VALUE FUNCTIONS AND DUBROVIN VALUATION RINGS ON SIMPLE ALGEBRAS

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Abstract. In this paper we prove relationships between two generalizations of commutative valuation theory for noncommutative central simple algebras: (1) Dubrovin valuation rings; and (2) the value functions called gauges introduced by Tignol and Wadsworth in [TW1] and [TW2]. We show that if \( v \) is a valuation on a field \( F \) with associated valuation ring \( V \) and \( v \) is defectless in a central simple \( F \)-algebra \( A \), and \( C \) is a subring of \( A \), then the following are equivalent: (a) \( C \) is the gauge ring of some minimal \( v \)-gauge on \( A \), i.e., a gauge with the minimal number of simple components of \( C/J(C) \); (b) \( C \) is integral over \( V \) with \( C = B_1 \cap \ldots \cap B_ξ \) where each \( B_i \) is a Dubrovin valuation ring of \( A \) with center \( V \), and the \( B_i \) satisfy Gräter’s Intersection Property. Along the way we prove the existence of minimal gauges whenever possible and we show how gauges on simple algebras are built from gauges on central simple algebras.

Introduction

Valuation theory has been a very useful tool in the study of finite-dimensional division algebras, particularly in the construction of examples, such as noncrossed products and division algebras with nontrivial reduced Whitehead group \( SK_1 \). (See [W] for a survey of valuation theory on division algebras.) But there has been some difficulty in applying valuation theory in noncommutative settings because division algebras do not have many valuations and simple algebras that are not division algebras do not have valuations. This has led to efforts to find structures similar to but less restrictive than valuations that would exist more widely.

One such approach was initiated by Dubrovin in [Du1] and [Du2]. By generalizing the idea of places in commutative valuation theory, he defined what are now called Dubrovin valuation rings. Such rings share many of the distinctive properties of commutative valuation rings, and it is known that for every central simple algebra \( A \) over a field \( F \) and every valuation ring \( V \) of \( F \) there is a Dubrovin valuation ring \( B \) of \( A \) with center \( V \), and \( B \) is unique up to isomorphism. However, there is in general no valuation associated with such a \( B \) (except in the integral case, see below). The substantial theory of Dubrovin valuation rings is the topic of the book [MMU].

Another approach was initiated rather recently by Tignol and the second author in [TW1] and [TW2] by the introduction of gauges, which are a kind of value function on a semisimple algebra \( S \).
finite-dimensional over a field $F$, but satisfying weaker axioms than for a valuation. (See §1 below for the definition of a gauge.) The theory of gauges is still developing, but it has already become a useful complement to the classical valuation theory of division algebras. A gauge $\alpha$ on $S$ always extends some valuation $v$ on $F$, so $\alpha$ is called a $v$-gauge. Just as the valuation $v$ induces a filtration on $F$, so $\alpha$ induces a filtration on $S$ yielding an associated graded ring $gr_{\alpha}(S)$, which is a finite-rank semisimple graded algebra over the graded field $gr_{\alpha}(F)$. The associated graded algebra captures much of the essential information about $\alpha$ on $S$, but $gr_{\alpha}(S)$ is often much easier to work with than with $S$ itself. Gauges are easy to construct in many cases, and they have good behavior with respect to tensor products and scalar extensions of algebras.

The question naturally arises what kind of connections there may be between Dubrovin valuation rings and gauges. A limited answer was provided in [TW] respect to tensor products and scalar extensions of algebras.

Let $\alpha$ be a $v$-gauge on the central simple $F$-algebra $A$. The degree zero piece of the associated graded ring $gr_{\alpha}(A)$, denoted $A^0_{\alpha}$, coincides with $R_{\alpha}/J(R_{\alpha})$, and is a semisimple ring finite-dimensional over the residue ring $F^v$ of the valuation $v$. Let

$$\omega(\alpha) = \text{the number of simple components of the semisimple ring } A^0_{\alpha}.$$

We show in Th. 3.5 that $\omega(\alpha) \geq \xi_{V,[A]}$, where $V$ is the valuation ring of $v$. We call $\alpha$ a minimal gauge if equality holds.

We show in this paper that there are still significant, and somewhat surprising, connections between the Dubrovin valuation theory and gauges even when the Dubrovin valuation rings are not integral over their centers. We use the special intersections of Dubrovin valuation rings analyzed by Gräter in [G1]. He showed that to every valuation ring $V$ of a field $F$ and every central simple $F$-algebra $A$ there is a subring $C$ of $A$, with $C$ integral over $V$ and determined uniquely up to isomorphism, such that $C = B_1 \cap \ldots \cap B_\xi$, where the $B_i$ are each Dubrovin valuation rings of $A$ with center $V$, and the $B_i$ are related by satisfying a special “Intersection Property.” We dub such a $C$ a Gräter ring for $V$ in $A$. Gräter proved that among other nice properties $C$ is a noncommutative Bézout ring and that the $B_i$ are determined from $C$ as the localizations of $C$ with respect to its maximal ideals.

The number $\xi$ of $B_i$ in the intersection (= the number of maximal ideals of $C$) is an invariant $\xi = \xi_{V,[A]}$ of $V$ and the Brauer class $[A]$ of $A$. Gräter called this number the “extension number.” The same number had appeared earlier in [W1] in the “Ostrowski Theorem” for Dubrovin valuation rings. The extension number equals 1 if and only if some (hence every) Dubrovin valuation ring of $A$ with center $V$ is integral over $V$.

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$$\omega(\alpha) = \text{the number of simple components of the semisimple ring } A^0_{\alpha}.$$

We show in Th. 3.5 that $\omega(\alpha) \geq \xi_{V,[A]}$, where $V$ is the valuation ring of $v$. We call $\alpha$ a minimal gauge if equality holds. We show in Th. 3.9 that if $\alpha$ is a minimal gauge on $A$ then its gauge ring $R_{\alpha}$ is a
Gräter ring. Conversely, we prove in Th. 3.11, that if $v$ is defectless in $A$ then a Gräter ring of $A$

Defect in valuation theory refers to the failure of equality in the Fundamental Inequality. The
background results we need on defect are given in §1.b. The defect is trivial for valuations of residue
characteristic 0 or of prime residue characteristic not dividing the index of a central simple algebra.
It is not hard to prove that if a semisimple $F$-algebra $S$ has a $v$-gauge, then the valuation $v$
must be defectless in $S$ (see Prop. 1.9). In Th. 4.3 we prove the very nontrivial converse that if $v$
is defectless in $S$, then $S$ necessarily has a minimal $v$-gauge. This is of interest in itself, and is also
essential for the proof of Th. 3.11.

Our approach to proving results about gauges and Gräter rings with respect to a central valuation $v$
often involves working back from corresponding objects for a coarser valuation $w$. A $v$-gauge
$\alpha$ on a central simple $F$-algebra $A$ has a coarsening to a $w$-gauge $\beta$ on $A$ with an induced
$v/w$-gauge on the residue ring $A_0^\beta$, see Prop. 1.5. However, the semisimple $\overline{F}^w$-algebra $A_0^\beta$
need not be central simple. Therefore, it has been necessary to determine how gauges on semisimple
algebras are related to gauges for central simple algebras. We do this in §2. Reduction from the
semisimple case to the simple case is very easy. The simple case turns out to have a nice description
that takes some work to prove: If $S$ is a finite-dimensional simple $F$-algebra, and $v$ is a valuation
on $F$, let $v_1, \ldots, v_r$ be the extensions of $v$ to the center $Z(S)$. If $\alpha$ is a $v$-gauge on $F$, we show
in Th. 2.2 that there are uniquely determined $v_i$-gauges $\alpha_i$ on $S$ such that $\alpha = \min(\alpha_1, \ldots, \alpha_r)$.
Moreover, in Th. 2.8 we give a necessary and sufficient compatibility condition on $v_i$-gauges $\alpha_i$ so
that $\min(\alpha_1, \ldots, \alpha_r)$ is a $v$-gauge. This is a notable result in its own right for the still-developing
theory of gauges.

For valuations on fields, the valuation is determined (up to equivalence) by the valuation ring.
This is likewise true for the Morandi value functions determined by Dubrovin valuation rings integral
over their centers. We give an example in §5 to show that the corresponding property does not
hold for minimal gauges. In our example there are infinitely many nonisomorphic minimal gauges
on a central simple algebra all with the same gauge ring $R_\alpha$.

There is a book in preparation [TW$_3$] that will give a more extensive treatment of gauges than
is available in the currently published literature. We have adapted §1.b below on defect and the
result in the Appendix from that book because the material is essential for this paper and there is
no adequate published reference available. Also, the existence of gauges for defectless algebras was
first proved in that book. What is needed here to complete the results in §3 below is the existence
of minimal gauges for defectless algebras. This is proved in §4 by an argument that is substantially
different from the one in [TW$_3$].

1. Value functions on semisimple algebras

For the convenience of the reader, we recall a few basic properties of gauges proved in [TW$_1$] and
[TW$_2$]. First, a few words of notation: If $R$ is any ring (with 1), we write: $R^\times$ for the group of units
of $R$; $Z(R)$ for the center of $R$; $J(R)$ for the Jacobson radical of $R$; and $M_k(R)$ for the $k \times k$-matrix
ring over $R$. Fix for now a divisible totally ordered abelian group $\Gamma$, chosen to be sufficiently large
to contain the values of all valuations and the degrees of all gradings we consider. If $v : F \to \Gamma \cup \{\infty\}$
is a valuation on a field $F$, let $V$ be the corresponding valuation ring, $V = \{x \in F | v(x) \geq 0\}$;
then $J(V) = \{x \in F | v(x) > 0\}$, which is the unique maximal ideal of $V$. We write $\overline{F}^v$ for the
residue field $V/J(V)$ and $\Gamma_v$ for the value group $v(F^\times)$, which is a subgroup of $\Gamma$. We say that
$v$ and $V$ are trivial if $V = F$. For basic assumed background on valuations on fields we refer to Bourbaki [B, Ch. 6] or Engler-Prestel [EP]. For valuations on division rings we use analogous notation and terminology to that for fields. A good reference for valuation theory on division rings is the paper [JW].

Let $D$ be a division ring finite-dimensional over its center. A valuation $v : D \to \Gamma \cup \{\infty\}$ defines a filtration on $D$: for each $\gamma \in \Gamma$, we set

$$D_{\geq \gamma} = \{d \in D \mid v(d) \geq \gamma\}, \quad D_{> \gamma} = \{d \in D \mid v(d) > \gamma\}, \quad \text{and} \quad D_{\gamma} = D_{\geq \gamma}/D_{> \gamma}.$$  

The associated graded ring for $v$ on $D$ is

$$\text{gr}_v(D) = \bigoplus_{\gamma \in \Gamma} D_{\gamma}.$$  

It is a graded division ring because every nonzero homogeneous element in $\text{gr}(D)$ is invertible. (If $D$ is commutative, then $\text{gr}(D)$ is also commutative and is then called a graded field.) The grade set $\Gamma_{\text{gr}(D)}$ of $\text{gr}(D)$ is $\{\gamma \in \Gamma \mid D_{\gamma} \neq \{0\}\}$, which is a subgroup of $\Gamma$ coinciding with $\Gamma_v$. Note that $D_0$ is a division ring. Also, since $\Gamma_v$ is a torsion-free, $D^\times = \bigcup_{\gamma \in \Gamma_v} D_{\gamma} \setminus \{0\}$. Now, let $M = \bigoplus_{\gamma \in \Gamma} M_{\gamma}$ be a graded right $\text{gr}(D)$-module, i.e., $M$ is a right $\text{gr}(D)$-module with $M_{\gamma} \cdot D_{\delta} \subseteq M_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Then, $M$ is a free $\text{gr}(D)$-module with well-defined rank and a homogeneous base. We therefore call $M$ a right graded $\text{gr}(D)$-vector space, and write $\dim_{\text{gr}(D)} M$ for the rank of $M$ as a $\text{gr}(D)$-module. If $N$ is another graded $\text{gr}(D)$-vector space, then a $\text{gr}(D)$-homomorphism $\psi : M \to N$ is called a graded homomorphism if $\psi(M_{\gamma}) \subseteq N_{\gamma}$ for all $\gamma \in \Gamma$. We write $M \cong g N$ if there is a graded isomorphism $M \to N$. If $m \in M \setminus \{0\}$ is homogeneous, then there is a unique $\gamma \in \Gamma$ with $m \in M_{\gamma}$; we call $\gamma$ the degree of $m$, and write $\gamma = \deg m$. For any $\delta \in \Gamma$, we write $M(\delta)$ for the $\text{gr}(D)$-vector space obtained from $M$ by shifting the grading by $\delta$:

$$M(\delta) = \bigoplus_{\gamma \in \Gamma} M(\delta)_{\gamma} \quad \text{where} \quad M(\delta)_{\gamma} = M_{\gamma+\delta}. \quad (1.1)$$

Now let $M$ be a right $D$-vector space. A $v$-value function on $M$ is a map $\alpha : M \to \Gamma \cup \{\infty\}$ such that

(i) $\alpha(x) = \infty$ if and only if $x = 0$;

(ii) $\alpha(x + y) \geq \min (\alpha(x), \alpha(y))$ for $x, y \in M$;

(iii) $\alpha(xd) = \alpha(x) + v(d)$ for all $x \in M$ and $d \in D$.

We write $\Gamma_{\alpha}$ to denote the value set $\alpha(A \setminus \{0\})$, which need not be a group, but it is a union of cosets of $\Gamma_v$. The $v$-value function $\alpha$ is called a $v$-norm if $M$ is finite-dimensional and contains a $D$-vector space base $(x_i)_{i=1}^n$ such that

$$\alpha \left( \sum_{i=1}^n x_i d_i \right) = \min_{1 \leq i \leq n} \left( v(d_i) + \alpha(x_i) \right) \quad \text{for} \quad d_1, \ldots, d_n \in D. \quad (1.2)$$

Such a base $(x_i)_{i=1}^n$ is called a splitting base of $M$ for $\alpha$. Just as with valuations, the value function $\alpha$ defines a filtration on $M$: for each $\gamma \in \Gamma$, we set

$$M_{\geq \gamma} = \{ x \in M \mid \alpha(x) \geq \gamma \}, \quad M_{> \gamma} = \{ x \in M \mid \alpha(x) > \gamma \}, \quad \text{and} \quad M_{\gamma} = M_{\geq \gamma}/M_{> \gamma}. \quad (1.2)$$

(When we need to specify the value function, we write $M^\alpha_{\geq \gamma}, M^\alpha_{> \gamma}, \text{and} M^\alpha_{\gamma}$.) The associated graded object

$$\text{gr}_{\alpha}(M) = \bigoplus_{\gamma \in \Gamma} M_{\gamma}$$
is a graded right gr(D)-vector space with \( \dim_{gr(D)} gr_\alpha(M) \leq \dim_D M \) (see [TW, Cor. 2.3]). We will write \( gr(M) \) instead of \( gr_\alpha(M) \) when the value function \( \alpha \) is clear. Every nonzero element \( x \in M \) has an image \( \tilde{x}^\alpha \) in \( gr(M) \), defined by \( \tilde{x}^\alpha = x + M_{>\alpha(x)} \subseteq M_{\alpha(x)} \). We will often write simply \( \tilde{x} \) instead of \( \tilde{x}^\alpha \) when \( \alpha \) is clear. A \( D \)-base \( (x_i)^k_{i=1} \) of \( M \) is a splitting base of \( M \) for \( \alpha \) if and only if \( (\tilde{x}_i)^k_{i=1} \) is a \( gr(D) \)-base of \( gr(M) \) (see [RTW, Cor. 2.3]). Hence, \( \alpha \) is a \( v \)-norm on \( M \) if and only if \( \dim_{gr(D)} gr(M) = \dim_D M \).

A \( v \)-value function \( \alpha \) on a finite-dimensional \( F \)-algebra \( A \) is surmultiplicative if \( \alpha(1) = 0 \) and \( \alpha(xy) \geq \alpha(x) + \alpha(y) \) for \( x, y \in A \). For such an \( \alpha \), set

\[
R_\alpha = \{ x \in A \mid \alpha(x) \geq 0 \} \quad \text{and} \quad J_\alpha = \{ x \in A \mid \alpha(x) > 0 \}.
\]

It is clear from the axioms for \( \alpha \) that \( R_\alpha \) is a subring of \( A \) and \( J_\alpha \) is a two-sided ideal of \( R_\alpha \). One can check further that \( 1 + J_\alpha \subseteq R_\alpha^{\geq} \); hence \( J_\alpha \) lies in the Jacobson radical \( J(R_\alpha) \). Thus, if \( R_\alpha/J_\alpha \) is semisimple, then \( J_\alpha = J(R_\alpha) \). Also, if \( F = Z(A) \) and \( \Gamma_\alpha \) lies in the divisible hull of \( \Gamma_v \) then \( Z(R_\alpha) = R_\alpha \cap F = V \), the valuation ring of \( v \), and \( J(R_\alpha) \cap F = J(V) \). Moreover, if in addition \( \alpha \) is a \( v \)-norm, then it follows from [TW1, Lemma 1.20 and Th. 3.1] that \( R_\alpha \) is integral over \( V \). Because \( \alpha \) is surmultiplicative, the multiplication in \( A \) induces a multiplication on the graded \( gr(F) \)-vector space \( gr_\alpha(A) \) such that for \( x, y \in A \)

\[
\tilde{x} \tilde{y} = xy + A_{>\alpha(x)+\alpha(y)} = \begin{cases} \tilde{xy} & \text{if } \alpha(xy) = \alpha(x) + \alpha(y), \\ 0 & \text{if } \alpha(xy) > \alpha(x) + \alpha(y). \end{cases}
\]

Thus, \( gr_\alpha(A) \) is a graded algebra over \( gr(F) \). We write \([gr_\alpha(A); gr(F)]\) for \( \dim_{gr(F)} gr_\alpha(A) \).

Let \( F \) be a field with a valuation \( v \) and valuation ring \( V \). A surmultiplicative \( v \)-value function \( \alpha \) on a finite-dimensional \( F \)-algebra \( A \) is called a \( v \)-gauge if \( \alpha \) is a \( v \)-norm and \( gr_\alpha(A) \) is a graded semisimple \( gr(F) \)-algebra, i.e., \( gr_\alpha(A) \) has no nonzero homogeneous nilpotent ideal. In this case, \( R_\alpha \) is called the gauge ring of \( \alpha \). If \( \alpha \) is a \( v \)-gauge on a semisimple \( F \)-algebra \( A \) and \( \alpha' \) is a \( v \)-gauge on a semisimple \( F \)-algebra \( A' \) we say that \( \alpha \) and \( \alpha' \) are isomorphic \( v \)-gauges if there is an \( F \)-algebra isomorphism \( \eta: A \to A' \) such that \( \alpha' \circ \eta = \alpha \).

For finite-dimensional graded algebras over a graded field there are structure theorems analogous to the Wedderburn theorems in the ungraded context. The graded theory is developed in [HW, §1].

**Example 1.1.** One of the basic constructions of gauges is that of \( End \)-gauges on endomorphism rings determined by norms on vector spaces, which we now recall. Let \( D \) be a finite-dimensional division \( F \)-algebra and let \( M \) be a finite-dimensional right \( D \)-vector space. Suppose the valuation \( v \) on \( F \) extends to a valuation \( w \) on \( D \) and let \( \alpha \) be a \( w \)-norm on \( M \). Then there is a well-defined surmultiplicative \( v \)-value function \( End(\alpha) \) on the endomorphism ring \( End_D(M) \), given by

\[
End(\alpha)(f) = \min_{m \in M \setminus \{0\}} \left( \alpha(f(m)) - \alpha(m) \right)
\]

(see [TW1, Prop. 1.19]). Moreover,

\[
gr_{End(\alpha)}(End_D(M)) \cong_{g} End_{gr(D)} \left( gr_\alpha(M) \right).
\]

(The endomorphism ring \( End_{gr(D)} \left( gr_\alpha(M) \right) \) has a natural grading in which a map \( g: gr_\alpha(M) \to gr_\alpha(M) \) has degree \( \gamma \) if \( g(M_\delta) \subseteq M_{\gamma+\delta} \) for all \( \delta \in \Gamma_\alpha \).) Thus \( gr_{End(\alpha)}(End_D(M)) \) is graded simple, i.e., it has no proper nonzero homogeneous ideals. It follows by dimension count that \( End(\alpha) \) is a \( v \)-gauge if and only if \( w \) on \( D \) is defectless over \( v \), i.e., \([gr_\alpha(D); gr_\alpha(F)] = [D:F] \) (see [TW1, Prop. 1.19]). Since \( \dim_D M < \infty \) the grade set \( \Gamma_\alpha = \alpha(M \setminus \{0\}) \) consists of finitely many cosets of \( \Gamma_v \), say
The graded ring $\text{End}_{\text{gr}(D)}(\text{gr}(M))$ can be viewed as a matrix ring with shifted grading as follows: Let $(m_1, \ldots, m_n)$ be any splitting base of $M$ with respect to $\alpha$, and let $\gamma_i = \alpha(m_i)$, for each $i$. Then $(\bar{m}_1, \ldots, \bar{m}_n)$ is a homogeneous base of the $(D)$-vector space $\text{gr}(M)$ with $\deg \bar{m}_i = \gamma_i$. Let $D = \text{gr}(D)$. Each $\bar{m}_i$ spans a 1-dimensional graded $D$-subspace of $M$, and $\bar{m}_iD \cong_g D(\gamma_i)$, which is $D$ with its grading shifted by $\gamma_i$, as in (1.1). Thus, as graded $D$-vector spaces, $\text{gr}(M) = \bigoplus_{i=1}^n \bar{m}_iD \cong_g \bigoplus_{i=1}^n D(\gamma_i)$. Then, as graded $(F)$-algebras,

$$\text{End}_{\text{gr}(D)}(\text{gr}(M)) \cong_g \text{End}_D (D(\gamma_1) \oplus \ldots \oplus D(\gamma_n)) \cong_g \mathbb{M}_n(D)(\gamma_1, \ldots, \gamma_n),$$

where $\mathbb{M}_n(D)(\gamma_1, \ldots, \gamma_n)$ is the matrix ring $\mathbb{M}_n(D)$ but with grading shifted so that its $\delta$-component consists of the matrices with each $ij$-entry $D_{\gamma_j - \gamma_i + \delta}$, for all $\delta \in \Gamma$. Since $D_{\gamma_j - \gamma_i + \delta} \neq \{0\}$ if and only if $\delta \in \gamma_i - \gamma_j + \Gamma_w$, we have

$$\Gamma_{\text{End}(\alpha)} = \Gamma_{\text{End}_{\text{gr}(D)}(\text{gr}(M))} = \bigcup_{i=1}^n \bigcup_{j=1}^n \gamma_i - \gamma_j + \Gamma_w.$$

We will need in §4 the following more general construction of End-gauges:

**Lemma 1.2.** Let $A$ be a semisimple $F$-algebra with a $v$-gauge $\alpha$. Let $M$ be a free right $A$-module with base $(m_1, \ldots, m_n)$. Take any $\gamma_1, \ldots, \gamma_n \in \Gamma$, and let $\eta: M \to \Gamma \cup \{\infty\}$ be the “$\alpha$-$v$-norm” defined by

$$\eta \left( \sum_{i=1}^n m_i a_i \right) = \min_{1 \leq i \leq n} \left( \gamma_i + \alpha(a_i) \right), \quad \text{for all } a_1, \ldots, a_n \in A.$$

Let $E = \text{End}_A(M)$ and let $\psi$ be the $v$-value function $\text{End}(\eta)$ on $E$ defined by

$$\psi(f) = \min_{1 \leq i \leq n} \left( \eta(f(m_i)) - \eta(m_i) \right).$$

Then,

$$\psi(f) = \min_{m \in M \setminus \{0\}} \left( \eta(f(m)) - \eta(m) \right),$$

and $\psi$ is a $v$-gauge on $E$. Moreover, $\text{gr}_\eta(M)$ is a free right $\text{gr}_\alpha(A)$-module with base $(\bar{m}_1, \ldots, \bar{m}_n)$, and $\text{gr}_\psi(E) \cong_g \text{End}_{\text{gr}_\alpha(A)}(\text{gr}_\eta(M))$. 

Proof. Note that for all \( m, m' \in M \) and \( a \in A \),

\[
\eta(ma) \geq \eta(m) + \alpha(a) \quad \text{and} \quad \eta(m + m') \geq \min(\eta(m), \eta(m')).
\]

Now, take any \( f \in E \) and nonzero \( m \in M \), say \( m = \sum_{i=1}^{n} m_i a_i \) with \( a_i \in A \). Then,

\[
\eta(f(m)) - \eta(m) = \eta\left( \sum_{i=1}^{n} f(m_i)a_i \right) - \eta(m) \geq \min_{1 \leq i \leq n} \left( \eta(f(m_i)) + \alpha(a_i) \right) - \eta(m)
\]

\[
= \min_{1 \leq i \leq n} \left[ \left( \eta(f(m_i)) - \eta(m_i) \right) + \left( \eta(m_i) + \alpha(a_i) \right) \right] - \eta(m)
\]

\[
\geq \min_{1 \leq i \leq n} \left( \eta(f(m_i)) - \eta(m_i) \right) + \min_{1 \leq i \leq n} \left( \eta(m_i) + \alpha(a_i) \right) - \eta(m)
\]

\[
= \min_{1 \leq i \leq n} \left( \eta(f(m_i)) - \eta(m_i) \right) = \psi(f).
\]

Thus, \( \eta(f(m)) - \eta(m) \geq \psi(f) \). Hence,

\[
\psi(f) = \min_{1 \leq i \leq n} \left( \eta(f(m_i)) - \eta(m_i) \right) \geq \min_{m \in M \setminus \{0\}} \left( \eta(f(m)) - \eta(m) \right) \geq \psi(f),
\]

so equality holds throughout, proving (1.10).

Note that for the identity map \( id_M \), clearly \( \psi(id_M) = 0 \). Also, let \( f, g \in E \). Then,

\[
\psi(f \circ g) = \min_{m \in M \{0\}} (\eta(f(g(m))) - \eta(m)) = \min_{m \in M \{0\}} \left[ \eta(f(g(m))) - \eta(g(m)) \right] + \min_{m \in M \{0\}} \left( \eta(g(m)) - \eta(m) \right)
\]

\[
\geq \psi(f) + \psi(g).
\]

Thus, \( \psi \) is surmultiplicative.

Now consider the graded structures. Note that as \( \alpha \) is a \( \nu \)-norm, \( \eta \) is a \( \nu \)-value function on \( M \). So, \( gr_\eta(M) \) is a graded \( gr_\nu(F) \)-vector space. Formula (1.9) shows \( \eta(ma) \geq \eta(m) + \alpha(a) \) for all \( m \in M \) and \( a \in A \). Hence, the right \( A \)-module structure of \( M \) induces a well-defined \( gr_\alpha(A) \)-module action on \( gr_\eta(M) \) such that

\[
\hat{m} \hat{a} = ma \text{ if } \eta(ma) = \eta(m) + \alpha(a);
\]

\[
0 \text{ if } \eta(ma) > \eta(m) + \alpha(a).
\]

In particular, \( \hat{m} \hat{a} = \hat{a} \hat{m} \). Let \( (a_j)_{j=1}^{r} \) be a splitting base of \( A \) for \( \alpha \). By formula (1.9), \( (m_i a_j)_{i,j=1}^{n,r} \) is a splitting base of \( M \) for \( \eta \). Hence,

\[
gr_\eta(M) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{r} \hat{m}_i \hat{a}_j gr_\nu(F) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{r} \hat{m}_i \hat{a}_j gr_\nu(F) = \bigoplus_{i=1}^{n} \hat{m}_i \bigoplus_{j=1}^{r} \hat{a}_j gr_\nu(F) = \bigoplus_{i=1}^{n} \hat{m}_i gr_\alpha(A).
\]

Therefore, \( gr_\eta(M) \) is a free \( gr_\alpha(A) \)-module of rank \( n \) with base \( \{\hat{m}_1, \ldots, \hat{m}_n\} \).

Take any nonzero \( f \in E \). Formula (1.10) above shows that \( \eta(f(m)) \geq \psi(f) + \eta(m) \), for all \( m \in M \), with equality for some \( m \in M \setminus \{0\} \). From this one can see that \( f \) induces a well-defined map \( \hat{f} : gr_\eta(M) \to gr_\eta(M) \) defined on homogeneous elements by

\[
\hat{f}(\hat{m}) = \left\{ \begin{array}{ll}
\hat{f}(\hat{m}) & \text{if } \eta(f(m)) = \psi(f) + \eta(m); \\
0 & \text{if } \eta(f(m)) > \psi(f) + \eta(m).
\end{array} \right.
\]
Routine calculations show that \( \hat{f} \) is a \( \text{gr}_v(F) \)-vector space endomorphism of \( \text{gr}_\eta(M) \) which maps each \( M_i \) into \( M_{\gamma + \psi(f)} \). The definition of \( \psi \) shows that \( \hat{f}(m_i) \neq 0 \) for some \( i \); so \( \hat{f} \neq 0 \). Moreover, for any \( m \in M \) and \( a \in A \), it is easy to check that

\[
\hat{f}(m \bar{a}) = \begin{cases} 
\hat{f}(m a) & \text{if } \eta(f(ma)) = \psi(f) + \eta(m) + \alpha(a) \\
0 & \text{if } \eta(f(ma)) > \psi(f) + \eta(m) + \alpha(a) 
\end{cases} = \hat{f}(m)\bar{a},
\]

from which it follows that \( \hat{f} \in \text{End}_{\text{gr}_\alpha(A)}(\text{gr}_\eta(M)) \) and \( \hat{f} \) is homogeneous of degree \( \psi(f) \). Moreover, for nonzero \( g \in E \) and any \( m \in M \),

\[
\hat{g} \circ \hat{f} = \begin{cases} 
g(\hat{f}(m)) & \text{if } \eta(g(\hat{f}(m))) = \psi(g) + \psi(f) + \eta(m), \\
0 & \text{if } \eta(g(\hat{f}(m))) > \psi(g) + \psi(f) + \eta(m), 
\end{cases}
\]

hence,

\begin{equation}
(1.11)
\hat{g} \circ \hat{f} = \begin{cases} 
g \circ \hat{f} & \text{if } \psi(g \circ f) = \psi(g) + \psi(f); \\
0 & \text{if } \psi(g \circ f) > \psi(g) + \psi(f).
\end{cases}
\end{equation}

Also, if \( \psi(g) > \psi(f) \), then \( \hat{f} + g = \hat{f} \). Therefore, \( \hat{f} \) depends only on the image \( \tilde{f} \) of \( f \) in \( \text{gr}_\psi(E) \). Thus, there is a well-defined map \( \iota: \text{gr}_\psi(E) \to \text{End}_{\text{gr}_\alpha(A)}(\text{gr}_\eta(M)) \) given on homogeneous elements by \( \iota(\tilde{f}) = \hat{f} \) for \( f \in E \). Using (1.11), one can check that \( \iota \) is a graded \( \text{gr}_v(F) \)-algebra homomorphism, and \( \iota \) is injective since it is injective on each homogeneous component of \( \text{gr}_\psi(E) \).

For surjectivity of \( \iota \), take any \( j, k \in \{1, \ldots, n\} \) and any nonzero \( a \in A \). Define \( h \in E \) by \( h(m_j) = m_j a \) and \( h(m_k) = 0 \) for \( i \neq k \). Then, as \( \eta(m_j a) = \gamma_j + \alpha(a) \) and \( \eta(h(m_k)) = \eta(m_j a) = -\eta(m_j) = 0 \) for \( i \neq k \). Of course, \( \iota(h) = \iota(\hat{h}) = \iota(\tilde{h}) \in \text{im}(\iota) \). Because maps such as \( \tilde{h} \) generate \( \text{End}_{\text{gr}_\alpha(A)}(\text{gr}_\eta(M)) \) as an abelian group, the map \( \iota \) is surjective, so a graded isomorphism. Because \( \text{gr}_\alpha(A) \) is graded semisimple (as \( \alpha \) is a \( v \)-gauge) and \( \text{gr}_\eta(M) \) is a free \( \text{gr}_\alpha(A) \)-module, \( \text{End}_{\text{gr}_\alpha(A)}(\text{gr}_\eta(M)) \) is graded semisimple by the graded Wedderburn theory (see [HW, §1, especially Prop. 1.3]). Hence, \( \text{gr}_\psi(E) \) is graded semisimple. Furthermore,

\[
[\text{gr}_\psi(E):\text{gr}_v(F)] = [\text{End}_{\text{gr}_\alpha(A)}(\text{gr}_\eta(M)):\text{gr}_v(F)] = n^2[\text{gr}_\alpha(A):\text{gr}_v(F)]
\]

Thus, \( \psi \) is a surmultiplicative \( v \)-norm on \( E \) with \( \text{gr}_\psi(E) \) graded semisimple, showing that \( \psi \) is a \( v \)-gauge on \( E \). \( \square \)

The following proposition shows how to construct gauges on direct products of algebras.

