Strategy-proofness on the Non-Paretian Subdomain

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Abstract

Let $g$ be a strategy-proof rule on the domain $NP$ of profiles where no alternative Pareto-dominates any other. Then we establish a result with a Gibbard-Satterthwaite flavor: $g$ is dictatorial if its range contains at least three alternatives.

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1 Introduction

This paper shows that dictatorial rules are the only ones satisfying strategy-proofness on the domain $NP$, the set of all profiles of strong preferences at which no alternative Pareto-dominates any other. All existing proofs of Gibbard-Satterthwaite employ profiles that are not in $NP$.

There are two main reasons for studying strategy-proofness on $NP$. The first is that profiles $u$ not in $NP$ may be quite unlikely. The society may be large and the set of alternatives small. Then for no pair of alternatives is a unanimous agreement likely.

Alternatively, in a two-stage group decision process the probability of Pareto domination would be zero if the first stage narrows the set of alternatives by rejecting, among others, alternatives that are Pareto-dominated. The domain

\[1\] Of the many proofs of the Gibbard-Satterthwaite theorem on the domain of all profiles of strong orders (e.g., Gibbard (1973), Satterthwaite (1975), Schmeidler and Sonnenschein (1998), Barberà (1980, 1983a, 1983b, 2001), Svensson (1999), Benoît (2000), Reny (2001), Sen (2001), Larsen and Svensson (2006)), every single one makes critical use of profiles in which some alternatives Pareto-dominate others. For example, many invoke Arrow’s theorem after first using a choice function to construct a social ordering by seeing which of two alternatives is chosen when they are both brought to the top of everyone’s ordering, and so Pareto dominate everything else. Aswal, Chatterji, and Sen (2003) characterize the family of domains on which every strategy-proof social choice function with full range is dictatorial, but only for domains that are product sets.
NP of surviving alternatives comprise the feasible set, from which the second stage group chooses.

The second reason is that we might start with a (manipulable) rule \( g \) defined on all profiles of strong preferences but where the restriction \( g|_{NP} \) of \( g \) to \( NP \) is strategy-proof. Then the conclusion that \( g|_{NP} \) is dictatorial will help determine the behavior of \( g \). This paper stems from a companion paper (Campbell and Kelly, 2014), which treated universally beneficial manipulation (UBM) rules defined by the property that if anyone manipulates, everyone gains and no one is hurt. Such UBM rules must be strategy-proof on \( NP \). Completing our classification analysis in that earlier paper requires proving a Gibbard-Satterthwaite result on \( NP \).

This domain \( NP \) is very large, even as a fraction of all profiles of strong preferences, and so our result may not seem unexpected, but the difficulty of our proof is surprising. Of course, whenever we take up a proper subdomain, we must reconsider strategy-proofness, as we will have excluded manipulations that would have been possible on the larger domain. But strategy-proofness is more difficult to analyze here because \( NP \) is not a product set.

It should also be noted that just because \( NP \) is large is no guarantee of dictatorship for full-range strategy-proof rules. There are supersets of \( NP \) for which there do exist non-dictatorial strategy-proof rules: Suppose the number of individuals, \( n \), is odd, the set of alternatives is \( \{x, y, z\} \) and the domain is \( D \), the union of \( NP \) and the collection of all the profiles for which \( z \) is the top-ranking alternative for every individual. The rule on \( D \) that selects the majority winner between \( x \) and \( y \) at every profile in \( NP \) and selects \( z \) elsewhere in \( D \) is non-dictatorial and has full range. It is strategy-proof because, for example, if \( x \) is selected then someone ranks \( x \) above \( z \) and thus no individual who prefers \( z \) to \( x \) can unilaterally cause \( z \) to be selected.

After introducing terminology and notation in Section 2, we prove some intermediate results in Section 3. Then our main result is proved by induction. The basis step, for three individuals and three alternatives is established in Section 4. Induction on the number of individuals takes place in Section 5. Then induction on the number of alternatives takes place in Section 6.

2 Framework.

For given \( m, n \geq 3 \), we consider a finite set \( X \) of alternatives where \( |X| = m \) and finite set \( N = \{1, 2, \ldots, n\} \) of individuals. A (strong) ordering on \( X \) is a complete, asymmetric, transitive relation on \( X \) and the set of all such orderings is \( L(X) \). For \( R \in L(X) \) and \( Y \subset X \) let \( R|_{Y} \) denote the relation \( R \cap Y \times Y \) on \( Y \), the restriction of \( R \) to \( Y \). If \( R \) is a member of \( L(X) \) we let \( R^{-1} \) denote the inverse of \( R \): That is, \( (x, y) \in R^{-1} \) if and only if \( (y, x) \in R \).
A profile $p$ is a map from $N$ to $L(X)$, where $p = (p(1), p(2), \ldots, p(n))$ and we write $x \succ_p y$ if individual $h$ strongly prefers $x$ to $y$ at profile $p$. The set of all profiles is $L(X)^N$. A domain $\varphi$ is subset of $L(X)^N$. For each subset $Y$ of $X$ and each profile $p$ in $\varphi$, let $p | Y$ denote the restriction of profile $p \in \varphi$ to $Y$. That is, $p | Y$ represents the function $q : L(Y)^N$ satisfying $q(i) = p(i) | Y$ for all $i \in N$. A social choice rule on $\varphi$ is a function $g : \varphi \to X$, where $\varphi$ is a nonempty subset of $L(X)^N$. Rule $g$ is dictatorial if there exists an individual $i$ such that for each profile $p$ in $\varphi$, alternative $g(p)$ is the top-ranked element in $p(i)$ restricted to $\text{Range}(g)$. A rule $g$ is full-range if $\text{Range}(g) = X$.

In this paper, we consider social choice rules on the Non-Paretian domain, $NP(n, m)$, the set of all profiles $p \in L(X)^N$ such that no alternative is Pareto-dominates any other.

$$NP(n, m) = \{ p \in L(X)^N : \text{for every } x, y \in X, \text{ if } x \neq y, \text{ then we have } x \succ_p y \text{ for some } i \in N \text{ and } y \succ_p x \text{ for some } j \in N \}$$

It is very important to note that $NP(n, m)$ is not a Cartesian product set. This will greatly complicate our analysis.

Two profiles $p$ and $q$ are $h$-variants, where $h \in N$, if $q(i) = p(i)$ for all $i \neq h$. Individual $h$ can manipulate the social choice rule $g : \varphi \to X$ at $p$ via $p^*$ if $p$ and $p^*$ belong to $\varphi$, $p$ and $p^*$ are $h$-variants, and $g(p^*) \succ_{p(h)} g(p)$. And $g$ is strategy-proof if no one can manipulate $g$ at any profile.

We will repeatedly use the following simple consequence of strategy-proofness without explicitly alluding to it:

If we switch adjacent alternatives $a$ and $b$ in some individual $i$’s preference ordering, without changing anyone else’s preferences, then the selected alternative will not change unless $a$ is selected before the switch and $b$ is selected after and individual $i$ preferred $a$ to $b$ initially, or $b$ is selected before and $a$ is selected after and $i$ preferred $b$ to $a$ initially.

Rule $g$ on $L(X)^N$ satisfies universally beneficial manipulation (UBM) if for every profile $u$ and individual $h$ such that there exists an $h$-variant profile $u^*$ with $g(u^*) \succ_{u(h)} g(u)$, we have $g(u^*) \succ_{u(j)} g(u)$ for every individual $j$. Our primary motivation for studying strategy-proofness on $NP(n, m)$ is that if $g$ satisfies UBM on $L(X)^N$, then the restriction of $g$ to $NP(n, m)$ is strategy-proof on that subdomain. The classification theorems (Campbell and Kelly, 2014) characterizing all UBM rules rely on the claim that if $g : NP(n, m) \to X$ is strategy-proof and has a range of at least three alternatives, then $g$ is dictatorial. We will exclude $n = 2$, for in that case every rule is strategy-proof on $NP(n, m)$ for the trivial reason that no two distinct profiles in $NP(n, m)$ are $i$-variants of one another.

3 Paths in NP(n, m) and an equivalence theorem.
From a global perspective, our proof will use two inductions. First, we show strategy-proofness plus full range implies dictatorship for \( m = 3 \) and \( n = 3 \). We show in Section 5 that the result for \( m = 3 \) and \( n = 3 \) implies the result for \( m = 3 \) and \( n + 1 \). Finally, Section 7 contains the proof that the result for \( m \) and \( n \) implies the result for \( m + 1 \) and \( n \).

