The Sphere Covering Inequality and Its Applications

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October 28, 2016

Abstract

In this paper, we show that the total area of two distinct surfaces with Gaussian curvature equal to 1, which are also conformal to the Euclidean unit disk with the same conformal factor on the boundary, must be at least $4\pi$. In other words, the areas of these surfaces must cover the whole unit sphere after a proper rearrangement. We refer to this lower bound of total area as the Sphere Covering Inequality. The inequality and its generalizations are applied to a number of open problems related to Moser-Trudinger type inequalities, mean field equations and Onsager vortices, etc, and yield optimal results.

1 Introduction

A large number of important second order nonlinear elliptic equations involve exponential nonlinearities. These equations arise, for example, in the study of Gaussian curvature of surfaces with metrics conformal to Euclidean metric ([12], [14], [15], [18], [19], etc.), Moser-Trudinger type inequalities ([1], [2], [7], [21], [22], [24], [28], [34], [35], [36], [39]), the mean field theory of statistical mechanics of classical vortices and thermodynamics ([4], [6], [10], [11], [13], [16], [29], [32]), and self gravitating cosmic string configurations in the framework of Einstein’s general relativity ([17], [37], [40]). In this article, we shall prove a basic and important inequality which becomes a crucial tool for tackling several open problems in the above mentioned areas.

Let us consider the equation

$$\Delta v + e^{2v} = 0, \quad y \in \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a $C^2$ simply-connected bounded region. It is well-known that for a solution $v \in C^2(\Omega)$ of (1.1), the two dimensional Riemannian manifold with boundary $(\Omega, g)$ with a conformal Euclidean metric $dg = e^{2v} dy$ has Gaussian curvature equal to 1 everywhere. The
total area as well as the total curvature of such manifold is equal to \( A = \int_\Omega e^{2\nu}dy \). The well-known Gauss-Bonnet Theorem states that
\[
A = \int_\Omega e^{2\nu}dy = \int_\Omega dg = 2\pi - \int_{\partial\Omega} \kappa_g dg_s
\]
where \( \kappa_g \) is the geodesic curvature and \( dg_s \) the length parameter of \( \partial\Omega \). From the equation, it is also easy to see that
\[
A = -\int_{\partial\Omega} \frac{\partial \nu}{\partial r} ds.
\]
These formulas, though very useful in general, do not impose any restriction on the area of the surface, as the uniformization theorem says that every simply-connected Riemann surface is conformally equivalent to one of the three domains: the open unit disk, the complex plane, or the Riemann sphere. However, if there is another surface \( (\Omega, \tilde{g}) \) with a distinct conformal metric \( \tilde{g} = e^{2\tilde{v}}dy \in \Omega \), where \( \tilde{v} \in C^2(\Omega) \) is a solution of (1.1) and \( \tilde{g} = g \) on \( \partial\Omega \), we shall show
\[
\tilde{A} + A = \int_\Omega (e^{2\tilde{v}} + e^{2\nu})dy \geq 4\pi.
\]
(1.2)
Since the standard sphere has Gaussian curvature 1 and area 4\( \pi \), and these two surfaces have total area bigger than or equal to that of the standard sphere, one may think that these two surfaces could cover the standard sphere if they are properly arranged (this will be made more rigorous later in Section 2.1). The equality obviously hold when the two surfaces are isometric to two complementing spherical caps on the standard sphere. We therefore refer to (1.2) as the Sphere Covering Inequality. Let us give an explicit example to better demonstrate the Sphere Covering Inequality. Set \( \lambda > 0 \) and define
\[
V_\lambda(y) := -\ln(1 + \frac{\lambda^2|y|^2}{8}) + \ln(\lambda) - \frac{1}{2} \ln(2),
\]
satisfying
\[
\Delta V_\lambda + e^{2V_\lambda} = 0, \quad y \in \mathbb{R}^2.
\]
For every \( \lambda_2 > \lambda_1 \), there exists a unique \( R \in \mathbb{R} \) such that \( V_{\lambda_1}(\partial B_R) = V_{\lambda_2}(\partial B_R) \) and \( V_{\lambda_2} > V_{\lambda_1} \) in \( B_R \). Now define two surfaces \( S_1 \) and \( S_2 \) with constant Gaussian curvature 1 as follows
\[
S_1 = (B_R, e^{2V_{\lambda_1}}dy) \quad \text{and} \quad S_2 = (B_R, e^{2V_{\lambda_2}}dy).
\]
Notice that the metrics \( g_i = e^{2V_{\lambda_i}}dy, i = 1, 2 \) have the same conformal factor on \( \partial B_R \), which implies that \( \lambda_1\lambda_2 = \frac{8}{R^2} \). By scaling in \( y \in \mathbb{R}^2 \) and using the stereographic projection \( \Pi: S^2 \to \mathbb{R}^2 \) with respect to the north pole \( N = (1, 0, 0) \):
\[
y = \Pi(x) := \left( \frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right),
\]
we can see that the surface \( S_1 = (B_R, e^{2V_{\lambda_1}}dy) \) is isometric to
\[
(B_{\lambda_1 R/\sqrt{8}}, e^{2V_1}dy) = \left( B_{\lambda_1 R/\sqrt{8}}, \frac{1}{(1 + |y|^2)^2}dy \right),
\]
which is isometric to a disc $C_1$ around the south pole. Similarly, the surface $S_2 = (B_R, e^{2V_1}dy)$ is isometric to
\[(B_{\lambda_2 R/\sqrt{8}}, e^{2V_1}dy) = (B_{\lambda_2 R/\sqrt{8}}, \frac{1}{(1 + |y|^2)^2}dy),\]
which is isometric to a disc $C_2$ around the south pole.

Using the Kelvin transformation $z = y/|y|^2$ and the fact $\lambda_1 \lambda_2 = \frac{8}{R}$, one can see that
\[(B_{\lambda_2 R/\sqrt{8}}, \frac{1}{(1 + |y|^2)^2}dy),\]
is isometric to
\[(\mathbb{R}^2 \setminus B_{\lambda_1 R/\sqrt{8}}, \frac{1}{(1 + |z|^2)^2}dz),\]
which is isometric to $S^2 \setminus C_1$. This implies that $S_1$ and $S_2$ are indeed isometric to two complimenting spherical caps on the unit sphere, and therefore their total areas are exactly $4\pi$.

We will prove the inequality (1.2) in a more general setting as follows.

**Theorem 1.1 (The Sphere Covering Inequality)** Let $\Omega$ be a simply-connected subset of $\mathbb{R}^2$ and assume $v_i \in C^2(\overline{\Omega})$, $i = 1, 2$ satisfy
\[\Delta v_i + e^{2v_i} = f_i(y), \tag{1.4}\]
where $f_2 \geq f_1 \geq 0$ in $\Omega$. If $v_2 \geq v_1, v_2 \not\equiv v_1$ in $\Omega$ and $v_2 = v_1$ on $\partial \Omega$, then
\[\int_{\Omega} e^{2v_1} + e^{2v_2} dy \geq 4\pi. \tag{1.5}\]
Moreover, the equality only holds when $f_2 \equiv f_1 \equiv 0$ and $(\Omega, e^{2v_i}dy), i = 1, 2$ are isometric to two complimenting spherical caps on the standard unit sphere.

For the simplicity of the equation, we may replace $2v$ by $u - \ln 2$ and consider
\[\Delta u + e^u = 0, \quad y \in \Omega,\]
Geometrically, this is equivalent to multiplying the conformal factor by $\sqrt{2}$ so the sphere in comparison has radius $\sqrt{2}$ and total area $8\pi$. Indeed Theorem 1.1 is equivalent to Theorem 3.1 in Section 3.

The Sphere Covering Inequality is closely related to the symmetry of solutions of elliptic equations with exponential nonlinearity in $\mathbb{R}^2$. To see the connection, consider the equation
\[\Delta w + e^w = f \geq 0 \quad \text{in} \quad \mathbb{R}^2, \tag{1.6}\]
and let $w$ be a classical solution with a critical point located at $P \in \mathbb{R}^2$. Assume that $f$ is evenly symmetric about a line passing through $P$. It follows from the Sphere Covering Inequality that if $\int_{\mathbb{R}^2} e^w dy < 16\pi$, then $u$ must be symmetric about the line. More precisely, suppose $P = (p, 0)$ and $f(y_1, y_2) = f(y_1, -y_2)$ in $\mathbb{R}^2$. Define $\bar{w}(y_1, y_2) = w(y_1, -y_2)$, and set $\bar{v} := w - \bar{w}$.
Then $\tilde{v}$ satisfies
\[ \Delta \tilde{v} + c \tilde{v} = 0, \]
where
\[ c = e^w - e^{\bar{w}}. \]
Suppose $\tilde{v} \not\equiv 0$. It follows from the Hopf’s lemma that the nodal line of $\tilde{v}$ divides a neighborhood of $P$ into at least four regions, and consequently there exist at least two simply-connected regions $\Omega_1, \Omega_2 \subset \mathbb{R}^2_+$ such that $\tilde{v} > 0$ in $\Omega_1$, $\tilde{v} < 0$ in $\Omega_2$, and $\tilde{v} = 0$ on $\partial \Omega_1 \cup \partial \Omega_2$. Therefore on each $\Omega_i$, $i = 1, 2$, the equation (1.6) has two distinct solutions, $w$ and $\bar{w}$, satisfying the assumptions of Theorem 3.1 which is an equivalent form of Theorem 1.1 for (1.6). Thus
\[ \int_{\mathbb{R}^2} e^w dy \geq \int_{\Omega_1} e^w + e^{\bar{w}} dy + \int_{\Omega_2} e^w + e^{\bar{w}} dy \geq 16\pi, \]
which is a contradiction and leads to the symmetry of $w$.

