Extremal Type I $\mathbb{Z}_k$-codes and $k$-frames of odd unimodular lattices

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Abstract

For some extremal (optimal) odd unimodular lattice $L$ in dimensions 12, 16, 20, 28, 32, 36, 40 and 44, we determine all integers $k$ such that $L$ contains a $k$-frame. This result yields the existence of an extremal Type I $\mathbb{Z}_k$-code of lengths 12, 16, 20, 32, 36, 40 and 44, and a near-extremal Type I $\mathbb{Z}_k$-code of length 28 for positive integers $k$ with only a few exceptions.

1 Introduction

Self-dual codes and unimodular lattices are studied from several viewpoints (see [11] for an extensive bibliography). Many relationships between self-dual codes and unimodular lattices are known and there are similar situations between two subjects. As a typical example, it is known that a unimodular lattice $L$ contains a $k$-frame if and only if there is a self-dual $\mathbb{Z}_k$-code $C$ such that $L$ is isomorphic to the lattice obtained from $C$ by Construction A, where $\mathbb{Z}_k$ is the ring of integers modulo $k$ with $k \geq 2$.

Type II $\mathbb{Z}_{2k}$-codes were defined in [2] as a class of self-dual codes, which are related to even unimodular lattices. If $C$ is a Type II $\mathbb{Z}_{2k}$-code of length $n \leq 136$, then we have the bound on the minimum Euclidean weight $d_E(C)$ of $C$ as follows: $d_E(C) \leq 4k \left\lfloor \frac{n}{24} \right\rfloor + 4k$ for every positive integer $k$ (see [21]).
A Type II $\mathbb{Z}_{2k}$-code meeting the bound with equality is called extremal ($n \leq 136$). It was shown in [6, 14] that the Leech lattice, which is one of the most remarkable lattices, contains a $2k$-frame for every integer $k \geq 2$. This result yields the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length 24 for every positive integer $k$. Recently, the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of lengths $n = 32, 40, 48, 56, 64$ was established in [21] for every positive integer $k$. This was done by finding a $2k$-frame in some extremal even unimodular lattices in these dimensions $n$.

Recently, it was shown in [28] that the odd Leech lattice contains a $k$-frame for every integer $k$ with $k \geq 3$. This motivates our investigation of the existence of a $k$-frame in extremal odd unimodular lattices. In this paper, for some extremal (optimal) odd unimodular lattices $L$ in dimensions 12, 16, 20, 28, 32, 36, 40 and 44, we determine all integers $k$ such that $L$ contains a $k$-frame. This result yields the existence of an extremal Type I $\mathbb{Z}_k$-code of lengths 12, 16, 20, 32, 36, 40 and 44, and a near-extremal Type I $\mathbb{Z}_k$-code of length 28 for positive integers $k$ with only a few small exceptions.

This paper is organized as follows. In Section 2, we give definitions and some basic properties of self-dual codes and unimodular lattices used in this paper. The notion of extremal Type I $\mathbb{Z}_k$-codes of length $n$ is given for $n \leq 48$ and $k \geq 2$. Lemma 2.2 gives a reason why we consider unimodular lattices only in dimension $n$ divisible by 4. In Section 3, we provide a method for constructing $m$-frames in unimodular lattices, which are constructed from some self-dual $\mathbb{Z}_k$-codes by Construction A (Proposition 3.1). This method is a slight generalization of [21, Propositions 3.3 and 3.6]. Using Proposition 3.1, we give $k$-frames in the unique extremal odd unimodular lattice in dimensions 12, 16, some extremal (optimal) odd unimodular lattices in dimensions 20, 28, 32, 36, 40 and 44, which are listed in Table 2 for all integers $k$ satisfying the condition ($\ast$) in Table 2 (Lemma 3.3). In Section 4, several extremal (near-extremal) Type I $\mathbb{Z}_k$-codes are explicitly constructed for some positive integers $k$. Then we establish the existence of a $k$-frame in the extremal (optimal) unimodular lattices $L$ in dimensions 12, 16, 20, 28, 32, 36, which are listed in Table 2 (except only lattices $A_3(C_{20,3}(D_{10}^1))$ and $A_5(C_{20,5}(D_{10}^1))$), for every integer $k$ with $k \geq \min(L)$, where $\min(L)$ denotes the minimum norm of $L$. As a consequence, by considering the even unimodular neighbors of the above extremal odd unimodular lattice in dimension 32, it is shown that the 32-dimensional Barnes–Wall lattice $BW_{32}$ contains a $2k$-frame if and only if $k$ is an integer with $k \geq 2$. When $n = 40, 44$, we show that there is an extremal odd unimodular lattice in dimension $n$ containing a $k$-frame if and only if $k$
is an integer with \( k \geq 4 \). Using the above existence of \( k \)-frames, the existence of an extremal Type I \( \mathbb{Z}_k \)-code of lengths \( n = 12, 16, 20, 32, 36, 40, 44 \), and a near-extremal Type I \( \mathbb{Z}_k \)-code of length \( n = 28 \) is established for a positive integer \( k \), where \( k \neq 1, 3 \) if \( n = 32 \) and \( k \neq 1 \) otherwise. At the end of Section 4, we examine the existence of both \( k \)-frames in optimal odd unimodular lattices in dimension 48 and near-extremal Type I \( \mathbb{Z}_k \)-codes of length 48.

All computer calculations in this paper were done by Magma [4].

2 Preliminaries

In this section, we give definitions and some basic properties of self-dual codes and unimodular lattices used in this paper. The notion of extremal Type I \( \mathbb{Z}_k \)-codes of length \( n \) is given for \( n \leq 48 \) and \( k \geq 2 \).

2.1 Self-dual codes

Let \( \mathbb{Z}_k \) be the ring of integers modulo \( k \), where \( k \) is a positive integer. In this paper, we always assume that \( k \geq 2 \) and we take the set \( \mathbb{Z}_k \) to be \( \{0, 1, \ldots, k - 1\} \). A \( \mathbb{Z}_k \)-code \( C \) of length \( n \) (or a code \( C \) of length \( n \) over \( \mathbb{Z}_k \)) is a \( \mathbb{Z}_k \)-submodule of \( \mathbb{Z}_n^k \). A \( \mathbb{Z}_2 \)-code and a \( \mathbb{Z}_3 \)-code are called binary and ternary, respectively. The Euclidean weight of a codeword \( x = (x_1, \ldots, x_n) \) of \( C \) is \( \sum_{\alpha=1}^{\lfloor k/2 \rfloor} n_\alpha(x) \alpha^2 \), where \( n_\alpha(x) \) denotes the number of components \( i \) with \( x_i \equiv \pm \alpha \pmod{k} \) (\( \alpha = 1, 2, \ldots, \lfloor k/2 \rfloor \)). The minimum Euclidean weight \( d_E(C) \) of \( C \) is the smallest Euclidean weight among all nonzero codewords of \( C \).

A \( \mathbb{Z}_k \)-code \( C \) is self-dual if \( C = C^\perp \), where the dual code \( C^\perp \) of \( C \) is defined as \( \{x \in \mathbb{Z}_n^k \mid x \cdot y = 0 \text{ for all } y \in C\} \) under the standard inner product \( x \cdot y \). For only even positive integers \( 2k \), a Type II \( \mathbb{Z}_{2k} \)-code was defined in [2] as a self-dual \( \mathbb{Z}_{2k} \)-code with the property that all Euclidean weights are congruent to 0 modulo \( 4k \). It is known that a Type II \( \mathbb{Z}_{2k} \)-code of length \( n \) exists if and only if \( n \) is divisible by 8 [2]. A self-dual code which is not Type II is called Type I. Two self-dual \( \mathbb{Z}_k \)-codes \( C \) and \( C' \) are equivalent if there exists a monomial \((\pm 1, 0)\)-matrix \( P \) with \( C' = C \cdot P \), where \( C \cdot P = \{xP \mid x \in C\} \).
2.2 Unimodular lattices

A (Euclidean) lattice $L \subseteq \mathbb{R}^n$ in dimension $n$ is unimodular if $L = L^*$, where the dual lattice $L^*$ of $L$ is defined as $\{ x \in \mathbb{R}^n \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L \}$ under the standard inner product $(x,y)$. Two lattices $L$ and $L'$ are isomorphic, denoted $L \cong L'$, if there exists an orthogonal matrix $A$ with $L' = L \cdot A$, where $L \cdot A = \{ xA \mid x \in L \}$. The automorphism group $\text{Aut}(L)$ of $L$ is the group of all orthogonal matrices $A$ with $L = L \cdot A$.

