CONFORMAL ANOMALY OF SUBMANIFOLD OBSERVABLES IN ADS/CFT CORRESPONDENCE

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Abstract. We analyze the conformal invariance of submanifold observables associated with \( k \)-branes in the AdS/CFT correspondence. For odd \( k \), the resulting observables are conformally invariant, and for even \( k \), they transform with a conformal anomaly that is given by a local expression which we analyze in detail for \( k = 2 \).

1. Introduction

There has been much recent interest in the correspondence [1] between conformal field theory and string theory on negatively curved or Anti de Sitter spacetimes. In this correspondence, correlation functions of local conformal fields on an \( n \)-manifold \( M \) are computed in terms of the asymptotic behavior of fields on \( n + 1 \)-dimensional Einstein manifolds \( X \) of conformal boundary \( M \) [2],[3]. (Here we are actually suppressing the role of some compactified dimensions that will be unimportant for the present paper. The string theory is actually formulated on \( X \times W \) for a suitable compact manifold \( W \); the dimension of \( X \times W \) is 10 or 11 depending on whether we are doing string theory or \( M \)-theory. We will generally suppress mention of \( W \).)

The relevant notion of conformal boundary is that of conformal infinity in the sense introduced by Penrose (see [1]). \( X \) is the interior of an \( n + 1 \)-dimensional manifold-with-boundary \( \overline{X} \) whose boundary is \( M \). The metric \( g_+ \) on \( X \) is complete and has a double pole on the boundary in the following sense. If \( r \) is a smooth function on \( \overline{X} \) with a first order zero on the boundary of \( \overline{X} \), and positive on \( X \), then \( r \) is called a defining function. The requirement on \( g_+ \) is that for any defining function \( r \), \( \overline{g} = r^2 g_+ \) extends as a smooth metric on \( \overline{X} \). Clearly, if so, the restriction of \( \overline{g} \) to \( M \) gives a metric on \( M \). This metric changes by a conformal transformation if the defining function is changed, so \( M \) has a well-defined conformal structure but not a well-defined metric. If \( X \) is an Einstein manifold, then with a suitable choice of coordinates the metric of \( X \) looks like

\[
g_+ = \frac{1}{r^2} \left( g_r + dr^2 \right),
\]

with \( g_r \) an \( r \)-dependent family of metrics on \( M \); the metric on \( M \) is just \( g_0 \).

The converse problem – given a conformal structure on \( M \), find an Einstein metric on \( X \) which induces it at infinity – also has a solution under certain conditions. For example, if \( M = S^n \), endowed with a conformal structure sufficiently close to the standard one, and \( X \) is an \( n + 1 \)-ball, then [3] there is a unique Einstein metric on \( X \) with a prescribed negative curvature that is close to the standard hyperbolic metric on the ball and has \( M \) as conformal infinity.

In relating conformal field theory on \( M \) to quantum gravity on \( X \), ultraviolet divergences on \( M \) are typically related to infrared divergences on \( X \). This is briefly described in [3]; the

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relation of the statement to ideas about holography is explained somewhat more fully in [6]. For example, to compute the partition function of the conformal field theory on $M$, as a function of the metric of $M$, one must, in the supergravity approximation, evaluate the gravitational action for an Einstein metric on $X$ that induces at infinity the given conformal structure on $M$. For an Einstein metric, the gravitational action is proportional to the volume. Thus, to compute the partition function of the conformal field theory, one formally must evaluate the volume

$$\text{Vol}(X) = \int_X dv_X,$$

where $dv_X = \sqrt{\det g_+} dx^{n+1}$ is the Riemannian volume form of $X$. This integral clearly diverges in view of the form (1.1) of the metric on $X$. One regularizes it by letting $X_\epsilon$ be the subset of $X$ with $r \geq \epsilon$, and

$$\text{Vol}_\epsilon(X) = \int_{X_\epsilon} dv_X.$$

$\text{Vol}_\epsilon(X)$ has terms that diverge as negative powers of $\epsilon$, and also a logarithmic term if $n$ is even. After subtracting the divergent terms, one gets a renormalized functional $\text{Vol}_R(X)$, which, for a suitable choice of $X$, determines the supergravity limit of the conformal field theory partition function on $M$. As shown in detail in [7], the logarithmic term in $\text{Vol}_\epsilon(X)$ leads to a conformal anomaly in $\text{Vol}_R(X)$ that reproduces the expected conformal anomaly of the conformal field theory on $M$.

The conformal anomaly is possible because the regularization violates conformal invariance. Indeed, the choice of a particular defining function $r$ is built into the definition of $X_\epsilon$. But the choice of $r$ violates conformal invariance, because it fixes a particular metric (namely $r^2 g_+$) in the conformal class of metrics on $M$. The anomaly means that, for even $n$, conformal invariance is not restored after renormalization.

