SETS OF APPROXIMATION AND INTERPOLATION IN $\mathbb{C}$ FOR
MANIFOLD-VALUED MAPS

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1. Introduction

1.1. Notation. This article is devoted to the study of approximation and interpolation of maps from a compact $K \subset \mathbb{C}$ into a complex manifold $\mathcal{M}$, and in particular to giving examples of sets $K$ for which certain types of approximation and interpolation are possible. For brevity, we introduce three properties $A_1$, $A_2$ and $A_3$ that compact subsets of the plane may possess. We first define property $A_2$.

For a compact $K \subset \mathbb{C}$, let $K^c$ be the interior of $K$. We will let $\mathcal{A}(K, \mathcal{M})$ denote the continuous maps from $K$ into the complex manifold $\mathcal{M}$ which are holomorphic on $K^c$. Let $\mathcal{O}(K, \mathcal{M}) \subset \mathcal{A}(K, \mathcal{M})$ denote the subspace of those maps $f$ which extend to a holomorphic map from some neighborhood $U_f$ of $K$ to $\mathcal{M}$. We endow $\mathcal{A}(K, \mathcal{M})$ with an arbitrary metric, which makes $\mathcal{A}(K, \mathcal{M})$ and $\mathcal{O}(K, \mathcal{M})$ into metric spaces with the uniform metric on maps. We will say that a compact $K \subset \mathbb{C}$ has the property $A_2$, if, for every complex manifold $\mathcal{M}$, every finite set $\mathcal{P} \subset \mathcal{M}$, and every $\epsilon > 0$, we can approximate any $f \in \mathcal{A}(K, \mathcal{M})$ by a map $f_\epsilon \in \mathcal{O}(K, \mathcal{M})$ such that $\text{dist}_{\mathcal{M}}(f, f_\epsilon) < \epsilon$, and for each $p \in \mathcal{P}$, we have $f_\epsilon(p) = f(p)$. That is, $f_\epsilon$ is an uniform approximation to $f$ which interpolates the values of $f$ on $\mathcal{P}$.

We now define the property $A_1$, which will be the special case of $A_2$ with $\mathcal{M} = \mathbb{C}$. More precisely, we will say that $K \subset \mathbb{C}$ has the property $A_2$, if, for any finite $\mathcal{P} \subset K$, any function in $\mathcal{A}(K) := \mathcal{A}(K, \mathbb{C})$ can be uniformly approximated by functions in $\mathcal{O}(K) := \mathcal{O}(K, \mathbb{C})$ which interpolate the values of $f$ on $\mathcal{P}$. Obviously for a set $K$, $A_2 \Rightarrow A_1$.

It is easy to see that a set $K$ has property $A_1$ iff $\mathcal{O}(K)$ is dense in $\mathcal{A}(K)$. Note the trivial fact that there is a constant $C > 0$ such that for $g \in \mathcal{C}(K)$, if $L_\mathcal{P}(g)$ is the Lagrange polynomial which interpolates the values of $g$ on $\mathcal{P}$, we have $\|L_\mathcal{P}(g)\|_K < C \|g\|_K$. If $\tilde{f} \in \mathcal{O}(K)$ be such that $|f - \tilde{f}| < \frac{\epsilon}{C}$, then $|f - f_\epsilon| < \epsilon$, and $f(p) = f_\epsilon(p)$ for $p \in \mathcal{P}$, where $f_\epsilon = \tilde{f} + L_\mathcal{P}(f - \tilde{f})$.

The compact sets $K \subset \mathbb{C}$ such that $\mathcal{O}(K)$ is dense in $\mathcal{A}(K)$ can be characterized by Vituškin’s theorem (\cite{15}). A sufficient condition is that $\mathbb{C} \setminus K$ has finitely many connected components. Also, such approximation can be localized, in the sense that $\mathcal{O}(K)$ is dense in $\mathcal{A}(K)$ iff every point $z \in K$ has a neighborhood $U_z$ in $\mathbb{C}$ such that $\mathcal{O}(K \cap \overline{U_z})$ is dense in $\mathcal{A}(K \cap \overline{U_z})$.

We now define property $A_3$. Let $\phi : \mathcal{M} \to \mathbb{C}$ be a holomorphic submersion such that $\phi(K) \supset \mathcal{M}$. Let $\mathcal{A}_\phi(K, \mathcal{M})$ (resp. $\mathcal{O}_\phi(K, \mathcal{M})$) be the subspace of $\mathcal{A}(K, \mathcal{M})$ (resp. $\mathcal{O}(K, \mathcal{M})$) consisting of sections of $\phi$ over $K$, i.e. maps $s : K \to \mathcal{M}$ such that $\phi \circ s = \text{Id}_K$. We will say that a compact
$K \subset \mathbb{C}$ has property $A_3$ if for every complex manifold $\mathcal{M}$ and every holomorphic submersion $\phi$ such that $\phi(\mathcal{M}) \supset K$, and every finite set $\mathcal{P} \subset K$, every section $\sigma \in \mathcal{A}_\phi(K,\mathcal{M})$ can be uniformly approximated by sections in $\mathcal{O}_\phi(K,\mathcal{M})$ which interpolate the values of $\sigma$ on $\mathcal{P}$.

We also note the following elementary fact:

**Lemma 1.** For a compact $K \subset \mathbb{C}$, $A_3 \Rightarrow A_2$.

**Proof.** Assume that $K$ satisfies $A_3$, and let $\mathcal{P}$ and $\mathcal{M}$ have the same meaning as above, and let $f \in \mathcal{A}(K,\mathcal{M})$. Consider the complex manifold $\mathcal{N} = \mathcal{M} \times \mathbb{C}$ and let $\phi$ and $\pi$ be the projections onto $\mathbb{C}$ and $\mathcal{M}$ respectively. If $F : K \to \mathcal{N}$ is defined by $F(z) = (f(z), z)$, then clearly $F \in \mathcal{A}_\phi(K,\mathcal{N})$ and since $K$ has the property $A_3$, we can approximate $F$ by maps $G \in \mathcal{O}_\phi(K,\mathcal{N})$ such that $G(p) = F(p) = (f(p), p)$ for each $p \in \mathcal{P}$. Then $g = \pi \circ G$ is in $\mathcal{O}(K,\mathcal{M})$ and an approximation to $f$ with $g(p) = f(p)$ for each $p \in \mathcal{P}$, which shows that $K$ has property $A_2$. □

(We may remark here that while holomorphic submersions $\phi : \mathcal{M} \to \mathbb{C}$ always exist if $\mathcal{M}$ is Stein ([7]), in general such $\mathcal{M}$ may be highly nontrivial, see e.g. [3].)

It is natural to ask the question: which are the compact sets in the plane which satisfy property $A_2$ or property $A_3$? Given that approximation is local in nature, one can conjecture that using patching arguments in the manifold $\mathcal{M}$ we may be able to show that every set with property $A_1$ has property $A_2$, or even $A_3$. We do not know if this program can be carried out. In this article, we confine ourselves to the much more modest goal of giving examples of sets $K$ for which properties $A_2$ and $A_3$ hold (see Theorems [12] and [3] below.)

1.2. Known results. Some results have been obtained recently regarding the approximation and interpolation of manifold-valued maps.

The following was proved in [2]. Define a Jordan domain to be an open set $\Omega \subset \mathbb{C}$, such that the boundary $\partial \Omega$ has finitely many connected components, each of which is homeomorphic to a circle. (We explicitly allow $\Omega$ to be multiply connected.)

**Proposition 1.** Let $K = \overline{\Omega}$, where $\Omega \subset \mathbb{C}$ is a Jordan domain. For any complex manifold $\mathcal{M}$, the subspace $\mathcal{O}(K,\mathcal{M})$ is dense in $\mathcal{A}(K,\mathcal{M})$.

This is purely a statement about approximation, no interpolation being involved. The following result is contained in what was proved by Drnovšek and Forstnerič in [6], Theorem 5.1:

**Proposition 2.** Let $\Omega \subset \mathbb{C}$ have $C^2$ boundary, $\mathcal{M}$ be a complex manifold, and let $f : \overline{\Omega} \to \mathcal{M}$ be a map of class $C^r$ ($r \geq 2$) which is holomorphic in $\Omega$. Given finitely many points $z_1, \ldots, z_l \in \Omega$, and an integer $k \in \mathbb{N}$, there is a sequence of maps $f_\nu \in \mathcal{O}(K,\mathcal{M})$ such that $f_\nu$ agrees with $f$ to order $k$ at $z_j$ for $j = 1, \ldots, l$ and $\nu \in \mathbb{N}$, and the sequence $f_\nu$ converges to $f$ in $C^r(\overline{\Omega})$ as $\nu \to \infty$.

This can in fact be proved when $\Omega \subset S$, where $S$ is a non-compact Riemann surface, and $\mathcal{M}$ is a complex space. Observe that there is a stronger assumption on the smoothness of the map than in property $A_2$, and also a stronger conclusion regarding approximation and interpolation. However, the set of points of interpolation is restricted to the interior of $\overline{\Omega}$. The authors subsequently proved the following:
Proposition 3. If \( \Omega \subseteq \mathbb{C} \) is a domain with \( C^2 \) boundary, \( M \) a complex manifold, and \( \phi : M \to \mathbb{C} \) a holomorphic submersion such that \( \phi(M) \supseteq \Omega \), then \( \mathcal{O}_\phi(\Omega, M) \) is dense in \( \mathcal{A}_\phi(\Omega, M) \).

This is a (very) special case of [5], Theorem 5.1 regarding the approximation of sections of submersions over strongly pseudoconvex sets in Stein manifolds, which may be thought of as a far-reaching generalization of the Henkin-Ramírez-Kerzman approximation theorem for functions on such domains ([9], Theorem 2.9.2.)

