Improved Lower Bound for Frankl’s Union-Closed Sets Conjecture

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Abstract

We verify an explicit inequality conjectured in [8], thus proving that for any nonempty union-closed family \( F \subseteq 2^n \), some \( i \in [n] \) is contained in at least a \( \frac{3 - \sqrt{5}}{2} \approx 0.38 \) fraction of the sets in \( F \). One case, an explicit one-variable inequality, is checked by computer calculation. \textbf{Mathematics Subject Classifications:} 05D05

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1 Introduction

Let \( \mathcal{M}_\phi \) be the set of probability measures \( \mu \in \mathcal{P}([0, 1]) \) with expectation \( \phi \). Define

\[ F(\mu) = \mathbb{E}_{(x,y) \sim \mu \times \mu} H(xy) - \mathbb{E}_{x \sim \mu} H(x) \quad (1) \]

where \( H(x) = -x \log x - (1 - x) \log(1 - x) \) is the entropy function and \( \log \) denotes the natural logarithm. Note that \( F \) is continuous in the weak topology and \( \mathcal{M}_\phi \) is compact, so \( F \) has a minimizer over \( \mathcal{M}_\phi \). In this note, we will show the following results.

\textbf{Theorem 1.} For all \( \phi \in [0, 1] \), the minimum of \( F(\mu) \) over \( \mathcal{M}_\phi \) is attained at some \( \mu \) supported on at most two points. Furthermore, if a minimizer is supported on exactly two points, then one of the points is 0.

The case of \( \mu \) supported on \( \{0, x\} \) leads to the following definition:

\[ S = \{ \phi \in [0, 1] : \phi H(x^2) \geq xH(x) \ \forall x \in [\phi, 1] \}, \quad \phi^* = \min(S). \]

Note that the condition defining \( S \) is monotone in \( \phi \) and \( S \) is clearly closed, so \( \min(S) \) is well defined. As in the recent breakthrough [8] by Gilmer, a bound on Frankl’s union-closed conjecture follows from the above.

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Theorem 2. The union-closed conjecture holds with constant $1 - \phi^*$, i.e. for any non-empty union-closed family $F \subseteq 2^{[n]}$, some $i \in [n]$ is contained in at least $1 - \phi^*$ fraction of the sets in $F$.

Throughout this paper we set $\varphi = \frac{\sqrt{5} - 1}{2}$. In the Appendix, we give a numerical verification of the following claim. We require certain computer calculations (detailed in an attached Python file) to be accurate to within margin of error $10^{-3}$, which can be made completely rigorous using interval arithmetic.

Claim 3. If $x \in [\varphi, 1]$, then $\varphi H(x^2) \geq x H(x)$, with equality if and only if $x \in \{\varphi, 1\}$.

Assuming Claim 3, the following claim identifies the value of $\phi^*$. Then, Theorem 2 implies that the union-closed conjecture holds with constant $1 - \varphi = \frac{3 - \sqrt{5}}{2}$. This is a natural barrier for the method of [8] as explained therein. Interestingly, Claim 3 has been mentioned previously in a different context by [3].

Claim 4. We have that $\phi^* = \varphi$.

Related Work. The union-closed conjecture has been the subject of much study, see [1, 10, 14, 2, 9] or the survey [4]. The recent breakthrough [8] by Gilmer showed that this conjecture holds with constant 0.01.

Concurrently with and independently of this work, Chase and Lovett [6], Sawin [12], and Pebody [11] also proved the union-closed conjecture with constant $\frac{3 - \sqrt{5}}{2}$. [12] also outlined an argument to improve this bound by an additional small constant, which was subsequently made explicit in [15] (using Lemma 5 below) and [5]. Moreover, [12] and Ellis [7] found counterexamples to [8, Conjecture 1], which would have implied the full union-closed conjecture with constant $\frac{1}{2}$.

