SYMMETRIC POLYNOMIALS VANISHING ON THE SHIFTED DIAGONALS AND MACDONALD POLYNOMIALS

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Abstract. For each pair \((k, r)\) of positive integers with \(r \geq 2\), we consider an ideal \(I_n^{(k,r)}\) of the ring of symmetric polynomials. The ideal \(I_n^{(k,r)}\) has a basis consisting of Macdonald polynomials \(P_\lambda(x_1, \cdots, x_n; q, t)\) at \(t^{k+1} q^{r-1} = 1\), and is a deformed version of the one studied earlier in the context of Jack polynomials. In this paper we give a characterization of \(I_n^{(k,r)}\) in terms of explicit zero conditions on the \(k\)-codimensional shifted diagonals of the form \(x_2 = t q^s x_1, \cdots, x_{k+1} = t q^s x_k\).

The ideal \(I_n^{(k,r)}\) may be viewed as a deformation of the space of correlation functions of an abelian current of the affine Lie algebra \(\widehat{\mathfrak{sl}}_r\). We give a brief discussion about this connection.

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1. Introduction

Recall that integrable representations of Kac-Moody Lie algebras can be characterized in terms of vanishing ideals. Let us formulate this fact for the current algebra $\hat{sl}_r$. Denote by $e_{ij}(z)$ ($i \neq j$) the currents corresponding to root vectors $e_{ij}$ of $\hat{sl}_r$. A representation of $\hat{sl}_r$ in category $\mathcal{O}$ is integrable if and only if $e_{ij}(z)^{k+1}$ acts as 0 for some $i \neq j$, where $k$ is a non-negative integer called the level. In this case $e_{ij}(z)^{k+1} = 0$ holds for all $i \neq j$. Actually there are more relations which follow from these. Consider the commutative currents $e_{12}(z), e_{13}(z), \ldots, e_{1r}(z)$. On level $k$ integrable representations they satisfy the relations

$$e_{12}(z)\nu_1 e_{13}(z)\nu_2 \cdots e_{1r}(z)\nu_{r-1} = 0$$

for all $\nu_1, \ldots, \nu_{r-1} \in \mathbb{Z}_{\geq 0}$ such that $\nu_1 + \cdots + \nu_{r-1} = k + 1$. Let us combine them into a single abelian current

$$e(z) = e_{12}(z^{-\nu_{r-1}}) + z^{-\nu_{r-1}} e_{13}(z^{-\nu_{r-2}}) + \cdots + z^{-\nu_{r-1}+2} e_{1r}(z^{-\nu_{r-1}})$$

$$= \sum_{i \in \mathbb{Z}} e_{i} z^i.$$

If we fix a primitive root of unity $\tau$ of order $r - 1$, then the relations (1.1) can be rewritten in terms of $e(z)$ as follows.

$$e(z)\nu_1 e(\tau z)\nu_2 \cdots e(\tau^{r-2} z)\nu_{r-1} = 0$$

$$(\nu_1, \ldots, \nu_{r-1} \in \mathbb{Z}_{\geq 0}, \nu_1 + \cdots + \nu_{r-1} = k + 1).$$

Using (1.3), one can obtain a monomial basis of integrable representations $[\mathcal{P}]$. For simplicity, let us consider the vacuum representation $L$ of level $k$ with highest weight vector $v$ such that $e_i v = 0$ for $i \leq 0$. We call $W = \mathbb{C}[e_1, e_2, \ldots] v \subset L$ the principal subspace of $L$. Then the following set of monomials constitutes a basis of $W$:

$$e_{i_1}^{a_1} e_{i_2}^{a_2} \cdots e_{i_l}^{a_l} \quad (l \in \mathbb{Z}_{\geq 0}),$$

where $a_i \in \mathbb{Z}_{\geq 0}$, $a_i + a_{i+1} + \cdots + a_{i+r-1} \leq k$ for all $i \geq 1$.

We present here a deformation of the relations (1.3) which preserves the structure of the monomial basis (1.4). Namely, let $q, t$ be complex numbers such that $t^{k+1} q^{r-1} = 1$. Consider an abelian current $e(z) = \sum_{i \in \mathbb{Z}} e_i z^i$ satisfying the relations

$$e(z) e(t q^{s_1} z) \cdots e(t^k q^{s_k} z) = 0$$

for all $0 \leq s_1 \leq \cdots \leq s_k \leq r - 2$. For $t = 1$ and $q = \tau$ we get back to the relations (1.3). If we replace $e(z)$ by $\sum_{i \geq 1} e_i z^i$ in (1.3), then the Fourier coefficients of the left hand side are well defined elements of $\mathbb{C}[[e_i]]$. Let $\mathfrak{g}(q, t)$ be the ideal generated by them. In this paper we prove that the quotient $\mathbb{C}[[e_i]] / \mathfrak{g}(q, t)$ has the same set (1.4) as a monomial basis, if $q, t$ are ‘generic’ (see (2.9), (2.10) below for the precise condition). From this fact it follows that the current $e(z)$ of $\hat{sl}_r$ acting on $L$ can be deformed to a current satisfying the relations (1.3). We also give an analogous result for integrable representations of $\hat{sl}_r$ other than the vacuum module.
From a slightly different point of view, our result can be described as follows. Let $D$ be an element of the Weyl group of $\mathfrak{sl}_r$ such that $De_iD^{-1} = e_{i+r}$. Then, on level $k$ integrable representations of $\widehat{\mathfrak{sl}}_r$, we have an action of the algebra $E_{k,r} = \mathbb{C}[D, D^{-1}] \ltimes \mathbb{C}[\{e_i\}_{i \in \mathbb{Z}}]/\mathfrak{g}(\tau, 1)$, where $\mathfrak{g}(q, t)$ denotes the ideal of $\mathbb{C}[\{e_i\}_{i \in \mathbb{Z}}]$ generated by the Fourier coefficients of the left hand side of (1.5). (To be precise a completion is necessary, but we do not discuss such details here.) It is possible to show that each level $k$ integrable irreducible representation of $\widehat{\mathfrak{sl}}_r$ remains irreducible upon restriction to $E_{k,r}$. (Actually $E_{k,r}$ has more irreducible representations than $\mathfrak{sl}_r$, but their classification is not known.) Our result shows that for generic $q, t$ the algebra $E_{k,r}(q, t) = \mathbb{C}[D, D^{-1}] \ltimes \mathbb{C}[\{e_i\}_{i \in \mathbb{Z}}]/\mathfrak{g}(q, t)$ has irreducible representations which are ‘deformations’ of representations of $\mathfrak{sl}_r$.

