CENTRAL LIMIT THEOREM AND MODERATE DEVIATIONS FOR A STOCHASTIC CAHN-HILLIARD EQUATION

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Abstract: In this paper, we prove a central limit theorem and a moderate deviation principle for a perturbed stochastic Cahn-Hilliard equation defined on $[0,T] \times [0,\pi]^d$ with $d \in \{1,2,3\}$. This equation is driven by a space-time white noise. The weak convergence approach plays an important role.

Keyword: Stochastic Cahn-Hilliard equation; Large deviations; Moderate deviations; Central limit theorem.

MSC: 60H15, 60F05, 60F10.

1. Introduction

Since the pioneer works of Freidlin and Wentzell [19], the theory of small perturbation large deviation principle (LDP for short) for stochastic (partial) differential equations has been extensively developed, see monographs [12, 13, 16], and papers [6, 7, 8, 27, 33] and references therein for further developments. Like the large deviations, the moderate deviation problems arise in the theory of statistical inference quite naturally. The moderate deviation principle (MDP for short) can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [14, 18, 22] and references therein.

The problem of moderate deviations for stochastic partial differential equations has been receiving much attention in very recent years, such as Wang and Zhang [32] for the stochastic reaction-diffusion equations, Wang et al. [31] for the stochastic Navier-Stokes equations, Li et al. [26] for the fractional stochastic heat equation, Yang and Jiang [34] for the fourth-order stochastic heat equations with fractional noises, Gao and Wang [21] for the forward-backward stochastic differential equations. By using a new weak convergence approach, Budhiraja et al. [5] and Dong et al. [15] studied the MDP for stochastic dynamics with jumps.

Since the work of Da Prato and Debussche [11], it has attracted much attention in the study of stochastic Cahn-Hilliard equation, which describes phase separation in a binary alloy in the presence of thermal fluctuations (see [1]). For example, see Cardon-Weber [9] for the stochastic Cahn-Hilliard equation driven by the Guassian space-time white noise, Bo and Wang [3] for the Lévy space-time white noise case, Bo et al. [2] for the fractional noises case, and so on. The ergodic properties and invariant measures of the stochastic Cahn-Hilliard equation with degenerate noise were studied in Goudenège and Manca [23]. The sharp interface limits for the stochastic Cahn-Hilliard equation are extensively studied, see Funaki [20] and references therein for further developments.
Wentzell-Freidlin’s large deviation results for stochastic Cahn-Hilliard equation have been established in [28] and [4]. In this paper, we shall study the central limit theorem and MDP for this equation.

The rest of this paper is organized as follows. In Section 2, we give the framework of the stochastic Cahn-Hilliard equation. In Section 3, we prove the central limit theorem. Section 4 is devoted to the proof of the MDP by using the weak convergence method.

Throughout the paper, the generic positive constant $C$ may change from line to line. If it is essential, the dependence of a constant $C$ on some parameters, say $T$, will be written by $C_{p,T}$. We denote the space variable by $x$ and the space integral by $\int \cdot dx$, and for any $p \geq 1$ denote $\| \cdot \|_p$ the $L^p$-norm with respect to $dx$.

Let $\mathcal{D} = [0, \pi]^d$ with $d \in \{1, 2, 3\}$. For any $T > 0, p \geq 1, \alpha \in (0, 1)$, let $C^\alpha([0, T], L^p(\mathcal{D}))$ be the Hölder space equipped with the norm defined by

$$
\| f \|_{\alpha,p} := \sup_{t \in [0,T]} \| f(t, \cdot) \|_p + \sup_{s \neq t, s, t \in [0,T]} \frac{\| f(t, \cdot) - f(s, \cdot) \|_p}{|t - s|^\alpha},
$$

and

$$
C^{\alpha,0}([0, T], L^p(\mathcal{D})) := \left\{ f \in C^\alpha([0, T], L^p(\mathcal{D})); \lim_{\delta \to 0} O_f(\delta) = 0 \right\},
$$

where $O_f(\delta) := \sup_{|t-s| < \delta} \frac{\| f(t, \cdot) - f(s, \cdot) \|_p}{|t-s|^\alpha}$. Then $C^{\alpha,0}([0, T], L^p(\mathcal{D}))$ is a Polish space, which is denoted by $\mathcal{E}_\alpha$.

2. Framework and main results

2.1. Framework. For any $T > 0, \varepsilon > 0$, consider

$$
\begin{align*}
\frac{\partial u^\varepsilon}{\partial t}(t, x) &= -\Delta (\Delta u^\varepsilon(t, x) - f(u^\varepsilon(t, x))) + \sqrt{\varepsilon} \sigma(u^\varepsilon(t, x)) \dot{W}(t, x), \text{ in } [0, T] \times \mathcal{D}, \\
u^\varepsilon(0, x) &= u_0(x), \\
\frac{\partial u^\varepsilon}{\partial n}(t, x) &= \frac{\varepsilon \Delta u^\varepsilon}{\varepsilon n}(t, x) = 0, \text{ on } [0, t] \times \partial \mathcal{D},
\end{align*}
$$

(1)

where $\Delta u^\varepsilon$ denotes the Laplacian of $u^\varepsilon$ in the $x$-variable; $\dot{W}$ is a Gaussian space-time white noise on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$; $n$ is the outward normal vector; the functions $\sigma, f$ and $u_0$ satisfy the following Hypothesis (H):

(H.1) $\sigma$ is a bounded and Lipschitz function;
(H.2) $f$ is a polynomial of degree 3 with positive dominant coefficient;
(H.3) $u_0$ is a continuous function on $\mathcal{D}$ which belongs to $L^p(\mathcal{D})$ for $p \geq 4$;
(H.4) $u_0$ is a $\gamma$-Hölder continuous function on $\mathcal{D}$ with $\gamma \in (0, 1)$. 

According to Cardon-Weber [9], under hypothesis (H.1)-(H.3), Eq.(1) admits a unique solution $u^\varepsilon$ in the following mild form:

$$u^\varepsilon(t, x) = \int_D G_t(x, y) u_0(y) dy + \int_0^t \int_D \Delta G_{t-s}(x, y) f(u^\varepsilon(s, y)) dy ds$$

$$+ \sqrt{\varepsilon} \int_0^t \int_D G_{t-s}(x, y) \sigma(u^\varepsilon(s, y)) W(ds, dy),$$

where $G_t(\cdot, \cdot)$ is the Green function corresponding to the operator $\frac{\partial^2}{\partial t^2} + \Delta^2$ with Neumann boundary conditions, which satisfies

$$\sup_{\varepsilon \leq 1} \sup_{t \in [0, T]} \mathbb{E} \left[ \left\| u^\varepsilon(t, \cdot) \right\|_p^q \right] < +\infty,$$

for $q \geq p$, see [9, Theorem 1.3]. Moreover, Theorem 1.4 in [9] asserts that under hypothesis (H), the trajectories of the solution are a.s. $\alpha$-Hölder continuous in $t$ with $\alpha \leq \frac{\gamma}{4}$, $\alpha < \frac{1}{2} (1 - \frac{d}{4})$. Consequently, the random field solution $\{u^\varepsilon(t, x); (t, x) \in [0, T] \times D\}$ to Eq.(1) lives in space $E_\alpha$.

