Centrally generated primitive ideals of $U(\mathfrak{n})$ for exceptional types

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It was proved in [IP, Theorem 3.1] and [EG1, Theorem 2.4] that, when $\Phi$ is of classical type (i.e., $\Phi = A_n, B_n, C_n$ or $D_n$), $J \in \text{Prim} U(n)$ is centrally generated if and only if $J = J(f)$ for a certain Kostant form $f$. In this paper, we prove that this fact is also true when $\Phi$ is of exceptional type, i.e., $\Phi = E_6, E_7, E_8, F_4$ or $G_2$. Namely, let $\Delta, \beta \in B$, be the set of canonical generators of $Z(n)$ (see Section 2). Let $J$ be a primitive ideal of $U(n)$. Since $J$ is the annihilator of a simple $n$-module, given $\beta \in B$, there exists the unique $c_\beta \in \mathbb{C}$ such that $\Delta_\beta - c_\beta \in J$. Our main result, Theorem 5.1, claims that the following conditions are equivalent:

i) $J$ is centrally generated;

ii) all scalars $c_\beta, \beta \in B \setminus \Delta$, are nonzero;

iii) $J = J(f)$ for a Kostant form $f$.

If these conditions are satisfied, then we present an explicit way how to reconstruct $f$ by $J$. As a corollary, we conclude that the same is true for arbitrary (probably, reducible) root system.

The paper is organized as follows. In Section 2 we briefly recall the Kostant’s characterization of $Z(n)$ and present a (more or less) explicit description of the canonical generators of $Z(n)$ based on A. Panov’s work [Pa2]. Using this description, in Section 3 we prove that certain centrally generated ideals are primitive (in fact, it is the key ingredient in the proof of the main result, see Proposition 3.3). Section 4 is devoted to some particular classes of coadjoint orbits. Namely, we prove that certain orbits are primitive (in fact, it is the key ingredient in the proof of the main result, see Proposition 4.3). Finally, in Section 5 combining our results form two previous sections, we prove the main result, Theorem 5.1. As an immediate corollary, we obtain that the similar result is true for an arbitrary semisimple Lie algebra, see Theorem 5.2.

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2. The center of $U(n)$

Let $G$ be a complex semisimple algebraic group, $H$ be a Cartan subgroup of $G$, $B$ be a Borel subgroup of $G$ containing $H$, and $N$ be the unipotent radical of $B$. We denote by $\Phi$ the root system of $G$ with respect to $B$, and by $\Phi^+$ the set positive roots with respect to $B$. Let $\mathfrak{g}$ (respectively, $\mathfrak{h}, \mathfrak{b}$ and $\mathfrak{n}$) be the Lie algebra of $G$ (respectively, of $H, B$ and $N$), so that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ as vector spaces. The Lie algebra $\mathfrak{n}$ has a basis consisting of root vectors $e_\alpha, \alpha \in \Phi^+$. We denote the dual basis of the dual space $\mathfrak{n}^*$ by $e^*_\alpha, \alpha \in \Phi^+$.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the standard inner product $(\cdot, \cdot)$, and $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. If $\Phi$ is irreducible then we identify $\Phi^+$ with the following subset of $\mathbb{R}^n$ [Bo]:

$$A_{n-1}^+ = \{e_i - e_j, 1 \leq i < j \leq n\},$$

$$B_n^+ = \{e_i - e_j, 1 \leq i < j \leq n\} \cup \{e_i + e_j, 1 \leq i < j \leq n\} \cup \{e_i, 1 \leq i \leq n\},$$

$$C_n^+ = \{e_i - e_j, 1 \leq i < j \leq n\} \cup \{e_i + e_j, 1 \leq i < j \leq n\} \cup \{2e_i, 1 \leq i \leq n\},$$

$$D_n^+ = \{e_i - e_j, 1 \leq i < j \leq n\} \cup \{e_i + e_j, 1 \leq i < j \leq n\},$$

$$E_6^+ = \{\pm e_i + e_j, 1 \leq i < j \leq 5\} \cup \left\{\frac{1}{2} \left(\bar{e}_8 - e_7 - e_6 + \sum_{i=1}^5 (-1)^{\nu(i)} e_i\right), \sum_{i=1}^5 \nu(i) \text{ is even}\right\},$$

$$E_7^+ = \{\pm e_i + e_j, 1 \leq i < j \leq 6\} \cup \{e_8 - e_7\} \cup \left\{\frac{1}{2} \left(\bar{e}_8 - e_7 + \sum_{i=1}^6 (-1)^{\nu(i)} e_i\right), \sum_{i=1}^6 \nu(i) \text{ is even}\right\},$$

$$E_8^+ = \{\pm e_i + e_j, 1 \leq i < j \leq 8\} \cup \left\{\frac{1}{2} \left(\bar{e}_8 + \sum_{i=1}^7 (-1)^{\nu(i)} e_i\right), \sum_{i=1}^7 \nu(i) \text{ is even}\right\},$$

$$F_4^+ = \{e_i \pm e_j, 1 \leq i < j \leq 4\} \cup \{e_i + e_2 \pm e_3 \pm e_4\}/2 \cup \{e_i, 1 \leq i \leq 4\},$$

$$G_2^+ = \{e_1 - e_2, -2e_1 + e_2 + e_3, -e_1 + e_3, -e_2 + e_3, e_1 - 2e_2 + e_3, -e_1 - e_2 + 2e_3\}.$$
Under this identification, the set $\Delta \subset \Phi^+$ of the simple roots has the following form:

$$
\Delta = \begin{cases} 
\bigcup_{i=1}^{n-1} \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \} & \text{for } A_{n-1}, \\
\bigcup_{i=1}^{n-1} \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \} \cup \{ \alpha_n = \epsilon_n \} & \text{for } B_n, \\
\bigcup_{i=1}^{n-1} \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \} \cup \{ \alpha_n = 2\epsilon_n \} & \text{for } C_n, \\
\bigcup_{i=1}^{n-1} \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \} \cup \{ \alpha_n = \epsilon_{n-1} + \epsilon_n \} & \text{for } D_n, \\
\{ \alpha_1 = (\epsilon_1 + \epsilon_8 - \sum_{k=2}^7 \epsilon_k)/2, \\
\alpha_2 = \epsilon_1 + \epsilon_2 \} \cup \bigcup_{i=1}^3 \{ \alpha_{i+2} = \epsilon_{i+1} - \epsilon_i \} & \text{for } E_6, \\
\{ \alpha_1 = (\epsilon_1 + \epsilon_8 - \sum_{k=2}^7 \epsilon_k)/2, \\
\alpha_2 = \epsilon_1 + \epsilon_2 \} \cup \bigcup_{i=1}^3 \{ \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i \} & \text{for } E_7, \\
\{ \alpha_1 = (\epsilon_1 + \epsilon_8 - \sum_{k=2}^7 \epsilon_k)/2, \\
\alpha_2 = \epsilon_1 + \epsilon_2 \} \cup \bigcup_{i=1}^3 \{ \alpha_{i+2} = \epsilon_{i+1} - \epsilon_i \} & \text{for } E_8, \\
\{ \alpha_1 = \epsilon_1 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_1, \alpha_3 = \epsilon_4, \\
\alpha_4 = (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2 \} & \text{for } F_4, \\
\{ \alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3 \} & \text{for } G_2.
\end{cases}
$$

(1)

Recall that there exists a natural partial order on $\Phi$: by definition, $\alpha < \beta$ if $\beta - \alpha$ can be represented as a sum of positive roots. Denote by $B$ the subset of $\Phi^+$ constructed by the following inductive procedure. Let $B_1$ be the set consisting of the maximal roots of all irreducible components of $\Phi$. For $n \geq 2$, we denote $\Phi_n = \{ \alpha \in \Phi \mid \alpha \perp \beta \text{ for all } \beta \in B_1 \cup \ldots \cup B_{n-1} \}$, and set $B_n$ to be the set of the maximal roots of all irreducible components of $\Phi_n$. Finally, we denote by $B$ the union of all $B_n$’s. Note that $B$ is a maximal strongly orthogonal subset of $\Phi^+$, i.e., $B$ is maximal with the property that if $\alpha, \beta \in B$ then neither $\alpha - \beta$ nor $\alpha + \beta$ belongs to $\Phi^+$.

**Definition 2.1.** We call $B$ the *Kostant cascade* of orthogonal roots in $\Phi^+$.

If $\Phi$ is irreducible then $B$ has the following form:

$$
B = \begin{cases} 
\bigcup_{i=1}^{n/2} \{ \beta_i = \epsilon_i - \epsilon_{n-i+1} \} & \text{for } A_{n-1}, \\
\bigcup_{i=1}^{n/2} \{ \beta_{2i-1} = \epsilon_{2i-1} + \epsilon_{2i}, \beta_{2i} = \epsilon_{2i-1} - \epsilon_{2i} \} & \text{for } B_n, \text{ n even}, \\
\bigcup_{i=1}^{n/2} \{ \beta_{2i-1} = \epsilon_{2i-1} + \epsilon_{2i}, \beta_{2i} = \epsilon_{2i-1} - \epsilon_{2i} \} \cup \{ \beta_n = \epsilon_n \} & \text{for } B_n, \text{ n odd}, \\
\bigcup_{i=1}^{n/2} \{ \beta_i = 2\epsilon_i \} & \text{for } C_n, \\
\bigcup_{i=1}^{n/2} \{ \beta_{2i-1} = \epsilon_{2i-1} + \epsilon_{2i}, \beta_{2i} = \epsilon_{2i-1} - \epsilon_{2i} \} \cup \{ \beta_n = \epsilon_n \} & \text{for } D_n, \\
\{ \beta_2 = (-\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8)/2, \\
\beta_3 = -\epsilon_1 + \epsilon_4, \beta_4 = -\epsilon_2 + \epsilon_3 \} & \text{for } E_6, \\
\{ \beta_1 = -\epsilon_7 + \epsilon_8, \beta_2 = \epsilon_5 + \epsilon_6, \beta_3 = \epsilon_3 + \epsilon_4, \beta_4 = -\epsilon_5 + \epsilon_6, \\
\beta_5 = \epsilon_1 + \epsilon_2, \beta_6 = -\epsilon_1 + \epsilon_2, \beta_7 = -\epsilon_3 + \epsilon_4 \} & \text{for } E_7, \\
\{ \beta_1 = \epsilon_7 + \epsilon_8, \beta_2 = -\epsilon_7 + \epsilon_8, \beta_3 = \epsilon_5 + \epsilon_6, \beta_4 = \epsilon_3 + \epsilon_4, \\
\beta_5 = -\epsilon_5 + \epsilon_6, \beta_6 = \epsilon_1 + \epsilon_2, \beta_7 = -\epsilon_1 + \epsilon_2, \beta_8 = -\epsilon_3 + \epsilon_4 \} & \text{for } E_8, \\
\{ \beta_1 = \epsilon_1 + \epsilon_2, \beta_2 = \epsilon_1 - \epsilon_2, \beta_3 = \epsilon_3 + \epsilon_4, \beta_4 = \epsilon_3 - \epsilon_4 \} & \text{for } F_4, \\
\{ \beta_1 = -\epsilon_1 - \epsilon_2 + 2\epsilon_3, \beta_2 = \epsilon_1 - \epsilon_2 \} & \text{for } G_2.
\end{cases}
$$

Denote by $U(n)$ the enveloping algebra of $\mathfrak{n}$, and by $S(n)$ the symmetric algebra of $\mathfrak{n}$. Then $\mathfrak{n}$ and $S(n)$ are $B$-modules as $B$ normalizes $N$. Denote by $Z(n)$ the center of $U(n)$. It is well-known that the restriction of the symmetrization map

$$
\sigma: S(n) \to U(n), \ x^k \mapsto x^k, \ x \in \mathfrak{n}, \ k \in \mathbb{Z}_{\geq 0},
$$

to the algebra $Y(n) = S(n)^N$ of $N$-invariants is an algebra isomorphism between $Y(n)$ and $Z(n)$.
We next present a canonical set of generators of \( Z(n) \) (or, equivalently, of \( S(n)^N \)), whose description goes back to J. Dixmier, A. Joseph and B. Kostant [D3], [Jo1], [Ko1], [Ko2]. We can consider \( \mathbb{Z}\Phi \), the \( \mathbb{Z} \)-linear span of \( \Phi \), as a subgroup of the group \( \mathcal{X} \) of rational multiplicative characters of \( H \). Recall that a vector \( \lambda \in \mathbb{R}^n \) is called a \textit{weight} of \( H \) if \( c(\alpha, \lambda) = 2(\alpha, \lambda)/(\alpha, \alpha) \) is an integer for any \( \alpha \in \Phi^+ \). A weight \( \lambda \) is called \textit{dominant} if \( c(\alpha, \lambda) \geq 0 \) for all \( \alpha \in \Phi^+ \). An element \( a \) of an \( H \)-module is called an \textit{H-weight vector}, if there exists \( \nu \in \mathcal{X} \) such that \( h \cdot a = \nu(h)a \) for all \( h \in H \). By [Ko2] Theorems 6, 7], every \( H \)-weight occurs in \( S(n)^N \) with multiplicity at most 1. Furthermore, there exist unique (up to scalars) prime polynomials \( \xi_\beta \in S(n)^N \), \( \beta \in \mathcal{B} \), such that each \( \xi_\beta \) is an \( H \)-weight polynomial of a dominant weight \( \mu_\beta \) belonging to the \( \mathbb{Z} \)-linear span \( \mathbb{Z}\mathcal{B} \) of \( \mathcal{B} \). A remarkable fact is that

\[
\xi_\beta, \ \beta \in \mathcal{B}, \text{ are algebraically independent generators of } Y(n),
\]

so \( S(n)^N \) and \( Z(n) \) are polynomial rings. We call \( \{\xi_\beta, \ \beta \in \mathcal{B}\} \) the set of \textit{canonical generators} of \( S(n)^N \). It turns out that the weights \( \mu_\beta \)'s have the following form [La2 Theorem 2.12].

