Reassessing nuclear matter incompressibility and its density dependence

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Experimental giant monopole resonance energies are now known to constrain nuclear incompressibility of symmetric nuclear matter \( K \) and its density slope \( M \) at a particular value of sub-saturation density, the crossing density \( \rho_c \). Consistent with these constraints, we propose a reasonable way to construct a plausible equation of state of symmetric nuclear matter in a broad density region around the saturation density \( \rho_0 \). Help of two additional empirical inputs, the value of \( \rho_0 \) and that of the energy per nucleon \( e(\rho_0) \) are needed. The value of \( K(\rho_0) \) comes out to be 211.9 ± 24.5 MeV.

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I. INTRODUCTION

The nuclear incompressibility parameter \( K_0 \) defined for symmetric nuclear matter (SNM) at saturation density \( \rho_0 \) stands out as an irreducible element of physical reality. It has an umbilical association with the isoscalar giant monopole resonances (ISGMR) in microscopic nucleli; it also underlies in a proper understanding of supernova explosion in the cosmic domain [1]. From careful microscopic analysis of ISGMR energies with suitably constructed energy density functional (EDF) \( E(\rho) \) in a non relativistic framework as applicable to finite and infinite nuclear systems, its value had initially been fixed at \( K_0 \approx 210 \pm 30 \) MeV [2,3]. In microscopic relativistic approaches on the other hand, a higher value of \( K_0 \approx 260 \) MeV was obtained [4]. After several revisions from different corners, however, its value settled to \( K_0 \approx 230 \pm 20 \) MeV [3,5]. It gives good agreement with the experimentally determined centroids of ISGMR, in particular, for \(^{208}\)Pb, \(^{90}\)Zr and \(^{144}\)Sn nuclei, calculated both with non-relativistic [6] and relativistic [7] energy density functionals. The near-settled problem was, however, left open with the apparent incompatibility of the said value of \( K_0 \) with the recent ISGMR data for Sn and Cd-isotopes [8,9,10,11]. These nuclei showed remarkable softness towards compression, the ISGMR data appeared explained best with \( K_0 \approx 200 \) MeV [8].

A plausible explanation was recently put forward by Khan et al. [18] for the apparent discrepancy. It is argued that there may not be an unique relation between the value of \( K_0 \) associated with an effective force and the monopole energy of a nucleus predicted by the force [19]. The region between the center and the surface of the nucleus is the most sensitive towards displaying the compression as manifested in the ISGMR. The ISGMR centroid \( E_G \) is related to the integral of incompressibility \( \left( \int K(\rho) d\rho \right) \) over the whole density range [20]. As a result, a larger value of \( K(\rho_0) \) for a given EDF can be compensated by lower values of \( K(\rho) \) at sub-saturation densities so as to predict a similar value of ISGMR energy in nuclei. It is seen that the incompressibility \( K(\rho) \) calculated with a multitude of energy density functionals when plotted against density cross close to a single density point [18], this universality possibly arising from the constraints encoded in the EDF from empirical nuclear observables. This crossing density \( \rho_c = (0.71 \pm 0.005)\rho_0 \) [21] seems more relevant as an indicator for the ISGMR centroid. Because of the incompressibility integral, the centroid seems more intimately correlated to the derivative of the compression modulus (defined as \( M = 3pK'(\rho) \) ) at the crossing density rather than to \( K_0 \). The value of \( K_c (= K(\rho_c)) \) is seen to be \( \simeq 35 \pm 4 \) MeV [21]. From various functionals, the calculated values of \( M_c (= M(\rho_c)) \) are found to be linearly correlated with the correspondingly calculated values of ISGMR centroids for \(^{208}\)Pb and also for \(^{120}\)Sn. From the known experimental ISGMR data for these nuclei, a value of \( M_c \simeq 1050 \pm 100 \) MeV [21] is then obtained, revised from an earlier estimate of \( 1100 \pm 70 \) MeV [18]. Using a further assumption of a linear correlation between \( K_0 \) and \( E_G \) calculated from different EDF, a value for \( K_0 \simeq 230 \) MeV with an uncertainty of \( \simeq 40 \) MeV is reported, the uncertainty being inferred from the spread of \( K_0 \) values obtained with the different functionals used.

