Exact Effective Superpotential for $SO(N_c)$ Gauge Theory with $N_f$ Flavors

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ABSTRACT: Motivated by the duality conjecture of Dijkgraaf and Vafa between supersymmetric gauge theories and matrix models, we derive the effective superpotential of $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SO(N_c)$ and arbitrary tree level polynomial superpotential of one chiral superfield in the adjoint representation and $N_f$ fundamental matter multiplets.

KEYWORDS: Supersymmetric gauge theory, Matrix Models.
1. Introduction

Large $N$ topological duality relating $U(N)$ Chern-Simons gauge theory on $S^3$ to $A$-model topological string [1] and its embedding in the superstring context [2] has led to interesting interconnections between geometry of Calabi-Yau three-folds (CY$_3$) and $\mathcal{N}=1$ supersymmetric gauge theories. Strong coupling dynamics of supersymmetric gauge theories can be studied within the superstring duality [3] by geometrically engineering $D$-branes. Using the geometric considerations of dualities in IIB string theory, Cachazo et al [4] have obtained low energy effective superpotential for a class of CY$_3$ geometries whose singular limit is given by

$$W'(x)^2 + y^2 + z^2 + w^2 = 0 \, ,$$

where $W(x)$ is a polynomial of degree $n+1$. In fact, the low energy effective superpotential corresponds to a $\mathcal{N}=1$ supersymmetric $U(N)$ Yang-Mills with adjoint scalar $\Phi$ and tree level superpotential $W_{\text{tree}}(\Phi) = \sum_{k=1}^{n+1}(g_k/k)Tr\Phi^k$.

The mirror version of the large $N$ topological duality conjecture [1] was considered in ref. [4] relating topological $B$ strings on the CY$_3$ geometries [5] to matrix models. The potential of the matrix model $W(\Phi) = (1/g_s)W_{\text{tree}}(\Phi)$ where $\Phi$ denotes a hermitean matrix. Further, Dijkgraaf-Vafa have conjectured that the low-energy effective superpotential can be obtained from the planar limit of these matrix models [4, 5, 6]. The Dijkgraaf-Vafa conjecture was later on proved by various methods: (i) by factorization of Seiberg-Witten curves [7], (ii) using perturbative field theory arguments [8] and (iii) generalized Konishi anomaly approach [9].
The extension of topological string duality relating Chern-Simons theory with $SO/Sp$ gauge groups to $A$-model closed string on an orientifold of the resolved conifold was studied by Sinha-Vafa [10]. Generalizing the geometric procedure considered for $U(N)$ [3], the effective superpotential for $\mathcal{N} = 1$ supersymmetric theories with $SO/Sp$ gauge groups with $W_{\text{tree}} = \sum_k (g_{2k}/2k) Tr \Phi^{2k}$ where $\Phi$ is adjoint scalar superfield were derived for the orientifolds of the $CY_3$ geometries [1]. These effective superpotentials have also been computed within perturbative gauge theory [12], using matrix model techniques in [13] and using the factorization property of $\mathcal{N} = 2$ Seiberg-Witten curves [14]. Related works involving second rank tensor matter fields have been considered in refs. [15, 16, 17].

So far, the effective superpotential computation involved $\mathcal{N} = 1$ supersymmetric gauge theories with either adjoint matter or second rank tensor matter. The inclusion of matter transforming in the fundamental representation of these gauge groups can also be studied within the Dijkgraaf-Vafa setup [18, 19, 20, 21, 22, 23, 24]. For $U(N)$ gauge group, it was shown that the effective superpotential gets contributions from the matrix model planar diagrams with zero or one boundary [18, 20]. In [22], it has been shown that the $U(N)$ effective superpotential for theories with $N_f$ fundamental flavors can be calculated in terms of quantities computed in the pure gauge theory.

Further, geometric engineering of these supersymmetric theories with $N_f$ fundamental flavors was considered in [25] by placing D5 branes at locations given by the the mass $m_a$ ($a = 1, 2, \ldots N_f$) which are not the zeros of $W'(x) = 0$. Though a formal expression for effective potential is derived, explicit form is given only for quadratic potential of the adjoint matter. The $SO/Sp$ effective superpotential computations from matrix model approach have been presented for tree level superpotential of adjoint matter up to quartic terms [23, 24]. In this paper we consider $\mathcal{N} = 1$ supersymmetric $SO(N_c)$ gauge theory with arbitrary tree level superpotential of one chiral superfield in the adjoint representation and $N_f$ fundamental matter multiplets. We use the technique developed in [22] to calculate the exact effective superpotential of this theory.