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2 Reduction to Two Point Masses

Lemma 5. $F$ is concave on $\mathcal{M}_\phi$ for any $\phi \in [0, 1]$, i.e.

$$pF(\mu_1) + (1 - p)F(\mu_2) \leq F(p\mu_1 + (1 - p)\mu_2) \quad \forall \mu_1, \mu_2 \in \mathcal{M}_\phi, \ p \in [0, 1].$$

Proof. Let $\gamma(x) = \mu([0, x])$ be the cumulative distribution function of $\mu$. Thus $\gamma(1) = 1$ and

$$\phi = \int_0^1 x \mu(dx) = 1 - \int_0^1 \gamma(x) \ dx,$$
so
\[ \int_0^1 \gamma(x) \, dx = 1 - \phi. \] (3)

Using integration by parts,
\[ \int_0^1 H(x) \mu(dx) = H(x) \gamma(x) \bigg|_0^1 - \int_0^1 H'(x) \gamma(x) \, dx = \int_0^1 \left( \log \frac{x}{1-x} \right) \gamma(x) \, dx. \]

Similarly,
\[ \int_0^1 H(xy) \mu(dy) = H(xy) \gamma(y) \bigg|_0^1 - \int_0^1 xH'(xy) \gamma(y) \, dy \\
= H(x) + \int_0^1 \left( x \log \frac{xy}{1-xy} \right) \gamma(y) \, dy; \]
\[ \int_0^1 \left( x \log \frac{xy}{1-xy} \right) \mu(dx) = \left( x \log \frac{xy}{1-xy} \right) \gamma(x) \bigg|_0^1 - \int_0^1 \frac{d}{dx} \left( x \log \frac{xy}{1-xy} \right) \gamma(x) \, dx, \]
\[ = \log \frac{y}{1-y} - \int_0^1 \left( \frac{1}{1-xy} + \log \frac{xy}{1-xy} \right) \gamma(x) \, dx; \]
\[ \int_{[0,1]^2} H(xy) \mu(dx) \mu(dy) = \int_0^1 H(x) \mu(dx) + \int_0^1 \gamma(y) \int_0^1 x \log \frac{xy}{1-xy} \mu(dx) \, dy, \]
\[ = 2 \int_0^1 \left( \log \frac{x}{1-x} \right) \gamma(x) \, dx \\
- \int_{[0,1]^2} \left( \frac{1}{1-xy} + \log \frac{xy}{1-xy} \right) \gamma(x) \gamma(y) \, dx \, dy. \]

So, letting \( F(\gamma) = F(\mu) \) by slight abuse of notation, we have
\[ F(\gamma) = \int_0^1 \left( \log \frac{x}{1-x} \right) \gamma(x) \, dx \\
- \int_{[0,1]^2} \left( \log x + \log y + \frac{1}{1-xy} + \log \frac{1}{1-xy} \right) \gamma(x) \gamma(y) \, dx \, dy. \]

We will show this is concave in \( \gamma \). The first integral is manifestly linear in \( \gamma \), and the contributions of \( \log x \) and \( \log y \) are linear because, in light of (3),
\[ \int_{[0,1]^2} \log x \gamma(x) \gamma(y) \, dx \, dy = (1 - \phi) \int_0^1 (\log x) \gamma(x) \, dx. \]

After removing these terms, we are reduced to showing convexity of
\[ \int_{[0,1]^2} \left( \frac{1}{1-xy} + \log \frac{1}{1-xy} \right) \gamma(x) \gamma(y) \, dx \, dy. \]
Note that both $\frac{1}{1-xy}$ and $\log \frac{1}{1-xy}$ are of the form $\sum_{k \geq 0} a_k x^k y^k$ for constants $a_k \geq 0$. Hence it suffices to prove convexity of

$$\iint_{[0,1]^2} x^k y^k \gamma(x) \gamma(y) \, dx \, dy = \left( \int_0^1 x^k \gamma(x) \, dx \right)^2$$

for any $k \geq 0$. This is the square of a linear function of $\gamma$, and hence is convex. (Note that all integrands are in $L^1$ and so there are no convergence issues.)

**Lemma 6.** $\arg\min_{\mu \in \mathcal{M}_\phi} F(\mu)$ contains some $\mu$ supported on at most two points.

**Proof.** This follows immediately from Lemma 5 and the Krein-Milman theorem since $\mathcal{M}_\phi$ is compact in the weak-* topology and convex, and all extreme measures in $\mathcal{M}_\phi$ are supported on 1 or 2 points (see e.g. [13] for more on the latter point).

