A proposed Optimized Spline Interpolation

Ramtin Madani, Student Member, IEEE, Ali Ayremlou, Student Member, IEEE, Arash Amini, Farrokh Marvasti, Senior Member, IEEE,

Abstract—The goal of this paper is to design compact support basis spline functions that best approximate a given filter (e.g., an ideal Lowpass filter). The optimum function is found by minimizing the least square problem ($\ell_2$ norm of the difference between the desired and the approximated filters) by means of the calculus of variation; more precisely, the introduced splines give optimal filtering properties with respect to their time support interval. Both mathematical analysis and simulation results confirm the superiority of these splines.

Index Terms—Spline, Interpolation, Filter Design

I. INTRODUCTION

The conversion of continuous-time signals such as multimedia data with discrete and digitized samples is a common trend nowadays. This is mainly due to the existence of powerful tools in the discrete domain. However, the conversion of continuous-time signals into the discrete form by means of sampling may destroy all or some parts of the data. Under certain conditions on the continuous signal, such as bandlimitedness [1], the sampling process is guaranteed to be one to one; i.e., there should be a priori a continuous model. In spite of the technological movement toward digital signal processing, by the advances in wavelet theory [2]–[4], a revival of continuous-time modeling for the digital data has been triggered. Multiresolution analysis [5], [6], self-similarity [2], [8], and singularity analysis [9] are inseparable from a continuous-time interpretation. It is therefore crucial to have efficient mathematical tools that allow easy switching from the digital domain to the continuous, and this is precisely the niche that splines, and, to some extent, wavelets, are trying to fill.

In this field, polynomial splines, such as B-splines, are particularly popular, mainly due to their simplicity, compact support, and excellent approximation capabilities compared to other methods. Spline-based methods have spread to various applications since the development of B-splines [10]–[12].

Though B-splines generate remarkable results in many applications, they are not the optimum solutions for filtering problems such as interpolation. This paper, focuses on the problem of designing optimal compact support splines which best approximate a given filter such as the ideal lowpass filter. In fact, the desired filter reflects the characteristics of the continuous-time model and can be arbitrary.

The remainder of the paper is organized as follows: The next section briefly describes the spline interpolation method. In section III, a novel scheme is proposed to produce new optimized splines for interpolation regardless of the type of filtering. The performance of the proposed method is evaluated in section IV by comparing the interpolation results of the proposed method on standard test images to those of well-known interpolation techniques. Section V concludes the paper.

II. PRELIMINARIES

In this paper, the following notation and definitions are used:

Definition 1. For a continuous-time signal $x(t)$, a continuous-time signal $x_p(t)$ and a discrete-time signal $x_d[n]$ are defined as follows,

$$x_d[n] \triangleq x(nT)$$
$$x_p(t) \triangleq x(t)p(t) = \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT)$$

where $p(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$ is the periodic impulse train that is referred as the sampling function. (Fig. 1)

The sampling period $T \triangleq 1$ is normalizes throughout the paper without any loss of generality.

Definition 2. For a continuous-time signal $x(t)$ and any odd integer $m$, $x^m_s(t)$ is a polynomial spline of order $m$ if:

1) For any $n \in \mathbb{Z}$, $x^m_s(t)$ would be a polynomial of the (at most) order $m$, in the interval $[n, n + 1]$.
2) For any $n \in \mathbb{Z}$, $x^m_s(n) = x_d[n]$ (Interpolation property)
3) $x^m_s \in C^{m-1}(\mathbb{R}, \mathbb{R})$ (Smoothness)

According to the first property, $m+1$th derivation of $x^m_s$ is zero in non-integer points, and is equal to an impulse train.

