Identification and Estimation of Dynamic Games with Unknown Information Structure*

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Abstract

This paper studies the identification and estimation of dynamic games when the underlying information structure is unknown to the researcher. To tractably characterize the set of Markov perfect equilibrium predictions while maintaining weak assumptions on players’ information, we introduce Markov correlated equilibrium, a dynamic analog of Bayes correlated equilibrium. The set of Markov correlated equilibrium predictions coincides with the set of Markov perfect equilibrium predictions that can arise when the players can observe more signals than assumed by the analyst. Using Markov correlated equilibrium as the solution concept, we propose tractable computational strategies for informationally robust estimation, inference, and counterfactual analysis that deal with the non-convexities arising in dynamic environments. We use our method to analyze the dynamic entry game between Starbucks and Dunkin’ in the US and the role of informational assumptions.

Keywords: Dynamic games, Markov, correlated equilibrium, information, partial identification

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1 Introduction

This paper develops a computationally tractable econometric framework for estimating empirical dynamic discrete game models with weak assumptions on players’ information. Let

$$f : (\Theta, \mathcal{S}) \Rightarrow \mathcal{D}$$

describe the data-generating process under Markov perfect equilibrium, where $\Theta$ is the set of structural parameters, $\mathcal{S}$ is the set of information structures, and $\mathcal{D}$ is the data (e.g., conditional choice probabilities and state transition probabilities); the correspondence in (1) reflects the possibility of multiple equilibria. The standard econometric approach obtains the identified set by inverting (1) after assuming that the true information structure takes a particular form, namely that players observe their payoff shocks but not their opponents’ (Rust, 1994; Aguirregabiria and Mira, 2007; Bajari, Benkard, and Levin, 2007; Pakes, Ostrovsky, and Berry, 2007; Pesendorfer and Schmidt-Dengler, 2008; Bugni and Bunting, 2021; Dearing and Blevins, 2024).

However, assuming the true information structure is known to the researcher is far from innocuous. While such an assumption has facilitated computationally tractable estimation of dynamic games, researchers often have little evidence to defend the assumptions. Recent works have documented that misspecification of players’ information may create substantial bias in both static and dynamic settings (Fershtman and Pakes, 2012; Grieco, 2014; Pakes et al., 2015; Aguirregabiria and Magesan, 2020; Syrgkanis, Tamer, and Ziani, 2021; Gualdani and Sinha, 2021; Magnolfi and Roncoroni, 2023; Koh, 2023). The information structure misspecification problem appears especially serious in dynamic game settings because fewer informational assumptions have been entertained in econometric works; we are unaware of an econometric framework that considers an alternative information assumption other than Ericson and Pakes (1995). Unfortunately, attempts to invert (1) directly are likely futile since the set of information structures is large and infinite-dimensional.

In this paper, we develop a tractable empirical framework for analyzing dynamic discrete games while treating the underlying information structure as unknown to the analyst. We assume that the researcher only knows minimal information available to

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1See Aguirregabiria, Collard-Wexler, and Ryan (2021) for a recent survey of the literature.
the players but remains agnostic about the possibility that the players might observe more signals.\textsuperscript{2} The framework typically yields a set of partially identified parameters robust to misspecification on the information structure.

The main innovation of this paper is the development of Markov correlated equilibrium, a dynamic (Markovian) analog of Bayes correlated equilibrium (Bergemann and Morris, 2016). We show that Markov correlated equilibrium can capture the set of Markov perfect equilibrium predictions (the joint distribution of actions and states) that can arise when the players might observe more signals than assumed by the analyst. We establish the informational robustness property of Markov correlated equilibrium by formulating Markov perfect equilibrium as the Bayes Nash equilibrium of the associated reduced one-shot game via the one-shot deviation principle and invoking Theorem 1 of Bergemann and Morris (2016).

We build on the theory of Markov correlated equilibrium and propose computationally tractable approaches for estimation and inference. Roughly speaking, we prove

$$\Theta_{I}^{MCE} = \bigcup_{\tilde{S} \in \mathcal{S}} \Theta_{I}^{MPE}(\tilde{S}),$$

where $\Theta_{I}^{MCE}$ is the identified set of parameters after assuming that a Markov correlated equilibrium generated the data, and $\Theta_{I}^{MPE}(\tilde{S})$ is the Markov perfect equilibrium identified set when the information structure is assumed to be $\tilde{S}$. Directly inverting (1) requires constructing the right-hand side of (2), which is computationally infeasible when $\mathcal{S}$ is large. However, we show that constructing the left-hand side of (2) is computationally tractable. In sum, treating the data as if they were generated by a Markov correlated equilibrium enables an informationally robust econometric analysis. Identified set (2) is usually non-singleton. We propose a computationally attractive strategy for estimation and inference by formulating the estimation problem as mathematical programs with equilibrium constraints (MPEC) (Su and Judd, 2012) and applying the inference strategy of Horowitz and Lee (2022) by nesting the confidence set for the conditional choice probabilities into the MPEC problem.\textsuperscript{3}

\textsuperscript{2}Note that this assumption restricts the set of possible information structure $\mathcal{S}$ in (1). Our framework allows $\mathcal{S}$ to span the entire set of information structures, but the corresponding identified set tends to be large and uninformative. We focus on more restrictive sets in our empirical application.

\textsuperscript{3}The standard estimation strategies for dynamic games are inapplicable (e.g., the nested fixed-
We apply our framework to study the dynamic entry and exit decisions of Starbucks and Dunkin’, the two largest coffee chains in the US. We estimate a tight identified set by leveraging the large support idea of Tamer (2003). Specifically, we estimate the firms’ profit parameters by identifying markets where they operate as single players and applying the standard dynamic discrete choice estimation methods. Next, we estimate the competitive effects parameters under a Markov correlated equilibrium assumption using the data from the markets where the two firms compete. The single-agent profit parameters are point-estimated, but the competitive effects parameters are set-estimated. We also estimate the parameters under the standard incomplete information Markov perfect equilibrium assumption.

Comparison between the estimates from the standard incomplete information Markov perfect equilibrium assumption and our informationally robust Markov correlated equilibrium assumption suggests that the standard incomplete information assumption may be misspecified. The estimates from the standard approach suggest Dunkin’ likes Starbucks’ presence, while Starbucks is not affected by Dunkin’s presence. However, the Markov correlated equilibrium identified set shows that these conclusions are not robust to the misspecification of information structures. The Markov correlated equilibrium identified set does not rule out both positive and negative spillover effects.

We also conduct a counterfactual experiment in which we cut the firms’ entry costs and simulate the evolution of the number of active firms in our sample period. We consider two parameter values that capture the opposite extremes of the Markov correlated equilibrium identified set estimates. The two cases capture scenarios in which firm entry decisions are strategic substitutes and complements. We find that the counterfactual outcomes can vary widely depending on the true value of the competitive effects parameters (which we cannot determine under the Markov correlated equilibrium assumption in the parameter estimation). However, we find that for a fixed parameter value, informationally robust bounds on counterfactual outcomes are tight, indicating that an information designer cannot affect the outcomes much. These tight counterfactual outcome bounds occur because the competitive effects parameters are small in absolute value, leaving the information designer with little room point algorithm of Rust (1987) or the conditional choice probability inversion of Hotz and Miller (1993)).
to affect the firms’ decisions relative to the case where they operate as single agents who do not account for the opponents’ entry decisions strategically.

Our work advances the literature on the econometric analysis of game-theoretic models in several ways. First, we contribute to the dynamic games literature by developing a new solution concept with attractive computational properties. Markov correlated equilibrium is a natural extension of Bayes correlated equilibrium (Bergemann and Morris, 2013, 2016) to a class of dynamic games that build on Maskin and Tirole (2001). While various versions of correlated equilibrium in dynamic games have been studied (e.g., see Nowak and Raghavan (1992), Duffie et al. (1994), Solan (2001), Solan and Vieille (2002), Mailath and Samuelson (2006), Von Stengel and Forges (2008), He and Sun (2017), Forges (2020) and the works cited therein), our version is new as we extend Bayes correlated equilibrium to a class of dynamic stochastic Markov games with incomplete information. Doval and Ely (2020) and Makris and Renou (2023) also extend Bayes correlated equilibrium to dynamic environments, but our setting is different as we focus on a class of dynamic discrete games common in empirical works.

This paper also contributes to the literature on misspecification-robust econometric analysis of game-theoretic models. Our work is most closely related to the econometric frameworks that pursue robustness to misspecification on information structure (Grieco, 2014; Pakes et al., 2015; Syrgkanis, Tamer, and Ziani, 2021; Gualdani and Sinha, 2021; Magnolfi and Roncoroni, 2023; Koh, 2023; Han, Kaido, and Magnolfi, 2024). We contribute by developing a novel framework for estimating a class of dynamic games, building on the aforementioned works on the econometric analysis of dynamic Markovian games. Our work also relates to works that develop econometric approaches for estimating incomplete models with multiple equilibria (e.g., Tamer (2003), Ciliberto and Tamer (2009), Beresteanu, Molchanov, and Molinari (2011), and Galichon and Henry (2011)) since our approach does not require an assumption on equilibrium selection.

The rest of the paper is organized as follows. In Section 2, we define Markov correlated equilibrium and establish a connection between Markov correlated equilibria and Markov perfect equilibria in a class of dynamic games and discuss its properties. In Section 3, we introduce an econometric model and discuss informationally robust
identification. Section 4 proposes strategies for computing the identified set. Section 5 discusses our approach to inference. Section 6 applies our framework to the dynamic entry/exit game by Starbucks and Dunkin’ in the US. Section 7 concludes. All proofs are in Appendix A.

2 Theory of Markov Correlated Equilibrium

We consider a class of dynamic Markovian games that has been used as a standard framework for empirical works. We introduce the concept of Markov correlated equilibrium for informationally robust analysis and discuss its properties.

2.1 Setup

Let \( t = 1, 2, ..., \infty \) denote discrete time. A stationary dynamic Markov game of incomplete information is defined by a pair of basic game \( G \) and information structure \( S \). A basic game \( G = (I, (A_i, u_i)_{i \in I}, \mathcal{X}, \mathcal{E}, \psi, f, \delta) \) specifies the payoff-relevant primitives of the model: \( i \in I = \{1, 2, ..., I\} \) indexes the players; \( a_{it} \in A_i \) denotes player \( i \)'s action at time \( t \), where \( A_i \) is a finite set of actions available to player \( i \); \( a_t = (a_{1t}, ..., a_{It}) \in A = \times_{i \in I} A_i \) denotes an action profile; \( x_t \in \mathcal{X} \) denotes a state variable that is publicly observed by the players, and \( \mathcal{X} \) is assumed to be finite; \( \varepsilon_t \in \mathcal{E} \) denotes a latent state variable that is not directly observed by the players, and \( \mathcal{E} \) is assumed to be finite; \( \delta \in [0, 1) \) is the players’ discount factor.

An information structure \( S = ((T_i)_{i \in I}, \pi) \) specifies the information-relevant primitives; \( \tau_{it} \in T_i \) denotes player \( i \)'s private signal at period \( t \), where \( T_i \) represents a finite set of private signals; \( \tau_t = (\tau_{1t}, ..., \tau_{It}) \in T \equiv \times_{i \in I} T_i \) is a signal profile; \( \pi : \mathcal{X} \times \mathcal{E} \rightarrow \Delta(T) \) is a signal distribution that maps the state of the world to a signal

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4 The finite state space assumption is non-material and simplifies the notation. The assumption also facilitates the transition from theory to econometrics since the state space will be discretized for feasible estimation. There is no conceptual difficulty in extending the theory to a more general space; see Syrgkanis, Tamer, and Ziani (2021) and Magnolfi and Roncoroni (2023) for the case of Bayes correlated equilibrium.
profile. Note that players’ signals are allowed to be correlated. The interpretation is that each player $i$ does not directly observe the latent state $\varepsilon_t$ but receives a private signal $\tau_{it}$ whose informativeness about $\varepsilon_t$ depends on $\pi$.

