An invariant classification of cubic integrals of motion

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Abstract

We employ an isometry group invariants approach to study Killing tensors of valence three defined in the Euclidean plane. The corresponding invariants are found to be homogeneous polynomials of the parameters of the vector space of the Killing tensors. The invariants are used to classify the non-trivial first integrals of motion which are cubic in the momenta of Hamiltonian systems defined in the Euclidean plane. The integrable cases isolated by Holt and Fokas-Lagerström are investigated from this viewpoint.

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1 Introduction

The study of Hamiltonian systems with two degrees of freedom admitting first integrals of motion which are cubic in the momenta has been for many years an active area of research [1, 2, 3, 4]. It is an integral part of the well-established theory of superintegrable systems which originated in the pioneering works by Winternitz et al [5, 6] (see, for example, [7, 8, 9]).

The main problem for Hamiltonian systems with two degrees of freedom can be characterized as follows. Let \((M, g)\) be a two-dimensional pseudo-Riemannian manifold. Consider a Hamiltonian system on \(M\) defined by a natural Hamiltonian of the form

\[
H(q, p) = \frac{1}{2} g^{ij}(q) p_i p_j + V(q),
\]

via the canonical Poisson bi-vector \(P_0 = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i}, \ i = 1, 2\), where \(g^{ij}\) are the components of the metric tensor \(g\) and \((q, p) \in T^* M\) are the standard position-momenta coordinates. We wish to describe the Hamiltonian system defined by (1.1), or, more specifically, by its potential function \(V\), whose complete integrability is assured by the existence of an additional first integral, whose component

\[
F(q, p) = L^{ijk}(q)p_i p_j p_k + B^k(q)p_k, \quad i, j, k = 1, 2,
\]

where \(L^{ijk}\) denote the components of a Killing tensor \(L\) of valence three defined on \((M, g)\) (see below). The problem appears to be very difficult, since there are three quantities involved, namely the Killing tensor \(L\), vector field \(B\) and potential \(V\) related via the partial differential equations stemming from the vanishing of the Poisson bracket \(\{H, F\} = P_0 dH dF = 0\), all of which may be varied. Most of the known Hamiltonian systems with the designated property have been found by fixing the form of the potential \(V\) in (1.1) and then trying to solve the system of equations determined by \(\{H, F\} = 0\) under this condition (see, for instance, [2, 4]).

In this paper we consider the problem from a different perspective, which is based on the classification of all admissible, non-trivial Killing tensors defining the leading term in (1.2). This can be done by making use of a new method developed by the authors in [10, 11, 12, 13] which is based on group invariants of Killing tensors defined on pseudo-Riemannian spaces of constant curvature under the action of the isometry group. The group of
isometries of $M$ is the most natural choice for such a classification, since the action of the group preserves the dynamics of the system defined by (1.1)-(1.2). In this paper we are mainly concerned with the case $M = \mathbb{R}^2$, where $\mathbb{R}^2$ denotes the Euclidean plane. Accordingly, we find the isometry group invariants of Killing tensors of valence three defined in $\mathbb{R}^2$ and split the space of Killing tensors into invariant subspaces whose respective Killing tensors are characterized by the invariants of the isometry group. This procedure entails a classification of first integrals of motion which are cubic in the momenta of Hamiltonian systems defined by (1.1) in $\mathbb{R}^2$.

2 Isometry group invariants of vector spaces of Killing tensors

Recall that the classical theory of algebraic invariants emerged in the 19th century as the study of invariant properties of algebraic polynomials (see [14, 15] for more details). Loosely speaking, its main problem is to find invariants and covariants of vector spaces of homogeneous polynomials under a change of variables induced by a group action.

The authors have incorporated these classical ideas into the study of Killing tensors defined in pseudo-Riemannian spaces of constant curvature [10, 11, 12, 13]. More specifically, let $(M, g)$ be an $n$-dimensional pseudo-Riemannian manifold of constant curvature. A vector space $K^p(M)$ of Killing tensors of valence $p \geq 1$ defined on $(M, g)$ can be considered as a natural counterpart of a vector space of homogeneous polynomials of fixed degree in the classical invariant theory. Indeed, for a fixed $p \geq 1$ the dimension $d$ of $K^p(M)$ is determined by the Delong-Takeuchi-Thompson formula [16, 8, 18]:

$$d = \dim K^p(M) = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p}, \quad p \geq 1,$$

(2.3)

where $\dim M = n$. From this point of view, we treat each element $K \in K^p(M)$ as an algebraic object determined by its $d$ parameters $a_1, \ldots, a_d$. This approach to the study of Killing tensors differs from the more conventional approach according to which a Killing tensor is viewed as a sum of symmetrized tensor products of Killing vectors. Our primary focus has been on considering the behaviour of $K^p(M)$ under the induced action of the isometry group, which reveals invariant properties of the former. Thus, it has been possible
to describe the space of invariants, or, polynomial functions of the parameters \(a_1, \ldots, a_6\) that remain fixed under the action of the isometry group for the vector spaces \(\mathcal{K}^2(\mathbb{R}^2)\) and \(\mathcal{K}^2(\mathbb{R}^2_1)\) (see [10] and [13] respectively). The new invariants turn out to be very useful in classification problems involving Killing tensors. In this paper we turn our attention to the vector space \(\mathcal{K}^3(\mathbb{R}^2)\) and the corresponding invariants of the isometry group to be used in classification of cubic integrals of motion.