For self-containedness, we also include an explicit and elementary version of this argument. First let $\mu \in \mathcal{M}_\phi$ be any minimizer of $F$ and note that $\mu$ can be approximated arbitrarily well in the weak topology by $\hat{\mu}$ with finite support. In particular for any $\varepsilon > 0$, there exists $\hat{\mu} \in \mathcal{M}_\phi$ with $F(\hat{\mu}) \leq F(\mu) + \varepsilon$ of the form

$$\hat{\mu}(a_i) = b_i - b_{i-1}, \quad 1 \leq i \leq k$$

for constants $0 \leq a_1 < \cdots < a_k \leq 1$ and $0 = b_0 < b_1 < \cdots < b_k = 1$. We claim that for any $\varepsilon > 0$, the minimal $k$ such that such a $\hat{\mu}$ exists is at most two. Indeed given such a $\hat{\mu}$ with $k \geq 3$, we may consider $\hat{\mu}_\eta$ defined by

$$\hat{\mu}_\eta(a_1) = b_1 - b_0 + \eta(a_3 - a_2),$$
$$\hat{\mu}_\eta(a_2) = b_2 - b_1 - \eta(a_3 - a_1),$$
$$\hat{\mu}_\eta(a_3) = b_3 - b_2 + \eta(a_2 - a_1),$$
$$\hat{\mu}_\eta(a_i) = \hat{\mu}(a_i) = b_i - b_{i-1}, \quad \forall i \in \{4, 5, \ldots, k\}.$$

Then there exist $c_1, c_2 > 0$ such that $\hat{\mu}_\eta \in \mathcal{M}_\phi$ if and only if $-c_1 \leq \eta \leq c_2$; moreover the map $\eta \mapsto F(\hat{\mu}_\eta)$ is concave by Lemma 5. It is easy to see that both $\hat{\mu}_{-c_1}, \hat{\mu}_{c_2}$ have support size at most $k - 1$, and at least one of $F(\hat{\mu}_{-c_1}), F(\hat{\mu}_{c_2})$ is at most $F(\hat{\mu})$ by concavity. Iterating this argument, we find a $\tilde{\mu} \in \mathcal{M}_\phi$ with support size at most 2 and with $F(\tilde{\mu}) \leq F(\hat{\mu}) \leq F(\mu) + \varepsilon$. Taking a subsequential weak limit of the resulting $\tilde{\mu}$ as $\varepsilon \to 0$ completes the proof. \[\square\]

### 3 Optimization over Two Point Masses

**Lemma 7.** If $\mu$ is supported on exactly two points, neither of which is 0, then $\mu$ is not a minimizer of $F$ over $\mathcal{M}_\phi$.

**Proof.** Suppose $\mu = p\delta_x + (1-p)\delta_y$ is a minimizer for $F$ over $\mathcal{M}_\phi$ for $0 < y < x < 1$ distinct and $0 < p < 1$. Then any $z \in [0,1]$ can be written as $z = qx + (1-q)y$ for some $q \in \mathbb{R}$ (which may be negative). We have

$$\mu + t\delta_z - tq\delta_x - t(1-q)\delta_y \in \mathcal{M}_\phi$$

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for sufficiently small $t \geq 0$ and so
\[
\lim_{t \to 0^+} \frac{F(\mu + t\delta_z - tq\delta_x - t(1-q)\delta_y) - F(\mu)}{t} \geq 0.
\]
It is not difficult to see from the definition (1) of $F$ that the left-hand limit equals
\[
f(z) - qf(x) - (1-q)f(y) \geq 0,
\]
for
\[
f(w) := 2[pH(xw) + (1-p)H(yw)] - H(w).
\]
Equation (4) implies that $f$ lies above the line passing through $(x, f(x))$ and $(y, f(y))$. Since $f$ is a smooth function and $x, y$ are in the interior of $[0, 1]$, we deduce that

(a) $f'(x) = f'(y) = \frac{f(x) - f(y)}{x - y}$, and

(b) $f''(x), f''(y) \geq 0$.

