Global Quantization in Gauge Orbit Space with Magnetic Monopoles As a Solution to Strong CP Problem and the Relevance to $U_A(1)$ Problem

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Abstract

We generalize our discussions and give more general physical applications of a new solution to the strong CP problem with magnetic monopoles as originally proposed by the author\textsuperscript{1}. Especially, we will discuss about the global topological structure in the relevant gauge orbit spaces to be clarified. As it is shown that in non-abelian gauge theories with a $\theta$ term, the induced gauge orbit space with gauge potentials and gauge functions restricted on the space boundary $S^2$ has a magnetic monopole structure and the gauge orbit space has a vortex structure if there is a magnetic monopole in the ordinary space. The Dirac’s quantization conditions in the quantum theories ensure that the vacuum angle $\theta$ in the gauge theories must be quantized. The quantization rule is given by $\theta = 2\pi/n$ ($n \neq 0$) with $n$ being the topological charge of the magnetic monopole. Therefore, the strong CP problem is automatically solved in the presence of a magnetic monopole of charge $\pm 1$ with $\theta = \pm 2\pi$, or magnetic monopoles of very large total topological charge ($|n| \geq 10^92\pi$) if it is consistent with the abundance of magnetic monopoles. Where in the first case with a magnetic monopole of topological charge 1 or -1, we mean the strong CP-violation can be only very small by the measurements implemented so far. Since $\theta = \pm 2\pi$ correspond to different monopole sectors, the CP can not be conserved exactly in strong interactions in this case. In the second case, the strong CP cannot be conserved either for large but finite $n$. The fact that the strong CP-violation measured so far can be only so small or vanishing may be a signal for the existence of magnetic monopoles. We also conjecture that the parity violation and CP violation in weak interaction fundamentally may intimately connected to the magnetic monopoles. The relevance to the $U_A(1)$ problem is also discussed. The existence of colored magnetic monopoles may also solve the $U_A(1)$ problem. In the presence of U(1) or monopoles as color singlets, the ’t Hooft’s solution to
the $U_A(1)$ problem is expected. The quantization formula for the vortex structure is also derived. In the presence of a magnetic monopole of topological charge $n \neq 0$ in non-abelian gauge theories, the relevant integral for the vortex along a closed loop in the gauge orbit space is quantized as $4N\pi/n$ with integer $N$ being the Pontryagin topological number for the relevant gauge functions.
1 Introduction

Since the discovery of Yang-Mills theories, particle physics has gained great development in the framework of non-abelian gauge theories. One of the most interesting features of particle physics is the non-perturbative effects in gauge theories such as instanton effects and magnetic monopoles. One of the other most interesting features in non-abelian gauge theories is the strong CP problem in QCD. It is known that, in non-abelian gauge theories a Pontryagin or \( \theta \) term,

\[
L_\theta = \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu}^a F_{\lambda\sigma}^a,
\]

(1)
can be added to the Lagrangian density of the system due to instanton effects. With an arbitrary value of \( \theta \), it can induce CP violations. However, the interesting fact is that the \( \theta \) angle in QCD can be only very small \((\theta \leq 10^{-9})\) or vanishing. Where in our discussions of QCD, \( \theta \) denotes \( \theta + \text{arg}(\text{det}M) \) effectively with \( M \) being the quark mass matrix, with the effects of electroweak interactions are included. One of the most interesting approaches to solve the strong CP problem has been the assumption of an additional Peccei-Quinn \( U(1)_{PQ} \) symmetry. In this approach, the vacuum angle is ensured vanishing due to the axions introduced. But there has not been observational support to the axions which are needed in this approach. Therefore, it is of fundamental interest to consider other possible solutions to the strong CP problem.

One of the main purposes of this paper is to generalize the discussions of a non-perturbative approach proposed by the author to solve the strong CP problem and some relevant applications. This is due to its physical importance as well as the other physical relevance. The section 3 has some overlapping with our brief note Ref. 5, this is essential for the completeness due to its intimate connection to the other aspects of the gauge theories in our discussions. Our
approach is to show that the existence of magnetic monopoles can ensure the quantization of the $\theta$ angle and thus can provide the solution to the strong CP problem. We will extend the formalism of Wu and Zee\textsuperscript{10} for discussing the effects of the Pontryagin term in pure Yang-Mills theories in the gauge orbit spaces in the Schrodinger formulation. This formalism is useful to the understanding of topological effects in gauge theories, it has also been used with different methods to derive the mass parameter quantization in three-dimensional Yang-Mills theory with Chern-Simons term\textsuperscript{10–11}. Wu and Zee showed\textsuperscript{10} that the Pontryagin term induces an abelian background field in the gauge configuration space of the Yang-Mills theory. In our discussions, we will consider the case with the existence of a magnetic monopole. Especially, we will show that magnetic monopoles in the space will induce an abelian gauge field with non-vanishing field strength in gauge configuration space, and there can be non-vanishing magnetic flux through a two-dimensional sphere in the gauge orbit space. Then, the Dirac quantization conditions\textsuperscript{4–5} in the corresponding quantum theories ensure that the relevant vacuum angle $\theta$ must be quantized. The quantization rule is derived as $\theta = 2\pi / n$ with $n$ being the topological charge of the monopole to be given. Therefore, the strong CP problem is automatically solved with the existence of magnetic monopoles of charge $\pm 1$, or monopoles with very large total magnetic charges ($n \geq 10^9 2\pi$). As we will see that an interesting feature in our derivation is that the Dirac quantization condition both in the ordinary space and the relevant induced gauge orbit space will be used. The relevance to the $U_A(1)$ problem will also be discussed. We will also discuss about the vortex structure in the gauge configuration space in this case. As we will show that the vortex in the gauge orbit space must be quantized also intimately connected to the quantization rule for the vacuum angle $\theta$. In the presence of a magnetic monopole of topological charge $n \neq 0$, the relevant integrals for the vortex along
a closed loops in the gauge orbit space are quantized as $4N\pi/n$ with integers $N$ being the Pontryagin topological numbers for the relevant gauge functions.

