On $n$-saturated Closed Graphs

Szymon Głab and Przemysław Gordinowicz

Abstract. Geschke proved in [2] that there is a clopen graph on $2^\omega$ which is 3-saturated, but the clopen graphs on $2^\omega$ do not even have infinite subgraphs that are 4-saturated; however there is $F_\sigma$ graph that is $\omega_1$-saturated. It turns out that there is no closed graph on $2^\omega$ which is $\omega$-saturated, see [3]. In this note we complete this picture by proving that for every $n \in \mathbb{N}$ there is an $n$-saturated closed graph on the Cantor space $2^\omega$. The key lemma is based on probabilistic argument. The final construction is an inverse limit of finite graphs.

Mathematics Subject Classification. Primary 05C63; Secondary 05C60.

Keywords. $n$-saturated graph, topological graph, closed graph, inverse limit.

For any cardinal number $\kappa$, a graph $G$ is $\kappa$-saturated if for any set $A \subseteq V(G)$ of its vertices, $|A| < \kappa$, and any subset $B \subseteq A$ there is a vertex $w \in V(G) \setminus A$, adjacent to all vertices from $A \setminus B$ and to no vertex of $B$. The random graph [6] is known to be a countable graph which is $\aleph_0$-saturated, or, in other words, it is $n$-saturated for every $n \in \mathbb{N}$.

In the last few decades there have been various results relating graph-theoretic properties of definable graphs on Polish spaces to their descriptive complexity. The most important result in this area is probably the $G_0$-dichotomy for analytic graphs due to Kechris, Solecki and Todorcevic [5]. At the other end of the spectrum we have the clopen graphs that have been studied by Geschke who proved in [2] that there is clopen graph on $2^\omega$ which is 3-saturated, but the clopen graphs on $2^\omega$ do not even have infinite subgraphs that are 4-saturated. Increasing the complexity a little bit, there is an $F_\sigma$ graph that is $\aleph_1$-saturated. In between we have the closed graphs. It turns out that there is no closed graph on $2^\omega$ which is $\aleph_0$-saturated, see [3]. So there is a natural question whether there exist $n$-saturated closed graphs on $2^\omega$, for finite $n > 3$? We answer this question in positive. This makes our knowledge...
of topological graph saturation more complete. Our construction uses different means comparing to that in [2]. It utilizes a probabilistic argument in the key lemma and an inverse limit of finite graphs in the final construction. Our result proves also the existence of \( n \)-saturated open graphs on \( 2^{\omega} \), for \( n \in \mathbb{N} \). The latter follows from the fact that graph \( G \) is \( n \)-saturated if and only if its complement \( \overline{G} \) is.

1. Introduction

By a graph we understand a pair \( G = (V(G), E(G)) \), where \( V(G) \) is a non-empty set of vertices and \( E(G) \) is a symmetric and reflexive relation on \( V(G) \). The reflexivity of \( E(G) \) means that each vertex of the graph \( G \) has a loop. A homomorphism of graphs is a map \( h: V(G) \rightarrow V(H) \) that preserves edges. A graph homomorphism \( h: V(G) \rightarrow V(H) \) is called strict if for every edge \( (p, q) \in E(H) \) such that \( p, q \in h(V(G)) \) there is an edge \( (a, b) \in E(G) \) such that \( h(a) = p \) and \( h(b) = q \). A surjective strict homomorphism is called a quotient map. Note that having loops in considered graphs is needed to being able to map an edge onto a single vertex without violating edge preserving property. That is basically the only reason to consider graphs with loops.

Let \( G \) be a graph, \( A \subseteq V(G) \). A type over \( A \) is a function \( f \in \{0, 1\}^A \) (that means \( f: A \rightarrow \{0, 1\} \)). A vertex \( v \in V(G) \backslash A \) realizes type \( f \) over \( A \) provided that for every \( a \in A \), \( a \) and \( v \) are adjacent if and only if \( f(a) = 1 \). We say that \( G \) is an \( n \)-saturated graph if for every \( A \subseteq V(G) \), \( |A| < n \) and any type \( f \in \{0, 1\}^A \) there is \( x \in V \) that realizes type \( f \); in other words \( (a, x) \in E(G) \iff f(a) = 1 \) for every \( a \in A \). Further, we say that \( G \) is a weakly \( n \)-saturated graph if for every \( A \subseteq V(G) \), \( |A| < n \), there is \( x \in V(G) \) which is adjacent to every \( a \in A \).

