A Component-Based Dual Decomposition Method for the OPF Problem

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Abstract—This paper proposes a component-based dual decomposition of the nonconvex AC optimal power flow (OPF) problem, where the modified dual function is solved in a distributed fashion. The main contribution of this work is that it demonstrates that a distributed method with carefully tuned parameters can converge to globally optimal solutions despite the inherent nonconvexity of the problem and the absence of theoretical guarantees of convergence. This paper is the first to conduct extensive numerical analysis resulting in the identification and tabulation of the algorithmic parameter settings that are crucial for the convergence of the method on 72 AC OPF test instances. Moreover, this work provides a deeper insight into the geometry of the modified Lagrange dual function of the OPF problem and the absence of theoretical guarantees of convergence. This paper is the first to conduct extensive numerical analysis resulting in the identification and tabulation of the algorithmic parameter settings that are crucial for the convergence of the method on 72 AC OPF test instances. Moreover, this work provides a deeper insight into the geometry of the modified Lagrange dual function of the OPF problem and the absence of theoretical guarantees of convergence. This work makes it well suited for smart grid applications such as multi-period OPF with demand response (DR) and security constrained unit commitment (SCUC) with contingency constraints and multiple transmission system operators (TSOs).

Index Terms—Optimal power flow, distributed methods, component-based dual decomposition, Augmented Lagrangian relaxation, ADMM, smoothing methods.

NOTATION

A. Input data and operators

| Symbol | Description |
|--------|-------------|
| $B$ | Set of buses in the power network. |
| $B_i$ | Set of buses connected to bus $i$. |
| $b_{ij}^{sh}$ | Shunt susceptance (p.u.) at bus $i$. |
| $g_{ij}^{sh}$ | Shunt conductance (p.u.) at bus $i$. |
| $b_{ij}^{ch}$ | Charging susceptance (p.u.) in the $\pi$-model of line $ij$. |
| $c_0^g$ | Constant coefficient ($/)$ term of generator $g$’s cost function. |
| $c_1^g$ | Coefficient ($/$MW) of the linear term of generator $g$’s cost function. |
| $c_2^g$ | Coefficient ($/$MW$^2$) of the quadratic term of generator $g$’s cost function. |
| $G$ | Set of all generators $(g,i)$ in the power network such that $g$ is the generator and $i$ is the bus connected to it. |
| $G_i$ | Set of all generators connected to bus $i$. |
| $i$ | Imaginary unit. |
| $L$ | Set of all transmission lines $ij$ where $i$ is the “from” bus. |
| $L_t$ | Set of all transmission lines $ij$ where $i$ is the “to” bus. |
| $p_{ij}^d/q_{ij}^d$ | Active/reactive power demand (MW/MVAr) at bus $i$. |
| $P_{ij}$ | Apparent power rating (MVA) of line $ij$. |

| Symbol | Description |
|--------|-------------|
| $\theta_{ij}^\Delta$ | Lower limit of the difference of voltage angles of buses $i$ and $j$. |
| $\bar{\theta}_{ij}$ | Upper limit of the difference of voltage angles of buses $i$ and $j$. |
| $\theta_{ij}^{\text{shift}}$ | Phase shift (Radian) of phase shifting transformer connected between buses $i$ and $j$ ($\theta_{ij}^{\text{shift}} = 0$ for a transmission line). |
| $\tau_{ij}$ | Tap ratio magnitude of phase shifting transformer connected between buses $i$ and $j$ ($\tau_{ij} = 1$ for a transmission line). |
| $T_{ij}$ | Complex tap ratio of a phase shifting transformer ($T_{ij} = \tau_{ij}e^{\theta_{ij}^{\text{shift}}}$. |
| $Y_{ij}$ | Series admittance (p.u.) in the $\pi$-model of line $ij$. |
| $\Re{\{\cdot\}}$ | Imaginary value operator. |
| $\Im{\{\cdot\}}$ | Real value operator. |
| $\min_{\cdot}/\max_{\cdot}$ | Minimum/maximum magnitude operator. |
| $\cdot^\ast$ | Conjugate operator. |
| $\bullet$ | Conjugate operator/Cardinality of a set. |
| $k$ | Iteration number. |
| $\rho_{\nu \theta}$ | ADMM penalty parameter. |
| $\rho_{pq}$ | Penalty parameter. |
| $\nu$ | Proximal penalty parameter. |

B. Decision variables

| Symbol | Description |
|--------|-------------|
| $p_i^d/q_i^d$ | Active/reactive power (MW/MVAr) generation of generator $g$ at bus $i$. |
| $p_{ij}/q_{ij}$ | Active/reactive power (MW/MVAr) flow along transmission line $ij$. |
| $V_i$ | Complex phasor voltage (p.u.) at bus $i$ ($V_i = |V_i| e^{\theta_i}$). |
| $v_{i(j)}$ | Duplicate of $v_i$ at line $ij$ such that $j \in B_i$. |
| $\theta_{i(j)}$ | Duplicate of $\theta_i$ at line $ij$ such that $j \in B_i$. |
| $\lambda$ | Vector of Lagrange multipliers. |

C. Acronyms

| Symbol | Description |
|--------|-------------|
| AC | Alternating current. |
| ADMM | Alternating direction method of multipliers. |
| DR | Demand response. |
| IPM | Interior-point optimization methods. |
| GNLP | Global nonlinear programming. |
| KKT | Karush-Kuhn-Tucker. |
| NLP | Nonlinear programming. |
| OCD | Optimality conditions decomposition. |
| OPF | Optimal power flow. |
| SCUC | Security-constrained unit commitment. |
| SDP | Semidefinite programming. |
| SOCP | Second-order cone programming. |
TSO  Transmission system operator.

I. INTRODUCTION

The alternating current (AC) power flow equations, which model the steady-state physics of power flows, are the linchpins of a broad spectrum of optimization problems in electrical power systems. Unfortunately, these nonlinear equations are the main sources of nonconvexity, which makes these problems notorious for being extremely challenging to solve using global nonlinear programming (GNLP) solvers. Therefore, the research community has focused on improving interior-point nonlinear optimization methods (IPM) to compute feasible solutions efficiently [1], [2]. Although these methods only (theoretically) guarantee local optimality, they have been shown, thanks to tight convex relaxations [3]–[7], to reach near-optimal (if not globally optimal) solutions on all the known test cases in the literature. This paper capitalizes on this to numerically show that the proposed distributed method solves the modified dual problem of the nonconvex AC OPF problem to near optimality, if not to global optimality.