The organization of the paper as follows: In section 2, we briefly discuss the relevant matrix model and its free energy. Then we discuss the $SO(N_c)$ effective superpotential for $\mathcal{N} = 1$ supersymmetric theory with fundamental matter in section 3. In particular, we obtain a neat expression for the effective superpotential for a most general tree level superpotential involving adjoint matter field. This is the main result of the paper. In section 4, we recapitulate the geometric considerations of dualities. Then, we evaluate the effective superpotential for a quartic tree level potential and show that the results agree with our expressions in section 3. We conclude with summary and discussions in section 5.

2. Relevant Matrix Model

Let us consider $\mathcal{N} = 1$ supersymmetric $SO(N_c)$ gauge theory with one adjoint field $\Phi$ and $N_f$ flavors of quarks $Q^I$’s with mass $m_I$’s ($I = 1, 2, \ldots N_f$) in the vector(fundamental)
representation. The tree level superpotential of this theory is given by

\[ W_{\text{tree}} = W(\Phi) + \sum_{l=1}^{N_f} \left( \tilde{Q} \Phi Q + m_l \tilde{Q}Q \right), \tag{2.1} \]

where \( W(\Phi) \) is a polynomial with even powers of \( \Phi \):

\[ W(\Phi) = \sum_{k=1}^{n+1} \frac{g_{2k}}{2k} Tr \Phi^{2k}. \tag{2.2} \]

According to Dijkgraaf-Vafa conjecture, the effective superpotential of this theory can be obtained from the planar limit of the matrix model whose tree level potential is proportional to \( W_{\text{tree}} \). Hence the partition function of the matrix model is \[ Z = e^{-F} = \int D\Phi DQ \exp \left\{ - \frac{1}{gs} \left[ W(\Phi) + \sum_{l=1}^{N_f} \left( \tilde{Q} \Phi Q + m_l \tilde{Q}Q \right) \right] \right\}, \tag{2.3} \]

where \( \Phi \) is \( M \times M \) real antisymmetric matrix and \( Q, \tilde{Q} \) are \( M \) dimensional vectors. For this theory, the Dijkgraaf-Vafa conjecture can be generalized \[ \] to obtain effective superpotential as a function of glueball field \( S = TrW^\alpha W^\alpha \):

\[ W_{\text{eff}} = N_c \frac{\partial F_{s^2}}{\partial S} + 4F_{RP^2} + F_{D^2}, \tag{2.4} \]

where \( F_{s^2} \) is a free energy of a diagram with topology of sphere, \( F_{RP^2} \) is free energy of the diagram with cross-cap one (topology of \( RP^2 \)) and \( F_{D^2} \) is the free energy of the diagram with one boundary (topology of \( D^2 \)). It is also known that \[ \]

\[ F_{RP^2} = -\frac{1}{2} \frac{\partial F_{s^2}}{\partial S}. \tag{2.5} \]

Using this \( W_{\text{eff}} \) becomes

\[ W_{\text{eff}} = (N_c - 2) \frac{\partial F_{s^2}}{\partial S} + F_{D^2} \]
\[ = (N_c - 2) \frac{\partial F_{\chi=2}}{\partial S} + F_{\chi=1} \]
\[ = W_{\text{VY}} + (N_c - 2) \frac{\partial F_{\chi=2}^{\text{pert}}}{\partial S} + F_{\chi=1}, \tag{2.6} \]

where \( W_{\text{VY}} \) denotes the Veneziano-Yankielowicz potential \[ \]. Here we have absorbed \( F_{RP^2} \) in \( F_{\chi=2} \) and \( F_{\chi=1} \) contains the contribution to the free energy coming from the fundamental matter. As proposed by Dijkgraaf-Vafa, we need to take the planar limit of the matrix model. The planar limit can be obtained by taking \( (M, N_f \to \infty) \) as well as \( g_s \to 0 \) such that \( S = g_s M \) and \( S_f = g_s N_f \) are finite. The total free energy of this matrix model can be expressed as an expansion in genus \( g \) and the number of quark loops \( h \):

\[ F = \sum_{g,h} g_s^{2g-2} S_f^h F_{g,h}(S). \tag{2.7} \]
Assuming that the fundamental quarks are massive compared to adjoint matter, we can integrate out the fundamental matter fields appearing quadratically in the partition function \((2.3)\) to give

\[
Z = e^{-F} = \int \mathcal{D}\Phi \exp \left[-\frac{1}{g_s} Tr \left(W(\Phi) + S_f \sum_{I=1}^{N_f} \log(\Phi + m_I)\right)\right].
\]  

We are now in a position to calculate \(F_{\chi=1}\) and \(F_{\chi=2}\) contributions to the superpotential. In the following section we apply the method developed in [22] for the \(SO(N_c)\) gauge theory with one adjoint matter field and \(N_f\) fundamental flavors.