Intuitively, as the parameter $\varepsilon$ tends to zero, the solution $u^\varepsilon$ of (1) will tend to the solution of the deterministic equation

$$u^0(t, x) = \int_D G_t(x, y) u_0(y) dy + \int_0^t \int_D \Delta G_{t-s}(x, y) f(u^0(s, y)) dy ds.$$

Particularly, taking $\varepsilon = 0$ in Eq.(2), the solution of (4) has the following estimate

$$\sup_{t \in [0, T]} \left\| u^0(t, \cdot) \right\|_p < +\infty.$$

In this paper, we shall investigate deviations of $u^\varepsilon$ from $u^0$, as $\varepsilon$ decreases to 0. That is the asymptotic behavior of the trajectories,

$$Z^\varepsilon(t, x) := \frac{1}{\sqrt{\varepsilon} h(\varepsilon)} (u^\varepsilon - u^0)(t, x), \quad (t, x) \in [0, T] \times D.$$

(LDP) The case $h(\varepsilon) = 1/\sqrt{\varepsilon}$ provides some large deviation estimates. Shi et al. [28], Boulamba and Mellouk [4] proved that the law of the solution $u^\varepsilon$ satisfies an LDP.

(CLT) If $h(\varepsilon) \equiv 1$, we are in the domain of the central limit theorem (CLT for short). We will show that $(u^\varepsilon - u^0)/\sqrt{\varepsilon}$ converges to a random field as $\varepsilon \to 0$, see Theorem [2.1] below.

(MDP) To fill in the gap between the CLT scale and the large deviations scale, we will study moderate deviations, that is when the deviation scale satisfies

$$h(\varepsilon) \to +\infty \text{ and } \sqrt{\varepsilon} h(\varepsilon) \to 0, \text{ as } \varepsilon \to 0.$$

In this case, we will prove that $Z^\varepsilon$ satisfies an LDP, see Theorem [2.2] below. This special type of LDP is called the MDP for $u^\varepsilon$, see [13, Section 3.7].

Throughout this paper, we assume that (7) is in place.
2.2. Main results. Let
\[
\begin{cases}
\frac{\partial Y}{\partial t}(t,x) = -\Delta (\Delta Y(t,x) - f'(u^0(t,x))Y(t,x)) + \sigma(u^0(t,x))\dot{W}(t,x), & \text{in } [0,T] \times \mathcal{D}, \\
Y(0,x) = 0, \\
\frac{\partial Y}{\partial n}(t,x) = \frac{\partial \Delta Y}{\partial n}(t,x) = 0, & \text{on } [0,T] \times \partial \mathcal{D}.
\end{cases}
\] (8)

Following the proofs of Theorems 1.3 and 1.4 in [9], it is easy to obtain that under \((H)\), Eq. (8) admits a unique solution \(Y\), which satisfies
\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \|Y(t,\cdot)\|_p^q \right] < +\infty,
\] (9)
for \(q \geq p\). Furthermore, for any \(t,t' \in [0,T]\),
\[
\mathbb{E} \left[ \|Y(t',\cdot) - Y(t,\cdot)\|_p^q \right] \leq C|t' - t|^\alpha, \quad \mathbb{P}\text{-a.s.}
\] (10)
with \(\alpha \leq \frac{1}{4}, \alpha < \frac{1}{2}(1 - \frac{d}{4})\).

Our first main result is the following central limit theorem.

**Theorem 2.1.** Under \((H)\), for any \(\alpha \leq \frac{1}{4}\) and \(\alpha < \frac{1}{2}(1 - \frac{d}{4})\), the random field \((u^\varepsilon - u^0)/\sqrt{\varepsilon}\) converges in probability to a random field \(Y\) defined by (8) on \(\mathcal{E}_\alpha\).

Noticing that the coefficients \(f'(u^0)\) and \(\sigma(u^0)\) are bounded and independent of \(Y\), Eq. (8) becomes a fourth-order stochastic heat equation, by using the classical Freidlin-Wentzell’s large deviation theory (e.g., [24]), it is easily to obtain that \(Y/\sqrt{\varepsilon}\) obeys an LDP on \(\mathcal{E}_\alpha\) with the speed \(h^2(\varepsilon)\) and with the good rate function:
\[
I(g) := \begin{cases}
\inf_{v} \left\{ \frac{1}{2} \int_0^T \int_{\mathcal{D}} v^2(t,x)dxdt; Z^v = g \right\}, & \text{if } g \in \text{Im}(Z^v); \\
+\infty, & \text{otherwise},
\end{cases}
\] (11)
where the infimum is taken over all \(v \in L^2([0,T] \times \mathcal{D})\) and the functional \(Z^v\) is the solution of the following deterministic partial differential equation
\[
Z^v(t,x) = \int_0^t \int_{\mathcal{D}} G_{t-s}(x,y)\sigma(u^0(s,y))v(s,y)dyds \\
+ \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x,y)f'(u^0(s,y))Z^v(s,y)dyds.
\] (12)

Our second main result is that the law of \\{(u^\varepsilon - u^0)/[\sqrt{\varepsilon}h(\varepsilon)]; \varepsilon \in (0,1]\} obeys the same LDP with \(Y/\sqrt{\varepsilon}\), that is the following theorem.

**Theorem 2.2.** Under \((H)\), the family \\{(u^\varepsilon - u^0)/[\sqrt{\varepsilon}h(\varepsilon)]; \varepsilon \in (0,1]\} satisfies a large deviation principle on \(\mathcal{E}_\alpha\) with the speed function \(h^2(\varepsilon)\) and with the good rate function \(I\) given by (11). More precisely, the following statements hold:
(a) for each \(M < \infty\), the level set \(\{f \in \mathcal{E}_\alpha; I(f) \leq M\}\) is a compact subset of \(\mathcal{E}_\alpha\);
(b) for each closed subset \(F\) of \(\mathcal{E}_\alpha\),
\[
\limsup_{\varepsilon \to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P} \left( \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}h(\varepsilon)} \in F \right) \leq -\inf_{f \in F} I(f);
\]
(c) for each open subset $G$ of $\mathcal{E}_\alpha$,
\[ \liminf_{\varepsilon \to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P} \left( \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon h(\varepsilon)}} \in G \right) \geq - \inf_{f \in G} I(f). \]

**Remark 2.3.**

(1) It is well-known that the Freidlin-Wentzell’s theory is very powerful for the studies of various asymptotics and their applications to problems of the behavior of a random process on large time intervals, such as the problem of the limit behavior of the invariant measure, the problem of exit of a random process from a domain, and the problem of stability under random perturbations, see [19]. The reader is referred to the recent work by Tao [20] for a lot of literature on the development of numerical methods.

