TOPOLOGICAL PROPERTIES OF NEUMANN DOMAINS

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Abstract. A Laplacian eigenfunction on a two-dimensional manifold dictates some natural partitions of the manifold; the most apparent one being the well studied nodal domain partition. An alternative partition is revealed by considering a set of distinguished gradient flow lines of the eigenfunction - those which are connected to saddle points. These give rise to Neumann domains. We establish complementary definitions for Neumann domains and Neumann lines and use basic Morse homology to prove their fundamental topological properties. We study the eigenfunction restrictions to these domains. Their zero set, critical points and spectral properties allow to discuss some aspects of counting the number of Neumann domains and estimating their geometry.

1. Introduction

Topological properties of eigenfunctions of the Laplacian on domains and manifolds are of essential interest to mathematical physics in recent years [35, 17]. The nodal patterns of eigenfunctions form a major and well developed research direction in this field. Nodal sets of eigenfunctions have been studied with respect to their volume [9, 7, 14, 11, 19] and geometry [5, 6] and nodal domains of eigenfunctions have been studied with respect to their count [12, 3, 18, 24, 8] and metric properties [28, 29, 27]. The study of a related object has been recently suggested in two independent works by Zelditch [37] and McDonald, Fulling [30]. This object is a partition of the manifold dictated by some gradient flow lines of an eigenfunction \( f \) on the manifold. The specific gradient flow lines which form the partition are those which are connected to saddle points of \( f \) and they are called Neumann lines [30], for reasons to be explained below. Similarly to nodal lines which are specific level sets of \( f \) (of zero value), Neumann lines are selected gradient flow lines of \( f \). In this note we deal only with two dimensional manifolds and further restrict to the case of Morse eigenfunctions, for the sake of simplicity. In fact, for generic metrics eigenfunctions are in this class as shown in [36].

The complement of the union of all Neumann lines decomposes into connected components which are called (connected) Neumann domains. The definitions of Neumann lines and Neumann domains appear separately (and unrelatedly) in the recent works mentioned above [37, 30]. We rephrase these previous definitions (see definitions 3.1, 3.3 in the current paper) and then show that both sets are indeed complementary (propositions 3.9 and 4.9). The definitions and proofs distinguish between the cases of Riemannian 2-manifolds without boundary (section 3) and simply connected Euclidean domains with sufficiently regular boundary (section 4). The results, however, are common for all of these manifolds.

The main purpose of this note is to introduce the rigorous study of Neumann domains and their topological and spectral properties. In particular, we point out some similarities and differences between Neumann domains and nodal domains and argue that the former have a simpler topological structure. Using the notations \( f \) for an eigenfunction and \( \Omega \) for a connected Neumann domain we now describe the main results of this paper.

1.1. Topology of Neumann domains. We prove that a connected Neumann domain is simply connected. This is done in theorem 3.12(iv) for the non-boundary case and in theorem 4.10(v) for domains with Dirichlet boundary. This is one major difference between Neumann domains and nodal domains, as the latter are in general multiply connected (see 32, 33 for results on random waves).

1.2. Critical points and the nodal set of \( f|_{\overline{\Omega}} \). We show that the critical points of \( f|_{\overline{\Omega}} \) lie on the boundary of \( \overline{\Omega} \) and contain exactly a single minimum, a single maximum and
at least one saddle point (theorem 3.12(i),(ii)). If \( f \) is also a Morse-Smale function then there are at most two saddle points on the boundary of \( \Omega \) (theorem 3.12(iii)). In addition, we prove that each connected Neumann domain, \( \Omega \), contains precisely one nodal line without self-intersections, which intersects the boundary of \( \Omega \) at two different points (theorems 3.12(v),(vi),(vii),(viii) and theorem 4.10). These facts imply the structural simplicity of \( f|_{\Omega} \) (see figure 3.2).

1.3. The spectral position of \( f|_{\Omega} \). The normal derivative of \( f \) on the Neumann lines is zero, i.e. for \( x \in \partial \Omega \), \( \hat{n} \cdot \nabla f|_x = 0 \), following from the definition. Therefore, the restriction of the eigenfunction \( f \) to a connected Neumann domain, \( \Omega \), is a solution of an eigenvalue problem on this domain, with Neumann boundary conditions, hence the use of the term Neumann lines and domains. It is tempting to believe (for reasons to be pointed later on) that this restriction equals the first non-constant eigenfunction of this domain with Neumann boundary conditions. This is the case for nodal domains with the corresponding Dirichlet eigenvalue problem. It is argued in [37] that this might be ‘often’ the case for Neumann domains as well. In this note, we point out in chapter 5.2 a prototypical example where this property does not hold (even more severely, we show that the restricted eigenfunction can be located arbitrarily high in the spectrum of the Neumann domain). This non-trivial problem of determining the position in the spectrum of the restricted eigenfunction is highly connected to the number of Neumann domains and forms one motivation of the current work.

1.4. On the collapse of Neumann domains. It was pointed out in [30] that there exist eigenspaces, in which continuous families of eigenfunctions have Neumann domains that collapse to a line. This does not occur for nodal domains due to the lower bound on their volume in terms of the eigenvalue. In section 5.1 we relate this collapse to a lower bound on the outer radius of Neumann domains.

2. Preliminaries

Let \( M \) be a 2-dimensional, connected, compact and orientable surface without boundary with a smooth Riemannian metric \( g \) and let \( \Delta_g \) be the corresponding Laplace-Beltrami operator. Consider the eigenvalue problem
\[
-\Delta_g f = \lambda f.
\]
We assume in the following that the eigenfunctions \( f \) are Morse functions, i.e. have no degenerate critical points. We call such an \( f \) a Morse-eigenfunction. The smooth gradient vector field of \( f \), \( \nabla f \) defines a smooth flow, \( \varphi \), along the integral curves of \( -\nabla f \):
\[
\varphi : \mathbb{R} \times M \to M,
\]
\[
\partial_t \varphi(t, x) = -\nabla f|_{\varphi(t,x)},
\]
\[
\varphi(0, x) = x.
\]
As critical points play a crucial role throughout this paper, we introduce the following notations. Let \( \mathcal{C}(f) \) denote the set of critical points of \( f \), \( \mathcal{S}(f) \) and \( \mathcal{X}(f) \) the sets of saddle points and extrema of \( f \), respectively. In addition, let \( \mathcal{M}_-(f) \) and \( \mathcal{M}_+(f) \) denote the sets of minima and maxima of \( f \), respectively.

For a critical point \( x \in \mathcal{C}(f) \), we denote by \( \lambda_x \) its index (the number of negative eigenvalues of the Hessian of \( f \) at \( x \)) and define its stable and unstable manifolds by
\[
W^s(x) = \{ y \in M | \lim_{t \to \infty} \varphi(t,y) = x \} \text{ and } \nabla f|_{\varphi(t,x)},
\]
\[
W^u(x) = \{ y \in M | \lim_{t \to -\infty} \varphi(t,y) = x \},
\]
respectively.

A fundamental theorem in Morse homology is the stable/unstable manifold theorem and we quote here part of it.
Lemma 2.1. (part of theorem 4.2 in [2]) Let $f$ be a real Morse function on an $m$-dimensional, compact Riemannian manifold and let $x$ be a critical point of $f$. The stable and unstable manifolds of $x$ are smoothly embedded open disks of dimension $m - \lambda_x$ and $\lambda_x$, respectively, where $\lambda_x$ is the Morse index of $f$ at $x$.

In two dimensions the non-degenerate critical points are maxima, minima and saddle points. According to the lemma above the stable manifold of a maximum $q \in M$ is $\{q\}$, and the unstable manifold is the embedding of a two dimensional open disk. The converse holds for a minimum. For a saddle point, $r$, the stable and unstable manifolds are embeddings of open one dimensional intervals. Another useful tool is the decomposition of the manifold $M$ into a union of stable (or unstable) manifolds.

Lemma 2.2. [Proposition 4.22 in [2]] Let $f : M \to \mathbb{R}$ be a Morse function on a compact, smooth, closed Riemannian manifold $(M, g)$, then $M$ is a disjoint union of the stable manifolds of $f$, i.e.

\[(2.4) \quad M = \bigsqcup_{x \in \mathcal{C}(f)} W^s(x).\]

Similarly,

\[(2.5) \quad M = \bigsqcup_{x \in \mathcal{C}(f)} W^u(x).\]

Finally, we recall the following relevant definition. A Morse-Smale function is a Morse function, which in addition fulfills the Morse-Smale transversality condition, saying that stable and unstable manifolds intersect transversally (cf. [2]). In two dimensions the Morse-Smale transversality condition means that there are no two saddle points which are connected by gradient flow lines.