Those induction steps will require some preliminary theorems. One goal of this section is to show that in our proof we can restrict attention to rules that satisfy a full range assumption. Before that, we need to show that on \( NP(n, m) \), any strategy proof rule \( g \) with range \( S \) has the property that \( u \mid S = u^* \mid S \) implies \( g(u) = g(u^*) \). On \( L(X)^N \), this is easy, just use a standard sequence argument in the manner of Gibbard (1973) and Satterthwaite (1975). But here, because \( NP(n, m) \) is not a Cartesian product, those standard sequences can take you outside \( NP(n, m) \). So we must find sequences of profiles all of which are in \( NP(n, m) \). In addition, we may have to vary the order of individuals whose preferences are to be changed (an order which is fixed in standard sequence arguments). To deal with these problems, we start with a definition and then a lemma.

Given two profiles \( u \) and \( u^* \) in \( NP(n, m) \), consider a sequence of profiles \( u_1, u_2, ..., u_T \), all in \( NP(n, m) \), such that

1. \( u_1 = u \);
2. \( u_T = u^* \);
3. For each \( t, 1 \leq t < T \), there exists an \( h \in N \) such that \( u_t \) and \( u_{t+1} \) are \( h \)-variants;
4. For each \( t, 1 \leq t < T \), \( u_t \mid S = u_{t+1} \mid S \).

We call such a sequence an \( S \)-path in \( NP(n, m) \) from \( u \) to \( u^* \). Note two obvious properties of \( S \)-paths:

1. If \( u_1, u_2, ... \) is an \( S \)-path from \( u \) to \( v \), then \( u_T, u_{T-1}, ... \) is an \( S \)-path from \( v \) to \( u \);
2. If \( u_1, u_2, ... \) is an \( S \)-path from \( u \) to \( v \), and \( v_1, v_2, ... \) is an \( S \)-path from \( v \) to \( w \), then \( u_1, u_2, ... \) is an \( S \)-path from \( u \) to \( w \).

We next prove two lemmas regarding \( S \)-paths in \( NP(n, m) \).

**Lemma 3-1.** Suppose there exists an \( x \) in \( X \) such that profiles \( u \) and \( u^* \) in \( NP(n, m) \) agree on \( S = X \setminus \{x\} \). Then for each of \( u \) and \( u^* \), there is a \( S \)-path in \( NP(n, m) \) from \( u \) to \( u^* \).

**Proof:**

We show that for each of \( u \) and \( u^* \), there is an \( S \)-path in \( NP(n, m) \) from \( u \) to \( u^* \).

\[
\begin{array}{|c|c|c|c|}
\hline
& 1 & 2 & \cdots & n \\
\hline
\text{x} & u(2) \mid S & \cdots & u(n) \mid S \\
\hline
u(1) \mid S & x & \cdots & x \\
\hline
\end{array}
\]
Then an $S$-path from $u$ to $u^*$ in $NP(n, m)$ will be found by first following the path from $u$ to $u^{**}$ and then following - in reverse - the path from $u^*$ to $u^{**}$ (see properties 1 and 2 above).

If $x$ is individual #1’s top alternative, then create the path to $u^{**}$ by taking $x$ down to the bottom for each $i > 1$ in turn.

Otherwise, we proceed by steps, raising $x$ one rank in #1’s ordering until it is #1’s top. We describe one such step. Without loss of generality, suppose #1’s ordering, $1 : a...bex...$, with $x$ in the position just below $c$. If anyone else has $c \succ u(j)x$, we could bring $x$ up just above $c$ in #1’s ordering and still be in $NP(n, m)$. So assume $x \succ u(j)c$ for all $j > 1$.

\[
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & \ldots & n \\
\hline
a & : & : & \\
\vdots & x & \ldots & x \\
b & : & : & \\
c & c & \ldots & c \\
x & : & : & \\
\vdots & & & \\
\hline
\end{array}
\]

Whatever is between $x$ and $c$ (if anything) for #2, is either below $x$ for #1 or in \{a, ..., b\}. But if we are in $NP(n, m)$, each of the alternatives in \{a, ..., b\} above $c$ for #2 must be below $c$, and so below $x$, for some individual $j > 2$. Then we can bring $x$ down just below $c$ for #2 and stay in $NP(n, m)$. That allows us to bring $x$ up just above $c$ in #1’s ordering and still be in $NP(n, m)$. This continues until $x$ has been raised to 1’s top. □

Now we extend that result in the $S$-Path Lemma. Note that this lemma is not about a (strategy-proof) rule $g$ - it is solely about the structure of $NP(n, m)$.

**Lemma 3-2** For $S$ any subset of $X$, let $u$ and $u^*$ be two profiles in $NP(n, m)$ with $u \upharpoonright S = u^* \upharpoonright S$. Then there exists an $S$-path in $NP(n, m)$ from $u$ to $u^*$.

Proof: To establish the existence of a sequence of profiles in $NP(n, m)$ from $u$ to $u^*$, we will show that there are $S$-path sequences in $NP(n, m)$ from each of $u$ and $u^*$ to a profile $u^{**}$ we describe shortly. Then an $S$-path from $u$ to $u^*$ in $NP(n, m)$ will be found by first following the path from $u$ to $u^{**}$ and then following - in reverse - the path from $u^*$ to $u^{**}$ (see properties 1 and 2 above). We suppose $X \setminus S = \{x, y, ..., z\}$ and let $xy...z$ be a fixed ordering on $X \setminus S$. Then $u^{**}$ is given by
So individual #1 ranks everything in $X \setminus S$ above everything in $S$ and ranks the elements of $S$ the same way as they are ranked in $u(1)$. Individuals 2,...,n rank everything in $S$ above everything in $X \setminus S$ and rank the elements of $S$ the same way they rank those elements at $u$. Finally individuals 2,...,n rank the elements of $X \setminus S$ as $z...yx$, the opposite of their ordering by #1.

We show that there is an $S$-path in $NP(n,m)$ from both $u$ and $u^*$ (which agree on $S$) to $u^{**}$ by a series of applications of Lemma 3-1. In this case with $X \setminus S = \{x, y, ..., z\}$, we first set $S_1 = X \setminus \{x\}$ and so $X \setminus S_1 = \{x\}$. Then, by Lemma 3-1, there is a path from $u$ to the following profile $u_1$:

|   | 1    | 2    | ... | n    |
|---|------|------|-----|------|
| $u$ | $x$  | $u(2)$ | $S$ | $u(n)$ | $S$ |
|   | $y$  |       |    |       |    |
|   | $\vdots$ |     |    |       |    |
|   | $z$  |       |    |       |    |
| $u(1)$ | $S$  |       |    |       |    |

Next, take $X = S_1$ and $S_2 = X \setminus \{y\}$. By Lemma 3-1, there is an $S_1$-path and so an $S$-path from $u_1|S_1$ to the following profile:

|   | 1    | 2    | ... | n    |
|---|------|------|-----|------|
| $u_2$ | $x$  | $u(2)$ | $S_1$ | $n$    |
| $u_1(1)$ | $S_2$ |       |    |       |    |

|   | 1    | 2    | ... | n    |
|---|------|------|-----|------|
| $u_3$ | $x$  | $u(2)$ | $S_2$ | $n$    |
| $u_1(1)$ | $S_2$ |       |    |       |    |

If we take each profile in that path and insert an $x$ at the top of #1’s ordering and $x$ at the bottom for everyone else, we get an $S$-path from $u_1$ to $u_3$.

|   | 1    | 2    | ... | n    |
|---|------|------|-----|------|
| $u_3$ | $x$  | $u(2)$ | $S_2$ | $n$    |
| $u_1(1)$ | $S_2$ |       |    |       |    |

Combining these two paths sequentially yields a path from $u$ to $u_3$. Continuing in this fashion we get an $S_2$-path and so an $S$-path from $u$ to $u^{**}$. Continuing in this pattern yields a path from $u^*$ to $u^{**}$, which was our goal. □
Remark 1. The $S$-path lemma does not require that $S$ be Range($g$) for a strategy-proof $g$, though that will be our primary application, as in the Equivalence Theorem just below.

Remark 2. In the $S$-path lemma, $S$ can even be a singleton, showing that there is a path in $NP(n,m)$ from any profile in $NP(n,m)$ to any other.

These two remarks will not be used in this paper, but they serve to emphasize that the $S$-path lemma is not about a social choice rule, but rather about the domain $NP(n,m)$.

**Equivalence Theorem 3-3.** For $m,n \geq 3$, and any rule $g$ that is strategy-proof on $NP(n,m)$ and has range $S$, if $u | S = u^* | S$, then $g(u) = g(u^*)$.

**Proof.** Let $u_1, u_2, \ldots, u_T$ be an $S$-path in $NP(n,m)$ from $u$ to $u^*$ as guaranteed by the $S$-path Lemma. If $g(u) \neq g(u^*)$, there must be a $t$, $1 \leq t < T$, such that $g(u_t) \neq g(u_{t+1})$, where $u_t$ and $u_{t+1}$ are $h$-variants. But individual $h$ orders $g(u_t)$ and $g(u_{t+1})$ the same at $u_t$ and $u_{t+1}$ since $u_t | S = u_{t+1} | S$, so $g$ must be manipulable by $h$ at either $u_t$ or $u_{t+1}$. Therefore, $g(u) = g(u^*)$. □

This theorem will be the key at the conclusion to extending our analysis from full-range rules to the more general case.