| \( \Phi = A_{n-1} \) | \( \mu_{\beta_i} = \epsilon_1 + \ldots + \epsilon_i - \epsilon_{n-i+1} - \ldots - \epsilon_n, \ 1 \leq i \leq m \) |
| \( \Phi = B_n \) | \( \mu_{\beta_i} = \begin{cases} 2\epsilon_1 + \ldots + 2\epsilon_{i-1} & \text{for even } i, \\ \epsilon_1 + \ldots + \epsilon_{i+1} & \text{for odd } i < n, \\ \epsilon_1 + \ldots + \epsilon_i & \text{for odd } i = n, \end{cases} \) |
| \( \Phi = C_n \) | \( \mu_{\beta_i} = 2\epsilon_1 + \ldots + 2\epsilon_i, \ 1 \leq i \leq m \) |
| \( \Phi = D_n \) | \( \mu_{\beta_i} = \begin{cases} 2\epsilon_1 + \ldots + 2\epsilon_{i-1} & \text{for even } i < n, \\ \epsilon_1 + \ldots + \epsilon_{n-1} - \epsilon_n & \text{for even } i = n, \\ \epsilon_1 + \ldots + \epsilon_{i+1} & \text{for odd } i \end{cases} \) |
| \( \Phi = E_6 \) | \( \mu_{\beta_1} = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8)/2, \\ \mu_{\beta_2} = \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8, \\ \mu_{\beta_3} = (-\epsilon_1 + \epsilon_2 + \epsilon_3 + 3\epsilon_4 + 3\epsilon_5 - 3\epsilon_6 - 3\epsilon_7 + 3\epsilon_8)/2, \\ \mu_{\beta_4} = 2\epsilon_3 + 2\epsilon_4 + 2\epsilon_5 - 2\epsilon_6 - 2\epsilon_7 + 2\epsilon_8 \) |
| \( \Phi = E_7 \) | \( \mu_{\beta_1} = -\epsilon_7 + \epsilon_8, \)
| \( \Phi = E_8 \) | \( \mu_{\beta_1} = \epsilon_7 + \epsilon_8, \\ \mu_{\beta_2} = 2\epsilon_8, \)
| \( \Phi = F_4 \) | \( \mu_{\beta_1} = \epsilon_1 + \epsilon_2, \ \mu_{\beta_2} = 2\epsilon_1, \\ \mu_{\beta_3} = 3\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4, \ \mu_{\beta_4} = 4\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 \) |
| \( \Phi = G_2 \) | \( \mu_{\beta_1} = -\epsilon_1 - \epsilon_2 + 2\epsilon_1, \ \mu_{\beta_2} = -2\epsilon_2 + 2\epsilon_3 \) |
For the sequel, we need to express the weights $\mu_{\beta_i}$’s as linear combination of simple roots. Such expressions are presented in the table below.

| $\Phi$ | $\mu_{\beta_i}$ |
|--------|----------------|
| $A_{n-1}$ | $\mu_{\beta_i} = \sum_{1 < k < \lfloor n/2 \rfloor} k\alpha_k + \sum_{\lfloor n/2 \rfloor < k < n-1} (n-k)\alpha_k$, $1 \leq i \leq m$ |
| $B_n$ | $\mu_{\beta_i} = \begin{cases} 2 \sum_{k=1}^{i-1} k\alpha_k + 2(i-1) \sum_{k=i}^{n} \alpha_k & \text{for even } i, \\ \sum_{k=1}^{i-1} k\alpha_k + (i+1) \sum_{k=i+1}^{n} \alpha_k & \text{for odd } i \end{cases}$ |
| $C_n$ | $\mu_{\beta_i} = \begin{cases} 2 \sum_{k=1}^{i-1} k\alpha_k + 2i \sum_{k=i+1}^{n-1} \alpha_k + i\alpha_n & \text{for } i < n, \\ 2 \sum_{k=1}^{n-1} k\alpha_k + n\alpha_n & \text{for } i = n \end{cases}$ |
| $D_n$ | $\mu_{\beta_i} = \begin{cases} 2 \sum_{k=1}^{i-1} k\alpha_k + 2(i-1) \sum_{k=i}^{n-2} \alpha_k + i(\alpha_{n-1} + \alpha_n) & \text{for even } i < n, \\ \sum_{k=1}^{i-1} k\alpha_k + n\alpha_{n-1}/2 + (n-2)\alpha_n/2 & \text{for even } i = n, \\ \sum_{k=1}^{i-1} k\alpha_k + (i+1) \sum_{k=i+1}^{n-2} \alpha_k + (i+1)(\alpha_{n-1} + \alpha_n)/2 & \text{for odd } i < n-1, \\ \sum_{k=1}^{i-1} k\alpha_k + (n-2)\alpha_{n-1}/2 + n\alpha_n/2 & \text{for odd } i = n-1 \end{cases}$ |
| $E_6$ | $\mu_{\beta_1} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$, $\mu_{\beta_2} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$, $\mu_{\beta_3} = 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 3\alpha_6$, $\mu_{\beta_4} = 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6$, $\mu_{\beta_5} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7$, $\mu_{\beta_6} = 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7$, $\mu_{\beta_7} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7$, $\mu_{\beta_8} = 4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7$, $\mu_{\beta_9} = 6\alpha_1 + 8\alpha_2 + 12\alpha_3 + 16\alpha_4 + 12\alpha_5 + 8\alpha_6 + 4\alpha_7$, $\mu_{\beta_{10}} = 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7$, (4) |
| $E_7$ | $\mu_{\beta_1} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, $\mu_{\beta_2} = 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$, $\mu_{\beta_3} = 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8$, $\mu_{\beta_4} = 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8$, $\mu_{\beta_5} = 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 16\alpha_6 + 12\alpha_7 + 6\alpha_8$, $\mu_{\beta_6} = 10\alpha_1 + 16\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8$, $\mu_{\beta_7} = 14\alpha_1 + 20\alpha_2 + 28\alpha_3 + 40\alpha_4 + 32\alpha_5 + 24\alpha_6 + 16\alpha_7 + 8\alpha_8$, $\mu_{\beta_8} = 16\alpha_1 + 24\alpha_2 + 32\alpha_3 + 48\alpha_4 + 40\alpha_5 + 30\alpha_6 + 20\alpha_7 + 10\alpha_8$ |
| $E_8$ | $\mu_{\beta_1} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, $\mu_{\beta_2} = 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$, $\mu_{\beta_3} = 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8$, $\mu_{\beta_4} = 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8$, $\mu_{\beta_5} = 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 16\alpha_6 + 12\alpha_7 + 6\alpha_8$, $\mu_{\beta_6} = 10\alpha_1 + 16\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8$, $\mu_{\beta_7} = 14\alpha_1 + 20\alpha_2 + 28\alpha_3 + 40\alpha_4 + 32\alpha_5 + 24\alpha_6 + 16\alpha_7 + 8\alpha_8$, $\mu_{\beta_8} = 16\alpha_1 + 24\alpha_2 + 32\alpha_3 + 48\alpha_4 + 40\alpha_5 + 30\alpha_6 + 20\alpha_7 + 10\alpha_8$, $\mu_{\beta_9} = 18\alpha_1 + 28\alpha_2 + 36\alpha_3 + 54\alpha_4 + 48\alpha_5 + 40\alpha_6 + 28\alpha_7 + 20\alpha_8$ |
| $F_4$ | $\mu_{\beta_1} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\mu_{\beta_2} = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4$, $\mu_{\beta_3} = 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 6\alpha_4$, $\mu_{\beta_4} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$, $\mu_{\beta_5} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$, $\mu_{\beta_6} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$, $\mu_{\beta_7} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$, $\mu_{\beta_8} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$, $\mu_{\beta_9} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$, $\mu_{\beta_{10}} = 6\alpha_1 + 12\alpha_2 + 16\alpha_3 + 8\alpha_4$ |
| $G_2$ | $\mu_{\beta_1} = 3\alpha_1 + 2\alpha_2$, $\mu_{\beta_2} = 4\alpha_1 + 2\alpha_2$ |
Further, we also need to express the weights $\mu_{\beta_i}$’s as linear combinations of roots from $\mathcal{B}$ (it is possible since $\mu_{\beta_i} \in \mathbb{Z}\mathcal{B}$ for all $i$). Such expressions are presented in the table below.

| $\Phi$ = $A_{n-1}$ | $\mu_{\beta_i} = \beta_1 + \ldots + \beta_i$, $1 \leq i \leq m$ |
|---------------------|--------------------------------------------------|
| $\Phi$ = $B_n$ | $\mu_{\beta_i} = \left\{ \begin{array}{ll} 2\beta_1 + 2\beta_3 + \ldots + 2\beta_{i-2} + \beta_{i-1} + \beta_i & \text{for even } i, \\ \beta_1 + \beta_3 + \ldots + \beta_i & \text{for odd } i < n \end{array} \right.$ |
| $\Phi$ = $C_n$ | $\mu_{\beta_i} = \beta_1 + \ldots + \beta_i$, $1 \leq i \leq m$ |
| $\Phi$ = $D_n$ | $\mu_{\beta_i} = \left\{ \begin{array}{ll} 2\beta_1 + 2\beta_3 + \ldots + 2\beta_{i-2} + \beta_{i-1} + \beta_i & \text{for even } i < n, \\ \beta_1 + \beta_3 + \ldots + \beta_{n-3} + \beta_n & \text{for even } i = n, \\ \beta_1 + \beta_3 + \ldots + \beta_i & \text{for odd } i \end{array} \right.$ |
| $\Phi$ = $E_6$ | $\mu_{\beta_1} = \beta_1$, $\mu_{\beta_2} = \beta_1 + \beta_2$, $\mu_{\beta_3} = 2\beta_1 + \beta_2 + \beta_3$, $\mu_{\beta_4} = 3\beta_1 + \beta_2 + 3\beta_3 + \beta_4$ |
| $\Phi$ = $E_7$ | $\mu_{\beta_1} = \beta_1$, $\mu_{\beta_2} = \beta_1 + \beta_2$, $\mu_{\beta_3} = 2\beta_1 + \beta_2 + \beta_3$, $\mu_{\beta_4} = \beta_1 + \beta_2 + 3\beta_3 + \beta_4$, $\mu_{\beta_5} = 2\beta_1 + \beta_2 + 3\beta_3 + \beta_5$, $\mu_{\beta_6} = 3\beta_1 + \beta_2 + 3\beta_3 + \beta_6$, $\mu_{\beta_7} = 3\beta_1 + 2\beta_2 + 3\beta_3 + \beta_7$ |
| $\Phi$ = $E_8$ | $\mu_{\beta_1} = \beta_1$, $\mu_{\beta_2} = \beta_1 + \beta_2$, $\mu_{\beta_3} = 2\beta_1 + \beta_2 + \beta_3$, $\mu_{\beta_4} = 3\beta_1 + 2\beta_2 + 3\beta_3 + \beta_4$, $\mu_{\beta_5} = 3\beta_1 + 2\beta_2 + 3\beta_3 + \beta_5$, $\mu_{\beta_6} = 3\beta_1 + 2\beta_2 + 3\beta_3 + \beta_6$, $\mu_{\beta_7} = 4\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4 + \beta_7$, $\mu_{\beta_8} = 5\beta_1 + 3\beta_2 + 2\beta_3 + 2\beta_4 + \beta_8$ |
| $\Phi$ = $F_4$ | $\mu_{\beta_1} = \beta_1$, $\mu_{\beta_2} = \beta_1 + \beta_2$, $\mu_{\beta_3} = 2\beta_1 + \beta_2 + \beta_3$, $\mu_{\beta_4} = 3\beta_1 + \beta_2 + 3\beta_3 + \beta_4$ |
| $\Phi$ = $G_2$ | $\mu_{\beta_1} = \beta_1$, $\mu_{\beta_2} = \beta_1 + \beta_2$ |

**Remark 2.2.**

i) In fact, we will use Tables (3), (4), (5) only for exceptional root systems, but for the reader’s convenience we describe the weights $\mu_{\beta_i}$ for all irreducible root systems.

ii) Note that the correspondence between $\beta_i$ and $\mu_{\beta_i}$ is uniquely determined by the fact that $\mu_{\beta_i} \in \langle \beta_1, \ldots, \beta_i \rangle_{\mathbb{R}} \setminus \langle \beta_1, \ldots, \beta_{i-1} \rangle_{\mathbb{R}}$, where $\langle \cdot \rangle_{\mathbb{R}}$, as usual, denotes the linear span over $\mathbb{R}$. Furthermore, each $\beta \in \mathcal{B}$ occurs in $\mu_{\beta_i}$ with coefficient 1. Note also that if $\beta \in \mathcal{B} \cap \Delta$ then $\mu_{\beta_i}$ is the unique weight in Table 5 in which expression $\beta$ occurs. Our numeration of the weights $\mu_{\beta_i}$ here slightly differs from [Pa2], [IG1] and [IP].

iii) Recall that $\{e^*_\alpha, \alpha \in \Phi^+\}$ is the basis of $n^*$ dual to the basis $\{e_\alpha, \alpha \in \Phi^+\}$ of $n$. Put $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $R = \left\{ t = \sum_{\beta \in \mathcal{B}} t^*_\beta e^*_\beta, \ t_\beta \in \mathbb{C}^\times \right\}$ and denote by $X$ the union of all $N$-orbits in $n^*$ of elements of $R$. In fact, $X$ is a single $B$-orbit in $n^*$, and the $N$-orbits of two distinct point of $R$ are disjoint. Kostant [Ko3] Theorems 1.1, 1.3 proved that $X$ is a Zariski dense subset of $n^*$, and for $t \in R$, up to scalar multiplication, for each $\beta_i \in \mathcal{B}$,

$$\xi_{\beta_i}(t) = \prod_{\beta \in \mathcal{B}} t^*_\beta r^*_{\beta_i}(\beta), \ 1 \leq i \leq m,$$

where $r^*_{\beta_i}(\beta) = \frac{\langle \mu_{\beta_i}, \beta \rangle}{\langle \beta, \beta \rangle}$.

Clearly, $r^*_{\beta_i}(\beta)$ is nothing but the coefficient at $\beta$ in the expression of $\mu_{\beta_i}$ in Table (5).

We fix the generators $\xi_{\beta_i}, \beta \in \mathcal{B}$, so that the formulas (5) are satisfied (without any additional scalars). For $\beta \in \mathcal{B}$, we denote $\Delta_\beta = \sigma(\xi_{\beta}) \in Z(\mathfrak{n})$. Explicit formulas for $\xi_{\beta}$ and $\Delta_\beta$ for classical root systems can be found in [IP Subsection 2.1].

**Definition 2.3.** We call $\Delta_\beta$ (respectively, $\xi_{\beta}$), $\beta \in \mathcal{B}$, the canonical generators of the algebra $Z(\mathfrak{n})$ (respectively, of the algebra $Y(\mathfrak{n})$).
3. Centrally generated ideals

Let \( g, n, \Phi, \Delta \), etc., be as in Section 2. A (two-sided) ideal \( J \subset U(n) \) is called primitive if \( J \) is the annihilator of a simple \( n \)-module. An ideal \( J \) is called centrally generated if \( J \) generated (as an ideal) by its intersection \( J \cap Z(n) \) with the center \( Z(n) \) of \( U(n) \).

In the 1960s A. Kirillov, B. Kostant and J.-M. Souriau discovered that the orbits of the coadjoint action play a crucial role in the representation theory of \( B \) nilpotent Lie algebra (in particular, of \( n \) orbit method provides a nice description of primitive ideals of the universal enveloping algebra of a nilpotent Lie algebra (in particular, of \( n \)). Below we briefly recall this description.