Definition 3. For the polynomial spline $x^m_s(t)$, the polynomial spline coefficients $\dot{x}^m_d[n]$ are defined as,

$$\dot{x}^m_p(t) = \sum_{n=-\infty}^{\infty} \dot{x}^m_d[n]\delta(t - n) \triangleq \frac{d^{m+1}}{dt^{m+1}}x^m_s(t)$$

To determine each polynomial of order $m$ that is forming the $x^m_s(t)$, its $m+1$ unknown coefficients should be found in order to satisfy the conditions 2 and 3 (Fig. 2). If the goal is to discover a piecewise polynomial signal that is $m-1$ times

![](https://example.com/image1.png)

Fig. 1. Sampling process modeled by multiplying an impulse train into a primary time continuous signal

All authors are with Advanced Communications Research Institute, the Department of Electrical Engineering, Sharif University of Technology, Tehran, IRAN, e-mails: madani@ee.sharif.edu, a_ayremlou@ee.sharif.edu, marvasti@sharif.ir
differentiable with continuous derivatives, a natural way is to derive \( \dot{x}^m_d[n] \) according to \( x_d[n] \) and then calculate the integral of \( \dot{x}_x(t) \), \( m + 1 \) times, i.e.,
\[
x^m_x(t) = \int_t^{t_m} \ldots \int_{t_1}^{t_m} \dot{x}^m_x(t_0) dt_0 \ldots dt_{m-1} dt_m \\
= (u^{m+1}_m * \dot{x}^m_p)(t)
\]  
(4)
where \( u^1(t) \) is the unity step function and for any \( k \in \mathbb{N} \),
\[ u^{k+1}(t) = (u^k * u^1)(t). \]

Proposition 1. If the ROC of \( X^m_d(z) \) is not bounded by the unit circle (i.e., there exist \( z \in \text{ROC} \{X^m_d \} \) such that \( |z| > 1 \), then \( x^m_x(t) \) will be uniquely derivable according to \( x_d[n] \). And,
\[
x^m_x(t) = \left( (u^{m+1}_m * (u^{m+1}_m)^{-1}) * x_p(t) \right) \tag{5}
\]
where \( (u^{m+1}_m)^{-1}(t) \) is defined as the inverse of \( u^{m+1}_m(t) \), i.e., \( \left( (u^{m+1}_m)^{-1} * u^{m+1}_m \right)(t) = \delta(t) \). And \( X^m_d(z) \) is the z-transform of \( x_d[n] \).

Proof:
\[
x^m_p(t) = x^m(t)p(t) = \left( (u^{m+1}_m * x^m_p)(t) \right)p(t) = \left( y^{m+1}_m * x^m_p \right)(t) \tag{6}
\]
Hence,
\[
x^m_d[n] = \left( u^{m+1}_m * \dot{x}^m_d \right) \tag{7}
\]
The ROC of \( U^{m+1}_d(z) \) is \( |z| > 1 \) and there is no zeros in this region either. Since the ROC of \( X^m_d(z) \) in not bounded by the unit circle, \( \left( U^{m+1}_d(z) \right)^{-1} \) and \( X^m_d(z) \) have a region in common. Thus,
\[
\dot{x}^m_d(t) = \left( (u^{m+1}_m)^{-1} * x^m \right)(t) \tag{8}
\]
And according to (4),
\[
x^m_x(t) = \left( (u^{m+1}_m * \dot{x}^m_p)(t) \right) = \left( u^{m+1} * (u^{m+1}_m)^{-1} * x^m_p \right)(t) \tag{9}
\]

Definition 4. A discrete-time signal \( y_d[n] \) is called an appropriate signal if and only if it will be stable and have a unique and stable inverse \( y_{d}^{-1}[n] \).

Definition 5. For any continuous-time signal \( y(t) \), if \( y_d[n] \) was an appropriate signal, then \( \tilde{y}(t) \) is defined as follows,
\[ \tilde{y}(t) = ((y_p)^{-1} * y)(t) \tag{10} \]

Proposition 2. \( \tilde{y}(t) \) is the impulse response of a filter with interpolation property, in the other word:
\[ \tilde{y}_p(t) = \delta(t) \tag{11} \]

Proof:
\[ \tilde{y}_p(t) = \left( ((y_p)^{-1} * y) \right)(p(t) = \left( (y_p)^{-1} * y_p \right)(t) = \delta(t) \tag{12} \]

Proposition 3. For any polynomial spline \( y^m_d(t) \) if \( y^m_d(t) \) was an appropriate signal, \( \tilde{y}^m_d(t) \) is independent of \( y(t) \) and is only a function of \( m \).