4 **Induction basis: 3 individuals and 3 alternatives.**

Now let $g$ be a strategy-proof social choice rule on $NP(n,m)$ with $|\text{Range}(g)| \geq 3$. We would like to show $g$ is dictatorial, i.e., there exists an individual $i$ such that for each profile $u$ in $NP(n,m)$, alternative $g(u)$ is the top-ranked element in $u(i)$ restricted to $\text{Range}(g)$. In this and the next two sections, we do this for the special case where $\text{Range}(g) = X$. Our goal, then, is to prove, for $m,n \geq 3$, the proposition $SP(n,m)$: every strategy-proof rule on $NP(n,m)$ is dictatorial if it has full range.

Our analysis proceeds by induction on $n$ first and then by induction on $m$. The basis step deals with $m = 3$ and $n = 3$.

**Theorem 4-3 (Basis).** $SP(3,3)$. That is, let $g$ be a strategy-proof social choice rule on $NP(3,3)$. If $\text{Range}(g) = X$, then $g$ is dictatorial: there exists an individual $i$ such that for each profile $u$ in $NP(3,3)$, $g(u)$ is the top-ranked element in $u(i)$.

**Proof:** Assume $X = \{a,b,c\}$ and $N = \{1,2,3\}$. The proof consists of three steps.

(Step 1) For any strategy-proof rule $g$ on $NP(3,3)$, we show how choice at a profile in $VP$, where $VP$ is the subdomain of $NP(3,3)$ consisting of voting
paradox profiles (where each alternative loses to some other under majority voting), will lead to individual decisiveness for one alternative over another, on all of $NP(3,3)$.

**Step 2** Then we show that if the range of strategy-proof $g$, restricted to $VP$, is all of $X$, there must be a dictator for $g$.

**Step 3** Last, we show that if the range of $g$ on $NP(3,3)$ is $X$ then the range of $g$ restricted to $VP$ is also $X$.

We begin with Step 1, the decisiveness result.

**Lemma 4-4.** If rule $g$ is strategy-proof on $NP(3,3)$ then, for every profile $u$ in $VP$, if alternative $\alpha \in X = \{a, b, c\}$ is chosen at $u$, and $\alpha$ is top for individual $j$ while $\beta \in X$ is $j$’s bottom at $u$, then $g$ never chooses $\beta$ at any profile in $NP$ where $j$ prefers $\alpha$ to $\beta$; in this case we say $j$ is *decisive* for $\alpha$ against $\beta$ on $NP$.

(If $j$ is decisive for $\alpha$ against $\beta$, then we write $\alpha D_j \beta$.)

**Proof:** Consider a voting paradox profile $u$:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\hline \\
a & b & c \\
b & c & a \\
c & a & b \\
\end{array}
$$

Without loss of generality, assume $g(u) = a$. We want to show for every profile $v$ in $NP$ with $a \succ_v c$ will have $g(v) \neq c$. Such profiles have $v(1) = abc, acb$, or $bac$.

**Case 1.** $v(1) = abc$. So $v$ is:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\hline \\
a & b & c \\
v(2) & v(3) \\
c & a & b \\
\end{array}
$$

Note $g(u^1) = a$ at $u^1$, a 2-variant of $u$:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\hline \\
a & c & c \\
b & b & a \\
c & a & b \\
\end{array}
$$

or #2 will manipulate to $u^1$ from $u$. Then look at $u^2$, a 3-variant of $u^1$: 8
If \( g(u^2) = c \), then \#3 would manipulate from \( u^1 \) to \( u^2 \). So \( g(u^2) \neq c \). But then \( g(v) \neq c \) or \#2 would manipulate from \( u^2 \) to \( v \). Therefore \( c \) is not chosen at any profile in \( NP \) at which individual \#1’s ordering is \( abc \).

**Case 2.** \( v(1) = acb \). One profile with person \#1 having ordering \( acb \) is \( u' \), a simple 1-variant of \( u \) above:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
a & c & c \\
b & b & a \\
c & a & v(3)
\end{array}
\]

\( g(u') = a \) or \#1 would manipulate to \( u \) with \( u(1) = abc \).

We trace out the consequences of this by considering the next figure, where individual \#1 has ordering \( acb \), \#2’s possible orderings are in different rows, and \#3’s preference orderings are in different columns. [Black cells are not in \( NP(3,3) \); gray background cells are in \( VP \). “\( \neg \) c” in a cell means \( c \) is not chosen there.] There is an “\( a \)” in cell (4,5), i.e., row 4 and column 5, in the main body of the table (i.e., not including the preference label row and preference label column) indicating the outcome at that cell, which is the profile \( u' \). There is an “\( a \)” above that in (3,5) or \#2 (the row player) would manipulate from (4,5) to (3,5). Similarly, there are “\( a \)”s in cells (4,1) and (4,2) or \#3 (the column player) would manipulate from there to (4,5). An “\( a \)” appears in (6,1) or \#2 would manipulate there from (4,1). Three cells in row 4 are labeled “\( \neg \) c” (i.e., “not \( c \)” because \( c \) in such a cell would lead \#3 to manipulate there from (4,5). Similarly for two cells in row 3.

1: acb
We extend this analysis in the next table, where the results just obtained are emboldened and in regular (non-Italic). Now $c$ can’t be chosen at $6,3$ or #3 will manipulate to $(6,1)$. And that means $c$ isn’t chosen at $5,3$ or #2 would manipulate from $(6,3)$ to $(5,3)$. At cells $(1,4)$ and $(1,6)$, alternative $c$ won’t be chosen because $c$ is #2’s worst there and #2 would manipulate to row 3 (or 4).
In the next table, again all the preceding results are emboldened. Suppose $c$ is chosen at cell (6, 4) indicated by “$c$?”. Then $b$ is not chosen in (6, 3) or #3 will manipulate there from (6, 4). Therefore $a$ is chosen in (6, 3). In turn, since $a$ is #2’s bottom at (6, 3), $a$ must also be chosen at (5, 3) and (4, 3). Since $c$ is chosen at (6, 4), $a$ can’t be chosen at (4, 4), or #2 would manipulate from there to (6, 4). Hence $b$ is chosen at (4, 4). But that leads to a contradiction, since now #3 would manipulate from (4, 3) to (4, 4). From all that, we can conclude $c$ is not chosen at (6, 4) and that implies $c$ is chosen nowhere in column 4 or #2 would manipulate up from (6, 4). We have determined that $c$ is chosen in none of the $NP(3, 3)$ profiles with individual #1’s ordering $acb$. 
1: acb

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 3 | a  | b  | a  | b  | c  |
| 2 | b  | c  | c  | a  | c  |
| a  | b  | c  | a  | c  | b  |
| a  | c  | b  | a  | c  | b  |
| a  | c  | b  | c  | a  | b  |
| ¬c | ¬c | ¬c | ¬c | ¬c | ¬c |
| a  | b  | c  | a  | c  | b  |
| a  | c  | b  | a  | c  | b  |
| a  | c  | b  | c  | a  | b  |
| a  | c  | b  | c  | a  | b  |
| a  | c  | b  | a  | c  | b  |
| a  | c  | b  | ¬c | ¬c | ¬c |
| a  | c  | b  | ¬c | ¬c | ¬c |

**Case 3.** $v(1) = bac$. In the next figure, we show the $NP(3, 3)$ profiles for which individual #1 has ordering $bac$. In twelve cells, we have entered "$\neg c$" because, were $c$ to be chosen at one of those cells, #1 (for whom $c$ is worst) would manipulate by changing his ordering to $abc$ or $acb$, whichever would yield a profile still in $NP(3, 3)$ because we know that $c$ wouldn’t be chosen at the resulting profile.
Since c is not chosen in cell (5, 6) where c is at the top of #3’s ordering, c must not be chosen anywhere in row 5 or else #3 would manipulate there from (5, 6). But in row 5, c is at the top of #2’s ordering and so c can’t be chosen anywhere in the table or #2 would manipulate there from row 5. □

Step 2 of the proof consists of proving the following corollaries.

**Corollary 4-4-1.** Suppose $g$ is a strategy-proof rule on $NP(3, 3)$: then if $g$ is dictatorial on $VP$, it is dictatorial on all of $NP(3, 3)$.

**Corollary 4-4-2.** Suppose $g$ is a strategy-proof rule on $NP(3, 3)$: then if $g$ has singleton range on $VP$, it has singleton range on all of $NP(3, 3)$.

Proof of these first two corollaries is easy.

**Corollary 4-4-3.** Suppose $g$ is a strategy-proof rule on $NP(3, 3)$: then if $g$ has range $X$ on $VP$, then it is dictatorial on all of $NP(3, 3)$.

**Proof of Corollary 4-4-3:** It suffices, by Corollary 4-4-1, to show that if $g$ has range $X$ on $VP$, then it is dictatorial on $VP$.

