GROUND STATES OF TWO-COMPONENT ATTRACTIVE
BOSE-EINSTEIN CONDENSATES II: SEMI-TRIVIAL LIMIT
BEHAVIOR

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Abstract. As a continuation of [14], we study new pattern formations of
ground states \((u_1, u_2)\) for two-component Bose-Einstein condensates (BEC)
with homogeneous trapping potentials in \(\mathbb{R}^2\), where the intraspecies interaction
\((-a, -b)\) and the interspecies interaction \(-\beta\) are both attractive, i.e., \(a, b\) and
\(\beta\) are all positive. If \(0 < b < a^* := \|w\|^2_2\) and \(0 < \beta < a^*\) are fixed, where
\(w\) is the unique positive solution of \(\Delta w - w + w^3 = 0\) in \(\mathbb{R}^2\), the semi-trivial
behavior of \((u_1, u_2)\) as \(a \rightarrow a^*\) is proved in the sense that \(u_1\) concentrates
at a unique point and while \(u_2 \equiv 0\) in \(\mathbb{R}^2\). However, if \(0 < b \leq a^*\) and
\(a^* \leq \beta < \beta^* = a^* + \sqrt{(a^* - a)(a^* - b)}\), the refined spike profile and the
uniqueness of \((u_1, u_2)\) as \(a \rightarrow a^*\) are analyzed, where \((u_1, u_2)\) must be unique,
\(u_1\) concentrates at a unique point, and meanwhile \(u_2\) can either blow up or
vanish, depending on how \(\beta\) approaches \(a^*\).

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1. Introduction

In this paper, we study the following coupled nonlinear Gross-Pitaevskii system

\[
\begin{aligned}
&-\Delta u_1 + V_1(x)u_1 = \mu u_1 + au_1^3 + \beta u_1^2 u_2 	ext{ in } \mathbb{R}^2, \\
&-\Delta u_2 + V_2(x)u_2 = \mu u_2 + bu_2^3 + \beta u_1^2 u_2 	ext{ in } \mathbb{R}^2,
\end{aligned}
\]

where \((u_1, u_2) \in \mathcal{X} = \mathcal{H}_1(\mathbb{R}^2) \times \mathcal{H}_2(\mathbb{R}^2)\) and the space

\[
\mathcal{H}_i(\mathbb{R}^2) = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V_i(x)|u(x)|^2 \, dx < \infty \right\}
\]

is equipped with the norm \(\|u\|_{\mathcal{H}_i} = \left( \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + V_i(x)|u(x)|^2 \right] \, dx \right)^{\frac{1}{2}}\) for \(i = 1, 2\).

The system (1.1) is used (see [11, 6, 8, 9, 19, 24, 25, 26, 32, 34]) to describe two-component Bose-Einstein condensates (BEC) with trapping potentials \(V_1(x)\) and \(V_2(x)\), where \(\mu \in \mathbb{R}\) is a chemical potential. From the physical point of view, we assume that the trapping potentials \(V_i(x) \geq 0\) (\(i = 1, 2\)) satisfy

\[
\lim_{|x| \to \infty} V_i(x) = \infty, \quad \inf_{x \in \mathbb{R}^2} V_i(x) = 0 \quad \text{and} \quad \inf_{x \in \mathbb{R}^2} \left( V_1(x) + V_2(x) \right) \text{ are attained.}
\]

In the system (1.1), \(a > 0\) and \(b > 0\) (resp. \(< 0\)) represent that the intraspecies interaction of the atoms inside each component is attractive (resp. repulsive), and \(\beta > 0\) (resp. \(< 0\)) denotes that the interspecies interaction between two components is attractive (resp. repulsive).

As a continuation of [14], in this paper we study ground states of (1.1) for the case where the intraspecies interaction and interspecies interaction are both attractive, i.e. \(a, b, \beta > 0\). As illustrated in [14] Proposition A.1], ground states of (1.1) in this case can be described equivalently by the minimizers of the following \(L^2\)-critical constraint variational problem

\[
\begin{aligned}
\text{e}(a, b, \beta) := \inf_{\{(u_1, u_2) \in \mathcal{X} : \int_{\mathbb{R}^2} (u_1^2 + u_2^2) \, dx = 1\}} E_{a,b,\beta}(u_1, u_2), \quad a > 0, \quad b > 0, \quad \beta > 0,
\end{aligned}
\]

where the Gross-Pitaevskii (GP) energy functional \(E_{a,b,\beta}(u_1, u_2)\) is given by

\[
E_{a,b,\beta}(u_1, u_2) = \int_{\mathbb{R}^2} \left( |\nabla u_1|^2 + |\nabla u_2|^2 \right) \, dx + \int_{\mathbb{R}^2} \left( V_1(x)u_1^2 + V_2(x)u_2^2 \right) \, dx
\]

\[
- \int_{\mathbb{R}^2} \left[ \frac{a}{2}|u_1|^4 + \frac{b}{2}|u_2|^4 + \beta |u_1|^2|u_2|^2 \right] \, dx, \quad (u_1, u_2) \in \mathcal{X}.
\]

To discuss equivalently ground states of (1.1), throughout the whole paper we shall therefore focus on investigating (1.3), instead of (1.1). Since the GP energy functional \(E_{a,b,\beta}(u_1, u_2)\) is even in \((u_1, u_2)\), any minimizer \((u_1, u_2)\) of \(\text{e}(a, b, \beta)\) must be either nonnegative or nonpositive. Without loss of generality, in this paper we therefore restrict to study nonnegative minimizers of \(\text{e}(a, b, \beta)\), which are called ground states of (1.1).

Besides the assumption (1.2), for the physical correlation we shall consider the trapping potentials \(V_1(x)\) and \(V_2(x)\) in the class of homogeneous functions, for which we define

**Definition 1.1.** \(h(x) \geq 0\) in \(\mathbb{R}^2\) is homogeneous of degree \(p \in \mathbb{R}^+\) (about the origin), if \(h(x)\) satisfies

\[
h(tx) = t^p h(x) \quad \text{in } \mathbb{R}^2 \quad \text{for any } t > 0.
\]
The above definition implies that the homogeneous function $h(x)$ of degree $p \in \mathbb{R}^+$ satisfies
\begin{equation}
0 \leq h(x) \leq C|x|^p \text{ in } \mathbb{R}^2,
\end{equation}
where $C > 0$ denotes the maximum of $h(x)$ on $\partial B_1(0)$. Note that $\nabla h(x) = 0$ if and only if $x = 0$ for the case where $\lim_{|x| \to \infty} h(x) = +\infty$. We also use $w = w(|x|)$ to denote (cf. \cite{2 11 21 22 31 33}) the unique (up to translations) positive radially symmetric solution of the following nonlinear scalar field equation
\begin{equation}
\Delta w - w + w^3 = 0, \quad w \in H^1(\mathbb{R}^2). \tag{1.7}
\end{equation}
We remark that $w$ satisfies (cf. \cite{15}) the following identities
\begin{equation}
\|w\|_2^2 = \|\nabla w\|_2^2 = \frac{1}{2}\|w\|_4^4,
\end{equation}
and it follows from \cite{10} Proposition 4.1 that $w(x)$ also satisfies
\begin{equation}
w(x), \ |\nabla w(x)| = O(|x|^{-\frac{1}{2}} e^{-|x|^\frac{1}{2}}) \text{ as } |x| \to \infty. \tag{1.9}
\end{equation}
Recall from \cite{14} that the analysis of $c(a, b, \beta)$ depends strongly on the following Gagliardo-Nirenberg type inequality
\begin{equation}
\int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2)^2 \, dx \leq 2 \|u\|_2^2 \int_{\mathbb{R}^2} (|\nabla u_1|^2 + |\nabla u_2|^2) \, dx \int_{\mathbb{R}^2} (u_1^2 + u_2^2) \, dx,
\end{equation}
where $(u_1, u_2) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. It is proved in \cite{14} Lemma A.2 that $\frac{2}{\|w\|_2^2}$ is the best constant of $\cite{11 10}$, where the equality is attained at $(w \sin \theta, w \cos \theta)$ for any $\theta \in [0, 2\pi)$.

When $V_i(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p_i \geq 2$ and satisfies $\cite{12}$ for $i = 1$ and 2, it then follows immediately from \cite{14} the following existence and nonexistence.

**Theorem A (Theorems 1.1 and 1.2 in \cite{14})** Suppose $V_i(x) \geq 0$ satisfies $\cite{12}$ and there exists at least one common point $x_0 \in \mathbb{R}^2$ such that $V_i(x_0) = \inf_{x \in \mathbb{R}^2} V_i(x) = 0$, where $i = 1, 2$. Set
\begin{equation}
\beta^* = \beta^*(a, b) := a^* + \sqrt{(a^* - a)(a^* - b)}, \quad \text{where} \quad 0 < a, b < a^* := \|w\|_4^2.
\end{equation}
Then $e(a, b, \beta)$ admits minimizers if and only if $0 < a < a^*$, $0 < b < a^*$ and $0 < \beta < \beta^*$.

The above Theorem A shows that $e(a, b, \beta)$ admits minimizers if and only if the point $(a, b, \beta)$ lies within the cuboid defined by Figure 1(a) below. Following \cite{14} Proposition A.1 on the equivalence between ground states of $\cite{11}$ and minimizers of $e(a, b, \beta)$, one can further obtain that for any given $(a, b, \beta)$, a minimizer of $e(a, b, \beta)$ is a ground state of $\cite{11}$ for some $\mu \in \mathbb{R}$; conversely, a ground state of $\cite{11}$ for some $\mu \in \mathbb{R}$ is a minimizer of $e(a, b, \beta)$.

By employing the energy method and blow up analysis, the uniqueness and the refined blow up behavior of nonnegative minimizers $(u_1, u_2)$ for $e(a, b, \beta)$ are investigated in \cite{14} under different types of trapping potentials, where we consider $0 < a < a^*$, $0 < b < a^*$ and $\beta = \beta^* := a^* + \sqrt{(a^* - a)(a^* - b)}$. In such a limit case, it turns out in \cite{14} that $(u_1, u_2)$ must be unique and blows up at a unique point. This further implies the strict positivity of $(u_1, u_2)$ in such a limit case.
The main purpose of this paper is to investigate new pattern formations of nonnegative minimizers \((u_1, u_2)\) for \(e(a, b, \beta)\), where \(0 < b < a^*, \beta \in (0, \beta^*) = (0, a^* + \sqrt{(a^* - a)(a^* - b)})\) and \(a \not\sim a^*\). Different from those studied in [14], we shall analyze that \((u_1, u_2)\) may admit the semi-trivial limit behavior for this case, depending on how \(\beta\) approaches to \(a^*,\) in the sense that \(u_1 > 0\) and \(u_2 \equiv 0\) in \(\mathbb{R}^2.\)

1.1. Main results. In this subsection, we shall introduce the main results of this paper. Stimulated by [12] Theorem 1.1, we define

\[
H_i(y) := \int_{\mathbb{R}^2} V_i(x + y)w^2(x)\,dx > 0, \quad \text{where} \quad i = 1, 2.
\]

We remark that our analysis also makes full use of the following classical Gagliardo-Nirenberg type inequality

\[
\frac{\|w\|_2^2}{2} = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \frac{\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx \int_{\mathbb{R}^2} |u(x)|^2\,dx}{\int_{\mathbb{R}^2} |u(x)|^4\,dx},
\]

where the equality is attained at \(w\) (cf. [38]). Our first result is concerned with the following interesting limit behavior of nonnegative minimizers.

**Theorem 1.1.** Suppose \(0 \leq V_i(x) \in C^2(\mathbb{R}^2)\) is homogeneous of degree \(p_i\) with \(2 \leq p_1 \leq p_2\), where \(V_i(x)\) satisfies (1.12) and

\[
y_0 \text{ is the unique and non-degenerate critical point of } H_i(y).
\]

Let \((u_{1k}, u_{2k})\) be a nonnegative minimizer of \(e(a_k, b, \beta_k)\), where \(0 < b < a^* := \|w\|_2^2, a_k \not\sim a^*\) as \(k \to \infty\) and

\[
a^* < \beta_k < \beta_k^* = a^* + \sqrt{(a^* - a_k)(a^* - b)} \quad \text{and} \quad a^* - a_k = o(\beta_k - a^*)
\]

as \(k \to \infty.\) Then there exists a subsequence, still denoted by \(\{a_k\}\), of \(\{a_k\}\) such that

\[
\begin{align*}
\sqrt{a^*}\varepsilon_{k}u_{1k}(\varepsilon_{k}x + x_{1k}) & \to w(x) \quad \text{and} \quad \sqrt{a^* - b}_{\beta_k - a^*}\varepsilon_{k}u_{2k}(\varepsilon_{k}x + x_{2k}) \to w(x)
\end{align*}
\]

uniformly in \(\mathbb{R}^2\) as \(k \to \infty,\) where \(\varepsilon_k > 0\) is given by

\[
\varepsilon_k := \frac{1}{\lambda} \left[ (a^* - a_k)(a^* - b) - (\beta_k - a^*)^2 \right]^\frac{1}{2}, \quad \lambda = \left[ \frac{p_1(a^* - b)}{2} H_1(y_0) \right]^\frac{1}{2}
\]

and \(x_{ik}\) is the unique maximum point of \(u_{ik}\) satisfying

\[
\begin{align*}
\lim_{k \to \infty} \frac{x_{ik}}{\varepsilon_k} & = y_0, \quad i = 1, 2.
\end{align*}
\]

We remark that the similar estimate of (1.18) appeared earlier in [12], where a singular perturbation problem was studied. Even though Theorem 1.1 is proved mainly by the variational methods and blow up analysis as employed in [14] [15] [27] [35] [36], there are some new difficulties appearing in its proof. Firstly, since the blow up rate (1.17) of Theorem 1.1 is different from those in [15] [14] [18] [29] [30], one needs to seek for a different type of test functions so that the optimal upper estimate of \(e(a_k, b, \beta_k)\) can be derived. Secondly, since the existing argument only gives that \(\varepsilon_ku_{2k}(\varepsilon_kx + x_{1k}) \to 0\) uniformly in \(\mathbb{R}^2\) as \(k \to \infty,\) one needs to investigate an approach of addressing that \(u_{2k} \neq 0\) for sufficiently large \(k > 0.\) As shown in Lemma 3.3, we shall achieve this purpose by analyzing the more refined energy estimates of \(e(a_k, b, \beta_k),\) for which we make full
use of the refined spike profiles proved in [13, Theorem 1.2]. Once \( u_{2k} \neq 0 \) holds for sufficiently large \( k > 0 \), we define

\[
\bar{u}_{1k}(x) = \sqrt{a^*} \varepsilon_k u_{1k}(\varepsilon_k x + x_{1k}) \quad \text{and} \quad u_{2k}(\varepsilon_k x + x_{1k}) = C_\infty \sigma_k \bar{u}_{2k}(x),
\]

where \( \sigma_k = \|u_{2k}\|_\infty > 0 \) and \( C_\infty = \frac{1}{\|w\|_\infty} > 0 \),

and \( x_{1k} \) is the unique maximum point of \( u_{1k} \). To complete the proof of Theorem 1.1, the rest key point is thus to analyze the limit behavior of \( \bar{u}_{2k} \) and \( \sigma_k \) as \( k \to \infty \), for which we shall carry out a very delicate analysis of the PDE system associated to \( (\bar{u}_{1k}, \bar{u}_{2k}) \). We also remark that if \( \beta_k \) is close enough to \( a^* \), the limit behavior (1.16) still holds without the non-degeneracy assumption of (1.14), see Theorem 3.6 for more details.

Under the assumptions of Theorem 1.1 one can note from (1.19) that the non-negative minimizers of \( e(a_k, b, \beta_k) \) exhibit interesting new pattern formations where \( u_{1k} \) blows up at a unique point and however \( u_{2k} \) can either blow up or vanish, depending on how \( \beta_k \) approaches to \( a^* \). More precisely, for given \( (a_k, b) \), if \( \beta_k \) goes closer to \( \beta^*_k \), then \( u_{2k} \) prefers to blow up at a unique point; conversely, if \( \beta_k \) goes far away from \( \beta^*_k \), then \( u_{2k} \) tends to decrease its height. Especially, if \( \beta_k \leq a^* \) we then have the following semi-trivial limit behavior of nonnegative minimizers, in the sense that \( u_{1k} \) blows up at a unique point and however \( u_{2k} \equiv 0 \) for sufficiently large \( k > 0 \). We also comment that the authors in [3, 4] analyzed recently three-component Schrödinger systems in which some similar semi-trivial limits were found.

**Theorem 1.2.** Suppose \( 0 \leq V_i(x) \in C^2(\mathbb{R}^2) \) is homogeneous of degree \( p_i \geq 2 \) and satisfies (1.2) for \( i = 1 \) and 2. Assume also that

\[
y_0 \text{ is the unique critical point of } H_1(y).
\]
Let \((u_{1k}, u_{2k})\) be a nonnegative minimizer of \(e(a_k, b, \beta_k)\), where \(0 < b < a^*\), \(a_k \nearrow a^*\) as \(k \to \infty\) and \(0 < \beta_k < a^*\) satisfies 

1. either \(\beta_k \to \beta_* \in (0, a^*)\) as \(k \to \infty\), or 
2. \(\beta_k \nearrow a^*\) and \(a^* - a_k = o(a^* - \beta_k)\) as \(k \to \infty\).

Then, up to a subsequence if necessary, we have

\[
\begin{aligned}
&\lim_{k \to \infty} \sqrt{a^*} \varepsilon_k u_{1k}(\varepsilon_k x + x_{1k}) = w(x) \text{ uniformly in } \mathbb{R}^2, \\
&u_{2k}(x) \equiv 0 \text{ in } \mathbb{R}^2, \text{ when } k > 0 \text{ is large enough,}
\end{aligned}
\]

where

\[
\varepsilon_k := \frac{1}{\lambda_1} (a^* - a_k)^{\frac{1}{p_1 - 2}} > 0, \quad \lambda_1 := \left[ \frac{p_1}{2} H_1(y_0) \right]^{\frac{1}{p_1 - 2}},
\]

and the point \(x_{1k}\) is the unique maximum point of \(u_{1k}\) satisfying

\[
\lim_{k \to \infty} \frac{x_{1k}}{\varepsilon_k} = y_0.
\]

The challenging point of proving Theorem 1.2 is to prove that \(u_{2k} \equiv 0\) for sufficiently large \(k > 0\). Roughly speaking, by contradiction if \(u_{2k} \neq 0\) for the case where \(0 < \beta_k < a^*\) satisfies \(\beta_k \to \beta_* \in (0, a^*)\) as \(k \to \infty\), a suitable transform of \(u_{2k}\) then approaches to a nontrivial nonnegative solution of \(\Delta u - u + \frac{b}{a^*} w^2 u = 0\) in \(\mathbb{R}^2\), which is however a contradiction in view of [37, Lemma 4.1], see Theorem 2.1 for details. However, if \(u_{2k} \neq 0\) for the case where \(0 < \beta_k < a^*\) satisfies \(\beta_k \nearrow a^*\) and \(a^* - a_k = o(a^* - \beta_k)\) as \(k \to \infty\), we shall consider \((u_{1k}, u_{2k})\) as a transform of \(u_{2k}\), from which the argument of proving Theorem 1.1 finally leads to a contradiction. As illustrated by Figure 1(b), we also mention that for any given \(0 < b < a^*\), if \((a_k, \beta_k)\) approaches to \((a^*, a^*)\) within Region I (resp. Region III), then the limit behavior of \((u_{1k}, u_{2k})\) can be described by Theorem 1.1 (resp. Theorem 1.2). However, for any \(0 < b < a^*\), if \((a_k, \beta_k)\) approaches to \((a^*, a^*)\) within Region II, we expect that \(u_{2k}\) can either blow up or vanish, depending on \(V_i(x)\) and how \(\beta_k\) approaches to \(a^*\).

Under the non-degeneracy assumption of (1.14), we finally address the following uniqueness of nonnegative minimizers for \(e(a, b, \beta)\).

**Theorem 1.3.** Suppose \(0 \leq V_i(x) \in C^2(\mathbb{R}^2)\) is homogeneous of degree \(p_i\) and satisfies (1.2) and (1.14), where \(2 \leq p_1 \leq p_2\) and \(H_2(y_0) \neq H_1(y_0)\) for the case \(p_1 = p_2\). Then there exists a unique nonnegative minimizer for \(e(a, b, \beta)\), where \((a, b, \beta)\) satisfies

\[
0 < b < a^*, \quad a^* \leq \beta < \beta^* := a^* + \sqrt{(a^* - a)(a^* - b)}, \quad a^* - a = o(\beta - a^*)
\]

as \(a \nearrow a^*\).

Even though the uniqueness of nonnegative minimizers for \(e(a, b, \beta)\) is also tackled in [14] Theorem 1.5, there are some essential differences in the proof of Theorem 1.3. To prove Theorem 1.3 by contradiction we suppose \((u_{1, k}, v_{1, k})\) and \((u_{2, k}, v_{2, k})\) to be two different nonnegative minimizers of \(e(a_k, b, \beta_k)\). The proof of Theorem 1.1 then motivates us to define

\[
\bar{u}_{i, k}(x) = \sqrt{a^*} \varepsilon_k u_{i, k}(\varepsilon_k x + x_{2, k}) \quad \text{and} \quad v_{i, k}(\varepsilon_k x + x_{2, k}) = C_{\infty} \sigma_k \bar{v}_{i, k}(x),
\]

where \(i = 1, 2, \quad \sigma_k = \|v_{2, k}\|_\infty > 0\) and \(C_{\infty} = \frac{1}{\|u\|_\infty} > 0\).
Here $\varepsilon_k > 0$ is given by Proposition 3.2 and $x_{2,k}$ is the unique maximum point of $u_{2,k}$. Different from [14, Theorem 1.5], we then need to consider the following difference function

\begin{align}
\xi_{1,k}(x) &= \frac{\bar{u}_{2,k}(x) - \bar{u}_{1,k}(x)}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \|\bar{v}_{2,k} - \bar{v}_{1,k}\|_{L^2(\mathbb{R}^2)}}, \\
\xi_{2,k}(x) &= \frac{\bar{v}_{2,k}(x) - \bar{v}_{1,k}(x)}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \|\bar{v}_{2,k} - \bar{v}_{1,k}\|_{L^2(\mathbb{R}^2)}}.
\end{align}

(1.26)

By using more delicate analysis, the limit behavior $(\xi_{10}, \xi_{20})$ of $(\xi_{1,k}, \xi_{2,k})$ as $k \to \infty$ further turns out to satisfy the following non-degenerate system: as proved in (4.21), the solution set of

\[
\begin{cases}
L_1 \xi_{10} := -\Delta \xi_{10} + [1 - 3w^2] \xi_{10} = \frac{2}{a^*}(\int_{\mathbb{R}^2} u_0^3 \xi_{10})w \text{ in } \mathbb{R}^2, \\
L_2(\xi_{10}) \xi_{20} := -\Delta \xi_{20} + (1 - w^2) \xi_{20} - 2w^2 \xi_{10} = \frac{2}{a^*}(\int_{\mathbb{R}^2} u_0^3 \xi_{10})w \text{ in } \mathbb{R}^2,
\end{cases}
\]

(1.27)

satisfies

\[
\begin{pmatrix}
\xi_{10} \\
\xi_{20}
\end{pmatrix} = b_0 \begin{pmatrix}
0 \\
w
\end{pmatrix} + \sum_{j=1}^2 b_j \begin{pmatrix}
\frac{\partial w}{\partial x_j} \\
\frac{\partial w}{\partial x_j}
\end{pmatrix} + c_0 \begin{pmatrix}
w + x \cdot \nabla w \\
w + x \cdot \nabla w
\end{pmatrix}
\]

(1.28)

for some constants $c_0$ and $b_j$ with $j = 0, 1, 2$, which is more involved than those in [13, 14]. By deriving local Pohozaev identities (cf. [5, 7, 12, 13]), we shall first prove that $c_0 = 0$, based on which we shall derive that $b_1 = b_2 = 0$ in (1.28). Following these, we shall prove that $\xi_{1,k}(x) \to \xi_{10}$ uniformly in $\mathbb{R}^2$ as $k \to \infty$. To reach a contradiction by further showing $b_0 = 0$, one then needs to derive a refined expansion of $\xi_{1,k}$ in terms of $\sigma_k$ and $\varepsilon_k$.

