Topological quantum D-branes and wild embeddings from exotic smooth $\mathbb{R}^4$

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This is the next step of uncovering the relation between string theory and exotic smooth $\mathbb{R}^4$. Exotic smoothness of $\mathbb{R}^4$ is correlated with D6 brane charges in IIA string theory. We construct wild embeddings of spheres and relate them to a class of topological quantum D$p$-branes as well to KK theory. These branes emerge when there are non-trivial NS-NS $H$-fluxes where the topological classes are determined by wild embeddings $S^2 \to S^3$. Then wild embeddings of higher dimensional $p$-complexes into $S^n$ correspond to D$p$-branes. These wild embeddings as constructed by using gropes are basic objects to understand exotic smoothness as well Casson handles. Next we build $C^*$-algebras corresponding to the embeddings. Finally we consider topological quantum D-branes as those which emerge from wild embeddings in question. We construct an action for these quantum D-branes and show that the classical limit agrees with the Born-Infeld action such that flat branes = usual embeddings.

Keywords: quantum D-branes; wild embeddings; non-commutative geometry; exotic $\mathbb{R}^4$.

1. Introduction

Despite the substantial effort toward quantizing gravity in 4 dimensions, this issue is still open. One of the best candidates till now is the superstring theory formulated in 10 dimensions. A way from superstring theory to 4-dimensional quantum gravity or standard model of particle physics (minimal supersymmetric extension thereof) is, at best, highly nonunique. Many techniques of compactifications and flux stabilization along with specific model-building branes configurations and dualities, were worked out toward this end within the years. Possibly some important data of a fundamental character are still missing enabling the connection with physics in dimension 4.

In this paper we follow the idea from [11,19] that different smoothings of Euclidean $\mathbb{R}^4$ are presumably crucial for the program of QG and string theory. These structures are footed certainly in dimension 4 and have great importance to physics.
Here we again try to consider exotic $\mathbb{R}^4$’s as serving a link between higher dimensional superstring theory and 4-dimensional “physical” theories. String theory D- and NS-branes in some backgrounds are correlated naturally with exotic smoothness on $\mathbb{R}^4$ appearing in these backgrounds\textsuperscript{11}. Moreover, when taking quantum limit of D-branes and spaces, such that these become represented by separable $C^*$-algebras, the connection with exotic $\mathbb{R}^4$’s extends naturally. This is due to the representing exotic $\mathbb{R}^4$’s by convolution $C^*$-algebras of the codimension-one foliations of certain 3-sphere. In this paper we focus on the topological level underlying the quantum branes and exotic $\mathbb{R}^4$’s connection. We show that there exists topological counterparts of D-branes in a quantum regime. Namely, by the use of $C^*$-algebra approach to quantum D-branes the manifold model of a quantum D-brane as wild embedding is constructed. Then we show that the $C^*$-algebra of the wild embedding is isomorphic to the $C^*$-algebra of the quantum D-brane. We call the wild embeddings representing quantum D-branes as topological quantum branes. Moreover, the low dimensional wild embedding, i.e. $S^2 \rightarrow S^3$ expresses the existence of the non-trivial $B$-field on the quantum level. Next we construct a quantum version of an action using cyclic cohomology of $C^*$-algebra. In the classical limit this action reduces to the Born-Infeld one for flat branes given by tame embedding.

The basic technical ingredient of the analysis of small exotic $\mathbb{R}^4$’s enabling uncovering many applications also in string theory is the relation between exotic (small) $\mathbb{R}^4$’s and non-cobordant codimension-1 foliations of the $S^3$ as well gropes and wild embeddings as shown in\textsuperscript{7}. The foliation are classified by the Godbillon-Vey class as element of the cohomology group $H^3(S^3, \mathbb{R})$. By using the $S^1$-gerbes it was possible to interpret the integral elements $H^3(S^3, \mathbb{Z})$ as characteristic classes of a $S^1$-gerbe over $S^3$\textsuperscript{13}.

## 2. Small exotic $\mathbb{R}^4$, gropes and foliations

In this short section we will only give a rough overview about a relation between small exotic $\mathbb{R}^4$ and foliations. Some of the details can be found in\textsuperscript{7} and more detailed approach will appear here\textsuperscript{10}. At first we will start with some facts about exotic 4-spaces.

An exotic $\mathbb{R}^4$ is a topological space with $\mathbb{R}^4$—topology but with a different (i.e. non-diffeomorphic) smoothness structure than the standard $\mathbb{R}^4_{\text{std}}$ getting its differential structure from the product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The exotic $\mathbb{R}^4$ is the only Euclidean space $\mathbb{R}^n$ with an exotic smoothness structure. The exotic $\mathbb{R}^4$ can be constructed in two ways: by the failure to arbitrarily split a smooth 4-manifold into pieces (large exotic $\mathbb{R}^4$) and by the failure of the so-called smooth h-cobordism theorem (small exotic $\mathbb{R}^4$). Here we will use the second method.

Consider the following situation: one has two topologically equivalent (i.e. homeomorphic), simple-connected, smooth 4-manifolds $M, M'$, which are not diffeomorphic. There are two ways to compare them. First one calculates differential-topological invariants like Donaldson polynomials\textsuperscript{26} or Seiberg-Witten invariants...
But there is another possibility: It is known that one can change a manifold $M$ to $M'$ by using a series of operations called surgeries. This procedure can be visualized by a 5-manifold $W$, the cobordism. The cobordism $W$ is a 5-manifold having the boundary $\partial W = M \sqcup M'$. If the embedding of both manifolds $M, M'$ in to $W$ induces homotopy-equivalences then $W$ is called an h-cobordism. Furthermore we assume that both manifolds $M, M'$ are compact, closed (no boundary) and simply-connected. As Freedman showed a h cobordism implies a homeomorphism, i.e. h-cobordant and homeomorphic are equivalent relations in that case. Furthermore, for that case the mathematicians are able to prove a structure theorem for such h-cobordisms:

