CUT-OFF FUNCTION LEMMA IN $\mathbb{P}^k$

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Abstract. In this note, we compute a cut-off function over $\mathbb{P}^k$. Let sufficiently small $\delta > 0$ be given. When we are given a compact set $K$ in $\mathbb{P}^k$ and a prescribed open neighborhood $K_\delta$ of $K$, we find a smooth cut-off function $\chi_\delta \equiv 1$ over $K$ and $\text{supp}(\chi_\delta) \subseteq K_\delta$, where $K_\delta$ denotes the set of points whose distance to $K$ is less than $\delta$ with respect to the Fubini-Study metric of $\mathbb{P}^k$. Moreover, we estimate the bound of the derivatives of $\chi_\delta$ in terms of $\delta$. It seems to be well-known, but we want to provide detailed computations. They are very elementary.

1. Introduction

In this note, our space is $\mathbb{P}^k$ and we assume that the distance is measured with respect to the Fubini-Study metric if we do not specify.

Let $\delta_0 > 0$ be given. We consider $0 < \delta < \delta_0$. Let $K \subseteq \mathbb{P}^k$ be compact and $K_\delta$ a $\delta$-neighborhood of $K$, that is, the set of points whose distance to $K$ is less than $\delta$ with respect to the Fubini-Study metric. We want to prove the following lemma:

Lemma 1.1. There exists a smooth cut-off function $\chi_\delta : \mathbb{P}^k \to [0, 1]$ such that $\chi_\delta \equiv 1$ over $K$ and $\text{supp}(\chi_\delta) \subseteq K_\delta$. Moreover, $\|\chi_\delta\|_{C^\alpha} \lesssim |\delta|^{-\alpha}$ as $\delta$ varies.

Here, $\|\cdot\|_{C^\alpha}$ denotes the $C^\alpha$-norm of the function. The idea is simply to smooth out a characteristic function by convolution (of the Lie group of automorphisms over $\mathbb{P}^k$).

2. Family of Local Coordinate Charts of $\mathbb{P}^k$

It suffices to prove the lemma for a fixed family of local coordinate charts. Thus, we will fix one as follows.

For $\mathbb{P}^k$, we can find $k$ natural affine coordinate charts covering $\mathbb{P}^k$ of the form $\{[z_0 : \ldots : z_{i-1} : z_{i+1} : \ldots : z_k] | j \in \mathbb{C} \text{ for } j \neq i \}$ for $i = 0, \ldots, k$, which we will call the $Z_i$-coordinate chart. For this chart, there is a natural coordinate map $\zeta_i : Z_i \to \mathbb{C}^i \times \{1\} \times \mathbb{C}^{k-i}$ defined by $\zeta_i([z_0 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_k]) = (z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_k)$.

We defined a norm $\|\cdot\|_i$ defined by

$$\|(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_k)\|_i = (|z_0|^2 + \ldots + |z_{i-1}|^2 + |z_{i+1}|^2 + \ldots + |z_k|^2)^{\frac{1}{2}}$$

for each $\mathbb{C}^i \times \{1\} \times \mathbb{C}^{k-i}$.
3. Automorphism group of $\mathbb{P}^k$

The group $\text{Aut}(\mathbb{P}^k) = \text{PGL}(k+1, \mathbb{C})$ of automorphisms of $\mathbb{P}^k$ is a complex Lie group of complex dimension $k^2 + 2k$. An element of $\text{Aut}(\mathbb{P}^k)$ can be understood as an equivalence class of the complex $(k+1) \times (k+1)$ matrix group under the equivalence relation given by scaling.

Without loss of generality, we may consider a point $z \in \mathbb{Z}$ and its coordinates $\zeta \in \{1\} \times \mathbb{C}^k$. Let $h = \{0, h_1, h_2, ..., h_k\}$ with $|h_i| < \epsilon$ for sufficiently small $\epsilon > 0$. Then $\zeta + h \in \{1\} \times \mathbb{C}^k$ is a very close point near $\zeta \in \{1\} \times \mathbb{C}^k$, where the addition is coordinatewise and we can find a unique linear map $G_h : \{1\} \times \mathbb{C}^k \rightarrow \{1\} \times \mathbb{C}^k$ defined by

$$G_h = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ h_1 & 1 & 0 & \cdots & 0 \\ h_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & 0 & 0 & \cdots & 1 \end{pmatrix}$$

such that $G_h(\zeta) = \zeta + h$. Note that $G_h \circ G_{-h} = G_{-h} \circ G_h = \text{Id}$.

