Moving and merging of Dirac points on a square lattice and hidden symmetry protection

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First, we study a square fermionic lattice that supports the existence of massless Dirac fermions, where the Dirac points are protected by a hidden symmetry. We also consider two modified models with a staggered potential and the diagonal hopping terms, respectively. In the modified model with a staggered potential, the Dirac points exist in some range of magnitude of the staggered potential, and move with the variation of the staggered potential. When the magnitude of the staggered potential reaches a critical value, the two Dirac points merge. In the modified model with the diagonal hopping terms, the Dirac points always exist and just move with the variation of amplitude of the diagonal hopping. We develop a mapping method to find hidden symmetries evolving with the parameters. In the two modified models, the Dirac points are protected by this kind of hidden symmetry, and their moving and merging process can be explained by the evolution of the hidden symmetry along with the variation of the corresponding parameter.

PACS numbers: 71.10.Fd, 71.10.Pm, 02.20.-a, 03.65.Vf

I. INTRODUCTION

Success in the preparation of graphene has led to an enormous amount of interests in massless Dirac fermions in condensed matter physics. Many schemes on the simulation of massless Dirac fermions in optical lattices have been proposed theoretically and verified experimentally. The recent discovery of topological insulators and Weyl semimetals further facilitated the research on massless Dirac fermions in condensed matter systems.

In condensed matter materials, Dirac fermions appear as emergent particles near Dirac points in the Brillouin zone, where the band degeneracies occur. Around Dirac points, the dispersion relation is linear and can be described by the Dirac equation. Sometimes, the chirality can be defined for massless Dirac fermions, which can be considered as Weyl fermions. In three-dimensional materials, the band degeneracy at Dirac points can be accidental. The von Neumann-Wigner theorem tells us that, to achieve such a two-band accidental degeneracy, three real parameters are required to be tuned. Thus, accidental band degeneracies are vanishingly improbable in two dimensions if there are not additional symmetry constraints. Therefore, in two dimensions, the band degeneracy at Dirac points must be protected by some symmetry. In general, the band degeneracy is protected by point groups or time-reversal symmetry. Recently, the author has shown that touching points of some two-dimensional lattices are protected by a kind of hidden symmetry. These hidden symmetries are discrete symmetries with antiunitary composite operators, which, in general, consist of a translation, a complex conjugation and a sublattice exchange, and sometimes they also include a local gauge transformation and a rotation.

In this paper, we first consider a square fermionic lattice supporting the existence of massless Dirac fermions, where the Dirac points are protected by a hidden symmetry. We regard this model as the original model. We also consider two modified models with a staggered potential and the diagonal hopping terms, respectively. In the modified model with a staggered potential, the two Dirac points with opposite topological charges move away from or towards each other with increasing magnitude of the staggered potential. When the magnitude of the staggered potential arrives at a critical value, the two Dirac points merge.

II. MODEL

First, we consider the original model that consists of two sublattices denoted as A and B, respectively, as shown in Fig.1(a). The two sublattices have the lattice spacings 2d and d in the x and y directions, respectively. For sublattice B, along the y direction, there exists an accompanying phase π of hopping between two neighbor lattice sites. For each sublattice, the primitive lattice vectors are defined as \( \mathbf{a}_1 = (2d, 0) \) and \( \mathbf{a}_2 = (0, d) \). In the following process, for simplicity, we assume \( d = 1 \). The tight-binding Hamiltonian for the original model can be...
written as,

\[ H_0 = -\sum_{i \in A} [t_x a_i^\dagger b_{i+\hat{x}} + t_x a_i^\dagger b_{i-\hat{x}} + t_y a_i^\dagger a_{i+\hat{y}} + t_y e^{-i\pi b_i^\dagger b_{i+\hat{x}+\hat{y}} + H.c.}], \quad (1) \]

where \( a_i \) is the annihilation operator that destructs a particle in the Wannier state \( w^A_i(r) = w^B_0(r - R^A_i) \) located at the site \( i \) in sublattice \( A \), and \( b_j \) is the annihilation operator destructing a particle in the Wannier state \( w^B_j(r) = w^B_0(r - R^B_j) \) located at the site \( j \) in sublattice \( B \); the subscript \( i \equiv (i_x, i_y) \) is the coordinate for the lattice sites; \( \hat{x} = (1, 0) \) and \( \hat{y} = (0, 1) \) represent the unit vectors in the \( x \) and \( y \) directions, respectively; \( t_x \) and \( t_y \) are the amplitudes of hopping along the \( x \) and \( y \) directions, respectively. Without loss of generality, we assume that \( t_x \) and \( t_y \) are positive in the following process.

Based on the original model, we will consider two modified models with different adding terms, respectively. In the first modified model, we add a staggered potential to the original model. The total Hamiltonian can be described by \( H_1 = H_0 + H_s \), where \( H_s \) is the staggered potential Hamiltonian as

\[ H_s = -v \sum_{i \in A} a_i^\dagger a_i + v \sum_{j \in B} b_j^\dagger b_j, \quad (2) \]

where \( v \) is the magnitude of the staggered potential. In the second modified model, we add the diagonal hopping terms to the original model, which is shown in Fig.1(b). The Hamiltonian is \( H_2 = H_0 + H_d \), where \( H_d \) is the diagonal hopping Hamiltonian as

\[ H_d = -t_{xy} \sum_{i \in A} [a_i^\dagger b_{i+\hat{x}+\hat{y}} + a_i^\dagger b_{i-\hat{x}-\hat{y}} - a_i^\dagger b_{i+\hat{x}+\hat{y}} + H.c.], \quad (3) \]

where \( t_{xy} \) is the amplitude of hopping along the diagonal direction.

