THE QUANTUM TEICHMÜLLER SPACE
AS A NONCOMMUTATIVE ALGEBRAIC OBJECT

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Abstract. We consider the quantum Teichmüller space of the punctured surface introduced by Chekhov-Fock-Kashaev, and formalize it as a noncommutative deformation of the space of algebraic functions on the Teichmüller space of the surface. In order to apply it in 3-dimensional topology, we put more attention to the details involving small surfaces.

Let $S$ be an oriented surface of finite topological type, with at least one puncture. A quantization of the Teichmüller space $\mathcal{T}(S)$ of $S$ was developed by L. Chekhov and V. Fock [7, 8, 5] and, independently, by R. Kashaev [11] (see also [14]) as an approach to quantum gravity in $2+1$ dimensions. This is a deformation of the $C^*$-algebra of functions on the usual Teichmüller space $\mathcal{T}(S)$ of $S$, depending on a parameter $\hbar$, in such a way that the linearization of this deformation at $\hbar = 0$ corresponds to the Weil-Petersson Poisson structure on $\mathcal{T}(S)$.

In this paper, we develop a slightly different version of this quantization, which has a more algebraic flavor. It essentially is the image under the exponential map of the quantization of Chekhov-Fock-Kashaev. The original quantization was expressed in terms of self-adjoint operators on Hilbert spaces and made strong use of the holomorphic function

$$\phi^\hbar(z) = -\frac{\pi\hbar}{2} \int_{-\infty}^{\infty} \frac{e^{-itz}}{\sinh \pi t \sinh \pi\hbar t} dt$$

called the quantum dilogarithm function. Our exponential version enables us to leave the realm of analysis and to focus on the algebraic aspects of the construction.

From a mathematical point of view, the main benefit of the quantum Teichmüller space $\mathcal{T}_q(S)$ which we construct here is that it admits a rich finite-dimensional representation theory. This representation theory is investigated in [3], where we show that it is strongly connected to 3-dimensional hyperbolic geometry through the space of representations $\pi_1(S) \to \text{PSL}_2(\mathbb{C})$. In particular, we use this connection to construct in $\mathbb{R}$ quantum invariants for diffeomorphisms of $S$. The present paper is devoted to laying down the foundations of the theory, by showing that the quantum Teichmüller space $\mathcal{T}_q(S)$ is well-defined as an object in non-commutative algebraic geometry.

Our construction parallels that of Chekhov and Fock. We start from Thurston’s exponential shear coordinates for Teichmüller space, associated to an ideal triangulation of $S$. A fundamental property of these exponential shear coordinates is that, as one shifts from one ideal triangulation to another, the corresponding coordinate changes are rational. In particular, there is a well-defined notion of rational function on $\mathcal{T}(S)$. In general, the quantization of a space is a deformation of the algebra of continuous functions on this space. Here we construct a deformation of the algebra $\text{Rat}(\mathcal{T}(S))$ of all rational functions on $\mathcal{T}(S)$. The main strategy is, first to define a deformation of the expression of $\text{Rat}(\mathcal{T}(S))$ in the set of shear coordinates associated to an ideal triangulation, and then to construct appropriate “coordinate change isomorphisms” to make the construction independent of any choice of ideal triangulation. The technical challenge is to check that these coordinate change isomorphisms are compatible with each other, which is accomplished in $\mathbb{B}$.

Many aspects of this construction are already implicitly or explicitly present in the work of Chekhov-Fock. In particular, the algebraic coordinate change isomorphisms originate from the symmetries of the quantum dilogarithm function $\phi^\hbar$. Our main contribution is to systematically develop the theory from an algebraic point of view, and to carefully check the details of the construction. From a technical point of view, this

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leads us to be a little more careful with non-embedded diagonal exchanges, and to explicitly describe the formulas which occur in these cases.

So far, we have avoided defining the Teichmüller space of the surface $S$. The reason is that there are several competing definitions, leading to different objects. One possibility is to define the Teichmüller space $\mathcal{T}(S)$ as the space of isotopy classes of complete hyperbolic metrics on $S$, possibly with infinite area. This is a manifold-with-corners, of dimension $6g - 6 + 3p$ where $g$ is the genus of $S$ and $p$ is the number of its punctures. This space can be expanded to the enhanced Teichmüller space $\mathcal{E}\mathcal{T}(S)$ defined in [2] which has the same space of rational functions as $\mathcal{T}(S)$. This is the set-up of [8] and of the first part of this article. However, a more commonly used object is the cusped Teichmüller space $\mathcal{C}\mathcal{T}(S) \subset \mathcal{E}\mathcal{T}(S)$, consisting of all isotopy classes of finite area complete hyperbolic metrics on $S$; this is a manifold of dimension $6g - 6 + 2p$. We construct a quantization of this cusped Teichmüller space in the last section [3] of the article.

There is one property which we would like to mention before closing this introduction. The original definition of the quantum Teichmüller space was grounded in the geometry of hyperbolic metrics on $S$. However, a recent result of Hua Bai [1] shows that it is intrinsically tied to the combinatorics of the Harer-Penner simplicial complex of ideal triangulations [10, 13]. Indeed, Bai proved that the coordinate change formulas which occur in these cases.

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1. Ideal Triangulations

Let $S$ be an oriented surface of genus $g$ with $p \geq 1$ punctures, obtained by removing $p$ points $\{v_1, \ldots, v_p\}$ from the closed oriented surface $\bar{S}$ of genus $g$. An ideal triangulation of $S$ is a triangulation of the closed surface $\bar{S}$ whose vertex set is exactly $\{v_1, \ldots, v_p\}$. An Euler characteristic argument shows that any ideal triangulation has $n = 6g - 6 + 3p$ edges.

Two ideal triangulations are considered the same if they are isotopic. In addition, we require that each ideal triangulation $\lambda$ is endowed with an indexing $\lambda_1, \lambda_2, \ldots, \lambda_n$ of its edges by the index set $\{1, 2, \ldots, n\}$. Let $\Lambda(S)$ denote the set of isotopy classes of such indexed ideal triangulations $\lambda$.

The set $\Lambda(S)$ admits a natural action of the group $\mathfrak{S}_n$ of permutations of $n$ elements, acting by permuting the indices of the edges of $\lambda$. Namely $\lambda' = \alpha(\lambda)$ for $\alpha \in \mathfrak{S}_n$ if its $i$-th edge $\lambda'_i$ is equal to $\lambda_{\alpha(i)}$.

Another important transformation of $\Lambda(S)$ is provided by the $i$-th diagonal exchange map $\Delta_i : \Lambda(S) \to \Lambda(S)$ defined as follows. The $i$-th edge $\lambda_i$ of an ideal triangulation $\lambda \in \Lambda(S)$ is adjacent to two triangles. If these two triangles are distinct, their union forms a square $Q$ with diagonal $\lambda_i$. Then $\Delta_i(\lambda)$ is obtained from $\lambda$ by replacing the edge $\lambda_i$ by the other diagonal $\lambda'_i$ of the square $Q$, as illustrated in Figure 2. By convention, $\Delta_i(\lambda) = \lambda$ when the two sides of $\lambda_i$ belong to the same triangle; this happens exactly when $\lambda_i$ is the only edge of $\lambda$ leading to a certain puncture of $S$.

The reindexings and diagonal exchanges satisfy the following relations:

1. The Composition Relation: $(\alpha \beta)(\lambda) = \alpha(\beta(\lambda))$ for every $\alpha, \beta \in \mathfrak{S}_n$;
2. The Reflexivity Relation: $(\Delta_i)^2 = \text{Id}$;
3. The Reindexing Relation: $\Delta_i \circ \alpha = \alpha \circ \Delta_{\alpha(i)}$ for every $\alpha \in \mathfrak{S}_n$;

![Figure 1. An ideal triangulation of a genus 2 surface with 3 punctures](image-url)
Theorem 1. Given two ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \), there exists a finite sequence of ideal triangulations \( \lambda = \lambda(0), \lambda(1), \ldots, \lambda(m) = \lambda' \) such that each \( \lambda(k+1) \) is obtained from \( \lambda(k) \) by a diagonal exchange or by a reindexing of its edges.

Theorem 2. Given two ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \) and given two sequences \( \lambda = \lambda(0), \lambda(1), \ldots, \lambda(m) = \lambda' \) and \( \lambda = \lambda'(0), \lambda'(1), \ldots, \lambda'(m') = \lambda' \) of diagonal exchanges and reindexings connecting them as in Theorem 1, these two sequences can be related to each other by successive applications of the following moves and of their inverses.