The game $(G, S)$ is common knowledge to the players. The model proceeds as follows. At the beginning of period $t$, the common knowledge state $x_t \in \mathcal{X}$ is given and publicly observed. Conditional on $x_t$, the latent state $\varepsilon_t \in \mathcal{E}$ is drawn from the probability distribution $\psi_{\varepsilon_t|x_t}$. Next, a profile of private signals $\tau_t = (\tau_{1t}, ..., \tau_{It}) \in \mathcal{T}$ is drawn from the signal distribution $\pi_{\tau_t|x_t, \varepsilon_t}$. At this point, each player $i$ observes $(x_t, \tau_{it})$ and uses Bayes’ rule to infer $\varepsilon_t$. Then, the players simultaneously choose actions $a_{it}, i = 1, ..., I$, and each player $i$ receives period $t$ payoff $u_i(a_{it}, x_t, \varepsilon_t) \in \mathbb{R}$. Finally, the observable state $x_t$ transitions to $x_{t+1}$ via the probability kernel $f_{x_t+1|x_t, \varepsilon_t}$. Given $x_{t+1}$, period $t+1$ begins.

Note that the primitives imply that the transition probability of state variables factors as $\Pr(x_{t+1}, \varepsilon_{t+1}|a_t, x_t, \varepsilon_t) = \psi_{\varepsilon_{t+1}|x_{t+1}} f_{x_{t+1}|a_t, x_t, \varepsilon_t}$. Assuming that the latent variables are generated independently of the previous-period states is standard in the empirical literature. The assumption also plays a vital role in this paper as it simplifies the characterization of correlated equilibria in dynamic settings.

The players are rational and forward-looking. In period $t$, each player $i$ chooses action plans to maximize the expected discounted sum of intertemporal payoffs

$$E \left[ \sum_{s=t}^{\infty} \delta^{s-t} u_i(a_s, x_s, \varepsilon_s)|x_t, \tau_{it} \right].$$

The conditioning on $(x_t, \tau_{it})$ reflects the assumption that $i$ observes the public state $x_t$ and her signal $\tau_{it}$ before choosing period $t$ action. Since the environment is stationary, we suppress time subscripts unless necessary.

**Example 1** (Two-player dynamic entry game). As a running example, we consider the two-player dynamic entry/exit model of Pesendorfer and Schmidt-Dengler (2008), which has served as a representative example in the literature (Egesdal, Lai, and Su, 2015; Aguirregabiria et al., 2021; Dearing and Blevins, 2024). We also adopt a similar

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5When the latent variables are correlated over time so that information asymmetries become persistent, computing the perfect Bayesian equilibria easily becomes intractable because the players need to keep track of the entire history and apply Bayes rule for each contingent. See Fershtman and Pakes (2012) for a discussion.
model in our empirical application. There are two players, \( i = 1, 2 \). In each period, firms simultaneously decide whether to be active \((a_{it} = 1)\) or inactive \((a_{it} = 0)\). The per-period payoff is

\[
u_i(a_{it}, a_{jt}, z_{it}, \varepsilon_{it}) =
\begin{cases}
(1 - a_{jt})\pi^m + a_{jt}\pi^d + (1 - z_{it})c + \varepsilon_{it} & \text{if } a_{it} = 1 \\
z_{it}\kappa & \text{if } a_{it} = 0
\end{cases}
\]

Here, \( \pi^m, \pi^d, c, \) and \( \kappa \) represent monopoly profit, duopoly profit, entry cost, and scrap value, respectively. The public state variable is \( x_t = (z_{1t}, z_{2t}) \), where \( z_{it} \) represents whether firm \( i \) is an incumbent \((z_{it} = 1)\) or not \((z_{it} = 0)\). The firms’ previous actions determine the incumbency states as \( z_{it} = a_{i,t-1} \). In empirical applications, \( x_t \) usually contains additional observable market characteristics. The latent state variable is \( \varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}) \), where \( \varepsilon_{it} \in \mathcal{E}_i \subseteq \mathbb{R} \) only enters player \( i \)'s payoff. Each \( \varepsilon_{it} \) independently follows the standard normal distribution.

Standard empirical dynamic discrete game models assume that each player observes \( \varepsilon_{it} \) but not \( \varepsilon_{jt} \) for \( j \neq i \). Formally, \( \mathcal{T}_i = \mathcal{E}_i \), and the private signal is \( \tau_{it} = \varepsilon_{it} \) with probability one. Alternatively, if the information is complete, players publicly observe \( \varepsilon_t \), so \( \mathcal{T}_i = \mathcal{E} \), and \( \tau_{it} = \varepsilon_t \) with probability one.

### 2.2 Markov Perfect Equilibrium

A Markov strategy of player \( i \) is a mapping \( \beta_i : \mathcal{X} \times \mathcal{T}_i \rightarrow \Delta(A_i) \) that specifies a probability distribution over actions at each realization of observable state and private signal.\(^6\) A strategy profile \( \beta = (\beta_1, \ldots, \beta_I) \) is a Markov perfect equilibrium if it is a subgame perfect equilibrium.

To leverage the results in static games to a dynamic environment, it is useful to characterize the Markov perfect equilibrium conditions using the one-shot deviation principle.\(^7\) The one-shot deviation principle formulation has two advantages. First,
we can leverage results established in static environments as it translates a Markov perfect equilibrium as a Bayes-Nash equilibrium of the associated normal-form game. Second, it facilitates equilibrium computation during econometric analysis.

Suppose a strategy profile \( \beta = (\beta_1, ..., \beta_I) \) is given. Define the \textit{ex-ante value function} \( V_i^\beta(x) \in \mathbb{R} \) as the expected payoff to player \( i \) at state \( x \) prior to the realization of \( \varepsilon \) when all players follow the prescriptions in \( \beta \). For each \( i \in I \), \( V_i^\beta \in \mathbb{R}^{|X|} \) is the unique solution to

\[
V_i^\beta(x) = \sum_{\varepsilon \in \mathcal{E}, \tau \in \mathcal{T}, a \in \mathcal{A}} \psi_{\varepsilon\mid x, \tau} \beta_{a|\varepsilon} \left\{ u_i(a, x, \varepsilon) + \delta \sum_{x' \in \mathcal{X}} V_i^\beta(x') f_{x'|a,x,\varepsilon} \right\}, \quad \forall x \in \mathcal{X} \quad (4)
\]

where \( \beta_{a|\varepsilon} \equiv \prod_{i=1}^I \beta_i(a_i|\varepsilon, \tau_i) \) denotes the conditional distribution over action profiles induced by the strategy profile \( \beta \). Next, define the \textit{outcome-specific payoff function} given strategy profile \( \beta \) as

\[
v_i^\beta(a, x, \varepsilon) \equiv u_i(a, x, \varepsilon) + \delta \sum_{x' \in \mathcal{X}} V_i^\beta(x') f_{x'|a,x,\varepsilon},
\]

which represents the continuation payoff to player \( i \) when \((a, x, \varepsilon)\) is realized today, and all players follow the prescriptions in \( \beta \) from tomorrow onward.

A strategy profile \( \beta \) in game \((G, S)\) induces a reduced normal-form basic game \( G^\beta = (I, \mathcal{A}, v_i^\beta)_{i \in I}, \mathcal{X}, \mathcal{E}, \psi) \). Then \((G^\beta, S)\) describes a “static” game in which player \( i \)’s “static” payoff function is given by \( v_i^\beta : \mathcal{A} \times \mathcal{X} \times \mathcal{E} \rightarrow \mathbb{R} \). The following lemma allows us to import the results in static environments into dynamic environments.

\textbf{Lemma 1} (One-shot deviation principle formulation of Markov perfect equilibrium). 

\textit{A strategy profile \( \beta \) is a Markov perfect equilibrium of \((G, S)\) if and only if \( \beta \) is a Bayes Nash equilibrium of \((G^\beta, S)\).}
2.3 Markov Correlated Equilibrium

We introduce a dynamic analog of Bayes correlated equilibrium for dynamic discrete games. A stationary Markov decision rule in \((G, S)\) is a mapping

\[
\sigma: \mathcal{X} \times \mathcal{E} \times \mathcal{T} \rightarrow \Delta(\mathcal{A}),
\]

which specifies a probability distribution over action profiles at each realization of state and signals. It is instructive to think of \(\sigma\) as a recommendation strategy of an omniscient mediator who observes \((x, \varepsilon, \tau)\) and privately recommends action to each player. Suppose the mediator commits to a recommendation strategy \(\sigma\) and announces it to the players. After the state and players’ signal \((x, \varepsilon, \tau)\) are realized, an action profile \(a = (a_1, \ldots, a_I)\) is drawn from the probability distribution \(\sigma_{a|x,\varepsilon,\tau}\), and each \(a_i\) is privately recommended to each player \(i\). Each player \(i\), having observed \((x, \tau_i, a_i)\), decides whether to obey (play \(a_i\)) or not (deviate to \(a'_i \neq a_i\)). If the decision rule \(\sigma\) is such that the players are always obedient, we call \(\sigma\) a Markov correlated equilibrium of \((G, S)\).

We formalize the equilibrium condition as follows. Let \(V^\sigma_i \in \mathbb{R}^{|X|}\) denote the vector of ex-ante value functions induced by \(\sigma\), obtained as a unique solution to

\[
V^\sigma_i(x) = \sum_{\varepsilon \in \mathcal{E}, \tau \in \mathcal{T}, a \in \mathcal{A}} \psi_{\varepsilon|x,\tau} \pi_{\tau|x,\varepsilon} \sigma_{a|x,\varepsilon,\tau} \left\{ u_i(a, x, \varepsilon) + \delta \sum_{x' \in \mathcal{X}} V^\sigma_i(x') f_{x'|a,x,\varepsilon} \right\}, \quad \forall x \in \mathcal{X}.
\]

(5)

Define the outcome-specific value function associated with \(\sigma\) as

\[
v^\sigma_i(a, x, \varepsilon) \equiv u_i(a, x, \varepsilon) + \delta \sum_{x' \in \mathcal{X}} V^\sigma_i(x') f_{x'|a,x,\varepsilon},
\]

which represents the payoff to player \(i\) if \((a, x, \varepsilon)\) is realized today and \(\sigma\) determines the players’ actions in the future. Let \(\mathbb{E}^\sigma[v^\sigma_i(a'_i, a_{-i}, x, \varepsilon)|x, \tau_i, a_i]\) be the expected payoff to player \(i\) from choosing \(a'_i\) when \(i\) observes \((x, \tau_i)\) and receives recommendation \(a_i\).

The following definition states that \(\sigma\) is a Markov correlated equilibrium of \((G, S)\) if the players do not deviate from the mediator’s recommendations.

**Definition 1.** A decision rule \(\sigma\) is a Markov correlated equilibrium of \((G, S)\) if for
each $i \in I$, $x \in X$, $\tau_i \in T_i$, and $a_i \in A_i$, we have

$$E^\sigma[v_i^\sigma(a_i, a_{-i}, x, \varepsilon)|x, \tau_i, a_i] \geq E^\sigma[v_i^\sigma(a_i', a_{-i}, x, \varepsilon)|x, \tau_i, a_i]$$  \hspace{1cm} (6)

for each $a_i' \neq a_i$ whenever $Pr^\sigma(x, \tau_i, a_i) > 0$.

