New oscillation results for higher order nonlinear differential equations with a nonlinear neutral terms

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Abstract

The paper deals with the oscillation of higher order nonlinear differential equations with a nonlinear neutral term. The main results are proved via utilizing an integral criterion as well as a comparison theorem with first-order delay differential equation whose oscillatory properties are known. The proposed theorems improve, extend, and simplify existing ones in the literature. The results are associated with four numerical examples.

Keywords: Oscillation, asymptotic behavior, neutral differential equation, comparison method.

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1. Introduction

Consider the higher order nonlinear differential equations with a nonlinear neutral term of the form

\begin{align*}
\left( \tau (\theta) \left( \eta^{(n-1)} (\theta) \right)^{\alpha} \right)' + b(\theta) x^\gamma (\tau (\theta)) + q(\theta) x^\mu (\omega (\theta)) &= 0, \quad \theta \geq \theta_0,
\end{align*}

where $n > 0$ is an even natural number, and the neutral part $\eta (\theta)$ is defined as $\eta (\theta) = x(\theta) - p_1 (\theta) x^\beta (p(\theta))$. A solution of (1.1) is a function $x(\theta)$ which is continuous on $[T_x, \infty)$, $T_x \geq \theta_0$, and satisfies (1.1) on $[T_x, \infty)$. The solutions which are vanishing identically in some neighborhood of infinity will be excluded from our consideration. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative, and to be nonoscillatory otherwise.

Throughout the remaining part of the paper, we make use of the following assumptions:

\begin{enumerate}[(A_1)]
\item $\alpha, \beta, \gamma, \text{ and } \mu$ are quotients of positive odd integers, $\alpha \geq 1$;
\item $b(\theta), p_1 (\theta), \text{ and } q(\theta) \in C ([\theta_0, \infty), \mathbb{R}_+)$ and $b(\theta) \neq 0$;
\end{enumerate}

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(A₃) \( \rho, \tau, \omega \in C ([\theta_0, \infty), \mathbb{R}) \) such that \( \rho(\theta) \leq \theta, \tau(\theta) \leq \theta, \omega(\theta) \geq \theta \) and \( \lim_{\theta \to \infty} \rho(\theta) = \infty = \lim_{\theta \to \infty} \tau(\theta) \);

(A₄) \( \delta(\theta) = \rho^{-1}(\tau(\theta)) \leq \theta, \delta^*(\theta) = \rho^{-1}(\omega(\theta)) \geq \theta \) and \( \delta(\theta) \to \infty \) as \( \theta \to \infty \).

Furthermore, we let
\[
\mathcal{R}(\nu, u) = \int_{u}^{\nu} \frac{ds}{\tau^{1/\alpha}(s)} \quad \text{and} \quad \mathcal{R}^*(\theta, \theta_0) = \int_{\theta_0}^{\theta} (\theta - s)^{n-2} \mathcal{R}(s, \theta_0) ds \to \infty \quad \text{as} \quad \theta \to \infty. \tag{1.2}
\]

Deviating differential equations are thought to be the best tool for modelling a variety of phenomena in science and engineering. In particular, neutral higher order differential equations (NHDEs) arise in the modelling of electric networks containing lossless transmission lines, vibrating masses attached to elastic bars, in some other variational problems, and to mention a few, see [15–19, 21, 22, 31, 34, 36, 37, 56, 60]. Consequently, a great interest has been paid to the study of the qualitative behaviors of NHDEs, such as the existence property, stability property, asymptotic behaviors, oscillatory properties, [3–6, 8, 11, 31–33, 40, 50, 52, 57, 58].

Despite the growing interest in the study of NHDEs, the oscillation and nonoscillation of solutions to these equations remain an ongoing challenge. Starting from the oscillation theory developed by Sturm in 1836, there has been a substantial literature on the oscillation theory of functional differential equations; see [2, 3, 28, 29, 31, 33, 39, 41, 42] and the references cited therein. In this context, one can figure out that many scholars have studied various generalizations of NHDEs and improved the oscillation conditions by using different methods such as the Riccati transformation technique, integral averaging method, comparison method, and inequality technique. The last decades have witnessed the development of the oscillation theory for second-order as well as third-order nonlinear differential equations; the reader is asked to consult the monograph [7] and papers [4–6, 23–27, 30]. However, a little work has been done on the oscillation of higher order neutral differential equations; see the papers [12–14, 20, 38, 46–49, 51, 55]. The motivation behind this paper was initiated upon reading the paper [48], where the authors examined the asymptotic properties and oscillation of the following even-order neutral differential equations of the form
\[
\left( \tau(\theta) \left( u^{(n-1)}(\theta) \right)^{\alpha} \right) + p(\theta) \left( u^{(n-1)}(\theta) \right)^{\alpha} + q(\theta) u^{(\alpha)}(\delta(\theta)) = 0, \tag{1.3}
\]
where \( n \geq 4 \) is an even number, \( u(\theta) = y(\theta) + c(\theta) y(g(\theta)), 0 \leq c(\theta) \leq c_0 < 1, g(\theta) \leq \theta, \delta(\theta) \leq \theta \) and \( \alpha \) is a ratio of odd natural numbers.

