The multiplicity of pairs of modules and hypersurface singularities

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INTRODUCTION

In [10] the author introduced the notion of the multiplicity of a pair of modules as a way of working with modules of non-finite colength. Some applications of this notion to equisingularity problems were described in [11]. The invariants introduced using this tool have the advantage that they must be independent of the parameters in the family when the stratification condition they describe holds. These invariants provide a framework for studying the equisingularity conditions $W$, $W_f$ and $A_f$ for very general families of spaces and functions. In this paper we will illustrate the use of these invariants in the study of families of functions with non-isolated singularities and show how the invariants arise naturally in the work of Pellikaan ([27], [28]) and Zaharia ([33], [34]). Pellikaan studied functions $f$ whose singular set was an isolated complete intersection singularity (ICIS) of dimension 1, Zaharia those of of dimension 2.

The principal tool for connecting the multiplicity of the pair with geometry is the multiplicity polar theorem (Theorem 1.1) which we review in section 1. This theorem is used to relate multiplicity information at the special fiber of a family with information at the generic fiber. As an illustration of the theorem we use it to give a geometric interpretation of the multiplicity of a module (Theorem 1.2). This interpretation is then used in remark 1.3 to connect the multiplicity of a module with Fulton’s k-th degeneracy class. In section 2 we show how the multiplicity of the pair $(J(f), I)$ appears naturally in the work of Pellikaan and give two formulas for it. The first formula relates this multiplicity to the number of $D_\infty$ and $A_1$ points appearing in a deformation of $f$. The second formula shows that the multiplicity of the pair $(J(f), I)$ if defined is actually the length of $I/J(f)$. This length is Pellikaan’s invariant $j(f)$. Both formulas are contained in Theorem 2.3 and its proof. These formulas are used to give a new formula for the Lê number of dimension 0 (Proposition 2.4). (Cf. [24] for details on the Lê numbers.)

In section 3, we extend the results of Pellikaan for singular sets of dimension 1 to ICIS of dimension $d$, then use these results to prove extensions of the theorems of section 2. The computation of the formula for the Lê number of dimension 0 uses Zaharia’s computation of the homology of the Milnor fiber.

These formulae suggest in general that the Lê number of dimension 0 is the sum of the invariant which controls the $A_f$ condition, (which in turn is related the multiplicity of a pair of ideals), and invariants of dimension 0 related to the other singularity types in the singular set of $f$. 
Section 3 also shows that the condition that $j(f)$ is finite imposes strong restrictions on $f$—there must exist a set of generators of $I$, \{$g_1, \ldots, g_p$\} such that $f = \sum g_i^2$. This implies that every such function is the composition of a function $h$ with a Morse singularity at the origin and the map $G$ whose components are generators of $I$. In particular, all of the germs of type $D(d,p)$, with $d > 1$, studied by Pellikaan have $j(f) = \infty$, contrary to assertions made in Remark 5.3 on page 52 of [27] and in Remark 5.4 on page 373 of [28].

In section 4 we then use the multiplicity of the pair to give a necessary and sufficient condition for the $A_f$ condition to hold for a family of functions $f_y$ (Theorem 4.5). The proof of this result involves a new trick which is used to pass information from strata in the singular set of $f$ to the ambient geometry of $f_0$. This enables us to drop the hypothesis that the “natural” stratification of the singular set of $f$ satisfies Whitney A.

In the case that the singular locus of $f_0$ is an ICIS of dimension 1, we use the relation between our invariant and the Lê numbers, to show that a strong form of the $A_f$ condition also implies that the Lê numbers are constant as well (Corollary 4.7). This is used to show that in this situation the strong form of the $A_f$ condition implies the triviality of the Milnor fibrations (Corollary 4.8). In example 4.9, by modifying the example of Briancon-Speder we show that both the $A_f$ condition and topological triviality of the family may hold, yet the Lê numbers may not be constant. It remains open whether the strong form of the $A_f$ condition implies the Lê number of dimension 0 is constant in general, or if the strong form of $A_f$ is needed if the dimension of $S(f) = 1$.

We then discuss the $W_f$ condition for the situation of Theorem 4.5. Here we show that the independence from parameter of a single invariant is all that is required for a $W_f$-Whitney stratification of a family of functions, which implies the topological triviality of the family (Theorem 4.10). This invariant is then related to the relative polar multiplicities of the members of the family and the multiplicity of the pair that is used to control the $A_f$ condition (Corollary 4.13). In turn, this implies that the $A_f$ condition combined with the independence from parameter of the relative polar multiplicities implies that we have a $W_f$-Whitney stratification (Corollary 4.14).

The application of the multiplicity of the pair to equisingularity problems grew out of a long series of conversations with Steven Kleiman; the author thanks him for his encouragement. The author also thanks David Massey and James Damon for helpful conversations, and the referee for his careful reading of the paper, and helpful suggestions.

1. The multiplicity polar theorem

In this paper we work with complex analytic sets and maps. Let $\mathcal{O}_X$ denote the structure sheaf on a complex analytic space $X$. If a module $M$ has finite colength in $\mathcal{O}_{X,x}^p$, it is possible to attach a number to the module, its Buchsbaum-Rim multiplicity ([3]). We can also define the multiplicity of a pair of modules
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Let $M \subset N$, $M$ of finite colength in $N$, as well, even if $N$ does not have finite colength in $\mathcal{O}_X^p$. We recall how to construct these numbers following the approach of Kleiman and Thorup ([20]).

Given a submodule $M$ of a free $\mathcal{O}_X$ module $F$ of rank $p$, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_X$ algebra on $p$ generators. This is known as the Rees algebra of $M$. If $(m_1, \ldots, m_p)$ is an element of $M$ then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$. Then $\text{Proj}(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of $M$ at points where the rank of a matrix of generators of $M$ is maximal. Denote the projection to $X$ by $c$, or by $c_M$ where there is ambiguity.

If $M$ is a submodule of $N$ or $h$ is a section of $N$, then $h$ and $M$ generate ideals on $\text{Proj}(\mathcal{R}(N))$; denote them by $\rho(h)$ and $\rho(M)$. If we can express $h$ in terms of a set of generators $\{n_i\}$ of $N$ as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i / T_1$. Having defined the ideal sheaf $\rho(M)$, we blow up by it.

On the blowup $B_{\rho(M)}(\text{Proj}(\mathcal{R}(N)))$ we have two tautological bundles, one the pullback of the bundle on $\text{Proj}(\mathcal{R}(N))$, the other coming from $\text{Proj}(\mathcal{R}(M))$; denote the corresponding Chern classes by $l_M$ and $l_N$, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of $N$ (and hence of $M$) is $e$. Then the multiplicity of a pair of modules $M, N$ is:

$$e(M, N) = \sum_{j=0}^{d+e-2} \int D_{M,N} \cdot l_M^{d+e-2-j} \cdot l_N^j.$$

The multiplicity of the pair is well defined as long as the set of points where $N$ is not integrally dependent on $M$ is of finite colength and the multiplicity of $M$ is the multiplicity of the pair $(M, \mathcal{O}_X^p)$. Later in this section we will give a new geometric interpretation of this number based on polar methods.

If $\mathcal{O}_{X^s,x}$ is Cohen-Macaulay, and $M$ has $d+p-1$ generators then there is a useful relation between $M$ and its ideal of maximal minors; the multiplicity of $M$ is the colength of $M$, is the colength of the ideal of maximal minors, by some theorems of Buchsbaum and Rim [3], 2.4 p. 207, 4.3 and 4.5 p. 223. In section 2 we will see a first generalization of this result to pairs of modules.

We next develop the notion of polar varieties which is the other term in the multiplicity polar theorem.

Assume we have a module $M$ which is a submodule of a free module on $X^d$, an equidimensional, analytic space, reduced off a nowhere dense subset of $X$, and the generic rank of $M$ is $e$ on each component of $X$. The hypothesis on the equidimensionality of $X$ and on the rank of $M$ ensures that $\text{Proj}(\mathcal{R}(M))$ is equidimensional of dimension $d+e-1$. Note that $\text{Proj}(\mathcal{R}(M))$ can be embedded in $X \times \mathbb{P}^{r-1}$, provided we can chose a set of generators of $M$ with $r$ elements. The polar variety of codimension $k$ of $M$ in $X$ denoted $\Gamma_k(M)$ is constructed
by intersecting Projan $\mathcal{R}(M)$ with $X \times H_{e+k-1}$ where $H_{e+k-1}$ is a general plane of codimension $e + k - 1$, then projecting to $X$. This notion was developed by Teissier in the case where $M = JM(F)$, $X = F^{-1}(0)$ (\cite{32}). Think of $H$ as the projectivised row space of a linear submersion $\pi$. Then $\Gamma_k(\mathcal{R}(M))$ consists of the set of points where the matrix formed from $D\pi$ and $DF$ has less than maximal rank, hence greater than minimal kernel rank. These are the points where the restriction to $X$ of $\pi$ is singular. In general, think of $\Gamma_k(M)$ as the set of points where the module whose matrix of generators consists of the matrix of generators of $M$ augmented by the rows of the linear submersion $\pi$, has less than maximal rank $n - k + 1$. When we consider $M$ as part of a pair of modules $M, N$, where the generic rank of $M$ is the same as the generic rank of $N$, then other polar varieties become interesting as well. In brief, we can intersect $B_{\rho(M)}(\text{Projan} \mathcal{R}(N)) \subset X \times \mathbb{P}^{N-1} \times \mathbb{P}^{p-1}$ with a mixture of hyperplanes from the two projective spaces which are factors of the space in which the blowup is embedded. We can then push these intersections down to Projan $\mathcal{R}(N)$ or $X$ as is convenient, getting mixed polar varieties in Projan $\mathcal{R}(N)$ or in $X$. These mixed varieties play an important role in the proof of the multiplicity-polar theorem, the theorem we next describe.

Setup: We suppose we have families of modules $M \subset N$, $M$ and $N$ submodules of a free module $F$ of rank $p$ on an equidimensional family of spaces with equidimensional fibers $X^{d+k}$, $X$ a family over a smooth base $Y^k$. We assume that the generic rank of $M, N$ is $e \leq p$. Let $P(M)$ denote Projan $\mathcal{R}(M)$, $\pi_M$ the projection to $X$. let $C(M)$ denote the locus of points where $M$ is not free, i.e. the points where the rank of $M$ is less than $e$, $C(\text{Projan} \mathcal{R}(M))$ its inverse image under $\pi_M$, $C(M)$ the cosupport of $\rho(M)$ in $P(\text{Projan} \mathcal{R}(N))$.