1. Use the Composition Relation to replace \( \ldots, \lambda(1), \beta(\lambda(1)), \alpha \circ \beta(\lambda(1)), \ldots \) by \( \ldots, \lambda(1), (\alpha \beta)(\lambda(1)), \ldots \) where \( \alpha, \beta \in \mathcal{S}_n \).
2. Use the Reflexivity Relation to replace \( \ldots, \lambda(1), \ldots \) by \( \ldots, \lambda(1), \Delta_i(\lambda(1)), \lambda(1), \ldots \).
3. Use the Reindexing Relation to replace \( \ldots, \lambda(1), \Delta_i \circ \alpha(\lambda(1)), \lambda(1), \ldots \) by \( \ldots, \lambda(1), \alpha \circ \Delta_i(\alpha(\lambda(1))), \lambda(1), \ldots \) where \( \alpha \in \mathcal{S}_n \).
4. Use the Distant Commutativity Relation to replace \( \ldots, \lambda(1), \ldots \) by \( \ldots, \lambda(1), \Delta_i(\lambda(1)), \Delta_j(\lambda(1)), \Delta_i \Delta_j(\lambda(1)), \Delta_j \Delta_i(\lambda(1)), \Delta_j(\lambda(1)), \lambda(1), \ldots \) where \( \lambda_i, \lambda_j \) are two edges which do not belong to a same triangle of \( \lambda(1) \).
5. Use the Pentagon Relation to replace \( \ldots, \lambda(1), \ldots \) by \( \ldots, \lambda(1), \Delta_i(\lambda(1)), \Delta_j(\lambda(1)), \Delta_i \Delta_j(\lambda(1)), \Delta_j \Delta_i(\lambda(1)), \Delta_i \Delta_j(\lambda(1)), \lambda(1), \ldots \) where \( \lambda_i, \lambda_j \) are two diagonals of a pentagon of \( \lambda(1) \).
2. THE EXPONENTIAL SHEAR COORDINATES FOR TEICHMÜLLER SPACE

W. Thurston associated to an ideal triangulation $\lambda$ a certain system of shear coordinates for the Teichmüller space $\mathcal{T}(S)$. See [10] for the dual notion of length coordinates, and [2][7] for details on these shear coordinates.

Consider a complete hyperbolic metric $m \in \mathcal{T}(S)$. It is well-known that the ends of the complete hyperbolic surface $(S, m)$ can be of two types: finite area cusps bounded on one side by a horocycle; and infinite area funnels bounded on one side by a simple closed geodesic. This can also be expressed in terms of the convex core $\text{Conv}(S, m)$ of $(S, m)$, which is the smallest non-empty closed convex subset of $(S, m)$, and is bounded in $S$ by a family of disjoint simple closed geodesics. The cusp ends of $(S, m)$ are those which are also ends of $\text{Conv}(S, m)$, while each funnel end of $S$ faces a boundary component of $\text{Conv}(S, m)$. Note that the interior $\text{Int}(\text{Conv}(S, m))$ is homeomorphic to $S$, by the homeomorphism which is uniquely determined up to isotopy by the property that it is homotopic to the inclusion map.

It is convenient to enhance the hyperbolic metric $m \in \mathcal{T}(S)$ with some additional data, consisting of an orientation for the boundary $\partial \text{Conv}(S, m)$ of its convex cores. Let the enhanced Teichmüller space $\widetilde{\mathcal{T}}(S)$ consist of all isotopy classes of hyperbolic metrics $m \in \mathcal{T}(S)$ enhanced with an orientation of $\partial \text{Conv}(S, m)$. Since the convex core $\text{Conv}(S, m)$ depends continuously on the metric $m$, the enhanced Teichmüller space $\widetilde{\mathcal{T}}(S)$ inherits from the topology of $\mathcal{T}(S)$ a topology for which the natural projection $\widetilde{\mathcal{T}}(S) \to \mathcal{T}(S)$ is a branched covering map.

![A hyperbolic surface](image)

Consider an enhanced hyperbolic metric $m \in \widetilde{\mathcal{T}}(S)$ together with an ideal triangulation $\lambda$. Each edge $\lambda_i$ specifies a proper homotopy class of paths going from one end of $\text{Int}(\text{Conv}(S, m)) \cong S$ to another end. This proper homotopy class is also realized by a unique $m$–geodesic $g_i$ such that each end of $g_i$, either converges towards a cusp end of $S$, or spirals around a component of $\partial \text{Conv}(S, m)$ in the direction specified by the enhancement of $m$. The union of $\partial \text{Conv}(S, m)$ and of the $g_i$ forms an $m$–geodesic lamination $g$ contained in the convex core $\text{Conv}(S, m)$.

The enhanced hyperbolic metric $m \in \widetilde{\mathcal{T}}(S)$ now associates to the edge $\lambda_i$ of $\lambda$ a positive number $x_i$ defined as follows. The geodesic $g_i$ separates two triangle components $T^1_i$ and $T^2_i$ of $\text{Conv}(S, m) - g$. Identify the universal covering of $S$, endowed with the metric $m$, to the hyperbolic plane $\mathbb{H}^2$. Lift $g_i$, $T^1_i$ and $T^2_i$ to a geodesic $\tilde{g}_i$ and two triangles $\tilde{T}^1_i$ and $\tilde{T}^2_i$ in $\mathbb{H}^2$ so that the union $\tilde{g}_i \cup \tilde{T}^1_i \cup \tilde{T}^2_i$ forms a square $\tilde{Q}$ in $\mathbb{H}^2$. In the standard upper half-space model for $\mathbb{H}^2$, let $z_-, z_+, z_r, z_l$ be the vertices of $\tilde{Q}$ in such a way that $\tilde{g}_i$ goes from $z_-$ to $z_+$ and, for this orientation of $\tilde{g}_i$, $z_r, z_l$ are respectively to the right and to the left of $\tilde{g}_i$ for the orientation of $\tilde{Q}$ given by the orientation of $S$. Then,

$$x_i = - \text{cross-ratio}(z_r, z_l, z_-, z_+).$$

Note that $x_i$ is positive since the points $z_-, z_r, z_+, z_l$ occur in this order in the real line bounding the upper half-space $\mathbb{H}^2$.

The real numbers $x_i$ are the exponential shear coordinates of the enhanced hyperbolic metric $m \in \widetilde{\mathcal{T}}(S)$. The more standard shear coordinates are just their logarithms $\log x_i$, but the $x_i$ turn out to be better behaved for our purposes.
There is an inverse construction which associates a hyperbolic metric to each system of positive weights \( x_i \) attached to the edges \( \lambda_i \) of the ideal triangulation \( \lambda \): Identify each of the components of \( S - \lambda \) to a triangle with vertices at infinity in \( \mathbb{H}^2 \), and glue these hyperbolic triangles together in such a way that adjacent triangles form a square whose vertices have cross-ratio \(-x_i\) as above. This defines a possibly incomplete hyperbolic metric on the surface \( S \). An analysis of this metric near the ends of \( S \) shows that its completion is a hyperbolic surface \( S' \) with geodesic boundary, and that each end of an edge of \( \lambda \) either spirals towards a component of \( \partial S' \) or converges towards a cusp end of \( S' \). Extending \( S' \) to a complete hyperbolic surface without boundary and identifying this surface to \( S \), we now have found a complete hyperbolic metric \( m \) on \( S \) whose convex core is isometric to \( S' \). In addition, the spiralling pattern of the ends of \( \lambda \) provides an enhancement for the hyperbolic metric \( m \). See [17] \([3.4]\) or [2] for details.

The \( x_i \) then define a homeomorphism \( \phi_\lambda : T(S) \to \mathbb{R}_+^n \) between the enhanced Teichmüller space \( \tilde{T}(S) \) and \( \mathbb{R}_+^n \).

For the enhanced hyperbolic metric \( m \in \tilde{T}(S) \) associated to the exponential shear parameters \( x_i \), the geometry of \( m \) near the \( j \)-th puncture \( v_j \) is completely determined by the exponential length parameter \( p_j = x_1^{k_{j1}}x_2^{k_{j2}} \cdots x_n^{k_{jn}} \) where \( k_{ij} \in \{0, 1, 2\} \) is the number of end points of the edge \( \lambda_i \) that are equal to \( v_j \). The end of \( S \) corresponding to \( v_j \) is a cusp for \( m \) exactly when \( p_j = 1 \). Otherwise, this end faces a boundary component of \( \text{Conv}(S, m) \) which is a closed geodesic of length \( |\log p_j| \), and the orientation of this boundary component defined by the enhancement of \( m \) coincides with the boundary orientation exactly when \( p_j < 1 \).

The exponential shear coordinates associates a parametrization \( \phi_\lambda : \tilde{T}(S) \to \mathbb{R}_+^n \) to each ideal triangulation \( \lambda \in \Lambda(S) \) (endowed with an indexing of its edges). We now investigate the coordinate changes \( \phi_{\lambda'} \circ \phi_{\lambda}^{-1} \) associated to two ideal triangulations.

If \( \lambda' = \alpha(\lambda) \) is obtained by reindexing the edges of \( \lambda \) by \( \alpha \in \mathfrak{S}_n \), it is immediate that \( \phi_{\lambda'} \circ \phi_{\lambda}^{-1} \) is just the permutation of the coordinates by \( \alpha \). For a diagonal exchange, the expression for \( \phi_{\lambda'} \circ \phi_{\lambda}^{-1} \) depends on the possible identifications between the sides of the square where the diagonal exchange takes place.