1.3. Sets with property \( A_2 \) or \( A_3 \). We recall some definitions from elementary point-set topology. Let \( X \) be a topological space. For an integer \( n \geq 0 \), we say that \( \dim(X) \leq n \) if the following holds: given any open cover \( A \) of \( X \), there is a refinement \( B \) of \( A \) such that any point of \( X \) is contained in at most \( n+1 \) of the sets in the cover \( B \). If \( \dim(X) \leq n \) holds, but \( \dim(X) \leq n-1 \) does not, we say that the dimension \( \dim(X) \) of \( X \) is \( n \). The dimension is clearly a topological invariant of \( X \). Also, if \( \dim(X) = n \), then each connected component of \( X \) (or more generally, any closed subset) has dimension less than or equal to \( n \).

We say that a topological space is **locally contractible** if there is a basis of the topology consisting of contractible open sets. We can now state the following:

**Theorem 1.** Let \( K \subset \mathbb{C} \) be a connected compact set, such that \( \dim(K) = 1 \), and suppose that \( K \) is locally contractible. Then \( K \) has property \( A_3 \).

Examples of sets that satisfy the hypotheses are

1. arcs in \( \mathbb{C} \), where By an **arc** \( \alpha \) in a topological space \( X \) we mean a continuous injective map \( \alpha : [0,1] \to X \) from the unit interval. The injectivity avoids pathologies like space filling curves. By standard abuse of language, we will refer to the image \( \alpha([0,1]) \) as the arc \( \alpha \).
2. planar realizations of connected **graphs** i.e. finite simplicial complexes having only 0-simplices (“vertices”) and 1-simplices (“edges”). We can think of these as unions of arcs in the plane, any two of which meet at at most one endpoint of each.

Let \( K \subset \mathbb{C} \), then \( \dim(K) \leq 1 \) iff \( K^c = \emptyset \) ([1], p.133, Theorem 20.) However, it is possible for a set \( K \) with \( K^c = \emptyset \), to be not locally contractible, and yet to satisfy \( A_1 \) (recall that \( A_1 \) is equivalent to \( \mathcal{O}(K) \) being dense in \( \mathcal{A}(K) \).) An example is the “bouquet of circles” \( K = \bigcup_{n=1}^{\infty} C_n \), where \( C_n \) is the circle in the plane with center at \( \frac{1}{n} + i0 \), and radius \( \frac{1}{n} \). It can be shown that this \( K \) has property \( A_1 \), yet Theorem 1 does not apply to it. (We don’t know if \( K \) has property \( A_3 \) or \( A_2 \).)

For completeness, we record the following easy fact:

**Lemma 2.** Let \( K \) be a compact subset of \( \mathbb{C} \) with \( \dim(K) = 0 \). Then \( K \) has property \( A_3 \).

We now go on to give examples of two-dimensional sets which have properties \( A_2 \) or \( A_3 \). Let \( k \geq 0 \) be an integer, \( \infty \) or \( \omega \). We introduce a class of sets in the plane denoted by \( \mathfrak{C}_k \). We will let \( \mathfrak{C}_k \) denote the class of compact sets \( K \subset \mathbb{C} \) such that there is an integer \( N \) and domains \( \Omega_j \), where \( j = 1, \ldots, N \) with the following properties:

- For \( k \neq 0 \), each \( \Omega_j \) is a domain with \( \mathcal{C}^k \) boundary, and for \( k = 0 \), each \( \Omega_j \) is a Jordan domain.
• The $\Omega_j$’s are pairwise disjoint: $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$.
• $K = \bigcup_{i=1}^{N} \Omega_j$. We will refer to each $\Omega_j$ as a summand of $K$.
• whenever $i \neq j$, the set
  \[ P_{ij} := \overline{\Omega_i} \cap \overline{\Omega_j} = \partial \Omega_i \cap \partial \Omega_j \]
  is finite.
• If $k \neq 0$, at each point of $P_{ij}$, the boundaries $\partial \Omega_i$ and $\partial \Omega_j$ are tangent to each other.

We can now provide a supply of sets which have property $A_2$:

**Theorem 2.** Each compact set $K$ of class $C_0$ has property $A_2$.

This can be thought of as a generalization of Proposition 1. The second result gives examples of sets with property $A_3$:

**Theorem 3.** Each compact set $K$ of class $C_2$ has property $A_3$.

This again can be thought of as a generalization of Proposition 2.

1.4. In §2 below, we prove a general result (Theorem 4) regarding approximation of sections of submersions over a set $K$ which admits a decomposition $K = K_1 \cup K_2$ into a “good pair” $(K_1, K_2)$ of compact sets. This is a direct generalization of the results in §4.1 for maps into manifolds. Proofs of all new statements are given in detail, although they are similar to the proofs in the case of maps. This is the tool for patching maps which is used to prove Theorems 1 and 3.

In §3, we give proofs of Lemma 2, Theorem 1, and Theorem 3. For the last two, the crucial ingredient is three successive applications of Theorem 4 to produce a section which is holomorphic in a neighborhood of the given compact $K$. For Theorem 3 we also require a result from §4 regarding conformal mappings continuous to the boundary.

It should be noted that the results proved here have analogs for the approximation and interpolation of $A^k$ maps, i.e., maps which are $C^k$ and holomorphic in the interior. The proofs are exactly the same. For notational simplicity we stick to the case of maps which are only continuous to the boundary. Also, this article is largely self contained, with the exception of the proofs of Theorem 5 and Proposition 5.

## 2. Approximation on Good Pairs

2.1. **Some definitions.** We will call a set $K \subset \mathbb{C}$ nicely contractible if there is a homotopy $c : [0, 1] \times K \to K$ with the following properties:

1. for each $t$, the map $z \mapsto c(t, z)$ is in $A(K, \mathbb{C})$,
2. $z \mapsto c(1, z)$ is the identity map on $K$,
3. there is a $z_0 \in K$ such that $c(0, z) \equiv z_0$,
4. for $t \neq 0$, $z \mapsto c(t, z)$ maps $K^\circ$ into $K^\circ$. 

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Of course, convex sets are nicely contractible, as are strongly star-shaped sets (these are sets \( K \)

such that the maps \( c(t,z) \) may be taken as dilations with stretch factor \( t \) and center \( z_0 \in K \).

Another important class of nicely contractible sets are arcs. The property of nicely contractible

sets which is used in the proof of Lemma 3 (and therefore Theorem 4) is the following: if \( K \) is nicely

correctable and \( M \) is a connected complex manifold, then \( \mathcal{A}(K,M) \) is in fact contractible. We

can take the contraction to be the map \( \phi_t \) from \( \mathcal{A}(K,M) \) to itself given by \( \phi_t(f)(z) = f(c(t,z)) \).

Let \( K_1 \) and \( K_2 \) be compact subsets of \( C \). We will say that \((K_1, K_2)\) is a good pair if the following

hold:

1. \( \overline{K_1 \setminus K_2} \cap \overline{K_2 \setminus K_1} = \emptyset \).
2. \( K_{1,2} := K_1 \cap K_2 \) has finitely many connected components, each of which is nicely con-

tractible.

Let \( M \) be a complex manifold of complex-dimension \( n \), and \( \phi : M \to C \) be a submersion. We

will say that an open set \( U \subset M \) is \( \phi \)-adapted, if \( U \) is biholomorphic to an open set in \( C^n \), and

there are holomorphic coordinates \((z_1, \cdots, z_n)\) on \( U \) such that with respect to these coordinates

(and the standard coordinate on \( C \)), the map \( \phi \) takes the form \((z_1, \cdots, z_n) \mapsto z_n \). (This is of

course the same as saying \( z_n = \phi|_U \).)

2.2. The main result. We introduce the following notation and definitions:

1. Let \( K = K_1 \cup K_2 \), where \((K_1, K_2)\) is a good pair.
2. Let \( M \) be a complex manifold and \( \phi : M \to C \) be a holomorphic submersion such that

\( \phi(M) \supset K \).
3. Let \( B \subset C \) be compact and such that \( B \cap K_1 = \emptyset \), and each function \( g \in A(K_2) \) can be

approximated uniformly on \( K_2 \) by functions in \( A(K_2 \cup B) \) i.e., if \( g \in A(K_2) \) and \( \epsilon > 0 \),

then there is a \( g_\eta \in A(K_2 \cup B) \) such that \( |g - g_\eta| < \eta \) on \( K_2 \).
4. Let \( P \) be a finite subset of \( K \).

We now state the following result regarding the approximation of sections of \( \phi \) over \( K \):

Theorem 4. With \( K_1, K_2, M, \phi, B, \) and \( P \) as above, let \( s \in A_\phi(K,M) \) be a section of \( \phi \) such

that each \( s(K_j) \) has a \( \phi \)-adapted neighborhood in \( M \) for \( j = 1, 2 \). Then, given \( \eta > 0 \), there is an

\( s_\eta \in A_\phi(K,M) \) such that \( \text{dist}(s, s_\eta) < \eta \) on \( K \), \( s_\eta \) extends as a holomorphic section of \( \phi \) to a

neighborhood \( B_\eta \) of \( K_2 \cap B \), and for each \( p \in P \), we have \( s_\eta(p) = s(p) \). Moreover, \( s_\eta(K_2 \cup B_\eta) \) has a \( \phi \)-adapted neighborhood in \( M \).

