Four-Loop Collinear Anomalous Dimension in $\mathcal{N} = 4$ Yang-Mills Theory

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We report a calculation in $\mathcal{N} = 4$ Yang-Mills of the four-loop term $g^{(4)}$ in the collinear anomalous dimension $g(\lambda)$ which governs the universal subleading infrared structure of gluon scattering amplitudes. Using the method of obstructions to extract this quantity from the $1/\epsilon$ singularity in the four-gluon iterative relation at four loops, we find $g^{(4)} = -1240.9$ with an estimated numerical uncertainty of 0.02%. We also analyze the implication of our result for the strong coupling behavior of $g(\lambda)$, finding support for the string theory prediction computed recently by Alday and Maldacena using AdS/CFT.

I. INTRODUCTION

Gluon scattering amplitudes in QCD and supersymmetric gauge theories are notoriously difficult to compute, yet possess remarkably simple hidden structure. We expect the greatest simplicity in the maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory, where planar $L$-loop amplitudes are believed to satisfy iterative relations [1, 2, 3, 4, 5, 6, 7] in $L$. These relations allow the complete, planar all-loop $n$-particle MHV amplitude $A$ to be written in an exponential form due to Bern, Dixon and Smirnov [8]. Their ansatz for the $n = 4$ particle amplitude is

$$\frac{A}{A_{\text{tree}}} = (A_{\text{div}}(s,t))^2 \exp \left[ \frac{f(\lambda)}{8} \log^2(t/s) + c(\lambda) \right], \quad (1)$$

where $\lambda = g^2_{YM} N$ is the 't Hooft coupling, $s$ and $t$ are the usual four-particle Mandelstam invariants, and the infrared divergences are encoded in the prefactors $A_{\text{div}}$. In dimensional regularization to $D = 4 - 2\epsilon$ the divergences take the form

$$A_{\text{div}}(s,t) = \exp \left[ \left( -\frac{1}{8\epsilon^2} f^{(-2)}(\lambda \mu^{2\epsilon}/s^\epsilon) \right) \right]$$

$$+ \left( -\frac{1}{4\epsilon} g^{(-1)}(\lambda \mu^{2\epsilon}/s^\epsilon) \right) + (s \leftrightarrow t)
+ \mathcal{O}(\epsilon) \quad (2)$$

in terms of an IR cutoff scale $\mu$ (required on dimensional grounds) and two functions $f^{(-2)}(\lambda)$ and $g^{(-1)}(\lambda)$ which respectively are related to the $f(\lambda)$ appearing in (1) and to a second function called $g(\lambda)$ according to

$$f(\lambda) = \left( \frac{\lambda}{d\lambda} \right)^2 f^{(-2)}(\lambda), \quad g(\lambda) = \frac{d}{d\lambda} g^{(-1)}(\lambda). \quad (3)$$

The three functions $f(\lambda)$, $g(\lambda)$ and $c(\lambda)$ are universal; the same functions appear in the exponential ansatz for any planar $n$-particle MHV amplitude [3].

The function $f(\lambda)$ is well-known from another role: it is the cusp anomalous dimension, which also governs the scaling of twist-two operators in the limit of large spin $S$ [8, 9, 10, 11].

$$\Delta \left( \text{Tr} \left( Z D^8 Z \right) \right) - S = f(\lambda) \log S + \mathcal{O}(S^{0}). \quad (4)$$

At weak coupling it has been computed through four loops [12, 13, 14, 15, 16],

$$f(\lambda) = 8 \left( \frac{\lambda}{16\pi^2} \right) - \frac{8\pi^2}{3} \left( \frac{\lambda}{16\pi^2} \right)^2 + \frac{88\pi^4}{45} \left( \frac{\lambda}{16\pi^2} \right)^3
- \left( \frac{584\pi^6}{315} - 64(1 + r) \xi^2 \right) \left( \frac{\lambda}{16\pi^2} \right)^4 + \mathcal{O}(\lambda^5) \quad (5)$$

with $r = -2.00002(3)$ (the quantity in parentheses denotes the current best numerical uncertainty in the last digit), while AdS/CFT calculations [10, 17] indicate the strong coupling behavior

$$f(\lambda) = 4 \sqrt{\frac{\lambda}{16\pi^2}} - \frac{3 \log 2}{\pi} + \mathcal{O}(1/\sqrt{\lambda}). \quad (6)$$

The recently proposed dressing phase [18, 19] for the asymptotic $S$-matrix [20, 21, 22, 23, 24, 25] of the spin chain description of planar $\mathcal{N} = 4$ Yang-Mills implies that at finite $\lambda$, $f(\lambda)$ satisfies a certain integral equation [26] whose solution is compatible with the limits [27, 28, 29] and moreover predicts that $r = -2$ in (5).

