AN UNBOUNDED OPERATOR WITH SPECTRUM IN A STRIP AND MATRIX DIFFERENTIAL OPERATORS

MICHAEL GIL’

(Received 16 February 2021; accepted 1 March 2021; first published online 16 April 2021)

Abstract

Let $A$ and $\tilde{A}$ be unbounded linear operators on a Hilbert space. We consider the following problem. Let the spectrum of $A$ lie in some horizontal strip. In which strip does the spectrum of $\tilde{A}$ lie, if $A$ and $\tilde{A}$ are sufficiently ‘close’? We derive a sharp bound for the strip containing the spectrum of $\tilde{A}$, assuming that $\tilde{A} - A$ is a bounded operator and $A$ has a bounded Hermitian component. We also discuss applications of our results to regular matrix differential operators.

2020 Mathematics subject classification: primary 47A10; secondary 47A55, 47B10, 47E05.

Keywords and phrases: differential operator, Hilbert space, spectrum localisation.

1. Introduction and statement of the main result

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(\cdot, \cdot)$, norm given by $\| \cdot \| = \sqrt{(\cdot, \cdot)}$ and unit operator $I$. By $L(\mathcal{H})$ we denote the set of all bounded operators in $\mathcal{H}$. For an operator $A$ on $\mathcal{H}$, $D(A)$ is its domain, $A^*$ and $A^{-1}$ are the adjoint and inverse operators, respectively, $\sigma(A)$ is the spectrum, $R_z(A) = (A - zI)^{-1}$ ($z \notin \sigma(A)$) is the resolvent, and $\lambda_j(A)$ ($j = 1, 2, \ldots$) denote the eigenvalues of $A$ taken with their multiplicities. In addition, for $\omega > 0$, we denote by

$$H_\omega := \{ z \in \mathbb{C} : \| \Im z \| < \omega \}$$

the horizontal strip of height $2\omega$ which is symmetric with respect to the real axis. Following [10, Section 4.1], we will say that an operator $A$ on $\mathcal{H}$ is a strip-type operator of height $\omega$ (in short, $A \in \text{Strip}(\omega)$) if $\sigma(A) \subset H_\omega$ and $\sup_{\Im z \geq \omega} \| R_z(A) \| < \infty$ for all $\omega' > \omega$. Finally,

$$\omega_\text{st}(A) := \inf \{ \omega \geq 0 : A \in \text{Strip}(\omega) \}$$

is called the spectral height of $A$.

We consider the following problem. Let $A$ and $\tilde{A}$ be strip-type operators on $\mathcal{H}$. In which strip does the spectrum of $\tilde{A}$ lie if $\omega_\text{st}(A)$ is known and $\tilde{A}$ and $A$ are sufficiently ‘close’? We also discuss applications of our results to matrix differential operators.

© Australian Mathematical Publishing Association Inc. 2021. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.
The strip-type operators form a wide class of unbounded operators in a Banach space. The important example here is the logarithm of a sectorial operator, arising in various applications (see [10, 16]). The natural functional calculus for strip-type operators appears first in [2]. It is discussed in [11] in a general setting and used in [3]. The theory of strip-type operators is developed in [9, 16, 17] and the references given therein. For more details, see [10, Ch. 4]. To the best of our knowledge, the above-mentioned problem has not been considered in the literature, although it is important for the localisation of spectra and in various applications.

Furthermore, $A$ is said to be a strong strip-type operator of height $\omega$, if for any $\omega' > \omega$ there is an $L_{\omega'}$ such that

$$\|R_z(A)\| \leq \frac{L_{\omega'}}{|\text{Im} \, z| - \omega'} \quad \text{for} \quad |\text{Im} \, z| > \omega'.$$

