GEOMETRIC PROPERTIES OF HOMOGENEOUS PARABOLIC GEOMETRIES WITH GENERALIZED SYMMETRIES

JAN GREGOROVIČ AND LENKA ZALABOVÁ

Abstract. We investigate geometric properties of homogeneous parabolic geometries with generalized symmetries. We show that they can be reduced to a simpler geometric structures and interpret them explicitly. For specific types of parabolic geometries, we prove that the reductions correspond to known generalizations of symmetric spaces. In addition, we illustrate our results on an explicit example and provide a complete classification of possible non-trivial cases.

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1. Introduction

The aim of this article is to study homogeneous parabolic geometries. The reader should be familiar with the theory of parabolic geometries, cf. [1]. We show here that the existence of a distinguished class of automorphisms, which we call generalized symmetries, imposes additional geometric properties that the homogeneous parabolic geometries with generalized symmetries satisfy. We introduced and investigated the generalized symmetries in detail in the article [7], but for convenience of the reader we recall the details about the generalized symmetries in the introduction.

We will always consider the parabolic geometry \((\mathcal{G} \to M, \omega)\) of type \((G, P)\) on the connected smooth manifold \(M\) satisfying the following assumptions: We assume that the parabolic geometry \((\mathcal{G} \to M, \omega)\) is regular and normal, and that its automorphism group \(\text{Aut}(\mathcal{G}, \omega)\) acts transitively on \(M\), i.e., the parabolic geometry is homogeneous. We assume that the group \(G\) is simple (not necessarily connected) Lie group with the Lie algebra \(\mathfrak{g}\). We fix the restricted root system of \(\mathfrak{g}\), and denote

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by $\alpha$, the positive simple restricted roots ordered according to the convention from \cite{[12] [1]}. We denote by $g$, the root space of the root $\alpha$. We assume that $P$ is a standard parabolic subgroup of $G$ with the Lie algebra $\mathfrak{p}$, which is given by the admissible subset $\Xi$ of the set of simple restricted roots. We denote by $|g|$ the $|k|$-grading of $g$ by $\Xi$-heights, and we use the usual notation $g_-$ and $g_+$ for the negative and positive part of the grading. As usual, we denote by $G_0 \subset P$ the Lie group with the Lie algebra $g_0$ of elements preserving the grading. Moreover, we assume that $G/P$ is connected and that the maximal subgroup of $P$ normal in $G$ is trivial, and thus $\text{Ad} : P \to G\text{L}(g)$ is injective. Let us point out that these additional assumptions on the pair $(G, P)$ do not restrict the possible $g$ and $\Xi$. In fact, they only impose additional topological conditions on $G$ and $P$.

Let us recall that each homogeneous parabolic geometry $(\mathcal{G} \to M, \omega)$ admits a completely algebraic description, and we follow the concepts and notation of \cite{[1], Sections 1.5. and 3.1.} with a slight simplification which is possible due to our assumptions.

**Definition 1.** Let $K$ be a Lie group, let $H$ be a closed subgroup of $K$ which is simultaneously a subgroup of $P$, and denote by $i$ the inclusion $H \subset P$. Let $\alpha : \mathfrak{k} \to \mathfrak{g}$ be an (injective) linear map satisfying:

1. $\alpha|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}}$,
2. $\alpha$ induces a linear isomorphism of $\mathfrak{k}/\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{p}$,
3. $\text{Ad}_h \circ \alpha = \alpha \circ \text{Ad}_h$ for all $h \in H$.

Then we say that $(\alpha, i)$ is the extension of $(K, H)$ to $(G, P)$.

Let us remark that in the general approach, the map $i$ can be a Lie group homomorphism. However, our assumptions allow us to identify the subgroup $H$ of $K$ with its image $i(H) \subset P$ and use the above simplified notation.

If $(\alpha, i)$ is an extension of $(K, H)$ to $(G, P)$, then there is $K$-invariant Cartan connection $\omega_{\alpha}$ of type $(G, P)$ on $K \times_H P \to K/H$ induced by the $\alpha$–image of the Maurer–Cartan form of $K$.

**Definition 2.** We say that the parabolic geometry $(K \times_H P \to K/H, \omega_{\alpha})$ of type $(G, P)$ is the parabolic geometry given by extension $(\alpha, i)$ of $(K, H)$ to $(G, P)$.

One of the crucial results about homogeneous parabolic geometries, see \cite{[1], Theorem 1.5.15.}, is that for any subgroup $K$ of $\text{Aut}(\mathcal{G}, \omega)$ acting transitively on $M$ and any point $u_0 \in \mathcal{G}$, there is extension $(\alpha, i)$ of $(K, H)$ to $(G, P)$ such that the map $K \times_H P \to \mathcal{G}$ defined as $[k, p] \mapsto k(u_0)p$ is isomorphism of parabolic geometries $(K \times_H P \to K/H, \omega_{\alpha})$ and $(\mathcal{G} \to M, \omega)$.

**Definition 3.** We say that $(\mathcal{G} \to M, \omega)$ is given by the extension $(\alpha, i)$ of $(K, H)$ to $(G, P)$ at $u_0$ if there is an inclusion of $K$ into $\text{Aut}(\mathcal{G}, \omega)$ such that the map $[k, p] \mapsto k(u_0)p$ induced by this inclusion is an isomorphism of $(K \times_H P \to K/H, \omega_{\alpha})$ and $(\mathcal{G} \to M, \omega)$.

Thus we will always consider the parabolic geometry $(\mathcal{G} \to M, \omega)$ to be given by an extension $(\alpha, i)$ of $(K, H)$ to $(G, P)$ at $u_0$. We denote by $x_0$ the point of $M$ underlying $u_0$ and thus corresponding to the point $eH$. In this setting, it is easy to define the generalized symmetries (at $x_0$) in the following way:

**Definition 4.** Let $s$ be an element of the center $Z(G_0)$ of $G_0$. We say that the automorphism $h \in H$ is $s$–symmetry (in $K$) at the point $eH$ if $h$ is $P$–conjugated to $s$. All $s$-symmetries at $eH$ (in $K$) for all possible elements $s$ in $Z(G_0)$ together are called generalized symmetries at $eH$ (in $K$).
Let us point out that the definition does not depend on the chosen extension. However, if \( K \neq \text{Aut}(G, \omega) \), then \( H \) does not have to contain all generalized symmetries at \( eH \). Moreover, it is enough to consider generalized symmetries only at one point due to homogeneity.

Each \( s \)-symmetry at \( x_0 \) has a natural (frame independent) action on \( T_{x_0}^{-1}M \) induced by the adjoint action of \( s \) on \( (g_{-1} \oplus p) \cong T_{x_0}^{-1}M \). In fact, \( \text{Ad}_s \) has single eigenvalue on each indecomposable \( g_0 \)-module and thus, the eigenvalues of each \( s \)-symmetry at \( x_0 \) on \( T_{x_0}^{-1}M \) are precisely given by eigenvalues of \( \text{Ad}_s \) on the root spaces \( g_{-\alpha} \), for all \( \alpha \in \Xi \). Moreover, there is a natural frame in which the eigenvalues of the \( s \)-symmetry at \( x_0 \) on \( T_{x_0}M \) are given by eigenvalues of \( \text{Ad}_s \) on the root spaces in \( g_{-\alpha} \). It is simply the \( u_0 \) for which is the \( s \)-symmetry at \( eH \) represented by \( s \in H \).

Remark 1. We investigated in [7] a more general class of automorphisms that share the same action as generalizes symmetries at \( x_0 \) on \( T_{x_0}^{-1}M \). However, we proved that the existence of such automorphism ensures the existence of generalized symmetry with the same action. Thus we will only consider generalized symmetries in this article.

We have shown in [7] that, we can algebraically compute the set of all local generalized symmetries. We recall the following result:

**Proposition 1.1.** Let \( h \in P \) be such that \( \text{Ad}_h(\alpha(\mathfrak{k})) \subset \alpha(\mathfrak{k}) \) and \( \text{Ad}_h, \kappa(u_0) = \kappa(u_0) \) (for the action on \( \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \)). Then there is a local automorphism \( \phi \) of \( \hat{G} \to M, \omega \) such that \( \phi(u_0) = u_0 h \). In particular, such \( h \) induces an automorphism of \( \mathfrak{k} \) and if \( M \) is simply connected, then \( \phi \) is a global automorphism.

We recall that the simply connected covering \( \hat{M} \) of \( M = K/H \) carries the pull-back parabolic geometry of type \( (G, P) \), which we denote \( \hat{G} \to \hat{M}, \hat{\omega} \) and call it the simply connected covering of \( G \to M, \omega \). In the second section of the article, we construct explicitly extension \( (\hat{K}, \hat{H}) \) to \( (G, P) \) giving \( \hat{G} \to \hat{M}, \hat{\omega} \) at point covering \( u_0 \) and having additional distinguished properties, see Theorem 2.2. Namely, we can require that the stabilizer \( \hat{H} \) satisfies the following conditions:

- \( \hat{H} \) is a linear algebraic subgroup of \( P \) containing \( H \),
- \( \hat{H} \) contains (coverings of) all local automorphisms of \( \hat{G} \to M, \omega \) with the fixed point \( x_0 \) that preserve \( \mathfrak{k} \), and
- the subgroups \( \exp(\mathfrak{h} \cap \mathfrak{p}_+) \) of \( K \) and \( \exp(\mathfrak{h} \cap \mathfrak{p}_+) \) of \( \hat{K} \) coincide.

Further, in the third section, we relate the generalized symmetries on \( (\hat{G} \to M, \hat{\omega}) \) with generalized symmetries on \( (\hat{G} \to \hat{M}, \hat{\omega}) \). In fact, the results of the second and third section allow us to prove our first main result (Theorem 1.2). However, we need to introduce the following notations before we formulate the result. Let us point out that the defined sets do not depend on the point \( u_0 \), but only on the Lie group \( K \).

- We denote by \( J(\mathfrak{k}) \) the set consisting of all elements \( s \in Z(G_0) \) such that there is a local \( s \)-symmetry of \( \hat{G} \to \hat{M}, \hat{\omega} \) preserving \( \mathfrak{k} \).
- We prove in the third section that the set \( J(\mathfrak{k}) \) can be algebraically computed, because there always is \( u_0 \in \mathcal{G} \) such that \( J(\mathfrak{k}) = \hat{H} \cap Z(G_0) \), see Theorem 3.1.
- We denote by \( \Phi(\mathfrak{k}) \) the set consisting of all roots \( \alpha \in \Xi \) such that there is an element of \( \mathfrak{h} \cap \mathfrak{p}_+ \) with non-trivial component in the \( g_0 \)-submodule of the root space \( g_{\alpha} \) in \( p_+ \).

We prove in the second section that if \( \kappa \neq 0 \), then the sets \( \Phi(\mathfrak{k}) \) are bounded by the sets \( I_\mu \) from the tables in the Appendix C, see Theorem 2.2 and Appendix A.
• Finally, we denote
\[ Θ(τ) := \{ α_s ∈ Ξ : Ad_s G_{θ−α_s} = id \text{ for each } s ∈ J(τ) \}. \]

This means that the set Θ(τ) corresponds to the common eigenspace of all local generalized symmetries at \( x_0 \) preserving \( τ \) with eigenvalue 1 in \( T_{x_0}^{-1} M \).

Simultaneously, for arbitrary admissible subset \( Ξ' \) of the set of simple restricted roots of \( g \), we denote by \( g_{Ξ',-} \) the grading of \( g \) given by \( Ξ' \)-heights and we use \( g_{Ξ',0} \) and \( g_{Ξ',+} \) for the negative and positive parts. We denote by \( p_{Ξ'} = g_{Ξ',0} ⊕ g_{Ξ',+} \) the standard parabolic subalgebra of \( g \), and by \( q_{Ξ'} = p_{Ξ',-} ⊕ g_{Ξ',0} \) the standard opposite parabolic subalgebra of \( g \) given by \( Ξ' \). We assume that the corresponding subgroups \( P_{Ξ'} \) and \( Q_{Ξ'} \) with the Lie algebras \( p_{Ξ'} \) and \( q_{Ξ'} \) satisfy the same topological restrictions as the subgroup \( P \) and use the analogous notations, e.g. we denote by \( G_{Ξ',0} \) the Levi part of \( P_{Ξ'} \).

Having all these notations at hand, we can formulate the main result.

**Theorem 1.2.** Suppose \( κ ≠ 0 \) and \( J(τ) ≠ \{ e \} \) and denote

\[ Λ(τ) : = Ξ − Φ(τ) − Θ(τ). \]

Then there is \( u_0 \) such that \( H ⊂ Q_{Λ(τ)} \cap P \) holds for the extension \((α, i)\) of \((K, H)\) to \((G, P)\) giving \((G → M, ω)\) at \( u_0 \).

The fourth section of the article then consists of the proof of the Theorem 1.2. Let us remark that if \( J(τ) = \{ e \} \), then \( Λ(τ) = \emptyset \) and the claim in the Theorem 1.2 is trivial. On the other hand, for the most of the types of the parabolic geometries, the classification results in the Appendix C ensures that \( Λ(τ) ≠ \emptyset \). Moreover, the example in the Appendix E provides a counter-example for a stronger statement to hold.

One of the most important consequences of the Theorem 1.2 is that there is a reduction of the \( P \)-bundle \( G \) to the \( (Q_{Λ(τ)} \cap P) \)-bundle \( K ×_H (Q_{Λ(τ)} \cap P) \). In the fifth section, we show that we can reduce our parabolic geometry \((G → M, ω)\) to a Cartan geometry of type \((Q_{Λ(τ)} \cap P)\) on \( K ×_H (Q_{Λ(τ)} \cap P) \), see Theorem 5.1. In particular, we prove the following statement as the special case of the Theorem 5.4 for \( K = Aut(G, ω) / Λ(τ) \):

**Theorem 1.3.** Assume \( κ ≠ 0 \) and assume there is a local \( s \)-symmetry for \( s ≠ e \). Denote \( Λ := Λ(τ) \) in the case \( K = Aut(G, ω) \). Then there is a (unique up to isomorphism) Cartan geometry \((G_Λ → M, ω^Λ)\) of type \((Q_Λ \cap P)\) such that \((G_Λ → M, ω^Λ)\) determines the same underlying geometric structure as \((G → M, ω)\), and \( Aut(G, ω) = Aut(G_Λ, ω^Λ) \).

We interpret the reduction to \((Q_{Λ(τ)} \cap P)\) in detail in the fifth section. Namely, we show that \( TM \) decomposes into two invariant subbundles \( T_{Λ(τ)} M \), which is graded, and \( VM \), which is filtered, see Proposition 5.5. Further, we describe a particular class of Weyl structures naturally compatible with these reductions, see Proposition 5.6. Clearly, Weyl connections corresponding to the class provide tools for further computations on these geometries.

In the sixth section, we discuss the correspondence and twistor spaces related to the geometry. Let us introduce the following notation:

• We denote by \( Ψ \) the maximal subset of \( Ξ \) such that the harmonic curvature does not have entries in \( g_{Ξ−Σ,0} \).

The set \( Ψ \) characterizes possible (local) twistor spaces for the parabolic geometries of type \((G, P)\) by results of [2]. Let us point out that the set \( Ψ \) can be computed explicitly from \( κ_H \), see Appendix A. We investigate, when the twistor space can be constructed globally, and we discuss the compatibility of this construction with generalized symmetries, see Theorem 5.6.
In the seventh section, we investigate the additional geometric properties for important types of parabolic geometries. In particular, it turns out that for specific types of parabolic geometries, \( M \) is a (locally) symmetric or (locally) \( \mathbb{Z}_3 \)-symmetric space, or a fiber bundle over it, respectively, and there often is an invariant Weyl connection on \( M \), which corresponds to the canonical connection of the (locally) symmetric or (locally) \( \mathbb{Z}_3 \)-symmetric space.

