Teleparallel equivalent of general relativity: a critical review

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Abstract

After reminder some facts concerning general relativity (GR) we pass to teleparallel gravity. We are confining to the special model of the teleparallel gravity, which is popular recently, called the teleparallel equivalent of general relativity (TEGR). We are finishing with conclusion and some general remarks.

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1 Introduction and standard formulation of GR

As it is known GR is a modern geometrical theory of gravity which simultaneously gives a mathematical model of the physical spacetime.

The mathematical model of the physical spacetime in GR is given by a pseudo-Riemannian differential manifold (Haussdorff, paracompact, connected, inextensible, orientable) \((M_4, g_L)\). Here \(g_L\) means a Lorentzian metric which satisfies Einstein equations

\[ G_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu \]

\((\alpha, \beta, \gamma, ..., \mu, \nu, ..., = 0, 1, 2, 3)\)

So, \(g_L\) is a dynamical object.

Here \(G_\mu^\nu\) is the so-called Einstein tensor, \(T_\mu^\nu\) is the matter energy-momentum tensor (the source of the gravitational field), \(c\) is the velocity of light in vacuum, and \(G\) means Newtonian gravitational constant.

The mathematical model of the physical spacetime in GR originated from Einstein Equivalence Principle (EEP). The main ingredient of this Principle is universality of the free falls of the test bodies in a given gravitational field.

GR reduces the gravitational interactions to some geometric aspects of the spacetime. Namely, we have:

1. \(g_L\) = gravitational potentials,
2. \(\{a_\beta \gamma\}\) = gravitational strengths, and
3. \(R^\alpha_{\beta\gamma\delta}(\{\})\) = strengths of the gravitational tidal forces.

The symmetry group of the GR is the infinite group DiffM_4.

The Levi-Civita connection \(\{a_\beta \gamma\}\) is symmetric, metric and torsion-free.

Usually one uses in GR a maximal atlas of the local charts (local maps, coordinate patches) and implicite coordinate frames (natural frames, holonomic frames) and coframes \(\{\partial_\mu\}, \{dx^\alpha\}\) and coordinate components of the geometrical objects.

\[\text{We will identify geometrical objects with the sets of their components. Greek indices mean coordinate components of the geometrical objects.}\]
Every coordinate transformation
\[ x^{\alpha'} = x^{\alpha'}(x^\beta), \quad \text{det} \left[ \frac{\partial x^{\alpha'}}{\partial x^\beta} \right] \neq 0 \] (2)
changes coordinate frames and coframes, and coordinate components of the geometrical objects in standard way.

In the introductory relativity textbooks \([2]\) one usually says about coordinate transformations and about transformations of the coordinate components of the geometrical objects. In fact, it is sufficient. Also some conservative specialist on tensor analysis follow this way \([3]\). But one can use in GR (and in tensor calculus also) arbitrary frames, especially non-holonomic (or anholonomic) frames and coframes \(\{ h^a_\mu(x) \}, \{ h^b_\alpha(x) \} : h^a_\mu(x)h^b_\mu(x) = \delta^b_a, \) \((a, b, c, d, ..., = 0, 1, 2, 3)\). Latin indices (= anholonomic indices) numerate vectors and covectors.

The anholonomic frames and coframes \emph{are not connected with local coordinates}, e.g., they are neutral under coordinate transformations. Instead of we have
\[ \partial_a = h^b_\alpha(x)\partial_b, \quad dx^a = h^a_\alpha(x)dx^\alpha, \] (3)
or, equivalently,
\[ \bar{e}_a := \partial_a = h^b_\beta(x)\partial_\beta, \quad \vartheta^b := dx^b = h^b_\mu(x)dx^\mu. \] (4)

Here \((x) := \{x^\alpha\}\) are \emph{spacetime coordinates}, and \(\{x^a\}\) mean \emph{tangent space coordinates}. \footnote{In GR every tangent space is endowed with Minkowski structure.}

For coordinate frames and coframes one has
\[ \bar{e}_a = \delta^a_\beta\partial_\beta, \quad \vartheta^b = \delta^b_\mu dx^\mu. \] (5)

Some remarks are in order:

1. \(\{\bar{e}_a(x)\} \equiv \{\partial_a(x)\}\) is a coordinate frame in tangent space \(T_x(M_4, g_L)\), and \(\{\vartheta^b\} \equiv \{dx^b\}\) is a coordinate coframe in the dual space space \(T_x^* (M_4, g_L)\).