The norm of a vector $x$ is defined as $(x,x)$. The minimum norm $\min(L)$ of a unimodular lattice $L$ is the smallest norm among all nonzero vectors of $L$. The theta series $\theta_L(q)$ of $L$ is the formal power series $\theta_L(q) = \sum_{x \in L} q^{(x,x)}$. The kissing number of $L$ is the second nonzero coefficient of the theta series. A unimodular lattice with even norms is said to be even, and that containing a vector of odd norm is said to be odd. An even unimodular lattice in dimension $n$ exists if and only if $n$ is divisible by 8, while an odd unimodular lattice exists for every dimension.

It was shown in [34] that a unimodular lattice $L$ in dimension $n$ has minimum norm $\min(L) \leq 2\lfloor \frac{n}{24} \rfloor + 2$ unless $n = 23$ when $\min(L) \leq 3$ (see [36] for the case that $L$ is even). A unimodular lattice meeting the bound with equality is called extremal. Any extremal unimodular lattice in dimension $24k$ has to be even [12]. Hence, an odd unimodular lattice $L$ in dimension $24k$ satisfies $\min(L) \leq 2k + 1$. We say that an odd unimodular lattice with the largest minimum norm among all odd unimodular lattices in that dimension is optimal.

Let $L$ be a unimodular lattice. Define $L_0 = \{ x \in L \mid (x,x) \equiv 0 \pmod{2} \}$. Then $L_0$ is a sublattice of $L$ of index 2 if $L$ is odd and $L_0 = L$ if $L$ is even. The shadow $S$ of $L$ is defined as $S = L_0^* \setminus L$ if $L$ is odd and as $S = L$ if $L$ is even [10]. Now suppose that $L$ is an odd unimodular lattice. Then there are cosets $L_1, L_2, L_3$ of $L_0$ such that $L_0^* = L_0 \cup L_1 \cup L_2 \cup L_3$, where $L = L_0 \cup L_2$ and $S = L_1 \cup L_3$. Two lattices are neighbors if both lattices contain a sublattice of index 2 in common. If the dimension is divisible by 8, then $L$ has two even unimodular neighbors of $L$, namely, $L_0 \cup L_1$ and $L_0 \cup L_3$.

2.3 Construction A and $k$-frames

We give a method to construct unimodular lattices from self-dual $\mathbb{Z}_k$-codes, which is referred to as Construction A (see [2, 22]). If $C$ is a self-dual $\mathbb{Z}_k$-code
of length \( n \), then the following lattice
\[
A_k(C) = \frac{1}{\sqrt{k}} \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid (x_1 \mod k, \ldots, x_n \mod k) \in C\}
\]
is a unimodular lattice in dimension \( n \). The minimum norm of \( A_k(C) \) is
\[
\min\{k, d_E(C)/k\}. 
\]
Moreover, \( C \) is a Type II \( \mathbb{Z}_{2k} \)-code if and only if \( A_{2k}(C) \) is an even unimodular lattice [2].

A set \( \{f_1, \ldots, f_n\} \) of \( n \) vectors \( f_1, \ldots, f_n \) of a unimodular lattice \( L \) in dimension \( n \) with \( (f_i, f_j) = k\delta_{i,j} \) is called a \( k \)-frame of \( L \), where \( \delta_{i,j} \) is the Kronecker delta. It is trivial that if a unimodular lattice in dimension \( n \) contains a \( k \)-frame then the number of vectors of norm \( k \) is greater than or equal to \( 2n \). It is known that a unimodular lattice \( L \) contains a \( k \)-frame if and only if there exists a self-dual \( \mathbb{Z}_k \)-code \( C \) with \( A_k(C) \cong L \) (see [22]). Therefore, we have the following:

**Lemma 2.1.** Suppose that there is a unimodular lattice \( L \) in dimension \( n \) containing a \( k \)-frame. Then there is a self-dual \( \mathbb{Z}_k \)-code \( C \) such that
\[
d_E(C) \geq k \min(L).
\]
The above lemma is useful when establishing the existence of extremal Type I \( \mathbb{Z}_k \)-codes in Section 4.

By the following lemma, it is sufficient to consider the existence of a \( p \)-frame in a unimodular lattice for each prime \( p \). The lemma also gives a reason why we consider unimodular lattices only in dimension \( n \) divisible by 4.

**Lemma 2.2** ([6, Lemma 5.1]). Let \( n \) be a positive integer divisible by 4. If a lattice \( L \) in dimension \( n \) contains a \( k \)-frame, then \( L \) contains a \( km \)-frame for every positive integer \( m \).

### 2.4 Upper bounds on the minimum Euclidean weights

It is known [25, 33, 34] that a self-dual \( \mathbb{Z}_k \)-code \( C \) of length \( n \) satisfies the following bound:

\[
d_E(C) \leq \begin{cases} 
4\left\lfloor \frac{n}{24} \right\rfloor + 4 & \text{if } k = 2, n \not\equiv 22 \pmod{24}, \\
4\left\lfloor \frac{n}{24} \right\rfloor + 6 & \text{if } k = 2, n \equiv 22 \pmod{24}, \\
3\left\lfloor \frac{n}{12} \right\rfloor + 3 & \text{if } k = 3, \\
8\left\lfloor \frac{n}{24} \right\rfloor + 8 & \text{if } k = 4, n \not\equiv 23 \pmod{24}, \\
8\left\lfloor \frac{n}{24} \right\rfloor + 12 & \text{if } k = 4, n \equiv 23 \pmod{24}. 
\end{cases}
\]
Note that a binary self-dual code of length divisible by 24 meeting the bound with equality must be Type II.

Although the following two lemmas are somewhat trivial, we give proofs for the sake of completeness.

**Lemma 2.3.** Let $C$ be a self-dual $\mathbb{Z}_k$-code of length $n$.

(a) If $n \neq 23$ and $k \geq 2\lfloor \frac{n}{24} \rfloor + 3$, then $d_E(C) \leq 2k\lfloor \frac{n}{24} \rfloor + 2k$.

(b) If $n = 23$ and $k \geq 4$, then $d_E(C) \leq 3k$.

**Proof.** Since both cases are similar, we only give a proof of (a). Note that the Euclidean weight of a codeword of $C$ is divisible by $k$. Suppose that $d_E(C) \geq 2k\lfloor \frac{n}{24} \rfloor + 3$. Since $\min(A_k(C)) = \min\{k, d_E(C)/k\}$, $\min(A_k(C)) \geq 2\lfloor \frac{n}{24} \rfloor + 3$, which is a contradiction to the upper bound on the minimum norms of unimodular lattices.

**Lemma 2.4.** If $C$ is a self-dual $\mathbb{Z}_k$-code of length 48, then $d_E(C) \leq 6k$.

**Proof.** By the bound (1) and Lemma 2.3, it is sufficient to consider the cases only for $k = 5, 6$. Assume that $k = 5, 6$ and $d_E(C) \geq 7k$. Since $k < d_E(C)/k$, $\min(A_k(C)) = k$ and the kissing number of $A_k(C)$ is 96. Note that unimodular lattices $L$ with $\min(L) = 6$ and 5 are extremal even unimodular lattices and optimal odd unimodular lattices, respectively. However, the kissing numbers of such lattices are 52416000 (see [11, Chap. 7, (68)]) and 385024 or 393216 [20], respectively. This is a contradiction.

By the bound (1) along with Lemmas 2.3 and 2.4, a self-dual $\mathbb{Z}_k$-code $C$ of length $n \leq 48$ satisfies the following bound:

$$d_E(C) \leq \begin{cases} 
3k & \text{if } n = 23 \text{ and } k \geq 4, \\
4\lfloor \frac{n}{24} \rfloor + 6 & \text{if } n = 22, 46 \text{ and } k = 2, \\
20 & \text{if } n = 47 \text{ and } k = 4, \\
2k\lfloor \frac{n}{24} \rfloor + 2k & \text{otherwise}.
\end{cases}$$

We say that a self-dual $\mathbb{Z}_k$-code meeting the bound (2) with equality is extremal for length $n \leq 48$. We say that a self-dual $\mathbb{Z}_k$-code $C$ is near-extremal

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1For $k = 3$, a self-dual code meeting the bound (1) is usually called extremal. However, we here adopt this definition since we simultaneously consider the existence of extremal self-dual $\mathbb{Z}_k$-codes for all integers $k$ with $k \geq 2$. 

