Linear instability criterion for the Korteweg–de Vries equation on metric star graphs

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Abstract
The aim of this work is to establish a novel linear instability criterion for the Korteweg–de Vries (KdV) model on metric graphs. In the case of balanced graphs with a structure represented by a finite collection of semi-infinite edges and with boundary condition of δ-type interaction at the graph-vertex, we show that the continuous tail and bump profiles are linearly unstable. In this case, the use of the analytic perturbation theory of operators as well as the extension theory of symmetric operators is fundamental in our stability analysis. The arguments shown in this investigation have prospects in the study of the instability of stationary waves solutions for nonlinear evolution equations on metric graph.

Keywords: Korteweg–de Vries model, star graph, instability, δ-type interaction, extension theory, perturbation theory
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1. Introduction

A quantum graph is a metric graph (i.e., a network-shaped structure of vertices connected by edges) with a linear Hamiltonian operator suitably defined on functions that are supported on the edges. Examples of such operators, we have Schrödinger-like operator or Airy-like operator. This specific framework appears in the simplification of models in wave propagation, for instance, in a quasi one-dimensional (e.g. meso- or nanoscale) system that looks like a thin neighborhood of a graph. Thus, we have the description of a variety of physical applications

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in chemistry and engineering (see [14, 16, 20, 45, 49] for details and references). Recently, quantum graphs have attracted much attention in the context of soliton transport in branched structures (see [60, 61]) since wave dynamics in these structures can be modeled by nonlinear evolution equations suitably defined on the edges. Specific branched systems appear in condensed matter, Josephson junction networks, polymers, optics, neuroscience, DNA, blood pressure waves in large arteries, and in shallow water models for describing a fluid network (see [3, 11–14, 17, 19–21, 31, 45, 49, 52, 54] and references therein). To address these issues of transport, in general the problem is difficult to tackle because both the nonlinear equation of motion and the ‘geometry of the graph’ can be complex. A first focus in this study is to look at what happens with the dynamic of the linear equation associated to the nonlinear model (see [14, 50]), but in many cases, however the nonlinearity cannot be neglected and so the study can become more challenging.

In recent years, the study of nonlinear dispersive models on metric graph has attracted a lot of attention of mathematicians and physicists. In particular, we have the following nonlinear vectorial Schrödinger model (see Adami et al [1, 2], Angulo et al [7, 8]),

\[ iU_t + U_{xx} + |U|^p U = 0, \]  

which has been studied on a star graph, namely, a metric graph with \(N\) half-lines of the form \((0, +\infty)\) at a common vertex \(\nu = 0\), and the vectorial Benjamin–Bona–Mahoney equation (see Bona et al [17] and Mugnolo et al [51]),

\[ (1 - \frac{\partial_x^2}{\partial t}) U_t + \partial_t U + U \partial_x U = 0, \]  

on a metric graph of the \(Y\)-junction’s type (namely, \(Y = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty)\)) or on trees.

It is important to note that with the introduction of the nonlinearity in the case of a dispersive model, the network provides a nice field where one can look for interesting soliton propagations or nonlinear dynamics in general. In the literature few analytic studies of soliton propagation through networks are known. In particular, the stability or instability mechanism of these profiles are still unclear. So, one of the main objectives of this work is to shed light on these themes. A central point that makes this analysis a delicate problem, it is the presence of a vertex (or several) where the underlying one-dimensional metric graph should bifurcate. Indeed, a soliton-profile coming into the vertex along one of the bonds shows a complicated motion around the vertex such as reflection and emergence of the radiation there, in particular, one cannot see easily how energy travels across the network. Thus, the study of the dynamics of evolution equations on metric graphs can become a challenge and as an example of this situation we have the cases of the models in (1.1) and (1.2) (see [1, 7, 8, 17]). We note that these studies show us that to obtain a fruitful study of some specific dynamic, it will depend heavily on the conditions at the vertex (or vertices) of the metric graph.

The focus of our study here will be the following vectorial Korteweg–de Vries equation (KdV henceforth)

\[ \partial_t u_e(x,t) = \alpha_e \partial_x^3 u_e(x,t) + \beta_e \partial_x u_e(x,t) + 2u_e(x,t) \partial_x u_e(x,t), \]  

on a metric graph \(\mathcal{G}\) with a structure represented by the set \(E = E_+ \cup E_-\) where \(E_+\) and \(E_-\) are finite or countable collections of semi-infinite edges \(e\) parameterized by \((0, +\infty)\) or \((-\infty, 0)\), respectively. The half-lines are connected at a unique vertex \(\nu = 0\). In this case, \(\mathcal{G}\) is also called a star-shaped metric graph or a metric star graph (see figure 1). Here \((\alpha_e)_{e \in E}\) and \((\beta_e)_{e \in E}\) are
two sequences of real numbers. We recall that the KdV equation on all the line
\[ \partial_t u + \partial_x^3 u + \partial_x u + u \partial_x u = 0, \quad u = u(x, t), \quad x \in \mathbb{R}, \quad (1.4) \]
was derived by Korteweg and de-Vries [42] in 1895 as a model for long waves propagating on a shallow water surface. Recently, the KdV equation has been appearing in other context. More precisely, this equation has been used as a model to study blood pressure waves in large arteries. In this way, for example, Chuiko and Dvornik in [27] proposed a new computer model for systolic pulse waves within the cardiovascular system based on the KdV equation. Also, Crépeau and Sorine in [29] showed that some particular solutions of the KdV equation, more exactly, the two and three-soliton well-known solutions, seem to be good candidates to match the observed pressure pulse waves. For the precise definition of $N$-solitons for the KdV model, we refer to the reader to see Lamb [46], Martel, Merle and Tsai [48] and references therein.

In a mathematical context, the Cauchy problem for the KdV in (1.4) has been studied deeply in the last years, for instance, on all the line, torus, half-lines and on a finite interval (see [18, 28, 30, 32, 33, 36, 38, 39, 41, 43] and references therein). In particular on the right half-line, we have the stability of solitons profiles established in Cavalcante and Muñoz in [24] and the study of boundary values problems via the unified transform method of Fokas in [32, 33] (we recall that this unified method can be considered as a generalization of the so-called inverse scattering transform (IST) method to initial-boundary value problems formulated on the half-line). At this point, naturally, arise several questions related to the study of the KdV model on metric graphs via the IST method, such as, Lax pair analysis, explicit soliton asymptotics for solutions, etc. As far as we know, there are no specific works in the literature on these issues associated to the KdV on graphs. This study will not be pursued here. But, we think that using the unified method of Fokas in the matrix case can be a good option for this study. We recommend the interested reader to see Caudrelier [22] for the case of the nonlinear Schrödinger equation (1.1) on a star graph.

Now, studies for the linearized KdV equation (1.3) (the Airy-type evolution equation)
\[ \partial_t u_e(x, t) = \alpha_e \partial_x^3 u_e(x, t) + \beta_e \partial_x u_e(x, t), \]
\[ x \neq 0, t \in \mathbb{R}, e \in E, \quad (1.5) \]
on star-shaped metric graphs have recently appeared in the literature. The existence and uniqueness of solutions by using potential theory have been studied in [59] and the characterization of boundary conditions under which the dynamic of (1.5) is generated either by a contraction semigroup or by a unitary group was established in Mugnolo, Noja and Seifert in [50].

As far as we know, the study of the KdV model in (1.3) in metric graphs is relatively underdeveloped. We have the recent work of Cavalcante in [23], about the local well-posedness for the Cauchy problem associated to KdV model on a $\mathcal{Y}$-junction framework with specific...
boundary conditions at the vertex. We also have the results of stabilization and boundary controllability on metric graphs with bounded edges, obtained by Ammari and Crépeau [5], Cerpa, Crépeau and Moreno [25] and Cerpa, Crépeau and Valein [26].

Up to our knowledge, there are no rigorous analytical studies of the stability of stationary solutions to the KdV model (1.3) on a metric graph. Our principal interest here will be to establish a linear instability criterion for this type of solutions. A stationary solution for the KdV model (1.3) has the following representation

$$(u_e(x, t))_{e \in E} = (\phi_{e}(x))_{e \in E},$$

where for $e \in E_-$ the profile $\phi_e : (-\infty, 0) \to \mathbb{R}$ satisfies $\phi_e(-\infty) = 0$, and for $e \in E_+$, $\phi_e : (0, +\infty) \to \mathbb{R}$ satisfies $\phi_e(+\infty) = 0$. The existence of profiles of stationary type, namely, solutions of the following nonlinear elliptic equation

$$\alpha_e \frac{d^2}{dx^2} \phi_e(x) + \beta_e \phi_e(x) + \phi_e^2(x) = 0, \quad e \in E,$$  

depend on the following solitons profiles,

$$\phi_e(x) = c(\alpha_e, \beta_e) \text{sech}^2(d(\alpha_e, \beta_e)x + p_e), \quad e \in E.$$  

For instance, for $\alpha_e > 0$ and $\beta_e < 0$, for each $e \in E$, we get several families of profiles satisfying the conditions $\phi_e(\pm \infty) = 0, e \in E_\pm$ (see section 3 below). The specific value of the shift $p_e$ in (1.7) will depend on which other conditions are given for the profile $\phi_e$ on the vertex of the graph $\nu = 0$.

The main interest of our study here is to establish a linear instability criterion for stationary profiles of the KdV model on a star-shaped metric graph $G$. A key point in our stability theory will be to determine when the Airy-type operator

$$A_0 : (u_e)_{e \in E} \mapsto \left(\alpha_e \frac{d^3}{dx^3} u_e + \beta_e \frac{d}{dx} u_e\right)_{e \in E}$$

has extensions $A_{\text{ext}}$ on $L^2(G)$, such that the dynamics induced by the following linear evolution problem

$$\begin{cases}
\dot{z} = A_{\text{ext}} z, \\
z(0) = u_0 \in D(A_{\text{ext}})
\end{cases}$$

is given by a $C_0$-group, $z(t) = e^{tA_{\text{ext}}} u_0$. Here we are interested when the extension operator $A_{\text{ext}}$ is a skew-self-adjoint operator (see (2.7) and (6.2)). Therefore, from Stone’s theorem we obtain that the dynamics in (1.9) is given by a unitary group. We note from Mugnolo, Noja and Seifert in [50] that the extension $A_{\text{ext}}$ and $A_{\text{ext}}^*$ can also be considered as dissipative, and so from Lumer–Phillips’s theorem we obtain that these extensions are the infinitesimal generator of a $C_0$-semigroup of contractions on $L^2(G)$.

Now, two delicate issues emerge at this point of the analysis, which we will need in our stability study. The first one is, how to determine the possible skew-self-adjoint extensions of the Airy operator $A_0$, and the second is related to find some kind of representation for the unitary groups associated to those extensions. With regard to the first issue, a characterization of all skew-self-adjoint extensions of $A_0$ was obtained recently by Mugnolo, Noja and Seifert in [50] via Krein spaces (see also Schubert, Seifert, Voigt and Waurick in [58]). By convenience of the reader, in section 2 we give a brief description of the results to be used from this theory.
and in this way, in proposition 2.3 we construct (initially, in the case of a metric graph with two half-lines) a one-parameter family of skew-self-adjoint extensions \((AZ, D(AZ))\), \(Z \in \mathbb{R}\), of \(\delta\)-type interaction with \(A_Z = A_0\) and

\[
D(A_Z) = \{ u = (u_-, u_+) \in H^1(-\infty, 0) \oplus H^1(0, +\infty) : u_-(0-) = u_+(0+),
\]

\[
u = 0
\]

\[
u = 0
\]

\[
\begin{align*}
&u'_+(0+) - u'_-(0-) = Zu_- (0-), \quad \frac{Z^2}{2} u_- (0-) + Zu'_-(0-) = u'_+(0+) - u'_-(0-) \}.
\end{align*}
\]

(1.10)

With regard to our second issue of specific representations of unitary groups associated to skew-self-adjoint extensions of the Airy-operator \(A_0\), it is an open problem in general. From the extension theory (see proposition 2.1) we should have \(9|E_+|^2\) families of skew-self-adjoint extensions of \(A_0\) in the case of a balanced star-shaped graph with \(|E_+|\)-positive edges, hence we need to have the same number of families of unitary groups. In our appendix A, using Green’s functions, we establish the basic informations to get, possibly, all of these representations (see lemmas A.7 and A.8). Thus, in proposition A.10 (appendix A) we establish a representation for the unitary groups \(\{e^{itAZ}\}_{t \in \mathbb{R}}\) associated to the one-parameter family of skew-self-adjoint extensions \((AZ, D(AZ))\) of \(\delta\)-type in (1.10). The case of \(\delta\)-type interactions on a general balanced star-shaped graph will be established in proposition A.12. These representation formulas will be a key point in our instability analysis of stationary profiles on balanced graphs.

Next, following our analysis, we determine which stationary solutions \((\phi_e)_{e \in E}\), with \(\phi_e\) defined in (1.7), belong to the domain \(D(A_Z)\). Indeed, we found that the only possible profiles will be of type either tail or bump (see section 3 and figures 2 and 3 above). In section 6, we extend our analysis to the case of domains of skew-self-adjoint extensions of \(\delta\)-type interaction for \(A_0\) on a star-shaped graph \(\mathcal{G}\) with \(|E_+| = |E_-| = n\) and \(n \geq 2\) (the so-called balanced star-shaped metric graphs).

In theorem 4.4 below we establish our linear instability criterion for stationary solutions of the KdV model (1.3) on metric star-shaped graphs not necessarily balanced. This instability criterion can be seen as an extension of Lopes’s result in [47] (We note that the classical stability theories in Grillakis, Shatah and Strauss [34, 35] and Pego and Weinstein [56], cannot be applied in all its extension in the case of KdV type models on graphs because the translation
symmetric property is broken on graph structures). Theorem 4.4 will be applied to the family of stationary profiles of tail and bump type that we found with vertex conditions of $\delta$-type (see figures 2–5), and so we obtain that they are linearly unstable when $|E_+| = |E_-| = n$, with $n \geq 1$, namely, on balanced graphs (see theorems 5.1 and 6.1). In the case $n = 1$, our linear instability result is based on the analytic perturbations theory of operators, while the case $n \geq 2$ it required analytic perturbation and the extension theory of symmetric operators of Krein and von Neumann. We have divided our stability study of the tails and bumps profiles into the last two cases ($n = 1$ and $n \geq 2$) to highlight how the ‘geometry of the metric graph’ induces the addition of new tools in the analysis.

We believe that our linear unstable results for tails and bumps profiles are in fact nonlinearly unstable type in the ‘energy space’ $H^1(\mathcal{G})$, and this is the focus of a work currently in progress. We recall that the implication of a linear instability result to nonlinear instability property is not an easy deal, especially when dealing with KdV models. We hope that following some ideas in Henry, Perez and Wreszinski [37], Angulo, Lopes and Neves [10] and Angulo and Natali [9], we can obtain the nonlinear instability property of these tails and bumps profiles.

We note that the arguments introduced in this work, were successfully applied in the study of the instability properties of the kink and anti-kink profiles for the following sine-Gordon model on the framework of a $Y$-junction (see Angulo and Plaza [11, 12])

$$\begin{cases}
\partial_t u_e = v_e, & e \in E = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty), \\
\partial_t v_e = c_e^2 \partial_x^2 u_e - \sin(u_e), & e \in \mathbb{R} - \{0\}.
\end{cases}$$

(1.11)

Lastly, the paper is organized as follows. In the preliminaries (section 2) we give a brief description of the existence of skew-self-adjoint extensions for the Airy-type operator in (1.8). In particular, we consider the case of extensions with boundary conditions of $\delta$-type interaction at the vertex $\nu = 0$ for two half-lines. The existence of stationary solutions with profiles of type tail or bump is given in sections 3 and 6. Our linear instability criterion on a general metric graph is established in section 4. Sections 5 and 6 are dedicated to apply the instability criterion in the cases of tail and bump profiles for the KdV model (1.3) on two half-lines and on general balanced graphs, respectively. In appendix A we briefly discuss some tools of the extension theory of Krein and von Neumann which we used in our linear instability study, as well as,
we give a unitary group representation associated to the one-parameter families of skew-self-adjoint extensions \((A_2, D(A_2))\) and \((H_2, D(H_2))\), defined in (2.10) and (6.2), respectively.