In particular, the second-order cone programming (SOCP) and the semidefinite programming (SDP) relaxations have garnered considerable attention. The increased interest in this line of research stems from the fact that the SDP relaxation is shown to be exact, i.e., yields a zero optimality gap, on a variety of case studies [8]. However, in many practical OPF instances, the SDP relaxation yields inexact solutions [9], [10]. In these scenarios, an AC feasible solution cannot be recovered from the SDP relaxed solution. Nonetheless, the SDP relaxation can be strengthened by solving a hierarchy of moment relaxations [11]–[13] or by a combination of lifted nonlinear cuts, valid inequalities and bound tightening methods [5], [6], at the cost of larger SDP problems. In even more recently, increased attention was given to the computationally less demanding SOCP relaxation initially proposed in [14]. The SOCP relaxation in its classical form in [14] is shown to be dominated by the SDP relaxation but recent strengthening techniques [3], [4], [7] have shifted this paradigm.

A. State-of-the-art

There is a plethora of existing works on distributed OPF. These can be broadly classified into three categories, dual decomposition methods [15]–[24], optimality conditions decomposition (OCD) methods [25]–[29] and sparse SDP decomposition methods [30], [31]. The dual decomposition techniques underlying the dual-decomposition-based distributed OPF methods in the literature can in turn be classified into two categories: region-based decompositions [15]–[20], [23], [32] and component-based decompositions [21], [22], [24]. The focus of this study revolves around the latter decomposition techniques because they preserve privacy with respect to all components (generators, transformers, loads, buses, lines etc.) and are flexible enough to incorporate discrete decision variables to suit a wide variety of optimization applications in power system operations such as optimal transmission switching, capacitor placement, transmission and distribution network expansion planning, optimal feeder reconfiguration, power system restoration, and vulnerability analysis, to name a few. On the downside, dual-decomposition-based AC OPF methods have no theoretical guarantee of convergence because the (primal) OPF problem is nonconvex. Nonetheless, this paper numerically shows that under the right conditions, the proposed distributed method can converge to near-optimal (possibly globally optimal) solutions. Unlike [21]–[23], [30], [31], [31], which solve a convexified version of the OPF problem, this paper tackles the nonlinear nonconvex AC OPF directly. Convex relaxations are appealing because they are computationally conducive but their main disadvantage is that they do not always yield feasible solutions. Furthermore, in contrast to [24], the work in this paper conducts extensive numerical analysis and specifies the algorithmic parameter settings that are crucial for the convergence of the proposed component-based dual decomposition method on a vast array of test instances. On the other hand, OCD methods [25]–[29] rely on matrix factorization [33] to parallelize the computation of the Karush-Kuhn-Tucker (KKT) conditions. However, as of yet, these methods are not amenable to decompositions in terms of components.

B. Contributions of this work

In contrast to most distributed AC OPF algorithms in the existing literature, the algorithm in this paper is not only tested on the classical MATPOWER [2] instances but also on the more challenging NESTA v6 [34] test cases, which are designed specifically to incorporate key network parameters such as line thermal limits and small angle differences, which are critical in optimization applications. To get a grasp on how difficult the problem is, the methods in [20], [30], with the exception of [19], are only tested on MATPOWER cases with at most 118-buses. The method in [19] is tested on MATPOWER’s 300-bus system but does not converge after 10,000 iterations.

Against this background, this paper is the first to conduct extensive numerical analysis on the application of a distributed algorithm to solve the modified Lagrange dual function of the AC OPF problem. In more detail, this paper advances the state of the art in the following ways:

- Extensive numerical simulations on 72 test cases from MATPOWER [2], PEGASE [35] and NESTA v6 [34] instances show that the proposed algorithm converges to the same near-optimal (possibly globally optimal) solutions obtained from the centralized IPMs.
- The algorithmic parameter settings that are crucial for convergence are identified and tabulated.
- A deeper insight into the geometry of the modified Lagrange dual function of the OPF problem shows that this function can be nonsmooth for small values of the penalty parameters.

This type of distributed OPF analysis has not been conducted in the existing literature, let alone on a component-based dual
decomposition of the OPF. Therefore the techniques developed in this paper can serve as a basis for a myriad of smart grid optimization methods that are based on AC OPF, such as security constrained unit commitment (SCUC) with contingency constraints and multiple transmission system operators (TSOs), stochastic OPF, probabilistic OPF, and multi-period OPF with demand response (DR), to name a few.

C. Notation

All vectors are column vectors unless otherwise specified, and 1 is an all-ones vector of length depending on the context. The inner product of two vectors \( x, y \in \mathbb{R}^n \) is delineated by \( \langle x, y \rangle := x^T y \), where \( x^T \) is the transpose of \( x \). The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is denoted by \( \| x \| := \sqrt{x^T x} \) and the nonnegative orthant in \( \mathbb{R}^n \) is denoted by \( \mathbb{R}^n_+ \). Also, the Hadamard product of two vectors \( x \) and \( y \) is denoted by \( x \circ y \). Moreover, complex variables and parameters are in upper case whereas real variables and parameters are in lower case.

D. Organization of the paper

The paper starts with a formal description of the polar form OPF in general networks in Section II, followed by the component-based dual decomposition in Section III. Section IV describes the modified dual function and the proposed distributed algorithms. Section V shows the numerical evaluation of the algorithms and Section VI concludes the paper. Finally, Appendices A and B supplement the paper with valuable examples that provide a deeper insight on the optimality of dual-decomposition methods on nonconvex problems.

II. The OPF problem

In a power network, the OPF problem consists of finding the least-cost dispatch of power from generators to satisfy the load at all buses in a way that is governed by physical laws, such as Ohm’s Law and Kirchhoff’s Law, and other technical restrictions, such as transmission line thermal limit constraints. Knowing that \( \mathbb{R} \{ V_i V_j^* \} := v_i v_j \cos (\theta_i - \theta_j) \) and \( \mathbb{R} \{ V_i V_j^* \} := v_i v_j \sin (\theta_i - \theta_j) \), the OPF problem in polar form can be written as

\[
\begin{align}
\text{minimize} & \quad \sum_{p^g_i,q^d_i,g_i \in \mathcal{G}} f_p^g (p^g_i) & (1a) \\
\text{subject to} & \quad p^g_i \leq p^d_i \leq \bar{p}^d_i, & (1b) \\
& \quad q^g_i \leq q^d_i \leq \bar{q}^d_i, & (1c) \\
& \quad v_i \leq v_i \leq \bar{v}_i, & (1d) \\
& \quad g^\Delta_{ij} \leq \theta_i - \theta_j \leq \bar{g}^\Delta_{ij}, & (1e) \\
& \quad \sum_{(g,i) \in \mathcal{G}} p^g_i - p^d_i = \sum_{j \in \mathcal{B}} p_{ij} + g^\Delta_{ij} v_i^2, & (1f) \\
& \quad \sum_{(g,i) \in \mathcal{G}} q^g_i - q^d_i = \sum_{j \in \mathcal{B}} q_{ij} - b^\Delta_{ij} v_i^2, & (1g) \\
& \quad p_{ij} = g^\Delta_{ij} v_i^2 - g_{ij} v_i v_j \cos (\theta_i - \theta_j) \\
& \quad + b_{ij} v_i v_j \sin (\theta_i - \theta_j), & (i,j) \in \mathcal{L} & (1h)
\end{align}
\]