**Proposition 1.3.** Let \( A_1, \ldots, A_k \) be finite-dimensional simple \( F \)-algebras with respective \( v \)-gauges \( \alpha_1, \ldots, \alpha_k \), and let \( A = A_1 \times \cdots \times A_k \). The map \( \alpha: A \to \Gamma \cup \{\infty\} \) defined by

\begin{equation}
(1.12)
\alpha(x_1, \ldots, x_k) = \min(\alpha_1(x_1), \ldots, \alpha_k(x_k)), \quad \text{for } x_1 \in A_1, \ldots, x_k \in A_k,
\end{equation}

is a \( v \)-gauge on \( A \) and there exists a canonical identification of \( \text{gr}(F) \)-algebras

\[
\text{gr}_\alpha(A) = \text{gr}_{\alpha_1}(A_1) \times \cdots \times \text{gr}_{\alpha_k}(A_k).
\]

**Proof.** It is easily checked that \( \alpha \) is a surmultiplicative \( v \)-value function on \( A \). For \( \gamma \in \Gamma \), we have \( \alpha(x_1, \ldots, x_k) \geq \gamma \) if and only if \( \alpha_i(x_i) \geq \gamma \) for \( i = 1, \ldots, k \). Hence \( A_{\geq \gamma} = A_{1, \geq \gamma} \times \cdots \times A_{k, \geq \gamma} \). Similarly, \( A_{> \gamma} = A_{1, > \gamma} \times \cdots \times A_{k, > \gamma} \). Therefore,

\begin{equation}
(1.13)
\text{gr}_\alpha(A) = \text{gr}_{\alpha_1}(A_1) \times \cdots \times \text{gr}_{\alpha_k}(A_k).
\end{equation}
By counting dimensions, we have

$$[\text{gr}_\alpha(A) : \text{gr}(F)] = \sum_{i=1}^{k} [\text{gr}_{\alpha_i}(A_i) : \text{gr}(F)].$$

Since $[\text{gr}_{\alpha_i}(A_i) : \text{gr}(F)] = [A_i : F]$ for $i = 1, \ldots, k$, it follows that $[\text{gr}_\alpha(A) : \text{gr}(F)] = [A : F]$. Finally, the projections of a homogeneous nilpotent two-sided ideal of $\text{gr}_\alpha(A)$ are homogeneous nilpotent two-sided ideals of each $\text{gr}_{\alpha_i}(A_i)$, which is trivial since $\text{gr}_{\alpha_i}(A_i)$ is assumed semisimple. Therefore, $\text{gr}_\alpha(A)$ is also semisimple. \(\square\)

Every $v$-gauge on $A$ as in Prop. 1.3 is given by the formula (1.12); more precisely, if $\beta : A \to \Gamma \cup \{\infty\}$ is a $v$-gauge, then for each $i = 1, \ldots, k$ the map $\beta_i : A_i \to \Gamma \cup \{\infty\}$ defined by

$$\beta_i(x) = \beta(0, \ldots, 0, x, 0, \ldots, 0) \quad (x \text{ in the } i\text{-th position})$$

is a $v$-gauge on $A_i$ and

$$\beta(x_1, \ldots, x_k) = \min(\beta_1(x_1), \ldots, \beta_k(x_k)), \text{ for } x_1 \in A_1, \ldots, x_k \in A_k,$$

(see [TW1, Prop. 1.6] for a proof).

1.a. **Composition of gauges.** Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation on a field $F$, where $\Gamma$ is a divisible totally ordered abelian group, and let $V$ be the valuation ring of $v$. A valuation $v'$ on $F$ is said to be equivalent to $v$ if its valuation ring is also $V$. Recall that a valuation $w$ on $F$ is said to be a coarsening of $v$ if its valuation ring $W$ contains $V$. When this occurs, the maximal ideal $J(W)$ of $W$ is a prime ideal of $V$, and $W$ is the localization $V_{J(W)}$ of $V$ at $J(W)$. As is well-known, the map $w \mapsto J(W)$ gives a one-to-one correspondence between the equivalence classes of coarsenings of $v$ and the set of prime ideals of $V$. Given $v$ and $w$, the ring $U = V/J(W)$ is a valuation ring of $\overline{F}^w = W/J(W)$, and a valuation $u$ on $\overline{F}^w$ with ring $U$ is called the residue valuation determined by $v$ and $w$; we sometimes denote $u$ by $v/w$. Its residue field is $\overline{F}^w = \overline{F}^v$. From the perspective of $w$, the valuation $v$ is called a refinement of $w$, and $v$ is determined up to equivalence by $w$ and $u$, since $V = \pi_w^{-1}(U)$ where $\pi_w : W \to \overline{F}^w$ is the canonical projection; we call $v$ the composite of $w$ and $u$, and write $v = u \ast w$.

We now look at coarsenings from the perspective of the valuation $v : F \to \Gamma \cup \{\infty\}$. Let $\Delta \subseteq \Gamma$ be any convex subgroup, i.e., a subgroup of $\Gamma$ satisfying for all $\gamma \in \Gamma$, $\delta \in \Delta$, if $0 \leq \gamma \leq \delta$ then $\gamma \in \Delta$. So, $\Delta$ is a divisible group. Let $\Lambda = \Gamma/\Delta$, which is a divisible totally ordered abelian group under the ordering induced from $\Gamma$. Let $\varepsilon : \Gamma \to \Lambda$ be the canonical map, which we extend to $\Gamma \cup \{\infty\}$ by setting $\varepsilon(\infty) = \infty$. By composing $v$ with $\varepsilon$, we obtain a valuation $w$ on $F$

$$w = \varepsilon \circ v : F \to \Lambda \cup \{\infty\},$$

which is a coarsening of $v$. Let $W$ be the valuation ring of $w$. The residue valuation $u = v/w$ is

$$u : \overline{F}^w \to \Delta \cup \{\infty\} \quad \text{given by } u(c + J(W)) = \begin{cases} v(c) & \text{if } w(c) = 0, \\ \infty & \text{if } w(c) > 0, \end{cases} \quad \text{for all } c \in W.$$ 

Note that all coarsenings of $v$ (up to equivalence) are obtainable this way: If $w'$ is a coarsening of $v$, let $\Delta_0 = \{v(c) \mid w(c) = 0\}$, and let $\Delta'$ be the convex hull of $\Delta_0$ in $\Gamma$. Then $w'$ is equivalent to the coarsening of $v$ determined by $\Delta'$.

Let

$$\mathbb{H}(\Gamma_v) = \Gamma_v \otimes \mathbb{Z} \mathbb{Q},$$

(1.15)
which is the divisible hull of $\Gamma_v$. Recall that the ordering on $\Gamma_v$ extends uniquely to $\mathbb{H}(\Gamma_v)$, making the latter into a divisible totally ordered abelian group. Moreover, we may view $\mathbb{H}(\Gamma_v)$ as a subgroup of $\Gamma$, since there is a unique monomorphism $\mathbb{H}(\Gamma_v) \to \Gamma$ extending the inclusion $\Gamma_v \to \Gamma$. Since valuations extending $v$ on algebraic extensions of $F$ all have value groups lying in $\mathbb{H}(\Gamma_v)$, there is generally no loss for us to assume $\Gamma = \mathbb{H}(\Gamma_v)$. Recall that the convex subgroups of $\mathbb{H}(\Gamma_v)$ are in one-to-one correspondence with the convex subgroups of $\Gamma_v$, which are in turn in one-to-one correspondence with prime ideals of $V$, which are in turn in one-to-one correspondence with equivalence classes of coarsenings of $v$.

Now let $A$ be a finite-dimensional $F$-algebra and let $\alpha: A \to \Gamma \cup \{\infty\}$ be a surmultiplicative $v$-value function. With $\Delta$, $\Lambda$, $\varepsilon$, and $w$ as above, the composition of $\alpha$ with $\varepsilon$ yields a surmultiplicative $w$-value function

$$\beta = \varepsilon \circ \alpha: A \to \Lambda \cup \{\infty\}.$$  

If $\alpha$ is a $v$-gauge, then $\beta$ is a $w$-gauge, by [TW$_2$, Prop. 4.3]. In this case, $\beta$ is called a coarsening of $\alpha$. If $\Gamma_{\alpha} \subseteq \mathbb{H}(\Gamma_v)$, then $w = \beta|_F$ determines $\Delta \cap \mathbb{H}(\Gamma_v)$, which determines $\beta$; we then call $\beta$ the $w$-coarsening of $\alpha$.

**Proposition 1.4.** Let $\alpha$ be a $v$-gauge on a central simple $F$-algebra $A$ such that $\Gamma_{\alpha}$ lies in the divisible hull $\mathbb{H}(\Gamma_v)$ of $\Gamma_v$. Let $w$ be any valuation on $F$ which is a coarsening of $v$, and let $W$ be the valuation ring of $w$. Let $\beta$ be the $w$-coarsening of $\alpha$. Then the gauge ring $R_{\beta}$ is a central localization of $R_{\alpha}$ by $P = J(W)$, that is, $R_{\beta} = R_{\alpha} \cdot V_P$.

**Proof.** For each $x \in A$, if $\alpha(x) \geq 0$, then $\beta(x) = \varepsilon(\alpha(x)) \geq 0$. Thus, $R_{\alpha} \subseteq R_{\beta}$. Since $W = V_P \subseteq R_{\beta}$, we have $R_{\alpha} \cdot V_P \subseteq R_{\beta}$. For the reverse inclusion, let $b \in R_{\beta}$. Since $\beta(x) > 0$ implies $\alpha(x) > 0$, we only have to consider the case $\beta(x) = 0$. In this case, $\alpha(x) \in \Delta$. Since $\Gamma_{\alpha} \subseteq \mathbb{H}(\Gamma_v)$, there exists a positive integer $n$ such that $na(x) \in \Gamma_v$. Thus, $na(x) \in \Gamma_v \cap \Delta$. Let $c \in F$ such that $v(c) = n|\alpha(x)|$. Hence, we have $w(c) = \varepsilon(v(c)) = 0$. Since $-n|\alpha(x)| \leq \alpha(x) \leq n|\alpha(x)|$, it follows that $\alpha(cx) = v(c) + \alpha(x) \geq 0$. Therefore, $x = (xc)^{-1} \in R_{\alpha} \cdot V_P$. \[\square\]

For the coarsening $\beta$ of the $v$-gauge $\alpha$ on $A$ as above, let $A^\beta_0$ be the degree zero part of $\text{gr}_\beta(A)$, which is a finite-dimensional semisimple $\mathbb{F}^w$-algebra. Thus,

$$A^\beta_0 = A^\beta_{\geq 0} / A^\beta_{> 0} = \{x \in A \mid \alpha(x) \in \Delta \text{ or } \alpha(x) > \delta \text{ for all } \delta \in \Delta\} / \{x \in A \mid \alpha(x) > \delta \text{ for all } \delta \in \Delta\}.$$

For $u = v/w$, we can define a $u$-value function on $A^\beta_0$:

$$\alpha_0: A^\beta_0 \to \Delta \cup \{\infty\} \text{ by } x + A^\beta_{> 0} \mapsto \begin{cases} \alpha(x) & \text{if } \beta(x) = 0, \\ \infty & \text{if } \beta(x) > 0. \end{cases}$$

(1.16)  

This is well-defined by [TW$_2$, Lemma 4.1]. Note that $\Gamma_{\alpha_0} = \Delta \cap \Gamma_v$. For any $\delta \in \Delta$ we have

$$(A^\beta_0)_\delta = (A^\beta_{\geq 0})_{> \delta} / (A^\beta_{> 0})_{> \delta} = (A^\alpha_{\geq 0} / A^\beta_{> 0})_{> \delta} / (A^\alpha_{> 0} / A^\beta_{> 0})_{> \delta} \cong A^\alpha_{\geq 0} / A^\alpha_{> 0} = A^\alpha_{> \delta}. $$

Hence,

(1.17)  

$$\text{gr}_{\alpha_0}(A^\beta_0) \cong_{g} \bigoplus_{\delta \in \Delta} A^\alpha_{\delta},$$

which is easily checked to be a graded ring isomorphism. Thus, we may view $\text{gr}_{\alpha_0}(A^\beta_0)$ as a graded subring of $\text{gr}_{\alpha}(A) = \bigoplus_{\gamma \in \Gamma} A^\alpha_{\gamma}$. 


Proposition 1.5. The value function \( \alpha_0 \) is a \( u \)-gauge on \( A_0^\beta \).

Proof. We know from [TW2, Prop. 4.3] that \( \alpha_0 \) is a \( u \)-norm. Moreover, \( \alpha_0 \) is surmultiplicative since \( \alpha \) is surmultiplicative. Thus, it remains only to prove that \( \text{gr}_{\alpha_0}(A_0^\beta) \) is graded semisimple. Since \( \text{gr}_\alpha(A) \) is graded semisimple, it suffices to consider the case when \( \text{gr}_\alpha(A) \) is graded simple. In this case, by the graded version of Wedderburn’s Theorem (see [HW, Prop. 1.3]), we can identify \( \text{gr}_\alpha(A) \) with \( \text{End}_D(V) \), where \( D \) is a graded division ring and \( V \) is a finite-dimensional right graded \( D \)-vector space. (The grading on \( \text{End}_D(V) \) is given by: for \( \eta \in \Gamma \), an \( f \in \text{End}_D(V) \) is homogeneous of degree \( \eta \) if and only if \( f(V_\gamma) \subseteq V_{\gamma+\eta} \) for all \( \gamma \in \Gamma \).) Let \{\( b_1, \ldots, b_n \)\} be a homogeneous \( D \)-base of \( V \), and let \( \gamma_i = \text{deg}(b_i) \). Since \( \Gamma_V \) is a finite union of cosets of \( \Gamma_D \), we can write \{\( b_1, \ldots, b_n \)\} as a disjoint union \( \bigcup_{i=1}^k S_i \), where \( b_i \) and \( b_j \) are in the same \( S_i \) if and only if \( \gamma_i - \gamma_j \in \Gamma_D + \Delta \). Let \( V_i \) be the graded \( D \)-vector subspace of \( V \) generated by \( S_i \). We have a direct sum decomposition \( V = \bigoplus_{i=1}^k V_i \). By (1.17), each nonzero homogeneous \( f \in \text{gr}_{\alpha_0}(A_0^\beta) \) is a graded division ring and \( A_0^\beta \) is graded simple. Let \( \text{End}_D(V_i) \) be the graded \( D \)-vector subspace of \( V_i \) generated by \( S_i \). We have a direct sum decomposition \( V = \bigoplus_{i=1}^k V_i \). By (1.17), each nonzero homogeneous \( f \in \text{gr}_{\alpha_0}(A_0^\beta) \) is a graded division ring. Let \( \alpha \) be a graded simple algebra. Let \( B_i = \bigoplus_{\gamma \in \Delta} \text{End}_D(V_i)_\gamma \), which is a graded \( D \)-vector subring of \( D \). Now fix an index \( i \in \{1, \ldots, k\} \). The grade set of \( V_i \) lies in some coset \( \lambda_i + (\Gamma_D + \Delta) \). Let \( V_i = \bigoplus_{\gamma \in \lambda_i + \Delta} V_\gamma \subseteq \bigoplus_{\gamma \in \lambda_i + \Gamma_D + \Delta} V_\gamma = V_i \), so \( V_i \) is a graded \( D' \)-vector subspace of \( V_i \). Let \( \Gamma_D = \bigcup_{\gamma \in J} \gamma \cup (\Delta \cap \Gamma_D) \), a disjoint union of cosets of \( \Delta \cap \Gamma_D \). Then we also have \( \Gamma_D + \Delta = \bigcup_{\gamma \in J} \gamma + \Delta \), which is again a disjoint union. For each \( j \in J \), pick some \( d_j \in D_\delta_j \setminus \{0\} \). Then, as \( \Gamma_D d_j = \delta_j + (\Delta \cap \Gamma_D) \), we have \( D = \bigoplus_{j \in J} D'd_j \). Also, as \( V_i d_j = \bigoplus_{\gamma \in \lambda_i + \delta_j + \Delta} V_\gamma \), \( V_i = \bigoplus_{j \in J} V_i d_j = \bigoplus_{j \in J} V_i \otimes_{D'} D'd_j = V_i \otimes_{D'} D \).

Any map in \( B_i \) sends \( V_i \) to itself, since \( \Gamma_{B_i} \subseteq \Delta \). Thus, there exists a homomorphism of graded rings \( \psi: B_i \to \text{End}_{D'}(V_i') \), given by \( g \mapsto g|_{V_i'} \). This map \( \psi \) has an inverse given by sending \( h \in \text{End}_{D'}(V_i') \) to \( h \otimes \text{id}_D \in \text{End}_{D'}(V_i' \otimes_{D'} D) \). Note that if \( h \) is homogeneous, then \( \text{deg}(h) \in \Delta \), as \( \Gamma_{V_i'} \subseteq \lambda_i + \Delta \). Hence, \( h \otimes \text{id}_D \) is homogeneous with \( \text{deg}(h \otimes \text{id}_D) = \text{deg}(h) \in \Delta \), showing that \( h \otimes \text{id}_D \in B_i \). Therefore, \( B_i \cong \text{End}_{D'}(V_i') \), which is a simple graded algebra.

1.b. Defectlessness of valuations in semisimple algebras. We now develop the notion of defectlessness of a valuation \( v \) on \( F \) in a finite-dimensional semisimple \( F \)-algebra \( A \). We will see in §4 that defectlessness is the condition required for the existence of \( v \)-gauges on \( A \). First, we review the concept of defect on division algebras. A good reference for the division algebra case is [M1].
Let $D$ be a finite-dimensional division algebra over a field $F$. For a valuation $w$ on $D$, extending a valuation $v$ on $F$, we have the “fundamental inequality”

\[(1.19) \quad [D:F] \geq [D:F] \cdot |\Gamma_w:\Gamma_v|.
\]

When equality holds in (1.19), we say the valuation $w$ on $D$ is defectless over $F$. If $v$ extends uniquely to $Z(D)$, we define the defect $\partial_{D/F}$ of $D$ over $F$ by

\[(1.20) \quad \partial_{D/F} = \frac{[D:F]}{[D:F] \cdot |\Gamma_w:\Gamma_v|}.
\]

In particular, $\partial_{D/Z(D)}$ is always defined; we call it simply the defect of $D$ and use the simpler notation $\partial_D$ for $\partial_{D/Z(D)}$. When $\partial_{D/F}$ is defined we have

\[(1.21) \quad \partial_{D/F} = \partial_D \cdot \partial_{Z(D)/F}
\]

by the transitivity formulas for residue degrees and and indices of value groups. In fact, for $\overline{p} = \text{char } F$, we have $\partial_{D/F} = \overline{p}^\ell$ for some integer $\ell \geq 0$ if $\overline{p} \neq 0$, and $\partial_{D/F} = 1$ if $\overline{p} = 0$. This result is known as Ostrowski’s Theorem and was proved by Draxl in [D, Th. 2] for $v$ Henselian and in general by Morandi in [M1, Th. 3]. Hence in particular, if $\overline{p} = 0$ or if $\overline{p} \neq 0$ and $\overline{p} \not| [D:F]$, then $\partial_{D/F} = 1$.

Let $A$ be a (finite-dimensional) semisimple $F$-algebra, let $(F_h, v_h)$ be a Henselization of $(F, v)$, and let $A_h = A \otimes_F F_h$. Since $F_h$ is a separable extension of $F$, the $F_h$-algebra $A_h$ is semisimple; hence, it has a decomposition into simple components

\[A_h \cong \mathbb{M}_{n_1}(D_1) \times \ldots \times \mathbb{M}_{n_r}(D_r)
\]

for some integers $n_1, \ldots, n_r$ and some division algebras $D_1, \ldots, D_r$ over $F_h$. We say that $v$ is defectless in $A$ if for each $i = 1, \ldots, r$ the unique valuation on $D_i$ extending $v_h$ is defectless over $F$, i.e., $\partial_{D_i/F_h} = 1$ for $i = 1, \ldots, r$.

It is clear from the definition that $v$ is defectless in $A$ if and only if $v_h$ is defectless in each simple component of $A$, and that this condition holds if and only if $v_h$ is defectless in $A_h$. We single out two particular cases:

- If $K$ is a finite-degree field extension of $F$ and $v_1, \ldots, v_n$ are all the extensions of $v$ of $F$ to $K$, then $v$ is defectless in $K$ if and only if equality holds in the Fundamental Inequality, i.e.,

\[ [K:F] = \sum_{i=1}^n [K^{(v_i)}:F] \cdot |\Gamma_{v_i}:\Gamma_v|.
\]

This follows readily from Th. A.1 in the Appendix.

- If $A$ is a central simple $F$-algebra, then $v$ is defectless in $A$ if and only if the valuation on the division algebra associated to $A_h$ is defectless over $F_h$. In particular, if $v$ is defectless in $A$, it is also defectless in every algebra Brauer-equivalent to $A$.

In simple algebras that are not central, we have the following reduction to the central case:

**Proposition 1.6.** Let $A$ be a simple $F$-algebra, and let $v_1, \ldots, v_r$ be the valuations on $Z(A)$ extending $v$. The following conditions are equivalent:

(a) $v$ is defectless in $A$;

(b) $v$ is defectless in $Z(A)$ and $v_1, \ldots, v_r$ are each defectless in $A$. 

Proof. For each $i = 1, \ldots, r$, let $(Z_i, v_{i,h})$ be a Henselization of $(Z(A), v_i)$. From Th. A.1 in the Appendix, we have 

$$Z(A)_h \cong Z_1 \times \ldots \times Z_r.$$ 

Since $Z(A_h) = Z(A)_h$, the number of prime components of $A_h$ is $r$, and we have division algebras $D_1, \ldots, D_r$ with centers $Z_1, \ldots, Z_r$ respectively such that 

$$A_h \cong \mathbb{M}_{n_1}(D_1) \times \ldots \times \mathbb{M}_{n_r}(D_r)$$ 

for some integers $n_1, \ldots, n_r$. Now, we have $\partial_{D_i/F_h} = \partial_{D_i} \cdot \partial_{Z_i/F_h}$ (cf. (1.21)); hence $\partial_{D_i/F_h} = 1$ if and only if $\partial_{D_i} = 1$ and $\partial_{Z_i/F_h} = 1$. The equivalence of (a) and (b) follows. 

When $\Gamma_F \cong \mathbb{Z}$, $v$ is defectless in any central simple $F$-algebra and $v$ is defectless in a semisimple $F$-algebra $A$ if and only if it is defectless in $Z(A)$. Also, by Ostrowski’s Theorem, if $\text{char } \overline{F} = 0$, then $v$ is defectless in every semisimple $F$-algebra; if $\text{char } \overline{F} = \overline{p} \neq 0$, then $v$ is defectless in every central simple $F$-algebra $A$ whose index $\text{ind}(A)$ is not divisible by $\overline{p}$.

**Lemma 1.7.** Let $w$ be any coarsening of the valuation $v$ on $F$, and let $K$ be a finite-degree field extension of $F$. If $v$ is defectless in $K$, then $w$ is defectless in $K$.

Proof. Let $w_1, \ldots, w_\ell$ be the extensions of $w$ to $K$. For $j \in \{1, \ldots, \ell\}$, let $v_{1j}, v_{2j}, \ldots, v_{kj}$ be the extensions of $v$ to $K$ that are refinements of $w_j$. Thus, the $v_{ij}$ are all the extensions of $v$ to $K$. Let $v/w$ denote the valuation on $\overline{F}^w$ induced by $v$ and likewise $v_{ij}/w_j$ the valuation on $\overline{K}^{w_j}$ induced by $v_{ij}$. Note that $v_{1j}/w_j, v_{2j}/w_j, \ldots, v_{kj}/w_j$ are all the extensions of $v/w$ to $\overline{K}^{w_j}$. For each $i, j$ there is a commutative diagram of value groups with exact rows:

$$\begin{array}{cccc}
0 & \to & \Gamma_{v/w} & \to & \Gamma_v & \to & \Gamma_w & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma_{v_{ij}/w_j} & \to & \Gamma_{v_{ij}} & \to & \Gamma_{w_j} & \to & 0
\end{array}$$

Because the map $\Gamma_w \to \Gamma_{w_j}$ is injective, the Snake Lemma yields a short exact sequence of cokernels of the columns:

$$0 \longrightarrow \Gamma_{v_{ij}/w_j}/\Gamma_{v/w} \longrightarrow \Gamma_{v_{ij}}/\Gamma_v \longrightarrow \Gamma_{w_j}/\Gamma_w \longrightarrow 0$$

Hence,

$$\text{ind} \Gamma_{v_{ij}} : \Gamma_v = [\Gamma_{v_{ij}/w_j} : \Gamma_{v/w}] [\Gamma_{w_j} : \Gamma_w].$$

(1.22)

Since $v$ is defectless in $K$, we have

$$[K:F] = \sum_{j=1}^{\ell} \sum_{i=1}^{k_j} [\overline{K}^{v_{ij}} : \overline{F}^v] [\Gamma_{v_{ij}} : \Gamma_v].$$
Equation (1.22) together with the Fundamental Inequality for each \( K^{w_j} \) over \( \overline{F}^w \) and for \( K \) over \( F \) then yield,
\[
[K:F] = \sum_{j=1}^{\ell} \left( \sum_{i=1}^{k_j} \left[ \frac{K^{v_{ij}}}{F} : \overline{F}^w \right] \left| \Gamma_{v_{ij}/w_j} : \Gamma_{v/w} \right| \left| \Gamma_{w_j} : \Gamma_w \right| \right)
\]
\[
\leq \sum_{j=1}^{\ell} \left[ \frac{K^{w_j}}{F} : \overline{F}^w \right] \left| \Gamma_{w_j} : \Gamma_w \right| \leq [K:F].
\]

The last inequality must therefore be an equality, showing that \( w \) is defectless in \( K \). \( \square \)

**Proposition 1.8.** Let \( w \) be any coarsening of the valuation \( v \) on \( F \). Let \( A \) be a semisimple \( F \)-algebra. If \( v \) is defectless in \( A \), then \( w \) is defectless in \( A \).

**Proof.** Assume first that \( A \) is central simple over \( F \). Let \( (F_{h,v}, v_h) \) be a Henselization of \( (F, v) \). Let \( w' \) be the valuation on \( F_{h,v} \) with \( W \cdot V_h \). So, \( w' \) is the extension of \( w \) which is a coarsening of \( v_h \). Since \( v_h \) is Henselian, its coarsening \( w' \) is also Henselian, by [EP, Cor. 4.1.4, p. 90], so \( (F_{h,v}, w') \) contains a Henselization \( (F_{h,w}, w_h) \) of \( (F, w) \). It follows from [M1, Th. 2] that \( w' \) is inertial (= unramified) over \( w_h \). Let \( D_{h,v} \) (resp. \( D_{h,w} \)) be the central division algebra over \( F_{h,v} \) (resp. \( F_{h,w} \)) associated to \( A \otimes_F F_{h,v} \) (resp. \( A \otimes_F F_{h,w} \)). Since \( v \) is defectless in \( A \), \( D_{h,v} \) is defectless for \( v_h \); it is then also defectless for the coarser valuation \( w' \) by [M4, Lemma 1]. Then, by [JW, Remark 3.4] applied to the inertial extension \( (F_{h,v}, w') \) of \( (F_{h,w}, w_h) \), \( D_{h,w} \) is defectless for \( w_h \). Hence, \( w \) is defectless in \( A \), as desired.

Now assume only that \( A \) is simple. Let \( K = Z(A) \), and let \( v_1, \ldots, v_r \) be the extensions of \( v \) to \( K \), and \( w_1, \ldots, w_\ell \) the extensions of \( v \) to \( K \). For \( i \in \{1, \ldots, r\} \), let \( j(i) \in \{1, \ldots, \ell\} \) be the index such that \( w_{j(i)} \) is the \( w \)-coarsening of \( v_i \). Since \( v \) is defectless in \( A \), \( v \) is defectless in \( K \) and each \( v_i \) is defectless in \( A \). Then, \( w \) is defectless in \( K \) by Lemma 1.7, and each \( w_{j(i)} \) is defectless in \( A \) by the central simple case just considered. Since \( w_{j(1)}, \ldots, w_{j(r)} \) are all the extensions of \( w \) to \( K \), it follows by Prop. 1.6 that \( w \) is defectless in \( A \). This completes the proof for \( A \) simple, and the general case for \( A \) semisimple follows easily by considering the simple components of \( A \). \( \square \)

The following result says that defectlessness is a necessary condition for the existence of a \( v \)-gauge on an arbitrary semisimple \( F \)-algebra. We will see in Th. 4.3 below that this necessary condition is also sufficient.

**Proposition 1.9.** Let \( A \) be any semisimple (finite-dimensional) \( F \)-algebra. If \( A \) has a \( v \)-gauge, then \( v \) is defectless in \( A \).

**Proof.** \( A \) has a \( v \)-gauge if and only if each simple component of \( A \) has a \( v \)-gauge, by Prop. 1.3. Also, by definition, \( v \) is defectless in \( A \) if and only if \( v \) is defectless in each simple component of \( A \). Thus, we may assume that \( A \) is simple. Let \( K = Z(A) \), and let \( v_1, \ldots, v_r \) be the extensions of \( v \) to the field \( K \). Let \( (F_h, v_h) \) be a Henselization of \( (F, v) \), and let \( (K_{h,v_i}, v_{i,k}) \) be a Henselization of \( (K, v_i) \). Let \( \alpha \) be a \( v \)-gauge on \( A \). Then the restriction \( \alpha|_K \) is a surmultiplicative \( v \)-norm on \( K \) with \( \text{gr}_{\alpha|_K}(K) \) graded semisimple, since it is a central graded subalgebra of the graded semisimple algebra \( \text{gr}_{\alpha}(A) \). Hence \( \alpha|_K \) is a \( v \)-gauge on \( K \), so \( v \) is defectless in \( K \) by [TW1, Cor. 1.9]. Moreover,
\[ \alpha \otimes v_h \] is a \( v_h \)-gauge on \( A \otimes_F F_h \) and
\[ A \otimes_F F_h \cong (A \otimes_K K_{h,v_1}) \times \cdots \times (A \otimes_K K_{h,v_r}). \]
By Prop. 1.3 each \( A \otimes_K K_{h,v_i} \) carries a gauge, hence the corresponding central division algebra over \( K_{h,v_i} \) is defectless by [TW1, Th. 3.1]. \hfill \square

2. Gauges on simple algebras

Throughout this section, let \( v \) be an arbitrary valuation on a field \( F \), and let \( A \) be a (finite-dimensional) simple \( F \)-algebra. Let \( K \) be the center of \( A \), so \( K \) is a finite-degree field extension of \( F \). Let \( v_1, \ldots, v_r \) be all the extensions of \( v \) to \( K \), and let \( V_i \) be the valuation ring of \( v_i \). Our goal is to characterize \( v \)-gauges on \( A \) in terms of \( v_i \)-gauges for \( i = 1, \ldots, r \). This will be achieved in Th. 2.2 and Th. 2.8.

**Proposition 2.1.** Let \( L \) be a field with \( F \subseteq L \subseteq K \), and suppose \( v \) has a unique extension to a valuation \( v_L \) of \( L \). If \( A \) has a \( v \)-gauge \( \alpha \), then \( \alpha|_L = v_L \), which is a \( v \)-gauge on \( L \), and \( \alpha \) is a \( v_L \)-gauge. Thus, whenever \( v_L \) is a \( v \)-gauge on \( L \), the \( v \)-gauges on \( A \) are the same as the \( v_L \)-gauges on \( A \).

