Chern-Simons Theory in the Temporal Gauge and Knot Invariants through the Universal Quantum R-Matrix

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ABSTRACT

In temporal gauge $A_0 = 0$ the 3$d$ Chern-Simons theory acquires quadratic action and an ultralocal propagator. This directly implies a 2$d$ $R$-matrix representation for the correlators of Wilson lines (knot invariants), where only the crossing points of the contours projection on the $xy$ plane contribute. Though the theory is quadratic, $P$-exponents remain non-trivial operators and $R$-factors are easier to guess then derive. We show that the topological invariants arise if additional flag structure $\mathbb{R}^3 \supset \mathbb{R}^2 \supset \mathbb{R}^1$ ($xy$ plane and an $y$ line in it) is introduced, $R$ is the universal quantum $R$-matrix and turning points contribute the "enhancement" factors $q^\rho$.

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1 Introduction

Since the seminal work of E.Witten [1] (see also [2]) it is known that knot invariants should be described as correlators of Wilson lines in 3$d$ Chern-Simons theory (CST), provided they are somehow regularized in a way, which preserves topological invariance of the classical CST [3]. Enormous efforts were applied since then [3]--[36] in order to realize this idea and associate various existing representations of knot invariants [37]--[60] with different gauge choices in CST. Detailed overview of entire situation and references is beyond the scope of this
paper, where we are going to concentrate on one distinguished implication of the Witten's conjecture. Namely, it is clear that CS theory with the action

$$S_{CS} = k \int \text{tr} \left( AdA + \frac{2}{3} A^3 \right) d^3x = k \epsilon^{\mu
u\lambda} \int \left( A_x^a \partial_\mu A_\mu^a + \frac{2}{3} f_{abc} A^a_\mu A^b_\nu A^c_\lambda \right) dt dx dy$$

(1)

becomes exactly solvable not in a transcendental, but in straightforward and constructive way in the temporal gauge $A_0^a = 0$, when the cubic term disappears and the action becomes quadratic \cite{24, 30, 34}. This free-field representation of CST is the most natural from the point of view of QFT and string theory \cite{61}, and it immediately implies the $R$-matrix kind of representation of knot invariants a la \cite{62}.

Amusingly, even this description remains unexplored in exhaustive way, despite a number of promising attempts \cite{28, 29}, see \cite{34} for a recent review. The problem is not so simple because the relevant observables – Wilson $P$-exponents – are in no way simplified in the temporal gauge and various regularization problems still need to be resolved. The best way to proceed here, as usual, is to first guess the answer, which possesses all the features it should have in the temporal gauge, and then explain how this answer can be technically deduced from original theory (by developing a 3d analogue of the 2d free-field calculus \cite{63} for the WZNW model).

The present paper is devoted to the first step – the "educated guess". The "obvious" properties the answer should have are:

- it is formulated in terms of projection of original 3d contour on the $xy$ plane,
- it gets contributions only from the crossing and turning points of this projection,
- each contribution should arise in a universal – representation-independent – form, i.e. contain only the generators $T_R^a$ of the algebra $G$ (or its deformation) in representation $R$,
- this contribution should be some clever regularization (deformation) of the naive expression $\prod q^{T_R^a \otimes T_R^b}$ with $q = \exp \frac{2\pi i}{k}$ where $k$ is the renormalization if the bare constant $k$.
- the full answer should be topological invariant, i.e. invariant under the Reidemeister moves of the projected contours, and independent of the choice of auxiliary direction on the $xy$ plane, needed to define the "turning points".

Actually an answer with these properties exists and its ingredients are well known, see, for example, \cite{64}-\cite{66}: it involves the quantum universal $R$-matrix \cite{67, 69}, associated with the quantum version $G_q$ of the original group $G$, and the relevant generators $T_R^a$ are those of the quantum algebra \cite{70}-\cite{73}. The only problem is that this construction is naturally associated with the operators, which are group elements of $G_q$ \cite{65}, rather than the Wilson $P$-exponents in the temporal gauge. Thus, if one believes in this answer, the next step should be the explanation how the group elements arise in "bosonization" of Wilson loops. This step, however, will be left for the future research, while in the present text we just add a little more to the motivation in \S2 find equations that describes topological properties of vevs of Wilson loops in the temporal gauge in \S3 and finally, describe their possible solution in \S4. We also give several examples of explicit computations of knot invariants in \S5.

This paper can be considered as completion of the guessing process, originated long ago in \cite{15, 16} and \cite{30}.

\section{Motivation}

In the $A_0 = 0$ gauge the CS action \cite{11} becomes quadratic,

$$S_{CS} = k \int (A_x^a \dot{A}_y^a - A_y^a \dot{A}_x^a) dt dx dy$$

(2)

and the propagator becomes

$$\langle A_x^a(\vec{x}), A_y^b(\vec{x}') \rangle = \delta^{ab} \epsilon_{ij} \text{sign}(t-t') \delta^{(2)}(\vec{x} - \vec{x}')$$

(3)

This means that the correlator of the Wilson lines in representation $q$ of the gauge group $G$

$$W_q(C) = \text{tr}_q P \exp \int_C A_\mu dx^\mu = \text{tr}_q P \exp \int_C (A_x dx + A_y dy)$$

(4)

gets contribution only from the (self)-intersections of the contours $\tilde{C}$, which are the 2d projections of the 3d contours $C$ onto the $xy$ plane.

Let us pick up a direction in the $xy$ plane, and let it be the $y$ axis. Then instead of the closed Wilson lines we can consider the matrix (representation)-valued "open" lines

$$U(\tilde{C}, y) = P \exp \int_{C}^{y} (A_x dx + A_y dy)$$

(5)
and study the "evolution" of the ordered product $\prod U(\tilde{C}_i, y)$ with the change of $y$. Actually there is no evolution, the product does not depend on $y$, except for the "moments" when some two lines intersect (crossing points) or some line turns backwards (turning point). Then some factors $R$ and $Q$ appear, acting in the product of two representations in the first case and in the single representation in the second case. The factor $R$ is independent of the angle between the intersecting lines, this property can be observed in the Abelian situation, see [34] for definiteness. Denote the state at the slice (moment) 1 in Fig.1(a) by $U \otimes V$.

Figure 1:

Then at the moment 2 it will turn into $R_{UV}(U \otimes V)$. At the same time it should be something made from reordered product $V \otimes U$. The natural guess is

$$R_{UV}(U \otimes V) = (V \otimes U)R_{UV}$$

and, looking at the propagator [3], we understand that

$$R_{UV} \approx \left(q^{T^a_R \otimes T^a_R}\right)_{\text{reg}}$$

with $q = \exp(2\pi i/k_1)$ and somehow deformed r.h.s., because regularization can and do modify the naive answer. In the last expression Similarly, if the third operator $\bar{W}$ is involved, then we get

$$R_{UV}(U \otimes V) = (V \otimes U \otimes \bar{W})R_{UV}$$

and so on. At the turning point we have a transition Fig.1(b)

$$Q_U(U \otimes \bar{U}) = 1$$

from a pair of operators into an empty state.

Entire knot can be described as a transition from an empty state at $y = -\infty$ to an empty state at $y = \infty$ with the associated amplitude, made from a product of $R$ and $Q$ factors. For this product to transform under Reidemeister moves as vev of a Wilson loop the elementary factors should satisfy some bilinear and trilinear relations see s.3 the main one being the Yang-Baxter equation for $R$. Then (7) implies that $R$ is the quantum universal $R$-matrix of $G_q$ [64]-[73]. For such choice of $R$ the operators $U, V, W$ with the property (6) do indeed exist: the simplest example being the group elements of $G_q$ [65]. As explained in the introduction, it remains to understand why the regularized open Wilson lines [14] have the same commutation relations (7) – and this is beyond the scope of the present paper. Note that understanding of this fact is important for Chern-Simons theory, but is not really needed for building knot invariants: just instead of referring to the "obvious" topological invariance of Wilson-line correlators in CST one needs to prove invariance of the product explicitly - and this will be done explicitly in s.4 below.

3 Operators arising from CST in temporal gauge

The invariance of vevs of Wilson loops under topological deformations imposes several conditions on the operators arising from CST in the temporal gauge. The purpose of this section is to argue that there are only two
independent operators from which the vevs of Wilson loops can be constructed. We also derive the equations for these operators that ensures the topological invariance of vevs.

3.1 Projected knots

If additional flag structure \( \mathbb{R}^3 \supset \mathbb{R}^2 \supset \mathbb{R}^1 \) on the space is fixed, then according to sections 1 and 2 we suppose that the answer for vev of a Wilson loop has the following properties:

- The answer is formulated in terms of a projection of a knot to a fixed plane \( \mathbb{R}^2 \). For definiteness, we specify this plane to be \( xy \)-plane in some reference frame.

- The answer gets angle-independent contributions only from crossings of the resulting two-dimensional curve and its turning points with respect to the fixed direction \( \mathbb{R} \) in the plane. We specify this direction to be \( y \)-direction in our reference frame. This means that we need the universally defined operators corresponding to the crossings and the turning points. The vevs of Wilson loops are just appropriate contractions of these operators.

