Abstract. Manin’s conjecture is proved for a split del Pezzo surface of degree 5 with a singularity of type \(A_2\).

1. Introduction

Let \(S \subset \mathbb{P}^5\) be the del Pezzo surface of degree 5 defined by
\[
\begin{align*}
    x_0x_2 - x_1x_5 &= x_0x_2 - x_3x_4 = x_0x_3 + x_1^2 + x_1x_4, \\
    &= x_0x_5 + x_1x_4 + x_4^2 = x_3x_5 + x_1x_2 + x_2x_4 = 0.
\end{align*}
\]
(1.1)

It contains a unique singularity of type \(A_2\) and four lines, all of them defined over \(\mathbb{Q}\). Let \(U \subset S\) be the complement of these lines.

We define the height of any rational point \(x \in S(\mathbb{Q})\) that is represented by integral and relatively coprime coordinates \((x_0, \ldots, x_5)\) as
\[
    H(x) := \max\{|x_0|, \ldots, |x_5|\}.
\]

For any \(B \geq 1\), let
\[
    N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) \mid H(x) \leq B\}
\]
be the number of rational points in \(U\) whose height is at most \(B\).

We prove the following result:

Theorem. We have
\[
    N_{U,H}(B) = c_{S,H}B(\log B)^4 + O(B(\log B)^{4-1/5}),
\]

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where
\[ c_{S,H} = \frac{1}{864} \cdot \omega_\infty \cdot \prod_p \left( 1 - \frac{1}{p} \right)^5 \left( 1 + \frac{5}{p} + \frac{1}{p^2} \right) \]
and
\[ \omega_\infty = \int_{|t_5|,|t_1|,|t_1^2t_6^2+t_7^2t_6|,|t_1t_5t_6|,|t_2^2t_6+t_1|,|t_3^2t_6^2+t_1t_5t_6| \leq 1, \ t_5 > 0} dt_1 dt_5 dt_6. \]

Manin’s conjecture [PM89] predicts that \( N_{U,H}(B) \) grows as \( cB(\log B)^{k-1} \) for \( B \to \infty \) where \( k \) is the rank of the Picard group of the minimal desingularization \( \tilde{S} \) of \( S \). As \( S \) is a del Pezzo surface of degree 5 whose lines are defined over \( \mathbb{Q} \), we have \( k = 5 \), so our result agrees with this conjecture.

Peyre [Pey95] predicts that \( \omega_\infty \) agrees with the density at each prime \( p \), and that \( \omega_\infty \) agrees with the real density, but we do not check this here.

Note that \( S \) is neither toric nor an equivariant compactification of \( \mathbb{G}_m^n \), so our theorem is not a consequence of [BT98] or [CLT02].

For the proof of the theorem, we use the basic strategy of [BB07], [BBD07] and [DT07] together with the techniques introduced in [BD07]. In Section 2, we translate the counting problem to the question of integral points on a universal torsor and split their counting into three parts. As outlined at the end of Section 2 these parts are handled separately in Sections 3 to 7 and put together again in Section 8 to complete the proof of the theorem.

2. A universal torsor

We use the notation
\[ \eta = (\eta_1, \ldots, \eta_4), \quad \eta' = (\eta_1, \ldots, \eta_6), \quad \alpha = (\alpha_1, \alpha_2) \]
and, for \( (n_1, \ldots, n_4) \in \mathbb{Q}^4 \),
\[ \eta^{(n_1,n_2,n_3,n_4)} = \eta_1^{n_1} \eta_2^{n_2} \eta_3^{n_3} \eta_4^{n_4}. \]

By the method of [DT07] and using the data of [Der06] on the geometry of \( S \) and its minimal desingularization \( \tilde{S} \), we obtain a bijection \( \Psi : \mathcal{T} \to U(\mathbb{Q}) \) with
\[ \mathcal{T} := \{(\eta', \alpha) \in \mathbb{Z}^5 \times \mathbb{Z}_{>0} \times \mathbb{Z}^2 \mid \text{coprimality conditions hold}\} \]
where
\[ (\eta_4 \eta_5^2 \eta_6 + \eta_1 \alpha_1 + \eta_2 \alpha_2 = 0 \]
and the coprimality conditions are described by the extended Dynkin diagram of \( E_1, \ldots, E_6, A_1, A_2 \) in Figure 1, using the rule that two variables are coprime unless the corresponding divisors in the diagram are connected by an edge. The map \( \Psi \) sends \((\eta', \alpha) \in \mathcal{T} \) to
\[ (\eta^{(2,2,3,2)} \eta_5, \eta^{(2,1,2,1)} \alpha_1, \eta_6 \alpha_1 \alpha_2, \eta^{(1,0,1,1)} \eta_5 \eta_6 \alpha_1, \eta^{(1,2,2,1)} \alpha_2, \eta^{(0,1,1,1)} \eta_5 \eta_6 \alpha_2) \]
in \( U(\mathbb{Q}) \).
Note that these coprimality conditions imply that the formula above for \( \Psi(\eta', \alpha) \) results in relatively coprime coordinates \( \Psi(\eta', \alpha)_i \), so
\[
H(\Psi(\eta', \alpha)) = \max_i \{|\Psi(\eta', \alpha)_i|\}.
\]
With (2.1), \( H(\Psi(\eta', \alpha)) \leq B \) implies
\[
\eta^{(1,1,2,2)}_{\eta_5^2}|\eta_6| \leq 2B, \quad \eta^{(0,0,1,2)}_{\eta_5^3}|\eta_6|^2 \leq 2B.
\]
Using (2.1), the coprimality conditions can be rewritten as
\[
\gcd(\alpha_2, \eta_3 \eta_5) = 1, \quad \gcd(\alpha_1, \eta_3 \eta_4) = 1, \quad \gcd(\eta_6, \eta_1 \eta_2 \eta_3 \eta_4) = 1, \quad \gcd(\eta_5, \eta_1 \eta_2 \eta_3) = 1, \quad \gcd(\eta_1, \eta_2) = 1, \quad \gcd(\eta_1, \eta_4) = 1, \quad \gcd(\eta_2, \eta_4) = 1.
\]
Therefore, the number \( N_{U,H}(B) \) coincides with the number of \((\eta', \alpha) \in \mathbb{Z}_5^5 \times \mathbb{Z}_{\neq 0} \times \mathbb{Z}^2 \) which satisfy the torsor equation (2.1), the coprimality conditions (2.3)–(2.7) and the height condition \( H(\Psi(\eta', \alpha)) \leq B \).