It is worth mentioning that the aforementioned equation (1.3) is a particular case of equation (1.1). However, the obtained results in [48] fail to apply to Eq. (1.1). To the best of the authors’ knowledge, moreover, there are no existing results for higher order differential equations involving a nonlinear neutral term, especially of the type (1.1). Inspired by this, the objective of this paper is to study the oscillation problem of (1.1) via comparison with the oscillatory behavior of first order differential equations. The obtained results improve and correlate many of the known oscillation criteria existing in the literature, even for the case of (1.1) with \( p_1(\theta) = 0 \).

For simplicity in what follows, we define:
\[
\psi_1(\theta) = y(\theta), \quad \psi_{k+1}(\theta) = \psi_k(\psi(\theta)), \quad J_1(\theta) = \theta - \psi(\theta), \quad J_{k+1}(\theta) = \int_{\psi(\theta)}^{\theta} I_k(s) ds \quad \text{for} \quad k = 1, 2, \ldots, n-1,
\]
and
\[
\eta_1(\theta) = \eta(\theta), \quad \eta_{k+1}(\theta) = \eta_k(\eta(\theta)), \quad I_1(\theta) = \eta(\theta) - \theta, \quad I_{k+1}(\theta) = \int_{\eta(\theta)}^{\theta} I_k(s) ds \quad \text{for} \quad k = 1, 2, \ldots, n-1,
\]
where \( \eta(\theta), \psi(\theta) \in C ([\theta_0, \infty)) \). We also set
\[
\Theta(\theta) = \frac{b(\theta)}{p_1(\rho^{-1}(\tau(\theta))))} = \frac{b(\theta)}{(p_1(\delta(\theta))))} \quad \text{and} \quad P(\theta) = \frac{q(\theta)}{(p_1(\rho^{-1}(\omega(\theta))))} = \frac{q(\theta)}{(p_1(\delta^*(\theta)))}.
\]
2. Some essential lemmas

To prove our main results, we need the following lemmas.

**Lemma 2.1.** Let $b \in C([\theta_0, \infty), \mathbb{R}_+)$, $g \in C([\theta_0, \infty), \mathbb{R})$ such that $g(\theta) \to \infty$ as $\theta \to \infty$ and $f \in C(\mathbb{R}, \mathbb{R})$ such that $f'(u) > 0$, $\lim f(u) > 0$ for $u \neq 0$.

(I) The associated delay differential equation (i.e., $g(\theta) \leq \theta$)

$$U'(\theta) + b(\theta)f(U(g(\theta))) = 0$$

has an eventually positive solution if the first-order delay inequality

$$U'(\theta) + b(\theta)f(U(g(\theta))) \leq 0$$

does.

(II) The associated advanced differential equation (i.e., $g(\theta) \geq \theta$)

$$U'(\theta) - b(\theta)f(U(g(\theta))) = 0$$

has an eventually positive solution if the first-order advanced inequality

$$U'(\theta) - b(\theta)f(U(g(\theta))) \geq 0$$

does.

**Proof.** This Lemma is an extension of known results in [39, 45, Corollary 1] and the proof is an immediate consequence.

In the sequel, we present the following preliminary lemmas to obtain sign properties of possible nonoscillatory solutions of delayed and/or advanced differential equations. We adopt the method of contradiction to prove the following results:

**Lemma 2.2.** Let $(A_1)$–$(A_4)$ and (1.2) hold. Then the equation

$$\left(\tau(\theta)(U^{(n-1)}(\theta))^\alpha\right)' + b(\theta)\tau(\theta) = 0$$

(2.1)

has no eventually positive solution satisfying $U(\theta) > 0$ and $U^{(n-1)}(\theta) < 0$ eventually.

**Proof.** Let $U(\theta)$ be a nonoscillatory solution of (2.1) such that $U(\theta) > 0$, $\tau(\theta) > 0$ for $\theta \geq \theta_1 > \theta_0$ satisfying $U^{(n-1)}(\theta) < 0$ for $\theta \geq \theta_2$. Since $U^{(n-1)}(\theta) < 0$ for $\theta \geq \theta_2$, then there exists a $\theta_3 > \theta_2$ such that

$$\tau(\theta_3)(U^{(n-1)}(\theta_3))^\alpha = c < 0$$

and

$$\tau(\theta)(U^{(n-1)}(\theta))^\alpha \leq -c < 0$$

for $\theta \geq \theta_3$.

or,

$$U^{(n-1)}(\theta) \leq \left(\frac{-c}{\tau(\theta)}\right)^{1/\alpha} \text{ for } \theta \geq \theta_3.$$

Integrating the preceding inequality $(n - 1)$ times and then applying the condition (1.2) results in the conclusion that $\lim_{\theta \to \infty} U(\theta) = -\infty$, which is a contradiction.