We will be interested in computing the change in the multiplicity of the pair $(M, N)$, denoted $\Delta(e(M, N))$. We will assume that the integral closures of $M$ and $N$ agree off a set $C$ of dimension $k$ which is finite over $Y$, and assume we are working on a sufficiently small neighborhood of the origin, that every component of $C$ contains the origin in its closure. Then $e(M, N, y)$ is the sum of the multiplicities of the pair at all points in the fiber of $C$ over $y$, and $\Delta(e(M, N))$ is the change in this number from 0 to a generic value of $y$. If we have a set $S$ which is finite over $Y$, then we can project $S$ to $Y$, and the degree of the branched cover at 0 is $\text{mult}_y S$. (Of course, this is just the number of points in the fiber of $S$ over our generic $y$.)

We can now state our theorem.

**Theorem (1.1)** Suppose in the above setup we have that $\overline{M} = \overline{N}$ off a set $C$ of dimension $k$ which is finite over $Y$. Suppose further that $C(\text{Projan} \mathcal{R}(M))(0) = C(\text{Projan} \mathcal{R}(M(0)))$ except possibly at the points which project to $0 \in X(0)$. Then, for $y$ a generic point of $Y$,

$$\Delta(e(M, N)) = \text{mult}_y \Gamma_d(M) - \text{mult}_y \Gamma_d(N).$$
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**Proof.** The proof in the ideal case appears in [11]; the general proof will appear in [12].

Now we describe an application of the result to the simple case where \( N \) is free.

The following geometric interpretation of the multiplicity of an ideal is well known. Given an ideal \( I \) of finite colength in \( \mathcal{O}_{X,x} \), \( X \) is equidimensional, choose \( d \) elements \( (f_1, \ldots, f_d) \) of \( I \) which generate a reduction of \( I \). (Recall that if \( M \) is a submodule of \( N \), then \( M \) is reduction of \( N \) if they have the same integral closure.) Then the multiplicity of \( I \) is the degree at \( x \) of \( F \) where \( F \) is the branched cover defined by the map-germ with components \( (f_1, \ldots, f_d) \), or the number of points in a fiber of \( F \) over a regular value close to 0.

We wish to give a similar interpretation of the multiplicity of a module.

**Theorem (1.2)** Given \( M \) a submodule of \( \mathcal{O}_{X,x}^p \), \( X \) is equidimensional, choose \( d + p - 1 \) elements which generate a reduction \( K \) of \( M \). Denote the matrix whose columns are the \( d + p - 1 \) elements by \( [K] \); \( [K] \) induces a section of \( \text{Hom}(\mathbb{C}^{d+p-1}, \mathbb{C}^p) \) which is a trivial bundle over \( X \). Stratify \( \text{Hom}(\mathbb{C}^{d+p-1}, \mathbb{C}^p) \) by rank. Let \( [e] \) denote a \( p \times (d + p - 1) \) matrix, whose entries are small, generic constants. Then, on a suitable neighborhood \( U \) of \( x \) the section of \( \text{Hom}(\mathbb{C}^{d+p-1}, \mathbb{C}^p) \) induced from \( [K] + [e] \) has at most kernel rank 1, is transverse to the rank stratification, and the number of points where the kernel rank is 1 is \( e(M) \).

**Proof.** The first step is to explain by construction what we mean by “generic constants”. Consider the family of maps \( G_a \) from \( X \), parametrised by \( \mathbb{C}^{p(d+p-1)} \) to \( \text{Hom}(\mathbb{C}^{d+p-1}, \mathbb{C}^p) \) defined by \( G_a(x) = G(x,a) = [K(x)] + [A] \), where \([A]\) is the \( p \times (d+p-1) \) matrix whose entries are coordinates \( a_{i,j} \) on \( \mathbb{C}^{p(d+p-1)} \). Let \( \tilde{X} \) be a resolution of \( X \), so we have an induced family of maps \( \tilde{G} \) on \( \tilde{X} \). Since the map \( \tilde{G}(x,a) \) is a submersion, it follows that for a \( \mathbb{Z} \)-open subset \( V \) of \( \mathbb{C}^{p(d+p-1)} \), that for \( a \in V \), the map \( \tilde{G}_a \) is transverse to the rank stratification. We claim that the points of \( V \) are the generic constants in the theorem. Note that the points of \( \text{Hom}(\mathbb{C}^{d+p-1}, \mathbb{C}^p) \) of kernel rank 1 have codimension \( 1 \cdot ((d+p-1)-(p-1)) = d \); so since \( \tilde{G}_a \) is transverse it can only hit points of the rank stratification of kernel rank 1, and only if \( D\tilde{G}_a \) has maximal rank at such points which implies \( X \) is smooth at the projection of such points. Let \( \tilde{K} \) be the submodule of \( \mathcal{O}_{X \times \mathbb{C}^{p(d+p-1)}}^p \) defined by the matrix \([K(x)] + [A]\).

Now apply the multiplicity-polar theorem to \( X \times \mathbb{C}^{p(d+p-1)} \), thought of as a family parametrised by \( \mathbb{C}^{p(d+p-1)} \), and \( (\tilde{K}, \mathcal{O}_{X \times \mathbb{C}^{p(d+p-1)}}^p) \). Use a point of \( V \) as the generic parameter value \( \epsilon \). Then \( \mathcal{O}_{X \times \mathbb{C}^{p(d+p-1)}}^p \) has no polar, because it is free, \( \tilde{K} \) has no polar, because \( \tilde{K} \) is generated by \( d + p - 1 \) elements. Choose \( U \) a neighborhood of \( x \times \mathbb{C}^{p(d+p-1)} \) sufficiently small such that every component of the cosupport of \( \tilde{K} \) which meets \( U \) has \((x,0)\) in its closure. Now at \( \epsilon \) the cosupport of \( \tilde{K}_\epsilon \) is just the points where \([K] + [\epsilon]\) has less than maximal rank. At such points \( e(\tilde{K}_\epsilon) \) is 1, because since we are at a smooth point of \( X \), the local
ring of $X$ is Cohen-Macaulay, so $e(\tilde{K}_x)$ is just the colength, which is 1. Hence $e(M) = e(K) = e(\tilde{K}_0) = e(\tilde{K}_x)$, which is the number of points where the kernel rank of $[K] + [\epsilon]$ is 1.

Remark (1.3) In [5] p254 Fulton describes the $k$-th degeneracy class associated to $\sigma$ a homomorphism of vector bundles over $X^d$. The support of the class is the set of points where the rank of $\sigma$ is less than or equal to $k$. Suppose $\sigma : E \to F$ where the rank of $E$ is $e$ and the rank of $F$ is $f$, $e \geq f$, $e - f + 1 = d$. Then the $f - 1$ degeneracy class is supported at isolated points. Fulton shows that if $X$ is Cohen-Macaulay at $x$, the contribution to the class at $x$ is the colength of the ideal of maximal minors of the matrix of $\sigma$ at $x$ for some suitable local trivializations of $E$ and $F$. Note that this is just the Buchsbaum-Rim multiplicity of the module generated by the columns of the matrix associated to $\sigma$. Theorem 1.2 shows that in this situation if $X$ is pure dimensional, the contribution to the degeneracy locus is always the Buchsbaum-Rim multiplicity associated to $\sigma$ at $x$, the Cohen-Macaulay hypothesis is unnecessary. (Just use the proof of 1.2 to construct a rational equivalence to go back to Fulton’s case close to $x$.)

2. Hypersurface singularities with 1-dimensional singular locus.

In his thesis ([27]) Pellikaan studied non-isolated hypersurface singularities. This is the setup for his work. He assumed that $f : \mathbb{C}^{n+1} \to \mathbb{C}$, $f$ had a 1-dimensional singular locus $\Sigma$, which is a complete intersection curve defined by an ideal $I$. He assumed that $f \in I^2$. This ensured that $J(f)$, the jacobian ideal of $f$ was in $I$ as well. (In fact for the singular locus a complete intersection Pellikaan proved that if $f$ and its partials were in $I$ then $f$ was in $I^2$.) One of the key invariants of $f$ was

$$j(f) = \dim_{\mathbb{C}} \frac{I}{J(f)}$$

which plays the same role in Pellikaan’s work as the dimension of $\frac{O_{n+1}}{J(f)}$ does in the case of isolated singularities.

Two important examples of non-isolated singularities are germs of type $A_\infty$ which have the normal form $f(z_1, \ldots, z_{n+1}) = \sum_{i=1}^{n} z_i^2$ and germs of type $D_\infty$ which have normal form $f(z_1, \ldots, z_{n+1}) = z_1z_2^2 + \sum_{i=3}^{n+1} z_i^2$. Note that if $n=2$ then $D_\infty$ is just a Whitney umbrella. For $A_\infty$ germs $j(f) = 0$ while for $D_\infty$ germs $j(f) = 1$.

Using these building blocks, Pellikaan was able to give a nice geometric description of $j(f)$.

Theorem (2.1) Suppose $f$ is as above and $j(f)$ finite. Then $f$ has a deformation $F$ such that $F_y$ has $\Sigma_y$ as singular locus for generic $y$ where $\Sigma_y$ is the
Milnor fiber of $\Sigma$, with only $A_1$ singularities off $\Sigma_y$ and only $A_\infty$ singularities at points of $\Sigma_y$, except for isolated $D_\infty$ points. Moreover

$$j(f) = \# \{ D_\infty(F_y) \} + \# \{ A_1(F_y) \}.$$

**Proof.** Cf. [27] p 87 proposition 7.20.

In applying the theory of integral closure to ambient stratification conditions like $A_f$ or $W_f$ in Pellikaan’s situation, we see that there are three strata—the open stratum, $\Sigma - 0$ and the origin. So, there are two pairs of ideals $(I, O_{n+1})$ and $(J(f), I)$ that we are interested in. We wish to give a geometric interpretation of $(J(f), I)$ using Pellikaan’s theorem and the multiplicity-polar theorem. First we look at our building block germs.

**Proposition (2.2)** If $f$ is a germ of type $A_\infty$, then $e(I, J(f)) = 0$, if $f$ is a germ of type $D_\infty$, then $e(I, J(f)) = 1$.

**Proof.** If $f$ is a germ of type $A_\infty$, then $I = J(f)$, so $e(J(f), I) = 0$. So suppose $f$ is a germ of type $D_\infty$. We may assume $f$ is in normal form, as changes of coordinates do not affect the multiplicity of the pair. We have to compute a sum of intersection numbers:

$$e(J(f), I) = \sum_{j=0}^{n} \int D_{J(f), I} \cdot l_{J(f)}^{n-j} \cdot l_{I}^{j}.$$

Consider the part of the sum of form:

$$\sum_{j=1}^{n} \int D_{J(f), I} \cdot l_{J(f)}^{n-j} \cdot l_{I}^{j} = \sum_{j=0}^{n-1} \int (D_{J(f), I} \cdot l_{I}) \cdot l_{J(f)}^{n-1-j} \cdot l_{I}^{j}.$$

This is $e(J(f), I)$ where both ideals are restricted to the codimension 1 polar variety of $I$. Consider the family of candidate polar varieties defined by $z_2 = \sum_{i=3}^{n+1} a_i z_i$. Since this a Z-open subset of all potential polar varieties, if we show that for a Z-open subset of them that the multiplicity of the pair of the restriction of the ideals to each candidate in the set is zero then we will have shown that all of terms in this second sum are zero and all these candidates are actually polars. Now it is obvious from the normal form of $f$ that when we restrict our two ideals to any element of this set the two ideals become equal so all of the terms in the second sum are zero.

