Nonlinear Evolution and Breakup of the Cavitating Liquid Jet Surrounded by the Rotary Compressible Air

Shuang Xi Liu† and Ming Lu*‡

†National Engineering Laboratory for Mobile Source Emission Control Technology, China Automotive Technology & Research Center, Tianjin 300300, China
‡School of Mechanical, Electronic and Control Engineering, Beijing Jiaotong University, Beijing 100044, China

ABSTRACT: A nonlinear dispersion relationship has been established to study the surface evolution and breakup of the cavitating liquid jet with cavitation bubbles surrounded by the rotary air, and the built dispersion relationship and its solution are validated by comparing with the results in the reference. The effects of air rotation, fluid compressibility, and bubble volume fraction on jet morphology are investigated mainly. Air rotation changes the dominant mode of perturbation wave on the jet surface, and more uneven corrugated flows are formed at the interface with the increase of gas rotational strength. The fluid compressibility has little impact on jet morphology in the circumferential direction, while it has some impacts on the axial morphology, especially on the arrangement of droplets. Cavitation bubbles will affect the jet morphology, while the effect is smaller than gas rotation. Both swirling gas and fluid compressibility promote the jet breakup, while the influence of compressibility on jet breakup is obviously greater than gas rotation. In addition, the bubble volume fraction will promote the breakup of the cavitating liquid jet; however, this kind of promoting impact decreases with the increase of the cavitation bubble volume fraction.

1. INTRODUCTION

Liquid jets generally exist in nature and engineering, such as diesel engines, cylinder direct injection gasoline engines, gas turbines, and other power machinery.1,2 The fuel injection process has an important impact on the power, economy, and emission characteristics of the power machinery. The breakup and atomization process of fuel jets is actually the instability process of liquid jets. In the liquid injection process, the disturbance wave on the liquid surface will continue to develop, which will cause jet morphology change and splitting after instability. The stability of liquid jets has always been a research hotspot in the field of fluid mechanics.

Now, there are two main theories to study the instability of liquid jets, linear instability analysis and nonlinear instability analysis.

The research on the instability of liquid jets used by linear instability analysis is always the focus of attention.3–6 The purpose of linear instability analysis is to obtain the maximum temporal or spatial disturbance growth rate, the oscillation frequency, the dominant wavenumber, and other jet instability conditions. However, linear instability analysis only considers that the disturbance at the interface of liquid jets is of the first order and neglects the second order and more high order disturbances. Therefore, the calculation results about jet morphology and jet breakup are inaccurate but these parameters are the key indicators to determine jet atomization.

Compared with linear instability analysis, nonlinear instability analysis considers high order disturbances and can better reflect the changes of jet morphology and breakup situation. For now, a lot of research studies have reported on the instability of liquid jets by using the nonlinear instability analysis, and the liquid jet morphology and breakup have also been analyzed under some assumed conditions.8–20 Nonlinear instability analysis has been used to study the liquid jet morphology and breakup with swirling air, but the effect of air rotation on liquid surface morphology has not been found. Due to the complexity, the effect of air compressibility on liquid jet morphology has been ignored.19

Received: August 5, 2019
Accepted: December 5, 2019
Published: December 13, 2019

This is an open access article published under an ACS AuthorChoice License, which permits copying and redistribution of the article or any adaptations for non-commercial purposes.
In fact, in internal combustion engines, airflow is always swirling when diesel ejects out of the nozzle. In addition, with the increasingly stringent emission control requirements and energy shortage conditions, people want to increase fuel injection pressure as a kind of technical measure to meet the requirements of lower emission and higher combustion efficiency of diesel engines. Therefore, a study on the influences of fluid compressibility and air rotation on liquid jet morphology and instability has become more and more important. To our knowledge, studying the nonlinear evolution and instability of the cavitating liquid jet surrounded by the rotary compressible air has not been reported.

Based on our previous research, which studied the temporal instability and spatial instability of a cavitating liquid jet surrounded by the rotary compressible air by using linear instability analysis, this paper has analyzed the nonlinear evolution and instability of a cavitating liquid jet surrounded by the rotary compressible air by the use of three-dimensional nonlinear spatial instability theory. In Section 2, the first-order dispersion equation and second-order dispersion equation have been established, respectively, and the built dispersion equations and their solutions are validated by comparing with the results from the reference in Section 3. Then, in Section 4, the influences of air rotation, fluid compressibility, and bubble volume fraction on jet morphology are investigated, respectively.

2. PHYSICAL MODEL AND GOVERNING EQUATIONS

The research object is the ejection of a cavitating liquid jet into the rotary air. Liquid and air are inviscid and compressible. The nozzle radius is \( a \), the jet velocity is \( U_0 \), and the rotary velocity of air is \( W_0 \) in the \( \theta \) direction. Figure 1 gives the schematic of a cavitating liquid jet surrounded by the rotary air.