**Lemma 2.3.** Let $(A_1)$–$(A_4)$ and (1.2) hold. If the delay equation

$$U'(\theta) + Cb(\theta)\left(\frac{\tau^{n-1}(\theta)}{\tau^{1/\alpha}(\tau(\theta))}\right)^\gamma U^{\gamma}(\tau(\theta)) = 0$$

(2.2)

is oscillatory for any constant $C > 0$, then (2.1) has no eventually positive solution satisfying $U(\theta) > 0$, $U'(\theta) > 0$ and $U^{(n-1)}(\theta) > 0$ eventually.
Proof. Suppose that \( u(\theta) \) is a nonoscillatory solution of (2.1) such that \( u(\theta) > 0, u(\tau(\theta)) > 0 \) for \( \theta > \theta_1 > \theta_0 \) satisfying \( u'(\theta) > 0 \) and \( u^{(n-1)}(\theta) > 0 \) for \( \theta > \theta_2 \). Following [29, Lemma 2.2], there exists a constant \( \epsilon \in (0, 1) \) such that
\[
U(\tau(\theta)) > \frac{\epsilon}{(n-1)!} \tau^{n-1}(\theta) U^{(n-1)}(\tau(\theta)) \quad \text{for} \quad \theta \geq \theta_1.
\]
Setting \( W(\theta) = r(\theta)(U^{(n-1)}(\theta))^{\alpha} \) in the preceding inequality, we get
\[
U(\tau(\theta)) \geq \left( \frac{\epsilon}{(n-1)!} \right) \left( \frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))} \right) \ W^{1/\alpha}(\tau(\theta)). \tag{2.3}
\]
Using (2.3) in (2.1), we have
\[
W'(\theta) \leq -b(\theta) \left( \frac{\epsilon}{(n-1)!} \right)^{\gamma} \left( \frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))} \right)^{\gamma} \ W^{\frac{\gamma}{\alpha}}(\tau(\theta)) = -C b(\theta) \left( \frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))} \right) \ W^{\frac{\gamma}{\alpha}}(\tau(\theta)), \tag{2.4}
\]
where \( C = \left( \frac{r^{1/\alpha}(\gamma)}{(n-1)!} \right)^{\gamma} \). According to [52, Corollary 1], there is a positive solution \( W(\theta) \) of (2.4) with \( \lim_{\theta \to \infty} W(\theta) = 0 \), despite the fact that (2.2) is oscillatory. \( \square \)

Lemma 2.4. Let \((A_1)-(A_4)\) and (1.2) hold. Assume that there exists a nondecreasing function \( \eta(\theta) \in C([\theta_0, \infty)) \) such that
\[
\eta(\theta) > \theta \text{ and } \eta^{(n-1)}(\theta) < 0. \tag{2.5}
\]
If the delay equation
\[
X'(\theta) + b(\theta) I_{\eta-1}^{\gamma}(\tau(\theta)) r^{-\tilde{x}}(\eta^{(n-1)}(\tau(\theta))) X^{\tilde{x}}(\eta^{(n-1)}(\tau(\theta))) = 0 \tag{2.6}
\]
is oscillatory, then (2.1) has no eventually positive solution satisfying \( U(\theta) > 0, U'(\theta) < 0 \) and \( U^{(n-1)}(\theta) > 0 \) eventually.

Proof. Let \( u(\theta) \) be a nonoscillatory solution of (2.1) such that \( u(\theta) > 0, u(\tau(\theta)) > 0 \) for \( \theta > \theta_1 > \theta_0 \) satisfying \( u'(\theta) < 0 \) and \( u^{(n-1)}(\theta) > 0 \) for \( \theta > \theta_2 \). This is the case when \( n \) is odd, see [48, Lemma 3]. It is not difficult to see that \( u(\theta) \) satisfies
\[
(-1)^m u^m > 0, \quad m = 1, 2, \ldots, n-1.
\]
Consequently,
\[
-u^{(n-2)}(\theta) > u^{(n-2)}(\eta(\theta)) - u^{(n-2)}(\theta) = \int_{\eta(\theta)}^{\theta} u^{(n-1)}(s) ds \geq (\eta(\theta) - \theta) u^{(n-1)}(\eta(\theta)) = I_1(\theta) u^{(n-1)}(\eta(\theta)).
\]
The repeated integrations of the preceding inequality from \( \theta \) to \( \eta(\theta) \) yields
\[
U(\theta) \geq I_{n-1}(\theta) u^{(n-1)}(\eta_{n-1}(\theta)). \tag{2.7}
\]
Using (2.7) in (2.1), we have
\[
X'(\theta) = (\tau(\theta)(U^{(n-1)}(\theta))^{\alpha})' \leq -b(\theta) I_{n-1}^{\gamma}(\tau(\theta)) u^{(n-1)}(\eta^{(n-1)}(\tau(\theta))) \tilde{x} \leq -b(\theta) I_{n-1}^{\gamma}(\tau(\theta)) r^{-\tilde{x}}(\eta^{(n-1)}(\tau(\theta))) X^{\tilde{x}}(\eta^{(n-1)}(\tau(\theta))),
\]
where \( X(\theta) = r(\theta)(U^{(n-1)}(\theta))^{\alpha} \). The rest of the proof is similar to that of Lemma 2.3 and hence is omitted. \( \square \)
Next, we consider the equation

\[ (r(\theta)(U^{(n-1)}(\theta))^\alpha)' = b(\theta)\|U(\theta)\|^{\gamma} + q(\theta)U^\mu(\omega(\theta)). \]  

(2.8)

**Lemma 2.5.** Let \((A_1)-(A_4)\) and (1.2) hold. Assume that there exists a nondecreasing function \(\xi(\theta) \in C([\theta_0, \infty))\) such that \(\tau(\theta) \leq \xi(\theta)\) for \(\theta \geq \theta_0\). If the delay first order equation

\[ y'(\theta) + \left(\frac{\varepsilon}{(n-1)!}\right)^\gamma b(\theta)\tau^{(n-2)\gamma}(\theta) \left(\frac{\xi(\theta) - \tau(\theta)}{r(\tau(\theta))}\right)^\gamma (y(\xi(\theta)))^{\frac{\gamma}{r(\xi(\theta))}} = 0 \]

is oscillatory for \(\varepsilon \in (0, 1]\), then (2.8) has no eventually positive solution satisfying \(U(\theta) > 0\), \(U'(\theta) > 0\) and \(U^{(n-1)}(\theta) < 0\) eventually.