### 3 Cubic integrals of motion

Consider a Hamiltonian system defined by (1.1). Assume it admits an additional first integral of the form

\[
G(q, p) = L^{ijk}(q)p_ip_jp_k + K^{ij}p_ip_j(q) + B^l(q)p_l + U(q),
\]

where \(i, j, k = 1, 2\). The vanishing of the Poisson bracket \(\{H, G\} = 0\) decouples into the following conditions expressed in component (contravariant and covariant) and coordinate-free forms respectively:

\[
L_{,f}^{(ijk)g}f - \frac{3}{2}L_{,f}^{(ij)k}g_f = 0 \iff [L, g] = 0, \quad (3.5)
\]

\[
K^{(ij), fg}k_j - K^{(i)fg}k_j = 0 \iff [K, g] = 0, \quad (3.6)
\]

\[
B^{(ig)f}_{,f} - \frac{1}{2}B^{f}_{,f}g_{ij} - 3L^{f}_{,f}V_f = 0 \iff [B, g] = 3LdV, \quad (3.7)
\]

\[
U_{,f}g^{fi} - 2K^{f}iV_f = 0 (U_{,i} = 2K^{f}iV_j) \iff dUg = 2KdV, \quad (3.8)
\]

\[
B^fV_f = 0 \iff B(V) = 0, \quad (3.9)
\]

where \(, f\), (, ) and [ , ] denote partial differentiation, symmetrization and the Schouten bracket respectively. Note \([B, g] = -\mathcal{L}_B g\), where \(\mathcal{L}\) denotes the Lie derivative operator. It follows immediately from (3.5) and (3.6) that \(L\) and \(K\) are Killing tensors of valence three and two respectively, while \(B\) - in general - is not. Equation (3.9) reveals that the potential function \(V\) is preserved by the vector field \(B\). We observe next that Equations (3.5)-(3.9) separate into two groups, namely (3.5), (3.7), (3.9) and (3.6), (3.8), involving only components of \(G\) which are odd and even in the momenta respectively.
This fact implies that the first integral can be written as $G = G_{\text{odd}} + G_{\text{even}}$, where

$$
G_{\text{odd}}(q, p) = L_{ijk}(q)p_ip_jp_k + B^k(q)p_k,
$$

$$
G_{\text{even}}(q, p) = K^{ij}(q)p_ip_j + U(q),
$$

and furthermore that $\{H, G_{\text{odd}}\} = \{H, G_{\text{even}}\} = 0$.

**Definition 3.1** We say that a Hamiltonian system defined by (1.1) is trivial if it admits a first integral $G = X^i p_i$ linear in the momenta, where $X$ is a Killing vector, satisfying $X(V) = 0$. Accordingly, we say that a Killing tensor $L$ of valence three is trivial if it is a product of the metric $g$ and a Killing vector $X$, i.e. $L = g \odot X$.

These arguments confirm that (1.2) is the most general form of a first integral cubic in the momenta and a Hamiltonian system (1.1) admitting a first integral of the form (3.4) is in fact superintegrable, provided it is not trivial, for in this case one can combine the metric $g$ with $X$ and square of $X$ to obtain cubic and quadratic terms in the momenta respectively in the representation (3.4). Note the decomposition (3.10) was proven in [4] by employing a time reflection symmetry assumption that does not seem to be required.

We note that this observation can be easily extended to the case of Hamiltonian systems of the type (1.1) admitting first integrals of order $r > 3$ in the momenta of the form:

$$
F_r(q, p) =
$$

$$
K^{a_1...a_r}_r(q)p_{a_1}...p_{a_r} + \cdots + K^{a_1...a_i}_i(q)p_{a_1}...p_{a_i} + \cdots + K_1(q)^{a_1} p_{a_1} + U(q),
$$

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where $1 \leq a_1, \ldots, a_r \leq n$, $r > 3$. Indeed, the vanishing of the poisson bracket \( \{ F_r, H \} = 0 \) yields in this case the following tensor equations:

\[
\begin{align*}
[K_r, g] &= 0, \\
[K_{r-1}, g] &= 0, \\
[K_{r-2}, g] &= rK_r dV, \\
&\vdots \\
[K_{i}, g] &= (i + 2)K_{i+2} dV, \\
&\vdots \\
[K_1, g] &= 3K_3 dV,
\end{align*}
\]

\[dU g = [U, g] = 2K_2 dV,
\]

\[K_1(V) = 0,
\]

where the tensorial quantities $K_r$, $r \geq 1$ are determined by the corresponding components of (3.11). It is clear therefore that a general first integral $F_r$ of order $r$ given by (3.11) can also be written as $F_r = F_{r(\text{odd})} + F_{r(\text{even})}$, immediately implying $\{ H, F_{r(\text{odd})} \} = \{ H, F_{r(\text{even})} \} = 0$. Accordingly, we arrive at the following result:

**Proposition 3.1** A Hamiltonian system with two degrees of freedom determined by (1.1) which admits a general first integral (3.11) of order $r \geq 3$ having both even and odd terms in the momenta is necessarily superintegrable.