The proof of Theorem 4 will be reduced to the solution of a non-linear patching problem in Euclidean

space by the use of coordinate in neighborhoods of \( s(K_j) \). We now describe the ingredients required in

the proof. The most important is the following version (resembling that of Douady in \[\text{[4]}, \text{pp.47-48}\]of

H. Cartan's lemma on holomorphic matrices.

Lemma 3. Let \( \mathcal{O} \) be a complex connected Lie Group, and let \((K_1, K_2)\) be a good pair, and let

\( g \in A(K_{1,2}, \mathcal{O}) \). Then, for \( j = 1, 2 \) there are \( g_j \in A(K_j, \mathcal{O}) \) such that \( g = g_2 \cdot g_1 \) on \( K_{1,2} \).
For a proof, see [2], Lemma 4.4, where it is assumed that \( G = GL_n(\mathbb{C}) \), and each component of the intersection is star shaped, but the proof is valid in general.

If \( P \subset \mathbb{C} \) is a finite set, we will let \( A^P(K, \mathbb{C}^n) \) denote the closed subspace of the Banach space \( A(K, \mathbb{C}^n) \) consisting of those functions which vanish at each \( p \in K \cap P \). We will require the following version of the solution of the additive Cousin problem continuous to the boundary (the case with \( P = \emptyset \) is in fact used in the proof of Lemma 3):

**Lemma 4.** Let \( K_1, K_2 \) be a compact subsets of the plane such that \( K_1 \setminus K_2 \cap K_2 \setminus K_1 = \emptyset \), and let \( P \) be a finite subset of the plane. There exist bounded linear maps \( T_j : A^P(K_{1,2}, \mathbb{C}) \to A^P(K_j, \mathbb{C}) \) such that for any function \( f \) in \( A^P(K_{1,2}, \mathbb{C}) \) we have on \( K_{1,2} \),

\[
T_1f + T_2f = f,
\]

(1)

**Proof.** We reduce the problem to a \( \partial \) equation in the standard way. Let \( \chi \) be a smooth cutoff which is 1 near \( K_1 \setminus K_2 \) and 0 near \( K_2 \setminus K_1 \). Let \( \lambda := f \frac{\partial \chi}{\partial z} \), so that \( \lambda \in A(K_{1,2}, \mathbb{C}) \). Let

\[
\Lambda_f(z) = \frac{1}{2\pi i} \int_{K_{1,2}} \frac{\lambda(\zeta)}{\zeta - z} d\zeta \wedge d\zeta
\]

Let \( q_f \) be the Lagrange interpolation polynomial with the property that \( q_f(p) = \Lambda_f(p) \) for \( p \in P \). Observe that both \( \Lambda_f \) and \( q_f \) are linear in \( f \) and continuous in the sup norm when restricted to compact sets.

We can now define (assuming \( (1 - \chi).f = 0 \) where \( \chi = 1 \) even if \( f \) is not defined):

\[
(T_1f)(z) = (1 - \chi(z)).f(z) + \Lambda_f(z) - q_f(z)
\]

and (assuming \( \chi.f = 0 \) where \( \chi = 0 \) even if \( f \) is not defined):

\[
(T_2f)(z) = \chi(z).f(z) - \Lambda_f(z) + q_f(z).
\]

Since \( T_jf \) is clearly holomorphic (resp. continuous) where \( f \) is, the result follows. \( \square \)

We will use the following standard result regarding Banach spaces, which can be proved by iteration (see [10] pp. 397-98):

**Lemma 5.** In a metric space \( X \), let \( B_X(p,r) \) denote the open ball in \( X \) of radius \( r \) centered at \( p \). Let \( E \) and \( F \) be Banach Spaces and let \( \Phi : B_E(p,r) \to F \) be a \( C^1 \) map. Suppose there is a constant \( C > 0 \) such that:

- for each \( h \in B_E(p,r) \), the linear operator \( \Phi'(h) : E \to F \) is surjective and the equation \( \Phi'(h)u = g \) can be solved for \( u \) in \( E \) for all \( g \) in \( F \) with the estimate \( \|u\|_E \leq C\|g\|_F \).
- for any \( h_1 \) and \( h_2 \) in \( B_E(p,r) \) we have \( \|\Phi'(h_1) - \Phi'(h_2)\| \leq \frac{1}{2C} \).

Then,

\[
\Phi(B_E(p,r)) \supset B_F \left( \Phi(p), \frac{r}{2C} \right).
\]

We will use the three lemmas above to give a proof of the following result (the main component of which is due to Rosay, [12], and comments in [13]) regarding the solution of a non-linear Cousin problem (see Lemma 4.5 of [2]). It is simply a translation of Theorem 4 to coordinates.
Lemma 6. Let \( \omega \) be an open subset of \( \mathbb{C}^n \) and let \( \tilde{\mathcal{F}} : \omega \to \mathbb{C}^n \) be a holomorphic immersion. Assume that \( \tilde{\mathcal{F}} \) preserves the last coordinate, i.e., \( \tilde{\mathcal{F}} \) is of the form \( \tilde{\mathcal{F}}(z_1, \ldots, z_n) = (F(z_1, \ldots, z_n), z_n) \) for some map \( F : \omega \to \mathbb{C}^{n-1} \).

Let \((K_1, K_2)\) denote a good pair of compact subsets of \( \mathbb{C} \), let \( \mathcal{P} \) be a finite subset of \( K = K_1 \cup K_2 \), and suppose for each \( p \in K_2 \setminus K_1 \) we are given a point \( q(p) \in \mathbb{C}^{n-1} \). Let \( u_1 \in \mathcal{A}(K_1, \mathbb{C}^n) \) be such that

- \( u_1(K_{1,2}) \subset \omega \), and
- \( u_1 \) is of the form \( u_1(z) = (t_1(z), z) \) where \( t_1 : K_1 \to \mathbb{C}^{n-1} \)

Given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( u_2 \in \mathcal{A}(K_2, \mathbb{C}^n) \) be such that

- \( \|u_2 - \tilde{\mathcal{F}} \circ u_1\| < \delta \) on \( K_2 \cap K_2 \),
- for \( p \in \mathcal{P} \cap (K_{1,2}) \), we have \( u_2(p) = \tilde{\mathcal{F}}(u_1(p)) \),
- for \( p \in K_2 \setminus K_1 \) we have \( u_2(p) = (q(p), p) \in \mathbb{C}^n \), and
- \( u_2(z) = (t_2(z), z) \), where \( t_2 : K_2 \to \mathbb{C}^{n-1} \)

then for \( j = 1, 2 \) there exist \( v_j \in \mathcal{A}^p(K_j, \mathbb{C}^n) \) such that

- \( \|v_j\| < \varepsilon \),
- \( u_2 + v_2 = \mathcal{F}(u_1 + v_1) \), and
- the last coordinate function of each of \( v_1 \) and \( v_2 \) is 0.

We now give a proof of Lemma 6.

Proof. In order to apply Lemma 6, we choose the Banach spaces \( \mathcal{E}, \mathcal{F} \) and the map \( \Phi \) as follows:

- For \( j = 1, 2 \), let \( \mathcal{B}_j \) be the closed subspace of the Banach space \( \mathcal{A}^p(K_j, \mathbb{C}^n) \) consisting of those maps whose last coordinate function is 0, i.e., \( \mathcal{B}_j = \mathcal{A}^p(K_j, \mathbb{C}^n) \oplus 0 \). We now let \( \mathcal{E} \) the Banach space \( \mathcal{B}_1 \oplus \mathcal{B}_2 \), which we endow with the norm \( \|\cdot\|_{\mathcal{E}} := \max\left(\|\cdot\|_{\mathcal{A}(K_1, \mathbb{C}^n)}, \|\cdot\|_{\mathcal{A}(K_2, \mathbb{C}^n)}\right) \).
- Let \( \mathcal{F} \) be the Banach space \( \mathcal{A}^p(K_{1,2}, \mathbb{C}^{n-1}) \).
- Let \( \pi : \mathbb{C}^n \to \mathbb{C}^{n-1} \) denote the projection on the first \( n-1 \) coordinates, and let the open subset \( \mathcal{U} \) of \( \mathcal{E} \) be given by \( \{(w_1, w_2) : (u_1 + w_1)(K_{1,2}) \subset \omega\} \). (Observe that \( \{0\} \times \mathcal{B}(K_2, \mathbb{C}^n) \subset \mathcal{U} \).) Let \( p = (P_1, \ldots, P_n) \) be an \( n \)-tuple of polynomials such that \( P_n(z) \equiv z \), for \( p \in K_{1,2} \cap \mathcal{P} \), we have \( P(p) = \tilde{\mathcal{F}}(u_1(p)) \), and for \( p \in K_2 \setminus K_1 \) we have \( (P_1(p), \ldots, P_{n-1}(p)) = q(p) \).

Let the map \( \Phi : \mathcal{U} \to \mathcal{F} \) be given by

\[
\Phi(w_1, w_2) := \pi \circ [(P + w_2)|_{K_{1,2}} - \tilde{\mathcal{F}} \circ ((u_1 + w_1)|_{K_{1,2}})].
\]

(Observe that, since \( \tilde{\mathcal{F}} \) preserves the last coordinate, the last coordinate function of \( (P + w_2)|_{K_{1,2}} - \tilde{\mathcal{F}} \circ ((u_1 + w_1)|_{K_{1,2}}) \) is actually 0. So the precomposition with \( \pi \) simply drops a coordinate which is identically 0.)

A computation shows that \( \Phi(w_1, w_2) \) is the bounded linear map from \( \mathcal{E} \) to \( \mathcal{F} \) given by

\[
(w_1, w_2) \mapsto \pi \circ [w_2|_{K_{1,2}} - \tilde{\mathcal{F}}'((u_1 + w_1)|_{K_{1,2}})(w_1|_{K_{1,2}})].
\]
Observe that \( w_2 \) plays no role whatsoever in this expression, and therefore \( \Phi'(w_1, w_2) \in BL(\mathcal{E}, \mathcal{F}) \) is in fact a smooth function of \( w_1 \) alone, and we will henceforth denote it by \( \Phi'(w_1, \ast) \).