**Proposition 3.** Suppose that the ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \) are obtained from each other by a diagonal exchange, namely that \( \lambda' = \Delta_i(\lambda) \). Label the edges of \( \lambda \) involved in this diagonal exchange as \( \lambda_j, \lambda_k, \lambda_l, \lambda_m \) as in Figure 2. If \( (x_1, x_2, \ldots, x_n) = \phi_\lambda(m) \) and \( (x'_1, x'_2, \ldots, x'_n) = \phi_{\lambda'}(m) \) are the exponential shear coordinates associated to the same enhanced hyperbolic metric \( m \in \tilde{T}(S) \), then \( x'_h = x_h \) for every \( h \not\in \{i, j, k, l, m\} \), \( x'_{i} = x_i^{-1} \) and:

**Case 1:** if the edges \( \lambda_j, \lambda_k, \lambda_l, \lambda_m \) are distinct, then
\[
x'_j = (1 + x_j)x_j, \quad x'_k = (1 + x_i^{-1})^{-1}x_k, \quad x'_l = (1 + x_i)x_l, \quad x'_m = (1 + x_i^{-1})^{-1}x_m;
\]

**Case 2:** if \( \lambda_j \) is identified with \( \lambda_k \), and \( \lambda_l \) is distinct from \( \lambda_m \), then
\[
x'_j = x_ix_j, \quad x'_l = (1 + x_i)x_l, \quad x'_m = (1 + x_i^{-1})^{-1}x_m;
\]

**Case 3:** (the inverse of Case 2) if \( \lambda_j \) is identified with \( \lambda_m \), and \( \lambda_k \) is distinct from \( \lambda_l \), then
\[
x'_j = x_ix_j, \quad x'_k = (1 + x_i^{-1})^{-1}x_k, \quad x'_l = (1 + x_i)x_l;
\]

**Case 4:** if \( \lambda_j \) is identified with \( \lambda_l \), and \( \lambda_k \) is distinct from \( \lambda_m \), then
\[
x'_j = (1 + x_i)^2x_j, \quad x'_k = (1 + x_i^{-1})^{-1}x_k, \quad x'_m = (1 + x_i^{-1})^{-1}x_m;
\]

**Case 5:** (the inverse of Case 4) if \( \lambda_k \) is identified with \( \lambda_l \), and \( \lambda_j \) is distinct from \( \lambda_m \), then
\[
x'_j = (1 + x_j)x_j, \quad x'_k = (1 + x_i^{-1})^{-2}x_k, \quad x'_l = (1 + x_i)x_l;
\]

**Case 6:** if \( \lambda_j \) is identified with \( \lambda_k \), and \( \lambda_l \) is identified with \( \lambda_m \) (in which case \( S \) is a 3-times punctured sphere), then
\[
x'_j = x_ix_j, \quad x'_l = x_ix_l;
\]

**Case 7:** (the inverse of Case 6) if \( \lambda_j \) is identified with \( \lambda_m \), and \( \lambda_k \) is identified with \( \lambda_l \) (in which case \( S \) is a 3-times punctured sphere), then
\[
x'_j = x_ix_j, \quad x'_k = x_ix_k;
\]
Case 8: if $\lambda_j$ is identified with $\lambda_l$, and $\lambda_k$ is identified with $\lambda_m$ (in which case $S$ is a once punctured torus), then

$$x_j' = (1 + x_i)^2 x_j \quad x_k' = (1 + x_i^{-1})^{-2} x_k.$$  

Proof. This immediately follows from the combinatorics of cross-ratios. For instance, let us focus on Case 4. The other cases are similar.

Isometrically identify $S$, endowed with the metric $m$, to the quotient of the upper half-plane model for $\mathbb{H}^2$ under an action of the fundamental group $\pi_1(S)$. Lift the square $Q \subset S$ bounded by $\lambda_j$, $\lambda_k$, $\lambda_l$ and $\lambda_m$ to a square $\tilde{Q} \subset \mathbb{H}^2$ bounded by lifts $\tilde{\lambda}_j$, $\tilde{\lambda}_k$, $\tilde{\lambda}_l$ and $\tilde{\lambda}_m$, and with diagonal $\tilde{\lambda}_i$. The fact that $\lambda_i = \lambda_j$ means that there exists a covering translation $\gamma \in \pi_1(S)$ such that $\gamma \tilde{\lambda}_i = \tilde{\lambda}_j$. Label the corners of the square $\tilde{Q}$ clockwise as $\ldots, z_1, z_+, z_-$, starting from the corner $\lambda_j \cap \lambda_k$. Then, by definition,

$$x_i = \frac{(z_t - z_-)(z_+ - z_1)}{(z_+ - z_1)(z_- - z_t)}$$

and

$$x_i' = \frac{(z_1 - z_-)(z_+ - z_t)}{(z_1 - z_t)(z_- - z_1)} = x_i^{-1}$$

since the component $\tilde{\lambda}_i'$ goes from $z_1$ to $z_t$.

The geodesic $\tilde{\lambda}_k$ is the diagonal of a square with vertices $z_+, z_-, z_1$ and a fourth vertex $z_k$ outside of $\tilde{Q}$. Then

$$x_k = \frac{(z_k - z_1)(z_+ - z_-)}{(z_k - z_-)(z_+ - z_1)}.$$  

With respect to the preimage $\tilde{\lambda}_k \subset \mathbb{H}^2$ of $\lambda'$, $\tilde{\lambda}_k' = \tilde{\lambda}_k$ is the diagonal of a square with vertices $z_-, z_t, z_1$ and $z_k$. Consequently,

$$x_k' = \frac{(z_k - z_1)(z_1 - z_-)}{(z_k - z_-)(z_1 - z_1)} = (1 + x_i^{-1})^{-1} x_k.$$  

The same argument shows that $x_m' = (1 + x_i^{-1})^{-1} x_m$.

For $x_j$, there is a new twist because $\gamma \in \pi_1(S)$ sends $\tilde{\lambda}_i$ to $\tilde{\lambda}_j$, so that $\tilde{\lambda}_j$ is the diagonal of a square with vertices $z_+ = \gamma z_1$, $z_+, z_t = \gamma z_+$ and $\gamma z_-$. Switching from $\lambda$ to $\lambda'$ now changes two vertices of this square in the sense that, with respect to $\lambda'$, $\tilde{\lambda}_j' = \tilde{\lambda}_j$ is the diagonal of a square with vertices $z_-, z_1, z_+ z_+$ and $\gamma z_1$. Therefore,

$$x_j = \frac{(\gamma z_1 - z_1)(z_+ - z_1)}{(z_+ - z_1)(\gamma z_1 - z_1)}$$

and, since $\rho(\gamma) \in \text{Isom}^+(\mathbb{H}^3)$ respects cross-ratios,

$$x_j' = \frac{(\gamma z_1 - z_1)(z_+ - z_1)}{(\gamma z_1 - z_1)(z_+ - z_1)} x_j = \left(\frac{(z_1 - z_1)(z_+ - z_1)}{(z_1 - z_1)(z_+ - z_1)}\right)^2 x_j = (1 + x_i)^2 x_j.$$  

The fact that $x_h' = x_h$ for $h \notin \{i,j,k,l,m\}$ is obvious from the definition of the shear coordinates. \qed

Using Theorem 1 an immediate corollary of Proposition 3 is the following.

Corollary 4. For two ideal triangulations $\lambda$, $\lambda' \in \Lambda(S)$, the coordinate change map $\phi_{\lambda'} \circ \phi_{\lambda}^{-1} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is rational. \qed

In particular, there is a well-defined notion of rational function on $\mathcal{T}(S)$. Namely, the (partially defined) function $f : \mathcal{T}(S) \to \mathbb{C}$ is rational if, for an arbitrary ideal triangulation $\lambda$, $f \circ \phi_{\lambda}^{-1} : \mathbb{R}_+^n \to \mathbb{C}$ is a rational function in the usual sense. If $\text{Rat} \mathcal{T}(S)$ denotes the algebra of such rational functions, every ideal triangulation $\lambda$ specifies an algebra isomorphism $\Phi_{\lambda}$ from $\text{Rat} \mathcal{T}(S)$ to the algebra of rational fractions.
\[ \mathbb{C}(X_1, X_2, \ldots, X_n). \] In particular, at the rational function algebra level, two ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \) induce a coordinate change isomorphism
\[ (2.1) \quad \Phi_{\lambda\lambda'} : \mathbb{C}(X_1, X_2, \ldots, X_n) \to \mathbb{C}(X_1, X_2, \ldots, X_n) \]
defined by \( g \mapsto g \circ \phi_{\lambda'} \circ \phi_\lambda^{-1} \).

### 3. The Chekhov-Fock Algebra

The goal of this section is to quantize the enhanced Teichmüller space \( \mathcal{T}(S) \) by defining a deformation of the algebra \( \operatorname{Rat} \mathcal{T}(S) \), depending on a parameter \( q \).

Fix an ideal triangulation \( \lambda \in \Lambda(S) \). The complement \( S - \lambda \) has \( 2n \) spikes converging towards the punctures, and each spike is delimited by one \( \lambda_i \) on one side and one \( \lambda_j \) on the other side, with possibly \( i = j \). For \( i, j \in \{1, \ldots, n\} \), let \( a_{ij}^\lambda \) denote the number of spikes of \( S - \lambda \) which are delimited on the left by \( \lambda_i \) and on the right by \( \lambda_j \), and set
\[
\sigma_{ij}^\lambda = a_{ij}^\lambda - a_{ji}^\lambda.
\]
Note that \( \sigma_{ij}^\lambda \) can only belong to the set \( \{-2, -1, 0, +1, +2\} \), and that \( \sigma_{ji}^\lambda = -\sigma_{ij}^\lambda \). It turns out that the coefficients \( \sigma_{ij}^\lambda \) are related to the expression of the Weil-Petersson symplectic form of \( \mathcal{T}(S) \) in the exponential shear coordinates, but this plays only a historical rôle in this paper.

