The capacity of hybrid quantum memory

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The general stable quantum memory unit is a hybrid consisting of a classical digit with a quantum digit (qudit) assigned to each classical state. The shape of the memory is the vector of sizes of these qudits, which may differ. We determine when $N$ copies of a quantum memory $\mathcal{A}$ embed in $N(1+\alpha(1))$ copies of another quantum memory $\mathcal{B}$. This relationship captures the notion that $\mathcal{B}$ is as at least as useful as $\mathcal{A}$ for all purposes in the bulk limit. We show that the embeddings exist if and only if for all $p \geq 1$, the $p$-norm of the shape of $\mathcal{A}$ does not exceed the $p$-norm of the shape of $\mathcal{B}$. The log of the $p$-norm of the shape of $\mathcal{A}$ can be interpreted as the maximum of $S(p)+H(p)/p$ (quantum entropy plus discounted classical entropy) taken over all mixed states $\rho$ on $\mathcal{A}$. We also establish a noiseless coding theorem that justifies these entropies. The noiseless coding theorem and the bulk embedding theorem together say that either $\mathcal{A}$ blindly bulk-encodes into $\mathcal{B}$ with perfect fidelity, or $\mathcal{A}$ admits a state that does not visibly bulk-encode into $\mathcal{B}$ with high fidelity.

In conclusion, the utility of a hybrid quantum memory is determined by its simultaneous capacity for classical and quantum entropy, which is not a finite list of numbers, but rather a convex region in the classical-quantum entropy plane.

1. INTRODUCTION

Many questions in quantum information theory involve both quantum and classical information. The usual computational model for such dual information is independent quantum and classical memory. The measurement algebra of a combined memory consisting of an $a$-state qudit and a $b$-state classical digit is

$$\mathcal{M}_a \otimes \mathbb{C}^b = \bigoplus_{k=1}^b \mathcal{M}_a,$$

where $\mathcal{M}_a$ is the set of $a \times a$ matrices. But this is not the most general possible hybrid of classical and quantum memory. Rather the measurement algebra of a finite memory could be any direct sum of matrix algebras of possibly different dimensions:

$$\mathcal{A} \cong \bigoplus_{k=1}^n \mathcal{M}_{\lambda_k}.$$

The partition (i.e., non-negative integral vector) $\lambda = \lambda(\mathcal{A})$ is a list of the dimensions of the matrix algebras called the shape of the memory $\mathcal{A}$. Section 2 discusses why this is a reasonably general quantum memory model.

For example, the simplest hybrid memory is a hybrid trit, with shape $(2,1)$. It consists of matrices of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$ 

This memory models a three-state system in which one state is observed by the environment but the other two remain coherent relative to each other. It is easy to compare the capacity of the hybrid trit to any other quantum memory: It is between a qubit and a qutrit, more than a classical trit, less than any larger memory that contains a qubit, and neither more nor less than a classical digit with at least 4 states.

It turns out that there is more than one notion by which one memory unit has more capacity than another. (Atypically, all such notions are equivalent for the hybrid trit.) The strictest relevant relationship between memories is given by algebra embeddings. If $\mathcal{A} \hookrightarrow \mathcal{B}$ is an algebra embedding (which need not be unit-preserving, or unital), then the memory $\mathcal{B}$ can simulate the memory $\mathcal{A}$. In other language, an algebra embedding is a blind, perfect-fidelity decoding. Section 3 also explains that although other blind, perfect-fidelity encodings are possible, any such encoding can be replaced by an algebra embedding. As Section 3 explains, the question of whether $\mathcal{A}$ embeds in $\mathcal{B}$ is a computable (but NP-hard) bin-packing problem.

In this article we will consider a more relaxed comparison, namely whether many copies of $\mathcal{A}$ embed in slightly more copies of $\mathcal{B}$. More precisely we say that $\mathcal{A}$ bulk-embeds in $\mathcal{B}$, or $\mathcal{A} \rightarrow_b \mathcal{B}$, if for every rational $\epsilon > 0$, there exists an $N$ such that

$$\mathcal{A}^\otimes N \hookrightarrow \mathcal{B}^\otimes (N+\epsilon).$$

If $\mathcal{A}$ bulk-embeds in $\mathcal{B}$, there is no reason to pay more for $\mathcal{A}$ than $\mathcal{B}$ when buying large quantities of the two memories with equal performance. Our first main result is a characterization of when $\mathcal{A}$ bulk-embeds in $\mathcal{B}$:

**Theorem 1.1.** If $\mathcal{A}$ and $\mathcal{B}$ are two hybrid memories, then $\mathcal{A} \rightarrow_b \mathcal{B}$ if and only if

$$||\lambda(\mathcal{A})||_p \leq ||\lambda(\mathcal{B})||_p$$

for all $p \in [1,\infty]$.

One direction of Theorem 1.1 is straightforward. The $p$-norm of a partition $\lambda$ is defined as

$$||\lambda||_p = \left(\sum_k \lambda_k^p\right)^{1/p}.$$
It is easy to check that the $p$-norm is multiplicative:
\[ ||\lambda(\mathcal{A} \otimes \mathcal{B})||_p = ||\lambda(\mathcal{A})||_p ||\lambda(\mathcal{B})||_p,\]
for any pair of memories $\mathcal{A}$ and $\mathcal{B}$. On the other hand the
bin-packing model implies that if $\mathcal{A}$ embeds in $\mathcal{B}$, then
\[ ||\lambda(\mathcal{A})||_p \leq ||\lambda(\mathcal{B})||_p.\]
It follows that this inequality also holds when $\mathcal{A}$ bulk-embeds in $\mathcal{B}$. The proof of the other direction of Theorem 1.2 is the
topic of Section 3.

The $p$-norm has an interesting information-theoretic interpretation. In Section 4 we will define the classical entropy $H(\rho)$ and the quantum entropy $S(\rho)$ of a state $\rho$ of a quantum memory $\mathcal{A}$. Their definitions are justified by a capacity estimate, Theorem 1.2 and by a noless noiseless coding theorem, Theorem 1.3.

**Theorem 1.2.** Every state $\rho$ of a memory $\mathcal{A}$ satisfies inequality
\[ \frac{H_{\mathcal{A}}(\rho)}{p} + S_{\mathcal{A}}(\rho) \leq \log ||\lambda(\mathcal{A})||_p,\]
where $\rho$ has classical entropy $H(\rho)$ and quantum entropy $S_{\mathcal{A}}(\rho)$. For each $p \geq 1$ there exists a $\rho$ that achieves equality. Any non-negative pair $(H, S)$ satisfying the inequality for all $p$ can be expressed as
\[ (H, S) = (H_{\mathcal{A}}(\rho) + t, S_{\mathcal{A}}(\rho) - t)\]
for some $\rho$ and some $t \in [0, 1]$.

![Figure 1: The capacity region of a memory $\mathcal{A}$ with shape (2,1,1), and its 3-norm bounding line.](image)

Note that the three most common $p$-norms are also significant for quantum information theory. The logarithm of the 1-norm, $\log ||\lambda(\mathcal{A})||_1$, is the purely classical capacity of $\mathcal{A}$. The logarithm of the $\infty$-norm, $\log ||\lambda(\mathcal{A})||_\infty$, is the purely quantum capacity. And the logarithm of the 2-norm,
\[ \log ||\lambda(\mathcal{A})||_2 = \log \frac{\log \dim \mathcal{A}}{2},\]
is half of the dense coding capacity of $\mathcal{A}$.