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if \( d_E(C) + k \) meets the bound (2). We only consider near-extremal self-dual \( \mathbb{Z}_k \)-codes when there is no extremal self-dual \( \mathbb{Z}_k \)-code of that length.

The following lemma shows that an extremal self-dual \( \mathbb{Z}_k \)-code of lengths 24 and 48 must be Type II for every even positive integer \( k \).

**Lemma 2.5.** Let \( C \) be a Type I \( \mathbb{Z}_k \)-code of length \( n \).

(a) If \( n = 24 \), then \( d_E(C) \leq 3k \).

(b) If \( n = 48 \), then \( d_E(C) \leq 5k \).

**Proof.** We give a proof of (b). By the bound (1), it is sufficient to consider only \( k \geq 4 \). Assume that \( d_E(C) \geq 6k \). If \( k \geq 6 \), then \( A_k(C) \) has minimum norm 6 from the upper bound on the minimum norms of unimodular lattices. In addition, \( A_k(C) \) must be even \([12]\), that is, \( C \) is Type II. If \( k = 5 \), then \( A_5(C) \) is an optimal odd unimodular lattice with kissing number 96, which contradicts that the kissing number is 385024 or 393216 \([20]\). Finally, suppose that \( k = 4 \). Since \( d_E(C) \geq 24 \), \( A_4(C) \) satisfies the condition that \( \min(A_4(C)) = 4 \), the kissing number is 96 and there is no vector of norm 5. By \([10]\) (2) and (3)], one can determine the possible theta series of \( A_4(C) \) and its shadow \( S \) as follows:

\[
\begin{align*}
\theta_{A_4(C)}(q) &= 1 + 96q^4 + (35634176 + 16777216\alpha)q^6 + \cdots, \\
\theta_S(q) &= \alpha + (96 - 96\alpha)q^2 + (-4416 + 4512\alpha)q^4 + \cdots,
\end{align*}
\]

respectively, where \( \alpha \) is an integer. From the coefficients of \( q^2 \) and \( q^4 \) in \( \theta_S(q) \), it follows that \( \alpha = 1 \). Hence, \( A_4(C) \) must be even, that is, \( C \) is Type II.

The proof of (a) is similar to that of (b), and it can be completed more easily. So the proof is omitted. \( \square \)

**Remark 2.6.** The odd Leech lattice contains a \( k \)-frame for every integer \( k \) with \( k \geq 3 \) \([28]\). The binary odd Golay code is a near-extremal Type I code of length 24. Hence, by Lemma 2.1 there is a near-extremal Type I \( \mathbb{Z}_k \)-code of length 24 if and only if \( k \) is an integer with \( k \geq 2 \).

### 2.5 Negacirculant matrices

Throughout this paper, let \( A^T \) denote the transpose of a matrix \( A \) and let \( I_k \) denote the identity matrix of order \( k \). An \( n \times n \) matrix is **circulant** and
negacirculant if it has the following form:

$$\begin{pmatrix}
  r_0 & r_1 & \cdots & r_{n-2} & r_{n-1} \\
  cr_{n-1} & r_0 & \cdots & r_{n-3} & r_{n-2} \\
  cr_{n-2} & cr_{n-1} & \cdots & r_{n-4} & r_{n-3} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  cr_1 & cr_2 & \cdots & cr_{n-1} & r_0
\end{pmatrix},$$

where $c = 1$ and $-1$, respectively. Most of matrices constructed in this paper are based on negacirculant matrices. In Section 4, in order to construct self-dual $\mathbb{Z}_k$-codes of length $4n$, we consider generator matrices of the following form:

$$(3) \quad \left( I_{2n} \begin{array}{cc} A & B \\ -B^T & A^T \end{array} \right),$$

where $A$ and $B$ are $n \times n$ negacirculant matrices. It is easy to see that the code is self-dual if $AA^T + BB^T = -I_n$.

In Section 3, in order to find $k$-frames in some unimodular lattices, we need to construct matrices $M$ satisfying the condition (5) in Proposition 3.1. Suppose that $p$ is a prime, which is congruent to 3 modulo 4. Let $Q_p = (q_{ij})$ be a $p \times p$ matrix over $\mathbb{Z}$, where $q_{ij} = 0$ if $i = j$, $-1$ if $j - i$ is a nonzero square modulo $p$, and 1 otherwise. We consider the following matrix:

$$P_{p+1} = \begin{pmatrix}
  0 & 1 & \cdots & 1 \\
  -1 & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  -1 & \cdots & 0 & 1
\end{pmatrix}.$$

Then it is well known that $P_{p+1}^TP_{p+1} = pI_{p+1}$, $P_{p+1}^T = -P_{p+1}$, and $P_{p+1} + I_{p+1}$ is a Hadamard matrix of order $p + 1$. Hence, these matrices $P_{p+1}$ satisfy (5).

In Section 3 we construct more $2m \times 2m$ matrices $M$ satisfying (5) using the following form:

$$(4) \quad \begin{pmatrix}
  A_1 & A_2 \\
  -A_2^T & A_1^T
\end{pmatrix},$$

where $A_1$ and $A_2$ are $m \times m$ negacirculant matrices.
2.6 Number theoretical results

In order to give infinite families of $k$-frames by Proposition 3.1, the following lemma is needed. The proofs are given by Miezaki in private communication [29], which are similar to those in [6, 21, 28].

Lemma 2.7 (Miezaki [29]). (a) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - d \pmod{3}$, $d \equiv a + b \pmod{3}$ and $p = \frac{1}{3}(a^2 + 25b^2 + c^2 + 25d^2)$ for each prime $p \neq 2, 5, 7, 13, 23$.

(b) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{4}$, $d \equiv a + 2b \pmod{4}$ and $p = \frac{1}{4}(a^2 + 7b^2 + c^2 + 7d^2)$ for each prime $p \neq 2, 7$.

(c) There are integers $a, b, c$ and $d$ satisfying $b \equiv c \pmod{5}$, $d \equiv a \pmod{5}$ and $p = \frac{1}{5}(a^2 + 49b^2 + c^2 + 49d^2)$ for each prime $p \neq 3, 7, 11, 19, 29$.

(d) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{5}$, $d \equiv a + 2b \pmod{5}$ and $p = \frac{1}{5}(a^2 + 25b^2 + c^2 + 25d^2)$ for each prime $p \neq 2, 3, 17$.

(e) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{4}$, $d \equiv a + 2b \pmod{4}$ and $p = \frac{1}{4}(a^2 + 15b^2 + c^2 + 15d^2)$ for each prime $p \neq 2, 3$.

(f) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{6}$, $d \equiv a + 2b \pmod{6}$ and $p = \frac{1}{6}(a^2 + 49b^2 + c^2 + 49d^2)$ for each prime $p \neq 2, 3, 5, 7$.

(g) There are integers $a, b, c$ and $d$ satisfying $b \equiv c \pmod{4}$, $d \equiv a \pmod{4}$ and $p = \frac{1}{4}(a^2 + 19b^2 + c^2 + 19d^2)$ for each prime $p \neq 3, 13, 19$.

(h) There are integers $a, b, c$ and $d$ satisfying $b \equiv c \pmod{5}$, $d \equiv a \pmod{5}$ and $p = \frac{1}{5}(a^2 + 39b^2 + c^2 + 39d^2)$ for each prime $p \neq 2, 3, 7, 17$.

3 Construction of $m$-frames in some unimodular lattices

In this section, we provide a method for constructing $m$-frames in unimodular lattices, which are constructed from some self-dual $\mathbb{Z}_k$-codes by Construction A. Combining Lemma 2.7 with the method, we construct $m$-frames in odd unimodular lattices.

