Electrostatic and Magnetic Fields in Bilayer Graphene

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Abstract

We compute the transmission probability through rectangular potential barriers and p-n junctions in the presence of a magnetic and electric fields in bilayer graphene taking into account the full four bands of the energy spectrum. For energy $E$ higher than the interlayer coupling $\gamma_1$ ($E > \gamma_1$) two propagation modes are available for transport giving rise to four possible ways for transmission and reflection probabilities. However, when the energy is less than the height of the barrier the Dirac fermions exhibits transmission resonances and only one mode of propagation is available. We study the effect of the interlayer electrostatic potential $\delta$ and the different geometry parameters of the barrier on the transmission probability.

PACS numbers: 73.22.Pr, 72.80.Vp, 73.63.-b
Keywords: bilayer graphene, barriers, scattering, transmission, conductance.

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1 Introduction

Graphene is a one atom thick single layer of carbon material, which takes the form of a planar honeycomb lattice of $sp^2$ bonded carbon atoms. It is the first two-dimensional (2D) crystalline material which has been experimentally realized [1]. This new material has peculiar electronic properties, among them, an unusual quantum Hall effect [2,3] and is a transparent conductor [4]. The equation describing the electronic excitations in graphene is formally similar to the Dirac equation for massless fermions which travel at a speed of the order on $10^6 m/s$ [5,6]. As a result graphene has a number of physical properties which makes it attractive for several different applications. Its conductivity can be modified over a wide range either by chemical doping or by applying an electric field. The mobility of graphene is very high [7] which makes the material very interesting for electronic high speed applications [8].

Bilayer graphene consists of two A-B stacking (Bernal form [9]) single layer graphene sheets, this material emerged as another attractive two-dimensional carbon material and many of the properties of bilayer graphene are similar to those of a single layer graphene [10,11]. However, while the energy spectrum of a single layer graphene consists of two cone shape bands, in bilayer graphene it possess four bands where the lowest conduction band and highest valence band have quadratic spectra and are tangent to each other near the K-points [12–15]. On the other hand, a single layer graphene has two atoms per unit cell while bilayer graphene has four atoms per unit cell. Even though atoms in different layers interact with each other the most important interaction between the two layers is represented by an overlap integral $\gamma_1$ [12] connecting atoms on top of each other in A-B Bernal stacking, higher order interactions between layers will have minor effect on the properties of the bilayer system and hence will be neglected in the present work.

Recently there have been some theoretical investigations of bilayer graphene in particular the work of Van Duppen [16] and our recent work [17], where we developed a theoretical framework to deal with bilayer graphene in the presence of a perpendicular electric and magnetic fields. Systematic study revealed that interlayer interaction is essential, in particular the direct interlayer coupling parameter $\gamma_1$, for the study of transmission properties. Actually this interlayer coupling $\gamma_1$ sets the main energy scale in the problem. For incident energies $E$ we found that for $E < \gamma_1$ there is only one channel of transmission exhibiting resonances while for $E > \gamma_1$ two propagating modes are available for transport resulting in four possible ways of transmission. Subsequently, we used the transfer matrix method to determine the transmission probability and associated current density. This work allowed us to investigate the current density and transmission through a double barrier system in the presence of a perpendicular electric and magnetic fields and allowed us to compare our numerical results with existing literature on the subject.

The present paper is organized as follows. In section 2, we formulate our model by setting the Hamiltonian system and computing the associated energy eigenvalues in each potential region. In section 3, we consider the three potential regions for the bilayer, we obtain the spinor solution corresponding to each region in terms of barrier parameters and applied fields. The boundary conditions enabled us to calculate the transmission and reflection probabilities. We then studied two interesting cases where the incident electron energy is either smaller or greater than the interlayer coupling parameter, $E < \gamma_1$ or $E > \gamma_1$. In section 4 we consider the first situation where $E < \gamma_1$, we have a two band tunneling in the absence of interlayer potential difference which results in one transmission
and one reflection channel. Then in section 5 we consider the case $E > \gamma_1$ which leads to a four band tunneling, this results in four transmission and four reflection channels. In section 6, we show the numerical results for the conductance and investigate the contribution of each transmission channel. Finally, in section 7, we conclude our work.

2 Theoretical model

We consider a bilayer graphene consisting of two layers of graphene having the structure A-B stacking of 3D graphite. Each layer has two independent atoms $(A_1,B_1)$ and $(A_2,B_2)$, respectively, as shown in Figure 1, with the two indices (1,2) corresponding to the lower and upper graphene layer, respectively, the letters $A$ and $B$ Label the two basis atoms in each plane. Every $B_1$ site in the bottom layer lies directly below an $A_2$ site in the upper layer while $A_1$ and $B_2$ sites do not lie directly below or above each other. Our theoretical model is based on the well established tight binding Hamiltonian of graphite [18] and adopt the Slonczewski-Weiss-McClure parametrization of the relevant intralayer and interlayer couplings [19] to model the bilayer graphene system. The in-plane hopping parameter, due to near neighbor overlap, is called $\gamma_0$ and gives rise to the in-plan carrier velocity. The strongest interlayer coupling between pairs of $A_2 - B_1$ orbitals that lie directly below and above each other is called $\gamma_1$, this coupling is at the origin of the high energy bands and plays an important role in our present work. A much weaker coupling between the $A_1 - B_2$ sites, which are not on top of each other, and hence is considered as a higher order near neighbor interaction and leads to an effective interlayer coupling called $\gamma_3$ the effect of which will be substantial only at very low energies. The last coupling parameter $\gamma_4$ represents the interlayer coupling between the same kind atoms $A_1 - A_2$ and $B_1 - B_2$. These last two coupling parameters $\gamma_4$ and $\gamma_3$ have negligible effect at high energy and consequently will be neglected in our present work [12,20]. The numerical values of these parameters have been estimated to be $\gamma_0 \approx 1.4 \text{ eV}$ for the intralayer coupling and $\gamma_1 \approx 0.4 \text{ eV}$ for the most relevant interlayer coupling while $\gamma_3 \approx 0.3 \text{ eV}$ and $\gamma_4 \approx 0.1 \text{ eV}$.

