Dynamics of super-horizon photons during inflation with vacuum polarization

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We study asymptotic dynamics of photons propagating in the polarized vacuum of a locally de Sitter Universe. The origin of the vacuum polarization is fluctuations of a massless, minimally coupled, scalar, which we model by the one-loop vacuum polarization tensor of scalar electrodynamics. We show that late time dynamics of the electric field on superhorizon scales approaches that of an Airy oscillator. The magnetic field amplitude, on the other hand, asymptotically approaches a nonvanishing constant (plus an exponentially small oscillatory component), which is suppressed with respect to the initial (vacuum) amplitude. This implies that the asymptotic photon dynamics is more intricate than that of a massive photon obeying the local Proca equation.

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I. INTRODUCTION

Even though we have recently witnessed significant progress in understanding the dynamics of gauge fields (photons) during inflation, our understanding remains quite rudimentary. In earlier work [1, 2, 3] we computed the one loop vacuum polarization from charged, massless, minimally coupled scalars in a locally de Sitter background. We also evaluated the integral of the position-space vacuum polarization tensor against a tree order photon wave function and found a nonzero result which grows with the same number of scale factors as the mass term of the Proca equation.

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What we have so far done has a direct analogue in flat space quantum field theory. Because the background possesses spacetime translation invariance it is much simpler to work in momentum space. Gauge and Lorentz invariance imply that the vacuum polarization is proportional to the transverse projection operator, \( \Pi^{\mu\nu}(p) = (p^2 \eta^{\mu\nu} - p^\mu p^\nu)\Pi(p^2) \). Here \( \eta^{\mu\nu} \) is the Minkowski metric,

\[
\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \quad \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1),
\]  

and \( p^2 \equiv p^\mu p^\nu \eta_{\mu\nu} \). One checks that the photon remains massless by evaluating \( p^2 \Pi(p^2) \) on the tree order mass shell, \( p^2 = 0 \). A nonzero result — as in the Schwinger model \([4]\) — proves that quantum corrections change the tree order mass shell, but it does not determine the fully corrected mass shell. For that one must find the pole of the full propagator, that is, solve for \( p^2 \) such that,

\[
p^2 \left[ 1 - \Pi(p^2) \right] = 0. 
\]  

What we have done for photons during inflation is the analogue of computing \( \Pi(p^2) \) at one loop and showing that \( p^2 \Pi(p^2)|_{p^2=0} \neq 0 \). What we do in this paper is the analogue of solving equation \((2)\).

To introduce the problem, we remind the reader that we work in the conformal coordinate system of a locally de Sitter space-time whose metric and inverse are,

\[
g_{\mu\nu} = a^2 \eta_{\mu\nu}, \quad g^{\mu\nu} = a^{-2} \eta^{\mu\nu}.
\]  

Here \( a \) is the scale factor,

\[
a(\eta) = -\frac{1}{H \eta}, \quad (\eta < 0).
\]  

The Lagrangean density of massless minimally coupled scalar electrodynamics in a general metric reads,

\[
\mathcal{L}_{\Phi QED} = -\frac{1}{4} \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - \sqrt{-g} \eta^{\mu\nu} (\partial_\mu - ieA_\mu)\phi^* (\partial_\nu + ieA_\nu)\phi,
\]  

where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \). Its reduction to the locally de Sitter background \([3,4]\) is,

\[
\mathcal{L}_{\Phi QED} \rightarrow -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - a^2 \eta^{\mu\nu} (\partial_\mu - ieA_\mu)\phi^* (\partial_\nu + ieA_\nu)\phi,
\]  

where we made use of \( \sqrt{-g} = a^4 \). From this form of the Lagrangean, it is obvious that the photon field couples conformally to gravity, with the trivial rescaling,

\[
F_{\mu\nu} \rightarrow F_{\mu\nu} \quad (A_\mu \rightarrow A_\mu).
\]
An important consequence is that tree level photon wave functions take the same form in conformal coordinates as they do in flat space, $A_\mu \propto \epsilon_\mu e^{ik\cdot x}$.

On the other hand, the minimally coupled scalar field $\phi$ does not couple conformally to gravity, which can be easily seen upon the conformal rescaling, $\phi \rightarrow a^{-1}\phi$ in Eq. (6). This has as a consequence the extensively studied particle creation during inflation, known as superadiabatic amplification. At one loop this conformal symmetry breaking is communicated to the photon sector by the cubic and quartic terms in (6). The main objective of this work is to study how this changes photon wave functions.