**Notation.** Let \(-\infty \leq a < b \leq \infty\). We denote by \(L^2(a, b)\) the Hilbert space equipped with the inner product \((u, v) = \int_a^b u(x) \overline{v(x)} \, dx\). By \(H^p(\Omega)\) we denote the classical Sobolev spaces on \(\Omega \subset \mathbb{R}\) with the usual norm. We denote by \(\mathcal{G}\) a metric graph parameterized by \(E = E_- \cup E_+\), where \(E_-\) and \(E_+\) are sets of half-lines of the form \((-\infty, 0)\) and \((0, +\infty)\), respectively, attached to a common vertex \(\nu = 0\). On the graph \(\mathcal{G}\) we define the classical spaces \(L^p(\mathcal{G}) = \bigoplus_{e \in E} L^p(-\infty, 0) \oplus \bigoplus_{e \in E_+} L^p(0, +\infty), p > 1\), and \(H^p(\mathcal{G}) = \bigoplus_{e \in E_-} H^p(-\infty, 0) \oplus \bigoplus_{e \in E_+} H^p(0, +\infty)\), with the natural norms. Also, for \(u = (u_e)_{e \in E}, v = (v_e)_{e \in E} \in L^2(\mathcal{G})\), the inner product is defined by

\[
\langle u, v \rangle = \sum_{e \in E_-} \int_{-\infty}^0 u_e \overline{v_e} \, dx + \sum_{e \in E_+} \int_0^\infty u_e \overline{v_e} \, dx.
\]

We also denote sometimes the element \((u_{\pm e})_{e \in E}\), as \((u_{\pm e})_{e \in E} = (u_{-1}, \ldots, u_{m\pm}, u_{1, +}, \ldots, u_{n, +})\), where \(u_{\pm e} : (-\infty, 0) \to \mathbb{R}\) and \(u_{\pm e} : (0, +\infty) \to \mathbb{R}\). Depending on the context we will use the following notations for different objects. By \(\| \cdot \|\) we denote the norm in \(L^2(\Omega)\) (\(\Omega = (-\infty, 0)\) or \((0, +\infty)\)), with \(\mathcal{G}\) or in \(L^p(\mathcal{G})\). By \(\| \cdot \|_p\) we denote the norm in \(L^p(\Omega)\) or in \(L^p(\mathcal{G})\). By \(\Pi\) we denote the context we identify \(u = (u_{-}, u_{+}) \in L^2(\mathcal{G})\) as an element in \(\Pi_{m=1}^n L^2(-\infty, 0) \times \Pi_{m=1}^n L^2(0, +\infty)\), with \(m = |E_-|\) and \(n = |E_+|\) or as \((m + n) \times 1\)-matrix column.

Let \(A\) be a closed densely defined symmetric operator in the Hilbert space \(H\). The domain of \(A\) is denoted by \(D(A)\). The deficiency indices of \(A\) are denoted by \(n_{\pm}(A) = \dim \ker(A^* \mp iI)\), with \(A^*\) denoting the adjoint operator of \(A\). The number of negative eigenvalues counting multiplicities (or Morse index) of \(A\) is denoted by \(n(A)\).

## 2. Preliminaries

For convenience of the reader we give some brief description about the characterization of skew-self-adjoint extensions of the Airy operator in (1.8). Our strategy for that will be to follow the theory established by Mugnolo, Noja and Seifert in [50] in the case of balanced graphs. We also give an example of that approach for the case of two half-lines with a \(\delta\)-type interaction. In section 6, we consider this interaction on general balanced metric graphs.

### 2.1. Airy operators and the existence of unitary groups

In the following, we will define properly the following Airy operator

\[
A_0 : (u_e)_{e \in E} \mapsto \left( \alpha_{\pm e} \frac{d^3}{dx^3} u_e + \beta_{\pm e} \frac{d}{dx} u_e \right)_{e \in E}
\]

as an unbounded operator on a certain Hilbert space, in such a way that the possible extensions \(A_{\text{ext}}\) of \(A_0\) induce that the solution of the following linear equation

\[
z_\pm = A_{\text{ext}} z_\pm
\]

is given by a \(C_0\)-unitary group. So, by Stone’s theorem we need to verify that \(A_0\) has skew-self-adjoint extensions \(A_{\text{ext}}\). We note that the existence of those extensions will be a key point
in our stability study. Now, since the Airy operator $A_0$ is of odd order, by changing the sign of each constant $\alpha_e$ it is equivalent to exchange the positive and negative half-lines, and so without loss of generality, we can choose $\alpha_e > 0$ for every $e \in E$. The following proposition from Mugnolo, Noja and Seifert in [50] gives us an answer about the previously established problem associated to (2.2).

\textbf{Proposition 2.1.} Let $\mathcal{G}$ be a metric graph determined by $E \equiv E_- \cup E_+$ and let $(\alpha_e)_{e \in E}$, $(\beta_e)_{e \in E}$ be two sequences of real numbers with $\alpha_e > 0$ for all $e \in E$. Consider the Airy operator $A_0$ in (2.1) with

$$D(A_0) \equiv \bigoplus_{e \in E_-} C^\infty_c (\infty, 0) \oplus \bigoplus_{e \in E_+} C^\infty_c (0, +\infty).$$

Then, $iA_0$ is a densely defined symmetric operator on the Hilbert space $L^2(\mathcal{G}) = \bigoplus_{e \in E} L^2(\mathcal{G})$, with deficiency indices $(n_+(iA_0), n_-(iA_0)) = (2|E_-| + |E_+|, |E_-| + 2|E_+|)$. Therefore, $A_0$ has skew-self-adjoint extension on $L^2(\mathcal{G})$ if and only if $|E_+| = |E_-|$, i.e. the number of incoming half-lines is the same of outgoing half-lines, the metric graph $\mathcal{G}$ is called \textit{balanced}.

Some comments about the former proposition deserve to be made which will be very useful in our stability study.

\textbf{Remark 2.2.} From proposition 2.1 in the case of balanced metric graphs, and from the classical Krein–von Neumann extension theory for symmetric operators (see chapter 4 in Naimark [53] and theorem X.2 in Reed and Simon [57]), we obtain that operator $(A_0, D(A_0))$ admits a 9$|E_+|^2$-parameter families of skew-self-adjoint extensions, such that each of them generates a unitary dynamics in $L^2(\mathcal{G})$ for (2.2). Moreover, every skew-self-adjoint extension $(A_{ext}, D(A_{ext}))$ is obtained as a restriction of the operator $(-A_0^*, D(A_0^*))$ with $-A_0^* = A_0$ and

$$D(A_0^*) \equiv \bigoplus_{e \in E_-} H^3(-\infty, 0) \oplus \bigoplus_{e \in E_+} H^3(0, +\infty).$$

We note that the action of $A_0$ in (2.1) can be seen as the one made by the matrix-diagonal operator

$$A_0 = \text{diag} \left( \left( \alpha_e \frac{d}{dx} u_e + \beta_e \frac{d}{dx} u_e \right) \delta_{i,j} \right), \quad 1 \leq i, j \leq |E_+| + |E_-|,$$

with $\delta_{i,j}$ being the Kronecker delta.

\subsection*{2.2. Construction of skew-self-adjoint extensions for Airy operators}

The complete characterization of skew-self-adjoint extensions of the Airy operator $(A_0, D(A_0))$ is a bit complex and one strategy for finding these was obtained by Mugnolo, Noja and Seifert in [50] via Krein spaces (see also Schubert, Seifert, Voigt and Waurick in [58]). The central idea of the process is given in theorems 3.7 and 3.8 in [50], where these extensions are parameterized through relations between boundary values at the vertex $\nu = 0$. Next, we will use this approach and for convenience of the reader we will briefly explain this one for a balanced metric graph $\mathcal{G}$. For abbreviating our notations, for $u = (u_e)_{e \in E} \in D(A_0^*)$ in (2.3), we denote

$$u(0-) \equiv (u_e(0-))_{e \in E_-} \quad \text{and} \quad u(0+) \equiv (u_e(0+))_{e \in E_+}.$$
and consider the space of boundary values in $C^{3n}$ ($n = |E_{\pm}|$).

$$U(0-): (u(0-), u'(0-), u''(0-)) \quad \text{and} \quad U(0+): (u(0+), u'(0+), u''(0+)),$$

spanning respectively subspaces $G_-$ and $G_+$ in $C^{3n}$. Thus, the boundary form of the operator $A_0$ is easily seen for $u, v \in D(A_0^*)$ to be (where we are identifying a vector with its transpose)

$$\langle A_0^* u, v \rangle + \langle u, A_0^* v \rangle = \begin{pmatrix} -I \beta_- & 0 & -I \alpha_- \\ 0 & I \alpha_- & 0 \\ -I \alpha_- & 0 & 0 \end{pmatrix} B_- \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix},$$

$$- \begin{pmatrix} -I \beta_+ & 0 & -I \alpha_+ \\ 0 & I \alpha_+ & 0 \\ -I \alpha_+ & 0 & 0 \end{pmatrix} B_+ \begin{pmatrix} v(0+) \\ v'(0+) \\ v''(0+) \end{pmatrix},$$

where

$$B_- = \begin{pmatrix} -I \beta_- & 0 & -I \alpha_- \\ 0 & I \alpha_- & 0 \\ -I \alpha_- & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_+ = \begin{pmatrix} -I \beta_+ & 0 & -I \alpha_+ \\ 0 & I \alpha_+ & 0 \\ -I \alpha_+ & 0 & 0 \end{pmatrix},$$

with $I = I_{n \times n}$ represents the identity matrix of order $n \times n$ and $\alpha \pm = (\alpha_\pm)_{e \in E_{\pm}}, \beta \pm = (\beta_\pm)_{e \in E_{\pm}}$. Thus, by considering the (indefinite) inner product $\langle \cdot \mid \cdot \rangle_{\pm} : G_\pm \times G_\pm \to \mathbb{C}$ by $\langle x \mid y \rangle_{\pm} = (B_\pm x, y)_{G_\pm}, x, y \in G_\pm$, we obtain that $(G_\pm, \langle \cdot \mid \cdot \rangle_{\pm})$ are Krein spaces and $\langle \cdot \mid \cdot \rangle_{\pm}$ is non-degenerate (for $x \in G_\pm$ with $\langle x \mid x \rangle_{\pm} = 0$ follows $x = 0$). Then, from theorem 3.8 in [50] we have that for a linear operator $L : G_- \to G_+$, the operator $(A_L, D(A_L))$ defined by

$$\begin{cases} A_L u = -A_0^* u = A_0 u, \\ D(A_L) = \{ u \in D(A_0^*) : L(u(0-), u'(0-), u''(0-)) = (u(0+), u'(0+), u''(0+)) \} \end{cases}$$

represents a skew-self-adjoint extension of $(A_0, D(A_0))$ if and only if $L$ is $(G_-, G_+)$-unitary, namely,

$$\langle Lx \mid Ly \rangle_+ = (B_+ L x, L y)_{G_+} = \langle x \mid y \rangle_- = (B_- x, y)_{G_-}.$$  \tag{2.8}$$

In other words, $L^* B_+ L = B_-$. Indeed, for $u, v \in D(A_L)$ we get from (2.5) the relations

$$\langle -A_L u, v \rangle + \langle u, -A_L v \rangle = \langle A_0^* u, v \rangle + \langle u, A_0^* v \rangle = \langle U(0-) | V(0-) \rangle_- - \langle U(0+) | V(0+) \rangle_+$$

$$= \langle U(0-) | V(0-) \rangle_- - \langle L U(0-) | L V(0-) \rangle_+.$$
the triplets \((u_-(0-), u'_-(0-), u''_-(0-)) \subset \mathbb{C}^3\) and \((u_+(0+), u'_+(0+), u''_+(0+)) \subset \mathbb{C}^3\). Then we have the next result.

**Proposition 2.3.** For \(Z \in \mathbb{R} \setminus \{0\}\) we consider the linear operator \(L_Z : G_- \to G_+\) given by

\[
L_Z = \begin{pmatrix}
1 & 0 & 0 \\
Z & 1 & 0 \\
\frac{Z^2}{2} & Z & 1
\end{pmatrix}.
\] (2.9)

Then we have a family \((A_Z, D(A_Z))\) of skew-self-adjoint extensions of \((A_0, D(A_0))\) parameterized by \(Z\) and which are defined by

\[
\begin{align*}
A_Z u &= A_0 u, \\
D(A_Z) &= \left\{ u = (u_-, u_+) \in H^3(-\infty, 0) \oplus H^3(0, +\infty) : u_-(0-) = u_+(0+) \right. \\
&\quad \left. + u'_-(0-) = Zu_-(0-) + \frac{Z^2}{2} u_-(0-) + Zu'_+ (0-) \right\}.
\end{align*}
\] (2.10)

Moreover, for \(\alpha_+ = (\alpha_-, \alpha_+) \in \mathbb{R}^+ \times \mathbb{R}^+\) and \(\beta_+ = (\beta_-, \beta_+) \in \mathbb{R} \times \mathbb{R}\) we need to have \(\alpha_- = \alpha_+\) and \(\beta_- = \beta_+\). Let us note, that each element in \(D(A_Z)\) can be seen as an element in \(H^4(\mathbb{R})\).

**Proof.** From the extension theory framework established above, we see from (2.8) that \(L_Z B_i L_Z = B_i\) if and only if \(\alpha_- = \alpha_+\) and \(\beta_- = \beta_+\). Then the operator \(A_Z u \equiv A_0 u, \ Z \in \mathbb{R} \setminus \{0\}\), defined for \(u = (u_-, u_+)\) such that \(L_Z (u_-(0-), u'_-(0-), u''_-(0-)) = (u_+(0+), u'_+(0+), u''_+(0+))\), will represent a skew-self-adjoint extension family of \((A_0, D(A_0))\). This finishes the proof. \(\square\)

**Remark 2.4.** In proposition 2.3, there is not a priori connection between the parameters \(Z, \alpha\) and \(\beta\) from relation \(L_Z^2 B_i L_Z = B_i\). In proposition A.10 appendix A we give a representation formula for the unitary group generated by \((A_Z, D(A_Z))\). Moreover, in (6.1) and (6.2) we extend to general balanced metric graphs the definitions in (2.9) and (2.10).

### 3. Stationary solutions in the case of two half-lines

In this section, we show the existence of stationary solutions for the KdV model (1.3) on a metric graph \(\mathcal{G}\) represented by \(E = (-\infty, 0) \cup (0, \infty)\) attaching at a common vertex \(\nu = 0\). As we will see, the possibility of different stationary profiles is very varied (discontinuous, positive and negative, etc). Now, which profiles may be viable for a fruitful stability study is not very clear for us. Here we will consider the case when they are in some domain of skew-self-adjoint extension for \(A_0\).

We consider the following stationary profile for the KdV model (1.3) on two half-lines,

\[
(u_\phi(x, t))_{x \in E} = (\phi_\phi(x))_{x \in E} \equiv (\phi_- \phi_+) \in H^3(-\infty, 0) \oplus H^3(0, +\infty)
\] (3.1)
for all \( t \in \mathbb{R} \). Then, by substituting this profile in (1.3) and integrating once we obtain the following nonlinear system of elliptic equations

\[
\begin{aligned}
\alpha_+ \phi_+''(x) + \beta_+ \phi_+(x) + \phi_+^2(x) &= 0, \quad x > 0 \\
\alpha_- \phi_-''(x) + \beta_- \phi_-(x) + \phi_-^2(x) &= 0, \quad x < 0.
\end{aligned}
\tag{3.2}
\]

Next, it is well known that the equation \( a \psi''(x) + b \psi(x) + \psi^2(x) = 0 \) for all \( x \in \mathbb{R} \) has nontrivial solutions for \( \psi(\pm \infty) = 0 \) in the cases either \( a > 0 \) and \( b < 0 \) or \( a < 0 \) and \( b > 0 \). The first case represents the classical positive-soliton for the KdV model (1.4)

\[
\psi(x) = -\frac{3b}{2} \operatorname{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{b}{a}} x + p \right), \quad x \in \mathbb{R}, \quad p \in \mathbb{R}.
\tag{3.3}
\]

The second case produces a depression soliton \((-\psi \text{ modulo translation})\). Next, we establish some specific profiles for \( \phi_\pm \) which will be the focus of our stability study here: for \( \alpha_\pm > 0 \) and \( \beta_\pm < 0 \) we obtain

\[
\phi_\pm(x) = -\frac{3}{2} \beta_\pm \operatorname{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{\beta_\pm}{\alpha_\pm}} x + p_\pm \right),
\tag{3.4}
\]

where the shift parameters \( p_\pm \) depend on boundary conditions for \( \phi_\pm \) at the vertex \( \nu = 0 \).

Our first example of solutions for (1.3) will belong to the family of skew-self-adjoint extension of \( A_0 \) via the operator \( L_2 \) in (2.9). Thus, for \( \phi_Z = (\phi_-, \phi_+) \) in (3.4) belonging to \( D(A_2) \) we need to have \( \alpha_+ = \alpha_- > 0 \) and \( \beta_+ = -\beta_- < 0 \), by proposition 2.3. Then we have that the profiles \( \phi_\pm \) satisfy the same equation in (3.2), and from (3.4) for \( -\frac{\beta_+}{\alpha_+} > \frac{Z^2}{4} \), we obtain

\[
\phi_+(x) = -\frac{3}{2} \beta_+ \operatorname{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{\beta_+}{\alpha_+}} x + p_+ \right), \quad x > 0
\tag{3.5}
\]

and \( \phi_-(x) \equiv \phi_+(x) \) for \( x < 0 \). Since \( \phi_-(0-) = \phi_+(0+) \) (continuity in zero), we get the condition in (2.10).

\[
\phi_+''(0+) - \phi_-''(0-) = \frac{Z^2}{2} \phi_+(0-) + Z \phi_-'(0+).
\tag{3.6}
\]

Figures 2 and 3 below show the profiles of \( \phi_Z \) for \( Z \neq 0 \). For \( Z < 0 \), we have the so-called tail profile, and for \( Z > 0 \), the so-called bump profile. We note that it is not difficult to show that the only stationary solutions (modulo sign) in \( D(A_2) \) for the KdV model (1.3) are exactly the tail and bump profiles defined by formula (3.5).

**Remark 3.1.** In section 6, we extend the tail and bump profiles above to the case of general balanced graphs (see figures 4 and 5).

