ON THE Q CONSTRUCTION FOR EXACT ∞-CATEGORIES

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Abstract. We prove that the algebraic $K$-theory of an exact $\infty$-category can be computed via an $\infty$-categorical variant of the $Q$ construction. This construction yields a quasicategory whose weak homotopy type is a delooping of the $K$-theory space. We show that the direct sum endows this homotopy type with the structure of a infinite loop space, which agrees with the canonical one. Finally, we prove a proto-dévissage result, which gives a necessary and sufficient condition for a nilimmersion of stable $\infty$-categories to be a $K$-theory equivalence. In particular, we prove that a well-known conjecture of Ausoni–Rognes is equivalent to the weak contractibility of a particular $\infty$-category.

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Exact $\infty$-categories, which we introduced in [2], are a natural $\infty$-categorical generalization of Quillen’s exact categories. They include a large portion of those $\infty$-categories to which one wishes to apply the machinery of Waldhausen’s algebraic $K$-theory. The algebraic $K$-theory of any ordinary exact category, the $K$-theory of arbitrary schemes and stacks, and Waldhausen’s $A$-theory of spaces can all be described as the $K$-theory of exact $\infty$-categories.

Quillen showed that the algebraic $K$-theory of an ordinary exact category can be described as the loopspace of the nerve of a category — the $Q$ construction. In perfect analogy with this, we prove that the algebraic $K$-theory of an exact $\infty$-category can be computed as the loopspace of the classifying space of an $\infty$-category — given by an $\infty$-categorical $Q$ construction. This construction yields a quasicategory whose weak homotopy type is a delooping of the $K$-theory space (Th. 3.10). Moreover, we show that the direct sum endows this homotopy type with the structure of a infinite loop space, which agrees with the canonical one (Th. 4.6).

Finally, we discuss consequences of Quillen’s Theorem A and Theorem B for $\infty$-categories (the latter of which we prove — Th. 5.3) for the algebraic $K$-theory of exact $\infty$-categories. In particular, we prove a “proto-dévissage” theorem (Th. 5.4), which gives a necessary and sufficient condition for a nilimmersion (Df. 5.6) of stable $\infty$-categories to be a $K$-theory equivalence. In particular, we show (Ex. 5.9)
that the well-known conjecture of Ausoni–Rognes [1, (0.2)] can be expressed as the weak contractibility of a relatively simple $\infty$-category.

## 1. Recollections on exact $\infty$-categories

We briefly recall the relevant definitions.

### 1.1. Definition

An $\infty$-category $C$ will be said to be **additive** if its homotopy category $hC$ is additive (as a category enriched in the homotopy category $hKan$ of spaces).

### 1.2. Example

The nerve of any ordinary additive category is additive $\infty$-category, and any stable $\infty$-category is additive.

### 1.3. Definition

Suppose $\mathcal{C}$ an $\infty$-category, and suppose $\mathcal{C}_0, \mathcal{C}^\dagger \subset \mathcal{C}$ subcategories that contain all the equivalences. We call the morphisms of $\mathcal{C}_0$ **ingressive**, and we call the morphisms of $\mathcal{C}^\dagger$ **egressive**.

(1.3.1) A pullback square

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & Y',
\end{array}
\]

is said to be **ambigressive** if $X' \rightarrow Y'$ is ingressive and $Y \rightarrow Y'$ is egressive. Dually, a pushout square

\[
\begin{array}{ccc}
X & \leftarrow & Y \\
\downarrow & & \downarrow \\
X' & \leftarrow & Y',
\end{array}
\]

is said to be **ambigressive** if $X \leftarrow Y$ is ingressive and $X \leftarrow X'$ is egressive.

(1.3.2) We will say that the triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{C}^\dagger)$ is an **exact $\infty$-category** if it satisfies the following conditions.

- (1.3.2.1) The underlying $\infty$-category $\mathcal{C}$ is additive.
- (1.3.2.2) The pair $(\mathcal{C}_0, \mathcal{C}^\dagger)$ is a Waldhausen $\infty$-category.
- (1.3.2.3) The pair $(\mathcal{C}_0, \mathcal{C}^\dagger)$ is a coWaldhausen $\infty$-category.
- (1.3.2.4) A square in $\mathcal{C}$ is an ambigressive pullback if and only if it is an ambigressive pushout.

(1.3.3) An **exact sequence** in $\mathcal{C}$ is an ambigressive pushout/pullback square

\[
\begin{array}{ccc}
X' & \leftarrow & X \\
\downarrow & & \downarrow \\
0 & \leftarrow & X''
\end{array}
\]

in $\mathcal{C}$. The cofibration $X' \leftarrow X$ will be called the **fiber** of $X \leftarrow X''$, and the fibration $X \rightarrow X''$ will be called the **cofiber** of $X' \rightarrow X$.

### 1.4. Example

(1.4.1) The nerve $NC$ of an ordinary category $C$ can be endowed with a triple structure yielding an exact $\infty$-category if and only if $C$ is
an ordinary exact category, in the sense of Quillen, wherein the admissible monomorphisms are exactly the cofibrations, and the admissible epimorphisms are exactly the fibrations. This is proved by appealing to the “minimal” axioms of Keller [6, App. A].

1.4.2) Any stable ∞-category is an exact ∞-category in which all morphisms are both egressive and ingressive.

Suppose ℱ a stable ∞-category equipped with a t-structure, and suppose a and b integers.

1.4.3) The ∞-category ℱ[a,∞) := ℱ[≥a] admits an exact ∞-category structure, in which every morphism is ingressive, but a morphism Y → Z is egressive just in case the induced morphism πaX → πaY is an epimorphism of ℱ⊙.

1.4.4) Dually, the ∞-category ℱ(−∞,b] := ℱ[≤b] admits an exact ∞-category structure, in which every morphism is egressive, but a morphism X → Y is ingressive just in case the induced morphism πbX → πbY is a monomorphism of ℱ⊙.

1.4.5) We may intersect these subcategories to obtain the full subcategory

ℱ[a,b] := ℱ[≥a] ∩ ℱ[≤b] ⊂ ℱ,

and we may intersect the subcategories of ingressive and egressive morphisms described to obtain the following exact ∞-category structure on ℱ[a,b]. A morphism X → Y is ingressive just in case the induced morphism πaX → πaY is a monomorphism of the abelian category ℱ⊙. A morphism Y → Z is egressive just in case the induced morphism πbX → πbY is an epimorphism of ℱ⊙.

More generally, suppose ℱ any stable ∞-category,

1.4.6) Suppose ℶ ⊂ ℱ any full additive subcategory that is closed under extensions. Declare a morphism X → Y of ℶ to be ingressive just in case its cofiber in ℱ lies in ℶ. Dually, declare a morphism Y → Z of ℶ to be egressive just in case its fiber in ℱ lies in ℶ. Then ℶ is exact with this triple structure.

1.5. Definition. Suppose ℶ and ℷ two exact ∞-categories. A functor ℶ → ℷ that preserves both cofibrations and fibrations will be said to be exact if both the functor

(Č′, Č′) → (Ď, Ď)

of Waldhausen ∞-categories and the functor

(Č′, Č′) → (Ď, Ď)

of coWaldhausen ∞-categories are exact.

It turns out that either of these conditions suffices. See [2, Pr. 3.4].

Exact ∞-categories and exact functors between them organize themselves into an ∞-category Exact∞.

2. THE TWISTED ARROW ∞-CATEGORY

In this section, we construct, for any exact ∞-category ℶ, an ∞-category Q(Č′) whose underlying simplicial set is a (single) delooping of K(Č′). The construction proceeds along the same principles as the ones used by Quillen; namely, we construct
an ∞-category $Q(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$, whose morphisms from $X$ to $Y$ are “spans”

\[
\begin{array}{ccc}
X & \xrightarrow{U} & Y \\
& \searrow & \\
& & W
\end{array}
\]

in which the morphism $U \to X$ is egressive and the morphism $U \to Y$ is ingressive, and whose composition law is given by the formation of pullbacks:

\[
\begin{array}{ccc}
X & \xrightarrow{U} & Y \\
& \swarrow & \searrow \\
& & Z
\end{array}
\]

Some new machinery is involved in relating the $Q$ construction to our $\mathcal{S}$ construction. To codify their relationship, we form a Reedy fibrant straightening $Q_*(\mathcal{C})$ of an edgewise subdivision of the left fibration $i_{N\Delta^{op}} \mathcal{S} \to N\Delta^{op}$; then we show that $Q_*(\mathcal{C})$ is in fact a complete Segal space in the sense of Charles Rezk \[12\], whence by a theorem of André Joyal and Myles Tierney \[5\], the simplicial set $Q_n(\mathcal{C})$, whose $n$-simplices may be identified with the vertices of the simplicial set $Q_n(\mathcal{C})$, is a quasicategory whose underlying simplicial set is equivalent to the geometric realization of $Q_*(\mathcal{C})$, which is in turn equivalent to $i_{N\Delta^{op}} \mathcal{S} \to N\Delta^{op}$.

