AN ASSESSMENT OF DIRECT MULTISEARCH WHEN ENRICHED WITH FIRST-ORDER INFORMATION FOR MULTIOBJECTIVE OPTIMIZATION

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Abstract. Direct MultiSearch (DMS) is a robust and efficient derivative-free optimization algorithm, able to generate approximations to the complete Pareto front of a given multiobjective optimization (MOO) problem. When first (or higher) order derivatives of the different components of the objective function are available, typical approaches for MOO problems are based on generating a single sequence of iterates that converges to a point with corresponding image lying on the Pareto front (one at a time). The purpose of this work is to assess the potential enrichment of adding first-order information, when derivatives are available, to the DMS framework. For that, we describe and analyze several different combined techniques that maintain the search/poll paradigm of DMS, while adding in a suitable way gradient information to the poll step. To properly evaluate the new proposed schemes, we provide numerical results for a set of benchmark MOO problems, in the form of performance profiles, where common performance metrics considered in the MOO community are reported. The proposed schemes are compared with the original DMS and also with the recently developed MOSQP approach that approximates the entire Pareto front at once, using first and second order information.

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1. Introduction. Multiobjective optimization (MOO) problems appear frequently in engineering and scientific applications, in such diverse areas such as civil engineering, environment, medicine or aerospace engineering [2, 27, 31, 32], just to cite a few. The major feature of a MOO problem is the presence of finitely many components in the objective function that have to be simultaneously optimized. The several objectives associated to the different components of the objective function are usually conflicting among each others. Hardly a single point will optimize all of them at once, hence a nonstandard notion of optimality is required. The fundamental optimality concept is that of Pareto optimal point, which is a point such that no improvement in all the components of the objective function can be achieved by moving to another feasible point. The image set of all Pareto optimal points (also called the Pareto front) is usually a continuum that may have disjoint components. In general, for a problem with \( p > 1 \) objectives, the Pareto front is a manifold of dimension \( p - 1 \). For example, if \( p = 2 \) the Pareto front will be a curve (or a set of curve segments), which provides in a compact way all the information required for a user to choose an appropriate Pareto optimal point as a compromise solution between the usually conflicting components of the objective function. Like in classical single objective optimization,
finding global Pareto optimal points is difficult, unless that additional information is available about the objective function. Thus, MOO algorithms typically try to find local Pareto optimal points for the problems, meaning that the definition of Pareto optimality is satisfied in a neighborhood of the current point.

There are several classes of MOO algorithms, depending on the level of smoothness of the objective function but also on the time when the user establishes an order preference for the different components of the objective function [28]. In this work, we will focus on methods with a posteriori articulation of preferences, which attempt to capture the whole Pareto front of the problem, never establishing preferences among the several components of the objective function. Evolutionary algorithms, or other similar heuristics, belong to this class. However, these algorithms miss a well-established convergence analysis and are usually slow in converging to the Pareto front of the problem, requiring a large number of iterations and function evaluations [18]. On the other hand, when derivatives of the different components of the objective function are available, typical approaches approximate one point at a time in the Pareto front [21, 30, 20, 6, 19]. Multistart approaches [29] or scalarization techniques [17], can help in finding approximations to the complete Pareto front of a given MOO problem. Although, the first can be computational expensive and the latter generally fails in detecting nonconvex parts of it [13].

Recently, a novel approach has been developed to approximate the entire Pareto front using first and second order information [22]. The so-called MOSQP method keeps a list of nondominated points, which approximates the Pareto front of the MOO problem, that is improved both for spread along the Pareto front and optimality by solving single-objective constrained optimization problems derived as SQP problems.

In derivative-free optimization, Direct MultiSearch (DMS) [12] is a well-established algorithm, able to compute approximations to the Pareto front of a given MOO problem, with theoretical results established regarding convergence, and consistently used with good results both for benchmark of new solvers [8, 26] or in real applications [5, 24].

The purpose of the current work is to assess the potential enrichment of adding first-order information, when derivatives are available, to the DMS framework and to compare it with derivative-based MOO methods, also able to generate approximations to the complete Pareto front of a MOO problem. For that, we describe and analyze several different combined techniques that maintain the search/poll paradigm of DMS, adding in a suitable way gradient information to the poll step.