**Corollary 3.1** Given a general first integral of order $r \geq 3$ of the type (3.11) for a Hamiltonian system (1.1). Then only the tensorial quantities $K_r$ and $K_{r-1}$ that define the first two terms of (3.11) are Killing tensors: $K_r \in C^r(M), K_{r-1} \in C^{r-1}(M)$.

We conclude therefore that for a Hamiltonian system (1.1) admitting a cubic first integral (1.2) determined by its Killing tensor $L$ and vector field $B$, the
following tensorial equations are essential:

\[
[L, g] = 0, \quad \text{(Killing tensor equation for } L) \quad (3.13)
\]

\[
[B, g] = 3LdV, \quad \text{(recursion tensor equation for } B) \quad (3.14)
\]

\[
B(V) = 0. \quad \text{(compatibility condition for } V) \quad (3.15)
\]

Let us fix \(L, B\) and \(V\) in (1.2) and (1.1) respectively. Since \((M, g)\) is of constant curvature, the dimension of the corresponding isometry group \(I(M)\) is three. Therefore the space \(\mathcal{K}^1(M)\) of Killing vectors (infinitesimal isometries) of \(M\) is spanned by three Killing vectors \(X_1, X_2\) and \(X_3\). Clearly, in view of linearity of the Schouten bracket, the vector field \(\tilde{B}\) given by

\[
\tilde{B} = B + \sum_{i=1}^{3} k_i X_i,
\]

is also a solution to the tensor equation (3.14), where \(k_1, k_2, k_3 \in \mathbb{R}\) are arbitrary constants. Therefore as far as the equation (3.14) is concerned, each \(B\) generates a space \(S_B \subset TM\) of vector fields satisfying (3.14), \(S_B = \{B\} \cup \mathcal{K}^1(M)\). This is not true however for the equation (3.15).

We also note that the right hand side of (3.14) measures the deviation of the vector field \(B\) from a Killing vector.

**Remark 3.1** The space of vector fields \(S_B\) is not closed under the Lie commutator. Indeed, let \(B_1, B_2 \in S_B\). Hence, in view of (3.16) \(B_1 = B + Y_1\) and \(B_2 = B + Y_2\), where \(Y_1, Y_2 \in \mathcal{K}^1(M)\). Taking into account (3.14) and (3.15) and using the standard properties of Lie derivatives, we get

\[
[[B_1, B_2], g] = 3LdV + 3Ld\tilde{V},
\]

where \(L = \mathcal{L}_{Y_1-Y_2} L\) and \(\tilde{V} = (Y_1-Y_2)(V)\).

**Remark 3.2** Lie derivative deformations of \(B\) with respect to Killing vectors do not preserve the form of \(B\). Indeed, using the same arguments as in Remark 3.1, it is easy to see, that \([[B, X], g] = 3LdV + 3Ld\tilde{V}, \text{ where } X \in \mathcal{K}^1(M)\) and \(\tilde{L} = \mathcal{L}_X L, \tilde{V} = X(V)\). Note the right hand side of the last expression is different from that of (3.14).

**Remark 3.3** Let \(B \in \mathcal{K}^1(M)\). This immediately entails that the corresponding Hamiltonian system is trivial and superintegrable, enjoying a second additional first integral of the form \(G = L^{ijk} p_i p_j p_k\). The right hand side of (3.14) in this case is also changed.
In what follows, we find and use isometry group invariants of Killing tensors of valence three defined in the Euclidean plane $\mathbb{R}^2$ to classify cubic integrals of motion (1.2) up to the leading term represented by a Killing tensor of valence three for the Hamiltonian systems with two degrees of freedom defined in $\mathbb{R}^2$.

4 Isometry group invariants of Killing tensors of valence three

Let $(M, g)$ be the Euclidean plane $\mathbb{R}^2$. In this section we apply the method developed recently by the authors in [10, 11, 12, 13] to the problem of classification of Killing tensors belonging to the linear space $\mathcal{K}^3(\mathbb{R}^2)$.

Consider the linear space $\mathcal{K}^3(\mathbb{R}^2)$. The dimension of the space can be computed by employing the Delong-Takeuchi-Thompson formula [16, 17, 18]:

$$\dim \mathcal{K}^3(\mathbb{R}^2) = \frac{1}{2} \left( \frac{2 + 3}{3 + 1} \right) \left( \frac{2 + 3 - 1}{3} \right) = 10.$$  

Alternatively, we can solve the Killing tensor equation $[L, g] = 0$, $L \in \mathcal{K}^3(\mathbb{R}^2)$, for the metric tensor of $\mathbb{R}^2$ in (say) Cartesian coordinates $(x, y)$, to get

\begin{align*}
L^{111} &= a_1 + 3\alpha_1 y + 3\beta_1 y^2 + \gamma y^3, \\
L^{112} &= a_2 + \alpha_2 y - \alpha_1 x - 2\beta_1 xy + \beta_2 y^2 - \gamma xy^2, \\
L^{122} &= a_3 - \alpha_2 x - \alpha_3 y - 2\beta_2 xy + \beta_1 x^2 + \gamma yx^2, \\
L^{222} &= a_4 + 3\alpha_3 x + 3\beta_2 x^2 - \gamma x^3. 
\end{align*}