The above argument is at the core of the proof of the symmetry results in this paper, and consists of two main ingredients: the Sphere Covering Inequality and the nodal set analysis. The idea of using nodal sets to prove symmetry results for elliptic equations with exponential nonlinearity was used by Lin and others to obtain symmetry results for mean field equations in $\mathbb{R}^2$ and on $S^2$ and flat tori (see, e.g., [31], [33], [24], [5], etc). The key in their arguments is Proposition 3.2 in Section 3, which has a geometrical interpretation in terms of the extremal first eigenvalue (see Remark 3.4) and usually yields a lower bound of $2\pi$ for (1.4) instead of $4\pi$ in the Sphere Covering Inequality. Note that Proposition 3.2 may be regarded as a special limiting case of the Sphere Covering Inequality, although the geometric meaning of the Sphere Covering Inequality itself still remains unclear and worth further exploring. In this paper, the Sphere Covering Inequality will be used to solve several important open problems. Below we introduce some of the problems.

### 1.1 Best Constant in a Moser-Trudinger Type Inequality

Let $S^2$ be the unit sphere and for $u \in H^1(S^2)$ define
\[ J_\alpha(u) = \frac{\alpha}{4} \int_{S^2} |\nabla u|^2 d\omega + \int_{S^2} u d\omega - \log \int_{S^2} e^u d\omega, \quad (1.7) \]
where the volume form $d\omega$ is normalized so that $\int_{S^2} d\omega = 1$. The well-known Moser-Trudinger inequality [34] says that $J_\alpha$ is bounded below if and only if $\alpha \geq 1$. Onofri [35] showed that for $\alpha$ close to $1$ the best lower bound is equal to zero. On the other hand, Aubin [2] proved that if $J_\alpha$ is restricted to
\[ \mathcal{M} := \{ u \in H^1(S^2) : \int_{S^2} e^u x_i = 0, \quad i = 1, 2, 3 \}, \]
then for $\alpha \geq \frac{1}{2}$, $J_\alpha$ is bounded below and the infimum is attained in $\mathcal{M}$. In 1987 Chang and Yang [14], in their work on prescribing Gaussian curvature on $S^2$ (see also [15]), showed that for $\alpha$ close to $1$ the best constant again is equal to zero, and this led to the following conjecture.
Conjecture A. For $\alpha \geq \frac{1}{2}$

$$\inf_{u \in \mathcal{M}} J_\alpha(u) = 0.$$ 

In 1998, Feldman, Froese, Ghoussoub and the first author [23] proved that this conjecture is true for axially symmetric functions when $\alpha > \frac{16}{25} - \epsilon$. Later the first author and Wei [28], and independently Lin [32] proved Conjecture A for radially symmetric function, but the problem remained open for non-axially symmetric functions.

In [24] Ghoussoub and Lin, showed that Conjecture A holds true for $\alpha \geq \frac{2}{3} - \epsilon$, for some $\epsilon > 0$. See Chapter 19 in [25] for a complete history of the problem. In this paper, among other results, we will prove that Conjecture A is true.

**Theorem 1.2** For $\alpha \geq \frac{1}{2}$

$$\inf_{u \in \mathcal{M}} J_\alpha(u) = 0.$$ 

Indeed we apply Theorem 3.1 to show that the corresponding Euler-Lagrange equation

$$\frac{\alpha}{2} \Delta u + \frac{e^u}{\int_{S^2} e^u d\omega} - 1 = 0 \quad \text{on} \quad S^2$$

(1.8)

has only constant solutions.

### 1.2 A Mean Field Equation with singularity on $S^2$

Consider the mean field equation

$$\Delta_g u + \lambda \left( \frac{e^u}{\int_{S^2} e^u d\omega} - \frac{1}{4\pi} \right) = 4\pi (\delta(P) - \frac{1}{4\pi}) \quad \text{on} \quad S^2,$$

(1.9)

where $g$ is the standard metric on $S^2$ with the corresponding volume form $d\omega$, $\alpha > -1$, $\lambda > 0$, and $P \in S^2$. In [4], motivated by the study of vortices in self-dual gauge field theory, Bartolucci, Lin and Tarentello studied symmetry of solutions of (1.9) under the assumption

$$\lambda = 4\pi (3 + \alpha)$$

(1.10)

and showed that (1.9) admits a solution if and only if $\alpha \in (-1, 1)$. Then, via a new bubbling phenomenon, they proved that there exists $\delta > 0$ such that for $\alpha \in (1 - \delta, 1)$ the equation (1.9) admits a unique solution that is in addition is axially symmetric about the direction $\overrightarrow{OP}$, and proposed the following

**Conjecture B.** All solutions of (1.9)-(1.10) are axially symmetric about $\overrightarrow{OP}$ for every $\alpha \in (-1, 1)$.

In Section 5, we shall use the Sphere Covering Inequality to provide an affirmative answer to the above question. Indeed we will prove the following result.

**Theorem 1.3** For every $\alpha \in (-1, 1)$ equation (1.9)-(1.10) has a unique solutions that in addition is axially symmetric about $\overrightarrow{OP}$.
1.3 A Mean Field Equation for the Spherical Onsager Vortex

Consider the following equation

\[ \Delta_g u(x) + \frac{\exp(\alpha u(x) - \gamma \langle n, x \rangle)}{\int_{S^2} \exp(\alpha u(x) - \gamma \langle n, x \rangle) d\omega} - \frac{1}{4\pi} = 0 \text{ on } S^2, \]  

(1.11)

where $g$ is the standard metric on $S^2$ with the corresponding volume form $d\omega$, $\vec{n}$ is a unique vector in $\mathbb{R}^3$, $\alpha \geq 0$, and $\gamma \in \mathbb{R}$. Since $\gamma < 0$ can be changed to $-\gamma$ by replacing the north pole with the south pole, we only need to consider the case $\gamma \geq 0$. This equation is invariant up to adding a constant and we seek a normalized solution with

\[ \int_{S^2} u d\omega = 0. \]  

(1.12)

In [32], Lin showed that if $\alpha < 8\pi$, then for $\gamma \geq 0$ the equation (1.11) has a unique solution that in addition is axially symmetric with respect to $\vec{n}$. In this case the coefficient in the equation is decreasing and therefore the moving plane method applies (see (5.16)). He also conjectured the following

**Conjecture C.** Let $\gamma \geq 0$ and $\alpha \leq 16\pi$. Then every solution $u$ of (1.11) is axially symmetric with respect to $\vec{n}$.

In an attempt to prove this conjecture, in [31], C.S. Lin proved the following theorems for $\alpha > 8\pi$.

**Theorem A.** ([31]) For every $\gamma > 0$, there exists $\alpha_0 = \alpha_0(\gamma) > 8\pi$ such that, for $8\pi < \alpha \leq \alpha_0$, any solution $u$ of (1.11) is axially symmetric.

**Theorem B.** ([31]) Let $u_i$ be a solution of (1.11) with $\gamma = 0$ and $\alpha_i \to 16\pi$. Suppose $\lim_{i \to \infty} \sup u_i(x) = +\infty$. Then $u_i$ is axially symmetric with respect to some direction $\vec{n}_i$ in $\mathbb{R}^3$ for $i$ large enough.

In Section 5, we shall apply the Sphere Covering Theorem to prove the following result.

**Theorem 1.4** Suppose $8\pi \leq \alpha \leq 16\pi$ and

\[ 0 \leq \gamma \leq \frac{\alpha}{8\pi} - 1. \]  

(1.13)

Then every solution of (1.11) is axially symmetric with respect to $\vec{n}$. In particular if $\gamma = 0$ and $8\pi \leq \alpha \leq 16\pi$, then the trivial solution $u \equiv 0$ is the only solution of (1.11).

In all problems above and many others, there exists a critical number $8\pi$ for a quantity which may be interpreted as total area literally. In (1.7), the quantity is $4\pi/\alpha$, with $\alpha$ being a parameter; In (1.9), the quantity is $\lambda$; While in (1.11), the quantity is $\alpha$. Note that the parameter $\alpha$ has different meanings in these three equations which should be clear from the context. The work of Brezis and Merle [9] and Y.Y. Li [30] as well as others showed that
8\pi m, m \in N are values where solutions of these type of equations may lose compactness and blow-up phenomena may happen. The critical level 8\pi also separates two significantly different cases in terms of the coerciveness of associate functionals and the positiveness of linearized operators. A crucial tool is required to deal with the supercritical cases of many important problems in related research. The Sphere Covering Inequality provides exactly such a much needed tool. Besides the applications in this paper, other applications of the Sphere Covering Inequality will also be discussed in forthcoming papers [26] and [27].

