FOUR-DIMENSIONAL ZERO-HOPF BIFURCATION FOR A LORENZ-HAKEN SYSTEM

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ABSTRACT. In this work we study the periodic orbits which bifurcate from a zero-Hopf bifurcations that a Lorenz-Haken system in $\mathbb{R}^4$ can exhibit. The main tool used is the averaging theory.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The Lorenz–Haken equation named after the fluid dynamist Lorenz and laser theorist Haken [1] describe the dynamics of a homogeneously broadened gain medium in an unidirectional ring cavity. In the notation given in the Reference [4], the Lorenz-Haken equations is given by

\begin{align*}
\dot{x} &= -\sigma(x - y) + iq|x|^2, \\
\dot{y} &= -(1 - i\delta)y + (r - z)x, \\
\dot{z} &= -bz + \text{Re}(xy),
\end{align*}

where $x$, $y$ and $z$ are complex variables, and $\sigma, b, q, r, \delta$ are the real parameters. In 2019, Hayder Natiq [2] derived a new 4D chaotic laser system with three equilibrium points from (1.1), since both $x$ and $z$ can be chosen to be real and $y$ a complex variable.

In this paper, we study a four-dimensional system of differential equations which is a generalization of the system introduced in [2]. We want to study the periodic orbits of the Lorenz-Haken systems of $\mathbb{R}^4$ with five parameters, in which bifurcate in the zero-Hopf bifurcations of the singular points given by

\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= -cy - dz + (e - w)x, \\
\dot{z} &= dy - cz, \\
\dot{w} &= -bw + xy,
\end{align*}

where $x, y, z, w$ are state variables and $a, b, c, d$ and $e$ are real parameters.

In the first instance we are going to compute the equilibrium points of Lorenz–Haken system (1.2).

Proposition 1.1. Let $\Delta = \frac{ec - c^2 - d^2}{c}$ and $c \neq 0$. The following statements are true:

1. If $\Delta \leq 0$ and $b \neq 0$, system (1.2) has an unique equilibrium point $p_0 = (0,0,0,0)$.

2010 Mathematics Subject Classification. 34C23, 34C25, 37G10.

Key words and phrases. Zero-Hopf Bifurcation, Periodic solutions, Averaging theory.
(2) If $\Delta > 0$ and $b \neq 0$, we have two equilibrium points
\[ p_{\pm} = \left( \pm \sqrt{b\Delta}, \pm \sqrt{b\Delta}, \pm \frac{\sqrt{b\Delta}}{c}, \Delta \right). \]

(3) If $b = 0$ and $\Delta \neq 0$ we has a straight line of equilibria
\[ p = \left( 0, 0, 0, \Delta \right). \]

Proposition 1.2 follows easily by direct computations.

We observe that the two equilibria $p_{\pm}$ tends to the equilibrium point $p$ when $b \to 0$. In short, the equilibrium point of system $(1.2)$ can be $p_{+}$, $p_{-}$, $p$ and the origin. Additionally, the system $(1.2)$ has invariance under the coordinate transformation $(x, y, z, w) \to (-x, -y, -z, w)$. Consequently, the system $(1.2)$ has rotational symmetry around the $w$-axis.

Due to that, in what follows we consider the only equilibrium $p_{+}$ in order to verify its possibility of being a zero–Hopf equilibrium for some values of the parameter, and clearly the same will occur for the other equilibrium $p_{-}$.

In the next result we characterize when the equilibrium $p_{+}$ and $p_{-}$ and the origin are zero–Hopf equilibrium of the system $(1.2)$.

**Proposition 1.2.** For the hyperchaotic system $(1.2)$, the following statements hold:

(i) $p_{0}$ is a zero-Hopf equilibrium if only if $a = -2c, b = 0, d = -\frac{\sqrt{c^2 + \omega^2}}{3}$ and $e = \frac{2c^2 + \omega^2}{3}$.

(ii) $p$ is a zero-Hopf equilibrium if only if $a = -2c, b = 0$ and $3d^2 - c^2 > 0$.

(iii) $p_{+}$ and $p_{-}$ are zero-Hopf equilibrium if only if $a = -2c, b = 0, d = -\frac{\sqrt{c^2 + \omega^2}}{\sqrt{3}}$.

In the rest of this section, we will study the zero-Hopf bifurcation and periodic solutions of the hyperchaotic system $(1.2)$ at all the equilibrium points.

**Theorem 1.3.** For the hyperchaotic system $(1.2)$. The following statements hold.

(i) Let
\[ (a, b, d, e) = \left( -2c + \varepsilon a, \varepsilon b, -\frac{\sqrt{c^2 + \omega^2}}{3} + \varepsilon d, \frac{4c^2 + \omega^2}{3c} + \varepsilon e \right) \]
where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameters. If $a_{1} \neq 0, b_{1} \neq 0, c \neq 0$, $\eta = 3c_{1} + 2\sqrt{3d_{1}}c_{1} + \omega_{1} \neq 0$ and $\eta_{1} = 3a_{1}c_{1} - 2c_{1} \neq 0$, then for $\varepsilon > 0$ sufficiently small, the hyperchaotic system $(1.2)$ has a zero-Hopf bifurcation at the equilibrium point located at $p_{0}$, and at most four periodic orbits can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, the periodic solutions are stable if $a_{1} > 0, b_{1} > 0, 16\eta + 3b_{1}c_{1} < 0$ and $4\eta_{1} + 3b_{1}c_{1} < 0$.

(ii) Let
\[ (a, b) = (-2c + \varepsilon a, \varepsilon b), \]
where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameter. If $a_{1} \neq 0, b_{1} \neq 0, c \neq 0, d \neq 0, (c^{2} - d^{2})(c^{2} + d^{2} - ce) \neq 0, 2(c^{2} - d^{2}) - ce \neq 0, 3d^{2} - c^{2} > 0, c^{4} - 8c^{2}d^{2} + 7d^{4} + 2cd^{2}e < 0$ and $(c^{4} - 4c^{2}d^{2} + 3d^{4})(c^{2} + d^{2} - ce) < 0$, then for $\varepsilon > 0$ sufficiently small, the hyperchaotic system $(1.2)$ has a zero-Hopf bifurcation at the equilibrium point located at $p$, and at most five periodic orbits.
can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, the periodic solution are stable if $a_1 > 0, b_1 > 0, (c^4 - 8c^2d^2 + 7d^4 + 2cd^2e) < 0, 2c^2 - 2d^2 - ce < 0$ and $c^4 - d^4 - c^3e + cd^2e > 0$.

(iii) Let

$$(a, b, d) = (-2c + \varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2 + \omega^2}}{\sqrt{3}} + \varepsilon d_1),$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameter. If $c \neq 0, a_1 \neq 0,$ and $\kappa = b_1(4c^2 - 3ce + 3\omega^2) < 0,$ then for $\varepsilon > 0$ sufficiently small, the hyperchaotic system (1.2) has a zero-Hopf bifurcation at the equilibrium point located at $p_\pm$, and at most two periodic orbits can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, the periodic solutions are unstable.