**Proof.** Let \(U(\theta)\) be a nonoscillatory solution of (2.8) such that \(U(\theta) > 0\), \(U(\tau(\theta)) > 0\), and \(U(\omega(\theta)) > 0\) for \(\theta \geq \theta_1 \geq \theta_0\) satisfying \(U'(\theta) > 0\) and \(U^{(n-1)}(\theta) < 0\) for \(\theta \geq \theta_2\). Following [29, Lemma 2.2], there exists a constant \(\varepsilon \in (0, 1]\) such that

\[ U(\tau(\theta)) \geq \left(\frac{\varepsilon}{(n-1)!}\right)\tau^{(n-2)}(\theta)\|U^{(n-2)}(\tau(\theta))\| \text{ for } \theta \geq \theta_1. \]

Using this inequality in (2.8), we have

\[ (r(\theta)(U^{(n-1)}(\theta))^\alpha)' \geq b(\theta)\|U(\tau(\theta))\| \geq b(\theta)\left(\frac{\varepsilon}{(n-1)!}\right)^\gamma \tau^{(n-2)\gamma}(\theta) \left(\frac{\xi(\theta) - \tau(\theta)}{r(\tau(\theta))}\right)^\gamma U(\tau(\theta))^\gamma, \]

or

\[ (r(\theta)(Z'(\theta))^\alpha)' \geq \left(\frac{\varepsilon}{(n-1)!}\right)^\gamma b(\theta)\tau^{(n-2)\gamma}(\theta)Z^\gamma(\tau(\theta)), \]

(2.9)

where \(Z(\theta) = U^{(n-2)}(\theta)\). Because \(Z'(\theta) = U^{(n-1)}(\theta)\) is nondecreasing, then we can see that for \(v \geq u \geq \theta_1\),

\[ Z(u) - Z(v) \geq (v - u)(-Z'(\theta)). \]

Setting \(u = \tau(\theta)\) and \(v = \xi(\theta)\) in the last inequality yields

\[ Z(\tau(\theta)) \geq Z(\xi(\theta)) \geq (\xi(\theta) - \tau(\theta))(-Z'(\xi(\theta))). \]

As a result, the inequality (2.9) can be written as

\[ (r(\theta)(Z'(\theta))^\alpha)' \geq \left(\frac{\varepsilon}{(n-1)!}\right)^\gamma b(\theta)\tau^{(n-2)\gamma}(\theta) \left(\frac{\xi(\theta) - \tau(\theta)}{r(\tau(\theta))}\right)^\gamma (-Z'(\xi(\theta)))^\gamma, \]

which implies that

\[ -y'(\theta) \geq \left(\frac{\varepsilon}{(n-1)!}\right)^\gamma b(\theta)\tau^{(n-2)\gamma}(\theta) \left(\frac{\xi(\theta) - \tau(\theta)}{r(\tau(\theta))}\right)^\gamma y(\xi(\theta)), \]

where \(y(\theta) = -r(\theta)(Z'(\theta))^\alpha\). The rest of the proof is similar to the proof of Lemma 2.4 and hence is omitted. \(\Box\)

**Lemma 2.6.** Let \((A_1)-(A_4)\) and (1.2) hold. Assume that there exists a nondecreasing function \(\eta(\theta) \in C([\theta_0, \infty))\) such that

\[ \eta(\theta) > 0 \text{ and } \eta_{n-2}(\tau(\theta)) < 0. \]

If the delay equation

\[ Y'(\theta) + r^{-\frac{1}{\alpha}}(\theta) \left(\int_0^\infty b(s)ds\right)^\frac{1}{\alpha} \frac{r^\frac{\gamma}{\alpha}(\tau(\theta))Y^{\frac{\gamma}{\alpha}}(\eta_{n-2}(\tau(\theta)))}{r^\frac{\gamma}{\alpha}(\eta_{n-2}(\tau(\theta)))} = 0, \]

(10.20)

is oscillatory, then (2.8) has no eventually positive solution satisfying \(U(\theta) > 0\), \(U'(\theta) < 0\) and \(U^{(n-1)}(\theta) < 0\) eventually.
Proof. Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.8) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$, and $\mathcal{U}(\omega(\theta)) > 0$ for $\theta \geq \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) < 0$ and $\mathcal{U}^{(n-1)}(\theta) < 0$ for $\theta \geq \theta_2$. This is the case when $n$ is even, see [48, Lemma 3]. It is easy to see that $\mathcal{U}(\theta)$ satisfies

$$(-1)^m \mathcal{U}^m > 0, \quad m = 1, 2, \ldots, n - 1.$$  

Proceeding exactly as in the proof of Lemma 2.4, we get

$$\mathcal{U}(\theta) \geq I_{n-2}(\theta) \mathcal{U}^{(n-2)}(\eta_{n-2}(\theta)). \quad (2.11)$$