So consider the twelve profiles in $VP$:
|   | 1       | 5       | 9       |
|---|---------|---------|---------|
| 1 | abc     | bac     | cab     |
| 2 | bca     | a       | not c   |
| 3 | cab     | cba     | not c   |

|   | 2       | 6       | 10      |
|---|---------|---------|---------|
| 1 | abc     | bac     | cab     |
| 2 | cab     | a       | not c   |
| 3 | bca     | cba     | not c   |

|   | 3       | 7       | 11      |
|---|---------|---------|---------|
| 1 | acb     | bca     | cab     |
| 2 | cba     | a       | not c   |
| 3 | bac     | abc     | not c   |

|   | 4       | 8       | 12      |
|---|---------|---------|---------|
| 1 | acb     | bca     | cab     |
| 2 | bac     | a       | not c   |
| 3 | cba     | abc     | not c   |

Without loss of generality, we assume that \( g \) yields \( a \) at the first profile. Therefore, by Lemma 4, individual #1 is decisive for \( a \) against \( c \), which we write \( aD_1c \). That implies \( c \) is not chosen at profiles 2, 3, 4, 5, and 6. Also \( g \) can not yield \( b \) at profile 10 or else \( bD_2a \), contrary to the outcome at profile 1.
Profile 2. Suppose b were chosen at profile 2:

|   | 1 | 5 | 9 |
|---|---|---|---|
| 1 | abc | bac | cab |
| 2 | bca | acb | abc |
| 3 | cab | cba | not a; :; b | bca |

|   | 2 | 6 | 10 |
|---|---|---|-----|
| 1 | abc | bac | cab |
| 2 | cab | acb | not c |
| 3 | bca | cba | not c |

|   | 3 | 7 | 11 |
|---|---|---|-----|
| 1 | acb | bca | cba |
| 2 | cba | cab | bac |
| 3 | bac | not a; :; b | abc |

|   | 4 | 8 | 12 |
|---|---|---|-----|
| 1 | acb | bca | cba |
| 2 | bac | abc | acb |
| 3 | cba | not a; :; b | cab |


Then by Lemma 4, bD3a, which implies a is not chosen at profiles 3, 4, 5, 9, and 12. But that implies b is chosen at each of profiles 3, 4, and 5. These results imply bD3c, bD2c, and bD1c, respectively. But at any profile in NP, at least one of the individuals must prefer b to c, so c would never be chosen in NP(3, 3). So now we may assume b is not chosen at profile 2. Since we already know c isn’t chosen there, a must be chosen. Now b can’t be chosen at profile 9 or bD3a, contrary to the choice of a at profile 2. All results up to this point are emboldened in the next display.
 Profiles 3 and 4. Suppose b were chosen at profile 3. Then \( bD3c \), and so c is not chosen at profiles 7, 9, 10, and 12.

|   | 1 |   | 5 |   | 9 |
|---|---|---|---|---|---|
| 1 | abc | 1 | bac | 1 | cab |
| 2 | bca | 2 | acb | 2 | abc | not b |
| 3 | cab | 3 | cba | 3 | bca | not c; \( \therefore \) a |

|   | 2 |   | 6 |   | 10 |
|---|---|---|---|---|----|
| 1 | abc | 1 | bac | 1 | cab |
| 2 | cab | 2 | cba | 2 | bca | not b; |
| 3 | bca | 3 | acb | 3 | abc | not c; \( \therefore \) a |

|   | 3 |   | 7 |   | 11 |
|---|---|---|---|---|----|
| 1 | acb | 1 | bca | 1 | cba |
| 2 | cba | 2 | cab | 2 | bac | not c |
| 3 | bac | 3 | abc | 3 | acb | |

|   | 4 |   | 8 |   | 12 |
|---|---|---|---|---|----|
| 1 | acb | 1 | bca | 1 | cba |
| 2 | bac | 2 | abc | 2 | acb | not c |
| 3 | cba | 3 | cab | 3 | bac | |

At profile 9, neither b nor c is chosen, so a must be. Therefore, \( aD2c \) which implies c is not chosen at profiles 8 or 11. Then c would not be in the range of \( g \) restricted to \( VP \). All that was a consequence of assuming b was chosen at profile 3. So b isn’t chosen there which means a is. But then \( aD1b \), so b is not chosen at profile 4, and therefore a must be chosen there.
Profile 5.

Assume that $a$ is also chosen at profile 5. Then $aD_2b$ and so $b$ is not chosen at profiles 7, 8, and 12.

If $a$ were also chosen at profile 6, everyone would be decisive for $a$ against $b$ and $b$ would never be chosen on $NP(3,3)$. So $a$ is not chosen there which means that $b$ is chosen there. But then $bD_1c$ and so $c$ is not chosen at profiles 7 and 8. This means $a$ is chosen at each of those profiles and that implies everyone is decisive for $a$ against $c$ and $c$ would never be chosen on $NP(3,3)$. Therefore our assumption that $a$ is chosen at profile 5 fails and $b$ must be chosen there. Since $b$ is chosen at profile 5, $bD_1c$ and thus $c$ is not chosen at profiles 7 and 8.

If $a$ were also chosen at profile 6, everyone would be decisive for $a$ against $b$ and $b$ would never be chosen on $NP(3,3)$. So $a$ is not chosen there which means that $b$ is chosen there. But then $bD_1c$ and so $c$ is not chosen at profiles 7 and 8. This means $a$ is chosen at each of those profiles and that implies everyone is decisive for $a$ against $c$ and $c$ would never be chosen on $NP(3,3)$. Therefore our assumption that $a$ is chosen at profile 5 fails and $b$ must be chosen there. Since $b$ is chosen at profile 5, $bD_1c$ and thus $c$ is not chosen at profiles 7 and 8.
Suppose \( a \) is chosen at profile 6, so that \( aD_3b \). This implies \( b \) is not chosen at profiles 7 and 8, so \( a \) is chosen there. But then everyone is decisive for \( a \) against \( c \) and \( c \) would never be chosen on \( NP(3,3) \). Accordingly, \( a \) must not be chosen at profile 6, so \( b \) must be. Because \( b \) is chosen at profiles 5 and 6, neither individual 2 nor 3 can be decisive for \( a \) against \( b \), and thus \( a \) cannot be selected at profiles 11 or 12.

|   |   |   |
|---|---|---|
| 1 | abc | 1 | bac |
| 2 | bca | a | 2 | acb | b |
| 3 | cab |   | 3 | cba |   |
| 6 |   |   |   |   |   |
| 1 | abc | 1 | bac | 1 | cab |
| 2 | cab | a | 2 | acb |   |
| 3 | bca |   | 3 | acb |   |
| 7 |   |   |   |   |   |
| 1 | acb | 1 | bca | 1 | cba |
| 2 | cba | a | 2 | cab |   |
| 3 | bac |   | 3 | abc |   |
| 8 |   |   |   |   |   |
| 1 | acb | 1 | bca | 1 | cba |
| 2 | bac | a | 2 | abc |   |
| 3 | cba |   | 3 | cab |   |
| 9 |   |   |   |   |   |
| 1 | cab |   | 1 | acb |   |
| 2 | acb | not b |
| 3 | bca |   |   |   |

| 10 |   |   |
| 1 | cab |   |
| 2 | bca |   |
| 3 | abc |   |

| 11 |   |   |
| 1 | cba |   |
| 2 | bac |   |
| 3 | abc |   |

| 12 |   |   |
| 1 | cba |   |
| 2 | acb |   |
| 3 | bac |   |
Profiles 7 and 8.

|   | 5   | 9   |
|---|-----|-----|
| 1 | abc | 1 | cab |
| 2 | bca | 2 | abc | not b |
| 3 | cab | 3 | cba |

|   | 6   | 10  |
|---|-----|-----|
| 1 | abc | 1 | cab |
| 2 | cab | 2 | cba | not b |
| 3 | bca | 3 | abc |

|   | 7   | 11  |
|---|-----|-----|
| 1 | acb | 1 | bca |
| 2 | cba | 2 | cab | not c; a? |
| 3 | bac | 3 | abc |

|   | 8   | 12  |
|---|-----|-----|
| 1 | acb | 1 | bca |
| 2 | bac | 2 | abc | not c |
| 3 | cba | 3 | cab |

Now suppose $a$ is chosen at profile 7 so that, by Lemma 4, $aD_3c$. Then $c$ is not chosen at profiles 10, 11, and 12. But $b$ being chosen at 11 and 12 implies $bD_2c$ and $bD_3c$, which in turn imply $c$ is never chosen on $NP(3,3)$. So $a$ is not chosen at profile 7, so $b$ must be. Thus $bD_1a$ and $a$ is not chosen at profile 8, so $b$ is chosen there.
Profile 9.

