1 Introduction

Abstract. We define a Hopf algebra of motivic iterated integrals on the line and prove an explicit formula for the coproduct $\Delta$ in this Hopf algebra. We show that this formula encodes the group law of the automorphism group of a certain noncommutative variety. We relate the coproduct $\Delta$ with the coproduct in the Hopf algebra of decorated rooted plane trivalent trees, which is a plane decorated version of the one defined by Connes and Kreimer [CK]. As an application we derive explicit formulas for the coproduct in the motivic multiple polylogarithm Hopf algebra. These formulas play a key role in the mysterious correspondence between the structure of the motivic fundamental group of $\mathbb{P}^1 - \{0, \infty \} \cup \mu_N$, where $\mu_N$ is the group of all $N$-th roots of unity, and modular varieties for $GL_m ([G1–2])$.

In Chapter 7 we discuss some general principles relating Feynman integrals and mixed motives. They are suggested by Chapter 4 and the Feynman integral approach for multiple polylogarithms on curves given in [G2]. Chapter 8 contains background material.

1. The Hopf algebra of motivic iterated integrals. Consider the iterated integral

$$I(\gamma(a_0; a_1, ..., a_n; a_{n+1}) := (2\pi i)^{-n} \int_{\Delta_n,\gamma} \frac{dt_1}{t_1 - a_1} \wedge \frac{dt_2}{t_2 - a_2} \wedge ... \wedge \frac{dt_n}{t_n - a_n}$$

$$1$$
The regularization depends on the choice of tangent vectors in the points \(a_0\) and \(a_{n+1}\). Below we will assume that these tangent vectors are \(\partial/\partial t\), where \(t\) is the standard coordinate on \(A^1\).

The numbers \(I\) are periods of \(\mathbb{Q}\)-rational framed Hodge-Tate structures. A Hodge-Tate structure is a mixed Hodge structure such that the Hodge numbers \(h^{p,q} = 0\) if \(p \neq q\). The framing is an additional data which, together with a splitting of the weight filtration, allows to consider a specific period of a mixed Hodge structure, which is simply a complex number. If we want to keep the information about this period only we are led to an equivalence relation on the set of all framed Hodge-Tate structures. The equivalence classes form a commutative graded Hopf algebra.

We review these definitions in the Appendix. The product structure on \(\mathcal{H}_\bullet\) is compatible with the product of the periods. The coproduct is something really new: it is invisible on the level of numbers.

Let us choose an embedding \(\mathbb{Q} \hookrightarrow \mathbb{C}\). Let us assume that the parameters \(a_i\) of the iterated integral \(I\) are algebraic numbers. Then we can do even better, and upgrade \(I\) to a framed mixed Tate motive over \(\mathbb{Q}\), called **motivic iterated integral**:

\[
I^M(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{A}_n(\mathbb{Q})
\]  

By its very definition it lies in a commutative, graded Hopf algebra \(\mathcal{A}_\bullet(\mathbb{Q})\) with a coproduct \(\Delta\), defined in the Appendix.

More precisely, let \(F\) be a number field. Then there is a graded, commutative Hopf algebra \(\mathcal{A}_\bullet(F)\), the fundamental Hopf algebra of the abelian category \(\mathcal{M}_T(F)\) of mixed Tate motives over \(F\), see the Appendix for the background. It is isomorphic to the Hopf algebra of regular functions on the unipotent part of the motivic Tate Galois group of \(\mathbb{Q}\) (loc. cit.).

The Hopf algebra \(\mathcal{A}_\bullet(F)\) depends functorially on \(F\), and have the following structure. Let \(T(V_\bullet)\) denotes the tensor algebra of a graded \(\mathbb{Q}\)-vector space \(V_\bullet\). It has a natural commutative graded Hopf algebra structure. Then there is an isomorphism

\[
\mathcal{A}_\bullet(F) \cong T \left( \oplus_{n \geq 1} K_{2n-1}(F) \otimes \mathbb{Q} \right)
\]

where \(K_{2n-1}\) on the right sits in the degree \(n\).

Given an embedding \(\sigma : F \hookrightarrow \mathbb{C}\) of \(F\) into \(\mathbb{C}\), we define in Chapter 8.5 a filtered algebra \(\mathcal{P}_\sigma(F)\) over \(\mathbb{Q}\) of periods of mixed Tate motives over \(F\) in the Hodge realization provided by \(\sigma\):

\[
\mathbb{Q} = \mathcal{P}_\leq n(F) \subset \mathcal{P}_\leq 1(F) \subset \mathcal{P}_\leq 2(F) \subset \ldots; \quad \mathcal{P}_\sigma(F) = \bigcup \mathcal{P}_\leq n(F)
\]

Let \(\mathcal{P}_\sigma(F)\) be its associate graded. We prove in Theorem 1.3 that there is canonical surjective homomorphism of graded commutative algebras

\[
p_\sigma : \mathcal{A}_\bullet(F) \longrightarrow \mathcal{P}_\bullet(\mathcal{F})
\]  

If the parameters of the iterated integral \(I\) are in the subfield \(\sigma(F) \subset \mathbb{C}\), one can prove ([G3]) that the projection of the value of integral \(I\) to \(\mathcal{P}_n(\mathcal{F})\) does not depend on the path \(\gamma\), i.e. on the monodromy. Thus the values of the iterated integral \(I\) provide a well defined element

\[
I(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{P}_n(\mathcal{F}), \quad a_i \in \sigma(F) \subset \mathbb{C}
\]

**Theorem 1.1** Suppose that \(a_i\) are elements of a number field \(F\). Then there exist motivic iterated integrals

\[
I^M(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{A}_n(F)
\]  

such that

\[
p_\sigma(I^M(a_0; a_1, \ldots, a_n; a_{n+1})) = I(\sigma(a_0); \sigma(a_1), \ldots, \sigma(a_n); \sigma(a_{n+1}))
\]
Example. Let $n = 1$. Then

$$I(a, b, c) = (2\pi i)^{-1} \int_a^c \frac{dt}{t - b} = (2\pi i)^{-1} \log \frac{c - b}{a - b}$$

This integral diverges if $a = b$ or $b = c$. To regularize it we calculate the asymptotic expansion when $\varepsilon \to 0$ of the integral

$$(2\pi i)^{-1} \int_{a+\varepsilon}^{c+\varepsilon} \frac{dt}{t - b} = (2\pi i)^{-1} \log \frac{c - b + \varepsilon}{a - b + \varepsilon},$$

getting a polynomial in $\log \varepsilon$, and take its free term as the regularized value of the integral. We get the following answer. Let

$$\tilde{r}(a, b, c) := \begin{cases} \frac{(c-b)}{(a-b)} & \text{if } a \neq b, b \neq c \\ c - b & \text{if } a = b, \text{ but } b \neq c \\ (a - b)^{-1} & \text{if } b = c, \text{ but } a \neq b \\ 1 & \text{if } a = b = c \end{cases}$$

(5)

Then

$$I(a, b, c) = (2\pi i)^{-1} \log \tilde{r}(a, b, c)$$

The right hand side provides a well defined element $I(a, b, c) \in \mathbb{C}/\mathbb{Q}$. There is a canonical isomorphism $A_1(F) = F^* \otimes \mathbb{Q}$, and

$$M(a, b, c) = \tilde{r}(a, b, c)$$

One of our results is the following explicit formula for the coproduct of the elements $M$.

**Theorem 1.2** The coproduct $\Delta$ is computed by the formula

$$\Delta M(a_0; a_1, a_2, \ldots, a_n; a_{n+1}) = \sum_{0=i_0<i_1<\ldots<i_k<i_{k+1}=n+1} M(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^k M(a_{i_p}; a_{i_{p+1}}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

(6)

Here $0 \leq k \leq n$ and $a_i \in \overline{\mathbb{Q}}$.

We prove a similar result for the Hodge and $l$-adic analogs of the motivic iterated integrals, see Theorem 3.3 for a precise statement. Another proof in the Hodge set-up was given in Sections 5-6 of [G3].

The terms in the formula (6) are in one–to–one correspondence with the subsequences

$$\{a_{i_1}, \ldots, a_{i_k}\} \subset \{a_1, \ldots, a_n\}$$

(7)

If we locate the ordered sequence $\{a_0, \ldots, a_{n+1}\}$ on a semicircle then the terms in (6) correspond to the polygons with vertices at the points $a_i$, containing $a_0$ and $a_{n+1}$, inscribed into the semicircle.
The picture illustrates the term

\[ I^M(a_0; a_3, a_5, a_7; a_{10}) \otimes I^M(a_0; a_1, a_2; a_3)I^M(a_3; a_4; a_5)I^M(a_5; a_6; a_7)I^M(a_7; a_8, a_9; a_{10}) \]

**Example.** For \( m = 2 \) formula [1] gives

\[
\begin{align*}
\Delta I^M(a_0; a_1, a_2; a_3) & = 1 \otimes I^M(a_0; a_1, a_2; a_3) + \\
I^M(a_0; a_1; a_3) \otimes I^M(a_1; a_2; a_3) + I^M(a_0; a_2; a_3) \otimes I^M(a_0; a_1; a_2) + I^M(a_0; a_1; a_2; a_3) \otimes 1
\end{align*}
\]

The \( m = 3 \) case is worked out at the end of Chapter 3.

Our proof penetrates some algebraic structures staying behind this formula. They include automorphism groups of certain noncommutative varieties discussed in Chapters 2 and 3, and a Hopf algebra of decorated rooted plane trees in Chapter 4.

**2. Why do we care about motivic iterated integrals?** In other words, what do we gain by upgrading the iterated integrals [11] to the elements [12] of the motivic Hopf algebra \( A_\bullet(F) \)?

First, by doing this we **conjecturally** do not lose any information.

**Conjecture 1.3** The map [3] is an isomorphism of \( \mathbb{Q} \)-vector spaces.

Our point is that it is much easier to deal with motivic iterated integrals than with numbers!

Here is what we gain:

A) Unlike the numbers, the motivic iterated integrals form a Hopf algebra.

B) Working with the more sophisticated motivic iterated integrals we eliminated the transcendental aspect of the problem.

C) Motivic iterated integrals serve as a bridge between their Hodge (i.e. analytic) and \( l \)-adic (i.e. arithmetic) realizations. This allows to use arithmetic insights in analytic problems and vice versa.

Here is a simplest example. The Leibniz formula expresses special values of the Riemann \( \zeta \)-function via iterated integrals:

\[
\zeta(m) = \int_{0 \leq t_1 \leq \ldots \leq t_m \leq 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge \ldots \wedge \frac{dt_m}{t_m}
\]

According to Euler \( (2\pi i)^{-2k}\zeta(2k) \in \mathbb{Q} \). In accordance with this one has \( \zeta^M(2k) = 0 \). Nobody can prove so far that the numbers \( \zeta(2k+1) \) are linearly independent over \( \mathbb{Q} \), e.g. that \( \zeta(5) \notin \mathbb{Q} \). However it is easy to show that the motivic elements

\[
\zeta^M(2k+1) \in A_{2k+1}(\mathbb{Q})
\]

are non-zero. Moreover they are linearly independent over \( \mathbb{Q} \). Indeed, they belong to components of different degrees in \( A_\bullet(\mathbb{Q}) \).

Another example is worked out in Chapter 6.7, where the motivic double zeta’s are investigated. Recall that the double zeta values \( \zeta(m,n) \) can be presented both as power series and iterated integrals:

\[
\zeta(m,n) = \sum_{0 < k_1 < k_2} \frac{1}{k_1^m k_2^n} = I(0; 1, 0, \ldots, 0, 1, 0, \ldots, 0; 1)
\]

They satisfy the (regularized) double shuffle relations, obtained by expressing the product of two classical \( \zeta \)-values by using either the power series or the iterated integral presentations. The iterated integral presentation leads to the definition of motivic double zeta’s \( \zeta^M(m,n) \). It follows from [G8] that the motivic double \( \zeta \)'s satisfy the same double shuffle relations. We will reprove in Chapter 6.7 a version of this result, and prove that any relation between the motivic double \( \zeta \)'s is a consequence of the motivic double shuffle relations.
I think that an understanding of the transcendental aspects of the iterated integrals is impossible without investigation of the corresponding motivic objects.

3. Motivic iterated integrals unramified outside of a finite set $S$ of prime ideals. Given the parameters $a_i \in F$, one can describe rather precisely the subspace in $A_n(F)$ where the motivic iterated integral land. Namely, let $S$ be a collection of primes in a number field $F$, and $O_{F,S}$ the ring of $S$-integers in $F$. Then there exists the category of mixed Tate motives over $O_{F,S}$, ([DG]). Let $A_*(O_{F,S})$ be its fundamental Hopf algebra (Section 8.4). We prove the following result (Theorem 6.7). Recall $\tilde{r}(a, b, c)$ from [5].

**Theorem 1.4** Let $a_0, ..., a_{n+1}$ be elements of a number field $F$. If for any $0 \leq i < j < k \leq n + 1$ one has $\tilde{r}(a_i, a_j, a_k) \in O_{F,S}$, then

$$I^M(a_0; a_1, ..., a_n; a_{n+1}) \in A_n(O_{F,S})$$

(8)

**Corollary 1.5** Suppose that $a_0, ..., a_{n+1} \in \{\zeta_N^k\} \cup \{0\}$, where $\zeta_N$ is a primitive root of unity. Then

$$I^M(a_0; a_1, ..., a_n; a_{n+1}) \in A_n(\mathbb{Z}[\zeta_N][\frac{1}{N}])$$

(9)

In particular $\zeta^M(n_1, ..., n_m) \in A_w(\mathbb{Z})$, where $w = n_1 + ... + n_m$.

Indeed, $\zeta^a_N - \zeta^b_N = \zeta^c_N(1 - \zeta^{c-a}_N) \in (\mathbb{Z}[\zeta_N][\frac{1}{N}])^*$. So the corollary follows from Theorem 1.4.

**Example.**

The Hopf algebra $A_*(O_{F,S})$ has the following structure. There exists an isomorphism of commutative graded Hopf algebras, see Section 8.5:

$$A_*(O_{F,S}) \cong T\left(\oplus_{n>1} K_{2n-1}(O_{F,S}) \otimes \mathbb{Q}\right)$$

where $K_{2n-1}$ sits in the degree $n$. In particular, if $S$ is a finite set, then $A_n(O_{F,S})$ is a finite-dimensional $\mathbb{Q}$-vector space! Indeed, if $n > 1$ then for any $S K_{2n-1}(O_{F,S}) \otimes \mathbb{Q} = K_{2n-1}(F) \otimes \mathbb{Q}$ is finite dimensional by Borel’s theorem. But $K_{2n-1}(O_{F,S}) = O_{F,S}$ is of finite rank only if $S$ is finite. This combined with Theorem 1.4 give a strong estimate from above on the dimension of the subspaces in $C$ generated by the values of iterated integrals, see Section 6.8.

Here is a typical example. Combining Corollary 1.5 with 1.4, we get an estimate from above on the $\mathbb{Q}$-vector space $Z_n(\mu_N \cup \{0\})$ spanned by the iterated integrals $I(a_0; a_1, ..., a_n; a_{n+1})$, where $a_0, ..., a_{n+1} \in \{\zeta_N^a\} \cup \{0\}$, modulo similar integrals of weight $< n$:

$$\dim Z_n(\mu_N \cup \{0\}) \leq \dim T(n)\left(\oplus_{n>1} K_{2n-1}(\mathbb{Z}[\zeta_N][\frac{1}{N}])\right)$$

(10)

Here on the right stays the dimension of the degree $n$ part of the tensor algebra. A bit different way to get the same estimate see in [DG].

To clarify the relationship between the numbers and their more sophisticated motivic counterparts we discuss below the Hodge and $l$-adic versions of iterated integrals.

4. The Hodge iterated integrals. By definition (see the Appendix) an $n$-framing on a Hodge-Tate structure $H$ is a choice of nonzero morphisms of Hodge-Tate structures

$$v : \mathbb{Q}(0) \longrightarrow \text{gr}_0^W H; \quad f : \text{gr}_{2m}^WH \longrightarrow \mathbb{Q}(m)$$

Consider the coarsest equivalence relation $\sim$ on the set of all $n$-framed Hodge-Tate structures such that $(H_1, v_1, f_1) \sim (H_2, v_2, f_2)$ if there is a morphism of Hodge-Tate structures $H_1 \rightarrow H_2$ respecting the frames. The equivalence classes of $n$-framed Hodge-Tate structures form an abelian group $H_n$, and $H_* := \oplus_{n \geq 0} H_n$ has a commutative graded Hopf algebra structure.
The framed Hodge-Tate structure corresponding to the iterated integrals (11) arises as follows. Let $S := \{z_1, \ldots, z_m\}$. Denote by $v_a, v_b$ the tangent vectors at the points $a, b$ dual to the canonical differential $dt$ on $C$. Then the pronilpotent completion

$$P^H(C - S; v_a, v_b)$$

of the topological torsor of paths from $v_a$ to $v_b$ (see s. 3.1) is equipped with the structure of a projective limit of Hodge-Tate structures ([D]). In particular it carries a weight filtration $W_\bullet$ such that

$$\text{gr}_{-2m}P^H(C - S; v_a, v_b) \cong \otimes^m H_1(C - S)$$

(12)
The forms $(2\pi i)^{-1}d\log(t - z_i)$ provide a basis of $H^1(C - S)$. We define

$$I^H(a; z_1, \ldots, z_m; b) \in \mathcal{H}_m$$

(13)
as the Hodge-Tate structure (11) framed by $1 \in \mathbb{Q}$ and

$$(2\pi i)^{-1}d\log(t - z_1) \otimes \ldots \otimes (2\pi i)^{-1}d\log(t - z_m) \in \text{gr}_{-2m}P^H(C - S; v_a, v_b)$$

(14)

If we choose in addition a splitting of the weight filtration in (11), one gets a complex number, period, as the Hodge-Tate structure (11) framed by $1 \in \mathbb{Q}$.

To see why the coproduct appears, let us discuss the coproduct $\Delta$.

Example: Let $\mathbb{Q}$-valued functions on the Galois group form a Hopf algebra. Its coproduct $\Delta(\mathfrak{g})$ is provided by the group multiplication. We prove that the coproduct $\Delta(\mathfrak{g})I_{F_p}(a; z_1, \ldots, z_m; b)$ is calculated by the same formula (3). It follows that when $a, b, z_1, \ldots, z_m$ run through the elements of a finite set $S$, the functions (16) span over $\mathbb{Q}_l$ a commutative graded Hopf algebra $\mathcal{H}_l^*(S)$.

Definition 1.6 $I_{F_p}^{(l)}(a; z_1, \ldots, z_m; b)$ is the function on $\text{Gal}(\overline{\mathbb{Q}}/F(\zeta_{l\infty}))$ given by the matrix element

$$g \mapsto < d\log(t - z_1) \otimes \ldots \otimes d\log(t - z_m) | g >$$

(16)

The $\mathbb{Q}_l$-valued functions on the Galois group form a Hopf algebra. Its coproduct $\Delta^{(l)}$ is provided by the group multiplication. We prove that the coproduct $\Delta^{(l)}I_{F_p}^{(l)}(a; z_1, \ldots, z_m; b)$ is calculated by the same formula (3). It follows that when $a, b, z_1, \ldots, z_m$ run through the elements of a finite set $S$, the functions (16) span over $\mathbb{Q}_l$ a commutative graded Hopf algebra $\mathcal{H}_l^{(l)}(S)$.

Example: By Kummer’s theory an element $a \in F^*$ gives rise to a $\mathbb{Z}_l$-valued function $\chi_a(g)$ on $\text{Gal}(\overline{\mathbb{Q}}/F(\zeta_{l\infty}))$ given by $\zeta_{l\alpha}(g) := g(a^{1/l\alpha})/a^{1/l\alpha}$. One has $\Delta^{(l)}\chi_a = \chi_a \otimes 1 + 1 \otimes \chi_a$. We have

$$\chi_a = I^{(l)}(1; 0, a) = I^{(l)}(0; 0, a)$$
It turns out to be independent of a choice of Frobenius or splitting.

On the other hand, the Galois module \([15]\), framed by 1 and \(d \log(t - z_1) \otimes \cdots \otimes d \log(t - z_m)\), provides an element \(I^q(a; z_1, \ldots, z_m; b)\) of the commutative graded algebra of framed mixed Tate \(l\)-adic representations of the Galois group, see the Appendix. They are the \(l\)-adic counterparts of the elements \(I^H(a; z_1, \ldots, z_m; b)\). The functions \(I^q_F(a; z_1, \ldots, z_m; b)\) on the Galois group are the \(l\)-adic counterparts of periods. One can avoid making the choice ii), getting \(\mathbb{Q}(l)(n)\)-valued functions on the Galois group. However, just as in the analytic case, without the choice i) we can not get functions on the Galois group. The elements \(I^q(a)\) are canonical: their definition does not depend on the choices i) and ii). They span a Hopf algebra, which is canonically isomorphic to \(\mathcal{H}_q(S)\), see the Appendix.

The definition of the motivic iterated integrals as framed objects is similar. It uses the motivic version of the pro-nilpotent torsor of paths \([11]\).

A version of Definition \([16]\) is obtained if one does not use the choice ii). Then formula \([16]\) yields a \(\mathbb{Q}(l)(n)\)-valued function on the Galois group \(\text{Gal}(\overline{\mathbb{Q}}/F)\).

6. The algebraic structures underlying formula \([6]\). The integrals \([11]\) satisfy the following basic properties ([Chen]):

The shuffle product formula. Let \(\Sigma_{m,n}\) be the set of all shuffles of the ordered sets \(\{1, \ldots, m\}\) and \(\{m + 1, \ldots, m + n\}\). Then

\[
I_{\gamma}(a; z_1, \ldots, z_m; b) \cdot I_{\gamma}(a; z_{m+1}, \ldots, z_{m+n}; b) = \sum_{\sigma \in \Sigma_{m,n}} I_{\gamma}(a; z_{\sigma(1)}, \ldots, z_{\sigma(m+n)}; b)
\]

The path composition formula. If \(\gamma = \gamma_1 \gamma_2\), where \(\gamma_1\) is a path from \(a\) to \(x\), and \(\gamma_2\) is a path from \(x\) to \(b\) in \(\mathbb{C} - \{z_1 \cup \ldots \cup z_n \cup a \cup b\}\), then

\[
I_{\gamma}(a; z_1, \ldots, z_m; b) = \sum_{k=0}^{m} I_{\gamma_1}(a; z_1, \ldots, z_k; x) \cdot I_{\gamma_2}(x; z_{k+1}, \ldots, z_m; b)
\]

Let \(S\) be an arbitrary set. In Chapter 2 we define a commutative, graded Hopf algebra \(\mathcal{I}_* (S)\). As a commutative algebra it is generated by symbols \((s_0; s_1, \ldots, s_n; s_{n+1})\) with \(s_i \in S\) subject to the relations as for \(I_x\) above. We use formula \([3]\) to define the coproduct in \(\mathcal{I}_* (S)\). If \(S\) is a subset of points of the affine line \(\mathbb{A}^1\) the Hopf algebra \(\mathcal{I}_* (S)\) reflects the basic properties of the iterated integrals on \(\mathbb{A}^1_S := \mathbb{A}^1 - S\) between the tangential base points at \(S\). To show that \(\mathcal{I}_* (S)\) is indeed a Hopf algebra we interpret it as the algebra of regular functions on a certain pro-unipotent group scheme. Namely, let \(\Gamma_S\) be the graph whose vertices are elements of \(S\), and every two vertices are connected by a unique edge. Let \(P(S)\) be the path algebra of this graph. Its basis over \(\mathbb{Q}\) is formed by the paths in the graph. The product of two basis elements is given by the composition of paths, provided that the end of the first coincides with the beginning of the second. Otherwise the product is zero.