For the original model and the two modified models, the lattice vectors can be expressed as \( R_m = n_1 a_1 + n_2 a_2 \) with \( n_1 \) and \( n_2 \) being integers. We chose a lattice site in sublattice \( A \) as the origin, then the lattice sites in sublattice \( A \) and sublattice \( B \) can be written as \( R^A_m = R_m \) and \( R^B_m = R_m + a_1/2 \). In the reciprocal lattice, the reciprocal lattice vectors are defined as \( K_m = m_1 b_1 + m_2 b_2 \), where \( m_1 \) and \( m_2 \) are integers, and \( b_1 = (\pi, 0) \) and \( b_2 = (0, 2\pi) \) are the corresponding primitive reciprocal-lattice vectors. Thus, the Brillouin zone is \(-\pi/2 \leq k_x \leq \pi/2, -\pi \leq k_y \leq \pi \), as shown in Fig.2(b).

III. MASSLESS DIRAC FERMIONS, MOVING AND MERGING OF DIRAC POINTS

In this section, we show that Dirac points exist in the original model. In the modified model with a staggered potential, the Dirac points in the Brillouin zone move when the magnitude of the staggered potential changes, and they merge when the magnitude of the staggered potential arrives at a critical value. In the modified model with the diagonal hopping terms, the two Dirac points in the Brillouin zone move towards two opposite directions as the amplitude of the diagonal hopping increases, and they never vanish or merge for any value of the amplitude of the diagonal hopping.

A. The original model

![Figure 1](image1.png)

![Figure 2](image2.png)
and define the two-component annihilation operator as \( \eta_k \equiv [a_k, b_k]^T \). The total Hamiltonian \( H = H_0 + H_1 \) can be rewritten as \( H = \sum_k \eta_k^\dagger H_0(k) \eta_k \) with
\[
H_0(k) = -2t_x \cos k_x \sigma_x - 2t_y \cos k_y \sigma_z,
\]
and \( \sigma_x \) and \( \sigma_z \) are the Pauli matrices.

Diagonalizing Eq. (6), we obtain the dispersion relation as
\[
E_0(k) = \pm \sqrt{4t_x^2 \cos^2 k_x + 4t_y^2 \cos^2 k_y},
\]
The corresponding Bloch functions can be expressed as
\[
\Psi^{(0)}_k(r) \equiv \begin{pmatrix} u^{(0)}_{1,k}(r) \\ u^{(0)}_{2,k}(r) \end{pmatrix} e^{i k \cdot r},
\]
where \( u^{(0)}_{i,k}(r) = u^{(0)}_{i,k}(r + R_m) \). In the momentum space, the Bloch function \( \Psi^{(0)}_k(r) \) and eigenenergy \( E_0(k) \) are periodic for reciprocal lattice vectors, i.e. \( \Psi^{(0)}_k(r) = \Psi^{(0)}_{k+K_m}(r) \) and \( E_0(k) = E_0(k + K_m) \).

The conduction and valence bands touch at \((\pm \pi/2, \pm \pi/2)\), which are located at the boundary of the Brillouin zone as shown in Fig. 2(a). Among these degenerate points, there are only two distinct ones. Near these degenerate points, the single-particle Hamiltonian (6) can be linearized as
\[
h(p) = 2t_x p_x \sigma_x \pm 2t_y p_y \sigma_z,
\]
where the signs \( \pm \) representing the linearized Hamiltonian around the different touching points, respectively. Around the touching points, the quasiparticles behave like massless Dirac fermions. For these massless Dirac fermions, a chirality can be defined as
\[
w = \text{sgn}[\text{det}(v_{ij})] = \pm 1,
\]
for a two-dimensional Hamiltonian \( h(k) = \sum_{ij} v_{ij} k_i \sigma_j \), with \( k \) and \( \sigma \) being the wave vector and the Pauli matrix in two dimensions, respectively. If we use \( \sigma_y \) to rede-note \( \sigma_z \) as defined in Eq. (9), the corresponding quasiparticles have a chirality \( \pm 1 \) as defined above. The quasiparticles are massless Dirac fermions with a chirality, so they can be considered as two-dimensional Weyl fermions. The chirality of Dirac points can be considered as a topological charge.

**B. The modified model with a staggered potential**

For the modified model with a staggered potential, the Bloch Hamiltonian can be written as
\[
H_1^\prime(k) = -2t_x \cos k_x \sigma_x - (v + 2t_y \cos k_y) \sigma_z,
\]
and the corresponding dispersion relation is
\[
E_1(k) = \pm \sqrt{4t_x^2 \cos^2 k_x + (v + 2t_y \cos k_y)^2}.
\]

**FIG. 3:** (Color online). The dispersion relation of the modified model with a staggered potential with \( t_x = t_y = t \) and (a) \( v = t_y \), (b) \( v = 2t_y \), (c) \( v = 3t_y \), (d) \( v = -t_y \), (e) \( v = -2t_y \), (f) \( v = -3t_y \).