**Submanifolds**

Our goal in the present paper is to analyze a somewhat similar problem concerning submanifolds of $X$. Consider, for example, Type IIB superstring theory on $X \times S^5$, where $X$ is an Einstein manifold (of negative scalar curvature) with conformal boundary a four-manifold $M$. It has been argued [3,4] that to compute the expectation value of a Wilson line on a circle $N \subset M$, one must consider a path integral with a string whose worldsheet $Y \subset X$ has $N$ as boundary. In the supergravity approximation, $Y$ should obey the equation of a minimal area surface. Formally speaking, if $A(Y)$ is the area of $Y$ and $T$ the string tension, then the expectation value of the Wilson line on $N$ is

$$\langle W(N) \rangle = \exp(-TA(Y))$$

in the supergravity approximation. Here $W(N)$ is the Wilson line operator and $\langle W(N) \rangle$ is its expectation value.

Similarly, one can consider examples in which the string theory contains a $k$-brane for some $k$. Then one associates an observable $W(N)$ to a $k$-dimensional submanifold $N$ of $M$. Its expectation value is computed by a path integral with a brane wrapped on a $k+1$-dimensional

\[ Y \text{ is actually a submanifold of } X \times W. \text{ For simplicity, we will consider only the case that } Y \text{ projects to a point in } W \text{ and so can be regarded as a submanifold of } X. \]
submanifold $Y \subset \bar{X}$ whose boundary is $N$. In the supergravity approximation, the expectation value of the surface observable $W(N)$ is again given formally by (1.4), with $A(Y)$ now the volume of $Y$.

The trouble with this formula is that, given the form of the metric (1.1), $A(Y)$ is always infinite. Hence, one proceeds just as one does in renormalizing the volume $\text{Vol}(X)$. One lets $Y_\epsilon$ be the part of $Y$ with $r \geq \epsilon$, and one denotes the volume of $Y_\epsilon$ as $A_\epsilon(Y)$. As $\epsilon \to 0$, $A_\epsilon(Y)$ has divergent terms that are negative powers of $\epsilon$, plus a logarithmic term if $k$ is even. As explained in [3], the divergent terms correspond to ultraviolet divergences in conformal field theory. After subtracting the divergent terms, one gets a renormalized volume functional $A_R(Y)$ which should be used instead of $A(Y)$ in (1.4). However, when $k$ is even, $A(Y)$ is not conformally invariant.

The violation of conformal invariance is given by a local expression that we will analyze. The anomaly is possible, of course, because the definition of $Y_\epsilon$ depends on the choice of the defining function $r$ and so violates conformal invariance.

Note that the problem of defining the volume of a submanifold is in a very precise sense a generalization of the problem of defining $\text{Vol}(X)$. Indeed, in the special case $k = n$, $N = M$, $Y = X$, $A_R(Y)$ coincides with $\text{Vol}_R(X)$.

Questions of existence and regularity of minimal area submanifolds (usually of hyperbolic space) with prescribed boundary at infinity have been studied in the mathematical literature; see [10], [11], [12], [13], [14], [15]. (An error in [13] is corrected in [15].)

**Examples**

We conclude this introduction with comments on a few of the basic examples to which the discussion can be applied.

First we consider examples with zero branes, that is with $k = 0$. For example, in Type IIB on $X \times S^3 \times T^4$, with $X$ a hyperbolic three-manifold, zero branes on $X$ arise from one branes wrapped on a one-cycle in $T^4$. Let $\mu$ be the mass of the zero brane (in units in which the Einstein equations on $X$ read $R_{ij} = -3 g_{ij}$). In the regime in which supergravity formulas such as (1.4) are valid, one has $\mu \gg 1$. The conformal boundary of $X$ is a Riemann surface $M$.

Since $k = 0$, the brane world-volume is a curve $Y \subset X$; its boundary consists of a pair of points $P, Q \in M$. The operator associated with the endpoint of the brane worldvolume is thus a local operator $\Psi(P)$. Actually, assuming that the zerobrane worldvolume is oriented (as in the example noted in the last paragraph), the points $P, Q$ are endowed with opposite orientations and conjugate operators $\Psi(P), \overline{\Psi}(Q)$. To compute the two point function $\langle \Psi(P)\overline{\Psi}(Q) \rangle$ in the supergravity approximation, one takes $Y$ to be a minimum length geodesic connecting $P$ and $Q$. One then has asymptotically

$$\langle \Psi(P)\overline{\Psi}(Q) \rangle = \exp(-\mu L(Y)), \quad (1.5)$$

with $L(Y)$ the “length” of $Y$.

Here, because $k = 0$ is even, we encounter a conformal anomaly. In fact, it is clear without any analysis at all that there must be an anomaly. If the correlation function (1.3) did not require some renormalization leading to an anomaly, then this correlation function would be conformally invariant. Hence the operators $\Psi$, $\overline{\Psi}$ would have conformal dimension zero, a behavior which is impossible (for non-constant local operators) in a unitary conformal field theory.

