Multiple critical points
for a class of nonlinear functionals

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Abstract

In this paper we prove a multiplicity result concerning the critical points of a class of functionals involving local and nonlocal nonlinearities. We apply our result to the nonlinear Schrödinger-Maxwell system in $\mathbb{R}^3$ and to the nonlinear elliptic Kirchhoff equation in $\mathbb{R}^N$ assuming on the local nonlinearity the general hypotheses introduced by Berestycki and Lions.

1 Introduction

In the celebrated papers [8, 9], Berestycki and Lions proved the existence of a ground state and a multiplicity result for the equation

$$-\Delta u = g(u), \quad u : \mathbb{R}^N \to \mathbb{R}, \quad (1)$$

for $N \geq 3$, assuming that

(g1) $g \in C(\mathbb{R}, \mathbb{R})$ and odd;

(g2) $-\infty < \liminf_{s \to 0^+} g(s)/s \leq \limsup_{s \to 0^+} g(s)/s = -m < 0$;

(g3) $-\infty \leq \limsup_{s \to +\infty} g(s)/|s|^{2^*-1} \leq 0$, with $2^* = 2N/(N-2)$;

(g4) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s) \, ds > 0$.

Modifying, if necessary, in a suitable way the nonlinearity $g$ (without losing the generality of the problem), it can be proved that equation (1) possesses a variational structure, namely its solutions can be found as critical points of the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).$$

Solutions of several nonlinear elliptic equations involving local and nonlocal nonlinearities can be found looking for critical points of a suitable perturbation of $I$, namely

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + qR(u) - \int_{\mathbb{R}^N} G(u), \quad u \in H^1(\mathbb{R}^N), \quad (2)$$

where $q > 0$ is a small parameter and $R : H^1(\mathbb{R}^N) \to \mathbb{R}$. In order to define the functional $I_q$ we need to replace (g3) with the stronger assumption

$$(g3’) \lim_{s \to +\infty} g(s)/|s|^{2^*-1} = 0.$$

In this paper we are interested in providing a multiplicity result in critical point theory for $I_q$. To this end we suppose that $R = \sum_{i=1}^k R_i$ and, for each $i = 1, \ldots, k$ the functional $R_i$ satisfies:

(R1) $R_i$ is $C^1(H^1(\mathbb{R}^N), \mathbb{R})$, nonnegative and even;

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(R2) there exists $\delta_i > 0$ such that $R_i'(u)[u] \leq C\|u\|^\delta_i$, for any $u \in H^1(\mathbb{R}^N)$;

(R3) if $\{u_j\}_j$ is a sequence in $H^1(\mathbb{R}^N)$ weakly convergent to $u \in H^1(\mathbb{R}^N)$, then
$$\limsup_j R_i'(u_j)[u - u_j] \leq 0;$$

(R4) there exist $\alpha_i, \beta_i \geq 0$ such that if $u \in H^1(\mathbb{R}^N)$, $t > 0$ and $u_t = u(. / t)$, then
$$R_i(u_t) = t^{\alpha_i} R_i(u^\beta_i);$$

(R5) $R_i$ is invariant under the action of $N$-dimensional orthogonal group, i.e. $R_i(u(g \cdot)) = R_i(u(\cdot))$ for every $g \in O(N)$.

The effect deriving from the presence of the perturbation $qR$ is to modify the structure of the functional $I$ both as regards the geometrical properties, and as regards compactness properties. In particular two remarkable difficulties arise: the first is related with the problem of applying classical min-max arguments to find Palais-Smale sequences at suitable levels, the second is concerned with the compactness of these sequences. If, on one hand, just assuming the positiveness of the functional $R$ we overcome the difficulty of finding suitable min-max levels, on the other, the problem of boundedness of Palais-Smale sequences is not nearly trivial. This is a consequence of the fact that no Ambrosetti-Rabinowitz hypothesis like
$$0 < \nu G(t) \leq t q(t), \text{ for } \nu > 2,$$

is assumed on $q$. The monotonicity trick based on an idea of Struwe [29] and formalized by Jeanjean [17] has turned out to be a powerful method to overcome this difficulty. By means of the monotonicity trick and a truncation argument based on an idea of Berti and Bolle [10] and of Jeanjean and Le Coz [18] (see also [21]), in [5] we have proved an existence result for a functional which is included in the class we are treating. The same arguments have been used also in [4] to prove a similar existence result also for another functional of the type described in (2). In both the results it is required that the parameter $q$ is sufficiently small. The well known fact proved in [9] and more recently in [15] that $I$ possesses infinitely many critical points has led us to wonder if, at least for small $q$, a multiplicity result on the number of critical points keeps holding for $I_q$. In this direction a fundamental contribution comes from the recent paper [15], where, developing some ideas of [16], a new method to find multiple solutions to equations involving general local nonlinearities has been introduced. Here we will get our multiplicity result by a suitable combination of the new method described in [15] with the truncation argument of [18].

