Measures of the full Hausdorff dimension for a general Sierpiński carpet

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Abstract

The measure of the full dimension for a general Sierpiński carpet is studied. In the first part of this study, we give a criterion for the measure of the full Hausdorff dimension of a Sierpiński carpet. Meanwhile, it is the conditional equilibrium measure of zero potential

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with respect to some Gibbs measure $\nu_\alpha$ of matrix-valued potential $\alpha N$ (defined later). On one hand, this investigation extends the result of [17] without condition (H). On the other hand, it provides a checkable condition to ensure the existence and uniqueness of the measure of the full Hausdorff dimension for a general Sierpiński carpet.

In the second part of this paper we give a criterion for the Markov projection measure and estimate its number of steps by means of the induced matrix-valued potential. The results enable us to answer some questions which arise from [4] and [1] on the projection measure and factors.

**Keywords:** Sofic measure, Sierpiński carpet, matrix-valued potential, Gibbs measure, $\alpha$-weighted thermodynamic formalism

**MSC:** 37D35, 37C45

1 Introduction and main results

Let $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ which is invariant under the endomorphism

$$T = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$$

and corresponds to a shift of finite type. Denote the set of digits as follows:

$$D = \{1, \ldots, m\} \times \{1, \ldots, n\} \quad (1)$$

For $(d_k)_{k=1}^\infty \in D^\mathbb{N}$ there corresponds a point in $T^2$ via what may be called ”base $T$ representation” [12].

$$R_T((d_k)) = \sum_{k=1}^\infty \begin{pmatrix} m^{-k} & 0 \\ 0 & n^{-k} \end{pmatrix} d_k.$$ 

Any 0-1 matrix $A$ with rows and columns indexed by $D$ defines a shift of finite type (SFT, for short) and let $K_T(A)$ be its image under $R_T$, i.e.,

$$K_T(A) = \{R_T((d_k)) \mid A(d_k, d_{k+1}) = 1 \text{ for } k \geq 1\}.$$ 

We say $K_T(A)$ is a Sierpiński carpet and denote by $Z = Z_{(m, n)}(A)$ the Sierpiński carpet for a given pair $(m, n)$ and transition matrix $A$. To avoid confusion we will call this a Markov Sierpiński carpet. McMullen [16] computes the Hausdorff dimension according to the following formula.
Theorem 1 ([16]). Let $Z$ be a Markov Sierpiński carpet. Construct $n$ matrices $A_1, \cdots, A_n$ which are indexed by $D$ as follows: $A_j(d, d') = A(d, d')$, if the second coordinate of $d' \in D$ is $j$ and $A_j(d, d') = 0$, otherwise. Then the Hausdorff dimension of $Z$ can be formulated as the following formula.

$$\dim_H Z = \frac{1}{\log n} \lim_{k \to \infty} \frac{1}{k} \log \sum_{1 \leq i_0, \cdots, i_{k-1} \leq n} \|A_{i_0} \cdots A_{i_{k-1}}\|^\alpha,$$  \hspace{1cm} (2)

where $\alpha = \frac{\log n}{\log m} \leq 1$.

Notably, Kenyon and Peres [12] extend formula (2) to a sofic Sierpiński carpet. In view of (2), since $\{A_1, \ldots, A_n\}$ are collections of matrices, our goal in this investigation is to look more closely at how it relates to the thermodynamic formalism with the matrix-valued potential function. On the other hand, the more we know about the structure of $\{A_1, \ldots, A_n\}$ also enables us to establish more information about the projection space (defined later).

First, we recall the results of Olivier [17] in the study of the full Hausdorff dimension of sofic or Markov Sierpiński carpets. Let $Z$ be a Markov or sofic shift, let $\sigma_Z : Z \to Z$ be its shift map, and the author defines the so-called $(H)$ condition on $\sigma_Z$. $\sigma_Z$ is said to satisfy the condition $(H)$ if the $y$-axis projection $\pi_y \mu$ of the Parry measure $\mu$ on $Z$ is a $\phi$-conformal measure of some normalized potential $\phi : Y \to \mathbb{R}$. The condition therein was imposed to ensure that the invariant measures of the full Hausdorff dimension are the equilibrium states of some potential function, and the Hausdorff dimension formula (3) on $Z$ holds.

$$\dim_H Z = \frac{\log m}{\log n} \cdot \frac{h_{top}(Z)}{\log m} + \frac{P(\sigma_Y, \alpha \phi)}{\log n}. \hspace{1cm} (3)$$

In the first part of this paper, we define an induced matrix-valued potential $N$ on $Y$, and present the criterion for the existence and uniqueness for the full Hausdorff dimension on $Z$. Meanwhile, we derive the analogous formula for the Hausdorff dimension (see (6)). We emphasize here that the $(H)$ condition may not be satisfied in our assumption, however, (6) still holds. That is, (6) holds under extensive conditions, namely, the irreducibility of the induced matrix-valued potential function.

Before formulating our main first result, we give some definitions. Let $(m, n) \in \mathbb{Z}^2_+$ and define two orders $\prec_x$ and $\prec_y$ on $D$: we say $d \prec_x d'$ if
$d_1 < d'_1$ or $d_1 = d'_1$ and $d_2 < d'_2$. According to this order, every $d \in D$ has a unique number on $\{1, \ldots, mn\}$, we denote by $\Xi^{(z)} : D \to \{1, \ldots, mn\}$ the map which assigns each element in $D$ to the unique number in $\{1, \ldots, mn\}$. Define the order $<_{y}$ in the same fashion: $d <_{y} d'$ if $d_2 < d'_2$ or $d_2 = d'_2$ and $d_1 < d'_1$. Let $\Xi^{(y)} : D \to \{1, \ldots, mn\}$ be also defined similarly, we denote by $\tau_{(m,n)}$ the permutation on $D$: $\tau(d) = d'$ if $\Xi^{(x)}(d) = \Xi^{(y)}(d')$ and denote by $P_{(m,n)}$ the permutation matrix associated with the permutation $\tau_{(m,n)}$.

Let $P = P_{(m,n)}$, define $B = PAP^{-1}$ and regard $B$ as $n \times n$ system with entries are $m \times m$ matrices. That is

$$B = \begin{bmatrix}
B^{(1,1)} & \cdots & B^{(1,n)} \\
\vdots & \ddots & \vdots \\
B^{(n,1)} & \cdots & B^{(n,n)}
\end{bmatrix}
$$

$$= \begin{bmatrix}
(PAP^{-1})^{(1,1)} & \cdots & (PAP^{-1})^{(1,n)} \\
\vdots & \ddots & \vdots \\
(PAP^{-1})^{(n,1)} & \cdots & (PAP^{-1})^{(n,n)}
\end{bmatrix} \tag{4}
$$

The matrix-valued potential function $N = (N_{ij})_{i,j=1}^{n}$ arising from $B$ is defined:

$$N = (N_{ij})_{i,j=1}^{n} = (B^{(i,j)})_{i,j=1}^{n} = (PAP^{-1})^{(i,j)}_{i,j=1}^{n}. \tag{5}
$$

We adapt the name from [4, 5] to call $N$ the induced (matrix-valued) potential on $Y$. The normalized induced (matrix-valued) potential $\bar{N} = (\bar{N}_{ij})_{i,j=1}^{n}$ is also defined by $\bar{N}_{ij} = \rho_A^{-1} N_{ij}$ for all $1 \leq i, j \leq n$, where $\rho_A$ denotes the maximal eigenvalue of matrix $A$.

A family of $n \times n$ matrices $(N_i)_{i \in S}$ with entries in $\mathbb{R}$ is said to be irreducible over $\mathbb{R}^n$ if there is no non-zero proper linear subspace $V$ of $\mathbb{F}^n$ such that $N_i V \subseteq V$ for all $i \in S$. The first result of this investigation is the following.

**Theorem 2.** Let $Z$ be a Markov Sierpiński carpet and $N = (N_{ij})_{i,j=1}^{n}$ be the induced potential from $A$. Assume $N$ is irreducible, then

(i) The following statements are equivalent.

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1 We note here that we use the index form of $A^{(i,j)}$ to denote the $(i, j)$-coordinate of $A$ and $A^{(i,j)}$ is a matrix. And we use the standard form $A(i, j)$ to denote the $(i, j)$-coordinate of $A$ if it is a real value.
(a) $\mu$ is the unique measure of the full Hausdorff dimension.

(b) $\mu$ is the unique conditional equilibrium measure (defined in Section 3.1) of the zero potential function on $Z$ with respect to $\nu_\alpha$, where $\nu_\alpha$ is the unique equilibrium measure of the matrix-valued potential $\alpha N = (||N_J||^\alpha)_{J \in Y}$.

(ii) The following Hausdorff dimension formula holds:

$$\dim_H Z = \frac{h_{top}(Z)}{\log m} + \frac{P(\sigma_Y, \alpha N)}{\log n}$$  \hspace{1cm} (6)

The essential ingredient of the proof in Theorem 2 is that the irreducibility of $N$ ensures the existence of the Gibbs measure $\nu$. Since the Gibbs measure $\nu$ may have infinite memory (cf. [4, 5]), the question arises: which conditions ensure that the measure $\nu$ has finite memory? The structure of a $k$-th higher block induced (matrix-valued) potential $N^{[k]}$ (defined later) plays an important role in answering this question. We denote by $Y_k$ the collection of all possible words in $Y$ of length $k$. For $k \geq 2$, let

$$D^{[k]} = \left\{ \left( \left( d_1^{(1)} \ldots d_1^{(k)}, d_2^{(1)} \ldots d_2^{(k)} \right) : (d_1^{(i)}, d_2^{(i)}) \in D \text{ for all } i = 1, \ldots, k \right) \right\},$$

$$d^{[k]} = (d_1^{[k]}, d_2^{[k]}) = \left( d_1^{(1)} \ldots d_1^{(k)}, d_2^{(1)} \ldots d_2^{(k)} \right)$$

and $A^{[k]} \in \mathbb{R}^{d^{[k]} \times d^{[k]}}$ ($d = m \times n$) be the $k$-th higher block transition matrix from $A$ which is indexed by $D^{[k]}$. We define the permutation matrix $P^{[k]} = P^{[k]}_{(m,n)}$ in the same fashion as $P = P^{[1]}_{(m,n)}$. Set $B^{[k]} = P^{[k]} A^{[k]} (P^{[k]})^{-1}$ and regards $B^{[k]}$ as $n^k \times n^k$ system with entries are $m^k \times m^k$ matrices. The matrix-valued potential function $N^{[k]} = \left(N_J^{[k]} \right)_{J \in Y_{k+1}}$ is defined by

$$N_J^{[k]} = (B^{[k]})^{(J(0,k-1),J(1,k))} \text{ for all } J = J(0,k) \in Y_{k+1}.$$  

We call $N^{[k]}$ the $k$-th higher block induced (matrix-valued) potential on $Y^{[k]}$. Note that $N = N^{[1]}$, and $N^{[i]}$ is defined by $A^{[i]}$ and $P^{[i]}$, for $i = 1, \ldots, k$.
Figure 1: For every matrix $A^{[i]}$, there exists a permutation matrix $P^{[i]}$ such that the induced matrix $N^{[i]}$ is obtained by applying $P^{[i]}$ on $A^{[i]}$.

If $J \in Y_k$ and $0 \leq m, n \leq k$, we use the notation $J(m,n)$ to denote the subword of $J$ from coordinate $m$ to $n$, i.e., $J(m,n) = (j_m \ldots j_n)$ if $J = (j_0 j_1 \ldots j_k)$. For $k \geq 1$ and $J \in Y_n$ with $n \geq k$, $N^{[k]}_J$ stands for the product of matrices of $N^{[k]}$ along the path of $J$, i.e.,

$$N^{[k]}_J = N^{[k]}_{J(1,n)} = \prod_{i=1}^{n-k} N^{[k]}_{J(i,i+k)}$$

for all $J = J(1,n) \in Y_n$.

We say that $N$ satisfies the Markov condition from left of order $k$ if there exists a non-zero linear subspace $\{V_J\}_{J \in Y_k} \subseteq \mathbb{R}^m$ such that $V_{J(0,k-1)} N^{[k]}_{J(0,k)} \subseteq V_{J(1,k)}$ for all $J(0,k-1)$ and $J(1,k) \in Y_k$. $N$ satisfies the Markov condition from right of order $k$ if $(N^{[k]})^t = \left((N^{[k]}_J)^t\right)_{J \in Y_{k+1}}$ satisfies the Markov condition from left, where $A^t$ denotes the transpose of $A$. Finally, say $N$ satisfies the Markov condition if it satisfies either the Markov condition from the left or right for some order $k \in \mathbb{N}$. The following theorem provides a criterion for checking whether $\nu$ is a Markov measure.

**Theorem 3.** Let $Z$ be a Markov Sierpiński carpet and $N = (N_{ij})_{i,j=1}^n$ be the induced matrix-valued potential from $A$. Then, $\nu$ is a $k$-step Markov measure on $Y$ if and only if $N$ satisfies the Markov condition of order $k$. Furthermore, if $\nu$ is a $k$-step Markov measure, then $k \leq m - n$.

We mention here that the inequality $k \leq m - n$ in Theorem 3 is sharp. More precisely, we examine the well-known example of a McMullen carpet (Example 28) in which the induced matrix-valued potential satisfies the Markov condition of order 1. It also follows from Theorem 3 and the fact that $m = 3$ and $n = 2$, that the Markov measure induced from $N$ can only be 1-step.
Compared to Theorem 2, Theorem 3 reveals that the more structured the vector space of $V_J$ from $N^{[k]}$, the more it implies about the property of $\nu$. In other words, Theorem 3 illustrates that the projection measure $\nu$ is Markov if and only if the collection of $(N^{[k]}V_J)_{J \in Y_k}$ is a finite set, which guarantees that the Gibbs measure $\nu$ falls into the finite range. (Readers may consult \cite{[4], [5]} for more detail.)