2.1. Proposition. The following are equivalent for a functor $\theta: \Delta \to \Delta$.

2.1.1) The functor $\theta^{op}: N\Delta^{op} \to N\Delta^{op}$ is cofinal in the sense of Joyal \[7, Df. 4.1.1.1\].

2.1.2) The induced endofunctor $\theta^*: s\text{Set} \to s\text{Set}$ on the ordinary category of simplicial sets (so that $(\theta^*X)_n = X_{\theta(n)}$) carries every standard simplex $\Delta^m$ to a weakly contractible simplicial set.

2.1.3) The induced endofunctor $\theta^*: s\text{Set} \to s\text{Set}$ on the ordinary category of simplicial sets is a left Quillen functor for the usual Quillen model structure.

Proof. By Joyal’s variant of Quillen’s Theorem A \[7, Th. 4.1.3.1\], the functor $\theta^{op}$ is cofinal just in case, for any integer $m \geq 0$, the nerve $N(\theta/\mathcal{m})$ is weakly contractible. The category $(\theta/\mathcal{m})$ is clearly equivalent to the category of simplices of $\theta^*(\Delta^m)$, whose nerve is weakly equivalent to $\theta^*(\Delta^m)$. This proves the equivalence of the first two conditions.

It is clear that for any functor $\theta: \Delta \to \Delta$, the induced functor $\theta^*: s\text{Set} \to s\text{Set}$ preserves monomorphisms. Hence $\theta^*$ is left Quillen just in case it preserves weak equivalences. Hence if $\theta^*$ is left Quillen, then it carries the map $\Delta^n \to \Delta^0$ to an equivalence $\theta^*\Delta^n \simeq \theta^*\Delta^0 \cong \Delta^0$, and, conversely, if $\theta^{op}: N\Delta^{op} \to N\Delta^{op}$ is cofinal, then for any weak equivalence $X \simeq Y$, the induced map $\theta^*X \to \theta^*Y$
factors as
\[
\theta^* X \simeq \hocolim_n X_{\theta(n)} \simeq \hocolim_n X_n \\
\simeq X \\
\xrightarrow{\sim} Y \\
\simeq \hocolim_n Y_n \\
\simeq \hocolim_n Y_{\theta(n)} \simeq \theta^* Y,
\]
which is a weak equivalence. This proves the equivalence of the third condition with
the first two. \(\square\)

2.2. **Definition.** Let us call any functor \(\theta: \Delta \to \Delta\) satisfying the equivalent
conditions above a **combinatorial subdivision**.

2.3. As pointed out to us by Katerina Velcheva, one may completely classify combinatorial subdivisions: they are generated by the functors \(id\) and \(op\) under the concatenation operation \(*\) on \(\Delta\). Her work, which we hope will appear soon, also
classifies more general sorts of subdivisions.

Any combinatorial subdivision can be used to doctor our construction of \(K\)-theory in the following manner.

2.4. **Construction.** Suppose \(\theta: \Delta \to \Delta\) a combinatorial subdivision. Then pull-
back along \(\theta\) defines an endofunctor \(\theta^*: \Wald_{\infty,N\Delta^{op}} \to \Wald_{\infty,N\Delta^{op}}\); since \(\theta^{op}\)
is cofinal, it is clear that the following diagram commutes, up to equivalence:

\[
\begin{array}{ccc}
\Wald_{\infty,N\Delta^{op}} & \xrightarrow{\theta^*} & \Wald_{\infty,N\Delta^{op}} \\
|\cdot|_{N\Delta^{op}} & & |\cdot|_{N\Delta^{op}} \\
& \V Wald_{\infty} &
\end{array}
\]

In particular, the additivization of any pre-additive theory \(\phi\) with left derived func-
tor \(\Phi\) can be computed as
\[
D\phi(\mathcal{E}) \simeq \Omega \Phi(\theta^*\mathcal{F}(\mathcal{E})].
\]

The motivating example of a combinatorial subdivision is the following.

2.5. **Example.** Denote by \(\epsilon: \Delta \to \Delta\) the combinatorial subdivision given by the
concatenation \(op \star id\):
\[
\epsilon: [n] \longrightarrow [n]^{op} \star [n] \cong [2n + 1].
\]
Including \([n]\) into either factor of the join \([n]^{op} \star [n]\) (either contravariantly or
covariantly) defines two natural transformations \(op \longrightarrow \epsilon\) and \(id \longrightarrow \epsilon\). This functor
induces an endofunctor \(\epsilon^*\) on the ordinary category of simplicial sets, together with
natural transformations \(\epsilon^* \longrightarrow op\) and \(\epsilon^* \longrightarrow id\).

For any simplicial set \(X\), the **edgewise subdivision** of \(X\) is the simplicial set
\[
\tilde{\theta}(X) := \epsilon^* X.
\]
That is, \(\tilde{\theta}(X)\) is given by the formula
\[
\tilde{\theta}(X)_n = \text{Mor}(\Delta^{n, op} \star \Delta^n, X) \cong X_{2n+1}.
\]
The two natural transformations described above give rise to a morphism
\[ \bar{\delta}(X) \to X^{\text{op}} \times X, \]
functorial in \( X \).

2.6. For any simplicial set \( X \), the vertices of \( \bar{\delta}(X) \) are edges of \( X \); an edge of \( \bar{\delta}(X) \) from \( u \to v \) to \( x \to y \) can be viewed as a commutative diagram (up to chosen homotopy)

\[
\begin{array}{ccc}
& x & \\
\downarrow & & \downarrow \\
v & \leftarrow & y
\end{array}
\]

When \( X \) is the nerve of an ordinary category \( C \), \( \bar{\delta}(X) \) is isomorphic to the nerve of the twisted arrow category of \( C \) in the sense of [3]. When \( X \) is an \( \infty \)-category, we will therefore call \( \bar{\delta}(X) \) the \textit{twisted arrow \( \infty \)-category} of \( X \). This terminology is justified by the following.

2.7. Proposition (Lurie, [3] Pr. 4.2.3]). If \( X \) is an \( \infty \)-category, then the functor \( \bar{\delta}(X) \to X^{\text{op}} \times X \) is a left fibration; in particular, \( \bar{\delta}(X) \) is an \( \infty \)-category.

2.8. Example. To illustrate, for any object \( p \in \Delta \), the \( \infty \)-category \( \bar{\delta}(\Delta^p) \) is the nerve of the category

\[
\begin{array}{c}
0^p \\
\vdots \\
0^2 \downarrow \\
0^1 \downarrow \\
0^0
\end{array}
\]

\[
\begin{array}{c}
1^p \\
\vdots \\
1^2 \downarrow \\
1^1 \downarrow \\
1^0
\end{array}
\]

\[
\begin{array}{c}
2^p \\
\vdots \\
2^2 \downarrow \\
2^1 \downarrow \\
2^0
\end{array}
\]

(Here we write \( n \) for \( p - n \).)

3. The \( \infty \)-categorical \( Q \) construction

We now use the edgewise subdivision to define a quasicategorical variant of Quillen’s \( Q \) construction. To do this, it is convenient to use the theory of marked simplicial sets and the cocartesian model structure. As a result, in the definition and proposition that follow, we will use some of the notation of [7].