**Theorem 1.2** implies that the set of possible pairs
\[ (H_{\mathcal{A}}(\rho) + t, S_{\mathcal{A}}(\rho) - t),\]
where $0 \leq t \leq S_{\mathcal{A}}(\rho)$, forms a convex capacity region $C(\mathcal{A})$ in the first quadrant of the plane. Figure 1 shows an example. The constant $t$ expresses the fact that quantum entropy can be used classically. Since the $S$-intercept of the line tangent to $C(\mathcal{A})$ with slope $-\frac{1}{q}$ is log $||\lambda(\mathcal{A})||_p$, another way to state Theorem 1.1 is that memory $\mathcal{A}$ bulk-embeds in another memory $\mathcal{B}$ if and only if $C(\mathcal{A}) \subseteq C(\mathcal{B})$. In other words, $\mathcal{A}$ bulk-embeds in $\mathcal{B}$ if and only if it has no state $\rho$ with too much entropy to fit in $\mathcal{B}$.

Our second main result is the following noiseless coding theorem, which generalizes a result of Barnum, Hayden, Jozsa, and Winter [1]. The terms of the theorem and a self-contained proof appear in Section 4.2.

**Theorem 1.3.** Let $\mathcal{A}$ be a quantum memory with a state $\rho$ and let $\mathcal{B}$ be another quantum memory. Then there is a reliable noiseless coding sequence
\[ \mathcal{A} \otimes \mathcal{N} \xrightarrow{\mathcal{Y}} \mathcal{B} \otimes \mathcal{N}(1+\varepsilon) \xrightarrow{\mathcal{X}_n} \mathcal{A} \otimes \mathcal{N} \]
for every rational $\varepsilon > 0$ if and only if $(H_{\mathcal{A}}(\rho), S_{\mathcal{A}}(\rho)) \in C(\mathcal{B})$. Here “reliable” means that the complete fidelity $F(\rho \otimes \mathcal{N}, \mathcal{X}_n \circ \mathcal{Y}_n) \rightarrow 1$ as $N \rightarrow \infty$.

The “no-go” direction of Theorem 1.3 depends on an interesting Hölder inequality for fidelity of encodings, Theorem 4.1. In simplified form, our inequality says that if
\[ \mathcal{A} \xrightarrow{\mathcal{X}} \mathcal{B} \xrightarrow{\mathcal{Y}} \mathcal{A} \]
are two quantum operations and $\frac{1}{p} + \frac{1}{q} = 1$, then
\[ \text{Tr}(\mathcal{X} \circ \mathcal{Y}) \leq \log \frac{\lambda(\mathcal{A})}{||\lambda(\mathcal{A})||_p} \]
This inequality is a broad generalization of the following elementary combinatorial fact: If a (uniformly) random number $x$ from 1 to $a$ is encoded into a random number from 1 to $b$ with $b < a$ and decoded back again, then the probability that $x$ is recovered is at most $\frac{b}{a}$.

In conclusion, Theorem 1.3 is an important converse to Theorem 1.1. Together they say that if $\mathcal{A}$ and $\mathcal{B}$ are two hybrid quantum memories, then, then either $\mathcal{A}$ blindly bulk-encodes into $\mathcal{B}$ with perfect fidelity, or $\mathcal{A}$ has a state $\rho$ that does not visibly bulk-encode into $\mathcal{B}$ with high fidelity.

## 2. MEMORY

As explained in the introduction, the first question is whether our model of a hybrid memory is adequately general. One justification comes from viewing a quantum system not as a Hilbert space, but as an abstract operator algebra $\mathcal{A}$. If $\mathcal{A}$ is infinite-dimensional, it should satisfy some analytic
axioms in order to be useful for quantum probability theory; usually it is assumed to be either a C*-algebra or a von Neumann algebra \( \mathcal{A} \). But if it is finite-dimensional, it suffices to require that \( \mathcal{A} \) be a (positive-definite) *-algebra; it is then also a C*-algebra and a von Neumann algebra. This means that in addition to the fact that \( \mathcal{A} \) is a complex vector space with associative multiplication, it has an abstract *-operation which is anti-linear, product-reversing, and suitably positive-definite:

\[
(\lambda AB)^* = \overline{\lambda} B^* A^* \quad A^* A = 0 \implies A = 0.
\]

Positive definiteness leads to an important partial ordering on \( \mathcal{A} \). By definition \( X \geq Y \) if \( X - Y = A^* A \) for some \( A \).

For example, the matrix algebra \( \mathcal{M}_n \) is a *-algebra. Despite their abstraction, *-algebras have all of the necessary structure for quantum information theory. The elements of a *-algebra \( \mathcal{A} \) of the form \( A^* A \) are called positive. A state \( \rho \) on a *-algebra \( \mathcal{A} \) is defined as a dual vector \( \rho \in \mathcal{A}^* \) which is positive on positive elements and which is normalized by \( \rho(1) = 1 \). Consequently we write \( \rho(A) \) for the expectation of \( A \) rather than \( \text{Tr}(\rho A) \). (The latter notation is of course equivalent when \( \mathcal{A} \) is a matrix algebra; it expresses \( \rho \) as a density operator.) A quantum operation from a system with *-algebra \( \mathcal{A} \) to a system with *-algebra \( \mathcal{B} \) is defined as a unital, completely positive (UCP) linear map \( \mathcal{E} : \mathcal{A} \to \mathcal{B} \). Here completely positive means that \( \mathcal{E} \) sends positive elements to positive elements after tensoring with the identity on a third *-algebra. Note that the transpose \( \mathcal{E}^T : \mathcal{B}^* \to \mathcal{A}^* \) is the corresponding map on states. It is completely positive and trace-preserving if we take \( \rho(I) \) to be the trace of \( \rho \).

It will be useful to consider a larger class of maps than traditional quantum operations. A completely positive map \( \mathcal{E} : \mathcal{A} \to \mathcal{B} \) is subunital (or SUCP) if \( \mathcal{E}(I) \leq I \). Whereas a UCP map conserves probability, an SUCP map either conserves or diminishes it. An SUCP map can be physically realized in the same way as a UCP map, with the extra interpretation that missing probability corresponds to ending the experiment. An SUCP map can also be called a decay quantum operation.

A standard classification theorem \([3]\) says that every finite-dimensional *-algebra \( \mathcal{A} \) is a direct sum of matrix algebras,

\[
\mathcal{A} \cong \bigoplus_{k=1}^n \mathcal{M}_{k}\mathbb{C}.
\]

Thus a quantum memory of shape \( \lambda \) is the most general possible finite-dimensional complex algebra of observables satisfying reasonable algebraic axioms. (However abandoning \( \mathbb{C} \) as the field of scalars leads to other possibilities \([4]\).)

Another justification comes from the interaction of a physical memory with its environment. Consider a physical device whose state is defined by a *-algebra \( \mathcal{M} \). Realistically \( \mathcal{M} \) is very large, but almost all of it is thermally coupled to the environment. Its decoherence on the thermal time scale is given by some decay quantum operation \( \mathcal{E} : \mathcal{M} \to \mathcal{M} \). If the thermal time scale is much shorter than the computational time scale, then the information retained by \( \mathcal{E}^n \) in the limit \( n \to \infty \) is the reliable memory of \( \mathcal{M} \).

Certainly any finite-dimensional *-algebra \( \mathcal{A} \) is the reliable memory retained by some quantum operation on a matrix algebra \( \mathcal{M}_d \). In the minimal construction, let \( d = ||\lambda(\mathcal{A})||_1 \) be the total size of all blocks of \( \mathcal{A} \). We realize \( \mathcal{A} \subseteq \mathcal{M}_d \) as matrices with a diagonal block of size \( \lambda_k(\mathcal{A}) \) for each \( k \). The algebra \( \mathcal{M}_d \) has a POVM whose kth element \( P_k \) is the identity of the kth summand \( \mathcal{M}_k \). The corresponding quantum operation

\[
\mathcal{P}(A) = \sum_{k=1}^n P_k A P_k
\]

is a projection, meaning \( \mathcal{P}^2 = \mathcal{P} \), and its image is \( \mathcal{A} \). If the thermal evolution of \( \mathcal{M}_d \) is given by \( \mathcal{P} \), the algebra \( \mathcal{A} \) measures the retained information.