Provided it couples minimally to gravity, an obvious candidate for the scalar field $\phi$ is the charged component of the fundamental Higgs scalar, which at low energies manifests as the longitudinal component of the $W^\pm$ boson. Of course, there may be many more charged scalars of this type whose masses lie between that of the Higgs field, $m_H \sim 10^2$ GeV, and the energy scale of inflation, $H \sim 10^{13}$ GeV. An important class of such particles would be the supersymmetric scalar partners of the standard model quarks and charged leptons.

In the presence of charged scalar fluctuations the vacuum becomes polarized, such that in conformal space times $\mathbb{R}$, Maxwell’s equations generalize to $\mathbb{R}$, 

$$\eta^{\mu\nu}\partial_\mu F_{\sigma\nu}(x) + \int d^4 x' \, [^\mu \Pi'_{\text{ret}}](x, x') A_\nu(x') = 0.$$  (8)

Here $[^\mu \Pi'_{\text{ret}}](x, x')$ denotes the retarded vacuum polarization bi-tensor, which has the general form,

$$[^\mu \Pi'_{\text{ret}}](x, x') \equiv -[^\mu P'] \, \chi_e(x, x') + [^\mu \bar{P}'] \, \delta n^2(x, x').$$  (9)

The two transverse projectors are defined by,

$$[^\mu P'] \equiv [\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}] \partial'_\rho \partial_\sigma, \quad [^\mu \bar{P}'] \equiv \eta^{\mu i} \eta^{\nu j} [\delta_{ij} \nabla' \cdot \nabla - \partial'_i \partial'_j],$$  (10)

with $\partial_\mu \equiv \partial/\partial x^\mu$, $\partial'_\mu \equiv \partial/\partial x'^\mu$. The coefficient of $[^\mu \bar{P}']$ in (9) is,

$$\delta n^2(x, x') = \chi_e(x, x') + \frac{\chi_m}{1 + \chi_m}(x, x').$$  (11)

where $\chi_e(x, x')$ and $\chi_m(x, x')$ denote the electric and magnetic susceptibilities of the vacuum. An important consistency check of equations (8)–(9) is obtained by taking a derivative $\partial_\mu$ of Eq. (8).

The first term vanishes by the antisymmetry of $F_{\sigma\nu}$, while the integral term vanishes because of the transverse structure of the vacuum polarization tensor $\mathbb{R}$,

$$\partial_\mu[^\mu \Pi'] = 0 = \partial'_\nu[^\mu \Pi'].$$  (12)
This is an immediate consequence of $\partial_\mu [\mu P^\rho] = 0$, $\partial_\mu [\mu \bar{P}^\rho] = 0$, $\partial'_\nu [\mu P^\rho] = 0$, and $\partial'_\nu [\mu \bar{P}^\rho] = 0$.

In a locally de Sitter space-time (14) there is a simple relation between $\ell(x; x')$, the invariant length from $x^\mu$ to $x'^\mu$, and the conformal coordinate interval $\Delta x^2(x; x')$:

$$4 \sin^2 \left( \frac{1}{2} H \ell(x; x') \right) = a a' H^2 \Delta x^2(x; x') \equiv y(x; x').$$

(13)

We refer to $y(x; x')$ as the de Sitter length function. The scalar propagator can be expressed in terms of $y \equiv y(x; x')$ and the two scale factors,

$$i \Delta(x; x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ 2^{D-4} \Gamma(D - 1) \ln(aa') - \pi \cot(\frac{\pi y}{4}) \frac{\Gamma(D - 1)}{\Gamma(\frac{D}{2})} \right.$$ 

$$\left. + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\Gamma(D - 1 + n)}{\Gamma(D + n)} \left( \frac{y}{4} \right)^n - \frac{1}{n - \frac{D}{2}} \frac{\Gamma(D - 1 + n)}{\Gamma(n)} \left( \frac{y}{4} \right)^{n - \frac{D}{2}} \right\}. \quad (14)$$

Note that the homogeneous terms (i.e., $y^n$) and the constant terms have slightly different proportionality constants from our previous expression (3, 5). This has been done to make (14) valid for regulating a spacetime in which $D$ will ultimately be taken to some dimension other than four (14). In $D = 4$, the propagator (14) reduces to,

$$i \Delta(x; x') \overset{D \to 4}{\rightarrow} \frac{H^2}{4\pi^2} \left\{ \eta \frac{\eta'}{\Delta x^2} - \frac{1}{2} \ln(H^2 \Delta x^2) - \frac{1}{4} + \ln(2) \right\}. \quad (15)$$