**Proof.** The restriction \( \alpha|_L \) of \( \alpha \) to \( L \) is clearly surmultiplicative, and is a \( v \)-norm on \( L \) by [RTW, Prop. 2.5] since \( L \) is an \( F \)-subspace of \( A \). Moreover, as \( L \subseteq Z(A) \), we have \( gr_{\alpha|_L}(L) \subseteq Z(gr_{\alpha}(A)) \).

Because \( gr_{\alpha}(A) \) is graded semisimple, it contains no nonzero central homogeneous nilpotent elements. Therefore, the commutative \( gr(F) \)-algebra \( gr_{\alpha|_L}(L) \) is semisimple, and hence \( \alpha|_L \) is a \( v \)-gauge on \( L \). Because \( v_L \) is the unique extension of \( v \) to \( L \), [TW1, Cor. 1.9] shows that \( \alpha|_L = v_L \). Hence, for \( c \in L^\times \), we have
\[ \alpha(c^{-1}) = v_L(c^{-1}) = -v_L(c) = -\alpha(c). \]
So, a short computation (cf. [TW1, Lemma 1.3]) shows that \( \alpha(ca) = \alpha(c) + \alpha(a) \) for all \( a \in A \).

This proves that the \( v \)-value function \( \alpha \) on \( A \) is actually a \( v_L \)-value function. Since \( v_L \) and \( \alpha \) are \( v \)-norms, we have
\[ [L:F] [gr_{\alpha}(A):gr_{v_L}(L)] = [gr_{v_L}(L):gr_v(F)] [gr_{\alpha}(A):gr_{v_L}(L)] \]
\[ = [gr_{\alpha}(A):gr_v(F)] = [A:F] = [L:F][A:L]. \]
Hence, \( [gr_{\alpha}(A):gr_{v_L}(L)] = [A:L] \), showing that \( \alpha \) is a \( v_L \)-norm. The other conditions needed for \( \alpha \) to be a \( v_L \)-gauge hold because it is a \( v \)-gauge. Conversely, whenever \( v_L \) is a \( v \)-gauge (hence a \( v \)-norm) and \( \beta \) is a \( v_L \)-gauge on \( A \), then
\[ [gr_{\beta}(A):gr_v(F)] = [gr_{\beta}(A):gr_{v_L}(L)] [gr_{v_L}(L):gr_v(F)] = [A:L][L:F] = [A:F], \]
so \( \beta \) is also a \( v \)-norm and hence a \( v \)-gauge on \( A \). \hfill \square

**Theorem 2.2.** Let \( \alpha \) be a \( v \)-gauge on \( A \). Then there exist \( v_i \)-gauges \( \alpha_i \) on \( A \) for \( i = 1, \ldots, r \) such that
\[ \alpha(a) = \min(\alpha_1(a), \ldots, \alpha_r(a)) \quad \text{for all } a \in A. \]

Furthermore,
\[ gr_{\alpha}(A) \cong gr_{\alpha_1}(A) \times \cdots \times gr_{\alpha_r}(A). \]
Hence, the semisimple $\overline{F}$-algebra $\text{gr}_\alpha(A)_0$ has at least $r$ simple components. Moreover, the $\text{gr}_\alpha(A)$ are the graded simple components of $\text{gr}_\alpha(A)$ and
\[
[\text{gr}_\alpha(A):\text{gr}(F)] = [A:K][\text{gr}_{v_i}(K):\text{gr}(F)].
\]

We call the $\alpha_i$ of the theorem the $v_i$-component of $\alpha$, for $i = 1, \ldots, r$. We will see in Cor. 2.5 below that the $\alpha_i$ are uniquely determined by $\alpha$.

**Proof.** Let $(F_h, v_h)$ be the Henselization of $(F, v)$, and let
\[
B = A \otimes_F F_h \quad \text{and} \quad L = Z(B) = K \otimes_F F_h.
\]
Then, $L$ is a direct product of finitely many fields. Let $e_1, \ldots, e_r$ be the primitive idempotents of $L$, so
\[
L = L_1 \times \ldots \times L_r \quad \text{where} \quad L_i = e_iL,
\]
and each $L_i$ is a field. The $L_i$ are indexed by the $v_i$, as we will explain below. Since $B \cong A \otimes_K L$, the ring $B$ is a product of algebras
\[
B = B_1 \times \ldots \times B_r \quad \text{where} \quad B_i = e_iB \cong A \otimes_K L_i.
\]
So, each $B_i$ is a central simple $L_i$-algebra. We identify $K, A, F_h$ with their isomorphic copies $K \otimes 1, A \otimes 1, 1 \otimes F_h$ in $B$. But, we do not identify them with their isomorphic copies $e_iK, e_iA, e_iF_h$ in $B_i$. For each $i$ we have canonical inclusions
\[
p_i: A \hookrightarrow B_i, \quad a \mapsto e_i(a \otimes 1) \quad \text{and} \quad q_i: F_h \twoheadrightarrow B_i, \quad c \mapsto e_i(1 \otimes c).
\]
Thus, $B_i$ has subalgebras $p_i(A), p_i(K)$, and $L_i$, with
\[
B_i = p_i(A) \otimes_{p_i(K)} L_i \cong A \otimes_K L_i, \quad \text{hence,} \quad [B_i:L_i] = [A:K].
\]
Each field $L_i$ is a compositum of fields, $L_i = p_i(K) \cdot q_i(F_h)$. The Henselian valuation $v_h$ on $F_h$ has an isomorphic (Henselian) valuation $v_h \circ q_i^{-1}$ on $q_i(F_h)$, which extends uniquely to a Henselian valuation $w_i$ on $L_i$. This pulls back to a valuation $w_i \circ p_i$ on $K$ which extends $v$ on $F$. The proof of Th. A.1 in the Appendix shows that the valuations $w_i \circ p_1, \ldots, w_r \circ p_r$ are all distinct and are all the extensions of $v$ to $K$. Thus, after renumbering the $e_i$ if necessary, we can assume $w_i \circ p_i = v_i$ for $i = 1, \ldots, r$. That is,
\[
(2.2) \quad v_i(d) = w_i(e_i(d \otimes 1)) \quad \text{for all} \quad d \in K.
\]
From Th. A.1, we have also that $(L_i, w_i)$ is a Henselization of $(K, v_i)$.

Let $\beta = \alpha \otimes v_h$, which is a $v_h$-gauge on $B$ with
\[
(2.3) \quad \text{gr}_\beta(B) \cong_g \text{gr}_\alpha(A) \otimes_{\text{gr}(F)} \text{gr}(F_h) \cong_g \text{gr}_\alpha(A)
\]
by [TW$_1$, Cor. 1.26]. Let $\beta_i = \beta|_{B_i}$, which is a $v_h$-gauge on $B_i$ via the embedding $q_i: F_h \to B_i$. By Prop. 1.3,
\[
(2.4) \quad \beta(b) = \min_{1 \leq i \leq r} (\beta_i(e_ib)) \quad \text{for all} \quad b \in B,
\]
and
\[
(2.5) \quad \text{gr}_\beta(B) \cong_g \prod_{i=1}^r \text{gr}_{\beta_i}(B_i).
\]
Since $w_i$ is the unique extension of the Henselian valuation $v_h$ to $L_i$, Prop. 2.1 above shows that each $\beta_i$ is a $w_i$-gauge. The structure theorem [TW$_1$, Th. 3.1] for gauges on simple algebras with
respect to Henselian valuations shows that $\beta_i$ is an End-gauge as in Ex. 1.1; hence, $\text{gr}_{\beta_i}(B_i)$ is graded simple.

Define $v$-value functions $\alpha_1, \ldots, \alpha_r$ on $A$ by

$$\alpha_i(a) = \beta(p_i(a)) = \beta_i(e_i(a \otimes 1)).$$

Then, each $\alpha_i$ is surmultiplicative as $\beta_i$ is surmultiplicative, and for all $a \in A$,

$$\alpha(a) = \beta(a \otimes 1) = \min_{1 \leq i \leq r} (\beta_i(e_i(a \otimes 1))) = \min_{1 \leq i \leq r} (\alpha_i(a)).$$

Furthermore, as $\beta_i$ is a $w_i$-value function, for all $c \in K$ and $a \in A$,

$$\alpha_i(ca) = \beta_i(e_i(ca \otimes 1)) = \beta_i(e_i(c \otimes 1) \cdot e_i(a \otimes 1)) = w_i(e_i(c \otimes 1) + \beta_i(e_i(a \otimes 1))) = v_i(c) + \alpha_i(a).$$

Thus, $\alpha_i$ is a $v_i$-value function on $A$. The following diagram shows the algebras related to $B_i$ and the associated value functions being considered here.

(2.7)

Now, for all $\gamma \in \Gamma$, the definition of the $\beta_i$ and (2.4) and (2.6) above show that for each $i$ we have a commutative diagram

$$
\begin{array}{ccc}
A_{\geq \gamma} & \longrightarrow & A_{\geq \gamma} \\
\downarrow & & \downarrow \\
B_{\geq \gamma}^{\beta_i} & \longrightarrow & B_{i, \geq \gamma}^{\beta_i} \\
\downarrow & & \downarrow \\
\alpha_i(a) & \longrightarrow & a \\
\end{array}
$$

given by

$$
\begin{array}{ccc}
a & \longrightarrow & a \\
\downarrow & & \downarrow \\
\alpha_i(a) & \longrightarrow & a(\alpha \otimes 1) \\
\end{array}
$$

There is a corresponding commutative diagram with $> \gamma$ replacing $\geq \gamma$, hence an induced commutative diagram of corresponding factor groups; these together yield a commutative diagram of graded $\text{gr}(F)$-algebra homomorphisms:

$$
\begin{array}{ccc}
\text{gr}_\alpha(A) & \longrightarrow & \prod_{i=1}^{r} \text{gr}_{\alpha_i}(A) \\
\downarrow & & \downarrow \\
\text{gr}_\beta(B) & \longrightarrow & \prod_{i=1}^{r} \text{gr}_{\beta_i}(B_i) \\
\end{array}
$$

Here, the top map is injective by (2.6); the left map is the isomorphism of (2.3); the right map is injective since for each $i$, the definition of $\alpha_i$ shows that $\text{gr}_{\alpha_i}(A) \to \text{gr}_{\beta_i}(B)$ is injective; and the bottom map is the isomorphism of (2.5). Therefore, all the maps in this diagram must be isomorphisms. Hence, for each $i$, $\text{gr}_{\alpha_i}(A) \cong_{s} \text{gr}_{\beta_i}(B_i)$, which is graded simple as we saw above after (2.5). Since the Henselization $(L_i, w_i)$ (resp. $(F_h, v_h)$) is an immediate extension of $(K, v_i)$ (resp. $(F, v)$), we have

$$[L_i:F_h] \geq [\text{gr}_{w_i}(L_i):\text{gr}_{v_h}(F_h)] = [\text{gr}_{v_i}(K):\text{gr}(F)].$$
So, as $\beta_i$ is a $v_i$-norm,
\[
[\text{gr}_{\alpha_i}(A)]:\text{gr}_{v_i}(K)] [\text{gr}_{\alpha_i}(K)]:\text{gr}(F) = [\text{gr}_{\alpha_i}(A)]:\text{gr}(F)
\]
(2.9)
\[
= [\text{gr}_{\beta_i}(B_i)]:\text{gr}_{v_i}(F_h)] = [B_i:F_h]
\]
\[
= [B_i:L_i] [L_i:F_h] = [A:K] [L_i:F_h]
\]
\[
\geq [A:K] [\text{gr}_{v_i}(K)]:\text{gr}(F)],
\]
and hence $[\text{gr}_{\alpha_i}(A)]:\text{gr}_{v_i}(K)] \geq [A:K]$. Since the reverse inequality holds for any $v_i$-value function, we have $[\text{gr}_{\alpha_i}(A)]:\text{gr}_{v_i}(K)] = [A:K]$. Hence, $\alpha_i$ is a $v_i$-norm on $A$; with the graded simplicity noted above, this yields that $\alpha_i$ is a $v_i$-gauge. Furthermore, equality holds in (2.9), yielding $[\text{gr}_{\alpha_i}(A)]:\text{gr}(F)] = [A:K] [\text{gr}_{v_i}(K)]:\text{gr}(F)]$. The isomorphism for $\text{gr}_{\alpha_i}(A)$ in the theorem is the top isomorphism in the commutative diagram (2.8). Since each $\text{gr}_{\alpha_i}(A)$ is graded simple, the $\text{gr}_{\alpha_i}(A)$ are the graded simple components of $\text{gr}_{\alpha_i}(A)$. Because $\text{gr}_{\alpha_i}(A)$ has $r$ graded simple components, its degree zero part must have at least $r$ simple components. \hfill $\square$

Let $v$ be a valuation on some division algebra $D$, and let $\alpha, \beta, \eta_1, \ldots, \eta_r$ be $v$-value functions on a finite-dimensional $D$-vector space $M$. We write $\alpha \leq \beta$ if $\alpha(z) \leq \beta(z)$ for all $z \in M$. Likewise, we write $\alpha = \min (\eta_1, \ldots, \eta_r)$ if $\alpha(z) = \min (\eta_1(z), \ldots, \eta_r(z))$ for all $z \in M$. It is easy to construct examples of $v$-norms $\alpha, \beta$ on $M$ with $\alpha \leq \beta$ and $\alpha \neq \beta$. By contrast, we will see in Cor. 2.6 below for gauges on semisimple algebras that $\alpha \leq \beta$ implies $\alpha = \beta$. This will be proved by showing a minimality property characterizing the components of a gauge on a simple algebra.

**Lemma 2.3.** Let $A$ and $B$ be $F$-algebras with respective surmultiplicative $v$-value functions $\alpha$ and $\beta$. Suppose there is an $F$-algebra homomorphism $f: A \rightarrow B$ such that
\[
\beta(f(a)) \geq \alpha(a) \quad \text{for all } a \in A.
\]
Then $f$ induces a graded $\text{gr}(F)$-algebra homomorphism
\[
\hat{f}: \text{gr}_{\alpha}(A) \rightarrow \text{gr}_{\beta}(B)
\]
such that $\hat{f}(a) = f(x) + B_{>\alpha(x)} \in B_{\alpha(x)}$ for all $x \in A$. Moreover, $\hat{f}$ is injective if and only if equality holds in (2.10).

**Proof.** For any $\delta \in \Gamma$, we have $f(A_{>\delta}) \subseteq B_{>\delta}$ and $f(A_{>\delta}) \subseteq B_{>\delta}$, hence $f$ induces a map $(\hat{f})_{\delta}: A_{\delta} \rightarrow B_{\delta}$ given by $(\hat{f})_{\delta}(x + A_{>\delta}) = f(x) + B_{>\delta}$. Then set $\hat{f} = \bigoplus_{\delta \in \Gamma} (\hat{f})_{\delta}: \text{gr}_{\alpha}(A) \rightarrow \text{gr}_{\beta}(B)$. Clearly, $\hat{f}$ is a graded $\text{gr}(F)$-vector space homomorphism. To see that $\hat{f}$ is multiplicative, it suffices to check this for homogeneous elements. That is, for any nonzero $x, y \in A$ we need
\[
\hat{f}(x \ y) = \hat{f}(x) \hat{f}(y).
\]
(2.11)
When the left expression in (2.11) is nonzero, it equals $\hat{f}(xy)$, and when the right expression is nonzero it equals $\hat{f}(x)\hat{f}(y) = f(x)f(y) = \hat{f}(xy)$. So, equality indeed holds in (2.11) when each side is nonzero. Now we have
\[
\beta(f(xy)) \geq \alpha(xy) \geq \alpha(x) + \alpha(y)
\]
and
\[
\beta(f(xy)) = \beta(f(x)f(y)) \geq \beta(f(x)) + \beta(f(y)) \geq \alpha(x) + \alpha(y).
\]
The left expression in (2.11) is nonzero if and only if $\alpha(xy) = \alpha(x) + \alpha(y)$ (so $\hat{x} \ y = \hat{x} \ y \neq 0$) and $\beta(f(xy)) = \alpha(xy)$, i.e., equality holds throughout (2.12). The right expression in (2.11) is nonzero if
and only if $\beta(f(x)) = \alpha(x)$ and $\beta(f(y)) = \alpha(y)$, and $\beta(f(x)f(y)) = \beta(f(x)) + \beta(f(y))$, i.e., equality holds throughout (2.13). Each of these conditions holds if and only if $\beta(f(xy)) = \alpha(x) + \alpha(y)$. Thus, we have equality in (2.11) in all cases. Now note that $\hat{f}(\bar{x}) = 0$ if and only if $\beta(f(x)) > \alpha(x)$. Since $\ker(\hat{f})$ is a homogeneous two-sided ideal of $\gr_{\alpha}(A)$, the stated condition for injectivity of $\hat{f}$ holds.

**Theorem 2.4.** Let $\alpha$ be any $v$-gauge on the simple $F$-algebra $A$, and, as in Th. 2.2 above, let $\alpha_i$ be the $v_i$-component of $\alpha$ for $i = 1, \ldots, r$. Suppose $\eta$ is a $v_k$-gauge on $A$, for some $k$. If $\alpha \leq \eta$, then $\eta = \alpha_k$.

**Proof.** Pick a homogeneous base $(b_1, \ldots, b_n)$ of $\gr_{\alpha_k}(A)$ as a graded $\gr_{v_k}(K)$-vector space, and let $\gamma_j = \deg(b_j)$. Then, pick $a_1, \ldots, a_n \in A$ with each $\hat{a}_j^\alpha \mapsto (0, \ldots, 0, b_j, 0, \ldots, 0)$ ($b_j$ in the $k$-th position) under the isomorphism $\gr_{\alpha}(A) \cong y\prod_{i=1}^{\alpha} \gr_{\alpha_i}(A)$ of Th. 2.2. This means that for all $j = 1, \ldots, n$ and $i = 1, \ldots, r$,

$$\alpha_i(a_j) > \gamma_j \quad \text{for} \quad i \neq k, \quad \alpha(a_j) = \alpha_k(a_j) = \gamma_j, \quad \text{and} \quad \hat{a}_j^\alpha = b_j \text{ in } \gr_{\alpha_k}(A).$$

Since $(\hat{a}_1^\alpha, \ldots, \hat{a}_n^\alpha)$ is a homogeneous base of the graded $\gr_{\alpha_k}(K)$-vector space $\gr_{\alpha_k}(A), (a_1, \ldots, a_n)$ is a splitting base of the $K$-vector space $A$ for the $v_k$-gauge $\alpha_k$ (see the comments preceding (1.3) above). Furthermore, $\alpha_k(a_j) = \alpha(a_j) \leq \eta(a_j)$, for all $j$. Therefore, for any $a \in A$, writing $a = \sum_{j=1}^{n} a_j c_j$ with $c_j \in K$, we have

$$\alpha_k(a) = \min_{1 \leq j \leq n} (\alpha_k(a_j) + v_k(c_j)) \leq \min_{1 \leq j \leq n} (\eta(a_j) + v_k(c_j)) \leq \eta(\sum_{j=1}^{n} a_j c_j) = \eta(a).$$

Thus, $\alpha_k \leq \eta$ as $v_k$-gauges on $A$.

Since $\alpha_k \leq \eta$, Lemma 2.3 (with $f = id_A$) shows that there is a well-defined graded $\gr_{v_k}(K)$-algebra homomorphism $\varphi: \gr_{\alpha_k}(A) \rightarrow \gr_{\eta}(A)$ given by $\hat{a}_i^\alpha \mapsto a + A_r^\eta(\alpha_k(a))$ for all $a \in A$.

But, as $\alpha_k$ is a $v_k$-gauge on the central simple $K$-algebra $A$, [TW1, Cor. 3.7] shows that $\gr_{\alpha_k}(A)$ is a simple graded algebra. Hence $\varphi$ must be injective. Again by Lemma 2.3 we have $\alpha_k = \eta$.

**Corollary 2.5.** Let $\alpha$ be any $v$-gauge on the simple $F$-algebra $A$, and, as in Th. 2.2 above, let $\alpha_i$ be the $v_i$-component of $\alpha$ for $i = 1, \ldots, r$. Suppose $\alpha = \min(\eta_1, \ldots, \eta_r)$ for some $v_i$-gauges $\eta_i$. Then each $\eta_i = \alpha_i$.

**Proof.** For each $i$, we have $\alpha \leq \eta_i$. Hence, $\eta_i = \alpha_i$ by the preceding theorem.

**Corollary 2.6.** Let $\alpha$ and $\eta$ be $v$-gauges on a semisimple $F$-algebra $C$. If $\alpha \leq \eta$, then $\alpha = \eta$.

**Proof.** It suffices to check this for the restrictions of $\alpha$ and $\eta$ on the simple components of $C$. Therefore, we may assume that $C$ is a simple $F$-algebra. Then, let $v_1, \ldots, v_r$ be all the extensions of $v$ to $Z(C)$, and let $\alpha_i$ (resp. $\eta_i$) be the $v_i$-component of $\alpha$ (resp. $\eta$). Since, for each $i$ we have $\alpha \leq \eta \leq \eta_i$, Th. 2.4 shows that $\alpha_i = \eta_i$. Hence,

$$\alpha = \min(\alpha_1, \ldots, \alpha_r) = \min(\eta_1, \ldots, \eta_r) = \eta.$$

**Corollary 2.7.** Let $\alpha$ be a $v$-gauge on the simple $F$-algebra $A$, with $v_i$-components $\alpha_i$, for $i = 1, \ldots, r$, and suppose $\Gamma_\alpha$ lies in the divisible hull of $\Gamma_v$. Let $w$ be any valuation on $F$ which is a coarsening of $v$, and let $w_1, \ldots, w_\ell$ be all the extensions of $w$ to $K$. Let $\beta$ be the $w$-coarsening of $\alpha$, and let $\beta_j$
be the $w_j$-component of $\beta$ for $j = 1, \ldots, \ell$. For $i \in \{1, \ldots, r\}$, let $j(i) \in \{1, \ldots, \ell\}$ be the index such that $w_{j(i)}$ is the $w$-coarsening of $v_i$ (i.e., $W_{j(i)} = W \cdot V_i$). Then, $\beta_{j(i)}$ is the $w_{j(i)}$-coarsening of $\alpha_i$.

**Proof.** Let $\alpha_{i,w}$ denote the $w_{j(i)}$-coarsening of $\alpha_i$. Let $\Delta$ be the convex subgroup of $\Gamma$ associated to $w$. Recall that $\beta = \varepsilon \circ \alpha$, where $\varepsilon : \Gamma \to \Gamma/\Delta$ is the canonical surjection. This $\varepsilon$ is compatible with the orderings on $\Gamma$ and $\Gamma/\Delta$. Likewise, $w_{j(i)} = \varepsilon \circ v_i$ and $\alpha_{i,w} = \varepsilon \circ \alpha_i$. Now fix any $i$. Since $\alpha \leq \alpha_i$, $\beta = \varepsilon \circ \alpha \leq \varepsilon \circ \alpha_i = \alpha_{i,w}$, i.e., $\beta \leq \alpha_{i,w}$. Since $\alpha_{i,w}$ is a $w_{j(i)}$-gauge, Th. 2.4 shows that $\alpha_{i,w} = \beta_{j(i)}$. \hfill \square

We next determine when given $v_i$-gauges $\eta_1, \ldots, \eta_r$ yield a $v$-gauge as $\min(\eta_1, \ldots, \eta_r)$. For $i, j \in \{1, \ldots, r\}$ let $v_{ij}$ denote the finest common coarsening of $v_i$ and $v_j$. That is, $v_{ij}$ is the valuation on $K$ associated valuation ring $V_{ij} = V_i \cdot V_j$.

**Theorem 2.8.** Suppose $v$ is defectless in the simple $F$-algebra $A$. For $i = 1, \ldots, r$, let $\eta_i$ be a $v_i$-gauge on $A$ with $\Gamma_{\eta_i} \subseteq \mathbb{H}(\Gamma_v)$. Let $\alpha = \min(\eta_1, \ldots, \eta_r)$. Then, $\alpha$ is a $v$-gauge on $A$ if and only if $\eta_i$ and $\eta_j$ have the same $v_{ij}$-coarsening for all pairs $i, j$. When this occurs, each $\eta_i$ is the $v_i$-component of $\alpha$.

**Proof.** Suppose our $\alpha$ is a $v$-gauge on $A$. By Cor. 2.5, each $\eta_i$ is the $v_i$-component of $\alpha$. Fix indices $i, j$, let $w = v_{ij}|_F$, and let $\beta$ be the $w$-coarsening of $\alpha$. The $v_{ij}$-coarsenings of $\eta_i$ and $\eta_j$ must be the same, since by Cor. 2.7 they coincide with the $v_{ij}$-component of $\beta$.

Suppose each $\eta_i$ and $\eta_j$ have the same $v_{ij}$-coarsening. Now, $\alpha = \min(\eta_1, \ldots, \eta_r)$ is clearly a surmultiplicative $v$-value function on $A$. Consider the graded $\text{gr}(F)$-algebra homomorphism

$$\Psi: \text{gr}_\alpha(A) \longrightarrow \prod_{i=1}^r \text{gr}_{\eta_i}(A) \quad \text{given by} \quad a + A_{>\alpha(a)}^\alpha \mapsto (a + A_{>\alpha(a)}^{\eta_1}, \ldots, a + A_{>\alpha(a)}^{\eta_r}).$$

Then, $\Psi$ is well-defined and injective because $\alpha = \min(\eta_1, \ldots, \eta_r)$. We will show below that $\Psi$ is an isomorphism. It then follows that $\text{gr}_\alpha(A)$ is semisimple, as each $\text{gr}_{\eta_i}(A)$ is semisimple. Moreover, $v$ is defectless in $K$ by Prop. 1.6 since it is defectless in $A$, whence

$$[\text{gr}_\alpha(A) : \text{gr}(F)] = \sum_{i=1}^r [\text{gr}_{\eta_i}(A) : \text{gr}(F)] = \sum_{i=1}^r [\text{gr}_{\eta_i}(A) : \text{gr}_{\eta_i}(K)] [\text{gr}_{\eta_i}(K) : \text{gr}(F)]$$

$$= \sum_{i=1}^r [A : K] [\text{gr}_{\eta_i}(K) : \text{gr}(F)] = [A : K] [K : F] = [A : F].$$

Hence, $\alpha$ is a $v$-gauge.

To prove surjectivity of $\Psi$, we use the following approximation lemma, for which we use the notation: let $\Delta_{ij}$ be the convex subgroup of the divisible hull $\Gamma$ of $\Gamma_v$ associated to $v_{ij}$; so $v_{ij}$ is a map $K \to \Gamma/\Delta_{ij} \cup \{\infty\}$. For any $\delta_1, \ldots, \delta_n \in \Gamma$ we say that the $n$-tuple $(\delta_1, \ldots, \delta_n)$ is compatible in $\Gamma^n$ if for all $i, j$ we have $\delta_i - \delta_j \in \Delta_{ij}$.

**Lemma 2.9.** With the notation just above, fix some $k \in \{1, \ldots, n\}$, and let $(\delta_1, \ldots, \delta_n)$ be a compatible $n$-tuple in $\Gamma^n$ with $\delta_k = 0$. Then, there is $c \in K^\times$ with $v_i(c) > \delta_i$ for each $i \neq k$, $v_k(c) = 0$, and $v_k(c - 1) > 0$.

**Proof.** For each pair of indices $i, j$, we have

$$0 \leq |\delta_i | - |\delta_j | \leq |\delta_i - \delta_j |.$$
Since $\delta_i - \delta_j \in \Delta_{ij}$ and $\Delta_{ij}$ is convex, it follows that $|\delta_i| - |\delta_j| \in \Delta_{ij}$. Furthermore, as $\Gamma/G_v$ is a torsion group, there is $m \in \mathbb{N}$ with each $m|\delta_i| \in \Gamma_v$. Hence, each $m|\delta_i| \in \Gamma_v$ and $m|\delta_i| - m|\delta_j| \in \Delta_{ij}$. These are the precise conditions needed for $(m|\delta_1|, \ldots, m|\delta_n|)$ to be compatible in $\Gamma_{v_1} \times \ldots \times \Gamma_{v_r}$ in the terminology of Ribenboim’s paper [R]. Since this compatibility holds, the general approximation theorem for incomparable valuations on $K$ [R, Th. 5] says that there exists $d \in K^*$ with

$$v_i(d) = m|\delta_i|, \quad \text{for all } i.$$ 

In particular, $v_k(d) = m|\delta_k| = 0$. Now, let $V_i$ be the valuation ring of $v_i$, and let $T = V_1 \cap \ldots \cap V_r \subseteq K$. A weaker approximation theorem for incomparable valuations on $K$ (see [EP, Th. 3.2.7(3), p. 64]) says that the canonical map $\rho: T \to \mathbf{K}^{|V_1|} \times \ldots \times \mathbf{K}^{|V_r|}$ is surjective. Therefore, there is $t \in T$ with $\rho(t) = (\overline{t}, \overline{0}, \ldots, \overline{d}^{-1}, \ldots, \overline{0})$, i.e., $v_i(t) > 0$ for $i \neq k$ and $v_k(t) = 0$ with $\overline{t} = \overline{d}^{-1} \in \mathbf{K}^{v_k}$. Let $c = td$. Then for $i \neq k$ we have

$$v_i(c) = v_i(t) + v_i(d) > 0 + m|\delta_i| \geq \delta_i,$$

hence $v_i(c) > \delta_i$. Also, $v_k(c) = v_k(t) + v_k(d) = 0$, and in $\mathbf{K}^{v_k}$,

$$\overline{c} = \overline{t} \cdot \overline{d} = \overline{d}^{-1} \overline{d} = \overline{1}.$$

Thus, $c$ has all the required properties.

Proof of Th. 2.8 completed: It remains only to prove the surjectivity of the map $\Psi$ in (2.14). Fix any $k \in \{1, \ldots, r\}$, and take any $b \in A \setminus \{0\}$. Let

$$\delta_i = \eta_k(b) - \eta_i(b) \quad \text{for } i = 1, \ldots, r.$$

For each pair of indices $i, j$, let $\Delta_{ij}$ be the convex subgroup of $\mathbb{H}(\Gamma_F)$ associated to the finest common coarsening $v_{ij}$ of $v_i$ and $v_j$ on $K$. Then, since $\eta_i$ and $\eta_j$ are assumed to have the same $v_{ij}$-coarsening, $\eta_i(b)$ and $\eta_j(b)$ have the same image in $\Gamma/\Delta_{ij}$; hence,

$$\delta_i - \delta_j = \eta_j(b) - \eta_i(b) \in \Delta_{ij}.$$

Thus, $(\delta_1, \ldots, \delta_r)$ is a compatible $r$-tuple in $\mathbb{H}(\Gamma_F)^r$. Since $\delta_k = 0$, Lemma 2.9 yields $c \in K^*$ with $v_i(c) > \delta_i$ for all $i \neq k$ and $v_k(c) = 0$ with $\overline{c} = \overline{1}$ in $\mathbf{K}^{v_k}$. Let $a = cb \in A$. Then, for $i \neq k$,

$$\eta_i(a) = v_i(c) + \eta_i(b) > \delta_i + \eta_i(b) = \eta_k(b),$$

so $\eta_i(a) > \eta_k(b)$. But

$$\eta_k(a) = v_k(c) + \eta_k(b) = \eta_k(b).$$

Hence, $\alpha(a) = \min(\eta_1(a), \ldots, \eta_r(a)) = \eta_k(b)$. Moreover,

$$\eta_k(a - b) = \eta_k(cb - b) = v_k(c - 1) + \eta_k(b) > \eta_k(b).$$

Thus, for $\tilde{a} = a + A^\alpha_{> \alpha(a)} \subseteq \mathfrak{gr}_\alpha(A)$ and $\tilde{b} = b + A^\eta_{> \eta_k(b)} \subseteq \mathfrak{gr}_{\eta_k}(A)$, we have

$$\Psi(\tilde{a}) = (a + A^m_{> \alpha(a)}, \ldots, a + A^{nr}_{> \alpha(a)}) = (0, \ldots, 0, \tilde{b}, 0, \ldots, 0) \in \prod_{i=1}^r \mathfrak{gr}_{\eta_i}(A).$$

Since for arbitrary $k$ and $b$ such elements generate $\prod_{i=1}^r \mathfrak{gr}_{\eta_i}(A)$, the map $\Psi$ is surjective. This completes the proof. \[ \square \]

**Corollary 2.10.** Suppose $v$ is defectless in $A$, and suppose the extensions $v_1, \ldots, v_r$ of $v$ to $K$ are pairwise independent. Take any $\eta_i$-gauges $\eta_i$ on $A$, $i = 1, \ldots, r$. Then $\min(\eta_1, \ldots, \eta_r)$ is a $v$-gauge on $A$ with components $\eta_1, \ldots, \eta_r$. In particular, this holds whenever $v$ has rank 1.

\[ \square \]
Proof. This is immediate from the preceding theorem. For when \( v_i \) and \( v_j \) are independent valuations, their finest common coarsening \( v_{ij} \) is the trivial valuation, so the compatibility condition on \( \eta_i \) and \( \eta_j \) holds automatically. \( \square \)

3. Dubrovin valuation rings and Gräter rings

In this section we study the connection between gauges and Dubrovin valuation rings. The best general reference for Dubrovin theory is the book [MMU]. Let \( F \) be a field and let \( A \) be a central simple \( F \)-algebra. A subring \( B \) of \( A \) is called a Dubrovin valuation ring of \( A \) if there is an ideal \( J \) of \( B \) such that the following hold:

1. \( B/J \) is simple Artinian;
2. for each \( s \in A \setminus B \) there exist \( b, c \in B \) such that \( bs, sc \in B \setminus J \).