The actual computation of the anomaly is straightforward for $k = 0$. The length is regularized by replacing $L(Y)$ by $L_\epsilon(Y)$, the length of the part of $Y$ with $r \geq \epsilon$. As $\epsilon \to 0$, $L_\epsilon(Y)$ receives a
divergent contribution from each of the two ends of \( Y \). With the metric in (1.1), the divergent contribution of either end is precisely \(+1 \cdot \ln 1/\epsilon\). Hence the two point function in (1.5), after regularization, has the form
\[
\langle \Psi(P) \overline{\Psi}(Q) \rangle_\epsilon = \epsilon^{2\mu} S(\epsilon),
\]
where \( S(\epsilon) \) has a limit as \( \epsilon \to 0 \). One defines the renormalized two point function \( \langle \Psi(P) \overline{\Psi}(Q) \rangle_R \) to be the limit \( S(0) \). Since \( \epsilon \) has dimensions of length or (mass)\(^{-1}\), \( S \) has dimensions of (mass)\(^2\)\(^\mu\), and hence \( \Psi \) and \( \overline{\Psi} \) have conformal dimension \( \mu \). By contrast, in the general AdS/CFT correspondence, an AdS particle of mass \( \mu \) should (in \( n \) dimensions) correspond to a conformal field of dimension \( d(\mu) = (n + \sqrt{n^2 + 4\mu^2})/2 \). This reduces to \( d(\mu) = \mu \) in the large \( \mu \) limit, where (1.5) is valid, so the anomaly reproduces the expected conformal dimension of \( \Psi \) and \( \overline{\Psi} \) in this limit.

The next case is \( k = 1 \). The most familiar example is that of ’t Hooft and Wilson loops in four-dimensional gauge theory. There is no anomaly for odd \( k \), so these operators have no conformal anomaly.

The next case – and the last one that we will consider specially – is \( k = 2 \). Examples are known for various values of \( n \) up to \( n = 6 \). For instance, \( k = 2, n = 6 \) arises in the case of \( M \)-theory on \( AdS_7 \times S^4 \). By replacing \( S^4 \) with a quotient, one can also build other \( k = 2, n = 6 \) examples with less supersymmetry. For \( k = 2 \) and \( n = 6 \), the operators \( W(N) \) are “surface” observables in a six-dimensional conformal field theory. As \( k = 2 \) is even, an anomaly occurs; it will be analyzed in detail in the next section. Because of our limited understanding of the six-dimensional conformal field theories, no independent way to compute this anomaly is presently known, so there is no theory for us to compare to.

The computation of the anomaly for \( k = 2 \) thus gives new information about the conformal field theories that arise from these constructions. In section 2 of this paper, we will define the anomaly precisely, and describe some of its general properties, for all \( k \). Then we will do a detailed computation for \( k = 2 \).

The anomaly for \( k = 2 \) has also been briefly discussed, in the case that \( M \) is conformally flat, in [10], which appeared while the present paper was in gestation.

2. The Computation

Let \( X^{n+1} \) be the interior of a compact manifold with boundary \( \overline{X} \), let \( M = \partial X \), and let \( g_+ \) be a conformally compact metric on \( X \). This means that if \( r \) is a defining function for \( M \subset \overline{X} \) (in a sense explained in the introduction), then \( \overline{g} = r^2 g_+ \) extends smoothly to \( \overline{X} \). The conformal class of the restriction of \( \overline{g} \) to \( TM \) is independent of the choice of defining function. The function \( |dr|^2_{\overline{g}} \) extends smoothly to \( \overline{X} \), and its restriction to \( M \) is independent of the choice of \( r \), so is an invariant of \( g_+ \). We will assume that \( g_+ \) satisfies the Einstein condition \( \text{Ric}(g_+) = -ng_+ \). It follows upon conformally transforming the Einstein equation that in this case one has \( |dr|^2_{\overline{g}} = 1 \) on \( M \).

Conformally compact metrics with \( |dr|^2_{\overline{g}} = 1 \) on \( M \) may be put in a special form near the boundary using a special class of defining functions. The following Lemma is taken from [3] (see Lemma 5.2).

**Lemma 2.1.** A metric on \( M \) in the conformal infinity of \( g_+ \) determines a unique defining function \( r \) in a neighborhood of \( M \) in \( \overline{X} \) such that \( \overline{g}|_{TM} \) is the prescribed boundary metric and such that \( |dr|^2_{\overline{g}} = 1 \).
Proof. Given any choice of defining function $r_0$, let $\overline{g}_0 = r_0^2 g_+$ and set $r = r_0 e^\omega$, so $\overline{g} = e^{2\omega} \overline{g}_0$ and $dr = e^\omega (dr_0 + r_0 d\omega)$. Thus
\begin{equation}
|dr|^2_{\overline{g}} = |dr_0 + r_0 d\omega|^2_{\overline{g}_0} = |dr_0|^2 + 2r_0 (dr_0, d\omega) + r_0^2 |d\omega|^2_{\overline{g}_0}
\end{equation}
(where $(\ ,\ )$ is the inner product in the metric $g_0$), so the condition $|dr|^2_{\overline{g}} = 1$ is equivalent to
\begin{equation}
2(dr_0, d\omega) + r_0 |d\omega|^2_{\overline{g}_0} = \frac{1 - |dr_0|^2_{\overline{g}_0}}{r_0}.
\end{equation}
This is a non-characteristic first order PDE for $\omega$, so there is a solution near $M$ with $\omega|_M$ arbitrarily prescribed.