Moreover, (a) implies

(c) $f''(z) \leq 0$ for some $z \in [y, x]$.

However we compute using $H'(w) = \log \frac{1-w}{w}$ that:
\[
f'(z) = 2 \left[ px \log \frac{1-xz}{xz} + (1-p)y \log \frac{1-yz}{yz} \right] - \log \frac{1-z}{z},
\]
\[
f''(z) = -2 \left[ \frac{px}{z(1-xz)} + \frac{(1-p)y}{z(1-yz)} \right] + \frac{1}{z(1-z)}.
\]

Note that $g(z) := z(1-z)(1-xz)(1-yz)f''(z)$ has the same sign as $f''(z)$ and is a quadratic function in $z$ with leading coefficient
\[-2pxy - 2(1-p)xy + xy = -xy < 0.\]

Hence the inequalities $g(x), g(y) \geq 0$ and $g(z) \leq 0$ can hold only if $g$ and hence $f''$ vanishes on the entire interval $[x, y]$. This is impossible since we just saw $g$ has non-zero leading coefficient.

The case $x = 1, y > 0$ is very similar. While we have $f''(y) \geq 0$ as above, since 1 is not in the interior of $[0, 1]$, we cannot immediately deduce that $f''(1) \geq 0$. However in this case $g(z)$ is a multiple of $1-z$, and so $g(1) = 0 \geq 0$. Then the same argument applies: $g(z)$ is a quadratic polynomial with negative leading coefficient $-y < 0$. Because $g$ takes non-negative values at $y$ and 1, it takes positive values in between. However since $f$ is continuous on $[0, 1]$ and smooth on $(0, 1)$, and stays above the line segment through $(y, f(y))$ and $(1, f(1))$, it must have non-positive second derivative at some $z \in (y, 1)$. Since $g$ and $f''$ have the same sign on $(0, 1)$, this is a contradiction. (Note that $f''(1)$ does not actually exist if $x = 1$ and is not used in this argument.)
4 Conclusion

Proof of Theorem 1. Follows from Lemmas 6 and 7.

Lemma 8. We have that $\phi^* \geq \varphi$.

Proof. Note that $H(\varphi^2) = H(\varphi)$. If $\varphi < \varphi$, then $\phi H(\varphi^2) < \varphi H(\varphi)$, and so $\phi \notin S$.

Corollary 9. If $\phi \geq \phi^*$, then $F(\mu) \geq 0$ for all $\mu \in M_\phi$.

Proof. By Theorem 1, it suffices to check $F(\mu) \geq 0$ for $\mu = p\delta_x + (1 - p)\delta_0$ with $p = \phi/x$ and $x \in [\varphi, 1]$ (this includes the case $\mu = \delta_\varphi$, corresponding to $x = \varphi$). By monotonicity of the condition defining $S$, $\phi \in S$. So,

$$F(\mu) = \frac{\phi^2}{x^2} H(x^2) - \frac{\phi}{x} H(x) = \frac{\phi}{x^2} (\phi H(x^2) - xH(x)) \geq 0.$$  

From Theorem 1, we deduce the following tight version of Gilmer’s [8, Lemma 1]. Theorem 2 follows from Corollary 10 by the same argument as in [8, Proof of Theorem 1]. We recall Gilmer’s ingenious insight was that given a union-closed family $F \subseteq 2^\mathcal{S}$, if $A, A'$ are independent uniformly random samples from $F$, then $A \cup A' \in F$ is not uniformly random and thus has strictly smaller entropy. On the other hand, Corollary 10 can be applied element-by-element to show that $A \cup A'$ actually has equal or larger entropy.