Let \( \mathfrak{G} \subset GL_n(\mathbb{C}) \) be the complex Lie subgroup of matrices of the form

\[
\begin{pmatrix}
A & b \\
0 & 1
\end{pmatrix},
\]

where \( A \in GL_{n-1}(\mathbb{C}) \), \( b \) is an \( 1 \times (n - 1) \) column vector, and \( 0 \) is the zero row vector of size \( (n - 1) \times 1 \). It is easy to see that \( \mathfrak{G} \) is connected.

We construct a right inverse to \( \Phi'(u_1, \ast) \). Let \( \gamma = \mathfrak{G}' \circ (u_1|_{K_{1,2}}) \). Since \( \mathfrak{G} \) preserves the last coordinate, \( \gamma \in \mathcal{A}(K_{1,2}, \mathfrak{G}) \), and thanks to Lemma 8 above, we may write \( \gamma = \gamma_2 \cdot \gamma_1 \), where \( \gamma_j \in \mathcal{A}(K_j, \mathfrak{G}) \). (We henceforth suppress the restriction signs.) Let \( j : \mathcal{F} \rightarrow A(K_{1,2}, \mathbb{C}^n) \) be the continuous inclusion induced by the map \( \mathbb{C}^n \rightarrow \mathbb{C}^n \) given by \((z_1, \cdots, z_{n-1}) \mapsto (z_1, \cdots, z_{n-1}, 0) \). For \( g \in \mathcal{F} \), let

\[
S(g) = (-\gamma_1^{-1}T_1(\gamma_2^{-1}j(g)), \gamma_2T_2(\gamma_2^{-1}j(g))),
\]

where the \( T_j \) are as in equation 1 above (and extended to \( \mathbb{C}^n \) componentwise). Observe that

\[
\begin{align*}
-\gamma_1^{-1}T_1(\gamma_2^{-1}j(g)) & \quad \text{and} \quad 
\gamma_2T_2(\gamma_2^{-1}j(g)) \quad \text{vanish at each}
\end{align*}
\]

\( p \in \mathcal{P} \) and they belong to \( B_1 \) and \( B_2 \) respectively. It is easy to see that \( S : \mathcal{F} \rightarrow \mathcal{E} \) is a bounded linear operator, and a computation shows that

\[
(1) \quad \|\Phi'(u_1, \ast) - \Phi'(u_1, \ast)\|_{\text{op}} < \frac{1}{\|S\|} \quad \text{(possible by continuity)},
\]

and

\[
(2) \quad \text{the equation } \Phi'(u_1, \ast)u = g \text{ can be solved with the estimate } \|u\| \leq 2\|S\|\|g\|, \quad \text{(possible from the fact that small perturbations of a surjective linear operator is still surjective.)}
\]

Consequently, if \( \epsilon < \theta \) and \( \tilde{u}_2 \in B(K_2, \mathbb{C}^n) \), for the ball \( B_{\mathbb{C}^n}(0, \tilde{u}_2, \epsilon) \) the hypothesis of Lemma 5 are verified with \( C = 2\|S\| \). We have therefore,

\[
\Phi(B_{\mathbb{C}^n}(0, \tilde{u}_2, \epsilon)) \supset B_F\left(\Phi(0, \tilde{u}_2), \frac{\epsilon}{2C}\right) = B_F\left(P + \tilde{u}_2 - \mathfrak{G}(u_1), \frac{\epsilon}{2C}\right).
\]

So, if \( \|P + \tilde{u}_2 - F(u_1)\| < \frac{\epsilon}{2C} \), we have 0 \( \in \Phi(B_{\mathbb{C}^n}(0, \tilde{u}_2, \epsilon)) \). This is exactly the conclusion required, since any \( u_2 \) such that \( u_2(p) = P(p) \) for each \( p \in K_2 \cap \mathcal{P} \) can be written as \( u_2 = P + \tilde{u}_2 \) for some \( \tilde{u}_2 \in B(K_2, \mathbb{C}^n) \). \( \square \)

We will now prove Theorem 4.

\textbf{Proof.} We omit the restriction signs on maps for notational clarity. For \( j = 1, 2 \) let \( V_j \) be a \( \phi \)-adapted neighborhood of \( s(K_j) \), and let \( \mathfrak{G}_j : V_j \rightarrow \mathbb{C}^n \) be a coordinate system such that \( \mathfrak{G}_j(z) = (F_j(z), \phi(z)) \). Let \( \mathfrak{G} = \mathfrak{G}_2 \circ \mathfrak{G}_1^{-1} \) be the associated transition function. Then \( \mathfrak{G} \) is a biholomorphism from the open set \( \omega = \mathfrak{G}_1(V_1 \cap V_2) \) onto the open set \( \mathfrak{G}_2(V_2 \cap V_1) \), and \( \mathfrak{G} \) preserves the last coordinate, i.e., \( \mathfrak{G} \) is of the form \( \mathfrak{G}(z_1, \cdots, z_{n}) = (F(z_1, \cdots, z_{n}), z_{n}) \) for some map \( F : \omega \rightarrow \mathbb{C}^{n-1} \). Any section of \( \phi \) over \( K_j \) is represented in the coordinate system \( \mathfrak{G} \) by a map of the form \( t_j : K_j \rightarrow \mathbb{C}^n \), where \( t_j(z) = (\hat{t}_j(z), z) \), with \( \hat{t}_j \) a map from \( K_j \) to \( \mathbb{C}^{n-1} \). Also, for \( j = 1, 2 \), given maps \( t_j : K_j \rightarrow \mathbb{C}^n \) of
the form \( t_j = (t_j, z) \), they glue together to form a section over \( K_1 \cup K_2 \) (i.e. there is a section \( \lambda \) of \( \phi \) over \( K_1 \cup K_2 \) such that \( t_j = \tilde{\lambda}_j \circ \lambda \) iff \( t_2 = \tilde{\lambda} \circ t_1 \).

Since \( B \cap K_1 = \emptyset \) by hypothesis, the pair of compact sets \( (K_1, K_2 \cup B) \) is good. Let \( \epsilon > 0 \), and let \( u_1 = \tilde{\lambda}_1 \circ s \). Observe that \( u_1(K_{1,2}) \subset \omega \), and \( u_1 \) is of the form \( u_1(z) = (t_1(z), z) \), where \( t_1 : K_1 \rightarrow \mathbb{C}^{n-1} \). Then Lemma 2 gives a \( \delta > 0 \) corresponding to \( u_1 \), the good pair \( (K_1, K_2 \cup B), \omega \) and \( \tilde{\lambda} \).

Let \( w_2 = \tilde{\lambda}_2 \circ (s|_{K_2}) \), then \( w_2 \in \mathcal{A}(K_2, \mathbb{C}^n) \), and is of the form \( w_2(z) = (\tilde{w}_2(z), z) \). Thanks to the hypothesis regarding uniform approximation of functions in \( \mathcal{A}(K_2, \mathbb{C}) \) by functions in \( \mathcal{A}(K_2 \cup B, \mathbb{C}) \), we can find a \( u_2 \in \mathcal{A}(K_2 \cup B, \mathbb{C}^n) \) of the same form as \( w_2 \) such that \( \|w_2 - \tilde{\lambda}(u_1)\| < \delta \) on \( K_{1,2} \) (Since \( \tilde{\lambda}(u_1) = w_2 \) on \( K_{1,2} \).) Further we may assume that for \( p \in \mathcal{P} \cap K_2 \), we have \( u_2(p) = w_2(p) \) and that the last coordinate function of \( w_2 \) is \( z \).

Then, by Proposition 3 there is a \( v_1 \in \mathcal{A}^p(K_1, \mathbb{C}^n) \) and a \( v_2 \in \mathcal{A}^p(K_2 \cup B, \mathbb{C}^n) \), such that \( \|v_j\| < \epsilon \) and \( u_2 + v_2 = \tilde{\lambda}(u_1 + v_1) \), and the last coordinate functions of the \( v_j \) are 0’s. Hence the maps \( u_1 + v_1 \) and \( u_2 + v_2 \) glue together to form a section of \( \phi \) (which we call \( \tilde{s}_\epsilon \) given by

\[
\tilde{s}_\epsilon := \begin{cases} 
\tilde{\lambda}_1^{-1}(u_1 + v_1) & \text{on } K_1 \\
\tilde{\lambda}_2^{-1}(u_2 + v_2) & \text{on } K_2 \text{ and near } K_2 \cap B
\end{cases}
\]

Clearly, \( \tilde{s}_\epsilon \) is in \( \mathcal{A}_\phi(K_1 \cup K_2, \mathcal{M}) \), and extends to a holomorphic map near \( K_2 \cap B \). Moreover, \( \text{dist}(\tilde{s}_\epsilon, s) = O(\epsilon) \). By construction, we have \( s_\eta(p) = s(p) \). Therefore, given \( \eta > 0 \), we can find \( s_\eta \) with required properties. \( \square \)

3. Sets with property \( A_3 \)

In this section we prove Lemma 2 and Theorems 1 and 3. In each we show that a certain set \( K \) has property \( A_3 \). We will let \( \mathcal{M} \) be a complex manifold, \( \phi : \mathcal{M} \rightarrow \mathbb{C} \) a holomorphic submersion such that \( \phi(\mathcal{M}) \supset K \), \( \mathcal{P} \) a finite subset of \( K \), and \( s \) a section of \( \phi \), \( s \in \mathcal{A}_\phi(K, \mathcal{M}) \). We let \( \epsilon > 0 \), and want to show that there is an \( s_\epsilon \in \mathcal{O}_\phi(K, \mathcal{M}) \) such that \( \text{dist}(s, s_\epsilon) < \epsilon \) and \( s_\epsilon(p) = s(p) \) for each \( p \in \mathcal{P} \).