Since

$$E^\sigma[v_i(\tilde{a}_i, a_{-i}, x)|x, \tau_i, a_i] = \sum_{\varepsilon, a_{-i}} v_i^\sigma(\tilde{a}_i, a_{-i}, x, \varepsilon)Pr^\sigma(\varepsilon, a_{-i}|x, \tau_i, a_i)$$

$$= \sum_{\varepsilon, a_{-i}} v_i^\sigma(\tilde{a}_i, a_{-i}, x, \varepsilon) \left( \frac{\sum_{\tau_{-i}} \psi|_{x, \tau}|x, \varepsilon, \sigma_{a}|x, \varepsilon, \tau}{\sum_{\varepsilon, \tau_{-i}, a_{-i}} \psi|_{x, \tau}|x, \varepsilon, \sigma_{a}|x, \varepsilon, \tau} \right),$$

canceling out the denominator—which is constant across all possible realizations of $(\varepsilon, \tau_{-i}, a_{-i})$—lets us rewrite the obedience condition (6) as, for each $i$,

$$\sum_{\varepsilon, \tau_{-i}, a_{-i}} \psi|_{x, \tau}|x, \varepsilon, \sigma_{a}|x, \varepsilon, \tau v_i^\sigma(a, x, \varepsilon) \geq \sum_{\varepsilon, \tau_{-i}, a_{-i}} \psi|_{x, \tau}|x, \varepsilon, \sigma_{a}|x, \varepsilon, \tau v_i^\sigma(a_i', a_{-i}, x, \varepsilon), \quad \forall x, \tau_i, a_i, a_i'.$$  \hspace{1cm} (7)

Equation (7) serves as the operational definition of Markov correlated equilibrium in our econometric analysis.

A Markov correlated equilibrium always exists. Nowak and Raghavan (1992) establishes the existence of a stationary correlated equilibrium in a discounted stochastic game with complete information under standard regularity conditions. We will show in the following sections that the set of Markov correlated equilibria under arbitrary information structure always includes a complete information stationary correlated equilibrium. Thus, the set of Markov correlated equilibria is always non-empty.

### 2.4 Relationship to Bayes Correlated Equilibrium

As in Section 2.2, a decision rule $\sigma$ in game $(G, S)$ induces a reduced normal-form basic game $G^\sigma = \langle I, (A_i, v_i^\sigma)_{i \in I}, X, E, \psi \rangle$. Then (7) shows that if $\sigma$ is a Markov correlated equilibrium of $(G, S)$, it is a Bayes correlated equilibrium of $(G^\sigma, S)$.

**Lemma 2** (One-shot deviation principle characterization of Markov correlated equilibrium). A decision rule $\sigma$ is a Markov correlated equilibrium of $(G, S)$ if and only
if it is a Bayes correlated equilibrium of \((G^\sigma, S)\).

When players fully discount the future \((\delta = 0)\), a Markov correlated equilibrium collapses to a Bayes correlated equilibrium of the stage game: for each \(i\),

\[
\sum_{\varepsilon, \tau - i, a_{-i}} \psi_{\varepsilon|a} \pi_{\tau|x, \varepsilon} \sigma_{a|x, \varepsilon, \tau} u_i(a, x, \varepsilon) \geq \sum_{\varepsilon, \tau - i, a_{-i}} \psi_{\varepsilon|a} \pi_{\tau|x, \varepsilon} \sigma_{a'|x, \varepsilon, \tau} u_i(a'_i, a_{-i}, x, \varepsilon), \quad \forall x, \tau, a_i, a'_i.
\]

Thus, Markov correlated equilibrium is a proper dynamic analog of Bayes correlated equilibrium.

Unfortunately, contrary to its static analog, the Markov correlated equilibrium conditions are not linear with respect to the decision rule, which creates two challenges that are absent in the static case. First, Markov correlated equilibrium is more difficult to compute. While computing a Bayes correlated equilibrium amounts to solving a linear program, computing a Markov correlated equilibrium generally requires solving a non-convex program. Second, the set of Bayes correlated equilibria is convex, but the set of Markov correlated equilibria is generally non-convex. The convexity of the set of Bayes correlated equilibria implies that a mixture of multiple equilibria is observationally equivalent to a single equilibrium; Syrgkanis, Tamer, and Ziani (2021) and Magnolfi and Roncoroni (2023) use this property to stay agnostic about the true equilibrium selection rule in the data generating process. However, since Markov correlated equilibrium does not share this property, assuming a single equilibrium selection rule is not without loss.

### 2.5 Informational Robustness

We establish the informational robustness property of Markov correlated equilibrium. Let an equilibrium prediction refer to the joint distribution on actions, states, and signals that are implied by either an equilibrium strategy profile or decision rule. We prove the following claim.

**Claim 1** (Markov correlated equilibrium is informationally robust). The set of Markov correlated equilibrium predictions is equal to the set of Markov perfect equilibrium predictions that can arise when players can observe more signals that are unknown to the analyst.
In the empirical application, we consider a scenario where the analyst knows that each firm observes its firm-specific payoff shock but cannot rule out the possibility that they receive some signals about opponents’ shocks. Claim 1 says that the analyst can capture the set of all predictions that can arise in such a scenario by resorting to Markov correlated equilibrium. Thus, as in Bayes correlated equilibrium, Markov correlated equilibrium attains a form of informational robustness that allows the analyst to be agnostic about the players observing additional signals after specifying players’ minimal information. We formalize Claim 1 by following Bergemann and Morris (2016)’s framework and connecting Markov correlated equilibrium to Bayes correlated equilibrium via the one-shot deviation principle.

First, we introduce a partial order on the set of information structures using the notion of expansion as in Bergemann and Morris (2016). Let \( \omega \in \Omega = \mathcal{X} \times \mathcal{E} \) denote the state of the world.

**Definition 2 (Expansion).** Let \( S = (\mathcal{T}, \pi) \) be an information structure. Information structure \( S^* = (\mathcal{T}^*, \pi^*) \) is an expansion of \( S \), or \( S^* \succeq_E S \), if there exists \( (\tilde{\mathcal{T}}_i)_{i \in \mathcal{I}} \) and \( \lambda : \Omega \times \mathcal{T} \to \Delta(\tilde{\mathcal{T}}) \) such that \( \mathcal{T}^*_i = \mathcal{T}_i \times \tilde{\mathcal{T}}_i \) for all \( i \in \mathcal{I} \) and \( \pi^* \tau, \tilde{\tau} | \omega = \pi \tau | \omega \lambda_{\tilde{\tau}} | \omega, \tau \).

Intuitively, if \( S^* \succeq_E S \), the players receive extra signals in \( S^* \) than they do in \( S \). Specifically, complete information (players observe the latent state \( \varepsilon \) with probability one) is an expansion of arbitrary information structure.

Next, we claim that if a strategy profile induces the same joint distribution over states and actions as a decision rule, the two induce the same reduced normal-form games. Let \( S^* \) be an expansion of \( S \). A strategy profile \( \beta \) in \( (G, S^*) \) induces a decision rule \( \sigma \) for \( (G, S) \) if

\[
\sigma_a|x,\varepsilon,\tau = \sum_{\tilde{\tau} \in \tilde{T}} \lambda_{\tilde{\tau}|x,\varepsilon,\tau} \prod_{i=1}^{I} \beta_i(a_i|x, \tau_i, \tilde{\tau}_i), \quad \forall a, x, \varepsilon, \tau.
\]

The strategy profile \( \beta \) in a game with more signals induces the same joint distribution over actions, states, and signals as the decision rule \( \sigma \) in a game with fewer signals.

**Lemma 3** (Identical reduced normal-form basic games). If \( \beta \) induces \( \sigma \), then the associated reduced normal-form basic games, \( G^\beta \) and \( G^\sigma \), are identical.
Lemma 3 is intuitive: if \( \beta \) induces \( \sigma \), then \( \beta \) and \( \sigma \) induce identical distributions over action profiles at each state, so the associated ex-ante value functions must be identical to each other, i.e., \( V_\beta^i = V_\sigma^i \). Then, the outcome-specific payoff functions are also identical to each other, i.e., \( v_\beta^i = v_\sigma^i \), making all primitives of the basic games identical to each other.

Finally, a solution concept generates a prediction defined as a probability distribution over actions at each state and signal. Let \( \mathcal{P}^{MPE}_{a|x,\varepsilon,\tau}(G, S) \) represent the set of predictions that can be induced by a Markov perfect equilibrium in \((G, S)\).\(^8\) Let \( \mathcal{P}^{MCE}_{a|x,\varepsilon,\tau}(G, S) \) be defined similarly for a Markov correlated equilibrium.

We are ready to state our main theorem. Suppose the analyst knows that the players observe signals specified by \( S \) but does not know whether the players have access to extra signals. What is the set of feasible Markov perfect equilibrium predictions? The following theorem says that the set of Markov correlated equilibria of \((G, S)\) captures all Markov perfect equilibrium predictions when the players can observe more signals than in \( S \).

**Theorem 1** (Markov correlated equilibrium is informationally robust). For any basic game \( G \) and information structure \( S \),

\[
\mathcal{P}^{MCE}_{a|x,\varepsilon,\tau}(G, S) = \bigcup_{S^* \models E\, S} \mathcal{P}^{MPE}_{a|x,\varepsilon,\tau}(G, S^*).
\]

Theorem 1 is attractive because \( \mathcal{P}^{MCE}_{a|x,\varepsilon,\tau}(G, S) \) is far easier to characterize and compute than \( \bigcup_{S^* \models E\, S} \mathcal{P}^{MPE}_{a|x,\varepsilon,\tau} \); the former does not require searching over the set of information structures. Note that when \( \delta = 0 \), Theorem 1 reduces to

\[
\mathcal{P}^{BCE}_{a|x,\varepsilon,\tau}(G, S) = \bigcup_{S^* \models E\, S} \mathcal{P}^{BNE}_{a|x,\varepsilon,\tau}(G, S^*),
\]

which is analogous to Bergemann and Morris (2016) Theorem 1.

In empirical works, it is common to assume that the analyst observes the conditional choice probabilities that represent the probability of each action profile at each state that is observable to the econometrician. Thus, it is useful to character-

\(^8\)We slightly abuse the notation and use \( \mathcal{P}^{MPE}_{a|x,\varepsilon,\tau}(G, S) \) to denote a set of conditional distributions over \( \mathcal{A} \) defined at each \((x, \varepsilon, \tau)\).
ize the implications of Theorem 1 in terms of conditional choice probabilities. Let \( \mathcal{P}_{a|x}^{MPE} (G, S) \) denote the set of feasible conditional choice probabilities that can arise under a Markov perfect equilibrium of \((G, S)\). Let \( \mathcal{P}_{a|x}^{MCE} (G, S) \) be defined similarly.

**Corollary 1.** For any basic game \( G \) and information structure \( S \),

\[
\mathcal{P}_{a|x}^{MCE} (G, S) = \bigcup_{S^* \subseteq E S} \mathcal{P}_{a|x}^{MPE} (G, S^*).
\]

As in the case of Bayes correlated equilibrium, if players observe more signals, the set of Markov correlated equilibria becomes smaller. In other words, more information implies a smaller set of Markov correlated equilibria.

**Theorem 2.** Let \( G \) be an arbitrary basic game. If \( S^1 \succsim_E S^2 \), then \( \mathcal{P}_{a|x, \varepsilon, \tau}^{MCE} (G, S^1) \subseteq \mathcal{P}_{a|x, \varepsilon, \tau}^{MCE} (G, S^2) \).

### 2.6 Numerical Example

We continue Example 1 to illustrate the Markov correlated equilibrium identified set. The true parameters in (3) are set to \((\pi^m, \pi^d, c, \kappa) = (1.2, -1.2, -0.2, 0.1)\), and we use the discount factor \( \delta = 0.9 \). We generate data by finding a Markov perfect equilibrium under the standard assumption that \( \varepsilon_i \) is private information to player \( i \). Pesendorfer and Schmidt-Dengler (2008) finds five equilibria, two of which are symmetric to other equilibria. We report the conditional choice probabilities of equilibria (i), (ii), and (iii) of Pesendorfer and Schmidt-Dengler (2008) in Table 1.\(^9\)

To characterize the bounds on the game outcomes that can arise in a Markov correlated equilibrium, we compute the Markov correlated equilibria that maximize and minimize the expected number of active firms.\(^{10}\) Using the associated conditional choice probabilities, we simulate the number of active firms over \( T = 6 \) periods, assuming that the initial state has no incumbent, i.e., \( x_1 = (0, 0) \).