Our main result is the following.

**Theorem 1.1.** Let us suppose (g1), (g2), (g3), (g4) and (R1)–(R5). Then for any $h \in \mathbb{N}, h \geq 1$, there exists $q(h) > 0$ such that for any $0 < q < q(h)$ the functional $I_q$ admits at least $h$ couples of critical points in $H^1(\mathbb{R}^N)$ with radial symmetry.

Some nonlinear mathematical physics problems can be solved looking for critical points of functionals strictly related with $I_q$. Among them, we recall, for instance, the electrostatic Schrödinger-Maxwell equations. This system constitutes a model to describe the interaction between a nonrelativistic charged particle and a truncation argument based on an idea of Berti and Bolle [72]. Among them, we recall, for instance, the electrostatic Schrödinger-Maxwell equations. This system constitutes a model to describe the interaction between a nonrelativistic charged particle and a truncation argument based on an idea of Berti and Bolle [72].

Finding solutions to the previous system is equivalent to look for critical points of the functional
$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * u^2 \right) u^2 - \int_{\mathbb{R}^3} G(u).$$

In [2], the authors study (3) with $g(u) = -u + |u|^{p-1} u$ and $1 < p < 5$ and use an abstract tool, based on the monotonicity trick, to prove a multiplicity result.

As a consequence of Theorem 1.1 we prove...
Theorem 1.2. Let us suppose (g1), (g2), (g3), (g4). Then for any \( h \in \mathbb{N}, h \geq 1 \), there exists \( q(h) > 0 \) such that for any \( 0 < q < q(h) \) system (3) admits at least \( h \) couples of solutions in \( H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) with radial symmetry.

Another variational problem related with our abstract result is the following. Let us consider the multidimensional Kirchhoff equation

\[
\frac{\partial^2 u}{\partial t^2} - \left( p + q \int_{\Omega} |\nabla u|^2 \right) \Delta u = 0 \quad \text{in} \; \Omega,
\]

where \( \Omega \subset \mathbb{R}^N, p > 0 \) and \( u \) satisfies some initial or boundary conditions. It arises from the following Kirchhoff’ nonlinear generalization (see [22]) of the well known d’Alembert equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0,
\]

and it describes a vibrating string, taking into account the changes in length of the string during the vibration. Here, \( L \) is the length of the string, \( h \) is the area of the cross section, \( E \) is the Young modulus of the material, \( \rho \) is the mass density and \( P_0 \) is the initial tension.

If we look for static solutions, the equation we have to solve is

\[
- \left( p + q \int_{\Omega} |\nabla u|^2 \right) \Delta u = 0.
\]

In the same spirit of [1, 4] we consider the semilinear perturbation

\[
- \left( p + q \int_{\Omega} |\nabla u|^2 \right) \Delta u = g(u), \quad \text{in} \; \Omega \subset \mathbb{R}^N. \tag{4}
\]

Recently this equation has been extensively treated by many authors in bounded domains, assuming Dirichlet conditions on the boundary (see for example [1, 14, 23, 24, 25, 26, 31]).

Here we are interested in showing an application of our abstract result to the equation (4) in all the space \( \mathbb{R}^N, N \geq 3 \). The solutions are the critical points of the functional

\[
I_q(u) = \frac{p}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{q}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^N} G(u).
\]

We prove the following result.

Theorem 1.3. Let us suppose (g1), (g2), (g3), (g4). Then for any \( h \in \mathbb{N}, h \geq 1 \), there exists \( q(h) > 0 \) such that for any \( 0 < q < q(h) \) equation (4) admits at least \( h \) couples of solutions in \( H^1(\mathbb{R}^N) \) with radial symmetry.

The paper is organized as follows: in Section 2 we prove Theorem 1.1; in Section 3 we show as it can be applied to the nonlinear Schrödinger-Maxwell system and the nonlinear elliptic Kirchhoff equation in order to prove Theorems 1.2 and 1.3.