Our further strategy is as follows. For fixed \( \eta' \), we estimate the number of \( \alpha \) satisfying the torsor equation, the coprimality conditions and the height condition. We sum this number over all suitable \( \eta' \) afterwards. To get a hold of the error terms in these summations, it will be useful to do this summations in different orders depending on the relative size of \( \eta_1, \ldots, \eta_6 \).

We denote the number of \((\eta', \alpha)\) contributing to \( N_{U,H}(B) \) that fulfill
\[
|\eta_5| \geq |\eta_6|
\]
by \( N_a(B) \), and the number of those satisfying
\[
|\eta_5| < |\eta_6|
\]
by \( N_b(B) \).

We split the elements contributing to \( N_b(B) \) further into two subsets: For some \( A > 0 \) to be chosen in Section 8, let \( N_{b_1}(B; A) \) be the number of \((\eta', \alpha)\) satisfying (2.9) and
\[
\eta^{(2,2,3,2)} \leq \frac{B}{(\log B)^A},
\]
while \( N_{b_2}(B; A) \) is the number of the remaining ones.
We deal with $N_{b_2}(B; A)$ in the following Section \[3\]. As a first step for both $N_{b_1}(B; A)$ and $N_{b_2}(B; A)$, we estimate the number of $\alpha$ in Section \[5\]. For $N_{a}$, we sum first over the bigger $\eta_1$ and then over $\eta_6$ in Section \[8\] while for $N_{b_1}(B; A)$, we sum in the reverse order in Section \[7\] The resulting main terms are put together and summed over the remaining variables $\eta_1, \ldots, \eta_4$ in Section \[8\] to complete the proof of the theorem.

3. Estimating $N_{b_2}(B; A)$

Our strategy is to estimate the number of $(\eta', \alpha)$ lying in dyadic intervals first, and to sum over all possible intervals in a second step.

**Lemma 1.** We have $N_{b_2}(B; A) \ll_A B(\log B)^3(\log \log B)^2$.

**Proof.** Let $\mathcal{N} = \mathcal{N}(N_1, \ldots, N_6, A_1, A_2)$ be the number of $(\eta', \alpha)$ subject to $N_i/2 < |\eta_i| \leq N_i$ for $i \in \{1, \ldots, 6\}$ and $A_j/2 < |\alpha_j| \leq A_j$ for $j \in \{1, 2\}$.

Because of the height conditions and using the notation $N^{(n_1,n_2,n_3,n_4)} := N_1^{n_1} N_2^{n_2} N_3^{n_3} N_4^{n_4}$, we have, if $\mathcal{N} > 0$,

\begin{align*}
(3.1) & \quad B(\log B)^{-A} \ll N^{(2,2,3,2)} \ll B, \\
(3.2) & \quad N_6 A_1 A_2 \ll B, \\
(3.3) & \quad N^{(1,0,1,1)} N_5 N_6 A_1 \ll B, \\
(3.4) & \quad N^{(0,1,1,1)} N_5 N_6 A_2 \ll B, \\
(3.5) & \quad N^{(1,1,2,2)} N^2_5 N_6 \ll B, \\
(3.6) & \quad N_5 \ll (\log B)^A. 
\end{align*}

Here, (3.5) follows from (2.2). As in [BD07 Lemma 5, 6], we obtain by estimating the number of $\alpha_1, \alpha_2$ in two ways first and summing over $\eta_1, \ldots, \eta_6$ afterwards:

\[ \mathcal{N} \ll N_3 N_4 N_5 N_6 (N_1 A_1)^{1/2} (N_2 A_2)^{1/2} + N_1 N_2 N_3 N_4 N_5 N_6. \]

Next, we sum this estimate for $\mathcal{N}(N_1, \ldots, N_6, A_1, A_2)$ over all possible dyadic intervals, with $N_1, \ldots, N_6, A_1, A_2$ subject to (3.1)–(3.6).

For the first term, we have using (3.2)–(3.5)

\[
\sum_{N_1, \ldots, N_6, A_1, A_2} N^{(1/2,1/2,1,1)} N_5 N_6 A_1^{1/2} A_2^{1/2} \ll B^{1/4} \sum_{N_1, \ldots, N_6, A_1, A_2} N^{(1/2,1/2,1,1)} N_5 N_6^{3/4} A_1^{1/4} A_2^{1/4} \ll B^{3/4} \sum_{N_1, \ldots, N_6} N^{(1/4,1/4,1/2,2)} N_5^{1/2} N_6^{1/4} \ll B \sum_{N_1, \ldots, N_5} 1 \ll_A B(\log B)^3(\log \log B)^2.
\]

Here we have used that for fixed $N_2, N_3, N_4$, there are only $O_A(\log \log B)$ possibilities for $N_1$ and $N_5$ by (3.1) and (3.6).
For the second term, we use (3.5) to obtain
\[
\sum_{N_1, \ldots, N_6, A_1, A_2} N^{(1,1,1,1)} N_5 N_6 \ll B \sum_{N_1, \ldots, N_5, A_1, A_2} \frac{1}{N_3 N_4 N_5} \ll_A B (\log B)^3 (\log \log B),
\]
which completes the proof.

4. Real-valued functions

Let
\[
h(t_0, t_1, t_5, t_6) := \max \left\{ \frac{|t_0^4 t_5|, |t_0^4 t_1|, |t_1^2 t_5 t_6 + t_6^2 t_1 t_6|, |t_0^2 t_1 t_5 t_6|}{|t_0^2 t_5^2 t_6 + t_6^2 t_1|, |t_5^2 + t_0^2 t_1 t_5 t_6|} \right\}.
\]
Defining
\[
Y_0 := \left( \frac{\eta(2,2,3,2)}{B} \right)^{1/5}, \quad Y_1 := \left( \frac{B}{\eta(2,3,2,3)} \right)^{1/5},
\]
\[
Y_5 := Y_0^{-1}, \quad Y_6 := \left( \frac{B}{\eta(-3,-3,-2,2)} \right)^{1/5},
\]
we note that the height condition \( H(\Psi(\eta', \alpha)) \leq B \) is equivalent to
\[
h(Y_0, \alpha_1/Y_1, \eta_5/Y_5, \eta_6/Y_6) \leq 1.
\]