At the end of the article, there are three appendixes attached. The Appendix [A] recalls the details about the harmonic curvature \( \kappa_H \) and the notation derived from it. The Appendix [B] contains a concrete example of a homogeneous parabolic geometry on which we demonstrate our theory. The Appendix [C] then contains the complete classification of types of parabolic geometries that can satisfy \( \kappa \neq 0 \) and \( J(\ell) \neq \{e\} \) for \( \ell \) simple.

2. Extensions giving simply connected coverings of homogeneous parabolic geometries

Firstly, let us show that there always is an extension giving the simply connected covering \( (\tilde{G} \to \tilde{M}, \tilde{\omega}) \) of \( (\bar{G} \to \bar{M}, \bar{\omega}) \) at a point covering \( u_0 \), which is induced naturally by the extension \((\alpha, i) \) of \((K, H) \) to \((G, P) \).

**Proposition 2.1.** Let \( K^c \) be the connected simply connected Lie group with the Lie algebra \( \mathfrak{k} \), and \( K^c/H^c \) the simply connected covering of \( K/H = M \). Let \((K', H') \) be the effective quotient of \((K^c \rtimes H^c \rtimes H) \), where the actions of \( H \) are naturally induced by the adjoint action. Then \( H' = H \) and \( \mathfrak{k}' = \mathfrak{k} \mathfrak{t} \), and \((\alpha, i) \) is extension of \((K', H') \) to \((G, P) \) giving the simply connected covering \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) at any \( \bar{u}_0 \) covering \( u_0 \).

In particular, the covering map \( \tilde{G} \to \bar{G} \) is a local isomorphism of parabolic geometries of type \((G, P) \), \( K' \) is contained in the automorphisms group of \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) and acts transitively on \( \tilde{M} \), and any automorphism of \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) gives a local automorphism of \((G \to M, \omega) \).

**Proof.** Since \( K^c \) is simply connected and \( H^c \) connected, the semidirect products are well-defined. Then the maximal subgroup of \( H^c \times H \) normal in \( K^c \times H \) consists of elements \((h, h') \) such that \( \text{Ad}_h \text{Ad}_{h'} = \text{id}_H \), because exactly such elements act as \( \text{id} \) on \( K^c \times H/H^c \times H \), and thus it follows from [H Section 1.4.1.]. Since \((K, H) \) is effective, the effective quotient of \((K^c \rtimes H, H^c \times H) \) is of the form \((K', H) \), and \( \mathfrak{k}' = \mathfrak{k} \mathfrak{t} \). Then the remaining claims follow, because \( K'/H \) is simply connected covering of \( M \), and thus the obvious local isomorphism of \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) and \((K'/H, \omega_0) \) extends to the global isomorphism. \( \square \)

If we apply the construction of the simply connected covering \( (\tilde{G} \to \tilde{M}, \tilde{\omega}) \) in the case \( K = \text{Aut}(\bar{G}, \bar{\omega}) \), then \( K \) does not have to be the full automorphism group of \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \). In particular, there are further automorphisms of \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) given by \( h \in P \) satisfying the conditions of the Proposition [H]. Let us denote \( N_P(\alpha) := \{ h \in P : \text{Ad}_h(\alpha(\mathfrak{t})) \subset \alpha(\mathfrak{t}), \text{Ad}_h.\kappa(u_0) = \kappa(u_0) \} \)
the set of all of such \( h \in P \). We show that we can use the set \( N_P(\alpha) \) to modify the construction of the extension giving \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) to obtain the extension with the distinguished properties.

**Theorem 2.2.** There is the extension \((\tilde{\alpha}, \tilde{i}) \) of \((\bar{K}, \bar{H}) \) to \((G, P) \) giving \((\tilde{G} \to \tilde{M}, \tilde{\omega}) \) at \( \bar{u}_0 \) covering \( u_0 \) such that the following statements hold:

1. The pair \((\bar{K}, \bar{H}) \) is the effective quotient of \((K^c \rtimes N_P(\alpha), H^c \rtimes N_P(\alpha)) \), \( \bar{H} = N_P(\alpha), \tilde{i} \) is the natural inclusion of \( N_P(\alpha) \) into \( P \), and \( \tilde{\alpha} = (\alpha + \text{id}) \).
By definition, α such that the construction in the Proposition 2.1 by taking N in different and simultaneously (\( G \rightarrow M, \), \( \omega \)) and on (\( G \rightarrow M, \), \( \omega \)) a non–trivial projection into the decomposition of \( H \) to (\( G \rightarrow M, \), \( \omega \)) at \( \tilde{u}_0 \) such that the stamens (1), (2) are hold. The extension giving (\( G \rightarrow M, \), \( \omega \)) at \( \tilde{u}_0 \) such that the stamens (1), (2) are hold. The statement (3) follows directly from [7 Lemma 3.6].

Let us decompose \( Z \in h \cap (g_\alpha + \cdots + g_\varepsilon) \) as \( Z = Z_1 + Z_2 + R \), where \( Z_1 \) and \( Z_2 \) are in different \( g_\alpha \)-submodules of \( g_\alpha \) and \( R \in g_{\alpha+1} + \cdots + g_\varepsilon \). Then there are \( X, Y \in \mathfrak{t} \) such that \( \alpha(X) = X_1 + S \) for suitable \( S \in p \), and \( \alpha(Y) = X_2 + T \) for suitable \( T \in p \), and simultaneously \( (X_1, [X_1, Z_1, Z_2]) \) and \( (X_2, [X_2, Z_2], Z_2) \) are \( \mathfrak{sl}(2) \)-triples. Then the projection of \( [aX + bY, Z] \in h \) into \( h_0 \) is of the form \( a[X_1, Z_1] + b[X_2, Z_2] \), and gives a two dimensional subspace of the Cartan subalgebra of \( g_\alpha \). This two–dimensional subspace acts on \( Z_1 \) and \( Z_2 \) by different weights. Indeed, if the elements of the subspace act by eigenvalue 2 on both \( Z_1 \) and \( Z_2 \), then [5 Theorem 10.10] (after complexification) gives the contradiction with \( Z_1 \) and \( Z_2 \) being in different \( g_\alpha \)-submodules of \( g_\alpha \). Since \( Z \) and the decomposition of it is arbitrary, we get the claim (4). Then the claim (5) follows from refinement of results in [9], see the Appendix A.

In particular, the condition (6) follows in the case \( \kappa \neq 0 \), because if \( Z \in h \cap p_+ \), has a non–trivial projection into the \( g_\alpha \)-submodule outside \( p_{\Phi(t^+)\alpha} \), then one can make brackets of \( Z \) with \( X \in \alpha(\mathfrak{t}) \) having a non–trivial projection into root spaces with zero \( \Phi(\mathfrak{t})\)-height to get a non–trivial element in \( p_+ \), which is the contradiction. \( \Box \)

3. Generalized symmetries on simply connected coverings

Let us now investigate the relation between generalized symmetries on (\( G \rightarrow M, \), \( \omega \)) on (\( G \rightarrow M, \), \( \omega \)) in detail. Firstly, we need to recall that if \( (\alpha, i) \) is extension of \( (K, H) \) to \( (G, P) \) giving (\( G \rightarrow M, \omega \)) at \( u_0 \), then there is the extension \( (Ad_{p^{-1}} \alpha, i) \) of \( (K, p^{-1} H_p) \) giving (\( G \rightarrow M, \omega \)) at \( u_0 p \). Let us further recall that the linear algebraic group \( H \) can be decomposed into the maximal reductive subgroup \( H_0 \) and the unipotent radical \( N \). If we fix the reductive Levi decomposition \( H = H_0 \cdot N \), then we shall find \( \tilde{u}_0 \) in which this decomposition is compatible with the decomposition of \( P = G_0 \exp(p_+) \).

Theorem 3.1. There is \( u_0 \in G \) such that the extension \( (\tilde{\alpha}, i) \) of \( (\tilde{K}, \tilde{H}) \) to \( (G, P) \) from the Theorem 2.2 satisfies the following additional conditions.

1. For the reductive Levi decomposition \( \tilde{H} = H_0 \cdot \tilde{N} \), the reductive part \( H_0 \) is contained in \( G_0 \), and the unipotent radical \( \tilde{N} \) is of the form \( \tilde{N} = \exp(\tilde{n}) \) for some subalgebra \( \tilde{n} \) of the nilpotent radical of the Borel subalgebra of \( p \).
2. It holds \( \mathcal{J}(\mathfrak{t}) = \tilde{H} \cap Z(G_0) \).
Let us fix the decomposition $N_P(\alpha) = H_0', N'$ into the reductive part $H_0'$, and the unipotent radical $N'$. Then it follows from the general theory of linear algebraic groups and Borel subgroups, see [8], that there is $\exp(Y) \in p_+ \subset p$ such that $\exp(Y)H_0'\exp(-Y) \subset G_0$, and there is $c$ in the maximal compact subgroup of $G_0$ such that $c\exp(Y)N'\exp(-Y)c^{-1}$ is in the unipotent radical of the Borel subgroup of $P$. Thus $c\exp(Y)N'\exp(-Y)c^{-1} = \exp(\tilde{n})$ for some subalgebra $\tilde{n}$ of the Borel subalgebra of $p$. Then $u_0c\exp(Y)$ for arbitrary $u_0$ is the point of $G$ for which the first claim of the Theorem holds. Then the subsequent claims follow from the definition of $H$. Clearly, $c\exp(Y)p$ has the same properties as $c\exp(Y)$ and the last claim follows.

Let us show that the structure of $1$–eigenspaces of $s \in J(t)$ provides a basic obstruction for non–flat geometries to a priory have nice geometric properties.

**Proposition 3.2.** Let $(G \to M, \omega)$ be given by extension $(\alpha, i)$ of $(K, H)$ to $(G, P)$ at $u_0$ from the Theorem 2.2. Assume $\kappa \neq 0$ and denote by $m$ the intersection of all $1$–eigenspaces in $p_+$ for all $s \in J(t)$.

1. Then the nilpotent radical $\tilde{n}$ is contained in the subalgebra of $p$ generated by $m$–orbits of $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \cap \mathfrak{g}_{\Phi(t), 1}$.

2. If $m = 0$, then $H \subset Q_{\Xi - \Theta(t)} \cap P$.

**Proof.** The claim (1) is a direct consequence of the part (5) of the Theorem 2.2 and the fact that $\tilde{n}$ splits according to the eigenvalues of $Ad_s$ for any $s \in J(t)$ at $u_0$. The claim (2) follows from the claim (1) for $m = 0$, because $H_0 \subset G_0$ from the Theorem 2.2. □

Thus the Theorem 1.2 is a generalization of the claim (2) of the previous proposition. Now, we are ready to start the proof of the Theorem 1.2.

4. The proof of Theorem 1.2.

**Proof.** Let us continue in the setting of the previous section. Namely, assume that $u_0$ is from the Theorem 3.1 and consider the subspace $m$ from the Proposition 3.2. If $m = 0$, the claim follows from the Proposition 3.2. Thus we assume $m \neq 0$ and we prove that $n \subset \mathfrak{u}_A(t) \cap p$ after conjugation by $p \in P$ satisfying the conditions of the Theorem 3.4. Moreover, we can assume that $m \subset \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$. This gives no additional restriction, because $m_1 \subset \mathfrak{g}_{\Xi - \Theta(t), 0} \oplus \mathfrak{g}_{\Theta(t), 0}$, and if we consider $\mathfrak{g}_{\Xi - \Theta(t), 0}$–modules instead of $\mathfrak{g}_0$–modules, then each $s \in J(t)$ will have a unique eigenvalue on each indecomposable $\mathfrak{g}_{\Xi - \Theta(t), 0}$–module. In particular, the assumption $m \subset \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ implies $\Theta(t) = 0$ and simplifies notation.

Suppose $Z \subset \tilde{n}$. Let $H_0 = H_a \cdot H_s$ be the decomposition into the Levi subgroup $H_s$ contained in the semisimple part of $G_0$ and the abelian Lie subgroup $H_a$ consisting of semisimple elements, see [12]. Without loss of generality, we can assume that $Z$ is in irreducible $H_a$–invariant subspace of $\tilde{n}$, and if $Z$ has non–trivial projection
to \( g_0 \), then \( Z \in g_0 \oplus m \). The first assumption on \( Z \) is obvious. The last assumption is possible, too, because according to the classification in the Appendix C, the \( m \)-orbits of all possible 1-eigenspaces in \( \hat{n} \cap p_+ \) are trivial under our assumptions.

Now, we can divide the proof into the following two cases:

1. \( Z \in g_0 \oplus m \).
2. \( Z \in n_+ := \hat{n} \cap p_+ \).

The method of the proof is as follows: We show that the existence of a non-trivial projection of \( Z \) into a \( g_0 \)-submodule outside of \( q_{\mathbb{Z} \cdot \Theta(t)} \cap p \) implies that there is an element of \( n_+ \) with a non-trivial projection outside of the subalgebra of \( p \) generated by \( m \)-orbits of \( g_0 \oplus g_1 \cap g_{\Theta(t),1} \), which is the contradiction with the Proposition 3.2.

(1) Let us start with the case \( Z \in g_0 \oplus m \). Suppose there are the following indecomposable \( g_0 \)-modules:

- \( V_2 \) is a submodule of \( m \) outside of \( q_{\mathbb{Z} \cdot \Theta(t)} \cap p \),
- \( V_{-1} \) is the submodule of \( g_{-1} \) such that \( 0 \neq [V_{-1}, V_2] \) is outside of the subalgebra of \( p \) generated by \( m \)-orbits of \( g_0 \oplus g_1 \cap g_{\Theta(t),1} \),
- \( V_1 \) is the submodule \([V_{-1}, V_2]\).

We recall that each indecomposable \( g_0 \)-submodule in \( g \) is determined by \( g \)-weight (which is unique up to \( g_0 \)-weights). Thus it is clear that for each indecomposable \( g_0 \)-submodule \( V_2 \) of \( m \), there exists \( V_{-1} \) with claimed properties such that the projection of \([V_{-1}, m]\) into \( V_1 \) vanishes on the other \( g_0 \)-submodules of \( m \). This allows us to resolve this case by considering all possible \( g_0 \)-modules \( V_2 \) case by case.

Consider \( X_- \in V_{-1} \). There exists \( X \in \alpha(t) \) such that \( X_- \) is the projection of \( X \) to \( g_- \), and we denote by \( X_+ \) the projection of \( X \) to \( V_1 \). The component \( X_+ \) is uniquely determined by \( X_- \), because the difference of two such elements \( X, X' \in \alpha(t) \) with the same projection to \( g_- \) is an element of \( n_+ \).

Further, let us consider the decomposition \( Z = Z_0 + Z_+ + Z' \), where \( Z_0 \in g_0 \) and \( Z_+ \in V_2 \) and \( Z' \) is in the remaining \( g_0 \)-submodules in \( m \). Then the components of \([Z, X]\) in \( V_{-1} \) and \( V_1 \) are of the form

\[
[Z_0, X_-] \in V_{-1},
\]

\[
[Z_0, X_+] + [Z_+, X_-] \in V_1.
\]

We will view all submodules \( V_i \) as the modules of the semisimple part of \( g_0 \), and we will treat separately the following two possible cases:

(A) There is no simple submodule of the semisimple part of \( g_0 \) acting non-trivially on all \( V_i \).

(B) There is such a submodule.

(1A) If there is no simple submodule of the semisimple part of \( g_0 \) acting non-trivially on all submodules \( V_i \), then considering the non-degeneracy results from \( \mathbb{I} \), we can write

\[
V_{-1} = U_1 \otimes U_2 \otimes W,
\]

\[
V_2 = U_1^* \otimes W \otimes U_3^*,
\]

\[
V_1 = W \otimes U_2 \otimes U_3^*,
\]

where \( W \) is always the trivial representations, and \( U_1, U_2, U_3 \) are representations of the appropriate semisimple submodule of the semisimple part of \( g_0 \). We have to discuss the situations, when \( U_2 \) is the only non-trivial representation, or \( U_3 \) is the only non-trivial representation. The situation when \( U_1 \) is the only non-trivial representation...
representation is dual (with the same $V_2$) to the situation when $U_3$ is the only non–trivial representation via the Killing form of $\mathfrak{g}$.