The universality of the crossing point \( \rho_c \) and the values of \( K_0 \) and \( M_c \) can be readily acknowledged; \( M_c \) is seen to be well correlated to \( E_G \). The Pearson correlation coefficient \( r \) of \( M_c \) with \( E_G \) for \(^{120}\)Sn is 0.80 and is 0.94 for \(^{208}\)Pb. However, assumption of a linear correlation between \( K_0 \) and \( E_G \) may not be justified, they seem to be very weakly correlated (\( r = 0.67 \) for \(^{120}\)Sn and 0.79 for \(^{208}\)Pb) [21]. The inferred value of incompressibility around saturation may then be called into question. One can see that a linear Taylor expansion \( K_0(\rho_0) = K(\rho_0) + (\rho_0 - \rho_c)K'(\rho_c) \) yields for \( K_0 \simeq 185 \pm 14.3 \) MeV, noting that \( K'(\rho_c) = M_c/(3\rho_c) \).

The absence of a strong linear correlation between \( K_0 \) and \( E_G \) calculated from different effective forces prompts one to think that \( K_c \) and \( M_c \) alone are not sufficient to yield the correct value of \( K_0 \). Further empirical information is possibly needed to arrive at that. In this paper, we show that with given values of only \( K_c \) and \( M_c \) along with some time-tested values of empirical nuclear constants, it is possible to address to a proper assessment of the value of incompressibility \( K \) and its density depen-
dence. The empirical constants are the saturation density \( \rho_0 \), taken as \( 0.155 \pm 0.008 \) fm\(^{-3} \) for SNM and the energy per nucleon at that density \( e(\rho_0) \), taken as \(-16.0\pm0.1\) MeV \(^{25,26}\). An acceptable value of the effective nucleon mass \( m^*/m \), which lies in the range \( m^*/m \sim 0.8 \pm 0.2 \) \(^{27}\) at saturation density is also used.

This paper is structured as follows. In Sec. II, we introduce the theoretical elements to calculate the nuclear equation of state from \( K_c \) and \( p_c \) with the aid of empirical inputs mentioned. Results and discussions are presented in Sec. III. Sec. IV contains the concluding remarks.

II. THEORETICAL EDIFICE

We keep the discussions pertinent for SNM at any density \( \rho \) at zero temperature \( (T = 0) \). The chemical potential of a nucleon is given by the single-particle energy at the Fermi surface,

\[
\mu = \varepsilon_F = \frac{p_F^2}{2m} + U
\]

where \( p_F(\rho) \) is the Fermi momentum and \( U(\rho) \) the single-particle potential. Assuming the nucleonic interaction to be momentum and density dependent, the single-particle potential separates into three parts \(^{22}\)

\[
U = V_0 + p_F^2 V_1 + V_2.
\]

The last term \( V_2 \) is the rearrangement potential that arises only for density-dependent interactions, and the second is the momentum-dependent term that defines the effective mass \( m^* \),

\[
\frac{p_F^2}{2m^*} = \frac{p_F^2}{2m} + p_F^2 V_1
\]

so that

\[
\frac{1}{m^*} = \frac{1}{m} + 2V_1.
\]

The energy per nucleon at density \( \rho \) is given by,

\[
e(\rho) = < \frac{p^2}{2m} > + \frac{1}{2} < p^2 > V_1 + \frac{1}{2} V_0
\]

\[
= \frac{1}{2} (1 + \frac{m^*}{m}) < \frac{p^2}{2m^*} > + \frac{1}{2} V_0
\]

From Gibbs-Duhem relation,

\[
\mu = e + \frac{P}{\rho},
\]

where \( P \) is the pressure. Keeping this in mind, from Eqs. (1), (5) and (6), we get

\[
e(\rho) = \frac{p_F^2}{10m} [3 - \frac{2m}{m^*}] - V_2 + \frac{P}{\rho},
\]

where we have put \( < p^2 > = \frac{3}{5} p_F^2 \).