3.1. Definition. For any marked simplicial set \( X \), denote by \( R_\ast(X) : \Delta^{\text{op}} \to s\text{Set} \) the functor given by the assignment

\[ [n] \mapsto \text{Map}^\delta(\Delta^{\ast})^{\text{op}} \times X. \]

3.2. Proposition. The functor \( R_\ast : s\text{Set}_{\text{cocart}}^{\ast} \to \text{Fun}(\Delta^{\text{op}}, s\text{Set})_{\text{Reedy}} \) preserves fibrant objects and weak equivalences between them.
Proof. We first show that for any ∞-category $C$, the simplicial space $R_\ast(C^\natural)$ is Reedy fibrant. This is the condition that for any monomorphism $K \hookrightarrow L$, the map

$$\text{Map}^d(\tilde{\Theta}(L)^{\text{op},b}, C^\natural) \to \text{Map}^d(\tilde{\Theta}(K)^{\text{op},b}, C^\natural)$$

is a Kan fibration of simplicial sets. This follows immediately from Pr. 2.1 and [7, Lm. 3.1.3.6]. To see that $R_\ast$ preserves weak equivalences between fibrant objects, we note that for any ∞-category $C$, the simplicial set $\text{Map}^\flat(\tilde{\Theta}(\Delta^n)^{\text{op}}, C^\natural)$ can be identified with the Kan complex $\text{ιFun}(\tilde{\Theta}(\Delta^n)^{\text{op}}, C)$, which clearly respects weak equivalences in $C$. □

3.3. Definition. Suppose $\mathcal{C}$ an exact ∞-category. For any integer $n \geq 0$, let us say that a functor $X : \tilde{\Theta}(\Delta^n) \to \mathcal{C}$ is ambigressive if, for any integers $0 \leq i \leq k \leq \ell \leq j \leq n$, the square

$$\begin{array}{ccc}
X_{ij} & \longrightarrow & X_{kj} \\
\downarrow & & \downarrow \\
X_{i\ell} & \longrightarrow & X_{k\ell}
\end{array}$$

is an ambigressive pullback.

Write $Q_\ast(\mathcal{C}) \subset R_\ast(\mathcal{C})$ for the subfunctor in which $Q_n(\mathcal{C})$ is the full simplicial subset of $R_n(\mathcal{C})$ spanned by the ambigressive functors $X : \tilde{\Theta}(\Delta^n) \to \mathcal{C}$. Note that since any functor that is equivalent to an ambigressive functor is itself ambigressive, the simplicial set $Q_n(\mathcal{C})$ is a union of connected components of $R_n(\mathcal{C})$.

3.4. Proposition. For any exact ∞-category $\mathcal{C}$, the simplicial space $Q_\ast(\mathcal{C})$ is a complete Segal space.

Proof. The Reedy fibrancy of $Q_\ast(\mathcal{C})$ follows easily from the Reedy fibrancy of $R_\ast(\mathcal{C})$.

To see that $Q_\ast(\mathcal{C})$ is a Segal space, it is necessary to show that for any integer $n \geq 1$, the Segal map

$$Q_n(\mathcal{C}) \to Q_1(\mathcal{C}) \times Q_0(\mathcal{C}) \times \cdots \times Q_0(\mathcal{C})$$

is an equivalence. Let $L_n$ denote the ordinary category

$$00 \longrightarrow 01 \longrightarrow 11 \leftarrow 12 \longrightarrow \cdots (n-1)(n-1) \leftarrow (n-1)n \to nn;$$

equip $NL_n$ with the triple structure in which the maps $(i-1)i \to ii$ are ingressive, and the maps $ii \to i(i+1)$ are egressive. The target of the Segal map can then be identified with the maximal Kan complex contained in the full subcategory of $\text{Fun}(NL_n, \mathcal{C})$ spanned by those functors $NL_n \to \mathcal{C}$ that preserve both cofibrations and fibrations. The Segal map is therefore an equivalence by the uniqueness of limits in ∞-categories [7, Pr. 1.2.12.9].

Finally, to check that $Q_\ast(\mathcal{C})$ is complete, let $E$ be the nerve of the contractible ordinary groupoid with two objects; then completeness is equivalent to the assertion that the Rezk map

$$Q_0(\mathcal{C}) \to \lim_{[n] \in (\Delta^1)^{\text{op}}} \text{Q}_n(\mathcal{C})$$

is a weak equivalence. The source of this map can be identified with $\iota\mathcal{E}$; its target can be identified with the full simplicial subset of $\iota\text{Fun}(\tilde{\Theta}(E)^{\text{op}}, \mathcal{C})$ spanned by those functors $X : \tilde{\Theta}(E)^{\text{op}} \to \mathcal{C}$ such that for any simplex $\Delta^n \to E$, the induced functor $\tilde{\Theta}(\Delta^n)^{\text{op}} \to \mathcal{C}$ is ambigressive. Note that the twisted arrow category of
the contractible ordinary groupoid with two objects is the contractible ordinary groupoid with four objects. Hence the image of any functor $X : \tilde{\mathcal{O}}(E)^{\text{op}} \to \mathcal{C}$ is contained in $\iota \mathcal{C}$, whence $X$ is automatically ambigressive. Thus the target of the Rezk map can be identified with $\iota \text{Fun}(\tilde{\mathcal{O}}(E)^{\text{op}}, \mathcal{C})$ itself, and the Rezk map is an equivalence. □

Note that this result does not require the full strength of the condition that $\mathcal{C}$ be an exact $\infty$-category; it requires only that $\mathcal{C}$ admit a triple structure in which ambigressive pullbacks exist, the ambigressive pullback of an ingressive morphism is ingressive, and the ambigressive pullback of an egressive morphism is egressive.

3.5. It is now clear that $Q_*$ defines a relative functor $\text{Exact}_\infty^0 \to \text{CSS}^0$, where $\text{CSS}^0 \subset \text{Fun}(\Delta^{\text{op}}, s\text{Set})$ is the full simplicial subcategory spanned by complete Segal spaces (and $\text{CSS}^0$ is its category of 0-simplices). It therefore defines a functor of $\infty$-categories $Q_* : \text{Exact}_\infty \to \text{CSS}$. We may also regard $Q_*$ as a functor $\Delta^{\text{op}} \times \text{Exact}_\infty^0 \to s\text{Set}$.

Now we aim to show that for an exact $\infty$-category $\mathcal{C}$, the simplicial space $Q_*(\mathcal{C})$ is a straightening of the left fibration $\iota \circ \epsilon \circ \tilde{S}^* \to T_*$. Thus the target of the Rezk map can be identified with $\iota \text{Fun}(\tilde{\mathcal{O}}(E)^{\text{op}}, \mathcal{C})$ itself, and the Rezk map is an equivalence.

3.7. Proposition. The natural transformation $\tau$ induces an equivalence $\iota \circ \epsilon^* \tilde{S}_* \rightarrow T_*$, where $\tilde{S}_*$ is the functor described in [2] 4.16.

3.7.1. Corollary. There is a natural equivalence $K(\mathcal{C}) \rightarrow \Omega|Q_*(\mathcal{C})|$ for any exact $\infty$-category $\mathcal{C}$.

Joyal and Tierney show that the functor that carries a simplicial space $X$ to the simplicial set whose $n$-simplices are the vertices of $X_n$ induces an equivalence of relative categories $\text{CSS}^0 \to \text{Cat}_\infty^0$. This leads us to the following definition and theorem.

3.8. Definition. For any exact $\infty$-category $\mathcal{C}$, denote by $Q(\mathcal{C})$ the $\infty$-category whose $n$-simplices are vertices of $Q_n(\mathcal{C})$, i.e., ambigressive functors $\tilde{\mathcal{O}}(\Delta^n)^{\text{op}} \to \mathcal{C}$. This defines a relative functor $Q : \text{Exact}_\infty^0 \to \text{Cat}_\infty^0$ and hence a functor of $\infty$-categories $Q : \text{Exact}_\infty \to \text{Cat}_\infty$. 

3.9. For any exact $\infty$-category, an $n$-simplex of $Q(\mathcal{C})$ is a diagram

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]

of $\mathcal{C}$ in which every square is an ambigressive pullback/pushout. (Here we write $\overline{n}$ for $p - n$.)

3.10. **Theorem.** There is a natural equivalence $K(\mathcal{C}) \xrightarrow{\sim} \Omega Q(\mathcal{C})$ for any exact $\infty$-category $\mathcal{C}$.

As a final note, let us note that the $\infty$-categorical $Q$ construction we have introduced here is an honest generalization of Quillen’s original $Q$ construction.

3.11. **Proposition.** If $C$ is an ordinary exact category, then $Q(\text{NC})$ is canonically equivalent to the nerve $\text{N}(Q\text{C})$ of Quillen’s $Q$ construction.