3.1. Proof of Lemma 2 \( \square \) If \( K \) is not finite, it can be written as a disjoint union of finitely many singletons and a closed subset \( C \) homeomorphic to the Cantor middle-third set. (See e.g., [1], pp. 108-109.) In particular, \( \mathbb{C} \setminus K \) is connected, and \( K \) has property \( A_1 \).

Observe that \( s \in \mathcal{A}_\phi(K, \mathcal{M}) \) is simply a continuous section of \( \phi \) over \( K \). We can cover \( K \) by finitely many sets \( \{U_j\} \) such that each \( s(U_j) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). We can choose a refinement \( \{V_j\} \) of this cover, such that the \( V_j \)’s are pairwise disjoint. Observe that \( K_j := K \cap \overline{V_j} \) has property \( A_1 \). Now, with respect to the \( \phi \)-adapted coordinate system around \( s(K_j) \), the map \( s \) has a representation on \( K_j \) of the form \( s(z) = (\tilde{s}(z), z) \), where \( \tilde{s} \) is continuous and takes values in \( \mathbb{C}^{n-1} \). Approximating \( \tilde{s} \) by a holomorphic map in a neighborhood of \( K_j \) our result follows.

3.2. Proof of Theorem 1 \( \square \) We first introduce some combinatorial preliminaries. Let \( vw \) be an edge in a graph \( \Gamma \). We can construct a new graph \( \Gamma' \) with one more vertex by “splitting the edge
More formally, if \( V \) is the vertex set of \( \Gamma \), and \( E \) its edge set, the vertex set of \( \Gamma' \) is \( V \cup \{ u \} \) (where \( u \not\in V \)), and the edge set is \( (E \setminus \{ uv \}) \cup \{ vu, uw \} \).

Recall that a graph is \( n \)-colorable, if there is an assignment of \( n \) colors to its vertices so that no two adjacent vertices have the same color. By a classical result of König (see [3], Theorem 12.1), the condition that a graph is 2-colorable is that it does not have a circuit (closed non-self intersecting path) of odd length. We now have the following:

**Lemma 7.** Given any finite graph \( G \), there are edges \( e_1, \ldots, e_k \) such that the graph \( G' \) obtained after successively splitting these edges is 2-colorable.

**Proof.** Let \( V \) be the vector space over the field \( \mathbb{Z}/2\mathbb{Z} \) with basis the set of edges of \( G \). Given a circuit \( C \) in \( G \), traversing the edges \( x_1, x_2, \ldots, x_m \), associate with it an element \( c \) of \( V \) given by \( c = x_1 + x_2 + \cdots + x_m \). Let \( Z \) be the subspace of \( V \) spanned by all \( c \) for each circuit \( C \). Pick a basis \( c_1, \ldots, c_k \). Since any circuit can be written as a linear combination of the \( c_j \)'s it is sufficient to split any edge of those \( c_j \) which have an odd number of summands. Moreover, from the fact that the \( c_j \) are linearly independent over \( \mathbb{Z}/2\mathbb{Z} \) it follows that each \( c_j \) has an edge not contained in any \( c_k, k \neq j \). The result follows.

Now we can prove Theorem 1. Note that as in the proof of Lemma 1 above, \( K^o = \emptyset \), so \( s \) is simply a continuous section of \( \phi \) over \( K \).

For \( z \in K \), there is a neighborhood \( U_z \) of \( z \) in \( K \) such that \( s(U_z) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). Select a finite subcover \( \mathcal{F} \) of \( \{ U_z \}_{z \in K} \), and let \( \mathcal{G} \) be a refinement of \( \mathcal{F} \) which has only double intersections (this is possible since \( \dim(K) = 1 \), and \( K \) is connected.) We can write \( \mathcal{G} = \{ V_1, \ldots, V_M \} \).

We associate to \( \mathcal{G} \) a graph \( N_{G'} \) by taking the nerve: the vertices \( V_i \) and \( V_j \) are joined by an edge in \( N_{G'} \) iff \( V_i \cap V_j \neq \emptyset \). Let \( V \) and \( W \) be open sets in \( \mathcal{G} \) such that \( V \cap W \neq \emptyset \). We define a new open cover \( S_{VW}(\mathcal{G}) \) of \( K \) in the following way. Replace the sets \( V \) and \( W \) by \( V', U \) and \( W' \), where \( V' \subset V \), \( W' \subset W \), and \( U \) is a neighborhood of \( V \cap W \) such that \( V' \cup U \cup W' = V \cup W \), and \( V \cap U \neq \emptyset \), \( U \cap W \neq \emptyset \) but \( V \cap W = \emptyset \). If \( U \) is chosen small enough, then \( (G \setminus \{ V, W \}) \cup \{ V', U, W' \} \) is again an open cover of \( K \) with only double intersections. This is our \( G' = S_{VW}(\mathcal{G}) \). At the level of nerves \( N_{G'} \) is obtained by splitting the edge \( VW \) in the graph \( N_{G} \), as in Lemma 7. Thanks to Lemma 7 we can obtain after finitely many rounds of edge-splitting a new cover \( \mathcal{H} \), with only double intersections such that the corresponding nerve \( N_{H} \) is 2-colorable. It also follows that for each \( U \in \mathcal{H} \), the set \( s(U) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). Further, by shrinking the \( U \)'s we can assume that whenever an intersection \( U \cap U' \) of two sets in \( \mathcal{H} \) is non-empty, it is contractible. Let us call the colors used in coloring \( N_{H} \) red and blue. We can now define

\[
K_1 = \bigcup_{U \in \mathcal{U}} \mathcal{U}_{red}, \quad \text{and} \quad K_2 = \bigcup_{U \in \mathcal{U}} \mathcal{U}_{blue}.
\]

It is easy to verify that \( (K_1, K_2) \) is a good pair, and for \( j = 1, 2 \) the set \( s(K_j) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \).
We also define sets $K_1' \subset K_1$ and $K_2' \subset K_2$ in the following way. For $U, U' \in H$, let $W_{UU'}$ be a simply connected neighborhood of $U \cap U'$ in $C$ with $C^2$ boundary. Further we can assume that $\overline{W_{UU'}} \cap K$ is contractible and $s(\overline{W_{UU'}} \cap K)$ is contained in a $\phi$-adapted open set of $M$. We set
\[
W = \bigcup_{\substack{U, U' \in H \setminus \emptyset \, \forall U \cap U' \neq \emptyset}} W_{UU'},
\]
so that $W$ is a neighborhood in $C$ of the points in $K$ which are contained in two sets of the cover $H$.

Let $V \in H$. We set
\[
V^s = V \setminus W
\]
so that $V^s \subset V$, and points of $V^s$ do not belong to any other set of $H$ apart from $V$. We set:
\[
K_1' = \bigcup_{U \in N \, \text{red}} U^s, \quad \text{and} \quad K_2' = \bigcup_{U \in N \, \text{blue}} U^s.
\]
We now apply Theorem 3 to the good pair $(K_1, K_2)$, and obtain an $s_1 \in O_\phi (K \cup B_1, M)$, where $B_1$ is a neighborhood of $K_1'$ in $C$, such that $\text{dist}(s, s_1) < \varepsilon$, and $s(p) = s_1(p)$ for $p \in P$. We can further assume that $B_1$ is a disjoint union of simply connected neighborhoods of the sets $V^s$ for $V$ red, $\partial B_1$ is smooth, and $\partial B_1$ and $\partial W$ meet transversely at each point of intersection.

Observe now that $(K_1 \cup B_1, K_2)$ is a good pair, and we can apply Theorem 4 to it. We obtain a $s_2 \in A_\phi (K \cup B_1 \cup B_1, M)$ with $\text{dist}(s_2, s_1) < \varepsilon$, and $s_2(p) = s_1(p)$ for $p \in P$. As before, $B_2$ is a disjoint union of simply connected neighborhoods of the sets $V^s$ for $V$ blue, $\partial B_2$ is smooth, and $\partial B_2$ and $\partial W$ meet transversely at each point of intersection.

We set $B = B_1 \cup B_2$ so that $\partial B = \partial B_1 \cup \partial B_2$ meets $\partial W$ transversely, so that $\partial B \cap \partial W$ is a finite set. Let $U$ be a simply connected neighborhood of $K$ in $C$ such that all points of $\partial B \cap \partial W$ lie outside $U$ and $\partial U$ meets both $\partial B$ and $\partial W$ transversely. Note that $K_1 \cup K_2 \subset W \setminus B$. Let
\[
L_1 = \partial U \cap \partial B,
\]
and
\[
L_2 = \partial U \cap \partial W.
\]
It is easy to verify that $(L_1, L_2)$ is a good pair, and each of $s(L_j)$ has a $\phi$-adapted neighborhood in $M$. Setting $L = L_1 \cup L_2$, we apply Theorem 4 to $(L_1, L_2)$ to obtain a map $s_3 \in A_\phi (L \cup C, M)$, where $C$ is a neighborhood of $K_1 \cap K_2$ in $C$, $\text{dist}(s_3, s_2) < \varepsilon$, and $s_3(p) = s_2(p)$ for all $p \in P$. Clearly, $s_3 \in O_\phi (K, M)$, and we are done.