This lemma means that not only does a defining function $r$ determine a metric on $M$ in its conformal class, but conversely given such a metric on $M$, there is a natural way to determine a distinguished defining function $r$, at least in a neighborhood of $M$. Since we will only be interested in the behavior of $r$ near $M$, it follows that for our purposes, the choice of a metric on $M$ in its conformal class is equivalent to the choice of a defining function.

A defining function determines for some $\epsilon > 0$ an identification of $M \times [0, \epsilon)$ with a neighborhood of $M$ in $\overline{X}$: $(p, \lambda) \in M \times [0, \epsilon)$ corresponds to the point obtained by following the integral curve of $\nabla \pi^r$ emanating from $p$ for $\lambda$ units of time. For a defining function of the type given in the lemma, with $|dr|^2_{\overline{g}} = 1$, the $\lambda$-coordinate is just $r$, and $\nabla \pi^r$ is orthogonal to the slices $M \times \{\lambda\}$. Hence, identifying $\lambda$ with $r$, on $M \times [0, \epsilon)$ the metric $\overline{g}$ takes the form $\overline{g} = g_r + dr^2$ for a 1-parameter family $g_r$ of metrics on $M$, and
\begin{equation}
g_+ = r^{-2}(g_r + dr^2).
\end{equation}

One can explicitly calculate the Ricci curvature of a metric of the form (2.3) so as to express the Einstein condition directly in terms of $g_r$. From an analysis of the formal asymptotics of solutions of the resulting equations (see [17] or [7]), one deduces that for $n$ odd, the expansion of $g_r$ is of the form
\begin{equation}
g_r = g^{(0)} + g^{(2)} r^2 + (\text{even powers}) + g^{(n-1)} r^{n-1} + g^{(n)} r^n + \ldots,
\end{equation}
where the $g^{(j)}$ are tensors on $M$, and $g^{(n)}$ is trace-free with respect to a metric in the conformal class on $M$. For $j$ even and $0 \leq j \leq n - 1$, the tensor $g^{(j)}$ is locally formally determined by the conformal representative, but $g^{(n)}$ is formally undetermined, subject to the trace-free condition. For $n$ even the analogous expansion is
\begin{equation}
g_r = g^{(0)} + g^{(2)} r^2 + (\text{even powers}) + kr^n \log r + g^{(n)} r^n + \ldots,
\end{equation}
where now the $g^{(j)}$ are locally determined for $j$ even and $0 \leq j \leq n - 2$, $k$ is locally determined and trace-free, the trace of $g^{(n)}$ is locally determined, but the trace-free part of $g^{(n)}$ is formally undetermined. Moreover, so long as $n \geq 3$, one has
\begin{equation}
g^{(2)}_{ij} = -P_{ij},
\end{equation}
where $(n - 2) P_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$, and $R_{ij}$ and $R$ denote the Ricci tensor and scalar curvature of the chosen representative $g_{ij}$ of the conformal infinity.

Later we will need to use the following Lemma.
Lemma 2.2. Let \( r \) and \( \hat{r} \) be special defining functions as in Lemma 2.1 associated to two different conformal representatives. Then

\[
\hat{r} = re^\omega
\]

for a function \( \omega \) on \( M \times [0, \epsilon) \) whose expansion at \( r = 0 \) consists only of even powers of \( r \) up through and including the \( r^{n+1} \) term.

Proof. We have \( \hat{r} = e^\omega r \) where \( \omega \) is determined by (2.2), which in this case becomes

\[
2\omega_r + r(\omega_r^2 + |d_M\omega|^2_{g_r}) = 0.
\]

The Taylor expansion of \( \omega \) is determined inductively by differentiating this equation at \( r = 0 \). Clearly \( \omega_r = 0 \) at \( r = 0 \). Consider the determination of \( \partial_r^{k+1}\omega \) resulting from differentiating (2.8) an even number \( k \) times and setting \( r = 0 \). The term \( \omega_r^2 \) gets differentiated \( k - 1 \) times, so one of the two factors ends up differentiated an odd number of times, so by induction vanishes at \( r = 0 \). Now \( |d_M\omega|^2_{g_r} = g_r^{ij}\omega_i \omega_j \), so the \( k - 1 \) differentiations must be split between the three factors, so one of the factors must receive an odd number of differentiations. When an odd number of derivatives hits a \( \omega_i \), the result again vanishes by induction. But by (2.4) and (2.5), so long as \( k - 1 < n \), the odd derivatives of \( g_r \) vanish at \( r = 0 \). □

We will calculate near \( M \) the minimal surface equation for a submanifold of \( (X, g_+) \). We begin by deriving the minimal surface equation for a graph in a Riemannian manifold.