- Under topological deformations these contractions are transformed exactly as vevs of Wilson loops in CST.

With these desired properties in mind, let us consider a two-dimensional projection of a knot to the fixed \( xy \) plane, for example as in the Fig.2\( (a) \). The arrows denote the orientation of the knot. Without loss of generality, we assume that all crossings of the resulting two-dimensional curve have two upper and two lower legs about fixed \( y \)-direction. For example in the Fig.2\( (b) \) \( a, b \) are upper and \( c, d \) are lower legs. Obviously, every crossing can be reduced to this form by a small deformation in the plane.

![Figure 2](image)

We can see that for every two-dimensional projection only 8 types of crossings, which differ from each other by orientation and 4 types of turning points about \( y \) direction can appear (see Fig.3).

![Figure 3](image)

According to the second property from of our list, we define 8 corresponding crossing operators \( R_{\alpha \beta}^{i \epsilon \delta} \), \( i = 1...8 \) and 4 operators for each type of turning point \( M_{j}^{\alpha} \), \( j = 1...4 \). The indices of these operators correspond
to free legs of the crossings and the turning points in the two-dimensional picture. We assume that the upper index denotes an incoming leg and the lower denotes an outgoing one. Therefore, the crossing operators have two upper and two lower indices, the turning point operators have one index of each type Fig.4.

The vev of a Wilson loop $< W(c) >$ can be represented in the form of contractions of these operators. More precisely, to find the vev $< W(c) >$, we need to peak a two dimensional projection of the closed curve $c$, then attach to every crossing and turning points corresponding operators and contract the indices according to the two dimensional diagram. For example, for unknot $U_0$ with the two-dimensional projection represented in the Fig.5 the result is:

$$< W(U_0) > = M_3^a M_1^b M_3^c M_2^d = \text{tr}(M_3 M_1 M_3 M_2)$$  \hspace{1cm} (10)
are related by the inversion:

\[ M_3 = M_1^{-1}, \quad M_2 = M_4^{-1} \quad (11) \]

Therefore, in fact, there are only two independent turning point operators \( M \) and \( \overline{M} \), and the others can be expressed through them:

\[ M_1 = M, \quad M_4 = \overline{M}, \quad M_3 = M^{-1}, \quad M_2 = \overline{M}^{-1} \quad (12) \]

### 3.3 Crossing operators

Similar topological arguments show that there are only two independent crossing operators. Indeed, let us consider the topological deformations represented in the Fig.\( \text{\textit{7}} \). The topological invariance of vevs of Wilson loops in CST implies that the crossing operators \( R_2, R_3 \) and \( R_4 \) can be expressed through the first one \( R_1 := R \):

\[
\begin{align*}
R_{a,b}^{2,c,d} &= R_{c,b}^{m,k} M^{-1} M_{b}^{a} M_{d}^{b} & \Rightarrow & & R_2 = (1 \otimes M) R (M \otimes 1)^{-1} \\
R_{a,b}^{3,c,d} &= R_{m,a}^{k,b} \overline{M}^{-1} M_{k}^{b} M_{c}^{m} & \Rightarrow & & R_3 = (\overline{M} \otimes 1) R (1 \otimes \overline{M})^{-1} \\
R_{a,b}^{4,c,d} &= R_{k,b}^{h,f} M^{-1} M_{a}^{h} M_{b}^{f} M_{c}^{k} & \Rightarrow & & R_4 = (M \otimes \overline{M}) R (M \otimes M)^{-1}
\end{align*}
\]  

(13)

If in Fig.\( \text{\textit{7}} \) under-crossings are changed to over-crossings then we have:

\[
\begin{align*}
R_{6} &= (1 \otimes M) \overline{R} (M \otimes 1)^{-1} \\
R_{7} &= (\overline{M} \otimes 1) \overline{R} (1 \otimes \overline{M})^{-1} \\
R_{8} &= (M \otimes \overline{M}) \overline{R} (M \otimes M)^{-1}
\end{align*}
\]  

(14)
i.e. the crossing operators $R_6$, $R_7$ and $R_8$ can be expressed through $R_5 := \overline{R}$.

### 3.4 Reidemeister movies. Ambient topological invariants.

Two knots in $\mathbb{R}^3$ are considered to be equivalent if one of them can be transformed into the other via smooth deformations in $\mathbb{R}^3$. This is the definition of *ambient isotopy equivalence* and two such knots are referred to as *ambient isotopy equivalent*. Relations (12)-(14) for crossing and turning point operators ensure that the vevs constructed from them do not depend on homotopic deformations of two-dimensional projection in the plane, i.e. on smooth deformations in the plane that do not change the total number of crossings of this projection. This is not enough, however, for the vev to be topological invariant. Indeed, different two-dimensional projections of some knot can have different topologies, for example different numbers of crossings. Therefore, different, non-homotopic two-dimensional projections can represent the same knot.

Fortunately, there is a simple way to find out if the two knots represented by their two-dimensional projections are ambient isotopy equivalent or not [37]:

**Theorem (Reidemeister)**

*Two knots in $\mathbb{R}^3$ represented by their two-dimensional projections $P_1$ and $P_2$ are ambient isotopy equivalent if and only if $P_1$ can be deformed into $P_2$ via smooth deformations in the two-dimensional plane and finite set of Reidemeister moves, shown in Fig.8.*

![Figure 8:](image)

The quantities that depend only on equivalence classes of ambient isotopy, or equivalently, do not vary under smooth two-dimensional deformations and Reidemeister moves are referred to as the *ambient isotopy invariants*.

### 3.5 Temporal framing. Regular isotopy invariants.

Despite the topological nature of CST, the expectation values of Wilson loops in this theory are not ambient isotopy invariants [8]-[19]. This feature of CST is closely related to the self-linking problem and the *framing* procedure. In CST the vevs of Wilson loops $< W(c) >$ are ill-defined quantities, and to define these quantities properly one can introduce a second, auxiliary contour $c'$, called framing of $c$, which can be considered as a slight displacement of contour $c$ along a normal to $c$ vector field [1]. More concretely, the word "slight" means that we replace the contour $c$ representing the knot by a narrow-width rope bounded by $c$ and $c'$. Two different choices of framing contour for straight line are presented in the Fig.9(a). When auxiliary contour is chosen, then it makes sense to calculate the linking number between contours $c$ and $c'$ and its non-Abelian generalizations - the normalized vevs of product of two Wilson loops in CST:

$$< W(c) > \overset{\text{def}}{=} < : W(c) :: W(c') : >$$  \hspace{1cm} (15)
The normalization means that in perturbation theory only propagators with different ends attached to different contours $c$ and $c'$ are taken into account. Therefore, there are no collapsible propagators, and the quantity (15) turns out to be well defined.

Roughly speaking, the CST describes the properties of knots that are made not of a "rope" but of a "ribbon" bounded by $c$ and $c'$. In this case an additional parameter comes into the game. Namely, a "rope" can be replaced by a "ribbon" in many different ways with different number of twists $n$ of the "ribbon" Fig(9(a)). The answer in CST depends only on this $n$. To fix the number of the ribbon twists, different framing fixing are used. In the temporal gauge there exists a natural choice for framing fixing. Namely, one can fix two dimensional plane $\mathbb{R}^2 \subset \mathbb{R}^3$ and then only ribbons which have non-perpendicular to this plane tangent planes at every point are considered. We call this procedure - temporal framing. The examples of knots in the temporal framing are represented in the Fig(9(b), here the fixed two-dimensional plane is the plane of the page.

Figure 9:

The "Wilson-loop" calculations in CST depend on the definition of the framing. In what follows we assume that all quantities are represented in the temporal framing.

Now, it is obvious that in the temporal framing the first Reidemeister move from Fig(8) is not respected, because the ribbon with loop on it in the temporal framing is not equivalent to a straight ribbon, but to a ribbon with additional twist, see Fig(10)

Figure 10:

The equivalence of knot projections defined only by the second and the third Reidemeister moves form Fig(8) is usually referred to as regular isotopy invariance, and quantities that does not change under these moves are the so called regular isotopy invariants. In this way, we see, that the presence of the framing result in the fact
that the vevs of Wilson loops are not ambient but only the regular invariants of knots.

It was shown in the Witten’s work [1] that the shift of the framing contour by \( n \) twists results in a simple multiple for the vev of a Wilson loop:

\[
< W(c) > \rightarrow \alpha^n < W(c) >
\]

This is a crucial observation that enables the CST to work. It means that although we must pick additional framing contour to make the vev well defined, we do not lose the information since we know how this vev changes under change of framing. Explicit perturbative computations performed in [8] show that the first terms of \( 2 \pi/k_1 \) expansion of \( \alpha \) are:

\[
\alpha = 1 + \frac{2 \pi}{k_1} \varrho(\Omega_2) + \frac{1}{2} \left( \frac{2 \pi}{k_1} \varrho(\Omega_2) \right)^2 + O\left( \frac{1}{k_1^3} \right)
\]

where \( k_1 \) is renormalization of the bare constant \( k \), \( \Omega_2 \) is the value of quadratic Casimir of the gauge Lie algebra and \( \varrho \) is the representation carried by the Wilson loop. We see that the results of the calculations are in agreement with the following expression:

\[
\alpha = q^{\varrho(\Omega_2)}, \quad q = \exp\left( \frac{2 \pi i}{k_1} \right)
\]  

(16)

The difference between \( k_1 \) and \( k \) is somewhat controversial point, usually one assumes that \( k_1 = k + \varrho_{Ad}(\Omega_2) \) (here \( \varrho_{Ad}(\Omega_2) \) stands for the value of quadratic Casimir in the adjoint representation of the gauge group) like in 2d WZNW theory [63]. For our guessing this is not very essential: in the following we use directly parameter \( q \) irrespective of its exact dependence of the bare \( k \).