Differential forms \(\vartheta^b = dx^b = h^b_\mu(x)dx^\mu\) are \textit{not integrable} for anholonomic frames \(\{ h^b_\mu(x) \} : d\vartheta^b \neq 0.\)
2. Henceforth we will consequently use an old tensorial terminology of J.A. Schouten, and S. Gołab, i.e., we will call \( h^\alpha_\beta(x) \) “frame” instead of \( \vec{e}_a(x) \), and \( h^b_\mu(x) \) “coframe” instead of \( \vartheta^b \). It will be useful in passing to teleparallel gravity because majority of the authors working in this field uses this terminology.

3. We permanently use standard Einstein summation convention. As we see, anholonomic frames and coframes in our terminology connect the partial derivatives \( \partial_\alpha \) and \( \partial_\beta \), and differentials \( dx^\alpha \) with \( dx^a \). They also connect anholonomic components of the geometrical objects (denoted by Latin indices) with their coordinate components (denoted by Greek indices). Namely, one has (coordinates \( \{x^\mu\} \) are fixed) for a tensor field of the type \((r,s)\)

\[
T^{a_1\ldots a_r}_{b_1\ldots b_s}(x) = h^{a_1\mu_1}(x)\ldots h^{a_r\mu_r}(x)h^{b_1\nu_1}(x)\ldots h^{b_s\nu_s}(x)T^{\mu_1\ldots \mu_r}_{\nu_1\ldots \nu_s}(x),
\]

and, conversely

\[
T^{\mu_1\ldots \mu_r}_{\nu_1\ldots \nu_s}(x) = h_{a_1\mu_1}(x)\ldots h_{a_r\mu_r}(x)h^{b_1\nu_1}(x)\ldots h^{b_s\nu_s}(x)T^{a_1\ldots a_r}_{b_1\ldots b_s}(x).
\]

For a linear and metric connection \( \omega \) one obtains:

\[
\omega^a_{bc}(x) = h^c_\nu(x)\omega^a_{b\nu}(x),
\]

where

\[
\omega^a_{b\nu}(x) = h^a_\lambda(x)\Gamma^\lambda_{\mu\nu}(x)h^\mu_\rho(x) + h^a_\rho(x)\partial_\nu h^{\rho}_b(x)
\]

is so-called spin connection. Conversely, we have

\[
\Gamma^\rho_{\mu\nu}(x) = h_{a\rho}(x)h^{b}_\mu(x)\omega^a_{b\nu}(x) + h_{a\rho}(x)\partial_\nu h^{a}_\mu(x).
\]

In \textbf{GR} one usually uses the anholonomic frames \( \{h^\mu_\alpha(x)\} \) and dual coframes \( \{h^b_\mu(x)\} \) which form the so-called orthonormal tetrad and cotetrad fields. These fields are defined as follows

\[
h^a_\mu(x)h^b_\nu(x)\eta_{ab} = g_{\mu\nu}(x),
\]

or, equivalently

\[
h^\mu_\alpha(x)h^\nu_\beta(x)g_{\mu\nu}(x) = \eta_{ab}.
\]

\footnote{From here we confine to anholonomic tetrads and cotetrad (See below).}
Here $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric of the tangent spaces $T_x(M_4, g_L)$ and $g_{\mu\nu}(x)$ means the spacetime metric $g_L$.

The transformations of the spacetimes coordinates act only on spacetime indices (Greek indices) in standard way, whereas on the tangent space indices (Latin indices) act only local or global Lorentz transformations, e.g.,

$$h^{ba}_\mu = \Lambda^a_b(x)h^{b}_\mu(x),$$

where

$$\Lambda^a_b(x)\eta_{ac}\Lambda^c_d(x) = \eta_{bd}.$$  

(14)

For a global Lorentz transformation one has $\Lambda^a_b = \text{const}$.

Tetrads are not uniquely determined by the given spacetime metric $g_{\mu\nu}(x)$ but only up to local Lorentz transformations, i.e., up to six arbitrary functions. It is because a metric has only ten independent components and a tetrad field has sixteen independent components. So, for a given metric $g_{\mu\nu}(x)$ there exists $\infty^6$ different classes of tetrad fields $\{h_\mu^a(x)\}$ which satisfy (11)-(12).

Contrary, given tetrad field $\{h_\mu^a(x)\}$ determines unique metric

$$g_{\mu\nu}(x) = h^a_\mu(x)h^b_\nu(x)\eta_{ab},$$

(15)

where

$$h^a_\mu(x)h^\mu_b(x) = \delta^a_b.$$  

(16)

In GR fundamental role plays the spacetime metric $g_{\mu\nu}(x)$ (it is an observable), whereas the orthonormal tetrads (they are not observables) play only an auxiliary role: they simplify calculations and they enable us to introduce spinors into spacetime structure.