\(^9\)We use Dearing and Blevins (2024)’s MATLAB code to generate the equilibrium conditional choice probabilities.

\(^{10}\)Specifically, let \( \phi^\sigma (a_1, a_2|x) \) be the conditional choice probability implied by decision rule \( \sigma \). The expected number of active firms at state \( x \) is \( 0 \cdot \phi^\sigma (0, 0|x) + 1 \cdot (\phi^\sigma (0, 1|x) + \phi^\sigma (1, 0|x)) + 2 \cdot \phi^\sigma (1, 1|x) \). We find the Markov correlated equilibria that maximize and minimize the unweighted average over \( x \).
Table 1: Probability of entry in three Markov perfect equilibria

|       | (i) P1 | (i) P2 | (ii) P1 | (ii) P2 | (iii) P1 | (iii) P2 |
|-------|--------|--------|---------|---------|---------|---------|
| Out/Out | 0.733  | 0.276  | 0.615   | 0.528   | 0.576   | 0.576   |
| Out/In  | 0.613  | 0.420  | 0.312   | 0.840   | 0.305   | 0.842   |
| In/Out  | 0.800  | 0.223  | 0.831   | 0.303   | 0.842   | 0.305   |
| In/In   | 0.752  | 0.294  | 0.606   | 0.578   | 0.595   | 0.595   |

Note: Each value represents the entry probability of each player at each state. P1 and P2 represent player 1 and player 2.

Figure 1: Average number of active firms over time

Figure 1 plots the simulated average number of active firms for $T = 6$ periods. Note that the outcomes of Pesendorfer and Schmidt-Dengler (2008)’s three Markov perfect equilibria are bounded by those of the Markov correlated equilibria. The Markov correlated equilibrium bounds characterize the sharp bounds on the Markov perfect equilibrium outcomes that can arise when players can receive information about the opponent’s payoff shock. \(^{11}\)

3 Informationally Robust Identification

In this section, we specialize the model for econometric analysis and discuss informationally robust identification in dynamic discrete games using Markov correlated

\(^{11}\)An alternative interpretation is that the Markov correlated equilibrium bounds characterize the limits to which an omniscient information designer who observes players’ information can manipulate the outcome.
equilibrium.

3.1 Setup

Econometric Assumptions

To facilitate econometric analyses, we impose the following assumptions on the basic game primitives. As commonly assumed in empirical works, we let \( \varepsilon \equiv (\varepsilon_1, \varepsilon_2, ..., \varepsilon_I) \) where \( \varepsilon_i \in \mathcal{E}_i \) only enters player \( i \)'s payoff and \( f_{x'|a,x,\varepsilon} = f_{x'|a,x} \), i.e., the transition probability of public state \( x \) is independent of latent state \( \varepsilon \). We assume that the payoff functions and the prior distribution are parameterized by a finite-dimensional vector \( \theta \) so that \( u_i = u_i^\theta \) and \( \psi = \psi^\theta \).

The econometrician observes data

\[
\{a_{m,t}, x_{m,t} : m = 1, 2, ..., M, t = 1, 2, ..., T\},
\]

which have players’ actions \( a_{m,t} \) and common knowledge state variables \( x_{m,t} \) across \( M \) independent markets over \( T \) periods. We assume that \( M \) is large and that the conditional choice probabilities \( \phi : \mathcal{X} \to \Delta(\mathcal{A}) \) and the transitional probability function \( f : \mathcal{A} \times \mathcal{X} \to \Delta(\mathcal{X}) \) can be non-parametrically identified from the data. Finally, we assume that the common discount factor \( \delta \in [0, 1) \) is known to the econometrician. In sum, we treat the basic game \( G^\theta = (\mathcal{I}, (A_i, u_i^\theta)_{i \in \mathcal{I}}, \mathcal{X}, \mathcal{E}, \psi^\theta, f, \delta) \) as known to the econometrician up to \( \theta \).

We summarize the baseline assumptions for econometric analysis as follows.

**Assumption 1** (Baseline assumptions for econometric analysis).

1. The set of covariates \( \mathcal{X} \) and the set of latent states \( \mathcal{E} \) are finite.

2. The prior distribution \( \psi^\theta \) and the payoff functions \( u_i^\theta \) are known up to a finite-dimensional parameter \( \theta \).

3. The latent state is a vector of player-specific payoff shocks, i.e., \( \varepsilon = (\varepsilon_1, ..., \varepsilon_I) \) and \( u_i^\theta(a, x, \varepsilon) = u_i^\theta(a, x, \varepsilon_i) \), and does not affect the transition probability of public states, i.e., \( f_{x'|a,x,\varepsilon} = f_{x'|a,x} \).
4. The conditional choice probabilities \( \phi : \mathcal{X} \rightarrow \Delta (\mathcal{A}) \) and the transition probability function \( f : \mathcal{A} \times \mathcal{X} \rightarrow \Delta (\mathcal{A}) \) are identified from the data.

The only "unconventional" assumption is that \( \mathcal{E} \) is finite. Although unnecessary for the identification arguments, the discrete support assumption is necessary for feasible estimation and makes the connection to the previous section straightforward. Other econometric papers that use Bayes correlated equilibrium also discretize the state space for feasible estimation (Gualdani and Sinha, 2021; Syrgkanis, Tamer, and Ziani, 2021; Magnolfi and Roncoroni, 2023). The player-specific payoff shock assumption, which is a conventional assumption, makes the role of the latent state clear, but it is also not required for our identification arguments.

**Identified Set**

Given a game \( (G^\theta, S) \), a solution concept determines a set of feasible predictions, which in turn induces a set of feasible conditional choice probabilities. A standard econometric approach is to assume that the information structure \( S \) is known and identify the model parameters by inverting the mapping from the model parameters to the observed conditional choice probability vector \( \phi \).

**Definition 3** (Identified set). Given Assumption 1, a solution concept \( SC \), and an information structure \( S \), the identified set of parameters is defined as:

\[
\Theta^{SC}_{I}(S) \equiv \{ \theta \in \Theta : \phi \in \mathcal{P}^{SC}_{\alpha|x}(G^\theta, S) \}.
\]

In words, a candidate parameter \( \theta \) enters the identified set if the observed conditional choice probabilities \( \phi \) can be rationalized by some equilibrium (defined by the solution concept) of the model.

**3.2 Informationally Robust Identified Set**

We consider a scenario where a Markov perfect equilibrium generates the data, but the true information structure is unknown to the analyst; misspecifying the information structure leads to biased parameter estimates. We assume that the analyst only knows that the players minimally observe signals in \( S \) but may observe more. For
instance, as assumed in Magnolfi and Roncoroni (2023), the analyst might know that each player $i$ observes at least $\varepsilon_i$ but does not know whether they can receive extra information about opponents’ payoff shocks.

**Assumption 2** (Identification under Markov perfect equilibrium). *The data are generated by a Markov perfect equilibrium of $(G^{\theta_0}, S^0)$, where $S^0$ is an expansion of $S$.*

It is well-understood that working directly with Assumption 2 is computationally infeasible; when the analyst knows $S$ but does not know the true information structure $S^0$, the analyst has to consider the set of all information structures that are expansions of $S$, but the set is too large. We can make the identification problem computationally feasible by replacing Assumption 2 with the following.

**Assumption 3** (Identification under Markov correlated equilibrium). *The data are generated by a Markov correlated equilibrium of $(G^{\theta_0}, S)$.*

**Theorem 3** (Equivalence of identified sets). *The identified set under Assumptions 1 and 2 is equal to the identified set under Assumptions 1 and 3.*

Theorem 3 says that an econometrician who does not know the true information structure underlying the data-generating process can proceed by treating the data as if they were generated by a Markov correlated equilibrium. Gualdani and Sinha (2021), Syrgkanis, Tamer, and Ziani (2021), Magnolfi and Roncoroni (2023), and Koh (2023) use similar results in static settings. Theorem 3 extends the results to a class of dynamic games. Note that a Markov correlated equilibrium identified set always includes the parameters that can be obtained under the complete information Markov perfect equilibrium assumption with mixed strategies (Ericson and Pakes, 1995; Doraszelski and Satterthwaite, 2010).

### 3.3 Properties of Informationally Robust Identified Set

#### Identified Set Shrinks with Stronger Assumptions on Players’ Information

The Markov correlated equilibrium identified set depends on the baseline information structure set by the analyst, which specifies what players minimally observe. Theorem 2 has shown that a stronger assumption on players’ information leads to a tighter set of predictions. The following translates this intuition to identified sets.
Theorem 4. If $S_1 \succeq_E S_2$, then $\Theta^{MCE}_I(S_1) \subseteq \Theta^{MCE}_I(S_2)$.

Theorem 4 implies a trade-off between informational robustness and the tightness of the identified set. Using weaker assumptions on information is more robust but increases the size of the identified set. The econometrician obtains the tightest identified set when players’ baseline information is complete; the econometrician obtains the loosest identified set when players minimally observe nothing about $\varepsilon$. Magnolfi and Roncoroni (2023) and Koh (2023) have shown that fully robust identified sets (i.e., ones with no assumption on the players’ information) can be quite large and empirically uninformative.

**Markov Correlated Equilibrium is Not Robust to Multiple Equilibria Issue**

In the static case, assuming that a single Bayes correlated equilibrium generates the data is innocuous because any mixture of Bayes correlated equilibria is a Bayes correlated equilibrium. However, this property does not carry over to the dynamic case because a mixture of Markov correlated equilibria need not be a Markov correlated equilibrium. This occurs because the Bayes correlated equilibrium obedience condition is linear in the decision rule while the Markov correlated equilibrium obedience condition is nonlinear (non-convex) in the decision rule. Thus, although we remain agnostic about the equilibrium selection rule, we assume that the data are generated by a single equilibrium. Identification with data generated by multiple equilibria is possible (e.g., by using Beresteanu, Molchanov, and Molinari (2011)) but far more computationally challenging.

**Excluded Variables with Large Support Ensure Point-Identification**

While the tightness of the identified set is largely an empirical question, the identified set can shrink substantially when excluded variables have large support. Magnolfi and Roncoroni (2023) prove that players’ payoff parameters are point-identified under Bayes correlated equilibrium when there are player-specific excluded variables with large support.\footnote{Parameter governing the correlation of players’ payoff types may not be point-identified.} Consider, for example, a two-player entry game. Driving a variable that enters only player $j$’s payoff to either positive or negative infinity ensures that
the player always stays in or stays out, rendering the other player’s decision problem a single-agent binary choice problem. The econometrician can use the single-agent data to identify the payoff parameters, including the competition effect parameters (Tamer, 2003). The same intuition holds in the dynamic case: in markets where one player always stays in or out, the other player solves a single-agent dynamic discrete choice problem while taking the other player’s action as fixed. The players’ payoff parameters can be estimated using the single-agent dynamic discrete choice data.

In our empirical application, we leverage the intuition from the large support strategy. We can identify many local markets where one of two firms is highly unlikely to enter so that we can exclude the firm as a potential entrant. Then, the other firm is solving a single-agent dynamic discrete choice problem, which is not subject to the identification problem associated with the econometrician not knowing the true information structure underlying the strategic interactions.\textsuperscript{13} Thus, we can apply the standard single-agent dynamic discrete choice estimation methods to identify the parameters that govern the single-agent dynamic decision problem.\textsuperscript{14} After estimating the “single-agent” parameters, we can use the Markov correlated equilibrium to identify the competitive effects parameters. This estimation strategy that mimics the large support strategy helps us shrink the identified set dramatically.

### 3.4 Numerical Example

Following Pesendorfer and Schmidt-Dengler (2008), we assume that the econometrician knows the values of the scrap value $\kappa = 0.1$ and the discount factor $\delta = 0.9$. However, instead of assuming that the researcher knows the information structure exactly, we assume that the researcher only knows that $i$ minimally observes $\varepsilon_i$ and leaves open the possibility that $i$ receives extra signals about $\varepsilon_{-i}$.