The **Chekhov-Fock algebra** associated to the ideal triangulation \( \lambda \) is the algebra \( \mathcal{T}_\lambda^q \) defined by generators \( X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1} \), with each pair \( X_i^\pm 1 \) associated to an edge \( \lambda_i \) of \( \lambda \), and by the relations
\[
\begin{align*}
X_i X_j &= q^{2\sigma_{ij}^\lambda} X_j X_i \\
X_i X_i^{-1} &= X_i^{-1} X_i = 1.
\end{align*}
\]

This is an iterated skew Laurent polynomial algebra. In particular, it is a Noetherian ring and a right Ore domain, so that we can introduce its *fraction division algebra* \( \mathcal{T}_\lambda^q \). See for instance [6, 9, 4]. The algebra \( \mathcal{T}_\lambda^q \) consists of all formal fractions \( PQ^{-1} \) with \( P, Q \in \mathcal{T}_\lambda^q \) and \( Q \neq 0 \), and two such fractions \( P_1 Q_1^{-1} \) and \( P_2 Q_2^{-1} \) are identified if there exists \( S_1, S_2 \in \mathcal{T}_\lambda^q - \{0\} \) such that \( P_1 S_1 = P_2 S_2 \) and \( Q_1 S_1 = Q_2 S_2 \).

In practice, the Chekhov-Fock algebra \( \mathcal{T}_\lambda^q \) consists of all Laurent polynomials in variables \( X_1, X_2, \ldots, X_n \) satisfying the skew-commutativity relations (3.1). Its fraction division algebra \( \mathcal{T}_\lambda^q \) consists of all rational fractions in the variables \( X_1, X_2, \ldots, X_n \) satisfying the same skew-commutativity relations. In particular, when \( q = 1 \), \( \mathcal{T}_\lambda^q \) and \( \mathcal{T}_\lambda^1 \) respectively coincide with the Laurent polynomial algebra \( \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}] \) and the rational fraction algebra \( \mathbb{C}(X_1, X_2, \ldots, X_n) \). The general \( \mathcal{T}_\lambda^q \) and \( \mathcal{T}_\lambda^q \) can be considered as deformations of this case.

The algebras \( \mathcal{T}_\lambda^q \) and \( \mathcal{T}_\lambda^q \) strongly depend on the ideal triangulation \( \lambda \). To define a triangulation independent deformation of the algebra \( \operatorname{Rat} \mathcal{T}(S) \), we need to generalize the coordinate change isomorphism of (2.1) to this non-commutative context, by introducing appropriate algebra isomorphisms \( \Phi_{\lambda\lambda'}^q : \mathcal{T}_\lambda^q \to \mathcal{T}_{\lambda'}^q \).

There is no geometry to guide us here, so we will do this stepwise, using Theorems 1 and 2. To ease the exposition, we will denote by \( X_1', X_2', \ldots, X_n' \) the generators of \( \mathcal{T}_\lambda^q \) associated to the edges \( \lambda_1', \lambda_2', \ldots, \lambda_n' \) of \( \lambda' \), and by \( X_1, X_2, \ldots, X_n \) the generators of \( \mathcal{T}_\lambda^q \) associated to the edges \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( \lambda \). In particular, the isomorphism \( \Phi_{\lambda\lambda'}^q : \mathcal{T}_{\lambda'}^q \to \mathcal{T}_\lambda^q \) will be completely determined once we specify the images \( \Phi_{\lambda\lambda'}^q(X_i') \).

**Proposition 5.** Suppose that the ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \) are obtained from each other by a diagonal exchange, namely that \( \lambda' = \Delta_i(\lambda) \). Label the edges of \( \lambda \) involved in this diagonal exchange as \( \lambda_i, \lambda_j, \lambda_k, \lambda_l, \lambda_m \) as in Figure 3. Then there is a unique algebra isomorphism
\[
\Phi_{\lambda\lambda'}^q : \mathcal{T}_{\lambda'}^q \to \mathcal{T}_\lambda^q
\]
such that \( X_h' \mapsto X_h \) for every \( h \notin \{i, j, k, l, m\} \), \( X_i' \mapsto X_i^{-1} \) and:

**Case 1:** if the edges \( \lambda_j, \lambda_k, \lambda_l, \lambda_m \) are distinct, then
\[
\begin{align*}
X_j' &\mapsto (1 + qX_i)X_j & X_k' &\mapsto (1 + qX_i^{-1})^{-1}X_k \\
X_l' &\mapsto (1 + qX_i)X_l & X_m' &\mapsto (1 + qX_i^{-1})^{-1}X_m.
\end{align*}
\]
Case 2: if $\lambda_j$ is identified with $\lambda_k$, and $\lambda_l$ is distinct from $\lambda_m$, then
\[
X'_j \mapsto X_iX_j \quad X'_l \mapsto (1 + q X_i)X_l \quad X'_m \mapsto (1 + q X_i^{-1})^{-1} X_m
\]

Case 3: (the inverse of Case 2) if $\lambda_j$ is identified with $\lambda_m$, and $\lambda_k$ is distinct from $\lambda_l$, then
\[
X_j \mapsto X_iX_j \quad X'_l \mapsto (1 + q X_i^{-1})^{-1} X_k \quad X'_m \mapsto (1 + q X_i)X_l
\]

Case 4: if $\lambda_j$ is identified with $\lambda_l$, and $\lambda_k$ is distinct from $\lambda_m$, then
\[
X'_j \mapsto (1 + q X_i)(1 + q^3 X_i)X_j \quad X'_l \mapsto (1 + q X_i^{-1})^{-1} X_k
\]

Case 5: (the inverse of Case 4) if $\lambda_k$ is identified with $\lambda_l$, and $\lambda_j$ is distinct from $\lambda_l$, then
\[
X'_j \mapsto (1 + q X_i)X_j \quad X'_l \mapsto (1 + q X_i)X_l
\]

Case 6: if $\lambda_j$ is identified with $\lambda_k$, and $\lambda_l$ is identified with $\lambda_m$ (in which case $S$ is a 3-times punctured sphere), then
\[
X'_j \mapsto X_iX_j \quad X'_l \mapsto X_iX_l
\]

Case 7: (the inverse of Case 6) if $\lambda_j$ is identified with $\lambda_m$, and $\lambda_k$ is identified with $\lambda_l$ (in which case $S$ is a 3-times punctured sphere), then
\[
X'_j \mapsto X_iX_j \quad X'_l \mapsto X_iX_k
\]

Case 8: if $\lambda_j$ is identified with $\lambda_l$, and $\lambda_k$ is identified with $\lambda_m$ (in which case $S$ is a once punctured torus), then
\[
X'_j \mapsto (1 + q X_i)(1 + q^3 X_i)X_j \quad X'_k \mapsto (1 + q X_i^{-1})^{-1}(1 + q^3 X_i^{-1})^{-1} X_k
\]

Proof. By inspection, these formulas are compatible with the skew-commutativity relations \textcircled{3}. For instance, in Case 4, $X'_j X'_l = q^4 X_j X'_l X_i X'_k = q^{-2} X'_k X'_j$ and $X'_j X_k = q^2 X'_k X'_j$ in $\hat{T}'_\Lambda$, and
\[
[X_i^{-1}] [(1 + q X_i)(1 + q^3 X_i)X_j] = q^4 [(1 + q X_i)(1 + q^3 X_i)X_j] [X_i^{-1}]
\]
\[
[X_i^{-1}] ([1 + q X_i^{-1}]^{-1} X_k] = q^{-2} ([1 + q X_i^{-1}]^{-1} X_k] [X_i^{-1}]
\]
\[
[(1 + q X_i)(1 + q^3 X_i)X_j] [1 + q X_i^{-1}]^{-1} X_k] = q^2 [(1 + q X_i^{-1})^{-1} X_k] [1 + q X_i(1 + q^3 X_i)X_j]
\]
in $\hat{T}'_\Lambda$ since $X_iX_j = q^{-4} X_j X_i$, $X_iX_k = q^2 X_k X_j$ and $X_jX_k = X_k X_j$. It follows that these formulas extend to a unique algebra homomorphism $\phi'_{\Lambda_{\hat{T}'}_\Lambda} : \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \rightarrow \hat{T}'_\Lambda$.

Extending $\phi'_{\Lambda_{\hat{T}'}_\Lambda}$ to the fraction division algebra $\hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$ will require a little care. Indeed, for a formal fraction $PQ^{-1} \in \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$ with $P, Q \in \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$, we want to define $\Phi'_{\Lambda_{\hat{T}'}_\Lambda}(PQ^{-1}) = \phi'_{\Lambda_{\hat{T}'}_\Lambda}(P)\phi'_{\Lambda_{\hat{T}'}_\Lambda}(Q)^{-1} \in \hat{T}'_\Lambda$. For this we need $\phi'_{\Lambda_{\hat{T}'}_\Lambda}(Q)$ to be non-zero. In other words, we need to show that $\phi'_{\Lambda_{\hat{T}'}_\Lambda}$ is injective.