Much less is known about the second function $g(\lambda)$ which governs the subleading infrared divergence and may be called the “collinear” anomalous dimension [30]. Perturbative calculations through three loops [5] have established that

$$g(\lambda) = -4\zeta_3 \left( \frac{\lambda}{16\pi^2} \right)^2 + 8(4\zeta_5 + \frac{9}{5} \pi^2 \xi \zeta_3) \left( \frac{\lambda}{16\pi^2} \right)^3$$

$$+ g^{(4)} \left( \frac{\lambda}{16\pi^2} \right)^4 + \mathcal{O}(\lambda^5), \quad (7)$$

and the purpose of this note is to report a numerical calculation of the four-loop coefficient

$$g^{(4)} = -1240.9(3). \quad (8)$$

It is an important outstanding problem to relate $g(\lambda)$ to more familiar observables which could perhaps be computed using integrability techniques. In particular it
would be interesting to derive an integral equation satisfied by \( g(\lambda) \).

An important step forward was recently taken by Alday and Maldacena [31], who gave a prescription for computing gluon amplitudes at strong coupling using AdS/CFT and found perfect agreement with the ansatz (1). A consequence of their calculation is the prediction

\[
g(\lambda) = 2(1 - \log 2)\sqrt{\frac{\lambda}{16\pi^2}} + O(1) \tag{9}
\]

for the leading strong-coupling behavior of \( g(\lambda) \). It is important to mention that while \( f(\lambda) \) is independent of the IR renormalization scale, the same is not true for \( g(\lambda) \). Rescaling \( \mu \) to \( k\mu \) sends \( g(\lambda) \) to \( g(\lambda) + 2 \log(k) f(\lambda) \).

In section II we review our calculational method, which is the same as the one used in [16] to calculate \( f \) being AdS/CFT and found perfect agreement with the prediction (1). A consequence of their calculation is the pre-

\[
M^{(4)}(\epsilon) - M^{(3)}(\epsilon)M^{(1)}(\epsilon) + M^{(2)}(\epsilon)(M^{(1)}(\epsilon))^2 - \frac{1}{2}(M^{(2)}(\epsilon))^2 - \frac{1}{4}(M^{(1)}(\epsilon))^4 = (f_0^{(4)} + f_1^{(4)})(M^{(1)}(\epsilon)) + O(\epsilon^0), \tag{10}
\]

where \( M^{(L)}(\epsilon) \) is the ratio of the \( L \)-loop amplitude to the tree amplitude and \( f_0^{(4)} \), \( f_1^{(4)} \) are two numbers. According to [5], whose conventions we follow except as noted below, these are related to the four-loop coefficients \( f^{(4)} \) and \( g^{(4)} \) in \( f(\lambda) \) and \( g(\lambda) \) by

\[
f_0^{(4)} = \frac{1}{2^4} f^{(4)}, \quad f_1^{(4)} = \frac{1}{2^4} 2 g^{(4)} \tag{11}
\]

The factor of \( 1/2^4 \) arises here at four loops because the expansion parameter used by [5] differs by a factor of 2 from that of [31] which we use in [16] and [17] below.

Each of the five terms on the left-hand side of (10) is an extremely complicated function of the ratio \( x = t/s \) with singular behavior starting at \( O(1/\epsilon^8) \). It is therefore rather remarkable that the five terms conspire to add up to such a simple object involving only two free constants [2, 3, 4]. The leading singularity of \( M^{(1)}(4\epsilon) \) on the right-hand side is \( O(1/\epsilon^2) \), so we can determine \( f_1^{(4)} \) by reading off the coefficient of the \( 1/\epsilon^2 \) pole on both sides of (10).

