\(\eta\)-invariant and Chern-Simons current

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Abstract

We show that the \(\mathbb{R}/\mathbb{Z}\) part of the analytically defined \(\eta\) invariant of Atiyah-Patodi-Singer for a Dirac operator on an odd dimensional closed spin manifold can be expressed purely geometrically through a stable Chern-Simons current on a higher dimensional sphere. As a preliminary application, we discuss the relation with the Atiyah-Patodi-Singer \(\mathbb{R}/\mathbb{Z}\) index theorem for unitary flat vector bundles, and prove an \(\mathbb{R}\) refinement in the case where the Dirac operator is replaced by the Signature operator. We also extend the above discussion to the case of \(\eta\) invariants associated to Hermitian vector bundles with non-unitary connection, which are constructed by using a trick due to Lott.

§1. Introduction

The \(\eta\) invariant of Atiyah-Patodi-Singer was introduced in [APS1] as the correction term on the boundary of the index theorem for Dirac operators on manifolds with boundary. Since then it has appeared in many parts of geometry, topology as well as physics. We first recall the definition of this \(\eta\) invariant.

Let \(M\) be an odd dimensional closed oriented spin Riemannian manifold. Let \(S(TM)\) be the associated bundle of spinors. Let \(E\) be a Hermitian vector bundle over \(M\) carrying with a Hermitian connection. Then one can define canonically a Dirac operator \(D_E : \Gamma(S(TM) \otimes E) \rightarrow \Gamma(S(TM) \otimes E)\). It is a formally self-adjoint first order elliptic differential operator.

Let \(s \in \mathbb{C}\) with \(\text{Re}(s) > \frac{\dim M}{2}\). Following [APS1], one defines the \(\eta\) function of \(D_E\) by

\[
\eta(D_E, s) = \sum_{\lambda \in \text{Spec}(D_E) \setminus \{0\}} \frac{\text{sgn}(\lambda)}{|\lambda|^s}. \tag{1.1}
\]

It is shown in [APS1] that \(\eta(D_E, s)\) is a holomorphic function for \(\text{Re}(s) > \frac{\dim M}{2}\), and can be extended to a meromorphic function on \(\mathbb{C}\). Moreover, it is holomorphic at \(s = 0\). The value of \(\eta(D_E, s)\) at \(s = 0\) is called the \(\eta\) invariant of \(D_E\) and is denoted by \(\eta(D_E)\). Let \(\tilde{\eta}(D_E)\) be the reduced \(\eta\) invariant of \(D_E\) which is also defined in [APS1]:

\[
\tilde{\eta}(D_E) = \frac{\dim(\ker D_E) + \eta(D_E)}{2}. \tag{1.2}
\]

It turns out that this analytically defined invariant may jump by integers as the metrics and connections on \(TM\) and \(E\) change. These jumps can be detected by spectral flows

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introduced in [APS3]. On the other hand, the mod \( \mathbb{Z} \) component of \( \bar{\eta}(D^E) \) is smooth with respect to the involved metrics and connections, and its variation can be expressed through Chern-Simons forms. However, whether \( \bar{\eta}(D^E) \mod \mathbb{Z} \) itself can be expressed geometrically, without passing to the spectral set of \( D^E \), remains a question for long time. Here we only mention that such a formula for \( \bar{\eta}(D^E) \mod \mathbb{Q} \) was proved in [CS, Theorem 9.1], expressing the \( \mathbb{R}/\mathbb{Q} \) component of \( \bar{\eta}(D^E) \) through the Cheeger-Simons differential characters.

The purpose of this short article is to show that there is indeed a purely geometric formula for \( \bar{\eta}(D^E) \mod \mathbb{Z} \). More precisely, if we embed \( M \) into a higher odd dimensional sphere, then \( \bar{\eta}(D^E) \mod \mathbb{Z} \) can be expressed through a Chern-Simons current on the sphere. Comparing with Cheeger-Simons’ mod \( \mathbb{Q} \) one, such a formula is more of \( K \)-theoretic nature, and should be viewed as an index theorem in some geometric \( K \)-theory (compare with [AS] and [L]).

In fact, this formula, which will be stated in its precise form in Theorem 2.3, can be obtained as an immediate application of a localization formula for \( \eta \) invariants proved by Bismut and Zhang in [BZ]. Our simple observation is that if one applies the Bismut-Zhang formula to an embedding into a higher dimensional sphere, then a simple application of the Bott periodicity will lead us to a geometric formula for the \( \mathbb{R}/\mathbb{Z} \) component of \( \bar{\eta}(D^E) \).

Thus in the next section we will recall the Bismut-Zhang localization formula for \( \eta \) invariants and prove the geometric formula for \( \bar{\eta}(D^E) \mod \mathbb{Z} \).

Also recall that the proof given by Bismut-Zhang in [BZ] for their localization formula relies heavily on the difficult paper of Bismut-Lebeau [BL], and might not be easy to follow. So in Section 3 we will give an alternate proof of this localization formula by making use of the Freed-Melrose index theorem for \( \mathbb{Z}/k \) manifolds [FM] instead.

In Section 4, we present some preliminary applications of our formula to the case of flat vector bundles. In particular, we show that our formula leads to an intrinsic formulation of the Atiyah-Patodi-Singer \( \mathbb{R}/\mathbb{Z} \) index theorem for unitary flat vector bundles [APS3, Theorem 5.3]. Moreover, we show that when considering the Signature operator instead of the Dirac operator, one can refine the above index theorem to an \( \mathbb{R} \) valued one.

Now note that it is pointed out in [APS3] that the above mentioned index theorem for flat vector bundles [APS3, Theorem 5.3] indeed holds for all flat connections not necessarily unitary, and that one can extend the definition of the \( \eta \) invariant for these not necessarily unitary connections.

In Section 5 of this paper, we will propose a definition of the \( \eta \) invariant for Dirac operators coupled with Hermitian vector bundles with non-unitary connection, by using a trick of Lott in [L, Section 5]. When applying to flat connections, we show that this invariant verifies the Atiyah-Patodi-Singer index theorem [APS3, Theorem 5.3], which implies that when mod \( \mathbb{Z} \), our construction agrees with what indicated in [APS3]. We will discuss further applications of this invariant elsewhere.