It remains to compute $\int D_{J(f), I} \cdot l_{J(f)}^{n}$. Our approach is to choose a Z-open set of candidate polar curves of $J(f)$, then show that each candidate gives the same value for the computation of the desired intersection number. Consider the family of curves defined by ideals $J_{a,b,c} = (b_1(z_1 z_2) + c_1 z_2^2 + \sum_{i=3}^{n+1} a_{1,i} z_i, \ldots, b_n(z_1 z_2) + \ldots$
For a $Z$-open set of coefficients, we can re-write the ideals defining these curves as

$$J_{a,b,c} = (z_1z_2 + cz_2^2, \ldots, z_i + b_iz_1z_2, \ldots)$$

where $3 \leq i \leq n + 1, c \neq 0$. Each curve in this family has two components; one of which (given by $z_2 = 0$) lies in $V(J(f))$. The other component is the candidate polar curve. So we get the family of parameterizations $\phi(t) = (-ct, t, \ldots, b_i et^2, \ldots)$ for the candidate polar curves. Now the intersection number we want is just the multiplicity of the pair restricted to a polar curve; by the additivity of the multiplicity ([20]) this is just $e(J(f)) - e(I)$ restricted to the polar curve; given a parameterization this is just the order of vanishing of $\phi^*(J(f))$ less the order of vanishing of $\phi^*(I)$. Now $\phi^*(J(f)) = (t^2)$ and $\phi^*(I) = (t)$ for all parameterizations, so the value of this intersection number is $2 - 1 = 1$, so $e(J(f), I) = 1$.

For our basic building block germs we have seen that $j(f) = e(J(f), I)$. The next theorem shows that this is true in general. If $F$ depends on coordinates $(y, z)$, let $J_z(F)$ denote the ideal generated by the partials of $F$ with respect to $z$.

**Theorem (2.3)** Suppose $f : \mathbb{C}^{n+1} \to \mathbb{C}$, $f$ has a 1-dimensional singular locus $\Sigma$, which is a complete intersection curve defined by an ideal $I$, $f \in I^2$ and $j(f)$ finite. Then

$$j(f) = \dim_{\mathbb{C}} \frac{I}{J(f)} = e(J(f), I).$$

**Proof.** Let $F$ be the deformation of Theorem 2.1. Denote the parameter space by $Y^k$. The singular set of $F$ is given by a complete intersection $\bar{I}$. We are interested in the family of pairs of ideals given by $(J_z(F), \bar{I})$ as these restrict to $(J(f), I)$ at $y = 0$. Since $\bar{I}$ defines a complete intersection it has no polar variety of dimension $k$. Since $J_z(F)$ is generated by $n + 1$ generators it has no polar of dimension $k$ either. This means that the multiplicity of the pair at the origin is same as the sum of the multiplicities over a generic parameter value by the multiplicity-polar formula. Pick a generic $y$. We have $(J_z(F))_y = J(F_y)$, so the cosupport of $(J_z(F))_y$ consists of $A_1$ points off $\Sigma_y$, isolated $D_\infty$ points on $\Sigma_y$ and $A_\infty$ points. Off $\Sigma_y$, $(\bar{I})_y = O_{n+1}$, so off $\Sigma_y$, at $A_1$ points, $e(J(F_y), \bar{I}_y) = e(J(F_y), O_{n+1}) = 1$ and 0 elsewhere off $\Sigma_y$. On $\Sigma_y$, $e(J(F_y), \bar{I}_y) = 1$ at $D_\infty$ points, otherwise it is 0 by proposition 2.2. So the sum of the $e(J(F_y), \bar{I}_y), z)$ at points where it is non-zero is just $\#D_\infty(F_y) + \#A_1(F_y)$.

Then, by the multiplicity-polar formula we know that

$$e(J(f), I) = \#D_\infty(F_y) + \#A_1(F_y)$$

which proves the theorem.

If $R$ is Cohen-Macaulay of dimension $d$, $M$ a submodule of a rank $p$ free module $F$ of finite colength, then by a theorem of Buchsbaum and Rim ([3]),
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\( e(M, F) \), which is \( e(M) \), is just the colength of \( M \) if \( M \) has \( d + p - 1 \) generators. Theorem 3 can be viewed as a first step in generalizing this result to pairs of modules.

Using some other results of Pellikaan, we can link \( e(J(f), I) \) and the Lê numbers introduced by Massey. In the situation of Theorem 2.3 there are two Lê numbers – \( \lambda^0(f) \) and \( \lambda^1(f) \); denote the number of \( D_\infty \) points of \( f \) by \( \delta(f) \).

**Proposition (2.4)** Assume the hypotheses of Theorem 2.3, then

\[
\lambda^0(f) = e(J(f), I) + e(JM(\Sigma)) + \delta(f)
\]

**Proof.** If \( F \) is the Milnor fiber of \( f \), we have that

\[
\chi(F) = 1 + (-1)^{n-1} \lambda^1(f) + (-1)^n \lambda^0(f) = 1 + (-1)^n (j(f) + \delta_f + \mu(\Sigma) - 1).
\]

The first equality is due to Massey ([24]), while the second is due to Pellikaan ([27], p113, proposition 10.11). In the present situation, since the transverse Milnor number is 1, \( \lambda^1(f) = \text{mult}(\Sigma) \), while \( e(JM(\Sigma)) = \mu(\Sigma) + \text{mult}(\Sigma) - 1 \).

Therefore, substituting and canceling we get

\[
\lambda^0(f) = e(J(f), I) + e(JM(\Sigma)) + \delta(f).
\]

Now we turn to the extension of these ideas to hypersurface singularities with a higher dimensional singular locus.

3. Hypersurface singularities with d-dimensional singular locus.

In this section we assume that \( I = (g_1, \ldots, g_p) \subset O_n \) defines a complete intersection of dimension \( d > 1 \), and \( S(f) = V(I) \), hence we can write \( f \) as \( f = \sum_{i,j} h_{i,j} g_i g_j \), where \( h_{i,j} = h_{j,i} \), for some \( h_{i,j} \). Let \([H]\) denote the symmetric matrix with entries \( h_{i,j} \). We will want to study those germs \( f \) for which \( j(f) < \infty \). Basic examples of such germs are those of type \( A(d) \). For these germs up to a change of coordinates, \( I = (z_1, \ldots, z_{n-d}) \), \( f = \sum_{i=1}^{n-d} z_i^2 \), \( z_i \) part of a coordinate system on \( \mathbb{C}^n \). It turns out that the condition that \( j(f) < \infty \) is much more restrictive than in the case where dimension of \( V(I) = 1 \). Pellikaan already showed that \( j(f) < \infty \) implies \( I \) defines an ICIS. The next proposition gives a further restriction.

**Proposition (3.1)** Suppose \( f, I \) as above, then if \([H]\) has less than maximal rank at the origin, the set of points on \( V(I) \) where the singularity type is not \( A(d) \) is of codimension 1 in \( V(I) \), hence \( j(f) \) is not finite.

**Proof.** If \( f \) has an \( A(d) \) singularity at \( x \in V(I) \) then \( V(I) \) is smooth at \( x \) and the matrix \([H(x)]\) must have rank \( n - d \). But the points where \( \det[H] = 0 \) defines a non-empty hypersurface in \( V(I) \), since \( \det[H(0)] = 0 \) and the dimension of \( V(I) > 1 \). Hence, at these points \( f \) does not have an \( A(d) \) singularity. Since at these points \( I \neq J(f) \), it follows that \( j(f) = \infty \).

There are two types of Lê cycles; those which are the images in \( \mathbb{C}^n \) of components of the exceptional divisor of the jacobian blow up, called fixed cycles, and the polar varieties of the fixed cycles called moving cycles.
**Corollary (3.2)** Suppose $f$, $I$, $[H]$ as above. Then $V(I)$ contains a fixed Lé cycle of dimension $d - 1$.

**Proof.** We can deform $f$ so that the $D^\infty$ points are dense in the zero set of $\det[H] = 0$. These points are clearly the image of a component of the exceptional divisor by Proposition 2.4, and by the properties of the Lé numbers. Then when we specialize, the component of $E$ will specialize as well.

In [27] and in [28] Pellikaan defines the singularities of type $D(d, p)$; here $d$ is the dimension of $S(f)$, while $p$ is the dimension of the kernel rank of $[H]$ at the point in question. Then $f : \mathbb{C}^n, x \to \mathbb{C}, 0$ has type $D(d, p)$ at $x$ if local coordinates can be chosen so that $f$ has the local form

$$ f = z_1^2 + \ldots + z_q^2 + \sum_{1 \leq i \leq j \leq p} x_{i,j} y_i y_j $$

where $z, x, y$ are part of a coordinate system on $\mathbb{C}^n$ at $x$, $n - d = q + p$. From 3.1 it follows that if $f$ has singularity type $D(d, p)$ at the origin, and $d > 1$, then, since $\det[H(0)] = 0$, it follows that $j(f) = \infty$, contrary to remark 5.3 of [27] and Remark 5.4 of [28]. This shows that $j(f)$ fails to be finite in what seems to be the next most simple case to the $A(d)$ singularities when $d > 1$. Instead, the structure of $S(f)$ seems more like a discriminant, in that the non-generic points appear in codimension 1.

In the next lemma we begin to look at those germs where $[H]$ has maximal rank, so we can characterize those germs where $j(f) < \infty$.

**Lemma (3.3)** Suppose $f = \sum_{i,j} h_{i,j} g_i g_j$, $\det[H(0)] \neq 0$, $I = (g_1, \ldots, g_p) \subset \mathcal{O}_n$. Then one can chose a set of generators $(g'_1, \ldots, g'_p)$ of $I$ such that $f = \sum_{i} (g'_i)^2$.

**Proof.** The proof is standard, so we just sketch the details. Given an invertible matrix $[R]$ with entries in $\mathcal{O}_n$, it is clear that if $[g] = [R][g']$, where $[g]$ is the column vector whose entries are the $g_i$, $[g']$ another column vector, that the entries of $[g']$ are also a set of generators of $I$. Given

$$ [f] = [g]^t [H] [g] $$

and

$$ [g] = [R][g'] $$

it follows that

$$ [f] = [g']^t ([R]^t [H] [R]) [g'] $$

Hence, we need to show that by choice of $[R]$ we can reduce $[H]$ to the identity matrix. This is done in two steps—first we can chose $[R] \in \text{Gl}(p, \mathbb{C})$ so that we
can assume \([H(0)] = I\). (This follows because the action of \(Gl(p, C)\) clearly preserves rank, the orbits of \(Gl(p, C)\) are connected constructible sets, and the orbits of non-singular matrices are open, by a tangent space calculation.)