Let $W$ be a h-cobordism between $M, M'$. Then there are contractible submanifolds $A \subset M, A' \subset M'$ together with a sub-cobordism $V \subset W$ with $\partial V = A \sqcup A'$, so that the h-cobordism $W \setminus V$ induces a diffeomorphism between $M \setminus A$ and $M' \setminus A'$. Thus, the smoothness of $M$ is completely determined (see also) by the contractible submanifold $A$ and its embedding $A \hookrightarrow M$ determined by a map $\tau : \partial A \rightarrow \partial A$ with $\tau \circ \tau = id_{\partial A}$ and $\tau \neq \pm id_{\partial A}$ ($\tau$ is an involution). One calls $A$, the Akbulut cork. According to Freedman, the boundary of every contractible 4-manifold is a homology 3-sphere. This theorem was used to construct an exotic $\mathbb{R}^4$. Then one considers a tubular neighborhood of the sub-cobordism $V$ between $A$ and $A'$. The interior $\text{int}(V)$ (as open manifold) of $V$ is homeomorphic to $\mathbb{R}^4$. If (and only if) $M$ and $M'$ are homeomorphic, but non-diffeomorphic 4-manifolds then $\text{int}(V) \cap M$ is an exotic $\mathbb{R}^4$. As shown by Bizaca and Gompf, one can use $\text{int}(V)$ to construct an explicit handle decomposition of the exotic $\mathbb{R}^4$. We refer for the details of the construction to the papers or to the book. The idea is simply to use the cork $A$ and add some Casson handle $CH$ to it. The interior of this construction is an exotic $\mathbb{R}^4$. Therefore we have to consider the Casson handle and its construction in more detail. Briefly, a Casson handle $CH$ is the result of attempts to embed a disk $D^2$ into a 4-manifold. In most cases this attempt fails and Casson looked for a substitute, which is now called a Casson handle. Freedman showed that every Casson handle $CH$ is homeomorphic to the open 2-handle $D^2 \times \mathbb{R}^2$ but in nearly all cases it is not diffeomorphic to the standard handle. The Casson handle is built by iteration, starting from an immersed disk in some 4-manifold $M$, i.e. a map $D^2 \rightarrow M$ with injective differential. Every immersion $D^2 \rightarrow M$ is an embedding except on a countable set of points, the double points. One can kill one double point by immersing another disk into that point. These disks form the first stage of the Casson handle. By iteration one can produce the other stages. Finally consider not the immersed disk but rather a tubular neighborhood $D^2 \times D^2$ of the immersed disk, called a kinky handle, including each stage. The union of all neighborhoods of all stages is the Casson handle $CH$. So, there are two input data involved with the construction of a $CH$: the number of double points in each stage and their orientation $\pm$. Thus we can visualize the Casson handle $CH$ by a tree: the root is the immersion $D^2 \rightarrow M$ with $k$ double points, the first stage forms the next
level of the tree with \( k \) vertices connected with the root by edges etc. The edges are evaluated using the orientation \( \pm \). Every Casson handle can be represented by such an infinite tree.

The main idea of the relation between small exotic \( \mathbb{R}^4 \) is the usage of a radial family of small exotic \( \mathbb{R}^4 \), i.e. a continuous family of exotic \( \{ \mathbb{R}^4_{\rho} \}_{\rho \in [0, +\infty]} \) with parameter \( \rho \) so that \( \mathbb{R}^4_{\rho} \) and \( \mathbb{R}^4_{\rho'} \) are non-diffeomorphic for \( \rho \neq \rho' \). This radial family has a natural foliation (see Theorem 3.2 in [25] which can be induced by a polygon \( \mathbb{P} \) in the two-dimensional hyperbolic space \( \mathbb{H}^2 \). The area of \( \mathbb{P} \) is a well-known invariant, the Godbillon-Vey class as element in \( H^3(S^3, \mathbb{R}) \), determining a codimension-one foliation on the 3-sphere (firstly constructed by Thurston [37], see also the book [36] chapter VIII for the details). This 3-sphere is part of the boundary \( \partial A \) of the Akbulut cork \( A \) (or better there is an embedding \( S^3 \to \partial A \)). Furthermore one can show that the codimension-one foliation of the 3-sphere induces a codimension-one foliation of \( \partial A \) so that the area of the corresponding polygons (and therefore the invariants) agree. The Godbillon-Vey invariant \([GV] \in H^3(S^3, \mathbb{R})\) of the foliation is related to the parameter of the radial family by \( \langle GV, [S^3] \rangle = \rho^2 \) using the pairing between cohomology and homology (the fundamental class \([S^3] \in H_3(S^3)\)).

Thus we are able to obtain a relation between an exotic \( \mathbb{R}^4 \) (of Bizaca as constructed from the failure of the smooth h-cobordism theorem) and codimension-one foliation of the \( S^3 \). Two non-diffeomorphic exotic \( \mathbb{R}^4 \) implying non-cobordant codimension-one foliations of the 3-sphere described by the Godbillon-Vey class in \( H^3(S^3, \mathbb{R}) \) (proportional to the are of the polygon). This relation is very strict, i.e. if we change the Casson handle then we must change the polygon. But that changes the foliation and vice versa. Finally we obtained the result:

*The exotic \( \mathbb{R}^4 \) (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in \( H^3(S^3, \mathbb{R}) \) of a 3-sphere seen as submanifold \( S^3 \subset \mathbb{R}^4 \). We say: the exoticness is localized at a 3-sphere inside the small exotic \( \mathbb{R}^4 \).*

3. RR charges of D6-Branes in the presence of B-field

In this section, we will describe the direct reference of 4-dimensional structures to the dynamics of special higher dimensional branes, the D6-brane, in flat spacetime. This D6-brane is usually involved in building various „realistic“ 4-dimensional models derived from brane configurations. We will analyze this case separately along with the discussion of compactifications in string theory in a forthcoming paper.

Let us consider the D6-brane of IIA string theory in flat 10 dimensional spacetime and assume a vanishing B-field. The world-volumes of flat Dp-branes are classified by \( K_1(\mathbb{R}^{p+1}) \) where this K-homology group is understood as \( K_1(C_0(\mathbb{R}^{p+1})) \). Then there is the isomorphism \( K_1(\mathbb{R}^{p+1}) = K_1(S^{p+1}) \) between the K-groups induced by the isomorphism of the reduced \( C^* \) algebra of functions \( C_0(\mathbb{R}^{p+1}) = C(S^{p+1}) \). Their charges, constraining the dynamics of the brane, are dually described by \( K^1(\mathbb{R}^{9-p}) = K^1(S^{9-p}) \). In the case of the D6-branes, the group \( K^1(S^3) \)
classifies the RR charges of flat D6-branes in flat 10-dimensional spacetime.