Corollary 10. Suppose $\{p_c\}_{c \in \mathcal{S}} \subset [0, 1]$ is a finite sequence of real numbers and $c$ is a random variable supported on $\mathcal{S}$ such that $\mathbb{E}_c[p_c] \leq 1 - \phi^*$. If $c'$ is an independent copy of $c$, then

$$\mathbb{E}_{c,c'}[H(p_c + p_c' - p_c p_c')] \geq \mathbb{E}_c[H(p_c)].$$

Proof. Let $\mu$ be the distribution of $x = 1 - p_c$. Let $\phi = \mathbb{E}_{x \sim \mu}[x]$, so $\phi > \phi^*$. By Corollary 9,

$$\mathbb{E}_{c,c'}[H(p_c + p_c' - p_c p_c')] - \mathbb{E}_c[H(p_c)] = \mathbb{E}_{(x,y) \sim \mu \times \mu} H(xy) - \mathbb{E}_{x \sim \mu} H(x) = F(\mu) \geq 0.$$  

Finally, we verify Claim 4 assuming Claim 3.

Proof of Claim 4. Claim 3 implies $\varphi \in S$, so $\phi^* \geq \varphi$ by definition of $\phi^*$. On the other hand, Lemma 8 gives $\phi^* \geq \varphi$. 

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A Proof of Claim 3

In this appendix, we prove Claim 3. Throughout this appendix, we use Claims to indicate results requiring the correctness of computer outputs within margin of error $10^{-3}$ or greater. The only computations which rely on a computer are the entries in Tables 1 and 2. Figure 1 plots the function

$$G(x) = \varphi H(x^2) - xH(x),$$

from which Claim 3 can be checked visually. We show below that, assuming correctness of certain computer calculations to within margin of error $10^{-3}$,

$$G(x) \geq 0, \quad \forall x \in \varphi, 1].$$

We verify this separately on the intervals $I_1 = [\varphi, 0.77], I_2 = [0.76, 0.98], I_3 = [0.98, 1]$. 

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Figure 1: Plot of $G(x)$ for $x \in [0.6, 1]$. Claim 3 states the minimum value of 0 on $x \in [\varphi, 1]$ is achieved precisely at the endpoints $x \in \{\varphi, 1\}$.

A.1 Verification on $I_1$

We first compute the derivative of $G$:

$$G'(x) = 2x\varphi \log \frac{1-x^2}{x^2} - H(x) - x \log \frac{1-x}{x}$$

$$= 2x\varphi \log \frac{1-x^2}{x^2} + x \log x + (1-x) \log(1-x) + x \log x - x \log(1-x)$$

$$= 2x\varphi \log \frac{1-x^2}{x^2} + 2x \log x + (1-2x) \log(1-x)$$

Note that $G(\varphi) = G'(\varphi) = 0$, the latter since

$$G'(\varphi) = 2\varphi^2 \log(1/\varphi) + 2\varphi \log(\varphi) + (1 - 2\varphi) \log(\varphi^2)$$

$$= (-2\varphi^2 + 2\varphi + 2(1 - 2\varphi)) \log \varphi$$

$$= 2(1 - \varphi - \varphi^2) \log(\varphi) = 0.$$

Claim 11. Claim 3 holds on $I_1 = [\varphi, 0.77]$.

Proof. As $G(\varphi) = G'(\varphi) = 0$, it suffices to verify that $G$ is convex on $I_1$. It is not hard to check that its second derivative equals $G''(x) = L(x)/(1-x^2)$, where

$$L(x) := 2\varphi(1-x^2) \log(x^2 - 1) - 4\varphi - 2x^2 \log x + 2(x^2 - 1) \log(1-x) + x + 2 \log(x) + 1.$$  

We now estimate the Lipschitz constant of each non-constant term of $L$ on $x \in I_1$. For the first term,

$$\left| \frac{d}{dx} \left( 2\varphi(1-x^2) \log(x^2 - 1) \right) \right| \leq 2\varphi \sup_{x \in I_1} \left( |2x^3| + 2|x \log(x^2 - 1)| \right)$$

$$\leq 2\varphi(1.1 + 1.6 \cdot \log(2))$$

$$\leq 2\varphi \cdot 2.3 \leq 3.$$  

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since $\log(2) \leq 0.75$ and $\varphi \leq 5/8$. Next,

$$\left| \frac{d}{dx} (2x^2 \log(x)) \right| \leq \sup_{x \in I_1} |4x \log(x) + 2x|$$