The idea behind the method of obstructions, explained in detail in [16], is that we do not need to fully compute all of the \( M^{(L)}(\epsilon) \). Rather, it is sufficient to compute what are in some sense the ‘constant pieces’, since the \( x \)-dependent pieces are guaranteed to cancel each other out on the left-hand side of (10). This idea is made precise by writing each amplitude as a Mellin transform, in the form

\[
\int_{-\infty}^{+\infty} dy \ x^y F(y, \epsilon) \tag{12}
\]

for some \( F(y, \epsilon) \). It turns out that \( f_1^{(4)} \) multiplies \( \delta(y) \) in Mellin space, so in order to isolate this term it is sufficient to read off just the coefficient of \( \delta(y) \) on both sides of (10), throwing away the rest. We call terms proportional to \( \delta \)-functions obstructions. The utility of this method benefits significantly from the fact that we can work directly in Mellin space, where it is relatively easy to construct explicit formulas for Feynman integrals [33, 34].

It was further shown in [16] that obstructions obey a product algebra structure, meaning that the obstruction in any product of amplitudes is given by the product of their obstructions. This statement is obvious in \( y \)-space where it hinges on the simple fact that \( \delta(y) \) convolved with itself is again \( \delta(y) \). Consequently we can calculate the obstruction \( P^{(L)}(\epsilon) \) in each \( M^{(L)}(\epsilon) \) separately and then insert those results into the polynomial in (10).

The \( L = 1, 2, 3, 4 \) loop amplitudes we require may be expressed in terms of the 12 scalar Feynman diagrams depicted in Fig. 1 [3, 12, 33, 36]. The reader may find all necessary details in [3, 13, 15]. Our convention is that each loop momentum integral comes with a factor

\[
-\frac{1}{2} (-s)^{\epsilon/2} (-t)^{\epsilon/2} \left[-i\pi^{-D/2} e^{\gamma_E}\right] \int d^D p. \tag{13}
\]

The standard convention includes only the factor in brackets. The additional factor \( (-s)^{\epsilon/2} (-t)^{\epsilon/2} \) renders all amplitudes dimensionless (when the appropriate numerator factors are included), but it does not alter the form of (10) since each term in that equation has a common factor \( (-s)^{-2\epsilon} (-t)^{-2\epsilon} \). The factor of \( -1/2 \) in front here eliminates the need for a factor of \( (-1/2)^{\epsilon} \) in (14) below.
FIG. 1: The 12 integrals appearing in the $L \leq 4$-loop four-particle amplitudes in $\mathcal{N} = 4$ Yang-Mills theory. We refer the reader to [15] for all necessary details.

We have computed the obstructions in the 12 separate integrals through the first eight orders in $\varepsilon$ and present the results in equation (19) at the end of the paper. The calculation was performed using the algorithm explained in [16] based on Czakon’s MB program [37]. Numerical integrations were performed using CUBA’s Cuhe algorithm [38], and the digit in parentheses denotes the reported uncertainty in the final digit.

An exact formula for $I^{(1)}(c)$ was given in [16], as were the first seven terms in the two- and three-loop amplitudes (though here we indicate separately the contributions from the two three-loop integrals). No significant effort was spent on attempting to improve the numerical accuracy of the $\mathcal{O}(1/c^3)$ and $\mathcal{O}(1/c^2)$ terms in the individual four-loop integrals because these values play no role in this paper.

To find the full obstruction $P^{(L)}$ in the $L$-loop amplitude we add together the contributions from the individual integrals according to [16, 27, 35, 36]

\[
P^{(1)} = I^{(1)}
\]
\[
P^{(2)} = 2I^{(2)}
\]
\[
P^{(3)} = 2I^{(3)a} + 4I^{(3)b}
\]
\[
P^{(4)} = 2I^{(4)a} + 4I^{(4)b} + 4I^{(4)c} + 2I^{(4)d} + 8I^{(4)e} + 4I^{(4)f} - 4I^{(4)g} - I^{(4)h}.
\]

Plugging the resulting expressions for $P^{(L)}$ into (10) in place of $M^{(L)}$ and using the relation (11) leads to the advertised value [8].