The paper is organized as follows. In Section 2, we shall discuss some preliminary results about the classical Bol's inequality and prove a counterpart of Bol's inequality for radially symmetric functions which is needed for the proof of the Sphere Covering Inequality. In Section 3, we will prove the Sphere Covering Inequality. In Section 4, the Sphere Covering Inequality shall be applied to (1.7) to show the best constant. Finally, we will present a general symmetry result regarding Gaussian curvature equations on \( \mathbb{R}^2 \) which leads to optimal results for (1.9), (1.11) and others.

## 2 Bol’s Isoperimetric Inequality

Bol’s isoperimetric inequality plays a crucial role in the proof of our main results. In this section we present some preliminary results on Bol’s inequality that will be used in subsequent sections. First we recall the classical Bol’s isoperimetric inequality [3, 8, 38]:

**Proposition 2.1** Let \( \Omega \subset \mathbb{R}^2 \) be a simply-connected and assume \( u \in C^2(\Omega) \) satisfies

\[
\Delta u + e^u \geq 0 \quad \text{and} \quad \int_{\Omega} e^u \leq 8\pi.
\] (2.1)

Then for every \( \omega \Subset \Omega \) of class \( C^1 \) the following inequality holds

\[
\left( \int_{\partial \omega} e^{\frac{u}{2}} \right)^2 \geq \frac{1}{2} \left( \int_{\omega} e^u \right) \left( 8\pi - \int_{\omega} e^u \right).
\] (2.2)

We first show an example for which the equality in Bol’s inequality holds. For \( \lambda > 0 \), let define \( U_\lambda \) by

\[
U_\lambda := -2 \ln(1 + \frac{\lambda^2 |y|^2}{8}) + 2 \ln(\lambda).
\] (2.3)

Then

\[
\Delta U_\lambda + e^{U_\lambda} = 0,
\]

and

\[
\int_{B_r} e^{U_\lambda} dy = \frac{8\pi \lambda^2 r^2}{8 + \lambda^2 r^2},
\]

for all \( r > 0 \), where \( B_r \) denotes the ball of radius \( r \) centered at the origin in \( \mathbb{R}^2 \). One can check that

\[
\left( \int_{\partial B_r} e^{\frac{U_\lambda}{2}} \right)^2 = \frac{1}{2} \left( \int_{B_r} e^{U_\lambda} \right) \left( 8\pi - \int_{B_r} e^{U_\lambda} \right),
\]
for all $r > 0$ and $\lambda > 0$. Indeed, $e^{U_\lambda(y)}dy$ corresponds to the metric in a standard sphere with radius $\sqrt{2}$.

By examining the proof of Bol’s inequality (see, e.g., [38]), it can be seen that if the equality holds for some $\omega$ in (2.2), then $\Delta u + e^u = 0$ in $\omega$, and $e^{u(z)}dz = e^{U_\lambda(\xi)}d\xi$, where $z = g^{-1}(\xi)$ for some analytic function $g: \Omega \to B_R$, and $\lambda > 0$. More precisely, let us consider the case that $\omega = \Omega$ is simply-connected, and follow the arguments in [38] by considering the harmonic lifting $h$ of boundary value of $u$ in $\Omega$, i.e.,

$$\Delta h(z) = 0, \quad z \in \Omega; \quad h = u, \quad z \in \partial \Omega.$$ 

It is known that there is an analytic function $\xi = g(z)$ such that $e^h = |g'(z)|^2$. The equality in (2.2) implies that $g(\Omega) = B_R$ and $g(\partial \Omega) = g(\partial B_R)$ for some $R > 0$. Furthermore, letting

$$q(\xi) := e^u(g^{-1}(\xi)) |g'(g^{-1}(\xi))|^{-2},$$

we have $e^u(z)dz = q(\xi)d\xi$ and $q(\xi)$ is radially symmetric. Therefore $q(\xi) = e^{U_\lambda(\xi)}$ for some $\lambda > 0$. In general, if $\omega$ is not simply-connected, Bartolucci and Lin showed in [5] that strict inequality holds in (2.2). So the equality may hold only for simply-connected $\omega$. Note that if $\Omega$ is not simply-connected, Proposition 2.1 is not valid in general. Indeed, (2.2) does not hold for certain annulus regions as shown in [5].

For the proof of our main results, we shall need the following counterpart of the Bol’s inequality for radial functions. The proof is a modification of an argument by Suzuki [38] and we present it here for the sake of completeness.

**Proposition 2.2** Let $B_R$ be the ball of radius $R$ in $\mathbb{R}^2 \psi \in C^{0,1}(\overline{B_R})$ be a strictly decreasing, radial, Lipschitz function satisfying

$$\int_{\partial B_r} |\nabla \psi|ds \leq \int_{B_r} e^\psi dy \text{ a.e. } r \in (0, R), \text{ and } \int_{B_R} e^\psi \leq 8\pi. \quad (2.4)$$

Then the following inequality holds

$$\left( \int_{\partial B_R} e^{\frac{\psi}{2}} \right)^2 \geq \frac{1}{2} \left( \int_{B_R} e^\psi \right) \left( 8\pi - \int_{B_R} e^\psi \right). \quad (2.5)$$

Moreover if $\int_{\partial B_r} |\nabla \psi|ds \neq \int_{B_r} e^\psi dy$ in $(0, R)$, then the inequality in (2.2) is strict.

**Proof.** Let $\beta := \psi(R)$ and define

$$k(t) = \int_{\{\psi > t\}} e^\psi dy, \quad \text{and} \quad \mu(t) = \int_{\{\psi > t\}} dy,$$

for $t > \beta$. Then

$$-k'(t) = \int_{\{\psi = t\}} e^\psi |\nabla \psi| = -e^t \mu'(t), \text{ for a.e. } t > \beta.$$
Hence

\[-k(t)k'(t) \geq \int_{\{\psi = t\}} |\nabla \psi| \cdot \int_{\{\psi = t\}} \frac{e^\psi}{|\nabla \psi|} \]  

(2.6)

\[= (\int_{\{\psi = t\}} e^{\psi/2})^2 = e^t (\int_{\{\psi = t\}} ds)^2 \]

\[= 4\pi e^t \int_{\{\psi > t\}} dy = 4\pi e^t \mu (t), \text{ for a.e. } t > \beta.\]

Therefore

\[\frac{d}{dt}[e^t \mu(t) - k(t) + \frac{1}{8\pi} k^2(t)] = e^t \mu(t) + \frac{1}{4\pi} k'(t)k(t) \leq 0, \text{ for a.e. } t > \beta.\]

Integrating on $(\beta, \infty)$ we get

\[\left[e^t \mu(t) - k(t) + \frac{1}{8\pi} k^2(t)\right]_\beta^\infty = -\left(e^\beta \mu(\beta) - k(\beta) + \frac{1}{8\pi} k^2(\beta)\right) \leq 0. \]  

(2.7)

Now notice that

\[k(\beta) = \int_{B_R} e^\psi dy\]

and

\[e^\beta \mu(\beta) = e^\beta \int_{B_R} dy = \frac{1}{4\pi} e^\beta (\int_{\partial B_R} ds)^2 = \frac{1}{4\pi} (\int_{\partial B_R} e^\psi ds)^2.\]

Thus (2.5) follows from the inequality (2.7). Finally if \(\int_{\partial B_R} |\nabla \psi| ds \neq \int_{B_R} e^\psi \in (0, R)\), then the inequality (2.6) will be strict in a set with a positive measure in \(\{t > \beta\}\), and consequently (2.7) and (2.5) will also be strict. \(\Box\)

2.1 Rearrangement with respect to two measures

Let \(\Omega \subset \mathbb{R}^2\) and \(\lambda > 0\), and suppose that \(w \in C^2(\Omega)\) satisfies

\[\Delta w + e^w \geq 0.\]

Then any function \(\phi \in C^2(\Omega)\) can be equimeasurably rearranged with respect to the measures \(e^w dy\) and \(e^{U_\lambda} dy\) (see [3], [38], and [33]), where \(U_\lambda\) is defined in (2.3). More precisely, for \(t > \min_{y \in \Omega} \phi\) define

\[\Omega_t := \{\phi > t\} \subset \subset \Omega,\]

and define \(\Omega_t^*\) be the ball centered at the origin in \(\mathbb{R}^2\) such that

\[\int_{\Omega_t^*} e^{U_\lambda} dy = \int_{\Omega_t} e^w dy := a(t).\]
Then \(a(t)\) is a right-continuous function, and \(\phi^*: \Omega^* \to \mathbb{R}\) defined by \(\phi^*(y) := \sup\{t \in \mathbb{R} : y \in \Omega^*_t\}\) provides an equimeasurable rearrangement of \(\phi\) with respect to the measure \(e^w dy\) and \(e^{U_\lambda} dy\), i.e.

\[
\int_{\{\phi^* > t\}} e^{U_\lambda} dy \leq \int_{\{\phi > t\}} e^w dy, \quad \forall t > \min_{y \in \Omega} \phi. \tag{2.8}
\]

Let

\[
j(t) := \int_{\{\phi > t\}} |\nabla \phi|^2 dy, \quad j^*(t) := \int_{\{\phi^* > t\}} |\nabla \phi^*|^2 dy, \quad \forall t > \min_{y \in \Omega} \phi;
\]

\[
J(t) := \int_{\{\phi > t\}} |\nabla \phi| dy, \quad J^*(t) := \int_{\{\phi^* > t\}} |\nabla \phi^*| dy, \quad \forall t > \min_{y \in \Omega} \phi.
\]

It is easy to see that both \(j(t)\) and \(J(t)\) are absolutely continuous and decreasing in \(t > \min_{y \in \Omega} \phi\).