Let us first consider the situation when $U_2$ is the only non–trivial representation. Then $\text{ad}_{Z_0}$ commutes with $\text{ad}_{Z_1}$, because $\text{ad}_{Z_1}$ is in the trivial representation. This implies that the components of $\text{ad}_{Z_2}(X)$ in $V_1$ are $\text{ad}_{Z_0}(X_+)$, $\text{ad}_{Z_0}(X_-) \in V_1$ and $\text{ad}_{Z_0}(X_+ + i[Z_+, \text{ad}_{Z_0}^{-1}(X_-)]) \in V_1$. Moreover, since $\text{ad}_{Z_1}$ is non–degenerate, it induces isomorphism $V_1 \cong V_{-1}$ (of representations of the semisimple part of $\mathfrak{g}_0$), and since the chains of $\text{ad}_{Z_n}$ in $V_{-1}$ and $V_1$ have the same length, we get $Z_+ = 0$ and $X_+ \in \text{Ker}(\text{ad}_{Z_0})$ for $0 \neq \text{ad}_{Z_0}(X_-) \in \text{Ker}(\text{ad}_{Z_0})$, because $\text{Im}(\text{ad}_{Z_0})$ is one step further in the chain than $\text{Im}(\text{ad}_{Z_0})$, and the chains have finite length.

Let us now consider the situation when $U_3$ is the only non–trivial representation. Then we get $\text{ad}_{Z_3}(X_-) = 0$, because $V_{-1}$ is the trivial module, and the components of $\text{ad}_{Z}(X)$ in $V_{-1}$ and $V_1$ are $0 \in V_{-1}$ and $[Z_0, X_+ + [Z_+, X_-] \in V_1$. Moreover, since $\text{ad}_{X_-}$ is non–degenerate, and there is $\mathfrak{g}_0$–submodule $V_{-1}^\ast$ in $p_-$ dual to $V_{-1}$ via the Killing form of $\mathfrak{g}$, the Jacobi identity implies that

$$Z_+ = [X_+, [Z_0, X_+]] = [Z_0, [X_+, X_+]]$$

holds for suitable $X_+ \in V_{-1}^\ast$.

Since $X_+$ and $X_+$ does not depend on the choice of $Z \in \mathfrak{g}_0 \oplus \mathfrak{m}$, it is enough to show that $p = \exp([X_+, X_+])$ satisfies the conditions of Theorem 3.1. The condition (b) is clearly satisfied. Since $X_-$ and $X_+$ are in trivial modules for the semisimple part of $\mathfrak{g}_0$ and $X_+$ is unique, $H_\ast$ commutes with $p$ and $H_\ast$ commutes with $p$, because $Z$ is in an irreducible invariant subspace of $\mathfrak{n}$. Thus the condition (a) is satisfied, too. Moreover, it follows from the uniqueness that after the conjugation, the only parts of $\mathfrak{g}_0 \oplus \mathfrak{m}$ that change are the submodule $V_2$ and the submodules in the higher part of the filtration than $V_2$.

Thus we can assume $Z_+ = 0$ without loss of generality. Under this assumption, $X_+$ is in the intersection of all $\text{Ker}(Z_0)$ for $Z \in \mathfrak{n} \cap (\mathfrak{g}_0 \oplus \mathfrak{m})$ due to the uniqueness of $X_+$.

(1B) Assume there is a simple submodule of the semisimple part of $\mathfrak{g}_0$ acting non–trivially on all spaces $V_i$. Then the non–trivial action can be viewed as the combination of two cases from the previous part, and we can use the results from above. In fact, there are the following three possibilities:

Firstly, let $U_1$ be trivial and $U_2 \cong U_3$, and we consider the (anti)symmetrization. Then we clearly obtain $Z_+ = 0$ and $X_+$ is in the intersection of all $\text{Ker}(\text{ad}_{Z_0})$ for $0 \neq \text{ad}_{Z_0}(X_-) \in \text{Ker}(\text{ad}_{Z_0})$ and $Z \in \mathfrak{n} \cap (\mathfrak{g}_0 \oplus \mathfrak{m})$.

Secondly, let $U_2$ be trivial and $U_1^\ast \cong U_3$, and we consider the (conjugate) (anti)symmetrization. Then we can assume that $Z_+ = 0$ and $X_+$ is in the intersection of all $\text{Ker}(\text{ad}_{Z_0})$ for $Z \in \mathfrak{n} \cap (\mathfrak{g}_0 \oplus \mathfrak{m})$, because as above, we can find $p \in \exp(p_\ast)$ satisfying the conditions of Theorem 3.1 such that $Z_+ = [Z_0, \ln(p)]$ for all $Z \in \mathfrak{n} \cap (\mathfrak{g}_0 \oplus \mathfrak{m})$.

Finally, let $U_2$ be trivial and $U_1 \cong U_2$, and we consider the (anti)symmetrization. This case is dual to the first one by the Killing form of $\mathfrak{g}$.

(2) Let us now consider the case, when $Z \in \mathfrak{n}_+$. If $n_+ = 0$ or $\mathfrak{n}_+ \subset \mathfrak{q}_{\Xi - \Phi(t)} \cap \mathfrak{p}$, then the claim follows. Otherwise, let us consider the analogy of the previous situation in the setting of $\mathfrak{Z} - \Phi(t)$–heights. So consider $Z \in \mathfrak{g}_{\Xi - \Phi(t), 0} \oplus \mathfrak{m}'$, where $\mathfrak{m}'$ is the subalgebra of $\mathfrak{p}_+$ generated by the $\mathfrak{m}$–orbits of $\mathfrak{g}_1 \cap \mathfrak{g}_{\Phi(t), 1}$ of positive $\Xi - \Phi(t)$–height, which is at least 2 by our assumptions. Thus we can consider $\mathfrak{g}_{\Xi - \Phi(t), 0}$–submodules induced by the following $\mathfrak{g}_0$–modules:

$$V_2$$

is a submodule of $\mathfrak{m}'$ outside of $\mathfrak{q}_{\Xi - \Phi(t)} \cap \mathfrak{p}$. 
\textit{V. }is the submodule of \( g_{\mathbb{E} - \phi(t), -1} \) such that \( 0 \neq [V_1, V_2] \) is outside of subalgebra of \( \mathfrak{p}_+ \) generated by \( \mathfrak{m} \)-orbits of \( \mathfrak{g}_1 \cap \mathfrak{g}_{\phi(t)} \).

\( V_1 \) is the submodule \([V_1, V_2]\).

From the discussion analogous to the discussion in the case (1) follows that \( \mathfrak{n}_+ \subset \mathfrak{q}_{\mathbb{E} - \phi(t)} \cap \mathfrak{p} \) up to conjugation by the element of \( p \in \exp(\mathfrak{p}_+) \) commuting with \( H_0 \). Moreover, from the above construction of the element \( p = \exp(Y) \), the element \( Y \) is contained in the trivial representation of the semisimple part of \( \mathfrak{g}_{\mathbb{E} - \phi(t), 0} \), and thus commutes with all elements \( Z \in \mathfrak{n} \cap (\mathfrak{g}_0 \oplus \mathfrak{m}) \). This completes the proof. \( \square \)

5. Geometric properties of geometries with generalized symmetries

In this section, we assume that the parabolic geometry \((G \to M, \omega)\) is given by extension \((\alpha, i)\) of \((K, H)\) to \((G, P)\) at \( w_0 \) such that \( H \subset Q_{\Lambda(t)} \cap P \). Such extension always exists due to the Theorem 1.2. We investigate here the geometric consequences of this assumption, i.e., geometric consequences of the Theorem 1.2.

Firstly, we show that we can always reduce the parabolic geometry to a simpler geometric structure. It turns out that this can be done via the theory of holonomy reductions for parabolic geometries from \([3]\), but we construct the holonomy reduction directly.

\textbf{Theorem 5.1.} Consider the decomposition \( \alpha = \alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0} + \alpha_{\Lambda(t), +} \) according to the values in \( \mathfrak{g}_{\Lambda(t), -} \oplus \mathfrak{g}_{\Lambda(t), 0} \oplus \mathfrak{g}_{\Lambda(t), +} \). There are the following objects available:

- the smooth section \( \tau \) of \( \mathcal{G} \times_P G/Q_{\Lambda(t)} \) given by the projection onto the quotient of \( Ku_0 \subset \mathcal{G} \times_P G \),
- the parabolic geometry \((\mathcal{G} \to M, \omega_{\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}})\) given at \( w_0 \) by extension \((\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}, i)\) of \((K, H)\) to \((G, P)\),
- the natural (non–linear) connection \( \nabla \) induced by the Cartan connection \( \omega_{\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}} \) of the above geometry,

and the following claims hold:

1. The parabolic geometry \((\mathcal{G} \to M, \omega_{\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}})\) from above and the parabolic geometry \((\mathcal{G} \to M, \omega)\) provide equivalent underlying geometric structures.
2. The section \( \tau \) is a holonomy reduction of \((\mathcal{G} \to M, \omega_{\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}})\) of \( G^-\)-type \( G/Q_{\Lambda(t)} \), i.e., \( \tau \) is parallel with respect to \( \nabla \).
3. All points of \( M \) have the same \( P\)-type with respect to \( \tau \). Let us denote \( \mathcal{G}_{\Lambda(t)}(w_0) := Ku_0(Q_{\Lambda(t)} \cap P) \), \( \omega_{\Lambda(t)} := \iota^*\omega_{\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}} \), where \( \iota \) is the natural inclusion \( \mathcal{G}_{\Lambda(t)}(w_0) \hookrightarrow \mathcal{G} \). Then

\[ (\mathcal{G}_{\Lambda(t)}(w_0) \to M, \omega_{\Lambda(t)}) \]

is a homogeneous Cartan geometry of type \((Q_{\Lambda(t)}, Q_{\Lambda(t)} \cap P)\).
4. The parabolic geometry \((\mathcal{G} \to M, \omega_{\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}})\) is obtained by applying the extension functor induced by the inclusion \( Q_{\Lambda(t)} < G \) on the geometry \((\mathcal{G}_{\Lambda(t)}(w_0) \to M, \omega_{\Lambda(t)})\).

\textbf{Proof.} Since \( H \subset Q_{\Lambda(t)} \cap P \), the projection on the quotient of \( Ku_0 \subset \mathcal{G} \times_P G \) is a correctly defined smooth section of \( \mathcal{G} \times_P G/Q_{\Lambda(t)} \). Moreover, the corresponding function \( \mathcal{G} \to G/Q_{\Lambda(t)} \) has its values in the class \( P e Q_{\Lambda(t)} \) in \( P \mathcal{G}/Q_{\Lambda(t)} \). Thus if \( \tau \) is a holonomy reduction, then all points of \( M \) have the same \( P\)-type with respect to \( s \).

Further, since \( H \subset Q_{\Lambda(t)} \cap P \), the above decomposition of \( \alpha \) is \( H \)-invariant, and clearly, \((\alpha_{\Lambda(t), -} + \alpha_{\Lambda(t), 0}, i)\) is extension of \((K, H)\) to \((G, P)\). By the definition,
the parabolic geometry \((G \to M, \omega_{\alpha_{\Lambda(t)},- + \alpha_{\Lambda(t)},0})\) given by the extension \((\alpha_{\Lambda(t)},-, + \alpha_{\Lambda(t)},0, i)\) at \(u_0\) induces the same filtration of \(TM\) and the same reduction to \(G_0\). Thus the claim about the equivalence of underlying geometric structures follows for all types of parabolic geometries except projective and contact projective structures, for which the claim follows from the existence and uniqueness of invariant (partial) Weyl connection, see [3, 4].

Then by the construction of \(\hat{\nabla}\), it is clear that the section \(\tau\) is parallel, i.e. it is a holonomy reduction, and the remaining claims follow from [3, Theorem 2.6]. □

Remark 2. We remark that we are not reducing the normal Cartan connection, but a Cartan connection describing the same underlying geometric structure with different kind of normalization.

Let us now prove that this holonomy reduction does not depend on the choice of the point \(u_0\) satisfying the Theorem 1.2.

**Proposition 5.2.** Suppose \(p \in P\) is such that \(p^{-1}Hp \subset Q_{\Lambda(t)} \cap P\).

1. The section of \(G \times_P G/Q_{\Lambda(t)}\) given by the projection onto the quotient of \(Ku_0p \subset G \times_P G\) coincides with \(\tau\) (from the Theorem 5.1).
2. The above section is parallel with respect to the natural (non-linear) connection induced by the Cartan connection \(\omega_{\text{Ad}_p(\alpha_{\Lambda(t)},-, + \alpha_{\Lambda(t)},0)}\) on the geometry given by extension
   \[
   (\text{Ad}_p(\alpha_{\Lambda(t)},-, + \alpha_{\Lambda(t)},0), \text{conj}_{p^{-1}} \circ i)
   \]
   of \((K, H)\) to \((G, P)\) at \(u_0\), where \(\text{Ad}_p\) is the truncated adjoint action on \(\mathfrak{q}_{\Lambda(t)}\).
3. The parabolic geometry
   \[
   (G \to M, \omega_{\text{Ad}_p(\alpha_{\Lambda(t)},-, + \alpha_{\Lambda(t)},0)})
   \]
   provides an underlying geometric structure which is equivalent to the underlying geometric structure of the geometry \((G \to M, \omega)\).
4. The Cartan geometries \((G_{\Lambda(t)}(u_0) \to M, \omega_{\Lambda(t)})\) and \((G_{\Lambda(t)}(u_0p) \to M, \text{Ad}_p \circ \omega_{\Lambda(t)})\) are equivalent Cartan geometries of type \((Q_{\Lambda(t)}, Q_{\Lambda(t)} \cap P)\).

**Proof.** The claims follow from the property \(p^{-1}Hp \subset Q_{\Lambda(t)} \cap P\) in the same way as in the Theorem 5.1. Then the principal fibre bundle map

\[
Ku_0(Q_{\Lambda(t)} \cap P) \to Ku_0p(Q_{\Lambda(t)} \cap P)
\]

induces the equivalence of the Cartan geometries \((G_{\Lambda(t)}(u_0) \to M, \omega_{\Lambda(t)})\) and \((G_{\Lambda(t)}(u_0p) \to M, \text{Ad}_p \circ \omega_{\Lambda(t)})\). □

Thus let us fix the reduction \((G_{\Lambda(t)} \to M, \omega_{\Lambda(t)})\) given by the fixed choice of \(u_0\). The previous results have many crucial consequences. To write them explicitly, it remains to interpret the underlying geometric structure for Cartan geometries of type \((Q_{\Lambda(t)}, Q_{\Lambda(t)} \cap P)\).

**Proposition 5.3.** Let \(\omega_{\Lambda(t)} = \omega_{\Lambda(t)}^- + \omega_{\Lambda(t)}^0\) be the decomposition of \(\omega_{\Lambda(t)}\) according to the values in \(\mathfrak{g}_{\Lambda(t),-} \oplus \mathfrak{g}_{\Lambda(t),0}\). Denote

\[
VM := G_{\Lambda(t)} \times_{(Q_{\Lambda(t)} \cap P)} \mathfrak{g}_{\Lambda(t),0}/(\mathfrak{g}_{\Lambda(t),0} \cap \mathfrak{p}).
\]

1. The part \(\omega_{\Lambda(t)}^-\) provides the complement of \(VM\) in \(TM\) of the form
   \[
   T_{\Lambda(t),-}M := G_{\Lambda(t)} \times_{(Q_{\Lambda(t)} \cap P)} \mathfrak{g}_{\Lambda(t),-}.
   \]
2. The part \(\omega_{\Lambda(t)}^0\) defines a connection on \(T_{\Lambda(t),-}M\).
Proof. The claims follow directly from the inclusion \( H \subset Q_{\Lambda(t)} \cap P \) and the reductivity of the pair \((Q_{\Lambda(t)}, G_{A(t),0})\).