For the second step we assume \([H(0)] = I\), consider the linear homotopy from \(I\) to \([H]\); this stays inside the set of invertible symmetric matrices. The congruence transformation gives an action of the group \(C\) of invertible \(p \times p\) matrices with entries in \(O_n\) on the \(p \times p\) symmetric matrices. Applying the techniques of Mather-Damon produces a homotopy in \(C\) which trivializes our linear homotopy, which finishes the proof.

Lemma 3.3 also appears as a remark without proof in [33] (see page 87).

Given a set of generators \(\{g_1, \ldots, g_p\}\) for an ideal, we can form the function \(G\) whose components are the \(g_i\). If \(\{g_1, \ldots, g_p\}\) define an ICIS, then the map \(G\) is said to be of finite singularity type.

**Corollary (3.4)** Suppose \(f = \sum_{i,j} h_{i,j}g_ig_j\), \(\det[H(0)] \neq 0\), \(I = (g_1, \ldots, g_p) \subset O_n\), \(\{z_i\}\) coordinates on \(C^p\) then generators \((g'_1, \ldots, g'_p)\) of \(I\) can be chosen so that

\[
f = \sum_{i=1}^{p} z_i^2 \circ G'\]

**Proof.** By lemma 3.3 we have there exists generators \((g'_1, \ldots, g'_p)\) of \(I\) such that

\[
f = \sum_{i=1}^{p} (g'_i)^2 = \sum_{i=1}^{p} z_i^2 \circ G'.\]

Thus the study of functions with \(j(f) < \infty\) is intimately tied up with the study of functions on the discriminant of a map germ of finite singularity type as we shall see below.

We wish to describe a condition which will ensure that the pullback by \(G\) of a function on \(C^p\) with a Morse singularity at the origin gives a function on \(C^n\) with \(j(f) < \infty\) for the ideal defined by the components of \(G\). This completes our geometric description of the meaning of \(j(f)\) finite.

Our condition is based on the intersection of the levels of the Morse function in the target with the discriminant, \(\Delta(G)\), of \(G\). At this point we assume that \(I\) defines an ICIS. This implies that if \(G\) comes from a minimal set of generators of \(I\), then \(G|S(G)\) is a finite map.

We can partition \(S(G)\) by the \(S_i(G)\) which denotes points of \(S(G)\) where the kernel rank of \(G\) is \(i\). We can also partition \(\Delta(G)\) as follows. For each point \(z\) of \(\Delta(G)\), list the points \(S_z\) of \(S(G)\) mapped to \(z\). The points \(z\) and \(z'\) are in the same element of the partition if there is a bijection between \(S_z\) and \(S_{z'}\) which preserves components of the \(S_i(G)\). It is easy to see that the elements of this partition are constructible sets since \(G|S(G)\) is finite. Given an element of the partition of \(\Delta(G)\), we now associate a collection of systems of linear subspaces of \(TC^p\) over the underlying set \(P\) of the partition element. Since \(G\) has constant rank on each \(S_i(G)\), \(D(G)|_{S_i(G)\cap G^{-1}(P)}(TC^n|_{S_i(G)\cap G^{-1}(P)})\) is a well defined sub bundle of \(G^*TC^p\) over \(S_i(G)\cap G^{-1}(P)\). Since the restriction of \(G\) to
each component of $G^{-1}(P)$ is a homeomorphism or finite cover, the push forward by $G$ of these sub bundles gives the desired collection of systems of linear spaces. We call the partition of $\Delta(G)$ together with the collection of linear spaces on each element of the partition an enriched partition. A smooth subset $V$ of $C^p$ is enriched transverse to the enriched partition if at every point of intersection with the elements of the partition the tangent space of $V$ is transverse to each of the linear spaces we have associated to the element of the partition at that point. Since the restriction of $G$ to each component of $G^{-1}(P)$ is a homeomorphism or finite cover, all of the linear spaces at a smooth point in a partition element contain the tangent space to the partition element. So if $V$ is transverse to each element of the partition it is enriched transverse. The next proposition describes a situation in which transversality and enriched transversality are equivalent.

**Proposition (3.5)** Suppose there exists an element $P$ of the partition which is a Z-open subset of $\Delta(G)$ whose pre-images lie in the Z-open subset $S_{n-p+1}(G)$ on which $G$ is immersive. Then all of the systems of linear spaces associated to $P$ are just the tangent bundle to $P$.

**Proof.** Suppose $y \in P$, $z$ a preimage in $S_{n-p+1}(G)$. Since $G$ restricted to $S_{n-p+1}(G)$ is immersive at $z$, the dimension of $DG(TS_{n-p+1}(G))$ is $p - 1$ which is the dimension of $D(G)(z)TC^n$, so these spaces are equal; further $DG(TS_{n-p+1}(G))$ is the tangent space to $\Delta(G)$ at $y$, which is the tangent space to $P$ at $y$.

Now we give our condition for $j(f)$ finite.

**Theorem (3.6)** Suppose $I = (g_1, \ldots, g_p) \subset O_n$ defines an ICIS of dimension $d > 1$, $G$ the map germ whose components are the $g_i$, $h : C^{p}, 0 \to C, 0$ a function with an isolated singularity at the origin, $f = h \circ G$. Then

$$j(f, G^*(J(h))O_n) := \dim \frac{G^*(J(h))O_n}{J(f)} < \infty$$

if and only if $h^{-1}(0)$ is enriched transverse to the enriched partition of $\Delta(G)$ except possibly at the origin.

**Proof.** Suppose $j(f, G^*(J(h))O_n)$ finite. Then, except possibly at the origin, $J(f) = G^*(J(h))O_n$. If the enriched transversality condition fails, there must be a curve $\phi : C \to \Delta G$, such that the image of $\phi$ lies in an element of the partition, and the tangent space to $h^{-1}(0)$ contains one of the systems of linear spaces along the partition element. This implies that contained system is in the kernel of $Dh$ along $\phi$. Then $\phi$ has a lift to the component of $S_i(G)$ associated to the contained system, denoted $\psi$. Along the image of $\psi$ we have

$$Df \circ \psi(TC^n) = Dh \circ (G \circ \psi)DG \circ \psi(TC^n) = 0$$

Hence $V(J(f)) \supset \text{im}\psi$, while $V(G^*(J(h))O_n) = V(I)$ which is a contradiction.

Suppose enriched transversality holds. If $j(f, G^*(J(h))O_n)$ is not finite, there exists a curve $\psi$ whose image properly contains the origin in $C^n$, such that
\[ J(f) \neq G^*(J(h))\mathcal{O}_n \text{ along } \psi. \] At points of \( \mathbb{C}^n \) off \( S(G) \), \( G \) is a submersion, hence \( J(f) = G^*(J(h))\mathcal{O}_n \). If \( \psi \) lies in \( S(G) \), then the image of \( \psi \) lies in \( S(f) \) since \( G^*(J(h))\mathcal{O}_n = \mathcal{O}_n \) at such points. Then \( \psi \) lies in the zero set of \( F \), hence \( G \circ \psi \) lies in the zero set of \( h \). Then enriched transversality fails along \( G \circ \psi \).

**Corollary (3.7)** Suppose \( h \) has a Morse singularity at the origin in the set-up of Theorem 3.6, then \( j(f) \) is finite if and only if \( h^{-1}(0) \) is enriched transverse to the enriched partition of \( \Delta(G) \) except possibly at the origin.

**Proof.** If \( h \) has a Morse singularity, then \( G^*(J(h))\mathcal{O}_n = I \).

**Corollary (3.8)** Suppose \( I = (g_1, g_2) \) in the setup of Theorem 3.6. Then \( j(f, G^*(J(h))\mathcal{O}_n) \) is finite iff \( f^{-1}(0) \cap S(G) \) is the origin.

**Proof.** If \( p = 2 \), then \( \Delta(G) \) is a curve, and \( G \) restricted to each branch of \( S(G) \) is an immersion except at the origin. Then enriched transversality becomes ordinary transversality, so \( h^{-1}(0) \) must miss \( \Delta(G) \) off the origin, so \( f^{-1}(0) \cap S(G) \) is the origin.

Theorem 3.6 introduces an interesting class of functions. Given an ICIS, by using appropriate \( h \) we can construct examples of non-isolated singularities in which the singular locus is the ICIS, but the transverse singularity type is constant and is that of \( h \). In studying the equisingularity of families of such examples, the key invariant is the multiplicity of the pair \( J(f), G^*(J(h))\mathcal{O}_n \). This number should also be linked to the way \( h^{-1}(0) \) meets the discriminant of \( G \) at the origin.

Now we show that such functions with \( j(f) \) finite are plentiful.

**Proposition (3.9)** Suppose \( G : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0, G^{-1}(0) \) an ICIS, \( p > 1 \). Then if \( h_a(x) = \sum a_i x_i^2 \), for \( a \in U \), \( U \) a Z-open subset of \( \mathbb{C}^p \), \( h^{-1}(0) \) is transverse to the enriched partition of \( \Delta(G) \) except perhaps at the origin.

**Proof.** Consider \( H(a, z) = \sum a_i x_i^2 \circ G(z) \). We have

\[ DH = \langle \ldots, x_i^2 \circ G, \ldots, 2a_i x_i \circ G, \ldots \rangle \]

This implies that \( H \) is a submersion except along \( \mathbb{C}^p \times G^{-1}(0) \). Denote by \( \pi \) the projection of \( H^{-1}(0) \) to \( \mathbb{C}^p \). By Sard’s lemma for varieties (prop. 3.7 p42 [25]) there exists a Z-open subset \( U \subset \mathbb{C}^p \) such that \( \pi \) is smooth at \( z \in H^{-1}(0) \cap \pi^{-1}(U) / \mathbb{C}^p \times G^{-1}(0) \). This implies that the fiber of \( \pi \), which is the fiber of \( h_a \circ G \) over \( 0 \) is smooth at \( z \); in addition since \( \pi \) maps \( T_z(H^{-1}(0)) = \ker DH_z \) surjectively to \( \mathbb{C}^p \), the \( \ker DH_z \) does not contain \( \mathbb{C}^n \), thus \( h_a \circ G \) is a submersion at \( z \) as well, hence \( h_a \) is enriched transverse to the enriched partition of \( \Delta(G) \), except perhaps at the origin.

Now that we know that it is worth proving results about functions with \( j(f) \) finite for \( V(I) \) an ICIS of dimension \( > 1 \), we prove the analogue of 2.3. To do this we first study a special deformation of \( f = \sum z_i^2 \).