Integrating the inequality $(\tau(\theta)(\mathcal{U}^{(n-1)}(\theta))^\alpha)' \geq b(\theta)\mathcal{U}^{\gamma}(\tau(\theta))$ from $\theta$ to $u$ and letting $u \to \infty$, we obtain

$$-\mathcal{U}^{(n-1)}(\theta) \geq \mathcal{U}^{\gamma}(\tau(\theta)) \tau^{\frac{\gamma}{\alpha}}(\theta) \left( \int_0^\infty b(s) \, ds \right)^{\frac{1}{\alpha}}. \quad (2.12)$$

Using (2.11) in the preceding inequality, we get

$$-\mathcal{U}^{(n-1)}(\theta) \geq \left( I_{n-2}(\tau(\theta)) \mathcal{U}^{(n-2)}(\eta_{n-2}(\tau(\theta))) \right)^{\frac{\gamma}{\alpha}} \tau^{\frac{\gamma}{\alpha}}(\theta) \left( \int_0^\infty b(s) \, ds \right)^{\frac{1}{\alpha}},$$

or

$$-\mathcal{U}'(\theta) \geq \tau^{\frac{\gamma}{\alpha}}(\theta) \left( \int_0^\infty b(s) \, ds \right)^{\frac{1}{\alpha}} I_{n-2}(\tau(\theta)) \mathcal{U}^{\gamma}(\eta_{n-2}(\tau(\theta))),$$

where $\mathcal{U}(\theta) = \mathcal{U}^{(n-2)}(\theta)$. The rest of the proof is similar to that of Lemma 2.3 and hence is omitted. \qed

**Lemma 2.7.** Let $(A_1)$-$(A_4)$ and (1.2) hold. Assume that there exist nondecreasing function $\psi(\theta) \in C([\theta_0, \infty))$ such that

$$\psi(\theta) < \theta, \psi_{n-1}(\omega(\theta)) > \theta \text{ and } \delta(\theta) \leq \xi(\theta) \leq \psi(\theta) \text{ for } \theta \geq \theta_0. \quad (2.13)$$

If the advanced equation

$$\mathcal{W}'(\theta) - q(\theta) \mathcal{I}^\mu_{n-1}(\omega(\theta)) r^{-\frac{\mu}{\alpha}}(\psi_{n-1}(\omega(\theta))) \mathcal{W}^{\mu}(\psi_{n-1}(\omega(\theta))) = 0, \quad (2.14)$$

is oscillatory, then (2.8) has no eventually positive solution satisfying $\mathcal{U}(\theta) > 0$, $\mathcal{U}'(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$ eventually.

**Proof.** Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.8) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$, and $\mathcal{U}(\omega(\theta)) > 0$ for $\theta \geq \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$ for $\theta \geq \theta_2$. Since $\mathcal{U}(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$, then we only have the case

$$\mathcal{U}^m(\theta) > 0 \text{ for } m = 0, 1, 2, \ldots, n - 1.$$  

Consequently,

$$\mathcal{U}^{(n-2)}(\theta) \geq \mathcal{U}^{(n-2)}(\theta - \mathcal{U}^{(n-2)}(\psi(\theta))) = \int_0^{\psi(\theta)} \mathcal{U}^{(n-1)}(s) \, ds \geq J_1(\theta) \mathcal{U}^{(n-1)}(\psi(\theta)).$$

The repeated integrations of this inequality from $\psi(\theta)$ to $\theta$ yields

$$\mathcal{U}(\theta) \geq J_{n-1}(\theta) \mathcal{U}^{(n-1)}(\psi_{n-1}(\theta)). \quad (2.15)$$

Using (2.14) in the inequality $(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^\alpha)' \geq q(\theta)\mathcal{U}^{\mu}(\omega(\theta))$, we have

$$\mathcal{W}'(\theta) \geq q(\theta) \left( J_{n-1}(\omega(\theta)) \mathcal{U}^{(n-1)}(\psi_{n-1}(\omega(\theta))) \right)^{\mu} \geq q(\theta) J_{n-1}(\omega(\theta)) r^{-\frac{\mu}{\alpha}}(\psi_{n-1}(\omega(\theta))) \mathcal{W}^{\mu}(\psi_{n-1}(\omega(\theta))),$$

where $\mathcal{W}(\theta) = r(\theta)(\mathcal{U}^{(n-1)}(\theta))^\alpha$. It follows from Lemma 2.1 (II) that the corresponding differential equation (2.13) also has a positive solution, which is a contradiction. This completes the proof. \qed
3. Oscillation results

We are ready to present our novel comparison theorem, which reduces the oscillation problem of higher order neutral differential equations to a set of first order delayed and/or advanced differential equations.

