Conditional symmetries for systems of PDEs: new definitions and their application for reaction–diffusion systems

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Abstract
New definitions of $Q$-conditional symmetry for systems of PDEs are presented, which generalize the standard notation of non-classical (conditional) symmetry. It is shown that different types of $Q$-conditional symmetry of a system generate a hierarchy of conditional symmetry operators. A class of two-component nonlinear reaction–diffusion systems is examined to demonstrate the applicability of the definitions proposed and it is shown when different definitions of $Q$-conditional symmetry lead to the same operators.

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1. Introduction
Since 1952, when Turing published the remarkable paper [1], in which a revolutionary idea about the mechanism of morphogenesis (the development of structures in an organism during its life) has been proposed, nonlinear reaction–diffusion (RD) systems have been extensively studied by means of different mathematical methods, including group-theoretical methods (see [2–5] and the papers cited therein). The main attention was paid to the investigation of the two-component RD systems of the form
\begin{align}
U_t &= [D^1(U)U_x]_x + F(U, V), \\
V_t &= [D^2(V)V_x]_x + G(U, V),
\end{align}
where $U = U(t, x)$ and $V = V(t, x)$ are the two unknown functions representing the densities of populations (cells, chemicals, etc), $F(U, V)$ and $G(U, V)$ are the two given functions describing the interaction between them and environment, the functions $D^1(U)$ and $D^2(V)$
are the relevant diffusivities (hereafter they are assumed positive functions) and the subscripts 
t and x denote differentiation with respect to these variables. Note that the nonlinear system
(1) generalizes many well-known nonlinear second-order models used to describe various
processes in physics [6], biology [7, 8] and ecology [9].

At present, one can claim that all possible Lie symmetries of (1) with the constant
diffusivities are completely described in [2, 3], while it has been done for the non-constant
diffusivities in [5]. However, the problem of construction of Q-conditional symmetries (non-classical symmetries) for (1) is still not solved. Moreover, to the best of our knowledge,
there are only a few papers devoted to the search of conditional symmetries of evolution
systems, which contain RD systems as a subclass [10–13]. Notably, some general results
about Q-conditional symmetries of systems with power diffusivities of the form
\[ U_t = (U^k U_x)_x + F(U, V), \]
\[ V_t = (V^l V_x)_x + G(U, V) \]
(2)
have been obtained in the recent paper [13]. However, the results obtained in [13] cannot
be adopted for any system of the form (1). Moreover, the authors clearly indicated that two
interesting cases, \( l = k = 0 \) and \( l = k = -1 \), were not studied therein.

It should be noted that, starting from the pioneering work [14], many papers were devoted
to the construction of such symmetries for the scalar nonlinear RD equations of the form
[15–23] (the reader may find more references in the recent book [24])

\[ U_t = [D(U)U_x]_x + F(U) \]
(3)

with the convective term \( B(U)U_x \). (Here, \( B(U) \), \( D(U) \) and \( F(U) \) are the arbitrary smooth
functions.) It is well known that conditional symmetries can be applied for finding exact
solutions of the relevant equations, which are not obtainable by the classical Lie method.
Moreover, the solutions obtained in such a way may have a physical or biological interpretation
(see, e.g., examples in [19, 22, 23]). Thus, the same should be expected in the case of RD
systems.

In this paper, we consider a general multi-component system of evolution PDEs and its
particular case, system (1). The paper is organized as follows. In section 2, we present
new definitions of Q-conditional symmetry, which create a hierarchy of conditional symmetry
operators. In section 3, the definitions are used to construct the systems of determining
equations (DEs) for the RD system (1). An analysis of the derived systems is carried
out; particularly, an example is presented, which illustrates different types of Q-conditional
symmetries. In section 4, the definition of Q-conditional symmetry is applied to find new exact
solutions of the classical Lotka–Volterra system with diffusive terms. Finally, we present some
conclusions in section 5.

2. Definitions of conditional symmetry for systems of PDEs

Here, we present new definitions of Q-conditional symmetry, which naturally arise for systems
of PDEs. To avoid possible difficulties that can occur in the case of an arbitrary system of PDEs,
we restrict ourself to systems of evolution PDEs (the RD system (1) is a typical example).

Consider a system of \( m \) evolution equations \((m \geq 2)\) with two independent \((t, x)\) and \( m \)
dependent \( u = (u_1, u_2, \ldots, u_m) \) variables. Let us assume that the \( k \)-th order \((k \geq 2)\) equations
of evolution type
\[ u_i^t = F^l(t, x, u, u_x, \ldots, u^{(k)}_x), \quad i = 1, 2, \ldots, m, \]
(4)
are defined on a domain $\Omega \subset \mathbb{R}^2$ of independent variables $t$ and $x$. Hereafter, $F^i$ are the smooth functions of the corresponding variables, the subscripts $t$ and $x$ denote differentiation with respect to these variables, $u_i^k = \frac{\partial u_i}{\partial x^k}$ and $u_i^{(k)} = \left( \frac{\partial^k u_i}{\partial x^k} \right)$, $j = 1, 2, \ldots, k$.