Table 2 reports the projection intervals of the Markov correlated equilibrium identified sets obtained under each equilibrium. In all cases, the Markov correlated equilibrium identified set contains the true parameter vector, as expected. However, the

\textsuperscript{13}Specifically, we assume that each firm observes its $\varepsilon_i$ but allow for the firms to observe arbitrary signals about $\varepsilon_j$. However, if firm $i$ knows $j$ always stays out, then $\varepsilon_j$ is payoff-irrelevant to firm $i$ under our econometric assumption.\textsuperscript{14}We cannot apply this idea to identify the competitive effects parameters since we cannot identify markets where one of two firms is always present.
size of the identified set can vary depending on which equilibrium generated data. We find that the Markov correlated equilibrium identified set obtained from equilibrium (i)—which is the only nested pseudo-likelihood-stable equilibrium and generates highly asymmetric choice probabilities across players—is the tightest. The identified sets under equilibria (ii) and (iii) are indistinguishable because the equilibrium conditional choice probabilities are similar to each other.

Table 2: Markov correlated equilibrium identified sets

|       | True  | (i)               | (ii)               | (iii)               |
|-------|-------|-------------------|-------------------|-------------------|
| \(\pi^m\) | 1.2   | [0.93, 1.83]      | [0.88, 2.52]      | [0.88, 2.52]      |
| \(\pi^d\) | -1.2  | [-1.87, -1.01]    | [-3.16, -0.65]    | [-3.16, -0.65]    |
| \(c\)   | -0.2  | [-0.51, 0.37]     | [-0.67, 1.02]     | [-0.67, 1.02]     |

Note: The table reports the projections of the Markov correlated equilibrium identified set assuming that the data were generated from Pesendorfer and Schmidt-Dengler (2008)’s Markov perfect equilibrium (i), (ii), or (iii).

Figure 2: Projections of the identified set

Figure 2 visualizes the results by plotting the convex hull of the sharp identified set corresponding to equilibrium (ii). In each subfigure, the shaded area represents the convex hull of the projected identified sets, and the star represents the location of the true parameters. As expected, the identified set is non-singleton and contains the true parameter.

The nested pseudo-likelihood-stability has been the subject of investigation in the dynamic games estimation literature (see Dearing and Blevins (2024) for a review). Equilibrium (i) is nested pseudo-likelihood-stable while equilibria (ii) and (iii) are not. Our Markov correlated equilibrium estimator is not sensitive to the nested pseudo-likelihood-stability of the underlying equilibrium.
Our numerical example shows that a strong assumption on information is far from innocuous: removing arbitrary assumptions on who observes what may result in a significant loss of identifying power. Robustness to misspecification on players’ information structure expands the identified set, but the degree to which the identified set expands can vary depending on which equilibrium is selected in the data generating process. Our framework provides a computationally feasible approach to calculating the sharp identified set associated with a weak assumption on players’ information.

4 Estimation

We propose computationally feasible estimation strategies by formulating a mathematical program with equilibrium constraints à la Su and Judd (2012). We assume the econometrician knows the true conditional choice probability vector \( \phi \) (we relax this assumption in the next section). We also assume that the baseline information structure \( S \) is fixed and denote the sharp identified set as \( \Theta_I \equiv \Theta_{I}^{MCE}(S) \).

4.1 Mathematical Program

For a candidate parameter \( \theta \), we have \( \theta \in \Theta_I \) if and only if there exists a decision rule \( \sigma \) that satisfies the Markov correlated equilibrium obedience conditions and induces the observed conditional choice probabilities. We characterize the Markov correlated equilibrium as a set of constraints that the econometrician can plug into optimization software.

**Theorem 5** (Sharp Identified Set). Suppose Assumption 1 holds. Then \( \theta \in \Theta_I \) if and only if there exists a pair \((\sigma, V)\) that satisfies

\[
\begin{align*}
\sigma_{a|x,\varepsilon,\tau} &\geq 0 \text{ for each } a \text{ and } \sum_a \sigma_{a|x,\varepsilon,\tau} = 1, & \forall x, \varepsilon, \tau \quad (8) \\
\phi_{a|x} &= \sum_{\varepsilon,\tau} \psi_{\varepsilon|x}\pi_{\tau|x,\varepsilon}\sigma_{a|x,\varepsilon,\tau}, & \forall a, x \quad (9) \\
\sum_{\varepsilon,\tau, a \neq a_i} \psi_{\varepsilon|x}\pi_{\tau|x,\varepsilon}\sigma_{a_i|x,\varepsilon,\tau} &\partial u_i^\theta(a_i', a, x, \varepsilon_i) \leq 0, & \forall i, x, \tau_i, a_i, a_i' \quad (10) \\
V_{i,x} &= \sum_{\varepsilon,\tau,a} \psi_{\varepsilon|x}\pi_{\tau|x,\varepsilon}\sigma_{a|x,\varepsilon,\tau} u_i^\theta(a, x, \varepsilon_i) + \delta \sum_{a,a'} \phi_{a|x} V_{i,x'} f_{x'|a,x}, & \forall i, x \quad (11)
\end{align*}
\]
where $v^\theta_i(a, x, \varepsilon_i) \equiv u^\theta_i(a, x, \varepsilon_i) + \delta \sum_{x'} V_{i,x'} f_{x'|a,x} \varepsilon_i$ is the outcome-specific value function, and $\partial v^\theta_i(a_i', a, x, \varepsilon_i) \equiv v^\theta_i(a_i', a_{-i}, x, \varepsilon_i) - v^\theta_i(a_i, a_{-i}, x, \varepsilon_i)$ represents the deviation payoff from $a_i$ to $a_i'$.

Constraints (8) mean that $\sigma$ is a proper conditional probability distribution, as required by the definition of a decision rule; (9) requires that $\sigma$ induces the observed conditional choice probabilities $\phi$; (10) is the Markov correlated equilibrium obedience condition; (11) is the ex-ante value function equation described in (5) but one that makes the equation linear in $(\sigma, V)$ by imposing (9). Note that $\psi$, $\pi$, $\phi$, $\delta$, and $f$ are treated as known objects in the mathematical program.

Theorem 5, which casts the identification problem as a nonlinear feasibility program, provides a basis for estimating $\Theta_I$. First, one may approximate $\Theta_I$ by checking feasibility for a large number of candidate points that approximate the parameter space. This approach is straightforward but may be computationally intensive. Second, one may find a projection interval of $\Theta_I$ by minimizing $p^\top \theta$ subject to the constraints above, where $p \in \mathbb{R}^{\dim(\theta)}$ is a direction vector (see, e.g., Kaido, Molinari, and Stoye (2019)). The projection approach alleviates the need for a grid search over the entire parameter space but can be numerically difficult.

### 4.2 Fully Robust Identified Set

In the special case where the analyst sets the baseline information structure to players having no minimal signals, $S^{null}$, the corresponding identified set—"fully robust identified set"—allows for the underlying information structure to be completely arbitrary. It turns out that determining whether a candidate parameter enters the fully robust identified set is computationally easier because the associated program is linear. Since $\Theta_I^{MCE}(S) \subseteq \Theta_I^{MCE}(S^{null})$ for any information structure $S$, the analyst can take advantage of the computational tractability to restrict the search area.

---

16 Note (11) also allows us to express each $V_i$ as a closed-form function of $\sigma$. This is similar to expressing the ex-ante value function as a closed-form function of conditional choice probabilities in Aguirregabiria and Mira (2007).

17 In practice, it is useful to transform the feasibility program—which only provides a binary conclusion—into one that measures the degree of constraint violations, e.g., a criterion function approach of Chernozhukov, Hong, and Tamer (2007).

18 For example, $p = (1, 0, ..., 0)$ and $(-1, 0, ..., 0)$ give the endpoints of $\Theta_I$'s projection interval in the first component of $\theta$. 
Theorem 6 (Fully Robust Identified Set). Suppose 1 holds. \( \theta \in \Theta_1^{MCE}(S^{null}) \) if and only if there exists \((\sigma, V)\) such that

\[
\sigma_{a|x,\varepsilon} \geq 0 \text{ for each } a \text{ and } \sum_a \sigma_{a|x,\varepsilon} = 1, \quad \forall x, \varepsilon
\]

\[
\phi_{a|x} = \sum_{\varepsilon} \psi_{\varepsilon|x} \sigma_{a|x,\varepsilon}, \quad \forall a, x \tag{12}
\]

\[
\sum_{\varepsilon, a-i} \psi_{\varepsilon|x} \sigma_{a|x,\varepsilon} \partial u_1^\theta(a_i', a, x, \varepsilon_i) + \delta \sum_a \phi_{a|x} V_{i,x'} \partial f_{x'|a_i',a,x} \leq 0, \quad \forall i, x, a_i, a_i' \tag{13}
\]

\[
V_{i,x} = \sum_{\varepsilon, a} \psi_{\varepsilon|x} \sigma_{a|x,\varepsilon} u_1^\theta(a, x, \varepsilon_i) + \delta \sum_a \phi_{a|x} V_{i,x'} f_{x'|a,x}, \quad \forall i, x \tag{14}
\]

where \( \partial u_1^\theta(a_i', a, x, \varepsilon_i) \equiv u_1^\theta(a_i', a, x, \varepsilon_i) - u_1^\theta(a, x, \varepsilon_i) \) represents the deviation flow payoff from \( a_i \) to \( a_i' \), and \( \partial f_{x'|a_i',a,x} \equiv f_{x'|a_i',a,x} - f_{x'|a,x} \) is the change of the transition probability from \( a_i \) to \( a_i' \).

The expressions above reflect the assumption that the players receive no signals about the latent state \( \varepsilon \). They only rely on their common prior to update their beliefs after observing private recommendations by the information designer. Since the constraints in Theorem 6 are linear in \((\sigma, V)\), checking whether a candidate \( \theta \) enters the fully robust identified set amounts to solving a linear program.

5 Inference

The inference problem in this paper is characterized by a moment inequality model with a large number of moment inequalities and a high-dimensional nuisance parameter, making it difficult to apply many inference methods for partially identified models.\(^{19}\) Inference methods used by Gualdani and Sinha (2021), Magnolfi and Roncoroni (2023), Syrgkkanis, Tamer, and Ziani (2021) (e.g., Chernozhukov, Hong, and Tamer (2007)’s subsampling approach) are computationally challenging to implement in our setting because the Markov correlated equilibrium constraints are non-convex.

We propose a computationally attractive strategy for inference based on the key

\(^{19}\)See Canay and Shaikh (2017) and Canay, Illanes, and Velez (2023) for recent surveys of the inference methods for partially identified models.
insights from Horowitz and Lee (2022). Our approach is intended to minimize the need for repeatedly solving non-convex optimization problems. For simplicity, let us assume that the transition probability function is known to the researcher. However, an extension to the case where \( f \) needs to be estimated is straightforward. Our approach is also applicable to Bayes correlated equilibrium.

In our setting, the sampling errors are only associated with the reduced-form parameters, namely the conditional choice probabilities (CCPs) \( \phi \). If \( \phi \) is known, the identified set can be estimated without statistical uncertainty. Thus, we propose controlling for the sampling uncertainty using a confidence set for the CCPs.\(^{21}\)

### 5.1 Confidence Set

Suppose \( \Phi_\alpha \) is a confidence set for the CCPs \( \phi \) with the property that \( \phi \in \Phi_\alpha \) with probability at least \( 1 - \alpha \) as the sample size \( n \) goes to infinity. Leading examples of \( \Phi_\alpha \) are box constraints or ellipsoids. Let \( \Theta_I(\phi) \) be the identified set when the CCPs are \( \phi \). Define the confidence set as

\[
\hat{\Theta}_I^\alpha = \bigcup_{\phi \in \Phi_\alpha} \Theta_I(\phi).
\]

If \( \Phi_\alpha \) covers the true \( \phi \) with high probability, then \( \hat{\Theta}_I^\alpha \) covers the true \( \Theta_I \) with high probability. Furthermore, checking whether \( \theta \in \hat{\Theta}_I^\alpha \) amounts to solving a nonlinear program in Theorem 5, but with \( \phi \) as part of the optimization variables subject to the constraint \( \phi \in \Phi_\alpha \).