![Figure 1. Schematic of a cavitating liquid jet surrounded by the rotary air.](image)

The basic flow field when the jet is not disturbed (jet velocity, swirling airflow, the pressure relationship between liquid and air) are listed as follows:

\[ \mathbf{V}_1 = (0, 0, -U_0) \quad (0 \leq r \leq a) \]

\[ \mathbf{V}_2 = (0, W_0/r, 0) \quad (a < r < \infty) \]

\[ p_1 = \text{constant}, \quad p_2(r) = \frac{\sigma}{a} + \frac{1}{2} p_2 W_0^2 \left( \frac{1}{a^2} - \frac{1}{r^2} \right) \]

\[ (a < r < \infty) \]

where \( \mathbf{V}_1 \) is the velocity of the liquid jet, \( \mathbf{V}_2 \) is the air velocity. \( \bar{p}_1 \) is the liquid-side pressure, \( \bar{p}_2 \) is the air-side pressure, \( \sigma \) is the factor of surface tension, \( \bar{p}_2 \) is the density of the surrounding air.

Cavitating liquid jet density and sound speed are listed as follows:

\[ \bar{p}_1 = \alpha \rho_1 + (1 - \alpha) \rho_1 \]

\[ \bar{c}_1 = \left( \frac{E \rho_2 k_1}{\bar{p}_2 (1 - \alpha) + E \alpha} \right)^{1/2} \]

where \( \bar{p}_1 \) represents the density of the cavitating liquid jet, \( \rho_1 \) is the bubble density, \( \rho_1 \) is the liquid density, \( \bar{c}_1 \) represents the sound speed of the cavitating liquid jet, \( k_1 \) is the adiabatic index of the bubble, \( z_0 \) is the compressibility factor of the bubble, \( R \) represents the air constant, \( E_1 \) represents the liquid elasticity modulus, and \( \alpha \) represents the bubble volume fraction \((0 \leq \alpha \leq 0.2)\) in the study.

2.1. Disturbance Governing Equations. The research object meets the continuity equation and Euler equations:

\[
\frac{\partial \rho_i}{\partial t} + \frac{\partial \rho_i \mathbf{v}_i}{\partial r} + \frac{\partial \rho_i v_i \theta}{\partial \theta} + \frac{\partial \rho_i v_i z}{\partial z} = -\rho_i \left( \frac{1}{r} \frac{\partial}{\partial r} (r u_i) + \frac{1}{r \cos \theta} \frac{\partial}{\partial \theta} (r v_i \cos \theta) + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right)
\]

\[
\frac{\partial u_i}{\partial t} + \frac{\partial \mathbf{v}_i}{\partial r} + \frac{\partial \mathbf{v}_i \theta}{\partial \theta} + \frac{\partial \mathbf{v}_i z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (r \rho_i u_i) + \frac{1}{r \cos \theta} \frac{\partial}{\partial \theta} (r \rho_i v_i \cos \theta) + \frac{\partial}{\partial z} \frac{\partial}{\partial z} - \frac{\sigma_i}{\rho_i} \frac{\partial}{\partial z}
\]

where \( i = 1 \) represents the jet parameters, and \( i = 2 \) represents the air parameters.

The flow parameters are given as follows:

\[ \rho_1 = \bar{p}_1 + \rho_1', \mathbf{V}_1 = \mathbf{V}_1 + \mathbf{V}_1', \rho_2 = \bar{p}_2 + \rho_2' \]

where the horizontal line is the unperturbed parameter, and the apostrophe is the perturbed parameter.

Using the disturbance expansion method, the perturbed parameters can be listed as follows:

\[ \rho_1' = \sum_{n=1}^{\infty} \eta_{\rho_1} \rho_1^{n,n}, \rho_2' = \sum_{n=1}^{\infty} \eta_{\rho_2} \rho_2^{n,n}, \mathbf{V}_1' = \sum_{n=1}^{\infty} \eta_{\mathbf{V}_1} \mathbf{V}_1^{n,n} \]

where \( \eta_{\rho_1} \) is the initial amplitude of perturbed parameters.

In addition, the cavitating liquid jet requires first- and second-order governing equations as follows:

First order \((\eta_{\rho_0})\)

For the cavitating jet

\[
M_i \left\{ \frac{\partial v_{ii}'}{\partial t} - \frac{\partial v_{i1}'}{\partial z} \right\} = \left( \frac{v_{i1,1}}{r} + \frac{\partial v_{i1,1}}{\partial r} + \frac{\partial v_{i1,1}}{\partial \theta} + \frac{\partial v_{i1,1}}{\partial z} \right)
\]
\[
\frac{\partial v'_{1,1}}{\partial t} + \frac{\partial v'_{0,1}}{\partial z} = -\frac{1}{r} \frac{\partial p'_1}{\partial r}
\]
\[
\frac{\partial v'_{0,1}}{\partial t} + \frac{\partial v'_{0,1}}{\partial z} = -\frac{1}{r} \frac{\partial p'_1}{\partial r}
\]
\[
\frac{\partial v'_{1,1}}{\partial t} + \frac{\partial v'_{1,1}}{\partial z} = -\frac{1}{r} \frac{\partial p'_1}{\partial r}
\]  

(11)