2. The Averaging Theory of First Order

The averaging theory is a classical and mature tool for studying the dynamic behavior of nonlinear dynamical systems, especially for the study of periodic solutions. This will be the main tool for proving Theorem 1.3.

Consider differential system:

$$\dot{x} = \varepsilon F(t, x) + \varepsilon^2 G(t, x, \varepsilon),$$

where $x \in D$ is an open subset of $\mathbb{R}^n$, $t \geq 0$. We assume that $F(t, x)$ and $G(t, x, \varepsilon)$ are $T$-periodic in $t$. We define averaged function

$$f(x) = \frac{1}{T} \int_0^T F(t, x) dt.$$

**Theorem 2.1.** Make the following assumptions:

(i) $F$, its Jacobian $\frac{\partial F}{\partial x}$ and its Hessian $\frac{\partial^2 F}{\partial x^2}$, $G$, its Jacobian $\frac{\partial G}{\partial x}$ are defined, continuous and bounded by a constant independent of $\varepsilon$ in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.

(ii) $T$ is a constant independent of $\varepsilon$.

Then the following conclusions can be obtained:

(a) If $p$ is the zero of the averaged function $f(x)$, and

$$\det \left( \frac{\partial f}{\partial x} \right)_{x=p} \neq 0,$$

then there exists a $T$-periodic solution $x(t, \varepsilon)$ of system (2.1) such that $x(0, \varepsilon) \to p$ as $\varepsilon \to 0$.

(b) If the eigenvalue of the Jacobian matrix $\left( \frac{\partial f}{\partial x} \right)$ has a negative real part, the periodic solution $x(t, \varepsilon)$ is asymptotically stable.

For more information about the averaging theory see [3] and [5].

3. Proof of results

In this section we will provide the proofs of Proposition 1.2 and Theorem 1.3.
Proof of Proposition 1.2. The characteristic polynomial $P(\lambda)$ of the linear part of the differential systems (1.2) at the equilibrium point $p_0 = (0, 0, 0, 0)$ is

$$P(\lambda) = \lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D,$$

where

\begin{align*}
A &= a + b + 2c, \\
B &= 2bc + c^2 + d^2 + a(b + 2c - e), \\
C &= b(c^2 + d^2) + a(2bc + c^2 + d^2 - (b - c)e), \\
D &= ab(c^2 + d^2 - ce).
\end{align*}

The equilibrium point $p_0$ is a zero hopf equilibrium if and only if $P(\lambda) = \lambda^2(\lambda^2 + \omega^2)$ with $\omega > 0$, the parameter must be satisfied,

\begin{align*}
a &= -2c, \quad b = 0, \quad d = -\frac{\sqrt{c^2 + \omega^2}}{3} \quad \text{and} \quad e = \frac{4c^2 + \omega^2}{3c}.
\end{align*}

(ii) The characteristic polynomial $P(\lambda)$ of the linear part of the differential systems (1.2) at the equilibrium point $p$ is

$$P(\lambda) = \lambda^4 + (a + 2c)\lambda^3 + \left(c^2 + d^2 + a(e - \frac{d^2}{c})\right)\lambda^2.$$

The equilibrium point $p$ is a zero hopf equilibrium if and only if $P(\lambda) = \lambda^2(\lambda^2 + \omega^2)$ with $\omega > 0$, the parameter must be satisfied,

\begin{align*}
a &= -2c, \quad b = 0,
\end{align*}

in this case, Eq. (5.2) has roots $\lambda_{1,2} = 0$, $\lambda_{3,4} = \pm \sqrt{3d^2 - c^2}i$.

(iii) The Jacobian matrix of systems (1.2) evaluated at $p_+$ is

\[
\begin{pmatrix}
-a & 0 & 0 \\
\frac{d^2}{c} & a & -d - \sqrt{b(c e - c^2 - d^2)} \\
0 & -c & -d \\
\frac{\sqrt{b(c e - c^2 - d^2)}}{\sqrt{c}} & \frac{\sqrt{b(c e - c^2 - d^2)}}{\sqrt{c}} & 0
\end{pmatrix}
\]

ant its characteristic polynomial is

$$P(\lambda) = \lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D,$$

where

\begin{align*}
A &= a + b + 2c, \\
B &= c^2 + d^2 + a(b + c - \frac{d^2}{c}) + b(c - \frac{d^2}{c} + e), \\
C &= b\left(c e + a(-c - \frac{3d^2}{c} + 2e)\right), \\
D &= -2ab(c^2 + d^2 - ce).
\end{align*}
The equilibrium point $p_+$ is a zero hopf equilibrium if and only if $P(\lambda) = \lambda^2(\lambda^2 + \omega^2)$ with $\omega > 0$, the parameter must be satisfied,

$$a = -2c, \quad b = 0, \quad d = -\frac{\sqrt{c^2 + \omega^2}}{\sqrt{3}}.$$ 

This completes the Proof of Proposition 1.2. □

Proof. of statement (i) of Theorem 1.3 Let

$$(a, b, d, e) = \left( -2c + \varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2 + \omega^2}}{3} + \varepsilon d_1, \frac{4c^2 + \omega^2}{3c} + \varepsilon e_1 \right)$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameters. Then, the differential systems (1.2) becomes

$$\begin{align*}
\dot{x} &= 2c(x - y) - a_1(x - y)\varepsilon, \\
\dot{y} &= \left(e_1x - d_1z\right)\varepsilon - \frac{-4c^2x + 3cwz + 3c^2y - x\omega^2 - \sqrt{3cz\sqrt{c^2 + \omega^2}}}{3c}, \\
\dot{z} &= d_1y\varepsilon + \frac{1}{3}\left(-3cz - \sqrt{3y\sqrt{c^2 + \omega^2}}\right), \\
\dot{w} &= xy - b_1w\varepsilon.
\end{align*}$$

(3.4)

Performing the rescaling of variables

$$(x, y, z, w) \rightarrow (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$$

system (3.4) can be written as

$$\begin{align*}
\dot{x} &= 2c(x - y) - a_1(x - y)\varepsilon, \\
\dot{y} &= \left(e_1x - wx - d_1z\right)\varepsilon - \frac{-4c^2x + 3c^2y - x\omega^2 - \sqrt{3cz\sqrt{c^2 + \omega^2}}}{3c}, \\
\dot{z} &= d_1y\varepsilon + \frac{1}{3}\left(-3cz - \sqrt{3y\sqrt{c^2 + \omega^2}}\right), \\
\dot{w} &= (-b_1w + xy)e_1.
\end{align*}$$

(3.5)