Using the exponential map of Lie algebra to Lie group, we can find holomorphic coordinates $\psi : \text{sl}(k+1, \mathbb{C}) \rightarrow \text{PGL}(k+1, \mathbb{C})$ near $\text{Id} \in \text{PGL}(k+1, \mathbb{C})$ where $\text{sl}(k+1, \mathbb{C})$ is the special linear Lie algebra, which is the set of $(k+1) \times (k+1)$ matrices with zero trace. Near the $\text{Id} \in \text{PGL}(k+1, \mathbb{C})$, we can also find a representation $\text{PGL}(k+1, \mathbb{C}) \rightarrow \text{GL}(k+1, \mathbb{C})$ by picking a $(k+1) \times (k+1)$ matrix with the $(1,1)$-component being 1. Let $\phi$ denote this representation. We consider the following diagram

$$\begin{array}{ccc} \text{sl}(k+1, \mathbb{C}) & \rightarrow^{H_h} & \text{sl}(k+1, \mathbb{C}) \\ \psi \downarrow & & \psi \downarrow \\ \text{PGL}(k+1, \mathbb{C}) & \rightarrow^{[G_h]} & \text{PGL}(k+1, \mathbb{C}) \\ \phi \downarrow & & \phi \downarrow \\ \text{GL}(k+1, \mathbb{C}) & \rightarrow^{\overline{G_h}} & \text{GL}(k+1, \mathbb{C}) \end{array}$$

where in the second line, $[]$ means the equivalence class that contains the inside element, $[G_h][A] = [A \cdot G_h]$ for $[A] \in \text{PGL}(k+1, \mathbb{C})$, and $\overline{G_h}$ is defined as follows:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,k+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k+1} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k+1} \end{pmatrix}$$

$$\downarrow \overline{G_h}$$

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,k+1} \\ a_{2,1} + \sum_{i=2}^{k+1} a_{2,i} \cdot h_{i-1} & a_{2,2} & \cdots & a_{2,k+1} \\ a_{3,1} + \sum_{i=2}^{k+1} a_{3,i} \cdot h_{i-1} & a_{3,2} & \cdots & a_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} + \sum_{i=2}^{k+1} a_{k+1,i} \cdot h_{i-1} & a_{k+1,2} & \cdots & a_{k+1,k+1} \end{pmatrix}$$
Note that \(H_h, [G_h]\) and \(\overline{G_h}\) in the diagram are not defined over the entire space. However, there exists a sufficiently small \(\epsilon > 0\) such that for all \(\{h_i\}_{i=1}^n\) with \(|h_i| < \epsilon\) for \(i = 1, \ldots, n\), \(\overline{G_h}\) is well-defined over all \(A \in \text{GL}(n + 1, \mathbb{C})\) with \(\|A - \text{Id}\| < \epsilon\) and with the \((1,1)\)-component of \(A\) being 1, where \(\|\cdot\|\) is the standard matrix norm. Since \(\phi\) and \(\psi\) are local biholomorphisms, we can also find corresponding subsets in \(\text{PGL}(k + 1, \mathbb{C})\) and \(sl(k + 1, \mathbb{C})\).

We identify \(sl(k + 1, \mathbb{C})\) with \(\mathbb{C}^{k^2 + 2k}\) and the set of representations of \(\text{PGL}(k + 1, \mathbb{C})\) with \(\mathbb{C}^{k^2 + 2k}\). For convenience, we use \(x = (x_1, \ldots, x_{k^2 + 2k})\) for \(sl(k + 1, \mathbb{C})\) and \(\xi = (\xi_1, \ldots, \xi_{k^2 + 2k})\) for the other. Then

\[
\mathbb{C}^{k^2 + 2k} \xrightarrow{\psi} \mathbb{C}^{k^2 + 2k} \quad \text{with \(\psi\)} \quad \text{and \(\psi\downarrow\)}
\[
PGL(k + 1, \mathbb{C}) \xrightarrow{[\Phi]} PGL(k + 1, \mathbb{C}) \quad \text{with \(\Phi\)} \quad \text{and \(\phi\downarrow\}}
\]

\[
PGL(k + 1, \mathbb{C}) \xrightarrow{\Phi^{-1} \circ \overline{G_h} \circ \Phi} \mathbb{C}^{k^2 + 2k}.
\]