For the surrounding air

\[
Ma \left( \frac{\partial p'_{2,1}}{\partial t} + \frac{E}{r^2} \frac{\partial p'_{2,1}}{\partial \theta} \right) = -Q \left( \frac{v'_{2,1}}{r} + \frac{\partial v'_{2,1}}{\partial r} + \frac{1}{r} \frac{\partial v'_{2,1}}{\partial \theta} \right)
\]

(12)

Second order (\( \eta_6^2 \))

For the cavitating jet

\[
Ma \left( \frac{\partial p'_{2,1}}{\partial t} - \frac{\partial p'_{1,2}}{\partial z} + \frac{v'_{0,1}}{r} \frac{\partial p'_{2,1}}{\partial r} \right) + \frac{v'_{2,1}}{r} \frac{\partial p'_{2,1}}{\partial \theta} + \frac{1}{r} \frac{\partial v'_{0,1}}{\partial \theta} + \frac{\partial v'_{1,2}}{\partial z} = 0
\]

(14)

\[
\frac{\partial v'_{12}}{\partial t} + \frac{v'_{11} v'_{01}}{r} \frac{\partial v'_{12}}{\partial \theta} + \frac{v'_{01}}{r} \frac{\partial v'_{12}}{\partial z} + v'_{11} \frac{v'_{12}}{r} \frac{\partial v'_{12}}{\partial \theta} + v'_{11} \frac{v'_{12}}{r} \frac{\partial v'_{12}}{\partial \theta} = 0
\]

(15)

\[
\frac{\partial v'_{02}}{\partial t} + \frac{v'_{11} v'_{01}}{r} \frac{\partial v'_{02}}{\partial \theta} + \frac{v'_{01}}{r} \frac{\partial v'_{02}}{\partial z} + v'_{11} \frac{v'_{02}}{r} \frac{\partial v'_{02}}{\partial \theta} + v'_{11} \frac{v'_{02}}{r} \frac{\partial v'_{02}}{\partial \theta} = 0
\]

(16)

\[
\begin{align*}
\left( \frac{\partial v'_{12}}{\partial t} + v'_{11} \frac{\partial v'_{01}}{\partial \theta} + v'_{01} \frac{\partial v'_{12}}{\partial z} &+ v'_{11} \frac{v'_{01}}{r} \frac{\partial v'_{12}}{\partial \theta} + v'_{11} \frac{v'_{01}}{r} \frac{\partial v'_{12}}{\partial \theta} \\
\frac{\partial v'_{02}}{\partial t} + v'_{11} \frac{\partial v'_{01}}{\partial \theta} + v'_{01} \frac{v'_{12}}{r} \frac{\partial v'_{02}}{\partial \theta} + v'_{11} \frac{v'_{02}}{r} \frac{\partial v'_{02}}{\partial \theta} &+ v'_{11} \frac{v'_{02}}{r} \frac{\partial v'_{02}}{\partial \theta} \\
\end{align*}
\]

(17)

For the surrounding air

\[
Ma \left( \frac{\partial p'_{2,1}}{\partial t} + \frac{E}{r^2} \frac{\partial p'_{2,1}}{\partial \theta} \right) + \frac{v'_{2,1}}{r} \frac{\partial p'_{2,1}}{\partial \theta} + \frac{1}{r} \frac{\partial v'_{0,1}}{\partial \theta} + \frac{\partial v'_{1,2}}{\partial z} = 0
\]

(18)

\[
\begin{align*}
\left( \frac{\partial p'_{2,2}}{\partial t} + v'_{2,1} \frac{\partial v'_{2,1}}{\partial \theta} + E \frac{\partial v'_{2,2}}{\partial \theta} + v'_{0,1} \frac{\partial v'_{2,1}}{\partial z} &+ v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} \\
\frac{\partial p'_{2,2}}{\partial t} + v'_{0,1} \frac{\partial v'_{2,1}}{\partial \theta} + E \frac{\partial v'_{2,2}}{\partial \theta} + v'_{0,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} &+ v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} \\
\end{align*}
\]

(19)

\[
\begin{align*}
\left( \frac{\partial p'_{2,2}}{\partial t} + v'_{2,1} \frac{\partial v'_{2,1}}{\partial \theta} + E \frac{\partial v'_{2,2}}{\partial \theta} + v'_{0,1} \frac{\partial v'_{2,1}}{\partial z} &+ v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} \\
\frac{\partial p'_{2,2}}{\partial t} + v'_{0,1} \frac{\partial v'_{2,1}}{\partial \theta} + E \frac{\partial v'_{2,2}}{\partial \theta} &+ v'_{0,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + \frac{1}{r} \frac{\partial p'_{2,2}}{\partial \theta} \\
\end{align*}
\]

(20)

\[
\begin{align*}
\left( \frac{\partial p'_{2,2}}{\partial t} + v'_{2,1} \frac{\partial v'_{2,1}}{\partial \theta} + E \frac{\partial v'_{2,2}}{\partial \theta} &+ v'_{0,1} \frac{\partial v'_{2,1}}{\partial z} + v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + \frac{1}{r} \frac{\partial p'_{2,2}}{\partial \theta} \\
\frac{\partial p'_{2,2}}{\partial t} + v'_{0,1} \frac{\partial v'_{2,1}}{\partial \theta} + E \frac{\partial v'_{2,2}}{\partial \theta} &+ v'_{0,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + v'_{2,1} \frac{v'_{2,2}}{r} \frac{\partial v'_{2,1}}{\partial \theta} + \frac{1}{r} \frac{\partial p'_{2,2}}{\partial \theta} \\
\end{align*}
\]

(21)

where \( E = \frac{W_0}{(U_0 \alpha)} \), \( Q = \varphi_2 / \varphi_1 \), and \( Ma = U_0 / \varepsilon_0 \).