It is well known that to find Lie invariance operators, one needs to consider system (4) as the manifold $\mathcal{M} = \{ S_1 = 0, S_2 = 0, \ldots, S_m = 0 \}$, where

$$S_i \equiv u_i^j - F^j(t, x, u, u_x, \ldots, u^{(k)}_i) = 0, \quad i = 1, 2, \ldots, m,$$

in the prolonged space of the variables $t, x, u, u_t, \ldots, u_k$.

Here, $k = \max(k_i, \ i = 1, \ldots, m)$ and the symbol $^j$ ($j = 1, 2, \ldots, k$) denotes totalities of the $j$th-order derivatives w.r.t. the variables $t$ and $x$.

According to the definition, system (4) is invariant (in Lie sense!) under the transformations generated by the infinitesimal operator

$$Q = \xi^0(t, x, u) \partial_t + \xi^1(t, x, u) \partial_x + \eta^1(t, x, u) \partial_{u_1} + \cdots + \eta^m(t, x, u) \partial_{u_m},$$

if the following invariance conditions are satisfied:

$$Q S_i = Q\left( u_i^j - F^j(t, x, u, u_x, \ldots, u^{(k)}_i) \right)|_{\mathcal{M}} = 0, \quad i = 1, 2, \ldots, m. \quad (7)$$

The operator $Q$ is the $k$th-order prolongation of the operator $Q$ and its coefficients are expressed via the functions $\xi^0, \xi^1, \eta^1, \ldots, \eta^m$ by well known formulae (see, e.g., [25, 26]).

The crucial idea used for introducing the notion of $Q$-conditional symmetry (non-classical symmetry) is to change the manifold $\mathcal{M}$, namely the operator $Q$ is used to reduce $\mathcal{M}$. It is important to note that there are several different possibilities to realize this idea in the case of systems containing $m$ PDEs. In fact, different definitions can be formulated for such systems depending on the number of complementary conditions generated by the operator $Q$.

**Definition 1.** Operator (6) is called the $Q$-conditional symmetry of the first type for an evolution system of the form (4) if the following invariance conditions are satisfied:

$$Q S_i = Q\left( u_i^j - F^j(t, x, u, u_x, \ldots, u^{(k)}_i) \right)|_{\mathcal{M}_1} = 0, \quad i = 1, 2, \ldots, m, \quad (8)$$

where the manifold $\mathcal{M}_1$ is one $\{ S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u_i) = 0 \}$ with a fixed number $i_1 (1 \leq i_1 \leq m)$.

**Definition 2.** Operator (6) is called the $Q$-conditional symmetry of the $p$-th type for an evolution system of the form (4) if the following invariance conditions are satisfied:

$$Q S_i = Q\left( u_i^j - F^j(t, x, u, u_x, \ldots, u^{(k)}_i) \right)|_{\mathcal{M}_p} = 0, \quad i = 1, 2, \ldots, m, \quad (9)$$

where the manifold $\mathcal{M}_p$ is one $\{ S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u_{i_1}) = 0, \ldots, Q(u_{i_p}) = 0 \}$ with any given numbers $i_1, \ldots, i_p (1 \leq p \leq i_p \leq m)$.

**Definition 3.** Operator (6) is called the $Q$-conditional symmetry (non-classical symmetry) for an evolution system of the form (4) if the following invariance conditions are satisfied:

$$Q S_i = Q\left( u_i^j - F^j(t, x, u, u_x, \ldots, u^{(k)}_i) \right)|_{\mathcal{M}_m} = 0, \quad i = 1, 2, \ldots, m, \quad (10)$$

where the manifold $\mathcal{M}_m$ is one $\{ S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u_1) = 0, \ldots, Q(u_m) = 0 \}$.

Obviously, all three definitions coincide in the case of $m = 1$, i.e. a single-evolution equation. If $m > 1$, then one obtains a hierarchy of conditional symmetry operators. It can easily be seen that $\mathcal{M}_m \subset \mathcal{M}_p \subset \mathcal{M}_1 \subset \mathcal{M}_1$; hence, each Lie symmetry is automatically
a $Q$-conditional symmetry of the first and $p$-th types, while the $Q$-conditional symmetry of the first type is that of the $p$-th type. From the formal point of view it is enough to find all the $Q$-conditional symmetry (non-classical symmetry) operators. Having the full list of $Q$-conditional symmetries one may simply check which of them is the Lie symmetry or/and $Q$-conditional symmetry of the $p$-th type.

On the other hand, to construct any $Q$-conditional symmetry for a system of PDEs, one needs to solve another nonlinear system, so-called system of DEs, which usually is much more complicated and cumbersome. This problem arises even in the case of a single linear PDE and it was the reason why Bluman and Cole in their well-known work [14] were unable to describe all $Q$-conditional symmetries in explicit form even for the linear heat equation. Thus, all three definitions are important from theoretical and practical point of view. Notably, definition 1 should be more applicable for solving systems of DEs when one examines a multi-component system containing three and more PDEs.

