SU(N) Evolution of a Frustrated Spin Ladder

Miraculous J. Bhaveen and Alexei M. Tsvelik
Department of Physics, Brookhaven National Laboratory, Upton, NY 11973-5000, USA
(Dated: March 22, 2022)

Recent studies indicate that the weakly coupled, $J_{\perp} \ll J_{\parallel}$ spin-1/2 Heisenberg antiferromagnet with next nearest neighbor frustration, $J_{N} \perp J_{\parallel}$, supports massive spinons for $J_{N} = J_{\perp}/2$. The straightforward SU(N) generalization of the low energy ladder Hamiltonian yields two independent SU(N) Thirring models with $N-1$ multiplets of massive "spinon" excitations. We study the evolution of the complete set of low energy dynamical structure factors using form factors. Those corresponding to the smooth (staggered) magnetizations are qualitatively different (the same) in the $N=2$ and $N > 2$ cases. The absence of single-particle peaks preserves the notion of spinons stabilized by frustration. In contrast to the ladder, we note that the $N \to \infty$ limit of the four chain model is not a trivial free theory.

PACS numbers: 71.10.Pm, 75.10.Jm, 75.10.Pq

I. INTRODUCTION

Frustrated quantum antiferromagnets are a source of considerable theoretical and experimental attention — see for example reference [1]. Their characteristics include enhanced classical ground state degeneracies and the suppression of long-range Néel order. In addition to their intrinsic interest, their prominence is fueled by the high-$T_c$ superconducting cuprates, where hole doping frustrates, and ultimately destroys the long-range Néel order of the parent compounds — see for example [2]. This motivates the quest for simple models of frustrated quantum magnets, and a detailed understanding of their properties.

Important examples include nearest neighbor antiferromagnets on frustrated lattices, such as the triangular [3] pyrochlore, and Kagomé [4] lattices, and further neighbor models on regular lattices. The second variety embraces frustrated chains [3] and ladders [3,7,8] the planar pyrochlore [9,10,11] and the square lattice antiferromagnet with next-nearest neighbor interactions. Indeed, the latter model was suggested by Anderson in his influential work [12] on La$_2$CuO$_4$, as a means to realize his "resonating-valence-bond" (RVB) or "quantum spin-liquid" state. With isotropic nearest neighbor exchange, $J_1$, this is often referred to as the $J_1 - J_2$ model — for an introduction to spin-liquids see Chapter 6 of the book by Fradkin [13]. Other examples include multispin exchange models, and those of dimers [14]. Although enormous progress continues to be made, frustrated quantum magnetism remains theoretically challenging. In general one must resort to $1/S$ or $1/N$ expansions, numerical simulations, or other approximation schemes — see for example reference [15].

Building on the work of reference [8] Nersesyan and Tsvelik have made considerable advances in the so-called confederate flag model [16]. This is an anisotropic version of the much studied $J_1 - J_2$ model, in which the nearest neighbor exchange has a strongly preferred chain direction — see figure 1. The limit $J_{\parallel} \leq J_{\perp} \ll J$ may be viewed as a collection of weakly coupled, but nevertheless interacting chains, and field theory methods may be employed. In general, the massless spinons of the spin-1/2 chain [17] are confined by the interchain interactions. However, along the line, $J_{\parallel} = J_{\perp}/2$, massive spinons emerge in pairs, as the elementary spin excitations of the coupled system [8,16]. In general they are neither bosons nor fermions, but have momentum dependent scattering. There have been many speculations about the existence of such excitations in two-dimensional frustrated antiferromagnets, and their possible rôle in high-$T_c$ superconducting systems [18]. The developments of reference [16] deserve further investigation.

In this paper we return to an SU(N) generalization of the ladder introduced in reference [8]. Our motivation is twofold: firstly, the large-N approach is known to miss qualitative features in this case [19] and we wish to track its evolution in detail. Large-N results will be important in two-dimensions, and we hope to gain expertise in all the solvable cases. Secondly, we calculate the dynamical structure factors of the staggered magnetizations. These involve correlation functions of interacting WZNW fields, and their evaluation beyond the ladder, is a highly challenging and open problem [20,21].

The layout of this paper is as follows: in section [22] we reacquaint the reader with the spin-1/2 model, and it’s mapping on to two different “parity” sectors [8,16]. We introduce the SU(N) variant of the low energy action and
II. MODEL

In this section we reacquaint the reader with the spin-1/2 confederate flag model, and it’s mapping on to two different “parity” sectors. We shall specialize to the ladder in due course. Consider a Heisenberg antiferromagnet on a two-dimensional square lattice (of spacing $a_0$) with next nearest neighbor exchange interaction $0 < J_x < J_\perp < J$ as depicted in figure 1.

$$H = \sum_{i=1}^{L} \sum_{n} \left[ J_S i, n \cdot S_{i,n+1} + J_\perp S_{i,n} \cdot S_{i+1,n} + J_x (S_{i,n} \cdot S_{i+1,n+1} + S_{i,n+1} \cdot S_{i+1,n}) \right].$$

(1)

It is well established, that the low energy dynamics of a single spin-1/2 (isotropic) Heisenberg chain

$$H_{i}^{1D} = \sum_{n} J_S i, n \cdot S_{i,n+1},$$

(2)

are described by the $\mathfrak{su}(2)$ Wess-Zumino-Novikov-Witten (WZNW) model for a review see. This WZNW model has conserved currents $J = L_i^a t_{a} L_\beta$ and $\mathcal{J} = R_i^a t_{a} R_{\beta}$, which generate the $\mathfrak{su}(2)_1$ Kac-Moody current algebra, and the Hamiltonian density, $H = \int dx H_i$, may be written in the following (Sugawara) form:

$$H_i^{1D} = \mathcal{N} \hbar v (J_1 \cdot \mathbf{J}_i + J_2 \cdot \mathbf{\bar{J}}_i + \cdots)$$

(3)

Here $v$ is the spin velocity, $\mathcal{N}$ is a normalization constant, and the ellipsis stand for less relevant operators. We replace the perturbing lattice spin operators by their continuous, slowly varying, uniform and staggered components:

$$S_{i,n} \rightarrow S \left( x \right) = M_i \left( x \right) + (-1)^n N_i \left( x \right),$$

(4)

where $x \equiv n a_0$ measures the distance along chain $i$. Neglecting oscillatory and derivative terms, Hamiltonian (2) becomes $H = \int dx H_i$, where

$$H = \sum_{i=1}^{N} H_i^{1D} + J_\perp a_0 (M_i \cdot M_{i+1} + N_i \cdot N_{i+1}) + 2 J_x a_0 (M_i \cdot M_{i+1} - N_i \cdot N_{i+1})$$

(5)