### 2.2. Solution of a Set of Equations

Employing eq 9, the perturbed pressure is listed as follows

\[
p'_1 = \eta_6 P'_1 + \eta_6^2 P'_{1,2}
\]

\[
\eta_6 P'_1 (r) \exp\{i(k_1 \zeta + m_1 \theta)\}
\]

\[
\eta_6^2 P'_{1,2} (r) \exp\{2(2k_1 \zeta + 2m_1 \theta)\}
\]

where \( m \) represents the angular wavenumber, \( m_1 \) is the first-order angular wavenumber, \( m_2 \) is the second-order angular wavenumber, \( k_1 = k_1 + ik_1 \), \( k_2 = k_2 + ik_2 \) is the first-order axial wavenumber, \( k_3 = k_3 + ik_3 \) is the second-order axial wavenumber, \( \omega_1 \) and \( \omega_2 \) are the first and second-order complex wave frequencies, \( \omega_1 = \omega_{1,1} + i \omega_{1,2} \), \( \omega_2 = \omega_{2,1} + i \omega_{2,2} \).
The solution of the first-order pressure disturbance is the modified Bessel function as follows

\[
p_{11}' = \frac{-n_1}{\omega_1 - ik_1} d_{11}'L_m(n_1r)e^{i(k_1z + m_1\theta)}
\]

where \( d_{11} \) is the constant. \( I \) is the first kind of modified Bessel function. \( n_1 = (k_1^2 + M\omega_1^2(\omega_1 - ik_1))^0.5 \).

Substituting eq 23 into eq 11, the first-order velocity disturbance is solved as follows

\[
\begin{align*}
\nu_{11}' &= \frac{-n_1}{\omega_1 - ik_1} d_{11}'L_m(n_1r)e^{i(k_1z + m_1\theta)} \\
\nu_{01}' &= \frac{-im}{(\omega_1 - ik_1)^2} d_{11}'L_m(n_1r)e^{i(k_1z + m_1\theta)} \\
\nu_{10}' &= \frac{-i}{\omega_1 - ik_1} d_{11}'L_m(n_1r)e^{i(k_1z + m_1\theta)}
\end{align*}
\]

(24)

For the surrounding air, substituting eq 22 into eqs 12 and 13, the first-order pressure and velocity disturbance are listed as follows

\[
p_{22}' = Q\left(\frac{\omega_1 + imE}{r^2}\right) d_{22}'K_m(n_2r)e^{i(k_2z + m_2\theta)}
\]

(25)

\[
\begin{align*}
\nu_{22}' &= -n_2 d_{22}'K_m(n_2r)e^{i(k_2z + m_2\theta)} \\
\nu_{02}' &= \frac{-im}{r} d_{22}'K_m(n_2r)e^{i(k_2z + m_2\theta)} \\
\nu_{20}' &= -\frac{i}{r} d_{22}'K_m(n_2r)e^{i(k_2z + m_2\theta)}
\end{align*}
\]

(26)

where \( d_{22} \) is the integration constant, \( n_2 = (k_2^2 + M\omega_1^2(\omega_1 + imE))^0.5 \).

For the solving process of the second-order disturbance, it is difficult to get the solution as the first-order disturbance. However, the second-order disturbance can be solved by using the symbolic mathematical computation program Mathematica. Therefore, we do not discuss the solution of the second-order disturbances here.

2.3. Boundary Conditions. The kinematic and dynamic boundary conditions at the air–liquid interface will be discussed next.

Kinematic boundary condition satisfies

\[
dF/dt = 0
\]

(27)

where \( F \) is the air–liquid interface equation

\[
F(t, r, \theta, z) = r - a - \eta(t, \theta, z) = 0
\]

(28)

where \( \eta \) is the amplitude of interface disturbance

\[
\eta(t, \theta, z) = \sum_{n=1}^{\infty} \eta_n(t, \theta, z) = \eta_0(t, \theta, z) + \eta_1(t, \theta, z) + ...
\]

\[
= \eta_0\hat{h}_0(r)e^{i(\omega_1t + i(k_1z + m_1\theta))} \\
+ \eta_1\hat{h}_1(r)e^{i(2k_1z + 2m_1\theta)} \\
+ \eta_0\hat{h}_{20}(r)e^{i(2\omega_1t + i(2k_1z + 2m_1\theta))} + ...
\]

(29)

where \( \eta_0 \) and \( \eta_1 \) are the first-order and the second-order disturbance components, respectively.