Proof:
\[ \tilde{y}^m_d(t) = \left( ((y^m_p)^{-1} * y^m) \right)(t) = ((y_p)^{-1} * y^m) \tag{13} \]

Hence \( \tilde{y}^m_d(t) \) is only a function of \( m \).

Proposition 4. Suppose that \( k \) is the least positive integer in which there exist a polynomial spline of order \( m \) like \( \tilde{y}^m_d(t) \), that takes zero outside the interval \( (0, k) \), i.e.,
\[ \forall t; \quad t \notin (0, k) \Rightarrow \tilde{y}^m_d(t) = 0 \tag{15} \]
then \( k = m + 1 \).

Proof: Suppose that \( \tilde{y}^m_d(t) \) satisfies the equation (13) and \( \bar{Y}^m_d(z) \) is the z-transform of \( \tilde{y}^m_d[n] \). Where \( \tilde{y}^m_d[n] \) is the polynomial spline coefficients signal of \( \tilde{y}^m_d(t) \) according to the definition (3). It can be claimed that,
\[ (z - 1)^{m+1} |\bar{Y}^m_d(z^{-1}) \tag{16} \]
In order to prove (16), define a sequence of polynomials \( \{Q_n\}_{n=0}^m \) such that \( Q_0(z) = Y_d^m(z^{-1}) \) and for \( 1 \leq n \leq m, \)
\[
Q_n = z \frac{d}{dz} Q_{n-1} = \sum_{n=0}^k \gamma_m^n [n](n!)z^n
\]  
(17)

Also polynomial \( H \) is defined as follows,
\[
H(t) = \frac{1}{m!} \sum_{i=0}^m (-1)^i Q_i(1) \left( \begin{array}{c} m \\ i \end{array} \right) t^{m-i}
\]
\[
= \frac{1}{m!} \sum_{i=0}^m (-1)^i \left( \sum_{n=0}^k \gamma_d^n [n](n!) \right) \left( \begin{array}{c} m \\ i \end{array} \right) t^{m-i}
\]
\[
= \frac{1}{m!} \sum_{n=0}^k \gamma_d^n [n] \sum_{i=0}^m \left( \begin{array}{c} m \\ i \end{array} \right) (-1)^i t^{m-i}
\]
\[
= \frac{1}{m!} \sum_{n=0}^k \gamma_d^n [n](t-n)^m
\]  
(18)

Thus according to the equation (4) for any \( t > k, \) \( H(t) \) is equal to \( y_s^m(t), \) i.e,
\[
\forall t; t > k \Rightarrow H(t) = y_s^m(t) = 0
\]  
(19)

Since \( H \) is a polynomial and is equals to zero for infinite amount of \( t, \) all of its coefficients are equal to zero, hence,
\[
H(t) = 0 \Rightarrow Q_m(1) = Q_{m-1}(1) = \cdots = Q_0(1) = 0
\]  
(20)

from the above equations it is concluded directly by induction that,
\[
\forall n \in \mathbb{N}; 0 \leq n \leq m \Rightarrow \frac{d^n}{dz^n} Y_d^m(z^{-1})|_{z=1} = 0
\]  
(21)

Hence \( (z-1)^{m+1} Y_d^m(z^{-1}) \). On the other hand since \( Y_d^m(z^{-1}) = \sum_{n=0}^k \gamma_d^n [n]z^n, \) (16) cites that \( \gamma_d^m [m+1] \neq 0 \) hence,
\[
y_s^m(t)|_{t \in (m+1-\epsilon,m+1+\epsilon)} \neq 0
\]  
(22)

thus \( k \geq m+1. \)

Finally it must be shown that there exist a polynomial spline of order \( m \) that is bounded by the interval \( (0, m+1). \) Suppose that \( Y_d^m(z) = (z^{-1} - 1)^{m+1} \) then,
\[
\Rightarrow \gamma_d^m [n] = (-1)^n \left( \begin{array}{c} m+1 \\ n \end{array} \right)
\]  
(23)

\[
y_s^m(t) = u^{m+1} \sum_{n=0}^{m+1} \gamma_d^n [n] \delta(t-n)
\]  
(24)

\[
y_s^m(t) = \sum_{n=0}^{m+1} \gamma_d^n [n] u^{m+1}(t-n)
\]  
(25)

Thus for all \( t \geq m+1, y_s^m(t) = 0 \) and the proof completed.