In the presence of charged scalar fluctuations described by this propagator, the photon in a locally de Sitter space-time sees a polarized vacuum. This effect is characterized by the renormalized vacuum polarization bi-tensor, the one loop result for which is (2, 3),

$$i \left[ {}^{\mu \Pi^\nu}_{\text{ren}} \right](x, x') = \frac{\alpha}{32\pi^3} \left\{ - [\mu P^\rho] \left[ \frac{1}{12} \partial^4 \left( \ln^2(\mu^2 \Delta x^2) - 2 \ln(\mu^2 \Delta x^2) \right) \right. \right.$$

$$+ i \frac{16\pi^2}{3} \ln(a) \delta^4(x - x')

$$+ \frac{1}{\eta' \eta} \partial^2 \left( \ln^2(H'^2 \Delta x^2) + 2 \ln(H'^2 \Delta x^2) \right) \right. \right.$$

$$\left. + \left[ {}^{\mu \bar{P}^\nu} \right] \left[ \frac{2}{\eta' \eta} \left( \ln^2(H'^2 \Delta x^2) + 4 \ln(H'^2 \Delta x^2) \right) \right] \right\}. \quad (16)$$

Here $\mu$ is the renormalization scale, $\alpha \equiv e^2/4\pi$ is the fine structure constant, and $H' \equiv 2^{-1} e^{1/4} H$. The polarization tensor (16) was obtained using dimensional regularization with the scalar propagator (14). It is fully renormalized, which means that it is an integrable function of $x'^\mu$ when the derivatives are taken after the integration. It is also gauge invariant on account of its transversality on both indices (12).

The renormalized vacuum polarization (16) is properly an in-out matrix element. To convert it into the retarded vacuum polarization $[{}^{\mu \Pi^\nu}_{\text{ret}}]$ of Eq. (8) we employ the Schwinger-Keldysh
formalism as has been described at length elsewhere [3]. The result takes the same form as (9) with electric susceptibility,

\[ \chi_e = \chi_e|_{\text{flat}} + \chi_e|_{\text{anomaly}} + \chi_e|_{\text{de Sitter}} \]

\[ \chi_e|_{\text{flat}} = \frac{\alpha}{96\pi^2} \partial^4 \left[ \left( \ln(\mu^2 \Delta \tau^2) - 1 \right) \theta(\Delta \eta) \theta(\Delta \tau^2) \right] \]

\[ \chi_e|_{\text{anomaly}} = \frac{\alpha}{6\pi} \delta^4(x - x') \]

\[ \chi_e|_{\text{de Sitter}} = \frac{\alpha}{8\pi^2} H^2 a a' \partial^2 \left[ \left( \ln(H'^2 \Delta \tau^2) + 1 \right) \theta(\Delta \eta) \theta(\Delta \tau^2) \right] . \] (17)

The change in the index of refraction is,

\[ \delta n^2 = \frac{\alpha}{4\pi^2} H^4 a^2 a'^2 \left( \ln(H'^2 \Delta \tau^2) + 2 \right) \theta(\Delta \eta) \theta(\Delta \tau^2) . \] (18)

In these formulae \( a \equiv a(\eta), \ a' \equiv a(\eta') \), \( H' = 2^{-1} e^{1/4} H \), \( \Delta \tau^2 = (\eta - \eta')^2 - ||\vec{x} - \vec{x}'||^2, \Delta \eta = \eta - \eta', \) and \( \theta(x) = 1 \) for \( x > 0, \theta(x) = 0 \) for \( x < 0 \).

The electric susceptibility in (17) can be neatly split into three contributions. The first contribution \( \chi_e|_{\text{flat}} \) is conformally invariant. When expressed in terms of conformal time, this contribution becomes identical to the one-loop electric susceptibility in Minkowski (flat) space-time. This term contains no scale factors, and it is renormalized precisely as in flat space. The second contribution, \( \chi_e|_{\text{anomaly}} \), comes from the trace anomaly, which arises as a consequence of imperfect cancellation between the counterterm and the local one-loop diagram in dimensional regularization [3]. The anomalous contribution depends only weakly on the scale factor \( a \). The trace anomaly was first considered in the context of electrodynamics in conformal space-times in Ref. [6], while for a discussion of scalar electrodynamics and nonabelian gauge theories see Refs. [7] and [3, 8]. The third contribution to the electric susceptibility, \( \chi_e|_{\text{de Sitter}} \) in (17) is purely inflationary, completely finite and grows linearly with the scale factors \( a \) and \( a' \). And finally, \( \delta n^2 \) in (18) is the second inflationary contribution, which grows quadratically with the scale factors \( a \) and \( a' \).