Define
\[
g_0(t_0, t_5, t_6) := \int_{h(t_0, t_1, t_5, t_6) \leq 1} 1 \, dt_1,
\]
\[
g_1^a(t_0, t_6; \eta; B) := \int_{Y_5 t_5 \geq \max\{Y_6 t_6, t_5\} \neq 0} g_0(t_0, t_5, t_6) \, dt_5,
\]
\[
g_1^b(t_0, t_5; \eta; B) := \int_{Y_6 t_6 \geq \max\{Y_5 t_5, 1\}} g_0(t_0, t_5, t_6) \, dt_6,
\]
\[
g_2^a(t_0; \eta; B) := \int_{Y_5 t_5 > 1} g_1^a(t_0, t_6; \eta; B) \, dt_6,
\]
\[
g_2^b(t_0; \eta; B) := \int_0^\infty g_1^b(t_0, t_5; \eta; B) \, dt_5.
\]
We have
\[
g_2(t_0; \eta; B) := g_2^a(t_0; \eta; B) + g_2^b(t_0; \eta; B)
\]
\[
= \int_{h(t_0, t_1, t_5, t_6) \leq 1, |Y_6 t_6| > 1, t_5 > 0} \frac{1}{t_0 |t_5|^{3/4}} dt_1 dt_5 dt_6.
\]

Lemma 2. Let \( \eta \in \mathbb{Z}_{>0}^4 \) be given. Then we have:

1. \( g_0(t_0, t_5, t_6) \ll \frac{1}{t_0^{1/2} |t_5|^{3/4}} \).
2. \( g_1^a(t_0, t_6; \eta; B) \ll \int_0^\infty g_0(t_0, t_5, t_6) \, dt_5 \ll \min\left\{ \frac{1}{t_0^{1/2} |t_5|^{3/4}}, \frac{1}{t_0^{1/4}} \right\} \).
3. \( g_1^b(t_0, t_5; \eta; B) \ll \int_{-\infty}^\infty g_0(t_0, t_5, t_6) \, dt_6 \ll \frac{1}{t_0^{3/4}} \).
Proof. Since \( h(t_0, t_1, t_5, t_6) \leq 1 \) implies \( t_1 \leq t_0^{-4} \) and \( t_5 \leq t_0^{-4} \), the second bound of (2) holds.

It is not hard to check that given \( a, b \in \mathbb{R} \setminus \{0\} \), the condition \( |at^2 + bt| \leq 1 \) describes a set of \( t_1 \) whose length is \( \ll |a|^{-1/2} \) for \( b^2 \leq 8|a| \), while its length is \( \ll |b|^{-1} \ll |a|^{-1/2} \) for \( b^2 > 8|a| \).

We apply this for \( a = t_0^2 t_6 \) and \( b = t_5^2 t_6 \), which gives \( g_0(t_0, t_5, t_6) \ll (t_0^2 t_6)^{-1/2} \) which is (1). Integrating it over \( t_6 \ll t_0^{3/2} \) (which holds since \( |t_0^2 t_1 t_5 t_6| \leq 1 \) and \( |t_0^2 t_6^2 + t_5^2 t_1 t_5 t_6| \leq 1 \) imply \( |t_0^2 t_6^2| \leq 2 \)) results in (3).

For the first bound of (2), we distinguish the case \( t_0^4 t_6 \leq 8t_0^2 t_6 \) and its opposite. In the first case, we combine \( t_5 \ll t_0^{1/2} t_6^{-3/4} \) with (1). In the second case, we integrate \( g_0(t_0, t_5, t_6) \ll t_5^{-2} t_6^{-2} \) over \( t_5 \gg t_0^{1/2} t_6^{-3/4} \).

Finally, we define

\[
G_2(t_0) := \int_{h(t_0, t_1, t_5, t_6) \leq 1, t_5 > 0} dt_1 dt_5 dt_6
\]

which is related to \( \omega_{\infty} \) defined in the statement of our theorem:

**Lemma 3.** For any \( t_0 > 0 \), we have \( G_2(t_0) = \frac{\omega_{\infty}}{t_0^5} \).

Proof. Similar to [BD07, Lemma 7]. \( \square \)

5. Estimating \( N_a(B) \) and \( N_b(B; A) \) – first step

For fixed \( \eta' \) subject to the coprimality conditions (2.5)–(2.7), let \( N_0 \) be the number of \( \alpha_1, \alpha_2 \) subject to (2.1), \( h(Y_0, \alpha_1/Y_1, \eta_5/Y_5, \eta_6/Y_6) \leq 1 \) and the coprimality conditions (2.3), (2.4).

We remove (2.3) by a Möbius inversion and obtain

\[
N_0 = \sum_{k_2 \mid \eta_3 \eta_5} \mu(k_2) \# \left\{ \alpha_1 \mid \eta_4 \eta_5^2 \eta_6 \equiv -\eta_1 \alpha_1 \pmod{k_2 \eta_2}, \ h(Y_0, \alpha_1/Y_1, \eta_5/Y_5, \eta_6/Y_6) \leq 1, \right\}
\]

The summand vanishes unless \( \gcd(k_2, \eta_1 \eta_4) = 1 \). Since \( \eta_3, \eta_5 \) are coprime, we write \( k_2 = k_{23} k_{25} \) uniquely such that \( k_{2i} \mid \eta_i \) for \( i \in \{3, 5\} \). We check that \( k_{25} \mid \alpha_1 \). We write \( \eta_5 = k_{25} \eta_5' \), \( \alpha_1 = k_{25} \alpha_1' \) and obtain

\[
N_0 = \sum_{k_2 \mid \eta_3 \eta_5} \mu(k_2) \mu(k_{23}) \mu(k_{25}) N_0(k_{23}, k_{25})
\]

where

\[
N_0(k_{23}, k_{25}) = \# \left\{ \alpha_1' \mid k_{25} \eta_4 \eta_5^2 \eta_6 \equiv -\eta_1 \alpha_1' \pmod{k_{23} \eta_2}, \ h(Y_0, \alpha_1' k_{25}/Y_1, \eta_5/Y_5, \eta_6/Y_6) \leq 1, \right\}
\]

Note that \( \gcd(k_{25}, \eta_3 \eta_4) = 1 \) holds automatically, so we may remove this condition. We remove the coprimality condition for \( \alpha_1' \) by another Möbius inversion and obtain, writing \( \alpha_1' = k_1 \alpha_2' \),

\[
N_0(k_{23}, k_{25}) = \sum_{k_1 \mid \eta_3 \eta_4} \mu(k_1) \# \left\{ \alpha_1' \mid k_{25} \eta_4 \eta_5^2 \eta_6 \equiv -k_1 \eta_1 \alpha_2' \pmod{k_{23} \eta_2}, \ h(Y_0, \alpha_1' k_{25} k_1/Y_1, \eta_5/Y_5, \eta_6/Y_6) \leq 1, \right\}
\]
Note that the summand vanishes unless \( \gcd(k_1, k_{23}) = 1 \), so we may restrict the summation over \( k_1 | \eta_3 \eta_4 \) subject to \( \gcd(k_1, k_{23}) = 1 \). Since then \( \gcd(k_1 \eta_1, k_{23} \eta_2) = 1 \), the number of \( \alpha''_1 \) is

\[
\frac{Y_1}{k_1 k_{23} k_{25} \eta_2} g_0(Y_0, \eta_5 / Y_5, \eta_6 / Y_6) + O(1).
\]

Define \( \phi^*(n) := \prod_{p | n} (1 - 1/p) \).