We denote \(\phi \circ \psi\) by \(\Phi\). Then, \(\xi_i = \Phi_i(x_1, \ldots, x_{k^2 + 2k})\) for \(i = 1, \ldots, k^2 + 2k\) and the map \(H_h = \Phi^{-1} \circ \overline{G_h} \circ \Phi\) is a map from \(\mathbb{C}^{k^2 + 2k}\) to \(\mathbb{C}^{k^2 + 2k}\). Note that in our case, \(\psi, \phi\) are smooth and \(\overline{G_h}\) is smooth with respect to \(h\).

\[\text{4. Measures on } sl(k + 1, \mathbb{C})\]

Recall that \(x\) is used for \(sl(k + 1, \mathbb{C})\). Let \(\lambda\) denote the standard Euclidean measure on \(sl(k + 1, \mathbb{C})\). We assign the standard matrix norm \(|x|_s\) to each \(x \in sl(k + 1, \mathbb{C})\). We consider a smooth radial probability measure \(\mu\) over the coordinate \(sl(k + 1, \mathbb{C})\) centered at \(O \in sl(k + 1, \mathbb{C})\) with its support \(|x|_s < \sigma\) for sufficiently small \(\sigma > 0\), which makes \(\Phi\left(\{||x||_s < \sigma\}\right) \subseteq \{||A - \text{Id}\| < \epsilon\}\). Then, \(d\mu = M(x)d\lambda\) where \(M\) is a smooth function defined on \(sl(k + 1, \mathbb{C})\) and has support in \(|x|_s < \sigma\).

Let \(h_{\theta} : sl(k + 1, \mathbb{C}) \longrightarrow sl(k + 1, \mathbb{C})\) be a scaling map by \(\theta\) for \(|\theta| \leq 1\). We define \(\mu_{\theta} := (h_{\theta})_*(\mu)\). Then, \(\mu_{\theta}\) is a smooth measure for \(\theta \neq 0\) and a Dirac measure at \(O \in sl(k + 1, \mathbb{C})\) for \(\theta = 0\). Note that the support of \(\mu\) is in \(\{||x||_s \leq \theta\sigma\} \subseteq \{||x||_s \leq \sigma\}\).

For the better terminology, by the derivatives of \(\mu_{\theta}\), we mean the derivatives of the Radon-Nikodym derivative of \(\mu_{\theta}\) with respect to the standard Euclidean measure \(\lambda\).

\[\text{5. Regularization}\]

In this section, we define a regularization of a bounded function and provide the estimate of the regularity.

Let \(f\) be a bounded complex-valued function over \(\mathbb{P}^k\) with compact support. Without loss of generality, we may assume that \(0 \leq |f| \leq 1\). Then, we define the \(\theta\)-regularization \(f_{\theta}\) of \(f\) as being

\[f_{\theta}(z) = \int_{\text{Aut}(\mathbb{P}^k)} ((\tau_x)_*)f)(z)d\mu_{\theta}(x)\]
Without loss of generality, we may assume that \( z \in \mathbb{Z}_0 \). Let \( \zeta \in \{1\} \times \mathbb{C}^k \) be the coordinates of \( z \) and \( F \) the representation of \( f \) with respect to \( \{1\} \times \mathbb{C}^k \). With respect to the coordinate \( \{1\} \times \mathbb{C}^k \), we have the following representation:

\[
F_\theta(\zeta + h) = \int_{s\mathcal{I}(k+1,\mathbb{C})} ((\Phi(x))_s F)(G_h(\zeta))d\mu_\theta(x)
\]

Note that \( H_h \) is holomorphic and injective over the support of the measure \( \mu_\theta \). By change of coordinates, we have

\[
F_\theta(\zeta + h) = \int_{s\mathcal{I}(k+1,\mathbb{C})} ((\Phi(H_h(x)))_s F)(\zeta)d\mu_\theta(x)
\]

With \( \zeta \) fixed, the differentiation of the right hand side with respect to \( h_i \)'s makes sense since the measure is smooth. By the direct application of the definition of the derivative, the partial derivative of \( F_\theta(\zeta) \) with respect to \( \zeta_i \) at \( \zeta \) is the same as the partial derivative of \( F_\theta(\zeta + h) \) with respect to \( h_i \) at 0. Thus, we can see that \( F_\theta \) is smooth. Moreover, we can estimate its regularity.