When $H_1(y)$ has $N$ non-degenerate critical points, it was proved in [12] that the number of single peak solutions for some scalar equations equals exactly to $N$, where $N \geq 1$. Our results show that the uniqueness of Theorem 1.3 is true for the case where $N = 1$, and it seems more complicated for the general case where $N > 1$.

This paper is organized as follows. The main purpose of Section 2 is to establish Theorem 2.1. In Section 3 we shall first establish Proposition 3.2 based on which we then finish the proof of Theorems 1.1 and 1.2 in Subsection 3.1. Following Proposition 3.2 in Section 4 we shall complete the proof of Theorem 1.3. The proofs of Lemma 4.1 and 4.11 are given in Appendix A.

### 2. Limit Behavior of Nonnegative Minimizers: $0 < \beta < a^*$

In this section, we mainly establish the following Theorem 2.1 on the semi-trivial limit behavior of nonnegative minimizers for $\epsilon(a, b, \beta)$.

**Theorem 2.1.** Suppose $V_i(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p_i \geq 2$ and satisfies (1.2) and (4.20) for $i = 1$ and 2. Let $(u_{1k}, u_{2k})$ be a nonnegative minimizer of $e(a_k, b, \beta_k)$, where $0 < b < a^*$, $a_k \nearrow a^*$ and $0 < \beta_k < a^*$ satisfies $\beta_k \to \beta^* \in (0, a^*)$ as $k \to \infty$. Then, up to a subsequence if necessary, we have

\[
\lim_{k \to \infty} \sqrt{a^*} \varepsilon_k u_{1k}(\varepsilon_k x + x_{1k}) = w(x) \text{ uniformly in } \mathbb{R}^2,
\]

\[
u_{2k}(x) \equiv 0 \text{ in } \mathbb{R}^2, \quad \text{when } k \text{ is large enough},
\]

(2.1)
where
\begin{equation}
(2.2) \quad \varepsilon_k := \frac{1}{\lambda_1} (a^* - a_k)^{\frac{1}{4+2}} > 0, \quad \lambda_1 := \left[ \frac{p}{2} H_1(y_0) \right]^{\frac{1}{p+1}},
\end{equation}
and the point \( x_{1k} \) is the unique maximum point of \( u_{1k} \) satisfying
\begin{equation}
(2.3) \quad \lim_{k \to \infty} \frac{x_{1k}}{\varepsilon_k} = y_0
\end{equation}
for \( y_0 \in \mathbb{R}^2 \) given in \((1.20)\).

In order to prove Theorem 2.1,

let \( (u_{1k}, u_{2k}) \) be a nonnegative minimizer of \( e(a_k, b, \beta_k) \) in view of Theorem A, so that the expression of \( e(a_k, b, \beta_k) \) can be rewritten as
\begin{align*}
e(a_k, b, \beta_k) &= E_{a_k, b, \beta_k}(u_{1k}, u_{2k}) \\
&= \int_{\mathbb{R}^2} |\nabla u_{1k}(x)|^2 + |\nabla u_{2k}(x)|^2 \, dx - \frac{a^*}{2} \int_{\mathbb{R}^2} |u_{1k}(x)|^2 + |u_{2k}(x)|^2 \, dx \\
&+ \int_{\mathbb{R}^2} V_1(x)|u_{1k}(x)|^2 \, dx + \int_{\mathbb{R}^2} V_2(x)|u_{2k}(x)|^2 \, dx \\
&+ \frac{a^* - a_k}{2} \int_{\mathbb{R}^2} |u_{1k}(x)|^4 \, dx + \frac{a^* - b}{2} \int_{\mathbb{R}^2} |u_{2k}(x)|^4 \, dx \\
&+ (a^* - \beta_k) \int_{\mathbb{R}^2} |u_{1k}(x)|^2 |u_{2k}(x)|^2 \, dx.
\end{align*}

Applying [14, Theorem 1.2],

we have \( e(a_k, b, \beta_k) \to 0 \) as \( k \to \infty \) by choosing \( k \) large enough that \( a^* > \beta_k \). It then follows from \((1.10)\) and \((2.4)\) that
\begin{equation}
(2.5) \quad \lim_{k \to \infty} \int_{\mathbb{R}^2} V_1(x)u_{1k}^2 \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^2} V_2(x)u_{2k}^2 \, dx = 0,
\end{equation}
and
\begin{equation}
(2.6) \quad \lim_{k \to \infty} \int_{\mathbb{R}^2} u_{2k}^4 \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^2} u_{1k}^2 u_{2k}^2 \, dx = 0.
\end{equation}

Recall that
\begin{align*}
e(a_k, b, \beta_k) &= \int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx - \frac{a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx + \int_{\mathbb{R}^2} V_1(x)|u_{1k}|^2 \, dx \\
&+ \int_{\mathbb{R}^2} |\nabla u_{2k}|^2 \, dx - \frac{b}{2} \int_{\mathbb{R}^2} |u_{2k}|^4 \, dx + \int_{\mathbb{R}^2} V_2(x)|u_{2k}|^2 \, dx \\
&- \beta_k \int_{\mathbb{R}^2} |u_{1k}|^2 |u_{2k}|^2 \, dx.
\end{align*}

Hence, using the Gagliardo-Nirenberg inequality \((1.13)\),

two equalities that
\begin{align*}
\lim_{k \to \infty} \int_{\mathbb{R}^2} |\nabla u_{2k}|^2 \, dx &= 0 \quad \text{and} \quad \lim_{k \to \infty} \left( 1 - \frac{a_k}{a^*} \|u_{1k}\|_2^4 \right) \int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx = 0.
\end{align*}

On the other hand, the argument of proving [14, Lemma 3.1(1)] gives that
\( \int_{\mathbb{R}^2} (|\nabla u_{1k}|^2 + |\nabla u_{2k}|^2) \, dx \to +\infty \) as \( k \to \infty \). Together with the fact \( \int_{\mathbb{R}^2} (|u_{1k}|^2 + |u_{2k}|^2) \, dx = 1 \),

one can derive from the above two equalities that
\begin{align*}
(2.8) \quad \lim_{k \to \infty} \int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx &= +\infty, \quad \lim_{k \to \infty} \int_{\mathbb{R}^2} |u_{1k}|^2 \, dx = 1.
\end{align*}
Moreover, using (2.5)-(2.9), we deduce from (2.7) that

\[ \lim_{k \to \infty} \int_{\mathbb{R}^2} |\nabla u_{2k}|^2 \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^2} |u_{2k}|^2 \, dx = 0. \]

Moreover, using (2.5)-(2.9), we deduce from (2.7) that

\[ \lim_{k \to \infty} \frac{\int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx}{\int_{\mathbb{R}^2} |u_{1k}|^4 \, dx} = \frac{a^*}{2}. \]

Following above estimates, we now address the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Setting \( \bar{\varepsilon}_k := (\int_{\mathbb{R}^2} |\nabla u_{1k}(x)|^2 \, dx)^{-\frac{1}{2}} > 0 \), it then follows from (2.5) that \( \bar{\varepsilon}_k \to 0 \) as \( k \to \infty \). Denote

\[ \bar{w}_{1k}(x) := \bar{\varepsilon}_k u_{1k}(\bar{\varepsilon}_k x + x_{1k}), \quad i = 1, 2, \]

where \( x_{1k} \) is a global maximum point of \( u_{1k} \). Since \( (u_{1k}, u_{2k}) \) satisfies the system (1.1), \( (\bar{w}_{1k}, \bar{w}_{2k}) \) satisfies

\[ \begin{align*}
-\Delta \bar{w}_{1k} + \varepsilon_k^2 V_1(\bar{\varepsilon}_k x + x_{1k}) \bar{w}_{1k} &= \varepsilon_k^2 \mu_k \bar{w}_{1k} + a_k \bar{w}_{1k}^3 + \beta_k \bar{w}_{2k}^2 \bar{w}_{1k} \quad \text{in} \; \mathbb{R}^2, \\
-\Delta \bar{w}_{2k} + \varepsilon_k^2 V_2(\bar{\varepsilon}_k x + x_{1k}) \bar{w}_{2k} &= \varepsilon_k^2 \mu_k \bar{w}_{2k} + b \bar{w}_{2k}^3 + \beta_k \bar{w}_{1k}^2 \bar{w}_{2k} \quad \text{in} \; \mathbb{R}^2,
\end{align*} \]

where \( \mu_k \in \mathbb{R} \) is a suitable Lagrange multiplier. Note from (2.8) and (2.9) that for any sequence \( \{a_k\} \) with \( a_k \not\to a^* \) as \( k \to \infty \), \( \bar{w}_{1k} \) is bounded uniformly in \( H^1(\mathbb{R}^2) \) and \( \bar{w}_{2k} \to 0 \) in \( H^1(\mathbb{R}^2) \) as \( k \to \infty \). By the argument of proving (4.6) and (4.7) in [14], one can also obtain that \( \bar{w}_{1k} \) and \( \nabla \bar{w}_{1k} \) decay exponentially as \( |x| \to \infty \) for \( i = 1, 2 \). Using the standard elliptic regularity theory, one can further derive from (2.12) that

\[ \bar{w}_{2k}(x) \to 0 \quad \text{in} \; L^\infty(\mathbb{R}^2). \]

Therefore, the system (2.12) must degenerate into a single equation of the form

\[ -\Delta \bar{w}_{1k} + \varepsilon_k^2 V_1(\bar{\varepsilon}_k x + x_{1k}) \bar{w}_{1k} = (\varepsilon_k^2 \mu_k + o(1)) \bar{w}_{1k} + a_k \bar{w}_{1k}^3 \quad \text{in} \; \mathbb{R}^2 \]

as \( k \to \infty \). Following the proof of [13] Theorem 1.2] (see also [14] Theorem 1.3), one can conclude from (2.5) and (2.10) that, passing to a subsequence if necessary, \( \bar{w}_{1k} \) satisfies

\[ \bar{w}_{1k}(x) \to \frac{w(x)}{\|w\|_2} \quad \text{strongly} \; \text{in} \; H^1(\mathbb{R}^2), \]

and \( x_{1k} \) is the unique maximum point of \( u_{1k} \).

In order to determine the convergence rate \( \bar{\varepsilon}_k \to 0 \), motivated by [13]-[18], we next analyze a refined estimate of the energy \( e(a_k, b, \beta_k) \) as \( k \to \infty \). Specifically, here we claim that

\[ \lim_{k \to \infty} \frac{e(a_k, b, \beta_k)}{(a^* - a_k)^{\frac{2}{3}}} = \frac{\lambda^2}{a^*} p_1 + \frac{2}{p_1}. \]

Actually, by taking the following test function

\[ \phi_1(x) = \frac{\tau}{\|w\|_2} w(\tau x - y_0) \quad \text{and} \quad \phi_2(x) = 0, \]
where \( \tau = (\lambda_1(a^* - a_k))^{\frac{1}{p_1 + 2}} > 0 \), \( \lambda_1 > 0 \) is defined in (2.2) and \( y_0 \) is the unique critical point of \( H_1(y) := \int_{\mathbb{R}^2} V_1(x + y) u_k^2(x) \, dx \), the calculations yield the following upper bound

\[
(2.17) \quad e(a_k, b, \beta_k) \leq E_{a_k, b, \beta_k}(\phi_1, \phi_2) = \frac{\lambda_1^2}{a^*} p_1 + 2 \frac{(a^* - a_k)^{\frac{1}{p_1 + 2}}}{\lambda_1} \text{ as } k \to \infty.
\]

On the other hand, let \((w_{1k}, u_{2k})\) be a nonnegative minimizer of \(e(a_k, b, \beta_k)\) as \(k \to \infty\). It follows from (2.17) and (2.10) that

\[
e(a_k, b, \beta_k) \geq \frac{a^* - a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx + \int_{\mathbb{R}^2} V_1(x) |u_{1k}|^2 \, dx
= \frac{a^* - a_k}{a^*} (\varepsilon_k)^{-2} + (\varepsilon_k)^{p_1} \int_{\mathbb{R}^2} V_1 \left( \frac{x_{1k}}{\varepsilon_k} \right) \bar{w}_{2k}^2 \, dx.
\]

By the argument of proving (3.35) in [14], it then yields from above that

\[
\liminf_{k \to \infty} \frac{e(a_k, b, \beta_k)}{(a^* - a_k)^{\frac{1}{p_1 + 2}}} \geq \frac{\lambda_1^2}{a^*} p_1 + 2 \frac{(a^* - a_k)^{\frac{1}{p_1 + 2}}}{\lambda_1},
\]

where the equality holds if and only if

\[
(2.19) \quad \lim_{k \to \infty} \frac{x_{1k}}{\varepsilon_k} = y_0, \text{ where } y_0 \in \mathbb{R}^2 \text{ is defined in (1.20),}
\]

and

\[
(2.20) \quad \lim_{k \to \infty} \varepsilon_k/\varepsilon_k = 1, \text{ where } \varepsilon_k = \frac{1}{\lambda_1} (a^* - a_k)^{\frac{1}{p_1 + 2}} > 0 \text{ is defined in (2.2).}
\]

Therefore, we conclude (2.16) from (2.17) and (2.18).

The above proof of (2.16) implies that the equality of (2.18) holds true. This further implies that both (2.19) and (2.20) are true, and therefore (2.15) follows. Furthermore, we obtain from (2.15) and (2.20) that

\[
\lim_{k \to \infty} \sqrt{a^* \varepsilon_k} u_{1k}(\varepsilon_k x + x_{1k}) = u(x) \text{ strongly in } H^1(\mathbb{R}^2).
\]

Since we have as before that \( u(x) \) and \( w_{1k} \) decay exponentially as \( |x| \to \infty \), the standard elliptic regularity theory yields that the first limit of (2.11) holds uniformly in \( \mathbb{R}^2 \) (see [28, Lemma 4.9] for similar arguments).

The rest is to prove that \( u_{2k}(x) \equiv 0 \) in \( \mathbb{R}^2 \) when \( k > 0 \) is large enough. On the contrary, suppose this is false. Let \( y_k \) be a global maximum point of \( u_{2k} \), and set \( \bar{u}_{2k}(x) := \frac{1}{\delta_k} u_{2k}(\varepsilon_k x + y_k) \), where \( \delta_k := \|u_{2k}\|_\infty \) and \( \varepsilon_k > 0 \) is given in (2.2). Then \( \delta_k > 0 \) and \( \bar{u}_{2k}(x) \) satisfies

\[
- \Delta \bar{u}_{2k} + \varepsilon_k^2 V_2(\varepsilon_k x + y_k) \bar{u}_{2k}
= \mu_k \varepsilon_k^2 u_{2k} + b \varepsilon_k^2 u_{2k}^3 \bar{w}_{1k}^2 \left( x + \frac{y_k - x_{1k}}{\varepsilon_k} \right) \bar{u}_{2k} \text{ in } \mathbb{R}^2,
\]

where \( w_{1k}(x) := \varepsilon_k u_{1k}(\varepsilon_k x + x_{1k}) \) and \( x_{1k} \) is the unique maximum point of \( u_{1k} \).

Note from (2.11), (2.13) and (2.20) that

\[
(2.22) \quad \delta_k \varepsilon_k \to 0 \text{ as } k \to \infty.
\]

It also follows from (2.4) and (2.14) that

\[
(2.23) \quad \varepsilon_k^2 \mu_k = \varepsilon_k^2 e(a_k, b, \beta_k) - o(1) - \frac{a_k}{2} \int_{\mathbb{R}^2} w_{1k}^4 \, dx \to -1 \text{ as } k \to \infty.
\]
Since the origin is a global maximum point of \( \bar{u}_{2k} \) and \( \bar{u}_{2k}(0) = \frac{u_{2k}(y)}{\|u_{2k}\|_{\infty}} = 1 \), we then derive from (2.21) that \( w_{1k}^{2} \left( \frac{y_{k} - x_{1k}}{\varepsilon_{k}} \right) \geq \frac{1}{2\beta^{*}} \). Since \( w_{1k} \) decays exponentially as \( |x| \to \infty \), applying the maximum principle to (2.21) then gives that \( \{\frac{y_{k} - x_{1k}}{\varepsilon_{k}}\} \) is bounded uniformly in \( k \), where (2.23) is also used. Thus, passing to a subsequence if necessary, one can get that

(2.24) \[ \lim_{k \to \infty} \frac{y_{k} - x_{1k}}{\varepsilon_{k}} = y^{0} \text{ for some } y^{0} \in \mathbb{R}^{2}. \]

Furthermore, the standard elliptic regularity implies that \( \|\bar{u}_{2k}\|_{C^{2,\alpha}(\mathbb{R}^{2})} \leq C \) for some \( \alpha \in (0, 1) \), where the constant \( C > 0 \) is independent of \( k \). Then there exist a subsequence of \( \{\bar{u}_{2k}\} \) (still denoted by \( \{\bar{u}_{2k}\} \)) and some \( \bar{u}_{20} \in C^{2}_{loc}(\mathbb{R}^{2}) \) such that \( \bar{u}_{2k} \to \bar{u}_{20} \) in \( C^{2}_{loc}(\mathbb{R}^{2}) \) as \( k \to \infty \). Especially, we have

(2.25) \[ \bar{u}_{20}(y^{0}) = \lim_{k \to \infty} \bar{u}_{2k} \left( \frac{y_{k} - x_{1k}}{\varepsilon_{k}} \right) = 1. \]

On the other hand, one can derive from (2.15) and (2.22)-(2.24) that \( \bar{u}_{20} \) satisfies

(2.26) \[ -\Delta \bar{u}_{20} + \bar{u}_{20} - \frac{\beta^{*}}{a^{*}} w^{2}(x + y^{0}) \bar{u}_{20} = 0 \text{ in } \mathbb{R}^{2}, \]

where \( 0 < \frac{\beta^{*}}{a^{*}} < 1 \) and \( u \) is the unique positive solution of (1.7). However, since it follows from [37, Lemma 4.1] that

\[
\int_{\mathbb{R}^{2}} |\nabla u|^{2} \, dx + \int_{\mathbb{R}^{2}} u^{2} \, dx \geq \int_{\mathbb{R}^{2}} w^{2} u^{2} \, dx \text{ for any } u \in H^{1}(\mathbb{R}^{2}),
\]

we then reduce from (2.26) that

\( \bar{u}_{20} \equiv 0 \text{ in } \mathbb{R}^{2}, \)

which however contradicts to (2.25). Therefore, we conclude that \( u_{2k}(x) \equiv 0 \text{ in } \mathbb{R}^{2} \) when \( k > 0 \) is large enough. This completes the proof of Theorem 2.1. \( \Box \)

3. Limit Behavior of Nonnegative Minimizers: \( a^{*} \leq \beta < \beta^{*} \)

In this section we shall prove Theorem 1.1 for the case where \((a_{k}, b, \beta_{k})\) satisfies (1.15). As a byproduct, we then complete the proof of Theorem 1.2. We begin with the following lemma under the general assumption (1.2).