By [MMU, Lemma 5.2, p. 22], the ideal \( J \) is the only maximal ideal of \( B \); therefore, \( J = J(B) \), the Jacobson ideal of \( B \). Moreover, the center \( Z(B) = F \cap B \) is a valuation ring of \( F \) and \( J \cap F = J(Z(B)) \) (see [MMU, Lemma 7.1, p. 35]). We review a few of the nontrivial properties of Dubrovin valuation rings that we will use repeatedly below. Proofs of all of them can be found in Chapters 5 and 6 of [MMU]. Let \( S \) be an overring of \( B \) in \( A \), i.e., a ring with \( B \subseteq S \subseteq A \). Then, \( S \) is a Dubrovin valuation ring of \( A \); \( J(S) \) is a prime ideal of \( B \); \( S \) is the left (and right) localization of \( B \) with respect to the elements of \( B \) regular mod \( J(S) \); and \( B/J(S) \) is a Dubrovin valuation ring of \( S/J(S) \). Also, \( S \) is the central localization \( S = B \otimes_{Z(B)} W = B \cdot W \), where the valuation ring \( W = Z(S) \) is the localization of \( Z(B) \) at its prime ideal \( J(S) \cap F \). Moreover, every prime ideal of \( B \) has the form \( J(S) \) for some such \( S \). If \( R \) is a subring of \( B \) with \( J(B) \subseteq R \), then \( R \) is a Dubrovin valuation ring of \( A \) if and only if \( R/J(B) \) is a Dubrovin valuation ring of \( B/J(B) \). Examples of Dubrovin valuation rings include

- total valuation ring in division algebras \( D \), i.e., subrings \( T \) of \( D \) such that \( d \) or \( d^{-1} \) lies in \( T \) for every \( d \in D^{\times} \);
- matrix rings \( M_n(B) \) for any Dubrovin valuation ring \( B \);
- rings \( eBe \) for any Dubrovin ring \( B \) of \( A \) and any nonzero idempotent \( e \) of \( A \);
- Azumaya algebras over commutative valuation rings.

Associated to a Dubrovin valuation ring \( B \) there is its residue ring \( \overline{B} = B/J(B) \), which is simple Artinian. This \( B \) also has a value group \( \Gamma_B = st(B)/B^{\times} \), where \( st(B) = \{ x \in A^{\times} \mid xBx^{-1} = B \} \); the value group \( \Gamma_B \) is a totally ordered abelian group. It was proved in [W1] that there is a strong connection between Dubrovin valuation rings and invariant valuation rings (i.e., the valuation rings associated to a valuation on a division algebra), which shows up with passage to the Henselization of the valuation of the center: Let \( (F_h, v_h) \) be a Henselization of \( (F, v) \). The ring \( A \otimes_F F_h \) is a central simple \( F_h \)-algebra; hence,

\[
A \otimes_F F_h \cong M_{n_B}(D_h),
\]

where \( n_B \) is a positive integer and \( D_h \) is a central division \( F_h \)-algebra. Thus, \( n_B \) is the matrix size of the algebra \( A \otimes_F F_h \). The Henselian valuation \( v_h \) has a unique extension to a valuation \( w \) on \( D_h \). By [W1, Th. B] or [MMU, Lemma 11.4, p. 59],

\[
\Gamma_B = \Gamma_w \quad \text{in} \quad \mathbb{H}(\Gamma_v)
\]
and
\[(3.3) \quad \overline{B} \cong M_{t_B}(D_n^w),\]
where the positive integer $t_B$ is the matrix size of $\overline{B}$. Moreover, $n_B/t_B$ is always an integer that appears in the Ostrowski Theorem for Dubrovin valuation rings, which says that
\[(3.4) \quad [A : F] = [\overline{B} : F^w] \cdot [\Gamma_B : \Gamma_v] \cdot (n_B/t_B)^2 \cdot \partial_B,
\]
where the defect $\partial_B = \overline{p}^d$, for $\overline{p} = \text{char}(F^w)$ and $d$ a non-negative integer, with $\overline{p}^d = 1$ if $\overline{p} = 0$ (see [W1, Th. C]). It follows from (1.20) applied to $D_h$ and (3.4) that $\partial_B = \partial_{D_h/F_h}$. Hence, $v$ is defectless in $A$ if and only if $\partial_B = 1$ for any Dubrovin valuation ring $B$ of $A$ extending $V$.

Dubrovin valuation rings have good properties with respect to extension from the center. More precisely, if $V$ is a valuation ring of $F$ then there always exists a Dubrovin valuation ring $B$ of the central simple $F$-algebra $A$ extending $V$, i.e., $V = B \cap F$, see [MMU, Th. 9.4, p. 50]. Moreover, by [W1, Th. A] or [MMU, Th. 9.8, p. 52] there is as much uniqueness as possible, i.e., all Dubrovin ring extensions of $V$ to $A$ are conjugate in $A$. But the number of extensions of $V$ to Dubrovin valuation rings of $A$ is usually infinite; the exception occurs only when $A$ is a division algebra and $V$ can be extended to a total valuation ring $T$ of $A$. In this special case, the number of total valuation rings of $A$ extending $V$ is given by $n_B/t_B$. In order to obtain a better understanding of the Ostrowski Theorem (equivalently, a better interpretation of the integer $n_B/t_B$ for arbitrary Dubrovin valuation rings), Gräter introduced in [G1] the Intersection Property for a finite number of Dubrovin valuation rings as follows: Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of $A$ and let $R = B_1 \cap \ldots \cap B_n$. Let $\mathcal{B}(B_i)$ denote the set of all overrings of $B_i$ in $A$. Then $B_1, \ldots, B_n$ have the Intersection Property (IP) if the map
\[(3.5) \quad \varphi: \mathcal{B}(B_1) \cup \ldots \cup \mathcal{B}(B_n) \to \text{Spec}(R)
\]
\[S \quad \mapsto \quad J(S) \cap R
\]
is a well-defined order-reversing bijection, where Spec$(R)$ is the set of prime ideals of $R$. Actually, if one supposes only that $\varphi$ is well-defined (i.e., each ideal $J(S) \cap R$ is a prime ideal of $R$), then in fact $\varphi$ is an order-reversing bijection (see [Z]). Note that if $B_j \supseteq B_i$ for some $i, j$, then we may delete $B_j$ from the list of $B$’s and the ring $R$ and the domain and target of $\varphi$ are unchanged. Thus, in working with the IP, we may delete all such redundant $B_j$ and assume that the $B_i$ are pairwise incomparable.

At the same time that Intersection Property was introduced by Gräter, Morandi was working independently on a general approximation theorem for a finite set of Dubrovin valuation rings (see [M3]). He found the following condition: Let $B_1, \ldots, B_n$ be pairwise incomparable Dubrovin valuation rings. For each $i \neq j$, let $B_{ij}$ be the subring of $A$ generated by $B_i$ and $B_j$; so $B_{ij}$ is a Dubrovin valuation ring, since it is an overring of $B_i$. Set $\overline{B}_i = B_i/J(B_{ij})$ and $\widetilde{B}_j = B_j/J(B_{ij})$, which are Dubrovin valuation rings of $\overline{B}_{ij} = B_{ij}/J(B_{ij})$. The condition needed for the approximation theorem to hold for $B_1, \ldots, B_n$ is that the valuation rings $Z(\overline{B}_i)$ and $Z(\widetilde{B}_j)$ be independent in the field $Z(\overline{B}_{ij})$, i.e., $Z(\overline{B}_i) \cdot Z(\widetilde{B}_j) = Z(\overline{B}_{ij})$. Gräter proved that this condition is equivalent to the Intersection Property for $B_1, \ldots, B_n$ (see [G1, Cor. 6.2, Prop. 6.3, Cor. 6.7] or [MMU, Cor. 16.9, p. 93]). It follows that
\[(3.6) \quad B_1, \ldots, B_n \text{ satisfy the IP if and only if each pair } B_i, B_j \text{ satisfies the IP,}
\]
for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. 

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We call a subring \( R \) of a central simple \( F \)-algebra \( A \) a Grätter ring if \( R = B_1 \cap \ldots \cap B_n \) where \( B_1, \ldots, B_n \) is a family of incomparable Dubrovin valuation rings of \( A \) with the IP and \( R \) is integral over \( Z(R) \). This name is appropriate because of the remarkable properties Grätter proved about such rings in \([G_1]\) and \([G_2]\). (His results appear also in \([MU]\).) He showed that a Grätter ring is a semilocal Bezout ring whose center is a finite intersection of valuation rings of \( F \). When \( R = B_1 \cap \ldots \cap B_n \) as above, then the localizations of the \( R \) with respect to its maximal ideals exist and coincide with the \( B_i \). Thus in particular,

\[
\text{(3.7) \quad \text{the number } n \text{ of } B_i = \text{the number of maximal ideals of } R.}
\]

Moreover, if \( T \) is any finite intersection of valuation rings of \( F \) then there exists a Grätter ring \( R \) of \( A \) with \( Z(R) = T \). Further, Grätter proved a very strong uniqueness property: If \( R' \) is another Grätter ring of \( A \) with \( Z(R') = T = Z(R) \), then there is \( q \in A^\times \) with \( R' = qRq^{-1} \). See \([MU, \text{Th. 16.14, p. 94; Th. 16.15, p. 96}] \) for proofs of these properties.

There are also striking properties when the center of a Grätter ring is a valuation ring \( V \): Let \( R = B_1 \cap \ldots \cap B_n \) be a Grätter ring with center \( V \) (and the \( B_i \) incomparable); then, each \( B_i \) has center \( V \), by \([MU, \text{Lemma 16.13, pp. 93–94}] \). Moreover, if \( B'_1, \ldots, B'_k \) is another family of Dubrovin valuation rings of \( A \) each with center \( V \) and having the IP, then the \( B'_i \) are incomparable (since overrings of \( B'_i \) are central localizations) and this family can be enlarged with further Dubrovin valuation rings \( B'_{k+1}, \ldots, B'_m \) each with center \( V \) so that \( R' = B'_1 \cap \ldots \cap B'_m \) is a Grätter ring with center \( V \) (see \([MU, \text{Th. 16.14, p. 94}]\)). The conjugacy result noted above shows that \( R' \cong R \), hence \( m = n \) by (3.7). Thus, this number \( n \) depends only on \( V \) and \( A \), and Grätter defined it to be the extension number of \( V \) in \( A \). Thus, the extension number is the number of maximal ideals in any Grätter ring of \( A \) with center \( V \), and it equals the number of incomparable Dubrovin valuation rings in any family whose intersection is a Grätter ring with center \( V \) (see (3.7)); it is also the maximum number of Dubrovin rings with center \( V \) which have the IP. It turns out that the extension number also equals the quotient \( n_B/t_B \) of the integers \( n_B \) and \( t_B \) given in (3.1) and (3.3), for any Dubrovin valuation ring \( B \) of \( A \), see \([MU, \text{Prop. 19.2, p. 108}] \). From the conjugacy of such rings \( B \), it is clear that \( n_B \) and \( t_B \) do not depend on the choice of \( B \). Moreover, it follows from the definition of \( n_B \) and \( t_B \) that the extension number depends only on the valuation ring \( V \) of the center \( F \) of \( A \) and the Brauer equivalence class of \( A \). Let

\[
\text{(3.8) \quad \xi_{V,[A]} = \text{the extension number of } V \text{ to } A = n_B/t_B.}
\]

When the algebra \( A \) in question is clear, we write \( \xi_V \) for \( \xi_{V,[A]} \).

Now let \( W \) be any (valuation) ring with \( V \subseteq W \subseteq F \), and let \( S = W \cdot B \). Then \( S \) is a Dubrovin valuation ring of \( A \) containing \( B \) and \( W = S \cap F \). Let \( \tilde{V} = V/J(W) \), which is a valuation ring of \( \overline{W} = W/J(W) \), and let \( \tilde{B} = B/J(S) \), which is a Dubrovin valuation ring of the simple \( \overline{W} \)-algebra \( \overline{S} = S/J(S) \). Let

\[
\text{(3.9) \quad \ell_{V,W} = \text{the number of extensions of } \tilde{V} \text{ to valuation rings of } Z(\overline{S}).}
\]

This number does not depend on the choice of the Dubrovin extensions \( B \) of \( V \) and \( S \) of \( W \), by the conjugacy of \( B \) with center \( V \) and the property that overrings of \( B \) are obtained by central localization. The following fundamental relation was given in \([W_1, \text{Th. E}]\):

\[
\text{(3.10) \quad \xi_{V,[A]} = \xi_{W,[A]} \left( n_B/t_B \right) \ell_{V,W} = \xi_{W,[A]} \xi_{Z(\overline{B}),[\overline{S}]} \ell_{V,W}.}
\]
Hence,
\begin{equation}
(3.11) \quad \xi_{W[A]} \mid \xi_{V[A]} \quad \text{for any valuation ring } W \text{ with } V \subseteq W \subseteq F.
\end{equation}

There is another interpretation of }_{L,W} \text{ that will be needed later: Let } w \text{ be the valuation on } F \text{ associated to } W, \text{ let } (F_{h,w}, w_h) \text{ be the Henselization of } (F, w), \text{ let } D_h \text{ be the associated division algebra of } A \otimes_F F_{h,w}, \text{ and let } D_h^{w'} \text{ be the residue division algebra of } D_h \text{ for the valuation } w' \text{ on } D_h \text{ extending } w_h \text{ on } F_{h,w}. \text{ Let } u \text{ be the valuation of } \overline{V} \text{ on } F^w. \text{ Then,}
\begin{equation}
(3.12) \quad \ell_{V,W} = \text{the number of extensions of } u \text{ to valuations of } Z(D_h^{w'}).
\end{equation}

This holds because } S \cong M_{\ell_S}(D_h^{w'}), \text{ see (3.3), so the } F^w\text{-algebras } S \text{ and } D_h^{w'} \text{ have isomorphic centers.}

Note that the following conditions are equivalent:

(a) Some (so every) Dubrovin valuation ring } B \text{ of } A \text{ extending } V \text{ is integral over } V.

(b) } V=1.

(c) Every Gräter ring } C \text{ of } A \text{ with } C \cap F = V \text{ is a Dubrovin valuation ring.}

These conditions do not hold in general, but for given } V \text{ and } A, \text{ they hold for some nontrivial coarsening of } V, \text{ see Prop. (3.3) below.}

**Example 3.1.** Let } A \text{ be a central simple } F\text{-algebra, and let } V \text{ be a rank 2 valuation ring of } F. \text{ Let } W \text{ be the rank 1 valuation ring with } V \subseteq W \not\subseteq F. \text{ Then } \xi_{W[A]} = 1 \text{ by Prop. 3.3(iv) below. Let } S \text{ be a Dubrovin valuation ring of } A \text{ extending } W. \text{ So, } S \text{ is integral over } W. \text{ Let } S = S/J(S), \text{ which is a simple } F = W/J(W)\text{-algebra. Suppose the valuation ring } U = V/J(W) \text{ of } F \text{ has exactly } k \text{ different extensions } U_1, \ldots, U_k \text{ to } Z(S). \text{ The map } B \mapsto B/J(S) \text{ gives a one-to-one correspondence between the Dubrovin valuation rings } B \text{ of } A \text{ extending } V \text{ and lying in } S, \text{ and the Dubrovin valuation rings } T \text{ of } S \text{ with } T \cap F = U. \text{ For each such } T, \text{ the intersection } T \cap Z(S) \text{ is a valuation ring of } J(S) \text{ extending } U, \text{ so it is one of the } U_j. \text{ Moreover, } T \text{ is integral over } U_j \text{ since } U_j \text{ has rank 1. Let } B_1, \ldots, B_n \text{ be Dubrovin valuation rings of } A \text{ extending } V \text{ with each } B_j \subseteq S. \text{ Let } B_i = B_i/J(S), \text{ and let } B_i \cap Z(S) = U_j(i). \text{ Then, } B_1, \ldots, B_n \text{ have the IP if and only if } j(1), \ldots, j(n) \text{ are all different. The ring } C = \bigcap_{i=1}^n B_i \text{ is a Gräter ring if and only if further } C \text{ is integral over } V, \text{ if and only if } \bigcap_{i=1}^n B_i \text{ is integral over } U, \text{ if and only if } n = k \text{ and } \{j(1), j(2), \ldots, j(n)\} = \{1, \ldots, n\}. \text{ Thus, } \xi_{V[A]} = k.

In general, there is no valuation associated to a Dubrovin valuation ring } B \text{ of } A. \text{ However, a significant exception occurs when } B \text{ is integral over } Z(B): \text{ Morandi introduced in [M_2] a type of value function associated to any Dubrovin valuation ring integral over its center. Let } \alpha: A \to \Gamma \cup \{\infty\} \text{ be a surmultiplicative } v\text{-value function. We say that } \alpha \text{ is a Morandi value function if}

(1) } A_0 \text{ is a simple ring;}

(2) } \Gamma_\alpha = \alpha({\text{st}}(\alpha)), \text{ where } \text{st}(\alpha) = \{a \in A^\times \mid \alpha(a^{-1}) = -\alpha(a)\}.

If } v = \alpha|_F \text{, then } v \text{ is a valuation on } F \text{ and we call } \alpha \text{ a } v\text{-Morandi value function. Morandi showed in [M_2, Th. 2.4] (or see [MMU, Th. 23.3, p. 135]) the following: If } \alpha \text{ is a Morandi value function on } A, \text{ then its associated ring}
\begin{equation}
R_\alpha = \{x \in A \mid \alpha(x) \geq 0\}
\end{equation}
is a Dubrovin valuation ring integral over its center and
\begin{equation}
(3.13) \quad \Gamma_{R_\alpha} = \Gamma_\alpha.
\end{equation}
Conversely, to every Dubrovin valuation ring \( B \) of \( A \) integral over its center, there exists a Morandi value function \( \alpha \) on \( A \) such that \( R_{\alpha} = B \); moreover, \( \alpha \) is uniquely determined by \( B \) (see [M2, Th. 2.3, Prop. 2.6] or [MMU, Th. 23.2, p. 134]). (By contrast, we will see in Example 5.1 below that if \( \beta \) is a gauge on \( A \), then its gauge ring \( R_{\beta} \) does not always determine \( \beta \).) The connection between gauges and Morandi value functions was shown in [TW1, Prop. 2.5]: If \( A \) is a central simple \( F \)-algebra and \( v \) is a valuation on \( F \) defectless in \( A \), then a surmultiplicative \( v \)-value function \( \alpha \) is a Morandi value function if and only if \( \alpha \) is a gauge with \( A_0 \) simple. The following result generalizes this for simple but not necessarily central simple algebras. If \( \alpha \) is a gauge on a semisimple algebra \( A \), then since \( \text{gr}_{\alpha}(A) \) is graded semisimple, the degree zero part \( A_0 \) of \( \text{gr}_{\alpha}(A) \) must be semisimple (cf. [TW1, Prop. 2.1]). But \( A_0 \) is not necessarily simple, even if \( A \) is simple, as is illustrated in Ex. 1.1. Let

\[
\omega(\alpha) = \text{the number of simple components of } A_0.
\]

**Theorem 3.2.** Suppose \( v \) is a valuation on \( F \) defectless in a simple \( F \)-algebra \( A \). Let \( v_1, \ldots, v_r \) be all the extensions of \( v \) to \( K = Z(A) \).

(i) For \( i \in \{1, \ldots, r\} \), let \( \alpha_i \) be a \( v_i \)-Morandi value function on \( A \) and let \( B_i \) be the Dubrovin valuation ring associated to \( \alpha_i \). Let \( \alpha = \min (\alpha_1, \ldots, \alpha_r) \). Then \( \alpha \) is a \( v \)-gauge on \( A \) if and only if \( B_1, \ldots, B_r \) have the IP.

(ii) Let \( \alpha \) be a \( v \)-gauge on \( A \) with \( \omega(\alpha) = r \). Then there are uniquely determined \( \alpha_i \)-Morandi value functions \( \alpha_i \) for \( i \in \{1, \ldots, r\} \) such that \( \alpha = \min (\alpha_1, \ldots, \alpha_r) \).

**Proof.** For each \( i \) and \( j \), let \( V_i \) be the valuation ring of \( v_i \), and let \( V_{ij} = V_i \cdot V_j \), with its associated valuation \( v_{ij} \), which is the finest common coarsening of \( v_i \) and \( v_j \). Let \( \bar{V}_i = V_i/J(V_i) \subseteq V_i \), and define \( \bar{V}_j \) analogously. Note that \( \bar{V}_i \) and \( \bar{V}_j \) are independent in \( V_{ij} \), i.e., \( \bar{V}_i \cdot \bar{V}_j = \bar{V}_i \bar{V}_j \), since \( V_i \cdot V_j = V_{ij} \).

(i) Suppose \( \alpha \) is a \( v \)-gauge on \( A \). By Th. 2.8, for every pair \( i, j \), the gauges \( \alpha_i \) and \( \alpha_j \) have the same \( v_{ij} \)-coarsening, call it \( \alpha_{ij} \). So, for the gauge rings, we have \( R_{\alpha_i} \subseteq R_{\alpha_{ij}} \) and \( R_{\alpha_j} \subseteq R_{\alpha_{ij}} \). Since \( B_i = R_{\alpha_1} \) and likewise for \( j \), we must have \( B_i \subseteq R_{\alpha_{ij}} \).

\[
Z(B_{ij}) = B_{ij} \cap K \subseteq R_{\alpha_{ij}} \cap K = Z(R_{\alpha_{ij}}) = V_{ij}.
\]

Since we always have \( V_{ij} = V_i \cdot V_j \subseteq Z(B_{ij}) \), we obtain \( V_{ij} = Z(B_{ij}) \). The field \( Z(B_{ij}) \) is a finite-degree extension of the field \( V_{ij} \). Since the valuation rings \( \bar{V}_i \) and \( \bar{V}_j \) are independent in \( V_{ij} \subseteq Z(B_{ij}) \) and \( Z(\bar{B}_i) \cap V_{ij} = \bar{V}_i \), it follows that the valuation rings \( Z(\bar{B}_i) \) and \( Z(\bar{B}_j) \) are independent in \( Z(B_{ij}) \). Hence, \( B_i \) and \( B_j \) have the IP. Therefore, \( B_1, \ldots, B_r \) have the IP by (3.6). Moreover, since by Th. 2.2

\[
\text{gr}_{\alpha}(A) \cong g \text{ gr}_{\alpha_1}(A) \times \ldots \times \text{gr}_{\alpha_r}(A)
\]

and each \( A_0^{\alpha_i} \) is the simple residue ring of \( R_{\alpha_i} \), we have \( \omega(\alpha) = \sum_{i=1}^r \omega(\alpha_i) = \sum_{i=1}^r 1 = r \).

Conversely, suppose \( B_1, \ldots, B_r \) have the IP. For any distinct \( i, j \in \{1, \ldots, r\} \) let \( \Delta_{ij} \) be the convex subgroup of the divisible hull \( \Gamma \) of \( \Gamma \) associated to \( V_{ij} \) and let \( \theta_{ij} : \Gamma \to \Gamma/\Delta_{ij} \) be the canonical map. Since \( B_i \) is the gauge ring \( R_{\alpha_i} \), by Prop. 1.4, \( R_{\theta_{ij} \circ \alpha_i} = B_i \cdot V_{ij} \); likewise, \( R_{\theta_{ij} \circ \alpha_j} = B_j \cdot V_{ij} \).

Since \( B_i \cdot V_{ij} \) and \( B_j \cdot V_{ij} \) are Dubrovin valuation rings of \( A \) integral over \( V_{ij} \), it follows that \( \theta_{ij} \circ \alpha_i \) and \( \theta_{ij} \circ \alpha_j \) are Morandi value functions. Moreover, since \( B_i \) and \( B_j \) have the IP, their overrings \( B_i \cdot V_{ij} \) and \( B_j \cdot V_{ij} \) also have the IP by [MMU, Th. 16.8, p. 92]. But the integrality of \( B_i \cdot V_{ij} \) over \( V_{ij} \) implies that \( \xi_{V_{ij},[A]} = 1 \). By using this or [MMU, Lemma 16.5, p. 90], it follows that \( B_i \cdot V_{ij} = B_j \cdot V_{ij} \). Therefore, \( \theta_{ij} \circ \alpha_i = \theta_{ij} \circ \alpha_j \), because a Morandi value function is completely.
determined by its associated Dubrovin valuation ring (see [MMU, Prop. 23.6, p. 135]). Thus, it follows from Th. 2.8 that \( \alpha \) is a \( v \)-gauge.

(ii) By Th. 2.2 there exist \( v_i \)-gauges \( \alpha_i \) on \( A \) for \( i \in \{1, \ldots, r\} \) such that \( \alpha = \min (\alpha_1, \ldots, \alpha_r) \) and (3.15) holds. Thus \( \omega(\alpha) = \omega(\alpha_1) + \ldots + \omega(\alpha_r) \). Since \( \omega(\alpha) = r \), we must have each \( \omega(\alpha_i) = 1 \); it follows from [TW, Prop. 2.5] that each \( \alpha_i \) is a \( v_i \)-Morandi value function. The uniqueness of the \( \alpha_i \) follows from Cor. 2.5. \( \square \)

For proofs of some of the following theorems we need the notions of jump rank and jump prime ideals, defined as follows: Let \( A \) be a central simple \( F \)-algebra and let \( V \) be the valuation ring of a nontrivial valuation \( v \) on \( F \). For each nonzero prime ideal \( P \) of \( V \), let \( v_P \) be the valuation on \( F \) with valuation ring \( V_P \), let field \( F_{h,P} \) be a Henselization of \( F \) with respect to \( v_P \), and let \( n_P \) be the matrix size of \( A \otimes F_{h,P} \). So, \( 1 \leq n_P \leq \deg A \). For prime ideals \( P \subseteq Q \) we have \( n_P \leq n_Q \) since \( F_{h,P} \) embeds in \( F_{h,Q} \). We say that \( P \) is a jump prime ideal of \( v \) for \( A \) if \( n_P < n_Q \) for every prime ideal \( Q \supseteq P \); we then say that \( v_P \) is a jump valuation of \( v \) for \( A \). The jump rank of \( v \) for \( A \) is defined to be

\[
j(v,A) = \text{the number of jump prime ideals of } v \text{ for } A.
\]

Note that if a positive integer \( m = n_P \) for some \( P \), and \( \mathfrak{P} \) is the union of all the prime ideals \( Q \) with \( n_Q = m \), then \( \mathfrak{P} \) is a prime ideal of \( V \), since the \( Q \)'s are linearly ordered, and \( n_{\mathfrak{P}} = m \) since \( F_{h,\mathfrak{P}} \) is the direct limit of the \( F_{h,Q} \); so \( \mathfrak{P} \) is the unique jump prime ideal of \( v \) with \( n_{\mathfrak{P}} = m \). Thus,

\[
j(v,A) = \left| \{n_P \mid P \text{ is a nonzero prime ideal of } V \} \right|,
\]

and \( 1 \leq j(v,A) \leq \deg A \). The jump rank is useful for induction arguments, since it is always finite even when the valuation \( v \) has infinite rank. If \( P_1 \supseteq P_2 \supseteq \ldots \supseteq P_j(v,A) \) are all the distinct jump prime ideals of \( v \) for \( A \), we call \( P_i \) the \( i \)-th jump prime of \( v \) for \( A \). Note that \( P_j(v,A) = J(V) \). If \( v \) is the trivial valuation on \( F \), we set \( j(v,A) = 0 \). More information on the jump rank can be found in [W] or [MMU].

**Proposition 3.3.** Let \( F \) be a field, and let \( v \) be a nontrivial valuation on \( F \) with valuation ring \( V \). Let \( A \) be a central simple \( F \)-algebra, and let \( B \) be a Dubrovin valuation ring of \( A \) with center \( V \).

(i) There is a unique nonzero prime ideal \( \mathfrak{P} \) of \( V \) maximal with the property that \( B \cdot V_{\mathfrak{P}} \) is integral over \( V_{\mathfrak{P}} \).

(ii) Let \( Q \) be any prime ideal of \( V \). Then, \( B \cdot V_Q \) is integral over \( V_Q \) if and only if \( Q \subseteq \mathfrak{P} \).

(iii) \( \mathfrak{P} \) is a jump prime ideal of \( v \) for \( A \).

(iv) If \( j(v,A) = 1 \), then \( \mathfrak{P} = J(V) \), so \( B \) is integral over \( V \).

(v) If \( \mathfrak{Q} \neq J(V) \), then \( \ell_{V,\mathfrak{Q}} > 1 \).

**Proof.** (i) and (v) The existence of \( \mathfrak{P} \) with the maximal property is given in [MMU, Prop. 12.4, p. 72], where it is also proved that if \( \mathfrak{Q} \neq J(V) \) (so \( B \) is not integral over \( V \)), then \( \ell_{V,\mathfrak{Q}} > 1 \). The prime ideal \( \mathfrak{P} \) is unique with the maximal property since the prime ideals of the valuation ring \( V \) are linearly ordered.

(ii) For a prime ideal \( Q \) of \( V \), if \( Q \supsetneq \mathfrak{P} \), then \( B \cdot V_Q \) is not integral over \( V_Q \) by the maximality of \( \mathfrak{P} \). But if \( Q \subseteq \mathfrak{P} \), then \( V_Q \supseteq V_{\mathfrak{P}} \), so by (3.11) \( \xi_{V_Q,[A]} \leq \xi_{V_{\mathfrak{P}},{[A]} = 1} \). Hence, \( \xi_{V_Q,[A]} = 1 \), showing that \( B \cdot V_Q \) is integral over \( V_Q \).

(iii) If \( \mathfrak{Q} = J(V) \), then \( \mathfrak{Q} \) is a jump prime ideal of \( v \) for \( A \). Assume now that \( \mathfrak{Q} \neq J(V) \). Let \( W = V_{\mathfrak{Q}} \) and let \( S = B \cdot W \), which is a Dubrovin valuation ring of \( A \) with center \( W \). Let
\( \overline{B} = B/J(S) \), which is a Dubrovin valuation ring of \( \overline{S} = S/J(S) \). Note that
\[
(3.16) \quad \overline{B} = (B/J(S)) / J(B/J(S)) \cong B/J(B) = \overline{B}.
\]
Hence, these residue rings have the same matrix size, i.e., \( t_{\overline{B}} = t_B \). We have \( \overline{S} = \mathbb{M}_{t_S}(E) \) for some division ring \( E \). If \( C \) is a Dubrovin valuation ring of \( E \) with center the valuation ring \( Z(B) \), then \( \mathbb{M}_{t_S}(C) \) and \( \overline{B} \) are Dubrovin valuation rings of \( \overline{S} \) with the same center. Hence, \( \overline{B} \cong \mathbb{M}_{t_S}(C) \), which implies that \( \overline{B} \cong \mathbb{M}_{t_S}(C) \).