We call the following pair of deformations a smoothing of \( f \).
\[ F(u, b, z) = \sum_i (1 + \sum_j b_{i,j}z_j)(g_i - u_i)^2 \]

\[ \tilde{G}(u, z) = (g_1(z) - u_1, \ldots, g_p(z) - u_p) \]

This is called a smoothing because of the following lemma:

**Lemma (3.10)** For a Z-open subset \( U \) of \( \mathbb{C}^p \times \mathbb{C}^m \), \( f_{u,b} \) has only \( A_1 \) singularities off \( G^{-1}(u) \), \( G^{-1}(u) \) is smooth and \( f_{u,b} \) has only \( A(d) \) singularities on \( G^{-1}(u) \).

**Proof.** Let \( V \subset \mathbb{C}^p \) be the complement of \( \Delta(G) \) in \( \mathbb{C}^p \), then \( G^{-1}(u) \) is smooth for \( u \in V \).

We claim \( D_z F(u, b, z) \) is a submersion off \( \mathbb{C}^m \times \Gamma(G) \), where \( \Gamma(G) \subset \mathbb{C}^n \times \mathbb{C}^p \) denotes the graph of \( G \).

Let \( e_i \), where \( 1 \leq i \leq n \) denote the unit vectors in \( \mathbb{C}^n \). Then we have

\[ \frac{\partial D_z(F)}{\partial b_{i,j}} = (g_i - u_i)^2 e_j. \]

This implies \( D_z F(u, b, z) \) has maximal rank when some \( g_i - u_i \) is not zero which proves the claim.

Now consider \( D_z F(u, b, z)^{-1}(0) \). The claim shows this is smooth off \( \mathbb{C}^m \times \Gamma(G) \).

As in the proof of 3.9 we consider the projection of this set to \( \mathbb{C}^p \times \mathbb{C}^m \), let \( W \) be the Z-open subset of the base over which \( \pi \) is smooth off \( \mathbb{C}^m \times \Gamma(G) \). Now the tangent space to \( D_z F(u, b, z)^{-1}(0) \) at a point \( x \) is just the kernel at \( x \) of \( D(D_z F(u, b, z)) \), which has dimension \( p + pn \) and which surjects to \( \mathbb{C}^p \times \mathbb{C}^m \). Hence \( D_z^2 F(u, b, z) \) has maximal rank, so \( f_{u,b} \) has only Morse singularities off \( G = u \). Let \( U = W \cap \mathbb{C}^m \times V \), then for \( (u, b) \in U \), we have \( g_u \) has a smooth fiber over zero. Since the set of points where the matrix \( H \) with entries \( h_{i,i} = \sum_j 1+b_{i,j}z_j \), \( h_{i,j} = 0 i \neq j \) has maximal rank on some Z-open subset of \( \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p \) which contains zero, we can ensure that each of the \( f_{u,b} \) has only \( A(d) \) singularities on some fixed neighborhood of the origin in \( \mathbb{C}^n \) on \( g_u = 0 \).

**Remark** It was pointed out to the author by the referee that this lemma also follows from the statement and proof of theorem 1 of Now we extend Theorem 2.3 to ICIS of dimension greater than 1.

**Theorem (3.11)** Suppose \( f : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \), \( f \) has a d-dimensional singular locus \( \Sigma \), \( d > 1 \), which is an ICIS defined by an ideal \( I \), \( f \in I^2 \) and \( j(f) \) finite. Then

\[ j(f) = \dim \mathbb{C} \frac{I}{J(f)} = e(J(f), I) = \#A_1(f). \]

where \( \#A_1(f) \) is the number of \( A_1 \) singularities appearing in a smoothing of \( f \).

**Proof.** The proof is similar to that of 2.3. By [27] Theorem 3.1 p145 the quotient of the ideals \((g_1(z) - u_1, \ldots, g_p(z) - u_p)/J_z(F)\) is perfect, where \( F \) is part
of a smoothing of \( f \), hence the length of the quotients \( (g_u)/J(f_{u,b}) \) is independent of parameter, and for generic parameter value is just \( \#A_1(f) \). Meanwhile, \( J_z(F) \) and \( (g_1(z) - u_1, \ldots, g_p(z) - u_p) \) have no polar varieties of dimension \( p + p(n + 1) \), so as in Theorem 2.3, the multiplicity polar theorem implies that \( e(J(f_{u,b}, g_u)) \) is independent of parameter, so again is \( \#A_1(f) \), hence the theorem follows.

Now we wish to extend proposition 2.4 to ICIS of dimension greater than 1.

In [33], prop 5.5.5, p86, (cf. also [26]), Zaharia computed the homology of the Milnor fiber, \( \hat{f} \), of a function germ \( f \) defined on \( \mathbb{C}^{n+1} \) whose singular set \( \Sigma \) was an ICIS of codimension \( p \) such that \( j(f) < \infty \). His result was:

\[
H_*(\hat{f}) = \begin{cases} 
Z, & \text{if } * = 0, p - 1 \\
Z^{u \Sigma + \sigma} & \text{if } * = n \\
0, & \text{otherwise}
\end{cases}
\]

Here \( \sigma \) is the number of \( A_1 \) points appearing in a smoothing, which we have shown is \( j(f) \).

**Proposition (3.12)** Assume the hypotheses of Theorem 3.11, then

\[
\lambda^0(f) = e(J(f), I) + e(JM(\Sigma))
\]

**Proof.** By Massey ([24]) we have that

\[
\chi(\hat{f}) = 1 + \sum_{i=0}^{d} (-1)^{n-i} \lambda^i(f)
\]

\[
= 1 + (-1)^n \lambda^0(f) + \sum_{i=1}^{d} (-1)^{n-i} \lambda^i(f)
\]

Now, for \( i > 0 \), since \( \Sigma \) is the only fixed Lé cycle of dimension greater than 0, and \( f \) has transverse Milnor number 1, since the type of \( f \) is \( A(d) \) generically on \( \Sigma \),

\[
\lambda^i(f) = m_{d-i}(\Sigma),
\]

where \( m_{d-i}(\Sigma) \) is the \( d - i \) polar multiplicity of the ICIS \( \Sigma \). In turn, \( m_{d-i}(\Sigma) = \mu^{d-i}(\Sigma) + \mu^{d-1}(\Sigma) \) ([9]) where \( \mu^{d-i}(\Sigma) \) is the Milnor number of \( \Sigma \cap H_i \) where \( H_i \) is a generic plane of codimension \( i \), and where \( \mu^{-1} = 1 \).

Substituting, the sum telescopes to:

\[
\chi(\hat{f}) = 1 + (-1)^n \lambda^0(f) + (-1)^{n-d} + (-1)^{n-1} \mu^{d-1}(\Sigma).
\]

Calculating \( \chi(\hat{f}) \) from the homology calculation of [33] we get:

\[
1 + (-1)^n \lambda^0(f) + (-1)^{n-d} + (-1)^{n-1} \mu^{d-1}(\Sigma) = 1 + (-1)^{n-d} + (-1)^n(\mu_\Sigma + \sigma),
\]

Hence
\[ \lambda^0(f) = \sigma + \mu_\Sigma + \mu(\Sigma \cap H_1) \]

\[= e(J(f), I) + e(JM(\Sigma)).\]

**Remark (3.13)** There are two other general calculations of the homolgy of the Milnor fiber in [33] (Theorem 5.5.4 and Proposition 5.5.6). (Note, however the typo in the formula of 5.5.4—the coefficients of \(\mu_\Delta\) and \(\mu_\Sigma\) should be exchanged.) Using these calculations, it is possible to prove by the same methods as 3.12, two other formulas for \(\lambda^0(f)\).

In the first case, assume \(V(I) = \Sigma\) is an ICIS of dimension 2, write \([f] = [g]^t[H][g]\) as we did earlier, let \(H\) denote the ideal generated by \(I^2\) and the entries of \([H][g]\), assume \(\text{dim}_C H/J(f)\) is finite, \(V(\det[H]) \cap \Sigma = \Delta\), where \(\Delta\) is an ICIS of dimension 1. We can consider the smoothing used by Zaharia to study this situation, and the ideal \(H\) extends to \(\tilde{H}\) in a natural way, to the space of the smoothing. Then the polar of \(\tilde{H}\) may be non-empty if the kernel rank of \([H]\) is > 2. Call the multiplicity of the polar of \(\tilde{H}\) over the base \(m(\Gamma(\tilde{H}))\). Then the multiplicity polar theorem applied to the smoothing gives

\[e(J(f), H) + m(\Gamma(\tilde{H})) = \#(A_1(f))\]

and hence,

\[\lambda^0(f) = e(J(f), H) + m(\Gamma(\tilde{H})) + e(JM(\Sigma)) + 2e(JM(\Delta)).\]

In the second case, assume \(V(I)\) has dimension \(d > 1\), assume the rank of \([H(0)]\) is \(p - 1\) (one less than maximal). Then, as Zaharia remarks ([33] p. 87), generators \((g_1, \ldots, g_p)\) for \(I\) can be found so that \(f = \det([H])[g_1^2 + g_2^2 + \cdots + g_p^2].\) Then the ideal \(H\) of the last paragraph is just \((\det([H])[g_1, g_2, \ldots, g_p]).\) Since \(H\) has only \(p + 1\) generators as does \(\tilde{H}\) the polar of \(\tilde{H}\) is empty and

\[\lambda^0(f) = e(J(f), H) + e(JM(\Sigma)) + 2e(JM(\Delta)).\]

The form of these formulae makes it likely that they are special cases of a more general theorem.

It has long been known that in cases like those considered here, that the independence from parameter of the Lê numbers implies that the families \(\Sigma(t)\) and \(\Delta(t)\) are Whitney equisingular (See for example [13] prop 4.6, for the case where \(I = J(f)\), and use the fact that the components of the exceptional divisor of the blowup of \(C^{n+1}\) by \(J(f)\) which project to \(\Sigma\) and \(\Delta\) are the conormals of \(\Sigma\) and \(\Delta\). ) Thus, a relation between the Lê numbers and the invariants used to control the Whitney equisingularity of \(\Sigma\) and \(\Delta\) is not unexpected. That the formulae relate \(\lambda_0\) so simply to the zero dimensional invariants of the strata and to the \(A_f\) invariant is surprising.

Now we develop some results which shows how well \(e(J(f), I)\) is linked to the \(A_f\) and \(W_f\) conditions.
4. Conditions $A_f$ and $W_f$

In this section, we’ll study Thom’s Condition $A_f$, and Henry, Merle and Sabbah’s Condition $W_f$, which concern limiting tangent hyperplanes at a singular point of a complex analytic space. First we recall the notions of integral dependence and strict dependence.