![Diagram](image)

It has some additional structures: two algebra structures \(\circ\) and \(\ast\), plus a coproduct \(\Delta\). Let \(G(S)\) be the group of automorphisms of \(P(S)\) preserving these structures, and acting as the identity on \(P_+ (S)/P^2_+ (S)\). Here \(P_+ (S)\) is given by paths of positive length. In other words it is the group of
automorphisms of the corresponding noncommutative variety preserving some natural structures on it. We prove in Theorem 2.5 that
\[ G(S) = \text{Spec}(I_\bullet(S)) \]

In Chapter 3 we show that the algebra \( P(S) \) is provided by the motivic fundamental groupoid
\[ P^M(A^1_S, S) \] (17)
of paths on the affine line punctured at \( S \), between the chosen tangential base points at \( S \). We consider (17) as a pro–object in the abelian category of mixed Tate motives over a number field or in one of the realization categories, see the Appendix for the background. Such a category is equipped with a canonical fiber functor \( \omega \). The motivic Galois group acts on \( \omega(P^M(A^1_S, S)) \). Let \( G_M(S) \) be its image. The motivic fundamental groupoid carries some additional structures provided by the composition of paths and “canonical loops” near the punctures. Using these structures we identify \( \omega(P^M(A^1_S, S)) \) with the path algebra \( P(S) \). Therefore \( G_M(S) \) is realized as a subgroup of \( G(S) \):
\[ G_M(S) \hookrightarrow G(S) \] (18)

Theorem 1.2 follows immediately from this.

In Chapter 4 we show how the collection of all plane trivalent rooted trees decorated by the ordered set \( \{a_0, a_1, ..., a_m, a_{m+1}\} \) governs the fine structure of the motivic object (2). Namely, in Sections 4.1-4.2 we define a commutative Hopf algebra \( T_\bullet(S) \) of \( S \)-decorated planar rooted trivalent trees, and relate the groups \( G(S) \) and \( \text{Spec}(T_\bullet(S)) \). Precisely, let \( \tilde{T}_\bullet(S) \) be the commutative graded algebra freely generated by the elements \( I(s_0; s_1, ..., s_m; s_{m+1}) \). We use formula (6) to define the coproduct in \( \tilde{T}_\bullet(S) \), and show that it is indeed a Hopf algebra. Consider the map
\[ t : I(s_0; s_1, ..., s_m; s_{m+1}) \mapsto \text{formal sum of all plane rooted trivalent trees} \] (19)
decorated by the ordered set \( \{s_0, s_1, ..., s_m, s_{m+1}\} \)

**Example.** Here is how the map \( t \) looks in the two simplest cases

\[ I(s_0; s_1, s_2) \rightarrow \begin{array}{c}
\begin{array}{c}
 s_0 \\
 s_1 \\
 s_2 
\end{array}
\end{array} \]

\[ I(s_0; s_1, s_2, s_3) \rightarrow \begin{array}{c}
\begin{array}{c}
 s_0 \\
 s_1 \\
 s_2 \\
 s_3 
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
 s_0 \\
 s_1 \\
 s_2 \\
 s_3 
\end{array}
\end{array} \]

We prove in Chapter 4 that the map \( t \) provides a Hopf algebra homomorphism
\[ t : \tilde{T}_\bullet(S) \rightarrow T_\bullet(S) \]

**Remark.** This map does not descend to a Hopf algebra map from \( I_\bullet(S) \) to \( T_\bullet(S) \). I do not know whether there exists a quotient of the Hopf algebra \( T_\bullet(S) \) such that \( t \) maps \( I_\bullet(S) \) to this quotient.

7. **An application: motivic multiple polylogarithm Hopf algebras.** Recall the multiple polylogarithms ([G0], [G6]) where \( n_i \in \mathbb{N}_+, |x_i| < 1 \):
\[ \text{Li}_{n_1, ..., n_m}(x_1, ..., x_m) = \sum_{0 < k_1 < k_2 < ... < k_m} \frac{x_1^{k_1} x_2^{k_2} ... x_m^{k_m}}{k_1^{n_1} k_2^{n_2} ... k_m^{n_m}} \] (20)
Theorem 2.2 in [G3] allows to write them as iterated integrals. Namely, if $|x_i| < 1$, then setting

$$a_1 := (x_1...x_m)^{-1}, a_2 := (x_2...x_m)^{-1}, \ldots, a_m := x_m$$

we get for a path $\gamma$ inside of the unit disc

$$L_{n_1,\ldots,n_m}(x_1,\ldots,x_m) = (-1)^m I_{n_1,\ldots,n_m}(a_1,\ldots,a_m) := (-1)^m I_\gamma(0; a_1,0,\ldots,0,a_2,0,\ldots,0,a_3,\ldots,a_m,0,\ldots,0;1)$$ \hspace{1cm} (21)

Let $\mathcal{M}$ be the category of mixed Tate motives over a number field $F$ or one of the mixed Tate categories described in section 3.1. Let $G$ be a subgroup of the multiplicative group $F^*$ of the field $F$. Upgrading iterated integrals (21) to their motivic counterparts (2), and using identity (21) as a definition, we arrive at motivic multiple polylogarithms

$$L_{n_1,\ldots,n_m}^M(x_1,\ldots,x_m), \quad x_i \in G \subset F^* \hspace{1cm} (22)$$

Adding the motivic logarithms $\log^M(x)$ to them, we get a graded, commutative Hopf algebra $Z^M_\bullet(F^*)$, see Theorem 5.4. In particular if $G = \mu_N$ we get the cyclotomic Hopf algebra $Z^M_\bullet(\mu_N)$. According to (9), there is an inclusion of graded Hopf algebras

$$Z^M_\bullet(\mu_N) \subset A^M_\bullet(\mathbb{Z}[[\zeta_N]]^{1/N})$$

The structure of the corresponding cyclotomic Lie coalgebra

$$C^M_\bullet(\mu_N) := Z^M_{\geq 0}(\mu_N)/(Z^M_{\geq 0}(\mu_N))^2$$

is related to the geometry of modular varieties, see [G1-2] and the Section 6.7 for an example.

According to Conjecture 5.9 the category of all finite dimensional graded comodules over the Hopf algebra $Z^M_\bullet(F^*)$ should be equivalent to the conjectural abelian category of all mixed Tate motives over $F$. It follows from Theorem 1.2 that the Hopf algebra $Z^M_\bullet(F^*)$ has an additional structure: the depth filtration, such that the depth of (22) is $m$. We suggest a conjectural intrinsic description of the depth filtration in Chapter 5.4.

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2 Hopf algebras of iterated integrals

In this chapter we introduce the Hopf algebras $I_\bullet(S)$ and $\tilde{I}_\bullet(S)$ and show that they appear naturally as the algebras of functions on the automorphism groups of certain non-commutative varieties.

1. The Hopf algebra $\tilde{I}_\bullet(S)$ and $I_\bullet(S)$. Let $S$ be a set. We will define a commutative Hopf algebra $I_\bullet(S)$ over $\mathbb{Q}$, graded in an obvious way by the integers $n \geq 0$.

As a commutative $\mathbb{Q}$-algebra, $I_\bullet(S)$ is generated by the elements

$$\mathbb{I}(s_0; s_1, \ldots, s_m; s_{m+1}), \quad s_i \in S, \quad m \geq 0 \hspace{1cm} (23)$$

The generator (23) is homogeneous of degree $m$. The relations are the following. For any $s_i, a, b, x \in S$ one has:
The counit is determined by the condition that it kills \( I \).

**Proposition 2.1** In \( I_\star(S) \) one has, for any \( m \geq 0 \) and \( a_0, \ldots, a_{m+1} \in S \):

\[
\mathbb{I}(a_0; a_1, \ldots, a_m; a_{m+1}) = (-1)^m \mathbb{I}(a_{m+1}; a_m, \ldots, a_1; a_0)
\]

**Proof.** We use induction on \( m \). When \( m = 1 \) the path composition formula iii) plus i) gives

\[
0 = \mathbb{I}(a_0; a_1; a_0) = \mathbb{I}(a_0; a_1; a_2) + \mathbb{I}(a_2; a_1; a_0)
\]

Let us assume we proved the claim for all \( k < m \). The path composition formula with \( x := a_{m+1} \) gives

\[
0 = \mathbb{I}(a_0; a_1, \ldots, a_m; a_0) = \mathbb{I}(a_0; a_1, \ldots, a_m; a_{m+1}) + \sum_{1 \leq k \leq m-1} \mathbb{I}(a_0; a_1, \ldots, a_k; a_{m+1}) \cdot \mathbb{I}(a_{m+1}; a_{k+1}, \ldots, a_m; a_0) + \mathbb{I}(a_{m+1}; a_1, \ldots, a_m; a_0)
\]

Applying the induction assumption to the second factors in the sum we get

\[
0 = \mathbb{I}(a_0; a_1, \ldots, a_m; a_{m+1}) + \sum_{1 \leq k \leq m-1} (-1)^{m-k} \mathbb{I}(a_0; a_1, \ldots, a_k; a_{m+1}) \cdot \mathbb{I}(a_{m+1}; a_{k+1}, \ldots, a_m; a_0) + \mathbb{I}(a_{m+1}; a_1, \ldots, a_m; a_0)
\]

We claim that

\[
\mathbb{I}(a_0; a_1, \ldots, a_m; a_{m+1}) + \sum_{1 \leq k \leq m-1} (-1)^{m-k} \mathbb{I}(a_0; a_1, \ldots, a_k; a_{m+1}) \cdot \mathbb{I}(a_{m+1}; a_{k+1}, \ldots, a_m; a_0)
\]

**Theorem 3.1** The coproduct \( \Delta(a_0; a_1, \ldots, a_m; a_{m+1}) \) on \( I_\star(S) \) is defined by

\[
\Delta(a_0; a_1, \ldots, a_m; a_{m+1}) = \sum_{i=0}^{m} \mathbb{I}(a_0; a_1, \ldots, a_i; a_{i+1}, \ldots, a_{m+1}) \cdot \mathbb{I}(a_{i+1}; a_{i+2}, \ldots, a_{m+1})
\]
Indeed, this equality is equivalent to

\[
I(a_0; a_1, ..., a_m; a_{m+1}) + \sum_{0 \leq k \leq m-1} (-1)^{m-k}I(a_0; a_1, ..., a_k; a_{m+1}) \cdot I(a_m; a_m, ..., a_{k+1}; a_{m+1}) = 0
\]

To prove (26) we use the shuffle product formula to rewrite the sum. Then the claim is rather obvious (see the beginning of the proof of Theorem 4.1 in [G1] for details). It remains to notice that the proposition follows immediately from (25). The proposition is proved.

Let \( \tilde{I}_\bullet(S) \) be defined in the same way as \( I_\bullet(S) \), except that the relations ii)–iv) are omitted. By abuse of notation, we will denote the generators of \( \tilde{I}_\bullet(S) \) by the same symbol as for \( I_\bullet(S) \).

**Proposition 2.2** The coproduct \( \Delta \) provides \( I_\bullet(S) \), as well as \( \tilde{I}_\bullet(S) \), with the structure of a commutative, graded Hopf algebra.

The algebra of regular functions \( \mathcal{O}(G) \) on a group scheme \( G \) is a commutative Hopf algebra with the coproduct \( \Delta : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G) \) induced by the group multiplication map \( G \times G \rightarrow G \).

To prove Proposition 2.2 which will follow from Theorem 2.5 below, we interpret \( I_\bullet(S) \) and \( \tilde{I}_\bullet(S) \) as the algebras of regular functions on certain pro-unipotent group schemes, defined as automorphism groups of certain non-commutative objects.

2. The path algebra \( P(S) \). Let \( S \) be a set. Let \( K \) be a field. Let \( P(S) \) be the \( K \)-vector space with basis

\[
p_{s_0, ..., s_n}, \quad n \geq 1, s_k \in S \text{ for } k = 0, ..., n
\]

It has a grading such that the degree of a typical basis element \( p_{s_0, ..., s_n} \) is \(-2(n-1)\). Let us equip it with the following structures.

i) The \( \circ \)-product. There is an associative product

\[
\circ : P(S) \otimes_K P(S) \rightarrow P(S)
\]

\[
p_{a,x,b} \circ p_{c,y,d} := \begin{cases} p_{a,x,y,d} : & b = c; \\ 0 : & b \neq c \end{cases}
\]

Here the small letters denote elements, and the capital ones subsequences, possibly empty, of the set \( S \). In particular \( p_{a,b} = p_{a,x} \circ p_{x,b} \). The element \( e_0 := \sum_{i \in S} p_{i,i} \) is the unit for this product. The algebra \( P(S) \) is decomposed into a sum

\[
P(S) = \oplus_{i,j \in S} P(S)_{i,j}
\]

where \( P(S)_{i,j} \) is spanned by the elements (27) with \( s_0 = i, s_n = j \). Below we consider only the automorphisms \( F \) of \( P(S) \) preserving this decomposition:

\[
F(P(S)_{i,j}) \subset P(S)_{i,j}
\]

The algebra \( P(S) \) has an interpretation as a tensor algebra in a certain monoidal category, see section 2.3, which makes this restriction on \( F \) natural.

ii) The \( * \)-product. We define another associative product

\[
* : P(S)(1) \otimes P(S)(1) \rightarrow P(S)(1)
\]
by the formula

\[ p_{X,b} \ast p_{c,Y} = \begin{cases} p_{X,b,Y} & : b = c; \\ 0 & : b \neq c \end{cases} \]  

(30)

where \( b, c \) are elements, and \( X, Y \) are ordered collections of elements of \( S \). Here \( M(1) \) means \( M \) with the grading shifted down by 2. Then \( P(S)(1) \) is an associative algebra generated by the elements \( p_{i,j} \).

One can add to this algebra the elements \( p_i, i \in S \), whose composition with the other elements is given by formula (30) where \( X \) or/and \( Y \) can be empty. Then the \( p_i \) are orthogonal projectors:

\[ p_i^2 = p_i, p_i \ast p_j = 0 \quad \text{if} \quad i \neq j, \quad \text{and} \quad e := \sum_{i \in S} p_i \quad \text{is the unit with respect to} \ast. \]

Using these projectors we can describe decomposition (28) as

\[ P(S) = \oplus_{i,j \in S} P(S)_{i,j}; \quad P(S)_{i,j} := p_i \ast P(S) \ast p_j \]

The automorphisms of the algebra \( P(S) \) preserving the projectors \( p_i \) respect this decomposition.

Recall the graph \( \Gamma_S \) from s. 1.2. The \( \ast \)-algebra \( P(S) \) is the path algebra of this graph. Namely, \( p_{i_0,..,i_n} \) corresponds to the path \( i_0 \rightarrow i_1 \rightarrow ... \rightarrow i_n \) in \( \Gamma_S \). It turns out to be isomorphic to the free product, denoted \( \ast_Q \), and not to be confused with \( \ast \), of the algebra \( Q[S] \) of functions on \( S \) with finite support with the polynomial algebra \( Q[x] \);

\[ P(S) \xrightarrow{\sim} Q[S] \ast_Q Q[x] \]

iii) The coproduct \( \delta \). Let us define a coproduct

\[ \delta : P(S) \rightarrow P(S) \otimes_K P(S) \]

such that

\[ \delta : P(S)_{a,b} \mapsto P(S)_{a,b} \otimes P(S)_{a,b} \]  

(31)

by the formula

\[ \delta(p_{a,x_1,...,x_n,b}) := \sum p_{a,x_1,...,x_k,b} \otimes p_{a,x_{j_1},...,x_{j_{n-k}},b} \]  

(32)

where \( n \geq 0 \) and the sum is over all decompositions

\[ \{1, ..., n\} = \{i_1, ..., i_k\} \cup \{j_1, ..., j_{n-k}\}, \quad i_1 < ... < i_k; \quad j_1 < ... < j_{n-k} \]

It is easy to see that it is cocommutative and coassociative.

The compatibilities. The expression \( A \ast B \circ C \ast D \circ E \) does not depend on the bracketing.

\( \delta \) is an algebra morphism for the \( \circ \)-algebra structure:

\[ \delta(X \circ Y) = \delta(X) \circ \delta(Y) \]  

(33)

The compatibility with the \( \ast \)-structure is this:

\[ \delta(X \ast Y) = \delta(X) \ast_2 \delta(Y); \]  

(34)

where the operation \( \ast_2 \) is defined by

\[ (X_1 \otimes Y_1) \ast_2 (X_2 \otimes Y_2) := (X_1 \ast Y_1) \otimes (X_2 \circ Y_2) + (X_1 \circ Y_1) \otimes (X_2 \ast Y_2) \]

It follows that \( \delta \) is uniquely determined by (33), (34) and

\[ \delta(p_{a,b}) = p_{a,b} \otimes p_{a,b} \]  

(35)
Observe that
\[ \delta(P(S)_{a,b}) = \delta(p_{a,a} \circ P(S) \circ p_{b,b}) \subset (p_{a,a} \otimes p_{a,a}) \circ (P(S) \otimes P(S)) \circ (p_{b,b} \otimes p_{b,b}) = P(S)_{a,b} \otimes P(S)_{a,b} \]

So [33] plus [35] imply [31].

**Remark.** \( \delta(e_0) \neq e_0 \otimes e_0 \). Apart from this, the coproduct \( \delta \) has all the properties of a coproduct in a Hopf \( \circ \)-algebra.

For every \( a \in S \) the \( \circ \)-algebra \( P(S)_{a,a} \) is a cocommutative Hopf algebra with the unit \( p_{a,a} \). It is the universal enveloping algebra of the Lie algebra \( L(S)_a \), reconstructed as the subspace of primitives in \( P(S)_{a,a} \). The Lie algebra \( L(S)_a \) is free with generators labeled by the set \( S \). The canonical Lie algebra isomorphism \( i_{a,b} : L(S)_a \rightarrow L(S)_b \) is given by \( l \mapsto p_{b,a} \circ l \circ p_{a,b} \).

3. **The path algebra as a tensor algebra in a monoidal category.** Let \( S \) be a finite set. Let \( \mathcal{C} \) be a tensor category. Consider the category \( Q(\mathcal{C})(S) \) whose objects

\[ V = \{ V_{i,j} \}, \quad (i, j) \in S \times S \]

are objects \( V_{i,j} \) of \( \mathcal{C} \) labeled by elements of \( S \times S \). Let \( V = \{ V_{i,j} \} \) and \( W = \{ W_{i,j} \} \). Then

\[ \text{Hom}_{Q(\mathcal{C})(S)}(V, W) := \oplus_{(i,j) \in S \times S} \text{Hom}_{\mathcal{C}}(V_{i,j}, W_{i,j}) \]

We define \( V \otimes W \) by setting

\[ (V \otimes W)_{i,j} = \oplus_{k \in S} V_{i,k} \otimes_{\mathcal{C}} W_{k,j} \]

From now on we assume that \( \mathcal{C} \) is the category of \( K \)-vector spaces, and denote by \( Q(S) \) the corresponding monoidal category. There is a unit object \( \mathbb{1} \) defined as

\[ \mathbb{1}_{i,j} := \begin{cases} 0 & : i \neq j; \\ K & : i = j \end{cases} \]

Consider an object \( \mathbb{E}_S \) such that

\[ \dim(\mathbb{E}_S)_{i,j} = 1 \quad \text{for any} \quad (i, j) \in S \times S \]

Let \( T(\mathbb{E}_S) \) be the free associative algebra with unit in the category \( Q(S) \) generated by \( \mathbb{E}_S \).

**Lemma 2.3** \( T(\mathbb{E}_S) \) is isomorphic to the path \( * \)-algebra \( P(S)(1) \).

**Proof.** For any \( i, j \in S \) choose a nonzero vector \( p_{i,j} \) in \( (\mathbb{E}_S)_{i,j} \). Set

\[ p_{i_0,\ldots,i_n} := p_{i_0,i_1} \otimes p_{i_1,i_2} \otimes \ldots \otimes p_{i_{n-1},i_n} \quad (36) \]

Then

\[ T(\mathbb{E}_S) = \bigoplus_{n \geq 0} \mathbb{E}_S^n = \mathbb{1} \oplus \bigoplus_{n \geq 1} \bigoplus_{i_0,\ldots,i_n \in S} K \cdot p_{i_0,\ldots,i_n} \]

Writing \( \mathbb{1}_{i,i} = K \cdot p_i \) we get the needed isomorphism. The lemma is proved.

Below we identify \( p_{i,j} \) with the object of \( Q(S) \) whose \((i',j')\)-component is zero unless \( i = i' , j = j' \), when it is \( K \). Then we have the analogous objects to [36] in the category \( Q(S) \).

There is the second product

\[ \circ : T(\mathbb{E}_S) \otimes T(\mathbb{E}_S) \rightarrow T(\mathbb{E}_S); \quad \text{deg}(\circ) = -2 \]

4. **The pro–unipotent algebraic group scheme \( G(S) \).**
Definition 2.4  
a) $\text{Aut}_0(P(S))$ is the pro–unipotent group scheme of automorphisms of the $*$–algebra $P(S)$ which act as the identity on the quotient

$$P_+(S)/(P_+(S))^2$$  \hspace{1cm} (37)

b) $G(S) \subset \text{Aut}_0(P(S))$ is the subgroup of all automorphisms $F$ such that:

i) $F$ commutes with $\delta$.

ii) $F$ is an automorphism of the $\circ$–algebra structure.

The grading of $P(S)$ provides natural gradings of the algebras of regular functions on $G(S)$ and $\text{Aut}_0(P(S))$.

Theorem 2.5  
a) The commutative, graded algebra $\mathcal{O}\left(\text{Aut}_0(P(S))\right)$ is identified with the polynomial algebra in an infinite number of variables $I_{s_0,\ldots,s_{n+1}}$ with $n \geq 0$ and $s_i \in S$:

$$\mathcal{O}\left(\text{Aut}_0(P(S))\right) = K[I_{s_0,\ldots,s_{n+1}}]; \quad \deg(I_{s_0,\ldots,s_{n+1}}) = -2n$$  \hspace{1cm} (38)

b) The identification in a) provides an isomorphism of commutative, graded Hopf algebras

$$\mathcal{O}\left(\text{Aut}_0(P(S))\right) \xrightarrow{\sim} \mathcal{I}_\bullet(S)$$  \hspace{1cm} (39)

c) The isomorphism (38) induces an isomorphism of commutative, graded Hopf algebras

$$\mathcal{O}(G(S)) \xrightarrow{\sim} \mathcal{I}_\bullet(S)$$  \hspace{1cm} (40)

Proof. We consider first the case when $S$ is finite, and then take the inductive limit over finite subsets of $S$.

An automorphism $F$ of $P(S)$ satisfying (29), being a $*$–algebra automorphism, is uniquely determined by its values on the generators $p_{a,b}$. Indeed, condition (29) plus Lemma 2.3 imply that it is an automorphism of the free $*$–algebra in the monoidal category $Q(S)$ generated by the elements $p_{a,b}$.

