THE GRAPH, RANGE AND LEVEL SET SINGULARITY SPECTRA OF $b$-ADIC INDEPENDENT CASCADE FUNCTION

XIONG JIN

Abstract. With the “iso-Hölder” sets of a function we naturally associate subsets of the graph, range and level set of the function. We compute the associated singularity spectra for a class of statistically self-similar multifractal functions, namely the $b$-adic independent cascade function.

1. Introduction

1.1. The singularity spectra of multifractal function. Let $f$ be a real-valued function defined on an interval $I$. For any subset $E$ of $I$ let

$$G_f(E) = \{(x, f(x)) : x \in E\}, \quad R_f(E) = \{f(x) : x \in E\}$$

be the graph and the range of $f$ over the set $E$; for any $y \in R_f(E)$ let

$$L_y^f(E) = \{(x, f(x)) : x \in E, f(x) = y\}$$

be the level set of $f$ over the set $E$ at level $y$. When $(f(t))_{t \in I}$ is a stochastic process, to find the Hausdorff dimension of these sets, denoted by $\dim_H S_f(E)$ for $S \in \{G, R, L_y\}$, is a classical and important question in probability and geometric measure theory. The original works on these questions could be traced back to 1953, [33] by Lévy or [47] by Taylor, regarding the Hausdorff dimension and the Hausdorff measure of the range of Brownian motion. Since then, many progresses have been made in this subject for fractional Brownian motions, stable Lévy processes and many other processes and functions [13, 14, 48, 23, 44, 10, 9, 22, 10, 27, 39, 45, 49, 8, 11, 31, 24, 25, 46, 20, 29, 16] (see also the survey paper [52] and the references therein).

As a typical example, in [27] Kahane studies the fractional Brownian motion $(X(t))_{t \in \mathbb{R}_+}$, i.e., given $\beta \in (0, 1)$, the unique centered continuous Gaussian process satisfying $X(0) = 0$ and $\mathbb{E}(|X(s) - X(t)|^2) = |s - t|^{2\beta}$ for any $s, t \in \mathbb{R}_+$. He shows that for any compact set $E \subset \mathbb{R}_+$, almost surely

$$\dim_H G_X(E) = \frac{\dim_H E}{\beta} \land (\dim_H E + 1 - \beta), \quad \dim_H R_X(E) = \frac{\dim_H E}{\beta} \land 1$$

2000 Mathematics Subject Classification. Primary: 26A30; Secondary: 28A78, 28A80.

Key words and phrases. Hausdorff dimension; Multifractal; Random function; Graph; Range; Level set; Independent cascades.
and if \( \dim_H E > \beta \), then there exists a random open set \( G \subset R_X(E) \) such that \( \mathbb{P}(G \neq \emptyset) > 0 \) and

\[
y \in G \Rightarrow \dim_H L^y_X(E) = \dim_H E - \beta.
\]

Notice that here \( \beta \in (0, 1) \) is the (single) Hölder exponent of the function \( X \) and it is uniform on \( E \), indeed \( X \) is a monofractal on \( \mathbb{R}_+ \). When the function is not monofractal, a natural parallel to the above formulas is to take \( E = E_f(h) \), the set of points at which the pointwise Hölder exponents of \( f \) are all equal to a constant \( h > 0 \), and to verify formulas like

\[
\dim_H G_f(E_f(h)) = \frac{\dim_H E_f(h)}{h} \wedge \left( \dim_H E_f(h) + 1 - h \right)
\]

and

\[
\dim_H R_f(E_f(h)) = \frac{\dim_H E_f(h)}{h} \wedge 1.
\]

The set \( E_f(h) \) and its Hausdorff dimension \( \dim_H E_f(h) \) naturally appeared in the multifractal analysis of functions, which consists in computing the singularity spectrum \( d_f : h \geq 0 \mapsto \dim_H E_f(h) \). This, together with the above parallel formulas, leads us to consider the following graph, range and level set singularity spectra:

\[
d^S_f : h \geq 0 \mapsto \dim_H S_f(E_f(h)), \ S \in \{G, R, L^y\}.
\]

To our best knowledge, such singularity spectra have not been considered before.

1.2. A general upper bound. At first, it is natural to seek for general upper bounds for these new singularity spectra. Such bounds can be found thanks to the following generalization of Lemma 8.2.1 in [1], Theorem 6 of Chapter 10 in [27] and Lemma 2.2 in [51].

First we note that the pointwise Hölder exponent considered in this paper is defined by

\[
h_f(x) := \liminf_{r \to 0^+} \frac{1}{\log r} \log \left( \sup_{s, t \in B(x, r)} |f(s) - f(t)| \right),
\]

which covers the definition of the lower local dimension

\[
h_\mu(x) := \liminf_{r \to 0^+} \frac{1}{\log r} \log \mu(B(x, r))
\]

of a measure \( \mu \) support on an interval \([a, b] \) if we set \( f(x) = \mu([a, x]) \) for \( x \in [a, b] \).

Also, we should introduce the level set of \( f \) over set \( E \) in \( \theta \)-direction (since in this paper we can only show the level set singularity spectrum in “Lebesgue almost every direction”): For \( \theta \in (-\pi/2, \pi/2) \) denote by \( l_\theta \) the line in \( \mathbb{R}^2 \) passing through the origin and making an angle \( \theta \) with the \( y \)-axis (clockwise). For any \( y \in l_\theta \), denote by \( l_{y, \theta}^\perp \) the line perpendicular to \( l_\theta \), passing through \( y \). Denote by \( \text{Proj}_\theta \) the orthogonal projection from \( \mathbb{R}^2 \) onto \( l_\theta \). Define \( R_{f, \theta}(E) = \text{Proj}_\theta(G_f(E)) \). Then for each \( y \in R_{f, \theta}(E) \), the level set of \( f \) over the set \( E \) in \( \theta \)-direction is defined by \( L^y_{f, \theta}(E) = G_f(E) \cap l_{y, \theta}^\perp \). Notice that the typical level set \( L^y_f(E) \) is just the level set \( L^y_{f, \theta}(E) \) when \( \theta = 0 \).
Denote by $\dim_P$ the packing dimension. We have the following theorem:

**Theorem 1.1.** Let $E$ be any subset of $I$. Suppose that $\inf_{x \in E} h_f(x) = h > 0$.

(a) For $D \in \{H, P\}$ we have

$$\dim_D G_f(E) \leq \left( \frac{\dim_D E}{h} \wedge (\dim_D E + 1 - h) \right) \vee \dim_D E,$$

$$\dim_D R_f(E) \leq \frac{\dim_D E}{h} \wedge 1.$$

(b) Suppose $h \leq 1$. Fix $\theta \in (-\pi/2, \pi/2)$. Let $\mu$ be any positive Borel measure defined on $l_\theta$. For any $\gamma > 0$ define the set $R^\mu_{f, \theta}(E) := \{y \in R_f, \theta(E) : h_\mu(y) \geq \gamma\}$. If $\mu(R^\mu_{f, \theta}(E)) > 0$ and $\dim_H E - h \cdot \gamma > 0$, then for $\mu$-almost every $y \in R^\mu_{f, \theta}(E)$,

$$\dim_H L^\mu_{f, \theta}(E) \leq \dim_H E - h \cdot \gamma.$$

If we replace $E$ by the set $E_f(h) = \{x \in I : h_f(x) = h\}$ for $h > 0$, then Theorem 1.1 provides us with general upper bounds of the graph, range and level set singularity spectra. These upper bounds are strongly related to the classical singularity spectrum $d_f$ (see Corollary 1.1).

From the multifractal analysis of functions we know that $d_f$ has a general upper bound given by the Legendre transform of the so-called scaling function or $L^q$-spectrum of $f$, defined by

$$\tau_f(q) = \lim_{r \to 0^+} \frac{1}{\log r} \log \sup_i \sum Osc_f(B_i)^q, \quad q \in \mathbb{R},$$

where $Osc_f(B_i) = \sup_{s,t \in B_i} |f(s) - f(t)|$ denotes the oscillation of $f$ over $B_i$ and the supremum is taken over all the families of disjoint closed intervals $B_i$ of radius $r$ with centers in $\{x \in I : \forall \gamma > 0, \text{ Osc}_f(B(x, \gamma)) > 0\}$. Due to [26, 5], we have

$$d_f(h) \leq \tau_f^*(h) := \inf_{q \in \mathbb{R}} qh - \tau_f(q) \quad (\forall h \geq 0),$$

a negative dimension meaning that $E_f(h)$ is empty. Then as a direct consequence, we have the following corollary of Theorem 1.1.

**Corollary 1.1.** For any $h > 0$ we have

$$\dim_H G_f(E_f(h)) \leq \left( \frac{d_f(h)}{h} \wedge (d_f(h) + 1 - h) \right) \vee d_f(h)$$

$$\dim_H R_f(E_f(h)) \leq \frac{d_f(h)}{h} \wedge 1 \leq \frac{\tau_f^*(h)}{h} \wedge 1$$

and with the same notations as in Theorem 1.1(b), for $\mu$-almost every $y \in R^\mu_{f, \theta}(h)$,

$$\dim_H L^\mu_{f, \theta}(E_f(h)) \leq d_f(h) - h \cdot \gamma \leq \tau_f^*(h) - h \cdot \gamma.$$
1.3. Main result. Corollary [11] naturally raises the question: Do these upper bounds provide the exact dimensions, especially when \( f \) obeys the multifractal formalism, i.e. \( d_f(h) = \tau_f^*(h) \) for \( h > 0 \)?

In general the answer is negative. We can easily find a counterexample in case when \( f \) is a monofractal function, that is \( h_f(x) \) is equal to a constant \( \beta \in (0,1) \) for all \( x \in I \). Suppose, moreover, that the whole graph \( G_f(I) = G_f(E_f(\beta)) \) is irregular, that is its Hausdorff and lower box-counting dimensions are different. We have:

\[
1 - \beta + \tau_f^*(\beta) = 1 - \beta + d_f(\beta) = 2 - \beta > \dim_H G_f(E_f(\beta)) > \dim_H G_f(E_f(\beta)).
\]

Such examples can be found in [19, 20]. To the contrary, if the whole graph is regular (like for the fractional Brownian motion mentioned before), then \( \dim_H G_f(E_f(\beta)) = 1 - \beta + \tau_f^*(\beta) \).

However, monofractal examples clearly represent a very restrictive class for our purpose. Simple multifractal examples are the following: Consider \( f(x) = \mu([0,x]) \) for \( x \in [0,1] \), where \( \mu \) is a probability measure fully supported by \([0,1]\), and assume that \( f \) obeys the multifractal formalism with the exponent \( \hat{h}_f(x) \) defined by

\[
(4) \quad \hat{h}_f(x) = \lim_{r \to 0^+} \frac{1}{\log r} \log \left( \sup_{s,t \in [x-r,x+r]} |f(s) - f(t)| \right),
\]

whenever it exists. This property holds whenever \( \mu \) is a Gibbs or a random cascade measure [15, 3]. Then by using the results in [36] on the multifractal analysis of the inverse measure \( \mu^* = \mu \circ f^{-1} \) carried by the range of \( f \), it is easy to check that the upper bounds in Corollary [11] give the exact dimensions. But there, the graph and range singularity spectra are always a combination of the singularity spectra of \( \mu \) and \( \mu^* \), and the level set spectrum is trivial since \( f \) is an increasing function.

It is an interesting question to find examples of multifractal functions whose graph, range and level set singularity spectra can be calculated and are not trivial in the above sense. In this paper we consider the so-called b-adic independent cascade function introduced in [4] as an extension to random functions of statistically self-similar measures introduced in [35]. We obtain the graph and range singularity spectra for this class of random functions, in the so-called non-conservative case. The level set singularity spectrum is still open. However, inspired by the classical Marstrand theorem (see [37]), we obtain the level set singularity spectrum in Lebesgue almost every direction.

In a brief, the result is the following: Let \( F \) be the b-adic independent cascade function (see Figure [1] for an illustration and see Section 2.2 for definition). For \( h > 0 \) denote \( S_F(h) = S_F(E_F(h)) \) for \( S \in \{G,R\} \). For \( \theta \in (-\pi/2, \pi/2) \), recall that \( l_\theta \) is the line in \( \mathbb{R}^2 \) passing through the origin and making an angle \( \theta \) with the y-axis, and for any \( y \in l_\theta \), \( l_{y,\theta}^\perp \) is the line perpendicular to \( l_\theta \), passing through \( y \). For \( h > 0 \) define \( R_{F,\theta}(h) = \text{Proj}_{y}(G_F(h)) \), and for each \( y \in R_{F,\theta}(h) \) define \( L_{F,\theta}^y(h) = G_F(h) \cap l_{y,\theta}^\perp \).

We have the following theorem (the assumptions (A1)-(A3) will be given later):
**Theorem 1.2.** Suppose that assumptions (A1)-(A3) hold.

(a) Almost surely for all \( h \in J_F = \{ h > 0 : \tau^*_F(h) > 0 \} \),
\[
\dim_H G_F(h) = \left( \frac{\tau^*_F(h)}{h} \wedge (\tau^*_F(h) + 1 - h) \right) \lor \tau^*_F(h),
\]
\[
\dim_H R_F(h) = \frac{\tau^*_F(h)}{h} \wedge 1.
\]
Moreover, denote \( G_F \) the whole graph, then almost surely
\[
\dim_H G_F = \dim_P G_F = \dim_B G_F = 1 - \tau_F(1).
\]

(b) Almost surely for Lebesgue almost every \( \theta \in (-\pi/2, \pi/2) \), for all \( h \in (0,1) \) such that \( \tau^*_F(h) - h > 0 \), for \( \mu^R_{h,\theta} \) almost every \( y \in R_{F,\theta}(h) \),
\[
\dim_H L^R_{F,\theta}(h) = \tau^*_F(h) - h,
\]
where \( \mu^R_{h,\theta} \) is a positive Borel measure carried by \( R_{F,\theta}(h) \) and it is absolutely continuous with respect to the one-dimensional Lebesgue measure on \( l_\theta \).

---

**Figure 1.** \( F \) in the non-conservative case.
Figure 2. $d^G_F$ (Left) and $d^R_F$ (Right) in case of $\sup J_F < 1$.

Figure 3. $d^G_F$ (Left) and $d^R_F$ (Right) in case of $\sup J_F > 1$.

Figure 4. $d^G_F$ (Left) and $d^R_F$ (Right) in case of $\sup J_F = \infty$. 
2. Coding space. Let $b \geq 2$ be an integer and $\mathcal{A} = \{0, \ldots, b-1\}$ be the alphabet. Let $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ (by convention $\mathcal{A}^0 = \{\emptyset\}$ the set of empty word) and $\mathcal{A}^{N+} = \{0, \ldots, b-1\}^{N+}$.

Denote the length of $w$ by $|w| = n$ if $w \in \mathcal{A}^n$, $n \geq 0$ and $|w| = \infty$ if $w \in \mathcal{A}^{N+}$.

The word obtained by concatenation of $w \in \mathcal{A}^*$ and $t \in \mathcal{A}^* \cup \mathcal{A}^{N+}$ is denoted by $w \cdot t$ and sometimes $w t$.

For every $w \in \mathcal{A}^*$, the cylinder with root $w$, i.e. $\{w \cdot t : t \in \mathcal{A}^{N+}\}$ is denoted by $[w]$. The set $\mathcal{A}^{N+}$ is endowed with the standard metric distance

$$d(s, t) = \inf\{b^{-n} : n \geq 0, \exists w \in \mathcal{A}^n, s, t \in [w]\}.$$

If $n \geq 1$ and $w = w_1 \cdots w_n \in \mathcal{A}^n$ then for every $1 \leq k \leq n$, the word $w_1 \cdots w_k$ is denoted by $w|_k$, and if $k = 0$ then $w|_0$ stands for $\emptyset$. Also, for any infinite word $t = t_1 t_2 \cdots \in \mathcal{A}^{N+}$ and $n \geq 1$, $t|_n$ denotes the word $t_1 \cdots t_n$ and $t|_0$ the empty word. Let

$$\lambda : t \in \mathcal{A}^* \cup \mathcal{A}^{N+} \mapsto \sum_{k=1}^{|t|} t_k \cdot b^{-k}$$

be the canonical projection from $\mathcal{A}^* \cup \mathcal{A}^{N+}$ onto $[0, 1]$. For any $x \in [0, 1]$ and $n \geq 1$, we define $x|_n = x_1 \cdots x_n$ the unique element of $\mathcal{A}^n$ such that $\lambda(x|_n) \leq x < \lambda(x|_n) + b^{-n}$ if $t < 1$, as well as $1|_n = b-1 \cdots b-1$.

2.2. $b$-adic independent cascade function. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space. Let

$$W = (W_0, \cdots, W_{b-1}) : \Omega \mapsto \mathbb{R}^b \quad \text{and} \quad L = (L_0, \cdots, L_{b-1}) : \Omega \mapsto (0, 1)^b$$
be two random vectors such that \( \mathbb{E}(\sum_{j=0}^{b-1} W_j) = \mathbb{E}(\sum_{j=0}^{b-1} L_j) = 1 \). Let
\[
\{(W, L)(w) = (W(w), L(w)) : w \in \mathcal{A}^* \}
\]
be a family of independent copies of \((W, L)\).