In terms of the currents, $M_i \equiv J_i + \bar{J}_i$, the Hamiltonian density (3) may be written

$$H = \sum_{i=1}^{N} \mathcal{H}_i^{1D} + \lambda_1 (J_i + \bar{J}_i) \cdot (J_i + \bar{J}_{i+1})$$

$$+ \lambda_2 N_i \cdot N_{i+1} + \cdots$$

(6)

where

$$\lambda_1 = (J_\perp + 2J_x) a_0, \quad \lambda_2 = (J_\perp - 2J_x) a_0.$$

(7)

In particular, for $J_x = J_\perp / 2$, the strongly relevant inter-chain coupling, $\lambda_2$, between the staggered magnetizations vanishes. Setting $J_x = J_\perp / 2$, and neglecting velocity renormalizing terms, the Hamiltonian splits into two independent pieces, or “parity” sectors:

$$H = H_+ + H_-,$$

(8)

where

$$H_+ = \sum_i \mathcal{N} \hbar v (J_{2i} \cdot \mathbf{J}_{2i} + \tilde{J}_{2i+1} \cdot \tilde{\mathbf{J}}_{2i+1}) + \lambda_1 J_{2i} \cdot \tilde{\mathbf{J}}_{2i+1},$$

(9)

and $H_-$ is obtained from $H_+$ by the (parity) transformation $J \leftrightarrow \mathbf{J}$. In the sector of positive parity, the even (odd) chains carry left (right) moving fields; in the sector of negative parity the situation is reversed (see figure 2). Equivalently, $H_+$ and $H_-$ are interchanged under a shift by $a_0$ transverse to the chains. Specializing to the ladder:

$$H_+ = \mathcal{N} \hbar v (\mathbf{J}_1 \cdot J_{1\uparrow} + \mathbf{J}_{1\downarrow} \cdot J_{1\downarrow}) + \lambda_1 (\mathbf{J}_1 \cdot J_{1\downarrow}),$$

(10)

where we label the chains by Roman numerals to avoid subsequent confusion with space-time indices. The Hamiltonian (10) may be brought into a more familiar form by introducing a spinor, the left component of which resides on one chain and the right resides on the other:

$$\psi_+ = \left( \begin{array}{c} R_i^1 \\ L_{1\downarrow} \end{array} \right),$$

(11)

In terms of this spinor, the Hamiltonian (10) becomes

$$H_+ = \mathcal{N} \hbar v (J_{1\uparrow} \cdot J_{1\uparrow} + \mathbf{J}_{1\downarrow} \cdot \mathbf{J}_{1\downarrow}) + \lambda_1 (J_{1\uparrow} \cdot J_{1\downarrow})$$

(12)

and similarly for $H_-$. (Equivalently one may perform the chiral interchange $J_1 \leftrightarrow J_{1\downarrow}$ on the original Hamiltonians.) We see that $H_+$ is nothing but an SU(2) Thirring model. That is to say, the frustrated ladder may be reformulated as the sum of two decoupled SU(2) Thirring models, labelled by their parity. We emphasize that each of these decoupled models captures the behavior of the coupled ladder, as highlighted in section B, and not just a single chain. In particular, the elementary excitations of the ladder are those of the SU(2) Thirring model, namely massive spinons. These correspond to domain walls separating regions of different spontaneous dimerization.
In this section we compute the dynamical structure factors of the generalized model. These are a direct probe of the elementary excitations.

III. DYNAMICAL STRUCTURE FACTOR

In this section we compute the dynamical structure factor (as may be seen by neutrons) for momentum transfers close to the “soft modes” at 0 and π. This is nothing but a Fourier transform of the spin-spin correlation functions:

\[ S(\omega, q, q_\perp) \propto \text{Im} \int_{-\infty}^{\infty} dx \int_0^\infty dt \, e^{i(\omega+i\delta)t-qhx} \langle [S^\alpha_1(t,x) \pm S^\alpha_1(0,0) \pm S^\alpha_1(0,0)] \rangle. \] (13)

The plus (minus) sign corresponds to \( q_\perp = 0 \) (\( q_\perp = \pi \)), and \( \delta \) ensures convergence of the temporal integral. The longitudinal momentum transfers in the vicinity of \( q = 0 \) \((q = \pi)\) probe the smooth (staggered) components of the spin operators. The task is to relate the spin operators entering \([13]\) to the operators of the Thirring models, and to evaluate their matrix elements.

A. Smooth Components

The smooth component of the sum of the chain spin densities may be expressed in terms of the two Thirring models as follows:

\[ S_1 + S_{1\perp} |_{\text{smooth}} = J_{1\perp} + J_{1} + J_{11} + \bar{J}_{1\perp} \]

\[ = J_{0, +} + J_{0, -}, \]

where \( J_{0, +} = \bar{J}_{1\perp} + J_{11} \) \((J_{0, -} = J_{1} + \bar{J}_{1\perp})\) is the temporal component of the Thirring current in the model of positive (negative) parity. Simply put, the structure factor \( S(w, q \sim 0, 0) \) of the frustrated ladder, may be obtained from the correlators of \( J_0 \) in the SU(N) Thirring model.

\[ S(\omega, q \sim 0, 0) \propto \text{Im} \int_{\mathcal{P} = \pm} \int_{-\infty}^{\infty} dx \int_0^\infty dt \, e^{i(\omega+i\delta)t-qhx} \langle |J^\alpha_{0, \mathcal{P}}(t,x) J^\alpha_{0, \mathcal{P}}(0,0)| \rangle \] (14)

where the summation is over parity sectors. The elementary excitations of the SU(N) Thirring model consist of \( N-1 \) multiplets of massive particles, corresponding to the fundamental representations of SU(N). The length of the Young tableau is termed the “rank” of the particle Ind and their masses are given by \([13]\). It is convenient to move to a basis of such particles and to parametrize their energy and momentum in terms of rapidity:

\[ E_i = m_i \cosh \theta_i \quad P_i = m_i \sinh \theta_i. \] (15)