For this, we take advantage of the reflexivity of the above formulas. Exchanging the roles of $\lambda$ and $\lambda'$, we also have an algebra homomorphism $\phi'_{\Lambda_{\hat{T}'}_\Lambda} : \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \rightarrow \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$. Note that the elements of $\phi'_{\Lambda_{\hat{T}'}_\Lambda}(\hat{T}'_{\Lambda_{\hat{T}'}_\Lambda})$ are all polynomials in the $X_k^{-1}$, with $h = 1, 2, \ldots, n$. In the quantity $(1 + q X_i^{-1})^{-1}$. After checking the skew-commutativity relations and because $\phi'_{\Lambda_{\hat{T}'}_\Lambda}(X_i) = (X_i')^{-1}$, we can therefore extend $\phi'_{\Lambda_{\hat{T}'}_\Lambda}$ to $\phi'_{\Lambda_{\hat{T}'}_\Lambda}(\hat{T}'_{\Lambda_{\hat{T}'}_\Lambda})$ by defining $\phi'_{\Lambda_{\hat{T}'}_\Lambda}((1 + q X_i^{-1})^{-1}) = (1 + q X_i)^{-1}$.

We now have a composition
\[
\hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \xrightarrow{\phi'_{\Lambda_{\hat{T}'}_\Lambda}} \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \xrightarrow{\phi'_{\Lambda_{\hat{T}'}_\Lambda}} \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}
\]
of two algebra homomorphisms. By inspection of the formulas defining these homomorphisms, $\phi'_{\Lambda_{\hat{T}'}_\Lambda} \circ \phi'_{\Lambda_{\hat{T}'}_\Lambda}(X_h') = X_h'$ for every generator $X_h'$ of $\hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$. It follows that $\phi'_{\Lambda_{\hat{T}'}_\Lambda} \circ \phi'_{\Lambda_{\hat{T}'}_\Lambda}$ is the inclusion map. In particular, it is injective, which proves that the first homomorphism $\phi'_{\Lambda_{\hat{T}'}_\Lambda} : \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \rightarrow \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$ is injective.

As indicated before, this enables us to extend $\phi'_{\Lambda_{\hat{T}'}_\Lambda} : \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \rightarrow \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$ to a map $\Phi'_{\Lambda_{\hat{T}'}_\Lambda} : \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda} \rightarrow \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$ by setting $\Phi'_{\Lambda_{\hat{T}'}_\Lambda}(PQ^{-1}) = \phi'_{\Lambda_{\hat{T}'}_\Lambda}(P)\phi'_{\Lambda_{\hat{T}'}_\Lambda}(Q)^{-1}$ if $P, Q \in \hat{T}'_{\Lambda_{\hat{T}'}_\Lambda}$. It can be shown that this map $\Phi'_{\Lambda_{\hat{T}'}_\Lambda}$ is an algebra homomorphism. See [3] Chap. 9.
A symmetric argument provides an algebra homomorphism $\Phi^q_{\lambda \lambda'} : \tilde{T}^q_{\lambda} \to \tilde{T}^q_{\lambda'}$. From the fact that $\phi^q_{\lambda \lambda'}(X^q_k) = X^q_k$ for every generator $X^q_k$ of $\mathbb{T}^q$, we conclude that $\Phi^q_{\lambda \lambda'} \circ \Phi^q_{\lambda \lambda'}$ is the identity. By symmetry, it follows that $\Phi^q_{\lambda \lambda'} : \tilde{T}^q_{\lambda} \to \tilde{T}^q_{\lambda'}$ is an isomorphism, with inverse $\Phi^q_{\lambda \lambda'}$. \hfill \qed

When $\lambda'$ is obtained from $\lambda$ by an edge reindexing, we also have the immediate relation.

**Proposition 6.** Suppose that the ideal triangulations $\lambda$, $\lambda' \in \Lambda(S)$ are obtained from each other by an edge reindexing, namely that $\lambda'_j = \lambda_{\alpha(i)}$ for some permutation $\alpha \in S_n$. Then there exists a unique isomorphism

$$\Phi^q_{\lambda \lambda'} : \tilde{T}^q_{\lambda} \to \tilde{T}^q_{\lambda'}$$

such that $\Phi^q_{\lambda \lambda'}(X^q_j) = X^q_{\alpha(i)}$ for every generator $X^q_j$ of $\tilde{T}^q_{\lambda'}$. \hfill \qed

The following facts are straightforward from the definitions.

**Proposition 7.** The isomorphisms $\Phi^q_{\lambda \lambda'} : \tilde{T}^q_{\lambda} \to \tilde{T}^q_{\lambda'}$ defined by Propositions 5 and 6 satisfy the following properties.

1. Composition Relation: $\Phi^q_{\lambda \beta(\lambda)} \circ \Phi^q_{\beta(\lambda) \alpha(\beta(\lambda))} = \Phi^q_{\lambda \alpha(\beta(\lambda))}$ for every $\alpha, \beta \in S_n$.
2. Reflexivity Relation: $\Phi^q_{\lambda \lambda} \circ \Phi^q_{\lambda \lambda} = \text{Id}$
3. Reindexing Relation: $\Phi^q_{\lambda \alpha(\lambda)} \circ \Phi^q_{\alpha(\lambda) \alpha(\alpha(\lambda))} = \Phi^q_{\lambda \alpha(\alpha(\lambda))} \circ \Phi^q_{\alpha(\alpha(\lambda)) \alpha(\alpha(\alpha(\lambda)))}$ for every $\alpha \in S_n$.
4. Distant Commutativity Relation: $\Phi^q_{\lambda \Delta_1(\lambda)} \circ \Phi^q_{\Delta_1(\lambda) \Delta_1(\Delta_1(\lambda))} = \Phi^q_{\lambda \Delta_1(\Delta_1(\lambda))} \circ \Phi^q_{\Delta_1(\Delta_1(\lambda)) \Delta_1(\Delta_1(\Delta_1(\lambda)))}$ if the edges $\lambda_j, \lambda_j$ of $\lambda$ do not belong to a same triangle. \hfill \qed

Dealing with the Pentagon Relation will require more efforts.

### 4. The Pentagon Relation

The goal of this section is to show that the isomorphisms $\Phi^q_{\lambda \lambda'}$ constructed in the previous section are compatible with the quantum relation satisfied by the diagonal exchanges $\Delta_1$.

The main tool to achieve this is borrowed from [8]. It consists of a certain order 5 automorphism of the quantum torus. The quantum torus $\mathcal{W}^q$ is the algebra defined by the generators $U^{\pm 1}, V^{\pm 1}$ and by the relation $VU = q^2UV$.

Set $U(0) = U, V(0) = V$ and inductively define

$$U(k+1) = (1 + qU(k))V(k), \quad V(k+1) = U^{-1}(k)$$

From this definition we deduce the following relations.

$$U(k+1)U(k) = q^2U(k)U(k+1), \quad V(k)U(k) = q^2U(k)V(k)$$

This provides us with a few combinatorial tricks to deal with noncommutativity. One of them is that the products $U(k+1)U(k-1)$ and $U(k-1)U(k+1)$ both commute with $U(k)$. Also, the two equations of (4.3) deform to:

$$q^{-1}U^{-1}(k)U(k+1)U(k-1) = 1 + q^{-1}U^{-1}(k)$$
$$q^{-1}U^{-1}(k-1)U(k+1) = 1 + qU^{-1}(k)$$

What we have accomplished is decomposing the polynomial factors $1 + q^{\pm 1}U^{\pm 1}(k)$ into monomials. This decomposition will turn out to be very convenient later on.

**Lemma 8** (Chekhov-Fock). $U(k+5) = U(k)$ for every $k \in \mathbb{Z}$.

**Proof.** This immediately follows from iterated applications of (4.1):

$$U(k+5) = (1 + qU(k+4))U^{-1}(k+3)$$
$$= U^{-1}(k+3) + q(U(k+3)U^{-1}(k+2))U^{-1}(k+3)$$
$$= U^{-1}(k+3) + qU^{-1}(k+2)U^{-1}(k+3) + q^2U(k+3)U^{-1}(k+2)U^{-1}(k+3)$$
$$= U^{-1}(k+3) + qU^{-1}(k+2)U^{-1}(k+3) + U^{-1}(k+2)$$
Consider a pentagon cycle of geodesic laminations \( \lambda(0) \), \( \lambda(1) = \Delta_i(\lambda(0)) \), \( \lambda(2) = \Delta_j(\lambda(1)) \), \( \lambda(3) = \Delta_i(\lambda(2)) \), \( \lambda(4) = \Delta_j(\lambda(3)) \), \( \lambda(5) = \Delta_i(\lambda(4)) \) as in Figure 3. We want to show that

\[
\Phi^q_{\lambda(0)\lambda(1)} \circ \Phi^q_{\lambda(1)\lambda(2)} \circ \Phi^q_{\lambda(2)\lambda(3)} \circ \Phi^q_{\lambda(3)\lambda(4)} \circ \Phi^q_{\lambda(4)\lambda(5)} = \Phi^q_{\lambda(0)\lambda(5)}
\]

where the \( \Phi^q_{\lambda(k)\lambda(k+1)} \) are defined by Proposition 5 and \( \Phi^q_{\lambda(0)\lambda(5)} \) by Proposition 6.