For any \( w \in \mathcal{A}^* \), \( u = u_1 \cdots u_n \in \mathcal{A}^n \) and \( n \geq 1 \) we define the products:
\[
\begin{align*}
W_u(w) &= W_{u_1}(w) \cdot W_{u_2}(w \cdot u_1) \cdots W_{u_n}(w \cdot u_1 \cdots u_{n-1}); \\
L_u(w) &= L_{u_1}(w) \cdot L_{u_2}(w \cdot u_1) \cdots L_{u_n}(w \cdot u_1 \cdots u_{n-1}).
\end{align*}
\]

For \( w \in \mathcal{A}^* \), \( n \geq 1 \) we define two continuous functions
\[
F_{W,n}^u(t) = \int_0^t b^n \cdot W_{u|n}(w) \, du \quad \text{and} \quad F_{L,n}^u(t) = \int_0^t b^n \cdot L_{u|n}(w) \, du, \quad t \in [0, 1].
\]

For \( p \in \mathbb{R} \) and \( U \in \{W, L\} \) let
\[
\varphi_U(p) = -\log_b \mathbb{E}(\sum_{j=0}^{b-1} 1_{\{U_j \neq 0\}} |U_j|^p).
\]

From 28, 17, 4 we know that if the following assumption holds:

(A1) There exists \( p > 1 \) such that \( \varphi_W(p) > 0 \), with \( p \in (1, 2] \) if \( \mathbb{P}(\sum_{j=0}^{b-1} W_j = 1) < 1 \). Moreover, \( \mathbb{E}(\sum_{j=0}^{b-1} L_j \log L_j) < 0 \),

then for \( U \in \{W, L\} \) and for any \( w \in \mathcal{A}^* \), \( F_{U,n}^w \) converges uniformly, almost surely and in \( L^p \) norm for \( p > 1 \) such that \( \varphi_U(p) > 0 \), as \( n \) tends to \( \infty \). Moreover, the limit of \( F_{L,n}^w \) is almost surely increasing.

We denote by \( F_W \) and \( F_L \) the limits of \( F_{W,n}^w \) and \( F_{L,n}^w \). Then, the \( b \)-adic independent cascade function considered in this paper is
\[
F = F_W \circ F_L^{-1} : [0, F_L(1)] \mapsto \mathbb{R}.
\]

We say that we are in the conservative case if \( \mathbb{P}(\sum_{j=0}^{b-1} W_j = 1) = 1 \), and in the non-conservative case otherwise.

**Remark 2.1.**

(a) If we set \( L = (b^{-1}, \cdots, b^{-1}) \) and let the entries of \( W \) only take positive values then \( F \) becomes the indefinite integral of the multiplicative cascades measure \( \mu_W \) constructed in 35, 28. If \( W \) is also deterministic, then \( \mu_W \) is nothing but the multinomial measure on \([0, 1]\) associated with the probability vector \( W \).

(b) If \((W, L)\) is a deterministic pair, then we are in the conservative case and \( F \) is the self-affine function studied by Bedford 7 and Kono 30, whose multifractal analysis is a consequence of the study of the digit frequency by Besicovitch 12 and Eggleston 18. The graph and range singularity spectra in this case are still unknown. Even for the dimension of the whole graph, to our best knowledge, there are only results for the box-counting dimension 7 and in some special cases the Hausdorff dimension 8, 21, 49. When \( W \) is
Proposition 2.1. We have the Hausdorff dimension is defined as \( \dim_A \) on the coding space with the construction of an uncountable family of statistically self-similar measures \( \mu_q \) defined on the coding space \( \mathcal{A}^N \) with desired Hausdorff dimension. More precisely, for \( (q, t) \in \mathbb{R}^2 \) we define

\[
\Phi(q, t) = E\left( \sum_{j=0}^{b-1} 1_{\{W_j \neq 0\}} |W_j|^q \cdot L_j^{-t} \right).
\]

Clearly \( \Phi(q, t) \) is analytic on the rectangle \( \{(q, t) \in \mathbb{R}^2 : |\Phi(q, t)| < \infty \} \). Since \( L_j \in (0, 1) \) for \( 0 \leq j \leq b - 1 \), for each \( q \in \tilde{J} := \{ q \in \mathbb{R} : (q, t) \in \mathbb{R}^2, |\Phi(q, t)| < \infty \} \) there is a unique \( \tau(q) \) such that \( \Phi(q, \tau(q)) = 1 \), and the function \( \tau \) is easily seen to be concave and analytic over \( \tilde{J} \).

Define the interval \( J = \{ q \in \tilde{J} : q\tau'(q) - \tau(q) > 0 \} \).

For any \( q \in J \) and \( w \in \mathcal{A}^* \) define the random vector

\[
W_q(w) = (W_{q,j}(w) = 1_{\{W_{j}(w) \neq 0\}} |W_j(w)|^q \cdot L_j(w)^{-\tau(q)})_{0 \leq j \leq b - 1}.
\]

Due to (5) and (6), for \( w \in \mathcal{A}^* \), \( u \in \mathcal{A}^n \), \( n \geq 1 \) we can define the product

\[
W_{q,u}(w) = 1_{\{W_u(w) \neq 0\}} |W_u(w)|^q \cdot L_u(w)^{-\tau(q)}.
\]

For \( q \in J \), \( w \in \mathcal{A}^* \) and \( n \geq 1 \) define

\[
Y_{q,n}(w) = \sum_{u \in \mathcal{A}^n} W_{q,u}(w).
\]

For \( q \in J \), let

\[
\xi(q) = -\frac{\partial}{\partial q} \Phi(q, \tau(q)) \quad \text{and} \quad \tilde{\xi}(q) = \frac{\partial}{\partial t} \Phi(q, \tau(q)).
\]

By construction \( \tau'(q) = \xi(q)/\tilde{\xi}(q) \). We present our second assumption:

(A2) There exists \( q < 0 \) such that \( \varphi_W(q) > -\infty \); \( \varphi_L \) is finite on \( \mathbb{R} \). Moreover, \( \mathbb{P}(\sum_{j=0}^{b-1} 1_{\{W_j \neq 0\}} \geq 2) = 1 \).

Recall that if \( \mu \) is a positive Borel measure on a compact metric space, its lower Hausdorff dimension is defined as \( \dim_H(\mu) = \inf \{ \dim_H E : \mu(E) > 0 \} \). From (5) we have

**Proposition 2.1.** Suppose that (A1) and (A2) hold.

(a) With probability 1, for all \( q \in J \) and \( w \in \mathcal{A}^* \), the sequence \( Y_{q,n}(w) \) converges to a positive limit \( Y_q(w) \). For any \( w \in \mathcal{A}^* \setminus \{\emptyset\} \), the function \( J \ni q \mapsto Y_q(w) \) is a copy of \( J \ni q \mapsto Y_q(\emptyset) := Y_q \).
(b) For every compact subset $K$ of $J$ and $w \in \mathcal{A}^*$ define
\begin{equation}
Y_K(w) = \sup_{q \in K} Y_q(w).
\end{equation}

For any $w \in \mathcal{A}^* \setminus \{\emptyset\}$, $Y_K(w)$ is a copy of $Y_K(\emptyset) := Y_K$ and there exists $p_K > 1$ such that $\mathbb{E}(Y_K^p) < \infty$.

(c) With probability 1, for all $q \in J$, the function
\begin{equation}
\mu_q([w]) = W_{q,w}(\emptyset) \cdot Y_q(w), \ w \in \mathcal{A}^*
\end{equation}
defines a Borel measure on $\mathcal{A}^{N+}$ with
\begin{equation}
\dim_H(\mu_q) = \frac{\gamma(q)}{\log(b)}, \ \text{where} \ \gamma(q) = q\xi(q) - \tau(q)\tilde{\xi}(q).
\end{equation}

These statistically self-similar measures are the effective tools to study the multifractal behavior of $F = F_L \circ F_W^{-1}$. In fact, with each $\mu_q$ we can induce a measure $\mu^D_q$ on the domain $[0, F_L(1)]$: for any Borel set $A \subset \mathbb{R}$,
\begin{equation}
\mu^D_q(A) = \mu_q\left(\left\{t \in \mathcal{A}^{N+} : F_L \circ \lambda(t) \in A\right\}\right).
\end{equation}
It is proved in [3] that, with probability 1, for all $q \in J$, the measure $\mu^D_q$ is carried by the set $E_F(\tau'(q))$ and for $\mu^D_q$-almost every $x \in E_F(\tau'(q))$,
\begin{equation}
h_{\mu^D_q}(x) = \liminf_{r \to 0^+} \frac{\log \mu^D_q(B(x,r))}{\log r} = q\tau'(q) - \tau(q) = \tau^*(\tau'(q)).
\end{equation}
Consequently, $\dim_H E_F(\tau'(q)) \geq \dim_H(\mu^D_q) = \tau^*(\tau'(q))$. This is used to obtain the following result in [3].

**Proposition 2.2.** Suppose that (A1) and (A2) hold.

(a) With probability 1, $\tau_F = \tau$ on the interval $J$. Moreover, if $\varphi_W$ is finite on $\mathbb{R}$, then if $\overline{q} = \sup J < \infty$ (resp. $q := \inf J > -\infty$) we have $\tau_F(q) = \tau'(\overline{q})q$ (resp. $\tau'(q)q$) over $[\overline{q}, \infty)$ (resp. $(-\infty, q]$).

(b) With probability 1, $d_F = \tau^*_F = \tau^*$ on the interval $\tau'(J)$.

3. **Proof of Theorem 1.2**

3.1. **Proof of Theorem 1.2(a).** Since using $\mu_q$ is successful to describe the singularity spectrum of $F$, it is also worth trying to use these measures to study the graph and range singularity spectra of $F$.

For each $q \in J$, like $\mu^D_q$, associated with $F_L$ and $F_W$ we can induce:

- a measure $\mu^G_q$ carried by the graph: for any Borel set $A \subset \mathbb{R}^2$,
  \begin{equation}
  \mu^G_q(A) = \mu_q\left(\left\{t \in \mathcal{A}^{N+} : (F_L \circ \lambda(t), F_W \circ \lambda(t)) \in A\right\}\right);
  \end{equation}

- a measure $\mu^R_q$ carried by the range: for any Borel set $A \subset \mathbb{R}$,
  \begin{equation}
  \mu^R_q(A) = \mu_q\left(\left\{t \in \mathcal{A}^{N+} : F_W \circ \lambda(t) \in A\right\}\right).\end{equation}
We focus on the lower Hausdorff dimension of the measures $\mu_q^G$ and $\mu_q^R$. We show that these dimensions provide the graph and range singularity spectra. Our approach is based on the estimation of the energy of these two measures restricted on suitable random sets (see Remark 3.3). Before presenting the results, let us introduce our third assumption:

(A3) $P(\sum_{j=0}^{b-1} W_j = 1) < 1$, $P(\forall j, |W_j| > 0) = 1$, and $\varphi_W$ is finite on $\mathbb{R}$.

Remark 3.1. The first condition in (A3) ensures that $F$ has enough randomness: We have to avoid the conservative case where $P(\sum_{j=0}^{b-1} W_j = 1) = 1$. The second condition $P(\forall j, |W_j| > 0) = 1$ implies that almost surely $F$ is nowhere locally constant. The last condition ensures that the probability distribution of $F_W(1)$ has a bounded density, in fact this condition can be weaken to the existence of a real number $q < -1$ such that $\varphi_W(q) > -\infty$. The existence of the bounded density of $F_W(1)$ is a key property in the proof.

As an essential intermediate result, we have the following theorem:

Theorem 3.1. Suppose that (A1)-(A3) hold. With probability 1, for all $q \in J$ we have $\dim_H(\mu_q^G) = \gamma^G(q)$ and $\dim_H(\mu_q^R) = \gamma^R(q)$, where

\begin{align}
\gamma^G(q) &= \left(\frac{\tau^*(\tau'(q))}{\tau'(q)} \wedge (\tau^*(\tau'(q)) + 1 - \tau'(q))\right) \vee \tau^*(\tau'(q)), \\
\gamma^R(q) &= \frac{\tau^*(\tau'(q))}{\tau'(q)} \wedge 1.
\end{align}

Remark 3.2. Notice that for $q \in J$ we have $\dim_H(\mu_q^D) = \tau^*(\tau'(q))$, and we can write

$$\gamma^G(q) = \tau^*(\tau'(q)) + \gamma^R(q) \cdot (1 - \tau'(q)) \vee 0.$$ 

So Theorem 3.1 actually provides us with a Ledrappier-Young like formula [32] for the uncountable family of statistically self-similar measure $(\mu_q)_{q \in J}$ uniformly: with probability 1, for all $q \in J$,

$$\dim_H(\mu_q^G) = \dim_H(\mu_q^D) + \dim_H(\mu_q^R) \cdot (1 - \tau'(q)) \vee 0.$$ 

Similar formula also appear in Theorem 12 in [8], Theorem 3 in [21] and Corollary 5.2 in [6] in the study of the Hausdorff dimension of self-affine measures and sets.

Remark 3.3. It is worth noting that to prove Theorem 3.1 we are forced to calculate the energy of $\mu_q$ restricted to suitable Cantor-like random sets (see Section 5.1). If we did not use this restriction, for example for the measure $\mu_q^G$ in the case where its dimension is greater than 1, we would have to estimate the expectation of

$$\int_{s,t \in \mathbb{R}^+} \frac{d\mu_q(s)d\mu_q(t)}{|F_L \circ \lambda(s) - F_L \circ \lambda(t)|^2 + |F_W \circ \lambda(s) - F_W \circ \lambda(t)|^2}^\gamma/2, \quad \gamma > 1,$$

which turns out to be finite only if $\Phi(2q - 1, \gamma - 1 + 2\tau(q)) < 1$, which is equivalent to saying that $\gamma < 1 + \tau(2q - 1) - 2\tau(q)$. So the best lower bound we would get is:

$$\dim_H(\mu_q^G) \geq 1 + \tau(2q - 1) - 2\tau(q).$$
Comparing this value with the exact dimension $1 + (q - 1)\tau'(q) - \tau(q)$, we find that (10) always provides a strict lower bound unless $q = 1$. Thus, such an approach only provides the Hausdorff dimension of the whole graph.

Since $\mu_q^D$ is carried by the set $E_F(\tau'(q))$, by definition the measure $\mu_q^S$ is carried by the set $S_F(E_F(\tau'(q)))$ for $S \in \{G, R\}$. Then combining the results in Proposition 2.1, Proposition 2.2 and Theorem 3.1 we prove the results on the singularity spectra part of Theorem 1.2(a).

For the result on the dimension of the whole graph, let $I = [0, F_L(1)]$. For each $n \geq 1$ we divide $I$ into $b^n$ semi-open to the right intervals of the same length denoted by $I_{n,k}$, for $k = 1, \cdots, b^n$.

Recall that $\text{Osc}_F(I_{n,k}) = \sup_{x,y \in I_{n,k}} |F(x) - F(y)|$, so for each interval $I_{n,k}$ we will need at most $\left\lceil \frac{\text{Osc}_F(I_{n,k})}{|I_{n,k}|} \right\rceil + 1$ many squares whose side length is $|I_{n,k}|$ to cover $G_F(I_{n,k})$. Then, by definition of the upper box-counting dimension and the definition of $\tau_F$ in (3) we get

$$\dim_B G_F \leq \limsup_{n \to \infty} \frac{\log \sum_{j=1}^{b^n} \left(\left\lceil \frac{\text{Osc}_F(I_{n,k})}{|I_{n,k}|} \right\rceil + 1\right)}{-\log(b^{-n} \cdot F_L(1))} \leq 1 + (-\tau_F(1)) \vee 0.$$

From Proposition 2.2 we know that almost surely $\tau_F(1) = \tau(1) \leq 0$ and applying Theorem 3.1 to $q = 1$ we get with probability 1,

$$\dim_H G_F \geq \dim_H G_F(\tau_F'(1)) = 1 - \tau_F(1).$$

Consequently, with probability 1,

$$\dim_H G_F = \dim_P G_F = \dim_B G_F = 1 - \tau_F(1).$$

Remark 3.4. If we consider the exponent $\tilde{\tau}_F(x)$ defined in (4) and consider the following smaller iso-Hölder sets

$$\tilde{E}_F(h) := \{x \in I : \tilde{\tau}_F(x) = h\},$$

then we claim that the results in Theorem 1.2(a) also works for the packing dimension if we replace the mono-Hölder set $E_F(h)$ by the set $\tilde{E}_F(h)$, since in this case we have $\dim_P \tilde{E}_F(h) \leq \tau_F^*(h)$ for $h > 0$ and, moreover, under the assumption $\mathbb{P}(\forall j, |W_j| > 0) = 1$ in (A3), the measure $\mu_q^D$ is actually carried by the set $\tilde{E}_F(\tau'(q))$ for $q \in J$ (see [5]).

3.2. Proof of Theorem 1.2(b). To get measures on the level sets, in the same spirit as when one constructs the local times of certain stochastic processes, we could disintegrates the measures $\mu_q^G$ with respect to $\mu_q^R$ in order to obtain Radon measures $\mu_q^g$ carried by $L_y^q$ for $\mu_q^R$-almost every $y$, but such a disintegration turns out to be difficult to study. The reason is that the energy method we use does not provide the exact gauge function needed to describe the density of the measure $\mu_q^R$ with respect to Lebesgue. It only yields the lower Hausdorff dimension of $\mu_q^R$. However, inspired by what is done in [12] to calculate the Hausdorff dimension of the level sets of
Gaussian process by using classical Marstrand theorem, and in \(38\) to deal with the
the Hausdorff dimension of slices of sets, it is possible to solve this problem for for
Lebesgue almost every direction.