3. The structure of Neumann lines and Neumann domains - manifolds without boundary

In this section, we define Neumann domains and Neumann lines. We then prove that they form complementary sets on the manifold and eventually prove a theorem on the topological properties of Neumann domains. Throughout this section we assume that $M$ is a 2-dimensional compact, orientable surface without boundary and $f$ is a Morse eigenfunction on $M$. The following definition is motivated by Zelditch in [37].

Definition 3.1. A Neumann domain is any set, $\Omega_{p,q}(f)$, which has the form

\[(3.1) \quad \Omega_{p,q}(f) = W^s(p) \cap W^u(q),\]

where $p \in \mathcal{M}_{-}(f)$, $q \in \mathcal{M}_{+}(f)$. Any connected component of $\Omega_{p,q}$ is termed connected Neumann domain.

Remark 3.2. A connected Neumann domain is also path connected. This can be seen for example as $W^s(p)$ and $W^u(q)$ are smooth embeddings of two-dimensional balls, by lemma 2.1.

In the following we tend to omit the indices and denote a connected Neumann domain by $\Omega$. The next definition owes to the recent paper of McDonald and Fulling [30]. We allow a certain modification of the definition, so that it fits the framework of this paper.

Definition 3.3. The Neumann line set of $f$ is

\[(3.2) \quad N(f) := \bigcup_{r \in \mathcal{C}(f)} W^s(r) \cup W^u(r).\]

Figure 3.1 demonstrates the definitions above by showing the Neumann lines and the Neumann domains for two eigenfunctions on the flat torus. As saddle points play a major role in defining Neumann lines, it is useful in what follows to understand the local behavior of $N(f)$ in the vicinity of saddle points. The following lemma summarizes results of that kind, some of which appear in [30].
Lemma 3.4. Let $M$ and $f$ be as above and let $r \in \mathcal{N}(f) \cap \operatorname{int} M$. Then

1. There exists a neighbourhood, $U$, of $r$ such that $N(f) \cap U$ consists of four curves which meet with right angles at $r$.
2. There exists a neighbourhood $V$, of $r$ such that the previous claim holds in $V$ and in addition, $f^{-1}(f(r)) \cap V$ consists of four curves which meet at $r$ and interlace with the four curves $N(f) \cap U$.

Remark 3.5. The case $f(r) = 0$ is particularly interesting as it relates the nodal lines and the Neumann lines in the vicinity of $r$.

Proof. The first claim of the lemma is proved in [30] by examining the Taylor expansion of $f$ around $r$. The second claim follows similarly. \hfill \Box

We start by providing three basic lemmata which are relevant for proving that Neumann lines and Neumann domains form complementary sets (cf. proposition 3.9).

Lemma 3.6. [Proposition 3.19 in [24]] \textbf{∀} $x \in M$, both limits $\lim_{t \to \pm \infty} \varphi(t, x)$ exist and they are both critical points of $f$, i.e., $\lim_{t \to \pm \infty} \varphi(t, x) \in \mathcal{C}(f)$.

Lemma 3.7. Let $r \in \mathcal{N}(f)$. Then $q \in \overline{W^s(r)} \setminus W^s(r)$ if and only if $q \in \mathcal{C}(f)$ and $W^u(q) \neq \emptyset$. Similarly, $p \in \overline{W^u(r)} \setminus W^u(r)$ if and only if $p \in \mathcal{C}(f)$ and $W^u(p) \cap W^s(p) \neq \emptyset$.

Proof. We start by proving the direction $\Rightarrow$. By lemma 2.1 we know that $W^s(r)$ is homeomorphic to an embedded open one dimensional interval. Let $x_1, x_2 \in W^s(r)$ two points in different connected components of $W^s(r) \setminus \{r\}$. Each of the sets $X_1 := \{\varphi(t, x_1)\}_{t \in \mathbb{R}}, X_2 := \{\varphi(t, x_2)\}_{t \in \mathbb{R}}$ is also homeomorphic to an embedded open one dimensional interval, and we have the disjoint decomposition $W^s(r) = X_1 \cup \{r\} \cup X_2$. As $\lim_{t \to \infty} \varphi(t, x_1) = r$ we get that $\overline{W^s(r)} \setminus W^s(r) = \{\lim_{t \to -\infty} \varphi(t, x_1), \lim_{t \to -\infty} \varphi(t, x_2)\}$. In particular we conclude that

\begin{equation}
q \in \overline{W^s(r)} \setminus W^s(r) \iff q \in \left\{ \lim_{t \to -\infty} \varphi(t, x_1), \lim_{t \to -\infty} \varphi(t, x_2) \right\}.
\end{equation}

Combining this with the implication

\begin{equation}
q \in \left\{ \lim_{t \to -\infty} \varphi(t, x_1), \lim_{t \to -\infty} \varphi(t, x_2) \right\} \Rightarrow q \in \mathcal{C}(f) \land \{x_1 \in W^u(q) \lor x_2 \in W^u(q)\},
\end{equation}

which follows from lemma 3.6 we get

\begin{equation}
q \in \overline{W^s(r)} \setminus W^s(r) \Rightarrow q \in \mathcal{C}(f) \land W^s(r) \cap W^u(q) \neq \emptyset.
\end{equation}

For the other direction, we choose some $x \in W^s(r) \cap W^u(q)$. For any sequence, $\{t_n\}$ such that $t_n \to -\infty$ we get that $\varphi(t_n, x) \to q$ and thus $q \in \overline{W^s(r)}$ as an accumulation point of a
sequence in $W^s(r)$. The assumption $q \in \mathcal{C}(f)$ gives $\nabla f \big|_q = 0$, which implies $q \notin W^s(r)$, so that $q \in W^s(r) \setminus W^s(r)$. The second part of the lemma is proven similarly. \hfill \Box

**Lemma 3.8.** If $N(f) \neq \emptyset$ then $N(f) = \left\{ \bigcup_{r \in \mathcal{S}(f)} W^s(r) \cup W^u(r) \right\} \coprod \mathcal{X}(f)$.

*Proof.* We observe that

\[
N(f) \subseteq \left\{ \bigcup_{r \in \mathcal{S}(f)} W^s(r) \cup W^u(r) \right\} \coprod \mathcal{C}(f)
\]

(3.6)

\[
= \left\{ \bigcup_{r \in \mathcal{S}(f)} W^s(r) \cup W^u(r) \right\} \coprod \mathcal{X}(f),
\]

where the first line is a deduction from lemma 3.7 and the second holds as $\mathcal{C}(f) \setminus \mathcal{X}(f) = \mathcal{S}(f) \subset \bigcup_{r \in \mathcal{S}(f)} W^s(r)$.

We proceed to show that the inclusion above is an exact equality. Let $q \in M$ be a maximum of $f$. We show that if $N(f) \neq \emptyset$ then $\exists r \in \mathcal{S}(f)$ such that $q \in W^s(r)$. Similar arguments can be used to show that if $p \in M$ is a minimum of $f$ then $\exists r \in \mathcal{S}(f)$ such that $p \in W^u(r)$ and in combination this proves the lemma.

We consider now the maximum $q \in M$ in view of the second decomposition stated in lemma 2.2. According to it, $M$ can be decomposed into (a) stable manifolds of minima, which are two dimensional simply connected subsets of $M$, (b) stable manifolds of the saddle points, which are open one-dimensional subsets of $M$ and (c) the set of all maxima. We assume that there is no saddle point, such that $q$ belongs to the closure of its stable manifold and show that this implies $N(f) = \emptyset$. By the assumption and lemma 3.7 there is an open neighborhood $U$ of $q$, which does not intersect with stable manifolds of saddle points and does not contain any other maxima. By the decomposition of $M$ from lemma 2.2 the punctured neighbourhood, $U \setminus \{q\}$, can be covered by a finite number of stable manifolds of minima. However, as these stable manifolds are open and disjoint this is only possible if $U \setminus \{q\}$ is covered by exactly one stable manifold of some minimum $p$. As $W^s(p)$ is homeomorphic to an open two-dimensional disk we conclude that $q$ is a single connected component of this stable manifold’s boundary, $\partial W^s(p)$ and this implies that $W^s(p) = M$ and $M = S^2$. In particular, this leaves no saddle points of $f$ on $M$ and therefore $N(f) = \emptyset$. \hfill \Box

Following lemma 3.8 as long as the set of Neumann lines, $N(f)$, is non-empty, we get that it is complementary to the union of the Neumann domains, as states the following proposition.