Let \( (x^\alpha, u^\alpha) \) denote coordinates in \( \mathbb{R}^m \times \mathbb{R}^l \), let \( g \) be a metric on \( \mathbb{R}^m \times \mathbb{R}^l \), and let \( Y \) denote the graph \( \{u = u(x)\} \). Then \( g \) restricts to \( Y \) to the metric \( h \) given in the coordinates \( x \) on \( Y \) by

\[
h_{\alpha\beta} = g_{\alpha\beta} + 2g_{\alpha'\beta'} + g_{\alpha'\beta'}u_{\alpha'}^\alpha u_{\beta'}^\beta,
\]

where the indices after a comma indicate coordinate differentiation. The area of \( Y \) is \( A = \int \sqrt{\det h} \, dx \), so \( \delta A = \frac{1}{2} \int \sqrt{\det h} h^{\alpha\beta} \delta h_{\alpha\beta} \, dx \). Since

\[
\delta h_{\alpha\beta} = \left[ g_{\alpha\beta,\gamma} + 2g_{\alpha'\beta'} + g_{\alpha'\beta'}u_{\alpha'}^\alpha u_{\beta'}^\beta \right] \delta u_{\gamma'} + 2g_{\alpha'\beta'}u_{\alpha'}^\alpha \delta u_{\beta'}^\beta + 2g_{\alpha'\beta'}u_{\beta'}^\beta \delta u_{\alpha'}^\alpha,
\]

it follows that \( Y \) is stationary for area iff \( u \) satisfies

\[
2 \left[ \sqrt{\det h} h^{\alpha\beta} \left( g_{\alpha\gamma'} + g_{\alpha\gamma'}u_{\alpha'}^\alpha \right) \right]_{\beta} = 0,
\]

which may be rewritten as

\[
\partial_{\beta} + \frac{1}{2} \left( \log(\det h) \right)_{\beta} \left[ h^{\alpha\beta} \left( g_{\alpha\gamma'} + g_{\alpha\gamma'}u_{\alpha'}^\alpha \right) \right] - \frac{1}{2} h^{\alpha\beta} \left[ g_{\alpha\beta,\gamma'} + 2g_{\alpha\gamma'}u_{\beta'}^\beta + 2g_{\alpha\gamma'}u_{\alpha'}^\alpha \right] = 0.
\]

Let now \( Y^{k+1} \) be a submanifold of our conformally compact Einstein manifold \( (X, g_+) \), where \( 0 \leq k \leq n - 1 \). Suppose that \( Y \) extends smoothly to \( \overline{X} \) and set \( N = Y \cap M \). Locally near a point of \( N \), coordinates \( (x^\alpha, u^\alpha) \) for \( M \) may be chosen, where \( 1 \leq \alpha \leq k \) and \( 1 \leq \alpha' \leq n - k \), so that \( N = \{u = 0\} \) and so that \( \partial_{x^\alpha} \perp \partial_{u^\alpha} \) on \( N \) with respect to a metric in the conformal infinity of \( g_+ \). Under the identification discussed above, the choice of such a metric determines an extension of the \( x^\alpha \) and \( u^\alpha \) to a neighborhood of \( M \), and together with \( r \) these form a coordinate system on \( \overline{X} \). We consider submanifolds \( Y \) which in these coordinates may be written as a graph
\{u = u(x, r)\}. The minimal surface equation for \(Y\) is therefore given by (2.10), where however \(x^\alpha\) must be replaced by \((x^\alpha, r)\) and \(g\) by \(g_+\). Set \(\overline{h} = r^2 h\) and \(L = \log(\det h)\). Recalling (2.3) and writing simply \(g\) for \(g_r\), the minimal surface equation for \(Y\) becomes \(\mathcal{M}(u) = 0\), where

\[
\mathcal{M}(u)_{\gamma'} = \left[ r \frac{\partial_r}{r} - (k + 1) + \frac{1}{2} rL_{,r} \right] \left[ \overline{h}^{\alpha\gamma'} g_{\alpha\gamma'} u_{,\gamma'}^\alpha + \overline{h}^{\alpha\beta} \left( g_{\alpha\gamma'} + g_{\alpha\gamma'} u_{,\gamma'} u_{,\alpha}^\alpha \right) \right]
+ \frac{r}{2} \left[ \frac{\partial_\beta}{\beta} + \frac{1}{2} L_{,\beta} \right] \left[ \overline{h}^{\beta\gamma'} g_{\beta\gamma'} u_{,\gamma'}^\beta + \overline{h}^{\alpha\beta} \left( g_{\alpha\gamma'} + g_{\alpha\gamma'} u_{,\gamma'} u_{,\alpha}^\alpha \right) \right]
\]