**Lemma 4.** We have

\[
N_0 = \frac{Y_1}{\eta_2} g_0(Y_0, \eta_5 / Y_5, \eta_6 / Y_6) \vartheta_0(\eta) \frac{\phi^*(\eta_5)}{\phi^*(\gcd(\eta_5, \eta_4))} + O(R_0(\eta, \eta_5, \eta_6))
\]

with

\[
\vartheta_0(\eta) := \sum_{\substack{k_{23} | \eta_3 \\ \gcd(k_{23}, \eta_5 \eta_4) = 1}} \frac{\mu(k_{23}) \phi^*(\eta_3 \eta_4)}{k_{23} \phi^*(\gcd(\eta_3, k_{23} \eta_2))}
\]

and

\[
\sum_{\eta_1, \ldots, \eta_6} R_0(\eta, \eta_5, \eta_6) \ll B (\log B)^2.
\]

**Proof.** For the main term, note that

\[
\sum_{\substack{k_{23} | \eta_3, k_{25} | \eta_5 \\ \gcd(k_{23} k_{25}, \eta_1 \eta_4) = 1}} \frac{\mu(k_{23}) \mu(k_{25})}{k_{23} k_{25}} \sum_{\substack{k_1 | \eta_5 \eta_4 \\ \gcd(k_1, k_{23} \eta_2) = 1}} \frac{\mu(k_1)}{k_1} = \sum_{\substack{k_{23} | \eta_3 \\ \gcd(k_{23}, \eta_1 \eta_4) = 1}} \frac{\mu(k_{23})}{k_{23}} \phi^*(\eta_5) \frac{\phi^*(\gcd(\eta_5, \eta_1 \eta_4))}{\phi^*(\gcd(\eta_5 \eta_4, k_{23} \eta_2))}.
\]

Using \( \gcd(\eta_5, \eta_1) = 1 \) and \( \gcd(\eta_1, k_{23} \eta_2) = 1 \), we obtain \( \vartheta_0 \).

We have

\[
R_0(\eta, \eta_5, \eta_6) \ll 2^{\omega(\eta_5) + \omega(\eta_5) + \omega(\eta_3 \eta_4)}
\]

We sum this over all suitable \( \eta_1, \ldots, \eta_6 \) and use (2.2) to obtain

\[
\sum_{\eta_1, \ldots, \eta_6} R_0(\eta, \eta_5, \eta_6) \ll \sum_{\eta_1, \ldots, \eta_5} 2^{\omega(\eta_5) + \omega(\eta_5) + \omega(\eta_3 \eta_4)} B \eta_5^{1/2} \eta_6^2
\]

\[
\ll B (\log B)^2,
\]

completing the proof of this lemma. \( \square \)

6. Estimating \( N_a(B) \) – Second Step

Let \( N_1^a := N_1^a(\eta; B) \) be the sum of the main term of Lemma 4 over \( \eta_5 \) subject to (2.6) and (2.8). We sum the main term of \( N_1^a \) over \( \eta_6 \) afterwards to obtain \( N_2^a(\eta; B) \).
Using [BD07, Lemma 2] with $\alpha = 0, q = 1$, we obtain (where $f_{a,b}(n)$ is
defined to be $\phi^*(n)/\phi^*(\gcd(n,a))$ if $\gcd(n,b) = 1$ and to be zero otherwise)

\[ N_1^a = \frac{Y_1}{\eta_2} \vartheta_0(\eta) \sum_{\eta_5 \gg |\eta_6|} f_{\eta_4,\eta_5,\eta_6,\eta_7}(\eta_5) g_0(Y_0, \eta_5/Y_5, \eta_6/Y_6; \eta; B) \]

\[ = \frac{Y_1 Y_5}{\eta_2} g_1^a(Y_0, \eta_6/Y_6; \eta; B) \vartheta_0(\eta) \frac{\phi^*(\eta_1 \eta_2 \eta_3)}{\zeta(2)} \prod_{p|\eta_1 \eta_2 \eta_3} \left(1 - \frac{1}{p^2}\right)^{-1} \]

\[ + O \left( \frac{Y_1}{\eta_2} |\vartheta_0(\eta)||\log B|2^{\omega(\eta_1 \eta_2 \eta_3)} \sup_{t_5} g_0(Y_0, t_5, \eta_6/Y_6; \eta; B) \right), \]

where the supremum is taken over $t_5 \gg |\eta_6|/Y_5$.

**Lemma 5.** We have

\[ N_1^a = \frac{Y_1 Y_5}{\eta_2} g_1^a(Y_0, \eta_6/Y_6; \eta; B) \vartheta_0^a(\eta) + O(R_1^a(\eta, \eta_6; B)) \]

with

\[ \vartheta_0^a(\eta) := \vartheta_0(\eta) \frac{\phi^*(\eta_1 \eta_2 \eta_3)}{\zeta(2)} \prod_{p|\eta_1 \eta_2 \eta_3} \left(1 - \frac{1}{p^2}\right)^{-1} \]

and

\[ \sum_{\eta_1, \eta_6} R_1^a(\eta, \eta_6; B) \ll B \log B. \]

**Proof.** The main term is clear. Define $\phi^+(n) := \prod p|n(1 + 1/p)$. For the error term, we use Lemma [241] to estimate its sum over $\eta, \eta_6$ as

\[ \ll \sum_{\eta, \eta_6} 2^{\omega(\eta_1 \eta_2 \eta_3)} \phi^+(\eta_3) Y_1 Y_6^{1/2} \log B \]

\[ = \sum_{\eta, \eta_6} 2^{\omega(\eta_1 \eta_2 \eta_3)} \phi^+(\eta_3) B^{1/2} \log B \]

\[ \ll \sum_{\eta_1, \eta_2, \eta_3, \eta_6} 2^{\omega(\eta_1 \eta_2 \eta_3)} \phi^+(\eta_3) B \log B \]

\[ \ll B \log B. \]

Here, we use

\[ \eta_4 \leq \left( \frac{B}{\eta_{(2,2,3,0)}|\eta_6|} \right)^{1/4} \cdot \left( \frac{B}{\eta_{(1,1,2,0)}|\eta_6|} \right)^{1/4} = \frac{B^{1/2}}{\eta_{(3/4,3/4,5/4,0)|\eta_6|}}, \]

which is obtained with (2.35).