In case of a non-vanishing $B$-field for a stable D6-brane, the $B$-field needs be non-trivial on the space $\mathbb{R}^3$ transversal to the world-volume (on the brane we have $dB = H$). Hence the $B$-field must be nontrivial on the space $S^3$. It is convenient to adopt (see [17], p. 654) the definition of $B$-field on a manifold $X$ as a class of gerbe (with connection), which refers directly to the topologically non-trivial $H^3(S^3, \mathbb{Z})$ classes instead of local 2-form $B$:

Definition 3.1.

A $B$-field $(X, H)$ is a gerbe with one-connection over $X$ and characteristic class $[H] \in H^3(X, \mathbb{Z})$ which is an NS–NS H-flux.

Then the classification of D6-brane charges in IIA type superstring theory in flat space is influenced by the presence of non-trivial B-field and now is given by the twisted K-theory $K_H(S^3)$, so that $K^1(S^3, H) = \mathbb{Z}_k$ with $0 \neq [H] = k \in H^3(S^3, \mathbb{Z})$. Hence the dynamics of D6-branes in type IIA superstring theory on flat spacetime is influenced by a non-zero B-field.

Following our philosophy already implicitly present in our previous work, the source of the non-trivial $B$-field on $S^3$ (hence $H \neq 0$) is the exoticness of the ambient $\mathbb{R}^4$. This result is motivated by our work that some (small) exotic $\mathbb{R}^4_H$’s correspond to non-trivial classes $[H] \in H^3(S^3, \mathbb{Z})$ and conversely, where $S^3$ is part of the boundary of the Akbulut cork [77]. Moreover, exotic smoothness of $\mathbb{R}^4_H$ twists the K-theory groups $K^*(S^3)$ [8] where the 3-sphere $S^3$ lies at the boundary of the Akbulut cork. Hence the dynamics induced by D6-branes in the spacetime $\mathbb{R}^4_H \times \mathbb{R}^5.1$ is equivalent to the dynamics induced by a D6-brane with a non-zero B-field on the transversal $\mathbb{R}^3$ compactified to $S^3$. Finally we get:

Theorem 3.1.

$RR$ charges of D6-branes in string theory IIA in the presence of a non-trivial $B$-field ($H \neq 0$), (these charges are classified by $K_H(S^3)$, and $[H] \in H^3(S^3, \mathbb{Z})$), are related to exotic smoothness of small $\mathbb{R}^4_H$. This exotic $\mathbb{R}^4_H$ corresponds to $[H]$ which twists $K(S^3)$ [8], where $S^3 \subset \mathbb{R}^4$ lies at the boundary of the Akbulut cork and $S^3$ is transverse to the branes. Thus, changing the smoothness of $\mathbb{R}^4$ gives rise to the change of the allowed charges for D6 branes inducing a change of the dynamics.

We saw that the geometric realization of (classical) D-branes in certain backgrounds of string theory is correlated with small exotic $\mathbb{R}^4$’s which can be all embedded in the standard smooth $\mathbb{R}^4$. As was shown in our previous paper [11], quantum D-branes correspond to the net of exotic smooth $\mathbb{R}^4$’s embedded in certain exotic smooth $\mathbb{R}^4$. An intriguing interpretation for this correspondence can be given by: in some limit of IIA superstring theory, small exotic smooth $\mathbb{R}^4$’s can be considered as carrying the $RR$ charges of D6 branes. A generalization of this concept will be studied in the next section.
4. D6-brane charges and embeddings of \((4k-1)\)- into \(6k\)-manifolds

In the usual definition of Dp-branes, one considers embedded p-dimensional objects in some higher-dimensional space. In the presence of NS-NS H-fluxes, curved (twisted) classical branes are defined still as submanifolds with some extra topological condition (cancelling the anomaly) (see \(\text{Def. 1.14, p. 654}\)):

**Definition 4.1.**

Twisted D-brane in a \(B\)-field \((X, H)\) is a triple \((W, E, \phi)\), where \(\phi : W \to X\) is a closed, embedded oriented submanifold with \(\phi^* H = W_3(W)\), and \(E \in K_0(W)\)

where \(W_3(W) \in H^3(W, \mathbb{Z})\) is the third Stiefel-Whitney class of the normal bundle of \(W\) in \(X, N(X/W)\). This case of non-trivial H-flux directly refers to the non-commutative geometry tools hence quantum D-branes are considered naturally here \(\text{[17]}\). It is known \(\text{[17]}\) that in the presence of a topologically non-trivial \(B\)-field world-volumes of D-branes are rather described as noncommutative spaces in Conne’s sense. We try to find a topological characterisation for D-branes as a kind of embedding also in the quantum regime. These branes emerge in case of the non-trivial H-flux too, and we call them *topological quantum branes*. However the reason for the existence of this H-flux is deeply rooted in the geometry of some exotic \(\mathbb{R}^4\). Exotic \(\mathbb{R}^4\) itself refers to non-commutative spaces and tools from non-commutative geometry, which can be seen as one reasons behind the quantum description of topological branes .

As explained in the previous section, the charges of D-branes are given by some (twisted) K-theory classes also related to (twisted) cohomology. In case of a D6-brane, the charge is classified by the twisted K-theory \(K_H(S^3)\) with \([H] \in H^3(S^3, \mathbb{Z})\) (Čech 3-cocycle). Therefore one has a 3-sphere determining the charge of a 6-dimensional object. To simplify the discussion, we will compactify the (possible) infinite D6-brane to a 6-sphere \(S^6\) in the following.

Now we start with a short discussion of embeddings \(S^3 \to S^6\) as an example \(k = 1\) of a general map \(S^{4k-1} \to S^{6k}\) to understand the charge classification. As Haefliger \(\text{[32]}\) showed, the isotopy classes of these embeddings are determined by the integer classes (Hopf invariant) in \(H^3(S^3, \mathbb{Z})\). Thus the \(4k-1\) space is knotted in the \(6k\) space. This phenomenon depends strongly on smoothness, i.e. it disappears for continuous or PL embeddings. Usually every \(n\)-sphere or every homology \(n\)-sphere unknots (in PL or TOP) in \(\mathbb{R}^m\) for \(m \geq n + 3\), i.e. for codimension \(m - n = 3\) or higher. Of course, one has the usual knotting phenomena in codimension 2 and the codimension 1 was shown to be unique for embeddings \(S^n \to S^{n+1}\) (for \(n \geq 6\)) but is hard to solve in other cases.