### 4. Linear instability criterion for KdV on a metric graph

In this section, we establish a novel linear instability criterion of stationary solutions for the KdV model (1.3) on a start graph \( G \) with \( |E_+| = n \) and \( |E_-| = m \). Hence, initially we consider an extension \((A_{ext}, D(A_{ext}))\) of the Airy operator \( A_0 \) in (1.8), and \((\phi_\nu)_{\nu \in \mathcal{E} \subset D(A_{ext})} \) such that
\((\tilde{u}_e(x,t))_{e \in E} = (\phi_e(x))_{e \in E}\) is a nontrivial solution of \((1.3)\). Thus we obtain the following set of equations
\[
\begin{align*}
\alpha_e \frac{d^3}{dx^3} \phi_e + \beta_e \frac{d}{dx} \phi_e + 2 \phi_e \phi_e' = 0, \\
\text{e } \in \text{ E } = \text{ E }_- \cup \text{ E }_+ .
\end{align*}
\] (4.1)

Then, since \(\phi_e(\pm \infty) = 0\) we obtain for \(e \in E\) that each component satisfies the elliptic equation
\[
\alpha_e \frac{d^2}{dx^2} \phi_e + \beta_e \phi_e + \phi_e^2 = 0. 
\] (4.2)

Next, we suppose for \(e \in E\) that \(u_e\) satisfies formally equality in \((1.3)\) and we also define
\[
v_e(x,t) \equiv u_e(x,t) - \phi_e(x). 
\] (4.3)

Then, for \((v_e)_{e \in E} \in D(A_{\text{ext}})\) we have for each \(e \in E\) the equation
\[
\partial_t v_e = \alpha_e \partial_x^3 v_e + \beta_e \partial_x v_e + 2 \partial_x(v_e \phi_e) + \partial_x(v_e^2). 
\] (4.4)

Thus, one has that the system (abusing the notation)
\[
\partial_t v_e(x,t) = \alpha_e \partial_x^3 v_e(x,t) + \beta_e \partial_x v_e(x,t) + 2 \partial_x(v_e(x,t) \phi_e(x)) 
\] (4.5)
represents the linearized equation for \((1.3)\) around \(\phi_e\). In the sequel, our objective is to provide sufficient conditions for the trivial solution \(v_e \equiv 0, e \in E\), to be unstable by the linear flow of \((4.5)\). More exactly, we are interested in finding a growing mode solution of \((4.5)\) with the form \(v_e(x,t) = e^{\lambda t} \psi_e\) and \(\text{Re}(\lambda) > 0\). In other words, we need to solve the system
\[
\lambda \psi_e = -\partial_x L_e \psi_e, \\
L_e = -\alpha_e \frac{d^2}{dx^2} - \beta_e - 2 \phi_e, \\
e \in E 
\] (4.6)

with \((\psi_e)_{e \in E} \in D(A_{\text{ext}})\).

Next, we write our eigenvalue problem in \((4.6)\) in an Hamiltonian matrix form. Indeed, for \(\psi = (\psi_e)_{e \in E} \equiv (\psi_-, \psi_+)\) with \(\psi_- = (\psi_e)_{e \in E_-}\) and \(\psi_+ = (\psi_e)_{e \in E_+}\), we write \((4.6)\) as
\[
\lambda \begin{pmatrix} 
\psi_- \\
\psi_+ 
\end{pmatrix}
= 
\begin{pmatrix} 
-\partial_x L_- & 0 \\
0 & -\partial_x L_+ 
\end{pmatrix}
\begin{pmatrix} 
\psi_- \\
\psi_+ 
\end{pmatrix}
= N E \begin{pmatrix} 
\psi_- \\
\psi_+ 
\end{pmatrix} 
\] (4.7)

with
\[
L_+ = \text{diag} \left( -\alpha_1 \frac{d^2}{dx^2} - \beta_1 - 2 \phi_1, \ldots, -\alpha_m \frac{d^2}{dx^2} - \beta_m - 2 \phi_m \right), 
\] (4.8)

where \((\alpha_e)_{e \in \text{E }_-} \equiv (\alpha_1, \ldots, \alpha_m), (\beta_e)_{e \in \text{E }_-} \equiv (\beta_1, \ldots, \beta_m),\) and \((\phi_e)_{e \in \text{E }_-} \equiv (\phi_1, \ldots, \phi_m). \) \(L_+\) being defined similarly for \((\alpha_e)_{e \in \text{E }_+}, (\beta_e)_{e \in \text{E }_+}\) and \((\phi_e)_{e \in \text{E }_+}\). Thus, we have that \(N \) and \(E\) are \((m + n) \times (m + n)\)-diagonal matrix defined by
\[
N = \begin{pmatrix}
-\partial_x I_m & 0 \\
0 & -\partial_x I_n 
\end{pmatrix}, \\
E = \begin{pmatrix}
L_- & 0 \\
0 & L_+ 
\end{pmatrix},
\] (4.9)

where \(I_k\) denotes the identity matrix of order \(k\).
If we denote by \( \sigma(NE) = \sigma_{\text{disc}}(NE) \cup \sigma_{\text{ess}}(NE) \) the spectrum of \( NE \) (namely, \( \lambda \in \sigma_{\text{disc}}(NE) \) if \( \lambda \) is an isolated eigenvalue and with finite multiplicity, \( \sigma_{\text{ess}}(NE) \) represents the essential spectrum), the latter discussion suggests the utility of the following definition:

**Definition 4.1.** The stationary vector solution \( (\psi, e_{\text{loc}}) \in D(A_{\text{ext}}) \) is said to be **spectrally stable** for model (1.3) if the spectrum of \( NE, \sigma(NE) \), satisfies \( \sigma(NE) \subset i\mathbb{R} \). Otherwise, the stationary solution \( (\psi, e_{\text{loc}}) \) is said to be **spectrally unstable**.

It is standard to show that \( \sigma(NE) \) is symmetric with respect to both the real and imaginary axes and that \( \sigma_{\text{ess}}(NE) \subset i\mathbb{R} \) by supposing \( N \) skew-symmetric and \( E \) self-adjoint (see, for instance, [35, lemma 5.6 and theorem 5.8]). These cases on \( N \) and \( E \) will be considered in our theory. Hence, it is equivalent to say that \( (\psi, e_{\text{loc}}) \in D(A_{\text{ext}}) \) is **spectrally stable** if \( \sigma_{\text{disc}}(NE) \subset i\mathbb{R} \), and it is spectrally unstable if \( \sigma_{\text{disc}}(NE) \) contains point \( \lambda \) with \( \Re(\lambda) > 0 \).

### 4.1. Linear instability criterion

Let \( G \) be a star graph \( \mathcal{G} \) with a structure represented by the set \( E \equiv E_- \cup E_+ \) where \( E_- \) and \( E_+ \) are finite or countable collections of semi-infinite edges \( e \) parameterized by \((-\infty, 0]\) or \([0, +\infty)\), respectively. The half-lines are connected at a unique vertex \( \nu = 0 \).

From (4.7), our eigenvalue problem to solve is reduced to

\[
NE\psi = \lambda \psi, \quad \Re(\lambda) > 0, \quad \psi \in D(NE).
\]

Next, we establish our theoretical framework and assumptions for obtaining a nontrivial solution to problem in (4.10):

(S1) Let \( (A_{\text{ext}}, D(A_{\text{ext}})) \) be an extension of \( (A_0, D(A_0)) \) such that the solution of the linear problem in (1.9) is given by a strongly continuous \( C_0 \)-group on \( L^2(\mathcal{G}) \) with generator \( (A_{\text{ext}}, D(A_{\text{ext}})) \).

(S2) Suppose \( 0 \neq \phi = (\phi_e)_{e \in E} \in D(A_{\text{ext}}) \) such that \( (\nu_\phi(x, t))_{x \in \mathcal{E}} = (\phi_e(x))_{e \in E} \) is a stationary solution for the KdV model (1.3). Thus, by (2.3) and from the Sobolev’s embedding theorem we obtain \( \phi, \phi' \in L^\infty(\mathcal{G}) \).

(S3) Let \( E : D(E) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G}) \) be a self-adjoint operator with \( D(A_{\text{ext}}) \subset D(E) \).

(S4) Let \( N : D(N) \subset H^1(\mathcal{G}) \to L^2(\mathcal{G}) \) be a closed linear operator with dense domain \( D(N) \). We assume that \( N \) is one-to-one operator, skew-symmetric, and \( D(E) \subset D(N) \).

(S5) Since for every \( u \in D(A_{\text{ext}}) \) the expression \( NEu \) has sense, we require that \( \langle NEu, \phi \rangle = 0 \).

(S6) Suppose \( E : D(E) \to L^2(\mathcal{G}) \) is invertible with Morse index \( n(E) \) such that:

(a) For \( n(E) = 1 \), \( \sigma(E) = \{\lambda_0\} \cup J_0 \) with \( J_0 \subset [r_0, +\infty) \), for \( r_0 > 0 \), and \( \lambda_0 < 0 \),

(b) For \( n(E) = 2 \), \( \sigma(E) = \{\lambda_1, \lambda_2\} \cup J \) with \( J \subset [r, +\infty) \), for \( r > 0 \), and \( \lambda_1, \lambda_2 < 0 \). Moreover, for \( \Phi_1, \Phi_2 \in D(E) - \{0\} \) with \( E\Phi_1 = \lambda_1\Phi_1, E\Phi_2 = \lambda_2\Phi_2 \) (\( i = 1, 2 \)) we have \( \langle N\phi, \Phi_1 \rangle \neq 0 \) or \( \langle N\phi, \Phi_2 \rangle \neq 0 \).

(S7) For \( \psi \in D(E) \) with \( E\psi = \phi \), we have \( \langle \psi, \phi \rangle \neq 0 \).

The former assumptions deserve specific comments which will be very useful in the development of our linear instability theory.

**Remark 4.2.**

(a) About assumption (S1), we note that in the case of unbalanced metric graphs it is not possible to obtain a strongly continuous \( C_0 \)-unitary group generated for any extension of
we consider here $N(b)$. The property of $D$ follows the self-adjoint property of $D$. Consequently, we use it for finding an eigenfunction-solution of (4.10) in the orthogonal subspace to the stationary profile $\psi$. We note that the decomposition in (4.11) shows obviously that $N\psi$ is well defined for every $\psi \in \text{D}(A_{\text{ext}})$. Moreover, this assumption will be combined with item (a) of assumption (S5) for obtaining solution to (4.10) (see theorem 4.4). Lastly, assumption (S5) and the densely property of $D(A_{\text{ext}})$ in $L^2(\mathcal{G})$ imply the property $\phi \in \text{Ker}(E''N)$. 

(c) In contrast to the classical stability theories for solitary waves solutions on all line (see [6]), in the case of a metric graph we have in general that $N\phi \notin D(E)$ (for a specific situation, see lemma 5.2 below). But from (4.2) we always have the relations $\mathcal{L}_+ \phi'_+(x) = 0$ for $x > 0$, and $\mathcal{L}_- \phi'_-(x) = 0$ for $x < 0$, where we are writing $(\phi_e)e\in E = (\phi_-, \phi_+)$, with $\phi_- = (\phi_e)e\in E_-$ and $\phi_+ = (\phi_e)e\in E_+$. 

(d) Assumption (S2) has a technical spirit in the sense that we use it for finding an eigenfunction-solution of (4.10) in the orthogonal subspace to the stationary profile $\phi$. We note that the decomposition in (4.11) shows obviously that $N\psi$ is well defined for every $\psi \in \text{D}(A_{\text{ext}})$. Moreover, this assumption will be combined with item (a) of assumption (S6) for obtaining solution to (4.10) (see theorem 4.4). Lastly, assumption (S5) and the densely property of $D(A_{\text{ext}})$ in $L^2(\mathcal{G})$ imply the property $\phi \in \text{Ker}(E''N)$. 

(e) In the following we give a specific example to illustrate assumptions (S1)–(S4). We consider the operator $(A_2, D(A_2))$ defined in (2.10), the tail or bump profile $\phi = (\phi_-, \phi_+) \in D(A_2)$ given in (3.5), and $E$ in (4.9) with the domain 

$$D(E) = \{(u_-, u_+) \in H^2(-\infty, 0) \oplus H^2(0, +\infty) : u_-(0-) = u_+(0+)\}.$$ 

Then, we have obviously that $D(A_2) \subset \text{D}(E)$ and from proposition A.6 (appendix A) follows the self-adjoint property of $E$. For $D(N) = H^1(\mathcal{G}) \cap \{(u_-, u_+) \in L^2(\mathcal{G}) : u_-(0-) = u_+(0+)\}$, we get immediately assumption (S5). 

Next, for $u = (u_-, u_+) \in D(A_2)$ follows from integration by parts (without loss of generality we consider here $\alpha_- = \alpha_+ = 1$ and $\beta_- = \beta_+ = -1$ in (4.1)),
\[ \langle \partial^3_x u_-, \phi_- \rangle + \langle \partial^3_x u_+, \phi_+ \rangle \equiv \int_{-\infty}^{0} \partial_x (\partial^2_x u_-) \phi_- \, dx + \int_{0}^{+\infty} \partial_x (\partial^2_x u_+) \phi_+ \, dx \]

\[ = - \int_{-\infty}^{0} u_- \phi_-' \, dx - \int_{0}^{\infty} u_+ \phi_+' \, dx \]

\[ + [u_-''(0-) - u_+'(0+)] \phi_+(0+) + u_+'(0+) \phi_+(0+) - u_-'(0-) \phi_-'(0-) \]

\[ = - \int_{-\infty}^{0} u_- \phi_-' \, dx - \int_{0}^{\infty} u_+ \phi_+' \, dx + \left[ -\frac{Z^2}{2} u_-'(0-) - Z u_-''(0-) \right] \phi_+(0+) \]

\[ + Z u_+'(0-) \phi_+(0+) + Z u_-''(0-) \phi_-'(0+) \]

\[ = - \int_{-\infty}^{0} u_- \phi_-' \, dx - \int_{0}^{\infty} u_+ \phi_+' \, dx - \left[ Z \phi_+(0+) - \frac{Z^2}{2} \phi_+(0+) \right] \]

\[ = - \int_{-\infty}^{0} u_- \phi_-' \, dx - \int_{0}^{\infty} u_+ \phi_+' \, dx - \langle u_-, \partial^3_x \phi_- \rangle - \langle u_+, \partial^3_x \phi_+ \rangle, \quad (4.14) \]

where in the last equality we use the 'even-property' of \( (\phi_-, \phi_+) \), namely, \( \phi_+(0+) = \frac{Z}{2} \phi_+(0+) \). Next, since \( u_-(0-) = u_+(0+) \) and \( \phi_-(0-) = \phi_+(0+) \) we obtain

\[ \int_{-\infty}^{0} \partial_x (u_- - 2 \phi_- u_-) \phi_- \, dx + \int_{0}^{+\infty} \partial_x (u_+ - 2 \phi_+ u_+) \phi_+ \, dx \]

\[ = - \int_{-\infty}^{0} u_- (1 - 2 \phi_-) \phi_-' \, dx - \int_{0}^{+\infty} u_+ (1 - 2 \phi_+) \phi_+' \, dx. \quad (4.15) \]

Thus from (4.2)–(4.14) and (4.15), we obtain for \( u \in D(A_2) \)

\[ \langle Neu, \phi \rangle = \langle -\partial_x L_- u_-, \phi_- \rangle_{L^2(-\infty, 0]} + \langle -\partial_x L_+ u_+, \phi_+ \rangle_{L^2(0, +\infty)} \]

\[ = \int_{-\infty}^{0} u_- (-\phi_-' + \phi'_-) \, dx \]

\[ + \int_{0}^{+\infty} u_+ (-\phi_+' + \phi'_+) - 2 \phi_+ \phi'_+ \, dx = 0. \quad (4.16) \]

Now, we give the preliminaries for establishing our instability criterion described in theorem 4.4 below. The main idea is to reduce initially our eigenvalue problem (4.10) to the orthogonal subspace \( [\phi]^c \) (see assumption (S1)). Thus we consider the orthogonal projection \( Q : L^2(\mathcal{G}) \rightarrow L^2(\mathcal{G}) \)

\[ Q(u) = u - \langle u, \phi \rangle \frac{\phi}{\| \phi \|^2}, \quad (4.17) \]

associated to the nontrivial stationary solution \( \phi \), and we define.

\[ X_2 \equiv Q(L^2(\mathcal{G})) = \{ f \in L^2(\mathcal{G}) : f \perp \phi \} = [\phi]^c. \]

We also define the closed skew-adjoint operator \( N_0 : D(N_0) \subset X_2 \rightarrow X_2, \quad D(N_0) \equiv D(N) \cap X_2, \) for \( f \in D(N_0) \) by

\[ N_0 f \equiv QNf = Nf - \langle Nf, \phi \rangle \frac{\phi}{\| \phi \|^2}, \quad (4.18) \]
and the reduced self-adjoint operator for $E, F : D(F) \to X_2, D(F) = D(E) \cap X_2$ by
\[
Ff \equiv QEf = Ef - \langle Ef, \phi \rangle \frac{\phi}{\|\phi\|^2}. \tag{4.19}
\]

Now, for $f \in D(NE) \cap X_2 = D(A_{\text{ext}}) \cap X_2$ (see (4.11)), from assumptions $(S_4)$ and $(S_5)$, we get the relation
\[
N_f Ef = NEf - \langle Ef, \phi \rangle \frac{N \phi}{\|\phi\|^2} = \langle Ef, \phi \rangle \frac{N \phi}{\|\phi\|^2} \tag{4.20}
\]

\textbf{Proposition 4.3.}  \textit{N}_f F : D(N_f F) \subset X_2 \to X_2,\ D(N_f F) = D(A_{\text{ext}}) \cap X_2 \subset D(E) \cap X_2, \text{ is the infinitesimal generator of a strongly continuous } C_0 \text{-group of operators } S_0(t) \text{ in the space } X_2.