Let $(X, 0)$ be the germ of a complex analytic space, and $E := \mathcal{O}_X^k$ a free module of rank $p$ at least 1. Let $M$ be a coherent submodule of $E$, and $h$ a section of $E$. Given a map of germs $\varphi : (C, 0) \to (X, 0)$, denote by $h \circ \varphi$ the induced section of the pullback $\varphi^* E$, or $\mathcal{O}_{C}^p$, and by $M \circ \varphi$ the induced submodule. Call $h$ integrally dependent (resp., strictly dependent) on $M$ at 0 if, for every $\varphi$, the section $h \circ \varphi$ of $\varphi^* E$ is a section of $M \circ \varphi$ (resp., of $m_1(M \circ \varphi)$, where $m_1$ is the maximal ideal of 0 in $C$). The submodule of $E$ generated by all such $h$ will be denoted by $\overline{M}$, resp., by $\overline{M}^\dagger$.

In the context of hypersurface singularities, given a family of map-germs $F(y, z)$ parametrised by $Y = C^n$, where $F : C^k \times C^{n+1}, C^k \times 0, 0 \to C, 0, 0$ Thom’s $A_f$ condition holds for the pair $(C^k \times C^{n+1} - S(F), C^k \times 0)$ at $y$ if and only if every limit of tangent hyperplanes to the fibers of $F$ on $C^k \times C^{n+1} - S(F)$ contains $TY$ at $y$. The condition holds for the pair if it holds for the pair at every $y$. Although this condition looks like it says nothing about strata other than the open stratum, this can be deceiving, as we shall see.

**Proposition (4.1)** Suppose $F : C^k \times C^{n+1}, C^k \times 0, 0 \to C, 0, 0$ then the following are equivalent:

1) The $A_F$ condition holds for the pair $(C^k \times C^{n+1} - S(F), C^k \times 0)$ at 0.

2) The fiber over 0 of the exceptional divisor $E$ of the blowup of $C^k \times C^{n+1}$ by $J(F)$, denoted $B_{J(F)}(C^k \times C^{n+1})$ is contained in $C(Y)$, the conormal of $Y$.

3) $\frac{\partial E}{\partial y_i} \in J(F)^\dagger$ for $1 \leq i \leq k$.

4) $\frac{\partial E}{\partial y_i} \in J_z(F)^\dagger$ for $1 \leq i \leq k$.

**Proof.** The fiber over 0 of the exceptional divisor $E$ of $B_{J(F)}(C^k \times C^{n+1})$ is exactly the set of limiting tangent hyperplanes at 0 to the fibers of $F$ on $C^k \times C^{n+1} - S(F)$; saying that this fiber lies in the conormal of $Y$ just says that each limit contains the tangent space to $Y$ at 0. This shows 1) and 2) are equivalent. The equivalences of 1) and 3) and 4) can be found in [14].

The $W_F$ condition holds for the pair $(C^k \times C^{n+1} - S(F), C^k \times 0)$ at 0 if there exist a (Euclidean) neighborhood $U$ of 0 in $C^k \times C^{n+1}$ and a constant $C > 0$ such that, for all $y$ in $U \cap Y$ and all $x$ in $U \cap (C^k \times C^{n+1} - S(F))$, we have

$$\text{dist}(T_y Y(F(y)), T_x(C^k \times C^{n+1})(F(x))) \leq C \text{ dist}(x, Y)$$

where $T_y Y(F(y))$ and $T_x(C^k \times C^{n+1})(F(x))$ are the tangent spaces to the indicated fibers of $F$ and the restriction $F|Y$.

**Proposition (4.2)** Suppose $F : C^k \times C^{n+1}, C^k \times 0, 0 \to C, 0, 0$ then the following are equivalent:
1) The WP condition holds for the pair \((C^k \times C^{n+1} - S(F), C^k \times 0)\) at 0.

2) \(\frac{\partial F}{\partial y_i} \in \text{m}_y J(F)\) for \(1 \leq i \leq k\).

3) \(\frac{\partial F}{\partial y_i} \in \text{m}_y J_2(F)\) for \(1 \leq i \leq k\).

**Proof.** This follows from proposition 1.1 of [15]

Now we want to look at the connection between the multiplicity of the pair, \(e(J(f), I)\), and the \(A_F\) condition. At this point we no longer assume that \(I\) defines a curve singularity. We do need two simple lemmas first.

**Lemma (4.3)** Suppose \(I\) is an ideal generated by \(d\) elements in an equidimensional local ring \(R\) of dimension \(n\) such that \(R/I\) has dimension \(n - d\). Suppose \(J \subset I\) is a reduction of \(I\). Then \(J = I\).

**Proof.** The proof is by induction on \(d\). Assume \(d = 1\), denote the generator of \(I\) by \(p_1\). Let \(J = (f_1p_1, \ldots, f_kp_1)\). If some \(f_i\) is a unit, then we are done. Suppose no \(f_i\) is, and denote the ideal they generate by \(F\). If \(p_1\) satisfies a relation of integral dependence, then we get

\[
(p_1)^k + \sum_{i=0}^{k-1} g_i p_1^i = 0
\]

where \(g_i \in J^{k-i}\). Then \(g_i \in F^{k-i}(p^{k-i})\), so the equation of integral dependence implies that there exists a unit \(u\) such that \(u \cdot p^k = 0\) which is a contradiction.

Assume \(I\) is generated by \(d\) elements; work on \(R' = R/(p_1)\), then applying the induction hypothesis to the homomorphic images of \(J\) and \(I\) in \(R'\) we have that these images are equal, hence \(p_i = g_i + r_i p_1\) where \(g_i \in J\). Notice that \(\{p_1, p_2 - r_2 p_1, p_3, \ldots, p_d\}\) is a set of generators for \(I\). Now mod out by \(p_2 - r_2 p_1 = g_i\), and again apply the induction hypothesis. This shows that \(\{p_1, p_3, \ldots, p_d\}\) are in \(J\) hence \(I\) is in \(J\) since the missing generator of \(I\) is already in \(J\).

Note that it is not necessary for \(I\) to be radical.

We say that \(f : C^{n+1}, x \to C, 0\) has singularity of type \(A(d)\) at \(x\), if local coordinates \((z_1, \ldots, z_d, w_1, \ldots, w_r)\) can be found such that

\[
f(z, w) = w_1^2 + \ldots + w_r^2.
\]

If \(f\) has singularity of type \(A(d)\) at \(x\) then \(S(f) = V(w_1, \ldots, w_r) = J(f)\) so \(j(f) = 0\). There is a partial converse.

**Lemma (4.4)** Suppose \(f : C^{n+1}, 0 \to C, 0\). Suppose \(I\) defines a complete intersection \(\Sigma^d\) at 0 with reduced structure, and suppose \(j(f) = 0\). Then \(f\) has a singularity of type \(A(d)\) at 0.

**Proof.** If \(d = 0\) the hypothesis implies that \(J(f) = m_{n+1}\), and the result is implied by the Morse lemma. Suppose \(d > 0\), then Theorem 5.14 p59 of [27] implies that \(\Sigma\) is an ICIS, and \(f\) is \(A(d)\) except perhaps at 0. Further, the formula of 5.14 implies that the Tjurina number of \(\Sigma\) is 0, hence \(\Sigma\) is smooth at the origin. Then proposition 3.13 p35 of [27], the formula cited above, and remark 5.3 on p52 imply that \(f\) is \(A(d)\) at the origin as well.

Now we are ready to prove our result about \(A_f\).
Theorem (4.5) Suppose $F : \mathbb{C}^k \times \mathbb{C}^{n+1}, \mathbb{C}^k \times 0, 0 \to \mathbb{C}, 0, 0$, suppose the singular set of $F$, $S(F)$ is $V(I)$ where $I$ defines a family of complete intersections with isolated singularities of fiber dimension $d$, and every component of $V(I)$ contains $Y = \mathbb{C}^k \times 0$. Suppose further that $J(F) = I \text{ off } Y$. Then:

1) If the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), \mathbb{C}^k \times 0)$ satisfies the $A_F$ condition then $e(J(f_y), I_y, (y, 0))$ is independent of $y$.

2) If $e(J(f_y), I_y, (y, 0))$ is independent of $y$, then $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), V(I) - Y, Y)$ has the $A_F$ property.

Proof. To start the proof of 1), assume the $A_F$ condition; this implies that

$$\frac{\partial F}{\partial y_i} \in J_z(F)^d$$

for $1 \leq i \leq k$, by proposition 3.1. Now

$$e(J(F)(y), I(y), (y, z)) = e(J_z(F)(y), I(y), (y, z)) = e(J(f_y), I(y), (y, z))$$

for all $(y, z)$ in some neighborhood of $(0, 0)$. Since $J(F) = I \text{ off } Y$, this implies $e(J(f_y), I(y), (y, z)) = 0 \text{ off } Y$.

Since $\Gamma^k(I) = \Gamma^k(J_z(F)) = \emptyset$, by the multiplicity-polar theorem,

$$e(J(f_0), I(0), (0, 0)) = e(J(f_y), I(y), (y, 0))$$

for all $y$.

Now we prove 2). By hypothesis we have $I = J(F) \text{ off } Y$. So by lemma 3.4 off of $Y$ we have that $V(I)$ is smooth and $F$ has only $A(k + d)$ singularities. So the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), V(I) - Y)$ has the $A_F$ property.

Since $e(J(f_y), I_y, (y, 0))$ is independent of $y$, and $I$ and $J_z(F)$ have no polars of dimension $k$, it then follows from the multiplicity-polar theorem that $e(J(f_y), I_y, (y, z)) = 0$, for $z \neq 0$. This implies that $J(f_y) = J_y$. By lemma 3.3, $J(f_y) = I_y$. In turn this implies by lemma 3.5 that $V(I_y)$ is smooth off the origin and $f$ has an $A(d)$ singularity at points of $V(I_y)$ off the origin. Now we have that $J_z(F) \subset I$ and at a point $(y, z)$ of $\Sigma$ off $Y$,

$$\dim_{\mathbb{C}} J(f_y, z)/(J(f_y, z) \cap m_{y,z}^2) = n + 1 - d \leq \dim J_z(F)/(J_z(F) \cap m_{y,z}^2) \leq \dim I/(I \cap m_{y,z}^2) = n + 1 - d$$

Hence $J_z(F) = I$ at points of $\Sigma$ off $Y$.

Using what we have learned about $F$ above, we can describe the components of the exceptional divisor $E$ of $(B_{J_z(F)}(\mathbb{C}^k \times \mathbb{C}^{n+1}), \pi)$; we do this in order to get ready to apply 2) of 3.1, which will finish the proof.