The relations of the type (1.5) can be put in more general context as follows. Fix a function $\lambda(x, y)$ in two variables. Following [FO] we define two associative algebras $S$ and $G$. The algebra $S$ is a graded algebra $S = \oplus_{n\geq 0}S_n$, each graded component $S_n$ being the space of symmetric functions in $n$ variables. For $F \in S_m$ and $G \in S_n$, the product $F \ast G \in S_{m+n}$ is defined by the formula

$$F \ast G(x_1, \ldots, x_{m+n}) = \text{Sym} \left( F(x_1, \ldots, x_m)G(x_{m+1}, \ldots, x_{m+n}) \prod_{1 \leq i \leq m, m+1 \leq j \leq m+n} \lambda(x_i, x_j) \right). \quad (1.6)$$

In (1.6), the symbol $\text{Sym}$ stands for the symmetrization. The algebra $A = \oplus_{n \geq 0}A_n$ is defined similarly, where $A_n$ consists of anti-symmetric functions in $n$ variables and symmetrization in (1.6) is replaced by anti-symmetrization. Let $K \subset \mathbb{C} \times \mathbb{C}$ be the set of zeroes of the function $\lambda(x, y)$.

We say that $F \in S_n$ (or $F \in A_n$) satisfies the wheel condition if the following holds:

$$F = 0 \text{ whenever } (x_1, x_2, \ldots, (x_l, x_{l+1}), (x_{l+1}, x_1) \in K \text{ for some } 1 \leq l \leq n. \quad (1.7)$$

If $F, G$ satisfy the wheel condition, then so does $F \ast G$. Therefore, functions satisfying the wheel condition (1.7) constitute a subalgebra $S^w \subset S$ (resp. $A^w \subset A$). Note that the subalgebra generated by $S_1$ (resp. $A_1$) in $S$ (resp. $A$) is contained in $S^w$ (resp. $A^w$).

Set

$$\lambda(x, y) = \frac{(x-t_1y)\cdots(x-t_my)}{(x-y)^s}$$

where $S = \{t_1, \ldots, t_s\}$ is a set of non-zero complex numbers. We will refer to $S$ as the wheel set.

Let $B(S)$ consist of rational functions of the form

$$F(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n}(x_i - x_j)^{s-1}}.$$
where \(f(x_1, \cdots, x_n)\) is a symmetric Laurent polynomial satisfying
\[
f(x_1, \cdots, x_n) = 0 \quad \text{if } \frac{x_2}{x_1} = t_{i_1}, \cdots, \frac{x_{l+1}}{x_l} = t_{i_l}, \quad \frac{x_1}{x_{l+1}} = t_{i_{l+1}} \quad \text{for some } i_1, \cdots, i_{l+1}.
\] (1.8)

Then \(B(S)\) is a subalgebra of \(S^w\) if \(s\) is odd and of \(A^w\) if \(s\) is even.

Note that the wheel condition is non-trivial only when the parameters \(t_1, \cdots, t_s\) satisfy
\[
t_1^{\kappa_1} t_2^{\kappa_2} \cdots t_s^{\kappa_s} = 1
\]
for some integers \(\kappa_1, \cdots, \kappa_s\). We call such an equation a resonance condition.

Returning to the abelian current \(e(z)\) and (1.3), we take \(s = r, l = k\), and the wheel set
\[
S = \{t, tq, \cdots, tq^{r-1}\}
\] (1.9)
where \(t^{k+1}q^{r-1} = 1\) is assumed. Then the relation (1.3) coincides with (1.4) in the following sense.

Suppose we have a representation \(W\) of the current \(e(z)\) satisfying (1.5). (For the relations (1.3) to make sense, we consider only such representations that for any vector \(v \in W\) there is an integer \(N\) satisfying \(e_i v = 0\) for \(i < N\).) Then the matrix elements
\[
f(z_1, \cdots, z_n) = \langle v^\vee, e(z_1) \cdots e(z_n)v\rangle.
\] (1.10)

where \(v \in W\) and \(v^\vee \in W^*\), are Laurent polynomials satisfying the wheel condition (1.3).

Therefore the study of the representation \(W\) is closely related to that of the algebra \(B(S)\). Now let \(W = \mathbb{C}[e_1, e_2, \cdots] / \mathfrak{J}(q, t), v = 1 \mod \mathfrak{J}(q, t)\) and \(e_i v = 0\) \((i \leq 0)\). Then the dual space \(W^*\) can be identified with the space \(J^{(k,r)} = \oplus_{n \geq 0} J_n^{(k,r)}\) of all symmetric polynomials satisfying the wheel condition relative to (1.9). Here \(J_n^{(k,r)}\) denotes the subspace of \(n\) variables.

We use the theory of symmetric functions to find a basis in \(J^{(k,r)}\). Namely, let \(I_n^{(k,r)}\) be spanned by Macdonald polynomials \(P_\lambda(x; q, t)\), where \(q, t\) satisfy \(t^{k+1}q^{r-1} = 1\), and \(\lambda\) ranges over a set of \((k, r, n)\)-admissible (see (2.11) for the definition) partitions. Our main result is that for ‘generic’ \(q, t\) we have \(I_n^{(k,r)} = J_n^{(k,r)}\). In other words, the above Macdonald polynomials constitute a basis of \(J_n^{(k,r)}\).

We do not understand well the reason why the particular choice (1.3) of the wheel set is exactly what we need to deform the integrability condition (1.1) for \(\tilde{sl}_r\).

This paper is organized as follows. In Section 2, we review known facts about Macdonald polynomials. We discuss briefly their regularity properties when the parameters \(q, t\) satisfy the relation \(q^a t^b = 1\) with some \(a, b \in \mathbb{Z}_{\geq 1}\), following the work [PJM] on Jack polynomials. Our main result is stated as Theorem 2.4. Section 3 is devoted to its proof. In Section 4 we give a monomial base for the analogs of (non-vacuum) integrable representations of \(\tilde{sl}_r\). In Section 5, we discuss an expected link between the present work and representations of \(W_k\) algebras.

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1) In the main text, we shift the index of \(e_i\) by one and consider a quotient space of \(\mathbb{C}[e_0, e_1, \cdots]\).
2. Macdonald polynomials and wheel condition

2.1. Preliminaries. In this subsection, we review basic facts about the Macdonald polynomials which we use in the text. Our basic reference is Macdonald’s book [M].

Let \( n \) be a non-negative integer. The Macdonald operators \( \{D_r^r\}_{0 \leq r \leq n} \) are mutually commuting \( q \)-difference operators acting on the ring of symmetric polynomials \( \mathbb{C}(q,t)[x_1, \cdots, x_n]^{S_n} \), where \( S_n \) stands for the symmetric group on \( n \) letters. Explicitly they are given by the formula

\[
D_r^r = \sum_{|I|=r} A_I(x; t) T_I,
\]

where \( I \subset \{1, \cdots, n\} \) runs over subsets of cardinality \( r \),

\[
A_I(x; t) = t^{r(r-1)/2} \prod_{i \in I} (t x_i - x_j),
\]

\[
T_I = \prod_{i \in I} T_{q,x_i},
\]

and \( (T_{q,x}, f)(x_1, \cdots, x_n) = f(x_1, \cdots, qx_i, \cdots, x_n) \). Let \( D_n(X; q, t) = \sum_{r=0}^{n} D_r^r X^r \) be their generating function.