Fig. 1: A 3-bus system showing the duplication of the coupling variables and the resulting component-based decomposition.

\[
\begin{align}
q_{ij} &= b_{ij} v_i^2 - g_{ij} v_i v_j \cos (\theta_i - \theta_j) \\
&\quad - g_{ij} v_i v_j \sin (\theta_i - \theta_j), & (i,j) \in \mathcal{L} & (1i) \\
p_{ji} &= g_{ji} v_i^2 - g_{ji} v_i v_j \cos (\theta_j - \theta_i) \\
&\quad + b_{ji} v_i v_j \sin (\theta_j - \theta_i), & (i,j) \in \mathcal{L} & (1j) \\
q_{ji} &= b_{ji} v_i^2 - b_{ji} v_i v_j \cos (\theta_j - \theta_i) \\
&\quad - g_{ji} v_i v_j \sin (\theta_j - \theta_i), & (i,j) \in \mathcal{L} & (1k)
\end{align}
\]

where, \( g^\Delta_{ij} := \mathbb{R} \{ Y_{ij}^* - \frac{b^{sh}_{ij}}{v_i^2} \} \), \( b^\Delta_{ij} := \mathbb{R} \{ Y_{ij}^* - \frac{b^{ch}_{ij}}{v_i^2} \} \), \( g_{ij} := \mathbb{R} \{ Y_{ij}^* \} \), \( b_{ij} := \mathbb{R} \{ Y_{ij}^* \} \), \( g_{ji} := \mathbb{R} \{ Y_{ji}^* - \frac{b^{sh}_{ji}}{v_i^2} \} \), \( b_{ji} := \mathbb{R} \{ Y_{ji}^* - \frac{b^{ch}_{ji}}{v_i^2} \} \), \( g_{ij} := \mathbb{R} \{ Y_{ij}^* \} \), and \( b_{ij} := \mathbb{R} \{ Y_{ij}^* \} \). and \( f^3 (p^g_i) := c_{21}^3 (p^g_i)^2 + c_{12}^3 (p^g_i) + c_{03}^3 \). The OPF in (1) is a nonconvex nonlinear optimization problem that is known to be NP-hard [8]. The nonconvexities stem from equality constraints (1f)–(1k), which include nonconvex voltage bilinear terms multiplied by nonconvex sine and cosine functions of the angles, and a quadratic function of the voltage, which is also nonconvex in this equality constraint setting as it describes the boundary of the set \( \{ v^2 | v \in [\underline{v}, \bar{v}] \}^3 \).

III. Component-based dual decomposition

The OPF problem in its native form in (1) is not separable in terms of components, as relaxing the coupling constraints in (1f) and (1g) is not enough to bestow a component-based

\[\begin{array}{l}
\text{minimize} \quad \sum_{(g,i) \in \mathcal{G}} f_p^g (p^g_i) \\
\text{subject to} \quad p^g_i \leq p^d_i \leq \bar{p}^d_i, \\
q^g_i \leq q^d_i \leq \bar{q}^d_i, \\
v_i \leq v_i \leq \bar{v}_i, \\
g^\Delta_{ij} \leq \theta_i - \theta_j \leq \bar{g}^\Delta_{ij}, \\
\sum_{(g,i) \in \mathcal{G}} p^g_i - p^d_i = \sum_{j \in \mathcal{B}} p_{ij} + g^\Delta_{ij} v_i^2, \\
\sum_{(g,i) \in \mathcal{G}} q^g_i - q^d_i = \sum_{j \in \mathcal{B}} q_{ij} - b^\Delta_{ij} v_i^2, \\
p_{ij} = g^\Delta_{ij} v_i^2 - g_{ij} v_i v_j \cos (\theta_i - \theta_j) \\
+ b_{ij} v_i v_j \sin (\theta_i - \theta_j),
\end{array}\]

\[\begin{array}{l}
\text{if} \quad (i,j) \in \mathcal{L}.
\end{array}\]

The method in this paper was also applied to the OPF in rectangular form but the results are not documented here because they were not significantly different than the polar form ones.
separability. Towards this aim, the following variables are duplicated

\[ v_i = v_{i(j), \nu}, \quad (i, j) \in \mathcal{L} \cup \mathcal{L}_t, \]  

(2)

\[ \theta_i = \theta_{i(j), \nu}, \quad (i, j) \in \mathcal{L} \cup \mathcal{L}_t, \]  

(3)

and the OPF problem now becomes

\[
\begin{align*}
\text{minimize} & \quad \sum_{(i,j) \in \mathcal{G}} f_{ij}^g(p_i^g) \\
\text{subject to} & \quad (1b), (1c), (1f), (1g), (1l), (2), (3)
\end{align*}
\]

(4a)

\[
\begin{align*}
& \sum_{i,j}^n p_i^g - \nu_i \leq v_{i(j), \nu} \leq \sum_{i,j}^n p_i^g \\
& -\nu_j \leq v_{j(j), \nu} - \sum_{i,j}^n p_i^g \leq \nu_j \\
& \sum_{i,j}^n p_i^g \leq \theta_{i(j), \nu} - \theta_{j(j), \nu} \leq \sum_{i,j}^n p_i^g
\end{align*}
\]

(4b)

\[
\begin{align*}
& \sum_{i,j}^n p_i^g - \nu_i \leq \theta_{i(j), \nu} - \theta_{j(j), \nu} \leq \sum_{i,j}^n p_i^g \\
& -\nu_j \leq \theta_{j(j), \nu} - \theta_{i(j), \nu} \leq \nu_j
\end{align*}
\]

(4c)

\[
\begin{align*}
& \sum_{i,j}^n p_i^g \leq \lambda_{ij} \cos(\theta_{i(j), \nu} - \theta_{j(j), \nu}) \\
& + b_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}), \quad (i, j) \in \mathcal{L}
\end{align*}
\]

(4d)

\[
\begin{align*}
& \sum_{i,j}^n p_i^g - \nu_i \leq \lambda_{ij} \cos(\theta_{i(j), \nu} - \theta_{j(j), \nu}) \\
& + b_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}), \quad (i, j) \in \mathcal{L}
\end{align*}
\]

(4e)

\[
\begin{align*}
& \sum_{i,j}^n p_i^g \leq \lambda_{ij} \cos(\theta_{i(j), \nu} - \theta_{j(j), \nu}) \\
& - g_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}), \quad (i, j) \in \mathcal{L}
\end{align*}
\]

(4f)

\[
\begin{align*}
& \sum_{i,j}^n p_i^g \leq \lambda_{ij} \cos(\theta_{i(j), \nu} - \theta_{j(j), \nu}) \\
& + b_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}), \quad (i, j) \in \mathcal{L}
\end{align*}
\]

(4g)

The main reasons for solving the Lagrange dual function instead of the primal (4) are that, first, the former is the pointwise infimum of a family of affine functions in \( \lambda \) and is therefore concave, even though the primal problem (4) is nonconvex. Subsequently, first-order methods from convex optimization can be applied to solve the dual to optimality. Second, if the problem has zero duality gap, a feasible and optimal primal solution can be recovered from the dual solution. Third, the dual is separable in terms of components and can therefore be solved in a distributed fashion, thus preserving privacy. However, in this case, since the objective functions in (6) are neither finite nor strictly convex,\(^5\) the dual function in (5) is unbounded.