**Proof.** Unwinding the definitions, we find that the simplicial set $\text{Hom}_{Q(\text{NC})}^R(X,Y)$ of [7, §1.2.2] is isomorphic to the nerve of the groupoid $\text{G}(X,Y)$ in which an object $Z$ is a diagram

\[
\begin{array}{c}
Z \\
\xrightarrow{X} X, \\
\xrightarrow{Y} Y,
\end{array}
\]

where $Z \twoheadrightarrow X$ is an admissible epimorphism and $Z \hookrightarrow Y$ is an admissible monomorphism, and in which a morphism $Z' \hookrightarrow Z$ is a diagram

\[
\begin{array}{c}
Z' \\
\xleftarrow{X} X, \\
\xrightarrow{Y} Y, \\
\xleftarrow{Z} Z,
\end{array}
\]

It is now immediate that $\text{N}(Q\text{C})$ is equivalent to the homotopy category of $Q(\text{NC})$, and so it suffices by [7] Pr. 2.3.4.18 to verify that every connected component of the groupoid $\text{G}(X,Y)$ is contractible. For this, we simply note that if $f, g$ are two morphisms $Z' \hookrightarrow Z$ of $\text{G}(X,Y)$, then $f = g$ since $Z \hookrightarrow Y$ is a monomorphism. \[\square\]
4. AN INFINITE DELOOPING OF $Q$

In this section, we describe an infinite delooping of the $Q$ construction of the previous section. In effect, the direct sum on an exact $\infty$-category $\mathcal{C}$ induces a symmetric monoidal structure on the $\infty$-category $Q(\mathcal{C})$. Segal’s delooping machine then applies to give an infinite delooping of the $Q$ construction. We will then show that this delooping coincides with the canonical one from [3, Cor. 7.4.1] by appealing to the Additivity Theorem [3, Th. 7.2]. We thank Lars Hesselholt for his questions about such a delooping; his hunch was right all along.

It is convenient to introduce some ordinary categories that control the combinatorics of direct sums.

4.1. Notation. Denote by $\Lambda(F)$ the following ordinary category. An object of $\Lambda(F)$ is a finite set; a morphism $J \rightarrow I$ of $\Lambda(F)$ is a map $J \rightarrow I_+$, or equivalently a pointed map $J_+ \rightarrow I_+$. Clearly $\Lambda(F)$ is isomorphic to the category of pointed finite sets, but we shall regard the objects of $\Lambda(F)$ simply as finite (unpointed) sets.

Recall that $\Lambda(F)$ admits a symmetric monoidal structure that carries a pair of finite sets to their product. We shall denote this symmetric monoidal structure $(I,J) \mapsto I \wedge J$.

For any finite set $I$, denote by $L_I$ the following ordinary category. An object of $L_I$ is a subset $J \subset I$. A morphism from an object $K \subset I$ to an object $J \subset I$ is a map $\psi: K \rightarrow J$ such that the square

$$
\begin{array}{ccc}
\psi^{-1}(J) & \rightarrow & J \\
\downarrow & & \downarrow \\
K & \leftarrow & I
\end{array}
$$

commutes.

Any morphism $\phi: I' \rightarrow I$ of $\Lambda(F)$ induces a functor $\phi^*: L_I \rightarrow L_I'$ that carries $J \subset I$ to $\phi^{-1}(J) \subset I'$. With this, one confirms easily that the assignments $I \mapsto L_I$ and $\phi \mapsto \phi^*$ together define a functor $L: \Gamma \rightarrow \text{Cat}$.

If $K \subset J$, then write $i_{K \subset J}$ for the morphism $K \rightarrow J$ of $\Lambda(F)$ given by the inclusion $K \hookrightarrow J_+$, and write $p_{K \subset J}$ for the morphism $J \rightarrow K$ of $\Lambda(F)$ given by the map $J \mapsto K_+$ that carries every $j \in J \setminus K$ to the basepoint, and every element $j \in K$ to itself.

We now proceed to show that an exact $\infty$-category $\mathcal{C}$ admits an essentially unique symmetric monoidal structure in which all the multiplication functors are exact.

4.2. Notation. We write $\text{Exact}\_\infty$ for the full subcategory of $\text{Fun}(N\Lambda(F), \text{Exact}\_\infty)$ consisting of those functors $\mathcal{X}: N\Lambda(F) \rightarrow \text{Exact}\_\infty$ such that for any finite set $I$, the functors $\{\mathcal{X}(I) \rightarrow \mathcal{X}(\{i\})\}_{i \in I}$ induced by the morphisms $p_{\{i\} \subset I}: I \rightarrow \{i\}$ together exhibit the exact $\infty$-category $\mathcal{X}(I)$ as a product of the exact $\infty$-categories $\mathcal{X}(\{i\})$. Write $U: \text{Exact}\_\infty \rightarrow \text{Exact}\_\infty$ for the evaluation functor $\mathcal{X} \mapsto \mathcal{X}(\{1\})$.

In the opposite direction, let us construct a functor $\text{DS}: \text{Exact}\_\infty \rightarrow \text{Exact}\_\infty$. For any finite set $I$ and any exact $\infty$-category, denote by $\text{DS}(I; \mathcal{C})$ the full subcategory of $\text{Fun}(NL_I, \mathcal{C})$ spanned by those functors $X: NL_I \rightarrow \mathcal{C}$ such that, for any subset $J \subset I$, 

\[
\begin{array}{ccc}
\end{array}
\]
the set of morphisms
\[ \{ X(j) \to X(\{j\}) \}_{j \in J} \]
induced by the morphisms \( p_{(j) \in J} : J \to \{j\} \) of \( L_1 \) exhibit \( X(j) \) as a product of the objects \( X(\{j\}) \), and,
\[(4.3.2) \text{ dually, the morphisms } \{ X(\{j\}) \to X(J) \}_{j \in J} \text{ induced by the morphisms } i_{(j) \in J} : \{j\} \to J \text{ of } L_1 \text{ exhibit } X(J) \text{ as a coproduct of the objects } X(\{j\}). \]

It is clear that \((I, \mathcal{C}) \mapsto DS(I; \mathcal{C})\) defines a simplicial functor
\[ N \Lambda(F) \times \text{Exact}_\infty \to \text{Cat}_\infty. \]

Now suppose \( \mathcal{C} \) an exact \( \infty \)-category. It follows from the uniqueness of limits and colimits in \( \infty \)-categories [7, Pr. 1.2.12.9] that for any finite set \( I \), the functors
\[ \{ DS(I; \mathcal{C}) \to DS(\{i\}; \mathcal{C}) \}_{i \in I} \]
induced by the morphisms \( p_{(i) \in I} : I \to \{i\} \) of \( \Lambda(F) \) together exhibit the \( \infty \)-category \( DS(I; \mathcal{C}) \) as the product of the \( \infty \)-categories \( DS(\{i\}; \mathcal{C}) \), which are each in turn equivalent to \( \mathcal{C} \). Consequently the \( \infty \)-categories \( DS(I; \mathcal{C}) \) are exact \( \infty \)-categories, and since direct sum in \( \mathcal{C} \) preserves ingressives and any pushouts that exist, one easily confirms that \( DS(\_; \mathcal{C}) \) is an object of \( \text{Exact}_\infty^\oplus \); this therefore defines a functor \( DS : \text{Exact}_\infty \to \text{Exact}_\infty^\oplus \).

4.3. Proposition. The functor \( U \) exhibits an equivalence \( \text{Exact}_\infty^\oplus \equiv \text{Exact}_\infty \), and the functor \( DS \) exhibits a quasi-inverse to it.