Now we shall write the linear part at the origin of the system (3.5) when $\varepsilon = 0$ into its real Jordan normal form, i.e. as

$$\begin{pmatrix}
0 & -\omega & 0 & 0 \\
\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$
For doing that we consider the linear change $(x, y, z, w) \mapsto (X, Y, Z, W)$

$$
x = \frac{2c(\sqrt{3}cY \omega + \sqrt{3}X \omega^2 - 3cZ \sqrt{c^2 + \omega^2})}{3\omega^2 \sqrt{c^2 + \omega^2}},
$$

$$
y = \frac{\sqrt{3}cX \omega^2 + \sqrt{3}Y \omega^3 + 2c^2 (\sqrt{3}Y \omega - 3Z \sqrt{c^2 + \omega^2})}{3\omega^2 \sqrt{c^2 + \omega^2}},
$$

$$
z = \frac{1}{3} \left( X + \frac{c(\sqrt{3}Y \omega^2 - 3Z \sqrt{c^2 + \omega^2})}{\omega^2} \right),
$$

$$
w = W.
$$

By using the new variables $(X, Y, Z, W)$, the system \((3.5)\) can be written as follows:

\[(3.6)\]

\[
\dot{X} = -Y \omega + \frac{1}{3} \varepsilon \left( a_1 (X - Y \omega) + \frac{d_3}{\omega^2 \sqrt{c^2 + \omega^2}} (\sqrt{3}cY \omega + \sqrt{3}X \omega^2 - 3cZ \sqrt{c^2 + \omega^2}) \right),
\]

\[
\dot{Y} = X \omega + \frac{\varepsilon}{3\omega^3 \sqrt{c^2 + \omega^2}} \left( 6\sqrt{3}c^4 (-e_1 + W)Z - 6c^2 (e_1 - W) \omega (\sqrt{3}Z \omega - Y \sqrt{c^2 + \omega^2}) \right.
\]

\[
\left. - \omega^3 (\sqrt{3}d_1 X \omega + 2a_1 Y \sqrt{c^2 + \omega^2}) + 4c^3 d_1 (\sqrt{3}Y \omega - 3Z \sqrt{c^2 + \omega^2}) \right.
\]

\[
+ \omega^2 \left( 2(a_1 + 3e_1 - 3W) X \sqrt{c^2 + \omega^2} + 3d_1 (\sqrt{3}Y \omega - 2Z \sqrt{c^2 + \omega^2}) \right) \bigg),
\]

\[
\dot{Z} = \frac{\varepsilon}{18c^2 \omega^2 \sqrt{c^2 + \omega^2}} \left( -24\sqrt{3}c^3 d_1 Z - 4\sqrt{3}a_1 c^2 Y \omega^3 - \sqrt{3}a_1 Y \omega^5 + \omega^3 (\sqrt{3}a_1 X \omega + 6d_1 Y \sqrt{c^2 + \omega^2}) \right.
\]

\[
+ 4c^3 \omega \left( \sqrt{3}((a_1 + 3e_1 - 3W) X - 6d_1 Z) \omega + 6d_1 Y \sqrt{c^2 + \omega^2} \right)
\]

\[
+ 12c^4 (e_1 - W) (\sqrt{3}Y \omega - 3Z \sqrt{c^2 + \omega^2}) \bigg),
\]

\[
\dot{W} = \varepsilon \left( -b_1 W + \frac{2c}{\theta_0 c^2 (e_1 + \sqrt{3}cX \omega^2) + 3cZ \sqrt{c^2 + \omega^2}) (\sqrt{3}(cX \omega^2 + Y \omega^3) + 2c^2 (\sqrt{3}Y \omega - 3Z \sqrt{c^2 + \omega^2}) \bigg).\right.
\]
Then we use the cylindrical coordinates $X = r \cos \theta$, $Y = r \sin \theta$, and obtain (3.7)

\[
\ddot{r} = \frac{\varepsilon}{3cw^3c^2 + \omega^2} \left( crω (\sqrt{3}c_e - a_1 \sqrt{c^2 + \omega^2}) \cos^2 \theta + c \sin \theta(6cZ(-2c^2 + ω^2) + ω^2)d_1 \sqrt{c^2 + \omega^2} - \sqrt{3}c(e_1 - W)(c^2 + ω^2) ) + rω(\sqrt{3}(4c^3 + 3c_1ω^2)d_1 + (6c^3(e_1 - W) - 2a_1ω^2) \sqrt{c^2 + ω^2} \sin \theta + ω \cos \theta(-6c^3_1Z \sqrt{c^2 + \omega^2} + rω(2\sqrt{3}c^3d_1 + 2c^3(a_1 + 3e_1 - 3W) \sqrt{c^2 + \omega^2} + a_1ω^2 \sqrt{c^2 + \omega^2} \sin \theta ) + ω \sqrt{3}c_1r_1ω^2(-ω + c \sin 2θ)) \right),
\]

\[
\dot{θ} = \frac{1}{3cw^3c^2 + \omega^2} \left( crω^2 \sqrt{c^2 + \omega^2}(2c(a_1 + 3e_1 - 3W)e + 3ω^2) \cos^2 \theta + cε \cos θ(6cZ(-2c^2 + ω^2)d_1 \sqrt{c^2 + \omega^2} - \sqrt{3}c(e_1 - W)(c^2 + ω^2) ) + rω(4\sqrt{3}c^3d_1 + 6c^2(e_1 - W) \sqrt{c^2 + ω^2} - a_1ω^2 \sqrt{c^2 + \omega^2} \sin \theta ) + ω(6c^3d_1Zε \sqrt{c^2 + ω^2} \sin θ + rω(-2\sqrt{3}c^3d_1ε + (3c - a_1ε)ω^2 \sqrt{c^2 + ω^2} \sin θ ) + ω \sqrt{3}c_1r_1ω^2(-ω + c \sin 2θ)) \right),
\]

\[
\dot{Z} = \frac{ε}{18c^2ω^2 \sqrt{c^2 + ω^2}} \left( -12c^3Z(2\sqrt{3}(c^2 + ω^2)d_1 + 3c(e_1 - W) \sqrt{c^2 + ω^2} ) + \sqrt{3}crω^2(4c^2(a_1 + 3e_1 - 3W) + a_1ω^2) \cos \theta + rω((24c^3 + 6c_1ω^2)d_1 \sqrt{c^2 + ω^2} - \sqrt{3}(12c^4(-e_1 + W) + 4a_1c^2ω^2 + a_1ω^4) \sin \theta ) \right),
\]

\[
\dot{W} = \frac{ε}{3ω^4(c^2 + ω^2)} \left( (12c^4Z^2 + 2c^2r_1^2ω^2 - 3b_1Wω^4)(c^2 + ω^2) + crω(-2c^3r_1ω \cos 2θ - 2\sqrt{3}cZ \sqrt{c^2 + ω^2}(3cω \cos θ + (4c^2 + ω^2) \sin θ ) + 3c^2r_1ω^2 \sin 2θ + rω^4 \sin 2θ ) \right).
\]
We take $\theta$ as a new independent variable and obtain the system

\[(3.8)\]

\[\frac{dr}{d\theta} = \frac{\varepsilon}{3\omega^4 \sqrt{c^2 + \omega^2}} \left( cr \omega^3 (\sqrt{3}a_1 - a_1 \sqrt{c^2 + \omega^2}) \cos^2 \theta + \omega \sqrt{c^2 + \omega^2} \cos \theta \right)

\[\left(-6c^3 d_1 Z + r \omega (6c^2 (e_1 - W) + a_1 \omega^2) \sin \theta \right) + \varepsilon \sin \theta \left(6c \varepsilon (2c^2 + \omega^2) d_1 \sqrt{c^2 + \omega^2} \right)

\[+ \omega^2 d_1 \sqrt{c^2 + \omega^2} - \sqrt{3} \varepsilon (e_1 - W) (c^2 + \omega^2) \right) + r \omega \left(2 \omega \varepsilon \sqrt{3} a_1 \right)

\[+ a_1 \sqrt{c^2 + \omega^2} \cos \theta + (4c^3 + 3c \omega^2) \sqrt{3} d_1 + (6c^2 (e_1 - W) - 2a_1 \omega^2) \sqrt{c^2 + \omega^2} \right)