IV. MODIFIED DUAL FUNCTION AND THE DISTRIBUTED METHOD

To make the Lagrangian function finite and strictly convex, it is modified as follows

\[
L_{\nu}(x, \lambda_k^k) := \sum_{(i,j) \in \mathcal{L}} L_{\nu,i,j}^1 (x_{i,j}^k, \lambda_{ij}^k), \quad \Lambda_k = \left[ \lambda_{ij}^k \right]_{i,j \in \mathcal{L}}
\]

where

\[
\begin{align*}
& L_{\nu,i,j}^1 (x_{i,j}^k, \lambda_{ij}^k) := \frac{1}{2} \| x_{i,j}^k - x_{i,j}^k \|^2, \\
& + \nu \| x_{i,j}^k - x_{i,j}^k \|^2, \\
& + \nu \| x_{i,j}^k - x_{i,j}^k \|^2
\end{align*}
\]

Consequently, the modified Lagrange dual function is

\[
D_{\nu}(\lambda_k^k) := \text{minimize}_{x} L_{\nu}(x, \lambda_k^k)
\]

(9a)

subject to (1b), (1c), (1l), (4c)–(4h).

(9b)

Particularly, in (9), generators solve

\[
D_{\nu,i}^b(\lambda_k^k) := \text{minimize}_{x_i^k} L_{\nu,i}^b (x_{i,j}^k, \lambda_{ij}^k)
\]

(10a)

subject to (1b) and (1c),

(10b)

whereas buses solve

\[
D_{\nu,i}^b(\lambda_k^k, (\lambda_{ij}^k)_{j \in \mathcal{B}_i}) := \inf_{x_i^k} L_{\nu,i}^b (x_{i,j}^k, \lambda_{ij}^k, (\lambda_{ij}^k)_{j \in \mathcal{B}_i}),
\]

(11)

and lines solve

\[
D_{\nu,i,j}^b(\lambda_k^k, \lambda_{ij}^k) := \text{minimize}_{x_{i,j}^k} L_{\nu,i,j}^b (x_{i,j}^k, \lambda_{ij}^k, \lambda_{ij}^k)
\]

(12a)

subject to

\[
\begin{align*}
& \sum_{i,j}^n p_i^g - \nu_i \leq v_{i(j), \nu} \leq \sum_{i,j}^n p_i^g, \quad \sum_{i,j}^n p_i^g \leq \theta_{i(j), \nu} - \theta_{j(j), \nu} \leq \sum_{i,j}^n p_i^g
\end{align*}
\]

(12b)

\[
\begin{align*}
& g_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}) + b_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}) \leq \theta_{i(j), \nu} - \theta_{j(j), \nu}
\end{align*}
\]

(12c)

\[
\begin{align*}
& g_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}) + b_{ij} v_{i(j), \nu} v_{j(j), \nu} \sin(\theta_{i(j), \nu} - \theta_{j(j), \nu}) \leq \theta_{i(j), \nu} - \theta_{j(j), \nu}
\end{align*}
\]

(12d)

\(^5\)Equivalently, the Lagrangian in (5) is unbounded below in \( x \).
where
\[
\sum_{(g,i) \in G} p_{i,j}^k - \sum_{j \in B_i} p_{i,j}^{k+1} - \sum_{j \in B_i} p_{i,j}^{k+1} - g_i^k (v_i^{k+1})^2
\]
and
\[
\sum_{(g,i) \in G} q_{i,j}^k - \sum_{j \in B_i} q_{i,j}^{k+1} + b_i^k (v_i^{k+1})^2
\]
and
\[
g_{\nu}^k := \left[ \begin{array}{c}
g_{\nu}^{k+1} - g_{\nu}^{k} \\
g_{\nu}^{k+1} - g_{\nu}^{k} \\
\end{array} \right].
\]

The effect of adding the proximal regularization term (with \( \nu > 0 \)) is twofold. First, it makes the local cost functions finite and strictly convex and therefore the modified dual function bounded. Second, it makes the modified dual function differentiable for large values of \( \nu \). For small values of \( \nu \), the concave modified dual function \( D_\nu(\lambda^k) \) is typically nondifferentiable. Indeed, using Danskin’s theorem (See Appendices A and B), the subdifferentials of \( D_\nu(\lambda^k) \) are \( \partial D_\nu(\lambda^k) := \{ A_\nu x : D_\nu(\lambda^k), x \in \lambda^k \} \), where \( \lambda^k \) is the feasible set defined by constraints (1b), (1c), (11), (4e)–(4h) and A\( \nu \), is the coupling constraint matrix associated with coupling constraints (1f) and (1g) and consensus constraints (2) and (3). More specifically, as the (nonconvex) transmission line subproblems in (12) can have multiple (global) optimal solutions for a given vector \( \lambda^k \), the subdifferentials \( \partial D_\nu(\lambda^k) \) may be not be unique and the modified dual function \( D_\nu(\lambda^k) \) can be nonsmooth. In more detail,
\[
g_{\nu}^k := \left[ \begin{array}{c}
g_{\nu}^{k+1} - g_{\nu}^{k} \\
g_{\nu}^{k+1} - g_{\nu}^{k} \\
\end{array} \right].
\]

which is a subgradient of \( D_\nu(\lambda^k) \), may not be unique when \( \nu \) is small. On the other hand, for large values of \( \nu \), \( g_{\nu}^k \) is unique and is therefore a gradient of \( D_\nu(\lambda^k) \), i.e., \( g_{\nu}^k = \nabla D_\nu(\lambda^k) \). The component-based modified dual decomposition algorithm is described in Algorithm 1.