The physical foundations and standard formulation of the GR have very good observational evidence. Observational consequences of the Einstein equations were confirmed up to 0.003% in Solar System (weak gravitational field), and up to 0.05% in binary pulsars (strong gravitational field). Universality of the free falls was confirmed up to $10^{-14}$ and some other consequences of the EEP were confirmed up to $10^{-23}$ (See, e.g., [1]).

So, up to now, we needn’t modify or generalize GR. (Ockham razor).

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5 One class of the tetrad $\{h_\mu^a(x)\}$ means these tetrads which are connected by a global Lorentz transformation.

6 We mean here EEP, Einstein equations and mathematical model $(M_4, g_L)$ of the physical spacetime.
We would like to emphasize that we have no free parameter in GR.
Fascinating is that despite this the theory has passed all the stringent tests with favour.

2 Teleparallel gravity

This is a gravity with an absolute parallelism, i.e., with curve independent parallelism of distant vectors and tensors.

In this old approach (since 1928; renewed recently) the mathematical model of the physical spacetime is based on Weitzenböck geometry (= teleparallel geometry or geometry with absolute parallelism).

The geometry of such a kind is uniquely determined by the given tetrad field \( \{ h_\mu^a \}(x) \). Namely, one has (Coordinates \( \{ x^\alpha \} \) are fixed):

1. Metric \( g_{\mu\nu}(x) := h_\mu^a(x)h_\nu^b(x)\eta_{ab} \).

2. Teleparallel connection (Weitzenböck’s connection) \( \Gamma^\rho_{\mu\nu} := h_\rho^a(x)\partial_\nu h^a_\mu(x) \).

Here \( h_\mu^a(x)h_\nu^b(x) = \delta^b_a \).

The teleparallel Weitzenböck connection\(^8\) has non-vanishing torsion \( T^\rho_{\mu\nu} := \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} \) iff the tetrads \( \{ h_\mu^a(x) \} \) are anholonomic, and it has identically vanishing curvature \( R^\theta_{\rho\mu\nu}(\Gamma) \), where

\[
R^\theta_{\rho\mu\nu}(\Gamma) := \partial_\mu \Gamma^\rho_{\nu\theta} - \partial_\nu \Gamma^\rho_{\theta\mu} + \Gamma^\rho_{\sigma\mu} \Gamma^\sigma_{\theta\nu} - \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\theta\mu}.
\]  

Important remarks are in order:

1. Weitzenböck connection is metric, i.e.,

\[
\nabla^\rho g_{\mu\nu} := \partial^\rho g_{\mu\nu} - \Gamma^\alpha_{\mu\rho} g_{\alpha\nu} - \Gamma^\alpha_{\nu\rho} g_{\mu\alpha} \equiv 0 \tag{19}
\]

\(^7\)In the proposed generalized gravity theories one has many free parameters, e.g., one has 28 free parameters in metric-affine gravity. These parameters can be adjusted in order to have agreement with experience.

\(^8\)In the following we will call it “Weitzenböck connection”.

\(^9\)But the other possible covariant derivative

\[
\tilde{\nabla} g_{\mu\nu}(x) := \partial_\rho g_{\mu\nu} - \Gamma^\alpha_{\rho\mu} g_{\alpha\nu} - \Gamma^\alpha_{\rho\nu} g_{\mu\alpha},
\]

is different from zero because Weitzenböck connection is not symmetric.
2. Torsion of the Weitzenböck connection is entirely determined by the Schouten-Van Danzig anholonomy object $\Omega^a_{bc}(x)$, where

\[
\Omega^a_{bc}(x) := h^b_\beta(x)h^\gamma_c(x)[\partial_\gamma h^a_\beta(x) - \partial_\beta h^a_\gamma(x)].
\]

Namely, we have

\[
T^\rho_{\mu\nu}(x) = h^\rho_a(x)h^b_\mu(x)h^c_\nu(x)\Omega^a_{bc}(x).
\]

3. One has the following relation between the components of the Weitzenböck connection $\Gamma^\rho_{\mu\nu}(x)$ and between the components $\{^\rho_{\mu\nu}\}(x)$ of the Levi-Civita connection for the metric $g_{\mu\nu}(x)$

\[
\Gamma^\rho_{\mu\nu}(x) = \{^\rho_{\mu\nu}\}(x) + K^\rho_{\mu\nu}(x),
\]

where

\[
K^\rho_{\mu\nu}(x) := \frac{1}{2}(T^\rho_{\mu\nu} + T^\rho_{\nu\mu} - T^\rho_{\mu\nu})
\]

is the contortion tensor.