The density dependence of the effective mass \(^{27}\) can be cast as \( \frac{m^*}{m} = 1 + k_1 \rho \), the rearrangement potential can be written in the form \( V_2 = a \rho^\alpha \). This is the form that emerges for finite range density-dependent forces \(^{26}\) in a non relativistic framework or for Skyrme interactions. The quantities \( a \), \( \alpha \) and \( k_1 \) are numbers. If \( \frac{m^*}{m}(\rho_0) \) is chosen, \( k_1 \) is known.

At \( \rho = \rho_0 \), \( P = 0 \), then from Eq. (7), writing for \( \frac{p_F^2}{2m} = b \rho^{2/3} \) with \( b = \left( \frac{2\pi^2}{2m} \right)^{2/3} \),

\[
e_0 = e(\rho_0) = \frac{b}{5} \rho_0^{2/3} [1 - 2k_1 \rho_0] - a \rho_0^\alpha.
\]

Since \( P = \rho^2 \frac{\partial e}{\partial \rho} \), from Eq. (7) again we get,

\[
P = \frac{b}{15} \rho_0^{5/3} - \frac{1}{3} b k_1 \rho_0^{8/3} - \frac{1}{2} a \rho_0^{\alpha+1} + \frac{1}{2} \rho \frac{\partial P}{\partial \rho}.
\]

At \( \rho_0 \), this yields (since \( K_0 = \frac{9b^2}{\alpha \rho_0} \)),

\[
\frac{1}{2} a \rho_0^{\alpha} + \frac{1}{3} b k_1 \rho_0^{5/3} - \left( \frac{K_0}{18} + \frac{b}{15} \right) \rho_0^{2/3} = 0.
\]

Furthermore, Eq. (9) gives

\[
K(\rho) = \frac{\partial P}{\partial \rho} = 2b \rho^{2/3} - 16bk_1 \rho^{5/3} - 9a(\alpha + 1) \rho^\alpha + 9\rho \frac{\partial^2 P}{\partial \rho^2}.
\]

Defining \( M = 3\rho \frac{4k_1}{\alpha \rho_0} = 27 \rho \frac{2}{2}\rho \), this leads, at \( \rho = \rho_c \) to

\[
9a(\alpha + 1) \rho_c^\alpha + 16bk_1 \rho_c^{5/3} - (2b \rho_c^{2/3} + \frac{M_c}{3} - K_c) = 0.
\]

Since \( k_1 \) is a given entity and \( \rho_c \) and \( (M_c/3 - K_c) \) are known, eqs. (8) and (12) can be solved for \( \alpha \) and \( \rho_c \), eq. (10) then gives the value of the nuclear incompressibility \( K_0 \). Once \( K_0 \) is obtained, \( M_0(= M(\rho_0)) \) is evaluated from eq. (12) by choosing \( \rho_0 \) for \( \rho_c \). Then \( Q_0 = 27 \rho_0 \frac{\partial^3 e}{\partial \rho^3} \) is also known from \( M_0 = 12K_0 + Q_0 \).

The structure of eq. (9) shows that the pressure and its first derivative are interrelated. One can then get higher density derivatives of \( P \) or of energy \( e \) recursively from eq. (9) as is evident from eq. (11). For the present, we show that

\[
9\rho \frac{\partial^3 P}{\partial \rho^3} = 9a^2(\alpha + 1) \rho^{\alpha-1} + \frac{80}{3} b k_1 \rho^{2/3} - \frac{4}{3} b \rho^{-1/3}.
\]