**Definition 1:** Let \( P_{\text{IPM}}^1 \) be a feasible primal solution computed centrally by an IPM solver, and let \( D_{\text{AMD}}(\lambda^1) \) be a solution of the approximate modified dual function computed in a distributed fashion by Algorithms 1, 2 or 3, initialized with the same algorithmic starting point used to find \( P_{\text{IPM}}^1 \). Then the gap between the feasible primal solution \( P_{\text{IPM}}^1 \) and its associated approximate modified dual function optimal value \( D_{\text{AMD}}(\lambda^1) \) is given by
\[
\text{AMDgap} := \left( \frac{P_{\text{IPM}}^1 - D_{\text{AMD}}(\lambda^1)}{P_{\text{IPM}}^1} \right) \times 100. \tag{14}
\]

Algorithm 1: Distributed algorithm

1. Initialization: \( \lambda^1 = 0 \), \( \nu > 0 \), \( \epsilon \leq 10^{-4} \), \( x_{ij}^{1} = [1, 0] \) for all \( i \in B \), \( x_{\nu}^{1} = \left[ \frac{\nu - x_{ij}^0}{2} \right] \) for all \( (g, i) \in G \), and \( x_{ij}^{1} = [0, 0, 0, 1, 0, 1, 0, 1] \) for all \( (i, j) \in L \).
2. while \( \|g_{\nu}^k\| \geq \epsilon \) do
3. Generators, buses and lines solve (10), (11) and (12) respectively in parallel, and send \( x_{\nu}^{k,b+1} \) and \( x_{ij}^{k,b+1} \) to adjacent buses.
4. Each bus \( i \in B \) updates its (local) Lagrange multipliers as in (13) and sends \( \lambda_{ij}^{k+1} \) and \( \lambda_{ij}^{k+1} \) to corresponding adjacent lines and generators.
5. \( k \leftarrow k + 1 \).
6. end while

Note that in Definition 1, if \( P_{\text{IPM}}^1 \) is globally optimal and \( \text{AMDgap} = 0 \), then \( D_{\text{AMD}}(\lambda^1) \) is an accurate approximation of the modified dual function. Also, note that unlike the classical dual function, the modified dual function \( D_{\text{AMD}}(\lambda^1) \) is not a lower bound on the optimal solution \( P^* \) and thus the definition in (14) instead of the classical definition
\[
\text{Duality gap} := \left( \frac{(P^* - D(\lambda^1))}{P^*} \right) \times 100.
\]

**Definition 2:** Let \( P_{\text{AMD}}^1 = f \left( P_{\text{IPM}}^1 \right) \) be a feasible primal solution computed in a distributed fashion by Algorithms 1, 2 or 3, initialized with the same algorithmic starting point used to find \( P_{\text{IPM}}^1 \). Then the gap between \( P_{\text{IPM}}^1 \) and \( P_{\text{AMD}}^1 \) is given by
\[
\text{ROgap} := \left( \frac{P_{\text{IPM}}^1 - P_{\text{AMD}}^1}{P_{\text{IPM}}^1} \right) \times 100.
\]

However, Algorithm 1 exhibits a very slow convergence due to an oscillatory behaviour witnessed across all the considered test cases. These oscillations are illustrated in Figure 2, which shows the evolution of \( \|g_{\nu}^k\| \) when Algorithm 1 is applied to MATPOWER’s case 14 with \( \nu = 100000 \), \( \alpha_i = 100 \) and \( \alpha_{ij} = 10000 \).
Algorithm 2: Distributed algorithm

1. Initialization: $\lambda^k = 0$, $\nu > 0$, $\rho > 0$, $0 < \epsilon < 10^{-4}$, $x^{k,1}_{i} = [1, 0]$ for all $i \in \mathcal{B}$, $x^{k,1}_{g,i} = \frac{\eta^2}{\rho_b^2}$ for all $(g,i) \in \mathcal{G}$, and $x^{k,1}_{i,j} = [0, 0, 0, 1, 0, 1, 0]$ for all $(i,j) \in \mathcal{L}$.

2. while $\|g^k\|_2 \geq \epsilon$ do
   3. Generators and lines solve (10) and (15) respectively in parallel, and send $x^{k+1}_{i} \in \mathcal{B}$ and $x^{k+1}_{i,j} \in \mathcal{L}$ to adjacent buses.
   4. Buses solve (16) in parallel and update their (local) Lagrange multipliers as in (13).
   5. Buses send $x^{k+1}_{i} \in \mathcal{B}$ and $\lambda^{k+1}_{i,j}$ to corresponding adjacent lines and generators.
   6. $k \leftarrow k + 1$.

7: end while

\[
\sum_{(g,i) \in \mathcal{G}} L^g_{i,k} \left( x^g_i, \lambda^k_i \right) + \sum_{i \in \mathcal{B}} L^b_{i,k} \left( x^b_i, \lambda^k_i, \left( \lambda^k_{i,j} \right)_{j \in \mathcal{B}_i} \right),
\]

where

\[
L^b_{i,k} \left( x^b_i, \lambda^k_i, \left( \lambda^k_{i,j} \right)_{j \in \mathcal{B}_i} \right) := L^b \left( x^b_i, \lambda^k_i, \left( \lambda^k_{i,j} \right)_{j \in \mathcal{B}_i} \right) + \frac{\rho_{\theta}}{2} \sum_{j \in \mathcal{B}_i} \left( \left( v_i - v_{s(i,j)}^{k+1} \right)^2 + \left( \theta_i - \theta_{s(i,j)}^{k+1} \right)^2 \right),
\]

and

\[
L^\nu_{\nu,p,i,j} \left( x^\nu_{i,j}, \lambda^k_{i,j} \right) := L^\nu_{\nu,p,i,j} \left( x^\nu_{i,j}, \lambda^k_{i,j} \right) + \frac{\rho_{\theta}}{2} \sum_{(l,m) \in \{(i,j)\cup(j,i)\}} \left( \left( v_i^{\nu} - v_{l,m} \right)^2 + \left( \theta_i^{\nu} - \theta_{l,m} \right)^2 \right).
\]

Consequently, generators now solve (10), transmission lines solve

\[
\min_{x^\nu_{i,j}, \lambda^k_{i,j}} L^\nu_{\nu,p,i,j} \left( x^\nu_{i,j}, \lambda^k_{i,j} \right)
\]

subject to (12b)–(12h).

and buses solve

\[
D^b_{i,k} \left( \lambda^k_i, \left( \lambda^k_{i,j} \right)_{j \in \mathcal{B}_i} \right) := \inf_{x^b_i} L^b \left( x^b_i, \lambda^k_i, \left( \lambda^k_{i,j} \right)_{j \in \mathcal{B}_i} \right).
\]

The component-based modified dual decomposition algorithm with the ADMM penalty term is described in Algorithm 2.

The key behind the superior convergence of Algorithm 2 is the ADMM penalty term which controls the stability of the iterates. This is illustrated in Figure 3, which shows the evolution of $\|g^*_{\nu,\rho}\|$ when Algorithm 2 is applied to MATPOWER’s case 14 with $\nu = 1000$, $\rho_{\theta} = 100000$, $\alpha_i = 100$ and $\alpha_{ij} = 100000$.