Let \(\Sigma \to S^6\) be an embedding of a homology 3-sphere \(\Sigma\) (containing the case \(S^3\)). Then the normal bundle of \(F\) is trivial (definition of an embedding) and homotopy classes of trivialisations of the normal bundle (normal framings) are classified by the homotopy classes \([\Sigma, SO(3)]\) with respect to some fixed framing. There is an isomorphism \([\Sigma, SO(3)] = [\Sigma, S^2]\) (so-called Pontrjagin-Thom construction) and
[\Sigma, S^2] \text{ can be identified with } H^3(\Sigma, \mathbb{Z}) = \mathbb{Z}. \text{ That is one possible way to get the classification of isotopy classes of embeddings } \Sigma \rightarrow S^6 \text{ by elements of } H^3(\Sigma, \mathbb{Z}) = \mathbb{Z}. \text{ A class } [H] \text{ in } H^3(\Sigma, \mathbb{Z}) \text{ determines via an injective homomorphism a (deRham-)cocycle } H \in H^3(\Sigma, \mathbb{R}). \text{ If } H \text{ is the field strength for the B-field then the 3-form } H \text{ must be a multiply of the volume form on } \Sigma \text{ and we have }
\int_{\Sigma} H = Q \neq 0
\text{ by the usual pairing between homology and cohomology. By the cellular approximation theorem, the elements in } H^3(\Sigma, \mathbb{Z}) \text{ are determined by } H^3(S^3, \mathbb{Z}). \text{ Combined with our result that } H^3(S^3, \mathbb{Z}) \text{ determines some exotic } \mathbb{R}^4 \text{ we have shown: }

\textbf{Theorem 4.1.} \textit{(The topological origins of the allowed D6-brane charges)}

Let \( \mathbb{R}^4_H \) be some exotic \( \mathbb{R}^4 \) determined by some 3-form \( H \), i.e. by a codimension-1 foliation on the boundary \( \partial A \) of the Akbulut cork \( A \). The codimension-1 foliation on \( \partial A \) is determined by \( H^3(\partial A, \mathbb{R}) \). Each integer class in \( H^3(\partial A, \mathbb{Z}) \) determines the isotopy class of an embedding \( \partial A \rightarrow S^6 \). Hence, the group of allowed charges of D6-branes in the presence of B-field \( H \), i.e. \( K_H^\ast(S^3) \) is determined equivalently by the isotopy classes of embeddings \( \partial A \rightarrow S^6 \). The classes of H-field are topologically determined by the isotopy classes of the embeddings, which affects the allowed charges of D6-branes.

But more is true. Given two embeddings \( F_i : \Sigma_i \rightarrow S^6 \) between two homology 3-spheres \( \Sigma_i \) for \( i = 0, 1 \). A homology cobordism is a cobordism between \( \Sigma_0 \) and \( \Sigma_1 \). This cobordism can be embedded in \( S^6 \times [0, 1] \) determining the homology bordism class of the embedding. Then two embeddings of an oriented homology 3-sphere in \( S^6 \) are isotopic if and only if they are homology bordant.

\textbf{5. From wild embeddings to topological quantum D-branes}

In this section we try to give a geometric approach to quantum D-branes using wild embeddings of trivial complexes into \( S^n \) or \( \mathbb{R}^n \). This point of view is supported by the Theorem 4.1 above. Here we will describe a dimension-independent way: every wild embedding \( j \) of a \( p \)-dimensional complex \( K \) into the \( n \)-dimensional sphere \( S^n \) is determined by the fundamental group \( \pi_1(S^n \setminus j(K)) \) of the complement. This group is perfect and uniquely representable by a 2-dimensional complex, a singular disk or grope (see [19]). As we showed in [17], the exotic \( \mathbb{R}^4 \) is related to a grope. Thus, these constructed topological quantum D-branes are determined by exotic \( \mathbb{R}^4 \)'s which act as a kind of germ for the branes.

\textbf{5.1. Wild and tame embeddings}

We call a map \( f : N \rightarrow M \) between two topological manifolds an embedding if \( N \) and \( f(N) \subset M \) are homeomorphic to each other. From the differential-topological
point of view, an embedding is a map $f : N \to M$ with injective differential on each point (an immersion) and $N$ is diffeomorphic to $f(N) \subset M$. An embedding $i : N \hookrightarrow M$ is tame if $i(N)$ is represented by a finite polyhedron homeomorphic to $N$. Otherwise we call the embedding wild. There are famous wild embeddings like Alexander's horned sphere or Antoine's necklace. In physics one uses mostly tame embeddings but as Cannon mentioned in his overview [18], one needs wild embeddings to understand the tame one. As shown by us [7], wild embeddings are needed to understand exotic smoothness. As explained in [18] by Cannon, tameness is strongly connected to another topic: decomposition theory (see the book [24]).

Two embeddings $f, g : N \to M$ are said to be isotopic, if there exists a homeomorphism $F : M \times [0, 1] \to M \times [0, 1]$ such that

1. $F(y, 0) = (y, 0)$ for each $y \in M$ (i.e. $F(\cdot, 0) = id_M$)
2. $F(f(x), 1) = g(x)$ for each $x \in N$, and
3. $F(M \times \{t\}) = M \times \{t\}$ for each $t \in [0, 1]$.

If only the first two conditions can be fulfilled then one call it concordance. Embeddings are usually classified by isotopy. An important example is the embedding $S^1 \to \mathbb{R}^3$, known as knot, where different knots are different isotopy classes.