Conversely, the following two results show that if \( \mathcal{E} \) is a (decay) quantum operation on a finite-dimensional *-algebra, the information retained by \( \mathcal{E}^n \) in the limit \( n \to \infty \) is measured by a smaller *-algebra of effective observables. (See also Zurek \([14]\).)

**Theorem 2.1.** Let \( \mathcal{E} : \mathcal{M} \to \mathcal{M} \) be an SUCP map on a finite-dimensional *-algebra \( \mathcal{M} \). Then there exists a sequence of integers \( n_k \to \infty \) such that \( \mathcal{E}^{n_k} \) converges to a unique projection \( \mathcal{P} \).

**Proof.** (Sketch) Choose a basis of \( \mathcal{M} \) that puts \( \mathcal{E} \) in Jordan canonical form. Since \( \mathcal{E}^{n_k} \) is SUCP, its matrix entries are bounded. Therefore \( \mathcal{E} \) has no eigenvalues \( \lambda \) with \( |\lambda| > 1 \), and if \( |\lambda| = 1 \), the \( \lambda \)-isotypic part of \( \mathcal{E} \) is diagonal. Choose a sequence of exponents \( n_k \to \infty \) such that the phases of these diagonal entries of \( \mathcal{E}^{n_k} \) are aligned with 1 in the limit. The rest of the matrix of \( \mathcal{E}^{n_k} \) decays to 0 as \( n \to \infty \). The map \( \mathcal{P} \) is unique because if the phases do not align with 1, the limiting map is not a projection. \( \square \)

Finally a result of Choi and Effros \([3\] pp. 166-7\) completes our justification for the *-algebra model.

**Theorem 2.2 (Choi, Effros).** If \( \mathcal{M} \) is a finite-dimensional *-algebra and \( \mathcal{P} \) is an SUCP projection on \( \mathcal{M} \), then the image of \( \mathcal{P} \) is a *-algebra \( \mathcal{A} \) with a modified product \( A \circ B = \mathcal{P}(AB) \).

The non-trivial part of Theorem \([2,2]\) (which more generally holds for C*-algebras) is the fact that the modified product \( A \circ B \) is associative. The modified product structure is consistent with applying \( \mathcal{P} \) between any two computational manipulations of \( \mathcal{M} \). Technically speaking, Choi and Effros prove Theorem \([2,2]\) for UCP maps, but the proof for SUCP maps is the same.

A quantum operation \( \mathcal{E} : \mathcal{B} \to \mathcal{A} \) is a blind, perfect-fidelity encoding if it has a right inverse \( \mathcal{V} : \mathcal{A} \to \mathcal{B} \), which is then called the decoding. In this case the reverse composition \( \mathcal{V} \circ \mathcal{E} \) is a CPU projection \( \mathcal{P} \). Moreover, \( \mathcal{P} \) identifies \( \mathcal{A} \) with the Choi-Effros algebra structure on \( \mathcal{P} \). This construction is reversible: Given \( \mathcal{P} \), we can define \( \mathcal{A} \) to be im \( \mathcal{P} \) with its Choi-Effros structure. Certainly if \( \mathcal{V} \) embeds \( \mathcal{B} \) into \( \mathcal{B} \), then a corresponding \( \mathcal{E} \) exists. (If \( \mathcal{V} \) is not unital, then it is a decay quantum operation, but \( \mathcal{E} \) can always be made
non-decay.) Generally, even when $\mathcal{A}$ and $\mathcal{B}$ are abelian, $\mathcal{Y}$ is not an algebra embedding, but another argument of Choi and Effros [5, pp.202-3] says that it always yields one.

**Theorem 2.3 (Choi, Effros).** If $\mathcal{M}$ is a finite-dimensional $*$-algebra and $\mathcal{P}$ is an SUCP projection on $\mathcal{M}$, then im$\mathcal{P}$ also embeds (non-unitaly) as a subalgebra of $\mathcal{M}$.

Theorem 2.3 more generally holds for von Neumann algebras. The proof adjusts $\mathcal{P}$ in a canonical way. It is not hard to show that every algebra embedding is a blind, perfect-fidelity decoding $\mathcal{Y}$; there exists an $\mathcal{X}$ to match it.

### 3. EMBEDDINGS

#### 3.1. Bin packing

Besides embeddability and bulk embeddability, we will also compare memories using a partial ordering on partitions which resembles dominance [13, Ch.7], or majorization, but is stricter. The partition $\lambda$ supermajorizes the partition $\mu$, or $\mu \trianglelefteq \lambda$, if for every $n$, the sum of all parts of $\lambda$ that are at least $n$ exceeds the same sum for $\mu$. Lemma 3.1 below and Theorem 1.1 imply that supermajorization lies between embeddability and bulk embeddability:

$$\mathcal{A} \hookrightarrow \mathcal{B} \Rightarrow \lambda(\mathcal{A}) \trianglelefteq \lambda(\mathcal{B}) \Rightarrow \mathcal{A} \rightarrow b \mathcal{B}$$

$$\mathcal{A} \rightarrow b \mathcal{B} \Rightarrow \lambda(\mathcal{A}) \trianglelefteq \lambda(\mathcal{B}) \Rightarrow \mathcal{A} \hookrightarrow \mathcal{B}.$$

We can view the parts of a partition $\lambda$ as an unordered multiset $\{\lambda_i\}$. It is sometimes convenient to assume a specific order on the parts. In this case we follow the usual convention that the parts of $\lambda$ are non-increasing:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1.$$

Given a partition $\lambda$, let $\lambda_{\geq x}$ denote the sum of all parts of $\lambda$ that are at least $x$. Thus $\lambda \trianglelefteq \mu$ means that

$$\lambda_{\geq x} \leq \mu_{\geq x},$$

for all $x$. Obviously integer values of $x$ suffice, but it will be convenient later to allow non-integer values. Also $\ell \lambda$ denotes $\lambda$ with each part repeated $\ell$ times. (This is not to be confused with magnifying each part by a factor of $\ell$.)

In order to analyze bulk embeddings and prove Theorem 1.1 we first analyze ordinary embeddings $\mathcal{A} \hookrightarrow \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional $*$-algebras, then any algebra homomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is characterized by a Bratteli diagram $\Gamma$ whose vertices are the summands of $\mathcal{A}$ and $\mathcal{B}$. Let $S_k$ be the $k$th summand of $\mathcal{A}$, so that $S_k \cong M_{\lambda_k}$, and likewise for $\mathcal{B}$. If we denote the adjacency matrix of $\Gamma$ by $\Gamma$ as well, then the diagram’s interpretation is that $f$ embeds $\Gamma_{j,k}$ copies of $\mathcal{A}$ in $\mathcal{B}$. (The matrix $\Gamma$ is the adjacency matrix of the diagram $\Gamma$.) The matrix $\Gamma$ must satisfy the inequality

$$\sum_j \Gamma_{j,k} \lambda(\mathcal{A})_j \leq \lambda(\mathcal{B})_k$$

for all $k$. (Bratteli diagrams often describe unital homomorphisms, which require equality.) The homomorphism $f$ is an embedding if and only if each summand of $\mathcal{A}$ has at least one edge, or equivalently that

$$\sum_k \Gamma_{j,k} \geq 1$$

for all $j$.

Thus we can think of $\mathcal{A}$ as a set of 1-dimensional blocks, $\mathcal{B}$ as a set of 1-dimensional bins, and the embedding as a way to pack the blocks of $\mathcal{A}$ in the bins of $\mathcal{B}$. The packing might repeat some of the summands of $\mathcal{A}$, but if there is any embedding, there is one with no repetition. (Repetition in this sense has nothing to do with cloning as in the no-cloning theorem. In representation theory this kind of repetition is usually called multiplicity.)