To any linear form \( \lambda \in n^* \) one can assign a bilinear form \( \beta_\lambda \) on \( n \) by putting \( \beta_\lambda(x, y) = \lambda([x, y]) \). A subalgebra \( p \subseteq n \) is a polarization of \( n \) at \( \lambda \) if it is a maximal \( \beta_\lambda \)-isotropic subspace. By \[V_n\], such a subalgebra always exists. Let \( p \) be a polarization of \( n \) at \( \lambda \), and \( W \) be the one-dimensional representation of \( p \) defined by \( x \mapsto \lambda(x) \). Then the annihilator \( J(\lambda) = \text{Ann}_{U(n)} V \) of the induced representation \( V = U(n) \otimes_U (p) W \) is a primitive two-sided ideal of \( U(n) \). It turns out that \( J(\lambda) \) depends only on \( \lambda \) and not on the choice of polarization. Further, \( J(\lambda) = J(\mu) \) if and only if the coadjoint \( N \)-orbits of \( \lambda \) and \( \mu \) coincide. Finally, the Dixmier map

\[
D: n^* \to \text{Prim} U(n), \ \lambda \mapsto J(\lambda),
\]

induces a homeomorphism between \( n^*/N \) and \( \text{Prim} U(n) \), where the latter set is endowed with the Jacobson topology. (See \[Di2\], \[Di4\], \[BGR\] for the details.)

In addition, it is well known that the following conditions on an ideal \( J \subset U(n) \) are equivalent \[Di4\] Proposition 4.7.4, Theorem 4.7.9]:

1. \( J \) is primitive;
2. \( J \) is maximal;
3. the center of \( U(n)/J \) is trivial;
4. \( U(n)/J \) is isomorphic to a Weyl algebra of finitely many variables.

Recall that the Weyl algebra \( A_s \) of \( 2s \) variables is the unital associative algebra with generators \( p_i, q_i \) for \( 1 \leq i \leq s \), and relations \([p_i, q_i] = 1, \ [p_i, q_j] = 0 \) for \( i \neq j \), \([p_i, p_j] = q_i, q_j] = 0 \) for all \( i, j \).

Furthermore, in conditions (1) we have \( U(n)/J \cong A_s \) where \( s \) equals one half of the dimension of the coadjoint \( N \)-orbit of \( \lambda \), given that \( J(\lambda) \).

**Definition 3.1.** To a map \( \xi: B \to \mathbb{C} \) we assign the linear form \( f_\xi = \sum_{\beta \in B} \xi(\beta) e_\beta^* \in n^* \). We call a form \( f_\xi \) a Kostant form if \( \xi(\beta) \neq 0 \) for any \( \beta \in B \setminus \Delta \).

Let \( V \) be a simple \( n \)-module and \( J = \text{Ann}_{U(n)} V \) be the corresponding primitive ideal of \( U(n) \). By a version of Schur’s Lemma \[Di4\], each central element of \( U(n) \) acts on \( V \) as a scalar operator. Given \( \beta \in B \), let \( c_\beta \) be the scalar corresponding to \( \Delta_\beta \). We denote by \( J_c \) the ideal of \( U(n) \) generated by all \( \Delta_\beta - c_\beta, \beta \in B \). Clearly, \( J_c \subseteq J \). Further, since \( Z(n) \) is a polynomial ring and the center of \( U(n)/J \) is trivial, \( J \) is centrally generated if and only if \( J = J_c \).

The following result was proved in \[ILP\] Theorem 3.1 and \[ILG1\] Theorem 2.4.

**Theorem 3.2.** Suppose \( \Phi \) is an irreducible root system of classical type, i.e., \( \Phi = A_{n-1}, B_n, C_n \) or \( D_n \). The following conditions on a primitive ideal \( J \subset U(n) \) are equivalent:

1. \( J \) is centrally generated (or, equivalently, \( J = J_c \));
2. the scalars \( c_\beta, \beta \in B \setminus \Delta \), are nonzero;
3. \( J = J(f_\xi) \) for a Kostant form \( f_\xi \in n^* \).

If these conditions are satisfied, then the map \( \xi \) can be reconstructed by \( J \).
The main result of the paper is to prove that this is also true for exceptional root systems, see Theorem 8.1 in Section 8. One of the key ingredients in the proof of Theorem 8.2 was to check that if condition (ii) is satisfied then $J_c$ is primitive. To do this for $A_{n-1}$ and $C_n$, in $[11]$ an explicit set of generators of the quotient algebra $U(n)/J_c$ was constructed. It turns out that these generators satisfy (up to scalars) the defining relations of the Weyl algebra $A_s$ for $s = \{\Phi^+ \setminus \emptyset/2$. Since $A_s$ is simple and, as one can check, $J \neq U(n)$, we conclude that $U(n)/J_c \cong A_s$, and, consequently, $J_c$ is primitive. On the other hand, for $B_n$ and $D_n$, in $[12]$ an explicit set of generators for $U(n)/J$ was constructed a posteriori (see $[12]$ Theorem 2.9], while primitivity of $J_c$ was established by another argument. In this section we modify the idea from $[12]$ Proposition 2.5] to check that $J_c$ is primitive if $c_3 \neq 0$ for $-1 \leq \Delta$.

To do this, we need some additional notation. From now on an to the end of this section we assume that $\Phi$ is an irreducible root system of exceptional type, i.e., $\Phi = E_6, E_7, E_8, F_4$ or $G_2$. Recall that $\beta_1$ is the maximal root with respect to the natural order on $\Phi$. It is obvious that $(\alpha, \beta_1) \geq 0$ for all $\alpha \in \Phi^+$. We put $\tilde{\Phi} = \{\alpha \in \Phi \mid (\alpha, \beta_1) = 0\}$ and $\tilde{\Phi}^+ = \tilde{\Phi} \cap \Phi^+$, $\Delta = \tilde{\Phi} \cap \Delta$. Then $\tilde{\Phi}$ is of respective type $D_5, E_6, E_7, C_3$ or $A_1$. Denote

$$\tilde{n} = \langle e_\alpha, \alpha \in \Phi^+, (\alpha, \beta_1) = 0 \rangle_\mathcal{C} = \langle e_\alpha, \alpha \in \tilde{\Phi}^+ \rangle_\mathcal{C},$$

$$\mathfrak{f} = \langle e_\alpha, \alpha \in \Phi^+, (\alpha, \beta_1) > 0 \rangle_\mathcal{C}.$$ 

Then $\tilde{n}$ is a Lie subalgebra of $n$ isomorphic to the nilradical of the Borel subalgebra $\tilde{b} = \tilde{g} \cap b$ of the simple Lie algebra $g$ with the root system $\tilde{\Phi}$, where $\tilde{g}$ is the subalgebra of $g$ generated by the root vectors $e_\alpha, \alpha \in \Phi$.

On the other hand, $\mathfrak{f}$ is an ideal of $n$ isomorphic to the Heisenberg Lie algebra $\mathfrak{hei}_n$, where $s = (\Phi^+ \setminus \tilde{\Phi}^+) \setminus 0$, with the center $C_{\beta_1}$. (This follows from the fact that if $\alpha \in \Phi^+$ and $(\alpha, \beta_1) > 0$ then $\beta_1 - \alpha$ is again a positive root, see $[102]$ Corollary 2.3 for the details.) Recall that $\mathfrak{hei}_n$ is the $(2s + 1)$-dimensional Lie algebra with basis $\{x, x_i, y_i, 1 \leq i \leq n\}$ and relations $[x_i, y_i] = z$ for all $i, [x_i, z] = [y_j, z] = [x_i, y_j] = 0$ for all $i \neq j$.

Given $c_1 \in \mathbb{C}_x$, denote by $J_1$ the ideal of $U(\mathfrak{f})$ generated by $e_{\beta_1} - c_1$, then, clearly, $U(\mathfrak{f})/J_1 \cong A_s$. Since $\mathfrak{f}$ is an ideal of the Lie algebra $n$, given $x \in \tilde{n}$, one can consider $ad_x$ as a derivation of $\mathfrak{f}$. Since $e_{\beta_1} - c_1$ is a central element of $U(n)$, one has $ad_x(J_1) \subseteq J_1$, so $ad_x$ can be considered as a derivation of $A_s$. It is well known (see, e.g., $[12]$, 10.1.4) that there exist the unique element $\theta(x) \in A_s$ such that $ad_x(y) = \theta(x), y$ for all $y \in A_s$, and $\theta: \tilde{n} \rightarrow A_s$ is a morphism of Lie algebras. Furthermore, there exist the unique epimorphism of associative algebras $r: U(n) \rightarrow U(\tilde{n}) \otimes A_s$ such that $r(y) = 1 \otimes \tilde{y}$ for $y \in \mathfrak{f}$ and $r(x) = x \otimes 1 + 1 \otimes \theta(x)$ for $x \in \tilde{n}$. Here $\mathfrak{f}$ is the image of an element $a \in U(\mathfrak{f})$ under the canonical projection $U(\mathfrak{f}) \rightarrow U(\mathfrak{f})/J_1 \cong A_s$. It turns out that the kernel of the epimorphism $r$ coincides with the ideal $J_0$ of $U(n)$ generated by $e_{\beta_1} - c_1$ $[12]$, Lemma 10.1.5].

Proposition 3.3. Let $J_c$ be the ideal of $U(n)$ generated by $\Delta_c = \tilde{c}_3, \beta \in \mathcal{B}, \beta \neq 0$ for $\beta \in \mathcal{B} \setminus \Delta$. Then $J_c$ is primitive.

Proof. Put $c_1 = e_{\beta_1}$. Since $r$ is surjective, $r(J_c)$ is an ideal of $U(\tilde{n}) \otimes A_s$ generated by $r(\Delta_c) - c_3, \beta \in \tilde{B} = \mathcal{B} \setminus \{\beta_1\}$. Note that $\tilde{B}$ is the Kostant cascade of $\tilde{\Phi}$. Denote by $\tilde{\Delta}_c, \beta \in \tilde{\Phi}$, the set of canonical generators of $Z(\tilde{n})$. Our first goal is to check that, up to nonzero scalar, $r(\Delta_c)$ coincides with $\tilde{\Delta}_c \otimes 1$ for all $\beta \in \tilde{B}$.

To check this fact, we will use Tables $[19], [20], [21]$. Pick a root $\beta \in \tilde{B}$. Since $r$ is surjective, $r(\Delta_c)$ is central in $U(\tilde{n}) \otimes A_s$. The center of this algebra has the form $Z(\tilde{n}) \otimes \mathbb{C}$, so in fact $r(\Delta_c) \in Z(\tilde{n}) \otimes \mathbb{C}$. Denote $\tilde{h} = \tilde{n} \cap \mathfrak{h}$, then $\tilde{h}$ is a Cartan subalgebra of $\tilde{g}$ and $B = \tilde{h} \oplus \mathfrak{n}$ as vector spaces. By $[102]$ Theorem 6] (see also $[11]$, Lemma 4.4), $Z(n)$ (respectively, $Z(\tilde{n})$) is a direct sum of 1-dimensional weight spaces of $\mathfrak{h}$ (respectively, of $\tilde{h}$) with respect to the adjoint action of the corresponding Cartan subalgebras. Since $[h, e_{\beta_1}] = 0$ for all $h \in \mathfrak{h}$, the algebra $\tilde{h}$ naturally acts on $A_s$, and so on $U(\tilde{n}) \otimes A_s$. We define the result of this action by $h.x, h \in \tilde{h}, x \in U(\tilde{n}) \otimes A_s$. Hence it is enough to check that, given $\beta \in \tilde{B}$, $r(\Delta_c)$ is a nonzero $\tilde{h}$-weight element of weight $\tilde{\mu}_c = \mu_\beta - \beta(\beta_1)\beta_1$.
To prove that \( r(\Delta_\beta) \) is an \( \tilde{\mathfrak{h}} \)-weight element of weight \( \tilde{\mu}_\beta \), denote the result of the natural (adjoint) action of \( \tilde{\mathfrak{h}} \) on \( U(n) \) by \( h \cdot x, h \in \tilde{\mathfrak{h}}, x \in U(n) \). As above, since \( \tilde{\mathfrak{h}} \cdot e_{\beta_1} = 0 \), the algebra \( \tilde{\mathfrak{h}} \) naturally acts on \( U(n)/J_0 \) by the formula \( h \cdot r(x) = r(h \cdot x) \). We claim that this action coincides with the action of \( \tilde{\mathfrak{h}} \) on \( U(\tilde{n}) \otimes A_\xi \) defined above, i.e., that \( h \cdot x = h \cdot x \) for all \( h \in \tilde{\mathfrak{h}}, x \in U(\tilde{n}) \otimes A_\xi \).

Indeed, if \( y \in \mathfrak{k} \), then
\[
\begin{align*}
h \cdot r(y) &= r([h, y]) = 1 \otimes [h, y] = h.(1 \otimes y) = h \cdot r(y),
\end{align*}
\]
as required. On the other hand, if \( e_\alpha \in \tilde{n} \) for some root \( \alpha \in \Phi^+ \), then, by [LO1], Subsection 4.8, \( \theta(e_\alpha) \) is a linear combination of elements of the form \( \bar{e}_{\alpha+\gamma} \bar{e}_{\beta_1-\gamma} \), \( \gamma \in \Phi^+ \setminus \Phi^+ \) (i.e., \( (\gamma, \beta_1) > 0 \)).

We conclude that
\[
h(1 \otimes \theta(e_\alpha)) = (\beta_1 + \alpha)(h)1 \otimes \theta(\alpha) = \alpha(h)1 \otimes \theta(e_\alpha),
\]
because \( \beta_1(\bar{h}_c) = 0 \). Thus, we obtain
\[
h \cdot r(e_\alpha) = [h, e_\alpha] \otimes 1 + 1 \otimes \alpha(h) \theta(e_\alpha) = \alpha(h)1 \otimes r(e_\alpha) = h \cdot r(e_\alpha).
\]
It remains to note that \( \Delta_\beta \) is an \( \mathfrak{h} \)-weight element of \( U(n) \) of weight \( r_\beta(\beta_1) \beta_1 + \bar{\mu}_\beta \), but \( \beta_1(\tilde{\mathfrak{h}}) = 0 \).

To show that \( r(\Delta_\beta) \neq 0 \), recall that the kernel of \( r \) is \( J_0 \). If \( \Delta_\beta \in J_0 \) (i.e., if \( \Delta_\beta = (e_{\beta_1} - c_{\beta_1})a \) for some \( a \in U(n) \)), then, clearly, \( a \in Z(n) \). But this contradicts the fact that \( \Delta_\beta \) and \( \Delta_{\beta_1} \) are algebraically independent, because, as one can deduce from [Pa2, \( \Delta_{\beta_1} = c_{\beta_1} \) for all irreducible root systems.