This paper will be organized as follows. Next, we will first give a brief description of the Schrödinger formulation for our purpose. Then in section 2, we will clarify the topological results relevant to our discussions. In section 3, we will show the existence of the monopole structure in the relevant gauge orbit space and realize the relevant topological results explicitly. In section 4, we will discuss about the monopole structure as a solution to the strong CP problem and its relevance to the $U_A(1)$ problem. The section 5 will be mainly discussions of the vortex structure in the gauge orbit space in the presence of a magnetic monopole. Our conclusions will be summarized in section 6.

We will now consider the Yang-Mills theory with the existence of a magnetic monopole at the origin. The Lagrangian of the system with the $\theta$ term is given by

$$\mathcal{L} = \int d^4x \{ -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu}^a F_{\lambda\sigma}^a \}.$$  

(2)

We will choose Weyl gauge $A_0 = 0$. This is convenient since effectively $A_0$ is not relevant to abelian gauge structure in the gauge configuration space with the $\theta$ term included. The conjugate momentum corresponding to $A_i^a$ is then

$$\pi_i^a = \frac{\delta \mathcal{L}}{\delta \dot{A}_i^a} = \dot{A}_i^a + \frac{\theta}{8\pi^2} \epsilon_{ijk} F_{jk}^a.$$  

(3)

In the Schrödinger formulation, the system is similar to the quantum system of a particle with the coordinate $q_i$ moving in a gauge field $A_i(q)$ with the correspondence$^{10-11}$

$$q_i(t) \rightarrow A_i^a(x, t),$$  

(4)

$$A_i(q) \rightarrow A_i^a(A(x)),$$  

(5)
where
\[ A_i^a(A(x)) = \frac{\theta}{8\pi^2} \epsilon_{ijk} F^a_{jk}. \] (6)

Thus there is a gauge structure with gauge potential \( A \) in this formalism within a gauge theory with the \( \theta \) term included. According to this, the system can be described by a Hamiltonian equation\(^{10}\) or in the path integral formalism\(^{11}\).

We will not discuss about this here, since we only need the Dirac quantization condition for our purpose. For details, see Ref. 10 and 11.

### 2 Relevant Topological Results

In our discussions, We will use the convention in Ref. 10 and differential forms\(^{12}\) where \( A = A_i^a L^a dx^i, F = \frac{1}{2} F_{jk}^a L^a dx^j dx^k \) with \( F = dA + A^2 \), and \( tr(L^a L^b) = -\frac{1}{2} \delta^{ab} \) in a basis \( \{L^a \mid a = 1, 2, ..., \text{rank}(G)\} \) for the Lie algebra of the gauge group \( G \). In quantum theory, the Schrödinger formulation is described in the gauge orbit space with the constraint of Gauss' law. Let \( U \) denote the gauge configuration space consisting of all the well-defined gauge potentials \( A \) that transform as \( A^g = g^{-1} A g + g^{-1} dg \) under a gauge transformation with gauge function \( g \). Denoting by \( G \) the space of all the continuous gauge transformations, the gauge orbit space \( U/G \) is the quotient space of the gauge configuration space with gauge potentials connected by continuous gauge transformations as equivalent. In the presence of a magnetic monopole, generally a singularity-free gauge potential may need to be defined in each local coordinate region. The separate gauge potentials in an overlapping region can only differ by a continuous gauge transformation\(^{5}\). In fact, the single-valuedness of the gauge function in the overlapping regions corresponds to the Dirac quantization condition\(^{5}\). For a monopole at the origin, one can actually divide the space outside the monopole into two overlapping regions. At a given \( r \), the regions are two extended semi-
spheres around the monopole, with $\theta \in [\pi/2 - \delta, \pi/2 + \delta](0 < \delta < \pi/2)$ in the overlapping region, where the $\theta$ denotes the $\theta$ angle in the spherical polar coordinates.

As we will see that our equation for our quantization rule for the $\theta$ is determined by the integration on the space boundary which is topologically a 2-sphere $S^2$ for non-singular monopoles. Thus for the quantization of $\theta$, the relevant case is that gauge potentials and gauge functions are restricted on the space boundary $S^2$. We will call the induced spaces of $\mathcal{U}$, $\mathcal{G}$ and $\mathcal{U}/\mathcal{G}$ with $A$ and $g$ restricted on the space boundary as restricted gauge configuration space, restricted space of gauge transformations and the restricted gauge orbit space respectively. Collectively, they will be called as the restricted spaces, and the unrestricted ones will be called as usual spaces. We will use the same notation $\mathcal{U}$, $\mathcal{G}$ and $\mathcal{U}/\mathcal{G}$ for both of them for convenience, there should not be confusing. Our discussions for the monopole and vortex structures will be on the restricted and usual spaces respectively.

The topological discussions and the results we will now give are true both for the usual spaces and the restricted spaces. Since $\mathcal{U}$ is topologically trivial both for the usual and restricted gauge configuration spaces as we will see.

