FERMIONIZATION AND CONVERGENT PERTURBATION EXPANSIONS IN CHERN-SIMONS GAUGE THEORY

JONATHAN WEITSMAN

Abstract. We show that Chern-Simons gauge theory with appropriate cutoffs is equivalent, term by term in perturbation theory, to a Fermionic theory with a nonlocal interaction term. When an additional cutoff is placed on the Fermi fields, this Fermionic theory gives rise to a convergent perturbation expansion. This leads us to conjecture that Chern-Simons gauge theory also gives rise to convergent perturbation expansions, which would give a mathematically well-defined construction of the theory.

1. Introduction

Chern-Simons gauge theory was studied by Witten [9] as a geometric context for the Jones polynomial using formal path integrals as follows. Let $M$ be a compact three-manifold, and let $G$ be a compact simple Lie group. Choose an invariant inner product on $\mathfrak{g} = \text{Lie}(G)$. The space $\Omega^1(M, \mathfrak{g})$ of $\mathfrak{g}$-valued one-forms on $M$ can be identified with the space $\mathcal{A}(M)$ of connections on the trivialized principal $G$-bundle on $P = M \times G \to M$. In these terms, the Chern-Simons invariant of a connection $A \in \mathcal{A}(M)$ is given by

$$CS(A) = \frac{1}{4\pi} \text{tr} \int_M A \wedge dA + \frac{2}{3} A^3.$$ 

Given $\lambda \in \mathbb{Z}$, the partition function of the Chern-Simons quantum field theory is given schematically by

$$\int_{\mathcal{A}(M)} D A e^{-i\lambda CS(A)},$$

where integration on $\mathcal{A}(M)$ is a formal—and mysterious—operation.

The integrand in (1.1) is invariant under the group $G = \text{Aut}(P) = \text{Map}(M, G)$ of automorphisms of the bundle $P$. The gauge fixed action was studied by Axelrod and Singer [1]. Suppose there exists a flat connection $A_0 \in \mathcal{A}(M)$ such that $H^*(\Omega^*(M, \mathfrak{g}), d_{A_0})$ vanishes. Choose a Riemannian metric on $M$. Choose also an orthonormal basis $e_\alpha$ for $\mathfrak{g}$, and denote by $f_{\alpha\beta\gamma}$ the corresponding structure constants. The gauge-fixed action is a function of a connection $A \in \ker d^*_{A_0}$ and of two Fermi fields $c \in \Omega^0(M, \mathfrak{g})$ and $C \in \ker (d^*_{A_0}) \subset \Omega^2(M, \mathfrak{g})$. It is given by [11]

$$S(A, c, C) = \frac{1}{2\pi} \int_M \sum_\alpha \frac{1}{2} (A_\alpha \wedge (d_{A_0} A)_\alpha) - C_\alpha \wedge (d_{A_0} c)_\alpha + \frac{1}{6} \sum_{\alpha, \beta, \gamma} f_{\alpha\beta\gamma} (A_\alpha \wedge A_\beta \wedge A_\gamma - 6 C_\alpha \wedge A_\beta \wedge c_\gamma),$$

and the gauge fixed partition function is given by

2000 Mathematics Subject Classification. 57R56,81T13,81T08.
Supported in part by NSF grant DMS 04/05670.
February 22, 2009.

1Here $d_{A_0}$ denotes the de Rham operator in the twisted de Rham complex corresponding to the bundle $M \times \mathfrak{g} = ad(P)$ and the connection $A_0$. 

1
Now the formal path integral appearing in (1.3) is not in any sense well-defined. However, it does give rise to a perturbation series by a variant of the usual Feynman procedure. Axelrod and Singer show that each of the terms in this series is finite—in other words that the usual divergences appearing in perturbative quantum field theory do not appear in this case. They also show that appropriate combinations of the terms in the perturbation series give rise to topological invariants of the three-manifold $M$. The methods of [1] do not address convergence of the perturbation series, and hence their results do not give a mathematical definition of the path integral. Indeed the general expectation in Bosonic quantum field theories is that the perturbation series has radius of convergence equal to zero.

However, in [7] we showed that a cut-off version of Yang-Mills theory in four dimensions is equivalent, term-by-term in perturbation theory, to a Fermionic theory with nonlocal interactions. This Fermionic theory, when given a further cutoff, gives rise to a convergent perturbation series. The purpose of the present paper is to show that the methods of [7] apply also to Chern-Simons theory. That is, a cut-off version of the action (1.2) is equivalent, term-by-term in perturbation theory, to a theory where the connection $A$ is replaced by a bilinear in Fermion fields (there is obviously no need to Fermionize $c$ and $C$ since they are already Fermions); and a further momentum cutoff placed on the Fermion fields yields a convergent perturbation series. Since the perturbation series of Chern-Simons gauge theory, unlike that of Yang-Mills theory, is finite, we conjecture that it, too, is convergent. However, our estimates are not uniform in the cutoff and are therefore not able to address this problem.

1.1. The results of Axelrod and Singer. We first describe in some more detail the results of Axelrod and Singer [1]; we refer the reader to [1] for more information.

Let $\Delta = d^* A_0 d A_0 + d A_0 d^* A_0$ be the Laplacian on $\Omega^*(M, g)$, and let $L : \Omega^*(M, g) \to \Omega^*(M, g)$ be the operator defined by

$$L = d^* A_0 (\Delta A_0)^{-1}.$$ 

Denote the component of $L \circ \star$ (where $\star$ denotes the Hodge star operator) acting on $p$–forms by $L_p$.

If we choose an orthonormal framing of the tangent bundle $TM$, we may view the kernels of $L_0$ and $L_1$ as a smooth functions on $M \times M - \Delta$ with values in $(\mathbb{R}^3 \otimes g) \otimes (\mathbb{R}^3 \otimes g)$; here $\Delta \subset M \times M$ denotes the diagonal. Denote these functions by $L_0(x, y)$ and $L_1(x, y)$ for $x, y \in M$.