Now we can define distinguished classes of Weyl structures compatible with the above geometric structures. As usual, we denote by \( G_0 \simeq G/\exp(p_+) \) the underlying \( G_0 \)–bundle.

**Definition 5.** We call \( G_0 \)–equivariant sections \( \sigma : G_0 \rightarrow G_{\Lambda(t)} \subset G \) almost \( \Lambda(t) \)–invariant Weyl structures. If \( G_0 = G_{\Lambda(t)} \), then we call the Weyl structure \( \sigma = t : G_0 \rightarrow G \) invariant.

The situation is simple in the case \( G_0 = G_{\Lambda(t)} \). Then \( T^{\Lambda(t),-}M = TM \), and the soldering form and the connection from the Proposition correspond to the soldering form and the connection given by the invariant Weyl structure. In the general situation, we prove the existence of almost \( \Lambda(t) \)–invariant Weyl structures and investigate their properties.

**Proposition 5.4.** Almost \( \Lambda(t) \)–invariant Weyl structures always exist (on geometries satisfying our assumptions), and form an affine space over sections of \( gr(V^*M) := G_0 \times_{AdG_0} (q_{\Lambda(t)} \cap p_+) \).

Moreover, the pull–back \( \sigma^*\omega \) coincides on \( q_{\Lambda(t)} \) with the pull–back \( \sigma^*\omega^{\Lambda(t)} \) for any almost \( \Lambda(t) \)–invariant Weyl structure \( \sigma \).

Proof. The proof follows in the same way as the proof of the existence of global Weyl structures in [1, Section 5.1.1]. The only difference is that we consider \( G_0\exp(p_+ \cap q_{\Lambda(t)}) \) instead of \( G_0 \exp(p_+) \).

The above Proposition particularly clarifies the name almost \( \Lambda(t) \)–invariant Weyl structure \( \sigma \), because the pullback of \( \sigma \) by arbitrary automorphism of the parabolic geometry equals to \( \sigma \exp(T) \) for suitable \( T \) satisfying \( Im(T) \subset (q_{\Lambda(t)} \cap p_+) \).

Let \( \sigma \) be an arbitrary fixed almost \( \Lambda(t) \)–invariant Weyl structure. We can decompose the pullback \( \sigma^*\omega \) into \( G_0 \)–invariant parts which can be interpreted as follows:

**Proposition 5.5.** In the setting as above:

1. The part of \( \sigma^*\omega \) valued in \( g_{\Lambda(t),-} \) provides isomorphism \( T^{\Lambda(t),-}M \simeq K \times_H g_{\Lambda(t),-} \simeq G_0 \times_{G_0} g_{\Lambda(t),-} \).

Moreover, the part of \( \sigma^*\omega \) valued in arbitrary \( g_{\Lambda(t),0} \)–submodule \( \mathfrak{d} \) of \( g_{\Lambda(t),-} \) provides isomorphism of \( G_0 \times_{G_0} \mathfrak{d} \) with the invariant subbundle \( K \times_H \mathfrak{d} \) of \( T^{\Lambda(t),-}M \).

2. The part of \( \sigma^*\omega \) valued in \( g_{\Lambda(t),0} \) provides a unique affine connection on each \( G_0 \times_{G_0} \mathfrak{d} \) from (1).

3. The part of \( \sigma^*\omega \) valued in \( g_{\Lambda(t),0} \cap g_- \) provides isomorphism of \( gr(VM) \) with \( VM \).

4. The part of \( \sigma^*\omega \) valued in \( g_{\Lambda(t),0} \cap g_+ \) provides an affine connection on \( VM \).

5. The part of \( \sigma^*\omega \) valued in \( g_{\Lambda(t),0} \cap p_+ \) provides isomorphism of \( gr(V^*M) \) and \( V^*M \).

6. The part of \( \sigma^*\omega \) valued in \( g_{\Lambda(t),+} \) provides isomorphism \( (T^{\Lambda(t),-}M)^* \simeq K \times_H g_{\Lambda(t),+} \simeq G_0 \times_{G_0} g_{\Lambda(t),+} \).

Moreover, the part of \( \sigma^*\omega \) valued in arbitrary \( g_{\Lambda(t),0} \)–submodule \( \mathfrak{d}^* \) of \( g_{\Lambda(t),+} \), which dual (via Killing form of \( g \)) to \( g_{\Lambda(t),0} \)–submodule \( \mathfrak{d} \) of \( g_{\Lambda(t),-} \) provides isomorphism of \( G_0 \times_{G_0} \mathfrak{d}^* \) with \( K \times_H \mathfrak{d}^* \).

Proof. The claim follows directly from the previous Propositions and from the definition and properties of Weyl structures, see [1, Section 5.1.].
6. Generalized symmetries on correspondence and twistor spaces

Let us first consider arbitrary correspondence space of our parabolic geometry \((\mathcal{G} \to M, \omega)\) given by extension from Theorem 6.2. Clearly, the correspondence space does not have to satisfy our assumptions anymore. Indeed, it does not have to be homogeneous anymore. However, the generalized symmetries of the parabolic geometry downstairs are lifted to generalized symmetries of appropriate type on the correspondence space. Moreover, there can be generalized symmetries, that do not cover generalized symmetries downstairs. It is easy to characterize the lifted symmetries.

**Proposition 6.1.** A generalized symmetry at \(x\) on a correspondence space to \((\mathcal{G} \to M, \omega)\) is a lift of an underlying generalized symmetry of the parabolic geometry \((\mathcal{G} \to M, \omega)\) if and only if the vertical bundle \(\mathcal{V}_xM\) is contained inside the 1–eigenspace of the generalized symmetry.

**Proof.** Clearly, the condition that \(\mathcal{V}_xM\) is inside the 1–eigenspace of the generalized symmetry is necessary. Since \(\text{Ker}(\text{Ad}_{G_0}|_{\mathfrak{g}_o}) = Z(G_0)\), it is also sufficient. \(\square\)

Now, let us focus on (local) twistor spaces of the geometry \((\mathcal{G} \to M, \omega)\). Let us fix \(s \in Z(G_0)\). Then we can characterize, when \(s\)-symmetry is (locally) a lift of a (local) generalized symmetry from a suitable (local) twistor space.

**Proposition 6.2.** Suppose there is a local \(s\)-symmetry and denote by \(\Psi(1)\) the set of all simple restricted roots of \(s\) such that \(\mathfrak{g}_o\) is in 1–eigenspace of \(\text{Ad}_s\). Then the following statements hold:

1. The parabolic geometry \((\mathcal{G} \to M, \omega)\) is locally equivalent to an open subset of a correspondence space to a parabolic geometry of type \((G, \mathcal{P}_{\mathcal{E}_s-\Psi(1)})\).
2. The parabolic geometry of type \((G, \mathcal{P}_{\mathcal{E}_s-\Psi(1)})\) from (1) is locally homogeneous with local generalized symmetries covered by local \(s\)-symmetries.
3. The preimage \(\mathfrak{l}\) of \(\mathfrak{p}_{\mathcal{E}_s-\Psi(1)}\) in \(\mathfrak{sl}\) is a Lie subalgebra.

**Proof.** By the definition of \(\Psi\), the harmonic curvature vanishes on insertions of an element of \(\mathfrak{p}_{\mathcal{E}_s-\Psi(1)}\). Then the existence of the twistor space follows by the results in 6.4.1, and the existence of local generalized symmetries follows from the Proposition. Moreover, since the whole curvature vanishes on the insertions of entries of \(\mathfrak{p}_{\mathcal{E}_s-\Psi(1)}\), the last claim follows. \(\square\)

Let us remark that although the parabolic geometry on the twistor space is normal, it does not have to be regular. However, it follows from the classification in Appendix C that this behavior applies only for the following cases:

**Lemma 6.3.** The parabolic geometries of type \((G, \mathcal{P}_{\mathcal{E}_s-\Psi(1)})\) from the Proposition 6.2 are not regular in the following cases:

- \(\mathfrak{sl}(n+1, \{\mathbb{R}, \mathbb{C}\})\) with \(\Xi = \{1, 2, p, q\}\) and \(\Psi(1) = \{1\}\),
- \(\mathfrak{sp}(2n, \{\mathbb{R}, \mathbb{C}\})\) with \(\Xi = \{1, 2, p\}\), where \(p < n\), and \(\Psi(1) = \{1\}\).

Let us now show, when it is possible to construct the twistor space globally. Suppose \(s \in \mathcal{J}(\mathfrak{g})\), and let \(\mathfrak{l}\) be the preimage of \(\mathfrak{p}_{\mathcal{E}_s-\Psi(1)}\) in \(\mathfrak{sl}\). Then \(\mathfrak{l}\) decomposes (not directly) as

\[ \mathfrak{l} = (\mathfrak{l}(1) + \{h \cap \mathfrak{p}_{\mathcal{E}_s-\Psi(1)}, +\}) \]

where \(\mathfrak{l}(1)\) is the 1–eigenspace of \(\text{Ad}_s\) in \(\mathfrak{l}\), and each summand is \(\mathfrak{l}(1)\)–module. As usual, \(K^\circ\) is the connected simply connected Lie group with Lie algebra \(\mathfrak{g}\), and let \(\Gamma\) be the discrete central subgroup of \(K^\circ\) such that \(\Gamma \backslash K^\circ\) is the component of...
identity of $K$. Let $L^c$ be the (virtual) Lie subgroup of $K^c$ generated by $\exp((1))$, $\exp(h \cap \mathfrak{p}_{\mathfrak{g}-\mathfrak{g}(1),+})$ and $\Gamma$. The problem is to show, when $L^c$ is closed in $K^c$.

**Lemma 6.4.** Suppose there is $s' \in \mathcal{J}(\mathfrak{g})$ such that $\text{Ad}_{s'}|_{\mathfrak{g}(1)} = \text{id}$ and $\text{Ad}_{s'}|_{\mathfrak{g}_{-k}} \neq \text{id}$. Then $L^c$ is closed in $K^c$.

**Proof.** Since $\exp(1)$ is stable under conjugation by elements of $\Gamma$, the Lie algebra of $L^c$ is $\mathfrak{l}$. Further, $\exp(\mathfrak{h} \cap \mathfrak{p}_{\mathfrak{g}-\mathfrak{g}(1),+})$ is a Lie subgroup of $K^c$ stable under conjugation by elements of $L^c$. Thus it is enough to show that the virtual Lie subgroup of $K$ generated by $\exp((1))$ is closed. By our assumption, $\exp((1))$ is virtual Lie subgroup of the fixed point set $K^{\text{Ad}_{s'}}$ of the automorphism of $K^c$ induced by $\text{Ad}_{s'}$. Consider the pullback (via $\alpha$) of the filtration induced by $(\Xi - \Psi(1))$–heights on $\mathfrak{g}$, and the intersection of all normalizers of filtration steps of this filtration of $\mathfrak{g}$ in $K^{\text{Ad}_{s'}}$. Then the connected component of identity of this intersection is $\exp((1))$, because on the Lie algebra level, the claim follows in the regular case from the regularity. Indeed, if $Z$ from the Lie algebra of the intersection has a non–trivial projection to $\mathfrak{g}_-$, then $i \neq -k$ by assumption, and if $i < 0$, then there is $X \in \alpha(\mathfrak{g})$ such that $[Z,X]$ is outside of the filtration step of $X$. In the two non–regular cases from above, the claim follows by the same arguments, because either $X$ or $Z$ are in $\mathfrak{g}_{\mathfrak{g}-\mathfrak{g}(1)}$, where the curvature does not have entries. Thus $\exp((1))$ is closed in $K^c$, and the claim follows.

Let us remark that if there is no such $s' \in \mathcal{J}(\mathfrak{g})$ as in the previous Lemma, then the centralizer of $\mathfrak{l}$ in $\mathfrak{g}$ can have a non–trivial projection to $\mathfrak{g}_{-k}$. Since such elements in the centralizer can preserve the filtration steps, the subgroup $L^c$ does not have to be closed in general situation.

If $L^c$ is closed, then $K^c/L^c$ is a smooth manifold and can be identified with $K/L$, where the stabilizer $L$ of the point $eL^c$ has the Lie algebra $\mathfrak{l}$. Then the fiber of the submersion $K/H \to K/L$ is diffeomorphic to $L/H$ and connected by the construction. Let us show that the parabolic geometry $(\mathcal{G} \to M, \omega)$ descends to the twistor space $K/L$.

**Proposition 6.5.** Suppose $L$ is a closed Lie subgroup of $K$ with the Lie algebra $\mathfrak{l}$. Then the pair $(\alpha, \text{Ad})$ defines an extension of (non–effective) $(K, L)$ to $(\mathcal{G}, P_{\mathfrak{g}-\mathfrak{g}(1)})$, and the parabolic geometry $(\mathcal{G} \to M, \omega)$ covers an open subset in the correspondence space to the parabolic geometry

$$(K \times_{\text{Ad}(L)} P_{\mathfrak{g}-\mathfrak{g}(1)} \to K/L, \omega_\alpha)$$

given by this extension. In particular, there is a natural inclusion of the Lie group $Ker(\text{Ad}) \backslash K$ into the group of automorphisms of $(K \times_{\text{Ad}(L)} P_{\mathfrak{g}-\mathfrak{g}(1)} \to K/L, \omega_\alpha)$.

**Proof.** To prove the first claim, it suffices to show that $\text{Ad}(L)$ is naturally contained inside $P_{\mathfrak{g}-\mathfrak{g}(1)} \cong \text{Ad}(P_{\mathfrak{g}-\mathfrak{g}(1)}) \subset GL(\mathfrak{g})$. By definition, inclusions $\text{ad}(\mathfrak{l}) \subset \mathfrak{p}_{\mathfrak{g}-\mathfrak{g}(1)}$ and $\text{Ad}(H) \subset P_{\mathfrak{g}-\mathfrak{g}(1)}$ hold, and $L/H$ is connected. Thus the first claim follows.

Of course, the map $\text{Ad}$ does not have to be injective. In fact, the kernel of $\text{Ad}$ characterizes the covering $L/H \to \text{Ad}(L)/H \subset P_{\mathfrak{g}-\mathfrak{g}(1)}/P$ and the second claim follows. In particular, $Ker(\text{Ad})$ is the maximal normal subgroup of $K$ contained in $L$ and the last claim clearly follows.

The intersection of the sets $\Theta(\mathfrak{g})$ and $\Psi$ is contained in $\Psi(1)$ for any $s \in \mathcal{J}(\mathfrak{g})$. We can generalize the previous Proposition in the following way.

**Theorem 6.6.** Suppose there is $s \in \mathcal{J}(\mathfrak{g})$ such that $\mathfrak{g}_{-k} \not\subset \mathfrak{g}(1)$. Then the subgroup

$L_{\Theta(\mathfrak{g})\cap\Psi} := \{l \in L : \text{Ad}_l \in P_{\mathfrak{g}-\Theta(\mathfrak{g})\cap\Psi}\}$

of the Lie subgroup $L$ (defined for $s$ as above) is a Lie subgroup of $K$. The pair $(\alpha, \text{Ad})$ defines an extension of $(K, L_{\Theta(\mathfrak{g})\cap\Psi})$ to $(\mathcal{G}, P_{\mathfrak{g}-\Theta(\mathfrak{g})\cap\Psi})$, and the parabolic
geometry \((\mathcal{G} \to M, \omega)\) covers an open subset in the correspondence space to the parabolic geometry

\[
(K \times_{\mathcal{L}_{\Theta(t)}} P_{\Xi^-(\Theta(t) \cap \Psi)} \to K/L_{\Theta(t) \cap \Psi}, \omega_\alpha)
\]
given by the extension. Moreover, the following claims hold.