The following method is a slight generalization of [21] Propositions 3.3 and 3.6. Also, the cases $(k, m, \ell) = (4, 11, 2)$ and $(4, 11, 0)$ of the following method are used in [6, 28], respectively.
Proposition 3.1. Let \( k \) be an integer with \( k \geq 2 \), and let \( \ell \) be an integer with \( 0 \leq \ell \leq k - 1 \). Let \( M \) be an \( n \times n \) matrix over \( \mathbb{Z} \) satisfying

\[ M^T = -M \text{ and } MM^T = mI_n, \]

where \( m + \ell^2 \equiv -1 \pmod{k} \). Let \( C_{2n,k}(M) \) be the \( \mathbb{Z}_k \)-code of length \( 2n \) with generator matrix \( ( I_n \ M + \ell I_n ) \), where the entries of the matrix are regarded as elements of \( \mathbb{Z}_k \). Let \( a, b, c \) and \( d \) be integers with \( b \equiv c - \ell d \pmod{k} \) and \( d \equiv a + \ell b \pmod{k} \). Then \( C_{2n,k}(M) \) is self-dual, and the set of \( 2n \) rows of the following matrix

\[ F(M) = \frac{1}{\sqrt{k}} \begin{pmatrix} aI_n + bM & cI_n + dM \\ -cI_n + dM & aI_n - bM \end{pmatrix} \]

forms a \( \frac{1}{k}(a^2 + mb^2 + c^2 + md^2) \)-frame in the unimodular lattice \( A_k(C_{2n,k}(M)) \).

Proof. Since \( MM^T = mI_n \), \( M^T = -M \) and \( m + \ell^2 \equiv -1 \pmod{k} \), \( C_{2n,k}(M) \) is a self-dual \( \mathbb{Z}_k \)-code of length \( 2n \). Thus, \( A_k(C_{2n,k}(M)) \) is a unimodular lattice. Since \( C_{2n,k}(M) \) is self-dual and \( M^T = -M \), both \( G = ( I_n \ M + \ell I_n ) \) and \( H = ( M - \ell I_n \ I_n ) \) are generator matrices of \( C_{2n,k}(M) \).

Let \( s, t \) be integers. Here, we regard the entries of the matrices \( G, H \) as integers. Then

\[ \begin{pmatrix} sG + tH \\ -tG + sH \end{pmatrix} = \begin{pmatrix} (s - \ell t)I_n + tM & (\ell s + t)I_n + sM \\ -((\ell s + t)I_n + sM) & (s - \ell t)I_n - tM \end{pmatrix}. \]

Hence, if \( b \equiv c - \ell d \pmod{k} \) and \( d \equiv a + \ell b \pmod{k} \), then all rows of the matrix \( F(M) \) are vectors of \( A_k(C_{2n,k}(M)) \). Since \( F(M)F(M)^T = \frac{1}{k}(a^2 + mb^2 + c^2 + md^2)I_{2n} \), the result follows. \( \square \)

Remark 3.2. (i) It follows from the assumption that \( a^2 + mb^2 + c^2 + md^2 \equiv 0 \pmod{k} \).

(ii) By [13, Proposition 2.12], if \( n \equiv 2 \pmod{4} \) then \( m \) must be a square.

The matrices \( P_{p+1} \) \((p = 7, 19)\), which are given in Section 2.5, satisfy the assumptions in Proposition 3.1 for the integers \( k, m \) and \( \ell \) listed in Table 1. Using the form \( (4) \), we have found matrices \( D_n \) \((n = 6, 10, 14, 16, 18, 22, 24)\), \( D'_n \) \((n = 10, 14)\) and \( D''_{10} \) satisfying the assumptions in Proposition 3.1 for the integers \( k, m \) and \( \ell \) listed in Table 1 where the first rows \( r_{A_1} \) and \( r_{A_2} \) of negacirculant matrices \( A_1 \) and \( A_2 \) in (4) are listed in Table 1.
Table 1: Matrices satisfying the assumptions in Proposition 3.1

| \( M \) | \((k, m, \ell)\) | \(r_{A_1}\) | \(r_{A_2}\) |
|--------|----------------|------------|------------|
| \( D_6 \) | \((3, 25, 1)\) | \((0, 2, 2)\) | \((0, 1, -4)\) |
| \( P_8 \) | \((4, 7, 2)\) | \((0, 0, 2, 0)\) | \((1, 2, 2, -2)\) |
| \( D'_{10} \) | \((3, 25, 1)\) | \((0, 0, 0, 0)\) | \((-3, -2, 2, -2)\) |
| \( D''_{10} \) | \((5, 49, 0)\) | \((0, 0, 3, 3)\) | \((-2, -3, 4, -1)\) |
| \( D_{14} \) | \((3, 25, 1)\) | \((0, 2, 1, 0, 0, 1, 2)\) | \((-1, -2, 1, -2, 1, 0)\) |
| \( D_{14}' \) | \((5, 25, 2)\) | \((0, 0, 2, -1, -1, 2, 0)\) | \((-2, -1, -2, 0, -1, -1, -2)\) |
| \( D_{16} \) | \((4, 15, 2)\) | \((0, 1, 1, 0, 1, 0, 1, 1)\) | \((1, 1, 1, -1, -1, 2, 1, 0)\) |
| \( D_{18} \) | \((6, 49, 2)\) | \((0, 1, -3, 0, 2, 2, 0, -3, 1)\) | \((-2, -1, 2, 1, 2, 1, 1, 1)\) |
| \( P_{20} \) | \((4, 19, 0)\) | \(\star\) | \(\star\) |
| \( D_{22} \) | \((5, 25, 2)\) | \((0, 0, -1, 1, 0, 0, 0, 1, -1, 0)\) | \((1, 0, -2, 1, 1, 1, 2, 1, 0, 2, -2)\) |
| \( D_{24} \) | \((5, 39, 0)\) | \((0, 1, 1, 1, 2, -1, 1, -1, 2, 1, 1, 1)\) | \((-2, -1, 2, -1, -1, -2, 0, 1, 0, 2, -1, -1)\) |

By Proposition 3.1 for each of the matrices \( M \) given in Table 1, the odd unimodular lattice \( A_k(C_{2n,k}(M)) \), which is constructed from the Type I \( \mathbb{Z}_k \)-code \( C_{2n,k}(M) \), contains a \( \frac{1}{3}(a^2 + mb^2 + c^2 + md^2) \)-frame for integers \( a, b, c \) and \( d \) with \( b \equiv c - \ell d \pmod{k} \) and \( d \equiv a + \ell b \pmod{k} \). The minimum norms \( \min(L) \) of the lattices \( L = A_k(C_{2n,k}(M)) \) listed in Table 2, which have been determined by MAGMA, are also listed in the table.

**Lemma 3.3.** Suppose that \( L \) is any of the lattices listed in Table 2. Then \( L \) contains a \( k \)-frame for an integer \( k \) satisfying the conditions (\( \star \)) listed in Table 2 where \( m_i \) in (\( \star \)) is a non-negative integer.

**Proof.** All cases are similar, and we only give details for the lattice \( A_3(C_{12,3}(D_6)) \).

Let \( a, b, c \) and \( d \) be integers with \( b \equiv c - d \pmod{3} \) and \( d \equiv a + b \pmod{3} \).

By Proposition 3.1 \( A_3(C_{12,3}(D_6)) \) contains a \( \frac{1}{3}(a^2 + 25b^2 + c^2 + 25d^2) \)-frame.

By Lemma 2.7 (a), there are integers \( a, b, c \) and \( d \) satisfying \( b \equiv c - d \pmod{3} \), \( d \equiv a + b \pmod{3} \) and \( p = \frac{1}{3}(a^2 + 25b^2 + c^2 + 25d^2) \) for each prime \( p \neq 2, 5, 7, 13, 23 \). The result follows from Lemma 2.2. For the other lattices, Table 2 lists the cases of Lemma 2.7 which are needed in the proof. \( \square \)
Table 2: Unimodular lattices by Proposition 3.1