![Figure 1: Lattice structure of bilayer graphene with $(A,B)$ atoms within the same layer](image)

We consider bilayer graphene in the presence of a perpendicular electric and magnetic fields. The charge carriers are scattered by a single barrier potential along the $x$-direction which results in three different regions denoted by I, II and III. Based on the tight binding approach we can write the
Hamiltonian of the system in the long wavelength limit [21,22], and the associated eigenstates $\psi(x,y)$ as follows

$$ H = \begin{pmatrix} V^+ & v_F \pi^+ & -v_4 \pi^+ & v_3 \pi \\ v_F \pi & V^+ & \gamma_1 & -v_4 \pi \\ -v_4 \pi & \gamma_1 & V^- & v_4 \pi \\ v_3 \pi^+ & -v_4 \pi & v_F \pi & V^- \end{pmatrix}, \quad \psi(x,y) = \begin{pmatrix} \psi_{A1}(x,y) \\ \psi_{B1}(x,y) \\ \psi_{A2}(x,y) \\ \psi_{B2}(x,y) \end{pmatrix}. \tag{1} $$

Here $\pi = p_x + ip_y$, $p_{x,y} = -i\hbar \nabla + eA(x,y)$ is the in-plane momentum relative to the Dirac point, $v_F = \frac{3a_2}{2\pi} = 10^6 m/s$ is the Fermi velocity for electrons in monolayer graphene, $V^+$ and $V^-$ are the potentials on the first and second layer, and $v_{3,4} = \frac{v_F \gamma_3,4}{\gamma_0}$ are the effective velocities. We first choose the following potential barrier in each region as shown in Figure 2, the system is infinite along the $y$-axis

$$ V^\tau = \begin{cases} 0 & \text{if } x < d_1 \\ V + \tau \delta & \text{if } d_1 < x < d_2 \\ 0 & \text{if } x > d_2 \end{cases} \tag{2} $$

where $\tau = +1$ for the first layer and $\tau = -1$ for the second layer and $V$ is the barrier potential strength, and $\delta$ is the interlayer electrostatic potential difference between the two layers. Choosing the magnetic field to be perpendicular to the graphene sheet, along the $z$-direction and defined by $B(x,y) = B\Theta[(d_1 - x)(d_2 - x)]$ (with constant $B$), where $\Theta$ is the Heaviside step function. In the Landau gauge, the corresponding vector potential $A(x,y) = (0,A_y(x))$ gives rise to the above uniform magnetic field and takes the form

$$ A_y(x) = \frac{\hbar}{e l_B} \begin{cases} d_1 & \text{if } x < d_1 \\ x & \text{if } d_1 < x < d_2 \\ d_2 & \text{if } x > d_2 \end{cases} \tag{3} $$

where $l_B = \sqrt{\hbar/eB}$ is the magnetic length and $e$ is the electronic charge. In order to solve the eigenvalue problem we can separate variables and write the eigenspinors as plane waves in the $y$-direction. Since $[H,p_y] = 0$ requires conservation of momentum along the $y$-direction so that we can write

$$ \psi(x,y) = e^{ik_y y} \psi(x,k_y) \tag{4} $$

At low energies the effect of parameters $v_3$ and $v_4$ in our original Hamiltonian are negligible on the transmission coefficient [16], hence we neglect them in our present calculations. Therefore, our
Hamiltonian (1) and its associated wavefunction become

\[
H = \begin{pmatrix} V^+ & v_F \pi^+ & 0 & 0 \\
 v_F \pi & V^+ & \gamma_1 & 0 \\
 0 & \gamma_1 & V^- & v_F \pi^+ \\
 0 & 0 & v_F \pi & V^- \end{pmatrix}, \quad \psi(x, y) = \begin{pmatrix} \psi_{A1}(x, y) \\
 \psi_{B1}(x, y) \\
 \psi_{A2}(x, y) \\
 \psi_{B2}(x, y) \end{pmatrix}.
\]

(5)

In the Appendix we solve explicitly our eigenvalue equations and obtain the following expression for the energy inside the barrier region

\[
E = V + \frac{1}{\sqrt{6}} \left[ \pm \left( \mu + (A^2 + 3C)\mu^{-\frac{1}{2}} + 2A \right)^{\frac{1}{2}} \\
\pm \left[ -6B\sqrt{6} \left( \mu + (A^2 + 3C)\mu^{-\frac{1}{2}} + 2A \right)^{-\frac{1}{2}} - \left( \mu + (A^2 + 3C)\mu^{-\frac{1}{2}} - 4A \right) \right]^{\frac{1}{2}} \right]
\]

where we have set the quantities

\[
\mu = -A^3 + 27B^2 + 9AC + 3\sqrt{3} \left[ (A^2 + 3C)^3 + (-A^3 + 27B^2 + 9AC)^2 \right]^{\frac{1}{2}}
\]

(7)

\[
A = \delta^2 + (2n + 1)\vartheta_0^2 + \gamma_1^2
\]

(8)

\[
B = \vartheta_0^2 \delta
\]

(9)

\[
C = \left( 2n + 1 \right) \vartheta_0^2 - \delta^2 - \vartheta_0^4 + \gamma_1^2 \delta^2
\]

(10)

with \(\vartheta_0 = \frac{\hbar v_F}{\sqrt{B}}\) is the energy scale and \(n\) is an integer number. To show the main properties of our four bands (6), we plot the energy in terms of the magnetic field \(B\) in Figure 3. For \(\delta = 0\) and the Landau levels \((n = 1, 2, 3)\), we observe in Figure 3(a) that for the first layer and the second one we have \(E = V\) and \(E = V \pm \gamma_1\), respectively, which are corresponding to \(B = 0\). The situation changes in Figure 3(b) when we consider \(\delta \neq 0\) where the energy becomes \(E = V \pm \delta\) for \(B = 0\) and therefore \(\Delta E = 2\delta\) represents the gap of the energy spectrum. While in both cases, the energy increases/decreases as long as \(B\) and the Landau levels increases inside the barrier.