\textbf{Proof.}  \text{ We divide the proof in two steps:}

(a) Define $C = N(QEQ : D(C) \subset L^2(G) \to L^2(G), \quad D(C) = D(A_{\text{ext}})$, then, from $D(A_{\text{ext}}) \subset D(E)$, the self-adjoint property of $E$ and from assumptions $(S_4)$ and $(S_5)$, we obtain for $f \in D(A_{\text{ext}})$
\[
Cf = NEf - \langle f, \phi \rangle \frac{NE \phi}{\|\phi\|^2} - \langle Ef, \phi \rangle \frac{N \phi}{\|\phi\|^2} + \langle f, \phi \rangle \frac{E \phi}{\|\phi\|^2} \frac{\phi}{\|\phi\|^2} = NEf - Bf, \tag{4.21}
\]

where $B : L^2(G) \to L^2(G)$ defined by,
\[
Bf = \langle f, \phi \rangle \frac{NE \phi}{\|\phi\|^2} + \langle f, E \phi \rangle \frac{N \phi}{\|\phi\|^2} - \langle f, \phi \rangle \frac{E \phi}{\|\phi\|^2} \frac{\phi}{\|\phi\|^2}
\]

is a bounded operator. Thus, from the theory of semigroups (see [55], section 3.1) and remark 4.2-item (a), we obtain that $C$ generates a strongly continuous $C_0$-group of operators $S_1(t)$ on $L^2(G)$. Since $C$ commutes with $Q$ (because we have $Cf \in X_2$), $S_1(t)$ also commutes with $Q$.

(b) Define $S_0(t) : X_2 \to X_2$ by $S_0(t) = QS_1(t)$. Then $\{S_0(t)\} t \in \mathbb{R}$ is a strongly continuous $C_0$-group of linear operators on $X_2$ and it is not difficult to see that its infinitesimal generator is $N_f F$.

This finishes the proposition. \hfill \Box

Next, we have the following basic assumption for our linear instability criterion in the case $n(E) = 2$ in assumption $(S_6)$.

(H) There is a real number $\eta$, satisfying $\eta > 0$, such that $F : D(F) \to X_2, D(F) = D(E) \cap X_2$, it is invertible and with Morse index equal to one. Moreover, all the remainder of the spectrum is contained in $[\eta, + \infty)$.

\textbf{Theorem 4.4.}  \textit{Suppose the basic assumption (H) and the assumptions $(S_1)$–$(S_7)$ hold with $n(E) = 2$ in assumption $(S_5)$. Then the operator $NE$ has a real positive and a real negative eigenvalue.}

The proof of theorem 4.4 is based in the following Krasnoselskii result on closed convex cone [44], chapter 2, section 2.2.6.
Theorem 4.5. Let $K$ be a closed convex cone of a Hilbert space $(X, \| \cdot \|)$ such that there is a continuous linear functional $\Phi$ and a constant $a > 0$ such that $\Phi(u) \geq a\|u\|$ for any $u \in K$. If $T : X \to X$ is a bounded linear operator that leaves $K$ invariant, then $T$ has at least an eigenvector in $K$ associated to a nonnegative eigenvalue.

Proof (Proof of theorem 4.4). We divide the proof in two steps:

(a) Our first step is to show that the operator $N_0F : D(N_0F) \subset X_2 \to X_2$ has a real positive and a real negative eigenvalue. Indeed, from assumption (H) we consider $\psi_0 \in D(F) = D(E) \cap X_2$, $\|\psi_0\| = 1$ and $\lambda_0 < 0$ such that $F\psi_0 = \lambda_0\psi_0$. We define

$$K = \{ z \in D(F) : \langle Fz, z \rangle \leq 0 \quad \text{and} \quad \langle z, \psi_0 \rangle \geq 0 \},$$

then $K$ is a nonempty closed convex cone in $L^2(\mathcal{G})$. Moreover, this cone is invariant under the group $\{ S_0(t) \}$ with infinitesimal generator $N_0F$ (see proposition (4.3)). Indeed, we will use a density argument based in the existence of a core for $A_{\text{ext}}$. Thus, from semi-group theory it follows that the space

$$D(A_{\text{ext}}^\infty) = \bigcap_{n \in \mathbb{N}} D(A_{\text{ext}}^n)$$

with $D(A_{\text{ext}}^n) = \{ f \in D(A_{\text{ext}}^{-n-1}) : A_{\text{ext}}^{-1}f \in D(A_{\text{ext}}) \}$, is dense in $L^2(\mathcal{G})$ and it also is a $\{ S_0(t) \}_{t \in \mathbb{R}}$-invariant subspace of $D(A_{\text{ext}})$. Thus, $D(A_{\text{ext}}^\infty)$ is a core for $A_{\text{ext}}$. Therefore, it is enough to consider the case $f \in K \cap D(A_{\text{ext}}^\infty)$ and so we obtain that the reduced Hamiltonian equation

$$\begin{cases}
\dot{z} = N_0Fz \\
z(0) = f
\end{cases}$$

has solution $z(t) = S_0(t)f \in D(A_{\text{ext}}^\infty)$. Then, from the self-adjoint property of $F$ and the skew-symmetric property of $N_0$ we obtain

$$\frac{d}{dt} \langle Fz(t), z(t) \rangle = \langle FN_0Fz(t), z(t) \rangle + \langle Fz(t), N_0Fz(t) \rangle = 0,$$

so, for all $t$ we obtain $\langle Fz(t), z(t) \rangle = \langle Ff, f \rangle \leq 0$. Next, we suppose $\langle f, \psi_0 \rangle > 0$ and that there is $I_0$ such that $\langle S_0(t_0)f, \psi_0 \rangle < 0$. Then by continuity of the flow $t \to S_0(t)f$ there is $r \in (0, t_0)$ with $\langle S_0(r)f, \psi_0 \rangle = 0$. Now, from assumption (H) and from the spectral theorem for self-adjoint operators, we obtain the following orthogonal decomposition for $f_r = S_0(r)f$,

$$f_r = \sum_{i=1}^m a_i h_i + g, \quad g \perp h_i, \quad \text{for all } i,$$

where $Fh_i = \lambda_i h_i$, $\|h_i\| = 1$, $\lambda_i \in \sigma_d(F)$ with $\lambda_i \geq \eta$, and $\langle Fr, g \rangle \geq \theta \|g\|^2$, $\theta > 0$. Therefore,

$$0 \geq \langle Fr, fr \rangle \geq \sum_{i=1}^m a_i^2 \lambda_i + \theta \|g\|^2 \geq \eta \sum_{i=1}^m a_i^2 + \theta \|g\|^2 \geq 0.$$

Thus, it follows $g = 0$ and $a_i = 0$ for $i$. Therefore $S_0(r)f = 0$ and since $S_0(t)$ is a group we obtain $f = 0$ and so $\langle f, \psi_0 \rangle = 0$ which is a contradiction. Now we suppose $\langle f, \psi_0 \rangle = 0,$

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then the former analysis shows \( f = 0 \) and so \( S_0(t)f \equiv 0 \) for all \( t \). This shows the invariance of \( K \) by \( S_0(t) \). Then, for \( \mu \) large, we obtain from semigroup’s theory the following integral representation of the resolvent

\[
T_z \equiv (\mu I - N_0F)^{-1}(z) = \int_0^\infty e^{-\mu t}S_0(t)z dt
\]  

(4.23)

and so \( T \) also leaves \( K \) invariant. Next, for \( \Phi : L^2(G) \to \mathbb{R} \) defined by \( \Phi(z) = \langle z, \psi_0 \rangle \) we will see that there is \( \alpha > 0 \) such that \( \Phi(z) \geq \alpha \|z\| \) for any \( z \in K \). Indeed, suppose for \( \|g\| = 1 \), that \( \langle g, \psi_0 \rangle = \gamma > 0 \) and \( \langle Fg, g \rangle \leq 0 \). Since \( \text{Ker}(\Phi) \) is a hyperplane we obtain \( g = z + \gamma \psi_0 \) with \( \langle z, \psi_0 \rangle = 0 \). So, \( -\lambda \gamma^2 \geq \langle Fz, z \rangle \). Now, from the orthogonal decomposition \( z = \sum_{i=1}^m \langle z, h_i \rangle h_i + g, g \perp h_i \), for all \( i \), follows for \( \eta, \theta > 0 \), \( \langle Fz, z \rangle = \min \{\eta, \theta\}(1 - \gamma^2) \). Then,

\[
\langle g, \psi_0 \rangle = \gamma \geq \sqrt{\frac{\min \{\eta, \theta\}}{-\lambda + \min \{\eta, \theta\}}} \equiv a.
\]

Therefore, by the analysis above and theorem 4.5 there are an \( \alpha \geq 0 \) and a nonzero element \( \omega_0 \in K \) such that \( (\mu I - N_0F)^{-1}(\omega_0) = \alpha \omega_0 \). It is immediate that \( \alpha > 0 \) and so \( N_0F\omega_0 = \zeta \omega_0 = 0 \) with \( \zeta = \frac{\omega_0}{a} \). Next we see that \( \zeta \neq 0 \). In fact, suppose that \( \zeta = 0 \), then from (4.20) and the injectivity of \( N \) (assumption (S6)) we obtain

\[
E\omega_0 = \langle E\omega_0, \phi \rangle \frac{\phi}{\|\phi\|^2}.
\]

From assumption (S7), let \( \psi \in D(E) \) with \( E\psi = \phi \), then since \( E \) is invertible it follows

\[
\omega_0 = \frac{\langle E\omega_0, \phi \rangle}{\|\phi\|^2} \psi \quad \text{and} \quad 0 = \langle \omega_0, \phi \rangle = \frac{\langle E\omega_0, \phi \rangle}{\|\phi\|^2} \langle \psi, \phi \rangle.
\]

Since \( \langle \psi, \phi \rangle \neq 0 \) it follows \( \langle E\omega_0, \phi \rangle = 0 \). Hence \( E\omega_0 = 0 \) and so \( \omega_0 = 0 \), which is a contradiction. Then, \( N_0F \) has a nonzero real eigenvalue \( \zeta \).

Now, from assumption (H), we have \( \sigma(N_0F) = \sigma((N_0F)^*) = -\sigma(FN_0) = -\sigma(FN_0FF^{-1}) = -\sigma(N_0F) \) and so \( -\zeta \) also belongs to \( \sigma(N_0F) \). Now, from theorem 5.8 of [35] we obtain that the essential spectrum of \( N_0F \) lies on the imaginary axis and so we need to have that \( -\zeta \) is also an eigenvalue of \( N_0F \) and this proves our initial claim.

(b) Let \( \omega_0 \in D(N_0F) \), \( \omega_0 \neq 0 \), such that \( N_0F\omega_0 = \zeta \omega_0 \), with \( \zeta > 0 \). Then, from (4.20) we obtain

\[
NE\omega_0 = \langle E\omega_0, \phi \rangle \frac{N\phi}{\|\phi\|^2} + \zeta \omega_0.
\]  

(4.24)

Next we consider two cases:

(a) Suppose \( \langle E\omega_0, \phi \rangle = 0 \), then \( NE\omega_0 = \zeta \omega_0 \) and the proof of the criterion finishes.

(b) Suppose \( r \equiv \frac{1}{\|\phi\|^2} \langle E\omega_0, \phi \rangle \neq 0 \). From assumption (S6)-(b), we consider

\[
u = \omega_0 + a\Phi_1 + b\Phi_2, \quad E\Phi_i = \lambda_i \Phi_i, \quad 1 \leq i \leq 2,
\]

with \( \|\Phi_i\| = 1 \), \( \Phi_1 \perp \Phi_2 \). We will find \( a, b \in \mathbb{R} \), not both zero, such that \( NEu = \zeta u \) and \( u \neq 0 \). Thus, we obtain initially the relation (we recall that \( D(E) \subset D(N) \))

\[n\phi + a\lambda_1 N\Phi_1 + b\lambda_2 N\Phi_2 = a\zeta \Phi_1 + b\zeta \Phi_2.
\]  

(4.25)
Therefore, from the skew-symmetric property of \( N \) we obtain the system

\[
\begin{align*}
    a\zeta + b\lambda_2 \langle N\Phi_1, \Phi_2 \rangle &= r \langle N\phi, \Phi_1 \rangle \\
    a\lambda_1 \langle N\Phi_1, \Phi_2 \rangle - \zeta b &= -r \langle N\phi, \Phi_2 \rangle.
\end{align*}
\]

(4.26)

Thus, since \( \zeta^2 + \lambda_1 \lambda_2 \langle N\Phi_1, \Phi_2 \rangle^2 \neq 0 \) (namely, the determinant of the coefficients in (4.26)), \( r \neq 0 \), and assumption (S6)-(b), we obtain a nontrivial solution for (4.26).

Next, we see \( u \neq 0 \). Indeed, suppose \( u = 0 \). Then, from relation \( \omega_0 = -a\Phi_1 - b\Phi_2 \) and by substituting in (4.24) we obtain

\[
a\lambda_1 r \langle N\phi, \Phi_1 \rangle + b\lambda_2 r \langle N\phi, \Phi_2 \rangle = 0.
\]

(4.27)

Hence, by using system (4.26) in (4.27) we arrive to the relation \( \zeta (a^2 \lambda_1 + b^2 \lambda_2) = 0 \), which is a contradiction.

This finishes the theorem. \( \square \)

Next, we consider the case \( n(E) = 1 \) in the assumption (S6).

**Theorem 4.6.** Suppose the assumptions (S1)–(S6) hold with \( n(E) = 1 \). Then the operator \( NE \) has a real positive and a real negative eigenvalue.

**Proof.** In this case we do not need to reduce the eigenvalue problem (4.10) to the orthogonal subspace \( \{\phi\}^\perp \). Indeed, from assumption (S1) we have that \( NE \) is the infinitesimal generator of a \( C_0 \)-group \( \{S(t)\}_{t \in \mathbb{R}} \) (see remark 4.2, item (a)). For \( \psi_0 \in D(E), \|\psi_0\| = 1 \) and \( \lambda_0 < 0 \) such that \( E\psi_0 = \lambda_0 \psi_0 \), we consider the following nonempty closed convex cone

\[
K_0 = \{ z \in D(E) : \langle Ez, z \rangle \leq 0, \quad \text{and} \quad \langle z, \psi_0 \rangle \geq 0 \}.
\]

Similarly as in the proof of theorem 4.4, \( K_0 \) is invariant under the group \( S(t) \). Thus, for \( T = (\mu I - NE)^{-1}, \mu \) large, \( T \) leaves \( K_0 \) invariant. Then, by using theorem 4.5 with this \( T \) and \( \Phi(z) = \langle z, \psi_0 \rangle \), we can see that \( NE \) has a real positive and a real negative eigenvalue. This finishes the proof. \( \square \)

### 4.2. One application of theorem 4.4

The following framework will be used in the study of linear instability of bump and tail profiles on metric graph in the following two sections. Suppose that assumptions (S1)–(S7) hold with \( n(E) = 2 \) in assumption (S6), and for \( \phi \) such that \( E\phi = \phi \) we suppose \( \langle \psi, \phi \rangle < 0 \). Then we can conclude that assumption (H) is true. Indeed, from assumption (S6) and from the invertibility property of \( E \) we obtain that \( F \) is invertible. Next, there are \( a, b \in \mathbb{R} \) (not both zeros) with \( \langle a\Phi_1 + b\Phi_2, \phi \rangle = 0 \) and \( \langle F(a\Phi_1 + b\Phi_2), a\Phi_1 + b\Phi_2 \rangle < 0 \). Then via min-max principle we have \( n(F) \geq 1 \). Next, suppose that \( n(F) = 2 \) and consider \( z_1, z_2 \in X_2, z_1 \perp z_2, \mu_1, \mu_2 < 0, \) and \( Fz_i = \mu_i z_i \). Then we get

\[
\langle Fz_i, z_i \rangle = \langle Ez_i, z_i \rangle = \mu_i \|z_i\|^2 < 0 \quad \text{and} \quad \langle Ez_1, z_2 \rangle = 0.
\]

Moreover, since \( \psi \notin X_2 = \{\phi\}^\perp \) it follows that set \( \{\psi, z_1, z_2 \} \subset E \) is linearly independent and we have the relations

\[
\langle Ez_i, \psi \rangle = \langle z_i, \phi \rangle = 0 \quad \text{and} \quad \langle E\psi, \psi \rangle = \langle \phi, \psi \rangle < 0.
\]
Therefore \((E(\alpha \psi + \beta z_1 + \theta z_2), \alpha \psi + \beta z_1 + \theta z_2) < 0\) for any \(\alpha, \beta, \theta,\) and so \(n(E) \geq 3\), which is not true. Then \(n(F) = 1\) and all other eigenvalues (and the remain of the spectrum) are contained in \([\eta, +\infty)\), \(\eta > 0\), because of the spectral structure of \(E\) and the equality \((Ef, f) = \langle Ef, f \rangle\) for \(f \in D(F)\). Thus, from theorem 4.4 it follows that \(NE\) has a real positive and a real negative eigenvalue.