1. It holds

\[
\mathcal{J}(\mathfrak{k}) = \mathcal{L}_{\Theta \cap \Psi} \cap Z(G_{\Xi^-(\Theta \cap \Psi), 0}),
\]

where \(\mathcal{L}_{\Theta \cap \Psi} = N_{P_{\Xi^-(\Theta(t) \cap \Psi)}}(\alpha)\).

2. All the local generalized symmetries on \(M\) preserving \(\mathfrak{t}\) are lifts of local generalized symmetries on the twistor space preserving \(\mathfrak{t}\).

**Proof.** It follows from Lemma 6.3 that \(L\) is closed in \(K\), and the first part of the Theorem is a direct consequence of the Proposition 6.1, because we are just taking the correspondence space to \(P_{\Xi^-(\Theta(t) \cap \Psi)}\). Further, the claims (1) and (2) follow from Proposition 6.1 and the definition of \(\Theta(\mathfrak{k})\).

Let us now investigate the relation between the set \(\Psi\) and the decomposition \(TM = T^\Lambda(\mathfrak{k}) - M \oplus VM\) given by the holonomy reduction from Section 5. Firstly, we can decompose \(T^\Lambda(\mathfrak{k}) - M\) into subbundles corresponding to \(g_{\Lambda_1(\mathfrak{k})}\) submodules of \(g_{\Lambda_1(\mathfrak{k})}\). In particular, we consider subbundles corresponding to \(g_{\Xi^-(\Theta) \cap g_{\Lambda_1(\mathfrak{k})}}\) submodules.

**Corollary 6.7.** Suppose \(\mathfrak{d}\) is a \(g_{\Lambda_1(\mathfrak{k})}\) subbundle of \(g_{\Xi^-(\Theta) \cap g_{\Lambda_1(\mathfrak{k})}}\). Then the subbundle \(K \times_H (\mathfrak{d} \oplus g_{\Lambda_1(\mathfrak{k})}/g_{\Lambda_1(\mathfrak{k})})\) is integrable. If we denote by \(\mathcal{D}\) the leaf of \(K \times_H \mathfrak{d}\), then it is initial submanifold of \(M\) and carries a flat homogeneous geometry of type \((\exp \mathfrak{d} \times (G_{\Lambda_1(\mathfrak{k})} \cap P), G_{\Lambda_1(\mathfrak{k})} \cap P)\).

Clearly, \(\mathcal{D}\) does not have to be embedded and its stabilizer in \(K\) does not have to be closed in \(K\) or form reductive pair with \(H\).

Let us define the following complementary subbundles of \(TM\), which together generate the bundle \(V M = \mathcal{G}_{\Lambda(\mathfrak{k})} \times (\mathfrak{Q}_{\Lambda(\mathfrak{k})} \cap P) \mathfrak{g}_{\Lambda(\mathfrak{k})}/(\mathfrak{g}_{\Lambda(\mathfrak{k})} \cap \mathfrak{p})\):

- \(V^+ M := \mathcal{G}_{\Lambda(\mathfrak{k})} \times (\mathfrak{Q}_{\Lambda(\mathfrak{k})} \cap P) \mathfrak{g}_{\Lambda(\mathfrak{k})}/(\mathfrak{g}_{\Lambda(\mathfrak{k})} \cap (\Xi^-(\Theta) \cap P) \mathfrak{g}_{\Lambda(\mathfrak{k})} \cap (\Xi^-(\Theta) \cap P) \cap \mathfrak{p})\),
- \(V^- M := \mathcal{G}_{\Lambda(\mathfrak{k})} \times (\mathfrak{Q}_{\Lambda(\mathfrak{k})} \cap P) \mathfrak{g}_{\Lambda(\mathfrak{k})}/(\mathfrak{g}_{\Lambda(\mathfrak{k})} \cap (\Theta^- \cap P) \mathfrak{g}_{\Lambda(\mathfrak{k})} \cap (\Theta^- \cap P) \cap \mathfrak{p})\).

**Corollary 6.8.** The subbundle \(V^- M\) is integrable. If we denote by \(V^-\) the leaf of \(V^- M\), then it is initial submanifold of \(M\), and any almost \(\Lambda(\mathfrak{k})\) invariant Weyl structure restricted to \(V^-\) prolongs to a flat homogeneous parabolic geometry of (non–effective) type \((G_{\Lambda(\mathfrak{k}) \cup (\Xi^-(\Theta))}, 0, G_{\Lambda(\mathfrak{k}) \cup (\Xi^-(\Theta))}, 0) \cap P)\).

Again, \(V^-\) does not have to be embedded and its stabilizer in \(K\) does not have to be closed in \(K\).

If we consider, where can the harmonic curvature have its entries for the cases in tables in appendix A then we obtain the following claim.

**Corollary 6.9.** The subbundle \(V^+ M\) is integrable. If we denote by \(V^+\) the leaf of \(V^+ M\), then it is initial submanifold of \(M\) and any almost \(\Lambda(\mathfrak{k})\) invariant Weyl structure restricted to \(V^+\) prolongs to a flat homogeneous parabolic geometry of (non–effective) type \((G_{\Lambda(\mathfrak{k}) \cup (\Theta^-)}, 0, G_{\Lambda(\mathfrak{k}) \cup (\Theta^-)}, 0) \cap P)\).

Again, \(V^+\) does not have to be embedded and its stabilizer does not have to be closed in \(K\). Moreover, even if the stabilizer is closed, the curvature generally takes values in \(V^+ M\), and thus there is no induced parabolic geometry on the twistor space.
7. Parabolic geometries with invariant Weyl structures in generic situation

There are many geometries for which is generically \( \Lambda(\mathfrak{t}) = \Xi \). We investigate them in this section in detail. In particular, it turns out that the most studied types of parabolic geometries are among them.

**Theorem 7.1.** Assume \( \kappa \neq 0 \) and assume there is a (local) \( s \)-symmetry of \((G \to M, \omega)\) at \( x_0 \) for some \( s \neq e \). Then (under our assumptions) \( \Lambda(\mathfrak{t}) = \Xi \) holds for any subgroup \( K \) of \( \text{Aut}(G, \omega) \) acting transitively on \( M \) if and only if one of the following conditions holds for the parabolic geometry:

- \( \mathfrak{g}_{-1} \) is indecomposable \( \mathfrak{g}_0 \)-module, or
- the pair \((G, P)\) and the (subset of the set of) non-trivial components of \( \kappa_H \) correspond to one of the entries in the table.

| \( \mathfrak{g} \)          | \( \Xi \)          | \( \kappa_H \)          |
|---------------------------|--------------------|-------------------------|
| \( \text{sl}(n+1, \mathbb{R}, \mathbb{C}) \) | \( \{1, n\} \)  | \( (\alpha_1, \alpha_n) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, n\} \)  | \( (\alpha_1, \alpha_2)(\alpha_n, \alpha_n-1) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, n\} \)  | \( (\alpha_1, \alpha_n')(\alpha_n, \alpha_n') \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, n\} \)  | \( (\alpha_1, \alpha_2')(\alpha_1, \alpha_2') \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, n\} \)  | \( (\alpha_1, \alpha_2)(\alpha_1, \alpha_2') \) |
| \( \text{so}(3, 4) \)    | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_2, \alpha_1) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2')(\alpha_1, \alpha_2') \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_2', \alpha_1') \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_1', \alpha_2) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_1, \alpha_2') \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_2, \alpha_1)(\alpha_1, \alpha_2) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_2, \alpha_1')(\alpha_1, \alpha_2) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, p\} \)  | \( (\alpha_1, \alpha_2)(\alpha_1, \alpha_2) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{2, n-1\} \) | \( (\alpha_2, \alpha_1)(\alpha_n-1, \alpha_n) \) |
| \( \text{sl}(n+1, \mathbb{C}) \)  | \( \{1, 2, n\} \) | \( (\alpha_1, \alpha_2)(\alpha_2, \alpha_1)(\alpha_1, \alpha_n) \) |
| \( \text{sp}(4, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_1, \alpha_2') \) |
| \( \text{sp}(4, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_1', \alpha_2') \) |
| \( \text{sp}(4, \mathbb{C}) \)  | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2')(\alpha_1', \alpha_2) \) |
| \( \text{sp}(2n, \mathbb{C}) \) | \( \{1, 2\} \)  | \( (\alpha_1, \alpha_2)(\alpha_2, \alpha_1) \) |

In such case, the holonomy reduction from the Theorem 5.1 is a Cartan geometry \((G_0 \to M, \omega^2)\) of type \((Q_\Xi, G_0)\).

Moreover, the following claims hold for such parabolic geometries:

1. For each \( s \), there is at most one \( s \)-symmetry at each point of \( M \).
2. There always exists a (local) \( s \)-symmetry of \((G \to M, \omega)\) at \( x_0 \) (and thus at each point) for some \( s \neq e \) of a finite order, i.e., there is \( n \in \mathbb{N} \) such that \( s^n = e \).
3. There is a smooth manifold \( N \) and surjective submersion \( \pi : M \to N \) such that \( \text{Ker}(T \pi) \) consists of the 1–eigenspace[s] of the (local) \( s \)-symmetries of finite order from (2). The \( s \)-symmetries descend to \( N \) and determine a structure of a (locally) \( \mathbb{Z}_n \)-symmetric space on \( N \) (see (19)).
4. There always exist invariant Weyl structures, and the corresponding invariant Weyl connections descend to \( N \) and correspond to the canonical connection of the \( \mathbb{Z}_n \)-symmetric space \( N \).

**Proof.** Clearly, if \( \mathfrak{g}_{-1} \) is indecomposable \( \mathfrak{g}_0 \)-module, then \( \Phi(\mathfrak{t}) = \emptyset \) and if \( s \neq e \), then \( \Theta(\mathfrak{t}) = \emptyset \). By looking in the tables in Appendix C we get the remaining cases, when \( \Lambda(\mathfrak{t}) = \Xi \) generically holds.
Since \( \Phi(\mathfrak{t}) = \emptyset \), the claim (1) holds. It follows from the classification in the Appendix \[\text{C}\] that either the claim (2) holds for \( s \), or that \( Q_\Xi \) and \( \wedge^2 \mathfrak{p}_+ \otimes q_\Xi \) decomposes into indecomposable \( \mathfrak{g}_0 \)-modules according to eigenvalues of \( s \) and thus there is \( h \in Z(G_0) \) of finite order satisfying assumptions of the Proposition \[\text{I}\] and the claim (2) follows. Then by uniqueness of (local) \( s \)-symmetries and homogeneity, the 1–eigenspaces form a subbundle of \( TM \), which is invariant and integrable, because it corresponds to 1–eigenspace of \( \text{Ad}_s \) in \( \mathfrak{g}_- \). If we denote \( L \) the subgroup of \( K \) fixing the leaf of this integrable distribution through \( x_0 \), then \( L \) corresponds to \( l \in K \) such that \( \text{Ad}_l \) commutes with \( \text{Ad}_s \) and thus it is a closed subgroup of \( K \) with the Lie algebra corresponding to 1–eigenspace of \( \text{Ad}_s \) in \( \mathfrak{t} \). Thus \( N = K/L \) and the claim (3) holds. Finally, the claim (4) is a consequence of the invariance of the splitting of \( \mathfrak{g}_- \) given by the invariant Weyl structure, which cleary has to exist. \( \square \)

Let us point out that the list in the previous Theorem is complete and there are no other types of parabolic geometries for which \( \Lambda(\mathfrak{t}) = \Xi \) generically hold. Let us now formulate the results for the distinguished classes of parabolic geometries in detail.

**Proposition 7.2.** Let \((G, P)\) be type of a \([-1]\)-graded parabolic geometry. Assume \( \kappa \neq 0 \) and assume there is a (local) \( s \)-symmetry of \((G \to M, \omega)\) at \( x_0 \) for some \( s \neq e \). Then \( M = N \) is either (locally) symmetric or (locally) \( \mathbb{Z}_3 \)-symmetric space, and the invariant Weyl structure is unique.

**Proof.** All claims follow directly from the previous Theorem and the classification in the Appendix \[\text{C}\]. \( \square \)

We remark that we investigated the case of \( s \)-symmetries of \([-1]\)-graded parabolic geometries order \( 2 \) in detail in \[13\] \[15\]. The theory developed in the article does not provide any new results for \([-1]\)-graded geometries with these symmetries except the fact that only local existence of \( s \)-symmetries is sufficient.

**Proposition 7.3.** Let \((G, P)\) be type of a parabolic contact geometry. Assume \( \kappa \neq 0 \) and assume there is a (local) \( s \)-symmetry of \((G \to M, \omega)\) at \( x_0 \) for some \( s \neq e \). If \( \Lambda(\mathfrak{t}) = \Xi \), then one of the following facts applies:

- \( \ker(T_x \pi) \cong \mathfrak{g}_{-2} \) for each \( x \) and \( N \) is (locally) symmetric space. This happens in the following cases:

  \[
  \begin{array}{|c|c|c|}
  \hline
  \mathfrak{g} & \Xi & \kappa_H \\
  \hline
  \mathfrak{s}(n + 1, \mathbb{R}, \mathbb{C}) & \{1, n\} & (\alpha_1, \alpha_n) \\
  \mathfrak{su}(q, n + 1 - q) & \{1, n\} & (\alpha_1, \alpha_n) \\
  \mathfrak{sp}(2n, \mathbb{R}, \mathbb{C}) & \{1\} & (\alpha_1, \alpha_2) \\
  \mathfrak{sp}(2n, \mathbb{C}) & \{1\} & (\alpha_1, \alpha_1') \\
  \hline
  \end{array}
  \]

- \( \ker(T_x \pi) \cong \mathfrak{g}_{-2} \) for each \( x \) and \( N \) is (locally) \( \mathbb{Z}_3 \)-symmetric space. This happens in the following cases:

  \[
  \begin{array}{|c|c|c|}
  \hline
  \mathfrak{g} & \Xi & \kappa_H \\
  \hline
  \mathfrak{s}(n + 1, \mathbb{C}) & \{1, n\} & (\alpha_1, \alpha_2, \alpha_n, \alpha_{n-1}) \\
  \mathfrak{s}(n + 1, \mathbb{C}) & \{1, n\} & (\alpha_1, \alpha_{n-1}), (\alpha_n, \alpha_1') \\
  \mathfrak{s}(n + 1, \mathbb{C}) & \{1, n\} & (\alpha_1, \alpha_{n-1}), (\alpha_n, \alpha_1') \\
  \mathfrak{su}(q, n + 1 - q) & \{1, n\} & (\alpha_1, \alpha_2) \\
  \hline
  \end{array}
  \]

- \( M = N \) is (locally) \( \mathbb{Z}_3 \)-symmetric space. This happens in the following cases:
Which case happens in the case of \( \mathfrak{sp}(2n, \mathbb{C}) \) depends on the component of the curvature of the reduced geometry \( \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2} \otimes \mathfrak{g}_0 \).

Proof. All claims follow from the previous Theorem, the classification in the Appendix \[\Xi\] and the fact there are only the following possible components of curvature (in the \( \mathfrak{g}_0 \)-submodules of \( \wedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{g}_\Xi \)) of the reduced geometry:

- \( \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \) for the eigenvalue \( e^{i\theta} \), \( \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2} \otimes \mathfrak{g}_0 \) for the eigenvalue \( \sqrt{T} \) in \( \mathfrak{sp}(2n, \mathbb{C}) \)-case, \( \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \) for the eigenvalue \( \sqrt{T} \), and \( \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \) for eigenvalue \(-1\).

We remark that we investigated the case of \( s \)-symmetries of the first kind in \[\Xi\] \[\Xi\] \[\Xi\]. The theory developed in this article provides new results in this case. Moreover, if there are more generalized symmetries, we can strengthen the results of \[\Xi\] in the following way:

**Proposition 7.4.** Consider the first case from the Proposition \[\Xi\] and assume there is in addition a generalized symmetry other than the one of order 2 and the identity. Then the following holds for the parabolic geometries in question:

1. In the case of Lagrangean contact geometries, there is invariant para–complex structure \( \hat{\mathcal{I}} \) on \( TN \) induced by the para–complex structure \( \mathcal{I} \) on \( T^{-1}M \).
2. In the case of CR–geometries, there is invariant complex structure \( \hat{\mathcal{I}} \) on \( TN \) induced by the complex structure \( \mathcal{I} \) on \( T^{-1}M \).
3. In the case of complex Lagrangean contact geometries, there are both invariant para–complex and complex structures.