To establish the topological results we need, we note first that $\mathcal{U}$ is topologically trivial, thus $\Pi_N(\mathcal{U}) = 0$ for any $N$. This is due to the fact that the interpolation between any two gauge potentials $A_1$ and $A_2$

$$A_t = tA_1 + (1 - t)A_2$$

(7)

for any real $t$ is also a gauge potential, thus $A_t \in \mathcal{U}$ (Theorem 7 in Ref.9, and Ref.6). since $A_t$ is transformed as a gauge potential in each local coordinate region, and in an overlapping region, both $A_1$ and $A_2$ are gauge potentials may be defined up to a gauge transformation, then $A_t$ is a gauge potential which may be defined up to a gauge transformation in the overlapping regions, or $A_t \in \mathcal{U}$. 

7
The space $\mathcal{U}$ can be considered as a bundle over the base space $\mathcal{U}/\mathcal{G}$ with fiber $\mathcal{G}$. More generally for a bundle $\beta = \{B, P, X, Y, \bar{G}\}$ with bundle space $B$, base space $X$, fiber $Y$, group $\bar{G}$, and projection $P$, let $Y_0$ be the fiber over $x_0 \in X$, and let $i : Y_0 \to B$ and $j : B \to (B, Y_0)$ be the inclusion maps. Then we have the homotopy sequence\(^{13}\) of $(B, Y_0, y_0)$ given by

$$
\Pi_N(Y_0) \xrightarrow{i_*} \Pi_N(B) \xrightarrow{j_*} \Pi_N(B, Y_0) \xrightarrow{\partial_*} \Pi_{N-1}(Y_0) \xrightarrow{i_*} \Pi_{N-1}(B) \quad (N \geq 1),
$$

(8)

where $\partial$ is the natural boundary operator, $i_*, j_*$ and $\partial_*$ are maps induced by $i, j$ and $\partial$ respectively. Let $P_0$ denote the restriction of $P$ as a map $(B, Y_0, y_0) \to (X, x_0, x_0)$. Then $P_0j$ is the projection $p : (B, y_0) \to (X, x_0, B)$. We have the isomorphism relation

$$p_* : \Pi_N(B, Y_0) \cong \Pi_N(X, x_0).$$

(9)

Defining $\Delta_* = \partial(P_0*)^{-1} : \Pi_N(X, x_0) \to \Pi_{N-1}(Y_0, y_0)$, the exact homotopy sequence can be written as

$$\Pi_N(Y_0, y_0) \xrightarrow{i_*} \Pi_N(B) \xrightarrow{P_*} \Pi_N(X, x_0) \xrightarrow{\Delta_*} \Pi_{N-1}(Y_0, y_0) \xrightarrow{i_*} \Pi_{N-1}(B) \quad (N \geq 1).$$

(10)

Now for our purpose with $B = \mathcal{U}, X = \mathcal{U}/\mathcal{G}$, $Y = \mathcal{G}$, and $\bar{G} = G$ for the gauge group $G$. The choice of the base points $x_0$ and $y_0$ are irrelevant in our discussions, since all the relevant homotopy groups based on different points are isomorphic. Note that homotopy theory has also been used to study the global gauge anomalies\(^{14-22}\), especially by using extensively the exact homotopy sequences of fiber bundles and in terms of James numbers of Stieffel manifolds.

More explicitly, we can now consider the following exact homotopy sequence\(^{13}\):

$$\Pi_N(\mathcal{U}) \xrightarrow{P_*} \Pi_N(\mathcal{U}/\mathcal{G}) \xrightarrow{\Delta_*} \Pi_{N-1}(\mathcal{G}) \xrightarrow{i_*} \Pi_{N-1}(\mathcal{U}) \quad (N \geq 1).$$

(11)

Since as we have seen that $\Pi_N(\mathcal{U}) = 0$ for any $N$, we have

$$0 \xrightarrow{P_*} \Pi_N(\mathcal{U}/\mathcal{G}) \xrightarrow{\Delta_*} \Pi_{N-1}(\mathcal{G}) \xrightarrow{i_*} 0 \quad (N \geq 1).$$

(12)
This implies that
\[ \Pi_N(U/G) \cong \Pi_{N-1}(G) \quad (N \geq 1). \quad (13) \]
As shown by Wu and Zee for the usual spaces in pure Yang-Mills theory in four dimensions,
\[ \Pi_1(U/G) \cong \Pi_0(G) \quad (14) \]
is non-trivial, and thus \( \theta \) term induces a vortex structure in gauge orbit space. This isomorphism will also be used in our discussions of the vortex structure in the presence of a magnetic monopole, but as we will see that it’s explicit realization is more non-trivial. It was also showed in Ref. 10 that the field strength \( F \) associated with the gauge potential \( \mathcal{A} \) is vanishing, and thus there is no flux corresponding to \( F \) in the pure Yang-Mills theory.

However, as we will show in the next section that in the presence of a magnetic monopole, the relevant topological properties of the system are drastically different. This will give interesting consequences in the quantum theory. One of the main topological result we will use for the restricted spaces in the presence of a magnetic monopole is
\[ \Pi_2(U/G) \cong \Pi_1(G). \quad (15) \]

Now \( \Pi_2(U/G) \neq 0 \) corresponds to the condition for the existence of a magnetic monopole in the restricted gauge orbit space. In the next section, we will realize the above topological results. We will first show that in this case \( \mathcal{F} \neq 0 \), and then demonstrate explicitly that the magnetic flux \( \int_{S^2} \hat{F} \neq 0 \) can be non-vanishing in the restricted gauge orbit space, where \( \hat{F} \) denotes the projection of \( F \) into the restricted gauge orbit space.
3 Monopole Structure in the Restricted Gauge Orbit Space in the Presence of Magnetic Monopoles