It should be stressed that definition 3 was only used in [10–13] devoted to the search $Q$-conditional symmetries for the systems of PDEs. Moreover, to the best of our knowledge, nobody has noted that several definitions producing a hierarchy of conditional symmetry operators can be defined for systems of PDEs.

3. Conditional symmetries for the RD systems

Consider the RD system (1). According to the definitions presented above, two types of conditional symmetry operators can be derived: the $Q$-conditional symmetry of the first type and $Q$-conditional symmetry of the second type. The second type coincides with the standard non-classical symmetry. It turns out that systems of DEs corresponding to both types essentially have a different structure.

First of all, the RD system (1) can be simplified if one applies the Kirchhoff substitution

$$u = \int D^1(U) \, dU, \quad v = \int D^1(V) \, dV,$$

where $u(t, x)$ and $v(t, x)$ are the new unknown functions. Hereafter, we assume that there exist inverse functions to those arising on the right-hand sides of (11). Substituting (11) into (1), one obtains

$$u_{xx} = d^1(u)u_t + C^1(u, v),$$
$$v_{xx} = d^2(v)v_t + C^2(u, v),$$

where the functions $d^1$, $d^2$ and $C^1$, $C^2$ are uniquely defined via $D^1$, $D^2$ and $F$, $G$, respectively.

Consider a conditional symmetry operator of system (1)

$Q_s = \xi^0_s(t, x, U, V)\partial_t + \xi^1_s(t, x, U, V)\partial_x + \eta^1_s(t, x, U, V)\partial_U + \eta^2_s(t, x, U, V)\partial_V,$

where $\xi^0_s$, $\xi^1_s$, $\eta^1_s$ and $\eta^2_s$ are to-be-found functions. The Kirchhoff substitution transforms (13) to the same form

$Q = \xi^0(t, x, u, v)\partial_t + \xi^1(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v,$

where the operator coefficients are uniquely expressed via those with stars using formulae (11).

Let us use definition 1 to construct the system of DEs for finding $Q$-conditional symmetry operators of the first type. According to the definition the following invariance conditions must be satisfied:

$$\frac{\partial}{\partial t}(d^1(u)u_t + C^1(u, v) - u_{xx})|_{S_1} = 0,$$
$$\frac{\partial}{\partial t}(d^2(v)v_t + C^2(u, v) - v_{xx})|_{S_1} = 0,$$

where $S_1$ is the solution of the system of DEs corresponding to definition 1.
where the manifold
\[ \mathcal{M}_1 = \{ S_1 = 0, S_2 = 0, Q(u) = \xi^0(t, x, u, v)xu^2 + \xi^1(t, x, u, v)xu - \eta^1(t, x, u, v) = 0 \}. \]

Note that the condition \( Q(v) = 0 \) instead of \( Q(u) = 0 \) can be also used; however, it will lead to such conditional symmetry operators, which are obtainable from those generated by the invariance conditions (15) and (16). In fact, the simple renaming \( u \rightarrow v, v \rightarrow u \) (trivial discrete transformations) and the corresponding coefficients’ renaming preserve the form (12) and transform the invariance conditions (15) into that with the condition \( Q(v) = 0 \) because of arbitrariness of the functions \( d^k \) and \( C^k, k = 1, 2 \).

**Remark 1.** In the case of system (12) with the fixed functions \( d^k \) and \( C^k, k = 1, 2 \), the discrete invariance \( u \rightarrow v, v \rightarrow u \) can be broken, so that definition 1 in two cases should be examined.

Now we apply the rather standard procedure for obtaining a system of DEs, using the invariance conditions (15). From the formal point of view, the procedure is the same as for the Lie symmetry search; however, three (not two!) different derivatives, for example \( u_{xx}, v_{xx}, u_t \), can be excluded using the manifold \( \mathcal{M}_1 \). After rather cumbersome calculations, one arrives at the nonlinear system of DEs

\[
\begin{align*}
&1. \ (\xi^0_{xx} - \xi^0_x = \xi^0_{vv} = \xi^0_v = \xi^1 = 0, \\
&2. \ (\eta^1_u = \eta^1_{uu} = \eta^2_{uu} = \eta^2_v = \eta^2_{vv} = 0, \\
&3. \ (\xi^1 \eta^2_{xx} = 0 + 2\xi^0 \eta^2_{uv} = 0, \\
&4. \ (\xi^0 \xi^1 - \xi^0 \xi^1 - 2\xi^1 \xi^1) d^1 - \xi^1 \eta^1 d^1 - 2\xi^0 \eta^1 = 0, \\
&5. \ (2\xi^1 - \xi^1) d^1 = \eta^2_{ux} = 0, \\
&6. \ (\xi^1 d^2 + 2\eta^1_v - \xi^1_{xx} = 0, \\
&7. \ (\eta^1 u + (\eta^1 + 2\xi^1 \eta^1 - \xi^0 \eta^1) ) d^1 - \eta^1_{xx} + \eta^1 C^1 + \eta^2 C^1 + (2\xi^1 - \eta^1) C^1 = 0, \\
&8. \ (2\eta^1_{xx} + \eta^1 \eta^2_{xx} - \eta^1 C^2 + \eta^2 C^2 - \eta^2 C^1 + (2\xi^1 - \eta^1) C^2 = 0, \\
\end{align*}
\]

if \( \xi^0 \neq 0 \) and \( d^1(u) \neq d^2(v) \). (The special cases \( \xi^0 = 0 \) and \( d^1(u) = d^2(v) \) can be treated in the similar way; however, the results are omitted here to avoid cumbersome formulae.)