**Lemma 3.1.** Suppose \( V_{1}(x) \) and \( V_{2}(x) \) satisfy (1.2). Let \((u_{1k}, u_{2k})\) be a nonnegative minimizer of \( e(a_{k}, b, \beta_{k}) \) satisfying

(3.1) \[ 0 < b < a^{*}, \quad a^{*} \leq \beta_{k} < \beta^{*} = a^{*} + \sqrt{(a^{*} - a_{k})(a^{*} - b)}, \]

where \( a_{k} \not< a^{*} \) as \( k \to \infty \). Then we have

(i). \((u_{1k}, u_{2k})\) satisfies

(3.2) \[ \lim_{k \to \infty} \int_{\mathbb{R}^{2}} V_{1}(x) u_{1k}^{2} \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^{2}} V_{2}(x) u_{2k}^{2} \, dx = 0, \]

(3.3) \[ \lim_{k \to \infty} \int_{\mathbb{R}^{2}} |u_{1k}|^{4} \, dx = \infty. \]
(ii). \((u_{1k}, u_{2k})\) also satisfies

\[
\lim_{k \to \infty} \frac{\int_{\mathbb{R}^2} |u_{2k}|^4 \, dx}{|u_{1k}|^4} = 0, \quad \lim_{k \to \infty} \frac{\int_{\mathbb{R}^2} |u_{1k}|^2 |u_{2k}|^2 \, dx}{|u_{1k}|^4} = 0,
\]

(3.4)

\[
\lim_{k \to \infty} \frac{\int_{\mathbb{R}^2} |\nabla u_{2k}|^2 \, dx}{|u_{1k}|^4} = \lim_{k \to \infty} \int_{\mathbb{R}^2} |u_{2k}|^2 \, dx = 0,
\]

and

(3.5)

\[
\lim_{k \to \infty} \frac{\int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx}{|u_{1k}|^4} = \frac{a^*}{2}, \quad \lim_{k \to \infty} \int_{\mathbb{R}^2} |u_{1k}|^2 \, dx = 1.
\]

Proof. (i). We first note that \(e(a_k, b, \beta_k)\) can be rewritten as

\[
e(a_k, b, \beta_k) = \int_{\mathbb{R}^2} \left( |\nabla u_{1k}|^2 + |\nabla u_{2k}|^2 \right) \, dx - \frac{a^*}{2} \int_{\mathbb{R}^2} (|u_{1k}|^2 + |u_{2k}|^2)^2 \, dx
\]

\[
+ \int_{\mathbb{R}^2} V_1(x)|u_{1k}|^2 \, dx + \int_{\mathbb{R}^2} V_2(x)|u_{2k}|^2 \, dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^2} (\sqrt{a^* - a_k}|u_{1k}|^2 - \sqrt{a^* - b}|u_{2k}|^2)^2 \, dx
\]

\[
+ (\beta_k - \beta) \int_{\mathbb{R}^2} |u_{1k}|^2 |u_{2k}|^2 \, dx.
\]

(3.7)

From \([14\text{ Theorem 1.2}]\), one can get that \(e(a_k, b, \beta_k) \to 0\) as \(k \to \infty\), and hence follows directly from (1.10) and (3.7). As for (3.3), we prove it by contradiction. Suppose that \(\int_{\mathbb{R}^2} |u_{1k}|^4 \, dx \leq C\) uniformly for all \(k\). By the following Hölder inequality

\[
\int_{\mathbb{R}^2} |u_{1k}|^2 |u_{2k}|^2 \, dx \leq \left( \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |u_{2k}|^4 \, dx \right)^{\frac{1}{2}},
\]

we then deduce from (3.7) that

\[
\lim_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^2} \left( \sqrt{a^* - a_k} ||u_{1k}||_{L^4(\mathbb{R}^2)} - \sqrt{a^* - b} ||u_{2k}||_{L^4(\mathbb{R}^2)} \right)^2 \right)
\]

\[
\leq \lim_{k \to \infty} \int_{\mathbb{R}^2} \left( \sqrt{a^* - a_k} |u_{1k}|^2 - \sqrt{a^* - b} |u_{2k}|^2 \right)^2 \, dx = 0,
\]

(3.9)

which implies that \(\lim_{k \to \infty} \int_{\mathbb{R}^2} |u_{2k}|^3 \, dx = 0\), and thus \(\lim_{k \to \infty} \int_{\mathbb{R}^2} |u_{1k}|^2 |u_{2k}|^2 \, dx = 0\). Following this, one can derive from (2.7) that

\[
\lim_{k \to \infty} \left( \int_{\mathbb{R}^2} (|\nabla u_{1k}|^2 + |\nabla u_{2k}|^2) \, dx - \frac{a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx \right) = 0,
\]

which then implies that \(\int_{\mathbb{R}^2} (|\nabla u_{1k}|^2 + |\nabla u_{2k}|^2) \, dx \leq C\) uniformly for all \(k\). On the other hand, similar to \([14\text{ Lemma 3.1(1)}]\), one can verify that \(\int_{\mathbb{R}^2} (|\nabla u_{1k}|^2 + |\nabla u_{2k}|^2) \, dx \to \infty\) as \(k \to \infty\). This is however a contradiction, and therefore (3.3) holds true.

(ii). It directly follows from (3.9) that the first equality of (3.4) holds, and then the second one can be obtained by using the Hölder inequality (3.2). As for (3.5) and (3.6), we note from (3.3) that

\[
e(a_k, b, \beta_k) \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx \to 0 \text{ as } k \to \infty.
\]
Applying (3.2)–(3.4), it then follows from (2.7) and above that
\begin{equation}
\lim_{k \to \infty} \left( \frac{\int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx}{\int_{\mathbb{R}^2} |u_{1k}|^4 \, dx} + \frac{\int_{\mathbb{R}^2} |\nabla u_{2k}|^2 \, dx}{\int_{\mathbb{R}^2} |u_{1k}|^4 \, dx} \right) = \frac{a^*}{2}.
\end{equation}

On the other hand, one can obtain from (1.13) that
\begin{equation}
\frac{\int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx}{\int_{\mathbb{R}^2} |u_{1k}|^4 \, dx} \geq \frac{\int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx}{\int_{\mathbb{R}^2} |u_{1k}|^4 \, dx} \geq \frac{a^*}{2},
\end{equation}

since \( \|u_{1k}\|^2 + \|u_{2k}\|^2 = 1 \). Thus, (3.5) and (3.6) follow from (3.10) and the above inequality, and the lemma is proved. \( \square \)

For any sequence \( \{a_k\} \) satisfying \( a_k \not\to a^* \) as \( k \to \infty \), define
\begin{equation}
\bar{\varepsilon}_k := \left( \int_{\mathbb{R}^2} |u_{1k}(x)|^4 \, dx \right)^{-\frac{1}{4}} > 0,
\end{equation}

and by (3.9) we then have \( \bar{\varepsilon}_k \to 0 \) as \( k \to \infty \). From (3.5), we know that \( \varepsilon_{k} u_{2k}(\bar{\varepsilon}_k x) \to 0 \) strongly in \( H^1(\mathbb{R}^2) \) as \( k \to \infty \). Similar to [16, Theorem 1.2] (see also [14, Theorem 1.3]), one can obtain from Lemma 3.1 that, passing to a subsequence if necessary, \( \bar{w}_{1k} \) satisfies
\begin{equation}
\bar{w}_{1k}(x) := \varepsilon_{k} u_{1k}(\bar{\varepsilon}_k x + x_{1k}) \to \sqrt{\frac{1}{2}} \frac{w(\sqrt{\frac{a^*}{2}} x)}{\varepsilon_k} \quad \text{strongly in } H^1(\mathbb{R}^2),
\end{equation}

where \( x_{1k} \) is the unique maximum point of \( u_{1k} \). Under some further assumptions on the trapping potentials, the following proposition gives the explicit limit behavior of \( u_{1k} \) as \( k \to \infty \).

**Proposition 3.2.** Suppose \( V_i(x) \in C^2(\mathbb{R}^2) \) is homogeneous of degree \( p_i \) and satisfies (1.2) and (1.29), where \( i = 1, 2 \) and \( 2 \leq p_1 \leq p_2 \). Let \( (u_{1k}, u_{2k}) \) be a nonnegative minimizer of \( e(a_k, b, \beta_k) \) satisfying (7.1). Then there exists a subsequence, still denoted by \( \{a_k\} \), of \( \{a_k\} \) such that
\begin{equation}
\varepsilon_{k} u_{1k}(\bar{\varepsilon}_k x + x_{1k}) \to w(x) \quad \text{and} \quad \varepsilon_{k} u_{2k}(\bar{\varepsilon}_k x) \to 0 \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \mathbb{R}^2,
\end{equation}

where \( x_{1k} \) is the unique maximum point of \( u_{1k} \) satisfying
\begin{equation}
\lim_{k \to \infty} \frac{x_{1k}}{\varepsilon_k} = y_0,
\end{equation}

and
\begin{equation}
\varepsilon_k := \frac{1}{\lambda} \left[ (a^* - a_k)(a^* - b) - (\beta_k - a^*)^2 \right]^{\frac{1}{2p_1}} > 0, \quad \lambda = \frac{p_1}{2} \left( a^* - b \right) H_1(y_0) \right]^{\frac{1}{2p_1}}
\end{equation}

for \( y_0 \in \mathbb{R}^2 \) given by (1.20). Moreover, \( \bar{u}_{1k} \) decays exponentially in the sense that
\begin{equation}
|\nabla \bar{u}_{1k}| \leq C e^{-\frac{|x|}{C}} \quad \text{in} \quad \mathbb{R}^2,
\end{equation}

and
\begin{equation}
|\nabla \bar{u}_{1k}| \leq C e^{-\frac{|x|}{C}} \quad \text{in} \quad \mathbb{R}^2,
\end{equation}

where the constant \( C > 0 \) is independent of \( k \).

**Proof.** We first prove that the energy \( e(a_k, b, \beta_k) \) satisfies
\begin{equation}
\lim_{k \to \infty} e(a_k, b, \beta_k) = \frac{\lambda^2}{a^*(a^* - b)} \frac{p_1 + 2}{p_1},
\end{equation}

where \( (a_k, b, \beta_k) \) satisfies (3.1) and \( \lambda > 0 \) is given in (3.15).
To derive (3.18), we take a test function of the form

\[
\begin{aligned}
\phi_1(x) &= A \frac{\tau}{\|w\|_2} u(\tau x - y_0), \\
\phi_2(x) &= A \frac{\sqrt{b_0 - b}}{\|w\|_2} \frac{\tau}{\|w\|_2} u(\tau x - y_0),
\end{aligned}
\]

where \( y_0 \in \mathbb{R}^2 \) is given by (1.20), \( \tau > 0 \) and \( A > 0 \) is chosen so that \( \int_{\mathbb{R}^2} (\phi_1^2 + \phi_2^2) \, dx = 1 \). One can check that \( A = \left( \frac{a^*-b}{\beta_k - b} \right)^{\frac{1}{2}} \leq 1 \), since \( (a_k, b, \beta_k) \) satisfies (5.1). Using (1.8) and (1.5), some calculations yield that as \( \tau \to \infty \),

\[
\begin{aligned}
\int_{\mathbb{R}^2} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) \, dx &- \int_{\mathbb{R}^2} \left( \frac{a_k}{2} |\phi_1|^4 + \frac{b}{2} |\phi_2|^4 + \beta_k |\phi_1|^2 |\phi_2|^2 \right) \, dx \\
&= \tau^2 - A^4 a^{-1} \tau^2 \left[ a_k + b \left( \frac{\beta_k - a^*}{a^* - b} \right)^2 + 2 \beta_k \frac{\beta_k - a^*}{a^* - b} \right] \\
&= \frac{A^4}{a^*} \tau^2 \left[ (a^* - a_k) + \frac{(\beta_k - a^*)}{a^* - b} \right]^2 - 2 (\beta_k - a^*) \frac{(\beta_k - a^*)}{a^* - b} \\
&= \frac{1}{a^*} \frac{a^* - b}{(\beta_k - b)^2} \left[ (a^* - a_k)(a^* - b) - (\beta_k - a^*) \right] \tau^2,
\end{aligned}
\]

and

\[
\begin{aligned}
\int_{\mathbb{R}^2} (V_1(x)|\phi_1|^2 + V_2(x)|\phi_2|^2) \, dx \\
&= \frac{A^2}{a^*} \int_{\mathbb{R}^2} V_1 \left( \frac{x + y_0}{\tau} \right) w^2 \, dx + \frac{A^2 \beta_k - a^*}{a^* - b} \int_{\mathbb{R}^2} V_2 \left( \frac{x + y_0}{\tau} \right) w^2 \, dx \\
&= \left( \frac{1}{a^* p_1} \frac{1}{\beta_k - b} \lambda^{p_1+2} + o(1) \right) \tau^{-p_1},
\end{aligned}
\]

where \( \lambda > 0 \) is as in (3.15). Thus, by taking

\[
\tau = \lambda \left( \frac{\beta_k - b}{a^* - b} \right)^{\frac{1}{p_1 - 2}} \left[ \frac{1}{(a^* - a_k)(a^* - b) - (\beta_k - a^*)} \right]^{\frac{1}{p_1 - 2}},
\]

we derive from (3.20) and (3.21) that

\[
e(a_k, b, \beta_k) \leq E_{a_k, b, \beta_k}(\phi_1, \phi_2) \\
\leq \frac{1}{a^* (\beta_k - b)^2} \left[ (a^* - a_k)(a^* - b) - (\beta_k - a^*) \right] \tau^2 + \left( \frac{1}{a^* p_1} \frac{1}{\beta_k - b} \lambda^{p_1+2} + o(1) \right) \tau^{-p_1} \\
\leq \left( \frac{\lambda^2}{a^*(a^* - b)} \frac{p_1 + 2}{p_1} + o(1) \right) \left[ (a^* - a_k)(a^* - b) - (\beta_k - a^*) \right]^{\frac{p_1}{p_1 - 2}} \text{ as } k \to \infty.
\]

Hence, this estimate implies that

\[
\limsup_{k \to \infty} \frac{e(a_k, b, \beta_k)}{[(a^* - a_k)(a^* - b) - (\beta_k - a^*)]^{\frac{p_1}{p_1 - 2}}} \leq \frac{\lambda^2}{a^*(a^* - b)} \frac{p_1 + 2}{p_1}.
\]
Let \((u_{1k}, u_{2k})\) be now a nonnegative minimizer of \(e(a_k, b, \beta_k)\), where \((a_k, b, \beta_k)\) satisfies (3.11). Since \(a^* \leq \beta_k \leq \beta_k^*\), we then have

\[
\frac{a^* - a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx + \frac{a^* - b}{2} \int_{\mathbb{R}^2} |u_{2k}|^4 \, dx
+ (a^* - \beta_k) \int_{\mathbb{R}^2} |u_{1k}|^2 |u_{2k}|^2 \, dx
\geq \frac{a^* - a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx \left[ 1 + \frac{a^* - b}{a^* - a_k} \frac{\|u_{2k}\|_4^4}{\|u_{1k}\|_4^4} \right]
- 2 \frac{\beta_k - a^*}{a^* - a_k} \left( \frac{\|u_{2k}\|_4^4}{\|u_{1k}\|_4^4} \right)^{\frac{1}{4}} \left( \frac{\|u_{2k}\|_4}{\|u_{1k}\|_4} \right)^{\frac{1}{4}}
\]

(3.23)

\[
\geq \frac{a^* - a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx \left[ 1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} \right]
+ \frac{a^* - b}{a^* - a_k} \left( \frac{\|u_{2k}\|_4^4}{\|u_{1k}\|_4^4} \right)^{\frac{1}{4}} (\beta_k - a^*)^2
\geq \left[ 1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} \right] \frac{a^* - a_k}{2} \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx
= \frac{a^* - a_k}{2(a^* - b)} (\beta_k - a^*)^2 \int_{\mathbb{R}^2} |u_{1k}|^4 \, dx
\]

as \(k \to \infty\), where the first inequality follows from the Hölder inequality (3.8). On the other hand, similar to proving (3.33) in [14], one can verify from (3.12) that

\[
\liminf_{k \to \infty} \bar{\varepsilon}_k^{-p_1} \int_{\mathbb{R}^2} V_1(x) |u_{1k}|^2 \, dx = \liminf_{k \to \infty} \int_{\mathbb{R}^2} V_1 \left( x + \frac{x_{1k}}{\bar{\varepsilon}_k} \right) |\tilde{w}_{1k}|^2 \, dx
\geq \frac{1}{a^*} \int_{\mathbb{R}^2} V_1 \left( \sqrt{\frac{2}{a^*}} x + y^{10} \right) |w|^2 \, dx
\geq \left( \frac{2}{a^*} \right) \frac{\lambda^{2+p_1}}{(a^* - b)p_1}
\]

(3.24)

where \(\varepsilon_k > 0\) is defined by (3.11), \(\lambda > 0\) is given in (3.13) and \(y^{10} := \lim_{k \to \infty} \frac{x_{1k}}{\bar{\varepsilon}_k}\).

Note that the last equality of (3.24) holds, if and only if

(3.25)

where \(y_0 \in \mathbb{R}^2\) is given in (1.20). Hence, together with (2.4) and (1.10), it follows from (3.24) and (3.23) that

\[
e(a_k, b, \beta_k) \geq \frac{(a^* - a_k)(a^* - b) - (\beta_k - a^*)^2}{2(a^* - b)} \bar{\varepsilon}_k^2
+ \left[ (\frac{2}{a^*}) \frac{\lambda^{2+p_1}}{(a^* - b)p_1} + o(1) \right] \bar{\varepsilon}_k^{p_1}
\]

as \(k \to \infty\).
Taking the infimum of (3.26) over $\varepsilon_k > 0$ yields that

$$(3.27) \quad \liminf_{k \to \infty} \frac{e(a_k, b, \beta_k)}{[(a^* - a_k)(a^* - b) - (\beta_k - a^*)^2]^{1/2}} \geq \frac{\lambda^2}{a^*(a^* - b)} \frac{p_1 + 2}{p_1},$$

where the equality holds if and only if (3.25) holds and

$$(3.28) \quad \lim_{k \to \infty} \varepsilon_k/\varepsilon_k = \sqrt{\frac{a^*}{2}} \text{ with } \varepsilon_k > 0 \text{ given by (3.15)}.\]$$

We thus conclude from (3.22) and (3.27) that (3.18) holds, which implies that all equalities in (3.24) and (3.27) hold. Therefore, both (3.25) and (3.28) hold true. Thus, it follows from (3.12), (3.25) and (3.28) that (3.14) is true and (3.13) holds strongly in $H^1(\mathbb{R}^2)$. Furthermore, similar to the proof of [14, Lemma 4.1], we have the exponential decay (3.16) and (3.17). Finally, applying the standard elliptic regularity theory, the argument similar to proving Theorem 2.1 (see also [13, Proposition 2.1]) implies that (3.13) holds uniformly in $L^\infty(\mathbb{R}^2)$. This therefore completes the proof of Proposition 3.2.

In the following we address some sufficient conditions ensuring that $u_{2k} \neq 0$ in $\mathbb{R}^2$ for sufficiently large $k > 0$.

**Lemma 3.3.** Suppose $V_i(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p_i$ with $2 \leq p_1 \leq p_2$, where $V_i(x)$ satisfies (1.2) and

$$(3.29) \quad y_0 \text{ is a unique and non-degenerate critical point of } H_1(y).$$

Let $(u_{1k}, u_{2k})$ be a nonnegative minimizer of $e(a_k, b, \beta_k)$ satisfying (3.7). If $\beta_k$ also satisfies $a^* - a_k = o(\beta_k - a^*)$ as $k \to \infty$, then we have

$$(3.30) \quad u_{2k} \neq 0 \text{ in } \mathbb{R}^2 \text{ for sufficiently large } k > 0.$$ 

**Proof.** We shall prove (3.30) by contradiction. On the contrary, suppose $u_{2k} \equiv 0$ in $\mathbb{R}^2$ for sufficiently large $k > 0$, from which we first derive a refined lower estimate of the energy $e(a_k, b, \beta_k)$ satisfying (3.1). Under the assumption (3.29), since $u_{2k} \equiv 0$ in $\mathbb{R}^2$ for sufficiently large $k > 0$, we then derive from [13, Theorem 1.2] that $u_{1k}$ solves a single elliptic equation and admits the following refined spike profile

$$(3.31) \quad u_{1k} = \frac{1}{\|w\|^2_{\varepsilon_k}} \left\{ w \left( \frac{x - x_k}{\varepsilon_k} \right) + (a^* - a_k)\psi \left( \frac{x - x_k}{\varepsilon_k} \right) + o(a^* - a_k) \right\} \text{ as } k \to \infty$$

uniformly in $\mathbb{R}^2$ for some $\psi \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, where $\varepsilon_k > 0$ satisfies

$$(3.32) \quad \varepsilon_k = \frac{1}{\lambda_0} (a^* - a_k)^{1+\frac{1}{p_1+1}}, \quad \lambda_0 := \left( \frac{p_1}{2} \right) H_1(y_0)^{\frac{1}{p_1+1}} > 0,$$

and $x_k$ is the unique maximum point of $u_{1k}$ satisfying

$$(3.33) \quad \left| \frac{x_k}{\varepsilon_k} - y_0 \right| = (a^* - a_k)O(|y_0|) \text{ as } k \to \infty$$

for some $y_0 \in \mathbb{R}^2$. We then derive from (3.31) that

$$\int_{\mathbb{R}^2} |\nabla u_{1k}|^2 \, dx = \frac{1}{\varepsilon_k^2} + \frac{2\lambda_0 p_1 + 1}{a^*} \int_{\mathbb{R}^2} \nabla w \nabla \psi \varepsilon_k^{p_1} + o(\varepsilon_k^{p_1}) \text{ as } k \to \infty,$$
and

\[
\int \mathbb{R}^2 V_1(x)u_{1k}^2 \, dx = \int \mathbb{R}^2 V_1(x)\left(x + \frac{x_k}{\tilde{\varepsilon}_k}\right)^2 u_{1k}^2 (x + x_k) \, dx
\]
\[
= \frac{1}{a^*} \tilde{\varepsilon}_k \int \mathbb{R}^2 V_1(x) \left(x + \frac{x_k}{\tilde{\varepsilon}_k}\right) \left[u^2 + 2\lambda_0^{p_1+1}\tilde{\varepsilon}_k^{p_1+2} \psi w + o(\tilde{\varepsilon}_k^{p_1+2})\right]
\]
\[
= \frac{H_1(y_0)\tilde{\varepsilon}_k^{p_1}}{a^*} + \frac{2}{a^*} \lambda_0^{p_1+1}\tilde{\varepsilon}_k^{p_1+2} \int \mathbb{R}^2 V_1(x + y_0)\psi w
\]
\[
+ \frac{1}{a^*} \tilde{\varepsilon}_k \left[H_1(x_k) - H_1(y_0)\right] + o(\tilde{\varepsilon}_k^{p_1+2}) \text{ as } k \to \infty.
\]

Note from (3.29) and (3.33) that \(H_1(x_k) - H_1(y_0) = o(a^* - a_k)\) as \(k \to \infty\). Hence, we reduce from above that

\[
e(a_k, b, \beta_k) \geq \frac{1}{a^*} \left(a^* - a_k\right) \int \mathbb{R}^2 |\nabla u_{1k}|^2 \, dx + \int \mathbb{R}^2 V_1(x)u_{1k}^2 \, dx
\]
\[
= \frac{\lambda_0^2}{a^*} p_1 + \frac{2}{p_1} \left(a^* - a_k\right)^{p_1+2} + \frac{2\lambda_0}{a^*} \int \mathbb{R}^2 |\nabla w| \psi \left(a^* - a_k\right)^{p_1+2} + \frac{2}{a^*} \int \mathbb{R}^2 V_1(x + y_0)\psi w
\]
\[
+ \frac{1}{a^*} \tilde{\varepsilon}_k \left[H_1(x_k) - H_1(y_0)\right] + o\left((a^* - a_k)^{2+2p_1}\right) \text{ as } k \to \infty,
\]

where (1.13) is used in the first inequality. Therefore, under the assumption (3.29) we conclude from above that

\[
e(a_k, b, \beta_k) \geq \frac{\lambda_0^2}{a^*} p_1 + \frac{2}{p_1} \left(a^* - a_k\right)^{p_1+2} \left[1 + O\left(a^* - a_k\right)\right] \text{ as } k \to \infty,
\]

where \(\lambda_0 > 0\) is defined by (3.32).