**Proposition 3.9.** If $N(f) \neq \emptyset$ then we get the following disjoint decomposition of the manifold

(3.7)

\[
M = \coprod_{p \in \mathcal{M}_-(f)} \coprod_{q \in \mathcal{M}_+(f)} \Omega_{p,q}(f) \coprod N(f).
\]

*Proof.* Note the following disjoint decomposition of the manifold

(3.8)

\[
M = \left\{ \coprod_{p \in \mathcal{M}_-(f)} \left[ W^s(p) \cap W^u(q) \right] \right\} \coprod \left\{ \coprod_{r \in \mathcal{S}(f)} \left\{ W^s(r) \cup W^u(r) \right\} \right\} \coprod \mathcal{X}(f).
\]

One can check the validity of this decomposition by separation to cases. For every $x \in M$, we get from lemma 3.6 that $\lim_{t \to \pm \infty} \varphi(t,x) \in \mathcal{C}(f)$. If $x$ is a critical point itself then both limits are equal to $x$ and $x \in \mathcal{C}(f) = \mathcal{X}(f) \setminus \mathcal{S}(f)$. Otherwise, if both limits $(\lim_{t \to \pm \infty} \varphi(t,x))$ are different and they are obtained at extremal points then

(3.9)

\[
x \in \coprod_{p \in \mathcal{M}_-(f)} \coprod_{q \in \mathcal{M}_+(f)} \left[ W^s(p) \cap W^u(q) \right].
\]
Finally, there is also the case where at least one of the limits is obtained at a saddle point and then $x \in \bigcup_{r \in \mathcal{S}(f)} \{W^s(r) \cup W^u(r)\}$. The proposition is now proven as the last two terms of the union equal $N(f)$ by lemma 3.8. 

**Remark 3.10.** Proposition 3.9 justifies that the Neumann lines and the Neumann domains define a partition of $M$.

**Remark 3.11.** Let $f$ be a Morse function with no Neumann lines, $N(f) = \emptyset$. This means that $\mathcal{S}(f) = \emptyset$ and from the proof of lemma 3.8 we conclude that $M = S^2$ and $f$ has precisely one extremum of each kind. All other cases ($N(f) \neq \emptyset$) are treated by proposition 3.9.

We are now in a position to prove a theorem which establishes the fundamental topological properties of Neumann domains.

**Theorem 3.12.** Let $M$ be a smooth, compact, two dimensional, orientable manifold without boundary and $g$ a smooth Riemannian metric on $M$. Let $f$ be a Morse eigenfunction with $\mathcal{S}(f) \neq \emptyset$. Let $p \in \mathcal{M}_-(f)$, $q \in \mathcal{M}_+(f)$ and $\Omega$ be a connected component of $W^s(p) \cap W^u(q)$, i.e., $\Omega$ is a connected Neumann domain. The following properties hold.

**Critical points location**

(i) $\mathcal{C}(f) \subset N(f)$
(ii) $\mathcal{X}(f) \cap \partial \Omega = \{p, q\}$
(iii) If $f$ is in addition a Morse-Smale function then $\partial \Omega$ consists of Neumann lines connecting saddle points with extrema. In particular, the boundary, $\partial \Omega$, contains either one or two saddle points.

**Neumann domain topology**

(iv) $\Omega$ is simply connected.

**Nodal set of $f_\Omega$**

(v) $\Omega \cap f^{-1}(0) \neq \emptyset$
(vi) Each connected component of $\Omega \cap f^{-1}(0)$ has a non-empty intersection with $\partial \Omega$.
(vii) $\Omega \cap f^{-1}(0) \cap (\nabla f)^{-1}((0,0)) = \emptyset$ (no singular points of $f$ lie in $\Omega$).
(viii) $\Omega \cap f^{-1}(0)$ is an embedding of a closed one dimensional interval whose endpoints lie on $\partial \Omega$.

**Proof.** [Proof of (i)]. From lemma 3.8, we have that $\mathcal{X}(f) \subset N(f)$. The saddle points are included in $N(f)$ by definition.

[Proof of (ii)] First we show that $p, q \in \partial \Omega$. It is easy to see that $p, q \notin \Omega$ and we therefore just need to show that $p, q \in \partial \Omega$. Start from any $x \in \Omega$. Consider the flow line which passes through $x$, $X = \{\varphi(t,x)\}_{t \in R}$. As $x \in W^s(p)$, we get by definition that $X \subset W^s(p)$. Similarly, $X \subset W^u(q)$ and therefore $X \subset \Omega$. As $\lim_{t \to \infty} \varphi(t,x) = p$, each neighborhood of $p$ has a non-empty intersection with $X$ (and hence with $\Omega$) and therefore $p \in \partial \Omega$. A similar argument shows that $q \in \partial \Omega$. Now assume by contradiction that there is some other minimum, $\tilde{p} \neq p$ such that $\tilde{p} \in \partial \Omega$. Being on the boundary, we have that $W^s(\tilde{p}) \cap \Omega \neq \emptyset$. From the definition of $\Omega$, we get $W^s(\tilde{p}) \cap W^u(p) \neq \emptyset$, which gives a contradiction. A similar argument shows that $q$ is the only maximum of $f$ which belongs to $\partial \Omega$.

[Proof of (iii)] This is an immediate deduction from the definition of a Morse-Smale function.

[Proof of (iv)] Examine the following sequence of homomorphisms between homology groups $H_n$, $H_{n-1}$.

\begin{equation}
H_n \left(W^s(p) \cup W^u(q)\right) \longrightarrow H_{n-1} \left(W^s(p) \cap W^u(q)\right) \longrightarrow H_{n-1} \left(W^s(p) \oplus H_{n-1} \left(W^u(q)\right)\right).
\end{equation}

This sequence is exact, being part of the Mayer-Vietoris sequence (cf. [5]) and using that the sets $W^s(p)$, $W^u(q)$ are open. For $n \geq 2$ we have that $H_n \left(W^s(p) \cup W^u(q)\right) = 0$ as $M$ is two-dimensional and also as $W^s(p) \cup W^u(q) \subseteq M$ (which holds as $\mathcal{S}(f) \neq \emptyset$). For $n \geq 2$ we also have that $H_{n-1} \left(W^s(p)\right) = H_{n-1} \left(W^u(q)\right) = 0$, as $W^s(p)$, $W^u(q)$ are both embeddings of an open disk. We thus conclude from the exact sequence above that $H_{n-1} \left(W^s(p) \cap W^u(q)\right) = 0$ for $n \geq 2$. In particular, for $n = 2$ we conclude that, $\Omega$, being a path connected component
of $W^s(p) \cap W^u(q)$ is simply connected.

[Proof of (v)] Since $f$ is an eigenfunction of the Laplacian, its maxima are positive and minima are negative. In particular, $f(p) > 0$ and $f(q) < 0$ and by continuity $f$ must vanish somewhere in $\Omega$.

[Proof of (vi)] Assume by contradiction that there is a connected component of $\Omega \cap f^{-1}(0)$ which does not intersect $\partial \Omega$. As $\Omega$ is simply connected, this means that there is some nodal domain $\omega$, which is fully contained in $\Omega$, $\omega \subset \Omega$. Each nodal domain contains at least one extremal point, which implies the existence of an extremal point in $\Omega$. This contradicts (i).

[Proof of (vii)] This is actually a particular case of (i) as $(\nabla f)^{-1}((0,0)) = \partial \omega(f)$. The statement is worthwhile to be mentioned as the set $f^{-1}(0) \cap (\nabla f)^{-1}((0,0))$ has a special role, being the set of the crossing points of the nodal lines of $f$.

[Proof of (viii)] We deduce from (iv) that $\Omega$ has a single boundary connected component. This boundary, $\partial \Omega$, decomposes to two curves, $\gamma_1, \gamma_2$, whose endpoints are $p, q$. Namely, $\partial \Omega = \gamma_1 \cup \gamma_2$ and $\gamma_1 \cap \gamma_2 = \{p, q\}$. The restriction, $f|_{\partial \Omega}$, is monotonic on $\gamma_1$ and $\gamma_2$. As $f(p) < 0$, $f(q) > 0$ we conclude that $f|_{\partial \Omega}$ vanishes exactly at two points, $x \in \gamma_1$, $y \in \gamma_2$. The nodal set, $f^{-1}(0)$, on a two dimensional manifold is topologically equivalent to an embedded graph, i.e., a union of one-dimensional manifolds which intersect at some points [10]. These points are where nodal lines cross and they do not belong to the interior of the Neumann domain following (vii). We therefore get that the mentioned graph consists of a union of curves whose endpoints are $x, y$. Yet, if there is more than one curve in this union, this implies the existence of a nodal domain $\omega$, fully contained within $\Omega$, which was already ruled out in (vi).