Usually, the relatively simple form of the corresponding rate function in Theorem 2.2 suggests that many of the constructions needed to implement an effective importance sampling scheme would be simpler than in the corresponding large deviation context, e.g., Klebaner and Liptser [25], Dupuis and Wang [17]. Refer to Budhiraja et al. [5] for further development and references.

(2) By using the classical exponential approximation method, Yang and Jiang [34] proved a MDP for a fourth-order stochastic heat equation with fractional noises, in which the drift term $f$ together with its derivative are assumed to be Lipschitz and the diffusion coefficient $\sigma = 1$. Due to the non-Lipschitz and nonlinearity of $\Delta f$ and the multiplicative noise in (1), it is difficult to get some exponential estimates and the classical exponential approximation method would become rather complicated for stochastic Cahn-Hilliard equation (1). Here, we use the weak convergence approach [6, 16], in which one only needs some moment estimates.

3. Central Limit Theorem

3.1. **Preparation.** First, let us recall some estimations on the Green function $G$ which will be used quite often later. These properties can be found in [2, 9].

**Lemma 3.1.**

1. There exists some positive constant $C$ such that for any $t \in (0, T]$ and $x \in \mathcal{D}$,
\[ \int_{\mathcal{D}} |G_t(x, y)|^2 dy \leq Ct^{-\frac{d}{2}}. \]  
(13)

2. For $\gamma < 1 - \frac{d}{4}$, there exists $C > 0$ such that for any $0 \leq t_0 < t \leq T$ and $x \in \mathcal{D}$,
\[ \int_{t_0}^t \int_{\mathcal{D}} |G_{t-s}(x, y)|^2 dy ds \leq C|t - t_0|^{-\gamma}. \]  
(14)

3. Suppose that $\rho \in [1, \infty), p \in [\rho, \infty), \beta \geq 1$ and $\kappa = \frac{1}{p} - \frac{1}{\rho} + 1 \in [0, 1]$ (if $d = 3$, we also need $\frac{1}{\rho} < 3$; and if $d = 2$, $\frac{1}{\rho} \neq \infty$). For $v \in L^\beta([0, T], L^\rho(\mathcal{D}))$, $0 \leq t_0 < t \leq T$ and $x \in \mathcal{D}$, define
\[ J(v)(t_0, t, x) := \int_{t_0}^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y)v(s, y)dy ds. \]
Then $J$ is a bounded operator from $L^\beta([0, T], L^p(D))$ into $L^\infty([0, T], L^p(D))$ and there exists a constant $C > 0$ such that the following inequalities hold:

(1) for all $\beta > \frac{1}{2} + \frac{1}{q} - \frac{1}{2}$,

$$
\| J(v)(t_0, t, \cdot) \|_p \leq C t^{\frac{1}{2} + \frac{1}{4}(\kappa - 1) - \frac{1}{2}} \| v(\cdot, \cdot) \|_{L^\beta([0, t], L^p(D))},
$$

(15) with $0 \leq t_0 < t \leq T$.

(2) for any $\gamma \in (0, \frac{1}{2} + \frac{1}{q} - \frac{1}{2})$ and $\beta > \frac{1}{2} + \frac{1}{4}(\kappa - 1) - \gamma$,

$$
\| J(v)(0, t', \cdot) - J(v)(0, t, \cdot) \|_p \leq C |t' - t|^{\gamma} \| v(\cdot, \cdot) \|_{L^\beta([0, T], L^p(D))},
$$

(16) with $0 \leq t < t' \leq T$.

Next we give an estimate about the convergence of $u^\varepsilon$ as $\varepsilon \to 0$.

For any $q \geq p, t \in [0, T]$ and $M > 0$, let

$$
\mathcal{A}^\varepsilon(t, M) := \left\{ \omega \in \Omega; \sup_{s \in [0, t]} \| u^\varepsilon(s, \cdot, \cdot) \|_p^q \leq M \right\}.
$$

(17)

Proposition 3.2. Under (H), for any $q \geq p$, there exists some positive constant $C$ such that for any $M > 0$,

$$
\sup_{t \in [0, T]} \mathbb{E} \left[ J_{\mathcal{A}^\varepsilon(t, M)} \| u^\varepsilon(t, \cdot) - u^0(t, \cdot) \|_p^q \right] \leq C \varepsilon^{\frac{q}{2}} \to 0, \quad \text{as} \ \varepsilon \to 0.
$$

(18)

Proof. Since for any $t \in [0, T]$, $u^\varepsilon(t, x) - u^0(t, x) = \sqrt{\varepsilon} \int_0^t \int_D G_{t-s}(x, y) \sigma(u^\varepsilon(s, y)) W(ds, dy)$

$$
+ \int_0^t \int_D \Delta G_{t-s}(x, y) \left[ f(u^\varepsilon(s, y)) - f(u^0(s, y)) \right] dyds
$$

=: T_1^\varepsilon(t, x) + T_2^\varepsilon(t, x),

we obtain that for any $q \geq p$,

$$
\| u^\varepsilon(t, \cdot) - u^0(t, \cdot) \|_p^q \leq C \left( \| T_1^\varepsilon(t, \cdot) \|_p^q + \| T_2^\varepsilon(t, \cdot) \|_p^q \right),
$$

(19)

for some positive constant $C$.

For $T_1^\varepsilon(t, x)$, firstly, it follows from H"older's inequality that

$$
\mathbb{E} \left[ \| T_1^\varepsilon(t, \cdot) \|_p^q \right] = \mathbb{E} \left( \int_D |T_1^\varepsilon(t, x)|^p dx \right)^{\frac{q}{p}} \leq C \int_D \mathbb{E} \left[ |T_1^\varepsilon(t, x)|^q \right] dx.
$$

Then BDG inequality, the boundedness of $\sigma$ and Eq. (14) imply that

$$
\mathbb{E} \left[ |T_1^\varepsilon(t, x)|^q \right] \leq C \varepsilon^{\frac{q}{2}} \mathbb{E} \left( \left[ \int_0^t \int_D G_{t-s}^2(x, y) \sigma^2(u^\varepsilon(s, y)) dyds \right]^{\frac{q}{2}} \right)
$$

$$
\leq C \varepsilon^{\frac{q}{2}} \left[ \int_0^t \int_D G_{t-s}^2(x, y) dyds \right]^{\frac{q}{2}} \leq C(T) \varepsilon^{\frac{q}{2}}.
$$

(20)
Therefor, we have
\[ \mathbb{E} \left[ \| T_1(t, \cdot) \|^q \right] \leq C(T)\varepsilon^{\frac{q}{2}}. \]  
(21)