(4.17)

The ten parameters $a_1, \ldots, \gamma$ are the constants of integration which also imply that the dimension of $\mathcal{K}^3(\mathbb{R}^2) = 10$. Accordingly, we can view each element of $\mathcal{K}^3(\mathbb{R}^2)$ as an algebraic object determined by its ten parameters $a_1, \ldots, \gamma$. Furthermore, for the linear space $\mathcal{K}^3(\mathbb{R}^2)$, we can formulate an analogue of the main problem of the classical theory of algebraic invariants (see [14, 13] for more details). Indeed, recall that the main problem of the classical theory of algebraic invariants developed by Hilbert [14] in modern mathematical language reads:

**Problem 4.1** Determine the linear action of a group $G$ on a $K$-vector space $V$. Then in the ring of polynomial functions $K[V]$ describe the subring $K[V]^G$. 

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of all polynomial functions on \( V \) which are invariant under the action of \( G \) (i.e., the invariants).

In the present study, we formulate the problem as follows:

**Problem 4.2** Let \( \Sigma \) be the space spanned by the ten parameters \( a_1, \ldots, \gamma \) that appear in (4.17). Determine the action induced by the isometry group \( I(\mathbb{R}^2) \) on \( \Sigma \). Then in the space of functions defined in \( \Sigma \) describe the subspace of all functions in \( \Sigma \) which are invariant under the action induced by the isometry group.

Note that the linear spaces \( \Sigma \) and \( K^3(\mathbb{R}^2) \) are isomorphic. The solution of Problem 4.2 consists of two essential parts: first determining the action of \( I(\mathbb{R}^2) \) in \( \Sigma \) and second finding the invariants. Unlike the classical theory of algebraic invariants [14], where the group acts naturally in a space of polynomials by linear substitutions, the determination of the action of the isometry group in the present situation is more complex. To make the first step, we proceed as in the analogous problems considered by the authors in [10, 12, 13] for the linear spaces \( K^2(\mathbb{R}^2) \) and \( K^2(\mathbb{R}^2_1) \), where \( \mathbb{R}^2_1 \) denotes the Minkowski plane. Observe that the generators of the Lie algebra \( \mathfrak{i}(\mathbb{R}^2) \) of the isometry group \( I(\mathbb{R}^2) \) with respect to Cartesian coordinates \( (x, y) \) are the vector fields \( \mathbf{X} = \frac{\partial}{\partial x}, \mathbf{Y} = \frac{\partial}{\partial y} \) and \( \mathbf{R} = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \) whose flows represent translations along \( x, y \) and a rotation respectively. They enjoy the following commutator relations:

\[
[X, Y] = 0, \quad [X, R] = -Y, \quad [Y, R] = X. \quad (4.18)
\]

Now we introduce the projection map \( \pi : K^3(\mathbb{R}^2) \to T\Sigma \) defined for a fixed \( L^0 \in K^3(\mathbb{R}^2) \) as follows:

\[
\pi(L^0) = \sum_{i=1}^{4} a_i^0 \frac{\partial}{\partial a_i} + \sum_{i=1}^{3} \alpha_i^0 \frac{\partial}{\partial \alpha_i} + \sum_{i=1}^{2} \beta_i^0 \frac{\partial}{\partial \beta_i} + \gamma^0 \frac{\partial}{\partial \gamma}, \quad (4.19)
\]

where \( a_1^0, \ldots, \gamma^0 \) are the ten parameters that determine \( L^0 \) via the representation (4.17) with respect to Cartesian coordinates \( (x, y) \) and \( a_1, \ldots, \gamma \) are the ten coordinate functions of the space \( \Sigma \). In order to determine the induced action of \( I(\mathbb{R}^2) \) in \( \Sigma \), we use the composition map \( \pi \circ L \) of \( \pi \) and the Lie derivative operator \( \mathcal{L} \). Define the following vector fields in \( T\Sigma \):

\[
V_1 := \pi \mathcal{L}_X L, \quad V_2 := \pi \mathcal{L}_Y L, \quad V_3 := \pi \mathcal{L}_R L, \quad (4.20)
\]
where \( L \) is the general Killing tensor, whose components with respect to Cartesian coordinates \((x, y)\) are given in (4.17). Note that the Lie derivative deformations of \( L \) with respect to Killing vectors are themselves Killing tensors of valence three and as such represent elements of the linear space \( \mathcal{K}^3(\mathbb{R}^2) \). This fact is a consequence of the Jacobi identity for the Schouten bracket acting in the space of symmetric contravariant tensorial quantities. We conclude therefore that the formulas (4.20) are well-defined. Employing the formulas (4.20) in conjunction with (4.17), we arrive at the following representations of \( V_1, V_2, V_3 \in T\Sigma \) with respect to the coordinate functions \( a_1, \ldots, \gamma \):

\[
V_1 = -\alpha_1 \frac{\partial}{\partial a_2} - \alpha_2 \frac{\partial}{\partial a_3} + 3\alpha_3 \frac{\partial}{\partial a_4} - 2\beta_1 \frac{\partial}{\partial \alpha_2} + 2\beta_2 \frac{\partial}{\partial \alpha_3} - \gamma \frac{\partial}{\partial \beta_2}
\]