Under the additional assumption that \(\beta_k\) also satisfies \(a^* - a_k = o(\beta_k - a^*)\) as \(k \to \infty\), we next derive a refined upper estimate of \(e(a_k, b, \beta_k)\) as \(k \to \infty\). Take a test function of the form (3.19), where \(y_0 \in \mathbb{R}^2\) is given by (3.29), \(A = \left(\frac{a^* - b}{\beta_k - b}\right)^{\tau} < 1\), and \(\tau = \lambda_0 \left[\frac{1}{\left(1 - \frac{\|a_k - a^*\|^2}{\lambda_0^{1+\tau}}\right)}\right]^{1+\tau} \left(a^* - a_k\right)^{1+\tau} > 0\) for \(\lambda_0 > 0\) defined by (3.32).

Similar to (3.20) and (3.21), some calculations then yield that

\[
\int \mathbb{R}^2 \left(|\nabla \phi_1|^2 + |\nabla \phi_2|^2\right) \, dx - \int \mathbb{R}^2 \left(\frac{a_k}{2} |\phi_1|^4 + \frac{b}{2} |\phi_2|^4 + \beta_k |\phi_1|^2 |\phi_2|^2\right) \, dx
\]
\[
= \frac{1}{a^*} \left(a^* - b\right)^2 \left[1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)}\right](a^* - a_k)^{\tau^2}
\]
\[
\leq \frac{\lambda_0^2}{a^*} \left(a^* - a_k\right)^{p_1+2} \left[1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)}\right] \left(a^* - a_k\right)^{p_1+2} \text{ as } k \to \infty,
\]
implies that 0 case. We next consider the case where lim inf

\[
\int_{\mathbb{R}^2} (V_1(x)|\phi_1|^2 + V_2(x)|\phi_2|^2) \, dx
\]

\[
= \frac{1}{a^*} \left( \frac{\beta_k - a^*}{a^* - a_k} \right)^{p_1 + 2} + \frac{1}{\alpha^*} \left( \frac{\beta_k - a^*}{\alpha^* - a_k} \right)^{p_1 + 2} H_2(y_0) - \frac{p_1}{p_2} \left( \frac{\beta_k - a^*}{a^* - a_k} \right)^{p_1 + 2} (a^* - a_k)^{p_1 + 2}
\]

(3.36)

as \( k \to \infty \). Thus, we derive from (3.35) and (3.36) that

\[
e(a_k, b, \beta_k) = E_{a_k,b,\beta_k}(\phi_1, \phi_2) \leq \frac{\lambda_0^2}{a^*} \left( \frac{p_1}{p_1} \right) (a^* - a_k)^{p_1 + 2} I_k \quad \text{as} \quad k \to \infty,
\]

where \( \lambda_0 > 0 \) is defined by (3.32) and \( I_k > 0 \) satisfies

\[
I_k := \left[ 1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} \right]^{p_1 + 2} \left( 1 + \frac{2}{p_1 + 2} \frac{H_2(y_0)}{H_1(y_0)} \right)
\]

\[
\leq \left[ 1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} \right]^{p_1 + 2} \left( 1 + \frac{2}{p_1 + 2} \frac{H_2(y_0)}{H_1(y_0)} \beta_k - a^* \right)
\]

since \( 2 \leq p_1 \leq p_2 \). Under the assumption that \( a^* - a_k = o(\beta_k - a^*) \) as \( k \to \infty \), we next derive a contradiction by two cases.

We first consider the case where \( \liminf_{k \to \infty} \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} := \gamma > 0 \), which then implies that \( 0 < \gamma \leq 1 \) in view of (3.31). We further reduce from above that

\[
0 < I_k < I_0 := \left( 1 - \frac{\gamma}{2} \right)^{p_1 + 2} < 1 \quad \text{as} \quad k \to \infty.
\]

This estimate and (3.37) then give that

\[
e(a_k, b, \beta_k) < \frac{\lambda_0^2}{a^*} \left( \frac{p_1}{p_1} \right) (a^* - a_k)^{p_1 + 2} I_0 \quad \text{as} \quad k \to \infty,
\]

which however contradicts to (3.34), and the lemma is therefore proved in the first case. We next consider the case where \( \lim_{k \to \infty} \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} = 0 \). In this case, since \( a^* - a_k = o(\beta_k - a^*) \) as \( k \to \infty \), we then have

\[
\beta_k - a^* = o \left( \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} \right) \quad \text{as} \quad k \to \infty,
\]

from which we have

\[
0 < I_k \leq \left[ 1 - \frac{p_1}{p_1 + 2} \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} + o \left( \frac{(\beta_k - a^*)^2}{a^* - a_k} \right) \right]
\]

\[
\cdot \left( 1 + \frac{2}{p_1 + 2} \frac{H_2(y_0)}{H_1(y_0)} \beta_k - a^* \right)
\]

\[
< 1 - \frac{p_1}{2(p_1 + 2)(a^* - b)} \left( \frac{(\beta_k - a^*)^2}{(a^* - a_k)} \right)^2 (a^* - a_k) \quad \text{as} \quad k \to \infty.
\]
This estimate and (3.37) then give that
\[ e(ak, b, β_k) < \frac{λ^2_p 2p_1 + 2(a^* - ak)^2}{p_1} \left[ 1 - \frac{p_1}{2(p_1 + 2)(a^* - ak)} \right] \]
as \( k \to \infty \), which also contradicts (3.34) in view of the assumption that \( a^* - ak = o(β_k - a^*) \) as \( k \to \infty \). This therefore finishes the proof of the lemma.

**Remark 3.4.** Under the assumptions that \( V_i(x) \in C^2(\mathbb{R}^2) \) is homogeneous of degree \( p_i \) with \( 2 \leq p_1 \leq p_2 \) and satisfies (1.2) and (1.20), instead of the non-degeneracy (3.29), the proof of Lemma 3.3 implies that if \( β_k \) also satisfies \( \lim_{k \to \infty} \frac{(β_k - a^*)^2}{(a^* - ak)(a^* - b)} > 0 \), we also have \( u_{2k} \neq 0 \) in \( \mathbb{R}^2 \) for sufficiently large \( k > 0 \).

3.1. **Refined spike profiles of \( u_{2k} \).** Based on Proposition 3.2, the first purpose of this subsection is to derive the refined spike profiles of \( u_{2k} \) as \( k \to \infty \) for the case where \( u_{2k} \neq 0 \) in \( \mathbb{R}^2 \), by which we then complete the proof of Theorems 1.1 and 1.2. Recall that \((u_{1k}, u_{2k})\) solves the following PDE system
\[
\begin{aligned}
-∆u_{1k} + V_1(x)u_{1k} &= μ_k u_{1k} + ak u_{1k}^3 + β_k u_{2k}^2 u_{1k} \quad \text{in } \mathbb{R}^2, \\
-∆u_{2k} + V_2(x)u_{2k} &= μ_k u_{2k} + bu_{2k}^3 + β_k u_{1k}^2 u_{2k} \quad \text{in } \mathbb{R}^2,
\end{aligned}
\]
where \( μ_k \in \mathbb{R} \) is a suitable Lagrange multiplier.

If \( u_{2k} \neq 0 \) in \( \mathbb{R}^2 \), define
\[
\bar{u}_{1k}(x) = \sqrt{a^*} ε_k u_{1k}(ε_k x + x_{1k}) \quad \text{and} \quad u_{2k}(ε_k x + x_{1k}) = C_∞ \sigma_k \bar{u}_{2k}(x),
\]
where \( σ_k = \|u_{2k}\|_∞ > 0 \) and \( C_∞ = \frac{1}{\|w\|_∞} > 0 \), and \( x_{1k} \) is the unique maximum point of \( u_{1k} \), so that
\[
\bar{u}_{1k}(x) \to w(x) \quad \text{and} \quad σ_k ε_k \to 0 \quad \text{as} \quad k \to \infty,
\]
where (3.13) is used. Then \((\bar{u}_{1k}, \bar{u}_{2k})\) solves the following PDE system
\[
\begin{aligned}
-∆\bar{u}_{1k} + ε_k^2 V_1(ε_k x + x_{1k})\bar{u}_{1k} &= μ_k ε_k^2 \bar{u}_{1k} + \frac{a_k}{a^*} \bar{u}_{1k}^3 + β_k C_∞^2 σ_k^2 ε_k^2 \bar{u}_{2k}^2 \bar{u}_{1k} \quad \text{in } \mathbb{R}^2, \\
-∆\bar{u}_{2k} + ε_k^2 V_2(ε_k x + x_{1k})\bar{u}_{2k} &= μ_k ε_k^2 \bar{u}_{2k} + bC_∞^2 σ_k^2 ε_k^2 \bar{u}_{1k}^3 + \frac{β_k}{a^*} \bar{u}_{1k}^2 \bar{u}_{2k} \quad \text{in } \mathbb{R}^2.
\end{aligned}
\]
The following lemma gives the fundamental limit behavior of \( u_{2k} \) as \( k \to \infty \).

**Lemma 3.5.** Suppose \( V_i(x) \in C^2(\mathbb{R}^2) \) is homogeneous of degree \( p_i \) and satisfies (1.2) and (1.20), where \( i = 1, 2 \) and \( 2 \leq p_1 \leq p_2 \). Let \((u_{1k}, u_{2k})\) be a nonnegative minimizer of \( e(ak, b, β_k) \) satisfying (3.1). Suppose that \( u_{2k} \neq 0 \) in \( \mathbb{R}^2 \) and define
\[
u_{2k}(ε_k x + x_{2k}) = C_∞ σ_k \bar{u}_{2k}(x),
\]
where \( σ_k = \|u_{2k}\|_∞ > 0 \), \( C_∞ = \frac{1}{\|w\|_∞} > 0 \), and \( x_{1k} \) is the unique maximum point of \( u_{ik} \) for \( i = 1, 2 \). Then there exists a subsequence of \( \{\bar{u}_{2k}\} \) (still denoted by \( \{\bar{u}_{2k}\} \)) such that
\[
\bar{u}_{2k}(x) \to w(x) \quad \text{uniformly in } \mathbb{R}^2 \quad \text{as} \quad k \to \infty,
\]
and
\[
\lim_{k \to \infty} \frac{x_{2k} - x_{1k}}{ε_k} = 0.
\]
Proof. Consider (3.44), where \( x_{2k} \in \mathbb{R}^2 \) is a global maximum point of \( u_{2k} \). We then obtain that
\[
\tilde{u}_{2k}(0) = \| \tilde{u}_{2k}(x) \|_{\infty} = \| w \|_{\infty} > 0, \tag{3.46}
\]
and \((\tilde{u}_{1k}, \tilde{u}_{2k})\) solves the elliptic PDE system
\[
\begin{aligned}
\begin{cases}
-\Delta \tilde{u}_{1k} + \varepsilon_k^2 V_1(\varepsilon_k x + x_{1k}) \tilde{u}_{1k} \\
= \mu_k \varepsilon_k^2 \tilde{u}_{1k} + \frac{a_k}{a^*} \tilde{u}_{1k}^3 + \beta_k C_2 \varepsilon_k^2 \tilde{u}_{2k}^2 (x + \frac{x_{1k} - x_{2k}}{\varepsilon_k}) \tilde{u}_{1k} & \text{in } \mathbb{R}^2,
\end{cases}
\end{aligned}
\tag{3.47}
\]
\[-\Delta \tilde{u}_{2k} + \varepsilon_k^2 V_2(\varepsilon_k x + x_{2k}) \tilde{u}_{2k} \\
= \mu_k \varepsilon_k^2 \tilde{u}_{2k} + b C_2 \varepsilon_k^2 \tilde{u}_{2k}^3 + \frac{\beta_k}{a^*} \tilde{u}_{1k}^2 (x + \frac{x_{2k} - x_{1k}}{\varepsilon_k}) \tilde{u}_{2k} & \text{in } \mathbb{R}^2,
\]
where the Lagrange multiplier \( \mu_k \in \mathbb{R} \) satisfies \( \mu_k \varepsilon_k^2 \to -1 \) as \( k \to \infty \). Using the elliptic regularity theory, we thus deduce from (3.47) that there exists \( 0 \leq u_0(x) \in H^1(\mathbb{R}^2) \) such that
\[
\tilde{u}_{2k} \to u_0(x) \text{ in } C^2_{\text{loc}}(\mathbb{R}^2) \text{ as } k \to \infty. \tag{3.48}
\]
Also, we have
\[
u_0(0) = \| u_0(x) \|_{\infty} = \| w \|_{\infty} > 0. \tag{3.49}
\]

Similar to those in [14] and references therein, one can further derive from (3.47) that \((u_{1k}, u_{2k})\) admits a unique maximum global point \((x_{1k}, x_{2k})\), and satisfies the exponential decay (3.16) and (3.17).

We now show that
\[
\left\{ \frac{x_{2k} - x_{1k}}{\varepsilon_k} \right\} \text{ is bounded uniformly in } \mathbb{R}^2. \tag{3.50}
\]

Indeed, since \( \beta_k \searrow a^* \) as \( k \to \infty \), if (3.50) is false, we then obtain from (3.41) and (3.47) that \( u_0 \) satisfies \(-\Delta u_0(x) + u_0 = 0 \) in \( \mathbb{R}^2 \). This implies that \( u_0(x) \equiv 0 \) in \( \mathbb{R}^2 \), which however contradicts to (3.49). Therefore, the estimate (3.50) holds true. Up to a subsequence if necessary, we then deduce from (3.50) that there exists an \( x_0 \in \mathbb{R}^2 \) such that
\[
\lim_{k \to \infty} \frac{x_{2k} - x_{1k}}{\varepsilon_k} = x_0 \in \mathbb{R}^2. \tag{3.51}
\]

Moreover, it follows from (3.41), (3.47) and (3.51) that \( u_0(x) \) satisfies
\[
- \Delta u_0(x) + u_0(x) = w^2(x + x_0) u_0(x) \text{ in } \mathbb{R}^2. \tag{3.52}
\]

We thus obtain from (3.46) and [37, Lemma 4.1] that
\[
u_0(x) \equiv w(x + x_0) \text{ in } \mathbb{R}^2.
\]

Since \( x = 0 \) is a maximum point of \( \tilde{u}_{2k}(x) \) for each \( k \in \mathbb{N} \), it is also a maximum point of \( w(x + x_0) \). However, \( w(x) \) admits a unique maximum point \( x = 0 \), from which we conclude that (3.51) holds for \( x_0 = 0 \). Therefore, this implies that (3.49) holds.

Finally, since \((u_{1k}, u_{2k})\) satisfies the exponential decay (3.16) and (3.17), by (3.48) we can follow the proof of Theorem 2.1 to conclude that (3.44) holds uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \). This completes the proof of the lemma. \( \square \)
Under the assumptions of Lemma 3.3 if \( u_{2k} \neq 0 \) in \( \mathbb{R}^2 \), we next define

\[
(3.53) \quad w_{ik}(x) = \bar{u}_{ik}(x) - w(x), \quad i = 1, 2, \]

so that \( w_{ik}(x) \to 0 \) uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \) in view of \( (3.41), (3.44) \) and \( (3.45) \).

We also denote the linearized operator

\[
(3.54) \quad \mathcal{L}_{1k} w_{1k} = -\Delta w_{1k} + \left[ 1 - (\bar{u}_{1k}^2 + \bar{u}_{1k} w + w^2) \right] w_{1k} \quad \text{in} \quad \mathbb{R}^2,
\]

\[
\mathcal{L}_{2k} (w_{1k}) w_{2k} = -\Delta w_{2k} + \left[ 1 - \bar{u}_{1k}^2 \right] w_{2k} - w (\bar{u}_{1k} + w) w_{1k} \quad \text{in} \quad \mathbb{R}^2,
\]

and the associated limit operator

\[
(3.55) \quad \begin{cases} 
\mathcal{L}_1 \phi_1 = -\Delta \phi_1 + [1 - 3w^2] \phi_1 & \text{in} \quad \mathbb{R}^2, \\
\mathcal{L}_2 (\phi_1) \phi_2 = -\Delta \phi_2 + (1 - w^2) \phi_2 - 2w^2 \phi_1 & \text{in} \quad \mathbb{R}^2.
\end{cases}
\]

Then \((w_{1k}, w_{2k})\) satisfies \( \nabla w_{1k}(0) = 0 \) and

\[
(3.56) \quad \begin{aligned}
\mathcal{L}_{1k} w_{1k} &= - (\alpha^* - \alpha_k) \frac{1}{\alpha^*} \bar{u}_{1k}^3 (x) - \varepsilon_k^{2+\gamma_1} V_1 (x + \frac{x_{1k}}{\varepsilon_k}) \bar{u}_{1k} (x) \\
&\quad + \delta_k \bar{u}_{1k} (x) + \beta_k \sigma_{\infty}^{2+\gamma_2} \varepsilon_k^{2+\gamma_2} \bar{u}_{1k} w_{1k} := h_{1k} (x) \quad \text{in} \quad \mathbb{R}^2,
\end{aligned}
\]

\[
\begin{aligned}
\mathcal{L}_{2k} (w_{1k}) w_{2k} &= (\beta_k - \alpha^*) \frac{1}{\alpha^*} \bar{u}_{1k}^3 \bar{u}_{2k} - \varepsilon_k^{2+\gamma_2} V_2 (x + \frac{x_{1k}}{\varepsilon_k}) \bar{u}_{2k} \\
&\quad + \delta_k \bar{u}_{2k} (x) + b \sigma_{\infty}^{2+\gamma_2} \varepsilon_k^{2+\gamma_2} \bar{u}_{2k} := h_{2k} (x) \quad \text{in} \quad \mathbb{R}^2,
\end{aligned}
\]

where we denote

\[
(3.57) \quad \delta_k = 1 + \mu_k \varepsilon_k^2 \to 0 \quad \text{as} \quad k \to \infty.
\]

Taking the limit of \( (3.56) \) and using \( (3.14) \), let \((w_1, w_2)\) solve the following system

\[
(3.58) \quad \begin{cases} 
\mathcal{L}_1 w_1 = \delta_k w + \alpha^* \sigma_{\infty}^{2+\gamma_2} \varepsilon_k^{2+\gamma_2} \bar{u}_{2k}^3 \bar{u}_{2k} w + \frac{\alpha^* - \alpha_k}{\alpha^*} \bar{u}_{2k}^3 - \varepsilon_k^{2+\gamma_1} V_1 (x + y_0) w \\
= h_1 (x) \quad \text{in} \quad \mathbb{R}^2, \quad \nabla w_1 (0) = 0,
\end{cases}
\]

\[
\begin{aligned}
\mathcal{L}_2 (w_1) w_2 &= \delta_k w + b \sigma_{\infty}^{2+\gamma_2} \varepsilon_k^{2+\gamma_2} \bar{u}_{2k}^3 \bar{u}_{2k} w + \beta_k - \alpha^* \bar{u}_{2k}^3 - \varepsilon_k^{2+\gamma_2} V_2 (x + y_0) w \\
&\quad := h_2 (x) \quad \text{in} \quad \mathbb{R}^2, \quad \nabla w_2 (0) = 0,
\end{aligned}
\]

where \( w_i \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) for \( i = 1, 2 \). Here \( \nabla w_i (0) = 0 \) is due to \( (3.45), (3.53) \) and the fact that \( \nabla \bar{u}_{2k} (\frac{\bar{x}_{2k} - \bar{x}_{1k}}{\varepsilon_{2k}}) \equiv 0 \). We then obtain that \( w_i \) exists and satisfies \( w_i \to 0 \) uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \) for \( i = 1, 2 \). Following above results, we are now ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Under the assumptions of Theorem 1.1 by Proposition 3.2 the rest is to further prove that \( u_{2k} \) satisfies

\[
(3.59) \quad \sqrt{\frac{\alpha^* (a^* - b)}{\beta_k - a^*}} \varepsilon_k u_{2k} (\varepsilon_k x + x_{2k}) \to w(x)
\]

uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \), where \( \varepsilon_k > 0 \) is given by \( (3.15) \).

Actually, under the assumptions of Theorem 1.1 we note from Lemma 3.3 that \( u_{2k} \neq 0 \) for sufficiently large \( k > 0 \). By considering \( \bar{u}_{ik} \) defined in \( (3.40) \) for \( i = 1, 2, \ldots \),
we then get that Lemma 3.5 holds true. Following these, we thus deduce from (3.58) that $w_1$ exists and
\[
-2 \int_{\mathbb{R}^2} w^3 \, w_1 \\
= \int_{\mathbb{R}^2} w \left[ -\Delta + (1 - w^2) \right] w_1 - 2 \int_{\mathbb{R}^2} w^3 \, w_1 \\
= \int_{\mathbb{R}^2} w \mathcal{L}_1 w_1 = \int_{\mathbb{R}^2} w \mathcal{L}_1 w \\
= a^* \delta_k + 2(a^*)^2 C^2 \sigma_k \varepsilon_k^2 - 2(a^* - a_k) - H_1(y_0) \varepsilon_k^{2+p_1} \quad \text{as } k \to \infty.
\] (3.60)

On the other hand, we also get from (3.58) that $w_2$ exists and
\[
-2 \int_{\mathbb{R}^2} w^3 \, w_1 \\
= \int_{\mathbb{R}^2} w^4 \left[ -\Delta + (1 - w^2) \right] w_2 - 2 w^2 \, w_1 \\
= \int_{\mathbb{R}^2} w \mathcal{L}_2(w_1) \, w_2 = \int_{\mathbb{R}^2} w \mathcal{L}_2(w) \\
= 2(\beta_k - a^*) - H_2(y_0) \varepsilon_k^{2+p_2} + a^* \delta_k + 2a^* b C^2 \sigma_k \varepsilon_k^2 \quad \text{as } k \to \infty,
\] where $H_2(y)$ is defined by (1.12). Therefore, above two identities give that
\[
2a^*(a^* - b) C^2 \sigma_k \varepsilon_k^2 \\
= 2(\beta_k - a^*) + 2(a^* - a_k) + H_1(y_0) \varepsilon_k^{2+p_1} - H_2(y_0) \varepsilon_k^{2+p_2} \quad \text{as } k \to \infty.
\] (3.62)

Since the assumption (1.10) implies that $a^* - a_k = o(\beta_k - a^*)$ as $k \to \infty$, we then derive from above that
\[
a^*(a^* - b) C^2 \sigma_k \varepsilon_k^2 \sim (\beta_k - a^*) \quad \text{as } k \to \infty.
\] (3.63)

Applying Lemma 3.5 we therefore conclude (3.59) from (3.40) and (3.63), and we are done.