At this point, a further question arises: If $N$ satisfies the Markov condition of order $k$, what kind of Markov measure is $\nu$? To answer this question, we may assume $m(J(0,k-1),J(1,k)) \in \mathbb{R}$ such that

$$V_{J(0,k-1)}N^{[k]}_{J(0,k)} = m(J(0,k-1),J(1,k))V_{J(1,k)},$$

the following theorem illustrates that the coefficient of $m(J,J')$ helps us to determine what kind of Markov measure $\nu$ is.

**Theorem 4.** If $N$ satisfies the Markov condition of order $k$, then $\nu$ is the unique maximal measure of the subshift of finite type $X_M$ with adjacency matrix $M = [m(J,J')]_{J,J' \in Y_k}$.

Let us return to the Markov or sofic Sierpiński carpet. We recall the two following interesting problems:

(i) When are the Hausdorff and Minkowski dimensions coincident?

(ii) What is the exact value of the Hausdorff dimension?

These two problems seems to have satisfactory answers when $Z$ is a Markov Sierpiński carpet. For (i), Kenyon and Peres show that if $A$ is primitive, then $\dim_H Z = \dim_M Z$ if and only if the unique invariant measure of maximal entropy on $Z$ projects via $\pi_y$ to the unique measure of maximal entropy on $\pi_y(Z)$ (Theorem 20). For problem (ii), if $Z$ is a Markov Sierpiński carpet, let $D' \subseteq D = \{1, \ldots, m\} \times \{1, \ldots, n\}$ be the non-empty subset of $D$. Define

$$K(T, D') = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} m^{-k} & 0 \\ 0 & n^{-k} \end{pmatrix} d_k : d_k \in D' \text{ for all } k \right\}. \quad (7)$$

The Hausdorff dimension of $Z' = K(T, D')$ has a closed form: let $z(j)$ be the number of rectangles in row $j$

$$\dim_H Z' = \frac{1}{\log n} \log \sum_{j=1}^{n} z(j)^\alpha, \text{ where } \alpha = \frac{\log n}{\log m}. \quad (8)$$
In the following, the structure of $N$ helps us to derive the closed formula for a more general Sierpiński carpet and the explicit value for the Hausdorff dimension. Assume that $N$ satisfies the Markov condition of order 1. Define the induced graph and the corresponding induced transition matrix as follows:

Let $T$ be given and $N$ be the induced matrix-valued potential from $T$, then let $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} = \{(i,j)\}_{i,j=1}^{n}$, where $(i,j) = 1$ if $N_{ij}$ is non-zero matrix. We call $G = (\mathcal{V}, \mathcal{E})$ the induced graph. Define

\[
T_{G}(i,j) = \begin{cases} 
1, & \text{if } N_{ij} \neq 0_{m \times m}; \\
0, & \text{otherwise}
\end{cases}
\]

the induced transition matrix corresponds to $G$. Finally, we define $G^{[k]}$ and $T^{[k]} := T_{G^{[k]}}$ in the same fashion if $N$ satisfies the Markov condition of order $k > 1$.

**Theorem 5.** Let $Z$ be a Markov Sierpiński carpet. Assume $N$ satisfies the Markov condition of order $k$ and let $T^{[k]}$ be the induced transition matrix which corresponds to induced graph $G^{[k]}$. Then

(i) $\rho_{M} = \rho_{T^{[k]}}$ if and only if $\dim_{H} Z = \dim_{M} Z$, where $M$ is defined in Theorem 4.

(ii) Let $D' \subseteq D$ and $K(T, D')$ as defined in (7), then (8) holds.

(iii) Define $M^{\alpha} = [m^{\alpha}(J, J')]_{J, J' \in \mathcal{Y}_{k}}$. Then $\dim_{H} Z = \log n \rho_{M^{\alpha}}$, where $\rho_{M}$ is the maximal eigenvalue of $M$ and $\alpha = \log n / \log m$.

**Remark 6.** If $N = (N_{ij})_{i,j=1}^{n}$ is reducible, Proposition 1.4 of [10] demonstrates that one can decompose $N$ to the irreducible components. This reveals that the equilibrium measures for $N$ may not be unique. On the other hand, it can be easily checked whether or not the reducibility of $N = (N_{ij})_{i,j=1}^{n}$ implies the reducibility of $A$ (Since $N$ is extracted from $B$ which is the permutation of $A$). This illustrates that the non-uniqueness for the equilibrium measure of $\nu_{\alpha}$ on $Y$ relates to the non-uniqueness for the maximal measure on $Z$ of $A$.

The rest of the paper is organized as follows. Since the space $Y$ with the induced potential $N$ is no longer $p$-specification (cf. [1, 6, 7]), it is weak $p$-specification instead. We review some known results in [9] for weak $p$-specification shift in Section 2. The detailed proofs for Theorem 2, Theorem 3 and Theorem 4 are presented in Section 3.
In Section 4, the established results for the induced potential $N$ enable us to answer problems raised by Chazottes and Ugaldes [4], and Boyle and Petersen [2]. To be precise, Chazottes and Ugaldes use the ansatz of the induced potential to prove the existence of well-defined potential function, and the corresponding Gibbs measure (BGM [4]) on the projection space under $(H1)$ and $(H2)$. They raise the following problem: When is the factor map not a topological Markov map? On the other hand, Boyle and Petersen raise the following question ([2, Problem 3.3]): Given a procedure to decide, and given a factor map $\pi : \Omega_A \to \Omega_B$, where $\Omega_A$ and $\Omega_B$ are the Markov system induced by the transition matrices $A$ and $B$, how can we know whether $\pi$ is Markovian? Theorem 26 is presented in Section 4 to provide a criterion for determining whether such $\pi$ is Markovian. Finally, we also list some interesting examples, namely, the Blackwell and McMullen examples therein.

## 2 Preliminaries

Let $Z$ be a Markov Sierpiński carpet introduced in Section 1, and define the sliding block code $\Pi_y : D \to \{0, \ldots, n - 1\}$ by

$$\Pi_y(d) = d_2 \text{ if } d = (d_1, d_2) \in D.$$ 

Denote by $G_A$ the graph associated with the adjacent matrix $A$. Then the pair $\mathcal{G} = (G_A, \Pi_y)$ forms a one-block factor map from alphabet $Z$ to $Y = (\Pi_y)_\infty (Z)$ as follows:

$$(\Pi_y)_\infty (Z) = \left\{ y \in \{0, \ldots, n - 1\}^\mathbb{N} : y = (\Pi_y)_\infty (z) \text{ for some } z \in Z \right\}, \quad (9)$$

where $(\Pi_y)_\infty (z) = \Pi_y(z_0)\Pi_y(z_1) \cdots \in \{1, \ldots, n\}^\mathbb{N}$. In the following, we write $\pi_y$ instead of $(\Pi_y)_\infty$.

We say that $X$ satisfies the criterion for weak specification [9] if there exists $p \in \mathbb{N}$ such that, for any two words $I$ and $J \in X^* = \cup_{n \in \mathbb{N}} X_n$, where $X_n$ is collection admissible words in $X$ of length $n$, there is a word $K$ of length not exceeding $p$ such that the word $IKJ \in X^*$.

Denote by $\mathcal{D}_w(X, p)$ [9] the collection of functions $f : X^* \to [0, \infty)$ such that $f(I) > 0$ for at least one $I \in X^*$ and there exists $0 < c \leq 1$ so that

1. $f(IJ) \leq c^{-1} f(I) f(J)$ for all $I, J \in X^*$. 

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(2) For all $i, j \in X^*$, there exists $K \in \mathcal{U}^p_{i=0}X_i$ such that $IKJ \in X^*$ and $f(IKJ) \geq cf(I) f(J)$.

One can easily check that $D_w(X, p) \neq \emptyset$ if and only if $X$ satisfies the weak $p$-specification. Let $N = (N_{ij})_{i,j=1}^n$ be the induced potential from $A$ on $Y$ and $f(J) = \|N_J\|$, by Proposition 2.8 and Lemma 2.1 of [3], we see that $f \in D_w(Y, p)$. It also follows from Theorem 5.5 [9], $\alpha N = (f(J)^\alpha)_{j \in Y}$ has a unique equilibrium $\nu_\alpha$. Finally, Theorem 6.1 of [9] shows that if $\nu_\alpha$ is the unique equilibrium measure of $\alpha N$, the zero potential function on $Z$ has a unique $a$-weighted equilibrium state which is the conditional equilibrium states of $\nu_\alpha$ with respect to $\Phi$. We present some useful Lemmas as follows.

**Lemma 7** ([9, Lemma 5.2]). Suppose $f \in D_w(X, p)$. Then the following two properties hold:

(i) There exists a constant $\gamma > 0$ such that for each $I \in X^*$, there exists $i, j \in A(X) \text{ such that } f(iI) \geq \gamma f(I)$ and $f(Ij) \geq \gamma f(I)$, where $A(X)$ denotes the symbol set on $X$.

(ii) Let $u_n = \sum_{j \in X} f(J)$. Then the limit $u = \lim_{n \to \infty} (1/n) \log u_n$ exists and $u_n \approx \exp(nu)$.

**Lemma 8.** Let $Z = Z_{(m,n)}(A)$ with $A \in \mathbb{R}^{d \times d}$ be irreducible, where $d = m \times n$. Let $A_1, \ldots, A_n$ be as defined in Theorem [7] and $B_k = P A_k P^{-1}$, for $k = 1, \ldots, n$, and we write $B_k = \left(B_k^{(i,j)}\right)_{i,j=1}^n$. Then $B_k^{(i,j)} = N_{ij}$, if $j = k$ and $1 \leq i \leq n$, and $B_k^{(i,j)} = 0_{m \times m}$ otherwise, where $0_{m \times m}$ denotes the $m \times m$ matrix with all entries being 0’s.

**Proof.** It follows from the definition of $N$ and the permutation $P$, it can be easily checked that the index of the matrix $B = \left(B^{(i,j)}\right)_{i,j=1}^n$ equals to the $D$ with the order $\prec_{y}$. Define

$$E_{d_2} = \{d_1 : (d_1, d_2) \in D \text{ with } \Pi_y(d) = d_2\}.$$

We see that $B^{(i,j)}$ is indexed by $E_i \times E_j$ and $B^{(i,j)}(p, q) = 1$ if $A((p, i), (q, j)) = 1$ with $p \in E_i$ and $q \in E_j$. Take $1 \leq k \leq n$, it follows from the definition of $A_k$: $A_k(d, d') = A(d, d')$ if the second coordinate of $d' \in D$ is $k$, it means that $B_k^{(i,j)}$ is indexed by $E_i \times E_j$ for which $B_k^{(i,j)}(p, q) = 1$ if and only if $A((p, i), (q, j)) = 1$ and $j = k$. Therefore,

$$B_k^{(i,j)}(p, q) = \left(P A_k P^{-1}\right)(p, q) = N_{ik} \text{ for all } 1 \leq i \leq n.$$
The proof is thus completed.

**Theorem 9.** Let $Z = Z_{(m,n)}(A)$ be a Markov Sierpiński carpet with $A$, assume that $N$ the induced potential from $A$ is irreducible. Then,

$$\dim_H Z = \frac{1}{\log n} \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in Y_n} \|N_J\|^\alpha$$

where $\alpha = \log n / \log m$.

**Proof.** Let $Z = Z_{(m,n)}(A)$ be given and $\pi_y : Z \to Y$ be a sliding block code from $Z$ to $Y$ as in (9) and recall that $d = m \times n$. We first show that there exists $c > 0$ such that for all $k \in \mathbb{N}$ and $J = (j_0, \ldots, j_{n-1}) \in Y_n$

$$c^{-1} \left\| N_{j_0 j_1 \ldots j_{n-2} j_{n-1}} \right\| \leq \| B_J \| = \| A_J \| \leq c \left\| N_{j_0 j_1 \ldots j_{n-2} j_{n-1}} \right\|,$$

where $B_J = B_{j_0} B_{j_1} \cdots B_{j_{n-1}}$ and $B_k$ is defined in Lemma 8. Indeed, since $P$ is a permutation, we have

$$\| A_J \| = 1_d^t A_{j_0} A_{j_1} \cdots A_{j_{n-1}} 1_d = 1_d^t P^{-1} B_{j_0} B_{j_1} \cdots B_{j_{n-1}} P 1_d$$

$$= 1_d^t B_{j_0} B_{j_1} \cdots B_{j_{n-1}} 1_d = \| B_J \|.$$

Therefore

$$\| A_J \| = \| B_J \| \text{ for all } J \in Y_n.$$  \hfill (13)

Since $A$ is irreducible we conclude that $B$ is also irreducible, we have $c_1 = \max_{0 \leq u, v \leq m - 1} \max_{0 \leq i \leq n - 1} (\sum_{i=1}^{n} N_{ij}) (u, v) > 0$. According to Lemma 8 we also have

$$\| B_J \| = 1_d^t B_{j_0} B_{j_1} \cdots B_{j_{n-1}} 1_d = 1_d^t \left[ \begin{array}{cccc} 0 & \cdots & N_{j_0} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & N_{j_{n-1}} & \cdots & 0 \end{array} \right] \times \ldots$$

$$\times B_{j_{n-2}} \times \left[ \begin{array}{cccc} 0 & \cdots & N_{1j_{n-1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & N_{n_{j_{n-1}}} & \cdots & 0 \end{array} \right] 1_d$$

$$= 1_d^t \left( \sum_{i=1}^{n} N_{ij_0} \right) N_{j_0 j_1} \cdots N_{j_{n-2} j_{n-1}} 1_d^t.$$
Therefore,
\[
\|B_J\| = 1^t_m \left( \sum_{i=1}^{n} N_{ij} \right) N_{j_0j_1} \cdots N_{j_{n-2}j_{n-1}} 1_m \leq c_1 \|N_J\|. \tag{15}
\]

On the other hand, since \(N\) is irreducible, then Lemma 7-(i) is applied to show that there exists a \(1 \leq i \leq n\) such that
\[
\|B_J\| \geq \gamma 1^t_m N_{j_0j_1} \cdots N_{j_{n-2}j_{n-1}} 1_m = \gamma \|N_J\|. \tag{16}
\]

Combining (16), (13), (15), Lemma 7-(ii), Theorem 1 and (11) yields
\[
\dim_H Z = \frac{1}{\log n} \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in Y} \|A_J\|^\alpha = \frac{1}{\log n} \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in Y} \|B_J\|^\alpha = \frac{1}{\log n} \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in Y} \|N_J\|^\alpha \quad \text{(Lemma 7(ii)).}
\]

The theorem is thus proved. \(\square\)

**Remark 10.** 1. We define the limit in (10) as the topological pressure
\[ P(\sigma_Y, \alpha N) \text{ on } Y \] with respect to the potential function \(N\).