For \(h > 0\) and \(\theta \in (-\pi/2, \pi/2)\), recall that \(R_{F,\theta}(h) = \text{Proj}_{\theta}(G_{F}(h))\), and for each
\(y \in R_{F,\theta}(h)\) recall that \(L_{F,\theta}^{R}(h) = G_{F}(h) \cap l_{y,\theta}^{R}\).

For \(q \in J\), let \(\mu_{q,\theta}^{R}\) be the orthogonal projection of the measure \(\mu_{q}^{G}\) onto \(l_{\theta}\):
\(\mu_{q,\theta}^{R}(A) = \mu_{q}^{G} \circ \text{Proj}_{\theta}^{-1}(A)\) for any Borel set \(A \subset l_{\theta}\). Since \(\mu_{q}^{G}\) is carried by \(G_{F}(\tau'(q))\),
so \(\mu_{q,\theta}^{R}\) is carried by \(R_{F,\theta}(\tau'(q))\).

We have the following theorem:

**Theorem 3.2.** Suppose that (A1)-(A3) hold. With probability 1, for Lebesgue almost
every \(\theta \in (-\pi/2, \pi/2)\), for all \(q \in J\) such that \(\dim_{H}(\mu_{q}^{G}) = \gamma^{G}(q) > 1:\)

(a) The projected measure \(\mu_{q,\theta}^{R}\) is absolutely continuous with respect to the one-
dimensional Lebesgue measure on the \(l_{\theta}\).

(b) For \(\mu_{q,\theta}^{R}\)-almost every \(y \in l_{\theta}\), the following limit:
\[
\lim_{r \to 0^{+}} \frac{1}{r} \int_{x \in \mathbb{R}^{2}, |x - l_{y,\theta}| \leq r} \psi(x) \, d\mu_{q}^{G}(x)
\]
exists for any continuous function \(\psi : \mathbb{R}^{2} \to \mathbb{R}_{+}\), so it defines a measure \(\mu_{q,\theta}^{y}\)
carried by \(L_{F,\theta}^{R}(\tau'(q))\).

(c) There exists a random set \(R_{q,\theta} \subset R_{F,\theta}(\tau'(q))\) of full \(\mu_{q,\theta}^{R}\)-measure such that
for any \(y \in R_{q,\theta}\), the measure \(\mu_{q,\theta}^{y}\) has lower Hausdorff dimension
\[
\dim_{H}(\mu_{q,\theta}^{y}) = \dim_{H}(\mu_{q}^{G}) - 1 = \tau^{*}(\tau'(q)) - \tau'(q).
\]

Theorem 1.2(b) is almost a direct consequence of Theorem 1.1(b) and Theorem
3.2, we only remark that \(\gamma^{G}(q) > 1\) if and only if \(\tau'(q) < 1\) and \(\tau^{*}(\tau'(q)) - \tau'(q) > 0\).

**Remark 3.5.** As mentioned in Remark 3.7, the condition “\(\varphi_{W}\) is finite on \(\mathbb{R}\)” can
be weakened to “there exists \(q < -1\) such that \(\varphi_{W}(q)\) is finite”. Under this weaker
assumption, the results in Theorem 1.2 will still hold, but only for \(h \in \{h > 0 : \tau^{*}(h) > 0\}\). The reason why we cannot conclude for \(h > 0 : \tau^{*}(h) > 0\) is that
under this weaker assumption we do not know the value of \(\tau_{F}\) outside the interval
\(J = \{q : q\tau'(q) - \tau(q) > 0\}\). But if we assume “there exist \(q < \overline{q} \in J\) such that
\(q\tau'(q) - \tau(q) = \overline{q}(\tau'(q) - \tau(\overline{q}) = 0\)” then we will also obtain the same results as in
Theorem 1.2.

4. Proof of Theorem 1.1

4.1. Results on the packing dimension.

**Proof.** Without loss of generality we suppose \(I = [0, 1]\). For any \(x \in [0, 1]\) and \(n \geq 1\),
we denote by \(I_{x,n}\) the unique interval \([k \cdot 2^{-n}, (k + 1) \cdot 2^{-n})\), \(k \in \mathbb{Z}\) that contains \(x\).
Let \( \{E_i \subset E, i \geq 1\} \) be any countable covering of \( E \). The families \( \{R_i = R_f(E_i) \subset R_f(E), i \geq 1\} \) and \( \{G_i = G_f(E_i) \subset G_f(E, i \geq 1\} \) are countable coverings of \( R_f(E) \) and \( G_f(E) \) respectively, so
\[
\dim_P R_f(E) = \sup_i \dim_P R_i, \quad \dim_P G_f(E) = \sup_i \dim_P G_i.
\]

For any \( i \geq 1, k \geq 0 \) and \( \epsilon > 0 \) with \( h - \epsilon > 0 \) we define
\[
E_{i,k} = \{x \in E_i : \forall n \geq k, O_f(I_{x,n}) \leq |I_{x,n}|^{h-\epsilon}\},
\]
as well as \( R_{i,k} = R_f(E_{i,k}) \) and \( G_{i,k} = G_f(E_{i,k}) \). By the fact that for any \( x \in E_i \subset E \),
\[
\lim_{n \to \infty} \inf \frac{\log O_f(I_{x,n})}{\log |I_{x,n}|} \geq h_f(x) \geq h,
\]
we have \( \bigcup_{k \geq 0} E_{i,k} = E_i \), thus \( \bigcup_{k \geq 0} R_{i,k} = R_i \) and \( \bigcup_{k \geq 0} G_{i,k} = G_i \).

For \( n \geq 1 \) and \( A \subset \mathbb{R}^2 \), let \( N_n(A) \) be the minimal number of dyadic squares of the form \([k \cdot 2^{-n}, (k+1) \cdot 2^{-n}) \times [l \cdot 2^{-n}, (l+1) \cdot 2^{-n})\), \( 0 \leq k < 2^n - 1, l \in \mathbb{Z} \) necessary to cover \( A \). By the construction of \( E_{i,k} \), for any \( n \geq k \) we have
\[
N_n(R_{i,k}) \leq 2 \cdot N_{\left\lceil \frac{n}{h} \right\rceil + 1}(E_{i,k}), \quad N_n(G_{i,k}) \leq \left\{ \begin{array}{ll}
2^{n-((1-h+\epsilon)\vee 0)} \cdot N_n(E_{i,k})
& \text{if } (1-h+\epsilon) \vee 0 \leq n/2^n - 1 \leq 1 \leq \left\lceil \frac{n}{h} \right\rceil + 1,
2 \cdot N_{\left\lceil \frac{n}{h} \right\rceil + 1}(E_{i,k})
& \text{otherwise}.
\end{array} \right.
\]
This gives us
\[
\overline{\dim}_B R_{i,k} \leq \frac{1}{h - \epsilon} \overline{\dim}_B E_{i,k}, \quad \overline{\dim}_B G_{i,k} \leq \left\{ \begin{array}{ll}
(1 - h + \epsilon) \vee 0 + \overline{\dim}_B E_{i,k}
& \text{if } (1-h+\epsilon) \vee 0 \leq \overline{\dim}_B E_{i,k}/((h - \epsilon) \wedge 1).
\end{array} \right.
\]

We know that the packing dimension is equal to the upper modified box-counting dimension (see Chapter 3, [19]), that is, for any set \( A \subset \mathbb{R}^2 \),
\[
\dim_P A = \overline{\dim}_MB A := \inf \left\{ \sup_i \overline{\dim}_B A_i : A \subset \bigcup_i A_i \right\},
\]
where the infimum is taken over all possible countable coverings of \( A \). Since \( \{R_{i,k}, k \geq 0\} \) and \( \{G_{i,k}, k \geq 0\} \) are countable coverings of \( R_i \) and \( G_i \), we have
\[
\dim_P R_i = \overline{\dim}_MB R_i \leq \sup_k \overline{\dim}_B R_{i,k} \leq \frac{1}{h - \epsilon} \sup_k \overline{\dim}_B E_{i,k};
\]
\[
\dim_P G_i = \overline{\dim}_MB G_i \leq \sup_k \overline{\dim}_B G_{i,k} \leq \left\{ \begin{array}{ll}
(1 - h + \epsilon) \vee 0 + \sup_k \overline{\dim}_B E_{i,k}
& \text{if } (1-h+\epsilon) \vee 0 \leq \sup_k \overline{\dim}_B E_{i,k}/((h - \epsilon) \wedge 1).
\end{array} \right.
\]

Since for any \( k \geq 0 \) we have \( E_{i,k} \subset E_i \) so that \( \sup_k \overline{\dim}_B E_{i,k} \leq \overline{\dim}_B E_i \). Then we have shown that for any countable covering \( \{E_i, i \geq 1\} \),
\[
\dim_P R_f(E) = \sup_i \dim_P R_i \leq \frac{1}{h - \epsilon} \sup_i \overline{\dim}_B E_i;
\]
\[
\dim_P G_f(E) = \sup_i \dim_P G_i \leq \left\{ \begin{array}{ll}
(1 - h + \epsilon) \vee 0 + \sup_i \overline{\dim}_B E_i
& \text{if } (1-h+\epsilon) \vee 0 \leq \overline{\dim}_B E_i/((h - \epsilon) \wedge 1).
\end{array} \right.
\]
By taking the infimum over all the possible \( \{E_i, i \geq 1\} \) we get

\[
\dim P R_f(E) \leq \frac{1}{h - \epsilon} \dim P E, \quad \dim P G_f(E) \leq \left\{ \begin{array}{ll}
(1 - h + \epsilon) \vee 0 + \dim P E; & \\
\dim P E / ((h - \epsilon) \wedge 1). & 
\end{array} \right.
\]

Letting \( \epsilon \) tend to 0 yields the conclusion. \( \square \)

4.2. Results on the Hausdorff dimension.

Proof. We fix \( \epsilon \in (0, h) \) and \( \theta \in (-\pi/2, \pi/2) \). Recall that \( \mbox{Proj}_\theta \) is the orthogonal projection of \( \mathbb{R}^2 \) onto \( l_\theta \) and \( R_{f,\theta}(E) = \mbox{Proj}_\theta(G_f(E)) \).

For \( x \in \mathbb{R}, r > 0 \) and \( k \in \mathbb{Z} \) we define the interval \( I_k(x, r) = [x + (2k - 1)r, x + (2k + 1)r] \) and the square \( Q_k(x, r) = [x - r, x + r] \times I_k(f(x), r) \).

For any \( r > 0 \) let \( n(r) = [r^{h-1}] + 1 \) and \( \mathcal{N}(r) = \{ k \in \mathbb{Z} : 2|k| \leq [r^{h-1}] \} \). We have \( \max\{r, r^{h-1}\} \leq n(r) \cdot r \) and \( \#\mathcal{N}(r) \leq n(r) \).

For any \( x \in E \) and \( r > 0 \) define the family \( \mathcal{Q}(x, r) = \{Q_k(x, r) : k \in \mathcal{N}(r)\} \) and for any \( y \in R_{f,\theta}(E) \) define the family

\[
\mathcal{Q}_\theta(x, r) = \left\{ Q_k(x, r) : k \in \mathcal{N}(r), \ Q_k(x, r) \cap l_{y,\theta}^\perp \neq \emptyset \right\}.
\]

By simple calculation we know \( \#\mathcal{Q}_\theta(x, r) \leq |\tan \theta| + 1 \).

Let \( d \) stand for \( \dim_H E \). By definition of the Hausdorff dimension we can find a decreasing sequence \( (\delta_i)_{i \geq 1} \) tending to 0 and for each \( i \geq 1 \),

\[
\mathcal{B}_i := \left\{ B_j^{(i)} = B \left( x_j^{(i)}, r_j^{(i)} \right) \right\}_{j \in \mathcal{J}_i},
\]

a countable \( \delta_i \)-covering of \( E \) such that \( x_j^{(i)} \in E \) for \( j \in \mathcal{J}_i \), and \( \sum_{j \in \mathcal{J}_i} (r_j^{(i)})^{d+\epsilon} \leq 2^{-i} \).

For \( x \in E \) and \( r > 0 \) denote the rectangle:

\[
R(x, r) = B(x, r) \times [f(x) - n(r) \cdot r, f(x) + n(r) \cdot r],
\]

and for any \( B_j^{(i)} = B \left( x_j^{(i)}, r_j^{(i)} \right) \in \mathcal{B}_i \), we use the convention \( R_j^{(i)} = R \left( x_j^{(i)}, r_j^{(i)} \right) \).

Let \( \mu \) be a positive Borel measure defined on \( l_\theta \). For \( \gamma > 0 \) recall that

\[
R_{f,\theta}^{\mu,\gamma}(E) = \{ y \in R_{f,\theta}(E) : h_{\mu}(y) \geq \gamma \}.
\]

Suppose that \( \mu(R_{f,\theta}^{\mu,\gamma}(E)) > 0 \). Then define a subset of \( \mathcal{J}_i \):

\[
\mathcal{J}_{i,\theta}^{\mu,\gamma} = \left\{ j \in \mathcal{J}_i : \mu \left( \mbox{Proj}_\theta(R_j^{(i)}) \right) \leq \left( (2\pi r_j^{(i)} \cdot \cos \theta + |\sin \theta|) \cdot r_j^{(i)} \right)^{\gamma - \epsilon} \right\}.
\]

We have the following lemma:

**Lemma 4.1.** For any \( N \geq 1 \), let \( \mathcal{C}_N^R = \bigcup_{i \geq N} \bigcup_{j \in \mathcal{J}_i} \left\{ I_0(f(x_j^{(i)}), (r_j^{(i)})^{h-\epsilon}) \right\} \),

\[
\mathcal{C}_N^G = \bigcup_{i \geq N} \bigcup_{j \in \mathcal{J}_i} Q(x_j^{(i)}, r_j^{(i)}), \quad \mathcal{C}_N^G = \bigcup_{i \geq N} \bigcup_{j \in \mathcal{J}_i} \left\{ Q_0(x_j^{(i)}, n(r_j^{(i)}) \cdot r_j^{(i)}) \right\},
\]

and for any \( y \in R_{f,\theta}^{\mu,\gamma}(E) \) let \( \mathcal{C}_{\theta,N}^y = \bigcup_{i \geq N} \bigcup_{j \in \mathcal{J}_{i,\theta}^{\mu,\gamma}} Q_j^{(i)}(x_j^{(i)}, r_j^{(i)}). \)
Then $C_N^R$, $C_N^G$, $\widetilde{C}_N^G$ and $C_N^{\theta,N}$ form respectively a $(\delta_N)^{h-\epsilon}$-covering of $R_f(E)$, a $\delta_N$-covering of $G_f(E)$, an $n(\delta_N) \cdot \delta_N$-covering of $G_f(E)$, and a $\delta_N$-covering of $L_{f,\theta}^u(E)$.

**Proof.** Fix $N \geq 1$. For any $x \in E$, since for any $i \geq N$ there are balls in $B_i$ covering $x$ and $\delta_i \searrow 0$, we can find a sequence of balls \( \{B_i = B(x_i, r_i)\}_{i \geq 1} \subset \bigcup_{i \geq N} B_i \) such that $x \in B_i$ for all $i \geq 1$ and $r_i \searrow 0$ as $l \to \infty$. For each $l \geq 1$, let $\tilde{r}_i = |x - x_i| \leq r_i$ and $\tilde{r}_i = (2n(r_i) \cos \theta + |\sin \theta|) \cdot r_i$. Since

\[
\liminf_{l \to \infty} \frac{\log O_f(B(x, \tilde{r}_i))}{\log \tilde{r}_i} \geq h_f(x) \geq h \quad \text{and} \quad \liminf_{l \to \infty} \frac{\log \mu(l_\theta \cap B(\Proj f(x), \tilde{r}_i))}{\log \tilde{r}_i} \geq h(\Proj f(x)) \geq \gamma,
\]

if $\Proj f(x) \in R_{f,\theta}^{\gamma}(E)$, so we can find $l_*$ (depending on $x$) such that for all $l \geq l*$,

\[
O_f(B(x, \tilde{r}_i)) \leq (\tilde{r}_i)^{h-\epsilon} \quad \text{and} \quad \mu(l_\theta \cap B(\Proj f(x), \tilde{r}_i)) \leq (\tilde{r}_i)^{\gamma-\epsilon}.
\]

Now for any $l \geq l_*$ we have

\[
|f(x_i) - f(x)| \leq O_f(B(x, \bar{x}_i)) \leq (\tilde{r}_i)^{h-\epsilon} \leq (r_i)^{h-\epsilon} \leq n(r_i) \cdot r_i.
\]

This implies that:

- $x \in B(x_i, r_i)$;
- $f(x) \in I_0(f(x_i), (r_i)^{h-\epsilon}) \subset I_0(f(x_i), n(r_i) r_i)$;
- $(x, f(x)) \in Q_0(x_i, n(r_i) r_i)$;
- $l_\theta \cap B(\Proj f(x), \tilde{r}_i) \supset \Proj f(R(x_i, r_i))$,

which gives us the conclusion. \( \square \)

Now we are going to show that the coverings constructed in Lemma 4.1 lead to the expected upper bounds. In order to simplify the proof, we use the convention $|Q| = \frac{1}{2} \sup_{x,y \in Q} |x - y|$, the half-diameter of the set $Q$.