**Lemma 3.1.** If $\mathcal{A} \hookrightarrow \mathcal{B}$, then $\lambda(\mathcal{A}) \trianglelefteq \lambda(\mathcal{B})$. If $2\lambda(\mathcal{A}) \trianglelefteq \lambda(\mathcal{B})$, then $\mathcal{A} \hookrightarrow \mathcal{B}$.

**Proof.** Both statements follow by induction on the number of parts of $\lambda(\mathcal{A})$. They both hold trivially when $\lambda(\mathcal{A})$ is empty. To prove the first assertion, suppose that in some embedding, $\mathcal{A}_k$ embeds in $\mathcal{B}_k$. Let $\mathcal{A}$ be $\mathcal{A}$ with $\mathcal{A}_k$ removed and let $\mathcal{B}$ be $\mathcal{B}$ with $\mathcal{B}_k$ reduced by $\lambda(\mathcal{A})_k$, or removed if $\lambda(\mathcal{A})_k = \lambda(\mathcal{A})_1$. By construction, $\mathcal{A} \hookrightarrow \mathcal{B}$. Thus by induction,

$$\lambda(\mathcal{A}) \triangleright x \leq \lambda(\mathcal{B}) \triangleright x$$

for all $x \geq 1$. By the definition of $\mathcal{A}$ and $\mathcal{B}$,

$$\lambda(\mathcal{A}) \triangleright x = \lambda(\mathcal{A}) \triangleright x - \lambda(\mathcal{A})_1$$

$$\lambda(\mathcal{B}) \triangleright x \leq \lambda(\mathcal{B}) \triangleright x - \lambda(\mathcal{A})_1$$

for $x \leq \lambda(\mathcal{A})_1$, while $\lambda(\mathcal{A}) \triangleright x$ vanishes for $x > \lambda(\mathcal{A})_1$. Thus

$$\lambda(\mathcal{A}) \triangleright x \leq \lambda(\mathcal{B}) \triangleright x,$$

as desired.

To prove the second assertion, suppose that $2\lambda(\mathcal{A}) \trianglelefteq \lambda(\mathcal{B})$, or equivalently that

$$2\lambda(\mathcal{A}) \triangleright x \leq \lambda(\mathcal{B}) \triangleright x$$

for all $x$. We can greedily put $\mathcal{A}_1$ in any $\mathcal{B}_k$ in which it fits and make $\mathcal{A}$ and $\mathcal{B}$ as before. (In this greedy algorithm it is important to start with the largest summand of $\mathcal{A}$, not an arbitrary one.) If $\lambda(\mathcal{B}_k) \leq 2\lambda(\mathcal{A})_1$, then

$$\lambda(\mathcal{A}) \triangleright x = \lambda(\mathcal{A}) \triangleright x - \lambda(\mathcal{A})_1$$

$$\lambda(\mathcal{B}) \triangleright x \geq \lambda(\mathcal{B}) \triangleright x - 2\lambda(\mathcal{A})_1$$

for all $x \leq \lambda(\mathcal{A})_1$, while $\lambda(\mathcal{A}) \triangleright x$ vanishes for $x > \lambda(\mathcal{A})_1$. On the other hand if $\lambda(\mathcal{B}_k) \geq 2\lambda(\mathcal{A})_1$, then bin $k$ remains larger than any block even after block 1 is subtracted. In this case

$$\lambda(\mathcal{A}) \triangleright x = \lambda(\mathcal{A}) \triangleright x - \lambda(\mathcal{A})_1$$

$$\lambda(\mathcal{B}) \triangleright x \geq \lambda(\mathcal{B}) \triangleright x - \lambda(\mathcal{A})_1$$
be the logarithm of the Laplace transform of

either way, so the bin packing exists by induction.

### 3.2. Large deviations

The proof of Theorem 3.2 combines Lemma 3.1 with the Chernoff-Cramér theorem on large deviations (6). The theorem is usually stated in terms of sums of independent random variables, but it is more convenient here to formulate it in terms of convolutions of measures.

**Theorem 3.2 (Chernoff, Cramér).** Let μ be a measure on an interval \([0, u]\), let

\[
\ell(\beta) = \log \int_0^\infty e^{\beta x} d\mu(x)
\]

be the logarithm of the Laplace transform of μ and let \(t > 0\). Then for all \(n \in \mathbb{Z}_+\) and all \(\beta > 0\),

\[
\int_{n(t-s)}^\infty d\mu^n \geq e^{n(\ell(\beta) - \beta t)} \left( 1 - \frac{\ell''(\beta)}{n s^2} \right).
\]

If \(\ell'(0) \leq t < u\) and \(\beta\) minimizes \(\ell(\beta) - \beta t\), then for all \(0 < s < t\),

\[
\int_{n(t-s)}^\infty d\mu^n \geq e^{n(\ell(\beta) - \beta t - \beta s)} \left( 1 - \frac{\ell''(\beta)}{ns^2} \right).
\]

Here \(\mu^n\) denotes the \(n\)-fold convolution of \(\mu\) with itself. When \(\ell'(0) < t < u\), the expression

\[
\hat{\ell}(t) = \min_{\beta} \ell(\beta) - \beta t
\]

is the Legendre transform of \(\ell(\beta)\). Note that a unique \(\beta\) achieves the minimum because the mininamd is concave up, increases as \(\beta \to \infty\), and does not increase at \(\beta = 0\).

**Proof.** (Sketch) For any \(\beta\),

\[
\int_{n(t-s)}^\infty d\mu^n \leq e^{-n\beta t} \int_0^\infty e^{\beta x} d\mu^n(x) = e^{-n\beta t} e^{n\ell(\beta)}.
\]

This establishes the upper bound, Chernoff’s inequality.

If \(\beta\) is chosen to minimize \(\ell(\beta) - \beta t\), then \(t = \ell'(\beta)\). In this case

\[
\int_{n(t-s)}^\infty d\mu^n \geq e^{-n\beta(s+t)} \int_{n(t-s)}^{n(t+s)} e^{\beta x} d\mu^n(x) \geq e^{-n\beta(s+t)} \int_0^\infty \left( 1 - \frac{(x-n t)^2}{(ns)^2} \right) e^{\beta x} d\mu^n(x) = e^{-n\beta(s+t)} \left( 1 - \frac{\ell''(\beta)}{n s^2} \right) e^{n\ell(\beta)}.
\]

The equality uses the identities

\[
\int_0^\infty xe^{\beta x} d\mu^n(x) = (e^{\ell(\beta)})' = n\ell'(\beta) e^{n\ell(\beta)}
\]

\[
\int_0^\infty x^2 e^{\beta x} d\mu^n(x) = (e^{n\ell(\beta)})'' = (n\ell''(\beta) + n^2\ell'(\beta)^2) e^{n\ell(\beta)}.
\]

This establishes the lower bound, Cramér’s theorem.

**Proof of Theorem 3.2.** In brief, without loss of generality

\[
||\lambda(\mathcal{A})||_p < ||\lambda(\mathcal{B})||_p
\]

for all \(p \in [1, \infty]\). In this case we apply Theorem 3.2 to the measures

\[
\mu_\mathcal{A} = \sum_k \lambda_k(\mathcal{A}) \delta_{\log \lambda_k(\mathcal{A})} \\
\mu_\mathcal{B} = \sum_k \lambda_k(\mathcal{B}) \delta_{\log \lambda_k(\mathcal{B})}
\]

where \(\delta_x\) denotes a delta function (or atom) at \(x\). For sufficiently large \(n\), Chernoff’s bound for \(\mu_\mathcal{A}\) and Cramér’s inequality for \(\mu_\mathcal{B}\) together imply the criterion

\[
2^\lambda(\mathcal{A}^\otimes n) \geq x \leq 2^\lambda(\mathcal{B}^\otimes n) \geq x
\]

of Lemma 3.1 uniformly for \(x \in [1, \infty]\).