The remaining of this document is organized as follows. In Section 2 we present the MOO problem and briefly revise the derivative-free optimization method DMS. Section 3 details the use of first-order information to eliminate directions in the poll step of DMS and reports extensive numerical results in an academic test set. In Section 4 the usefulness of ascent directions is motivated by illustrating their performance on one properly chosen biobjective problem. Results are then reported in the complete test set. Finally, in Section 5 we present some concluding remarks.

2. DMS at a glance. We consider multiobjective minimization problems of the form

\[
\min_{x \in \Omega} F(x) = (f_1(x), \ldots, f_p(x))^\top,
\]

where \( p \geq 2 \), \( \Omega \subseteq \mathbb{R}^n \) represents the feasible region, typically defined as a box \( \Omega = \{x \in \mathbb{R}^n : l \leq x \leq u\} \), and for each \( i \) (\( 1 \leq i \leq p \)) \( f_i : \Omega \to \mathbb{R} \cup \{+\infty\} \) denotes a
component of the objective function, which we assume to be strictly differentiable in \( \Omega \) (continuity of the partial derivatives is not required).

The Direct MultiSearch (DMS) method was originally proposed in [12], generalizing directional direct search to multiobjective derivative-free optimization. For a review on single objective derivative-free optimization methods we recommend [3, 9]. The algorithm has also been successfully extended to global multiobjective derivative-free optimization [11], by coupling it with a multistart initialization technique, where not all the initialized searches are conducted until the end.

Being a directional direct search method, each iteration of DMS conforms to the search/poll paradigm. The search step is optional, since the convergence results derive from the poll step of the algorithm. In fact, in the original presentation of the method [12], it was left empty and this will be the approach followed in the present work. Recently, the minimization of quadratic polynomial models, which have always played a key role in derivative-free methods for single objective optimization, was used for successfully defining a search step for DMS [4]. First-order information can surely be used to define appropriate search steps for the algorithm in [4], but that will not be the subject of the present work, which will focus on the poll step.

We present a simplified description of the DMS framework, where only the poll step is considered, and where the globalization strategy is based on the use of integer lattices, meaning that all the points generated by the algorithm lie on an implicit mesh. For a more general description, we refer to the original work [12].

The algorithm initializes with a list of feasible, nondominated points (eventually just one) and corresponding stepsizes parameters. Making use of the strict partial order induced by the cone \( \mathbb{R}^p_+ \), we say that point \( x \) dominates point \( y \) when \( F(x) \prec_F F(y) \), i.e., when \( F(y) - F(x) \in \mathbb{R}^p_+ \setminus \{0\} \). If \( x \) does not dominate \( y \) and \( y \) does not dominate \( x \), \( x \) and \( y \) are said to be nondominated. The list, representing the current approximation to the Pareto front of the MOO problem, will be updated at every iteration by generating new feasible points which are compared with the points already stored in it, only keeping the nondominated ones.

At each iteration, a feasible nondominate point stored in the list and the associated stepsizes parameter, will be selected. Different strategies can be considered in the selection of this poll center. Currently it is based on a spread metric, in an attempt of reducing the gaps between consecutive points lying in the current approximation to the Pareto front of the problem.

The poll step of the algorithm consists on a local search around the selected poll center, by testing a set of directions with an adequate geometry, scaled by the corresponding stepsizes parameter. Typically, positive spanning sets are considered [14], that should conform to the geometry of the nearby active constraints of the current poll center [25].

For convergence purposes, the poll step can be performed either in a complete or an opportunistic way. In the latter, the polling procedure is stopped once a new feasible nondominated point is found. The complete approach tests all the poll directions, only adding to the list the new feasible nondominated points found (and removing from the list all the dominated ones). We will follow this last approach, which is the one corresponding to the original algorithmic implementation of DMS [12], in an attempt of maximizing the number of feasible nondominated points generated at each iteration.

The final step of each iteration is the update of the stepsizes parameter, which is increased or kept constant for successful iterations and decreased for unsuccessful
ones. An iteration is said to be successful if the list changes, meaning that at least a new feasible nondominated point was found. Unsuccessful iterations keep the list unchanged.

A simplified description of the DMS framework is provided in Algorithm 1. For a complete description see [12].

Algorithm 1: A simplified description of Direct MultiSearch (DMS).

**Initialization**

Choose a set of nondominated points \( x_{0i} \in \Omega, \ 0i \in I \) with \( f_j(x_{0i}) < +\infty, \forall j \in \{1, \ldots, p\}, \forall 0i \in I, \ a_{0i} > 0, \ 0i \in I \) an initial stepsize, \( 0 < \beta_1 \leq \beta_2 < 1 \) the coefficients for stepsize contraction and \( \gamma \geq 1 \) the coefficient for stepsize expansion. Let \( D \) be a set of positive spanning sets.