**Theorem 7.** Let \( \Phi_\alpha \) be a confidence region for \( \phi \) such that

\[
\lim_{n \to \infty} \inf \Pr (\phi \in \Phi_\alpha) \geq 1 - \alpha.
\]

\(^{20}\)Koh (2023) uses a similar approach in static discrete games settings.

\(^{21}\)Horowitz and Lee (2022) proposes methods for carrying out (non-asymptotic) inference when the partially identified parameters are characterized by constraints that involve unknown population means that can be estimated from data. Our approach follows their insights in treating the CCPs as unknown parameters that determine the partially identified set. A similar insight has been employed by Kline and Tamer (2016), who propose a Bayesian approach.
Then

\[
\liminf_{n \to \infty} \Pr \left( \Theta \subseteq \hat{\Theta}_I^\alpha \right) \geq 1 - \alpha.
\]

Moreover, \( \theta \in \hat{\Theta}_I^\alpha \) if and only if there exists \((\sigma, V, \phi)\) subject to \(\phi \in \Phi_\alpha\), (8), (10), and (11).

## 5.2 Implementation

We briefly discuss possible approaches for constructing \(\Phi_\alpha\). While there are many possible approaches available, it is helpful to construct \(\Phi_\alpha\) as linear constraints because they are computationally easy to handle. For example, one can construct simultaneous confidence intervals of the form

\[
\Phi \equiv \left\{ \phi : \hat{L}_{a|x} \leq \phi_{a|x} \leq \hat{U}_{a|x}, \quad \forall a, x \right\}
\]

where the vectors \(\hat{L}, \hat{U} \in \mathbb{R}^{|A| \times |X|}\) are determined by data. Examples include simultaneous confidence intervals for multinomial probabilities proposed by Fitzpatrick and Scott (1987) or sup-\(t\) band by Montiel Olea and Plagborg-Møller (2019); the former is easier to compute since the endpoints of the intervals can be computed in closed-form, but the latter can be tighter.

An alternative is to take a parametric approach. Suppose \(\phi = \phi(\gamma)\), where \(\gamma\) is the parameter vector that governs the CCPs. One may estimate \(\gamma\) and construct simultaneous confidence intervals of the form

\[
\Phi \equiv \left\{ \gamma : \hat{L}_k \leq \gamma_k \leq \hat{U}_k, \quad \forall k \right\}
\]

where \(\hat{L}, \hat{U} \in \mathbb{R}^{\dim(\gamma)}\) can be computed using the asymptotic distribution of \(\hat{\gamma}\). One may also consider replacing \(\phi\) with its linear approximation \(\phi(\gamma) \approx \phi(\hat{\gamma}) + \nabla \phi(\hat{\gamma})^T (\gamma - \hat{\gamma})\).

---

\textsuperscript{22}When the number of observations is small relative to the dimension of observable covariates, estimating CCPs nonparametrically can be challenging. For example, there might be a covariate bin with no data points. Employing a parametric approach can alleviate this challenge. In particular, a flexibly-specified multinomial logit model can be useful. While such an estimator is generally inconsistent because of model misspecification, it has a lower variance than nonparametric frequency estimates and, thus, may perform better in small samples. Aguirregabiria and Mira (2007) test a similar approach for implementing the two-step pseudo maximum likelihood estimation and report that it works quite well.
γ) to further simplify the computation. In the identified set search, we also add the constraints defining ϕ as proper conditional distributions, i.e., \( \phi_{a|x} \geq 0, \forall a, x \), and \( \sum_a \phi_{a|x} = 1, \forall x \).

6 Empirical Application

We study a dynamic entry game by Starbucks and Dunkin’ in the US and illustrate the usefulness of our framework.

6.1 Background

The US coffee chain industry has grown significantly in recent decades due to the growing demand for specialty coffee and the rise of a culture emphasizing quality, convenience, and socialization. The coffee chain industry is highly competitive, with companies competing for market share through pricing, product differentiation, marketing strategies, and in-store service. In particular, Starbucks and Dunkin’ have grown to be the two largest players in the industry, significantly outpacing other competitors by a wide margin. In 2019, Starbucks and Dunkin’ accounted for 65% and 28% (resp. 55% and 35%) of total sales (resp. outlets) from the top 8 coffee chains in the US (Technomic, 2019).\(^{23}\) Figure 3 shows that the number of Starbucks and Dunkin’ stores in the US increased dramatically from 1997 to 2023. Figure 4 illustrates how the firms dramatically expanded their geographic footprint over time.

![Figure 3: Number of Starbucks and Dunkin’ stores in the US over time](image)

\(^{23}\)Starbucks and Dunkin’ outperform other chains by a significant margin. The third and fourth chains were Tim Hortons and Dutch Bros., which accounted for 2.2% and 1.7% (resp. 2.6% and 1.3%) of the sales (resp. outlets) out of the top 8 chains.
Little is publicly known about how coffee chains predict opponents' strategic entry decisions. It is reasonable to expect that firms have good information about their stores’ (potential) profitability. Still, less is known about how firms learn about their opponents’ profitability. Resorting to the standard “incomplete” information assumption—firms observe their payoff shocks but observe nothing about their opponents’ shocks—will likely result in a model misspecification. Whether this misspecification is an empirical concern depends on how much it may change the structural estimates and counterfactual outcomes. We apply our Markov correlated equilibrium framework to estimate the parameters and conduct a counterfactual experiment with weaker assumptions on players’ information.

6.2 Data

Our primary dataset is Data Axle’s U.S. Historic Business Database, which has annual location information of all business establishments in the U.S. from 1997 to 2023. We define markets as four-digit 2010 census tracts.\textsuperscript{24} Our geographic market definition

\textsuperscript{24}Four-digit census tracts are census-defined geographic units that are contiguous and strike a balance between county (which is too large) and six-digit census tracts (which is too small).
assumes that coffee chains compete in small geographic areas.\(^{25}\) We supplement
the firm-location data with the demographic/geographic data from the Longitudinal
Track Data Base (Logan, Xu, and Stults, 2014) and the neighborhood characteristics
data from the National Neighborhood Data Archive. We limit our sample period to
2003–2017 to match the sample period of the neighborhood characteristics data. We
define a firm as a potential entrant in each market if it ever operated an outlet in the
market or had at least one outlet in the corresponding county and adjacent markets
during the sample period.

Table 3 reports the summary statistics. Our final sample comprises a balanced
panel of \(M = 35,686\) tracts observed over \(T = 15\) years. Starbucks and Dunkin’ are
defined as potential entrants in 83% and 71% of the markets. The firms are active in
15% and 12% of the observations, respectively. Entries and exits are rare (around 1% for entry and 0.2% for exit), indicating a high correlation in the incumbency status.
The number of eating places is the key common market characteristic that controls for
the overall market attractiveness (customer traffic increases with the number of food
establishments in the neighborhood). Finally, we use own-chain outlets in the county
and adjacent markets as firm-specific excluded variables that capture the network
effects associated with economies of density (Guler, 2018).

6.3 Model

Our model follows the standard dynamic oligopoly entry/exit model with hetero-
genous firms (Aguirregabiria and Mira, 2007; Pakes, Ostrovsky, and Berry, 2007;
Pesendorfer and Schmidt-Dengler, 2008). In each market \(m\) and time period \(t\), firm
\(i\) decides whether to operate in the market \((a_{imt} = 1)\) or not \((a_{imt} = 0)\). An active
firm’s flow profit is

\[
\begin{align*}
\varphi_i(a_{imt} = 1, a_{jmt}, x_{mt}, \varepsilon_{imt}) &= \theta_i^{\top} w_{imt} + \theta_i,cc(1 - z_{imt}) + \theta_i,ce a_{jmt} + \varepsilon_{imt}.
\end{align*}
\]

\(^{25}\)Guler (2018) shows that demand for Starbucks is highly localized. Thomadsen (2005) shows
evidence that only fast-food chains within approximately 0.5 miles—which amounts to 10 minutes
walking distance—compete as close substitutes in California; it is likely that coffee consumers travel
even less.
Table 3: Summary statistics of 35,686 coffee chain markets from 2003 to 2017

| Variable                  | Mean  | Median | Std. Dev. | Min | Max |
|---------------------------|-------|--------|-----------|-----|-----|
| Potential Entrant         |       |        |           |     |     |
| SB is player              | 0.831 | 1      | 0.375     | 0   | 1   |
| DK is player              | 0.708 | 1      | 0.455     | 0   | 1   |
| Entry/Exit Decisions      |       |        |           |     |     |
| SB is active              | 0.151 | 0      | 0.358     | 0   | 1   |
| DK is active              | 0.122 | 0      | 0.328     | 0   | 1   |
| SB entered                | 0.012 | 0      | 0.107     | 0   | 1   |
| DK entered                | 0.007 | 0      | 0.086     | 0   | 1   |
| SB exited                 | 0.002 | 0      | 0.043     | 0   | 1   |
| DK exited                 | 0.002 | 0      | 0.043     | 0   | 1   |
| Market Characteristics    |       |        |           |     |     |
| Number of eating places   | 16.508| 9      | 26.667    | 0   | 910 |
| SB outlets in county      | 1021.233 | 252   | 2127.375  | 0   | 10923 |
| DK outlets in county      | 668.877 | 107   | 1346.954  | 0   | 7050 |
| SB outlets in adjacent markets | 1.733 | 1     | 3.413     | 0   | 108 |
| DK outlets in adjacent markets | 1.109 | 0     | 2.142     | 0   | 41  |

Note: SB and DK stand for Starbucks and Dunkin’.

Here, public state $x_{mt}$ includes two components, observable market characteristics $w_{mt}$ and incumbency status $z_{imt} \equiv a_{im,t-1}$. $w_{mt}$ includes a constant, (log) number of eating places, (log) own-chain outlets in the county, and (log) own-chain outlets in adjacent markets. Parameters $\theta_{i,w}$, $\theta_{i,ec}$, and $\theta_{i,ce}$ represent coefficients on market characteristics, entry cost, and competitive effects, respectively. Finally, $\varepsilon_{imt}$ is the idiosyncratic per-period payoff shock assumed to be independently drawn from the standard logistic distribution. We set the flow payoff from being inactive to zero.\(^{26}\)

We assume the agents take the market characteristics as exogenous and fixed over time. The state transition is deterministic since the only dynamic state variable is the incumbency status, determined by the firms’ actions in the previous period.

\(^{26}\)Normalizing the flow profit from the outside option to zero is not without loss of generality for counterfactual analysis (Aguirregabiria and Suzuki, 2014; Kalouptsidi, Scott, and Souza-Rodrigues, 2017, 2021; Aguirregabiria, Collard-Wexler, and Ryan, 2021). We show that our counterfactual experiment of reducing a proportion of entry cost is equivalent to providing an entry subsidy equal to the same proportion of the entry cost net of scrap value without this normalization in Online Appendix A.
6.4 Estimation Strategy

*Baseline Information Structure.* We estimate the Markov correlated equilibrium identified set assuming that the players observe at least their idiosyncratic shocks $\varepsilon_{imt}$. Our identified set is robust to the assumption that the players may observe more information about opponents’ payoff shocks before making decisions.

*Single-Agent Dynamic Discrete Choice Estimation.* We leverage the intuition from the literature that having excluded variables with large support can substantially tighten the identified set (Tamer, 2003; Magnolfi and Roncoroni, 2023). Recall that our definition of potential entrant requires that a firm has had at least one outlet in the county during the sample period. In markets where a firm did not operate even a single outlet in the relevant county, the firm is highly unlikely to enter. In markets where Dunkin’ (resp. Starbucks) is not a potential entrant, Starbucks (resp. Dunkin’) solves a single-agent dynamic entry problem. The econometrician can then apply single-agent discrete choice estimation methods to identify the payoff parameters except the competition effects parameters. We apply the two-step pseudo-likelihood method.

