Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies

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The recently established formalism of a worldline quantum field theory, which describes the classical scattering of massive bodies (black holes, neutron stars or stars) in Einstein gravity, is generalized up to quadratic order in spin, revealing an alternative $N = 2$ supersymmetric description of the symmetries inherent in spinning bodies. The far-field time-domain waveform of the gravitational waves produced in such a spinning encounter is computed at leading order in the post-Minkowskian (weak field, but generic velocity) expansion, and exhibits this supersymmetry. From the waveform we extract the leading-order total radiated angular momentum in a generic reference frame, and the total radiated energy in the center-of-mass frame to leading order in a low-velocity approximation.

In this Letter we calculate gravitational waveforms — the primary observables of GW detectors — produced in the parameter-space region of highly eccentric (scattering) spinning BHs and neutron stars (NSs), to leading order in the weak-field, or post-Minkowskian (PM), approximation. Following the above strategy, this is a valuable input for future eccentric waveform models. Indeed, the extension of contemporary quasi-circular (non-eccentric) waveform models for spinning binaries to eccentric orbits (including scattering) is under active investigation [10]. This is motivated, for instance, by the potential insight gained on the formation channels or astrophysical environments of binary BHs (BBHs) through measurements of eccentricity [11] and spins [12], or the search for scattering BHs [13] in our universe. Accurate predictions for GWs from BBHs should crucially also account for the BHs’ spins [14], and this is an important aspect of the present work. The gravitational waveforms presented here are valid up to quadratic order in angular momenta (spins) of the compact stars; that is, we extend Crowley, Kovacs and Thorne’s seminal non-spinning result [15]. We also improve on our earlier reproduction of the non-spinning result [16] by presenting results in a compact Lorentz-covariant form, using an improved integration strategy.

To obtain these results we generalize the recently introduced worldline quantum field theory (WQFT) formalism [16, 17] to spinning particles on the worldline. This is achieved by including anticommuting worldline fields carrying the spin degrees of freedom, building upon Refs. [18–20]. Our formalism manifests an $N = 2$ extended worldline supersymmetry (SUSY) which holds up to the desired quadratic order in spin. The SUSY implies conservation of the covariant spin-supplementary condition (SSC), and thus represents an alternative formulation of the symmetries inherent to spinning bodies. It also operates on the spinning waveform.

The spinning WQFT innovates over previous approaches to classical spin based on corotating-frame variables [21, 22] in the effective field theory (EFT) of compact objects [23, 24] — see Refs. [25] for the construction of PM integrands and Refs. [26, 27] for worldline and spin deflections (in agreement with scattering amplitude results [28, 29]). The worldline EFT was applied to radiation also in the weak-field and slow-motion, i.e. post-Newtonian (PN), approximation [30] — see Refs. [31] for more traditional methods. Other approaches to PM spin effects can be found in Refs. [32].

Spinning Worldline Quantum Field Theory. — It has been known since the 1980s [18] that the relativistic wave equation for a massless or massive spin-$N/2$ field in flat spacetime (generalizing the Klein-Gordon, Dirac and Maxwell or Proca equations) may be obtained by quantization of an extended supersymmetric particle model where one augments the bosonic trajectory $x^\mu(\tau)$ by $N$ anticommuting, real worldline fields. Generalizing this to a curved background spacetime comes with consistency problems beyond $N = 2$. Yet the situation for spins up to one is well understood [20], and sufficient for our purposes of describing two-body scattering up to quadratic order in spin.
We therefore augment the worldline trajectories $x_{\mu}^{i}(\tau_{i})$ ($i = 1, 2$) of our two massive bodies by anticommuting complex Grassmann fields $\psi^{a}_{i}(\tau_{i})$. These are vectors in the flat tangent Minkowski spacetime connected to the curved spacetime via the vierbein $e_{a}^{\mu}(x)$. The worldline action in the massive case for each body takes the form (suppressing the $i$ subscripts) [20, 33]

$$S = -m \int d\tau \left[ \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} + i \bar{\psi}_{a} \frac{D^{a}}{D\tau} + \frac{1}{2} R_{abcd} \bar{\psi}^{a} \psi^{b} \bar{\psi}^{c} \psi^{d} \right],$$

(1)

where $g_{\mu\nu} = e_{a}^{\mu} e_{b}^{\nu} \eta_{ab}$ is the metric in mostly minus signature, $D^{a}/D\tau = \dot{\psi}^{a} + \dot{x}^{\mu} \omega_{a}^{\mu} \psi^{b}$ includes the spin connection $\omega_{\mu ab}$ and the Riemann tensor is $R_{\mu\nu ab} = e_{a}^{\mu} e_{b}^{c} R_{\mu\nu cd} = 2(\partial_{\mu} \omega_{\nu ab} + \omega_{\mu}^{\nu} \omega_{ab})$. This theory enjoys a global $N = 2$ SUSY: it is invariant under

$$\delta x^{\mu} = i \epsilon \dot{x}^{\mu} + i \epsilon \bar{\psi}^{a}, \quad \delta \bar{\psi}^{a} = -i \epsilon \dot{x}^{\mu} \omega_{\mu}^{a} \psi^{b},$$

(2)

with constant SUSY parameters $\epsilon$ and $\bar{\epsilon} = \epsilon^{\dagger}$.