$$\leq 1.6 \sup_{x \in I_1} |2 \log(x) + 1|$$

$$\leq 1.6$$

since $\log(x) \in [-1, 0]$ for all $x \in I_1$. Continuing, using $\log(5) \leq 2$,

$$\left| \frac{d}{dx} (2(x^2 - 1) \log(1 - x)) \right| \leq 2 \sup_{x \in I_1} |2x \log(1 - x) - \frac{x^2 - 1}{1 - x}|$$

$$\leq 2 \sup_{x \in I_1} |2x \log(1 - x) + x + 1|$$

$$\leq 2 \cdot \max(1.6 \log(5), 1.8)$$

$$\leq 2 \cdot 1.6 \cdot 2 = 6.4.$$ 

Finally $\frac{d}{dx} (x) = 1$ and $\frac{d}{dx} (2 \log x) = 2/x \leq 3.5$. Combining, we find that $L(x)$ restricted to $I_1$ has Lipschitz constant at most

$$1.6 + 6.4 + 1 + 3 + 3.5 \leq 15.5.$$ 

Therefore to show $G$ is convex and hence non-negative on $I_1 = [\varphi, 0.77]$ it suffices to exhibit a $\frac{1}{400}$-dense subset of $I_1$ on which $L(x) = (1 - x^2)G''(x) \geq 0.04 \geq \frac{15.5}{400}$. In Table 1 below we compute the values of $L$ on each multiple of $\frac{1}{400}$ from 0.6 to 0.77 inclusive. We find that $L(x) \geq 0.09$ holds at all of these points, completing the numerical verification on $I_1$.

| x   | L(x) | x   | L(x) | x   | L(x) | x   | L(x) | x   | L(x) |
|-----|------|-----|------|-----|------|-----|------|-----|------|
| 0.600 | 0.1020 | 0.630 | 0.1117 | 0.660 | 0.1173 | 0.690 | 0.1182 | 0.720 | 0.1137 | 0.750 | 0.1032 |
| 0.605 | 0.1039 | 0.635 | 0.1130 | 0.665 | 0.1178 | 0.695 | 0.1178 | 0.725 | 0.1124 | 0.755 | 0.1009 |
| 0.610 | 0.1057 | 0.640 | 0.1141 | 0.670 | 0.1182 | 0.700 | 0.1173 | 0.730 | 0.1109 | 0.760 | 0.0983 |
| 0.615 | 0.1074 | 0.645 | 0.1151 | 0.675 | 0.1184 | 0.705 | 0.1167 | 0.735 | 0.1093 | 0.765 | 0.0955 |
| 0.620 | 0.1089 | 0.650 | 0.1159 | 0.680 | 0.1185 | 0.710 | 0.1159 | 0.740 | 0.1075 | 0.770 | 0.0925 |
| 0.625 | 0.1104 | 0.655 | 0.1167 | 0.685 | 0.1184 | 0.715 | 0.1149 | 0.745 | 0.1054 | 0.745 | 0.1054 |

Table 1: Evaluations of $L$ to precision $10^{-4}$. All values appear to be at least 0.09, and it suffices for all values to be at least 0.04.

### A.2 Verification on $I_2$

Our verification for $x \in I_2$ is based on evaluating $G$. We write $G(x) = g_1(x) - g_2(x)$ for

$$g_1(x) = \varphi H(x^2),$$

$$g_2(x) = x H(x).$$

Note that $g_1$ is clearly decreasing on $I_2$. The next lemma shows the same for $g_2$. 

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Lemma 12. $g_2$ is decreasing on $[5/7, 1] \supseteq I_2$.

Proof. First we claim that it suffices to show $g'_2(5/7) \leq 0$. This is because

$$g'_2(x) = H(x) + x \log \frac{1-x}{x} = 2x \log \frac{1}{x} - (2x - 1) \log \frac{1}{1-x}$$

so $g'_2(x) \leq 0$ if and only if

$$\left(1 - \frac{1}{2x}\right) \log \frac{1}{1-x} \geq \log \frac{1}{x}. \quad (6)$$

Indeed both terms on the left-hand side are increasing while the right-hand side is decreasing.