Let $\chi \in C^\infty(\mathbb{R})$ satisfy

- $1 \geq \chi \geq 0$.
- $\chi' \geq 0$.
- $\chi(x) = 0$ if $x \leq 1$.
- $\chi(x) = 1$ if $x \geq 2$.

For $\epsilon > 0$ let $\chi_\epsilon(x) := \chi(\epsilon x)$, and for all $x, y \in M \times M - \Delta$, and $i = 0, 1$, let

$$L_i^\epsilon(x, y) := L_i(x, y) \chi_\epsilon(d(x, y)),$$

where $d(x, y)$ denotes the distance between $x$ and $y$ given by the Riemannian metric on $M$. Then the functions $L_i^\epsilon$ extend to smooth functions on $M \times M$, which we continue to denote by $L_i^\epsilon$.

The cut-off perturbation series of the action (1.2) is given by the formal power series

$$(1.4)\quad Z_{sc}(A_0; \lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} Z_n(\epsilon).$$
where \( Z_{sc}(A_0; \lambda) \) is the semi-classical approximation to the partition function, involving Chern-Simons and torsion invariants of \( A_0 \) (see [9]), and where

\[
\Xi_n(\epsilon) := \frac{R_0 3^n R_I 2^n}{(3n)! (2n)!} |A=0,c=0,C=0|,
\]

and \( R_0, R_I \) are defined as follows. In terms of formal even variables \( A_\alpha^i(x) \) and formal odd variables \( c_\alpha(x), C_\alpha^{i,j}(x), x \in M, i,j = 1,2,3, i < j, \alpha = 1, \ldots, \dim g \), the polynomial \( R_I(A,c,C) \) is given by

\[
R_I(A,c,C) := -\frac{i}{2\pi} \sum_{i,j,k,\alpha,\beta,\gamma} \int_M dx \epsilon_{ijk} f_{\alpha\beta\gamma} \left( \frac{1}{6} A_\alpha^i(x) A_\beta^j(x) A_\gamma^k(x) - A_\alpha^i(x) c_\beta(x) C_\gamma^{i,j}(x) \right),
\]

and the formal differential operator \( R_0 \) is given by

\[
R_0 := -2\pi i \sum_{i,j,\alpha,\beta} \int_{M \times M} dxdy \left( \left( L_0^i(x,y) \right)_{i,j,\alpha,\beta} \frac{\delta}{\delta A_\alpha^i(x)} \frac{\delta}{\delta A_\beta^j(y)} - 2 \left( L_0^i(x,y) \right)_{i,j,\alpha,\beta} \frac{\delta}{\delta c_\alpha(x)} \frac{\delta}{\delta C_\beta^{i,j}(y)} \right);
\]

here we have used the notation \( (L_k^i(x,y))_{i,j,\alpha,\beta, k=0,1} \) for the matrix elements of \( L_k^i(x,y) \) in the basis given by the framing of the tangent bundle and the chosen basis \( e_\alpha \) of \( g \).

Then the key result of Axelrod and Singer is the following

**Theorem 1** (Axelrod and Singer [1] 1995). The limit

\[
\Xi_n = \lim_{\epsilon \to 0} \Xi_n(\epsilon)
\]

is finite for every \( n \).

Axelrod and Singer then show that the quantities \( \Xi_n \) are topological invariants of \( M \). These “finite-type” invariants have been the focus of intensive research since the publication of [1].

1.2. Fermionization. We now Fermionize Chern-Simons gauge theory, by a method similar to the one used in [7] in the case of Yang-Mills theory. Morally, we replace the connection \( A_\alpha^i \) with a bilinear in Fermi fields. Let \( H_i(x), i = 1,2,3 \), and \( \Psi_\alpha(x), \alpha = 1, \ldots, \dim g \), be complex Fermi fields. The Fermi action is given by

\[
S_F^\epsilon(H_i, \Psi_\alpha, c_\alpha, C_\alpha^{i,j}) = S_{F,0}(H_i, \Psi_\alpha, c_\alpha, C_\alpha^{i,j}) + S_{F,1}(H_i, \Psi_\alpha, c_\alpha, C_\alpha^{i,j}) + \frac{i}{2\pi \sqrt{\hbar}} \sum_{i,j,k,\alpha,\beta,\gamma} \int_M dx \epsilon_{ijk} f_{\alpha\beta\gamma} \left( \frac{1}{(3\hbar)^2} H_i(x) \Psi_\alpha(x) H_j(x) \Psi_\beta(x) H_k(x) \Psi_\gamma(x) - H_i(x) \Psi_\alpha(x) \bar{c}_\beta(x) C_\gamma^{i,j}(x) \right)
\]

and

\[
S_{F,1}(H_i, \Psi_\alpha, c_\alpha, C_\alpha^{i,j}) = \frac{i}{2\pi \sqrt{\hbar}} \sum_{i,j,\alpha,\beta} \int_M dx \epsilon_{ijk} f_{\alpha\beta} \left( \frac{1}{(3\hbar)^2} H_i(x) \Psi_\alpha(x) H_j(x) \Psi_\beta(x) - H_i(x) \Psi_\alpha(x) \bar{c}_\beta(x) \bar{C}_\gamma^{i,j}(x) \right)
\]