So, given \( \beta \in \bar{B} \), there exists the unique \( a_{\beta} \in \mathbb{C}^x \) such that \( r(\Delta_\beta) = a_{\beta} \Delta_\beta \otimes 1 \). Consequently, \( r(J_c) \) is generated by \( \Lambda_\beta \otimes 1 - \bar{c}_\beta \), \( \beta \in \bar{B} \), where \( \bar{c}_\beta = a_{\beta}^{-1}c_{\beta} \). In particular, \( \bar{c}_\beta \neq 0 \) if \( \beta \in \bar{B} \) is not a simple root of \( \Phi^+ \).

Now we will use the induction on \( rk \Phi \) to prove that \( J_c \) is primitive. The base (i.e., the case of classical \( \Phi \) of low rank) immediately follows from [IP Theorem 3.1] and [Ig1 Theorem 2.4]. Denote by \( \bar{J}_c \) the ideal of \( U(\tilde{n}) \) generated by \( \Lambda_\beta \otimes 1 - \bar{c}_\beta, \beta \in \bar{B} \). By the inductive assumption, \( \bar{J}_c \) is a primitive ideal of \( U(\tilde{n}) \), so \( U(\tilde{n})/\bar{J}_c \cong A_t \) for certain \( t \). We conclude that
\[
U(n)/J_c \cong (U(n)/J_0)/r(J_c) \cong (U(\tilde{n}) \otimes A_\xi)/r(J_c) = (U(\tilde{n})/\bar{J}_c) \otimes A_\xi \cong A_t \otimes A_\xi \cong A_{t+s}.
\]
Thus, \( J_c \) is primitive. The proof is complete. \( \square \)

### 4. Distinct coadjoint orbits

Recall that, given a primitive ideal \( J \) in \( U(n) \), there exist the unique scalars \( c_\beta \in \mathbb{C} \) such that \( \Delta_\beta - c_\beta \in J \) for all \( \beta \in B \). To prove our main result, Theorem 5.1 we need to check that if \( J \) is centrally generated then \( c_\beta \neq 0 \) for \( \beta \in B \setminus \Delta \). To do this, we will prove that certain coadjoint \( N \)-orbits on \( \mathfrak{n}^* \) are distinct.

Namely, let \( D \) be a subset of \( \Phi^+ \). To each map \( \xi: D \rightarrow \mathbb{C}^x \) one can assign the linear form
\[
f_{D,\xi} = \sum_{\beta \in D} \xi(\beta) e_\beta^* \in \mathfrak{n}^*.
\]

Denote by \( \Omega_{D,\xi} \) the coadjoint \( N \)-orbit of \( f_{D,\xi} \). We say that \( f_{D,\xi} \) and \( \Omega_{D,\xi} \) are associated with the subset \( D \). For example, \( f_{D,\xi} \) is a Kostant form if and only if \( \Delta \setminus D \subset \Delta \setminus B \).

It was proved in [Pa1 Corollary 1.4] that if \( \Phi = A_{n-1} \), \( D \) is an orthogonal subset (i.e., \( (\alpha, \beta) = 0 \) for all \( \alpha, \beta \in D, \alpha \neq \beta \) and \( \xi_1, \xi_2 \) are two distinct maps from \( D \) to \( \mathbb{C}^x \) then \( \Omega_{D,\xi_1} \neq \Omega_{D,\xi_2} \). It is not hard to deduce from this result that the same is true for all classical root systems, see the proof of [IP Theorem 3.1] and the proof of [Ig1 Theorem 2.4]. But for exceptional types this is not an immediate consequence of the result for \( A_{n-1} \). In this section, we prove that if \( \xi_1 \neq \xi_2 \) then \( \Omega_{D,\xi_1} \) and \( \Omega_{D,\xi_2} \) are distinct for some particular orthogonal subsets \( D \) and some particular maps \( \xi_1, \xi_2 \), which will be used in the next section in the proof of our main result.
To do this, we need to introduce the notion of singular roots.

**Definition 4.1.** Let \( \beta, \alpha \) be positive roots. We say that \( \alpha \) is \( \beta \)-singular (or singular for \( \beta \)) if there exists \( \gamma \in \Phi^+ \) such that \( \beta = \alpha + \gamma \). The set of all \( \beta \)-singular roots is denoted by \( S(\beta) \).

Note that if \( \Phi \) is irreducible and simple-laced (i.e., if all roots in \( \Phi \) have the same length) then, given \( \beta > \alpha \), \( \alpha \) is \( \beta \)-singular if and only if \((\alpha, \beta) > 0 \). It turns out that if \( D \) is an orthogonal subset of \( \Phi^+ \), \( \xi \) is a map from \( D \) to \( \mathbb{C}^\times \), and \( \beta, \beta' \in D \) are such that \( \beta' \in S(\beta) \) then \( \Omega_{D,\xi} = \Omega_{D,\xi'} \), where \( D' = D \setminus \beta' \) and \( \xi' \) is the restriction of \( \xi \) to \( D' \) \cite[Lemma 1.3]{[12]}

**Proposition 4.2.** Let \( \Phi \) be an irreducible root system, and \( D \) be a subset of \( \Phi^+ \) such that if \( \beta_1, \beta_2 \in D \) then \( \beta_1 \notin S(\beta_2) \). Let \( \beta_0 \) be a root in \( D \), \( \xi_1 \) and \( \xi_2 \) be maps from \( D \) to \( \mathbb{C}^\times \) for which \( \xi_1(\beta_0) \neq \xi_2(\beta_0) \). Assume that there exists a simple root \( \alpha_0 \in \Delta \) satisfying \((\alpha_0, \beta_0) \neq 0 \) and \((\alpha_0, \beta) = 0 \) for all \( \beta \in D \) such that \( \beta \neq \beta_0 \). Then \( \Omega_{D,\xi_1} \neq \Omega_{D,\xi_2} \).

**Proof.** As usual, given a vector space \( V \), we denote by \( \mathfrak{gl}(V) \) the Lie algebra of all linear operators on \( V \). Denote by \( \text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) be the adjoint representation of the Lie algebra \( \mathfrak{g} \), i.e., \( \text{ad}(x) = \text{ad}_x \). It is well known that the adjoint representation is exact, so \( \text{ad}(\mathfrak{g}) \) and \( \mathfrak{g} \) are isomorphic as Lie algebras.

Let \( \text{GL}(V) \) be the group of all invertible linear operators on a vector space \( V \). Since we fixed a basis in \( \mathfrak{g} \), the group \( \text{GL}(\mathfrak{g}) \) is identified with the group \( \text{GL}_{\text{dim} \mathfrak{g}}(\mathbb{C}) \), and \( \exp \text{ad}(\mathfrak{n}) \cong N \) is identified with a subgroup of the group \( U \) of all upper-triangular matrices from \( \text{GL}_{\text{dim} \mathfrak{g}}(\mathbb{C}) \) with 1’s on the diagonal. Furthermore, using the Killing form on \( \mathfrak{g} \) and the trace form on \( \mathfrak{gl}(\mathfrak{g}) \), one can identify \( \mathfrak{n}^* \) with the space \( \mathfrak{n}_- = \{e_{-\alpha}, \alpha \in \Phi^+ \}_\mathbb{C} \) and \( \mathfrak{u}^* \) with the space \( \mathfrak{u}_- = \mathfrak{u}^T \), where the superscript \( T \) denote the transposed matrix. Under all these identifications, it is enough to check that the coadjoint \( U \)-orbits of the linear forms \( \tilde{f}_{D,\xi} \) and \( \tilde{f}_{D,\xi'} \) are distinct. Here, given a map \( \xi : D \to \mathbb{C}^\times \), we denote by \( \tilde{f}_{D,\xi} \) the matrix

\[
\tilde{f}_{D,\xi} = \left( \frac{\sum_{\beta \in D} \xi(\beta)e_\beta}{\sum_{\beta \in D} \xi(\beta)e_\beta} \right)^T \in \mathfrak{u}_- \cong \mathfrak{u}^*
\]

To do this, we will study the matrix \( f = \tilde{f}_{D,\xi} \) in more details. The rows and the columns of matrices from \( \mathfrak{gl}(\mathfrak{g}) \) are now indexed by the elements of the Chevalley basis fixed above. Given a matrix \( x \) from \( \mathfrak{gl}(\mathfrak{g}) \) and two basis elements \( a, b \), we will denote by \( x_{a,b} \) the entry of \( x \) lying in the \( a \)th row and the \( b \)th column. Since

\[
\text{ad}_{e_{\beta_0}}(h_{\alpha_0}) = [e_{\beta_0}, h_{\alpha_0}] = -\frac{2(\alpha_0, \beta_0)}{(\alpha_0, \alpha_0)}e_{\beta_0},
\]

we obtain \( f_{h_{\alpha_0}, e_{\beta_0}} = -\xi(\beta_0)\frac{2(\alpha_0, \beta_0)}{(\alpha_0, \alpha_0)}e_{\beta_0} \neq 0 \). One may assume without loss of generality that \( h_{\alpha_0} >_t h_{\alpha_i} \) for all \( \alpha_i \neq \alpha_0 \). We claim that

\[
f_{h_{\alpha_0}, e_\alpha} = f_{e_{-\gamma}, e_{\beta_0}} = 0 \text{ for all } e_\alpha <_t e_{\beta_0} \text{ and all } e_{-\gamma}, \alpha, \gamma \in \Phi^+.
\]

Indeed, if \( \alpha \notin D \) then, evidently, \( f_{h_{\alpha_0}, e_\alpha} = 0 \). If \( \alpha = \beta \in D \) and \( e_\beta <_t e_{\beta_0} \) then \( \beta \notin \beta_0 \), hence

\[
f_{h_{\alpha_0}, e_\beta} = -\xi(\beta)\frac{2(\alpha_0, \beta)}{(\alpha_0, \alpha_0)} = 0,
\]

because \( (\alpha_0, \beta) = 0 \). On the other hand, if \( f_{e_{-\gamma}, e_{\beta_0}} \neq 0 \) for some \( \gamma \in \Phi^+ \) then \( \beta_0 = \beta - \gamma \). This contradicts the condition \( \beta_0 \notin S(\beta) \).
Thus, \((f_{D,ξ_1})_{h_{α_0},e_{α}}\) and \((f_{D,ξ_2})_{h_{α_0},e_{α}}\) are different nonzero scalars, and \(f\) is satisfied both for \(f = \tilde{f}_{D,ξ_1}\) and for \(f = \tilde{f}_{D,ξ_2}\). Now it follows immediately from the proof of [An, Proposition 3] that the coadjoint \(U\)-orbits of these matrices are distinct, and, consequently, \(Ω_{D,ξ_1} \neq Ω_{D,ξ_2}\), as required. □

Now, for exceptional \(Φ\), we present a list of certain subsets \(D \subset Φ^+\). To each \(D\) from this list we assign its subset \(D' \subset D\). Using Proposition 4.2 we will show that if \(ξ_1\) and \(ξ_2\) are two maps from \(D\) to \(C^x\) such that \(ξ_1(β_0) \neq ξ_2(β_0)\) for some root \(β_0 \in D'\) then \(Ω_{D,ξ_1} \neq Ω_{D,ξ_2}\), see Proposition 4.3 below. We will consider all exceptional root systems subsequently. For brevity, we use the following short notation. If \(β = \sum_{i=1}^n m_iα_i \in D\) then we write \(m_1 \ldots m_n\) instead of \(β\) (our numeration of simple roots is as in (1)). Below one can find the table for the root system \(E_6\). All other tables are presented at the end of the paper (see Appendix A). Note that in all cases, except cases 11, 12 and 14 for \(F_4\), \(D\) is an orthogonal subset of \(Φ\), while in cases 11, 12, 14 for \(F_4\) all inner products of distinct roots from \(D\) are non-positive. Note also that all these subsets are linearly independent.

| \(D\) | \(D'\) | \(D\) | \(D'\) | \(D\) | \(D'\) |
|---|---|---|---|---|---|
| 1 010000 | 010000 | 2 011211, 111221, 112210 | 112210 | 3 011111, 111110, 112321 | 111110, 112321 |
| 4 011211, 111210, 112221 | 011211, 111210 | 5 111111, 112321 | 112321 | 6 011210, 111221, 112211 | 111221, 112211 |
| 7 011110, 111111, 112321 | 011110, 111111, 112321 | 8 010000, 011210, 111211, 112211 | 112211 | 9 001000, 122321 | 001000 |
| 10 001100, 000111, 122321, 101110 | 001100, 000111 | 11 001111, 122321, 101110 | 101110 | 12 001111, 000100, 122321, 101110 | 001111, 101110 |
| 13 001000, 101111, 122321 | 001000 | 14 001100, 000110, 122321, 101111 | 000110 | 001100, 000110 |

**Proposition 4.3.** Let \(Φ\) be an irreducible root system of exceptional type, and \(D \subset Φ^+\) be an subset from the list above or from one of the lists in Appendix A. Let \(ξ_1, ξ_2\) be maps from \(D\) to \(C^x\). Assume that there exists a root \(β_0 \in D'\) such that \(ξ_1(β_0) \neq ξ_2(β_0)\). Then \(Ω_{D,ξ_1} \neq Ω_{D,ξ_2}\).

**Proof.** Accordingly to Proposition 4.2 it is enough to check that there exists a simple root \(α_0\) such that \((α_0, β_0) \neq 0\) and \((α_0, β) = 0\) for all \(β \in D \setminus D'\) (one can pick \(β_0\) to be maximal among all roots from \(D'\) which are non-orthogonal to \(α_0\)). This can be done straightforward case-by-case for all subsets \(D\).

For example, consider case 10 for \(Φ = E_6\). Here
\[
D = \{α_3 + α_4, α_4 + α_5 + α_5, α_1 + 2α_2 + 2α_3 + 3α_4 + 2α_5 + α_6, α_1 + α_3 + α_4 + α_5\},
\[
D' = \{α_3 + α_4, α_4 + α_5 + α_5\}.
\]
One can immediately check that \(β_0 = α_3 + α_4\) and \(α_0 = α_4\) satisfy all the conditions of Proposition 4.2 hence \(Ω_{D,ξ_1} \neq Ω_{D,ξ_2}\). All other cases can be considered similarly. □
5. Proof of the main result

We are now ready to formulate and, using the previous sections, prove our main result, Theorem 5.1 (cf. Theorem 3.2). Note that each element of $S(\mathfrak{n})$ can be considered as a polynomial function on $\mathfrak{n}^*$ via the natural isomorphism $(\mathfrak{n}^*)^* \cong \mathfrak{n}$.