It remains to show that $g'_2(5/7) \leq 0$ which in light of (6) is equivalent to showing

$$\frac{3}{10} \log(7/2) \geq \log(7/5),$$

i.e. $(7/5)^{10/3} \leq 7/2$. This holds since $(7/5)^3 \leq 2(7/5) = 14/5$ and $7/5 \leq \left(\frac{5}{4}\right)^3 = \left(\frac{7/2}{14/5}\right)^3$. \hfill \qed

Claim 13. Claim 3 holds for $x \in I_2$.

Proof. We computer-evaluate $g_1, g_2$ at a finite set of values $x_1 < x_2 < \cdots < x_{97}$ with $5/7 < x_1 < 0.76$ and $x_{97} = 0.98$ and verify that $g_1(x_{i+1}) \geq g_2(x_i)$ for each $i$. The values are shown in Table 2; note that in all cases $g_1(x_{i+1}) - g_2(x_i) \geq \frac{2}{1000}$ holds, modulo rounding to four decimal places. The intervals $[x_i, x_{i+1}]$ cover $I_2$, and for all $x \in [x_i, x_{i+1}]$ we have $g_2(x) \leq g_2(x_i) \leq g_1(x_{i+1}) \leq g_1(x)$. \hfill \qed

A.3 Verification on $I_3$

Proposition 14. Claim 3 holds for $x \in I_3$.

Proof. Taylor expansion of $\log(1-e)$ gives that for all $\varepsilon \in (0, 1)$,

$$\varepsilon \left( \log \frac{1}{\varepsilon} + 1 - \varepsilon \right) \leq H(\varepsilon) \leq \varepsilon \left( \log \frac{1}{\varepsilon} + 1 \right).$$

Let $x = 1 - \varepsilon$ for $\varepsilon \in [0, 0.02]$. Then

$$g_1(x) = \varphi H(2\varepsilon - \varepsilon^2) \geq \varphi \varepsilon (2 - \varepsilon) \left( \log \frac{1}{\varepsilon} - \log(2 - \varepsilon) + (1 - \varepsilon)^2 \right),$$