Finally, the modified Lagrange dual function would be

\[
D^\nu_{\nu,p,i,j} \left( \lambda^k_{i,j} \right) := \sum_{(i,j) \in \mathcal{L}} D^\nu_{\nu,p,i,j} \left( \lambda^k_{i,j} \right) + \sum_{(g,i) \in \mathcal{G}} D^g_{\nu,p,i} \left( \lambda^k_i \right) + \sum_{i \in \mathcal{B}} D^b_{\nu,p,i} \left( \lambda_i, \left( \lambda_{i,j} \right)_{j \in \mathcal{B}_i} \right),
\]

and the associated algorithm is described in Algorithm 3.

The convergence of Algorithm 3 on MATPOWER’s case 14 is illustrated in Figure 4, which shows the evolution of $\|g^*_{\nu,\rho}\|$ with $\nu = 1000$, $\rho_{\theta} = 100000$, $\rho_{\theta} = 1000$, $\alpha_i = 100$ and $\alpha_{ij} = 100000$. In this case, Algorithm 3 converges to a solution with an ROgap = 0.0008% and an AMDgap = $-7 \times 10^{-5}$% in 857 iterations as compared to 923 iterations when applying Algorithm 2 (see Figure 3).

The evolution from Algorithm 1 to Algorithm 3 results in a faster convergence but comes at the expense of more message exchanges. In fact, on many test instances, Algorithm 2 can
Algorithm 3: Distributed algorithm

1. **Initialization:** Same as in Algorithm 2.
2. while $\|g_{\nu,\rho}\| \geq \epsilon$ do
3. Generators and lines solve (17) and (18) respectively in parallel, and send $x_i^{g,k+1}$ and $x_{ij}^{l,k+1}$ to adjacent buses.
4. Buses solve (16) in parallel and update their (local) Lagrange multipliers as in (13).
5. Buses send $x_i^{g,k+1}$, $\lambda_{ij}^{k+1}$, $\lambda_{ij}^{k+1}$, $x_{ij}^{g,k}$ and $x_{ij}^{c,k}$ to corresponding adjacent lines and generators.\(^7\)
6. $k \leftarrow k + 1$.
7. end while

![Graph](image)

Fig. 4: Evolution of $\|g_{\nu,\rho}\|$ when Algorithm 3 is applied to MATPOWER’s case 14 with $\nu = 1000$, $\rho_{c0} = 100000$, $\rho_{pq} = 1000$, $\alpha_i = 100$ and $\alpha_{ij} = 100000$.

The method can easily be extended to incorporate discrete variables such as transformer taps and shunt capacitor banks.

TABLE I: Convergence of Algorithm 3 on MATPOWER instances.

| Case | $P_{\text{IPM}}$ | $P_{\text{AM}}$ | $P_{\text{RO}}$ | $\text{AMDgap}$ (%) |
|------|------------------|----------------|----------------|---------------------|
| 5    | 17554.89         | 17551.16       | 17552.02       | -3.28E-04           |
| 6    | 3143.82          | 3148.82        | 3146.97        | -2.01E-04           |
| 14   | 8081.16          | 8082.16        | 8081.53        | -1.00E-04           |
| 24   | 36332.20         | 36335.33       | 36327.21       | -6.47E-04           |
| 30   | 576.89           | 576.97         | 576.89         | -1.10E-04           |
| 30   | 8906.14          | 8906.14        | 8906.14        | -1.11E-04           |
| 39   | 41864.18         | 41864.22       | 41864.23       | -1.01E-04           |
| 57   | 41737.79         | 41737.99       | 41737.99       | -1.51E-04           |
| 89   | 5819.91          | 5819.91        | 5819.91        | -1.84E-03           |
| 118  | 129660.69        | 129660.27      | 129660.75      | -8.27E-05           |
| 300  | 719725.10        | 719724.47      | 719725.38      | -3.88E-05           |

results are shown in Tables I and II for MATPOWER [2], PEGASE [35] and NESTA v6 [34] instances respectively. The NESTA test cases are designed specifically to incorporate key network parameters such as line thermal limits and small angle differences, which are critical in optimization applications.

Tables I and II show that for $\epsilon = 10^{-4}$, and after careful individualized tuning of parameters (see Table III), Algorithm 3 converges to feasible solutions with negligible AMDgap and ROgap on all the 72 test cases.\(^9\)

The glimmerings of a principled way of setting the parameters of Algorithm 3 are not apparent in Tables I and II. However, extensive simulations show that they can be clustered in a summarizing table (Table III) of plausible parameter settings. Some parameter settings, like B, K and P for example, seem to work on the most number of test cases. This stands in contrast to settings A, C, D, G, I, J, M, N, O, Q, R, S and T which are tailored specifically to their respective test cases in Table II. There are three key contributors behind the convergence of Algorithm 3 on all the 72 cases. First, parameters $\nu$, $\rho_{pq}$ and $\rho_{c0}$ are set to high values, typically in the ranges $[100, 80000]$, $[1, 8000]$ and $[1000, 800000]$, respectively. Second, most test cases require setting $\rho_{c0}$ to at least 2 orders of magnitude larger than $\nu$ and 3 orders of magnitude larger than $\rho_{pq}$. Third, this disproportion in setting $\nu$, $\rho_{pq}$ and $\rho_{c0}$ is reflected exactly in setting the values of the step size in the multiplier update (13). More specifically, the step size $\alpha_{ij}$ is also 2 orders of magnitude larger than $\alpha_i$ in these test cases. In fact, $\alpha_i$ is set to 0.1$\nu$ and $\alpha_{ij}$ is set equal to $\rho_{c0}$. To see the significance of this, all the test instances with this specific parameter tuning would otherwise either diverge or require more than $10^6$ iterations to converge. Some notoriously difficult cases are MATPOWER’s case 5, NESTA’s cases 30\_f3r\_API and 189\_API for which very few other parameter settings (which are not shown here due space limitations), besides the corresponding ones in Table I, seem to make Algorithm 3 converge. Furthermore, even after exhaustive parameter tuning, the convergence on some test instances (like NESTA’s 300 bus systems) is substantially slower than others. Nonetheless, this still suggests that