Our final graph will be constructed as an inverse limit of finite graphs. Let us briefly recall the definition of the inverse limit in the graph context. Assume that \( \{G_n: n \in \mathbb{N}\} \) is a family of finite graphs and \( p_n : V(G_{n+1}) \rightarrow V(G_n) \) are quotient maps for \( n \in \mathbb{N} \). Let \( p^n_k = p_k \circ p_{k+1} \circ \cdots \circ p_{n-1} \) for \( n > k \geq 1 \). Then \( p^n_k : V(G_n) \rightarrow V(G_k) \) is a quotient map as a composition of quotient maps. Let \( G := \lim G_n \) be the inverse limit of \( \{(G_n)_{n\in\mathbb{N}}, (p^n_k)_{n>k}\} \), that is the graph with the set of vertices

\[
V(G) = \left\{ a \in \prod_{n \in \mathbb{N}} V(G_n) : a(k) = p^n_k(a(n)) \text{ for } n > k \right\}
\]

and the edge relation \( E(G) \) given by

\[
(a, b) \in E(G) \iff \forall n \in \mathbb{N} \ (a(n), b(n)) \in E(G_n).
\]

Clearly \( E(G) \) is symmetric and reflexive, and therefore \( G \) is a graph. Assume that each finite graph \( G_n \) has the discrete topology. On the product \( \prod V(G_n) \) we consider the product topology, that is the topology given by basic sets of the form
Moreover, we consider the following random construction. For a fixed $n \in \mathbb{N}$, we define $B(x_1, \ldots, x_n) := \left\{ a \in \prod_{i \in \mathbb{N}} V(G_n) : a(i) = x_i \text{ for } i \leq n \right\}$

where $x_i \in V(G_i)$ for $i \leq n$. The space $\prod V(G_n)$ is metrizable, compact, zero-dimensional and perfect (i.e. it has no isolated points). Therefore by the Brouwer Theorem [4, 7.4] it is homeomorphic to the Cantor space. We consider $G$ with the topology inherited from $\prod V(G_n)$. It turns out that $V(G)$ is a closed subset of $\prod V(G_n)$ and $E(G)$ is closed subset of $V(G) \times V(G)$. Moreover $q_n : V(G) \rightarrow V(G_n)$ given by $q_n(a) = a(n)$ is a quotient map for every $n \in \mathbb{N}$. Graphs of the form $\lim_{n \rightarrow \infty} G_n$, where $G_n$ are finite, are called profinite graphs. It turns out that profinite graphs coincide with compact graphs on zero-dimensional metrizable compact spaces, see [3].

A closed subset of a compact metrizable zero-dimensional space is again compact metrizable and zero-dimensional. Note that $a$ is an isolated point in the compact metrizable zero-dimensional topological space $V(G)$ if there is $n$ such that $\{a\} \cap B(x_1, \ldots, x_n) \cap V(G) = \{a\}$ where $x_i = a(i)$. We say that every vertex in $(G_n)$ eventually splits, that is for any $n \in \mathbb{N}$ and $v \in V(G_n)$ there are $m > n$ and two distinct $x, y \in V(G_m)$ with $p_n^m(x) = p_n^m(y) = v$. This condition ensures us that if a basic set $B(x_1, \ldots, x_n) \cap V(G)$ is non-empty, it is not a singleton. Finally, if every vertex in $(G_n)$ eventually splits, then $\lim_{n \rightarrow \infty} G_n$ is perfect, and consequently it is homeomorphic to the Cantor space.

2. Construction of an $n$-saturated Closed Graph on the Cantor Space

Consider the following random construction. For a fixed $n \in \mathbb{N}$ we start with a weakly $n$-saturated finite graph $H_0$ having vertex set $V(H_0) = \{1, 2, \ldots, k\}$ for some $k \geq n$. By $H_m$ we denote a random graph with vertex set $V(H_m) = \{1, 2, \ldots, k\} \times \{0, 1, \ldots, m\}$ and the edge relation defined as follows

(i) For every $i < j \leq k$ $(i, 0)$ and $(j, 0)$ are adjacent in $H_m \iff i$ and $j$ are adjacent in $H_0$;

(ii) For any $i, j \leq k$ if $i$ and $j$ are not adjacent in $H_0$, then $(i, s)$ and $(j, t)$ are not adjacent as well for any $s, t \leq m$;

(iii) If $i$ and $j$ are adjacent in $H_0$, $s, t \leq m$ with $s^2 + t^2 > 0$ (at least one of $s$ and $t$ is greater than zero) and $(i, s) \neq (j, t)$, then $(i, s)$ and $(j, t)$ are adjacent in $H_m$ with probability $1/2$, and the decision – whether there is such an edge or not – is made independently to the others.

(iv) $H_m$ is reflexive.

It is well known (see eg. [1, Thm. 10.4.5]) that, given $n \in \mathbb{N}$ and $p \in (0, 1)$, almost all (sufficiently large) random graphs in are $n$-saturated, where by random graphs we mean here graphs in which each pair of vertices is joined by an edge independently, with probability $p$.