(i) Since we took $\epsilon \in (0, h)$, we have

\[
\sum_{Q \in \mathcal{B}_N^R} |Q|^{\frac{d+\epsilon}{h-\epsilon}} = \sum_{i \geq N} \sum_{j \in J_i} ((r_j^{(i)})^{h-\epsilon})^{\frac{d+\epsilon}{h-\epsilon}} = \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^{d+\epsilon} \leq 2^{-N+1}.
\]

(ii) If $h > 1$, if we take $\epsilon$ small enough so that $h - \epsilon > 1$, then $n(r) = 1$ for all $r < 1$, and for $N$ large enough so that $\delta_N < 1$,

\[
\sum_{Q \in \mathcal{B}_N^R} |Q|^{d+\epsilon} = \sum_{i \geq N} \sum_{j \in J_i} (n(r_j^{(i)}) \cdot r_j^{(i)})^{d+\epsilon} = \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^{d+\epsilon} \leq 2^{-N+1}.
\]

(iii) If $h \leq 1$, then $h - \epsilon - 1 \leq -\epsilon < 0$, thus for $r < 1$ we have $r^{h-\epsilon-1} > 1$, and this implies that $n(r) \leq 2 \cdot r^{h-\epsilon-1}$, hence for $N$ large enough so that $\delta_N < 1$,

\[
\sum_{Q \in \mathcal{B}_N^R} |Q|^{\frac{d+\epsilon}{h-\epsilon}} = \sum_{i \geq N} \sum_{j \in J_i} (n(r_j^{(i)}) \cdot r_j^{(i)})^{\frac{d+\epsilon}{h-\epsilon}} \leq 2^{\frac{d+\epsilon}{h-\epsilon}} \cdot \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^{d+\epsilon} \leq 2^{\frac{d+\epsilon}{h-\epsilon}} \cdot 2^{-N+1}.
\]
(iv) If \( h \leq 1 \), for the same reason as (c), for \( N \) large enough such that \( \delta_N < 1 \),
\[
\sum_{Q \in \mathcal{E}} |Q|^{d+1-h+2\epsilon} = \sum_{i \geq N} \sum_{j \in J_i} \sum_{k \in \mathcal{N}(r_j^{(i)})} |Q_k(x_j^{(i)}, r_j^{(i)})|^{d+1-h+2\epsilon} \\
\leq \sum_{i \geq N} \sum_{j \in J_i} n(r_j^{(i)}) \cdot (r_j^{(i)})^{d+1-h+2\epsilon} \leq 2 \cdot \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^{d+\epsilon} \leq 2^{-N+2}.
\]

Now by letting \( N \) tend to infinity and then \( \epsilon \) to 0 we obtain the desired upper bound for \( \dim_H R_f(E) \) and \( \dim_H G_f(E) \).

Next we prove the upper bound for the Hausdorff dimension of the level sets.

If \( s = d + \epsilon - (\gamma - \epsilon)(h - \epsilon) > 0 \) and \( h \leq 1 \), then for any \( N \) large enough we have
\[
\int_{y \in R_f^\mu(\gamma)} \sum_{Q \in \mathcal{E}} |Q|^s d\mu(y) \\
\leq \sum_{i \geq N} \sum_{j \in J_i} \sum_{Q \in \mathcal{Q}_\theta(x_j^{(i)}, r_j^{(i)})} |Q|^s \cdot \mu(\text{Proj}_\theta(Q)) \\
\leq \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^s \cdot (|\tan \theta| + 1) \cdot \mu(\text{Proj}_\theta(R(x_j^{(i)}, r_j^{(i)}))) \\
\quad \left( \text{since } |Q| = r_j^{(i)} \text{ and } \# \mathcal{Q}_\theta(x_j^{(i)}, r_j^{(i)}) \leq |\tan \theta| + 1 \right) \\
\leq (|\tan \theta| + 1) \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^s \cdot \left( (2n(r_j^{(i)}) \cdot \cos \theta + |\sin \theta|) \cdot r_j^{(i)} \right)^{\gamma-\epsilon} \\
\leq C \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^s \cdot \left( (r_j^{(i)})^{h-\epsilon} \cdot r_j^{(i)} \right)^{\gamma-\epsilon} \left( \text{we used } h \leq 1 \text{ and } N \text{ large enough} \right) \\
= C \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^{d+\epsilon} \leq C \sum_{i \geq N} \sum_{j \in J_i} (r_j^{(i)})^{d+\epsilon} \leq 2^{-N+1+2\gamma},
\]

where \( C = 2^{2(\gamma-\epsilon)} (|\tan \theta| + 1) \). Due to Borel-Cantelli lemma we get for \( \mu \)-almost every \( y \in R_f^\mu(\gamma) \),
\[
\dim_H L_f^\mu(\gamma) \leq s = d + \epsilon - (\gamma - \epsilon)(h - \epsilon).
\]

Applying this with a sequence \( (\epsilon_n)_{n \geq 1} \) we get the conclusion. \( \Box \)

5. Proof of Theorem 3.1 and Theorem 3.2

From now on we assume that (A1)-(A3) hold.

5.1. Cantor-like subsets of \( \mathcal{A}^\mathbb{N} + \) carrying \( \mu_\theta \). For any \( w \in \mathcal{A}^* \), define the \( b \)-adic interval \( I_w = \lambda([w]) \). By construction we know that the limit functions \( F_W \)
and $F_L$ satisfy the following functional equation: For any $w \in \mathcal{A}^*$, $x, y \in I_w$ and $U \in \{W, L\}$,

$$F_U(x) - F_U(y) = U_w(\emptyset) \cdot \left( F_U^{[w]}(b^w \cdot (x - \lambda(w))) - F_U^{[w]}(b^w \cdot (y - \lambda(w))) \right).$$

(17) For $w \in \mathcal{A}^*$ and $U \in \{W, L\}$ we define the oscillations $O_U(w) = O_{F_U^{[w]}([0, 1])}$. Then from (17) we get that for any $w \in \mathcal{A}^*$ and $U \in \{W, L\}$,

$$O_{F_U}(I_w) = U_w(\emptyset) \cdot O_U(w).$$

For $w \in \mathcal{A}^*$ denote by $w^-$ (resp. $w^+$) the unique element of $\mathcal{A}^{[w]}$ such that $\lambda(w^-) = \lambda(w) - b^{-|w|}$ (resp. $\lambda(w^+) = \lambda(w) + b^{-|w|}$) whenever $\lambda(w) \neq 0$ (resp. $\lambda(w) \neq 1 - b^{-|w|}$).

Recall (10), the definition of $\xi(q)$ and $\widetilde{\xi}(q)$. For any $q \in J$, $\epsilon > 0$, $u, v \in \mathcal{A}^*$ we define the following subsets of $\Omega$:

$$1_{[u]}^{[w]}(q, \epsilon) = 1_{w^v \in \mathcal{A}^*[u] \cap \mathcal{A}^{[w]}(q, \epsilon) \cap \mathcal{A}^{[w]}(q, \epsilon)}.$$  

(19) and for $q \in J$ and $\epsilon > 0$ we define random subsets of $\mathcal{A}^N$:

$$\mathcal{A}^N_n(q, \epsilon) = \{ t \in \mathcal{A}^N : 1_{[q]}^{[t]}(q, \epsilon) = 1 \} \quad \text{and} \quad \mathcal{A}^N_n(q, \epsilon) = \mathcal{A}^N_n \setminus \mathcal{A}^N_n(q, \epsilon).$$

In [5] we proved the following result:

**Proposition 5.1.** Let $K$ be a compact subset of $J$. Then for any $\epsilon > 0$ there exist constants $C = C(K) > 0$ and $\delta = \delta(\epsilon, K) > 0$ such that for any $n \geq 1$,

$$\mathbb{E} \left( \sup_{q \in K} \sum_{w \in \mathcal{A}^n} \mu_q([w] \cap \mathcal{A}^N_n(q, \epsilon)^c) \right) \leq C \cdot n \cdot b^{-n \delta}.$$  

(21) Now for $n \geq 1$ and $\epsilon > 0$ we define the random Cantor-like sets in $\mathcal{A}^N$.

$$C_n(q, \epsilon) = \bigcap_{q \geq n} \mathcal{A}^N_n(q, \epsilon) \quad \text{and} \quad C(q) = \lim_{\epsilon \to 0} \lim_{n \to \infty} C_n(q, \epsilon).$$

(22) Then we can deduce from Proposition 5.1 that, with probability 1, for all $q \in K$, $\mu_q$ is carried by $C(q)$, that is, $\mu_q(C(q)) = \|\mu_q\| = Y_q > 0$.

It worth noting that by construction, for any $q \in C(q)$, we have

$$\lim_{n \to \infty} \frac{\log W_{t|n}(\emptyset)}{-n} = \lim_{n \to \infty} \frac{\log L_{t|n}(\emptyset)}{-n} = \xi(q),$$

$$\lim_{n \to \infty} \frac{\log W_{t|n}^- (\emptyset)}{-n} = \lim_{n \to \infty} \frac{\log L_{t|n}^- (\emptyset)}{-n} = \lim_{n \to \infty} \frac{\log W_{t|n}^+ (\emptyset)}{-n} = \tilde{\xi}(q),$$

$$\lim_{n \to \infty} \frac{\log W_{t|n}^+ (\emptyset)}{-n} = \lim_{n \to \infty} \frac{\log L_{t|n}^+ (\emptyset)}{-n} = \lim_{n \to \infty} \frac{\log W_{t|n}^- (\emptyset)}{-n} = \tilde{\xi}(q).$$
\[ \lim_{n \to \infty} \frac{\log O_U(t_n)}{-n} = \lim_{n \to \infty} \frac{\log O_U(t_{n-})}{-n} = \lim_{n \to \infty} \frac{\log O_U(t_{n+})}{-n} = 0, \quad U \in \{W, L\}. \]

Moreover, due to (18), the above equalities imply that

\[ \lim_{r \to \infty} \frac{\log \text{Osc}_F(B(\lambda(t), r))}{\log r} = \frac{\xi(q)}{\tilde{\xi}(q)} = \tau'(q). \]

### 5.2. Proof of Theorem 3.1

**Proof.** From now on, for \( U \in \{W, L\} \), \( w \in \mathcal{A}^* \) and \( t \in \mathcal{A}^* \cup \mathcal{A}^N + \), we will use the convention \( F_U[w] \circ \lambda(t) = F_U[w](t) \).

For any \( s, t \in \mathcal{A}^* \cup \mathcal{A}^N + \) and \( \gamma > 0 \) we define the Riesz-like kernels:

\[
K_\gamma(s, t) = \begin{cases} 
(\|F_L(s) - F_L(t)\|^2 + |F_W(s) - F_W(t)|^2)^{-\gamma/2} & \text{if } \gamma \geq 1; \\
|F_W(s) - F_W(t)|^{-\gamma} & \text{if } \gamma < 1.
\end{cases}
\]

Recall the definitions of \( \gamma^G(q) \) and \( \gamma^R(q) \) in (14) and (15) (see also Remark 5.1). For \( q \in J, S \in \{G, R\} \) and \( \delta > 0 \) we will use the notation

\[
K_{q,\delta}^S(s, t) = K_{\gamma^S(q) - \delta}(s, t).
\]

Recall the definition of \( C_n(q, \epsilon) \) in (22). For \( q \in J, \epsilon > 0 \) and \( \delta > 0 \) define the \( n \)-th energy for \( n \geq 1 \) and \( S \in \{G, R\} \):

\[
I_{n,\delta}^S(q, \epsilon) = \int \int_{s, t \in C_n(q, \epsilon), s \neq t} K_{q,\delta}^S(s, t) \, d\mu_q(s) \, d\mu_q(t).
\]

Let \( K \) be any compact subset of \( J \). We assume for a while that we have proved that there exists \( \delta_K > 0 \) such that for any \( \delta \in (0, \delta_K) \), there exists \( \epsilon_\delta > 0 \) such that for any \( n \geq 1, \epsilon \in (0, \epsilon_\delta) \) and \( S \in \{G, R\} \),

\[
E \left( \sup_{q \in K} I_{n,\delta}^S(q, \epsilon) \right) < \infty.
\]

The following lemma is a slight modification of Theorem 4.13 in [19] regarding the Hausdorff dimension estimate through the potential theoretic method.

**Lemma 5.1.** Let \( \mu \) be a Borel measure on \( \mathbb{R}^m \) and let \( E \subset \mathbb{R}^m \) be a Borel set such that \( \mu(E) > 0 \). For any \( \gamma > 0 \), if

\[
\int \int_{x, y \in E, x \neq y} |x - y|^{-\gamma} \, d\mu(x) \, d\mu(y) < \infty,
\]

then

\[
\mu \left( \left\{ x \in E : h_\mu(x) = \liminf_{r \to 0^+} \frac{\log \mu(B(x, r))}{\log r} < \gamma \right\} \right) = 0.
\]

Then, it easily follows from Proposition 5.1 (25) and Lemma 5.1 that, with probability 1, for all \( q \in K \):
for $\mu^G_q$-almost every $x \in G_q := \{(F_L(t), F_W(t)) : t \in \mathcal{C}(q)\} \subset G_F(\tau'(q))$,  

\[ h_{\mu^G_q}(x) = \liminf_{r \to 0^+} \frac{1}{\log r} \log \mu^G_q(B(x, r)) \geq \gamma^G(q) - \delta; \]

for $\mu^R_q$-almost every $y \in R_q := \{F_W(t) : t \in \mathcal{C}(q)\} \subset R_F(\tau'(q))$,  

\[ h_{\mu^R_q}(x) = \liminf_{r \to 0^+} \frac{1}{\log r} \log \mu^R_q(B(y, r)) \geq \gamma^R(q) - \delta. \]

We can consider a countable sequence of compact subintervals $K_n \subset J$ such that $\bigcup K_n = J$ and a corresponding sequence $\delta_n \in (0, \delta_K)$. Then the above facts imply that with probability 1, for any $q \in J$ and $S \in \{G, R\}$, for $\mu^S_q$-almost every $x \in S_q$, $h_{\mu^S_q}(x) \geq \gamma^S(q)$, hence $\dim_H(\mu^S_q) \geq \gamma^S(q)$ (we use the mass distribution principle, see [19]).

To complete the proof, we use the fact that, with probability 1, for all $q \in J$, $\mu^D_q$ is carried by the set $E_F(\tau'(q))$. Then, applying Theorem [11] to any set $E \subset \text{Supp}(\mu^D_q) \cap E_F(\tau'(q))$ yields

\[
\dim_H(\mu^G_q) \leq \left( \frac{\dim_H(\mu^D_q)}{\tau'(q)} \right) \land \left( \dim_H(\mu^D_q) + 1 - \tau'(q) \right) \lor \dim_H(\mu^D_q),
\]

\[
\dim_H(\mu^R_q) \leq \frac{\dim_H(\mu^D_q)}{\tau'(q)} \land 1,
\]

and the conclusion comes from the fact that $\dim_H(\mu^D_q) = \tau^*(\tau'(q))$ for all $q \in J$.

Now we prove (25).

For any $\bar{q} \in K$ and $\epsilon > 0$ we define the neighborhood of $\bar{q}$ in $K$:

\[
U_\epsilon(\bar{q}) = \left\{ q \in K : \max_{\alpha \in \{\xi, \tau, \tau', \gamma, \gamma', \gamma'' \}} |\alpha(q) - \alpha(\bar{q})| < \epsilon \right\}.
\]

By continuity of these functions, the set $U_\epsilon(\bar{q})$ is open in $K$.

For any $u, v \in \mathcal{S}_* \cup \mathcal{S}^{N_+}$ and $p \geq 2$, we define the indicator function

\[ 1_p(u, v) = 1_{\{b^{p+2} \leq |\lambda(u) - \lambda(v)| < b^{p+2}\}}. \]

For any $w \in \mathcal{S}_*$ we use the notation

\[ [w]_{q, \epsilon}^n = [w] \cap \mathcal{C}_n(q, \epsilon). \]

Notice that for $q \in K$, $\delta > 0$ and $S \in \{G, R\}$ the Riez-like kernels $\mathcal{K}^S_{q, \delta}$ is a positive function and, moreover, by the continuity of $F_W$ and $F_L$ we have for any $s, t \in \mathcal{S}^{N_+}$,

\[ \lim_{m \to \infty} \mathcal{K}^S_{q, \delta}(s|_m, t|_m) = \mathcal{K}^S_{q, \delta}(s, t). \]

Then by applying Fatou’s lemma we get

\[
I_{n, \delta}(q, \epsilon) = \iint_{s, t \in \mathcal{C}_n(q, \epsilon), s \neq t} \lim_{m \to \infty} \mathcal{K}^S_{q, \delta}(s|_m, t|_m) \, d\mu_q(s) \, d\mu_q(t)
\]
Remark 5.1. Let 

\[ A_{p,m} = \sum_{u,v \in \mathcal{S}^m, 1_p(u,v)=1} \mathcal{K}_{q,\delta+\epsilon}^S(u,v) \cdot \mu_q([u]_{q,\epsilon}) \mu_q([v]_{q,\epsilon}) \]

Then,

\[
\sup_{q \in U_n(q)} I_{n,\delta}^S(q, \epsilon) \leq \sup_{q \in U_n(q)} \sum_{p \geq 2} \liminf_{m \to \infty} A_{p,m} \leq \sup_{q \in U_n(q)} \sum_{p \geq 2} \left(A_{p,m_p} + \sum_{m \geq m_p} |A_{p,m+1} - A_{p,m}|\right) \leq \sum_{p \geq 2} \left(\sup_{q \in U_n(q)} A_{p,m_p} + \sum_{m \geq m_p} \sup_{q \in U_n(q)} |A_{p,m+1} - A_{p,m}|\right),
\]

(27)

where for \( p \geq 2 \), we can choose \( m_p \geq 3 \) to be any integer. We have

\[
(\mathcal{I}_{n,q,\epsilon}^S)_{p,m} = \sum_{u,v \in \mathcal{S}^m, 1_p(u,v)=1} \mathcal{K}_{q,\delta+\epsilon}^S(u,v) \sup_{q \in U_n(q)} \mu_q([u]_{q,\epsilon}) \mu_q([v]_{q,\epsilon}),
\]

and \( (\Delta \mathcal{I}_{n,q,\epsilon}^S)_{p,m} = \sum_{u,v \in \mathcal{S}^m, u',v' \in \mathcal{S}; 1_p(u,v)=1} \left|\mathcal{K}_{q,\delta+\epsilon}^S(uu', vv') - \mathcal{K}_{q,\delta+\epsilon}^S(u, v)\right| \cdot \sup_{q \in U_n(q)} \mu_q([uu']_{q,\epsilon}) \mu_q([vv']_{q,\epsilon}), \)

and we have used the equality \( \mu_q([u]_{q,\epsilon}) = \sum_{u' \in \mathcal{S}} \mu_q([uu']_{q,\epsilon}) \) to get the second inequality.