The \( C^\alpha \)-norm of \( F_\theta \) completely depends on the value of \( F \) near \( \zeta \) and the derivatives of the measure with respect to \( h \). It is not hard to see that \( (H_h)_s [(h_\theta)_s] d\lambda = |\theta|^{-2k^2-4k} d\lambda \). Indeed, \( \Phi \) is a coordinate change map and \( G_h \) is a linear shear map. Thus, it remains to estimate the \( C^\alpha \)-norm of \( M \). So, since \( (H_h)_s [(h_\theta)_s] M = M((\frac{1}{2}\Phi^{-1} o G_h o \Phi)) \), the \( C^\alpha \)-norm of \( (H_h)_s [(h_\theta)_s] M \) is bounded by the product of \( |\theta|^{-\alpha} \) and a constant multiple of \( C^\alpha \)-norms of \( M, \Phi \) and \( \Phi^{-1} \). Note that the latter is independent of \( \theta \).

Putting all together, since \( F \) is bounded, the support of the measure is \( ||x|| \leq \theta \sigma \) and \( dim_{\mathcal{I}} \mathcal{I}(k+1,\mathbb{C}) = k^2 + 2k \),

\[
f_\theta C^\alpha \lesssim |\theta|^{-2k^2-4k-\alpha} |\theta|^{2k^2+4k} \|f\|_{C^\alpha} = |\theta|^{-\alpha} \|f\|_{C^\alpha}.
\]

Note that it can be more precise when we estimate the absolute value at a point in terms of its neighborhood with compact closure.

6. MAIN CUT-OFF FUNCTION LEMMA

We consider two kinds of open balls in \( \{1\} \times \mathbb{C}^k \). One is induced from the Fubini-Study metric of \( \mathbb{P}^k \) and the other is from the standard Euclidean metric \( ||\cdot||_0 \). The open ball centered at \( \zeta \in \{1\} \times \mathbb{C}^k \) and of radius \( r > 0 \) of first kind is denoted by \( B_F(\zeta, r) \) and that of second kind is denoted by \( B_E(\zeta, r) \). Then, by comparison of the infinitesimal versions of the two metrics, we know that \( B_E(\zeta, \frac{r}{2} ||\zeta||_0) \subseteq B_F(\zeta, r) \).

The proof of Lemma [1.7]. Note that \( \Phi \) is holomorphic near the closure of the neighborhood of \( \{||x||_s < \sigma\} \), we can find a constant \( C > 0 \) such that \( \frac{1}{C} ||\Phi(x) - Id|| < ||x||_s < C ||\Phi(x) - Id|| \) for \( \{||x||_s < \sigma\} \). Here, \( C \) is independent of \( \delta \) and \( \theta \). Recall that \( ||\Phi(x)(\zeta) - \zeta||_0 \leq ||\Phi(x) - Id|| ||\zeta||_0 \). We take a \( \theta \) such that \( |\theta| \leq 1 \) and such
that $C\theta \sigma \leq \frac{\delta_0}{4}$. Let $C' := \frac{C\theta \sigma}{\delta_0/4} \leq 1$. Then, for all $0 < \delta < \delta_0$, we take its corresponding $\theta$ to satisfy $C\theta \sigma = C' \frac{\delta}{4}$. Note that $C'$ is fixed with respect to $\theta$ and $\delta$. Then, for each $0 < \delta < \delta_0$ and for its $\theta$, we have that for $\{\|x\|_s < \sigma\}$,

\begin{equation}
\|\Phi(x)(\zeta) - \zeta\|_0 \leq \|\Phi(x) - Id\| \|\zeta\|_0 \leq C \|x\|_s \|\zeta\|_0 \leq C\theta \sigma \|\zeta\|_0 = \frac{C'\delta}{2} \frac{\|\zeta\|_0}{2} \leq \frac{\delta}{2} \frac{\|\zeta\|_0}{2}.
\end{equation}

Consider $K \subseteq K_\frac{\delta}{2} \subseteq K_\delta$. Let $\chi_K$ be the characteristic function whose support is exactly $K_\frac{\delta}{2}$. Then $(\chi_K)_\theta$ is the desired function with the desired estimate. Indeed, the estimate is straightforward by plugging-in $C\theta \sigma = C' \frac{\delta}{4}$ into Estimate 5.1. Equation 6.1 proves the support of the function and its region over which the function is identically 1.

So far, we have considered over $Z_0$ only. The above argument can be directly applied to each $Z_i$ for $i = 0, ..., k$ in the exactly same way. Indeed, we use the same measure on $\text{Aut}(\mathbb{P}^k)$ and the same constants $C, C'$ and $\theta$ to $Z_i$ for $i = 1, ..., k$ as in the case of $Z_0$. Thus, we have just proved the lemma.

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