Denote by \( \pi_n \) the set of partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i \) are non-negative integers satisfying \( \lambda_i \geq \lambda_{i+1} \) (\( 1 \leq i \leq n-1 \)). The Macdonald polynomials \( \{P_\lambda\}_{\lambda \in \pi_n} \) constitute a unique \( \mathbb{C}(q,t) \)-basis \( \mathbb{C}(q,t)[x_1, \cdots, x_n]^{S_n} \) characterized by the following defining properties.

\[
D_n(X; q, t)P_\lambda = \prod_{i=1}^{n} (1 + X q^{\lambda_i} t^{n-i}) \cdot P_\lambda,
\]

\[
P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda \mu} m_\mu \quad (u_{\lambda \mu} \in \mathbb{C}(q,t)).
\]

In the second line, \( m_\lambda = \sum_{\alpha \in S_n, \lambda} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) stands for the monomial symmetric function. The dominance ordering \( \mu < \lambda \) on \( \pi_n \) is defined by \( \mu \neq \lambda \) and \( \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \) for all \( i = 1, \cdots, n \).

Along with (\ref{2.1}), we also consider the following operators \([4]\). For \( m \geq 0 \), set

\[
E_m = \sum_{i=1}^{n} x_i^{m} A_i(x; t) \frac{\partial}{\partial q x_i},
\]

where \( A_i(x; t) = A_{\{i\}}(x; t) \) and

\[
\frac{\partial}{\partial q x_i} = \frac{1}{(q-1)x_i} (T_{q,x_i} - 1).
\]
For $m = 1$, (2.4) is related to $D_n^1$ by $(q - 1)E_1 = D_n^1 - (1 - t^n)/(1 - t)$. Set further $e_1 = \sum_{i=1}^{n} x_i$. Then we have

$$e_1 P_\lambda = \sum_{j=1}^{\ell(\lambda) + 1} \psi'_{\lambda(j)/\lambda} P_{\lambda(j)},$$

(2.5)

$$E_0 P_\lambda = \sum_{j=1}^{\ell(\lambda)} \psi''_{\lambda/\lambda(j)} P_{\lambda(j)},$$

(2.6)

$$E_2 P_\lambda = \frac{\ell(\lambda) + 1}{1 - q} \sum_{j=1}^{m-1} (1 - q^{\lambda_i t^1 - j}) \psi'_{\lambda(j)/\lambda} P_{\lambda(j)}.$$

(2.7)

Here $\lambda^{(j)}$ (resp. $\lambda_{(j)}$) denotes the partition obtained by adding one node to (resp. removing one node from) the $j$-th row of $\lambda$, and $\ell(\lambda) = \max\{j \mid \lambda_j > 0\}$ signifies the length of $\lambda$. The coefficients are given by

$$\psi'_{\lambda(j)/\lambda} = \prod_{i=1}^{j-1} \frac{1 - q^{\lambda_i - \lambda_j + 1 - j - i + 1}}{1 - q^{\lambda_i - \lambda_j + 1 - j - i}},$$

$$\psi''_{\lambda/\lambda(j)} = \frac{1 - q^{\lambda_j - \lambda_j + 1 - j - i + 1}}{1 - q^{\lambda_j - \lambda_j + 1 - j - i}} \prod_{i=j+1}^{n} \frac{1 - q^{\lambda_i - \lambda_j + 1 - j - i + 1}}{1 - q^{\lambda_i - \lambda_j + 1 - j - i}}.$$

When $\lambda_j = \lambda_{j-1}$ and thus $\lambda^{(j)}$ is not defined, the corresponding term is absent in the right hand sides of (2.5), (2.7), because $\psi'_{\lambda(j)/\lambda} = 0$. The same remark applies to (2.6).

Eq.(2.5) is a special case of the Pieri formula ([M], eq.(6.24)), while (2.6), (2.7) are due to [JMM].

2.2. Regularity of Macdonald polynomials. The coefficients $u_{\lambda\mu}$ in (2.3) are rational functions of $q$ and $t$. For a partition $\lambda \in \pi_n$, denote by $\lambda'$ the conjugate partition. It is known [INT] that if we set

$$c_\lambda(q, t) = \prod_{(i,j) \in \lambda} (1 - q^{\lambda_i - j t^{\lambda'_j - i + 1}}),$$

then $c_\lambda P_\lambda$ is a polynomial in $q, t$. In particular, all possible poles of $u_{\lambda\mu}$ are of the form

$$q^a t^b = 1 \quad (a, b \in \mathbb{Z}, \ a \geq 0, b > 0).$$

(2.8)

For given $n$ and $q, t$ satisfying (2.8), it is natural to ask which $P_\lambda$ remain well defined.

We have studied a sufficient condition in [FJMM] in the limit $t = q^3$, $q \to 1$, where Macdonald polynomials reduce to Jack polynomials. As we show below, the results of [FJMM] have straightforward extensions to the setting of Macdonald polynomials.

Throughout this paper, we fix integers $k, r$ where $k \geq 1$ and $r \geq 2$. As opposed to [FJMM], we do not assume that $k + 1, r - 1$ are coprime. Let $m$ be the greatest common divisor of $k + 1$ and $r - 1$, and let $\omega$ be a primitive $m$-th root of unity. Let further $\omega_1 \in \mathbb{C}$ be such that $\omega_1^{(r-1)/m} = \omega$. For an indeterminate $u$ we consider the
specialization
\[ t = u^{\frac{1}{m}}, \quad q = \omega_1 u^{\frac{k+1}{m}}, \]
so that \( t^{\frac{k+1}{m}} q^{\frac{-1}{m}} = \omega \). For integers \( a, b \in \mathbb{Z} \), we have then

\[ q^a t^b = 1 \text{ if and only if } a = (r - 1)s, \quad b = (k + 1)s \text{ for some } s \in \mathbb{Z}. \tag{2.10} \]

As in [FJMM], we say that a partition \( \lambda \in \pi_n \) is \((k, r, n)\)-admissible if
\[ \lambda_i - \lambda_{i+k} \geq r \quad (i = 1, \cdots, n-k). \tag{2.11} \]

The following two Lemmas can be verified by noting (2.10) and repeating the working of Lemma 2.1–2.3 in [FJMM].