Lemma 1. There is $m \in \mathbb{N}$ such that
Theorem 3. Let \( n \in \mathbb{N} \). There exists \( n \)-saturated closed graph on \( 2^\omega \).
Proof. We will construct a profinite graph which is $n$-saturated as an inverse limit of finite graphs. Let $G_0$ be a complete graph with $n$ vertices. Clearly $G_0$ is weakly $n$-saturated. Using Lemma 1 and Lemma 2 there is a finite $n$-saturated graph $G_1$ and a quotient map $p_0 : V(G_1) \to V(G_0)$. Proceeding inductively we find sequences $(G_\ell)_{\ell \in \mathbb{N}}$ and $(p_\ell)_{\ell \in \mathbb{N}}$ of finite $n$-saturated graphs and of quotient mappings, respectively, such that $p_\ell : V(G_{\ell+1}) \to V(G_\ell)$. More precisely at each step we utilize our random construction for $H_0 = G_k$, then we find $m$ and $H_m$ such that the assertion of Lemma 1 holds, finally, we define $G_{\ell+1}$ as $H_m$ and $p_\ell : V(G_{\ell+1}) \to V(G_\ell)$ as a quotient map from Lemma 2. Let $G$ be its inverse limit. In the construction of $H_m$, $p(i, 0) = p(i, 1) = i$ for any $i \in V$. This means that every vertex in $(G_\ell)$ eventually splits, and therefore $G$ is, as a topological space, homeomorphic to the Cantor space $2^\omega$. We will show that $G$ is $n$-saturated.

Let $A \subseteq V(G)$, $|A| = n - 1$ and $f \in \{0, 1\}^A$. Then $A = \{x_1, \ldots, x_{n-1}\}$, $x_i(m) \in V(G_m)$, $p_m(x_i(m + 1)) = x_i(m)$, for every $m \in \mathbb{N}$. Setting $k = \sum_i f(x_i)$ we may assume that $f(x_i) = 1$ if and only if $i \leq k$. There is $m$ such that the set $p_m(A) = \{x_1(m), \ldots, x_{n-1}(m)\} \subseteq V(G_m)$ has $n - 1$ elements. There is $v_m \in G_m$ which realizes $f$, that means $v_m$ is adjacent to vertices $x_1(m), \ldots, x_k(m)$ and not adjacent to $x_{k+1}(m), \ldots, x_{n-1}(m)$. By Lemma 2 there is $v_{m+1} \in G_{m+1}$ which is adjacent to $x_1(m+1), \ldots, x_k(m+1)$ such that $p_m(v_{m+1}) = v_m$. Proceeding inductively for any $i > m$ we find $v_i$ such that $v_i \in V(G_i)$ which is adjacent to $x_1(i), \ldots, x_k(i)$ and $p_{i-1}(v_i) = v_{i-1}$.

Define $v \in V(G)$ as follows

$$v(i) = \begin{cases} v_i & \text{for } i \geq m, \\ p_i^m(v_m) & \text{for } i < m. \end{cases}$$

Then $v(i)$ is adjacent to $x_1(i), \ldots, x_k(i)$ for every $i$ (for $i < m$ it follows from the fact that the projection $p_i^m$ preserves edges). Therefore $v$ is adjacent to $x_1, \ldots, x_k$. Note that $v$ is not adjacent to $x_{k+1}, \ldots, x_n$, since $v(m)$ is not adjacent to $x_{k+1}(m), \ldots, x_{n-1}(m)$.

\begin{acknowledgements}
We would like to thank the anonymous referee for her or his valuable comments, some of them were used in the introduction to this article.
\end{acknowledgements}

\begin{references}
[1] Alon, N., Spencer, J.H.: The probabilistic method, 3rd ed., John Wiley & Sons, (2008)
[2] Geschke, S.: Clopen Graphs. Fund. Math. \textbf{220}, 155–189 (2013)
[3] S. Geschke, S. Głąb, W. Kubiś, Inverse limits of finite graphs, in preparation
\end{references}
[4] Kechris, A.S.: Classical descriptive set theory, *Graduate Texts in Mathematics* 156, Springer-Verlag, New York, (1995), xviii+402

[5] Kechris, A.S., Solecki, S., Todorčević, S.: Borel chromatic numbers. Adv. Math. 141(1), 1–44 (1999)

[6] Rado, R.: Universal graphs and universal functions. Acta Arith. 9, 331–340 (1964)

Szymon Głąb and Przemysław Gordinowicz
Institute of Mathematics, Łódź University of Technology
ul. Wólczańska 215
93-005 Łódź
Poland
e-mail: szymon.glab@p.lodz.pl; pgordin@p.lodz.pl

Received: January 26, 2022.
Accepted: June 17, 2022.

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