5. Linear instability of tail and bump solutions for the KdV on two half-lines

The focus of this section is to apply the linear instability criterion in theorems 4.4 and 4.6 to the KdV on a metric graph with two half-lines and a \(\delta\)-interaction-type at the vertex \(\nu = 0\) induced by matrix \(L_z\) in proposition 2.3. In the next section we consider the general case of balanced metric graphs. By using the notations in subsection 2.2 and section 3, our main instability result about the tail and bumps profiles is the following,

**Theorem 5.1.** For \(Z \neq 0\), \(\alpha_- = \alpha_+ > 0\), \(\beta_- = \beta_+ < 0\), and \(-\frac{\beta}{\alpha} > \frac{2}{4}\). Let \(\phi \equiv (\phi_-, \phi_+) \in D(A_\nu)\) defined for \(\phi_+(x)\) by the formula (3.5) with \(x > 0\) and \(\phi_-(x) = \phi_+(x)\) for \(x < 0\). On the metric graph \(G\) determined by \(E = (-\infty, 0) \cup (0, +\infty)\), we consider the following family of stationary solutions for the KdV model (1.3)

\[
U_x(t, \nu) = (\phi_-(\nu), \phi_+(\nu)), \quad t \in \mathbb{R}.
\]

Then, this family of tail \((Z < 0)\) and bump \((Z > 0)\) profiles are linearly unstable.

The proof of theorem 5.1 will be divided in several steps by verifying the assumptions in subsection 4.1. Indeed, from remark 4.2-item (5) we have the assumptions (S1)–(S5). The linear eigenvalue problem in (4.10) is determined by the matrices \(N\) and \(E \equiv E_z = \text{diag}(L_-, L_+)\) in (4.9) with the Schrödinger operators

\[
L_\pm = -\alpha_\pm \frac{d^2}{dx^2} - \beta_\pm - 2\phi_\pm.
\]

Now, for studying the spectrum structure of the self-adjoint operators \((E_z, D(E_z))\) is sufficient to consider the cases \(\alpha_+ = 1\), \(\beta_+ = -1\) and \(1 > \frac{2}{4}\), without loss of generality (see appendix B–B.4 in [6]). We recall that

\[
D(E_z) = \{(u_-, u_+) \in H^2(G) : u_-(0-) = u_+(0+), u'_-(0+) = u'_+(0-) \}
\]

hence \((E_z, D(E_z))\) represents a self-adjoint family of point-interaction type on all the line by identifying \(D(E_z)\) as \(H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R})\) (see [4]).

The following two lemmas show assumption (S6) for operator \((E_z, D(E_z))\).

**Lemma 5.2.** For every \(Z \neq 0\) we have \(\text{Ker}(E_z) = \{0\}\). Moreover, since \(\sigma_{\text{ess}}(E_z) = [1, +\infty)\) we obtain \(E_z : D(E_z) \to L^2(G)\) is invertible.

**Proof.** Let \(u = (u_-, u_+) \in D(E_z), E_z u = 0\). Since \(L_\pm \phi_\pm = 0\), we obtain from classical Sturm–Liouville theory that \(u_-(x) = a \phi'_-(x)\), \(x < 0\), and \(u_+(x) = b \phi'_+(x)\), \(x > 0\) (see [15], chapter 2, theorem 3.3). Next, from the continuity property at zero for \(u\), the conditions \(\phi'_-(0+) = -\phi'_+(0-)\) and \(\phi'_+(0+) = \phi'_-(0-).\) Then, we have that

\[
a = -b \quad \text{and} \quad -2a \phi''_+(0+) = Z \phi_+(0+) = Z \phi_-(0-) = \frac{Z}{2} \phi'_+(0-) = -Z \phi'_+(0+).
\]

(5.2)
Suppose $a \neq 0$. Then, from (5.2) we obtain $\phi'_0(0+) = \frac{Z^2}{4} \phi_+(0+)$. Next, from (3.2) we arrive to $1 - \phi_+(0+) = \frac{Z^2}{4}$ and therefore $Z^2 < 4$ (it does not happen because $1 > \frac{Z^2}{4}$). Then, $a = b = 0$ and $u \equiv 0$.

Next, by Weyl's theorem (see theorem XIII.14 of [57]), the essential spectrum of $E_Z$ coincides with $[1, +\infty)$. Then $E_Z$ is an invertible operator. This finishes the proof. 

Lemma 5.3. For $Z > 0$ we have $n(E_Z) = 2$ and for $Z < 0$ that $n(E_Z) = 1$.

Proof. Our strategy is to use an analytic perturbation theory approach as in [7, 8]. Thus, we will give the main points of the analysis. For this purpose, we define the following classical self-adjoint operator on $L^2(\mathbb{R})$

$$L_0 = -\frac{d^2}{dx^2} + 1 - 2\phi_0, \quad D(L_0) = H^2(\mathbb{R}),$$

where $\phi_0$ denotes the classical soliton solution for the KdV equation on the full line, namely,

$$\phi_0(x) = \frac{3}{2} \text{sech}^2 \left( \frac{1}{2} x \right), \quad x \in \mathbb{R}. \quad (5.4)$$

From classical Sturm–Liouville theory we obtain $\text{Ker}(L_0) = [\phi'_0, ]$, $n(L_0) = 1$ and $\sigma_{\text{ess}}(L_0) = [1, +\infty)$ (see chapter 2 in [15] or theorem B.61 in [6]). Now, we consider the domain

$$D(E_0) = \{(u_-, u_+) \in H^2(\mathcal{G}); u_-(0-) = u_+(0+), u'_-(0-) = u'_+(0+)\}, \quad (5.5)$$

on which the following ‘limit’ operator $E_0$ (associated with $E_Z$) is self-adjoint.

$$E_0 = \text{diag} \left( \frac{d^2}{dx^2} + 1 - 2\phi_{0-}, -\frac{d^2}{dx^2} + 1 - 2\phi_{0+} \right), \quad (5.6)$$

with $\phi_{0-} = \phi_0|_{(-\infty, 0)}$ and $\phi_{0+} = \phi_0|_{(0, +\infty)}$. Thus, by considering the unitary operator $\hat{U}: D(E_0) \to H^2(\mathbb{R})$ defined for $u = (u_-, u_+) \in E_0$ by $\hat{U}(u) = \hat{u} \in H^2(\mathbb{R})$, where

$$u(x) = \begin{cases} u_-(x), & x < 0, \\ u_+(x), & x > 0, \\ u_+(0+), & x = 0, \end{cases} \quad (5.7)$$

e we obtain $\sigma(E_0) = \sigma(L_0)$ and $\lambda \in \sigma_{\text{ess}}(E_0)$ if and only if $\lambda \in \sigma_{\text{ess}}(L_0)$ with the same multiplicity. Moreover, $\sigma_{\text{ess}}(E_0) = [1, +\infty)$. Therefore, we conclude that $\text{Ker}(E_0) = [\Phi'_0, ]$, $\Phi_0 = (\phi_{0-}, \phi_{0+})$, and $n(E_0) = 1$.

Next, by using a similar strategy as in [7, 8] we have the following:

(a) $\phi_Z = (\phi_-, \phi_+) \to \Phi_0$, as $Z \to 0$, in $H^1(\mathcal{G})$

(b) The family $\{E_Z\}_{Z \in \mathbb{R}}$ represents a real-analytic family of self-adjoint operators of type (B) in the sense of Kato (see [40]).

(c) Since $E_Z$ converges to $E_0$ as $Z \to 0$ in the generalized sense, we obtain from theorem IV-3.16 from Kato [40] and from Kato–Rellich theorem ([57], theorem XII.8) the existence of two analytic functions $\Omega, \Pi$ defined in a neighborhood of zero with $\Omega: (0, \mathbb{R}) \to \mathbb{R}$ and $\Pi: (-\mathbb{R}, Z_0) \to L^2(\mathcal{G})$ such that $\Omega(0) = 0$ and $\Pi(0) = \Phi'_0$. For all $Z \in (-Z_0, Z_0)$, $\Omega(Z)$ is the simple isolated second eigenvalue of $E_Z$, and $\Pi(Z)$ is the associated eigenvector.
for $\Omega(Z)$. Moreover, $Z_0$ can be chosen small enough to ensure that for $Z \in (-Z_0, Z_0)$ the spectrum of $E_Z$ in $L^2(\mathcal{G})$ is positive, except at most the first two eigenvalues.

(d) From an ODE’s analysis we have that if $\lambda$ is an simple eigenvalue for $E_Z$ then the eigenfunction associated is either even or odd. Therefore, since $\Pi(Z) \to \Phi_0^+$, as $Z \to 0$, and $N\Phi_0$ is odd, we can see that $\Pi(Z) \in H^2(\mathbb{R})$ and it is an odd function. Thus we obtain the relation

$$\langle N\phi_Z, \Pi(Z) \rangle \neq 0, \quad Z \approx 0. \quad (5.8)$$

Indeed, since $\lim_{Z \to 0} \langle N\phi_Z, \Pi(Z) \rangle = ||N\Phi_0||^2 > 0$ we have for $Z$ small the property $(5.8)$. Next, from a continuation argument we obtain that $(5.8)$ holds for all $Z \in (-\infty, \infty)$.

(e) From Taylor’s theorem we see that there exists $0 < Z_1 < Z_0$ such that $\Omega(Z) > 0$ for any $Z \in (-Z_1, 0)$, and $\Omega(Z) < 0$ for any $Z \in (0, Z_1)$. Thus, in the space $L^2(\mathcal{G})$ for $Z$ small, we have $n(E_Z) = 1$ as $Z < 0$, and $n(E_Z) = 2$ as $Z > 0$.

(f) Lastly, we define $Z_\infty$ by

$$Z_\infty = \sup \{ \bar{Z} > 0 : E_Z \text{ has exactly two negative eigenvalues for all } Z \in (0, \bar{Z}) \}.$$

Then, item (e) above implies that $Z_\infty$ is well defined and $Z_\infty \in (0, \infty]$. Then, by using that $\text{Ker}(E_Z) = \{0\}$ for $Z \neq 0$, items (b) and (c) above, and Riesz-projection theory, we obtain $Z_\infty = \infty$ (see [7, 8]).

Analogously, we can prove that $n(E_Z) = 1$ in the case $Z < 0$. This finishes the proof of the lemma.

The following argument shows assumption $(S_7)$. Indeed, by returning to the variable $\beta_+$ defining the profiles $\phi_\pm$ in $(3.5)$ with $\alpha_+ = 1$, we have that these profiles represent a differentiable family of stationary solutions a one-parameter dependence as $\omega \equiv \omega(\beta_+ > 0)$. Thus, by denoting this dependence as $\phi_{Z,\omega} = (\phi_{-\omega, \beta_+}, \phi_{+\omega, \beta_+})$ we obtain from $(3.2)$ (after derivation in $\omega$) that

$$\left( -\frac{d^2}{dx^2} + \omega - 2\phi_{\pm\omega} \right) \left( \frac{d}{d\omega} \phi_{\pm\omega} \right) = -\phi_{\pm\omega}. \quad (5.9)$$

Now, for $\psi_\omega \equiv (-\frac{d}{dx} \phi_{-\omega}, -\frac{d}{dx} \phi_{+\omega})$ is not difficult to see that $\psi \equiv \psi_\omega|_{\omega=1} \in D(E_Z)$ and so we obtain $E_Z \psi = \phi_Z$. Therefore, with the notation above, we obtain the following result.

**Lemma 5.4.** Let $Z \neq 0$. The smooth curve of profiles $\omega \in (\mathbb{R}^2, +\infty) \to \phi_{Z,\omega}$ satisfies for $\psi \equiv -\frac{d}{dx} \phi_{Z,\omega} |_{\omega=1} \in D(E_Z)$, $E_Z \psi = \phi_Z$ and $\langle \psi, \phi_Z \rangle < 0$.

**Proof.** From proposition 3.19 in [8] (item (b), $p = 2$) we have for every $Z \in \mathbb{R}$ the relation $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{Z,\omega}(x)dx + \int_{0}^{+\infty} \phi_{Z,\omega}(x)dx > 0$. Therefore $\langle \psi, \phi_Z \rangle < 0$. This finishes the proof.

**Proof of Theorem 5.1** Let $Z > 0$. From lemmas 5.2, 5.3 and 5.4, subsection 4.2 and $(5.8)$, we obtain from theorem 4.4 that the profiles of bump type for the KdV are linear unstable.

Let $Z < 0$. From lemmas 5.2 and 5.3, we obtain that theorem 4.6 implies the linear instability of the tail profile for the KdV model. This finishes the proof.

6. Linear instability of tail and bump solutions for a general balanced graph $\mathcal{G}$

In the following we consider a balanced metric graph $\mathcal{G}$ with a structure $E \equiv E_+ \cup E_-$, where $|E_+| = |E_-| = n, n \geq 2$. For $I = I_n \times n$ being the identity matrix of order $n \times n$ we consider the
matrix \( L : \mathbb{G}_- \rightarrow \mathbb{G}_+ \) of order \( 3n \times 3n \), for \( Z \in \mathbb{R} \), as
\[
L \equiv \begin{pmatrix}
I & 0 & 0 \\
\frac{Z^2}{2}I & ZI & I \\
ZI & I & 0
\end{pmatrix}.
\]

Thus, from (2.6) and with the notations \( \alpha_\pm = (\alpha_e)_{e \in \mathcal{E}_\pm}, \beta_\pm = (\beta_e)_{e \in \mathcal{E}_\pm} \), we obtain \( L^* B_+ L = B_- \) if and only if \( \alpha_+ = \alpha_- \) and \( \beta_+ = \beta_- \). Then, in this case (and only in this one) we obtain that \( L \) is \((\mathbb{G}_-, \mathbb{G}_+)-\)unitary. Therefore, the following operators \((H_Z, D(H_Z))\) defined by
\[
\begin{align*}
H_Zu = -A_0^* u = A_0 u \\
D(H_Z) = \{ u \in D(A_0^*): L(u(0^-), u'(0^-), u''(0-)) \}
\end{align*}
\]

are a skew-self-adjoint family of extension for \((A_0, D(A_0))\), where for \( u = (u_0)_{e \in \mathcal{E}} \in D(H_Z) \) we have used the abbreviations \( u(0^-) = (u_e(0^-))_{e \in \mathcal{E}_-}, u'(0^-) = (u'_e(0^-))_{e \in \mathcal{E}_-}, u''(0-) = (u''_e(0^-))_{e \in \mathcal{E}_-} \) (similarly for the terms \( u(0+), u'(0+) \) and \( u''(0+) \)). Thus, we obtain the following system of conditions
\[
u(0-) = u(0+), \quad u'(0+) - u'(0-) = Zu(0-),
\]
\[(\delta - \text{interaction type on each two oriented half-lines})
\]
and
\[
\frac{Z^2}{2}u(0-) + Zu'(0-) = u''(0+) - u''(0-).
\]

Next, we build a family of stationary profiles for the KdV model (1.3) on a balanced graph \( \mathcal{G} \) with the ‘continuity at zero’ property. For the convenience of the reader and clarity in the exposition, we consider the constants sequences \((\alpha_e)_{e \in \mathcal{E}} = (\alpha_+), (\beta_e)_{e \in \mathcal{E}} = (\beta_+)\), with \( \alpha_+ > 0 \) and \( \beta_+ < 0 \) (in remark 6.7 we study the general case). Then, for \(-\frac{\alpha_+}{\beta_+} > \frac{Z^2}{4}\) we can consider the half-soliton profile \( \phi_+ \) defined in (3.5) and \( \phi_-(\cdot x) \equiv \phi_+(-\cdot x) \) for \( x < 0 \). Hence, we define the following constants sequences of functions \( u_+ = (\phi_+)_{e \in \mathcal{E}_+} \) and \( u_- = (\phi_+)_{e \in \mathcal{E}_-} \). Therefore, \( U_Z \equiv U_{Z, \alpha_+, \beta_+} \equiv (u_-, u_+) \) represents a family of stationary profiles for the KdV model satisfying the boundary conditions (6.3) for \( Z > 0 \) we have the bump-type profile showed in figure 4, for \( Z < 0 \) the corresponding tail-type profile in figure 5.

With the notations above, the main instability result of this section is the following.

**Theorem 6.1.** Let \( Z \neq 0 \). For \( \alpha_+ > 0 \) and \( 0 > \beta_+, -\frac{\alpha_+}{\beta_+} > \frac{Z^2}{4} \), we consider the profiles \( \phi_\pm \) in (3.5). Define \( U_Z = (\phi_e)_{e \in \mathcal{E}} \in D(H_Z) \) with \( \phi_e = \phi_- \) for \( e \in \mathcal{E}_- \) and \( \phi_e = \phi_+ \) for \( e \in \mathcal{E}_+ \). Then,
\[
\Phi_Z(x, t) = U_Z(x)
\]
defines a family of linearly unstable stationary solutions for the KdV model (1.3).

For the general case of sequences \((\alpha_e)_{e \in \mathcal{E}}\) and \((\beta_e)_{e \in \mathcal{E}}\), in remark 6.7 below we establish the necessary conditions for obtaining the linear instability of the corresponding bump and tail profiles.
The linear instability of the continuous (at zero) tail and bump profiles $U_Z$, $Z \neq 0$, is obtained from theorems 4.4–4.6 with a framework determined by the spaces $C$ and $D(H_Z) \cap C$ where

$$C = \{(u_e)_{e \in E} \in L^2(G) : u_{1,-}(0-) = \ldots = u_{n,-}(0-) = u_{1,+}(0+) = \ldots = u_{n,+}(0+)\}$$

(6.4)

represents the set of elements of $L^2(G)$ continuous at the graph-vertex $\nu = 0$. We start with the verification of assumption $(S_1)$ on $C$. Indeed, if $\{V(t)\}_{t \in \mathbb{R}}$ represents the unitary group generated by $H_Z$ (see proposition A.12 in appendix A) we have that $V(t) : C \rightarrow C$ is well-defined and $D(H_Z) \cap C$ is invariant by this group.