When \(\phi \equiv C\) on \(\partial \Omega\), it can be shown that

\[
\int_{\{\phi = t\}} |\nabla \phi| ds \geq \int_{\{\phi^* = t\}} |\nabla \phi^*| ds, \quad \text{for a.e. } t > \min_{y \in \Omega} \phi. \tag{2.9}
\]

Indeed it follows from Cauchy-Schwarz and Bol’s inequalities that

\[
\int_{\{\phi = t\}} |\nabla \phi| ds \geq \left( \int_{\{\phi = t\}} e^\pi \right)^2 \left( \int_{\{\phi = t\}} \frac{e^w}{|\nabla \phi|} \right)^{-1}
\]

\[
= \left( \int_{\{\phi = t\}} e^{\pi \over 2} \right)^2 \left( -\frac{d}{dt} \int_{\Omega_t} e^w \right)^{-1}
\]

\[
\geq \frac{1}{2} \left( \int_{\Omega_t^*} e^w \right)^2 \left( 8\pi - \int_{\Omega_t^*} e^w \right) \left( -\frac{d}{dt} \int_{\Omega_t} e^w \right)^{-1}
\]

\[
= \frac{1}{2} \left( \int_{\Omega_t^*} e^{U_\lambda} \right) \left( 8\pi - \int_{\Omega_t^*} e^{U_\lambda} \right) \left( -\frac{d}{dt} \int_{\Omega_t^*} e^{U_\lambda} \right)^{-1}
\]

\[
= \int_{\{\phi^* = t\}} |\nabla \phi^*| ds, \quad \text{for a.e. } t > \min_{y \in \Omega} \phi.
\]

It also follows that \(j^*(t), J^*(t)\) are absolutely continuous and decreasing in \(t > \min_{y \in \Omega} \phi\), since both functions are right-continuous by definition and

\[
0 \leq j^*(t) - j^*(t - 0) \leq j(t) - j(t - 0) = \int_{\{\phi = t\}} |\nabla \phi|^2 dy = 0, \quad t > \min_{y \in \Omega} \phi.
\]

\[
0 \leq J^*(t) - J^*(t - 0) \leq J(t) - J(t - 0) = \int_{\{\phi = t\}} |\nabla \phi| dy = 0, \quad t > \min_{y \in \Omega} \phi.
\]

Therefore we have the following proposition.

**Proposition 2.3** Let \(w \in C^2(\overline{\Omega})\) satisfy

\[
\Delta w + e^w \geq 0 \quad \text{in} \quad \Omega,
\]
and let $U_\lambda$ be given by (2.3). Suppose $\phi \in C^1(\overline{\Omega})$ and $\phi \equiv C$ on $\partial \Omega$. Define the equimeasurable symmetric rearrangement $\phi^*$ of $\phi$, with respect to the measures $e^{w_1}dy$ and $e^{U_\lambda}dy$, by (2.3). Then $j^*(t), J^*(t)$ are absolutely continuous and decreasing in $t > \min_{y \in \Omega} \phi$ and

$$\int_{\{\phi^* = t\}} |\nabla \phi^*| ds \leq \int_{\{\phi = t\}} |\nabla \phi| ds, \text{ for a.e. } t > \min_{y \in \Omega} \phi.$$

### 3 The Sphere Covering Inequality

The main objective of this section is to prove the following theorem.

**Theorem 3.1** Let $\Omega$ be a simply-connected subset of $\mathbb{R}^2$ and assume $w_i \in C^2(\overline{\Omega}), i = 1, 2$ satisfy

$$\Delta w_i + e^{w_i} = f_i(y), \tag{3.1}$$

where $f_2 \geq f_1 \geq 0$ in $\Omega$. If $w_2 \geq w_1, w_2 \neq w_1$ in $\Omega$ and $w_2 = w_1$ on $\partial \Omega$, then

$$\int_{\Omega} e^{w_1} + e^{w_2} dy \geq 8\pi. \tag{3.2}$$

Moreover, the equality only holds when $f_2 \equiv f_1 \equiv 0$ and there is an analytic function $\xi = g(y)$ such that $g(\Omega) = B_R$ and $g(\partial \Omega) = \partial B_R$ for some $R > 0$, and $e^{w_1}(y)dy = e^{U_{\lambda_1}}(\xi)d\xi, e^{w_2}(y)dy = e^{U_{\lambda_2}}(\xi)d\xi$ with $e^{U_{\lambda_1}}(R) = e^{U_{\lambda_2}}(R) = 1$.

**Remark 3.2** Note that if $f_1 \equiv f_2$, the condition $w_2 \geq w_1, w_2 \neq w_1$ in the theorem can just be replaced by $w_2 \neq w_1$, since there must be a simply connected subset $\omega$ of $\Omega$ such that $w_2 > w_1$ in $\omega$ and $w_2 = w_1$ on $\partial \omega$ after switching the indices. If $w_2 - w_1$ changes sign in $\Omega$, the inequality has indeed a lower bound as $16\pi$.

Before proving the above theorem, let us first show that Theorem 3.1 holds when $w_1, w_2$ are both radial. Choose $\lambda_2 > \lambda_1$ and let $U_{\lambda_1}, U_{\lambda_2}$ be given by (2.3). Suppose $U_{\lambda_1} = U_{\lambda_2}$ on $\partial B_R$ for some $R > 0$. Then

$$\frac{\lambda_1}{1 + \frac{\lambda_1^2 R^2}{8}} = \frac{\lambda_2}{1 + \frac{\lambda_2^2 R^2}{8}} = \kappa.$$

Hence $\lambda_1, \lambda_2$ are positive real roots of the quadratic equation

$$R^2 \lambda^2 + 8 = \frac{8}{\kappa} \lambda.$$

This implies $\kappa \leq 2/R^2$,

$$\lambda_1 + \lambda_2 = \frac{8}{\kappa R^2}, \text{ and } \lambda_1 \lambda_2 = \frac{8}{R^2}. \tag{3.3}$$

Direct computations yield

$$\int_{B_R} e^{U_{\lambda_1}} + e^{U_{\lambda_2}} dy = 8\pi \left( \frac{\lambda_1^2 R^2}{8 + \lambda_1^2 R^2} + \frac{\lambda_2^2 R^2}{8 + \lambda_2^2 R^2} \right)$$

$$= 8\pi \left( \frac{\lambda_1^2 R^2}{8\lambda_1} + \frac{\lambda_2^2 R^2}{8\lambda_2} \right) = 8\pi \left[ \frac{\kappa R^2}{8} (\lambda_1 + \lambda_2) \right]$$

$$= 8\pi.$$

Thus we have the following
Proposition 3.1 Let \( \lambda_2 > \lambda_1 \), and \( U_{\lambda_1} \) and \( U_{\lambda_2} \) be radial solutions of the equation 
\[
\Delta u + e^u = 0,
\]
defined in (2.3) with \( U_{\lambda_2} > U_{\lambda_1} \) in \( B_R \), and \( U_{\lambda_1} = U_{\lambda_2} \) on \( \partial B_R \), for some \( R > 0 \). Then 
\[
\int_{B_R} (e^{U_{\lambda_1}} + e^{U_{\lambda_2}}) dy = 8\pi.
\]

To understand the above inequality geometrically, we may scale the conformal factor by \( 1/2 \) and consider two surfaces \( S_1 \) and \( S_2 \) with constant Gaussian curvature 1 as follows
\[
S_1 = (B_R, e^{2V_{\lambda_1}} dy) \quad \text{and} \quad S_2 = (B_R, e^{2V_{\lambda_2}} dy).
\]
where \( 2V_{\lambda} = U_{\lambda} - \ln 2 \). Notice that the metrics \( g_i = e^{2V_{\lambda_i}} dy \) have the same conformal factor on \( \partial B_R \) and hence (3.3) holds and
\[
\kappa \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) = \frac{\kappa(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} = 1. \tag{3.4}
\]

Next we explain that areas of \( S_1 \) and \( S_2 \) are equal to the areas of two complimenting spherical caps on the unit sphere, and consequently the total area must be
\[
A_1 + A_2 = \int_{B_R} e^{2V_{\lambda_1}} dy + \int_{B_R} e^{2V_{\lambda_2}} dy = 4\pi.
\]
Indeed, we have
\[
\int_{B_R} e^{2V_{\lambda_i}} dy = \int_{B_R} \frac{\lambda_i^2}{2(1 + \frac{\lambda_i^2 |x|^2}{8})} dy = \int_{B_{\lambda_iR}/\sqrt{8}} \frac{4}{(1 + |x|^2)} dy, \quad i = 1, 2.
\]

Hence by the stereographic projection, \( A_i \) is equal to the area of the spherical cap \( C_i \) of the unit sphere that lies below the plane
\[
z = z_i := \frac{\lambda_i^2 R^2}{8} - 1 = \frac{\lambda_i - 1}{\frac{\lambda_i}{\kappa}} = 1 - \frac{2\kappa}{\lambda_i}, \quad i = 1, 2.
\]
It follows from (3.4) that \( z_1 < 0 \) while \( z_2 > 0 \). Moreover
\[
|z_i| = \frac{2\kappa}{\lambda_i} - 1 = 1 - \frac{2\kappa}{\lambda_2} = z_2, \quad i = 1, 2.
\]
Thus \( C_1 \) and \( C_2 \) are two complimenting spherical caps on the unit sphere. Note that the area of the smaller cap \( C_1 \) can be arbitrarily close to 0 or \( 2\pi \).