From [10, Example 4.1.1.2, page 92], if $iA$ generates a $C_0$-group $e^{itA_R}$ in a Hilbert space, then $A$ is a strong strip-type operator of height $\theta(e^{itA_R})$, where $\theta(e^{itA_R})$ is the group type of $e^{itA_R}$. In particular,

$$\omega_s(A) = \theta(e^{itA_R}). \quad (1.1)$$

Throughout the paper it is assumed that $D(A)$ is dense in $\mathcal{H}$, $A = A_R + iA_I$, where $A_R$ and $A_I$ are self-adjoint operators, and

$$A_I \in \mathcal{L}(\mathcal{H}). \quad (1.2)$$

According to the Stone theorem (see [10, Section 4.1]), the operator $iA_R$ generates a $C_0$-group $e^{itA_R}$ ($-\infty < t < \infty$) of unitary operators. In particular, for $t \geq 0$ it is a semigroup. Moreover, by [5, Theorem II.4.6], $iA_R$ generates a bounded analytic semigroup. Hence, by [5, Proposition III.1.12], $iA$ generates a bounded analytic semigroup, since $A_I$ is bounded. Thus, under condition (1.2), $A$ is a strip-type operator and therefore (1.1) holds.

Let

$$D(\tilde{A}) = D(A) \quad \text{and} \quad q := \|A - \tilde{A}\| < \infty. \quad (1.3)$$

Then $\|\tilde{A}_I\| \leq q + \|A_I\|$ and therefore $\tilde{A}$ is also a strip-type operator.

We introduce the notation $x(t) = e^{itA}x_0 \quad (x_0 \in D(A))$, $\alpha(A_I) = \sup \sigma(A_I)$ and $\beta(A_I) = \inf \sigma(A_I)$. Then

$$\frac{d}{dt}(x(t), x(t)) = 2\text{Re} \, (iAx(t), x(t)) = -2(A_I x, x) \leq -2\beta(A_I) \leq 2\|A_I\|\|x(t)\|^2$$

and

$$\frac{d}{dt}(x(t), x(t)) = -2(A_I x, x) \geq -2\alpha(A_I)(A_I x, x).$$
Consequently, $\|e^{iAt}x_0\| \leq \|x_0\|e^{\|A\|t}$ for $t \geq 0$. Thus, from (1.1), $\omega_{st}(A) \leq \|A\|$.

Similarly,

$$\omega_{st}(\tilde{A}) \leq \|\tilde{A}\|. \quad (1.4)$$

This inequality is rather rough. Below, we present a considerably sharper estimate.

To this end, note that according to (1.1), $\|e^{iAt}\| \leq \text{const.} e^{\omega_{st} t} (t \geq 0)$, and thus the operators $-(cI \pm iA)$, for $c \in \mathbb{R}$, generate the exponentially stable semigroups $e^{-(cI + iA)t}$, provided $c > \omega_{st}$. Hence, the integral

$$X_c := \int_0^\infty e^{-(iA + cI)t} e^{-(iA + cI)t} \, dt \quad (c > \omega_{st}) \quad (1.5)$$

strongly converges and

$$\|X_c\| \leq \int_0^\infty e^{-2ct}\|e^{iA}\|^2 \, dt.$$ 

We are now in a position to formulate our main result, which we prove in Section 2.

**Theorem 1.1.** Let conditions (1.2) and (1.3) hold. Let $X_c$ be defined by (1.5) for some $c > \omega_{st}$. Then

$$\omega_{st}(\tilde{A}) < c, \text{ provided } q\|X_c\| < 1/2.$$ 

Now put

$$w_c(A) := \frac{1}{2\pi} \int_{-\infty}^\infty \|iA + (is + c)I\|^{-1} \|ds.$$ 

By the classical Parseval–Plancherel equality [1, Theorem 5.2.1], for any $x \in \mathcal{H}$,

$$(X_c x, x) = \left( \int_0^\infty e^{-(ic + iA)t} e^{-(ic + iA)t} x \, dt, x \right) = \int_0^\infty \|e^{-(Ai + Is)t}\|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|(iA + (is + c)I)^{-1}x\|^2 \, ds.$$ 

Hence,

$$\|X_c\| \leq w_c(A). \quad (1.6)$$

If $A$ is normal, that is, $AA^* = A^*A$, then by the spectral representation (see, for instance, [12]), we easily see that $\|e^{iAt}\| = e^{-\|A\|t}$, where $\beta(A) := \inf \text{ Im } \sigma(A)$ and $t \geq 0$. But $\beta(A) \geq -\omega_{st}(A)$. Therefore,

$$\|X_c\| \leq \int_0^\infty e^{-2(c + \beta(A))} \, dt = \frac{1}{2(c + \beta(A))} = \frac{1}{2(c - \omega_{st}(A))} \quad (c > \omega_{st}(A)).$$

Making use of Theorem 1.1, we obtain $\omega_{st}(\tilde{A}) \leq \omega_{st}(A) + q + \epsilon$ for $\epsilon > 0$. Hence, letting $\epsilon \to 0$, we arrive at the following result.