### III. BRIDGING WEAK AND STRONG COUPLING

As mentioned above, string theory provides predictions for how $f(\lambda)$ and $g(\lambda)$ behave at strong coupling. It is therefore tempting to try to make a connection between the weak and strong coupling regimes. An approximation scheme that has been very successful for the weak and strong coupling regimes. An approximating function $\hat{f}(\lambda)$ as an appropriate solution to the polynomial equation [39]

\[
\left( \frac{\lambda}{16\pi^2} \right)^n = \sum_{r=n}^{2n} c_r \hat{f}(\lambda)^r,
\]

where the coefficients $c_r$ are determined by imposing that $\hat{f}(\lambda)$ agree with $f(\lambda)$ through $\mathcal{O}(\lambda^{n+1})$.

The form of (15) incorporates a strong coupling expansion for $\hat{f}(\lambda)$ in the parameter $4\pi/\sqrt{\lambda}$ with leading term $\alpha \sqrt{\lambda}/4\pi$ where $\alpha = 1/c^2$.

Using the expansion of $f(\lambda)$ up to four loops one finds an approximating function $\hat{f}(\lambda)$ that agrees, within a few percent, with the string theory strong coupling prediction and with the BES ansatz [26] for all positive values of the coupling.

The success of this approximation scheme makes its application to $g(\lambda)$ a natural thing to attempt. However, since $g(\lambda)$ is negative for small $\lambda$ but becomes positive at strong coupling, it must have at least one zero for $\lambda > 0$. Clearly, any approximating function of the form (15) (with an appropriate modification because $g(\lambda)$ only starts at two loops)

\[
\left( \frac{\lambda}{16\pi^2} \right)^{2n} = \sum_{r=n}^{4n} c_r \hat{g}(\lambda)^r,
\]

leads to a $\hat{g}(\lambda)$ that cannot possibly have a zero for $\lambda \neq 0$ and hence a contradiction. One way to find a consistent approximating function was introduced in [31]. The idea is to perform a change of the IR renormalization scale $\mu \rightarrow \exp(\xi/2)\mu$. Then, as discussed above, $g(\lambda) \rightarrow g(\lambda) + \xi f(\lambda)$. The new function $\hat{g}(\lambda, \xi) = g(\lambda) + \xi f(\lambda)$ can now be approximated by using (15) for any $\xi > 0$.

Consider the approximation (15) for $\hat{g}(\lambda, \xi)$ with $n = 2,$

\[
\left( \frac{\lambda}{16\pi^2} \right)^2 = c_2 \hat{g}(\lambda, \xi)^2 + c_3 \hat{g}(\lambda, \xi)^3 + c_4 \hat{g}(\lambda, \xi)^4.
\]

This equation depends on four parameters, namely, $c_2, c_3, c_4$ and $\xi$. Imposing that the approximating function $\hat{g}(\lambda, \xi)$ agrees with $\hat{g}(\lambda, \xi)$ through $\mathcal{O}(\lambda^3)$ determines
the three coefficients $c_2, c_3, c_4$ in terms of $\xi$. In [31] the resulting approximating function was extrapolated to strong coupling and compared to the string theory prediction for a range of $\xi$. At the special value $\xi = \frac{1}{2} \log 2$, for example, the approximation gives $\hat{g}(\lambda) = 1.37 \sqrt{\lambda}/4\pi$ compared to the string theory prediction $\tilde{g}(\lambda) = 2\sqrt{\lambda}/4\pi$.

Here we would like to use the results of Alday and Maldacena to make a prediction for $g^{(4)}$ to compare our result to. Imposing that $\hat{g}$ agrees with the string theory prediction at strong coupling fixes the parameter $\xi \approx 0.73679$. Then all four parameters $c_2, c_3, c_4$ and $\xi$ are completely fixed and we can expand the resulting $\hat{g}$ in $\lambda$ to $O(\lambda^4)$ to find the predicted value $g^{(4)} \approx -1336.9$. Remarkably our result is only 7% away from this value. We interpret this as good evidence for the string theory strong coupling prediction of Alday and Maldacena [31].

Finally we present also a more conventional Padé approximant for $g(\lambda)$ based upon all currently available data. Here we follow the approach described in [15] where a $[3/2]$ Padé approximant in terms of the auxiliary variable $u = \sqrt{1 + \lambda/\pi^2}$ was considered for $f(\lambda)$. Specifically we consider the ansatz

$$G(\lambda) = (u - 1)^2 \frac{N_0 + N_1 u}{1 + D_1 u + D_2 u^2}$$

with four parameters $N_0, N_1, D_1, D_2$. The choice $u = \sqrt{1 + \lambda/\pi^2}$ was motivated by evidence that $f(\lambda)$ should have a branch point at $\lambda = -\pi^2$. It seems reasonable to guess that the same might be true for $g(\lambda)$, though we have no direct evidence for this guess.