**Lemma 2.1.** Suppose \( 1 \leq i < j \leq n \) and \( \lambda \in \pi_n \) is \((k, r, n)\)-admissible. Then
\[
q^{\lambda_i - \lambda_j} t^{j-i} \neq 1, \\
q^{\lambda_i - \lambda_j - 1} t^{j-i+1} \neq 1, \\
q^{\lambda_i - \lambda_j - 1} t^{j-i} \neq 1.
\]

If in addition \( \lambda_j < \lambda_{j-1} \), then
\[ q^{\lambda_i - \lambda_j} t^{j-i} \neq 1. \]

**Lemma 2.2.** If \( \lambda \) is \((k, r, n)\)-admissible, then \( \psi_{\lambda^{(\beta)}}/\lambda \) is well defined. It is zero if and only if \( \lambda_{j-1} = \lambda_j \).

**Lemma 2.3.** Assume either \( \lambda \) is \((k, r, n)\)-admissible, or else \( \lambda \) is obtained from a \((k, r, n)\)-admissible partition by adding or removing one node. Then \( P_\lambda \) has no pole at \((t, q) = (u^{\frac{1}{m}}, \omega_1 u^{\frac{k+1}{m}})\).

**Proof.** In the context of Jack polynomials, an analogous statement is given as Proposition 2.6 in [FJMM]. The same proof applies by using (2.10), Lemma 2.1 and the formula (2.2) for the eigenvalue of the Macdonald operators. We omit further details. \( \square \)

In the rest of this paper, we fix the specialization of \( t, q \) as in (2.9).

### 2.3. Statement of the result.
We set \( \Lambda_n = K[x_1, \cdots, x_n]^S_n \), where the ground field is \( K = \mathbb{C}(u) \). Define a subspace \( I_n^{(k,r)} \) of \( \Lambda_n \) by
\[
I_n^{(k,r)} = \text{span}_K \{ P_\lambda(x_1, \cdots, x_n; q, t) \mid \lambda \text{ is } (k, r, n)\text{-admissible} \}.
\]
Our goal is to characterize this space in terms of the wheel condition.

Consider the subspace \( J_n^{(k,r)} \subset \Lambda_n \) of all symmetric polynomials \( f(x_1, \cdots, x_n) \) satisfying the wheel condition (1.8) relative to the wheel set \( S = \{ t, t q, \cdots, t^q - 1 \} \) relative to the wheel set \( S = \{ t, t q, \cdots, t^q - 1 \} \). Equivalently, \( f \in \Lambda_n \) belongs to \( J_n^{(k,r)} \) if and only if
\[
f = 0 \quad \text{if} \quad x_i = t^{i-1} q^{s_i} + \cdots + s_i x_1 \quad (2 \leq i \leq k+1)
\]
for all \( s_1, \cdots, s_{k+1} \in \mathbb{Z}_{\geq 0} \) satisfying \( s_1 + \cdots + s_{k+1} = r - 1 \). \tag{2.12}

Here we require the vanishing of \( f \) on the wheel of length \( k+1 \). Since we have the resonance of the form (2.10), we have wheels of larger length of a multiple of \( k+1 \).
However, the vanishing of $f$ for such a wheel follows from (2.12) because the larger wheel necessarily contains a wheel of length $k+1$.

The following is our main result.

**Theorem 2.4.** For all $n \geq 0$ we have an equality of ideals of symmetric polynomials

$$I_n^{(k,r)} = J_n^{(k,r)}.$$  

(2.13)

Moreover these ideals are stable under the action of $D(X; q, t)$ and $E_m$ ($m \geq 0$).

For $r = 2$, the condition (2.12) simplifies to

$$f = 0 \text{ if } x_j = t^{j-1}x_1 \text{ for } j = 1, \cdots, k + 1.$$  

Theorem 2.4 in this case was stated in [FJMM].

2) The next Section is devoted to the proof of Theorem 2.4.

3. Proof of Theorem 2.4

3.1. Stability by Macdonald type operators. From the proof of Proposition 3.4–3.6 in [FJMM] we see that, if $\lambda$ is $(k, r, n)$-admissible, then the formulas (2.5)–(2.7) remain valid if only admissible partitions are retained in the right hand side. In particular, the space $I_n^{(k,r)}$ is invariant under the action of the operators $D_n(X; q, t)$, $E_m$ ($m = 0, 1, 2$) and multiplication by $e_1$. We will prove that an analogous statement holds also for $J_n^{(k,r)}$.

**Lemma 3.1.** The ideal $J_n^{(k,r)}$ is invariant under the action of the operators $D_n(X; q, t)$, $E_m$ ($m \geq 0$).

**Proof.** We show that for any $f \in J_n^{(k,r)}$ and $I \subset \{1, \cdots, n\}$, $A_f(x; t)(T_I f)(x)$ satisfies the condition (2.12). The assertion of the lemma is a corollary of this fact.

Let $x_i$, $s_i$ be as in (2.12). Because of (2.10), the denominator of $A_f(x; t)$ does not vanish. Set $\tilde{x}_i = qx_i$ if $i \in I$ and $\tilde{x}_i = x_i$ otherwise. Then $\tilde{x}_{i+1}/\tilde{x}_i = t^{\tilde{s}_i}$ for $1 \leq i \leq k + 1$, where $\tilde{i} = i$ ($1 \leq i \leq k + 1$), $k + 2 = 1$, and

$$\tilde{s}_i = \begin{cases} 
  s_i + 1 & (i \notin I, i + 1 \notin I), \\
  s_i - 1 & (i \in I, i + 1 \notin I), \\
  s_i & (\text{otherwise}).
\end{cases}$$

If we have $s_i = 0$ and $i \in I$, $i + 1 \notin I$ for some $1 \leq i \leq k + 1$, then $tx_i - x_{i+1} = 0$ and $A_f(x; t) = 0$. Otherwise $\tilde{s}_i \geq 0$ for all $i$ and $\sum_{i=1}^{k+1} \tilde{s}_i = r - 1$. Hence the wheel condition (2.12) implies $(T_I f)(x) = f(\tilde{x}) = 0$. \hfill \Box

2) In [FJMM], 5 lines above Theorem 4.4, the condition ‘$x_j = t^{j-1}$’ should read ‘$x_j = t^{j-1}x_1$’.
3.2. Proof of an inclusion relation. In this subsection we prove the inclusion
\[ J_n^{(k,r)} \subset J_n^{(k,r)}, \]  
(3.1)
If \( n \leq k \), then both sides are equal to \( \Lambda_n \). Hence it suffices to consider the case \( n \geq k+1 \).

a) The case \( n = k + 1 \). For \( i = 1, \cdots, k+1 \), let us call \( (C_i) \) the following statement:
\[ \lambda_1 - \lambda_{i+1} \geq r, \ell(\lambda) \leq i \implies P_\lambda \in J_n^{(k,r)} . \]
We prove \( (C_i) \) by induction on \( i \).