One may now insert a complete set of states between the current operators in \([13]\):

\[ \mathbb{I} = \sum_n \sum \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} \begin{vmatrix} |\theta_n \ldots \theta_1\rangle \end{vmatrix}_{\epsilon_n \ldots \epsilon_1} \epsilon_1 \ldots \epsilon_n |\theta_1 \ldots \theta_n\rangle \] (16)

where the \( \epsilon_i \) are the internal (or isotopic) indices carried by the members of each multiplet. Using

\[ \epsilon'_1 \ldots \epsilon'_n |\theta'_1 \ldots \theta'_n\rangle |O(t,x)\rangle |\theta_n \ldots \theta_1\rangle \epsilon_n \ldots \epsilon_1 \]

\[ \equiv e^{i \sum_j (E'_j - E_j)(t-x'_j - t_j)} \times \]

\[ \epsilon'_1 \ldots \epsilon'_n |\theta'_1 \ldots \theta'_n\rangle |O(0,0)\rangle |\theta_n \ldots \theta_1\rangle \epsilon_n \ldots \epsilon_1 \] (17)

one obtains

\[ S(\omega, q \sim 0, 0) \propto \cdots \]
\[-2\pi \text{ Im} \sum_{n=0}^{\infty} \sum_{\epsilon_n} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} |F_{J_{0}}(\theta_1 \cdots \theta_n)_{\epsilon_1 \cdots \epsilon_n}|^2 \]

\[
\left[ \frac{\delta(vq - \sum_j m_j \sinh \theta_j)}{\omega - \sum_j m_j \cosh \theta_j + i\delta} - \frac{\delta(vq + \sum_j m_j \sinh \theta_j)}{\omega + \sum_j m_j \cosh \theta_j + i\delta} \right]
\]

where \(F_{J_{0}}(\theta_1 \cdots \theta_n)_{\epsilon_1 \cdots \epsilon_n}\) is a multiparticle form factor of the temporal Thirring current:

\[
F_{J_{0}}(\theta_1 \cdots \theta_n)_{\epsilon_1 \cdots \epsilon_n} \equiv \langle 0|J_{0}^{a}(0,0)|\theta_n \cdots \theta_1\rangle_{\epsilon_n \cdots \epsilon_1}.
\]

The dominant contributions to \(F_{J_{0}}\) come from the states with the lowest mass. In the case at hand these are two particle states of the (rank-1) fundamental \(\Box\), and it’s (rank-N-1) conjugate \(\Box\). In particular, the current operator couples to the adjoint representation occurring in the \(SU(N)\) tensor product \(\Box \otimes \Box\); for \(N = 2\), \(\Box\) is \(\Box\). As we discuss in appendix B, this form factor is

\[
F_{J_{0}}(\theta_1, \theta_2)_{\Box, \Box} \propto m \sinh \left( \frac{\theta_1 + \theta_2}{2} \right) f_{\adj}(\theta_{12})
\]

where

\[
f_{\adj}(\theta_{12}) = \exp \left\{ \int_{0}^{\infty} dx \left[ \frac{2 \exp(x/N) \sinh(x/N) \sin^2(x\theta/2\pi)}{x \sin^2 x} \right] \right\}
\]

and \(\theta = i\pi - \theta\); see equations (134) and (127). We have supressed the isotopic and component information in \(f_{\adj}\) and concentrated solely on the rapidity dependence. Inserting this into (18) and performing the \(\theta\) integrations one obtains

\[
S(\omega, q \sim 0, 0) \propto \frac{m^2 v^2 q^2}{s^3 \sqrt{s^2 - 4m^2}} |f_{\adj}^{\Box}(\theta(s))|^2
\]

where \(s^2 = \omega^2 - v^2 q^2\) and \(\theta(s) = \text{arccosh}(s/2m)\) and

\[
4m^2 < s^2 < \begin{cases} 16m^2 & N = 2, \\ 9m^2 & N = 3, \\ 16m^2 \cos^2(\pi/N) & N > 3. \end{cases}
\]

This result is plotted in figure 3 and is exact, provided \(\sqrt{s^2 - 4m^2}\) is fulfilled. For larger energy transfers there are small corrections due to higher mass states; the upper thresholds correspond to four rank-1 solitons, three rank-1 (or rank-2) solitons, and a rank-2 bound state and its conjugate, respectively. In particular, there are no single-particle bound states appearing below the gap; the elementary Thirring excitations correspond to fundamental \(SU(N)\) representations, and do not couple to the current directly, which span the adjoint.

The result \(\sqrt{s^2 - 4m^2}\) interpolates between two known limits. For \(N = 2\), it coincides with equation (34) of reference 8, and in the limit \(N = \infty\), where \(F_{J_{0}}\) tends to unity, we recover the result for free massive fermions.\(^{6,8}\) In particular, the \((\theta = 0)\) threshold behavior of \(\sqrt{s^2 - 4m^2}\) is quite instructive: for \(N = 2\) it vanishes like \(\sinh(\theta/2)\), as may be seen from (132), whereas it is finite and non-vanishing for any \(N > 2\). As a result, the structure factor \(\sqrt{s^2 - 4m^2}\) vanishes as \(s^2 - 4m^2\) in the physical case of \(N = 2\), but diverges as \(1/\sqrt{s^2 - 4m^2}\) for any \(N > 2\) — see figure 3. Solely on the basis of the \(N = 2\) and \(N = \infty\) limits,\(^{6,8}\) one might have expected the threshold to get steeper and narrower with increasing \(N\), but to remain qualitatively correct for \(N < \infty\). The actual evolution, and the departure even for \(N = 3\), is a sobering example of how \(SU(N)\) treatments may miss simple features over the entire range of \(N\).

Likewise, the smooth component of the difference of the chain spin densities may be expressed in terms of the two Thirring models as follows:

\[
S_{I} - S_{II}_{\text{smooth}} = J_{1} + J_{-1} - J_{11} - J_{-11}
\]

where \(J_{1+} = J_{1} - J_{11}\) \((J_{1-} = J_{1} - J_{-11})\) is the spatial component of the Thirring current in the model of positive (negative) parity. Simply put, the structure factor \(S(w, q \sim 0, \pi)\) of the frustrated ladder, may be obtained from the correlators of \(J_{1}\) in the \(SU(N)\) Thirring model:

\[
F_{J_{1}}(\theta_1, \theta_2)_{\Box, \Box} \propto m \cos \left( \frac{\theta_1 + \theta_2}{2} \right) f_{\adj}(\theta_{12}).
\]

We obtain

\[
S(\omega, q \sim 0, \pi) \propto \frac{m^2 \omega^2}{s^3 \sqrt{s^2 - 4m^2}} |f_{\adj}^{\Box}(\theta(s))|^2.
\]

Once again, this result interpolates between the known \(N = 2\) and \(N = \infty\) results\(^{6,8}\) and the \(SU(N)\) approach leads to qualitatively incorrect results over the entire range of \(N > 2\).