2. We note here that Yayama [18] derived a similar result as in Theorem 9 ([18, Theorem 4.4-(1)]). To be precise, the author proves the following:
\[
\dim_H Z = \frac{1}{\log n} \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in Y} \left| \pi^{-1}_y (y_1 \cdots y_n) \right|^\alpha \tag{17}
\]
\[
= \frac{1}{\log n} \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in Y} \|N_J\|^\alpha.
\]

The second equality comes from Lemma 8. We emphasize here that the potential \(\alpha N = f^\alpha(J)\) on \(Y\) is not necessary continuous.

### 2.1 Existence and uniqueness of Gibbs measures for matrix-valued potential

This section presents the existence and uniqueness of the Gibbs measure for the matrix-valued potential function with some irreducible condition. Feng
and Kaenmaki characterize the structure of equilibrium and the Gibbs measure for matrix-valued potentials for irreducible $N$.

**Theorem 11.** Let $N = (N_i)_{i \in S}$ be a family of $d \times d$ matrices with entries in $\mathbb{R}$. If $N = (N_i)_{i \in S}$ is irreducible. Then for each $\alpha > 0$, $P(\sigma_Y, \alpha N)$ has a unique $\alpha$-equilibrium measure $\mu_\alpha$ which satisfies the Gibbs property: $\forall n \in \mathbb{N}$ and $J \in Y_n$, there exists $c > 0$ such that

$$c^{-1} \exp(-nP(\sigma_Y, \alpha N)) \|N_J\|^\alpha \leq \mu_\alpha([J]) \leq c \exp(-nP(\sigma_Y, \alpha N)) \|N_J\|^\alpha.$$  

(18)

The following theorem illustrates the existence of the Gibbs measure on $Y$ with respect to the induced potential function $N$. Suppose $X$ is a shift space, denote by $\mathcal{M}(X, \sigma_X)$ the collection of all $\sigma_X$-invariant measures on $X$.

**Corollary 12.** Under the same assumptions of Theorem 11, then, for any $\alpha > 0$, there exists a unique $\alpha$-equilibrium $\nu_\alpha \in \mathcal{M}(Y, \sigma_Y)$ which satisfies (18). Furthermore, if $\bar{\nu}_\alpha \in \mathcal{M}(Y, \sigma_Y)$ is the equilibrium measure of $\alpha \bar{N}$, then $\nu_\alpha = \bar{\nu}_\alpha$.

**Proof.** Since $N$ is irreducible, the existence of the unique $\alpha$-equilibrium $\nu_\alpha$ which satisfies (18) is the immediate consequence of Theorem 11. From the definition of $N$ and $\bar{N}$ we know that $N$ is irreducible if and only if $\bar{N}$ is irreducible. Then there exists a unique $\alpha$-equilibrium measure $\bar{\nu}_\alpha$ which satisfies (18) for $\alpha \bar{N}$, i.e., $\forall J \in Y_n$ and $n \in \mathbb{N}$, we have

$$d^{-1} \exp(-nP(\sigma_Y, \alpha \bar{N})) \|\bar{N}_J\|^\alpha \leq \bar{\nu}_\alpha([J]) \leq d \exp(-nP(\sigma_Y, \alpha \bar{N})) \|\bar{N}_J\|^\alpha,$$

for some $d > 0$. We claim that $\nu_\alpha = \bar{\nu}_\alpha$. Indeed, for each $n \in \mathbb{N}$ and $J \in Y_n$ we have

$$\nu_\alpha([J]) \leq c \exp(-nP(\sigma_Y, \alpha N)) \|N_J\|^\alpha \quad \text{(From (18))}$$

$$= c \exp(-nP(\sigma_Y, \alpha N)) \rho_A^{-\alpha n} \|\bar{N}_J\|^\alpha \quad \text{(since $\bar{N} = \rho_A^{-1} N$)}$$

$$= c \exp(-nP(\sigma_Y, \alpha \bar{N})) \rho_A^{-\alpha n} \|\bar{N}_J\|^\alpha$$

$$= c \exp(-nP(\sigma_Y, \alpha \bar{N})) \|\bar{N}_J\|^\alpha$$

$$\leq cd\bar{\nu}_\alpha([J]).$$  

(19)

Similarly, we have

$$\nu_\alpha([J]) \geq c^{-1}d^{-1} \bar{\nu}_\alpha([J]).$$  

(20)

The claim follows by combining (19), (20) and the fact that $\nu_\alpha$ and $\bar{\nu}_\alpha$ are both ergodic. This completes the proof.
2.2 Sofic measures and linear representable measures

Let \((X, \sigma_X)\) and \((Y, \sigma_Y)\) be subshifts and \(\pi : X \to Y\) be a sliding block code, each measure \(\mu \in \mathcal{M}(X, \sigma_X)\) determines a measure \(\pi \mu \in \mathcal{M}(Y, \sigma_Y)\) by

\[(\pi \mu)(E) = \mu(\pi^{-1}(E)), \quad \forall E \subset Y.\]

If \(\mu\) is a Markov measure, then \(\pi \mu\) is called a sofic measure (cf. [2]). Let \(B \in \mathbb{R}^d\) be an irreducible matrix with spectral radius \(\rho_B\) and positive right eigenvector \(r\), the stochasticization of \(B\) is the stochastic matrix

\[B := \text{stoch}(B) = \frac{1}{\rho_B} D^{-1} BD,\]

where \(D\) is the diagonal matrix with diagonal entries \(D(i,i) = r(i)\). A measure \(\mu\) on \(X\) is called linear representable with dimension \(d\) if there exists a triple \((x, P, y)\) with \(x \in \mathbb{R}^n\) being a \(1 \times d\) row vector, \(y \in \mathbb{R}^n\) is a \(d \times 1\) column vector and \(P = (P_i)_{i \in A(X)}\), where \(P_i \in \mathbb{R}^{d \times d}\) such that for all \(I = (i_0, \ldots, i_{n-1}) \in X_n\), the measure \(\mu\) can be characterized as the following form:

\[\mu([I]) = x P_I y,\]

where \(P_I = P_{i_0} P_{i_1} \cdots P_{i_{n-1}}\) (readers may refer to [2] for more detail). The triple \((x, P, y)\) is called the linear representation of the measure \(\mu\).

**Proposition 13** ([2, Theorem 4.20]). Let \(X = X_A\) be a Markov shift with adjacent matrix \(A \in \mathbb{R}^{n \times n}\) which is irreducible and \(\pi : X \to Y\) be a factor induced from one block map \(\Pi : A(X) \to A(Y)\), i.e., \(\pi = \Pi_{\infty}\). Let \(A = \text{stoch}(A)\) and \(l\) be the left eigenvector of \(A\). Then

(i) The Markov measure \(\mu_A\) on \(X\) is the linear representable measure with respect to the triple \((x, P, y)\), where \(x = l, y = 1_n\), where \(P\) is generated by \((P_i)_{i \in A(X)} = (\mathbb{A}_i)_{i \in A(X)}\) for which

\[P_I = P_{i_0} \cdots P_{i_{n-1}} = \mathbb{A}_{i_0} \cdots \mathbb{A}_{i_{n-1}}, \quad \text{for all } I = (i_0, \ldots, i_{n-1}) \in X_n\]

where \(\mathbb{A}_k(i, j) = A(i, j)\) if \(j = k\) and \(\mathbb{A}_k(i, j) = 0\) otherwise.

(ii) The push forward measure \(\nu = \pi \mu\) is the linear representable with respect to the triple \((x, Q, y)\), where \(x = l, y = 1_n\) and \(Q\) is generated by \((Q_j)_{j \in A(Y)} = (\mathbb{A}_j)_{j \in A(Y)}\) for which \(\mathbb{A}_k(u, v) = A(u, v)\) if \(\Pi(v) = k\) and \(\mathbb{A}_k(u, v) = 0\) otherwise.
The following Proposition presents that the push forward measure of maximal measure on $Z$ is the equilibrium measure with $N$. Recall that $A_1, \ldots, A_n$ are induced from $A$ in Theorem \([1]\) define $\hat{A}_j = (stoch(A))_j$ for $j = 1, \ldots, n$.

**Proposition 14.** Let $Z$ be a Markov Sierpiński carpet with $A$ being irreducible and the induced potential $N$ also being irreducible. Let $\mu_A$ be the unique Markov measure of $A$, then $\nu = \bar{\nu} = \pi_y \mu_A$.

**Proof.** Since $\nu = \bar{\nu}$ from Corollary \([12]\) it suffices to show that $\nu = \pi_y \mu_A$. Since $\pi_y \mu_A$ is a linear representable sofic measure by Proposition \([13]\) Let $\hat{A} = stoch(A)$ and $l$ be the $1 \times d$ ($d = m \times n$) left eigenvector of $\hat{A}$ with respect to the maximal eigenvalue 1. It follows from Proposition \([13]\) that the triple $(l, \hat{A}, 1_d)$ defines a linear representable measure $\pi_y \mu_A$, where $\hat{A} = \left(\hat{A}_j\right)_{j=1}^n$ with $\hat{A}_j(u, v) = \hat{A}(u, v)$ if $\pi_y(v) = j$, $\hat{A}_j(u, v) = 0$ otherwise. That is, 

$$
\pi_y \mu_A(J) = l \hat{A}_{j_0} \hat{A}_{j_1} \ldots \hat{A}_{j_{n-1}} 1_d, \text{ for all } J = (j_0 \ldots j_{n-1}) \in Y_n.
$$

Under the same argument of the proof in Theorem \([9]\) and log $\rho_A = P(\sigma_Y, N)$. For $J = (j_0 \ldots j_{n-1}) \in Y_n$ we have 

$$
\pi_y \mu_A([J]) = l \hat{A}_{j_0} \hat{A}_{j_1} \ldots \hat{A}_{j_{n-1}} 1_d
$$

$$
= \rho_A^{-n} lD^{-1}A_{j_0}A_{j_1} \ldots A_{j_{n-1}} 1_d
$$

$$
\leq \max_{1 \leq j \leq n} \{r^{-1}(j)\} \max_{1 \leq j \leq n} \{r(j)\} \rho_A^{-n} 1_d A_{j_0} A_{j_1} \ldots A_{j_{n-1}} 1_d
$$

$$
= c_1 \rho_A^{-n} \|A_{j_0} A_{j_1} \ldots A_{j_{n-1}}\|
$$

$$
\leq c_2 \rho_A^{-n} \|N_{j_0 j_1} N_{j_1 j_2} \ldots N_{j_{n-2} j_{n-1}}\|
$$

$$
= c_2 \exp \left(-nP(\sigma_Y, N)\right) \|N_{j_0 j_1} N_{j_1 j_2} \ldots N_{j_{n-2} j_{n-1}}\|
$$

$$
\leq c_3 \nu([J]) \text{ (Corollary \([12]\))}
$$

Similarly, since $N$ is irreducible

$$
\pi_y \mu_A([J]) = l \hat{A}_{j_0} \hat{A}_{j_1} \ldots \hat{A}_{j_{n-1}} 1_d
$$

$$
= \rho_A^{-n} lD^{-1}A_{j_0}A_{j_1} \ldots A_{j_{n-1}} 1_d
$$

$$
\geq \min_{1 \leq j \leq n} \{r^{-1}(j)\} \min_{1 \leq j \leq n} \{r(j)\} \rho_A^{-n} 1_d A_{j_0} A_{j_1} \ldots A_{j_{n-1}} 1_d
$$

$$
\geq c_5 \rho_A^{-n} \|N_{j_0 j_1} N_{j_1 j_2} \ldots N_{j_{n-2} j_{n-1}}\|
$$

$$
\geq c_6 \nu([J])
$$

Since $\pi_y \mu_A$ and $\nu$ are ergodic, $\pi_y \mu_A = \nu$. The proof is completed. \[\square\]
3 Proofs

This section presents the detailed proofs for Theorem 2, Theorem 3, Theorem 4 and Theorem 5.