3. Effective Superpotential

We wish to compute the exact effective superpotential of \(N = 2\) supersymmetric \(SO(N_c)\) gauge theory with \(N_f\) flavors of quark loops in the fundamental representation, broken to \(N = 1\) by addition of a tree level superpotential \(W(\Phi)\) given by eqn.(2.2). Following the arguments in [22], for the effective superpotential evaluation we can still look at a point in the quantum moduli space of \(N = 2\) pure gauge theory where \(r = [N_c/2] \) (rank of \(SO(N_c)\)) monopoles become massless [14]. This corresponds to the point where the Seiberg-Witten curve factorizes completely.

We have seen that the effective superpotential of this theory gets contributions from free energies \(F_{\chi=2}\) and \(F_{\chi=1}\) of the matrix model described in the previous section. We shall first calculate the contribution to the superpotential coming from \(F_{\chi=2}\) using the moduli associated with Seiberg-Witten factorization.

3.1 Contribution of \(F_{\chi=2}\)

In this section we compute the contribution of \(F_{\chi=2}\) to the effective superpotential. This contains free energies of the diagrams having topology of \(S^2\) and \(RP^2\). Taking derivative of eqn.(2.7) with respect to \(g_s\)

\[
\frac{\partial F}{\partial g_s} = \sum_{g,h} g_s^{2g-3}(S_f)^h \left( (2g - 2)F_{g,h} + S \frac{\partial F_{g,h}}{\partial S} \right) + \sum_{g,h} h g_s^{2g-3}(S_f)^h F_{g,h}(S). \tag{3.1}
\]

According to Dijkgraaf-Vafa, one should take the planar limit on the matrix model side. Also we take number of quark loops, \(h = 0\) for \(\chi = 2\) free-energy computation. Planar limit of the above equation gives

\[
\frac{\partial F}{\partial g_s} = g_s^{-3} \left( S \frac{\partial F_{\chi=2}}{\partial S} - 2 F_{\chi=2} \right). \tag{3.2}
\]

We can also differentiate eqn.(2.8) with respect to \(g_s\) to give

\[
\frac{\partial F}{\partial g_s} = -g_s^{-2} \langle Tr W(\Phi) \rangle. \tag{3.3}
\]
From the above two equations
\[ gs\langle TrW(\Phi)\rangle = 2F_{\chi=2} - S \frac{\partial F_{\chi=2}}{\partial S}. \] (3.4)

The form of \( W(\Phi) \) shows that the LHS contains the vacuum expectation values \( \langle Tr\Phi^{2p}\rangle \). It is clear from the above equation that once we obtain the vevs \( \langle Tr\Phi^{2p}\rangle \), we can easily compute \( F_{\chi=2} \). In the case of \( \mathcal{N} = 2 \) \( SO(N_c) \) gauge theory, the moduli are given by \( u_{2p} = \frac{1}{2p} Tr\Phi^{2p} \). We are interested in the complete factorization of the Seiberg-Witten curve. The moduli that factorizes the Seiberg-Witten curve are given by \[ \langle u_{2p} \rangle = \frac{N_c - 2}{2p} c_{2p}^p \Lambda^{2p}, \] (3.5)

where \( C_{ij} = \frac{i!}{j!(j-i)!} \) and \( \Lambda \) is the scale governing the running of the gauge coupling constant.

The matrix model calculation of the vevs of the moduli done in the context of \( SU(N_c) \) [28, 29] can be extended to \( SO(N_c) \) giving
\[ \langle u_{2p} \rangle = (N_c - 2) \frac{\partial}{\partial S} g_s \langle Tr\Phi^{2p}\rangle . \] (3.6)

It is obvious from the above two equations that
\[ \frac{\partial}{\partial S} g_s \langle Tr\Phi^{2p}\rangle = C_{2p}^p \Lambda^{2p}. \] (3.7)

We denote the effective superpotential of pure \( SO(N_c) \) gauge theory by \( W_{eff}^0 \). From eqn. (2.6) we can write
\[ W_{eff}^0 = (N_c - 2) \frac{\partial F_{\chi=2}}{\partial S} + \frac{(N_c - 2)}{2} S \left( \log \frac{S}{\Lambda^3} + 1 \right) + (N_c - 2) \frac{\partial F_{pert}^{\chi=2}}{\partial S}. \] (3.8)