B. Staggered Components

We denote the staggered component of the spin on chain I, \(S_{I}(t, x)_{\text{stagg}}\) by \(N_{I}(t, x)\). In the UV limit (cor-
responding to decoupled chains and $m = 0$ $N(t, x)$ is a spinless $su(N)_{1}$ primary field with (full) scaling dimension $\Delta = 1 - 1/N$. For the ladder we propose the following formula for the long distance asymptotics of the real space correlation functions:

$$
\langle [N_{1}(t, x) \pm N_{\Pi}(t, x)], [N_{1}(0, 0) \pm N_{\Pi}(0, 0)] \rangle \\
\propto \langle N_{1}(t, x), N_{1}(0, 0) \rangle \pm \langle N_{1}(t, x), N_{\Pi}(0, 0) \rangle \\
\propto m^{2\Delta} [K_{\Delta}^{\pm}(mr) \pm K_{\Delta}^{0}(mr)] + \cdots
$$

(26)

where $r = \sqrt{z^2} = \sqrt{x^2 - t^2}$ ($v = 1$) and $K_{\nu}(x)$ is MacDonald’s function, also known as the modified Bessel function of the third kind. The dots stand for more rapidly decaying terms. In order to get a feel for this result we begin by studying a few limits. In the limit $N \to \infty$, $\Delta \to 1$, each parity sector reduces to non-interacting massive fermions. More specifically, $N_{1}$ may be replaced by the fermion bilinear $L_{1}^{\alpha} t_{\alpha \beta} R_{1}^{\beta} + \bar{R}_{1}^{\alpha} t_{\alpha \beta} L_{1}^{\beta}$ and one obtains

$$
\langle N_{1} N_{1} \rangle \propto \langle L_{1}^{\dagger} L_{1} \rangle \langle R_{1}^{\dagger} R_{1} \rangle
$$

(27)

$$
\langle N_{1} N_{\Pi} \rangle \propto \langle L_{1}^{\dagger} R_{\Pi} \rangle \langle R_{1}^{\dagger} L_{\Pi} \rangle
$$

(28)

with the usual massive Dirac fermion correlators:

$$
\langle L_{1}^{\dagger} L_{1} \rangle = 2m \sqrt{2} K_{1}(mr)
$$

(29)

$$
\langle L_{1}^{\dagger} R_{1} \rangle = 2m K_{0}(mr)
$$

(30)

--- see for example chapter 13 of the book. In equations (27) and (28) we see quite clearly that the correlators of staggered magnetizations are products of correlators from the sectors of different parity by definition the left and right moving fields on a given chain belong to different sectors. In coupling to the staggered magnetizations, the solitons are still created in pairs, but belong to different sectors. In a given sector (i.e. Thirring model) we thus require the matrix elements of single-soliton creation operators. The matrix elements of such operators have only recently become available. The free fermions appearing in (27) and (28) for $N \to \infty$, are replaced by chiral fields $L_{s}, R_{s}$, which are non-local single soliton creation operators and carry the Lorentz spin, $\pm \Delta/2$, of a Thirring soliton, we take the plus (minus) sign for left (right) movers. These chiral fields are the components of an interacting $su(N)_{1}$ primary field, and the Lorentz spin is nothing but the UV conformal dimension. The single-soliton form factors of such operators are governed (upto normalization) solely by their Lorentz transformation properties:

$$
\langle 0 | L_{s} | \theta \rangle = m^{\Delta/2} e^{\Delta\theta/2}, \quad \langle 0 | R_{s} | \theta \rangle = m^{\Delta/2} e^{-\Delta\theta/2},
$$

(31)

and their two-point functions are now readily computed:

$$
\langle L_{s}^{\dagger} L_{s} \rangle = m^{\Delta} \int d\theta e^{\Delta\theta} e^{-\tau m_{ch} \theta + i x m_{sh} \theta}
$$

(32)

$$
= m^{\Delta} \left( \frac{\tau}{2} \right)^{\Delta/2} 2K_{\Delta}(m\sqrt{z})
$$

(33)

$$
\langle L_{s}^{\dagger} R_{s} \rangle = m^{\Delta} \int d\theta e^{\Delta\theta} e^{-\tau m_{ch} \theta + i x m_{sh} \theta}
$$

(34)

$$
= m^{2} 2K_{0}(m\sqrt{z})
$$

(35)

where $z = \tau - ix$ and $\tau = it$. The results for $\langle R_{s}^{\dagger} R_{s} \rangle$ and $\langle R_{s}^{\dagger} L_{s} \rangle$ follow by interchanging $z$ and $\bar{z}$. In particular, equation (33) first appeared in the study of weakly coupled one-dimensional Mott insulators. Replacing the correlators in (27) and (28) with these more general expressions, the result (26) follows immediately.

Further, the Macdonald function has the asymptotic expansion given by equation 9.7.2 of reference. The free fermions appearing in (27) and (28) are replaced by chiral fields $L_{t,x}, R_{t,x}$ associated to decoupled chains and

$$
\langle N_{s}(t, x), N_{s}(0, 0) \rangle \sim m^{2\Delta - 1}/r e^{-2mr}
$$

(37)

Coupling the chains together not only generates exponentially decaying interchain correlations, but also modifies the $1/r^{2\Delta}$ behavior within the chains. Substituting (29) into the definition (26) and effecting the Fourier transforms we obtain the following structure factors:

$$
S(\omega, q, \pi, 0) \propto \left[ s + \sqrt{s^{2} - 4m^{2}} \right]^{2\Delta} \frac{(2m)^{2\Delta}}{s^{2} - 4m^{2}}
$$

(38)

$$
S(\omega, q, \pi, \pi) \propto \left[ s + \sqrt{s^{2} - 4m^{2}} \right]^{2\Delta} \frac{(2m)^{2\Delta}}{s^{2} - 4m^{2}}
$$

(39)

where $s^{2} = \omega^{2} - (q - \pi)^{2}$. In deriving these expressions the reader may find the integral representations and (41) more convenient. At threshold, $S(\omega, q, \pi, 0)$ diverges as $1/\sqrt{s^{2} - 4m^{2}}$ for all $N$, and we plot this behavior in figure 4. The large $s$ behavior is $s^{-2/3}$. Similarly, at threshold, $S(\omega, q, \pi, \pi)$ tends to a constant for all $N$. In contrast to the magnetization correlators, we obtain qualitatively similar results over the entire range of $N$.

IV. CONCLUSIONS

In this paper we have studied the SU($N$) evolution of a frustrated spin ladder. The dynamical structure factors corresponding to the smooth (staggered) magnetizations are shown to be qualitatively different (the same) in the $N = 2$ and $N > 2$ cases. A robust feature which survives however, is the absence of coherent single-particle excitations at low energies. This is in stark contrast to the unfrustrated ladder, and reinforces the notion of spinons stabilized by frustration.
In closing we note that the N = ∞ limit of the two chain model is a free theory, but this is not so in general. In particular, each parity sector of the four chain model may be viewed as two (non-chiral) $\mathfrak{s}u(N)_1$ WZNW models coupled by currents:

$$\mathcal{H}_+ = \mathfrak{s}u(N)_1 + \mathfrak{s}u(N)_1 + \lambda J_+ J_+ .$$

With transverse periodic boundary conditions the currents $\mathbf{J}_+ = \mathbf{J}_1 + \mathbf{J}_{II}$, $\mathbf{J}_- = \mathbf{J}_{II} + \mathbf{J}_{IV}$ generate $\mathfrak{s}u(N)_2$ Kac–Moody algebras, and it is convenient to write $\lambda \rightarrow \infty$.