**Corollary 1.2.** Let conditions (1.2) and (1.3) hold and let $A$ be normal. Then

$$\omega_{st}(\tilde{A}) \leq \omega_{st}(A) + q.$$ 

In particular, if $A$ is self-adjoint, then $\omega_{st}(\tilde{A}) \leq q$. 
Let us show that Theorem 1.1 is sharp. To this end, assume that \( K \in \mathcal{L}(\mathcal{H}) \) and \( A \) are self-adjoint commuting operators and \( \tilde{A} = A + iK \). Suppose also that \( \sigma(A) \) and \( \sigma(K) \) are discrete. Then \( \sigma(\tilde{A}) \) consists of the eigenvalues \( \lambda_{jk}(\tilde{A}) = \lambda_j(A) + i\lambda_k(K) \) (\( j, k = 1, 2, \ldots \)).

Hence, \( \omega_{st}(\tilde{A}) = \sup_k |\lambda_k(K)| = q \), since \( q = ||\tilde{A} - A|| = ||K|| = \sup_k |\lambda_k(K)| \). But due to Corollary 1.2, \( \omega_{st}(\tilde{A}) \leq q \), since \( \omega_{st}(A) = 0 \). So the bound in Theorem 1.1 is attained in this case.

### 2. Proof of Theorem 1.1

We need the following well-known theorem (see [4, Theorem 5.1.3, page 217]).

**Theorem 2.1.** Suppose that \( B \) is the infinitesimal generator of the \( C_0 \)-semigroup \( T(t) \) on a Hilbert space \( \mathcal{H} \). Then \( T(t) \) is exponentially stable if and only if there exists a bounded positive definite operator \( P \) such that

\[
(Bz, Pz) + (Pz, Bz) = -(z, z) \quad (z \in D(B)).
\]  

Moreover, if \( B \) is the infinitesimal generator of an exponentially stable \( C_0 \)-semigroup then from [4, Section 5.5.3a, Equation (5.62)], for any \( Q \in \mathcal{L}(\mathcal{H}) \) the equation

\[
(Bz_1, Pz_2) + (Pz_1, Bz_2) = -(z_1, Qz_2)
\]  

has a solution \( P \in \mathcal{L}(\mathcal{H}) \) which, again by [4, Section to 5.5.3a], is representable as

\[
P = \int_0^\infty e^{B^t}Qe^{Bt} \, dt.
\]  

For a self-adjoint operator \( S \) we write \( S > 0 \) (\( S < 0 \)), if \( S \) is positive (negative) definite. Let \( D(B) = D(B^*) \) and \( B^*P + PB = -C^2 \) (with \( C > 0 \)) on \( D(B) \) for some positive definite \( P \in \mathcal{L}(\mathcal{H}) \). Then

\[
C^{-1}B^*PC^{-1} + C^{-1}PBC^{-1} = C^{-1}B^*CC^{-1}PC^{-1} + C^{-1}PC^{-1}C^{-1} - I.
\]

That is, \( M^*Y + YM = -I \), where \( M = CBC^{-1} \) and \( Y = C^{-1}PC^{-1} \).

According to Theorem 2.1, \( M \) generates an exponentially stable semigroup. Since \( M \) and \( B \) are similar, we arrive at the following result.

**Corollary 2.2.** Let \( D(B) = D(B^*) \) and \( B^*P + PB < 0 \) on \( D(B) \) for some positive definite \( P \in \mathcal{L}(\mathcal{H}) \). Then \( \sup \text{Re} \sigma(B) < 0 \).