In the case \( i = 1 \), \( \lambda \) has only one row. The corresponding Macdonald polynomials have a generating function given by a special case of the Cauchy identity (eq.(4.13), [M])
\[ \sum_{l \geq 0} P_{\{l\}}(x_1, \cdots, x_n; q,t) \left( \frac{(t; q)_l}{(q)_l} \right) y^l = \prod_{i=1}^{n} \left( \frac{(tx_i; q)_\infty}{(x_i; q)_\infty} \right), \]
(3.2)
where \((z; q)_m = \prod_{i=0}^{m-1}(1-zq^i)\). Let \( s_1, \cdots, s_{k+1} \in \mathbb{Z}_{\geq 0} \) be integers satisfying \( s_1 + \cdots + s_{k+1} = r - 1 \). Let \( x_i, s_i \) be as in (2.12), and specialize \( (3.2) \) accordingly, taking \( n = k + 1 \). The right hand side becomes \( \prod_{i=1}^{k+1} (t^i q^{s_1 + \cdots + s_i - 1} x_1 y; q)_s \), which is a polynomial of degree \( r - 1 \) in \( y \). This implies that \( P_{\{l\}} \) for \( l \geq r \) vanishes under the condition (2.12).

Suppose that for some \( i \) such that \( 2 \leq i \leq k+1 \), the condition \( (C_{i-1}) \) is true. We show \( (C_i) \) by induction on \( \lambda_i > 0 \). Let \( \lambda \) be as in \( (C_i) \) and set \( \mu = \lambda_{(i)}, \nu = \mu^{(i+1)} \).

We have \( P_{\mu} \in J_{k+1}^{(k,r)} \) by the induction hypothesis. From the formulas (2.3), (2.4) and (2.7), we have modulo \( J_{k+1}^{(k,r)} \)
\[ \varepsilon_1 P_{\mu} = \psi_{\lambda/\mu}^P P_{\lambda} + \psi_{\nu/\mu}^P P_{\nu} , \]
\[ c \varepsilon_2 P_{\mu} = (1 - q^\mu t^{-1}) \psi_{\lambda/\mu}^P P_{\lambda} + (1 - q^{-\mu} t^{-1}) \psi_{\nu/\mu}^P P_{\nu} , \]
where \( c = t^{n-1}/(1 - q) \). By Lemma 3.1, the left hand sides belong to \( J_{k+1}^{(k,r)} \). By Lemma 2.2 we have \( \psi_{\lambda/\mu}^P \neq 0 \), and \( 1 - q^\mu t^{-1} \neq 1 - t^{-1} \) by (2.10). We conclude that \( P_\lambda \) belongs to \( J_{k+1}^{(k,r)} \).

b) The case \( n \geq k + 2 \). Let
\[ \rho : \Lambda_n \to \Lambda_{n-1}, \quad \rho(f)(x_1, \cdots, x_{n-1}) = f(x_1, \cdots, x_{n-1}, 0) \]
denote the specialization map. To prove (3.3) for \( n \geq k + 2 \), it suffices to show that
\[ f \in I_n^{(k,r)} \implies \rho(\partial_n f) \in J_{n-1}^{(k,r)} \quad (j \geq 0), \]
(3.3)
where \( \partial_n = \partial/\partial x_n \).

Since \( P_\lambda(x_1, \cdots, x_{n-1}, 0; q,t) = P_\lambda(x_1, \cdots, x_{n-1}; q,t) \), (3.3) holds for \( j = 0 \). Suppose it is true for \( j - 1 \). Set
\[ \partial_n^{-1} E_0 f = \sum_{i=1}^{n} X_i, \]
(3.4)
where

$$X_i = A'_i \partial_n^{-1} \left( \frac{tx_i - x_n}{x_i - x_n} \frac{\partial f}{\partial q x_i} \right), \quad A'_i = \prod_{l(\neq i, n)} \frac{tx_i - x_l}{x_i - x_l},$$

$$X_n = \partial_n^{-1} \left( A_n \frac{\partial f}{\partial q x_n} \right).$$

Since $E_0 I_{n}^{(k,r)} \subset I_{n}^{(k,r)}$, by induction hypothesis the image by $\rho$ of the left hand side of (3.4) belongs to $J_{n-1}^{(k,r)}$. Consider the terms with $i \leq n - 1$,

$$\rho(X_i) = \sum_{s=0}^{j-1} \binom{j-1}{s} \rho \left( \partial_n^{j-1-s} \frac{tx_i - x_n}{x_i - x_n} \right) A'_i \frac{\partial}{\partial q x_i} \rho(\partial_n^s f).$$

Since $g = \rho(\partial_n^s f)$ belongs to $J_{n-1}^{(k,r)}$, $A'_i \frac{\partial}{\partial q x_i} g$ satisfies (2.12), as we have seen in the proof of Lemma 3.1. This shows that $\rho(X_i) \in J_{n-1}^{(k,r)} (i \leq n - 1)$. Noting that

$$\rho \left( \partial_n^{s-1} \frac{\partial f}{\partial q x_n} \right) = a_s \rho(\partial_n^s f)$$

with $a_s = (1 - q^s)/(s(1 - q))$, we find

$$\rho(X_n) = \sum_{s=0}^{j-1} \binom{j-1}{s} \rho \left( \partial_n^{j-1-s} A_n \right) \rho \left( \partial_n^s \frac{\partial f}{\partial q x_n} \right) \equiv a_j \rho(\partial_n^j f) \mod J_{n-1}^{(k,r)}.$$

The assertion (3.3) follows from these.

3.3. Comparison of dimensions. In this subsection we finish the proof of Theorem (2.4) by comparing dimensions. Counting $\deg x_i = 1$ for all $i$, we denote by $\Lambda_{n,d} \subset \Lambda_n$ the subspace of polynomials of homogeneous degree $d$. The spaces $J_{n,d}^{(k,r)}$, $I_{n,d}^{(k,r)}$ are homogeneous with respect to the bi-grading of $\Lambda = \oplus \Lambda_n$. We set $J_{n,d}^{(k,r)} = J^{(k,r)} \cap \Lambda_{n,d}$, $I_{n,d}^{(k,r)} = I^{(k,r)} \cap \Lambda_{n,d}$. Let also $\pi_{n,d}$ be the set of partitions $\lambda \in \pi_n$ satisfying $\sum_{i=1}^n \lambda_i = d$. We denote by $e_{n,d}^{(k,r)}$ the set of $(k, r, n)$-admissible partitions, and $e_{n,d}^{(k,r)} = e^{(k,r)} \cap \pi_n$.