![Figure 3](image-url)

Figure 3: The four energy eigenvalues inside the barrier region as a function of the magnetic field \(B\), with \(V = 10 \gamma_1\). (a) and (b) for \(\delta = 0 \gamma_1\) and \(\delta = 3 \gamma_1\), respectively.

For \(\delta = 0\), that is in the absence of electric field, (6) reduces to [23]

\[
E = V \pm \sqrt{(2n + 1)\vartheta_0^2 + \frac{\gamma_1^2}{2}} \pm \sqrt{(2n + 1)\vartheta_0^2 \gamma_1^2 + \vartheta_0^4 + \gamma_1^4}.
\]

(11)
These energy eigenvalues inside the barrier region will reduce to the case of a single layer graphene where $\gamma_1 \rightarrow 0$, to give $E = V \pm \vartheta_0 \sqrt{2n + 1 \pm 1}$.

Outside the barrier region, the energy expression can be defined as follows

$$\epsilon = \pm \sqrt{k_{1,2}^2 + \frac{\Gamma_1^2}{2}} \pm \sqrt{\Gamma_1^2 k_{1,2}^2 + \frac{\Gamma_4^2}{4}}$$

where $\epsilon = E/\hbar v_F$, $\Gamma_1 = \gamma_1/\hbar v_F$ and

$$k_{1,2} = \sqrt{\left(\alpha_{1,2}^\pm\right)^2 + \left(k_y + \frac{d_{1,2}}{l_B^2}\right)^2}$$

with $\alpha_{1}^\pm$ being the wave vector of the propagating wave in the first region where there are two right-going (incident) propagating modes and two left-going (reflected) propagating modes, and $\alpha_{2}^\pm$ is the wave vector of the propagating wave in the third region with two right-going (transmission) propagating modes. We plot the energy (12) in Figure 4 to show its behavior in each region according the propagating modes. It is clear that the behaves differently in region I (red line) and region III (dashed line), which absent in the cases of simple and double barrier in the absence of magnetic field [16,17].

Figure 4: The four energy eigenvalues in outside the barrier region as a function of the wave vector $k_y$ along the $y$-direction for $l_B = 13.5 \ nm$ and $d_2 = -d_1 = 7.5 \ nm$, where red (dashed) line correspond to the two layer for region I (region III).

Next we will calculate the transmission and reflection probabilities of electrons across the potential barrier in our bilayer graphene system.

### 3 Transmission probability and conductance

The transmission and reflection coefficients are obtained by imposing the continuity of the wave function at each potential interface. The wave function given in the Appendix can be used in each region denoted by the integer $j$. The wave function can then be rewritten in a matrix notation as follows

$$\psi_j = G_j \cdot M_j \cdot A_j$$

(14)
where the index $j$ denotes each potential region, $j = I$ for the incident region, $j = II$ for the potential barrier region and $j = III$ for the transmission region. Outside the barrier region, $A^+_I$ and $A^+_III$ are defined by

$$A^+_I = \begin{pmatrix} \delta_{+,1} \\ r^+_+ \\ \delta_{+,1} \\ r^-_- \end{pmatrix}, \quad A^+_III = \begin{pmatrix} t^+_+ \\ 0 \\ t^-_- \\ 0 \end{pmatrix}$$

(15)

where $\pm$ indicates the wave vector $\alpha_{I,II}^\pm$ as defined in the Appendix and $\delta_{+,1}$ is the Kronecker delta function, $G_{I,III}$ and $M_{I,III}$ are defined by

$$G_{I,III} = \begin{pmatrix} f^{++}_{I,II} & f^{+-}_{I,II} & f^{-+}_{I,II} & f^{--}_{I,II} \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -g^{++}_{I,II} & -g^{+-}_{I,II} & g^{-+}_{I,II} & g^{--}_{I,II} \end{pmatrix}$$

(16)

and

$$M_{I,III} = \begin{pmatrix} e^{i\alpha_{I,II}^+ x} & 0 & 0 & 0 \\ 0 & e^{-i\alpha_{I,II}^+ x} & 0 & 0 \\ 0 & 0 & e^{i\alpha_{I,II}^+ x} & 0 \\ 0 & 0 & 0 & e^{-i\alpha_{I,II}^+ x} \end{pmatrix}.$$ 

(17)

Inside the barrier region, we have $A_{II} = (c_+, c_-, d_+, d_-)^T$, $G_{II}$

$$G_{II} = \begin{pmatrix} \eta_{-} \lambda_+ \chi^+_I & \eta_{-} \lambda_+ \chi^-_I & \eta_{-} \lambda_- \chi^+_I & \eta_{-} \lambda_- \chi^-_I \\ \lambda_0^+ & \lambda_0^- & \lambda_0^+ & \lambda_0^- \\ \zeta^+ \chi_0^+ & \zeta^+ \chi_0^- & \zeta^- \chi_0^+ & \zeta^- \chi_0^- \\ \eta_{+} \zeta^+ \chi_1^+ & \eta_{+} \zeta^+ \chi_1^- & \eta_{+} \zeta^- \chi_1^+ & \eta_{+} \zeta^- \chi_1^- \end{pmatrix}$$

(18)

where $\chi^\pm_I = D[\lambda_\pm + l, \pm Z]$ and $M_{II} = I_4$. The boundary conditions at $x = d_1$ and $x = d_2$ can be written in a matrix notation as follows

$$G_I \cdot M_I(x = d_1) \cdot A^+_I = G_{II}(x = d_1) \cdot M_{II} \cdot A_{II}$$