\sin \theta \left) \right) + O(\varepsilon^2) \]

\[= \varepsilon F_1(\theta, r, Z, W) + O(\varepsilon^2), \]

\[\frac{dZ}{d\theta} = \frac{\varepsilon}{18c^2 \omega^3 \sqrt{c^2 + \omega^2}} \left(-12c^3 Z \left(2 \sqrt{3} (c^2 + \omega^2) d_1 + 3c (e_1 - W) \sqrt{c^2 + \omega^2} \right) \right.

\[+ \sqrt{3} c \omega \left(4c^2 (a_1 + 3c_1 - 3W) + a_1 \omega^2 \right) \cos \theta + r \omega \left(6c^3 + c \omega^2 \right) d_1 \sqrt{c^2 + \omega^2} \right)

\[+ \sqrt{3} (12c^4 (-e_1 + W) + 4a_1 c^2 \omega^2 + a_1 \omega^4) \right) \sin \theta \right) \right) + O(\varepsilon^2) \]

\[= \varepsilon F_2(\theta, r, Z, W) + O(\varepsilon^2), \]

\[\frac{dW}{d\theta} = \frac{\varepsilon}{3 \omega^3 \sqrt{c^2 + \omega^2}} \left( (c^2 + \omega^2) \left(12c^4 Z^2 + 2c^2 r^2 \omega^2 - 3b_1 W \omega^4 \right) \right.

\[+ cr \left(-2c^3 r \omega \cos 2\theta \right)

\[+ r \omega^4 \sin 2\theta \right) \right) + O(\varepsilon^2) \]

\[= \varepsilon F_3(\theta, r, Z, W) + O(\varepsilon^2). \]

Using the notation of averaging theory introduced in Theorem 2.1, we get $t = \theta$, $T = 2\pi$, $x = (r, Z, W)$ and

\[F(\theta, r, Z, W) = \begin{pmatrix} F_1(\theta, r, Z, W) \\ F_2(\theta, r, Z, W) \\ F_3(\theta, r, Z, W) \end{pmatrix}, \text{ and } f(r, Z, W) = \begin{pmatrix} f_1(r, Z, W) \\ f_2(r, Z, W) \\ f_3(r, Z, W) \end{pmatrix}. \]

Then we compute the integrals, i.e.

\[f_1(r, Z, W) = \frac{1}{2\pi} \int_0^{2\pi} F_1(\theta, r, Z, W) d\theta = \frac{r \left(6c^2 (e_1 - W) - 3a_1 \omega^2 + 4 \sqrt{3} d_1 \sqrt{c^2 + \omega^2} \right)}{6\omega^3}, \]

\[f_2(r, Z, W) = \frac{1}{2\pi} \int_0^{2\pi} F_2(\theta, r, Z, W) d\theta = -\frac{2\omega \left(3c (e_1 - W) + 2 \sqrt{3} d_1 \sqrt{c^2 + \omega^2} \right)}{3\omega^3}, \]

\[f_3(r, Z, W) = \frac{1}{2\pi} \int_0^{2\pi} F_3(\theta, r, Z, W) d\theta = \frac{12c^4 Z^2 + 2c^2 r^2 \omega^2 - 3b_1 W \omega^4}{3\omega^5} . \]
Solving the equations $f_1(r, Z, W) = f_2(r, Z, W) = f_3(r, Z, W) = 0$, we can get the following five solutions:

\[ s_0 = (0, 0, 0), \]
\[ s_{1,2} = \left( 0, \pm \frac{\sqrt{b_1 \omega^4 (3c e_1 + 2\sqrt{3} d_1 \sqrt{c^2 + \omega^2})}}{2\sqrt{3} c^{5/2}}, e_1 + \frac{2d_1 \sqrt{c^2 + \omega^2}}{\sqrt{3} c} \right), \]
\[ s_{3,4} = \left( \mp \frac{\sqrt{b_1 \omega^2 (6c^2 e_1 - 3a_1 \omega^2 + 4\sqrt{3} d_1 \sqrt{c^2 + \omega^2})}}{2c^2}, 0, e_1 + \frac{1}{6c^2}(-3a_1 \omega^2 \right. \]
\[ + 4\sqrt{3} d_1 \sqrt{c^2 + \omega^2} \). \]

The first solution $s_0$ corresponds to the equilibrium at the origin. For other four solutions, we get

(I) For the solution $s_{1,2}$ when $c \neq 0$, $s_{1,2}$ are real solutions. The Jacobian of solution $s_{1,2}$ is

\[ \det \left( \frac{\partial f}{\partial x}(s_1) \right) = \det \left( \frac{\partial f}{\partial x}(s_2) \right) = \frac{2a_1 b_1 c (3c e_1 + 2\sqrt{3} d_1 \sqrt{c^2 + \omega^2})}{3\omega^5}. \]

(II) For the solution $s_{3,4}$ when $c \neq 0$, $s_{3,4}$ are real solutions. The Jacobian of solution $s_{3,4}$ is

\[ \det \left( \frac{\partial f}{\partial x}(s_3) \right) = \det \left( \frac{\partial f}{\partial x}(s_4) \right) = \frac{a_1 b_1 (-6c^2 e_1 + 3a_1 \omega^2 - 4\sqrt{3} d_1 \sqrt{c^2 + \omega^2})}{3\omega^5}. \]

When $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$, $\eta = 3c e_1 + 2\sqrt{3} d_1 \sqrt{c^2 + \omega^2} \neq 0$ and $\eta_1 = 3a_1 \omega^2 - 2c \eta \neq 0$, then $\det \left( \frac{\partial f}{\partial x}(s_j) \right) \neq 0$, $j = 1, \ldots, 4$. Then according to Theorem 2.1 we see that the system (3.8) has one periodic solution $x_j(\theta, \varepsilon)$ such that $x_j(0, \varepsilon) = s_j + O(\varepsilon)$, $j = 1, \ldots, 4$. Bring the solution back to the system (3.4), and we have one periodic solution $\Phi_j(t, \varepsilon) = (X_j(t, \varepsilon), Y_j(t, \varepsilon), Z_j(t, \varepsilon), W_j(t, \varepsilon))$. Then the system (3.4) has the periodic solution $\varepsilon \Phi_j(t, \varepsilon)$, $j = 1, \ldots, 4$.

To determine the stability of the periodic solution $\varepsilon \Phi_j(t, \varepsilon)$, $j = 1, \ldots, 4$, one needs to calculate eigenvalues of the Jacobian matrix $\left( \frac{\partial f}{\partial x}(s_{2,3}) \right)$. 

