a synthetic mode and comparing the modeling results calculated by the wavelet Nyström fast numerical method with that calculated by the finite difference method, we verified the accuracy of our modeling. From our other numerical examples that is not shown in this paper, we are confirmed that the algorithm has lower computational complexity and requires less computational effort than traditional numerical algorithms.

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