In our discussions, we denote the differentiation with respect to space variable \( x \) by \( d \), and the differentiation with respect to parameters \( \{ t_i \mid i = 1, 2, \ldots \} \) which \( A(x) \) may depend on in the gauge configuration space by \( \delta \), and assume \( d\delta + \delta d = 0 \). Then, similar to \( A = A_\mu dx^\mu \) with \( \mu \) replaced by \( a, i, x \), the gauge potential in the gauge configuration space can be written as a 1-form given by

\[
\mathcal{A} = \int d^3x A_i^a(A(x)) \delta A_i^a(x). \quad (16)
\]

Using Eq.(6), this gives

\[
\mathcal{A} = \frac{\theta}{8\pi^2} \int d^3x \epsilon_{ijk} F_{jk}^a(x) \delta A_i^a(x) = -\frac{\theta}{2\pi^2} \int_M tr(\delta AF), \quad (17)
\]

with \( M \) being the space manifold. Since \( \mathcal{A} \) is an abelian, then the field strength is given by

\[
\mathcal{F} = \delta \mathcal{A}. \quad (18)
\]

With \( \delta F = -D_A(\delta A) = -\{d(\delta A) + A\delta A - \delta AA\} \), we have

\[
\mathcal{F} = \frac{\theta}{2\pi^2} \int_M tr[\delta AD_A(\delta A)] = \frac{\theta}{4\pi^2} \int_M dtr(\delta A\delta A) = \frac{\theta}{4\pi^2} \int_{\partial M} tr(\delta A\delta A), \quad (19)
\]

up to a local term with vanishing projection to the relevant gauge orbit space.

Usually, one may assume \( A \to 0 \) faster than \( 1/r \) as \( x \to 0 \), then this would give \( \mathcal{F} = 0 \) as in the case of pure Yang-Mills theory\(^{10}\). However, it is more subtle in the presence of a magnetic monopole. Asymptotically as \( r \to 0 \) with a monopole at the origin, the monopole may generally give a field strength of the form\(^{4-5,22}\)

\[
F_{ij} = \frac{1}{4\pi^2} \epsilon_{ijk} (\hat{r}) G_0(\hat{r}), \quad (20)
\]

with \( \hat{r} \) being the unit vector for \( r \), and this gives \( A \to O(1/r) \) as \( x \to 0 \). Thus, one can see easily that a magnetic monopole can give a nonvanishing field strength \( \mathcal{F} \) in the gauge configuration space.
To evaluate $\mathcal{F}$, one needs to specify the space boundary $\partial M$ in the presence of a magnetic monopole. We now consider the case that the magnetic monopole does not generate a singularity in the space. Then the effects in the case that monopoles are singular will be discussed. In fact, non-singular monopoles may be more relevant in the unification theory since there can be monopoles as a smooth solution of a spontaneously broken gauge theory similar to 't Hooft Polyakov monopole\textsuperscript{4}. For example, it is known that\textsuperscript{23} there are monopole solutions in the minimal SU(5) model. When the monopole is non-singular, the space boundary then may be regarded as a large 2-sphere $S^2$ at the spatial infinity. For our purpose, we actually only need to evaluate the projection of $\mathcal{F}$ into the gauge orbit space. But the evaluation of $\mathcal{F}$ can give more explicit understanding of the topological properties of the system. The $\mathcal{F}$ is similar to a constant $F$ in the ordinary space, it does not give any flux through a closed surface in the space $\mathcal{U}$. However, the quantum theory is based on the gauge orbit space in Schrodinger formulation, the relevant magnetic flux needs to considered in the gauge orbit space. In fact, as we will see that the corresponding magnetic flux in the gauge orbit space can be non-vanishing. A gauge potential in the gauge orbit space can be written in the form of

$$A = g^{-1}ag + g^{-1}dg,$$  \hspace{1cm} (21)

for an element $a \in \mathcal{U}/\mathcal{G}$ and a gauge function $g \in \mathcal{G}$. Then the projection of a form into the gauge orbit space contains only terms proportional to $(\delta a)^n$ for integers $n$. We can now write

$$\delta A = g^{-1}[\delta a - D_a(\delta gg^{-1})]g.$$  \hspace{1cm} (22)

Then we obtain

$$\mathcal{A} = -\frac{\theta}{2\pi^2} \int_M tr(f\delta a) + \frac{\theta}{2\pi^2} \int_M tr[fD_a(\delta gg^{-1})],$$  \hspace{1cm} (23)
where \( f = da + a^2 \). With some calculations, this can be simplified as

\[
\mathcal{A} = \hat{\mathcal{A}} + \frac{\theta}{2\pi^2} \int_{S^2} tr[f \delta gg^{-1}],
\]

(24)

where

\[
\hat{\mathcal{A}} = -\frac{\theta}{2\pi^2} \int_M tr(f \delta a),
\]

(25)

is the projection of \( \mathcal{A} \) into the gauge orbit space. Similarly, we have

\[
\mathcal{F} = \frac{\theta}{4\pi^2} \int_{S^2} tr\{[\delta a - D_a(\delta gg^{-1})][\delta a - D_a(\delta gg^{-1})]\}
\]

(26)

or

\[
\mathcal{F} = \hat{\mathcal{F}} - \frac{\theta}{4\pi^2} \int_{S^2} tr\{\delta a D_a(\delta gg^{-1}) + D_a(\delta gg^{-1}) \delta a - D_a(\delta gg^{-1}) D_a(\delta gg^{-1})\},
\]

(27)

where

\[
\hat{\mathcal{F}} = \frac{\theta}{4\pi^2} \int_{S^2} tr(\delta a \delta a),
\]

(28)

is the projection of the \( \mathcal{F} \) to the gauge orbit space or the restricted gauge orbit space based on the space boundary \( S^2 \).