Remark 5.1.
(a) For technical reasons, we need to divide $J$ into three parts, in which $K$ will be chosen:

$$
\begin{align*}
J_1 &= \{ q \in J : \gamma^G(q) > 1 \}, \\
J_2 &= \{ q \in J : \gamma^G(q) \leq 1, \tau'(q) < 1 \}, \\
J_3 &= \{ q \in J : \gamma^G(q) \leq 1, \tau'(q) \geq 1 \}.
\end{align*}
$$

Then, due to (14) and (15), with $h = \tau'(q)$ we have

$$
\gamma^G(q) = \begin{cases} 
\tau^*(h) + 1 - h & \text{if } q \in J_1, \\
\tau^*(h) / h & \text{if } q \in J_2, \\
\tau^*(h) & \text{if } q \in J_3.
\end{cases}
$$

(b) Here we briefly explain why $\text{dim}_H(\mu^G_q) = \gamma^G(q) = \tau^*(\tau'(q))$ when $q \in J_3$. Thus we will not consider this case in the rest of the proof.

From [4] we have $\text{dim}_H(\mu^D_q) = \tau^*(\tau'(q))$ and $\mu^D_q$ is the orthogonal projection of $\mu^G_q$ onto the $x$-axis, so we automatically have $\text{dim}_H(\mu^G_q) \geq \text{dim}_H(\mu^D_q) = \tau^*(\tau'(q))$. Then to prove $\text{dim}_H(\mu^G_q) = \gamma^G(q)$ we only need an upper bound estimate, but this estimate is actually a direct consequence of Theorem 1.1.

For any compact subset $K$ of $J$ there exists $c_K \in (0, 1)$ such that for any $c < c_K$, $\gamma^G(q) - c > 1$ if $K \subset J_1$ and $\gamma^R(q) - c > 0$ if $K \subset J_2 \cup J_3$. Let $\delta_K = \epsilon_K = c_K / 2$.

An essential tool in this paper is the following proposition, whose proof is given in Section 6.

**Proposition 5.2.** Let $S \in \{G, R\}$. Suppose that $K$ is a compact subset of $J_1$ or $J_2$ if $S = G$, or a compact subset of $J_1$ or $J_2 \cup J_3$ if $S = R$. Then there exists $q_s \in (0, \epsilon_K)$ such that for any $0 < \delta < \delta_K$, we can find constants $\kappa_1, \kappa_2, \eta_1, \eta_2 > 0$ and $C > 0$ such that for any $q \in K$, $0 < \epsilon \leq \epsilon_s$, $n \geq 1$, $p \geq 2$, and $m \geq 3 \cdot (n \vee p)$

\[
\mathbb{E} \left( \mathcal{I}^S_{n, q, \epsilon}(p, m) \right) \leq C \cdot b^{(n \vee p) - p + 1} \cdot e^{\epsilon \sum_{p \leq n} \kappa_1 \cdot n} \cdot e^{-\eta_1 \delta (n \vee p) + \kappa_1 \epsilon \cdot m},
\]

\[
\mathbb{E} \left( \Delta \mathcal{I}^S_{n, q, \delta}(p, m) \right) \leq C \cdot b^{(n \vee p) - p + 1} \cdot e^{\kappa_2 \delta (n \vee p) - \eta_2 \cdot m}.
\]

Now we may choose $m_p = \frac{\kappa_2 + \frac{\eta_1}{\eta_2}}{\delta}$ \cdot $(n \vee p)$ (by modifying a little $\eta_2$ we can always assume that $\frac{\kappa_2 + \frac{\eta_1}{\eta_2}}{\delta} > 3$) and $\epsilon_s = \epsilon_s \wedge \frac{\eta_1 \cdot \eta_2}{\kappa_1 \cdot (\kappa_2 + \frac{\eta_1}{\eta_2})}$. Then by using Proposition 5.2 and (27), (28), for any $\delta < \delta_K$, $q \in K$, $\epsilon < \epsilon_\delta$ and $S \in \{G, R\}$ we have

\[
\mathbb{E} \left( \sup_{q \in U(q)} \mathcal{I}^S_{n, \delta}(q, \epsilon) \right) \leq \sum_{p \geq 2} \left( \mathbb{E} \left( \mathcal{I}^S_{n, q, \epsilon}(p, m_p) \right) + \sum_{m \geq m_p} \mathbb{E} \left( \Delta \mathcal{I}^S_{n, q, \epsilon}(p, m) \right) \right) \leq \sum_{p \geq 2} C \cdot b^{(n \vee p) - p + 1} \cdot e^{\epsilon \sum_{p \leq n} \kappa_1 \cdot n} \cdot e^{-\eta_1 \delta (n \vee p) + \kappa_1 \epsilon \cdot m_p} + \sum_{m \geq m_p} e^{\kappa_2 \delta (n \vee p) - \eta_2 \cdot m}.
\]
\[ C \cdot \sum_{p \geq 2} b^{(n vp) - p + 1} \cdot e^{1_{\{p < n\} \cdot \kappa_1 \cdot n} - \eta_1 \cdot \delta_{n vp} + \kappa_1 \cdot \epsilon_1 \cdot \eta_2 \cdot m_p + e^{\kappa_2 \cdot (n vp) - \eta_2 \cdot m_p}} \cdot \left( \frac{1}{1 - e^{-\eta_2}} \right) \]
\[ \leq C \cdot \sum_{p \geq 2} b^{(n vp) - p + 1} \cdot e^{1_{\{p < n\} \cdot \kappa_1 \cdot n} \cdot \left( e^{-\eta_1 \cdot \delta_{n vp} + \kappa_1 \cdot \epsilon_1 \cdot \eta_2} + e^{\kappa_2 \cdot (n vp) - \eta_2} \right)} \]
\[ = \frac{2C}{1 - e^{-\eta_2}} \cdot \sum_{p \geq 2} b^{(n vp) - p + 1} \cdot e^{1_{\{p < n\} \cdot \kappa_1 \cdot n} \cdot \left( \frac{1}{1 - e^{-\frac{\delta_{n vp}}{2}}} \right)} \]
\[ = \frac{2C}{1 - e^{-\eta_2}} \cdot \left( b^{n+1} \cdot e^{\kappa_1 \cdot n} \cdot \frac{1 - b^{-n}}{1 - b^{-1}} + b \cdot e^{-\frac{\delta_{n vp}}{2}} \cdot \frac{1}{1 - e^{-\frac{\eta_2}{2}}} \right) < \infty. \]

Since for any \( 0 < \epsilon < \epsilon_\delta \), the family \( \{ U_\epsilon(\bar{q}) \}_{\bar{q} \in K} \) forms an open covering of \( K \), there exist \( \bar{q}_1, \ldots, \bar{q}_N \) such that \( \{ U_\epsilon(\bar{q}_i) \}_{1 \leq i \leq N} \) also covers \( K \). This gives us the conclusion. \( \square \)

5.3. Proof of Theorem 3.2

Proof. The proof of Theorem 3.2 exploits the main idea developed in [38] to study the dimension of projections and sections of sets. Some complications come from the fact that we want results holding for uncountably many sets and measures simultaneously.

Through the proof we use the same notation as in Section 5.2. Moreover, for \( n \geq 1, q \in J \) and \( \epsilon > 0 \) we define
\[ G_n(q, \epsilon) = \{ (F_L(t), F_W(t)) : t \in C_n(q, \epsilon) \} \]
and for \( \theta \in (-\pi/2, \pi/2) \) we define \( R_n,\theta(q, \epsilon) = \text{Proj}_\theta(G_n(q, \epsilon)) \subset l_\theta. \)

For any \( y \in l_\theta \) we define the lower derivative of the measure \( \mu_{q,\theta}^R \big|_{R_n,\theta(q, \epsilon)} \) with respect to the one-dimensional Lebesgue measure on \( l_\theta \) at \( y \):
\[ D(\mu_{q,\theta}^R \big|_{R_n,\theta(q, \epsilon)}, y) = \lim \inf_{r \to 0^+} \frac{1}{r} \cdot \mu_{q,\theta}^R \left( B(y, r) \cap R_n,\theta(q, \epsilon) \right). \]

We fix a compact subset \( K \subset J_1 \) (recall Remark 5.1). For any \( \bar{q} \in K \), we can choose \( \delta \in (0, \delta_K) \) and \( \epsilon_* \) such that the conclusions of Proposition 5.2 hold. Notice that for such \( \delta \) and \( \epsilon \in (0, \epsilon_*) \) we always have \( \gamma^G(\bar{q}) - \delta - \epsilon > 1. \)

For \( s, t \in A^* \cup A^{N+} \) recall
\[ K_1(s, t)^{-1} = \left( |F_L(s) - F_L(t)|^2 + |F_W(s) - F_W(t)|^2 \right)^{1/2} \wedge 1, \]
and for \( \theta \in (-\pi/2, \pi/2) \) and \( \gamma > 0 \) we define
\[ d_{\theta,\gamma}^2(s, t) = K_\gamma(s, t)^{-1} \cdot |\sin(\theta + \theta_{s,t})|, \]
where $\theta_{s,t}$ stands for the angle between $(F_L(t), F_W(t)) - (F_L(s), F_W(s))$ and $x$-axis (clockwise). Notice that for any $r > 0$ and $\gamma > 1$ we always have
\[
1 \{d_{s,t}^{G}(s,t) < r\} \leq \liminf_{m \to \infty} 1 \{d_{s,m}^{G}(s,m,t) < 2r\} \leq \liminf_{m \to \infty} 1 \{d_{s,m}^{G}(s,m,t) < 2r\}.
\]

Now, recall (26). For simplicity we will also use the notation
\[
M^n_{q,\epsilon}(u, v) = \sup_{q \in U(\epsilon)} \mu_q([u]_{q,\epsilon}^n)\mu_q([v]_{q,\epsilon}^n).
\]

Then an integration similar to that used in the proof of Theorem 9.7 of [38], as well as arguments similar to that used in Section 5.2 yield for any $\gamma > 1$ (by using Fatou's lemma repetitively)
\[
\begin{align*}
\int_0^\infty \sup_{q \in U(\epsilon)} \int_{y \in R_{n,\epsilon}(q)} \frac{\mathcal{D}(\mu_{q,\epsilon}^{R})|_{R_{n,\epsilon}(q,\epsilon)}(y)}{\mu_{q,\epsilon}^{R}(y)} d\mu_{q,\epsilon}^{R}(y) \, d\theta \\
= \int_0^\infty \sup_{q \in U(\epsilon)} \int_{y \in R_{n,\epsilon}(q)} \liminf_{r \to 0^+} \int_{x \in C_n(q,\epsilon)} 1 \{|x-y| \leq r\} d\mu_q(x) d\mu_{q,\epsilon}^{R}(y) \, d\theta \\
\leq \int_0^\infty \sup_{q \in U(\epsilon)} \int_{y \in R_{n,\epsilon}(q)} \liminf_{r \to 0^+} \int_{x \in C_n(q,\epsilon)} 1 \{d_{s,t}^{G}(s,t) \leq r\} d\mu_{q,\epsilon}^{R}(y) \, d\theta \\
= \int_0^\infty \sup_{q \in U(\epsilon)} \int_{y \in R_{n,\epsilon}(q)} \liminf_{r \to 0^+} \int_{x \in C_n(q,\epsilon)} 1 \{d_{s,t}^{G}(s,t) \leq r\} d\mu_{q,\epsilon}^{R}(y) \, d\theta \\
\leq \int_0^\infty \sup_{q \in U(\epsilon)} \int_{y \in R_{n,\epsilon}(q)} \liminf_{r \to 0^+} \sum_{p \geq 2} \sum_{1_p(s,t) = 1} \liminf_{m \to \infty} 1 \{d_{s,m}^{G}(s,m,t) < 2r\} d\mu_{q,\epsilon}^{R}(y) \, d\theta \\
\leq \int_0^\infty \sup_{q \in U(\epsilon)} \int_{y \in R_{n,\epsilon}(q)} \liminf_{r \to 0^+} \sum_{p \geq 2} \sum_{1_p(s,t) = 1} \liminf_{m \to \infty} 1 \{d_{s,m}^{G}(s,m,t) < 2r\} d\mu_{q,\epsilon}^{R}(y) \, d\theta \\
\leq \liminf_{r \to 0^+} \frac{1}{r} \sum_{p \geq 2} \sum_{1_p(s,t) = 1} \int_0^\infty \sup_{q \in U(\epsilon)} \max_{\theta \in [0,\pi]} \sum_{u,v \in \mathbb{R}^m_{p,s,t}} 1 \{d_{s,t}^{G}(u,v) < 2r\} d\theta \cdot \cdot M^n_{q,\epsilon}(u, v) + \sum_{m \geq m_p} \\
\sum_{u,v \in \mathbb{R}^m_{p,s,t}} \int_0^\pi 1 \{d_{s,t}^{G}(uu', uu'') < 2r\} - 1 \{d_{s,t}^{C}(u,v) < 2r\} d\theta \cdot \cdot M^n_{q,\epsilon}(uu', vv') \\
\leq \sum_{p \geq 2} \left( \sum_{u,v \in \mathbb{R}^m_{p,s,t}} \limsup_{r \to 0^+} \frac{1}{r} \int_0^\pi 1 \{d_{s,t}^{G}(u,v) < 2r\} d\theta \cdot \cdot M^n_{q,\epsilon}(u, v) + \sum_{m \geq m_p} \right)
\end{align*}
\]
\[ \sum_{u,v \in \mathbb{R}^+} \lim_{r \to 0^+} \frac{1}{r} \int_0^\pi \left| \mathbf{1}_{\{d_{q,\gamma}^R(uu',vv') < 2r\}} - \mathbf{1}_{\{d_{q,\gamma}^L(u,v) < 2r\}} \right| d\theta \cdot M^r_{q,\epsilon} (uu', vv') \],

where \( m_p \) is taken as in the proof of Theorem 3.1.

Notice that there exists a universal constant \( C' > 0 \) such that for all \( r > 0 \),

\[ \int_0^\pi \mathbf{1}_{\{d_{q,\gamma}^R(u) \leq 2r\}} d\theta = \int_0^\pi \mathbf{1}_{\{|\sin(\theta + \theta, v)\| \leq 2r \cdot \mathcal{K}_\gamma(u,v)\}} d\theta \leq C' \cdot r \cdot \mathcal{K}_\gamma(u,v) \]

and

\[ \int_0^\pi \left| \mathbf{1}_{\{d_{q,\gamma}^R(uu', vv') \geq 2r\}} - \mathbf{1}_{\{d_{q,\gamma}^L(u,v) \leq 2r\}} \right| d\theta \leq C' \cdot r \cdot \mathcal{K}_\gamma(uu', vv') - \mathcal{K}_\gamma(u,v) \].

Thus by taking \( \gamma = \gamma^G(q) - \delta - \epsilon > 1 \), we deduce from Proposition 5.2 that

\[ \mathbb{E} \left( \int_0^\pi \sup_{q \in U(q)} \int_{y \in \mathbb{R}_{R_n,\theta}(q, \epsilon)} \mathcal{D}(\mu^R_{q,\theta}|_{R_n,\theta(q, \epsilon)}, y) d\mu^R_{q,\theta}(y) d\theta \right) \leq C' \cdot \mathbb{E} \left( \sum_{p \geq 2} (\mathcal{T}^{S, \delta}_{n,q,\epsilon})_{p, m_p} + \sum_{m \geq m_p} (\Delta \mathcal{T}^{S, \delta}_{n,q,\epsilon})_{p, m} \right) < \infty. \]

Then by using the same argument as in Section 5.2, we can conclude that with probability 1, for Lebesgue almost every \( \theta \in (-\pi/2, \pi/2) \), for all \( q \in J_1 \), for \( \mu^R_{q,\theta} \), almost every \( y \in l_\theta \), the lower derivative \( \mathcal{D}(\mu^R_{q,\theta}, y) \) is finite, which is equivalent to saying that \( \mu^R_{q,\theta} \) is absolutely continuous with respect to the one-dimensional Lebesgue measure on \( l_\theta \). This ensures that for \( \mu^R_{q,\theta} \)-almost every \( y \in l_\theta \), the following limit:

\[ \lim_{r \to 0^+} \frac{1}{r} \int_{|x - l_{y,\theta}^+| \leq r} \psi(x) d\mu^G_q(x) \]

exists for any continuous function \( \psi : \mathbb{R}^2 \to \mathbb{R}_+ \) and thus defines a measure \( \mu^y_{q,\theta} \) carried by \( l_{y,\theta}^+ \cap \{ (F_L(t), F_W(t)) : t \in \mathcal{C}(q) \} \).