Let $\Sigma_i$ be the $i$th component of $\Sigma$; then there exists a component $V_i$ of $E$ which surjects to $\Sigma_i$. Suppose $V$ is a component of $E$ such that $\pi(V)$ is not contained in $Y$. Let $x$ be a point off $Y$ in $\pi(V)$. Then there is a neighborhood $U$ of $x$ in $\mathbb{C}^k \times \mathbb{C}^{n+1}$ such that on $U$, $J(F) = J_z(F) = I$, and only one component of $\Sigma$ intersects $U$. Hence over $U$ the corresponding blowups are isomorphic; in particular there is only one component of each exceptional divisor which projects to $\Sigma \cap U$. So the $V_i$ are the only components of $E$ whose image does not lie in $Y$.

Suppose $W$ is a component of $E$ whose image lies in $Y$. Then $W^{n+k} \subset Y^k \times \mathbb{P}^n$, hence $W = Y^k \times \mathbb{P}^n$ if $W$ exists. We have shown that every component of $E$
projects to a set which contains \( Y \) in its closure. (This uses the fact that every \( \Sigma_i \) contains \( Y \) in its closure.)

Since \( A_F \) is true generically, there exists a \( Z \)-open set \( U \) which contains a \( Z \)-open subset of \( Y \), and on \( U \) we have \( \frac{\partial E}{\partial z_i} \in J_z(F)^\dagger \) for \( 1 \leq i \leq k \). This implies that if we pull back \( J_z(F) \) and \( J(F) \) to the normalization of \( B_{J_z(F)}(C^k \times C^{n+1}) \), then along every component of the exceptional divisor \( E_N \) which meets \( \pi_N^{-1}(U) \) in a \( Z \)-open set, that \( \pi_N^*(J(F)) = \pi_N^*(J_z(F)) \). But this is true for all components of \( E_N \), since every component of \( E \) of \( B_{J_z(F)}(C^k \times C^{n+1}) \) projects to a set which contains \( Y \) in its closure. This implies that \( J_z(F) = J(F) \) at all points of \( Y \) ([23]).

The last equality implies that \( E_J \), the exceptional divisor of \( B_{J(F)}(C^k \times C^{n+1}) \), is finite over \( E \). The components of \( E_J \) which are in \( \pi_N^{-1}(Y) \), have dimension \( k + n \) and have fiber dimension \( n \), which is the fiber dimension of \( W \), since they are finite over \( W \). Hence they surject onto \( W \), and hence \( Y \). Since \( A_F \) holds generically, these components are in \( C(Y) \), the conormal of \( Y \), which also has dimension \( n + k \), hence they are equal to the conormal, so there is only 1 such component.

Over each \( V_i \) as we have seen there is only one component of \( E_J \); since \( A_F \) holds between the open stratum and these components, a dimension count shows that this unique component is \( C(\Sigma_i) \). The proof will be complete if we can show that each component of \( \Sigma \) satisfies Whitney A over \( Y \). (This is also what it means for \( A_f \) to hold for the pair \( (\Sigma, Y) \).)

Claim: For every \( i \), \( C(\Sigma_i) \cap \pi_N^{-1}(Y \cap U) \) is dense in \( C(\Sigma_i) \cap \pi_N^{-1}(Y) \).

Since \( C(\Sigma_i) \cap \pi_N^{-1}(Y \cap U) \) lies in \( C(Y) \) this will finish the proof by 2) of 3.1.

By Lemma 5.7 p230 of [16], we know that each component of \( C(\Sigma_i) \cap \pi_N^{-1}(Y) \) has dimension \( n + k - 1 \), that is, must be a hypersurface in \( C(\Sigma_i) \). (This uses the fact that \( I \) defines a complete intersection.) If the claim fails there must be a component for some \( i \) of \( C(\Sigma_i) \cap \pi_N^{-1}(Y) \) which does not surject onto \( Y \). Since \( E_N|Y \) is finite over \( E|Y \subset Y \times \mathbb{P}^n \), this component must map to a subset of \( Y \) of dimension \( k - 1 \), and must have constant fiber dimension \( n \).

Let \( C \) be the fiber of the bad component over 0. Consider \( B_{J(F)(0)}(0 \times C^{n+1}) \). This must contain \( C \) as a component of its exceptional divisor, as \( C \) is a subset of \( B_{J(F)}(C_k \times C^{n+1}) \cap 0 \times C^{n+1} \times \mathbb{P}^{n+k} \), and its dimension is too small to be a component of the intersection. Construct a polar variety of \( J(F) \) of dimension \( k + 1 \). This is a family of curves over \( Y \); the fiber over 0 contains a curve which is the projection of the intersection of the plane defining the polar with \( B_{J(F)(0)}(0 \times C^{n+1}) \) forced by the existence of \( C \). Let \( \Gamma \) be the component of our polar which contains this curve.

We choose the plane of codimension \( n \) of \( \mathbb{P}^{n+k} \) so that it misses the points of \( C \cap C(Y) \). On some sufficiently small metric neighborhood of the origin in \( \Gamma \), then we know that \( \Gamma \) intersects \( Y \) only at \( (0, 0) \). Restrict \( I \) and \( J(F) \) to \( \Gamma \). Now we apply the multiplicity-polynomial theorem again. \( J(F) \) has no polar, because it is integrally dependent on \( J_z(F) \) which has no polar. Over a generic \( y \) value, the
only points where $J(F)$ has support are on $\Sigma - Y$ hence $e(J(F)(y), I_y) = 0$ at such points. We claim that the multiplicity of the pair $(J(F)(0), I_0)$ on $\Gamma(0)$ at $(0, 0)$ is not zero. This number has an alternate meaning. It is part of the intersection number $\int D_{J(F)(0), I(0)} \cdot l^n_{J(F)(0)}$, which in turn is part of $e((J(F)(0), I_0), (0, 0))$ on $C^{n+1}$. We know that $B_{\rho(J(F)(0))}(\text{Proj} \mathcal{R}(I_0))$, dominates both $B_{I_0}(C^{n+1})$ and $B_{J(F)(0)}(C^{n+1})$; corresponding to $C$ there is a component of the exceptional divisor of $B_{J(F)(0)}(\text{Proj} \mathcal{R}(I_0))$. The map to $B_{I_0}(C^{n+1})$ cannot be finite on this component, because the component projects to the origin in $C^{n+1}$, and the fiber dimension of the exceptional divisor of $B_{I_0}(C^{n+1})$ over the origin must have dimension less than $n - d < n$, hence this component over $C$ makes a non-zero contribution to $\int D_{J(F)(0), I(0)} \cdot l^n_{J(F)(0)}$, so the multiplicity of the pair $(J(F)(0), I_0)$ on $\Gamma(0)$ at $(0, 0)$ is not zero, so the multiplicity-polar theorem gives a contradiction—the change in multiplicity from the special fiber to the generic fiber is positive, but there is no polar variety of dimension $k$ of $J(F)$. So $C$ does not exist, which implies Whitney A holds for $(\Sigma - Y, Y)$ and the theorem is proved.

**Remark (4.6)** The key point in the last proof, was the ability to take information about the $k + d$ dimensional strata of the total space, and relate it to the open stratum of $f_0$. This was possible because we had good control on the conormals of the $k + d$ dimensional strata.

The above proof shows that it is easy to show that a stratification condition implies that the associated invariants are independent of parameter. To prove that the independence from parameter implies the stratification condition requires in general the principle of specialization of integral dependence developed in [12].

As we shall see in general (remark 4.9) the $A_F$ condition does not imply that the Lê numbers are independent of parameter. We can introduce a stronger notion of $A_F$ which does imply that the Lê numbers are constant if our ICIS is a curve. In the situation of Theorem 4.5 we say the strong $A_F$ condition holds if the $A_F$ condition holds, and for a generic linear function $l$ the $A_l$ condition holds for the pair $V(I) - Y, Y$.

From Theorem 4.5, and the formula for $\lambda^0$ in Theorem 2.4, we can now show that the strong $A_F$ condition implies that the Lê numbers are constant in the setup originally considered by Pellikaan.

**Corollary (4.7)** Suppose $F : C^k \times C^{n+1}, C^k \times 0, 0 \rightarrow C, 0, 0$, suppose the singular set of $F$, $S(F)$ is $V(I)$ where $I$ defines a family of complete intersection curves with isolated singularities, and every component of $V(I)$ contains $Y = C^k \times 0$. Suppose further that $J(F) = I$ off $Y$. Suppose the pair $(C^k \times C^{n+1} - S(F), C^k \times 0)$ satisfies the $A_F$ condition, and the pair $V(I) - C^k \times 0, C^k \times 0$ satisfies the $A_l$ condition for a generic linear function $l$, then the Lê numbers of $f_y$ at the origin are independent of $y$.

**Proof.** Theorem 4.5 1) and Theorem 2.3 imply that $j(f_y)$ is constant along
The condition that the singular set of $F$ is $V(I)$ implies that $F$ is in $I^2$ (p8, prop 1.9 [27]), hence $F = \sum_{i,j} h_{i,j}g_i g_j$ where \{g_i\} are a set of generators of $I$, and $h_{i,j} = h_{j,i}$ ([27], p54). Let $\Delta$ be the determinant of the matrix with entries $h_{i,j}$. Then the number of $D_\infty$ points at $(y, z)$ is just the colength of $(\Delta, p)$ in $O(V(I_y), z)$ ([27], p81 lemma 7.17). This number is just the local degree at $(y, z)$ of the map with components $(\Delta, p)$ where $p$ is projection to the parameter space $Y$ on $V(I)$. Thus if $\delta(f_y)$ varies along $Y$ it must be upper semicontinuous, and if the value for generic $y$ is less than the value over $y = 0$, there must be other points in the fiber over $y$ where $\delta(f_y)$ is non-zero. However as the proof of Theorem 3.5 2) shows off $Y f_y$ has only $A_\infty$ singularities on $V(I_y)$. Hence, $\delta(f_y)$ is constant along $Y$.

Since the pair $V(I) - C^k \times 0, C^k \times 0$ satisfies the $A_F$ condition for a generic linear function $l$, by Theorem 5.8 p232 of [16] the Milnor numbers of $V(I_y)$ and $V(I(y)) \cap V(l)$ are constant. Since $l$ is generic, the sum of these Milnor numbers is just $e(JM(\Sigma_y))$, which is then independent of $y$. The result for $\lambda^0$ now follows from the formula for $\lambda^0$ in proposition 2.4.

Since the Milnor number of $V(I(y)) \cap V(l)$ is just the multiplicity of $\Sigma_y$, less 1, the multiplicity of $V(I_y)$ is independent of $Y$. Since the transverse Milnor number is always 1, and the multiplicity of $V(I_y)$ constant, it follows that $\lambda^1$ is independent of $Y$ as well.