Now all our discussions will be based on the restricted spaces. To see that the flux of \( \hat{\mathcal{F}} \) through a closed surface in the gauge orbit space \( U/G \) can be nonzero, we will construct a 2-sphere in it. Consider an given element \( a \in U/G \), and a loop in \( G \). The set of all the gauge potentials obtained by all the gauge transformations on \( a \) with gauge functions on the loop then forms a loop \( C^1 \) in the gauge configuration space \( U \). Obviously, the \( a \) is the projection of the loop \( C^1 \) into \( U/G \). Now since \( \Pi_1(U) = 0 \) is trivial, the loop \( C^1 \) can be continuously extended to a two-dimensional disc \( D^2 \) in the \( U \) with the boundary \( \partial D^2 = C^1 \). Obviously, the projection of the \( D^2 \) into the gauge orbit space with the boundary \( C^1 \) identified as a single point is topologically a 2-sphere \( S^2 \subset U/G \). With the Stokes’ theorem in the gauge configuration space, We now have

\[
\int_{D^2} \mathcal{F} = \int_{D^2} \delta \mathcal{A} = \int_{C^1} \mathcal{A}.
\]

(29)
Using Eqs. (24) and (29) with $\delta a = 0$ on $C^1$, this gives

$$\int_{D^2} F = \int_{C^1} A = \frac{\theta}{2\pi^2} \text{tr} \int_{S^2} \int_{C^1} [f\delta gg^{-1}].$$

(30)

Thus, the projection of the Eq(30) to the gauge orbit space gives

$$\int_{S^2} \hat{F} = \frac{\theta}{2\pi^2} \text{tr} \int_{S^2} \{f \int_{C^1} \delta gg^{-1}\},$$

(31)

where note that in the two $S^2$ are in the restricted gauge orbit space and the ordinary space respectively. This can also be obtained by

$$\int_{D^2} \text{tr} \int_{S^2} \text{tr} \{\delta a D_a (\delta gg^{-1}) + D_a (\delta gg^{-1}) \delta a - D_a (\delta gg^{-1}) D_a (\delta gg^{-1})\} = 0,$$

(32)

or the projection of $\int_{D^2} F$ gives $\int_{S^2} \hat{F}$. We have verified this explicitly or the topological result that the projection of $\int_{D^2} F$ gives $\int_{S^2} \hat{F}$. For this one needs to use Stokes theorem in the ordinary space and the gauge configuration space with $d\delta + \delta d = 0$, $a \in U/G$ or $a$ is a constant on $C^1$, and $\int_{D^2} \hat{F} = \int_{S^2} \hat{F}$ in the gauge orbit space since $\hat{F}$ is the projection of the $F$ into the gauge orbit space.

In quantum theory, Eq. (31) corresponds to the topological result $\Pi_2(U/G) \simeq \Pi_1(G)$ for the restricted spaces. This feature in the gauge orbit space has some similarity to that given in Refs. 10 and 11 for the discussions of three-dimensional Yang-Mills theories with a Chern-Simons term. We only need the Dirac quantization condition here for our purpose. In the restricted gauge orbit space, the Dirac quantization condition gives

$$\int_{S^2} \hat{F} = 2\pi k,$$

(33)

with $k$ being integers. We will now determine the quantization rule for the $\theta$. Now let $f$ be the field strength 2-form for the magnetic monopole. There may be many ways to obtain non-vanishing results for the right-hand side. For our purpose, one way is to restrict $g$ to a $U(1)$ subgroup of the gauge group, and
obtain a non-zero topological number. Then the quantization rule for the $\theta$ will be obtained.

Let $\{H_i \mid i = 1, 2, ..., r = \text{rank}(G)\}$ denote a basis of the Cartan subalgebra for the gauge group $G$. The corresponding simple roots and fundamental weights are denoted by $\{\alpha_i \mid i = 1, 2, ..., r\}$ and $\{\lambda_i \mid i = 1, 2, ..., r\}$ respectively. Then we have

$$\frac{2 < \lambda_i, \alpha_j >}{< \alpha_j, \alpha_j >} = \delta_{ij}, \quad (34)$$

where the $< \lambda_i, \alpha_j >$ denotes the inner product in the root vector space. By the theorem which states that for any compact and connected Lie group $G$, any element in the Lie algebra is conjugate to at least one element in its Cartan subalgebra by a group element in $G$, the quantization condition for the magnetic monopole is given by

$$\exp\{\int_{S^2} f\} = \exp\{G_0\} = \exp\{4\pi \sum_{i=1}^{r} \beta^i H_i\} \in \mathbb{Z}. \quad (35)$$

Where

$$G_0 = \int_{S^2} f = 4\pi \sum_{i=1}^{r} \beta^i H_i \quad (36)$$

is the magnetic charge up to a conjugate transformation by a group element.

Now let $g(t) \in [0, 1]$ be in the following $U(1)$ subgroup on the $C^1$

$$g(t) = \exp\{4\pi mt \sum_{i,j=1}^{r} \frac{\alpha_i}{< \alpha_i, \alpha_j >} \}, \quad (37)$$

with $m$ being integers. In fact, $m$ should be identical to $k$ according to our topological result $\Pi_2(U/G) \cong \Pi_1(G)$ for the restricted spaces. In this case, the relevant homotopy groups obtained are isomorphic to $\mathbb{Z}$ which may be only a subgroup of the homotopy groups generally for a non-abelian gauge group $G$. In fact for this case, the $k$ and $m$ should be identical since they correspond to the topological numbers on each side. Using $tr(H_iH_j) = -\frac{1}{2} \delta_{ij}$ and

$$\int_{C^1} \delta gg^{-1} = 4\pi m \sum_{i,j=1}^{r} \frac{\alpha_i}{< \alpha_i, \alpha_i >} \quad (38)$$
we obtain
\[ \theta = \frac{2\pi}{n} \quad (n \neq 0). \]  
(39)