**Theorem 5.1.** Suppose $\Phi$ is an irreducible root system of exceptional type, i.e., $\Phi = E_6$, $E_7$, $E_8$, $F_4$ or $G_2$. The following conditions on a primitive ideal $J \subset U(\mathfrak{n})$ are equivalent:

i) $J$ is centrally generated;

ii) the scalars $c_\beta$, $\beta \in \mathcal{B} \setminus \Delta$, are nonzero;

iii) $J = J(f_\xi)$ for a Kostant form $f_\xi \in \mathfrak{n}^*$.

*If these conditions are satisfied, then the map $\xi$ can be reconstructed by $J$.*

*Proof.* (ii) $\implies$ (iii). Recall that each two $N$-orbits of distinct linear form from

$$R = \left\{ t = \sum_{\beta \in \mathcal{B}} t_\beta e_\beta^*, \ t_\beta \in \mathbb{C}^* \right\}$$

are disjoint, and the union $X$ of all such orbits is an open dense subset of $\mathfrak{n}^*$; in fact, $X$ is a single $B$-orbit. Given $\beta_i \in \mathcal{B}$ and $t \in R$, one has

$$\xi_{\beta_i}(t) = \prod_{\beta \in \mathcal{B}} t^r_{\beta_i} = r_{\beta_i}(\beta), \text{ where } \mu_{\beta_i} = \sum_{\beta \in \mathcal{B}} r_{\beta_i}(\beta)\beta,$$

see Remark 2.2 (iii). Furthermore, from Remark 2.2 (i), (ii) we see that actually

$$\mu_{\beta_i} = \beta_i + \sum_{j<i} r_{\beta_i}(\beta_j)\beta_j, \text{ so } \xi_{\beta_i}(t) = t_{\beta_i} \prod_{j<i} r_{\beta_j}(\beta_j).$$

We claim that there exists the unique map $\xi : \mathcal{B} \to \mathbb{C}^*$ such that $\xi(\beta) \neq 0$ for $\beta \in \mathcal{B} \setminus \Delta$ and the Kostant form $f_\xi$ satisfies $\xi_{\beta_i}(f_\xi) = c_\beta$ for all $\beta_i \in \mathcal{B}$. Indeed, since

$$f_\xi = \sum_{j=1}^m \xi(\beta_j)e_{\beta_j}^*, \ m = |\mathcal{B}|,$$

belongs to $\overline{R}$, the Zariski closure of $R$ in $\mathfrak{n}^*$, we obtain that

$$\xi_{\beta_i}(f_\xi) = \xi(\beta_i) \prod_{j<i} \xi(\beta_j)^{r_{\beta_i}(\beta_j)}$$

for all $i$ from 1 to $m$. Since $\xi_{\beta_i} = c_{\beta_i}$, we must set $\xi(\beta_1) = c_{\beta_1}$. Now, assume that $i > 1$ and that $\xi_{\beta_i}$ is already defined for all $j < i$ so that $\xi_{\beta_j}(f_\xi) = c_{\beta_j}$. Then one can put $\xi(\beta_i)$ to be equal to $c_{\beta_i} \prod_{j<i} \xi(\beta_j)^{-r_{\beta_i}(\beta_j)}$, so that $\xi_{\beta_i}(f_\xi) = c_{\beta_i}$, as required. Note that, accordingly to Remark 2.2 (i), if $\xi(\beta_j) = 0$ for some $j < i$ then $\beta_j \in \mathcal{B} \cap \Delta$, and, consequently, $r_{\beta_i}(\beta_j) = 0$. Thus, $\beta_j$ does not actually occurs in the expression of $\mu_{\beta_i}$ and we do not divide by zero in the definition of $\xi(\beta_i)$.

Now, let $\xi : D \to \mathbb{C}^*$ be such that $\xi(\beta_i)(f_\xi) = c_{\beta_i}$ for all $i$. Then, by [Di1], 6.6.9 (c)], $J(f_\xi)$ contains $\Delta_{\beta_i} - c_{\beta_i}$ for all $i$, hence $J(f_\xi)$ contains the centrally generated ideal $J_c$, which is generated by definition by all $\Delta_{\beta_i} - c_{\beta_i}$, $1 \leq i \leq m$. But, thanks to Proposition 3.3, $J_c$ is primitive. Thus, both $J(f_\xi)$ and $J_c$ are primitive (and so maximal), hence $J(f_\xi) = J_c$.

(iii) $\implies$ (i). Again by [Di1], 6.6.9 (c)],

$$c_{\beta_i} = \xi_{\beta_i}(f_\xi) = \xi(\beta_i) \prod_{j<i} \xi(\beta_j)^{r_{\beta_i}(\beta_j)} \quad (9)$$

for all $\beta_i \in \mathcal{B}$, hence $c_{\beta_i} \neq 0$ for $\beta_i \in \mathcal{B} \setminus \Delta$. Both $J$ and $J(f_\xi)$ contain the centrally generated ideal $J_c$, and condition (ii) is satisfied, so $J_c$ is primitive and $J = J_c = J(f_\xi)$, as required.
Let $c$ be an $m$-tuple as above, and $D$ be the corresponding subset of $\Phi^+$. Recall that in the previous section we assigned to $D$ a subset $D' \subset D$. Our next claim is to prove that there exists a map $\eta$ from $D$ to the field of rational functions $\mathbb{C}(x)$ such that

i) $\eta(\gamma)$ is non-constant for all $\gamma \in D'$,

ii) $\eta(\gamma)(x)$ is well-defined and nonzero for all $x \neq 0$, $\gamma \in D$.

iii) $\xi_{\beta_i}(f_{D,\eta(x)}) = c_{\beta_i}$ for all $\beta_i \in B$ and $x \neq 0$.

Here, given $x \in \mathbb{C}^*$, we denote by $\eta(x)$ the map from $D$ to $\mathbb{C}^*$ defined by $D \ni \gamma \mapsto \eta(\gamma)(x)$.

To construct such a map $\eta$, we firstly note that if $c_{\beta_i} \neq 0$ then the weight $\mu_{\beta_i}$ can be uniquely expressed as a $\mathbb{Z}_{\geq 0}$-linear combination of the roots from $D$, i.e., there exist unique $a_{\gamma,\beta_i} \in \mathbb{Z}_{\geq 0}$, $\gamma \in D$, satisfying

$$
\mu_{\beta_i} = \sum_{\gamma \in D} a_{\gamma,\beta_i} \gamma, \quad 1 \leq i \leq m, \quad c_{\beta_i} \neq 0.
$$

Indeed, the uniqueness follows from the linear independence of $D$, while the existence can be checked directly. For instance, if the root system $\Phi$ is of type $E_6$ and $D = \{\gamma_1 = \alpha_2, \gamma_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \gamma_3 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \gamma_4 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \gamma_5 = \gamma_6 = 0\}$ is the 8th subset from the table above then $\mu_{\beta_2} = \gamma_7 + \gamma_4$, $\mu_{\beta_3} = \gamma_2 + \gamma_3 + 2\gamma_4$, $\mu_{\beta_4} = 2\gamma_2 + 2\gamma_3 + 2\gamma_4$. This means that, given a map $\xi: D \to \mathbb{C}^*$, the value of at most one monomial of $\xi_{\beta_i}$ at the linear form $f_{D,\xi}$ is non-zero (precisely, of the monomial $e_D = \prod_{\gamma \in D} x_1^{a_{\gamma,\beta_i}}$). Note, however, that a priori we do not know that the monomial $e_D$ in fact occurs in $\xi_{\beta_i}$.

To check that $e_D$ really occurs in $\xi_{\beta_i}$, we consider the subset

$$
D_1 = \begin{cases} 
D, & \text{if } \beta_1 \in D, \\
D \cup \{\beta_1\}, & \text{if } \beta_1 \notin D.
\end{cases}
$$

(Actually, $\beta_1 \in D$ if and only if $c_{\beta_1} \neq 0$.) Given a map $\xi: D \to \mathbb{C}^*$, we consider a map $\xi_1: D_1 \to \mathbb{C}^*$ such that $\xi_1(\gamma) = \xi(\gamma)$ for $\gamma \in D$, and the linear form $f_1 = f_{D_1,\xi_1} = f_{D,\xi} + \xi_1(\beta_1)e_{\beta_1}^*$. Clearly, $e_D(f_{D,\xi}) = e_D(f_1)$. On the other hand, one can easily check that the condition $f_1(e_{\beta_1}) \neq 0$ implies that there exists a linear form $\lambda$ in the coadjoint $N$-orbit of $f_1$ such that $\lambda(e_{\beta_1}) \neq 0$ for all $\beta_1 \in B$. By Remark 2.2 (iii), $e_B = \prod_{\beta \in B} e_{\beta}^{r_{\beta}(\beta)}$ enters $\xi_1$, with coefficient 1, and $e_B(\lambda) = 0$. But $\lambda = g.f_1$ for a certain $g \in N$, where $g.f_1$ denotes the result of the coadjoint action of $N$ on $n^*$. The adjoint action of $N$ on the algebra $S(n)$ has the form $(z.s)(\mu) = s(z^{-1}.\mu)$, $z \in N$, $s \in S(n)$, $\mu \in n^*$. We see that $(g^{-1}.e_B)(f_1) = e_B(\lambda) \neq 0$. But $\xi_{\beta_i}$ in $N$-invariant, so the monomial $g^{-1}.e_B$ really occurs in $\xi_{\beta_i}$ (with coefficient 1). Thus, the latter monomial coincides with $c_{\beta_i}^D e_D$ for certain $c_{\beta_i}^D \in \mathbb{C}^*$.  

![Table: Correspondence between $c$ and $D$ for $E_6$ and $F_4$](image)
Next, we note that the (affine) solution space for the system of linear equations
\[ \sum_{\gamma \in D} a_{\gamma,\beta} y_{\gamma} = b_i, \quad 1 \leq i \leq m, \quad c_{\beta_i} \neq 0. \] (10)
is at least one-dimensional for all possible \( b_i \in \mathbb{C} \). Indeed, \( k = |D| \) is in fact not less than the number of non-zero scalars in \( c \) plus one, so this system on \( k \) variables contains at most \((k-1)\) equations. It follows from Panov’s description of the weights \( \mu_{\beta_i} \) [Pa2, p. 8] that the equations are linearly independent, and the rank of the system is at most \( (k-1) \). Hence the solution space is at least one-dimensional, as required. Note that \( D' \) is exactly the set of roots \( \gamma \in D \) for which the solution space is not orthogonal to the axis \( y_{\gamma} \).

We are ready to construct a map \( \eta \) satisfying the above conditions. Assume that such a map is already constructed. Then, given \( x \in \mathbb{C}^x \), one has
\[
\xi_{\beta_i}(f_{D,\eta(x)}) = c_D(\prod_{\gamma \in D} c_{\beta_i}) (f_{D,\eta(x)})
\]
Let \( b_i \in \mathbb{C} \) be such that \( \exp(b_i) = c_{\beta_i} - c_D \), and \( y = (y_{\gamma})_{\gamma \in D} \) be a solution of system (10). Then, clearly, \( \prod_{\gamma \in D} x_{\gamma}^{a_{\gamma,\beta}} = c_{\beta_i} (c_D)^{-1} \) for all \( i \), where \( x_{\gamma} = \exp(y_{\gamma}) \). Thus, we may set \( \eta(\gamma) = x_{\gamma} \) for \( \gamma \in D \). It is evident that \( \eta(\gamma) \) is a rational function on \( x \), where \( x \) is the exponent of a variable \( y_{\gamma} \) for \( \gamma \in D' \). Condition (iii) is satisfied by the definition of \( \eta \), condition (ii) is obvious. Finally, condition (i) is satisfied because the solution space is not orthogonal to the axis \( y_{\gamma} \) for \( \gamma \in D' \).

For example, if \( \Phi = E_6 \) and \( D = \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \} \) is, as above, the 8th subset from the table, then one can put
\[
\eta(\gamma_1) = x, \quad \eta(\gamma_2) = c_2 (c_D)^{-1/2}, \quad \eta(\gamma_3) = c_3 (c_D)^{-1/2}, \quad \eta(\gamma_4) = c_4 (c_D)^{-1/2}.
\]
(Here, given \( a \in \mathbb{C} \), we denote by \( a^{1/2} \) a complex number such that \( (a/2)^2 = a \).) Another example: if \( \Phi = E_6 \) and \( D = \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \} \) is the 11th subset from the table, then \( D' = \{ \gamma_1, \gamma_3 \} \) and
\[
\mu_{\beta_1} = \gamma_2, \quad \mu_{\beta_3} = \gamma_1 + 2 \gamma_2 + \gamma_3, \quad \eta(\gamma_1) = x, \quad \eta(\gamma_2) = c_1 (c_D)^{-1}, \quad \eta(\gamma_3) = c_3 (c_D)^{-1} c_2^{-1} x^{-1}.
\]
In general, let \( \eta: D \to \mathbb{C}(x) \) be a map satisfying conditions (i), (ii), (iii). Since \( \xi_{\beta_i}(f_{D,\eta(x)}) = c_{\beta_i} \) for all \( i \), one has \( J \subset J(f_{D,\eta(x)}) \) for all \( x \in \mathbb{C}^x \). But both \( J \) and \( J(f_{D,\eta(x)}) \) are maximal, hence \( J = J(f_{D,\eta(x)}) \) for all \( x \in \mathbb{C}^x \). Pick a root \( \beta_0 \in D' \). Since \( \eta(\beta_0) \) is non-constant, there exist \( x_1, x_2 \in \mathbb{C}^x \) such that \( \eta(\beta_0)(x_1) \neq \eta(\beta_0)(x_2) \). By Proposition 4.3, the coadjoint orbits of \( f_{D,\eta(x_1)} \) and \( f_{D,\eta(x_2)} \) do not coincide. Hence \( J(f_{D,\eta(x_1)}) \neq J(f_{D,\eta(x_2)}) \), a contradiction.

Finally, if conditions (i)–(iii) are satisfied then formula (9) and the equality \( \Delta_{\beta_i} = c_{\beta_i} \) together imply that the map \( \xi \) can be reconstructed by \( J \). The proof is complete.

As an immediate corollary, we obtain that a similar result is true for all (probably, reducible) root systems.

**Theorem 5.2.** Let \( \Phi \) be an arbitrary root system. The following conditions on a primitive ideal \( J \subset U(n) \) are equivalent:

i) \( J \) is centrally generated;

ii) the scalars \( c_{\beta}, \beta \in \mathcal{B} \setminus \Delta \), are nonzero;

iii) \( J = (f_z) \) for a Kostant form \( f_z \in n^* \).