(19)

$$G_{III} \cdot M_{III}(x = d_2) \cdot A^+_III = G_{II}(x = d_2) \cdot M_{II} \cdot A_{II}.$$ 

(20)

Using the transfer matrix method we can connect $A^+_I$ with $A^+_III$ through the matrix $N$ defined by

$$N = M_{I}^{-1}(x = d_1) \cdot G_{I}^{-1}(x = d_1) \cdot G_{II}(x = d_1) \cdot G_{f,I}^{-1}(x = d_2) \cdot G_{III} \cdot M_{III}(x = d_2)$$

(21)

with the help of the relation $A^+_I = N A^+_III$, the transport coefficients can then be extracted from

$$\begin{pmatrix} t^+_+ \\ r^+_+ \\ t^-_- \\ r^-_- \end{pmatrix} = \begin{pmatrix} N_{11} & 0 & N_{13} & 0 \\ N_{21} & -1 & N_{23} & 0 \\ N_{31} & 0 & N_{33} & 0 \\ N_{41} & 0 & N_{43} & -1 \end{pmatrix}^{-1} \begin{pmatrix} \delta_{+,1} \\ 0 \\ \delta_{-,1} \\ 0 \end{pmatrix}.$$ 

(22)
where \( N_{ij} \) are the elements of the matrix \( N \). The transmission and reflection coefficients take the following forms

\[
t^\pm = \frac{N_{13}\delta_{\pm,-1} - N_{33}\delta_{\pm,1}}{N_{13}N_{31} - N_{33}N_{11}}
\]

(23)

\[
t^\pm = \frac{-N_{11}\delta_{\pm,-1} + N_{31}\delta_{\pm,1}}{N_{31}N_{13} - N_{11}N_{33}}
\]

(24)

\[
r^\pm = N_{21}t^\pm + N_{23}r^\pm
\]

(25)

\[
r^\pm = N_{41}t^\pm + N_{43}r^\pm.
\]

(26)

Finally, the transmission and the reflection probabilities can be obtained by using the relation

\[
\vec{J} = \pm iv(x,k_y)\hat{\sigma}\psi(x,k_y)
\]

(27)

where \( J \) defines the electric current density for our system. Computing explicitly equation (27) gives for the incident, reflected and transmitted current densities

\[
J_{inc}^x = \pm 4i\alpha^\pm \frac{\epsilon}{\alpha^1} |t^\pm|^2.
\]

(28)

\[
J_{ref}^x = \pm 4i\alpha^\pm \frac{\epsilon}{\alpha^1} (r^\pm)^*r^\pm.
\]

(29)

\[
J_{tra}^x = \pm 4i\alpha^\pm \frac{\epsilon}{\alpha^1} (t^\pm)^*t^\pm.
\]

(30)

The transmission and reflection probabilities, are expressed as follows

\[
T^\pm = \frac{|J_{tra}^x|}{|J_{inc}^x|} = \frac{\alpha^\pm}{\alpha^1} |t^\pm|^2
\]

(31)

\[
R^\pm = \frac{|J_{ref}^x|}{|J_{inc}^x|} = \frac{\alpha^\pm}{\alpha^1} |r^\pm|^2.
\]

(32)

Therefore, we ended up with four channels for transmissions and reflections probabilities, because we have four bands.

Since the electrons can be scattered into a four propagation modes and then we need to take into account the change in their wave velocities. The conductance of our system can then be expressed in terms of the transmission probability using the Landauer-Büttiker formula [24]

\[
G = G_0 \frac{L_y}{2\pi} \int_{-\infty}^{\infty} dk_y \sum_{\pm,\pm} T_{\pm}^\pm (E, k_y)
\]

(33)

where \( G_0 = Ne^2/(2\pi h) \approx 3.87 \times 10^{-5} N \Omega^{-1} \), \( N \) is the number of transverse channels and \( L_y \) is the width of the sample in the \( y \)-direction.

We will study numerically two interesting cases depending on the value of the incident energies, \( E \), as compared with the interlayer coupling parameter \( \gamma_1 \). The two band tunneling leads to one transmission and one reflection channel and takes place at energies less than the interlayer coupling \((E < \gamma_1)\), we have just one mode of propagation \( \alpha^+ \). On the other hand, for energies higher than the interlayer coupling parameter \( \gamma_1 \((E > \gamma_1)\), the four band tunneling takes place giving rise to four transmission and four reflection channels. We denote them as \( T_+^+ \) and \( T_-^- \) for scattering from the \( \alpha^+ \) and \( \alpha^- \), respectively. Therefore, we have two transmission channels \((T_+^+\) and \( T_-^-)\) of electrons moving in opposite direction (from \( \alpha^+ \) to \( \alpha^- \) and \( \alpha^- \) to \( \alpha^+ \)). In the next sections we will study each of these regimes separately. We fix \( \vartheta_0/\gamma_1 = 1.64/l_B \) in the rest of the paper.
4 Two Band Tunneling