The following lemma will play a key role in the proof of Theorem 3.1

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Lemma 3.3 Assume that $\psi \in C^{0,1}(\overline{B_R})$ is a strictly decreasing, radial, Lipschitz function, and satisfies
\begin{equation}
\int_{\partial B_r} |\nabla \psi| \leq \int_{B_r} e^\psi \tag{3.5}
\end{equation}
a.e. $r \in (0, R)$ and $\psi = U_{\lambda_2} = U_{\lambda_1}$ for some $\lambda_2 > \lambda_1$ on $\partial B_R$, and $R > 0$. Then there holds
\begin{equation}
\text{either } \int_{B_R} e^\psi \leq \int_{B_R} e^{U_{\lambda_1}} \quad \text{or} \quad \int_{B_R} e^\psi \geq \int_{B_R} e^{U_{\lambda_2}}. \tag{3.6}
\end{equation}
Moreover if the inequality in (3.5) is strict in a set with positive measure in $(0, R)$, then the inequalities in (3.6) are also strict.

**Proof.** Let $m_1 := \int_{B_R} e^{U_{\lambda_1}}$, $m_2 := \int_{B_R} e^{U_{\lambda_2}}$, and $m := \int_{B_R} e^\psi$. Also define
\[ \beta := \left( \int_{\partial B_R} e^{\frac{\psi}{2}} \right)^2 = \left( \int_{\partial B_R} e^{\frac{U_{\lambda_1}}{2}} \right)^2 = \left( \int_{\partial B_R} e^{\frac{U_{\lambda_2}}{2}} \right)^2. \]
It follows from Proposition 2.2 that
\[ \beta \geq \frac{1}{2} m (8\pi - m). \]
On the other hand,
\[ \beta = \frac{1}{2} m_1 (8\pi - m_1) = \frac{1}{2} m_2 (8\pi - m_2). \]
Hence $m_1$ and $m_2$ are roots of the quadratic equation
\[ x^2 - 8\pi x + 2\beta = 0. \]
Since $m$ satisfies
\[ m^2 - 8\pi m + 2\beta \geq 0, \]
we have
\[ \text{either } m \leq m_1 \quad \text{or} \quad m \geq m_2. \]
Equality holds only when the equality in (3.5) holds for a.e. $r \in (0, R)$. This completes the proof. \[\square\]

Now we are ready to prove Theorem 3.1

**Proof of Theorem 3.1.** Suppose $w_1$ and $w_2$ satisfy the assumptions of Theorem 3.1. Then
\[ \Delta (w_2 - w_1) + e^{w_2} - e^{w_1} = f_2 - f_1 \geq 0. \]
Now we can choose $\lambda_2 > \lambda_1$ such that $U_{\lambda_1}$ and $U_{\lambda_2}$ are as described in Proposition 3.1 and
\begin{equation}
\int_{\Omega} e^{w_1} = \int_{B_1} e^{U_{\lambda_1}}. \tag{3.7}
\end{equation}
Let \( \varphi \) be the symmetrization of \( w_2 - w_1 \) with respect to the measures \( e^{w_1} dy \) and \( e^{U_{\lambda_1}} dy \). Then by Proposition 2.3,

\[
\int_{\{\varphi = t\}} |\nabla \varphi| \leq \int_{\{w_2 - w_1 = t\}} |\nabla (w_2 - w_1)| \leq \int_{\Omega} (e^{w_2} - e^{w_1}) = \int_{\{\varphi > t\}} e^{U_{\lambda_1} + \varphi} - \int_{\{\varphi > t\}} e^{U_{\lambda_1}} = \int_{\{\varphi > t\}} e^{U_{\lambda_1} + \varphi} - \int_{\{\varphi = t\}} |\nabla U_{\lambda_1}|, \quad \text{for a.e. } t > 0.
\]

Hence

\[
\int_{\{\varphi = t\}} |\nabla (U_{\lambda_1} + \varphi)| \leq \int_{\{\varphi > t\}} e^{(U_{\lambda_1} + \varphi)}, \quad \text{for a.e. } t > 0. \tag{3.8}
\]

Since \( \varphi \geq 0 \) is decreasing in \( r \), \( \psi := U_{\lambda_1} + \varphi \) is a strictly decreasing function, and

\[
\int_{\partial B_r} |\nabla \psi| \leq \int_{B_r} e^{\psi} dy, \quad \text{a.e. } r \in (0, 1), \tag{3.9}
\]

by Proposition 2.3 and the above inequality we know that \( \psi \) belongs to \( C^{0,1}(B_R) \). It follows from Lemma 3.3 that

\[
\int_{B_1} e^{\psi} dy = \int_{B_1} e^{U_{\lambda_1} + \varphi} dy \geq \int_{B_1} e^{U_{\lambda_2}}.
\]

Hence

\[
\int_{B_1} e^{w_1} + e^{w_2} dy = \int_{B_1} (e^{U_{\lambda_1}} + e^{U_{\lambda_1} + \varphi}) dy \geq \int_{B_1} e^{U_{\lambda_1}} + e^{U_{\lambda_2}} dy = 8\pi.
\]

Moreover, it is clear that if the equality holds, then \( f_2 \equiv f_1 \) and the equality in (3.9) holds for all \( r > 0 \). This leads to the equality in Bol’s inequality for \( w_2 \) in \( \Omega \), therefore \( f_2 \equiv 0 \) and there is an analytic function \( \xi = g(y) \) such that \( g(\Omega) = B_R \) and \( g(\partial \Omega) = \partial B_R \) for some \( R > 0 \) and \( e^{w_1(y)} dy = e^{U_{\lambda_1}(\xi)} d\xi \). Furthermore, \( \psi = U_{\lambda_1} + \varphi \equiv U_{\lambda_2}(\xi) \). This proof is complete. \( \square \)

Note that the following consequence of Bol’s inequality and the equimeasurable symmetric rearrangement (see Lemma 3.1 in [24] or Proposition 3.3 in [33] for a proof) may be regarded as a limiting case of the Sphere Covering Inequality.

**Proposition 3.2** Let \( \Omega \subset \mathbb{R}^2 \) be a simply-connected domain and assume that \( w \in C^2(\overline{\Omega}) \) satisfies \( \Delta w + e^w \geq 0 \) in \( \overline{\Omega} \) and \( \int_{\Omega} e^w \leq 8\pi \). Consider an open set \( \omega \subset \Omega \) and define the first eigenvalue of the operator \( \Delta + e^w \) in \( H^1_0(\omega) \) by

\[
\lambda_{1,w}(\omega) := \inf_{\phi \in H^1_0(\omega)} \left\{ \int_{\omega} |\nabla \phi|^2 - \int_{\omega} \phi^2 e^w \right\} \leq 0.
\]

Then \( \int_{\Omega} e^w \geq 4\pi \) if \( \lambda_{1,w}(\Omega) \leq 0 \).
Suppose that $w^k_1, w^k_2$ are solutions of (3.1) with $f^k_1, f^k_2$ in $\Omega_k$, $k = 1, 2, \cdots$ and the conditions of Theorem 3.1 hold for each $k$ large. Further assume that $\Omega_k \to \Omega$ in $C^2$, $w^k_i \to w$ in $C^2$, $f^k_i \to f_i$ in $C^0$, $i = 1, 2$ as $k \to \infty$. Then $w$ satisfies the condition in Proposition 3.2 and the Sphere Covering Inequality gives the same conclusion as Proposition 3.2.

**Remark 3.4** It can be seen from its proof that Proposition 3.2 has a geometrical interpretation as follows: Given a simply connected region $\omega$ on a surface with Gaussian curvature less than $1/2$, if the first eigenvalue of Laplacian in $\omega$ with zero Dirichlet boundary condition 

$$
\lambda_1(\omega) := \inf_{\phi \in H^1_0(\omega)} \frac{\int_{\omega} |\nabla \phi|^2}{\int_{\omega} \phi^2 e^w} \leq 1,
$$

then the area of $\omega$ must be bigger than or equal to $4\pi$, which is the area of a hemisphere with Gaussian curvature $1/2$. Note that such a hemisphere has first eigenvalue equal to $1$ as the height function from the boundary equator of the hemisphere is the first eigenfunction. In other words, Proposition 3.2 is an immediate consequence of the extremal eigenvalue theorem which says that a geodesic disc on the sphere achieves the smallest first eigenvalue of Laplacian among all surfaces with the same area and the same Gaussian curvature upper bound. It would be very interesting to see if there is a simple but deep geometric explanation of Theorem 3.1.