The first term in the above equation is the Veneziano-Yankielowicz superpotential [31] and \( \Lambda^{3(N_c-2)} \) is the strong coupling scale of the \( \mathcal{N} = 1 \) theory. The second term is perturbative in glueball superfield \( S \) with
\[ F_{pert}^{\chi=2} = \sum_{n \geq 1} f_{n}^{\chi=2}(g_{2p}) S^{n+2}. \] (3.9)

Once we compute the functions \( f_{n}^{\chi=2}(g_{2p}) \), we will have the exact superpotential of \( SO(N_c) \) pure gauge theory. In order to compute these functions we need to take the derivative of (3.4) with respect to \( S \)
\[ \frac{\partial}{\partial S} g_s \langle TrW(\Phi)\rangle = \frac{\partial F_{\chi=2}}{\partial S} - S \frac{\partial^2 F_{\chi=2}}{\partial S^2}. \] (3.10)

Substituting eqns. (3.8, 3.9) in the above equation, we get
\[ (N_c - 2) \frac{\partial}{\partial S} g_s \langle TrW(\Phi)\rangle = W_{eff}^0 - S \frac{\partial W_{eff}^0}{\partial S} = (N_c - 2) \left[ \frac{S}{2} - \sum_{n \geq 1} n(n + 2) f_{n}^{\chi=2} S^{n+1} \right]. \] (3.11)
At the critical point of the superpotential that is when $\partial W^0_{\text{eff}}/\partial S = 0$ we have

$$ W^0_{\text{eff}} = (N_c - 2) \frac{\partial}{\partial S} g \langle Tr W(\Phi) \rangle = \sum_p g_{2p} \langle u_{2p} \rangle. \quad (3.12) $$

The glueball superfield can be obtained at the critical point by the following relation [14]:

$$ S = \frac{\partial W^0_{\text{eff}}}{\partial \log \Lambda} = \sum_{p \geq 1} g_{2p} C_{2p}^p \Lambda^{2p}. \quad (3.13) $$

Inserting eqn.(3.7) and eqn.(3.13) in eqn.(3.11) one gets

$$ \sum_{p \geq 1} \frac{1}{2p} g_{2p} C_{2p}^p \Lambda^{2p} = \frac{1}{2} \sum_{p \geq 1} g_{2p} C_{2p}^p \Lambda^{2p} - \sum_{n \geq 1} (n + 1) f_{n=2}^\chi (g_{2p}) S^{n+1}. \quad (3.14) $$

Substituting glueball field $S$ in terms of $\Lambda$ (3.13) and equating the powers of $\Lambda$ on both sides of the above equation, we can extract the functions $f_{n=2}^\chi (g_{2p})$:

$$ f_{n=2}^\chi = \frac{1}{8} g_4 \frac{C_{2(n+1)}^{n+1}}{g_2^{n+1}} - \frac{1}{2n+2} \frac{g_2^{n+1}}{g_2^{n+1}} - \sum_{l=1}^{n-1} \frac{l(l+2)}{n(n+2)} f_{l=2}^\chi \sum_{p_1+\ldots+p_{l+1}=n+1} C_{2p_1}^{p_1} \ldots C_{2p_{l+1}}^{p_{l+1}} \frac{g_{2p_1} \ldots g_{2p_{l+1}}}{2^{n+1} g_2^{2n+1}}. \quad (3.15) $$

Now that we have computed the functions $f_{n=2}^\chi (g_{2p})$, the $\chi = 2$ contribution to the effective superpotential of the $SO(N_c)$ theory with one adjoint chiral superfield with arbitrary tree level superpotential is known exactly. From eqn.(3.8) and eqn.(3.9), it is given by

$$ W^0_{\text{eff}} = (N_c - 2) \left[ \frac{S}{2} \left( -\log \frac{S}{\Lambda^3} + 1 \right) + \sum_{n \geq 1} (n + 1) f_{n=2}^\chi (g_{2p}) S^{n+1} \right]. \quad (3.16) $$