Proof. In light of the discussion above, it suffices to prove that for any object \( \mathcal{X} \in \text{Exact}_\infty^\oplus \), the cocartesian fibration \( \mathcal{X}^{\oplus} \to N \Lambda(F) \) classified by the composite
\[ N \Lambda(F) \to \text{Exact}_\infty \to \text{Cat}_\infty \]
is a cartesian symmetric monoidal \( \infty \)-category [8, Df. 2.4.0.1]. It is obvious that it is symmetric monoidal. Furthermore, since the map \( \Delta^0 \simeq \mathcal{X}(\emptyset) \to \mathcal{X}(\{1\}) \) is exact, the unit object is a zero object, and since the functor
\[ \otimes : \mathcal{X}(\{1\}) \times \mathcal{X}(\{1\}) \simeq \mathcal{X}(\{1, 2\}) \to \mathcal{X}(\{1\}) \]
induced by the unique map \( \{1, 2\} \to \{1\}_+ \) that does not hit the basepoint is exact, when applied to pullback squares
\[
\begin{array}{ccc}
X & \to & 0 \\
\downarrow & & \downarrow \\
X & \to & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Y & \to & Y \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]
it yields a pullback square
\[
\begin{array}{ccc}
X \otimes Y & \to & Y \\
\downarrow & & \downarrow \\
X & \to & 0
\end{array}
\]
Thus \( \mathcal{X}^{\oplus} \) is cartesian. \( \square \)
4.4. We may now lift the $Q$ construction to $\text{Exact}_\infty^{\oplus}$. Composition with the functors

$$Q: \text{Exact}_\infty \longrightarrow \text{Cat}_\infty \quad \text{and} \quad Q_*: \text{Exact}_\infty \longrightarrow \text{CSS}$$

defines functors

$$\text{Fun}(\Lambda(\mathcal{F}), \text{Exact}_\infty) \longrightarrow \text{Fun}(\Lambda(\mathcal{F}), \text{Cat}_\infty)$$

and

$$\text{Fun}(\Lambda(\mathcal{F}), \text{Exact}_\infty) \longrightarrow \text{Fun}(\Lambda(\mathcal{F}), \text{CSS}),$$

which restrict to functors

$$Q^\oplus: \text{Exact}_\infty^{\oplus} \longrightarrow \text{Cat}_\infty^{\oplus} \quad \text{and} \quad Q_*^\oplus: \text{Exact}_\infty^{\oplus} \longrightarrow \text{CSS}^{\oplus}$$

where $\text{Cat}_\infty^{\oplus}$ (respectively, $\text{CSS}_\infty^{\oplus}$) denotes the full subcategory of the $\infty$-category $\text{Fun}(\Lambda(\mathcal{F}), \text{Cat}_\infty)$ (respectively, of $\text{Fun}(\Lambda(\mathcal{F}), \text{CSS})$) spanned by those functors $X: \Lambda(\mathcal{F}) \longrightarrow \text{Cat}_\infty$ (respectively, by those functors $X: \Lambda(\mathcal{F}) \longrightarrow \text{CSS}$) such that for any finite set $I$, the morphisms $\{X(I) \longrightarrow X(\{i\})\}_{i \in I}$ induced by the morphisms $p_{i \subseteq I}: I \longrightarrow \{i\}$ together exhibit $X(I)$ as a product of $X(\{i\})$. We thus have commutative squares

$$\begin{array}{ccc}
\text{Exact}_\infty^{\oplus} & \xrightarrow{Q^\oplus} & \text{Cat}_\infty^{\oplus} \\
\downarrow U & & \downarrow U \\
\text{Exact}_\infty & \xrightarrow{Q} & \text{Cat}_\infty
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\text{Exact}_\infty^{\oplus} & \xrightarrow{Q_*^\oplus} & \text{CSS}_\infty^{\oplus} \\
\downarrow U & & \downarrow U \\
\text{Exact}_\infty & \xrightarrow{Q_*} & \text{CSS}
\end{array}$$

in which the vertical maps are evaluation at the vertex $\{1\} \in \Lambda(\mathcal{F})$.

For any object $\mathcal{X} \in \text{Exact}_\infty^{\oplus}$, the object $Q^\oplus \mathcal{X} \simeq [Q_*^\oplus \mathcal{X}]$ is in particular a $\Gamma$-space that satisfies the Segal condition and, since the underlying space of $Q\mathcal{X}(\{1\})$ is connected, it is grouplike. Segal’s delooping machine therefore provides an infinite delooping of $Q\mathcal{X}(\{1\})$.

Let us now prove that the infinite delooping provided by $Q^\oplus$ agrees with the one guaranteed by [3, Cor. 7.4.1]. Roughly speaking, the approach is the following. For any $\Gamma$-space $X$ that satisfies the Segal condition that is grouplike, the $n$-fold delooping provided in this way coincides with the iterated $\mathcal{X}$ construction.

4.5. Notation. Write $u: \Delta^{op} \longrightarrow \Lambda(\mathcal{F})$ for the following functor. For any nonempty totally ordered finite set $S$, let $u(S)$ be the set of surjective morphisms $S \longrightarrow [1]$ of $\Delta$. For any map $g: S \longrightarrow T$ of $\Delta$, define the map $u(g): u(T) \longrightarrow u(S)$ by

$$u(g)(\eta) = \begin{cases} 
\eta \circ g & \text{if } \eta \circ g \text{ is surjective;} \\
\ast & \text{otherwise.}
\end{cases}$$

Furthermore, for any nonnegative integer $n$, we may define $u^{(n)}: (\Delta^{op})^n \longrightarrow \Lambda(\mathcal{F})$ as the composite

$$(\Delta^{op})^n \xrightarrow{u^n} (\Lambda(\mathcal{F}))^n \xrightarrow{\Delta^n} \Lambda(\mathcal{F}).$$

For any nonempty totally ordered finite set $S$ and any morphism $\alpha: [1] \longrightarrow S$ of $\Delta$, let $\rho_S(\alpha) \subset u(S)$ be the set of retractions of $\alpha$, i.e., the morphisms $\beta: S \longrightarrow [1]$. 

such that $\beta \circ \alpha = \text{id}$. For any nonnegative integer $n$, denote by $O: \Delta^{\times n} \rightarrow \text{Cat}$ for the multicosimplicial category given by

$$O(S_1, S_2, \ldots, S_n) := \text{Fun}(\{1\}, S_1) \times \text{Fun}(\{1\}, S_2) \times \cdots \times \text{Fun}(\{1\}, S_n),$$

and write $\rho: O \rightarrow u^{(n), \text{op}} \circ L$ for the natural transformation given by

$$\rho_{S_1, S_2, \ldots, S_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) := \rho_{S_1}(\alpha_1) \wedge \rho_{S_2}(\alpha_2) \wedge \cdots \wedge \rho_{S_n}(\alpha_n).$$

This natural transformation induces a natural transformation $\rho^*: u^{(n), \text{op}} \circ DS \rightarrow \overline{S}_\ast^{(n)}$ between functors $\Delta^{\text{op}} \times \text{Exact}_\infty \rightarrow \text{Wald}_\infty$. Combined with the natural transformation $\tau$ from Def. 3.6, we obtain a natural transformation $R: u^{(n), \ast} \circ Q^\oplus \circ DS \rightarrow \iota \circ \epsilon^* \overline{S}_\ast \circ \overline{S}_\ast^{(n)}$ between functors $\Delta^{\text{op}} \times (\Delta^{\text{op}})^{\times n} \times \text{Exact}_\infty \rightarrow \text{Wald}_\infty$.

The following is an immediate consequence of the Additivity Theorem [3, Th. 7.2].

4.6. Proposition. For any positive integer $n$, the natural transformation $R$ above is a natural equivalence

$$|u^{(n), \ast} \circ Q^\oplus \circ DS| \simeq |\iota \circ \epsilon^* \overline{S}_\ast \circ \overline{S}_\ast^{(n)}|$$

of functors $\text{Exact}_\infty \rightarrow \text{Kan}$.

In other words, the delooping of algebraic $K$-theory provided by the $Q^\oplus$ construction is naturally equivalent to the canonical delooping obtained in [3, Cor. 7.4.1].

5. Nilimmersions and a relative $Q$ construction

One of the original uses of the $Q$ construction was Quillen’s proof of the Dévissage Theorem, which gives an effective way of determining whether an exact functor $\psi: \mathcal{B} \rightarrow \mathcal{A}$ induces a $K$-theory equivalence [10, Th. 4]. This theorem made it possible for Quillen to identify the “fiber term” in his Localization Sequence for higher algebraic $K$-theory [10, Th. 5]. The technical tool Quillen introduced for this purpose was his celebrated Theorem A. Joyal proved the following $\infty$-categorical variant of Quillen’s Theorem A.

5.1. Theorem (Theorem A for $\infty$-categories [7, Th. 4.1.3.1]). Suppose $G: C \rightarrow D$ a functor between $\infty$-categories. If, for any object $X \in D$, the $\infty$-category

$$G_X/ := D_{X/} \times_D C$$

is weakly contractible, then the map of simplicial sets $G$ is a weak homotopy equivalence.

This form of Theorem A (or rather its opposite) directly implies a recognition principle for $K$-theory equivalences in the $\infty$-categorical context:

5.2. Proposition. An exact functor $\psi: \mathcal{B} \rightarrow \mathcal{A}$ between exact $\infty$-categories induces an equivalence of $K$-theory spectra if, for every object $X \in \mathcal{B}$, the simplicial set

$$Q(\psi)/X := Q(\mathcal{B}) \times_{Q(\mathcal{A})} Q(\mathcal{A})/X$$

is weakly contractible.
There is also the following variant of Quillen’s Theorem B.