**NOTATION**

We will use the following notations:

- for any \( 1 \leq s \leq +\infty \), we denote by \( \| \cdot \|_s \) the usual norm of the Lebesgue space \( L^s(\mathbb{R}^N) \);
- \( H^1(\mathbb{R}^N) \) is the usual Sobolev space endowed with the norm
  \[
  \| u \|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + u^2;
  \]
- \( D^{1,2}(\mathbb{R}^N) \) is completion of \( C_0^\infty(\mathbb{R}^N) \) (the compactly supported functions in \( C^\infty(\mathbb{R}^N) \)) with respect to the norm
  \[
  \| u \|^2_{D^{1,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla u|^2;
  \]
- \( C, C', C_i \) are various positive constants which may also vary from line to line.
2 The abstract result

We set for any $s \geq 0$,

\[ g_1(s) := (g(s) + ms)^+, \]
\[ g_2(s) := g_1(s) - g(s), \]

and we extend them as odd functions. Since

\[
\lim_{s \to 0} g_1(s) = 0, \\
\lim_{s \to \pm \infty} \frac{g_1(s)}{|s|^{2^* - 1}} = 0,
\]

and

\[ g_2(s) \geq ms, \quad \forall s \geq 0, \]

by some computations, we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

\[ g_1(s) \leq C_\varepsilon |s|^{2^* - 1} + \varepsilon g_2(s), \quad \forall s \geq 0. \]

(7)

If we set

\[ G_i(t) := \int_0^t g_i(s) \, ds, \quad i = 1, 2, \]

then, by (6) and (7), we have

\[ G_2(s) \geq \frac{m}{2} s^2, \quad \forall s \in \mathbb{R} \]

and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

\[ G_1(s) \leq C_\varepsilon |s|^{2^*} + \varepsilon G_2(s), \quad \forall s \in \mathbb{R}. \]

(9)

Since, for any $u \in H^1(\mathbb{R}^N)$, $R_i(u) - R_i(0) = \int_0^1 \frac{d}{dt} R_i(tu) \, dt$, by (R2) we have that

\[ R_i(u) \leq C_1 + C_2 \|u\|^\delta. \]

(10)

The hypothesis (R5) assures that all functionals that we will consider in this paper are invariant under rotations. Then

\[ H^1_r(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) \mid u \text{ radial} \} \]

is a natural constraint to look for critical points, namely critical points of the functional restricted to $H^1_r(\mathbb{R}^N)$ are true critical points in $H^1(\mathbb{R}^N)$. Therefore, from now on, we will directly define our functionals in $H^1_r(\mathbb{R}^N)$.

As in [18], we consider a cut-off function $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ such that

\[
\begin{cases}
\chi(s) = 1, & \text{for } s \in [0, 1], \\
0 \leq \chi(s) \leq 1, & \text{for } s \in [1, +\infty[,
\chi(s) = 0, & \text{for } s \in [2, +\infty[, \\
\|\chi'\|_\infty \leq 2,
\end{cases}
\]

and we introduce the following truncated functional $I^T_q : H^1_r(\mathbb{R}^N) \to \mathbb{R}$

\[ I^T_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + qk_T(u)R(u) - \int_{\mathbb{R}^N} G(u), \]

where

\[ k_T(u) = \chi \left( \frac{\|u\|^2}{T^2} \right). \]

Of course, any critical point $u$ of $I^T_q$ with $\|u\| \leq T$ is a critical point of $I_q$.

The $C^1$–functional $I^T_q$ has the symmetric mountain pass geometry:
Lemma 2.1. There exist \( r_0 > 0 \) and \( \rho_0 > 0 \) such that

\[
I_q^T(u) \geq 0, \quad \text{for } \|u\| \leq r_0, \tag{11}
\]

\[
I_q^T(u) \geq \rho_0, \quad \text{for } \|u\| = r_0. \tag{12}
\]

Moreover, for any \( n \in \mathbb{N}, n \geq 1 \), there exists an odd continuous map

\[
\gamma_n : S^{n-1} = \{ \sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| = 1 \} \rightarrow H^1_\tau(\mathbb{R}^N),
\]

such that

\[
I_q^T(\gamma_n(\sigma)) < 0, \quad \text{for all } \sigma \in S^{n-1}.
\]

Proof By (8), (9) and the positivity of the map \( R \),

\[
I_q^T(u) \geq C_1 \|u\|^2 - C_2 \|u\|^2
\]

from which we obtain (11) and (12).

Moreover, arguing as in \([9, \text{Theorem 10}]\), for every \( n \geq 1 \) we can consider an odd continuous map \( \pi_n : S^{n-1} \rightarrow H^1_\tau(\mathbb{R}^N) \) such that

\[
0 \notin \pi_n(S^{n-1}), \quad \int_{\mathbb{R}^N} G(\pi_n(\sigma)) \geq 1 \quad \text{for all } \sigma \in S^{n-1}.
\]

Then, for \( t \) sufficiently large, we take

\[
\gamma_n(\sigma) = \pi_n(\sigma) / t
\]

and we obtain

\[
I_q^T(\gamma_n(\sigma)) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \pi_n(\sigma)|^2 + q \chi \left( \frac{t^{N-2} \|\nabla \pi_n(\sigma)\|^2}{2} + \int_{\mathbb{R}^N} G(\pi_n(\sigma)) \right) R(\gamma_n(\sigma))
\]

\[
\leq \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \pi_n(\sigma)|^2 - t^N < 0.
\]