Next, by following the same strategy as in section 5 (with $(\alpha_e)_{e \in E} = (1)_{e \in E}$, $(\beta_e)_{e \in E} = (-1)_{e \in E}$, without loss of generality), we consider the $2n \times 2n$-matrix derivative operator $N$ in (4.9) with domain $D(H) = H^1(G) \cap C$, and the $2n \times 2n$-matrix Schrödinger operator

$$E_Z = \text{diag} \left( L_{Z,-}, L_{Z,+} \right)$$

(6.5)

with

$$L_{Z,\pm} = \text{diag} \left(-\frac{d^2}{dx^2} + 1 - 2\phi_x, \ldots, -\frac{d^2}{dx^2} + 1 - 2\phi_x\right),$$

(6.6)

being $n \times n$-diagonal matrices. Then, from the proof of lemma 6.4 below (see also proposition A.6), we have that $E_Z$ is a family of self-adjoint operators with domain $D(E_Z) = D_{Z,\beta} \cap C \subset H^2(G)$ where

$$u \in D_{Z,\beta} \Leftrightarrow u(0-) = u(0+) \quad \text{and} \quad \sum_{e \in E_+} u'_e(0+) - \sum_{e \in E_-} u'_e(0-) = Znu_{1,+}(0+).$$

(6.7)

Then, it is immediate from (6.3) that $D(H_Z) \cap C \subset D(E_Z)$ and so assumption $(S_1)$ holds. By following an argument similar to that in remark 4.2-item (4) we obtain assumptions $(S_4)$ and $(S_5)$.

The proof of the following result (associated to assumption $(S_4)$) follows the same strategy as in lemma 5.2.

**Lemma 6.2.** Let $Z \neq 0$ and the operator $E_Z : D(E_Z) \rightarrow L^2(G)$ defined in (6.5) with $D(E_Z) = D_{Z,\beta} \cap C$. Then, $E_Z$ is invertible with $\sigma_{ess}(E_Z) = [1, +\infty)$.

In the following proposition we determine the Morse index for $E_Z$ on a specific subspace of $L^2(G)$ which is sufficient for our instability study.

**Proposition 6.3.** Let $E_Z : D(E_Z) \rightarrow L^2(G)$ defined in (6.5) with $D(E_Z) = D_{Z,\beta} \cap C$. Define the following closed subspace in $L^2(G)$.

$$L^2_0(G) = \{ u = (u_e)_{e \in E} : u_e = f, \quad \text{for all} \ e \in E_-, u_e = g, \quad \text{for all} \ e \in E_+ \}$$

Then, $n(E_Z|_{L^2_0(G)}) = 2$ for $Z > 0$, and $n(E_Z|_{L^2_0(G)}) = 1$ for $Z < 0$.

The proof of proposition 6.3 is based in the analytic perturbation theory and in the extension theory of symmetric operators. We note that in the case $Z < 0$ (tail case), it can be given an argument based exclusively in the extension theory of symmetric operators for obtaining that $n(E_Z) = 1$ on $D(E_Z)$ without the constrain $D(E_Z) \cap L^2_0(G)$ (see [7, 8]).
Following the same strategy as in the proof of lemma 5.3, the cornerstone for obtaining proposition 6.3 is the following.

**Lemma 6.4.** It considers the following self-adjoint matrix Schrödinger operator in $L^2(\mathcal{G})$

$$\mathcal{E}_0 = \text{diag } (\mathcal{L}_{0,-}, \mathcal{L}_{0,+})$$

with

$$\mathcal{L}_{0,\pm} = \text{diag } \left(-\frac{d^2}{dx^2} + 1 - 2\phi_0, \ldots, -\frac{d^2}{dx^2} + 1 - 2\phi_0\right),$$

being $n \times n$-diagonal matrices and $\phi_0$ the soliton profile defined in (5.4), and the Kirchhoff’s type condition at $v = 0$, namely,

$$D(\mathcal{E}_0) = \left\{ u \in H^2(\mathcal{G}) \cap \mathcal{C} : u(0-)=u(0+), \sum_{e \in \mathcal{E}_+} u'_e(0+) - \sum_{e \in \mathcal{E}_-} u'_e(0-) = 0 \right\}.$$ (6.10)

Then,

(a) In the space $L^2(\mathcal{G})$ we have $\text{Ker}(\mathcal{E}_0) = [\Phi'_0]$, where $\Phi'_0 = (\phi'_0)_{e \in \mathcal{E}}$.

(b) The operator $(\mathcal{E}_0, D(\mathcal{E}_0))$ has one simple negative eigenvalue in $L^2(\mathcal{G})$. Moreover, we also have $\eta(\mathcal{E}_0|_{L^2(\mathcal{G})}) = 1$.

(c) The rest of the spectrum of $\mathcal{E}_0$ is positive and bounded away from zero.

**Proof.** The proof of item (a) follows from a similar analysis as in lemma 5.2. Indeed, let $v = (v_e)_{e \in \mathcal{E}} \in \text{Ker}(\mathcal{E}_0) \cap L^2(\mathcal{G})$, then

$$-v''_e + v_e - 2\phi_0 v_e = 0, \quad e \in \mathcal{E}. \quad (6.11)$$

Therefore, $v_e = c_e \phi_0'$ for $e \in \mathcal{E}$ and so $v_e(0-) = v_e(0+) = 0$. Now, since $v \in L^2(\mathcal{G})$, we obtain for $e \in \mathcal{E}_-$ that $v_e = c_0 \phi_0'$ with $c_0 = c_e$, and for $e \in \mathcal{E}_+$ that $v_e = c_1 \phi_0'$ with $c_1 = c_e$. Then from (6.10) we obtain $nc_1 \phi_0''(0) = nc_0 \phi_0''(0)$. Therefore, since $\phi_0''(0) \neq 0$ we get $v = c_0 \Phi'_0$.

For item (b), we will use extension theory for symmetric operators. Indeed, we consider the $2n \times 2n$-diagonal matrix operator

$$\mathcal{F}_0 = \text{diag } \left(-\frac{d^2}{dx^2}, \ldots, -\frac{d^2}{dx^2}\right),$$

with domain

$$D(\mathcal{F}_0) = \left\{ u \in H^2(\mathcal{G}) : u(0-) = u(0+) = 0, \sum_{e \in \mathcal{E}_+} u'_e(0+) - \sum_{e \in \mathcal{E}_-} u'_e(0-) = 0 \right\}.$$ (6.13)

Then $(\mathcal{F}_0, D(\mathcal{F}_0))$ represents a closed symmetric operator densely defined on $L^2(\mathcal{G})$ (we note that $\bigoplus_{e \in \mathcal{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{e \in \mathcal{E}_+} C_c^\infty(0, +\infty) \subset D(\mathcal{F}_0)$). Moreover, the adjoint operator $(\mathcal{F}_0^*, D(\mathcal{F}_0^*))$ is given by

$$\mathcal{F}_0^* = \mathcal{F}_0, \quad D(\mathcal{F}_0^*) = \{ u \in H^2(\mathcal{G}) : u \in \mathcal{C} \}, \quad \text{(6.14)}$$
and so the deficiency indices for \((\mathcal{F}_0, D(\mathcal{F}_0))\) are \(n_{\pm}(\mathcal{F}_0) = 1\) (see proposition A.6 in appendix A below). Then, from the Krein–von Neumann extension theory for symmetric operators (see [4], theorem A.1) and from proposition A.6 we obtain that all self-adjoint extension of \((\mathcal{F}_0, D(\mathcal{F}_0))\), denoted by \((\mathcal{E}_Z, D(\mathcal{E}_Z))\), can be parameterized by \(Z \in \mathbb{R}\) as \(\mathcal{E}_Z = \mathcal{F}_0\), and \(u \in D(\mathcal{E}_Z)\) if and only if \(u \in \tilde{C}\) and \(u\) satisfying (6.7). Next, we define the following bounded operator on \(L^2(G)\)

\[
B_0 = \begin{pmatrix} M_{0,+} & 0 \\ 0 & M_{0,-} \end{pmatrix}, \quad M_{0,\pm} = \text{diag}(1 - 2\phi_0, \ldots, 1 - 2\phi_0)
\]

with \(M_{0,\pm}\) being \(n \times n\)-diagonal matrices. Then, from [53]-chapter IV-theorem 6 it follows that the symmetric operators \(\mathcal{F}_0\) and \(\tilde{\mathcal{F}}_0 = \mathcal{F}_0 + B_0\), with \(D(\tilde{\mathcal{F}}_0) = D(\mathcal{F}_0)\), have the same deficiency indices, \(n_{\pm}(\mathcal{F}_0) = n_{\pm}(\tilde{\mathcal{F}}_0) = 1\). Thus \((\mathcal{E}_0, D(\mathcal{E}_0))\) belongs to the family of the self-adjoint extensions of \(\mathcal{F}_0\) \((Z = 0\) in (6.7)).

Next we see that the symmetric operator \(\tilde{\mathcal{F}}_0\) with domain \(D(\tilde{\mathcal{F}}_0) = D(\mathcal{F}_0)\) in (6.13), it is non-negative. Indeed, it is easy to verify that for \(u = (u_e)_{e \in E} \in H^2(G)\) the following holds

\[
-\frac{1}{\phi_0} \frac{d}{dx} \left( \phi_0 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right) \right) = -\left( \frac{\phi_0}{\phi_0} \right)^2 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right) + \left( \frac{\phi_0}{\phi_0} \right)^2 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right) \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right)
\]

for \(x < 0\) if \(e \in E_-\), \(x > 0\) if \(e \in E_+\). Using the above equality and integrating by parts, we get for \(u = (u_e)_{e \in E} \in D(\tilde{\mathcal{F}}_0)\),

\[
\langle \tilde{\mathcal{F}}_0 u, u \rangle = \sum_{e \in E_{-\infty}} \int_0^0 \left( \phi_0 \right)^2 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right)^2 dx + \sum_{e \in E_{+\infty}} \int_0^0 \left( \phi_0 \right)^2 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right)^2 dx - \sum_{e \in E_+} \int_{-\infty}^0 \left( \phi_0 \right)^2 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right)^2 dx - \sum_{e \in E_-} \int_0^0 \left( \phi_0 \right)^2 \frac{d}{dx} \left( \frac{u_e}{\phi_0} \right)^2 dx.
\]

The integral terms in (6.16) are non-negative and equal zero if and only if \(u \equiv 0\). Due to the conditions \(u(-\infty) = u(+\infty) = 0\) and \(\phi_0(0 \pm) \neq 0\), non-integral terms vanish, and we get \(\tilde{\mathcal{F}}_0 \geq 0\).

Due to proposition A.3 (appendix A), we have that the self-adjoint extension \(\mathcal{E}_0\) of \(\mathcal{F}_0\) satisfies \(n(\mathcal{E}_0) \leq 1\). Taking into account the notation \(\Phi_0 = (\phi_0)_{e \in E}\) for the solitary wave profile we have that \(\Phi_0 \in D(\mathcal{E}_0)\) and \(\mathcal{E}_0 \Phi_0 = \Psi\), with \(\Psi = (-\phi_0^2)_{e \in E}\). Then,

\[
\langle \mathcal{E}_0 \Phi_0, \Phi_0 \rangle = -n \int_{-\infty}^0 \phi_0^2(x) dx - n \int_0^{+\infty} \phi_0^2(x) dx < 0
\]

and so from minimax principle we arrive at \(n(\mathcal{E}_0) = 1\) on \(D(\mathcal{E}_0)\). Lastly, since \(\Phi_0 = (\phi_0)_{e \in E} \in L^2(G)\) we get \(n(\mathcal{E}_0)_{|L^2(G)} = 1\).

Item (c) is an immediate consequence of Weyl’s theorem (see theorem XIII.14 in Reed and Simon [57]). This finishes the proof.

\[\Box\]

**Remark 6.5.** We observe that, when we deal with deficiency indices, the operator \(\mathcal{E}_0\) is assumed to act on complex-valued functions which however does not affect the analysis of negative spectrum of \(\mathcal{E}_0\) acting on real-valued functions (see proposition A.6).

**Proof of Proposition 6.3.** For \(u = (\phi_-)_{e \in E_-}, u_+ = (\phi_+)_{e \in E_+}\), it is not difficult to see the convergence \(U_Z = (\ldots, u_+) \rightarrow \Phi_0 = (\phi_0)_{e \in E}\) as \(Z \rightarrow 0\), in \(H^1(G)\). Then, from lemmas 6.2 and
and from perturbation theory, we show that $n(\mathcal{E}_Z) = 1$ as $Z < 0$, and $n(\mathcal{E}_Z) = 2$ as $Z > 0$, on $D(\mathcal{E}_Z) \cap L^2(\mathcal{G})$. This finishes the proof.

Similarly to the case of two half-lines, for $\alpha_+ = 1$ and $\omega = -\beta_+$, we have the differentiable family of stationary solutions a one-parameter $U_{Z\omega} = (u_{-\omega}, u_{+\omega})$, with $u_{-\omega} = (\phi_{-\omega})$ and $u_{+\omega} = (\phi_{+\omega})$. Thus, for $\varphi_{\omega} = (-\frac{d}{dx} u_{-\omega}, -\frac{d}{dx} u_{+\omega})$ we have $\varphi \equiv \varphi_{\omega}|_{\omega = 1} \in D(\mathcal{E}_Z)$ and $\mathcal{E}_Z \varphi = U_Z$. Hence, we obtain the following result (assumption $S_7$).

**Lemma 6.6.** Let $Z \neq 0$. The smooth curve of profiles $\omega \in (\frac{2\pi}{T}, +\infty) \to U_{Z\omega} = (\phi_{-\omega}, \phi_{+\omega}) \in D(\mathcal{E}_Z) \cap L^2(\mathcal{G})$ satisfies for $\varphi \equiv -\frac{d}{dx} U_{Z\omega} |_{\omega = 1}$ the relations $\mathcal{E}_Z \varphi = U_Z$ and $\langle \varphi, U_Z \rangle < 0$.

**Proof of Theorem 6.1.** From lemmas 6.2–6.6, proposition 6.3, subsection 4.2, and theorems 4.4 and 4.6, we obtain the linear instability property of the tails and bump profiles $U_Z$ for any $Z \neq 0$.

**Remark 6.7.** The extension of theorem 6.1 for arbitrary sequences $(\alpha_k)_{k \in \mathbb{E}} = (\alpha_-, \alpha_+)$ and $(\beta_k)_{k \in \mathbb{E}} = (\beta_-, \beta_+)$, with $\alpha_- = \alpha_+$, $\beta_- = \beta_+$, it can be obtained via the following steps:

(a) Let $n \geq 2$. For $\alpha_+ = (\alpha_1, \ldots, \alpha_n)$ and $\beta_+ = (\beta_1, \ldots, \beta_n)$ we consider the associated either bumps or tails profiles $U_{Z\alpha+, \beta_+} = (u_-, u_+)$, where for $\phi_{\pm, \alpha_\beta}$ defined by (3.5) and $\phi_{-\alpha_\beta}(x) = \phi_{-\alpha_\beta}(-x)$ for $x < 0$, we have

$$u_- = (\phi_{-\alpha_\beta})_{1 \leq i \leq n}, \quad u_+ = (\phi_{+\alpha_\beta})_{1 \leq i \leq n}.$$  \hspace{1cm} (6.17)

In other words, we have $n$-profiles of either bump or tail type as in figures 2 and 3 on a balanced graph $\mathcal{G}$. We note that $a \text{ priori}$ they do not need to be continuous at the graph-vertex. Thus, we obtain that $U_{Z, \alpha+, \beta_+} \in D(\mathcal{H}_Z) \cap \mathcal{C} \subset D(\mathcal{E}_{Z, \alpha+, \beta_+}) = \mathcal{C} \cap D_{Z, \alpha+, \beta_+}$ if and only if

$$\beta_{1+} + \frac{Z^2}{4}\alpha_{1+} = \beta_{2+} + \frac{Z^2}{4}\alpha_{2+} = \ldots = \beta_{n+} + \frac{Z^2}{4}\alpha_{n+} \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = n.$$ \hspace{1cm} (6.18)

Here, the $2n \times 2n$-matrix self-adjoint Schrödinger operator $\mathcal{E}_{Z, \alpha+, \beta_+} = \text{diag} \left( \mathcal{L}_{Z-}, \mathcal{L}_{Z+} \right)$ is defined by the $n \times n$-diagonal matrices,

$$\mathcal{E}_{Z+,} = \text{diag} \left( -\alpha_{1+} \frac{d^2}{dx^2} - \beta_{1+} - 2\phi_{\pm, \alpha_\beta}, \ldots, -\alpha_{n+} \frac{d^2}{dx^2} - \beta_{n+} - 2\phi_{\pm, \alpha_\beta} \right).$$ \hspace{1cm} (6.19)

and $D_{Z, \alpha+, \beta_+} \subset H^2(\mathcal{G})$ is defined by

$$u \in D_{Z, \alpha+, \beta_+} \iff u(0-) = u(0+), \quad \sum_{i=1}^{n} \alpha_i u_i'_{\pm}(0+) - \sum_{i=1}^{n} \alpha_i u_i'(0-) = Z\alpha_{i+}(0+).$$ \hspace{1cm} (6.20)

(b) $\mathcal{E}_{Z, \alpha+, \beta_+} : D(\mathcal{E}_{Z, \alpha+, \beta_+}) \to L^2(\mathcal{G})$ is invertible: In fact, let $u = (u_1, \ldots, u_n, u_1+, \ldots, u_n+) \in D(\mathcal{E}_{Z, \alpha+, \beta_+})$ and $\mathcal{E}_{Z, \alpha+, \beta_+} u = 0$. Then, for $\phi_{\pm, j} \equiv \phi_{\pm, \alpha_j, \beta_j}$, Sturm–Liouville
theory implies \( u_{i\pm} = a_{i\pm} \phi_{i\pm}^0, i = 1, \ldots, n \). Hence, since \( \phi_{i+}^0(0+) = -\phi_{i-}^0(0-) \) and \( u \in C \), we obtain

\[
a_{i+} = -a_{i-}, \quad a_{1+} = \ldots = a_{n+}, \quad a_{1-} = \ldots = a_{n-}, \quad i = 1, \ldots, n.
\]

Next, since \( u \in D_{Z_{\alpha+}, \beta+} \) and \( U_{Z_{\alpha+}, \beta+} \in C \) we obtain from (3.2) the relation

\[
n a_{1+} Z_2^2 \phi_{i+}(0+) = 2a_{1+} \sum_{i=1}^n a_i \phi_{i+}(0+) = 2a_{1+} \phi_{i+}(0+)
\]

\[
\times \sum_{i=1}^n [-\beta_{i+} - \phi_{i+}(0+)].
\]

Suppose \( a_{1+} \neq 0 \). Then, we have the following chain of equalities

\[
n Z_4^2 = -\sum_{i=1}^n \left[ \bar{\beta}_{i+} - \frac{3}{2} \left( \beta_{i+} + \frac{Z_4^2}{4} \alpha_{i+} \right) \right] = \frac{1}{2} \sum_{i=1}^n \left[ \beta_{i+} + \frac{Z_4^2}{4} \alpha_{i+} \right] + n Z_4^2. \quad (6.21)
\]

Therefore, from (6.17) we get \( \beta_{1+} + \frac{Z_4^2}{2} \alpha_{1+} = 0 \) and so we obtain a contradiction because of \( -\beta_{1+} > \frac{Z_4^2}{2} \alpha_{1+} \). Hence, \( a_{1+} = 0 \) and so \( u \equiv 0 \).