In this model, the energy dispersion relation (12) possesses Dirac points at \((\pi/2, \pm \arccos(-v/2t_y))\) in the Brillouin zone for \(|v| < 2t_y\), as shown Fig. 3(a) and (d). We find that the two distinct Dirac points move as \( v \) changes. For the positive \( v \), with increasing \( v \), the two Dirac points move away from each other. When \( v = 0 \), the Dirac points are located at points \((\pi/2, \pm \pi/2)\) as shown in Fig 2(a). When \( v \) changes from 0 to \( t_y \), the two distinct Dirac points move to \((\pi/2, \pm 2\pi/3)\), respectively, as shown in Fig 3(a). When \( v \) arrives at \( 2t_y \), the two Dirac points move to \((\pi/2, \pm \pi)\), which are the same point on the boundary of the Brillouin zone as shown in Fig 2(b), that is to say, the Dirac points merge. If one continues to increase \( v \) to \( v > 2t_y \), a gap opens, as shown in Fig 3(c). Then the system turns into an insulator. For the negative \( v \), with increasing \(|v|\), the two Dirac points move towards each other. When \( v \) changes from 0 to \( -t_y \), the two distinct Dirac points move to \((\pi/2, \pm 3\pi/3)\), respectively, as shown in Fig 3(d). When \( v \) arrives at \( -2t_y \), the two Dirac points merge at the point \((\pi/2, 0)\) as shown in Fig 3(e). When \( v \) is less than \(-2t_y \), a gap opens and the systems turns into an insulator, as shown in Fig 3(f).

The above merging process of Dirac points can be interpreted from the topological view. Two Dirac points have opposite chirality; that is to say, they have opposite topological charges. As long as the band touching points are protected by a symmetry, the topolog-
charges can not be destroyed, and the system is a topological semimetal. However, when the Dirac points with opposite topological charges meet, they merge and the opposite topological charges annihilate each other. A further increase of $|t|$ makes a gap open and the system turns into an insulator.

IV. EXPLANATION FROM HIDDEN SYMMETRY PROTECTION

In this section, we prove that the Dirac points are protected by a kind of hidden symmetry in the original model. In the two modified models, the additive terms violate the hidden symmetry respected by the original model. Thus, we develop a mapping method to find hidden symmetries evolving with the parameters for the two modified model. We explain the moving of Dirac points in the two modified models by the evolution of the hidden symmetries along with the variation of the parameters, and we explain the merging of Dirac points in the modified model with a staggered potential by the disappearance of the hidden-symmetry-invariant points in the Brillouin zone.

A. The original model

The original model supports the existence of massless Dirac fermions with the Dirac points located at $(\pi/2, \pm \pi/2)$ in the Brillouin zone. We will show that the band degeneracies at the Dirac points are protected by a hidden symmetry. Here, we define a hidden symmetry with the operator as follows,

$$\Upsilon = (e^{i\pi})^i \sigma_x KT_{a_1/2},$$

(15)

where $T_{a_1/2} = \{E|a_1/2\}$ is a translation operator that moves the lattice by $a_1/2$ along the $x$ direction, $K$ is the complex conjugate operator, $\sigma_x$ is the Pauli matrix representing sublattice exchange, and $(e^{i\pi})^i$ is a local $U(1)$ gauge transformation. Obviously, the hidden symmetry operator $\Upsilon$ is an antiunitary operator. The corresponding inverse operator is $\Upsilon^{-1} = (e^{i\pi})^i \sigma_x KT_{a_1/2}^{-1}$. It is easy to verify that the Hamiltonian of the original model is invariant under the hidden symmetry transformation, i.e., $H_0 = \Upsilon H_0 \Upsilon^{-1}$.

The hidden symmetry operator $\Upsilon$ acts on the Bloch functions $\Psi_k$ as follows

$$\Upsilon \Psi_k^{(0)}(r) = \begin{cases} u_{2,k}^{(0)}(r - a_1/2) e^{i k_x} & \text{if } k_y > 0, \\ u_{1,k}^{(0)}(r - a_1/2) e^{i k_x} & \text{if } k_y < 0 \end{cases},$$

(16)

Because $\Upsilon$ is the symmetry operator for the original model, $\Psi_k^{(0)}(r)$ must be a Bloch wave function of the original model. Thus, we obtain $u_{1,k}^{(0)}(r) = u_{2,k}^{(0)*} (r - a_1/2) e^{i k_x}$ and $u_{2,k}^{(0)}(r) = u_{1,k}^{(0)*} (r - a_1/2) e^{i k_x}, k_x = -k_x$ and $k'_y = \pi - k_y$. The square of the hidden symmetry operator is

$$\Upsilon^2 = T_{a_1},$$

(17)

where $T_{a_1} = \{E|a_1\}$. Therefore, we have

$$\Upsilon^2 \Psi_k^{(0)}(r) = T_{a_1} \Psi_k^{(0)}(r) = e^{-2i k_x} \Psi_k^{(0)}(r).$$