where

\[
\Xi_n(\epsilon) := \frac{R_0 3^n R_I 2^n}{(3n)! (2n)!} |A=0,c=0,C=0|,
\]
To make further progress, we impose a cutoff on the Fermi fields, as in [7]. It is convenient to do this by convolutions with approximate delta functions and step functions, as follows. Let \( \zeta \in C^\infty(\mathbb{R}) \) be an even function satisfying
\[
\begin{align*}
&\zeta \geq 0, \\
&\int_0^\infty x^2 \zeta(x) dx = \frac{1}{4\pi}, \\
&\zeta'(x) \leq 0 \text{ for } x > 0, \\
&\zeta(x) = 0 \text{ for } x \geq 1.
\end{align*}
\]
Given \( h > 0 \), define \( \delta_h : M \times M \to \mathbb{R} \) by
\[
\delta_h(x, y) := (2/h)^3 \zeta(2d(x, y)/h).
\]
Similarly let \( Z \in C^\infty(\mathbb{R}) \) satisfy
\[
\begin{align*}
&Z \geq 0, \\
&Z(x) = 1 \text{ if } x \in [-1, 1], \\
&Z'(x) \leq 0 \text{ if } x > 0, \\
&Z(x) = 0 \text{ for } |x| \geq 2.
\end{align*}
\]
For all \( h > 0 \), define \( D_h : M \times M \to \mathbb{R} \) by
\[
D_h(x, y) := Z(d(x, y)/2h).
\]
We now define the cut-off Fermi fields by
\[
\begin{align*}
\Psi^h_\alpha(x) &= \int_M \Psi_\alpha(y) \delta_h(y, x) dy, \\
H^h_i(x) &= \int_M H_i(y) \delta_h(y, x) dy, \\
c^h_\alpha(x) &= \int_M c_\alpha(y) \delta_h(y, x) dy, \\
C^{i,j,h}_\alpha(x) &= \int_M C^{i,j}_\alpha(y) \delta_h(y, x) dy, \\
\bar{\Psi}^h_\alpha(x) &= \int_M \bar{\Psi}_\alpha(y) \delta_h(y, x) dy, \\
\bar{H}^h_i(x) &= \int_M \bar{H}_i(y) D_h(y, x) dy, \\
\bar{c}^h_\alpha(x) &= \int_M \bar{c}_\alpha(y) \delta_h(y, x) dy, \\
\bar{C}^{i,j,h}_\alpha(x) &= \int_M \bar{C}^{i,j}_\alpha(y) \delta_h(y, x) dy,
\end{align*}
\]
and define the cut-off Fermi action by
\[
(1.8) \quad S_{F}^{\epsilon,h}(H_i, \Psi_\alpha, c_\alpha, C^{i,j}_\alpha, \bar{H}_i, \bar{\Psi}_\alpha, \bar{c}_\alpha, \bar{C}^{i,j}_\alpha) =
S_{F,0}(H_i, \Psi_\alpha, c_\alpha, C^{i,j}_\alpha, \bar{H}_i, \bar{\Psi}_\alpha, \bar{c}_\alpha, \bar{C}^{i,j}_\alpha) + S_{F,1}(H^h_i, \Psi^h_\alpha, c^h_\alpha, C^{i,j,h}_\alpha, \bar{H}^h_i, \bar{\Psi}^h_\alpha, \bar{c}^h_\alpha, \bar{C}^{i,j,h}_\alpha)
\]
Then the analog of the Fermionization theorem of [7] is the following

**Theorem 2.** Each term of the perturbation series of the Fermi action \( S_{F}^{\epsilon,h} \) coincides in the limit \( h \to 0 \) with the corresponding term \( \Xi_n(\epsilon) \) of the perturbation series \( (1.4) \) of the cut-off, gauge-fixed Chern-Simons gauge theory.
Remark 1.9. As in [7], it is possible to write down a Fermi theory which gives rise to a perturbation series identical with that of (1.4). However this theory does not arise from a Lagrangian; the free correlation functions for the Fermi fields $\tilde{H}_i$ in this theory are given by

$$<\tilde{H}_i(x)H_j(y)> = \delta_{ij} \text{ if } x = y;$$
$$<\tilde{H}_i(x)H_j(y)> = 0 \text{ if } x \neq y,$$

as might be expected from the limiting behavior of the correlations of the cut-off fields $\tilde{H}_i^h, H_i^h$.

1.3. Convergent perturbation theory. As in other examples of purely Fermionic theories [3, 4, 7], we expect the cut-off Fermionic action (1.8) to give rise to a convergent perturbation series. The analog of Theorem 2 of [7] is the following.

Theorem 3. The perturbation series corresponding to the action $S_{F,h}^\epsilon$ converges for all $\lambda \neq 0$.

Remark 1.10. In fact it is not necessary to place an additional cut-off on the fields $c$ and $C$; the action

$$S_{F,0}(H_1, \Psi_{\alpha}, c_{\alpha}, C_{i,j}^{\alpha}, \tilde{H}_1, \tilde{\Psi}_{\alpha}, \tilde{c}_{\alpha}, \tilde{C}_{i,j}^{\alpha}) + S_{F,1}(H_1^h, \Psi_{\alpha}^h, c_{\alpha}^h, C_{i,j}^{\alpha}, \tilde{H}_1^h, \tilde{\Psi}_{\alpha}^h, \tilde{c}_{\alpha}^h, \tilde{C}_{i,j}^{\alpha})$$

also gives rise to a convergent perturbation series.

The convergence estimates used to prove Theorem 3 are not uniform in $\epsilon$ and $h$, as is indeed the case for the Gross-Neveu model [3, 4] and for Yang-Mills theory [7]. In those asymptotically free theories, the coupling constant is adjusted to approach zero as the ultraviolet cutoff is removed, which in our case would correspond to the limit where $\epsilon$ and $h$ approach zero. It is this fact which makes non-uniform estimates useful for the construction of the theory in [3, 4]. The situation is different for Chern-Simons gauge theory, which is finite to all orders in perturbation theory. So we make the following conjecture.

Conjecture 1.11.

(a) The perturbation series

$$\sum_{n=0}^{\infty} \frac{1}{\lambda^n} \xi_n(\epsilon)$$

is convergent for all $\epsilon$ and all $\lambda \neq 0$.