Therefore, the CST generalizes the first Reidemeister move form Fig.8 and the rest two remains unchanged. Finally, we can infer that the vevs of Wilson loops have the transformation properties summarized in the Fig.11

\[
\begin{align*}
(1) \quad \infty & \Rightarrow q^{\varrho(\Omega_2)} \infty \quad ; \quad \infty & \Rightarrow q^{-\varrho(\Omega_2)} \infty \\
(2) \quad & = \\
(3) \quad & =
\end{align*}
\]

Figure 11:

3.6 The fundamental equations

Let us analyze the restrictions imposed by generalized Reidemeister moves Fig.11 on the crossing and the turning point operators. First of all, we note that using the relations (12)-(14) we can express the crossing operators through \( R \) and \( \overline{R} \), similarly, the turning point operators are expressed through \( \mathcal{M} \) and \( \overline{\mathcal{M}} \). Therefore, the resulting restrictions on the crossing and turning point operators can be reduced to equations on \( R, \overline{R}, \mathcal{M} \) and \( \overline{\mathcal{M}} \). In terms of \( R \) and \( \overline{R} \), the second Reidemeister relation in Fig.12 can be expressed as:

\[
\overline{R}^{a b}_{c d} R^{c d}_{e f} = \delta^a_e \delta^b_f, \quad \text{or} \quad \overline{R} = R^{-1}
\]  

(17)
The first relation in Fig.12 means that:

\[ R_{a} e c e M_{d} e c = q^{e(\Omega_{2})} \delta_{b}^{a}, \quad R_{a} e c e M_{d} e c = q^{-e(\Omega_{2})} \delta_{b}^{a} \] (18)

or, equivalently:

\[ \text{tr}_2 (R^{\pm 1} 1 \otimes Q) = q^{\pm e(\Omega_{2})} \] (19)

where \( \text{tr}_2 \) denotes the trace over the second space in the tensor product and we denote:

\[ Q = M \overline{M}. \] (20)

In terms of the "down-directed" crossings corresponding to the operator \( R \) in Fig.12(3) third Reidemeister move implies that \( R \) satisfies the following trilinear equation:

\[ R_{a b} d e R_{d f} e g R_{f k} k m = R_{b c} d e R_{a d} k m R_{g e} m g \] (21)

It is easy to check that all other possible orientations of the crossings in the third Reidemeister relation Fig.11(3) can be reduced to (21) by transformations (13) and (14). For this reason only (21) should be considered.

Introducing the notations \( R_{12} = R \otimes 1, \ R_{23} = 1 \otimes R \) for the operators, acting in the triple product \( V \otimes V \otimes V \), we can rewrite this equation in the form:

\[ R_{12} R_{23} R_{12} R_{12} R_{23} = R_{23} R_{12} R_{12} R_{23} \] (22)

This is the famous quantum Yang-Baxter equation (QYBE), arising in a number of physical and mathematical topics, from exactly solvable statistical models [67] to quantum groups [64]-[70]. Therefore, to describe the properties of vevs of Wilson loops in the CST the crossing and the turning point operators must satisfy two fundamental equations:

\[
\begin{align*}
\text{tr}_2 (R^{\pm 1} 1 \otimes Q) = q^{\pm e(\Omega_{2})} \\
R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}
\end{align*}
\] (23)

Solution of these equations is a pair \((R, Q)\) of operators acting in appropriate tensor products of gauge group representations. With such a solution at hand, one can construct all crossing operators with the help of relations (13), (14) and (17). On the other hand side the pair \((R, Q)\) satisfying (23) does not define the turning point operators but only the product of two operators (20). There is no additional relation between \( M \) and \( \overline{M} \) imposed by topological moves and arbitrary choice of \( M \) and \( \overline{M} \) satisfying (20) is possible. For instance, one can consider the following choice:

\[ M = Q, \quad \overline{M} = 1 \]

what means that the turning point operators \( M_2 \) and \( M_4 \) are identity operators due to (11) and do not give contribution to vev of Wilson loops and therefore, the corresponding turning points are not taken into account.

In what follows we, however, assume the following relation:

\[ M = \overline{M}, \quad M^2 = Q \] (24)

This choice of additional relation is for the sake of symmetry as it makes all the four turning point operators and turning points equivalent. When added to (13), (14), (11), the relation (24) allows to construct all crossing and turning point operators from given solution of fundamental equations.
4 Universal quantum $R$-matrix

As it was shown in the previous paper [34], the use of the naive propagator (3) leads to the crossing operator of the form:

$$R = \sum_{m=0}^{\infty} \frac{h^{m}}{m!} (T^{a_1}_{g} T^{a_2}_{g} ... T^{a_m}_{g}) \otimes (T^{a_1}_{g} T^{a_2}_{g} ... T^{a_m}_{g}), \quad \{T^{a_1}_{g} T^{a_2}_{g} ... T^{a_m}_{g}\} = \frac{1}{m!} \sum_{\sigma \in S_m} T^{\sigma(1)}_{g} T^{\sigma(2)}_{g} ... T^{\sigma(m)}_{g} \quad (25)$$

where $S_m$ is the symmetric group of $m$-elements, $g$ is a representation carried by a Wilson loop and $h = 2\pi i/k_1$. The universal, representation independent, form of (25) is:

$$R = \sum_{k=0}^{\infty} \frac{h^{k}}{k!} \{T^{a_1} T^{a_2} ... T^{a_m}\} \otimes \{T^{a_1} T^{a_2} ... T^{a_m}\} \in U(g) \otimes U(g) \quad (26)$$

where $U(g)$ is the universal enveloping algebra of $g$. Unfortunately, such crossing operator (26) does not satisfy the QYBE (22), what indicates that more accurate regularization of propagator is necessary. Obviously, such a regularization of the naive answer should deform (26) somehow. In this section we make an obvious suggestion that this deformation of (26) is given by the well known quantum universal $R$-matrix that seems to be in accordance with [36, 42, 62]. The quantum universal $R$-matrix is not the element of $U(g) \otimes U(g)$ anymore but the element of $U_{h}(g) \otimes U_{h}(g)$, where the algebra $U_{h}(g)$ is the $h$-deformed, or the quantized universal enveloping algebra of $g$. In the subsections 4.1-4.3 we remind explicitly definition of $U(g)$ and $U_{h}(g)$. In the subsection 4.4 we give the explicit expression for the universal quantum $R$-matrix $R \in U_{h}(g) \otimes U_{h}(g)$, then define the element $Q \in U_{h}(g)$ and prove that these elements satisfy the fundamental equations (23).

4.1 Universal enveloping algebras

Let $g$ be a simple Lie algebra of rank $r$ with the root system $\Phi$. Denote by $\Phi^{\pm}$ the subset of positive roots and by $n = |\Phi^{\pm}|$ be the number of positive roots. Fix the set of simple roots $\{\alpha_1, ..., \alpha_r\} = \Delta$, and let $\{\alpha'_1, ..., \alpha'_r : \alpha \in \Delta\}$ be the set of elements dual to the roots elements:

$$(\alpha', \beta) = \delta_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta \text{ and } 1 \text{ otherwise} \quad (27)$$
Here \((, )\) is the Killing form reduced to the Cartan subalgebra. Let \(\{e_\alpha, f_\alpha, h_\beta : \alpha \in \Phi^+, \beta \in \Delta\}\) be a Cartan-Weil basis of \(g\). The Cartan matrix of the algebra is defined as \(a_{\alpha\beta} = 2 (\alpha \beta) / (\alpha \alpha)\). For arbitrary element \(\gamma\) from the Cartan subalgebra, we define the corresponding elements of the algebra as follows:

\[
h_\gamma = \sum_{\alpha \in \Delta} h_\alpha (\alpha^\vee, \gamma) \tag{28}
\]

In particular, we will need the elements \(h_\beta, \beta \in \Phi^+\), and the element \(h_\rho\) where \(\rho\) is the half-sum of all positive roots:

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \tag{29}
\]

The universal enveloping algebra of \(g\) is the quotient of the tensor algebra \(T(g)\) by the ideal generated by the elements \(x \otimes y - y \otimes x = [x, y]\):

\[
U(g) = T(g)/I, \quad T(g) = \bigoplus_{m=0}^{\infty} g^{\otimes_m}, \quad I = \{x \otimes y - y \otimes x = [x, y]\} \tag{30}
\]