| $L$                      | min($L$) | Condition (*) | Case |
|--------------------------|----------|---------------|------|
| $A_3(C_{12,3}(D_6))$    | 2        | $k \geq 2$, $k \neq 2^{m_1}5^{m_2}7^{m_3}13^{m_4}23^{m_5}$ | (a)  |
| $A_4(C_{16,4}(P_8))$    | 2        | $k \geq 2$, $k \neq 2^{m_1}7^{m_2}$ | (b)  |
| $A_3(C_{20,3}(D_{10}))$ | 2        | $k \geq 2$, $k \neq 2^{m_1}5^{m_2}7^{m_3}13^{m_4}23^{m_5}$ | (a)  |
| $A_3(C_{20,3}(D_{10}'))$ | 2        | $k \geq 2$, $k \neq 2^{m_1}5^{m_2}7^{m_3}13^{m_4}23^{m_5}$ | (a)  |
| $A_5(C_{20,5}(D_{10}'))$ | 2        | $k \geq 2$, $k \neq 2^{m_1}3^{m_2}7^{m_3}11^{m_4}19^{m_5}29^{m_6}$ | (c)  |
| $A_3(C_{28,3}(D_{14}))$ | 3        | $k \geq 3$, $k \neq 2^{m_1}5^{m_2}7^{m_3}13^{m_4}23^{m_5}$ | (a)  |
| $A_5(C_{28,5}(D_{14}'))$ | 3        | $k \geq 3$, $k \neq 2^{m_1}3^{m_2}17^{m_3}$ | (d)  |
| $A_4(C_{32,4}(D_{16}))$ | 4        | $k \geq 4$, $k \neq 2^{m_1}3^{m_2}$ | (e)  |
| $A_6(C_{36,6}(D_{18}))$ | 4        | $k \geq 4$, $k \neq 2^{m_1}3^{m_2}5^{m_3}7^{m_4}$ | (f)  |
| $A_4(C_{40,4}(P_{20}))$ | 4        | $k \geq 4$, $k \neq 2^{m_1}3^{m_2}13^{m_3}19^{m_4}$ | (g)  |
| $A_5(C_{44,5}(D_{22}))$ | 4        | $k \geq 4$, $k \neq 2^{m_1}3^{m_2}17^{m_3}$ | (d)  |
| $A_5(C_{48,5}(D_{24}))$ | 5        | $k \geq 5$, $k \neq 2^{m_1}3^{m_2}7^{m_3}17^{m_4}$ | (h)  |

4 Frames of some extremal odd unimodular lattices and extremal Type I $\mathbb{Z}_k$-codes

In this section, for each $L$ of the extremal (optimal) odd unimodular lattices listed in Table 2, we determine all integers $k$ such that $L$ contains a $k$-frame. This yields the existence of some extremal (near-extremal) Type I $\mathbb{Z}_k$-codes.

4.1 Frames of $D_{12}^+$ and Type I $\mathbb{Z}_k$-codes of length 12

There is a unique extremal odd unimodular lattice in dimension 12, up to isomorphism (see [11, Table 16.7]), where the lattice is denoted by $D_{12}^+$. There is a unique binary extremal Type I code of length 12, up to equivalence [31], where the code is denoted by $B_{12}$ in [31, Table 2]. It is known that $D_{12}^+ \cong A_2(B_{12})$. Hence, by Lemma 3.3, it is sufficient to investigate the existence of a $k$-frame in $D_{12}^+$ for $k = 5, 7, 13, 23$.

There are 16 inequivalent Type I $\mathbb{Z}_5$-codes of length 12 [24]. We have verified by MAGMA that $D_{12}^+ \cong A_5(C_i)$ for $i = 8, 11, 13, 16$, where $C_i$ denotes the $i$th code in [24, Table III]. There are 64 inequivalent Type I $\mathbb{Z}_7$-codes of length 12 [23], where these codes are denoted by $C_{12,i}$ ($i = 1, 2, \ldots, 64$) in [23, Table 1]. We have verified by MAGMA that $D_{12}^+ \cong A_7(C_{12,i})$ for $i = 11, 12,$
For $k = 13$ and 23, let $C_{k,12}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (3), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are as follows:

$$(r_A, r_B) = ((0,1,6),(2,3,1)) \text{ and } ((0,1,18),(7,4,0)),$$

respectively. Since $AA^T + BB^T = -I_3$, these codes are Type I. Moreover, we have verified by Magma that $A_k(C_{k,12}) \cong D_{12}^+(k = 13, 23)$. Hence, combining with Lemma 3.3 we have the following:

**Theorem 4.1.** $D_{12}^+$ contains a $k$-frame if and only if $k$ is an integer with $k \geq 2$.

By Lemma 2.1, we have the following:

**Corollary 4.2.** There is an extremal Type I $\mathbb{Z}_k$-code of length 12 if and only if $k$ is an integer with $k \geq 2$.

### 4.2 Frames of $D_8^2$ and Type I $\mathbb{Z}_k$-codes of length 16

There is a unique extremal odd unimodular lattice in dimension 16, up to isomorphism (see [11, Table 16.7]), where the lattice is denoted by $D_8^2$. There is a unique binary extremal Type I code of length 16, up to equivalence [31], where the code is denoted by $F_{16}$ in [31, Table 2]. It is known that $D_8^2 \cong A_2(F_{16})$. Hence, by Lemma 3.3 it is sufficient to investigate the existence of a 7-frame in $D_8^2$.

Let $C_{7,16}$ be the $\mathbb{Z}_7$-code with generator matrix of the form (3), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are as follows:

$$r_A = (0,0,1,1) \text{ and } r_B = (1,3,1,0),$$

respectively. Since $AA^T + BB^T = -I_4$, $C_{7,16}$ is Type I. We have verified by Magma that $A_7(C_{7,16}) \cong D_8^2$. Hence, combining with Lemma 3.3 we have the following:

**Theorem 4.3.** $D_8^2$ contains a $k$-frame if and only if $k$ is an integer with $k \geq 2$.

By Lemma 2.1, we have the following:

**Corollary 4.4.** There is an extremal Type I $\mathbb{Z}_k$-code of length 16 if and only if $k$ is an integer with $k \geq 2$. 


4.3 Frames of $D_4^5$, $A_5^4$, $D_{20}$ and Type I $\mathbb{Z}_k$-codes of length 20

There are 12 non-isomorphic extremal odd unimodular lattices in dimension 20 (see [11, Table 16.7]). We denote the $i$th lattices ($i = 1, 11, 12$) in dimension 20 in [11, Table 16.7], by $D_{20}$, $D_{4}^5$, $A_5^4$, respectively. We have verified by MAGMA that $D_4^5 \cong A_3(C_{20,3}(D_{10}))$ in Table 2, $A_5^4 \cong A_3(C_{20,3}(D_{10}))$ in Table 2 and $D_{20} \cong A_5(C_{20,5}(D_{10}'))$ in Table 2. By Lemma 3.3, it is sufficient to investigate the existence of a $k$-frame in $D_4^5$ and $A_5^4$ for $k = 2, 5, 7, 13, 23$, and a $k$-frame in $D_{20}$ for $k = 2, 3, 7, 11, 19, 29$.

Table 3: Extremal Type I $\mathbb{Z}_k$-codes of length 20

| Code  | $r_A$     | $r_B$     | Code  | $r_A$     | $r_B$     |
|-------|-----------|-----------|-------|-----------|-----------|
| $C_{5,20}$ | (0, 0, 0, 1, 1) | (1, 4, 2, 1, 0) | $C_{7,20}$ | (0, 0, 0, 1, 6) | (3, 0, 1, 1, 0) |
| $C_{13,20}$ | (0, 0, 0, 1, 1) | (10, 3, 2, 1, 0) | $C_{23,20}$ | (0, 0, 0, 1, 18) | (7, 4, 0, 0, 0) |
| $C_{5,20}'$ | (0, 0, 0, 1, 4) | (3, 1, 4, 1, 0) | $C_{7,20}'$ | (0, 0, 0, 1, 5) | (1, 5, 3, 1, 0) |
| $C_{13,20}'$ | (0, 0, 0, 1, 4) | (4, 0, 3, 3, 0) | $C_{23,20}'$ | (0, 0, 0, 1, 12) | (3, 5, 7, 1, 0) |
| $C_{11,20}'$ | (0, 0, 0, 1, 4) | (1, 3, 2, 3, 1) | $C_{9,20}'$ | (0, 0, 0, 1, 3) | (1, 2, 4, 2, 0) |
| $C_{29,20}'$ | (0, 0, 0, 1, 8) | (5, 6, 6, 3, 2) | $C_{19,20}'$ | (0, 0, 0, 1, 12) | (14, 12, 11, 1, 0) |

Figure 1: A generator matrix of $C_{4,20}'$

There are 7 binary extremal Type I codes of length 20, up to equivalence [31]. The unique code containing 5 (resp. 45) codewords of weight 4 is denoted by $M_{20}$ (resp. $J_{20}$) in [31, Table 2]. We have verified by MAGMA that $A_2(M_{20}) \cong D_4^5$ and $A_2(J_{20}) \cong D_{20}$. It is known that there is no binary
Type I code \( C \) such that \( A_2(C) \cong A_4^2 \). There are 6 inequivalent ternary self-dual codes of length 20 and minimum weight 6 \[32\]. We have verified by MAGMA that \( L \) is an extremal odd unimodular lattice in dimension 20 such that \( L \cong A_3(C) \) for some ternary self-dual code \( C \) if and only if \( L \) is isomorphic to \( D_5^5 \) or \( A_4^5 \).