By Eq. (15) in Lemma 3.1 for any \( \rho \in [1, p] \), \( \kappa = 1/p - 1/\rho + 1 \in [0, 1] \) and \( \beta \in \left( \frac{1}{2} + \frac{1}{q}(\kappa - 1), q \right) \), we obtain that
\[ \mathbb{E} \left[ I_{A^\varepsilon(t, M)} \left( \int_D \left| T_2^\varepsilon(t, x) \right|^p dx \right) \right]^{\frac{q}{p}} \leq C(T) \mathbb{E} \left[ I_{A^\varepsilon(t, M)} \left( \int_0^t \| f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot)) \|_\rho^\beta ds \right) \right]^{\frac{q}{p}} \]
\[ \leq C(T) \mathbb{E} \left[ I_{A^\varepsilon(t, M)} \left( \int_0^t \| f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot)) \|_\rho^q ds \right) \right] \leq C(T) \mathbb{E} \left[ I_{A^\varepsilon(s, M)} \| f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot)) \|_\rho^q ds \right], \]  
(22)
where the last inequality is by the fact that \( A^\varepsilon(t, M) \subset A^\varepsilon(s, M) \) for any \( 0 \leq s \leq t \).

Since \( f \) is a polynomial function of degree 3, taking \( \rho = \frac{p}{3} \), by Hölder’s inequality, we have
\[ \| f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot)) \|_\rho^q \leq C \| u^\varepsilon(s, \cdot) - u^0(s, \cdot) \|_\rho^q \left( 1 + \| u^\varepsilon(s, \cdot) \|_p^{2q} + \| u^0(s, \cdot) \|_p^{2q} \right). \]  
(23)

Then the definition of \( A^\varepsilon(t, M) \) and estimation (5) yield that
\[ \mathbb{E} \left[ I_{A^\varepsilon(t, M)} \| T_2^\varepsilon(t, \cdot) \|_\rho^q \right] \leq C(T, M) \int_0^t \mathbb{E} \left[ I_{A^\varepsilon(s, M)} \| u^\varepsilon(s, \cdot) - u^0(s, \cdot) \|_\rho^q \right] ds. \]  
(24)

Combining (21) with (24), we have
\[ \mathbb{E} \left[ I_{A^\varepsilon(t, M)} \| u^\varepsilon(t, \cdot) - u^0(t, \cdot) \|_\rho^q \right] \leq C(T)\varepsilon^{\frac{q}{2}} + C(T, M) \int_0^t \mathbb{E} \left[ I_{A^\varepsilon(s, M)} \| u^\varepsilon(s, \cdot) - u^0(s, \cdot) \|_\rho^q \right] ds. \]

By Gronwall’s inequality, we obtain the desired estimate.

The proof is complete. \( \square \)

3.2. Proof of Theorem 2.1

Proof of Theorem 2.1. Denote \( Y^\varepsilon := (u^\varepsilon - u^0)/\sqrt{\varepsilon} \). We will prove that
\[ \| Y^\varepsilon - Y \|_{\alpha, p}^q \rightarrow 0 \quad \text{in probability as } \varepsilon \rightarrow 0, \]  
(25)
with \( \alpha \leq \frac{3}{4} \) and \( \alpha < \frac{1}{2}(1 - \frac{d}{4}) \) and \( q \geq p \).

By (3), we have \( P(\mathcal{A}^\varepsilon(T, M)^c) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) and \( M \rightarrow \infty \), thus we only need to show that
\[ \lim_{\varepsilon \rightarrow 0} \| Y^\varepsilon - Y \|_{\alpha, p}^q = 0 \quad \text{in probability on } \mathcal{A}^\varepsilon(T, M). \]  
(26)
The task of the following part is to prove that

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[ I_{A^\varepsilon(T,M)} \| Y^\varepsilon - Y \|_{\alpha,p}^q \right] = 0, \quad (27)$$

which is stronger than (26). Denote

$$V^\varepsilon := Y^\varepsilon - Y.$$

By Lemma A.1 in [10], to prove Eq. (27), we only need to verify the following two conditions for $V^\varepsilon$:

(A1). for all $t \in [0, T]$,

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| V^\varepsilon(t, \cdot) \|_{p}^q \right] = 0;$$

(A2). there exists $\gamma > 0$, such that for all $t', t \in [0, T]$,

$$\mathbb{E} \left[ I_{A^\varepsilon(T,M)} \| V^\varepsilon(t', \cdot) - V^\varepsilon(t, \cdot) \|_{p}^q \right] \leq C|t' - t|^\gamma,$$

where $C$ is a positive constant independent of $\varepsilon$.

**Step 1.** First, we prove (A1) for $V^\varepsilon(t, x)$. Notice that

$$Y^\varepsilon(t, x) - Y(t, x) = \int_0^t \int_D G_{t-s}(x, y) \left[ \sigma(u^\varepsilon(s, y)) - \sigma(u^0(s, y)) \right] W(ds, dy)$$

$$+ \int_0^t \int_D \Delta G_{t-s}(x, y) \left( \frac{f(u^\varepsilon(s, y)) - f(u^0(s, y))}{\sqrt{\varepsilon}} - f'(u^0(s, y))Y^\varepsilon(s, y) \right) dyds$$

$$+ \int_0^t \int_D \Delta G_{t-s}(x, y) f'(u^0(s, y))(Y^\varepsilon(s, y) - Y(s, y))dyds$$

$$= I_1^\varepsilon(t, x) + I_2^\varepsilon(t, x) + I_3^\varepsilon(t, x),$$

thus

$$\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| V^\varepsilon(t, \cdot) \|_{p}^q \right] \leq C \sum_{i=1}^3 \mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| I_i^\varepsilon(t, \cdot) \|_{p}^q \right]. \quad (28)$$

By Hölder’s inequality, BDG’s inequality and the Lipschitz continuity of $\sigma$, we can deduce that

$$\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| I_1^\varepsilon(t, \cdot) \|_{p}^q \right]$$

$$\leq C \mathbb{E} \left[ I_{A^\varepsilon(t,M)} \int_D |I_1^\varepsilon(t, \cdot)|^q dx \right]$$

$$= C \mathbb{E} \left[ I_{A^\varepsilon(t,M)} \int_D \left( \int_0^t \int_D G_{t-s}(x, y) \left[ \sigma(u^\varepsilon(s, y)) - \sigma(u^0(s, y)) \right] W(ds, dy) \right|^q dx \right]$$

$$\leq C \int_D \mathbb{E} \left[ I_{A^\varepsilon(t,M)} \left( \int_0^t \int_D G_{t-s}^2(x, y) |u^\varepsilon(s, y) - u^0(s, y)|^2 dyds \right)^{\frac{q}{2}} \right] dx. \quad (29)$$
By Hölder’s inequality, for some $0 < r < 1$, the above inequality implies that

$$
\mathbb{E} \left[ I_{A^v(t,M)} \right] \leq C \int_D \mathbb{E} \left[ I_{A^v(t,M)} \left( \int_0^t \int_D \left| G_{t-s}^2(x,y) - u^0(s,y) \right|^{\frac{q}{r}} \, dy \, ds \right)^{\frac{q}{r}} \right] \, dx
$$

where Eq. (14) is used in the last inequality.