Where we define generally the topological charge of the magnetic monopole as
\[ n = -2 < \delta, \beta > = -2 \sum_{i=1}^{r} < \lambda_i, \beta >, \]  
(40)

which must be an integer by the quantization condition\(^{23}\) for the magnetic monopoles. Where the \(\delta\) is given by
\[ \delta = \sum_{i=1}^{r} \frac{2\alpha_i}{< \alpha_i, \alpha_i>} = \sum_{i=1}^{r} \lambda_i. \]  
(41)

The minus sign is due to our normalization convention for the Lie algebra generators. Actually, the fundamental weights \(\{\lambda_i \mid i = 1, 2, \ldots, r\}\) and \(\{\frac{2\alpha_i}{< \alpha_i, \alpha_i>} \mid i = 1, 2, \ldots, r\}\) form the Dynkin basis and its dual basis in the root vector space respectively.

In our definition, the topological charge of the magnetic monopole can be understood as follows. Up to a conjugate transformation, the magnetic charge of the monopole is contained in a Cartan subalgebra of the gauge group. Restricting to each U(1) subgroup generated by a generator \(H_i (i=1, 2, \ldots, r)\) in the basis of the Cartan subalgebra, the monopole has a topological number \(n_i\) corresponding to the Dirac quantization condition. Then generally the topological number \(n\) in our definition is given by
\[ n = \sum_{i=1}^{r} n_i. \]  
(42)

Obviously, we expect that this is the natural generalization of the topological charge to the non-abelian magnetic monopole. To the knowledge of the author, such an explicit general definition Eq.(40) in terms of the fundamental weights of the Lie algebra for the topological charge of non-abelian magnetic monopoles is first obtained by the author.
As a remark, our derivation has been topological. Our quantization rule can also be obtained by using constraints of Gauss’ law. This more physical approach and the discussion of its physical relevance will be given elsewhere.

4 Magnetic Monopoles as A Solution to the Strong CP Problem and the Relevance to the $U_A(1)$ Problem

As we have seen that in the presence of magnetic monopoles, the vacuum angle $\theta$ must be quantized. The quantization rule is given by Eq.(39). Therefore, we conclude that the existence of magnetic monopoles can provide a solution to the strong CP problem. In the presence of magnetic monopoles with topological charge $\pm 1$, the vacuum angle of non-abelian gauge theories must be $\pm 2\pi$, the existence of such magnetic monopoles gives a solution to the strong CP problem. The existence of many monopoles can ensure $\theta \to 0$, and the strong CP problem may also be solved. In this possible solution to the strong CP problem with $\theta \leq 10^{-9}$, the total magnetic charges present are $|n| \geq 2\pi 10^9$. This may possibly be within the abundance allowed by the ratio of monopoles to the entropy$^{26}$, but with the possible existence of both monopoles and anti-monopoles, the total number of magnetic monopoles may be larger than the total magnetic charges. Generally, one needs to ensure that the total number is consistent with the experimental results on the abundance of monopoles.

In the above discussions, we consider the case that magnetic monopole generates no singularity in the space, for example, with monopole as a smooth solution in a spontaneously broken gauge theory. If we consider the magnetic monopoles as a singularity similar to the Wu-Yang monopole$^{27}$ which is the first non-abelian monopole solution found, then the space boundary can be regarded as consisting of an infinitesimal inward 2-sphere around each magnetic monopole.
and a large 2-sphere at the spatial infinity. In the space outside the monopole, each infinitesimal inward sphere effectively gives a contribution equivalent to a monopole of opposite topological charge. Then the total contribution of the infinitesimal spheres and the contribution from the large sphere at the spatial boundary are all cancelled in the relevant integrations. Therefore, only the existence of non-singular magnetic monopoles may provide solution to the strong CP problem.

Moreover, note that our conclusions are also true if we add an additional $\theta$ term in QED with the $\theta$ angle the same as the effective $\theta$ in QCD if there exist Dirac monopoles as color singlets, or a non-abelian monopoles with magnetic charges both in the color SU(3) and electromagnetic U(1). Then the explanation of such a QED $\theta$ term is needed. The effect of the term proportional to $\varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$ in the presence of magnetic charges was first considered relevant to chiral symmetry. Then, the effect of a similar U(1) $\theta$ term was discussed for the purpose of considering the induced electric charges as quantum excitations of dyons associated with the 't Hooft Polyakov monopole and generalized magnetic monopoles. Especially, the generalized magnetic monopoles are used to consider the possibility of quarks as dyons in a spontaneously broken gauge theory. An interesting feature is that if quarks are dyons in a spontaneously broken gauge theory, then their electric charges will not be exactly fractionally quantized, instead they will carry extra charges proportional to $\theta$. Moreover, two meson octets, one baryon octet and one baryon decuplet free of magnetic charges were constructed from quarks as dyons. For our purpose, we expect that if a QED $\theta$ term is included, it may be possibly an indication of unification for the color gauge symmetry and electromagnetic U(1) symmetry. A $\theta$ term needs to be included in the unification gauge theory since $\Pi_3(G) = Z$ for the unification group G, monopoles with magnetic charges involving the QED U(1)
symmetry are generated through spontaneous gauge symmetry breaking. Generally, such an arbitrary $\theta$ term in QED may not be discarded since it is not a total divergence globally in the presence of magnetic charges and as we have seen that the $\theta$ term physically can have non-perturbative effects.