Let \( C_{k,20}, C'_{k,20} \) \((k = 5, 7, 13, 23)\) and \( C''_{k,20} \) \((k = 7, 9, 11, 19, 29)\) be the \( \mathbb{Z}_k \)-codes with generator matrices of the form \[3\], where the first rows \( r_A \) and \( r_B \) of \( A \) and \( B \) are listed in Table \[3\]. Since \( AA^T + BB^T = -I_5 \), these codes are Type I. Let \( C'_{4,20} \) be the \( \mathbb{Z}_4 \)-code with generator matrix of the following form:

\[
G_{20} = \begin{pmatrix}
I_9 & A & B_1 + 2B_2 \\
O & 2I_2 & 2D
\end{pmatrix},
\]

where we list in Figure \[1\] the matrices \( A \) (1, 0)-matrices and \( O \) denotes the zero matrix of appropriate size. It follows from \( G_{20}G_{20}^T = O \) and \( \#C'_{4,20} = 4^{10} \) that \( C'_{4,20} \) is self-dual. The code \( C'_{4,20} \) has been found by directly finding a 4-frame in \( A_5^4 \) using MAGMA. In a similar way, some other (new) self-dual \( \mathbb{Z}_4 \)-codes are also constructed in this paper. We have verified by MAGMA that \( A_k(C_{k,20}) \cong D_4^5 \) \((k = 5, 7, 13, 23)\), \( A_k(C''_{k,20}) \cong A_5^4 \) \((k = 4, 5, 7, 13, 23)\) and \( A_k(C''_{k,20}) \cong D_{20} \) \((k = 7, 9, 11, 19, 29)\). Hence, combining with Lemma \[3.3\], we have the following:

**Theorem 4.5.** \( D_5^5 \) contains a \( k \)-frame if and only if \( k \) is an integer with \( k \geq 2 \). \( A_5^4 \) contains a \( k \)-frame if and only if \( k \) is an integer with \( k \geq 3 \). \( D_{20} \) contains a \( k \)-frame if and only if \( k \) is an integer with \( k \geq 2, k \neq 3 \).

**Remark 4.6.** \( D_{20} \) has theta series \( 1 + 760q^2 + 77560q^4 + 524288q^5 + \cdots \).

By Lemma \[2.1\], we have the following:

**Corollary 4.7.** There is an extremal Type I \( \mathbb{Z}_k \)-code of length 20 if and only if \( k \) is an integer with \( k \geq 2 \).

### 4.4 Type I \( \mathbb{Z}_k \)-codes of length 28

There is no extremal odd unimodular lattice in dimension 28 and the largest minimum norm among odd unimodular lattices in dimension 28 is 3. There are 38 non-isomorphic optimal odd unimodular lattices in dimension 28 \[1\]. In \[1\], the 38 lattices are denoted by \( R_{28,1}(\emptyset), R_{28,2}(\emptyset), \ldots, R_{28,36}(\emptyset), R_{28,37}(\emptyset) \),
We have verified by MAGMA that \( \mathbf{R}_{28,32}(\emptyset) \cong A_3(C_{28,3}(D_{14})) \) in Table 2 and \( \mathbf{R}_{28,15}(\emptyset) \cong A_5(C_{28,5}(D'_{14})) \) in Table 2. By Lemma 3.3, it is sufficient to investigate the existence of a \( k \)-frame in \( \mathbf{R}_{28,32}(\emptyset) \) for \( k = 4, 5, 7, 13, 23 \) and a \( k \)-frame in \( \mathbf{R}_{28,15}(\emptyset) \) for \( k = 3, 4, 17 \).

| Code   | \( r_A \)          | \( r_B \)          |
|--------|---------------------|---------------------|
| \( C_{5,28} \) | (0, 0, 0, 1, 3, 4, 2) | (3, 1, 2, 0, 3, 4, 0) |
| \( C_{7,28} \) | (0, 1, 2, 4, 2, 3) | (2, 2, 4, 0, 4, 1, 2) |
| \( C_{13,28} \) | (0, 0, 0, 1, 0, 9, 1) | (5, 1, 3, 7, 7, 1, 4) |
| \( C_{23,28} \) | (0, 0, 0, 1, 12, 1, 1) | (3, 19, 7, 5, 14, 21, 17) |
| \( C'_{17,28} \) | (0, 0, 0, 1, 13, 14, 2) | (10, 1, 1, 9, 16, 11, 15) |

Let \( C_{k,28} (k = 5, 7, 13, 23) \) and \( C'_{17,28} \) be the \( \mathbb{Z}_k \)-codes with generator matrices of the form (3), where the first rows \( r_A \) and \( r_B \) of \( A \) and \( B \) are listed in Table 4. Since \( AA^T + BB^T = -I_7 \), these codes are Type I. Let \( C_{4,28} \) and \( C'_{4,28} \) be the \( \mathbb{Z}_4 \)-codes with generator matrices of the following form:

\[
\begin{pmatrix}
I_{13} & A & B_1 + 2B_2 \\
O & 2I_2 & 2D
\end{pmatrix},
\]

where we list in Figure 2 the matrices \( \begin{pmatrix} A & B_1 + 2B_2 \\ 2I_2 & 2D \end{pmatrix} \). Then these codes are Type I. For \( k = 4, 5, 7, 13, 23 \), we have verified by MAGMA that \( A_k(C_{k,28}) \cong \mathbf{R}_{28,32}(\emptyset) \). For \( k = 4, 17 \), we have verified by MAGMA that \( A_k(C'_{4,28}) \cong \mathbf{R}_{28,15}(\emptyset) \). It is known that \( \mathbf{R}_{28,15}(\emptyset) \) contains a 3-frame (see [22] for a classification of 3-frames in the 38 lattices). Hence, combining with Lemma 3.3 we have the following:

**Theorem 4.8.** \( \mathbf{R}_{28,i}(\emptyset) (i = 15, 32) \) contains a \( k \)-frame if and only if \( k \) is an integer with \( k \geq 3 \).

**Lemma 4.9.** Let \( C \) be a Type I \( \mathbb{Z}_k \)-code of length 28. Then \( d_E(C) \leq 3k \).

**Proof.** As described above, the largest minimum norm among odd unimodular lattices in dimension 28 is 3. Since \( d_E(C) \leq 3k \) for \( k = 2, 3 \) (see [8, 25]), it is sufficient to consider the cases \( k \geq 4 \) only. Assume that \( d_E(C) \geq 4k \). Since \( \min(A_k(C)) = \min\{k, d_E(C)/k\} \), \( \min(A_k(C)) \geq 4 \), which is a contradiction. \( \square \)
There are three inequivalent binary Type I codes of length 28 and minimum weight 6 \[8\]. Hence, by Lemma 2.1, we have the following:

**Corollary 4.10.** There is a near-extremal Type I $Z_k$-code of length 28 if and only if $k$ is an integer with $k \geq 2$.

\[
\begin{pmatrix}
00 & 3221032113010 \\
00 & 2312302202000 \\
01 & 1011113132031 \\
01 & 2021011201031 \\
10 & 303332032202 \\
00 & 2220031132311 \\
00 & 1102312302202 \\
11 & 2213122020013 \\
01 & 320020111201 \\
01 & 3133230220230 \\
10 & 3110000202123 \\
10 & 3011332120200 \\
10 & 3331011112112 \\
20 & 222002020000 \\
02 & 002222000000
\end{pmatrix}
\begin{pmatrix}
01 & 1023301203302 \\
01 & 1022200000021 \\
01 & 1130203022312 \\
11 & 1202303012212 \\
00 & 2321232113032 \\
01 & 002112332213 \\
11 & 113323310300 \\
00 & 33210002311 \\
00 & 1210231221321 \\
11 & 2012013002211 \\
11 & 1010001123020 \\
11 & 2203101320001 \\
00 & 3302011030033 \\
20 & 0002202020200 \\
02 & 2220222222000
\end{pmatrix}
\]

Figure 2: Generator matrices of $C_{4,28}$ and $C'_{4,28}$

### 4.5 Type I $Z_k$-codes of length 32

There are 5 non-isomorphic extremal odd unimodular lattices in dimension 32, and these 5 lattices are related to the 5 inequivalent binary extremal Type II codes of length 32 \[10\]. The 5 codes are denoted by $C81, C82, \ldots, C85$ in \[8\, Table A\]. We denote the extremal odd unimodular lattice related to $Ci$ by $L_{32,i}$ ($i = 81, \ldots, 85$). We have verified by MAGMA that $L_{32,82} \cong A_4(C_{324}(D_{16}))$ in Table 2. Since $A_4(C_{324}(D_{16}))$ contains a 4-frame, it is sufficient to investigate the existence of a $k$-frame in $L_{32,82}$ for $k = 6, 9$ by Lemma 3.3.