3.1 Proof of Theorem 2

We first review some background knowledge of $a$-weighted thermodynamic formalism proposed by Barral and Feng [1] for $p$-specification shift space and by Feng [9] for the weak $p$-specification case. For $a = (a, b)$, the $a$-weighted pressure is defined as follows:

$$P_a(\sigma_X, \Phi) = \sup \{ \Phi_*(\eta) + ah_\eta(\sigma_X) + bh_{\pi \eta}(\sigma_Y) : \eta \in M(X, \sigma_X) \}.$$  (21)

Define the collection of equilibrium measures and $a$-weighted equilibrium as follows:

$$I(\Phi) \ = \\{ \mu \in M(X, \sigma_X) : \mu \text{ is an equilibrium measure of } \Phi \},$$

$$I(\Phi, a) \ = \\{ \mu \in M(X, \sigma_X) : \mu \text{ attain the supremum of (21)} \}. $$

Let $\pi : X \to Y$ be a factor, the conditional equilibrium measure $\mu \in M(X, \sigma_X)$ of $\Phi$ with respect to $\nu$ if $\pi \mu = \nu$, and $\mu$ satisfies the conditional variational principle, i.e.,

$$\Phi_*(\mu) + h_\mu(\sigma_X) = \sup \{ \Phi_*(\eta) + h_\eta(\sigma_X) : \eta \in M(X, \sigma_X), \pi \eta = \nu \}. $$

Denote by $I_\nu(\Phi)$ the collection of all conditional equilibrium measure of $\Phi$ with respect to $\nu$

**Theorem 15** ([9, Corollary 3.11], [1, Theorem 1.1]). Let $\Phi = (\log \phi_n)_{n=1}^\infty$ be a subadditive potential function on $X$. For all $J \in Y_n$, $n \in \mathbb{N}$, define $\psi_n : X_n \to \mathbb{R}$ as follows

$$\psi_n(J) = \sum_{I \in X_n : \pi(I) = J} \phi(I).$$

Let $\Psi = (\log \psi_n)_{n=1}^\infty$ be the collection of $\psi_n$. Then

(i) $P_a(\sigma_X, \Phi) = (a + b)P(\sigma_Y, (\frac{a}{a+b})\Psi)$
(ii) \( \mu \in \mathcal{I}(\Phi, a) \) if and only if \( \mu \circ \pi^{-1} \in \mathcal{I}\left(\frac{a}{a+b}\right) \) and \( \mu \in \mathcal{I}_{\mu \circ \pi^{-1}}(\frac{1}{a}\Phi) \), where \( \frac{a}{a+b}\Psi = \left(\frac{a}{a+b}\log \left(\psi_{n}\right)_{n=1}^{\infty}\right) \) and \( \frac{1}{a}\Phi = \left(\frac{1}{a}\log \left(\phi_{n}\right)_{n=1}^{\infty}\right) \).

**Proof of Theorem 2.** Step 1. \((i) : (a) \Rightarrow (b)\) We assume that \( \mu \) is the invariant measure of the full Hausdorff dimension, i.e., \( \dim_{H} Z = \dim_{H} \mu \). We claim that \( \mu \) attains the supremum of \((21)\) with \( a = (\alpha, 1 - \alpha) \in \mathbb{R}^{2} \) and zero potential \( \Phi \). Indeed, under the same argument of Theorem 9 there exists a constant \( c > 0 \) such that for all \( y \in Y \), we have

\[
c^{-1} \| N_{y|n} \| \leq \psi_{n}(y|n) \leq c \| N_{y|n} \|, \tag{22}\]

where \( y|n = (y_{0}, \ldots, y_{n-1}) \in Y_{n} \). Combining Theorem 9 and Theorem 15 we have

\[
\dim_{H} \mu = \dim_{H} Z = \frac{1}{\log n} P(\sigma_{Y}, \alpha N) \quad \text{(Theorem 9)}
= \frac{1}{\log n} P(\sigma_{Y}, \Psi) = \frac{1}{\log n} P^{a}(\sigma_{Z}, \Phi). \tag{23}
\]

Combining \((23)\) and the Ledrappier-Young formula for the Hausdorff dimension of measure \( \mu \) \([13, 14]\) yields

\[
\frac{1}{\log n} P^{a}(\sigma_{Z}, \Phi) = \dim_{H} \mu
= \frac{h_{\mu}(\sigma_{Z})}{\log m} + \left(\frac{1}{\log n} - \frac{1}{\log m}\right) h_{\pi_{\mu}}(\sigma_{Y})
= \frac{h_{\mu}(\sigma_{Z})}{\log m} + \frac{1}{\log n} \left(1 - \frac{1}{\log m}\right) h_{\pi_{\mu}}(\sigma_{Y})
= \frac{h_{\mu}(\sigma_{Z})}{\log m} + \frac{1}{\log n} (1 - \alpha) h_{\pi_{\mu}}(\sigma_{Y}).
\]

Therefore,

\[
P^{a}(\sigma_{Z}, \Phi) = \frac{\log n}{\log m} h_{\mu}(\sigma_{Z}) + (1 - \alpha) h_{\pi_{\mu}}(\sigma_{Y})
= \alpha h_{\mu}(\sigma_{Z}) + (1 - \alpha) h_{\pi_{\mu}}(\sigma_{Y}).
\]

This shows that \( \mu \in \mathcal{I}(\Phi, a) \) with \( a = (\alpha, 1 - \alpha) \in \mathbb{R}^{2} \).
Step 2. \((i) : (b) \Rightarrow (a)\) It follows from the Ledrappier-Young formula of measure \(\mu\) and it is the \(a\)-weighted equilibrium measure with \(a = (\alpha, 1 - \alpha) \in \mathbb{R}^2\) by Theorem 23. Up to a minor modification of Proposition 2.6 of [1]

\[
N_* (\nu_\alpha) + h_{\nu_\alpha} (\sigma_Y) = \sup \{ h_\eta (\sigma_Z) : \pi \eta = \nu_\alpha \}, \tag{24}
\]

where

\[
N_* (\nu_\alpha) = \lim_{n \to \infty} \frac{1}{n} \int_Y \log \| N_{g_n} \| \, d\nu_\alpha (y).
\]

It follows from the variational principle we have

\[
(\alpha N)_* (\nu_\alpha) + h_{\nu_\alpha} (\sigma_Y) = P (\sigma_Y, \alpha N). \tag{25}
\]

Combining Theorem 15 (ii), (24), (25) and the Ledrappier-Young formula of measure \(\mu\) obtains

\[
\dim_H Z - \dim_H \mu = \frac{1}{\log n} (P (\sigma_Y, \alpha N) - \alpha h_\mu (\sigma_Z) - (1 - \alpha) h_{\nu_\alpha} (\sigma_Y))
\]

\[
= \frac{1}{\log n} (P (\sigma_Y, \alpha N) - h_{\nu_\alpha} (\sigma_Y) - \alpha N_* (\nu_\alpha))
\]

\[
= \frac{1}{\log n} (P (\sigma_Y, \alpha N) - (h_{\nu_\alpha} (\sigma_Y) + (\alpha N)_* (\nu_\alpha)))
\]

\[
= \frac{1}{\log n} (P (\sigma_Y, \alpha N) - P (\sigma_Y, \alpha N)) = 0.
\]

This shows that \(\mu\) is the invariant measure of the full Hausdorff dimension. This completes the proof of (i).

Step 3. It remains to prove the dimension formula (6). Indeed, take \(a = (\alpha, 1 - \alpha) \in \mathbb{R}^2\) and \(\nu_\alpha = \pi \mu \in \mathcal{I} (\alpha N)\) and \(\Phi\) is zero potential. It follows from Theorem 24 (i) and the definition of \(a\)-weighted pressure (21) we obtain that

\[
P^a (\sigma_Z, \Phi) = \sup \left\{ \alpha h_\eta (\sigma_Z) + (1 - \alpha) h_{\pi_\eta \mu} (\sigma_Y) : \eta \in \mathcal{M} (Z, \sigma_Z) \right\}.
\]

\[
= \alpha h_\mu (\sigma_Z) + (1 - \alpha) h_{\pi_\mu \mu} (\sigma_Y) \tag{26}
\]
Combining (26) and the fact of $h_{\text{top}}(\sigma_Z) = \log \rho_A$ we have

$$\dim_H H = \frac{1}{\log n} P(\sigma_Y, \alpha N) = \frac{1}{\log n} (h_{\sigma_{\nu}}(\sigma_Y) + (\alpha N)_*(\nu_\alpha))$$

$$= \frac{1}{\log n} (h_{\sigma_{\nu}}(\sigma_Y) + \alpha N_*(\nu_\alpha)) \text{ (Proposition 14)}$$

$$= \frac{1}{\log n} \left( \alpha \log \rho_A + h_{\sigma_{\nu}}(\sigma_Y) + (\alpha N)_*(\nu_\alpha) \right) \text{ (N is normalized)}$$

$$= \frac{1}{\log n} \left( \alpha h_{\text{top}}(\sigma_Z) + P(\sigma_Y, \alpha N) \right)$$

$$= \frac{\alpha h_{\text{top}}(Z)}{\log n} + \frac{P(\sigma_Y, \alpha \bar{N})}{\log n}$$

$$= \frac{h_{\text{top}}(Z)}{\log m} + \frac{P(\sigma_Y, \alpha \bar{N})}{\log n}.$$

This establishes the formula (6). □

### 3.2 Proof of Theorem 3

For the proof of Theorem 3 we give some useful lemmas first.

**Lemma 16.** Let $A \in \mathbb{R}^{n \times n}$, $A$ be irreducible and $L = ([L(i)]_{i=1}^n)\text{T}$, $R = [R(i)]_{i=1}^n$ be the left and right eigenvector of $A$ corresponding to the maximal eigenvalue $\rho_A$. If \( \text{rank}(A) = 1 \), then $L = ([C(j)]_{j=1}^n)\text{T}$ and $R = [D(i)]_{i=1}^n$, where

$$C(j) = \begin{cases} A(i,j)/A(i,1), & \text{if } A(i,1) \neq 0; \\ 0, & \text{otherwise}. \end{cases}$$

and

$$D(i) = \begin{cases} A(i,j)/A(1,j), & \text{if } A(1,j) \neq 0; \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** Without loss of generality, we may assume that $L(1) = 1$. Since \( \text{rank}(A) = 1 \), $C(j)$ is well-defined for all $j = 1, \ldots, n$. It follows from the fact that $L$ is the left eigenvector of $A$ with respect to the eigenvalue $\rho_A$, we have for all $j = 2, \ldots, n$.

$$\sum_{i=1}^n L(i) A(i,j) = \sum_{i=1}^n L(i) C(j) A(i,1) = C(j) \rho_A L(1).$$

19
Hence \( L(j) = C(j) L(1) \) for all \( j = 1, \ldots, n \) and \( L = ([C(j)]_{j=1}^n)^t \). It follows from the same argument that we have \( R = [D(i)]_{i=1}^n \). This completes the proof. \[\square\]

**Lemma 17.** Let \( \hat{L} = \left( \left[ \hat{L}(k) \right]_{k=1}^n \right)^t \), where \( \hat{L}(k) = ([L(d_1, k)]_{d_1=1}^m)^t \) and \( B_k = PA_kP^{-1} \) for all \( k = 1, \ldots, n \), where \( B_k^{(i,j)} = N_{ij} \) if \( j = k \), and \( B_k^{(i,j)} = 0_{m \times m} \), otherwise. Then for all \( k = 1, \ldots, n \),

\[
\hat{L}B_k = \left[ 0_m, 0_m, \ldots, \rho_A \hat{L}(k), 0_m, \ldots, 0_m \right],
\]

where \( 0_m \in \mathbb{R}^{1 \times m} \) with all entries being 0's.

**Proof.** Since \( LA = \rho_A L \), we have \( \sum_{d \in D} L(d) A(d, d') = \rho_A L(d') \) for all \( d' \in D \). For each \( 1 \leq k \leq n \), construct \( A_k \) as follows:

\[
A_k(d, d') = \begin{cases} 
A(d, d'), & \text{if } d' = (d'_1, d'_2) \text{ with } d'_2 = k; \\
0, & \text{otherwise}.
\end{cases}
\]

We obtain

\[
\sum_{d \in D} L(d) A_k(d, d') = \begin{cases} 
\rho_A L(d'), & \text{if } d' = (d'_1, d'_2) \text{ with } d'_2 = k; \\
0, & \text{otherwise}.
\end{cases}
\]

On the other hand, since \( \hat{L} = LP^{-1} \) and \( LA_k = L(P^{-1}B_kP) \) for all \( k = 1, \ldots, n \),

\[
\hat{L}B_k = LA_kP^{-1} = \rho_A \left[ 0_m, \ldots, 0_m, \hat{L}(k), 0_m, \ldots, 0_m \right].
\]

The proof is thus completed. \[\square\]

Notably, if we let \( \hat{R} = PR = \left[ \hat{R}(k) \right]_{k=1}^n \), where \( \hat{R}(k) = [R(d_1, k)]_{d_1=1}^m \), we also have

\[
B_k \hat{R} = \left( \left[ 0_m, \ldots, 0_m, \rho_A \hat{R}(k), 0_m, \ldots, 0_m \right] \right)^t \text{ for all } k = 1, \ldots, n.
\]

**Proof of Theorem 3.** For clarity, we prove theorem 3 for the cases of \( k = 1 \) and \( k \geq 2 \).

**Step 1** \((k = 1)\).
1. (⇒) We claim that if $Z$ is a Markov Sierpiński carpet, then the induced potential $N$ satisfies the Markov condition of order 1, that is, there exists $V_1, \ldots, V_n$ such that $V_i N_{ij} \subseteq V_j$ for all $1 \leq i, j \leq n$.

Since $Z$ is a Markov Sierpiński carpet, we obtain $\text{rank}(A) = 1$. Hence, for all $1 \leq j \leq m, 1 \leq k \leq n$ and $d \in D$,

$$
C(j, k) = \begin{cases} 
A(d, (j, k)) / A(d, (1, k)), & \text{if } A(d, (1, k)) \neq 0; \\
0, & \text{otherwise}
\end{cases}
$$

is well-defined. Let $L = ([L(d)]_{d \in D})^t$ be the left eigenvector of $A$ corresponding to the maximal eigenvalue $\rho_A$, we have $\hat{L} := LP^{-1} = \left(\left[\hat{L}(k)\right]_{k=1}^n\right)^t$, where $\hat{L}(k) = ([L(d, k)]_{d=1}^m)^t$. It follows from (27) and Lemma 16, we obtain

$$
\hat{L}(k) = \left([C(j, k)]_{j=1}^m\right)^t.
$$

Choose $V_k = \hat{L}(k)$ for $1 \leq k \leq n$. Combining the fact that $N_{ij} = [A((p, i), (q, j))]_{p, q=1}^m$ and (27), yields

$$A((p, i), (q, j)) = C(q, j) A((p, i), (1, j))$$

for all $1 \leq q \leq m, 1 \leq j \leq n$ and $(p, i) \in D$.