**Corollary (4.8)** Suppose $F : C^k \times C^{n+1}, C^k \times 0, 0 \to C, 0, 0$, suppose the singular set of $F$, $S(F)$ is $V(I)$ where $I$ defines a family of complete intersection curves with isolated singularities, and every component of $V(I)$ contains $Y = C^k \times 0$. Suppose further that $J(F) = I$ off $Y$. Suppose the pairs $(C^k \times C^{n+1} - S(F), C^k \times 0), V(I) - C^k \times 0, C^k \times 0$ satisfy the strong $A_F$ condition at $(0, 0)$ then

1) The homology of the Milnor fibre of $f_y$ at the origin is independent of $y$ for all $y$ small.
   
   If $n \geq 3$

2) The fibre homotopy-type of the Milnor fibrations of $f_y$ at the origin is independent of $y$ for all $y$ small.
   
   If $n \geq 4$

3) The diffeomorphism-type of the Milnor fibrations of $f_y$ at the origin is independent of $y$ for all $y$ small.

**Proof.** Since the strong $A_F$ condition holds, Corollary 4.7 implies that the Lê numbers are constant, then theorem 9.4 of [24] p90 gives the result. (Although Massey states his theorem for the case where the dimension of the parameter stratum is 1, it also applies to the case at hand.)

This raises the interesting question of whether a strong $A_F$ stratification or an $A_F$ stratification implies the triviality (in the sense of the last corollary) of the
Milnor fibrations. The formulae in proposition 3.12 and remark 3.13 show that a strong $A_F$ stratification implies that $\lambda^0(f_t)$ is independent of $t$ in these cases.

As the next example shows, in the $A_F$ case, this problem cannot be tackled by hoping that the existence of an $A_F$ stratification implies that the Lê numbers are constant.

**Remark (4.9)** This example shows that neither the $A_f$ condition nor topological triviality imply that the Lê numbers are constant. Let

$$f_t = z^5 + t y^6 z + y^7 x + x^{15}.$$ 

This family of functions was introduced by Briançon-Speder, ([2]) who showed that $\mu^3(f_t) = 364$ for all $t$, while the Milnor number of a generic hyperplane slice $\mu^2(f_t)$ is 28 when $t = 0$ and 26 otherwise. Historically, this example was important, because it showed that the $\mu^*$ constant condition was stronger than topological triviality.

Now consider $F_t = f_t^2 + w^2$ where $w$ is a disjoint variable. Then

$$J_z(F) = \langle w, 2f_t \frac{\partial f_t}{\partial x}, 2f_t \frac{\partial f_t}{\partial y}, 2f_t \frac{\partial f_t}{\partial z} \rangle.$$ 

So the singular locus of $F$ is defined by $\langle w, f_t \rangle$, hence is a family of complete intersections with isolated singularities. A computation shows that:

$$e(J(F_t), (w, f_t)) = j(F_t) = \mu^3(f_t).$$

Now, the only Lê cycle of dimension 2 is $V(w, f_t)$, so

$$\lambda^2(F_t) = m(X_t) = 5,$$

while

$$\lambda^1(F_t) = m(\Gamma_1^1(X_t, 0)) = \mu^2(f_t) + \mu^1(f_t).$$

Now by 3.12

$$\lambda^0(F_t) = e(J(F_t), (w, f_t)) + e(JM(V(f_t, f_t))) = \mu^3(f_t) + (\mu^2(f_t) + \mu^3(f_t)).$$

The first equality shows that the $A_F$ condition holds by Theorem 3.5. However $\lambda^0(F_t)$ and $\lambda^1(F_t)$ vary with $t$. It is not hard to check by a vector field argument that the family of functions $F_t$ are topologically trivial; however this can be seen directly by the following argument which was pointed out to me by J. N. Damon.

We know that there exists a topological trivialization $\phi(z, t) : \mathbb{C}^4 \to \mathbb{C}^3$ of $f_t$, by [2], so $f_t(\phi(z, t)) = f_0(z)$ Then, we can define $\Phi(z, w, t) = (\phi(z, t), w) : \mathbb{C}^5 \to \mathbb{C}^4$, which gives a topological trivialization of $F_t$ since

$$F_t(\Phi(x, w, t)) = f_t(\phi(x, t)) + w^2 = f_0(x) + w^2 = F_0(x).$$

Now we turn to the $W_f$ condition. It is a paradox, but because this condition is stronger, it is easier to prove results about it.
Theorem (4.10) Suppose $F : \mathbb{C}^k \times \mathbb{C}^{n+1}, \mathbb{C}^k \times 0, 0 \to \mathbb{C}, 0, 0$, suppose the singular set of $F$, $S(F)$ is $V(I)$ where $I$ defines a family of complete intersections with isolated singularities of fiber dimension $d$, and every component of $V(I)$ contains $Y = \mathbb{C}^k \times 0$. Suppose further that $J(F) = I$ off $Y$. Then:

1) If the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), \mathbb{C}^k \times 0)$ satisfies the $W_F$ condition then $e(m_{n+1}J(f_y), I_y, (y, 0))$ is independent of $y$.

2) If $e(m_{n+1}J(f_y), I_y, (y, 0))$ is independent of $y$, then the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), \mathbb{C}^k \times 0)$ satisfies the $W_F$ condition, and $\{\mathbb{C}^k \times \mathbb{C}^{n+1} - V(F), V(F) - V(I), V(I) - Y, Y\}$ is a Whitney stratification on some neighborhood of $Y$.

Proof.

1) Suppose the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), \mathbb{C}^k \times 0)$ satisfies the $W_F$ condition, then by Theorem 2.1 p.23 of [8], the dimension of the fiber of the exceptional divisor over $Y$ of $B_{m_Y J(F)}(\mathbb{C}^k \times \mathbb{C}^{n+1})$ is independent of $y$ and is $n$. This implies that the polar of dimension $k$ of $m_Y J(F)$ is empty; hence by the multiplicity polar theorem $e(m_{n+1}J(f_y), I_y, (y, 0))$ is independent of $y$.

2) Suppose $e(m_{n+1}J(f_y), I_y, (y, 0))$ is independent of $y$. Off $Y$, $m_{n+1}J(f_y) = J(f_y)$, so off $Y$ by the same arguments found in the proof of 3.6, $J(f_y) = I_y$, so by the multiplicity polar theorem, $\Gamma^k(m_Y J(F))$ is empty, hence the dimension of the fiber of the exceptional divisor over $Y$ of $B_{m_Y J(F)}(\mathbb{C}^k \times \mathbb{C}^{n+1})$ is $n$, hence is constant over $Y$. Then by Corollary 2.1, p.19 of [8], the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), \mathbb{C}^k \times 0)$ satisfies the $W_F$ condition. This implies $V(F) - V(I)$ is Whitney over $Y$.

Since $F$ is of type $A_{\infty}$ off $Y$ it follows that $V(F) - V(I)$ is Whitney over $V(I) - Y$. It remains to show $V(I) - Y$ is Whitney over $Y$. Suppose not; then for each $C$ and neighborhood $U$ of the origin there exists a sequence of points $x_i \in U$ on some component of $V(I)$, converging to the origin, and hyperplanes $H_i$ which are tangent hyperplanes to $V(I)$ at $x_i$ such that

$$\text{dist}(Y, H_i) > C \text{ dist}(x, Y).$$

From the proof of theorem 3.6, we have $C(V(I)) \subset B_{J(F)}(\mathbb{C}^k \times \mathbb{C}^{n+1})$. This implies we can find points $\tilde{x}_i \in U \cap (\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F))$ and hyperplanes $\tilde{H}_i$ tangent to the fibers of $F$ at $x_i$, such that the distance between $x_i$ and $\tilde{x}_i$, $H_i$ and $\tilde{H}_i$ is as small as desired. Then a similar inequality holds for $\tilde{x}_i$ and $\tilde{H}_i$, hence $W_F$ fails, which is a contradiction.

Corollary (4.11) Suppose in the above setup $e(m_{n+1}J(f_y), I_y, (y, 0))$ is independent of $y$, then the family of functions $\{f_y\}$ is topologically trivial.

Proof. Since $e(m_{n+1}J(f_y), I_y, (y, 0))$ is independent of $y$, we have the pair $(\mathbb{C}^k \times \mathbb{C}^{n+1} - S(F), \mathbb{C}^k \times 0)$ satisfies the $W_F$ condition, and $\{\mathbb{C}^k \times \mathbb{C}^{n+1} - V(F), V(F) - V(I), V(I) - Y, Y\}$ is a Whitney stratification on some neighborhood of $Y$. Then we can lift the constant fields over $V(F)$, to the ambient space in such a way that the resulting fields can be integrated to give homeomorphisms.
There is a nice geometric interpretation of the number \(e(m_{n+1}J(f_y))\) which we now describe. We denote the multiplicity of the relative polar variety of \(f_y\) of dimension \(i\) by \(m^i(f_y)\).

**Theorem (4.12)** Suppose \(f : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}, 0, J\) any ideal in \(\mathcal{O}_{n+1}\) such that \(\dim_{\mathbb{C}} J/J(f) < \infty\), then
\[
e(m_{n+1}J(f), J) = e(J(f), J) + 1 + \sum_{i=1}^{n} \binom{n+1}{i} m^i(f_y).
\]

**Proof.** This is exactly the content of the formula in Theorem 9.8 (i) p221 [20].

**Corollary (4.13)** Suppose \(f : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}, 0, S(f)\) is \(V(I)\) where \(I\) defines a complete intersection with isolated singularities of dimension \(d\), and suppose further that \(J(f) = I\) off \(Y\). Then
\[
e(m_{n+1}J(f), I) = e(J(f), I) + 1 + \sum_{i=1}^{n} \binom{n+1}{i} m^i(f_y).
\]

**Proof.** Follows immediately from Theorem 3.12

**Corollary (4.14)** Suppose \(F : \mathbb{C}^k \times \mathbb{C}^{n+1}, \mathbb{C}^k, 0 \rightarrow \mathbb{C}, 0, 0\), suppose the singular set of \(F\), \(S(F)\) is \(V(I)\) where \(I\) defines a family of complete intersections with isolated singularities of fiber dimension \(d\), and every component of \(V(I)\) contains \(Y = \mathbb{C}^k \times 0\). Suppose further that \(J(F) = I\) off \(Y\). Then the following are equivalent:
1) \(e(J(f_y), I_y)\) and the relative polar multiplicities of \(f_y\) are independent of \(y\).
2) \(A_F\) holds for the pair \((\mathbb{C}^k \times \mathbb{C}^{n+1} - V(I), Y)\), and the relative polar multiplicities of \(f_y\) are independent of \(y\).
3) The pair \((\mathbb{C}^k \times \mathbb{C}^{n+1} - V(I), \mathbb{C}^k \times 0)\) satisfies the \(W_F\) condition.

**Proof.** 1) and 2) are equivalent by Theorem 3.5, while 2 and 3 are equivalent by Corollary 3.13 and Theorem 3.9.

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