Let us write

$$F(p_{a,b}) = p_{a,b} + \sum I_{a,s_1,\ldots,s_m,b}(F) \cdot p_{a,s_1,\ldots,s_m,b}$$

where the summation is over all nonempty ordered collections of elements $s_1,\ldots,s_m$ of $S$. Here $I_{a,s_1,\ldots,s_m,b}(F) \in K$ are the coefficients, providing a regular function

$$I_{a,s_1,\ldots,s_m,b} \in \mathcal{O}(\text{Aut}_0(P(S)))$$

Observe that $I_{a,b}(F) = 1$ just means that $F$ acts as the identity on $[S]$. We claim that the map

$$I_{a,s_1,\ldots,s_m,b} \rightarrow \mathbb{I}(a; s_1,\ldots,s_m; b)$$

provides the isomorphism (38). Indeed, it is obviously an isomorphism of commutative, graded $K$–algebras. So we get a).

c) The condition that $F$ commutes with the $\circ$–product is equivalent to iii) plus iv) in the definition of the Hopf algebra $\mathcal{I}_\bullet(S)$. Indeed, the path composition formula iii) is equivalent to $F(p_{a,b}) = F(p_{a,x}) \circ F(p_{x,b})$. The $\circ$–unit is given by $\sum_{i \in S} p_{i,i}$. The condition iv) just means that $F$ preserves the $\circ$–unit: $F(\sum p_{i,i}) = \sum p_{i,i}$.
Given that \( F \) preserves both products, the fact that \( F \) commutes with \( \delta \) is equivalent to the shuffle product formula ii). Indeed, the condition
\[
\delta F(p_{a,b}) = F(\pi_{a,b}) \quad \text{is just equivalent to the shuffle product formula. Since } \delta \text{ is completely determined by } \pi \text{ and compatibilities with } *, \text{ the statement follows.}
\]
The part c) and hence the theorem are proved.

Proposition 2.2 follows from Theorem 2.5.

3 The motivic fundamental groupoid and its Galois group

In this chapter we show that the basic properties of the motivic fundamental groupoid \( \mathcal{P}^\mathbb{C}(\mathbb{A}_S^1; S) \) in one of the mixed Tate categories \( \mathcal{C} \) described in s. 3 imply that the canonical fiber functor on \( \mathcal{C} \) sends it to the path algebra \( P(S) \). This immediately implies Theorem 1.2.

1. The Betti realization of the fundamental groupoid. Let \( S \) be a subset of \( \mathbb{C} \). Let \( t \) be a standard coordinate on \( \mathbb{C} \). Choose a tangent vector \( v_s \) at every \( s \in S \) such that \( dt(v_s) = 1 \).

For \( a \in S \), let \( \mathcal{C}_S := \mathbb{C} \setminus S \). Let us recall the pro-nilpotent completion of the topological torsors of paths associated with \( \mathbb{C}_S \). Let \( I_a \) be the augmentation ideal of the group ring \( \mathbb{Z}[\pi_1(\mathbb{C}_S; a)] \) of the topological fundamental group \( \pi_1(\mathbb{C}_S; a) \). Denote by \( \mathcal{P}(\mathbb{C}_S; a, b) \), where \( a, b \in S \), the free abelian group generated by the set of homotopy classes of paths between the tangential base points \( v_a, v_b \).

Then
\[
\pi_1^{\mathbb{B}}(\mathbb{C}_S; a) := \overline{\lim} \mathbb{Z}[\pi_1(\mathbb{C}_S; a)]/I_a^n; \quad \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, b) := \overline{\lim} \mathcal{P}(\mathbb{C}_S; a, b)
\]

There are the path composition morphisms: for any \( a, b, c \in S \)
\[
\circ : \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, b) \otimes \pi_1^{\mathbb{B}}(\mathbb{C}_S; b, c) \rightarrow \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, c)
\]

They provide \( \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, b) \) with the structure of a principal homogeneous space over \( \pi_1^{\mathbb{B}}(\mathbb{C}_S; a) \). There is a coproduct map
\[
\delta : \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, b) \rightarrow \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, b) \otimes \pi_1^{\mathbb{B}}(\mathbb{C}_S; a, b)
\]

provided by the map given on the generators by \( \gamma \mapsto \gamma \otimes \gamma \). It is obviously compatible with the composition of paths. It makes \( \pi_1^{\mathbb{B}}(\mathbb{C}_S; a) \) into a cocommutative Hopf algebra.

We have an increasing filtration \( W_\bullet \) of \( \mathcal{P}(\mathbb{C}_S; a, b) \) indexed by \( \mathbb{Z} = \{0, -2, -4, \ldots\} \):
\[
W_{-2n}\mathcal{P}(\mathbb{C}_S; a, b) := I_a^n \circ \mathcal{P}(\mathbb{C}_S; a, b)
\]

There are canonical isomorphisms
\[
\text{gr}_0 W_\mathcal{P}(\mathbb{C}_S; a, b) = \mathbb{Z}(0); \quad \text{gr}_2 W_\mathcal{P}(\mathbb{C}_S; a, b) = H_1(\mathbb{C}_S, \mathbb{Z}) = \mathbb{Z}[S]
\]

Denote by \( p_{a,b} \) the canonical generator of \( \text{gr}_0 W_\mathcal{P}(\mathbb{C}_S; a, b) \). The second isomorphism is provided by the map
\[
s \in S \mapsto p_{a,s,b} := p_{a,s} \circ ([\gamma_s] - [1]) \circ p_{b,b} \in \text{gr}_2 W_\mathcal{P}(\mathbb{C}_S; a, b)
\]
where \( \gamma_s \) is a simple loop around \( s \) based at \( v_s \), and 1 is the identity loop at \( v_s \), see the left picture below.

Set
\[
\mathcal{P}(\mathbb{C}_S; S) := \bigoplus_{a,b \in S} \mathcal{P}(\mathbb{C}_S; a, b)
\]

The \( * \)-algebra structure on \( \mathcal{P}(\mathbb{C}_S; S)(1) \) is given by
\[
\alpha_{a,b} * \alpha_{b,c} := \alpha_{a,b} \circ ([\gamma_b] - [1]) \circ \alpha_{b,c}
\]
Lemma 3.1 There is a canonical isomorphism

\[ P(S) \xrightarrow{\sim} \text{gr}^W \mathcal{P}^{\text{nil}}(\mathbb{C}S; S) \]

It respects the grading, the \( \ast \)-algebra and \( \circ \)-algebra structures and the coproduct \( \delta \).

Proof. The isomorphism is given by

\[ p_{a,s_1,\ldots,s_n,b} \mapsto p_{a,s_1} \circ ([\gamma_{s_1}] - [1]) \circ p_{s_1,s_2} \circ ([\gamma_{s_2}] - [1]) \circ \ldots \circ p_{s_n,b} \]

The picture below illustrates this formula in the case when \( n = 1 \) and \( n = 2 \).

The proof follows easily from the fact that, if \( S \) is finite, \( \pi_1(\mathbb{C}S; a) \) is a free group with \( |S| \) generators, and hence \( \pi_1^{\text{nil}}(\mathbb{C}S; a) \) is the completion of the tensor algebra generated by the elements \( p_{a,s,a}, s \in S \). The lemma is proved.

Now let us discuss the motivic fundamental groupoids.

2. The set–up. Let \( F \) be a field. In the rest of this Chapter we work in one of the following categories \( \mathcal{C} \). Their properties and the corresponding formalism are presented in the Appendix.

i) the abelian category of mixed Tate motives over a number field \( F \) ([L1], [G5], [DG]).

ii) \( F = \mathbb{C} \), and \( \mathcal{C} \) is the category of Hodge-Tate structures. (Appendix, s. 3.2).

iii) \( F \) is an arbitrary field such that \( \mu_{\infty} \notin F \), and \( \mathcal{C} \) is the mixed Tate category of \( l \)-adic Tate \( \text{Gal}(\overline{F}/F) \)-modules. (Appendix, s. 3.3, and the references there).

There is also one hypothetical set–up:

iv) \( F \) is an arbitrary field, \( \mathcal{C} \) is the hypothetical abelian category of mixed Tate motives over \( F \).

Any category \( \mathcal{C} \) from the list above is a mixed Tate \( K \)-category, where \( K = \mathbb{Q} \) in i), ii) and \( K = \mathbb{Q}_l \) in iii), see [BD] or the Appendix, subsections 1 and 2, for the background. Let us briefly recall the main features which will be used.

\( \mathcal{C} \) is generated as a tensor category by a simple object \( K(1) \). Each object carries a canonical weight filtration \( W_\bullet \), and morphisms in \( \mathcal{C} \) are strictly compatible with this filtration. There is a canonical fiber functor to the category of finite dimensional graded vector spaces:

\[ \omega : \mathcal{C} \rightarrow \text{Vect}_\bullet, \quad X \mapsto \bigoplus_n H\text{om}_\mathcal{C}(K(-n), \text{gr}^W_n X) \]

Forgetting the grading we get a fiber functor \( \tilde{\omega} \). The space \( \text{End}(\tilde{\omega}) \) of its endomorphisms is a graded Hopf algebra. Its graded dual is canonically isomorphic to the commutative graded Hopf algebra \( \mathcal{A}_\bullet(\mathcal{C}) \) of the framed objects in \( \mathcal{C} \)

\[ \mathcal{A}_\bullet(\mathcal{C}) \cong \text{graded dual } \text{End}(\tilde{\omega})^\vee, \]

see Theorem 8.2 in the Appendix. The functor \( \omega \) provides a canonical equivalence between the category \( \mathcal{C} \) and the category of finite dimensional graded \( \mathcal{A}_\bullet(\mathcal{C}) \)-comodules. \( \text{Spec}(\mathcal{A}_\bullet(\mathcal{C})) \) is a pro–unipotent group scheme. The grading on \( \mathcal{A}_\bullet(\mathcal{C}) \) encodes a natural semidirect product of \( \mathbb{G}_m \) and \( \text{Spec}(\mathcal{A}_\bullet(\mathcal{C})) \).

3. The fundamental groupoid. Let \( S \) be any subset of \( F = \mathbb{A}^1(F) \). A particular interesting case is \( S = F \). Choose a tangent vector \( v_s \) at every point \( s \in S \). We assume \( v_s \) is defined over \( F \). The differential \( dt \) provides a canonical choice of \( v_s \) for all \( s \in F \) which is used below.

Let \( \mathcal{P}^\mathcal{C}(\mathbb{A}^1_S; S) \) be the fundamental groupoid of paths on \( \mathbb{A}^1_S := \mathbb{A}^1 - S \) between the tangential base points \( v_s \). It is a pro–object in \( \mathcal{C} \).
The fundamental groupoid in the situations ii) and iii) above was defined by Deligne [D]. In the situation i) it is defined in [DG] (another construction of the same object is given in Chapter 6 of [G8]). In particular there are path composition morphisms in \( \mathcal{C} \)

\[
\circ : \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b) \otimes \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; b, c) \longrightarrow \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, c)
\]

(40)

They provide \( \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b) \) with a structure of principal homogeneous space over the fundamental group \( \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, a) \), understood as a Hopf algebra in \( \mathcal{C} \).

4. The structures on the fundamental groupoid. Let us assume that \( S \) is finite. Recall the monoidal category \( Q_\mathcal{C}(S) \), see s. 2.3. We define an object \( \mathcal{P}^\mathcal{C}(S) \) in \( Q_\mathcal{C}(S) \) by

\[
\mathcal{P}^\mathcal{C}(S)_{a, b} := \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b)
\]

There are the following structures on this object ([DG]).

i) The path composition morphisms \( \circ \) provide \( \mathcal{P}^\mathcal{C}(S) \) with the structure of an algebra in \( Q_\mathcal{C}(S) \), called the \( \circ \)-algebra structure.

ii) There are canonical “loop around \( s \)” morphisms

\[
\gamma_s : \mathbb{Q}(1) \longrightarrow \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; s, s)
\]

They provide morphisms

\[
* : \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b)(1) \otimes \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; b, c)(1) \longrightarrow \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, c)(1)
\]

\[
\alpha_{a,b} \ast \alpha_{b,c} := \alpha_{a,b} \circ \gamma_b \circ \alpha_{b,c}
\]

These morphisms make \( \mathcal{P}^\mathcal{C}(S)(1) \) into an algebra in the category \( Q_\mathcal{C}(S) \). We call it the \( \ast \)-algebra structure.

iii) For any \( a, b \in S \) there is a coproduct given by a morphism in \( \mathcal{C} \)

\[
\delta : \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b) \longrightarrow \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b) \otimes \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b)
\]

It is a \( \circ \)-algebra morphism. One has

\[
\delta(\gamma_b) = \gamma_b \otimes 1 + 1 \otimes \gamma_b;
\]

(41)

It follows from this that the compatibility of \( \delta \) with the \( \ast \)-algebra product is given by [DG].

5. The path algebra provided by the fundamental groupoid and \( G_\mathcal{C}(S) \). Let us apply the fiber functor \( \omega \) to \( \mathcal{P}^\mathcal{C}(S) \), getting

\[
\mathcal{P}_\omega(S) := \omega(\mathcal{P}^\mathcal{C}(S))
\]

Remark. Since \( \mathcal{P}^\mathcal{C}(S) \) is a pro-object in \( \mathcal{C} \), \( \mathcal{P}_\omega(S) \) is a projective limit of finite dimensional \( K \)-vector spaces. However its graded components are finite dimensional. Denote by \( \mathcal{P}^\mathcal{C}_{gr}(S) \) the direct sum of the graded components of \( \mathcal{P}_\omega(S) \).

Then \( \mathcal{P}^\mathcal{C}_{gr}(S) \) is an algebra in the monoidal category \( Q_{\mathcal{A}_\bullet(c) \text{--mod}}(S) \), and \( \mathcal{P}^\mathcal{C}_{gr}(S)(1) \) is a \( \ast \)-algebra in the same category.

The space

\[
\text{Hom}(K(0), \text{gr}^W \mathcal{P}^\mathcal{C}(\mathbb{A}_S^1; a, b))
\]

is one-dimensional. It has a natural generator \( p_{a,b} \) such that

\[
\delta(p_{a,b}) = p_{a,b} \otimes p_{a,b}
\]

(42)

Set

\[
p_{a,s_1,\ldots,s_m,b} := p_{a,s_1} \ast p_{s_1,s_2} \ast \cdots \ast p_{s_{m-1},s_m} \ast p_{s_m,b}
\]

(43)

It follows from [11] and [12] that \( \delta \) is given by formula [12].
Proposition 3.2 There is a natural isomorphism of $K$-vector spaces

$$\mathcal{P}^\oplus_S(S) \rightarrow P(S)$$

respecting the grading, the $*$-algebra and $\circ$-algebra structures, and the coproduct on both objects.

Proof. The statement boils down to the fact that the elements \(\{a_1, \ldots, a_m\}\), when the set \(\{s_1, \ldots, s_m\}\) runs through all elements of \(S^m\), form a basis in \(\mathcal{P}_{2m}^\oplus(\mathbb{A}_S^1; a, b)\). Since \(\mathcal{P}_{\text{nll}}(\mathbb{C}_S; S)\) is the Betti realization of the Hodge version \(\mathcal{P}^H(\mathbb{A}_S^1; S)\), there is an isomorphism

$$\mathcal{P}_{\omega}(S) \cong \mathcal{P}_{\text{nll}}(\mathbb{C}_S; S)$$

This plus Lemma 3.1 implies the statement in the Hodge setting. Hence (by the injectivity of the regulators, see, for example, \cite{DG}) it is true in the motivic situation i). The $l$-adic case is handled similarly using the comparison theorem with the Betti realization. The proposition is proved.

We define an element

$$\Gamma^C(a_0; a_1, a_2, \ldots, a_m; a_{m+1}) \in \mathcal{A}_m(C)$$

as the linear functional on \(\text{End}(\omega)\) given by the matrix element

$$F \in \text{End}(\omega) \mapsto \langle F(p_{a,b}^*), p_{a,s_1,\ldots,s_m;b} \rangle$$

One can describe the elements of \(\mathcal{A}_m(C)\) by framed objects in \(C\), see Subsection 2 of the Appendix. Then element \(43\) is represented by the following framed object in \(C\):

$$\left(\mathcal{P}^C(\mathbb{A}_S^1; a, b); p_{a,b}, p^*_{a,s_1,\ldots,s_m;b}\right)$$

Here \(p^*_{a}\) are the elements of the basis dual to \(p_{a}\).

Recall the group \(G(S)\) defined in Chapter 2. The group scheme \(\text{Spec}(\mathcal{A}_m(C))\) acts on \(\mathcal{P}_{\omega}(S)\) through its quotient, denoted \(G_C(S)\). The semidirect product of \(\mathbb{G}_m\) and \(G_C(S)\) is called the Galois group of the fundamental groupoid \(\mathcal{P}^C(\mathbb{A}_S^1; S)\).

Theorem 3.3 a) \(G(S)\) is the group of all automorphisms of the $*$-algebra \(\mathcal{P}^\oplus_S(1)\) in the monoidal category \(\mathcal{Q}_{\mathcal{A}_m(C) - \text{mod}}(S)\) preserving the $\circ$-algebra structure on \(\mathcal{P}_{\omega}^\oplus(S)\), commuting with the coproduct \(\delta\) and acting as the identity on

$$\mathcal{P}_{\omega}(S)(1)/(\mathcal{P}_{\omega}(S)(1))^2$$

b) The map

$$I(a_0; a_1, a_2, \ldots, a_m; a_{m+1}) \rightarrow \Gamma^C(a_0; a_1, a_2, \ldots, a_m; a_{m+1})$$

provides a surjective morphism of Hopf algebras \(O(G(S)) \rightarrow O(G_C(S))\), and hence an inclusion of pro-unipotent group schemes

$$G_C(S) \hookrightarrow G(S)$$

c) The coproduct is computed by the formula

$$\Delta \Gamma^C(a_0; a_1, a_2, \ldots, a_m; a_{m+1}) =$$

$$\sum_{0=i_0<i_1<\ldots<i_k<i_{k+1}=m+1} \Gamma^C(a_0; a_{i_1}, \ldots, a_{i_k}; a_{m+1}) \otimes \prod_{p=0}^{k} \Gamma^C(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})$$
Proof. Part c) means simply that (46) commutes with the coproduct. Part b) follows immediately from a). And a) is a direct consequence of Theorem 2.5 and Proposition 3.2. The theorem is proved.

Example. When \( m = 3 \) the formula (47) gives

\[
\Delta I^c(a_0; a_1, a_2, a_3; a_4) = 1 \otimes I^c(a_0; a_1, a_2, a_3; a_4) + \\
I^c(a_0; a_1; a_4) \otimes I^c(a_1; a_2, a_3; a_4) + \\
I^c(a_0; a_2; a_4) \otimes I^c(a_0; a_1; a_2) \cdot I^c(a_2; a_3; a_4) + \\
I^c(a_0; a_3; a_4) \otimes I^c(a_0; a_1, a_2; a_3) + I^c(a_0; a_1, a_2; a_4) \otimes I^c(a_2; a_3; a_4) + \\
I^c(a_0; a_1, a_3; a_4) \otimes I^c(a_1; a_2; a_3) + I^c(a_0; a_2, a_3; a_4) \otimes I^c(a_0; a_1; a_2) + \\
I^c(a_0; a_1, a_2, a_3; a_4) \otimes 1 
\]

There is an equivalent version of Theorem 3.3a):

\[ G(S) \] is the group of all automorphisms of the \( \circ \)–algebra \( P^\infty(S) \) in the monoidal category \( Q_{\mathbb{A}^*} - \text{mod}(S) \) preserving the elements \( \gamma_s \), the coproduct \( \delta \), and acting as the identity on (45).

Indeed, the \( * \)–product is determined by the \( \circ \)–product and the elements \( \gamma_s \).

4 Iterated integrals and plane trivalent rooted trees

In this chapter we show how the Hopf algebra of decorated rooted plane trivalent trees encodes the properties of the motivic iterated integrals.

1. Terminology. A tree can have two types of edges: internal edges and legs. Both vertices of an internal edge are of valency \( \geq 3 \). A valency 1 vertex of a leg is called an external vertex.

We may picture plane trees inscribed into a circle such that the external vertices divide this circle into a union of arcs. Let \( S \) be a set. An \( S \)–decoration of a plane tree is an \( S \)–valued function on the set of open arcs.

A rooted tree is a tree with one distinguished leg called the root. We picture plane rooted trees growing down from the root, and put the external vertices of all the legs but the root on a line. These vertices divide this line into a union of arcs. They correspond to the arcs defined above, so we can talk about \( S \)–decorated plane rooted trees.

We say that a plane rooted tree is decorated by an ordered set \( \{ s_0, s_1, \ldots, s_m, s_{m+1} \} \) if the tree has \( m + 1 \) bottom legs, and the corresponding \( m + 2 \) arcs are decorated, in their natural order from left to right, by the set above. We decorate the arcs, not legs – see however the remark after Theorem 4.2.

2. The Hopf algebra \( T_*(S) \) of \( S \)–decorated plane rooted trivalent trees. It is a commutative, graded Hopf algebra. As a vector space it is generated by disjoint unions of \( S \)–decorated plane rooted trivalent trees. More precisely, consider a graded vector space with a basis given by \( S \)–decorated plane rooted trivalent trees with \( n + 2 \) arcs, where \( n \geq 1 \) provides the grading. The vector space \( T_*(S) \) is its symmetric algebra. In other words, \( T_*(S) \) is the commutative algebra generated
by $S$–decorated plane rooted trivalent trees with the only relation that a decorated plane tree with just one edge equals 1 (regardless of the decoration).

Let us define the coproduct $\Delta_T$ on $\mathcal{T}_*(S)$. Since $\Delta_T$ has to be an algebra morphism, we have to define it only on the generators. Let $T$ be an $S$–decorated plane rooted trivalent tree. Consider the set

$$\mathcal{E}_T := \{ \text{all internal edges of } T \} \cup \{ \text{the root of } T \}$$

An element $E \in \mathcal{E}_T$ determines a plane trivalent rooted tree $T_E$ growing down from $E$: the edge $E$ serves as the root of this tree. The tree $T_E$ inherits a natural $S$–decoration: take the arcs containing the endpoints of the tree $T_E$ and keep their decorations. A subset $\{E_1, ..., E_k\}$ of $\mathcal{E}_T$ is called admissible if for any $i \neq j$ the edges $E_i$ and $E_j$ do not lie on the same path going down from the root. In other words, $E_i$ is not contained in the tree $T_{E_j}$ if $i \neq j$. Such an admissible subset determines a connected $S$–decorated plane trivalent tree $T/(T_{E_1} \cup ... \cup T_{E_k})$. Namely, this tree is obtained by shrinking each of the rooted trees $T_{E_1}, ..., T_{E_k}$ into a leg of a new tree: we shrink the domain encompassed by each of these trees and the bottom line. In particular we shrink to points all the arcs on the bottom line located under these trees. The remaining arcs with the inherited decorations are the arcs of the new tree. See the picture where we shrink the two trees under the thick edges:

Now we set, for a $T$ as above,

$$\Delta_T(T) := \sum_{T_{E_1} \cup ... \cup T_{E_k}} \frac{T}{T_{E_1} \cup ... \cup T_{E_k}} \otimes \prod_{i=1}^{k} T_{E_i}$$

where the sum is over all admissible subsets of $\mathcal{E}(T)$, including the empty subset and the subset formed by the root.

**Lemma 4.1** $\Delta_T$ provides $\mathcal{T}_*(S)$ with the structure of a commutative, graded Hopf algebra.

**Proof.** Left to the reader as an easy exercise. It is essentially the same as in [CK].

**Remark.** The Hopf algebra $\mathcal{T}_*(S)$ is a Hopf subalgebra of the similar Hopf algebra of all decorated rooted plane trees (not necessarily trivalent). The latter is the plane decorated version of the one defined by Connes and Kreimer [CK]. On the other hand, as was pointed out to me by J.-L. Loday, the Hopf algebra $\mathcal{T}_*(S)$ is closely related to the structures studied in [Lo1], [Lo2].