Now for the lower bound of the Hausdorff dimension of \( \mu^y_{q,\theta} \), we notice that like in Theorem 10.7 of [38], we have the following equality for the \( \gamma^G(q) - \delta - 1 \) energy

\[ \mathcal{I}_{q,\theta} := \int_{y \in \mathbb{R}_{R_n,\theta}(q, \epsilon)} \int_{x_1, x_2 \in \mathbb{R}_{l_{y,\theta}^+} \cap G_n(q, \epsilon)} 1 \vee |x_1 - x_2|^{-\gamma^G(q) - \delta - 1} d\mu^y_{q,\theta}(x_1) d\mu^y_{q,\theta}(x_2) d\text{Leb}_\theta(y) \]

\[ = \lim_{r \to 0^+} \frac{1}{r} \int_{s, t \in \mathbb{C}(q, \epsilon)} 1_{\{d_{q,\theta}(s, t) \leq r\}} d\mathcal{G}^G(s, t)^{-\gamma^G(q) - \delta - 1} d\mu_q(t) d\mu_q(s), \]

where \( \text{Leb}_\theta \) stands for the one-dimensional Lebesgue measure on \( l_\theta \). By using the same method as above to establish the finiteness of \( \mathcal{D}(\mu^R_{q,\theta}, y) \), we can show that

\[ \int_0^\pi \sup_{q \in U(q)} \mathcal{I}_{q,\theta} d\theta \]
\[ \leq \sum_{p \geq 2} \left( \sum_{u, v \in \mathcal{A}^m_p, \ 1_p(u, v) = 1} \limsup_{r \to 0^+} \frac{1}{r} \int_0^\pi 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} K_{\tilde{\gamma}}(u, v) d\theta \cdot M^n_{\tilde{\gamma}, \epsilon}(u, v) + \sum_{m \geq m_p} \sum_{u, v \in \mathcal{A}^m_u, v' \in \mathcal{A}^m_{u'}} 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} K_{\tilde{\gamma}}(u, v) \right) \]

\[
\limsup_{r \to 0^+} \frac{1}{r} \int_0^\pi 1_{\{d_{\theta,1}^G(u', v') \leq 2r\}} K_{\tilde{\gamma}}(u', v') - 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} K_{\tilde{\gamma}}(u, v) d\theta \cdot M^n_{\tilde{\gamma}, \epsilon}(u, v),
\]

where \( \tilde{\gamma} = \gamma - 1 = \gamma^G(\tilde{q}) - \delta - \epsilon - 1 > 0 \).

Notice that for the same universal constant \( C' \), we have

\[
\int_0^\pi 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} K_{\tilde{\gamma}}(u, v) d\theta \leq C' \cdot r \cdot K_{\tilde{\gamma} + 1}(u, v) = C' \cdot r \cdot K_{\gamma}(u, v)
\]

and

\[
\int_0^\pi \left| 1_{\{d_{\theta,1}^G(u', v') \leq 2r\}} K_{\tilde{\gamma}}(u', v') - 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} K_{\tilde{\gamma}}(u, v) \right| d\theta \\
\leq \left( K_{\tilde{\gamma}}(u', v') \lor K_{\tilde{\gamma}}(u, v) \right) \cdot \int_0^\pi \left| 1_{\{d_{\theta,1}^G(u', v') \leq 2r\}} - 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} \right| d\theta \\
+ \left| K_{\tilde{\gamma}}(u', v') - K_{\tilde{\gamma}}(u, v) \right| \int_0^\pi 1_{\{d_{\theta,1}^G(u', v') \leq 2r\}} \lor 1_{\{d_{\theta,1}^G(u, v) \leq 2r\}} d\theta \\
\leq C' \cdot r \cdot \Delta K_{\tilde{\gamma}}(u', v'),
\]

where

\[
\Delta K_{\tilde{\gamma}}(u', v') = \left( \tilde{\gamma}(u', v') \lor \tilde{\gamma}(u, v) \right) \cdot \left| K_1(u', v') - K_1(u, v) \right| \\
+ \left( K_1(u', v') \lor K_1(u, v) \right) \cdot \left| \tilde{\gamma}(u', v') - \tilde{\gamma}(u, v) \right|
\]

Then

\[
(30) \quad \mathbb{E} \left( \int_0^\pi \sup_{q \in U_\epsilon(q)} I_{q, \theta} \ d\theta \right) \leq \mathbb{E} \left( \sum_{p \geq 2} \left( T_{n, q, \epsilon}^\delta \right)_{p, m_p} + \sum_{m \geq m_p} \left( \Delta T_{n, q, \epsilon}^\delta \right)_{p, m} \right) < \infty,
\]

where

\[
(\Delta T_{n, q, \epsilon}^\delta)_{p, m} = \sum_{u, v \in \mathcal{A}^m_u, v' \in \mathcal{A}^m_{u'}} \Delta K_{\gamma}(u', v') \sup_{q \in U_\epsilon(q)} \mu_q([u']_{q, \epsilon}^n) \mu_q([v']_{q, \epsilon}^n).
\]

The justification of the finiteness in (30) is postponed to Remark 6.1.

Now, by using the same arguments as in Section 5.2 again, we deduce that, with probability 1, for Lebesgue-almost every \( \theta \in (-\pi/2, \pi/2) \), for all \( q \in J_1 \), for \( \mu_{q, \theta}^R \)-almost every \( y \in l_\theta \), we have

\[
\dim_H(\mu_{q, \theta}^y) \geq \dim_H(\mu_{q, \theta}^G) - 1 = \tau^*(\tau'(q)) - \tau'(q).
\]

Then we get the conclusion by applying Theorem 1.1(b) to the measure \( \mu_{q, \theta}^R \) since we know that the for \( \mu_{q, \theta}^R \)-almost every \( y \in l_\theta \), the lower local dimension \( h_{\mu_{q, \theta}^y}(y) \) is equal to 1. \( \square \)
6. Proof of Proposition \[5,2\]

6.1. Main Proof.

**Proof.** Due to (26) we always have

\[
\bigcup_{q \in U_\epsilon(q)} C_n(q, \epsilon) \subset C_n(q, 2\epsilon).
\]

Let \( \rho_K = 1 \vee \sup_{q \in K} \{|q| + \xi(q) + \bar{\xi}(q) + |\tau(q)| + \gamma(q)\} < \infty \).

Due to (29), (12), (20), (26) and (111) we have

\[
M_{\bar{q}, \epsilon}(u, v) = \sup_{q \in U_\epsilon(q)} \mu_q([u]_q^n) \mu_q([v]_q^n)
\]

\[
= \sup_{q \in U_\epsilon(q)} 1_{\{[u]_q^n \neq \emptyset\}} 1_{\{[v]_q^n \neq \emptyset\}} |W_{q,u}(\emptyset)||W_{q,v}(\emptyset)| Y_q(u) Y_q(v)
\]

\[
\leq \sup_{q \in U_\epsilon(q)} 1_{\{[u]_q^n \neq \emptyset\}} 1_{\{[v]_q^n \neq \emptyset\}} e^{-[(|u| + |v|)(\gamma(q) - 4\rho_K \epsilon)]} Y_q(u) Y_q(v)
\]

\[
\leq 1_{\{[u]_q^n \neq \emptyset\}} 1_{\{[v]_q^n \neq \emptyset\}} e^{-[(|u| + |v|)(\gamma(q) - 4\rho_K \epsilon)]} Y_K(u) Y_K(v).
\]

This gives us

\[
\sum_{u,v \in \mathcal{A}^m, 1_p(u,v) = 1} K_{\bar{q}, \delta + \epsilon}(u, v) \cdot 1_{\{[u]_q^n, [v]_q^n \neq \emptyset\}} Y_K(u) Y_K(v),
\]

\[
\sum_{u,v \in \mathcal{A}^m, u', v' \in \mathcal{A}, 1_p(u,v) = 1} [K_{\bar{q}, \delta + \epsilon}(uu', vv') - K_{\bar{q}, \delta + \epsilon}(u, v)] \cdot 1_{\{[u]_q^n, [v]_q^n \neq \emptyset\}} Y_K(uu') Y_K(vv').
\]

Now we deal with the individual terms of the above sums.

Fix \( p \) and \( n \) in \( \mathbb{N}_+ \), let \( r = p \vee n \), and fix \( m \geq 3r \).

Fix a pair \( u, v \in \mathcal{A}^m \) with \( 1_p(u,v) = 1 \), so \(|\lambda(u) - \lambda(v)| \in [b^{-p+1}, b^{-p+2}]\).

Without loss of generality we suppose that \( \lambda(u) < \lambda(v) \).

Since \( 1_p(u,v) = 1 \), we have \( \lambda(u) < \lambda(v|_p) \leq \lambda(v|_r) \).

Let

\[
V := K_{\bar{q}, \delta + \epsilon}(u, v) 1_{\{[u]_q^n, [v]_q^n \neq \emptyset\}} Y_K(u) Y_K(v);
\]

\[
\Delta V := [K_{\bar{q}, \delta + \epsilon}(uu', vv') - K_{\bar{q}, \delta + \epsilon}(u, v)] 1_{\{[uu']_q^n, [vv']_q^n \neq \emptyset\}} Y_K(uu') Y_K(vv').
\]

Let us state two elementary claims.

**Claim 1.** Recall that \([u]_q^n = [u] \cap C_n(q, 2\epsilon)\). Due to (20) and (22), if \([u]_q^n \neq \emptyset\), then for \( l = r, \cdots, m \) we have \( 1_{\mathcal{W}_u((q,2\epsilon))} 1_{\mathcal{L}_u((q,2\epsilon))} 1_{\mathcal{O}_u((q,2\epsilon))} = 1 \). Define

\[
1_{u,v}^{(1)}(\bar{q}, \epsilon) = 1_{\{O_F(W_1) \cap O_F(W_2) \leq e^{-r(\xi(q) + 6\epsilon)} \}};
\]

\[
1_{u,v}^{(2)}(\bar{q}, \epsilon) = 1_{\mathcal{W}_u((q,2\epsilon))} 1_{\mathcal{L}_u((q,2\epsilon))} 1_{\mathcal{O}_u((q,2\epsilon))};
\]
and $1_{u,v}^{(2)}(\bar{q}, \epsilon) = 1_u^{(2)}(\bar{q}, \epsilon) \cdot 1_v^{(2)}(\bar{q}, \epsilon)$. Also, $[uu']^{n}_{\bar{q},2\epsilon} \neq \emptyset$ implies $[u]^{n}_{\bar{q},2\epsilon} \neq \emptyset$. Then, due to (18) and (19), we have

$$1_{[uu']^{n}_{\bar{q},2\epsilon}, [vu']^{n}_{\bar{q},2\epsilon} \neq \emptyset} \leq 1_{[uu']^{n}_{\bar{q},2\epsilon}, [vu']^{n}_{\bar{q},2\epsilon} \neq \emptyset} \leq 1_{u,v}^{(1)}(\bar{q}, \epsilon) \cdot 1_{u,v}^{(2)}(\bar{q}, \epsilon).
$$

**Claim 2.** For $w \in \mathcal{A}^*$ we define $Z_W(w) = F_W^{[w]}(1)$. Then from (17) we have

$$F_W(\lambda(w) + b^{-|w|}) - F_W(\lambda(w)) = W_w(\emptyset) \cdot Z_W(w), \quad \forall w \in \mathcal{A}^*.
$$

Due to (36) we have

$$F_W(v) - F_W(u) = F_W(v) - F_W(v|_r) + F_W(v|_r) - F_W(v|_r^+) + F_W(v|_r^+) - F_W(u) = W_{u|_r^+} v_r^+ - Z_W(v|_r) + F_W(v|_r) - F_W(v|_r^+) + F_W(v|_r^+) - F_W(u).$$

By construction we have $Z_W(v|_r^+)$ is measurable with respect to

$$\mathcal{A}(v|_r^+) := \sigma((W,L)(v|_r^+ \cdot w) : w \in \mathcal{A}^*)$$

and is independent of

$$\mathcal{A}^c(v|_r^+) := \sigma((W,L)(w) : w \in \mathcal{A}^*, |w| < r \text{ or } w|_r \neq w|_r^+).$$

Also due to the statistical self-similarity (17) we have

$$\sigma \left( \left\{ W_{u|_r^+}, F_W(v|_r) - F_W(v|_r^+) + F_W(v|_r^+) - F_W(u), 1_{u,v}^{(2)}(\bar{q}, \epsilon), Y_L(u), Y_K(uu')Y_K(v), Y_K(vv') \right\} \right) \subset \mathcal{A}^c(v|_r^+).$$

Now, due to Claim 1 and (33) we have

$$V \leq K_{\delta + \epsilon}(u,v) \cdot 1_{u,v}^{(2)}(\bar{q}, \epsilon) \cdot Y_K(u)Y_K(v), \text{ and}
$$

$$\Delta V \leq \Delta K_{\delta + \epsilon}^{S}(uu', vv') \cdot 1_{u,v}^{(2)}(\bar{q}, \epsilon) \cdot Y_K(uu')Y_K(vv'),$$

where

$$K_{\delta + \epsilon}(u,v) = K_{\delta + \epsilon}^{S}(u,v) \cdot 1_{u,v}^{(1)}(\bar{q}, \epsilon);$$

$$\Delta K_{\delta + \epsilon}^{S}(uu', vv') = |K_{\delta + \epsilon}^{S}(uu', vv') - K_{\delta + \epsilon}^{S}(u,v)| \cdot 1_{u,v}^{(1)}(\bar{q}, \epsilon).$$

Then due to Claim 2 we have

$$\mathbb{E}(V | \mathcal{A}(v|_r^+)) \leq \mathbb{E}(K_{\delta + \epsilon}^{S}(u,v) \cdot 1_{u,v}^{(2)}(\bar{q}, \epsilon) \cdot Y_K(u)Y_K(v),$$

$$\mathbb{E}(\Delta V | \mathcal{A}(v|_r^+)) \leq \mathbb{E}(\Delta K_{\delta + \epsilon}^{S}(uu', vv') \cdot 1_{u,v}^{(2)}(\bar{q}, \epsilon) \cdot Y_K(uu')Y_K(vv').$$

Recall in Remark 5.1 we distinguished the cases $K \subset J_i$, $i = 1, 2, 3$ according to whether or not the corresponding power on the kernel is greater than 1. Then, due to (23), once we have taken $\delta < \delta_K$ and $\epsilon < \epsilon_K$, only two situations are left:

$$K_{\delta + \epsilon}^{S}(u,v) = \left\{ \begin{array}{ll}
(F_L(u) - F_L(v))^2 + |F_W(u) - F_W(v)|^2 & \text{if } \gamma > 1; \\
|F_W(u) - F_W(v)|^{-\gamma} & \text{if } \gamma < 1,
\end{array} \right.$$
where \( \gamma = \gamma^S(q) - \delta - \epsilon \).

We have the following lemma, whose proof is given in Section 6.2.

**Lemma 6.1.** There exists a constant \( C_\gamma \) such that

\[
1_{w,u}^{(2)}(q, \epsilon) \cdot \mathbb{E}\left( \mathbb{K}_{q,\delta+\epsilon}^S(u, v) \big| A^c(v|_r) \right) \leq C_\gamma \cdot 1_{w,u}^{(2)}(q, \epsilon) \cdot \left\{ \begin{array}{ll}
\frac{e^{r(\xi(q)+6\epsilon-(\xi(q)+6\epsilon)(1-\gamma))}}{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}}, & \text{if } \gamma > 1 \\
\frac{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}}{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}, & \text{if } \gamma < 1
\end{array} \right.
\]

\[
1_{w,u}^{(2)}(q, \epsilon) \cdot \mathbb{E}\left( \Delta \mathbb{K}_{q,\delta+\epsilon}^S(uu', vv') \big| A^c(v|_r) \right) \leq C_\gamma \cdot 1_{w,u}^{(2)}(q, \epsilon) \cdot \left\{ \begin{array}{ll}
\frac{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}}{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}, & \text{if } \gamma > 1 \\
\frac{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}}{e^{r(\xi(q)+6\epsilon-1_{p(u)}(\xi(q)-6\epsilon)(1-\gamma))}, & \text{if } \gamma < 1
\end{array} \right.
\]

To complete the proof, it remains to count the average number of pairs \((u, v)\) in \((S^m)^2\) such that \(1_p(u, v) = 1\) and \(1_{w,u}^{(2)}(q, \epsilon) = 1\). This is done in the next lemma, whose proof is given in Section 6.3.