In section II we argue that only the one loop terms \( \chi_e|_{\text{de Sitter}} \) and \( \delta n^2 \) need be kept in equation (8). We also give the integro-differential equation which describes a spatial plane wave with arbitrary time dependence. (The derivation is reserved for the Appendix.) In section III we analyze the integro-differential equation for superhorizon photons at asymptotically late times. We show analytically that while the vector potential approaches a nonzero constant, its time derivative behaves like an Airy oscillator. Numerical analysis is also presented. In section IV we translate these results into statements about the physical electric and magnetic fields.
II. PHOTON DYNAMICS WITH VACUUM POLARIZATION IN INFLATION

We shall now study the dynamics of the photon field by including only the genuinely new inflationary terms contributing at one-loop to the vacuum polarization tensor, $\chi_e|_{\text{de Sitter}}$ and $\delta n$, since their contribution to the vacuum polarization grows rapidly with the scale factor. Indeed, $\delta n \propto a^2$, and the electric susceptibility, $\chi_e|_{\text{de Sitter}} \propto a$. From $\frac{\partial_0 a}{a} = \frac{\delta_n}{H a}$ it follows however, that this term can also give an $a^2$ in Eq. 17 and has to, therefore, be treated on an equal footing as $\delta n^2$. Moreover, we expect that, when compared with the one-loop contributions to $\chi_e|_{\text{de Sitter}}$ and $\delta n^2$, the contributions of higher loops can be neglected, since they are down by more powers of $\alpha$, but can give no extra powers of $a$.

A general solution can be expressed as a linear combination of spatial plane waves in Coulomb gauge,

$$A_\mu(x) = \epsilon_\mu(\vec{k}, \eta) \, e^{i\vec{k} \cdot \vec{x}}, \quad \epsilon_i k_i = 0.$$  \hfill (19)

In the Appendix we show that $\epsilon_0(\vec{k}, \eta)$ vanishes in the absence of sources, just as it does in flat space. The Appendix also describes how to perform the integral over $\vec{x}'$ to obtain the following integro-differential equation for the time dependent polarization vector,

$$-(\partial_0^2 + k^2)\epsilon_i(\vec{k}, \eta) + \frac{\alpha H^2}{\pi k} \left[ a(\partial_0^2 + k^2) \int_{\eta_0}^\eta d\eta' a' Y_1(\kappa \Delta \eta, \frac{k}{H'}) \epsilon_i(\vec{k}, \eta') \right. 
+ \left. H^2 a^2 \int_{\eta_0}^\eta d\eta' a'^2 \sin(k \Delta \eta) \epsilon_i(\vec{k}, \eta') \right] = 0.$$  \hfill (20)

Here $Y_1(x, \zeta)$ is,

$$Y_1(x, \zeta) = \sin(x) \left[ 2 \ln(2x/\zeta) + 2 + ci(2x) - \gamma_E - \ln(2x) \right] - \cos(x) \left[ \sin(2x) + \pi/2 \right].$$  \hfill (21)

Upon acting the time derivatives upon the first integral, Eq. (20) can be recast as,

$$(\partial_0^2 + k^2)\epsilon_i(\vec{k}, \eta) + a^2 \frac{2\alpha H^2}{\pi} \left\{ \left( \ln(a) - \frac{5}{4} \right) \epsilon_i(\vec{k}, \eta) - \frac{H^2}{k} \int_{\eta_0}^\eta d\eta' a'^2 \sin(k \Delta \eta) \epsilon_i(\vec{k}, \eta') 
+ \frac{1}{a} \int_{\eta_0}^\eta d\eta' a'^2 \frac{1 - \cos(k \Delta \eta)}{\Delta \eta} \epsilon_i(\vec{k}, \eta') 
- \int_{\eta_0}^\eta d\eta' \ln \left( 1 - \frac{\eta}{\eta'} \right) \partial_0 \epsilon_i(\vec{k}, \eta') \right\} = 0.$$  \hfill (22)