(c) The relations \( m(\mathcal{E}_{Z_{\alpha+}, \beta+}) = 2 \) for \( Z > 0 \) and \( m(\mathcal{E}_{Z_{\alpha+}, \beta+}) = 1 \) for \( Z < 0 \) follow via a perturbation analysis (or extension theory approach for \( Z < 0 \)). Moreover, from (6.17) and (3.5) we obtain for \( \psi = (\frac{d}{d\lambda} u_{1+}, \frac{d}{d\lambda} u_{1+}) \) that \( \mathcal{E}_{Z_{\alpha+}, \beta+} \psi \equiv U_{Z_{\alpha+}, \beta+} \) and \( \langle \psi, U_{Z_{\alpha+}, \beta+} \rangle < 0 \).

Therefore, from items (a)-(c) above and from theorems 4.4 and 4.6 we obtain the linear instability property of the solitons profiles \( U_{Z_{\alpha+}, \beta+} \) for every \( Z \neq 0 \).

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Appendix A

A.1. Extension theory

For the sake of completeness, in this part of the appendix we develop the extension theory of symmetric operators suitable for our needs. For further information on the subject the reader is referred to the monograph by Naimark [53]. The following classical result, known as the von-Neumann decomposition theorem, can be found in [57], p 138.

**Theorem A.1.** Let \( A \) be a closed, symmetric operator, then

\[
D(A^*) = D(A) \oplus \mathcal{N}_{\alpha^-} \oplus \mathcal{N}_{\beta^-}, \quad (A.1)
\]

with \( \mathcal{N}_{\alpha^\pm} = \text{Ker}(A^* \mp iI) \). Therefore, for \( u \in D(A^*) \) and \( u = x + y + z \in D(A) \oplus \mathcal{N}_{\alpha^-} \oplus \mathcal{N}_{\beta^-}, \)

\[
A^* u = Ax + (-i)y + i\zeta. \quad (A.2)
\]
Remark A.2. The direct sum in (A.1) is not necessarily orthogonal.

The following propositions provide a strategy for estimating the Morse-index of the self-adjoint extensions and the characterization of these extensions in a particular case (see Naimark [53] (theorem 16, p 44) and Reed and Simon, vol 2, [57] (theorem X.2, p 140)).

Proposition A.3. Let $A$ be a densely defined lower semi-bounded symmetric operator (that is, $A \geq \mu$) with finite deficiency indices $n_{\pm}(A) = k < \infty$ in the Hilbert space $\mathcal{H}$, and let $\tilde{A}$ be a self-adjoint extension of $A$. Then the spectrum of $\tilde{A}$ in $(-\infty, \mu)$ is discrete and consists of at most $k$ eigenvalues counting multiplicities.

Remark A.2. Nonlinearity 34 (2021) 3373 J Angulo Pava and M Cavalcante

Proposition A.4. Let $A$ be a densely defined, closed, symmetric operator in some Hilbert space $\mathcal{H}$ with deficiency indices equal $n_{\pm}(A) = 1$. All self-adjoint extensions $A_\theta$ of $A$ may be parametrized by a real parameter $\theta \in [0, 2\pi)$ where

$$D(A_\theta) = \{ x + c\phi_+ + \zeta e^{i\theta} \phi_- : x \in D(A), \zeta \in \mathbb{C} \},$$

$$A_\theta(x + \zeta \phi_+ + \zeta e^{i\theta} \phi_-) = Ax + i\zeta \phi_+ - i\zeta e^{i\theta} \phi_-,$$

with $A_\star \phi_\pm = \pm \phi_\pm$, and $\| \phi_\pm \| = \| \phi_- \|$.

Next proposition can be found in Naimark [53] (theorem 9, p 38).

Proposition A.5. All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and the same continuous spectrum.

The following result was used in the proof of lemma 6.4.

Proposition A.6. It considers the closed symmetric operator densely defined on $L^2(G)$, $(\mathcal{F}_0, D(\mathcal{F}_0))$, by (6.11) and (6.12). Then, the deficiency indices are $n_{\pm}(\mathcal{F}_0) = 1$. Therefore, we have that all the self-adjoint extensions of $(\mathcal{F}_0, D(\mathcal{F}_0))$ can be parametrized by $Z \in \mathbb{R}$, namely, $(\mathcal{L}_Z, D(\mathcal{L}_Z))$, with the action $\mathcal{L}_Z \equiv \mathcal{F}_0$ and $u \in D(\mathcal{L}_Z)$ if and only if $u \in \mathcal{C}$ and $D(\mathcal{C}_0)$ (see (6.3)–(6.6)).

Proof. We show initially that the adjoint operator $(\mathcal{F}_0^*, D(\mathcal{F}_0^*))$ of $(\mathcal{F}_0, D(\mathcal{F}_0))$ is given by

$$\mathcal{F}_0^* = \mathcal{F}_0, \quad D(\mathcal{F}_0^*) = \{ u \in H^2(G) : u \in \mathcal{C} \}. \quad (A.3)$$

Indeed, for $u, v \in H^2(G)$ we have

$$\langle \mathcal{F}_0^* v, u \rangle = -\sum_{\mathbf{e} \in \mathcal{E}_-} v_{\mathbf{e}}(0)u_{\mathbf{e}}(0) + \sum_{\mathbf{e} \in \mathcal{E}_+} v_{\mathbf{e}}(0)u_{\mathbf{e}}(0) + \sum_{\mathbf{e} \in \mathcal{E}_-} v_{\mathbf{e}}^0(0)u_{\mathbf{e}}(0) - \sum_{\mathbf{e} \in \mathcal{E}_+} v_{\mathbf{e}}^0(0)u_{\mathbf{e}}(0) + \langle v, \mathcal{F}_0u \rangle. \quad (A.4)$$

It denotes by $D_0 = \{ u \in H^2(G) : u \in \mathcal{C} \}$. Then we will show $D_0^* = D(\mathcal{F}_0^*)$. Indeed, we see initially $D_0^* \supseteq D(\mathcal{F}_0^*)$. So, for $u \in D_0^*$ and $v \in D(\mathcal{F}_0)$ it follows from (A.4) that

$$\langle \mathcal{F}_0^* v, u \rangle = u_{1,+}(0+) - \sum_{\mathbf{e} \in \mathcal{E}_-} v_{\mathbf{e}}^0(0) + \sum_{\mathbf{e} \in \mathcal{E}_+} v_{\mathbf{e}}^0(0) + \langle v, \mathcal{F}_0u \rangle = \langle v, \mathcal{F}_0u \rangle. \quad (A.5)$$

Hence $u \in D(\mathcal{F}_0)$ and $\mathcal{F}_0^* u = \mathcal{F}_0u$. Let us show the inverse inclusion $D_0^* \supseteq D(\mathcal{F}_0^*)$. Take $u \in D(\mathcal{F}_0^*)$, then for any $v \in D(\mathcal{F}_0)$ we have from (A.4)

$$\langle \mathcal{F}_0 v, u \rangle = -\sum_{\mathbf{e} \in \mathcal{E}_-} v_{\mathbf{e}}(0)u_{\mathbf{e}}(0) + \sum_{\mathbf{e} \in \mathcal{E}_+} v_{\mathbf{e}}^0(0)u_{\mathbf{e}}(0) + \langle v, \mathcal{F}_0u \rangle$$
Thus, for any \( v \in D(F_0) \) we obtain the equality
\[
\sum_{e \in E_+} v'_e(0)u_e(0) - \sum_{e \in E_-} v'_e(0)u_e(0) = 0.
\] (A.7)

Next, it considers \( v = (v_{1,-}, v_{2,-}, \ldots, v_{n,-}, 0, \ldots, 0) \in D(F_0) \) with \( v'_{1,-}(0-) = \ldots = v'_{n,-}(0-) = v_{1,-}(0-) = 0 \) and \( v_{1,-}(0-) \neq 0 \). Then from (A.7) we obtain \( v'_{1,-}(0-) (u_{1,-}(0-) - u_{2,-}(0-)) = 0 \) and so \( u_{1,-}(0-) = u_{2,-}(0-) = \ldots = u_{n,-}(0-) \). Repeating similar arguments we get \( u_{1,-}(0-) = u_{2,+}(0-) = \ldots = u_{n,-}(0+) \). Similarly, we see \( u_{1,+}(0-) = u_{2,+}(0+) = u_{3,+}(0+) = \ldots = u_{n,+}(0+) \). Lastly, for obtaining \( u_{1,-}(0-) = u_{1,+}(0-) \) we consider \( v \in D(F_0) \) such that \( v'_{1,-}(0-) = \ldots = v'_{n,-}(0-) = v'_{1,+}(0+) = \ldots = v'_{n,+}(0+) = 0 \) and \( v'_{1,+}(0+) \neq 0 \). Then, from (A.7) and relation \( v'_{1,+}(0+) (u_{1,+}(0+) - u_{1,-}(0-)) = 0 \) we have \( u \in D_0 \). Therefore, (A.3) holds.

From (A.3) we obtain that the deficiency indices for \( (F_0, D(F_0)) \) is \( n_\pm(F_0) = 1 \). Indeed, \( \text{Ker}(F_0^* \pm iI) = \{ \Psi \} \) with \( \Psi = (\Psi_{x,\pm})_{x \in E} \) defined by
\[
\Psi_{x,\pm} = \begin{cases} 
\left( \begin{array}{c} i e^{\pm ik_1 x} \\
\vdots \\
i e^{\pm ik_n x} 
\end{array} \right), & x < 0, \ e \in E_- \\
\left( \begin{array}{c} i e^{\pm ik_1 x} \\
\vdots \\
i e^{\pm ik_n x} 
\end{array} \right), & x > 0, \ e \in E_+,
\end{cases}
\] (A.8)

with \( k_{\pm}^2 = \mp i, \text{Im}(k) < 0 \) and \( \text{Im}(k) > 0 \).

Next, we show that the domain of any self-adjoint extension \( \hat{F} \) of the operator \( (F_0, D(F_0)) \) (and acting on complex-valued functions) is given by \( D_{2, \pm} \) in (6.6). Indeed, from von-Neumann decomposition theorem above and proposition A.4 we have that \( D(\hat{F}) \subset C \) and
\[
D(\hat{F}) = \left\{ u \in H^2(G) : u = u_0 + c \Psi_- + c e^{i\theta} \Psi_+ : u_0 \in F_0, c \in \mathbb{C}, \theta \in [0, 2\pi) \right\}.
\]

Thus, it is easily seen that for \( u \in D(\hat{F}) \) we have
\[
\sum_{e \in E_+} u'_e(0+) - \sum_{e \in E_-} u'_e(0-) = 2cn(1 - e^{i\theta}) = u_{1,+}(0+) = -c \left( e^{i\theta} - e^{i(\theta - \pi)} \right),
\] (A.9)

and hence
\[
\sum_{e \in E_+} u'_e(0+) - \sum_{e \in E_-} u'_e(0-) = Zn_{1,+}(0+) \text{ where}
\]
\[
Z = \frac{-2(1 - e^{i\theta})}{e^{i\theta} - e^{i(\theta - \pi)}} \in \mathbb{R} \cup \{ \pm \infty \}. \tag{A.10}
\]

This finishes the proof.

\[\square\]

### A.2. Unitary representation for Airy operators

The main idea of this part of the appendix is to establish a representation formula for the unitary groups generated by the family of skew-self-adjoint operators \( (H_2, D(H_2)) \) defined in (6.2) for the case of a balanced metric graph (see proposition A.12 below). This result is
essential in the linear instability theory of tail and bumps profiles established in section 6.
We call the attention that the arguments established below can be used for obtaining a
representation of unitary groups of general skew-self-adjoint extensions for the Airy operator
in (2.1) (see proposition 2.1).

We start our study by establishing initially a representation formula for the unitary group
associated to the linear evolution equation
\[
\begin{cases}
  u_t = AZu, & t \in \mathbb{R} \\
  u(0) = u_0 \in D(A_Z),
\end{cases}
\]
with \((A_Z, D(A_Z))\) defined in proposition 2.3, namely, the case of two half-lines. Now, for clarity
in the exposition we will consider without loss of generality the cases
\[\alpha_- = \alpha_+ = \beta_- = \beta_+ = 1\] in (2.1) (see remark A.11). Since \(A_Z\) is a skew-self-adjoint operator it follows from
Stone’s theorem that the solution of (A.11), \(u(t) = W(t)u_0\), is given by a unitary group
\(\{W(t)\}_{t \in \mathbb{R}}\) on \(L^2(G)\) with associated infinitesimal generator \(AZ\). Thus, for denoting \(W(t) = W_-(t) \oplus W_+(t)\) and \(w = (p, q) \in L^2(G) = L^2(-\infty, +0) \oplus L^2(0, +\infty)\) we can see the action of \(W(t)\) on \(w\) as
\[
W(t)w \equiv (W_-(t)p, W_+(t)q).
\]

The purpose of the following results is to establish initially explicit formulas for every \(W_\pm\).

**Lemma A.7.** Let \(q \in L^2(0, +\infty)\) and \(\text{Re } \lambda > 0\). The non-homogeneous linear problem
\[
\begin{cases}
  Nv(x) = q(x), & 0 \leq x < +\infty \\
  v(0) = a_0, & v(x), v'(x) \to 0 \quad \text{as } x \to +\infty.
\end{cases}
\]
with \(Nv(x) \equiv \lambda v(x) + v'(x) + v''(x)\), has the representation
\[
v_+(x) = a_0 e^{\gamma_1 x} + \int_0^x G_+(x, \zeta, \lambda) q(\zeta) d\zeta, \quad x \geq 0,
\]
where \(G_+(x, \zeta, \lambda)\) is the associated Green’s function for (A.12) and \(\text{Re } \gamma_1 < 0\).