In the case of quadratic tree level superpotential, that is when $g_{2p} = 0$ for $p \geq 2$, the functions $f_{n=2}^\chi (g_{2p})$ vanish for all $n$. And we can fix the coupling scale $\Lambda^3$ to $2g_2 \Lambda^2$ by the requirement that $W^0_{\text{eff}}$ satisfies equation (3.13). The $W^0_{\text{eff}}$ for the quartic tree level superpotential can be obtained by substituting $g_{2p} = 0$ for $p \geq 3$ in the above result (3.16):

$$ W^0_{\text{eff}} = W_{VY} + (N_c - 2) \left( \frac{3}{2} \left( \frac{g_4}{4g_2} \right) S^2 - \frac{9}{2} \left( \frac{g_4^2}{8g_2^2} \right) S^3 + \frac{45}{2} \left( \frac{g_4^3}{16g_2^3} \right) S^4 + \ldots \right). \quad (3.17) $$

This is in perfect agreement with the result of [24] where it has been evaluated in terms of the matrix model as well as IIB closed string theory on Calabi-Yau with fluxes. We now compare the result (3.16) with the corresponding result in the $SU(N_c)$ gauge theory with one adjoint matter. The exact effective superpotential of $SU(N_c)$ theory has been
obtained in [22]. Comparison of the effective superpotentials of these two theories provides
the following equivalence:

\[ W_{\text{eff}}^{0 \text{SO}(N_c)}(g_{2p}) = \frac{N_c - 2}{2N_c} W_{\text{eff}}^{0 \text{SU}(N_c)}(g_{2p} = 2g_{2p}), \quad (3.18) \]

which agrees with the relation obtained in [14]. We shall now address the fundamental matter contribution \( F_{\chi=1} \) to the effective potential.

### 3.2 Contribution of \( F_{\chi=1} \)

We differentiate the free energy given by eqn.(2.7) with respect to \( S_f \)

\[ \frac{\partial F}{\partial S_f} = \sum_{g,h} h g_s^{2g-2} (S_f)^{h-1} F_{g,h}(S). \quad (3.19) \]

We are interested in genus \( g = 0 \) and one quark loop \( h = 1 \) contribution in the planar limit \( g_s \to 0 \). The dominant term from eqn.(3.19) is \( \partial S_f F = g_s^{-2} F_{\chi=1} \).

Differentiation of eqn.(2.8) with respect to \( S_f \) gives

\[ \frac{\partial F}{\partial S_f} = g_s^{-1} \sum_{\ell=1}^{\mathcal{N}_f} \langle \text{Tr} \log(\Phi + m_{\ell}) \rangle, \quad (3.20) \]

This implies

\[ F_{\chi=1} = g_s \sum_{\ell=1}^{\mathcal{N}_f} \langle \text{Tr} \log(\Phi + m_{\ell}) \rangle. \quad (3.21) \]

Expanding the above equation around the critical point \( \Phi = 0 \), we get

\[ F_{\chi=1} = \sum_{\ell=1}^{\mathcal{N}_f} \left( S \log m_{\ell} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k m_{\ell}^k} g_s \langle \text{Tr} \Phi^k \rangle \right). \quad (3.22) \]

Differentiating with respect to \( S \) and using eqn.(3.7) we get

\[ \frac{\partial F_{\chi=1}}{\partial S} = \sum_{\ell=1}^{\mathcal{N}_f} \left( \log m_{\ell} - \sum_{k=1}^{\infty} \frac{1}{2 k m_{\ell}^k} C_{2k}^k \Lambda^{2k} \right). \quad (3.23) \]

Integrate above equation with respect to \( S \)

\[ F_{\chi=1} = \sum_{\ell=1}^{\mathcal{N}_f} S \log m_{\ell} - \sum_{\ell=1}^{\mathcal{N}_f} \sum_{k,l \geq 1} \frac{l g_{2l} C_{2l}^l C_{2k}^k}{2k(k+l)m_{\ell}^{2k}} \Lambda^{2(k+l)} + A, \quad (3.24) \]

where \( A \) is the constant of integration. We postulate

\[ A = \sum_{\ell=1}^{\mathcal{N}_f} W_{\text{tree}}(m_{\ell}), \quad (3.25) \]
and we will see in the next section that the result agrees with the one obtained from Calabi-Yau geometry with fluxes. In order to write this expression in powers of $S$, we write $F_{\chi=1}$ as

$$F_{\chi=1} = S \sum_{I=1}^{N_f} \log m_I + \sum_{n \geq 1} f_{\chi=1}^{n=1}(g_{2p}) S^{n+1} + \sum_{I=1}^{N_f} W_{\text{tree}}(m_I). \quad (3.26)$$