$$g_2(x) = (1 - \varepsilon) H(\varepsilon) \leq \varepsilon (1 - \varepsilon) \left( \log \frac{1}{\varepsilon} + 1 \right).$$
\[
\begin{array}{cccccccc}
 x & g_1(x) & g_2(x) & x & g_1(x) & g_2(x) & x & g_1(x) \\
 0.7598 & 0.4210 & 0.4189 & 0.7797 & 0.4210 & 0.4111 & 0.8472 & 0.3678 \\
 0.7600 & 0.4209 & 0.4188 & 0.7814 & 0.4131 & 0.4103 & 0.8507 & 0.3643 \\
 0.7603 & 0.4208 & 0.4187 & 0.7832 & 0.4124 & 0.4095 & 0.8543 & 0.3606 \\
 0.7606 & 0.4207 & 0.4186 & 0.7851 & 0.4115 & 0.4085 & 0.8579 & 0.3567 \\
 0.7609 & 0.4206 & 0.4185 & 0.7871 & 0.4106 & 0.4075 & 0.8615 & 0.3528 \\
 0.7613 & 0.4205 & 0.4184 & 0.7892 & 0.4095 & 0.4064 & 0.8651 & 0.3486 \\
 0.7617 & 0.4204 & 0.4183 & 0.7913 & 0.4085 & 0.4053 & 0.8688 & 0.3442 \\
 0.7621 & 0.4203 & 0.4181 & 0.7935 & 0.4074 & 0.4041 & 0.8725 & 0.3397 \\
 0.7626 & 0.4201 & 0.4180 & 0.7958 & 0.4062 & 0.4028 & 0.8762 & 0.3350 \\
 0.7631 & 0.4200 & 0.4178 & 0.7982 & 0.4048 & 0.4014 & 0.8799 & 0.3301 \\
 0.7637 & 0.4198 & 0.4176 & 0.8007 & 0.4034 & 0.3999 & 0.8836 & 0.3251 \\
 0.7643 & 0.4196 & 0.4174 & 0.8033 & 0.4019 & 0.3983 & 0.8873 & 0.3201 \\
 0.7650 & 0.4194 & 0.4171 & 0.8060 & 0.4003 & 0.3965 & 0.8909 & 0.3146 \\
 0.7657 & 0.4191 & 0.4169 & 0.8088 & 0.3985 & 0.3947 & 0.8945 & 0.3092 \\
 0.7665 & 0.4189 & 0.4166 & 0.8116 & 0.3967 & 0.3927 & 0.8981 & 0.3035 \\
 0.7673 & 0.4186 & 0.4163 & 0.8145 & 0.3948 & 0.3907 & 0.9017 & 0.2977 \\
 0.7682 & 0.4183 & 0.4159 & 0.8175 & 0.3927 & 0.3884 & 0.9052 & 0.2919 \\
 0.7692 & 0.4179 & 0.4156 & 0.8206 & 0.3904 & 0.3861 & 0.9087 & 0.2859 \\
 0.7702 & 0.4176 & 0.4152 & 0.8237 & 0.3881 & 0.3836 & 0.9122 & 0.2797 \\
 0.7713 & 0.4172 & 0.4147 & 0.8269 & 0.3857 & 0.3810 & 0.9156 & 0.2734 \\
 0.7725 & 0.4167 & 0.4142 & 0.8301 & 0.3831 & 0.3783 & 0.9190 & 0.2670 \\
 0.7738 & 0.4163 & 0.4137 & 0.8334 & 0.3803 & 0.3754 & 0.9223 & 0.2606 \\
 0.7752 & 0.4157 & 0.4131 & 0.8368 & 0.3777 & 0.3723 & 0.9256 & 0.2539 \\
 0.7766 & 0.4152 & 0.4125 & 0.8402 & 0.3744 & 0.3691 & 0.9288 & 0.2473 \\
 0.7781 & 0.4145 & 0.4119 & 0.8437 & 0.3711 & 0.3657 & 0.9319 & 0.2406 \\
\end{array}
\]

Table 2: Evaluations of \(g_1\) and \(g_2\) to precision \(10^{-4}\). We require that for consecutive inputs \(x_i < x_{i+1}\) in the table, \(g_1(x_{i+1}) - g_2(x_i) \geq 0\). The values shown in fact satisfy \(g_1(x_{i+1}) - g_2(x_i) \geq \frac{2}{1000}\) modulo rounding.

Dividing by \(\varepsilon\), it suffices to prove

\[
(2\varphi - 1) + (1 - \varphi)\varepsilon \log \frac{1}{\varepsilon} \geq (1 - \varepsilon)(1 - \varphi(1 - \varepsilon)(2 - \varepsilon)) + \varphi(2 - \varepsilon) \log(2 - \varepsilon).
\]

Noting \(\varphi(1 - \varepsilon)(2 - \varepsilon) \geq 1\) in the first line below, we next find

\[
(1 - \varepsilon)(1 - \varphi(1 - \varepsilon)(2 - \varepsilon)) + \varphi(2 - \varepsilon) \log(2 - \varepsilon) \leq 2\varepsilon \log 2 = (\sqrt{5} - 1) \log 2,
\]

\[
(2\varphi - 1) + (1 - \varphi)\varepsilon \log \frac{1}{\varepsilon} \geq (2\varphi - 1) \log \frac{1}{\varepsilon} \geq (\sqrt{5} - 2) \log 50.
\]

Finally \((\sqrt{5} - 2) \log 50 \geq (\sqrt{5} - 1) \log 2\) because

\[
\log_2(50) \geq \log_2(2^5 \cdot 1.5) \geq 5.5
\]

\[
\geq 3 + \sqrt{5} = (\sqrt{5} - 1)/(\sqrt{5} - 2).
\]

Hence the proof is complete. Equality holds if and only if \(\varepsilon = 0\), i.e. \(x = 1\). \(\square\)

**Proof of Claim 3.** Follows by combining Claims 11, 13 and Proposition 14. \(\square\)

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