Choosing $1 - r = \frac{2}{q}$ for some $q > 2$, by Eq. (13) in Lemma 3.1 and Hölder’s inequality,

$$
\mathbb{E} \left[ I_{A^v(t,M)} \right] \leq C(T) \mathbb{E} \left[ I_{A^v(t,M)} \left( \int_0^t \int_D \left| G_{t-s}^2(x,y) \right|^{\frac{q}{2}} \, dy \, ds \right)^{\frac{q}{2}} \right] \leq C(T) \mathbb{E} \left[ \left( \int_0^t \int_D \left| u^\varepsilon(s,y) - u^0(s,y) \right|^{\frac{q}{2}} \, dy \, ds \right)^{\frac{q}{2}} \right].
$$

Taking $p = q$, Proposition 3.2 yields that

$$
\mathbb{E} \left[ I_{A^v(t,M)} \right] \leq C(T) \varepsilon^\frac{q}{2}.
$$

Notice that $u^\varepsilon = u^0 + \sqrt{\varepsilon} Y^\varepsilon$. By the mean theorem for derivatives, there exists a random field $\xi^\varepsilon(t,x)$ taking values in $(0,1)$ such that

$$
\frac{f(u^\varepsilon) - f(u^0)}{\sqrt{\varepsilon}} = f'(u^0 + \sqrt{\varepsilon} \xi^\varepsilon Y^\varepsilon) Y^\varepsilon.
$$

By (H.2), we have

$$
\left| \frac{f(u^\varepsilon) - f(u^0)}{\sqrt{\varepsilon}} - f'(u^0) Y^\varepsilon \right| = \left| f'(u^0 + \sqrt{\varepsilon} \xi^\varepsilon Y^\varepsilon) - f'(u^0) \right| Y^\varepsilon
\leq C \sqrt{\varepsilon} \left( 2\left| u^0 \right| + \left| \sqrt{\varepsilon} Y^\varepsilon \right| + 1 \right) \left| Y^\varepsilon \right|^2.
$$
By the same calculation as that in Eq. (22), we have for any $1 \leq \rho \leq p$

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| I_2^\varepsilon(t, \cdot) \|_{p,\rho}^q \right]
\leq C(T) \mathbb{E} \left[ \int_0^t I_{A^\varepsilon(s,M)} \| f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot)) \|_{\rho}^q \frac{ds}{\sqrt{\varepsilon}} \right]
\leq \varepsilon^{\frac{q}{2}} C(T) \mathbb{E} \left[ \int_0^t I_{A^\varepsilon(s,M)} \left( |2u^0(s, \cdot)| + \sqrt{\varepsilon} |Y^\varepsilon(s, \cdot)| + 1 \right) |Y^\varepsilon(s, \cdot)|^2 \right] ds.
$$

(34)

Therefor, Hölder’s inequality, estimation (5) and Proposition 3.2 jointly yield that

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| I_2^\varepsilon(t, \cdot) \|_{p,\rho}^q \right] \leq \varepsilon^{\frac{q}{2}} C(T, M).
$$

(35)

Similarly, we have for any $1 \leq \rho \leq p$

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| I_2^\varepsilon(t, \cdot) \|_{p,\rho}^q \right] \leq C(T) \mathbb{E} \left[ \int_0^t I_{A^\varepsilon(s,M)} \| f'(u^0(s, \cdot))V^\varepsilon(s, \cdot) \|_{\rho}^q \right] ds.
$$

(36)

Putting (28), (32), (35) and (36) together, we have

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| V^\varepsilon(t, \cdot) \|_{p,\rho}^q \right] \leq C(T, M) \left( \varepsilon^{\frac{q}{2}} + \int_0^t \mathbb{E} \left[ I_{A^\varepsilon(s,M)} \| V^\varepsilon(s, \cdot) \|_{p,\rho}^q \right] ds \right).
$$

By Gronwall’s inequality, we have

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| V^\varepsilon(t, \cdot) \|_{p,\rho}^q \right] \leq C(T, M) \varepsilon^{\frac{q}{2}} \to 0, \quad \text{as } \varepsilon \to 0,
$$

(37)

which implies (A1).

**Step 2.** Notice that for any $0 \leq t < t' \leq T$ and $x \in D$,

$$
V^\varepsilon(t', x) - V^\varepsilon(t, x) = (Y^\varepsilon(t', x) - Y^\varepsilon(t, x)) - (Y(t', x) - Y(t', x)).
$$

To verify (A2) for $V^\varepsilon(t', \cdot) - V^\varepsilon(t, \cdot)$, it is sufficient to prove that for $Y^\varepsilon(t', \cdot) - Y^\varepsilon(t, \cdot)$ and $Y(t', \cdot) - Y(t, \cdot)$, respectively. Since the Hölder regularity of $Y(t', \cdot) - Y(t, \cdot)$ is given in Eq. (10), we only need to give the proof for $Y^\varepsilon(t', \cdot) - Y^\varepsilon(t, \cdot)$, that is there exists a constant $\gamma > 0$ such that, for all $t, t' \in [0, T]$,

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| Y^\varepsilon(t', \cdot) - Y^\varepsilon(t, \cdot) \|_{p,\rho}^q \right] \leq C|t' - t|^\gamma.
$$

(38)

Since for any $t \in [0, T], \frac{t}{T} \in [0, 1]$, we only need to prove the existence of $\gamma$ for any $t, t' \in [0, 1]$.

Notice that

$$
Y^\varepsilon(t, x) = \int_0^t \int_D G_{t-s}(x, y)\sigma(u^\varepsilon(s, y))W(ds, dy)
+ \int_0^t \int_D \Delta G_{t-s}(x, y)\frac{f(u^\varepsilon(s, y)) - f(u^0(s, y))}{\sqrt{\varepsilon}}dyds
=: J^\varepsilon_1(t, x) + J^\varepsilon_2(t, x).
$$

(39)
By the proof of Theorem 1.4 in [4], there exists a constant $\gamma_1 \in \left(0, \frac{1}{2} \left(1 - \frac{1}{q}\right)\right)$ such that

$$
\mathbb{E} \left[ \| J_1^\varepsilon(t', \cdot) - J_2^\varepsilon(t, \cdot) \|_p^q \right] \leq C |t' - t|^{\gamma_1 q}.
$$

(40)