**Theorem 4.2** The map

$$t : I(s_0; s_1, ..., s_m; s_{m+1}) \mapsto \text{sum of all plane rooted trivalent trees}$$

decorated by the ordered set $\{s_0, s_1, ..., s_m, s_{m+1}\}$

provides an injective homomorphism of commutative, graded Hopf algebras

$$t : \mathcal{I}_*(S) \hookrightarrow \mathcal{T}_*(S) \quad (48)$$
Theorem. The element $I^C(s_0; s_1, ..., s_m; s_{m+1})$ is invariant under the action of the translation group of the affine line. Therefore it is natural to encode it by orbits of the action of the additive group $A^1$ on the set of $(m+2)$-tuples $\{s_0, s_1, ..., s_m, s_{m+1}\}$. These orbits are described by the $(m+2)$-tuples

$$\{s_0 - s_{m+1}, s_1 - s_0, s_2 - s_1, ..., s_{m+1} - s_m\}$$

which naturally sit not at the arcs, but rather at the legs of the corresponding trees.

3. The $\otimes^m$-invariant of variations of mixed Tate structures ([G5], s. 5.1). Let $V$ be a variation of mixed Tate objects over a smooth base $B$ in one of our set-ups ii) – iv). So $V$ is a unipotent variation of Hodge–Tate structures in ii), a lisse $l$-adic mixed Tate sheaf in iii), and a (yet hypothetical) variation of mixed Tate motives in iv). We assume that $\mu_{p^\infty} \not\subset O^*(B)$ in iii).

Then $V$ is an object of an appropriate mixed Tate category $C_B$ of variations of mixed Tate objects over $B$. The Tate object $K(m)_B$ is given by $K(m)_B := p^*K(m)$ where $p : B \to \text{Spec}(F)$ is the structure morphism. We can use the standard formalism of mixed Tate categories (see [BD] or Appendix). In particular there is a commutative, graded Hopf $K$-algebra $A^C_*(B)$ with a coproduct $\Delta$, called the fundamental Hopf algebra of this category. One has

$$A^C_1(B) = O^*(B)_K := O^*(B) \otimes_{\mathbb{Q}} K$$
For any positively graded (coassociative) Hopf algebra $A$, there is a canonical map

$$\Delta^{[m]} : A_m \longrightarrow \otimes^m A_1$$

Namely, it is dual to the multiplication map $\otimes^m A_1^\vee \longrightarrow A_m^\vee$, and can be defined as the composition

$$A_m \xrightarrow{\Delta} A_{m-1} \otimes A_1 \xrightarrow{\Delta \otimes \text{Id}} A_{m-2} \otimes A_1 \otimes A_1 \xrightarrow{\Delta \otimes \text{Id} \otimes \text{Id}} \ldots \xrightarrow{\Delta \otimes \text{Id} \otimes \text{Id} \otimes \text{Id}} \otimes^m A_1,$$

Let us choose a $K(m)_B$-framing of the variation $\mathcal{V}$. Then, by the very definition, we get an element

$$[\mathcal{V}] \in A_m^c(B)$$

**Definition 4.3** The element

$$\Delta^{[m]}([\mathcal{V}]) \in \otimes^m \mathcal{O}^*(B)_K$$

is called the $\otimes^m$-invariant of a framed variation of mixed Tate objects over $B$.

**4. The $\otimes^m$-invariant of the multiple logarithm variation and plane rooted trivalent trees.** Let $\mathcal{M}_{m+2}(A^1)$ be the space of $m + 2$ ordered distinct points $(a_0, ..., a_{m+1})$ on the affine line over a field. Every internal vertex $v$ of a plane trivalent rooted tree $T$ provides an invertible function $f_v^T$ on $\mathcal{M}_{m+2}(A^1)$. Indeed, if we picture a plane trivalent tree inscribed into a circle such that the external vertices lie on this circle, the complement to the tree in the disc bounded by this circle is a union of several connected domains. These domains are in bijective correspondence with the arcs on the circle defined by the tree.

An internal vertex $v$ determines three such domains $D_v^1, D_v^2, D_v^3$, so that $v$ shares the boundaries of these domains. The plane structure of the tree plus the orientation of the plane provide a cyclic order of these domains, and the root of the tree provides a natural order of these domains. Namely, among the edges sharing the vertex $v$ one edge is closer to the root than the others. We count the domains going counterclockwise from this edge. We will assume that the enumeration $D_v^1, D_v^2, D_v^3$ reflects this order. Let $a_v^1, a_v^2, a_v^3$ be the labels of the arcs assigned to the domains $D_v^1, D_v^2, D_v^3$. Set

$$f_v^T := \frac{a_v^3 - a_v^2}{a_v^1 - a_v^2} \in \mathcal{O}^*(\mathcal{M}_{m+2}(A^1))$$

The function corresponding to an internal vertex of decorated rooted plane trivalent tree

On the internal vertices of a plane trivalent rooted tree we have a *canonical partial order* $\prec$: we have $v_1 \prec v_2$ if and only if there is a path going down from the root such that both vertices are located on this path, and $v_1$ is closer to the root than $v_2$. We say that an ordering $(v_1, ..., v_m)$ of internal vertices $v_i$ of a plane rooted tree $T$ is *compatible* with the canonical order if $v_i \prec v_j$ implies $i < j$.

**Definition 4.4**

$$\Omega_m := \sum_T \sum_{\{v_1, ..., v_m\}} f_{v_1}^T \otimes ... \otimes f_{v_m}^T \in \otimes^m \mathcal{O}^*(\mathcal{M}_{m+2}(A^1))$$

22
Here the first sum is over all different plane trivalent rooted trees $T$ with $m+1$ leaves, and the second sum is over all orderings $\{v_1, ..., v_m\}$ of the set of internal vertices of the tree $T$ compatible with the canonical partial order on this set.

**Proposition 4.5** In the notations as above, we have

$$\Omega_m = \Delta^{[m]} t^C (a_0; a_1, ..., a_m; a_{m+1})$$

**Proof.** Follows easily from formula (47) by induction on $m$.

5 The motivic multiple polylogarithm Hopf algebra

In this section we use formula (47) to calculate explicitly the coproduct for the motivic multiple polylogarithms. Then we define the cyclotomic Hopf and Lie algebras.

1. The basic formula for the coproduct. In this section we work with the Hopf algebra $I_\bullet(S)$ where $S$ is any set containing a distinguished element $0$. Set, for $m \geq 0$ and $a_i \in S$,

$$I_{n_0, n_1 + 1, ..., n_m + 1}(a_0; a_1, ..., a_m; a_{m+1}) := \sum_{n_i \geq 0} \Delta \left( \sum_{n_i \geq 0} I_{n_0, n_1 + 1, ..., n_m + 1}(a_0; a_1, ..., a_m; a_{m+1}) t_0^{n_0} ... t_m^{n_m} \right)$$

Let $V$ be a vector space. Denote by $V[[t_1, ..., t_m]]$ the vector space of formal power series in the variables $t_i$ whose coefficients are vectors of $V$. Let us form the generating series

$$I(a_0; a_1, ..., a_m; a_{m+1}|t_0; t_1; ...; t_m) := \sum_{n_i \geq 0} I_{n_0, n_1 + 1, ..., n_m + 1}(a_0; a_1, ..., a_m; a_{m+1}) t_0^{n_0} ... t_m^{n_m} \in I_\bullet(S)[[t_0, ..., t_m]]$$

To visualize them consider a line segment with the following additional data, called *decoration*:

i) The beginning of the segment is labeled by $a_0$, the end by $a_{m+1}$.

ii) There are $m$ points inside of the segment labeled by $a_1, ..., a_m$ from left to right.

iii) These points cut the segment into $m+1$ arcs labeled by $t_0, t_1, ..., t_m$.

A decorated segment

![Decorated Segment]

**Remark.** The way in which the $t$’s sit between the $a$’s reflects the shape of the iterated integral $I_{n_0, n_1 + 1, ..., n_m + 1}(a_0; a_1, ..., a_m; a_{m+1})$.

Our goal is to calculate the coproduct of the generating series (50). Here, by definition, the coproduct acts on the coefficients of the series, leaving the $t$’s untouched:

$$\Delta \left( \sum_{n_i \geq 0} I_{n_0, n_1 + 1, ..., n_m + 1}(a_0; a_1, ..., a_m; a_{m+1}) t_0^{n_0} ... t_m^{n_m} \right) := \sum_{n_i \geq 0} \Delta \left( I_{n_0, n_1 + 1, ..., n_m + 1}(a_0; a_1, ..., a_m; a_{m+1}) \right) t_0^{n_0} ... t_m^{n_m}$$

As we show in Theorem 5.1 below, the terms of the coproduct of the element (50) correspond to the decorated segments equipped with the following additional data, called *marking*:

![Marked Decorated Segment]

A marked decorated segment
a) Mark (by making them fat points on the picture) points $a_0; a_{i_1}; ..., a_{i_k}; a_{m+1}$ so that

$$0 = i_0 < i_1 < ... < i_k < i_{k+1} = m + 1 \quad (51)$$

b) Mark (by crosses) segments $t_{j_0}, ..., t_{j_k}$ such that there is just one marked segment between any two neighboring marked points.

![Marked decorated segment](image)

The conditions on the crosses for a marked decorated segment just mean that

$$i_\alpha \leq j_\alpha < i_{\alpha + 1} \quad \text{for any } 0 \leq \alpha \leq k \quad (52)$$

The marks provide a new decorated segment:

$$(a_0 | t_{j_0} | a_{i_0} | t_{j_1} | a_{i_1} | ... | a_{i_k} | t_{j_k} | a_{m+1}) \quad (53)$$

**Theorem 5.1** Let $a_i \in S, m \geq 0$. Then

$$\Delta \mathbb{I}(a_0; a_1, ..., a_m; a_{m+1} | t_0; ...; t_m) = \sum \mathbb{I}(a_0; a_{i_1}, ..., a_{i_k}; a_{m+1} | t_{j_0}; t_{j_1}; ...; t_{j_k}) \otimes \prod_{\alpha=0}^{k} \left( \mathbb{I}(a_{i_\alpha}; a_{i_\alpha + 1}; ..., a_{j_\alpha}; 0 | t_{i_\alpha}; ...; t_{j_\alpha}) \cdot \mathbb{I}(0; a_{j_{\alpha + 1}}; ..., a_{i_{\alpha + 1} - 1}; a_{i_{\alpha + 1}} | t_{j_{\alpha}}; t_{j_{\alpha} + 1}; ...; t_{i_{\alpha + 1} - 1}) \right)$$

where the sum is over all marked decorated segments, i.e. over all sequences $\{i_\alpha\}$ and $\{j_\alpha\}$ satisfying inequality (52).

**Proof.** Recall that by iv) in section 2.1 one has

$$\mathbb{I}(0; a_1, ..., a_m; 0) = 0 \quad (55)$$

To calculate (54) we apply the coproduct formula (24) to the element (49) and then keep track of the nonzero terms using (55).

The left hand side factors of the nonzero terms in the formula for the coproduct correspond to certain subsequences

$$A \subset \{a_0; 0, ..., 0; a_1; 0, ..., 0; ...; a_m; 0, ..., 0; a_{m+1}\}$$

containing $a_0$ and $a_{m+1}$, and called the admissible subsequences. Such a subset $A$ determines the subsequences $I = \{i_1 < ... < i_k\}$ where $a_0, a_{i_1}, ..., a_{i_k}, a_{m+1}$ are precisely the set of all $a_i$’s contained in $A$. A subsequence $A$ is called *admissible* if it satisfies the following properties:

i) $A$ contains $a_0$ and $a_{m+1}$.

ii) The sequence of 0’s in $A$ located between $a_{i_\alpha}$ and $a_{i_{\alpha + 1}}$ must be a string of *consecutive* 0’s located between $a_{j_\alpha}$ and $a_{j_{\alpha + 1}}$ for some $i_\alpha \leq j_\alpha < i_{\alpha + 1}$.
In other words, the factors in the coproduct are parametrized by

- a marked decorated segment and a connected string of 0’s in each of the crossed arcs.

The connected string of 0’s in some of the crossed arcs might be empty.

The string of zeros between \( a_{i,\alpha} \) and \( a_{i,\alpha+1} \) satisfying ii) looks as follows:

\[
\{a_{i,\alpha}, \tilde{\alpha}_{j,\alpha}, 0, \ldots, 0, \tilde{\alpha}_{j,\alpha+1}, a_{i,\alpha+1}\}
\]

(56)

where \( p_{j,\alpha} + q_{j,\alpha} + s_{j,\alpha} = n_{j,\alpha} \). This notation emphasizes that all 0’s located between \( a_{i,\alpha} \) and \( a_{i,\alpha+1} \) are in fact located between \( a_{j,\alpha} \) and \( a_{j,\alpha+1} \), and form a connected segment of length \( s_{j,\alpha} \).

An admissible subset \( A \) provides the following element, which is the left hand side of the corresponding term in the coproduct:

\[
\mathbb{I}(a_0; \tilde{\alpha}_{j,\alpha}, 0, \ldots, 0; a_{i,\alpha}, \tilde{\alpha}_{j,\alpha+1}, a_{i,\alpha+1}, 0, \ldots, 0; \tilde{\alpha}_{j,\alpha+1}, \ldots; a_{m+1})
\]

\( s_{j,\alpha} \) times

\( s_{j,\alpha} \) times

The right hand side of the term in the coproduct corresponding to the subset \( A \) is a product over \( 0 \leq \alpha \leq k \) of the following shape:

\[
\mathbb{I}(a_{i,\alpha}; 0, \ldots, 0; a_{i,\alpha+1}; 0, \ldots, 0; \ldots; a_{j,\alpha}; 0, \ldots, 0; \tilde{\alpha}_{j,\alpha+1}; 0, \ldots, 0; a_{j,\alpha+1})
\]

where the middle factor \( \mathbb{I}(0; 0) \cdot \mathbb{I}(0; 0) \) is equal to 1 since \( \mathbb{I}(0; 0) = 1 \) according to (55). Translating this into the language of the generating series we get the promised formula for the coproduct. The case \( s_{j,\alpha} = 0 \) needs a special treatment: the expression (50) must be decomposed using the path composition formula with 0 as one of the endpoints. This produces the correct formula for all admissible \( j_\alpha \). The theorem is proved.

A geometric interpretation of formula (53). It is surprisingly similar to the one for the multiple logarithm element. Recall that the expression (50) is encoded by a decorated segment

\[
(a_0 | t_0 | a_1 | t_1 | \ldots | a_m | t_m | a_{m+1})
\]

(57)

The terms of the coproduct are in bijection with the marked decorated segments \( C \) obtained from a given one (57). Denote by \( L_C \otimes R_C \) the term in the coproduct corresponding to \( C \). The left factor \( L_C \) is encoded by the decorated segment (53) obtained from the marked points and arcs. For example for the marked decorated segment on the picture we get \( L_C = \mathbb{I}(a_0 | t_0 | a_2 | t_3 | a_4 | t_4 | a_6) \).

The marks (which consist of \( k + 1 \) crosses and \( k + 2 \) boldface points) determine a decomposition of the segment (57) into \( 2(k + 1) \) little decorated segments in the following way. Cutting the initial segment in all the marked points and crosses we get \( 2(k + 1) \) little segments. For instance, the very right one is the segment between the last cross and point \( a_m \), and so on. Each of these segments either starts from a marked point and ends by a cross, or starts from a cross and ends at a marked point.
There is a natural way to make a decorated segment out of each of these little segments: mark the "cross endpoint" of the little segment by the point 0, and for the arc which is next to this marked point use the letter originally attached to the arc containing it. For example the marked decorated segment on the picture above produces the following sequence of little decorated segments:

$$(a_0 | t_0 | 0), \ (0 | t_0 | a_1 | t_1 | a_2), \ (a_2 | t_2 | a_3 | t_3 | 0), \ (0 | t_3 | a_4), \ (a_4 | t_4 | 0), \ (0 | t_4 | a_5 | t_5 | a_6)$$

Then the factor $R_C$ is the product of the generating series for the elements corresponding to these little decorated segments. For example for the marked decorated segment on the picture we get

$$R_C = I(a_0; 0 | t_0) \cdot I(0; a_1; a_2 | t_1) \cdot I(a_2; a_3; 0 | t_2; t_3) \cdot I(0; a_4 | t_3) \cdot I(a_4; 0 | t_4) \cdot I(0; a_5; a_6 | t_4; t_5)$$

To check that the formula for the coproduct of the multiple logarithms fits into this description, we use the path composition formula iii) in Section 2.1, provided by Theorem \ref{thm:3.3}, together with the fact that each term on the right hand side of this formula corresponds to a marked colored segment shown in the picture:

\[ \begin{array}{c}
\text{a} \quad \text{p} \quad \text{a} \quad \text{p+k} \quad \times \quad \text{a} \quad \text{p+k+1} \\
\text{a} \quad \text{p+q} \\
\end{array} \]

2. The multiple polylogarithm Hopf algebra of a subgroup $G \subset F^*$. Now let us suppose that we work in one of the set-ups i) - iv), and $F$ is the corresponding field. Let $S := F$. We define the $\mathcal{I}^c$-elements as the images of the corresponding $\mathcal{I}$-elements by the homomorphism established in Theorem \ref{thm:3.3}. Below we choose a coordinate $t$ on the affine line and use the standard tangent vectors $v_s$ dual to $dt$. By Theorem \ref{thm:3.3}, the results of the previous section are valid for the corresponding $\mathcal{I}^c$-elements.

The $\mathcal{I}^c$-analogs of elements \ref{eq:4.9} are clearly invariant under translation. In general they are not invariant under the action of $G_m$ given by $a_i \mapsto \lambda a_i$. However, if the iterated integral $I_{n_0, n_1 + 1, \ldots, n_m + 1}(a_0; a_1, \ldots, a_m; a_{m+1})$ is convergent, then the corresponding element is invariant under the action of the affine group, see Lemma \ref{lem:5.3} below.

Let $a \in F^*$. We set

$$[a] := \log^c(a) := \mathcal{I}^c(0; 0; a) \in \mathcal{A}_1(C)$$

In any of the set-ups i) - iii), and hypothetically in iv), we have an isomorphism $F^* \otimes Q K \cong \mathcal{A}_1(C)$ materialized by the "motivic logarithm":

$$\log^c : F^* \otimes Q K \longrightarrow \text{Ext}^1_C(K(0), K(1)) \cong \mathcal{A}_1(C); \quad a \mapsto \log^c(a)$$

Its $l$-adic realization is given by the Kummer extension, and if $F \subset \mathbb{C}$ its Hodge realization is $\log^H(a)$. See Appendix, subsection 3, especially formula \ref{eq:1.01}. Here it is essential that we use the tangential base point $\partial/\partial t$, where $t$ is the standard parameter on the line. Then the fact that we got the above isomorphism is a basic standard fact in each of the set-ups. One also has $\log^c(a) = \mathcal{I}^c(1; 0; a)$, but we will not use it. Observe that we are working here in realizations, rather then with the formal elements $\mathcal{I}$. Set ($\hat{\otimes}$ stands for the completed tensor product)

$$a^t := \exp(\log^c(a) \cdot t) \in \mathcal{A}_*(C) \otimes k[[t]]$$

The shuffle product formula implies

$$\mathcal{I}^c(0; a | t) = a^t$$

One has

$$\Delta(1) = 1 \otimes 1, \quad \Delta([a]) = [a] \otimes 1 + 1 \otimes [a]$$
It follows that

$$\Delta(a^t) = a^t \otimes a^t$$  \hfill (58)

Suppose that $a_i \neq 0$. Set

$$I^C_{n_1, \ldots, n_m}(a_1, \ldots, a_m) :=
I^C(0; a_1, 0, \ldots, 0, a_2, 0, \ldots, 0, \ldots, a_m, 0, \ldots; 1)
\quad n_1 - 1 \quad n_2 - 1 \quad n_m - 1
$$

Let us package them into the generating series

$$I^C(a_1, \ldots, a_m|t_1, \ldots, t_m) :=
\sum_{n_i \geq 1} I^C_{n_1, \ldots, n_m}(a_1, \ldots, a_m)t_1^{n_1-1} \cdots t_m^{n_m-1} \in \mathcal{A}_\bullet(C)[[t_1, \ldots, t_m]]$$  \hfill (59)

**Lemma 5.2** Suppose that $a_i \neq 0$. Then

$$I^C(0; a_1, \ldots, a_m; a_{m+1}|t_0; \ldots; t_m) =
a_{m+1}^{t_0}I^C(a_1, \ldots, a_m|t_1 - t_0, \ldots, t_m - t_0)$$  \hfill (60)

**Proof.** It follows by induction using the shuffle product formula for

$$\sum_{n_i \geq 1} I^C(0; 0, \ldots, 0; a_{m+1})t_0^n \cdot \sum_{n_i > 0} I^C_{n_1, \ldots, n_m}(0; a_1, \ldots, a_m; a_{m+1})t_1^{n_1-1} \cdots t_m^{n_m-1}$$

just as in the second proof of Proposition 2.15 in [G3]. The lemma is proved.

**Lemma 5.3** One has

$$I^C(a_1; a_2, \ldots, a_m; 0|t_1; \ldots; t_m) =
(-1)^{m-1}I^C(0; a_m, \ldots, a_2; a_1| - t_m; -t_{m-1}; \ldots; -t_1)$$  \hfill (61)

In particular $I^C(a; 0|t) = a^{-t}$.

**Proof.** This is a special case of the equality

$$I^C(a_1; a_2, \ldots, a_m; 0) = (-1)^{m-1}I^C(0; a_m, \ldots, a_2; a_1)$$

provided by Proposition 2.1 and Theorem 3.3. The lemma is proved.

A marked decorated segment is *special* if the first cross is marking the segment $t_0$. A decorated segment with such a data is called a *special marked decorated segment*. See an example on the picture after formula 51. The conditions on the crosses for a marked decorated segment just mean that

$$i_\alpha \leq j_\alpha < i_{\alpha+1} \quad \text{for any } 0 \leq \alpha \leq k, \quad j_0 = i_0 = 0$$  \hfill (62)

We will employ the notation

$$I^C(a_1; \ldots; a_{m+1}|t_0; \ldots; t_m) := I^C(0; a_1, \ldots, a_m; a_{m+1}|t_0; \ldots; t_m)$$
\textbf{Theorem 5.4} One has, for any \(a_i\)’s,
\[
\Delta^C(a_1 : \ldots : a_{m+1}|t_0 : \ldots : t_m) =
\sum I^C(a_{i_1} : \ldots : a_{i_k} : a_{m+1}|t_{j_0} : t_{j_1} : \ldots : t_{j_k}) \otimes 
\prod_{\alpha=0}^{k} \left( (-1)^{j_{\alpha}-i_{\alpha}} I^C(a_{j_\alpha} : a_{j_\alpha-1} : \ldots : a_{i_{\alpha}} - t_{j_\alpha} : -t_{j_{\alpha}-1} : \ldots : -t_{i_{\alpha}}) \right). \tag{63}
\]
where the sum is over all special marked decorated segments, i.e. over all sequences \(\{i_\alpha\}\) and \(\{j_\alpha\}\) such that
\[
i_\alpha \leq j_\alpha < i_{\alpha+1} \quad \text{for any } 0 \leq \alpha \leq k, \quad j_0 = i_0 = 0, i_{k+1} = i_{m+1}\tag{64}
\]
\textbf{Proof.} It follows immediately from Theorem 5.1 using Lemmas 5.2 and 5.3.