(3.9) \[ P(s_{2,3}) = c_0 \lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 \]
where \( c_0, c_1, c_2 \) and \( c_3 \) are
\[
\begin{align*}
c_0 &= -1, \\
c_1 &= -\frac{a_1 + 2b_1}{2\omega}, \\
c_2 &= \frac{b_1 \left( -3a_1\omega^2 - 8c(3ce_1 + 2\sqrt{3d_1\sqrt{c^2 + \omega^2}}) \right)}{6\omega^4}, \\
c_3 &= \frac{2a_1b_1 \left( 3ce_1 + 2\sqrt{3d_1\sqrt{c^2 + \omega^2}} \right)}{3\omega^2}.
\end{align*}
\]

The eigenvalues are given as follows:
\[
\lambda_1 = -\frac{a_1}{2\omega}, \quad \lambda_{2,3} = -\frac{3b_1 \pm \sqrt{3\omega} \left( 48c^2e_1 + 3b_1\omega^2 + 32\sqrt{3cd_1\sqrt{c^2 + \omega^2}} \right)}{6\omega^2}.
\]

On the other hand, the characteristic polynomial and its eigenvalues of the Jacobian matrix \( \frac{\partial f}{\partial x}(s_3,4) \) are
\[
(3.10) \quad P(s_3,4) = c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3
\]
where \( c_0, c_1, c_2 \) and \( c_3 \) are
\[
\begin{align*}
c_0 &= -1, \\
c_1 &= -\frac{a_1 + b_1}{\omega}, \\
c_2 &= -\frac{2b_1c(3ce_1 + 2\sqrt{3d_1\sqrt{c^2 + \omega^2}})}{3\omega^4}, \\
c_3 &= \frac{a_1b_1(-6c^2e_1 + 3a_1\omega^2 - 4\sqrt{3cd_1\sqrt{c^2 + \omega^2}})}{3\omega^5}.
\end{align*}
\]

The eigenvalues are given as follows:
\[
\tilde{\lambda}_1 = -\frac{a_1}{\omega}, \quad \tilde{\lambda}_{2,3} = -\frac{3b_1\omega^3 \pm \sqrt{3} \omega^4 \left( 3(4a_1 + b_1)\omega^2 - 8c(3ce_1 + 2\sqrt{3d_1\sqrt{c^2 + \omega^2}}) \right)}{6\omega^3}.
\]

We have that \( \lambda_1, \tilde{\lambda}_1 \) is real and \( \lambda_{2,3}, \tilde{\lambda}_{2,3} \) are complex numbers if \( 16\eta + 3b_1\omega^2 < 0 \) and \( 4\eta_1 + 3b_1\omega^2 < 0 \). In this case, the periodic solution \( \varepsilon\Phi_j(t, \varepsilon) \) is stable if \( a_1 > 0, b_1 > 0 \). \( \square \)

**Proof. of statement (ii) of Theorem 1.3** Let
\[
(a, b) = (-2c + \varepsilon a_1, \varepsilon b_1),
\]
where \( \omega > 0 \) and \( \varepsilon > 0 \) are sufficiently small parameter. Then, we translate \( p \) to the origin the coordinates doing system (1.2) becomes \((x, y, z, w) = (\bar{x}, \bar{y}, \bar{z}, \bar{w}) + p\), then we introduce the scaling of variables \((\bar{x}, \bar{y}, \bar{z}, \bar{w}) = (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)\), with these
changes of variables system (1.2) can be written as

\[
\begin{align*}
\dot{x} &= 2c(x - y) - a_1(x - y)\varepsilon, \\
\dot{y} &= \frac{c^2x + d^2x - c^2y - cdz}{c} - wx\varepsilon, \\
\dot{z} &= dy - cz, \\
\dot{w} &= \frac{b_1(c^2 + d^2 - ce)}{c} + (-b_1w + xy)\varepsilon.
\end{align*}
\]

(3.11)

After the linear change in variables \((x, y, z, w) \mapsto (X, Y, Z, W)\),

\[
\begin{align*}
x &= -6d^2X + 2c^2(3W + X) - 2c\sqrt{-c^2 + 3d^2Y}, \\
y &= \frac{1}{3(c^3 - 3c^2d)} \left(3cd^2X - c^3(6W + X) + \sqrt{-c^2 + 3d^2(c^2 + 3d^2)Y}\right), \\
z &= d\left(-3d^2X + c^2(-6W + X) + 2c\sqrt{-c^2 + 3d^2Y}\right), \\
w &= Z.
\end{align*}
\]

(3.12)

the linear part at the origin of system (3.11) for \(\varepsilon = 0\) can be transformed into its real Jordan normal form,

\[
\begin{pmatrix}
0 & \frac{-\sqrt{3d^2 - c^2}}{c} & 0 & 0 \\
\frac{\sqrt{3d^2 - c^2}}{c} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Under the change in variable (3.12), where we have written \((x, y, z, w)\) instead of \((X, Y, Z, W)\) the system (3.11) can be written as (3.13)

\[
\begin{align*}
\dot{x} &= \frac{1}{3} \left(-a_1 x\varepsilon + \frac{\sqrt{-c^2 + 3d^2y(-3c + a_1\varepsilon)}}{c}\right), \\
\dot{y} &= \frac{1}{3} \left(-2a_1 y\varepsilon + 6c^2 \left(\frac{3cw}{(-c^2 + 3d^2)^{3/2}} + \frac{y}{c^2 - 3d^2}\right) z\varepsilon + \frac{x}{\sqrt{-c^2 + 3d^2}} \left(-3d^2x + c^2(3w + x) - c\sqrt{-c^2 + 3d^2y}\right) \right), \\
\dot{z} &= b_1 \left(c + \frac{d^2}{c} - e\right) + \frac{2\varepsilon}{9c(c^2 - 3d^2)^2} \left(-3d^2x + c^2(3w + x) - c\sqrt{-c^2 + 3d^2y}\right) \left(-3cd^2x + c^3(6w + x) - (c^2 + 3d^2)\sqrt{-c^2 + 3d^2y}\right) - b_1 z\varepsilon, \\
\dot{w} &= \frac{1}{6} \left(\frac{a_1(c^2 + d^2)}{c^3} \left(cx - \sqrt{-c^2 + 3d^2y}\right) + 4 \left(\frac{3d^2x - c^2(3w + x) + c\sqrt{-c^2 + 3d^2y}}{c^2 - 3d^2}\right) z\varepsilon.\right.
\end{align*}
\]

(3.14)