Definition 7. The polynomial bspline of order \( m \) is defined as follows,
\[
\beta^m(t) = \sum_{n=0}^{m+1} (-1)^n \left( \begin{array}{c} m+1 \\ n \end{array} \right) u^{m+1}(t-n)
\]  
(26)

In order to have an FIR continuous-time calculation during polynomial spline interpolation process, it can be implemented by the polynomial bsplines, i.e,
\[
\dot{x}_s^m[n] = (\beta_d^m)^{-1} \cdot x_d^m[n]
\]  
(27)

\[
x_s^m(t) = \sum_{n=-\infty}^{\infty} \dot{x}_d^m[n] \beta^m(t-n)
\]  
(28)

III. PROPOSED OPTIMIZED B-SPLINE

In many applications, it is desirable that the interpolation filter be depicted as an ideal filter, and the second and third conditions of definition [1] may not be important. In this section an optimized basis splines will be introduced to be replaced by the polynomial basis splines in order to have an interpolation process with the most possible coincidence with a desired filter.

Definition 8. Let \( D \) denote the set of all continuous-time signals that satisfy the dirichlet conditions, i.e for any \( y(t) \in D \)

1) \( y(t) \) have a finite number of extrema in any given interval.
2) \( y(t) \) have a finite number of discontinuities in any given interval
3) \( y(t) \) be absolutely integrable over a period.
4) \( y(t) \) be bounded.

First of all, an affine subspace of all signals that satisfy the dirichlet conditions will be defined, and then the optimized solution will be obtained in this set by the calculus of variation.

Definition 9. Let \( y_d[n] \) be an appropriate signal that takes zero for all \( n \leq 0 \) and \( n \geq m+1, \) then \( y^m(y_d) \) is the set of all continuous-time signals \( y(t) \) that satisfy the following conditions,

1) \( y \in D \)
2) \( \forall n \in \mathbb{N}; y(n) = y_d[n] \)
3) \( \forall t \notin (0, m+1); y(t) = 0 \)

Using \( y(t) \in \chi^m(y_d) \) as a basis spline to interpolate \( x_d[n] \) according to the equations (27) and (28) is a linear time invariant process with the impulse response \( \gamma(t). \)

Definition 10. The error function \( e_z: \chi^m(y_d) \rightarrow \mathbb{R} \) is defined as follows,
\[
e_z(y) = \int_{-\infty}^{\infty} |\mathcal{F}\{\hat{y} * x_p\} - \mathcal{F}\{x\}|^2 df
\]
\[
= \int_{-\infty}^{\infty} |\mathcal{F}\{(y_d)^{-1} * x_p\} - \mathcal{F}\{x\}|^2 df
\]
\[
= \int_{-\infty}^{\infty} |\frac{\mathcal{F}\{x_p\}}{\mathcal{F}\{y_d\}}| \mathcal{F}\{y\} - \mathcal{F}\{x\}|^2 df
\]  
(29)

Where \( \mathcal{F} \) is defined as the continuous time Fourier transform operator.

Definition 11. According to the above definition, if \( \rho_d^m \) be an appropriate signal that takes zero for all \( n \leq 0 \) and \( n \geq m+1 \) an optimized basis spline \( \rho^m[x, \rho_d^m] \) is defined as follows,
\[
\rho^m[x, \rho_d^m] = \arg \min_{y \in \chi^m(\rho_d^m)} e_z(y)
\]  
(30)
Now, calculus of variation may be used to evaluate the optimum $\rho^m$ which minimizes the error $\epsilon_x(\rho^m)$.