Suppose $a$ is chosen at profile 9. Then $aD_2c$ and so $c$ is also not chosen at profiles 11 or 12 so $b$ is chosen there. But then $bD_2c$ and $bD_3c$. Thus $c$ is not in the range of $g$ restricted to $NP(3,3)$, a contradiction. So we can’t have $a$ chosen at profile 9 and thus $c$ is chosen there.

|    | 1 | 5 | 9 |
|----|---|---|---|
| 1  | $abc$ | $bac$ | $cab$ |
| 2  | $bca$ | $acb$ | $b$ |
| 3  | $cab$ | $cba$ | $bca$ |

|    | 2 | 6 | 10 |
|----|---|---|----|
| 1  | $abc$ | $bac$ | $cab$ |
| 2  | $cab$ | $aca$ | $b$ |
| 3  | $bca$ | $acb$ | $bca$ |

|    | 3 | 7 | 11 |
|----|---|---|----|
| 1  | $acb$ | $bca$ | $cba$ |
| 2  | $cba$ | $cab$ | $b$ |
| 3  | $bac$ | $abc$ | $acb$ |

|    | 4 | 8 | 12 |
|----|---|---|----|
| 1  | $acb$ | $bca$ | $cba$ |
| 2  | $bac$ | $abc$ | $acb$ |
| 3  | $cba$ | $cab$ | $bac$ |
Profiles 10, 11, and 12.

Having $c$ chosen at profile 9 implies $cD_1b$, so $b$ is not chosen at profiles 11 or 12, so $c$ is also chosen there. And if $a$ were chosen at 10, we would have $aD_3c$, contrary to the choice of $c$ at profile 11. So $c$ is also chosen at profile 11 and #1 is a dictator on $VP$.

Step 3, the last stage in the proof of Theorem 3, is showing that if strategy-proof $g$ on $NP(3, 3)$ has range $X$, then so does the restriction of $g$ to $VP$. Equivalently, we show that if the range of $g$ restricted to $VP$ has range of just two alternatives, say $\{a, b\}$, then $c$ is not in the range of $g$.

Case A. Suppose $a$ is chosen at some profile $u$ in $VP$ where only one person prefers $a$ to $b$ but $c$ is not chosen in $VP$. Apart from permutations of individuals, $u$ must be

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
1 : abc & 1 : bac & 1 : cab \\
2 : bca & 2 : acb & 2 : abc \\
3 : cab & 3 : cba & 3 : bca \\
\end{array}
\]

Thus, $aD_1b$ and so $a$ is chosen at the 6 out of 12 profiles in $VP$ where #1 prefers $a$ to $b$.

\[
\begin{array}{ccc}
1 & 5 & 9 \\
\hline
1 : abc & 1 : bac & 1 : cab \\
2 : bca & 2 : acb & 2 : abc \\
3 : cab & 3 : cba & 3 : bca \\
\end{array}
\]

\[
\begin{array}{ccc}
2 & 6 & 10 \\
\hline
1 : abc & 1 : bac & 1 : cab \\
2 : cba & 2 : aba & 2 : bca \\
3 : abc & 3 : acb & 3 : abc \\
\end{array}
\]

\[
\begin{array}{ccc}
3 & 7 & 11 \\
\hline
1 : acb & 1 : bca & 1 : cba \\
2 : cba & 2 : cab & 2 : bac \\
3 : abc & 3 : abc & 3 : acb \\
\end{array}
\]

\[
\begin{array}{ccc}
4 & 8 & 12 \\
\hline
1 : acb & 1 : bca & 1 : cba \\
2 : bac & 2 : abc & 2 : acb \\
3 : cba & 3 : cab & 3 : bca \\
\end{array}
\]

By profile 2, $aD_1c$, by profile 9, $aD_2c$ and by profile 10, $aD_3c$. Lemma 4-4 implies we must have $c$ is not in the range of $g$. 

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Case B. Alternative $a$ is never chosen when only one person prefers $a$ to $b$ under our supposition that $c$ does not belong to the range of $g$ restricted to $VP$.

|   | 1  | 5  | 9 |
|---|----|----|---|
| 1 | $abc$ | $bac$ | $cab$ |
| 2 | $bca$ | $acb$ | $b$ |
| 3 | $cab$ | $cba$ | $bca$ |

|   | 2  | 6  | 10 |
|---|----|----|----|
| 1 | $abc$ | $bac$ | $cab$ |
| 2 | $cab$ | $cba$ | $bca$ |
| 3 | $bca$ | $acb$ | $abc$ |

|   | 3  | 7  | 11 |
|---|----|----|----|
| 1 | $acb$ | $bca$ | $c$ |
| 2 | $cba$ | $ab$ | $bac$ |
| 3 | $bac$ | $abc$ | $acb$ |

|   | 4  | 8  | 12 |
|---|----|----|----|
| 1 | $acb$ | $bca$ | $c$ |
| 2 | $bac$ | $ab$ | $abc$ |
| 3 | $cba$ | $cab$ | $bac$ |

By profile 5, $bD_{1c}$; by profile 4, $bD_{2c}$; and by profile 3, $bD_{3c}$. Then Lemma 4 implies $c$ is not in the range of $g$. □

5 Induction on $n$.

We have established $SP(n, m)$ for $n = 3$ and $m = 3$. We want to extend this to more individuals (but still for $m = 3$) by induction on $n$. The analysis of this section calls for defining rules on $NP(n, 3)$ as restrictions of rules on $NP(n + 1, 3)$ to subdomains. In particular, we want to define two rules on $NP(n, 3)$ by considering their images at a profile $p = (p(1), ..., p(n))$:

$$g^*(p) = g(p(1), p(2), ..., p(n - 1), p(n), p(n))$$
$$g^{**}(p) = g(p(1), p(1), p(2), ..., p(n - 1), p(n))$$

We will need these rules to be of full range when $g$ is of full range. Accordingly, we define two subdomains of $NP(n + 1, 3)$:

$NP^*(n + 1, 3)$ is the subdomain of $NP(n + 1, 3)$ consisting of all $p \in NP(n + 1, 3)$ such that $p(n) = p(n + 1)$;
$NP^{**}(n + 1, 3)$ is the subdomain of $NP(n + 1, 3)$ consisting of all $p \in NP(n + 1, 3)$ such that $p(1) = p(2)$.

Recall that $SP(n, 3)$ is the statement: "every strategy-proof rule on $NP(n, 3)$ is dictatorial if it has full range."

**Theorem 6-1.** $SP(n, 3)$ implies $SP(n + 1, 3)$.

**Proof:** Assume that $SP(n, 3)$ is true. Let $g$ be a given strategy-proof social choice function on $NP(n + 1, 3)$ that has full range.

We need to carry out several steps:

1. Given the set $N = \{1, 2, ..., n, n + 1\}$ of individuals, we let $N^* = \{1, 2, ..., n\}$.
2. Associate with each profile $p$ on $X$ in $NP(n, 3)$ two profiles, $p^*$ and $p^{**}$ on $X$ in $NP(n + 1, 3)$.
3. Define $g^*$ on $NP(n, 3)$ by relating $g^*(p)$ to $g(p^*)$ and define $g^{**}$ on $NP(n, 3)$ by relating $g^{**}(p)$ to $g(p^{**})$.
4. Show that strategy-proofness of $g$ on $NP(n + 1, 3)$ implies strategy-proofness of $g^*$ and $g^{**}$ on $NP(n, 3)$.
5. Show that $Range(g) = X$ implies $Range(g^*) = X$ and $Range(g^{**}) = X$. This step is carried out in companion paper Campbell and Kelly, 2014b).
6. Conclude by the induction hypothesis that $g^*$ and $g^{**}$ are dictatorial.
7. Show that the dictatorship of $g^*$ and $g^{**}$ implies dictatorship for $g$.

For the second step, given a profile $p = (p(1), p(2), ..., p(n))$ in $NP(n, 3)$, let

$p^* = (p(1), p(2), ..., p(n), p(n))$

where the last two orderings are the same, and

$p^{**} = (p(1), p(1), p(2), ..., p(n))$

where the first two orderings are the same.

**Remark:** $p$ is in $NP(n, 3)$ if and only if $p^*$ is in $NP(n + 1, 3)$; and $p$ is in $NP(n, 3)$ if and only if $p^{**}$ is in $NP(n + 1, 3)$.

For the third step, we define $g^*$ and $g^{**}$ as at the beginning of this section.

For the fourth step, we need to show $g^*$ and $g^{**}$ are strategy-proof. Consider rule $g*$: If individual $i < n$ can manipulate $g*$ at some profile $(p(1), p(2), ..., p(n))$ then that individual can obviously manipulate $g$ at profile $(p(1), p(2), ..., p(n), p(n))$ because

$g^*(p(1), p(2), ..., p(n)) = g(p(1), p(2), ..., p(n), p(n))$. 