(b) The limit series

$$\sum_{n=0}^{\infty} \frac{1}{\lambda^n} \xi_n$$

converges for all $\lambda \neq 0$.

Evidently a proof of Conjecture 1.11 would depend on finer determinant estimates than the ones used by [3, 4, 7] and applied to our case in the proof of Theorem 3.

Remark 1.14. There is strong evidence from topology that Conjecture 1.11 is true. This is because the Chern-Simons-Witten invariants of three-manifolds can be written in terms of invariants of finite type, which morally correspond to combinations of connected diagrams in the Axelrod-Singer expansion. For a fixed three-manifold $M$, all but finitely many of these invariants vanish, so that we might expect there to be only a finite number of nonvanishing connected diagrams in the Axelrod-Singer expansion; thus the sum over all diagrams should be convergent. A similar

---

A slight variation of our methods allows one to take the limit $\epsilon \to 0$ as long as $h$ remains finite; however, the theory remains a theory with cut-offs.
result holds for expectations of *gauge invariant* observables given by nonintersecting Wilson loops in \( M = S^3 \), where the resulting invariant, which is the Jones polynomial, is given as a sum of finite-type Vassiliev invariants, all but finitely many of which vanish for a given link (See for example [2].) Fermionization gives a potential quantum field theoretic context for the finite-type property of these invariants.

2. Proof of Theorem 2

Recall that the terms \( \Xi_n(\epsilon) \) of the perturbation series of Chern-Simons gauge theory with cutoff are given by

\[
\Xi_n(\epsilon) := \left. \frac{R_0^{3n}}{(3n)!} \frac{(R_I)^{2n}}{(2n)!} \right|_{A=0, c=0, C=0},
\]

where \( R_0 \) and \( R_I \) are given in terms of formal even variables \( A_i^\alpha(x) \) and formal odd variables \( c_\alpha(x), C_{ij}^\alpha(x) \) by

\[
R^I_i(A, c, C) := -\frac{i}{2\pi} \sum_{i,j,k,\alpha,\beta,\gamma} \int_M dx \epsilon_{ijk} f_{\alpha\beta\gamma} \left( \frac{1}{6} A_i^\alpha(x) A_j^\beta(y) A_k^\gamma(x) - A_i^\alpha(x) c_\beta(x) C_{jk}^\gamma(x) \right)
\]

and

\[
R_0 := -2\pi i \sum_{i,j,\alpha,\beta} \int_{M \times M} dx \frac{\delta}{\delta A_i^\alpha(x)} \frac{\delta}{\delta A_j^\beta(y)} - 2(L^e_i(x, y))_{i,j,\alpha,\beta} \frac{\delta}{\delta c_\alpha(x)} \frac{\delta}{\delta C_{ij}^\beta(y)}.
\]

Similarly the perturbation series of the Fermionic action \( S_F^{\epsilon, h} \) is given by

\[
\sum_{n=0}^\infty \frac{1}{\lambda^n} \Theta_n(\epsilon, h),
\]

where

\[
\Theta_n(\epsilon, h) := \int \mathcal{D}H \mathcal{D}\bar{H} \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}C \mathcal{D}\bar{C} \mathcal{D}e^{S_{F,0}} (T^e_i)^{3n} (T^h_i)^{2n} \frac{1}{(3n)!} \frac{1}{(2n)!},
\]

and where the polynomials \( T^h_i \) and \( T^e_i \) are given by

\[
T^h_i(H, \Psi, c, C) = \frac{i}{2\pi} \sum_{i,j,k,\alpha,\beta,\gamma} \int_M dx \epsilon_{ijk} f_{\alpha\beta\gamma} \left( \frac{1}{(3!)^2} \bar{H}^h_i(x) \Psi_\alpha^h(x) \bar{H}^h_j(x) \Psi_\beta^h(x) \bar{H}^h_k(x) \Psi_\gamma^h(x) - \bar{H}^h_i(x) \Psi_\alpha^h(x) \bar{H}^h_j(x) \Psi_\beta^h(x) C_{ij \beta}^\gamma(x) \bar{C}_{ij \beta}^\gamma(x) \right)
\]

and

\[
T^e_i := -2\pi i \sum_{i,j,\alpha,\beta} \int_{M \times M} dy \left( (L^e_i(x, y))_{i,j,\alpha,\beta} H^h_i(x) \Psi_\alpha^h(x) H^h_j(y) \Psi_\beta^h(y) - 2(L^e_i(x, y))_{i,j,\alpha,\beta} c_\alpha^h(x) C_{ij \beta}^\gamma(y) \right).
\]

Thus
We use this fact to write

\( \Theta_{n}(\epsilon, h) = \frac{1}{(2n)! (3n)!} \int \mathcal{D} H \mathcal{D} \bar{H} \mathcal{D} \Psi \mathcal{D} \bar{\Psi} \mathcal{D} dC \mathcal{D} \bar{C} \mathcal{D} e^{S_{F,0}} \)

\[ (-2\pi i)^{3n} \sum_{k=1}^{6n} \sum_{i_k,j_k=1,2,3 \alpha_k, \beta_k=1,\ldots, \text{dim} \, g} \int_{M^{6n}} dx_1 \ldots dx_{6n} \int_{M^{6n}} dz_1 \ldots dz_{6n} \]

\[ \left( \prod_{i=1}^{3n} \left( L^f_i(x_{2l}, x_{2l-1}) \right)_{i_2, i_2-1: \alpha_2, \alpha_{2l-1} \bar{H}^h_{i_2l}(x_{2l}) \Psi^h_{\alpha_2l}(x_{2l}) H^h_{i_2l-1}(x_{2l-1}) \Psi^h_{\alpha_{2l-1}}(x_{2l-1}) \right) \right) \]