*Markov Correlated Equilibrium Estimation.* We then use data from markets in which both firms are players to estimate the competitive effects parameters via our Markov correlated equilibrium framework outlined in Section 5. We estimate the first-stage CCPs using flexible logit. We construct simultaneous confidence intervals for CCPs with a plug-in sup-t implementation (Montiel Olea and Plagborg-Møller, 2019). We estimate a criterion function $Q(\theta_{ce})$ that measures the degree of violation in the Markov correlated equilibrium obedience condition (10) for given competitive effects parameters $\theta_{ce}$. We construct identified set estimates by collecting $\theta_{ce}$ that satisfy the estimated $Q(\theta_{ce}) \leq c$ for a small threshold $c > 0$. We use non-zero $c$.

---

27 Formally, let $\tilde{x}_{jmt}$ represent the log of the number of firm j’s outlets in the county. Suppose that $\tilde{x}_{jmt}$ is excluded from i’s payoff functions and that j’s payoff is increasing in $\tilde{x}_{jmt}$. Then markets in which firm j operates no outlet in the county has $\tilde{x}_{jmt} = \log 0 = -\infty$, naturally satisfying the large support assumption.

28 Aradillas-Lopez and Rosen (2022) also take a similar two-step approach in static ordered-response games setting. The authors point-identify the non-competitive effects parameters and use the estimates to set-identify the competitive effects parameters.

29 Nonparametric estimation is infeasible since observations do not exist in some covariate bins.

30 We use the covariance matrix for CCPs estimated by bootstrapping at the market level in the plug-in sup-t implementation.
to accommodate misspecifications in the structural and first-stage CCP logit models. The identified set estimates are more conservative as we use a larger $c$. We do not incorporate sampling error associated with the first-stage single-agent dynamic discrete choice estimation, so a larger $c$ may prevent us from making identified set estimates too tight.

We also estimate the parameters under the standard incomplete information Markov perfect equilibrium assumption for comparison. We set the discount factor to $\delta = 0.9$. We discretize each continuous variable into two bins by grouping the observations above and below the median and replacing the values with within-bin means.

6.5 Estimation Results

Table 4 reports our estimation results. Column MPE reports the estimates from the standard incomplete information Markov perfect equilibrium assumption. The competitive effects parameter estimates suggest the presence of Starbucks has a positive spillover effect on Dunkin’, while the presence of Dunkin’ does not affect Starbucks’ profitability. The market characteristics coefficient estimates show positive effects of attractive markets on profitability and economies of density. The entry cost estimates are quite large compared to the other parameter estimates, which rationalizes the low entry probability.

Column MCE reports the estimates from our Markov correlated equilibrium framework. The Markov correlated equilibrium identified set estimates of the competitive effects subsume their confidence intervals under the standard incomplete information assumption in column MPE and do not allow us to determine the signs for both firms. The parameter estimates from the single-agent dynamic discrete choice estimation are similar to those reported in column MPE, but the confidence intervals do not overlap for some parameters. The discrepancy of these parameter estimates can be attributed to either misspecification in the information structure of the standard incomplete information assumption or the differences in the parameters underlying separate segments of the markets.\footnote{We do not provide a formal test of misspecification in the information structure. See Han, Kaido, and Magnolfi (2024) for a related test in the Bayes correlated equilibrium context.}
## Table 4: Structural parameter estimates

|                | MPE |            | MCE |            |
|----------------|-----|------------|-----|------------|
|                | Estimate | 95% CI    | Estimate | 95% CI    |
| **Starbucks**  |       |            |       |            |
| Intercept      | -0.010 | [-0.032, 0.018] | -0.014 | [-0.045, 0.027] |
| Log eating places | 0.128 | [0.122, 0.135] | 0.130 | [0.123, 0.138] |
| Log outlets in county | 0.004 | [0.002, 0.007] | 0.008 | [0.005, 0.013] |
| Log adjacent outlets | 0.060 | [0.054, 0.067] | 0.053 | [0.041, 0.063] |
| Entry cost     | -8.064 | [-8.155, -7.976] | -8.115 | [-8.263, -7.984] |
| Competitive effect | 0.011 | [-0.001, 0.024] | - | [-0.056, 0.096] |
| **Dunkin’**    |       |            |       |            |
| Intercept      | 0.011 | [-0.010, 0.035] | 0.149 | [0.105, 0.183] |
| Log eating places | 0.071 | [0.065, 0.077] | 0.076 | [0.066, 0.086] |
| Log outlets in county | 0.018 | [0.016, 0.020] | 0.016 | [0.012, 0.023] |
| Log adjacent outlets | 0.083 | [0.075, 0.092] | 0.001 | [-0.015, 0.016] |
| Entry cost     | -8.106 | [-8.194, -8.012] | -8.543 | [-8.692, -8.395] |
| Competitive effect | 0.084 | [0.071, 0.098] | - | [-0.098, 0.132] |

*Note:* Parameters excluding the competitive effects parameters in column MCE are obtained using single-agent discrete choice estimation using data where the firms operate as single players. The Markov correlated equilibrium identified set for the competitive effects parameters does not account for sampling error associated with first-stage single-agent dynamic discrete choice estimations. Confidence intervals (CIs) are bootstrapped with resampling at the market level.

### 6.6 Counterfactual

As a counterfactual experiment, we reduce each firm’s entry cost by 40% and study how the number of active firms evolves beginning from 2003. We consider two competitive effects parameter values: lower and upper bounds of their estimated identified set in column MCE of Table 4. The lower bounds represent the most extreme case of strategic substitutes because both firms lose their profits most due to the competitor’s presence. The upper bounds are the other extreme of strategic complements, as both firms benefit most from the competitor’s presence. By considering the two extremes, we can investigate the bounds of counterfactual outcomes that can arise. We fix the parameters other than the competitive effects at their two-step pseudo maximum likelihood estimates in column MPE of Table 4.

At each candidate parameter, we find Markov correlated equilibria that maximize and minimize the expected number of active firms. The expected number of active firms...
firms at state \( x \) is 
\[
0 \times \phi(0, 0|x) + 1 \times (\phi(0, 1|x) + \phi(1, 0|x)) + 2 \times \phi(1, 1|x),
\]
where \( \phi(a_1, a_2|x) \) is the equilibrium CCPs. We take an unweighted average over \( x \) to obtain the (unconditional) expected number of active firms. We use the equilibrium CCPs to simulate the evolution of active firms. We allow the information structure in the counterfactual equilibrium to vary from the equilibrium in the estimation.\(^{32}\)

Figure 5 plots the counterfactual results. The black curve shows the evolution of active firms in the markets using the CCPs estimated from the data. The blue curves represent the bounds on counterfactual outcomes at the most strategic substitute case. The red curves represent the bounds on counterfactual outcomes at the most strategic complement case. As expected, the strategic complement case has a higher number of active firms than the strategic substitute case because the presence of opponents boosts firms’ profits when entries are strategic complements.

Figure 5 reveals that the counterfactual outcomes are not sensitive to assumptions on information structure in the counterfactual equilibrium. The differences between the maximum and minimum expected number of active firms are tight for a given set of parameter values. The region between the maximum and minimum includes

\(^{32}\)We do not apply the methodology of Bergemann, Brooks, and Morris (2022), which shows how to conduct counterfactual analysis with information structure fixed at the pre-counterfactual equilibrium.
the expected numbers of active firms arising from any Markov perfect equilibrium when the players minimally observe their payoff shocks and may observe more information about opponents’ payoff shocks. Thus, the tight counterfactual outcome bounds indicate that the ability to obtain more information about opponents’ payoff shocks has little impact on the outcome. The small competitive effects parameters (in absolute values) can explain the tight counterfactual bounds. If the competitive effects are null, then the firms do not care about the opponents’ behavior, and there is no room for an information designer to affect the outcomes. Here, the competitive effects parameters range between $-0.10$ and $0.14$, which is small considering the standard deviation of the idiosyncratic payoff shock (the standard deviation of the standard logistic random variable is $\sqrt{\pi^2/3} \approx 1.81$). Thus, the degree to which the outcomes can vary in counterfactual Markov correlated equilibrium depends largely on the competitive effects parameter values.

7 Conclusion

In this paper, we have studied the identification, estimation, and counterfactual analyses of a common class of dynamic games for empirical analysis, assuming the researcher knows some minimal information available to players. To facilitate informationally robust econometric analysis of dynamic games, we have defined Markov correlated equilibrium, a dynamic analog of Bayes correlated equilibrium, and studied its properties. We have discussed new challenges that arise in dynamic environments and proposed practical solutions. We have applied our proposed method to study the dynamic entry and exit decisions of the two largest coffee chains in the US.

We conclude this paper with a discussion of future research directions. First, we think further refinement of Markov correlated equilibrium (and thus Bayes correlated equilibrium) will attract larger interest from empiricists. The type of informational robustness offered by Markov/Bayes correlated equilibrium admits too many possibilities. Markov correlated equilibrium identified sets and counterfactual outcomes can span a wide range, which may not be empirically informative; Magnolfi and Roncoroni (2023) find similar results in their empirical application. Numerical examples from Koh (2023) and Magnolfi and Roncoroni (2023) show that even complete infor-
mation correlated equilibrium identified set can be large, indicating that correlation in actions can also weaken the identifying power substantially. In some settings, the researcher may be able to reasonably exclude some information structures from consideration. It will be fruitful to find a tractable approach that can accommodate further assumptions on the underlying information structure to narrow down the possibilities.

Second, estimation with Markov correlated equilibrium remains computationally challenging. A similar problem exists for Bayes correlated equilibrium since the analyst needs to run a grid search. Studying how to conduct computationally tractable estimation and inference in this class of partially identified models will be important to broaden the scope of applications.

A Proofs

We use BNE, BCE, MPE, and MCE for Bayes Nash equilibrium, Bayes correlated equilibrium, Markov perfect equilibrium, and Markov correlated equilibrium, respectively, for brevity in the following proofs.

A.1 Proof of Lemma 1

Proof. $(\Rightarrow)$ Let $\beta$ be a MPE of $(G, S)$. $\beta$ induces $(G^\beta, S)$. The BNE best-response condition is directly implied by the MPE condition of $(G, S)$.

$(\Leftarrow)$ Let $\beta$ be a BNE of $(G^\beta, S)$. The BNE best-response condition implies that there is no profitable one-shot deviation given that each player expects all players to follow the prescription of $\beta$ in the future. The absence of profitable one-shot deviation implies that $\beta$ is a subgame perfect equilibrium of $(G, S)$ by the one-shot deviation principle. Therefore, $\beta$ is a MPE of $(G, S)$. $\square$

A.2 Proof of Lemma 2

Proof. The statement follows by applying the one-shot deviation principle. $\square$
A.3 Proof of Lemma 3

Proof. It is enough to show that \( V_i^\beta (x) = V_i^\sigma (x) \) for all \( i \in \mathcal{I} \) and \( x \in \mathcal{X} \) because then it follows that \( v_i^\beta (a, x, \varepsilon) = v_i^\sigma (a, x, \varepsilon) \) for all \( i \in \mathcal{I}, a \in \mathcal{A}, x \in \mathcal{X}, \) and \( \varepsilon \in \mathcal{E}. \)

Recall that \( V_i^\beta \) and \( V_i^\sigma \) satisfy (4) for \((G, S^*)\) and (5) for \((G, S)\), respectively. Since \( \beta \) induces \( \sigma \), (4) for \((G, S^*)\) is equivalent to

\[
V_i^\beta (x) = \sum_{\varepsilon, \tau, \tilde{\tau}, a} \psi_{\varepsilon|x, \tau, x, \varepsilon, \varepsilon, \tilde{\tau}, \tilde{\tau}} \left( u_i(a, x, \varepsilon) + \delta \sum_{x'} V_i^\beta (x') f_{x'|a, x, \varepsilon} \right)
\]

where \( \beta_{a|x, \tau, \tilde{\tau}} \equiv \prod_{i=1}^I \beta_i (a_i|x, \tau_i, \tilde{\tau}_i). \) Comparison of the above equation for \( V_i^\beta \) and (5) for \((G, S)\) that defines \( V_i^\sigma \) implies that \( V_i^\beta = V_i^\sigma. \)

\(\square\)

A.4 Proof of Theorem 1

Proof. To prove the statement of the theorem, we make use of Theorem 1 of Berge-mann and Morris (2016) (BM), which we restate below.