To sum the main term of $N_1^a$ over $\eta_6$, we remove the coprimality condition (2.5) by a M"obius inversion and obtain, writing $\eta_6 = k_6 \eta_6'$ and applying
partial summation,
\[ N_2^\alpha = \frac{Y_1 Y_5}{\eta_2} \partial_2^\alpha(\eta) \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \mu(k_6) \sum_{|\eta_6| \geq 1} g_1^\alpha(Y_0, \eta_6 k_6/Y_6; \eta; B) \]
\[ = \frac{Y_1 Y_5 Y_6}{\eta_2} \partial_2^\alpha(\eta) \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \mu(k_6) \frac{k_6}{k_6} \int_{|t_6| \geq k_6 / Y_6} g_1^\alpha(Y_0, t_6; \eta; B) \, dt_6 \]
\[ + O\left( \frac{Y_1 Y_5}{\eta_2} |\partial_1^\alpha(\eta)| \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} |\mu(k_6)| \sup_{|t_6| \geq k_6 / Y_6} g_1^\alpha(Y_0, t_6; \eta; B) \right). \]

Lemma 6. We have
\[ N_2^\alpha = \frac{Y_1 Y_5 Y_6}{\eta_2} g_2^\alpha(Y_0; \eta; B) \partial_2^\alpha(\eta) + O(R_2^\alpha(\eta; B)) \]
with
\[ \partial_2^\alpha(\eta) := \partial_1^\alpha(\eta) \phi^*(\eta_1 \eta_2 \eta_3 \eta_4) \]
and
\[ \sum_{\eta} R_2^\alpha(\eta; B) \ll B (\log B)^{4-1/5}. \]

Proof. In order to replace the integral over \(|t_6| \geq k_6 / Y_6\) in the estimation before the statement of the lemma by \(g_2^\alpha(Y_0; \eta; B)\), we must add
\[ \frac{Y_1 Y_5 Y_6}{\eta_2} \partial_1^\alpha(\eta) \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \mu(k_6) \frac{k_6}{k_6} \int_{1/Y_6 < |t_6| < k_6 / Y_6} g_1^\alpha(Y_0, t_6; \eta; B) \, dt_6 \]
as a second error term.

We distinguish the case
\[ \eta^{(3,3,4,2)} < \lambda B \]
for some \(\lambda > 0\) to be chosen later, giving a total contribution \(E_1(\lambda)\) to the error term, and its opposite
\[ \eta^{(3,3,4,2)} \geq \lambda B, \]
contributing in total \(E_2(\lambda)\).

Starting with \(E_1(\lambda)\), we use the first bound of Lemma \([2][2]\). For the first error term, we obtain
\[ \ll \sum_{\eta} \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} |\mu(k_6)| \phi^*(\eta_3) Y_1 Y_5 Y_6^{5/4} \]
\[ \ll \sum_{\eta} \frac{\phi^*(\eta_3) B^{3/4}}{\eta^{(1/4,1/4,0,1/2)}} \]
\[ \ll \sum_{\eta_1, \eta_2, \eta_3} \frac{\phi^*(\eta_3) \lambda^{1/4} B}{\eta^{(1,1,1,0)}} \]
\[ \ll \lambda^{1/4} B (\log B)^3. \]
For the second error term, we use
\[ \int_{1/Y_0}^{k_6/Y_0} g_1^a(Y_0, t_6; \eta; B) \, dt_6 \ll \int_{1/Y_0}^{k_6/Y_0} \frac{1}{Y_0^{1/2} [t_6]^{5/4}} \, dt_6 \ll \frac{Y_6^{1/4}}{Y_0^{1/2}} \]
and obtain
\[
\ll \sum_{\eta} \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \frac{|\mu(k_6)| \phi^\perp(\eta_3) Y_1 Y_5 Y_6^{5/4}}{k_6 \eta_2 Y_0^{1/2}} \\
\ll \sum_{\eta} \frac{\phi^\perp(\eta_1 \eta_2 \eta_3 \eta_4) \phi^\perp(\eta_3) B^{3/4}}{\eta^{(1/4,1/4,0,1/2)}} \\
\ll \sum_{\eta_1 \eta_2 \eta_3} \frac{\phi^\perp(\eta_1 \eta_2 \eta_3) \phi^\perp(\eta_3) \lambda^{1/4} B}{\eta^{(1,1,1,0)}} \\
\ll \lambda^{1/4} B (\log B)^3.
\]
Therefore, \( E_1(\lambda) \ll \lambda^{1/4} B (\log B)^3 \).

For \( E_2(\lambda) \), we use the second bound of Lemma 2.3. For the first part of this error term, we get
\[
\ll \sum_{\eta} \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \frac{|\mu(k_6)| \phi^\perp(\eta_3) Y_1 Y_5}{\eta_2 Y_0^8} \\
\ll \sum_{\eta} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4)} \phi^\perp(\eta_3) B^2}{\eta^{(4,4,5,3)}} \\
\ll \sum_{\eta_1 \eta_2 \eta_3} \frac{2^{\omega(\eta_1 \eta_2 \eta_3)} \phi^\perp(\eta_3) B \log B}{\lambda \eta^{(1,1,1,0)}} \\
\ll \lambda^{-1} B (\log B)^7.
\]
For the second part of the error term, we use
\[ \int_{1/Y_0}^{k_6/Y_6} g_1^b(Y_0, t_6; \eta; B) \, dt_6 \ll \int_{1/Y_0}^{k_6/Y_6} \frac{1}{Y_0^{1/2} [t_6]^{5/4}} \, dt_6 \ll \frac{k_6}{Y_0^{3} Y_6} \]
and obtain
\[
\ll \sum_{\eta} \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \frac{|\mu(k_6)| \phi^\perp(\eta_3) Y_1 Y_5 Y_6}{k_6 \eta_2 Y_0^{8} Y_6} \\
\ll \sum_{\eta} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4)} \phi^\perp(\eta_3) B^2}{\eta^{(4,4,5,3)}} \\
\ll \sum_{\eta_1 \eta_2 \eta_3} \frac{2^{\omega(\eta_1 \eta_2 \eta_3)} \phi^\perp(\eta_3) B \log B}{\lambda \eta^{(1,1,1,0)}} \\
\ll \lambda^{-1} B (\log B)^7.
\]
In total, \( E_2(\lambda) \ll \lambda^{-1} B (\log B)^7 \).