4. For Weitzenböck connection $\Gamma^\rho_{\mu\nu}(x)$

\[
\omega^a_{b\nu}(x) \equiv 0 \Rightarrow \omega^a_{bc} \equiv 0,
\]

i.e., this connection identically vanishes in the tetrads $\{h^\mu_a(x)\}$ which have determined it.

Greek, i.e., holonomic indices are raised and lowered with the spacetime metric $g_{\mu\nu}$ and the Latin, i.e., anholonomic indices, are raised and lowered with the Minkowski metric $\eta_{ab}$.

The class of the tetrads $\{\{h^\mu_a(x)\}\}$ connected by global Lorentz transformations with $\Lambda^a_b = \text{const}$ determines the same Weitzenböck connection and geometry. On the other hand, the any two tetrad fields $\{h^a_\mu(x)\}$, $\{h^\prime_a_\mu(x)\}$ which are connected by a local Lorentz transformation

\[
h^a_\mu(x) = \Lambda^a_b(x)h^\prime_b_\mu(x)
\]

determine two different Weitzenböck connections, $\bar{\Gamma}^\rho_{\mu\nu}(x)$ and $\Gamma^\rho_{\mu\nu}(x)$ and two different Weitzenböck geometries.

\[\footnote{The anholonomy object measures anholonomy of the used tetrad field: for a holonomic tetrads $\{h^\mu_a(x)\}$ one has $\Omega^a_{bc}(x) \equiv 0.$} \]
So, the set of the all tetrads \(\{h_\mu^a(x)\}\) splits onto disjoint classes \(\infty^6\) classes\(^{11}\) which determine different Weitzenböck connections and geometries.

In consequence, the symmetry group of a teleparallel gravity consists of the group \(\text{DiffM}_4\) and the global Lorentz group.

In the following we will confine to the very special case of the teleparallel gravity, namely we will confine to the so-called teleparallel equivalent of general relativity (TEGR).

The TEGR is a recent approach to teleparallel gravity which is mainly developed by mathematicians and physicists from Brasil (See, e.g., [4]).

One can look on TEGR as a new trial to rescue torsion in theory of gravity because, up to now, no experiment confirmed the Riemann-Cartan torsion\(^{12}\).

The details of the standard approach to TEGR read.

One starts with the given metric \(g_{\mu\nu}(x)\). This metric determines (up to local Lorentz transformations) the anholonomic tetrad \(\{h_\mu^a(x)\}\) and dual cotetrad \(\{h^a_{\mu}(x)\}\) fields, which satisfy

\[
\begin{align*}
    h_\mu^a(x)h^b_{\nu}(x)\eta_{ab} & = g_{\mu\nu}(x), \\
    h^a_{\mu}(x)h^\mu_b(x) & = \delta^a_b.
\end{align*}
\]

Then, these fields determine the Weitzenböck connection

\[
\Gamma^\rho_{\mu\nu}(x) = h_\rho^a(x)\partial_\nu h^a_{\mu}(x),
\]

which satisfies

\[
\{^\rho_{\mu\nu}\}(x) = \Gamma^\rho_{\mu\nu}(x) - K^\rho_{\mu\nu}(x).
\]

Here \(\{^\rho_{\mu\nu}\}(x)\) is the Levi-Civita connection for the metric \(g_{\mu\nu}(x)\).

For the Weitzenböck connection \(\Gamma^\rho_{\mu\nu}(x)\) one has

\[
R^\rho_{\theta\mu\nu}(\Gamma) \equiv R^\rho_{\theta\mu\nu}(\{\}) + Q^\rho_{\theta\mu\nu} \equiv 0.
\]

Here

\[
R^\rho_{\theta\mu\nu}(\Gamma) := \partial^\rho_{\theta\mu\nu}h^\theta_{\mu\nu} - \partial^\rho_{\theta\nu}h^\theta_{\mu\mu} + \Gamma^\rho_{\sigma\mu\theta}h^\sigma_{\theta\nu} - \Gamma^\rho_{\sigma\nu\theta}h^\sigma_{\theta\mu},
\]

\(^{11}\)\(\infty^6\) classes because the local Lorentz transformations depend on six arbitrary functions.