This integral-differential equation is one of our main results. It should describe photon dynamics accurately from the beginning of inflation, when $a(\eta_0) = 1$ and the one loop terms are negligible, all the way to late times, when $a \gg 1$ and the one loop terms become important. Another of our main results is that, although certain aspects of this dynamical system can be described in
terms of local field equations, the system as a whole is essentially nonlocal. There is solid physics
behind this: the nonlocal terms reflect the influence of scalar fluctuations within the past light-
cone. Note that this is perfectly consistent with causality. Note further that, since \( \ln(1 - \eta/\eta') \)
diverges logarithmically at \( \eta' = \eta \), the last integral cannot be recast in the form of an integral over
a kernel \( K = K(\eta, \eta') \) multiplied by the polarization vector, \( \int_{\eta_0}^{\eta} K(\eta, \eta') \epsilon_i(\vec{k}, \eta') \). Instead, its form is
that of a nonlocal conductivity, \( \int_{\eta_0}^{\eta} K'(\eta, \eta') \partial_0' \epsilon_i(\vec{k}, \eta') \). We shall now address the relevance of these
nonlocal (potentially dissipative) contributions.

III. LATE-TIME DYNAMICS OF SUPERHORIZON PHOTONS

A. Analytical analysis

We begin by performing a partial integration on Eq. (22),

\[
(\partial^2_N + k^2) \epsilon_i(\vec{k}, \eta) + a^2 H^2 \frac{2\alpha}{\pi} \left\{ \left( \ln \left( \frac{k}{H} \right) - \frac{5}{4} + \gamma_E \right) \epsilon_i(\vec{k}, \eta)
- \int_{\eta_0}^{\eta} d\eta' \left[ \text{ci}(k \Delta \eta) + \frac{\sin(k \Delta \eta)}{k \eta'} \right] \partial_0' \epsilon_i(\vec{k}, \eta') \right\} = a^2 H^2 \frac{2\alpha}{\pi} \left[ \text{ci}(k \Delta \eta_0) + \frac{\sin(k \Delta \eta_0)}{k \eta_0} \right] \epsilon_i(\vec{k}, \eta_0). \tag{23}
\]

It is now convenient to rewrite this equation in terms of the number of e-foldings, \( N \equiv \ln(a) = -\ln(-H \eta) \),

\[
(\partial_N^2 + \partial_N + w^2 a^{-2}) \tilde{\epsilon}_i(w, N) + \frac{2\alpha}{\pi} \left\{ \left( \ln (w) - \frac{5}{4} + \gamma_E \right) \tilde{\epsilon}_i(w, N)
- \int_0^N dN' \left[ \text{ci}(w|a'^{-1} - a^{-1}|) - \frac{\sin(w|a'^{-1} - a^{-1}|)}{wa'^{-1}} \right] \partial_0' \tilde{\epsilon}_i(w, N') \right\} = \frac{2\alpha}{\pi} \left[ \text{ci}(w(1 - 1/a)) - \frac{\sin(w(1 - 1/a))}{w} \right] \tilde{\epsilon}_i(w, 0). \tag{24}
\]

The new dependent variable is, \( \tilde{\epsilon}_i(w, N) \equiv \epsilon_i(k, \eta) \), where \( w \equiv k/H \gg 1 \). The scale factors are
\( a = e^N \) and \( a' = e^{N'} \). When \( N \) becomes large the terms containing negative powers of \( a \) can be
dropped to give,

\[
(\partial_N^2 + \partial_N) \tilde{\epsilon}_i(w, N) + \frac{2\alpha}{\pi} \left\{ \left( \ln (w) - \frac{5}{4} + \gamma_E \right) \tilde{\epsilon}_i(w, N)
- \int_0^N dN' \left[ \text{ci}(wa'^{-1}) - \frac{\sin(wa'^{-1})}{wa'^{-1}} \right] \partial_N \tilde{\epsilon}_i(w, N') \right\} \approx \frac{2\alpha}{\pi} \left[ \text{ci}(w) - \frac{\sin(w)}{w} \right] \tilde{\epsilon}_i(w, 0). \tag{25}
\]

It is easy to see from equation (25) that \( \tilde{\epsilon}_i(w, N) \) approaches a constant plus an exponentially
small term. First note that the assumption is self consistent. If we make it then, for large enough
N', the integrand of the nonlocal term falls off. This means that the integral is independent of its upper limit and the constant is determined by the relation,
\[
\left(\ln(w) - \frac{5}{4} + \gamma_E\right) \tilde{\epsilon}_i(w, \infty) - \int_0^\infty dN' [ci(\omega a'^{-1}) - \sin(\omega a'^{-1})] \partial_N \tilde{\epsilon}_i(w, N') \simeq [ci(w) - \sin(w)] \tilde{\epsilon}_i(w, 0).
\] (26)

However, because the integral depends upon \(\partial_N \tilde{\epsilon}_i(w, N')\) for finite \(N'\), when it is still significant, the actual value of \(\tilde{\epsilon}_i(w, \infty)\) must be determined numerically.