§2. A geometric formula for \( \eta \) invariants

In this section, we recall the localization formula for \( \eta \) invariants of Bismut-Zhang [BZ] and use it to deduce a geometric formula for \( \eta \) invariants.

This section is organized as follows. In a), we recall the direct image construction of
Atiyah-Hirzebruch [AH] under real embeddings in a geometrical form. In b), we recall from [B2] and [BZ] the construction of the Chern-Simons current associated to the geometric direct image constructed in a). In c), we state the localization formula from [BZ]. In d), we apply this localization formula to get a geometric formula expressing the \( \eta \) invariants through Chern-Simons currents on spheres.

a). A geometric construction of direct images

Let \( i : Y \hookrightarrow X \) be an embedding between two smooth oriented manifolds. We make the assumption that \( \dim X - \dim Y \) is even and that if \( N \) denotes the normal bundle to \( Y \) in \( X \), then \( N \) is orientable, spin and carries an induced orientation as well as a (fixed) spin structure.

Let \( \mu \) be a complex vector bundle over \( Y \).

Atiyah and Hirzebruch have constructed in [AH] an element \( i_*\mu \in \widetilde{K}(X) \), called the direct image of \( \mu \) under \( i \). We here recall this construction in a geometric form.

Let \( g^N \) be a Euclidean metric on \( N \) and \( \nabla^N \) a Euclidean connection on \( N \) preserving \( g^N \).

Let \( S(N) \) be the vector bundle of spinors associated to \( (N, g^N) \). Then \( S(N) = S_+(N) \oplus S_-(N) \) (resp. its dual \( S^*(N) = S^*_+(N) \oplus S^*_-(N) \)) is a \( \mathbb{Z}_2 \)-graded complex vector bundle over \( Y \) carrying with an induced Hermitian metric \( g^S(N) = g^{S_+}(N) \oplus g^{S_-}(N) \) (resp. \( g^{S^*}(N) = g^{S^*_+}(N) \oplus g^{S^*_-}(N) \)) from \( g^N \), as well as a Hermitian connection \( \nabla^S(N) = \nabla^{S_+}(N) \oplus \nabla^{S_-}(N) \) (resp. \( \nabla^{S^*}(N) = \nabla^{S^*_+}(N) \oplus \nabla^{S^*_-}(N) \)) induced from \( \nabla^N \).

Let \( g^\mu \) be a Hermitian metric on \( \mu \) and \( \nabla^\mu \) a Hermitian connection on \( \mu \) preserving \( g^\mu \).

For any \( r > 0 \), set \( N_r = \{ Z \in N : |Z| < r \} \). We make the assumption that there is \( \varepsilon_0 > 0 \) such that \( N_{2\varepsilon_0} \) is diffeomorphic to an open neighborhood of \( Y \) in \( X \). Without confusion we now view directly \( N_{2\varepsilon_0} \) as an open neighborhood of \( Y \) in \( X \).

Let \( \pi : N \rightarrow Y \) denote the projection of the normal bundle \( N \) over \( Y \).

If \( Z \in N \), let \( \tilde{c}(Z) \in \text{End}(S^*(N)) \) be the transpose of \( c(Z) \) acting on \( S(N) \). Let \( \tau^N : \text{End}(S^*(N)) \rightarrow \text{End}(S^*(N)) \) be the transpose of \( \tau^N \) defining the \( \mathbb{Z}_2 \)-grading of \( S(N) = S_+(N) \oplus S_-(N) \).

Let \( \tau^N \tilde{c}(Z) : \pi^*(S^*_+(N))|_Z \rightarrow \pi^*(S^*_+(N))|_Z \) denote the corresponding pull back isomorphisms at \( Z \).

Let \( F \) be a complex vector bundle over \( Y \) such that \( S_- \otimes \mu \otimes F \) is a trivial complex vector bundle over \( Y \) (cf. [A]). Then

\[
\tau^N \tilde{c}(Z) \oplus \pi^*\text{Id}_F : \pi^*(S^*_+(N) \otimes \mu \otimes F) \rightarrow \pi^*(S^*_+(N) \otimes \mu \otimes F)
\]

induces an isomorphism between two trivial vector bundles over \( N_{2\varepsilon_0} \setminus Y \).

Let \( F \) admit a Hermitian metric \( g^F \) and a Hermitian connection \( \nabla^F \).

Clearly, \( \pi^*(S^*_+(N) \otimes \mu \otimes F)|_{\partial N_{2\varepsilon_0}} \) extend smoothly to two trivial complex vector bundles over \( X \setminus N_{2\varepsilon_0} \). Moreover, the isomorphism \( \tau^N \tilde{c}(Z) \oplus \pi^*\text{Id}_F \) over \( \partial N_{2\varepsilon_0} \) extends smoothly to an isomorphism between these two trivial vector bundles over \( X \setminus N_{2\varepsilon_0} \).

In summary, what we get is a \( \mathbb{Z}_2 \)-graded Hermitian vector bundle

\[
\xi = \xi_+ \oplus \xi_-,
\]

\[
g^\xi = g^{\xi_+} \oplus g^{\xi_-}
\]
over \( X \) such that
\[
\xi_{\pm|N_0} = \pi^*(S_\pm^*(N) \otimes \mu \oplus F)|_{N_0}, \quad g^{\xi_{\pm|N_0}} = \pi^*(g^{S_\pm^*(N) \otimes \mu} \oplus g^F)|_{N_0}, \tag{2.3}
\]
where \( g^{S_\pm^*(N) \otimes \mu} \) is the tensor product Hermitian metric on \( S_\pm^*(N) \otimes \mu \) induced from \( g^{S^*_\pm(N)} \) and \( g^\mu \). It is easy to see that there exists an odd self-adjoint automorphism \( V \) of \( \xi \) such that
\[
V|_{N_0} = \tau^{N*}c(Z) \oplus \pi^*\text{Id}_F. \tag{2.4}
\]
Moreover, there is a \( \mathbb{Z}_2 \)-graded Hermitian connection \( \nabla^\xi = \nabla^{\xi_+} \oplus \nabla^{\xi_-} \) on \( \xi = \xi_+ \oplus \xi_- \) over \( X \) such that
\[
\nabla^{\xi_{\pm|N_0}} = \pi^*(\nabla^{S_\pm_+^*(N) \otimes \mu} \oplus \nabla^F), \tag{2.5}
\]
where \( \nabla^{S_\pm_+^*(N) \otimes \mu} \) is the Hermitian connection on \( \nabla^{S_\pm_+^*(N) \otimes \mu} \) defined by \( \nabla^{S_\pm_+^*(N) \otimes \mu} = \nabla^{S_\pm_+^*(N)} \otimes \text{Id}_\mu + \text{Id}_\mu \otimes \nabla^\mu \).