Substituting eqs 28 and 29 into eq 27, we can get the nondimensional kinematic boundary conditions

\[
\begin{align*}
\hat{v}_{11}' &= (\omega_1 - ik_1)\hat{h}_1 \\
\hat{v}_{21}' &= (\omega_1 + imE)\hat{h}_1 \\
\hat{v}_{1,1}' &= (\omega_2 - 2ik_1)\hat{h}_{21} \\
\hat{v}_{2,1}' &= (\omega_2 + 2imE)\hat{h}_{21} \\
\hat{v}_{2,2}' &= (\omega_2 + 2imE)\hat{h}_{21} + \omega_2\hat{h}_{21}^2 + i\eta_1\hat{h}_{21}' \hat{h}_1 + ik\hat{v}_{2,1}' \hat{h}_1
\end{align*}
\]

(30)

The dynamic boundary condition satisfies

\[
\tau_1 \mathbf{n} - \tau_2 \mathbf{n} = \sigma \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \mathbf{n}
\]

(32)

Coupling eqs 3, 9, and 22, we can obtain the nondimensional dynamic boundary conditions

\[
\hat{p}_{1,1}' - \hat{p}_{2,1}' = \frac{We(k_1^2 + m_1^2 - 1)\hat{h}_1 + QE^2\hat{h}_1}{2} \\
\hat{p}_{1,21}' - \hat{p}_{2,21}' = -\frac{We(1 - 4k_2^2 - 4m_2^2)\hat{h}_{21} + QE^2\hat{h}_{21}}{2} \\
(\hat{p}_{1,1}' - \hat{p}_{2,1}'') + \eta_1\hat{h}_{21}' (\hat{p}_{1,1}' - \hat{p}_{2,1}')
\]

\[
= \frac{We\left(4k_1^2 + 4m_1^2 - 1\right)\hat{h}_{21} - \frac{5}{2}\left(1 - \frac{11}{5}k_1^2 - m_1^2\right)\hat{h}_1^2}{2} \\
+ QE^2\left(\hat{h}_{21}^2 - \frac{25}{2}\hat{h}_1^2\right)
\]

(35)

where \( We = \sigma/\rho_1 U_0 a \).

2.4. Dispersion Relationship. Based on the above discussions, the first- and second-order dispersion equations about the instability of the cavitating liquid jet surrounded by the rotary compressible air will be built. Substituting eqs 24–26 into eqs 30 and 33, we get the dispersion equation of the form \( \lambda(m_1, k_1, \omega_1) = 0. \) As to the second-order dispersion equation, it can be built by coupling the solutions of second-order disturbance parameters and eqs 31, 34, and 35.

In this paper, the secant method with superlinear convergence is used to solve the dispersion equations iteratively, and the used software is a symbolic mathematical computation program Mathematica.

The development state of surface disturbance can be obtained by taking the perturbation growth rate in spatial mode, axial wavenumber, and oscillation frequency obtained by the first- and second-order dispersion relationships into eq 27. In addition, combining \( r = a + \eta(t, \theta, z) = 0 \), the breakup length \( l_b \) of the cavitating liquid jet surrounded by the rotary compressible air will be gotten.

3. VERIFICATION

To verify the correctness of the above dispersion relationship of the cavitating liquid jet surrounded by the rotary compressible air, the experimental conditions are used for calculation, and the results are compared with the experimental.
data provided in the reference. The comparison result is shown in Figure 2. Abscissa We represents the reciprocal of Weber number, and ordinate nondimensional breakup length \( L_b \) is the ratio of actual breakup length \( L_0 \) and nozzle diameter \( d_0 \).

![Figure 2. Comparison of calculation and experimental results.\(^{24}\)](image)

From the comparison results given in Figure 2, it can be seen that the calculation results of the nondimensional breakup length obtained by using the built dispersion relationship in this paper are consistent with the changing trend of experimental results,\(^{24}\) which indicates the correctness of this dispersion relationship and its solution method based on nonlinear instability analysis.

### 4. RESULTS AND DISCUSSION

Here, Table 1 gives the used calculation parameters in the next discussion.\(^{25-29}\)

| parameters                      | value     |
|---------------------------------|-----------|
| liquid density (kg/m\(^3\))     | 848       |
| factor of surface tension (N/m) | 2.689 \times 10\(^{-2}\) |
| density of cavitation bubble (kg/m\(^3\)) | 1.087 \times 10\(^{-2}\) |
| speed of liquid jet (m/s)       | 200       |
| Nozzle radius (m)               | 1 \times 10\(^{-5}\) |
| sound speed in the liquid (m/s) | 1200      |

#### 4.1. Effect of Swirling Gas on Jet Morphology

Figure 3 shows that the spatial evolution (liquid jet morphology) at the jet surface under different nondimensional air rotational strengths.