As a byproduct, the argument of proving Theorem 1.1 leads us to complete the proof of Theorem 1.2.

\textbf{Proof of Theorem 1.2.} If $0 < b < a^*$, $a_k \not\rightarrow a^*$ and $\beta_k \rightarrow \beta_\ast \in (0, a^*)$ as $k \to \infty$, Theorem 1.2 then follows directly from Theorem 2.1.

We now address Theorem 1.2 for the case where $0 < b < a^*$, $a_k \not\rightarrow a^*$ and $\beta_k \not\rightarrow a^*$ satisfy $a^* - a_k = o(a^* - \beta_k)$ as $k \to \infty$. In this case, we first note that Proposition 3.2 still holds for $\varepsilon_k > 0$ satisfying (2.2), and hence the rest is to prove that $u_{2k} \equiv 0$ for sufficiently large $k > 0$. On the contrary, assume that $u_{2k} \not\equiv 0$ for sufficiently large $k > 0$. We then consider
\[
\tilde{u}_{1k}(x) = \sqrt{a^* \varepsilon_k} u_{1k}(\varepsilon_k x + x_{1k}) \quad \text{and} \quad u_{2k}(\varepsilon_k x + x_{1k}) = C_\infty \sigma_k \tilde{u}_{2k}(x),
\] (3.64)
where $\sigma_k = \|u_{2k}\|_\infty > 0$ and $C_\infty = \frac{1}{\|u\|_\infty} > 0$,

where $\varepsilon_k > 0$ satisfies (2.2), and $x_{1k} \in \mathbb{R}^2$ is the unique maximum point of $u_{1k}$. In this case, one can check that $\tilde{u}_{ik}$ ($i = 1, 2$) still satisfies Lemma 3.5. Following these,
the argument of (3.60)-(3.62) further gives that there exists a constant $M > 0$, independent of $k$, such that
\begin{equation}
0 < 2a^*(a^* - b)C^2 \sigma_k^2 \epsilon_k^2
\end{equation}
\begin{equation}
(3.65)
= 2(a^* - a_k) - 2(a^* - \beta_k) + H_1(y_0)\epsilon_k^{2+p_1} - H_2(y_0)\epsilon_k^{2+p_2}
\end{equation}
\begin{equation}
\leq M(a^* - a_k) - 2(a^* - \beta_k) \leq -(a^* - \beta_k) \text{ as } k \to \infty,
\end{equation}
a contradiction, where the last inequality follows from the assumption that $a^*-a_k = o(a^*-\beta_k)$ as $k \to \infty$. Therefore, we also have $u_{2k} \equiv 0$ for sufficiently large $k > 0$ in this case, and the proof is then complete. \hfill \Box

The rest part of this subsection is to derive the following theorem by a different approach, which shows that if $\beta_k$ is close enough to $\beta_k^*$, the refined spike behavior of $u_{2k}$ stated in Theorem 1.1 still holds without the non-degeneracy assumption of (1.14).

**Theorem 3.6.** Suppose $V_i(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p_i$ and satisfies (1.2) and (1.20), where $i = 1, 2$ and $2 \leq p_1 \leq p_2$. Let $(u_{1k}, u_{2k})$ be a nonnegative minimizer of $e(a_k, b, \beta_k)$ satisfying (3.1). If, additionally, $(a_k, b, \beta_k)$ also satisfies
\begin{equation}
(3.66)
\frac{\beta_k - a^*}{(a^* - a_k)(a^* - b)} \to \epsilon_0 \in (0, 1) \text{ as } k \to \infty,
\end{equation}
then (1.10) still holds.

We first remark that Proposition 3.2 holds under the assumptions of Theorem 3.6. Further, if $(a_k, b, \beta_k)$ also satisfies (3.66), it follows from Remark 3.4 that $u_{2k} \not\equiv 0$ in $\mathbb{R}^2$ for sufficiently large $k > 0$. Further, the following refined estimates are needed for the proof of Theorem 3.6.

**Lemma 3.7.** Under the assumptions of Theorem 3.6 let $(u_{1k}, u_{2k})$ be a nonnegative minimizer of $e(a_k, b, \beta_k)$ as $k \to \infty$. Then we have
\begin{equation}
(3.67)
\lim_{k \to \infty} \frac{\|u_{2k}\|_{L^4}^4}{\|u_{1k}\|_{L^4}^4} \left(\frac{a^* - b}{\beta_k - a^*}\right)^2 = 1
\end{equation}
\begin{equation}
(3.68)
\lim_{k \to \infty} \|u_{2k}\|_{L^2}^2 \frac{a^* - b}{\beta_k - a^*} = 1,
\end{equation}
and
\begin{equation}
(3.69)
\int_{\mathbb{R}^2} |\nabla u_{2k}|^2 dx \leq C \frac{\beta_k - a^*}{a^* - b} \int_{\mathbb{R}^2} |u_{1k}|^4.
\end{equation}

**Proof.** Under the assumptions of Theorem 3.6 we first note that Proposition 3.2 holds true, and its proof gives that the energy $e(a_k, b, \beta_k)$ satisfies
\begin{equation}
(3.70)
\lim_{k \to \infty} \frac{e(a_k, b, \beta_k)}{\epsilon_k^{p_1}} = \frac{\lambda^{2+p_1}}{a^*(a^* - b)p_1} \frac{p_1 + 2}{p_1},
\end{equation}
where $\epsilon_k > 0$ and $\lambda > 0$ are given by (5.13). By Proposition 3.2 one can check from (3.24) that
\begin{equation}
(3.71)
\lim_{k \to \infty} \frac{1}{\epsilon_k^{-p_1}} \int_{\mathbb{R}^2} V_i(x)|u_{1k}|^2 dx \geq \frac{\lambda^{2+p_1}}{a^*(a^* - b)p_1} \frac{2}{p_1}.
\end{equation}
Since $a^* \geq \beta_k \geq \beta_k^*$, we also derive from (3.23) and (3.66) that
we then deduce from (1.13) and (3.8) that

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where we have used that

$$\|u\|_2 \geq \lambda \|\nabla u\|_2$$

We next prove (3.68) as follows. Since

$$u_k \neq 0 \text{ in } \mathbb{R}^2$$

for sufficiently large $k > 0$, we then deduce from (1.13) and (3.8) that

$$II := \int_{\mathbb{R}^2} (|\nabla u_{1k}|^2 + |\nabla u_{2k}|^2) dx$$

and note that

$$\frac{a_k}{a^* - a_k} \left(1 - \frac{\|u_{2k}\|^2}{\|u_{1k}\|^2}\right) + \frac{a^* (1 - \|u_{2k}\|^2)}{(a^* - a_k) \|u_{2k}\|^2} \|u_{1k}\|^2$$

where we have used that $\|u_{1k}\|^2 + \|u_{2k}\|^2 = 1$. For simplicity, we now set $t_k := \frac{\|u_{2k}\|^2}{1 - \|u_{2k}\|^2}$ and note that

$$\frac{a_k}{a^* - a_k} \left(1 - \frac{\|u_{2k}\|^2}{\|u_{1k}\|^2}\right) + \frac{a^* (1 - \|u_{2k}\|^2)}{(a^* - a_k) \|u_{2k}\|^2} \|u_{1k}\|^2$$

where $\epsilon_k := \frac{\|u_{1k}\|^2}{1 - \|u_{2k}\|^2}$ and

$$\lim_{k \to \infty} \epsilon_k = \frac{\lambda^{2+p_1} \|(a^* - b) \|u_{1k}\|^2}{p_1}$$. Together with (3.70), this indicates that the identity in the last inequality of (3.72) holds, and (3.67) is thus proved.

We next prove (3.68) as follows. Since $u_{2k} \neq 0$ in $\mathbb{R}^2$ for sufficiently large $k > 0$, we then deduce from (1.13) and (3.8) that

$$II := \int_{\mathbb{R}^2} (|\nabla u_{1k}|^2 + |\nabla u_{2k}|^2) dx$$

and note that

$$\frac{a_k}{a^* - a_k} \left(1 - \frac{\|u_{2k}\|^2}{\|u_{1k}\|^2}\right) + \frac{a^* (1 - \|u_{2k}\|^2)}{(a^* - a_k) \|u_{2k}\|^2} \|u_{1k}\|^2$$

where we have used that $\|u_{1k}\|^2 + \|u_{2k}\|^2 = 1$. For simplicity, we now set $t_k := \frac{\|u_{2k}\|^2}{1 - \|u_{2k}\|^2}$ and note that

$$\frac{a_k}{a^* - a_k} \left(1 - \frac{\|u_{2k}\|^2}{\|u_{1k}\|^2}\right) + \frac{a^* (1 - \|u_{2k}\|^2)}{(a^* - a_k) \|u_{2k}\|^2} \|u_{1k}\|^2$$

where $\epsilon_k := \frac{\|u_{1k}\|^2}{1 - \|u_{2k}\|^2}$ and

$$\lim_{k \to \infty} \epsilon_k = \frac{\lambda^{2+p_1} \|(a^* - b) \|u_{1k}\|^2}{p_1}$$. Together with (3.70), this indicates that the identity in the last inequality of (3.72) holds, and (3.67) is thus proved.
In view of (3.14), (3.66) and (3.67), we have \( \sqrt{\frac{a^*}{t_k(a^*-a_k)}} \|u_{2k}\|_4^2 \to +\infty \). Therefore, the “=” in the last inequality of (3.74) holds true if and only if

\[
\sqrt{\frac{a_k t_k}{a^* - a_k}} = \sqrt{\frac{a^*}{t_k(a^*-a_k)}} \|u_{2k}\|_4^2 + o(1) \quad \text{as} \quad k \to \infty.
\]

By (3.73) and (3.74), we obtain that

\[
II \geq \frac{a^* - a_k}{2} \left[ 1 + \frac{2\sqrt{a_k a^* - 2\beta_k}}{a^* - a_k} \|u_{2k}\|_4^2 + \frac{a^* - b}{a^* - a_k} \|u_{2k}\|_4^2 \right] \int_{\mathbb{R}^2} |u_{1k}|^4 dx
\]

\[
= \frac{a^* - a_k}{2} \left[ 1 - \left( \frac{2(\beta_k - a^*)}{a^* - a_k} + \frac{2\sqrt{a^*}}{\sqrt{a^* + \sqrt{a_k}}} \right) \|u_{2k}\|_4^2 \right] \int_{\mathbb{R}^2} |u_{1k}|^4 dx
\]

\[
= \frac{a^* - a_k}{2} \left[ 1 - \frac{(\beta_k - a^*)^2}{(a^* - a_k)(a^* - b)} + o(1) \right] \int_{\mathbb{R}^2} |u_{1k}|^4 dx \quad \text{as} \quad k \to \infty,
\]

where (3.67) is used in the last equality. Using (3.71) and (3.76) we can obtain that

\[
\lim_{k \to \infty} \frac{e(a_k, b, \beta_k)}{e_k^{\beta_1}} \geq \frac{\lambda^2 + p_1}{a^* (a^* - b)^2} \frac{p_1 + 2}{p_1}.
\]

By (3.70), this yields that all equalities in (3.74) and (3.76) hold true. Therefore, (3.75) is true, and then (3.68) follows by applying (3.67).

As for (3.69), we note that \((u_{1k}, u_{2k})\) solves the system (3.39) with the Lagrange multiplier \(\mu_k\) satisfying \(\mu_k e_k^2 \to -1\) as \(k \to \infty\). By (3.67) and (3.68), we thus derive from above that

\[
\int_{\mathbb{R}^2} (|\nabla u_{2k}|^2 + V_2(x)|u_{2k}|^2) dx
\]

\[
\leq \mu_k \left( \int_{\mathbb{R}^2} |u_{2k}|^2 dx + b \int_{\mathbb{R}^2} |u_{1k}|^4 dx + \beta_k \|u_{1k}\|_4^2 \|u_{2k}\|^2_4 \right)
\]

\[
\leq C \|u_{1k}\|_4^4 \beta_k - a^* \|u_{1k}\|_4^4 \|u_{2k}\|^2_4 + b \left( \frac{\beta_k - a^*}{a^* - b} \right)^2 \|u_{1k}\|_4^4.
\]

This estimate then completes the proof of (3.69), and the proof of the lemma is therefore proved.

**Proof of Theorem 3.6.** Define

\[
\bar{u}_{2k}(x) := \sqrt{\frac{a^*(a^* - b)}{\beta_k - a^*}} e_k u_{2k}(\varepsilon_k x + x_{2k}),
\]

where \(x_{2k} \in \mathbb{R}^2\) is a global maximum point of \(u_{2k}\), and \(\varepsilon_k > 0\) is defined by (3.15). It then follows from (3.67)-(3.69) that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^2} \bar{u}_{2k}^2 dx = a^*, \quad \lim_{k \to \infty} \int_{\mathbb{R}^2} \bar{u}_{2k}^4 dx = 2a^* \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla \bar{u}_{2k}|^2 dx \leq C < \infty.
\]

Following these, one can derive (see [16, 17, 23]) that there exists \(C > 0\), independent of \(k\), such that

\[
\bar{u}_{2k}(0) = \|u_{2k}(x)\|_\infty > C > 0.
\]
We also recall from (3.39) that \( \bar{u}_{2k} \) satisfies
\[
- \Delta \bar{u}_{2k} + \varepsilon_k^2 V_2(\varepsilon_k x + x_{2k}) \bar{u}_{2k} = \mu_k \varepsilon_k^2 \bar{u}_{2k} + \frac{\beta_k - a^*}{a^*(a^* - b)} \bar{u}_{2k}^3 + \frac{\beta_k \varepsilon_k^2}{a^*} \bar{u}_{1k}^2 \left( x + \frac{x_{2k} - x_{1k}}{\varepsilon_k} \right) \bar{u}_{2k} \text{ in } \mathbb{R}^2,
\]
where \( \bar{u}_{1k} \) is given by (3.31). It then follows from the De Giorgi-Nash-Moser theory (c.f. [20] Theorem 4.1) that
\[
\int_{|x| < 1} |\bar{u}_{2k}|^2 dx \geq C \bar{u}_{2k}(0) \geq \tilde{C} > 0.
\]
Further, since it yields from (3.78) that \( \{ \bar{u}_{2k} \} \) is bounded uniformly in \( H^1(\mathbb{R}^2) \), then there exists \( 0 \leq \tilde{u}_0(x) \in H^1(\mathbb{R}^2) \) such that \( \bar{u}_{2k} \xrightarrow{k} \tilde{u}_0 \) in \( H^1(\mathbb{R}^2) \), and we derive from (3.80) that \( \tilde{u}_0(x) \neq 0 \) in \( \mathbb{R}^2 \). On the other hand, following the proof of (3.51), one can obtain that, up to a subsequence if necessary, \( \frac{x_{2k} - x_0}{\varepsilon_k} \rightarrow x_0 \) for some \( x_0 \in \mathbb{R}^2 \). Hence, it follows from (3.78) that \( \tilde{u}_0 \) solves the elliptic PDE
\[
- \Delta \tilde{u}_0(x) + \tilde{u}_0(x) = w^2(x + x_0) \tilde{u}_0(x) \text{ in } \mathbb{R}^2.
\]
By [37] Lemma 4.1, we thus conclude from above and (3.78) that
\[
\tilde{u}_0(x) = \gamma_0 w(x + x_0) \text{ in } \mathbb{R}^2, \quad 0 < \gamma \leq 1.
\]
We claim that \( \gamma_0 = 1 \) in (3.82). Actually, for any \( \delta > 0 \) one can choose \( R > 0 \) large that
\[
\int_{R \leq |x| < R + 1} |\bar{u}_{2k}|^2 dx \leq \delta^2 \quad \text{and} \quad 2 \int_{|x| \geq R} |w(x + x_0)|^4 dx \leq \delta^2.
\]
For above fixed \( R > 0 \), we then choose a cut-off function \( \varphi_R(x) \in C^2(\mathbb{R}^2) \) such that \( \varphi_R(x) \equiv 0 \) for \( |x| \leq R, \varphi_R(x) \in (0, 1] \) for \( R < |x| < R + 1 \) and \( \varphi_R(x) \equiv 1 \) for \( |x| \geq R + 1 \), where \( |\nabla \varphi_R(x)| \leq C \) holds for \( C > 0 \) independent of \( R \). Multiplying both sides of (3.79) by \( \varphi_R \bar{u}_{2k} \) and integrating over \( \{ x \in \mathbb{R}^2 : |x| \geq R \} \), it then follows that
\[
\int_{|x| \geq R} \varphi_R |\nabla \bar{u}_{2k}|^2 + \int_{R \leq |x| < R + 1} \bar{u}_{2k} \nabla \bar{u}_{2k} \nabla \varphi_R - \mu_k \varepsilon_k^2 \int_{|x| \geq R} \varphi_R \bar{u}_{2k}^2 \\
\leq \frac{\beta_k - a^*}{a^*(a^* - b)} \int_{\mathbb{R}^2} \bar{u}_{2k}^4 + \frac{\beta_k}{a^*} \left( \int_{|x| \geq R} \bar{u}_{1k}^2 \left( x + \frac{x_{2k} - x_{1k}}{\varepsilon_k} \right) \right)^\frac{1}{2} \left( \int_{\mathbb{R}^2} \bar{u}_{2k}^4 \right)^\frac{1}{2},
\]
where the Hölder inequality (3.8) is used. Since
\[
\bar{u}_{2k} \rightarrow \tilde{u}_0 \text{ strongly in } L^p_{\text{loc}}(\mathbb{R}^2) \text{ for any } 2 \leq p < \infty,
\]
it then follows from (3.78) and (3.83) that
\[
\int_{R \leq |x| < R + 1} \bar{u}_{2k} \nabla \bar{u}_{2k} \nabla \varphi_R dx \leq C \|
abla \bar{u}_{2k} \|_2 \left( \int_{R \leq |x| < R + 1} \bar{u}_{2k}^2 dx \right)^\frac{1}{2} \leq C \delta.
\]
We also derive from (3.13) and (3.83) that
\[
\int_{|x| \geq R} \bar{u}_{1k}^4 \left( x + \frac{x_{2k} - x_{1k}}{\varepsilon_k} \right) dx \leq 2 \int_{|x| \geq R} |w(x + x_0)|^4 dx \leq \delta^2.
\]
Since \( \beta_k \searrow a^* \) and \( \mu_k \varepsilon_k^2 \rightarrow -1 \) as \( k \rightarrow \infty \), it then yields from (3.84) and above that
\[
\int_{|x| \geq R + 1} \left( |\nabla \bar{u}_{2k}|^2 + |\bar{u}_{2k}|^2 \right) dx \leq C \delta \text{ if } k \text{ large enough},
\]
We then deduce from (3.78) that
\[ a = \mu \phi \]  
where \( \mu \) is the unique maximum point of \( L \). Therefore, we now conclude that \( \bar{u}_{2k} \) is the unique maximum point of \( u_{2k} \). Therefore, the solution set of (4.2) is the unique maximum point of \( u_{2k} \) about \( (a_k, b_k, \beta_k) \) satisfying (1.24). We first note that \( (u_0, v_0) = (w_0, w) \) is a positive solution of the following system
\[
\begin{align*}
\Delta u - u + u^3 &= 0 \text{ in } \mathbb{R}^2, \\
\Delta v - v + u^2v &= 0 \text{ in } \mathbb{R}^2,
\end{align*}
\]  
where \( w > 0 \) is a unique positive solution of (1.7). We claim that the positive solution \( (u_0, v_0) = (w_0, w) \) is non-degenerate, in the sense that the solution set of the linearized system for (1.1) about \( (u_0, v_0) \) satisfies
\[
\begin{align*}
\mathcal{L}_1(\phi_1) := \Delta \phi_1 - \phi_1 + 3u_0^2 \phi_1 &= 0 \text{ in } \mathbb{R}^2, \\
\mathcal{L}_2(\phi_1)\phi_2 := \Delta \phi_2 - \phi_2 + u_0^2 \phi_2 + 2u_0v_0\phi_1 &= 0 \text{ in } \mathbb{R}^2,
\end{align*}
\]  
for some constants \( b_j \), where \( j = 0, 1, 2 \). Actually, one can note from [37] Lemma 4.1 that the solution set of \( \mathcal{L}_1 \phi_1 = -\Delta \phi_1 + [1 - 3w^2] \phi_1 = 0 \) in \( \mathbb{R}^2 \) satisfies \( \phi_1 = \sum_{j=1}^2 b_j \frac{\partial w}{\partial x_j} \) for some constants \( b_j \), where \( j = 1, 2 \). Since it also follows from [37] Lemma 4.1 that the solution set of \( -\Delta \phi + (1 - w^2) \phi = 0 \) in \( \mathbb{R}^2 \) satisfies \( \phi = b_0w \) for some constant \( b_0 \), the claim (1.3) therefore follows.

For convenience, in the following we always suppose that \( (u_k, v_k) \) is a nonnegative minimizer of \( e(a_k, b_k) \) satisfying (1.24). Then \( (u_k, v_k) \) satisfies
\[
\begin{align*}
-\Delta u_k + V_1(x)u_k &= \mu_k u_k + a_k u_k^3 + \beta_k v_k^2 u_k \text{ in } \mathbb{R}^2, \\
-\Delta v_k + V_2(x)v_k &= \mu_k v_k + b_k v_k^3 + \beta_k u_k^2 v_k \text{ in } \mathbb{R}^2,
\end{align*}
\]  
where \( \mu_k \in \mathbb{R} \) is a suitable Lagrange multiplier and satisfies
\[
\mu_k = e(a_k, b_k) - \frac{a_k}{2} \int_{\mathbb{R}^2} u_k^4 - \frac{b_k}{2} \int_{\mathbb{R}^2} v_k^4 - \beta_k \int_{\mathbb{R}^2} u_k^2 v_k^2.
\]
Under the assumptions of Theorem 1.3 one can further check from (4.3) and previous sections that $\mu_k$ satisfies

$$\lim_{k \to \infty} \mu_k \varepsilon_k^2 = -1,$$

where $\varepsilon_k > 0$ is defined by

$$\varepsilon_k := \frac{1}{\lambda} \left[ (a^* - a_k)(a^* - b) - (\beta_k - a^*)^2 \right]^{\frac{1}{2} \pi}, \quad \lambda = \left[ \frac{\mu_1(a^* - b)}{2} H_1(y_0) \right]^{\frac{1}{2} \pi},$$

and $H_1(y_0) > 0$ is defined in (4.12).