$$\mathcal{H}_+ = Z_N + (\mathfrak{s}u(N)_2 + \lambda J_+ J_+).$$

The $Z_N$ parafermions describe gapless non-magnetic excitations with central charge $c = 2(N - 1)/(N + 2)$. They are unaffected by the interaction (due to the boundary conditions) and for $N \rightarrow \infty$ they reduce to two Gaus- sian models. The $\mathfrak{s}u(N)_2$ model on the other hand is rendered massive by the interaction, and is in fact integrable. The mass spectrum coincides with that of the (two chain) SU(N) Thirring model [41], but the scattering is notably different. The S-matrix can be extracted from a straightforward generalization of the Thermodynamic Bethe Ansatz (TBA) equations derived in reference [41]. At low temperatures ($T \ll M_1$) the free energy of the perturbed $\mathfrak{s}u(N)_k$ WZNW model is given by

$$F/L = -TN \sum_{j=1}^{N-1} M_j \int \frac{d\theta}{2\pi} \ln[1 + e^{-\epsilon_n^{(j)}(\theta)/T}]$$

where in this case $k = 2$. The excitation energies $\epsilon_n^{(j)}$ ($j = 1, \ldots, N - 1$, $n = 1, 2, \ldots$) satisfy

$$T \ln(1 + e^{\epsilon_n^{(j)}(\lambda)/T}) - T \lambda A_{ij} \ast C_{nm} \ast \ln(1 + e^{-\epsilon_m^{(n)}(\lambda)/T}) = \delta_{n,k} M_j \ln(2\pi N)$$

where $\ast$ denotes convolution, the kernels $C_{nm}(\lambda)$ and $A_{ij}(\lambda)$ are given in reference [41] and $\lambda = N\theta/2\pi$. We extract the Bethe equations $E = \sum_{a=1}^n m \ln \theta_a$ and

$$\exp(imL \ln \theta_a) = \prod_{b \neq a} S_0(\theta_a - \theta_b) e_{1}(\theta_a - \lambda_n) \prod_{\beta} \mathbf{E}(\theta_a - \mu_\beta)$$

where $S_0(\theta) = \exp \left\{- \int_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{-i\omega \theta} \times \right\}$

$$\left[-1 + \frac{1}{(1 + e^{-2|\omega|/N})^2} \left(1 + \frac{\text{sh}(1 - \frac{\omega}{N})}{\text{sh} \pi \omega} \right) \right] ,$$

and

$$e_n(x) = \frac{x - i\pi n/N}{x + i\pi n/N}, \quad \mathcal{E}(x) = \frac{e^{N\pi/2} - i}{e^{N\pi/2} + i}.$$

The rapdities $\lambda_n$ and $\mu_\beta$ are distributed according to the $A^{N-1}$ heirarchy, the details of which do not concern us here. Similar equations occur for the SU(N) invariant Thirring model ($k = 1$) but without the $\mu$ rapdities and with a different $S_0(\theta)$. In the limit $N \rightarrow \infty$ one obtains

$$\exp(imL \ln \theta_a) = \prod_{b \neq a} S_0^\infty(\theta_{ab})$$

where $S_0^\infty(\theta) = \exp(-i\frac{\pi}{2} \text{sign}(\theta))$. This is to be contrasted with the SU(N) invariant Thirring model where $S_0^\infty(\theta) = -1$. The absence of a simple $N \rightarrow \infty$ limit will be crucial for multiparticle form factors, and renders excitations with non-trivial statistics. In future publications we hope to study these pertinent issues in more detail. Recent progress on spinon propagation in the four chain model may be found in the work of Smirnov and Tsvelik.

Acknowledgements

We are extremely grateful to Fabian Essler and Feodor Smirnov. We are indebted to Philippe Lecheminant for valuable comments. We thank Sam Carr for proof reading this manuscript. We also acknowledge support from the US DOE under contract number DE-AC02 -98 CH 10886.

APPENDIX A: SPIN OPERATORS

In this appendix we comment on the connection between SU(N) spin representations and filling. At each lattice site (labelled by $n$) one may introduce the fermionic spin operators

$$S_n^a = \sum_{\alpha, \beta=1}^{N} c_{n,\alpha} t_{\alpha\beta}^a c_{n,\beta} \quad (A1)$$
where $c$ and $c^\dagger$ obey the canonical fermionic anticommutation relations
\[
\{c_{n,a}^\dagger, c_{m,b} \} = \delta_{n,m} \delta_{a,b} \quad \{c, c^\dagger \} = 0 \quad \{c^\dagger, c^\dagger \} = 0 \quad (A2)
\]
and the generators $t^a$ span the algebra $su(N)$: $[t^a, t^b] = i f^{abc} t^c$. It is readily verified that spins on different sites commute, whereas those on the same site satisfy the $su(N)$ algebra: $[S^a_n, S^b_m] = i \delta_{n,m} f^{abc} S^c_n$. In the fundamental representation, the generators are chosen to satisfy $tr(t^a t^b) = C \delta^{ab}$, $t^a t^a = C_2 I$, with $C = 1/2$ and $C_2 = (N^2 - 1)/2N$; see appendix A.3 of [12].

One may specify the $su(N)$ representation on which spin operators $S_n$ act by the relevant Young tableau — see for example 43. In particular, this fixes the value of the quadratic Casimir $S^2_n$, and thus by equation (A1), constrains fermion occupation numbers. As we shall demonstrate, the constraint
\[
\sum_{a=1}^N c_{n,a}^\dagger c_{n,a} = h; \quad \forall n. \quad (A3)
\]
corresponds to the vertical (i.e. antisymmetric) Young tableau of height $h$, as depicted in figure 5. The constraint (A3) fixes $h$ electrons per site, and the permissible states to be of the form:
\[
\psi_{\alpha_1, \alpha_2, \ldots, \alpha_h} = c_{n,\alpha_1}^\dagger c_{n,\alpha_2}^\dagger \cdots c_{n,\alpha_h}^\dagger |0\rangle, \quad (A4)
\]
where $\alpha_i \in \{1, \ldots, N\}$. By virtue of the fermion anticommutation relations (A2), this may be viewed as a tensor of rank $h$, antisymmetric under the interchange of any pair of labels $\alpha_i$; by the standard conventions for Young tableaux this corresponds to a vertical diagram of $h$ boxes. Moreover, it also follows from the anticommutation relations (A2), that there are $N(N-1) \cdots (N-h+1)/h!$ independent states of the form (A3); this coincides with the dimension of the representation corresponding to the Young tableau of fig 5, see §8.4 of [12]. Further, squaring equation (A4) and enforcing the constraint (A3) one obtains
\[
S^2_n = \frac{h(N^2 - h)}{2N} + \frac{h(1-h)}{2}. \quad (A5)
\]
This coincides with the quadratic Casimir of the $su(N)$ Young tableau depicted in fig 5; see eq. 2.19 of [12], e.g. for the fundamental $\Box$ of $su(2)$ ($h = 1, N = 2$) one obtains $S^2_2 = 3/4$, as appropriate for spin-1/2.