Proof. The existence of \( s \)-symmetry such that \( s \neq -\text{id}_{\mathfrak{g}_{-1}} \) implies that the eigenspaces \( \mathfrak{t}(j_1) \) and \( \mathfrak{t}(j_1^{-1}) \) or \( \mathfrak{t}(j_1) \), respectively, are complementary \( L \)-invariant subspaces in \( T_{\pi}(K/L) = T_{\pi(x_0)}N \) and provide invariant (para)–complex structure \( \hat{\mathcal{I}} \).

It is easy to check that it has the claimed properties.

Let us finally comment briefly the claims in Theorem \[\Xi\] for several remaining interesting types of parabolic geometries.

- In the case of (split)–quaternionic contact geometries and their complexifications, \( M \) is a (local) reflexion space with a three–dimensional fiber over a (locally) symmetric space \( N \).
- In the case of the \( (2,3,5) \)–geometry with \( \mathfrak{g} = \mathfrak{g}_2(\{2, \mathbb{C}\}) \), \( M \) is a (local) reflexion space with one–dimensional fiber over a (locally) symmetric space \( N \).
- In the case of the complex free \( (3,6) \)–distribution, \( M \) is a 6–dimensional (locally) \( \mathbb{Z}_3 \)–symmetric space with invariant polarization of \( TM \) given by the eigenspaces of \( \text{Ad}_s \).

**Appendix A. Notations related with the harmonic curvature**

For a parabolic geometry \( (\mathcal{G} \to M, \omega) \), we denote by \( \kappa \) its curvature, and by \( \kappa_H \) its harmonic curvature.

As usual, we assume that the geometries are regular and normal, which means for homogeneous parabolic geometries that at a point \( u_0 \in \mathcal{G} \) (and thus at each point), the curvature \( \kappa(u_0) \) viewed as an element of \( \wedge^2(\mathfrak{g}/p)^* \otimes \mathfrak{g} \) has positive homogeneity, and \( \partial^* \kappa(u_0) = 0 \) holds, where \( \partial^* \) is the Kostant’s codifferential.
Let us remind that the harmonic curvature \( \kappa_H(u_0) \) is the projection of the curvature \( \kappa(u_0) \) into the kernel of the Kostant Laplacian \( \square \), see [1] Section 3.1.12. According to the Kostant’s version of the Bott–Borel–Weyl theorem, the kernel \( \ker(\square) \) decomposes as \( \mathfrak{g}_0 \)-representation into the isotypical components which we represent by ordered pairs \( (a_{\alpha}, \alpha_b) \) meaning that the corresponding \( \mathfrak{p} \)-dominant and \( \mathfrak{p} \)-integral \( \mathfrak{g} \)-weight is obtained by the affine action of \( s_{\alpha_{\alpha}} s_{\alpha_b} \) on the highest root \( \mu^0 \) of \( \mathfrak{g} \), where \( s_{\alpha_i} \) denotes the simple reflection along \( \alpha_i \), see [1] Section 3.2.1. We use the notation \( (a_{\alpha}, \alpha_b) \) for the (highest) weight viewed as the element of \( H^2(\mathfrak{p}_+, \mathfrak{g}) \), too. The actual (lowest) weights representing indecomposable \( \mathfrak{g}_0 \)-submodules in \( H^2(\mathfrak{g}_+, \mathfrak{g}) \) are obtained via duality, i.e. the lowest weight vector corresponding to \( (a_{\alpha}, \alpha_b) \) is of the form \( X^{a_{\alpha}} \wedge X^{s_{\alpha} (\alpha_{\alpha})} \otimes X^{-s_{\alpha_{\alpha}} s_{\alpha_b} (\mu^0)} \), where \( X^{\alpha} \) denotes a root vector for \( \alpha \). Then the homogeneity of \( (a_{\alpha}, \alpha_b) \) with respect to \( \alpha_i \in \Xi \) for \( \mu^0 = \sum k_i \alpha_i = \sum r_i \lambda_i \), where \( \lambda_i \) denotes the corresponding fundamental weight, can be computed as follows:

- If \( i = a \), then the homogeneity is \(-k_a + 1 + r_a - \langle \alpha_b, \alpha_a \rangle (1 + r_b) \), where \( \langle , \rangle \) is the scalar product induced by the Killing form.
- If \( i = b \), then the homogeneity is \(-k_b + 1 + r_b \).
- If \( i = c \), where \( c \neq a, c \neq b \), then the homogeneity is \(-k_c \).

Let us remark that the length \( k \) of the grading of \( \mathfrak{g} \) given by \( \Xi \)-heights corresponds to \( \sum_{\alpha_i \in \Xi} k_i \).

Let us recall and refine several results of the article [9]. Firstly, the authors define the sets \( I_\mu \) as the sets of roots \( \alpha_i \in \Xi \) that satisfy \( \langle (\alpha_a, \alpha_b), \alpha_i \rangle = 0 \) for the highest weight \( \mu = (\alpha_a, \alpha_b) \) in \( H^2(\mathfrak{p}_+, \mathfrak{g}) \) representing a component of the harmonic curvature. The authors show in [9] Theorem 3.3.3 and Proposition 3.1.1 that each set \( I_\mu \) restricts the dimensions of projections of \( \mathfrak{k} \) into the associated grading of the filtration of \( \mathfrak{p} \). In fact, stronger results hold in the homogeneous setting with almost the same proofs:

**Proposition A.1.** In the case of homogeneous parabolic geometries, the Proposition 3.1.1 from [9] holds after the restriction to components in the indecomposable \( \mathfrak{g}_0 \)-submodules of \( \mathfrak{p} \). In particular, \( I_\mu \) characterizes the \( \mathfrak{g}_0 \)-submodules of \( \mathfrak{p} \), where the projection of \( \mathfrak{k} \) into the associated grading of \( \mathfrak{p} \) can be non-trivial.

**Proof.** Firstly, the [9] Theorem 3.3.3 gives results compatible with the decomposition to indecomposable \( \mathfrak{g}_0 \)-modules. Then the claim (4) of the Theorem 2.2 implies that the proof of [9] Proposition 3.1.1 can be applied to indecomposable \( \mathfrak{g}_0 \)-submodules of \( \mathfrak{p} \) without any change. Then the second claim is a consequence of [9] Theorem 3.3.3.

Finally, we prove a statement, which allows us to compute the set \( \Psi \) explicitly.

**Proposition A.2.** The set \( \Psi \) equals to the set of all simple roots \( \alpha_i \) such that \( \langle \alpha_i, (\alpha_a, \alpha_b) \rangle \geq 0 \) for all highest weights \( (\alpha_a, \alpha_b) \) representing non-trivial components of \( \kappa_H \). Moreover,

1. If \( \langle \alpha_a, \alpha_b \rangle \neq 0 \), then \( \alpha_a \) is the unique simple root \( \alpha \) such that \( \langle \alpha, (\alpha_a, \alpha_b) \rangle < 0 \).
2. If \( \langle \alpha_a, \alpha_b \rangle = 0 \), then \( \alpha_a \) and \( \alpha_b \) are the unique simple roots \( \alpha \) such that \( \langle \alpha, (\alpha_a, \alpha_b) \rangle < 0 \).

**Proof.** It follows by the general representation theory that the highest weight \( \mathfrak{g} \)-module of \( (\alpha_a, \alpha_b) \) is naturally both \( \mathfrak{p} \)-module and \( \mathfrak{p}_{\Xi - \Phi} \)-module and thus the component of the harmonic curvature represented by \( (\alpha_a, \alpha_b) \) does not take values in \( \mathfrak{g}_{\Xi - \Phi} \).
For arbitrary simple root $\alpha_i$, the direct computation implies

$$\langle \alpha_i, (\alpha_a, \alpha_b) \rangle = (\alpha_i, s_{\alpha_a}s_{\alpha_b}(\mu^g + \delta) - \delta)$$

$$= (\alpha_i, \mu^g) - (1 + \langle \mu^g, \alpha_b \rangle) \langle \alpha_i, \alpha_b \rangle$$

$$- (1 + \langle \mu^g, \alpha_a \rangle - (1 + \langle \mu^g, \alpha_b \rangle) (\alpha_b, \alpha_a) \langle \alpha_i, \alpha_a \rangle),$$

where $\delta$ denotes the lowest weight. Thus $\langle \alpha_i, (\alpha_a, \alpha_b) \rangle \geq 0$ holds for $i \neq a, b$.

If $i = a$, then

$$\langle \alpha_a, (\alpha_a, \alpha_b) \rangle = -\langle \alpha_a, \mu^g \rangle + (1 + \langle \mu^g, \alpha_b \rangle) \langle \alpha_a, \alpha_b \rangle - 2$$

$$= -k_a + (1 + k_b) \langle \alpha_a, \alpha_b \rangle - 2 < 0.$$ 

If $i = b$, then

$$\langle \alpha_b, (\alpha_a, \alpha_b) \rangle = -\langle \alpha_b, \mu^g \rangle - 2 - (1 + \langle \mu^g, \alpha_a \rangle) \langle \alpha_b, \alpha_a \rangle + (1 + \langle \mu^g, \alpha_b \rangle) \langle \alpha_b, \alpha_a \rangle^2.$$ 

Thus if $\langle \alpha_b, \alpha_a \rangle = 0$, then $\langle \alpha_b, (\alpha_a, \alpha_b) \rangle < 0$. Otherwise, the last term is greater than the first term, and the third term is greater than second term in the absolute value, and thus $\langle \alpha_b, (\alpha_a, \alpha_b) \rangle \geq 0$. □

**APPENDIX B. EXAMPLE**

Let us illustrate the theory on the following example. This example also provides a counter-example for the Theorem 1.2 to hold without considering the set $\Theta(\mathfrak{f})$.

Let us consider the seventeen–dimensional Lie subgroup $K$ of $Gl(8, \mathbb{R})$ consisting of the elements of the form

$$
\begin{pmatrix}
\pm \cos(x_9) & \mp \sin(x_9) & 0 & 0 & 0 & 0 & 0 & 0 \\
\sin(x_9) & \cos(x_9) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm \cosh(x_9) & \pm \sinh(x_9) & 0 & 0 & 0 & 0 \\
0 & 0 & \sinh(x_9) & \cosh(x_9) & 0 & 0 & 0 & 0 \\
x_1 & x_2 & 0 & 0 & x_{10} & x_{11} & 0 & 0 \\
x_3 & x_4 & 0 & 0 & x_{12} & x_{13} & 0 & 0 \\
0 & 0 & x_5 & x_6 & 0 & 0 & x_{14} & x_{15} \\
0 & 0 & x_7 & x_8 & 0 & 0 & x_{16} & x_{17}
\end{pmatrix},
$$

where $x_1, \ldots, x_{17} \in \mathbb{R}$ and $(x_{10}x_{13} - x_{11}x_{12})(x_{14}x_{17} - x_{15}x_{16}) > 0$, together with its five–dimensional (solvable) Lie subgroup $H$ consisting of the elements of the form

$$h = 
\begin{pmatrix}
\pm 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{10} & x_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_{14} & x_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_{13} & 0
\end{pmatrix},
$$

i.e., $x_{10}x_{13}^2x_{14} > 0$.

We will investigate the regular normal homogeneous parabolic geometry on $K/H$ of type $(PGL\langle 6, \mathbb{R} \rangle, P_{1,2,5})$ (and thus $\Xi = \{1, 2, 5\}$ and $\mathfrak{g} = sl(6, \mathbb{R})$) given by the extension $(\alpha, i)$ of $(K, H)$ to $(PGL\langle 6, \mathbb{R} \rangle, P_{1,2,5})$ defined as follows: We define $i(h)$ for $h \in H$ of the above form as:
Theorem 2.2. In particular, there are no other infinitesimal automorphisms of Ψ = \{x, y\}. Further, we define will turn out to be the usual (geodesic) symmetry for some underlying symmetric space. Further, we define

\[
\begin{pmatrix}
\frac{1}{\sqrt{210^2}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{210^2}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{210^2}} & 0 & x_{11} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{210^2}} & x_{15} & x_{15} \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{210^2}} & x_{15} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{210^2}}
\end{pmatrix}.
\]  

We have chosen \( PGl \) instead of \( Sl \) in order to allow \( \pm \) on the first position, which will turn out to be the usual (geodesic) symmetry for some underlying symmetric space. Further, we define

\[
\alpha = \begin{pmatrix}
0 & -X_9 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & X_9 & 0 & 0 & 0 & 0 \\
0 & 0 & X_9 & 0 & 0 & 0 & 0 & 0 \\
X_1 & X_2 & 0 & 0 & X_{10} & X_{11} & 0 & 0 \\
X_3 & X_4 & 0 & 0 & X_{12} & X_{13} & 0 & 0 \\
0 & 0 & X_5 & X_6 & 0 & 0 & X_{14} & X_{15} \\
0 & 0 & X_7 & X_8 & 0 & 0 & X_{16} & X_{17}
\end{pmatrix}
\]

\[
\begin{pmatrix}
C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_9 & C & 0 & 0 & 0 & 0 & 0 & 0 \\
X_1 & X_2 & C + X_{10} & 0 & X_{11} & 0 & 0 & 0 \\
X_5 & X_6 & 0 & C + X_{14} & X_{15} & X_{15} & 0 & 0 \\
X_3 & X_4 & X_{12} & 0 & C + X_{13} & 0 & 0 & 0 \\
X_7 - X_3 & X_8 - X_4 & -X_{12} & X_{16} & X_{17} - X_{13} & C + X_{17} & 0 & 0
\end{pmatrix},
\]

where \( C = -\frac{X_{10} + X_{11} + X_{14} + X_{17}}{6} \).

The curvature of this geometry at the origin for \( X, Y \in \mathfrak{g} \) as above is of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X_1Y_9 - X_9Y_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(X_5Y_9 - X_9Y_5) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X_3Y_9 - X_9Y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2(X_3Y_9 - X_9Y_3) - (X_7 - X_4)Y_9 + X_9(Y_7 - Y_3) & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where the bold entry corresponds to the harmonic part of the curvature. Thus \( \kappa_H \) has non-trivial part in the submodule represented by \( \{\alpha_1, \alpha_2\} \), \( \Phi(\mathfrak{g}) = \emptyset \) and \( \Psi = \{2, 5\} \).

It is easy computation to check that the extension satisfies all the properties of the Theorem 2.2. In particular, there are no other infinitesimal automorphisms of this geometry and \( \mathfrak{h} \cap \mathfrak{p}_+ = 0 \) as expected from the theory. Thus the set of all possible generalized symmetries consists of elements of the form:
is a simple computation to check that 
\[ \{ g \} \text{our theory. Since the eigenvalues of generalized symmetries on } L \text{ can apply the Theorem 6.6 for the whole curvature of this geometry is harmonic, which means that the } \text{it is not harmon ic.}

Our theory has the following geometrical consequences for this geometry: It is a simple computation to check that \( H \) is not \( P \)-conjugated to a subgroup of \( Q_{G - \Phi(t)} = Q_{1,2,5} \). On the other hand, \( H \subset Q_{\Lambda(t)} = Q_{1,2} \) holds consistently with our theory. Since the eigenvalues of generalized symmetries on \( g_{-k} \) are \( -j_2 \), we can apply the Theorem 6.6 for \( L_{\Theta(t) \cap \Phi} = L_5 \), which consists of the elements of the form

\[
\begin{pmatrix}
\pm 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{10} & x_{11} & 0 \\
0 & 0 & 0 & x_{12} & x_{13} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_{14} & x_{15} \\
0 & 0 & 0 & 0 & 0 & x_{16} & x_{17}
\end{pmatrix},
\]

i.e., the eigenvalues are \( j_1 = \pm 1, j_5 = 1 \) and \( j_2 \) can be arbitrary for the possible generalized symmetries. This means that this is not a generic situation, and \( \Theta(\mathfrak{t}) = \{ 5 \} \).