Let us specialize the parameters further to $t = 1, q = \tau$, where $\tau$ is a primitive $(r - 1)$-th root of unity. Under this specialization, the wheel condition (2.12) becomes

$$f = 0 \text{ if } x_i = \tau^p x_1 \text{ for all } p_i \in \mathbb{Z} \quad (2 \leq i \leq k + 1).$$

(3.5)

We consider the corresponding polynomial space $\overline{J}_{n}^{(k,r)} = \oplus_{n \geq 0} J_{n}^{(k,r)}$, where

$$\overline{J}_{n}^{(k,r)} = \{ f \in \mathbb{C}[x_1, \cdots, x_n]^{S_n} \mid f \text{ satisfies (3.3)} \}.$$

Let us determine the character of $\overline{J}_{n}^{(k,r)}$. For this purpose it is convenient to pass to the dual space. Let $\overline{R} = \mathbb{C}[e_0, e_1, \cdots]$ be the polynomial ring in indeterminates $\{e_i\}_{i \geq 0}$,
equipped with the bi-grading \( \deg e_i = (1, i) \). Let \( e(\zeta) = \sum_{i \geq 0} e_i \zeta^i \) be the generating series. Consider the ideal \( \overline{J} \subset \overline{R} \) generated by the Fourier coefficients of
\[
e(\zeta) e(\tau^{p_2} \zeta) \cdots e(\tau^{p_{k+1}} \zeta),
\]
where \( p_2, \ldots, p_{k+1} \) run through arbitrary integers. A standard argument shows (see e.g. [FJMMT], Lemma 3.3) that there is a non-degenerate coupling
\[
(\overline{R}/\overline{J}) \times \overline{J}^{(k,r)} \to \mathbb{C},
\]
through which each homogeneous component \( (\overline{R}/\overline{J})_{n,d} \) is isomorphic to \( \overline{J}_{n,d}^{(k,r)} \).

In the below, we write
\[
e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}
\]
for a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \pi_n \).

**Proposition 3.2.** The space \( \overline{R}/\overline{J} \) is spanned by the set of monomials
\[
B = \{ e_\lambda \mid \lambda \in \mathcal{G}^{(k,r)}_n \}.
\]

**Proof.** We use the lexicographic ordering \( \mu \succ \lambda \) defined by \( \mu_1 = \lambda_1, \ldots, \mu_{i-1} = \lambda_{i-1}, \mu_i > \lambda_i \) for some \( i \).

If we write
\[
e(\zeta) = \sum_{j=0}^{r-2} \zeta^j e_{1j+2}(\zeta^{r-1}), \quad e_{1j+2}(z) = \sum_{l \geq 0} e_{(r-1)l+j} z^l,
\]
then the defining relations \([3.6]\) for the ideal \( \overline{J} \) can be stated equivalently as
\[
e_{12}(z)^{\nu_0} e_{13}(z)^{\nu_1} \cdots e_{1r}(z)^{\nu_{r-2}} \equiv 0 \mod \overline{J}.
\]
Here \( \nu = (\nu_0, \ldots, \nu_{r-2}) \) runs over non-negative integers \( \nu_j \in \mathbb{Z}_{\geq 0} \) such that \( \sum_{j=0}^{r-2} \nu_j = k + 1 \). Hence \( \overline{J} \) is generated by elements of the form \( \sum_{\mu \in K_d(\nu)} C_\mu e_\mu \), where \( C_\mu \) are positive integers and
\[
K_d(\nu) = \{ \mu \in \pi_{k+1,d} \mid \sharp \{ i \mid \mu_i \equiv a \mod r - 1 \} = \nu_a \ (0 \leq a \leq r - 2) \}.
\]
It is easy to see that, for each \( d \) and \( \nu \), the set \( K_d(\nu) \) is either empty or contains a unique element \( \lambda \) satisfying \( \lambda_1 - \lambda_{k+1} = r - 1 \). If \( \mu \in K_d(\nu) \) with \( \mu \neq \lambda \), then we have \( \mu \succ \lambda \). Moreover, any \((k,r,k+1)\)-non-admissible partition \( \lambda \) is the minimal element of some \( K_d(\nu) \). Therefore, for such \( \lambda \), \( e_\lambda \) belongs to the linear span of \( B \).

Let us consider the general case \( \lambda \in \pi_{n,d} \). Suppose \( \lambda_i - \lambda_{i+k} \leq r - 1 \) for some \( i \). Set \( \mu = (\lambda_1, \ldots, \lambda_{i+k}) \), and let \( \tilde{\mu} \) be the partition obtained from \( \lambda \) by deleting \( \mu \). We have
\[
e_\lambda = e_{\tilde{\lambda}} e_\mu.
\]
Rewriting \( e_\mu \) as a linear span of elements of \( B \), we obtain
\[
e_\lambda = \sum_{\nu \in \pi_{n,d}} e_{\lambda \nu} e_\nu \quad (e_{\lambda \nu} \in \mathbb{C}).
\]
If non-admissible \( \nu \) appears in the right hand side, then we can repeat the same procedure for \( \nu \). Since \( \pi_{n,d} \) is a finite set, this process terminates after a finite number of steps, giving \( e_\lambda \) as a linear span of \( B \). \( \square \)
Let us return to the proof of Theorem 2.4.

From the definition of $I_n^{(k,r)}$, it is clear that

$$\#C_n^{(k,r)} = \dim_K I_n^{(k,r)}.$$  \hfill (3.9)

The inclusion (3.1) implies

$$\dim_K I_n^{(k,r)} \leq \dim_K J_n^{(k,r)}.$$  \hfill (3.10)

Since the defining relations (3.5) are obtained by specializing (2.12), we have

$$\dim_K J_n^{(k,r)} \leq \dim_C J_n^{(k,r)}. $$  \hfill (3.11)

From Proposition 3.2 we find

$$\dim_C J_n^{(k,r)} = \dim_C (R/J_n) \leq \#C_n^{(k,r)}.$$  \hfill (3.12)

Combining (3.9)–(3.12) we conclude that the equality takes place in (3.10). Proof of Theorem 2.4 is now complete.

**Corollary 3.3.** The set $B$ is a basis of $R/J$. We have

$$\dim_K J_n^{(k,r)} = \dim_C J_n^{(k,r)} = \#C_n^{(k,r)}.$$  


4. **Monomial basis**

In [P], a monomial base of the form (3.7) was constructed for arbitrary irreducible integrable representations of $\hat{\mathfrak{sl}}_r$. As we mentioned in Introduction, the space $R/J$ is a principal subspace of an integrable representation. We have given, for this special case, an alternative proof of the result of [P] and its deformation at the same time.

In this Section we present a counterpart of the monomial basis for general integrable representations of $\hat{\mathfrak{sl}}_r$.