4.1. Proof of Theorem 1.3. In this subsection we shall complete the proof of Theorem 1.3. Since $(a_k, b, \beta_k)$ satisfies (1.24), we then obtain from Proposition 3.2 that the nonnegative solution $(u_k, v_k)$ of $e(a_k, b, \beta_k)$ satisfies the limit behavior (3.13) as $k \to \infty$. It further follows from Theorem 1.1 that $v_k \not\equiv 0$ for sufficiently large $k > 0$. By Theorem 1.1 and Lemma 3.5 if $v_k(\varepsilon_k x + y_k) = C_\infty \sigma_k \bar{v}_k(x)$, where $\sigma_k = \|v_k\|_\infty > 0$, $C_\infty = \frac{1}{\|w\|_\infty} > 0$ and $y_k \in \mathbb{R}^2$ is a unique maximal point of $v_k$, then we have

$$\bar{v}_k(x) \to w(x)$$

uniformly in $\mathbb{R}^2$ as $k \to \infty$.

We now suppose that there exist two different nonnegative minimizers $(u_{1,k}, v_{1,k})$ and $(u_{2,k}, v_{2,k})$ of $e(a_k, b, \beta_k)$ satisfying (1.24). Let $(x_{1,k}, y_{1,k})$ and $(x_{2,k}, y_{2,k})$ be the unique maximum point of $(u_{1,k}, v_{1,k})$ and $(u_{2,k}, v_{2,k})$, respectively. Note from (4.3) that the nonnegative minimizer $(u_{i,k}, v_{i,k})$ solves the system

$$\begin{align*}
-\Delta u_{i,k} + V_1(x)u_{i,k} &= \mu_{i,k}u_{i,k} + a_k u_{i,k}^2 + \beta_k v_{i,k}^2 u_{i,k} \quad \text{in } \mathbb{R}^2, \\
-\Delta v_{i,k} + V_2(x)v_{i,k} &= \mu_{i,k}v_{i,k} + b v_{i,k}^2 + \beta_k u_{i,k}^2 v_{i,k} \quad \text{in } \mathbb{R}^2,
\end{align*}$$

where $\mu_{i,k} \in \mathbb{R}$ is a suitable Lagrange multiplier satisfying (4.5) and $\lambda$ with $\mu_k = \mu_{i,k}$ for $i = 1, 2$. Motivated by (4.8), we define

$$u_{i,k}(x) = \sqrt{\varepsilon_k} u_{\varepsilon_k}(x + 2x_2,k) \quad \text{and} \quad v_{i,k}(x) = \sqrt{\varepsilon_k} v_{\varepsilon_k}(x + 2x_2,k),$$

where $i = 1, 2$, $\varepsilon_k = \|v_{2,k}\|_\infty > 0$ and $C_\infty = \frac{1}{\|w\|_\infty} > 0$. By Theorem 1.1 we then obtain from (3.13), (3.35) and (4.8) that

$$\left( u_{i,k}(x), v_{i,k}(x) \right) \to (u_0, v_0) \equiv (w, w)$$

uniformly in $\mathbb{R}^2$ as $k \to \infty$, where $(u_0, v_0) = (w, w)$ is a positive solution of the system (4.1), and $(\bar{u}_{i,k}(x), \bar{v}_{i,k}(x))$ satisfies the system

$$\begin{align*}
-\Delta \bar{u}_{i,k} + \varepsilon_k^2 V_1(x) \bar{u}_{i,k} &= \mu_{i,k} \varepsilon_k^2 \bar{u}_{i,k} + \beta_k \sigma_\infty^2 \varepsilon_k^2 \bar{v}_{i,k}^2 \bar{u}_{i,k} \quad \text{in } \mathbb{R}^2, \\
-\Delta \bar{v}_{i,k} + \varepsilon_k^2 V_2(x) \bar{v}_{i,k} &= \mu_{i,k} \varepsilon_k^2 \bar{v}_{i,k} + b \sigma_\infty^2 \varepsilon_k^2 \bar{v}_{i,k}^3 \bar{v}_{i,k} \quad \text{in } \mathbb{R}^2,
\end{align*}$$

which completes the proof of Theorem 1.3.
One can check that \( \varepsilon_k^2 (\mu_{2,k} - \mu_{1,k}) \) satisfies

\[
\varepsilon_k^2 (\mu_{2,k} - \mu_{1,k}) = -\frac{a_k}{2} \varepsilon_k^2 \int_{\mathbb{R}^2} (u_{2,k}^4 - u_{1,k}^4) - \frac{b}{2} \varepsilon_k^2 \int_{\mathbb{R}^2} (v_{2,k}^2 - v_{1,k}^2)
- \beta \varepsilon_k^2 \int_{\mathbb{R}^2} [v_{2,k}^2 (u_{2,k}^2 - u_{1,k}^2) + u_{1,k}^2 (v_{2,k}^2 - v_{1,k}^2)]
\]

(4.13)

\[
= -\frac{a_k}{2(\alpha)^2} \int_{\mathbb{R}^2} (\tilde{u}_{2,k}^2 - \tilde{u}_{1,k}^2) - \frac{b}{2} C^\infty \sigma_k^2 \int_{\mathbb{R}^2} (\tilde{v}_{2,k}^2 - \tilde{v}_{1,k}^2)
- \beta_k \frac{C^2 \sigma_k^2 \varepsilon_k^2}{a^2} \int_{\mathbb{R}^2} [\tilde{v}_{2,k}^2 (\tilde{u}_{2,k}^2 - \tilde{u}_{1,k}^2) + \tilde{u}_{1,k}^2 (\tilde{v}_{2,k}^2 - \tilde{v}_{1,k}^2)],
\]

where we have used (4.9). Recall also from Proposition 3.2 that both \( \tilde{u}_{i,k}(x) \) and \( \nabla \tilde{u}_{i,k}(x) \) decay exponentially as \( |x| \to \infty \) for \( i = 1 \) and 2. Further, one can derive from (4.12) that both \( \tilde{v}_{i,k}(x) \) and \( \nabla \tilde{v}_{i,k}(x) \) also admit the similar exponential decay as \( |x| \to \infty \) for \( i = 1 \) and 2.

We also define

\[
(\hat{u}_{i,k}(\varepsilon_k x + x_{2,k}), \tilde{v}_{i,k}(\varepsilon_k x + x_{2,k})) := (\tilde{u}_{i,k}(x), \tilde{v}_{i,k}(x)), \quad \text{where } i = 1, 2,
\]

so that

\[
(\hat{u}_{i,k}(\varepsilon_k x + x_{2,k}), \tilde{v}_{i,k}(\varepsilon_k x + x_{2,k})) \to (u_0, v_0) \equiv (w, w)
\]

uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \) by (4.11). Note that \( (\hat{u}_{i,k}(x), \tilde{v}_{i,k}(x)) \) satisfies the system

\[
\begin{aligned}
egvarepsilon_k^2 \Delta \hat{u}_{i,k} + \varepsilon_k^2 V_1(x) \hat{u}_{i,k} \\
= \mu_{i,k} \varepsilon_k^2 \hat{u}_{i,k} + \frac{a_k}{\alpha^2} \hat{u}_{i,k}^3 + \beta_k \varepsilon_k^2 \sigma_k^2 \hat{v}_{i,k}^2 \hat{u}_{i,k} \quad \text{in } \mathbb{R}^2,

-\varepsilon_k^2 \Delta \tilde{v}_{i,k} + \varepsilon_k^2 V_2(x) \tilde{v}_{i,k} \\
= \mu_{i,k} \varepsilon_k^2 \tilde{v}_{i,k} + b C^\infty \sigma_k^2 \tilde{v}_{i,k}^3 \hat{v}_{i,k} + \frac{\beta_k}{a^2} \hat{u}_{i,k}^2 \tilde{v}_{i,k} \quad \text{in } \mathbb{R}^2.
\end{aligned}
\]

(4.16)

Since \( (u_{1,k}, v_{1,k}) \neq (u_{2,k}, v_{2,k}) \), we define

\[
\hat{\xi}_{1,k}(x) = \frac{\hat{u}_{2,k}(x) - \hat{u}_{1,k}(x)}{\|\hat{u}_{2,k} - \hat{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\varepsilon_k} \|\tilde{v}_{2,k} - \tilde{v}_{1,k}\|_{L^2(\mathbb{R}^2)}},
\]

\[
\hat{\xi}_{2,k}(x) = \frac{\hat{v}_{2,k}(x) - \hat{v}_{1,k}(x)}{\|\hat{u}_{2,k} - \hat{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\varepsilon_k} \|\tilde{v}_{2,k} - \tilde{v}_{1,k}\|_{L^2(\mathbb{R}^2)}},
\]

(4.17)

which is different from those used in [4]. We then have the following local estimates of \( (\hat{\xi}_{1,k}, \hat{\xi}_{2,k}) \).

**Lemma 4.1.** Assume that \( (a_k, b, \beta_k) \) satisfies (4.24). Then for any \( x_0 \in \mathbb{R}^2 \), there exists a small constant \( \delta > 0 \) such that

\[
\int_{\partial B_\delta(x_0)} (\varepsilon_k^2 \nabla \hat{\xi}_{i,k}^2 + \frac{1}{\varepsilon_k} \varepsilon_k^2 V_i(x) \hat{\xi}_{i,k}^2) dS = O(\varepsilon_k^2) \quad \text{as } k \to \infty, \quad i = 1, 2.
\]

(4.18)

The proof of Lemma 4.1 is given in Appendix A. Associated to \( (\hat{\xi}_{1,k}, \hat{\xi}_{2,k}) \), it is also convenient to define

\[
\xi_{1,k}(x) = \frac{\hat{u}_{2,k}(x) - \hat{u}_{1,k}(x)}{\|\hat{u}_{2,k} - \hat{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \|\tilde{v}_{2,k} - \tilde{v}_{1,k}\|_{L^2(\mathbb{R}^2)}},
\]

\[
\xi_{2,k}(x) = \frac{\hat{v}_{2,k}(x) - \hat{v}_{1,k}(x)}{\|\hat{u}_{2,k} - \hat{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \|\tilde{v}_{2,k} - \tilde{v}_{1,k}\|_{L^2(\mathbb{R}^2)}},
\]

(4.19)
so that

\[ \xi_{i,k}(x) = \xi_{i,k}(\epsilon_k x + x_{2,k}), \quad \text{where } i = 1, 2. \]

In the following we shall complete the proof of Theorem 1.3 by considering separately three different cases.

(1). We now consider the first case where \( u_{2,k} \neq u_{1,k} \) and \( v_{2,k} \neq v_{1,k} \) in \( \mathbb{R}^2 \), for which we shall continue the proof of Theorem 1.3 by the following six steps:

**Step 1.** There exists a subsequence (still denoted by \( \{a_k\} \)) of \( \{a_k\} \) such that

\[ (\xi_{1,k}, \xi_{2,k}) \to (\xi_{10}, \xi_{20}) \text{ in } C_{loc}(\mathbb{R}^2) \text{ in } k \to \infty, \]

where \( (\xi_{10}, \xi_{20}) \) satisfies

\[ \begin{pmatrix} \xi_{10} \\ \xi_{20} \end{pmatrix} = b_0 \begin{pmatrix} 0 \\ w \end{pmatrix} + \sum_{j=1}^{2} b_j \left( \frac{\partial w}{\partial x_j} \right) + c_0 \begin{pmatrix} w + x \cdot \nabla w \\ w + x \cdot \nabla w \end{pmatrix} \]

for some constants \( c_0 \) and \( b_j \) with \( j = 0, 1, 2 \).

Following (4.12), one can check from (4.19) that \( (\xi_{1,k}, \xi_{2,k}) \) satisfies

\[ \begin{cases} \Delta \xi_{1,k} - \epsilon_k^2 V_1(\epsilon_k x + x_{2,k}) \xi_{1,k} + \mu_{2,k} \epsilon_k^2 \xi_{1,k} \\ + \frac{a_k}{a} (\bar{u}_{2,k} + \bar{v}_{2,k} \bar{u}_{1,k} + \bar{u}_{1,k}^2) \xi_{1,k} \\ + \beta_k C_\infty^2 \sigma_k^2 \epsilon_k^2 [\bar{v}_{1,k}^2 \xi_{1,k} + \bar{u}_{2,k} (\bar{v}_{2,k} + \bar{v}_{1,k}) \xi_{2,k}] = c_k \bar{u}_{1,k} \text{ in } \mathbb{R}^2, \\ \Delta \xi_{2,k} - \epsilon_k^2 V_2(\epsilon_k x + x_{2,k}) \xi_{2,k} + \mu_{2,k} \epsilon_k^2 \xi_{2,k} \\ + b C_\infty^2 \sigma_k^2 \epsilon_k^2 [\bar{v}_{2,k} + \bar{v}_{2,k} \bar{v}_{1,k} + \bar{v}_{1,k}^2] \xi_{2,k} \\ + \frac{\beta_k}{a} (\bar{u}_{1,k}^2 \xi_{2,k} + \bar{v}_{2,k} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_{1,k}) = c_k \bar{v}_{1,k} \text{ in } \mathbb{R}^2. \end{cases} \]

Using (4.13), the coefficient \( c_k \) satisfies

\[ c_k := -\frac{\epsilon_k^2 (\mu_{2,k} - \mu_{1,k})}{\| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\mathbb{R}^2)} + \| \bar{v}_{2,k} - \bar{v}_{1,k} \|_{L^2(\mathbb{R}^2)}} = \frac{a_k}{2(a^*)^2} \int_{\mathbb{R}^2} (\bar{u}_{2,k}^2 + \bar{u}_{1,k}^2) (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_{1,k} \\ + b \frac{C_\infty^2 \sigma_k^2 \epsilon_k}{2} \int_{\mathbb{R}^2} [\bar{v}_{2,k}^2 + \bar{v}_{1,k}^2] (\bar{v}_{2,k} + \bar{v}_{1,k}) \xi_{2,k} \\ + \frac{\beta_k}{a} C_\infty^2 \sigma_k^2 \epsilon_k [\bar{u}_{1,k}^2 \xi_{2,k} + \bar{v}_{2,k} (\bar{u}_{1,k} + \bar{v}_{1,k}) \xi_{1,k} + \bar{u}_{1,k}^2 (\bar{v}_{2,k} + \bar{v}_{1,k}) \xi_{2,k}]. \]

Since \( \xi_{1,k} \) is bounded uniformly in \( \mathbb{R}^2 \) and \( \xi_{2,k} \) is bounded uniformly in \( L^2(\mathbb{R}^2) \), the standard elliptic regularity theory (cf. [11 Corollary 7.11]) then implies from (4.23) that \( \| \xi_{1,k} \|_{C_{loc}^2(\mathbb{R}^2)} \leq C \) for some \( \alpha \in (0, 1) \), where the constant \( C > 0 \) is independent of \( k \). Therefore, up to a subsequence if necessary, we have \( (\xi_{1,k}, \xi_{2,k}) \to (\xi_{10}, \xi_{20}) \) in \( C_{loc}(\mathbb{R}^2) \) as \( k \to \infty \), where the vector function \( (\xi_{10}, \xi_{20}) \) satisfies

\[ \begin{cases} \Delta \xi_{10} - \epsilon_{10}^2 \xi_{10} + 3u_0^2 \xi_{10} = \frac{2}{a^*} \left( \int_{\mathbb{R}^2} u_0^3 \xi_{10} \right) w \text{ in } \mathbb{R}^2, \\ \Delta \xi_{20} - \epsilon_{10}^2 \xi_{20} + u_0^2 \xi_{20} + 2u_0 \xi_{10} = \frac{2}{a^*} \left( \int_{\mathbb{R}^2} u_0^3 \xi_{10} \right) w \text{ in } \mathbb{R}^2, \end{cases} \]
and \((u_0, v_0) = (w, w)\). We then obtain from \([4.24]\) that there exist constants \(b_1, b_2\) and \(c_0\) such that

\[
\xi_{10} = b_1 \frac{\partial w}{\partial x_1} + b_2 \frac{\partial w}{\partial x_2} + c_0 (w + x \cdot \nabla w), \quad c_0 = \frac{1}{a^s} \int_{\mathbb{R}^2} w^2 \xi_{10}.
\]

We thus derive from \([4.3]\) and \([4.24]\) that \((\xi_{10}, \xi_{20})\) satisfies \([4.21]\) for some constants \(c_0\) and \(b_j\) with \(j = 0, 1, 2\), and Step 1 is thus established.

**Step 2.** We claim that if \(\delta > 0\) is small, we then have the following Pohozaev-type identities

\[
o(\varepsilon^4 \frac{\partial}{\partial x}) = \varepsilon^{3 + p_1} \int_{B_\delta(x_{2,k})} \frac{\partial V_1(x + \frac{x_{2,k}}{\varepsilon_k})}{\partial x_j} (\bar{u}_{2,k} + \bar{u}, k) \xi_{1,k} + \varepsilon^{3 + p_2} (a^* C^2_\delta \varepsilon^2_k) \int_{B_\delta(x_{2,k})} \frac{\partial V_2(x + \frac{x_{2,k}}{\varepsilon_k})}{\partial x_j} (\bar{v}_{2,k} + \bar{v}, k) \xi_{2,k} dx
\]

as \(k \to \infty\), where \(j = 1, 2\).

To prove the above claim, multiply the first equation of \([4.10]\) by \(\frac{\partial \bar{u}_{i,k}}{\partial x_j}\), where \(i, j = 1, 2\), and integrate over \(B_\delta(x_{2,k})\), where \(\delta > 0\) is small and given by \([4.18]\). It then gives that

\[
-\varepsilon_k \int_{B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial x_j} \Delta \bar{u}_{i,k} + \varepsilon_k \int_{B_\delta(x_{2,k})} V_1(x) \frac{\partial \bar{u}_{i,k}}{\partial x_j} \bar{u}_{i,k} = \mu_{i,k} \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial x_j} \bar{u}_{i,k} \bar{v}_{i,k}^2 + \frac{\alpha_k}{a^*} \int_{B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial x_j} \bar{u}_{i,k} \nu_j dS + \frac{1}{2} \beta_{i,k} C^2_\delta \varepsilon^2_k \int_{B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial x_j} \bar{u}_{i,k} \bar{v}_{i,k}^2,
\]

where \(\nu = (\nu_1, \nu_2)\) denotes the outward unit normal of \(\partial B_\delta(x_{2,k})\). Note that

\[
-\varepsilon_k \int_{B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial x_j} \Delta \bar{u}_{i,k} = -\varepsilon_k \int_{\partial B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial x_j} \bar{u}_{i,k} dS + \varepsilon_k \int_{B_\delta(x_{2,k})} \nabla \bar{u}_{i,k} \cdot \nabla \frac{\partial \bar{u}_{i,k}}{\partial x_j} dS
\]

and

\[
\varepsilon_k^2 \int_{B_\delta(x_{2,k})} V_1(x) \frac{\partial \bar{u}_{i,k}}{\partial x_j} \bar{u}_{i,k} = \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} V_1(x) \bar{u}_{i,k}^2 \nu_j dS - \frac{\varepsilon_k^2}{2} \int_{B_\delta(x_{2,k})} \frac{\partial V_1(x)}{\partial x_j} \bar{u}_{i,k}^2.
\]
Similarly, we derive from the second equation of (4.16) that

\[
\begin{align*}
\varepsilon^2 & \int_{B_k(x_{2,k})} \frac{\partial V_1(x)}{\partial x_j} \hat{u}_{i,k}^2 + \beta_k C^2 \sigma_k^2 \varepsilon_k^2 \int_{B_k(x_{2,k})} \frac{\partial \hat{u}_{i,k}^2}{\partial x_j} \hat{v}_{i,k}^2 \\
& = -2\varepsilon^2 k \int_{\partial B_k(x_{2,k})} \frac{\partial \hat{u}_{i,k}}{\partial x_j} \frac{\partial \hat{u}_{i,k}}{\partial \nu} dS + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} |\nabla \hat{u}_{i,k}|^2 \nu_j dS \\
& + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} V_1(x) \hat{u}_{i,k}^2 \nu_j dS - \mu_k \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \hat{u}_{i,k}^2 \nu_j dS \\
& - \frac{a_k}{2a^*} \int_{\partial B_k(x_{2,k})} \hat{u}_{i,k}^4 \nu_j dS := C_i.
\end{align*}
\]

(4.27)

Similarly, we derive from the second equation of (4.16) that

\[
\begin{align*}
\varepsilon^2 & \int_{B_k(x_{2,k})} \frac{\partial V_2(x)}{\partial x_j} \hat{v}_{i,k}^2 + \frac{\beta_k}{a^*} \int_{B_k(x_{2,k})} \frac{\partial \hat{v}_{i,k}^2}{\partial x_j} \hat{u}_{i,k}^2 \\
& = -2\varepsilon^2 k \int_{\partial B_k(x_{2,k})} \frac{\partial \hat{v}_{i,k}}{\partial x_j} \frac{\partial \hat{v}_{i,k}}{\partial \nu} dS + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} |\nabla \hat{v}_{i,k}|^2 \nu_j dS \\
& + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} V_2(x) \hat{v}_{i,k}^2 \nu_j dS - \mu_k \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^2 \nu_j dS \\
& - \frac{b}{2} C^2 \sigma_k^2 \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^4 \nu_j dS := D_i.
\end{align*}
\]

(4.28)

which then implies that

\[
\begin{align*}
\varepsilon^2 & \int_{B_k(x_{2,k})} \frac{\partial V_2(x)}{\partial x_j} \hat{u}_{i,k}^2 - \frac{\beta_k}{a^*} \int_{B_k(x_{2,k})} \frac{\partial \hat{u}_{i,k}^2}{\partial x_j} \hat{v}_{i,k}^2 \\
& = -2\varepsilon^2 k \int_{\partial B_k(x_{2,k})} \frac{\partial \hat{v}_{i,k}}{\partial x_j} \frac{\partial \hat{v}_{i,k}}{\partial \nu} dS + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} |\nabla \hat{v}_{i,k}|^2 \nu_j dS \\
& + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} V_2(x) \hat{v}_{i,k}^2 \nu_j dS - \mu_k \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^2 \nu_j dS \\
& - \frac{b}{2} C^2 \sigma_k^2 \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^4 \nu_j dS - \frac{\beta_k}{a^*} \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^4 \nu_j dS := D_i.
\end{align*}
\]

(4.29)