To get the rate at which \(\tilde{\epsilon}_i(w, N)\) approaches \(\tilde{\epsilon}_i(w, \infty)\) we differentiate (25) with respect to \(N\).

Neglecting exponentially small terms gives the following local equation for \(\partial_N \tilde{\epsilon}_i(w, N)\),
\[
\left\{ \partial_N^2 + \partial_N + \frac{2\alpha}{\pi} \left( N - \frac{1}{4} \right) \right\} \partial_N \tilde{\epsilon}_i(w, N) \simeq 0.
\] (27)

The general solution can be expressed in terms of Airy functions,
\[
\partial_N \tilde{\epsilon}_i(w, N) \simeq a^{-1/2} \left\{ C_A \text{Ai}\left( -\frac{2\alpha}{\pi} \right)^{1/3} \left( N - \frac{1}{4} - \frac{\pi}{8\alpha} \right) \right\} + C_B \text{Bi}\left( -\frac{2\alpha}{\pi} \right)^{1/3} \left( N - \frac{1}{4} - \frac{\pi}{8\alpha} \right) \right\}. \tag{28}
\]

Making use of the asymptotic expansion of the Airy functions [15], Eq. (28) can be integrated to give asymptotically,
\[
\tilde{\epsilon}_i(w, N) = \tilde{\epsilon}_i(w, \infty) + e^{-N/2} \left\{ C_A \sin\left( \varphi[N, \alpha] \right) - C_B \cos\left( \varphi[N, \alpha] \right) \right\} + O(N^{-4}), \quad N \gg \ln(w), \tag{29}
\]
where,
\[
\varphi[N, \alpha] = \frac{2}{3} \left( \frac{2\alpha}{\pi} \right)^{1/2} \left( N - \frac{1}{4} - \frac{\pi}{8\alpha} \right)^{3/2} - \frac{\pi}{4}. \tag{30}
\]
Of course the constants \(C_A\) and \(C_B\) are fixed by the behavior of \(\tilde{\epsilon}_i(w, N)\) before the asymptotic form obtains. They can only be determined numerically.

### B. Numerical analysis

By performing a numerical study of Eq. (22), we now confirm the analytical results of section III above. The following approximation turns out to be quantitatively justified. Upon neglecting the latter two integrals, Eq. (22) reduces to,
\[
(\partial^2 + k^2)\epsilon_i(\vec{k}, \eta) + a^2 \frac{2\alpha H^2}{\pi} \left\{ \ln(a) - \gamma_E + \ln\left( \frac{H}{2\lambda} \right) \right\} \epsilon_i(\vec{k}, \eta) - \zeta(\vec{k}, \eta) \simeq 0, \tag{31}
\]
where we define,
\[
\zeta(\vec{k}, \eta) \equiv \frac{H^2}{k} \int_{\eta_0}^{\eta} d\eta' a^2 \sin(k \Delta \eta) \epsilon_i(\vec{k}, \eta'). \tag{32}
\]
FIG. 1: A comparison of the exact (numerical) solution of the integro-differential equation (22) with that of Eqs. (31-33) for $k = 40H$. At late times the photon amplitude, $\varepsilon_i$, approaches a constant. The difference between the two solutions is smaller or of the order the thickness of the lines, and quantitatively, it reaches about 0.7%.

The function $\zeta(\vec{k}, \eta)$ obeys the following differential equation and initial conditions,

$$
\left( \frac{\partial^2}{\partial \eta^2} + k^2 \right) \zeta = a^2 H^2 \varepsilon_i \\
\zeta(\vec{k}, \eta_0) = 0, \quad \left. \frac{\partial}{\partial \eta} \zeta(\vec{k}, \eta) \right|_{\eta_0} = 0.
$$

(33)

To study how quantitative this approximation is, in figure 1 we compare the exact (numerical) solution of Eq. (22) with the approximate one obtained by solving Eqs. (31-33). As can be seen in figure 1, the agreement is quite good. This implies that the two final integrals in (22) are dynamically irrelevant.