## 4 Best Constant in a Moser-Trudinger Type Inequality

Let us consider the functional $J_\alpha(u)$ defined in (1.7) in

$$
\mathcal{M} := \{u \in H^1(S^2) : \int_{S^2} e^u x_j = 0 \text{ for } j = 1, 2, 3\}.
$$

In this section we shall prove that $\inf_{u \in \mathcal{M}} J_\alpha(u) = 0$ for $\alpha \geq \frac{1}{2}$. Critical points of $J_\alpha(u)$, up to an additive constant, satisfy

$$
\frac{\alpha}{2} \Delta u + e^u - 1 = 0 \text{ on } S^2. \tag{4.1}
$$

Following [24], let $\Pi$ be the stereographic projection $S^2 \to \mathbb{R}^2$ with respect to the north pole $N = (1, 0, 0)$:

$$
\Pi := \left( \frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).
$$

Suppose $u$ is a solution of (4.1), and let

$$
\bar{u}(y) := u(\Pi^{-1}(y)) \text{ for } y \in \mathbb{R}^2.
$$

Then $\bar{u}$ satisfies

$$
\Delta \bar{u} + \frac{8}{\alpha(1 + |y|^2)^2}(e^{\bar{u}} - 1) = 0 \text{ in } \mathbb{R}^2. \tag{4.2}
$$
Now if we let
\[ v = \bar{u} - \frac{2}{\alpha} \ln(1 + |y|^2) + \ln\left(\frac{8}{\alpha}\right), \]  
(4.3)
then \( v \) satisfies
\[ \Delta v + (1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^v = 0 \text{ in } \mathbb{R}^2, \]  
(4.4)
and
\[ \int_{\mathbb{R}^2} (1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^v \, dy = \frac{8\pi}{\alpha}. \]  
(4.5)

### 4.1 Uniqueness of Axially Symmetric Solutions

For the convenience of the reader, we first use a new method to prove Conjecture A for axially symmetric functions, which was originally proven in [28] and [32].

**Lemma 4.1** Let \( \alpha \geq \frac{1}{2} \) and \( u \in \mathcal{M} \) be a solution of (4.1). If \( u \) is axially symmetric, then \( u \equiv 0 \).

**Proof.** We may assume that \( u \) is symmetric about \( x_3 \)-axis, i.e. \( u = g(x_3), \, x_3 \in [-1, 1] \). Since \( \int_{S^2} e^u x_3 \, d\omega = 0 \), \( g \) could not be monotone in \( x_3 \) unless it is identically equal to a constant \( C \). Therefore, if \( u \not\equiv C \), then it must take either its absolute minimum or absolute maximum at some point \( x_3^0 \in (-1, 1) \). Without loss of generality we can assume \( x_3^0 \geq 0 \) and \( g(x_3^0) = \max_{[-1,1]} g(x_3) \). Now choose some point \( p = (0, x_2^p, x_3^p) \in S^2 \) with \( x_3^0 < x_3^p < 1 \) and let \( u_p(x) = u(R^{-1}(x)) \) for some \( R \in SO(3) \) with \( R(p) = (0, 0, 1) \). Define \( \bar{u}_p = u_p(\Pi^{-1}) \) and let
\[ \bar{v}_p = \bar{u}_p - \frac{2}{\alpha} \ln(1 + |y|^2) + \ln\left(\frac{8}{\alpha}\right). \]
the \( \bar{v}_p \) satisfies (4.4) and (4.3). Now let
\[ \varphi_p(y) := y_2^p \frac{\partial v_p}{\partial y_1} - y_1 \frac{\partial v_p}{\partial y_2}. \]
Note that the set of critical points of \( \bar{u}_p \) contains a closed simple curve \( C \subset \mathbb{R}^2 \) which contains the origin in its interior, and \( \bar{u}_p \) takes its absolute maximum on \( C \). On the other hand \( v_p \) and \( \varphi_p \) are evenly symmetric about \( y_1 \)-axis, therefore
\[ C \cup \{ y = (y_1, 0) : y \in \mathbb{R}^2 \} \subset \varphi_p^{-1}(0). \]
Hence \( \varphi_p^{-1}(0) \) divides \( \mathbb{R}^2 \) into at least four simply-connected regions \( \Omega_i, \, i = 1, 2, 3, 4 \). Now let \( w_p := \ln((1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^{v_p}) \). Then \( w_p \) satisfies
\[ \Delta w_p + e^{w_p} = \frac{8(\frac{1}{\alpha} - 1)}{(1 + |y|^2)^2} > 0 \text{ in } \mathbb{R}^2. \]
On the other hand \( \varphi_p \) satisfies
\[ \Delta \varphi_p + e^{w_p} \varphi_p = 0 \text{ in } \mathbb{R}^2. \]
Thus it follows from (3.2) that
\[ \frac{8\pi}{\alpha} = \int_{\mathbb{R}^2} (1 + |y|^2)^2(\frac{1}{\alpha}-1)e^{v_y}dy = \sum_{i=1}^{4} \int_{\Omega_i} e^{u_y} > 4\pi = 16\pi. \]
This implies \( \alpha < \frac{1}{2} \) which is a contradiction. Therefore \( \varphi_p \equiv 0 \) and consequently \( u \) is also axially symmetric about the line passing through \( p \) and the origin. Since \( p \neq (0,0,1) \), \( u \) must be identically equal to a constant, and therefore must be zero. \( \square \)

Next we prove that if \( u \) is evenly symmetric about a plane passing through the origin, then \( u \) is axially symmetric. Note that this result was remarked by Ghoussoub and Lin [24], we provide the details here since it is needed in the proof of the main result.

**Lemma 4.2** Let \( \alpha \geq \frac{1}{2} \) and \( u \) be a solution of (4.1). If \( u \) is evenly symmetric about a plane passing through the origin, then \( u \) is axially symmetric.

**Proof.** The proof is similar to the proof of Lemma 4.1. We may assume that \( u \) is evenly symmetric about \( x_1x_3 \)-plane. Let \( u_0 \) be the restriction of \( u \) to \( \{ x \in S^2 : x_2 = 0 \} \) and assume that \( p \in S^2 \) is a maximum point of \( u_0 \). Since \( u \) is symmetric about \( x_1x_3 \)-plane, \( p \) is also a critical point of \( u \) on \( S^2 \). Without loss of generality we may assume \( p = (0,0,-1) \). We claim that
\[ \varphi(y_1, y_2) = y_2 \frac{\partial v}{\partial y_1} - y_1 \frac{\partial v}{\partial y_2} \equiv 0, \]
where \( v \) is defined by (4.3). Suppose \( \varphi \not\equiv 0 \). Since \( v \) has a critical point at the origin, the nodal line of \( \varphi \) divides a neighborhood of the origin into at least four regions. On the other hand \( \varphi \) is symmetric with respect to \( y_1 \)-axis and the nodal line of \( \varphi \) contains the \( y_1 \)-axis. Therefore the nodal line of \( \varphi \) divides \( \mathbb{R}^2 \) into at least 4 simply-connected regions \( \Omega_i \), \( i = 1, 2, 3, 4 \). As before, we can show that \( \varphi \equiv 0 \) and consequently \( u \) is axially symmetric about the line passing through \( p \) and the origin. \( \square \)

### 4.2 The General Case

We shall prove the even symmetry of a solution to (4.1).

**Theorem 4.3** Let \( \alpha \geq \frac{1}{2} \) and assume \( u \) be a solution of (4.1). Then \( u \) is evenly symmetric about any plane passing through the origin and a critical point of \( u \). Therefore \( u \) must be axially symmetric and consequently \( u \equiv 0 \).

**Proof.** Without loss of generality we may assume that \( (1,0,0) \) is a critical point of \( u \), and that \( u \) is not symmetric about \( x_1x_2 \)-plane. To finish the proof, it is enough to prove that \( u \) is symmetric about the \( x_1x_2 \)-plane. Define \( u^*(x_1, x_2, x_3) := u(x_1, x_2, -x_3) \) and \( \hat{u}(x) = u(x) - u^*(x) \). Notice that \( \hat{u}(x_1, x_2, 0) = 0 \), for all \( (x_1, x_2, 0) \in S^2 \). Then \( \hat{u} \) satisfies
\[ \frac{\alpha}{2} \Delta \hat{u} + c(x)\hat{u} = 0, \quad \text{on} \ S^2, \quad (4.6) \]
where

\[ c(x) := \frac{e^u - e^{u^*}}{u - u^*}. \]

Since \((1,0,0)\) is a critical point of \(u\), it follows from the Hopf’s lemma that \(u\) must change sign in \(S^+ := \{(x) \in S^2 : x_3 > 0\}\). Therefore the nodal line of \(\tilde{u}\) divides \(S^+\) into at least two simply-connected regions and there exists \(S^+_+, S^+_+ \subset S^+\) such that \(u = u^*\) on \(\partial(S^+_+ \cup S^+_-)\), \(u > u^*\) on \(S^+_+\) and \(u < u^*\) on \(S^+_-\).

Define \(S^+_-, S^-_+\) to be the reflections of \(S^+_+, S^+_+\) with respect to the \(x_1x_2\)-plane. Then we also have \(u = u^*\) on \(\partial(S^+_+ \cup S^-_-)\), \(u < u^*\) on \(S^+_-\) and \(u > u^*\) on \(S^-_+\).