**Proof of Theorem 1.1.** From (2.3),

\[
(cI + iA)^*X_c + X_c(cI + iA) = I.
\]

Put \( E = \tilde{A} - A \). Then from (2.4),

\[
\begin{align*}
(i\tilde{A} + cI)^*X_c + X_c(i\tilde{A} + cI) &= (iA + cI)^*X_c + X_c(iA + cI) - iE^*X_c + iX_cE \\
&= I - iE^*X_c + iX_cE.
\end{align*}
\]
If $2q||X_c|| < 1$, then $(i\tilde{A} + cI)^*X_c + X_c(i\tilde{A} + cI) > 0$. By Corollary 2.2, it follows that $\sup \text{Re } \sigma(-i\tilde{A} - cI) < 0$. So $-c - \text{Re } (ix - y) = -c + y < 0$ for any $x + iy \in \sigma(\tilde{A})$. Thus $\sup \text{Re } \sigma(\tilde{A}) < c$. Replacing $\tilde{A}$ by $-\tilde{A}$ and proceeding in the same way, we find $-c + \text{Re } (ix - y) = -c - y < 0$. Thus $\inf \text{Re } \sigma(\tilde{A}) > -c$. This proves the theorem. 

3. Spectral strips of differential operators with matrix coefficients

Let $L^2 = L^2([0, 1], \mathbb{C}^n)$ be the space of functions defined on $[0, 1]$ with values in $\mathbb{C}^n$ and the scalar product

$$(f, h)_{L^2} = \int_0^1 (f(x), h(x))_n \, dx \quad (f, h \in L^2),$$

where $(\cdot, \cdot)_n$ means the scalar product in $\mathbb{C}^n$. On the domain

$$D(A) = \{u \in L^2 : u'' \in L^2 \text{ and } u(0) = u(1) = 0\},$$

consider the operator

$$\tilde{A} = -\frac{d^2}{dx^2} + C(x) \quad (x \in (0, 1)), \tag{3.1}$$

where $C(x)$ is an $n \times n$ matrix continuously dependent on $x$. We consider this operator as a perturbation of the operator

$$A = -\frac{d^2}{dx^2} + C_0 \quad (x \in (0, 1)) \tag{3.2}$$

with a constant $n \times n$ matrix $C_0$. By way of example, one can take $C_0 = C(0)$ or $C_0 = \int_0^1 C(x) \, dx$. Clearly,

$$(A\eta f)(x) = C_{\eta f} \eta f(x) \quad (f \in L^2, x \in [0, 1], C_{\eta f} = (C_0 - C_0^*)/2i)$$

and

$$q = ||A - \tilde{A}||_{L^2} \leq \sup_x ||C(x) - C_0||_n.$$ 

Here $||A - \tilde{A}||_{L^2}$ is the operator norm in $L^2$ of $A - \tilde{A}$ and $|| \cdot ||_n$ means the spectral matrix norm (the operator norm with respect to the Euclidean vector norm).

Take into account that the operator $S$ defined on $D(A)$ by $S := -d^2/dx^2$ commutes with constant matrices. Since the eigenvalues of $S$ are $\pi^2k^2 \ (k = 1, 2, \ldots)$, by simple calculations we can show that $\sigma(A)$ consists of the eigenvalues $\lambda_{jk}(A) = \pi^2k^2 + \lambda_j(C_0)$ $(k = 1, 2, \ldots, j = 1, \ldots, n)$, where $\lambda_j(C_0)$ are the eigenvalues of $C_0$ taken with their multiplicities. Thus,

$$\omega_q(A) = \omega_q(C_0) := \max_j |\text{Im } \lambda_j(C_0)|.$$
Since $S$ and $C_0$ commute, we have $e^{iAt} = e^{iC_0t}e^{iSt}$. Hence, taking into account that $S = S^*$ and therefore $\|e^{iS}\| = 1$, we can write $\|e^{iAt}\|_2 \leq \|e^{iC_0t}\|_n$ and

$$\|X_c\|_2 \leq \int_0^\infty e^{-2ct} \|e^{-iC_0t}\|_n^2 dt.$$  \hspace{1cm} (3.3)

To estimate $\|e^{iC_0t}\|_n$, for an $n \times n$ matrix $M$, introduce the quantity $g(M)$ which measures the departure from normality:

$$g(M) := \left[ N_2^2(M) - \sum_{k=1}^n |\lambda_k(M)|^2 \right]^{1/2},$$

where $N_2(M) := (\text{trace } (M^*M))^{1/2}$ is the Hilbert–Schmidt (Frobenius) norm of $M$ and $\lambda_k(M)$ ($k = 1, \ldots, n$) are the eigenvalues of $M$ taken with their multiplicities.