Figure 1 shows that, at asymptotically late times, $\tilde{\varepsilon}_i(w, N)$ is dominated by the constant value $\tilde{\varepsilon}_i(w, \infty)$, whose real part is slightly suppressed with respect to its classical asymptotic value of $\cos(40) \approx -0.67$. In figure 2 we subtract the asymptotic constant from the solution of Eqs. (31-33), multiply by $e^{N/2}$, and compare the result with the Airy function. The figure displays an oscillatory, slightly damped profile, in reasonably good agreement with our result (29).

IV. DISCUSSION

It is illuminating to translate our results for the vector potential, $A_i(x) = \varepsilon_i(\vec{k}, \eta)e^{i\vec{k} \cdot \vec{x}}$, into statements about the behavior of the electric and magnetic fields. In Coulomb gauge and using
conformal coordinates, the densities of electric and magnetic field lines per physical 3-volume are given by the following expressions,

$$E^i(x) = \frac{\partial_0 A^i(x)}{a^2(\eta)}, \quad B^i(x) = -\frac{\varepsilon^{ijk} \partial_j A^k(x)}{a^2(\eta)}.$$ (34)

For spatial plane waves with $a(\eta) = e^N$ and $w = k/H$ we have,

$$e^{-ik \cdot x} E^i(x) = He^{-N} \partial_N \bar{e}_i(w, N), \quad e^{-i\bar{k} \cdot x} B^i(x) = -ie^{ijk} k_j e^{-2N} \bar{e}_k(w, N).$$ (35)

Keeping track only of powers of the scale factor, we can summarize the results of section III by saying that the full quantum system results in the following asymptotic behavior,

$$\eta^{\mu\nu} \eta^{\rho\sigma} \partial_\rho F_{\sigma\nu}(x) + \int d^4x' [\eta^{\mu\nu} \Pi_{\mu\nu}^{\text{ret}}] (x, x') A_\nu(x') = 0 \implies E^i \sim a^{-\frac{3}{2}} \quad \text{and} \quad B^i \sim a^{-2}. \quad (36)$$

It is interesting to contrast the asymptotic behavior (36) of the full, nonlocal system with that of various local models. The tree order system gives,

$$\eta^{\mu\nu} \eta^{\rho\sigma} \partial_\rho F_{\sigma\nu} = 0 \implies E^i \sim a^{-2} \quad \text{and} \quad B^i \sim a^{-2}. \quad (37)$$

Although the behavior of the magnetic fields agree as regards powers of $a$, we saw in Figure I that the magnetic field of the actual system analyzed in section III B is smaller by a factor of about 2/3. On the other hand, the electric field of the actual system is enhanced by an enormous factor of $a^{\frac{1}{2}}$.

Adding a Proca term with fixed mass $m_\gamma = gH$ results in the following asymptotic behavior,

$$\eta^{\mu\nu} \left( \eta^{\rho\sigma} \partial_\rho F_{\sigma\nu} - g^2 H^2 a^2 A_\nu \right) = 0 \implies E^i \sim a^{-\frac{3}{2}} + \frac{1}{2} \sqrt{1-4g^2} \quad \text{and} \quad B^i \sim a^{-\frac{3}{2}} + \frac{1}{2} \sqrt{1-4g^2}. \quad (38)$$
The best perturbative fit to the actual system has \( g^2 = 2\alpha \ln(w)/\pi \) but we see that the agreement between (36) and (38) is not good. The constant mass system has too much electric field and not enough magnetic field.

If the mass term grows as \( m^2_\gamma = g^2H^2\ln(a) \) one finds,

\[
\eta^\mu^\nu \left( \eta^{\rho\sigma} \partial_\rho F_{\sigma\nu} - g^2H^2\ln(a)a^2 A_\nu \right) = 0 \quad \Rightarrow \quad E^i \sim a^{-\frac{3}{2}} \quad \text{and} \quad B^i \sim a^{-\frac{5}{2}}.
\]

This model is suggested by mean field theory \([9, 10, 11]\) with \( g^2 = 2\alpha/\pi \). It gives good agreement for the electric field but results in too little magnetic field. In particular note that, whereas the actual polarization vector approaches a nonzero constant, \( \bar{\epsilon}_i(w, \infty) \), the polarization vector of (39) approaches zero.