Lemma 4 (BM Theorem 1). Let \((G, S)\) be a static game of incomplete information. A decision rule \(\sigma\) is a BCE of \((G, S)\) if and only if, for some expansion \(S^*\) of \(S\), there is a BNE of \((G, S^*)\) that induces \(\sigma\).

\(\subseteq\) Suppose \(\sigma\) is a MCE of \((G, S)\). We want to show that there exists an expansion \(S^*\) and a strategy profile \(\beta\) in \((G, S^*)\) such that \(\beta\) is a MPE of \((G, S^*)\) and \(\beta\) induces \(\sigma\). Since \(\sigma\) is a MCE of \((G, S)\), it is a BCE of \((G^\sigma, S)\). By BM Theorem 1, there exists an expansion \(S^*\) of \(S\) and a strategy profile \(\beta\) of \((G^\sigma, S^*)\) such that \(\beta\) is a BNE of \((G^\sigma, S^*)\) and \(\beta\) induces \(\sigma\). But since \(G^\sigma = G^\beta\), \(\beta\) is also a BNE of \((G^\beta, S^*)\), which in turn implies that \(\beta\) is a MPE of \((G, S^*)\).

\(\supseteq\) Suppose \(\beta\) is a MPE of \((G, S^*)\). We want to show that if \(\beta\) induces \(\sigma\) in \((G, S)\), then \(\sigma\) is a MCE of \((G, S)\). Since \(\beta\) is a MPE of \((G, S^*)\), it is a BNE of \((G^\beta, S^*)\). By BM Theorem 1, if \(\sigma\) is induced by \(\beta\), then \(\sigma\) is a BCE of \((G^\beta, S)\). But since \(G^\beta = G^\sigma\), \(\sigma\) is also a BCE of \((G^\sigma, S)\), which in turn implies that \(\sigma\) is a MCE of \((G, S)\). \(\square\)
A.5 Proof of Corollary 1

Proof. \((\subseteq)\) Take \(\phi \in \mathcal{P}_{a|x}^\text{MCE} (G, S)\). By definition, there exists \(q \in \mathcal{P}_{a|x}^\text{MCE} (G, S)\) such that \(q\) induces \(\phi\), i.e., \(\phi_{a|x} = \sum_{e \in E, \tau \in T} \psi_{e|x} \pi_{x} q_{a|x, \tau}, \forall a \in A, x \in X\). Then there exists \(S^*\) such that \(S^* \succeq_E S\) and \(q \in \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)\), implying that \(\phi \in \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)\).

\((\supseteq)\) Take \(\phi \in \bigcup_{S \supseteq S} P_{a|x}^\text{MPE} (G, S^*)\). Then there exists \(S^*\) such that \(S^* \succeq_E S\) and \(\phi \in \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)\), which implies that there exists \(q \in \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)\) such that \(q\) induces \(\phi\). Then since \(q \in \mathcal{P}_{a|x,\xi,\tau}^\text{MCE} (G, S)\), we have \(q \in \mathcal{P}_{a|x}^\text{MCE} (G, S)\).

A.6 Proof of Theorem 2

Proof. Let \(\phi \in \mathcal{P}_{a|x,\xi,\tau}^\text{MCE} (G, S^1)\). We want to show \(\phi \in \mathcal{P}_{a|x,\xi,\tau}^\text{MCE} (G, S^2)\). From our Theorem 1, we have

\[
\mathcal{P}_{a|x,\xi,\tau}^\text{MCE} (G, S^1) = \bigcup_{S^* \succeq E S^1} \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*) \quad \text{and} \quad \mathcal{P}_{a|x,\xi,\tau}^\text{MCE} (G, S^2) = \bigcup_{S^* \succeq E S^2} \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)
\]

If \(\phi \in \mathcal{P}_{a|x,\xi,\tau}^\text{MCE} (G, S^1)\), then there exists \(S^*\) such that \(\phi \in \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)\). But since \(S^* \succeq E S^1\) and \(S^1 \succeq E S^2\), we have \(S^* \succeq E S^2\), which implies \(\phi \in \bigcup_{S^* \succeq E S^2} \mathcal{P}_{a|x,\xi,\tau}^\text{MPE} (G, S^*)\), which is what we wanted to show.

A.7 Proof of Theorem 3

Proof. We want to show that \(\Theta_{\text{MCE}}^I (S) = \bigcup_{S \succeq E S} \Theta_I^\text{MPE} (\tilde{S})\). Take \(\theta \in \Theta_{\text{MCE}}^I (S)\). By definition, \(\phi \in \mathcal{P}_{a|x}^\text{MCE} (G^\theta, S)\). By Corollary 1, there exists some \(S^* \succeq E S\) such that \(\phi \in \mathcal{P}_{a|x}^\text{MPE} (G, S^*)\). But this immediately implies \(\theta \in \Theta_I^\text{MPE} (S^*)\), and thus \(\theta \in \bigcup_{S \succeq E S} \Theta_I^\text{MPE} (\tilde{S})\). The proof of converse follows the above steps in reverse.

A.8 Proof of Theorem 4

Proof. The statement follows from Theorem 2.

A.9 Proof of Theorem 5

Proof. The statement directly follows from the fact that the constraints (8), (9), (10), and (11) characterize the MCE identification conditions as implied by the definition...
of identified sets. Note that the ex-ante value function equation (11) is equivalent to (5) because

\[ \sum_{\epsilon, \tau, a} \psi_{\epsilon} x^{\pi_{\tau}} \psi_{a|x, \epsilon} \sum_{x'} V_{i,x'} f_{x'|a,x} = \sum_{\epsilon, \tau} \left( \sum_{a,x'} \psi_{\epsilon} x^{\pi_{\tau}} \psi_{a|x, \epsilon} \right) V_{i,x'} f_{x'|a,x} = \sum_{a,x'} \phi_{a|x} V_{i,x'} f_{x'|a,x}. \]

\[ \square \]

A.10 Proof of Theorem 6

**Proof.** The ex-ante value function (14) can be derived by expanding (5) and imposing (12) similarly to the proof of Theorem 5 in Appendix A.9. The obedience condition (13) is derived as follows. The obedience condition in the fully robust case is

\[ \sum_{\epsilon, a_{-i}} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} \partial u_i^\theta (a'_i, a_i, x, \epsilon_i) \leq 0. \]  

But since

\[ \sum_{\epsilon, a_{-i}} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} V_{i,x'} f_{x'|a_i, a_{-i}, x} \]

\[ = \sum_{\epsilon, a_{-i}} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} \left( u_i^\theta (a'_i, a_{-i}, x, \epsilon_i) + \delta \sum_{x'} V_{i,x'} f_{x'|a_i, a_{-i}, x} \right) \]

\[ = \sum_{\epsilon, a_{-i}} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} u_i^\theta (a'_i, a_{-i}, x, \epsilon_i) + \sum_{a_{-i}} \left( \sum_{\epsilon} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} \right) \left( \delta \sum_{x'} V_{i,x'} f_{x'|a_i, a_{-i}, x} \right) \]

\[ = \sum_{\epsilon, a_{-i}} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} u_i^\theta (a'_i, a_{-i}, x, \epsilon_i) + \delta \sum_{a_{-i}, x'} \phi_{a|x} V_{i,x'} f_{x'|a'_i, a_{-i}, x}, \]

where the last equation is obtained by imposing (12), the obedience condition (15) can be replaced by

\[ \sum_{\epsilon, a_{-i}} \psi_{\epsilon} x^{\sigma_{a|x, \epsilon}} \partial u_i^\theta (a'_i, a, x, \epsilon_i) + \delta \sum_{a_{-i}, x'} \phi_{a|x} V_{i,x'} f_{x'|a'_i, a_{-i}, x} \leq 0, \forall i, x, a_i, a'_i \]
where $\partial u_\theta^i(a'_i, a, x, \varepsilon_i) \equiv u_\theta^i(a'_i, a_{-i}, x, \varepsilon_i) - u_\theta^i(a, x, \varepsilon_i)$ and $\partial f_{x'|a'_i,a,x} \equiv f_{x'|a'_i,a_{-i},x} - f_{x'|a,x}$.

A.11 Proof of Theorem 7

Proof. By construction, $\Pr\left(\Theta_I(\phi) \subseteq \bigcup_{\tilde{\phi} \in \Phi_\alpha} \Theta_I(\tilde{\phi})\right) \geq \Pr(\phi \in \Phi_\alpha)$. By taking the limit inferior on both sides, we obtain the desired result. The remaining statement holds by the definitions of $\Theta_I(\phi)$ and $\hat{\Theta}_I^\phi$.

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A Interpretation of Counterfactual Analysis

As noted in a series of papers (Aguirregabiria and Suzuki, 2014; Kalouptsidi, Scott, and Souza-Rodrigues, 2017, 2021; Aguirregabiria, Collard-Wexler, and Ryan, 2021), normalizing the flow payoff from the outside option to zero is not without loss of generality for counterfactual analysis. However, we provide a way to interpret our normalization and counterfactual analysis under the assumption that a parametric specification is correct.

Suppose that the flow payoff is given by

\[
\begin{align*}
    u_i^\theta(a_{it}, a_{jt}, x_t, \varepsilon_{it}) &= \begin{cases} 
    \theta_{i,0}^* + \theta_{i,w}^* w_t + \theta_{i,ec}^* (1 - z_{it}) + \theta_{i,sv}^* a_{jt} + \varepsilon_{it} & \text{if } a_{it} = 1 \\
    \theta_{i,sv}^* w_t & \text{if } a_{it} = 0
    \end{cases},
\end{align*}
\]

where \( \theta_{i,ec}^* < 0 \) is the true entry cost and \( \theta_{i,sv}^* > 0 \) is the true scrap value. A critical assumption in (1) is that the scrap value is independent of the market characteristics. Payoff (1) implies that the payoff difference is

\[
\begin{align*}
    u_i^\theta(1, a_{jt}, x_t, \varepsilon_{it}) - u_i^\theta(0, a_{jt}, x_t, \varepsilon_{it}) &= (\theta_{i,0}^* - \theta_{i,sv}^*) + \theta_{i,w}^* w_t + (\theta_{i,ec}^* + \theta_{i,sv}^*) (1 - z_{it}) + \theta_{i,sv}^* a_{jt} + \varepsilon_{it}.
\end{align*}
\]

Thus, the estimated intercept captures \( (\theta_{i,0}^* - \theta_{i,sv}^*) \) (i.e., fixed cost net of scrap value), and the estimated entry cost captures \( (\theta_{i,ec}^* + \theta_{i,sv}^*) \) (i.e., entry cost net of scrap value). This is consistent with Aguirregabiria and Suzuki (2014) that the fixed cost, entry cost, and scrap value are not separately identified.

Now consider the counterfactual experiment of reducing the entry cost as in Section 6.6. One can show that this is equivalent to holding \( \theta_{i,sv}^* \) fixed and reducing \( \theta_{i,ec}^* \) (making it less negative) to \( \theta_{i,ec}^{**} \) by

\[
\theta_{i,ec}^{**} = \theta_{i,ec}^* - \lambda (\theta_{i,ec}^* + \theta_{i,sv}^*),
\]

where \( \lambda < 0 \).
where $\lambda$ is the proportion of entry cost intended to be cut. Thus, our counterfactual experiment is equivalent to providing an entry subsidy equal to $\lambda$ of the entry cost net of scrap value.