5.3. **Theorem** (Theorem B for $\infty$-categories). Suppose $G : C \to D$ a functor between $\infty$-categories. If, for any morphism $f : X \to Y$ of $D$, the map $f^* : G_{X/} \to G_{Y/}$ is a weak homotopy equivalence, then for any object $X \in D$, the square

$$
\begin{array}{ccc}
G_{X/} & \to & C \\
\downarrow & & \downarrow \\
D_{X/} & \to & D
\end{array}
$$

is a homotopy pullback (for the Quillen model structure), and of course $D_{X/} \simeq \ast$.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
G_{X/} & \to & \tilde{\mathcal{O}}(D) \times_D C \to C \\
\downarrow & & \downarrow \\
D_{X/} & \to & \tilde{\mathcal{O}}(D) \to D \\
\downarrow & & \downarrow \\
\{X\} & \to & D^{\text{op}}
\end{array}
$$

in which every square is a pullback. The upper right and lower left squares are homotopy pullbacks because opposite maps in these squares are weak homotopy equivalences. It therefore remains to show that the large left-hand rectangle is a homotopy pullback. For this, it is enough to show that $\tilde{\mathcal{O}}(D) \times_D C \to D^{\text{op}}$ satisfies the conditions Waldhausen’s Theorem B for simplicial sets [13, Lm. 1.4.B] or, equivalently, is a sharp map in the sense of Hopkins and Rezk [11, §2].

To prove this, suppose $\sigma : \Delta^n \to D^{\text{op}}$ an $n$-simplex. Denote by $\sigma(0)$ its restriction to the $0$-simplex $\Delta^0 \subset \Delta^n$. The pullback

$$
G_{\sigma/} \simeq (\tilde{\mathcal{O}}(D) \times_D C) \times_{D^{\text{op}}} \Delta^n
$$

is naturally equivalent as an $\infty$-category to $G_{\sigma(0)/}$, and for any map $\eta : \Delta^m \to \Delta^n$, the pullback

$$
G_{\sigma \circ \eta/} \simeq (\tilde{\mathcal{O}}(D) \times_D C) \times_{D^{\text{op}}} \Delta^m
$$

is naturally equivalent to $G_{\sigma(\eta(0))/}$. Our hypothesis is precisely that the morphism $\sigma(\eta(0)) \to \sigma(0)$ induces a weak homotopy equivalence $G_{\sigma(0)/} \to G_{\sigma(\eta(0))/}$, whence $\eta$ induces an equivalence

$$
(\tilde{\mathcal{O}}(D) \times_D C) \times_{D^{\text{op}}} \Delta^n \simeq G_{\sigma/} \simeq G_{\sigma \circ \eta/} \simeq (\tilde{\mathcal{O}}(D) \times_D C) \times_{D^{\text{op}}} \Delta^m,
$$

as desired. \qed

The various conditions required in Quillen’s Dévissage Theorem [10, Th. 4] are quite stringent, even for abelian For stable $\infty$-categories, they are simply unreasonable. For example, the inclusion of the stable $\infty$-category of bounded complexes of finite-dimensional $F_p$-vector spaces into the stable $\infty$-category of $p$-torsion bounded complexes of finitely generated abelian groups is not full, and there is no meaningful sense in which it is closed under the formation of subobjects.
Nevertheless, one can make use of the sort of filtrations Quillen employed in his Dévissage Theorem. To this end, let us suppose that $\psi: B \rightarrow A$ is an exact functor between stable $\infty$-categories. Our aim is to reduce the study of the map $Q(B) \rightarrow Q(A)$ induced by $\psi$ to the study of the weak homotopy type of a single $\infty$-category $Z(\psi)$. The key lemma that makes this reduction possible is the following, which is analogous to the proof of Quillen’s Dévissage Theorem [10, Th. 4].

5.4. Lemma. For any object $U \in B$ and any pushout square

$$
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
0 & \rightarrow & \psi U
\end{array}
$$

of $\mathcal{A}$, the functor $\phi: Q(\psi)/Y \rightarrow Q(\psi)/X$ induced by $\phi$ is a weak homotopy equivalence.

Proof. We define a functor $\phi^*: Q(\psi)/X \rightarrow Q(\psi)/Y$, functors

$$
\lambda_\phi: Q(\psi)/X \rightarrow Q(\psi)/X \quad \text{and} \quad \mu_\phi: Q(\psi)/Y \rightarrow Q(\psi)/Y,
$$

and natural transformations

$$
\phi_\phi \circ \phi^* \xrightarrow{\alpha} \lambda_\phi \xleftarrow{\beta} \text{id}_{Q(\psi)/X} \\
\phi^* \circ \phi_\phi \xrightarrow{\gamma} \mu_\phi \xleftarrow{\delta} \text{id}_{Q(\psi)/Y}.
$$

These will exhibit $\phi^*$ as a homotopy inverse of $\phi_\phi$.

For any object $T$ of $\mathcal{A}$, an $n$-simplex $(W, Z, g)$ of $Q(\psi)/T$ may be said to consist of:

- a diagram $W$ in $B$ of the form

$$
\begin{array}{c}
W_0n \rightarrow W_1n \rightarrow \cdots \rightarrow W_{(n-1)n} \rightarrow W_nn \\
\downarrow \quad \downarrow \quad \downarrow \\
W_0(n-1) \rightarrow W_1(n-1) \rightarrow \cdots \rightarrow W_{(n-1)(n-1)} \\
\downarrow \quad \downarrow \\
\vdots \quad \vdots \\
\downarrow \quad \downarrow \\
W_{01} \rightarrow W_{11} \\
\downarrow \\
W_{00}
\end{array}
$$

in which every square is a pullback square,

- a sequence $Z$ of edges in $\mathcal{A}/T$ of the form

$$
Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_n \rightarrow T,
$$

and
— a diagram $g$ in $\mathcal{A}$ of the form

$$
\begin{array}{c}
Z_0 \\
g_0 \downarrow & \downarrow g_1 & \cdots & \downarrow g_n \\
\psi W_0 & \psi W_1 & \cdots & \psi W_n
\end{array}
$$

in which every square is a pullback.

In this notation, the functor $\phi_!$ carries a simplex $(W, Z, g) \in Q(\psi)/Y$ to $(W, Z, g) \in Q(\psi)/X$, where by an abuse of notation, we write $Z$ also for the image of $Z$ in $\mathcal{A}/X$.

Now we may define the remaining functors as follows.

— Let $\phi^*$ be the functor $Q(\psi)/X \to Q(\psi)/Y$ that carries a simplex $(W, Z, g) \in Q(\psi)/X$ to

$$(W, Z \times_X Y, g \circ pr_Z) \in Q(\psi)/Y,$$

where $(Z \times_X Y)_i := Z_i \times_X Y$, and $(pr_Z)_i$ is the projection $Z_i \times_X Y \to Z_i$.

— Let $\lambda_\oplus$ be the functor $Q(\psi)/X \to Q(\psi)/X$ that carries a simplex $(W, Z, g) \in Q(\psi)/X$ to

$$(W \oplus U, Z, j_U \circ g) \in Q(\psi)/X,$$

where $(W \oplus U)_i := W_i \oplus U$, and $(j_U)_i$ is the inclusion $\psi W_i \hookrightarrow \psi (W_i \oplus U)$.

— Let $\mu_{\psi}$ be the functor $Q(\psi)/Y \to Q(\psi)/Y$ that carries a simplex $(W, Z, g) \in Q(\psi)/Y$ to

$$(W \oplus \Omega U, Z \oplus \psi \Omega U, g \oplus \text{id}_{\psi \Omega U}) \in Q(\psi)/Y.$$

The composite $\phi_! \circ \phi^* : Q(\psi)/X \to Q(\psi)/X$ clearly carries $(W, Z, g) \in Q(\psi)/X$ to

$$(W, Z \times_X Y, g \circ pr_Z) \in Q(\psi)/X.$$

On the other hand, the canonical equivalence $Y \times_X Y \simeq Y \oplus \psi \Omega U$ permits us to express the composite $\phi^* \circ \phi_! : Q(\psi)/Y \to Q(\psi)/Y$ as the functor that carries $(W, Z, g) \in Q(\psi)/Y$ to

$$(W, Z \oplus \psi \Omega U, g \circ pr_Z) \in Q(\psi)/Y.$$

Now we are prepared to define the natural transformations $\alpha, \beta, \gamma, \delta$.