An analysis of system (17) shows that the finding of \( Q \)-conditional symmetry of the first type cannot be reduced to operators (14) with \( \xi^0 = 1 \). This is in contradiction to the well-known fact occurring in the case of single-evolution equations. Indeed, if one assumes that system (17) is locally equivalent to that with \( \xi^0 = 1 \), then equation (5) takes the form

\[ 2\xi^1 d^2 + \eta^2 d^2 = 0. \]

Therefore, the restriction \( \xi^1 = 0 \) is obtained provided \( d^2 = \text{const} \neq 0 \). On the other hand, there are several RD systems, which are invariant under the Lie symmetry operators of the form (see [2], table 3)

\[ 2t \delta_x + x \delta_x + \alpha_1 u \delta_u + \alpha_2 v \delta_v, \]

(here \( \alpha_1 \) and \( \alpha_2 \) are constants) i.e. those with \( \xi^1 \neq 0 \). Obviously, such operators will be lost if one assumes \( \xi^0 = 1 \) in the very beginning.
Now we apply definition 3 to construct the system of DEs for finding $Q$-conditional symmetry (non-classical) symmetry operators. Thus, the invariance conditions

\[
\begin{align*}
\frac{Q}{2}(d^2(u)u_x + C_1(u,v) - u_{xx})|_{\mathcal{M}_2} &= 0, \\
\frac{Q}{2}(d^2(v)v_x + C_2(u,v) - v_{xx})|_{\mathcal{M}_2} &= 0
\end{align*}
\]

(18)

must be satisfied. Here, the manifold is $\mathcal{M}_2 = \{ S_1 = 0, S_2 = 0, Q(u) = 0, Q(v) = 0 \}$, so that four different derivatives, for example $u_{xx}, v_{xx}, u_t, v_t$, can be excluded in this case. Finally, the system of DEs

\[
\begin{align*}
(1) \quad &\xi_0^0 = \xi_u^0 = \xi_v^0 = \xi_{uu}^1 = \xi_{vv}^1 = 0, \\
(2) \quad &\eta_{uv}^1 = \eta_{uu}^2 = 0, \\
(3) \quad &2\xi_0^0 \xi_u^1 d_1 + \eta_{uu}^1 - 2\xi_{xx}^1 = 0, \\
(4) \quad &2\xi_0^0 \xi_v^1 d_2 + \eta_{vv}^1 - 2\xi_{xx}^1 = 0, \\
(5) \quad &\xi_0^1 \xi_v^1 (d_1 + d_2) + 2\eta_{uv}^1 - 2\xi_{xx}^1 = 0, \\
(6) \quad &\xi_0^1 \xi_u^1 (d_1 + d_2) + 2\eta_{uu}^2 - 2\xi_{uu}^1 = 0, \\
(7) \quad &\xi_0^1 \eta_{u}^1 (d_1 - d_2) - 2\eta_{uv}^1 \xi_{v}^1 d_1 + 2\eta_{uv}^1 - 2\xi_{v}^1 C^1 = 0, \\
(8) \quad &\xi_0^1 \eta_{v}^2 (d_2 - d_1) - 2\eta_{uv}^1 \xi_{u}^1 d_2 + 2\eta_{uu}^2 - 2\xi_{u}^1 C^2 = 0, \\
(9) \quad &\left(\frac{\xi_0^1}{\xi_0^0} \xi_0^1 + 2\eta_{uv}^1 \xi_0^0 - \xi_1^1 - \frac{\eta_1^2}{\xi_0^0} \xi_0^1 - 2\xi_0^0 \xi_0^1 \right) d_1 - \frac{\xi_0^1}{\xi_0^0} \eta_1^1 d_1 \\
&+ \frac{\eta_1^1}{\xi_0^0} \xi_0^1 d_2 - 2\eta_{uu}^1 + \xi_{xx}^1 + 3\xi_{u}^1 C^1 + \xi_{v}^1 C^2 = 0, \\
(10) \quad &\left(\frac{\xi_0^1}{\xi_0^0} \xi_0^1 + 2\eta_{uv}^1 \xi_0^0 - \xi_1^1 - \frac{\eta_1^2}{\xi_0^0} \xi_0^1 - 2\xi_0^0 \xi_0^1 \right) d_2 - \frac{\xi_0^1}{\xi_0^0} \eta_1^2 d_2 \\
&+ \frac{\eta_1^1}{\xi_0^0} \xi_0^1 d_1 - 2\eta_{uv}^1 + \xi_{xx}^1 + 3\xi_{u}^1 C^2 + \xi_{v}^1 C^1 = 0, \\
(11) \quad &\left(\eta_1^1 - \eta_{uv}^1 \xi_0^0 + \frac{\eta_1^2}{\xi_0^0} \xi_0^1 - 2\xi_0^0 \xi_0^1 \right) d_1 + \frac{\eta_1^1}{\xi_0^0} \eta_1^1 d_1 - \eta_1^2 \xi_0^1 d_2 - \eta_{xx}^1 \\
&+ \eta_1^1 C^1 + \eta_1^2 C^1 + (2\xi_1^1 - \eta_1^1) C^1 - \eta_1^2 C^2 = 0, \\
(12) \quad &\left(\eta_1^2 - \eta_{uv}^1 \xi_0^0 + \frac{\eta_1^2}{\xi_0^0} \xi_0^1 - 2\xi_0^0 \xi_0^1 \right) d_2 + \frac{\eta_1^2}{\xi_0^0} \eta_1^2 d_2 - \eta_1^2 \xi_0^1 d_1 - \eta_{xx}^1 \\
&+ \eta_1^1 C^2 + \eta_1^2 C^2 - \eta_1^2 C^1 + (2\xi_1^1 - \eta_1^2) C^2 = 0
\end{align*}
\]