Let $R = K[[e_i]]_{i \geq 0}$. For an array $(b_0, \ldots, b_{r-2})$ of non-negative integers satisfying $0 \leq b_0 \leq \cdots \leq b_{r-2} \leq k$, introduce the quotient space

$$W_{b_0, b_1, \ldots, b_{r-2}} = R/I_{b_0, b_1, \ldots, b_{r-2}},$$

where $I_{b_0, b_1, \ldots, b_{r-2}}$ denotes the ideal of $R$ generated by

the Fourier coefficients of $e(z)e(tq^{a_1}z) \cdots e(tq^{a_r}z)$ \quad ($s_i \geq 0$, $\sum_{i=1}^{k} s_i \leq r - 1$) \hfill (4.1)

$$e_0^{a_0}e_1^{a_1} \cdots e_{r-2}^{a_{r-2}}, \quad a_0 + \cdots + a_i > b_i \text{ for some } 0 \leq i \leq r - 2.$$

(4.2)

The space $W_{b_0, b_1, \ldots, b_{r-2}}$ is bi-graded. We set

$$\lambda_{b_0, b_1, \ldots, b_{r-2}}(v, z) = \sum_{n,d} \dim_{n,d}(W_{b_0, b_1, \ldots, b_{r-2}}(v, z)) v^d z^n. $$

In order to parameterize the monomial basis in general, we find it convenient to represent a partition $\lambda$ by the numbers $a_i$ of parts $i$ of $\lambda$, where we set $a_i = 0$ for $i > \lambda_1$. We have a sequence of non-negative integers $a = (a_i)_{i=0}^{\infty}$ with $a_i = 0$ for $i$ large
Define its character by
\[
\chi^E_{b_0,b_1,\ldots,b_{r-2}}(v, z) = \sum_{a \in \mathcal{C}_{b_0,b_1,\ldots,b_{r-2}}} v^{\sum_{i=0}^{\infty} i a_i} z^{\sum_{i=0}^{\infty} a_i}.
\]

Our aim is to show the following.

**Proposition 4.1.** The space \( W_{b_0,b_1,\ldots,b_{r-2}} \) has a monomial basis
\[
B_{b_0,b_1,\ldots,b_{r-2}} = \left\{ \prod_{i=0}^{\infty} e_i^{a_i} \mid a = (a_i)_{i=0}^{\infty} \in \mathcal{C}_{b_0,b_1,\ldots,b_{r-2}} \right\}. \tag{4.3}
\]

From Corollary 3.3 we know that \( W_{k,k,\ldots,k} \) has the set \{3.7\} as monomial basis. Therefore the space \( W_{b_0,b_1,\ldots,b_{r-2}} \) is spanned by the monomials \{4.3\}, and we have
\[
\chi^W_{b_0,b_1,\ldots,b_{r-2}}(v, z) \leq \chi^E_{b_0,b_1,\ldots,b_{r-2}}(v, z). \tag{4.4}
\]

Here and in what follows, for two formal series \( f = \sum_{n,d} f_{n,d}q^d z^n \) and \( g = \sum_{n,d} g_{n,d}q^d z^n \), we write \( f \leq g \) to mean \( f_{n,d} \leq g_{n,d} \) for all \( n, d \).

The following Lemma is immediate.

**Lemma 4.2.** We have the recursion relation
\[
\chi^E_{b_0,b_1,\ldots,b_{r-2}}(v, z) = \chi^E_{b_0-1,b_1,\ldots,b_{r-2}}(v, z) + z^{b_0} \chi^E_{b_0,b_1,\ldots,b_{r-2}-b_0,k-b_0}(v, vz). \tag{4.5}
\]

**Lemma 4.3.** We have an exact sequence
\[
W_{b_1-b_0,\ldots,b_{r-2}-b_0,k-b_0} \xrightarrow{\varphi} W_{b_0,b_1,\ldots,b_{r-2}} \xrightarrow{\pi} W_{b_0-1,b_1,\ldots,b_{r-2}} \rightarrow 0.
\]

Here \( \pi \) is the canonical surjection and \( \varphi(u) = e_0^{b_0} T(u) \), where \( T : R \rightarrow R \) is a homomorphism of algebras defined by \( T(e_i) = e_{i+1} \).

**Proof.** Let us show that \( \varphi \) is well defined. Abusing the notation we use the same letter for the map \( \varphi : R \rightarrow R \) defined by the formula above. Noting that \( T(e(z)) = z^{-1}(e(z) - e_0) \), we have
\[
\varphi\left( \prod_{i=0}^{k} e(tq^i z) \right) = z^{-k-1} e_0^{b_0} \prod_{i=0}^{k} (e(tq^i z) - e_0) \equiv z^{-k-1} e_0^{b_0} \prod_{i=0}^{k} (tq^i z) \equiv 0 \mod I_{b_0,\ldots,b_{r-2}}.
\]
Similarly, if $a_0 + \cdots + a_i > b_i - b_0$ for some $i$ (where we set $b_{r-1} = k$), then

$$
\varphi\left(\prod_{i=0}^{r-2} e_i^{a_i}\right) = e_0^{b_0} \left(\prod_{i=1}^{r-1} e_i^{a_i-1}\right) \\
\equiv 0 \mod I_{b_0, \ldots, b_{r-2}}.
$$

This shows that \( \varphi(I_{b_1, b_2, \ldots, b_{r-2}, b_0, k-b_0}) \subset I_{b_0, \ldots, b_{r-2}}. \)

Clearly \( I_{b_0-1, \ldots, b_{r-2}}/I_{b_0, \ldots, b_{r-2}} \) is spanned by \( e_0^{b_0} \prod_{i \geq 1} e_i^{a_i} \) with \( b_0 + a_1 + \cdots + a_i \leq b_i \) for \( 1 \leq i \leq r-1 \). The exactness follows from this. \( \square \)

Proposition 4.1 is a consequence of the following.

**Proposition 4.4.** We have

$$
\chi_{b_0, b_1, \ldots, b_{r-2}}^W(v, z) = \chi_{b_0, b_1, \ldots, b_{r-2}}^c(v, z).
$$

**Proof.** Lemma 4.3 implies that

$$
\chi_{b_0, b_1, \ldots, b_{r-2}}^W(v, z) \\
\leq \chi_{b_0-1, b_1, \ldots, b_{r-2}}^W(v, z) + z^{b_0} \chi_{b_0-1, b_1, \ldots, b_{r-2}, b_0}^W(v, vz).
$$

Taking \( b_0 = l, b_1 = \cdots = b_{r-2} = k \) we find for \( 0 < l \leq k \) that

$$
\chi_{l, k, \ldots, k}^W(v, z) \\
\leq \chi_{l-1, k, \ldots, k}^W(v, z) + z^l \chi_{l-1, k, \ldots, k}^c(v, vz) \\
\leq \chi_{l-1, k, \ldots, k}^c(v, z) + z^l \chi_{l-1, k, \ldots, k}^c(v, vz) \\
= \chi_{l, k, \ldots, k}^c(v, z).
$$

In the last line we used (4.5). From Corollary 3.3 we have

$$
\chi_{k, k, \ldots, k}^W(v, z) = \chi_{k, k, \ldots, k}^c(v, z).
$$

Using (4.6) as a base of induction on \( l = k, k-1, \ldots, 0 \), we obtain

$$
\chi_{l, k, \ldots, k}^W(v, z) = \chi_{l, k, \ldots, k}^c(v, z), \\
\chi_{k-l, \ldots, k-l}^W(v, z) = \chi_{k-l, \ldots, k-l}^c(v, z).
$$