In detail, we assume that \(||\lambda(\mathcal{A})||_\infty > 1\); otherwise \(\mathcal{A}\) and \(\mathcal{B}\) are both entirely classical and Theorem 3.1 is easy. Since

\[
||\lambda(\mathcal{A})||_p \leq ||\lambda(\mathcal{B})||_p
\]

for all \(p \in [1, \infty]\), then for any \(k > 1\),

\[
||\lambda(\mathcal{A}^\otimes k)||_p < ||\lambda(\mathcal{B}^\otimes k+1)||_p.
\]

The \(\epsilon\) margin in Theorem 3.1 thus allows us to assume that

\[
||\lambda(\mathcal{A})||_p < ||\lambda(\mathcal{B})||_p
\]

for all \(p \in [1, \infty]\) by replacing \(\mathcal{A}\) by \(\mathcal{A}^\otimes k\) and \(\mathcal{B}\) by \(\mathcal{B}^\otimes k+1\).

The measure \(\mu_\mathcal{A}\) is defined so that

\[
\mu_\mathcal{A}^n = \mu_\mathcal{A} \otimes n
\]

and

\[
\lambda(\mathcal{A}^\otimes n) = \int_x^\infty d\mu_\mathcal{A}(x),
\]

and likewise for \(\mu_\mathcal{B}\). Therefore by Lemma 3.1, it suffices to show that there exists an \(n\) such that for all \(t \geq 0\),

\[
2 \int_0^{\infty} d\mu_\mathcal{A}^n \leq \int_0^{\infty} d\mu_\mathcal{B}^n.
\]

As in the statement of Theorem 3.2, let

\[
\ell_\mathcal{A}(\beta) = \log \int_0^{\infty} e^{\beta x} d\mu_\mathcal{A}(x) = \log ||\lambda(\mathcal{A})||_\beta + 1
\]

\[
\ell_\mathcal{B}(\beta) = \log \int_0^{\infty} e^{\beta x} d\mu_\mathcal{B}(x) = \log ||\lambda(\mathcal{B})||_\beta + 1.
\]
Observe that \( \ell_{\mathcal{A}}(\beta) \) is a smooth, concave function, and that
\[
\lim_{\beta \to \infty} \frac{\ell'_{\mathcal{A}}(\beta)}{\beta} = \log ||\lambda(\mathcal{A})||_\infty < \infty.
\]
It follows that \( \ell''_{\mathcal{A}}(\beta) \) has a finite maximum \( C \) for \( \beta \in [0, \infty) \). Note also that
\[
\frac{\ell_{\mathcal{A}}(\beta) - \ell_{\mathcal{A}}(\beta)}{\beta}
\]
achieves a positive minimum, since
\[
\lim_{\beta \to \infty} \frac{\ell_{\mathcal{A}}(\beta) - \ell_{\mathcal{A}}(\beta)}{\beta} = ||\lambda(\mathcal{A})||_\infty - ||\lambda(\mathcal{A})||_\infty
\]
\[
\lim_{\beta \to 0} \frac{\ell_{\mathcal{A}}(\beta) - \ell_{\mathcal{A}}(\beta)}{\beta} = \infty.
\]
Temporarily suppose that \( t \geq \ell_{\mathcal{A}}(0) \) and that \( \beta = \beta(t) \) minimizes \( \ell_{\mathcal{A}}(\beta) - \beta t \). Let
\[
s = \sqrt{\frac{2C}{n}}.
\]
Then
\[
\int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn} \leq e^{n(\ell_{\mathcal{A}}(\beta) - \beta t + \beta s)}
\]
\[
\int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn} \geq e^{n(\ell_{\mathcal{A}}(\beta) - \beta t - \beta s) - \log 2}.
\]
If \( n \) is large enough that
\[
2s + \frac{2 \log 2}{n} = 2 \sqrt{\frac{2C}{n}} + \frac{2 \log 2}{n} \leq \min_{\beta} \frac{\ell_{\mathcal{A}}(\beta) - \ell_{\mathcal{A}}(\beta)}{\beta},
\]
then
\[
2 \int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn} \leq \int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn}.
\]
Thus for some \( \varepsilon > 0 \), inequality (1) holds for all \( t \geq \ell_{\mathcal{A}}(0) - \varepsilon \). If \( t \leq \ell_{\mathcal{A}}(0) - \varepsilon \), let \( u = \ell_{\mathcal{A}}(0) \) and let \( \beta = 0 \). Then
\[
\int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn} \leq \int_{0}^\infty d\mu_{\mathcal{A}}^{sn} = e^{\ell_{\mathcal{A}}(0)},
\]
while
\[
\int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn} \geq \int_{n(t-s)}^\infty d\mu_{\mathcal{A}}^{sn} \geq e^{\ell_{\mathcal{A}}(0) - \log 2}
\]
provided that \( s \leq \varepsilon \). Since \( \ell_{\mathcal{A}}(0) < \ell_{\mathcal{A}}(0) \), inequality (1) holds when \( n \) is large enough.

4. ENTROPY

4.1. Capacity

Let \( \mathcal{A} \) be a finite-dimensional *-algebra, where as before
\[
\mathcal{A} = \bigoplus_{k=1}^{n} \mathcal{A}_k \cong \bigoplus_{k=1}^{n} \mathcal{H}_{\lambda_k}.
\]
Let \( \rho \) be a (mixed) state on \( \mathcal{A} \); as explained above we view \( \rho \) as a dual vector on \( \mathcal{A} \) rather than as an element of \( \mathcal{A} \). Let
\[
\rho_k = \rho|_{\mathcal{A}_k}
\]
be the restriction of \( \rho \) to \( \mathcal{A}_k \). Diagonalize each \( \rho_k \) and let \( r_{k,j} \) with \( 1 \leq j \leq \lambda_k \) be its diagonal entries. (In general a state \( \rho \) on matrices is diagonal if and only if \( \rho(A) \) depends only on the diagonal entries of \( A \). Equivalently in the present case we can interpret \( \rho \) as a density operator.) Let
\[
r_k = \rho_k(I) = \sum_{j=1}^{\lambda_k} r_{k,j}.
\]
be the total density of \( \rho \) in \( \mathcal{A}_k \); evidently
\[
\sum_{k=1}^{n} r_k = 1.
\]
We also define the normalized state \( \rho'_k \) on \( \mathcal{A}_k \) by
\[
\rho'_k = \frac{\rho_k}{r_k}
\]
with diagonal entries
\[
r'_{k,j} = \frac{r_{k,j}}{r_k}.
\]
The classical entropy of the state \( \rho \) on \( \mathcal{A} \) is defined as
\[
H_{\mathcal{A}}(\rho) = -\sum_{k=1}^{n} r_k \log r_k.
\]
The quantum entropy of \( \rho \) is defined as
\[
S_{\mathcal{A}}(\rho) = -\sum_{k=1}^{n} \sum_{j=1}^{\lambda_k} r_{k,j} \log r'_{k,j}.
\]
(Note that in the literature \( H \) is also sometimes used to denote quantum, or von Neumann, entropy. Here we follow the convention of Nielsen and Chuang (10).) These two entropies are supported by a number of elementary justifications: The classical entropy of \( \rho \) is the Shannon entropy of the restriction of \( \rho \) to the center of \( \mathcal{A} \), which is a classical system. The quantum entropy of \( \rho \) is the expected value of the von Neumann entropy of \( \rho_k \), where the index \( k \) is chosen randomly with probability \( r_k \). Finally the total entropy
\[
H_{\mathcal{A}}(\rho) + S_{\mathcal{A}}(\rho) = -\sum_{k=1}^{n} \sum_{j=1}^{\lambda_k} r_{k,j} \log r_{k,j}
\]
has the same formula as both the Shannon and the von Neumann entropy.