Initialize the list of feasible nondominated points and corresponding stepsize parameters \( L = \{ (x_{0i}; a_{0i}), i \in I \} \).

For \( k = 0, 1, 2, \ldots \)

1. **Selection of an iterate point:** Order the list \( L_k \) according to some criteria and select the first item \( (x; \alpha) \in L_k \) as the current iterate and stepsize parameter (thus setting \( (x_k; \alpha_k) = (x; \alpha) \)).

2. **Poll step:** Choose a positive spanning set \( D_k \) from the set \( D \).

   Evaluate \( F \) at the feasible poll points belonging to \( \{x_k + \alpha_k d : d \in D_k\} \). Compute \( L_{\text{trial}} \) by removing all dominated points from \( L_k \cup \{(x_k + \alpha_k d; \alpha_k) : d \in D_k \land x_k + \alpha_k d \in \Omega\} \). If \( L_{\text{trial}} \neq L_k \) declare the iteration (and the poll step) successful and set \( L_{k+1} = L_{\text{trial}} \). Otherwise, declare the iteration (and the poll step) unsuccessful and set \( L_{k+1} = L_k \).

3. **Stepsize parameter update:** If the iteration was successful then maintain or increase the corresponding stepsize parameters, by considering \( \alpha_{k, \text{new}} \in [\alpha_k, \gamma \alpha_k] \) and replacing all the new points \( (x_k + \alpha_k d; \alpha_k) \) in \( L_{k+1} \) by \( (x_k + \alpha_k d; \alpha_{k, \text{new}}) \). Replace also \( (x_k; \alpha_k) \), if in \( L_{k+1} \), by \( (x_k; \alpha_{k, \text{new}}) \).

   Otherwise, decrease the stepsize parameter, by choosing \( \alpha_{k, \text{new}} \in [\beta_1 \alpha_k, \beta_2 \alpha_k] \), and replace the poll pair \( (x_k; \alpha_k) \) in \( L_{k+1} \) by \( (x_k; \alpha_{k, \text{new}}) \).

The convergence of DMS has been established in [12], closely following the arguments used in the analysis of single objective directional direct search methods. The first step is to ensure that a subsequence of stepsize parameters converges to zero. For that, let \( D(L_0) \) be the image of the set of points dominated by \( L_0 \) and consider Assumption 2.1 and one of the Assumptions 2.2 or 2.3.

**Assumption 2.1.** The set \( \{x \in \Omega : F(x) \notin D(L_0)\} \) is compact.

**Assumption 2.2.** The set \( D = D \) of positive spanning sets is finite and the elements of \( D \) are of the form \( G \tilde{z}_j, \ j = 1, \ldots, |D|, \) where \( G \in \mathbb{R}^{n \times n} \) is a nonsingular matrix and each \( \tilde{z}_j \) is a vector in \( \mathbb{Z}^n \).

**Assumption 2.3.** Let \( D \) represent a finite set of positive spanning sets satisfying Assumption 2.2.

The set \( D \) is so that the elements \( d_k \in D_k \in D \) satisfy the following conditions:

1. \( d_k \) is a nonnegative integer combination of the columns of \( D \).
2. The distance between $x_k$ and the point $x_k + \alpha_k d_k$ tends to zero if and only if $\alpha_k$ does:

$$\lim_{k \to K} \alpha_k \|d_k\| = 0 \iff \lim_{k \to K} \alpha_k = 0,$$

for any infinite subsequence $K$.

3. The limits of all convergent subsequences of $\bar{D}_k = \{d_k/\|d_k\| : d_k \in D_k\}$ are positive spanning sets for $\mathbb{R}^n$.

The update of the stepsize parameter also needs to follow strict rules, ensuring that all the points generated by the algorithm lie in an implicit mesh (see Assumption 2.4).

**Assumption 2.4.** Let $\tau > 1$ be a rational number and $m^{\max} \geq 0$ and $m^{\min} \leq -1$ integers. If the iteration is successful, then the stepsize parameter is maintained or increased by considering $\alpha_{new} = \tau m^+ \alpha$, with $m^+ \in \{0, \ldots, m^{\max}\}$. If the iteration is unsuccessful, then the stepsize parameter is decreased by setting $\alpha_{new} = \tau m^- \alpha$, with $m^- \in \{m^{\min}, \ldots, -1\}$.