Let \( \lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e \) denote the sides of the pentagon, and let \( \lambda_i \) and \( \lambda_j \) be the diagonal edges of \( \lambda(0) \), all labelled as in Figure 3. Note that the \( \lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e \) are edges of all the \( \lambda(k) \), and in particular are associated to generators \( X_a, X_b, X_c, X_d, X_e \) of \( \mathcal{T}_{\lambda(0)} \). Let \( A(k), B(k), C(k), D(k), E(k) \in \mathcal{T}_{\lambda(0)} \) denote the respective images of \( X_a, X_b, X_c, X_d, X_e \) under \( \Phi^q_{\lambda(0)\lambda(1)} \circ \cdots \circ \Phi^q_{\lambda(k-1)\lambda(k)} \).

We do the same thing with diagonals, but with an additional twist. The diagonals of the pentagon in \( \lambda(k) \) are always its \( i \)-th and \( j \)-th edges, but their relative configuration with respect to other alternates.

In particular, the factor \( \sigma_{ij}(k) \) occurring in the skew-commutativity relations (3.1) is equal to \((-1)^{k+1}\). For this reason, we let \( U(k) \) and \( V(k) \) in \( \mathcal{T}_{\lambda(0)} \) be the respective images of \( X_{\alpha^{+}_{i+j}(k)} \) and \( X_{\alpha^{-}_{i+j}(k)} \) in \( \mathcal{T}_{\lambda(0)} \) under \( \Phi^q_{\lambda(0)\lambda(1)} \circ \cdots \circ \Phi^q_{\lambda(k-1)\lambda(k)} \), where \( \alpha^{+}_{i+j} \in \mathcal{S}_a \) is the transposition exchanging \( i \) and \( j \). This is specially designed so that \( V(k)U(k) = qU(k)V(k) \) for every \( k \). In addition, as one moves from \( \lambda(k) \) to \( \lambda(k+1) \), the diagonal exchange is always performed on the edge corresponding to \( U(k) \).

From the definition of \( \Phi^q_{\lambda(k)\lambda(k+1)} \), the elements \( U(k), V(k) \) satisfy the induction relation (4.1). In particular, it follows from Lemma 3 that \( U(k) = U(k) \) and \( V(k) = V(k) \).

The induction formulas for the other elements \( A(k), B(k), C(k), D(k), E(k) \) depend on whether there are identifications between the sides of the pentagon. We first consider the case of an embedded polygon, with no identification between its sides.

**Proposition 9 (Chekhov-Fock). The Pentagon Relation**

\[
\Phi^q_{\lambda(0)\lambda(1)} \circ \Phi^q_{\lambda(1)\lambda(2)} \circ \Phi^q_{\lambda(2)\lambda(3)} \circ \Phi^q_{\lambda(3)\lambda(4)} \circ \Phi^q_{\lambda(4)\lambda(5)} = \Phi^q_{\lambda(0)\lambda(5)}
\]

is satisfied in the case of an embedded pentagon.

**Proof.** Set \( \Psi = \Phi^q_{\lambda(0)\lambda(1)} \circ \Phi^q_{\lambda(1)\lambda(2)} \circ \Phi^q_{\lambda(2)\lambda(3)} \circ \Phi^q_{\lambda(3)\lambda(4)} \circ \Phi^q_{\lambda(4)\lambda(5)} \) to simplify the notation. We need to show that \( \Psi(X_k) = X_{\alpha^{+}_{i+j}(k)} \) for every \( k \). The property is immediate for those \( X_k \) which correspond to edges outside of the pentagon.

By definition of the \( U(k) \) and \( V(k) \) and by Lemma 3, \( \Psi(X_i) = V(0) = X_j \) and \( \Psi(X_j) = U(5) = U(0) = X_i \). This proves the property for \( X_i \) and \( X_j \).

For the sides of the pentagon, \( \Psi(X_a) = A(5) \), \( \Psi(X_b) = B(5) \), \( \Psi(X_c) = C(5) \), \( \Psi(X_d) = D(5) \) and \( \Psi(X_e) = E(5) \). From the definition of \( \Phi^q_{\lambda(k)\lambda(k+1)} \) in Proposition 5,

\[
\begin{align*}
A(5) &= (1 + qU_{(3)}^{-1})^{-1} (1 + qU_{(2)}^{-1})^{-1} (1 + qU_{(0)}) A(0) \\
B(5) &= (1 + qU_{(4)}^{-1})^{-1} (1 + qU_{(2)}) (1 + qU_{(0)}^{-1})^{-1} B(0) \\
C(5) &= (1 + qU_{(4)}) (1 + qU_{(2)})^{-1} (1 + qU_{(1)}^{-1})^{-1} C(0) \\
D(5) &= (1 + qU_{(4)}^{-1})^{-1} (1 + qU_{(3)}^{-1})^{-1} (1 + qU_{(1)}) D(0) \\
E(5) &= (1 + qU_{(3)})(1 + qU_{(1)}^{-1})^{-1} (1 + qU_{(0)}^{-1})^{-1} E(0)
\end{align*}
\]
Using (4.5),
\[
(1 + qU_{(k+3)}^{-1})^{-1}(1 + qU_{(k+2)}^{-1})^{-1} = q^{-1}U_{(k+3)}U_{(k+4)}U_{(k+2)}^{-1}q^{-1}U_{(k+2)}U_{(k+3)}U_{(k+1)}^{-1}
\]
\[
= U_{(k+4)}^{-1}U_{(k+1)}^{-1}
\]
\[
= U_{(k-1)}^{-1}U_{(k+1)}^{-1}
\]
\[
= (1 + qU_{(k)})^{-1}.
\]

Combining this relation with \(U_{(k'+5)} = U_{(k')}\), we get that \(A_{(5)} = A_{(0)} = X_a, B_{(5)} = B_{(0)} = X_b, C_{(5)} = C_{(0)} = X_c, D_{(5)} = D_{(0)} = X_d\) and \(E_{(5)} = E_{(0)} = X_e\), as required.

We now have to worry about possible identifications between the sides of the pentagon. Note that it suffices to prove the property for any cyclic permutation of the \(\lambda_{(k)}\). This reduces the analysis to 6 possible cases.

1. no identification (embedded pentagon);
2. \(\lambda_a = \lambda_b\), and the other sides are distinct;
3. \(\lambda_a = \lambda_c\), and the other sides are distinct;
4. \(\lambda_a = \lambda_b\) and \(\lambda_c = \lambda_d\);
5. \(\lambda_a = \lambda_b\) and \(\lambda_c = \lambda_e\);
6. \(\lambda_a = \lambda_c\) and \(\lambda_b = \lambda_e\).

We already considered Case 1.

In the other cases, note that the identifications between the sides of the pentagon have no impact on the images of \(X_i, X_j\). We therefore only need to consider \(X_a, X_b, X_c, X_d, X_e\).

**Case 2:** \(\lambda_a = \lambda_b\), and the other sides are distinct.

Using (4.6) and (4.3),
\[
\Psi(X_a) = A_{(5)} = (1 + qU_{(4)}^{-1})^{-1}(1 + qU_{(3)}^{-1})^{-1}U_{(2)}U_{(0)}A_{(0)}
\]
\[
= (1 + qU_{(1)})^{-1}U_{(2)}U_{(0)}A_{(0)}
\]
\[
= A_{(0)} = X_a
\]

The argument for \(X_c, X_d, X_e\) is identical to that of Proposition 9.

**Case 3:** \(\lambda_a = \lambda_c\), and the other sides are distinct.

Using (4.6) and (4.2),
\[
A_{(5)} = (1 + qU_{(4)})(1 + qU_{(3)}^{-1})^{-1}(1 + qU_{(2)}^{-1})^{-1}(1 + q^3U_{(2)}^{-1})^{-1}A_{(2)}
\]
\[
= (1 + qU_{(4)})U_{(0)}U_{(3)}U_{(4)}U_{(1)}^{-1}(1 + q^3U_{(2)}^{-1})^{-1}A_{(2)}
\]
\[
= U_{(0)}U_{(3)}U_{(4)}^{-1}U_{(1)}^{-1}U_{(2)}^{-1}U_{(0)}A_{(2)}
\]
\[
= U_{(0)}U_{(3)}U_{(4)}^{-1}q^{-1}U_{(3)}^{-1}U_{(1)}^{-1}U_{(2)}^{-1}U_{(0)}A_{(2)}
\]

and
\[
A_{(2)} = (1 + qU_{(1)}^{-1})^{-1}(1 + qU_{(0)})A_{(0)}
\]
\[
= q^{-1}U_{(0)}^{-1}U_{(1)}^{-1}U_{(2)}^{-1}U_{(3)}^{-1}U_{(4)}A_{(0)}
\]

Then
\[
A_{(5)}A_{(0)} = q^{-2}U_{(0)}U_{(3)}U_{(4)}^{-1}U_{(1)}^{-1}U_{(2)}^{-1}U_{(3)}^{-1}U_{(4)}U_{(1)}U_{(2)}U_{(0)}U_{(1)}U_{(2)} = 1.
\]

It follows that \(\Psi(X_a) = A_{(5)} = A_{(0)} = X_a\). The argument for \(X_b, X_d, X_e\) is identical to that for Case 1.