We next obtain a description of the simple components
\[
(3.19) \quad \text{lying in B over } V/S.
\]
So, for the matrix sizes, \( t_{\overline{B}} = t_B \). Thus, we have
\[
(3.17) \quad t_S \mid t_{\overline{B}} = t_B
\]
(cf. [W1, Th. E(ii)]). Since \( \xi_{V,[A]} = n_B/t_B \), formula (3.10) above can be restated
\[
n_B = n_S(t_{\overline{B}}/t_S)(n_{\overline{B}}/t_{\overline{B}}) \xi_{V,W}.
\]
Since \( t_B/t_S \geq 1 \) by (3.17) and \( n_B/t_{\overline{B}} \geq 1 \), and \( \xi_{V,W} > 1 \) from (v), we have \( n_B > n_S \).

Now let \( P' \) be any prime ideal of \( V \) with \( \mathfrak{q} \subsetneq P' \), and let \( V' = V_{P'} \) and \( B' = B \cdot V' \). The prime ideals \( Q \) of \( V' \) coincide with the prime ideals of \( V \) lying in \( P' \), and for any such \( Q \), we have \( V'_Q = V_Q \) and \( B' \cdot V'_Q = B \cdot V_Q \). Hence \( \mathfrak{q} \) is maximal among the prime ideals \( Q \) of \( V' \) with \( B' \cdot V'_Q \) integral over \( V' \). Therefore, the argument just given showing that \( n_B > n_S \) shows likewise that \( n_B > n_S \). Since this is true for every \( P' \supsetneq \mathfrak{q} \), this \( \mathfrak{q} \) is a jump prime ideal of \( v \) for \( A \).

(iv) If \( j(v,A) = 1 \), then \( J(V) \) is the only jump prime ideal of \( v \), so \( \mathfrak{q} = J(V) \) and \( B = B \cdot V_{\mathfrak{q}} \) is integral over \( V_{\mathfrak{q}} = V \). \( \square \)

The following general setup occurs repeatedly in the proofs of the next three theorems:

**Setup 3.4.** Let \( F \) be a field with a valuation \( v \) and let \( A \) be a central simple \( F \)-algebra with a \( v \)-gauge \( \alpha \). So, \( v \) is defectless in \( A \) by Prop. 1.9. Let \( w \) be a coarsening of \( v \). Then \( w \) is also defectless in \( A \) by Prop. 1.8. Let \( V \) be the valuation ring of \( v \), and let \( W \) be the valuation ring of \( w \). Let \( V = V/J(W) \), which is a valuation ring of \( V/W = W/J(W) \), and let \( u \) be the valuation associated to \( V \). Let \( \beta \) be the coarsening of \( \alpha \) such that \( \beta|_F = w \). By Prop. 1.5, \( \alpha \) induces a \( u \)-gauge \( \alpha_0 \) on \( A_0^\beta \). Let \( C_1, \ldots, C_\omega(\beta) \) be the simple components of \( A_0^\beta \). Let \( \alpha^i_0 \) be the restriction of \( \alpha_0 \) to \( C_i \) for \( i = 1, \ldots, \omega(\beta) \).

(3.18)
\[
\alpha_0(a_1, \ldots, a_\omega(\beta)) = \min(\alpha^0_0(a_1), \ldots, \alpha^\omega_0(\alpha(a_\omega(\beta))))
\]
for all \( a_1 \in C_1, \ldots, a_\omega(\beta) \in C_\omega(\beta) \). Moreover,
\[
(3.19) \quad \text{gr}_{\alpha_0}(A_0^\beta) \cong \text{gr}_{\alpha^0_0}(C_1) \times \ldots \times \text{gr}_{\alpha^\omega_0(\beta)}(C_\omega(\beta)).
\]
By (1.17), the associated graded algebras of \( \alpha \) and \( \alpha_0 \) have the same degree zero part. It then follows from (3.19) that
\[
(3.20) \quad \omega(\alpha) = \omega(\alpha_0) = \omega(\alpha^0_0) + \ldots + \omega(\alpha^\omega_0(\beta)).
\]
We next obtain a description of the simple components \( C_i \). Let \( (F_{h,w}, w_h) \) be a Henselization of \( (F,w) \). Consider the scalar extension \( A_h = A \otimes_F F_{h,w} \cong \mathbb{M}_h(D_h) \), where \( D_h \) is a central division \( F_{h,w} \)-algebra. Then, \( w_h \) is defectless in \( D_h \) by definition as \( w \) is defectless in \( A \), and the Henselian valuation \( w_h \) extends to a valuation \( w' \) on \( D_h \). Let \( R_{w'} \) be the invariant valuation ring of \( D_h \) associated to \( w' \) and let \( \overline{D}_h = R_{w'}/J(R_{w'}) \), the residue division ring. On the other hand, it follows from [TW1, Cor. 1.26] that \( \beta_h = \beta \otimes w_h \) is a \( w_h \)-gauge on \( A_h \) and \( \text{gr}_{\beta_h}(A_h) \cong \text{gr}_{\beta}(A) \).
Write $A_h = \text{End}_{D_h}(M)$ for some finite-dimensional $D_h$-vector space $M$. Since $w_h$ is Henselian and is defectless in $D_h$, [TW1, Th. 3.1] says that $\beta_h$ is an End-gauge as in Ex. 1.1, i.e., there is a $w'$-norm $\eta$ on $M$ such that $\text{gr}_{\beta_h}(A_h) = \text{End}_{\text{gr}_{w'}(D_h)}(\text{gr}_{\eta}(M))$. By [TW1, Prop. 2.1],
\[(3.21)\quad A_0^\beta = (A_h)_0^{\beta_h} = (\text{End}_{\text{gr}_{w'}(D_h)}(\text{gr}_{\eta}(M)))_0 \cong \prod_{i=1}^k \mathbb{M}_{r_i}(D_0),\]
where $D_0$ is the degree zero part of $\text{gr}_{w'}(D_h)$, which is $\overline{D_h}$; also, $k$ is the number of cosets of $\Gamma_w$ in $\Gamma_\eta$. By comparing (3.21) with the decomposition $A_0^\beta = C_1 \times \ldots \times C_{\omega(\beta)}$ from (3.19), we conclude that $k = \omega(\beta)$ and that each $C_i = \mathbb{M}_{r_i}(D_0)$, after re-indexing if necessary. Thus, $Z(C_1) = \ldots = Z(C_{\omega(\beta)}) = Z(\overline{D_h})$. Let $K$ denote this common field. (We also have $K = Z(S)$ for any Dubrovin valuation ring $S$ of $A$ with $Z(S) = W$.) As noted in (3.12), there are $\ell_{V,W}$ extensions of $u$ to $K$, which we denote $u_1, \ldots, u_{\ell_{V,W}}$. Let $U_j$ be the valuation ring of $u_j$. By Th. 2.2 applied to the $u$-gauge $\alpha_0^i$ on $C_i$, there exist $u_j$-gauges $\alpha_0^{ij}$ on $C_i$ for $j \in \{1, \ldots, \ell_{V,W}\}$ such that
\[(3.22)\quad \alpha_0^{ij}(a) = \min(\alpha_0^{i1}(a), \ldots, \alpha_0^{i\ell_{V,W}}(a)) \quad \text{for all } a \in C_i,\]
and also
\[\text{gr}_{\alpha_0^{ij}}(C_i) \cong \text{gr}_{\alpha_0^{i1}}(C_i) \times \ldots \times \text{gr}_{\alpha_0^{i\ell_{V,W}}}(C_i).\]
We thus have
\[(3.23)\quad \omega(\alpha_0^i) = \omega(\alpha_0^{i1}) + \ldots + \omega(\alpha_0^{i\ell_{V,W}}).\]
Hence, with (3.20),
\[(3.24)\quad \omega(\alpha) = \omega(\alpha_0) = \sum_{i=1}^{\omega(\beta)} \sum_{j=1}^{\ell_{V,W}} \omega(\alpha_0^{ij}).\]
Because the $C_i$ are Brauer equivalent central simple $K$-algebras, we have
\[(3.25)\quad \xi_{U_j,[C_i]} = \xi_{U_j,[C_i]} \quad \text{for all } i \in \{1, \ldots, \omega(\beta)\}, j \in \{1, \ldots, \ell_{V,W}\}.\]
We claim also that
\[(3.26)\quad \xi_{U_j,[C_i]} = \xi_{U_j,[C_i]} \quad \text{for all } i \in \{1, \ldots, \omega(\beta)\}, j \in \{1, \ldots, \ell_{V,W}\}.\]
We check this for $\overline{D_h}$, which is Brauer equivalent to $C_i$. Recall from [JW, Prop. 1.7] that $K = Z(\overline{D_h})$ is a normal field extension of $F_h^{\omega w} = F^{\omega}$, and hence the Galois group $G(K/F^{\omega})$ acts transitively on the set $\{u_1, \ldots, u_{\ell_{V,W}}\}$ of all extensions of $u$ on $F^{\omega}$ to $K$. Choose $\tau \in G(K/F^{\omega})$ with $u_j = u_1 \circ \tau$. By [JW, Prop. 1.7], there is $d \in D_h^\times$ whose associated automorphism $\tau_d: \overline{D_h} \rightarrow \overline{D_h}$ ($\pi \mapsto \overline{d\pi d^{-1}}$ for $a \in R_{w'}$) restricts to $\tau$ on $K$. Then, $\tau_d$ takes Dubrovin valuation rings of $\overline{D_h}$ with center $U_j$ to those with center $U_1$; so $\xi_{U_j,[\overline{D_h}]} = \xi_{U_1,[\overline{D_h}]}$. Thus, from the Brauer equivalence of $C_i$ and $\overline{D_h}$,
\[\xi_{U_j,[C_i]} = \xi_{U_j,[\overline{D_h}]} = \xi_{U_1,[\overline{D_h}]} = \xi_{U_1,[C_i]},\]
as claimed, proving (3.26).

The following diagram illustrates some of the objects considered here. We write $v \geq w$ to indicate that $v$ is a refinement of $w$; likewise for $\alpha \geq \beta$. 
Theorem 3.5. Let $F$ be a field with a valuation $v$ and associated valuation ring $V$, and let $A$ be a central simple $F$-algebra. Then, for any $v$-gauge $\alpha$ on $A$,

$$\omega(\alpha) \geq \xi_{V,[A]}.$$ 

Proof. We write $\xi_V$ for $\xi_{V,[A]}$. The proof is by induction on $\xi_V$. If $\xi_V = 1$, then we clearly have $\omega(\alpha) \geq \xi_V$. We can thus assume that $\xi_V > 1$. By [MMU, Th. 16.14, p. 94], there exist Dubrovin valuation rings $R_1, \ldots, R_{\xi_V}$ of $A$ having the IP such that each $Z(R_t) = V$ and $R_1 \cap \ldots \cap R_{\xi_V}$ is a Gräter ring integral over $V$. For each $t \in \{1, \ldots, \xi_V\}$, by Prop. 3.3(i) and (v) there exists a Dubrovin valuation ring $S_t$ of $A$ containing $R_t$ and minimal with the property that $S_t$ is integral over $W_t = Z(S_t)$ and we have $\ell_{t,v} \geq 2$. Now by [MMU, Cor. 16.6, p. 91], we have $S_1 = \ldots = S_{\xi_V}$, hence $W_1 = \ldots = W_{\xi_V}$. Let $S = S_1$ and $W = W_t$, for all $t$. The integrality of $S$ over $W$ yields that $\xi_{W,[A]} = 1$. We use the valuation $w$ of $W$ in Setup 3.4; so, $\beta$ is the $w$-coarsening of $\alpha$. Let $\widetilde{R_t} = R_t/J(S)$. Since $\bigcap_{t=1}^{\xi_V} R_t$ is integral over $V$, we have $\bigcap_{t=1}^{\xi_V} \widetilde{R_t}$ is integral over $V$. The valuation rings $Z(\widetilde{R_1}), \ldots, Z(\widetilde{R_{\xi_V}})$ must include all the extensions of $V$ to $K = Z(S)$, because $Z(\widetilde{R_1}) \cap \ldots \cap Z(\widetilde{R_{\xi_V}})$ is integral over $V$. These extensions are thus the valuation rings $U_1, \ldots, U_{\ell_{V,W}}$ of the extensions $u_1, \ldots, u_{\ell_{V,W}}$ of $v$ in Setup 3.4, but with possible repetitions. For any $j \in \{1, \ldots, \ell_{V,W}\}$, choose an $R_t$ with $Z(\widetilde{R_t}) = U_j$. It follows from (3.10) with $B = R_t$ that

$$\xi_V = \xi_{W,[A]} \xi_{U_j,[S]} \ell_{V,W} = \xi_{U_j,[S]} \ell_{V,W}. \tag{3.27}$$

Since $\ell_{V,W} \geq 2$, we conclude that each $\xi_{U_j,[S]} < \xi_V$. Since each $C_i$ is Brauer equivalent to $S$, we have $\xi_{U_j,[S]} = \xi_{U_j,[C_i]}$. Since each $\alpha_0^{ij}$ constructed in the Setup 3.4 is a $u_j$-gauge on $C_i$, we have by the induction hypothesis $\omega(\alpha_0^{ij}) \geq \xi_{U_j,[C_i]}$. The equality (3.27) (or (3.26)) shows that $\xi_{U_j,[C_i]} = \xi_{U_1,[C_i]} = \xi_{U_1,[S]}$ for all $i, j$. Thus, it follows from (3.24) that

$$\omega(\alpha) = \sum_{i=1}^{\omega(\beta)} \sum_{j=1}^{\ell_{V,W}} \omega(\alpha_0^{ij}) \geq \omega(\beta) \ell_{V,W} \xi_{U_1,[S]} \geq \ell_{V,W} \xi_{U_1,[S]} = \xi_V,$$

which completes the proof. \qed

Definition 3.6. Let $v$ be a valuation on a field $F$, and let $\alpha$ be a $v$-gauge on a central simple $F$-algebra $A$. Then, we call $\alpha$ a minimal gauge if $\omega(\alpha) = \xi_{V,[A]}$.

Remark 3.7. Note that if $\omega(\alpha) = 1$, i.e., $A_0^\beta$ is a simple ring, then by [TW1, Prop. 2.5] $\alpha$ is a Morandi value function on $A$ with associated Dubrovin valuation ring $R_\alpha$. Conversely, if $\alpha'$ is a Morandi value function on $A$ with $\alpha'|F = v$ and $v$ is defectless in $A$, then by [TW1, Prop. 2.5] $\alpha'$ is a $v$-gauge and $R_{\alpha'}$ is the Dubrovin valuation ring associated to $\alpha'$; so, $A_0^{\alpha'}$ is the simple ring $R_{\alpha'}/J(R_{\alpha'})$, whence $\omega(\alpha') = 1$. 

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (F) at (-2,-1) {$F$};
  \node (v) at (-1,-1) {$v$};
  \node (w) at (-1,-2) {$w$};
  \node (u) at (-1,-3) {$u$};
  \node (A0) at (2,0) {$A_0^\beta$};
  \node (Z) at (1,-1) {$Z(A_0^\beta)$};
  \node (W) at (2,-2) {$W$};
  \node (K) at (2,-3) {$K$};
  \node (C) at (5,0) {$C_i$};
  \node (C01) at (5,-1) {$\alpha_0^1, \ldots, \alpha_0^i_{V,W}$};
  \node (C0) at (5,-2) {$\alpha_0^0$};
  \node (C1) at (5,-3) {$u_1, \ldots, u_{\ell_{V,W}}$};

  \draw[->] (A) -- (F) node[midway,above] {$\alpha \geq \beta$};
  \draw[->] (A) -- (C) node[midway,above] {$\alpha_0^0 \geq \alpha_0^1 \geq \cdots \geq \alpha_0^i_{V,W}$};
  \draw[->] (F) -- (v) node[midway,above] {$v \geq w$};
  \draw[->] (v) -- (w) node[midway,above] {$u$};
  \draw[->] (Z) -- (W) node[midway,above] {$\geq$};
  \draw[->] (W) -- (K) node[midway,above] {$\geq$};
  \draw[->] (K) -- (C) node[midway,above] {$\geq$};

\end{tikzpicture}
\end{center}
Theorem 3.8. Let $F$ be a field with a valuation $v$ and associated valuation ring $V$. Let $A$ be a central simple $F$-algebra with a $v$-gauge $\alpha$. Let $\beta$ be a coarsening of $\alpha$, and let $w = \beta|_F$, which is a valuation on $F$ coarser than $v$. Let $W$ be the valuation ring of $w$. Then,

$$
\omega(\alpha)/\omega(\beta) \geq \xi_{V,[A]}/\xi_{W,[A]}.
$$

Consequently, if $\alpha$ is a minimal gauge, then so is $\beta$.

Proof. We use the notation of Setup 3.4 with the $v, \alpha, w, \beta$ given here. By Th. 3.5, we have $\omega(\alpha_i) \geq \xi_{U_i,[C_i]}$ for $i \in \{1, \ldots, \omega(\beta)\}$ and $j \in \{1, \ldots, \ell_{V,W}\}$. Equations (3.25) and (3.26) above show that $\xi_{U_i,[C_i]} = \xi_{U_i,[C_i]}$ for each $i \in \{1, \ldots, \omega(\beta)\}$ and $j \in \{1, \ldots, \ell_{V,W}\}$. It then follows from (3.24) that

$$
\omega(\alpha) \geq \omega(\beta) \ell_{V,W} \xi_{U_1,[C_1]}.
$$

Choose a Dubrovin valuation ring $B$ of $A$ extending $V$, and let $S = B \cdot W$. Then, for the Dubrovin valuation ring $\tilde{B} = B/J(S)$ the valuation ring $Z(\tilde{B})$ is an extension of $V/J(W)$ to $Z(S)$, so $Z(\tilde{B}) = U_j$ for some $j$. Then, (3.10) with this $B$ yields $\xi_{V,[A]} = \xi_{W,[A]} \ell_{V,W} \xi_{U_j,[S]} = \xi_{W,[A]} \ell_{V,W} \xi_{U_1,[C_1]}$. Hence,

$$
\omega(\alpha)/\omega(\beta) \geq \ell_{V,W} \xi_{U_1,[C_1]} = \xi_{W,[A]} \ell_{V,W} \xi_{U_1,[C_1]}/\xi_{W,[A]} = \xi_{V,[A]}/\xi_{W,[A]}.
$$

Suppose $\alpha$ is a minimal gauge. Then, $\omega(\alpha) = \xi_{W,[A]}$. The inequality just proved then shows that $\omega(\beta) \leq \xi_{W,[A]}$. The reverse inequality is given by Th. 3.5. Hence, $\omega(\beta) = \xi_{W,[A]}$, showing that $\beta$ is a minimal gauge. \hfill \Box

The next theorems give the fundamental connection between minimal gauges and Gräter rings.

Theorem 3.9. Let $F$ be a field with a valuation $v$ and let $A$ be a central simple $F$-algebra with a minimal $v$-gauge $\alpha$. Then, the gauge ring $R_\alpha = \{a \in A \mid \alpha(a) \geq 0\}$ is a Gräter ring of $A$ with center the valuation ring $V$ of $v$.

Proof. We may assume that the valuation $v$ on $F$ is nontrivial. The proof is by induction on the jump rank $j(v,A)$. If $j(v,A) = 1$, then any Dubrovin valuation ring of $A$ extending $V$ is integral over $V$ by Prop. 3.3(iv). Thus, $1 = \xi_{V,[A]} = \omega(\alpha)$, hence $\alpha$ is a Morandi value function by Remark 3.7. Therefore, $R_\alpha$ is a Dubrovin valuation ring integral over $V$, so $R_\alpha$ is a Gräter ring.

We can thus assume $j(v,A) > 1$. Let $Q$ be the $(j(v,A) - 1)$-st jump prime ideal of $v$ for $A$, let $W = V_Q$ be the corresponding valuation ring, and let $w$ be the valuation on $F$ associated to $W$. We write $\xi_V$ for $\xi_{V,[A]}$ and $\xi_W$ for $\xi_{W,[A]}$. The jump prime ideals of $w$ for $A$ are the same as the jump prime ideals of $v$ for $A$, except that $J(V)$ is excluded. Thus, $j(w,A) = j(v,A) - 1$. Let $\beta$ be a coarsening of $\alpha$ such that $\beta|_F = w$. We use these $v, w, \alpha, \beta$ in Setup 3.4. Since $\alpha$ is a minimal gauge, by Th. 3.8 $\beta$ is also a minimal gauge. Thus, by the induction hypothesis, $R_\beta$ is a Gräter ring with center $W$; hence, $R_\beta = \bigcap_{i=1}^{\xi_W} R_i$, where $R_1, \ldots, R_{\xi_W}$ are Dubrovin valuation rings having the IP such that each $Z(R_i) = W$. By [M3, Lemma 3.2], there is an isomorphism

$$
(3.28) \quad A_0^\beta \isom \prod_{i=1}^{\xi_W} R_i/J(R_i) \text{ given by } x + J(R_\beta) \leftrightarrow (x + J(R_1), \ldots, x + J(R_{\xi_W})).
$$

After re-indexing if necessary, we can write $C_i = R_i/J(R_i)$, as in Setup 3.4. By [W1, Cor. E], jump prime ideals of $u_j$ for $C_i$ pull back to jump primes ideals of $v$ for $A$ properly containing $Q$; hence,

$$
j(u_j,C_i) \leq j(v,A) - j(w,A) = 1.
$$
Therefore, $\xi_{ij},|c_i| = 1$ for all $i, j$. Thus, formula (3.10) reduces to $\xi_V = \xi_W \ell_V W$.

Since $\omega(\alpha) = \xi_V$ and $\omega(\beta) = \xi_W$, it follows from (3.24) that $\omega(\alpha_{ij}^0) = 1$ for all $i \in \{1, \ldots, \xi_W\}$ and $j \in \{1, \ldots, \ell_V W\}$. Hence each $\alpha_{ij}^0$ is a Morandi value function by Remark 3.7. Let $S_{ij} = \{ x \in C_i \mid \alpha_{ij}^0(x) \geq 0 \}$, which is a Dubrovin valuation ring of $C_i$ with $Z(S_{ij}) = U_i$. Let

$$B_{ij} = \{ a \in A \mid a + J(R_i) \in S_{ij} \},$$

which is a Dubrovin valuation ring of $A$ with $Z(B_{ij}) = V$. We prove that the set of Dubrovin valuation rings $B_{ij}$ for $i = 1, \ldots, \xi_W$ and $j = 1, \ldots, \ell_V W$ have the IP. By Th. 3.2, $S_{11}, \ldots, S_{\ell_V W}$ have the IP for any $i$. Thus, $B_{11}, \ldots, B_{\ell_V W}$ have the IP for each $i$, by [MMU, Prop. 16.4, p. 90].

Now let $i, q \in \{1, \ldots, \xi_W\}$ with $i \neq q$. Note that $B_{ij} \subseteq R_i$ and $B_{qr} \subseteq R_q$ for $j, r \in \{1, \ldots, \ell_V W\}$. Since $R_i$ and $R_q$ are incomparable and have the IP, $B_{ij}$ and $B_{qr}$ are also incomparable and have the IP by [MMU, Th. 16.8, p. 92]. Therefore, the Dubrovin valuation rings $\{B_{ij}\}_{i,j}$ are pairwise incomparable and have the IP.

We claim that $R_{\alpha} = \bigcap_{i=1}^{\xi_W} \bigcap_{j=1}^{\ell_V W} B_{ij}$. Since the gauge ring $R_{\alpha}$ is integral over its center $V$, it then follows that the intersection of the $B_{ij}$ is a Gräter ring, which completes the proof that $R_{\alpha}$ is a Gräter ring.

To prove the claim, note first that

$$J(R_\beta) \subseteq J(R_\alpha) \subseteq R_\alpha \subseteq R_\beta,$$

because $\alpha(x) \geq 0$ implies $\beta(x) \geq 0$ and $\beta(x) > 0$ implies $\alpha(x) > 0$. Likewise, for each $i \in \{1, \ldots, \xi_W\}$ and $j \in \{1, \ldots, \ell_V W\}$ we have

$$J(R_i) \subseteq J(B_{ij}) \subseteq B_{ij} \subseteq R_i.$$

Hence, using (3.28) for the first equality as $A_{\beta} = R_{\beta}/J(R_\beta)$

$$J(R_\beta) = \bigcap_{i=1}^{\xi_W} \bigcap_{j=1}^{\ell_V W} B_{ij} \subseteq \bigcap_{i=1}^{\xi_W} R_i = R_\beta.$$

Let $a \in R_\beta \setminus J(R_\beta)$. In view of (3.29) and (3.30), the proof of the claim will be completed by showing that $a \in R_{\alpha}$ if and only if $a \in \bigcap_{i=1}^{\xi_W} \bigcap_{j=1}^{\ell_V W} B_{ij}$. We identify $A_{\beta}^0$ with $R_1/J(R_1) \times \ldots \times R_{\xi_W}/J(R_{\xi_W})$, via the isomorphism (3.28). Thus, by the definition of the gauge $a_0$ on $A_{\beta} = R_{\beta}/J(R_\beta)$ and by (3.18),

$$\alpha(a) = \alpha_0(a + J(R_\beta)) = \alpha_0(a + J(R_1), \ldots, a + J(R_{\xi_W})) = \min_{1 \leq i \leq \xi_W} (\alpha_0^i(a + J(R_i))).$$

Hence, $\alpha(a) \geq 0$ if and only if $\alpha_0^i(a + J(R_i)) \geq 0$ for $i = 1, \ldots, \xi_W$. By (3.22), the gauge ring $R_{\alpha}^0 = \bigcap_{j=1}^{\ell_V W} S_{ij}$. Thus, $a \in R_{\alpha}$ if and only if $a + J(R_i) \in \bigcap_{j=1}^{\ell_V W} S_{ij}$, for $i = 1, \ldots, \xi_W$. But $a + J(R_i) \in S_{ij}$ if and only if $a \in B_{ij}$. Therefore, $a \in R_{\alpha}$ if and only if $a \in \bigcap_{i=1}^{\xi_W} \bigcap_{j=1}^{\ell_V W} B_{ij}$. This proves the claim, which, as noted above, implies that $R_{\alpha}$ is a Gräter ring.

**Remark 3.10.** Note that the Dubrovin valuation rings $B_{ij}$ of Th. 3.9 are uniquely determined by $\alpha$, because they are the localizations of $R_{\alpha}$ re its maximal ideals.

The following result shows that Th. 3.9 has a converse.

**Theorem 3.11.** Let $F$ be a field with a valuation $v$ and associated valuation ring $V$. Let $A$ be a central simple $F$-algebra such that $v$ is defectless in $A$. Then a subring $C$ of $A$ is a Gräter ring of $A$ with center $V$ if and only if $C = R_{\alpha}$ for some minimal $v$-gauge $\alpha$ on $A$. 

The proof requires the existence of minimal gauges on defectless $F$-algebras, which will be proved in §4.

**Proof.** If $\alpha$ is a minimal $v$-gauge on $A$, we have seen in Th. 3.9 that $R_\alpha$ is a Gräter ring with center $V$.

For the converse, let $C$ be a Gräter ring of $A = C$. Then, Th. 4.3 below shows that there exists a minimal $v$-gauge $\beta$ on $A$. By Th. 3.9, its gauge ring $R_\beta$ is a Gräter ring of $A$ with center $V$. By [MMU, Th. 16.15, p. 96], $C$ and $R_\beta$ are conjugate in $A$, i.e., $C = q R_\beta q^{-1}$ for some $q \in A^\times$. Composition of $\beta$ with the inner automorphism $\iota_q : A \to A$ defined by $x \mapsto q^{-1} x q$ yields a minimal $v$-gauge $\alpha = \beta \circ \iota_q$ on $A$ such that $R_\alpha = q R_\beta q^{-1} = C$. □

Note that the gauge $\alpha$ of the theorem is not uniquely determined by $C$, as the example in §5 below demonstrates.

### 4. Existence of minimal gauges

In this section we prove the existence of minimal gauges on defectless semisimple algebras. First, we extend the concept of minimal gauge to semisimple algebras.

**Proposition 4.1.** Let $(F, v)$ be a valued field and $A$ be a finite-dimensional simple $F$-algebra. Let $v_1, \ldots, v_r$ be all the extensions of $v$ to $Z(A)$, and let $V_i$ be the valuation ring of $v_i$. Let $\alpha$ be a $v$-gauge on $A$. Then,

\begin{equation}
\omega(\alpha) \geq \xi_{V_1[A]} + \ldots + \xi_{V_r[A]}.
\end{equation}

**Proof.** Let $\alpha_i$ be the $v_i$-component of $\alpha$ for $i = 1, \ldots, r$. Then, by the graded algebra isomorphism (2.1) and the inequality of Th. 3.5 for each $i$,

\begin{equation}
\omega(\alpha) = \omega(\alpha_1) + \ldots + \omega(\alpha_r) \geq \xi_{V_1[A]} + \ldots + \xi_{V_r[A]}.
\end{equation}

□

**Definition 4.2.** Let $v$ be a valuation on a field $F$, and let $\alpha$ be a $v$-gauge on a simple (finite-dimensional) $F$-algebra $A$. We say that $\alpha$ is a minimal $v$-gauge on $A$ if we have equality in (4.1). Note that (4.2) shows that $\alpha$ is a minimal $v$-gauge if and only if each component $\alpha_i$ of $\alpha$ is a minimal $v_i$-gauge. More generally, if $A$ is semisimple, say $A = A_1 \times \ldots \times A_k$ with each $A_i$ simple, and $\beta$ is a $v$-gauge on $A$, we say that $\beta$ is a minimal $v$-gauge on $A$ if each $\beta_i = \beta|_{A_i}$ (as in (1.14)) is a minimal $v$-gauge on $A_i$.

**Theorem 4.3.** If $v$ is a valuation on $F$ defectless in a finite-dimensional semisimple $F$-algebra $A$, then there exists a minimal $v$-gauge $\alpha$ on $A$ with $\Gamma_\alpha \subseteq \mathbb{H}(\Gamma_v)$.

The general method in proving the theorem is to build up $\alpha$ inductively from minimal gauges for $A$ for valuations on $F$ coarser than $v$. The proof will begin after Prop. 4.7 and be completed after Lemma 4.9.

**Lemma 4.4.** Let valuation $w$ be a coarsening of $v$ on $F$. Let $A$ be a semisimple $F$-algebra with a $w$-gauge $\beta$. Let $F'$ be a field containing $F$ with a valuation $w'$ which is an immediate extension of $w$, and let $v'$ be the extension of $v$ to $F'$ that refines $w'$, so $v'$ is an immediate extension of $v$. Let $A' = A \otimes_F F'$ and let $\beta' = \beta \otimes w'$, which is a $w'$-gauge on $A'$. Suppose $A'$ has a $v'$-gauge $\alpha'$ whose $w'$-coarsening is $\beta'$. Then $\alpha = \alpha'|_A$ is a $v$-gauge on $A$ with $w$-coarsening $\beta$, and $\text{gr}_\alpha(A) \cong_g \text{gr}_\alpha(A')$. 