The four parameters in [15] can be uniquely determined by fitting to all available data—the perturbative expansion through four loops as well as the strong-coupling limit [3]. The resulting approximant, displayed in Fig. 2, may be considered our best candidate picture of $g(\lambda)$ at the moment. Note that it is minimal in the sense that $G(\lambda)$ was designed to have a single zero along the positive real axis, whereas the true $g(\lambda)$ could potentially have any odd number of zeros. Curiously, the zero lies very close to $\lambda/16\pi^2 = 1$, although it is impossible to conclude based on the limited available data whether or not this is just a coincidence.

**Acknowledgments**

We have benefited from discussions with Z. Bern, L. Dixon and J. Maldacena. The research of FC at the Perimeter Institute is supported in part by funds from NSERC of Canada and MEDT of Ontario. The research of MS is supported by NSF grant PHY-0610259 and by an OJI award under DOE grant DE-FG02-91ER40688. The research of AV is supported by NSF CAREER Award PHY-0643150. This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET: www.sharcnet.ca).
\[ \mathcal{I}^{(4)\alpha} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1169}{2112 \epsilon^6} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 277 \pi^4 \frac{1}{\epsilon^4} + 6.204418(2) \frac{1}{\epsilon^3} - 63.9795(1) \frac{1}{\epsilon^2} + [34.479723(1)] \frac{1}{\epsilon} + \mathcal{O}(1), \]

\[ \mathcal{I}^{(4)\beta} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 277 \pi^4 \frac{1}{\epsilon^4} + 6.204418(2) \frac{1}{\epsilon^3} - 63.9795(1) \frac{1}{\epsilon^2} + [34.479723(1)] \frac{1}{\epsilon} + \mathcal{O}(1), \]

\[ \mathcal{I}^{(4)\gamma} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 11.1550(2) \frac{1}{\epsilon^2} + 12.30(1) \frac{1}{\epsilon} + [142.936(2)] \frac{1}{\epsilon} + \mathcal{O}(1), \]

\[ \mathcal{I}^{(4)\delta} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 277 \pi^4 \frac{1}{\epsilon^4} + 6.204418(2) \frac{1}{\epsilon^3} - 63.9795(1) \frac{1}{\epsilon^2} + [34.479723(1)] \frac{1}{\epsilon} + \mathcal{O}(1), \]

\[ \mathcal{I}^{(4)\epsilon} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 277 \pi^4 \frac{1}{\epsilon^4} + 6.204418(2) \frac{1}{\epsilon^3} - 63.9795(1) \frac{1}{\epsilon^2} + [34.479723(1)] \frac{1}{\epsilon} + \mathcal{O}(1), \]

\[ \mathcal{I}^{(4)\gamma} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 277 \pi^4 \frac{1}{\epsilon^4} + 6.204418(2) \frac{1}{\epsilon^3} - 63.9795(1) \frac{1}{\epsilon^2} + [34.479723(1)] \frac{1}{\epsilon} + \mathcal{O}(1), \]

\[ \mathcal{I}^{(4)\delta} (\epsilon) = + \frac{1}{36 \epsilon^8} - \frac{1}{36 \epsilon^8} - \frac{187 \pi^2}{6912 \epsilon^6} - \frac{1}{36 \epsilon^8} - \frac{1}{29 \epsilon^8} + \frac{3456}{1728 \epsilon^8} + \frac{1}{1175 \epsilon^8} + \frac{3456}{7 \pi^2 \epsilon^8} + \frac{521 \epsilon^6}{7 \pi^2 \epsilon^8} + \frac{1}{1728 \epsilon^8} - 277 \pi^4 \frac{1}{\epsilon^4} + 6.204418(2) \frac{1}{\epsilon^3} - 63.9795(1) \frac{1}{\epsilon^2} + [34.479723(1)] \frac{1}{\epsilon} + \mathcal{O}(1). \]
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