We would like to emphasize that in our approach with magnetic monopoles as a solution to the strong CP problem, the $U_A(1)$ problem can also be solved. The $U_A(1)$ problem was originally solved by t’Hooft\textsuperscript{3} with the fact that the $U_A(1)$ is not a symmetry in the quantum theory due to the axial anomaly\textsuperscript{32}, the conserved $U_A(1)$ symmetry is not gauge invariant and its spontaneous breaking does not generate a physical light meson. In our solution with the presence of colored magnetic monopoles, the $U_A(1)$ symmetry is explicitly broken\textsuperscript{28}. If the strong CP problem is solved by the pure electromagnetic $U(1)$ monopole with a $\theta$ term included with $\theta$ being the same as the effective $\theta_{QCD}$, then the $U_A(1)$ symmetry is not explicitly broken and the t’Hooft’s solution to the $U_A(1)$ problem can be applied. Thus $U_A(1)$ problem can be solved in our approach to solve the strong CP problem with the existence of magnetic monopoles.

\section{Vortex Structure in the Gauge Orbit Space}

In this section, we will discuss about the vortex structure in the gauge orbit space. It is known that\textsuperscript{33} there can be vortex structures in some three-dimensional field theories with the boundary of the space being topologically a circle. The discussions in Ref.\textsuperscript{10} in the gauge orbit space are for the pure Yang-Mills theory with a $\theta$ term. We will consider the case in the presence of magnetic monopoles.

In order to discuss about the vortex structure in the gauge orbit space in the gauge theories we are interested in, we need to consider the integration of $\hat{A}$
along a closed loop $\hat{C}$ in the gauge orbit space $U/G$. As in Ref. 10, such a loop $\hat{C}$ can be constructed by projection. Let $C$ denote an open path in the gauge configuration space $U$ with gauge potentials $A$ and $A^g$ as the two end points, where $g \in G$ is a gauge function. Obviously, the projection of $C$ into the $U/G$ gives a closed loop $\hat{C}$ with the two end points of $C$ identified as a single point. Thus we have\(^\text{10}\) topologically
\[
\int_{\hat{C}} \hat{A} \simeq \int_{C} A. \tag{43}
\]
In pure Yang-Mills theory, one can verify that\(^\text{10}\) the $A$ can be written as the differentiation of the Chern-Simons secondary topological invariant\(^\text{34}\) in the gauge configuration space. Thus in the case of pure Yang-Mills theory, the Chern-Simons secondary invariant can be regarded as a gauge function in the gauge configuration space. Now in the presence of a magnetic monopole, we obtain
\[
\delta \{ \theta W[A] \} = -\frac{\theta}{2\pi^2} \int_M \text{tr}(\delta AF) + \frac{\theta}{4\pi^2} \int_M d\text{tr}(A\delta A), \tag{44}
\]
or
\[
A = \delta \{ \theta W[A] \} - \frac{\theta}{4\pi^2} \int_M d\text{tr}(A\delta A), \tag{45}
\]
where
\[
W[A] = -\frac{1}{4\pi^2} \int_M \text{tr}(AdA + \frac{2}{3}A^3) \tag{46}
\]
is the Chern-Simons secondary topological invariant. This gives
\[
\int_C A = \int_C \delta \{ \theta W[A] \} - \frac{\theta}{4\pi^2} \int_C \int_{\partial M} d\text{tr}(A\delta A), \tag{47}
\]
or
\[
\int_C A = \theta \{ W[A^g] - W[A] \} - \frac{\theta}{4\pi^2} \int_C \int_{S^2} tr(A\delta A). \tag{48}
\]
Now
\[
W[A^g] - W[A] = \frac{1}{12\pi^2} \int_M tr(g^{-1} dg)^3 + \int_M d\alpha_2[A, g] = 2N[g] + \int_{S^2} \alpha_2[A, g]. \tag{49}
\]
where
\[ N[g] = \frac{1}{24\pi^2} \int_M tr(g^{-1}dg)^3, \]  
(50)
and
\[ \alpha_2[A, g] = -\frac{1}{4\pi^2} tr(Ag^{-1}dg). \]  
(51)
Thus
\[ \int_C A = 2\theta N[g] - \frac{\theta}{4\pi^2} \int_{S^2} tr(Ag^{-1}dg) - \frac{\theta}{4\pi^2} \int_C \int_{S^2} tr\{A\delta A\}. \]  
(52)
Now since C is an open path in \( \mathcal{U} \) from \( A \) to \( A^g \), the integral \( \int_C A \) generally contains two parts. The first part is topologically invariant as we will see, the second term depends only on the end points \( A \) and \( A^g \) or the gauge function \( g \).

The third term or the second and third term together is generally path dependent, but it does not contain any non-vanishing topological invariant, namely it is a path-dependent local term. This can be seen as follows. Since the space \( \mathcal{U} \) is topologically trivial, the open path C in \( \mathcal{U} \) with the two end points fixed can be continuously deformed into the straight interval
\[ A_t = tA^g + (1-t)A \quad t \in [0,1]. \]  
(53)
We only need to verify this by evaluating the integral with the C being the straight interval. Then topologically
\[ \int_C tr\{A\delta A\} \cong -\int_0^1 A_t(A^g - A)dt = -tr\{(A^g - A)\int_0^1 [(tA^g + (1-t)A)dt\}
= -tr\{\frac{1}{2}(A^g - A)(A^g + A)\} = -tr[(A^g)^2 - 2AA^g - A^2] = 2tr(AA^g). \]  
(54)
With \( A^g = g^{-1}Ag + g^{-1}dg \), this can be written as
\[ \int_C tr\{A\delta A\} \cong 2tr\{Ag^{-1}Ag + Ag^{-1}dg\}. \]  
(55)
Thus topologically
\[ \int_C A \cong 2\theta N[g] + I_2, \]  
(56)
where
\[ I_2 = -\frac{\theta}{4\pi^2} \int_{S^2} tr\{(2A g^{-1}A g + 3A g^{-1}dg\}. \] (57)