Comparison with eqn.(3.24) gives the following recursive relation for the coefficients $f_{\chi=1}^{n=1}(g_{2p})$

$$f_{\chi=1}^{n=1} = -\frac{1}{4} \sum_{l=1}^{N_f} \frac{1}{m_I^2 g_2},$$

$$f_{\chi=1}^{n \geq 1} = -\frac{1}{2^{n+1} g_2^{n+1}} \left( \sum_{l=1}^{N_f} \sum_{k=1}^{\infty} \frac{1}{2k(n+1)m_I^{2k}} \right) + \sum_{q=1}^{n-1} f_{\chi=1}^{q} \sum_{p_1 + \ldots + p_q+1 = n+1} C_{p_1} p_1 g_{2p_1} \ldots C_{p_q+1} p_q g_{2p_q+1} \right). \quad (3.27)$$

The eqn.(3.26) along with eqn.(3.27) gives the effective superpotential from fundamental matter, for the most general $W_{\text{tree}}$. If we substitute $g_{2p} = 0$ for $p \geq 2$ in the above result, we get $F_{\chi=1}$ for the gauge theory with quadratic superpotential. It is explicitly given by,

$$F_{\chi=1} = \sum_{I=1}^{N_f} \left[ S \log m_I - \frac{1}{4} \frac{S^2}{m_I^2 g_2} - \frac{1}{8} \frac{S^3}{m_I^4 g_2^2} - \frac{5}{48} \frac{S^4}{m_I^6 g_2^3} - \ldots \right] + A. \quad (3.28)$$

Also substituting $g_{2p} = 0$ for $p \geq 3$, we get $F_{\chi=1}$ for the theory with quartic superpotential.

$$F_{\chi=1} = \sum_{I=1}^{N_f} \left[ S \log m_I + \left( -\frac{1}{4m_I^2 g_2} \right) S^2 + \left( -\frac{1}{8m_I^4 g_2^2} + \frac{9g_4}{4m_I^2 g_2^2} \right) S^3 \right. \left. + \left( -\frac{5}{48m_I^6 g_2^3} + \frac{9g_6}{32m_I^4 g_2^4} - \frac{9g_8}{16m_I^2 g_2^5} \right) S^4 + \ldots \right] + A. \quad (3.29)$$

The total effective superpotential of the theory under consideration is

$$W_{\text{eff}} = W_{\text{eff}}^0 + F_{\chi=1}$$

$$= (N_c - 2) \left[ S \left( -\frac{1}{2} \log \frac{S}{2g_2 A^2} + 1 \right) + \sum_{n \geq 1} (n + 2) f_{\chi=1}^{n=1}(g_{2p}) S^{n+1} \right] \sum_{I=1}^{N_f} \log m_I + \sum_{n \geq 1} f_{\chi=1}^{n=1}(g_{2p}) S^{n+1} + \sum_{I=1}^{N_f} W_{\text{tree}}(m_I) \quad (3.30)$$

It is important to realize the power of assimilating Dijkgraaf-Vafa conjecture and the connections to factorization of Seiberg-Witten curves. As a consequence, we have obtained a concise expression for $SO(N_c)$ effective superpotential for arbitrary polynomials of tree
level superpotential. In order to make sure that the results are consistent, we need to compare with other approaches.

In the next section, we compare the results with explicit answers obtained from geometric approach of dualities for tree level superpotential upto quartic term.

4. Geometric Engineering and Effective $SO$ Superpotential

We will briefly recapitulate geometric dualities leading to the computation of $SO$ superpotential.

4.1 Geometric Transition

Consider type IIB String theory compactified on an orientifold of a resolved Calabi-Yau geometry whose singular limit is given by eqn. (1.1). For description of $SO$ gauge group, $W(x) \equiv W(x^1, x^2, x^3, x^4)$ must be even functions of $x$. Further, $W'(x) = 0$ determines the eigenvalues of $\Phi$ which can be $0, \pm ia_i$.

We are interested in $N = 1$ $SO(N_c)$ supersymmetric gauge theory in four dimensions. This can be realized by wrapping $N_c$ D5-branes on $\mathbb{RP}^2$ of the orientifolded resolved geometry - i.e., we place all the $N_c$ branes at $x = 0$ where eigenvalues of $\Phi$ are zero. Invoking large $N$ duality [2, 3, 11], the supersymmetric gauge theory is dual to IIB string theory on a deformed Calabi-Yau geometry with fluxes. The deformed geometry is described by

$$k \equiv W'(x)^2 + f_{2n-2}(x) + y^2 + z^2 + w^2 = 0, \quad (4.1)$$

where $f_{2n-2}(x)$ is a $n-1$ degree polynomial in $x^2$. The three-cycles in this geometry can be given in terms of basis cycles $A_i, B_i \in H^3(M, \mathbb{Z})$ ($i = 1, 2, \ldots, 2n + 1$) satisfying symplectic pairing