Since

\[
\frac{\partial^3 P}{\partial \rho^3} = 6 \frac{\partial^2 e}{\partial \rho^2} + 6 \frac{\partial^3 e}{\partial \rho^3} + \rho^2 \frac{\partial^4 e}{\partial \rho^4},
\]

we find

\[
9\rho_0 \frac{\partial^3 P}{\partial \rho^3} |_{\rho_0} = 6K_0 + 2Q_0 + \frac{1}{9} N_0.
\]
where we have defined $N_0 = 81 \rho_0^4 \partial_4^m |_{\rho_0}$. From eq. (13) and (15), knowing $K_0$ and $Q_0$, $N_0$ can be calculated. Similarly, one can calculate the fifth density derivative of energy ($R_0 = 243 \rho_0^5 \partial_5^m |_{\rho_0}$) by exploiting eqs. (13) and (14) from

$$9 \rho_0^5 \partial_5^4 |_{\rho_0} = 4Q_0 + \frac{8}{9}N_0 + \frac{1}{27}R_0.$$  

(16)

These help to find the density variation of the energy and also of the incompressibility, as is seen,

$$e(\rho) = e(\rho_0) + \frac{1}{2}K_0 \epsilon^2 + \frac{1}{6}Q_0 \epsilon^3$$

$$+ \frac{1}{24}N_0 \epsilon^4 + \frac{1}{120}R_0 \epsilon^5 + \ldots ,$$  

(17)

where $\epsilon = (\rho/\rho_0)$ (counting terms only up to $\epsilon^5$ is seen to be a very good approximation in the density range of $\sim \rho_0/4 < \rho < 2.0\rho_0$, we retain terms up to them). Eqs. (7) and (17) give

$$\frac{P(\rho)}{\rho} = e(\rho_0) + \frac{1}{2}K_0 \epsilon^2 + \frac{1}{6}Q_0 \epsilon^3 + \frac{1}{24}N_0 \epsilon^4$$

$$+ \frac{1}{120}R_0 \epsilon^5 - \frac{b}{5} \rho^{2/3} [1 - 2k \rho] + a \rho^2 .$$  

(18)

and eq. (9) gives

$$K(\rho) = \frac{9 dP}{d\rho} = 18 \left[ \frac{P}{\rho} - \frac{b}{15} \rho^{2/3} + \frac{1}{3} \rho \epsilon + \frac{1}{2} \alpha \rho^2 \right] .$$  

(19)

We have thus the equation of state (EOS) of symmetric nuclear matter in a reasonably spread-out density domain around the saturation density.

The incompressibility $K$ at any density $\rho$ can be calculated directly from eq. (19) or it may be calculated in terms of $K(\rho_c)$ and its higher density derivatives as

$$K(\rho) = K(\rho_c) + (\rho - \rho_c)K'(\rho_c) + \frac{(\rho - \rho_c)^2}{2}K''(\rho_c)$$

$$+ \frac{(\rho - \rho_c)^3}{6}K'''(\rho_c) + \ldots .$$  

(20)

The different derivatives can be calculated from eq. (19). With given values of $\rho_0, e_0, m^*(\rho_0)$, and $\rho_c$, one notes that the solutions for $\alpha$ and $\alpha$ do not depend separately on $K_c$ and $M_c$, but on $(M_c/3 - K_c)$.

III. RESULTS AND DISCUSSIONS

The values of the empirical constants $\rho_0, e_0$ and $m^*$ needed for our calculation have already been mentioned. As for the crossing density, we choose $\rho_c = 0.110 \pm 0.0008$ fm$^{-3}$. With given inputs of $M_c$ and $K_c$, it should be noted that the output values for $M_c$ and $K_c$ may come out to be different, but $(M_c/3 - K_c)$ remains invariant.