\(^{12}\)The Riemann-Cartan torsion is the torsion in the Riemann-Cartan geometry. This generalized metric geometry endowed with curvature and torsion was proposed by many authors since 1970 [5] as a geometric model of the physical spacetime. In our opinion lack of experimental evidence, many ambiguities to whose torsion leads, topological triviality of torsion and Ockham razor rather disqualify this model [6].
\[ R^\rho_{\theta \mu \nu}(\{} := \partial_{\mu}\{^\rho_{\theta \nu}\} - \partial_{\nu}\{^\rho_{\theta \mu}\} + \{^\rho_{\sigma \mu}\}\{^\sigma_{\theta \nu}\} - \{^\rho_{\sigma \nu}\}\{^\sigma_{\theta \mu}\}, \quad (32) \]

and

\[ Q^\rho_{\theta \mu \nu} := D_\mu K^\rho_{\theta \nu} - D_\nu K^\rho_{\theta \mu} + K^\rho_{\sigma \mu} K^\sigma_{\theta \nu} - K^\rho_{\sigma \nu} K^\sigma_{\theta \mu}. \quad (33) \]

\[ D_\mu \] is the Levi-Civita covariant derivative expressed in terms of the Weitzenböck connection, i.e.,

\[ D_\mu v^\mu := \partial_\mu v^\mu + (\Gamma^\mu_{\lambda \rho} - K^\mu_{\lambda \rho}) v^\lambda. \quad (34) \]

\[ R^\rho_{\theta \mu \nu}(\Gamma) \] is the main curvature tensor of the Weitzenböck geometry.

The Authors which work on T EGR, by use the fundamental formulas (26),(29),(30) of the Weitzenböck geometry, rephrase, step by step, all the formalism of the purely metric GR in terms of the Weitzenböck connection \( \Gamma^\rho_{\mu \nu}(x) \) and its torsion \( T^\rho_{\mu \nu}(x) \) (Mainly in terms of torsion).

For example:

1. The Einstein Lagrangian for GR

\[ L_E = (-)\alpha \sqrt{|g|} R(\{} + \partial_\mu w^\mu, \quad (35) \]

where \( g := \text{det}[g_{\mu \nu}] \), and

\[ w^\mu := \alpha \sqrt{|g|(g^{\alpha \beta}\{^\mu_{\alpha \beta}\} + g^{\alpha \mu}\{^\gamma_{\alpha \gamma}\})} \quad (36) \]

is rephrased to the form

\[ \alpha h S^{\rho \mu \nu} T_{\rho \mu \nu} =: L_{TEGR}, \quad (37) \]

where \( h = \text{det}[h^a_\mu] = \sqrt{|g|} \) and

\[ S^{\rho \mu \nu} = (-) S^{\rho \mu \nu} := \frac{1}{2}[K^{\mu \nu \rho} - g^{\rho \nu} T^{\alpha \mu}_\alpha + g^{\rho \mu} T^{\alpha \nu}_\alpha]. \quad (38) \]

2. The vacuum Einstein equations

\[ [R^\rho_\chi(\{} - \frac{1}{2}\delta^\rho_\chi R(\{}))]/\sqrt{|g|} = 0 \quad (39) \]

\[ ^{13} \text{Main curvature tensor because one can consider other curvatures in Weitzenböck geometry, e.g., Riemannian curvature.} \]

\[ ^{14} \text{One obtains in fact } \infty^6 \text{ different } L_{TEGR} \text{ because } L_{TEGR} \text{ like } L_E \text{ is invariant only under global Lorentz group. Despite that the field equations (39)-(40) are locally Lorentz invariant. We could get locally Lorentz invariant } L_{TEGR} \text{ if we rephrased } L = (-)\alpha \sqrt{|g|} R(\{}). \]
are rephrased to the form
\[ \partial_{\sigma}(hS_{\lambda}^{\sigma\rho}) - 4\alpha^{-1}(ht_{\lambda}^{\rho}) = 0, \] (40)
where
\[ t_{\lambda}^{\rho} = h_{\lambda}^{\alpha}J_{\alpha}^{\rho} + 4\alpha\Gamma_{\lambda\nu}^{\mu}S_{\mu}^{\nu\rho}, \] (41)
and
\[ J_{\alpha}^{\rho} = (-)4\alpha h_{\alpha}^{\lambda}S_{\mu}^{\nu\rho}T_{\nu\lambda}^{\mu} + 4\alpha h_{\rho}^{\alpha}S^{\alpha\beta\gamma}T_{\alpha\beta\gamma}, \] (42)
and so on.
\[ \alpha := \frac{c^4}{16\pi G}. \]