$$(A_i, B_j) = -(B_j, A_i) = \delta_{ij}, \quad (A_i, A_j) = (B_i, B_j) = 0.$$ 

Here the pairing $(A, B)$ of three-cycles $A, B$ is defined as the intersection number. For the deformed Calabi-Yau (4.1), these three-cycles are constructed as $\mathbb{P}^1$ fibration over the line segments between two critical points $x = 0^+, 0^-, \pm ia_i^+, \pm ia_i^- \ldots$ of $W'(x)^2 + f_{2n-2}(x)$ in $x$-plane. In particular, $A_0$ cycle corresponds to $\mathbb{P}^1$ fibration over the line segment $0^- < x < 0^+$ and $A_i$'s to be fibration over the line segments $ia_i^- < x < ia_i^+$. The three-cycles $B_0(B_i)$ are non-compact and are given be fibrations over line segments between $0 < x < \Lambda_0(ia_i^+ < x < i\Lambda_0)$ where $\Lambda_0$ is a cut-off. The deformed geometry (4.1) has $\mathbb{Z}_2$ symmetry and hence we can restrict the discussion to the upper half of $x$-plane. The holomorphic three-form $\Omega$ for the deformed geometry (4.1) is given by

$$\Omega = 2 \frac{dx \wedge dy \wedge dz}{\partial k/\partial \omega}. \quad (4.2)$$

The periods $S_i$ and the dual periods $\Pi_i$ for this deformed geometry are

$$S_i = \int_{A_i} \Omega, \quad \Pi_i = \int_{B_i} \Omega.$$
The dual periods in terms of prepotential $F(S_i)$ is $\Pi_i = \partial F/\partial S_i$. Using the fact that these three cycles can be seen as $\mathbb{P}^1$ fibrations over appropriate segments in the $x$-plane, the periods can be rewritten as integral over a one-form $\omega$ in the $x$-plane. That is, $S_0 = 1/(2\pi i) \int_0^{\Lambda_0} \omega$, $\Pi_0 = 1/(2\pi i) \int_{0}^{\Lambda_0} \omega$, ... where the one-form $\omega$ is given by

$$\omega = 2dx \left( W'(x)^2 + f_{2n-2}(x) \right)^{\frac{1}{2}} .$$

(4.3)

The effective superpotential $W^0_{eff}$ (recall the suffix 0 denotes the contribution from adjoint matter field $\Phi$) can be obtained as follows

$$-\frac{1}{2\pi i} W^0_{eff} = \int \Omega \wedge (H_R + \tau H_{NS}),$$

(4.4)

where $\tau$ is the complexified coupling constant of type IIB strings, the $H_R$ and $H_{NS}$ denotes the RR-three form and NS-NS three-form field strengths and their fluxes satisfy the following relations under the geometric transition:

$$N_c - 2 = \int_{A_0} H_R \ , \ \alpha = \int_{B_i} \tau H_{NS} .$$

(4.5)

For the classical solution $\Phi = 0$ with $N$ D5-branes at $x = 0$, the dual theory will require the above RR-flux over $A_0$ cycle alone and a non-zero period $S_0 \equiv S$. The prepotential $F(S)$ will be the $\chi = 2$ part of the matrix model free energy.

Inclusion of matter in fundamental representations in the geometric framework corresponds to placing D5 branes at locations $x = m_a$ where $m_a$'s are the masses of $N_f$ fundamental flavors. These locations are not the zeros of $W'(x) = 0$. The fundamental matter contribution to the effective potential is given by [25]:

$$W_{eff}^{flav} \equiv F_{\chi=1} = \frac{1}{2} \sum_{a=1}^{N_f} \int_{m_a}^{\Lambda_0} \omega .$$

(4.6)

It is important to work out explicitly these formal integrals for specific potentials and compare with our closed form expression obtained for arbitrary potentials in section 3.