If these conditions are satisfied, then the map \( \xi \) can be reconstructed by \( J \).
\[
\text{Proof.} \text{ Let } \Phi_i, 1 \leq i \leq k, \text{ be the irreducible components of the root system } \Phi, \text{ and } \mathfrak{n} = \bigoplus_{i=1}^{k} \mathfrak{n}_i \text{ be the corresponding division of } \mathfrak{n} \text{ into a direct sum of its nilpotent ideals, then } U(\mathfrak{n}) = U(\mathfrak{n}_1) \otimes \cdots \otimes U(\mathfrak{n}_k) \text{ as associative algebras.}
\]

(ii) \implies (iii). Denote \( \Delta_i = \Delta \cap \Phi_i \) and \( B_i = B \cap \Phi_i \) for \( 1 \leq i \leq k \), then \( \Delta_i \) is a basis for \( \Phi_i \) such that \( \Phi^+_i = \Phi^+ \cap \Phi_i \) and \( B_i \) is the Kostant cascade in \( \Phi^+_i \). Put also \( J_i = J \cap U(\mathfrak{n}_i) \), then \( J_i \) is an ideal of \( U(\mathfrak{n}_i) \) containing all \( \Delta_\beta - c_\beta, \beta \in B_i \). Since \( c_\beta \neq 0 \) for \( \beta \in B_i \setminus \Delta_i \), Theorem \[2,1,10,1\] and \[19\] Theorem 2.4] imply that \( J_i \) is a primitive ideal of \( U(\mathfrak{n}_i) \). Furthermore, for each \( i \), there exists a map \( \xi_i : B_i \to \mathbb{C} \) such that \( f_{\xi_i} \) is a Kostant form on \( \mathfrak{n}_i \) and \( J_i = J(f_{\xi_i}) \). Define \( \xi \) to be a map from \( \mathcal{B} \) to \( \mathbb{C} \) such that \( \xi(\beta) = \xi_i(\beta) \) for \( \beta \in B_i \), then \( f_\xi \) is a Kostant form on \( \mathfrak{n} \). Now, let \( J_\mathfrak{c} \) be, as above, the ideal of \( U(\mathfrak{n}) \) generated by \( \Delta_\beta - c_\beta, \beta \in B \). Then \( J_\mathfrak{c} \) is contained both in \( J \) and in \( J(f_\xi) \). But the quotient algebra \( U(\mathfrak{n}_i)/J_i \) is isomorphic to the Weyl algebra \( \mathcal{A}_{s_i} \) for certain \( s_i \geq 1 \). Thus,

\[
U(\mathfrak{n})/J \cong \mathcal{A}_{s_1} \otimes \cdots \otimes \mathcal{A}_{s_k} \cong \mathcal{A}_s,
\]

where \( s = s_1 + \cdots + s_k \), because \( \mathcal{A}_a \otimes \mathcal{A}_b \cong \mathcal{A}_{a+b} \). It follows that \( J_\mathfrak{c} \) is primitive (and so maximal), hence \( J = J_\mathfrak{c} = J(f_\xi) \).

(iii) \implies (i). Denote by \( \xi_i \) the restriction of \( \xi \) to \( \mathcal{B}_i \), \( 1 \leq i \leq k \). Again by Theorem \[2,1,10,1\] and \[19\] Theorem 2.4], each \( J(f_{\xi_i}) \) is a centrally generated primitive ideal of \( U(\mathfrak{n}_i) \), and \( c_\beta \neq 0 \) for all \( \beta \in B_i \setminus \Delta_i \) (and so for all \( \beta \in B \setminus \Delta \)). And we see again that the centrally generated ideal \( J_i \) is primitive and in the same time is contained in \( J = J(f_\xi) \), thus, \( J = J_\mathfrak{c} \).

(i) \implies (ii). Assume that there exists \( \beta \in \mathcal{B} \setminus \Delta \) such that \( c_\beta = 0 \). Let \( A \) be the set of all indices \( a \) between 1 and \( k \) such that there exists \( \beta \in B_a \setminus \Delta_a \) for which \( c_\beta = 0 \). It follows from the proofs of Theorem \[2,1,10,1\] and \[19\] Theorem 2.4] that, given \( a \in A \), there exist a subset \( D_a \subset \Phi^+_a \) and distinct maps \( \xi_1^a, \xi_2^a \) from \( D_a \) to \( \mathbb{C}^x \) such that

\[
(\xi_1^a(\beta_1), \xi_2^a(\beta_2)) = (\xi_1^a(\beta_3), \xi_2^a(\beta_4)) \Rightarrow (\beta_1 = \beta_3, \beta_2 = \beta_4)
\]

for all \( \beta \in B_a \), and the coadjoint orbits of \( f_{D_a, \xi_1^a} \) and \( f_{D_a, \xi_2^a} \) (in \( \mathfrak{n}_a^* \)) are distinct.

One the other hand, if \( i \in \{1, \ldots, k\} \setminus A \) then \( c_\beta \neq 0 \) for all \( \beta \in B_i \setminus \Delta_i \), hence, as above, \( J \cap U(\mathfrak{n}_i) = J(f_{\xi_i}) \) for a certain map \( \xi_i : D \to \mathbb{C} \) (in other words, \( f_{\xi_i} \) is a Kostant form on \( \mathfrak{n}_i \) and \( \xi_\beta(f_{\xi_i}) = c_\beta \) for all \( \beta \in B_i \)). Clearly, if \( f \in \mathfrak{n}^* \) is a linear form on \( \mathfrak{n} \) and \( f_j \) is its restriction to \( \mathfrak{n}_j \), \( 1 \leq j \leq k \), then

\[
\Omega_f = \Omega_{f_1} \times \cdots \times \Omega_{f_k},
\]

where \( \Omega_f \) (respectively, \( \Omega_{f_j} \)) is the coadjoint orbit of the form \( f \) in \( \mathfrak{n}^* \) (respectively, of the form \( f_j \) in \( \mathfrak{n}_j^* \)). Now, put

\[
D = \bigcup_{a \in A} D_a \cup \bigcup_{i \notin A} B_i
\]

and define \( \xi^j : D \to \mathbb{C}, j = 1, 2 \), by the rule

\[
\xi^j(\beta) = \begin{cases} 
\xi^j_1(\beta), & \text{if } \beta \in B_a \text{ for } a \in A, \\
\xi^j_2(\beta), & \text{if } \beta \in B_i \text{ for } i \notin A.
\end{cases}
\]

Put \( f^j = f_{D, \xi^j} \) for \( j = 1, 2 \), then

\[
\Omega_{f^1} = \prod_{a \in A} \Omega_{D_a, \xi^j_1} \times \prod_{i \notin A} \Omega_{\xi_i},
\]

where \( \Omega_{\xi_i} \) is the coadjoint orbit of the Kostant form \( f_{\xi_i} \) in \( \mathfrak{n}_i^* \). Since \( \Omega_{D_a, \xi^j_1} \neq \Omega_{D_a, \xi^j_2} \) for at least one \( a \in A \), one has \( \Omega_{f^1} \neq \Omega_{f^2} \), so \( J(f^1) \neq J(f^2) \). At the same time, both of these maximal ideals contain the maximal ideal \( J \), a contradiction.

Finally, if conditions (i)–(iii) are satisfied then the map \( \xi \) can be reconstructed by \( J \), because the restriction of \( \xi \) to \( B_i \) can be reconstructed by \( J_i \) for all \( i \). The proof is complete. \( \square \)
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### Appendix A

#### List of the subsets $D$ and $D'$ for $E_7$

|   | $D$ | $D'$ |   | $D$ | $D'$ |   | $D$ | $D'$ |   | $D$ | $D'$ |
|---|-----|-----|---|-----|-----|---|-----|-----|---|-----|-----|
| 1 | 0100000 | 0100000 | 2 | 0101111, 1234321, 1011111 | 0101111, 1234321, 1011111 | 3 | 0101110, 1112211, 1112211, 1234321, 1011110 |
| 4 | 0101111, 1234321, 1011111 | 0101111, 1234321, 1011111 | 5 | 1223210, 1122111, 1123321 | 1223210, 1122111, 1123321 | 6 | 0001111, 1223211, 1122221, 1234321 |
| 7 | 0001110, 1223321, 1122110, 1123211, 1123211 | 0001110, 1223321, 1122110, 1123211, 1123211 | 8 | 0001111, 1223321, 1122111, 1123210 | 0001111, 1223321, 1122111, 1123210 | 9 | 0101100, 1234321, 1112221, 1112221, 1234321 |
| 10 | 0111111, 1223210, 1122111, 1123221, 1123221 | 0111111, 1223210, 1122111, 1123221, 1123221 | 11 | 0111111, 1223211, 1122111, 1123211 | 0111111, 1223211, 1122111, 1123211 |
| 13 | 0101000, 1234321, 1234321, 1122110, 1123211 | 0101000, 1234321, 1234321, 1122110, 1123211 | 14 | 0101111, 1223210, 1122111, 1123221, 1123221 | 0101111, 1223210, 1122111, 1123221, 1123221 |
| 16 | 0101000, 1234321, 1234321, 1122110, 1123211 | 0101000, 1234321, 1234321, 1122110, 1123211 | 17 | 1112110, 1112221, 1112221, 1234321 | 1112110, 1112221, 1112221, 1234321 |
| 19 | 0001110, 1112110, 1112221, 1112221, 1234321 | 0001110, 1112110, 1112221, 1112221, 1234321 | 20 | 0001111, 1112111, 1112111, 1112221, 1112221, 1234321 | 0001111, 1112111, 1112111, 1112221, 1112221, 1234321 |
| 22 | 0000011, 1234321, 1122110, 1122211, 1122211 | 0000011, 1234321, 1122110, 1122211, 1122211 | 23 | 0000110, 1224321, 1221110, 1122111, 1122111, 1234321 | 0000110, 1224321, 1221110, 1122111, 1122111, 1234321 |
| 25 | 0100000, 1122110, 1122211, 1234321 | 0100000, 1122110, 1122211, 1234321 | 26 | 0100000, 1122110, 1122211, 1234321 | 0100000, 1122110, 1122211, 1234321 |
| 28 | 0100000, 1112111, 1112221, 1234321 | 0100000, 1112111, 1112221, 1234321 | 29 | 0100000, 1122110, 1122211 | 0100000, 1122110, 1122211 |
|   |   |   |   |   |   |   |   |   |   |   |

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| 31 | 0100000, 0000110, 1224321, 1122110, 1122111 | 32 | 0100000, 0000110, 1224321, 1122110, 1122111 | 33 | 1223221, 1123321, 1123321 |
|----|-----------------------------------------------|----|-----------------------------------------------|----|-----------------------------------------------|
| 34 | 000001, 1223221, 1123321 | 35 | 0101100, 112221, 1123321, 1234321, 1112221, 1111000 | 36 | 0101100, 0000001, 1112221, 1112221, 11234321, 1234321, 011100 |
| 37 | 1223221, 1122100, 1123321 | 38 | 000001, 1223221, 1122100, 0000110, 1234321 | 39 | 0001100, 1223321, 1122100, 1123221, 1123221 |
| 40 | 0001100, 1223221, 1122100, 1123321, 1223221 | 41 | 0101000, 111100, 1112221, 1123321 | 42 | 0101000, 0000001, 111100, 1112221, 11234321 |
| 43 | 0111100, 1223221, 1122100, 1123321 | 44 | 0111100, 1223221, 1122100, 1123321 | 45 | 0101000, 1223321, 1122100, 1123221, 1123221 |
| 46 | 0101000, 0000001, 1223221, 1122100, 1123321 | 47 | 0101000, 1223221, 1122100, 1123321 | 48 | 0101000, 0000001, 1223221, 1122100, 1123221 |
| 49 | 1112100, 1112221, 1112221, 1234321 | 50 | 0000001, 1122100, 1112221, 1234321 | 51 | 0001000, 1112100, 1112221, 1234321 |
| 52 | 0000100, 0000001, 1112100, 1112221, 1234321 | 53 | 1224321, 1000000, 1224321, 1122100, 1234321 | 54 | 0000001, 1224321, 1000000, 1122100, 1122221 |
| 55 | 0000100, 1224321, 1000000, 1122100, 1122221 | 56 | 0000100, 0000001, 1224321, 1000000, 1122100, 1122221 | 57 | 0100000, 1112100, 1112221, 1234321 |

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| 58 | 0100000, 0000001, 1112100, 1112221, 1234321 | 59 | 0100000, 0000100, 1112100, 1112221, 1234321 | 60 | 0100000, 0000100, 1112100, 1112221, 1234321 |
| 61 | 0100000, 1224321, 1000000, 1122100, 1122221 | 62 | 0100000, 0000001, 1224321, 1000000, 1122100, 1122221 | 63 | 0100000, 0000100, 1224321, 1000000, 1122100, 1122221 |
| 64 | 0100000, 0000100, 0000001, 1224321, 1000000, 1122100, 1122221 | 65 | 0100000, 2234321, 0100000, 0101111, 0011111, 0011111 |
| 67 | 0101110, 0112211, 0011110, 2234321 | 68 | 0101111, 0112211, 0011111, 2234321 | 69 | 0111000, 0112210, 0011111, 2234321 |
| 70 | 0111000, 0112211, 0011110, 0000011, 2234321 | 71 | 0111110, 0112211, 0011000, 0011000, 0000011 | 72 | 0111111, 0112210, 0011000, 0011000, 0011111 |
| 73 | 0101000, 0111110, 0112211, 2234321 | 74 | 0101000, 0111110, 0112211, 0000011, 2234321 | 75 | 0101000, 0111110, 0112211, 0000011, 2234321 |
| 76 | 0101000, 0111111, 0112210, 0011111, 2234321 | 77 | 0101000, 0111110, 0112211, 0000011, 2234321 | 78 | 0101111, 0111100, 0112210, 0011111, 2234321 |
| 79 | 0101110, 0111000, 0112211, 0011110, 2234321 | 80 | 0101000, 0111111, 0112210, 0011000, 0011111, 2234321 | 81 | 0112110, 0112211, 0000111, 2234321 |
| 82 | 0112110, 0112211, 0000011, 2234321 | 83 | 0112110, 0112211, 0000111, 2234321 | 84 | 0112110, 0112211, 0000111, 2234321 |
### List of the subsets $D$ and $D'$ for $E_8$

|   | $D$                      | $D'$                      |   | $D$                      | $D'$                      |   | $D$                      | $D'$                      |
|---|--------------------------|--------------------------|---|--------------------------|--------------------------|---|--------------------------|--------------------------|
| 1 | 010100000               | 010000000               | 2 | 23354321,               | 23354321,               |   | 23354321,               | 23354321,               |
|   | 0111100,            | 0111100,            |   | 1234321,               | 1234321,               |   | 1234321,               | 1234321,               |
|   | 0112221,            | 0112221,            |   | 1123321,               | 1123321,               |   | 1123321,               | 1123321,               |
|   | 0011000,            | 0011000,            |   | 1123210,               | 1123210,               |   | 1123210,               | 1123210,               |
|   | 2234321               | 2234321               |   | 1123221,               | 1123221,               |   | 1123221,               | 1123221,               |
| 112| 01010000,         | 01010000,         |   | 23354321,         | 23354321,         |   | 23354321,         | 23354321,         |
|   | 0111100,         | 0111100,         |   | 1234321,         | 1234321,         |   | 1234321,         | 1234321,         |
|   | 0112221,         | 0112221,         |   | 1123321,         | 1123321,         |   | 1123321,         | 1123321,         |
|   | 0011000,         | 0011000,         |   | 1123210,         | 1123210,         |   | 1123210,         | 1123210,         |
|   | 2234321         | 2234321         |   | 1123221,         | 1123221,         |   | 1123221,         | 1123221,         |