As shown in Figure 3, swirling gas has an important influence on the morphology of the liquid jet. When the surrounding gas is not swirling, as shown in Figure 3a, along the jet direction indicated by an arrow, the jet breakup proceeds in an axisymmetric manner. When the surrounding gas is swirling \( (E \neq 0) \), the jet morphology is changed, the cross section of the liquid jet will no longer be circular, and the disturbance waveform on the surface of the liquid jet also distorts from axisymmetric to increasingly asymmetric, which indicates that air rotation changes the morphology of the perturbation wave at the air–liquid interface, and the dominant mode changes from axisymmetric to asymmetric with the existence of swirling gas.

In addition, it is also found from Figure 3 that more and more uneven corrugated flows are formed on the jet surface with the increase of air rotary strength. When the amplitude of disturbance on the jet surface increases to a certain value, these ripples are likely to be torn apart to form some thin ligaments and then break up into tiny droplets.

Actually, jet morphology is mainly determined by the dominating azimuthal wave mode corresponding to the most unstable disturbance mode. To reflect the influence of swirling gas on jet morphology more clearly, Figure 4 gives the dominating azimuthal wave modes under different nondimensional gas rotational strengths.

As shown in Figure 4, with the increase of air rotary strength, the dominating azimuthal wave mode corresponding to the most unstable disturbance mode increases linearly, which suggests that the influence of swirling air on jet morphology is obvious.

#### 4.2. Effect of Compressibility on Jet Morphology

Note that \( \Delta Ma_1 \) and \( \Delta Ma_2 \) can reflect the liquid and air compressibility. Figure 5 shows that the three-dimensional spatial evolution of liquid jet under different liquid and air Mach numbers. Also, only the jet morphology close to breakup is shown here.

As shown in Figure 5, the compressibility of the liquid jet and surrounding gas has little effect on the morphology of the liquid jet in the circumferential direction. Considering the fluid compressibility or not, along the jet direction, the cross section of the liquid jet is always circular and the jet breakup proceeds in an axisymmetric manner. However, it is also displayed from Figure 5 that the axial morphology of the liquid jet changes with the increase of the jet and gas Mach numbers. As shown in Figure 5a, within a certain distance close to jet breakup, the axial morphology of the liquid jet is composed of “main droplet + satellite droplet + main droplet + satellite droplet”. Referring to Figure 5b, within a certain distance close to jet breakup, the axial morphology of the liquid jet changed to be composed of “satellite droplet + main droplet + satellite droplet + main droplet”. As shown in Figure 5c, within a certain distance close to jet breakup, the axial morphology of liquid jet is composed of “satellite droplet + main droplet + satellite droplet + main droplet + satellite droplet”. Therefore, a comparison of Figure 5a–c demonstrates that the fluid compressibility has some impacts on the axial morphology of the liquid jet, especially on the arrangement of droplets. The main reason is that the higher Mach number means the higher jet speed, which makes the effect of gas–liquid interaction on the axial morphology of the liquid jet more stronger and leads to more morphology elements (as shown in Figure 5c).

#### 4.3. Effect of Bubble Volume Fraction on Jet Morphology

Next, we will discuss the effect of bubble volume fraction on the morphology of the liquid jet surrounded by rotary compressible air. Figure 6 gives the spatial evolution of the cavitating liquid jet under different bubble volume fractions. Only the jet morphology close to breakup is shown.

As shown in Figure 6, compared with the jet morphology when the bubble volume fraction \( \alpha = 0 \), more corrugated flows are formed on the jet surface when the bubble volume fraction \( \alpha \neq 0 \), which indicates that the existence of the cavitation phenomenon will affect the jet morphology to some extent. However, the effect of the bubble volume fraction on jet morphology is smaller than swirling gas.

#### 4.4. Cavitating Jet Breakup

Based on the built second-order dispersion equation, the influences of the rotary air, the fluid compressibility, and the bubble volume fraction on the jet breakup length will be discussed, respectively.
Figure 7 shows the nondimensional jet breakup lengths under different nondimensional gas rotational strengths. As shown in Figure 7, when the gas rotational strength $E = 0$, the nondimensional jet breakup length $L_b = 101.0$. When gas rotational strength $E = 0.01$, the nondimensional jet breakup length $L_b = 100.8$. When the gas rotational strength $E = 0.05$, the nondimensional jet breakup length $L_b = 100.0$. When the gas rotational strength $E = 0.25$, the nondimensional jet breakup length $L_b = 90.6$. Therefore, with the 5-fold growth of gas rotational strength, the jet breakup length first decreases slowly and then decreases sharply, which suggests that swirling gas has a destabilizing effect on the liquid jet, but the destabilizing effect of swirling gas on liquid jet becomes obvious when the gas rotational strength increases to a certain value.

Figure 8 gives the nondimensional jet breakup lengths under different nondimensional jet and gas Mach numbers. It is seen from Figure 8 that the nondimensional jet breakup length decreases rapidly with the increase of jet and gas Mach numbers, which suggests that fluid compressibility promotes the breakup of the jet. In addition, comparing Figure 8 with Figure 7, it is also found that the effect of fluid compressibility on breakup length is obviously greater than swirling gas.