\[
V_2 = 3\alpha_1 \frac{\partial}{\partial a_1} + \alpha_2 \frac{\partial}{\partial a_2} - \alpha_3 \frac{\partial}{\partial a_3} + 2\beta_1 \frac{\partial}{\partial \alpha_1} + 2\beta_2 \frac{\partial}{\partial \alpha_2} + \gamma \frac{\partial}{\partial \beta_1}
\]

\[
V_3 = 3a_2 \frac{\partial}{\partial a_1} + (2a_3 - a_1) \frac{\partial}{\partial a_2} + (a_4 - 2a_2) \frac{\partial}{\partial a_3} - 3a_3 \frac{\partial}{\partial a_4} + \alpha_2 \frac{\partial}{\partial \alpha_1} - 2(\alpha_1 + \alpha_3) \frac{\partial}{\partial \alpha_2} + \alpha_2 \frac{\partial}{\partial \alpha_3} + \beta_2 \frac{\partial}{\partial \beta_1} - \beta_1 \frac{\partial}{\partial \beta_2}
\]

(4.21)

Our next observation is that \( V_1, V_2, V_3 \) satisfy the following commutator relations:

\[
[V_1, V_2] = 0, \quad [V_1, V_3] = -V_2, \quad [V_2, V_3] = V_1.
\]

(4.22)

Therefore \( V_1, V_2, V_3 \) form a basis for a Lie algebra \( i_{\mathcal{K}^3(\mathbb{R}^2)}(\Sigma) \), which is a Lie subalgebra of \( T\Sigma \). Furthermore, its generators \( V_1, V_2 \) and \( V_3 \) satisfy the same commutator relations as the generators \( X, Y \) and \( R \) of \( i(\mathbb{R}^2) = \mathcal{K}^1(\mathbb{R}^2) \) in \( T\mathbb{R}^2 \) (compare (4.18) with (4.21)). Therefore we have proven the following

**Proposition 4.1** The vector space \( i_{\mathcal{K}^3(\mathbb{R}^2)}(\Sigma) \subset T\Sigma \) is a Lie subalgebra of \( T\Sigma \) isomorphic to the Lie algebra of Killing vectors \( i(\mathbb{R}^2) = \mathcal{K}^1(\mathbb{R}^2) \).

Proposition 4.1 establishes the induced action of \( I(\mathbb{R}^2) \) in \( \Sigma \cong \mathcal{K}^3(\mathbb{R}^2) \). Therefore we have solved the first part of Problem 4.2.

Recall that invariance of an object under entire Lie group is equivalent to infinitesimal invariance under infinitesimal generators of the corresponding Lie algebra. This observation and the result of Proposition 4.1 prompts the following
Definition 4.1 A smooth function $F : \Sigma \to \mathbb{R}$ is said to be an $I(\mathbb{R}^2)$-invariant of $K^3(\mathbb{R}^2)$ iff

$$V_i(F) = 0, \quad i = 1, 2, 3,$$

where $V_i$, $i = 1, 2, 3$ are the generators of the Lie algebra $i_{K^3(\mathbb{R}^2)}(\Sigma)$ isomorphic to the Lie algebra $i(\mathbb{R}^2) = K^1(\mathbb{R}^2)$ of Killing vectors defined on $\mathbb{R}^2$.

Solving the second part of Problem 4.2 comes down to solving the system of partial differential equations (4.23) determined by the generators of $i_{K^3(\mathbb{R}^2)}(\Sigma)$. A simple counting argument reveals that there are $7 = 10 \ (\text{dimension of } \Sigma) - 3 \ (\text{dimension of the Lie group } I(\mathbb{R}^2))$ essential invariants that determine the space of all $I(\mathbb{R}^2)$-invariants of $K^3(\mathbb{R}^2)$. In what follows we may not need to determine all of them, since we wish to consider Problem 4.2 in conjunction with the dynamical problem envisaged in the Introduction. More specifically, we will exclude from further consideration all Killing tensors that are trivial, according to Definition 3.1. Ultimately, this will lead to a different (smaller) set of essential $I(\mathbb{R}^2)$-invariants of the linear space $K^3(\mathbb{R}^2)$.

5 Essential integral submanifolds of the space \( \Sigma \)

In this section we carry over the ideas exhibited in the previous section to the classification problem of the linear space $K^3(\mathbb{R}^2)$, that is find and use $I(\mathbb{R}^2)$-invariants of $K^3(\mathbb{R}^2)$ to classify Killing tensors of valence three and thus, the corresponding first integrals of motions of (1.1) up to their leading terms.