**4.2 Effective Superpotential for Quartic Potential**

In this subsection we consider the $\mathcal{N} = 1$ $SO(N_c)$ gauge theory with fundamental matter and quartic tree level superpotential:

$$W_{tree}(\Phi) = \frac{M}{2} Tr \Phi^2 + \frac{g}{4} Tr \Phi^4 .$$

(4.7)

The geometry corresponding to this gauge theory is given by

$$W'(x)^2 + f_2(x) + y^2 + z^2 + w^2 = 0 ,$$

(4.8)

where $f_2(x)$ is an even polynomial of degree 2. We concentrate on the special classical vacuum $\Phi = 0$, which is sometimes called as one cut solution in the context of matrix
models [24]. We require the critical points of \( W'(x)^2 + f_2(x) \) to be 0\(^+\), 0\(^-\). This is achieved by the following one form:

\[
\omega = 2\sqrt{W'(x)^2 + f(x)}dx = 2g(x^2 + \Delta + 2\mu^2)\sqrt{(x - 2\mu)(x + 2\mu)},
\]

where 0\(^\pm\) = \( \pm 2\mu \) and \( \Delta = \frac{M}{g} \). The period integral can be computed from

\[
S = \frac{1}{2\pi i} \int_{-2\mu}^{2\mu} \omega dx .
\]  

(4.10)

For the quartic \( W_{tree} \), it is explicitly given by

\[
S = 2g\mu^2(\Delta + 3\mu^2) .
\]

(4.11)

Equivalently, the above equation is quadratic in \( \mu^2 \) which can be solved to give the roots. Discarding the negative root, we take the other root

\[
\mu^2 = -\frac{\Delta}{6} + \frac{\Delta}{6} \sqrt{1 + \frac{6S}{g\Delta^2}} .
\]

(4.12)

In fact, it is this part of the computation which prevents generalization to potentials of powers higher than \( \Phi^4 \). For the given one-form, the \( \chi = 2 \) contribution to the free-energy is

\[
W_{eff}^0 = (N_c - 2) \frac{\partial F(S)}{\partial S} = (N_c - 2) \int_{2\mu}^{\Lambda} \omega dx .
\]

(4.13)

This result in the \( \Lambda_0 \to \infty \) limit agrees with eqn. (3.17).

The effective superpotential that comes from the contribution of flavors (4.6) is

\[
W_{eff}^{flavor} = -g \sum_{I=1}^{N_f} \left[ m_I \frac{\Delta + 3\mu^2}{2} \sqrt{m_I^2 - 4\mu^2} + \frac{m_I^2}{4} (m_I^2 - 4\mu^2)^{3/2} + \frac{\mu^2}{2} (2\Delta + 3\mu^2) \\
+ 2\mu^2 (\Delta + 3\mu^2) \log(2\Lambda_0) - 2\mu^2 (\Delta + 3\mu^2) \log \left( m_I + \sqrt{m_I^2 - 4\mu^2} \right) \right] .
\]

(4.14)

In obtaining the above result, we take the limit \( \Lambda_0 \to \infty \) and ignore the \( \Lambda_0 \) dependent terms. Substituting \( \mu^2 \) and rewriting in powers of \( S \) agrees with the our expansion (3.29).

If we take \( g \to 0 \) limit in the above equation, we get \( W_{eff}^{flavor} \) of the \( SO(N_c) \) gauge theory with quadratic tree level superpotential.

\[
W_{eff}^{flavor} = -\sum_{I=1}^{N_f} \left[ \frac{S}{2} + \frac{S}{2} \sqrt{1 - \frac{2S}{Mm_I^2}} \left( \frac{\Lambda_0}{m_I} \right) - S \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{2S}{Mm_I^2}} \right) \right] .
\]

(4.15)

If we replace \( M \) by \( M'/2 \), we get the Affleck-Dine-Seiberg \( SU(N) \) superpotential [31]. Expanding the above equation in powers of \( S \) agrees with eqn. (3.28).
5. Summary and Discussion

In this paper, we have derived $SO(N_c)$ effective superpotential for the supersymmetric theory with $N_f$ fundamental flavors (3.31). Using Dijkgraaf-Vafa conjecture and also the Sieberg-Witten factorization, we have obtained the effective superpotential for a most general tree level potential $W_{\text{tree}}(\Phi^2)$. We have shown agreement with the results from the geometric considerations of superstring dualities for a tree level potential up to quartic terms.

Though we have concentrated on the $SO$ gauge group, it appears that the fundamental matter contribution to the $Sp$ (symplectic) effective superpotential will be identical ($F_{\chi=1}$). The effective potential in the absence of matter is well-studied from various approaches which leads to the replacement of factor $N_c - 2$ in eqn. (3.30) by $N_c + 2$ to get $W_{\text{eff}}^{0}$ for $Sp(N_c)$ gauge group.

Within supersymmetric theories, the effective superpotentials for different regimes like $N_f = N_c$ or $N_f < N_c$ or $N_f > N_c$ could be addressed. It is still a challenging problem to see such distinction within the matrix model approach.
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