Following (4.27) and (4.28), we thus have

\[
\begin{align*}
\varepsilon^2 & \int_{B_k(x_{2,k})} \frac{\partial V_1(x)}{\partial x_j} (\hat{u}_{2,k} + \hat{u}_{1,k}) \hat{\xi}_{2,k} \\
& = -\varepsilon^2 \left( a^* C^2 \sigma_k^2 \varepsilon_k^2 \right) \int_{B_k(x_{2,k})} \frac{\partial V_2(x)}{\partial x_j} (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \\
& + \mathcal{I}_k^u + (a^* C^2 \sigma_k^2 \varepsilon_k^2) \mathcal{I}_k^v,
\end{align*}
\]

(4.29)
where we denote
\[
T_k^u = -2\epsilon_k^2 \int_{\partial B_k(x_{2,k})} \left[ \frac{\partial \hat{u}_{2,k}}{\partial x_j} \frac{\partial \xi_{1,k}}{\partial \nu} + \frac{\partial \hat{\xi}_{1,k}}{\partial x_j} \frac{\partial \hat{u}_{1,k}}{\partial \nu} \right] dS \\
+ \hat{c}_k \int_{\partial B_k(x_{2,k})} \nabla \xi_{1,k} \cdot \nabla (\hat{u}_{2,k} + \hat{u}_{1,k}) \nu_j dS + \hat{c}_k \int_{\partial B_k(x_{2,k})} V_1(x) (\hat{u}_{2,k} + \hat{u}_{1,k}) \xi_{1,k} \nu_j dS \\
- \hat{c}_k \int_{\partial B_k(x_{2,k})} \hat{u}_{2,k}^2 \nu_j dS - \mu_1 \epsilon_k \int_{\partial B_k(x_{2,k})} (\hat{u}_{2,k} + \hat{u}_{1,k}) \hat{\xi}_{2,k} \nu_j dS \\
- \frac{a_k}{2 a^*} \int_{\partial B_k(x_{2,k})} (\hat{u}_{2,k}^2 + \hat{u}_{1,k}^2) \nu_j dS,
\]
and
\[
T_k^v = -2\epsilon_k^2 \int_{\partial B_k(x_{2,k})} \left[ \frac{\partial \hat{v}_{2,k}}{\partial x_j} \frac{\partial \xi_{2,k}}{\partial \nu} + \frac{\partial \hat{\xi}_{2,k}}{\partial x_j} \frac{\partial \hat{v}_{1,k}}{\partial \nu} \right] dS \\
+ \hat{c}_k \int_{\partial B_k(x_{2,k})} \nabla \xi_{2,k} \cdot \nabla (\hat{v}_{2,k} + \hat{v}_{1,k}) \nu_j dS + \hat{c}_k \int_{\partial B_k(x_{2,k})} V_2(x) (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \nu_j dS \\
- \hat{c}_k \int_{\partial B_k(x_{2,k})} \hat{v}_{2,k}^2 \nu_j dS - \mu_1 \epsilon_k \int_{\partial B_k(x_{2,k})} (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \nu_j dS \\
- \frac{b}{2} C_\infty^2 \sigma_k^2 \epsilon_k \int_{\partial B_k(x_{2,k})} \left( \hat{v}_{2,k}^2 + \hat{v}_{1,k}^2 \right) \nu_j dS \\
- \frac{\beta_k}{a^*} \int_{\partial B_k(x_{2,k})} \left[ \hat{u}_{2,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{1,k} + \hat{v}_{1,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \right] \nu_j dS.
\]

Here the coefficient \( \hat{c}_k \) is defined by
\[
\hat{c}_k = \frac{-\epsilon_k^2 (\mu_{2,k} - \mu_{1,k})}{\| \hat{u}_{2,k} - \hat{u}_{1,k} \|_{L^\infty(\mathbb{R}^2)} + \frac{\epsilon_k}{\epsilon_k} \| \hat{v}_{2,k} - \hat{v}_{1,k} \|_{L^2(\mathbb{R}^2)}} \\
= \frac{a_k}{2 (a^*)^2 \epsilon_k} \int_{\mathbb{R}^2} (\hat{u}_{2,k}^2 + \hat{u}_{1,k}^2) (\hat{u}_{2,k} + \hat{u}_{1,k}) \hat{\xi}_{1,k} \\
+ \frac{b}{2} C_\infty^2 \sigma_k^2 \epsilon_k \int_{\mathbb{R}^2} (\hat{v}_{2,k}^2 + \hat{v}_{1,k}^2) (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \\
+ \frac{\beta_k}{a^*} C_\infty^2 \sigma_k^2 \epsilon_k \int_{\mathbb{R}^2} \left[ \hat{u}_{2,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{1,k} + \hat{v}_{1,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \right],
\]
due to (4.13). Since (4.17) gives that
\[
\| \xi_{1,k} \|_\infty \leq 1 \quad \text{and} \quad \int_{\mathbb{R}^2} |\hat{\xi}_{2,k}|^2 \leq \epsilon_k^2,
\]
we have
\[
\left| \int_{\mathbb{R}^2} \hat{u}_{1,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k} \right| \leq \left( \int_{\mathbb{R}^2} \hat{u}_{1,k}^4 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k}^2 \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^2} |\hat{\xi}_{2,k}|^2 \right)^{\frac{1}{2}} \leq C \epsilon_k.
\]
The above argument then yields that there exists a constant \( C > 0 \) such that
\[
|\hat{c}_k| \leq C \quad \text{uniformly in } k.
\]
Applying Lemma 4.1 if \( \delta > 0 \) is small, we then deduce that
\[
\varepsilon_k^2 \int_{\partial B_i(x_{2,k})} \left| \frac{\partial \hat{u}_{2,k}}{\partial x_j} \frac{\partial \hat{\xi}_{1,k}}{\partial \nu} \right| dS \\
\leq \varepsilon_k \left( \int_{\partial B_i(x_{2,k})} \left| \frac{\partial \hat{u}_{2,k}}{\partial x_j} \right|^2 dS \right)^\frac{1}{2} \left( \varepsilon_k^2 \int_{\partial B_i(x_{2,k})} \left| \frac{\partial \hat{\xi}_{1,k}}{\partial \nu} \right|^2 dS \right)^\frac{1}{2} \leq C \varepsilon_k^2 e^{-\frac{C\varepsilon_k}{k}} \quad \text{as} \quad k \to \infty,
\]
due to the fact that \( \nabla \hat{u}_{2,k}(\varepsilon_0 x + x_{2,k}) \) decays exponentially as mentioned soon after (4.13), where \( C > 0 \) is independent of \( k \). Similarly, we have
\[
\varepsilon_k^2 \int_{\partial B_i(x_{2,k})} \left| \frac{\partial \hat{\xi}_{1,k}}{\partial x_j} \frac{\partial \hat{u}_{1,k}}{\partial \nu} \right| dS \leq C \varepsilon_k^2 e^{-\frac{C\varepsilon_k}{k}} \quad \text{as} \quad k \to \infty,
\]
and
\[
\varepsilon_k^2 \left| \int_{\partial B_i(x_{2,k})} \nabla \hat{\xi}_{1,k} \cdot \nabla (\hat{u}_{2,k} + \hat{u}_{1,k}) \nu_j dS \right| \leq C \varepsilon_k^2 e^{-\frac{C\varepsilon_k}{k}} \quad \text{as} \quad k \to \infty.
\]
On the other hand, we also get that
\[
\left| \varepsilon_k^2 \int_{\partial B_i(x_{2,k})} V_i(x)(\hat{u}_{2,k} + \hat{u}_{1,k}) \hat{\xi}_{1,k} \nu_j dS \right| + \left| \int_{\partial B_i(x_{2,k})} (\hat{u}_{2,k} + \hat{u}_{1,k}) \hat{\xi}_{1,k} \nu_j dS \right| \\
+ \left| \int_{\partial B_i(x_{2,k})} (\hat{u}_{2,k} + \hat{u}_{1,k})(\hat{u}_{2,k} + \hat{u}_{1,k}) \hat{\xi}_{1,k} \nu_j dS \right| + \left| \int_{\partial B_i(x_{2,k})} \hat{u}_{2,k} \nu_j dS \right| \\
= o(e^{-\frac{C \varepsilon_k}{k}}) \quad \text{as} \quad k \to \infty,
\]
where the exponential decay of \( \hat{u}_{i,k} \) is also used. We thus conclude from above that
\[
(4.32) \quad \mathcal{I}_i^{\nu} + (a^* C^2 \sigma_{i,k}^2 \varepsilon_k^2) \mathcal{I}_k^{\nu} = o(e^{-\frac{C \varepsilon_k}{k}}) \quad \text{as} \quad k \to \infty,
\]
where \( C > 0 \) is independent of \( k \). It now follows from (4.29) and (4.32) that the claim (4.25) holds for \( j = 1, 2 \).

Step 3. The constants \( b_1 = b_2 = c_0 = 0 \) in (4.21), i.e., \( \xi_{10} = 0 \) and \( \xi_{20} = b_0 w \) for some constant \( b_0 \).

Using the integration by parts, we first note that
\[
-\varepsilon_k^2 \int_{B_i(x_{2,k})} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] \Delta \hat{u}_{i,k} \\
= -\varepsilon_k^2 \int_{\partial B_i(x_{2,k})} \frac{\partial \hat{u}_{i,k}}{\partial \nu} (x - x_{2,k}) \cdot \nabla \hat{u}_{i,k} \\
+ \varepsilon_k^2 \int_{B_i(x_{2,k})} \nabla \hat{u}_{i,k} \nabla [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] \\
= -\varepsilon_k^2 \int_{\partial B_i(x_{2,k})} \frac{\partial \hat{u}_{i,k}}{\partial \nu} (x - x_{2,k}) \cdot \nabla \hat{u}_{i,k} \\
+ \frac{\varepsilon_k^2}{2} \int_{\partial B_i(x_{2,k})} [(x - x_{2,k}) \cdot \nu] |\nabla \hat{u}_{i,k}|^2.
\]
Multiplying the first equation of (4.16) by \((x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}\), where \(i = 1, 2\), and integrating over \(B_{\delta}(x_{2,k})\), where \(\delta > 0\) is small as before, we deduce that for \(i = 1, 2\),

\[
-\varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] \Delta \hat{u}_{i,k} + \mu_{i,k} \int_{B_{\delta}(x_{2,k})} [V_{i}(x) - V_{1}(x)] \hat{u}_{i,k} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] + \frac{\alpha_{k}}{2a_{*}} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}] + \frac{\beta_{k}C_{\infty}^{2} \varepsilon_{k}^{2} \varepsilon_{k}^{2}}{2} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}]
\]

\[
= -\varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] \Delta \hat{u}_{i,k} + \mu_{i,k} \int_{B_{\delta}(x_{2,k})} [V_{i}(x) - V_{1}(x)] \hat{u}_{i,k} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] + \frac{\alpha_{k}}{2a_{*}} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}]
\]

\[
= \frac{\varepsilon_{k}^{2}}{2} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} \left\{ 2 [\mu_{i,k} - V_{1}(x)] - (x - x_{2,k}) \cdot \nabla V_{1}(x) \right\}
\]

\[
+ \frac{\varepsilon_{k}^{2}}{2} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot (x - x_{2,k})] \nu dS
\]

\[
- \frac{\alpha_{k}}{2a_{*}} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{4} + \frac{\alpha_{k}}{4a_{*}} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{4} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}] - \frac{\beta_{k}C_{\infty}^{2} \varepsilon_{k}^{2} \varepsilon_{k}^{2}}{2} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}]
\]

\[
= \frac{\varepsilon_{k}^{2}}{2} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} \left\{ 2 [\mu_{i,k} - V_{1}(x)] - (x - x_{2,k}) \cdot \nabla V_{1}(x) \right\}
\]

\[
+ \frac{\varepsilon_{k}^{2}}{2} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot (x - x_{2,k})] \nu dS
\]

Since \(x \cdot \nabla V_{1}(x) = p_{1}V_{1}(x)\), this yields that

\[
-\varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}] \Delta \hat{u}_{i,k} + \mu_{i,k} \int_{B_{\delta}(x_{2,k})} V_{i}(x) \hat{u}_{i,k}^{2} - \varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} [x_{2,k} \cdot \nabla V_{1}(x)] \hat{u}_{i,k}^{2}
\]

\[
(4.34)
\]

where the lower order term \(I_{i}\) satisfies

\[
I_{i} = \mu_{i,k} \varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} - \varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} V_{i}(x) \hat{u}_{i,k}^{2}
\]

\[
+ \frac{\alpha_{k}}{2a_{*}} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{4} + \frac{1}{2} \varepsilon_{k}^{2} \int_{B_{\delta}(x_{2,k})} [x_{2,k} \cdot \nabla V_{1}(x)] \hat{u}_{i,k}^{2}
\]

\[
+ \frac{\varepsilon_{k}^{2}}{2} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot (x - x_{2,k})] \nu dS
\]

\[
= \frac{\varepsilon_{k}^{2}}{2} \int_{B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} \left\{ 2 [\mu_{i,k} - V_{1}(x)] - (x - x_{2,k}) \cdot \nabla V_{1}(x) \right\}
\]

\[
+ \frac{\varepsilon_{k}^{2}}{2} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot (x - x_{2,k})] \nu dS + \frac{\alpha_{k}}{4a_{*}} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{4} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}] + \frac{\beta_{k}C_{\infty}^{2} \varepsilon_{k}^{2} \varepsilon_{k}^{2}}{2} \int_{\partial B_{\delta}(x_{2,k})} \hat{u}_{i,k}^{2} [(x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}^{2}]
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Similarly, we have
\begin{equation}
-\varepsilon_k^2 \int_{B_k(x_{2,k})} \left[(x - x_{2,k}) \cdot \nabla \hat{v}_{i,k}\right] \Delta \hat{v}_{i,k} \tag{4.36}
= -\varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \frac{\partial \hat{v}_{i,k}}{\partial \nu} (x - x_{2,k}) \cdot \nabla \hat{v}_{i,k}
+ \frac{\varepsilon_k^2}{2} \int_{\partial B_k(x_{2,k})} \left[(x - x_{2,k}) \cdot \nu\right] |\nabla \hat{v}_{i,k}|^2,
\end{equation}
and the second equation of (4.10) yields that
\begin{equation}
-\varepsilon_k^2 \int_{B_k(x_{2,k})} \left[(x - x_{2,k}) \cdot \nabla \hat{v}_{i,k}\right] \Delta \hat{v}_{i,k} \tag{4.37}
= -\mu_{i,k} \varepsilon_k^2 \int_{\mathbb{R}^4} \hat{v}_{i,k}^2 + \frac{2 + p_2}{2} \varepsilon_k^2 \int_{\mathbb{R}^4} V_2(x) \hat{v}_{i,k}^2 - \frac{\varepsilon_k^2}{2} \int_{\mathbb{R}^4} \left[x_{2,k} \cdot \nabla V_2(x)\right] \hat{v}_{i,k}^2
- \frac{b}{2} \gamma \sigma_k^2 \varepsilon_k^2 \int_{\mathbb{R}^4} \hat{v}_{i,k}^4 + \frac{\beta_k}{2a^*} \int_{B_k(x_{2,k})} \hat{u}_{i,k}^2 \left[(x - x_{2,k}) \cdot \nabla \hat{v}_{i,k}^2\right] + II_i,
\end{equation}
where the lower order term $II_i$ satisfies
\begin{equation}
II_i = \mu_{i,k} \varepsilon_k^2 \int_{\mathbb{R}^4 \setminus B_k(x_{2,k})} \hat{v}_{i,k}^2
+ \frac{b}{2} C^2 \sigma_k^2 \varepsilon_k^2 \int_{\mathbb{R}^4 \setminus B_k(x_{2,k})} \hat{v}_{i,k}^4
+ \frac{\varepsilon_k^2}{2} \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^2 \left[\mu_{i,k} - V_2(x)\right] (x - x_{2,k}) \nu dS
+ \frac{b}{4} C^2 \sigma_k^2 \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \hat{v}_{i,k}^4 (x - x_{2,k}) \nu dS, \quad i = 1, 2. \tag{4.38}
\end{equation}

Since it follows from (11.5) and (11.10) that
\begin{equation}
a^* \varepsilon_k^2 e(a_k, b, \beta_k) = \mu_{i,k} \varepsilon_k^2 \left[\int_{\mathbb{R}^4} \hat{u}_{i,k}^2 + a^* C^2 \sigma_k^2 \varepsilon_k^2 \int_{\mathbb{R}^4} \hat{v}_{i,k}^2\right] + \frac{a_k}{2a^*} \int_{\mathbb{R}^4} \hat{u}_{i,k}^4
+ \frac{b}{2} C^2 \sigma_k^2 \varepsilon_k^2 \int_{\mathbb{R}^4} \hat{v}_{i,k}^4 + \frac{\beta_k}{2a^*} C^2 \sigma_k^2 \varepsilon_k^2 \int_{\mathbb{R}^4} \hat{u}_{i,k}^2 \hat{v}_{i,k}^2,
\end{equation}
using (4.39) and (4.38), we then conclude from above that
\begin{equation}
a^* \varepsilon_k^2 e(a_k, b, \beta_k) = \frac{2 + p_2}{2} \varepsilon_k^2 \int_{\mathbb{R}^4} V_1(x) \hat{u}_{i,k}^2
- \frac{2 + p_2}{2} \varepsilon_k^2 (a^* C^2 \sigma_k^2 \varepsilon_k^2) \int_{\mathbb{R}^4} V_2(x) \hat{v}_{i,k}^2
+ \frac{\varepsilon_k^2}{2} \int_{\mathbb{R}^4} \left[x_{2,k} \cdot \nabla V_1(x)\right] \hat{u}_{i,k}^2 + \frac{\varepsilon_k^2}{2} (a^* C^2 \sigma_k^2 \varepsilon_k^2) \int_{\mathbb{R}^4} \left[x_{2,k} \cdot \nabla V_2(x)\right] \hat{v}_{i,k}^2
= I_i + (a^* C^2 \sigma_k^2 \varepsilon_k^2) II_i + \varepsilon_k^2 \int_{\partial B_k(x_{2,k})} \frac{\partial \hat{u}_{i,k}}{\partial \nu} (x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}
- \frac{\varepsilon_k^2}{2} \int_{\partial B_k(x_{2,k})} \left[(x - x_{2,k}) \cdot \nu\right] |\nabla \hat{u}_{i,k}|^2
+ \frac{\varepsilon_k^2}{2} (a^* C^2 \sigma_k^2 \varepsilon_k^2) \int_{\partial B_k(x_{2,k})} \frac{\partial \hat{v}_{i,k}}{\partial \nu} (x - x_{2,k}) \cdot \nabla \hat{u}_{i,k}
- \frac{\varepsilon_k^2}{2} (a^* C^2 \sigma_k^2 \varepsilon_k^2) \int_{\partial B_k(x_{2,k})} \left[(x - x_{2,k}) \cdot \nu\right] |\nabla \hat{u}_{i,k}|^2 := B_i, \quad i = 1, 2. \tag{4.39}
\end{equation}
which then implies that

\[
\frac{2 + p_1}{2} \varepsilon_k^2 \int_{\mathbb{R}^2} V_1(x)(\hat{u}_{2,k} + \hat{u}_{1,k})\xi_{1,k} \\
- \frac{\varepsilon_k^2}{2} \int_{\mathbb{R}^2} [x_{2,k} \cdot \nabla V_1(x)](\hat{u}_{2,k} + \hat{u}_{1,k})\xi_{1,k} \\
+ \frac{2 + p_2}{2} \varepsilon_k^2 (a^* C_\infty^2 \varepsilon_k^2) \int_{\mathbb{R}^2} V_2(x)(\hat{v}_{2,k} + \hat{v}_{1,k})\xi_{2,k} \\
- \frac{\varepsilon_k^2}{2} (a^* C_\infty^2 \varepsilon_k^2) \int_{\mathbb{R}^2} [x_{2,k} \cdot \nabla V_2(x)](\hat{v}_{2,k} + \hat{v}_{1,k})\xi_{2,k} := -T_k.
\]

(4.40)

We shall prove in the appendix that \( T_k \) satisfies

\[
T_k := \frac{B_2 - B_1}{\|\hat{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\varepsilon_k}\|\hat{v}_{2,k} - \bar{v}_{1,k}\|_{L^2(\mathbb{R}^2)}} = o(e^{-\frac{\varepsilon_k^2}{\tau}}),
\]

(4.41)

where \( B_i \) is defined in (4.39) for \( i = 1, 2 \). We thus conclude from (4.40) and (4.41) that

\[
\frac{2 + p_1}{2} \int_{\mathbb{R}^2} V_1(x + \frac{x_{2,k}}{\varepsilon_k})(\hat{u}_{2,k} + \bar{u}_{1,k})\xi_{1,k} \\
- \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{x_{2,k}}{\varepsilon_k} \cdot \nabla V_1(x + \frac{x_{2,k}}{\varepsilon_k}) \right](\hat{u}_{2,k} + \bar{u}_{1,k})\xi_{1,k} \\
+ \frac{2 + p_2}{2} \varepsilon_k^2 (a^* C_\infty^2 \varepsilon_k^2) \int_{\mathbb{R}^2} V_2(x + \frac{x_{2,k}}{\varepsilon_k})(\hat{v}_{2,k} + \bar{v}_{1,k})\xi_{2,k} \\
- \frac{\varepsilon_k^2}{2} (a^* C_\infty^2 \varepsilon_k^2) \int_{\mathbb{R}^2} \left[ \frac{x_{2,k}}{\varepsilon_k} \cdot \nabla V_2(x + \frac{x_{2,k}}{\varepsilon_k}) \right](\hat{v}_{2,k} + \bar{v}_{1,k})\xi_{2,k} \\
= o(e^{-\frac{\varepsilon_k^2}{\tau}}).
\]

(4.42)

We next establish Step 3 as follows. Since \( p_1 \leq p_2 \) and \( \sigma_k^2 \varepsilon_k^2 \rightarrow 0 \) as \( k \rightarrow \infty \), we then conclude from (1.25) and (4.42) that

\[
\int_{\mathbb{R}^2} V_1(x + \frac{x_{2,k}}{\varepsilon_k})(\hat{u}_{2,k} + \bar{u}_{1,k})\xi_{1,k} = o(1) \quad \text{as} \quad k \rightarrow \infty.
\]

Following this, we then obtain from (1.14) that

\[
0 = 2 \int_{\mathbb{R}^2} V_1(x + y_0)w_{10} \\
= 2c_0 \int_{\mathbb{R}^2} V_1(x + y_0)\left( w^2 + \frac{1}{2} x \cdot \nabla w^2 \right) \\
= 2c_0 \left\{ \int_{\mathbb{R}^2} V_1(x + y_0)w^2 - \frac{1}{2} \int_{\mathbb{R}^2} w^2 \left[ 2V_1(x + y_0) + x \cdot \nabla V_1(x + y_0) \right] \right\} \\
= -p_1c_0 \int_{\mathbb{R}^2} V_1(x + y_0)w^2 + c_0 \int_{\mathbb{R}^2} w^2 [y_0 \cdot \nabla V_1(x + y_0)] \\
= -p_1c_0 \int_{\mathbb{R}^2} V_1(x + y_0)w^2 = -p_1c_0H_1(y_0),
\]
which therefore implies that $c_0 = 0$. Using $c_0 = 0$, we further derive from \((4.21)\) and \((4.25)\) that

$$0 = 2 \int_{\mathbb{R}^2} \frac{\partial V_1(x+y_0)}{\partial x_j} u_0 \xi_{10} = 2 \int_{\mathbb{R}^2} \frac{\partial V_1(x+y_0)}{\partial x_j} u_0 \left( \sum_{i=1}^{2} b_i \frac{\partial u_0}{\partial x_i} \right)$$

$$= - \sum_{i=1}^{2} b_i \int_{\mathbb{R}^2} \frac{\partial^2 V_1(x+y_0)}{\partial x_j \partial x_i} u_0^2, \quad j = 1, 2,$$

which then gives that $b_1 = b_2 = 0$ in \((4.21)\), due to the non-degeneracy assumption \((1.14)\). Therefore, we have $c_0 = b_1 = b_2 = 0$, which implies that $\xi_{10} = 0$ and $\xi_{20} = b_0 w$ for some constant $b_0$.