is obtained if $\xi_0^0 \neq 0$. (The special case $\xi_0^0 = 0$ can be treated in a quite similar way.)

An analysis of system (19) shows that finding $Q$-conditional symmetry operators for the RD system (1) can be reduced to operators (14) with $\xi_0^0 = 1$ provided $\xi_0^0 \neq 0$. In fact, the substitution

\[
\xi_s = \frac{\xi_0^1}{\xi_0^0}, \quad \eta_s = \frac{\eta_0^1}{\xi_0^0}, \quad k = 1, 2,
\]

(20)
reduces system (19) to the form (the stars next to the functions $\xi$ and $\eta^k$ are skipped)

\begin{align}
(1) \quad & \xi_{uu} = \xi_{vv} = \xi_{uv} = 0, \\
(2) \quad & \eta_{vv}^1 = \eta_{uu}^2 = 0, \\
(3) \quad & 2\xi\xi_u d^1 + \eta_{uu}^1 - 2\xi v = 0, \\
(4) \quad & 2\xi\xi_v d^2 + \eta_{vu}^2 - 2\xi v = 0, \\
(5) \quad & \xi\xi_u (d^1 + d^2) + 2\eta_{uv}^1 - 2\xi u v = 0, \\
(6) \quad & \xi\xi_v (d^1 + d^2) + 2\eta_{uv}^2 - 2\xi u v = 0, \\
(7) \quad & \eta_{v}^1 (d^1 - d^2) - 2\xi u\eta^1 d^1 + 2\eta_{uv}^1 - 2\xi v^1 C^1 = 0, \\
(8) \quad & \eta_{v}^2 (d^2 - d^1) - 2\xi u\eta^2 d^2 + 2\eta_{uv}^2 - 2\xi u^2 C^2 = 0, \\
(9) \quad & (2\xi u \eta^1 - \xi_l - \xi_v \eta^2 - 2\xi \xi_v) d^1 - \xi \eta^1 d^1_u \\
& + \xi v \eta^2 d^2 - 2\eta_{uv}^1 + \xi_{xx} + 3\xi u C^1 + \xi u C^2 = 0, \\
(10) \quad & (2\xi u \eta^2 - \xi_l - \xi_u \eta^1 - 2\xi \xi_u) d^2 - \xi \eta^2 d^2_v \\
& + \xi u \eta^1 d^1 - 2\eta_{uv}^2 + \xi_{xx} + 3\xi u C^2 + \xi u C^1 = 0, \\
(11) \quad & (\eta^1_l + \eta^2_{uv} + 2\xi u \eta^1) d^1 + (\eta^2_{uv})^2 d^2_v - \eta^2_{xx} C^1 - \eta^2_{ux} d^2_v \\
& + \eta^1_{xx} C^1 + \eta^2_{xx} C^2 + (2\xi u - \eta^1_{xx}) C^1 - \eta^1_{xx} C^2 = 0, \\
(12) \quad & (\eta^1_v + \eta^2_{uv} + 2\xi u \eta^2) d^2 + (\eta^2_{uv})^2 d^2_u - \eta^1_{xx} d^1 - \eta^1_{xx} \\
& + \eta^2_{xx} C^2 - \eta_{xx} C^1 + (2\xi u - \eta_{xx}) C^2 = 0 \tag{21}
\end{align}

and $\xi^0$ is an arbitrary smooth function of the time variable. Now we realize that system (21) is not nothing else but the system of DEs to find the $Q$-conditional symmetry operators (14) with $\xi^0 = 1$. Thus, to find all $Q$-conditional symmetries of the RD system (1), one needs to solve only the particular case of system (19), i.e. system (21).

**Remark 2.** System (21) with $d^1(u) = u^m$, $d^2(v) = v^n$ coincides with the system of DEs obtained and analyzed in the recent paper [13].