— The component at $(W, Z, g)$ of the natural transformation $\alpha$ is the diagram

$$
\begin{array}{c}
Z \times_X Y \\
g \circ pr_Z \downarrow \downarrow j_U \circ g \\
\psi W & \psi \psi (W \oplus U) \\
\psi W
\end{array}
$$

the square is a pullback since

$$
\begin{array}{c}
Z \times_X Y \\
\downarrow \\
Z \\
\downarrow \\
Z \oplus \psi U
\end{array}
$$

is so.
— The component at \((W, Z, g)\) of the natural transformation \(\beta\) is the diagram

\[
\begin{array}{ccc}
Z & \rightarrow & Z \\
\downarrow \psi(W \oplus U) & & \downarrow \psi(W \oplus U) \\
\psi U.
\end{array}
\]

— The component at \((W, Z, g)\) of the natural transformation \(\gamma\) is the diagram

\[
\begin{array}{ccc}
Z \oplus \psi \Omega U & \rightarrow & Z \oplus \psi \Omega U \\
\downarrow \psi(W \oplus \Omega U) & & \downarrow \psi(W \oplus \Omega U) \\
\psi U.
\end{array}
\]

— Finally, the component at \((W, Z, g)\) of the natural transformation \(\delta\) is the diagram

\[
\begin{array}{ccc}
Z & \rightarrow & Z \oplus \Omega U \\
\downarrow g & & \downarrow g \oplus \text{id}_{\psi \Omega U} \\
\psi W & \rightarrow & \psi(W \oplus \Omega U)
\end{array}
\]

\[
\psi W.
\]

This completes the proof. \(\square\)

5.5. It’s worthwhile to note now some distinctions between our argument and Quillen’s. Our argument is more complicated in some ways and easier in others. (We suspect that one should be able to prove a result for more general exact \(\infty\)-categories that contains all the possible complications and specializes both to the lemma above and to Quillen’s result, but the additional complications are unnecessary for our work here.) The functor \(\mu_\phi\) above, for instance, does not make an appearance in Quillen’s argument. In effect, this is because on an ordinary abelian category, the loop space functor \(\Omega\) coincides the constant functor at 0. Furthermore, Quillen has to make use of closure properties of the full subcategory in order to check that suitable kernels and cokernels exist. In the stable setting, these issues vanish. Finally, in Quillen’s case, \(Q(\psi)/0\) is visibly contractible, and the lemma above along with the existence of suitable filtrations imply the Dévissage Theorem. In our situation, it is not always the case that \(Q(\psi)/0\) weakly contractible; this condition has to be verified separately. But we can find analogues of the filtrations sought by Quillen.

5.6. \textbf{Definition.} A filtration

\[
0 = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X
\]
of an object $X \in \mathcal{A}$ will be said to be $\psi$-admissible if the following conditions are satisfied.

(5.6.1) The diagram above exhibits $X$ as the colimit $\text{colim} X_i$.
(5.6.2) For any integer $i \geq 1$, the cofiber $C_i := X_i / X_{i-1}$ is equivalent to $\psi U_i$ for some object $U_i \in \mathcal{B}$.
(5.6.3) For any corepresentable functor $F : \mathcal{A} \to \text{Kan}$, the diagram

$$0 = FX_0 \to FX_1 \to \cdots \to FX$$

exhibits $FX$ as the colimit $\text{colim} FX_i$.

We will say that $\psi$ is a nilimmersion if every object of $\mathcal{A}$ admits a $\psi$-admissible filtration.

5.7. Example. Suppose $\Lambda$ an $E_1$ ring, and suppose $x \in \pi_* \Lambda$ a homogeneous element of degree $d$. Suppose $f : \Lambda \to \Lambda'$ a morphism of $E_1$ rings such that one has a cofiber sequence

$$\Lambda[d] \xrightarrow{x} \Lambda \xrightarrow{f} \Lambda'$$

of left $\Lambda$-modules. Assume that the induced functor $\text{Mod}_{\Lambda'}^f \to \text{Mod}_{\Lambda}^f$ preserves compact objects, so that it restricts to a functor $\text{Perf}_{\Lambda'}^f \to \text{Perf}_{\Lambda}^f$. If the multiplicative system $S \subset \pi_* \Lambda$ generated by $x$ satisfies the left Ore condition, then by [3] Lm. 8.2.4.13, the natural functor $\text{Perf}_{\Lambda'}^f \to \text{Nil}_{(\Lambda,S)}^f$ is a nilimmersion [3] Pr. 11.15.

5.8. Example. As a subexample, when $p$ is prime and $\Lambda = \text{BP}(n)$, we have a nilimmersion

$$\text{Perf}_{\text{BP}(n-1)}^f \to \text{Nil}_{(\text{BP}(n),S)}^f$$

where $S \subset \pi_* \Lambda$ is the multiplicative system generated by $v_n$. As suggested by [3] Ex. 11.16, this particular nilimmersion is of particular import for a well-known conjecture of Ausoni–Rognes [1] (0.2).

5.9. Theorem (“Proto-dévissage”). Suppose $\psi$ a nilimmersion. Then the square

$$\begin{array}{cc}
Q(\psi)/0 & \to & Q(\mathcal{B}) \\
\downarrow & & \downarrow \\
Q(\mathcal{A})/0 & \to & Q(\mathcal{A})
\end{array}$$

is a homotopy pullback (for the Quillen model structure), and of course $Q(\mathcal{A})/0 \simeq \ast$. In particular, the $\infty$-category $Q(\psi)/0$ is weakly contractible just in case the induced map $K(\mathcal{B}) \to K(\mathcal{A})$ is a weak equivalence.

Proof. We employ our variant of Theorem B (Th. 5.3). The conditions of this theorem will be satisfied once we check that for any object $X \in \mathcal{A}$, the $\infty$-category $Q(\psi)/_X$ is weakly equivalent to $Q(\psi)/_0$. For this, we apply Lm. 5.4 to a $\psi$-admissible filtration

$$0 = X_0 \to X_1 \to \cdots \to X$$

to obtain a sequence of weak homotopy equivalences of $\infty$-categories

$$Z(\psi) \simeq Q(\psi)/_0 = Q(\psi)/_{X_0} \xrightarrow{\sim} Q(\psi)/_{X_1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} Q(\psi)/_X.$$
The result now follows from the claim that the diagram above exhibits $Q(\psi)/X$ as the homotopy colimit (even in the Joyal model structure) of the $\infty$-categories $Q(\psi)/X_i$.

To prove this claim, it is enough to show the following: (1) that the set $\pi_0 Q(\psi)/X$ is exhibited as the colimit of the sets $\pi_0 Q(\psi)/X_i$, and (2) that for any objects $A, B \in Q(\psi)/X_i$, the natural map

$$\text{colim}_{j \geq i} \, \text{Map}_{Q(\psi)/X_j}(A,B) \to \text{Map}_{Q(\psi)/X}(A,B)$$

is an equivalence. The first claim follows directly, since any morphism $Y \to X$ factors through a morphism $Y \to X_i$ for some integer $i \geq 0$ by (5.6.3). For the second, note that for any object $Y \in \mathcal{A}$, one may identify a mapping space $\text{Map}_{Q(\psi)/Y}(A,B)$ as the homotopy fiber of the map $\text{Map}_{Q(\psi)/0}(A,B) \to \text{Map}_{\mathcal{A}}(s(A), s(Y))$ induced the source functor $s: Q(\psi)/0 \to \mathcal{A}$ and the given map $s(B) \to s(Y)$ over the point corresponding to the given map $s(A) \to s(Y)$. So the natural map above can be rewritten as the natural map

$$\text{colim}_{j \geq i} \left( \text{Map}_{Q(\psi)/0}(A,B) \times_{\text{Map}_{\mathcal{A}}(s(A), s(X_j))} \{\phi_j\} \right)$$

where $\phi_j: s(A) \to s(X_j)$ and $\phi: s(A) \to s(X)$ are the given maps. Now since filtered homotopy colimits commute with homotopy pullbacks, the proof is complete thanks to (5.6.3).

Let us study the $\infty$-category $Q(\psi)/0$ more explicitly.