For the second term, by Eq. (16) and Hölder inequality, for any $\rho \in [1, p]$, $\kappa = 1/q - 1/p + 1 \in [0, 1]$, there exist $\gamma_2 \in \left(0, \frac{1}{2} + \frac{1}{4}(\kappa - 1)\right)$ and $\beta \in \left(\frac{1}{\frac{1}{2} + \frac{1}{4}(\kappa - 1) - \gamma_2}, q\right)$, such that

$$
\mathbb{E} \left[ I_{A^\varepsilon(t,M)} \| J_2^\varepsilon(t', \cdot) - J_2^\varepsilon(t, \cdot) \|_p^q \right]
\leq C |t' - t|^{\gamma_2 q} \mathbb{E} \left[ I_{A^\varepsilon(t,M)} \left( \int_0^t \left\| \frac{f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot))}{\varepsilon} \right\|_\rho^\beta ds \right)^{q/\beta} \right]
\leq C |t' - t|^{\gamma_2 q} \mathbb{E} \left[ \int_0^t I_{A^\varepsilon(s,M)} \left\| \frac{f(u^\varepsilon(s, \cdot)) - f(u^0(s, \cdot))}{\varepsilon} \right\|_\rho^q ds \right]
\leq C(T, M) |t' - t|^{\gamma_2 q},
$$

(41)

where the last inequality is due to the same calculation as that of Eq. (23).

Combining (39)-(41), choosing $\gamma = \gamma_1 \wedge \gamma_2$, we get (38).

The proof is complete. $\square$

4. Moderate Deviation Principle

For any $N \in \mathbb{N}$, let

$$
\mathcal{H}_N := \left\{ v \in L^2([0, T] \times \mathcal{D}); \| v \| := \int_0^T \int_\mathcal{D} v(t, x)^2 dx dt \leq N \right\},
$$

which is a compact metric space of $L^2([0, T] \times \mathcal{D})$ in the weak topology. Denote $\mathcal{G}$ the set of predictable processes belonging to $L^2([0, T] \times \mathcal{D})$ a.s., and

$$
\mathcal{G}_N := \left\{ v \in \mathcal{G}; \ v(\omega) \in \mathcal{H}_N, \ \mathbb{P}\text{-a.s.} \right\}.
$$

For any $\varepsilon \in (0, 1)$ and $v \in \mathcal{G}_N$, consider the controlled equation $Z^{\varepsilon,v}$ defined by

$$
Z^{\varepsilon,v}(t, x) = \frac{1}{h(\varepsilon)} \int_0^t \int_\mathcal{D} G_{t-s}(x, y) \sigma(u^0(s, y) + \sqrt{\varepsilon} h(\varepsilon) Z^{\varepsilon,v}(s, y)) W(ds, dy)
+ \int_0^t \int_\mathcal{D} G_{t-s}(x, y) \sigma(u^0(s, y) + \sqrt{\varepsilon} h(\varepsilon) Z^{\varepsilon,v}(s, y)) v(s, y) dy ds
+ \int_0^t \int_\mathcal{D} \Delta G_{t-s}(x, y) \frac{f(u^0(s, y) + \sqrt{\varepsilon} h(\varepsilon) Z^{\varepsilon,v}(s, y)) - f(u^0(s, y))}{\sqrt{\varepsilon} h(\varepsilon)} dy ds,
$$

(42)

Following the proof of Theorem 3.2 in [4], one can prove that Eq. (12) admits a unique solution satisfying

$$
\sup_{\varepsilon \leq 1} \sup_{v \in \mathcal{G}_N} \mathbb{E} \left[ \| Z^{\varepsilon,v}(t, \cdot) \|_p^q \right] < +\infty,
$$

(43)

for $q \geq p$.  

Consider the following conditions which correspond to the weak convergence approach in our setting:

(a) For any family \( \{v^\varepsilon; \, \varepsilon > 0\} \subset \mathcal{G}_N \) which converges in distribution as \( H_N \)-valued random elements to \( v \in \mathcal{G}_N \) as \( \varepsilon \to 0 \),
\[
\lim_{\varepsilon \to 0} Z^{\varepsilon, v^\varepsilon} = Z^v
\]
in distribution,
as \( E_\varepsilon \)-valued random variables, where \( Z^v \) denotes the solution of Eq. (12) corresponding to the \( H_N \)-valued random variable \( v \) (instead of a deterministic function);

(b) The set \( \{Z^v; \, v \in H_N\} \) is a compact set of \( E_\varepsilon \), where \( Z^v \) is the solution of Eq. (12).

**The proof of Theorem 2.2.** According to [6, Theorem 6], we need to prove that Condition (a), (b) are fulfilled. Firstly, we verify Condition (a).

By the Skorokhod representation theorem, there exist a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})\), and, on this basis, a sequence of independent Brownian motions \( \tilde{W} \) and also a family of \( \mathcal{F}_t \)-predictable processes \( \{\tilde{v}^\varepsilon; \, \varepsilon > 0\}, \tilde{v} \) taking values in \( H_N \), \( \tilde{\mathbb{P}} \)-a.s., such that the joint law of \( (v^\varepsilon, v, \tilde{W}) \) under \( \mathbb{P} \) coincides with that of \( (\tilde{v}^\varepsilon, \tilde{v}, \tilde{W}) \) under \( \tilde{\mathbb{P}} \) and
\[
\lim_{\varepsilon \to 0} \int_0^T \int_D (\tilde{v}^\varepsilon(s, y) - \tilde{v}(s, y))g(s, y)dyds = 0, \quad \forall g \in L^2([0, T] \times D), \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{44}
\]

Let \( Z^{\varepsilon, \tilde{v}^\varepsilon} \) be the solution of a similar equation to (12) replacing \( v \) by \( \tilde{v}^\varepsilon \) and \( W \) by \( \tilde{W} \), and let \( \tilde{Z}^{\tilde{v}} \) be the solution of Eq. (12) corresponding to the \( H_N \)-valued random variable \( \tilde{v} \).

Now, we shall prove that for any \( q \geq p \) and \( \alpha \leq \frac{\gamma}{4}, \alpha < \frac{1}{2}(1 - \frac{\gamma}{4}) \),
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \|Z^{\varepsilon, \tilde{v}^\varepsilon} - Z^{\tilde{v}}\|^q_{\alpha, p} \right] = 0, \tag{45}
\]
which implies the validity of Condition (a). Here the expectation in (45) refers to the probability \( \tilde{\mathbb{P}} \).