The connection to a traditional description of spinning bodies in general relativity, using the spin field $S^{\mu\nu}$ and the Lorentz body-fixed frame $A^{A}_{\mu}$ [21, 22, 24, 34, 35], comes about upon identifying the spin field $S^{\mu\nu}(\tau)$ with the Grassmann bilinear:

$$S^{\mu\nu} = -2i e_{a}^{\mu} e_{b}^{\nu} \bar{\psi}^{a} \psi^{b}. \quad (3)$$

One can easily show that $S^{ab}$ obeys the Lorentz algebra under Poisson brackets $\{\bar{\psi}^{a}, \psi^{b}\}_{P.R.} = -i \eta^{ab}$. In fact, the spin-supplementary constraint (SSC) and preservation of spin length may be related to $N = 2$ SUSY-related constraints [33]. Finally, by deriving the classical equations of motion from the action these can be shown to match the Mathisson-Papapetrou equations [36] at quadratic spin order. This indicates a hidden $N = 2$ SUSY in the actions of Refs. [22, 34, 35].

The actions of Refs. [22, 34, 35] also carry a first spin-induced multipole moment term at quadratic order in spins with an undetermined Wilson coefficient $C_{E}$, where here $C_{E} = 0$ for a Kerr BH. Translating it to our formalism this term reads

$$S_{E^{SS}} := -m \int d\tau C_{E} E_{ab} \bar{\psi}^{a} \psi^{b} \bar{\psi} \psi,$$

(4)

where $E_{ab} := R_{\mu
u ab} \dot{x}^{\mu} \dot{x}^{\nu}$ is the “electric” part of the Riemann tensor. The $N = 2$ SUSY is now maintained only in an approximate sense [33]: it survives in the action for terms up to $O(\bar{\psi}^{5})$, i.e., quadratic order in spin.

In order to describe a scattering scenario we expand the worldline fields about solutions of the equations of motion along straight-line trajectories:

$$x_{\mu}^{i}(\tau_{i}) = b_{i}^{\mu} \tau_{i} + c_{i}^{\mu}(\tau_{i}),$$

$$\psi_{a}^{i}(\tau_{i}) = \psi_{a}^{i} + \psi_{a}^{i}(\tau_{i}),$$

(5)

where $S_{i}^{\mu\nu} := -2 \bar{\psi}_{a}^{i} [\psi_{a}^{i} \psi_{b}^{i}]$ captures the initial spin of the two massive objects. The weak gravity expansion of the vierbein reads

$$e_{a}^{\mu} = \eta^{\mu\nu} \left( \eta_{\mu\nu} + \frac{\kappa}{2} h_{\mu\nu} - \frac{\kappa^{2}}{8} h_{\mu\nu} h_{\rho\sigma} + O(\kappa^{3}) \right),$$

(6)

introducing the graviton field $h_{\mu\nu}(x)$ and the gravitational coupling $\kappa^{2} = 32\pi G$. Note that in this perturbative framework the distinction between curved $\mu, \nu, \ldots$ and tangent $a, b, \ldots$ indices necessarily drops.

The spinning WQFT has the partition function

$$Z_{WQFT} := \text{const} \times \int [D[h_{\mu\nu}]] e^{(S_{EH} + S_{E})}$$

$$\times \prod_{i=1}^{2} \int [D[\psi^{i}_{\mu}]] \exp \left[ i \sum_{i=1}^{2} S^{(i)} + S_{ESS}^{(i)} \right],$$

(7)

where $S_{EH}$ is the Einstein-Hilbert action and the gauge-fixing term $S_{E}$ enforces de Donder gauge. The SUSY variations (2) leave an imprint on the free energy (or eikonal) $F_{WQFT}(b_{i}, v_{i}, S_{i})$ [22] at quadratic order when the Wilson coefficients $C_{E,i}$ are included. As we shall see, this is also a symmetry of the wavefunction. Using a suitable shift of the proper times $\tau_{i}$ we may choose $b \cdot v_{i} = 0$, where $b^{\mu} = b_{i}^{\mu} - b_{i}^{\mu}$ is the relative impact parameter; by gauge fixing the SUSY transformations (8) we impose $v_{i}^{a} \mu S_{i}^{\mu\nu} = 0$ (the covariant SSC).

**Feynman rules.** — As the Feynman rules for the Einstein-Hilbert action are conventional we will not dwell on them; the only subtlety is our use of a retarded graviton propagator:

$$\frac{\mu\nu}{\omega} = \frac{m}{(k^{0} + i\omega)^{2} - k^{2}},$$

(9)

with $P_{\mu\nu, \rho\sigma} := \eta_{\mu(\rho} \eta_{\sigma)\nu} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma}$. On the worldline we work in one-dimensional energy (frequency) space: the propagators for the fluctuations $z^{\mu}(\omega)$ and anticommuting vectors $\psi^{\mu}(\omega)$ are respectively

$$\frac{\mu}{\omega} \rightarrow \frac{-i}{m(\omega + i\epsilon)^{2}},$$

$$\frac{\mu}{\omega} \rightarrow \frac{-i}{m(\omega + i\epsilon)}.$$

(10a, 10b)

which also both involve a retarded $ie$ prescription. The former was already used in Refs. [16, 17].
Illustrated in Fig. 1. Explicit expressions for the first Bremsstrahlung at 2PM order including spin effects we WQFT via the latter we sum over both routings of the fermion line. (d). Full expressions for these vertices are provided in Next we consider the worldline vertices. The simplest of these is the single-graviton emission vertex:

\[
\frac{1}{2} k_\mu k_\nu S^{\mu \nu} S^{\rho \sigma} + \frac{C_E}{2} v_\mu v_\nu (k \cdot S \cdot S \cdot k),
\]

where \( \delta(\omega) = (2\pi)\delta(\omega) \) and we have used \( S^{\mu \nu} = -2i\overline{\Psi}[\mu|q^\rho]. \) The other worldline-based vertices required for the 2PM Bremsstrahlung all appear in Fig. 1: the two-point interaction between a graviton and a single \( z^\mu \) mode in (b), the two-graviton emission vertex in (c), and the two-point interaction between a graviton and \( \psi^\mu \) in (d). Full expressions for these vertices are provided in the Supplementary Material.