**Remark 3.** The system of DEs to search Lie symmetries of RD systems with non-constant diffusivities (see equations (9)–(13) in [5]) differs essentially from systems (17) and (19). In contrast to systems of DEs for $Q$-conditional symmetries, one for Lie symmetries contains the subsystem of equations

$$\xi^0_u = \xi^0_v = \xi^1_u = \xi^1_v = \eta^1_u = \eta^1_v = 0$$

what essentially simplifies its solving.

Of course, the system of DEs (17) can be derived from system (19) because each $Q$-conditional symmetry of the first type is automatically a $Q$-conditional symmetry (but not vice versa!). However, system (19) is much complicated than (17) and its solving is a difficult problem even in particular cases, e.g. $d^1(u) = u^m$, $d^2(v) = v^n$ [13]. Thus, we believe that $Q$-conditional symmetries of the first type can be found much easily for many nonlinear RD systems (1) with correctly specified coefficients, arising in applications.

Now we present an example, which highlights when the $Q$-conditional symmetry (non-classical symmetry) is or is not of the first type.

**Example 1.** Consider system (19) assuming

$$\xi^1_u = \eta^1_u = \eta^2_u = 0. \tag{22}$$
In this case, the system can be immediately simplified and one obtains only two equations

\[
\left( \eta_1^2 - \frac{\eta_1^4}{\xi^2} \right) d^4 + \frac{\eta_1^2}{\xi^2} \eta_1^4 d^3_u - \eta_1^4 + \eta_1^4 C_u^1 + \eta_1^2 C_v^1 = 0,
\]

\[
\left( \eta_1^2 - \frac{\eta_2^4}{\xi^2} \right) d^2 + \frac{\eta_2^4}{\xi^2} \eta_2^2 d^2_v - \eta_2^4 + \eta_1^4 C_u^2 + \eta_2^2 C_v^2 = 0
\]

(23)

to find the functions

\[
\xi^0 = c(t),
\]

\[
\eta_1 = r^1(t) u + p^1(t, x),
\]

\[
\eta_2 = r^2(t) v + p^2(t, x),
\]

(24)

where the functions on the right-hand sides should be determined.

System (17) with restrictions (22) takes the form

\[
\left( \eta_1^2 - \frac{\eta_1^4}{\xi^2} \right) d^4 + \frac{\eta_1^2}{\xi^2} \eta_1^4 d^3_u - \eta_1^4 + \eta_1^4 C_u^1 + \eta_1^2 C_v^1 = 0,
\]

\[
\eta_1^2 d^2 - \eta_1^2 + \eta_1^4 C_u^2 + \eta_2^2 C_v^2 = 0,
\]

\[
\xi^0 d^2 = \eta_1^2 d_v^2
\]

(25)

where the unknown functions $\xi^0$ and $\eta^k$, $k = 1, 2$, must be of the form (24).

Now one notes that systems (23) and (25) are equivalent if $\eta^2 d_v^2 = 0$ (we remind the reader that substitution (20) reduces system (19) to that with $\xi^0 = 1$). Thus, we arrive at the statement: each $Q$-conditional symmetry operator (14) of the RD system (12) with $d^2(v) = \text{const}$ (or $d^1(u) = \text{const}$) is equivalent (up to the multiplier $c(t) \neq 0$) to a $Q$-conditional symmetry operator of the first type if restrictions (22) take place. Moreover, the statement is valid for all systems of the form (12) under the additional restriction $\eta^2 = 0$ (or $\eta^1 = 0$).

Taking into account the Kirchhoff substitution, the same statement can be formulated for the RD system (1) because substitution (11) preserves restrictions (22).

Finally, we briefly analyze the RD system (2) (the cases $l = k = 0$ and $l = k = -1$ are excluded) using table 1 of [13], which presents all possible $Q$-conditional symmetry (non-classical symmetry) operators of (2) under restrictions (22). There are five different cases according to theorem 1 [13]. Using the statement obtained, we conclude that the $Q$-conditional symmetry operator arising in the first case of table 1 is simultaneously a $Q$-conditional symmetry operator of the first type while the operator arising in the third case is not because $l = k = -\frac{3}{2} \neq 0$ and $\eta^i \eta^j \neq 0$. The other three operators of $Q$-conditional symmetry from table 1 are those of $Q$-conditional symmetry of the first type only under additional restrictions on coefficients of the corresponding RD systems. These restrictions can be easily derived using the statement formulated above. For example, the $Q$-conditional symmetry operator listed in the second case of table 1 is that of the first type under the restrictions either $\lambda_1 = 0$ or $l = 0$. (If $\lambda_1 = l = 0$, then it is the Lie symmetry operator of the corresponding RD system.)