Now let \(G\) be a subgroup of \(F^*\). We assume that \(F \subset \mathbb{C}\) in the Hodge case, and \(F\) is the base field otherwise. Denote by \(Z^C_w(G) \subset A_w(\mathcal{C})\) the \(\mathbb{Q}\)-vector subspace generated by the elements
\[
I^C(a_{n_1}, \ldots, a_m), \quad a_i \in G, \quad w = n_1 + \ldots + n_m\tag{65}
\]
Set
\[Z^C_w(G) := \oplus_{u \geq 1} Z^C_w(G)\]

\textbf{Definition 5.5} A depth filtration \(F^D_w\) on the space \(Z^C_w(G)\) is defined as follows:
- \(F^D_0 Z^C_w(G)\) is spanned by products of \(\log^C(a), a \in G\), and
- \(F^D_k Z^C_w(G)\) for \(k \geq 1\) is spanned by the elements \((65)\) with \(m \leq k\).

Since \(\log^C(a) = I^C(0; 0; a)\) is the depth zero multiple polylogarithm, this agrees with this definition.

\textbf{Theorem 5.6} Let \(G\) be any subgroup of \(F^*\). Then \(Z^C_w(G)\) is a graded Hopf subalgebra of \(A_w(\mathcal{C})\). The depth provides a filtration on this Hopf algebra.

\textbf{Proof.} The graded vector space \(Z^C_w(G)\) is closed under the coproduct by Theorem 5.4. The statement about the depth filtration is evident from the formula for the coproduct given in Theorem 5.4. It is a graded algebra by the shuffle product formula. The theorem is proved.

\textbf{Remark.} The depth filtration is not defined by a grading of the algebra \(Z^C_w(G)\) because of relations like
\[
\text{Li}^C_n(x) \cdot \text{Li}^C_m(y) = \text{Li}^C_{n+m}(x, y) + \text{Li}^C_{n,m}(x, y) + \text{Li}^C_{m,n}(x, y).
\]

3. \textbf{An example: the cyclotomic Lie algebras.} Let \(\mu_N\) be the group of \(N\)-th roots of unity. In this subsection we work with the abelian category \(\mathcal{M}_T(\mathbb{Q}(\zeta_N))\) of mixed Tate motives over the cyclotomic field. Thus we use the superscript \(\mathcal{M}\) instead of \(\mathcal{C}\). Set
\[
\mathcal{C}^M_\bullet(\mu_N) := \frac{Z^M_\bullet(\mu_N)}{Z^M_{\geq 0}(\mu_N) \cdot Z^M_{\geq 0}(\mu_N)}.
\]

\textbf{Corollary 5.7} a) \(\mathcal{C}^M_\bullet(\mu_N)\) is a graded Lie coalgebra. The depth provides a filtration on this Lie coalgebra.

b) \(\mathcal{C}^M_1(\mu_N) = Z^M_1(\mu_N) \simeq \left( \text{the group of cyclotomic units in } \mathbb{Z}[\zeta_N][1/N] \right) \otimes \mathbb{Q}.
\]
Proof. a). Follows from theorem \[5.6\]

b) Since

\[ I_i^M(a) = \log^M(1 - a^{-1}) \quad \text{and} \quad \log^M(1 - a) - \log^M(1 - a^{-1}) = \log^M(a) \]

the weight 1 component \( Z_1^M(G) \) is generated by \( \log^M(1 - a) \) and \( \log^M(a) \). Notice that if \( a^N = 1 \) then \( N \cdot \log^M(a) = 0 \). This proves b). The corollary is proved.

We call \( C^M(\mu_N) \) the cyclotomic Lie coalgebra. Its dual \( C^M_*(\mu_N) \) is the cyclotomic Lie algebra. The dual to the universal enveloping algebra of the cyclotomic Lie algebra is isomorphic to \( Z^M_*(\mu_N) \).

Let \( C^M_*(\mu_N) \) be the \( \mathbb{Q} \)-subspace of \( C^M_*(\mu_N) \) generated by all \( F^M(0; a_1, ..., a_m; 1) \) where \( a_i^N = 1 \).

Corollary 5.8 \( C^M_*(\mu_N) := \oplus_{m \geq 1} C^M_*(\mu_N) \) is a graded Lie coalgebra.

Proof. Clear from the Theorem \[5.4\]

The depth filtration on the cyclotomic Lie algebra plays a central role in the mysterious correspondence between the structure of this Lie algebra and the geometry of certain modular varieties for \( GL_m, m = 1, 2, 3, 4, ... \) ([G1-2]). However our definition of the depth filtration depends on the realization of the Hopf algebra \( Z^M_*(G) \) related to the projective line. In the next section we suggest that the depth filtration is induced, via the canonical embedding \( Z^M_*(G) \rightarrow A_*(F) \), by a natural filtration on the motivic Tate Hopf algebra of \( F \), given intrinsically in terms of the corresponding Lie algebra.

4. A hypothetical intrinsic definition of the depth filtration. In this subsection we assume the existence of the abelian category \( M_T(F) \) of mixed Tate motives over an arbitrary field \( F \) with all the standard properties. In particular \( M_T(F) \) is a mixed Tate category, so there are the corresponding fundamental Lie algebra \( L_*(F) \), its graded dual \( L_*(F) \), and \( A_*(F) \) is isomorphic to the universal enveloping algebra of \( L_*(F) \), see the Appendix.

We assume that for any \( a_1, ..., a_m \in F^* \) there exists an object \( I^M_{n_1, ..., n_m}(a_1, ..., a_m) \) of the category \( M_T(F) \) framed by \( \mathbb{Q}(0) \) and \( \mathbb{Q}(w) \), where \( w = n_1 + ... + n_m \), called the motivic multiple polylogarithm, and its coproduct \( \Delta I^M_{n_1, ..., n_m}(a_1, ..., a_m) \) is given by the formula from Theorem \[5.3\]. When \( F \) is a number field all this is already available.

The universality conjecture. It is the Conjecture 17a) in [G6], which tells us that every framed mixed Tate motive over \( F \) is equivalent to a \( \mathbb{Q} \)-linear combination of the ones \( I^M_{n_1, ..., n_m}(a_1, ..., a_m) \) with \( a_i \in F^* \). The universality conjecture can be reformulated as follows:

Conjecture 5.9 The Hopf algebra \( Z^M_*(F^*) \) is isomorphic to the fundamental Hopf algebra \( A_*(F) \) of the abelian category of mixed Tate motives over \( F \).

In other words every framed mixed Tate motive over \( F \) is a \( \mathbb{Q} \)-linear combination of the ones coming from the motivic torsor of paths \( P^M(\mathcal{A}_F; v_0, v_1) \) between the standard tangential base points at 0 and 1.

The main conjecture. Since \( Z^M_*(F^*) \) is a Hopf subalgebra in \( A_*(F) \), we can consider the depth filtration provided by Definition \[5.3\] on the former as a filtration on the latter. It follows from the formula in Theorem \[5.4\] that the depth filtration is a filtration by coideals in \( A_*(F) \). The depth filtration on \( A_*(F) \) induces the depth filtration on the Lie coalgebra \( L_*(F) \). In particular \( F^D_0 L_*(F) = L_1(F) = F^* \otimes \mathbb{Q} \).

Let us define another filtration on the Lie algebra \( L_*(F) \): \( I_*(F) := \oplus_{n=2}^{\infty} L_{-n}(F) \)

Proof: a). Follows from theorem \[5.6\]
We define the \textit{depth filtration} $\tilde{F}^D$ on the Lie algebra $L_\bullet(F)$ as an increasing filtration indexed by integers $m \leq 0$ and given by the powers of the ideal $I_\bullet(F)$

$$
\tilde{F}^D_0 L_\bullet(F) = L_\bullet(F); \quad \tilde{F}^D_1 L_\bullet(F) = I_\bullet(F); \quad \tilde{F}^D_{-m-1} L_\bullet(F) = [I_\bullet(F), \tilde{F}^D_{-m} L_\bullet(F)] 
$$

Thus there are two filtrations on the Lie coalgebra $L_\bullet(F)$: the dual to the depth filtration \eqref{66} and the filtration by the depth of multiple polylogarithms provided by the Definition \ref{5.5}.

**Conjecture 5.10** The dual to the depth filtration \eqref{66} coincides with the filtration provided by the Definition \ref{5.5}.

This conjecture, of course, implies Conjecture \ref{5.9}.

The role of the classical polylogarithms. Let $B_n(F)$ be the $\mathbb{Q}$-vector space in $L_n(F)$ spanned by the classical $n$-logarithm framed motives $I^M_n(a), a \in F^*$. (This definition differs from the usual one given in [G4], although the corresponding groups are expected to be isomorphic.) Let $H^i_n$ be the degree $n$ part of $H^n$. The following conjecture was stated in [G4]:

**Conjecture 5.11** a) $H^1_n I_\bullet(F) \cong B_n(F)$ for $n \geq 2$, i.e. $I_\bullet(F)$ is generated as a graded Lie algebra by the spaces $B_n(F)^\vee$ sitting in degree $-n$.

b) $I_\bullet(F)$ is a free graded (pro) - Lie algebra.

Conjecture \ref{5.10} obviously implies the part a) of Conjecture \ref{5.11}.

The following result, which is a motivic version of Theorem 2.22 in [G3], shows that all $n$-framed mixed Tate motives but perhaps a countable set are given by motivic multiple polylogarithms. This is strong support for the universality conjecture.

**Proposition 5.12** Let us assume the existence of the category of mixed motivic sheaves with all the expected properties. Let $\forall$ be a variation of $n$-framed mixed Tate motives over a connected rational variety $Y$. For an $F$-point $y \in Y$ the fiber $V_y$ provides a framed mixed Tate motive, and hence an element of $[V_y] \in A_\bullet(F)$. Then for any two points $y_1, y_2 \in Y$ the difference $[V_{y_1}] - [V_{y_2}]$ is a sum of motivic multiple polylogarithms.

**Proof.** One can suppose without loss of generality that there is a rational curve $X$ passing through $y_1$ and $y_2$. Let $\mathcal{P}^M(X; y_1, y_2)$ be the motivic torsor of paths from $y_1$ to $y_2$ on $X$. During the proof of Theorem 5.4 we calculated explicitly the mixed Tate motive corresponding to an arbitrary framing on $\mathcal{P}^M(X; y_1, y_2)$, and showed that it is equivalent to a sum of motivic multiple polylogarithms. This sum is given by the right hand side of the main formula in Theorem 5.4. Observe that product of motivic multiple polylogarithms is expressible as a sum of motivic multiple polylogarithms. The parallel transport along paths from $y_1$ to $y_2$ on $X$ provides a morphism of mixed Tate motives $p^X_{y_1, y_2} : V_{y_1} \otimes \mathcal{P}^M(X; y_1, y_2) \to V_{y_2}$. Let $\mathcal{A}$ be the kernel of the action of $L_\bullet(F)$ on $\mathcal{P}^M(P^1; v_0, v_1)$. Then $\mathcal{P}^M(X; y_1, y_2)$ is a trivial $\mathcal{A}$ - module. Therefore any $p \in \mathcal{P}^M(X; y_1, y_2)$ defines an isomorphism of $\mathcal{A}$ - modules $V_{y_1} \otimes p \to V_{y_2}$. This is equivalent to the statement of the proposition.

6 Examples and applications

1. The $L^c$–generators of the multiple polylogarithm Hopf algebra. Let us define several other generating series for the multiple polylogarithm elements. Consider the following two pairs of sets of variables:

i) $(x_0, ..., x_m)$ such that $x_0 ... x_m = 1$; ii) $(a_1 : ... : a_{m+1})$
Here the sum is over special marked decorated segments, i.e. sequences 
structure), and the (:)\_notation is used for those sets of var-
\begin{align*}
\sum_{i=1}^{m} a_{i+1} a_{i} - a_{i+1} a_{i} = 0 \quad \text{such that } u_{1} + \ldots + u_{m+1} = 0
\end{align*}

The relationship between them is given by

\begin{align}
x_{i} := \frac{a_{i+1}}{a_{i}}, \quad i = 1, \ldots, m; \quad x_{0} = \frac{a_{1}}{a_{m+1}} \\
u_{i} := t_{i} - t_{i-1}, \quad u_{m+1} := t_{0} - t_{m}
\end{align}

where the indices are taken modulo \( m + 1 \), and is illustrated on the picture below

\begin{center}
\begin{tikzpicture}
\draw (-5,0) -- (5,0);
\node at (0,0) {\(0\)};
\node at (-5,0) {\(x_{0}\)};
\node at (-4,0) {\(a_{0}\)};
\node at (-3,0) {\(u_{1}\)};
\node at (-2,0) {\(x_{1}\)};
\node at (-1,0) {\(a_{1}\)};
\node at (0,0) {\(t_{0}\)};
\node at (1,0) {\(u_{2}\)};
\node at (2,0) {\(x_{2}\)};
\node at (3,0) {\(a_{2}\)};
\node at (4,0) {\(t_{2}\)};
\node at (5,0) {\(u_{m+1}\)};
\node at (6,0) {\(x_{m+1}\)};
\end{tikzpicture}
\end{center}

Observe that \( x_{i}, a_{i} \) are multiplicative variables, and \( t_{i}, u_{i} \) are additive variables.

We introduce the \( \text{Li}^C \)-generating series

\begin{align}
\text{Li}^C (\ast, x_{1}, \ldots, x_{m} | t_{0} : \ldots : t_{m}) := \text{Li}^C (x_{0}, \ldots, x_{m} | t_{0} : \ldots : t_{m}) := \tag{67}
\end{align}

\begin{align*}
\text{Li}^C (x_{1}, \ldots, x_{m} | t_{0} : \ldots : t_{m}) & := (-1)^{m} \text{Li}^C (x_{1} \ldots x_{m}^{-1} : \ldots : x_{m}^{-1} | t_{0} : \ldots : t_{m}) \\
& := (-1)^{m} \text{Li}^C (a_{1} : a_{2} : \ldots : a_{m+1} | t_{0} : \ldots : t_{m}) \quad := \text{Li}^C (a_{1} : a_{2} : \ldots : a_{m+1} | a_{1} u_{1} \ldots a_{m+1})
\end{align*}

**Remark.** The \( (,) \)-notation is used for the variables which sum to zero (under the appropriate group structure), and the \( (,:) \)-notation is used for those sets of variables which are essentially homogeneous with respect to the multiplication by a common factor, see Lemma 5.3. The \( \ast \) in the left expression in \( (67) \) denotes \( (x_{1} \ldots x_{m})^{-1} \). This notation is handy in some situations.

**2. The coproduct in terms of the \( \text{Li}^C \)-generating series.** In this section we rewrite the formula for the coproduct using the \( \text{Li}^C \)-generators instead of the \( \text{I}^C \)-generators. Set

\begin{align*}
X_{a \rightarrow b} := \prod_{s=a}^{b-1} x_{s}
\end{align*}

**Proposition 6.1** Let us suppose that \( x_{i} \neq 0 \). Then

\begin{align}
\Delta \text{Li}^C (x_{0}, x_{1}, \ldots, x_{m} | t_{0} : t_{1} : \ldots : t_{m}) =
\sum_{i=1}^{m} \text{Li}^C (X_{i_{0} \rightarrow i_{1}}, X_{i_{1} \rightarrow i_{2}}, \ldots, X_{i_{k} \rightarrow m} | t_{j_{0}} : t_{j_{1}} : \ldots : t_{j_{k}}) \otimes \\
\prod_{p=0}^{k} (-1)^{j_{p}-i_{p}} X_{i_{p} \rightarrow i_{p+1}}^{t_{j_{p}}} \text{Li}^C (\ast, x_{j_{p}-1}, x_{j_{p}-2}, \ldots, x_{i_{p}}^{-1} | - t_{j_{p}} : - t_{j_{p}-1} : \ldots : - i_{p}) \tag{68}
\end{align}

\begin{align}
\text{Li}^C (\ast, x_{j_{p}+1}, x_{j_{p}+2}, \ldots, x_{i_{p+1}-1} | t_{j_{p}+1} : \ldots : t_{i_{p+1}-1}) \tag{69}
\end{align}

Here the sum is over special marked decorated segments, i.e. sequences \( \{i_{p}\}, \{j_{p}\} \) satisfying \( (64) \).

The \( p \)-th factor in the product on the right is encoded by the data on the \( p \)-th segment:
Namely, $X_{i_p \rightarrow i_{p+1}}$ is the product of all $x_i$ on this segment. The factor $\in j$ is encoded by the segment between $t_{j_p}$ and $t_{j}$, which we read from right to left. The factor $\in 0$ is encoded by the segment between $t_{j_p}$ and $t_{j}+1$ which we read from left to right.

**Proof.** Follows from Lemma 5.2 and Theorem 5.4.

3. The coproduct in the classical polylogarithm case. Recall that $A^\bullet(\mathcal{C})$ is the commutative Hopf algebra of the framed mixed objects in a mixed Tate category $\mathcal{C}$ with the coproduct $\Delta$ and the product $\ast$, see the Appendix, s.2 (where the notation $\mu$ for the product is used). Recall the restricted coproduct: $\Delta'(X) := \Delta(X) - (X \otimes 1 + 1 \otimes X)$. Notice that $\Delta$ is a homomorphism of algebras and $\Delta'$ is not.

**Corollary 6.2**

$$\Delta : \text{Li}^C(x|t) \mapsto \text{Li}^C(x|t) \otimes x^t + 1 \otimes \text{Li}^C(x|t)$$

(71)

**Proof.** This is a special case of Proposition 6.1. This formula just means that

$$\Delta' \text{Li}^C_n(x) = \text{Li}^C_{n-1}(x) \otimes \text{log}^C x + \text{Li}^C_{n-2}(x) \otimes \frac{\text{log}^C x}{2} + \ldots + \text{Li}^C_1(x) \otimes \frac{\text{log}^C x}{(n-1)!}$$

4. The coproduct for the depth two multiple polylogarithms. We will use both types of the I-notations for multiple polylogarithm elements, so for instance

$$\Gamma^C(a_1 : a_2 : 1|t_1, t_2) = \Gamma^C(0; a_1, a_2; 1|t_1; t_2)$$

Set $\zeta^C(t_1, \ldots, t_m) := \Gamma^C(1 : \ldots : 1|t_1, \ldots, t_m)$.

**Proposition 6.3**

a) One has

$$\Delta \Gamma^C(a_1 : a_2 : 1|t_1, t_2) = 1 \otimes \Gamma^C(a_1 : a_2 : 1|t_1, t_2)$$

$$\Gamma^C(a_1 : a_2 : 1|t_1, t_2) \otimes a_1^{-t_1} * a_2^{t_1-t_2} + \Gamma^C(a_1 : 1|t_1) \otimes a_1^{-t_1} * \Gamma^C(a_2 : 1|t_2 - t_1)$$

$$-\Gamma^C(a_1 : 1|t_2) \otimes a_1^{-t_2} * \Gamma^C(a_2 : a_1|t_2 - t_1) + \Gamma^C(a_2 : 1|t_1) \otimes \Gamma^C(a_1 : a_2|t_1) * a_2^{-t_2}$$

b) Let us suppose that $a_1^N = a_2^N = 1$. Then modulo $N$-torsion one has

$$\Delta' \Gamma^C(a_1 : a_2 : 1|t_1, t_2) = \Gamma^C(a_1 : 1|t_1) \otimes \Gamma^C(a_2 : 1|t_2 - t_1)$$

$$-\Gamma^C(a_1 : 1|t_2) \otimes \Gamma^C(a_2 : a_1|t_2 - t_1) + \Gamma^C(a_2 : 1|t_2) \otimes \Gamma^C(a_1 : a_2|t_1)$$

In particular

$$\Delta' \zeta^C(t_1, t_2) = \zeta^C(t_1) \otimes \zeta^C(t_2 - t_1) - \zeta^C(t_2) \otimes \zeta^C(t_2 - t_1) + \zeta^C(t_2) \otimes \zeta^C(t_1)$$

(72)

**Proof.** a) Since in our case $a_0 = 0$ and $t_0 = 0$, the nonzero contribution can be obtained only from those marked decorated segments where the $t_0$-arc is not marked. Let us call pictures where $a_0 = 0$, $a_{n+1} = 1$, $t_0 = 0$ and the $t_0$-arc is not marked, special marked decorated segments.

The five terms in the formula above correspond to the five special marked decorated segments presented in the picture.
Using the formulas from Lemma 5.3 we get the following four terms corresponding to the terms N1-N4 on the picture:

\[
\begin{align*}
\Gamma^C(a_1 : a_2 : 1|t_1, t_2) & \otimes \Gamma^C(a_1; 0|t_1) \cdot \Gamma^C(0; a_2|t_2) \cdot \Gamma^C(0; 1|t_2) = \\
\Gamma^C(a_1 : a_2 : 1|t_1, t_2) & \otimes a_1^{-t_1} \cdot a_2^{-t_2}
\end{align*}
\]

\[
\begin{align*}
\Gamma^C(a_1 : 1|t_1) & \otimes (I(a_1; 0|t_1) \cdot \Gamma^C(0; a_2; 1|t_1; t_2)) = \\
\Gamma^C(a_1 : 1|t_1) & \otimes a_1^{-t_1} \cdot \Gamma^C(a_2 : 1|t_2 - t_1)
\end{align*}
\]

\[
\begin{align*}
\Gamma^C(a_1 : 1|t_2) & \otimes (I(a_1; a_2; 0|t_1; t_2) \cdot \Gamma^C(0; 1|t_2)) = \\
-\Gamma^C(a_1 : 1|t_2) & \otimes a_1^{-t_2} \cdot \Gamma^C(a_2 : a_1|t_2 - t_1)
\end{align*}
\]

\[
\begin{align*}
\Gamma^C(0; a_2; 1|t_2) & \otimes (I(a_1; a_2; 0|t_1; t_2) \cdot \Gamma^C(0; 2|t_2) \cdot \Gamma^C(0; 1|t_2)) = \\
\Gamma^C(a_2 : 1|t_2) & \otimes \Gamma^C(a_1 : a_2; t_1) \cdot a_2^{-t_2}
\end{align*}
\]

Part b) follows from a) if we notice that \(a^N = 1\) provides \(a^t = 1\) modulo \(N\)-torsion. The proposition is proved.

Remark. In Theorem 4.5 of [G9] the reader can find a different way to write the formulas for the coproduct in depth 2 case. It is easy to see that the formulas given there are equivalent to the formulas above.