Remark 3.13. Note that the assumption that $f$ is an eigenfunction is not essential for some parts of the theorem above. In sections (iii)-(iv) we only need to assume that $f$ is a Morse function (and also Morse-Smale in section (iii)). In section (v) we use the observation that all maxima (minima) of an eigenfunction are positive (negative). In sections (vi)-(viii) we use structural properties of the nodal set of eigenfunctions.

Theorem 3.12 implies that the eigenfunction restriction to a Neumann domain, $f|_\Omega$, has a relatively simple structure. According to claim (ii), $f|_\Omega$ does not have any critical points. Claim (iii) shows that there are only two extremal points of $f|_\Omega$, which lie on the boundary, $\partial \Omega$, and they are exactly the defining minimum and maximum, $p, q$, of the Neumann domain, $\Omega_{p,q} = W^s(p) \cap W^u(q)$. According to claim (iv) a connected Neumann domain, $\Omega$, is also simply connected and this is used in proving claims (vi)-(vii), which deal with the nodal set contained within $\Omega$. These claims yield claim (viii), which shows that the nodal pattern of $f|_\Omega$ is simple; it is a single line without self intersections and with two endpoints on the boundary, $\partial \Omega$. For the additional assumption of Morse-Smale, a typical structure of $f|_\Omega$ is demonstrated in figure 3.2. For a non Morse-Smale function it is possible that there are more than two saddle points on the boundary of the Neumann domain.

The properties established above allow to give an estimate on the diameter of some Neumann domains - cf. theorem 5.2. In addition, as $f|_\Omega$ is a solution for a Neumann eigenvalue problem on $\Omega$, and due to the structural simplicity of $f|_\Omega$, we are tempted to conclude that the position of $f|_\Omega$ in the spectrum of the Neumann Laplacian on $\Omega$ is bounded. Furthermore, one might assume that $f|_\Omega$ is the first non constant eigenfunction of the Neumann eigenvalue problem on $\Omega$. This, however, is not always the case, as is shown and discussed in section 5.2.

4. Manifolds with Dirichlet boundary

In this section we discuss the structure of Neumann domains on simply-connected, compact euclidean 2-manifolds with boundary, i.e., subsets of $\mathbb{R}^2$. We are interested in this case, since many of the explicit examples which are used to study the characteristic structures of eigenfunctions are of this type (also known as billiards [3]). We will restrict, however, to a certain regularity class of boundaries, which avoids potential pathological behavior of the
eigenfunction near the boundary. In view of this, let \((M, g)\) be a simply-connected 2-manifold with \(g\) the flat metric with piecewise smooth boundary, such that at each non-smooth point \(p\) on the boundary, the left- and right-normal vector field meet at an angle of \(\pi/2\). The angle \(\pi/2\), is compatible with the structure of eigenfunctions at intersections of nodal lines so that the eigenfunction can still be a Morse function at these corners. This fails for angles of the form \(\pi/n\) for \(n \in \mathbb{N}\) with \(n > 2\) (cf. [10]). For the admissible angle \(\pi/2\), eigenfunctions on \(M\) of this type can be continued to an open set in \(\mathbb{R}^2\), which contains \(M\). This in turn implies that the stable and unstable manifolds of the extended eigenfunction on \(M\) are of similar form to the case without boundary, which allows us to use similar techniques as in the previous section. We use the notation \(\mathcal{J}(f)\) for the set of saddle points, which now also includes the saddle points of \(f\) on \(\partial M\), and similarly for \(\mathcal{C}(f)\). For extrema the sets remain the same; these cannot lie on the boundary since \(f\) is an eigenfunction with Dirichlet boundary conditions. In the following we present the analogue of definitions 3.1 and 3.3 for manifolds with boundary (which is definition 4.4 and its preceding discussion) and prove that the results stated in proposition 3.9 and theorem 3.12 hold in a slightly deformed form in this case as well (the analogues are proposition 4.9 and theorem 4.10).

4.1. Gradient flow in the boundary case. Before discussing the structure of stable and unstable manifolds, we introduce an adapted version of the gradient flow given in (2.2), for manifolds with boundary. For this purpose we need to separate the tangent space, \(T_xM\), at some boundary point \(x \in \partial M\) into its interior and exterior part. Specifically, if \(x \in \partial M\) is a point where the boundary is smooth and \(\hat{n}_x\) is the normal to the boundary at \(x\) (directed outwards), then we denote

\[
\text{Int} T_xM = \{X \in T_xM | \langle X, \hat{n}_x \rangle \leq 0 \}.
\]

If, however, the boundary \(\partial M\) is not smooth at \(x \in \partial M\), we should slightly modify the definition above using the limits of the normal vectors from both sides (as the boundary is piecewise smooth). Denote these limits as \(\hat{n}_x^{(+)}\) and \(\hat{n}_x^{(-)}\) (both belonging to \(T_xM\)) and define

\[
\text{Int} T_xM = \{X \in T_xM | \langle X, \hat{n}_x^{(-)} \rangle \leq 0 \text{ and } \langle X, \hat{n}_x^{(+)} \rangle \leq 0 \}.
\]
The gradient flow $\phi_x$ through a point $x \in M$, where $M$ is a manifold with boundary is defined by

$$
\phi_x : I_x \to M,
$$

where $I_x \subset \mathbb{R}$ is the maximal interval of existence for the flow $\phi_x$, which depends on $x$ and is in general not equal to $\mathbb{R}$. We have

$$
I_x = \begin{cases}
(0, \infty) & \text{if } x \in \partial M \text{ and } -\nabla f|_{\phi_x(t)} \in \text{Int}T_{\phi_x(t)}M \\
(0, \infty) & \text{if } x \in \partial M \text{ and } -\nabla f|_{\phi_x(t)} \notin \text{Int}T_{\phi_x(t)}M \\
[t_-(x), \infty) & \text{if } x \in \text{Int} M \text{ and the flow line through } x \text{ does not emanate from } \partial M
\end{cases}
$$

Here $t_-(x) < 0$. This definition of the flow allows to define the stable and unstable manifolds as

$$
W^s(x) = \{ y \in M \mid \lim_{t \to \sup I_y} \phi_y(t) = x \}
$$

$$
W^u(x) = \{ y \in M \mid \lim_{t \to \inf I_y} \phi_y(t) = x \}.
$$

Note that the above also holds for $x \in \partial M$. In order to prove properties of the stable and unstable manifolds, we need the following lemma.

**Lemma 4.1.** [follows from Proposition 3.18 in [2]] Let $x \in M$ and $t \in I_x$ be such that $\phi_x(t) \notin \partial M$. Then

$$
\frac{d}{dt} f(\phi_x(t)) = -\| (\nabla f)(\phi_x(t)) \|^2 \leq 0.
$$

The next lemma is analogous to lemma 2.1 but for the case with boundary.

**Lemma 4.2.** Let $f$ be a Morse eigenfunction on $M$ as above and let $p \in \mathcal{C}(f)$. Let $\lambda_p$ be the Morse index of $f$ at $p$. The intersection of the stable (unstable) manifold with the interior of $M$, $W^s(p) \cap \text{int} M$ ($W^u(p) \cap \text{int} M$) is an embedded open disk of dimension $2 - \lambda_p (\lambda_p)$.

**Remark 4.3.** If $p \in \mathcal{C}(f)$ is a saddle point it is possible that $p \in \partial M$. In that case the stable and unstable manifold of $p$ are defined as before.

**Proof.** We prove the statement above separately for the three different cases, $\lambda_p = 2, 1, 0$. We prove them only for $W^s(p) \cap \text{int} M$. The first case ($\lambda_p = 2$) holds due to the fact that extrema of eigenfunctions with Dirichlet boundary conditions belong to the interior of $M$, therefore lemma 2.1 applies to this case and $W^s(p) \cap \text{int} M = \{p\}$.

For the second and third cases ($\lambda_p = 1, 0$), we use the fact that under the assumption on the regularity of the boundary we can extend the eigenfunction to a smooth Morse function $\tilde{f}$ on $\mathbb{R}^2$ whose stable and unstable manifolds, that intersect with $M$ are bounded.