Clearly, \( \xi_+ - \xi_- \in \tilde{K}(X) \) is exactly the Atiyah-Hirzebruch direct image \( i\mu \) of \( \mu \) constructed in [AH]. We call \( (\xi, \nabla^\xi, V) \) constructed above a geometric direct image of \( (\mu, \nabla^\mu) \).

### b). A Chern-Simons current associated to a geometric direct image

We make the same assumptions and use the same notations as in a).

If \( E \) is a real vector bundle carrying with a connection \( \nabla^E \), we denote by \( \hat{A}(E, \nabla^E) \) the Hirzebruch characteristic form defined by
\[
\hat{A}(E, \nabla^E) = \det^{1/2} \left( \frac{\sqrt{-1}}{2\pi} \frac{\nabla^E}{\sinh \left( \frac{\sqrt{-1}}{2\pi} \frac{\nabla^E}{2} \right)} \right), \tag{2.6}
\]
where \( \nabla^E = \nabla^E_{2} \) is the curvature of \( \nabla^E \). While if \( E' \) is a complex vector bundle carrying with a connection \( \nabla^{E'} \), we denote by \( \text{ch}(E', \nabla^{E'}) \) the Chern character form associated to \( (E', \nabla^{E'}) \) (cf. [Z2, Section 1]).

Let \( i^{1/2} \) be a fixed square root of \( i = \sqrt{-1} \). The objects which will be considered in the sequel do not depend on this square root. Let \( \varphi \) be the map \( \alpha \in \Lambda^*(T^*X) \rightarrow (2\pi i)^{-\frac{\deg \alpha}{2}} \alpha \in \Lambda^*(T^*X) \).

We now use Quillen’s superconnection formalism. For \( T \geq 0 \), let \( C_T \) be the superconnection on the \( \mathbb{Z}_2 \)-graded vector bundle \( \xi \) defined by
\[
C_T = \nabla^\xi + \sqrt{T}V. \tag{2.7}
\]
The curvature \( C_T^2 \) of \( C_T \) is a smooth section of \( (\Lambda^*(T^*M) \otimes \text{End}(\xi))_{\text{even}} \). By [Q], we know that for any \( T > 0 \),
\[
\frac{\partial}{\partial T} \text{Tr}_s \left[ \exp \left( -C_T^2 \right) \right] = -\frac{d}{2\sqrt{T}} \text{Tr}_s \left[ V \exp \left( -C_T^2 \right) \right]. \tag{2.8}
\]

Clearly, the technical assumptions in [BZ, (1.10)-(1.12)] hold for our constructions in a). Thus one can proceed as in [B1], [B2] and [IB, Definition 1.3] to construct the Chern-Simons current \( \gamma^\xi, V \) as
\[
\gamma^\xi, V = \frac{1}{\sqrt{2\pi i}} \int_0^{+\infty} \varphi \text{Tr}_s \left[ V \exp \left( -C_T^2 \right) \right] \frac{dT}{2\sqrt{T}}. \tag{2.9}
\]
Let $\delta_Y$ denote the current of integration over the oriented submanifold $Y$ of $X$. Then by [BZ, Theorem 1.4], we have that

$$d\gamma_{\xi,V} = \text{ch}(\xi_+, \nabla\xi_+) - \text{ch}(\xi_-, \nabla\xi_-) - \tilde{A}^{-1}(N, \nabla^N)\text{ch}(\mu, \nabla\mu)\delta_Y. \quad (2.10)$$

Moreover, as indicated in [BZ, Remark 1.5], by proceeding as in [BGS, Theorem 3.3], one can prove that $\gamma_{\xi,V}$ is a locally integrable current.

c). A localization formula for $\eta$ invariants

We assume in this subsection that $i : Y \hookrightarrow X$ is an embedding between two odd dimensional closed oriented spin manifolds. Then the normal bundle $N$ to $Y$ in $X$ is even dimensional and carries a canonically induced orientation and spin structure. Let $g^{TX}$ be a Riemannian metric on $TX$. Let $g^{TY}$ be the restricted Riemannian metric on $TY$. Let $\nabla^{TX}$ (resp. $\nabla^{TY}$) denote the Levi-Civita connection associated to $g^{TX}$ (resp. $g^{TY}$).

Without loss of generality we may and we will make the assumption that the isometric embedding $(Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic. Let $N$ carry the canonically induced Euclidean metric as well as the Euclidean connection.

Thus, we may and we will make the same construction as in a), b).

Recall that the definition of the reduced $\eta$ invariant for a (twisted) Dirac operator on an odd dimensional spin Riemannian manifold has been recalled in Section 1.

Under our assumptions, we see easily that the localization formula for $\eta$ invariants proved in [BZ] holds in a slightly simplified form. We recall it as follows.

**Theorem 2.1** (Bismut-Zhang [BZ, Theorem 2.2]) *The following identity holds,*

$$\bar{\eta}(D\xi_+) - \bar{\eta}(D\xi_-) \equiv \bar{\eta}(D\mu) + \int_X \tilde{A}(TX, \nabla^{TX})\gamma_{\xi,V} \mod \mathbb{Z}. \quad (2.11)$$

**Remark 2.2** The extra Chern-Simons form in [BZ, Theorem 2.2] disappears here simply because we have made the simplifying assumption that the isometric embedding $(Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic.

d). A geometric formula for $\bar{\eta}(D\mu)$

We continue the discussion in c) and assume that $X = S^{2n-1}$, a higher odd dimensional sphere (but we do not assume that it admits the standard metric, this makes the isometric embedding $i : (Y, g^{TY}) \hookrightarrow (S^{2n-1}, g^{TS^{2n-1}})$ to be totally geodesic possible).