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Next, we show that the remaining individual, \( n \), cannot manipulate \( g^* \). Suppose to the contrary that \( n \) manipulates \( g^* \) at profile \( p = (p(1), p(2), \ldots, p(n - 1), p(n)) \) in \( NP(n, 3) \) with \( g^*(p) = x \), by submitting ordering \( v \) and getting profile \( q = (p(1), p(2), \ldots, p(n - 1), v) \) in \( NP(n, 3) \) with \( g^*(q) = y \neq x \) and \( y \succ_p x \).

\[
g(p(1), p(2), \ldots, p(n - 1), p(n)) = x; \quad \text{and} \quad g(p(1), p(2), \ldots, p(n - 1), v, v) = y.
\]

Note by the remark above, that these profiles are in \( NP(n + 1, 3) \).

Construct profile \( w = (p(1), p(2), \ldots, p(n - 1), p(n), v) \). Note that \( w \) is in \( NP(n + 1, 3) \). Now \( x \succ_p g(w) \) or \( g \) is manipulable by \( n + 1 \) at \( (p(1), p(2), \ldots, p(n - 1), p(n), p(n)) \), and \( g(w) \succ_p y \) or \( g \) is manipulable by \( n \) at \( w \). Transitivity then implies \( x \succ_p y \), contrary to our earlier assumption of manipulability of \( g^* \). Therefore, \( g^* \) is strategy-proof.

An analogous argument establishes the strategy-proofness of \( g^{**} \).

For the fifth step, as noted above, \( \text{Range}(g^*) = \text{Range}(g^{**}) = X \) is proven as the n-Range Lemma in (Campbell and Kelly, 2014b).

It follows from Steps 4 and 5, and our induction hypothesis, that \( g^* \) and \( g^{**} \) are dictatorial. We now want to prove that existence of a dictator for each of \( g^* \) and for \( g^{**} \) enables us to determine a dictator for \( g \).

**Case 1.** The dictator for \( g^* \) is an individual \( j < n \). Without loss of generality, \( j = 1 \). Let \( x \) be an arbitrary member of \( X \), and let

\[
q = (q(1), q(2), p(3), \ldots, q(n), q(n + 1))
\]

be any profile in \( NP(n + 1, 3) \) with the topmost element of \( q(1) \) being \( x \). Let \( p = (q(1), p(2), \ldots, p(n - 1), p(n), p(n + 1)) \) be a profile with \( p(1) = q(1) \) and for all \( i < 1 \), \( p(i) \) is the inverse of \( p(1) \). Note that \( p \in NP(n, 3) \). Now \( g(p) = x \) since \( p(n) = p(n + 1) \) and \#1 is a dictator for \( g^* \).

Next we consider a standard sequence from \( p \) to \( q \):

\[
\begin{align*}
p^1 &= p = (p(1), p(2), p(3), \ldots, p(n), p(n)) = (q(1), p(2), \ldots, p(n), p(n)) \\
p^2 &= (q(1), q(2), p(3), \ldots, p(n), p(n)) \\
& \vdots \\
p^n &= (q(1), q(2), q(3), \ldots, q(n), p(n)) \\
p^{n+1} &= q = (q(1), q(2), q(3), \ldots, q(n), q(n + 1)).
\end{align*}
\]

Note that every profile in this sequence is an element of \( NP(n + 1, 3) \), and that \( g(p) = x \). Then also \( g(p^2) = x \) since otherwise, with \( x \) at the bottom of \( p(2) \), \( g \) would be manipulable by \#2 at \( p^1 \). Similarly, \( g(p^3) = x \) or \( g \) would be manipulable by \#3 at \( p^2 \). Continuing in this fashion, \( g(q) = x \) and so \#1 is a dictator for \( g \).
Case 2. The dictator for $g^{**}$ is an individual $j > 1$. Using the argument of Case 1 as a template, we can show that $j + 1$ is a dictator for $g$.

Case 3. The dictator for $g^{*}$ is individual $n$ and the dictator for $g^{**}$ is #1. Consider a profile $u$ such that $u(1) = u(2) = xy...z$ and $u(n) = u(n + 1) = z...yx$. This profile is in $NP(n + 1, 3)$. Since #1 is a dictator for $g^{**}$, $g(u) = x$; since $n$ is a dictator for $g^{*}$, $g(u) = z$. This contradiction shows Case 3 isn't possible. □

6 Induction on m.

We have established that $SP(n, 3)$ holds for all $n \geq 3$. We now show

Theorem 6-1. For arbitrary $n \geq 3$ and arbitrary $m \geq 3$ the statement $SP(n, m)$ implies $SP(n, m + 1)$.

Proof: Assume that $SP(n, m)$ is true.

Let $g$ be a given strategy-proof social choice function on $NP(n, m + 1)$ that has full range. Now we define a rule $g^{*}$ based on $g$. Select arbitrary, but distinct, $w$ and $z$ in $X$. Let $NP^{wz}(n, m + 1)$ be the set of profiles in $NP(n, m + 1)$ such that alternatives $w$ and $z$ are contiguous in each individual ordering. Choose some alternative $x^{*}$ that does not belong to $X$ and set $X^{*} = \{x^{*}\} \cup X\{w, z\}$. Then $g^{*}$ will have domain $D^{*}$ by which we mean the domain $NP(n, m)$ when the feasible set is $X^{*}$. To define $g^{*}$ we begin by selecting arbitrary profile $p \in D^{*}$, and then we choose some profile $r \in NP^{wz}(n, m + 1)$ such that

1. $r|X\{w, z\} = p|X\{w, z\}$, and
2. for any $i \in \{1, 2, ..., n\}$, we have

$$\{x \in X\{w, z\} : x \succ r(i) w\} = \{x \in X\{w, z\} : x \succ p(i) x^{*}\}.$$ 

In words, we create $r$ from $p$ by replacing $x^{*}$ with $w$ and $z$ so that $w$ and $z$ are contiguous in each $r(i)$, and $r$ does not exhibit any Pareto domination, and in each $r(i)$ either $w$ or $z$ occupies the same rank as $x^{*}$ in $p(i)$. We next show that the selected alternative, which we can denote $f(p)$, is independent of the choice of profile $r$.

Suppose that $s$ and $t$ both belong to $NP^{wz}(n, m + 1)$ and both 1 and 2 hold for $r = s$ and $r = t$. We will show that if $t \neq s$ there is a profile $r'$ in $NP^{wz}(n, m + 1)$ and an individual $h$ such that

$$r'(i) = s(i) \text{ for all } i \neq h \text{ and } r'(h) = t(h) \neq s(h) \text{ and }$$
$$g(r') = g(s) \text{ if } g(s) \in X\{w, z\}, \text{ and } g(r') \in \{w, z\} \text{ if } g(s) \in \{w, z\}.$$ 

That will establish that we can create $t$ from $s$ in a serious of stages without changing the selected alternative unless it changes from $w$ to $z$ or from $z$ to $w$. We have shown that $g^{*}(p)$ is independent of the choice of $r$ in $NP^{wz}(n, m + 1)$, provided that 1 and 2 are both satisfied. Thus $g^{*}$ is well defined.
Choose some $j$ in $\{1, 2, ..., n\}$ such that $s(j) \neq t(j)$. Without loss of generality, assume that $w \succ_{t(j)} z$. Create $u$ from $s$ by switching $w$ and $z$ in $s(j)$. If $u \in NP^{uwz}(n, m + 1)$ then the fact that $w$ and $z$ are contiguous in $s$ and strategy-proofness of $g$ imply that $g(u) = g(s)$ if $g(s) \in X \setminus \{w, z\}$, and $g(u) \in \{w, z\}$ if $g(s) \in \{w, z\}$. Set $r' = u$.

Suppose that $u \notin NP(n, m + 1)$. Then we must have $z \succ_{s(i)} w$ for all $i \neq j$. Because $z$ does not Pareto dominate $w$ at $t$ and we have $t(j) \neq s(j)$, and $u(i) = s(i)$ for all $i \neq j$ there exists an individual $h$ such that $z \succ_{s(h)} w$ and $t(h) \neq s(h)$. Then switch $w$ and $z$ in $s(h)$ to form $v \in NP^{uwz}(n, m + 1)$. Strategy-proofness of $g$ implies that $g(v) = g(s)$ if $g(s) \in X \setminus \{w, z\}$, and $g(v) \in \{w, z\}$ if $g(s) \in \{w, z\}$. Set $r' = v$.

Therefore, we can use arbitrary $f$ satisfying 1 and 2 above to define $g^*$ as follows.

**Definition:** $g^*(p) = g(f(p))$ if $g(f(p)) \in X \setminus \{w, z\}$ and $g^*(p) = x^*$ if $g(f(p)) \in \{w, z\}$.

The rule $g^*$ is obviously strategy-proof if $g$ is. The **m-Range Lemma** in the companion paper (Campbell and Kelly, 2014b) establishes that the range of $g^*$ is $X^*$. We can now claim that $g^*$ is dictatorial using the induction hypothesis.