To allow for a suitable interpretation of our main results in the low energy regime \(E < \gamma_1\), we compute numerically the transmission probability under various conditions. First we plot the transmission probability at normal incidence \((k_y l_B = -d_1/l_B = 0)\) as a function of the Fermi energy \(E\), for \(V = 0.3 \gamma_1\) and three different values of the barrier width \(d = 25 \text{ nm}\) (red line), \(d = 30 \text{ nm}\) (blue line), and \(d = 40 \text{ nm}\) (green line), see Figure 5. Figures (a)/(b) for \(\delta = 0.0 \gamma_1/\delta = 0.1 \gamma_1\) and \(l_B = 13.5 \text{ nm}\). Figures (c)/(d) for \(\delta = 0.0 \gamma_1/\delta = 0.1 \gamma_1\) and \(l_B = 18.5 \text{ nm}\). We note that in Figure 5(a), when the energy is less than the height of the barrier potential, i.e \(E < V\), we have zero transmission, while, when the energy is more then the height of the barrier Dirac fermions exhibit transmission resonances. As usual the transmission probability is slightly displaced to the left as we increase the width of the barrier. Figure 5(b) shows the same parameters 5(a) but with \(\delta = 0.1 \gamma_1\). It is clearly shown that the transmission probability is related to the transmission gap \(\Delta E = 2 \delta\). To understand more our system and study the effect of magnetic length parameters \(l_B\) we plot the transmission as function of the Fermi energy \(E\) of the barrier using the same parameters used in Figure 5(a) with \(l_B = 18.5 \text{ nm}\), see in Figure 5(c) and 5(d) with zero gap \(\delta = 0.0 \gamma_1\) and finite gap \(\delta = 0.1 \gamma_1\), respectively. We can clearly see as we increase \(l_B\) the transmission resonances increase in number and the transmission probability for three different values of the barrier width exhibit a translation to left.

![Figure 5](image.png)

Figure 5: Plot of transmission probability as a function of the Fermi energy \(E\) at normal incidence, with \(V = 0.3 \gamma_1\). (a)/(b) for \(\delta = 0.0 \gamma_1/\delta = 0.1 \gamma_1\) and \(l_B = 13.5 \text{ nm}\). (c)/(d) for \(\delta = 0.0 \gamma_1/\delta = 0.1 \gamma_1\) and \(l_B = 18.5 \text{ nm}\).

Figures 6(a) and 6(c) shows a comparison of the density plot for the transmission probability at normal incidence \(k_y l_B = -d_1/l_B = 0\) and non-normal incidence \(k_y l_B \neq -d_1/l_B\) \((d_1 = 0 \text{ nm} \text{ and } k_y = 0.05 \text{ nm}^{-1})\), as a function of the barrier width \(d\) and energy \(E\), respectively, for \(\delta = 0.0 \gamma_1\) and
V = 0.3 γ₁ in both cases. In Figures 6(b) and 6(d) we use the same parameters as in 6(a) and 6(c), respectively, but with δ = 0.1 γ₁. One notices that, at normal incidence and for δ = 0.0 γ₁ the transmission probability shown in Figure 6(a) is zero and there are no resonances within a range of energy less than the height of the barrier potential, i.e E < V. While resonances are present at non-normal incidence as shown in Figure 6(c). When the energy is more than the height of the barrier potential exhibits the transmission resonances. As observed in Figures 6(b) and 6(d) the transmission probability is related to the transmission gap ΔE = 2 δ, for E > V + δ it remains the same. We also observe that the number of resonances in the transmission as shown in Figure 6(b) decreases for E < V − δ.

Figure 6: Density plot of transmission probability as a function of the barrier width d and its energy E, for V = 0.3 γ₁ and l_B = 18.5 nm. (a)/(b) for δ = 0.0 γ₁/δ = 0.1 γ₁ at normal incidence k_yl_B = −d₁/l_B = 0. (c)/(d) for δ = 0.0 γ₁/δ = 0.1 γ₁ at non-normal incidence k_yl_B ≠ −d₁/l_B(d₁ = 0 nm and k_y = 0.05 nm⁻¹).

In Figure 7, we show density plot of the transmission probability as a function of the transverse wave vector k_y and its energy E. For two values of the barrier width : d = 30 nm (d₂ = −d₁ = 15 nm) in Figures 7(a) and 7(b), and d = 40 nm (d₂ = −d₁ = 20 nm) in Figures 7(c) and 7(d). To see the effect the barrier width d on the transmission probability at non-normal incidence, in fact, from Figure 7(a) we can clearly show that when we increase d in figure 7(c) a new peak of resonance appear within the range of energy less than the height of the barrier potential, i.e E < V. The number of these resonance peaks depends on the distance between the barrier. At nearly normal incidence | k_y | ≈ | d₁/l_B | ≈ 0.04 nm⁻¹ in Figure 7(a), and | k_y | ≈ | d₁/l_B | ≈ 0.06 nm⁻¹ in Figure 7(c) we have zero transmission, when the energy is less than the height of the barrier potential. On the other hand, for energy more than the height of the barrier the Dirac fermions exhibit transmission resonances, from Figures 7(a) and 7(b) we see that the number of transmission resonances increase when we increase the barrier width d in Figures 7(c) and 7(d), respectively. As observed in Figures 7(b) and 7(d) the transmission probability is correlated to the transmission gap ΔE = 2 δ.
Figure 7: Density plot of transmission probability as a function of the transverse wave vector $k_y$ and its energy $E$, for $V = 0.3 \gamma_1$ and $l_B = 18.5 \text{ nm}$. (a)/(b) for $\delta = 0.0 \gamma_1/\delta = 0.1 \gamma_1$, respectively, and $d = 30 \text{ nm}$. (c)/(d) for $\delta = 0.0 \gamma_1/\delta = 0.1 \gamma_1$, respectively, and $d = 40 \text{ nm}$.

Figure 8: Density plot of transmission probability as a function of the transverse wave vector $k_y$ and the barrier width $d$, for $V = 0.3 \gamma_1$ and $l_B = 18.5 \text{ nm}$. (a)/(b) for $\delta = 0.0 \gamma_1/\delta = 0.02 \gamma_1$ and $E = \frac{9}{10} V$. (c)/(d) for $\delta = 0.0 \gamma_1/\delta = 0.05 \gamma_1$ and $E = \frac{13}{14} V$.