According to (28) and (29), for any $i, j = 1, \ldots, n$, we get

$$V_i N_{ij} = [C(1, i), C(2, i), \ldots, C(m, i)] \times
\begin{bmatrix}
A((1, i), (1, j)) & \cdots & A((1, i), (m, j)) \\
\vdots & \ddots & \vdots \\
A((m, i), (1, j)) & \cdots & A((m, i), (m, j))
\end{bmatrix}
\sum_{p=1}^m C(p, i) A((p, i), (1, j)) \left([C(q, j)]_{q=1}^m\right)^t = m(i, j) V_j,$$

where $m(i, j) = \sum_{p=1}^m C(p, i) A((p, i), (1, j))$. Therefore, $N$ satisfies the Markov condition of order 1.

2. (⇐) For the converse, we show that if $N$ satisfies the Markov condition of order 1, then $\nu$ is a 1-step Markov measure, i.e.,

$$
\frac{\nu([j_0 j_1 \ldots j_k])}{\nu([j_0])} = \frac{\nu([j_{-k} \ldots j_0 j_1])}{\nu([j_{-k} \ldots j_0])}, \text{ for all } k > 0.
$$
Assume \( N \) satisfies the Markov condition of order 1, it follows from the same argument as above, that we have
\[
\hat{L}(i) N_{ij} = m(i, j) \hat{L}(j), \quad \text{for all } 1 \leq i, j \leq n. \tag{30}
\]
Since for each \( J = [j_0 \ldots j_{n-1}] \in Y_n, \nu(J) = (1/\rho_A)^n L A_{j_0} A_{j_1} \cdots A_{j_{n-1}} R. \)
Hence, it follows from Lemma 17 and (30)
\[
\frac{\nu([j_0 j_1])}{\nu([j_0])} = \left( \frac{1}{\rho_A \left[ 0_m, \ldots, 0_m, \rho_A \hat{L}(j_0), 0_m, \ldots, 0_m \right] \hat{R}} \right) \times
\]
the only non-zero part is in \( j_0 \)-th coordinate
\[
\left[ 0_m, \ldots, 0_m, \rho_A \hat{L}(j_0), 0_m, \ldots, 0_m \right] \times
\]
the only non-zero column is in \( j_1 \)-th coordinate
\[
\left[ 0_m \cdots N_{1j_1} \cdots 0_m \\
\vdots \cdots \cdots \cdots \vdots \\
0_m \cdots N_{nj_1} \cdots 0_m \right] \times \hat{R}
\]
\[
= \frac{m(j_0, j_1) \hat{L}(j_1) \hat{R}(j_1)}{\rho_A \hat{L}(j_0) \hat{R}(j_0)}.
\]
On the other hand, for any \( k > 0, \)
\[
\frac{\nu([j_k j_{-k+1} \cdots j_0 j_1])}{\nu([j_k \cdots j_0])} = \left( \frac{1}{\rho_A \hat{L}B_{j_k} \cdots B_{j_0} \hat{R}} \right) \times
\]
the non-zero element is in the \( j_{-k+1} \)-th coordinate
\[
\left[ 0_m, \ldots, 0_m, \hat{L}(j_k) N_{j_k j_{-k+1}}, 0_m, \ldots, 0_m \right] \times B_{j_{-k+2}} \cdots B_{j_0} B_{j_1} \hat{R}
\]
\[
= \left( \frac{1}{\rho_A \hat{L}B_{j_k} \cdots B_{j_0} \hat{R}} \right) \frac{m(j_k, j_{-k+1})}{B_{j_{-k+2}} \cdots B_{j_0} B_{j_1} \hat{R}} 	imes
\]
\[
\left[ 0_m, \ldots, 0_m, \hat{L}(j_{-k+1}), 0_m, \ldots, 0_m \right] B_{j_{-k+2}} \cdots B_{j_0} B_{j_1} \hat{R}. \tag{31}
\]
Continuing the process as (31), we have
\[
\frac{\nu([j_k j_{-k+1} \cdots j_0 j_1])}{\nu([j_k \cdots j_0])} = \frac{m(j_0, j_1) \hat{L}(j_1) \hat{R}(j_1)}{\rho_A \hat{L}(j_0) \hat{R}(j_0)} = \frac{\nu([j_0 j_1])}{\nu([j_0])}
\]
for all \( k > 0 \). Hence, \( \nu \) is a 1-step Markov measure and the proof is thus completed.

**Step 2** \((k \geq 2)\). For this proof, we recall some setting first, since \( Z = Z_{(m,n)}(A) \) is a Markov Sierpiński carpet, denote \( Z[k] = Z_{(m^k,n^k)}(A[k]) \), where \( A[k] \in \mathbb{R}^{d^k \times d^k} \) \((d = m \times n)\) is the \( k \)-th higher block transition matrix from \( A \) which is indexed by \( D[k] \) and define as \( P[k] = P_{(m,n)^k}^{[k]} \). Let \( L[k] = \left( \left[ L[k](d[k]) \right]_{d[k] \in D[k]} \right)^t \) be the left eigenvector of \( A[k] \) corresponding to the maximal eigenvalue \( \rho_{A[k]} = \rho_A \). Set \( B[k] = P[k]A[k](P[k])^{-1} \). For \( J \in Y_k \), we construct \( A_j^k \) as follows:

\[
A_j^k(d[k], d'[k]) = \begin{cases} 
A[k](d[k], d'[k]), & \text{if } d[k] = (d_1[k], d_2[k]) \text{ with } d_2[k] = J; \\
0, & \text{otherwise}
\end{cases}
\]

and define an ordered set

\[
\Gamma_J := \left\{ (d[k])^{(r)} : (d[k])^{(r)} = \left( (d[k])^{(r)}_1, (d[k])^{(r)}_2 \right) \text{ with } (d[k])^{(r)}_2 = J \right\}.
\]

It can be easily checked that \#\( \Gamma_J = m^k \) for any \( J \in Y_k \).

We present two lemmas which are analogous to Lemma 8 and 17 for \( k = 1 \).

**Lemma 18** (Lemma 8 for \( k \geq 2 \)). Let \( Z = Z_{(m,n)}(A) \) be a Markov Sierpiński carpet with \( A \in \mathbb{R}^{d \times d} \) \((d = m \times n)\) be irreducible. For \( k \geq 2 \), \( A[k] \) is the \( k \)-th higher block transition matrix which is indexed by \( D[k] \). For all \( J \in Y_k \), let \( A_j^k \) be as defined in (32) and \( B_j^k = P[k]A_j^k(P[k])^{-1} \). Then

\[
\left( B_j^k \right)^{(J_2'(0,k-1), J_1'(1,k))} = \begin{cases} 
N_{J_2'(0,k)}, & \text{if } J'_1(1,k) = J; \\
0_{m^k \times m^k}, & \text{otherwise}.
\end{cases}
\]

**Lemma 19** (Lemma 17 for \( k \geq 2 \)). Let \( \tilde{L}[k] : = L[k](P[k])^{-1} = \left( \tilde{L}[k](J_i) \right)_{i=1}^{a_k} \) \((l_{r})\)

where

\[
\tilde{L}[k](J_i) = \left( \left[ L[k]\left( (d[k])^{(r)} \right) \right]_{r=1}^{m^k} \right)^t,
\]

for all \( (d[k])^{(r)} \in \Gamma_{J_i}, r = 1, \ldots, m^k \) and \( B_j^k = P_kA_j[A_j^k]P_k^{-1} \) for all \( i = 1, \ldots, a_k \). Then

\[
\tilde{L}[k]B_j^k = \left[ 0_{m^k}, \ldots, 0_{m^k}, \rho_A \tilde{L}(J_i), 0_{m^k}, \ldots, 0_{m^k} \right] \text{ for all } i = 1, \ldots, a_k.
\]
We continue the proof of Step 2 and divide it into two small parts.

1. (⇒) We claim that if \( Z \) is a Markov Sierpiński carpet, then \( N \) satisfies the Markov condition of order \( k \), that is, there exists \( \{ V_J \}_{J \in Y_k} \) such that \( V_{J(0,k-1)}N_{J(0,k)} \subseteq V_{J(1,k)} \) for all \( J(0,k-1), J(1,k) \in Y_k \). Indeed, since \( Z^{[k]} \) is also a Markov Sierpiński carpet, we have \( \text{rank}(A^{[k]}) = 1 \). Hence, for all \( J \in Y_k \),

\[
A^{[k]} \left( d^{[k]}(r), (d^{[k]}(r)) \right) = C(J, r)A^{[k]} \left( d^{[k]}(r), (d^{[k]}(r)) \right),
\]

where \( (d^{[k]}(r)) \in \Gamma_J, r = 1, \ldots, m^k \) and \( d^{[k]} \in D^{[k]} \). Let \( L^{[k]} = \left( \left[ L^{[k]}(d^{[k]}) \right]_{d^{[k]} \in D^{[k]}} \right)^t \) be the left eigenvector of \( A^{[k]} \) corresponding to the maximal eigenvalue \( \rho_{A^{[k]}} = \rho_A \). This implies \( \hat{L}^{[k]} := L^{[k]}P^{-1} = \left( \left[ \hat{L}^{[k]}(J) \right]_{J \in Y_k} \right)^t \), where \( \hat{L}^{[k]}(J) = \left( \left[ L^{[k]}(d^{[k]}(r)) \right]_{r=1}^{m^k} \right)^t \), for all \( J \in Y_k, (d^{[k]}(r)) \in \Gamma_J \). It follows from Lemma 16 and (33) we have

\[
\hat{L}^{[k]}(J) = \left( C(J, r) \right)_{r=1}^{m^k} \text{ for all } J \in Y_k.
\]

Taking \( V_J := \hat{L}^{[k]}(J) \) for all \( J \in Y_k \). For any \( J = J(0, k) \in Y_{k+1} \) we have

\[
N_J = N_{J(0,k)} = \left[ A^{[k]} \left( (d^{[k]}(p), (d^{[k]}(q)) \right)_{p,q=1}^{m^k} \right],
\]

where \( (d^{[k]}(p)) \in \Gamma_{J(0,k-1)}, (d^{[k]}(q)) \in \Gamma_{J(1,k)} \). It also follows from (33) we obtain that

\[
A^{[k]} \left( (d^{[k]}(p), (d^{[k]}(q)) \right) = C \left( (1, k), (1, k) \right) A^{[k]} \left( (d^{[k]}(p), (d^{[k]}(q)) \right),
\]

for all \( (d^{[k]}(p)) \in \Gamma_{J(0,k-1)}, (d^{[k]}(q)) \in \Gamma_{J(1,k)} \) and \( p, q = 1, \ldots, m^k \). We conclude from (34) and (35) that for any \( J = J(0, k) \in Y_{k+1} \) we have

\[
V_{J(0,k-1)}N_{J(0,k)} = \left[ 1, C \left( (0, k-1), (2) \right), \ldots, C \left( (0, k-1), m^k \right) \right] \times
\begin{bmatrix}
A^{[k]} \left( (d^{[k]}(1), (d^{[k]}(1)) \right) & \cdots & A^{[k]} \left( (d^{[k]}(1), (d^{[k]}(m^k)) \right) \\
\vdots & \ddots & \vdots \\
A^{[k]} \left( (d^{[k]}(1), (d^{[k]}(m^k)) \right) & \cdots & A^{[k]} \left( (d^{[k]}(m^k), (d^{[k]}(m^k)) \right)
\end{bmatrix}
\]

\[
= m \left( (0, k-1), (1, k) \right) V_{J(1,k)},
\]

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where \( m(J(0,k-1), J(1,k)) := \sum_{p=1}^{m_k} C(J(0,k-1), p) A^k((d^k)_1, (d^k)_1) \).

Hence, there exists \( \{V_J\}_{J \in Y_k} \) such that \( V_{J(0,k-1)} N_{J(0,k)} \subseteq V_{J(1,k)} \) for all \( J(0,k-1), J(1,k) \in Y_k \), i.e., \( N \) satisfies the Markov condition from left of order \( k \).