Various properties of $g(M)$ can be found in [8, Section 3.1]. In particular,

$$g^2(M) \leq N_2^2(M) - |\text{trace } M^2|$$

and

$$g^2(M) \leq 2N_2^2(M_I) \quad (\text{where } M_I = (M - M^*)/2i).$$

In addition, $g(zM) = |z|g(M)$ for $z \in \mathbb{C}$. If $M$ is a normal matrix, that is, $MM^* = M^*M$, then $g(M) = 0$. By [8, Theorem 3.5], for any $n \times n$ matrix $M$,

$$\|e^{Mt}\| \leq \exp[\alpha(M)t] \sum_{k=0}^{n-1} \frac{g^k(M)t^k}{(k!)^{3/2}} \quad (\alpha(M) = \max \text{Re } \lambda_k(M), t \geq 0).$$

But $\alpha(iC_0) \leq \omega_{st}(C_0)$ and $g(iC_0) = g(C_0)$. Thus,

$$\|e^{iC_0t}\| \leq \exp[\omega_{st}(C_0)t] \sum_{k=0}^{n-1} \frac{g^k(C_0)t^k}{(k!)^{3/2}} \quad (t \geq 0)$$

and from (3.3),

$$\|X_c\|_2 \leq \int_0^\infty \exp[-2(c - \omega_{st}(C_0))t] \left( \sum_{k=0}^{n-1} \frac{g^k(C_0)t^k}{(k!)^{3/2}} \right)^2 dt$$

$$= \int_0^\infty \exp[-2(c - \omega_{st}(C_0))t] \sum_{j,k=0}^{n-1} \frac{g^{j+k}(C_0)t^{k+j}}{(j!k!)^{3/2}} dt \quad (c > \omega_{st}(C_0)).$$

Since

$$\int_0^\infty \exp[-st]t^k dt = \frac{k!}{s^{k+1}} \quad (s > 0),$$

we find $\|X_c\| \leq \frac{1}{2} \zeta(c - \omega_{st}(C_0))$, where

$$\zeta(s) = \sum_{j,k=0}^{n-1} \frac{(j+k)! g^{j+k}(C_0)}{2^{j+k} s^{j+k+1}(j!k!)^{3/2}} \quad (s > 0).$$
Now Theorem 1.1 implies the following result.

**Corollary 3.1.** Let \( \tilde{A} \) be defined by (3.1) and, for some \( c > \omega_{st}(C_0) \), let the condition

\[
q \zeta(c - \omega_{st}(C_0)) < 1
\]

hold. Then \( \omega_{st}(\tilde{A}) < c \).

Let \( x_n \) be the unique nonnegative root of the equation

\[
q \zeta(y) = q \sum_{j,k=0}^{n-1} \frac{(j+k)! g^{j+k}(C_0)}{2^{j+k} (j! k!)^{3/2}} y^{2n-j-k-1} = 1 \quad (y > 0),
\]

which is equivalent to the equation

\[
y^{2n} = q \sum_{j,k=0}^{n-1} \frac{(j+k)! g^{j+k}(C_0)}{2^{j+k} (j! k!)^{3/2}} y^{2n-j-k-1} = 1.
\]

If \( y > x_n + \omega_{st}(C_0) \), then \( q \zeta(y) < q \zeta(x_n) = 1 \). Now Corollary 3.1 implies \( \omega_{st}(\tilde{A}) < y \). Letting \( y \to x_n + \omega_{st}(C_0) \), we obtain the following result.

**Corollary 3.2.** Let \( \tilde{A} \) be defined by (3.1). Then \( \omega_{st}(\tilde{A}) \leq \omega_{st}(C_0) + x_n \).

If \( C_0 \) is normal, then \( g(C_0) = 0 \), and with \( 0^0 = 1 \) we have \( \zeta(s) = 1/s \) and thus \( x_n = q \). The following lemma gives us an estimate for \( x_n \) in the case \( g(C_0) \neq 0 \).