(18)
From Eqs. [8] and [16], it is easy to show that the operator $\Upsilon$ has the following effect when acting on the wave vector $k$:

$$\Upsilon : k = (k_x, k_y) \rightarrow k' = (-k_x, -k_y + \pi). \tag{19}$$

If $k' = k + K_m$, then we can say that $k$ is an invariant point under the hidden symmetry transformation. In the Brillouin zone, the $\Upsilon$-invariant points are $M_{1,2} = (\pi/2, \pm \pi/2)$ and $M_{3,4} = (0, \pm \pi/2)$ as shown in Fig2(b). For a $\Upsilon$-invariant point $M_i$, we have $\Upsilon \Psi^{(0)}_M(r) = \Psi^{(0)}_{M_i}(r)$. Thus, $\Psi^{(0)}_M(r)$ and $\Psi^{(0)}_{M_i}(r)$ are both the eigenstates of Hamiltonian $H_0$ and have the same eigenenergy $E_0(M_i)$. Considering Eq. [17], we have $\Upsilon^2 \Psi^{(0)}_M(r) = T_{\delta t_y} \Psi^{(0)}_{M_i}(r) = e^{-2iM_x \delta t_y} \Psi^{(0)}_{M_i}(r)$, where $M_x$ is the $x$ component of $M_i$. We define $(\psi, \varphi)$ as the inner product of the two wave functions $\psi$ and $\varphi$. The antiunitary operator $\Upsilon$ has the property that $(\Upsilon \psi, \Upsilon \varphi) = (\psi, \varphi)^*$. Therefore, we have

$$\langle \Psi^{(0)}_{M_i} | \Psi^{(0)}_M \rangle = \langle \Upsilon \Psi^{(0)}_M | \Upsilon \Psi^{(0)}_{M_i} \rangle = \langle \Upsilon \Psi^{(0)}_M | \Upsilon^2 \Psi^{(0)}_{M_i} \rangle = e^{-2iM_x \delta t_y} \langle \Psi^{(0)}_M | \Psi^{(0)}_{M_i} \rangle. \tag{20}$$

Substituting the concrete $\Upsilon$-invariant points $M_i$, we have $\Upsilon^2 = -1$ at $M_{1,2}$, while $\Upsilon^2 = -1$ at $M_{3,4}$. Then we obtain the solution $(\Psi^{(0)}_{M_i}, \Psi^{(0)}_M) = 0$ at $M_{1,2}$, i.e., $\Psi^{(0)}_{M_i}$ and $\Psi^{(0)}_M$ are orthogonal to each other, while $(\Psi^{(0)}_{M_i}, \Psi^{(0)}_M)$ is unconstrained for Eq. [20] at $M_{3,4}$. Therefore, we arrive at the conclusion that the system must be degenerate at points $M_{1,2}$ in the Brillouin zone, which are just the positions where the Dirac points are located. We can conclude that the two Dirac points are protected by the hidden symmetry $\Upsilon$.

### B. The modified model with a staggered potential

For the modified model with a staggered potential, the total Hamiltonian violates the hidden symmetry, i.e., $[\Upsilon, H_1] \neq 0$. However, Dirac points still exist and just move to other points in the Brillouin zone before the magnitude of the staggered potential $\delta t_y$ arrives at the critical value. Due to the von Neumann-Wigner theorem, in the two-dimensional lattice, the band degeneracy must be protected by a symmetry. After the hidden symmetry $\Upsilon$ is violated, which symmetry protects the band degeneracy at the Dirac points? In the following, we will explain which symmetry is responsible for that.

Now we define a mapping as

$$\Omega_v : (k, H_1(k), \Psi^{(1)}_k(r)) \rightarrow (K, H_0(K), \Psi^{(0)}_K(r)), \tag{21}$$

which maps the Bloch Hamiltonian [11] into the form as

$$H_0(K) = -2t'_x \cos K_x \sigma_x - 2t'_y \cos K_y \sigma_z, \tag{22}$$

where $t'_x = t_x$ and $t'_y = t_y + |v|/2$. Eq. [22] is just the Bloch Hamiltonian [6] of the original model except for

![FIG. 5: (color online). The mapping of the Brillouin zone of the modified model with a staggered potential for (a) $v = t_y$, (b) $v = 2t_y$, (c) $v = 3t_y$, (d) $v = -t_y$, (e) $v = -2t_y$, and (f) $v = -3t_y$ into the Brillouin zone of the original model, respectively. The area surrounded by the solid line is the Brillouin zone of the original model. The yellow shaded area is the image of the Brillouin zone of the modified model with a staggered potential mapping into the Brillouin zone of the original model. The filled circles represent the $\Upsilon$-invariant points in the Brillouin zone of the original model.](image)

the different notations of the parameters. Eq. [22] must have the $\Upsilon$-invariant points in Brillouin zone as $M_{1,2} = (\pi/2, \pm \pi/2)$ and $M_{3,4} = (0, \pm \pi/2)$ as shown in Fig2(b). For the wave vectors, the transformation [21] is explicitly written as

$$K_x = k_x, \quad k_x \in [-\pi/2, \pi/2], \tag{23}$$

$$K_y = \begin{cases} -\arccos \left( \frac{v + 2t_y \cos K_y}{|v| + 2t_y} \right), & k_y \in [-\pi, 0] \\ \arccos \left( \frac{v + 2t_y \cos K_y}{|v| + 2t_y} \right), & k_y \in [0, \pi] \end{cases}. \tag{24}$$

and for the Bloch functions, it can be written as

$$\Psi^{(0)}_K(r) = \Psi^{(0)}_k(r), \tag{25}$$

where $\Psi^{(0)}_K(r)$ is the Bloch function of the original model. The shift from $k$ to $K$ after the transformation is $K - k = (0, \Delta k_y)$ with

$$\Delta k_y = \begin{cases} -k_y - \arccos \left( \frac{v + 2t_y \cos K_y}{|v| + 2t_y} \right), & k_y \in [-\pi, 0] \\ -k_y + \arccos \left( \frac{v + 2t_y \cos K_y}{|v| + 2t_y} \right), & k_y \in [0, \pi] \end{cases} \tag{26}$$