**Step 4.** There exist two constants $b_{11}$ and $b_{12}$ such that $\xi_{1,k}$ satisfies

$$\xi_{1,k} = \left[ - b_0 w + \sum_{i=1}^{2} b_{11} \frac{\partial w}{\partial x_i} \right] \left( a^* C^2_{\infty \sigma_k^2 \varepsilon_k^2} + o(\sigma_k^2 \varepsilon_k^2) \right) \text{ as } k \to \infty,$$

where the constant $b_0$ is the same as that of $\xi_{20} = b_0 w$ given in \((4.21)\).

Actually, similar to the proof of \((3.6)\) in \([13]\), one can obtain from \((4.22)\) that

$$\xi_{1,k} = \left( a^* C^2_{\infty \sigma_k^2 \varepsilon_k^2} \right) \times \xi_1 + o(\sigma_k^2 \varepsilon_k^2), \quad \text{as } k \to \infty,$$

where $\xi_1$ is a unique solution of $\nabla \xi_1(0) = 0$ and

$$\Delta \xi_1 - \xi_1 + 3w^2 \xi_1 - \frac{2}{a^*} \left( \int_{\mathbb{R}^2} w^3 \xi_1 \right) w = -2w^2 \xi_{20} + \frac{2}{a^*} \left( \int_{\mathbb{R}^2} w^3 \xi_{20} \right) w$$

$$= -2b_0 w^3 + 4b_0 w \text{ in } \mathbb{R}^2,$$

since $\xi_{20} = b_0 w$. One can check that $\xi_1$ satisfies

$$\xi_1 = -b_0 w + \sum_{i=1}^{2} b_{11} \frac{\partial w}{\partial x_i}$$

for some constants $b_{11}$ and $b_{12}$, where the constant $b_0$ is the same as that of $\xi_{20} = b_0 w$ given in \((4.21)\). Therefore, the estimate \((4.43)\) now follows from \((4.44)\) and \((4.45)\).

**Step 5.** $b_0 = 0$ in \((4.21)\), i.e., $\xi_{10} = \xi_{20} = 0$.

We shall consider separately the following two cases:

**Case 1:** $p_1 < p_2$. In this case, we follow from \((4.25)\) and Step 4 that

$$b_{11} \frac{\partial^2 H_1(y_0)}{\partial x_1 \partial x_j} + b_{12} \frac{\partial^2 H_1(y_0)}{\partial x_2 \partial x_j} = 0, \quad j = 1, 2,$$

which then implies that $b_{11} = b_{12} = 0$. It thus yields from \((4.43)\) and \((4.42)\) that

$$0 = 2(2 + p_1) H_1(y_0) b_0 + \left( y_0 \cdot \frac{\partial H_1(y_0)}{\partial x_1} \right) b_{11} + \left( y_0 \cdot \frac{\partial H_1(y_0)}{\partial x_2} \right) b_{12}$$

$$= -2(2 + p_1) H_1(y_0) b_0,$$

which gives that $b_0 = 0$, since $H_1(y_0) > 0$. Therefore, we have $\xi_{10} = \xi_{20} = 0$ in this case.

**Case 2:** $p_1 = p_2$. In this case, we deduce from \((4.25)\) and Step 4 that

$$2b_0 \frac{\partial H_2(y_0)}{\partial x_j} - b_{11} \frac{\partial^2 H_1(y_0)}{\partial x_1 \partial x_j} - b_{12} \frac{\partial^2 H_1(y_0)}{\partial x_2 \partial x_j} = 0, \quad j = 1, 2.$$
However, it yields from (4.43) and (4.42) that
\[
\left\{ 2(2 + p_1) \left[ H_2(y_0) - H_1(y_0) \right] - 2y_0 \cdot \nabla H_2(y_0) \right\} b_0
\]
\[
\left( y_0 \cdot \frac{\partial \nabla H_1(y_0)}{\partial x_1} \right) b_{11} + \left( y_0 \cdot \frac{\partial \nabla H_1(y_0)}{\partial x_2} \right) b_{12} = 0.
\]
(4.49)

We thus obtain from (4.48) and (4.49) that
\[
\begin{pmatrix}
M_{11} & y_0 \cdot \frac{\partial \nabla H_1(y_0)}{\partial x_1} & y_0 \cdot \frac{\partial \nabla H_1(y_0)}{\partial x_2} \\
\frac{\partial H_2(y_0)}{\partial x_1} & -\frac{\partial^2 H_1(y_0)}{\partial x_1 \partial x_1} & -\frac{\partial^2 H_1(y_0)}{\partial x_1 \partial x_2} \\
\frac{\partial H_2(y_0)}{\partial x_2} & -\frac{\partial^2 H_1(y_0)}{\partial x_1 \partial x_2} & -\frac{\partial^2 H_1(y_0)}{\partial x_2 \partial x_2}
\end{pmatrix}
\begin{pmatrix}
b_0 \\
b_{11} \\
b_{12}
\end{pmatrix}
= 0,
\]
(4.50)

where \(M_{11} = 2(2 + p_1) \left[ H_2(y_0) - H_1(y_0) \right] - 2y_0 \cdot \nabla H_2(y_0)\). In this case, if \(H_2(y_0) \neq H_1(y_0)\), we then derive from the non-degeneracy assumption (1.14) that
\[
2(2 + p_1) \left[ H_2(y_0) - H_1(y_0) \right] - 2y_0 \cdot \nabla H_2(y_0) y_0 \cdot \frac{\partial \nabla H_1(y_0)}{\partial x_1} y_0 \cdot \frac{\partial \nabla H_1(y_0)}{\partial x_2}
\]
\[
= 2(2 + p_1) \left[ H_2(y_0) - H_1(y_0) \right]
\]
\[
= 2(2 + p_1) \left[ H_2(y_0) - H_1(y_0) \right] \det \left( \frac{\partial^2 H_1(y_0)}{\partial x_i \partial x_j} \right) \neq 0,
\]

which thus implies that \(b_0 = 0\). Therefore, we also have \(\xi_{10} = \xi_{20} = 0\) in this case.

**Step 6.** \(\xi_{10} = \xi_{20} = 0\) cannot occur.

Finally, let \(x_k \in \mathbb{R}^2\) satisfy
\[
|\xi_{1,k}(x_k)| + \sqrt{\|\xi_{2,k}\|^2} = \|\xi_{1,k}(x)\|_\infty + \sqrt{\|\xi_{2,k}\|^2} = 1.
\]
(4.51)

If \(\xi_{1,k} \to \xi_{10} \neq 0\) uniformly on \(\mathbb{R}^2\) as \(k \to \infty\), it then contradicts to the fact that \(\xi_{10} \equiv 0\) on \(\mathbb{R}^2\).

We now assume that \(\xi_{1,k} \to \xi_{10} \equiv 0\) uniformly on \(\mathbb{R}^2\) as \(k \to \infty\). Since \((u_{i,k}, v_{i,k})\) decays exponentially as \(|x| \to \infty\) for \(i = 1, 2\), the linear elliptic theory applied to (1.22) gives that \(\xi_{2,k}\) is also bounded uniformly in \(\mathbb{R}^2\). Let \(y_k \in \mathbb{R}^2\) satisfy \(|\xi_{2,k}(y_k)| = \|\xi_{2,k}(x)\|_\infty\). Applying the maximum principle to (1.22) then yields that \(|x_k| \leq C \text{ and } |y_k| \leq C\) uniformly in \(k\), due to the exponential decay of \((u_{i,k}, v_{i,k})\).

By the comparison principle, one can get from (1.22) that \(\xi_{i,k}\) decays exponentially for \(i = 1, 2\), see [16, 17] for similar proofs. Following these, we then conclude from (4.51) that there exists a large \(R > 0\) such that \(\int_{B_R(y_k)} |\xi_{2,k}|^2 dx \geq \frac{1}{2}\) uniformly in \(k > 0\). This further implies that \(\xi_{2,k} \to \xi_{20} \neq 0\) uniformly on \(\mathbb{R}^2\) as \(k \to \infty\), a contradiction again. Therefore, the proof of Theorem 1.3 is now complete for the first case where \(u_{2,k} \neq u_{1,k}\) and \(v_{2,k} \neq v_{1,k}\) in \(\mathbb{R}^2\).

(2) We next consider the second case where \(u_{2,k} \equiv u_{1,k}\) and \(v_{2,k} \equiv v_{1,k}\) in \(\mathbb{R}^2\). In this case, we have \(\xi_{1,k} \equiv 0\) and \(\xi_{2,k} \neq 0\) in \(\mathbb{R}^2\), and the second equation of (4.22)
gives that
\[ \xi_{2,k} \rightarrow \xi_{20} := b_0w \] uniformly on \( \mathbb{R}^2 \) as \( k \rightarrow \infty \)
for some constant \( b_0 \). Moreover, one can also get that both (4.26) and (4.42) hold true with \( \xi_{1,k} \equiv 0 \), and it follows from (4.25) that
\[ 2 \int_{\mathbb{R}^2} \frac{\partial V_2(x + y_0)}{\partial x_j} w_{20} = 0, \quad j = 1, 2. \]

On the other hand, we derive from (4.42) that
\[ \text{which then implies that } b \text{ hold true with } \xi. \]

Following (4.14), one can check from (4.16) that (4.42) gives that
\[ \text{gives that } \]
\[ \text{as } k \rightarrow \infty. \]

Proof of Lemma 4.1. Following (4.29), we shall address the proofs of Lemma 4.1 and (4.41).

In this appendix we shall address the proofs of Lemma 4.1 and (4.41).

**Proof of Lemma 4.1** Following (4.14), one can check from (4.16) that \( (\hat{\xi}_{1,k}, \hat{\xi}_{2,k}) \) satisfies
\[ (A.1) \]
\[ \left\{ \begin{array}{l}
\varepsilon_k^2 \Delta \hat{\xi}_{1,k} - \varepsilon_k^2 V_1(x) \hat{\xi}_{1,k} + \mu_2 \varepsilon_k^2 \hat{\xi}_{1,k}^\ast \frac{\partial \hat{u}_2}{\partial x_1} + \frac{\partial \hat{u}_2}{\partial x_2} + \hat{\xi}_{1,k} \hat{\xi}_{2,k} + \hat{\xi}_{1,k} \hat{\xi}_{2,k} = \hat{c}_k \hat{u}_2 + \hat{u}_1 \hat{v}_1 \quad \text{in } \mathbb{R}^2, \\
\varepsilon_k^2 \Delta \hat{\xi}_{2,k} - \varepsilon_k^2 V_2(x) \hat{\xi}_{2,k} + \mu_2 \varepsilon_k^2 \hat{\xi}_{2,k} + b C_\infty \sigma_k^2 \hat{\xi}_{2,k} \hat{\xi}_{1,k} = \hat{c}_k \hat{v}_1 \quad \text{in } \mathbb{R}^2,
\end{array} \right. \]
where the coefficient $\hat{c}_k$ satisfies (4.30). Multiplying the first equation of (A.1) by $\hat{\xi}_{1,k}$ and integrating over $\mathbb{R}^2$, we obtain from (4.5) and (4.30) that

$$I_0 = \varepsilon_k^2 \int_{\mathbb{R}^2} |\nabla \hat{\xi}_{1,k}|^2 - \mu_{2,k} \varepsilon_k^2 \int_{\mathbb{R}^2} |\hat{\xi}_{1,k}|^2 + \varepsilon_k^2 \int_{\mathbb{R}^2} V_1(x)|\hat{\xi}_{1,k}|^2$$

$$= \frac{a_k}{a^*} \int_{\mathbb{R}^2} (\hat{u}_{2,k}^2 + \hat{u}_{2,k} \hat{u}_{1,k} + \hat{u}_{1,k}^2) |\hat{\xi}_{1,k}|^2$$

$$+ \beta_k C_\infty^2 \sigma_k^2 \varepsilon_k^2 \int_{\mathbb{R}^2} \left[ \hat{v}_{2,k}^2 |\hat{\xi}_{1,k}|^2 + \hat{u}_{2,k}(\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{1,k} \hat{\xi}_{2,k} \right]$$

$$- \int_{\mathbb{R}^2} \hat{u}_{1,k} \hat{\xi}_{1,k} \left\{ \frac{a_k}{2(a^*)^2 \varepsilon_k} \int_{\mathbb{R}^2} (\hat{u}_{2,k}^2 + \hat{u}_{1,k}^2) (\hat{\xi}_{1,k} + \hat{\xi}_{2,k}) \right\}.$$  

(A.2)

Recall that

$$\left| \int_{\mathbb{R}^2} \hat{u}_{2,k}(\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{1,k} \hat{\xi}_{2,k} \right| \leq \int_{\mathbb{R}^2} \hat{u}_{2,k}(\hat{v}_{2,k} + \hat{v}_{1,k}) |\hat{\xi}_{2,k}|$$

$$\leq \left( \int_{\mathbb{R}^2} \hat{u}_{2,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k})^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |\hat{\xi}_{2,k}|^2 \right)^{\frac{1}{2}} \leq C \varepsilon_k^2,$$

due to the fact that

$$||\hat{\xi}_{1,k}||_\infty \leq 1 \quad \text{and} \quad \int_{\mathbb{R}^2} |\hat{\xi}_{2,k}|^2 \leq \varepsilon_k^2.$$  

Using above estimates, we derive from (A.2) that

(A.3) \quad $I'_1 := \varepsilon_k^2 \int_{\mathbb{R}^2} |\nabla \hat{\xi}_{1,k}|^2 + \frac{1}{2} \int_{\mathbb{R}^2} |\hat{\xi}_{1,k}|^2 + \varepsilon_k^2 \int_{\mathbb{R}^2} V_1(x)|\hat{\xi}_{1,k}|^2 \leq I_0 < C_1 \varepsilon_k^2$

as $k \to \infty$ holds for some constant $C_1 > 0$. Applying [3 Lemma 4.5], we then conclude that for any $x_0 \in \mathbb{R}^2$, there exist a small constant $\delta > 0$ and $C_2 > 0$ such that

$$\int_{\partial B_{\delta}(x_0)} \left[ \varepsilon_k^2 |\nabla \hat{\xi}_{1,k}|^2 + \frac{1}{2} |\hat{\xi}_{1,k}|^2 + \varepsilon_k^2 V_1(x)|\hat{\xi}_{1,k}|^2 \right] dS \leq C_2 I'_1 \leq C_1 C_2 \varepsilon_k^2 \quad \text{as} \quad k \to \infty,$$

which therefore implies that (4.18) holds true for $i = 1$.

Similarly, applying the above argument to the second equation of (A.1), one can obtain that (4.18) holds true for $i = 2$, and we are therefore done. \qed
Proof of (4.41). Following (4.39), we have for small $\delta > 0$,

$$
T_k = \frac{B_2 - B_1}{2} \left[ \| \hat{u}_{2,k} - \hat{u}_{1,k} \|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\epsilon_k} \| \hat{v}_{2,k} - \hat{v}_{1,k} \|_{L^2(\mathbb{R}^2)} \right]
$$

$$
= (I_2 - I_1) + (\alpha C_\infty^2 \sigma_k^2 \epsilon_k^2)(I_2 - I_1)
$$

$$
\frac{\epsilon_k^2}{2} \int \left[ \langle x, -x, 2 \rangle \cdot \nabla \hat{u}_{2,k} + \nabla \hat{u}_{1,k} \right] \nabla \hat{\xi}_{1,k}
$$

$$
+ \frac{\epsilon_k^2}{2} \int \left[ \langle x, -x, 2 \rangle \cdot \nabla \hat{v}_{2,k} + \nabla \hat{v}_{1,k} \right] \nabla \hat{\xi}_{2,k}
$$

(A.4)

$$
\frac{\epsilon_k^2}{2} \int \left[ \langle x, -x, 2 \rangle \cdot \nabla \hat{u}_{2,k} + \nabla \hat{u}_{1,k} \right] \nabla \hat{\xi}_{1,k}
$$

$$
+ \frac{\epsilon_k^2}{2} \int \left[ \langle x, -x, 2 \rangle \cdot \nabla \hat{v}_{2,k} + \nabla \hat{v}_{1,k} \right] \nabla \hat{\xi}_{2,k}
$$

due to (4.18), where the second equality follows by applying the argument of estimating (4.32). Here $I_1$ and $I_1$ satisfy (4.35) and (4.36), respectively.

Using the arguments of estimating (4.32) again, along with the exponential decay mentioned soon after (4.13), we also derive from (4.32) that for small $\delta > 0$,

$$
II = (I_2 - I_1) + (\alpha C_\infty^2 \sigma_k^2 \epsilon_k^2)(I_2 - I_1)
$$

$$
\frac{\mu_2 k \epsilon_k^2}{2} \int \left[ \hat{u}_{2,k} + \hat{u}_{1,k} \right] \hat{\xi}_{1,k} - \frac{2 + p_1}{2} \epsilon_k^2 \int \left( \hat{u}_{2,k} + \hat{u}_{1,k} \right) V_1 \hat{\xi}_{1,k}
$$

$$
+ \frac{\alpha_k}{2 \alpha} \int \left[ \hat{u}_{2,k} + \hat{u}_{1,k} \right] \hat{\xi}_{1,k} + \frac{\beta_k C_\infty^2 \sigma_k^2 \epsilon_k^2}{2} \int \left[ \hat{v}_{2,k} + \hat{v}_{1,k} \right] \hat{\xi}_{1,k} + \frac{\epsilon_k^2}{2} \int \left[ \langle x, -x, 2 \rangle \cdot \nabla \hat{V}_1(x) \right] \left( \hat{u}_{2,k} + \hat{u}_{1,k} \right) \hat{\xi}_{1,k}
$$

$$
+ \frac{\epsilon_k^2}{2} \int \left[ \hat{u}_{2,k} + \hat{u}_{1,k} \right] \hat{\xi}_{1,k} + \frac{\beta_k C_\infty^2 \sigma_k^2 \epsilon_k^2}{2} \int \left[ \hat{u}_{2,k} + \hat{u}_{1,k} \right] \hat{\xi}_{1,k} (x - x, 2) \nu dS
$$

$$
- \frac{\epsilon_k^2}{2} \int \left[ \hat{u}_{2,k} + \hat{u}_{1,k} \right] \hat{\xi}_{1,k} \hat{V}_1(x) \langle x, -x, 2 \rangle \nu dS
$$

$$
+ \frac{\mu_2 k \epsilon_k^2}{2} \int \left[ \hat{u}_{2,k} + \hat{u}_{1,k} \right] \hat{\xi}_{1,k} (x - x, 2) \nu dS
$$

$$
+ \hat{\epsilon}_k \int \hat{u}_{2,k}^2 (x - x, 2) \nu dS + \frac{\beta_k C_\infty^2 \sigma_k^2 \epsilon_k^2}{2}
$$

$$
\int \hat{v}_{2,k}^2 (x - x, 2) \hat{\xi}_{1,k} + \hat{u}_{1,k}^2 (\hat{v}_{2,k} + \hat{v}_{1,k}) \hat{\xi}_{2,k}^2 \langle x, -x, 2 \rangle \nu dS
$$

$$
\int \hat{v}_{1,k}^2 (x - x, 2) \nu dS
$$
as \( k \to \infty \), and hence
\[
II = \hat{c}_k \left[ \int_{\mathbb{R}^2 \setminus B_\delta(x_{2,k})} \hat{u}_{1,k}^2 + \int_{\partial B_\delta(x_{2,k})} \hat{u}_{1,k}^2 (x - x_{2,k}) \nu dS \right] + \frac{1}{2} \epsilon_k^2 \int_{\mathbb{R}^2 \setminus B_\delta(x_{2,k})} \left[ x_{2,k} \cdot \nabla V_1(x) \right] \left( \hat{u}_{2,k} + \hat{u}_{1,k} \right) \hat{\xi}_{1,k} + o\left( \epsilon_k^2 \right),
\] (A.5)
as \( k \to \infty \), where \( \hat{c}_k \) is defined by (4.30) and satisfies (4.31). Therefore, we deduce from (A.4) and (A.5) that
\[
II = \frac{I_2 - I_1}{\| \hat{u}_{2,k} - \hat{u}_{1,k} \|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\epsilon_k} \| \hat{v}_{2,k} - \hat{v}_{1,k} \|_{L^2(\mathbb{R}^2)}} = o\left( \frac{\epsilon_k^2}{\epsilon_k^2} \right) \text{ as } k \to \infty,
\]
and similarly we also have
\[
\left( a^* C^2_\delta \sigma_k^2 \hat{u}_{2,k} \right)(II_2 - II_1) \| \hat{u}_{2,k} - \hat{u}_{1,k} \|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\epsilon_k} \| \hat{v}_{2,k} - \hat{v}_{1,k} \|_{L^2(\mathbb{R}^2)} = o\left( \frac{\epsilon_k^2}{\epsilon_k^2} \right) \text{ as } k \to \infty.
\]
Therefore, we conclude from (A.4) that \( T_k = o\left( \frac{\epsilon_k^2}{\epsilon_k^2} \right) \) as \( k \to \infty \), which completes the proof of (4.31). \( \square \)

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