We now prove the $\lambda_p = 1$ case. Denote the stable manifold of $p$ with respect to $\tilde{f}$ by $\tilde{W}^s(p)$. This stable manifold is defined with respect to the standard gradient flow on $\mathbb{R}^2$, which we denote by $\tilde{\phi} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. Note that the flows $\tilde{\phi}$ and $\phi_x$ for some $x \in M$ coincide in the interior of $M$. By lemma 2.1 the stable manifold $\tilde{W}^s(p)$ is an embedded open interval in $\mathbb{R}^2$. If its intersection with the interior of $M$ is connected, this would imply the claim. If $\tilde{W}^s(p) \cap \text{int} M$ is not connected, then one of the integral curves in $\tilde{W}^s(p)$ (i.e., $\{\tilde{\phi}(t, x)\}_{t \in \mathbb{R}}$ which is contained in $\tilde{W}^s(p)$) must cross $\partial M$ at least at two points. Denote these points on the boundary by $x_1, x_2$ and let $t_2 > 0$ such that $x_2 = \tilde{\phi}(t_2, x_1)$. According to lemma 4.1 the values of $\tilde{f}$ decrease monotonically along the flow line, $\{\tilde{\phi}(t; x_1)\}_{0 \leq t \leq t_2}$. As $\tilde{f}$ vanishes at both $x_1$ and
we conclude from lemma 4.1 that \( \forall 0 \leq t \leq t_2 \),
\[
\frac{d}{dt} f (\tilde{\varphi}(x; t)) = -\| (\nabla f) (\tilde{\varphi}(x; t)) \|^2 = 0.
\]
This continuum of critical points contradicts \( \tilde{f} \) being a Morse function and finishes the proof of the second claim.

To prove the third case \( (\lambda_p = 0) \), let \( p \in \mathcal{M}_-(f) \). Then the stable manifold \( \tilde{W}^s(p) \) is an embedded open disk in \( \mathbb{R}^2 \) and in particular simply connected. Since \( M \) is simply connected, we may use the same argument as in the proof of theorem 3.12 to conclude that \( \tilde{W}^s(p) \cap \text{int} M \) is simply connected if it is path connected. The set \( \tilde{W}^s(p) \cap \text{int} M \) is therefore a simply connected open set and hence homeomorphic to a two-dimensional disk. \( \square \)

4.2. Neumann domains in the boundary case. Let \( M \) be a connected compact 2-manifold with boundary as described in the beginning of this section and consider a Morse-eigenfunction \( f \) which obeys Dirichlet boundary conditions on \( \partial M \). The definition of the set of Neumann lines, \( \tilde{N}(f) \), in this case is unaltered and is still given by definition 3.3. The definition of Neumann domains, however, should be modified as follows.

**Definition 4.4.** A Neumann domain of a manifold \( M \) with boundary as above is any set, \( \Omega \), which has one of the following forms:

(i) \( \Omega_{p, q}(f) = \tilde{W}^s(p) \cap W^u(q) \), where \( p \in \mathcal{M}_+(f) \), \( q \in \mathcal{M}_-(f) \).

(ii) \( \Omega_{p, \partial}(f) = \tilde{W}^s(p) \cap \left( \bigcup_{y \in \partial M \setminus \mathcal{J}(f)} W^u(y) \right) \), where \( p \in \mathcal{M}_-(f) \).

(iii) \( \Omega_{\partial, q}(f) = \left( \bigcup_{y \in \partial M \setminus \mathcal{J}(f)} \tilde{W}^s(y) \right) \cap W^u(q) \), where \( q \in \mathcal{M}_+(f) \).

Neumann domains of type (1) are called inner Neumann domains and those of types (2) and (3) are called boundary Neumann domains. As in definition 3.3, any connected component of the above is called a connected Neumann domain.

Similarly to lemma 2.2, we have an analogue decomposition of \( M \).

**Lemma 4.5.** Let \( f \) be as above, then we have the following disjoint decompositions
\[
M = \bigsqcup_{x \in \mathcal{C}(f)} W^u(x) \bigsqcup_{y \in \partial M \setminus \mathcal{J}(f)} W^u(y).
\]

Similarly,
\[
M = \bigsqcup_{x \in \mathcal{C}(f)} \tilde{W}^s(x) \bigsqcup_{y \in \partial M \setminus \mathcal{J}(f)} \tilde{W}^s(y).
\]

Note that \( M \) above includes the boundary, \( \partial M \).

**Proof.** Both decompositions follow as each point belongs to a unique (un)stable manifold and there are no extremal points on a Dirichlet boundary. \( \square \)

The following lemma is the analogue of lemma 3.6.

**Lemma 4.6.** \( \forall x \in M \), both limits \( \lim_{t \to \sup I_x} \varphi_x(t) \) and \( \lim_{t \to \inf I_x} \varphi_x(t) \) exist and each is either a critical point of \( f \) or a boundary point of \( M \), i.e., \( \lim_{t \to \sup I_x} \varphi_x(t), \lim_{t \to \inf I_x} \varphi_x(t) \in \mathcal{C}(f) \cup \partial M \).

**Proof.** Let \( x \in M \). If \( \{ \varphi_x(t) \}_{t \in I_x} \cap \partial M = \emptyset \) then lemma 3.6 applies and we get that \( I_x = (-\infty, \infty) \) and that the limits \( \lim_{t \to \sup I_x} \varphi_x(t) \) and \( \lim_{t \to \inf I_x} \varphi_x(t) \) exist and both belong to \( \mathcal{C}(f) \) (and it might be that one this limits belongs to the boundary, \( \partial M \)). Otherwise, there exists \( t \in I_x \) such that \( \varphi_x(t) = y \) and \( y \in \partial M \). Assume that \( t < 0 \) (the \( t > 0 \) case can be proved similarly). Note that due to the reversibility of the flow, the gradient cannot vanish at \( y \), i.e., \( \nabla f|_y \neq 0 \). We conclude that \( y \) cannot be a corner of the boundary as this would imply \( y \in \mathcal{J}(f) \). Therefore \( y \) belongs to the smooth part of the boundary and \( \nabla f|_y \) is orthogonal
Lemma 4.7. Let \( r \in \mathcal{S}(f) \). Then \( q \in \overline{W^s(r)} \setminus W^s(r) \) if and only if \( q \in \mathcal{C}(f) \) and \( W^s(r) \cap W^u(q) \neq \emptyset \). Similarly, \( p \in \overline{W^u(r)} \setminus W^u(r) \) if and only if \( p \in \mathcal{C}(f) \) and \( W^u(r) \cap W^s(p) \neq \emptyset \).

Proof. The proof of direction \((\Leftarrow)\) is identical to that of lemma 3.7. The proof of the other direction is only slightly modified (using lemmata 4.2, 4.6 which are analogous to lemmata 2.1, 3.6) and is not repeated. The only proof ingredient which we do mention here concerns points on the boundary \( \partial M \). Let \( y \in \partial M \cap \overline{W^s(r)} \). Then we have that \( y \in W^s(r) \) if and only if \( y \notin \mathcal{C}(f) \). In particular if \( y \in \partial M \) then \( y \in \overline{W^s(r)} \setminus W^s(r) \) if and only if \( y \notin \mathcal{C}(f) \).

The main content of the following lemma (similarly to lemma 3.8) is that all extremal points belong to the set of Neumann lines, \( N(f) \). The proof is somewhat more involved here than the proof of the analogous lemma in the non-boundary case. Intuitively, this can be understood as following. Once an eigenfunction on a manifold without boundary, \( M \), is given, one can obtain a sub-manifold, \( M \) with a boundary by restricting oneself to the interior of some nodal line. This results with an eigenfunction \( \tilde{f} \) on the manifold \( \tilde{M} \) with a Dirichlet boundary. It might occur that \( N(\tilde{f})|_{\tilde{M}} \subsetneq N(f)|_{\tilde{M}} \). Namely, some Neumann lines of \( f \) might be absent from \( \tilde{f} \) even if originally they had a non-empty intersection with \( \tilde{M} \). This happens exactly if the saddle point attached to such a Neumann line does not belong to \( \tilde{M} \). In particular, having less Neumann lines means that it is harder to guarantee that all extremal points belong to the Neumann lines, in the boundary case. Hence the difference in the complexity of the proofs.

Lemma 4.8. If \( N(f) \neq \emptyset \) then

\[
N(f) = \left\{ \bigcup_{r \in \mathcal{S}(f)} W^s(r) \cup W^u(r) \right\} \bigcap \mathcal{S}^c(f).
\]

Proof. This proof partly follows the lines of the one for lemma 3.8 if we replace lemmata 2.1, 3.6 and 3.7 by the analogous lemmata 4.2, 4.6 and 4.7. However, this proof deviates at some point and additional arguments are supplied. We observe that

\[
N(f) \subseteq \left\{ \bigcup_{r \in \mathcal{S}(f)} W^s(r) \cup W^u(r) \right\} \bigcap \mathcal{C}(f)
\]

\[
= \left\{ \bigcup_{r \in \mathcal{S}(f)} W^s(r) \cup W^u(r) \right\} \bigcap \mathcal{S}^c(f),
\]

where the first line is a deduction from lemma 4.7 and the second equality holds as \( \mathcal{C}(f) \setminus \mathcal{S}^c(f) = \mathcal{S}(f) \subseteq \bigcup_{r \in \mathcal{S}(f)} W^s(r) \).