For $k = 6, 9$, let $C_{k,32}$ be the $Z_k$-code with generator matrix of the form \[3\], where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 5. Since $AA^T + BB^T = -I_8$, these codes are self-dual. Note that the code $C_{6,32}$ is not Type II, since $\text{wt}_E(r_A) + \text{wt}_E(r_B) = 41$, where $\text{wt}_E(x)$ denotes the Euclidean weight of $x$. For $k = 6, 9$, we have verified by MAGMA that $A_k(C_{k,32}) \cong L_{32,82}$. Hence, combining with Lemma 3.3, we have the following:
Table 5: Extremal Type I $Z_k$-codes of length 32

| Code | $r_A$       | $r_B$       |
|------|-------------|-------------|
| $C_{6,32}$ | (0, 0, 1, 2, 2, 1, 2) | (1, 0, 5, 5, 1, 3, 3) |
| $C_{9,32}$ | (0, 0, 1, 5, 6, 0, 1) | (0, 6, 2, 2, 6, 1, 7) |

**Theorem 4.11.** $L_{32,82}$ contains a $k$-frame if and only if $k$ is an integer with $k \geq 4$.

There are three inequivalent binary extremal Type I codes of length 32 [9]. Any ternary self-dual code of length 32 has minimum weight at most 9 [26]. Hence, by Lemma 2.1 we have the following:

**Corollary 4.12.** There is an extremal Type I $Z_k$-code of length 32 if and only if $k$ is a positive integer with $k \neq 1, 3$.

For each extremal odd unimodular lattice in dimension 32, one of the even unimodular neighbors is extremal [10]. Moreover, it follows from the construction in [10] that the extremal even unimodular neighbor of $L_{32,82}$ is the 32-dimensional Barnes–Wall lattice $BW_{32}$ (see e.g. [11, Chapter 8, Section 8] for $BW_{32}$). Since the even sublattice of $L_{32,82}$ contains a $2k$-frame for every integer $k$ with $k \geq 2$ by Theorem 4.11 we have the following:

**Proposition 4.13.** $BW_{32}$ contains a $2k$-frame if and only if $k$ is an integer with $k \geq 2$.

Since there are 5 inequivalent binary extremal Type II codes of length 32 [8], by Lemma 2.1 we have an alternative proof of the following:

**Corollary 4.14** (Harada and Miezaki [21]). There is an extremal Type II $Z_{2k}$-code of length 32 if and only if $k$ is a positive integer.

### 4.6 Type I $Z_k$-codes of length 36

Since $A_6(C_{36,6}(D_{18}))$ in Table 2 contains a 6-frame, it is sufficient to investigate the existence of a $k$-frame in $A_6(C_{36,6}(D_{18}))$ for $k = 4, 5, 7, 9$ by Lemma 3.3. For $k = 5, 7, 9$, let $C_{k,36}$ be the $Z_k$-code with generator matrix of the form (3), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 6.
Since $AA^T + BB^T = -I_9$, these codes are Type I. Let $C_{4,36}$ be the $\mathbb{Z}_4$-code with generator matrix of the following form:

$$
\begin{pmatrix}
I_{16} & A & B_1 + 2B_2 \\
O & 2I_4 & 2D
\end{pmatrix},
$$

where we list in Figure 3 the matrices $(A \ B_1 + 2B_2)$ and $2D$. It follows that $C_{4,36}$ is self-dual. For $k = 4, 5, 7, 9$, we have verified by Magma that $A_k(C_{k,36}) \cong A_6(C_{36,6}(D_{18}))$. Hence, combining with Lemma 3.3 we have the following:

**Theorem 4.15.** $A_6(C_{36,6}(D_{18}))$ contains a $k$-frame if and only if $k$ is an integer with $k \geq 4$.

**Remark 4.16.** We have verified by Magma that $A_6(C_{36,6}(D_{18}))$ has theta series $1 + 42840q^4 + 1916928q^5 + \cdots$ and automorphism group of order 288 (see [19] for details to distinguish $A_6(C_{36,6}(D_{18}))$ from the known lattices, and construction of more extremal odd unimodular lattices).

| Code  | $r_A$ | $r_B$ |
|-------|-------|-------|
| $C_{5,36}$ | (0,1,1,2,3,2,0,2,3) | (1,1,0,2,0,3,4,0,4) |
| $C_{7,36}$ | (0,1,6,2,3,3,6,4,5) | (4,3,3,6,2,4,3,0,3) |
| $C_{9,36}$ | (0,1,0,5,5,0,0,0,3) | (0,2,3,3,4,5,7,3) |

| Table 6: Extremal Type I $\mathbb{Z}_k$-codes of length 36 |

There are 41 inequivalent binary extremal Type I codes of length 36 [27]. There is a ternary extremal Type I code of length 36 [30]. Hence, by Lemma 2.1 we have the following:

**Corollary 4.17.** There is an extremal Type I $\mathbb{Z}_k$-code of length 36 if and only if $k$ is an integer with $k \geq 2$.

### 4.7 Type I $\mathbb{Z}_k$-codes of length 40

Since $A_4(C_{40,4}(P_{20}))$ in Table 2 contains a 4-frame, it is sufficient to investigate the existence of a $k$-frame in extremal odd unimodular lattices in dimension 40 for $k = 6, 9, 13, 19$ by Lemma 3.3. For $k = 9, 13, 19$, let $C_{k,40}$
be the $\mathbb{Z}_k$-code with generator matrix of the form (3), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 7. Since $AA^T + BB^T = -I_{10}$, these codes are Type I. Moreover, we have verified by Magma that $A_k(C_{k,40})$ is extremal ($k = 9, 13, 19$). An extremal Type I $\mathbb{Z}_6$-code of length 40 can be found in [15]. Hence, combining with Lemma 3.3, we have the following:

**Proposition 4.18.** There is an extremal odd unimodular lattice in dimension 40 containing a $k$-frame if and only if $k$ is an integer with $k \geq 4$.

**Remark 4.19.** The possible theta series of an extremal odd unimodular lattice in dimension 40 is given in [5]: $\theta_{40,0}(q) = 1 + (19120 + 256\alpha)q^4 + (1376256 - 4096\alpha)q^5 + \cdots$, where $\alpha$ is even with $0 \leq \alpha \leq 80$. There are 16470 non-isomorphic extremal odd unimodular lattices in dimension 40 having theta series $\theta_{40,80}(q)$, and these lattices are related to the 16470 inequivalent binary extremal Type II codes of length 40 [5]. We have verified by Magma that $A_4(C_{40,4}(P_{20}))$ has theta series $\theta_{40,80}(q)$ and automorphism group of order 7172259840, and $A_k(C_{k,40})$ ($k = 9, 13, 19$) have theta series $\theta_{40,0}(q)$ and automorphism group of order 40. Also, we have verified by Magma that three lattices $A_k(C_{k,40})$ ($k = 9, 13, 19$) are non-isomorphic.

There are 10200655 inequivalent binary extremal Type I codes of length 40 [5]. There is a ternary extremal Type I code of length 40 (see [35 Table XII]). Hence, by Lemma 2.1, we have the following:
Corollary 4.20. There is an extremal Type I $\mathbb{Z}_k$-code of length 40 if and only if $k$ is an integer with $k \geq 2$.