Choosing \( \lambda = (\log B)^{16/5} \) gives a total error term of \( O(B (\log B)^{4-1/5}) \).
7. Estimating $N_{b_1}(B; A)$ – second step

Let $N^b_1 := N^b_1(\eta, \eta_5; B)$ be the main term of $N_0$ in Lemma 4 summed over $\eta_6$ subject to (2.3) and (2.9). We denote the main term of this summed over all $\eta_5$ by $N^b_2 := N^b_2(\eta; B)$. We remove (2.5) by a Möbius inversion and get

$$N^b_1 = \frac{Y_1}{\eta_2} \vartheta_0(\eta) \phi^*(\eta_5) \sum_{k_6|\eta_1 \eta_2 \eta_3 \eta_4} \mu(k_6) A$$

where

$$A = \sum_{t_6 \in \mathbb{Z} \neq 0, k_6|\eta_6, |t_6| > \eta_5} g_0(Y_0, \eta_5/Y_5, k_6 \vartheta_0/Y_6; \eta; B).$$

By partial summation,

$$A = \frac{Y_6}{k_6} g_1^b(Y_0, \eta_5/Y_5; \eta; B) + O(\sup_{t_6} g_0(Y_0, \eta_5/Y_5, t_6)),$$

where the supremum is taken over $t_6$ subject to $|t_6| > \eta_5/Y_6$.

**Lemma 7.** We have

$$N^b_1 = \frac{Y_1 Y_6}{\eta_2} g_1^b(Y_0, \eta_5/Y_5; \eta; B) \vartheta_1^b(\eta) \phi^*(\eta_5) \phi^*(\gcd(\eta_5, \eta_4)) + O(R_1^b(\eta, \eta_5; B))$$

with

$$\vartheta_1^b(\eta) := \vartheta_0(\eta) \phi^*(\eta_1 \eta_2 \eta_3 \eta_4)$$

and

$$\sum_{\eta_5} R_1^b(\eta, \eta_5; B) \ll B \log B.$$

**Proof.** The main term is clear. We apply Lemma 23 to deduce that the error term can be estimated as

$$\sum_{\eta_5} R_1^b(\eta, \eta_5; B) \ll \sum_{\eta_5} 2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4)} \phi^*(\eta_5) Y_1 Y_6^{1/2} \eta_2 \eta_5 \eta_6^{1/2} Y_0$$

$$= \sum_{\eta_5} 2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4)} \phi^*(\eta_5) B^{1/2} \eta_1^{1/2} \eta_2^{1/2} \eta_5^{1/2}$$

$$\ll \sum_{\eta_1 \eta_2 \eta_3 \eta_5} 2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4)} \phi^*(\eta_5) B \log B \eta_5^{3/2} \eta_5^{3/2}$$

$$\ll B \log B.$$

In the last step, we have used

$$\eta_4 \leq \left( \frac{B}{\eta_6^{(2,2.3,0)} \eta_5} \right)^{1/4} \left( \frac{B}{\eta_6^{(1.1.2,0)} \eta_5^3} \right)^{1/4} = \frac{B^{1/2}}{\eta_6^{(3.4.3,4.3,4.5,4.0,4.0) \eta_5}},$$

which we obtain using (2.2) and (2.9).
Next, we sum the main term of Lemma 5 over all suitable \( \eta_5 \). Apply [BD07, Lemma 2] with \( \alpha = 0, q = 1 \) to obtain

\[
N_2^b = \frac{Y_1 Y_6}{\eta_2} \varphi_b^b(\eta) \sum_{\eta_5 \geq 1} f_{\eta_4, \eta_5, \eta_3} (\eta_5) g_b(Y_0, \eta_5/\eta; \eta; B)
= \frac{Y_1 Y_5 Y_6}{\eta_2} g_b^2(Y_0; \eta; B) \varphi_b^b(\eta) \frac{\phi^* (\eta \eta_5 \eta_3)}{\zeta(2)} \prod_{p \mid \eta_1 \eta_2 \eta_3} \left( 1 - \frac{1}{p^2} \right)^{-1}
+ O \left( \frac{Y_1 Y_6}{\eta_2} \varphi^b(\eta) (\log B)^{2 \omega(\eta \eta_5 \eta_3)} \sup_{t_5} g_1(Y_0, t_5; \eta; B) \right)
+ O \left( \frac{Y_1 Y_5 Y_6}{\eta_2} \varphi^b(\eta) \int_{0 \leq t_5 \leq 1/\eta_5} g_1(Y_0, t_5; \eta; B) dt_5 \right),
\]

where the supremum is taken over \( t_5 > 1/\eta_5 \).

**Lemma 8.** We have

\[
N_2^b = \frac{Y_1 Y_5 Y_6}{\eta_2} g_b^2(Y_0; \eta; B) \varphi_b^b(\eta) + O(R^b_2(\eta; B))
\]

with

\[
\varphi_b^b(\eta) := \varphi_1^b(\eta) \frac{\phi^*(\eta \eta_5 \eta_3)}{\zeta(2)} \prod_{p \mid \eta_1 \eta_2 \eta_3} \left( 1 - \frac{1}{p^2} \right)^{-1}
\]

and

\[
\sum_{\eta} R^b_2(\eta; B) \ll B (\log B)^{7 - A/4},
\]

where the sum is taken over \( \eta \) satisfying (2.10).