From now on, we drop the bars in the notation for the sake of simplicity, and we denote
\[
X^{\varepsilon, v, v} := Z^{\varepsilon, v^\varepsilon} - Z^v.
\]

By Lemma A1 in [7], in order to prove (45), it is sufficient to prove that:

1. For all \( t \in [0, T] \),
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \|X^{\varepsilon, v, v}(t, \cdot)\|^q_p \right] = 0. \tag{46}
\]

2. There exists a positive constant \( C \) and \( \gamma > 0 \) such that: for all \( t, t' \in [0, t] \)
\[
\sup_{\varepsilon \in (0, 1]} \mathbb{E} \left[ \|X^{\varepsilon, v, v}(t', \cdot) - X^{\varepsilon, v, v}(t, \cdot)\|^q_p \right] \leq C|t - t'|^{\gamma q}. \tag{47}
\]

According to the proof of Theorem 4.1 in [4], one can get the Hölder regularities for \( Z^v(t, \cdot), Z^{\varepsilon, v^\varepsilon}(t, \cdot) \), that is for some \( \gamma > 0 \),
\[
\mathbb{E} \left[ \|Z^v(t', \cdot) - Z^v(t, \cdot)\|^q_p \right] \leq C|t - t'|^{\gamma q},
\]
\[
\sup_{\varepsilon \in (0,1]} \mathbb{E} \left[ \left\| Z^{\varepsilon, v^\varepsilon}(t', \cdot) - Z^{\varepsilon, v^\varepsilon}(t, \cdot) \right\|_p^q \right] \leq C|t - t'|^{\gamma q}.
\]

Thus we get (47). Next we prove (46).

Notice that for any \((t, x) \in [0, T] \times \mathcal{D},\)

\[
Z^{\varepsilon, v^\varepsilon}(t, x) - Z^v(t, x)
= \frac{1}{h(\varepsilon)} \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y) \sigma(u^0(s, y) + \sqrt{h(\varepsilon)} Z^{\varepsilon, v^\varepsilon}(s, y)) W(ds, dy)
+ \left\{ \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y) \sigma(u^0(s, y) + \sqrt{h(\varepsilon)} Z^{\varepsilon, v^\varepsilon}(s, y)) v^\varepsilon(s, y) dy ds \right\}
+ \left\{ \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y) f(u^0(s, y) + \sqrt{h(\varepsilon)} Z^{\varepsilon, v^\varepsilon}(s, y)) - f(u^0(s, y)) \frac{dy ds}{\sqrt{h(\varepsilon)}} \right\}
- \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y) f'(u^0(s, y)) Z^v(s, y) dy ds \right\}
= : A_1^\varepsilon(t, x) + A_2^\varepsilon(t, x) + A_3^\varepsilon(t, x).
\]

**Step 1.** For the first term \(A_1^\varepsilon(t, x),\) following the same calculation as that for \(\mathbb{E} \left[ \left\| T_1^\varepsilon(t, \cdot) \right\|_p^q \right] \) in the proof of Proposition 3.2, we have that

\[
\mathbb{E} \left[ \left\| A_1^\varepsilon(t, \cdot) \right\|_p^q \right] \leq \frac{C(T)}{h(\varepsilon)^{\gamma q}},
\]

for some positive constant \(C(T).\)

**Step 2.** For the second term, we divide it into two terms:

\[
A_2^\varepsilon(t, x)
= \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y) \left[ \sigma(u^0(s, y) + \sqrt{h(\varepsilon)} Z^{\varepsilon, v^\varepsilon}(s, y)) - \sigma(u^0(s, y)) \right] v^\varepsilon(s, y) dy ds
+ \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y) \sigma(u^0(s, y)) [v^\varepsilon(s, y) - v(s, y)] ds dy
= : A_{2,1}^\varepsilon(t, x) + A_{2,2}^\varepsilon(t, x).
\]
By Cauchy-Schwarz inequality and the Lipschitz continuity of \( \sigma \), we obtain that
\[
\mathbb{E} \left[ \| A_{2,1}^\varepsilon(t, \cdot) \|_p^q \right] \leq C(\sqrt{\varepsilon} h(\varepsilon))^q \mathbb{E} \left[ \| v^\varepsilon \|_p^q \cdot \left( \int_0^t \int_D G_{t-s}^2(\cdot, y) |Z_{t-s}^{\varepsilon, v}(s, y)|^2 dy ds \right)^{\frac{1}{2}} \right]^{\frac{q}{2}} \\
\leq CN^{\frac{q}{2}}(\sqrt{\varepsilon} h(\varepsilon))^q \mathbb{E} \left[ \left( \int_0^t \int_D G_{t-s}^2(\cdot, y) |Z_{t-s}^{\varepsilon, v}(s, y)|^2 dy ds \right)^{\frac{1}{2}} \right]^{\frac{q}{2}}.
\]

Hölder’s inequality, Eq. (13) and Eq. (14) imply that
\[
\mathbb{E} \left[ \left( \int_0^t \int_D G_{t-s}^2(\cdot, y) |Z_{t-s}^{\varepsilon, v}(s, y)|^2 dy ds \right)^{\frac{1}{2}} \right]^{\frac{q}{2}} \\
\leq C(T) \mathbb{E} \left[ \int_D \left( \int_0^t \int_D G_{t-s}^2(x, y) dy ds \right)^{\frac{q}{2}} \right]^{\frac{q}{2}} \\
\leq C(T) \mathbb{E} \left[ \int_D \left( \int_0^t |Z_{t-s}^{\varepsilon, v}(s, \cdot)|^{p} ds \right)^{\frac{q}{2}} \right]^{\frac{q}{2}} \\
\leq C(T) \mathbb{E} \left( \int_0^t (t-s)^{-\frac{q}{4}} |Z_{t-s}^{\varepsilon, v}(s, \cdot)|^{q} ds \right) \\
\leq C(T) \mathbb{E} \left( \int_0^t (t-s)^{-\frac{q}{4}} |Z_{t-s}^{0, v}(s, \cdot)|^{q} ds \right).
\]

Thus by Hölder’s inequality and estimation (13), we get
\[
\mathbb{E} \left[ \| A_{2,1}^\varepsilon(t, \cdot) \|_p^q \right] \leq C(T, N)(\sqrt{\varepsilon} h(\varepsilon))^q,
\]

Next, we will show that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \| A_{2,1}^\varepsilon(t, \cdot) \|_p^q \right] = 0.
\]

For any \((t, x) \in [0, T] \times D\), by the boundedness of \( \sigma \) and Eq. (14), we have
\[
\int_0^T \int_D G_{t-s}^2(x, y) \sigma^2(u^0(s, y)) dy ds \leq C \int_0^T \int_D G_{t-s}^2(x, y) dy ds < +\infty,
\]
which implies that for any \((t, x) \in [0, T] \times D\), the function \( \{ G_{t-s}(x, y) \sigma(u^0(s, y)) \}; (s, y) \in [0, T] \times D \) takes its values in \( H_N \) for some \( N \in \mathbb{N} \). Since \( \nu^\varepsilon \to \nu \) weakly in \( H_N \), we obtain that:
\[
\lim_{\varepsilon \to 0} A_{2,2}^\varepsilon(t, x) = 0, \quad \text{a.s.}
\]
By Cauchy-Schwarz’s inequality on $\mathcal{H}_N$ and the facts that $v^\varepsilon, v \in \mathcal{H}_N$ a.s., we obtain that

$$|A_{2,2}^\varepsilon(t, x)| \leq \left( \int_0^t \int_D G_{t-s}(x,y)^2(u^0(s, \cdot))dyds \right)^{\frac{1}{2}} \left( \int_0^t \int_D (v^\varepsilon(s, y) - v(s, y))^2dyds \right)^{\frac{1}{2}}$$

$$\leq C(N) \left( \int_0^t \int_D G_{t-s}(x,y)^2(u^0(s, \cdot))dyds \right)^{\frac{1}{2}}$$

$$\leq C(N, T), \quad (56)$$

where $C(N, T)$ is independent of $(\varepsilon, t, x)$. This implies that $A_{2,2}^\varepsilon(t, x)$ is uniformly bounded a.s.. By the dominated convergence theorem, we obtain (54).