2. (\( \iff \)) we claim that if \( N \) satisfies the Markov condition from left of order \( k \), then \( \nu \) is a \( k \)-step Markov measure on \( Y \), i.e.,

\[
\frac{\nu(J(0,k))}{\nu(J(0,k-1))} = \frac{\nu(J(-n,k))}{\nu(J(-n,k-1))}, \text{ for all } J(-n,k) \in Y_{n+k+1}, n \in \mathbb{N}.
\]

Assume \( N \) satisfies the Markov condition from left of order \( k \), it follows from the same argument as above, we have \( V_J := \widehat{\mathcal{L}}^k(J) \) for all \( J \in Y_k \) and for all \( J(0,k) \in Y_{k+1} \),

\[
\widehat{\mathcal{L}}^k(J(0,k-1)) N_{J(0,k)} = m(J(0,k-1), J(1,k)) \widehat{\mathcal{L}}^k(J(1,k)). \quad (36)
\]

On the other hand, \( Z = Z_{(m,n)}(A) = Z_{(m^k,n^k)}(A^k) \) for \( k \geq 2 \), hence for any \( J = J(0,m-1) \in Y_m \) with \( m \geq k \),

\[
\nu(J(0,m-1)) = (1/\rho_A)^{m-k+1} \mathcal{L}^k \mathcal{A}^{[k]}_{J(0,k-1)} \mathcal{A}^{[k]}_{J(1,k)} \cdots \mathcal{A}^{[k]}_{J(m-k,m-1)} \mathcal{R}^{[k]}
\]

\[
= (1/\rho_A)^{m-k+1} \mathcal{L}^k \left( \left( \mathcal{P}^k \right)^{-1} \mathcal{B}^{[k]}_{J(0,k+1)} \mathcal{P}^{[k]} \right) \times
\]

\[
\cdots \times \left( \left( \mathcal{P}^k \right)^{-1} \mathcal{B}^{[k]}_{J(m-k,m-1)} \mathcal{P}^{[k]} \right)
\]

\[
= (1/\rho_A)^{m-k+1} \mathcal{L}^k \mathcal{B}^{[k]}_{J(0,k+1)} \cdots \mathcal{B}^{[k]}_{J(m-k,m-1)} \mathcal{R}^{[k]}.
\]

Combining the above computation, (36) and Lemma 19 yields

\[
\nu(J(0,m-1)) = (1/\rho_A)^{m-k} m(J(0,k-1), J(1,k)) \times
\]

\[
\left[ 0_{m^k}, \ldots, 0_{m^k}, \mathcal{L}^k(J(1,k)) \right] \times
\]

\[
\mathcal{B}^{[k]}_{J(2,k+1)} \cdots \mathcal{B}^{[k]}_{J(m-k,m-1)} \mathcal{R}^{[k]} \quad (37)
\]

Continuing the same process as (37), we obtain

\[
\nu(J(0,m-1)) = (1/\rho_A)^{m-k} m(J(0,k-1), J(1,k)) \cdots m(J(m-k-1,m-2), J(m-k,m-1))
\]

\[
\mathcal{L}^k(J(m-k,m-1)) \mathcal{R}(J(m-k,m-1))
\]

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Hence
\[
\frac{\nu(J(0, k))}{\nu(J(0, k - 1))} = \frac{m(J(0, k - 1), J(1, k)) \hat{L}^k(J(1, k)) \hat{R}^k(J(1, k))}{\rho_A \hat{L}^k(J(0, k - 1)) \hat{R}^k(J(0, k - 1))}.
\]

On the other hand, for any \( n > 0 \),
\[
\frac{\nu(J(-n, k))}{\nu(J(-n, k - 1))} = \frac{\rho_A^{-n(n+1)} \prod_{i=0}^{n-1} m(J(-n+i, -n+i+k), J(-n+i+1, -n+i+k)) \hat{L}^k(J(1, k)) \hat{R}^k(J(1, k))}{\rho_A \hat{L}^k(J(0, k - 1)) \hat{R}^k(J(0, k - 1))} \times \frac{m(J(0, k - 1), J(1, k)) \hat{L}^k(J(1, k)) \hat{R}^k(J(1, k))}{\rho_A \hat{L}^k(J(0, k - 1)) \hat{R}^k(J(0, k - 1))} = \nu(J(0, k)) \nu(J(0, k - 1)).
\]

Hence \( \nu \) is a \( k \)-step Markov measure.

**Step 3.** We finish the proof of Theorem 3 by setting some notation first. For \( k \geq 1 \), let \( A[k] \in \mathbb{R}^{d_k \times d_k} \) be the \( k \)-th higher block transition matrix which is indexed by \( D[k] \). Let \( A = A[1] = \text{stoch}(A) \) and \( A[k] = \text{stoch}(A[k]) \) for all \( k > 1 \), and let \( \mathbb{L}^k \) and \( 1_{a_k} \) be the left and right eigenvectors of \( A[k] \) corresponding to the eigenvalue \( \rho_{k|k} = 1 \).

For \( k \geq 1 \), we define \( S^{(0)} := \left\{ \hat{L}(i) : i \in A(Y) \right\} \) and
\[
S^{(k)} := \left\{ \hat{L}^k(J(0, k - 1)) N_{J(0, k)} : J(0, k) \in Y_{k+1} \right\} ,
\]
where \( \langle \; \rangle \) is used to denote span. Let \( \hat{S}^{(k)} \) be the following sets.
\[
\hat{S}^{(0)} = \left\{ \left[ 0_m, \ldots, 0_m, \hat{L}(i), 0_m, \ldots, 0_m \right] : i \in A(Y) \right\}
\]
and
\[
\hat{S}^{(k)} = \left\{ \left[ 0_{m^k}, \ldots, 0_{m^k}, \hat{L}^k(J(0, k - 1)) N_{J(0, k)}, 0_{m^k}, \ldots, 0_{m^k} \right] : \right. \left. J(0, k) \in Y_{k+1} \right\}.
\]

We show that if \( \nu \) is an \( n \)-step Markov measure, then \( n \leq m - n \). It is sufficient to prove that
\[
S^{(k)} \subseteq S^{(k+1)} \text{ for all } k \geq 0. \tag{38}
\]
This is true because if $\nu$ is an $n$-step Markov measure, then $S^{(n)} = S^{(n+1)}$.

Therefore,

$$S^{(n)} = S^{(n+1)} \text{ implies } S^{(n)} = S^{(t)} \text{ for all } t \geq n.$$  

Since $\dim(S^{(0)}) \leq n$, it follows that $n \leq m - n$.

To prove (38) we show that $\hat{S}^{(k)} \subseteq \hat{S}^{(k+1)}$ for all $k \geq 0$.

(a) For $k = 0$, we claim $\hat{S}^{(0)} \subseteq \hat{S}^{(1)}$. Indeed,

$$\hat{S}^{(1)} = \left\{ \left[ 0_m, \ldots, 0_m, \hat{L}(i)N_{ij}, 0_m, \ldots, 0_m \right] : j \in A(Y) \right\}.$$  

For all $j \in A(Y)$, since $L_{AB} = L$,

$$\hat{L}B = \left( LL(P[k])^{-1}BBP[k] (P[k])^{-1} = LL(A[k])^{-1} = LL(P[k])^{-1} = \hat{L}. $$

Thus,

$$\left[ 0_m, \ldots, 0_m, \hat{L}(j), 0_m, \ldots, 0_m \right] = \hat{L}B_j = (\hat{L}B)B_j = \sum_{i=1}^{n} \hat{L}B_iB_j = \sum_{i=1}^{n} \left[ 0_m, \ldots, 0_m, \hat{L}(i)N_{ij}, 0_m, \ldots, 0_m \right].$$

Hence $\hat{S}^{(0)} \subseteq \hat{S}^{(1)}$.

(b) For $k \geq 1$, we claim $\hat{S}^{(k)} \subseteq \hat{S}^{(k+1)}$. Indeed, since $A[k]$ and $A^{[k+1]}$ are the higher block transition matrices, it follows that $A[k]$ and $A^{[k+1]}$ are shift equivalent (see [15] for more detail), that is, there exists $F[k]$ where a row is indexed by $D[k+1]$ and a column is indexed by $D[k]$ such that

$$F[k]A[k] = A^{[k+1]}F[k].$$

By recalling the definition of $stoch(A[k])$, we have

$$A^{[k]} = stoch(A[k]) = \frac{1}{\rho_A}D_k^{-1}A[k]D_k$$

where $D_k$ is the diagonal matrix with diagonal entries $D_k(i, i) = R^{[k]}(i)$, $R^{[k]}$ is the right eigenvector of $A[k]$ with spectral radius $\rho_A = \rho_A$. Combining (39) and (40), yields

$$F[k]A[k] = A^{[k+1]}F[k]$$

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where $F^{[k]} = D_{k+1}^{-1}F^{[k]}D_k$. It follows from (41) that we have
\[
L^{[k]} = L^{[k+1]}F^{[k]} \quad \text{for all } k \geq 1 \quad (42)
\]
and
\[
F^{[k]}A^{[k]}_{J(1,k)} = \left( \sum_{J(0,k)} A^{[k+1]}_{J(0,k)} \right) F^{[k]} \quad \text{for all } k \geq 1. \quad (43)
\]
It follows from (42) and (43), for any $J(1,k+1) \in Y_{k+1}$, we obtain
\[
\begin{bmatrix}
0_{a_k}, \ldots, 0_{a_k}, \hat{L}^{[k]}(J(1,k))N_{J(1,k+1)}, 0_{a_k}, \ldots, 0_{a_k}
\end{bmatrix} = \left( L^{[k+1]}F^{[k]} \right) A^{[k]}_{J(1,k)}A^{[k]}_{J(2,k+1)}P^{[k]}
= \left( \sum_{J(0,k)} A^{[k+1]}_{J(0,k)} \right) \left( \sum_{J(1,k+1)} A^{[k+1]}_{J(1,k+1)} \right) F^{[k]}P^{[k]}
= \sum_{J(0,k)} L^{[k+1]}A^{[k+1]}_{J(0,k)}A^{[k+1]}_{J(1,k+1)}P^{[k+1]} \quad (44)
\]
\[
\begin{bmatrix}
0_{m^{k+1}}, \ldots, 0_{m^{k+1}}, \hat{L}^{[k+1]}(J(0,k))N_{J(0,k+1)}, 0_{m^{k+1}}, \ldots, 0_{m^{k+1}}
\end{bmatrix} = \left( \sum_{J(0,k)} L^{[k+1]}A^{[k+1]}_{J(0,k)}A^{[k+1]}_{J(1,k+1)}P^{[k+1]} \right).
\]
On account of the above (44) and (45), we thus get
\[
\begin{bmatrix}
0_{m^k}, \ldots, 0_{m^k}, \hat{L}^{[k]}(J(1,k))N_{J(1,k+1)}, 0_{m^k}, \ldots, 0_{m^k}
\end{bmatrix} = \sum_{J(0,k)} \begin{bmatrix}
0_{m^{k+1}}, \ldots, 0_{m^{k+1}}, \hat{L}^{[k+1]}(J(0,k))N_{J(0,k+1)}, 0_{m^{k+1}}, \ldots, 0_{m^{k+1}}
\end{bmatrix}.
\]
Hence $\hat{S}^{(k)} \subseteq \hat{S}^{(k+1)}$ for all $k \geq 1$.

According to (a) and (b), we have $\hat{S}^{(k)} \subseteq \hat{S}^{(k+1)}$ for all $k \geq 0$. Hence, (38) is proved, and so is the theorem.
3.3 Proof of Theorem 4

To increase the readability we only prove the theorem for \( k = 1 \), since the general case is similar.

Since \( X_M \) is SFT, we can construct the linear representation of measure \( \eta \) where \( \eta \) is the unique maximal measure on \( X_M \). Combining the fact of \( \hat{L} = \left( \left[ \hat{L}(k) \right]_{k=1}^{m} \right)^t \) is the left eigenvector of \( B \) corresponding to the maximal eigenvalue \( \rho_A \) and \( B^{(i,j)} = N_{ij} \) for all \( i, j = 1, \ldots, n \) yields

\[
\sum_{i=1}^{n} \hat{L}(i) N_{ij} = \rho_A \hat{L}(j) \quad \text{for all} \quad j = 1, \ldots, n. \tag{46}
\]

Since \( N \) satisfies the Markov condition from left of order 1, i.e., there exists \( V_1, \ldots, V_n \) such that \( V_i N_{ij} \subseteq V_j \) for \( i, j = 1, \ldots, n \). It can be checked that \( V_k = \hat{L}(k) \) for \( k = 1, \ldots, n \) and \( \hat{L}(i) N_{ij} = m(i, j) \hat{L}(j) \) for \( i, j = 1, \ldots, n \).

It follows from (46), for all \( j = 1, \ldots, n \), we have \( \rho_A \hat{L}(j) = \sum_{i=1}^{n} \hat{L}(i) N_{ij} = \sum_{i=1}^{n} m(i, j) \hat{L}(j) \), which implies \( \sum_{j=1}^{m} m(i, j) = \rho_A \) and \( 1_n^t M = \rho_A 1_n^t \), where \( 1_n \) is \( 1 \times n \) column vector with entries which are all 1’s. This means that \( 1_n^t \) is a left eigenvector of \( M \) corresponding to eigenvalues \( \rho_M = \rho_A \).

For any \( k = 1, \ldots, n \), we define \( M_k \) which is the matrix indexed by \( A(Y) = \{1, \ldots, n\} \) as follows. For \( i, j = 1, \ldots, n \),

\[
M_k(i,j) = \begin{cases} 
  m(i,j) & \text{if } j = k, \\
  0 & \text{otherwise.}
\end{cases}
\]

Choose \( \tilde{L} = 1_n^t \) and \( \tilde{R} = \left[ \hat{L}(i) \tilde{R}(i) \right]_{i=1}^{n} \), then the triple \( \left( \tilde{L}, \{M_i\}_{i=1}^{n}, \tilde{R} \right) \) represents the unique maximal measure on \( X_M \), that is, for any \( J = [j_0, \ldots, j_{n-1}] \in Y_n \),

\[
\eta(J) := (1/\rho_A)^n \tilde{L} M_{j_0} \cdots M_{j_{n-1}} \tilde{R}. \tag{47}
\]

We claim that \( \nu \) is the unique maximal measure on \( X_M \), i.e., \( \nu(J) = \eta(J) \) for all \( J \in Y_n, n \in \mathbb{N} \). According to (47), for any \( J = [j_0, \ldots, j_{n-1}] \in Y_n \), we get

\[
\eta(J) = \frac{1}{\rho_A}^{n-1} \rho_A \left[ 0, \ldots, 0, 1, 0, \ldots, 0 \right] M_{j_1} \cdots M_{j_{n-1}} \tilde{R} \quad \text{(by (47))}
\]

where 1 is at \( j_0 \)-th coordinate

\[
\eta(J) = \frac{1}{\rho_A}^{n-1} m(j_0, j_1) [0, \ldots, 0, 1, 0, \ldots, 0] M_{j_2} \cdots M_{j_{n-1}} \tilde{R}. \tag{48}
\]
Continue the same process as (48), we have
\[ \eta(J) = \frac{1}{\rho A} m(j_0, j_1) m(j_1, j_2) \cdots m(j_{n-2}, j_{n-1}) [0, \ldots, 0, 1, 0, \ldots, 0] \tilde{R} \]
\[ = \frac{1}{\rho A} m(j_0, j_1) m(j_1, j_2) \cdots m(j_{n-2}, j_{n-1}) \hat{L}(j_{n-1}) \hat{R}(j_{n-1}). \]

For any \( J \in Y_n \), we also have
\[ \nu(J) = \frac{1}{\rho A} \rho_A \hat{L}(j_0) N_{j_0j_1} N_{j_1j_2} \cdots N_{j_{n-2}j_{n-1}} \hat{R}(j_{n-1}) \]
\[ = \frac{1}{\rho A} m(j_0, j_1) \hat{L}(j_1) N_{j_1j_2} \cdots N_{j_{n-2}j_{n-1}} \hat{R}(j_{n-1}). \] (49)

Continue the same process as (49), we have
\[ \nu(J) = \frac{1}{\rho A} m(j_0, j_1) m(j_1, j_2) \cdots m(j_{n-2}, j_{n-1}) \hat{L}(j_{n-1}) \hat{R}(j_{n-1}) \]
\[ = \eta(J). \]

Hence, \( \nu \) is the unique maximal measure of \( X_M \). This completes the proof.