The proof of Theorem 1 is based on finding thermal states of \( \mathcal{A} \) with respect to a certain Hamiltonian. We define the energy \( E_k \) of the summand \( \mathcal{A}_k \) as the negative of its capacity for quantum entropy:
\[
E_k = -\log \lambda_k(\mathcal{A}).
\]
We retain the parameter $\beta$ from Section 3.2 setting $p = \beta + 1$, and we also define the temperature $T = 1/\beta$. The thermal state $\rho_T$ at temperature $T$ has the property that its restriction $\rho_k$ to each $\mathcal{A}_k$ is uniform. If $\rho$ is any state with this property, then its energy $E_{\mathcal{A}}(\rho)$ is, by definition, the negative of its quantum entropy:

$$E_{\mathcal{A}}(\rho) = -S_{\mathcal{A}}(\rho).$$

The free energy of $\rho$ is therefore

$$F_{\mathcal{A}}(\rho) = E_{\mathcal{A}}(\rho) - T(H_{\mathcal{A}}(\rho) + S_{\mathcal{A}}(\rho)) = -T(H_{\mathcal{A}}(\rho) + pS_{\mathcal{A}}(\rho)).$$

Since the thermal state minimizes the free energy, we have defined energy so that the thermal state $\rho_T$ maximizes quantum entropy plus classical entropy discounted by $p$. To compute the maximum, recall that for the thermal state $\rho_T$, the free energy is proportional to the log of the partition function:

$$F_{\mathcal{A}}(\rho_T) = -T \log Z_{\mathcal{A}}(\rho_T) = -T \log \left( \sum_{k=1}^{n} \lambda_k e^{\beta \log \lambda_k(\mathcal{A})} \right) = -T \log \left( \sum_{k=1}^{n} \lambda_k^{\beta+1} \right) = -T \log \left( \prod_{k=1}^{n} \lambda_k^p \right).$$

Therefore

$$H_{\mathcal{A}}(\rho_T) = S_{\mathcal{A}}(\rho_T) = \log \left( \prod_{k=1}^{n} \lambda_k^p \right),$$

as desired.

To prove the final claim of Theorem 1.2, observe that every state $\rho$ on $\mathcal{A}$ can be written in the form

$$(H_{\mathcal{A}}(\rho_T) + t, S_{\mathcal{A}}(\rho_T) - s - t)$$

with $0 \leq s, t$ and $s + t \leq S_{\mathcal{A}}(\rho_T)$. Starting with the state $\rho_T$, the quantum entropy in each block can be decreased to 0 without changing the total probability of that block, hence without changing the classical entropy. In this way we can absorb the constant $t$. The remaining constant $s$ just matches the one in the conclusion.

4.2. Noiseless coding

A final justification for quantum and classical entropies is Theorem 1.3 which we prove here. The theorem is a mutual generalization of, and entirely analogous to, Shannon’s classical and Schumacher’s purely quantum coding theorems Thms. 12.4 & 12.6) [11, 12].

Given an algebra $\mathcal{A}$ with a state $\rho$ and a second algebra $\mathcal{B}$, a noiseless coding is a pair of decay quantum operations

$$\mathcal{A} \xrightarrow{\mathcal{Y}} \mathcal{B} \xrightarrow{\mathcal{X}} \mathcal{A}.$$ 

Since these are maps on algebras rather than states spaces, the second map $\mathcal{X}$ is the encoding and the first map $\mathcal{Y}$ is the decoding.

We are interested in reliable noiseless coding, or in other words high-fidelity, visible bulk-encoding. But a rigorous definition of reliability is not obvious. Suppose that $\mathcal{E}$ is a decay quantum operation from a memory $\mathcal{A}$ to itself, and that $\mathcal{A}$ has a state $\rho$. If $\mathcal{E}$ is another memory, we define the $\mathcal{E}$-fidelity of $\mathcal{A}$ to be

$$F_\mathcal{E}(\rho, \mathcal{E}) = \min_{\sigma \in (\mathcal{C} \otimes \mathcal{A})^*} 1 - D(\sigma, (\mathcal{id} \otimes \mathcal{E}^T)(\sigma)), \tag{2}$$

where $D$ is the trace distance on states, and the minimum is taken over states $\sigma$ on $\mathcal{C} \otimes \mathcal{A}$ that project to the state $\rho$ on $\mathcal{A}$. In words, the $\mathcal{E}$-fidelity is the complement of the highest probability that the operation $\mathcal{E}$ leaves the larger system $\mathcal{C} \otimes \mathcal{A}$ in an erroneous state. We define the complete fidelity $F(\rho, \mathcal{E})$ to be the infimum of $\mathcal{E}$-fidelity over all $\mathcal{E}$. It is not hard to show that complete fidelity agrees with the classical non-error rate when $\mathcal{A}$ is classical, and with entanglement fidelity when $\mathcal{A}$ is purely quantum.

The more difficult half of Theorem 1.3 is the no-go direction. To review, the heart of the no-go direction of the classical encoding theorem is the following elementary fact about squeezing states: If a state $\rho$ of a classical memory is encoded into $b$ values, then it cannot be recovered with probability greater than $b||\rho||_\infty$, where $||\rho||_\infty$ is the probability of the most likely value of $\rho$. Or for simplicity, if $\rho$ is the uniform state on a memory with $q$ values, then the non-error rate is at most $\frac{q}{b}$. We will need a hybrid quantum generalization of this inequality. To state it, we replace $||\rho||_\infty$ with a different norm. If $\rho$ is a state on $\mathcal{A}$, define the dense-coding-based supremum of $\rho$ by

$$||\rho||_d = \max_{j,k} \frac{r^2}{k^2}.$$ 

in the notation of Section 4.1.

**Theorem 4.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be two hybrid quantum memories and let $\rho$ be a state on $\mathcal{A}$. If

$$\mathcal{A} \xrightarrow{\mathcal{Y}} \mathcal{B} \xrightarrow{\mathcal{X}} \mathcal{A}$$

are decay quantum operations and $\frac{b}{a} + \frac{q}{a} = 1$, then

$$F(\rho, \mathcal{X} \circ \mathcal{Y}) \leq ||\rho||_d ||\lambda(\mathcal{A})||_q ||\lambda(\mathcal{B})||_p.$$ 

Before proving Theorem 4.1 we discuss some special cases. If $\mathcal{A} = \mathcal{C}^a$ is classical and $\rho$ is the uniform state, then $||\rho||_d = \frac{1}{a^2}$. In this case, taking $p = 1$, Theorem 4.1 says that

$$F(\rho, \mathcal{X} \circ \mathcal{Y}) \leq \frac{||\lambda(\mathcal{B})||_1}{a^2}.$$ 

This generalizes the classical squeezing result, bounding the fidelity by the total number of independent states of $\mathcal{B}$ whether or not it is classical. On the other hand, if $\mathcal{A} = \mathcal{M}_a$ is purely quantum and $\rho$ is the uniform state, then $||\rho||_d = \frac{1}{a^2}$. In this case, taking $p = \infty$, Theorem 4.1 says that

$$F(\rho, \mathcal{X} \circ \mathcal{Y}) \leq \frac{\lambda_1(\mathcal{B})}{a^2}.$$
In other words, if \( \mathcal{A} \) is purely quantum, then the fidelity of squeezing is bounded by the largest quantum block of \( \mathcal{B} \), regardless of its classical part. (But if \( \mathcal{B} = \mathcal{M}_k \) is also purely quantum, then it can be shown that
\[
F(\rho, \mathcal{X} \circ \mathcal{Y}) \leq \frac{b^2}{a^2}
\]
when \( \rho \) is isometric. In this case if \( b \) divides \( a \), then multiplying \( \mathcal{B} \) by \( \frac{a}{b} \) classical states can boost fidelity to \( \frac{b^2}{a^2} \).