**Case 4:** \(\lambda_a = \lambda_b\) and \(\lambda_c = \lambda_d\).

The fact that \(\Psi(X_a) = X_a\) is proved as in Case 2. Shifting indices by 2 and rotating the picture, the argument of Case 2 gives \(C_{(7)} = C_{(2)}\). We can then backtrack to \(C_{(5)} = C_{(0)}\) by using the 5-periodicity of the \(U_{(k)}\). This proves that \(\Psi(X_c) = X_c\). The remaining generator \(X_e\) is treated as in Case 1.

**Case 5:** \(\lambda_a = \lambda_b\) and \(\lambda_c = \lambda_e\).
Again \( \Psi(X_a) = X_a \) as in Case 2, and \( \Psi(X_d) = X_d \) as in Case 1. The fact that \( \Psi(X_c) = X_c \) follows from Case 3 after shifting indices by 2 as above.

**Case 6**: \( \lambda_a = \lambda_c \) and \( \lambda_b = \lambda_c \). This again follows from a combination of the arguments of Cases 1, 2 and 3.

In conclusion, we have proved:

**Proposition 10. The Pentagon Relation**

\[
\Phi^{q(0)}_{\lambda(0)} \circ \Phi^{q(1)}_{\lambda(1)} \circ \Phi^{q(2)}_{\lambda(2)} \circ \Phi^{q(3)}_{\lambda(3)} \circ \Phi^{q(4)}_{\lambda(4)} \circ \Phi^{q(5)}_{\lambda(5)} = \Phi^{q(5)}_{\lambda(0)}
\]

is satisfied in all cases.

\( \square \)

5. The quantum Teichmüller space

We can now state and prove the main result of this paper.

**Theorem 11. There is a unique family of algebra isomorphisms**

\[
\Phi^q_{\lambda \lambda'} : \widehat{\mathcal{C}}_q \rightarrow \widehat{\mathcal{C}}^q
\]

defined as \( \lambda, \lambda' \in \Lambda(S) \) ranges over all pairs of ideal triangulations, such that:

1. \( \Phi^q_{\lambda \lambda'} = \Phi^q_{\lambda' \lambda} \circ \Phi^q_{\lambda \lambda'} \) for every \( \lambda, \lambda', \lambda'' \in \Lambda(S) \);
2. \( \Phi^q_{\lambda \lambda'} \) is the isomorphism of Proposition 6 when \( \lambda' \) is obtained from \( \lambda \) by a diagonal exchange;
3. \( \Phi^q_{\lambda \lambda'} \) is the isomorphism of Proposition 10 when \( \lambda' \) is obtained from \( \lambda \) by an edge reindexing.

**Proof.** Use Theorem 11 to connect \( \lambda \) to \( \lambda' \) by a sequence \( \lambda = \lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)} = \lambda' \) where each \( \lambda_{(k+1)} \) is obtained from \( \lambda_{(k)} \) by a diagonal exchange or by an edge reindexing, and define \( \Phi^q_{\lambda \lambda'} \) as the composition of the \( \Phi^q_{\lambda_{(k)} \lambda_{(k+1)}} \) provided by Propositions 5 and 6. Theorem 2 and Propositions 7 and 10 show that this \( \Phi^q_{\lambda_{(k)} \lambda_{(k+1)}} \) is independent of the choice of the sequence of \( \lambda_{(k)} \).

The uniqueness immediately follows from Theorem 11. \( \square \)

The quantum (enhanced) Teichmüller space of \( S \) can now be defined as the algebra

\[
\widehat{\mathcal{C}}^q_S = \left( \bigcup_{\lambda \in \Lambda(S)} \widehat{\mathcal{C}}^q_{\lambda \lambda'} \right) / \sim
\]

where the relation \( \sim \) is defined by the property that, for \( \lambda \in \widehat{\mathcal{C}}^q_{\lambda \lambda'} \) and \( \lambda' \in \widehat{\mathcal{C}}^q_{\lambda \lambda'} \),

\[
X \sim X' \iff X = \Phi^q_{\lambda \lambda'}(X').
\]

Note that the definition is specially designed so that, when \( q = 1 \), there is a natural isomorphism between \( \widehat{\mathcal{C}}_S \) and the algebra \( \text{Rat} \widehat{\mathcal{T}}(S) \) of rational functions on the enhanced Teichmüller space \( \widehat{\mathcal{T}}(S) \).

6. The quantum cusped Teichmüller space

The cusped Teichmüller space \( \mathcal{C}(S) \) is the set of isotopy classes of hyperbolic metrics on \( S \) for which all ends are of cusp type. Since such a metric admits a unique enhancement, \( \mathcal{C}(S) \) is a natural subspace of the enhanced Teichmüller space \( \widehat{\mathcal{T}}(S) \).

Consider the Thurston parametrization \( \phi_{\lambda} : \widehat{\mathcal{T}}(S) \rightarrow \mathbb{R}^n_+ \) of \( \widehat{\mathcal{T}}(S) \) by the shear coordinates \( x_i \) associated to the edges \( \lambda_i \) of an ideal triangulation \( \lambda \in \Lambda(S) \). Recall that we associated to the \( j \)-th puncture \( v_j \) of \( S \) the exponential length parameter \( p_j = x_1^{k_{j1}} x_2^{k_{j2}} \ldots x_n^{k_{jn}} \), where \( k_{ij} \in \{0, 1, 2\} \) is the number of end points of the edge \( \lambda_i \) that are equal to \( v_j \). We observed in § 2 that \( \mathcal{C}(S) \subset \widehat{\mathcal{T}}(S) \) corresponds under \( \phi_{\lambda} \) to the set of those \( x \in \mathbb{R}^n_+ \) such that \( p_j = 1 \) for every \( j = 1, \ldots, p \).

It is also natural to consider the product \( h = x_1 x_2 \ldots x_n \). Note that \( h^2 = p_1 p_2 \ldots p_p \) since every edge \( \lambda_i \) has two end points (so that \( \sum_j k_{ij} = 2 \) for every \( i \)). In particular, \( h \) is identically 1 on the image of \( \mathcal{C}(S) \) under \( \phi_{\lambda} \) since the \( x_i \) are all real positive.

In this non-quantum context, consider the elements \( P_j = X_1^{k_{j1}} X_2^{k_{j2}} \ldots X_n^{k_{jn}} \) and \( H = X_1 X_2 \ldots X_n \in \mathbb{C}(X_1, X_2, \ldots, X_n) \). Note that again \( H^2 = P_1 P_2 \ldots P_p \).

It can be shown that there exists an isomorphism

\[
\mathbb{C}(X_1, X_2, \ldots, X_n) \rightarrow \mathbb{C}(Y_1, Y_2, \ldots, Y_n)
\]
sending the first \( p - 1 \) elements \( P_1, P_2, \ldots, P_{p-1} \) to \( Y_1, Y_2, \ldots, Y_{p-1} \) and \( H \) to \( Y_p \), respectively. See for instance [3, §3]. It follows that the isomorphism

\[ \Phi_\lambda : \text{Rat} \, \mathcal{T}(S) \to \mathbb{C}(X_1, X_2, \ldots, X_n) \]

defined by \( \phi_\lambda \) induces an isomorphism

\[ \Psi_\lambda : \text{Rat} \, \mathcal{C}(S) \to \mathbb{C}(X_1, X_2, \ldots, X_n)/I \]

where \( I \) is the ideal generated by the elements \( P_j - 1 \) and \( H - 1 \). Note that it is important to include \( H - 1 \) to make sure that \( I \) is prime.

We extend this to the quantum set-up by introducing the elements

\[ P_j = q^{-\sum_{i < i'} \sigma_{ii'} k_{ii'} k_{i'j}} X_1^{k_{1j}} X_2^{k_{2j}} \cdots X_n^{k_{nj}} \]

and

\[ H = q^{-\sum_{i < i'} \sigma_{ii'} X_1 X_2 \cdots X_n} \]

of \( \mathcal{T}_\lambda^q \). The \( q \)-factor is introduced for the following property.

**Proposition 12.** The elements \( H \) and \( P_j \) in \( \mathcal{T}_\lambda^q \) are each invariant under reindexing of the edges of \( \lambda \).

**Proof.** It suffices to verify this for the transposition \( \alpha_{k+k+1} \). Exchanging \( k \) and \( k + 1 \) in the product \( X_1^{k_{1j}} X_2^{k_{2j}} \cdots X_n^{k_{nj}} \) of \( P_j \) results in multiplying this element by \( q^{-2\sigma_{k+1,k+k+1,j}} \). However, in the exponent \( -\sum_{i < i'} \sigma_{ii'} k_{ii'} k_{i'j} \) of the \( q \)-factor, \( \sigma_{k+1,k} \) is replaced by \( \sigma_{k+1,k} = -\sigma_{k,k+1} \), so that this exponent increases by \( 2\sigma_{k,k+1} k_{k+1,j} \). Consequently, the two contributions cancel out and \( P_j \) remains invariant. The argument is similar for \( H \).