Thus, equation (A1) supplemented by the constraint (A3) leads to spin operators $S_n$ described by the Young tableau of fig 5.

\section*{APPENDIX B: SU(N) THIRRING MODEL}

In this appendix we discuss the excitations, scattering matrices and form factors of the SU(N) Thirring (chiral Gross–Neveu) model. More details may be found in appendix A of Smirnov [26] and the literature [29,36,46,47,48].

\subsection*{1. Excitations}

The excitations of the $SU(N)$ invariant Thirring (chiral Gross–Neveu) model are $N - 1$ multiplets of fundamental particles, corresponding to the $N - 1$ fundamental representations of $SU(N)$. Their masses are given by
\[
M_a = m \frac{\sin \pi a/N}{\sin \pi/\theta}; \quad a = 1, 2, \ldots, N - 1, \quad (B1)
\]
and following Smirnov, we shall refer to the label $a$ as the "rank" of the particle.

\subsection*{2. S-Matrices}

The S-matrix describing the scattering of two (rank-1) fundamental particles in the SU(N) invariant Thirring (chiral Gross–Neveu) model is given by [29,37]
\[
S^{\Box_{12}}_{1,2}(\theta) \equiv \epsilon_{12} \langle \theta_2 \theta_1 | S^{\Box}(\theta) | \theta_1 \theta_2 \rangle \epsilon_{12} \quad (B2)
\]
where $\theta = \theta_1 - \theta_2$, $\epsilon \in \{1, \ldots, N\}$, and the S-matrix operator acts on the two body Hilbert space $\Box \otimes \Box$:
\[
S^{\Box_{12}}(\theta) = S_0(\theta) \left( \frac{\theta I - 2\pi i}{\theta - 2\pi i} \right), \quad (B3)
\]
$I$ and $P_{12}$ are the identity and permutation operator respectively with matrix elements
\[
\epsilon_{\epsilon'_{12}} \langle \theta_2 \theta_1 | I | \theta_1 \theta_2 \rangle \epsilon_{\epsilon_{12}} = \epsilon_{\epsilon'_{12}} \delta_{\epsilon'_{12}} \quad (B4)
\]
\[
\epsilon_{\epsilon'_{12}} \langle \theta_2 \theta_1 | P_{12} | \theta_1 \theta_2 \rangle \epsilon_{\epsilon_{12}} = \epsilon_{\epsilon_{12}} \delta_{\epsilon'_{12}} \epsilon'_{12} \quad (B5)
\]
and
\[
S_0(\theta) = \frac{\Gamma(1 - \frac{\theta}{2\pi} + \frac{\theta}{2\pi}) \Gamma(-\frac{\theta}{2\pi})}{\Gamma(1 - \frac{\theta}{2\pi} - \frac{\theta}{2\pi}) \Gamma(-\frac{\theta}{2\pi})}. \quad (B6)
\]
See equation (11a) of [37] or appendix A (p. 182) of [26], e.g. for $N = 2$ this reduces to equation (6) of [26]. Using the decompositions $I = P^+ + P^-$, and $P_{12} = P^+ - P^-$, one may also write (B6) in the form:
\[
S^{\Box_{12}}(\theta) = \sum_r S^{\Box_{12}}_r(\theta) P^{(r)} \quad (B7)
\]
\[
S_0(\theta) \left( P^+ + \frac{\theta + 2\pi i}{\theta - 2\pi i} P^- \right) \quad (B7)
\]
where $\mathcal{P}^{(+)}$ and $\mathcal{P}^{(-)}$ act on the symmetric and anti-symmetric representations occurring in the tensor product $\square \otimes \square$; e.g. $3 \otimes 3 = 6 + 3$ in SU(3). Bound states correspond to poles of the S-matrix, with masses

$$m_b = \sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cosh(\theta_{12})}.$$  \hspace{1cm} (B8)

Since $\Gamma(z)$ is free of zeros, and exhibits simple poles at $z = 0, -1, -2, \ldots$, it follows that (B7) has a single simple pole at $\theta = 2\pi i/N$ occurring within the physical strip, $0 < \theta < \pi i$. This yields the bound state mass of the second fundamental particle, $M_2 = m \sin(2\pi/N) / \sin(\pi/N)$, as given by equation (B1).

The S-matrix describing the scattering of a (rank-1) fundamental particle off its conjugate (rank-N - 1) may be obtained from (B7) by the crossing transformation:

$$S^{\square}(\theta) = C_{\square} S^{\square}(i\pi - \theta) C_{\square}$$  \hspace{1cm} (B9)

where $C_{\square}$ is the conjugation operator on $\square$. Utilizing $\mathcal{P}^{(0)} = C_{\square} \mathcal{P}_{12} C_{\square}/N$, and $I = \mathcal{P}^{(adj)} + \mathcal{P}^{(0)}$ one obtains

$$S^{\square}(\theta) = \sum_r S^{\square}_r(\theta) \mathcal{P}^{(r)}$$

$$= -S_1(\theta) \left( \mathcal{P}^{(adj)} + \frac{\theta + \pi i}{\theta - \pi i} \mathcal{P}^{(0)} \right)$$  \hspace{1cm} (B10)

where

$$S_1(\theta) = \frac{\Gamma(\frac{3}{2} + \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} - \frac{\theta}{2\pi i})}{\Gamma(\frac{3}{2} - \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\theta}{2\pi i})},$$  \hspace{1cm} (B11)

and $\mathcal{P}^{(adj)}$ and $\mathcal{P}^{(0)}$ act on the adjoint and singlet representations occurring in the tensor product $\square \otimes \square$; e.g. $3 \otimes 3 = 8 + 1$ in SU(3). In particular, for $N = 2$, (B7) and (B10) coincide (up to sign) as expected from the identification of $\square$ and $\square$ in SU(2). Moreover, equation (B1) is also in agreement with equation (1.6a) of reference 49.

The S-matrix (B10) has a pole at $\theta = \pi i - 2\pi i/N$ occurring within the physical strip, $0 < \theta < \pi i$. This is a cross channel pole.