**Waveform from WQFT.** — To describe the Bremsstrahlung at 2PM order including spin effects we compute the expectation value \( k^2 \langle h_{\mu \nu}(k) \rangle_{\text{WQFT}} \). This requires us to compute four kinds of Feynman graphs, illustrated in Fig. 1. Explicit expressions for the first two graphs (a) and (b) were given in the non-spinning case [16]; these are now modified by terms up to \( O(S^2) \). Graphs (c) and (d) are unique to the spinning case — for the latter we sum over both routings of the fermion line.

From this result we seek to obtain the waveform in spacetime in the wave zone, where the distance to the observer \( |x| = r \) is large compared to all other lengths. Following Ref. [16] the gauge-invariant frequency-domain waveform \( 4G e^{ik \cdot x} S_{\mu \nu}(k^\mu = \Omega (1, \hat{x})) \) is extracted from the WQFT via

\[
S_{\mu \nu}(k) = \frac{2}{k} k^2 \langle h_{\mu \nu}(k) \rangle_{\text{WQFT}},
\]

where \( \Omega \) is the GW frequency and \( \hat{x} = x/r \) points towards the observer. However, it is advantageous to study the time-domain waveform \( f(u, \hat{x}) \) which is given by a Fourier transform:

\[
\text{ke}^{i\mu} h_{\mu \nu} = \frac{f(u, \hat{x})}{r} = \frac{4G}{r} \int e^{-i(k \cdot x)} e^{i\mu} S_{\mu \nu}(k) |_{k^\mu = \Omega \mu \nu}.
\]

We have contracted with a polarization tensor \( e^{\mu \nu} = \frac{1}{2} \delta^{\mu \nu}, \int_{\Omega} = \int_{\infty}^{\frac{\Omega}{2\pi}}, \) and \( \rho^{\mu} = (1, \hat{x}) \); in a PM decomposition \( f = \sum_n G^n f(n) \) we seek the 2PM component \( f^{(2)} \). Note that \( k \cdot x = \Omega (t - r) \) yields the retarded time \( u = t - r \), and \( \epsilon \cdot \epsilon = \epsilon \cdot \rho = 0 \).

**Integration.** — Our integration procedure follows closely that used for the non-spinning calculation in Ref. [16], the main difference being that we maintain four-dimensional Lorentz covariance. Each diagram contributing to \( k^2 \langle h_{\mu \nu}(k) \rangle_{\text{WQFT}} \) carries the overall factor

\[
\mu_{1,2}(k) = e^{i(q_1 \cdot b_1 + q_2 \cdot b_2)} \delta(q_1 \cdot v_1) \delta(q_2 \cdot v_2) \delta(k - q_1 - q_2).
\]

We integrate over \( q_i \), the momentum emitted from each worldline (see Fig. 1). When we also integrate over \( \Omega \) — as in Eq. (13) — the full integration measure becomes

\[
\int_{\Omega, q_1, q_2} \mu_{1,2}(k) e^{-i(k \cdot x)} = \frac{1}{\rho \cdot v_2} \int_{q_1} \delta(q_1 \cdot v_1) e^{-iq_1 \cdot b_1},
\]

where \( \int_{q_1} := \int \frac{dq_1}{(2\pi)^3} \); the delta function constraints give \( \Omega = \frac{q_1 \cdot v_1}{v_2} \) and \( q_2 = k - q_1 \). The shifted impact parameter,

\[
\tilde{b}_\mu = b_\mu - b_\mu^0, \quad \tilde{b}_\mu = b_\mu + u_i v_i^\mu,
\]

extends the original impact parameter \( b_\mu = b_\mu^0 - b_\mu^0 \) along the undeflected trajectories of the two bodies. Finally, \( u_i \) is the retarded time in the \( i \)th rest frame:

\[
u_i = \frac{\rho \cdot (x - b_i)}{\rho \cdot v_i}, \]

This implies \( \rho \cdot \tilde{b}_i = \rho \cdot x = u \), so \( \rho \cdot \tilde{b} = 0 \).

Rewriting the integral measure as in Eq. (15) is convenient for performing the integrals of diagrams (b)–(d), in the rest frame of body 1. The mirrored counterparts to these diagrams are easily recovered after integration using the \( 1 \leftrightarrow 2 \) symmetry of the waveform. To integrate diagram (a) we insert the partial-fraction identity

\[
q_1^{-2} q_2^{-2} = q_1^{-2}(2k \cdot q_1)^{-1} - q_2^{-2}(2k \cdot q_2)^{-1} \quad \text{(valid for k on-shell)}
\]

and focus on the first term. The full 2PM waveform is then written schematically as (dropping the subscript on \( q_1 \))

\[
f^{(2)} = \frac{4\pi}{m_1 m_2} \int \delta(q \cdot v_1) e^{-iq \cdot \tilde{b}} \left( \frac{\mathcal{N}(q)}{q \cdot v_2 + i\epsilon} + \frac{\mathcal{M}(q)}{(q \cdot v_2)(q \cdot \rho)} \right),
\]

the \( \mathcal{N} \)- and \( \mathcal{M} \)-contributions corresponding to diagrams (b)–(d) and (a) in Fig. 1 respectively. The numerators
\[ N(q) \text{ and } M(q) \text{ have a uniform power counting in } q \text{ for each spin order:} \]
\[ N(q) = N_0 q^\nu + N_{\mu\nu} q^\nu \cdot q^\rho + N_{\mu\nu\rho\sigma} q^\nu \cdot q^\rho \cdot q^\sigma, \]
\[ M(q) = M_{\mu\nu} q^\nu + M_{\mu\nu\rho\sigma} q^\nu \cdot q^\rho \cdot q^\sigma + M_{\mu\nu\rho\sigma\tau} q^\nu \cdot q^\rho \cdot q^\sigma \cdot q^\tau, \]
and the non-spinning result involves only \( N_0 \) and \( M_{\mu\nu} \).
We present full expressions for \( N \) and \( M \) in the ancillary file attached to the arXiv submission of this Letter.