5.2. Real cohomology classes and wild embeddings

Wild embeddings are important to understand usual embeddings. Consider a closed curve in the plane. By common sense, this curve divides the plane into an interior and an exterior area. The Jordan curve theorem agrees with that view completely. But what about one dimension higher, i.e. consider the embedding $S^2 \to \mathbb{R}^3$? Alexander was the first who constructed a counterexample, Alexander's horned sphere [4], as wild embedding $I : D^3 \to \mathbb{R}^3$. The main property of this wild object $D^3_W = I(D^3)$ is the non-simple connected complement $\mathbb{R}^3 \setminus D^3_W$. This property is a crucial point of the following discussion. Given an embedding $I : D^3 \to \mathbb{R}^3$ which induces a decomposition $\mathbb{R}^3 = I(D^3) \cup (\mathbb{R}^3 \setminus I(D^3))$. In case, the embedding is tame, the image $I(D^3)$ is given by a finite complex and every part of the decomposition is contractable, i.e. especially $\pi_1(\mathbb{R}^3 \setminus I(D^3)) = 0$. For a wild embedding, $I(D^3)$ is an infinite complex (but contractable). The complement $\mathbb{R}^3 \setminus I(D^3)$ is given by a sequence of spaces so that $\mathbb{R}^3 \setminus I(D^3)$ is non-simple connected (otherwise the embedding must be tame) having the homology of a point (that is true for every embedding). Especially $\pi_1(\mathbb{R}^3 \setminus I(D^3))$ is non-trivial whereas its abelization $H_1(\mathbb{R}^3 \setminus I(D^3)) = 0$ vanishes. Therefore $\pi_1$ is generated by the commutator subgroup $[\pi_1, \pi_1]$ with $[a, b] = aba^{-1}b^{-1}$ for two elements $a, b \in \pi_1$, i.e. $\pi_1$ is a perfect group.

In the following we will concentrate on wild embeddings of spheres $S^n$ into spheres $S^m$ equivalent to embeddings of $\mathbb{R}^n$ into $\mathbb{R}^m$ relative to the infinity $\infty$ point or to relative embeddings of $D^n$ into $D^m$ (relative to its boundary). From the physical point of view, in the case of flat D-branes when B-field is trivial, branes are
seen as topological objects of a trivial type like \( \mathbb{R}^n, S^n \) or \( D^n \). Lets start with the case of a finite \( k \)-dimensional polyhedron \( K^k \) (i.e. a piecewise-linear version of a \( k \)-disk \( D^k \)). Consider the wild embedding \( i : K \to S^n \) with \( 0 \leq k \leq n-3 \) and \( n \geq 7 \). Then, as shown in [27], the complement \( S^n \setminus i(K) \) is non-simple connected with a countable generated (but not finitely presented) fundamental group \( \pi_1(S^n \setminus i(K)) = \pi \). Furthermore, the group \( \pi \) is perfect (i.e. generated by the commutator subgroup \([\pi, \pi] = \pi \) implying \( H_1(\pi) = 0 \) and \( H_2(\pi) = 0 \) (\( \pi \) is called a superperfect group).

With other words, \( \pi \) is a group where every element \( x \in \pi \) can be generated by a commutator \( x = [a, b] = aba^{-1}b^{-1} \) (including the trivial case \( x = a, b = e \)). By using geometric group theory, we can represent \( \pi \) by a grope (or generalized disk, see Cannon [19]), i.e. a hierarchical object with the same fundamental group as \( \pi \) (see the next subsection). In [17] the grope was used to construct a non-trivial involution of the 3-sphere connected with a codimension-1 foliation of the 3-sphere classified by the real cohomology classes \( H^3(S^3, \mathbb{R}) \). By using the suspension

\[
\Sigma X = X \times [0, 1]/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0, 1])
\]

of a topological space \((X, x_0)\) with base point \( x_0 \), we have an isomorphism of cohomology groups \( H^n(S^3) = H^{n+1}(\Sigma S^3) \). Thus the class in \( H^3(S^3, \mathbb{R}) \) induces classes in \( H^n(S^n, \mathbb{R}) \) for \( n > 3 \) represented by a wild embedding \( i : K \to S^n \) for some \( k \)-dimensional polyhedron. Then small exotic \( \mathbb{R}^4 \) determines also wild embeddings in higher dimensions, hence higher real cohomology classes of \( n \)-spheres:

**Theorem 5.1.**

Let \( \mathbb{R}^4_H \) be some exotic \( \mathbb{R}^4 \) determined by element in \( H^3(S^3, \mathbb{R}) \), i.e. by a codimension-1 foliation on the boundary \( \partial A \) of the Akbulut cork \( A \). Each wild embedding \( i : K^3 \to S^p \) for \( p > 6 \) of a 3-dimensional polyhedron (as part of \( S^3 \)) determines a class in \( H^n(S^n, \mathbb{R}) \) which represents a wild embedding \( i : K^p \to S^n \) of a \( p \)-polyhedron into \( S^n \).

Now we consider a class of topological quantum Dp-branes as these branes which are determined by the wild embeddings \( i : K^p \to S^n \) as above and in the classical and flat limit correspond to tame embeddings. The quantum character of such Dp-branes is driven by the presence of the non-trivial B-field which here is encoded in the wild embeddings \( i : K^3 \to S^p \). This in turn is derived from exotic \( \mathbb{R}^4 \) and is generated by the wild embedding \( S^2 \to S^4 \) [21,11]. In the next subsections we will examine the quantum character of wild embeddings and see how this is related to the class of quantum D-branes we deal with. Next we will show, directly from the action for such branes, how the tame embedding emerges in the classical limit.

### 5.3. \( C^* \)-algebras associated to wild embeddings

As described above, a wild embedding \( j : K \to S^n \) of a polyhedron \( K \) is characterized by its complement \( M(K, j) = S^n \setminus j(K) \) which is non-simply connected (i.e. the fundamental group \( \pi_1(M(K, j)) \) is non-trivial). The fundamental group \( \pi_1(M(K, j)) = \pi \) of the complement \( M(K, j) \) is a superperfect group,
i.e. \( \pi \) is identical to its commutator subgroup \( \pi = [\pi, \pi] \) (then \( H_1(\pi) = 0 \)) and 
\( H_2(\pi) = 0 \). This group is not finite in case of a wild embedding. Here we use 
gropes to represent \( \pi \) geometrically. The idea behind that approach is very sim-
ple: the fundamental group of the 2-diemsional torus \( T^2 \) is the abelian group 
\( \pi_1(T^2) = \langle a, b \mid [a, b] = aba^{-1}b^{-1} = e \rangle = \mathbb{Z} \oplus \mathbb{Z} \) generated by the two standard slopes 
a, b corresponding to the commuting generators of \( \pi_1(T^2) \). The capped torus \( T^2 \setminus D^2 \) has an additional element \( c \) in the fundamental group generated by the boundary 
\( \partial(T^2 \setminus D^2) = S^1 \). This element is represented by the commutator \( c = [a, b] \). In case 
of our superperfect group, we have the same problem: every element \( c \) is generated 
by the commutator \( [a, b] \) of two other elements \( a, b \) which are also represented by 
commutators etc. Thus one obtains a hierarchical object, a generalized 2-disk or a 
grope (see Fig. 1). Now we describe two ways to associate a \( C^* \)–algebra to this 
grope. This first approach uses a combination of our previous papers [7,8]. Then 
every grope determines a codimension-1 foliation of the 3-sphere and vice versa. 
The leaf-space of this foliation is a factor \( III_1 \) von Neumann algebra and we have a 
\( C^* \)–algebra for the holonomy groupoid. For later usage, we need a more direct way 
to construct a \( C^* \)–algebra from a wild embedding or grope. The main ingredient is 
the superperfect group \( \pi \), countable generated but not finitely presented group \( \pi \).