Let \(\Omega_1, \Omega_2, \Omega_3, \Omega_4 \subset \mathbb{R}^2\) be the images of \(S^+, S^+_+, S^+_-, S^+_+ \subset S^2\) under the stereographic projection, respectively. Define \(v_1, v_2\) as follows

\[ v_1(y) = u((\Pi^{-1}(y)) - 2 \frac{2}{\alpha} \ln(1 + |y|^2) + \ln(\frac{8}{\alpha}) \]

and

\[ v_2(y) = u^*(\Pi^{-1}(y)) - \frac{2}{\alpha} \ln(1 + |y|^2) + \ln(\frac{8}{\alpha}). \]

Then \(v_1\) and \(v_2\) both satisfy \(4.4\) and \(w_i\) defined by

\[ w_i := \ln((1 + |y|^2)^{(\frac{1}{\alpha} - 1)}e^{v_i}) \]

satisfies

\[ \Delta w_i + e^{w_i} = \frac{8(\frac{1}{\alpha} - 1)}{(1 + |y|^2)^2} \geq 0 \text{ in } \mathbb{R}^2, \ i = 1, 2. \]

Moreover \(w_1 = w_2\) on \(\partial \Omega_i, \ i = 1, 2, 3, 4\). Applying the Sphere Covering Inequality (Theorem 3.1) on \(\partial \Omega_i, \ i = 1, 2, 3, 4\), we obtain that

\[ 2 \times \frac{8\pi}{\alpha} = \int_{\mathbb{R}^2} (1 + |y|^2)^{(\frac{2}{\alpha} - 1)}e^{v_1} dy + \int_{\mathbb{R}^2} (1 + |y|^2)^{(\frac{2}{\alpha} - 1)}e^{v_2} dy \]

\[ \geq \sum_{i=1}^{4} \int_{\Omega_i} e^{w_1} + e^{w_2} dy > 4 \times 8\pi. \]

Hence \(\alpha < \frac{1}{2}\) which is a contradiction. Thus \(u\) is evenly symmetric about the \(x_1x_2\)-plane and the proof is complete.

5 Radial Symmetry of Solutions in \(\mathbb{R}^2\)

In this section, we shall consider solutions to a general class of equations in \(\mathbb{R}^2\) and prove radial symmetry of the solutions. Assume \(u \in C^2(\mathbb{R}^2)\) satisfies

\[ \Delta u + k(|y|)e^u = 0 \text{ in } \mathbb{R}^2, \]

(5.1)
and
\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} k(|y|)e^u \, dy = \beta \leq 8, \]  
(5.2)
where \( K(y) = k(|y|) \in C^2(\mathbb{R}^2) \) is a non constant positive function satisfying
\[
(K1) \quad \Delta \ln(k(|y|)) \geq 0, \quad y \in \mathbb{R}^2 \\
(K2) \quad k(|y|) \leq C(1 + |y|)^m, \quad y \in \mathbb{R}^2 
\]
for some constant \( C, m > 0 \). It is easy to see that \((K1)\) implies that both \( k(r) \) and \( \frac{r k'(r)}{k(r)} \) are nondecreasing. Let
\[ 2l = \lim_{r \to \infty} \frac{r k'(r)}{k(r)}. \]
From \((K2)\) we know that \( 0 \leq 2l \leq m \) and hence for any \( \epsilon > 0 \) there exists a positive constant \( C_\epsilon > 0 \) such that
\[ C_\epsilon (1 + |y|^2)^{l-\epsilon} \leq k(|y|) \leq C(1 + |y|^2)^l, \quad y \in \mathbb{R}^2. \]
Without loss of generality we may assume that \( m = 2l \). Then it follows from Theorem 1.1 in [20] that
\[ \beta \geq 2l + 2. \]
Following [24] and using Pohazaev identity, we can obtain the following result.

**Proposition 5.1** Suppose \( u \) is a solution to \((5.1)-(5.2)\), where \((K1)-(K2)\) hold with \( m = 2l \). Then, if \( \beta > 2l + 2 \), there holds
\[ 4 < \beta < 4l + 4. \]  
(5.3)

**Proof.** By Theorem 1.1 in [20], we have
\[ u(y) = -\beta \ln(|y|) + C + O(|y|^{-\gamma}) \]  
(5.4)
for some constants \( C \) and \( \gamma > 0 \) as \( x \to \infty \), if \( \beta > 2l + 2 \). Also if \( \beta = 2l + 2 \), then for any \( \epsilon > 0 \) there exists \( R(\epsilon) > 0 \) such that
\[ -\beta \ln(|y|) - C \leq u(y) \leq (\epsilon - \beta) \ln(|y|), \quad |y| \geq R(\epsilon) \]
for some constant \( C \). On the other hand, it is easy to see that when \( \beta > 2l + 2 \), we have
\[ \nabla u = (-\beta + o(1)) \frac{x}{|y|^2}, \quad \text{as} \ y \to \infty. \]

Multiplying \((5.1)\) by \( y \cdot \nabla u \) and integrating by parts on \( B_R = \{ y : |y| \leq R \} \), we obtain
\[
\int_{\partial B_R} (y \cdot \nabla u) \frac{\partial u}{\partial \nu} \, ds - \frac{1}{2} \int_{\partial B_R} (y \cdot \nu) |\nabla u|^2 \, ds = - \int_{B_R} k(|y|)y \cdot \nabla e^u \, dy \\
= \int_{B_R} (2k(|y|) + k'(|y|)|y|) e^u \, dy - \int_{\partial B_R} (y \cdot \nu) k(|y|) e^u \, ds.
\]
Letting \( R \to \infty \) and using \((5.4)\), we obtain that
\[ \int_{\mathbb{R}^2} (2k(|y|) + k'(|y|)|y|) e^u \, dy = \pi \beta^2. \]
Hence we derive (5.3) from
\[ 2k(|y|) \leq 2k(|y|) + k'(|y|)|y| \leq (2l + 2)k(|y|), \quad y \in \mathbb{R}^2, \]
and the fact that equality holds in the above inequalities only when \( l = 0 \) and \( k \) equals to a constant. Note that by our assumptions, \( k(|y|) = |y|^{2l} \) is not allowed for \( l > 0 \) since \( k(0) = 0 \) in this case, nor is \( k \) allowed to be equal to a positive constant. The proof is complete. \( \square \)

**Remark 5.1** In all applications considered in this paper, it holds that \( \beta > 2l + 2 \). We wonder if \( \beta > 2l + 2 \) is always true for all solutions to (5.1) - (5.2) under the general conditions (K1) – (K2).

It is shown in \([32]\) that

**Proposition 5.2** If \( 0 < l \leq 1 \), then there exists a radially symmetric solution \( u_\beta \) to (5.1) if and only if \( \beta \in (4, 4l + 4) \). The radial solution is also unique in this case. Also If \( l > 1 \) and \( \beta \in (4l, 4l + 4) \), then there exists a unique radially symmetric solution \( u_\beta \) to (5.1).

Now we are ready to prove the following general theorem.

**Theorem 5.2** Assume that \( K(y) = k(|y|) > 0 \) satisfies (K1) – (K2), and \( u \) is a solution to (5.1) - (5.2) with \( 2l + 2 < \beta \leq 8 \). Then \( u \) must be radially symmetric.

**Proof.** Since \( \lim_{|y| \to \infty} u(y) = -\infty \), \( u \) has a maximum point \( p \in \mathbb{R}^2 \). We first prove that \( u \) is even symmetric about the line passing through the origin and \( p \). In particular if \( p = (0,0) \), then the following argument guarantees that \( u \) is even symmetric about any line passing through the origin and hence \( u \) must be radially symmetric. Without loss of generality we may assume that \( p \) lies on \( y_1 \)-axis. Define
\[ v(y_1, y_2) = u(y_1, y_2) - u(y_1, -y_2). \quad (5.5) \]
Suppose \( v \neq 0 \). Then the nodal line of \( v \), \( v^{-1}(0) \), contains the \( y_1 \)-axis. On the other hand since the critical point \( p \) lies on \( y_1 \)-axis, the nodal line of \( v \) divides every small neighborhood of \( p \) into at least four regions. Therefore the nodal line of \( v \) divides \( \mathbb{R}^2 \) into at least four simply-connected regions \( \Omega_i, \ i = 1, 2, 3, 4 \). Now notice that on each \( \Omega_i \) the equation
\[ \Delta u + k(|y|)e^u = 0 \quad y \in \Omega_i \]
has two solutions \( u_1^i(y_1, y_2) = u(y_1, y_2) \) and \( u_2^i(y_1, y_2) = u(y_1, -y_2) \) with \( u_1^i|_{\partial \Omega} = u_2^i|_{\partial \Omega} \). Define \( w := u + \ln(k(|y|)) \). Then \( w \) satisfies
\[ \Delta w + e^w = \Delta (\ln(k(|y|))) \geq 0. \quad (5.6) \]
Thus on each \( \Omega_i \), the above equation has two solutions \( w_1^i, w_2^i \) with \( w_1^i|_{\partial \Omega} = w_2^i|_{\partial \Omega}, \ i = 1, 2, 3, 4 \). Hence it follows from Theorem [3.1] that
\[
4\pi \beta = 2\int_{\mathbb{R}^2} k(|y|)e^u dy = \int_{\mathbb{R}^2} k(|y|)e^{u(y_1, y_2)} dy + \int_{\mathbb{R}^2} k(|y|)e^{u(y_1, -y_2)} dy \\
\geq \sum_{i=1}^{4} \int_{\Omega_i} e^{w_1} + e^{w_2} dy > 4 \times 8\pi = 32\pi.
\]

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Consequently $\beta > 8$ which is a contradiction, and therefore $u$ is evenly symmetric about the $y_1$-axis.