According to the Serre’s theorem this algebra can be defined as the algebra generated by 3 \(r\) elements \(\{e_\alpha, f_\alpha, h_\alpha : \alpha \in \Delta\}\) with the following relations:

\[
[h_\alpha, h_\beta] = 0, \quad [h_\alpha, e_\beta] = (\alpha, \beta) e_\beta, \quad [h_\alpha, f_\beta] = -(\alpha, \beta) f_\beta \tag{31}
\]

\[
[e_\alpha, f_\beta] = \delta_{\alpha\beta} h_\alpha \tag{32}
\]

\[
(\text{ad} e_\alpha)^{1-a_{\alpha\beta}} e_\beta = \sum_{m=0}^{1-a_{\alpha\beta}} (-)^m \left[\frac{1-a_{\alpha\beta}}{m}\right] e_\alpha^{1-a_{\alpha\beta}-m} e_\beta^m = 0, \quad \alpha \neq \beta \tag{33}
\]

\[
(\text{ad} f_\alpha)^{1-a_{\alpha\beta}} f_\beta = \sum_{m=0}^{1-a_{\alpha\beta}} (-)^m \left[\frac{1-a_{\alpha\beta}}{m}\right] f_\alpha^{1-a_{\alpha\beta}-m} f_\beta^m = 0, \quad \alpha \neq \beta \tag{34}
\]

The possible choice of a basis in \(U(g)\) is given by the following Poincaré-Birchoff theorem: The elements

\[
eq_1^{\alpha_1} \ldots \eq_1^{\alpha_n} h_{\beta_1}^{t_1} \ldots h_{\beta_r}^{t_r}, f_{\alpha_1}^{s_1} \ldots f_{\alpha_n}^{s_n}, \quad \alpha_i \in \Phi^+, \quad \beta_i \in \Delta, \quad p_i, t_i, s_i \in \mathbb{N}_0
\]

form a basis of the universal enveloping algebra \(U(g)\).

### 4.2 Quantized universal enveloping algebras

Let \(h\) be indeterminate number, then the quantized universal enveloping algebra \(U_h(g)\) is the algebra with 1 over \(\mathbb{C}[q] = e^h\) generated by 3 \(r\) elements \(\{h_\alpha, E_\alpha, F_\alpha, \alpha \in \Delta\}\) subjected to the relations:

\[
[h_\alpha, h_\beta] = 0, \quad [h_\alpha, E_\beta] = (\alpha, \beta) E_\beta, \quad [h_\alpha, F_\beta] = (\alpha, \beta) F_\beta \tag{35}
\]

\[
[E_\alpha, F_\beta] = \delta_{\alpha\beta} \frac{q_\alpha h_\beta - q^{-h_\alpha}}{q_\alpha - q^{-1}} \tag{36}
\]

\[
\sum_{m=0}^{1-a_{\alpha\beta}} (-)^m \left[\frac{1-a_{\alpha\beta}}{m}\right] q_\alpha^{1-a_{\alpha\beta}-m} E_\alpha^{1-a_{\alpha\beta}-m} E_\beta^m = 0, \quad \alpha \neq \beta \tag{37}
\]

\[1\] Note, that our choice of Cartan elements \(h_\alpha\) is different from the conventional one by the factor \((\alpha, \alpha)/2\).
\[
\frac{1-a_{\alpha \beta}}{m} \left( \frac{1-a_{\alpha \beta}}{m} \right)_{q^m}^{F^1_{\alpha \beta} - m} F_{\alpha}^1 F_{\beta}^m = 0, \ i \neq j
\] (38)

where
\[
\left[ \frac{n}{m+1} \right] = \frac{[n]_q!}{[m]_q! [n-m]_q!}, \ [n]_q! = \prod_{k=1}^{n} [k]_q, \ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \ \text{and} \ q_\beta = q^{(n, \alpha)/2}
\] (39)

Note, that the definition of \( U_h(g) \) is similar to the definition of \( U(g) \) by the Serre relations (31)-(34). The relations (35) are formally equivalent to the relations (31), and the relations (36)-(38) are some \( h \)-deformations of the respective relations (32)-(34).

### 4.3 Braid groups and root elements

If \( a_{\alpha \beta} \) is the Cartan matrix of a Lie algebra \( g \), then the numbers \( a_{\alpha \beta}^2 a_{\beta \alpha} \) can take values 0, 1, 2 or 3. Let \( m_{\alpha \beta} \) be equal to 2, 3, 4 or 6 when \( a_{\alpha \beta} a_{\beta \alpha} = 0 \), \( 1 \), 2 or 3 respectively. The Weil group \( W_g \) is defined by reflections \( \{ \sigma_\alpha : \alpha \in \Delta \} \) corresponding to the simple roots of \( g \) satisfying the following relations:

\[
\sigma_\alpha^2 = 1, \ \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta \ldots = \sigma_\beta \sigma_\alpha \sigma_\beta \sigma_\alpha \ldots, \ \alpha \neq \beta
\] (40)

Let \( \sigma_0 \) be the longest element of the Weil group i.e. the element that has the longest length of its reduced decomposition \( \sigma_0 = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_n}, \ \alpha_{i_1} \in \Delta \) (length of this element is equal to the number of positive roots \( n \) for every Lie algebra). An important property of \( \sigma_0 \) is that the sequence:

\[
\beta_1 = \alpha_{i_1}, \ \beta_2 = \alpha_{i_2}, \ \ldots, \ \beta_n = \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_{n-1}} \alpha_{i_n}
\] (41)

exhaust all positive roots of \( g \) and defines the ordering \( \beta_1 < \beta_2 < \ldots < \beta_n \) on the set \( \Phi^+ \). Therefore, the Weil group and a fixed reduced decomposition of a longest element \( \sigma_0 \) allow to define all root elements of \( U(g) \) starting from simple roots.

In the definition of \( U_h(g) \) enter only elements \( h_\alpha, E_\alpha \) and \( F_\alpha \) corresponding to simple roots. To define the elements of \( U_h(g) \) corresponding to the non-simple roots we need a notion of quantum Weil group or Artin braid group.

The Artin braid group \( B_g \) associated with \( g \) is the group generated by elements \( \{ b_\alpha : \alpha \in \Delta \} \) with the following relations:

\[
b_\alpha b_\beta b_\alpha b_\beta \ldots = b_\beta b_\alpha b_\beta b_\alpha \ldots, \ \alpha \neq \beta
\] (42)

The braid group \( B_g \) acts by automorphisms on \( U_h(g) \) (40):

\[
b_\alpha(h_\beta) = h_\alpha - a_{\alpha \beta} h_\alpha, \ b_\alpha(E_\alpha) = -F_\alpha q_\alpha^{h_\alpha}, \ b_\alpha(F_\alpha) = -E_\alpha q_\alpha^{-h_\alpha}
\] (43)

\[
b_\alpha(E_\alpha) = -\frac{a_{\alpha \beta}}{m} \sum_{m=0}^{m} \left( \frac{q^m - q^{-m}}{q - q^{-1}} \right) q_\alpha^m (E_\alpha)^{-a_{\alpha \beta} - m} E_\beta (E_\alpha)^m
\] (44)

\[
b_\alpha(F_\beta) = -\frac{a_{\alpha \beta}}{m} \sum_{m=0}^{m} \left( \frac{q^m - q^{-m}}{q - q^{-1}} \right) q_\alpha^m (F_\alpha)^{m} F_\beta (F_\alpha)^{-a_{\alpha \beta} - m}
\] (45)

Let us define the roots of \( g \) as in (41), then we define the root elements of \( U_h(g) \) as follows:

\[
E_{\beta_k} = b_{i_k} b_{i_{k-1}} \ldots b_{i_1} E_{i_k} \quad \text{and} \quad F_{\beta_k} = b_{i_k} b_{i_{k-1}} \ldots b_{i_1} F_{i_k}, \quad k = 1, \ldots, n
\] (46)

The following analog of Poincaré-Birchhoff theorem gives a basis in \( U_h(g) \):

The elements

\[
F_{\alpha_1}^{p_1} \ldots F_{\alpha_k}^{p_k} h_{\beta_1}^{t_1} \ldots h_{\beta_t}^{t_t} E_{\alpha_1}^{s_1} \ldots E_{\alpha_k}^{s_k}, \ \alpha_i \in \Phi^+, \ \beta_i \in \Delta, \ \ p_i, t_i, s_i \in N_0
\]
form a basis of \( U_h(g) \).

For a proof see (70).
4.4 Universal quantum $R$-matrix

Let $E_\beta$ and $F_\beta$, $\beta \in \Phi^+$ be the root elements of $U_h(g)$ then the universal quantum $R$-matrix is defined as an element of $U_h(g) \otimes U_h(g)$ by the following explicit expression:

$$\mathcal{R} = \hat{P} \sum_{\beta \in \Phi^+} \prod_{\alpha \in \Phi^+} \exp_q \left( (q_\beta - q_\beta^{-1}) E_\beta \otimes F_\beta \right)$$

(47)

Here $\hat{P}$ is the permutation operator: $\hat{P} a \otimes b = b \otimes a \hat{P}$ and the arrow above the product implies that the factors appear in the order $\beta_n, \beta_{n-1}, \ldots, \beta_1$ defined by a reduced decomposition of longest elements of Weil group $\Phi^+$.