Now recall that by the Bott periodicity (cf. [1]), one has $K(S^{2n-1}) = \{0\}$. Thus, in our case, we have $i_*\mu = 0$. This means there is a trivial complex vector bundle $\theta$ over $S^{2n-1}$ such that $\theta \oplus \xi_+$ is isomorphic to $\theta \oplus \xi_-$. We equip $\theta$ with a Hermitian metric as well as a Hermitian connection $\nabla^\theta$.

Let $\xi' = \xi_+ \oplus \xi_-$ be the $\mathbb{Z}_2$-graded Hermitian vector bundle over $S^{2n-1}$ with $\xi'_\pm = \theta \oplus \xi_\pm$. Then $\xi'_\pm$ and $\xi'$ carry canonically induced Hermitian connections $\nabla^{\xi'_\pm}$ and $\nabla^{\xi'}$ respectively through direct sums.
Let $W$ be an odd self-adjoint automorphism of $\xi'$, which clearly exists by the above discussion.

For any $T \geq 0$, set
\[ C'_T = \nabla^{\xi'} + \sqrt{T}W. \tag{2.12} \]

Similarly as in (2.9), we define the Chern-Simons form $\gamma^{\xi',W}$ as
\[ \gamma^{\xi',W} = \frac{1}{\sqrt{2\pi i}} \int_0^{+\infty} \varphi \text{Tr}_s \left[ W \exp \left( -\left( C'_T \right)^2 \right) \right] \frac{dT}{2\sqrt{T}}. \tag{2.13} \]

Since $W$ is invertible, one has the following formula due to Bismut-Chéeger [BC, Theorem 2.28], which corresponds to the case with $Y = \emptyset$ in (2.11),
\[ \bar{\eta}(D_{\xi'}) - \bar{\eta}(D_{\xi'}) \equiv \int_{S^{2n-1}} \tilde{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\xi',W} \mod Z. \tag{2.14} \]

On the other hand, one clearly has that
\[ \bar{\eta}(D_{\xi'}) = \bar{\eta}(D_{\xi'}) + \bar{\eta}(D^\theta). \tag{2.15} \]

From (2.11), (2.14) and (2.15), one gets
\[ \bar{\eta}(D_{\mu}) \equiv \int_{S^{2n-1}} \tilde{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\xi',W} - \int_{S^{2n-1}} \tilde{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\xi',V} \mod Z. \tag{2.16} \]

We can reformulate (2.16) as follows.

Let $\tilde{\xi} = \tilde{\xi}_+ \oplus \tilde{\xi}_-$ be the $Z_2$-graded Hermitian vector bundle over $S^{2n-1}$ defined by
\[ \tilde{\xi}_+ = \xi_+ \oplus \xi'_-, \quad \tilde{\xi}_- = \xi_- \oplus \xi'_+, \tag{2.17} \]
carrying the canonically induced Hermitian connection $\nabla^{\tilde{\xi}}$ through direct sums. Let $\tilde{V} = V \oplus W^T$, where $W^T$ is the transpose of $W$, be the odd self-adjoint automorphism of $\tilde{\xi}$. Then $(\tilde{\xi}, \nabla^{\tilde{\xi}}, \tilde{V})$ forms a geometric direct image of $\mu$ in the sense of Section 2a).

Let $\gamma^{\tilde{\xi},\tilde{V}}$ be the associated Chern-Simons current defined by (2.9). By (2.10) and the construction of $(\tilde{\xi}, \nabla^{\tilde{\xi}}, \tilde{V})$, one verifies easily that
\[ d\gamma^{\tilde{\xi},\tilde{V}} = -\tilde{A}^{-1}(N, \nabla^N) \text{ch}(\mu, \nabla^\mu) \delta_Y. \tag{2.18} \]

We can now state the main result of this section as follows, which is simply a reformulation of (2.16).

**Theorem 2.3** The following identity holds,
\[ \bar{\eta}(D_{\mu}) \equiv -\int_{S^{2n-1}} \tilde{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\tilde{\xi},\tilde{V}} \mod Z. \tag{2.19} \]

**Remark 2.4** It is remarkable that the right hand side of (2.19) does not involve any spectral information of $D_{\mu}$. It is purely geometric/topological, although the existence of the invertible element $W$ is by no means trivial (as we have seen, it follows from the Bott periodicity). It
resembles well the $K$-theoretic proof of the Atiyah-Singer index theorem [AS]. On the other hand, the different choice of $W$ may cause the right hand side of (2.19) an integer jump. This partly explains that (2.19) is in general a mod $\mathbb{Z}$ formula. While conversely, one can always find a $W$ to make (2.19) a purely equality in $\mathbb{R}$. This may sound few sense as when $g^{TM}, g^\mu$ and $\nabla^\mu$ vary, dim(ker $D^\mu$) may jump. However, if we take $\mu = S(TY)$, then $D^{S(TY)}$ is the Signature operator of $(Y, g^{TY})$ and such an equality in $\mathbb{R}$ with a suitable choice of $W$ will not depend on the variation of $g^{TY}$.

**Remark 2.5** If dim $Y \equiv 3 \mod 8\mathbb{Z}$ and $\mu$ is a complexification of a Euclidean vector bundle carrying with a Euclidean connection, then one can embed $Y$ into a higher $8l + 3$ dimensional manifold and proceed as in [Z1, Section 3] to improve (2.19) to a mod $2\mathbb{Z}$ formula.

**Remark 2.6** As have been mentioned in Section 1, the proof of Theorem 2.1 given in [BZ] relies heavily on the difficult paper of Bismut-Lebeau [BL]. So in the next section, we will give an alternate proof of Theorem 2.1 for the case where $X$ is a higher dimensional sphere.

### §3. An alternate proof of Theorem 2.1

As in Section 2, let $Y$ be an odd dimensional closed oriented spin manifold carrying with a Riemannian metric $g^{TY}$ and the associated Levi-Civita connection $\nabla^{TY}$. Let $\mu$ be a complex vector bundle over $Y$ carrying with a Hermitian metric $g^\mu$ and a Hermitian connection $\nabla^\mu$.

In case when there will be no confusion, we will use the notations in Section 2 without further explanation.

Since dim $Y$ is odd, by a well-known result in bordism/cobordism theory, there is a positive integer $k$ such that the $k$ disjoint copies of $Y$ bound a compact oriented spin manifold $\hat{Y}$ of dimension dim $Y + 1$ such that the boundary $\partial \hat{Y}$ does not contain other components. Moreover, there is a complex vector bundle $\hat{\mu}$ over $\hat{Y}$ such that when restricted to boundary, it is just $\mu$ on each copy of $Y$.