Theorem 3.1. Let \( (A_1)-(A_4) \) and \( (1.2) \) hold. Assume that there exist nondecreasing functions \( \xi(\theta), \eta(\theta) \) and \( \psi(\theta) \in C([\theta_0, \infty)) \) such that

\[
\psi(\theta) < \theta, \psi_{n-1}(\delta^*(\theta)) > \theta \text{ and } \delta(\theta) \leq \xi(\theta) \leq \psi(\theta) \text{ for } \theta \geq \theta_0, \\
\eta(\theta) > \theta \text{ and } \eta_{n-1}(\tau(\theta)) < \theta.
\]

If the delay equations \( (2.2) \), \( (2.6) \), and \( (2.8) \) with \( b(\theta) \) and \( \tau(\theta) \) replaced by \( \Omega(\theta) \) and \( \delta(\theta) \), respectively and the advanced equation \( (2.13) \) with \( q(\theta) \) and \( \omega(\theta) \) replaced by \( \mathcal{P}(\theta) \) and \( \delta^*(\theta) \), respectively and \( (2.10) \) is oscillatory, then \( (1.1) \) is oscillatory.

Proof. On the contrary, assume that \( x(\theta) \) is a nonoscillatory solution of \( (1.1) \) such that \( x(\theta), x(\rho(\theta)), x(\tau(\theta)) \), and \( x(\omega(\theta)) \) positive eventually for \( \theta \geq \theta_1 > \theta_0 \). It follows from \( (1.1) \) that

\[
(\tau(\theta)(\mathbb{U}^{(n-1)}(\theta))^\alpha)' = -b(\theta)\mathcal{X}^\gamma(\tau(\theta)) - q(\theta)\mathcal{X}^\mu(\omega(\theta)) \leq 0.
\]

Hence \( \tau(\theta)(\mathbb{U}^{(n-1)}(\theta))^\alpha \) is nonincreasing and of one sign. That is, there exists a \( \theta_2 > \theta_1 \) such that \( \mathbb{U}^{(n-1)}(\theta) > 0 \) or, \( \mathbb{U}^{(n-1)}(\theta) < 0 \) for \( \theta \geq \theta_2 \). We shall distinguish the following four cases:

1. \( \mathbb{U}(\theta) > 0, \mathbb{U}^{(n-1)}(\theta) < 0; \)
2. \( \mathbb{U}(\theta) > 0, \mathbb{U}^{(n-1)}(\theta) > 0; \)
3. \( \mathbb{U}(\theta) < 0, \mathbb{U}^{(n-1)}(\theta) > 0; \)
4. \( \mathbb{U}(\theta) < 0, \mathbb{U}^{(n-1)}(\theta) < 0. \)

Case 1. Since \( \mathbb{U}^{(n-1)}(\theta) \) is nonincreasing and negative, we apply Lemma 2.2 and by condition \( (1.2) \), we conclude that \( \lim_{\theta \to \infty} \mathbb{U}(\theta) = -\infty \), which is a contradiction.

Case 2. Clearly, we see that either (i) \( \mathbb{U}'(\theta) > 0 \) or (ii) \( \mathbb{U}'(\theta) < 0 \) for \( \theta \geq \theta_2 \).

Subcase (i). From the definition of \( \mathbb{U}(\theta) \), we have

\[
\mathbb{U}(\theta) = x(\theta) - p_1(\theta)x^\beta(\rho(\theta)) \leq x(\theta).
\]

Thus, from \( (1.1) \) we have

\[
(\tau(\theta)(\mathbb{U}^{(n-1)}(\theta))^\alpha)' + b(\theta)\mathcal{X}^\gamma(\tau(\theta)) \leq 0.
\]

Applying Lemmas 2.1 and 2.3, we see that Eq. \( (2.2) \) is nonoscillatory, a contradiction.

Subcase (ii). By applying Lemma 2.4, we see that Eq. \( (2.6) \) is nonoscillatory, a contradiction.

After that, we’ll have a look at the situation \( \mathbb{U}(\theta) < 0 \) for \( \theta \geq \theta_2 \). Let

\[
z(\theta) = -\mathbb{U}(\theta) = -x(\theta) + p_1(\theta)x^\beta(\rho(\theta)) \leq p_1(\theta)x^\beta(\rho(\theta)),
\]

which implies that

\[
x(\rho(\theta)) \geq \left( \frac{z(\theta)}{p_1(\theta)} \right)^{1/\beta} \quad \text{or} \quad x(\theta) \geq \left( \frac{z(\rho^{-1}(\theta))}{p_1(\rho^{-1}(\theta))} \right)^{1/\beta}
\]

and so,

\[
(\tau(\theta)(z^{(n-1)}(\theta))^\alpha)' = b(\theta)\mathcal{X}^\gamma(\tau(\theta)) + q(\theta)\mathcal{X}^\mu(\omega(\theta))
\]

\[
\geq \frac{b(\theta)}{p_1(\rho^{-1}(\tau(\theta)))^\beta}z^\gamma(\rho^{-1}(\tau(\theta))) + \frac{q(\theta)}{p_1(\rho^{-1}(\omega(\theta)))^\beta}z^\mu(\rho^{-1}(\omega(\theta))).
\]

Therefore,

\[
(\tau(\theta)(z^{(n-1)}(\theta))^\alpha)' \geq \Omega(\theta)\mathcal{X}^\gamma(\delta(\theta)) + \mathcal{P}(\theta)z^\mu(\delta^*(\theta)).
\]

(3.1)
Case 3. Clearly, we see that \( z^{(n-1)}(\theta) < 0 \) and either (i) \( z'(\theta) > 0 \) or (ii) \( z'(\theta) < 0 \) for \( \theta \geq \theta_2 \).