Proof. Let \( u = v/w \), which is the valuation on \( \overline{F}w \) induced by \( v \) on \( F \); likewise let \( u' = v'/w' \) on \( \overline{F}w' \), which coincides with \( u \) under the canonical isomorphism \( \overline{F}w \cong \overline{F}w' \). Because \( w' \) is an immediate extension of \( w \), by [TW1, Cor. 1.26], \( \beta' = \beta \otimes w' \) is a \( u' \)-gauge on \( A' \) with

\[
\text{gr}_{\beta'}(A') \cong g \text{gr}_{\beta}(A) \otimes_{\text{gr}_{\alpha}(F)} \text{gr}_{w'}(F') \cong g \text{gr}_{\beta}(A).
\]

For \( \Gamma = \mathcal{H}(\Gamma_\alpha') \) and \( \Lambda = \mathcal{H}(\Gamma_\beta') \), let \( \varepsilon : \Gamma \to \Lambda \) be the map associated to the \( w \)-coarsening of \( v \). For each \( \lambda \in \Lambda \) we have the \( \lambda \)-component \( A_\lambda^{\beta'} \) of \( \text{gr}_{\beta'}(A') \), and the \( u' \)-value function \( \alpha'_\lambda \) on \( A_\lambda^\beta \) defined by

\[
\alpha'_\lambda(a + A_{>\lambda}^{\beta'}) = \begin{cases} 
\alpha'(a) & \text{if } \beta'(a) = \lambda, \\
\infty & \text{if } \beta'(a) > \lambda.
\end{cases}
\]

Since \( \alpha' \) is a \( u' \)-norm, each \( \alpha'_\lambda \) is a \( u' \)-norm on \( A_\lambda^\beta \) by [TW2, Prop. 4.3]. Likewise \( \alpha = \alpha'|_A \) induces the \( u \)-value function \( \alpha_\lambda \) on \( A_\lambda^\beta \). Since \( \beta = \beta'|_A \), we can view \( A_\lambda^\beta \subseteq A_\lambda^\beta' \) via the canonical inclusion; then clearly \( \alpha_\lambda = \alpha'_\lambda|_{A_\lambda^\beta} \). But since the canonical inclusion \( \text{gr}_{\beta}(A) \hookrightarrow \text{gr}_{\beta'}(A') \) is a graded isomorphism, we have \( A_\lambda^\beta = A_\lambda^\beta' \). So, \( \alpha_\lambda = \alpha'_\lambda \), which is a \( u \)-norm on \( A_\lambda^\beta \). Since in addition \( \beta \) is a \( w \)-norm on \( A \), by [TW2, Prop. 4.3] \( \alpha \) is a \( v \)-norm on \( A \). Moreover, \( \alpha = \alpha'|_A \) is surmultiplicative since \( \alpha' \) is surmultiplicative. There is a canonical algebra monomorphism \( \iota : \text{gr}_\alpha(A) \hookrightarrow \text{gr}_{\alpha'}(A') \). Take any \( \gamma \in \Gamma \). It follows from the definitions that \( A_\gamma^\alpha = (A_{\varepsilon(\gamma)}^{\beta'}|_{\varepsilon(\gamma)})^{\alpha_{\varepsilon(\gamma)}} \). Since \( A_\gamma^{\beta'} = A_{\varepsilon(\gamma)}^{\beta'} \) and \( \alpha_{\varepsilon(\gamma)} = \alpha'_{\varepsilon(\gamma)} \), we thus have

\[
A_\gamma^\alpha = (A_{\varepsilon(\gamma)}^{\beta'}|_{\varepsilon(\gamma)})^{\alpha_{\varepsilon(\gamma)}} = (A_{\varepsilon(\gamma)}^{\beta'})^{\alpha'_{\varepsilon(\gamma)}} = A_{\gamma}^{\alpha'},
\]

hence, \( \iota \) is a graded isomorphism. Thus, \( \text{gr}_\alpha(A) \) is graded semisimple, since this is true for \( \text{gr}_{\alpha'}(A') \), as \( \alpha' \) is a \( \gamma \)-gauge. Therefore, \( \alpha \) is a \( v \)-gauge. The \( w \)-coarsening of \( \alpha = \alpha'|_A \) is \( \beta'|_A = \beta \). \( \square \)

Lemma 4.5. Let \( A \) be a central simple \( F \)-algebra with \( j(v, A) = n > 1 \). Let \( P \) be the \( (n - 1) \)-st jump prime ideal of \( v \) for \( A \), and let \( W = V_P \) with its associated valuation \( w \). Let \( S \) be a Dubrovin valuation ring of \( A \) with \( Z(S) = W \), and let \( \overline{S} = S/J(S) \). Let \( u = v/w \), the residue valuation on \( \overline{F}w \) induced by \( v \), and let \( u_1, \ldots, u_r \) be the valuations on \( Z(\overline{S}) \) extending \( u \). Then \( u_1, \ldots, u_r \) are pairwise independent valuations.

Proof. Let \( T \) and \( B \) be Dubrovin valuation rings of \( A \) with \( B \subseteq T \subseteq S \) and \( Z(B) = V \). Let \( Y = Z(T) \) and let \( y \) be the valuation of \( Y \). We have \( V \subseteq Y \subseteq W \). Let \( n_B \) and \( t_B \) be as given in (3.1) and (3.3). Let \( \overline{B} = B/J(S) \) which is a Dubrovin valuation ring of \( \overline{S} = S/J(S) \). Recall from (3.17) that \( t_{\overline{B}} = t_B \). (The proof of (3.17) is valid for any overring \( S \) of a Dubrovin valuation ring \( B \).) Using this and \( \xi_{V, [A]} = n_B/t_B \), formula (3.10) yields

\[
n_B = \xi_{W, [A]} n_{\overline{B}} \ell_{Y, W}.
\]

Likewise, by replacing \( B \) by \( T \), we have

\[
n_T = \xi_{W, [A]} n_T \ell_{Y, W}.
\]

Hence,

\[
n_B/n_T = \left(n_{\overline{B}}/n_T\right) (\ell_{Y, W}/\ell_{Y, W}).
\]

Because \( y \) is a coarsening of \( v \), the Henselization \( F_{h, y} \) embeds in \( F_{h, v} \). Hence \( n_B/n_T \) is a positive integer, as likewise is \( n_{\overline{B}}/n_T \). Also, because the valuation ring \( \overline{V} = V/J(W) \) is a refinement of \( \overline{Y} = Y/J(W) \) in \( \overline{W} \), this \( \overline{V} \) has at least as many extensions to \( Z(\overline{S}) \) as \( \overline{W} \), i.e., \( \ell_{Y, W} \geq \ell_{Y, W} \). But,
since there are no jump prime ideals between \( J(Y) \) and \( J(V) \), we have \( n_B = n_T \). Thus, in (4.3) the left side equals 1 and the right side is a product of positive integers. Hence, \( \ell_{Y,W} = \ell_{Y,W} \).

The last equality says that the number of extensions \( U_i \) of \( U = V/J(W) \) to \( Z(S) \) equals the number of extensions of the coarser valuation \( Y/J(W) \) to \( Z(S) \). Hence, any two distinct \( U_i \) and \( U_j \) have distinct coarsenings to extensions of \( Y/J(W) \). Because this is true for every valuation ring \( Y \) with \( V \subseteq Y \subseteq W \), the finest common coarsening of \( U_i \) and \( U_j \) must be the trivial valuation ring. Hence, \( U_i \) and \( U_j \) are independent valuation rings in \( Z(S) \), so their corresponding valuations \( u_i \) and \( u_j \) are independent. \( \square \)

The next proposition is the most difficult step in the proof of Th. 4.3. Here is the setup for the proposition: Let \( A \) be a central simple \( F \)-algebra with \( v \) defectless in \( A \) and \( j(v,A) = 2 \). Let \( W = V_P \), where \( P \) is the first jump prime ideal of \( v \) for \( A \), and let \( w \) be the valuation of \( W \). Assume that \( w \) is Henselian. Let \( \beta \) be a \( w \)-gauge of \( A \) with \( \Gamma_\beta \subseteq \mathbb{H}(\Gamma_w) \). Write \( A = \text{End}_D(M) \), where \( D \) is the division algebra associated to \( A \), and \( M \) is a finite-dimensional right \( D \)-vector space. Let \( y \) be the valuation on \( D \) extending \( w \) on \( F \), and let \( D = \mathbb{D}^y \). Let \( u \) be the residue valuation \( v/w \) on \( \mathbb{F}^w \) induced by \( v \), and let \( u_1, \ldots, u_r \) be the extensions of \( u \) to \( Z(D) \). Let field \( S \) be the separable closure of \( \mathbb{F}^w \) in \( Z(D) \). Recall from [JW, Prop. 1.7] that \( Z(D) \) is normal over \( \mathbb{F}^w \) and that \( S \) is abelian Galois over \( \mathbb{F}^w \). Let \( K \) be the decomposition field of \( u_1 |_S \) over \( u \) (so \( K \) is also the decomposition field of each \( u_i |_S \) over \( u \), as \( \mathcal{G}(S/\mathbb{F}^w) \) is abelian). For basic properties of decomposition fields, see [Ef, pp. 133-136]. Let \( L \) be a subfield of \( D \) that is an inertial lift of \( K \) over \( F \). That is, \( \mathbb{F}^y = K \) and \([L:F] = [K: \mathbb{F}^w]\). Such an \( L \) exists (and is unique up to isomorphism) because \( w \) is Henselian and \( K \) is separable over \( \mathbb{F}^w \), cf. [JW, p. 135]. Let \( v_1, \ldots, v_r \) be the extensions of \( v \) to \( L \). Let \( C \) be the centralizer \( C_D(L) \).

**Proposition 4.6.** In the situation just described, let \( \alpha \) be a \( v_1 \)-gauge on \( C = C_D(L) \) with \( \Gamma_\alpha \subseteq \mathbb{H}(\Gamma_v) \). Then, \( A \) has a \( v \)-gauge \( \varphi \) with \( w \)-coarsening \( \beta \) such that \( \Gamma_\varphi \subseteq \mathbb{H}(\Gamma_v) \) and

\[
\omega(\varphi) = r \omega(\alpha) \omega(\beta),
\]

where \( r \) is the number of extensions of \( u \) to \( Z(D) \). Moreover \( v_1 \) on \( L \) has extension number 1 in \( C \). Hence, \( \alpha \) exists and can be chosen with \( \omega(\alpha) = 1 \).

**Proof.** Let \( A' = \text{End}_C(M) \cong A \otimes_F L \), which contains \( A = \text{End}_D(M) \) canonically, as an \( F \)-subalgebra. We will build \( \varphi \) as the restriction to \( A \) of a suitable End-gauge on \( A' \).

Since \( K \) is Galois over \( \mathbb{F}^w \), its inertial lift \( L \) is Galois over \( F \), with \( \mathcal{G}(L/F) \cong \mathcal{G}(K/\mathbb{F}^w) \), cf. [JW, p. 135]. Let \( G = \mathcal{G}(L/F) \). Since \( \mathcal{G}(S/K) \) is the decomposition group for \( u_1 |_S \) over \( u \) and every valuation on \( S \) has a unique extension to the purely inseparable field extension \( Z(D) \) of \( S \), we have

\[
r = |\{\text{extensions of } u \text{ from } \mathbb{F}^w \text{ to } Z(D)\}| = |\{\text{extensions of } u \text{ from } \mathbb{F}^w \text{ to } S\}|
\]

\[
= |\mathcal{G}(S/\mathbb{F}^w); \mathcal{G}(S/K)| = |G| = [L:F].
\]

For each \( i \in \{1,2,\ldots,r\} \), since \( K \) is the decomposition field of each \( u_i |_S \) over \( u \), this \( u_i |_S \) is the unique extension of \( u_i |_K \) to \( S \), cf. [Ef, Prop. 15.1.2(b), p. 134]. Hence, \( u_1 |_K, \ldots, u_r |_K \) are all distinct valuations of \( K \). Let \( w_L = y|_L \), which is the unique extension of the Henselian valuation \( w \) to \( L \). The valuations \( v_i \) of \( L \) extending \( v \) on \( F \) are the composite valuations \( v_i = u_i |_K \ast w_L \). Therefore, there are \( r \) distinct \( v_i \), since the \( u_i |_K \) are distinct. The group \( G \) acts transitively on the \( v_i \) since it acts transitively on the \( u_i |_K \), and this action is simply transitive as \( |G| = r \). Since \([L:F] = r \) the
Fundamental Inequality shows that each $v_i$ is an immediate extension of $v$. Thus,

\[ \text{gr}_{v_i}(L) \cong \text{gr}_v(F) \quad \text{for } i = 1, \ldots, r. \]

Note that since $D$ is Brauer equivalent to $A$, $j(v, D) = j(v, A) = 2$ and $v$ has the same jump prime ideals for $D$ as for $A$; these are $j(W)$ and $j(V)$. The valuation ring $Y$ of $y$ on $D$ is a Dubrovin valuation ring of $D$ with $Z(Y) = Y \cap F = W$ and $Y/J(Y) = \overline{D}$. By Lemma 4.5 (with $D$ (resp. $Y$) for the $A$ (resp. $S$) of the lemma) the valuation rings $u_1, \ldots, u_r$ of $Z(\overline{D})$ are pairwise independent; hence, $u_1|K, \ldots, u_r|K$ are pairwise independent; hence, the finest common coarsening of any distinct $v_i$ and $v_j$ is $w_L$. Let $\Gamma = \mathbb{H}(\Gamma_v)$ and $\Lambda = \mathbb{H}(\Gamma_w)$. So each $\Gamma_v \subseteq \Gamma$, and for the valuation $y$ on $D$ extending $w$ we have $\Gamma_y \subseteq \Lambda$. Since $w$ is a coarsening of $v$ on $F$, there is an epimorphism $\Gamma_v \to \Gamma_w$. Let $\varepsilon : \Gamma \to \Lambda$ be the unique extension of this map to $\Gamma$; then, $\varepsilon$ is surjective. Let $\Gamma_{C,y}$ be the value group of $y|_C$.

By Skolem-Noether, for each $\rho \in G$ there is $d_\rho \in D^\times$ with $d_\rho \ell d_\rho^{-1} = \rho(\ell)$ for all $\ell \in L$. Since $d_\rho L d_\rho^{-1} = L$, we have $d_\rho C d_\rho^{-1} = C$. Moreover, 

\[ D = \bigoplus_{\rho \in G} d_\rho C = \bigoplus_{\rho \in G} C d_\rho. \]

(This is a standard fact about generalized crossed product algebras (see [T, Th. 1.3] or [JW, p. 156])). Since $y$ is a valuation on the division algebra $D$, there is a canonical epimorphism $\Gamma_y \to \mathcal{G}(Z(\overline{D})/\overline{F}^w)$ induced by conjugation by elements of $D^\times$ (see [JW, Prop. 1.7]). Let $\zeta : \Gamma_y \to \mathcal{G}(K/\overline{F}^w)$ be the composition of epimorphisms 

\[ \zeta : \Gamma_y \longrightarrow \mathcal{G}(Z(\overline{D})/\overline{F}^w) \longrightarrow \mathcal{G}(K/\overline{F}^w). \]

Since $K = \overline{L}$, we have $\Gamma_{C,y} \subseteq \ker(\zeta)$. Thus,

\[ r = [L : F] = [D : C] \geq |\Gamma_y : \Gamma_{C,y}| \geq |\Gamma_y : \ker(\zeta)| = |\mathcal{G}(K/\overline{F}^w)| = [K : \overline{F}^w] = r. \]

So, equality holds throughout, showing that $D$ is totally ramified over $C$ and $\ker(\zeta) = \Gamma_{C,y}$. Since $\zeta(y(d_\rho)) = \rho$ for all $\rho \in G$, the values $y(d_\rho)$ are distinct modulo $\Gamma_{C,y}$. Thus, there is a disjoint union decomposition

\[ \Gamma_y = \bigsqcup_{\rho \in G} y(d_\rho) + \Gamma_{C,y}. \]

Let $\delta_\rho = y(d_\rho) \in \Gamma_y$ for all $\rho \in G$. The disjoint union decomposition for $\Gamma_y$ in (4.5) shows that for any $d = \sum_{\rho \in G} d_\rho c_\rho \in D$ with all $c_\rho \in C$,

\[ y\left( \sum_{\rho \in G} d_\rho c_\rho \right) = \min_{\rho \in G} \left( y(d_\rho) + y(c_\rho) \right) = \min_{\rho \in G} \left( y(c_\rho) + \delta_\rho \right). \]

Take any $\rho \in G$, and define $v_\rho : L \to \Gamma \cup \{\infty\}$ by

\[ v_\rho(\ell) = v_1(d_\rho^{-1} \ell d_\rho) = v_1(\rho^{-1}(\ell)) \quad \text{for all } \ell \in L. \]

Then, $v_\rho$ is a valuation of $L$ extending $v$ on $F$. Since $G$ acts simply transitively on $\{v_1, \ldots, v_r\}$ it follows that $\{v_1, \ldots, v_r\} = \{v_\rho \mid \rho \in G\}$ and the $v_\rho$ are distinct for distinct choices of $\rho$.

Because $w$ on $F$ is Henselian, the $w$-gauge on $A = \text{End}_D(M)$ is an End-gauge by [TW1, Th. 3.1], i.e., $\beta = \text{End}(\theta)$ as in Ex. 1.1 for some $y$-norm $\theta : M \to \Lambda \cup \{\infty\}$. Let $(m_1, \ldots, m_n)$ be a splitting base for $\theta$ of the $D$-vector space $M$, and let

\[ \pi_j = \theta(m_j) \quad \text{for } j = 1, \ldots, n. \]
So \( \theta(\sum_{j=1}^{n} m_j d_j) = \min_{1 \leq j \leq n} (\pi_j + y(d_j)) \) for all \( d_j \in D \). Hence, for any \( c_{j\rho} \in C \) for \( j = 1, \ldots, n \) and \( \rho \in G \),

\[
\theta\left( \sum_{j=1}^{n} \sum_{\rho \in G} m_j d_{\rho} c_{j\rho} \right) = \min_{1 \leq j \leq n} \left( \pi_j + y(\sum_{\rho \in G} d_{\rho} c_{j\rho}) \right)
= \min_{1 \leq j \leq n} \left( \pi_j + \min_{\rho \in G} (\delta_{\rho} + y(c_{j\rho})) \right)
= \min_{1 \leq j \leq n, \rho \in G} (\pi_j + \delta_{\rho} + y(c_{j\rho})).
\]

This shows that \( (m_j d_{\rho})_{j=1, \rho \in G}^{n} \) is a splitting base for \( \theta \) as a \( y|_{C} \)-norm on \( M \). Since we can adjust the \( m_j \) by multiplication by any element of \( D^\times \), we may assume that \( \pi_i = \pi_j \) whenever \( \pi_i + \Gamma_y = \pi_j + \Gamma_y \).

For each \( \rho \in G \), pick \( \gamma_\rho \in \Gamma \) with

\[
\varepsilon(\gamma_\rho) = \delta_{\rho}.
\]

For \( j = 1, \ldots, n \), pick \( \mu_j \in \Gamma \) with

\[
\varepsilon(\mu_j) = \pi_j.
\]

Choose the \( \mu_j \) so that \( \mu_i = \mu_j \) whenever \( \pi_i = \pi_j \). We now use the \( v_1 \)-gauge \( \alpha \) on \( C \) to define an “\( \alpha \)-\( v_1 \)-norm” \( \eta: M \to \Gamma \cup \{ \infty \} \) as in Lemma 1.2. For all \( c_{j\rho} \in C \), set

\[
\eta\left( \sum_{j=1}^{n} \sum_{\rho \in G} m_j d_{\rho} c_{j\rho} \right) = \min_{1 \leq j \leq n, \rho \in G} (\mu_j + \gamma_\rho + \alpha(c_{j\rho})).
\]

Since \( \alpha \) is a \( v_1 \)-gauge on \( C \), its coarsening \( \varepsilon \circ \alpha \) is a \( w_L \)-gauge on \( C \) by [TW2, Prop. 4.3]. But since the valuation \( w_L \) on \( L \) extends to a valuation on \( C \), by [TW1, Cor. 3.2] that valuation is the only \( w_L \)-gauge on \( C \); hence, \( \varepsilon \circ \alpha = y|_{C} \). Thus,

\[
\varepsilon \circ \eta\left( \sum_{j=1}^{n} \sum_{\rho \in G} m_j d_{\rho} c_{j\rho} \right) = \min_{1 \leq j \leq n, \rho \in G} (\pi_j + \delta_{\rho} + y(c_{j\rho})) = \theta\left( \sum_{j=1}^{n} \sum_{\rho \in G} m_j d_{\rho} c_{j\rho} \right),
\]

i.e., \( \varepsilon \circ \eta = \theta \). Now, let \( \psi = \text{End}(\eta) \), the \( v_1 \)-\( \text{End} \)-gauge on \( A' = \text{End}_{C}(M) \) determined by \( \eta \), as in Lemma 1.2. So, for \( f \in A' \),

\[
\psi(f) = \min_{m \in M \setminus \{0\}} \eta(f(m)) - \eta(m) = \min_{1 \leq j \leq n, \rho \in G} (\eta(f(m_d_{\rho})) - \mu_j - \gamma_\rho).
\]

Let \( \varphi = \psi|_{A} \). We will show that \( \varphi \) is the desired \( v \)-gauge on \( A \).

We claim first that \( \varphi \) is a \( v \)-norm. For this, take any \( f \in A \) and write

\[
f(m_j) = \sum_{i=1}^{n} m_i d_{ij} = \sum_{i=1}^{n} \sum_{\sigma \in G} m_i d_{\sigma} c_{ij\sigma},
\]

where each \( d_{ij} \in D \) and each \( c_{ij\sigma} \in C \). For any \( \sigma, \rho \in G \), we have \( d_{\sigma}^{-1} d_{\rho} \) centralizes \( L \), so lies in \( C \). That is, \( d_{\sigma} d_{\rho} = d_{\sigma \rho} t_{\sigma, \rho} \), for some \( t_{\sigma, \rho} \in C^{\times} \). Then,

\[
\varphi(f) = \psi(f) = \min_{1 \leq j \leq n, \rho \in G} \left( \eta\left( \left( \sum_{i=1}^{n} \sum_{\sigma \in G} m_i d_{\sigma} c_{ij\sigma} \right) d_{\rho} \right) - \mu_j - \gamma_\rho \right)
= \min_{1 \leq j \leq n, \rho \in G} \left( \eta\left( \left( \sum_{i=1}^{n} \sum_{\sigma \in G} m_i d_{\sigma} (t_{\sigma, \rho} d_{\rho}^{-1} c_{ij\sigma} d_{\rho}) \right) \right) - \mu_j - \gamma_\rho \right)
= \min_{1 \leq j \leq n, \rho \in G} \left( \min_{1 \leq i \leq n, \sigma \in G} \left( \alpha(t_{\sigma, \rho} d_{\rho}^{-1} c_{ij\sigma} d_{\rho}) + \mu_i + \gamma_\sigma \right) - \mu_j - \gamma_\rho \right)
= \min_{1 \leq i, j \leq n; \rho, \sigma \in G} \left( \alpha(t_{\sigma, \rho} d_{\rho}^{-1} c_{ij\sigma} d_{\rho}) + \mu_i - \mu_j + \gamma_\sigma - \gamma_\rho \right).
\]
The choice of $D$-base $(m_j)_{j=1}^n$ of $M$ gives an isomorphism $A = \text{End}_D(M) \cong \mathbb{M}_n(D)$, which we use to interpret formula (4.7). In $A$ we have the “matrix units” $e_{ij}$ for $i, j \in \{1, \ldots, n\}$, defined by

$$e_{ij}(m_j) = m_i \quad \text{and} \quad e_{ij}(m_k) = 0 \quad \text{for} \quad k \neq j.$$ 

We also have an embedding $\lambda: D \to A$ given by $d \mapsto \lambda_d$, where $\lambda_d(m_j) = m_jd$ for all $j$. (So, $\lambda_d(m_jb) = m_jdb$ for all $b \in D$, so $\lambda_d \circ \lambda_b = \lambda_{db}$. Clearly $\lambda_d \circ e_{ij} = e_{ij} \circ \lambda_d$ for all $i, j$ and all $d \in D$.) This $\lambda$ corresponds to the diagonal embedding of $D$ in $\mathbb{M}_n(D)$. The $f$ in (4.6) above can be described as $f = \sum_{i=1}^n \sum_{j=1}^n \sum_{\sigma \in G} e_{ij} \circ \lambda_{d_\sigma} \circ \lambda_{c_{ij,\sigma}}$. Thus, formula (4.7) becomes

$$f = \sum_{i=1}^n \sum_{j=1}^n \sum_{\sigma \in G} e_{ij} \circ \lambda_{d_\sigma} \circ \lambda_{c_{ij,\sigma}}.$$ 

This holds for all choices of $c_{ij,\sigma} \in \mathcal{C}$. In particular, fixing $i, j, \sigma$, we have

$$\varphi(e_{ij} \circ \lambda_{d_\sigma} \circ \lambda_{c}) = \min_{\rho \in G \cap \mathcal{C}} \left(\alpha(t_{\sigma,\rho}d_{\rho}^{-1}c_{ij,\sigma}d_{\rho}) + \mu_i - \mu_j + \gamma_{\sigma} - \gamma_{\rho}\right).$$

This shows that

$$\varphi(h \cdot c) = \varphi(h \circ \lambda_{c}) = \min_{\rho \in G \cap \mathcal{C}} \left(\alpha(t_{\sigma,\rho}d_{\rho}^{-1}c_{ij,\sigma}d_{\rho}) + \mu_i - \mu_j + \gamma_{\sigma} - \gamma_{\rho}\right).$$

Therefore, to show that $\varphi$ is a $v$-norm on $A$, it suffices to show that its restriction to each 1-dimensional $\mathcal{C}$-subspace $(e_{ij} \circ \lambda_{d_\sigma}) \cdot C$ is a $v$-norm. For this, fix $i, j \in \{1, \ldots, n\}$ and $\sigma \in G$, let $h = e_{ij} \circ \lambda_{d_\sigma}$, and let $H = h \cdot C$. For $c \in \mathcal{C}$, we have

$$\varphi(h \cdot c) = \varphi(h \circ \lambda_{c}) = \min_{\rho \in G \cap \mathcal{C}} \left(\alpha(t_{\sigma,\rho}d_{\rho}^{-1}c_{ij,\sigma}d_{\rho}) + \mu_i - \mu_j + \gamma_{\sigma} - \gamma_{\rho}\right).$$

where

$$\alpha_{\rho}(c) = \alpha(t_{\sigma,\rho}d_{\rho}^{-1}c_{ij,\sigma}d_{\rho}) + \mu_i - \mu_j + \gamma_{\sigma} - \gamma_{\rho}.$$ 

Hence, $\alpha_{\rho}$ is a $v_{\rho}$-value function on $C$. The function $g: C \to C$ given by $c \mapsto t_{\sigma,\rho}d_{\rho}^{-1}cd_{\rho}$ is an $F$-vector space isomorphism with $\alpha_{\rho}(c) = \alpha(g(c)) + \tau_{\rho}$ for all $c \in C$. So, for any $\gamma \in \Gamma$, $g$ maps $C_{\geq \gamma}$ bijectively to $C_{\geq \gamma - \tau_{\rho}}$, and $C\gamma_{\geq \gamma}$ bijectively to $C_{\geq \gamma - \tau_{\rho}}$. Hence, $g$ induces a bijection $\text{gr}(g): \text{gr}_{v_{\rho}}(C) \to \text{gr}(C)(-\tau_{\rho})$, in the grade shift notation of (1.1); clearly $g$ is a graded $\text{gr}_{v_{\rho}}(F)$-vector space isomorphism. Since $\text{gr}_{v_{\rho}}(L) = \text{gr}_{v_{\rho}}(F) = \text{gr}_{v_{\rho}}(L)$ by (4.4), and $\alpha$ is a $v_{\rho}$-norm on $C$, we have

$$\dim_{\text{gr}(g)} \text{gr}_{\alpha_{\rho}}(C) = \dim_{\text{gr}_{v_{\rho}}(F)} \text{gr}_{\alpha_{\rho}}(C) = \dim_{\text{gr}_{v_{\rho}}(F)} \left(\text{gr}(C)(-\tau_{\rho})\right) = \dim_{\text{gr}_{v_{\rho}}(L)} \text{gr}(C) = \dim_{\text{gr}_{v_{\rho}}(L)} \text{gr}_{\alpha_{\rho}}(C) = \dim_{\text{gr}_{v_{\rho}}(L)} C.$$ 

Thus, $\alpha_{\rho}$ is a $v_{\rho}$-norm on $C$. Hence, $\varphi_{\rho}: H \to \Gamma \cup \{\infty\}$ given by $\varphi_{\rho}(h \cdot c) = \alpha_{\rho}(c)$ is a $v_{\rho}$-norm on the 1-dimensional $\mathcal{C}$-vector space $H$, for every $\rho \in G$. 

We work back from the $\varphi_\rho$ to $\varphi|_H$. Since $\varphi|_H = \min_{\rho \in G}(\varphi_\rho)$ by (4.10), there is a graded $\text{gr}_v(F)$-vector space monomorphism

$$\Phi: \text{gr}_\varphi(H) \hookrightarrow \bigoplus_{\rho \in G} \text{gr}_\varphi_\rho(H)$$

given by

$$\bar{b}^\varphi = b + H^\varphi_{b(\varphi)} \mapsto (\ldots, b + H^\varphi_{b(\varphi)}, \ldots) \quad \text{for all } b \in H \setminus \{0\}.$$ 

Once we verify that $\Phi$ is surjective, we will have

$$\text{dim}_{\text{gr}_v(F)} \text{gr}_\varphi(H) = \sum_{\rho \in G} \text{dim}_{\text{gr}_v(F)} \text{gr}_\varphi_\rho(H) = \sum_{\rho \in G} \text{dim}_{\text{gr}_{v(L)}} \text{gr}_\varphi_\rho(H) = \sum_{\rho \in G} \text{dim}_L H = |G| \cdot \text{dim}_L H = \text{dim}_F H.$$ 

Hence, $\varphi|_H$ is a $v$-norm on $H$.

For the surjectivity of $\Phi$, consider the $w_L$-coarsening of $\varphi_\rho$. We have seen that $\varepsilon \circ \alpha = y|_C$ as $w_L$-gauges on $C$. Also the equation $d_\rho d_\rho = d_{\sigma_\rho} d_{\sigma_\rho}$ yields for the valuation $y$,

$$y(t_{\sigma_\rho}) + y(d_{\sigma_\rho}) - y(d_\rho) = y(d_\rho) \quad \text{for all } \sigma, \rho \in G.$$ 

Hence, for $c \in C$,

$$\varepsilon \circ \varphi_\rho(h \cdot c) = \varepsilon \circ \alpha(t_{\sigma_\rho} d_\rho^{-1} c d_\rho) + \varepsilon(\tau_\rho) = y(t_{\sigma_\rho} d_\rho^{-1} c d_\rho) + \pi_i - \pi_j + y(d_{\sigma_\rho}) - y(d_\rho) = y(c) + \pi_i - \pi_j + y(d_\rho).$$

Thus, $\varepsilon \circ \varphi_\rho$ is the same for each $\rho \in G$. Since $w_L$ is the finest common coarsening of $v_\rho$ and $v_{\rho'}$ for all distinct $\rho, \rho' \in G$, as noted at the beginning of the proof, it follows by an argument just like that for surjectivity of $\Psi$ in the proof of Th. 2.8 that $\Phi$ is surjective. Hence $\varphi|_H$ is a $v$-norm on $H$, as noted above. Since this is true for each subspace $H = (\varepsilon_{ij} \circ \lambda_{d_\rho}) \cdot C$ in the splitting decomposition of $A$ for $\varphi$, this $\varphi$ must be a $v$-norm on $A$, as claimed.

To see that $\varphi$ is not only a $v$-norm but actually a $v$-gauge, observe that $\varphi = \psi|_A$ is surmultiplicative on $A$ since the $v_1$-gauge $\psi$ is surmultiplicative on all of $A'$. Moreover, the inclusion $A \hookrightarrow A'$ yields a canonical graded monomorphism $\iota: \text{gr}_\varphi(A) \hookrightarrow \text{gr}_\psi(A')$. Because $\varphi$ is a $v$-norm and $\psi$ is a $v_1$-norm and $\text{gr}_v(F) = \text{gr}_{v_1}(L)$, we have

$$[\text{gr}_\varphi(A) : \text{gr}_v(F)] = [A : F] = [A' : L] = [\text{gr}_\psi(A') : \text{gr}_{v_1}(L)] = [\text{gr}_\psi(A') : \text{gr}_v(F)].$$

Hence, $\iota$ is a graded isomorphism. Therefore, $\text{gr}_\varphi(A)$ is graded semisimple, since this is true for $\text{gr}_\psi(A')$, as $\psi$ is a gauge. Thus, $\varphi$ is a $v$-gauge on $A$.