The crucial property of the universal quantum $R$-matrix is that this matrix represents suitable regularization of the crossing operator (23). More precisely:

**Proposition:** The pair $\{ \mathcal{R} \in U_h(g) \otimes U_h(g), Q \in U_h(g) \}$, where $\mathcal{R}$ is the universal quantum $R$-matrix (47) and $Q$ is defined by the following explicit expression:

$$Q = q^{h_\rho}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad h_\rho = \sum_{\alpha \in \Phi^+} h_\alpha$$

(48)

satisfies the fundamental equations, (23).

**Proof:** The proof of the quantum Yang-Baxter equation (the second fundamental equation) can be found in any textbook on quantum group theory, see for example [65] or [70]. The proof of the first fundamental equation in [23] is a relatively long but direct computation:

$$\text{tr}_2(\mathcal{R} I \otimes Q) = \text{tr}_2 \left\{ \hat{P} \exp(h \sum_{\alpha \in \Delta} h_\alpha \otimes h_{\alpha^\vee}) \prod_{\beta \in \Phi^+} \exp_q \left( (q_\beta - q_\beta^{-1}) E_\beta \otimes F_\beta \right) I \otimes q^{h_\rho} \right\}$$

$$= \text{tr}_2 \left\{ \hat{P} \sum_{k=0}^{\infty} \frac{h^k}{k!} \sum_{\alpha_1, \ldots, \alpha_k} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m \right\}$$

(49)

$$= \sum_{m, \beta \in \Phi^+} \sum_{\eta \in \Phi^+} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})m}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m$$

(50)

$$= q^{\sum_{\alpha} h_\alpha h_\rho} \prod_{m, \beta \in \Phi^+} \sum_{\eta \in \Phi^+} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})m}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m$$

(51)

$$= q^{\sum_{\alpha} h_\alpha h_\rho} \prod_{m, \beta \in \Phi^+} \sum_{\eta \in \Phi^+} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})m}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m$$

(52)

$$= q^{\sum_{\alpha} h_\alpha h_\rho} \prod_{\eta \in \Phi^+} \sum_{m, \beta \in \Phi^+} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})m}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m$$

(53)

$$= q^{\sum_{\alpha} h_\alpha h_\rho} \prod_{\eta \in \Phi^+} \sum_{m, \beta \in \Phi^+} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})m}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m$$

(54)

$$= q^{\sum_{\alpha} h_\alpha h_\rho} \prod_{\eta \in \Phi^+} \sum_{m, \beta \in \Phi^+} \prod_{m, \beta \in \Phi^+} \exp_q \left( \frac{(q_\beta - q_\beta^{-1})m}{[m]_q} q_\beta^m \delta^{m-1} \right) \prod_{\eta \in \Phi^+} h_\alpha \prod_{\mu \in \Phi^+} E^m \prod_{\nu \in \Phi^+} F^m$$

(55)
\[ q^{h_\rho} F^{m_\delta}_\delta q^{-h_\rho} = q^{(p, \delta) m_\delta} F^{m_\delta}_\delta, \quad \prod_{\delta \in \Phi^+} F^{m_\delta}_\delta h_\alpha = \left( h_\alpha + \sum_{\delta \in \Phi^+} (\alpha, \delta m_\delta) \right) \prod_{\delta \in \Phi^+} F^{m_\delta}_\delta \]

\[ \sum_{\alpha \in \Delta} h_\alpha \gamma (\alpha, \eta) = \sum_{\alpha \in \Delta} h_\alpha (\alpha^\gamma, \eta) = h_\eta \]

and \( \Omega_2 \) is by definition the quadratic Casimir element of the algebra \( U(g) \):

\[ \Omega_2 = \sum_{\alpha \in \Delta} h_\alpha \gamma h_\alpha + \sum_{\alpha \in \Phi^+} e_\alpha f_\alpha + f_\alpha e_\alpha \]  

(56)

The proof for \( R^{-1} \) is analogous. \( \Box \)

Now, having the solution of fundamental equations one can construct all the special point operators. The crossing operators are expressed through the universal \( R \)-matrix by \([12]\), \([13]\) and \([17]\). To construct the turning point operators we need to pick up the operators \( M \), and \( \overline{M} \) satisfying \([24]\). The obvious choice is:

\[ M = \overline{M} = q^{h_{h/2}}, \]

then the rest of the turning point operators are expressed through these two by \([12]\). Explicit expressions for these operators are summarized in \([58]\) and \([59]\).

5 Knot invariants

The crossing and turning point operators introduced in the previous sections allow to calculate the ambient and regular knot invariants. In this section we describe in detail the procedure for computation of these invariants and give examples of these calculations for several different choices of the gauge group and its representation.

5.1 Regular knot invariants.

Let \( D : S^1 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be a two-dimensional projection of a knot \( K \) (here \( P \) is the projector to the plane). Then the image \( D(S^1) \) is a closed planar curve with finite number of self-crossings and turning points about some distinguished direction in the plane. To compute the regular isotopy invariant \( < W_\rho(K) > \) we should reduce all crossings of \( D(S^1) \) to one of the 8 canonical crossings represented in Fig.4 (additional turning points can appear during this process). Then, we attach to every special point the corresponding operator, for example as in the Fig.4. The indices of these operators correspond to the incoming and outgoing lines of special points, therefore, the closed curve \( D(S^1) \) naturally defines the contraction of the indices of the operators. For instance such a contraction for the two-dimensional projection of the knot 52 represented in Fig.4 reads:

\[ < W(52) > = R_{1j}^{bi} R_{1d}^{ic} R_{2m}^{gk} R_{3n}^{de} R_{4n}^{bf} M_{1f}^{em} M_{2m}^{a} M_{3b}^{a} M_{4i}^{n} \]  

(57)

If a Wilson loop \( W_\rho(K) \) is in a representation \( \rho : G \rightarrow End(V) \) of a gauge group \( G \), then the operators of special points Fig.4 read:

\[ \begin{align*}
\mathcal{M}_1 &= \mathcal{M}_4 = \rho(q^{h_{h/2}}), \\
\mathcal{M}_2 &= \mathcal{M}_3 = \rho(q^{-h_{h/2}}),
\end{align*} \]  

(58)
\[ \begin{align*}
    R_1 &= \varrho \otimes \varrho \left( R \right), \\
    R_5 &= \varrho \otimes \varrho \left( R^{-1} \right), \\
    R_2 &= \varrho \otimes \varrho \left( (1 \otimes q^{h_\varrho/2}) R (q^{-h_\varrho/2} \otimes 1) \right), \\
    R_6 &= \varrho \otimes \varrho \left( (1 \otimes q^{h_\varrho/2}) R^{-1} (q^{-h_\varrho/2} \otimes 1) \right), \\
    R_3 &= \varrho \otimes \varrho \left( (q^{h_\varrho/2} \otimes 1) R (1 \otimes q^{-h_\varrho/2}) \right), \\
    R_7 &= \varrho \otimes \varrho \left( (q^{h_\varrho/2} \otimes 1) R^{-1} (1 \otimes q^{-h_\varrho/2}) \right), \\
    R_4 &= \varrho \otimes \varrho \left( (q^{h_\varrho/2} \otimes q^{h_\varrho/2}) R (q^{-h_\varrho/2} \otimes q^{-h_\varrho/2}) \right), \\
    R_8 &= \varrho \otimes \varrho \left( (q^{h_\varrho/2} \otimes q^{h_\varrho/2}) R^{-1} (q^{-h_\varrho/2} \otimes q^{-h_\varrho/2}) \right),
\end{align*} \]  

(59)

where \( R \) is the universal quantum \( R \)-matrix \((17)\) for \( U_\varrho(g) \). For example in the case of \( G = SU(2) \) and \( \varrho \) is the fundamental representation, the contraction \((57)\) gives:

\[ < W(5_2) > = q^{-17} (q + q^{-1}) (q^{10} - q^8 + 2q^6 - q^4 + q^2 - 1) \]

\( \begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13}
\caption{5.2 Writhe number. Ambient invariants.}
\end{figure} \]

As it was discussed in the section \ref{sec:ambient-invariants} the quantity \(< W_\varrho(K) >\) can be considered as an invariant of embedding \( K : S^1 \times [0, 1] \rightarrow \mathbb{R}^3 \) of a ribbon \([0, 1] \times S^1\) into three-dimensional space. The knot \( K \) is defined as \( K = \mathcal{K}(x, 0) : S^1 \rightarrow \mathbb{R}^3 \) and \( K' = \mathcal{K}(x, 1) : S^1 \rightarrow \mathbb{R}^3 \) is a framing contour of \( K \) introduced by regularization of CST. The quantity \(< W_\varrho(K) >\) depends on the number of twists on a ribbon Fig\ref{fig:writhe}(a), this results in the fact that it is not an ambient isotopy invariant of the knot, but only a regular invariant. Nevertheless, we can easily construct the ambient isotopy knot invariant using the quantity \(< W_\varrho(K) >\). Indeed, let \( w(K) \) is the total
number of twists of a framing ribbon embedded in $\mathbb{R}^3$, this quantity is referred to as a \textit{writhe} of the ribbon. According to \cite{10}, additional twist on the ribbon changes the vev of a Wilson loop $< W_\varrho(K) >$ for a factor $q^{\varrho(\Omega_2)}$. Therefore, the quantity $I(G, \varrho, K)$ defined as:

$$I(G, \varrho, K) = \frac{1}{< W(U_0) >} q^{-w(\mathcal{K}) \varrho(\Omega_2)} < W_\varrho(K)>$$  \hspace{1cm} (60)

is an ambient isotopy invariant. Here $< W(U_0) >$ stands for the vev of unknot and we divide by its value for normalization.