Figure 9 gives the nondimensional jet breakup lengths under different bubble volume fractions.

It can be found from Figure 9 that the nondimensional jet breakup length decreases and the decreasing extent declines gradually with the increase of the bubble volume fraction, which indicates that the bubble volume fraction has a big influence on promoting the breakup of the jet; however, this promoting effect reduces gradually with the increase of the bubble volume fraction. Note that when the value of the bubble volume fraction approaches 0.2, the instability analysis will lose its validity.
Figure 5. Spatial evolution of the liquid jet under different liquid and air Mach numbers. (a) $Ma_1 = Ma_2 = 0$, (b) $Ma_1 = 0.1$, $Ma_2 = 0.29$, and (c) $Ma_1 = 0.2$, $Ma_2 = 0.57$.

Figure 6. Spatial evolution of the cavitating liquid jet under different bubble volume fractions. (a) $\alpha = 0$, (b) $\alpha = 0.05$, and (c) $\alpha = 0.1$.

Figure 7. Jet breakup lengths under different gas rotational strengths.

Figure 8. Jet breakup lengths under different jet and gas Mach numbers.
5. CONCLUSIONS

A nonlinear dispersion relationship has been carried out to model the nonlinear evolution and instability of a cavitating liquid jet surrounded by rotary compressible air. Then, the effects of air rotation, fluid compressibility, and bubble volume fraction on jet morphology and breakup length are investigated mainly.

(1) The coaxial rotation of the surrounding gas changes the dominant mode of the perturbation wave on the surface of the liquid jet, and the dominant mode changes from axisymmetric to asymmetric with the existence of swirling gas. More and more uneven corrugated flows are formed on the jet surface with the increase of air rotational strength. When the amplitude of disturbance on the jet surface increases to a certain value, these ripples are likely to be torn apart to form some thin ligaments and then break up into tiny droplets. With the increase of air rotation strength, the dominating azimuthal wave mode corresponding to the most unstable disturbance mode increases linearly.

(2) The fluid compressibility has little effect on the morphology of the liquid jet in a circumferential direction. Considering the fluid compressibility or not, along the jet direction, the cross section of the liquid jet is always circular, and the jet breakup proceeds in an axisymmetric manner. However, the fluid compressibility has some impacts on the axial morphology of the liquid jet, especially on the arrangement of droplets.

(3) The existence of the cavitation phenomenon will affect the jet morphology to some extent. However, the effect of the bubble volume fraction on jet morphology is smaller than swirling gas.

(4) Breakup length decreases slowly with the increase of air rotational strength; however, when the gas rotational strength is further increased, the breakup length of the liquid jet decreases sharply. Therefore, swirling gas has a destabilizing influence on the liquid jet, and the destabilizing influence of swirling gas on the liquid jet becomes obvious when the gas rotational strength increases to a certain value.

(5) Breakup length decreases rapidly with the increase of liquid and gas Mach numbers, so the fluid compressibility promotes the liquid jet breakup. In addition, the influence of fluid compressibility on jet breakup is obviously greater than air rotation.

(6) Breakup length decreases and the decreasing extent declines gradually with the increase of the bubble volume fraction, which indicates that the bubble volume fraction has a big influence on promoting the jet breakup; however, this promoting influence decreases with the increase of the bubble volume fraction.

### REFERENCES

1. Turner, M. R.; Sazhin, S. S.; Healey, J. J.; et al. A breakup model for transient diesel fuel sprays. Fuel 2012, 97, 288–305.
2. Lü, M.; Ning, Z.; Sun, C. Numerical simulation of cavitation bubble collapse within a droplet. Comput. Fluids 2017, 152, 157–163.
3. Zhou, Z. W.; Lin, S. P. Effects of compressibility on the atomization of liquid jets. J. Propul. Power 1992, 8, 736–740.
4. Lü, M.; Ning, Z.; Yan, K.; et al. Temporal and spatial stability of liquid jet containing cavitation bubbles in coaxial swirling compressible flow. Meccanica 2016, 51, 2121–2133.
5. Miller, E.; Gibson, B.; McWilliams, E.; Rothstein, J. P. Collision of viscoelastic jets and the formation of fluid webs. Appl. Phys. Lett. 2005, 87, No. 014101.
6. Liang, X.; Deng, D. S.; Nave, J. C.; Johnson, S. G. Linear stability analysis of capillary instabilities for concentric cylindrical shells. J. Fluid Mech. 2011, 683, 235–262.

---

**AUTHOR INFORMATION**

**Corresponding Author**

*E-mail: lvming@bjtu.edu.cn.*

**ORCID**

Ming Lü: 0000-0002-7434-7395

**Notes**

The authors declare no competing financial interest.

**ACKNOWLEDGMENTS**

This project was supported by the National Natural Science Foundation of China (Grant Nos. 51606006, 51776016), the Beijing Natural Science Foundation (Grant Nos. 3172025, 3182030), and the National Engineering Laboratory for Mobile Source Emission Control Technology (Grant No. NELMS2017A10).