Since \( L_5 \) is reductive, \( K/H \) is open in the correspondence space to the parabolic geometry of type \( (PGL(6, \mathbb{R}), P_{1,2}) \) given by the extension \((\tilde{\alpha}, \tilde{i})\) of \((K, L_5)\) to \((PGL(6, \mathbb{R}), P_{1,2})\) defined as follows: The map \( \tilde{i} \) is \( \frac{1}{\sqrt{j_2}} \) multiple of the restriction of \( L_5 \) to the bottom right block, and \( \tilde{\alpha} \) has the following form for \( X \in \mathfrak{t} \) as above:

\[
\tilde{\alpha}(X) = \begin{pmatrix}
C & 0 & 0 & 0 & 0 & 0 & 0 \\
X_9 & C & 0 & 0 & 0 & 0 & 0 \\
X_1 & X_2 & C + X_{10} & X_{11} & 0 & 0 & 0 \\
X_3 & X_4 & X_{12} & C + X_{13} & 0 & 0 & 0 \\
X_5 & X_6 & 0 & 0 & C + X_{14} & X_{15} & 0 \\
X_7 & X_8 & 0 & 0 & X_{16} & C + X_{17} & 0
\end{pmatrix},
\]

where \( C = -\frac{X_{10} + X_{11} + X_{14} + X_{15}}{6} \). In particular, \( K/H \) is the \( K \)-orbit of \( pP_{1,2,5} \) for \( p \in P_{1,2} \) of the form

\[
p = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

Thus \( Ad_p(\alpha) = \tilde{\alpha} \) and \( p \tilde{i}(h)p^{-1} = \tilde{i}(h) \). Now, \( L_5 \subset G_{(1,2),0} \) holds and the whole curvature of this geometry is harmonic, which means that the \( L_5 \)-invariant complement of \( \mathfrak{l}_5 \) in \( \mathfrak{t} \) provides the integrable distribution in \( T(K/H) \). Let us point out that this is not the case when the whole curvature is not harmonic.
Now, we can apply the theory from the Section 5 on the geometry on $K/L_5$, because it satisfies all the necessary conditions. So there is the invariant Weyl connection on $K/L_5$, which is covered by a class of almost $\{1, 2\}$-invariant Weyl connections on $K/H$.

If $s \in J_2$ satisfies $j_2 = 1$, then $\Psi(1) \cap \Xi = \{2\}$, and we can apply the Theorem 6.5 for the subgroup $L = L_{2,5}$ consisting of the elements of the form

$$\begin{pmatrix}
\pm 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & x_{10} & x_{11} & 0 & 0 & 0 \\
x_4 & 0 & 0 & x_{12} & x_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & x_6 & 0 & 0 & x_{14} & x_{15} \\
0 & 0 & 0 & x_8 & 0 & 0 & x_{16} & x_{17}
\end{pmatrix}.$$ 

The pair $(K, L_{2,5})$ is reductive, and $K/L_5$ and $K/H$ are open in the correspondence spaces to the projective geometry given by the extension $(\alpha, \tilde{i})$ of $(K, L_5)$ to $(PGL(6, \mathbb{R}), P_1)$ defined as follows: The map $\tilde{i}$ is $\frac{1}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}}$ multiple of the restriction of $L_{2,5}$ to the bottom right block after the shifting the $x_2, x_4$ two columns to the right, and $\alpha$ is the same as above. In particular, $K/H$ is again the $K$–orbit of $pP_{1,2,5}$ and $K/L_5$ is the $K$–orbit of $eP_{1,2}$.

Moreover, the projective geometry on $K/L_{2,5}$ is symmetric. Thus $K/L_{2,5}$ is non–effective symmetric space, because the Lie algebra of the group generated by symmetries corresponds to the $L_5$–invariant complement of $l_5$ in $\mathfrak{g}$.

This geometry has further geometric properties, which does not have to occur in the general situation. Firstly, the distribution $T^{\lambda(\mathfrak{p})} \cdot (K/H)$ is integrable and corresponds to the Lie algebra of the group generated by generalized symmetries.

Further, for $\{2\} \subset \Psi$, the subgroup $L_2$ consisting of the elements of the form

$$\begin{pmatrix}
\pm 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & x_{10} & x_{11} & 0 & 0 & 0 \\
x_4 & 0 & 0 & x_{12} & x_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & x_6 & 0 & 0 & x_{14} & x_{15} \\
0 & 0 & 0 & x_8 & 0 & 0 & x_{16} & x_{17}
\end{pmatrix}$$

is closed in $K$, and we can apply the analogy of the Proposition 6.3 for this situation. Namely, there is the Lagrangean contact geometry on $K/L_2$ given by the extension $(\alpha, \tilde{i})$ of $(K, L_2)$ to $(PGL(6, \mathbb{R}), P_{1,5})$ defined as follows: We define $i(h)$ for $h \in L_2$ of the above form as:

$$\begin{pmatrix}
\pm \frac{1}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & x_{10} & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & \frac{x_{10}}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & 0 & x_{11} & 0 & 0 & 0 \\
0 & x_6 & 0 & \frac{x_{14}}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & x_{15} & x_{15} & 0 & 0 \\
0 & x_4 & 0 & 0 & \frac{x_{13}}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{x_{13}}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{x_{13}}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x_{13}}{\sqrt{x_{10}^2 x_{13}^2 x_{14}^2}}
\end{pmatrix}.$$
This geometry has only generalized symmetries of one type, and the generalized symmetries are the lifts of the symmetries of the above underlying symmetric projective geometry. So this geometry satisfies $\Theta(t) = \Psi(1) = \{5\}$, and $K/L_2$ is the $K$–orbit of $pP_{1.5}$ in the correspondence space to the projective geometry on $K/L_{2.5}$ for $p$ as above.

Finally, we can consider the subgroup $L_{1.5}$, which is the centralizer of $s \in Z(G_0) \cap H$ with eigenvalues $j_1 = 1, j_2 = -1, j_5 = 1$ in $K$, consisting of the elements of the form

$$
\begin{pmatrix}
\pm \cos(x_9) & \mp \sin(x_9) & 0 & 0 & 0 & 0 & 0 \\
\sin(x_9) & \cos(x_9) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm \cosh(x_9) & \pm \sinh(x_9) & 0 & 0 & 0 \\
0 & 0 & \sinh(x_9) & \cosh(x_9) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{10} & x_{11} & 0 \\
0 & 0 & 0 & 0 & x_{12} & x_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & x_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & x_{16} \\
0 & 0 & 0 & 0 & 0 & 0 & x_{17}
\end{pmatrix}
$$

Then $K/L_{1.5}$ is a symmetric space, but since there are entries containing $X_9, Y_9$ in the curvature, the parabolic geometry of type $(\mathrm{PGL}(6, \mathbb{R}), P_{1.2.5})$ on $M$ does not descend to a parabolic geometry of type $(\mathrm{PGL}(6, \mathbb{R}), P_2)$ on this underlying symmetric space.

**Appendix C. Tables of simple geometries admitting non–trivial curvature**

We present here tables classifying all possible generalized symmetries of non–flat parabolic geometries with $\mathfrak{g}$ simple. We recall that there is the construction in $\mathbb{Z}$, that allows to construct explicit examples of such geometries. The data in the tables can be divided into two parts. The first one describes the information on the parabolic geometry and consists of the following data:

- $\mathfrak{g}$ distinguishes the simple Lie algebra $\mathfrak{g}$, where $n \geq |\Xi|$ determines the rank and $q > 0$ the signature. We assume that $n > 1$ for type $A$, $n > 1$ for type $C$, $n > 2$ for type $B$, $n > 3$ for type $D$, and $n > 6$ for type $BD$.

- $\Xi$ distinguishes the parabolic subalgebra $\mathfrak{p}_\Xi$, where the parameter $p$ is such that all simple roots in $\Xi$ make sense. In the cases dealing with mixed types of curvature, where conjugate roots have different homogeneity, we list the conjugate roots (indicated by ‘) in $\Xi$, too.

- param. contains additional restrictions on parameters $p, q, n$.

- $\mu$ represents the $\mathfrak{g}_0$–submodule to which $\kappa_H$ can have non–trivial projection.

- homog. is the tuple characterizing homogeneity of $(\alpha_a, \alpha_b) \in I$ with respect to $\alpha_1 \in \Xi$ ordered in the same way as $\Xi$.

- $I_\mu$ characterizes all simple roots of $\Xi$ that can be contained in $\Phi(t)$ for this $\mu$, see Appendix A. Additional restrictions on parameters are listed to distinguish special situations.

- all info indicates, when it is convenient to put all the information in one column.

- The informations in rows (if they are not missing) are ordered as above.

Then for the given type of parabolic geometries, the second part describes the possible types of generalized symmetries by characterizing the possible eigenvalues $j_i$ of $\text{Ad}_s$ on the simple restricted root spaces $\mathfrak{g}_{\alpha_i}$ for $i \in \Xi$. Let us recall that in this setting, the only restrictions on the possible eigenvalues $j_i$ are the following:

- if $\alpha_{i_1}$ and $\alpha_{i_2}$ are complex conjugated, then $j_{i_1} = j_{i_2}$.
The first restriction follows from Proposition 3.2, and the second one from the definition of homogeneity. We split the different cases depending how the 1-
eigenspace $m\in\mathfrak{g}$ looks like.

In the case of mixed curvature, we write the eigenvalues $\alpha\in\mathfrak{g}$ and $\gamma\in\mathfrak{g}$

The first restriction follows from Proposition 3.2, and the second one from the definition of homogeneity. We split the different cases depending how the 1-
eigenspace $m\in\mathfrak{g}$ looks like.

The first restriction follows from Proposition 3.2, and the second one from the definition of homogeneity. We split the different cases depending how the 1-
eigenspace $m\in\mathfrak{g}$ looks like.

In both cases, $\mathfrak{g}$ represents the irreducible $\mathfrak{g}$-submodules of $P_+$ in the 1-
eigenspace $m\in\mathfrak{g}$. We use $\mathfrak{g}$ to shorten the entry, when the weight is

The first restriction follows from Proposition 3.2, and the second one from the definition of homogeneity. We split the different cases depending how the 1-
eigenspace $m\in\mathfrak{g}$ looks like.
| \(g\)       | \(\Xi\) | \(I_\mu\) | \(j_{i_1}\) | \(j_{i_2}\) | \(m\) |
|-------------|---------|-----------|-------------|-------------|------|
| param.      | \(\mu\) | homog.    |             |             |      |
| \(\mathfrak{sl}(n+1,\{R,C\})\) | \(\{1,2\}\) | \(\alpha_1,\alpha_2\) | 0 | 0 | \(\alpha_1 + \alpha_2\) |
| \(n > 2\)  |          |           | 0           | 0           |      |
| \(\mathfrak{sl}(n+1,\{R,C\})\) | \(\{1,2\}\) | \(\alpha_1\) | 0           | 0           | \(\alpha_1\) |
| \(n > 2\)  |          |           | 0           | 0           |      |
| \(\mathfrak{sl}(n+1,\{R,C\})\) | \(\{1,n\}\) | \(\alpha_1,\alpha_n\) | 0           | 0           | \(\alpha_1 + \cdots + \alpha_n\) |
| \(n > 2\)  |          |           | 0           | 0           |      |
| \(\mathfrak{su}(q, n+1-q)\) | \(\{1,p\}\) | \(\alpha_{p+1}\) | 0           | 0           | \(\alpha_{p+1}\) |
| \(n > 2\)  |          |           | 0           | 0           |      |
| \(\mathfrak{su}(q, n+1-q)\) | \(\{1,n\}\) | \(\alpha_1,\alpha_2\) | 0           | 0           | \(\alpha_1 + \cdots + \alpha_n\) |
| \(2 < p\)  |          |           | 0           | 0           |      |
| \(\mathfrak{su}(q, n+1-q)\) | \(\{1,n\}\) | \(\alpha_1,\alpha_2\) | 0           | 0           | \(\alpha_1 + \cdots + \alpha_n\) |
| \(n > 2\)  |          |           | 0           | 0           |      |
| $\Theta$ | $\Xi$ | $l_\mu$ | $j_1$ | $j_2$ | $m$ |
|--------|-------|---------|------|------|-----|
| $\text{sp}(4, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, 2\}$ | $\frac{1}{2}$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_2$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, 2\}$ | $\alpha_1$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + 2\alpha_2 + \ldots$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, n\}$ | $\alpha_1, \alpha_2$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + \cdots + \alpha_n$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, 2\}$ | $\alpha_1, \alpha_2$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + 2\alpha_2 + \ldots$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, n\}$ | $\alpha_1, \alpha_2$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + \cdots + \alpha_n$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, 2\}$ | $\alpha_1, \alpha_2$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + 2\alpha_2 + \ldots$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, n\}$ | $\alpha_1, \alpha_2$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + \cdots + \alpha_n$ |
| $\text{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$ | $\{1, n\}$ | $\alpha_1, \alpha_2$ | $\sqrt{1} \ j_1$ | $\sqrt{1} \ j_2$ | $\alpha_1 + 2\alpha_2 + \ldots$ |
| $\mathfrak{g}$ param. | $\Xi$ homog. | $I_\mu$ homog. | $j_1$ | $j_2$ | $m$ |
|----------------------|-------------|----------------|-------|-------|-----|
| $\mathfrak{so}(3, 4), \mathfrak{so}(7, \mathbb{C})$ | $\{(1, 3) \ (\alpha_3, \alpha_2)\}$ | $(-1, 3)$ | $1$ | $\sqrt{T}$ | $\alpha_1$ | $\alpha_1 + \alpha_2 + \alpha_3$ | $\alpha_1 + 2\alpha_2$ |
| $\mathfrak{so}(3, 4), \mathfrak{so}(7, \mathbb{C})$ | $\{(2, 3) \ (\alpha_3, \alpha_2)\}$ | $(0, 3)$ | $1$ | $1$ | $\alpha_2$ | $\alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3$ | $\alpha_3$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3$ | $\alpha_2 + 2\alpha_3$ |
| $\mathfrak{so}(3, 5)$ | $\{(3, 4) \ (\alpha_3, \alpha_2)\}$ | $(2, -1)$ | $\sqrt{T}$ | $\sqrt{T}$ | $\alpha_1 + \cdots + \alpha_4$ |
| $\mathfrak{so}(n, n), \mathfrak{so}(2n, \mathbb{C})$ | $\{1, n \ (\alpha_1, \alpha_2)\}$ | $(n, n)$ | $\alpha_n, n > 4$ | $(-1, -1)$ | $\sqrt{T}$ | $\sqrt{T}$ | $\alpha_n + \cdots + \alpha_n$ |
| $\mathfrak{so}(q, n-q), \mathfrak{so}(n, \mathbb{C})$ | $\{1, 2 \ (\alpha_1, \alpha_2)\}$ | $(0, 2)$ | $\alpha_2$ | $\alpha_2$ | $\alpha_1, \alpha_1 + 2\alpha_2 + \cdots$ | $\alpha_2$ | $\alpha_1 + \alpha_2$ | $\alpha_1 + 2\alpha_2 + \cdots$ | $\alpha_1$ |
| $\mathfrak{so}(q, n-q), \mathfrak{so}(n, \mathbb{C})$ | $\{1, 2 \ (\alpha_2, \alpha_1)\}$ | $(0, 1)$ | $\alpha_2$ | $\alpha_2$ | $\alpha_3, \alpha_1 + 2\alpha_2 + \cdots$ | $\alpha_3$ |
| $\mathfrak{so}(q, n-q), \mathfrak{so}(n, \mathbb{C})$ | $\{2, 3 \ (\alpha_3, \alpha_2)\}$ | $(0, 1)$ | $\alpha_2$ | $\alpha_2$ | $\alpha_2, 3\alpha_1 + 2\alpha_2$ | $\alpha_2, 2\alpha_1 + \alpha_2$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1, 3\alpha_1 + 2\alpha_2$ | $\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2$ |