To lowest order in \( q^\mu \), the first integral in eq. (18) is
\[
4\pi \int \frac{\delta(q \cdot v_1)}{q^2} \frac{q^\mu}{q^2} \, q \cdot v_2 + i\epsilon \quad \Rightarrow \quad \frac{P_{\mu\nu}^i v_{2,\nu}}{(\gamma^2 - 1)|b_1|} - \frac{\rho^\mu}{|b_1|^2} \left( \sqrt{\gamma^2 - 1} + \frac{u_2}{|b_1|} \right),
\]
where \( P_{\mu\nu}^i := \eta_{\mu\nu} - v^\mu v_\nu \) is a projector into the rest frame of the \( i \)th body; \( |b| = -\sqrt{\rho^\mu \rho_\mu} \) (the impact parameter is spacelike) and
\[
|\hat{b}|_{1,2} := \sqrt{-b_1 \rho_{1,2} b_\nu} = \sqrt{|b|^2 + (\gamma^2 - 1)u_{2,1}^2}
\]
are the lengths of the shifted impact parameter \( \hat{b}_\mu \) (16) in the two rest frames. The second integral in eq. (18) is
\[
4\pi \int \frac{\delta(q \cdot v_1)}{q^2} \frac{q^\mu q^\nu}{q^2} \, q \cdot v_2 \cdot \rho \quad \Rightarrow \quad \frac{K_{\mu\nu}^i v_2 \cdot K_i \cdot \rho - 2(v_2 \cdot K_1) (\rho \cdot K_i)^\nu}{(\gamma^2 - 1) (\rho \cdot v_1)^2 |b|^2 |\hat{b}|_1^2},
\]
where we have introduced the symmetric tensor
\[
K_{\mu\nu}^i := P_{\mu\nu}^i |\hat{b}|^2 + (P_1 \cdot \hat{b})^\mu (P_1 \cdot \hat{b})^\nu,
\]
with the property that \( K_{\mu\nu}^i v_{i,\nu} = K_{\mu\nu}^i \hat{b}_\nu = 0 \). Both integrals are derived in the Supplementary Material; one generalizes to higher powers of \( q^\mu \) in the numerators by taking derivatives with respect to \( \hat{b}_\mu \).

**Results.** — The 2PM waveform takes the schematic form
\[
\frac{f^{(2)}}{m_1 m_2} = \sum_{s=0}^2 \left( \frac{1}{|b_1|^{2s+1}} \left( \alpha_1^{(s)} + \frac{\beta_1^{(s)}}{|b_1|^{2s+2}} \right) + (1 \leftrightarrow 2) \right),
\]
where the coefficients \( \alpha_1^{(s)} \), \( \beta_1^{(s)} \), provided in the ancillary file, are associated with the \( N \)- and \( M \)-type contributions in Eq. (18) respectively; they are functions of \( u_i \), \( \hat{b}_\mu \), \( v_i^\nu \), \( \rho^\mu \), and \( S_{\mu\nu}^i \) and bi-linear in \( \rho^\nu \). The waveform \( f \) is invariant under the SUSY transformations in Eq. (8) to quadratic order in spin regardless of the values of \( C_E \).
To see this we expand the waveform at all PM orders in powers of spin:
\[
f = f_0 + \sum_{i=1}^2 S_{i,\mu\nu} f_{i,\mu\nu} + 2 \sum_{i,j=1}^2 S_{i,\mu\nu} S_{j,\rho\sigma} f_{ij,\mu\nu\rho\sigma} + O(S^3),
\]
\[
\text{FIG. 2. Total radiated angular momenta for the scattering of two Kerr-BHs with } v = 0.2 \text{ as a function of the angle between the total initial spins } a_3 = a_1 + a_2 \text{ and } b \text{ with } a_1 \cdot v_1 = 0 \text{ for a range of ratios } |a_3|/|b|. \text{ We show the normalized ratio of angular momenta emitted orthogonal to the } b, v \text{ plane (left plot) and in the } b \text{ direction (right plot), normalization is w.r.t. angular momentum emitted in the spinless case.}
\]
where \( f_{i,\mu\nu}^\mu \) and \( f_{ij,\mu\nu\rho\sigma}^\mu \) are defined modulo terms that vanish on support of \( v_{i,\mu} S_{ij}^\mu = 0 \). The SUSY links higher-spin to lower-spin terms:
\[
\frac{1}{2} \frac{\partial f_0}{\partial \delta_{i,\mu}} = v_{i,\nu} f_{i,\nu}^{[\mu\nu]} + \frac{1}{4} \frac{\partial f_i^{\mu\nu}}{\partial \delta_{j,\rho}} = v_{j,\sigma} f_{ij}^{[\mu\nu\rho\sigma]},
\]
and these identities are satisfied by the waveform (24).