Then, these authors call the obtained formal reformulation of GR in terms of the Weitzenböck geometry the teleparallel equivalent of the general relativity (TEGR) and conclude: “Gravitational interaction can be described alternatively in terms of curvature, as it is usually done in GR, or in terms of torsion, in which case we have the so-called teleparallel gravity. Whether gravitation requires a curved or torsional spacetime, therefore, turns out to be a matter of convention”. They also assert that TEGR “is better than the original GR” because, e.g., “in TEGR one can separate gravity from inertia (on the connection level) and this separation reads”
\[ \{^{\alpha}_{\beta\gamma}\} = \Gamma_{\beta\gamma}^{\alpha} - K_{\beta\gamma}^{\alpha}. \] (43)

Following the authors which work on TEGR, the left hand side term of the above “separation formula”, (\{^{\alpha}_{\beta\gamma}\}), represents gravity and inertia and the right hand side terms describe inertia, (\Gamma_{\beta\gamma}^{\alpha}), and gravitation, (K_{\beta\gamma}^{\alpha}), respectively.

Of course, such separation contradicts EEP and is impossible in standard formulation of the GR.

We cannot agree with such statements. In our opinion, the “teleparallel equivalent of GR” (What kind of equivalence?) is only formal and geometrically trivial, non-unique (See below) rephrase of GR in terms of the Weitzenböck geometry. Such rephrase is, of course, always possible not only with GR but also with any other purely metric theory of gravity.

In our opinion, we have no profound physical motivation for expression of the gravitational interaction in terms of the teleparallel torsion because the Weitzenböck torsion is entirely expressed in terms of the Van Danzig
and Schouten anholonomy object $\Omega^a_{\kappa}(x)$. So, the torsion of the teleparallel Weitzenböck connection describes only anholonomy of the used tetrad field and, therefore, it is not connected neither with the real geometry of the physical spacetime nor with real gravity, e.g., one can introduce Weitzenböck torsion already in flat Minkowski spacetime.

Weitzenböck torsion could only describe the inertial forces in the framework of the special relativity\textsuperscript{15}.

Contrary, the Levi-Civita part of the Weitzenböck connection, as independent of tetrad\textsuperscript{16}, can have and surely has the physical and geometrical meaning.

Further critical remarks on TEGR.

1. TEGR is nothing new. In fact, it is exactly the old tetrad formulation of GR given in the very distant past by C. Møller \textsuperscript{8} but expressed in terms of anholonomy of the tetrads instead of in terms of tetrads exclusively (As it was in Møller papers). For example, despite that the TEGR field equations are expressed in terms of torsion of the Weitzenböck geometry, they form the system of the 10 partial differential equations of the 2\textsuperscript{nd} order on 16 tetrads components, like the 10 field equations of the Møller’s tetrad formulation of GR. Solving the TEGR equations in vacuum (or in matter) we are looking for the tetrad components $\{h^a_{\kappa}(x)\}$ for apriori given general form of the metric $g_{\mu\nu}(x)$; not for the components of torsion. Weitzenböck connection and its torsion are calculated later \textsuperscript{9}.

Therefore, the notation of the Lagrangian and the field equations of TEGR in terms of Weitzenböck torsion is only a camouflage: TEGR is simply the Møller’s tetrad formulation of GR, and, like Møller’s formulation of GR, determines uniquely the metric only.

We would like to emphasize that one can find all the results of the TEGR including the TEGR energy-momentum tensor for pure gravity in the old Møller’s papers\textsuperscript{17}.

2. TEGR is not unique. This follows from the fact: given metric, $g_{\mu\nu}(x)\textsuperscript{17}$

\textsuperscript{15}In special relativity anholonomic tetrads really represent non-inertial frames.

\textsuperscript{16}The Levi-Civita connection depends only on metric. It is independent of the tetrads which determine the same spacetime metric.

\textsuperscript{17}This “tensor” is one of the most important results obtained in the framework of TEGR.
has 10 intrinsic components and determines only 10 components of the tetrad field \( \{ h_{\omega}^{\mu}(x) \} \) which has 16 intrinsic components. It is a consequence of the known fact that a given metric determines tetrad field up to local Lorentz transformations, which form the local, six-parameters, ortochronous Lorentz group \( L^+ \) defined as follows

\[
L^+ = \{ \Lambda^a_b(x) : \Lambda^a_b(x) \eta_{ac} \Lambda^c_d(x) = \eta_{bd}, \quad \det[\Lambda^a_b(x)] = 1, \quad \Lambda^0_0 \geq 1 \}. \tag{44}
\]

The ten field equations of GR (or TEGR) determine the metric and also determine only ten components of the tetrad field. The remaining six components are left arbitrary functions of the spacetime coordinates \( \{ x^a \} \) and can be arbitrarily established. It is a consequence of the local Lorentz invariance of the TEGR and GR field equations.