Next we shall prove that $u$ is indeed axially symmetric. Let $\phi = y_2 \cdot u_{y_1} - y_1 \cdot u_{y_2}$. Then $\phi$ satisfies

$$\Delta \phi + K(y)e^u \phi = 0, \quad y \in \mathbb{R}^2. \tag{5.7}$$

On the other hand, $u_{y_2}$ satisfies

$$\Delta u_{y_2} + K(y)e^u u_{y_2} = -y_2 \frac{k'(|y|)}{|y|} e^u, \quad y \in \mathbb{R}^2. \tag{5.8}$$

Note that both $u_{y_2}$ and $\phi$ are odd function in $y_2$. Let us multiply equation (5.7) by $u_{y_2}$ and equation (5.8) by $\phi$ and subtract. Then, integrating the resulting equation in $B_R^+ = \{y : y_2 > 0, |y| \leq R\}$, we obtain

$$\int_{\partial B_R^+} \phi \partial u_{y_2} \frac{\partial v}{\partial \nu} - u_{y_2} \partial \phi \frac{\partial v}{\partial \nu} ds = - \int_{B_R^+} y_2 \frac{k'(|y|)}{|y|} e^u \phi.$$

Applying the standard Schauder estimates for the elliptic equation satisfied by $u + \beta \ln(|y|)$ and using the fact that $\beta > 2l + 2$, we obtain

$$|\nabla u(y)| \leq \frac{C}{|y|}, \quad |\nabla^2 u(y)| \leq \frac{C}{|y|^2}, \quad |y| > 1$$

for some constant $C$. Letting $R \to \infty$, we derive

$$\int_{\mathbb{R}^2^+} y_2 \frac{k'(|y|)}{|y|} e^u \phi dy = 0.$$

We claim that $\phi \equiv 0$ in $\mathbb{R}^2$. Assume the contrary. Since $y_2 \frac{k'(|y|)}{|y|} e^u > 0$ in $\mathbb{R}^2^+$, there exist at least two regions $\Omega_1, \Omega_2 \subset \mathbb{R}^2^+$ such that $\phi > 0$ in $\Omega_1$ and $\phi < 0$ in $\Omega_2$ and $\phi = 0$ on $\partial \Omega_i, i = 1, 2$. Applying Proposition 3.2 to $\Omega_i, i = 1, 2$, we conclude that

$$\int_{\mathbb{R}^2^+} k(|y|) e^u dy \geq \int_{\Omega_1} e^u dy + \int_{\Omega_1} e^u dy > 8\pi,$$

and therefore $\beta > 8$. This contradiction shows $\phi \equiv 0$ in $\mathbb{R}^2$, and hence $u(y)$ is radially symmetric. \(\square\)

Now we consider several special cases of the equation (5.1)-(5.2). First, if $K(y) = (1 + |y|^2)^l$ for some $l \geq 0$, then the equations (5.1)-(5.2) read as

$$\Delta v + (1 + |y|^2)^l e^v = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{5.9}$$

and

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy = \beta. \tag{5.10}$$
The following result is conjectured in [24].

**Conjecture D.** For $0 < l \leq 2$ and $\beta = (2l + 4)$, solutions to (5.9)- (5.10) must be radially symmetric.

For $-2 < l \leq 0$, the radial symmetry of solutions to (5.9)-(5.10) was shown in [18] and [16] by the moving plane method; while for $l > 0$ the moving plane method does not seem to work, the conjecture was shown in [24] for $0 < l \leq 1$ by using the Alexandrov-Bol inequality. For $2 < l \neq (k-1)(k+2)$, where $k \geq 2$ is an integer, it is pointed out by Lin in [32] that there is a non-radial solution to (5.9)-(5.10). A direct application of Theorem 5.2 to (5.9)-(5.10) leads to an affirmative answer to Conjecture D. Indeed, all solutions to (5.9)-(5.10) must be radially symmetric as long as $\beta \leq 8$.

Another example is the following equation from the study of self-gravitating strings for a massive W-boson model coupled to Einstein theory in account of gravitational effects ([37], [40]).

\[
\Delta v + (1 + |y|^2)e^v = 0 \quad \text{in } \mathbb{R}^2, \tag{5.11}
\]

and

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |y|^2)e^v dy = \beta, \tag{5.12}
\]

where $l > 0$. It is shown in [37] that (5.11)-(5.12) admit a radial solution if and only if

\[
\beta \in \left(4 \max\{1, l\}, 4(l + 1)\right)
\]

and the corresponding radial solution is unique. Furthermore, for $0 < l \leq 1$, the interval above is also optimal for the solvability of (5.11)-(5.12) among non-radial functions. The main known difference between (5.11) and (5.9) is that the latter possesses radial solutions for a larger range of $\beta$ which is at least $(2l + 2, 4l + 4)$, and has multiple radial solutions when $l > 2$ and $\beta \in (\beta, 4l)$ for some $\beta \in (2l + 2, 2l + 4)$, which also implies the existence of non-radial solutions for (5.9) for $l > 2$ (see [21], [32]). While the former has a radial solution only for $\beta \in \left(4 \max\{1, l\}, 4(l + 1)\right)$, which is also unique. In particular, no non-radial solution is known in this case.

Theorem 5.2 implies that solutions to (5.11)-(5.12) must be radially symmetric when $\beta \leq 8$. As a consequence, the solvability range of $\beta$ among non-radial functions must be $\beta > 4 \max\{1, l\}$ when $l \leq 2$.

**Proof of Theorem 1.3** Let $u$ be a solution of (1.9)-(1.10). Without loss of generality we may assume $P = (0, 0, 1)$. Now let $\Pi : S^2 \to \mathbb{R}^2$ be the stereographic projection with north pole at $P = (0, 0, 1)$. Define

\[
v(y) = u(\Pi^{-1} y) - \ln \left( \int_{S^2} e^v \omega \right) + \ln(16\pi(3 + \alpha)) - 3 \ln(1 + |y|^2), \tag{5.13}
\]

where $y = \Pi(x)$. Then $u$ is a solution of (1.9)-(1.10) if and only if $v$ satisfies

\[
\begin{align*}
\Delta v + (1 + |y|^2)e^v &= 0 \quad \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} (1 + |y|^2)e^v dy &= 4\pi(\alpha + 3).
\end{align*}
\]
Therefore it follows from Theorem 5.2 that \( v \) is radially symmetric about the origin. Hence \( u \) is axially symmetric with respect to \( \overrightarrow{OP} \) and the proof is complete. \( \square \)

**Proof of Theorem 1.4.** Without loss of generality we may assume that \( \overrightarrow{n} = (0, 0, 1) \). Let \( \Pi : S^2 \to \mathbb{R}^2 \) be the stereographic projection with north pole at \( \overrightarrow{n} = (0, 0, 1) \). Define

\[
v(y) = u(\Pi^{-1}(y)) \quad \text{for} \quad y \in \mathbb{R}^2.
\]

Then \( v \) satisfies

\[
\Delta v + \frac{J^2(y) \exp(\alpha v - \gamma \psi(y))}{\int_{\mathbb{R}^2} J^2(y) \exp(\alpha v - \gamma \psi(y)) dy} - \frac{J^2(y)}{4\pi} = 0 \quad \text{for} \quad y \in \mathbb{R}^2, \tag{5.14}
\]

where

\[
J(y) = \frac{2}{1 + |y|^2} \quad \text{and} \quad \psi(y) = \frac{|y|^2 - 1}{|y|^2 + 1}.
\]

Now define

\[
w(y) := \frac{\alpha}{2} \left( v(y) - \frac{1}{4\pi} \ln(1 + |y|^2) \right) + c,
\]

with

\[
c = \frac{1}{2} \left( \gamma + \ln \left( \frac{2}{\alpha} \int_{\mathbb{R}^2} J^2(y) e^{\alpha v - \gamma \psi} \right) \right).
\]

Then we have

\[
\Delta w(y) + K(y) e^w = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{5.15}
\]

and

\[
\int_{\mathbb{R}^2} K(y) e^w dy = \alpha,
\]

where

\[
K(y) = 2(1 + |y|^2)^{-2 + \frac{\alpha}{4\pi}} e^{\gamma J(y)}. \tag{5.16}
\]

Now we compute

\[
\Delta (\ln K(y)) = \frac{4(-2 + \frac{\alpha}{4\pi})}{(1 + |y|^2)^3} + \frac{8\gamma(|y|^2 - 1)}{(1 + |y|^2)^3}.
\]

Since the right hand side of the above equation is nonnegative for \( 0 \leq \gamma \leq \frac{\alpha}{8\pi} - 1 \), it follows from Theorem 5.2 that \( w \) is radially symmetric about the origin. \( \square \)

**Acknowledgement** The authors would like to thank Professors Alice Chang, Nassif Ghoussoub, Yanyan Li, Fernando Marques, Richard Schoen, Paul Yang for their interests and helpful comments on the earlier draft of the paper. The first author is partially supported by a Simons Foundation Collaborative Grant (Award #199305) and NSFC grant No 11371128.
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