To illustrate the waveform we consider the gravitational wave memory \( \Delta f(\hat{\chi}) := f(+\infty, \hat{\chi}) - f(-\infty, \hat{\chi}) \).
The constant spin tensors are decomposed in terms of the Pauli-Lubanski vectors \( a_3^i \) as \( S_{ij}^\mu = \epsilon^{\mu\nu}_{\rho\sigma} v_i^\rho v_j^\sigma \), the latter satisfying \( a_3 \cdot v_1 = 0 \). In the aligned-spin case \( a_3 \cdot b = a_1 \cdot v_j = 0 \), i.e. the spin vectors are orthogonal to the plane of scattering. Writing \( |a_3| = \sqrt{-a_3^2} \) the wave memory is then proportional to the non-spinning result:
\[
\Delta f^{(2)}(\hat{\chi}) = \left( 1 + \frac{2v_3 |a_3|}{b_1 (1 + v_2^2)} + \frac{|a_3|^2}{|b|^2} - \sum_{i=1}^2 \frac{C_E |a_i|^2}{|b|^2} \right) \Delta f^{(2)}_{S=0},
\]
\[
\Delta f^{(2)}_{S=0} = \frac{4(2\gamma^2 - 1)\epsilon \cdot v_1 (2b \cdot \rho \cdot v_1 - b \cdot \rho \cdot v_1)}{|b|^2 \sqrt{\gamma^2 - 1} (\rho \cdot v_1)^2}, \quad (1 \leftrightarrow 2),
\]
where \( a_3^i = a_1^i + a_2^i \). For two Kerr black holes \( C_E,i = 0 \) with equal-and-opposite spins \( a_1^i = -a_2^i \) we see that \( \Delta f^{(2)} = \Delta f^{(2)}_{S=0} \), which we observe also when the spins are mis-aligned to the plane of scattering.

There is also a 1PM (non-radiating) contribution to the waveform consisting of single-graviton emission from either massive body:
\[
f^{(1)}(\hat{\chi}) = \frac{2m_1}{\rho \cdot v_1} (\epsilon \cdot v_1)^2 + \frac{2m_2}{\rho \cdot v_2} (\epsilon \cdot v_2)^2.
\]
At 1PM order there is manifestly no dependence on either the spins \( S_{ij}^\mu \) or impact parameters \( b_i^\nu \), so the SUSY identities in Eq. (26) are trivially satisfied.

Finally, the wave memory and 1PM part of the waveform contribute to the total radiated angular momentum.
e\_rad\_ij\). Using three-dimensional Cartesian basis vectors \( \hat{e}_i \), we choose a frame of reference with the initial velocities \( v_i^\mu \) restricted to the \( t-x \) plane; \( \mathbf{b} = [b] \hat{e}_x \) is orthogonal to these. Then we find two non-zero components of \( J_{ij}\): \( J_{xy}^{\text{initial}} \) and \( J_{xx}^{\text{initial}} \), which are conveniently arranged into

\[
\frac{J_{xy}^{\text{initial}}}{|S|=0} = 4G^2m_1m_2(2\gamma^2 - 1)\frac{\tau}{\sqrt{\gamma^2 - 1}} I(v)
\]

\[
\times \left( 1 - \frac{2\tau v}{|b|(1 + \tau v)} - \frac{(\tau v)^2}{|b|^2} + \frac{2}{\gamma^2} \sum_{i=1}^{2} \frac{C_{E,i}^2(a_i \cdot 1)^2}{|b|^2} \right)
\]

\[
+ \mathcal{O}(\gamma^3).
\]

We normalize with respect to \( J_{xy}^{\text{initial}}|_{S=0} \), the initial angular momentum in the non-spinning case. The spin vectors \( a_1 \) and \( a_2 \) are taken in the rest frame of each massive body; \( a_3 = a_1 + a_2 \), \( l = \hat{e}_2 + i\hat{e}_3 \), and

\[
I(v) = -\frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2 - 1)}{v^3} \arctanh(v)
\]

is a universal prefactor. Eq. (29) holds in the rest frame of either body or the center-of-mass (c.o.m.) frame; see Fig. 2 for plots. For a derivation we refer the reader to the Supplementary Material. There we also compute the total radiated energy in the c.o.m. frame. Due to the multi-scale nature of the waveform it is difficult to perform the necessary time and solid angle-integrals, so we performed a low velocity expansion. For terms up to \( \mathcal{O}(v^2) \) we find

\[
E_{\text{rad,LO}}^{\text{CoM}} = \frac{vG^3m_1^2m_2^2\pi}{|b|^3} \left[ \frac{37}{15} + \frac{v(65m_1 + 69m_2)(a_1 \cdot \hat{e}_3)}{10|b|(m_1 + m_2)} + \frac{1503(a_1 \cdot \hat{e}_1)(a_2 \cdot \hat{e}_1) - 3559(a_1 \cdot \hat{e}_2)(a_2 \cdot \hat{e}_2) + 1816(a_1 \cdot \hat{e}_3)(a_2 \cdot \hat{e}_3)}{320|b|^2} \\
+ \frac{9(185 - 176C_{E,1})(a_1 \cdot \hat{e}_1)^2 - (3385 - 3472C_{E,1})(a_1 \cdot \hat{e}_2)^2 + 8(245 - 236C_{E,1})(a_1 \cdot \hat{e}_3)^2}{320|b|^2} + (1 \leftrightarrow 2) + \mathcal{O}(v^2) \right],
\]

where the swap \((1 \leftrightarrow 2)\) does not affect the basis vectors \( \hat{e}_i \) or the constant term \( \frac{37}{15} \). It is straightforward to extend this result to higher orders in \( v \).