**Proposition 5.** Equation (30) has a unique solution that satisfies the following property,

$$[x_p * \varphi_p] * (\rho^m)^{-1} * (\varphi_p)^{-1} * \rho^m = \{\rho_p^m\}^{-1} * \varphi_p * x \quad (31)$$

for all $t \in (0, m + 1)$. Where $\varphi(t) \triangleq y(-t)$

The proof of (32) is presented at box [1]. Since $\chi^m(\rho^m)$ is boundless, in order to minimize $\epsilon_x(\rho^m)$, $\langle \rho(\rho^m), \gamma \rangle$ should be zero for all $\gamma \in \chi^m(0)$, which implies that the second term inside the integral should be zero for $t \in (0, m + 1)$, i.e.,

$$F^{-1} \left\{ \left[ \frac{F\{x_p\}}{F\{\rho_p^m\}} \right] * \left[ \frac{F\{\rho^m\}}{F\{\rho_p^m\}} \right] F\{x_p\} - F\{x\} \right\} = 0 \quad (33)$$

And this equation directly yealds (37).

Thus, it is proven that the optimized basis spline which could give the best estimation of $x$, should satisfy (31). By defining

$$v(t) \triangleq (x_p * \varphi_p) * \{[(\rho_p^m)^{-1} * (\varphi_p)^{-1}] \}$$

$$w(t) \triangleq [(\varphi_p)^{-1} * \varphi_p]$$

(31) can be written as $(v * \rho^m)(t) = w(t)|_{t \in (0,m+1)}$. Since this equation is only valid in a particular interval, $v^{-1}$ cannot be used to obtain $\rho^m$. But since $v$ is an impulse train, $\rho$ can be driven using matrix form, thus in order to derive $\rho^m$ from (31), two sequences of functions are defined such that for any $n \in \mathbb{Z}$,

$$R_n(t) = \begin{cases} \rho^m(t + n) & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

$$W_n(t) = \begin{cases} w(t + n) & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

Now, (31) could be written in matrix form as follows:

$$\begin{bmatrix} v[0] & v[-1] & \ldots & v[-m] \\ v[1] & v[0] & \ldots & v[-m+1] \\ \vdots & \vdots & \ddots & \vdots \\ v[m] & v[m-1] & \ldots & v[0] \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_m \end{bmatrix} \quad (38)$$

According to (38), $\{R_n\}^m_{n=1}$ is derived and thus the optimized basis spline is evaluated as

$$\rho^m(t) = \sum_{n=0}^{m} R_n(t - n) \quad (39)$$

This optimized basis spline that finally calculated, preforms the least interpolating mean square error for $x[n]$. The basis spline could be calculated by the signal before sampling $x$, portion of $x$ or statistic’s characteristics that are expected for $x$. Besides, smoothness of optimized basis spline and causality of prefilter are possible to be applied by setting $\beta_s$.

Another application and even more important of equation (31) is that by means of that it is possible to estimate any ideal interpolation filter by an optimized basis spline. In fact, spline dominance in against to other FIR windowed estimations is that an optimized basis spline with the same time interval can give more exact estimation for a desire ideal filter.

Now the goal is to design $\rho^m$ such that $\hat{\rho}^m$ would be the best estimation of $h$, which denotes the impulse response of a filter that has the interpolation property.

**Proposition 6.** (Estimating an ideal filter) Considering $h(t)$ as an impulse response satisfying the interpolation property and $\rho^m_d[n]$ as an appropriate signal, then

$$\arg \min_{y \in \chi^m(\rho^m_d)} \|h - \hat{y}\|_2 = \rho^m[h, \rho^m_d] \quad (40)$$

Proof:

$$e_h(y) = \int_{-\infty}^{\infty} |F\{\hat{y} * h_p\} - F\{h\}|^2 df = \int_{-\infty}^{\infty} |F\{\hat{y}\}F\{h_p\} - F\{h\}|^2 df$$

$$= \int_{-\infty}^{\infty} |F\{\hat{y}\} - F\{h\}|^2 df$$

$$= \|h - \hat{y}\|_2$$

Thus

$$\arg \min_{y \in \chi^m(\rho^m_d)} \|h - \hat{y}\|_2 = \arg \min_{y \in \chi^m(\rho^m_d)} e_h(y) = \rho^m[h, \rho^m_d] \quad (42)$$