The transformation [23] implies that $k_x$ mapping into $K_x$ is just an equivalence. From Eq. [24], we note that the mapping $k_y \rightarrow K_y$ is more complicated. When $v$ is positive, $\Omega_v$ maps the range $[-\pi, \pi]$ for $k_y$ into the range $-\arccos((v - 2t_y)/(|v| + 2t_y)), \arccos((v -
2t_y)/(|v| + 2t_y))$$ for $K_v$. For the points located on the top and bottom boundaries of the Brillouin zone of the modified model with a staggered potential, due to the equivalence between points on the boundary of the Brillouin zone, the mapping is not one-to-one. For the interior points of the Brillouin zone of the modified model with a staggered potential, the mapping is continuous and one-to-one. On the whole, the mapping is not surjective for non-vanishing $v$, i.e., the whole Brillouin zone of the modified model with a staggered potential maps into part of the Brillouin zone of the original model, as shown in Figs.5(a), (b) and (c). When $v$ is negative, $\Omega_v$ maps the ranges $[-\pi, 0]$ and $[0, \pi]$ for $k_y$ into the ranges $[-\pi, -\arccos((v + 2t_y)/(|v| + 2t_y))]$ and $[\arccos((v + 2)/(|v| + 2t_y)), \pi]$ for $K_y$, respectively. Especially, $\Omega_v$ maps $k_y = 0$ into $K_y = \pm \arccos((v + 2)/(|v| + 2t_y))$, which is not one-to-one. In the interior part of each range, the mapping is continuous and one-to-one. On the whole, the mapping is not surjective for non-vanishing $v$, as shown in Figs.5(d), (e) and (f).

We define a new hidden symmetry as $\Lambda_v = \Omega_v^{-1} \Theta \Omega_v$, which consists of three operators acting in order on the wave vector and the Bloch functions. Since the operator $\Lambda_v$ depends on $v$, the hidden symmetry evolves along with the magnitude of the staggered potential. When $v = 0$, the operator $\Lambda_v$ returns to $\Theta$. For the wave vectors, the operation is performed as $\Omega_v : k \rightarrow K$, $\Theta : K \rightarrow K'$ and $\Omega_v^{-1} : K' \rightarrow k'$. Considering the explicit form of these transformations as Eqs. (19), (23) and (24), we have

$$\Lambda_v : k = (k_x, k_y) \rightarrow k' = (-k_x, -k_y - \delta k_y - \delta k'_y + \pi)$$

(27)

If the condition $k' = k + K_m$ is satisfied, then we can say that $k$ is a $\Lambda_v$-invariant point in the Brillouin zone of the modified model with a staggered potential. Through Eqs. (26) and (27), we can show that the $\Lambda_v$-invariant points in the Brillouin zone of the modified model with a staggered potential have the form as $P_{1,2} = (\pi/2, \pm \arccos(-v/2t_y))$ and $P_{3,4} = (0, \pm \arcsin(-v/2t_y))$. We find that when $|v/t_y| = 2$, $\Lambda_v$-invariant points $P_1$ and $P_2$ are located at the $k_y = 0$ line on the or top and bottom boundaries of the Brillouin zone of the modified model with a staggered potential, so they meet together. That is so for $P_3$ and $P_4$. However, when $|v/t_y| > 2$, there is no solution for $\Lambda_v$-invariant points, i.e. there does not exist any $\Lambda_v$-invariant point.

For the square of the operator $\Lambda_v$, we have $\Lambda_v^2 = \Omega_v^{-1} \Theta \Omega_v$. The hidden symmetry operator $\Lambda_v$ acting on the Bloch function $\Psi_k^{(1)}(r)$ twice successively has following the effect:

$$\Lambda_v^2 \Psi_k^{(1)}(r) = e^{-2ik_x} \Psi_k^{(1)}(r),$$

(28)

which can be derived from Eqs. (18), (23), (24) and (25). Since $\Lambda_v$ is an antiunitary operator, similar to Eq. (20), we have the following equation

$$(\Psi_{P_1}^{(1)}, \Psi_{P_1}^{(1)}) = (\Lambda_v \Psi_{P_1}^{(1)}, \Lambda_v \Psi_{P_1}^{(1)}) = (\Lambda_v \Psi_{P_1}^{(1)}, \Lambda_v^2 \Psi_{P_1}^{(1)}) = e^{-2iP_y} (\Psi_{P_1}^{(1)}, \Psi_{P_1}^{(1)}),$$

(29)