So, for the given metric, \( g_{\mu\nu}(x) \), (GR) there exist \( \infty^6 \) different classes of tetrad fields (TEGR) and, in consequence, \( \infty^6 \), different Weitzenböck connections \( \Gamma^\rho_{\mu\nu}(x) \) (and geometries). Each of these connections satisfies the equations

\[
\{\rho_{\mu\nu}\}(x) = \Gamma^\rho_{\mu\nu}(x) - K^\rho_{\mu\nu}(x). \tag{45}
\]

In the above equations the left hand side is independent of tetrads; it depends only on metric \( g_{\mu\nu}(x) \), whereas the both terms on the right hand side depend on the class of the tetrad.\(^{18}\)

As a result we obtain \( \infty^6 \) different Lagrangians (37) for TEGR and \( \infty^6 \) different TEGR. This fact was already known C. Møller in context of his tetrad formulation of GR. Namely, Møller, in fact, also has obtained \( \infty^6 \) different tetrad formulations of GR because, the 10 field equations of his tetrad formulation of GR, identical with Einstein equations (1), determine the tetrad field up to local Lorentz transformations, i.e., up to six arbitrary functions. These field equations determine the metric only.\(^{19}\) In order to have field equations which would determine tetrad

\(^{18}\) (One) class of tetrads := the set of tetrads \( \{ h_{\omega}^{\mu}(x) \} \) which are connected by global Lorentz transformations. Class of tetrads determines the same Weitzenböck connection and geometry. Different classes of tetrads are connected by local Lorentz transformations and determine different Weitzenböck connections and geometries.

\(^{19}\) The same situation we have of course in the framework of the TEGR because the 10 field equations (40), like Møller’s equations, are locally Lorentz invariant.
field completely (apart from constant Lorentz rotations) Møller has developed tetrad theory of gravity in which one has sixteen field equations onto sixteen tetrad components.

3. The authors which work on TEGR assert that the formula (43) (or (45)) gives separation of inertia ($\Gamma_{\mu\nu}(x)$) from gravity ($K_{\mu\nu}(x)$).

Such speculative separation allows them, among other things, to introduce an energy-momentum tensor for gravity. But this separation is illusoric because there exist $\infty^6$ different separations of the form (43) (or (45)) for given $\{\alpha_{\beta\gamma}\}$, i.e., we have no separation inertia from gravity in TEGR. (In agreement with EEP).

In consequence, we have no unique gravitational energy-momentum tensor in TEGR.

4. The experts on TEGR transform trivially the geodesic equations of GR

$$\frac{d^2 x^\alpha}{ds^2} + \{\alpha_{\beta\gamma}\} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0$$

onto the forces equations

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = K^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds}$$

by putting in (46)

$$\{\alpha_{\beta\gamma}\} = \Gamma^\alpha_{\beta\gamma} - K^\alpha_{\beta\gamma}.$$  \hspace{1cm} (48)

The forces equations (47) remind the GR equations of motion for a charged test particle when the both fields, electromagnetic and gravitational, simultaneously act on the particle

$$\frac{d^2 x^\alpha}{ds^2} + \{\alpha_{\beta\gamma}\} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = \frac{Q}{m} F^\alpha_{\beta} \frac{dx^\beta}{ds}.$$  \hspace{1cm} (49)

Here $Q$, $m$ denote electric charge and mass of the particle respectively and $F^\alpha_{\beta}$ mean electromagnetic field acting on the particle. \hspace{1cm} (21)

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20 It is in fact a family of $\infty^6$ different tensors the same as the family of the tensors which has been obtained many years ago by C. Møller without any separation in his tetrad formulation of GR.

21 The right hand side of (49) is the electromagnetic force per unit mass which acts on the particle.
The specialists on TEGR try to attach some physical meaning to the force equations (47), namely following them, the right hand side of (47) describes gravitational force acting on the particle, whereas the term \( \Gamma^\alpha_{\beta\gamma} \frac{ds^\beta}{ds} \frac{ds^\gamma}{ds} \) describes inertial force.

But there exist \( \infty^6 \) different reformulations of the geodesic equations (46) to the form (47) with different \( \Gamma^\alpha_{\beta\gamma} \) and \( K^\alpha_{\beta\gamma} \). Which one of them is correct, i.e., which one of them gives correct inertial force and correct gravitational force?