In the temporal gauge $w(\mathcal{K})$ is the total number of twists of a ribbon in the temporal framing introduced in the section 3.5. The temporal framing is the procedure that naturally defines the class of ribbon embeddings $\{\mathcal{K}\}$ for a given two-dimensional projection $D : S^1 K \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^2$ of the knot $K$. Therefore, one can define the writhe of a knot projection as the writhe of its temporal framing ribbon:

$$w(D) = w(\mathcal{K})$$  \hspace{1cm} (61)

It is rather obvious, that this definition is correct, i.e. it does not depend on the choice of representative framing ribbon. As a function of $D$ the writhe $w(D)$ has a constructive representation in terms of $D$ itself. Indeed, every crossing of $D$ is equivalent up to the planar rotations to one of the two crossings represented in Fig.14

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{ }
\end{figure}

Define the writhes of these crossings as

$$w(c^+) = +1, \quad w(c^-) = -1$$  \hspace{1cm} (62)

Then the writhe $w(D)$ can be expressed as sum of writhes:

$$w(D) = \sum_i w(i)$$  \hspace{1cm} (63)

where sum runs over all crossing of $D$.

### 5.3 Braid representation of the knots.

A useful way for representing knots is to make use the braid group $B_n$. The Artin braid group $B_n$ (the braid group corresponding to the $A_n$ root system in terms of section 4.3) of $n$ strings is the group generated by $n-1$ generators $g_1, g_2, ..., g_{n-1}$ satisfying the following relations:

$$g_i g_j = g_j g_i \quad \text{for} \quad |i - j| > 1$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad \text{for} \quad i - 1 < n$$  \hspace{1cm} (64)

Let $\varrho : U_h(g) \rightarrow \text{End}(V)$ be a representation of some quantum universal enveloping algebra, then one can construct associated representation of the braid group $\varrho_{B_n} : B_n \rightarrow \text{End}(V^{\otimes n})$ by the following explicit definition:

$$\varrho_{B_n} : g_i \mapsto 1 \otimes 1 \otimes ... \otimes 1 \otimes R_{\varrho} \otimes 1 \otimes ... \otimes 1, \quad R_{\varrho} = \varrho \otimes \varrho (\mathcal{R})$$
The first defining relation (64) in this representation is obvious, and the second one simply follows from QYBE (22).

Each element of $B_n$ admits a graphical representation. One can associate to the element $g_i \in B_n$ the picture shown in Fig.15 (a), then the element $g_1 g_2 g_1^{-1} \in B_n$ can be represented as in the Fig.15 (b):

![Figure 15](image)

i.e. $g_i$ can be considered as a twist of $i$ and $i+1$ string on the set of $n$ strings. The closure $\hat{b}$ of the element $b \in B_n$ is a two-dimensional diagram obtained by connection the endpoints of the strings of $B_n$ in a order preserving way Fig.16 (a):

![Figure 16](image)

Every link and knot in particular, can be represented as a closure of an element $b \in B_n$ (this element and group $B_n$ are not uniquely defined). For instance the simplest knot $3_1$ can be represented as a closure of $g_1^3 \in B_2$ Fig.16 (b). The braid representation of knots provides useful tool for calculation of knot invariants (60). Indeed, applying the method for calculation of $I(G, \varrho, K)$ described in the sections 5.1-5.2 to a closure of $b \in B_n$ we arrive to the following proposition:

Let knot $K$ be represented as a closure $\hat{b}$ of some element of a braid group $b \in B_n$. Then an ambient isotopy invariant of a knot for gauge group $G$ and representation $\varrho : g \to \text{End}(V)$ is given by the following explicit formula:

$$I_{\varrho}(G, \varrho, K) = \frac{1}{\text{tr}(Q_{\varrho})} q^{-w(\hat{b})} e(\tilde{t}_2) \text{tr}(Q_{\varrho} \otimes b_{\varrho})$$

(65)

where $w(\hat{b})$ is a writhe of the diagram associated with $\hat{b}$ and

$$Q_{\varrho} = g(\varrho), \quad b_{\varrho} = g_{B_n}(b), \quad \text{tr}(Q_{\varrho}) = < W(U_0) >_{\varrho}$$

(66)

In the braid representation of a knot $\hat{b}$, every string of $b$ has only two turning points with respect to the vertical direction of the page. This is why the contribution of turning points appears in (65) in the form $Q = M \tilde{M} = M^2$. The calculation of (65) for some particular knot is just a matter of multiplication and taking a trace of relatively big matrices. In the next subsection we give several explicit examples of such calculations of the knot invariants for first five nontrivial knots. We also give explicit expressions for $R$ and $Q$ for different groups and their representations.

### 5.4 Explicit examples

In this section we give examples of explicit calculations of knot invariants and universal $R$-matrix (47) for several particular groups and their representations. The invariants are calculated for the first five non-trivial
knots from Rolfsen table \[60\] : 3\text{1}, 4\text{1}, 5\text{1}, 5\text{2} and 6\text{1}. The two dimensional projections of these knots with the temporal framing contour can be chosen as in the Fig.17:

\[
\begin{array}{c}
\text{3_1} \\
\text{4_1} \\
\text{5_1} \\
\text{5_2} \\
\text{6_1}
\end{array}
\]

Figure 17:

The braid representations of these knots and the writhe numbers of the corresponding closures are summarized in the table:

| Knot | Braid representation                                      | Writhe |
|------|----------------------------------------------------------|--------|
| 3\text{1} | \( b = g_1^3 \in B_2 \) | \( w(b) = 3 \) |
| 4\text{1} | \( b = g_2^2 g_1^{-1} g_3^{-1} \in B_3 \) | \( w(b) = 1 \) |
| 5\text{1} | \( b = g_1^5 \in B_2 \) | \( w(b) = 5 \) |
| 5\text{2} | \( b = g_1^5 g_2^{-1} g_3 \in B_3 \) | \( w(b) = 4 \) |
| 6\text{1} | \( b = g_1 g_2^{-1} g_3 g_2^{-1} g_3^{-2} \in B_4 \) | \( w(b) = -1 \) |

\[
(67)
\]

5.4.1 \( SU(2) \) representations of the weight \( \lambda \).

The polynomial knot invariants arising from vevs of Wilson loops carrying arbitrary representation of the group \( SU(2) \) is a very instructive example for application of general formulas of sections 3-4. For this reason let us consider in detail the construction of quantum generators for \( U_h(su(2)) \) and universal \( R \)-matrix (47) in the case of representation with highest weight \( \lambda \).

The algebra \( su(2) \) is the tree-dimensional Lie algebra with generators \( e, f \) and \( h \) subjected to the following relations:

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h
\]

All irreducible finite-dimensional representations of \( su(2) \) is the representations of highest weight \( \varrho_\lambda : su(2) \to End(V_\lambda), \quad \lambda \in \mathbb{N}_0 \). The dimension of the representation with highest weight \( \lambda \) is given by \( \dim(V_\lambda) = \lambda + 1 \). In a suitable basis in \( V_\lambda \) the generators \( e, f \) and \( h \) are represented by \((\lambda + 1) \times (\lambda + 1)\) matrices with the following elements:

\[
\begin{align*}
(\varrho_\lambda e)_{ij} & := (e_\lambda)_{ij} = (\lambda - i + 1) \delta_{i,j-1} \\
(\varrho_\lambda f)_{ij} & := (f_\lambda)_{ij} = i \delta_{i-1,j} \\
(\varrho_\lambda h)_{ij} & := (h_\lambda)_{ij} = (\lambda - 2(i - 1)) \delta_{i,j}
\end{align*}
\]

\[
(68)
\]
To get from Lie algebra generators of $su(2)$ to the quantum group generators $E$, $F$ and $h$ we replace the matrix elements of $e$ and $f$ by their quantum deformations:

\[
\begin{align*}
(E_\lambda)_{ij} &= [\lambda - i + 1] \delta_{i,j-1}, \\
(F_\lambda)_{ij} &= [i] \delta_{i-1,j}, \\
(h_\lambda)_{ij} &= (\lambda - 2(i - 1)) \delta_{i,j},
\end{align*}
\]

(69)

where

\[
[m] = \frac{q^m - q^{-m}}{q - q^{-1}}
\]

It is a simple exercise to check that for arbitrary $\lambda$ these matrices satisfy the relations (35)-(38) for $U_h(su(2))$:

\[
[h_\lambda, E_\lambda] = 2 E_\lambda, \quad [h_\lambda, F_\lambda] = -2 F_\lambda, \quad [E_\lambda, F_\lambda] = \frac{q^{h_\lambda} - q^{-h_\lambda}}{q - q^{-1}}
\]