\((x, y, z, w) \mapsto (r \cos \theta, r \sin \theta, z, w)\)

the system (3.13) becomes
\[
\frac{dr}{d\theta} = -\frac{\varepsilon}{3c(c^2 - 3d^2)^2} \left( a_1 c(-c^2 + 3d^2)^{3/2} r \cos^2 + (c^2 - 3d^2) a_1 c^2 + 3d^2) - 6c^2 z \right) \\
\cos \theta \sin \theta + 2c \sin \theta \left( -9c^3 w + \sqrt{c^2 + 3d^2} r(-a_1 c^2 + 3a_1 d^2 + 3c^2 z) \sin \theta \right) + O(\varepsilon^2) \\
= \varepsilon F_1(\theta, r, z, w) + O(\varepsilon^2),
\]
\[
\frac{dz}{d\theta} = \frac{\varepsilon}{9\sqrt{-c^2 + 3d^2(c^3 - 3cd^2)^2} r} \left( 3c^2 r \left( -2d^2(c^2 - 3d^2)r^2 + 12c^4 w^2 - 3b_1(c^2 - 3d^2)z \right) \\
+ 18c^4 w \left( (c^2 - 3d^2)r^2 - 3b_1(c^2 + d^2 - ce)z \right) \cos \theta + 6b_1 c^2(c^2 - 3d^2) c^2 + d^2 - ce)r \\
(a_1 - 3z) \cos^2 \theta + 2c^4(c^2 - 3d^2)^3 \cos 2\theta + c\sqrt{-c^2 + 3d^2}r \left( -(c^2 - 3d^2)(3a_1 b_1 c^2 + d^2 - ce) + 2(2c^2 + 3d^2)r^2 + 18b_1 c^2(c^2 + d^2 - ce)z \right) \cos \theta \sin \theta + 3r \sin \theta \left( -6c^3(c^2 + d^2 - ce) \sin \theta \right) + O(\varepsilon^2) \\
= \varepsilon F_2(\theta, r, z, w) + O(\varepsilon^2),
\]
\[
\frac{dw}{d\theta} = -\frac{\varepsilon}{6c^4(-c^2 + 3d^2)^{3/2}} \left( \sqrt{-c^2 + 3d^2}r \left( -a_1 c^2 - 3d^2)(c^2 + d^2) + 4c^4 z \right) \sin \theta \\
- 12c^5 w + c(c^2 - 3d^2)r(a_1 c^2 + d^2 - 4c^2 z) \cos \theta \right) + O(\varepsilon^2) \\
= \varepsilon F_3(\theta, r, z, w) + O(\varepsilon^2).
\]

System (3.15) is written in the normal form (2.1) for applying the averaging theory and satisfies all the assumptions of Theorem 2.1. Then, using the notations of the averaging theory described in Theorem 2.1, we have \( t = \theta, T = 2\pi, x = (r, z, w), \)

\[
F(\theta, r, z, w) = \begin{pmatrix} F_1(\theta, r, z, w) \\ F_2(\theta, r, z, w) \\ F_3(\theta, r, z, w) \end{pmatrix}, \quad \text{and} \quad f(r, z, w) = \begin{pmatrix} f_1(r, z, w) \\ f_2(r, z, w) \\ f_3(r, z, w) \end{pmatrix}
\]
Then we compute the integrals, i.e.

\[
f_1(r, z, w) = \frac{1}{2\pi} \int_0^{2\pi} F_1(\theta, r, z, w) d\theta = \frac{r(a_1(c^2 - 3d^2) - 2c^2 z)}{2(-c^2 + 3d^2)^{3/2}},
\]

\[
f_2(r, z, w) = \frac{1}{2\pi} \int_0^{2\pi} F_2(\theta, r, z, w) d\theta
= \frac{1}{6\sqrt{-c^2 + 3d^2}(c^3 - 3cd^2)^2} \left(3a_1b_1(c^4 - 4c^2d^2 + 3d^2)(c^2 + d^2 + 3d^4)2c^2
+ \left(-2d^2(c^2 - 3d^2)r^2 + 12c^4w^2 - 3b_1(c^3 - 3d^3)(2c^2 - 2d^2 - ce)z\right)\right),
\]

\[
f_3(r, z, w) = \frac{1}{2\pi} \int_0^{2\pi} F_3(\theta, r, z, w) d\theta = \frac{2c^2wz}{(-c^2 + 3d^2)^{3/2}}.
\]

Solving the equations \( f_1(r, z, w) = f_2(r, z, w) = f_3(r, z, w) = 0 \), we can get the following five solutions:

\[
s_1 = \left(0, \frac{a_1(c^2 - d^2)(c^2 + d^2 - ce)}{2c^2(2c^2 - 2d^2 - cc)}, \ 0\right),
\]

\[
s_{2,3} = \left(\pm \frac{\sqrt[3]{a_1}}{2c^2} \sqrt{-b_1\left(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e\right)} 1 \frac{1}{2}(1 - \frac{3d^2}{c^2}), 0\right),
\]

\[
s_{4,5} = \left(0, 0, \frac{\sqrt[3]{a_1}}{2\sqrt{2}c^3} \sqrt{-b_1\left(c^4 - 4c^2d^2 + 3d^4\right)(c^2 + d^2 - ce)}\right).
\]

The solution \( s_j, j = 1, \ldots, 5 \) exist if only if \( c \neq 0, d \neq 0, \) and \( 2(c^2 - d^2) - cc \neq 0 \). On the other hand, the solution \( s_1 \neq (0, 0, 0) \) if only if \( (c^2 - d^2)(c^2 + d^2 - ce) \neq 0 \), and the solutions \( s_{2,3} \) and \( s_{4,5} \) are real if only if \( c^4 - 8c^2d^2 + 7d^4 + 2cd^2e < 0 \) and \( (c^4 - 4c^2d^2 + 3d^4)(c^2 + d^2 - ce) < 0 \).

For the five solutions, we get

\[
\det \left(\frac{\partial f}{\partial x}(s_1)\right) = a_1^2b_1(c^2 - d^2)(c^2 + d^2 - cc)(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e),
\]

\[
\det \left(\frac{\partial f}{\partial x}(s_2)\right) = \det \left(\frac{\partial f}{\partial x}(s_3)\right) = a_1^2b_1(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e),
\]

\[
\det \left(\frac{\partial f}{\partial x}(s_4)\right) = \det \left(\frac{\partial f}{\partial x}(s_5)\right) = a_1^2b_1(c^2 - d^2)(c^2 + d^2 - 2ce).
\]

When \( a_1 \neq 0, b_1 \neq 0 \) and \( 3d^2 - c^2 > 0 \) then \( \det \left(\frac{\partial f}{\partial x}(s_j)\right) \neq 0 \), for each \( j = 1, \ldots, 5 \). Then according to Theorem 2.1, we see that the system (5.10) has one periodic solution \( x_j(\theta, \varepsilon) \) such that \( x_j(0, \varepsilon) = s_j + O(\varepsilon) \), for each \( j = 1, \ldots, 5 \).
Bring the solution back to the system (3.13) and we have one periodic solution $\Phi_j(t, \varepsilon) = (X_j(t, \varepsilon), Y_j(t, \varepsilon), Z_j(t, \varepsilon), W_j(t, \varepsilon))$. Then the system (3.11) has the periodic solution $\varepsilon \Phi_j(t, \varepsilon)$, $j = 1, \ldots, 5$.

To determine the stability of the periodic solution one needs to calculate eigenvalues of the Jacobian matrix $\partial F(s_j)/\partial x$, $j = 1, \ldots, 5$.