Subcase (i). From (3.1), we have
\[
(r(\theta)(z^{(n-1)}(\theta))^\alpha)' \geq \Omega(\theta)z^{\frac{\gamma}{\beta}}(\delta(\theta)).
\]

Since \( z^{(n-1)}(\theta) < 0 \) and \( z'(\theta) > 0 \), then following the line of proof of Lemma 2.5, we obtain the desired conclusion.

Subcase (ii). In this case, we apply Lemma 2.6 to Eq. (3.1) and get the desired conclusion.

Case 4. It is not difficult to see that \( z^{(n-1)}(\theta) > 0 \) and \( z(\theta) > 0 \). Following the line of proof of Lemma 2.7, we obtain the desired conclusion. This completes the proof of the theorem.

The oscillatory and asymptotic behavior of Eq. (1.1) is addressed in the following theorem.

**Theorem 3.2.** In Theorem 3.1, if we exclude the equations (2.6) and (2.10), then every solution \( x(\theta) \) of (1.1) is either oscillatory or converges to zero as \( \theta \) tends to \( \infty \).

**Proof.** Let \( x(\theta) \) be a nonoscillatory solution of (1.1) with \( \lim_{\theta \to \infty} x(\theta) \neq 0 \), \( x(\theta) > 0 \), \( x(\rho(\theta)) > 0 \), \( x(\tau(\theta)) > 0 \), and \( x(\omega(\theta)) > 0 \) for \( \theta \geq \theta_1 > \theta_0 \). In this case we exclude Subcase (ii) of Case 2 and Subcase (ii) of Case 3, that is, following the line of proof of Theorem 3.1, we proof all possible cases expect Subcase (ii) of Case 2 and Subcase (ii) of Case 3.

**Remark 3.3.** We may exclude the part of advanced argument of Case 4 of Theorem 3.2 by imposing the condition
\[
\int_{\theta_0}^\infty \Omega(s)(\mathbb{R}^\alpha(s, \theta_0))^\frac{1}{\alpha} ds = \infty \tag{3.2}
\]
and by replacing the proof of Case 4 of Theorem 3.1 by the following proof.

Clearly, we have
\[
r(\theta)(z^{(n-1)}(\theta))^\alpha > 0
\]
and so there exists a constant \( c > 0 \) such that \( r(\theta)(z^{(n-1)}(\theta))^\alpha \geq c \), or \( z^{(n-1)}(\theta) \geq \left( \frac{c}{r(\theta)} \right)^{\frac{1}{\alpha}} \). Thus,
\[
z(\theta) \geq c^{1/\alpha} \mathbb{R}^\alpha(\theta, \theta_2) \text{ for } \theta \geq \theta_3 > \theta_2. \tag{3.3}
\]

Using (3.3) in (3.1), we have
\[
(r(\theta)(z^{(n-1)}(\theta))^\alpha)' \geq \Omega(\theta)z^{\frac{\gamma}{\beta}}(\delta(\theta)) \geq \Omega(\theta) \left( c^{1/\alpha} \mathbb{R}^\alpha(\theta, \theta_2) \right)^{\frac{\gamma}{\beta}}.
\]

The rest of the proof is easy and hence is omitted.

From Theorem 3.2 and Remark 3.3, the following result follows immediately.

**Theorem 3.4.** Assume that all the hypotheses of Theorem 3.2 and condition (3.2) hold, then every solution of (1.1) is oscillatory, converges to zero or diverges to infinity as \( \theta \) tends to \( \infty \).

It is possible to generate sufficient conditions for oscillation of (1.1) by applying existing oscillation criteria to first order differential equations. The following corollary follows immediately from [41, Theorem 2], [42], and Theorem 3.2.
Corollary 3.5. Let \((A_1)-(A_4)\) and (1.2) hold. Assume that there exists a function \(\alpha : [\theta_0, \infty) \rightarrow (0, \infty)\) such that (2.3) is satisfied. Furthermore, assume that there exist nondecreasing functions \(\eta(\theta), \xi(\theta), \psi(\theta) \in C([\theta_0, \infty))\) such that (2.5) and (2.12) hold. If
\[
\liminf_{\theta \to \infty} \int_{\tau(\theta)}^{\theta} b(s) \left( \frac{x^{n-1}(s)}{r^\gamma(s)} \right)^\gamma \, ds = \infty, \quad \text{when } \gamma \leq \alpha,
\]
\[
\liminf_{\theta \to \infty} \int_{\xi(\theta)}^{\theta} Q(s) \left( \frac{(\xi(s) - \delta(s))}{r^\mu(s)} \right)^\mu \, ds = \infty, \quad \text{when } \gamma \leq \alpha \beta,
\]
and
\[
\liminf_{\theta \to \infty} \int_{0}^{\psi_n(\theta)} \beta(s) \left( \frac{(\xi(s) - \delta(s))}{r^\mu(s)} \right)^\mu \, ds \left\{ \begin{array}{ll}
\geq \frac{1}{r^\gamma}, & \text{when } \mu = \alpha \beta, \\
\rightarrow \infty, & \text{when } \mu < \alpha \beta,
\end{array} \right.
\]
hold, then every solution of (1.1) is oscillatory, or converges to zero as \(\theta \) tends to \(\infty\).

4. Examples

For the sake of completeness, the following examples are presented.