(70)

Note that for vev of the unknot we have:

\[
Q_\lambda = \varrho_\lambda(Q) = q^{h_\lambda} = \text{diag}[q^{\lambda - 2(i - 1)}], \quad i = 1...\lambda + 1
\]

(73)

For the quadratic Casimir element (56) we find:

\[
\varrho_\lambda(\Omega_2) = \frac{1}{2} h_\lambda h_\lambda + e_\lambda f_\lambda + f_\lambda e_\lambda = \lambda(\lambda + 2)
\]

(74)

Note that for vev of the unknot we have:

\[
< W(U_0) > = \text{tr} Q_\lambda = \text{tr} q^h = \sum_{m=1}^{\lambda+1} q^{\lambda - 2(m-1)}
\]

(75)

Therefore, the ambient isotopy invariant (65) for a knot, represented by the closure of $b \in B_n$, is:

\[
I(SU(2), \lambda, K) = \frac{1}{\sum_{m=1}^{\lambda+1} q^{\lambda - 2(m-1)}} q^{-w(K) \lambda(\lambda + 2)} \text{tr} \left( Q_\lambda^{\otimes n} b_\lambda(K) \right)
\]

(76)

Using the braid representations for first six non-trivial knots and their writhe numbers in the standard framing (67) we find that for fundamental representation $\lambda = 1$ this invariant gives the Jones polynomials of the knots:

\[
\begin{align*}
I(SU(2), 1, 3_1) &= (q^6 + q^2 - 1) q^{-8} \\
I(SU(2), 1, 4_1) &= (q^8 - q^6 + q^4 - q^2 + 1) q^{-4} \\
I(SU(2), 1, 5_1) &= (q^{10} + q^6 - q^4 + q^2 - 1) q^{-14} \\
I(SU(2), 1, 5_2) &= (q^{10} - q^8 + 2 q^6 - q^4 + q^2 - 1) q^{-12} \\
I(SU(2), 1, 6_1) &= (q^{12} - q^{10} + q^8 - 2 q^6 + 2 q^4 - q^2 + 1) q^{-4}
\end{align*}
\]

(77)
For a general value of the highest weight $\lambda$ we find:

$$I(SU(2), \lambda, 3_1) : \frac{1}{\lambda + 1} \sum_{m=0}^{\lambda} \left[ 2m + 1 \right] (-)^{\lambda - m} q^{3(\lambda + 2) - m(m+1)}$$

$$I(SU(2), \lambda, 4_1) : \frac{1}{\lambda + 1} \sum_{m, k=0}^{\lambda} \sqrt{2m + 1} \sqrt{2k + 1} a_{k m} q^{2(m+1) - k(k+1)}$$

$$I(SU(2), \lambda, 5_1) : \frac{1}{\lambda + 1} \sum_{m, k=0}^{\lambda} \sqrt{2m + 1} \sqrt{2k + 1} a_{k m} q^{2(\lambda + 1) - m(m+1) - 3k(k+1)/2}$$

$$I(SU(2), \lambda, 6_1) : \frac{1}{\lambda + 1} \sum_{m, k=0}^{\lambda} \sqrt{2m + 1} \sqrt{2k + 1} a_{k m} q^{2(m+1) - 2k(k+1)}$$

where the following conventions are used:

$$\begin{align*}
[m] &= \frac{q^m - q^{-m}}{q - q^{-1}}, \\
[a_{k m}] &= (-)^{k + m - \lambda} \sqrt{2k + 1} \sqrt{2m + 1} \binom{\lambda/2, \lambda/2, k}{\lambda/2, \lambda/2, m}
\end{align*}$$

The quantum Recah coefficients are:

$$\begin{align*}
\left[ \begin{array}{c} p_1, p_2, p_{12} \\ p_3, p_4, p_{23} \end{array} \right] &= \Delta(p_1, p_2, p_{12}) \Delta(p_3, p_4, p_{12}) \Delta(p_1, p_4, p_{23}) \Delta(p_3, p_2, p_{23}) \times \\
	imes \sum_{m > 0} (-)^m [m + 1]! \left[ \begin{array}{c} m - p_1 - p_2 - p_{12}! \end{array} \right] m - p_3 - p_4 - p_{12}! \times \\
	imes [m - p_1 - p_3 - p_2 - p_{23}]! \left[ \begin{array}{c} p_1 + p_2 + p_3 + p_4 - m! \end{array} \right] \times \\
	imes [p_1 + p_3 + p_12 + p_{23} - m]! \left[ \begin{array}{c} p_2 + p_4 + p_{12} + p_{23} - m! \end{array} \right]^{-1}
\end{align*}$$

where $m$ runs over all non-negative numbers, such that each q-factorial in the sum gets non-negative argument, and:

$$\Delta(a, b, c) = \sqrt{\frac{[-a + b + c]! [a - b + c]! [a + b - c]!}{[a + b + c + 1]!}}$$

The polynomials (78) were obtained in [31] as traces of monodromies for the correlators of the associated $SU(2)$ Wess-Zumino conformal field theory.

### 5.4.2 $SU(N)$ in fundamental representation

The irreducible representations of $su(N)$ are representations of the highest weight $\lambda = \{ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1} \}$, $\lambda_i \in \mathbb{N}_0$. Here we consider the case of the first fundamental representation corresponding to $\lambda_{f} = \{1, 0, 0, \ldots \}$. In this fundamental representation the root elements of the quantum group $U_q(su(N))$ have the following property:

$$\varrho_{\lambda_{f}}(E_{\alpha})^2 \varrho_{\lambda_{f}}(F_{\alpha})^2 = 0, \quad \alpha \in \Phi^+$$

and the universal $R$-matrix [17] takes the form:

$$R_{\lambda_{f}} = \hat{P} \varrho_{\lambda_{f}} \otimes \varrho_{\lambda_{f}}(R) = \hat{P} \sum_{\alpha \in \Phi^+} \varrho_{\lambda_{f}} \otimes \varrho_{\lambda_{f}}(h_{\alpha} \otimes h_{\alpha^v}) \prod_{\alpha \in \Phi^+} \left( 1 + (q - q^{-1}) \varrho_{\lambda_{f}} \otimes \varrho_{\lambda_{f}}(F_{\alpha} \otimes E_{\alpha}) \right)$$

In the standard basis $e_{ij}$ of $\text{End}(\mathbb{R}^N)$ this matrix can be expressed as:

$$R_{\lambda_{f}} = \hat{P} \left( q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \sum_{i \neq j} (q - q^{-1}) e_{ij} \otimes e_{ji} \right), \quad \text{where} \quad \hat{P} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}$$
The root system of \( so(N) \) corresponds to

\[ \theta_{5.4.3} \]

\[ \theta \]

The universal \( R \)-matrix (67) takes the form:

\[ R_{\lambda_f} = P (g_{\lambda_f} (R) = \sum_{i,j=1}^{N} e_{ij} e_{ji} = N \]

For the vev of the unknot in this representation we find:

\[ < W(U_0) > = \text{tr} Q_{\lambda_f} = \sum_{m=1}^{N} q^{-N+2m-1} = \frac{q^{N} - q^{-N}}{q - q^{-1}} \] (82)

Therefore the ambient isotopy invariant (65) in this case reads:

\[ I(SU(N), \lambda_f, K) = \frac{q - q^{-1}}{q^{N} - q^{-N}} q^{-N w(K)} \text{tr}(Q_{\lambda_f}^{\otimes n} b_{\lambda_f}(K)) \] (83)

Using the braid representations (67) for first six non-trivial knots and their writhe numbers in the temporal framing we get:

\[ I(SU(N), \lambda_f, 3_1) : \theta (1 + q^4 - \theta q^4) \]
\[ I(SU(N), \lambda_f, 4_1) : (1 - q^{-2} + \theta^{-1}q^2 - q^2 + \theta q^2) \]
\[ I(SU(N), \lambda_f, 5_1) : \theta^2 (1 + q^4 - \theta q^4 + q^8 - \theta q^8) \] (84)
\[ I(SU(N), \lambda_f, 5_2) : \theta (1 - q^2 + \theta q^2 + q^4 - \theta q^4 + \theta q^6 - \theta^2 q^6) \]
\[ I(SU(N), \lambda_f, 6_1) : \theta^{-2} q^{-4} (1 - \theta + \theta^2 q^2 - \theta^2 q^2 - \theta q^4 + 2 \theta^2 q^4 - \theta^2 q^6 + \theta^3 q^6) \]

where \( \theta = q^{-2N-2} \). In accordance with [1] these invariants are the HOMFLY polynomials [43] of the knots.