**Lemma 6.2.** For \( m \geq 3r \), we have

\[
\mathbb{E}\left( \sum_{u,v \in S^m} 1_p(u, v) \cdot 1_{w,u}^{(2)}(q, \epsilon) \right) \leq 2b^{r-p+1} \cdot e^{(2m-r)(\gamma(q)+8rK\epsilon)}.
\]

Now, by using Remark 5.1 and the definition of \( \gamma \), i.e. \( \gamma = \gamma^S(q) - \delta - \epsilon \) for \( S = G, R \), we have to deal with the following three cases (i), (ii), (iii):

\[
1 - \gamma = \left\{ \begin{array}{ll}
\frac{(\gamma(q) - \xi(q))/\xi(q) + \delta + \epsilon}{\xi(q) - \xi(q)}/\xi(q) + \delta + \epsilon, & \text{if } \gamma > 1, \\
\delta + \epsilon, & \text{if } \gamma < 1 \text{ and } K \subset J_1, \\
\end{array} \right.
\]

\[
\begin{array}{ll}
\text{case (i)} & \text{case (ii)} \\
\text{case (iii)}
\end{array}
\]

Then, due to (33), (32), (38) and (39), since \(1_{u,v}^{(2)}(q, \epsilon), Y_K(u) \) and \( Y_K(v) \) (resp. \( Y_K(uu') \) and \( Y_K(vv') \)) are independent, taking the expectation of \( Y_K(u) \) and \( Y_K(v) \) (resp. \( Y_K(uu') \) and \( Y_K(vv') \)), and using Lemmas 6.1 and 6.2 for cases (i), (ii), (iii) we have (\( C_K \) stands for \( \mathbb{E}(Y_K) \), which is finite by Proposition 2.1 (b)):

\[
\mathbb{E}\left( (T_{n,\xi,\epsilon})^{(2)}_{p,m} \right) \leq 2C_{\gamma}C_K^2 \cdot e^{-2m(\gamma(q)-4\rho_K\epsilon)} \cdot b^{r-p+1} \cdot e^{(2m-r)(\gamma(q)+8rK\epsilon)}
\]

\[
\leq \left\{ \begin{array}{ll}
e^{r(\xi(q)+6\epsilon-(\xi(q)+6\epsilon)(1-\gamma))}, & \text{(i)} \\
e^{r(\xi(q)+6\epsilon-(\xi(q)+6\epsilon)(1-\gamma))}, & \text{(ii)} \\
e^{r(\xi(q)+6\epsilon-(\xi(q)+6\epsilon)(1-\gamma))}, & \text{(iii)}
\end{array} \right.
\]

\[
= 2C_{\gamma}C_K^2 \cdot b^{r-p+1} \cdot e^{(2m-r)(\gamma(q)-4\rho_K\epsilon)}
\]

\[
\leq \left\{ \begin{array}{ll}
e^{-(\xi(q)-6\rho_K\xi(q) - (\xi(q)+6\epsilon)(1-\gamma))}, & \text{(i)} \\
e^{-(\xi(q)+\xi(q)+6\epsilon)(1-\gamma)}, & \text{(ii)} \\
e^{-(\xi(q)+6\rho_K\xi(q) - (\xi(q)+6\epsilon)(1-\gamma))}, & \text{(iii)}
\end{array} \right.
\]
\[ \mathbb{E} \left( (\Delta T_{n,q,e})_{p,m} \right) \leq 2C^2_\gamma K^2 \cdot e^{-2(m+1)(\gamma(q)-4\rho_K \epsilon)} \cdot b^2 \cdot b^{r-p+1} \cdot e^{(2m-r)(\gamma(q)+8\rho_K \epsilon)} \cdot e^{r(\xi(q)+6\epsilon)(1+\gamma)-m(\xi(q)-6\epsilon)}, \quad (i) \] 
\[ \leq 2C^2_\gamma K^2 e^{-2(\gamma(q)-4\rho_K \epsilon)} b^2 \cdot b^{r-p+1} \cdot e^{\left[\xi(q)+6\epsilon(1+\gamma)-\gamma(q)-8\rho_K \epsilon \cdot r - (\xi(q)+18\epsilon)\right] m}, \quad (\text{ii}) \] 
\[ \leq 2C^2_\gamma K^2 \cdot e^{-2(\gamma(q)-4\rho_K \epsilon)} b^2 \cdot b^{r-p+1} \cdot e^{\left[\xi(q)+6\epsilon-\gamma(q)-8\rho_K \epsilon \cdot r - (\xi(q)-6\epsilon)(1+\gamma)+24\rho_K \epsilon \cdot m\right]}, \quad (\text{ii}) \] 
\[ \leq 2C^2_\gamma K^2 \cdot e^{-2(\gamma(q)-4\rho_K \epsilon)} b^2 \cdot b^{r-p+1}, \quad (\text{ii}) \] 
\[ \text{and } \mathbb{E} \left( (\Delta T_{n,q,e})_{p,m} \right) \leq C \cdot b^{r-p+1} \cdot e^{\kappa_2 \cdot r - \eta_2 \cdot m}, \] 
which gives the conclusion. \[ \square \]

6.2. **Proof of Lemma 6.1**

**Step 1.** At first, we prove that the probability distribution of \( Z_W = F_W(1) \) has a bounded density function \( f_W \), with \( \|f_W\|_{\infty} = \tilde{C}_W < \infty \).

Let \( \phi(t) = \mathbb{E}(e^{itZ_W}) \) be the characteristic function of \( Z_W \). Since we have \( Z_W = \sum_{j=0}^{b-1} W_j \cdot Z_W(j) \), where \( \{W_j\}_j \) and \( \{Z_W(j)\}_j \) are independent, and the \( Z_W(j) \) are independent copies of \( Z_W \), we have

\[ \phi(t) = \mathbb{E} \left( \mathbb{E} \left( e^{it \sum_{j=0}^{b-1} W_j \cdot Z_W(j) \bigg| \sigma(Z_W(j), 0 \leq j \leq b-1) \right) \right) = \mathbb{E} \left( \prod_{j=0}^{b-1} \phi(W_j t) \right). \]

Since \( \mathbb{E}(|Z_W|) < \infty \), simultaneously we also get

\[ \phi'(t) = \mathbb{E}(iZ_W \cdot e^{itZ_W}) = \mathbb{E} \left( \sum_{k=0}^{b-1} iW_k \phi'(W_k t) \prod_{j \neq k} \phi(W_j t) \right). \]

Notice that \( |\phi(t)| = |\phi(-t)| \) and \( |\phi'(t)| = |\phi'(-t)| \), so we have \( |\phi(t)| = |\phi(-t)| \) and \( |\phi'(t)| = |\phi'(-t)| \), then

\[ |\phi(t)| \leq \mathbb{E} \left( \prod_{j=0}^{b-1} |\phi(W_j t)| \right) = \mathbb{E} \left( \prod_{j=0}^{b-1} |\phi(|W_j| t)| \right). \]
(41) \[ |\phi'(t)| \leq \mathbb{E} \left( \sum_{k=0}^{b-1} |W_k| |\phi'(|W_k|t)| \prod_{j\neq k} |\phi(|W_j|t)| \right); \]

Define \( l = \limsup_{t \to \infty} |\phi(t)|. \) Since \( |\phi(t)| \leq 1, \) we have \( l \leq 1. \) From Fatou’s lemma and the fact that \( \mathbb{P}(\forall j, W_j \neq 0) = 1, \) we have

\[
l \leq \limsup_{t \to \infty} \mathbb{E} \left( \prod_{j=0}^{b-1} |\phi(|W_j|t)| \right) \leq \mathbb{E} \left( \limsup_{t \to \infty} \prod_{j=0}^{b-1} |\phi(|W_j|t)| \right) = l^b.\]

This implies that \( l = 0 \) or \( 1. \) Since we are in the non-conservative case, \( Z_W \) is not almost surely a constant. Consequently, we can use the same approach as in the proof of Lemma 3.1 in [33] (which deals with the case \( W \geq 0, \)) and using the fact that \( \mathbb{E}(\max_{0 \leq j \leq b-1} |W_j|^p) \leq \mathbb{E}(\sum_{i=0}^{b-1} |W_i|^p) < 1 \) for some \( p > 1, \) we obtain \( l = 0. \)

Define \( N = \min_{0 \leq i \leq b-1, W_i \neq 0} |W_i|. \) Due to assumption (A3) there exists a \( q > 1 \) such that \( \mathbb{E}(N^{-q}) \leq \mathbb{E}(\sum_{i=0}^{b-1} 1_{W_i \neq 0} |W_i|^{-q}) < \infty. \) Then by using (III), (III), the same arguments as in the proofs of Theorem 2.1 and 2.2 in [33] we can get \( |\phi(t)| = O(t^{-q}) \) and \( |\phi'(t)| = O(t^{-(q+1)}) \) when \( t \to \infty. \) Now as a consequence (Lemma 3 in [2]) we have that \( Z_W \) has a density function, which is bounded by \( \int_{\mathbb{R}} |\phi(t)| dt < \infty. \)

**Step 2.** Recall that \( \gamma = \gamma^S(\bar{q}) - \delta - \epsilon \) and \( K_{\bar{q}, \delta+\epsilon}^S(u, v) = K_{\gamma}(u, v), \) as well as

\[
\begin{aligned}
\overline{K}_{\bar{q}, \delta+\epsilon}^S(u, v) &= K_{\gamma}(u, v) \cdot 1_{u, v}(\bar{q}, \epsilon) := \overline{K}_\gamma(u, v); \\
\overline{\Delta K}_{\bar{q}, \delta+\epsilon}^S(uu', vv') &= |K_{\gamma}(uu', vv') - K_{\gamma}(u, v)| \cdot 1_{u, v}(\bar{q}, \epsilon) := \overline{\Delta K}_\gamma(uu', vv').
\end{aligned}
\]

Let us prove the desired estimates, i.e., there exists a constant \( C_\gamma > 0, \) such that

\[
1_{(2)}(\bar{q}, \epsilon) \cdot \mathbb{E} \left( \overline{K}_\gamma(u, v) \bigg| \mathcal{A}^c(v^r) \right) \\
\leq C_\gamma \cdot 1_{(2)}(\bar{q}, \epsilon) \cdot \left\{ e^{r(\xi(\bar{q})+6\epsilon-(\xi(\bar{q})+6\epsilon)(1-\gamma))}, \right. \\
\left. e^{r(\xi(\bar{q})+6\epsilon-1(\mu \geq n) \cdot (\xi(\bar{q})-6\epsilon)(1-\gamma))}, \right. \quad \text{if } \gamma > 1; \\
\left. e^{r(\xi(\bar{q})+6\epsilon-1(\mu \geq n) \cdot (\xi(\bar{q})-6\epsilon)(1-\gamma))}, \right. \quad \text{if } \gamma < 1;
\]

\[
1_{(2)}(\bar{q}, \epsilon) \cdot \mathbb{E} \left( \overline{\Delta K}_\gamma(uu', vv') \bigg| \mathcal{A}^c(v^r) \right) \\
\leq C_\gamma \cdot 1_{(2)}(\bar{q}, \epsilon) \cdot \left\{ e^{r(\xi(\bar{q})+6\epsilon)(1+\gamma)-m(\xi(\bar{q})-6\epsilon)}}, \right. \\
\left. e^{r(\xi(\bar{q})+6\epsilon)-m(\xi(\bar{q})-6\epsilon)(1-\gamma))}, \right. \quad \text{if } \gamma > 1; \\
\left. e^{r(\xi(\bar{q})+6\epsilon)-m(\xi(\bar{q})-6\epsilon)(1-\gamma))}, \right. \quad \text{if } \gamma < 1.
\]

The \( \sigma \)-algebra \( \mathcal{A}^c(v^r) \) being defined as in **Claim 2**, we simplify the notations of the following quantities, which are measurable with respect to \( \mathcal{A}^c(v^r), \) hence
constant given \( \mathcal{A}^c(v|_r^-) \):

\[
\begin{align*}
A &= W_{v|_r^-}; \\
B &= F_W(v) - F_W(v|_r) + F_W(v|_r) - F_W(u); \\
C_1 &= \begin{cases} 1, & \text{if } p < n; \\ 2e^{-p(\xi(q)-6\varepsilon)}, & \text{if } p \geq n, \end{cases} \\
C_2 &= e^{-(n+1)(\xi(q)+6\varepsilon)}; \\
D_1 &= F_W(vv') - F_W(uu') - (F_W(v) - F_W(u)); \\
D_2 &= F_L(vv') - F_L(uu') - (F_L(v) - F_L(u)).
\end{align*}
\]

Let \( f_W \) stand for the bounded density of \( Z_W(v^-) \) obtained in \textbf{Step 1}. Let
\[
g_\gamma(s,t) = \left( |F_W(v) - F_W(u) + s|^2 + |F_L(v) - F_L(u) + t|^2 \right)^{-\gamma/2}, \ s,t \in \mathbb{R}.
\]

Define
\[
\begin{align*}
\zeta_1(\gamma) &= \int_{\mathbb{R}} \frac{f_W(x)}{|Ax + B|^2 + C_2^2} \gamma/2 \, dx; \\
\zeta_2(\gamma) &= \int_{|s| \leq |D_2|} \left| \frac{\partial}{\partial t} g_\gamma(0, t) \right| \, dt + \int_{|s| \leq |D_1|} \sup_{|t| \leq |D_2|} \left| \frac{\partial}{\partial s} g_\gamma(s, t) \right| \, ds; \\
\zeta_3(\gamma) &= \int_{|Ax + B| \leq C_1} \frac{f_W(x)}{|Ax + B|} \, dx; \\
\zeta_4(\gamma) &= \int_{\mathbb{R}} \left| \frac{1}{|Ax + B + D_1| \gamma} - \frac{1}{|Ax + B|} \right| f_W(x) \, dx.
\end{align*}
\]

From (34) and Claim 2, we have
\[
\begin{align*}
1_{u,v}(\bar{q}, \varepsilon) \cdot \left| F_W(v) - F_W(u) \right| &\leq C_1; \\
1_{u,v}(\bar{q}, \varepsilon) \cdot \left( F_L(v) - F_L(u) \right) &\geq 1_{u,v}(\bar{q}, \varepsilon) \cdot C_2.
\end{align*}
\]

This implies
\[
\begin{align*}
(42) \quad 1_{u,v}(\bar{q}, \varepsilon) \cdot \mathbb{E} \left( \mathcal{K}_\gamma(u, v) \mathcal{A}^c(v|_r^-) \right) &\leq 1_{u,v}(\bar{q}, \varepsilon) \cdot \left\{ 1 + \zeta_1(\gamma), \right. \quad \text{if } \gamma > 1; \\
&\quad 1 + \zeta_3(\gamma), \quad \text{if } \gamma < 1; \\
(43) \quad 1_{u,v}(\bar{q}, \varepsilon) \cdot \mathbb{E} \left( \mathcal{K}_\gamma(u, v) \mathcal{A}^c(v|_r^-) \right) &\leq 1_{u,v}(\bar{q}, \varepsilon) \cdot \zeta_2(\gamma), \quad \text{if } \gamma > 1; \\
(44) \quad 1_{u,v}(\bar{q}, \varepsilon) \cdot \mathbb{E} \left( \mathcal{K}_\gamma(u, v) \mathcal{A}^c(v|_r^-) \right) &\leq 1_{u,v}(\bar{q}, \varepsilon) \cdot \zeta_4(\gamma), \quad \text{if } \gamma < 1,
\end{align*}
\]

where we have used the inequality \(|x \lor 1 - y \lor 1| \leq |x - y|\) holds for any \( x, y \geq 0 \).