**Note:** Each row represents a subset $D$ or $D'$ with corresponding elements.
| 55 | 23465431, 00001000, 12243211, 11221000, 11222211 | 23465431, 12243211, 11221000, 11222211 | 56 | 23465421, 00001000, 12243221, 11221000, 11222211 | 00000001 | 57 | 01000000, 23465431, 11121000, 11222211, 12343211 | 23465431, 11121000, 11222211, 12343211 |
| 58 | 01000000, 23465421, 11121000, 11122211, 12343221 | 01121000, 11122211, 12343221 | 59 | 01000000, 23465431, 00001000, 11121000, 11122211, 12343221 | 23465431, 00001000, 11121000, 11122211, 12343221 |
| 58 | 01000000, 23465421, 11121000, 11122211, 12343221 | 01121000, 11122211, 12343221 | 60 | 01000000, 23465431, 00001000, 11121000, 11122211, 12343221 | 23465431, 00001000, 11121000, 11122211, 12343221 |
| 61 | 01000000, 23465431, 12243211, 12343221 | 00001000, 00000001, 12243221, 11221000, 11222211 | 62 | 01000000, 23465431, 00000100, 00000001, 12243221, 11221000, 11222211 | 23465431, 00000001 |
| 64 | 01000000, 23465421, 01122110, 01122211, 22454321 | 01122211, 22454321, 11222211 | 65 | 23354321, 22454321 | 23354321, 22454321 | 66 | 23354321, 22454321 | 23354321, 22454321 |
| 67 | 23354321, 01122110, 01122211, 22454321 | 01122211, 22454321, 11222211 | 68 | 23354321, 01122100, 01122211, 22454321 | 23354321, 22454321 |
| 70 | 01110000, 23465321, 01122221, 00111100, 23433211 | 23465321, 01122221, 00111100, 23433211 | 71 | 23454321, 01122211, 01122211, 22453421 | 23454321, 22453421 |
| 73 | 01010000, 01111100, 01122211, 23465431, 23433211 | 23465431, 01122211, 23433211 | 74 | 01010000, 01111100, 01111100, 22453421 | 23454321, 22453421 |
| 76 | 01010000, 23454321, 01122110, 01122211, 22354321 | 23454321, 01122110, 01122211, 22354321 | 77 | 23354321, 01111000, 01121110, 01122221, 22453421 | 23454321, 22453421 |
| 79 | 23354321, 01110000, 01122110, 01122211, 22454321 | 01122211, 01122211, 22453421 | 80 | 01010000, 23454321, 01122221, 01122221, 00110000, 22354321 | 01010000, 22354321 |
| 81 | 01121100, 01122221, 01122221, 22453421 | 01111000, 01121110, 01122221, 22453421 | 82 | 01121100, 01122221, 01122221, 22453421 | 01111000, 01121110, 01122221, 22453421 |
| 83 | 01121100, 01122221, 01122221, 22453421 | 01111000, 01121110, 01122221, 22453421 | 84 | 01121100, 01122221, 01122221, 22453421 | 01111000, 01121110, 01122221, 22453421 |
| 87 | 01121100, 01122221, 01122221, 22453421 | 01111000, 01121110, 01122221, 22453421 | 88 | 01121100, 01122221, 01122221, 22453421 | 01111000, 01121110, 01122221, 22453421 |
| 133 | 23465432, 12232100, 11221110, 11233210 | 134 | 23465432, 00011110, 12232110, 11222110, 11233210 | 135 | 23465432, 00011110, 12232110, 11223210, 11221110, 11233210 |
| 136 | 23465432, 00011110, 12233210, 11221110, 11233210 | 137 | 01010000, 23465432, 11122110, 11234210, 11223210 | 138 | 01111110, 23465432, 12232110, 11232110, 11232110 |
| 139 | 01111100, 23465432, 12232110, 11121110, 11233210 | 140 | 01111110, 23465432, 12232110, 11121110, 11233210 | 141 | 01010000, 23465432, 12232110, 11221110, 11232110 |
| 142 | 01011110, 23465432, 12232110, 11222110, 11232210 | 143 | 01011100, 23465432, 12232110, 11221100, 11233210 | 144 | 01010000, 23465432, 12232110, 11221110, 11232110 |
| 145 | 23465432, 11121100, 11122110, 12343210 | 146 | 23465432, 00001110, 11121100, 11122110, 112343210 | 147 | 23465432, 00001100, 11121100, 11122110, 112343210 |
| 148 | 23465432, 00001110, 11121110, 11221100, 12343210 | 149 | 23465432, 12232110, 11221100, 11222110, 11222110 | 150 | 23465432, 00001110, 11221100, 11222110, 11222110 |
| 151 | 23465432, 00001100, 11221100, 11221110, 12343210 | 152 | 23465432, 11221100, 11221110, 11222110, 11222110 | 153 | 01000000, 23465432, 11121100, 11122110, 12343210 |
| 154 | 01000000, 23465432, 00001110, 11121100, 11122110, 12343210 | 155 | 01000000, 23465432, 00001110, 11121100, 11122110, 12343210 | 156 | 01000000, 23465432, 11121100, 11122110, 12343210 |
| 157 | 01000000, 23465432, 12243210, 11221100, 11221110, 12343210 | 158 | 01000000, 23465432, 12243210, 11221100, 11221110, 12343210 | 159 | 01000000, 23465432, 12243210, 11221100, 11221110, 12343210 |
| 160 | 01000000, 23465432, 00001100, 00000110, 12243210, 11221100, 11222110 | 161 | 23465432, 12232210, 11233210 | 162 | 23465432, 00000010, 12232210, 11233210 |
| 163 | 01011000, 23465432, 11122210, 12343210, 10111000, 10111000 | 164 | 01011000, 23465432, 00000010, 11222110, 12343210, 10111000 | 165 | 23465432, 12232210, 11221000, 11233210 |
| 166 | 23465432, 00000010, 12232210, 11221000, 11233210 | 167 | 23465432, 00011000, 12233210, 11221000, 11233210 | 168 | 23465432, 00011000, 12233210, 11221000, 11233210 |
| 169 | 01010000, 23465432, 11111000, 11122210, 12343210 | 170 | 01010000, 23465432, 00000010, 11111000, 11122210, 12343210 | 171 | 01111000, 23465432, 12232210, 11211000, 11233210 |
| 172 | 01111000, 23465432, 00000010, 12232210, 11121000, 11233210 | 173 | 01010000, 23465432, 12233210, 11221000, 11233210 | 174 | 01010000, 23465432, 00000010, 11221000, 11233210 |
| 175 | 01010000, 23465432, 00110000, 12232210, 11221000, 11233210 | 176 | 01010000, 23465432, 00110000, 12232210, 11221000, 11233210 | 177 | 23465432, 11121000, 11122210, 12343210 |
| 178 | 23465432, 00000010, 11121000, 11122210, 12343210 | 179 | 23465432, 00010000, 11121000, 11122210, 12343210 | 180 | 23465432, 00010000, 11121000, 11122210, 12343210 |
| 181 | 23465432, 12232210, 10000000, 11221000, 11222210 | 182 | 23465432, 00000010, 12232210, 10000000, 11221000, 11222210 | 183 | 23465432, 00010000, 12232210, 10000000, 11221000, 11222210 | 184 | 10000000 |
| 184 | 23465432, 00001000, 00000010, 22434210, 10000000, 11221000, 11222210 | 10000000 | 185 | 01000000, 23465432, 11121000, 11122210, 12343210, 11210000, 11222100, 12343210 | 186 | 01000000, 23465432, 00000010, 11121000, 11122210, 12343210, 11210000, 12343210 |
| 187 | 01000000, 23465432, 00001000, 11121000, 11122210, 12343210 | 01000000, 23465432, 00001000, 00000010, 11121000, 11122210, 12343210 | 188 | 01000000, 23465432, 00001000, 00000010, 11121000, 11122210, 12343210 | 189 | 01000000, 23465432, 00001000, 00000010, 11121000, 11122210, 12343210 |
| 190 | 01000000, 23465432, 00000010, 12243210, 10000000, 11221000, 11222210 | 01000000, 23465432, 00001000, 12243210, 10000000, 11221000, 11222210 | 191 | 00111110, 23465432, 01111110, 2343210 | 192 | 00111110, 23465432, 01111110, 2343210 |
| 193 | 01000000, 23465432, 22343210 | 01000000, 23465432, 00111110, 22343210 | 194 | 01011110, 23465432, 00111110, 22343210 | 195 | 01011110, 23465432, 00111110, 22343210 |
| 196 | 01011110, 01122110, 23465432, 00111110, 22343210 | 01011110, 01122110, 23465432, 00111110, 22343210 | 197 | 01100000, 01122100, 23465432, 00111110, 22343210 | 198 | 01100000, 01122100, 23465432, 00111110, 22343210 |
| 199 | 01111110, 01122110, 23465432, 00011000, 00011110, 22343210 | 01111110, 01122110, 23465432, 00011000, 00011110, 22343210 | 200 | 01111110, 01122110, 23465432, 00011000, 00011110, 22343210 | 201 | 01010000, 01111110, 23465432, 22343210 |
| 202 | 01010000, 01111110, 01122110, 23465432, 00000110, 22343210 | 01010000, 01111110, 01122110, 23465432, 00000110, 22343210 | 203 | 01010000, 01111110, 01122110, 23465432, 00000110, 22343210 | 204 | 01010000, 01111110, 01122110, 23465432, 22343210 |
| 205 | 01010000, 01111110, 01122110, 23465432, 00110000, 22343210 | 01010000, 01111110, 01122110, 23465432, 00110000, 22343210 | 206 | 01010000, 01111110, 01122110, 23465432, 00110000, 22343210 | 207 | 01010000, 01111110, 01122110, 23465432, 00110000, 22343210 |
| 208 | 01010000, 01111110, 01122100, 23465432, 00110000, 00011110, 22343210 | 209 | 01121100, 01122110, 23465432, 22343210 | 210 | 01121100, 01122110, 23465432, 00000110, 22343210 | 01121100, 01122110, 00001110 |
| 211 | 01122110, 23465432, 00001110, 22343210 | 212 | 01121100, 01122110, 23465432, 00001110, 00001110, 22343210 | 213 | 01121100, 01122110, 23465432, 00100000, 22343210 | 01121100, 01122110 |
| 214 | 01122110, 23465432, 00000000, 00001110, 22343210 | 215 | 01121100, 01122110, 23465432, 00100000, 00001110, 22343210 | 216 | 01121100, 01122110, 23465432, 00001110, 00001110, 22343210 | 01121100, 01122110, 00001110 |
| 217 | 01000000, 01121100, 01122110, 23465432, 22343210 | 218 | 01000000, 01121100, 01122110, 23465432, 00001110, 22343210 | 219 | 01000000, 01121100, 01122110, 23465432, 00001110, 22343210 | 01121100, 01122110, 00001110 |
| 220 | 01000000, 01122110, 01122110, 23465432, 00001110, 00001110, 22343210 | 221 | 01000000, 01121100, 01122110, 23465432, 00100000, 22343210 | 222 | 01000000, 01121100, 01122110, 23465432, 00001110, 22343210 | 01121100, 01122110, 00001110 |
| 223 | 01000000, 01121100, 01122110, 23465432, 00001110, 00001110, 22343210 | 224 | 01000000, 01121100, 01122110, 23465432, 00100000, 00001110, 22343210 | 225 | 01000000, 01122210, 23465432, 00000100, 22343210 | 01000000 |
| 226 | 01000000, 01122210, 23465432, 00000100, 22343210 | 227 | 01011000, 01122210, 23465432, 00111000, 22343210 | 228 | 01011000, 01122210, 23465432, 00111000, 22343210 | 01111000, 01122110, 00111000 |
| 229 | 01110000, 01122210, 23465432, 00111000, 22343210 | 230 | 01110000, 01122210, 23465432, 00111000, 00000010, 22343210 | 231 | 01110000, 01122210, 23465432, 00110000, 00011000, 22343210 | 01111000, 01122110, 00110000 |

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List of the subsets $D$ and $D'$ for $F_4$

| $D$      | $D'$      | $D$      | $D'$      | $D$      | $D'$      |
|----------|-----------|----------|-----------|----------|-----------|
| 1        | 1000      | 1000     | 2         | 1231,    | 1231,     |
|          |           |          |           | 1222     | 1222      |
| 4        | 1000,     | 1000     | 5         | 1000,    | 1000      |
|          | 1221,     |          |           | 1232     | 1232      |
|          | 1242      |          | 6         | 1000     | 1000      |
|          |           |          |           | 1220,    | 1220      |
|          |           |          |           | 1222,    | 1222      |
|          |           |          |           | 1242     | 1242      |
| 7        | 1120,     | 1120,    | 8         | 1000,    | 1000      |
|          | 1122,     | 1122,    |           | 1220,    | 1220      |
|          | 1342      | 1342     |           | 1222,    | 1222      |
|          |           |           |           | 1242     | 1242      |
| 10       | 0111,     | 0111,    | 11        | 1121,    | 1121      |
|          | 0120,     | 0120     |           | 1342     | 1342      |
|          | 2342      | 2342     | 12        | 1100,    | 1100      |
|          |           |           |           | 1121,    | 1121      |
|          |           |           |           | 1342     | 1342      |
| 13       | 0100,     | 0100     | 14        | 1110,    | 1110      |
|          | 0122,     | 0122     |           | 1342     | 1342      |
|          | 2342      | 2342     | 31        | 31,      | 31,       |
|          |           |           | 31        | 31,      | 31,       |

List of the subsets $D$ and $D'$ for $G_2$

| $D$ | $D'$ | $D$ | $D'$ |
|-----|------|-----|------|
| 1   | 31   | 2   | 31,  |
|     |      |     |      |
|     |      |     | 11   |
|     |      |     |      |
|     |      |     | 11   |
|     |      |     |      |
|     |      |     |      |
|     |      |     |      |
|     |      |     |      |