\(^7\)Some parameter settings, like B, K and P for example, seem to work on the most number of test cases. This stands in contrast to settings A, C, D, G, I, J, M, N, O, Q, R, S and T which are tailored specifically to their respective test cases in Table II. There are three key contributors behind the convergence of Algorithm 3 on all the 72 cases. First, parameters $\nu$, $\rho_{pq}$ and $\rho_{c0}$ are set to high values, typically in the ranges $[100, 80000]$, $[1, 8000]$ and $[1000, 800000]$, respectively. Second, most test cases require setting $\rho_{c0}$ to at least 2 orders of magnitude larger than $\nu$ and 3 orders of magnitude larger than $\rho_{pq}$. Third, this disproportion in setting $\nu$, $\rho_{pq}$ and $\rho_{c0}$ is reflected exactly in setting the values of the step size in the multiplier update (13). More specifically, the step size $\alpha_{ij}$ is also 2 orders of magnitude larger than $\alpha_i$ in these test cases. In fact, $\alpha_i$ is set to 0.1$\nu$ and $\alpha_{ij}$ is set equal to $\rho_{c0}$. To see the significance of this, all the test instances with this specific parameter tuning would otherwise either diverge or require more than $10^6$ iterations to converge. Some notoriously difficult cases are MATPOWER’s case 5, NESTA’s cases 30\_f3r\_API and 189\_API for which very few other parameter settings (which are not shown here due space limitations), besides the corresponding ones in Table I, seem to make Algorithm 3 converge. Furthermore, even after exhaustive parameter tuning, the convergence on some test instances (like NESTA’s 300 bus systems) is substantially slower than others. Nonetheless, this still suggests that

\(^9\)Note that the stopping criterion in Algorithms 1, 2 and 3 requires a central authority to compute the norm of the subgradient; nonetheless, if a central authority is unavailable, the stopping criterion can be defined as in [19] or [21].
TABLE II: Convergence of Algorithm 3 on NESTA v5 instances.

| Case | Objective (B) | Gap (%) | Normal Operating Conditions |
|------|---------------|---------|----------------------------|
| 1    | 5812.64       | -9.09E-03 | 855 B                      |
| 2    | 156.43        | -1.87E-02 | 708 B                      |
| 3    | 17551.89      | -1.31E-03 | 3535 F                     |
| 4    | 23.21         | -3.96E-03 | 928 G                      |
| 5    | 3143.97       | -8.49E-03 | 918 B                      |
| 6    | 5296.30       | -7.24E-03 | 630 B                      |
| 7    | 244.05        | -3.83E-03 | 2544 B                     |
| 8    | 43352.20      | -3.00E-04 | 924 B                      |
| 9    | 29895.49      | -1.35E-04 | 45600 L                    |
| 10   | 803.13        | -9.62E-03 | 1512 B                     |
| 11   | 575.77        | -9.26E-03 | 1566 B                     |
| 12   | 204.97        | -2.20E-02 | 3725 B                     |
| 13   | 96505.52      | -1.57E-05 | 5915 B                     |
| 14   | 1143.27       | -2.34E-03 | 6204 B                     |
| 15   | 189764.44     | -3.35E-03 | 10532 B                    |
| 16   | 58198.1       | -3.35E-03 | 59703 I                    |
| 17   | 3718.64       | -1.66E-03 | 7432 B                     |
| 18   | 4230.23       | -3.35E-03 | 22387 E                    |
| 19   | 849.29        | -1.41E-03 | 26116 E                    |
| 20   | 16891.28      | -2.36E-03 | 97225 J                    |

TABLE III: Summarized parameter settings of Algorithm 3.

| Setting | $\nu$ | $P_{th} \%$ | $P_{th}$ | $\sigma_x$ | $\sigma_y$ |
|---------|------|-------------|----------|-----------|-----------|
| A       | 5,000| 30          | 300,000  | 300        | 300,000   |
| B       | 1,000| 100         | 10,000   | 100        | 10,000    |
| C       | 1,000| 1,000       | 10,000   | 100        | 10,000    |
| D       | 100  | 1           | 10,000   | 10         | 10,000    |
| E       | 5,000| 50          | 500,000  | 500        | 500,000   |
| F       | 5,000| 300,000     | 300,000  | 300,000    | 300,000   |
| G       | 100  | 1           | 10,000   | 10         | 10,000    |
| H       | 5,000| 500,000     | 500,000  | 500,000    | 500,000   |
| I       | 100  | 10,000      | 10,000   | 10         | 10,000    |
| J       | 10,000| 100,000     | 100,000  | 100,000    | 100,000   |
| K       | 10,000| 100,000     | 100,000  | 100,000    | 100,000   |
| L       | 1,000| 100,000     | 100,000  | 100,000    | 100,000   |
| M       | 8,000| 800,000     | 800,000  | 800,000    | 800,000   |
| N       | 5,000| 500,000     | 500,000  | 500,000    | 500,000   |
| O       | 80,000| 800,000     | 800,000  | 800,000    | 800,000   |
| P       | 10,000| 100,000     | 100,000  | 100,000    | 100,000   |
| Q       | 10,000| 100,000     | 100,000  | 100,000    | 100,000   |
| R       | 10,000| 100,000     | 100,000  | 100,000    | 100,000   |
| S       | 50,000| 500,000     | 500,000  | 500,000    | 500,000   |
| T       | 5,000| 100,000     | 100,000  | 100,000    | 100,000   |

Algorithm 3 converges even on these **difficult** test instances. 
Algorithm 3 is (theoretically) not guaranteed to converge to feasible solutions, let alone to globally optimal ones. However, as shown in Tables I and II and in Appendices A and B, the right starting point combined with the right parameter settings can result in a convergence to feasible near-optimal (possibly globally optimal) solutions (corroborated by tight convex relaxations [3]–[7]), despite having no guarantees that the subproblems in (15) are solved to global optimality. Moreover, as shown in Appendix B, case-specific parameter settings can lead to globally optimal solutions irrespective of the choice of algorithm starting point.

VI. CONCLUSION

The founding premise of this work is that, given the right algorithmic parameter settings, the method is numerically demonstrated to converge to feasible near-optimal (possibly globally optimal) solutions to the nonconvex AC OPF problem; corroborated by tight convex relaxations, on all the 72 considered test cases. Despite the absence of a principled way to set up the parameters of the algorithm, this work demonstrates that, first, the proximal and the ADMM penalty parameters should be set to at least 100. Second, the ADMM penalty parameter for the voltage and angle terms is set to at least 1 order of magnitude larger than the proximal penalty parameter in order to ensure differentiability of the modified dual function. Third, most test cases require setting the ADMM penalty parameter for the voltage and angle terms to at least 3 orders of magnitude larger than the ADMM penalty parameter for the active and reactive power terms to witness convergence. These three results do not only affect the speed of convergence, but can mean the difference between convergence and divergence. Future work will consist of investigating different accelerated subgradient methods to speed-up the convergence of the method.
The Lagrange dual function in (21) is concave, as it is the (partial) Lagrangian function of problem (20) as defined as

\[ L^\lambda(x, \lambda) := f(x) + \lambda(x_1 - x_2), \]

and when selecting the first suboptimal point from the list of KKT points of problem (21). Moreover, the same convergence behaviour is obtained by solving (21) using an IPM solver\(^{10}\) (IPOPT 3.125.3 [38], KNITRO 10.2 [37]) with a starting point \(x^0_{\text{IPM}} = [0.5, 0.5]\) at each iteration.

