Why can an electron mass vary from zero to infinity?

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April 1, 2022

Abstract

When a particle is in high speed or bound in the Coulomb potential of point nucleus, the variation of its mass can be ascribed to the variation of relative ratio of hiding antimatter to matter in the particle. At two limiting cases, the ratio approaches to 1.

The Einstein mass-energy relation $E = mc^2$ reveals the simple proportionality between energy $E$ and mass $m$ for any matter with $c$ being the speed of light. For a free particle moving with velocity $v$, the mass $m$ is related to its rest mass $m_0$ as $m = m_0(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, which approaches infinity when $v$ approaches to $c$. On the other hand, when an electron is bound in the Coulomb field of point nucleus with charge $Ze$ ($e > 0$) to form a hydrogen-like atom, the electron mass $m$ will decrease. It is also known that $m$ will approach to zero when $Z$ approaches 137. Here we show that all the above variation in particle mass can be ascribed to the variation of relative ratio of hiding antimatter to matter in a particle. That is the ratio of hiding positron ingredient to electron ingredient in an electron, which determines the mass of electron. At both sides $m \to \infty$ and $m \to 0$, the ratio approaches to 1.

For simplicity, we begin from a particle with mass $m_0$ but without spin. It is described in nonrelativistic quantum mechanics by the Schrödinger equation. Then its kinetic energy reads $\frac{1}{2}m_0v^2$ with velocity $v$ being unlimited and
$m_0$ unchanged. When it carries a charge ($-e$) and is bound in the Coulomb field of a hydrogenlike atom with potential energy

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \quad (1)$$

The binding energy $B$ is well known as ($\alpha = \frac{e^2}{4\pi\varepsilon_0 hc} \simeq \frac{1}{137}$ with $\hbar$ being the Plank constant)

$$B = \frac{Z^2\alpha^2}{2n^2}m_0c^2 \quad (2)$$

($n = 1, 2,...$). In other words, the mass of electron $m = m_0 - B/c^2$ would decrease without lower bound if the charge number of nucleus $Z$ is sufficiently large.

However, the situation becomes quite different in the theory of special relativity. Consider a meson $\pi^-$ binding in a point nucleus. Its wave function $\Phi(z, t)$ satisfies the Klein-Gordon (K-G) equation. [1],[2]

$$\left(i\hbar \frac{\partial}{\partial t} - V(r)\right)^2\Phi = m_0^2c^4\Phi - \frac{\hbar^2}{2m_0}\nabla^2\Phi \quad (3)$$

The main point of view in this paper is as follows. We should look at $\Phi$ being composed of two kinds of fields

$$\theta = \left(1 - \frac{V}{m_0c^2}\right)\Phi + i\frac{\hbar}{m_0c^2}\phi$$

$$\chi = \left(1 + \frac{V}{m_0c^2}\right)\Phi - i\frac{\hbar}{m_0c^2}\Phi \quad (4)$$

Then Eq.(3) can be recast into the form of coupling Schrödinger equations:

$$\left(i\hbar \frac{\partial}{\partial t} - V\right)\theta = m_0c^2\phi - \frac{\hbar^2}{2m_0}\nabla^2(\theta + \chi)$$

$$\left(i\hbar \frac{\partial}{\partial t} - V\phi\right) = -m_0c^2\chi + \frac{\hbar^2}{2m_0}\nabla^2(\chi + \theta) \quad (5)$$

Eq.(3) is invariant under the transformation ($\vec{x}' \rightarrow -\vec{x}', t \rightarrow -t$) and

$$\theta(-\vec{x}', -t) \rightarrow \chi(\vec{x}', t), V(-\vec{x}', -t) \rightarrow -V(\vec{x}', t) \quad (6)$$
The meaning of $\theta$ and $\chi$ can be seen from the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (7)$$

with the “probability density”

$$\rho = |\theta|^2 - |\chi|^2 = \theta^* \theta - \chi^* \chi \quad (8)$$

and the “current density”

$$\mathbf{j} = \frac{\mathbf{\pi}}{2m_0} [(\theta \nabla \theta^* - \theta^* \nabla \theta) + (\chi \nabla \chi^* - \chi^* \nabla \chi)$$

$$+ (\theta \nabla \chi^* - \chi^* \nabla \theta) + (\chi \nabla \theta^* - \theta^* \nabla \chi)] \quad (9)$$

We explain the field $\theta$ being the “particle (matter) ingredient” of a particle, whereas $\chi$ being the hiding “antiparticle (antimatter) ingredient” inside a particle.