Example 4.1. Consider the higher order differential equation
\[
\left( x^3 \left( x(\theta) - x^{1/3}(\theta/2) \right)^{(n-1)} \right) + b(\theta)x^{1/3}(\theta/2) + q(\theta)x^3(2\theta) = 0, \quad \theta \geq \theta_0,
\]
where \(n \in \mathbb{N}, \alpha = 3, \beta = 1/3, \tau(\theta) = \theta^3, \gamma = 1/3, \mu = 3, \rho(\theta) = \theta^2, \quad \tau(\theta) = \theta^3, \quad \omega(\theta) = 2\theta, \quad b(\theta) \text{ and } q(\theta) \text{ are appropriate positive continuous functions. It is easy to check that all conditions of Theorem 3.1 are satisfied and conclude that (4.1) is oscillatory.}

Example 4.2. Consider the higher order differential equation
\[
\left( e^{-\theta} \left( x(\theta) - x^{3}(\theta/2) \right)^\prime \right) + \frac{3}{4} x^{2}(\theta/2) = 0, \quad \theta \geq \theta_0,
\]
where \(\alpha = 1, \beta = 3, r(\theta) = e^{-\theta}, \gamma = 1, \mu = 3, r(\theta) = \theta^2, \quad \tau(\theta) = \theta^3, \quad \omega(\theta) = 2\theta, \quad b(\theta) = \frac{3}{4}, \quad q(\theta) = 0. \text{ It is not difficult to check that all conditions of Theorem 3.4 are satisfied. One such solution is } x(\theta) = e^\theta.

Example 4.3. Consider the higher order differential equation
\[
\left( e^{-\theta} \left( x(\theta) - \frac{1}{\theta^2} x^{3}(\theta/2) \right)^\prime \right) + b(\theta)x(\theta) + q(\theta)x(2\theta) = 0, \quad \theta \geq \theta_0 = 4,
\]
where \(\alpha = 1, \beta = 3, \gamma = 1, \mu = 1, \rho(\theta) = \frac{\theta}{2}, \quad \tau(\theta) = \frac{\theta}{2}, \quad \omega(\theta) = 2\theta, \quad r(t) = e^{-\theta}, \quad p_1(\theta) = \frac{1}{\theta^2}, \quad b(\theta) = \frac{4e^{\theta-24}}{b_0e^{\theta}}, \quad q(\theta) = \frac{20e^{\theta-72}}{9e^{\theta}}. \text{ It is not difficult to check that all conditions of Theorem 3.4 are satisfied. One such solution is } x(\theta) = \theta e^{2\theta}.

Example 4.4. Consider a second order differential equation
\[
\left( e^{-t} \left( x(t) - e^{-2\pi} x(t - 2\pi) \right)^\prime \right) + e^{3t-6\pi} x^3(\tau(t)) + e^{3t+9\pi} x^3(\omega(t)) = 0, \quad t \geq t_0 = 3\pi,
\]
where \(\alpha = 5, \beta = 1, \gamma = \mu = 3, \rho(\theta) = \theta - 2\pi, \tau(\theta) = \theta - 3\pi, \omega(\theta) = \theta + 2\pi, r(t) = e^{-t}, \quad p_1(t) = e^{-2\pi}, \quad b(t) = e^{3t-6\pi}, \quad q(\theta) = e^{3\theta+9\pi}. \text{ Clearly, } \delta(\theta) = \rho^{-1}(\tau(\theta)) = \theta - \pi \leq \theta \text{ and } \delta^*(\theta) = \rho^{-1}(\omega(\theta)) = \theta + 4\pi \geq t. \text{ A straightforward verification shows that all the conditions of Corollary 3.5 hold. Therefore, every solution of (4.2) is oscillatory, or converges to zero. Indeed, } x(\theta) = e^{-\theta} \sin \theta \text{ is such a solution of (4.2).}
5. Concluding remarks and open problems

In the present work, with the help of a comparison utility with the behavior of first order delay and/or advanced differential equations as well as an integral criterion, several results for the oscillation and asymptotic behavior of solutions of equation (1.1) are presented. The main results are formulated by the Theorems 3.1, 3.2, and 3.4. For this purpose, five auxiliary lemmas are given. As an application of the main results, a Corollary 3.5 as well as some examples are then presented. The examples are illustrative and support the theoretical results. Article [48] is concerned with the asymptotic behavior and oscillation of solutions to even order neutral differential equations, which is a topic very close to our investigations. Our obtained theorems not only generalize the existing results in the literature but also can be used to plan future research papers in a variety of directions. For example:

(Q1) One can consider equation (1.1) with
\[
\mathcal{L}(\theta) = x(\theta) - p_1(\theta)x^{\beta_1}(\rho_1(\theta)) + p_2(\theta)x^{\beta_2}(\rho_2(\theta))
\]
with \( p_1(\theta) \leq 0 \) and \( p_2(\theta) \leq 0 \).

(Q2) One can consider equation (1.1) under the case
\[
\lim_{\theta \to \infty} \mathcal{R}(\theta, \theta_0) = \lim_{\theta \to \infty} \int_{\theta_0}^{\theta} \frac{ds}{r_1/\alpha(s)} < \infty.
\]

(Q3) It would be of interest to extend the results of this paper for higher order equations of type (1.1), where \( n \geq 3 \) is an odd natural number.

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