First, we wish to exclude from further consideration the Killing tensors corresponding to the trivial first integrals (see Definition 3.1). In Cartesian coordinates the most general Killing tensor $L_{tr} \in K^3(\mathbb{R}^2)$ of this type can be obtained via multiplication of the metric $g$ of $\mathbb{R}^2$ by the most general Killing vector $X_{gen}$ of $\mathbb{R}^2$ spanned by $X$, $Y$ and $R$:

$$X_{gen} = (a + cy) \frac{\partial}{\partial x} + (b - cx) \frac{\partial}{\partial y}, \quad a, b, c \in \mathbb{R}.$$  

Thus, the components of $L_{tr}$ are given by $L_{tr}^{ijk} = g^{ij} X_{gen}^k$. Comparing the last formula with (4.17) and taking into account $g^{12} = 0$, $g^{11} = g^{22} = 1$,
we come to the conclusion that in $\Sigma$ the subspace $T \subset \Sigma$ corresponding to the space of trivial Killing tensors in $K^3(\mathbb{R}^2)$ is determined by the following conditions:

\[ a_1 - 3a_3 = 0, \quad 3a_2 - a_4 = 0, \quad \alpha_1 + \alpha_3 = 0, \quad \alpha_2 = 0, \quad \beta_1 = \beta_2 = 0, \quad \gamma = 0. \quad (5.24) \]

It is easy to see that the space $T \subset \Sigma$ defined by (5.24) is an integral submanifold in $\Sigma$ whose tangent space $T^T$ is spanned by the vector fields

\[ T_1 = \frac{3}{\partial a_1} + \frac{\partial}{\partial a_3}, \quad T_2 = \frac{\partial}{\partial a_2} + 3\frac{\partial}{\partial a_4}, \quad T_3 = \frac{\partial}{\partial a_1} - \frac{\partial}{\partial a_3}. \quad (5.25) \]

Indeed, the system $\{T_1, T_2, T_3\}$ is in involution ($\{T_1, T_2, T_3\}$ is a basis of an abelian Lie algebra in $T\Sigma$), and as such, in view of Frobenius’ theorem, is integrable. Interestingly, the system

\[ \{V_1, V_2, V_3, T_1, T_2, T_3\} \quad (5.26) \]

is a basis of a six-dimensional Lie algebra in $T\Sigma$ with the following commutator table:

\[
\begin{array}{c|cccccc}
 & V_1 & V_2 & V_3 & T_1 & T_2 & T_3 \\
\hline
V_1 & 0 & 0 & -V_2 & 0 & 0 & T_2 \\
V_2 & 0 & 0 & V_1 & 0 & 0 & -T_1 \\
V_3 & V_2 & -V_1 & 0 & T_2 & -T_1 & 0 \\
T_1 & 0 & 0 & -T_2 & 0 & 0 & 0 \\
T_2 & 0 & 0 & T_1 & 0 & 0 & 0 \\
T_3 & -T_2 & T_1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Therefore the system of vector fields (5.26) is also integrable. According to Definition 4.1, in order to find essential $I(\mathbb{R}^2)$-invariants of $K^3(\mathbb{R}^2)$ of non-trivial Killing tensors, we have to solve the following system of PDE’s:

\[ T_i(F) = 0, \quad V_i(F) = 0, \quad i = 1, 2, 3 \quad (5.27) \]

for a smooth function $F : \Sigma \rightarrow \mathbb{R}$. The equations $F = c$, where $c$ is a constant, will determine integral submanifolds of the integrable distribution of vector fields (5.26).

Although in principle the system (5.27) can be solved by the method of characteristics, the actual execution of the procedure is an arduous computational task. To alleviate the difficulties, we solve (5.27) in two steps.

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More specifically, let us first find all essential \( I(\mathbb{R}^2) \)-invariants of \( K^3(\mathbb{R}^2) \) for a subsystem of PDE's, namely the following.

\[
T_i(F) = 0, \quad V_j(F) = 0, \quad i = 1, 2, 3 \quad j = 1, 2.
\] (5.28)

Employing the method of characteristics, we find the following set of five essential \( I(\mathbb{R}^2) \)-invariants of \( K^3(\mathbb{R}^2) \) for the system (5.28) as functions of the ten parameters \( a_1, \ldots, \gamma \):

\[
\tilde{I}_1 = \gamma \alpha_2 - 2\beta_1 \beta_2, \\
\tilde{I}_2 = \gamma (\alpha_1 + \alpha_3) + \beta_2^2 - \beta_1^2, \\
\tilde{I}_3 = \gamma^2(a_1 - 3a_3) + 3\gamma(\beta_2 \alpha_2 - \beta_1(\alpha_1 + \alpha_3)) + 2\beta_1(\beta_1^2 - 3\beta_2^2), \\
\tilde{I}_4 = \gamma^2(3a_2 - a_4) - 3\gamma(\beta_1 \alpha_2 + \beta_2(\alpha_1 + \alpha_3)) + 2\beta_2(3\beta_1^2 - \beta_2^2), \\
\tilde{I}_5 = \gamma,
\] (5.29)

Note that all of the invariants (5.29) are homogeneous polynomials. Therefore we conclude that the most general \( I(\mathbb{R}^2) \)-invariant \( \tilde{F} : \Sigma \rightarrow \mathbb{R} \) satisfying (5.28) is given by

\[
\tilde{F} = \tilde{F}(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4, \tilde{I}_5). 
\] (5.30)