See first the free motion case $V = 0$. The particle is described by a plane wave function along $z$ axis:

$$\Phi \sim \exp \left\{ \frac{i}{\hbar} (pz - Et) \right\} \quad (10)$$

Beginning from $E = m_0 c^2$, $|\chi|$ increases from zero until the limit of momentum $p \to \infty$, i.e., $E \to \infty$, or

$$\lim_{v \to c} |\chi| \to |\theta| \quad (11)$$

Let us discuss the wave packet:

$$\Phi(z, t) = \int_{-\infty}^{\infty} \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} e^{-\frac{\sigma k^2}{2\hbar}} e^{i(kz - \omega t)} dk \quad (12)$$

with $\hbar \omega = \sqrt{\hbar^2 k^2 c^2 + m_0^2 c^4} \geq m_0 c^2 + \frac{\hbar^2 k^2}{2m_0} + \cdots$

Assume $\sqrt{\sigma} \ll \frac{mc}{\hbar}$, then

$$\Phi(z, t) \simeq \frac{\left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}}}{\left( 1 + \frac{i\sigma t}{m_0} \right)^{\frac{1}{2}}} \exp \left\{ -\frac{\sigma z^2}{2 \left( 1 + \frac{i\sigma t}{m_0} \right)^{\frac{1}{2}}} - \frac{im_0 c^2 t}{\hbar} \right\} \quad (13)$$

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If consider $\frac{v}{c} \ll 1$ to ignore the spreading of wave packet in low speed case \((v \ll c)\). Then we perform a “boost” transformation, i.e., to push the wave packet to high speed \((v \rightarrow c)\) case. Thus we see in the figure 1 that:

(i) The width of packet shrinks —— Lorentz contraction.
(ii) The amplitude of $\rho$ increases——“boost” effect.
(iii) The new observation is that both $|\theta|^2$ and $|\chi|^2$ in $\rho$ increase even more sharply while keeping $|\theta| > |\chi|$ to preserve $|\theta + \chi| \sim |\Phi|$ invariant.

The ratio of hiding $|\chi|^2$ to $|\theta|^2$ reads:

$$R_{KG}^{free} = \frac{\int_{-\infty}^{\infty} |\chi|^2 \, dz}{\int_{-\infty}^{\infty} |\theta|^2 \, dz} = \left[ \frac{1 - \sqrt{1 - (\frac{v}{c})^2}}{1 + \sqrt{1 - (\frac{v}{c})^2}} \right]^2$$

(14)

It is interesting to see the stationary $1S$ state (zero angular momentum state with principal quantum number $n=1$) in field $V(r)$ shown in Eq.(1).

Now the energy level is quantized to be

$$E_{1S}^{KG} = m_0 c^2 \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2 \alpha^2}}$$

(15)

which is a function of $Z$. When $Z \rightarrow \frac{1}{2\alpha} \approx \frac{137}{2}$, the energy $E_{1S}$ decreases to a lowest limit $\frac{m_0 c^2}{\sqrt{2}}$. Meanwhile, the ratio

$$R_{1S}^{KG} = \frac{\int |\chi|^2 \, d\overline{x}}{\int |\theta|^2 \, d\overline{x}} = 1 - 4 \left[ 2 + (y + \frac{1}{2})^2 + \frac{(y + \frac{1}{2})^3}{2y} \right]^{-1}$$

(16)

(\text{where } y = \sqrt{\frac{1}{4} - Z^2 \alpha^2},) \text{ increases from zero to the upper limit 1, as shown in the figure 2.}

Next turn to the electron case. Being a particle with spin $\frac{1}{2}$, it is described by a Dirac spinor wave function

$$\Psi = \begin{pmatrix} \theta \\ \chi \end{pmatrix}$$

(17)

with four components. Here $\theta$ and $\chi$, (each with two components) usually called as the “positive” and “negative” energy components in the literature
are just the counterpart of $\theta$ and $\chi$ in Eqs. (14) for particle without spin.