For the purpose of calculating form factors in section 3.3 it proves useful to have the S-matrices in an integral form. Taking the logarithm of (B10) one obtains

$$\ln \Gamma(z) = \int_0^\infty dt \left[ (z - 1) e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}} \right]$$  \hspace{1cm} (B12)

one obtains

$$S_a(\theta) = \exp \left\{ \int_0^\infty dx f_a(x) \sinh \left( \frac{x\theta}{\pi i} \right) \right\}$$  \hspace{1cm} (B13)

where

$$f_0(x) = \frac{2 \exp(x/N) \text{sinh}[x(1 - 1/N)]}{x \text{sinh} x},$$  \hspace{1cm} (B14)

$$f_1(x) = \frac{2 \exp(x/N) \text{sinh}[x(1/N)]}{x \text{sinh} x}.$$  \hspace{1cm} (B15)

In particular, the su(2) Thirring S-matrix coincides with the sine-Gordon S-matrix with $\beta^2 = 8\pi^2 / 29$.

$$S_a(\theta) \overset{N=2}{=} \exp \left\{ i \int_0^\infty dx \frac{e^{-(\pi i / 2) \sin \theta}}{\cosh \pi k / 2} \sin \theta \right\}.$$  \hspace{1cm} (B16)

## 3. Form Factors

In the previous paragraphs, we have discussed the elementary excitations of the SU(N) Thirring model. They are massive particles labeled by their rapdities, $\theta_i$, and carrying quantum numbers or isotopic indices, $\epsilon_i$. In order to compute correlation functions and dynamical susceptibilities, we will need the matrix elements of various physical operators, $O$, between the vacuum and the (lowest) multiparticle excited states. Such matrix elements are termed form factors, and their computation is an important enterprise; see for example 2D. As is discussed in Ch. 1 of Smirnov’s book, the two-particle form factors

$$F_{O}(\theta_1, \theta_2)_{\epsilon_1, \epsilon_2} = \langle \langle 0 | O (0, 0) | \theta_2 \theta_1 \rangle_{\epsilon_1, \epsilon_2} \rangle_{\epsilon_1, \epsilon_2}$$  \hspace{1cm} (B17)

satisfy a matrix (Riemann–Hilbert) problem, also known as Watson’s equations:

$$F(\theta_1, \theta_2 + 2\pi i)_{\epsilon_1, \epsilon_2} = F(\theta_1, \theta_2)_{\epsilon_1, \epsilon_2} S(\theta_1, \theta_2)$$  \hspace{1cm} (B18)

where $S(\theta)$ are the S-matrix eigenvalues. In particular, the Thirring current operator $J_{\mu}$ (with $N^2 - 1$ components) couples to the adjoint representation occurring in the tensor product $\square \otimes \square$; the relevant eigenvalue is

$$S(\theta) = S^{\square}(\theta) = -S_1(\theta).$$  \hspace{1cm} (B20)

Another constraint on the form factors comes from Lorentz invariance. Under a Lorentz boost, corresponding to a simultaneous shift of all rapdities by $\Lambda$, the two-particle form factor of an operator $O$ of spin $s$ satisfies:

$$F_{O}(\theta_1 + \Lambda, \theta_2 + \Lambda) = e^{s\Lambda} F_{O}(\theta_1, \theta_2).$$  \hspace{1cm} (B21)

In particular, the left (right) component of the Thirring current has spin $s = +1$ ($s = -1$) and one obtains:

$$F_{J_+}(\theta_1, \theta_2) \propto e^{(\theta_1 + \theta_2)/2} f^{\square}(\theta_1, \theta_2),$$  \hspace{1cm} (B22)

$$F_{J_-}(\theta_1, \theta_2) \propto e^{-(\theta_1 + \theta_2)/2} f^{\square}(\theta_1, \theta_2).$$  \hspace{1cm} (B23)

Note that $f^{\square}(\theta_1, \theta_2)$ is a function of $\theta_1 - \theta_2$, and is thus Lorentz invariant. The form factors corresponding to the
temporal and spatial components of the current may be written:

\[ F_{j_0}^{(1)}(\theta_1, \theta_2) \propto m \sinh \left( \frac{\theta_1 + \theta_2}{2} \right) f_{\text{adj}}(\theta_{12}), \quad (B24) \]

\[ F_{j_i}^{(1)}(\theta_1, \theta_2) \propto m \cosh \left( \frac{\theta_1 + \theta_2}{2} \right) f_{\text{adj}}(\theta_{12}). \quad (B25) \]

Substituting (B24) and either of (B24) and (B25) into (B19), one obtains a constraint on \( f_{\text{adj}}^{(1)}(\theta) \):

\[ f_{\text{adj}}^{(1)}(\theta - 2\pi i) = f_{\text{adj}}^{(1)}(\theta) S_1(\theta). \quad (B26) \]

Following the general arguments of Karowski and Weisz,\(^{20}\) (equations 2.18 and 2.19) equation (B26) may be solved by

\[ f_{\text{adj}}^{(1)}(\theta) = \exp \left\{ \int_0^\infty dx f_1(x) \frac{\sin^2(x(\theta/2\pi))}{\sinh x} \right\} \quad (B27) \]

where \( \theta = i\pi - \theta^{\text{adj}} \). Expanding the denominator factors in powers of \( e^{-2x} \), and employing the identity

\[ \exp \int_0^\infty dx x \frac{e^{-\beta x} \sinh \gamma x}{x} = \frac{\beta + \gamma}{\beta - \gamma}, \quad (B28) \]

one obtains the equivalent representation:

\[ f_{\text{adj}}^{(1)}(\theta) = \prod_{l,m=0}^{\infty} \left[ \frac{1 + l + m}{1 - \frac{\theta}{2\pi} + l + m} \right]^2 \times \left[ \frac{\frac{1}{2} - \frac{\theta}{2\pi} + l + m + \frac{\theta}{2\pi i}}{\frac{1}{2} + l + m + \frac{\theta}{2\pi i}} \right] \left[ \frac{\frac{1}{2} - \frac{\theta}{2\pi} + l + m - \frac{\theta}{2\pi i}}{\frac{1}{2} + l + m - \frac{\theta}{2\pi i}} \right] \quad (B29) \]

Application of Euler’s Formula yields:

\[ f_{\text{adj}}^{(1)}(\theta) = \prod_{l=0}^{\infty} \left[ \frac{\Gamma(1 - \frac{\theta}{2\pi} + l)}{\Gamma(1 + l)} \right]^2 \times \left[ \frac{\frac{1}{2} + l + m + \frac{\theta}{2\pi i}}{\frac{1}{2} - \frac{\theta}{2\pi} + l + m + \frac{\theta}{2\pi i}} \right] \left[ \frac{\frac{1}{2} + l + m - \frac{\theta}{2\pi i}}{\frac{1}{2} - \frac{\theta}{2\pi} + l + m - \frac{\theta}{2\pi i}} \right] \quad (B30) \]

As may be seen most clearly from (B29) and (B30), this form factor is free of poles in the physical strip \( 0 < \theta < i\pi \), and Watson’s minimal equations (in Karowski–Weisz form) are explicitly satisfied: It is indeed, a minimal form factor. Expressions (B27), (B29), and (B30) conform to the Karowski–Weisz normalization \( F(\pi) = 1 \), and have the asymptotic behaviour

\[ \lim_{\theta \to \pm \infty} f_{\text{adj}}^{(1)}(\theta) \sim \exp(\pm \theta/2N). \quad (B31) \]