\[ -2 \left( L^f_0(x_{2l}, x_{2l-1}) \right)_{i_2, i_2-1: \alpha_2, \alpha_{2l-1} \bar{c}^h_{i_2l}(x_{2l}) \mathcal{C}^{i_2l, i_2l-1}_{\alpha_2l-1}(x_{2l-1}) \right) \]

\[ \left( \frac{i}{2\pi} \right)^{2n} \left( \prod_{m=1}^{2n} \epsilon_{j_3m, j_3m-1, j_3m-2} \right) \int_{M^{6n}} dx_1 \ldots dx_{6n} \int_{M^{6n}} dz_1 \ldots dz_{6n} \]

\[ \frac{1}{(3!)^2} \bar{H}^h_{j_3m}(z_{3m}) \Psi^h_{\beta_3m}(z_{3m}) \bar{H}^h_{j_3m-1}(z_{3m-1}) \Psi^h_{\beta_3m-1}(z_{3m-1}) \bar{H}^h_{j_3m-2}(z_{3m-2}) \Psi^h_{\beta_3m-2}(z_{3m-2}) \]

\[ - \bar{H}^h_{j_3m}(z_{3m}) \bar{\Psi}^h_{\beta_3m}(z_{3m}) \bar{c}^h_{\beta_3m-1}(z_{3m-1}) \bar{c}^h_{\beta_3m-2}(z_{3m-2}) \].

Expanding the products, we have

\( \Theta_{n}(\epsilon, h) = \frac{(-2\pi i)^{3n}}{(2n)! (3n)!} \left( \frac{i}{2\pi} \right)^{2n} \int \mathcal{D} H \mathcal{D} \bar{H} \mathcal{D} \Psi \mathcal{D} \bar{\Psi} \mathcal{D} dC \mathcal{D} \bar{C} \mathcal{D} e^{S_{F,0}} \)

\[ \sum_{k=1}^{6n} \sum_{i_k,j_k=1,2,3 \alpha_k, \beta_k=1,\ldots, \text{dim} \, g} \int_{M^{6n}} dx_1 \ldots dx_{6n} \int_{M^{6n}} dz_1 \ldots dz_{6n} \]

\[ \sum_{q=1}^{2n} \binom{2n}{q} \binom{3n}{q+n} \prod_{l=1}^{q+n} \left( L^f_i(x_{2l}, x_{2l-1}) \right)_{i_2, i_2-1: \alpha_2, \alpha_{2l-1} \bar{H}^h_{i_2l}(x_{2l}) \Psi^h_{\alpha_2l}(x_{2l}) H^h_{i_2l-1}(x_{2l-1}) \Psi^h_{\alpha_{2l-1}}(x_{2l-1}) \right) \]

\[ \prod_{l=q+1}^{3n} \left[ -2 \left( L^f_0(x_{2l}, x_{2l-1}) \right)_{i_2, i_2-1: \alpha_2, \alpha_{2l-1} \bar{c}^h_{i_2l}(x_{2l}) \mathcal{C}^{i_2l, i_2l-1}_{\alpha_2l-1}(x_{2l-1}) \right) \right] \]

\[ \left( \prod_{m=1}^{2n} \epsilon_{j_3m, j_3m-1, j_3m-2} \right) \int_{M^{6n}} dx_1 \ldots dx_{6n} \int_{M^{6n}} dz_1 \ldots dz_{6n} \]

\[ \frac{1}{(3!)^2} \bar{H}^h_{j_3m}(z_{3m}) \Psi^h_{\beta_3m}(z_{3m}) \bar{H}^h_{j_3m-1}(z_{3m-1}) \Psi^h_{\beta_3m-1}(z_{3m-1}) \bar{H}^h_{j_3m-2}(z_{3m-2}) \Psi^h_{\beta_3m-2}(z_{3m-2}) \]

\[ - \bar{H}^h_{j_3m}(z_{3m}) \bar{\Psi}^h_{\beta_3m}(z_{3m}) \bar{c}^h_{\beta_3m-1}(z_{3m-1}) \bar{c}^h_{\beta_3m-2}(z_{3m-2}) \].

Standard Feynman diagram techniques allow us to write \( \Theta_{n}(\epsilon, h) \) in as a sum of terms corresponding to trivalent graphs. When the ghost lines in such a graph are cut, we obtain a pair of graphs, one of which corresponds to the \( \Psi \) and \( \bar{\Psi} \) fields and one of which to the \( H \) and \( \bar{H} \) fields. We use this fact to write

\[ \Theta_{n}(\epsilon, h) = \Theta^1_{n}(\epsilon, h) + \Theta^2_{n}(\epsilon, h) \]

where \( \Theta^1_{n}(\epsilon, h) \) is the sum of those terms in the diagrammatic expansion of \( \Theta_{n}(\epsilon, h) \) corresponding to pairs of identical Feynman diagrams—that is, Feynman diagrams where the combinatorics of the
The pairs of the \( \Psi \) and \( \bar{\Psi} \) fields are the same as those of the \( H \) and \( \bar{H} \) fields; we will prove that in the limit \( h \to 0 \) the terms appearing in \( \Theta_n^1(\epsilon, h) \) approach the corresponding terms in the diagrammatic expansion of \( \Xi_n(\epsilon) \). The sum of the remaining terms in the diagrammatic expansion of \( \Theta_n(\epsilon, h) \), which we denote by \( \Theta_n^2(\epsilon, h) \), consists of terms corresponding to pairs of Feynman diagrams where at least one \( H \) propagator is not matched by a corresponding \( \Psi \) propagator. We will see that in the limit \( h \to 0 \), such an “unmatched” propagator will give rise to a factor of order \( O(h^3) \), so that \( \lim_{h \to 0} \Theta_n^2(\epsilon, h) = 0 \). More precisely, we write