Conclusions. — In this Letter we extended the WQFT to describe spinning compact bodies to quadratic order in spin, and computed the leading-PM order waveform for highly eccentric (scattering) orbits. Our accompanying work [33] presents an application to further observables such as the spin kick and deflection at 2PM order and gives details on the approximate SUSY and its relation to the SSC. The radiated energy (31) should also be particularly useful for future studies. In Refs. [37, 38] the \( \mathcal{O}(S^3) \) energy loss from a scattering of non-spinning black holes was recently computed to all orders in velocity using the KMO formalism [39] (see also Ref. [40]); a similar result could conceivably be obtained at \( \mathcal{O}(S^3) \), and then checked against Eq. (31) in the low-velocity limit. Similarly, the remarkably simple result for radiated angular momentum (29) at 2PM order is intriguing; it may be important for understanding the high-energy limit, see Ref. [41, 42] for the non-spinning case.

The application of modern on-shell and integration techniques to compute scattering amplitudes [37, 43–47] holds great promise for pushing calculations to higher PM orders. This is demonstrated by the impressive calculation of the 4PM conservative dynamics in the potential region [47, 48] — see also Refs. [41, 42, 45, 49–53]. The connection between amplitudes and classical physics was studied in Refs. [39, 40, 54], and Refs. [27, 54] discussed the connection to bound orbits. Our WQFT framework [16, 17] provides an efficient, rather intuitive way to connect amplitude and (classical) worldline EFT calculations. It may therefore benefit from modern amplitude techniques at higher PM orders in future work, building on the compact Lorentz-covariant master integrals provided here.

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SUPPLEMENTARY MATERIAL

Feynman rules. — Here we give explicit expressions for the worldline Feynman rules used in the main calculation, the single-graviton emission vertex having already been given in Eq. (11). Adding an outgoing \( z^b \) line we have:
\[
\frac{z^\mu(\omega)}{h_{\mu\nu}(k)} = \frac{mK}{2} e^{ik \cdot \delta}(k \cdot v + \omega) \left( 2\omega \nu^{(\mu} \delta_{\nu')}^{\rho} + \nu^{\mu} v^{\nu'} k_{\rho} + i(k \cdot S) (\nu^{\mu} \nu'^{\rho} + \omega \delta_{\rho}^{\mu}) + \frac{1}{2} k_{\rho} (k \cdot S) S^{\mu} (S \cdot k)^{\nu} + C E \left( \left( 2\omega \nu^{(\mu} \delta_{\nu')}^{\rho} + \nu^{\mu} v^{\nu'} k_{\rho} \right) (k \cdot S \cdot S \cdot k) - \omega \nu^{(\mu} k_{\rho} S^{\mu} S^{\nu} + 2 \nu^{\mu} (k \cdot S) (\nu^{\rho} \delta_{\rho}^{\mu}) \right) \right),
\]

where we have adopted the shorthands \((k \cdot S)^{\mu} = k_{\rho} S^{\mu \rho}, (S \cdot S)^{\mu \nu} = S^{\mu \rho} S^{\nu \rho}\) and \((S \cdot k)^{\mu} = S^{\mu \nu} k_{\nu}\). Both this and the single-graviton emission vertex appear in the non-spinning case, and by setting \(S^{\mu \nu} = 0\) we recover the corresponding expressions from Refs. [16, 17]. New to the spinning case is the coupling with \(\psi^{\mu}\):

\[
\psi^{\mu}(\omega) = -i mK e^{ik \cdot \delta}(k \cdot v + \omega) \left( k_{\rho} \delta^{(\mu}_{\sigma} (v^{\nu}) - i (S \cdot k) (v^{\nu}) \right) + i C E \left( v^{(\mu} k_{\lambda} + \omega \delta^{\lambda}_{\rho} \right) \left( \nu^{\nu} k_{[\rho} + \omega \delta^{\nu}_{\rho} \right) S^{\lambda} \sigma \right) \tilde{\Psi}^{\sigma}.
\]

The vertex with \(\tilde{\Psi}^{\mu}(\omega)\) on an outgoing line is identical, except with \(\tilde{\Psi}^{\mu} \rightarrow \Psi^{\mu}\). Finally, starting at linear order in spin there is also the two-graviton emission vertex:

\[
\begin{align*}
&h_{\mu_1 \nu_1}(k_1) \ h_{\mu_2 \nu_2}(k_2) \ 
&= -\frac{mK^2}{4} e^{i(k_1 + k_2) \cdot \delta}((k_1 + k_2) \cdot v) \left((S \cdot k_1)^{\mu_1} (S \cdot k_2)^{\mu_2} - S^{\mu_1 \mu_2} (v^{\mu_1} k^{\mu_2} - \frac{1}{2} k^{\mu_1} v^{\mu_2}) + \frac{1}{2} k_1 \cdot k_2 S^{\mu_1 \nu_1} S^{\mu_2 \nu_2} - ik_2^{\nu_1} (S \cdot (k_1 + k_2))^\mu \eta^{\nu_1 \nu_2} + \frac{1}{2} k_1 \cdot k_2 S^{\mu_1 \nu_1} (S \cdot k_1 + k_2)^\mu \eta^{\nu_1 \nu_2} + i C E \left( 2k_1 \cdot v (S \cdot (k_1 + k_2))^{\mu_1} v^{\mu_2} - \frac{1}{2} (k_1 \cdot v)^2 (S \cdot S)^{\mu_1 \mu_2} - \frac{1}{2} (k_1 \cdot S \cdot k_2)^{\mu_1} v^{\mu_2} \right) \eta^{\nu_1 \nu_2} + i C E \left( -\frac{1}{2} k_1 \cdot k_2 (S \cdot S)^{\mu_1 \nu_1} v^{\mu_2} + k_2^{\nu_1} (S \cdot S)^{\mu_1 \nu_1} v^{\mu_2} + k_1^{\nu_1} (S \cdot S)^{\mu_1 \nu_1} v^{\mu_2} - k_2^{\nu_1} (S \cdot S)^{\mu_1 \nu_1} v^{\mu_2} \right) \right) + (1 \leftrightarrow 2),
\end{align*}
\]

with implicit symmetrization on \((\mu_1, \nu_1)\) and \((\mu_2, \nu_2)\).