### 5.4.3 SO(N) in fundamental representation

The root system of \( so(N) \) corresponds to \( B_{N/2} \) for even \( N \) and to \( D_{(N-1)/2} \) for even \( N \). In both cases the root system consist of \( N \) roots of two types with lengths \( (\alpha, \alpha) = 2 \) (long roots) and \( (\alpha, \alpha) = 1 \) (short roots). In the fundamental representation the root elements of the algebra \( U_h(so(N)) \) are nilpotent:

\[ g_{\lambda_f}(E_\alpha)^2 = g_{\lambda_f}(F_\alpha)^2 = 0, \text{ if } \alpha \text{ is long, } g_{\lambda_f}(E_\alpha)^3 = g_{\lambda_f}(F_\alpha)^3 = 0, \text{ if } \alpha \text{ is short,} \]

The universal \( R \)-matrix (67) takes the form:

\[ R_{\lambda_f} = P (g_{\lambda_f} \otimes g_{\lambda_f}(R) = \sum_{a, b} g_{\lambda_f} \otimes g_{\lambda_f} (h_a \otimes h_{\alpha, \gamma}) \prod_{\alpha \in \Phi^+} \sum_{m=0}^{2} \frac{q^{m(m-1)/2}}{[m]_{q^2}!} ((q - q^{-1})^m g_{\lambda_f}(F_\alpha)^m \otimes g_{\lambda_f}(E_\alpha)^m) \]

The explicit expression for quantum R-matrix is different for even and odd \( N \). In suitable basis for \( N = 2n + 1 \) we have:

\[ R_{\lambda_f} = P (\epsilon_{(N+1)/2(N+1)/2} \otimes e_{(N+1)/2(N+1)/2} + \sum_{i \neq N+1-i} (q e_{ii} \otimes e_{ii} + q^{-1} e_{ii} \otimes e_{N+1-iN+1-i}) + \sum_{i \neq N+1-j} (q - q^{-1}) e_{ij} \otimes e_{ij} + (q - q^{-1}) q^{\nu_i} e_{ij} \otimes e_{N+1-iN+1-j}) \] (85)

where

\[ \nu_i = \begin{cases} \frac{n+1}{2} - i, & i < n + 1 \\ 0, & i = n + 1 \\ n + 3/2 - i, & i > n + 1 \end{cases} \]
and

$$\hat{P} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}$$

In the case of even $N = 2n$ we get:

$$R_{\lambda_f} = \hat{P} \sum_{i \neq N+1-i}^{N} \left( q e_{ii} \otimes e_{ii} + q^{-1} e_{ii} \otimes e_{N+1-i N+1-i} \right) +$$

$$+ \sum_{i \neq N+1-j}^{N} e_{ii} \otimes e_{jj} + \sum_{i,j}^{N} \left( (q-q^{-1}) e_{ij} \otimes e_{ji} + (q-q^{-1}) q^{\nu_i-\nu_j} e_{ij} \otimes e_{N+1-i N+1-j} \right)$$

where

$$\nu_i = \begin{cases} n-i, & i < n+1 \\ n-i+1, & i \geq n+1 \end{cases}$$

The operator $[13]$ in both cases is given by the following expression:

$$Q_{\lambda_f} = \varrho_{\lambda_f}(\lambda) = \text{diag}(q^{2\nu N+1-i})$$

The quadratic Casimir element:

$$\varrho_{\lambda_f}(\Omega_2) = \sum_{\alpha \in \Delta} h_{\alpha} h_{\alpha} + \sum_{\alpha \in \Phi^+} e_{\alpha} f_{\alpha} + f_{\alpha} e_{\alpha} = N - 1$$

For the vev of unknot we get:

$$< W(U_0) > = \sum_{m=1}^{N} q^{2\nu N+1-i}$$

Therefore the ambient isotopy invariant $[65]$ in this case reads:

$$I(SO(N), \lambda_f, K) = \frac{1}{\sum_{m=1}^{N} q^{2\nu N+1-i}} q^{-(N-1)w(K)} \text{tr}(Q_{\lambda_f}^{\otimes n} b_{\lambda_f}(K))$$

(88)

Using the braid representations $[67]$ for first six non-trivial knots and their writhe numbers in the temporal framing we get:

$$I(3_1) : 2a^2 - a^4 + (-a^3 + a^5) z + (a^2 - a^4) z^2$$

$$I(4_1) : -1 + a^{-2} + a^2 + (a^{-1} - a) z + (-2 + a^{-2} + a^2) z^2 + (a^{-1} - a) z^3$$

$$I(5_1) : 3a^4 - 2a^6 + (-2a^5 + a^7 + a^9) z + (4a^4 - 3a^6 - a^8) z^2 + (-a^5 + a^7) z^3 + (a^4 - a^6) z^4$$

$$I(5_2) : a^2 + a^4 - a^6 + (-2a^5 + 2a^7) z + (a^2 + a^4 - 2a^6) z^2 + (a^3 - 2a^5 + a^7) z^3 + (a^4 - a^6) z^4$$

$$I(6_1) : a^{-4} - a^{-2} + a^2 + (2a^{-3} - 2a^{-1}) z + (3a^{-4} - 4a^{-2} + a^2) z^2 + (3a^{-3} - 2a^{-1} - a) z^3 +$$

$$+ \left( 1 + a^{-4} - 2a^{-2} \right) z^4 + (a^{-3} - a^{-1}) z^5$$

where $a = q^{N-1}$ and $z = q - q^{-1}$. In agreement with $[32]$ these invariants are the Kauffman polynomials $[43]$ of the knots.
5.4.4 \( Sp(2n) \) in fundamental representation

The case of the group \( Sp(2n) \) in fundamental representation is very similar to \( SO(2n) \) one. In appropriate basis for \( N=2n \) for universal \( R \)-matrix we have:

\[
R_{\lambda_f} = \hat{P} \sum_{i \neq N+1-i}^N \left( q e_{i\bar{i}} \otimes e_{i\bar{i}} + q^{-1} e_{N+1-i\bar{N+1}-i} \right) + \sum_{i \neq N+1-j}^N e_{i\bar{i}} \otimes e_{j\bar{j}} + \sum_{i > j} \left( (q-q^{-1}) e_{i\bar{j}} \otimes e_{j\bar{i}} + (q-q^{-1}) q^\nu e_{i\bar{j}} e_{j\bar{i}} \otimes e_{N+1-i\bar{N+1}-j} \right)
\]

(90)

where

\[
\nu_i = \begin{cases} 
    n-i+1, & i < n+1 \\
    n-i, & i > n+1
\end{cases}
\]

The operator \( Q_{\lambda_f} \) is given by the following expression:

\[
Q_{\lambda_f} = g_{\lambda_f}(Q) = \text{diag}(q^{2\nu_2n+1-i}), \quad i = 1, \ldots, 2n
\]

The quadratic Casimir element:

\[
g_{\lambda_f}(\Omega_2) = \sum_{\alpha \in \Delta} h_{\alpha\alpha} + \sum_{\alpha \in \Phi^+} e_{\alpha} f_{\alpha} + f_{\alpha} e_{\alpha} = N-1
\]

The vev of unknot in this case:

\[
< W(U_0) > = \frac{2n}{\sum_{m=1}^{2n} q^{2\nu m+1}} \quad (91)
\]

and for the ambient isotopy invariant [65] we get:

\[
I(Sp(2n), \lambda_f, K) = \frac{1}{\sum_{m=1}^{2n} q^{2\nu m+1}} \frac{q^{-(N-1)w(K)}}{\text{tr}(Q_{\lambda_f}^{\otimes n} b_{\lambda_f}(K))} \quad (92)
\]

In this case we again obtain the Kauffman polynomial [89] for the following values of parameters \( a = -q^{2n+1} \) and \( z = q - q^{-1} \).

6 Conclusion

In this paper we explicitly described the knot invariants, constructed from the universal quantum \( R \)-matrix of arbitrary simple (quantum) Lie algebra \( G_q \), and demonstrated that they are indeed invariant under all relevant Reidemeister moves. The operators, naturally associated with this construction are, however, the group elements of \( G_q \) in the sense of [65], rather than the Wilson \( P \)-exponents in the temporal gauge. The group elements [65] and special point operators discussed in this paper are made of the generators of quantum algebra. For this reason, one of the most important problem is to realize how the "primordial" generators of a Lie algebra are transformed into the quantum ones in the perturbation theory. This question and way the group elements arise as the free-field representation of these \( P \)-exponents, presumably \( a \) la [63], remains to be worked out. Also relation to many other descriptions of knot invariants, associated with other gauge choices in Chern-Simons theory, is left beyond the scope of the present paper.

We hope that the explicit expressions for crossing operators [59] will help to understand better the quantization of the CST in the temporal gauge. A possible step toward solution of this problem is to derive the analog of the Labastida-Pérez (LP) formula [24]. This formula expresses the \( m \)-th order of vev of Wilson loop in perturbation theory as the trace \( \text{tr} T_m(D) \), where the element \( T_m(D) \in U(g) \) is defined combinatorially by the crossings of two dimensional projection \( D \) of the knot. The summation in all orders was done in [54] and the result was represented in the form of contraction of the "wrong" crossing operator [25] over all crossings. Now, we are in a reverse situation. We know the "correct" answer for the crossing operator [17] depending on \( q = e^h \). Expanding this answer in powers of \( h \), one can find a proper analog of the LP formula. This formula should provide some combinatorial analogs of Kontsevich integral [58] for finite-type or Vassiliev invariants [57]. This would also help to solve a long-standing problem of combinatorial description of Vassiliev invariants. From the CST site, the correct analog of LP formula would give the properly regularized perturbation theory in the temporal gauge.
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