From Eq. (28), we have $\Lambda_v^2 = -1$ at $\Lambda_v$-invariant points $P_1$ and $P_2$, and $\Lambda_v^2 = 1$ at $\Lambda_v$-invariant points $P_3$ and $P_4$. Therefore, we have the solution $(\Psi_{P_1}^{(1)}, \Psi_{P_1}^{(1)}) = 0$ at the $\Lambda_v$-invariant points $P_1$ and $P_2$. We conclude that the bands must be degenerate at the points $P_1$ and $P_2$ but are not at the points $P_3$ and $P_4$, which is consistent with the dispersion relation calculated previously. That is to say, the Dirac points at $P_1$ and $P_2$ are protected by the hidden symmetry $\Lambda_v$. Since $\Lambda_v$ depends on the magnitude of the staggered potential $v$, the hidden symmetry protected degenerate points $P_1$ and $P_2$ evolve along with changing of the parameter $v$ for $|v/t_y| < 2$. For the case $|v/t_y| = 2$, $P_1$ and $P_2$ are the same points, so the Dirac points merge. When $|v/t_y| > 2$, the $\Lambda_v$-invariant points $P_1$ and $P_2$ do not exist, so a gap opens.

We can interpret the hidden symmetry protection in a more intuitive way. Although the modified model with a staggered potential violates the hidden symmetry $\Theta$, the operator $\Omega_v$ can map the Bloch Hamiltonian (11) into the Bloch Hamiltonian (6), which is just the Bloch Hamiltonian of the original model. However, the mapping $\Omega_v$ is not surjective. That is, the Brillouin zone of the modified model with a staggered model maps into part of the Brillouin zone of the original model, as shown in Fig. 5. When $|v/t_y| < 2$, the image of the Brillouin zone of the modified model with a staggered model includes the $\Theta$-invariant points in the Brillouin zone of the original model, as shown in Fig. 5(a) and (d). There always exist four points in the Brillouin zone of the modified model with a staggered model mapping into the four $\Theta$-invariant points in the Brillouin zone of the original model. When $|v/t_y| = 2$, the two points on the boundary or $k_y = 0$ line of the Brillouin zone of the modified model with a staggered potential map into the four $\Theta$-invariant points in the Brillouin zone of the original model, as shown in Fig. 5(b) and (e). The two Dirac points meet and merge. When $|v/t_y| > 2$, the Brillouin zone of the modified model with a staggered potential maps into part of the Brillouin zone of the original model, which does not include the $\Theta$-invariant points as shown in Fig. 5(c) and (f). This is to say, there does not exist any point in the Brillouin zone of the modified model with a staggered potential that can map into the $\Theta$-invariant points in the Brillouin zone of the original model. Thus, there are no symmetries to support the existence of Dirac points.

C. The modified model with the diagonal hopping terms

In the modified model with the diagonal hopping terms, the hidden symmetry $\Theta$ respected by the original model is violated due to the additive diagonal hopping terms. However, in this model, the Dirac points do not vanish, so there must be some symmetry to protect them.

Similarly, we define a mapping from the modified model
model with the hopping terms to the original model as
\[ \Omega'_{txy}(k, h_2(k), \Psi^{(2)}_k(r)) \rightarrow (K, h_0(K), \Psi^{(0)}_K(r)), \] (30)
where \( \Psi^{(2)}_k(r) \) is the Bloch function for the modified model with the diagonal hopping terms. Toward that end, we first rewrite the Bloch Hamiltonian \([13]\) as
\[ h_2(k) = -2t'_{xy} \cos(k_x - \alpha_{k_y}) \sigma_x - 2t'_{y} \cos k_y \sigma_z, \] (31)
where the parameter \( T_{k_y} \) is defined as \( T_{k_y} = \sqrt{t_y^2 + 4t_y^2 \sin^2 k_y} \), which depends on \( k_y \); the parameter \( \alpha_{k_y} \) also depends on \( k_y \) and is defined by \( \alpha_{k_y} = \arctan(2t_{xy} \sin k_y/t_x) \). If we suppose that the mapping \( \Omega'_{txy} \) has the effect as
\[ \Omega'_{txy}(k, y) \rightarrow (K, K_y) = (k_x - \alpha_{k_y}, k_y), \] (32)
and \( T_{k_y} \rightarrow t'_{x} = T_{k_y} \cos \alpha_{k_y} = t_x \) and \( t_y \rightarrow t'_{y} = t_y \), then \( \mathcal{H}_2(k) \) maps into \( \mathcal{H}_0(K) \) as
\[ \mathcal{H}_0(K) = -2t'_{xy} \cos K_x \sigma_x - 2t'_{y} \cos K_y \sigma_z, \] (33)
which is just the Bloch Hamiltonian \([6]\) except for the different notations of the parameters. For the Bloch functions, we have
\[ \Omega'_{txy} \Psi^{(2)}_k(r) = \Psi^{(0)}_K(r). \] (34)