Clearly, $(\hat{Y}, Y)$ is a $\mathbb{Z}/k$ manifold in the sense of Sullivan (cf. [FM] and [Z1]).

Let $g^{\hat{Y}}$ be a Riemannian metric on $T\hat{Y}$ which is of product nature near $\partial \hat{Y}$ and which on $\partial \hat{Y}$ is exactly $g^{TY}$ on each copy of the boundary. Let $\nabla^{\hat{Y}}$ be the associated Levi-Civita connection. Similarly, let $g^\hat{\mu}$ (resp. $\nabla^{\hat{\mu}}$) be a Hermitian metric on $\hat{\mu}$ such that it is of product nature near $\partial \hat{Y}$ and that on $\partial \hat{Y}$ it is exactly $g^\mu$ (resp. $\nabla^\mu$) on $\mu$ over each copy of $Y$.

Let $S^{2n,k}$ be the $\mathbb{Z}/k$ manifold obtained by removing $k$ balls $D^{2n}$ from the $2n$-sphere. Then the boundary $\partial S^{2n,k}$ consists of $k$ disjoint copies of $S^{2n-1}$. Let $i : \hat{Y} \hookrightarrow S^{2n,k}$ be a $\mathbb{Z}/k$ embedding (cf. [FM], [Z1]). The existence of such an embedding is clear when $n$ is sufficiently large.

Let $g^{TS^{2n,k}}$ be a $\mathbb{Z}/k$ metric on $TS^{2n,k}$ which is of product nature near $\partial S^{2n,k}$, such that $g^{TS^{2n,k}}|_{TY} = g^{TY}$ and that the isometric embedding $i : (\hat{Y}, g^{TY}) \hookrightarrow (S^{2n,k}, g^{TS^{2n,k}})$ is totally geodesic. Let $\nabla^{TS^{2n,k}}$ be the associated Levi-Civita connection.

Let $\hat{N}$ denote the normal bundle to $\hat{Y}$ in $S^{2n,k}$. Then $\hat{N}$ carries an induced orientation and spin structure, as well as a $\mathbb{Z}/k$ Euclidean metric (resp. connection) $g^{\hat{N}}$ (resp. $\nabla^{\hat{N}}$).
We can then apply the constructions in Sections 2a), b) to the embedding \( i : (\hat{Y}, g^{TY}) \to (S^{2n,k}, g^{TS^{2n,k}}) \) in a \( \mathbb{Z}/k \) manner (that is, preserving all the \( \mathbb{Z}/k \) structures), such that all the metrics, connections and maps involved are of product nature near boundary.

We denote the resulting \( \mathbb{Z}/k \) geometric direct image of \( (\hat{\mu}, \nabla\hat{\mu}) \) by \( (\hat{\xi} = \hat{\xi}_+ \oplus \hat{\xi}_-, \nabla\hat{\xi}, \hat{V}) \). When restrict each copy of the boundary, we denote the restricted geometric direct image of \( (\mu, \nabla\mu) \) by \( (\xi = \xi_+ \oplus \xi_-, \nabla\xi, V) \).

Let \( \text{ind}_k(D^\hat{\xi}_+) \), \( \text{ind}_k(D^\hat{\xi}_-) \) be the mod \( k \) indices defined by (cf. [FM, (5.2)], [Z1]),

\[
\text{ind}_k(D^\hat{\xi}_+) \equiv \int_{S^{2n,k}} \tilde{A}(TS^{2n,k}, \nabla^{TS^{2n,k}}) \text{ch}(\hat{\xi}_+, \nabla\hat{\xi}_+) - k \bar{\eta}(D^\hat{\xi}_+) \mod k\mathbb{Z}, \tag{3.1}
\]

\[
\text{ind}_k(D^\hat{\xi}_-) \equiv \int_{\hat{Y}} \tilde{A}(T\hat{Y}, \nabla\hat{Y}) \text{ch}(\hat{\mu}, \nabla\hat{\mu}) - k \bar{\eta}(D^\hat{\mu}) \mod k\mathbb{Z}. \tag{3.2}
\]

By the mod \( k \) index theorem of Freed and Melrose [FM, (5.5)], one knows that

\[
\text{ind}_k(D^\hat{\xi}_+) - \text{ind}_k(D^\hat{\xi}_-) = \text{ind}_k(D^\hat{\mu}) \tag{3.3}
\]

in \( \mathbb{Z}/k\mathbb{Z} \).

Now if we denote \( \gamma\hat{\xi}_V \) the Chern-Simons current constructed in Section 2b), then its restriction to the boundary consists of \( k \) copies of the Chern-Simons current \( \gamma\hat{\xi}_V \).

By applying the transgression formula (2.10) to \( \gamma\hat{\xi}_V \) and integrate over \( S^{2n,k} \), one then gets

\[
\int_{S^{2n,k}} \tilde{A}(TS^{2n,k}, \nabla^{TS^{2n,k}}) \text{ch}(\hat{\xi}_+, \nabla\hat{\xi}_+) - \int_{S^{2n,k}} \tilde{A}(TS^{2n,k}, \nabla^{TS^{2n,k}}) \text{ch}(\hat{\xi}_-, \nabla\hat{\xi}_-)
\]

\[
- \int_{\hat{Y}} \tilde{A}(T\hat{Y}, \nabla\hat{Y}) \text{ch}(\hat{\mu}, \nabla\hat{\mu}) = k \int_{S^{2n-1}} \tilde{A}(TY, \nabla^TY) \gamma\hat{\xi}_V. \tag{3.4}
\]

From (3.1)-(3.4), one deduces that

\[
\bar{\eta}(D^\hat{\xi}_+) - \bar{\eta}(D^\hat{\xi}_-) \equiv \bar{\eta}(D^\hat{\mu}) + \int_{S^{2n-1}} \tilde{A}(TY, \nabla^TY) \gamma\hat{\xi}_V \mod \mathbb{Z}, \tag{3.5}
\]

which is exactly the Bismut-Zhang formula (2.11) in the case where \( X = S^{2n-1} \).