One can now easily see that the second part \( I_2 \) of the integral contains no non-vanishing topological invariant. Since \( A \) and \( g \) are independent each other, the \( g \) as a mapping \( g : S^2 \rightarrow G \) can be continuously deformed into a constant mapping. The \( I_2 \) is topologically equivalent to an integral with the integrand proportional to \( trA^2 = 0 \). Thus, up to topologically trivial terms, we obtain
\[ \int \hat{C} \hat{A} \simeq 2\theta N[g]. \] (58)

It is known that the integral \( N[g] \) is topologically invariant when \( M \) is compactified as a 3-sphere \( S^3 \). It is straightforward to show that \( N[g] \) is topologically invariant with \( g \) as a mapping \( g : M \rightarrow G \) from the space manifold to the gauge group \( G \) in our discussion. The only change is that a small variation for the gauge function gives an additional boundary term which is vanishing due to the fact that the space boundary \( \partial M \) is topologically a 2-sphere and \( \Pi_2(G) = 0 \). To obtain non-vanishing results for \( N[g] \), we need to restrict to the gauge functions with \( g \rightarrow 0 \). Then for the space manifold is effectively compactified as a 3-sphere \( S^3 \), and \( N[g] \) is the Pontryagin topological number corresponding to the homotopy group \( \Pi_3(G) \). Thus
\[ \int \hat{C} \hat{A} \simeq 2\theta N, \] (59)

with \( N \) being integers. This corresponds to the isomorphism relation
\[ \Pi_1(U/G) \cong \Pi_0(G) = \Pi_3(G) = \mathbb{Z}, \] (60)

where \( G \) is the space of all the gauge transformations \( g \) satisfying \( g \rightarrow 0 \). With the quantization rule \( \theta = 2\pi/n \) we obtained, we can write
\[ \int \hat{C} \hat{A} \simeq \frac{4N\pi}{n} \quad (n \neq 0), \] (61)
with \( n \) being the topological charge of the monopole. Therefore, there can be vortex structure in the gauge orbit space. In the presence of magnetic monopoles the vortex must be topologically quantized with the quantization rule given by the above equation. In the presence of a monopole of topological charges \( \pm 1 \), the vortex is quantized as \( \pm 4\pi N \). In the presence of many monopoles with very large total topological charges \( n \), the vortex can be only very small or vanishing.

Our discussions in the presence of magnetic monopoles are more non-trivial than the case in pure Yang-Mills theory, especially for the explicit realization of the topological isomorphism due to the local terms involved.

In the above discussions, the magnetic monopoles are regarded non-singular in the space. In fact, one can easily see that in the presence of singular monopoles the quantization rule is given by Eq.(59), but as we have seen that in this case \( \theta \) can be arbitrary.

As a remark, note that in QED or more generally an abelian gauge theory with \( N = 0 \) since \( \Pi_3(U(1)) = 0 \), there is no corresponding topological vortex in the gauge orbit space even in the presence of magnetic monopoles.

6 Conclusions

We have discussed extensively about the topological structure in the relevant gauge orbit space of gauge theories with a \( \theta \) term. The presence of a magnetic monopole in the ordinary space can induce monopole and vortex structures in the restricted and usual gauge orbit spaces. The Dirac quantization conditions ensure that the vacuum angle \( \theta \) must be quantized. The quantization can provide a solution to the strong CP problem with the existence of one monopole of topological charge \( \pm 1 \), or many monopoles if it is consistent with the abundance of magnetic monopoles. The \( U_A(1) \) problem may also be solved with the exis-
tence of colored magnetic monopoles, or by t’ Hooft’s solution if the magnetic monopoles are of only U(1) charges as color singlets. Therefore, the fact that the strong CP-violation can be only so small or vanishing may be a signal for the existence of magnetic monopoles. An interesting feature is that in the presence of one magnetic monopole of charge ±1, θ = ±2π according to the quantization rule obtained. The cases of n = ±2 may also possibly solve strong CP problem. But when the vacuum angle is θ = ±π, other than the Strong CP problem it may have other effects different from the case of θ = ±2π or vanishing, for example, on quark masses, but these are usually discussed without the presence of monopoles. In the axion approach to solve the strong CP problem, the vacuum angle should be vanishing, and there has been argument that the vacuum energy is minimized at vanishing vacuum angle.

We have also derived the quantization formula for the vortex by using our quantization rule for the θ angle. Thus, as we have shown that the monopole structure and vortex structure in the restricted gauge orbit spaces and the usual gauge orbit spaces are connected through our quantization rules.

As a remark, note that usually if strong interaction conserves CP, then the 2π or π may be expected equivalent to −2π or −π since they are related by a CP operation and θ may be expected to be periodic with period 2π. However, according to our quantization rule, this is not true due to the fact that ±2π or ±π correspond to different monopole sectors. If the strong CP problem is solved by a monopole of topological charge ±1 or ±2, this means the CP-violation can be only very small in the measurements implemented so far, the CP can not be exactly conserved, since the θ = ±2π or θ = ±π correspond to two different physical systems. If the strong CP problem is solved due to the existence of many monopoles, then the observation of strong CP violation gives an indirect measurement of the abundance of magnetic monopoles. For
any finite number of magnetic monopoles, the CP cannot be exactly conserved in the strong interactions. The strong CP-violation may provide information about the structure of the universe.

As a conjecture, we expect that the parity violation and CP violation in weak interaction may intimately connected to magnetic monopoles also.

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