We define a new hidden symmetry as \( \Lambda'_{txy} = \Omega'_{txy}^{-1} \Upsilon \mathcal{O}'_{txy} \). For the wave vectors, the operation is performed as \( \Omega'_{txy} : k \rightarrow K, \Upsilon : K \rightarrow K', \Omega'_{txy}^{-1} : K' \rightarrow k' \). Considering the explicit form of these transformations as Eqs. \([19]\) and \([32]\), we have
\[ \Lambda'_{txy} : k = (k_x, k_y) \rightarrow k' = (-k_x + \alpha_{k_y} + \alpha'_{k_y}, -k_y + \pi) \] (35)
If the condition \( k' = k + K_\pi \) is satisfied, then we can say that \( k \) is a \( \Lambda'_{txy} \)-invariant point in the Brillouin zone of the modified model with the diagonal hopping terms. In this model, the \( \Lambda'_{txy} \)-invariant points in the Brillouin zone are \( Q_1 = (-\arctan(t_x/2t_{xy}), \pi/2), Q_2 = (\arctan(t_x/2t_{xy}), -\pi/2), Q_3 = (\arctan(2t_{xy}/t_x), \pi/2) \) and \( Q_4 = (-\arctan(2t_{xy}/t_x), -\pi/2) \), as shown in Fig. 6(a) for the case of \( t_{xy}/t_x > 0 \) and Fig. 6(d) for the case of \( t_{xy}/t_x < 0 \).

For the sake of the operator \( \Lambda'_{txy} \), we have \( \Lambda'_{txy}^2 = \Omega'_{txy}^{-1} \Upsilon^2 \Omega'_{txy} = \Omega'_{txy}^{-1} T_{a_\pi} \Upsilon \Omega'_{txy} \). The hidden symmetry operator \( \Lambda'_{txy} \) acting on the Bloch function \( \Psi^{(2)}_k(r) \) twice successively has the effect as
\[ \Lambda'_{txy}^2 \Psi^{(2)}_k(r) = e^{-2i(k_x - \alpha_{k_y})} \Psi^{(2)}_k(r) \] (36)
which can be derived from Eqs. \([18]\) and \([32]\). Since \( \Lambda'_{txy} \) is an antiunitary operator, similar to Eq. \([20]\), we have the following equation
\[ (\Psi^{(2)}_{Q_1}, \Psi^{(2)}_{Q_2}) = (\Lambda'_{txy} \Psi^{(2)}_{Q_1}, \Lambda'_{txy} \Psi^{(2)}_{Q_1}) = (\Lambda'_{txy} \Psi^{(2)}_{Q_1}, \Lambda'_{txy}^2 \Psi^{(2)}_{Q_1}) = e^{-2i(k_x - \alpha_{k_y})} (\Psi^{(2)}_{Q_1}, \Psi^{(2)}_{Q_2}). \] (37)
From Eq. \([36]\), we can obtain \( \Lambda^2_{txy} = -1 \) at the \( \Lambda'_{txy} \)-invariant points \( Q_1 \) and \( Q_2 \), and \( \Lambda^2_{txy} = 1 \) at the \( \Lambda'_{txy} \)-invariant points \( Q_3 \) and \( Q_4 \). Therefore, we have the solution \( (\Psi^{(2)}_{Q_1}, \Psi^{(2)}_{Q_2}) = 0 \) at the \( \Lambda'_{txy} \)-invariant points \( Q_1 \) and \( Q_2 \). We conclude that the bands must be degenerate at the points \( Q_1 \) and \( Q_2 \) while the band degeneracy is not guaranteed at the points \( Q_3 \) and \( Q_4 \), which is consistent with the dispersion relation calculated previously. That is to say, the Dirac points at \( Q_1 \) and \( Q_2 \) are protected by the hidden symmetry \( \Lambda'_{txy} \). The hidden symmetry \( \Lambda'_{txy} \) evolves along with the parameter \( t_{xy} \). It is easy to find that when \( t_{xy} = 0 \), the hidden symmetry operator \( \Lambda'_{txy} \) returns to the operator \( \Upsilon \) and the degenerate points \( Q_1 \) and \( Q_2 \) are just the points \( M_1 \) and \( M_2 \). When \( t_{xy} \) changes, the degenerate points \( Q_1 \) and \( Q_2 \) move towards
opposite directions, respectively. When \( t_{xy} \) approaches infinity, the degenerate points approach the \( k_y = 0 \) line from two sides, respectively. For any value of the parameter \( t_{xy} \), the degenerate points \( Q_1 \) and \( Q_2 \) do not merge and no gap opens. All these conclusions are consistent with the dispersion relation calculated previously.