3.3. Proof of Theorem 3. In 3.3.1 we record a result that will be used in the proof. In 3.3.2 we use Theorem 3 and Theorem 4 to give a proof of Theorem 3 in the special case when $N = 1$, i.e. $K$ is the closure of a smoothly bounded domain. As in the proof of Theorem 4 this uses a “Triple bumping.” Finally, in 3.3.3 we complete the proof by reducing the general case to the case considered in 3.3.2.
3.3.1. Arcs in Complex Manifolds. If $X$ is a differentiable manifold, and $\alpha$ is an arc (continuous injective map from $[0,1]$) which is at least $C^1$, we will say $\alpha$ is embedded if for each $t \in [0,1]$, we have $\alpha'(t) \neq 0$. A proof of the following result, required in the proof of Theorem 3, can be found in [2], Theorem 2:

**Theorem 5.** Let $M$ be a complex manifold, and $\alpha$ be an arc in $M$. Assume that

1. there is a complex-valued submersion $\phi$ defined in a neighborhood of $\alpha$ in $M$ such that $\phi \circ \alpha$ is a $C^1$ embedded arc in $\mathbb{C}$.
2. there is a finite subset $P \subset [0,1]$ such that $\alpha$ is $C^3$ on $[0,1] \setminus P$.

Then $\alpha$ has a $\phi$-adapted neighborhood in $M$.

3.3.2. Case of a smooth domain. In this section we prove the following special case of Theorem 3 (Compare with Proposition 2):

**Proposition 4.** Let $\Omega \Subset \mathbb{C}$ be a domain with $C^2$ boundary. Then $\overline{\Omega}$ has property $A_3$.

The first step in the proof is the following application of Sard’s Theorem:

**Lemma 8.** There is a unit vector $v$ in the plane such that:

- every straight line in the plane parallel to $v$ meets $\partial \Omega$ in only finitely many points.
- the number of straight lines parallel to $v$ which are tangent to $\partial \Omega$ is finite.

In fact these hold for almost all unit vectors $v$ in the unit circle $S^1$ (with respect to the standard measure on $S^1$.)

**Proof.** Fix an arbitrary orientation on $\partial \Omega$, and define the Gauss map $G : \partial \Omega \to S^1$ by mapping the point $z \in \partial \Omega$ to the unit tangent vector $G(z)$ to $\partial \Omega$ at the point $z$. This is a $C^1$ map, and the set of its critical values is of measure 0 by Sard’s Theorem. If $v$ is any regular value of $G$, such that $-v$ is also a regular value, it follows that the sets $G^{-1}(v)$ and $G^{-1}(-v)$ are discrete in $\partial \Omega$ (since $G$ is a diffeomorphism near each of them.) Since $\partial \Omega$ is compact, it follows that the sets $G^{-1}(\pm v)$ are finite. From this, the conclusions follow immediately. □

After a rotation if required, we will assume that $v$ is vertical, i.e., $v = \pm i$. Now we introduce some notation. For $c \in \mathbb{R}$, denote by $L(c)$ the vertical straight line $\Re(z) = c$ in $\mathbb{C}$, and for $a < b$, denote by $M[a,b]$ the vertical strip $\{z \in \mathbb{C} : a \leq \Re(z) \leq b\}$. Also set $l(c) = L(c) \cap \overline{\Omega}$ and $m[a,b] = M[a,b] \cap \overline{\Omega}$. We will assume without loss of generality that $\Omega \subset M[0,1]$.

Thanks to Lemma 8 and the choice $v = \pm i$, it follows that for each $c$, the set $l(c)$ has only finitely many components, each of which is either a point, or a line segment. Also, $l(c) \cap \partial \Omega$ is finite. We now make the following observation.

**Lemma 9.** There is a $\delta > 0$ such that for each $c \in [0,1]$, the set $s(m[c-\delta,c+\delta])$ has a $\phi$-adapted neighborhood in $M$.

**Proof.** First we show that for each $c \in [0,1]$, the set $s(l(c))$ has a $\phi$-adapted neighborhood in $M$. Since $s$ is injective (it has a left-inverse $\phi$), it is sufficient to show that each component of $l(c)$
has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). For a component of \( l(c) \) that reduces to a point, this is trivial. Therefore, consider a component which is a (vertical) line segment. We can think of this component as an arc in the plane, and parameterize it as \( \lambda(t) = z_0 + iat \), where \( z_0 \) is the lower end point of the segment, and \( a \) is a positive real number. Then \( s \circ \lambda \) is a continuous arc in \( \mathcal{M} \), which is real analytic except at finitely many points, and \( \phi \) is a submersion from \( \mathcal{M} \) to \( \mathbb{C} \) such that \( \phi \circ (s \circ \lambda) = \lambda \). Thanks to Proposition 3.3.2 above, \( s(\lambda) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). Therefore, \( s(l(c)) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \).

It follows that there is a \( \delta_c \) such that \( s(m[c - \delta_c, c + \delta_c]) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). The uniform choice of \( \delta \) follows by compactness. \( \square \)

We will also require the following simple fact (a proof may be found in [2], Observation 4.8)

**Lemma 10.** Let \( u \) and \( v \) be real valued \( C^1 \) functions defined on a neighborhood of \( 0 \) in \( \mathbb{R} \) such that for each \( x \), we have \( u(x) < 0 < v(x) \). Then there is an \( \eta > 0 \) such that for \( 0 < \theta < \eta \), the vertical strip

\[
S := \{(x, y) \in \mathbb{R}^2 : x \in [-\theta, \theta], u(x) \leq y \leq v(x)\}
\]

is star shaped with respect to the origin.

The proof of Proposition 3.3.2 will parallel that of Proposition 1 in that both require a “Triple bumping”, i.e., three successive applications of Theorem 4. However, the good pairs are obtained by different methods.

Let \( \mathcal{E} \subset [0, 1] \) be the set of \( c \) such that the line \( L(c) \) is tangent to some component of \( \partial \Omega \). \( \mathcal{E} \) is finite by Lemma 8. If \( c \notin \mathcal{E} \), \( L(c) \) meets \( \partial \Omega \) transversely at each point of intersection, so that

1. Each component of \( l(c) \) is a line segment, and
2. (by Lemma 10) there is a \( \theta_c \) such that for \( \epsilon \leq \theta_c \), each component of \( m([c - \epsilon, c + \epsilon]) \) is strongly star shaped.

By compactness, we can find \( c_1 < c_2 < \ldots < c_M \), and \( \eta_j > 0 \), \( j = 1, \ldots, M \), such that if \( m_j = m[c_j - \eta_j, c_j + \eta_j] \), we have \( \Omega = \bigcup_{j=1}^M m_j \). We can assume that the \( \eta_j < \frac{\delta}{10^j} \), where \( \delta \) is as in Lemma 10. We see that for each \( j \), \( s(m_j) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). We will further impose the following conditions on the \( m_j \)'s

1. Each point of \( \Omega \) is in at most two of the \( m_j \)'s (this can be done by shrinking the \( \eta_j \)'s.)
2. Each \( c \in \mathcal{E} \) occurs in the list \( \{c_j\}_{j=1}^M \).
3. Each component of \( m_j \cap m_{j+1} \) is strongly star shaped. Observe that by the previous step, \( m_j \cap m_{j+1} \) does not contain any \( l(c) \) for \( c \in \mathcal{E} \). Therefore, this can be achieved by shrinking the \( \eta_j \)'s.

Now let

\[
K_1 = \bigcup_{j \text{ odd}} m_j, \quad \text{and} \quad K_2 = \bigcup_{j \text{ even}} m_j.
\]

It is easy to see that \( (K_1, K_2) \) is a good pair. Since each \( s(m_j) \) has a \( \phi \)-adapted neighborhood, and \( K_1, K_2 \) are disjoint union of \( m_j \)'s, it follows that each of \( s(K_1) \) and \( s(K_2) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \).
Let $I_1 \subset \partial \Omega \cap K_1$ and $I_2 \subset \partial \Omega \cap K_2$ be such that (1) $I_1 \cap K_2 = I_2 \cap K_2 = \emptyset$, and (2) Each connected component $\partial \Omega \setminus (I_1 \cup I_2)$ is contained in a vertical strip of width $\frac{\delta}{2}$, where $\delta$ is as in Lemma 9. (This is possible, since $\eta_j < \frac{\delta}{100}$.)

Let $B$ be a neighborhood of $I_1$ such that $B \cap K_2 = \emptyset$. We apply Theorem 4 to the good pair $(K_1, K_2)$ to obtain an $s_1 \in \mathcal{A}_\phi (\overline{\Omega} \cup B_1, \mathcal{M})$, where $B_1$ is a neighborhood of $I_1$ contained in $B$, such that $\text{dist}(s, s_1) < \frac{\delta}{2}$, and $s(p) = s_1(p)$ for $p \in P$. Observe that for $B_1$ small enough, $s_1(K_1 \cup B_1)$ is contained in a $\phi$-adapted open set of $\mathcal{M}$, and $(K_1 \cup B_1, K_2)$ is again a good pair. We now apply Theorem 4 again to this good pair to obtain an $s_2 \in \mathcal{A}_\phi (\overline{\Omega} \cup B_1 \cup B_2, \mathcal{M})$ where $B_2$ is a neighborhood of $I_2$, such that $\text{dist}(s_1, s_2) < \frac{\delta}{2}$, and $s_1(p) = s_2(p)$ for $p \in P$.