Now, we have the following inequalities:

(1) By using the change of variable \( y = \frac{Ax + B}{C_2} \) we get
\[
1_{u,v}(\bar{q}, \varepsilon) \cdot \zeta_1(\gamma) \leq C_W |A|^{-1} C_1^{1-\gamma} \cdot \int_{\mathbb{R}} \frac{dy}{(y^2 + 1)^{\gamma}};
\]
(II) It is not difficult to check that when $F_L(v) - F_L(u) \geq C_2$ and $|t| \leq |D_2|$ we always have

$$\left| \frac{\partial}{\partial t} g_\gamma(0, t) \right| \vee \left| \frac{\partial}{\partial s} g_\gamma(s, t) \right| \leq \gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma - 1} \leq \gamma \cdot ((C_2 - |D_2|) \vee 0)^{-\gamma - 1}.$$ 

In fact, we have

$$\left| \frac{\partial}{\partial t} g_\gamma(0, t) \right| \leq \frac{\gamma \cdot |F_L(v) - F_L(u) + t|}{\left( |F_W(v) - F_W(u)|^2 + |F_L(v) - F_L(u) + t|^2 \right)^{1/2+1}}$$

$$\leq \frac{\gamma \cdot |F_L(v) - F_L(u) + t|}{|F_L(v) - F_L(u) + t|^\gamma + 1} = \gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma - 1}$$

$$\left| \frac{\partial}{\partial s} g_\gamma(s, t) \right| \leq \frac{\gamma \cdot |F_W(v) - F_W(u) + s|}{\left( |F_W(v) - F_W(u) + s|^2 + |F_L(v) - F_L(u) + t|^2 \right)^{1/2+1}}$$

$$\leq \frac{\gamma \cdot |F_W(v) - F_W(u) + s| \cdot |F_L(v) - F_L(u) + t|^{-\gamma}}{|F_W(v) - F_W(u) + s|^2 + |F_L(v) - F_L(u) + t|^2}$$

$$\leq \frac{\gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma}}{2 |F_L(v) - F_L(u) + t|} \leq \gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma - 1}$$

(where we have used that for any $a, b > 0$, $\frac{a}{a^2+b^2} \leq \frac{1}{2b}$). This together with the definition of $\zeta_2(\gamma)$ yields

$$1^{(2)}_{u,v}(\bar{q}, \epsilon) \cdot \zeta_2(\gamma) \leq 2((C_2 - |D_2|) \vee 0)^{-\gamma - 1}(|D_1| + |D_2|);$$

(III) By using the change of variable $y = Ax + B$ when $\gamma < 1$ we get

$$1^{(2)}_{u,v}(\bar{q}, \epsilon) \cdot \zeta_3(\gamma) \leq C_W |A|^{-1} \int_{|u| \leq C_1} \frac{dy}{|y|\gamma} = 2C_W |A|^{-1} C_1^{-1-\gamma};$$

(IV) By using the change of variable $y = \frac{Ax+B}{D_1}$ we get

$$1^{(2)}_{u,v}(\bar{q}, \epsilon) \cdot \zeta_4(\gamma) \leq C_W \frac{|D_1|^{1-\gamma}}{|A|} \int_{\mathbb{R}} \left| \frac{1}{|y + 1|\gamma} - \frac{1}{|y|\gamma} \right| dy.$$ 

Now we notice that

$$\int_{\mathbb{R}} (y^2 + 1)^{-\gamma/2} dy \quad (\gamma > 1) \quad \text{and} \quad \int_{\mathbb{R}} \frac{1}{|y + 1|\gamma} - \frac{1}{|y|\gamma} dy \quad (\gamma < 1)$$

are both finite and when $1^{(2)}_{u,v}(\bar{q}, \epsilon) = 1$, we have

$$\begin{cases} |A|^{-1} = |W_{\bar{q},v}|^{-1} \leq b^r(\xi(\bar{q})+2\epsilon), \\
D_2 \leq \text{Osc}_{F_L}(I_v) + \text{Osc}_{F_L}(I_u) \leq 2b^{-m(\tilde{\xi}(\bar{q})-6\epsilon)}, \\
D_1 \leq \text{Osc}_{F_W}(I_v) + \text{Osc}_{F_W}(I_u) \leq 2b^{-m(\tilde{\xi}(\bar{q})-6\epsilon)}. \end{cases}$$

Moreover, when $\gamma > 1$ we have $\xi(\bar{q}) < \tilde{\xi}(\bar{q})$, so if $m \geq 3r$ and $\tilde{\xi}(\bar{q}) - 12\epsilon > 0$ then

$$\zeta_2(\gamma) \leq \left[ b^{-r(\tilde{\xi}(\bar{q})+6\epsilon)} (1 - b^{-(m-r)(\tilde{\xi}(\bar{q})-6\epsilon)+2(r + \epsilon)}) \right]^{-1-1} 2(b^{-m(\xi(\bar{q})-6\epsilon)} + b^{-m(\tilde{\xi}(\bar{q})-6\epsilon)})$$
Then by applying these inequalities to (42), (44), we get the conclusion.

**Remark 6.1.** At the end of the proof of Theorem 3.2 we define two quantities $\Delta K_{KK}(uu', vv')$ and $(\Delta S_{n,q,e})_{p,m}$, and we claim (30). We justify this claim here. In fact, $(\Delta S_{n,q,e})_{p,m}$ can be estimated from above similarly: we have the first inequality is similar to (43)

$$1^{(1)}_{u,v}(\bar{q}, e) \cdot 1^{(2)}_{u,v}(\bar{q}, e) \cdot \Delta K_{KK}(uu', vv') \leq C_2^{-\gamma} \cdot \zeta_2(1) + C_2^{-1} \zeta_2(\bar{\gamma}) \leq 2(C_2 - |D_2|)^{-2\gamma}(|D_1| + |D_2|),$$

which is exactly the same bound as in (II) for $1^{(2)}_{u,v}(\bar{q}, e) \cdot \zeta_2(\gamma)$ (we have used (43) with $\gamma = 1$ and $\bar{\gamma} = \bar{\gamma}$). Then, the upper bound estimation of $(\Delta S_{n,q,e})_{p,m}$ can be treated like that of $(\Delta S_{n,q,e})_{p,m}$, and one obtains the same estimate as for $\mathbb{E}((\Delta S_{n,q,e})_{p,m})$:

$$\mathbb{E}((\Delta S_{n,q,e})_{p,m}) \leq C \cdot \bar{b}^{-p+1} \cdot e^{\kappa^2 r - n^2 m},$$

which is enough to get (30).

### 6.3. Proof of Lemma 6.2

**Proof.**

Let

$$S_{p,m} = \sum_{u,v \in \mathcal{A}^{p,m}} p(u, v) \cdot 1_{\mathcal{W}_{u|v}, (\bar{q}, 2e)} \cdot 1_{\mathcal{W}_{v|u}, (\bar{q}, 2e)} \cdot 1_{\mathcal{L}_{u|v}, (\bar{q}, 2e)} \cdot 1_{\mathcal{L}_{v|u}, (\bar{q}, 2e)} \cdot 1_{\mathcal{F}_{u|v}, (\bar{q}, 2e)} \cdot 1_{\mathcal{F}_{v|u}, (\bar{q}, 2e)} \cdot 1_{\mathcal{S}_{u|v}, (\bar{q}, 2e)} \cdot 1_{\mathcal{S}_{v|u}, (\bar{q}, 2e)}.$$

Then by (55) we have $\sum_{u,v \in \mathcal{A}^{p,m}} p(u, v) \cdot 1^{(2)}_{u,v}(\bar{q}, \bar{e}) \leq S_{p,m}$.

Recall that $r = p \vee n$ and $m \geq 3r$. For any $u \in \mathcal{A}^{m}$ we write $u = u_r \cdot u'$ with $u' \in \mathcal{A}^{m-r}$. From (55) we have $W_u = W_{u|r} \cdot W_{u'}(u_r)$, so $1_{\mathcal{W}_{u|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{W}_{u}, (\bar{q}, 2e)} = 1$ implies that

$$|W_{u'}(u_r)| \leq e^{-(m-r)[\xi(q)+2\frac{m+\epsilon}{m-r}]} \cdot e^{-(m-r)[\xi(q)-2\frac{m-\epsilon}{m-r}]}.$$

Thus, when $m \geq 3r$, we have

$$1_{\mathcal{W}_{u|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{W}_{u}, (\bar{q}, 2e)} \leq 1_{\mathcal{W}_{u|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{W}_{u|r|}, (\bar{q}, 4e)};$$

and, moreover, $1_{\mathcal{W}_{u|r}, (\bar{q}, 2e)}$ and $1_{\mathcal{W}_{u|r|}, (\bar{q}, 4e)}$ are independent. Simultaneously we also have

$$1_{\mathcal{W}_{v|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{W}_{v}, (\bar{q}, 2e)} \leq 1_{\mathcal{W}_{v|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{W}_{v|r|}, (\bar{q}, 4e)};$$

$$1_{\mathcal{L}_{u|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{L}_{u}, (\bar{q}, 2e)} \leq 1_{\mathcal{L}_{u|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{L}_{u|r|}, (\bar{q}, 4e)};$$

$$1_{\mathcal{L}_{v|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{L}_{v}, (\bar{q}, 2e)} \leq 1_{\mathcal{L}_{v|r}, (\bar{q}, 2e)} \cdot 1_{\mathcal{L}_{v|r|}, (\bar{q}, 4e)}.$$
We can drop the terms $1_{W_{v|p}(\bar{q},2\epsilon)}$ and $1_{L_{v|p}(\bar{q},2\epsilon)}$ so that the remaining indicator functions on the right hand side of the above inequalities are independent. Since for each $u \in \mathcal{A}^m$, there are at most $2b^{r-p+1}$ many $v|_r$ such that $1_p(u,v) = 1$, we get
\[
\mathbb{E}(N_{p,m}) \leq 2b^{-p+1}\mathbb{E}\left( \sum_{u \in \mathcal{A}^r} 1_{W_u(\bar{q},2\epsilon)} 1_{L_u(\bar{q},2\epsilon)} \right) \cdot \mathbb{E}\left( \sum_{u \in \mathcal{A}^{m-r}} 1_{W_u(\bar{q},4\epsilon)} 1_{L_u(\bar{q},4\epsilon)} \right)^2.
\]

For $\bar{q}$ with $|\bar{q}| + \xi(\bar{q}) + \tilde{\xi}(\bar{q}) + |\tau(\bar{q})| \leq \rho K$ we always have
\[
1_{W_u(\bar{q},2\epsilon)} \leq |W_u|^{\bar{q}}, \quad e^{\epsilon |u| \xi(\bar{q}) + 2\rho K \epsilon} \quad \text{and} \quad 1_{L_u(\bar{q},2\epsilon)} \leq L_u^{\tau(\bar{q})}, \quad e^{\epsilon |u| (\tau(\bar{q}) \tilde{\xi}(\bar{q}) + 2\rho K \epsilon)}.
\]

For $k \in \{r, m - r\}$ this yields
\[
\mathbb{E}\left( \sum_{u \in \mathcal{A}^k} 1_{W_u(\bar{q},4\epsilon)} 1_{L_u(\bar{q},4\epsilon)} \right) \leq e^{k(\gamma(\bar{q}) + 8\rho K \epsilon)},
\]
where recall that $\gamma(\bar{q}) = \bar{q}\xi(\bar{q}) - \tau(\bar{q})\tilde{\xi}(\bar{q})$. This gives us the conclusion. \square

Acknowledgement.

The author would like to thank gratefully his supervisor Julien Barral for having suggested him to study the new types of singularity spectra considered in this paper for $b$-adic independent cascade function, as well as for his help in achieving this work.

References

[1] Adler, R. J.: The Geometry of Random Fields. Wiley, New York. (1981)
[2] Athreya, K. R.: On the supercritical one dimensional age dependent branching processes. Ann. Math. Stat. 40(3), 743-763 (1969)
[3] Barral, J.: Continuity of the multifractal spectrum of a random statistically self-similar measure. J. Theoret. Probab. 13(4), 1027–1060 (2000)
[4] Barral, J., Jin, X. and Mandelbrot B.: Convergence of complex random cascades. To appear in Ann. Appl. Probab. arXiv:0812.3577v2
[5] Barral, J. and Jin, X.: Multifractal analysis of complex random cascades. To appear in Comm. Math. Phys. arXiv:0906.1501
[6] Barański, K.: Hausdorff dimension of the limit sets of some planar geometric constructions. Adv. Math. 210, 215–245 (2007)
[7] Bedford, T.: Hölder exponents and box dimension for self-affine fractal functions. Constr. Approx. 5(1), 33–48 (1989)
[8] Bedford, T.and Urbański, M.: The box and Hausdorff dimension of self-affine sets. Ergod. Th. & Dynam. Sys. 10 627–644 (1990)
[9] Berman, S. M.: Gaussian sample functions: uniform dimension and Hölder conditions nowhere. Nagoya. Math. J. 46 63-86 (1972)
[10] Berman, S. M.: Local nondeterminism and local times of general stochastic processes. Ann. Inst. H. Poincaré Sect. B 19 189–207 (1983)
[11] Bertoin, J.: Hausdorff dimension of the level sets for self-affine functions. Japan J. Appl. Math. 7 197–202 (1990)
[12] Besicovitch, A. S.: On the sum of digits of real numbers represented in the dyadic system. Math. Ann. 110 321–330 (1935)
[13] Blumenthal, R. M. and Getoor, R. K.: Sample functions of stochastic processes with stationary independent increments, J. Math. Mech. 10 493–516 (1961)
[14] Blumenthal, R. M. and Getoor, R. K.: The dimension of the set of zeroes and the graph of a symmetric stable process, Illinois J. Math. 6 308–316 (1962)
[15] Brown, G., Michon, G. and Peyrière, J.: On the multifractal analysis of measures. J. Stat. Phys. 66, 775–790 (1992)
[16] Demichel, Y. and Falconer, K. J.: The Hausdorff dimension of pulse-sum graphs, Math. Proc. Cambridge Philos. Soc. 143 145–155 (2007)
[17] Durrett, R. and Liggett, R.: Fixed points of the smoothing transformation. Z. Wahrsch. Verw. Gebiete 64, 275–301 (1983)
[18] Eggleston, H. G.: The fractional dimension of a set defined by decimal properties. Quart. J. Math. 20, 31-36 (1949)
[19] Falconer, K. J.: Fractal Geometry: Mathematical Foundations and Applications. 2nd Edition, Wiley, UK (2003)
[20] Feng, D. J.: The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers, Adv. Math. 195 24–101 (2005)
[21] Feng, D. J. and Wang, Y.: A class of self-affine sets and self-affine measures. J. Fourier Anal. Appl. 11, 107-124 (2005)
[22] Hawkes, J.: Local times and zero sets for processes with infinitely divisible distributions, J. London Math. Soc. 8(2) 517–525 (1974)
[23] Horowitz, J.: The Hausdorff dimension of the sample path of a subordinator, Israel J. Math. 6 176–182 (1968)
[24] Hu, T. Y. and Lau, K. S.: Fractal Dimensions and Singularities of the Weierstrass Type Functions, Trans. Amer. Math. Soc. 335 649–665 (1993)
[25] Hunt, B.: The Hausdorff dimension of graphs of Weierstrass functions, Proc. Amer. Math. Soc. 126 791–800 (1998)
[26] Jaffard, S.: Wavelets techniques in multifractal analysis. In Lapidus, M. and Frankenhuysen, M. V. ed. Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Proceedings of Symposia in Pure Mathematics. 72(2) 91–151 (2004)
[27] Kahane, J. P.: Some Random series of functions, 2nd edition. Cambridge University Press (1985)
[28] Kahane, J. P. and Peyrière, J.: Sur certaines martingales de B. Mandelbrot. Adv. Math. 22, 131–145 (1976)
[29] Khoshnevisan, G. and Xiao, Yimin: Lévy processes: Capacity and Hausdorff dimension, Ann. Probab. 33 841–878 (2005)
[30] Kôno, N.: On self-affine functions. Japan J. Appl. Math. 3 259–269 (1986)
[31] Ledrappier, F.: On the dimension of some graphs, Contemp. Math. 135 285–293 (1992)
[32] Ledrappier, F. and Young, L. S.: The metric entropy of diffeomorphisms, II. Relations between entropy, exponents and dimension. Ann. Math. (2) 122, 540574 (1985)
[33] Lévy, P.: La mesure de Hausdorff de la courbe du mouvement brownien. Giorn. Ist. Ital. Attuari 16, 137 (1953)
[34] Liu, Q.: Asymptotic properties and absolute continuity of laws stable by random weighted mean. Stochastic Process. Appl. 95, 83-107 (2001)
[35] Mandelbrot, B.: Intermittent turbulence in self-similar cascades: divergence of hight moments and dimension of the carrier. J. Fluid. Mech. 62 331–358 (1974)
[36] Mandelbrot, B. and Riedi, R.: Inverse measures, the inversion formula, and discontinuous multifractals. Adv. in Appl. Math. 18, 50-58 (1997)
[37] Marstrand, J. M.: Some fundamental geometrical properties of plane sets of fractional dimensions. Proc. London Math. Soc. 4(3), 257–302 (1954)
[38] Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability. Cambridge studies in advanced mathematics 44, Cambridge University Press (1995)
SINGULARITY SPECTRA OF $b$-ADIC INDEPENDENT CASCADE FUNCTION

[39] Mauldin, R. D. and Williams, S. C.: On the Hausdorff dimension of some graphs, Trans. Amer. Math. Soc. 298 793–803 (1986)

[40] Millar, P. W.: Path behavior of processes with stationary independent increments, Z. Wahrschein. Verw. Gebiete 17 53–73 (1971)

[41] Molchan, G. M.: Scaling exponents and multifractal dimensions for independent random cascades. Comm. Math. Phys. 179, 681–702 (1996)

[42] Orey, S.: Gaussian sample functions and the Hausdorff dimension of level crossings, Z. Wahrschein. Verw. Gebiete 17 39–47 (1971)

[43] Peyrière, J.: A singular random measure generated by splitting $[0,1]$. Z. Wahrsch. Verw. Gebiete 47(3), 289-297 (1979)

[44] Pruitt, W. E.: The Hausdorff dimension of the range of a process with stationary independent increments, J. Math. Mech. 19 371–378 (1969)

[45] Przytycki, F., and Urbański, M.: On the Hausdorff dimension of some fractal sets, Studia. Math. 93(2) 155–186 (1989)

[46] Roueff, F.: Almost sure Hausdorff dimension of graphs of random wavelet series, J. Fourier Anal. Appl. 9 237–260 (2003)

[47] Taylor, S. J.: The Hausdorff dimension of Brownian paths in $n$-space. Proc. Camb. Philos. Soc. 49, 3139 (1953)

[48] Taylor, S. J. and Wendel, J. G.: The exact Hausdorff measure of the zero set of a stable process, Z. Wahrsch. Verw. Gebiete 6 170–180 (1966)

[49] Urbański, M.: The Hausdorff dimension of the graphs of continuous self-affine functions, Proc. Amer. Math. Soc. 108 921–930 (1990)

[50] Waymire, E. C. and Williams, S. C.: Multiplicative cascades: dimension spectra and dependence. Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993). J. Fourier Anal. Appl. Special Issue, 589–609 (1995)

[51] Xiao, Yimin.: Dimension results for Gaussian vector fields and index-$\xi$ stable fields. Ann. Probab. 23(1) 273-291 (1995)

[52] Xiao, Yimin.: Random fractals and Markov processes, In Lapidus, M. and Frankenhuijsen, M. V. ed. Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Proceedings of Symposia in Pure Mathematics. 72(2), 261-338 (2004)

INRIA ROCQUENCOURT, B.P. 105, 78153 LE CHESNAY CEDEX, FRANCE
E-mail address: xiongjin82@gmail.com