The Jacobian matrices $\partial F(s_j)/\partial x$ have the same characteristic equation,

$$\lambda^3 + \Theta_1 \lambda^2 - \Theta_2 \lambda - \Theta_3$$

where $\Theta_1, \Theta_2$ and $\Theta_3$ are

$$\Theta_1 = \frac{a_1}{2(-c^2 + 3d^2)^{3/2}(2c^2 - 2d^2 - ce)} \left( a_1(c^2 - d^2)(c^2 + d^2 - ce)(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e) \right),$$

$$\Theta_2 = \frac{2(c^2 - 3d^2) \left( 2d^2 + c(-2c + e) \right)^2 \left( a_1(c^2 - d^2)(c^2 + d^2 - ce)(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e) \right)}{2(-c^2 + 3d^2)^{3/2}(2c^2 - 2d^2 - ce)}$$

The eigenvalues are given as follows:

$$\lambda_1 = \frac{-b_1 \left( 2d^2 + c(-2c + e) \right)}{\left( -c^2 + 3d^2 \right)^{3/2} \left( 2d^2 + c(-2c + e) \right)},$$

$$\lambda_2 = \frac{a_1 \left( c^4 - 8c^2d^2 + 7d^4 + 2cd^2e \right)}{2(-c^2 + 3d^2)^{3/2}(2c^2 - 2d^2 - ce)},$$

$$\lambda_3 = \frac{a_1 \left( -c^4 + d^4 + c^3e - cd^2e \right)}{\left( -c^2 + 3d^2 \right)^{3/2} \left( 2d^2 + c(-2c + e) \right)}.$$

The Jacobian matrices $\partial F(s_2)/\partial x$ and $\partial F(s_3)/\partial x$ have the same characteristic equation,

$$\lambda^3 - \Upsilon_1 \lambda^2 - \Upsilon_2 \lambda - \Upsilon_3$$

where $\Upsilon_1, \Upsilon_2$ and $\Upsilon_3$ are

$$\Upsilon_1 = \frac{a_1(c^2 - 3d^2) + b_1(2c^2 - 2d^2 - ce)}{\left( -c^2 + 3d^2 \right)^{3/2}},$$

$$\Upsilon_2 = \frac{a_1b_1(c^2 - d^2)(c^2 + d^2 - ce)}{\left( c^2 - 3d^2 \right)^3},$$

$$\Upsilon_3 = \frac{a_1^2b_1(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e)}{\left( -c^2 + 3d^2 \right)^{7/2}}.$$
where \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are

\[
\Gamma_1 = \frac{a_1 c^2 + 4b_1 c^2 - 3a_1 d^2 - 4b_1 d^2 - 2b_1 e c}{2(-c^2 + 3d^2)^{3/2}}, \\
\Gamma_2 = \frac{a_1 b_1 (2c^4 + 8c^2 d^2 - 10d^4 - 3c^3 e + cd^2 e)}{2(c^2 - 3d^2)^3}, \\
\Gamma_3 = \frac{a_1 b_1 (c^2 - d^2)(c^2 + d^2 - ce)}{(-c^2 + 3d^2)^{7/2}}.
\]

The eigenvalues are given as follows:

\[
\tilde{\lambda}_1 = -\frac{a_1}{2\sqrt{-c^2 + 3d^2}}, \\
\tilde{\lambda}_{4,5} = -\frac{2b_1 d^2 + b_1 c(-2c + e) \pm i \sqrt{b_1 \left( 8a_1(-c^4 + d^4 + c^3 e - cd^2 e) - b_1(2d^2 - c(-2c + e))^2 \right)}}{2(-c^2 + 3d^2)^{3/2}}.
\]

We have that \( \lambda_1, \tilde{\lambda}_1, \tilde{\lambda}_1 \) is real and \( \lambda_{2,3}, \tilde{\lambda}_{2,3}, \tilde{\lambda}_{4,5} \) are complex numbers if \( 16\eta + 3b_1\omega^2 < 0 \) and \( 4\eta + 3b_1\omega^2 < 0 \). In this case, since that \( a_1 > 0, b_1 > 0, (c^4 - 8c^2 d^2 + 7d^4 + 2cd^2 e) < 0, 2c^2 - 2d^2 - ce < 0 \) and \( c^4 - d^4 - c^3 e + cd^2 e > 0 \), then this implies that the periodic orbits \( \varepsilon \Phi_j(t, \varepsilon), j \in \{1, \ldots, 5\} \) are stable. \( \square \)

**Proof of statement (iii) of Theorem 1.3.** Let

\[
(a, b, d) = (-2c + \varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2 + \omega^2}}{\sqrt{3}} + \varepsilon d_1),
\]

where \( \omega > 0 \) and \( \varepsilon > 0 \) are sufficiently small parameter. Then, we translate \( p_\pm \) to the origin the coordinates doing system (1.2) becomes \( (x, y, z, w) = (\bar{x}, \bar{y}, \bar{z}, \bar{w}) + p_\pm \), then we introduce the scaling of variables \( (x, y, z, w) = (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w) \), with these changes of variables system (1.2) can be written as

\[
\begin{align*}
\dot{x} &= (x - y)(2c - a_1 \varepsilon), \\
\dot{y} &= \frac{1}{3} \left( c(4x - 3y) - 3w \varepsilon z - z(3d_1 \varepsilon + \sqrt{3} \sqrt{c^2 + \omega^2}) + \frac{b_1 w \sqrt{\varepsilon}}{\sqrt{c \sqrt{-b_1(4c^2 - 3ce + \omega^2)}}} \right), \\
\dot{z} &= -cz + d_1 y \varepsilon - \frac{y \sqrt{c^2 + \omega^2}}{\sqrt{3}}, \\
\dot{w} &= (-b_1 w + xy) \varepsilon - \frac{b_1 d_1 (x + y) \varepsilon^{3/2} \sqrt{c^2 + \omega^2}}{\sqrt{c \sqrt{-b_1(4c^2 - 3ce + \omega^2)}}} + \frac{(x + y) \sqrt{\varepsilon} \sqrt{-b_1(4c^2 - 3ce + \omega^2)}}{\sqrt{3} \sqrt{c}}.
\end{align*}
\]

After the linear change in variables \( (x, y, z, w) \mapsto (X, Y, Z, W) \),
the linear part at the origin of system (3.16) for \( \varepsilon = 0 \) can be transformed into its real Jordan normal form,

\[
\begin{pmatrix}
0 & -\omega & 0 & 0 \\
\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Under the change in variable (3.17), where we have written \((x, y, z, w)\) instead of \((X, Y, Z, W)\) the system (3.16) can be written as (3.18)