Now let us treat the functions \( \tilde{I}_i, i = 1, \ldots, 5 \) as new coordinates and apply the last vector field \( V_3 \) to \( \tilde{F} \). This leads to the first order linear partial differential equation

\[
0 = -2\tilde{I}_2 \frac{\partial \tilde{F}}{\partial \tilde{I}_1} + 2\tilde{I}_1 \frac{\partial \tilde{F}}{\partial \tilde{I}_2} + 3\tilde{I}_4 \frac{\partial \tilde{F}}{\partial \tilde{I}_3} - 3\tilde{I}_3 \frac{\partial \tilde{F}}{\partial \tilde{I}_4}
\] (5.31)

and four essential \( I(\mathbb{R}^2) \)-invariants of \( K^3(\mathbb{R}^2) \) as functions of \( \tilde{I}_i, i = 1, \ldots, 5 \) that are found to be:

\[
I_1 = 2(\tilde{I}_1^3 - 3\tilde{I}_2^2)\tilde{I}_1\tilde{I}_3\tilde{I}_4 + (3\tilde{I}_1^2 - \tilde{I}_2^2)\tilde{I}_2(\tilde{I}_3^2 - \tilde{I}_4^2), \\
I_2 = \tilde{I}_1^2 + \tilde{I}_2^2, \\
I_3 = \tilde{I}_3^2 + \tilde{I}_4^2, \\
I_4 = \tilde{I}_5.
\] (5.32)

This result puts in evidence that the function

\[
\tilde{F} = \tilde{F}(I_1, I_2, I_3, I_4)
\] (5.33)
is the solution to Problem 4.2 for the non-trivial elements of $\mathcal{K}^3(\mathbb{R}^2)$, that is the Killing tensors that determine non-trivial, according to Definition 3.1, first integrals of (1.1) cubic in the momenta. Now we can use the essential $I(\mathbb{R}^2)$-invariants of $\mathcal{K}^3(\mathbb{R}^2)$ $I_i, i = 1, \ldots, 4$ (5.32) to classify non-trivial Killing tensors of valence three up to the action of the isometry group $I(\mathbb{R}^2)$ according to whether $I_i, i = 1, \ldots, 4$ are equal to zero or not. Therefore there are $2^4 = 16$ distinct classes of non-trivial Killing tensors characterized by $I_i, i = 1, \ldots, 4$ (5.32). The elements of each class are equivalent up to the action of the isometry group $I(\mathbb{R}^2)$. We note however that the Killing tensors of valence three of the first integrals isolated by Holt [3] and Fokas-Lagerström [2] are non-trivial, yet the essential $I(\mathbb{R}^2)$-invariants $I_i, i = 1, \ldots, 4$ all vanish identically (see the next section). This happens due to the fact that for the Killing tensors of both Holt and Fokas-Lagerström first integrals, cubic in the momenta the parameters $\gamma, \beta_1$ and $\beta_2$ vanish identically, which, in turn, kills all of the essential $I(\mathbb{R}^2)$-invariants $I_i, i = 1, \ldots, 4$ given by (5.32). As might be expected, we need extra invariant(s) to distinguish the cases when $\gamma = \beta_1 = \beta_2 = 0$. Indeed, observe that $\gamma$ is an essential invariant of $\mathcal{K}^3(\mathbb{R}^2)$ for non-trivial Killing tensors of valence three. Therefore we can consider the integral hypersurface of the space $\Sigma \setminus \mathcal{T}$ determined by the condition $\gamma = 0$. Using the same arguments as above, we find that the homogeneous polynomial $\beta_1^2 + \beta_2^2$ is an $I(\mathbb{R}^2)$-invariant of $\mathcal{K}^3(\mathbb{R}^2)$ in this case. Therefore we can consider the integral submanifold $\Sigma_1 \setminus \mathcal{T}$ of $\Sigma \setminus \mathcal{T}$ determined by the conditions: $\gamma = 0, \beta_1^2 + \beta_2^2 = 0$. It is easy to see by using the same counting argument as that given in the end of Section 3 that there exists only one essential $I(\mathbb{R}^2)$-invariant of $\mathcal{K}^3(\mathbb{R}^2)$ in the subspace $\Sigma_1 \setminus \mathcal{T}$. Employing the procedure described above, we find it to be the following homogeneous polynomial of the parameters $\alpha_1, \alpha_2$ and $\alpha_3$:

$$I_* = (\alpha_1 + \alpha_3)^2 + \alpha_2^2.$$

(5.34)

Now we can classify Killing tensors of valence three in the space $\Sigma_1 \setminus \mathcal{T}$ up to the action of the isometry group $I(\mathbb{R}^2)$ according to whether $I_*$ zero or not. In view of homogeneity of the polynomial $I_*$ the Killing tensors of valence three determined by the condition $I_* \neq 0$ are equivalent up to rescaling $\tilde{x} = kx, \tilde{y} = ky, k \in \mathbb{R}$ (see (4.17)) that preserves the signature of the metric $g$. Note also that there exist no other essential $I(\mathbb{R}^2)$-invariants of $\mathcal{K}^3(\mathbb{R}^2)$ involving $\alpha_1, \alpha_2$ and $\alpha_3$. Thus, there are only two equivalence classes of Killing tensors in $\Sigma_1 \setminus \mathcal{T}$. Let $\Sigma_2 = \Sigma \setminus \Sigma_1$. We summarize our observations in the following table.
| Case                              | Integral submanifold in $\Sigma$ | Invariants | # of equivalence classes of elements of $K^3(\mathbb{R}^2)$ |
|----------------------------------|----------------------------------|------------|----------------------------------------------------------|
| $\gamma^2 + \beta_1^2 + \beta_2^2 \neq 0$ | $\Sigma_2 \setminus \mathcal{T}$ | $I_1, I_2, I_3, I_4$ | $2^4 - 6 = 10$ |
| $\gamma^2 + \beta_1^2 + \beta_2^2 = 0$ | $\Sigma_1 \setminus \mathcal{T}$ | $I_*$       | 2 |