**Proof.** The main term is clear from the discussion before the lemma. The first part of the error term makes the contribution

\[
\ll \sum_{\eta} 2^{2 \omega(\eta \eta_5 \eta_3)} \varphi^b(\eta_3) Y_1 Y_6 \log B \sup_{\eta_2} g_1(Y_0, t_5; \eta; B).
\]

We use Lemma 2 and (2.10) to obtain

\[
\ll \sum_{\eta} 2^{2 \omega(\eta \eta_5 \eta_3)} \varphi^b(\eta_3) Y_1 Y_5^{3/4} Y_6 \log B
= \sum_{\eta} 2^{2 \omega(\eta \eta_5 \eta_3)} \varphi^b(\eta_3) B^{3/4} \log B
\eta^{(1/2, 1/2, 1/4, 1/2)}
\ll \sum_{\eta_1, \eta_2, \eta_3} 2^{2 \omega(\eta \eta_5 \eta_3)} \varphi^b(\eta_3) B (\log B)^{1 - A/4}
\eta^{(1, 1, 0)}
\ll B (\log B)^{7 - A/4}.
\]
The contribution of the second term is, using Lemma 2(3) and (2.10) again,

\[ \ll \sum_{\eta} \frac{\phi^1(\eta_3)Y_1Y_5Y_6}{\eta_2} \int_0^{1/Y_5} \frac{1}{Y_0 t_5^{3/4}} dt_5 \]

\[ \ll \sum_{\eta} \frac{\phi^1(\eta_3)Y_1Y_6^{5/4}Y_6}{\eta_2 Y_0} \]

\[ = \sum_{\eta} \frac{\phi^1(\eta_3)B^{3/4}}{\eta^{(1/2,1/2,1/4,1/2)}} \]

\[ \ll \sum_{\eta_1, \eta_2, \eta_3} \frac{\phi^1(\eta_3)B(\log B)^{-A/4}}{\eta^{(1,1,1,0)}} \]

\[ \ll B(\log B)^{3-A/4}. \]

This completes the proof of the lemma. \[\square\]

8. The final step

By the discussion at the end of Section 2, we have, for any \( A > 0 \),

\[ N_{U,H}(B) = N_a(B) + N_{b_1}(B; A) + N_{b_2}(B; A). \]

By Lemma 1,

\[ N_{U,H}(B) = N_a(B) + N_{b_1}(B; A) + O_A(B(\log B)^3(\log \log B)^2). \]

Using Lemmas 4, 5 and 6 and combining their error terms shows that

\[ N_a(B) = \sum_{\eta \in \mathcal{E}(B)} \frac{Y_1Y_5Y_6}{\eta_2} g_2^a(Y_0; \eta; B) \vartheta_2^a(\eta) + O(B(\log B)^{4-1/5}), \]

where

\[ \mathcal{E}(t) := \{ \eta \in \mathbb{Z}_+^4 \mid (2.7), \eta^{(2,2,3,2)} \leq t \} \]

for any \( t \geq 1 \), while Lemmas 4, 7 and 8 give, choosing \( A := 28 \),

\[ N_{b_1}(B; 28) = \sum_{\eta \in \mathcal{E}(B/(\log B)^{28})} \frac{Y_1Y_5Y_6}{\eta_2} g_2^b(Y_0; \eta; B) \vartheta_2^b(\eta) + O(B(\log B)^2). \]

Recall the definition (4.7) of \( g_2 \).

**Lemma 9.** We have

\[ N_{U,H}(B) = \sum_{\eta \in \mathcal{E}(B)} \frac{Y_1Y_5Y_6}{\eta_2} g_2(Y_0; \eta; B) \vartheta(\eta) + O(B(\log B)^{4-1/5}) \]

where

\[ \vartheta(\eta) := \frac{\phi^*(\eta_3\eta_4)\phi^*(\eta_1\eta_2\eta_3)\phi^*(\eta_1\eta_2\eta_3\eta_4)}{\zeta(2)} \prod_{p|\eta_1\eta_2\eta_3\eta_4} \left( 1 - \frac{1}{p^2} \right)^{-1} \]

\[ \times \left( \sum_{k_{23} | \eta_3 \atop \gcd(k_{23}, \eta_1 \eta_4) = 1} \frac{\mu(k_{23})}{k_{23} \phi^*(\gcd(\eta_3, k_{23}\eta_2))} \right) \]
if the coprimality conditions \( (2.7) \) hold, and \( \vartheta(\eta) := 0 \) otherwise.

**Proof.** We easily check that \( \vartheta(\eta) \) agrees with \( \vartheta_2^a(\eta) \) and \( \vartheta_2^b(\eta) \) for \( \eta \) satisfying \( (2.7) \).

In view of the discussion before the lemma, it remains to show that

\[
\sum_{\eta \in \mathcal{E}(B) \setminus \mathcal{E}(B/\log B)^{28}} \frac{Y_1 Y_5 Y_6}{\eta_2} g_2^b(Y_0; \eta; B) \vartheta_2^b(\eta)
\]

makes a negligible contribution.

Indeed, we estimate this as

\[
\ll \sum_{\eta \in \mathcal{E}(B) \setminus \mathcal{E}(B/\log B)^{28}} \frac{\phi^a(\eta_3) Y_1 Y_5 Y_6}{\eta_2^2 Y_0^2}
\]

\[
= \sum_{\eta \in \mathcal{E}(B) \setminus \mathcal{E}(B/\log B)^{28}} \frac{\phi^a(\eta_3) B}{\eta^{(1,1,1,1)}}
\]

\[
\ll B(\log B)^3(\log \log B)
\]

since we have

\[
g_2^b(t_0; \eta; B) \ll \int_0^{1/t_5^3} g_1^b(t_0, t_5; \eta; B) \, dt_5 \ll \int_0^{1/t_5^3} \frac{1}{t_5 t_5^3/4} \, dt_5 \ll \frac{1}{t_5^2},
\]

using Lemma \(2\underline{[3]} \) and the fact that \( g_1^b(t_0, t_5; \eta; B) = 0 \) unless \( t_5 \ll 1/t_0^4 \) by \( (1.1) \).