Talking about equivalence of TEGR with GR is misleading because there exist \( \infty^6 \) different TEGR in consequence of the local Lorentz invariance of the field equations (40).

Here we have the same kind of “equivalence” as the “equivalence” between a given metric \( g_{\mu\nu}(x) \) (10 functions) and a tetrad field (16 functions), which satisfies \( h^a_\mu(x)h^b_\nu(x)\eta_{ab} = g_{\mu\nu}(x) \) i.e., we have no equivalence.

Incorrect is also statement of the specialists on TEGR that Weitzenböck geometry is flat, like Minkowski geometry. In fact, e.g., Riemannian curvature of such geometry is non-zero. Also the curvature tensor \( \tilde{R}^\alpha_{\beta\gamma\delta}(\Gamma) \) where

\[
\tilde{R}^\alpha_{\beta\gamma\delta}(\Gamma) := \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} + \Gamma^\sigma_{\gamma\delta} \Gamma^\alpha_{\sigma\beta} - \Gamma^\sigma_{\delta\sigma} \Gamma^\alpha_{\sigma\beta},
\]

is different from zero.

The tensor \( \tilde{R}^\alpha_{\beta\gamma\delta}(\Gamma) \) differs from the former main curvature tensor \( R^\alpha_{\beta\gamma\delta}(\Gamma) \) (See the formula (31)) by transposition lower indices in \( \Gamma^\alpha_{\beta\gamma}(x) \).

Resuming, in our opinion, TEGR is nothing new. It is camouflaged, the very old tetrad formulation of GR given by C. Møller, and it, by no means is better than standard GR. Contrary, standard GR is surely better than any TEGR because GR is invariant under any change of tetrads, whereas TEGR is not. TEGR, like any teleparallel gravity, is invariant only under global Lorentz rotations of tetrads.

We will finish with some general remarks about teleparallel gravity.

It should be emphasized that there exist many other approaches to teleparallel gravity, different from TEGR, and which generalize GR. At the

22 But we must emphasize that every TEGR determines unique and the same metric structure of the spacetime as GR does. So, from the metric point of view, the different TEGR are equivalent.

23 Remark also that metric and tetrads are different geometric objects.

24 For Riemannian geometry, owing to symmetry of the Levi-Civita connection, these both tensors are identically equal.
first time such approach to gravity was considered already by A. Einstein (“Fernparallelismus” in 1928 [10]) and then by C. Møller (1978), Pellegrini and Plebanski [11], Hayashi and Shirafuji [12], and others. Recently the teleparallel approach to gravity is developed by F.B. Estabrook, Y. Itin, and L. Schücking [13].

In these other approaches to teleparallel gravity the gravitational Lagrangian is built from irreducible torsion components or from tetrads immediately, and contains, in general, three free parameters to be determined by experiments. This Lagrangian is invariant under $\text{DiffM}_4$ and has also global Lorentz symmetry.

The fundamental geometric object are tetrads which determine spacetime metric and Weitzenböck connection, and, therefore, all the local Weitzenböck geometry of the physical spacetime.

In vacuum, we have in these approaches sixteen $2^{nd}$ order field equations on sixteen tetrad components. The field equations should determine the tetrads field $h_a^\mu(x)$ up to constant Lorentz rotations, i.e., up to global Lorentz group, and owing that, should determine a unique Weitzenböck geometry. But tetrads are not observables: they are very alike to the electromagnetic potentials. Moreover, there are problems with physical interpretation of the six additional tetrads components (10 components can describe gravitational field, but what about remaining 6 components?) and these theories suffer from badly posed Cauchy problem [14].

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Teleparalelny ekwiwalent ogólnej teorii względności: uwagi krytyczne

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Streszczenie

Po przedstawieniu podstawowych faktów z ogólnej teorii względności oraz z teleparalelnej grawitacji, ograniczam się do analizy specjalnego modelu teleparalelnej grawitacji nazwanego przez jego twórców teleparalelnym ekwiwalentem ogólnej teorii względności (w skrócie TEGR). Model ten był(i jest) ostatnio intensywnie badany głównie przez matematyków i fizyków z Brazylii.

W pracy pokazuję, że TEGR jest zakamuflowanym, starym, tetradowym sformułowaniem ogólnej teorii względności, dokonanym w latach 60-tych i 70-tych XX-go wieku przez C. Møllera i podkreślęm, że TEGR jest niejednoznacznym i trywialnym przeformułowaniem ogólnej teorii względności, które nie może dać nic lepszego od standardowe sformułowanie tej teorii (Moim zdaniem, przeformułowanie to jest gorsze).