(2.11)

Consider now the inductive determination of the expansion of \(u = u(x, r)\) from the equation \(\mathcal{M}(u) = 0\), beginning with the initial condition \(u(x, 0) = 0\). Since the coordinates were chosen so that \(g_{\alpha\gamma'} = 0\) on \(N\), the representative \(g_{ij}\) for the conformal structure decomposes at \(u = r = 0\) into two pieces \(g_{\alpha\beta}\) and \(g_{\alpha'\beta'}\). Upon setting \(r = 0\) and using the initial condition, all terms on the right hand side of (2.11) vanish except for the first, and one obtains \(u_r = 0\) at \(r = 0\). Thus a minimal \(Y\) must intersect the boundary orthogonally. Using this, one deduces that all terms on the right hand side of (2.11) are \(O(r^2)\) except the first and third, so that \((r \frac{\partial_r}{r} - (k + 1)) \left( \overline{h}^{\alpha\gamma'} g_{\alpha\gamma'} u_{,\gamma'}^\alpha \right) - \frac{1}{2} \overline{h}^{\alpha\beta} g_{\alpha\beta,\gamma'} = O(r^2)\). This gives \(k u_{rr} = -\frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\alpha'}\) at \(r = 0\). Of course when \(k = 0\) there are no unprimed indices so that this equation reads \(0 = 0\), recovering the fact familiar from hyperbolic geometry that at the boundary a geodesic may have any asymptotic curvature measured using the smooth metric \(\overline{g}\).

Recall that the second fundamental form \(B_{\alpha\beta}'\) of \(N\) with respect to \(g_{ij}\) is defined by \(B(X, Y) = (\nabla_X Y)\perp\) for vectors \(X, Y \in TN\); here \(\nabla\) denotes the Levi-Civita covariant derivative of \(g_{ij}\) and \(\perp\) the component in \(TN\). \(B\) is symmetric in its lower indices. The mean curvature vector of \(N\) is \(H' = g^{\alpha\beta} B_{\alpha\beta}'\). In our coordinates \((x, u)\) on \(M\), the definition immediately gives \(B_{\alpha\beta}' = \Gamma_{\alpha\beta}'\), where \(\Gamma\) denotes the Christoffel symbol. Since \(g_{\alpha\gamma'} = 0\) on \(N\), this gives

\[
B_{\alpha\beta}' = -\frac{1}{2} g^{\alpha\gamma'} g_{\alpha\beta,\alpha'},
\]

so that the above formula for \(u_{rr}'\) at \(r = 0\) becomes

(2.13)

\[
k u_{rr}' = H'.
\]

Inductively, suppose that \(v\) has been determined so that \(\mathcal{M}(v) = O(r^{m-1})\). Set \(u = v + wr^m\), where \(w\) is to be determined. Then it is not hard to see that (pressuring the raising and lowering of indices) \(\mathcal{M}(u) = \mathcal{M}(v) + m(m - k - 2)wr^{m-1} + O(r^m)\). It follows that so long as \(m < k + 2\) then \(w\) is uniquely determined. When \(m = k + 2\), \(w\) is formally undetermined and one must include a \(r^{k+2} \log r\) term in \(u\) if the corresponding \(\mathcal{M}(v)\) is not actually already \(O(r^{k+2})\).

Since by (2.4) and (2.5) the metric \(g_{ij}\) is even in \(r\) to high order, it follows by staring at (2.11) that the expansion determined for \(u\) will also be even in \(r\) up to the \(r^{k+2}\) term. (Alternatively, one may introduce \(r^{1/2}\) as a new variable and observe that (2.11) remains regular.) When \(k\) is odd, the same reasoning shows that the \(r^{k+2} \log r\) term in \(u\) mentioned above does not occur,
but instead the parity is broken at this point and \( u \) may have a formally undetermined \( r^{k+2} \) term. Thus we have for \( k \) odd
\[
(2.14) \quad u = u^{(2)} r^2 + (\text{even powers}) + u^{(k+1)} r^{k+1} + u^{(k+2)} r^{k+2} + \ldots,
\]
and for \( k \) even
\[
(2.15) \quad u = u^{(2)} r^2 + (\text{even powers}) + u^{(k)} r^k + v r^{k+2} \log r + u^{(k+2)} r^{k+2} + \ldots,
\]
where the \( u^{(j)} \) and \( v \) are functions of \( x \), all of which are locally determined except for \( u^{(k+2)} \).