### 3.4 Proof of Theorem 5

We present some results from [12] for the criterion of the equality for Hausdorff and Minkowski dimensions.

**Theorem 20** ([12, Theorem 1.3]). Under the same assumptions of Theorem 9 with \( A \) being irreducible. Then \( \dim_H Z = \dim_M Z \) iff the unique invariant measure of maximal entropy in \( Z \) projects via \( \pi_y \) to the unique measure of maximal entropy on \( \pi_y(Z) \).

**Proof of (i) of Theorem 5**. (\( \Rightarrow \)) Combining the fact that \( N \) satisfies the Markov condition of order \( k \) and Theorem 4, we have \( \pi_{\mu A} = \nu \) is the unique maximal measure of subshift of finite type \( X_M \) with adjacency matrix \( M \) where \( \mu_A \) is the measure of maximal entropy in \( Z \). If \( \rho_M = \rho_{T^k} \), then \( \nu \) is also the unique maximal measure of \( Y \). Hence \( \dim_H Z = \dim_M Z \) by Theorem 20.

(\( \Leftarrow \)) If \( \dim_H Z = \dim_M Z \), then the unique invariant measure of maximal entropy in \( Z \) projects via \( \pi_y \) to the unique measure of maximal entropy on \( \pi_y(Z) \). Hence, \( \rho_M = \rho_{T^k} \).

**Proof of (ii) of Theorem 5**. Without loss of generality, we may assume \( N \) satisfies the Markov condition of order 1 from left. Let \( A' \) be a transition matrix of \( Z' = K(\mathbb{T}, D') \). For all \( k = 1, \ldots, n \), we denote \( A'_k \) by
\[ A'_k(d, d') = \begin{cases} 1, & \text{if } d' = (d'_1, d'_2) \text{ with } d'_2 = k; \\ 0, & \text{otherwise}, \end{cases} \]
where \( d, d' \in D' \). Then

\[
\dim_H Z' = \dim_H K(\mathbb{T}, D')
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \log_n \left\| A_{j_1} A_{j_2} \cdots A_{j_N} \right\|^\alpha
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \log_n \sum_{J(1,N) \in Y_N} (LA_{j_1} A_{j_2} \cdots A_{j_N} R)^\alpha
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \log_n \sum_{J(1,N) \in Y_N} \left( \rho A m(j_1, j_2) \cdots m(j_{N-1}, j_N) \hat{L}(j_N) \hat{R}(j_N) \right)^\alpha
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \log_n \left\| \left( \rho A 1_n (M^\alpha)^{N-1} \right)^N \hat{L}(i) \hat{R}(i) \right\|_{i=1}^n
\]

From the proof of Theorem 3, we know \( N_{ij} = \left[ A ((p,i), (q,j)) \right]_{p,q=1}^m \) and \( V_i = \hat{L}(i) = \left[ \left[ L(d_1, i) \right]_{d_1=1}^m \right]^t \) for all \( i, j = 1, \cdots, n \). Recall that \( z(i) \) is the number of rectangles in row \( i \). Since

\[
L(d_1, i) = \begin{cases} 
1, & \text{if } (d_1, i) \in D' \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
A((p,i), (q,j)) = \begin{cases} 
1, & \text{if } (p,i), (q,j) \in D' \\
0, & \text{otherwise},
\end{cases}
\]

we obtain for all \( i = 1, \cdots, n \)

\[
\sum_{(p,i) \in D'} A((p,i), (q,j)) = z(i).
\]

(52)

According to (50), (51) and (52), we thus get for all \( i, j = 1, \cdots, n \)

\[
V_i N_{ij} = \hat{L}(i) N_{i,j}
\]

\[
= \left[ \sum_{p=1}^m L(p,i) A((p,i), (q,j)) \right]_{q=1}^m = \left[ \sum_{(p,i) \in D'} A((p,i), (q,j)) \right]_{q=1}^m
\]

\[
= \sum_{(p,i) \in D'} A((p,i), (q,j)) [L(q,j)]_{q=1}^m = z(i) [L(q,j)]_{q=1}^m = m(i, j) V_j,
\]

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that is, \( m(i, j) = z(i) \) for all \( i, j = 1, \ldots, n \). Thus, \( \log_n \| (M^\alpha)^{N-1} \| = \log_n \sum_{j=1}^{n} z(j)^\alpha \). The proof is thus completed.

**Proof of (iii) of Theorem 5** For \( k \geq 1 \), it follows from the same argument of \( k = 1 \), that we have

\[
\dim_H [K_T (A)] = \lim_{N \to \infty} \frac{1}{N} \log_n \sum_{J(0,N-1) \in Y_N} \left[ L^{[k]} \left( \prod_{i=0}^{N-k-1} A^ {[k]}_{J(i,i+k-1)} \right) R^{[k]} \right] ^\alpha.
\]

And for any \( J = J (0, N - 1) \in Y_N \) with \( N \geq k \), we have

\[
L^{[k]} \left( \prod_{i=0}^{N-k-1} A^ {[k]}_{J(i,i+k-1)} \right) R^{[k]} = \rho_A \left( \prod_{i=0}^{N-k-1} m \left( J (i, i + k - 1), J (i + 1, i + k) \right) \right) \times \tilde{L}^{[k]} (J (N - k - 1, N - 1)) \tilde{R}^{[k]} (J (N - k - 1, N - 1)) \).
\]

Hence,

\[
\dim_H Z = \lim_{N \to \infty} \frac{1}{N} \log_n \sum_{J(0,N-1) \in Y_N} \left[ \rho_A \left( \prod_{i=0}^{N-k-1} m \left( J (i, i + k - 1), J (i + 1, i + k) \right) \right) \times \tilde{L}^{[k]} (J (N - k - 1, N - 1)) \tilde{R}^{[k]} (J (N - k - 1, N - 1)) \right] ^\alpha
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \log_n \left( \rho_A \left( M^\alpha \right)^{N-k-2} \tilde{L}^{[k]} (J_p) \tilde{R}^{[k]} (J_p) \right) ^{a_k} \]

\[
= \lim_{N \to \infty} \frac{1}{N} \log_n \| (M^\alpha)^{N-k-2} \| = \log_n \rho_{M^\alpha}.
\]

The proof is completed.

**4 Application and examples**

**4.1 Criterion for Markov measure**

Assume \( \pi : X \to Y \) is an one-block code induced from \( \Pi : \mathcal{A}(X) \to \mathcal{A}(Y) \), we assume that \( \mathcal{A}(Y) = \{1, \ldots, n\} \) and \( X \) is a SFT with the transition matrix \( A \) which is irreducible. Suppose \( Y \) is a irreducible subshift of finite type, we
call $\pi : X \to Y$ Markovian (cf. [2, 3]) if for every Markov measure $\nu$ on $Y$, there is a Markov measure $\mu$ on $X$ with $\pi \mu = \nu$.

For $k = 1, \ldots, n$, let

$$E_k = \{ i \in A(X) : \Pi(i) = k \},$$

and denote by $e_k = \# E_k$ the number of $E_k$ and define $\hat{N}_{ij} \in \mathbb{R}^{e_i \times e_j}$ as follows:

$$\hat{N}_{ij}(k, l) = \begin{cases} 1, & \text{if } A(k, l) = 1 \text{ and } \Pi(k) = i \text{ and } \Pi(l) = j; \\ 0, & \text{otherwise}. \end{cases}$$

For $1 \leq i, j \leq n$, $\hat{N}_{ij}$ is called row allowable if for each $k \in \{1, \ldots, e_i\}$ there is an $l \in \{1, \ldots, e_j\}$ such that $\hat{N}_{ij}(k, l) \neq 0$. $\hat{N} = (\hat{N}_{ij})_{i,j=1}^n$ is called row allowable if $\hat{N}_{ij}$ is row allowable for all $1 \leq i, j \leq n$. In [4], the authors call such factors full row allowable. The following lemma shows that the full row allowability implies the projection space $Y$ is a subshift of finite type.

Lemma 21 ([4, Lemma 6]). If $\pi$ is full row allowable, then $Y$ is a subshift of finite type.

Define (H1) and (H2) as follows:

(H1) $\hat{N} = (\hat{N}_{ij})_{i,j=1}^n$ is row allowable.

(H2) For all $J \in \text{Per}_n(Y)$, the $n$-periodic orbit in $Y$, with $1 \leq n \leq \# A(Y)$, $\hat{N}_J$ is a positive matrix.

Under (H1) and (H2), Chazottes and Ugalde prove that there is a Gibbs measure of some well-defined potential on $Y$. In the following, we give another proof for this result and give a simple lemma first.

Lemma 22. Let $E_n \in R^{n \times n}$ be the full matrix and $C_1, C_2, \ldots, C_m$ be a sequence of row allowable matrices, then $C_1 C_2 \cdots C_m E_n \geq E_n$.

Proof. This is the immediate consequence of the observation that $A E_n \geq E_n$ if $A$ is row allowable. $\square$

Theorem 23. Let $\pi : X \to Y$ be an one-block code which satisfies (H1) and (H2), then there exists a unique Gibbs measure $\nu$ on $Y$. 

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Proof. Without loss of generality, we may assume that $\hat{N} = (\hat{N}_{ij})_{i,j=1}^{n}$ are all square matrices with the same size $m$, that is, $\hat{N}_{ij} \in \mathbb{R}^{m \times m}$ for all $1 \leq i, j \leq n$. We first claim that if $\pi$ satisfies (H1) and (H2), then $f \in D_{w}(Y,p)$ for some $p \in \mathbb{N}$, where $f(I) = \|\hat{N}_{I}\|$ for all $I \in Y$. Indeed, since $\|\hat{N}_{I,J}\| \leq \|\hat{N}_{I}\| \|\hat{N}_{J}\|$, we only need to prove that there is a $p \in \mathbb{N}$ such that for all $I$, $J \in Y^{*}$, there exists $K \in \bigcup_{i=0}^{p}Y_{i}$ with $IKJ \in Y^{*}$ and $\|\hat{N}_{IKJ}\| \geq c \|\hat{N}_{I}\| \|\hat{N}_{J}\|$ for some $c > 0$. For $I = (i_{0}, \ldots, i_{m-1}) \in Y_{m}$ and $J = (j_{0}, \ldots, j_{n-1}) \in Y_{n}$, take $\vec{i}_{m-1}, \vec{j}_{0} \in A(X)$ such that $\Pi(\vec{i}_{m-1}) = i_{m-1}$ and $\Pi(\vec{j}_{0}) = j_{0}$. Since $X$ is irreducible, there is a path in $X$ of the form $\vec{i}_{m-1} \to \vec{j}_{0}$ of length $m_{1}$, say $\vec{P} = (\vec{i}_{m-1}, \ldots, \vec{j}_{0})$, and a periodic path from $\vec{j}_{0}$ to $\vec{j}_{0}$ of length $m_{2}$, say $\vec{P} = (\vec{j}_{0}, \ldots, \vec{j}_{0})$. Denote by $\vec{I}^{'} \vec{P} = (\vec{i}_{m-1}, \ldots, \vec{j}_{0})$ the concatenation of $\vec{I}$ and $\vec{P}$ and let $I^{'} = \pi(\vec{I}^{'}) = \Pi(\vec{i}_{m-1}) \cdots \Pi(\vec{j}_{0}) \in Y_{m}$, and $P = \pi(\vec{P}) = \Pi(\vec{j}_{0}) \cdots \Pi(\vec{j}_{0}) \in Y_{m_{2}}$. Since $\pi(\vec{P})$ is a periodic path in $Y_{m_{2}}$, (H1) is applied to show that $\hat{N}_{P} \geq E_{m}$, then Lemma 22 and (H2) is applied to obtain

$$\|\hat{N}_{I^{'}P}\| = \|\hat{N}_{I}\hat{N}_{P}\hat{N}_{I^{'}P}\| \geq \|\hat{N}_{I}\hat{E}_{m}\hat{N}_{J}\| \geq c \|\hat{N}_{I}\| \|\hat{N}_{J}\|.$$  

Since both the length of $I^{'}$ and $P$ can be chosen so as to be less than $n$, i.e., the number of $A(Y)$. Then the claim follows if we take $p = 2m$. Hence, it follows from Theorem 5.5 of [9] that there is a unique equilibrium measure $\nu$ which is ergodic and satisfies the following Gibbs property, that is, there is a $c > 0$ such that for all $n \in \mathbb{N}$ and $J \in Y_{n}$

$$c^{-1} \leq \frac{\nu(J)}{\exp\left(-nP(\sigma_{Y}, \hat{N})\right)} f(J) \leq c.$$  

The proof is thus completed. \qed

A further question arose in [4]: When is the factor map not a Markov map? Recall the result of Boyle and Tuncel for the criterion of the Markovian factor $\pi$.