**Proposition 7.** Consider $h(t)$ as an impulse response with interpolation property and $\rho^m(t) \in \chi^m(\rho^m_d)$ as a basis spline which $\rho^m(t)$ best approximates $h(t)$ against $\chi^m(\rho^m_d)$, i.e., $\rho^m(t) = \arg \min_{y \in \chi^m(\rho^m_d)} \|h - \hat{y}\|_2$. Then $\rho^m(t)$ satisfies the following equation,

$$((\rho_p^m)^{-1} * (\varphi_p)^{-1}) * \rho^m = \{\rho_p^m\}^{-1} * \varphi_p * h \quad (43)$$

Fig. 3. Our Optimized Spline versus B-Spline, both of the order three. $\beta_s'$ is the optimized basis spline built for estimating ideal lowpass filter $h(t) = \sin(\pi t)$ with $b = (0.24, 0.48, 0.24)$.

**Fig. 3.** Our Optimized Spline versus B-Spline, both of the order three. $\beta_s'$ is the optimized basis spline built for estimating ideal lowpass filter $h(t) = \sin(\pi t)$ with $b = (0.24, 0.48, 0.24)$.
Proof: Follows directly from (31) and the fact that \( h(t) \) has interpolation property.

Fig. 3 shows the optimized B-Spline built for estimating ideal lowpass filter \( h(t) = \frac{\sin(\pi t)}{\pi t} \) with \( \rho_3^3(z) = 0.233z + 0.480z^2 + 0.233z^3 \) and cubic B-Spline. And Fig. 4 shows \( \hat{\rho}_3^3[h, \rho_3^3](t) \) in comparison to \( c^3(t) \).

IV. Simulation Results

The performance of the proposed method for an ideal lowpass filter has been compared to the B-spline and the results are depicted in Figs. 3 and 4. Fig. 3 shows the comparison of the optimized basis spline built for estimating an ideal lowpass filter and the cubic B-spline. Fig. 4 shows \( L_{\beta m} \) as compared to \( L_3 \). The optimized spline is superior to the B-spline method. The SNR values of these methods are 20.39dB and 13.15dB for the proposed method and the B-spline method, respectively, for \( m = 3 \).

To consider practical applications, the method was tested on several standard monochrome images. These images are down-sampled to provide the low solution images for interpolation. In image applications, splines can be used for zooming and enlargements. For comparison, three other image interpolation methods are also simulated: 1-bicubic interpolation, 2-wavelet-domain zero padding cycle-spinning [14] and 3-soft-decision estimation technique for adaptive image interpolation [15]. Table IV shows the Peak Signal-to-Noise Ratio (PSNR) performance of these three methods when applied to the seven well-known test images. In all cases, the proposed optimized spline interpolation algorithm performed best among all methods. For high frequency content images, such as Barbara and Baboon, the proposed algorithm outperforms other methods by 1dB.

Since PSNR is an average quality measure, the spatial locations where the proposed algorithm produces significantly smaller interpolation errors than the other competing methods are plotted in Fig. 5. The differences are more noticeable around the edge of the hat. The result of the present study compare favorably both subjectively and objectively. In addition, a wavelet scheme based on cycle-spinning interpolation has been included to provide a comparison with a powerful method operating in the wavelet domain.

V. Conclusion

This paper has introduced a method for optimizing a compact support interpolating spline for approximating a given filter in the least square sense. In particular, it demonstrated a newly proposed method for approximating the ideal lowpass filter. The interpolation results obtained by this method are better than those obtained by the conventional solutions, such as B-splines. Simulation results show about 1dB improvement in most of the cases. In the future, we plan to focus on

![Fig. 3. Comparison of different methods for the Lena image: (a) The original image, (b) bilinear interpolation, (c) bicubic Interpolation, (d) WZP Cycle-Spinning [14], (e) SAI [15], and (f) the proposed method.](image-url)
the application of these optimized splines for non-uniform sampling for 1-D and 2-D signals.

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