**Proof.** The operations \( \mathcal{X} \) and \( \mathcal{Y} \) admit Kraus representations
\[
\mathcal{X}(\bigoplus_k B_k) = \bigoplus_j \sum_{k,\ell} X^*_{j,k,\ell} B_k X_{j,k,\ell}
\]
\[
\mathcal{Y}(\bigoplus_k A_k) = \bigoplus_j \sum_{k,\ell} Y^*_{j,k,\ell} A_k Y_{j,k,\ell}
\]
subject to the subunital conditions
\[
\sum_{k,\ell} X^*_{j,k,\ell} X_{j,k,\ell} \leq I \in \mathcal{A} \quad \sum_{k,\ell} Y^*_{j,k,\ell} Y_{j,k,\ell} \leq I \in \mathcal{B}.
\]

Recall the definition of \( r_k \) and \( \rho_k \) of Section 2.1. The infimum in equation 2 is obtained by lifting the state \( \rho \) to the completely correlated, completely entangled state
\[
\sigma = \bigoplus_k r_k \psi_k
\]
on \( \mathcal{A} \otimes \mathcal{A} \), where \( \psi_k \) is a pure state that projects to \( \rho_k \). By a computation similar to one in Nielsen and Chuang [10, p. 421], the fidelity is then given by
\[
F = F(\rho, \mathcal{X} \circ \mathcal{Y}) = \sum_{j,k,\ell,m} r_j |\langle Y_{j,k,m} X_{j,k,\ell} | \rho \rangle|^2.
\]

Given any state \( \sigma \) on the matrix algebra \( \mathcal{M}_a \) and any matrices \( X \in \mathcal{M}_{b \times a} \) and \( Y \in \mathcal{M}_{a \times b} \), the Cauchy-Schwarz inequality and positivity together say that
\[
|\sigma(YX)|^2 \leq |\sigma(X^*X)| |\sigma(Y^*Y)| \leq ||\sigma||^2 \text{Tr}(X^*X) \text{Tr}(Y^*Y).
\]

Applying this to equation 2, we obtain the bound
\[
F \leq \sum_{j,k,\ell,m} ||\rho||_a \text{Tr}(X^*_{j,k,\ell} X_{j,k,\ell}) \text{Tr}(Y^*_{j,k,m} Y_{j,k,m}).
\]

Define the numbers
\[
x_{k,j} = \sum_{\ell,m} \text{Tr}(X^*_{j,k,\ell} X_{j,k,\ell}) \quad y_{j,k} = \sum_{m} \text{Tr}(Y^*_{j,k,m} X_{j,k,m})
\]
and define the vectors \( x = (x_{k,j}) \) and \( y = (y_{j,k}) \). Then we can restate inequality 3 as
\[
F \leq ||\rho||_a \sum_{j,k} x_{k,j} y_{j,k} = ||\rho||_a x \cdot y,
\]
while equation 3 implies that
\[
\sum_{j} x_{k,j} \leq \lambda_k(\mathcal{A}) \quad \sum_{k} y_{j,k} \leq \lambda_j(\mathcal{B}),
\]

Thus
\[
||x||_p \leq ||\lambda(\mathcal{A})||_p \quad ||y||_q \leq ||\lambda(\mathcal{B})||_q.
\]

Finally the Hölder inequality yields
\[
F \leq ||\rho||_a x \cdot y \leq ||\rho||_a ||x||_p ||y||_q
\]
\[
\leq ||\rho||_a ||\lambda(\mathcal{A})||_p ||\lambda(\mathcal{B})||_q
\]
when \( \frac{1}{p} + \frac{1}{q} = 1 \), as desired.

**Proof of Theorem** (Semi-sketch) As in the proofs of Shannon’s and Schumacher’s theorems as presented by Nielsen and Chuang [10], we first establish the existence of a \( \epsilon \)-typical subalgebra \( \mathcal{A}_{\text{typ}} \) of \( \mathcal{A} \otimes \mathcal{N} \) with respect to the state \( \rho \). (The \( \epsilon \) in the proof here is not the same as the one in the statement of the theorem, which we rename \( \delta \).) We will take \( \epsilon \) to implicitly depend on \( N \) with \( \epsilon \to 0 \) slowly as \( N \to \infty \). We will establish that \( \mathcal{A}_{\text{typ}} \) is approximately rectangular and that the restriction \( \rho_{\text{typ}} \) of \( \rho \otimes \mathcal{N} \). We will then confirm that if
\[
(S,H) = (S_{\text{typ}}(\rho),H_{\text{typ}}(\rho)) \in C(\mathcal{B}),
\]
then \( \mathcal{A}_{\text{typ}} \) embeds in \( \mathcal{B} \otimes \mathcal{N} \) for sufficiently large \( N \); in particular it reliably encodes. On the other hand, if \( (S,H) \not\in C(\mathcal{B}) \), we will confirm that \( \mathcal{A}_{\text{typ}} \) does not reliably encode in \( \mathcal{B} \otimes \mathcal{N} \); indeed the fidelity of any encoding-decoding converges to 0 exponentially.

Assume that the state \( \rho \in \mathcal{A} \) is diagonalized and that \( r_{k,j} \), with \( 1 \leq k \leq n(\mathcal{A}) \) and \( 1 \leq j \leq \lambda_k(\mathcal{A}) \), are its diagonal entries. Here \( n(\mathcal{A}) \) denotes the number of parts of \( \lambda(\mathcal{A}) \). This induces a diagonalization of the state \( \rho \otimes \mathcal{N} \) with a diagonal entry \( r_{K,J} \) for each pair of admissible sequences
\[
K = (k_1,k_2,\ldots,k_N) \quad J = (j_1,j_2,\ldots,j_N)
\]
is such that
\[
1 \leq \ell \leq N \quad 1 \leq k_\ell \leq n(\mathcal{A}) \quad 1 \leq j_\ell \leq \lambda_{k_\ell}(\mathcal{A}).
\]

Moreover, for each admissible \( K \), \( \mathcal{A} \otimes \mathcal{N} \) has an algebra summand \( (\mathcal{A} \otimes \mathcal{N})_K \). If \( (K,J) \) and \( (K,J') \) are two admissible pairs, the algebra summand \( (\mathcal{A} \otimes \mathcal{N})_K \) has an elementary matrix \( E_{K,J,J'} \); these matrices then form a basis of \( \mathcal{A} \otimes \mathcal{N} \). We will consider a set \( T \) of admissible pairs \( (K,J) \) called the typical set; momentarily it can be any set. The span of the matrices \( E_{K,J,J'} \) with \( (K,J),(K,J') \in T \) is a subalgebra \( \mathcal{A}_{\text{typ}} \). Another way to describe the algebra \( \mathcal{A}_{\text{typ}} \) is to define the projector
\[
P_{\text{typ}} = \sum_{(K,J) \in T} E_{K,J,J}
\]
and then let
\[
\mathcal{A}_{\text{typ}} = \rho_{\text{typ}}\mathcal{A} \otimes \mathcal{N} P_{\text{typ}}.
\]