In the limit \( N = 2 \), one may write (B24) in the form

\[ f_{\text{adj}}^{(1)}(\theta) \to -i \sinh(\theta/2) \times \exp \left\{ \int_0^\infty dx \frac{\sin^2(x(\theta/2\pi))}{x \sinh x} \right\} \left[ \tanh(x/2) - 1 \right] \quad (B32) \]

and expressions (B24) and (B29) coincide with the known results for the SU(2) invariant Thirring (or sine-Gordon) model; see equation (33) of Allen et al. or let \( \xi \to \infty \) in the formula for \( f_{\mu}(\beta_1, \beta_2) \) given on page 46 of Smirnov.\(^{20}\) and note the different definition of the physical strip. In the limit \( N \to \infty \), the SU(N) Thirring model maps onto a theory of free massive fermions, as reflected in the explicit S-matrices. In this limit \( f_{\text{adj}}^{(1)}(\theta) \to 1 \), and (B24) and (B30) coincide with the free fermion form factors given in equation (108) of Smirnov.\(^{20}\)

1. C. Lhuillier and G. Misguich, Frustrated Quantum Magnets, in *High Magnetic Fields*, edited by C. Berthier, L. P. Lévy and G. Martinez, Lecture Notes in Physics Vol 595, Springer, 2001, [cond-mat/0109146]
2. S. Sachdev, Order and quantum phase transitions in the cuprate superconductors, [cond-mat/0211005]
3. M. F. Collins and O. A. Petrenko, Triangular Antiferromagnets, [cond-mat/9706553]
4. P. Azaria, C. Hooley, P. Lecheminant, C. Lhuillier, and A. M. Tsvelik, Phys. Rev. Lett. 81, 1694 (1998), [cond-mat/9807228]
5. S. R. White and I. Affleck, Phys. Rev. B 54, 9862 (1996), [cond-mat/9602126]
6. D. Allen and D. Sénéchal, Phys. Rev. B 55, 299 (1997), [cond-mat/9606007]
7. A. A. Nersesyan, A. O. Gogolin, and F. H. L. Essler, Phys. Rev. Lett. 81, 910 (1998), [cond-mat/9804005]
8. D. Allen, F. H. L. Essler, and A. A. Nersesyan, Phys. Rev. B 61, 8871 (2000), [cond-mat/9907303]
9. S. E. Palmer and J. T. Chalker, Phys. Rev. B 64, 94412 (2001), [cond-mat/0102447]
10. J.-B. Fouet, M. Mambri, P. Sindzingre, and C. Lhuillier, Phys. Rev. B 67, 054411 (2003), [cond-mat/0108070]
11. O. Tchernyshyov, O. A. Starykh, R. Moessner, and A. G. Abanov, Bond order from disorder in the planar pyrochlore magnet, [cond-mat/0301030]
12. P. W. Anderson, Science 235, 1196 (1987).
13. E. Fradkin, *Field Theories of Condensed Matter Systems*, Frontiers in Physics Vol. 82 (Addison–Wesley Publishing Company, 1991).
14. R. Moessner and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 (2001), [cond-mat/0007378]
15. L. Capriotti, Int. J. Mod. Phys. B14, 3386 (2000), [cond-mat/0112207]
A. A. Nersesyan and A. M. Tsvelik, Phys. Rev. B 67, 024422 (2003), cond-mat/0206463.

L. D. Faddeev and L. H. Takhtajan, Phys. Lett. 85A, 375 (1981).

F. A. Smirnov and A. M. Tsvelik, cond-mat/0304634.

P. Lecheminant, Private Communication.

I. Affleck, Phys. Rev. Lett. 55, 1355 (1985).

I. Affleck, Nucl. Phys. B265, 409 (1996).

A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization in Strongly Correlated Systems (Cambridge University Press, 1999).

R. Assaraf, P. Azaria, M. Caffarel, and P. Lecheminant, Phys. Rev. B 60, 2299 (1999), cond-mat/9903057.

I. Affleck, J. Phys. (Cond. Matt.) 2, 405 (1990).

I. Affleck, Phys. Rev. Lett. 54, 966 (1985).

J. B. Marston and I. Affleck, Phys. Rev. B 39, 11538 (1989).

N. Read and S. Sachdev, Nucl. Phys. B316, 609 (1989).

F. A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory (World Scientific Publishing, 1990).

A. Erdeyli, Higher Transcendental Functions: The Bateman Manuscript Project Vol. 2 (McGraw-Hill Book Company, Inc, 1953).

A. M. Tsvelik, Quantum Field Theory in Condensed Matter Physics (Cambridge University Press, 1995).

S. Lukyanov and A. Zamolodchikov, Nucl. Phys. B607, 437 (2001), hep-th/0102079.

F. H. L. Essler and A. M. Tsvelik, Phys. Rev. B65, 115117 (2002), cond-mat/0108382.

F. H. L. Essler and A. M. Tsvelik, Phys. Rev. Lett. 90, 126401 (2003), cond-mat/0205294.

M. B. Halpern, Phys. Rev. D 12, 1684 (1976).

R. Köberle, V. Kurak, and J. A. Swieca, Phys. Rev. D 20, 897 (1979).

M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (Dover, 1965).

A. B. Zamolodchikov and V. A. Fateev, Sov. Nucl. Phys. 43, 657 (1986).

A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP 62, 215 (1985).

A. M. Tsvelik, Sov. J. Nucl. Phys. 47, 172 (1988).

M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley Publishing Company, 1995).

H. F. Jones, Groups, Representations and Physics (Institute of Physics Publishing, Bristol and Philadelphia, 1998).

S. Okubo, J. Math. Phys. 18, 2382 (1977).

A. Jerez, N. Andrei, and G. Zarand, Phys. Rev. B 8, 3814 (1998), cond-mat/9803137.

B. Berg and P. Weisz, Nucl. Phys. B146, 205 (1978).

V. Kurak and J. A. Swieca, Physics Letters B28, 289 (1979).

N. Andrei and J. H. Lowenstein, Phys. Rev. Lett. 43, 1698 (1979).

E. Ogievetsky, N. Reshetikhin, and P. Wiegmann, Nucl. Phys. B280, 45 (1987).

K. Karowski and P. Weisz, Nucl. Phys. B139, 455 (1978).

More accurately $f(\theta) = \delta_{ij}$ may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

Note the identity $t_{a}^{\alpha}_{b}t_{a}^{\beta}_{b} = \frac{1}{2}(\delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl}/N)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

Note the identity $t_{a}^{\alpha}_{b}t_{a}^{\beta}_{b} = \frac{1}{2}(\delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl}/N)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$. 

More accurately $f(\theta)$ as given by equation 199 may be seen to satisfy $f(\theta - 2\pi i) = f(\theta)S_1(\theta - 2\pi i)$, as follows from equation 2.13 of ref. 50; by definition $S_1(\theta - 2\pi i) := S_1(\theta)$.