Let us define

\[
b_n = b_n(q, T) = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I_q^T(\gamma(\sigma))
\]

where \( D_n = \{ \sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| \leq 1 \}, \)

\[
\Gamma_n = \left\{ \gamma \in C(D_n, H^1_\tau(\mathbb{R}^N)) \mid \begin{array}{l}
\gamma(-\sigma) = -\gamma(\sigma) \\
\gamma(\sigma) = \gamma_n(\sigma)
\end{array} \quad \text{for all } \sigma \in D_n \right\}
\]

and \( \gamma_n : \partial D_n \rightarrow H^1_\tau(\mathbb{R}^N) \) is given in Lemma 2.1.

Analogously to \([15] \), we set

\[
\tilde{I}_q(\theta, u) = I_q(u e^{-\theta}),
\]

\[
\tilde{I}_q^T(\theta, u) = I_q^T(u e^{-\theta}),
\]

\[
\tilde{I}_q'(\theta, u) = \frac{\partial}{\partial u} \tilde{I}_q(\theta, u),
\]

\[
(\tilde{I}_q^T)'(\theta, u) = \frac{\partial}{\partial u} \tilde{I}_q^T(\theta, u),
\]

\[
\tilde{b}_n = \tilde{b}_n(q, T) = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} \tilde{I}_q^T(\gamma(\sigma)),
\]

where

\[
\tilde{\Gamma}_n = \left\{ \tilde{\gamma} \in C(D_n, \mathbb{R} \times H^1_\tau(\mathbb{R}^N)) \mid \begin{array}{l}
\tilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \text{ satisfies} \\
(\tilde{\theta}(\sigma), \eta(\sigma)) = (\theta(\sigma), -\eta(\sigma)) \\
(\tilde{\theta}(\sigma), \eta(\sigma)) = (0, \gamma_n(\sigma))
\end{array} \quad \text{for all } \sigma \in D_n \right\}.
\]
By (R4) we have
\[ I_q(\theta, u) = \frac{e^{(N-2)\theta}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + q \sum_{i=1}^k \alpha_i R_i(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u), \]
\[ \tilde{I}_q(\theta, u) = \frac{e^{(N-2)\theta}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + q \chi \left( \frac{e^{(N-2)\theta}||\nabla u||^2 + e^{N\theta}||u||^2}{T^2} \right) \sum_{i=1}^k \alpha_i R_i(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u). \]

Arguing as in [15], the following lemmas hold.

Lemma 2.2. We have
1. there exists \( b > 0 \) such that \( b_n \geq \tilde{b} \) for any \( n \geq 1 \);
2. \( b_n \to +\infty \);
3. \( b_n = \tilde{b}_n \) for any \( n \geq 1 \).

Lemma 2.3. For any \( n \geq 1 \), there exists a sequence \( \{ \{\theta_j, u_j\} \} \subseteq \mathbb{R} \times H^1(\mathbb{R}^N) \) such that
1. \( \theta_j \to 0 \);
2. \( \tilde{I}_q(\theta_j, u_j) \to b_n \);
3. \( (\tilde{I}_q)'(\theta_j, u_j) \to 0 \) strongly in \( (H^1(\mathbb{R}^N))^{-1} \);
4. \( \frac{\partial}{\partial \theta} \tilde{I}_q(\theta_j, u_j) \to 0 \).

Now we prove that for a suitable choice of \( T \) and \( q \), the sequence \( \{u_j\} \) obtained in Lemma 2.3 actually is a bounded Palais-Smale sequence for \( I_q \).

Proposition 2.4. Let \( n \geq 1 \) and \( T_n > 0 \) sufficiently large. There exists \( q_n \) which depends on \( T_n \), such that for any \( 0 < q < q_n \), if \( \{ \{\theta_j, u_j\} \} \subseteq \mathbb{R} \times H^1(\mathbb{R}^N) \) is the sequence given in Lemma 2.3, then, up to a subsequence, \( \|u_j\| \leq T_n \) for any \( j \geq 1 \).