Without loss of generality, assume that person 1 is the dictator for $g^*$. We will show that person 1 is a dictator for $g$ and that will establish $SP(n, m+1)$.

We prove dictatorship of $g$ by showing that, for arbitrary $x \in X$, $g(p) = x$ for any profile in $NP(n, m + 1)$ that has $x$ at the top of person 1’s preference ordering. The first step is to show that if $x$ is selected at some profile that has $x$ at the top of person 1’s ordering and at the bottom of everyone else’s, then $g$ will select $x$ at any profile for which it is the top ranked alternative of person 1. Again, the analysis is complicated by the need to stay inside $NP(n, m + 1)$.

**Lemma 6-2:** For any $x \in X$, if there is a profile $r \in NP(n, m)$ such that $g(r) = x$, and $x$ is at the top of $r(1)$ but at the bottom of $r(i)$ for all $i > 1$ then $g(p) = x$ for any profile $p$ in $NP(n, m + 1)$ such that $x$ is at the top of $p(1)$.

**Proof:** Let $p$ be an arbitrary member of $NP(n, m + 1)$ such that $x$ is at the top of $p(1)$. By hypothesis, there exists a profile $r \in NP(n, m + 1)$ such that $g(r) = x$, and $x$ is at the top of $r(1)$ and the bottom of $r(i)$ for all $i > 1$. This justifies the first in the following sequence of statements, and each of the others follows from its predecessor by strategy-proofness of $g$, as we explain at the end of the list.

1. $g(r(1), r(2), r(3), ..., r(n-1), r(n)) = x$
2. $g(r(1), r(1)^{-1}, r(1)^{-1}, ..., r(1)^{-1}, r(1)^{-1}) = x$
3. $g(r(1), p(1)^{-1}, p(1)^{-1}, ..., p(1)^{-1}, r(1)^{-1}) = x$
4. $g(p(1), p(1)^{-1}, p(1)^{-1}, ..., p(1)^{-1}, r(1)^{-1}) = x$
5. $g(p(1), p(1)^{-1}, p(3), ..., p(n-1), p(n)) = x$
6. $g(p(1), p(2), p(3), ..., p(n-1), p(n)) = x$. 

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Note that each of the six profiles belongs to $NP(n, m + 1)$. Statement 1 implies 2 because we can replace $r(2)$ with $r(1)^{-1}$. Alternative $x$ will still be selected because $x$ is selected at the previous stage and $x$ is bottom ranked by $r(2)$. Then replace $r(3)$ with $r(1)^{-1}$. Alternative $x$ will still be selected because $x$ is selected at the previous stage and $x$ is bottom ranked by $r(3)$. And so on. Statement 2 implies 3 because we can replace person $i$’s ordering $r(i)^{-1}$ with $p(1)^{-1}$ one individual at a time, for $2 \leq i \leq n - 1$. Because $x$ is bottom ranked by $r(i)^{-1}$, if $x$ is not selected after replacing $r(i)^{-1}$ with $p(1)^{-1}$ for individual $i$ that person could manipulate at the profile for which $r(1)^{-1}$ is his true preference. Statement 3 implies 4 because the fourth profile is obtained from the third by replacing $r(1)$ with $p(1)$, and $x$ is top ranked by $p(1)$ so person 1 could manipulate at the fourth profile if $x$ were not selected there. Statement 4 implies 5 because we can replace person $i$’s ordering with $p(i)$ one individual at a time, for $i \geq 3$. Because $x$ is bottom ranked by $p(1)^{-1}$ and $r(1)^{-1}$, if $x$ is not selected after replacing $i$’s ordering at profile 4 that person could manipulate at the pre-replacement profile. Finally, statement 5 implies 6 because the last profile is obtained from its predecessor by changing person 2’s ordering, replacing $p(1)^{-1}$ with $p(2)$ and $x$ is at the bottom of $p(1)^{-1}$.

Now we establish that there is a profile $r$ in $NP(n, m + 1)$ such that $g(r) = x$ and $x$ is at the top of $r(1)$ and at the bottom of $r(i)$ for all $i > 1$.

**Lemma 6-3:** For each $x \in X$ there is a profile $p \in NP(n, m + 1)$ such that $g(p) = x$, and $x$ is at the top of $p(1)$ but at the bottom of $p(i)$ for all $i > 1$.

**Proof:** Step 1: We show that, for arbitrary $x \in X \setminus \{w, z\}$, we have $g(p) = x$ for any $p \in NP(n, m + 1)$ such that $x$ is at the top of $p(1)$.

Let $\alpha$ be any member of $L(X)$ such that $x$ is at the top of $\alpha$ and $w$ and $z$ are contiguous in $\alpha$. Set $r = (\alpha, \alpha^{-1}, \alpha^{-1}, ..., \alpha^{-1})$. We have $g(r) = x$ because person 1 dictates $g^*$. By Lemma 6-2, $g(p) = x$ for any $p \in NP(n, m + 1)$ such that $x$ is at the top of $p(1)$.

The previous paragraph will not suffice when $x \in \{w, z\}$ because the dictatorship of $g^*$ only establishes that $g$ selects either $w$ or $z$ when those two alternatives rank first and second (in either order) in person 1’s preference scheme. We need to show that $g$ selects $w$ (resp., $z$) when $w$ (resp., $z$) ranks at the top for person 1.

**Step 2:** We show that if person 1’s ordering has a member of $\{w, z\}$ at the top then $g$ will select 1’s first or second ranked alternative.

Let $q$ be any profile in $NP(n, m + 1)$ such that $z$ ranks first in $q(1)$ and $w$ ranks second, or $w$ ranks first in $q(1)$ and $z$ ranks second. (Note that $q$ does not have to belong to $NP^{wz}(n, m + 1)$. We show that $g(q) \in \{w, z\}$.) Let $r$ be the profile for which $r(1) = q(1)$, and $r(i) = q(1)^{-1}$ for all $i \geq 2$. We have $g(r) \in \{w, z\}$ because person 1 dictates $g^*$. Then $g(r(1), q(2), q(3), ..., q(n-1), r(n)) \in \{w, z\}$ by strategy-proofness of $g$. (For $2 \leq i \leq n-1$ replace $r(i)$ with $q(i)$ one individual at a time. The fact that $r(n) = r(1)^{-1}$ guarantees that we remain in
$NP(n, m + 1)$. We have $g(q) = g(r(1), q(2), q(3), ... , q(n - 1), q(n)) \in \{w, z\}$ by strategy-proofness of $g$.

Now, let $p$ be any profile in $N(n, m + 1)$ such that $p(1) = wz$... for some $x \in X \setminus \{w, z\}$. We prove $g(p) \in \{z, x\}$. Let $\beta$ be any member of $L(X)$ such that $x$ ranks first and $z$ ranks second. Let $r$ denote the profile $(\beta, \beta^{-1}, \beta^{-1}, ... , \beta^{-1})$. We have $g(r) = x$ by Step 1. This is the first in the following sequence of statements. Each of statements 2 through 5 follows from its predecessor by strategy-proofness of $g$, as we explain in the paragraph following the list.

1. $g(r(1), r(1)^{-1}, r(1)^{-1}, ... , r(1)^{-1}, r(1)^{-1}) = x$
2. $g(r(1), r(1)^{-1}, p(1)^{-1}, ... , p(1)^{-1}, p(1)^{-1}) = x$
3. $g(p(1), r(1)^{-1}, p(1)^{-1}, ... , p(1)^{-1}, p(1)^{-1}) \in \{x, z\}$
4. $g(p(1), p(2), p(3), ... , p(n - 1), p(1)^{-1}) \in \{x, z\}$
5. $g(p(1), p(2), p(3), ... , p(n - 1), p(n)) \in \{x, z\}$

Statement 1 implies 2 because, for $i \geq 3$, we can replace person $i$’s ordering $r(i) = r(1)^{-1}$ with $p(1)^{-1}$ one individual at a time. Then $x$ will still be selected at each stage because it is at the bottom of $r(i)$, and if it were not selected at the new profile then person $i$ could manipulate at the previous profile. Statement 2 implies 3 by strategy-proofness because we replace $r(1)$ with $p(1)$ which has $z$ ranked first and $x$ second. Statement 3 implies 4 because, for $2 \leq i \leq n - 1$, we replace $r(i) = r(1)^{-1}$ or $p(1)^{-1}$ with $p(i)$, one individual at a time. Then strategy-proofness of $g$ implies that $x$ or $z$ will still be selected because $x$ ranks at the bottom of $r(i) = r(1)^{-1}$ with $z$ second last and $p(1)^{-1}$ has $z$ ranked last and $x$ second last. Statement 4 implies 5 because we replaced person $n$’s ordering, $p(1)^{-1}$, which has $z$ ranked last and $x$ second last.

Similarly, if $p(1) = (w, x, ...)$ then $g(p) \in \{w, x\}$.