Let $I_3 = \partial \Omega \setminus (I_1 \cup I_2)$. By construction, each connected component of $I_3$ is contained in a vertical strip of width $\frac{\delta}{2}$. Let $\Omega'$ be a domain with $C^2$ boundary such that $\overline{\Omega} \cup B_1 \cup B_2 \supset \Omega' \supset \Omega \cup I_1 \cup I_2$ (i.e. $\Omega'$ is obtained by smoothly bumping $\Omega$ along $I_1$ and $I_2$.) We can assume that $\Omega'$ and $\Omega$ are so close that for $\delta$ as in Lemma 9 and any $c$, the set $s(\mathcal{M}[c-\delta, c+\delta] \cap \overline{\Omega})$ is contained in a $\phi$-adapted open set in $\mathcal{M}$. We can now repeat the constructions that gave us $K_1$ and $K_2$ to obtain a good pair $(L_1, L_2)$ such that (1) $L_1 \cup L_2 = \overline{\Omega'}$, (2) each of $s(L_1)$ and $s(L_2)$ has a $\phi$-adapted neighborhood in $\mathcal{M}$, and (3) $I_3 \subset L_1 \setminus L_2$. We can now apply Theorem 4 to the good pair $(L_1, L_2)$ to obtain an $s_3 \in \mathcal{A}_\phi (\overline{\Omega'} \cup C, \mathcal{M})$, where $C$ is a neighborhood of $I_3$ in $C$, such that $\text{dist}(s_3, s_2) < \frac{\delta}{2}$ and $s_3(p) = s_2(p)$, for each $p \in P$. Then $s_3 \in \mathcal{O}_\phi (\overline{\Omega'}, \mathcal{M})$, and we are done.

3.3.3. End of proof of Theorem 5. Let $K$ be of class $\mathcal{C}_2$. Recall that $K = \bigcup_{i=1}^N \overline{\Omega_i}$, and for $i \neq j$, we have $\partial \Omega_i$ and $\partial \Omega_j$ meet at a set of finitely many points $P_{ij}$. Set $P = \cup_{i \neq j} P_{ij}$. We can refer to the points in $P$ as nonsmooth points of $\partial K$.

The proof in this section is very similar to that in 3.3.2. The first step is to establish the following version of Lemma 8 for this case:

**Lemma 11.** Let $K$ be of class $\mathcal{C}_2$. Then there is a unit vector $\mathbf{v}$ in the plane with the following properties:

1. every straight line in the plane parallel to $\mathbf{v}$ meets $\partial K$ in only finitely many points.
2. The number of straight lines parallel to $\mathbf{v}$ which are tangent to $\partial K$ at smooth points is finite.
3. Let $p \in P_{ij}$ be a non-smooth point of $\partial K$. Then, the straight line through $p$ parallel to $\mathbf{v}$ is transverse to both $\partial \Omega_i$ and $\partial \Omega_j$ at $p$.

**Proof.** Applying Lemma 8 separately to each $\partial \Omega_j$, we conclude that properties (1) and (2) hold for almost all unit vectors $\mathbf{v}$. Let $p$ be a non-smooth point of $\partial K$, so that for some $i, j$, we have $p \in P_{ij} = \partial \Omega_i \cap \partial \Omega_j$. Let $(t)(p)$ be a common unit tangent vector to $\partial \Omega_i$ and $\partial \Omega_j$ at the point $p$. We can choose $\mathbf{v} \neq \pm t(p)$ for all $p \in P_{ij}$ for all $i$ and $j$. $\square$

As in 3.3.2 we can assume that $\mathbf{v} = \pm \mathbf{i}$. Let $L(c)$ and $M[a, b]$ have the same meaning as in the last section, and set $l'(c) = L(c) \cap K$, $m'[a, b] = M$. We can assume that $K \subset M[0, 1]$.

We let $E$ be the finite set of points $c \in [0, 1]$ such that either (1) $L(c)$ is tangent to $\partial \Omega_i$ for some $i$, or (2) $L(c)$ passes through a nonsmooth point of $\partial K$. Arguing as in the previous section, we
can find \( c_1 < c_2 < \ldots c_M \) and \( \eta_j > 0, j = 1, \ldots, M \), such that if \( m_j' = m'[c_j - \eta_j, c_j + \eta_j] \), we have \( K = \bigcap_{j=1}^{M} m_j' \) and each \( s(m_j') \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). We can further impose the following conditions, (the first three are just as in the last section, the last is a new condition):

1. Each point in \( K \) is contained in at most two of the \( m_j' \)'s, i.e., \( m_j' \cap m_k' = \emptyset \) if \( |j - k| > 1 \).
2. Each \( c \in \mathcal{E} \) occurs among the \( \{c_j\}_{j=1}^{M} \).
3. Each component of \( m_j' \cap m_{j+1}' \) is strongly star shaped (Note that, thanks to the last step, in this case we have ensured that \( m_j' \cap m_{j+1}' \) does not contain any nonsmooth points.)
4. Each nonsmooth point is contained in an \( m_j' \) with an even \( j \). This can be ensured by introducing additional \( c_j' \)'s. Observe that if a nonsmooth point \( p \in m_j' \), then \( p \notin m_{j-1}' \) and \( p \notin m_{j+1}' \).

As in the proof of Proposition \( \square \)

\[ K_1 = \bigcup_{j \text{ odd}} m_j', \quad \text{and} \quad K_2 = \bigcup_{j \text{ even}} m_j'. \]

It is easy to see that \((K_1, K_2)\) is a good pair. Since each \( s(m_j) \) has a \( \phi \)-adapted neighborhood, and \( K_1, K_2 \) are disjoint union of \( m_j' \)'s, it follows that each of \( s(K_1) \) and \( s(K_2) \) has a \( \phi \)-adapted neighborhood in \( \mathcal{M} \). Moreover, the set \( P \) of nonsmooth points of \( \partial K \) is contained in \( K_2 \setminus K_1 \).

Let \( B \) be a neighborhood of the nonsmooth points \( P \) such that \( B \cap K_1 = \emptyset \). Thanks to Theorem \( \square \) we can find an \( s' \in \mathcal{A}_\phi(K \cup B', \mathcal{M}) \) (where \( B' \) is a neighborhood of \( P \) contained in \( B \)), such that \( \operatorname{dist}(s, s') < \frac{\varepsilon}{2} \) and \( s'(p) = s(p) \) for each \( p \in \mathcal{P} \).

Now we can find an open set \( \Omega' \) with \( \mathcal{C}^2 \) boundary such that \( \overline{\Omega'} \subset K \cup B' \) but \( \Omega' \supset \Omega \cup P \). Then \( s' \in \mathcal{A}(\overline{\Omega'}, \mathcal{M}) \), and thanks to Proposition \( \square \) we can find \( s_\varepsilon \in \mathcal{O}_\phi(\overline{\Omega'}, \mathcal{M}) \) such that \( \operatorname{dist}(s_\varepsilon, s') < \frac{\varepsilon}{2} \) and \( s_\varepsilon(p) = s'(p) \) for \( p \in \mathcal{P} \). Since \( s_\varepsilon \in \mathcal{O}_\phi(K, \mathcal{M}) \), the proof is complete.

4. Proof of Theorem \( \square \)

4.1. \( \mathcal{A} \)-equivalence. Let \( K \) and \( K' \) be compact subsets of \( \mathbb{C} \). We will say that \( K \) and \( K' \) are \( \mathcal{A} \)-equivalent if there is a homeomorphism \( \chi : K \to K' \) such that \( \chi|_{K^c} \) is a conformal map of \( K^c \) onto \( K'^c \). We will call \( \chi \) an \( \mathcal{A} \)-equivalence from \( K \) to \( K' \). A well-known example of \( \mathcal{A} \)-equivalence is the following (\( \square \), Theorems IX.35 and IX.2):

**Lemma 12.** Let \( \Omega \) be a Jordan Domain, Then there is a domain \( \omega \) in the plane bounded by circles such that \( \overline{\Omega} \) and \( \overline{\omega} \) are \( \mathcal{A} \)-equivalent.

The significance of this notion in the current investigation is explained by the following observation:

**Lemma 13.** Suppose that two compact sets \( K \) and \( L \) in \( \mathbb{C} \) are \( \mathcal{A} \)-equivalent. If \( K \) has property \( A_2 \) and \( L \) has property \( A_1 \) then \( L \) has property \( A_2 \).

**Proof.** Let \( \mathcal{M} \) be a complex manifold, \( f \in \mathcal{A}(L, \mathcal{M}) \), and \( \mathcal{P} \) be a finite subset of \( K \). We want to approximate \( f \) by maps \( f_n \in \mathcal{O}(L, \mathcal{M}) \) such that \( f_n(p) = f(p) \) for \( p \in \mathcal{P} \). Let \( \chi : K \to L \) be an \( \mathcal{A} \)-equivalence, let \( \mathcal{Q} = \chi^{-1}(\mathcal{P}) \), and let \( g = f \circ \chi \). Then \( g \in \mathcal{A}(K, \mathcal{M}) \), and consequently there is
a sequence $g_n \in \mathcal{O}(K, \mathcal{M})$ such that $g_n \to g$ uniformly, and $g_n(q) = g(q)$ for $q \in \mathcal{Q}$. Let $\zeta = \chi^{-1}$, so that $\zeta \in A(L, \mathbb{C})$. Since $L$ has property $A_1$, we can find $\zeta_n \in \mathcal{O}(L, \mathbb{C})$ such that $\zeta_n \to \zeta$ on $L$, with $\zeta_n(p) = \zeta(p)$ for $p \in \mathcal{P}$. Then $f_n := g_n \circ \zeta_n \in \mathcal{O}(L, \mathcal{M})$, $f_n \to f$ uniformly, and $f_n(p) = f(p)$ for $p \in \mathcal{P}$. \qed

4.2. Proof of Theorem \ref{thm:boundary-interpolation}. Thanks to Lemma \ref{lem:finite-iteration} and Theorem \ref{thm:combinatorial} it is sufficient to prove the following result:

**Theorem 6.** For each $K$ in $\mathcal{E}_0$ there is an $L$ in $\mathcal{E}_\infty$ such that $K$ and $L$ are $A$-equivalent.

We will require two results, the first from Combinatorics. A vertex $v$ of a graph $G$ is said to be a cutpoint of $G$ if the graph $G^{(v)}$ obtained by removing from $G$ the vertex $v$ along with all edges incident at $v$ has at least one more connected component than $G$ has. (So for example, if $G$ is connected, $G^{(v)}$ is not connected.) We will need the following elementary fact.