Table 7: Extremal Type I $\mathbb{Z}_k$-codes of length 40

| Code   | $r_A$               | $r_B$               |
|--------|---------------------|---------------------|
| $C_{9,40}$ | $(0, 0, 1, 0, 5, 8, 3, 0, 4, 4)$ | $(0, 5, 0, 0, 5, 6, 7, 2, 5, 8)$ |
| $C_{13,40}$ | $(0, 0, 1, 4, 10, 5, 1, 10, 11, 4)$ | $(11, 4, 4, 6, 7, 12, 11, 7, 2, 8)$ |
| $C_{19,40}$ | $(0, 0, 1, 2, 14, 16, 17, 1, 0, 13)$ | $(10, 2, 15, 2, 18, 16, 9, 15, 12, 0)$ |

We have verified by Magma that at least one of the even unimodular neighbors of $L$ is extremal for $L = A_4(C_{40,4}(P_{20})), A_9(C_{9,40}), A_{13}(C_{13,40})$ and $A_{19}(C_{19,40})$. There are 16470 inequivalent binary extremal Type II codes of length 40 [3]. By Lemma 2.1, we have an alternative proof of the following:

Corollary 4.21 (Harada and Miezaki [21]). There is an extremal Type II $\mathbb{Z}_{2k}$-code of length 40 if and only if $k$ is a positive integer.

4.8 Type I $\mathbb{Z}_k$-codes of length 44

By considering $A_5(C_{44,5}(D_{22}))$ in Table 2, it is sufficient to investigate the existence of a $k$-frame in extremal odd unimodular lattices in dimension 44 for $k = 4, 6, 9, 17$ by Lemma 3.3. For $k = 9, 17$, let $C_{k,44}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (3), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 8. Since $AA^T + BB^T = -I_{11}$, these codes are Type I. Moreover, we have verified by Magma that $A_k(C_{k,44})$ is extremal ($k = 9, 17$). For $k = 4$ and 6, an extremal Type I $\mathbb{Z}_k$-code of length 44 can be found in [18, Table 1] and [15], respectively. Hence, combining with Lemma 3.3, we have the following:

Proposition 4.22. There is an extremal odd unimodular lattice in dimension 44 containing a $k$-frame if and only if $k$ is an integer with $k \geq 4$.

Remark 4.23. The possible theta series of an extremal odd unimodular lattice in dimension 44 is given in [17]:

$\theta_{44,1,\beta}(q) = 1 + (6600 + 16\beta)q^4 + (811008 - 128\beta)q^5 + \cdots$, $\theta_{44,2,\beta}(q) = 1 + (6600 + 16\beta)q^4 + (679936 - 128\beta)q^5 + \cdots$,

where $\beta$ is an integer. We have verified by Magma that $A_5(C_{44,5}(D_{22}))$ and $A_k(C_{k,44})$ ($k = 9, 17$) have theta series $\theta_{44,1,\beta}(q)$ ($\beta = 0, 88, 176$) and automorphism groups of orders 44, 88, 44, respectively.
Table 8: Extremal Type I $\mathbb{Z}_k$-codes of length 44

| Code      | $r_A$                   | $r_B$                   |
|-----------|-------------------------|-------------------------|
| $C_{9,44}$| $(0,0,0,0,1,0,1,4,0,8,0)$| $(7,0,7,1,8,8,2,8,1,5,1)$|
| $C_{17,44}$| $(0,0,0,0,1,13,7,13,11,16,13)$| $(12,14,8,14,7,12,14,7,14,14,7)$|

For $k = 2, 3$, there is an extremal Type I $\mathbb{Z}_k$-code of length 44 (see [35, Tables X and XII]). Hence, by Lemma 2.1, we have the following:

**Corollary 4.24.** There is an extremal Type I $\mathbb{Z}_k$-code of length 44 if and only if $k$ is an integer with $k \geq 2$.

### 4.9 Remarks on Type I $\mathbb{Z}_k$-codes of length 48

By considering $A_5(C_{48,5}(D_{24}))$ in Table 2 we examine the existence of a $k$-frame in optimal odd unimodular lattices in dimension 48 for $k = 6, 7, 8, 9, 17$ by Lemma 3.3. It was shown in [20] that an extremal even unimodular lattice in dimension 48 has an optimal odd unimodular neighbor. Using this result, we have the following:

**Lemma 4.25.** There is an optimal odd unimodular lattice in dimension 48 containing an $8k$-frame for every positive integer $k$.

**Proof.** Let $\Lambda$ be an extremal even unimodular lattice in dimension 48. Let $x$ be a vector of $\Lambda$ with $(x, x) = 8$. Note that there are vectors of norm 8 in $\Lambda$ (see [11, Chap. 7, (68)]). Put $\Lambda_x^+ = \{v \in \Lambda \mid (x, v) \equiv 0 \pmod{2}\}$. Since there is a vector $y$ of $\Lambda$ such that $(x, y)$ is odd, the following lattice

$$\Lambda_x = \Lambda_x^+ \cup \left(\frac{1}{2}x + y\right) + \Lambda_x^+$$

is an optimal odd unimodular neighbor of $\Lambda$ [20].

Some extremal even unimodular lattice in dimension 48 containing an 8-frame can be found in [7, Corollary 1]. We take this lattice as $\Lambda$ in the above construction. Let $\{f_1, \ldots, f_{48}\}$ be an 8-frame in $\Lambda$. Then $\Lambda_{f_1}$ is an optimal odd unimodular neighbor containing $\{f_1, \ldots, f_{48}\}$. The result follows from Lemma 2.2. \qed
Proposition 4.26. There is an optimal odd unimodular lattice in dimension 48 containing a $k$-frame for every integer $k$ with $k \geq 5$ and $k \neq 2^{m_1}3^{m_2}17^{m_3}$, where $m_i$ are integers $(i = 1, 2, 3)$ with $(m_1, m_2) \in \{(0, 0), (0, 1), (1, 0), (2, 0)\}$ and $m_3 \geq 1$.

Remark 4.27. $A_6(C_{6,48})$ has kissing number 393216 [20, p. 553]. In addition, we have verified by Magma that $A_k(C_{k,48})$ is optimal ($k = 7, 9$). Hence, we have the following:

Corollary 4.28. There is a near-extremal Type I $\mathbb{Z}_k$-code of length 48 for $k = 2, 3, 4$ and for integers $k$ with $k \geq 5$, $k \neq 2^{m_1}3^{m_2}17^{m_3}$, where $m_i$ are integers $(i = 1, 2, 3)$ with $(m_1, m_2) \in \{(0, 0), (0, 1), (1, 0), (2, 0)\}$ and $m_3 \geq 1$.

Table 9: Near-extremal Type I $\mathbb{Z}_k$-codes of length 48

| Code | $r_A$ | $r_B$ |
|------|-------|-------|
| $C_{7,48}$ | $(0, 1, 6, 3, 0, 2, 0, 2, 4, 2, 5, 3)$ | $(3, 6, 1, 5, 4, 6, 0, 5, 0, 5, 1, 5)$ |
| $C_{9,48}$ | $(0, 1, 2, 4, 6, 1, 6, 2, 2, 0, 3, 0)$ | $(7, 2, 5, 1, 6, 8, 4, 1, 2, 2, 8, 4)$ |

Some near-extremal Type I $\mathbb{Z}_6$-code $C_{6,48}$ of length 48 can be found in [20]. For $k = 7, 9$, let $C_{k,48}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (3), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 9. Since $AA^T + BB^T = -I_{12}$, these codes are Type I. Moreover, we have verified by Magma that $A_k(C_{k,48})$ is optimal ($k = 7, 9$). Hence, we have the following:

For $k = 2, 3$, there is a near-extremal Type I $\mathbb{Z}_k$-code of length 48 (see [35, Tables X and XII]). Let $C_{4,48}$ be the $\mathbb{Z}_4$-code with generator matrix:

$\begin{pmatrix}
0 & 1 & \cdots & 1 \\
1 \\
\vdots & & R \\
1
\end{pmatrix}$

where $R$ is the $23 \times 23$ circulant matrix with first row

$(1, 1, 3, 0, 3, 3, 1, 2, 0, 1, 3, 2, 3, 0, 0, 3, 3, 2, 1, 2, 1, 1, 0)$.

We have verified by Magma that $C_{4,48}$ is a near-extremal Type I $\mathbb{Z}_4$-code of length 48. Hence, by Lemma 2.1, we have the following:
Using the method in the previous subsections, we tried to construct a near-extremal Type I $\mathbb{Z}_{17}$-code of length 48. However, our extensive search failed to discover such a code, then we stopped our search at length 48. It is worthwhile to determine whether there is a near-extremal Type I $\mathbb{Z}_{17}$-code of length 48.

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