4. Conditional symmetries of the first type for the diffusive Lotka–Volterra system

Here, we consider the diffusive Lotka–Volterra (DLV) system

\[
\lambda_1 u_t = u_{xx} + u(a_1 + b_1 u + c_1 v),
\]

\[
\lambda_2 v_t = v_{xx} + v(a_2 + b_2 u + c_2 v),
\]

(26)
Theorem 1. In the case \( \lambda_1 \neq \lambda_2 \), the DLV system (26) is invariant under \( Q \)-conditional operators of the first type if and only if the corresponding system and \( Q \)-conditional symmetries (up to local transformations \( u \mapsto bu, v \mapsto cv, bc \neq 0 \)) have the forms

\[
\begin{align*}
\lambda_1 u_t &= u_{xx} + u(a_1 + a + v), \\
\lambda_2 v_t &= v_{xx} + v(a_2 + a + v), \\
Q_1 &= (\lambda_1 - \lambda_2) \partial_t + (a_1 - a_2) u(\partial_u - \partial_v), \\
Q_2 &= (\lambda_1 - \lambda_2) \partial_t + (a_1 - a_2) v(\partial_u - \partial_v).
\end{align*}
\]

(27) (28) (29)

Each other \( Q \)-conditional operator of the first type coincides with the Lie symmetry operator.

In the case \( \lambda_1 = \lambda_2 \), the DLV system (26) is invariant only under such \( Q \)-conditional operators of the first type, which coincide with the Lie symmetry operators.

Proof. Proof is based on solving the system of DEs (17) with \( d^2 = \lambda_\delta, C^1 = u(a_1 + b_1 u + c_1 v) \) and \( C^2 = v(a_2 + b_2 u + c_2 v) \). After rather simple calculations, one obtains that the DLV
system (26) is invariant only under a $Q$-conditional operator of the first type and the coefficient restrictions

$$\lambda_1 \neq \lambda_2, \quad b_1 = b_2 = b, \quad c_1 = c_2 = c.$$  

In this case the transformations $u \rightarrow bu, v \rightarrow cv$ reduce the DLV system to the form (27). The corresponding operator takes the form (28).

To find operator (29) one needs to proceed the same algorithm but with the invariance criteria (15) for the manifold

$$\mathcal{M}_1 = \{S_1 = 0, S_2 = 0, Q(v) \equiv \xi^0(t, x, u, v)u_t + \xi^1(t, x, u, v)v_t - \eta(t, x, u, v) = 0\},$$  

(30)

i.e. to use definition 1 with $i_1 = 2$.

**Remark 4.** Theorem 1 gives a complete description of $Q$-conditional symmetries of the first type in explicit form for the DLV system (26). It turns out that all possible $Q$-conditional symmetries of the second type cannot be derived in a similar way. In fact, the system of DEs (21) with $d^k = \lambda_k, C^1 = u(a_1 + b_1u + c_1v)$ and $C^2 = v(a_2 + b_2u + c_2v)$ has much more complicated structure and can be solved only under some additional assumptions.

The $Q$-conditional symmetry operators (28) and (29) can be applied for finding exact solutions of the DLV system (27) using the same algorithm as for classical Lie symmetries. Since both operators have the same structure, we examine only the second one. Thus, the corresponding ansatz reads

$$u(t, x) = \phi_1(x) - \phi_2(x) \exp\left(\frac{a_1 - a_2}{\lambda_1 - \lambda_2} t\right),$$  

$$v(t, x) = \phi_2(x) \exp\left(\frac{a_1 - a_2}{\lambda_1 - \lambda_2} t\right),$$  

(31)

and reduces system (27) to the ODE system

$$\phi''_1 + \phi''_2 + a_1 \phi_1 = 0,$$

$$\phi''_2 + \frac{a_2 \lambda_1 - a_1 \lambda_2}{\lambda_1 - \lambda_2} \phi_2 + \phi_1 \phi_2 = 0,$$

(32)

where $\phi_1(x)$ and $\phi_2(x)$ are the functions to be determined.

The general solution of this nonlinear ODE system cannot be found in an explicit form. However, we constructed its particular solutions, which lead to interesting solutions of the corresponding DLV system. In fact, the first ODE in (29) possesses the non-zero steady-state solution $\phi_1 = -a_1$. Substituting this solution into the second ODE in (29), one arrives at the linear ODE

$$\phi''_2 - \beta \lambda \phi_1 = 0,$$

(33)

where $\beta = \frac{a_1 - a_2}{\lambda_1 - \lambda_2} \neq 0$. Thus, using the general solution of (33) and ansatz (31), we obtain two families of exact solutions of the DLV system (27):

$$u(t, x) = -a_1 + \frac{1}{a_2 - a_1} (C_1 \exp(\sqrt{\beta \lambda_1} x) + C_2 \exp(-\sqrt{\beta \lambda_1} x)) e^{\beta t},$$  

$$v(t, x) = \frac{1}{a_1 - a_2} (C_1 \exp(\sqrt{\beta \lambda_1} x) + C_2 \exp(-\sqrt{\beta \lambda_1} x)) e^{\beta t},$$  

(34)
if $\beta > 0$, and
\[ u(t, x) = -a_1 + \frac{1}{a_2 - a_1} (C_1 \cos(\sqrt{-\beta \lambda_1} x) + C_2 \sin(\sqrt{-\beta \lambda_1} x)) e^{\beta t}, \]
\[ v(t, x) = \frac{1}{a_1 - a_2} (C_1 \cos(\sqrt{-\beta \lambda_1} x) + C_2 \sin(\sqrt{-\beta \lambda_1} x)) e^{\beta t}, \]  
(35)
if $\beta < 0$. (Hereafter, $C_1$ and $C_2$ are the arbitrary constants.)