\[
\begin{align*}
x &= \frac{2c(\sqrt{3}cZ + cY\omega + X\omega^2)}{\sqrt{3}\omega^2}, \\
y &= \frac{cX\omega^2 + Y\omega^3 + c^2(2\sqrt{3}Z - 2Y\omega)}{\sqrt{3}\omega^2}, \\
z &= \frac{\sqrt{c^2 + \omega^2}(2\sqrt{3}cZ - cY\omega + X\omega^2)}{3\omega^2}, \\
w &= W.
\end{align*}
\]
Performing the cylindrical change of variables, 
\[(x, y, z, w) \mapsto (r \cos \theta, r \sin \theta, z, w)\]
the system (3.18) becomes
\[
\frac{dr}{d\theta} = \frac{1}{3cr\omega_4 \sqrt{c^2 + \omega^2}} \left(-a_1 r \omega^3 \sqrt{c^2 + \omega^2} \cos^2 \theta + \cos \theta \left(-6c^3 d_1 r z \omega \right.ight.
\left.\left. + \left(-2 \sqrt{3} c d_1 r^2 \omega + \omega^2 \sqrt{c^2 + \omega^2} (3b_1 w^2 + a_1 r^2 \omega^2) + 2c^2 \sqrt{c^2 + \omega^2} (6b_1 w^2 + r^2 (a_1 - 3w) \omega^2) - c (4 \sqrt{3} d_1 r^2 \omega^4 + 9b_1 c w^2 \sqrt{c^2 + \omega^2}) \sin \theta \right) + cr \left(\sqrt{3} d_1 r^2 \omega^2 + 2c \omega \sqrt{c^2 + \omega^2} (3c \omega^2 r^2 - 3c \omega^2 \omega^2) \sin \theta - 2r \omega \sqrt{c^2 + \omega^2} (3c \omega^2 w + a_1 \omega^2) \sin^2 \theta \right) \right),
\]
\[
\frac{dz}{d\theta} = \frac{1}{18c^2 \sqrt{c^2 + \omega^2}} \left(c \left(6 \sqrt{3} b_1 w^2 (4c^2 - 3ce + \omega^2) + r^2 \omega^2 \left(-24cd_1 \sqrt{c^2 + \omega^2} \right.ight.ight.
\left.\left.\left. + \sqrt{3} (4c^2 (a_1 - 3w) + a_1 \omega^2) \right) \cos \theta + r \left(36c^4 wz + r \omega \left(6cd_1 \omega^2 \sqrt{c^2 + \omega^2} \right. \right.ight.
\left.\left.\left. - \sqrt{3} (12c^4 w + 4a_1 c^2 \omega^2 + a_1 \omega^4) \sin \theta \right) \right),
\]
\[
\frac{dw}{d\theta} = \frac{1}{3cr\omega_5} \left(12c^5 r z^2 + cr (2c^2 r^2 - 3b_1 w) \omega^4 + c^2 r^2 \omega \left(6c^2 \omega \cos \theta (-\sqrt{3} z + r \omega \sin \theta) \right. \right.
\left.\left. + cz (4c^2 + \omega^2) + r \omega^4 \cos \theta + 2c^3 r \omega \sin \theta \right) - b_1 \omega (4c^2 - 3ce + \omega^2) \cos \theta \left(-4 \sqrt{3} \omega^2 z + c \omega \cos \theta \left(4c^2 + \omega^2 \right) \sin \theta \right) \right) \right).
\]
System (3.19) is written in the normal form (2.1) for applying the averaging theory and satisfies all the assumptions of Theorem 2.1. Then, using the notations of the averaging theory described in Theorem 2.1 we have 
\[t = \theta, \ T = 2\pi, \ x = (r, z, w),\]
\[F(\theta, r, z, w) = \begin{pmatrix} F_1(\theta, r, z, w) \\ F_2(\theta, r, z, w) \\ F_3(\theta, r, z, w) \end{pmatrix}, \text{ and } f(r, z, w) = \begin{pmatrix} f_1(r, z, w) \\ f_2(r, z, w) \\ f_3(r, z, w) \end{pmatrix}.\]
Then we compute the integrals, i.e.
\[f_1(r, z, w) = \frac{r (2c^2 w + a_1 \omega^2)}{2 \omega^3},\]
\[f_2(r, z, w) = \frac{2c^2 wz}{\omega^3},\]
\[f_3(r, z, w) = \frac{24c^4 z^2 + c (4c^3 r^2 - 12b_1 cw + 9b_1 c w) \omega^2 + (4c^2 r^2 - 9b_1 w) \omega^4}{6 \omega^3}.\]
Solving the equations \[f_1(r, z, w) = f_2(r, z, w) = f_3(r, z, w) = 0,\] we can get the following three solutions:
\[ s_0 = (0, 0, 0), \]
\[ s_{1,2} = \left( \pm \frac{1}{2c^2} \sqrt{\frac{3}{2}} \sqrt{a_1 \omega} \sqrt{\frac{b_1(-4c^2 + 3ce - 3\omega^2)}{c^2 + \omega^2}}, 0, -\frac{a_1 \omega^2}{2c^2} \right). \]

For two solutions, we get
\[ \det \left( \frac{\partial f}{\partial x}(s_1) \right) = \det \left( \frac{\partial f}{\partial x}(s_2) \right) = \frac{a_1^2 b_1(4c^2 - 3ce + 3\omega^2)}{2\omega^5}. \]

When \( c \neq 0, a_1 \neq 0, \) and \( \kappa = b_1(4c^2 - 3ce + 3\omega^2) < 0 \) then \( \det \left( \frac{\partial f}{\partial x}(s_j) \right) \neq 0, j = 1, 2. \) Then according to Theorem 2.1 we see that the system (3.19) has one periodic solution \( x_j(\theta, \varepsilon) \) such that \( x_j(0, \varepsilon) = s_j + O(\varepsilon), j = 1, 2. \) Bring the solution back to the system (3.18), and we have one periodic solution \( \Phi_j(t, \varepsilon) = (X_j(t, \varepsilon), Y_j(t, \varepsilon), Z_j(t, \varepsilon), W_j(t, \varepsilon)). \) Then the system (3.10) has the periodic solution \( \varepsilon \Phi_j(t, \varepsilon), j = 1, 2. \)

The Jacobian matrices \( \partial F(s_j)/\partial x \) have the same characteristic equation,
\[ \lambda^3 + \frac{b_1(c(4c - 3e) + (2a_1 + 3b_1) \omega^2)}{2\omega^3} \lambda^2 - \frac{a_1^2 b_1(4c^2 - 3ce + 3\omega^2)}{2\omega^5}. \]

The eigenvalues are given as follows:
\[ \lambda_1 = \frac{-a_1}{\omega}, \]
\[ \lambda_2 = \frac{1}{4\omega^3} \left( b_1(4c^2 - 3ce + 3\omega^2) + \sqrt{\left( b_1(4c^2 - 3ce + 3\omega^2) \left( b_1c(4c - 3e) + (8a_1 + 3b_1) \omega^2 \right) \right)} \right), \]
\[ \lambda_3 = \frac{1}{4\omega^3} \left( b_1(4c^2 - 3ce + 3\omega^2) - \sqrt{\left( b_1(4c^2 - 3ce + 3\omega^2) \left( b_1c(4c - 3e) + (8a_1 + 3b_1) \omega^2 \right) \right)} \right). \]

We have that \( \lambda_1 \) and \( \lambda_{2,3} = \frac{1}{4\omega^3}(\kappa \pm \sqrt{\kappa^2 + 8a_1 \omega^2}) \) are reals, if \( a_1 > 0, \kappa > 0 \) and regardless of the sign assumed by \( \kappa(\kappa + 8a_1 \omega^2) \), at least one of the eigenvalues has a positive real part. In this case, the periodic solution \( \varepsilon \Phi_j(t, \varepsilon), j = 1, 2 \) is unstable.

\section*{Acknowledgments}

The first author was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

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