A SMALLER COUNTEREXAMPLE TO THE LANDO CONJECTURE

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Abstract. The following conjecture was proposed in 2010 by S. Lando.

Let $M$ and $N$ be two unions of the same number of disjoint circles in a sphere. Then there exist two spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as $M$ in one sphere and as $N$ in the other. Define union $M$ of disjoint circles to be situated in one sphere as union $M_1$ of disjoint circles in the other sphere if there is a homeomorphism between these two spheres which maps $M$ to $M_1$.

In this paper we prove that there exists pair of sets of 7 circles in sphere, that is a counterexample to the Lando conjecture. This is proved using the Avvakumov Theorem. We conjecture that there exists no pair $(M,N)$ that is counterexample and $M$ contains 6 or less circles.

Definitions.

Let $p$ and $q$ be two sets of edges of a tree $Y$.

The set $p$ is on the same side of $q$ (in this tree $Y$) if $p \cap q = \emptyset$ and for each two vertices of edges of $p$ there is a path in the tree connecting these two vertices, and containing an even number of edges of $q$. Sets $p$ and $q$ are unlinked (in this tree) if $p$ is on the same side of $q$ and $q$ is on the same side of $p$.

For vertex $P$ of graph we denote as $\delta P$ all edges whose end is $P$.

Let $K$ and $K'$ be two trees with the same number of edges. Let $h$ be a bijection (i.e. one-to-one correspondence) between their edges.

Then $h$ is called realizable if $h(\delta A)$ and $h(\delta B)$ are unlinked for each two vertices $A$ and $B$ in $K$ such that the path joining $A$ and $B$ contains even number of edges.

Graphs $K$ and $K'$ are friendly if such a bijection exists.

Figure 1: Graphs $H$ and $G$.

Let graph $G$ be a graph that has vertices $A, C_i, A', C'_i$ and edges $C_3 C'_3, AC_i, A'C'_i, i = 1, 2, 3$. Let graph $H$ be a graph that has vertices $B, D, P_i, Q_i$ and edges $BD, BP_i, P_i Q_i, i = 1, 2, 3$.

1 This paper is prepared under the supervision of Arkadiy Skopenkov and is submitted to the Moscow Mathematical Conference for High-School Students. Readers are invited to send their remarks and reports on this paper to mmks@mccme.ru
Theorem 1. Graphs $G$ and $H$ are unfriendly.

Let state a result that shows why this theorem is interesting. Suppose that $M$ is a union of disjoint circles in sphere $S^2$. Define (‘dual to $M$’) graph $G = G(S^2; M)$ as follows. The vertices are the connected components of $S^2 \setminus M$. Two vertices are connected by an edge if the corresponding connected components are neighbors.

Aavakumov Theorem. [A] Let $M$ and $N$ be two unions of the same number of disjoint circles in a sphere $S^2$. Then there exist two spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as $M$ in one sphere and as $N$ in the other if and only if the graph dual to $M$ and $N$ are friendly.

This theorem implies that friendliness is symmetric. This will be used in the proof.

Suppose $\phi$ is a realizable bijection between edges of $G$ and $H$. For edges $e_1,e_2,e_3,\ldots,e_n$ of graph $G$ by $h(e_1,e_2,e_3,\ldots,e_n)$ we denote subgraph formed by $\phi(e_1),\phi(e_2),\phi(e_3),\ldots,\phi(e_n)$ in graph $H$. And for edges $e_1,e_2,e_3,\ldots,e_n$ of graph $H$ by $g(e_1,e_2,e_3,\ldots,e_n)$ we denote subgraph formed by $\phi(e_1),\phi(e_2),\phi(e_3),\ldots,\phi(e_n)$ in graph $G$.

Proposition 1. Both graphs $H_1 := h(AC_1, AC_2, AC_3)$ and $H_2 := h(A'C'_1, A'C'_2, A'C'_3)$ are connected.

Proof. Let prove the connectedness for $H_1$, and for $H_2$ the proof is analogous.

If $H_1$ is not connected then one of edges from $H \setminus H_1 = h(AC_1', AC_2', AC_3', C_3C_3')$ belongs to path connecting two edges from $H_1$. Vertices $A$ and $C_3'$ are linked by a path of even length.

So $h(A'C_3')$ doesn’t belong to any path that joins a pair of edges of graph $H_1$. Analogically $h(A'C_3')$ doesn’t belong to any path that joins a pair of edges of graph $H_1$. Vertices $A$ and $C_3'$ are linked by a path of even length too. Hence,

1. Case 1. Neither $h(C_3C_3')$ nor $h(A'C_3')$ don’t belong to any path that joins a pair of edges of graph $H_1$;
2. Case 2. $h(C_3C_3')$ and $h(A'C_3')$ belong to path that joins a pair of edges $J_1,J_2$ of graph $H_1$.

In the first case the graph $H_1$ is connected.

In the second case $J_1,h(C_3C_3'),h(A'C_3'),J_2$ form a path of length 4. Without loss of generality this path is $Q_1P_1BP_2Q_2$. Hence path, that links edges $J_1$ and $H_1 - J_1 - J_2$, intersect only one of edges $h(C_3C_3'),h(A'C_3')$. Which is impossible.

QED

Proposition 2. Vertex $B$ is an endpoint of edge $h(C_3C_3')$.

Proof. Vertices $B$ and $Q_i$ are linked by a path of even length. $g(BD, BP_1, BP_2, BP_3) = g(\delta B)$ is unlinked with any edge $g(P_iQ_i)$ in $G$. This implies $g(\delta B)$ is connected. There are only 2 connected subgraphs with 4 edges in $G$ up to automorphism of $G$:

The first subgraph, say $X$, has vertices $C_1, C_2, C_3, A, C_3'$ and its edges are precisely the edges of $G$ with both ends in $X$.

The second subgraph, say $Y$, has vertices $C_1, A, C_3, C_3', A'$ and its edges are precisely the edges of $G$ with both ends in $Y$.

Since $C_3C_3'$ is fixed under Aut$(G)$ and $C_3C_3'$ is contained both in $X$ and $Y$, one of edges $BD, BP_1, BP_2, BP_3$ is $h(C_3C_3')$.

QED

Proof of theorem 1. Suppose graphs $G$ and $H$ are friendly. Then there exists a realizable bijection $\phi$ between edges of $G$ and $H$.

According to proposition 1 graph $H - h(C_3C_3')$ is a union of two connected graphs with 3 edges. According to proposition 2 one of these graphs contains at least two of the edges $P_iQ_i$.
as his edge. Without loss of generality let \( P_1Q_1 \) and \( P_2Q_2 \) be in \( h(AC_1, AC_2, AC_3) = H_1 \). But then the length of the path linking \( Q_1, Q_2 \) is 4. Since there are only 3 edges in \( H_1 \), this is impossible.

QED

References

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