In Figure 8 we show the density plot of transmission probability as function of the transfer wave vector $k_y$ and the barrier width $d$, for $V = 0.3 \gamma_1$ and $l_B = 18.5 \text{ nm}$. In Figures 8(a) and 8(b)
we fix the energy at \( E = \frac{9}{10} V \), for two different values of the interlayer potential \( \delta = 0.0 \, \gamma_1 \) and \( \delta = 0.02 \, \gamma_1 \). In Figures 8(c) and 8(d) we fix the energy at \( E = \frac{13}{10} V \) again for two different values of the interlayer electrostatic potential \( \delta = 0.0 \, \gamma_1 \) and \( \delta = 0.05 \, \gamma_1 \). For \( \delta = 0.0 \, \gamma_1 \) and for energy less than the height of the potential barrier, \( E < V \), we have a full transmission for a wide range of \( k_y \), by increasing the width \( d \), we create one resonance peak as depicted in Figure 8(a). However, the total transmission probability decreases for \( \delta = 0.02 \, \gamma_1 \) as shown in Figure 8(b). In Figure 8(c) most of the resonances disappeared while oscillations appear in the transmission, the number of oscillations decrease in presence of the interlayer electrostatic potential as reflected in Figure 8(d).

5 Four band tunneling

Once we allow for higher energies, \( E > \gamma_1 \), we will have four transmission and four reflection channels resulting in what we call the four band tunneling.

Figure 9: Density plot of transmission and reflection probabilities as a function of the transverse wave vector \( k_y \) and its energy \( E \) with \( V = 2.5 \, \gamma_1 \), \( \delta = 0.0 \, \gamma_1 \), \( l_B = 13.5 \, nm \), and \( d_2 = -d_1 = 7.5 \, nm \).

In Figure 9 we show the transmission and reflection probabilities, associated with different channels, as a function of the transverse wave vector \( k_y \) and the incident energy \( E \), we used \( V = 2.5 \, \gamma_1 \), \( \delta = 0.0 \, \gamma_1 \), \( l_B = 13.5 \, nm \), and \( d_2 = -d_1 = 7.5 \, nm \). For energies less than \( V - \gamma_1 \) the Dirac fermions
exhibit transmission resonance in $T_+^+$ in which the electrons propagate via $\alpha^+$ mode inside the barriers. For $V - \gamma_1 < E < V$, there are no available $\alpha^+$ states and the transmission is suppressed in this region. For nearly normal incidence, $k_y \approx -d_2 \approx -0.04 \text{ nm}^{-1}$ for $T_+^+$ and $k_y \approx -d_1 \approx 0.04 \text{ nm}^{-1}$ for $T_+^-$, the cloak effect [25] occurs in the energy region $V - \gamma_1 < E < V$, where the two modes $\alpha^+$ outside and inside barrier region are decoupled and therefore no scattering occurs between them [16] in the $T_+^+$ and $T_+^-$ channels. While for non-normal incidence the two modes $\alpha^+$ outside and inside barrier region are coupled, so that the transmission $T_+^+$ and $T_+^-$ channels in the same energy region are not zero. The transmission probabilities $T_+^+$ and $T_+^-$ are different ($T_+^+ \neq T_+^-$), which introduces an asymmetry for a single barrier due to the presence of a magnetic field. In addition, the reflection probabilities $R_+^+$ and $R_+^-$ are different ($R_+^+ \neq R_+^-$) and do not have the same number of resonances and anti-resonances.

For $T_-^-$ and $R_-^-$ the electrons propagate via $\alpha^-$ mode for $E < V$ and $E > V + \gamma_1$, which is blocked inside the barrier for $V < E < V + \gamma_1$ so that the transmission is suppressed in this region and this is equivalent to the cloak effect [16,17].

Figure 10: Density plot of transmission and reflection probabilities as a function of the transverse wave vector $k_y$ and its energy $E$ with $V = 2.5 \gamma_1$, $\delta = 0.3 \gamma_1$, $l_B = 13.5 \text{ nm}$, and $d_2 = -d_1 = 7.5 \text{ nm}$.

To probe the effect of the interlayer electrostatic potential $\delta$, we present the density plot of the transmission probability as function of the transfer wave vector $k_y$ and energy $E$, for the same param-
eters as in Figure 8 but for $\delta = 0.3 \gamma_1$ in Figure 10, we note that the transmission probability in the energy region $V - \delta < E < V + \delta$ is correlated to the transmission gap and shows a suppression due to cloak effect, as it was the case for the single barrier [16].

6 Conductance

In Figure 11 we show the conductance through a single barrier structure in the presence of a magnetic field as a function of the energy $E$ for $V = 2.5 \gamma_1$, $d_2 = -d_1 = 7.5$ nm for $l_B = 13.5$ nm (solid) and $l_B = 18.5$ nm (dotted). For energies smaller than the barriers height, the peaks in the conductance of the single barrier in the presence of a magnetic field, which are magnified in the inset of Figure 11(a), have shoulders due to the presence of resonances in the transmission probability $T^+_{+}$ in the region $0 < E < V$ and that of $T^+_{-}$, $T^+_0$, and $T^-_{-}$ in the region $\gamma_1 < E < V$ as depicted in Figure 9. The resonance peaks of the conductance resulting from propagation via $\alpha^+$ modes in the region $E < V - \gamma_1$, appear as shoulders of on other peaks [16]. Additional resonance peaks appear from propagation via $\alpha^-$ modes inside the barrier for energy larger than $\gamma_1$, $E > \gamma_1$. We should mention the inequality of the two channels $T^+_{+} \neq T^+_{-}$ due to the asymmetry in the presence of the magnetic field. For $V < E < V + \gamma_1$ the contribution of $T^-_{-}$ is zero due to the cloak effect [16, 17]. To see the effect of the interlayer electrostatic potential, we plot the conductance as function of the energy $E$ in Figures 11(b), we notice that the conductance in the energy region $\Delta E = 2\delta$ is correlated to the transmission gap.