**NOMENCLATURE**

- $a$: jet radius, m
- $c_1$: sound speed of liquid jet, m/s
- $c_2$: sound speed of surrounding gas, m/s
- $E$: nondimensional rotational strength
- $k_1$: perturbation growth rate in spatial mode
- $k_2$: axial wavenumbers
- $L_0$: nondimensional breakup length
- $m$: wavenumber in the $\theta$-direction
- $Ma_1$: Mach number of liquid
- $Ma_2$: Mach number of air
- $P_i$: pressure of liquid jet, Pa
- $P_g$: pressure of air, Pa
- $Q$: gas–liquid density ratio
- $U_0$: initial liquid velocity, m/s
- $W_o$: air rotation strength, m$^2$/s
- $W_e$: Weber number
- $\omega_0$: oscillation frequency
- $\omega_i$: temporal rates of exponential amplification
- $\alpha$: bubble volume fraction
- $\eta_0$: initial disturbance amplitude
- $\rho_1$: density of jet, kg/m$^3$
- $\rho_2$: density of air, kg/m$^3$
- $\rho_3$: density of liquid, kg/m$^3$
- $\rho_4$: density of cavitation bubbles, kg/m$^3$
- $\sigma$: factor of surface tension, N/m

---

Figure 9. Jet breakup lengths under different bubble volume fractions.
(7) Lin, S. P. Breakup of Liquid Sheets and Jets; Cambridge University Press: Cambridge, 2003; pp 60–100.
(8) Jazayeri, S. A.; Li, X. G. Nonlinear instability of plane liquid sheets. J. Fluid Mech. 2000, 406, 281–308.
(9) Yang, L. J.; Wang, C.; Fu, Q. F.; et al. Weakly nonlinear instability of planar viscous sheets. J. Fluid Mech. 2013, 735, 249–287.
(10) Yuen, M. C. Nonlinear capillary instability of a liquid jet. J. Fluid Mech. 1968, 33, 151–163.
(11) Sorokin, V. S.; Blekhman, I. I.; Vasilkov, V. B. Motion of a gas bubble in fluid under vibration. Nonlinear Dyn. 2012, 67, 147–158.
(12) Lafrance, P. Nonlinear breakup of a laminar liquid jet. Phys. Fluids 1975, 18, 428–432.
(13) Ibrahim, A. A.; Jog, M. A. Nonlinear breakup of a coaxial liquid jet in a swirling gas stream. Phys. Fluids 2006, 18, No. 114101.
(14) Ibrahim, A. A.; Jog, M. A. Nonlinear instability of an annular liquid sheet exposed to gas flow. Int. J. Multiphase Flow 2008, 34, 647–664.
(15) Rangel, R. H.; Sirignano, W. A. Nonlinear growth of Kelvin-Helmholtz instability: effect of surface view tension and density ratio. Phys. Fluids 1988, 31, 1845–1855.
(16) Zakaria, K. Stability of resonant interfacial waves in the presence of uniform magnetic field. Nonlinear Dyn. 2011, 66, 457–477.
(17) Ibrahim, E. A.; Lin, S. P. Weakly nonlinear instability of a liquid jet in a viscous gas. J. Appl. Mech. 1992, 59, S291–S296.
(18) Tharakan, T. J.; Ramamurthi, K.; Balakrishnan, M. Nonlinear breakup of thin liquid sheets. Acta Mech. 2002, 156, 29–46.
(19) Ibrahim, A. A.; Jog, M. A. Comprehensive Study of Internal Flow Field and Linear and Nonlinear Instability of an Annular Liquid Sheet Emanating from an Atomizer; University of Cincinnati: Cincinnati, 2006; pp 48–120.
(20) Yan, K.; Jog, M. A.; Ning, Z. Nonlinear spatial instability of an annular swirling viscous liquid sheet. Acta Mech. 2013, 224, 3071–3090.
(21) Hadji, L.; Schreiber, W. The stability of an inviscid liquid sheet containing vapor bubbles. J. Phys. Nat. Sci. 2007, 1, 1–11.
(22) Gao, Z. Y. A study of the propagation velocity of pressure wave in gas-liquid two phase mixtures. J. Eng. Therm. 1984, 5, 200–205.
(23) Lin, S. P.; Lian, Z. W. Mechanisms of the breakup of liquid jets. AIAA J. 1990, 28, 120–126.
(24) Sallam, K. A.; Dai, Z.; Faeth, G. M. Liquid breakup at the surface of turbulent round liquid jets in still gases. Int. J. Multiphase Flow 2002, 28, 427–449.
(25) Mulemane, A. S.; Subramaniyam, S.; Lu, P. H. Comparing cavitation in diesel injectors based on different modeling approaches. SAE Paper, 2004-01-0027.
(26) Jia, M.; Hou, D.; Li, J. A micro-variable circular orifice fuel injector for HCCI-conventional engine combustion — Part 1 numerical simulation of cavitation. SAE Paper, 2007-01-0249.
(27) Wang, X.; Su, W. H. A numerical study of cavitating flows in high-pressure diesel injection nozzle holes using a two-fluid model. Sci. Bull. 2009, 54, 1655–1662.