Table 1: Group invariant classification of Killing tensors of valence three of non-trivial cubic in the momenta first integrals of motion

**Remark 5.1** Note the conditions $I_2 = I_4 = 0$ and $I_3 = I_4 = 0$ incite the condition $\gamma = \beta_1 = \beta_2 = 0$. Therefore we have reduced the number of equivalence classes corresponding to the condition $\gamma^2 + \beta_1^2 + \beta_2^2 \neq 0$ by 6, which is the number of cases including the conditions $I_2 = I_4 = 0$ and/or $I_3 = I_4 = 0$.

In the next section we apply our classification scheme to study the cubic in the momenta first integrals of motion isolated by Holt [3] and Fokas-Lagerström [2].

### 6 The Holt and Fokas-Lagerström potentials

Consider the Holt and Fokas-Lagerström integrable systems characterized by the existence of the first integrals of motion

$$F_H(q, p) =$$

$$2p_1^3 + 3p_1p_2^2 + 3p_1(-3(q^2)^2 + 2(q^1)^2 + 2\delta)(q^2)^{-2/3} +$$

$$18p_2q^1(q^2)^{1/3},$$

(6.35)
and

\[ F_{FL}(q, p) = \]

\[ (p_1^2 - p_2^2)(q_1^1 p_2 - q_2^1 p_1) - \]

\[ 4(q_2^1 p_1 + q_1^1 p_2)((q_1^1)^2 - (q_2^1)^2)^{-2/3}, \]

respectively given in the standard position-momenta coordinates. Comparing the cubic terms in \( F_H \) and \( F_{FL} \) with the general formulas (4.17) and taking into account the conditions (5.24), we conclude that the corresponding Killing tensors of valence three \( L_H \) and \( L_{FL} \) determining the leading terms of (6.35) and (6.37) respectively are non-trivial, yet they both satisfy the invariant condition \( \gamma = \beta_1 = \beta_2 = 0 \). Therefore both \( L_H \) and \( L_{FL} \) belong to the space \( \Sigma_1 \) and as such can be characterized by the essential \( I(\mathbb{R}^2) \)-invariant \( I_* \) given by (5.34). Indeed, comparing (6.35) and (6.37) with (4.17), we get for \( L_H \): \( I_* = 0 \) and for \( L_{FL} \): \( I_* = 4/9 \). Hence, each of \( L_H \) and \( L_{FL} \) span the two equivalence classes of non-trivial Killing tensors of valence three in \( \Sigma_1 \). In other words, any Killing tensor belonging to \( \Sigma_1 \) can be obtained from either \( L_H \) or \( L_{HL} \) via the translations and rotations of the coordinates \((q^1, q^2)\) determined by the generators \( X, Y \) and \( R \) respectively or by rescaling.

Therefore any new integrable cases of (1.1) admitting first integrals of motion that are cubic in the momenta within the space \( \Sigma_1 \setminus \mathcal{T} \) can only be found by varying the respective vector fields \( B_H \) and \( B_{FL} \) in (6.35) and (6.37) and finding the corresponding potential functions \( V \) of (1.1) via the formulas (3.14) and (3.15).

7 Concluding remarks

In this paper we have extended the isometry group invariants approach developed in [10, 11, 12, 13] to the study of Killing tensors of valence three in the Euclidean plane, or elements of the linear space \( K^3(\mathbb{R}^2) \). More specifically, Problem 4.2, which is an analogue of the main problem of the classical theory of algebraic invariants [14], has been solved for the linear space of non-trivial Killing tensors of valence three defined in the Euclidean plane.

The approach allows us to classify first integrals cubic in the momenta for Hamiltonian systems defined in the Euclidean plane up to their leading cubic terms by employing essential \( I(\mathbb{R}^2) \)-invariants of \( K^3(\mathbb{R}^2) \). We have studied the
Killing tensors $L_H$ and $L_{FL}$ of valence three determined by the first integrals of motion found by Holt and Fokas-Lagerström respectively and described the integral submanifold in the space $\Sigma$ determined by the parameters of the generic Killing tensor of valence three defined in the Euclidean plane, that contains both of them. It has also been demonstrated that all other Killing tensors in this space are related to $L_H$ and $L_{FL}$ via the rigid motions of the Euclidean space $\mathbb{R}^2$.

We are convinced that the method used in this paper to study Hamiltonian systems can be successfully employed in other areas of mathematical physics. Recall, that Delong [16] studied the ten-parameter conformal symmetry group of the two-dimensional wave equation. He has proved that there are $(2k+1)(2k+2)(2k+3)/6$ independent $k$-order symmetries generated by the ten basic elements of this conformal symmetry group. This formula is an analogue of the Delong-Takeuchi-Thompson formula [16, 17, 18] used in the present study. Thus, one can derive invariants of the ten-parameter conformal symmetry group in the space, say, of second-order symmetries generated by the basic elements and use them in classification-type problems that arise in the study of conformal symmetries of the two-dimensional wave equation.

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