![Figure 11: Conductance of the single barrier structure in the presence of a magnetic field as a function of energy for $V = 2.5 \gamma_1$ and $d_2 = -d_1 = 7.5$ nm. (a) for $\delta = 0.0 \gamma_1$, $l_B = 13.5$ nm (solid) and $l_B = 18.5$ nm (dotted). (b) for $\delta = 0.3 \gamma_1$ and $l_B = 13.5$ nm (solid).](image)

7 Conclusion

In the present work we computed the transmission probability through rectangular potential barriers and p-n junctions in the presence of both electric and magnetic fields in bilayer graphene. The tight binding model that describes our system leads to the formation of four bands in the associated energy spectrum. The richness of the energy spectrum allows for two propagation modes whose energy scale is set by the interlayer coupling $\gamma_1$.

For energies higher than $\gamma_1$, $E > \gamma_1$, two propagation modes are available for transport, and four possible ways for transmission and reflection probabilities, while, when the energy is less than $\gamma_1$ the
Dirac fermions have only one mode of propagation available to them. The resulting conductance incorporates these new transport channels which manifest themselves by the presence of more resonances and higher values of the conductance at high energies. The presence of an externally controlled electrostatic potential $\delta$ created an asymmetry between the on-site energies in the two layers which then resulted in a tunable energy gap between the conduction and valence energy bands. Hence we studied the effect of the interlayer electrostatic potential $\delta$ and the different barrier geometry parameters on the transmission probability.

Acknowledgments

The generous support provided by the Saudi Center for Theoretical Physics (SCTP) is highly appreciated by all authors. HB and AJ acknowledges partial support by King Fahd University of petroleum and minerals under the theoretical physics research group project RG1306-1 and RG1306-2.

Appendix: Wavefunction of our system

Hamiltonian (5) used in the Schrodinger equation $H\psi(x, y) = E\psi(x, y)$ can be written as a four linear differential equations of the from

$$-i\hbar v_F \frac{\sqrt{3}}{l_B} a\psi_{B_1}(x, k_y) = (E - V - \delta)\psi_{A_1}(x, k_y) \quad (A-1a)$$

$$i\hbar v_F \frac{\sqrt{3}}{l_B} a^+\psi_{A_1}(x, k_y) = (E - V - \delta)\psi_{B_1}(x, k_y) - \gamma_1\psi_{A_2}(x, k_y) \quad (A-1b)$$

$$-i\hbar v_F \frac{\sqrt{2}}{l_B} a\psi_{B_2}(x, k_y) = (E - V + \delta)\psi_{A_2}(x, k_y) - \gamma_1\psi_{B_1}(x, k_y) \quad (A-1c)$$

$$i\hbar v_F \frac{\sqrt{2}}{l_B} a^+\psi_{A_2}(x, k_y) = (E - V + \delta)\psi_{B_2}(x, k_y) \quad (A-1d)$$

where $a = \frac{\hbar e}{\sqrt{2}} (\partial_x + k_y + \frac{e}{\hbar} A_y(x))$ and $a^+ = \frac{\hbar e}{\sqrt{2}} (-\partial_x + k_y + \frac{e}{\hbar} A_y(x))$ are the annihilation and creation operators. We find the expression of $\psi_{A_1}(x, k_y)$ in (A-1a) and $\psi_{B_2}(x, k_y)$ in (A-1d), then we replace both $\psi_{A_1}(x, k_y)$ and $\psi_{B_2}(x, k_y)$ in (A-1b) and (A-1c), respectively. This gives

$$\left[(2\vartheta_0^2 a^+ a - (E - V - \delta)^2)\psi_{B_1}(x, k_y) = -\gamma_1(E - V - \delta)\psi_{A_2}(x, k_y) \quad (A-2a)\right]$$

$$\left[(2\vartheta_0^2 a a^+ - (E - V + \delta)^2)\psi_{A_2}(x, k_y) = -\gamma_1(E - V + \delta)\psi_{B_1}(x, k_y) \quad (A-2b)\right]$$

where $\vartheta_0 = \frac{\hbar v_F}{m}\frac{1}{l_B}$ is the energy scale. Combining the above equations to obtain

$$\left[2\vartheta_0^2 a a^+ - (E - V + \delta)^2\right]\left[2\vartheta_0^2 a^+ a - (E - V - \delta)^2\right]\psi_{B_1}(x, k_y) = \gamma_1^2((E - V)^2 - \delta^2)\psi_{B_1}(x, k_y) \quad (A-3)$$

Solving the eigenvalue equation to end up with the eigenspinors outside ($x < d_1, x > d_2$) and inside ($d_1 < x < d_2$) the barrier region results in the following two situations:

a) Inside the barrier region

In region II ($d_1 < x < d_2$), the vector potential $A_y(x)$ is given by $\frac{\hbar}{m\vartheta_0} x$ which is then expressed in term of annihilation and creation operators ($a$ and $a^+$), using the envelope function $\psi_{B_1}(x, k_y) \equiv \psi_{B_1}(X)$
that depend on a combination of the variables, $X = \frac{x}{l_B^2} + ky/l_B$. We can rewrite $a$ and $a^+$ as follows:

$$a = \frac{1}{\sqrt{2}} (\partial_X + X)$$

and

$$a^+ = \frac{1}{\sqrt{2}} (-\partial_X + X)$$

then differential equation becomes

$$[-\partial_X^2 + X^2 - 1 - 2\lambda_+] [-\partial_X^2 + X^2 - 1 - 2\lambda_-] \psi_{B_1}(X) = 0$$

where

$$\lambda_+ = -\frac{1}{2} + \frac{(E - V)^2 + \delta^2}{2\theta_0^2} \pm \frac{\sqrt{(\theta_0^2 - 2(E - V)\delta)^2 + \gamma_1'(1(E - V)^2 - \delta^2)}}{2\theta_0^3}.$$  \hspace{1cm} (A-5)