5.10. Construction. The $\infty$-category $\mathcal{O}(\mathcal{A}) := \text{Fun}(\Delta^1, \mathcal{A})$ is also stable, and the cartesian and cocartesian fibration $t: \mathcal{O}(\mathcal{A}) \to \mathcal{A}$ induces a cartesian fibration $p: Q(\mathcal{O}(\mathcal{A})) \to Q(\mathcal{A})$.

One observes that $p$ is classified by a functor $Q(\mathcal{A})^\text{op} \to \text{Cat}_\infty$ that carries an object $X \in \mathcal{A}$ to the $\infty$-category $\mathcal{A}/X$. We may now extract the maximal right fibration

$$t_{Q(\mathcal{A})} P: t_{Q(\mathcal{A})} Q(\mathcal{O}(\mathcal{A})) \to Q(\mathcal{A})$$

contained in $p$. Write

$$E(\mathcal{A}) := t_{Q(\mathcal{A})} Q(\mathcal{O}(\mathcal{A})).$$

In light of [4, Cor. 3.3.4.6], the simplicial set $E(\mathcal{A})$ is the (homotopy) colimit of the functor $Q(\mathcal{A})^\text{op} \to \text{Kan}$ that classifies $t_{Q(\mathcal{A})} P$. This functor carries an object $X \in \mathcal{A}$ to the Kan complex $t(\mathcal{A}/X)$, and it carries a morphism

$$Z \quad \quad \quad \quad \quad \quad Y \quad \quad \quad \quad \quad \quad X$$

to the map $t(\mathcal{A}/X) \to t(\mathcal{A}/Y)$ given by $T \mapsto T \times_X Z$. 
Using the uniqueness of limits and colimits in $\infty$-categories \cite[Pr. 1.2.12.9]{7}, we obtain a trivial fibration $Q(\mathcal{A})_0 \rightarrow E(\mathcal{A})$; hence the $\infty$-category $Q(\psi)_0$ is naturally equivalent to the $\infty$-category 
\[ Z(\psi) := E(\mathcal{A}) \times_{Q(\mathcal{A})} Q(\mathcal{B}) \].

These equivalences are compatible with the maps to $Q(\mathcal{B})$ and $Q(\mathcal{A})$, so the square
\[
\begin{array}{ccc}
Z(\psi) & \longrightarrow & Q(\mathcal{B}) \\
\downarrow & & \downarrow \\
E(\mathcal{A}) & \longrightarrow & Q(\mathcal{A})
\end{array}
\]
is a homotopy pullback (for the Quillen model structure), and of course $E(\mathcal{A}) \simeq *$.

Again employing \cite[Cor. 3.3.4.6]{7}, we find that the simplicial set $Z(\psi)$ is the (homotopy) colimit of the functor $Q(\mathcal{B})^{\text{op}} \rightarrow \text{Kan}$ that classifies the pulled back right fibration $\iota_{Q(\mathcal{A})} \times_{Q(\mathcal{A})} Q(\mathcal{B})$. This functor carries an object $U \in \mathcal{B}$ to the Kan complex $\iota(\mathcal{A}/\psi U)$, and it carries a morphism of the form
\[
\begin{array}{ccc}
W & \begin{array}{c}
\nearrow \ \downarrow \\
V & U
\end{array} & \end{array}
\]
to the map $\iota(\mathcal{A}/\psi U) \rightarrow \iota(\mathcal{A}/\psi V)$ given by $T \mapsto T \times_{\psi U} \psi W$.

5.11. **Definition.** We call the $\infty$-category $Z(\psi)$ constructed above the **relative $Q$ construction** for the nilimmersion $\psi: \mathcal{B} \rightarrow \mathcal{A}$.

5.12. To unpack this further, we may think of the objects of the $\infty$-category $Z(\psi)$ as pairs
\[ (U, g) = (U, g: X \rightarrow \psi U), \]
where $U \in \mathcal{B}$, and $g$ is a map of $\mathcal{A}$. A morphism
\[ (V, h: Y \rightarrow \psi V) \rightarrow (U, g: X \rightarrow \psi U) \]
of this $\infty$-category is then a pair of diagrams
\[
\begin{array}{c}
\begin{array}{ccc}
W & \rightarrow & U \\
\downarrow & & \downarrow \\
V & & \psi V
\end{array} & , & \\
\end{array}
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
\psi W & \rightarrow & \psi U
\end{array}
\]
the first from $\mathcal{B}$ and the second from $\mathcal{A}$, in which the square in the second diagram is a pullback.

5.13. Waldhausen introduced a relative $S_*$ construction, which we described as a virtual Waldhausen $\infty$-category $\mathcal{K}(\psi)$ in \cite[Nt. 8.8]{3}. This has the property that the sequence
\[ K(\mathcal{K}(\psi)) \rightarrow K(\mathcal{B}) \rightarrow K(\mathcal{A}) \]
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is a (homotopy) fiber sequence. In effect, \( \mathcal{K}(\psi) \) is the geometric realization of the simplicial Waldhausen ∞-category \( K_*(\psi) \) whose \( m \)-simplices consist of a totally filtered object

\[
0 \rightarrow U_1 \rightarrow U_2 \rightarrow \ldots \rightarrow U_m
\]
of \( \mathcal{B} \), a filtered object

\[
X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_m
\]
of \( \mathcal{A} \), and a diagram

\[
\begin{array}{ccc}
X_0 & \rightarrow & X_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \psi(U_1)
\end{array}
\]

\[
\begin{array}{ccc}
X_0 & \rightarrow & X_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \psi(U_2)
\end{array}
\]

\[
\begin{array}{ccc}
X_0 & \rightarrow & X_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \psi(U_3)
\end{array}
\]

of \( \mathcal{A} \) in which every square is a pushout.

One may now point out that the relative Q construction \( Z(\psi) \) can be identified with the edgewise subdivision of (a version of) the simplicial space \( m \mapsto K_m(\psi)_0 \).

So, in these terms, the argument above essentially ensures that the natural map

\[
I(\mathcal{K}(\psi)) \rightarrow \Omega I(\mathcal{K}(\psi)) \simeq K(\mathcal{K}(\psi))
\]
is a weak homotopy equivalence, where \( I \) is the left derived functor of \( \iota \).

5.14. **Example.** Keep the notations and conditions of Ex. 5.7. In light of [3, Pr. 11.15], the resulting sequence

\[
K(\Lambda') \rightarrow K(\Lambda) \rightarrow K(\Lambda[x^{-1}])
\]
is a fiber sequence if and only if the ∞-category

\[
Z(f_*) = E(Nil_{(\Lambda,S)}^{t,\omega}) \times_{Q(Nil_{(\Lambda,S)}^{t,\omega})} Q(Perf_{A'})
\]
is weakly contractible.

5.15. **Example.** As a subexample, let us note that for a prime \( p \), when \( \Lambda = BP(n) \) is the truncated Brown–Peterson spectrum, we find, using [3, Ex. 11.16], that the conjecture of Ausoni–Rognes [1, (0.2)] that the sequence

\[
K(BP(n-1)) \rightarrow K(BP(n)) \rightarrow K(E(n))
\]
is a fiber sequence if and only if the weak contractibility of the ∞-category

\[
Z(f_*) = E(Nil_{(BP(n),v_n)}^{t,\omega}) \times_{Q(Nil_{(BP(n),v_n)}^{t,\omega})} Q(Perf_{BP(n-1)})
\]
is weakly contractible.

For the sake of clarity, let us state explicitly that the objects of \( Z(f_*) \) are pairs

\[
(U,g) = (U,g : X \rightarrow f_*U),
\]
in which \( U \) is a perfect left \( BP(n-1) \)-module, and \( g \) is a map of perfect \( v_n \)-nilpotent left \( BP(n) \)-modules. A morphism

\[
(V,h : Y \rightarrow f_*V) \rightarrow (U,g : X \rightarrow f_*U)
\]
of this $\infty$-category is then a pair of diagrams

$$
\begin{array}{ccc}
W & \longrightarrow & U \\
\downarrow & & \downarrow \\
V & \longrightarrow & f_*W \\
& & \downarrow \\
& & f_*V
\end{array}
\quad
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
& & \downarrow \\
& & f_*U
\end{array}
$$

the first in perfect left $\text{BP}(n-1)$-modules and the second in perfect $v_n$-nilpotent left $\text{BP}(n)$-modules, in which the square in the second diagram is a pullback. The conjecture is simply that this $\infty$-category is weakly contractible. The relatively concrete nature of this formulation of the Ausoni–Rognes conjecture is tantalizing but, so far, we are embarrassed to report, frustrating.

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