**Step 3:** Suppose that there exists $t \in NP(n, m + 1)$ such that $g(t) = z$ and $z$ is at the top of $t(1)$. We show that there exists a profile $u \in NP(n, m + 1)$ such that $g(u) = z$ and $z$ is at the top of $u(1)$ and at the bottom of $u(i)$ for all $i > 1$. And if there exists $t' \in NP(n, m + 1)$ such that $g(t') = w$ and $w$ is at the top of $t'(1)$ then there is a profile $u' \in NP(n, m + 1)$ such that $g(u') = w$ and $w$ is at the top of $u'(1)$ and at the bottom of $u'(i)$ for all $i > 1$.

By hypothesis, there exists a profile $t \in NP(n, m + 1)$ such that $g(t) = z$ and $t(1) = zxy...$. That is, $z$ ranks first in $t(1)$, and the second ranking alternative is denoted by $x$, with $y$ third. Without loss of generality we assume that $z \succ_{t(i)} x$ for all $i \leq k$ (possibly $k = 1$) and $x \succ_{t(i)} z$ for all $i > k$. We have $k < n$ because $z$ does not Pareto dominate $x$. Define $p \in NP(n, m + 1)$ by setting $p(i) = t(i)$ for all $i \leq k$ and $p(i) = t(1)^{-1}$ for all $i > k$. If we change $t(i)$ for $i > k$ one individual at a time, alternative $z$ will be selected at each stage. (Step 2 implies that either $z$ or $x$ will be selected, and if $x$ is selected then the individual whose ordering has changed could manipulate at the previous profile.) Therefore, $g(p) = z$. If $k = 1$ set $u = p$. 

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If \( k > 1 \) create \( q \) from \( p \) by setting \( q(2) = y...zx \), with \( q(i) = p(i) \) for all \( i \neq 2 \). (That is, \( q(2) \) is any ordering with \( y \) on top, \( x \) last, and \( z \) second last.) We have \( q \in NP(n, m + 1) \) because \( p(n) = t(1)^{-1} \). And \( g(q) \in \{z, x\} \) by Step 2. If \( g(q) = x \) then person 2 can manipulate at \( q \) via \( p(2) \). Thus, \( g(q) = z \).

Recall that \( g(n) = p(n) = t(1)^{-1} \). Create \( r \) from \( q \) by switching the order of \( y \) and \( x \) in \( q(n) \), with \( r(i) = q(i) \) for all \( i \neq n \). We have \( r(n) = ...xyz \) and \( y \succ_q 2 \). Therefore, \( r \in NP(n, m + 1) \). Step 2 implies that \( g(r) \in \{z, x\} \).

We have \( s \in NP(n, m + 1) \) because \( x \succ_{r(n)} y \). Strategy-proofness implies \( g(s) = z \).

Now create \( u^2 \) from \( s \) by switching the order of \( z \) and \( x \) in \( s(2) \). We have \( u^2(2) = y...xz \), and \( u^2 \in NP(n, m + 1) \) because \( z \succ_s 1 \) \( x \). Step 2 implies \( g(u^2) \in \{z, y\} \), but if \( g(u^2) = y \) then person 2 can manipulate at \( s \) via \( u^2(2) \). Therefore, \( g(u^2) = z \). If \( k = 2 \) we have \( z \) at the top of \( u^2(1) \) and at the bottom of \( u^2(i) \) for all \( i > 1 \), in which case we can set \( u = u^2 \). The following table specifies \( t(1) \) for the given profile \( t \) and also summarizes the changes that have been made. It only identifies the ordering for the individuals whose preferences have altered. We have established that alternative \( z \) is selected at each stage.

Sketch of proof that \( u \) exists when \( k = 2 \)

\[
\begin{align*}
t(1) & = zxy... \\
p(i) & = t(1)^{-1} \text{ for all } i > k \\
q(2) & = y...zx \\
r(n) & = ...xyz \\
s(1) & = zyx,... \\
u^2(2) & = y...xz
\end{align*}
\]

If \( k > 2 \) create \( u^3 \) from \( u^2 \) by replacing \( u^2(3) \) with \( u^2(2) \). Strategy proofness implies that \( g(u^3) = z \). Continue in this fashion until we have \( g(u^k) = z \), with \( z \) on top of \( u^k(1) \) and at the bottom of \( u^k(i) \) for all \( i > 1 \). Then set \( u = u^k \).

Similarly, we can prove the existence of a profile \( u' \) in \( NP(n, m + 1) \) such that \( g(u') = w \), with \( w \) at the top of \( u'(1) \) and at the bottom of \( u'(i) \) for all \( i > 1 \).

**Step 4:** We show that, for arbitrary \( a \in \{w, z\} \), there does in fact exist a profile \( t \in NP(n, m + 1) \) such that \( g(t) = a \) and \( a \) is at the top of \( t(1) \). Without loss of generality, \( a = z \).

Alternative \( z \) is in the range of \( g \) so there is a profile \( q \in NP(n, m + 1) \) such that \( g(q) = z \). Suppose that \( z \) is not the top alternative of \( q(1) \). Step 1 implies that the top alternative of \( q(1) \) does not belong to \( X \setminus \{w, z\} \). Therefore, \( q(1) = (w, ...) \). Step 2 implies that the alternative ranked second by \( g(1) \) must
be $z$. Therefore, $q(1) = wz...$ and $g(q) = z$. If $w \succ q(j) z$ for some $j > 1$ then let $r \in NP(n, m + 1)$ be the profile for which $r(i) = q(i)$ for all $i > 1$ and $r(1)$ is obtained from $q(1)$ by switching $w$ and $z$. Then $g(r) = z$ by strategy-proofness. Set $t = r$.

Suppose that $z \succ q(i) w$ for all $i > 1$. Create $p$ from $q$ by moving $z$ down in $q(2)$ until it’s just above $w$ in person 2’s ordering, preserving $p(2)|X\{z\} = q(2)|X\{z\}$, with $p(i) = q(i)$ for all $i \neq 2$. Then $p = q$ if $w$ and $z$ are contiguous in $q(2)$. (Then $p = q$ if $w$ and $z$ are contiguous in $q(2)$.)

We have $p \in NP(n, m + 1)$ because $z \succ q(1) x$ for all $x \in X\{w, z\}$. We have $g(q) = z$, and $g(p) \in \{w, z\}$ by Step 2. Therefore, $g(p) = z$ by strategy-proofness. Create profile $s$ from $p$ by switching $w$ and $z$ in $p(2)$, with $s(i) = p(i) = q(i)$ for all $i \neq 2$. Step 2 implies $g(s) \in \{w, z\}$. If $g(s) = w$ then we have a profile at which $w$ is at the top of person 1’s ordering and $w$ is selected. But then Lemma 6-2 and Step 3 of this lemma imply that $w$ is selected whenever it is at the top of person 1’s ordering, contradicting $g(q) = z$. Therefore, $g(s) = z$. Switch $w$ and $z$ in $s(1)$ to create profile $t$ for which $z$ is at the top of person 1’s ordering and $z$ is selected (by strategy-proofness).

Lemmas 6-2 and 6-3 imply that person 1 dictates $g$, and so we have shown $SP(n, m + 1)$. □

7 Conclusion

Combining the basis result, Theorem 5-1 and Theorem 6-1, we have

**Theorem 7-1.** For all $m, n \geq 3$, $SP(n, m)$.

Important as this result is, we are also interested in rules that have less than full range. Consider the rule $g$ defined as follows, where we assume $X$ is large: $m > 3$. Fix three alternatives, $\{a, b, c\} = S$ in $X$. At profile $u$, let $\psi(u)$ be the highest ranked alternative from $S$ according to ordering $u(1)$. Then $g(u) = \psi(u)$ unless there is a unique $x \in X$ that Pareto-dominates $\psi(u)$, in which case $g(u) = x$. This is a UBM rule with full range. But the restriction $g^*$ of $g$ to $NP$ has range that is only $S$. This rule $g^*$ is then of less than full range but is dictatorial on $NP$. That is a direct consequence of Theorem 7-1 and the Equivalence Theorem from Section 3.

**Theorem 7-2.** Let $g^*$ be a strategy-proof rule on $NP(n, m)$ for $n \geq 3$ and $m \geq 3$ with $\text{Range}(g^*) = S$ and $|S| \geq 3$. Then $S$ is dictatorial.

**Proof:** For a fixed ordering $x, y, ..., z$ of $X\setminus S$, define a rule $g^{**}$ on $NP(n, |S|)$ as follows: for profile $u$ in $NP(n, |S|)$, construct profile $u^*$ in $NP(n, m)$ as
Then define \( g^{**}(u) = g^*(u^*) \). It is easy to show that \( g^{**} \) is strategy-proof. By the Equivalence Theorem, \( g^{**} \) is of full range: \( \text{Range}(g^{**}) = S \). By Theorem 7-1, then, \( g^{**} \) is dictatorial. Then another application of the Equivalence Theorem shows \( g^* \) is dictatorial. □

8 References

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