Integration. — To compute the 2PM waveform we require explicit results for the following integrals:

\[
\begin{align*}
&\mathcal{J}^{\mu_1 \mu_2 \ldots \mu_n} = 4\pi \int q \delta(q \cdot v_1) e^{-i q \cdot b} g^{\mu_1 q^{\mu_2} \ldots q^{\mu_n}}, \\
&\mathcal{I}^{\mu_1 \mu_2 \ldots \mu_n} = 4\pi \int q \delta(q \cdot v_1) e^{-i q \cdot b} q^{\mu_1 q^{\mu_2} \ldots q^{\mu_n}},
\end{align*}
\]

with \(n = 1, 2, 3\) for the \(\mathcal{J}\)-integrals and \(n = 2, 3, 4\) for the \(\mathcal{I}\)-integrals. Expressions for \(\mathcal{J}^{\mu}\) and \(\mathcal{I}^{\mu}\) were presented in Eqs. (20) and (22) of the main text respectively — we derive these first, then generalize to higher-orders in \(q^{\mu}\) by taking derivatives with respect to the shifted impact parameter \(\tilde{b}^{\mu}\).

Our starting point for \(\mathcal{J}^{\mu}\) is

\[
4\pi \int q \delta(q \cdot v_1) e^{-i q \cdot b} g^{\mu} = -i P^{\mu \nu} \tilde{b}_{\nu},
\]

which is easily derived by specializing to the rest frame of massive body 1 — \(F_1 = b_{\nu}\), and \(|b| = (21)\) are the covariant “uplifts” of \(\tilde{b}^{\mu}\) and \(|b|\) from this frame. Using

\[
\int_\omega e^{-i \omega \tau} \frac{f(\omega)}{\omega + i \epsilon} = -i \int_\tau^{\Omega} d\tau' \int_\omega e^{-i \omega \tau'} f(\omega),
\]

the \(\mathcal{J}^{\mu}\) integral can be re-written as

\[
\mathcal{J}^{\mu} = -4\pi i \int_{-\infty}^{\infty} du_1 \int_q \delta(q \cdot v_1) e^{-i q \cdot b} q^{\mu} g^{\mu} = i \tilde{b}_{\nu} \mathcal{I}^{\nu} = 0, \\
\]

where the first equality is derived from the \(\mathcal{I}\)-type integrals definition (36). The integrand of \(\mathcal{I}^{\mu}\) is dimensionless in \(q^\mu\), so its integrated form depends only on dimensionless combinations of \(b^\mu\) — hence the second equality. \(\mathcal{I}^{\mu} = 0\) therefore lives in a two-dimensional subspace orthogonal
to \( v_i^\mu \) and \( \tilde{b}^\mu \), and we make an ansatz:

\[
\mathbf{T}_{\mu
u} = c_1 K_1^{\mu
u} + c_2 (v_2 \cdot K_1)^\mu (\rho \cdot K_1)^\nu .
\]  

(41)

\( K_1^{\mu\nu} \) was defined in Eq. (23) as the four-dimensional projector into this subspace. We solve for the coefficients by contracting \( \mathbf{T}_{\mu
u} \) with \( v_i^\mu \) and/or \( \rho^\nu \), evaluating the resulting scalar integrals to obtain

\[
\rho_{\mu} v_{2\mu} \mathbf{T}_{\mu\nu} = -\frac{1}{|b_1|}, \quad \rho_{\mu} \rho_{\nu} \mathbf{T}_{\mu\nu} = -\frac{v_2 \cdot K_1 \cdot \rho}{(\gamma^2 - 1)|b|^2 |b|} .
\]  

(42)

These allow us to fix \( c_1 \) and \( c_2 \), and we recover Eq. (22).

By differentiating these integrals with respect to \( b^\mu \) one can pull down additional factors of \( q^\mu \). For the \( \mathcal{J} \)-type integrals this procedure is unambiguous; special care should be taken for the \( \mathcal{I} \)-type integrals as \( b^\mu \) is constrained by \( \rho \cdot b = 0 \). However, provided one always works in the three-dimensional subspace defined by \( P_1^{\mu\nu} \) then one overcomes this problem, as all contractions involve \( P_1^{\mu\nu} \) and \( \rho \cdot P_1 \cdot b \neq 0 \).

**Radiated energy and angular momentum.** — In Ref. [16] we used the spin-less Bremsstrahlung waveform to compute expressions for the radiated energy and angular momentum, so here we extend these to include spin. The relevant starting points are the same \([42, 50]\):

\[
P_{\text{rad}}^\mu = \frac{1}{32 \pi G} \int d\sigma [f_{i j}]^2 \rho^\mu, \quad \text{where } f = f_{i j} \varepsilon^{i j}
\]  

(43)

\[
J_{i j}^\mu = \frac{1}{8 \pi G} \int d\sigma \left( f_{k[i} f_{j]k} - \frac{1}{2} x_{[i} \partial_{j]} f_{k i k} \right),
\]  

(44)

with \( \dot{f}_{i j} := \partial_t f_{i j} \) and \( d\sigma = \sin \theta d\theta d\phi \) is the unit sphere measure. Here we have introduced a spherical polar coordinate system via

\[
\hat{x} = \hat{e}_1 \cos \theta + \sin \theta (\hat{e}_2 \cos \phi + \hat{e}_3 \sin \phi),
\]  