\[
\Theta_n^1(\epsilon, h) = \frac{(-2\pi i)^{3n}}{(2\pi)!} \left( i \frac{2n}{2\pi} \right)^6 \prod_{k=1}^{6n} \sum_{i_k,j_k=1,2,3,\alpha_k,\beta_k=1} H \sigma \int_{M^{6n}} dx_1 \ldots dx_{6n} \int_{M^{6n}} dz_1 \ldots dz_{6n} 
\]

\[
\sum_{q=1}^{2n} \left( \frac{2n}{q} \right) \left( 3n \right) \prod_{\sigma \in S_{2q+2n}} \prod_{l=1}^{q+n} (L_l^x(x_{2l}, x_{2l-1}))_{i_{2l}, i_{2l-1}} \alpha_{2l, \alpha_{2l-1}} \prod_{l=q+n+1}^{3n} (-2L_0^x(x_{2l}, x_{2l-1}))_{i_{2l}, i_{2l-1}} \alpha_{2l, \alpha_{2l-1}} 
\]

\[
\prod_{m=1}^{2n} \epsilon_{j_{3m} j_{3m-1}} \beta_{3m-1} \beta_{3m-2} \delta(z_{3m}, z_{3m-1}) \delta(z_{3m}, z_{3m-2}) \left( \prod_{m=1}^q \left[ \frac{1}{(3!)^3} \int \mathcal{D}H \mathcal{D}\bar{H} \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{S_{F,0}} \right]
\right)
\]

\[
H_{\alpha(3m)}^{h} \left( x_{\sigma(3m)} \right) \Psi_{\alpha(3m)}^{h} \left( x_{\sigma(3m)} \right) H_{\alpha(3m-1)}^{h} \left( x_{\sigma(3m-1)} \right) \Psi_{\alpha(3m-1)}^{h} \left( x_{\sigma(3m-1)} \right) H_{\alpha(3m-2)}^{h} \left( x_{\sigma(3m-2)} \right) \Psi_{\alpha(3m-2)}^{h} \left( x_{\sigma(3m-2)} \right)
\]

\[
\left( \prod_{m=1}^{2n} \left[ \int \mathcal{D}H \mathcal{D}\bar{H} \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{S_{F,0}} \right] \right) \left( \prod_{l=q+n+1}^{3n} c_{\alpha_{2l}} \left( x_{2l} \right) \bar{c}_{\alpha_{2l-1}}^{h} \left( x_{2l-1} \right) \right) \left( \prod_{m=q+1}^{2n} \left[-c_{\beta_{3m-1}} \left( z_{3m-1} \right) \bar{c}_{\beta_{3m-2}}^{h} \left( z_{3m-2} \right) \right] \right).
\]

To obtain an explicit expression for \( \Theta_n^2(\epsilon, h) \), we proceed as in [7]. Given \( \sigma, \tau \in S_{2q+2n} \), we say \( \sigma \sim \tau \) if for every \( m = 1, \ldots, 3q \), there exists \( k \in 1, \ldots, q \) such that

\[
\sigma(m) \tau(m) \in \{3k, 3k+1, 3k+2\}
\]

and if for every \( m = 3q+1, \ldots, 2q+2n \), we have

\[
\sigma(m) = \tau(m).
\]

If \( \sigma \sim \tau \), there exists a smallest integer \( m = m(\sigma, \tau) \) such that either (2.4) or (2.5) is false. Then, as in [7],
Lemma 2.8. The functions $\tilde{\delta}_h$ and $\tilde{D}_h$ are positive. Furthermore, for $h$ sufficiently small,

- $\lim_{h \to 0} \tilde{\delta}_h = \delta$ (as elements of $\mathcal{D}'(M)$).
- $\tilde{\delta}_h \tilde{D}_h = \tilde{\delta}_h$.
- $\|\tilde{D}_h(x, \cdot)\|_\infty = 1$ for any $x \in M$.
- $\|\tilde{\delta}_h(x, \cdot)\|_1 \leq C$ for any $x \in M$, where $C$ is a constant independent of $h$ and $x$.
- $\|\tilde{\delta}_h \ast \tilde{D}_h(x, \cdot)\|_1 = O(h^3)$ for any $x \in M$.

where $\tilde{\delta}_h \ast \tilde{D}_h$ is the convolution

$$\tilde{\delta}_h \ast \tilde{D}_h(x, y) := \int_M \tilde{\delta}_h(x, z) \tilde{D}_h(z, y) dz.$$
Proposition 2.11. We have
\[ \lim_{h \to 0} \Theta_2^2(\epsilon, h) = 0. \]

**Proof.** By (2.6), and using the fact that \( M \) is compact and the \( L_i \)'s are bounded,

\[
|\Theta_2^2(\epsilon, h)| \leq C \sup_q (\sup_{z \in M} ||\tilde{\delta}_h(\cdot, z)||_1)^{6n-1}(\sup_{z \in M} ||\tilde{D}_h(\cdot, z)||_\infty)^{2q+2n-1} \sup_{z \in M} ||\delta_h \ast \tilde{D}_h(\cdot, z)||_1.
\]

By Lemma 2.8

\[ |\Theta_2^2(\epsilon, h)| = O(h^3). \]

It remains to show

**Proposition 2.11.**

\[ \lim_{h \to 0} \Theta_3^1(\epsilon, h) = \Xi_n(\epsilon). \]

**Proof.** We note that

\[
\lim_{h \to 0} \int D\mathcal{H}D\mathcal{H}D\mathcal{P}D\mathcal{P}e^{S_F,\nu}H_i^h(x)\Psi_\alpha^h(x)\bar{H}_j^h(y)\Psi_\beta(y) = -\frac{\delta}{\delta A_\alpha^1(x)}A_\beta^1(y).
\]