Under the previous assumptions, the first result required for establishing the convergence can be derived.

**Theorem 2.1.** (see [12]) Let Assumption 2.1 and Assumption 2.4 hold, combined with one of the Assumptions 2.2 or 2.3. Algorithm 1 generates a sequence of iterates satisfying

$$\lim \inf_{k \to +\infty} \alpha_k = 0.$$

Combining the previous result with Assumption 2.1, the existence of a convergent refining subsequence of iterates can be guaranteed.

**Definition 2.2.** A subsequence $\{x_k\}_{k \in K}$ of iterates corresponding to unsuccessful poll steps is said to be a refining subsequence if $\{\alpha_k\}_{k \in K}$ converges to zero.

The concept of refining direction is associated with convergent refining subsequences and is formalized in Definition 2.3.

**Definition 2.3.** Let $x_*$ be the limit point of a convergent refining subsequence $\{x_k\}_{k \in K}$. If the limit $\lim_{k \to K'} d_k/\|d_k\|$ exists, where $K' \subseteq K$ and $d_k \in D_k$, and if $x_k + \alpha_k d_k \in \Omega$, for sufficiently large $k \in K'$, then this limit is said to be a refining direction for $x_*$. Finally, assuming the density of the set of refining directions in the Clarke tangent cone to $\Omega$ computed at limit points of refining subsequences [7], the convergence of DMS is established.

**Definition 2.4.** A vector $d \in \mathbb{R}^n$ is said to be a Clarke tangent vector to the set $\Omega \subset \mathbb{R}^n$ at the point $x_*$ in the closure of $\Omega$ if for every sequence $\{y_k\}$ of elements of $\Omega$ that converges to $x_*$ and for every sequence of positive real numbers $\{t_k\}$ converging to zero, there exists a sequence of vectors $\{w_k\}$ converging to $d$ such that $y_k + t_k w_k \in \Omega$.

**Theorem 2.5.** (see [12]) Consider a refining subsequence $\{x_k\}_{k \in K}$ converging to $x_* \in \Omega$. Assume that $F$ is strictly differentiable at $x_*$ and that the interior of the tangent cone to $\Omega$ at $x_*$ is nonempty. If the set of refining directions for $x_*$ is dense in the Clarke tangent cone to $\Omega$ at $x_*$, then $x_*$ is a Pareto-Clarke-KKT critical point, i.e.,

$$\forall d \in T^C_{\Omega}(x_*), \exists j \in \{1, 2, \ldots, p\} : \nabla f_j(d)(x_*)^\top d \geq 0.$$
Recently, worst-case complexity bounds were provided for DMS, but considering a globalization strategy that requires sufficient decrease for accepting new points [10]. For a particular algorithmic instance, which considers a stricter criterion for accepting new nondominated points, DMS presents a worst-case complexity bound of $O(\varepsilon^{-2})$, similar to the one of steepest descent.

3. Pruning the poll set. At each iteration of DMS a positive spanning set is selected as poll set. The poll points correspond to directions in the poll step scaled by the stepsize parameter. The objective function will then be evaluated at all the feasible poll points, independently of corresponding or not to descent directions.

The following result is well-known for positive spanning sets (see Theorem 2.3 in [9]).

**Theorem 3.1.** If $\{v_1, \ldots, v_r\}$, with $v_j \neq 0$ for all $j \in \{1, \ldots, r\}$ positively spans $\mathbb{R}^n$ then for every vector $d \in \mathbb{R}^n$ there is an index $j \in \{1, \ldots, r\}$ such that $d^Tv_j > 0$.

Considering strict differentiability of each component of the objective function $F$, and setting $d = \nabla f_i(x)$ or $d = -\nabla f_i(x)$, for $i \in \{1, \ldots, p\}$, Theorem 3.1 allows us to conclude that in every positive spanning set, for each component of the objective function, we can find at least one ascent and one descent direction.