(45)

which defines the angles \( \theta \) and \( \phi \) towards the observer; \( \hat{e}_i \) are Cartesian spatial unit vectors (with Latin indices \( i, j, \ldots \)). Without loss of generality we assume that \( \hat{e}_2 \) and \( \hat{e}_3 \) are orthogonal to the initial velocities \( v_1^\mu \) and \( v_2^\mu \), and \( b = |b| \hat{e}_2 \) where \( b^\mu = (0, b) \). The waveform \( f_{i j}(u, \theta, \phi) \) is conveniently decomposed on a basis of transverse-traceless polarization tensors:

\[
f_{i j} = f_{\pi}(e_{+})_{i j} + f_{\times}(e_{\times})_{i j},
\]  

(46)

where \( f_{\pi, \times} = \frac{1}{2} (e_{\pi, \times})_{i j} f_{i j} \) and the polarization tensors are explicitly given as

\[
e_{i j} = \theta^i \theta^j - \phi^j \phi^i, \quad e_{i j} = \theta^i \phi^j + \phi^i \theta^j .
\]  

(47)

The two angular vectors orthogonal to \( \hat{x} \) are \( \hat{\theta} := \partial_\theta \hat{x} \) and \( \hat{\phi} := (\sin \theta)^{-1} \partial_\phi \hat{x} \).

Given our starting point of a fully Lorentz-covariant expression for the waveform \( f_{i j} \), we can make different choices of inertial frame for intermediate expressions. There are two of particular interest to us: the rest frame of the first massive body, and the center-of-mass (c.o.m.) frame. In either case, we decompose the velocities \( v_i^\mu \) and Pauli-Lubanski spin vectors \( a_i^\mu \); defined via \( S_i^{\mu\nu} = e^{\mu\nu\rho\sigma} v_i^\rho a_i^\sigma \); as

\[
v_i^\mu = (\gamma_i v_i^\mu), \quad a_i^\mu = \left( a_i + \frac{\gamma_i (v_i \cdot a_i)}{\gamma_i - 1} (v_i \cdot a_i) v_i \right), \quad \gamma_i = \frac{E_i}{m_i},
\]  

(48)

where \( v_i \parallel \hat{b}_i \). These choices manifestly ensure that \( a_i \cdot v_i = 0, \quad v_i^2 = 1, \quad \text{and } a_i^2 = -a_i^2 \). Note that \( a_i \) always denotes the spin vector of the ith body in its restframe. In the first rest frame \( v_1 = 0 \iff \gamma_1 = 1, \quad v_2 = v \iff \gamma_2 = \gamma \); in the c.o.m. frame \( v_1 = v_1 \hat{e}_1 \), \( v_2 = -v_2 \hat{e}_1 \), where

\[
v_i = \frac{p_{\infty}}{E_i}, \quad \gamma_i = \frac{E_i}{m_i},
\]  

(49)

and \( E_i = \sqrt{m_i^2 + p_i^2} \). The initial momenta are \( p_1^\mu = m_1 v_1^\mu = (E_1, p_{\infty}, 0, 0) \) and \( p_2^\mu = m_2 v_2^\mu = (E_2, -p_{\infty}, 0, 0) \). The c.o.m. momentum \( p_{\infty} \) is

\[
p_{\infty} = \frac{m_1 m_2 \sqrt{\gamma^2 - 1}}{\sqrt{m_1^2 + m_2^2 + 2\gamma m_1 m_2}}.
\]  

(50)

When working in the c.o.m. frame we prefer to express intermediate results in terms of \( \gamma_i \) and \( v_i \), then use

\[
v = \frac{v_1 + v_2}{1 + v_1 v_2}
\]  

(51)

to reassemble final expressions in terms of \( \gamma \) and \( v \).

We begin with the radiated angular momentum \( J_{i j}^{\text{rad}} \), which contributes at leading PM order \( G^2 \). There are two non-zero components: \( J_{\pi \pi}^{\text{rad}} \) and \( J_{\times \times}^{\text{rad}} \). As \( f^{(1)} \) is static the \( u \)-integration is trivially performed by expressing \( J_{\pi \pi}^{\text{rad}} \) and \( J_{\times \times}^{\text{rad}} \) in terms of the wave memories \( \Delta f_{\pi, \times} := f_{\pi, \times}|_{u=\infty} - f_{\pi, \times}|_{u=-\infty} \):

\[
J_{\pi \pi}^{\text{rad}} = i J_{\times \times}^{\text{rad}} = \frac{1}{8 \pi} \int d\sigma e^{-\phi} \left[ \Delta f_{\pi, \times}^{(1)} \Delta \right] + O(G^3).
\]  

(52)

The result after integration is Eq. (29). It holds in both the rest frame of the first body and the c.o.m. frame: in the former case \( J_{\pi \pi}^{\text{init}} \mid_{s=0} = m_2 \sqrt{\gamma^2 - 1} |b| \); in the latter \( J_{\pi \pi}^{\text{init}} \mid_{s=0} = p_{\infty} |b| \).

The radiated four-momentum \( P_{\pi \pi}^{\text{rad}} \) (43) contributes to leading PM order \( G^3 \). In the center-of-mass frame the radiated energy is \( E_{\pi \pi, \text{CoM}} = v_{\text{CoM}} P_{\pi \pi}^{\text{rad}} \), where

\[
v_{\text{CoM}} = \frac{m_1 v_1 + m_2 v_2}{\sqrt{m_1^2 + m_2^2 + 2\gamma m_1 m_2}}.
\]  

(53)

Due to the multi-scale nature of the waveform \( f_{i j} \), it is difficult to perform the time and solid-angle integrations in Eq. (43) directly; however, in a low velocity expansion we succeeded and the result is stated in Eq. (31).
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