With inputs $M_c=1050$ MeV and $K_c=35$ MeV, the output $M_c$ and $K_c$ are found to be $1051.8$ MeV and $35.46$ MeV, respectively. Since they are very close to the input values, they were not tinkered with for exact matching of the output and input values. The value of incompressibility at $\rho_0$ turns out to be $K_0=211.9 \pm 24.5$ MeV either from eq. (19) or eq. (20). We note that in eq. (20), at saturation, the value of the second term on the right hand side is $143.3$ MeV, the third term is $35.9$ MeV, the fourth term is $-3.2$ MeV, the fifth term (not shown in eq. (20)) is $0.55$ MeV and so on, which adds up to $\sim 211.9$ MeV.

The uncertainty in an observable $X$ (like $K, M$ etc) is calculated from $\Delta X = \sum_i (\frac{\partial X}{\partial y_i})^2 \Delta y_i$ where $\Delta y_i$ are the uncertainties in the empirically known entities $y_i$. The sensitivity of $K_0$ on these entities that influence the incompressibility most is displayed in Fig 1. The abscissa is scaled such that 0 refers to the central value of these entities $M_c, K_c, \rho_c$ and $\rho_0$; $\pm 1$ refer to the extrema of their domain ($\pm 100$ MeV, $\pm 5$ MeV, $\pm 0.005 \rho_0$ and $\pm 0.008$ fm$^{-3}$ from the central values of the entities, respectively). The value of $K_0$ is seen to be very sensitive with changes in either $M_c$ or $\rho_0$ when all other input entities are kept fixed. Its sensitivity to $K_c$ or $\rho_c$ is weak; on $m^*$ or to the energy per nucleon $e_0$, it is rather insensitive. The near-insensitivity of incompressibility to the effective mass is observed for Skyrme density functionals also. From the data base for these functionals as tabulated by Dutra et al [23], the correlation coefficient between $K_0$ and $m^*$ is calculated to be only $\sim -0.2.$
The near-perfect linear correlation of $K_0$ with $M_c$ as seen in Fig. 1 is very startling. From Eq. (20), one may expect that the second and higher order derivatives of $K(\rho_c)$ would destroy this correlation. However, we find that both $K''$ and $K'''$ are also linearly correlated with $M_c$ and thus $K(\rho_0)$ retains its linear correlation with $M_c$. This is displayed in Fig. 2 where we define $K_1 = (\rho_0 - \rho_c)K'(\rho_c)$, $K_2 = (\rho_0 - \rho_c)^2K''(\rho_c)$ and $K_3 = (\rho_0 - \rho_c)^3K'''(\rho_c)$. The weak correlation between $K_0$ and $M_c$ that can be inferred from the calculated correlation structure of $(M_c - E_G)$ and $(K_0 - E_G)$ in refs. [18, 21] possibly results from the use of different EDFs in getting the various relevant observables.

Figures 3 and 4 display the functional dependence of the nuclear EOS on density. The panels (a) and (b) in Fig. 3 show the energy per nucleon and the pressure, respectively in a selected range around the saturation density. As one sees, the uncertainty in energy and pressure grows as one moves away from the saturation density, similarly the uncertainty in incompressibility or its density derivative increases with distance from the crossing density.

IV. CONCLUSIONS

To sum up, we have made a modest attempt to re-assess the value of $K(\rho_0)$ consistent with the new-found constraint on the incompressibility $K(\rho_c)$ and its density slope $M(\rho_c)$ at a particular value of density at sub-saturation, the crossing density $\rho_c$. We have relied on some empirically well-known values of nuclear constants. We have further made the assumption of linear density dependence of the effective mass and the power law dependence of the rearrangement potential which happens to be generally true for non-relativistic momentum and density dependent interactions. In relativistic models, the density dependence of the effective mass may not be linear [28]. The rearrangement potential appears explicitly there only in the case of density dependent meson
The value of incompressibility $K(\rho_0)$ turns out to be $211.9 \pm 24.5$ MeV. This is somewhat lower than the current value in vogue, $K_0 \sim 230 \pm 20$ MeV. From recursive relations, our method allows also estimates of higher density derivatives of energy or of pressure and thus helps in constructing the nuclear EoS $e(\rho)$ at and around the saturation density.

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