5. Explicit formulas for the coproduct of the weight three, depth two multiple polylogarithm elements. Applying Proposition 6.3 or Proposition 6.1 we get

\[
\Delta' : \text{Li}_{2,1}^C(x, y) \quad \mapsto \quad \text{Li}_{1,1}^C(x, y) \otimes [x] + \text{Li}_{1}^C(y) \otimes \text{Li}_{2}^C(x) + \text{Li}_{2}^C(xy) \otimes \text{Li}_{1}^C(y)
\]

\[
\begin{align*}
-\text{Li}_{1}^C(xy) \otimes \left( \text{Li}_{2}^C(x) + \text{Li}_{2}^C(y) - \text{Li}_{1}^C(y) \cdot ([xy] + \frac{[x]^2}{2}) \right)
\end{align*}
\]

\[
\Delta' : \text{Li}_{1,2}^C(x, y) \quad \mapsto \quad \text{Li}_{1,1}^C(x, y) \otimes [y] - \text{Li}_{2}^C(xy) \otimes [x] - \text{Li}_{1}^C(xy) \otimes [xy]
\]

\[
\begin{align*}
+L_i^C(y) \otimes \text{Li}_{1}^C(x) + \text{Li}_{1}^C(y) \otimes \text{Li}_{1}^C(x) \cdot [y] + \text{Li}_{2}^C(xy) \otimes \text{Li}_{2}^C(y)
\end{align*}
\]

\[
\begin{align*}
-\text{Li}_{2}^C(xy) \otimes \text{Li}_{1}^C(x) - \text{Li}_{2}^C(xy) \otimes \text{Li}_{1}^C(x) \cdot [xy] - \text{Li}_{1}^C(xy) \otimes \text{Li}_{2}^C(x)
\end{align*}
\]
6. The depth two Lie coalgebra structure. Recall that the space of the indecomposables

\[ \mathcal{L}_*(C) := \frac{A_{>0}(C)}{A_{>0}(C) \cdot A_{>0}(C)} \]

inherits a structure of graded Lie coalgebra with the cobracket \( \delta \). I will describe

\[ \sum_{n,m>0} \delta I^C_{n,m}(a_1,a_2) \cdot t_1^{n-1} t_2^{m-1} \in \Lambda^2 A_*(C) \otimes \mathbb{Z}[[t_1,t_2]] \]

In the formulas below \( \delta \) acts on the first factor in \( A_*(C) \otimes \mathbb{Z}[[t_1,t_2]] \). Using Proposition \ref{prop}, we have

\[
\begin{align*}
\delta \left( \sum_{m>0,n>0} I_{m,n}(a:b:c) \cdot t_1^{m-1} t_2^{n-1} \right) = \\
\sum_{m>0,n>0} \left( I_{m,n}(a:b:c) \cdot t_1^{m-1} t_2^{n-1} \right) \wedge \left( \frac{b}{a} \cdot t_1 + \frac{c}{b} \cdot t_2 \right) - I_m(a:b) \cdot t_1^{m-1} \wedge I_n(b:c) \cdot t_2^{n-1} \\
+ I_m(a:c) \cdot t_1^{m-1} \wedge I_n(b:c) \cdot (t_2 - t_1)^{n-1} - I_n(a:c) \cdot t_2^{n-1} \wedge I_m(b:a) \cdot (t_2 - t_1)^{m-1}
\end{align*}
\]

Set \( I_{0,n} = I_{n,0} = 0 \). Here is a more concrete formula for \( \delta \):

\[
\delta I_{m,n}(a,b) = \\
I_{m-1,n}(a,b) \wedge \left[ \frac{b}{a} \right] + I_{m,n-1}(a,b) \wedge \left[ \frac{1}{b} \right] - I_m(a) \wedge I_n(b) + \\
\sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} I_{m-i}(a) \wedge I_{n+i}(b) \\
- (-1)^{m-1} \sum_{j=0}^{n-1} (-1)^j \binom{m+j-1}{j} I_{n-j}(a) \wedge I_{m+j}(\frac{b}{a})
\]

7. An application: the motivic double \( \zeta \)'s. Below we use the notation \( Z^* \) for the cyclotomic Hopf algebra \( Z^*(\mu_N) \) in the case \( N = 1 \). Recall the restricted coproduct map

\[ \Delta' : Z^* \longrightarrow Z^* \otimes Z^* \] \hspace{1cm} \text{(73)}

**Theorem 6.4** The degree \( n \) part of \( \ker \Delta' \) is zero if \( n = 1 \) or if \( n \) is even, and it is one dimensional otherwise. It is spanned by \( \zeta^M(n) \).

**Proof.** The motivic multiple zeta values are defined as the matrix coefficients of the motivic torsor of path \( \mathcal{P}^M(P^1 - \{0,1,\infty\};v_0,v_1) \), which itself is a mixed Tate motive over \( \text{Spec}(\mathbb{Z}) \) ([DG]). Therefore motivic multiple zeta's belong to \( A_*(\mathbb{Z}) \). So the kernel of the map \( \text{(73)} \) is a subspace of the kernel of the map \( \Delta' : A_*(\mathbb{Z}) \longrightarrow A_*(\mathbb{Z}) \otimes A_*(\mathbb{Z}) \). The degree \( n \) part of the latter is isomorphic to

\[
\text{Ext}^1_{\mathcal{M}_T(\mathbb{Z})}(\mathbb{Q}(0),\mathbb{Q}(n)) = K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 
0 & n: \text{even} \\
1 & n > 1: \text{odd}
\end{cases}
\] \hspace{1cm} \text{(74)}

On the other hand, the element \( \zeta^M(n) \) is non-zero for the same values of \( n \), and \( \Delta' \zeta^M(n) = 0 \). So it generates the degree \( n \) part of \( \ker \Delta' \) in \( \text{(73)} \). The theorem is proved.
Recall the depth filtration $F^D$ on the Hopf algebra $Z^M$. The $m$-th associate graded for the depth filtration $Z^M_m := \text{gr}_m Z^M$ and $C^M_m := \text{gr}_m C^M$ are graded by the weight.

Restricting the coproduct to the subspace of the depth two motivic multiple zeta’s and using Theorem 6.4 and formula (72) we get an embedding

\[ \Delta' : Z^M_{\bullet, 2} \hookrightarrow Z^M_{\bullet, 1} \otimes Z^M_{\bullet, 1} \]

Observe that if $\Delta'(x) = \Delta'(y) = 0$ then $\Delta'(xy) = x \otimes y + y \otimes x$. Therefore the product provides an inclusion $S^2 F^D Z^M_{\bullet} \hookrightarrow F^D Z^M_{\leq 2}$, as well as an inclusion $S^2 Z^M_{\bullet, 1} \hookrightarrow Z^M_{\bullet, 2}$. Recall

\[ C^M_{\bullet, 2} = Z^M_{\bullet, 2} / S^2 Z^M_{\bullet, 1}, \quad C^M_{\bullet, 1} := Z^M_{\bullet, 1} \]

Then $\Delta'$ induces an injective map

\[ \delta : C^M_{\bullet, 2} \hookrightarrow \Lambda^2 C^M_{\bullet, 1} \quad (75) \]

Denote by $\overline{\zeta}^M(m, n)$ the projection of $\zeta^M(m, n)$ onto $C^M_{m+n, 2}$. The formula (72) tells us that:

\[ \delta \overline{\zeta}^M(t_1, t_2) = \zeta^M(t_1) \wedge \zeta^M(t_2 - t_1) - \zeta^M(t_2) \wedge \zeta^M(t_2 - t_1) + \zeta^M(t_2) \wedge \zeta^M(t_1) \quad (76) \]

Since $\zeta^M(t) = \zeta^M(-t)$, we can write (76), employing a linear map $U : (t_1, t_2) \mapsto (t_1 - t_2, t_1)$, as

\[ \delta : \overline{\zeta}^M(t_1, t_2) \mapsto -(I + U + U^2) \zeta^M(t_1) \wedge \zeta^M(t_2) \quad (77) \]

Consider the generating series $\overline{\zeta}^M(t_0, t_1, t_2) := \overline{\zeta}^M(t_1, t_2)$, where $t_0 + t_1 + t_2 = 0$.

**Theorem 6.5**

a) The generating series $\overline{\zeta}^M(t_0, t_1, t_2)$ satisfy the dihedral symmetry relations:

\[ \overline{\zeta}^M(t_0, t_1, t_2) = \overline{\zeta}^M(t_1, t_2, t_0) = -\overline{\zeta}^M(t_0, t_2, t_1) = \overline{\zeta}^M(-t_0, -t_1, -t_2) \quad (78) \]

b) There are no other relations between the coefficients of the generating series $\overline{\zeta}^M(t_0, t_1, t_2)$.

**Proof.** a) The dihedral symmetry relations are evidently in the kernel of the map (75). Since this map is injective, we are done.

b). Consider the classical modular triangulation of the hyperbolic plane obtained by the action of the group $GL_2(\mathbb{Z})$ on the ideal triangle $T_{0,1,\infty}$ with the vertices at $0, 1, \infty$. Let

\[ M^1_{(2)} := M^1_{(2)} \xrightarrow{\partial} M^2_{(2)} \]

be the chain complex of this triangulation sitting in the degrees $[1, 2]$. It is a complex of $\mathbb{Z}[GL_2(\mathbb{Z})]$-modules where the first group is generated by the class $[T_{0,1,\infty}]$ of the oriented triangle $T_{0,1,\infty}$, and the second one by the class $[E_{0,\infty}]$ of the oriented geodesic from $0$ to $\infty$.

It is handy to introduce a new bigraded Lie coalgebra defined as a direct sum of Lie coalgebras

\[ \widehat{C}^M_{\bullet, 1} := C^M_{\bullet, 1} \oplus Q^M_{1,1} \]

where $Q^M_{1,1}$ is the one-dimensional Lie coalgebra in bidegree $(1, 1)$ spanned by a new formal element $\zeta^M(1)$. This element does not have any motivic meaning. The map (76) provides a map

\[ C^M_{\bullet, 2} \rightarrow \Lambda^2 \widehat{C}^M_{\bullet, 1} \quad (79) \]
Lemma 6.6  The formulas

\[
\mu^1 : [T_{0,1,\infty}] \otimes t_1^{m-1} t_2^{n-1} \mapsto \zeta^M(m,n) t_1^{m-1} t_2^{n-1}; \quad \mu^2 : [E_{0,\infty}] \otimes t_1^{m-1} t_2^{n-1} \mapsto \zeta^M(m) \wedge \zeta^M(n) t_1^{m-1} t_2^{n-1}
\]

provide an isomorphism of complexes \( \mathfrak{N} \), where \( \Lambda_w^2 \) means the degree \( w \) part of \( \Lambda^2 \):

\[
M_{12}^1 \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2 \xrightarrow{\partial} M_{12}^2 \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2
\]

\[
\begin{align*}
\mu^1 & \downarrow & \downarrow \mu^2 \\
C_{\cdot,2}^{\mathcal{M}} & \xrightarrow{\delta} & \Lambda_w^2 \mathfrak{C}_{\cdot,1}^{\mathcal{M}} \\
\end{align*}
\]

Proof. The dihedral group is the stabiliser of the triangle \( T_{0,1,\infty} \) in \( GL_2(\mathbb{Z}) \). Thus the relations \( \mathfrak{T} \) just mean that the first formula gives a well defined map of \( M_{12}^1 \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2 \). The relation \( \zeta^M(t) = \zeta^M(-t) \) is equivalent to a similar fact about the second formula. And formula \( \mathfrak{M} \) just means that the constructed maps commute with the differentials. The lemma is proved.

By its very definition the maps \( \mu^1 \) and \( \mu^2 \) are surjective. The map \( \mu^2 \) is an isomorphism. Indeed, the stabilizer of \( E_{0,\infty} \) in \( GL_2(\mathbb{Z}) \) is the subgroup generated by \( \left( \begin{array}{ll}
\pm 1 & 0 \\
0 & \pm 1
\end{array} \right) \) and \( \left( \begin{array}{ll}
0 & 1 \\
-1 & 0
\end{array} \right) \). So taking the coinvariants of the action of this subgroup on \( S^{w-2}V_2 \) amounts to the relation \( \zeta^M(t) = \zeta^M(-t) \) and the skew-symmetry of \( \Lambda^2 \). Since the map \( \mathfrak{T} \) is injective, to check that \( \mu^1 \) is injective it is sufficient to show that the top arrow \( \partial \) is injective. Let us prove this. Let \( \varepsilon_2 \) be the determinant representation of \( GL_2(\mathbb{Z}) \). By Lemma 2.3 from [G1] one has

\[
H^i \left( M_{12}^i \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2 \right) = H^{i-1}(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2), \quad i = 1, 2
\]

Since \( H^0(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) = 0 \), the map \( \partial \) is injective. The theorem is proved.

Remark. The relations \( \mathfrak{T} \) are the motivic counterparts of the (regularized) double shuffle relations for the double zeta values considered modulo products and depth one terms, see Section 2 of [G2] for the form of the double shuffle relations making this statement obvious. Therefore Theorem 6.5 plus Conjecture 1.3 imply that there should be no other relations between the double zeta’s. This was conjectured by Zagier [Z].

Example. Formula \( \mathfrak{T} \) tells us that

\[
\Delta' \zeta^M(3,5) = -5 \cdot \zeta^M(3) \otimes \zeta^M(5)
\]

Since \( \zeta^M(2n+1) \neq 0 \) this implies that \( \zeta^M(3,5) \neq 0 \). Moreover since \( \delta' \zeta^M(3,5) = -5 \cdot \zeta^M(3) \wedge \zeta^M(5) \) is also non-zero, \( \zeta^M(3,5) \) is irreducible, i.e. it is not a product of the classical motivic zeta’s. Thus it is a non-zero, non-classical, irreducible, motivic period over \( \mathbb{Z} \) of the smallest weight.

8. When does a motivic iterated integral unramified at \( \mathcal{P} \)?  Let \( \mathcal{O}_F \) be the ring of integers in a number field \( F \). Let \( S \) be a collection of ideals of \( \mathcal{O}_F \). Recall from Section 8.4 the fundamental Hopf algebra \( \mathcal{A}_\bullet(\mathcal{O}_{F,S}) \) of the category of mixed Tate motives over \( \text{Spec}(\mathcal{O}_{F,S}) \).

Recall that our definition of the motivic iterated integrals

\[
\Gamma^M(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{A}_n(F)
\]

assumes a choice of a tangent vector at every point \( a_i \). Below we assume that all these tangent vectors equal \( \partial / \partial t \), where \( t \) is the chosen coordinate on the affine line \( \mathbb{A}^1 \). To emphasise this choice we use the notation \( \Gamma^M_{\partial / \partial t}(a_0; a_1, \ldots, a_n; a_{n+1}) \). These elements are invariant under the shift \( a_i \rightarrow a_i + c \). Denote by \( v_{\mathcal{P}} : F^* \rightarrow \mathbb{Z} \) the valuation at the prime ideal \( \mathcal{P} \).
**Theorem 6.7** Let $a_0, ..., a_{n+1}$ be elements of a number field $F$. Then $I^M_{\partial/\partial t}(a_0; a_1, ..., a_n; a_{n+1})$ is unramified at a place $\mathcal{P}$ of $F$ if for any $0 \leq i < j < k \leq n + 1$ one has $v_\mathcal{P}(a_i, a_j, a_k) = 0$.

**Proof.** Recall the inductive definition of the space $A_n(\mathcal{O}_{F,S})$ given by (106). We prove the theorem by the induction on $n$. Let $n = 1$. Then one has

$$I^M_{\partial/\partial t}(a_0; a_1) = \tilde{r}(a_0; a_1)$$

(The right hand side is defined in the Example in Chapter 1.2). Since $A_1(\mathcal{O}_{F,S}) \simeq \mathcal{O}_{F,S}^* \otimes \mathbb{Q}$, this proves the theorem for $n = 1$. Let us show using the induction assumption that our condition implies

$$\Delta' I^M_{\partial/\partial t}(a_0; a_1, ..., a_n; a_{n+1}) \in \bigoplus_{k=1}^{n-1} A_k(\mathcal{O}_{F,S}) \otimes A_{n-k}(\mathcal{O}_{F,S})$$

(83)

Indeed, it is clear from the formula (2) that every factor in (83) is of type $I^M_{\partial/\partial t}(a_i; a_{p_1}, ..., a_{p_k}; a_j)$ where $0 \leq i < p_1 < ... < p_k < j \leq n + 1$. Moreover one has $k < n$ since we consider the restricted coproduct $\Delta'$. Therefore each of these factors belongs $A_n(\mathcal{O}_{F,S})$ by the induction assumption. The theorem is proved.

### 7 Feynman integrals, Feynman diagrams and mixed motives

**1. Motivic correlators.** Let us return to Chapter 4. Why do plane trivalent trees appear in the description of motivic iterated integrals, and what is the general framework for this relationship?

Previously, we defined in Chapters 8 and 9 of [G2] a real–valued version of multiple polylogarithms as correlators of a Feynman integral. In fact we gave there a more general construction which provides a definition of a real valued version of multiple polylogarithms on an arbitrary smooth curve. Moreover the Feynman diagram construction given in [G2] provides much more than just functions – it can be lifted to a construction of framed mixed motives whose periods are the correlators of the corresponding Feynman integral. A version of Conjecture 5.9 claims that all framed mixed Tate motives appear this way.

So why do Feynman integrals and Feynman diagrams appear in the theory of mixed motives, and what role do they play there? I think we see in the two examples above a manifestation of the following general principle.

Feynman integrals are described by their correlators. Correlators of Feynman integrals are often periods of framed mixed motives: see an example in Chapter 9 of [G2]. In this case, the correlators can be upgraded to more sophisticated objects: the equivalence classes of the corresponding framed mixed motives (see Appendix or Chapter 3 of [G7] for the background). These objects lie in a certain commutative Hopf algebra $H_{\text{Mot}}$ with the coproduct $\Delta_{\text{Mot}}$. One can loosely think about $H_{\text{Mot}}$ as of the algebra of regular functions on the motivic Galois group. More precisely, $H_{\text{Mot}}$ is a Hopf algebra in the hypothetical abelian tensor category $\mathcal{P}_M$ of all pure motives, see [G7]. The motivic Galois group is a pro–affine group scheme in this category, see [D2]. Although $H_{\text{Mot}}$ is still a hypothetical mathematical object, we can see it in different existing realizations, e.g. in the Hodge realization.

Climbing up the road

$$\text{correlators of Feynman integrals (numbers)} \quad \rightarrow \quad \text{motivic correlators (elements of the Hopf algebra } H_{\text{Mot}})$$

we gain a new perspective: one can now raise the question

$$\text{what is the coproduct of motivic correlators in } H_{\text{Mot}}?$$

(84)
Reflections on this theme occupy the rest of the Chapter.

2. The correspondence principle. Let us formulate question (84) more precisely. Let

\[ \text{Cor} = \langle \varphi(s_1), \ldots, \varphi(s_{m+1}) \rangle \]  

be a correlator of a certain Feynman integral \( F \). (Here \( \varphi \) is the traditional notation for fields over which we integrate). If we understand our Feynman integral by its perturbation series expansion then according to the Feynman rules correlator (85) is defined as a sum of finite dimensional integrals:

\[ \text{Cor} := \sum_{\Gamma \in \mathcal{S}_F(\text{Cor})} \int_{X_\Gamma} \omega_\Gamma \]  

The sum is over a (finite) set \( \mathcal{S}_F(\text{Cor}) \) of certain combinatorial objects \( \Gamma \), given by decorated graphs. It is determined by the Feynman integral \( F \) and the type of the correlator we consider. Such a \( \Gamma \) provides a real algebraic variety \( X_\Gamma \) and a differential form of top degree \( \omega_\Gamma \) on \( X_\Gamma(\mathbb{R}) \).

Let us assume that the integrals in (86) are convergent. This is often not the case in physically interesting examples. However it is the case for the Feynman integral considered in [G2]. Then correlator (85) is a well defined number.

Let us assume further that this number is a period of a mixed motive. This means that it is given by a sum of integrals of rational differential forms on certain varieties (which may differ from \( X_\Gamma \oplus \mathbb{C} \)) over certain chains whose boundaries lie in a union of divisors. Below we consider only Feynman integrals whose correlators have this property, called Feynman integrals of algebraic-geometric type. A nontrivial example of such a Feynman integral is given in [G2].

Then we conjecture that one can uniquely upgrade this number to the corresponding “motivic correlator”

\[ \text{Cor}_M \in \mathcal{H}_{\text{Mot}} \]  

It is a framed mixed motive. The period of its Hodge realization is given by (86). We ask in (84) about the coproduct

\[ \Delta_M(\text{Cor}_M) \in \mathcal{H}_{\text{Mot}} \otimes \mathcal{H}_{\text{Mot}} \]  

We suggest that the answer should be given combinatorially in terms of the decorated graphs \( \Gamma \) used in the Definition (86). Here is a more precise version of this guess.

Let \( H_1 \) and \( H_2 \) be Hopf algebras in tensor categories \( T_1 \) and \( T_2 \). A Hopf algebra homomorphism \( H_1 \to H_2 \) is given by a tensor functor \( F : T_1 \to T_2 \) and a Hopf algebra homomorphism \( F(H_1) \to H_2 \).

The correspondence principle. For a Feynman integral \( F \) of algebraic-geometric type there should exist a combinatorially defined commutative Hopf algebra \((\mathcal{H}_F, \Delta_F)\) in a tensor category \( T_F \) such that the right hand side of (86) provides an element

\[ \gamma := \sum_{\Gamma \in \mathcal{S}_F(\text{Cor})} [\Gamma] \in \mathcal{H}_F \]  

The map

\[ c_M : \gamma \mapsto \text{Cor}_M \]  

gives rise to a Hopf algebra homomorphism

\[ c_M : \mathcal{H}_F \to \mathcal{H}_{\text{Mot}} \]  

provided by a tensor functor \( F : T_F \to \mathcal{P}_M \).

In particular \( c_M \) is compatible with the coproducts:

\[ (c_M \otimes c_M)(\Delta_F(\gamma)) = \Delta_M(\text{Cor}_M) \]
This allows us to calculate the coproduct combinatorially as the left hand side in (91), providing an answer to the question.

**Remarks.**

1. If the motivic correlator is a pure motive, question is trivial: the coproduct is zero.

2. As we will see below, the correspondence principle imposes very strong constraints on the motivic correlators. It is not clear to me whether we can expect it for any algebraic-geometric Feynman integral. See s. 7.6.

3. An example. The story described in Chapter 4 should serve as an example of such a situation. In this case the motivic iterated integrals

\[ I^M(s_0; s_1, ..., s_m; s_{m+1}), \quad s_i \in S \]  

should be seen as the motivic correlators. So far there is no Feynman integral known providing these correlators, so we work directly with Feynman diagrams, given by trivalent plane rooted trees. Thanks to Theorem 4.2 there is a Hopf algebra map

\[ t : \mathcal{I}_*(S) \hookrightarrow \mathcal{T}_*(S) \]

where \( t \) is as in (19). The Hopf subalgebra

\[ t(\mathcal{I}_*(S)) \subset \mathcal{T}_*(S) \]

should be considered as the combinatorially defined Hopf algebra \( \mathcal{H}_F \) responsible for the motivic correlators. Then the map \( c_M \) is given by

\[ \sum \text{plane rooted trivalent trees decorated by } \{ s_0, s_1, ..., s_m, s_{m+1} \} \mapsto I^M(s_0; s_1, ..., s_m; s_{m+1}) \]

Since \( \mathcal{I}_*(S) \) is a quotient of \( \mathcal{I}_*(S) \), we arrive at the diagram

\[ \mathcal{O}(G_M(S)) \twoheadrightarrow \mathcal{I}_*(S) \twoheadrightarrow \mathcal{I}_*(S) =: \mathcal{H}_F \]

We define the affine group scheme \( G_F \) as the spectrum of \( \mathcal{H}_F \). Passing to the corresponding group schemes we get

\[ G_M(S) \xrightarrow{\text{def}} G(S) \xleftarrow{\text{def}} \text{Spec}(\mathcal{I}_*(S)) =: G_F \]

Observe that the coproduct in \( \mathcal{H}_F := t(\mathcal{I}_*(S)) \) is inherited from a bigger Hopf algebra \( \mathcal{T}_*(S) \).