Since $\psi$ and $\beta$ are End-gauges, we have, for all $f \in A$,

$$\varphi(f) = \psi(f) = \min_{m \in M \setminus \{0\}} (\eta(f(m)) - \eta(m)) \quad \text{and} \quad \beta(f) = \min_{m \in M \setminus \{0\}} (\phi(f(m)) - \phi(m)).$$

We observed earlier that $\varepsilon \circ \eta = \theta$. It follows that $\varepsilon \circ \varphi = \beta$, i.e., $\beta$ is the $w$-coarsening of $\varphi$. Also, from (4.8),

$$\Gamma_\varphi \subseteq \bigcup_{i,j=1}^n \bigcup_{\sigma, \rho \in G} \left( \mu_i - \mu_j + \gamma_{\sigma \rho} - \gamma_{\rho} + \Gamma_{\alpha} \right) \subseteq \Gamma = \mathbb{H}(\Gamma_v).$$

Since $\text{gr}_\varphi(A) \cong g_{\text{gr}} \text{gr}_\psi(A')$, we have $\omega(\varphi) = \omega(\psi)$. We compute $\omega(\psi)$. Partition $\{1, \ldots, n\}$ into a disjoint union $\bigcup_{j=1}^n S_j$ according to the coset of $\Gamma_y$ containing $\theta(m_j)$. That is, if $j \in S_i$, then $S_i = \{j' \in \{1, \ldots, n\} | \theta(m_{j'}) - \theta(m_j) \in \Gamma_y\}$. Recall that the $\pi_j = \theta(m_j)$ have been chosen so that
\[ \pi_j = \pi_{j'} \text{ if and only if } j \text{ and } j' \text{ lie in the same } S_\ell. \] Since \( \beta = \text{End}(\theta) \), by the comments with (1.6) the number of sets \( S_\ell \) equals \( \omega(\beta) \).

We use the \( S_\ell \) to decompose \( \mathfrak{gr}_\eta(M) \). Since \( (m_j \mathfrak{d}_\rho)^n_{j=1, \rho \in G} \) is the \( C \)-basis of \( M \) used in building \( \eta \) and \( m_j \mathfrak{d}_\rho = \bar{m}_j \mathfrak{d}_\rho \), by Lemma 1.2 \( (m_j \mathfrak{d}_\rho)^n_{j=1, \rho \in G} \) is a \( \mathfrak{gr}_\alpha(C) \)-basis of \( \mathfrak{gr}_\eta(M) \). For \( \ell = 1, \ldots, k \) and \( \rho \in G \), let

\[ N_{\ell\rho} = \bigoplus_{j \in S_\ell} \bar{m}_j \mathfrak{d}_\rho \mathfrak{gr}_\alpha(C). \]

So, \( \mathfrak{gr}_\eta(M) = \bigoplus_{\ell=1}^k \bigoplus_{\rho \in G} N_{\ell\rho} \), and each \( N_{\ell\rho} \) is a graded right \( \mathfrak{gr}_\alpha(C) \)-submodule of \( \mathfrak{gr}_\eta(M) \). We show that their grade sets do not overlap. If \( \gamma \in \Gamma_{N_{\ell\rho}} \), then \( \gamma = \text{deg}(\bar{m}_j \mathfrak{d}_\rho \mathfrak{c}) \) for some \( j \in S_\ell \) and \( c \in C^x \) with \( \bar{m}_j \mathfrak{d}_\rho \mathfrak{c} \neq 0 \). So, \( \gamma = \mu_j + \gamma_\rho + \alpha(c) \in \Gamma \). Then, as \( \varepsilon(\gamma_\rho) = \delta_\rho = y(d_\rho) \) and \( \varepsilon \circ \alpha = y|_C \),

\[ \varepsilon(\gamma) = \pi_j + \delta_\rho + y(c) = \pi_j + y(d_\rho \mathfrak{c}) \in \pi_j + \Gamma_y. \]

Now likewise let \( \gamma' \in \Gamma_{N_{\ell'\rho'}} \), say \( \gamma' = \text{deg}(\bar{m}_j \mathfrak{d}_\rho \mathfrak{c}') \) is \( \mu_j + \gamma_{\rho'} + \alpha'(c') \). If \( \gamma = \gamma' \), then \( \varepsilon(\gamma) = \varepsilon(\gamma') \), so \( \pi_j + \Gamma_y = \pi_j + \Gamma_{y'} \). Hence, \( S_\ell = S_{\ell'} \), so \( \ell = \ell' \) and \( \pi_j = \pi_{j'} \). Then the equality \( \varepsilon(\gamma) = \varepsilon(\gamma') \) yields \( y(d_\rho \mathfrak{c}) + y(c') = y(d_\rho \mathfrak{c}') \). Since the \( y(d_\rho) \) are distinct modulo \( \Gamma_{C,y} \), it follows that \( \rho = \rho' \).

Thus, \( N_{\ell\rho} = N_{\ell'\rho'} \) whenever their grade sets intersect. Hence, \( \Gamma_{\mathfrak{gr}_\alpha(M)} = \bigcup_{\ell=1}^k \bigcup_{\rho \in G} \Gamma_{N_{\ell\rho}} \), a disjoint union, so each homogeneous element of \( \mathfrak{gr}_\eta(M) \) lies in some \( N_{\ell\rho} \).

Let \( E = \text{End}_{\mathfrak{gr}_\alpha(C)}(\mathfrak{gr}_\eta(M)) \cong \mathfrak{gr}_\psi(A') \) by Lemma 1.2. If \( f \in E_0 \), then \( f \) is a degree-preserving map, so \( f \) must map each \( N_{\ell\rho} \) to itself, by the disjointness of the grade sets \( \Gamma_{N_{\ell\rho}} \). Thus,

\[ E_0 \cong \prod_{\ell=1}^k \prod_{\rho \in G} \left( \text{End}_{\mathfrak{gr}_\alpha(C)}(N_{\ell\rho}) \right)_0. \]

Since the \( \pi_j \) are the same for all \( j \) in \( S_\ell \), the \( \mu_j \) are likewise the same by hypothesis, hence the base elements \( \bar{m}_j \mathfrak{d}_\rho \) of \( N_{\ell\rho} \) all have the same degree \( \mu_j + \gamma_\rho \). Therefore, as graded \( \mathfrak{gr}_\alpha(C) \)-modules,

\[ N_{\ell\rho} \cong g \mathfrak{gr}_\alpha(C)^{|S_\ell||G|}(\mu_j + \gamma_\rho) \quad \text{for any} \ j \in S_\ell, \]

in the grade shift notation of (1.1). Hence,

\[ \text{End}_{\mathfrak{gr}_\alpha(C)}(N_{\ell\rho}) \cong g \text{End}_{\mathfrak{gr}_\alpha(C)}(\mathfrak{gr}_\alpha(C)^{|S_\ell||G|}) \cong g \mathbb{M}_{|S_\ell||G|}(\mathfrak{gr}_\alpha(C)). \]

So, in degree 0, \( (\text{End}_{\mathfrak{gr}_\alpha(C)}(N_{\rho}))_0 \cong \mathbb{M}_{|S_\ell||G|}(\mathfrak{gr}_\alpha(C)_0) \), and the number of its simple components coincides with the number of simple components of \( \mathfrak{gr}_\alpha(C) \), which is \( \omega(\alpha) \). So, from (4.11),

\[ \omega(\varphi) = \omega(\psi) = \text{number of simple components of } E_0 = k|G| \omega(\alpha) = \omega(\beta) \cdot \omega(\alpha). \]

To complete the proof, we show that \( \alpha \) can be chosen with \( \omega(\alpha) = 1 \). Let \( T \) be a Dubrovin valuation ring of \( D \) with \( Z(T) = V \). Then \( T \subset Y \) since the valuation ring \( Y \) of \( y \) on \( D \) is the unique Dubrovin valuation ring of \( D \) with center \( W \). Let \( \overline{T} = T/J(Y) \), which is a Dubrovin valuation ring of \( Y/J(Y) = \overline{D} \). The valuation of the valuation ring \( Z(T) \) is an extension to \( Z(\overline{D}) \) of \( u = v/y \). So this valuation is one of the \( u_i \); after renumbering if necessary, we may assume that it is \( u_1 \). Since jump prime ideals of \( u_1 \) for \( \overline{D} \) pull back to jump prime ideals of \( v \) for \( D \) strictly containing \( J(W) \) by [W1, Cor. E], and since by hypothesis \( J(V) \) is the only such jump prime ideal for \( D \), we have \( j(u_1, \overline{D}) = 1 \). Hence, by Prop. 3.3(iv) \( \overline{T} \) is integral over its center, which is the valuation ring \( U_1 \) of \( u_1 \). Now let \( B \) be a Dubrovin valuation ring of \( C \) with center \( Z(B) = V_1 \), the valuation ring of \( v_1 \) on \( L = Z(C) \). Then \( B \subset Y \cap C \), since the valuation ring \( Y \cap C \) is the unique Dubrovin valuation ring of \( C \) with center \( W_L \), the valuation ring of \( w_L \). Note that \( \overline{C}^y = \overline{D} \), since \( D \) is totally ramified.
over \(C \cong \mathbb{R}\). Let \(\tilde{B} = B/J(Y \cap C)\), which is a Dubrovin valuation ring of \(\overline{C}^V\), so of \(\overline{D}\). The valuation of the valuation ring \(Z(\tilde{B})\) restricts on \(\overline{T}' = K\) to \(v_1/u_L = u_1|_K\). Hence, \(Z(\tilde{B}) = U_1 = Z(\overline{T})\), as \(u_1\) is the unique extension of \(u_1|_K\) to \(Z(\overline{D})\). Because \(\tilde{B}\) and \(\overline{T}\) are Dubrovin valuation rings of \(\overline{D}\) with the same center, we have \(\tilde{B} \cong \overline{T}\). Therefore \(\tilde{B}\) is integral over \(U_1\), since this is true for \(\overline{T}\). But also \(U_1\) is integral over \(U_1 \cap K\), as \(u_1\) is the unique extension of \(u_1|_K\) to \(Z(\overline{D})\); hence, \(\tilde{B}\) is integral over \(U_1 \cap K\). Moreover, by [W2, p. 390] the valuation ring \(Y \cap C\) of the division ring \(C\) is integral over its center \(Y \cap L = W_L\). It follows by [MMU, Prop. 12.2, p. 70] that \(B\) is integral over \(V_1\). Let \(\alpha\) be the Morandi value function on \(C\) with \(R_\alpha = B\). To see that \(\alpha\) is a \(v_1\)-gauge we must check that \(v_1\) on \(L\) is defectless in \(C\). For this, note that by Th. A.1, there is an \(F_{h,v}\)-isomorphism

\[
L \otimes_F F_{h,v} \cong L_{h,v_1} \times \ldots \times L_{h,v_r}.
\]

Since \([L:F] = r\), each factor \(L_{h,v_i}\) must be 1-dimensional over \(F_{h,v}\), i.e., isomorphic to \(F_{h,v}\). Note that as \(C = C_{D}(L)\), the algebras \(D \otimes_F L\) and \(C\) are Brauer equivalent. Hence, \(C \otimes_L L_{h,v_1}\) is Brauer equivalent to \(D \otimes_F L_{h,v_1}\) \(\cong D \otimes_F F_{h,v}\). Since \(v\) on \(F\) is defectless in \(D\), \(v_1\) on \(L\) is defectless in \(C\).

It follows by Remark 3.7 that \(\alpha\) is a \(v_1\)-gauge on \(C\) with \(\omega(\alpha) = 1\). \(\square\)

There is a easier version of Prop. 4.6 for jump rank 1 that we will need later:

**Proposition 4.7.** Let \(A\) be a central simple \(F\)-algebra. Suppose the valuation \(v\) on \(F\) is defectless in \(A\). Let valuation \(w\) on \(F\) be a nontrivial coarsening of \(v\), and assume that \(w\) is Henselian. Suppose \(j(v,A) = 1\). Let \(\beta\) be a \(w\)-gauge on \(A\) with \(\Gamma_\beta \subseteq \mathbb{H}(\Gamma_w)\). Then, there is a \(v\)-gauge \(\varphi\) on \(A\) with \(w\)-coarsening \(\beta\) such that \(\omega(\varphi) = \omega(\beta)\) and \(\Gamma_\varphi \subseteq \mathbb{H}(\Gamma_v)\).

**Proof.** View \(A = \text{End}_D(M)\) where \(D\) is a division ring and \(M\) is a finite-dimensional right \(D\)-vector space. Since \(w\) is Henselian, it has a unique extension to a valuation \(y\) on \(D\). Moreover, by [TW1, Th. 3.1], \(\beta\) is an End-gauge, as in Ex. 1.1, say \(\beta = \text{End}(\theta)\) for some \(y\)-norm \(\theta\) on \(M\). Let \((m_1, \ldots, m_n)\) be a splitting base of \(M\) for \(\theta\), and let \(\pi_j = \theta(m_j)\) for \(j = 1, \ldots, n\). Let \(\Lambda = \mathbb{H}(\Gamma_w)\) and \(\Gamma = \mathbb{H}(\Gamma_v)\), and \(\varepsilon: \Lambda \to \Gamma\) the epimorphism induced by the canonical map \(\Gamma_v \to \Gamma_w\). From (1.8), \(\Gamma_\beta = \Gamma_\beta(A) = \bigcup_{1 \leq j \leq n} \pi_i - \pi_j + \Gamma_w\). By changing \(\theta\) by replacing each \(\pi_j\) by \(\pi_j - \pi_1\) (which does not change \(\text{End}(\theta)\)) we may assume that each \(\pi_j \in \Gamma_\beta \subseteq \mathbb{H}(\Gamma_w)\). Also, by adjusting the \(m_j\) by multiples in \(D^\times\), we may assume that \(\pi_i = \pi_j\) whenever \(\pi_i + \Gamma_y = \pi_j + \Gamma_y\). So, the number of distinct \(\pi_i\) equals the number of cosets of \(\Gamma_y\) in \(\Gamma_\beta\). This number equals \(\omega(\beta)\) by (1.6).

Because \(j(v,A) = 1\), we have \(\text{ind}(A \otimes_F F_{h,v}) = \text{ind}(A \otimes_F F_{h,w})\) where \(F_{h,v}\) (resp. \(F_{h,w}\)) is a Henselization of \(F\) with respect to \(v\) (resp. \(w\)). Since \(F_{h,w} = F\) as \(w\) is assumed Henselian, it follows that \(\text{ind}(A \otimes_F F_{h,v}) = \text{ind}(A)\). Hence, by Morandi’s theorem [M1, Th. 3] the valuation \(v\) on \(F\) extends to a valuation \(z\) on \(D\). Pick any \(\mu_1, \ldots, \mu_n \in \Gamma\) with \(\varepsilon(\mu_j) = \pi_j\) for all \(j\) and \(\mu_j = \mu_i\) whenever \(\pi_j = \pi_i\). Let \(\eta\) be the \(z\)-norm on \(M\) given by

\[
\eta(\sum_{j=1}^{n} m_j d_j) = \min_{1 \leq j \leq n} \left( \mu_j + z(d_j) \right) \quad \text{for all } d_1, \ldots, d_n \in D.
\]

Let \(\varphi = \text{End}(\eta)\), which is a \(v\)-gauge on \(A\) since \(v\) is defectless in \(A\) (see Ex. 1.1). Then \(\varepsilon \circ \varphi = \beta\) since \(\varepsilon \circ \eta = \theta\). Also, if \(\pi_i + \Gamma_y = \pi_j + \Gamma_y\), then \(\pi_i = \pi_j\), so \(\mu_i = \mu_j\) by the choice of the \(\mu_i\), so \(\mu_i + \Gamma_z = \mu_j + \Gamma_z\). Thus, by (1.6),

\[
\omega(\varphi) = |\{\text{cosets of } \Gamma_z \text{ in } \Gamma_\varphi\}| = |\{\text{cosets of } \Gamma_y \text{ in } \Gamma_\theta\}| = \omega(\beta).
\]
Because each $\mu_j \in \Gamma = \mathbb{H}(\Gamma_v)$ and $\Gamma_z \subseteq \mathbb{H}(\Gamma_v)$, we have by (1.8)

$$\Gamma_{\varphi} = \bigcup_{i,j=1}^n (\mu_i - \mu_j) + \Gamma_z \subseteq \mathbb{H}(\Gamma_v).$$

\[\square\]

**Proof of Th. 4.3 (Central simple case).** Suppose $A$ is central simple. We argue by induction on the jump rank $j(v, A)$. If $j(v, A) = 0$, then $v$ is the trivial valuation on $F$, and the trivial gauge on $A$ is a minimal $v$-gauge. If $j(v, A) = 1$, then $\xi_{v, [A]} = 1$ by Prop. 3.3(iv); so for any Dubrovin valuation ring $B$ of $A$ with $Z(B) = V$, $B$ is integral over $V$. Let $\alpha$ be the associated Morandi value function of $B$. Then $\Gamma_{\alpha} = \Gamma_B$ by (3.13), and $\Gamma_B \subseteq \mathbb{H}(\Gamma_v)$ by (3.2); hence $\Gamma_{\alpha} \subseteq \mathbb{H}(\Gamma_v)$. Since $v$ is defectless in $A$, by Remark 3.7 $\alpha$ is a $v$-gauge with $\omega(\alpha) = 1$, so $\alpha$ is a minimal $v$-gauge.

Now suppose $j(v, A) = n > 1$. Let $P$ be the $(n-1)$-st jump prime ideal of $v$ for $A$, and let $W = V_P$, with associated valuation $w$. Then $w$ is defectless in $A$ since $v$ is defectless in $A$, by Prop. 1.8. Since $j(w, A) = n - 1$, by induction there is a minimal $w$-gauge $\beta$ on $A$ with $\Gamma_{\beta} \subseteq \mathbb{H}(\Gamma_w)$. Let $(F', w_h)$ be the Henselization of $(F, w)$, let $v'$ be the valuation of $F'$ refining $w_h$ and restricting to $v$ on $F$. Let $A' = A \otimes_F F'$, which is a central simple $F'$-algebra, and let $\beta' = \beta \otimes w_h$, which is a $w_h$-gauge on $A'$ with $\text{gr}_{\beta'}(A') \cong_{\beta} \text{gr}_{\beta}(A)$ by [TW1, Cor. 1.26]. Note however that $\beta'$ need not be a minimal gauge even though $\beta$ is minimal.

We claim that $j(v', A') = 2$. To see this, let valuation $y$ on $F'$ be any coarsening of $v'$, and let $z = y|_F$, which is a coarsening of $v$. If $y = w_h$ or $y$ is coarser than $w_h$, then $y$ is Henselian, as $w_h$ is Henselian (see [EP, Cor. 4.1.4, p. 90]). Then the Henselization $F'_{h,y} = F'$, so of course $\text{ind}(A' \otimes_{F'} F'_{h,y}) = \text{ind}(A')$. Suppose instead that $y$ is properly finer than $w_h$, so $z$ is a refinement of $w$. We show that then the Henselizations $(F'_{h,y}, y_h)$ and $(F_{h,z}, z_h)$ are isomorphic. For this, note first that since the $w$-coarsening $w_1$ of $z_h$ in $F_{h,z}$ is Henselian and restricts to $w$ in $F$, there is an $F$-homomorphism $\eta_1: F' \to F_{h,z}$ with $w_h = w_1 \circ \eta_1$. The Henselian valuation $z_h$ on $F_{h,z}$ pulls back to a valuation $z'$ on $F'$ that refines $w_h$ with $z'|_F = z = y|_F$. Hence, $z' = y$. Because $z_h$ is Henselian and pulls back to $y$, there is an $F$-monomorphism $\eta_2: F'_{h,y} \to F_{h,z}$ with $y_h = z_h \circ \eta_2$. But also since $y_h$ is Henselian with $y_h|_F = y|_F = z$, there is an $F$-monomorphism $\eta_3: F_{h,z} \to F'_{h,y}$ with $z_h = y_h \circ \eta_3$. Thus, $\eta_2 \circ \eta_3$ is an $F$-monomorphism $F'_{h,z} \to F_{h,z}$ with $z_h = z_h \circ (\eta_2 \circ \eta_3)$. By the uniqueness in the universal property for the Henselization (recalled in the Appendix below), we must have $\eta_2 \circ \eta_3 = \text{id}_{F'_{h,z}}$, so $\eta_3$ is an isomorphism $(F_{h,z}, z_h) \cong (F'_{h,y}, y_h)$. In particular, $(F_{h,v'}, v_h) \cong (F'_{h,v'}, v_h')$. So, as there are no jump prime ideals of $v$ for $A$ between $P = J(W)$ and $J(V)$,

$$\text{ind}(A' \otimes_{F'} F'_{h,y}) = \text{ind}(A \otimes_F F_{h,v}) = \text{ind}(A \otimes_F F_{h,v'}) = \text{ind}(A' \otimes_{F'} F'_{h,v'})$$

this value is strictly smaller than $\text{ind}(A') = \text{ind}(A \otimes_F F')$ since $P$ is a jump prime ideal. Thus, $j(v', A') = 2$, as claimed. The calculation also shows that $w_h$ is the first jump valuation for $v'$ in $A'$. Note also that $v'$ is defectless in $A'$, since this depends on the defectlessness of the associated division algebra of $A' \otimes_{F'} F'_{h,v'}$ re $v_h$; but $A' \otimes_{F'} F'_{h,v'} \cong A \otimes_F F_{h,v}$ and $v$ is defectless in $A$. Thus, the hypotheses of Prop. 4.6 are satisfied for the field $F'$ with valuations $v'$ and $w_h$ and central simple $F'$-algebra $A'$ with $w_h$-gauge $\beta'$.

By Prop. 4.6, $A'$ has a $v'$-gauge $\varphi$ whose $w_h$-coarsening is $\beta'$ with $\omega(\varphi) = r\omega(\beta')$ and $\Gamma_{\varphi} \subseteq \mathbb{H}(\Gamma_{v'})$. Let $D'$ be the associated division algebra of $A'$, let $w_{D'}$ be the valuation on $D'$ extending $w_h$ on $F'$, and let $\overline{D} = \overline{D}_{w_{D'}}$. Let $u'$ be the residue valuation $v'/w_h$ on $\overline{D}_{w_{D'}}$ determined by $v'$. The integer $r$ in the formula for $\omega(\varphi)$ is the number of extensions of $u'$ to $Z(\overline{D})$. Hence, by (3.12), $r = \ell_{V;W}$. Let
\[ \alpha = \varphi|_{A}. \] By Lemma 4.4, \( \alpha \) is a \( v \)-gauge on \( A \) with \( w \)-coarsening \( \beta \), and \( \text{gr}_\alpha(A) \cong_{\beta} \text{gr}_\varphi(A') \). Hence, \( (4.12) \)
\[ \omega(\alpha) = \omega(\varphi) = r \omega(\beta) = \ell_{V,W} \omega(\beta) = \ell_{V,W} \xi_{W|A}. \]
The last equality in (4.12) holds since \( \beta \) is a minimal gauge. Since \( \xi_{V|A} \geq \xi_{W|A} \ell_{V,W} \) by (3.10), it follows from (4.12) that \( \omega(\alpha) \leq \xi_{V|A} \). But we always have \( \omega(\alpha) \geq \xi_{V|A} \) by Th. 3.5; so \( \omega(\alpha) = \xi_{V|A} \), showing that \( \alpha \) is a minimal gauge. Furthermore, \( \Gamma_\alpha = \Gamma_\varphi \subseteq \mathbb{H}(\Gamma_\nu) = \mathbb{H}(\Gamma_v) \). This completes the proof of the central simple case of Th. 4.3. The rest of the proof of the theorem will be given after Lemma 4.9 below. \( \square \)

**Proposition 4.8.** Let \( A \) be a central simple \( F \)-algebra with \( v \) defectless in \( A \). Let \( w \) be a valuation on \( F \) that is coarser than \( v \), and let \( \beta \) be a minimal \( w \)-gauge on \( A \) with \( \Gamma_\beta \subseteq \mathbb{H}(\Gamma_w) \). Then there is a minimal \( v \)-gauge \( \alpha \) with \( w \)-coarsening \( \beta \) and \( \Gamma_\alpha \subseteq \mathbb{H}(\Gamma_v) \).

**Proof.** Let \( V \) (resp. \( W \)) be the valuation ring of \( v \) (resp. \( w \)). We argue by induction on the jump rank \( j(v, A) \). Clearly, \( j(w, A) \leq j(v, A) \). If \( j(w, A) = 0 \), then \( w \) is the trivial valuation on \( F \) and \( \beta \) is the trivial \( w \)-gauge on \( A \). By the case of Th. 4.3 already proved, there is a minimal \( v \)-gauge \( \alpha \) on \( A \); then the trivial gauge \( \beta \) is a coarsening of \( \alpha \). Thus, we may assume that \( 1 \leq j(w, A) \leq j(v, A) \).

Suppose \( j(v, A) = 1 \). Then \( j(w, A) = 1 \), so \( \xi_{W|A} = 1 \) by Prop. 3.3(iv). Hence, \( \omega(\beta) = 1 \) because \( \beta \) is assumed to be a minimal gauge. Therefore, by Remark 3.7 \( R_3 \) is a Dubrovin valuation ring integral over its center, which is \( W \), and \( \beta \) is the Morandi value function of \( R_3 \). Let \( B \) be any Dubrovin valuation ring of \( A \) with \( Z(B) = V \) and \( B \subseteq R_3 \). Such a \( B \) is obtainable as the inverse image in \( R_3 \) of a Dubrovin valuation ring of \( R_3/J(R_3) \) whose center is a valuation ring \( U \) of \( Z(R_3/J(R_3)) \) satisfying \( U \cap (W/J(W)) = V/J(W) \). Since \( j(v, A) = 1 \), by Prop. 3.3(iv) \( B \) is integral over \( V \). Let \( \alpha \) be the Morandi value function of \( B \). Then \( \Gamma_\alpha \subseteq \mathbb{H}(\Gamma_v) \) (see (3.2)). Since \( v \) is assumed defectless in \( A \), by Remark 3.7 \( \alpha \) is a \( v \)-gauge with \( \omega(\alpha) = 1 \); so \( \alpha \) is minimal gauge. Because \( R_\alpha \subseteq R_3 \) and these Dubrovin valuation rings determine their associated gauges, \( \beta \) is a coarsening of \( \alpha \).

Now suppose \( j(v, A) = n > 1 \). Let \( y \) be the \((n - 1)\)-st jump valuation of \( v \) for \( A \). Suppose first that \( w \) is coarser than \( y \) or \( w = y \). Then, as \( j(y, A) = n - 1 \) and \( y \) is defectless in \( A \) since \( v \) is defectless in \( A \), by induction there is a minimal \( y \)-gauge \( \eta \) with \( \Gamma_\eta \subseteq \mathbb{H}(\Gamma_y) \) such that \( \beta \) is the \( w \)-coarsening of \( \eta \). The proof of Th. 4.3 (central simple case) shows that there is a minimal \( v \)-gauge \( \alpha \) with \( \Gamma_\alpha \subseteq \mathbb{H}(\Gamma_v) \) and \( y \)-coarsening \( \eta \). The \( w \)-coarsening of \( \alpha \) is then the \( w \)-coarsening of \( \eta \), which is \( \beta \).

Suppose instead that \( w \) is properly finer than \( y \). Let \( (F', w_h) \) be the Henselization of \( (F, w) \), and let \( v' \) be the valuation on \( F' \) refining \( w_h \) and restricting to \( v \) on \( F \). Let \( \beta' = \beta \otimes w_h \), which is a \( w_h \)-gauge on \( A' = A \otimes_F F' \), but not necessarily minimal, with \( \Gamma_{\beta'} = \Gamma_{\beta} \subseteq \mathbb{H}(\Gamma_w) = \mathbb{H}(\Gamma_{w_h}) \). Moreover, we claim that \( j(v', A') = 1 \). To see this, suppose \( z \) is any nontrivial valuation of \( F' \) with \( z \) coarser than \( v' \) or \( z = v' \). If \( z \) is coarser than \( w_h \) or \( z = w_h \), then \( z \) is Henselian, so the Henselization \( F'_{h,z} = F' \) and \( \text{ind}(A' \otimes_{F'} F'_{h,z}) = \text{ind}(A') \). If \( z \) is finer than \( w_h \), then as in the proof of Th. 4.3 (central simple case) above \( F'_{h,z} = F_{h,z|F}, \) the Henselization of \( F \) re \( z|_F \). Since there are no jump valuations between \( v \) and \( w \), so none between \( z|_F \) and \( v \),
\[ \text{ind}(A' \otimes_{F'} F'_{h,z}) = \text{ind}(A \otimes_{F} F_{h,z|F}) = \text{ind}(A \otimes_{F} F_{h,w}) = \text{ind}(A \otimes_{F} F') = \text{ind}(A'). \]
Since the indices are the same for all \( z \), \( j(v', A') = 1 \), as claimed.

By Prop. 4.7, applied to \( F', v', w', \beta' \), there is a \( v' \)-gauge \( \alpha' \) of \( A' \) with \( w_h \)-coarsening \( \beta' \), such that \( \omega(\alpha') = \omega(\beta') \), and \( \Gamma_{\alpha'} \subseteq \mathbb{H}(\Gamma_{\nu'}) = \mathbb{H}(\Gamma_v) \). Let \( \alpha = \alpha'|_{A} \). By Lemma 4.4, \( \alpha \) is a \( v \)-gauge on \( A \).
with \( \text{gr}_\alpha(A) \cong \text{gr}_{\alpha'}(A') \), so
\[
\omega(\alpha) = \omega(\alpha') = \omega(\beta') = \omega(\beta).
\]

Since \( \beta \) is a minimal gauge and \( \xi_{V_i[A]} \geq \xi_{W_i[A]} \), \( \alpha \) must also be a minimal gauge. Also, \( \Gamma_\alpha = \Gamma_{\alpha'} \subseteq \mathbb{H}(\Gamma_v) \).

Finally, since \( \beta' \) is the \( w_\beta \)-coarsening of \( \alpha' \), the \( w \)-coarsening of \( \alpha = \alpha'|_A \) is \( \beta'|_A = \beta \). \( \Box \)

Let \( F \subseteq L \) be fields with \( [L:F] \leq \infty \), and let \( v \) be a nontrivial valuation of \( F \) with valuation ring \( V \). For each nonzero prime ideal \( P \) of \( V \), let \( s(P) \) be the number of valuation rings of \( L \) extending the valuation ring \( V_P \) of \( F \). Clearly, \( 1 \leq s(P) \leq [L:F] \). Also for prime ideals \( P \subseteq P' \) we have \( s(P) \leq s(P') \). Moreover, if \( P = \bigcup_{j \in J} P_j \) for prime ideals \( P_j \), \( j \in J \), then \( s(P) = \max_{j \in J} s(P_j) \). We call \( P \) a splitting prime ideal of \( v \) in \( L \) if \( s(P) < s(P') \) for all prime ideals \( P' \supseteq P \). Define the splitting rank of \( v \) in \( L \) to be
\[
\text{srk}(v,L) = \text{the number of splitting prime ideals of } V \text{ in } L.
\]
Note that the maximal ideal \( J(V) \) is always a splitting prime ideal of \( V \) in \( L \), so \( \text{srk}(v,L) \geq 1 \). If \( P \) is a splitting prime ideal, we call the valuation of \( V_P \) a splitting valuation of \( V \) in \( L \). If \( v \) is the trivial valuation on \( F \), define \( \text{srk}(v,L) = 0 \).

**Lemma 4.9.** With the notation above, let \( v_1 \) and \( v_2 \) be two different extensions of \( v \) to \( L \). Then either \( v_1 \) and \( v_2 \) are independent or the finest common coarsening \( v_{12} \) of \( v_1 \) and \( v_2 \) restricts to a splitting valuation for \( v \) in \( L \).