**Proof.** Consideration is first directed to find the Green’s function \(G_+ \equiv G_+(x, \zeta, \lambda)\) associated to the non-homogeneous linear problem (A.12), namely, with \(a_0 = 0\) and \(q \equiv 0\). Indeed, let \(\gamma_1, \gamma_2, \gamma_3\) be the three roots of the characteristic equation
\[
\lambda + \gamma + \gamma^3 = 0, \quad \text{for } \text{Re } \lambda > 0,
\]
ordered so that
\[
\text{Re } \gamma_1 < 0, \quad \text{Re } \gamma_2 > 0, \quad \text{and } \text{Re } \gamma_3 > 0.
\]
As we know \(G_+\) is given as the unique solution of the problem
\[
\begin{cases}
  N_0 g(x, \zeta) = \delta(x - \zeta), & 0 < x, \zeta < +\infty, \\
  \text{for } 0 < x < \zeta \text{ we have } g(0) = 0, \quad \text{and } g(+\infty) = g'(+\infty) = 0, \\
  g, \frac{dg}{dx} \text{ are continuous at } x = \zeta; \quad \frac{d^2 g}{dx^2} \bigg|_{x=\zeta^+} - \frac{d^2 g}{dx^2} \bigg|_{x=\zeta^-} = 1.
\end{cases}
\]
Thus, since the equation $Nv(x) = 0$, for $x > 0$, has the following fundamental set of solutions \( \{e^{\gamma_1 x}, e^{\gamma_2 x}, e^{\gamma_3 x}\} \) we obtain that the conditions $g(\pm \infty) = g'(\pm \infty) = 0$ imply
\[
g(x, \zeta) = d(\zeta)e^{\gamma_k x} \quad \text{for} \quad \zeta < x < \pm \infty. \tag{A.17}
\]

Next, the condition $g(0, \zeta) = 0$ implies
\[
g(x, \zeta) = a(\zeta)(e^{\gamma_1 x} - e^{\gamma_3 x}) + b(\zeta)(e^{\gamma_1 x} - e^{\gamma_2 x}), \quad \text{for} \quad 0 < x < \zeta. \tag{A.18}
\]

Then, from the conditions of continuity and jump for $g$ we obtain after an application of Kramer’s rule that
\[
a(\zeta) = \frac{-1}{\Delta(\lambda)}(\gamma_2 - \gamma_1)e^{\gamma_1 \zeta}, \quad b(\zeta) = \frac{-1}{\Delta(\lambda)}((\gamma_2 - \gamma_1)e^{\gamma_1 \zeta} + (\gamma_1 - \gamma_3)e^{\gamma_3 \zeta}),
\]
and
\[
d(\zeta) = \frac{-1}{\Delta(\lambda)}((\gamma_2 - \gamma_1)e^{\gamma_1 \zeta} + (\gamma_1 - \gamma_3)e^{\gamma_3 \zeta} + (\gamma_3 - \gamma_2)e^{\gamma_2 \zeta}), \tag{A.19}
\]
where $\Delta(\lambda) = (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_3)$. Therefore, $G_+$ is given explicitly by
\[
G_+(x, s, \lambda) = \frac{1}{\Delta(\lambda)}[(\gamma_3 - \gamma_1)e^{\gamma_1 x - \gamma_2 \zeta} + (\gamma_1 - \gamma_2)e^{\gamma_2 x - \gamma_3 \zeta}]
\]
\[
+ Y(x, \zeta)(\gamma_2 - \gamma_3)e^{\gamma_3 (x - \zeta)}
\]
\[
+ (1 - Y(x, \zeta))(\gamma_1 - \gamma_3)e^{\gamma_1 (x - \zeta)}
\]
\[
+ (\gamma_2 - \gamma_1)e^{\gamma_2 (x - \zeta)}], \tag{A.20}
\]
with $Y(x, \zeta) = 1$ for $0 \leq \zeta \leq x$, and $Y(x, \zeta) = 0$ otherwise. Then, the solution for (A.12) is given immediately by the superposition principle as the formula in (A.13).

Next, we find the part $W_-$ of $W$ in (A.12).

**Lemma A.8** Let $p \in L^2(\mathbb{R}, 0)$ and $\Re \lambda > 0$. The non-homogeneous linear problem
\[
\begin{align*}
Nv(x) &= p(x), & -\infty < x < 0 \\
v(0^-) &= a_1, & v'(0^-) = a_2, \quad v(x) \to 0 \quad \text{as} \quad x \to -\infty.
\end{align*} \tag{A.21}
\]
with $Nv(x) \equiv \lambda v(x) + v'(x) + v''(x)$, has the representation
\[
v_-(x) = \alpha_1 e^{\gamma_1 x} + \alpha_2 e^{\gamma_3 x} + \int_{-\infty}^{x} G_-(x, \zeta, \lambda)p(\zeta)d\zeta, \quad x \leq 0, \tag{A.22}
\]
where $G_-(x, \zeta, \lambda)$ is the associated Green’s function for (A.21) and $\Re \gamma_2 > 0, \Re \gamma_3 > 0$. The constants $\alpha_i$ are chosen such that $v_-(0^-) = a_1$ and $v'_-(0^-) = a_2$.

**Proof.** We start by finding the Green’s function $G_- = G_-(x, \zeta, \lambda)$ associated to the non-homogeneous linear problem (A.12), namely, when $a_1 = a_2 = 0$ and $p \equiv 0$. Indeed, let $\gamma_1, \gamma_2,$
and \( \gamma_3 \) be the three roots of the characteristic equation \( \text{A.14} \) such that \( \text{Re } \gamma_1 < 0, \text{Re } \gamma_2 > 0, \) and \( \text{Re } \gamma_3 > 0. \) As we know \( G_- \) is given as the unique solution of the problem

\[
\begin{aligned}
  \text{Ng}(x, \zeta) &= \delta(x - \zeta), \quad -\infty < x, \zeta < 0, \\
  \text{for } \zeta < x < 0 \text{ we have } g(0-) &= g'(0-) = 0, \quad \text{and } g(-\infty) = 0, \\
  g, \frac{dg}{dx} \text{ are continuous at } x = \zeta; \quad \left. \frac{d^2 g}{dx^2} \right|_{x=\zeta^+} - \left. \frac{d^2 g}{dx^2} \right|_{x=\zeta^-} &= 1.
\end{aligned}
\]  

(A.23)

Thus, since the equation \( Nv(x) = 0, \) for \( x < 0, \) has the following fundamental set of solutions \( \{e^{\gamma_1 x}, e^{\gamma_2 x}, e^{\gamma_3 x}\} \) we obtain that condition \( g(-\infty) = 0 \) implies

\[
g(x, \zeta) = r(\zeta)e^{\gamma_1 x} \rightleftharpoons s(\zeta)e^{\gamma_3 x}, \quad \text{for } -\infty < x < \zeta < 0. \quad \text{A.24}
\]

Next, the condition \( g(0-, \zeta) = g'(0-, \zeta) = 0 \) implies

\[
g(x, \zeta) = n(\zeta) \left[ (\gamma_3 - \gamma_2)e^{\gamma_2 x} - (\gamma_3 - \gamma_1)e^{\gamma_1 x} \right]
+ (\gamma_2 - \gamma_1)e^{\gamma_1 x}), \quad \text{for } \zeta < x < 0. \quad \text{(A.25)}
\]

Then, from the conditions of continuity and jump for \( g \) we obtain

\[
r(\zeta) = \frac{1}{\Delta(\lambda)} \left[ (\gamma_3 - \gamma_1)e^{-\gamma_3 x} + (\gamma_1 - \gamma_3)e^{-\gamma_1 x} \right],
\]

and

\[
s(\zeta) = \frac{1}{\Delta(\lambda)} \left[ (\gamma_1 - \gamma_2)e^{-\gamma_2 x} + (\gamma_2 - \gamma_1)e^{-\gamma_1 x} \right],
\]

(A.26)

where \( \Delta(\lambda) = (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_3). \) Therefore, \( G_- \) is given by

\[
G_-(x, s, \lambda) = \frac{1}{\Delta(\lambda)} \left[ (\gamma_1 - \gamma_3)e^{\gamma_3 x - \gamma_1 x} + (\gamma_2 - \gamma_1)e^{\gamma_1 x - \gamma_2 x} \right]
+ Y_-(x, \zeta)(\gamma_3 - \gamma_2)e^{\gamma_1 x - \gamma_3 x} - (\gamma_2 - \gamma_1)e^{\gamma_3 x - \gamma_1 x})
+ (1 - Y_-(x, \zeta))(\gamma_3 - \gamma_1)e^{\gamma_1 x - \gamma_3 x} + (\gamma_1 - \gamma_2)e^{\gamma_3 x - \gamma_1 x})
\]

(A.27)

with \( Y_-(x, \zeta) = 1 \) for \( \zeta \leq x \leq 0, \) and \( Y_-(x, \zeta) = 0 \) otherwise. Then, the solution for \( \text{A.21} \) is given via the superposition principle by the formula in (A.22).

Next, we determine the resolvent operator for the skew-self-adjoint operator \( (A_Z, D(A_Z)) \) in (2.10).

**Proposition A.9.** Let \( \lambda \in \mathbb{C} \) such that \( \text{Re } \lambda > 0, \alpha_- = \alpha_+ = \beta_- = \beta_+ = 1 \) and \( Z \in \mathbb{R}. \) Then the resolvent operator for \( A_Z, R(\lambda; A_Z) = (\lambda I - A_Z)^{-1} : L^2(\mathcal{G}) \to D(A_Z) \) has the representation for \( \omega = (p, q) \in L^2(-\infty, 0) \oplus L^1(0, +\infty) \) as

\[
R(\lambda; A_Z) \omega = (R_-(\lambda; A_Z)p, R_+(\lambda; A_Z)q) = (v_-, v_+),
\]

with \( v_{\pm} \) defined by \( \text{A.13} \) and \( \text{A.22} \), respectively. The constants \( a_0, \alpha_1, \) and \( \alpha_2 \) in \( \text{A.13} \)–\( \text{A.22} \) are uniquely determined by the condition \( (v_-, v_+) \in D(A_Z). \)
Proof. Let $\omega = (p, q) \in L^2(-\infty, 0) \oplus L^2(0, +\infty)$ and $v = (v_-, v_+) = R(\lambda; A_2)\omega$. Then we obtain that $v_-, v_+$ satisfy the system

$$
\begin{align*}
&\begin{aligned}
\lambda v_-(x) + v_-'(x) + v_''(x) = p(x), &\quad -\infty < x < 0 \\
\lambda v_+(x) + v_+'(x) + v_''(x) = q(x), &\quad 0 < x < +\infty
\end{aligned} \\
v_-(0) = v_+(0+), &\quad v_+'(0+) - v_-(0-) = Zv_-(0-) \\
v_+'(0+) - v_''(0-) = \frac{Z^2}{2}v_-(0-) + Zv_-(0-),
\end{align*}
$$

(A.28)

and therefore $v_+, v_-$ are defined by the formulas in (A.13) and (A.22), respectively. The constants $a_0, a_1,$ and $a_2$ in (A.13)–(A.22) are the unique solution for the system

$$
A \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{d}{dx} \int_0^\infty G_+(x, \zeta, \lambda)q(\zeta)d\zeta_{|x=0^+} \\ -\frac{d^2}{dx^2} \int_{-\infty}^0 G_-(x, \zeta, \lambda)p(\zeta)d\zeta_{|x=0^-} - \frac{d^2}{dx^2} \int_0^\infty G_+(x, \zeta, \lambda)q(\zeta)d\zeta_{|x=0^+} \end{pmatrix}
$$

(A.29)

with

$$
A = \begin{pmatrix}
\gamma_1 & -1 & -1 \\
\gamma_2 & -(\gamma_2 + Z) & -1 \\
\gamma_3 & \frac{Z^2}{2} + Z\gamma_2 & -\gamma_2 \gamma_3
\end{pmatrix}.
$$

(A.30)

We note that $\det(A) = (\gamma_3 - \gamma_2) \left[ \frac{Z^2}{2} + Z(\gamma_2 + \gamma_3 - \gamma_4) + (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3) \right] \neq 0$ for all $Z \in \mathbb{R}$, because of the Girard’s relations

$$
\gamma_1 + \gamma_2 + \gamma_3 = 0, \quad \gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3 = 1, \quad \gamma_1 \gamma_2 \gamma_3 = -\lambda, \quad (A.31)
$$

imply that the second-degree polynomial equation $\frac{Z^2}{2} + Z(\gamma_2 + \gamma_3 - \gamma_4) + (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3) = 0$ does not have real roots. This finishes the proof. \qed

Proposition A.10. The unitary group $\{W(t)\}_{t \in \mathbb{R}}$ associated to equation (A.11) can be written for $\omega = (p, q) \in L^2(-\infty, 0) \oplus L^2(0, +\infty)$ as $W(t)\omega \equiv (W_-(t)p, W_+(t)q)$, with

$$
\begin{align*}
W_-(t)p(x) &= \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{\lambda t}R_-(\lambda; A_2)p(x)d\lambda, \quad x \leq 0, \\
W_+(t)q(x) &= \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{\lambda t}R_+(\lambda; A_2)q(x)d\lambda, \quad x \geq 0,
\end{align*}
$$

(A.32)

with

$$
\begin{align*}
R_-(\lambda; A_2)p(x) &= \alpha_1 e^{\gamma_2 x} + \alpha_2 e^{\gamma_3 x} + \int_0^\infty G_-(x, \zeta, \lambda)p(\zeta)d\zeta, \quad x \leq 0, \\
R_+(\lambda; A_2)q(x) &= \alpha_3 e^{\gamma_1 x} + \int_0^\infty G_+(x, \zeta, \lambda)q(\zeta)d\zeta, \quad x \geq 0,
\end{align*}
$$

(A.33)
where $G_\pm (x, \zeta, \lambda)$ are the associated Green’s functions for (A.12) and (A.21), respectively, and $\alpha_1, \alpha_2$, and $\alpha_3 \in \mathbb{R}$ are uniquely determined by the condition $(R_- (\lambda; AZ)p, R_+ (\lambda; AZ)q) \in D(AZ)$. 

**Proof.** Using the Laplace transform and proposition A.9, it follows from semigroup theory that for $\omega = (p, q) \in L^2(-\infty, 0) \oplus L^2(0, +\infty)$

$$W(t)w(x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{\lambda t} R(\lambda; AZ)w(x) d\lambda$$

$$= \frac{1}{2\pi i} \left( \int_{r-i\infty}^{r+i\infty} e^{\lambda t} R_- (\lambda; AZ)p(x) d\lambda, \int_{r-i\infty}^{r+i\infty} e^{\lambda t} R_+ (\lambda; AZ)q(x) d\lambda \right). \quad (A.34)$$

This finishes the proof. □

**Remark A.11.** We note that the three roots of the equation $\lambda + \beta \gamma + \alpha \gamma^3 = 0$, for $\Re \lambda > 0$, can also be ordered as in (A.15). Indeed, this is a consequence that the Girard’s relations implies that $\Re (\gamma_1 \gamma_2 + \gamma_3) = - \frac{\Re (\lambda)}{\alpha}$ and $\Re (\gamma_1 + \gamma_2 + \gamma_3) = - \frac{\alpha}{\alpha}$. Thus, proposition A.10 is also valid for the case $\alpha_- = \alpha_+ > 0$ and $\beta_- = \beta_+ < 0$, which arise in the stability study of tails and bumps in section 5.

The proposition A.10 provides the information for building a representation of the unitary group generated by ($H_Z, D(H_Z)$) in (6.2). This result gives us the invariance of the subspace $D(H_Z) \cap \mathcal{C}$ by that group, which was used in the instability theorem 6.1. We note that in the case of two half-lines, this invariance property for the domain $D(AZ)$ in (2.10) is obvious.

**Proposition A.12.** It considers the skew-self-adjoint operator $(H_Z, D(H_Z))$ in (6.2) on metric graph $G$ with a structure $\mathcal{E} \equiv \mathcal{E}_- \cup \mathcal{E}_+$, where $|\mathcal{E}_-| = |\mathcal{E}_+| = n$, $n \geq 2$. Let $\{V(t)\}_{t \in \mathbb{R}}$ be the unitary group associated to $H_Z$. Then, for $C$ defined by

$$C = \{(u_0)_{\mathcal{E}} \in L^2(G) : u_{1-} (0-) = \ldots = u_{n-} (0-) = u_{1+} (0+) = \ldots = u_{n+} (0+)\},$$

we have that $D(H_Z) \cap C$ is invariant by the group $\{V(t)\}_{t \in \mathbb{R}}$.

**Proof.** We consider the case $\alpha_{\mathcal{E}} = (1)_{\mathcal{E}}$ and $\beta_{\mathcal{E}} = (1)_{\mathcal{E}}$. From proposition A.10, for $(u_0)_{\mathcal{E}} = (u_1, \ldots, u_n)$, $u_{n-} \equiv u_{n+} \in L^2(G)$ we define

$$W_{j-} (t)u_{j-} (x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{\lambda t} R_- (\lambda; H_Z) u_{j-} (x) d\lambda,$$

\hspace{1cm} $x \leq 0, \ 1 \leq j \leq n,$

$$W_{j+} (t)u_{j+} (x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{\lambda t} R_+ (\lambda; H_Z) u_{j+} (x) d\lambda,$$

\hspace{1cm} $x \geq 0, \ 1 \leq j \leq n,$

with $R_{\pm} (\lambda; H_Z)$-components given by (A.33). Thus, we can write

$$V(t) = \bigoplus_{j=1}^{n} W_{j-} \bigoplus_{j=1}^{n} W_{j+},$$
Now, for \( u = (u_e)_{e \in E} \in D(H_2) \) it is obvious by definition of \( W_{j,\pm} \) that \( W(t)u \in D(H_2) \) (see proposition A.9). Moreover,
\[
W_{j,\pm}(t)u_{j,\pm}(0-) = W_{j,\pm}(t)u_{j,\pm}(0+) \quad \text{for all} \quad j \in \{1, 2, \ldots, n\}.
\]
Thus, for \( u = (u_e)_{e \in E} \in D(H_2) \cap C \) it follows immediate from (A.36) that \( V(t)u \in C \).

From remark A.11 it follows that for sequences \( (\alpha_e)_{e \in E} = (\alpha_-, \alpha_+) \) and \( (\beta_e)_{e \in E} = (\beta_-, \beta_+) \), with \( \alpha_- = \alpha_+ \) and \( \beta_- = \beta_+ \), we also obtain the conclusion of the proposition. This finishes the proof. □

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