It should be noted that solutions (34) and (35) cannot be constructed using Lie symmetries. In fact, the DLV system (27) admits only the trivial Lie symmetry [29]; hence, the plane wave solutions can only be found by Lie symmetry reductions (the reader may find examples of such solutions in [30] and [29]). Obviously, solutions (34) and (35) possess more complicated structures.

Example 2. Consider solution (35) with $C_1 = 0$. Using the substitution $u \to -bu$, $v \to -cv$ ($b > 0$, $c > 0$), one transforms the DLV system (27) to the standard system describing the competition of two species (see, e.g., [7])
\[ \lambda_1 u_t = u_{xx} + u(a_1 - bu - cv), \]
\[ \lambda_2 v_t = v_{xx} + v(a_2 - bu - cv), \]  
(36)
and solution (35) to the form
\[ u(t, x) = \frac{a_1 b}{a_1 - a_2} C_2 \sin(\sqrt{-\beta \lambda_1} x) e^{\beta t}, \]
\[ v(t, x) = \frac{1}{(a_2 - a_1) c} C_2 \sin(\sqrt{-\beta \lambda_1} x) e^{\beta t}, \]  
(37)
where the coefficient restrictions $\beta \equiv \frac{a_1 - a_2}{a_1 - a_2} < 0$, $a_1 > 0$, $a_2 > 0$ are assumed.

One notes that this solution satisfies the constant Dirichlet conditions
\[ x = 0 : u = \frac{a_1 b}{a_1 - a_2}, \quad v = 0, \]
\[ x = \frac{\pi}{\sqrt{\beta \lambda_1}} : u = \frac{a_1 b}{a_1 - a_2}, \quad v = 0, \]  
(38)
in the domain $\Omega = \{(t, x) \in (0, +\infty) \times \left(0, \frac{\pi}{\sqrt{\beta \lambda_1}} \right)\}$. Moreover, solution (37) has the time asymptotic
\[ (u, v) \to \left( \frac{a_1 b}{a_1 - a_2}, 0 \right), \quad t \to +\infty. \]  
(39)

Thus, this solution describes the competition between two species when the species $u$ eventually dominate while the species $v$ die.

5. Conclusions

In this paper, new definitions of $Q$-conditional symmetry for systems of PDEs are presented, which generalize the standard notation of non-classical (conditional) symmetry. It is shown that different types of $Q$-conditional symmetry generate a hierarchy of conditional symmetry operators. Since conditional symmetries can be applied for finding exact solutions of the relevant equations, which are not obtainable by the classical Lie method, we demonstrated this for constructing new exact solutions of the DLV system.

Systems of DEs to find $Q$-conditional symmetries of two types for the nonlinear RD system (1) are constructed. The case of conditional invariance under operator (13) with the
coefficients $\xi^1 = 0, \eta_1^1(t, x, U)$ and $\eta_2^1(t, x, V)$ is analyzed in detail. Using the recent paper [13], we established that there are nonlinear RD systems of five types, which possess $Q$-conditional symmetry (non-classical symmetry) operators satisfying definition 3. However, only one of them is simultaneously a $Q$-conditional symmetry operator of the first type.

Definitions 1–3 can be straightforwardly extended on an arbitrary $m$-component system of PDEs presented in a ‘canonical’ form (the system has a simplest form and there is no non-trivial differential consequence). However, new difficulties may arise because of differential consequences of additional conditions

$$Q(u_1) = 0, \ldots, Q(u_m) = 0, \quad (40)$$

generated by the conditional symmetry operator (6). In fact, there are examples of single-hyperbolic PDEs when the set of $Q$-conditional symmetry operators can be extended if one takes into account such differential consequences. For example, this occurs in the case of nonlinear hyperbolic equation $u_{tt} = uu_{xx}$. The reader may easily check that there is a much wider set of $Q$-conditional symmetry operators in [15] (chapter 5) than was found in [27] (supplement 7) neglecting differential consequences. It is the reason why there is the definition of $Q$-conditional invariance (non-classical invariance) for a $k$th-order single PDE, which requires to take into account all differential consequences of the additional condition $Q(u) = 0$ up to the order $k$ (see, e.g., [28]). From this point of view, definitions 1–3 can be formally extended by using differential consequences of (40) up to the order $k = \max\{k_i, i = 1, \ldots, m\}$. However, we believe that such extensions produce much more cumbersome formulae but do not lead to any new operators for evolution system (4) because all the above-mentioned differential consequences contain the mixed and/or high-order time derivatives, $u_{tx}, u_{tt}, u_{txx}, \ldots$, which are absent in system (4). For example, if one applies the ‘amended’ definition 1 with $M^* = \{\xi_1 = 0, \xi_2 = 0, Q(u) = 0, \partial_t Q(u) = 0, \partial_x Q(u) = 0\}$ (instead of (16)) to find $Q$-conditional symmetry operators of the first type for the RD system (12), then the system of DEs (17) is exactly obtained but nothing more.

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