Proof. By Lemmas 2.2 and 2.3, we infer that
\[ N \tilde{I}_q(\theta_j, u_j) - \frac{\partial}{\partial \theta} \tilde{I}_q(\theta_j, u_j) = Nb_n + o_j(1), \]
and so
\[ e^{(N-2)\theta_j} \int_{\mathbb{R}^N} |\nabla u_j|^2 = q \chi \left( \frac{\|u_j(e^{-\theta_j} \cdot)\|^2}{T^2} \right) \sum_{i=1}^k (\alpha_i - N) R_i(u_j(e^{-\theta_j} \cdot)) \]
\[ + q \chi \left( \frac{\|u_j(e^{-\theta_j} \cdot)\|^2}{T^2} \right) \sum_{i=1}^k e^{\alpha_i \theta_j} R_i(e^{\beta_i \theta_j} u_j) \]
\[ + q \chi \left( \frac{\|u_j(e^{-\theta_j} \cdot)\|^2}{T^2} \right) \frac{(N-2)e^{(N-2)\theta_j}||\nabla u_j||^2 + Ne^{N\theta_j}||u_j||^2}{T^2} R(u_j(e^{-\theta_j} \cdot)) \]
\[ + Nb_n + o_j(1). \]

We are going to estimate the right part of the previous identity. By the min-max definition of \( b_n \), if \( \gamma \in \Gamma_n \), we have
\[ b_n \leq \max_{\sigma \in D_n} I_q(T)(\gamma(\sigma)) \]
\[ \leq \max_{\sigma \in D_n} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \gamma(\sigma)|^2 - \int_{\mathbb{R}^N} G(\gamma(\sigma)) \right\} + \max_{\sigma \in D_n} \left\{ qk_T(\gamma(\sigma)) R(\gamma(\sigma)) \right\} \]
\[ = A_1 + A_2(T). \]
Multiple critical points

Now, if \( \|\gamma(\sigma)\|^2 \geq 2T^2 \) then \( A_2(T) = 0 \), otherwise, by (10), we have

\[
A_2(T) \leq q(C_1 + C_2\|\gamma(\sigma)\|^\delta) \leq q(C_1 + C_2T^\delta),
\]

for a suitable \( \delta > 0 \). Moreover we have that

\[
q\chi \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \sum_{i=1}^{k} (\alpha_i - N) R_i(u_j(e^{-\theta_j \cdot})) \leq q(C_1 + C_2T^\delta);
\]

\[
q\chi \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta_j} R_i(e^{\beta_i \theta_j} u_j)[\beta_i e^{\beta_i \theta_j} u_j] \leq CqT^\delta; \tag{14}
\]

\[
q\chi' \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \frac{(N - 2)e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + Ne^{N\theta_j}\|u_j\|_2^2 R(u_j(e^{-\theta_j \cdot}))}{T^2} \leq q(C_1 + C_2T^\delta). \tag{15}
\]

Then, from (13) we deduce that

\[
\int_{\mathbb{R}^N} |\nabla u_j|^2 \leq C' + q(C_1 + C_2T^\delta). \tag{16}
\]

On the other hand, since \( \frac{\partial}{\partial \theta} \tilde{I}_q^T(\theta_j, u_j) = o_j(1) \), by (9) we have that

\[
\frac{(N - 2)e^{(N-2)\theta_j}}{2} \int_{\mathbb{R}^N} |\nabla u_j|^2 + q\chi \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \sum_{i=1}^{k} \alpha_i R_u(u_j(e^{-\theta_j \cdot}))
\]

\[
+ q\chi \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta_j} R_i(e^{\beta_i \theta_j} u_j)[\beta_i e^{\beta_i \theta_j} u_j]
\]

\[
+ q\chi' \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \frac{(N - 2)e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + Ne^{N\theta_j}\|u_j\|_2^2 R(u_j(e^{-\theta_j \cdot}))}{T^2} \leq q(C_1 + C_2T^\delta) \tag{17}.
\]

Now, by (8), (14), (15), (16) and (17), we obtain

\[
\frac{Ne^{N\theta_j} m(1 - \varepsilon)}{2} \int_{\mathbb{R}^N} u_j^2 \leq (1 - \varepsilon) Ne^{N\theta_j} \int_{\mathbb{R}^N} G_2(u_j)
\]

\[
\leq Ne^{N\theta_j} C_e \int_{\mathbb{R}^N} |u_j|^2 - q\chi \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta_j} R_i(e^{\beta_i \theta_j} u_j)[\beta_i e^{\beta_i \theta_j} u_j]
\]

\[
- q\chi' \left( \frac{\|u_j(e^{-\theta_j \cdot})\|^2}{T^2} \right) \frac{(N - 2)e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + Ne^{N\theta_j}\|u_j\|_2^2 R(u_j(e^{-\theta_j \cdot}))}{T^2} + q_j(1)
\]

\[
\leq C \left( \int_{\mathbb{R}^N} |\nabla u_j|^2 \right)^{2^*/2} + q(C_1 + C_2T^\delta) + q_j(1)
\]

\[
\leq C(C' + q(C_1 + C_2T^\delta))^{2^*/2} + q(C_1 + C_2T^\delta) + o_j(1). \tag{18}
\]