However, in this case, instead of (8), we have

$$\rho_{\text{Dirac}} = \Psi^\dagger \Psi = \theta^\dagger \theta + \chi^\dagger \chi$$

(18)

Hence for a freely moving electron wave packet, instead of figure 1, we have figure 3. One sees that both $\theta^\dagger \theta$ and $\chi^\dagger \chi$ are increasing with the velocity $v$.

But they are constrained within the boosting $\rho$ and the invariant quantity during the boosting process is

$$\overline{\Psi} \Psi = \theta^\dagger \theta - \chi^\dagger \chi > 0$$

(19)

where the inequality ensures that the electron is always an electron though the hiding “antielectron (positron)” ingredient $|\chi|$ is already approaching $|\theta|$ when $v \to c$. The ratio reads

$$R_{\text{Dirac}}^{\text{free}} = \frac{1 - \sqrt{1 - \left(\frac{v}{c}\right)^2}}{1 + \sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

(20)

On the other hand, when the electron is bound inside a hydrogenlike atom, the energy level of $1S$ state is

$$E_{1S} = m_0 c^2 \left[1 + \frac{\alpha^2 Z^2}{\sqrt{1 - Z^2 \alpha^2}}\right]^{-1/2}$$

(21)

which decreases to zero when $Z \to \frac{1}{\alpha} \simeq 137$ as shown in figure 4. Meanwhile the ratio

$$R_{1S}^{\text{Dirac}} = \frac{1 - \sqrt{1 - Z^2 \alpha^2}}{1 + \sqrt{1 - Z^2 \alpha^2}}$$

(22)

increases from zero to 1, similar to Eq. (16) and the curve in figure 2.

In summary, some discussions are in order.

(a) In nonrelativistic quantum mechanics only the particle (e.g., the electron) is considered. The velocity of particle can enhance without a upper limit. On the other hand, the energy of a binding particle can decrease without a lower limit either. Its mass $m_0$ remains unchanged in any case.
(b) In relativistic quantum mechanics, a particle is always not pure. It is accompanied by its hiding antiparticle ingredient essentially. If a free rest particle with mass $m_0$ described by $\theta(\vec{x}, t)$, the accompanying $\chi(\vec{x}, t)$ will be excited coherently once the particle is set into motion or bound into an external field. Then its velocity $v$ is bound from above by a limiting speed $c$ while its energy $E$ or its changeable mass $m = E/c^2$ is bound from below by $m_0/\sqrt{2}$ (for KG particle) or zero (for Dirac particle).

(c) The common essence of any matter is the basic symmetry Eq.(6). It could be stated as a postulate that ”the space-time inversion ($\vec{x} \rightarrow -\vec{x}$, $t \rightarrow -t$) is equivalent to the transformation between particle and antiparticle”.  

(d) However, inside a particle, $\theta$ always dominates $\chi$, i.e., $|\theta| > |\chi|$. So they do not exhibit the symmetry Eq.(6) explicitly. Being the “slave” in the particle, $\chi$ has to obey the “master” $\theta$. In particular, the wave function for an electron in freely motion reads always as

$$\Psi_e^- \sim \theta \sim e^{i \frac{\hbar}{\pi} \left( \vec{p} \cdot \vec{x} - E \cdot t \right) } , \ (|\theta| > |\chi|)$$ \hspace{1cm} (23)

On the other hand, if we perform a space-time inversion, $\theta(\vec{x}, t) \rightarrow \theta (-\vec{x}, -t) = \chi_e (\vec{x}, t)$ becomes the “master”, whereas $\chi(\vec{x}, t) \rightarrow \chi (-\vec{x}, -t) = \theta_e (\vec{x}, t)$ reduces into the ”slave”. Then Eq.(23) turns into the wave function for a positron:

$$\Psi_{e^+} \sim \chi_e \sim \theta_e \sim e^{i \frac{\hbar}{\pi} \left( \vec{p} \cdot \vec{x} - E \cdot t \right) } , \ (|\chi_e| > |\theta_e|)$$ \hspace{1cm} (24)

(e) Note that, the “slave” $\chi$ in a particle can not display itself as $\chi_e$ in an antiparticle, so it does not show the opposite charge. What it can do is to pull back the motion of $\theta$, thus the inertial mass $m$ of particle enhances without a limit while its velocity has a limit $c$. Meanwhile, the instinct of $\chi$ demands the time evolution of phase in the wave function like that of $\chi_e$ in Eq.(24). But now it is forced to follow that in Eq.(23). They are in opposite directions. So a moving clock accompanying the particle is slower and slower with the enhancement of $\chi$ inside it.