Given a grope \( \mathcal{G} \) representing via \( \pi_1(\mathcal{G}) = \pi \) the (superperfect) group \( \pi \). Now 
we define the \( C^* \)–algebra \( C^*(\mathcal{G}, \pi) \) associated to the grope \( \mathcal{G} \) with group \( \pi \). The 
基本 elements of this algebra are smooth half-densities with compact supports on 
\( \mathcal{G} \), \( f \in C^\infty_c(\mathcal{G}, \Omega^{1/2}) \), where \( \Omega^{1/2} \) for \( \gamma \in \pi \) is the one-dimensional complex vector

![Fig. 1. An example of a grope](image-url)
space of maps from the exterior power $\Lambda^2 L$, of the union of levels $L$ representing $\gamma$ to $\mathbb{C}$ such that 

$$\rho(\lambda \nu) = |\lambda|^{1/2} \rho(\nu) \quad \forall \nu \in \Lambda^2 L, \lambda \in \mathbb{R}.$$  

For $f, g \in C^\infty_c(G, \Omega^{1/2})$, the convolution product $f * g$ is given by the equality 

$$(f * g)(\gamma) = \int_{[\gamma_1, \gamma_2] = \gamma} f(\gamma_1)g(\gamma_2)$$

Then we define via $f^*(\gamma) = \overline{f(\gamma^{-1})}$ a *operation making $C^\infty_c(G, \Omega^{1/2})$ into a *algebra. For each capped torus $T$ in some level of the grope $G$, one has a natural representation of $C^\infty_c(G, \Omega^{1/2})$ on the $L^2$ space over $T$. Then one defines the representation 

$$(\pi_x(f)\xi)(\gamma) = \int_{[\gamma_1, \gamma_2] = \gamma} f(\gamma_1)\xi(\gamma_2) \quad \forall \xi \in L^2(T).$$

The completion of $C^\infty_c(G, \Omega^{1/2})$ with respect to the norm 

$$||f|| = \sup_{x \in M} ||\pi_x(f)||$$

makes it into a $C^*$-algebra $C^\infty_c(G, \pi)$. Finally we are able to define the $C^*$-algebra associated to the wild embedding:

**Definition 5.1.** Let $j : K \to S^n$ be a wild embedding with $\pi = \pi_1(S^n \setminus j(K))$ as fundamental group of the complement $M(K, j) = S^n \setminus j(K)$. The $C^*$-algebra $C^\infty_c(K, j)$ associated to the wild embedding is defined to be $C^\infty_c(K, j) = C^\infty_c(G, \pi)$ the $C^*$-algebra of the grope $G$ with group $\pi$.

To get an impression of this superperfect group $\pi$, we consider a representation $\pi \to G$ in some infinite group. As the obvious example for $G$ we choose the infinite union $GL(\mathbb{C}) = \bigcup_{n=1}^{\infty} GL(n, \mathbb{C})$ of complex, linear groups (induced from the embedding $GL(n, \mathbb{C}) \to GL(n + 1, \mathbb{C})$ by an inductive limes process). Then we have a homomorphism 

$$U : \pi \to GL(\mathbb{C})$$

mapping a commutator $[a, b] \in \pi$ to $U([a, b]) \in [GL(\mathbb{C}), GL(\mathbb{C})]$ into the commutator subgroup of $GL(\mathbb{C})$. But every element in $\pi$ is generated by a commutator, i.e. we have 

$$U : \pi \to [GL(\mathbb{C}), GL(\mathbb{C})]$$

and we are faced with the problem to determine this commutator subgroup. Actually, one has Whitehead’s lemma (see [44]) which determines this subgroup to be the group of elementary matrices $E(\mathbb{C})$. One defines the elementary matrix $e_{ij}(a)$ in $E(n, \mathbb{C})$ to be the $(n \times n)$ matrix with 1’s on the diagonal, with the complex number
Theorem 5.2. Let \( j : K \to S^n \) be a wild embedding with \( \pi = \pi_1(S^n \setminus j(K)) \) as fundamental group of the complement \( M(K, j) = S^n \setminus j(K) \) and \( C^* \)-algebra \( C_c^\infty(K, j) \). Given another wild embedding \( i \) with \( C^* \)-algebra \( C_c^\infty(K, i) \). The elements of \( KK(C_c^\infty(K, j), C_c^\infty(K, i)) \) are the isometry classes of the wild embedding \( j \) relative to \( i \).

5.4. Isotopy classes of wild embeddings and KK theory

In section 5.1, we introduce the notion of isotopy classes for embeddings. Given two embeddings \( f, g : N \to M \) with a special map \( F : M \times [0, 1] \to M \times [0, 1] \) as deformation of \( f \) into \( g \), then both embeddings are isotopic to each other. The definition is independent of the tameness oder wilderness for the embedding. Now we specialize to our case of wild embeddings \( f, g \in M(K, f) \) and \( M(K, g) \). The map \( F : S^n \times [0, 1] \to S^n \times [0, 1] \) induces a homotopy of the complements \( M(K, f) \simeq M(K, g) \) giving an isomorphism of the fundamental groups \( \pi_1(M(K, g)) = \pi_1(M(K, f)) \). Thus, the isotopy class of the wild embedding \( f \) is completely determined by the \( M(K, f) \) up to homotopy. Using Connes work on operator algebras of foliation, our construction of the \( C^* \)-algebra for a wild embedding is functorial, i.e. an isotopy of the embeddings induces an isomorphism between the corresponding \( C^* \)-algebras. Given two non-isotopic, wild embeddings then we have a homomorphism between the \( C^* \)-algebras only. But every homomorphism (which is not a isomorphism) between \( C^* \)-algebras \( A, B \) gives an element of \( KK(A, B) \) and vice versa. Thus,

Theorem 5.2.