We proceed to show that the relation above is an exact equality. Let \( q \) be a maximum of \( f \). We show that \( \exists r \in \mathcal{S}(f) \) such that \( q \in \overline{W^s(r)} \) (the condition \( N(f) \neq \emptyset \) is not even needed in such a case). Similar arguments can be used to show that if \( p \) is a minimum of \( f \) then \( \exists r \in \mathcal{S}(f) \) such that \( p \in \overline{W^u(r)} \) and in combination this proves the lemma. Examine \( W^u(q) \). If its intersection with the boundary \( \partial M \) is empty, we may proceed as in the proof of lemma 3.8 to show the existence of \( r \in \mathcal{S}(f) \) such that \( q \in \overline{W^s(r)} \). Otherwise, we proceed as follows. If \( \partial M \subset W^u(q) \) we conclude \( \mathcal{S}(f) = \emptyset \) and therefore also \( N(f) = \emptyset \). This conclusion owes to \( W^u(q) \cap \text{int} M \) being simply connected (lemma 4.2), so that its boundary equals \( \partial M \) and therefore \( M = W^u(q) \), which implies \( \mathcal{S}(f) = \emptyset \).

Otherwise, we proceed by extending \( f \) to a Morse function \( \tilde{f} \) on \( \mathbb{R}^2 \) and use the notation \( \tilde{W}^u(q) \) for the unstable manifold of \( q \) with respect to \( \tilde{f} \) (as in the proof of lemma 4.2). Now, consider \( \partial\tilde{W}^u(q) \). Note that the boundary is taken with respect to the topology of \( \tilde{M} \), therefore \( \partial M \cap \partial\tilde{W}^u(q) = \partial M \cap \partial W^u(q) \) (In the RHS the boundary is taken with respect to the
Note that $(4.13)$ holds.

Proposition 4.9. If $N(f) \neq \emptyset$ then the following disjoint decomposition of the manifold holds.

$$
M = \bigsqcup_{p \in \mathcal{M}_+(f)} \{\Omega_{p,q}(f)\} \bigcup_{q \in \mathcal{M}_+(f)} \{\Omega_{p,q}(f)\} \bigcup_{q \in \mathcal{M}_+(f)} \{\Omega_{q,p}(f)\} \bigcup N(f) 
$$

Note that $M$ above includes the boundary $\partial M$. 

Proof. Note the following disjoint decomposition of the manifold

\[
M = \left\{ \prod_{p \in \mathcal{M}_-(f)} [W^s(p) \cap W^u(q)] \prod_{q \in \mathcal{M}_+(f)} \left[ W^s(p) \cap \left( \bigcup_{y \in \partial M \setminus \mathcal{E}(f)} W^u(y) \right) \right] \prod_{q \in \mathcal{E}(f)} \left\{ W^s(r) \cup W^u(r) \right\} \prod \mathcal{D}^+(f), \right. 
\]

(4.14)

whose validity follows from separation into cases (see beginning of the proof of proposition 3.9) together with \( f \) having no extremal points on the Dirichlet boundary. The proposition now follows from definition 4.4 and lemma 4.8. \( \square \)

The following structural theorem provides the same results as theorem 3.12 for inner Neumann domains and analogous results for the boundary Neumann domains.

**Theorem 4.10.** Let \( M \) be a smooth, simply-connected, two dimensional manifold with piecewise smooth boundary \( \partial M \) and possibly some corners of angle \( \pi/2 \) (in case of non-smooth boundary) and \( g \) the flat metric on \( M \). Let \( f \) be a Morse eigenfunction of \( -\Delta_g \) which obeys Dirichlet boundary conditions and is such that \( \mathcal{S}(f) \neq \emptyset \).

The following holds.

(i) \( \mathcal{E}(f) \subset N(f) \) (which is claim (i) of theorem 3.12)

**Inner Neumann domains**

(ii) Claims (ii)-(viii) of theorem 3.12 hold for all inner Neumann domains of \( f \).

**Boundary Neumann domains**

(iii) Let \( p \in \mathcal{M}_-(f) \) and let \( \Omega \) be a connected component of \( \Omega_{p, \partial M \setminus \mathcal{E}(f)} \). Then

(iv) \( \partial \overline{\Omega} \cap \mathcal{D}^-(f) = \{ p \} \).

(v) \( \Omega \) is simply connected.

(vi) \( f|_{\Omega \cap \partial M} < 0 \) and therefore \( \Omega \cap f^{-1}(0) = \Omega \cap \partial M \).

Analogous claims hold for boundary Neumann domains of maxima.

**Proof.** Claims (i)-(ii) here are proven identically as in theorem 3.12 (with lemma 4.8 as the analogue of lemma (3.8)). Claim (v) above for boundary Neumann domains is also proven as its analogue (claim (iv)) from theorem 3.12.

[Proof of (iv)] Assume by contradiction that there exists \( x \in \Omega \cap \partial M \) such that \( f(x) \geq 0 \). Consider the set \( \{ \varphi_x(t) \}_t \in (t_0, \infty)\). By definition of \( \Omega \), \( \lim_{t \to \inf I_x} \varphi_x(t) = \partial M \) and therefore \( \lim_{t \to \inf I_x} f(\varphi_x(t)) = 0 \). Similarly, \( \lim_{t \to \sup I_x} f(\varphi_x(t)) < 0 \), as \( f \) has negative sign at its minimum points, being an eigenfunction. Lemma 4.4 states that \( f \) cannot increase along gradient lines and therefore for all \( y \in \{ \varphi_x(t) \}_t \in I_x \), \( f(y) = 0 \) and we conclude that for \( 0 < t \in I_x \), \( \frac{d}{dt} f(\varphi_x(t)) = -\| (\nabla f)(\varphi_x(t)) \| ^2 = 0 \). We get a set \( \{ \varphi_x(t) \}_{t \in (t_0, \infty)} \) of non-isolated critical points of \( f \), in contradiction to \( f \) being a Morse function. We therefore have \( f|_{\Omega \cap \partial M} < 0 \) and conclude \( \Omega \cap f^{-1}(0) = \Omega \cap \partial M \).

[Proof of (vii)] Proving that \( \partial \overline{\Omega} \cap \mathcal{M}_-(f) = \{ p \} \) is done similarly to the analogue claim from theorem 3.12 (claim (ii)). It is then left to show that \( \partial \overline{\Omega} \cap \mathcal{M}_+(f) = \emptyset \). As \( f|_{\Omega \cap \partial M} < 0 \) by
claim (vii), we conclude $f|_{\Omega} = f|_{\Omega \setminus \partial M} \leq 0$. Since $f$ is positive at its maximum points, being an eigenfunction, we conclude that $\Omega \cap M_+ = \emptyset$ as needed. \hfill\Box

In addition to the theorem above, one can make some straightforward observations regarding Neumann lines which intersect with the boundary. We first note that every critical point on the boundary is a saddle point. In the vicinity of those saddle points the eigenfunction behaves locally as it does in a neighbourhood of nodal line intersection (which is like a harmonic polynomial, \cite{10}). The Neumann line structure near those boundary saddle points can be deduced from lemma \cite{3.4}. More explicitly, there are two types of saddle points at the boundary. Those that are located at corners of the manifold (each corner has such a saddle point) and those who are located at a point on the smooth part of the boundary. The former have a single Neumann line to which they are connected. The latter are connected to two perpendicular Neumann lines and to a nodal line which lies in between. In addition, there are also Neumann lines which perpendicularly intersect the boundary at points which are not saddles.

5. GEOMETRIC AND SPECTRAL PROPERTIES OF NEUMANN DOMAINS

We have discussed so far the topological structure of Neumann domains on both closed manifolds and a class of domains with Dirichlet boundary conditions. We proceed by pointing out a connection between the aforementioned results and geometric and spectral properties of Neumann domains.

5.1. On the outer radius of Neumann domains. The volume of nodal domains of an eigenfunction is bounded from below in terms of the eigenvalue (by Faber-Krahn inequality, \cite{15, 25}). A lower bound for Neumann domains does not exist. In particular, there are continuous families of eigenfunctions of a fixed multiple eigenvalue on the standard 2-torus which possess Neumann domains whose volumes go to zero (cf. \cite{30}).

There is, however, a geometric bound on the shape of Neumann domains which concerns their outer radius.