**Remark 3.1** The relation between the Bismut-Zhang formula (2.11) and the Freed-Melrose mod \( k \) index theorem [FM] was exploited in [Z1] where (2.11) is used to give an alternate proof of a mod \( k \) equality between the right hand sides of (3.1) and (3.2). Now such an equality can be proved directly by applying the Riemann-Roch property for Dirac operators on manifolds with boundary proved by Dai and Zhang in [DZ], without using the results in [BZ]. In fact, this can be done by first applying [DZ, Theorem 1.2 and Lemma 4.6] to get (3.3). The mod \( k \) equality between the right hand sides of (3.1) and (3.2) is then an easy consequence of the Atiyah-Patodi-Singer index theorem [APS1] (This observation grew out of discussions with Xianzhe Dai).

**Remark 3.2** One should also note that the proof of (3.5) given in this section holds only for those embeddings which can be obtained through an embedding between \( \mathbb{Z}/k \) manifolds,
while (2.11) is much more general. Still we hope the simplified proof in this section would be helpful for a good feeling of (2.11). While on the other hand, for many applications, the existence for such an embedding is suffice, as our main result (2.19) holds for it.

§4. Some applications

In this section, we discuss some immediate applications of formula (2.19).

This section is organized as follows. In a), we discuss briefly the relationship between the Chern-Simons current and the Cheeger-Simons differential character [CS]. In b), we discuss the Atiyah-Patodi-Singer $\mathbb{R}/\mathbb{Z}$ index theorem for unitary flat vector bundles [APS3] from the point of view of (2.19). In c), we show that by replacing the Dirac operator by the Signature operator, one may refined the above $\mathbb{R}/\mathbb{Z}$ formula to an $\mathbb{R}$ valued one.

We make the same assumptions and use the same notation as in Section 2.

a). Chern-Simons current and the Cheeger-Simons differential character

As was indicated by Bismut in [B2], the Chern-Simons currents constructed in Section 2 are closely related to the differential characters introduced by Cheeger and Simons in [CS]. This becomes clearer if we compare the transgression formula (2.18) with the one in [CS, (4.3)]. The difference is that in [CS, (4.3)], the transgression formula holds on different Stiefel manifolds, while our formula holds universally on a single sphere. Moreover, the differential characters for Chern character forms in [CS] were defined mod $\mathbb{Q}$, while our formula is clearly of an $\mathbb{R}/\mathbb{Z}$ nature (as the construction of the Chern-Simons current $\gamma_{\xi,\tilde{V}}$ in Section 2d) depends on the choice of an automorphism $W$).

More precisely, if we denote by $\hat{A}(TY,\nabla^{TY})$ (resp. $\hat{ch}(\mu,\nabla^{\mu})$) the Cheeger-Simons differential character associated to $A(TY,\nabla^{TY})$ (resp. $ch(\mu,\nabla^{\mu})$) constructed in [CS], then by [CS, Theorem 9.1], one has, in using the product notation as in [CS],

$$\tilde{\eta}(D^{\mu}) \equiv \left< \hat{A}(TY,\nabla^{TY}) \ast \hat{ch}(\mu,\nabla^{\mu}),[Y] \right> \mod \mathbb{Q}. \quad (4.1)$$

From (2.19) and (4.1), one gets

$$\int_{S^{2n-1}} \hat{A}(T S^{2n-1},\nabla^{TS^{2n-1}}) \gamma_{\xi,\tilde{V}} + \left< \hat{A}(TY,\nabla^{TY}) \ast \hat{ch}(\mu,\nabla^{\mu}),[Y] \right> \equiv 0 \mod \mathbb{Q}. \quad (4.2)$$

It would be interesting to give a direct proof of (4.2).

b). The Atiyah-Patodi-Singer index theorem for flat bundles revisited

In this section, we replace the Hermitian vector bundle $\mu$ in Section 2 by $\mu \otimes \rho$, where $\rho$ is a unitary flat vector bundle with the flat connection denoted by $\nabla^{\rho}$. We equip $\mu \otimes \rho$ with the induced tensor product Hermitian metric as well as the tensor product Hermitian connection $\nabla^{\mu \otimes \rho} = \nabla^{\mu} \otimes \text{Id}_{\rho} + \text{Id}_{\mu} \otimes \nabla^{\rho}$. 9
By [APS3],
\[ \tilde{\eta}_{\mu,\rho} := \tilde{\eta}(D^{\mu\otimes\rho}) - \text{rk}(\rho)\tilde{\eta}(D^\mu) \mod \mathbb{Z} \]  \tag{4.3}

is a smooth invariant with respect to \((g^T M, g^\mu, \nabla^\mu)\). Moreover, [APS3, Theorem 5.3] provides a topological interpretation for this invariant.

We now examine this invariant by using (2.19).

Let \((\tilde{\xi}_\rho, \nabla^{\tilde{\xi}_\rho}, \tilde{V}_\rho)\) be a geometric direct image of \((\mu \otimes \rho, \nabla^{\mu \otimes \rho})\) constructed similarly as that for \((\mu, \nabla^\mu)\) in Section 2d). Let \(\gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho}\) be the associated Chern-Simons current defined in (2.9). By (2.18), \(\gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho}\) verifies the transgression formula
\[ d\gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho} = -\hat{A}^{-1}(N, \nabla^N)\text{ch}(\mu, \nabla^\mu)\text{rk}(\rho)\delta_Y, \]  \tag{4.4}
as \(\rho\) is a flat bundle.

By (2.19), one also has
\[ \tilde{\eta}(D^{\mu\otimes\rho}) \equiv \int_{S^{2n-1}} \hat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}})\gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho} \mod \mathbb{Z}. \]  \tag{4.5}

From (2.19), (4.3) and (4.5), one gets
\[ \tilde{\eta}_{\mu,\rho} \equiv \int_{S^{2n-1}} \hat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \left( \text{rk}(\rho)\gamma^{\tilde{\xi}, \tilde{V}} - \gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho} \right) \mod \mathbb{Z}. \]  \tag{4.6}

On the other hand, from (2.18) and (4.4), one finds
\[ d \left( \text{rk}(\rho)\gamma^{\tilde{\xi}, \tilde{V}} - \gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho} \right) = 0. \]  \tag{4.7}

Thus, \(\text{rk}(\rho)\gamma^{\tilde{\xi}, \tilde{V}} - \gamma^{\tilde{\xi}_\rho, \tilde{V}_\rho}\) determines a cohomology class in \(H_{\text{odd}}^\text{dir}(S^{2n-1}, \mathbb{R})\). One verifies easily that this cohomology class does not depend on the choice of \(\nabla^\mu\). It only depends on the choices of the automorphisms \(W\) and \(W_\rho\) appearing in the construction of the geometric direct images of \((\xi, \nabla^\xi, V)\) and \((\tilde{\xi}_\rho, \nabla^{\tilde{\xi}_\rho}, \tilde{V}_\rho)\). The different choices of \(W\) and \(W_\rho\) cause a (possible) integer jump in integration term in the right hand side of (4.6).