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## Appendix B

**Correspondence between $c$ and $D$ for $E_7$**

| $D$  | Type of $c$ | $D$  | Type of $c$ | $D$  | Type of $c$ |
|------|-------------|------|-------------|------|-------------|
| 1    | $(0,0,0,0,0,0,0)$ | 2    | $(0,0,0,0,0,0,0)$ | 3    | $(0,0,0,0,0,0,0)$ |
| 4    | $(0,0,0,0,0,0,0)$ | 5    | $(0,0,0,0,0,0,0)$ | 6    | $(0,0,0,0,0,0,0)$ |
| 7    | $(0,0,0,0,0,0,0)$ | 8    | $(0,0,0,0,0,0,0)$ | 9    | $(0,0,0,0,0,0,0)$ |
| 10   | $(0,0,0,0,0,0,0)$ | 11   | $(0,0,0,0,0,0,0)$ | 12   | $(0,0,0,0,0,0,0)$ |
| 13   | $(0,0,0,0,0,0,0)$ | 14   | $(0,0,0,0,0,0,0)$ | 15   | $(0,0,0,0,0,0,0)$ |
| 16   | $(0,0,0,0,0,0,0)$ | 17   | $(0,0,0,0,0,0,0)$ | 18   | $(0,0,0,0,0,0,0)$ |
| 19   | $(0,0,0,0,0,0,0)$ | 20   | $(0,0,0,0,0,0,0)$ | 21   | $(0,0,0,0,0,0,0)$ |
| 22   | $(0,0,0,0,0,0,0)$ | 23   | $(0,0,0,0,0,0,0)$ | 24   | $(0,0,0,0,0,0,0)$ |
| 25   | $(0,0,0,0,0,0,0)$ | 26   | $(0,0,0,0,0,0,0)$ | 27   | $(0,0,0,0,0,0,0)$ |
| 28   | $(0,0,0,0,0,0,0)$ | 29   | $(0,0,0,0,0,0,0)$ | 30   | $(0,0,0,0,0,0,0)$ |
| 31   | $(0,0,0,0,0,0,0)$ | 32   | $(0,0,0,0,0,0,0)$ | 33   | $(0,0,0,0,0,0,0)$ |
| 34   | $(0,0,0,0,0,0,0)$ | 35   | $(0,0,0,0,0,0,0)$ | 36   | $(0,0,0,0,0,0,0)$ |
| 37   | $(0,0,0,0,0,0,0)$ | 38   | $(0,0,0,0,0,0,0)$ | 39   | $(0,0,0,0,0,0,0)$ |
| 40   | $(0,0,0,0,0,0,0)$ | 41   | $(0,0,0,0,0,0,0)$ | 42   | $(0,0,0,0,0,0,0)$ |
| 43   | $(0,0,0,0,0,0,0)$ | 44   | $(0,0,0,0,0,0,0)$ | 45   | $(0,0,0,0,0,0,0)$ |
| 46   | $(0,0,0,0,0,0,0)$ | 47   | $(0,0,0,0,0,0,0)$ | 48   | $(0,0,0,0,0,0,0)$ |
| 49   | $(0,0,0,0,0,0,0)$ | 50   | $(0,0,0,0,0,0,0)$ | 51   | $(0,0,0,0,0,0,0)$ |
| 52   | $(0,0,0,0,0,0,0)$ | 53   | $(0,0,0,0,0,0,0)$ | 54   | $(0,0,0,0,0,0,0)$ |
| 55   | $(0,0,0,0,0,0,0)$ | 56   | $(0,0,0,0,0,0,0)$ | 57   | $(0,0,0,0,0,0,0)$ |
| 58   | $(0,0,0,0,0,0,0)$ | 59   | $(0,0,0,0,0,0,0)$ | 60   | $(0,0,0,0,0,0,0)$ |
| 61   | $(0,0,0,0,0,0,0)$ | 62   | $(0,0,0,0,0,0,0)$ | 63   | $(0,0,0,0,0,0,0)$ |
| 64   | $(0,0,0,0,0,0,0)$ | 65   | $(0,0,0,0,0,0,0)$ | 66   | $(0,0,0,0,0,0,0)$ |
| 67   | $(0,0,0,0,0,0,0)$ | 68   | $(0,0,0,0,0,0,0)$ | 69   | $(0,0,0,0,0,0,0)$ |
| 70   | $(0,0,0,0,0,0,0)$ | 71   | $(0,0,0,0,0,0,0)$ | 72   | $(0,0,0,0,0,0,0)$ |
| 73   | $(0,0,0,0,0,0,0)$ | 74   | $(0,0,0,0,0,0,0)$ | 75   | $(0,0,0,0,0,0,0)$ |
| 76   | $(0,0,0,0,0,0,0)$ | 77   | $(0,0,0,0,0,0,0)$ | 78   | $(0,0,0,0,0,0,0)$ |
| 79   | $(0,0,0,0,0,0,0)$ | 80   | $(0,0,0,0,0,0,0)$ | 81   | $(0,0,0,0,0,0,0)$ |
| 82   | $(0,0,0,0,0,0,0)$ | 83   | $(0,0,0,0,0,0,0)$ | 84   | $(0,0,0,0,0,0,0)$ |
| 85   | $(0,0,0,0,0,0,0)$ | 86   | $(0,0,0,0,0,0,0)$ | 87   | $(0,0,0,0,0,0,0)$ |
| 88   | $(0,0,0,0,0,0,0)$ | 89   | $(0,0,0,0,0,0,0)$ | 90   | $(0,0,0,0,0,0,0)$ |
| 91   | $(0,0,0,0,0,0,0)$ | 92   | $(0,0,0,0,0,0,0)$ | 93   | $(0,0,0,0,0,0,0)$ |
| 94   | $(0,0,0,0,0,0,0)$ | 95   | $(0,0,0,0,0,0,0)$ | 96   | $(0,0,0,0,0,0,0)$ |
| 97   | $(0,0,0,0,0,0,0)$ | 98   | $(0,0,0,0,0,0,0)$ | 99   | $(0,0,0,0,0,0,0)$ |
| 100  | $(0,0,0,0,0,0,0)$ | 101  | $(0,0,0,0,0,0,0)$ | 102  | $(0,0,0,0,0,0,0)$ |
| 103  | $(0,0,0,0,0,0,0)$ | 104  | $(0,0,0,0,0,0,0)$ | 105  | $(0,0,0,0,0,0,0)$ |
| 106  | $(0,0,0,0,0,0,0)$ | 107  | $(0,0,0,0,0,0,0)$ | 108  | $(0,0,0,0,0,0,0)$ |
| 109  | $(0,0,0,0,0,0,0)$ | 110  | $(0,0,0,0,0,0,0)$ | 111  | $(0,0,0,0,0,0,0)$ |
| 112  | $(0,0,0,0,0,0,0)$ |
Correspondence between $c$ and $D$ for $E_8$

| $D$ | Type of $c$ | $D$ | Type of $c$ | $D$ | Type of $c$ |
|-----|-------------|-----|-------------|-----|-------------|
| 1   | $(0,0,0,0,0,0,0)$ | 2   | $(0,0,0,0,0,0,0)$ | 3   | $(0,0,0,0,0,0,0,0)$ |
| 4   | $(0,0,0,0,0,0,0,0)$ | 5   | $(0,0,0,0,0,0,0,0)$ | 6   | $(0,0,0,0,0,0,0,0)$ |
| 7   | $(0,0,0,0,0,0,0,0)$ | 8   | $(0,0,0,0,0,0,0,0)$ | 9   | $(0,0,0,0,0,0,0,0)$ |
| 10  | $(0,0,0,0,0,0,0,0)$ | 11  | $(0,0,0,0,0,0,0,0)$ | 12  | $(0,0,0,0,0,0,0,0)$ |
| 13  | $(0,0,0,0,0,0,0,0)$ | 14  | $(0,0,0,0,0,0,0,0)$ | 15  | $(0,0,0,0,0,0,0,0)$ |
| 16  | $(0,0,0,0,0,0,0,0)$ | 17  | $(0,0,0,0,0,0,0,0)$ | 18  | $(0,0,0,0,0,0,0,0)$ |
| 19  | $(0,0,0,0,0,0,0,0)$ | 20  | $(0,0,0,0,0,0,0,0)$ | 21  | $(0,0,0,0,0,0,0,0)$ |
| 22  | $(0,0,0,0,0,0,0,0)$ | 23  | $(0,0,0,0,0,0,0,0)$ | 24  | $(0,0,0,0,0,0,0,0)$ |
| 25  | $(0,0,0,0,0,0,0,0)$ | 26  | $(0,0,0,0,0,0,0,0)$ | 27  | $(0,0,0,0,0,0,0,0)$ |
| 28  | $(0,0,0,0,0,0,0,0)$ | 29  | $(0,0,0,0,0,0,0,0)$ | 30  | $(0,0,0,0,0,0,0,0)$ |
| 31  | $(0,0,0,0,0,0,0,0)$ | 32  | $(0,0,0,0,0,0,0,0)$ | 33  | $(0,0,0,0,0,0,0,0)$ |
| 34  | $(0,0,0,0,0,0,0,0)$ | 35  | $(0,0,0,0,0,0,0,0)$ | 36  | $(0,0,0,0,0,0,0,0)$ |
| 37  | $(0,0,0,0,0,0,0,0)$ | 38  | $(0,0,0,0,0,0,0,0)$ | 39  | $(0,0,0,0,0,0,0,0)$ |
| 40  | $(0,0,0,0,0,0,0,0)$ | 41  | $(0,0,0,0,0,0,0,0)$ | 42  | $(0,0,0,0,0,0,0,0)$ |
| 43  | $(0,0,0,0,0,0,0,0)$ | 44  | $(0,0,0,0,0,0,0,0)$ | 45  | $(0,0,0,0,0,0,0,0)$ |
| 46  | $(0,0,0,0,0,0,0,0)$ | 47  | $(0,0,0,0,0,0,0,0)$ | 48  | $(0,0,0,0,0,0,0,0)$ |
| 49  | $(0,0,0,0,0,0,0,0)$ | 50  | $(0,0,0,0,0,0,0,0)$ | 51  | $(0,0,0,0,0,0,0,0)$ |
| 52  | $(0,0,0,0,0,0,0,0)$ | 53  | $(0,0,0,0,0,0,0,0)$ | 54  | $(0,0,0,0,0,0,0,0)$ |
| 55  | $(0,0,0,0,0,0,0,0)$ | 56  | $(0,0,0,0,0,0,0,0)$ | 57  | $(0,0,0,0,0,0,0,0)$ |
| 58  | $(0,0,0,0,0,0,0,0)$ | 59  | $(0,0,0,0,0,0,0,0)$ | 60  | $(0,0,0,0,0,0,0,0)$ |
| 61  | $(0,0,0,0,0,0,0,0)$ | 62  | $(0,0,0,0,0,0,0,0)$ | 63  | $(0,0,0,0,0,0,0,0)$ |
| 64  | $(0,0,0,0,0,0,0,0)$ | 65  | $(0,0,0,0,0,0,0,0)$ | 66  | $(0,0,0,0,0,0,0,0)$ |
| 67  | $(0,0,0,0,0,0,0,0)$ | 68  | $(0,0,0,0,0,0,0,0)$ | 69  | $(0,0,0,0,0,0,0,0)$ |
| 70  | $(0,0,0,0,0,0,0,0)$ | 71  | $(0,0,0,0,0,0,0,0)$ | 72  | $(0,0,0,0,0,0,0,0)$ |
| 73  | $(0,0,0,0,0,0,0,0)$ | 74  | $(0,0,0,0,0,0,0,0)$ | 75  | $(0,0,0,0,0,0,0,0)$ |
| 76  | $(0,0,0,0,0,0,0,0)$ | 77  | $(0,0,0,0,0,0,0,0)$ | 78  | $(0,0,0,0,0,0,0,0)$ |
| 79  | $(0,0,0,0,0,0,0,0)$ | 80  | $(0,0,0,0,0,0,0,0)$ | 81  | $(0,0,0,0,0,0,0,0)$ |
| 82  | $(0,0,0,0,0,0,0,0)$ | 83  | $(0,0,0,0,0,0,0,0)$ | 84  | $(0,0,0,0,0,0,0,0)$ |
| 85  | $(0,0,0,0,0,0,0,0)$ | 86  | $(0,0,0,0,0,0,0,0)$ | 87  | $(0,0,0,0,0,0,0,0)$ |
| 88  | $(0,0,0,0,0,0,0,0)$ | 89  | $(0,0,0,0,0,0,0,0)$ | 90  | $(0,0,0,0,0,0,0,0)$ |
| 91  | $(0,0,0,0,0,0,0,0)$ | 92  | $(0,0,0,0,0,0,0,0)$ | 93  | $(0,0,0,0,0,0,0,0)$ |
| 94  | $(0,0,0,0,0,0,0,0)$ | 95  | $(0,0,0,0,0,0,0,0)$ | 96  | $(0,0,0,0,0,0,0,0)$ |
| 97  | $(0,0,0,0,0,0,0,0)$ | 98  | $(0,0,0,0,0,0,0,0)$ | 99  | $(0,0,0,0,0,0,0,0)$ |
| 100 | $(0,0,0,0,0,0,0,0)$ | 101 | $(0,0,0,0,0,0,0,0)$ | 102 | $(0,0,0,0,0,0,0,0)$ |
| 103 | $(0,0,0,0,0,0,0,0)$ | 104 | $(0,0,0,0,0,0,0,0)$ | 105 | $(0,0,0,0,0,0,0,0)$ |
| 106 | $(0,0,0,0,0,0,0,0)$ | 107 | $(0,0,0,0,0,0,0,0)$ | 108 | $(0,0,0,0,0,0,0,0)$ |
| 109 | $(0,0,0,0,0,0,0,0)$ | 110 | $(0,0,0,0,0,0,0,0)$ | 111 | $(0,0,0,0,0,0,0,0)$ |
| 112 | $(0,0,0,0,0,0,0,0)$ | 113 | $(0,0,0,0,0,0,0,0)$ | 114 | $(0,0,0,0,0,0,0,0)$ |
| 115 | $(0,0,0,0,0,0,0,0)$ | 116 | $(0,0,0,0,0,0,0,0)$ | 117 | $(0,0,0,0,0,0,0,0)$ |
| 118 | $(0,0,0,0,0,0,0,0)$ | 119 | $(0,0,0,0,0,0,0,0)$ | 120 | $(0,0,0,0,0,0,0,0)$ |
| 121 | $(0,0,0,0,0,0,0,0)$ | 122 | $(0,0,0,0,0,0,0,0)$ | 123 | $(0,0,0,0,0,0,0,0)$ |
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\(c\) & Type of \(c\) & \(D\) & Type of \(c\) \\
\hline
\(0, c_{3,2}\) & \((0,0)\) & \(0, c_{3,2}\) & \((0, c_{3,2})\) \\
\hline
\end{tabular}
\caption{Correspondence between \(c\) and \(D\) for \(G_2\)}
\end{table}