Arguing similarly, we find by induction that

$$
\chi_{b_0, b_1, b_2, \ldots, b_{r-2}}^W(v, z) = \chi_{b_0, b_1, b_2, \ldots, b_{r-2}}^c(v, z)
$$

for all \( 1 \leq s \leq r-2 \) and \( 0 < b \leq b_s \leq \cdots \leq b_{r-2} \leq k. \) \( \square \)

5. **Discussions**

In this Section we discuss a possible connection between this paper and the representations of the \( \mathcal{W}_k \) algebra associated with \( \widehat{\mathfrak{sl}}_k. \)

Recall the following well-known phenomenon in representation theory of \( \widehat{\mathfrak{sl}}_2 \). Let \( L \) be an irreducible integrable representation of \( \widehat{\mathfrak{sl}}_2 \) of level \( k \). Let \( \mathfrak{h} \) be the Heisenberg subalgebra of \( \widehat{\mathfrak{sl}}_2. \) Then we have a decomposition \( L = \bigoplus_{\alpha \in \mathbb{Z}} \pi_\alpha \otimes S_\alpha, \) where \( \pi_\alpha \) are irreducible representations of \( \mathfrak{h} \) and \( S_\alpha \) are irreducible representations of the \( \mathcal{W}_k \) algebra in the minimal series \( (k+1, k+2). \) In other words, we have on \( L \) an action of \( \mathcal{W}_k \)
commuting with $\hat{\mathfrak{h}}$. Therefore, in some sense $\hat{\mathfrak{sl}}_2$ on level $k$ is an “extension” of the product $\mathcal{W}_k \times \mathfrak{h}$.

We suggest a possible generalization of this construction. Consider a $\mathcal{W}_k$ minimal series representation labeled by relatively prime integers $(p, q)$. We will write them in the form $(p, q) = (k + s, k + r)$. There is a special set of primary fields $\varphi_0(z) = \text{id}, \varphi_1(z), \ldots, \varphi_{k-1}(z)$ with the operator product expansion
\[
\varphi_\alpha(z)\varphi_\beta(w) = (z - w)^{\Delta_{\alpha\beta}} \varphi_\gamma(w) + \ldots.
\]
Here $\gamma \equiv \alpha + \beta \mod k$, $\Delta_{\alpha\beta}$ are some rational numbers, and the higher terms denoted by the dots involve the descendants of $\varphi_\gamma(w)$. The operators $\{\varphi_\alpha(z)\}$ constitute a generalization of the parafermion algebra $[\mathcal{W}]$. Now consider a one-component Heisenberg algebra $A$ which commutes with $\mathcal{W}_k$. Let $V_+(z), V_-(z)$ be vertex operators for $\mathfrak{h}$ with the properties
\[
V_+(z)V_\pm(w) = (z - w)^{-\Delta_{11}} :V_\pm(z)V_\pm(w):,
\]
\[
V_\pm(z)V_\mp(w) = (z - w)^{\Delta_{11}} :V_\pm(z)V_\mp(w):.
\]

Set
\[
e(z) = V_+(z)\varphi_1(z),
\]
\[
f(z) = V_-(z)\varphi_{k-1}(z).
\]

It is easy to see that $[e(z), e(w)] = 0$, $[f(z), f(w)] = 0$, and that $[e(z), f(w)] = 0$ for $z \neq w$.

Let $A_{k+s,k+r}^k$ be the vertex operator algebra generated by $e(z), f(z)$. We view $s, r$ as parameters of the algebra and $k$ as the level of the representation. For instance, the algebra $A_{k+1,k+2}^k$ is $\mathfrak{sl}_2$ acting on integrable representations of level $k$. Not much is known about this algebra in general.

When $k$ is a non-negative integer, we can impose some additional integrability conditions. The integrability conditions can be reformulated as a statement about the matrix elements of the current $e(z)$.

For example, consider a level $k$ integrable representation $L$ of $\hat{\mathfrak{sl}}_2$. The matrix elements of $e(z)$,
\[
P(z_1, \ldots, z_n) = \langle v^\vee, e(z_1) \cdots e(z_n)v \rangle \quad (v \in L, v^\vee \in L^*)
\]
are symmetric Laurent polynomials in $(z_1, \ldots, z_n)$ satisfying
\[
P = 0 \text{ if } z_1 = \cdots = z_{k+1}.
\]

The integrable irreducible representations can be realized in a space of symmetric Laurent polynomials in infinite set of variables satisfying the zero condition (5.2) $[\mathcal{FS}]$. Therefore, for $\hat{\mathfrak{sl}}_2$, we can start from the relation $e(z)^{k+1} = 0$ and reconstruct the level $k$ vacuum representation. It is possible further to find the structure of a vertex operator algebra on it.

For general $A_{k+s,k+r}^k$, the matrix elements are also symmetric Laurent polynomials with some zero conditions on the diagonal of codimension $k + 1$. As in the case of $\hat{\mathfrak{sl}}_2$, one expects that these zero conditions determine the integrable representations, and even the algebra $A_{k+s,k+r}^k$ itself. When we try to establish these facts, the first obstacle
is that the integrability conditions are not known. In the special case $A_{k+1,k+r}$, we have presented in [FJMM] a conjectural description of the integrability condition in the language of matrix elements. More precisely, we considered certain subspaces (in fact, ideals) of symmetric Laurent polynomials which are spanned by Jack polynomials. We expect that they coincide with the space of all matrix elements of the current $e(z)$ in integrable representations of $A_{k+1,k+r}$.

In this paper, we found some evidence that all this can and should be ‘$q$-deformed’. An obvious counterpart of $\mathcal{W}_k$ would be its elliptic deformation. We conjecture that the algebra $A_{k+s,k+r}$ can be $q$-deformed, in such a way that the currents $e(z), f(z)$ remain commutative: $[e(z), e(w)] = 0$, $[f(z), f(w)] = 0$. In the undeformed case, the matrix elements of $e(z)$ and $f(z)$ have the structure

$$\langle w^\vee, e(z_1) f(z_2) w \rangle = F(z_1, z_2)(z_1 - z_2)^{-m},$$

where $F$ is a Laurent polynomial. If $s = 1$, then $m = r$. After deformation, we expect that they take the form $F(z_1, z_2) \prod_{i=1}^m (z_1 - q_i z_2)^{-1}$ with some $q_1, \ldots, q_m$.

In this paper we have found an explicit description of the integrability condition for the current $e(z)$ itself in the special case $s = 1$. The analogue description is only implicit in the undeformed case, see [FJMM]. This is one of the advantages of considering the $q$-deformation.

Note that for a special value of the parameters $(q, t) = (\tau, 1)$, the algebra $A_{k+1,k+r}$ is known - somewhat surprisingly, it is $\hat{\mathfrak{sl}}_r$ at level $k$.

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