**Lemma 14.** (\cite{H}, Theorem 3.4, p. 29) Let $G$ be a graph with more than one vertex. Then there are at least two vertices of $G$ which are not cutpoints.

The second result is the following boundary interpolation theorem for conformal maps, due to MacGregor and Tepper (\cite{MT}, Theorem 1). $\Delta \subset \mathbb{C}$ is the open unit disc, and $\{z_1, z_2, \ldots, z_n\} \subset \partial \Delta$ and $\{w_1, w_2, \ldots, w_n\} \subset \partial \Delta$ are given finite subsets of the unit circle.

**Proposition 5.** There is a function $f$ which is analytic and univalent in the union of $\Delta$ and a neighborhood of $\{z_1, z_2, \ldots, z_n\}$ and continuous on $\overline{\Delta}$ such that $f(z_k) = w_k$ for $k = 1, \ldots, n$. Furthermore, $|f(z)| = 1$ if $|z| = 1$ and $z$ is sufficiently near any of the points $z_k$, and also $f(\Delta) \subset \overline{\Delta}$.

For a compact connected set $K$ in the plane, by the outer boundary we mean the boundary of the unbounded component of $\mathbb{C} \setminus K$ (this is also a component of $\partial K$.) We use Proposition \ref{prop:boundary-interpolation} to prove the following lemma.

**Lemma 15.** Let $\Omega \subset \mathbb{C}$ be a Jordan domain, and let $\gamma$ be its outer boundary. Suppose we are given a finite set of points $\{z_1, \ldots, z_n\}$ on $\gamma$ and the same number of points $\{w_1, \ldots, w_n\}$ on the unit circle $\partial \Delta$. Then there is a continuous map $f : \overline{\Omega} \to \overline{\Delta}$ such that

1. $f|_{\Omega}$ is conformal,
2. $f(z_k) = w_k$, for $k = 1, \ldots, n$,
3. let $W = f(\Omega)$. Then $\partial W$ is $C^\infty$, and at each $w_k$, $\partial W$ is tangent to $\partial \Delta$.

**Proof.** By Lemma \ref{lem:connecting} there is a domain $D$ bounded by circles and an $A$-equivalence $f_0 : \overline{\Omega} \to \overline{\Delta}$. After applying an inversion of the plane if required, we can assume further than the outer boundary $\gamma$ of $\Omega$ is mapped onto the outer boundary of $D$, which we may assume is the unit circle $\partial \Delta$. Set $z'_k = f_0(z_k)$ for $k = 1, \ldots, n$.

Let $D'$ be a simply connected open set in $\mathbb{C}$ with $C^\infty$ boundary such that $\Delta \subset D'$, $\partial D' \cap \partial \Delta = \{z'_1, \ldots, z'_n\}$ where at each $z'_k$, the boundaries $\partial D'$ and $\partial \Delta$ are tangent to each other. Let $f_1 : D' \to \Delta$ be a conformal map of $D'$ onto $\Delta$. Since $\partial D'$ has $C^\infty$ boundary, $f_0$ extends to a
diffeomorphism of the closures. Set $z_k'' = f_1(z_k')$, and $D'' = f_1(D')$. Observe that $f_1 \circ f_0$ maps $\Omega$ to a subdomain $\Omega''$ of $\Delta$ and the boundary $\partial \Omega''$ is tangent to $\partial \Delta$ at each point of intersection $z_k''$.

We now apply Proposition 5 to obtain a continuous $f_2 : \overline{\Delta} \to \overline{\Delta}$ such that $f_2(z_k'') = w_k$, $f_2$ is conformal on the union of $\Delta$ with a neighborhood of \{ $z_1'', \ldots, z_n''$ \}, and $f_2$ maps a piece of $\partial \Delta$ near each $z_k''$ onto a piece of $\partial \Delta$ near $w_k$. It follows immediately that if $W = f_2(\Omega'')$, $\partial W$ meets $\partial \Delta$ tangentially at each $w_k$. We set $f := f_2 \circ f_1 \circ f_0$. The properties claimed are easily verified.

We now prove Theorem 6.

Proof. It is clear that we only need to consider the case in which $K$ is connected. We use induction on $N$, the number of summands of $K$.

When $N = 1$, the result is reduced to Lemma 12. Now suppose that the result has been proved for some $N \geq 1$, and let $K = \cup_{i=1}^{N+1} \overline{\Omega}_i$. Let $G$ be a graph whose vertices $v_i$ correspond to the sets $\Omega_i$, and there is an edge connecting $v_i$ and $v_j$ iff $\overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset$. Since we have assumed that $K$ is connected, it follows that $G$ is a connected graph. Thanks to Lemma 14 above, we can assume (after a renumbering of the vertices of $G$) that $v_{N+1}$ is not a cutpoint of $G$. Let $K' = \cup_{i=1}^N \overline{\Omega}_i$, and let $P \subset K'$ be the finite set $K' \cap \overline{\Omega}_{N+1} = K' \cap \partial \Omega_{N+1}$. Then $K'$ is connected, therefore contained in exactly one connected component $U$ of $\mathbb{C} \setminus \Omega_{N+1}$. Let $\gamma = \partial U$. Clearly $\gamma$ is a connected component of $\partial \Omega_{N+1}$. It follows that the set $P \subset \gamma$. Moreover, as $\overline{\Omega}_{N+1}$ is connected, it follows that $\Omega_{N+1}$ is contained in exactly one component of $\mathbb{C} \setminus K'$.

We claim that we can assume that $\gamma$ is the outer boundary of $\Omega_{N+1}$. To show this it is sufficient to show that for some $A$-equivalence $\Phi : K \to \hat{K} \subset \mathbb{C}$, $\Phi(\gamma)$ is the outer boundary of $\Phi(\Omega_{N+1})$.

If $\gamma$ not already the outer boundary of $\Omega_{N+1}$ let $z_0 \in U \setminus K'$, where $U \subset \mathbb{C}$ is the component of $\mathbb{C} \setminus \Omega_{N+1}$ which contains $K'$ (then $\gamma = \partial U$). Let $\rho > 0$ be small enough so that $B_{\mathbb{C}}(z_0, \rho) \subset U \setminus K'$, and define the inversion $\Phi : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ by

$$\Phi(z) = \frac{\rho^2}{z - z_0}.$$ 

Then $\Phi(K)$ is contained in the ball $B_{\mathbb{C}}(0, \rho)$, and since $z_0$ is mapped to the point at infinity, it follows that $U \setminus K'$ is mapped to the unbounded component of $\mathbb{C} \setminus K$. Since $\gamma = \partial U$, we see that $\gamma \subset \partial (U \setminus K')$, so that $\Phi(\gamma)$ is the outer boundary of $\Phi(\Omega_{N+1})$.

Now, by induction hypothesis, there is an $L' \in \mathcal{C}_\omega$, and an $A$-equivalence $\chi' : K' \to L'$. Using Lemma 5 we will extend $\chi'$ to an $A$-equivalence $\chi$ defined on $K = K' \cup \overline{\Omega}_{N+1}$.

Let us write $P = \{\zeta_1, \ldots, \zeta_n\}$, and let $\Phi_k = \chi'(\zeta_k)$. Then the $\zeta_k$'s lie at the boundary of a single connected component $U$ of $\mathbb{C} \setminus L'$. Let $U'$ be a simply connected domain, $U \subset U'$ such that $\partial U'$ is $C^\omega$ and passes through each $\zeta_k \in \partial U$, and further at each $\zeta_k$, $\partial U'$ is tangent to $\partial U$, i.e. to $\partial L$. Let $\theta$ be a conformal map of $U'$ onto the disc $\Delta$. Then $\theta$ extends to a holomorphic map of a neighborhood of $\overline{U'}$. We set $w_k = \theta(\zeta_k)$.

Thanks to Lemma 13 above, there is a map $\lambda \in \mathcal{A}(\overline{\Omega}_{N+1}, \mathbb{C})$ such that $\lambda |_{\Omega_{N+1}}$ is conformal, $\lambda(\Omega_{N+1}) \subset \Delta$, $f(\zeta_k) = w_k$, $\partial(\lambda(\Omega_{N+1}))$ is real analytic and tangent to $\partial \Delta$ at each point $w_k$. We
can now define
\[ \chi := \begin{cases} 
\chi' & \text{on } K', \\
\theta^{-1} \circ \lambda & \text{on } \Omega_{N+1}.
\end{cases} \]

Let \( \omega_{N+1} = \chi(\Omega_{N+1}) \), and \( L = L' \cup \omega_{N+1} \). Then \( L \) is in \( \mathcal{C}_\omega \), and \( \chi \) is an \( \mathcal{A} \)-equivalence between \( K \) and \( L \).

\[ \square \]

5. A Problem

We conclude this article by stating an open problem. An solution will lead to a clearer picture of sets with properties \( A_2 \) and \( A_3 \).

Is it possible to prove an analog of Theorem 4 for three sets \( K_1, K_2, K_3 \)? That is, given \( s : K \rightarrow M \), (where \( K = K_1 \cup K_2 \cup K_3 \)), such that \( s(K_j) \) lies in a subset of \( M \) homeomorphic to an open set in \( \mathbb{C}^n \), obtain an approximation to \( s \), after assuming reasonable hypotheses. In particular, we should have \( K_1 \cap K_2 \cap K_3 \neq \emptyset \).

Such a result will be necessary if we want to avoid the use of results like Theorem 5 two prove approximation results for two-dimensional sets. Observe that the use of Theorem 5 resulted in assumptions regarding the smoothness of the sets on which we want to do approximation.

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