In this notation, the map
\[
\mathcal{P}(X) = P_{\text{typ}}XP_{\text{typ}}
\]
is an SUCP projection on \( \mathcal{A} \otimes \mathcal{N} \) with image \( \mathcal{A}_{\text{typ}} \).
Given $\alpha > 0$, say that an admissible pair $(K,J)$ is $\alpha$-typical if the number of occurrences $N(K,J;k,j)$ of $(k,j)$ satisfies
\[
\frac{\left| \frac{N(K,J;k,j) - r_{k,j}}{N} \right|}{N} < \alpha.
\]
Let $T$ be the set of all $\alpha$-typical pairs. By repeated application of Chernoff’s inequality (Theorem 3.2 in a more traditional probabilistic context),
\[
\rho^\otimes N(P_{\text{typ}}) = \sum_{(K,J) \in T} r_{K,J} \rightarrow 1
\]
for any fixed $\alpha$ as $N \rightarrow \infty$. Moreover
\[
F(\rho^\otimes N(P_{\text{typ}}), \rho^\otimes N(P_{\text{typ}})^2),
\]
so for any fixed $\alpha$, $\rho^\otimes N(P_{\text{typ}})$ reliably encode into each other. At the same time, by a messy but straightforward calculation, if $\alpha$ is sufficiently small relative to $\epsilon$ (and depending on $\rho$ but not on $N$), $\tilde{\rho}_{\text{typ}}$ and $\rho_{\text{typ}}$ have the following properties:
\[
\begin{align*}
\left| \left( \log n(\tilde{\rho}_{\text{typ}}) - HN \right) \right| &< NE \\
\left| \left( \log \lambda(\tilde{\rho}_{\text{typ}}) - HS \right) \right| &< NE \\
\left( \log ||\rho'|| \right) + H + 2S &< NE.
\end{align*}
\]
Suppose that $(H,S) \in C(\mathcal{B})$. In this case, let $L = e^{N(S + \epsilon)}$; then
\[
\lambda(\tilde{\rho}_{\text{typ}}) \geq C = 0.
\]
Meanwhile equations 5 imply that
\[
\lambda(\mathcal{A}_{N,\epsilon}) > 0 < e^{N(H+S+2\epsilon)}.
\]
By a derivation using Crámer’s bound like the one in the proof of Theorem 1.1
\[
\lambda(\mathcal{B}^\otimes (N+1)\delta) \geq C > 2e^{N(H+S+\epsilon)}
\]
when $N$ is large enough, provided that $\epsilon$ is small compared to $\delta$. Thus by Lemma 3.1, $\mathcal{A}_{N,\epsilon}$ embeds in $\mathcal{B}^\otimes (N+1)\delta$ for large enough $N$, as desired.

Suppose that $(H,S) \notin C(\mathcal{B})$. In this case, suppose that
\[
\mathcal{A}_{\text{typ}} \xrightarrow{\mathcal{Y}} \mathcal{B}^\otimes (N+1)\delta \xrightarrow{\mathcal{X}} \mathcal{A}_{\text{typ}}
\]
are decay quantum operations and that $\frac{1}{p} + 1 = 1$. By the first two equations of 6,
\[
\log ||\lambda(\mathcal{A}_{\text{typ}})|| < \left( \frac{H}{q} + S + 2\epsilon \right)N.
\]
Combining this with Theorem 4.1 and the last equation of 6, we obtain Theorem 4.1
\[
\log F(\rho_{\text{typ}}, \mathcal{X} \circ \mathcal{Y}) < (\epsilon - H - 2S)N + \left( \frac{H}{q} + S + 2\epsilon \right)N + \log ||\lambda(\mathcal{B}^\otimes (N+1)\delta)||_p
\]
\[
= N((1 + \delta) \log ||\lambda(\mathcal{B})||)_p - \frac{H}{p} - S + 3\epsilon).
\]
Since $\delta$ must be sent to 0 and $\epsilon$ may be sent to 0, the fidelity therefore decays exponentially if there exists a $p$ such that
\[
\log ||\lambda(\mathcal{B})||_p < \frac{H}{p} + S.
\]
By the definition of $C(\mathcal{B})$, this inequality is equivalent to the assumed condition $(H,S) \notin C(\mathcal{B})$. Since $F(\rho_{\text{typ}}, \mathcal{X} \circ \mathcal{Y})$ decays exponentially, it cannot converge to 1.

5. DISCUSSION

Section 2 illustrates the principle that classical information theory is the abelian special case of quantum information theory. Many authors maintain a dichotomy between the two theories by considering ensembles of mixed states. But such formalism is ultimately redundant, because an ensemble is itself a classical probabilistic state. More precisely, let
\[
\rho = \sum_k p_k \rho_k \in \mathcal{A}
\]
be an ensemble of states in a memory $\mathcal{A}$. If the symbol $k$ is not recorded, then $\rho$ encodes all statistical information that can be extracted from the ensemble. But if each symbol $k$ is recorded as a state $\sigma_k$ in another memory $\mathcal{B}$, then we can let
\[
\rho' = \sum_k p_k \rho_k \otimes \sigma_k \in \mathcal{A} \otimes \mathcal{B}.
\]
If $\mathcal{B}$ is abelian and the $\sigma_k$’s are distinct pure states, then the state $\rho'$ denotes an ensemble with a record of its preparation. The term “ensemble” also typically implies that the memory $\mathcal{B}$ is hidden or untransmitted. This too is only a special case, because memory may be hidden whether or not it is abelian.

Theorems 1.1, 1.2, and 1.3 together suggest that all quantum information can be measured in the bulk limit by two numbers, classical entropy $H$ and quantum entropy $S$. By contrast information capacity has more structure than information itself. The capacity of a quantum memory is defined by a curve that represents trade-offs between classical and quantum entropy. The capacity of a general quantum channel could be even more complicated.

There are many interesting partial orderings on quantum memories besides embeddability, bulk embeddability, and supermajorization. One natural example is embeddability in the presence of an auxiliary memory, or stable embeddability. Given memories $\mathcal{A}$ and $\mathcal{B}$, when is there a memory $\mathcal{C}$ such that
\[
\mathcal{A} \otimes \mathcal{C} \twoheadrightarrow \mathcal{B} \otimes \mathcal{C}?
\]
We do not know when $\mathcal{A}$ stably embeds in $\mathcal{B}$. Stable embeddability implies bulk embeddability and is implied by embeddability, but we do not know how it compares to supermajorization order.

Theorem 1.1 is related to a much more general question in quantum information theory. Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be quantum operations representing two quantum channels
between general quantum memories. When are there operations \( \mathcal{X}_N \) and \( \mathcal{Y}_N \) that make the diagram

\[
\begin{array}{ccc}
\mathcal{A} \otimes N & \xrightarrow{\mathcal{E} \otimes N} & \mathcal{B} \otimes N \\
\downarrow \mathcal{X}_N & & \uparrow \mathcal{Y}_N \\
\mathcal{C} \otimes N(1+\epsilon) & \xrightarrow{\mathcal{F} \otimes N(1+\epsilon)} & \mathcal{D} \otimes N(1+\epsilon)
\end{array}
\]

commute with high fidelity? We can then say that the channel \( \mathcal{E} \) reliably bulk-encodes in the channel \( \mathcal{F} \). Theorems 1.1, 1.2, and 1.3 together answer the question when \( \mathcal{E} \) and \( \mathcal{F} \) are both the identity map, with the refinement that perfect fidelity is possible when high fidelity is possible. In light of Theorem 2.2, the cross-encoding question is also settled when \( \mathcal{E} \) and \( \mathcal{F} \) are SUCP projections.

Finally, it is well-understood that classical and quantum memory are inequivalent resources in quantum complexity theory. For example there is a quantum algorithm to find a collision of a 2-to-1 function with which uses \( \tilde{O}(N^{1/3}) \) classical space (and \( \tilde{O}(1) \) quantum space) [2]. But if the function only has a single repeated value, the best quantum algorithm uses \( \tilde{O}(N^{1/4}) \) quantum space [7]. It would be interesting to find an algorithm whose natural space complexity is hybrid quantum memory.

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