Define

\[ \mathcal{E}^*(B) := \{ \eta \in \mathbb{Z}_+^4 \mid \eta^{(2,2,3,2)} \leq B, \eta^{(3,3,4,2)} > B \}. \]

**Lemma 10.** We have

\[ N_{U,H}(B) = \omega_{\infty} B \sum_{\eta \in \mathcal{E}^*(B)} \frac{\vartheta(\eta)}{\eta^{(1,1,1,1)}} + O(B(\log B)^4 - 1/5). \]

**Proof.** By Lemma \(2\underline{[2]} \), we have

\[ g_2(Y_0; \eta; B) \ll \int_{|Y_0 t_6| > 1} \frac{1}{Y_0^{1/2} |t_6|^{5/4}} \, dt_6 \ll \frac{Y_6^{1/4}}{Y_0^{1/2}}. \]

Therefore,

\[
\sum_{\eta \in \mathcal{E}(B) \setminus \mathcal{E}^*(B)} \frac{Y_1 Y_5 Y_6}{\eta_2} g_2(Y_0; \eta; B) \vartheta(\eta) \ll \sum_{\eta^{(3,3,4,2)} \leq B} \frac{\phi^a(\eta_3) Y_1 Y_5 Y_6^{5/4}}{\eta_2 Y_0^{1/2}}
\]

\[
\ll \sum_{\eta^{(3,3,4,2)} \leq B} \frac{\phi^a(\eta_3) B^{3/4}}{\eta^{(1/4,1/4,0,1/2)}}
\]

\[
\ll \sum_{\eta_1, \eta_2, \eta_3} \frac{\phi^a(\eta_3) B}{\eta^{(1,1,1,0)}}
\]

\[ \ll B(\log B)^3. \]
This proves that
\[ N_{U,H}(B) = \sum_{\eta \in \mathcal{E}^*(B)} \frac{Y_1 Y_5 Y_6}{\eta_2} g_2(Y_0; \eta; B) \vartheta(\eta) + O(B(\log B)^{4-1/5}). \]

Comparing the definitions (4.7) and (4.8) of the functions \( g_2 \) and \( G \), together with the estimation
\[ \sum_{\eta \in \mathcal{E}^*(B)} \frac{Y_1 Y_5 Y_6}{\eta_2} \int_{h(Y_0, t_1, t_5, t_6) < 1, |Y_6 t_6| < 1, t_5 > 0} dt_1 dt_5 dt_6 \]
\[ 
\leq \sum_{\eta \in \mathcal{E}^*(B)} \frac{\phi^{1}(\eta_3) Y_1 Y_5 Y_6}{\eta_2} \int_{|Y_6 t_6| < 1} \frac{1}{Y_0} dt_6 
\leq \sum_{\eta \in \mathcal{E}^*(B)} \frac{\phi^{1}(\eta_3) Y_1 Y_5 Y_6}{\eta_2 Y_0^8} 
\leq \sum_{\eta_1 \eta_2 \eta_3} \frac{\phi^{1}(\eta_3) B}{\eta^{(1,1,1,0)}} 
\leq B(\log B)^3 
\]
shows that
\[ N_{U,H}(B) = \sum_{\eta \in \mathcal{E}^*(B)} \frac{Y_1 Y_5 Y_6}{\eta_2} G_2(Y_0) \vartheta(\eta) + O(B(\log B)^{4-1/5}). \]

Finally, we note that
\[ \frac{Y_1 Y_5 Y_6}{\eta_2} G_2(Y_0) = \frac{Y_1 Y_5 Y_6}{\eta_2 Y_0^6} \omega_{\infty} = \frac{B}{\eta^{(1,1,1,1)}} \omega_{\infty} \]
using Lemma 3.

For \( k = (k_1, k_2, k_3, k_4) \in \mathbb{Z}_+^4 \), let
\[ \Delta_k(n) := \sum_{\eta \in \mathbb{Z}^4_+, \eta^{(k_1, k_2, k_3, k_4)} = n} \frac{\vartheta(\eta)}{\eta^{(1,1,1,1)}}. \]
Consider the Dirichlet series
\[ F_k(s) = \sum_{n=1}^{\infty} \frac{\Delta_k(n)}{n^s} = \sum_{\eta \in \mathbb{Z}^4_+, \eta^{(k_1, k_2, k_3, k_4)} = n} \frac{\vartheta(\eta)}{\eta^{(k_1+1, k_2+1, k_3+1, k_4+1)}}. \]
It is absolutely convergent for \( \Re(s) > 0 \). We write it as an Euler product
\[ F_k(s) = \prod_p F_{k,p}(s), \]
where we compute that \( F_{k,p}(s) \) is
\[ (1 - 1/p) \cdot \left( (1 + 1/p) + \frac{1 - 1/p}{p^{k_1+1} - 1} + \frac{1 - 1/p}{p^{k_2+1} - 1} + \frac{1 - 1/p}{p^{k_4+1} - 1} \right) \]
\[ + \frac{1 - 1/p}{p^{k_3+1} - 1} \left( (1 - 2/p) + \frac{1 - 1/p}{p^{k_1+1} - 1} + \frac{1 - 1/p}{p^{k_2+1} - 1} + \frac{1 - 1/p}{p^{k_4+1} - 1} \right). \]
For $\varepsilon > 0$ and $k \in \{(2, 2, 3, 2), (3, 3, 4, 2)\}$ and all $s \in \mathbb{C}$ lying in the half-plane $\Re(s) \geq -1/8 + \varepsilon$, we have
\[ F_k, p(s) \prod_{j=1}^{4} \left( 1 - \frac{1}{p^{k_j s + 1}} \right) = 1 + O(\varepsilon^{1-\varepsilon}). \]
We define
\[ E_k(s) := \prod_{j=1}^{4} \zeta(k_j s + 1), \quad G_k(s) := \frac{F_k(s)}{E_k(s)} \]
and note that $F_k(s)$ has a meromorphic continuation to $\Re(s) \geq -1/8 + \varepsilon$ with a pole of order 4 at $s = 0$.

As in [BD07, Lemma 15], we use a Tauberian theorem to show that $M_k(t)$ can be estimated as
\[ G_k(0) \frac{P(\log t)}{4! \prod_{j=1}^{4} k_j} + O(t^{-\delta}) \]
for some $\delta > 0$ and $P$ a monic polynomial of degree 4.

Using Lemma 10 and the definitions of $\Delta_k$ and $M_k$,
\[ N_{U,H}(B) = \omega_{\infty} B \sum_{n \leq B} (\Delta_{2,2,3,2}(n) - \Delta_{3,3,4,2}(n)) + O(B(\log B)^{4-1/5}) \]
\[ = \omega_{\infty} G_k(0) \frac{1}{4!} \left( \frac{1}{23 \cdot 3} - \frac{1}{2 \cdot 3^2 \cdot 4} \right) B(\log B)^4 + O(B(\log B)^{4-1/5}) \]
Since
\[ \alpha(\bar{S}) = \frac{1}{864} = \frac{1}{4!} \left( \frac{1}{23 \cdot 3} - \frac{1}{2 \cdot 3^2 \cdot 4} \right) \]
and
\[ G_k(0) = \prod_{p} \left( 1 - \frac{1}{p} \right)^5 \left( 1 + \frac{5}{p} + \frac{1}{p^2} \right), \]
this completes the proof of the theorem.

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