We suppose by contradiction that there exists no subsequence of \( \{u_j\}_j \) which is uniformly bounded by \( T \) in the \( H^1 \)-norm. As a consequence, for a certain \( j_0 \) it should result that

\[
\|u_j\| > T, \quad \forall j \geq j_0. \tag{19}
\]

Without any loss of generality, we are supposing that (19) is true for any \( u_j \). Therefore, by (16) and (18), we conclude that

\[
T^2 < \|u_j\|^2 \leq C_3 + C_4 qT^{2^*/\delta}
\]
which is not true for $T$ large and $q$ small enough: indeed we can find $T_0 > 0$ such that $T_0^q > C_4 + 1$ and $q_0 = q_0(T_0)$ such that $C_4 q T_0^{2q} < 1$, for any $q < q_0$, and we find a contradiction. \hfill \Box

In our arguments, the following variant of the Strauss' compactness result [28] (see also [8, Theorem A.1]) will be a fundamental tool.

**Proposition 2.5.** Let $P$ and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying
\[
\lim_{s \to \infty} \frac{P(s)}{Q(s)} = 0,
\]
\[
\{v_j\}, \quad v \text{ and } w \text{ be measurable functions from } \mathbb{R}^N \text{ to } \mathbb{R}, \text{ with } w \text{ bounded, such that}
\sup_j \int_{\mathbb{R}^N} |Q(v_j(x))w| \, dx < +\infty,
\]
\[
P(v_j(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.
\]
Then $\|(P(v_j) - v)w\|_{L^1(B)} \to 0$, for any bounded Borel set $B$.
Moreover, if we have also
\[
\lim_{s \to 0} \frac{P(s)}{Q(s)} = 0,
\]
\[
\lim_{|x| \to \infty} |v_j(x)| = 0,
\]
then $\|(P(v_j) - v)w\|_{L^1(\mathbb{R}^N)} \to 0$.

In analogy with the well-known compactness result in [9], we state the following result.

**Lemma 2.6.** Let $n \geq 1$, $T_n, q_0 > 0$ as in Proposition 2.4 and $\{\theta_j, u_j\} \subset \mathbb{R} \times H^1_q(\mathbb{R}^N)$ be the sequence given in Lemma 2.3. Then $\{u_j\}$ admits a subsequence which converges in $H^1_q(\mathbb{R}^N)$ to a nontrivial critical point of $I_q$ at level $b_n$.

**Proof.** Since $\{u_j\}$ is bounded, up to a subsequence, we can suppose that there exists $u \in H^1_q(\mathbb{R}^N)$ such that
\[
u_j \to u \text{ weakly in } H^1_q(\mathbb{R}^N),
\]
\[
u_j \to u \text{ in } L^p(\mathbb{R}^N), \quad 2 < p < 2^*,
\]
\[
u_j \to u \text{ a.e. in } \mathbb{R}^N.
\]
(20)

By weak lower semicontinuity we have
\[
\int_{\mathbb{R}^N} |\nabla \nu_j|^2 \leq \liminf_j \int_{\mathbb{R}^N} |\nabla \nu_j|^2.
\]
(21)

Since $\|\nu_j\| \leq T_n$ we have
\[
\tilde{I}_q'\theta_j, \nu_j\nu_j = (\tilde{I}_q')\theta_j, \nu_j\nu_j\nu_j = e^{(N-2)\theta_j} \int_{\mathbb{R}^N} \nabla \nu_j \cdot \nabla v + \sum_{l=1}^k e^{(\alpha_l+\beta_l)\theta_j} R_l(\nu_j) \nu_j\nu_j\nu_j + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(\nu_j) v - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(\nu_j) v
\]
for every $v \in H^1_q(\mathbb{R}^N)$.