4. **The motivic Galois group of a Feynman integral and the correspondence principle.**

Let us reformulate the correspondence principle as a relationship between the groups \( G_F \) and the motivic Galois groups. We start with a somewhat more precise formulation of the correspondence principle:

i) A Feynman integral \( \mathcal{F} \) can be understood as an (infinite) collection of correlators. Correlators of Feynman integrals of algebraic–geometric type can be lifted to their motivic avatars. The commutative algebra they generate should form a Hopf subalgebra \( \mathcal{H}_{\text{Mot}}(\mathcal{F}) \) of the motivic Hopf algebra:

\[ \mathcal{H}_{\text{Mot}}(\mathcal{F}) \hookrightarrow \mathcal{H}_{\text{Mot}} \]  

We call it the *motivic Hopf algebra of the Feynman integral* \( \mathcal{F} \).

ii) A Feynman integral \( \mathcal{F} \) should determine its combinatorially defined Hopf algebra \( \mathcal{H}_F \).

iii) These two Hopf algebras are related by a canonical homomorphism of Hopf algebras, the *motivic correlator homomorphism*

\[ c_M : \mathcal{H}_F \rightarrow \mathcal{H}_{\text{Mot}}(\mathcal{F}) \]  

39
provided by a tensor functor $F : T_F \rightarrow P_M$. The map $c_M$ is surjective by the very definition of $\mathcal{H}_{\text{Mot}}(F)$.

Just like in s. 7.3, the coproduct in $\mathcal{H}_F$ is probably inherited from a bigger object: although only the sum $\sum[\Gamma]$ belongs to $\mathcal{H}_F$, we should be able to make sense of $\Delta_F[\Gamma]$ for every summand $[\Gamma]$.

We define the motivic Galois group $G_{\text{Mot}}(F)$ of a Feynman integral $F$ as the spectrum of the commutative algebra $\mathcal{H}_{\text{Mot}}(F)$. Then $c_M$ provides an injective homomorphism

$$c_M^* : G_{\text{Mot}}(F) \hookrightarrow G_F$$

It seems that the Feynman integral itself should be upgraded to an infinite dimensional mixed motive. Its matrix coefficients corresponding to different framings should provide us all its correlators.

5. The correspondence principle almost determines motivic correlators. To simplify the exposition we will demonstrate this in the particularly interesting mixed Tate case. Similar arguments work in general.

Let $F$ be a number field. Denote by $\mathcal{H}_{\text{Mot}}^{T}(F) := A_{\bullet}(F)$ the fundamental Hopf algebra of the category of mixed Tate motives over $F$. It is a commutative, graded Hopf algebra.

If all motivic correlators of a Feynman integral $F$ were mixed Tate motives, then $G_{\text{Mot}}(F)$ would be a semidirect product of $G_m$ and $\text{Spec}(\mathcal{H}_{\text{Mot}}(F))$, with the action of $G_m$ provided by the grading. In this case $\mathcal{H}_F$ should also be a graded Hopf algebra, and the semidirect product of $G_m$ and its spectrum provides us the group $G_F$ associated to a Feynman integral.

Let $\text{Cor}_M = c_M(\gamma)$, where $\gamma$ as in (89). Suppose that $\text{Cor}_M$ is a mixed motive framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$ defined over $F$. The calculation of $\Delta_{\mathcal{H}_F}(\gamma)$ is a combinatorial problem. So we may assume $\Delta_{\mathcal{H}_F}(\gamma)$ is given to us. Let us stress that we do not assume that $\text{Cor}_M$ is a mixed Tate motive. However let us assume that

$$(c_M \otimes c_M)(\Delta'_{\mathcal{H}_F}(\gamma)) \in \mathcal{H}_{\text{Mot}}^{T}(F) \otimes \mathcal{H}_{\text{Mot}}^{T}(F)$$

is mixed Tate. Then thanks to (94)

$$\Delta'_M(\text{Cor}_M) \in \mathcal{H}_{\text{Mot}}^{T}(F) \otimes \mathcal{H}_{\text{Mot}}^{T}(F)$$

Since we used the restricted coproduct in (93), all factors there are of smaller weight, and are mixed Tate by assumption. Therefore the following lemma implies that if the framing on $\text{Cor}_M$ is Tate then $\text{Cor}_M$ is (equivalent to) a framed mixed Tate motive over $F$!

**Lemma 7.1** Let $X \in \mathcal{H}_{\text{Mot}}(F)$ be the equivalence class of a mixed motive over $F$ framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$. Then

$$\Delta'_M X \in \mathcal{H}_{\text{Mot}}^{T}(F) \otimes \mathcal{H}_{\text{Mot}}^{T}(F)$$

implies $X \in \mathcal{H}_{\text{Mot}}^{T}(F)$.

**Proof.** Observe that $\Delta'_M(\Delta'_M X) = 0$. Since $\text{Ext}^2_{\text{Mot}/F}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ it follows that there exists an element $Y \in \mathcal{H}_{\text{Mot}}^{T}(F)$ such that $\Delta'_M Y = \Delta'_M X$. Therefore the lemma follows from the following easy but basic fact:

$$X, Y \in \mathcal{H}_{\text{Mot}}(F), \quad \Delta'_M X = \Delta'_M Y \implies X - Y \in \text{Ext}^1_{\text{Mot}/F}(\mathbb{Q}(0), \mathbb{Q}(n))$$

**Remark.** There is a similar lemma where the category of Tate motives is replaced by any pure tensor category of motives. However its proof uses a different idea.

If $\text{Cor}_M$ is defined over a number field $F$ then thanks to (95) it is determined by its restricted coproduct $\Delta'_M(\text{Cor}_M)$ up to an element of

$$\text{Ext}^1_{\text{Mot}/F}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(F) \otimes \mathbb{Q}$$
Therefore if a mixed motive representing $\text{Cor}_M$ is defined over $F$ (which is usually very easy to check), and the framing on it is Tate, the knowledge of (94) allows us to determine the equivalence class of $\text{Cor}_M$ up to an element of (96).

For example let $F = \mathbb{Q}$. Then (96) is spanned by $\zeta^M(n)$, see also (74). It is zero for even $n$ and one-dimensional for odd $n > 1$. So in this case $\text{Cor}_M$ is determined up to a rational multiple of $\zeta^M(n)$. In particular it is determined uniquely if $n$ is even.

Recall the inductive definition of the Hopf algebra $H^T_{\text{Mot}}(\mathbb{Z}) = A^\bullet(\mathbb{Z})$ given in Lemma 8.5 and (107) in the Appendix. Then induction on the weight shows that if we have the correspondence principle, and if the weight 2 correlators are zero, then, thanks to (107), in the situation above all motivic correlators must belong to $H^T_{\text{Mot}}(\mathbb{Z})$, i.e. must be framed mixed Tate motives over $\mathbb{Z}$. This plus Conjecture 17b) from [G6], which says that all framed mixed Tate motives over $\mathbb{Z}$ are multiple $\zeta$–motives, imply that $\text{Cor}_M$ is a multiple $\zeta$–motive.

More generally, let $O_S$ be the ring of $S$–integers in a number field $F$. Then if all weight two motivic correlators come from $O^\ast_S$, then in the situation above all motivic correlators must come from $H^T_{\text{Mot}}(O_S)$, i.e. are framed mixed Tate motives over $F$ with possible ramification at $S$.

6. An example: $\zeta^M(3,5)$. Physicists computed many correlators. D. Kreimer and D. Broadhurst discovered that, in small weights, these correlators are often expressed by the multiple $\zeta$–values ([BGK]). According to the example in Section 6.7, the simplest multiple $\zeta$–value which should not be expressed by the classical $\zeta$–values appears in weight 16 (the weights in the Tate case are even numbers, so it is customary to divide them by two; then the weight is 8). For example one can take $\zeta(3,5)$. Its motivic avatar has been discussed in the example mentioned above. According to [BGK] $\zeta(3,5)$ appears as the correlator corresponding to the following remarkable Feynman diagram:

![A Feynman diagram corresponding, according to [BGK], to $\zeta(3,5)$.](image)

This graph has some combinatorial properties which can not be found in any simpler graph. For instance it is the simplest graph which does not have Hamiltonian circles, i.e. cycles without self-intersections going through all vertices. So in this respect it is like $\zeta(3,5)$, which is the simplest non–classical $\zeta$–value. It would be very interesting to recover formula (81) for the coproduct from the combinatorics of this graph.

According to s. 7.5, the correspondence principle explains why the multiple $\zeta$–values appear in calculations of correlators. On the other hand, results of the paper [BB] suggest, although do not prove, that correlators of certain physically interesting Feynman integral might be any periods, and not only very specific multiple $\zeta$–values. It is not clear whether this, if true, contradicts the correspondence principle: the non–mixed Tate periods of smallest weight can appear because of non–Tate framing corresponding to the correlator, and then spread out into higher weights. The subject needs further investigation.

We assumed before that the correlators are given by convergent integrals, and thus are well defined numbers/motives. If these integrals are divergent, they still have well-defined regularizations, given by convergent finite dimensional Feynman integrals, which deliver the interesting numbers/functions. The renormalization group enters into the game, acting on possible regularizations. The Hopf algebra $H_F$ should extend the renormalization Hopf algebra of the type studied in [CK], since the latter
concentrates on the divergent correlators, but conceptually there should be no difference between $\mathcal{H}_F$ and the renormalization algebra.

8 Appendix. Mixed Tate categories and framed objects

We address the Tannakian formalism for mixed Tate categories in the language of framed objects. For a more general set-up see chapter 3 of [G7]. Let $\text{Spec} \mathcal{O}_{F,S}$ be the scheme obtained by deleting an arbitrary set $S$ of closed points from the spectrum of the ring of integers of a number field $F$. The abelian category of mixed Tate motives over $\text{Spec} \mathcal{O}_{F,S}$ was defined in [DG]. We give a simple description of its fundamental Hopf algebra.

1. A review of the Tannakian formalism for mixed Tate categories. Let $K$ be a characteristic zero field. Let $\mathcal{M}$ be a Tannakian $K$-category with an invertible object $K(1)$. So in particular $\mathcal{M}$ is an abelian tensor category. Set $K(n) := K(1)^{\otimes n}$, where $K(-1)$ is dual to $K(1)$. Recall ([BD]) that a pair $(\mathcal{M}, K(1))$ is called a mixed Tate category if the objects $K(n)$ are mutually nonisomorphic, any simple object is isomorphic to one of them, and $\text{Ext}_\mathcal{M}^1(K(0), K(n)) = 0$ if $n \leq 0$.

A pure functor between two mixed Tate categories $(\mathcal{M}_1, K(1)_1)$ and $(\mathcal{M}_2, K(1)_2)$ is a tensor functor $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ equipped with an isomorphism $\varphi(K(1)_1) = K(1)_2$.

It is easy to show that any object $M$ of a mixed Tate category has a canonical weight filtration $W^*M$ indexed by $2\mathbb{Z}$ such that $\text{gr}^{W^*}M$ is a direct sum of copies of $K(-n)$, and morphisms are strictly compatible with the weight filtration. The functor

$$\omega = \omega_{\mathcal{M}} : M \to \text{Vect}_\bullet, \quad M \mapsto \oplus_n \text{Hom}_{\mathcal{M}}(K(-n), \text{gr}^{W^*}M)$$

to the category of graded $K$-vector spaces is a fiber functor. Let $\tilde{\omega}$ be the fiber functor to the category of finite dimensional $K$-vector spaces obtained from $\omega$ by forgetting the grading. Let $L_\bullet(\mathcal{M}) := \text{Der}(\tilde{\omega}) := \{F \in \text{End}\tilde{\omega} | F_X \otimes Y = F_X \otimes id_Y + id_X \otimes F_Y \}$

be the space of its derivations. It is a pro-Lie algebra over $K$, called the fundamental Lie algebra of the mixed Tate category $\mathcal{M}$. It has a natural grading by the integers $n < 0$ provided by the original grading on $\omega(\mathcal{M})$.

The automorphisms of the fiber functor $\tilde{\omega}$ respecting the tensor structure provide a pro-algebraic group scheme over $K$, denoted $\text{Aut}^{\tilde{\omega}}$. It is a semidirect product of the multiplicative group scheme $\mathbb{G}_m$ and a pro-unipotent group scheme $U(\mathcal{M})$. The pro-Lie algebra $L_\bullet(\mathcal{M})$ is the Lie algebra of $U(\mathcal{M})$. The action of $\mathbb{G}_m$ provides a grading on $L_\bullet(\mathcal{M})$.

According to the Tannakian formalism the functor $\tilde{\omega}$ provides an equivalence between the category $\mathcal{M}$ and the category of finite dimensional modules over the pro-group scheme $\text{Aut}^{\tilde{\omega}}$. This category is naturally equivalent to the category of graded finite dimensional modules over the group scheme $U(\mathcal{M})$. Since $U(\mathcal{M})$ is pro-unipotent, the last category is equivalent to the category of graded finite dimensional modules over the graded pro-Lie algebra $L_\bullet(\mathcal{M})$.

Pure functors $(\mathcal{M}_1, K(1)_1) \to (\mathcal{M}_2, K(1)_2)$ between the mixed Tate categories are in bijective correspondence with the graded Lie algebra morphisms $L_\bullet(\mathcal{M}_2) \to L_\bullet(\mathcal{M}_1)$.

Let $L_\bullet(\mathcal{M}) := \text{End}(\tilde{\omega})$ be the space of all endomorphisms of the fiber functor $\tilde{\omega}$. It is a graded Hopf algebra isomorphic to the universal enveloping algebra of the Lie algebra $L_\bullet(\mathcal{M})$.

Recall that a Lie coalgebra is a vector space $\mathcal{D}$ equipped with a linear map $\delta : \mathcal{D} \to \Lambda^2 \mathcal{D}$ such that the composition $\mathcal{D} \xrightarrow{\delta} \Lambda^2 \mathcal{D} \xrightarrow{\Lambda^3 \delta} \Lambda^3 \mathcal{D}$ is zero. If $\mathcal{D}$ is finite dimensional then it is a Lie coalgebra if and only if its dual is a Lie algebra.

Let $G$ be a unipotent algebraic group over $\mathbb{Q}$. Then the ring of regular functions $\mathbb{Q}[G]$ is a Hopf algebra with the coproduct induced by the multiplication in $G$. The (continuous) dual of its completion at the group unit $e$ is isomorphic to the universal enveloping algebra of the Lie algebra.
there exists a unique antipode map $S$ on $A$. Let $\mu : A \otimes A \to A$ be a commutative bialgebra graded by the integers $n \geq 0$.

Removing the antipode $S$ is a Lie coalgebra. The cobracket $\delta$ on $L^\bullet(M)$ is induced by the restricted coproduct

$$\Delta'(X) := \Delta(X) - (X \otimes 1 + 1 \otimes X)$$

on $U^\bullet(M)^\vee$. The graded dual of the Lie coalgebra $L^\bullet(M)$ is identified with the fundamental Lie algebra $L^\bullet(M)$. Below we recall a more efficient way to think about it.

2. The Hopf algebra of framed objects in a mixed Tate category (cf. [BGSV], [G7]). Recall that a Hopf algebra over a field $K$ is an associative $K$-algebra $A$ with a product $\mu : A \otimes A \to A$ and a unit $i : A \to K$, equipped in addition with a comultiplication $\Delta : A \to A \otimes A$, a counit $\varepsilon : K \to A$, and an antipode $S : A \to A$. They must obey the following properties:

1. The maps $\Delta$ and $\varepsilon$ define a structure of a coassociative coalgebra on $A$.
2. The maps $\Delta : A \to A \otimes A$ and $\varepsilon : A \to K$ are homomorphisms of algebras.
3. The map $S$ is a linear isomorphism satisfying the relations

$$\mu \circ (S \otimes \text{id}) \circ \Delta = \mu \circ (\text{id} \otimes S) \circ \Delta = i \circ \varepsilon$$

(98)

Removing the antipode $S$ and condition (98) from the list of axioms we get a definition of a bialgebra.

**Lemma 8.1** Let $A^\bullet$ be a commutative bialgebra graded by the integers $n \geq 0$ such that $A_0 = k$. Then there exists a unique antipode map $S$ on $A^\bullet$.

**Proof.** The condition (98) determines uniquely the restriction of the map $S$ to $A_n$ by induction. One has $S|A_0 = \text{id}$. Then, for example $S|A_1 = -\text{id}$ and $S|A_2 = -\text{id} + \mu \Delta_{1,1}$ where $\Delta_{1,1}$ is the $A_1 \otimes A_1$ component of the coproduct. The lemma is proved.

Let $n \geq 0$. Say that $M$ is an $n$-framed object of $M$, denoted $(M, v_0, f_n)$, if it is supplied with non-zero morphisms $v_0 : K(0) \to gr^W_0 M$ and $f_n : gr^W_{-2n} M \to K(n)$. Consider the coarsest equivalence relation on the set of all $n$-framed objects for which $M_1 \sim M_2$ if there is a map $M_1 \to M_2$ respecting the frames. For example replacing $M$ by $W_0 M/W_{-2n} M$ we see that any $n$-framed object is equivalent to a one $M$ with $W_{-2-2n} M = 0$, $W_0 M = M$. Let $A_n(M)$ be the set of equivalence classes. It has the structure of an abelian group with the composition law defined as follows:

$$[M, v_0, f_n] + [M', v'_0, f'_n] = [M \oplus M', (v_0, v'_0), f_n + f'_n]$$

It is straightforward to check that the composition law is well defined on equivalence classes of framed objects. Indeed, if $\varphi : \tilde{M} \to M$ is a morphism providing an equivalence of the framed objects $[\tilde{M}, \tilde{v}_0, \tilde{f}_n] \sim [M, v_0, f_n]$ then $\varphi \oplus \text{id} : \tilde{M} \oplus M' \to M \oplus M'$ also provides an equivalence. The neutral element is $K(0) \oplus K(n)$ with the obvious frame. The inversion is given by

$$-[M, v_0, f_n] := [M, -v_0, f_n] = [M, v_0, -f_n]$$

See ch. 2 of [G8] for a proof of these two facts. The composition $f_0 \circ v_0 : K(0) \to K(0)$ provides an isomorphism $A_0(M) = K$. It follows from the very definitions that there is a canonical isomorphism

$$A_1(M) = \text{Ext}^1_M(K(0), K(1))$$
The tensor product induces the commutative and associative multiplication

\[ \mu : \mathcal{A}_k(\mathcal{M}) \otimes \mathcal{A}_\ell(\mathcal{M}) \to \mathcal{A}_{k+\ell}(\mathcal{M}) \]

One verifies that it is well defined on equivalence classes using an argument similar to the one used for checking the additive structure on \( \mathcal{A}_k(\mathcal{M}) \). The unit is given by \( 1 \in K = \mathcal{A}_0(\mathcal{M}) \). The counit is the projection of \( \mathcal{A}_*(\mathcal{M}) \) onto its 0-th component.

Let us define the comultiplication

\[ \Delta = \bigoplus_{0 \leq p \leq n} \Delta_{p,n-p} : \mathcal{A}_n(\mathcal{M}) \to \bigoplus_{0 \leq p \leq n} \mathcal{A}_p(\mathcal{M}) \otimes \mathcal{A}_{n-p}(\mathcal{M}) \]

Choose a basis \( \{b_i\} \), where \( 1 \leq i \leq m \), of \( \text{Hom}_\mathcal{M}(K(p), \text{gr}_W^W M) \) and the dual basis \( \{b'_i\} \) of \( \text{Hom}_\mathcal{M}(\text{gr}_W^{-2p} M, K(p)) \). Then

\[ \Delta_{p,n-p}[M, v_0, f_n] := \sum_{i=1}^m [M, v_0, b'_i] \otimes [M, b_i, f_n](-p) \]

In particular \( \Delta_{0,n} = id \otimes 1 \) and \( \Delta_{n,0} = 1 \otimes id \). Set \( \mathcal{A}_*(\mathcal{M}) := \oplus \mathcal{A}_n(\mathcal{M}) \).

**Theorem 8.2** a) \( \mathcal{A}_*(\mathcal{M}) \) has the structure of a graded Hopf algebra over \( K \) with the commutative multiplication \( \mu \) and the comultiplication \( \Delta \).

b) The Hopf algebra \( \mathcal{A}_*(\mathcal{M}) \) is canonically isomorphic to the Hopf algebra \( \mathcal{U}_*(\mathcal{M})^\vee \).

**Definition 8.3** The Hopf algebra \( \mathcal{A}_*(\mathcal{M}) \) is called the fundamental Hopf algebra of a mixed Tate category \( \mathcal{M} \).

**Proof.** a) Let us show that the coproduct is well defined on equivalence classes of framed objects. It is sufficient to prove this for equivalences given by injective and surjective morphisms in \( \mathcal{M} \). Indeed, if \( \varphi : M \to M' \) respects the frames in \( M \) and \( M' \) then the projection \( M \to M/\text{Ker}(\varphi) \) and injection \( M/\text{Ker}(\varphi) \to M' \) also respect the frames. Let us suppose that

\[ \varphi : [M, v_0, f_n] \leftrightarrow [M', v'_0, f'_n] \]

is an equivalence. Choose a basis \( \{b_i\} \) of \( \text{Hom}_\mathcal{M}(K(p), \text{gr}_W^W M') \) such that the first \( s \) basis vectors of \( \{b_i\} \) is a basis in \( \text{Hom}_\mathcal{M}(K(p), \text{gr}_W^W M) \). Denote by \( b'_i \) the dual basis. Then \( [M', v_0, b'_i] = 0 \) for \( i > s \). Indeed, we may assume that \( W_{-2p-2} M' = 0 \), and also \( \text{gr}_0^W M' = K(0) \). Then there is a natural injective morphism \( K(p) \oplus M \to M' \) respecting the frames, and a projection \( K(p) \oplus M/\text{W}_2 M = K(p) \oplus K(0) \). The statement is proved. The arguments in the case of the projection are similar (and can be obtained by dualization).

It is straightforward to show that \( \Delta \) is a homomorphism of algebras, and that \( \Delta \) is coassociative. The part a) of the theorem is proved.

b) We follow the proof of theorem 3.3 in [G7], making some necessary corrections. A canonical isomorphism \( \varphi : \mathcal{A}_*(\mathcal{M}) \to \mathcal{U}_*(\mathcal{M})^\vee \) is constructed as follows. Let \( F \in \text{End}(\mathcal{M})_n \) and \( [M, v_0, f_n] \in \mathcal{A}_n(\mathcal{M}) \). Denote by \( F_M \) the endomorphism of \( \omega(M) \) provided by \( F \). Then

\[ \langle \varphi([M, v_0, f_n]), F \rangle := \langle f_n, F_M(v_0) \rangle \]

It is obviously well defined on equivalence classes of framed objects. \( \varphi \) is evidently a morphism of graded Hopf algebras. Let us show that \( \varphi \) is surjective. Recall that for a commutative, non-negatively graded Hopf algebra \( \mathcal{A}_* \) the space of primitives \( \text{CoLie}(\mathcal{A}_*) := \mathcal{A}_{>0}/\mathcal{A}_{>0}^2 \) has a Lie coalgebra structure,
and the dual to the universal enveloping algebra of the dual Lie algebra is canonically isomorphic to $A_*$. Since $\varphi$ is a map of Hopf algebras it provides a morphism of the Lie coalgebras

$$\varphi : \text{CoLie}(A_*(M)) \rightarrow \text{CoLie}(U_*(M)^\vee)$$

Let us show that this map is surjective. Let $f$ be a functional on $\text{CoLie}(U_*(M)^\vee)$ which is zero on the graded components of degree $\neq n$. Consider $f$ as a functional on $U_*(M)^\vee$, and denote by $\text{Ker}(f)$ its kernel. Then $U_*(M)^\vee/\text{Ker}(f) =: K_{(-n)}$ is a one-dimensional space sitting in degree $-n$. The graded $K$-vector space $K_{(0)} \oplus U_*(M)^\vee/\text{Ker}(f)$ has a $U_*(M)^\vee$-module structure: an element $X \in U_{>0}(M)^\vee$ sends $1 \in K_{(0)} \mapsto f(X) \in K_{(-n)}$ and annihilates $K_{(-n)}$. Since by the definition of $f$ the square of the augmentation ideal acts by zero, we get a well defined action of $U_*(M)^\vee$. Therefore the map $\varphi$ is surjective. Dualizing $\varphi$ we get an injective map of Lie algebras, and hence an injective map of the corresponding universal enveloping algebras. Dualizing the latter map we prove the surjectivity of $\varphi$.