The induced metric \( h \) on \( Y \) is given by (2.14) with \( x^\alpha \) replaced by \( (x^\alpha, r) \) and \( g \) by \( g_+ \). Since the irregularities in \( g \), occur at order \( n \) and those in \( u \) at order \( k + 2 \), one concludes that up to terms vanishing to order greater than \( k \), the expansions of \( \overline{h}_{\alpha \beta} \) and \( \overline{h}_{rr} \) have only even terms in \( r \) and that of \( \overline{h}_{ar} \) has only odd terms. Thus the volume form \( dv_Y = \sqrt{\det \overline{h}} \, dx dr \) takes the form
\[
(2.16) \quad dv_Y = r^{-k-1} \left[ v^{(0)} + v^{(2)} r^2 + (\text{even powers}) + v^{(k)} r^k + \ldots \right] dv_N dr,
\]
where the \( \ldots \) indicates terms vanishing to higher order and \( dv_N \) denotes the volume form on \( N \) with respect to the chosen conformal representative on the boundary. All indicated \( v^{(j)} \) are locally determined functions on \( N \) and \( v^{(k)} = 0 \) if \( k \) is odd.

Consider now the asymptotics of \( \text{Area}_{g_+}(Y \cap \{ r > \epsilon \}) \) as \( \epsilon \to 0 \). Fix a small number \( \epsilon_0 \) and express \( \text{Area}(Y \cap \{ r > \epsilon \}) = C + \int_{Y \cap \{ \epsilon < r < \epsilon_0 \}} dv_Y \). By (2.16) we obtain for \( k \) odd
\[
(2.17) \quad \text{Area}(Y \cap \{ r > \epsilon \}) = c_0 \epsilon^{-k} + c_2 \epsilon^{-k+2} + (\text{even powers}) + c_{k-1} \epsilon^{-1} + c_k + o(1)
\]
and for \( k \) even
\[
(2.18) \quad \text{Area}(Y \cap \{ r > \epsilon \}) = c_0 \epsilon^{-k} + c_2 \epsilon^{-k+2} + (\text{even powers}) + c_{k-2} \epsilon^{-2} + d \log \frac{1}{\epsilon} + c_k + o(1).
\]
Observe that
\[
(2.19) \quad d = \int_N v^{(k)} dv_N.
\]

**Proposition 2.3.** If \( k \) is odd, then \( c_k \) is independent of the special defining function. If \( k \) is even, then \( d \) is independent of the special defining function.

**Proof.** Let \( r \) and \( \hat{r} \) be two special defining functions. On \( Y \), (2.7) becomes
\[
(2.20) \quad \hat{r} = r e^{\omega(x,u(x,r),r)}.
\]
From Lemma 2.2 and (2.14), (2.15), we see that the expansion of \( e^{\omega(x,u(x,r),r)} \) at \( r = 0 \) has only even powers of \( r \) up through and including the \( r^{k+1} \) term. So (2.20) can be solved for \( r \) to give \( r = \hat{r} b(x, \hat{r}) \) on \( Y \), where the expansion of \( b \) also has only even powers of \( \hat{r} \) up through the \( \hat{r}^{k+1} \) term. It is important to note that in this relation the \( x \) still refers to the identification associated with \( r \).

Set \( \hat{r}(x, \hat{r}) = e b(x, \hat{r}) \). Then on \( Y \), \( \hat{r} > \epsilon \) is equivalent to \( r > \hat{r}(x, \epsilon) \), so
\[
\text{Area}_{g_+}(Y \cap \{ r > \epsilon \}) - \text{Area}_{g_+}(Y \cap \{ \hat{r} > \epsilon \}) = \int_{\epsilon}^{\hat{r}} dv_Y =
\]
\[(2.21) \int_N \int_\epsilon^\rho \sum_{0 \leq j \leq k \atop j \text{ even}} v^{(j)}(x)r^{-k-1+j} drd\nu_N + o(1), \]

where we have used (2.16). For \(k \) odd this is
\[
\sum_{0 \leq j \leq k-1 \atop j \text{ even}} \epsilon^{-k+j} \int_N \frac{v^{(j)}(x)}{-k+j} (b(x, \epsilon)^{-k+j} - 1) dv_N + o(1).
\]

Since \(b(x, \epsilon)\) is even through terms of order \(k+1\) in \(\epsilon\), it follows that this expression has no constant term as \(\epsilon \to 0\). Similarly, when \(k\) is even, the \(r^{-1}\) term in (2.21) contributes \(\log b(x, \epsilon)\), so there is no \(\log \frac{1}{\epsilon}\) term as \(\epsilon \to 0\).

As explained in the introduction, we define the renormalized volume of \(Y\) to be \(A_R(Y) = c_k\).

The Anomaly For Surface Observables

As described in the introduction, the anomaly can be extracted rather trivially for \(k = 0\). So the first example that really illustrates this computation is \(k = 2\). In the remainder of the paper, we compute the anomaly as well as the log term coefficient \(d\) for \(k = 2\).