(f) The ratio $R < 1$ could be viewed as an order parameter characterizing the status of a “particle”. Formally, if we always define $R = \int |\chi|^2 \, d \vec{x}$
\[ \int |\theta|^2 \, d\vec{x}, \] then \( R > 1 \) will characterize the status of an “antiparticle”. In other words, we look at the “negative energy” state of a particle directly as the “positive energy” state of its antiparticle, either for KG particle or for Dirac particle. It seems to us that the historical mission of the concept of hole theory for electron is coming to an end.

(g) Actually, all the strange effects (including the Lorentz transformation) in special relativity can be derived by the symmetry Eq.(3) in combination with the principles of quantum mechanics. \[ \] (see also, G-j Ni and S-q Chen, Internet, hep.th/9508069 (1995) and G-j Ni, hep.th/9708156). The calculation shown in this paper provides the further support to the point of view by one of us (Ni) on contemporary physics as discussed in Ref \[ \]

Acknowledgments. This work was supported in part by the NSF in China.

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References

[1] Bjorken, J. D. and Drell, S. D., \textit{Relativistic Quantum Mechanics}, McGraw-Hill Book Company, 1964.

[2] Sakurai, J.J., \textit{Advanced Quantum Mechanics}, Addison-Wesley, 1967.

[3] Ni Guang-jiong, The relation between space-time inversion and particle-antiparticle transformation, Journal of Fudan University (Natural Science), 1974, No.3-4, 125-134.

[4] Ni G-j and Chen S-q, On the essence of special relativity, ibid, 35 (3),325-334 (1996)

[5] Ni G-j and Chen S-q, Relativistic stationary Schrödinger equation for many-particle system, ibid, 36 (3) 247-252 (1997)

[6] Ni G-j, To enjoy the morning flower in the evening——Is special relativity a classical theory? Kexue (Science) 50 (1) 29-33 (1998); Internet, quant-ph/9803034.

[7] Ni G-j, To enjoy the morning flower in the evening——Where is the subtlety of quantum mechanics? ibid,50 (2) 38-42 (1998); Internet, quant-ph/9804013.
[8] Ni G-j. To enjoy the morning flower in the evening——What does the appearance of infinity in physics imply? ibid, 50 (3) 36-40 (1998); Internet, quant-ph/9806009.

Figure legends:

Figure 1. The wave packet of Klein-Gordon particle (e.g. $\pi^-$) for four velocities. (a) $v=0.5c$. (b) $v=0.9c$. (c) $v=0.99c$. (d) $v=0.99999c$. The dash, dot, and solid curves denote the profiles of $|\theta|^2$, $|\chi|^2$, and $\rho = |\theta|^2 - |\chi|^2$ respectively. $\xi = m_0c(z - vt)/\hbar$ is a dimensionless quantity.

Figure 2. The dash and solid curve denote $E_{1S}^{KG}/m_0c^2$ and $R_{1S}^{KG}$ versus $Z/68.5$ respectively.

Figure 3. The wave packet of Dirac particle (e.g. the electron) for four velocities. (a) $v=0.5c$. (b) $v=0.9c$. (c) $v=0.99c$. (d) $v=0.99999c$. The dash, dot, and solid curves denote the profiles of $\theta^\dagger \theta$, $\chi^\dagger \chi$, and $\rho = \theta^\dagger \theta + \chi^\dagger \chi$ respectively. $\xi = m_0c(z - vt)/\hbar$ is a dimensionless quantity.

Figure 4. The dash and solid curve denote $E_{1S}^{Dirac}/m_0c^2$ and $R_{1S}^{Dirac}$ versus $Z/137$ respectively.