Let \( j : K \to S^n \) be a wild embedding with \( \pi = \pi_1(S^n \setminus j(K)) \) as fundamental group of the complement \( M(K, j) = S^n \setminus j(K) \) and \( C^* \)-algebra \( C_c^\infty(K, j) \). Given another wild embedding \( i \) with \( C^* \)-algebra \( C_c^\infty(K, i) \). The elements of \( KK(C_c^\infty(K, j), C_c^\infty(K, i)) \) are the isometry classes of the wild embedding \( j \) relative to \( i \).

6. Wild embeddings as quantum D-branes

Given a wild embedding \( f : K \to S^n \) with \( C^* \)-algebra \( C^*(K, f) \) and group \( \pi = \pi_1(S^n \setminus f(K)) \). In this section we will derive an action for this embedding to derive the D-brane action in the classical limit. The starting point is our remark above (see section 2) that the group \( \pi \) can be geometrically constructed by using a grope \( \mathcal{G} \) with \( \pi = \pi_1(\mathcal{G}) \). This grope was used to construct a codimension-1 foliation on the
3-sphere classified by the Godbillon-Vey invariant. This class can be seen as element of $H^3(BG, \mathbb{R})$ with the holonomy groupoid $G$ of the foliation. The strong relation between the grope $G$ and the foliation gives an isomorphism for the $C^*$-algebra which can be easily verified by using the definitions of both algebras. As shown by Connes [21,22], the Godbillon-Vey class $GV$ can be expressed as cyclic cohomology class (the so-called flow of weights)

$$GV_{HC} \in HC^2(C_c^\infty(G)) \simeq HC^2(C_c^\infty(G, \pi))$$

of the $C^*$-algebra for the foliation isomorphic to the $C^*$-algebra for the grope $G$. Then we define an expression

$$S = Tr_\omega(GV_{HC})$$

uniquely associated to the wild embedding ($Tr_\omega$ is the Dixmier trace). $S$ is the action of the embedding. Because of the invariance for the class $GV_{HC}$, the variation of $S$ vanishes if the map $f$ is a wild embedding. But this expression is not satisfactory and cannot be used to get the classical limit. For that purpose we consider the representation of the group $\pi$ into the group $E(\mathbb{C})$ of elementary matrices. As mentioned above, $\pi$ is countable generated and the generators can be arranged in the embeddings space. Then we obtain matrix-valued functions $X^\mu \in C_c^\infty(E(\mathbb{C}))$ as the image of the generators of $\pi$ w.r.t. the representation $\pi \to E(\mathbb{C})$ labelled by the dimension $\mu = 1, \ldots, n$ of the embedding space $S^n$. Via the representation $\iota: \pi \to E(\mathbb{C})$, we obtain a cyclic cocycle in $HC^2(C_c^\infty(E(\mathbb{C})))$ generated by a suitable Fredholm operator $F$. Here we use the standard choice $F = D|D|^{-1}$ with the Dirac operator $D$ acting on the functions in $C_c^\infty(E(\mathbb{C}))$. Then the cocycle in $HC^2(C_c^\infty(E(\mathbb{C})))$ can be expressed by

$$\iota_* GV_{HC} = \eta_{\mu\nu}[F, X^\mu][F, X^\nu]$$

using a metric $\eta_{\mu\nu}$ in $S^n$ via the pull-back using the representation $\iota: \pi \to E(\mathbb{C})$. Finally we obtain the action

$$S = Tr_\omega([F, X^\mu][F, X_\mu]) = Tr_\omega([D, X^\mu][D, X_\mu]|D|^{-2})$$

which can be evaluated by using the heat-kernel of the Dirac operator $D$. 

6.1. The classical limit

Similar to the case of the von Neumann algebra of a foliation, the non-commutativity of the $C^*$-algebra $C^*(K, f)$ is induced by the wild embedding $f: K \to S^n$. The complexity of the group $\pi = \pi_1(S^n \setminus f(K))$ is related to the complexity of the $C^*$-algebra constructed above. Therefore a tame embedding has a trivial group $\pi$ and we obtain for the $C^*$-algebra $C_c^\infty(K, f) = \mathbb{C}$, i.e. every operator is a multiplication operator (multiplication with a complex number).

From the physical point of view, the non-triviality of the $C^*$-algebra has an interpretation (via the GNS representation) as the observables algebra of a quantum
system. In our case, the non-triviality of the $C^*$-algebra is connected with the wildness of the embedding or the wild embedding is connected with a quantum system. But then the classical limit is equivalent to choose a tame embedding $f : K \to S^n$ of a $p$-dimensional complex $K$. The Dirac operator $D$ on $K$ acts on usual square-integrable functions, so that $[D, X^\mu] = dX^\mu$ is finite. The action (1) reduces to

$$ S = \text{Tr}_\omega(\eta_{\mu\nu}(\partial_k X^\mu \partial^k X^\nu)|D|^{-2}) $$

where $\mu, \nu = 1, \ldots, n$ is the index for the coordinates on $S^n$ and $k = 1, \ldots, p$ represents the index of the complex. From the physical point we expect to obtain an action which describes the embedding of the brane. For that purpose, we will choose a small fluctuation $\xi^k$ of a fixed embedding given by $X^\mu = (x^k + \xi^\mu)\delta^k_\mu$ with $\partial_k x^k = \delta^k_\mu$. Then we obtain

$$ \partial_k X^\mu \partial^k X^\nu = \delta^\mu_\xi \delta^\nu_\xi (1 + \partial_k \xi^\mu)(1 + \partial_k \xi^\nu) $$

and we use a standard argument to neglect the terms linear in $\partial \xi$; the fluctuation have no preferred direction and therefore only the square contributes. Then we have