Hence

\[
\lim_{h \to 0} \Theta_3^1(\epsilon, h) = \left( \prod_{m=1}^{2n} (2n!)^{3n} \right) \sum_{q=1}^{2n} (2n!) \sum_{\sigma \in S_{2q+2n}} \left( \prod_{l=1}^{3n} \left( L_i^l(x_2l, x_{2l-1}) \right) \delta_{2l, 2l-1} \delta_{\alpha_2l, \alpha_{2l-1}} \right)
\]

\[
\sum_{q=1}^{2n} \left( \prod_{q=1}^{2n} \left( \begin{array}{c} 3n \\ q \end{array} \right) \left( \begin{array}{c} 3n \\ q+n \end{array} \right) \right) \sum_{\sigma \in S_{2q+2n}} \left( \prod_{n=1}^{3n} \epsilon_{j_{3m}, j_{3m-1}, j_{3m-2}} \delta_{j_{3m}, j_{3m-1}, j_{3m-2}} \delta(z_{3m}, z_{3m-1}) \delta(z_{3m}, z_{3m-2}) \right)
\]

\[
\left( \prod_{m=1}^{q} \frac{1}{(3!)^2} \frac{\delta}{\delta A_{\sigma(3m)}(x_{\sigma(3m)})} \frac{\delta}{\delta A_{\sigma(3m-1)}(x_{\sigma(3m-1)})} \frac{\delta}{\delta A_{\sigma(3m-2)}(x_{\sigma(3m-2)})} \right) \left( \prod_{m=q+1}^{2n} \frac{\delta}{\delta A_{\sigma(2q+m)}(x_{\sigma(2q+m)})} \right)
\]

\[
\int DcDcDcDcDc e_{S_F,0}\left( \prod_{l=q+n+1}^{3n} c_{\alpha_2l}(x_{2l}) \right) \left( \prod_{m=q+1}^{2n} \left[ -\epsilon_{j_{3m-1}, j_{3m-2}} \sigma_{j_{3m-1}, j_{3m-2}} \right] \right)
\]

\[ = \Xi_n(\epsilon). \]

3. **Proof of Theorem 3**

Recall the explicit expression for the terms of the perturbation series of the action \( S_F^\epsilon, h \). This perturbation series is given by

\[
\sum_{n=0}^{\infty} \frac{1}{\lambda^n} \Theta_n(\epsilon, h)
\]

where (see (2.2) and (2.3))
\[ \Theta_n(\epsilon, h) = \frac{1}{(2n)!(3n)!} \int \mathcal{D}\Psi\mathcal{D}H\mathcal{D}c\mathcal{D}c\mathcal{D}\bar{\Psi}\mathcal{D}\bar{H}\mathcal{D}\bar{D}\mathcal{D}\bar{C} \exp(S_{F,0}) \]

\[
\left( \frac{i}{2\pi\sqrt{\lambda}} \sum_{i,j,k,\alpha,\beta,\gamma} \int_M dx \epsilon_{ijk} f_{\alpha\beta\gamma} \left[ \frac{1}{(3!)^2} H_i^h(x) \bar{H}_i^h(x) \bar{H}_j^h(x) \bar{H}_k^h(x) \bar{C}_{\alpha\beta}^i \bar{C}_{\gamma}^h (x) \right] \right)^{2n} 
\]

\[ (-2\pi i) \sum_{i,j,\alpha,\beta} \int_{M \times M} dx dy \left[ (L^i_1(x,y))_{i,j,\alpha,\beta} H_i^h(x) \Psi_{\alpha}^h(x) H_j^h(y) \Psi_{\beta}^h(y) - 2 (L^0_0(x,y))_{i,j,\alpha,\beta} c_{\alpha}^i c_{\beta}^h (y) \right]^{3n}. \]

Theorem 2 follows from the following estimate.

**Proposition 3.1.** There exists a constant \( C(\epsilon, h) > 0 \) such that

\[ |\Theta_n(\epsilon, h)| \leq \frac{C^n}{(2n)!(3n)!}. \]

To prove Proposition 3.1, we note that \( \Theta_n(\epsilon, h) \) is a sum of \( O(C^n) \) terms, each of which is (up to a constant of order \( C^n \)) of the form

\[
\frac{1}{(2n)!(3n)!} \int \mathcal{D}\Psi\mathcal{D}H\mathcal{D}c\mathcal{D}c\mathcal{D}\bar{\Psi}\mathcal{D}\bar{H}\mathcal{D}\bar{D}\mathcal{D}\bar{C} \exp(S_{F,0})
\]

\[
\int_{M^{6n}} dx_1 dy_1 \ldots dx_{3n} dy_{3n} \int_{M^{2n}} dz_1 \ldots dz_{2n} \prod_{l=1}^{3n+p} (L^i_1(x_l,y_l))_{i,j,\alpha,\beta} H_i^h(x_l) \Psi_{\alpha}^h(x_l) H_j^h(y_l) \Psi_{\beta}^h(y_l) 
\]

\[
\prod_{l=2n}^{3n} (-2L^0_0(x_l,y_l))_{i,j,\alpha,\beta} c_{\alpha}^i c_{\beta}^h (y_l) \prod_{m=1}^{q} \bar{H}_{pm}^h(z_m) \Psi_{\theta_m}^h(z_m) \bar{H}_{qm}^h(z_m) \bar{H}_{im}^h(z_m) \bar{H}_{rm}^h(z_m) \bar{\Psi}_{\sigma_m}^h(z_m) 
\]

\[
\prod_{m=q+1}^{2n} \bar{H}_{pm}^h(z_m) \Psi_{\theta_m}^h(z_m) c_{im}^h(z_m) C_{\sigma_m}^{qm,rm,h}(z_m), 
\]

where \( p + q = 2n \).