\[ \mathfrak{g}_2(2, \mathbb{C}) \]
| g param. | \( I_{\mu} \) | \( \mu \) | \( \Xi \) | \( j_1 \) | \( j_2 \) | \( j_3 \) | \( j_4 \) | \( m \) |
|----------|----------------|--------|--------|--------|--------|--------|--------|--------|
| \( sl(n + 1, \mathbb{C}) \) | \( \alpha_p \) \( p, p + 1, p', (p + 1)' \) | \( -1, -1, 1, 2 \) | 1 | \( \sqrt{T} \) | 1 | \( \pm e^{r/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( \alpha_p \) | \( \alpha_{p+1} \) |
| \( sl(n + 1, \mathbb{C}) \) | \( \alpha_p \) \( 1, p, 1', p' \) | \( 1, -1, 0, 1 \) | 1 | \( \sqrt{T} \) | \( \sqrt{T^2} \) | 1 | \( \pm e^{r/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( \alpha_1 \) | \( \alpha_1 + \cdots + \alpha_p \) |
| \( sl(n + 1, \mathbb{C}) \) | \( \alpha_p \) \( 1, p, 1', p' \) | \( 1, -1, 1, 0 \) | 1 | \( e^{i\phi} \) | \( e^{r^2/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( \alpha_p \) | |
| \( sp(2n, \mathbb{C}) \) | \( \alpha_{n-1}, \alpha_n \) \( n - 1, n, (n - 1)', n' \) | \( -2, -1, 3, 1 \) | 1 | \( \sqrt{T^3} \) | 1 | \( \pm e^{r/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( \alpha_{n-1} \) | \( \alpha_n \) | \( \alpha_{n-1} + \alpha_n \) |
| \( sp(2n, \mathbb{C}) \) | \( \alpha_{n} \) \( 1, n, 1', n' \) | \( 1, -1, 1, 0 \) | 1 | \( e^{i\phi} \) | \( e^{r^2/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( \alpha_n \) | \( 2\alpha_1 + \cdots + \alpha_n \) |
| \( sp(2n, \mathbb{C}) \) | \( \alpha_{n} \) \( 1, n, 1', n' \) | \( 1, -1, 0, 1 \) | 1 | \( \sqrt{T} \) | \( \pm e^{r/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( e^{r^2/3i\phi} \) | \( \alpha_n \) | \( 2\alpha_1 + \cdots + \alpha_n \) |
| all info | $j_1$ | $j_2$ | $j_3$ | $m$ |
|---------|------|------|------|-----|
| $\mathfrak{sl}(n + 1, \mathbb{R}, \mathbb{C})$ | 1 | -1 | 1 | $\alpha_1, \alpha_p$ |
| $\{1, 2, p\}$ | -1 | 1 | -1 | $\alpha_2, \alpha_1 + \cdots + \alpha_p$ |
| $(\alpha_2, \alpha_1)$ | -1 | -1 | -1 | $\alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_p$ |
| $(1, 2, -1)$ | 1 \(\sqrt{T}\) \(\sqrt{T}^2\) | $\alpha_1, \alpha_2 + \cdots + \alpha_p, \alpha_1 + \cdots + \alpha_p$ |
| $\alpha_1, \alpha_p, n > p > 3$ | $\bar{j}_2^2$ | $j_2$ | 1 | $\alpha_p$ |
| | $j_1$ | 1 | $j_1$ | $\alpha_2$ |
| | $j_2$ | $j_2^2$ | $j_1$ | $\alpha_2 + \cdots + \alpha_p$ |
| | $j_p^3$ | $j_p$ | $j_p^1$ | $\alpha_1 + \cdots + \alpha_p$ |
| | $j_1$ | $j_2$ | $\bar{j}_1 \bar{j}_2^2$ | $\alpha_1 + \alpha_2$ |
| $\mathfrak{sl}(n + 1, \mathbb{R}, \mathbb{C})$ | -1 | 1 | 1 | $\alpha_2, \alpha_p, \alpha_2 + \cdots + \alpha_p$ |
| $\{1, 2, p\}$ | $\sqrt{T}$ | 1 | $\sqrt{T}^2$ | $\alpha_2, \alpha_1 + \cdots + \alpha_p$ |
| $(\alpha_1, \alpha_2)$ | 1 | $j_2$ | 1 | $\alpha_1, \alpha_p$ |
| $(2, 0, -1)$ | $j_1$ | 1 | $j_1^2$ | $\alpha_2$ |
| $\alpha_p, n > p > 3$ | -1 | -1 | 1 | $\alpha_p, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \cdots + \alpha_p$ |
| | -1 | $j_2^2$ | 1 | $\alpha_p$ |
| | $j_1$ | $j_1^{-1}$ | $j_2^1$ | $\alpha_1 + \alpha_2$ |
| | $j_i$ | $j_i^{-1}$ | $j_2^1$ | $\alpha_2 + \cdots + \alpha_p$ |
| | $j_i$ | $j_i^{-3}$ | $j_2^1$ | $\alpha_1 + \cdots + \alpha_p$ |
| | $j_1$ | $j_2$ | $j_1^2 j_2^2$ | $\alpha_1 + \cdots + \alpha_p$ |
| $\mathfrak{sl}(n + 1, \mathbb{R}, \mathbb{C})$ | -1 | -1 | -1 | $\alpha_1, \alpha_p + \cdots + \alpha_n, \alpha_1 + \cdots + \alpha_n$ |
| $\{1, p, n\}$ | 1 | $j_p$ | $j_p$ | $\alpha_n, \alpha_1 + \cdots + \alpha_p, \alpha_1 + \cdots + \alpha_n$ |
| $(\alpha_1, \alpha_n)$ | $\bar{j}_i$ | $j_i$ | 1 | $\alpha_1$ |
| $(1, -1, 1)$ | $\sqrt{T}$ | $\sqrt{T}$ | $\sqrt{T}^2$ | $\alpha_1 + \cdots + \alpha_p, \alpha_p + \cdots + \alpha_n$ |
| $\alpha_p, p > 2$ | $j_1$ | $j_1^{-1}$ | $j_2^{-1}$ | $\alpha_p, \alpha_1 + \cdots + \alpha_n$ |
| | $j_1$ | $j_1^{-1}$ | $j_2$ | $\alpha_1 + \cdots + \alpha_n$ |
| | $j_1$ | $j_1^{-1}$ | $j_2^{-1}$ | $\alpha_1 + \cdots + \alpha_p$ |
| | $j_1$ | $j_1^{-1}$ | $j_2$ | $\alpha_2 + \cdots + \alpha_n$ |
| | $j_1$ | $j_1 j_n$ | $j_n$ | $\alpha_p + \cdots + \alpha_n$ |
| $\mathfrak{su}(2, 2)$ | $\sqrt{T}$ | 1 | $\sqrt{T}^2$ | $\alpha_1, \alpha_3$ |
| $\{1, 2, 3\}$ | $\bar{j}_2^1$ | $j_2$ | 1 | $\alpha_1, \alpha_3$ |
| $(\alpha_1, \alpha_2)$ | $\sqrt{T}$ | $j_2$ | $\sqrt{T}^2$ | $\alpha_1, \alpha_3$ |
| $(2, 0, -1)$ | $\sqrt{T}$ | 1 | $\sqrt{T}^2$ | $\alpha_2, \alpha_1 + \cdots + \alpha_3$ |
| $\mathfrak{su}(2, 2)$ | 1 | -1 | 1 | $\alpha_1, \alpha_3$ |
| $\{1, 2, 3\}$ | -1 | 1 | -1 | $\alpha_2, \alpha_1 + \cdots + \alpha_3$ |
| $(\alpha_2, \alpha_1)$ | -1 | -1 | -1 | $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3$ |
| $(1, 2, -1)$ | $e^e$ | 1 | $e^e$ | $\alpha_2$ |
| | $e^e$ | -1 | $e^e$ | $\alpha_3$ |
| $\mathfrak{su}(n, n)$ | $e^{\gamma n}$ | $j_1$ | $j_1 \gamma$ | $\gamma$ | $\alpha_n, \alpha_1 + \cdots + \alpha_{2n-1}$ |
| $\{1, n, 2n - 1\}$ | $\gamma$ | $j_1 \gamma$ | $\gamma$ | $\alpha_n, \alpha_1 + \cdots + \alpha_{2n-1}$ |
| $(\alpha_1, \alpha_{2n-1})$ | $\gamma$ | $j_1 \gamma$ | $\gamma$ | $\alpha_n, \alpha_1 + \cdots + \alpha_{2n-1}$ |
| $(1, -1, 1)$ | $\gamma$ | $j_1 \gamma$ | $\gamma$ | $\alpha_n, \alpha_1 + \cdots + \alpha_{2n-1}$ |
| $\alpha_n, n > 2$ | | | | |
| all info | \(j_1\) | \(j_2\) | \(j_3\) | \(m\) |
|----------|----------|----------|----------|----------|
| \(\text{sp}(2n, [\mathbb{R}, \mathbb{C}])\) | \(1\) | \(-1\) | \(1\) | \(\alpha_1, \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(\{1, 2, n\}\) | \(-1\) | \(1\) | \(-1\) | \(\alpha_2, \alpha_1 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \((\alpha_2, \alpha_1)\) | \(1\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\alpha_1, \alpha_2 + \cdots + \alpha_n, \alpha_1 + \cdots + \alpha_n\) |
| \((1, 2, -1)\) | \(\sqrt{\alpha}\) | \(1\) | \(\sqrt{\alpha}\) | \(\alpha_1, 2\alpha_2 + \ldots, \alpha_1 + 2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
| \(\alpha_1,\) \(\alpha_n, n > 3\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\alpha_1 + 2\alpha_2 + \ldots\) |
| \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\alpha_2 + \cdots + \alpha_n, 2\alpha_1 + \ldots\) |
| \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\alpha_1 + \cdots + \alpha_n, 2\alpha_2 + \ldots\) |
| \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
| \(-1\) | \(-1\) | \(-1\) | \(-1\) | \(\alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(j_2\) | \(j_2\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(j_1\) | \(1\) | \(1\) | \(j_1\) | \(\alpha_2\) |
| \(j_1\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\alpha_1 + \alpha_2\) |
| \(j_2\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\sqrt{\alpha}\) | \(\alpha_1 + 2\alpha_2 + \cdots + \alpha_n, 2\alpha_1 + \ldots\) |
| \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\alpha_2 + \cdots + \alpha_n, 2\alpha_1 + \ldots\) |
| \(j_2\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\alpha_1 + \cdots + \alpha_n, 2\alpha_2 + \ldots\) |
| \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
| \(j_2\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(j_2\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\alpha_2 + \cdots + \alpha_n, 2\alpha_1 + \ldots\) |
| \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\sqrt{j_2}\) | \(\alpha_1 + \cdots + \alpha_n, 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + \cdots + \alpha_n, 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + 2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + \cdots + \alpha_n, 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + 2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + \cdots + \alpha_n, 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(2\alpha_2 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_n, \alpha_1 + 2\alpha_2 + \ldots\) |
| \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_2\) | \(\alpha_1 + 2\alpha_2 + \ldots, 2\alpha_1 + \ldots\) |
\[\begin{array}{|c|c|c|c|c|}
\hline
\text{so}(3, 4), \text{so}(7, \mathbb{C}) & j_1 & j_2 & j_3 & m \\
\hline
\{1, 2, 3\} & \sqrt{3} & 1 & \sqrt{7} & \alpha_1, \alpha_2, \alpha_1 + \alpha_3 \\
(\alpha_3, \alpha_2) & \sqrt{1} & 1 & \sqrt{7} & \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 \\
(-1, 0, 3) & \sqrt{1} & 1 & \sqrt{7} & \alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3 \\
\hline
\text{so}(3, 5), \text{so}(2n, \mathbb{C}) & j_1 & j_2 & j_3 & m \\
\hline
\{2, 3, 4\} & \sqrt{1} & 1 & \sqrt{7} & \alpha_3, \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4 \\
(\alpha_4, \alpha_2) & \sqrt{1} & 1 & \sqrt{7} & \alpha_2, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3 + \alpha_4 \\
(0, -1, 2) & \sqrt{1} & 1 & \sqrt{7} & \alpha_1, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3 + \alpha_4 \\
\hline
\end{array}\]
| all info | $j_0$ | $j_2$ | $j_3$ | $j_4$ | $m$ |
|-----------|-------|-------|-------|-------|-----|
| $q+1 \{R.C\}$ | 1 | $j_2$ | $j_3$ | $j_4$ | $j_0$ |
| $1, 2, p, q$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3, a_4 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $(a_1, a_2)$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $(1, 2, -1, -1)$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $\omega_1, \omega_1$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $\alpha_0, \omega_q$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $\alpha_1, \alpha_1$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $\alpha_0, \alpha_0$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $\alpha_1, \alpha_1$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |
| $n > 4, p > 3$ | 1 | 1 | 1 | 1 | $a_1, a_2, a_3 + \cdots + a_n + a_1 + a_2 + \cdots + a_q + a_1 + \cdots + a_q$ |

References

[1] Čap A., and J. Slovák, “Parabolic Geometries I: Background and General Theory”, Amer. Math. Soc., 2009.
[2] Čap A., Correspondence spaces and twistor spaces for parabolic geometries, J. Reine Angew. Math. 582 (2005) 143-172.
[3] Čap A., M. Hammerl, A.R. Gover, Holonomy reductions of Cartan geometries and curved orbit decompositions, Duke Math. J. 163, no. 5 (2014) 1035-1070.
[4] Gregorovič J., General construction of symmetric parabolic geometries, Differential Geometry and its Applications 30, (2012), 450-476.
[5] Gregorovič J., “Geometric structures invariant to symmetries”, Masaryk University, 2012.
[6] J. Gregorovič, L. Zalabová, Symmetric parabolic contact geometries and symmetric spaces, Transformation Groups, Volume 18 (2013), Issue 3 (September), 711-737.
[7] J. Gregorovič, L. Zalabová, On automorphisms with natural tangent action on homogeneous parabolic geometries, arXiv:1312.7318, to appear in Journal of Lie Theory.
[8] Knapp A.W., “Lie Groups Beyond an Introduction”, Birkhäuser, 1996.
[9] Kruglikov B. and D. The, The gap phenomena in parabolic geometries, arXiv:1303.1307v4.
[10] O. Kowalski, Generalized Symmetric spaces, Lecture Notes in Mathematics, Vol. 865, Springer-Verlag 1980, 187 stran.
[11] O. Loos, An intrinsic characterization of fibre bundles associated with homogeneous spaces defined by Lie group automorphisms, Hamb. Math. Abb. 37 (1972), 160-179.
[12] A.L. Onishchik, E.B. Vinberg, (Eds.) Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras, Series: Encyclopaedia of Mathematical Sciences, Vol. 41, Springer, 1991, ISBN 978-3-540-54683-2

[13] Zalabová L., Symmetries of Parabolic Geometries, Differential Geometry and its Applications 27, (2009), 605-622.

[14] Zalabová L., Symmetries of Parabolic Contact Structures, Journal of Geometry and Physics 60, (2010), 1698-1709.

J.G. Department of Mathematics, Faculty of Science, Masaryk University, Kotlářská 2, Brno, 611 37, Czech Republic; L.Z. Institute of Mathematics and Biomathematics, Faculty of Science, University of South Bohemia in České Budějovice, Branišovská 31, České Budějovice, 370 05, Czech Republic

E-mail address: jan.gregorovic@seznam.cz, lzalabova@gmail.com