$$ S = \text{Tr}_\omega(\eta_{\mu\nu}(\delta^\mu_\xi \delta^\nu_\xi + \partial_k \xi^\mu \partial^k \xi^\nu)|D|^{-2}) $$

for the action. By using a result of [22] one obtains for the Dixmier trace

$$ \text{Tr}_\omega(|D|^{-2}) = 2 \int_K (\Phi_1) $$

with the first coefficient $\Phi_1$ of the heat kernel expansion [22]

$$ \Phi_1 = \frac{1}{6} R $$

and the action simplifies to

$$ S = \int_K \left( \eta_{\mu\nu}(\delta^\mu_\xi \delta^\nu_\xi + \partial_k \xi^\mu \partial^k \xi^\nu) \frac{1}{3} R \right) dvol(K) $$

for the main contributions where $R$ is the scalar curvature of $K$ (for $p > 2$). Usually we can assume a non-vanishing scalar curvature. Furthermore we can scale the fluctuation to get the action

$$ S = \int_K \left( \eta_{\mu\nu}(\partial_k \xi^\mu \partial^k \xi^\nu + \Lambda \eta^{\mu\nu}) \right) dvol(K) $$

for some number $\Lambda$ proportional to $R$. It is known that this action agrees with the usual (Born-Infeld) action

$$ S = \int_K \sqrt{\det (\eta_{\mu\nu}\partial_k \xi^\mu \partial^k \xi^\nu)} dvol(K) $$

of flat $p$-branes ($p > 2$) for $\Lambda > 0$ (i.e. $R > 0$) with vanishing $B$-field. Thus we obtain a (quantum) D-brane action by using wild embeddings for the description of
a quantum D-brane and the flux $H$ represented by the wild embedding $S^2 \rightarrow S^3$. These data, in the classical limit, reduce to the BI action for flat D$p$-brane. We will further investigate this point in a separate paper.

7. The 4-dimensional origin of quantum D-branes

The argumentation above can be simply resummed by the following arguments:

1. Given an embedding $f : K^p \rightarrow S^n$ of a $p$-complex $K$ into a $n$-sphere.
2. This embedding is wild, if the complement $S^n \setminus f(K)$ is non-simple connected, i.e. the fundamental group $\pi_1(S^n \setminus f(K)) \neq 0$ does not vanish.
3. We define a topological quantum $p$-brane as the wild embedding $f$.
4. The fundamental group $\pi = \pi_1(S^n \setminus f(K))$ is a perfect group, i.e. purely generated by the commutators $\pi = [\pi, \pi]$.
5. This group can be geometrically represented by a 2-complex, called a generalized disk or grope.
6. From this grope $G$ we constructed a non-trivial $C^*$-algebra $C^\infty_c(G, \pi)$.
7. Non-trivial B-field $H \in H^3(S^3, \mathbb{Z})$ is represented by the wild embedding $S^2 \rightarrow S^3$.

The grope is a 2-complex sometimes equipped with an embedding into the Euclidean space $\mathbb{E}^3$. As shown in [2], one can also use it to describe small exotic $\mathbb{R}^4$’s (see also some details in section [2]). At the first view we have two possible interpretations, the 2-dimensional grope and the 4-dimensional exotic $\mathbb{R}^4$, which are rather independent of each other. But in the derivation of the action above, we used implicitly the result that an exotic $\mathbb{R}^4$ (and the grope constructed from it) is (partly) classified by the Godbillon-Vey invariant. Therefore our topological quantum D-brane is generated by a small exotic $\mathbb{R}^4$ too.

8. Conclusions

Every small exotic $\mathbb{R}^4$ is a very rich many-facets hybrid object which links, among others, $C^*$ convolution algebras, K-theory, foliations and topology in particular. It can also be represented by a wild embedding $S^2 \rightarrow S^3$. When $\mathbb{R}^4$ is taken with its standard smooth structure, hence smoothness agrees with product topology, then all complexities of the structures disappear. In this paper we argue that exotic $\mathbb{R}^4$’s are involved in the formalism of string theory also at the non-perturbative domains where branes are considered as quantum objects. Especially, exotic $\mathbb{R}^4$’s determine a class of topological quantum D$p$-branes. On the other hand the presented results support our conjecture from [8], stating that:

The exotic small $\mathbb{R}^4$ lies at the heart of quantum gravity in dimension 4. Especially it is a quantized object.

The connections between 4-exotics and NS and D-branes in various string backgrounds were given in [11] and then extended formally to the quantum regime of
D-branes\textsuperscript{[5]}. Here we further extend this relation and propose a topological mechanism generating classes of branes and charges in some backgrounds. We study the case of quantum D-branes using $C^\ast$-algebras. The topological mechanism behind quantum branes is the wild embedding of 2-spheres into $S^3$ as well $S^3$ into higher dimensional spheres. These last embeddings generate D-branes which are considered as topological quantum D-branes whereas the non-trivial class, $H$, or $B$-field, is derived from the first wild embedding, i.e. $S^2 \to S^3$. The presented mechanism generates quantum topological Dp-branes when the non-trivial $B$-field on $S^3$ is given as a (quantum) wild embedding. On the other hand classical branes are considered as submanifolds or K-homology cycles. In case of the quantum regime they are usually described as K-theory classes on separable $C^\ast$-algebras\textsuperscript{[17]}. It appears that many kinds of this $C^\ast$-algebraic presentations have, in turn, topological origins and are again derived from the wild embeddings.

Taking the classical limit of such quantum Dp-branes, where $B$-field is confined on $S^3 \subset \text{WV}(Dp)$ corresponding to wild embeddings, one gets tame and flat embeddings of $p$-complexes. This follows in particular from the reduction of the quantum action to BI action. The results can be roughly summarized by:

The exotic small $\mathbb{R}^4$ as described by codimension-1 foliations on the 3-sphere is the germ of wide range of effects on D-branes. A topological quantum Dp-brane is related to a wild embedding of a $p$-dimensional complex into a $n-$dimensional space described by a two-dimensional complex, a grope. The grope is the main structure to get the relation between the exotic small $\mathbb{R}^4$ and the codimension-1 foliation on the 3-sphere\textsuperscript{[7][13][10]}.

The description of the wild embedding is rather independent of the dimension ($n > 6$, $p > 2$) which is the reason why small exotic $\mathbb{R}^4$’s appear in different dimensions as germs of higher dimensional topological quantum branes.

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