Now let us show that $\varphi$ is injective. Using the Tannaka theory we may assume that $M$ is the category of finite dimensional representations of the Hopf algebra $U_*(M)$. Suppose that $\varphi([M, v_0, f_n]) = 0$. Consider the cyclic submodule $U_*(M) \cdot K_{(0)}$. It has no non-zero components in degree $-n$, since otherwise $\varphi[M, v_0, f_n] \neq 0$. Thus there are maps

$$K_{(0)} \oplus K_{(-n)} \leftarrow U_*(M) \cdot K_{(0)} \oplus K_{(-n)} \rightarrow M/W_{<-n}M$$

respecting the frames. Thus $[M, v_0, f_n] = 0$. The part b), and hence the theorem, are proved.

Let $M^*$ be the object dual to $M$. Under the equivalence between the tensor category $\mathcal{M}$ and the category of graded finite dimensional comodules over the Hopf algebra $A_*(M)$ an object $M$ of $\mathcal{M}$ corresponds to the graded comodule $\omega(M)$ with $A_*(M)$-coaction $\omega(M) \otimes \omega(M^*) \rightarrow A_*(M)$ given by the formula

$$x_m \otimes y_n \rightarrow \text{the class of } M \text{ framed by } x_m, y_n \quad (99)$$

We call the right hand side the matrix coefficient of $M$ corresponding to $x_m, y_n$.

The restricted coproduct $\Delta'$ provides the quotient $A_*(M)/(A_{>0}(M))^2$ with the structure of a graded Lie coalgebra with cobracket $\delta$. It is canonically isomorphic to $L_*(M)$.

**Lemma 8.4** Each equivalence class of $n$-framed objects contains a unique minimal representative, which appears as a subquotient in any $n$-framed object from the given equivalence class.

**Proof.** Suppose we have morphisms $M_1 \leftarrow \overset{f}{N} \overset{g}{\rightarrow} M_2$ between the $n$-framed objects respecting the frames. For any $n$-framed object $X$ we can assume that $\text{gr}_{2n}^W X = 0$ unless $-n \leq m \leq 0$ as well as $\text{gr}_{-2n}^W X = Q(n)$; $\text{gr}_0^W X = Q(0)$. Taking the subquotient $\text{Im}(f)$ of $M_1$ we may suppose that $f$ is surjective. Then $\text{gr}_0^W \text{Ker}(f) = \text{gr}_{2n}^W \text{Ker}(f) = 0$. Therefore $g(\text{Ker}(f))$ has the same property, and hence $M_2$ is equivalent to $N' := M_2/\text{Ker}(f)$. The lemma follows from these remarks.

3. **Examples of mixed Tate categories.** Here is a general method of getting mixed Tate categories (cf. [BD]). Let $\mathcal{C}$ be any Tannakian $K$-category and $K(1)$ be a rank one object of $\mathcal{C}$ such that the objects $K(i) := K(1)^{\otimes i}$, $i \in \mathbb{Z}$, are mutually nonisomorphic. An object $M$ of $\mathcal{C}$ is called a mixed Tate object if it admits a finite increasing filtration $W_*$ indexed by $2\mathbb{Z}$, such that $\text{gr}_{2n}^W M$ is a direct sum of copies of $K(p)$. Denote by $\mathcal{T}\mathcal{C}$ the full subcategory of mixed Tate objects in $\mathcal{C}$. Then $\mathcal{T}\mathcal{C}$ is a Tannakian subcategory in $\mathcal{C}$ and $(\mathcal{T}\mathcal{C}, K(1))$ is a mixed Tate category. Further, if $(\mathcal{C}_i, K(1,i))$ are as above and $\varphi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a tensor functor equipped with an isomorphism $\varphi(K(1,1)) = K(1,2)$, then $\varphi(\mathcal{T}\mathcal{C}_1) \subset \mathcal{T}\mathcal{C}_2$ and the restriction of $\varphi$ to $\mathcal{T}\mathcal{C}_1$ is a pure functor.

1. **The category of $Q$-rational Hodge-Tate structures.** Applying the above construction to the category $\text{MHS}/\mathbb{Q}$ of mixed Hodge structures over $\mathbb{Q}$ we get the mixed Tate category $\mathcal{H}_T$ of $Q$-rational
Hodge-Tate structures. Namely, a $\mathbb{Q}$-rational Hodge-Tate structure is a mixed Hodge structure over $\mathbb{Q}$ with $h^{p,q} = 0$ if $p \neq q$. Equivalently, a Hodge-Tate structure is a $\mathbb{Q}$-rational mixed Hodge structure with weight $-2k$ quotients isomorphic to a direct sum of copies of $\mathbb{Q}(k)$. Set $\mathcal{H}_\bullet := A_\bullet(\mathcal{H}_T)$. Then

$$\mathcal{H}_1 \cong Ext^1_{\mathcal{H}_T}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \frac{\mathbb{C}}{2\pi i \mathbb{Q}} \cong \mathbb{C}^* \otimes \mathbb{Q} \quad (100)$$

**Example.** The extension corresponding via the isomorphism (100) to a given $z \in \mathbb{C}^*$ is provided by the $\mathbb{Q}$-rational Hodge-Tate structure $H(z)$, also denoted $\log^H(z)$, defined by the period matrix

$$\begin{pmatrix} 1 & 0 \\ \log(z) & 2\pi i \end{pmatrix}$$

Namely, denote by $H_\mathcal{C}$ the two dimensional $\mathbb{C}$-vector space with basis $e_0, e_{-1}$. The $\mathbb{Q}$-vector space $H(z)_\mathbb{Q}$ is the subspace generated by the columns of the matrix, i.e. the vectors $e_0 + \log z \cdot e_{-1}$ and $2\pi i e_{-1}$. The weight filtration on it is given by

$$W_0 H(z)_\mathbb{Q} = H(z)_\mathbb{Q}, \quad W_{-1} H(z)_\mathbb{Q} = W_2 H(z)_\mathbb{Q} = 2\pi i \cdot e_{-1}, \quad W_{-3} H(z)_\mathbb{Q} = 0$$

The Hodge filtration is defined by $F^1 H_\mathbb{C} = 0$, $F^0 H_\mathbb{C} = \langle e_0 \rangle$, $F^{-1} H_\mathbb{C} = H_\mathbb{C}$.

It is easy to see that

$$Ext^1_{\mathcal{H}_T}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \frac{\mathbb{C}}{(2\pi i)^n \mathbb{Q}} \cong \mathbb{C}^*(n-1) \otimes \mathbb{Q} \quad \text{for } n > 0$$

It was proved by Beilinson that the higher Ext groups are zero. So the fundamental Lie algebra of the category of Hodge-Tate structures is isomorphic to a free graded pro-Lie algebra over $\mathbb{Q}$ generated by the $\mathbb{Q}$-vector spaces $(\mathbb{C}^*(n-1) \otimes \mathbb{Q})^\vee$ sitting in degree $-n$ where $n \geq 1$.

2. The abelian category of mixed Tate motives over a number field. Let $F$ be a number field. Then, as explained in [L1], ch. 5 of [G2], and [DG], there exists an abelian category $\mathcal{M}_T(F)$ of mixed Tate motives over $F$ with all the needed properties. In particular it is a Tate category and

$$Ext^1_{\mathcal{M}_T(F)}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(F) \otimes \mathbb{Q}, \quad (101)$$

and the higher Ext groups vanish. Therefore the fundamental Lie algebra $L_\bullet(F)$ of the category $\mathcal{M}_T(F)$, is isomorphic to a free graded pro-Lie algebra over $\mathbb{Q}$ generated by the duals to the finite dimensional $\mathbb{Q}$-vector spaces

$$K_{2n-1}(F) \otimes \mathbb{Q} \cong [\mathbb{Z}^{\text{dim}(F,\mathcal{C})} \otimes \mathbb{Q}(n-1)]^+, \quad n > 1 \quad (102)$$

(where $+$ means the invariants under complex conjugation) sitting in degree $-n$, and by the infinite dimensional $\mathbb{Q}$-vector space $(F^* \otimes \mathbb{Q})^\vee$ for $n = -1$. The isomorphism (102) is given by the regulator map, and is due to Borel [Bo].

The category $\mathcal{M}_T(F)$ is equipped with an array of realization functors (see [DG]).

3. The mixed Tate category of lisse $\mathbb{Q}_l$-sheaves on a scheme $X$. Let $X$ be a connected coherent scheme over $\mathbb{Z}_{(l)}$ such that $\mathcal{O}^* (X)$ does not contain all roots of unity of order equal to a power of $l$. Denote by $\mathcal{F}_{\mathbb{Q}_l}(X)$ the Tannakian category of lisse $\mathbb{Q}_l$-sheaves on $X$. There is the Tate sheaf $\mathbb{Q}_l(1) := Q_l(1)_X$. Since $\mu_{l^\infty} \not\subset \mathcal{O}^* (X)$ the Tate sheaves $\mathbb{Q}_l(m)_X$ are mutually nonisomorphic. So the general construction above gives a mixed Tate category $\mathcal{T} \mathcal{F}_{\mathbb{Q}_l}(X)$ of lisse $\mathbb{Q}_l$-sheaves on $X$. It was considered by Beilinson and Deligne in [BD], see an exposition in s. 3.7 of [G1]. Let us stress that in the category $\mathcal{F}_{\mathbb{Q}_l}(X)$ one may not have $Ext^1(\mathbb{Q}_l(0), \mathbb{Q}_l(n)) = 0$ for $n < 0$. However in the category $\mathcal{T} \mathcal{F}_{\mathbb{Q}_l}(X)$ we force this to be true.
In particular, let $F$ be a field which does not contain all $l^\infty$ roots of unity. For example, $F$ can be a number field. Then we get a category $TF_{\mathrm{Q}}(F)$ of $l$-adic mixed Tate Galois modules. The underlying vector space of the representation provides another fiber functor on this category. It can be related to the canonical fiber functor as follows. Consider the subcategory $TF_{\mathrm{Q}}(F,p)$ of the Galois modules unramified at a prime $p$. Then making the choices i) and ii) as in Chapter 1.4, we get an isomorphism of the two fiber functors. It follows that for any finite dimensional mixed Tate Galois representation $V$ unramified at $p$ the Lie algebra $G_V$ of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_\infty))$ is isomorphic to the image of the fundamental Lie algebra of this mixed Tate category acting on $\omega(V)$. The weight filtration on $V$ induces a weight filtration on the Lie algebra $G_V$, and on the associated graded the above isomorphism is canonical. The isomorphism of fiber functors allows to consider the equivalence classes of the framed Galois representations as the functions on the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_\infty))$. This way we identify the $I^{(l)}$ and $I_{F,p}^{(l)}$-versions of the $l$-adic iterated integrals discussed in Chapter 1.4. This identification preserves the coproduct. So the description of the coproduct for $I_{F,p}^{(l)}$ follows from the corresponding result for their $I^{(l)}$-counterpart.

4. The hypothetical abelian category of mixed Tate motives over an arbitrary field $F$. It is supposed to be a mixed Tate category with the Ext groups given by Beilinson’s formula

$$\text{Ext}^i_{\mathcal{M}_T(F)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \text{gr}_n^\mathbb{Z}K_{2n-i}(F) \otimes \mathbb{Q},$$

In the number field case it is known that the right hand side is trivial for $i > 1$, so we arrive at (104). The corresponding motivic Lie algebra is not free in general. The formula (104) leads to the canonical isomorphisms

$$\mathcal{A}_1(\mathcal{M}_T(F)) = \text{Ext}^1_{\mathcal{M}_T(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) = K_1(F) \otimes \mathbb{Q} = F^* \otimes \mathbb{Q} \quad \text{(105)}$$

We will often employ a notation $\mathcal{A}_*(F)$ for $\mathcal{A}_*(\mathcal{M}_T(F))$, and call it the motivic Hopf algebra of a field $F$. The grading of the Hopf algebra $\mathcal{A}_*(F)$ provides an action of the multiplicative group $\mathbb{G}_m$ on the proalgebraic group scheme given by the spectrum of $\mathcal{A}_*(F)$ viewed as a commutative algebra. The corresponding semidirect product of $\mathbb{G}_m$ and $\text{Spec}(\mathcal{A}_*(F))$ is called the motivic Tate Galois group of $F$.

4. The fundamental Hopf algebra of the abelian category of mixed Tate motives over the ring of $S$-integers in a number field. Let $M$ be an object of a mixed Tate category $\mathcal{M}$. We say that a framed object $(M, x_n, y_m)$ is a matrix element of amplitude $n - m$ of $M$ if $x_n$ and $y_m$ are the framings at the weights $-2n$ and $-2m$ respectively. It is, by definition, an element of $\mathcal{A}_{n-m}(\mathcal{M})$.

Let $\mathcal{O}$ be the ring of integers in a number field $F$, $S$ a set of prime ideals in $\mathcal{O}$, and $\mathcal{O}_{F,S}$ the localization of $\mathcal{O}$ at the primes belonging to $S$. In [DG] an abelian category $\mathcal{M}_T(\mathcal{O}_{F,S})$ of mixed Tate motives over $\mathcal{O}_{F,S}$ is defined as a subcategory of $\mathcal{M}_T(F)$. Namely, an object $M$ of $\mathcal{M}_T(F)$ belongs to the subcategory $\mathcal{M}_T(\mathcal{O}_{F,S})$ if and only if all amplitude one matrix elements of $M$ provide the extension classes with the invariants in $\mathcal{O}_{F,S} \otimes \mathbb{Q} \hookrightarrow F^* \otimes \mathbb{Q}$. It was proved in [DG] that it is a mixed Tate category with

$$\text{Ext}^1_{\mathcal{M}_T(\mathcal{O}_{F,S})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \text{gr}_n^\mathbb{Z}K_{2n-1}(\mathcal{O}_{F,S}) \otimes \mathbb{Q} \quad \text{(106)}$$

and the higher Ext–groups vanish. We want to determine the fundamental Hopf algebra of $\mathcal{M}_T(\mathcal{O}_{F,S})$.

**Lemma 8.5** Let $\mathcal{A}_*(\mathcal{O}_{F,S})$ be the maximal Hopf subalgebra of $\mathcal{A}_*(\mathcal{M}_T(F)) := \mathcal{A}_*(\mathcal{M}_T(F))$ such that

$$\mathcal{A}_1(\mathcal{O}_{F,S}) = \mathcal{O}_{S}^* \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq \mathcal{A}_1(F) = F^* \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{(107)}$$

Then $\mathcal{A}_*(\mathcal{O}_{F,S})$ is canonically isomorphic to the fundamental Hopf algebra of the category $\mathcal{M}_T(\mathcal{O}_{F,S})$. 47
Proof. Observe that the Hopf algebra $A_n(O_{F,S})$ can be defined inductively:

$$A_n(O_{F,S}) = \{ x \in A_n(F) \mid \Delta'(x) \in \bigoplus_{k=1}^{n-1} A_k(O_{F,S}) \otimes A_{n-k}(O_{F,S}) \}$$ (107)

Consider the subcategory $M'_T(O_{F,S})$ of $M_T(F)$ determined by the following condition: an object $M$ lies in $M'_T(O_{F,S})$ if and only if all its matrix elements are in $A_n(O_{F,S})$. Clearly $M'_T(O_{F,S})$ is a subcategory of $M_T(O_{F,S})$. On the other hand it follows from (107) that any matrix coefficient of an object from $M_T(O_{F,S})$ lies in $A_n(O_{F,S})$. So these two subcategories coincide. The lemma is proved.

We say that an object $M$ of the category $M_T(F)$ is defined over $O_{F,S}$ if it belongs to the subcategory $M_T(O_{F,S})$.

Definition 8.6 Let $(v_0, f_n)$ be an $n$-framing on a mixed Tate motive $M$ over $F$. We say that the equivalence class of the $n$-framed mixed Tate motive $(M, v_0, f_n)$ is defined over $O_{F,S}$ if its image in $A_n(F)$ belongs to the subspace $A_n(O_{F,S})$.

Proposition 8.7 Let $(M, v_0, f_n)$ be an $n$-framed mixed Tate motive whose equivalence class is defined over $O_{F,S}$. Then its minimal representative $M$ is defined over $O_{F,S}$.

Of course $M$ itself is not necessarily defined over $O_{F,S}$.

Proof. Let us assume the opposite. Then a certain matrix element of $M$ does not belong to $A_n(O_{F,S})$. Since the minimal representative is a subquotient of every object within the equivalence class of $M$, every motive from the equivalence class has such a matrix coefficient. Thus no representative in this equivalence class is defined over $O_{F,S}$. On the other hand by Lemma 8.5 the Hopf algebra $A_n(O_{F,S})$ is the fundamental Hopf algebra of the category $M_T(O_{F,S})$. Thus there must be an $n$-framed object of the latter category which is equivalent to the one $(M, v_0, f_n)$. This contradiction proves the proposition.

So to produce a mixed Tate motive over $O_{F,S}$ it is enough to produce an element of $A_n(O_{F,S})$: then we take an $n$-framed mixed Tate motive representing this element and its minimal representative.

Remark. Let us recall the following rigidity result:

$$K_n(O_{F,S}) \otimes \mathbb{Q} = K_n(F) \otimes \mathbb{Q}, \quad \text{for } n > 1$$

Combined with (105), it implies that for $n > 1$ one has

$$\text{Ext}^1_{M_T(O_{F,S})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \text{Ext}^1_{M_T(F)}(\mathbb{Q}(0), \mathbb{Q}(n))$$ (108)

It shows that the only difference between the Ext’s in the categories of mixed Tate motives over $F$ and $O_{F,S}$ is the group $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$. This group is infinite dimensional for the motives over $F$ and finite dimensional for the motives over the $S$-integers. As a result the category of mixed Tate motives over the $S$-integers is “much smaller” than the one over $F$. For instance all graded components of $A_\bullet(F)$ are infinite dimensional $\mathbb{Q}$-vector spaces, while for $A_\bullet(O_{F,S})$ they are finite dimensional.

In the next subsection we show how to apply this remark to obtain estimates from above on the $\mathbb{Q}$-vector spaces spanned by periods of mixed Tate motives over $O_{F,S}$.

5. Periods of mixed Tate motives over $O_{F,S}$. Given an embedding $\sigma : F \hookrightarrow \mathbb{C}$, let us define an increasing family of $\mathbb{Q}$-vector subspaces in $\mathbb{C}$, depending on the choice of $\sigma$:

$$\mathbb{Q} = \mathcal{P}_{\leq 0}(O_{F,S}) \hookrightarrow \mathcal{P}_{\leq 1}(O_{F,S}) \hookrightarrow \mathcal{P}_{\leq 2}(O_{F,S}) \hookrightarrow \ldots$$

Let $(H, v_0, f_n)$ be a $\mathbb{Q}$-Hodge-Tate structure framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$. Let us choose a splitting of the weight filtration on $H_\mathbb{Q}$. Then there is a period $p(H, v_0, f_n) \in \mathbb{C}$ defined as follows. Using the splitting we lift the frame vector $v_0 \in \text{gr}_W^0 H_\mathbb{Q}$ to a vector $v_0' \in W_0 H_\mathbb{Q}$ which projects to $v_0$. Projecting

$$W_0 H_\mathbb{C} \rightarrow W_0 H_\mathbb{C}/W_{-2n-1} H_\mathbb{C} \rightarrow \text{gr}_W^{-n} H_\mathbb{C}$$
we get a vector \( v''_0 \in \text{gr}^{-n}_F H_C \) out of \( v'_0 \). Applying the frame functional \( \text{gr}^{-n}_F H_C \cong \text{gr}^{-2n}_W H_C \to \mathbb{C} \) to \( v''_0 \) we get the number \( p(H, v_0, f_n) \in \mathbb{C} \).

Let us denote by \( r^H_\sigma \) the Hodge realization functor on the category \( M_T(F) \) corresponding to an embedding \( \sigma : F \to \mathbb{C} \).

**Definition 8.8** The \( \mathbb{Q} \)-vector space \( P^\sigma_{\leq n}(O_{F,S}) \) is spanned over \( \mathbb{Q} \) by the periods of \( n \)-framed Hodge-Tate structures \( r^H_\sigma(M, v_0, f_n) \), where \( M \in M_T(O_{F,S}) \), for all possible splittings of the weight filtration on \( r^H_\sigma(M) \) and all \( n \)-framings on \( M \).

We call \( P^\sigma_{\leq n}(O_{F,S}) \) the space of weight \( \leq n \) periods of mixed Tate motives over \( O_{F,S} \). Since the period map on the splitted framed Hodge-Tate structures apparently commutes with the product (i.e. the period of the tensor product of the splitted framed Hodge-Tate structures is given by the product of the corresponding periods), the space

\[
P^\sigma(O_{F,S}) := \bigcup_{n \geq 0} P^\sigma_{\leq n}(O_{F,S})
\]

is a filtered algebra over \( \mathbb{Q} \). Consider its associate graded:

\[
P^\sigma_n(O_{F,S}) := \frac{P^\sigma_{\leq n}(O_{F,S})}{P^\sigma_{\leq n-1}(O_{F,S})}; \quad P^\sigma_\bullet(O_{F,S}) := \bigoplus_{n=0}^\infty P^\sigma_n(O_{F,S})
\]

**Theorem 8.9** There is a surjective homomorphism of commutative algebras

\[
p_\sigma : A_\bullet(O_{F,S}) \to P^\sigma_\bullet(O_{F,S}) \tag{109}
\]

**Proof.** The crucial claim is that the map \( p_\sigma \) is well-defined. Let us pick a representative \( (M, v_0, f_n) \) of an element of \( A_n(O_{F,S}) \). Choosing a splitting of the weight filtration of its Hodge realization we get its period. Changing a splitting we get another period, which differ from the first one by an element of \( P^\sigma_{\leq n-1}(O_{F,S}) \). So its projection to \( P^\sigma_n(O_{F,S}) \) is independent of the splitting. If \( (M', v'_0, f'_n) \) is a subquotient of \( (M, v_0, f_n) \), then one easily sees that \( p_\sigma(M', v'_0, f'_n) = p_\sigma(M, v_0, f_n) \).

Finally, the map \( p_\sigma \) commutes with the product. The theorem is proved.

**Corollary 8.10** One has \( \dim_\mathbb{Q} P^\sigma_n(O_{F,S}) \leq \dim_\mathbb{Q} A_n(O_{F,S}) \).

Recall the tensor algebra \( T(V_\bullet) \) generated by the graded \( \mathbb{Q} \)-vector space \( V_\bullet \). It has a graded Hopf algebra structure, with the commutative product given by the shuffle product formula, and the co-product given by the deconcatenation. Applying the functor \( T \) to the graded space \( \bigoplus_{n \geq 1} K_{2n-1}(O_{F,S})_{\mathbb{Q}} \), where \( K_{2n-1} \) is in the degree \( n \), and using \( 106 \), we get an isomorphism of Hopf algebras

\[
A_\bullet(O_{F,S}) \cong T \left( \bigoplus_{n \geq 1} K_{2n-1}(O_{F,S})_{\mathbb{Q}} \right) \tag{110}
\]

This isomorphism, combined with Corollary [S.10] implies an estimate from above for \( \dim_\mathbb{Q} P^\sigma_n(O_{F,S}) \).

The following conjecture tells us that this estimate is expected to be exact.

**Conjecture 8.11** The map \( 109 \) is an isomorphism.

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