Furthermore, the Lagrange dual function in Figure 6a is nonsmooth. The dual function in Figure 6a is nondifferentiable at \(\lambda = 0\) and \(\lambda = 0.1203\). This is because (21) can have multiple (globally) optimal solutions for a given \(\lambda\), and as a consequence, the subdifferentials \(\partial D(\lambda)\) may not be unique. Indeed, using Danskin’s theorem [39]–[41], the subdifferentials of \(D(\lambda)\) are \(\partial D(\lambda) := \{A_x : D : D(\lambda), x \in X\}\) (\(A_x = [1, -1]\)). The effect of nondifferentiability on the convergence of the subgradient projection method is apparent in Figure 7a which shows the oscillations of the dual (and primal) residuals when \(D(\lambda^k)\) approaches its maximum value of \(d^* = 1.7670\) at \(\lambda = 0.1203\).

Both issues of nonzero duality gap and nondifferentiability can be addressed by modifying the Lagrange function as follows

\[ L_\rho(x, \lambda) := L(x, \lambda) + \frac{\rho}{2} \|x_1 - x_2\|^2, \]

which is also known as the augmented Lagrange function, and the augmented Lagrange dual function would be

\[ D_\rho(\lambda) := \inf_{x \in X} L_\rho(x, \lambda). \]

As shown in Figure 8a, the problem now has a zero modified duality gap, which is the gap between the optimal primal value \(p^*\) and the optimal modified dual value \(d^*_M := D_\rho(\lambda^*).\)

This should not be surprising as for very large values of \(\rho\), the augmented Lagrangian regularization term would be

\(^{10}\)IPM solvers only guarantee local optimality.

\(^{11}\)The gap between the optimal primal value \(p^*\) and the optimal modified dual value \(d^*_M\) is called modified duality gap to distinguish it from the classical definition of duality gap, which is the gap between the optimal primal value \(p^*\) and the optimal dual value \(d^* := D(\lambda^*).\)
Fig. 8: The augmented Lagrange dual function of problem (20), with $\rho = 10$. The dashed lines in (a) and (b) show $p^*$ and $p^f$ respectively.

Fig. 9: Primal and dual residuals of ADMM ($\rho = 50$), for $x_1^* = -1$ (a) and $x_1^* = 1$ (b), and when (24) and (25) are solved to optimality.

Fig. 10: Primal and dual residuals of ADMM ($\rho = 10$), for $x_2^* = -1$ and $x_{10}^* = [1, 1]$.

Fig. 11: Primal and dual residuals of the proximal method with $\nu = 50$, for $x_1^* = [-1, -1]$ (a) and $x_1^* = [1, 1]$ (b).

function (Figure 8a), obtained by solving (23) to optimality, and the approximate modified dual function (Figure 8b), obtained by solving (23) to suboptimality. In this specific case, the oscillations recur and the subgradient projection method does not converge. In other cases with different $x_{10}^{ipm}$, the algorithm eventually converges but very slowly.\(^\text{12}\) The main reason why this is important is that in many cases, like the OPF problem, solving (24) and (25) to optimality can be time consuming (not ideal for real-time applications) and therefore IPM solvers are used instead of GNLP solvers. In these cases it is best to initialize both the algorithm and the IPM solvers at each iteration with the same starting point.\(^\text{13}\) Indeed, in this example, initializing both the algorithm and the IPM solver at each iteration with the same starting point results in a convergence to the same solutions obtained when the subproblems are solved to global optimality.

Finally, Figure 11 shows the convergence of the proximal method with $\nu = 50$, for $x_1^* = [-1, -1]$ (Figure 11a) and $x_1^* = [1, 1]$ (Figure 11b). In particular, given the current iterates $(x_1^k, x_2^k, \lambda^k)$, the proximal method generates a new iterate $(x_1^{k+1}, x_2^{k+1}, \lambda^{k+1})$ as follows

$$x_1^{k+1} = \arg\min_{x_1 \in X_1} L_\rho (x_1, x_2^k, \lambda^k),$$

$$x_2^{k+1} = \arg\min_{x_2 \in X_2} L_\rho (x_1^{k+1}, x_2, \lambda^k),$$

$$\lambda^{k+1} = \lambda^k + \rho (x_1^{k+1} - x_2^{k+1}).$$

\(^{12}\)Note that in this example we actually know which IPM solver starting point leads to convergence but in many other practical problems one does not have this information.

\(^{13}\)Note that this is not always obvious as most IPM solvers, when not given a starting point, select a trivial one $[0,0]$, which might not be an ideal starting point for the problem at hand.
The Lagrangian of (30) is written as in (21). The Lagrangian dual function is (theoretically) not guaranteed to converge to a global optimum if the primal problem is nonconvex [43]. Nonetheless, it is possible to witness a zero modified duality gap in such methods if the following conditions are witnessed when applying the proximal method. The only difference is that the proximal method takes longer than ADMM to converge due the oscillatory behaviour seen in Figure 11. However, the superior convergence of ADMM is problem-specific in practice. This should not be surprising as $\rho$ is also considered as the step size in the multiplier update in (26). The step size can certainly be adjusted separately but this will be at the expense of more parameter tuning, which results in the loss of generality and simplicity of the method.

Finally, to underscore the effect of solving (24) and (25) to suboptimality (as might be the case when using an IPM solver), an IPM solver is used in the following two cases for $\rho = 2$. In case 1, where the IPM solver is initialized with $x_{1,2}^0 = [−1, −1]$ at each iteration $k$ and the algorithm with $x_2^k = 1$, ADMM converges to the suboptimal point $d_M^* = p^1$ in 19 iterations, suggesting that (24) and (25) are consistently solved to suboptimality. In case 2, where the IPM solver is initialized with $x_{1,2}^0 = [1, 1]$ at each iteration $k$ and the algorithm with $x_2^k = 1$, ADMM converges to the optimal point $d_M^* = p^*$, similar to the convergence in Figure 14a.

The observations drawn from the examples above can be summarized as follows:

- The subproblems in the augmented Lagrangian relaxation have to be solved to global optimality in order to witness a zero modified duality gap.
- A distributed method that approximates the modified dual function is (theoretically) not guaranteed to converge to a global optimum if the primal problem is nonconvex [43]. Nonetheless, it is possible to witness a zero modified duality gap in such methods if the following conditions are not met.
hold:
- The subproblems are solved to global optimality at each iteration.
- The algorithm is initialized with the same starting point that leads the centralized IPM starting point to a globally optimal solution of the primal.
- The penalty parameters and the step sizes in the multiplier update are tailored specifically to the problem at hand, keeping in mind that some parameter settings can make the method insensitive to the choice of algorithmic starting point.

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