The Berezin integral appearing in (3.2) is

\[ B := \int \mathcal{D}\Psi\mathcal{D}H\mathcal{D}c\mathcal{D}c\mathcal{D}\bar{\Psi}\mathcal{D}\bar{H}\mathcal{D}\bar{D}\mathcal{D}\bar{C} \exp(S_{F,0}) \int_{M^{6n}} dx_1 dy_1 \ldots dx_{3n} dy_{3n} \int_{M^{2n}} dz_1 \ldots dz_{2n} \]

\[
\prod_{l=1}^{3n+p} H_i^h(x_l) \Psi_{\alpha_l}^h(x_l) H_j^h(y_l) \Psi_{\beta_l}^h(y_l) \prod_{l=2n+1}^{3n} c_{\alpha_l}^h(x_l) c_{\beta_l}^h(y_l) 
\]

\[
\prod_{m=1}^{q} \bar{H}_{pm}^h(z_m) \Psi_{\theta_m}^h(z_m) \bar{H}_{qm}^h(z_m) \bar{H}_{im}^h(z_m) \bar{H}_{rm}^h(z_m) \bar{\Psi}_{\sigma_m}^h(z_m) 
\]

\[
\prod_{m=q+1}^{2n} \bar{H}_{pm}^h(z_m) \Psi_{\theta_m}^h(z_m) c_{im}^h(z_m) C_{\sigma_m}^{qm,rm,h}(z_m). 
\]
This Berezin integral is the inner product of two elements of
\[ \bigwedge^{6q+4p}(L_2(M) \otimes (\mathbb{R}^3 \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathbb{R}^3 \otimes \mathbb{R}^3))), \]
and is bounded by
\[ |B| \leq (\sup_{x \in M}||\delta h(x, \cdot)||_{L_2(M)})^{9q+7p}(\sup_{x \in M}||D h(x, \cdot)||_{L_2(M)})^{3q+p}. \]

Since the kernels \( L^i \) are smooth, Proposition 3.1 follows.

4. Remarks

4.1. Correlation functions. As in [7], the generating function for correlation functions of the
gauge fields, which is obtained by adding a term of the form
\[ \int_M \sum_{i,\alpha} J^i(x) A^i_{\alpha}(x) dx \]
to the Chern-Simons Lagrangian, can be Fermionized by adding the term
\[ \int_M \sum_{i,\alpha} J^i(x) \bar{H}^i(x) \bar{\Psi}^\alpha(x) dx \]
to the Fermionized action.

4.2. Yang-Mills and QCD in three dimensions. Our techniques apply just as well to a La-
grangian obtained by adding a Yang-Mills term
\[ S_{A} = \frac{1}{\lambda^2} \int_M |F|^2 \]
to the Chern-Simons Lagrangian. As in [7], such a Lagrangian is equivalent to the Lagrangian
\[ S(A, F) = \frac{1}{\lambda^2} (||F||^2 + 2i\langle F, dA \rangle + 2i\langle F, [A, A] \rangle) + ikCS(A) \]
where \( F \in \Omega^2(M, \mathfrak{g}) \) is a conjugate field. The theory then has a cubic interaction term and can
be Fermionized by the same method we have used for pure Chern-Simons gauge theory. These
ideas work also for pure Yang-Mills theory (with no Chern-Simons term) in three dimensions. The
addition of Fermionic matter fields can likewise be accommodated by the same techniques.

Similar techniques should also apply to two-dimensional gauge theories.

4.3. String field theory. I believe that our techniques should also give a Fermionization of Wit-
ten’s string field theory. Recall that the string field theory Lagrangian is given by
\[ S_{sft}(A) = \int (A \ast QA + \frac{2}{3} A \ast A \ast A), \]
where \( A = A(\varphi, b, c) \) is the string field, which is a function of a bosonic field \( \varphi \) and two ghosts
\( b \) and \( c \), and the operator \( Q \) and the operations \( \int \) and \( \ast \) are defined in [10]. Imposing a gauge
condition reduces the quadratic part of \( S_{sft} \) (up to a constant) to a positive-definite form. One can
then write, as in this paper and in [7]
\[ A = H(\pi_+ \varphi, \pi_+ b, \pi_+ c) \Psi(\pi_- \varphi, \pi_- b, \pi_- c) \]
where \( H \) and \( \Psi \) are fermionic fields, \( \pi_+ \) is the operator on Fock space induced by the projection
\( \pi_+ : L_2([0, 1]) \rightarrow L_2([0, \frac{1}{2}]) \), and \( \pi_- \) is the operator induced on Fock space by the projection
\( \pi_- : L_2([0, 1]) \rightarrow L_2([\frac{1}{2}, 1]) \) [1]. There are various technical problems associated with the ghost current
anomaly, but I hope that with a proper cutoff (such as that of [8]) this theory can also be shown
to yield a convergent perturbation series.
References

[1] S. Axelrod, I. M. Singer, J. Differential Geom. 39 (1994), no. 1, 173–213
   – Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics,
   Vol. 1, 2 (New York, 1991), 3–45, World Sci. Publ., River Edge, NJ, 1992.

[2] D. Bar Natan, Topology 34 (1995) 423–472

[3] J. Feldman, J. Magnen, V. Rivasseau, R. Seneor, Commun. Math. Phys. 103, 67-105 (1986)

[4] K. Gawedzki, A. Kupiainen, Commun. Math. Phys. 102, 1-30 (1985)

[5] J. Glimm, A. Jaffe. Quantum Physics. Springer, 1987

[6] M. Salmhofer. Renormalization. Springer Verlag, 1999.

[7] J. Weitsman, Fermionization, Convergent Perturbation Theory, and Correlations in the Yang-Mills Quantum
   Field Theory in four dimensions. Preprint arXiv:0902.0096

[8] J. Weitsman, Measures on Banach Manifolds, Random Surfaces, and Nonperturbative String Field Theory
   with Cut-offs. arXiv:0807.2069

[9] E. Witten, Commun. Math. Phys. 121, 359 (1988)

[10] E. Witten, Nucl. Phys. B268 253 (1986)

Department of Mathematics, Northeastern University, Boston, MA 02115
E-mail address: j.weitsman@neu.edu