Using (2.9) with \(g\) again replaced by \(g + \epsilon\) and \(x^\alpha\) by \((x^\alpha, r)\) and recalling \(\mathcal{H} = r^2 h\), one concludes that the induced metric on \(Y\) satisfies
\[
\begin{align*}
\overline{h}_{\alpha\beta} &= g_{\alpha\beta} + O(r^3) \\
\overline{h}_{\alpha r} &= O(r^3) \\
\overline{h}_{rr} &= 1 + g^{\alpha'\beta'} u_{r}^\alpha u_{r}^{\beta'}.
\end{align*}
\]

If we denote by \(P_{\alpha\beta}\) the \(TN\)-component of \(P_{ij}\), then (2.9) and (2.13) give for \(k = 2\)
\[
\begin{align*}
g_{\alpha\beta}(x, u, r) &= g_{\alpha\beta}(x, u, 0) - P_{\alpha\beta}(x, u, 0)r^2 + O(r^3) \\
&= g_{\alpha\beta}(x, 0, 0) + g_{\alpha\beta, r'}(x, 0, 0)u^{r'}(x, r) - P_{\alpha\beta}(x, 0, 0)r^2 + O(r^3) \\
&= g_{\alpha\beta}(x, 0, 0) - \left(\frac{1}{2} \mathcal{H}_{r'} B_{\alpha\beta}^{r'} + P_{\alpha\beta}\right) r^2 + O(r^3),
\end{align*}
\]

and
\[
\overline{h}_{rr}(x, r) = 1 + g^{\alpha'\beta'}(x, 0, 0)u_{r}^\alpha(x, r)u_{r}^{\beta'}(x, r) + O(r^3)
\]
\[
= 1 + \frac{1}{4} |\mathcal{H}|^2 r^2 + O(r^3).
\]

Therefore, if we now use \(g_{\alpha\beta}\) to denote the values on \(N = \{u = r = 0\}\), we have
\[
\det \overline{h} = (\det \overline{h}_{\alpha\beta}) \overline{h}_{rr} + O(r^3)
\]
\[
= (\det g_{\alpha\beta}) \left[1 - \left(\frac{1}{4} |\mathcal{H}|^2 + g^{\alpha\beta} P_{\alpha\beta}\right) r^2\right] + O(r^3),
\]
so that
\begin{equation}
(2.22) \quad d\nu_Y = r^{-3} \left[ 1 - \frac{1}{8} \left( |H|^2 + 4g^{\alpha\beta}P_{\alpha\beta} \right) r^2 + O(r^3) \right] d\nu_N dr.
\end{equation}

By (2.13) we obtain for the log term coefficient

\[-8d = \int_N \left( |H|^2 + 4g^{\alpha\beta}P_{\alpha\beta} \right) d\nu_N.\]

By Proposition 2.3, this quantity is an invariant of the conformal structure on M and the submanifold N. In conformally flat spaces it is known in the mathematical literature as the Willmore functional of N and its conformal invariance is well-known. A problem that has received much attention is that of minimizing the Willmore functional over all embeddings of a 2-manifold of fixed topological type. In the physics literature, as noted in [16], this functional has been called the rigid string action [18].

To calculate the anomaly, let ̂g_{ij} = e^{2\Upsilon}g_{ij} be a second representative for the conformal infinity, where \( \Upsilon \in C^\infty(M) \). The associated defining functions ̂r and r are related by ̂r = e^\omega r, where \( \omega \) solves (2.8) and \( \Upsilon = \Upsilon \) on M. Differentiation of (2.8) gives

\[\omega_{rr} = -\frac{1}{2} \Upsilon_i \Upsilon_i \] at \( r = 0 \), so

\[\omega(x, u, r) = \Upsilon(x, 0) + \frac{1}{4} \Upsilon_i \Upsilon^i r^2 + O(r^3).\]

Solving the equation ̂r = e^{\omega(x, u(x, r), r)}r for r gives \( r = \hat{r}b(x, \hat{r}) \), where \( b = e^{-\Omega} \) and

\[\Omega(x, \hat{r}) = \Upsilon(x, 0) + \frac{1}{4} e^{-2\Upsilon} (\Upsilon_i H^i - \Upsilon_i \Upsilon^i) \hat{r}^2 + O(\hat{r}^3).\]

As previously observed, if \( c_2 \) and \( \hat{c}_2 \) denote the constant terms in (2.18), then \( c_2 - \hat{c}_2 \) is the constant term in (2.21). By (2.22), this is the same as the constant term in

\[\int_N \left[ -\frac{1}{2}(eb(x, \epsilon))^{-2} - \frac{1}{8}(|H|^2 + 4g^{\alpha\beta}P_{\alpha\beta}) \log b(x, \epsilon) \right] d\nu_N,
\]

which is easily calculated to give

**Proposition 2.4.** When \( k = 2 \), the anomaly is given by

\[8(c_2 - \hat{c}_2) = \int_N \left[ \Upsilon(|H|^2 + 4g^{\alpha\beta}P_{\alpha\beta}) - 2\Upsilon_i H^i + 2\Upsilon_i \Upsilon^i \right] d\nu_N.\]

It is interesting to observe that the linearization in \( \Upsilon \) of this anomaly involves derivatives of \( \Upsilon \)—terms which do not arise from the log term in (2.18) upon rescaling \( \epsilon \).

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