**Definition 5.1.** Let $(M, g)$ be a two-dimensional Riemannian manifold with or without boundary and $\Omega$ an open simply-connected subset of $M$. Let $B_r(x_0)$ denote a geodesic ball of radius $r$ around $x_0$. Then we define the outer radius $R(\Omega)$, by

\begin{equation}
R(\Omega) = \inf\{r > 0 : \exists x_0 \in M : \Omega \subset B_r(x_0)\}.
\end{equation}

**Theorem 5.2.** Let $(M, g)$ be a two-dimensional Riemannian manifold without boundary and $f$ a Morse-eigenfunction of the Laplace-Beltrami operator with eigenvalue $\lambda$. Let $\nu$ denote the number of nodal domains of $f$. Then there exists a real positive constant $C$ only depending on the metric $g$ such that for at least $[\nu/2]$ Neumann domains $\{\Omega_i\}_{1 \leq i \leq \lfloor \nu/2 \rfloor}$ of $f$

\begin{equation}
R(\Omega_i) \geq C\lambda^{-1/2}.
\end{equation}

**Proof.** This theorem follows from the structure of Neumann domains (theorem 3.12) and the bound on the inner radius of nodal domains by Mangoubi \cite{28} (see also \cite{31}). Each nodal domain of $f$ has a global extremum. Each of these $\nu$ extrema are members of the boundary of at least one Neumann domain each, by theorem 3.12. Let $q$ be one of those maxima and $D_q$, the corresponding nodal domain of $f$. By section 3 of \cite{28} there is a positive constant $C'$ independent of $\lambda$ and a geodesic ball $B_{C'/\sqrt{\lambda}}(q)$ such that $B_{C'/\sqrt{\lambda}}(q) \subset D_q$. Let $\Omega$ be a Neumann domain such that $q \in \partial \Omega$ and let $p$ be the unique minimum on $\partial \Omega$ (not necessarily a global minimum). Let $\gamma(p, q)$ be the geodesic ray between $p$ and $q$ and $d(p, q)$ its length. As $f(p) < 0$, $f(q) > 0$ and by continuity of $f$, there exists $x \in \gamma(p, q)$ such that $f(x) = 0$. We therefore get that

\begin{equation}
R(\Omega_i) \geq \frac{1}{2}d(p, q) \geq \frac{1}{2}d(x, q) \geq \frac{1}{2}C'\lambda^{-1/2}.
\end{equation}

\hfill\Box
Remark 5.3. The number of Neumann domains for which the theorem holds (currently $\lfloor \nu/2 \rfloor$) may be improved by studying the number of Neumann lines to which the extremal points are connected. This number equals the number of Neumann domains which share the same extremal point on their boundary and we call it the degree of the extremal point (see also section [6]).

The theorem above shows that there is always a certain number of “large-diameter” Neumann domains.

Remark 5.4. There is no general upper bound on the outer radius of Neumann domains. This can be demonstrated on the following family of separable eigenfunctions on the unit torus, \( \{ \cos (2\pi x) \cos (2\pi ny) \}_{n=1}^{\infty} \) (see figure 5.1). All of those eigenfunctions possess Neumann domains whose diameter equals \( 1/2 \).

5.2. On the restriction of eigenfunctions to Neumann domains. A fundamental feature of eigenfunctions is the fact that their restriction to any nodal domain equals the first eigenfunction of this domain. This has been used already in Pleijel’s asymptotic result for the nodal count [33]. It is therefore natural to ask whether a similar statement holds for Neumann domains. The restriction of an eigenfunction to one of its Neumann domains corresponds to an eigenfunction with Neumann boundary conditions on that domain. Due to section (iii) of theorem 3.12 the restriction of the eigenfunction to a Neumann domain \( \Omega \) cannot correspond to constant eigenfunctions. In addition, sections (i), (ii), (iv) and (viii) suggest that the restriction of an eigenfunction to one of its Neumann domains cannot have too rich a structure. This might lead one to conjecture for example that there exists a positive \( k \) such that for every Neumann domain \( \Omega \), the restricted eigenfunction, \( f|_{\Omega} \), is at most the \( k \)-th eigenfunction (of the restricted eigenproblem). Indeed, in [37] Zelditch conjectures “possibly it is ‘often’ the first non-constant Neumann eigenfunction.”

However, we provide here a counterexample, showing that the position of the ‘global’ eigenvalue in the spectrum of a single Neumann domain is not always (i.e., for all manifolds) bounded from above.

For a domain \( \Omega \) and an eigenvalue \( \lambda \) in the spectrum of the Laplacian on \( \Omega \) with Neumann boundary conditions, let the position of \( \lambda \) in the spectrum be denoted by \( \text{pos}(\lambda, \Omega) \) (we use here the convention \( \text{pos}(0, \Omega) = 0 \)). Let \( f \) be an eigenfunction on the manifold, \( M \), with eigenvalue \( \lambda \), then we define
\[
\text{p}(\lambda, f) := \max_{\Omega} \{ \text{pos}(\lambda, \Omega) : \Omega \text{ is Neumann domain of } f \}. \tag{5.4}
\]

We are now able to formulate the following.

Lemma 5.5. There exists a two-dimensional Riemannian manifold \((M, g)\) for which the set
\[
\{ \text{p}(\lambda, f) \mid (\lambda, f) \text{ is an eigenpair of } M \}
\]

is not bounded from above.

Before we present the counterexample we remark that it does not rule out that there are specific classes of manifolds or domains \( M \) for which there is an upper bound.

Proof. Let \((M, g)\) be a two-dimensional manifold and assume by contradiction that there exists an \( N \in \mathbb{N} \) such that
\[
\forall (\lambda, f) \quad \text{p}(\lambda, f) \leq N. \tag{5.6}
\]

By [26] we have the following bound
\[
\forall \lambda, f, \Omega \quad \lambda \leq \frac{8\pi N}{A(\Omega)}, \tag{5.7}
\]

where \( A(\Omega) \) denotes the area of the Neumann domain. Summing \( \lambda A(\Omega) \leq 8\pi N \) over all Neumann domains one gets \( \lambda A(M) \leq 8\pi N \mu \) and we obtain that the number of Neumann domains \( \mu \) obeys the following lower bound
\[
\mu \geq A(M) \frac{\lambda}{8\pi N}. \tag{5.8}
\]
We now point on an example which contradicts the bound (5.8). Consider the unit flat torus $T = [0, 1] \times [0, 1]$ with the euclidean metric and the following eigenfunction

$$f(x, y) = \cos (2\pi n_x x) \cos (2\pi n_y y),$$

with eigenvalue and number of Neumann domains

$$\lambda = 4\pi^2 (n_x^2 + n_y^2),$$
$$\mu = 8n_x n_y,$$

respectively (see figure 5.1). The contradiction with (5.8) can be easily seen if one chooses $n_x = 1, n_y \gg 1$.

One may get an insight on the contradiction in the proof above by investigating the shape of the Neumann domains obtained from the choice $n_x = 1, n_y \gg 1$. The eigenfunction (5.9) on the flat torus has Neumann domains of two distinguished shapes, which we call lense-like, and star-like (figure 5.1). We show in the following that for a sufficiently large value of $n_y/n_x$ the eigenfunction restriction to a lense-like Neumann domain does not equal the second eigenvalue of this domain.

**Lemma 5.6.** Let $(M, g)$ be the unit flat 2-dim. torus and $f_{(n_x, n_y)}(x, y) = \cos(2\pi n_x x) \cos(2\pi n_y y)$ the eigenfunction with $n_x, n_y \in \mathbb{Z}$. Let $\Omega$ be a lense-like Neumann domain of $f_{(n_x, n_y)}$. Then

$$\exists c > 0 \text{ such that } n_y/n_x > c \Rightarrow |x| < C\ell_y, \Omega \rangle > 1$$

**Proof.** The major and minor axes of the lense-like Neumann domain, $\Omega$, are of lengths $\ell_x = 1/2n_x, \ell_y = 1/2n_y$. This Neumann domain is convex and we may apply for it theorem 1.2(a) of [21]. According to that theorem, for a fixed value of $\ell_x$ (obeying $\ell_x > \ell_y$) there is a constant $C$ ($\ell_x$ dependent) such that the nodal set of the second eigenfunction is contained within a vertical strip of width $2C\ell_y$ around the centre of $\Omega$. Namely, if $\varphi(x, y) = 0 \Rightarrow |x| < C\ell_y$, with the origin taken at the centre of $\Omega$. Since in our case, the nodal set of $f_{(n_x, n_y)}(x, y) \Omega$ is horizontal along $\Omega$ (see figure 5.1), we conclude that for small enough value of $\ell_y$ the nodal set will not belong to the allowed strip and therefore $f_{(n_x, n_y)}(x, y) \Omega$ cannot be the second eigenfunction of $\Omega$. □

6. the number of Neumann domains

**Corollary 6.1.** [of theorem 3.12] Let $(M, g)$ be as in theorem 3.12 and $f$ a Morse eigenfunction on $M$, then the number of Neumann domains of $f$ is bounded from below by the number of nodal domains, i.e., $\mu \geq 2\nu$.