Thus, at each iteration, for $i \in \{1, \ldots, p\}$, if $\nabla f_i(x_k) \neq 0$, $d_k = -\nabla f_i(x_k)$ can be used to prune the positive spanning set, only keeping directions that are descent according to at least one component of the objective function. Since we are only discarding directions that are ascent according to all components of the objective function, the convergence results of Section 2 still hold. The pruned set of directions, $D_k^P$, to be considered as poll directions for DMS at Step 2 of Algorithm 1, will then be:

$$D_k^P = \bigcup_{i \in \{1, \ldots, p\}} \{d \in D_k : -\nabla f_i(x_k)^T d > 0\}.$$

The idea of pruning positive spanning sets was already proposed in single objective derivative-free optimization [11]. In this setting, it is easy to see that the cardinality of the pruned set will be $1 \leq |D_k^P| \leq |D_k| - 1$. The authors were even able to provide a particular enriched positive spanning set, that always reduces to a singleton after pruning.

If the goal is to generate an approximation to the complete Pareto front of a given problem, we do not wish to reduce the poll directions to a singleton, as we do not wish to use opportunistic approaches, which would generate at most a new feasible nondominated point at each iteration. Moreover, in multiobjective optimization, due to the presence of conflicting objectives, we cannot ensure the presence of an ascent direction, according to all the components of the objective function [12]. In fact, it is possible to build examples where the cone of the ascent directions, considering all components of the objective function, can be as narrow as one would desire.

The proposed strategy was implemented and numerically tested against the original DMS algorithm [12], and also against MOSQP [22]. The latter is a recent solver proposed for multiobjective derivative-based optimization which uses a SQP approach. MOSQP keeps a list of nondominated points that is improved both for spread along the Pareto front and optimality by solving single-objective constrained optimization problems. Default parameters were considered for the two solvers, with exception to the maximum number of function evaluations allowed, which was set to 20000.
Table 3.1: The test set considered in the numerical experiments. Here $n$ represents the number of variables and $p$ is the number of components of the objective function.

As test set, we considered the collection of 100 bound constrained multiobjective optimization problems available at http://www.mat.uc.pt/dms. This collection was previously used to test DMS and MOSQP, at the time of their first release [12, 22]. From this collection, we selected a total of 54 problems, for which we were able to guarantee the existence of derivatives. Table 3.1 reports the resulting test set, which comprises problems with 2 or 3 components in the objective function and a number of variables, $n$, between 1 and 30.

For performance assessment, we considered typical metrics from the multiobjective optimization literature, like is the case of purity and spread metrics, as defined in [12], and also the hypervolume indicator [34, 35].

In a simplified view, purity measures the percentage of nondominated points generated by a given solver. For problem $\hat{p} \in P$ and solver $s \in S$, purity is defined by the ratio

$$\tilde{t}_{\hat{p}, s} = \frac{|F_{\hat{p}, s} \cap F_{\hat{p}}|}{|F_{\hat{p}, s}|},$$

where $F_{\hat{p}, s}$ denotes the approximation to the Pareto front computed for problem $\hat{p} \in P$ by solver $s \in S$ and $F_{\hat{p}}$ is a reference Pareto front for problem $\hat{p} \in P$. This reference Pareto front is computed by joining the final approximations computed by any of the solvers tested and removing from it all the dominated points. A value of purity near 1 indicates that the majority of the points generated by the corresponding solver is nondominated. However, these could be concentrated in a single part of the true Pareto front. Spread metrics are required to have a fair assessment of the solver’s performance.

Since the goal is to build an approximation to the complete Pareto front of each problem, the computation of spread metrics initiates with the computation of the so-called ‘extreme points’ of the Pareto front (see [12]). The spread $\Gamma$ measures the maximum gap between consecutive points lying in the approximated Pareto front.
The metric \( \Gamma_{\hat{p},s} > 0 \) for problem \( \hat{p} \in \mathcal{P} \) and solver \( s \in \mathcal{S} \) is given by
\[
\Gamma_{\hat{p},s} = \max_{j \in \{1,...,p\}} \left( \max_{i \in \{0,...,N\}} \{ \delta_{i,j} \} \right),
\]
where \( \delta_{i,j} = (f_j(x_{i+1}) - f_j(x_i)) \), \( x_1, x_2, \ldots, x_N \) represent the points generated by solver \( s \in \mathcal{S} \) for problem \( \hat{p} \in \mathcal{P} \), and \( x_0, x_{N+1} \) correspond to the ‘extreme points’. Implicitly, we are assuming that the objective function values have been sorted by increasing order for each objective \( j \in \{1, \ldots, p\} \).