Then, by (iii) of Lemma 2.3
\[
\tilde{I}_q'\theta_j, \nu_j\nu_j = e^{(N-2)\theta_j} \int_{\mathbb{R}^N} \nabla \nu_j \cdot (\nabla \nu_j - \nabla \nu_j) + \sum_{l=1}^k e^{(\alpha_l+\beta_l)\theta_j} R_l(\nu_j) \nu_j\nu_j\nu_j + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(\nu_j)(u - \nu_j) - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(\nu_j)(u - \nu_j) = o_j(1).
\]
(22)
If we apply Proposition 2.5 for \( P(s) = g_1(s), i = 1, 2, \) \( Q(s) = |s|^2 - 1, (v_j)_j = (u_j)_j, v = g_i(u), i = 1, 2 \) and \( w \) a generic \( C^0_0(\mathbb{R}^N) \)-function, by (g3)', (5) and (20) we deduce that
\[
\int_{\mathbb{R}^N} g_i(u_j) w \to \int_{\mathbb{R}^N} g_i(u) w \quad i = 1, 2,
\]
and so
\[
\int_{\mathbb{R}^N} g_i(u_j) w \to \int_{\mathbb{R}^N} g_i(u) w \quad i = 1, 2.
\]
Moreover, applying Proposition 2.5 for \( P(s) = g_1(s), Q(s) = s^2 + |s|^2 - 1, (v_j)_j = (u_j)_j, v = g_1(u), \) and \( w = 1, \) by (g3)', (5) and (20), we deduce that
\[
\int_{\mathbb{R}^N} g_1(u_j) w \to \int_{\mathbb{R}^N} g_1(u) w.
\]
Moreover, by (20) and Fatou’s lemma
\[
\int_{\mathbb{R}^N} g_2(u) w \leq \liminf_j \int_{\mathbb{R}^N} g_2(u_j) w.
\]
By (22), (23), (24) (25) and (R3), we have
\[
\limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 = \limsup_j e^{(N-2)\theta_j} \int_{\mathbb{R}^N} |\nabla u_j|^2
\]
\[
= \limsup_j \left[ e^{(N-2)\theta_j} \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla u + q \sum_{i=1}^{k} e^{(\alpha_i + \beta_i)\theta_j} R_i' (e^{\beta_i \theta_j} u_j) (u - u_j) \right]
\]
\[
\quad \quad + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j) (u - u_j) - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j) (u - u_j) \right]
\]
\[
\leq \int_{\mathbb{R}^N} |\nabla u|^2.
\]
By (21) and (26), we get
\[
\lim_j \int_{\mathbb{R}^N} |\nabla u_j|^2 = \int_{\mathbb{R}^N} |\nabla u|^2,
\]
hence, by (22),
\[
\lim_j \int_{\mathbb{R}^N} g_2(u_j) w = \int_{\mathbb{R}^N} g_2(u) w.
\]
Since \( g_2(s) = ms^2 + h(s) \), with \( h \) a positive and continuous function, by Fatou’s Lemma we have
\[
\int_{\mathbb{R}^N} h(u) \leq \liminf_j \int_{\mathbb{R}^N} h(u_j),
\]
\[
\int_{\mathbb{R}^N} u^2 \leq \liminf_j \int_{\mathbb{R}^N} u_j^2.
\]
These last two inequalities and (28) imply that, up to a subsequence,
\[
\lim_j \int_{\mathbb{R}^N} u_j^2 = \int_{\mathbb{R}^N} u^2,
\]
which, together with (27), shows that \( u_j \to u \) strongly in \( H_0^1(\mathbb{R}^N) \). Therefore, since \( b_n > 0 \), \( u \) is a non-trivial critical point of \( I_q \) at level \( b_n \).

**Proof of Theorem 1.1**  Let \( h \geq 1 \). Since \( b_n \to +\infty \), up to a subsequence, we can consider \( b_1 < b_2 < \cdots < b_n \). By Lemma 2.6 we conclude, defining \( q(h) = q_{b_n} > 0 \). 

\( \square \)
3 Some applications

3.1 The nonlinear Schrödinger-Maxwell system

Let us consider the Schrödinger-Maxwell system:
\[
\begin{aligned}
-\Delta u + q\phi u &= g(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= qu^2 \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]
where \(q > 0\) and \(g\) satisfies (g1)-(g4). Arguing as in [5, 8], without loss of generality, we can suppose that \(g\) satisfies (g3). The solutions \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) of (SM) are the critical points of the action functional \(E_q: H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}\), defined as
\[
E_q(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{q}{2} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} G(u).
\]

The action functional \(E_q\) exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [7], by which we are led to study a one variable functional that does not present such a strongly indefinite nature. Indeed, for every \(u \in L^{\infty}(\mathbb{R}^3)\), there exists a unique \(\phi_u \in D^{1,2}(\mathbb{R}^3)\) solution of
\[
-\Delta \phi = qu^2, \quad \text{in } \mathbb{R}^3.
\]
Moreover it can be proved that \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a solution of (SM) (critical point of functional \(E_q\)) if and only if \(u \in H^1(\mathbb{R}^3)\) is a critical point of the functional \(I_q: H^1(\mathbb{R}^3) \to \mathbb{R}\) defined as
\[
I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} G(u),
\]
and \(\phi = \phi_u\).

According to our notations, in this case \(R(u) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2\). In order to check that \(R\) satisfies (R1)-(R5), we need some preliminary results on \(\phi_u\) (see for example [12]).