**Lemma 3.3.** Let \( q \zeta(1) \leq 1 \). Then

\[
x_n \leq \frac{2^{n}}{\sqrt{q \zeta(1)}}.
\]

**Proof.** By (3.4), \( q \zeta(x_n) = 1 \leq q \zeta(1) \). Since \( \zeta(s) \) is monotonically decreasing, it follows that \( x_n \leq 1 \). Now (3.5) proves the lemma. \( \square \)

Corollary 3.2 and the Lemma 3.3 yield the following result.

**Corollary 3.4.** Let \( \tilde{A} \) be defined by (3.1) and \( q \zeta(1) \leq 1 \). Then

\[
\omega_{st}(\tilde{A}) \leq \omega_{st}(C_0) + \frac{2^{n}}{\sqrt{q \zeta(1)}}.
\]

For recent results on the spectra of differential operators see, for instance, the works [6, 7, 13, 14, 15, 18, 19] and the references which are given therein.

**Acknowledgement**

I am very grateful to the referee of this paper for many helpful remarks.

**References**

[1] W. Arendt, C. J. K. Batty, F. Neubrander and M. Hieber, *Laplace Transforms and Cauchy Problems* (Springer, Basel, 2011).

[2] W. G. Bade, ‘An operational calculus for operators with spectrum in a strip’, *Pacific J. Math.* 3 (1953), 257–290.
[3] K. Boyadzhiev and R. deLaubenfels, ‘Spectral theorem for unbounded strongly continuous groups on a Hilbert space’, Proc. Amer. Math. Soc. 120(1) (1994), 127–136.

[4] R. Curtain and H. Zwart, Introduction to Infinite-Dimensional Systems Theory (Springer, New York, 1995).

[5] K.-J. Engel and R. Nagel, A Short Course on Operator Semigroups, Universitext (Springer, New York, 2006).

[6] M. I. Gil’, ‘Perturbations of operators on tensor products and spectrum localization of matrix differential operators’, J. Appl. Funct. Anal. 3(3) (2008), 315–332.

[7] M. I. Gil’, ‘Resolvent and spectrum of a nonselfadjoint differential operator in a Hilbert space’, Ann. Univ. Mariae Curie-Skłodowska Sect. A 66(1) (2012), 25–39.

[8] M. I. Gil’, Operator Functions and Operator Equations (World Scientific, Hackensack, NJ, 2018).

[9] M. Haase, ‘Spectral properties of operator logarithms’, Math. Z. 245(4) (2003), 761–779.

[10] M. Haase, The Functional Calculus for Sectorial Operators, Operator Theory: Advances and Applications, 169 (Birkhauser, Basel, 2006).

[11] M. A. Haase, ‘A characterization of group generators on Hilbert spaces and the $H^\infty$-calculus’, Semigroup Forum 66(2) (2003), 288–304.

[12] T. Kato, Perturbation Theory for Linear Operators (Springer-Verlag, Berlin, 1980).

[13] A. M. Kholkin and F. S. Rofe-Beketov, ‘On spectrum of differential operator with block-triangular matrix coefficients’, Zh. Mat. Fiz. Anal. Geom. 10(1) (2014), 44–63.

[14] J. Locker, Spectral Theory of Non-Self-Adjoint Two Point Differential Operators, Mathematical Surveys and Monographs, 73 (American Mathematical Society, Providence, RI, 1999).

[15] R. Ma, H. Wang and M. Elsansosi, ‘Spectrum of a linear fourth-order differential operator and its applications’, Math. Nachr. 286(17–18) (2013), 1805–1819.

[16] C. C. Martinez and A. M. Sanz, The Theory of Fractional Powers of Operators (North-Holland, Amsterdam, 2001).

[17] J. Pruss and H. Sohr, ‘On operators with bounded imaginary powers in Banach spaces’, Math. Z. 203(3) (1990), 429–452.

[18] A. Zettl and J. Sun, ‘Self-adjoint ordinary differential operators and their spectrum’, Rocky Mountain J. Math. 45(3) (2015), 763–886.

[19] M. Zhang, J. Sun and J. Ao, ‘The discreteness of spectrum for higher-order differential operators in weighted function spaces’, Bull. Aust. Math. Soc. 86(3) (2012), 370–376.

MICHAEL GIL’, Department of Mathematics, Ben Gurion University of the Negev, PO Box 653, Beer-Sheva 84105, Israel e-mail: gilmi@bezeqint.net