The spread metric \( \Delta \) measures the uniformity of the gaps across the approximation to the Pareto front:
\[
\Delta_{\hat{p},s} = \max_{j \in \{1,...,p\}} \left( \frac{\delta_{0,j} + \delta_{N,j} + \sum_{i=1}^{N-1} |\delta_{i,j} - \bar{\delta}_j|}{\delta_{0,j} + \delta_{N,j} + (N - 1)\bar{\delta}_j} \right),
\]
where \( \bar{\delta}_j \), for \( j = 1, \ldots, p \), represents the average of the distances \( \delta_{i,j} \), \( i = 1, \ldots, N - 1 \).

The fourth metric considered is the hypervolume indicator [35], which measures the volume of the portion of the objective function space that is dominated by the computed approximation to the Pareto front of the problem, and upper bounded by a given reference point \( U_{\hat{p}} \in \mathbb{R}^p \). This reference point should be dominated by all points belonging to the approximations computed for the Pareto front of a given problem \( \hat{p} \in \mathcal{P} \). Formally, it can be defined as:
\[
HV_{\hat{p},s} = Vol(\{ y \in \mathbb{R}^p | y \leq U_{\hat{p}} \cap \exists x \in F_{\hat{p},s} : x \leq y \}) = Vol \left( \bigcup_{x \in F_{\hat{p},s}} [x, U_{\hat{p}}] \right),
\]
where \( Vol(\cdot) \) denotes the Lebesgue measure of a \( p \)-dimensional set of points and \([x, U_{\hat{p}}] \) denotes the interval box with lower corner \( x \) and upper corner \( U_{\hat{p}} \). The approach proposed in [23] was used for its practical computation and the resulting hypervolume values were scaled to the interval \([0, 1]\).

Performance profiles [16] will be depicted for each of the four metrics considered. Let \( t_{\hat{p},s} \) denote the performance of solver \( s \in \mathcal{S} \) on problem \( \hat{p} \in \mathcal{P} \), assuming that lower values of \( t_{\hat{p},s} \) indicate a better performance. Each performance profile represents the curve
\[
\rho_s(\tau) = \frac{1}{|\mathcal{P}|} |\{ \hat{p} \in \mathcal{P} : r_{\hat{p},s} \leq \tau \}|,
\]
with \( r_{\hat{p},s} = t_{\hat{p},s} / \min \{ t_{\hat{p},\tilde{s}} : \tilde{s} \in \mathcal{S} \} \). In the case of purity and hypervolume metrics, larger values indicate better performance. Thus, when computing performance profiles for these two metrics, we set \( t_{\hat{p},s} = 1/t_{\hat{p},s} \), as proposed in [12].

Figures 3.1 and 3.2 compare DMS using derivative information for pruning the positive spanning sets used as poll directions, against MOSQP.

The two solvers present a similar performance in terms of purity and \( \Delta \) metrics. However, it is clear the advantage of using DMS with the pruned set of directions, when hypervolume or the \( \Gamma \) metrics are considered. This advantage in even clearer if the original implementation of DMS, without the pruning strategy, is used (see Figures 3.3 and 3.4).

It is common to say that if derivatives are available, or can be obtained at a reasonable cost (e.g. using finite-differences) then derivative-based optimization is
preferable to derivative-free optimization methods (see page 6 of [3]). If this is the case for single objective optimization, according to the numerical experience reported, the same does not necessarily apply to multiobjective optimization.

Although, it is surprising that ascent directions seem to have a role in increasing the performance of DMS. Figures 3.5 and 3.6 report the comparison between the original DMS implementation, and the variant that prunes the positive spanning sets before polling.

As we will see in Section 4, a richer set of directions, including some ascent directions, could bring benefits, when the goal is to compute an approximation to the complete Pareto front of a given MOO problem.

4. The role of ascent directions. In the presence of constraints, pruning ascent directions is not always a good strategy. Consider the biobjective minimization problem ZDT2, with $n = 30$ [33]. In this case, if we provide as initialization one Pareto critical point, the algorithm stalls, being unable to generate other Pareto critical points in the Pareto front. This behavior is accordingly to the convergence results derived for DMS, which only guarantee convergence to a single Pareto critical point. By providing ascent directions, that conform to the geometry of the nearby feasible region, the algorithm is able to proceed and generate a large number of Pareto critical points. Figure 4.1 illustrates the situation.
Thus, the approach taken was to return to the original positive spanning set $D_k$ (without pruning) at some iterations. Assume that at a given iteration the original positive spanning set was pruned and $D_k^p$ was used as poll set, but the algorithm was unable to proceed because every poll point was infeasible. At the next iteration pruning will not be applied, and the original positive spanning set $D_k$ will be considered as the set of poll directions. Again, since we are only disregarding directions that are ascent according to all components of the objective function, and only at some iterations, the convergence results of Section 2 continue to hold. Figures 4.2 and 4.3 report performance profiles comparing this new approach with the original implementation of DMS.