**Lemma 3.1.** The map \(u \in L^{\infty}(\mathbb{R}^3) \mapsto \phi_u \in D^{1,2}(\mathbb{R}^3)\) is \(C^1\). Moreover, for every \(u \in H^1(\mathbb{R}^3)\), we have
\[
i) \|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} = q \int_{\mathbb{R}^3} \phi_u u^2;
\]
\[
ii) \phi_u \geq 0;
\]
\[
iii) \phi_{-u} = \phi_u;
\]
\[
iv) \text{for any } t > 0: \phi_{u_t}(x) = t^2 \phi_u(x/t), \text{where } u_t(x) = u(x/t);
\]
\[
v) \text{there exist } C, C' > 0 \text{ independent of } u \in H^1(\mathbb{R}^3) \text{ such that}
\]
\[
\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \leq C q \|u\|_{L^4}^2,
\]
and
\[
\int_{\mathbb{R}^3} \phi_u u^2 \leq C' q \|u\|_{L^4}^4;
\]
\[
vi) \text{if } u \text{ is a radial function then } \phi_u \text{ is radial, too.}
\]

Now we use the previous lemma to deduce assumptions (R1)-(R5). Hypothesis (R1) is obvious. Since
\[
R'(u)[u] = \int_{\mathbb{R}^3} \phi_u u^2,
\]
(see for example [7]), then (R2) is again a consequence of (29). We pass to check (R3). Suppose that
\[
u_j \rightharpoonup u \text{ weakly in } H^1_0(\mathbb{R}^3).
\]
By compact embedding we deduce that

\[ u_j \to u \text{ in } L^{\frac{12}{5}}(\mathbb{R}^3) \]

and then, by continuity,

\[ \phi_{u_j} \to \phi_u \text{ in } D^{1,2}(\mathbb{R}^3). \]

Since \( R'(u)[v] = \int_{\mathbb{R}^N} \phi_u uv \), we have that

\[
\limsup_j R'(u_j)[u - u_j] = \limsup_j \int_{\mathbb{R}^3} \phi_{u_j}(u - u_j) \leq C \limsup_j \|\phi_{u_j}\|_{D^{1,2}(\mathbb{R}^3)} \|u - u_j\|_{L^2} = 0.
\]

Now in order to verify (R4), we consider \( u \in H^1(\mathbb{R}^3), u \neq 0 \) and the rescaled function \( u_t \). We compute

\[
R(u_t) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_t} u_t^2 = \frac{t^5}{4} \int_{\mathbb{R}^3} \phi_u u^2 = t^5 R(u)
\]

so (R4) holds true for \( \alpha = 5 \).

Finally (R5) follows from \( \text{vi) of Lemma 3.1.} \)

### 3.2 The elliptic Kirchhoff equation

In this subsection we treat the semilinear perturbation of the Kirchhoff equation

\[
- \left( p + q \int_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u = g(u) \quad \text{in } \mathbb{R}^N, \tag{K}
\]

where \( p > 0 \) and \( g \) satisfies (g1)-(g4). Arguing as in [4, 8], without loss of generality, we can suppose that \( g \) satisfies (g3'). We find the solution to (K) as the critical points of the functional

\[
I_g(u) = \frac{1}{2} \left( p + \frac{q}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \right) \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).
\]

It is easy to see that \( I_g \) is of the type (2), where \( R(u) = \frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 \).

Assumptions (R1)-(R2) are trivially satisfied as we can see by straight computations.

As to (R3), suppose that \( u_j \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \). By weak lower semicontinuity, we know that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \leq \liminf_j \int_{\mathbb{R}^N} |\nabla u_j|^2,
\]

and then

\[
\limsup_j R'(u_j)[u - u_j] = \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla (u - u_j) \leq \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \limsup_j \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla (u - u_j) \leq \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \left( \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla (u - \liminf_j \int_{\mathbb{R}^N} |\nabla u_j|^2) \right) \leq 0.
\]

By a simple computation, we have that

\[
R(u_t) = \frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u_t|^2 \right)^2 = \frac{t^{2(2N-2)}}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 = t^{2(2N-2)} R(u),
\]

and then also (R4) is satisfied.

Finally by a simple change of variable it can be proved that for any \( g \in O(N) \) we have

\[
R(u(gx)) = \frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u(gx)|^2 \right)^2 = \frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u(x)|^2 \right)^2 = R(u).
\]
Remark 3.2. Let us observe that we can easily apply Theorem 1.1 also to a sort of linear combination of the Schrödinger-Maxwell equation with the Kirchhoff one, namely we can find multiple critical points of the functional

\[ I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \left[ \lambda_1 \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * u^2 \right) u^2 + \lambda_2 \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 \right] - \int_{\mathbb{R}^3} G(u), \]

with \( \lambda_1, \lambda_2 \in \mathbb{R}_+ \) and \( q \) sufficiently small.

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