We can interpret the above conclusions from the mapping of the Brillouin zone of the modified model with the diagonal hopping terms to the Brillouin zone of the original model, which is shown in Figs. 6(a), (b), (c) for the case of \( t_{xy}/t_x > 0 \) and Figs. 6(d), (e), (f) for the case of \( t_{xy}/t_x < 0 \). Figs. 6(a) and (d) show the Brillouin zone of the modified model with the diagonal hopping terms. Figs. 6(b) and (e) show the image of the mapping \( \Omega'_{t_{xy}} \) of the Brillouin zone of the modified model with the diagonal hopping terms in the momentum space of the original model. If the image of the mapping \( \Omega'_{t_{xy}} \) is restricted in the Brillouin zone of the original model, it is like that shown in Figs. 6(c) and (f). It is easy to find that the mapping \( \Omega'_{t_{xy}} \) just shifts the points in the Brillouin zone along the \( x \) direction as shown in Figs. 6(b) and (e). The mapping \( \Omega'_{t_{xy}} \) is one-to-one and surjective, which can be found from Figs. 6(c) and (d). Specifically, the left and right boundaries of the Brillouin zone of the modified model with the diagonal hopping terms as shown in Figs. 6(a) and (b) map into the red solid lines in the Brillouin zone of the original model as shown in Figs. 6(c) and (f). The black curved lines in the Brillouin zone of the modified model with the diagonal hopping terms as shown in Figs. 6(a) and (b) map into the left and right boundary of the Brillouin zone of the original model, where the \( \Upsilon \)-invariant points \( M_1 \) and \( M_2 \) are located. The \( \Lambda'_{t_{xy}} \)-invariant points \( Q_i (i = 1, 2, 3, 4) \) in the Brillouin zone of the modified model with the diagonal hopping terms map into the \( \Upsilon \)-invariant points \( M_1 (i = 1, 2, 3, 4) \) in the Brillouin zone of the original model. Since the mapping is surjective, there always exist points \( Q_1 \) and \( Q_2 \) in the Brillouin zone of the modified model with the diagonal hopping terms mapping into the \( \Upsilon \)-invariant points \( M_1 \) and \( M_2 \) in the Brillouin zone of the original model. Therefore, the Dirac points always are protected by a hidden symmetry and no gap opens for any value of the parameter \( t_{xy} \). Because the corresponding hidden symmetry \( \Lambda'_{t_{xy}} \) evolves along with the parameter \( t_{xy} \), the Dirac points move as the parameter \( t_{xy} \) changes.

V. CONCLUSION

In summary, we have studied the original model, a fermionic square lattice with only the horizontal and vertical hopping terms, and the two modified models with a staggered potential and the diagonal hopping terms, respectively. All three models support the existence of massless Dirac fermions. In the original model, there are two Dirac points in the Brillouin zone, which are protected by a hidden symmetry. In the modified model with a staggered potential, the two Dirac points move away from or approach each other with increasing of the magnitude of the staggered potential. When the magnitude arrives at a critical value, the two Dirac points merge at the \( k_y = 0 \) line or the \( k_y = \pi \) line which is determined by the sign of the staggered potential. When the magnitude of the staggered potential is greater than the critical value, a gap opens, and the system becomes an insulator. In the modified model with the diagonal hopping terms, the two Dirac points in the Brillouin move with increasing amplitude of the diagonal hopping in two opposite directions, respectively. But the Dirac points never vanish and the system is always gapless for any amplitude of the diagonal hopping. For the two modified models, we have developed a mapping method that maps the modified models into the original model, to find hidden symmetries evolving with the parameters. The moving of the Dirac points in the Brillouin zone for the two modified models can be explained by the evolution of the hidden symmetries along with the parameters. The merging of Dirac points in the modified model with a staggered potential can also be explained by the disappearance of the hidden-symmetry-invariant points in the Brillouin zone when the parameter is beyond the critical value. The original model can be realized experimentally and detected in an optical lattice with laser-assisted tunneling as proposed in Reference [6]. Based on the original model, two modified models can also be realized with the existing techniques on optical lattices. The topological charge at Dirac points can be detected by the interferometric approach [22].

Acknowledgments

We thank W. Chen for helpful discussions. This work was supported by the National Natural Science Foundation of China under Grants No. 11274061 and No. 11004028.

1. K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, Y. Zhang, S.V. Dubonos, I.V. Grigorieva, and A. A. Firsov, Science 306, 666 (2004).

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K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos, and A.A. Firsov, Nature (London) **438**, 197 (2005).

Y. Zhang, Y.W. Tan, H.L. Stormer, and Philip Kim, Nature (London) **438**, 201 (2005).

V. P. Gusynin and S. G. Sharapov, Phys. Rev. Lett. **95**, 146801 (2005).

G. Li and E. Y. Andrei, Nat. Phys. **3**, 623 (2007).

J.M. Hou, W.X. Yang and X.J. Liu, Phys. Rev. A **79**, 043621 (2009).

S.L. Zhu, B. Wang, and L.M. Duan, Phys. Rev. Lett. **98**, 260402 (2007).

N. Goldman, A. Kubasiak, A. Bermudez, P. Gaspard, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. **103**, 035301 (2009).

D. Bercioux, D. F. Urban, H. Grabert, and W. Häusler, Phys. Rev. A **80**, 063603 (2009).

L. Fu, C.L. Kane, and E.J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).

J.E. Moore and L. Balents, Phys. Rev. B **75**, 121306 (2007).

R. Roy, Phys. Rev. B **79**, 195322 (2009).

L. Balents, Physics **4**, 36 (2011).

G.E. Volovik, Lect. Notes in Phys. **870**, 343 (2013).

J. von Neumann and E. Wigner, Z. Phys. **30**, 467 (1929).

J.H. Jiang, Phys. Rev. A **85**, 033640 (2012).

P. Delplace, J. Li, and D Carpentier, Europhys. Lett. **97**, 67004 (2012).

J.M. Hou, Phys. Rev. Lett. **111**, 130403 (2013).

L. Balents, Physics **4**, 36 (2011).

D.A. Abanin, T. Kitagawa, I. Bloch, and E. Demler, Phys. Rev. Lett. **110**, 165304 (2013).