Now the two variants of DMS are extremely close in terms of performance, but the new approach brings some slight advantage when comparing to MOSQP for the Δ metric (see Figures 4.4 and 4.5).

This good performance is not a result of the large budget allowed in terms of function evaluations (a maximum of 20000). If we reduce this budget to only 500 functions evaluations, the advantages of the new variant of DMS against the original implementation can still be observed with respect to purity and the Γ metrics in Figures 4.6 and 4.7.

Regarding MOSQP, there is an improvement in purity with a slight detriment in the performance profile associated to hypervolume. Figures 4.8 and 4.9 report the
Fig. 3.5: Performance profiles for purity and hypervolume metrics, comparing the original DMS implementation and a new version, where poll directions are pruned using first order information.

Fig. 3.6: Performance profiles for $\Gamma$ and $\Delta$ metrics, comparing the original DMS implementation and a new version, where poll directions are pruned using first order information.

5. Concluding remarks. DMS was proposed in [12] as a robust and efficient algorithm to solve derivative-free MOO problems. Surprisingly, it showed to be a strong competitor against the derivative-based solver MOSQP, evidencing that in MOO, when the goal is to generate an approximation to the complete Pareto front of a given problem, even if first-order derivatives are available, derivative-free solvers can be interesting alternatives to derivative-based approaches.

Derivatives can be used to prune the positive spanning sets to be considered as poll directions. However, care should be taken because ascent directions, that conform to the geometry of the nearby feasible region, can have an important role in the ability of generating a complete approximation to the Pareto front of a given problem.

The new variant of DMS, which prunes the poll set of directions, but that at some iterations considers its enrichment with ascent directions, showed to be competitive both with the derivative-based solver MOSQP and with the original implementation of DMS. For low computational budgets of function evaluations, it allows an increase in the percentage of nondominated points generated in the approximation to the Pareto front of the MOO problem and also a reduction in the largest gap across the generated Pareto front, when comparing with the original implementation of DMS. In the case
Fig. 4.1: Final approximations to the Pareto front of problem ZDT2, generated by two different algorithmic variants of DMS. On the left, positive spanning sets are pruned to sets only comprising descent directions. On the right, ascent directions are considered at some iterations.

Fig. 4.2: Performance profiles for purity and hypervolume metrics, comparing the original DMS implementation and a new version, where poll directions are pruned using first order information, but not at all the iterations.

Future work could include the definition of a search step taking advantage of first order information for building Taylor models, which will be minimized considering an approach similar to the one proposed and analyzed in [4].

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Fig. 4.3: Performance profiles for $\Gamma$ and $\Delta$ metrics, comparing the original DMS implementation and a new version, where poll directions are pruned using first order information, but not at all the iterations.

Fig. 4.4: Performance profiles for purity and hypervolume metrics, comparing MOSQP and the new version of DMS, where poll directions are pruned using first order information, but not at all the iterations.

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Fig. 4.5: Performance profiles for $\Gamma$ and $\Delta$ metrics, comparing MOSQP and the new version of DMS, where poll directions are pruned using first order information, but not at all the iterations.

Fig. 4.6: Performance profiles for purity and hypervolume metrics, comparing the original DMS implementation and a new version, where poll directions are pruned using first order information, but not at all the iterations (maximum budget of 500 function evaluations).

Fig. 4.7: Performance profiles for $\Gamma$ and $\Delta$ metrics, comparing the original DMS implementation and a new version, where poll directions are pruned using first order information, but not at all the iterations (maximum budget of 500 function evaluations).
Fig. 4.8: Performance profiles for purity and hypervolume metrics, comparing MOSQP and the new version of DMS, where poll directions are pruned using first order information, but not at all the iterations (maximum budget of 500 function evaluations).

Fig. 4.9: Performance profiles for $\Gamma$ and $\Delta$ metrics, comparing MOSQP and the new version of DMS, where poll directions are pruned using first order information, but not at all the iterations (maximum budget of 500 function evaluations).

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