**Step 3.** For the third term, it is also further divided into two terms

$$A_{3}^\varepsilon(t, x)$$

$$= \int_0^t \int_D \Delta G_{t-s}(x,y) \left[ f(u^0(s, y) + \sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s, y)) - f(u^0(s, y)) - f'(u^0(s, y))Z^{\varepsilon,v^\varepsilon}(s, y) \right]dyds$$

$$+ \int_0^t \int_D \Delta G_{t-s}(x,y)f'(u^0(s, y))(Z^{\varepsilon,v^\varepsilon}(s, y) - Z^v(s, y))dyds$$

$$= A_{3,1}^\varepsilon(t, x) + A_{3,2}^\varepsilon(t, x). \quad (57)$$

By the mean theorem for derivatives, for each $\varepsilon \in (0, 1]$, there exist random fields $\xi^\varepsilon(t, x)$ and $\eta^\varepsilon(t, x)$ taking values in $(0, 1)$ such that

$$\frac{f(u^0 + \sqrt{\varepsilon}h(\varepsilon)\xi^\varepsilon Z^{\varepsilon,v^\varepsilon}(s, y)) - f(u^0) - f'(u^0)Z^{\varepsilon,v^\varepsilon}}{\sqrt{\varepsilon}h(\varepsilon)}$$

$$= f'(u^0 + \sqrt{\varepsilon}h(\varepsilon)\xi^\varepsilon Z^{\varepsilon,v^\varepsilon})Z^{\varepsilon,v^\varepsilon} - f'(u^0)Z^{\varepsilon,v^\varepsilon}$$

$$= \sqrt{\varepsilon}h(\varepsilon)\xi^\varepsilon f''(u^0 + \sqrt{\varepsilon}h(\varepsilon)\eta^\varepsilon Z^{\varepsilon,v^\varepsilon})(Z^{\varepsilon,v^\varepsilon})^2.$$

Thus by the same calculation as Eq.(22), we have for any $\rho \in [1, p]$,

$$\mathbb{E} \left[ \|A_{3,1}^\varepsilon(t, \cdot)\|_p^q \right]$$

$$\leq (\sqrt{\varepsilon}h(\varepsilon))^q \mathbb{E} \left[ \left\| \int_0^t \int_D |\Delta G_{t-s}(x,y)|f''(u^0(s, y) + \sqrt{\varepsilon}h(\varepsilon)\eta^\varepsilon Z^{\varepsilon,v^\varepsilon}(s, y))(Z^{\varepsilon,v^\varepsilon}(s, y))^2 |dyds \right\|_p^q \right]$$

$$\leq C(T)(\sqrt{\varepsilon}h(\varepsilon))^q \mathbb{E} \left[ \int_0^t \left\| f''(u^0(s, \cdot) + \sqrt{\varepsilon}h(\varepsilon)\eta^\varepsilon Z^{\varepsilon,v^\varepsilon}(s, \cdot))(Z^{\varepsilon,v^\varepsilon}(s, \cdot))^2 \right\|_p^q ds \right]$$

$$\leq C(T)(\sqrt{\varepsilon}h(\varepsilon))^q, \quad (58)$$

where the last inequality is due to (H.2), Hölder’s inequality, Eq.(5) and Eq.(13).
Similarly to $A_{3,1}^\varepsilon$, one can prove that for any $\rho \in [1, p]$

$$
\mathbb{E} \left[ \left\| A_{3,2}^\varepsilon(t, \cdot) \right\|_p^q \right] \leq C(T) \mathbb{E} \left[ \int_0^t \left\| f'(u^0(s, \cdot))(Z_{\varepsilon,v}^\varepsilon(s, \cdot) - Z^v(s, \cdot)) \right\|_\rho^q \, ds \right] \\
\leq C(T) \int_0^t \mathbb{E} \left[ \left\| Z_{\varepsilon,v}^\varepsilon(s, \cdot) - Z^v(s, \cdot) \right\|_p^q \right] \, ds.
$$

(59)

Putting (48)-(53), (57)-(59) together, we have

$$
\mathbb{E} \left[ \left\| X_{\varepsilon,v}^\varepsilon(t, \cdot) \right\|_p^q \right] \leq C \left( h^{-q}(\varepsilon) + (\sqrt{\varepsilon} h(\varepsilon))^q + \mathbb{E} \left[ \left\| A_{2,2}^\varepsilon(t, \cdot) \right\|_p^q \right] \right)
$$

$$
+ (\sqrt{\varepsilon} h(\varepsilon))^q + \int_0^t \mathbb{E} \left[ \left\| X_{\varepsilon,v}^\varepsilon(s, \cdot) \right\|_p^q \right] \, ds.
$$

By Gronwall’s inequality and (54), we obtain that

$$
\mathbb{E} \left[ \left\| X_{\varepsilon,v}^\varepsilon(t, \cdot) \right\|_p^q \right] \leq C \left( h^{-q}(\varepsilon) + (\sqrt{\varepsilon} h(\varepsilon))^q + \mathbb{E} \left[ \left\| A_{2,2}^\varepsilon(t, \cdot) \right\|_p^q \right] \right) \longrightarrow 0, \quad \text{as } \varepsilon \to 0.
$$

Thus (46) is verified, which implies the Condition (a).

Replacing $\frac{1}{\varepsilon} \int_0^t \int_D G_{\varepsilon-s}(x, y) \sigma(u^0(s, y) + \sqrt{\varepsilon} h(\varepsilon) Z_{\varepsilon,v}^\varepsilon(s, y)) W(ds, dy)$ in the proof of the Condition (a), we obtain that $Z^v$ is a continuous mapping from $v \in \mathcal{H}_N$ into $\mathcal{E}_\alpha$.

Since $\mathcal{H}_N$ is compact in the weak topology of $L^2([0, T] \times D)$, $\{Z^v; v \in \mathcal{H}_N\}$ is a compact subset of $\mathcal{E}_\alpha$. That is the Condition (b).

The proof is complete. 

\[\square\]

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