**Theorem 24 ([3])**. For a factor map $\pi : X \to Y$ between irreducible SFTs, if there exists any fully supported Markov measure $\mu$ and $\nu$ with $\pi\mu = \nu$, then $\pi$ is Markovian.
We use the skill in Theorem 3 to answer the above question up to minor modification of the induced potential \( N \). Arrange the set \( E_k \) as an ordered set \( \{ i_k^j : \Pi (i_k^j) = k \}^{e_k}_{j=1} \), define \((j, k) = i_k^j\) for all \( 1 \leq k \leq n \) and \( 1 \leq j \leq e_k \). Let \( m = \max_{1 \leq k \leq n} \{ e_k \} \). Introduce new symbols \( D = \{ 1, \ldots, m \} \times \{ 1, \ldots, n \} \) and denote \( d = m \times n \), define the \textit{modified transition matrix} \( B \in \mathbb{R}^{d \times d} \) which is indexed by \( D \) as follows.

\[
B(d, d') = \begin{cases} 
1, & \text{if } d = (j, k) \in E_k, \; d' = (j', k') \in E_{k'} \text{ with } A(i_k^j, i_{k'}^{j'}) = 1; \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( N \) be the induced matrix-valued potential from \( B \), and call \( N \) the \textit{modified induced (matrix-valued) potential} on \( Y \). Note here that \( N \) is not full row allowable, however, \( \hat{N} = (N_{ij}(u, v) | u \in E_i, v \in E_j)_{i,j=1}^{n} \) is row allowable for each \( 1 \leq i, j \leq n \) from (H1). The following result comes from [11] which provides a criterion for Markov measures by means of a reduced module. To avoid the notation abuse, we omit the definitions of reduced module for measures and refer to [1, 11] and some references therein.

**Theorem 25** ([2, Theorem 5.1], [11, Proposition 3.2]). Let \((l, M, r)\) be a presentation of the reduced module of a sofic measure \( \nu \) on \( Y \), in which \( M_i \) denotes the matrix by which a symbol \( i \) of \( \mathcal{A}(Y) \) acts on the module. Suppose \( k \in \mathbb{N} \). Then the sofic measure \( \nu \) is \( k \)-step Markov if and only if every product \( M_{i_{(1)}} \cdots M_{i_{(k)}} \) of length \( k \) has a rank of \( 1 \) at most.

**Theorem 26.** If \( \hat{N} \) satisfies (H1) and (H2), then the projection measure \( \nu \) is Markov if and only if \( N \) satisfies the Markov condition where \( N = (N_{ij})_{i,j=1}^{n} \) is the modified induced matrix-valued potential on \( Y \).

**Proof.** According to Theorem 3, it is sufficient to show that if the projection measure \( \nu \) is Markov, then \( N \) satisfies the Markov condition. Without loss of generality, we may assume that \( \nu \) is a 1-step Markov measure. According to Proposition 13(ii), we have a natural presentation of a module \((L, \{ B_i \}_{i=1}^{n}, R)\) of a projection measure \( \nu \) on \( Y \), that is, let \( B \in \mathbb{R}^{d \times d} (d = m \times n) \) be the modified transition matrix and \( L, R \) be the left and right eigenvector of \( B \) corresponding to the maximal eigenvalue \( \rho_B \). Then \((L, \{ B_i \}_{i=1}^{n}, R)\) is a presentation of a module of a projection measure \( \nu \) on \( Y \), where \( B_i \) is defined by

\[
B_i (d, d') = \begin{cases} 
B(d, d'), & \text{if } d' = (d'_1, d'_2) \text{ with } d'_2 = i; \\
0, & \text{otherwise,}
\end{cases}
\]

\[1\]
for all $d, d' \in D$ and $i = 1, \ldots, n$.

If $\text{rank}(B_i) = 1$ for all $i \in \mathcal{A}(Y)$, by the same argument of the proof in Theorem 3, we have $N$ which satisfies the Markov condition. If not, we can construct a smaller module $(l, \{\tilde{B}_i\}_{i=1}^n, r)$ such that $(l, \{\tilde{B}_i\}_{i=1}^n, r)$ is the reduced module of $\nu$ on $Y$. This follows the same method as in [2].

Indeed, let $U$ be the vector space generated by vectors of the form

$$LB_{J_0(n-1)} = LB_{J_0}B_{j_1} \cdots B_{j_{n-1}},$$

for all $J_0(n-1) \in Y_n, n \in \mathbb{N}$. Let $\dim U = k$. If $k < d$, then construct a smaller module (presenting the same measure) as follows. Let $\mathcal{B} := \{u_i = LB_{J_0(n-1)}$ for some $J_0(n-1) \in Y_n, n \in \mathbb{N} : i = 1, \ldots, k\}$ be a basis of $U$. By Lemma 17 for each $a \in \mathcal{A}(Y)$, we define ordered sets

$$K_a := \left\{ u_l^{(a)} = \left[0, \ldots, 0, v_l^{(a)}, 0, \ldots, 0\right] \in \mathbb{R}^{1 \times d}, v_l^{(a)} \in \mathbb{R}^{1 \times m} : u_l^{(a)} = LB_{J_0(n-1)} \in \mathcal{B} \text{ and } j_{n-1} = a \right\}$$

and

$$K'_a := \left\{ v_l^{(a)} : l = 1, \ldots, k_a \right\}, \text{ where } k_a = \#K_a.$$

The matrix $U = (U^{(i,j)})_{i,j=1}^n$ arising from $K_a$ is defined by $U^{(i,j)} = 0_{k_i \times m}$ if $i \neq j$ and $U^{(i,j)}$ whose rows form an ordered set $K'_i$ if $i = j$. Then $UB_i = \tilde{B}_i U$ for all $i = 1, \ldots, n$, and $lU = L, r = UR$. Thus, $\rho_B = \rho_{\tilde{B}}$, where $\tilde{B} = \sum_{i=1}^n \tilde{B}_i$. According to the form of $U$, we obtain

$$\begin{align*}
[0_{k_1}, \ldots, 0_{k_{j-1}}, l(j), 0_{k_{j+1}}, \ldots, 0_{k_n}] U &= [0, \ldots, 0, (lU)(j), 0, \ldots, 0] \\
&= [0, \ldots, 0, L(j), 0, \ldots, 0] \quad (53)
\end{align*}$$

Without loss of generality, we may assume that $(l, \{\tilde{B}_i\}_{i=1}^n, r)$ is the reduced module of $\nu$ on $Y$. By Theorem 25, we obtain $\text{rank}(\tilde{B}_i) = 1$ for all $i = 1, \ldots, n$. Let $l = [l(1), \ldots, l(n)]$ where $l(i) \in \mathbb{R}^{1 \times k_i}$ for all $i$. Under the same argument of the proof in Theorem 3 the induced matrix-valued potential $\tilde{N}$ satisfies the Markov condition, that is,

$$l(i)\tilde{N}_{ij} = \tilde{m}(i,j)l(j), \quad (54)$$

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where $\tilde{N}_{ij} \in \mathbb{R}^{k_i \times k_j}$ and $\tilde{m}(i, j) \in \mathbb{R}$. Let $L = ([L(i)]_{i=1}^n)^t$ where $L(i) = ([L(d_1, i)]_{d_1 \in D})^t$. Combining (53), (54), for any $[ij] \in Y_2$, we have $\text{LB}_iB_j = (lU) B_iB_j = l\tilde{B}_i\tilde{B}_jU$

$$= \begin{bmatrix}
0_{k_1}, \ldots, 0_{k_{j-1}}, \rho_{\tilde{B}} l(i)\tilde{N}_{ij}, 0_{k_{j+1}}, \ldots, 0_{k_n}
\end{bmatrix} U$$

$$= \rho_{\tilde{B}} \begin{bmatrix} 0_{k_1}, \ldots, 0_{k_{j-1}}, \tilde{m}(i, j)l(j), 0_{k_{j+1}}, \ldots, 0_{k_n} \end{bmatrix} U$$

$$= \rho_{B} \tilde{m}(i, j) [0_{m}, \ldots, 0_{m}, L(j), 0_{m}, \ldots, 0_{m}]$$

On the other hand, since $LB_iB_j = \rho_{B} [0_{m}, \ldots, 0_{m}, L(i)N_{ij}, 0_{m}, \ldots, 0_{m}]$, then we have $L(i)N_{ij} = \tilde{m}(i, j)L(j)$ for all $[ij] \in Y_2$. Hence we pick $V_i = L(i)$ for all $i = 1, \ldots, n$ and the proof is thus completed.

4.2 Examples

We give two examples illustrating Theorem 3, Theorem 4 and the application of the Markovian property for a factor $\pi$.

In view of Theorem 25, readers may wonder whether $(L, \{B_i\}_{i=1}^n, R)$ is one reduced module of $Y$. The following example demonstrates that $N$ satisfies the Markov condition, however, $(L, \{B_i\}_{i=1}^n, R)$ is not a reduced module.

**Example 27** (Blackwell). Let $\mathcal{A}(X) = \{1, 2, 3\}$ and $\mathcal{A}(Y) = \{1, 2\}$ and the one-block map $\Pi : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$ be defined by $\Pi(1) = 1$, $\Pi(2) = 2$ and $\Pi(3) = 2$. Let $\pi : X \rightarrow Y$ be the factor from $X$ to $Y$ induced from $\Pi$ with $X = \Sigma_A$ for some

$$A = \begin{pmatrix}
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}.$$ 

This factor has been proven ([2, Example 2.7]) to be Markovian. Here we use Theorem 26 to give a criterion for this property. Since $E_1 = \{1\}$ and $E_2 = \{2, 3\}$ we see that $m = e_2 = 2$, and we introduce the new symbols and the corresponding sets $\hat{E}_1$ and $\hat{E}_2$ are as follows.

$$D = \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\},$$

$$\hat{E}_1 = \{1 = (1, 1), (2, 1)\}, \hat{E}_2 = \{2 = (1, 2), 3 = (2, 2)\}.$$
Therefore,

\[ B = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \]

\[ N_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_{21} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad N_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Take \( V_1 = (1, 0) \) and \( V_2 = (1, 1) \), one can easily check that \( N = (N_{ij})_{i,j=1}^2 \) satisfies the Markov condition of order 1. Thus Theorem 26 is applied to show that the factor is a Markov map. However, we remark here that \((L, \{B_i\}_{i=1}^n, R)\) is not a linear representation of the reduced module for sofic measure \( \nu \) on \( Y \) since \( \text{rank}(N_{22}) = 2 \). Let

\[ B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Since \( u = [1, 0, 1, 1] \) is the left eigenvector of \( B \) corresponding to \( \rho_B = 2 \), we have

\[ uB_1 = 2[1, 0, 0, 0] := 2u_1 \quad \text{and} \quad uB_2 = 2[0, 0, 1, 1] := 2u_2, \]

\[ u_1B_2 = u_2, \quad u_2B_2 = u_2, \quad u_2B_1 = 2u_1. \]

Therefore, \( U = \{u_1, u_2\} \) is the vector space generated by vectors of the form \( \{JB_J : J \in Y_n \text{ for } n \in \mathbb{N}\} \). Let \( k = \dim U = 2 < 4 \), and set

\[ L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \]

be the \( 2 \times 4 \) matrix with rows forming a basis of \( U \). Then

\[ LB_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} := \tilde{B}_1L, \]

\[ LB_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} := \tilde{B}_2L. \]

Since \( \dim U = 2 = \#\mathcal{A}(Y) \), this implies \( (l, \{\tilde{B}_i\}, r) \) is indeed a presentation of the reduced module of the sofic measure \( \nu \) on \( Y \), where \( l, r \) are the left and
right eigenvectors of $B$ corresponding to the maximal eigenvalue $\rho_B = \rho_B$. Moreover, $\text{rank}(\tilde{B}_1) = 1$ and $\text{rank}(\tilde{B}_2) = 1$.

**Example 28 ([16]).** Consider $(m, n) = (3, 2)$. Let the adjacent matrix $A \in \mathbb{R}^{6 \times 6}$ defining $Z$ as

$$
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The permutation $P_{(3,2)}$ induced from $\tau_{(3,2)}$ (see Section 1) is

$$
P_{(3,2)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

Denote $P = P_{(3,2)}$. Then the potential function $N$ extracted from $B = (B^{(i,j)})^2_{i,j=1}$ by [11] is as follows.

$$
\begin{bmatrix}
(PAP^{-1})^{(1,1)} & (PAP^{-1})^{(1,2)} \\
(PAP^{-1})^{(2,1)} & (PAP^{-1})^{(2,2)}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

One can easily check that $N = (N_{ij})^2_{i,j=1} = (B^{(i,j)})^2_{i,j=1}$

$$
N_{11} = \begin{pmatrix} 1 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1 \end{pmatrix}, \quad N_{12} = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix}, \quad N_{21} = \begin{pmatrix} 0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0 \end{pmatrix}, \quad N_{22} = \begin{pmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \end{pmatrix}.
$$
is irreducible with

\[ V_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}. \]

Since

\[ M = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \]

Theorem 3 and Theorem 4 are thus applied to show that the projection measure \( \nu \) is a Markov measure which is also the maximal measure of a full 2-shift. Finally, Theorem 5 is also applied to show that

\[ \dim_H Z = \log_2 \rho_{M^\alpha} = \log_2 (1 + 2^\alpha), \quad \text{where} \quad M^\alpha = \begin{pmatrix} 2^\alpha & 2^\alpha \\ 1 & 1 \end{pmatrix}. \]

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