Therefore, the general solution of (A-4), can be written in the form $\psi_{B_1}(Z) = \psi^+_{B_1}(Z) + \psi^-_{B_1}(Z)$ with

$$\psi^+_{B_1}(Z) = c_+ D[\lambda_+, Z] + c_- D[\lambda_+, -Z]$$

$$\psi^-_{B_1}(Z) = d_+ D[\lambda_-, Z] + d_- D[\lambda_-, -Z]$$

where $Z = \sqrt{2}X$. Using this result in equation (A-1a), gives $\psi_{A_1}(Z) = \psi^+_{A_1}(Z) + \psi^-_{A_1}(Z)$ with

$$\psi^+_{A_1}(Z) = c_+ \eta_- \lambda_+ D[\lambda_+ - 1, Z] + c_- \eta_+ \lambda_+ D[\lambda_+ - 1, -Z]$$

$$\psi^-_{A_1}(Z) = d_+ \eta_- \lambda_- D[\lambda_- - 1, Z] + d_- \eta_+ \lambda_- D[\lambda_- - 1, -Z]$$

where $\eta_\pm = \frac{\rho \theta_0}{E - V \pm \delta}$. Furthermore using both $\psi_{A_1}(x, k_y)$ and $\psi_{B_2}(x, k_y)$ in (A-1b) gives $\psi_{A_2}(Z) = \psi^+_{A_2}(Z) + \psi^-_{A_2}(Z)$ with

$$\psi^+_{A_2}(Z) = c_+ \zeta^+ D[\lambda_+, Z] + c_- \zeta^+ D[\lambda_+, -Z]$$

$$\psi^-_{A_2}(Z) = d_+ \zeta^- D[\lambda_-, Z] + d_- \zeta^- D[\lambda_-, -Z]$$

where $\zeta^\pm = \frac{E - V - \delta}{\gamma_1} - \frac{2\theta_0^2 \lambda_+}{\gamma_1(E - V - \delta)}$. Finally, using $\psi_{A_2}$ in (A-1d) gives $\psi_{B_2}(Z) = \psi^+_{B_2}(Z) + \psi^-_{B_2}(Z)$ with

$$\psi^+_{B_2}(Z) = c_+ \eta_+ \zeta^+ D[\lambda_+ + 1, Z] + c_- \eta_+ \zeta^+ D[\lambda_+ + 1, -Z]$$

$$\psi^-_{B_2}(Z) = d_+ \eta_+ \zeta^- D[\lambda_- + 1, Z] + d_- \eta_+ \zeta^- D[\lambda_- + 1, -Z].$$

b) Outside the barrier region

Solving the eigenvalue equation (A-3) to obtain the eigenspinor in region I ($x < d_1$) and in region III ($x > d_2$), where potential barrier $V$ and interlayer potential $\delta$ are equal to zero and the associated vector potential $A_y(x)$ is constant and equal to $\frac{h}{e d_1} d_1$ ($\frac{h}{e d_2} d_2$) in region I (region III). We obtain the general solution in a plane-wave form $\psi_{B_1}(x, k_y) = \psi^+_{B_1}(x, k_y) + \psi^-_{B_1}(x, k_y)$ with

$$\psi^+_{B_1}(x, k_y) = c_+ e^{i\alpha_{12}^+ x} + c_- e^{-i\alpha_{12}^+ x}$$

$$\psi^-_{B_1}(x, k_y) = d_+ e^{i\alpha_{12}^- x} + d_- e^{-i\alpha_{12}^- x}$$

where $\alpha_{12}^\pm = \sqrt{(E^2 + E\gamma_1)/(h\nu_F)^2 - (k_y + \frac{d_1}{2})^2}$ is the parallel wave vector component in the $x$-direction while indices 1 and 2 represent the two regions I and III, respectively. Using this result in (A-1a) gives $\psi_{A_1}(x, k_y) = \psi^+_{A_1}(x, k_y) + \psi^-_{A_1}(x, k_y)$ with

$$\psi^+_{A_1}(x, k_y) = c_+ f_1^{++} e^{i\alpha_{12}^{++} x} + c_- f_1^{+-} e^{-i\alpha_{12}^{++} x}$$

$$\psi^-_{A_1}(x, k_y) = d_+ f_1^{-+} e^{i\alpha_{12}^{-+} x} + d_- f_1^{--} e^{-i\alpha_{12}^{-+} x}$$
where $f_{1,2}^{\pm} = \left( \pm \alpha_{1,2}^{\pm} - i \left( k_y + \frac{d_{1,2}}{\hbar} \right) \right) \hbar v_F / E$. Replacing both $\psi_A(x, k_y)$ and $\psi_B(x, k_y)$ in equation (A-1b) gives

$$ \psi_{A_2}(x, k_y) = \psi_{\alpha_2}^A(x, k_y) + \psi_{\alpha_2}^{-A}(x, k_y) $$

with

$$ \psi_{\alpha_2}^A(x, k_y) = -c e^{i \alpha_{1,2}^+ x} - c e^{-i \alpha_{1,2}^- x} $$

$$ \psi_{\alpha_2}^{-A}(x, k_y) = d_+ e^{i \alpha_{1,2}^- x} + d_- e^{-i \alpha_{1,2}^+ x}. $$

Finally, we use $\psi_{A_2}$ in (A-1d) gives

$$ \psi_{B_2}(x, k_y) = \psi_{\alpha_2}^B(x, k_y) + \psi_{\alpha_2}^{-B}(x, k_y) $$

with

$$ \psi_{\alpha_2}^B(Z) = -c g_{1,2}^{++} e^{i \alpha_{1,2}^+ x} - c g_{1,2}^{+} e^{-i \alpha_{1,2}^- x} $$

$$ \psi_{\alpha_2}^{-B}(Z) = d_+ g_{1,2}^{-+} e^{i \alpha_{1,2}^- x} + d_- g_{1,2}^{-} e^{-i \alpha_{1,2}^+ x}, $$

where $g_{1,2}^{\pm} = \left( \pm \alpha_{1,2}^{\pm} + i \left( k_y + \frac{d_{1,2}}{\hbar} \right) \right) \hbar v_F / E$.

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