**Proof.** We argue by induction on \( n = \text{srk}(v,L) \). If \( n = 1 \), let \( w \) be the trivial valuation on \( F \). If \( n > 1 \), let \( w \) be the \((n-1)\)-st splitting valuation of \( F \) for \( v \) in \( L \). Let \( y \) be a valuation of \( F \) coarser than \( v \) and strictly finer than \( w \). Because there are no splitting valuations between \( y \) and \( v \), \( y \) must have the same number of extensions to \( L \) as \( v \). Hence, \( v_1 \) and \( v_2 \) have distinct \( y \)-coarsenings. If \( n = 1 \), this shows that \( v_{12} \) must be the trivial valuation, i.e., \( v_1 \) and \( v_2 \) are independent valuations. If instead \( n > 1 \), this shows that \( v_{12}|_F \) either equals or is coarser than \( w \). Since \( w \) is a splitting valuation, it suffices to consider the case when \( v_{12}|_F \) is strictly coarser than \( w \). Then the \( w \)-coarsenings \( w_1 \) of \( v_1 \) and \( w_2 \) of \( v_2 \) are distinct. Hence, the finest common coarsening \( w_{12} \) of \( w_1 \) and \( w_2 \) is coarser than or equals to \( v_{12} \). But \( v_{12} \) is coarser than each of \( w_1 \) and \( w_2 \), so coarser than or equal to \( v_{12} \). Hence, \( v_{12} = w_{12} \). Since \( \text{srk}(w,L) = n - 1 \), the conclusion of the lemma holds for \( w_1 \) and \( w_2 \) by induction; hence it also holds for \( v_1 \) and \( v_2 \). \( \Box \)

**Proof of Th. 4.3 (completed).** It suffices to prove the theorem for the simple components of a semisimple \( F \)-algebra. So, assume \( A \) is simple. Let \( L = Z(A) \), so \( [L:F] \leq \infty \). Let \( v_1, \ldots, v_r \) be the valuations on \( L \) extending \( v \) on \( F \) and let \( V_i \) be the valuation ring of \( v_i \). The argument is by induction on \( n = \text{srk}(v,L) \). If \( n = 0 \), then \( v \) is the trivial valuation, and the trivial gauge on \( A \) is a minimal \( v \)-gauge. Assume now that \( n = 1 \). Lemma 4.9 then shows that the \( v_i \) are pairwise independent. By the central simple case of Th. 4.3 proved above, for each \( i \) there is a minimal \( v_i \)-gauge \( \alpha_i \) of \( A \) with \( \Gamma_{\alpha_i} \subseteq \mathbb{H}(\Gamma_{v_i}) = \mathbb{H}(\Gamma_v) \). (Note that each \( v_i \) is defectless in \( L \), by Prop. 1.6.) Let \( \alpha = \min(\alpha_1, \ldots, \alpha_r) \). Because the \( v_i \) are pairwise independent, Cor. 2.10 shows that \( \alpha \) is a \( v \)-gauge on \( A \). Since each \( \alpha_i \) is a minimal \( v_i \)-gauge, \( \alpha \) is a minimal \( v \)-gauge (see the comments in Def. 4.2). Also, \( \Gamma_\alpha \subseteq \bigcup_{i=1}^r \Gamma_{\alpha_i} \subseteq \mathbb{H}(\Gamma_v) \).

Now assume that \( n > 1 \). Let \( P_1 \not\subseteq P_2 \not\subseteq \cdots \not\subseteq P_{n-1} \not\subseteq P_n \) be the splitting prime ideals of \( V \) in \( L \), and let \( W = V_{P_n-1} \) with associated valuation \( w \). Because \( v \) is defectless in \( A \), \( w \) is defectless in \( A \), by Prop. 1.8. Since \( \text{srk}(w,L) = n - 1 \), by induction there is a minimal \( w \)-gauge \( \beta \) on \( A \) with \( \Gamma_\beta \subseteq \mathbb{H}(\Gamma_w) \). Let \( w_1, \ldots, w_\ell \) be the valuations of \( L \) extending \( w \), and for each \( j \) let \( \beta_j \) be the
w_j-component of \( \beta \). The construction of the \( \beta_j \) in Th. 2.2 shows that \( \Gamma_\beta \subseteq \Gamma_\beta \subseteq \mathbb{H}(\Gamma_w) \) for each \( j \). By Th. 2.8, \( \beta_j \) and \( \beta_k \) have the same \( w_{jk} \)-coarsening for all \( j, k \in \{1, \ldots, \ell \} \). Since \( \beta \) is a minimal \( w \)-gauge and \( \beta = \min(\beta_1, \ldots, \beta_\ell) \), each \( \beta_j \) is a minimal \( w_j \)-gauge.

For each \( i \in \{1, \ldots, r \} \) let \( j(i) \in \{1, \ldots, \ell \} \) be the index such that \( w_{j(i)} \) is the \( w \)-coarsening of \( v_i \). For each \( i \), Prop. 4.8 applied to the valuations \( v_i \) and \( w_{j(i)} \) on \( L \) shows that there is a minimal \( v_i \)-gauge \( \alpha_i \) with \( w_{j(i)} \)-coarsening \( \beta_{j(i)} \) and \( \Gamma_{\alpha_i} \subseteq \mathbb{H}(\Gamma_{v_i}) = \mathbb{H}(\Gamma_v) \). Let \( \alpha = \min(\alpha_1, \ldots, \alpha_r) \). To see that \( \alpha \) is a \( v \)-gauge, we must check that the \( \alpha_i \) satisfy the compatibility condition of Th. 2.8. For this, take any distinct \( i, k \in \{1, \ldots, r \} \). If \( j(i) = j(k) \), then \( w_{j(i)} = w_{j(k)} \), which is coarser than both \( v_i \) and \( v_k \), so coarser than (or equal to) \( v_{ik} \). Since \( v_{ik} \mid F \) is a splitting valuation of \( v \) in \( L \), the choice of \( v_{ik} \) implies \( v_{ik} = w_{j(i)} = w_{j(k)} \). So, the \( v_{ik} \)-coarsening of \( \alpha_i \) is \( \beta_{j(i)} \), which is the same as the \( v_{ik} \)-coarsening \( \beta_{j(k)} \) of \( \alpha_k \). Suppose now instead that \( j(i) \neq j(k) \), so \( w_{j(i)} \neq w_{j(k)} \). The proof of Lemma 4.9 shows that in this case \( v_{ik} = w_{j(i)j(k)} \). Hence, the \( v_{ik} \)-coarsening of \( \alpha_i \) is the \( w_{j(i)j(k)} \)-coarsening of the \( w_{j(i)} \)-coarsening \( \beta_{j(i)} \) of \( \alpha_i \). By Th. 2.8, this coincides with the \( w_{j(i)j(k)} \)-coarsening of the \( w_{j(k)} \)-coarsening \( \beta_{j(k)} \) of \( \alpha_k \). Thus, \( \alpha_i \) and \( \alpha_k \) have the same \( v_{ik} \)-coarsening. Since the compatibility condition thus holds in all cases, Th. 2.8 shows that \( \alpha \) is a \( v \)-gauge. It is a minimal gauge since each \( \alpha_i \) is a minimal gauge. Moreover, \( \Gamma_\alpha \subseteq \bigcup_{i=1}^r \Gamma_{\alpha_i} \subseteq \mathbb{H}(\Gamma_v) \).

5. An example

In this section we construct an example of a central simple algebra with multiple non-isomorphic minimal gauges all having the same gauge ring.

Example 5.1. Let \( L \) be a field with \( \text{char}(L) \neq 2 \), let \( x \) be transcendental over \( L \), and let \( F = L(x)((y)) \), the Laurent series field in one variable over \( L(x) \). Let \( w \) be the complete discrete (so Henselian) \( y \)-adic valuation on \( F \), with \( \Gamma_w = \mathbb{Z} \) and \( F^w = L(x) \). Let \( W \) be the power series ring \( L(x)[[x]] \), which is the valuation ring of \( w \). Let \( v \) be the rank 2 valuation on \( F \) that is the composite of \( w \) with the discrete \( x \)-adic valuation on \( F^w \). Equivalently, \( v \) is the valuation on \( F \) obtained by restriction from the standard rank 2 Henselian valuation on \( L((x))((y)) \). Thus, \( F^v = L \), \( \Gamma_v = \mathbb{Z} \times \mathbb{Z} \) with right-to-left lexicographic ordering, \( v(x) = (1,0) \), \( v(y) = (0,1) \), and \( \text{gr}_v(F) = L[X, X^{-1}, Y, Y^{-1}] \), a twice iterated Laurent polynomial ring, where \( X = \bar{x} \) and \( Y = \bar{y} \). Let \( V \) be the valuation ring of \( v \). Note that \( w \) is the rank 1 coarsening of \( v \), and the epimorphism \( \varepsilon : \Gamma_v \rightarrow \Gamma_w \) given by \( v(c) \mapsto w(c) \) for \( c \in F^X \) is the projection \( (\ell, m) \mapsto m \). Since we will be working primarily with \( v \), we write \( F \) for \( F^v \).

Let

\[
D = (1 + x, y/F),
\]

a quaternion algebra over \( F \) with its standard \( F \)-base \( (1, i, j, k) \), where \( i^2 = 1 + x \), \( j^2 = y \), and \( k = ij = -ji \). Because \( w(1 + x) = 0 \) and \( \overline{1 + x}^w \) is not a square in \( F^w \), the valuation \( w \) has a unique and inertial extension to the field \( F(i) \). Therefore, every norm from \( F(i) \) to \( F \) has \( w \)-value in \( 2\Gamma_w \). Since \( w(y) = 1 \), \( y \) is not such a norm. Hence, \( D \) is a division algebra. The Henselian valuation \( w \) on \( F \) therefore has a unique extension to a valuation \( \beta \) on \( D \), with \( \beta(1) = \beta(\bar{i}) = 0 \) and \( \beta(j) = \beta(k) = \frac{1}{2} \). Since \( \beta(j) \notin \Gamma_w \), one can see that \( D \) is totally ramified over \( F(i) \) for \( \beta \), while \( F(i) \) is inertial over \( F \); indeed, \( \Gamma_\beta = \frac{1}{2}\mathbb{Z} \) and \( \overline{F(i)}^\beta = F(i)(\sqrt{1+x}) \), and for all \( a, b, c, d \in F \),

\[
(5.1) \quad \beta(a + bi + cj + dk) = \min \left( \beta(a + bi), \beta(cj + dk) \right) = \min \left( w(a), w(b), w(c) + \frac{1}{2}, w(d) + \frac{1}{2} \right).
\]
Let $K = F(t)$, where $t^2 = 1 + x$. Thus, $K$ is a quadratic extension field of $F$, and since $1 + x = \overline{1}$ in $\overline{F}$, $v$ has two extensions to $K$ that are distinguished by whether $\overline{t} = \overline{1}$ or $\overline{t}$ in $\overline{K}$. Let $v'$ denote the extension of $v$ to $K$ with $\overline{t} = 1$. Then, $\overline{K} = \overline{F} = L$ and $\Gamma_{v'} = \Gamma_v = Z \times Z$, so $\text{gr}(K) = \text{gr}(F)$. Note that as $x = (t - 1)(t + 1)$ and $v'(t + 1) = 0$, we have $v'(t - 1) = v(x) = (1, 0)$. The rank 1 coarsening of $v'$ is the unique, unramified, extension $v'$ of $w$ to $K$, with $\overline{K} = L(x)(\sqrt{1 + x})$ and $\Gamma_{w'} = Z$. Also, $K$ is a splitting field of $D$, as $K \cong F(i)$, which is a maximal subfield of $D$. Explicitly, let $S = \mathbb{M}_2(K)$, and view $D$ as an $F$-subalgebra of $S$ by identifying

$$1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad i = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \quad j = \left( \begin{array}{cc} 0 & \gamma \\ 1 & 0 \end{array} \right), \quad k = \left( \begin{array}{cc} 0 & ty \\ 1 & 0 \end{array} \right).$$

Give $\mathbb{Q} \times \mathbb{Q}$ the right-to-left lexicographic ordering. Fix any $\gamma \in \mathbb{Q}$ with $0 < \gamma < \frac{1}{2}$, and let $\delta = (\gamma, \frac{1}{2}) \in \mathbb{Q} \times \mathbb{Q}$. Let $\alpha'$ be the $v'$-gauge on $S$ given by

$$\alpha'(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}) = \min \left( \begin{array}{c} v'(p), v'(q) - \delta, v'(r) + \delta, v'(s) \end{array} \right).$$

Indeed, let $M = K$-span$\{\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\}$, and identify $S = \text{End}_K(M)$. Then $\alpha'$ is the $v'$-gauge $\text{End}(\eta)$ as in Ex. 1.1, where $\eta: M \rightarrow \mathbb{Q} \times \mathbb{Q} \cup \{\infty\}$ is the $v'$-norm on $M$ given by $\eta(\begin{smallmatrix} p \\ q \end{smallmatrix}) = \min (v'(p), v'(q) + \delta)$. Thus,

$$\text{gr}_{\alpha'}(S) = \text{End}_{\text{gr}(K)}(\text{gr}(M)) \cong_{g} \mathbb{M}_2(\text{gr}(K))(0, \delta),$$

in the notation of (1.7). Let $\alpha = \alpha'|_D$, which is a surmultiplicative $v$-value function on $D$ since the gauge $\alpha'$ on $S$ is surmultiplicative. While $\alpha'$ is a $v'$-gauge, we must still verify that $\alpha$ is a $v$-gauge. For this, note that for any $z = a + bi + cj + dk \in D$ with $a, b, c, d \in F$, we have $z = \left( \begin{array}{c} a+bt \\ c-dt \end{array} \right)$ in $S$. Thus, as $v(y) = (0, 1)$,

$$\alpha(z) = \min \left( \begin{array}{c} v'(a + bt), v'((c + dt)y) - \delta, v'(c - dt) + \delta, v'(a - bt) \end{array} \right) = \min \left( \begin{array}{c} v'(a + bt), v'(a - bt), v'(c + dt) + (-\gamma, \frac{1}{2}), v'(c - dt) + (\gamma, \frac{1}{2}) \end{array} \right).$$

So, $\alpha(1) = \alpha(i) = 0$ and

$$\alpha(j) = \alpha(k) = \min \left( \begin{array}{c} (-\gamma, \frac{1}{2}), (\gamma, \frac{1}{2}) \end{array} \right) = (-\gamma, \frac{1}{2}).$$

Since $\alpha'(1-t) = (1, 0)$, we have

$$\alpha(j - k) = \min \left( \begin{array}{c} (-\gamma, \frac{1}{2}) + (1, 0), (\gamma, \frac{1}{2}) \end{array} \right) = (\gamma, \frac{1}{2}),$$

as $\gamma < \frac{1}{2}$. So, in $\text{gr}_{\alpha}(D) \subseteq \text{gr}_{\alpha'}(S)$,

$$\overline{1+i} = \left( \begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array} \right) \in D_0, \quad \overline{1-i} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array} \right) \in D_0,$n

$$\overline{j} = k = \left( \begin{array}{cc} 0 & \gamma \\ 0 & 0 \end{array} \right) \in D(-\gamma, \frac{1}{2}), \quad \text{and} \quad \overline{j-k} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array} \right) \in D(\gamma, \frac{1}{2}).$$

Since $1 + i, 1 - i, j, j - k$ have images in $\text{gr}_{\alpha}(D)$ which are clearly $\text{gr}(F)$-independent, they comprise a splitting base of $\alpha$ as a $v$-value function; this shows that $\alpha$ is a $v$-norm on $D$. Moreover, $[\text{gr}_{\alpha}(D): \text{gr}(F)] = 4 = [\text{gr}_{\alpha'}(S): \text{gr}(K)] = [\text{gr}_{\alpha'}(S): \text{gr}(F)]$, since $\text{gr}(K) = \text{gr}(F)$. Hence, $\text{gr}_{\alpha}(D) = \text{gr}_{\alpha'}(S)$, which is graded simple. Thus, $\alpha$ is a $v$-gauge on $D$. Note that

$$\Gamma_{\alpha} = \Gamma_{\alpha'} = \mathbb{Z}^2 \cup (\delta + \mathbb{Z}^2) \cup (-\delta + \mathbb{Z}^2).$$

Also, $D_0 = S_0 = L \times L$, so $\omega(\alpha) = 2$.

From (5.2), we have

$$R_{\alpha} = \{a + bi + cj + dk \in D \mid v'(a + bt) \geq 0, v'(a - bt) \geq 0,$n

$$v'(c + dt) \geq (\gamma, -\frac{1}{2}), v'(c - dt) \geq (-\gamma, -\frac{1}{2})\}.$$
Let $v'(c + dt) = (\ell, m) \in \Gamma_{v'} = \mathbb{Z} \times \mathbb{Z}$. Then, $v'(c + dt) \geq (\gamma, -\frac{1}{2})$ if and only if $m \geq 0$, i.e., $w'(c + dt) \geq 0$ where $w'$ is the rank 1 coarsening of $v'$. Likewise, $v'(c - dt) \geq (\gamma, -\frac{1}{2})$ if and only if $w'(c - dt) \geq 0$. Therefore, each of the infinitely many choices of $\gamma$ yields the same gauge ring for the associated $v$-gauge $\alpha$. But different choices of $\gamma$ yield non isomorphic gauges since the gauges have different value sets (see (5.3)). Thus, the gauge ring $R_{\alpha}$ does not determine $\alpha$.

We still must show that $\alpha$ is a minimal gauge. We will show as well that $R_{\alpha}$ is an intersection of two total valuation rings $B_1, B_2$, which are the only Dubrovin valuation rings of $D$ with center $V$.

Since $\text{char}(F) \neq 2$ and $v'(t) = 0$, we have $\min (v'(a + bt), v'(a - bt)) = \min (v(a), v(b))$. Similarly, $\min(w'(c + dt), w'(c - dt)) = \min (w(c), w(d))$. Thus, the description of $R_{\alpha}$ simplifies to

$$R_{\alpha} = \{a + bi + cj + dk \mid a, b, c, d \in V\}.$$

The ring $R_{\alpha}$ lies in the invariant valuation ring $R_{\beta}$ of the valuation $\beta$ on $D$ extending $w$ on $F$. From (5.1), we have

$$R_{\beta} = \{a + bi + cj + dk \mid a, b, c, d \in W\} \quad \text{and} \quad J(R_{\beta}) = \{a + bi + cj + dk \mid a, b \in J(W), c, d \in W\}.$$



Let

$$\pi: R_{\beta} \rightarrow R_{\beta}/J(R_{\beta}) = L(x)(\sqrt{1 + x})$$

be the canonical projection. Let $U_1, U_2$ be the two valuation rings of $L(x)(\sqrt{1 + x})$ extending the valuation ring $U$ of the $x$-adic valuation on $L(x)$. The commutative valuation rings $U_{1/2}$ for $\ell = 1, 2$ are Dubrovin valuation rings. Since the invariant valuation ring $R_{\beta}$ is a Dubrovin valuation ring, the pullback rings $B_\ell = \pi^{-1}(U_{\ell})$ for $\ell = 1, 2$ are Dubrovin valuation rings of $D$ with center $V$ and $\tilde{B}_\ell = B_\ell/J(R_{\beta}) = U_{\ell}$. Moreover, since $R_{\beta}$ and $U_{\ell}$ are valuation rings, it is easy to check that $B_\ell$ is a total valuation ring, i.e., for any $d \in D \setminus \{0\}$, $d \in B_\ell$ or $d^{-1} \in B_\ell$. Let $B = B_1$. Since $B/J(B) \cong U_1/J(U_1)$, which is a field, $t_B = 1$. Also, if $(F_h, v_h)$ is a Henselization of $(F, v)$, then $1 + x \in F_h^2$, since $1 + x = 1$ in $F_h^{w} = F^w$. Therefore, $F_h$ splits $D$, which shows that $D \otimes_F F_h \cong M_2(F_h)$, hence $n_B = 2$. Thus,

$$\xi_{V, [D]} = n_B/t_B = 2 = \omega(\alpha),$$

showing that $\alpha$ is a minimal gauge. Another way to calculate the extension number $\xi_{V, [D]}$ is by using (3.10) with $R_{\beta}$ for $S$: Since the valuation rings $R_{\beta}$ and $U_1 = \tilde{B}$ have extension number 1 and the residue valuation ring $U = V/J(W)$ has two extensions to $R_{\beta}/J(R_{\beta})$,

$$\xi_{V, [D]} = \xi_{R_{\beta}, [D]} \xi_{V, W} \xi_{U_1, [R_{\beta}/J(R_{\beta})]} = 1 \cdot 2 \cdot 1 = 2.$$

Any inner automorphism $\iota$ of $D$ maps the invariant valuation ring $R_{\beta}$ to itself. The automorphism induced by $\iota$ on $R_{\beta}/J(R_{\beta})$ is one of the two $F^w = L(x)$-automorphisms of $L(x)(\sqrt{1 + x})$, so it either preserves or interchanges $U_1$ and $U_2$. Hence, the set of conjugates of $B_1$ in $D$ is $\{B_1, B_2\}$. The $B_\ell$ are therefore the only Dubrovin valuation rings of $D$ with center $V$. Since the $B_\ell$ are not integral over $V$ (as $\xi_{V, [D]} \neq 1$) the only possible Gräter ring of $D$ with center $V$ is $B_1 \cap B_2$. Since $R_{\alpha}$ is the gauge ring of a minimal $v$-gauge, it is a Gräter ring with center $V$ by Th. 3.9. Hence, $R_{\alpha} = B_1 \cap B_2$. (This equality can also be verified directly after showing that $V + V\iota$ is the integral closure of $V$ in $F(i)$ and that the $B_\ell \cap F(i)$ are the valuation rings of $F(i)$ extending $V$ in $F$.)

Note that our example required a valuation of rank at least 2. For if $v$ is a rank 1 valuation on a field $F$ and $A$ is a central simple $F$-algebra, then for the valuation ring $V$ of $v$, we have $j(v, A) = 1$, so $\xi_{V, [A]} = 1$ by Prop. 3.3(iv). Hence, for any minimal $v$-gauge $\alpha$ on $A$, we have $\omega(\alpha) = 1$, so by
Remark 3.7 the gauge ring $R_\alpha$ is a Dubrovin valuation ring integral over its center $V$, and $\alpha$ is the Morandi value function determined by $R_\alpha$.

**Appendix A. Tensor product and Henselization**

It is well known that if $L/F$ is a finite degree field extension, $v$ is a discrete (rank 1) valuation on $F$, and $v_1, \ldots, v_r$ are all the extensions of $v$ to $L$, then $L \otimes_F \hat{F} \cong \prod_{i=1}^r \hat{L}_i$, where $\hat{F}$ (resp. $\hat{L}_i$) is the completion of $F$ (resp. $L$) with respect to $v$ (resp. $v_i$) (see [B, Ch. VI, §8, no. 6, Prop. 11]). In this appendix we prove an analogous result for valuations of arbitrary rank, replacing the completion by the Henselization. For separable field extensions, this result is implicit in [E, Th. 17.17, p. 135]. We give a full proof here, since this result is essential for many of the arguments in this paper.

Let $F$ be a field with a valuation $v$. A Henselization of $(F, v)$ is a valued field extension $(F_h, v_h)$ of $(F, v)$ such that $v_h$ is Henselian and for any extension $(K, w)$ of $(F, v)$ with $w$ Henselian there is a unique $F$-homomorphism $\eta: (F_h, v_h) \to (K, w)$ such that $v_h = w \circ \eta$. We thus refer to $(\eta(F_h), w|_{\eta(F_h)})$ as the Henselization of $(F, v)$ within $(K, w)$. It is clear from the definition that a Henselization of $(F, v)$ is unique up to unique isomorphism. Thus, we sometimes say that $(F_h, v_h)$ is “the Henselization” of $(F, v)$. A proof of the existence of a Henselization can be found in [EP, Th. 5.2.2, p. 121].

**Theorem A.1.** Let $F$ be a field with a valuation $v$. Let $K$ be a finite-degree field extension of $F$, and let $v_1, \ldots, v_r$ be all the extensions of $v$ to $K$. Let $(F_h, v_h)$ be a Henselization of $(F, v)$, and let $(K_{h,i}, v_{i,h})$ be a Henselization of $(K_i, v_i)$ for $i = 1, \ldots, r$. Then,

$$K \otimes_F F_h \cong K_{h,1} \times \cdots \times K_{h,r}.$$

The proof will use the following two lemmas:

**Lemma A.2.** Let $F \subseteq N$ be fields with $N$ Galois over $F$ (possibly of infinite degree), and let $G = G(N/F)$. Let $K$ and $E$ be subfields of $N$ containing $F$, with $[K:F] < \infty$. Let $H = G(N/K) \subseteq G$ and $Z = G(N/E) \subseteq G$. Let $\tau_1, \ldots, \tau_r$ be representatives of the distinct $Z$-$H$ double cosets of $G$. (So, $G = \bigsqcup_{i=1}^r Z \tau_i H$, a disjoint union.) Then

$$K \otimes_F E \cong \tau_1(K) \cdot E \times \cdots \times \tau_r(K) \cdot E.$$

**Proof.** Since $K$ is separable over $F$, we have $K = F(a)$ for some $a$. Let $f$ be the minimal polynomial of $a$ over $F$. Then $f$ splits over $N$, as $N$ is normal over $F$, say $f = (X - a_1) \cdots (X - a_n) \in N[X]$ where the $a_i$ are distinct and $a_1 = a$. Let $A = \{a_1, \ldots, a_n\}$. Let $f = g_1 \cdots g_r$ be the irreducible factorization of $f$ in $E[X]$, and fix a root $b_i$ of $g_i$ for $i = 1, \ldots, r$. The Galois group $G$ acts transitively on $A$, but $A$ decomposes into $r$ disjoint $Z$-orbits, $A = \bigsqcup_{i=1}^r B_i$, where

$$B_i = \{Z \cdot b_i \mid \text{roots of } g_i \text{ in } N\}.$$

For each $i$, choose $\tau_i' \in G$ with $\tau_i'(a) = b_i$. Then, as $H = \{\sigma \in G \mid \sigma(a) = a\}$, we have

$$Z \tau_i' H = \{\sigma \in G \mid \sigma(a) \in B_i\}, \quad \text{for } i = 1, \ldots, r.$$

Thus, $Z \tau_1'H, \ldots, Z \tau_r'H$ are all the distinct $Z$-$H$ double cosets in $G$. Moreover, we may assume that the double coset representatives $\tau_i'$ coincide with the $\tau_i$ of the lemma, by replacing $b_i$ by $\tau_i(a)$. 

Since \( \gcd(g_i, g_j) = 1 \) for \( i \neq j \), the Chinese Remainder Theorem yields
\[
K \otimes_F E \cong F[X]/(f) \otimes_F E \cong E[X]/fE[X]
\]
\[
\cong E[X]/((g_1) \ldots (g_r)) \cong E[X]/(g_1) \times \ldots \times E[X]/(g_r)
\]
\[
\cong E(b_1) \times \ldots \times E(b_r) \cong \tau_1(K) \cdot E \times \ldots \times \tau_r(K) \cdot E.
\]
\[\square\]

**Lemma A.3.** Let \((F_h, v_h)\) be a Henselization of the valued field \((F, v)\). Let \(K\) be any extension field of \(F\) lying in the algebraic closure of \(F_h\), and let \(w\) be the unique extension of \(v_h\) to the compositum \(K \cdot F_h\). Then, \((K \cdot F_h, w)\) is a Henselization of \((K, w|_K)\).

**Proof.** The valuation \(w\) on \(K \cdot F_h\) is Henselian since \(v_h\) is Henselian. Let \((K_h, w_h)\) be a Henselization of \((K, w|_K)\). The universal property shows that \((K_h, w_h)\) embeds in \((K \cdot F_h, w)\). Thus, we may assume that \(K \subseteq K_h \subseteq K \cdot F_h\) and \(w_h = w|_{K_h}\). So, \(w_h|_F = w|_F = v\). Since \((F_h, v_h)\) is the Henselization of \((F, v)\) within \((K \cdot F_h, w)\), it is also the Henselization of \((F, v)\) within \((K_h, w_h)\), by the uniqueness in the universal property for the Henselization; so, \(F_h \subseteq K_h\). Since also \(K \subseteq K_h\), we have \(K \cdot F_h \subseteq K_h \subseteq K \cdot F_h\). Hence, \(K_h = K \cdot F_h\) and \(w = w_h\), showing that \((K \cdot F_h, w)\) is a Henselization of \((K, w|_K)\).
\[\square\]

**Proof of Th. A.1:** Assume first that \(K\) is separable over \(F\). Let \(F_{\text{sep}}\) be a separable closure of \(F\) containing \(F_h\), and let \(v_{\text{sep}}\) be the unique valuation on \(F_{\text{sep}}\) extending the Henselian valuation \(v_h\). Let \(G = \mathcal{G}(F_{\text{sep}}/F), H = \mathcal{G}(F_{\text{sep}}/K) \subseteq G, \) and \(Z = \mathcal{G}(F_{\text{sep}}/F_h) \subseteq G\). By [EP, Th. 5.2.2, p. 121] and the universal property of the Henselization, \(F_h\) is the decomposition field for \(v_{\text{sep}}\) over \(v\), so \(Z\) is the decomposition subgroup of \(G\), i.e.,
\[
Z = \{\sigma \in G \mid v_{\text{sep}} \circ \sigma = v_{\text{sep}}\}.
\]

Let \(\Omega\) be the set of all valuations on \(F_{\text{sep}}\) extending \(F\). Then, \(G\) acts transitively on \(\Omega\) (see [EP, Th. 3.2.14, p. 68]), while the distinct \(H\)-orbits of \(\Omega\) are \(\Omega_1, \ldots, \Omega_r\), where \(\Omega_i = \{w \in \Omega \mid w|_K = v_i\}\). For \(i = 1, \ldots, r\), choose \(\tau_i \in G\) with \(v_{\text{sep}} \circ \tau_i|_K = v_i\). Then,
\[
\{\sigma \in G \mid v_{\text{sep}} \circ \sigma|_K = v_i\} = \{\sigma \in G \mid v_{\text{sep}} \circ \sigma \in \Omega_i\} = Z \tau_i H.
\]

So, \(G = \bigsqcup_{i=1}^r Z \tau_i H\) is the disjoint \(Z\)-\(H\) double coset decomposition of \(G\). We now apply Lemma A.2 with \(N = F_{\text{sep}}\) and \(E = F_h\). (So, the \(K, H\) and \(Z\) of the lemma are the \(K, H\) and \(Z\) here.) By the lemma,

\[\text{(A.1) } K \otimes_F F_h \cong \prod_{i=1}^r \tau_i(K) \cdot F_h \cong \prod_{i=1}^r K \cdot \tau_i^{-1}(F_h),\]

where the second isomorphism follows by applying \(\tau_i^{-1}\) to the \(i\)-th factor. Note that the \(F\)-isomorphism \(\tau_i^{-1}\) maps \((F_h, v_h)\) to \((\tau_i^{-1}(F_h), v_h \circ \tau_i)\). Hence \((\tau_i^{-1}(F_h), v_h \circ \tau_i)\) is a Henselization of \((F, v)\). The unique extension of the Henselian valuation \(v_h \circ \tau_i\) to \(K \cdot \tau_i^{-1}(F_h)\) must be \(v_{\text{sep}} \circ \tau_i|_{K \cdot \tau_i^{-1}(F_h)}\), whose restriction to \(K\) is \(v_i\) by the choice of \(\tau_i\). Therefore, \(K \cdot \tau_i^{-1}(F_h) \cong K_{h,i}\) by Lemma A.3. The theorem (for \(K\) separable over \(F\)) then follows from (A.1).

If \(K\) is not separable over \(F\), let \(S\) be the separable closure of \(F\) in \(K\), and let \(y_i = v_i|_S\) for \(i = 1, \ldots, r\). Then, \(y_1, \ldots, y_r\) are all the extensions of \(v\) to \(S\). Since valuations extend uniquely from \(S\) to its purely inseparable extension \(K\), we have \(y_i \neq y_j\) for \(i \neq j\). As we just proved,
\( S \otimes_F F_h \cong \prod_{i=1}^r S_{h,i} \), where \((S_{h,i}, y_{i,h})\) is a Henselization of \((S, y_i)\) in \( F_{\text{alg}}\), the algebraic closure of \(F\). Therefore,

\[
K \otimes_F F_h \cong K \otimes_S (S \otimes_F F_h) \cong \prod_{i=1}^r K \otimes_S S_{h,i}.
\]

Because \(K\) is purely inseparable over \(S\) while \(S_{h,i}\) is separable over \(S\), the fields \(K\) and \(S_{h,i}\) are linearly disjoint over \(S\); so, \(K \otimes_S S_{h,i}\) is a field, which is isomorphic to the compositum \(K \cdot S_{h,i}\) in \(F_{\text{alg}}\). The Henselian valuation \(y_{i,h}\) on \(S_{h,i}\) has a unique extension to the field \(K \otimes_S S_{h,i}\) whose restriction to \(K\) is the unique extension of \(y_i\) to \(K\), which is \(v_i\). By Lemma A.3, \(K \otimes_S S_{h,i}\) is a Henselization of \(K\) with respect to \(v_i\). The theorem thus follows from (A.2).

\[\square\]

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