**Proof.** The statement follows from theorem 3.12(viii), where it was proved that a Neumann domain contains only a single nodal line. Hence, each Neumann domain intersects with exactly two nodal domains. Some of those nodal domains intersect with different Neumann domains, which gives the inequality. □
The number of Neumann domains is determined by the number of Neumann lines connected to each of the critical points of the eigenfunction. We therefore define the degree of a critical point, \( \deg(x) \), as the number of Neumann lines which are connected to \( x \). As each Neumann domain has a single minimum and a single maximum on its boundary we obtain that the Neumann domain count of a Morse eigenfunction is

\[
\mu = \sum_{p \in \mathcal{M}^{-}(f)} \deg(p) = \sum_{q \in \mathcal{M}^{+}(f)} \deg(q).
\]

The connection of the Neumann domain count to saddle points is more subtle. On the one hand, assuming a Morse eigenfunction, all saddles have degree equal to four (lemma 3.4). On the other hand, it is a priori not known how many saddles are located on the boundary of each Neumann domain. If we further assume that the eigenfunction is Morse-Smale, we obtain (theorem 3.12(iii)) that there are either one or two saddle points on the Neumann domain boundary. A Neumann domain with only a single saddle point on its boundary must have also an extrema of degree one on its boundary. Numerical explorations suggest that such domains either do not appear at all or are non-generic. If we indeed rule out this case, and assume that all Neumann domains have exactly two saddle points on their boundary, we obtain

\[
\mu = 2|\mathcal{S}(f)|.
\]

The relations above motivate the study of the degree of extremal points even if just in the distributional sense.

Finally, let us discuss the asymptotics of the Neumann domain count. Numerical experiments suggest that the number of Neumann domains goes to infinity as \( \lambda \to \infty \). This is the case even for sequences of eigenfunctions for which the number of nodal domains is bounded (see for example figure 6.1 which describes the well-known example by Stern given in [13]). However, the statement above does not hold for all metrics. There are known examples of metrics on the torus constructed by Jakobson and Nadirashvili [20], which have subsequences of eigenfunctions corresponding to eigenvalues \( \lambda \to \infty \) with uniformly bounded number of critical points. As the saddle points in this example are non-degenerate ones [1], exactly four Neumann lines are connected to each saddle and the boundedness of number of saddles implies that also the number of Neumann domains for these subsequences is uniformly bounded. In other words, eigenfunctions corresponding to arbitrarily high eigenvalues might have a small number of Neumann domains.

**Figure 6.1.** The Neumann lines for the eigenfunction \( \sin(2rx) \sin(y) + \mu \sin(2rx) \sin(y) \) with \( r = 5, \mu \approx 1 \) on a square of edge size \( \pi \) and Dirichlet boundary condition. It belongs to a family of eigenfunctions with only two nodal domains, but with number of Neumann domains which is proportional to \( r^2 \). Cf. the example in page 396 of [13].
7. Summary

This paper studies Laplacian eigenfunctions on surfaces by investigating their Neumann domains. Given an eigenfunction, we define Neumann lines and Neumann domains and show that they form a partition of the manifold. Furthermore, we claim that this partition is as natural as the partition dictated by the nodal set. However, numerous essential questions that are being investigated for nodal domains are open for Neumann domains. The current paper develops this study by discussing and answering some of those questions.

Let us specify some points of comparison between Neumann domains and nodal domains. From a topological point of view, Neumann domains are simply connected (theorems 3.12(iv) and 4.10(v)), whereas nodal domains are not in general [32, 34]. The simplicity of the Neumann partition is also apparent in the eigenfunction restriction to a Neumann domain, \( f|_{\Omega} \). Theorems 3.12 and 4.10 show that the structure of \( f|_{\Omega} \) cannot be too complex in terms of the position and number of critical points and the nodal set within \( f|_{\Omega} \). As \( f|_{\Omega} \) is also an eigenfunction of the domain \( \Omega \) with Neumann boundary conditions, its structural simplicity suggests that the position of \( f|_{\Omega} \) in the spectrum of \( \Omega \) cannot be too high. A similar question for nodal domains is easy to answer and for each nodal domain \( D \) of \( f \), it is known that \( f|_{D} \) is the first Dirichlet eigenfunction of \( D \). This observation was used by Pleijel [33] to obtain an asymptotic bound on the nodal domain count. Similarly, answering the analogous question for Neumann domains would help in estimating the number of Neumann domains, as is discussed in section 5.2. In particular, we already show that the number of Neumann domains is bounded from below by the number of nodal domains (corollary 6.1) for the types of manifolds we consider.

It is well known that the count of nodal domains is affected by the stability of the nodal set. This is apparent for example in the case of multiple eigenvalues. Such eigenvalues posses eigenfunctions where nodal lines intersect. Perturbations of these eigenfunctions prevent these crossings and the intersecting lines resolve into two separate nodal lines. The nature of this resolution of the intersection crucially affects the topology and number of nodal domains and makes their counting a difficult task. Neumann domains, however, show a different behaviour. A crossing of nodal lines always occurs at a saddle point of the function and therefore it also coincides with a Neumann line intersection. Such a Neumann line crossing is stable with respect to perturbations and thus there is no change in the number of Neumann domains when the eigenfunction is perturbed. This was already observed in [30] and it was suggested that the Neumann line pattern is relatively robust and hence the relative ease (in comparison with nodal domains) of the Neumann domain count. Yet, there is an additional phenomenon which complicates the count of Neumann domains. Considering a multiple eigenvalue and some non Morse-Smale eigenfunction which belongs to it, a perturbation might cause an appearance of a new Neumann domain. Such a domain appears at the Neumann line which connects some two saddle points and its volume may be arbitrarily small. The purpose of theorem 5.2 is to place a restriction on the number of such shrinking domains, by providing a lower bound on the outer radius of some of the Neumann domains.

Finally, we wish to point out open problems and possible exploration directions of Neumann domains. In the following \( M \) denotes a two dimensional compact manifold with or without boundary, \((\lambda, f)\) denotes an eigenpair of the Laplacian on \( M \) and \( \Omega \) is some Neumann domain of \( f \).

(i) Using the following notations (see also section 5.2)

\[
\text{pos}(\lambda, \Omega) : = \max \left\{ n \mid \lambda = \lambda_{n}^{(N)}(\Omega) \right\} \\
p(\lambda, f) : = \max_{\Omega} \left\{ \text{pos}(\lambda, \Omega) : \Omega \text{ is a Neumann domain of } f \right\},
\]

what conditions on \( M \) does one need to assure that \( \{ p(\lambda, f) \}_{(\lambda, f) \text{is an eigenpair}} \) is either bounded or possesses a bounded subsequence? Such boundedness imposes lower bounds and asymptotic results for the Neumann domain count (see proof of lemma 5.5).
(ii) What are the asymptotics of the Neumann domain count? More specifically, does the limit of \( \{ \mu_n/n \} \) exist, in general or for some classes of manifolds? If so, could it be bounded from below?

An easier task would be to bound \( \liminf_{n \to \infty} \frac{\mu_n}{n} \) from below. Note that the existence of subsequences of eigenfunctions whose nodal count goes to infinity was recently proved \cite{16, 22, 23}. In \cite{16} it was done for the arithmetic case and in \cite{22, 23} it was shown for a class of non positively curved manifolds. From corollary \cite{6.1} we conclude that in these cases there exists a subsequence of eigenfunctions whose Neumann domain count goes to infinity as well.

(iii) Is it possible to obtain a Courant-like bound? Namely, obtain an upper bound of the form \( \mu_n \leq h(n) \), with \( h \) being some function (possibly linear).

(iv) Improving the inequality established in corollary \cite{6.1} between the nodal count and the Neumann domain count. This can be done, for example, by bounding from below the degrees of extremal points (see discussion in section \cite{6}).

(v) Bounding the total length of the Neumann line set in terms of the eigenvalue.

(vi) Providing a global upper bound for the volume of a single Neumann domain in terms of the eigenvalue.

(vii) Is it possible to improve the lower bound on the outer radius of a Neumann domain in theorem \cite{5.2}? The main improvement might be to make this bound global, so that it applies to all Neumann domains of the eigenfunction.

(viii) Provide an upper bound on the inner radius of a Neumann domain.

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