**Remark 13.** The \( q \)-factor in the definition of \( H \) and the \( P_j \) is traditionally known as the quantum ordering in the physics literature. It is better explained in a situation when we can write \( X_i = \exp \xi_i \), where the \( \xi_i \) are symbols with central commutators \( \xi_i \xi_j - \xi_j \xi_i = 2\pi i \hbar \sigma_{ij} \in \mathbb{C} \). It then follows from the Campbell-Hausdorff formula that

\[
\exp(k_1 \xi_1 + k_2 \xi_2 + \cdots + k_n \xi_n) = e^{-\pi i \hbar \sum_{i < i'} \sigma_{ii'} k_{ii'} k_{i'j}} \exp(k_1 \xi_1) \exp(k_2 \xi_2) \cdots \exp(k_n \xi_n)
\]

if \( q = e^{\pi i \hbar} \). In particular, this makes the invariance under reindexing immediate.

**Proposition 14.** For any two ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \), the coordinate change isomorphism \( \Phi_{\lambda \lambda'}^q : \mathcal{T}_\lambda^q \to \mathcal{T}_{\lambda'}^q \) sends \( P_j \in \mathcal{T}_\lambda^q \) to \( P_j \in \mathcal{T}_{\lambda'}^q \), and \( H \in \mathcal{T}_\lambda^q \) to \( H \in \mathcal{T}_{\lambda'}^q \).

**Proof.** Because of Proposition 12, it suffices to check this when \( \lambda' \) and \( \lambda \) differ only by a diagonal exchange.

We will verify the property case-by-case, according to the type of the diagonal exchange. To help distinguishing the quantities associated to \( \lambda \) from those associated to \( \lambda' \), we will label with primes any data associated to \( \lambda' \). Thus, \( \mathcal{T}_\lambda^q \) is generated by the \( X_i' \), and \( P_j' \in \mathcal{T}_{\lambda'}^q \) is the element associated to the \( j \)-th puncture.

The diagonal exchange specifies two distinct triangle components of \( S - \lambda \). Let \( \overline{Q} \) be the closure of these two triangles in \( S \). Abstractly, \( \overline{Q} \) is obtained from the square \( Q \) where the diagonal exchange takes place by identifying some of its sides according to the case we are considering. Note that distinct vertices of \( \overline{Q} \) can correspond to the same puncture \( v_j \) of \( S \).

To simplify the notation (by saving the letter \( j \)), we focus attention on the element \( P_i \) associated to the first puncture \( v_1 \). Since (the argument of) Proposition 12 guarantees that the order in which we write its generators does not matter, we can write \( P_i = q^{-\sum_{k \leq n} \sigma_{k,k+1} X_{i_1} X_{i_2} \cdots X_{i_m}} \) where \( X_{i_1}, X_{i_2}, \ldots, X_{i_m} \) occur in this order as one goes counterclockwise around the puncture \( v_1 \). Similarly, \( P_i' = q^{-\sum_{k \leq n} \sigma_{k,k+1}' X_{i_1}' X_{i_2}' \cdots X_{i_m}'} \).

After a possible cyclic permutation of the \( X_{i_m} \), those touching the vertex \( v \) of \( \overline{Q} \) correspond to an interval \( a \leq k \leq b \). Let the contribution of \( v \) to \( P_i \) be

\[ c_v(P_i) = q^{-\sum_{a \leq k \leq b} \sigma_{k,k+1}' X_{i_a} X_{i_{a+1}} \cdots X_{i_b}}. \]

The contribution \( c_v(P_i') \) of \( v \) to \( P_i' \) is similarly defined.
The element $P_1$ may contain the contribution of several vertices of $\overline{Q}$. It is immediate that the part of $P_1$ which does not come from these contributions is invariant under the diagonal exchange. Therefore, we only need to show that $\Phi_{\lambda_\gamma}$ sends $c_v(P_1)$ to $c_v(P_1)$ for every vertex $v$ of $\overline{Q}$.

We label the data of the diagonal exchange according to Figure 2 and to the cases of Proposition 5.

Case 1: the edges $\lambda_j$, $\lambda_k$, $\lambda_l$, $\lambda_m$ are distinct (embedded diagonal exchange).

Because of the symmetries of the figure and of the fact that this case is its own inverse, it suffices to show this for an arbitrary corner of $\overline{Q} = Q$, for instance the upper right corner. If $v$ contributes to $P_4$ (and $P_4'$),

$$c_v(P_4) = q^{-1}X_jX_k \quad \text{and} \quad c_v(P_1) = q^{-2}X_jX_k.$$

It follows that

$$\Phi_{\lambda_\gamma}(c_v(P_4')) = q^{-1}(1 + qX_i)X_j(1 + qX_i^{-1})^{-1}X_k$$

$$= q^{-1}X_j(1 + q^{-1}X_i)(1 + qX_i^{-1})^{-1}X_k$$

$$= q^{-1}X_j q^{-1}X_k X_k = q^{-2}X_j X_k X_k = c_v(P_1).$$

Cases 2 (and Case 3): $\lambda_j$ is identified with $\lambda_k$, and $\lambda_l$ is distinct from $\lambda_m$.

We restrict to Case 2 since Case 3 is the inverse of Case 2.

In these cases, $\overline{Q}$ has three distinct vertices. The first one corresponds to the upper left and lower right vertices of $Q$. If this vertex $v$ contributes to $P_1$,

$$c_v(P_1) = q^{-1}X_mX_j X_j X_i X_j \quad \text{and} \quad c_v(P_1) = X_mX_j X_i.$$

Then,

$$\Phi_{\lambda_\gamma}(c_v(P_1')) = q^{-1}(1 + qX_i^{-1})^{-1}X_mX_j X_j X_j X_i^{-1}(1 + qX_i)X_k$$

$$= q^{-1}X_mX_j (1 + q^{-1}X_i^{-1})^{-1}X_i^{-1}(1 + qX_i)X_k$$

$$= X_mX_j X_k = c_v(P_1).$$

The second vertex of $\overline{Q}$ corresponds to the upper right vertex of $Q$. For this vertex $v$,

$$c_v(P_1) = X_j \quad \text{and} \quad c_v(P_1) = X_jX_i.$$

Then

$$\Phi_{\lambda_\gamma}(c_v(P_1')) = X_j X_j = c_v(P_1).$$

The argument for the third vertex of $\overline{Q}$, corresponding to the lower left vertex of $Q$, is identical to that of Case 1.

Cases 4 (and Case 5): $\lambda_j$ is identified with $\lambda_k$, and $\lambda_l$ is distinct from $\lambda_m$.

We again can restrict attention to Case 4 since Case 5 is the inverse of Case 4.

There are two vertices in $\overline{Q}$. This first one corresponds to the two left corners of $Q$. If this vertex contributes to $P_1$,

$$c_v(P_4) = q^2X_mX_j X_j X_j X_m \quad \text{and} \quad c_v(P_1) = q^2X_mX_j X_i X_m.$$

Then,

$$\Phi_{\lambda_\gamma}(c_v(P_1')) = q^2(1 + qX_i^{-1})^{-1}X_m X_j (1 + qX_i)(1 + q^3X_i)X_j (1 + qX_i^{-1})^{-1}X_m$$

$$= q^2X_m(1 + q^{-1}X_i^{-1})^{-1}X_i^{-1}(1 + qX_i)(1 + q^3X_i)(1 + q^{-3}X_i^{-1})^{-1}X_j X_m$$

$$= q^2X_m q^4 X_i X_j X_m = q^2X_m X_j X_i X_m = c_v(P_1).$$

The argument is similar for the second vertex of $\overline{Q}$, corresponding to the right corners of $Q$.

Cases 6-8: The arguments is similar to the above cases.

The same arguments apply to show that $\Phi_{\lambda_\gamma}(H') = H$. \hfill $\square$

Proposition 14 shows that $H$ and the $P_1$ are well-defined elements of the quantum enhanced Teichmüller space $\mathcal{T}_S^q$. It is not too hard to see that these elements are central. It is proved in 3 that they generate the center of $\mathcal{T}_S^q$. 
By analogy with the non-quantum case, we define the quantum cusped Teichmüller space $\mathcal{CT}_q^S$ as the quotient algebra

$$\mathcal{CT}_q^S = \mathcal{T}_q^S / I$$

where $I$ is the 2-sided ideal generated by $H - 1$ and the $P_j - 1$. As indicated at the beginning of this section, the algebra $\text{Rat}(\mathcal{CT}(S))$ on the cusped Teichmüller space $\mathcal{CT}(S)$ is naturally isomorphic to $\mathcal{CT}_q^S$ when $q = 1$.

In [3], we define invariants of a diffeomorphism $\varphi$ of $S$ by considering certain representations $\rho$ of the polynomial core (defined in that paper) of the quantum enhanced Teichmüller space $\mathcal{T}_q^S$. These representations are associated to the (unique) hyperbolic metric on the mapping torus $M_\varphi = S \times \mathbb{R} / \sim$, where $\sim$ identifies $(x, t)$ to $(\varphi^n(x), t + n)$. In particular, a property of $\rho$ is that $\rho(H) = \text{Id}$ and $\rho(P_j) = \text{Id}$. In other words, this representation $\rho$ is actually a representation of the polynomial core of the quantum cusped Teichmüller space.

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