Thus, (4.6) may be thought of in some sense as an intrinsic version of the Atiyah-Patodi-Singer index theorem for flat vector bundles stated in [APS3, Theorem 5.3]. Its conceptual novelty is that one need not divide the topological index into two parts (that is, a \(\mathbb{Q}/\mathbb{Z}\) part plus an \(\mathbb{R}\) part.)

**Remark 4.1** It is pointed out in [APS3] that the index theorem for flat vector bundles stated in [APS3, Theorem 5.3] indeed holds also for non-unitary flat vector bundles by suitable extension of the construction of the \(\eta\) invariants. In the next section, we will extend our intrinsic formula (4.6) to the case of non-unitary flat vector bundles along similar lines.

c). Signature operator and an \(\mathbb{R}\) valued index theorem for flat vector bundles

Now we set \(\mu = S(TY)\) in the above subsection. In this case, \(D^\mu\) is the Signature operator associated to \((TY, g^{TY})\), denoted by \(D_{\text{Sign}}\), while \(D^{\mu\otimes\rho}\) is now denoted by \(D_{\text{Sign}}^\rho\).
Set
\[ \bar{\eta}_{\text{Sign}, \rho} = \bar{\eta}(D_{\text{Sign}}^\rho) - \text{rk}(\rho)\bar{\eta}(D_{\text{Sign}}). \] (4.8)
Then \( \bar{\eta}_{\text{Sign}, \rho} \) is a smooth invariant equivalent to what defined in [APS2, Theorem 2.4].

On the other hand, as indicated in Remark 2.4, one can choose the automorphisms \( W \) and \( W_\rho \) in the construction of the Chern-Simons current so that
\[ \bar{\eta}(D_{\text{Sign}}^\rho) = \int_{S^{2n-1}} \hat{A}(TS^{2n-1}, \nabla T) \gamma \tilde{\xi}_\rho \tilde{V}_\rho, \] (4.9)
\[ \bar{\eta}(D_{\text{Sign}}) = \int_{S^{2n-1}} \hat{A}(TS^{2n-1}, \nabla T) \gamma \tilde{\xi} \tilde{V}. \] (4.10)
It is clear that (4.9), (4.10) are actually equalities not depending on the choice of \( g^{TM} \). Moreover, since \( K^1(S^{2n-1}) = \mathbb{Z} \), one sees that the choice of \( W \) and \( W_\rho \) is canonical (up to stable homotopy).

By (4.7), one also sees that \( \gamma \tilde{\xi}_\rho \tilde{V}_\rho - \text{rk}(\rho)\gamma \tilde{\xi} \tilde{V} \) determines canonically an element in \( H^{\text{odd}}(Y, \mathbb{R}) \).

We can state our \( \mathbb{R} \) valued refinement of (4.6) as follows.

**Theorem 4.2** The following \( K \)-theoretic formula for \( \bar{\eta}_{\text{Sign}, \rho} \) holds,
\[ \bar{\eta}_{\text{Sign}, \rho} = \int_{S^{2n-1}} \hat{A}(TS^{2n-1}, \nabla T) \left( \gamma \tilde{\xi}_\rho \tilde{V}_\rho - \text{rk}(\rho)\gamma \tilde{\xi} \tilde{V} \right). \] (4.11)

**Remark 4.3** Theorem 4.2 was proved for unitary flat vector bundles over spin manifolds. It is nature to ask whether there is still such a kind of formulas without the spin condition.

**Remark 4.4** If one could find a purely topological way to identify the automorphisms \( W \) and \( W_\rho \) (or the difference element of \( \text{rk}(\rho)W \) and \( W_\rho \) in \( K^1(S^{2n-1}) \)), then one would provide a positive answer to a question of Atiyah-Patodi-Singer stated implicitly in [APS2, page 406]. On the other hand, for any choice of \( W \) and \( W_\rho \), the right hand side of (4.11) provides a smooth invariant of \( Y \). So in some sense, \( \bar{\eta}_{\text{Sign}, \rho} \) becomes one example of a series of smooth invariants associated to the unitary flat vector bundle \( \rho \) over \( Y \).

**§5. Generalizations to Hermitian vector bundles with non-unitary connection**

In this section, we generalize the results in the previous sections to the case of non-unitary (i.e., non-Hermitian) connections on the twisted Hermitian vector bundle \( \mu \).

This section is organized as follows. In a), we propose a definition of the \( \eta \) invariants of Dirac operators coupled with Hermitian vector bundles with non-unitary connection. In b), we show that the generalized \( \eta \) invariants defined in a) can also be expressed as a Chern-Simons current on a higher dimensional sphere, extending Theorem 2.3. In c), we apply the formula obtained in b) to the case of general flat vector bundles.
a). \( \eta \) invariants for Hermitian vector bundles with non-unitary connection

We make the same assumptions and use the same notation as in Sections 1 and 2, except that we no longer assume that the connection \( \nabla^\mu \) is a Hermitian connection with respect to the Hermitian metric \( g^\mu \). Then the Dirac operator \( D^\mu \) need not be self-adjoint. To make clearer the dependence of the Dirac operator with respect to the connection, we will denote the Dirac operator coupled with connection \( \nabla^\mu \) on \( \mu \) by \( D^{\nabla^\mu} \).

Let \( \nabla^\mu \ast \) be the adjoint connection of \( \nabla^\mu \) with respect to \( g^\mu \). Then

\[
\tilde{\nabla}^\mu = \frac{1}{2} (\nabla^\mu + \nabla^{\mu \ast}) \quad (5.1)
\]

is a Hermitian connection on \( \mu \), and \( D^{\tilde{\nabla}^\mu} \) is self-adjoint.

Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( \psi(x) = 1 \) if \( x > 10 \), while \( \psi(x) = 0 \) if \( x < 2 \).

For any \( t \geq 0 \), set

\[
D(t) = \sqrt{t} \left( \psi(t)D^{\tilde{\nabla}^\mu} + (1 - \psi(t))D^{\nabla^\mu} \right). \quad (5.2)
\]

First of all, by using Duhamel principle, one knows easily that for any \( t > 0 \), the heat operator \( \exp(-D(t)^2) \) is well-defined.

By proceeding similarly as in [BF, Theorem 2.4], one knows that as \( t \to 0^+ \),

\[
\text{Tr} \left[ \frac{dD(t)}{dt} \exp \left( -D(t)^2 \right) \right] = O(1). \quad (5.3)
\]

Thus, one is able to define, by analogous to [L, (59)],

\[
\eta(D^{\nabla^\mu}, \psi) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \text{Tr} \left[ \frac{dD(t)}{dt} \exp \left( -D(t)^2 \right) \right]. \quad (5.4)
\]

By proceeding as in [L, Lemma 3], one knows that \( \eta(D^{\nabla^\mu}, \psi) \) does not depend on \( \psi \).

**Definition 5.1** Let \( \eta(D^{\nabla^\mu}) \) denote \( \eta(D^{\nabla^\mu}, \psi) \) and call it the \( \eta \) invariant of \( D^{\nabla^\mu} \). Let \( \tilde{\eta}(D^{\nabla^\mu}) \) be the reduced \( \eta \) invariant defined by

\[
\tilde{\eta}(D^{\nabla^\mu}) = \frac{\dim(\ker D^{\nabla^\mu}) + \eta(D^{\nabla^\mu})}{2}. \quad (5.5)
\]

Clearly, if \( \nabla^\mu \) preserves \( g^\mu \), then \( \tilde{\eta}(D^{\nabla^\mu}) \) is exactly the usual reduced \( \eta \) invariant discussed in Sections 1 and 2.

Let \( CS(\nabla^\mu, \tilde{\nabla}^\mu) \) be the Chern-Simons form verifying (cf. [Z2, Chapter 1])

\[
dCS(\nabla^\mu, \tilde{\nabla}^\mu) = \text{ch}(\mu, \nabla^\mu) - \text{ch}(\mu, \tilde{\nabla}^\mu). \quad (5.6)
\]
Proposition 5.2 The following identity holds,
\[ \tilde{\eta}(D^{\mu}) = \tilde{\eta}(D^{\tilde{\mu}}) + \int_Y \tilde{A}(T\mathcal{M}, \nabla^{TM}) CS(\nabla^{\mu}, \tilde{\nabla}^{\mu}). \] (5.7)

Proof. For any \( u \in [0, 1] \), set \( D^{\mu}(u) = (1 - u)D^{\mu} + uD^{\tilde{\mu}} \). By proceeding as in [BF, Theorem 2.10], one knows that the \( R/Z \) part of \( \tilde{\eta}(D^{\mu}(u)) \) verifies the variation formula as usual. This gives the local term in the right hand side of (5.7). On the other hand, it is clear that there is no integer jump in this deformation (Compare with [L, (65)]). Q.E.D.

b). A formula through Chern-Simons current

In this subsection, we show that Theorem 2.3 still holds in the current generalized situation. We state the result as follows.

Theorem 5.3 The following identity holds,
\[ \tilde{\eta}(D^{\mu}) \equiv - \int_{S^{2n-1}} \tilde{A}(T S^{2n-1}, \nabla^{TS^{2n-1}}) \tilde{\xi}, \tilde{V} \mod \mathbb{Z}. \] (5.8)

Proof. Let \( (\xi, \tilde{\nabla}\xi, \tilde{V}) \) be the geometric direct image constructed in Section 2d) with respect the connection \( \tilde{\nabla}^{\mu} \) on \( \mu \) instead of \( \nabla^{\mu} \). Let \( \tilde{\xi}, \tilde{V} \) be the associated Chern-Simons current. Then by Theorem 2.3, we know that
\[ \tilde{\eta}(D^{\tilde{\mu}}) \equiv - \int_{S^{2n-1}} \tilde{A}(T S^{2n-1}, \nabla^{TS^{2n-1}}) \tilde{\xi}, \tilde{V} \mod \mathbb{Z}. \] (5.9)

On the other hand, by the transgression formula (2.18), one deduces that
\[ \int_{S^{2n-1}} \tilde{A}(T S^{2n-1}, \nabla^{TS^{2n-1}}) \tilde{\xi}, \tilde{V} - \int_{S^{2n-1}} \tilde{A}(T S^{2n-1}, \nabla^{TS^{2n-1}}) \tilde{\xi}, \tilde{V} = \int_Y \tilde{A}(T\mathcal{M}, \nabla^{TM}) CS(\nabla^{\mu}, \tilde{\nabla}^{\mu}). \] (5.10)

From (5.7), (5.9) and (5.10), one gets (5.8). Q.E.D.

c). Applications to flat vector bundles

We will only be brief in this subsection as the arguments run parallel to those in Section 4.

Now we replace \( \mu \) by \( \mu \otimes \rho \) such that \( \mu \) is a Hermitian vector bundle carrying with a Hermitian connection, while \( \rho \) is a flat vector bundle carrying with a Hermitian metric. But we no longer assume that the flat connection \( \nabla^{\rho} \) is unitary.

With the definition given in Section 5a), it is clear that the invariant \( \tilde{\eta}_{\mu, \rho} \) is still well-defined. Moreover, by Theorem 5.3, one still has a formula of the same form as (4.6), giving a \( C/\mathbb{Z} \) valued \( K \)-theoretic formula for this analytically defined invariant. It is easy to check that the right hand side of such a formula is equal to the topological index given in [APS3,
Theorem 5.3]. This means that, at least on the mod $\mathbb{Z}$ level, our definition of the reduced $\eta$ invariant agrees with the one indicated in [APS3, Sections 5, 6]. This would be sufficient for many other applications as well.

Certainly one can also extend Theorem 4.2 to the case of non-unitary flat vector bundles.

We leave the details to the interested reader.

**Remark 5.4** Clearly, all the results of this paper can be extended to the case of spin$^c$ manifolds. We also leave this to the interested reader.

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