While none of the local models described above gives a perfect fit, the effect of vacuum polarization is to vastly enhance the asymptotic electric field with respect to its classical value in a manner most nearly described by the growing mass of (39). Without regard to local analogues, it is very apparent that vacuum polarization alters the kinematical properties of superhorizon photons during inflation. On the other hand, the fact that the asymptotic polarization vector is somewhat suppressed from its classical value seems to indicate that there is no significant production of photons.

Exactly the opposite results seem to pertain for massless fermions which are Yukawa coupled to a massless, minimally coupled scalar \([12]\). In that case chirality conservation protects the particle from gaining a mass, but the amplitude of the wave function grows faster than exponentially during inflation. The physical explanation seems to be copious particle production.

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Appendix: Reduction of the polarization integral

Here we show some details of the calculation of the inflationary contribution to the vacuum polarization integral in (8) with the general plane wave (19) for the photon field, $A_\nu(x') = \epsilon_\nu(k, \eta') e^{i\vec{k} \cdot \vec{x}'}$ ($k_i \epsilon_i = 0 = \epsilon_0$).

We begin our analysis by noting a useful identity,

$$- [\mu P^\nu] aa' = - a a' \delta_0^\mu \delta_0^\nu \nabla^2 + \bar{\eta}^{\mu \nu} [a a' (\nabla^2 - \partial_0^2) + H^2 a^2 a'^2 (1 - \Delta \eta \partial_0)] - H a a'^2 \delta_0^\nu + H a a' \bar{\delta}_0^\nu + aa' (\bar{\partial}_0^\mu \delta_0^\nu + \delta_0^\mu \bar{\partial}_0^\nu) \partial_0 - a a' \bar{\partial}^\mu \bar{\partial}_0^\nu. \quad (40)$$

In deriving this we made use of the relations,

$$\partial_\rho aa' = a a' \partial_\rho + H a a'^2 \delta_0^\rho \quad \text{and} \quad \partial_\sigma a a' = a a' \partial_\sigma + H a a'^2 \delta_0^\sigma, \quad (41)$$

which imply,

$$\partial_\rho' \partial_\sigma aa' = H^2 a^2 a'^2 \delta_0^\rho \delta_0^\sigma + H a a'^2 \delta_0^\rho \partial_\sigma - H a a'^2 \partial_\rho' \delta_0^\sigma - a a' \partial_\rho' \partial_\sigma. \quad (42)$$

Note also that we employ the notation,

$$\bar{\partial}^\mu \equiv \eta^{\mu i} \partial_i, \quad \bar{\eta}^{\mu \nu} \equiv \eta^{\mu \nu} + \delta_0^\mu \delta_0^\nu. \quad (43)$$

When a derivative operator stands on the right of $aa'$, then it acts on a function of $x^\mu - x'^\mu$ and we can write $\partial_\rho' = - \partial_\rho$. Moreover, due to the (spatial) transversality of the photon field, $\bar{\partial}_0^\nu A_\nu(x) = 0$, the terms in (40) containing $\bar{\partial}_0^\nu$ vanish identically.

Consider first the 0-th component of the photon field equation (8),

$$- \nabla^2 A_0 - \frac{\alpha H^2}{8 \pi^2} a \nabla^2 \partial^2 \int d^4 x' \theta(\Delta \eta) \theta(\Delta \tau^2) \left[ \ln(H^2 \Delta \tau^2) + 1 \right] a' A_0(x') = 0, \quad (44)$$

where we made use of Eqs. (17-18) and (40), and of the Coulomb gauge condition, $\nabla \cdot \vec{A} = 0$. Note that only the first term in (40) contributes to (44). Furthermore, due to the spatial transverse structure of the Lorentz breaking term, $[^\mu \bar{P}^\nu] \delta n^2$, it does not contribute to (44).

At $\eta = \eta_0$ the integral drops out of (44) and we conclude that $\nabla^2 A_0(\vec{x}, \eta_0) = 0$. We can make use of residual gauge freedom to make the unique solution be $A_0(\vec{x}, \eta_0) = 0$. The same thing can be done for the first conformal time derivative. But then the equation implies that $A_0(\vec{x}, \eta)$ vanishes for all $\eta$. Therefore, it is consistent to assume $\epsilon_0(\vec{k}, \eta) = 0$. This discussion also implies that the terms in (40) which contain $\delta_0^\nu$ cannot contribute to the photon dynamical equation (8).
Consider now the spatial components of equation (8). We start with the de Sitter contribution to the electric susceptibility, \( \chi_e \) of de Sitter \( [17] \), which with the help of \( [10] \) (only the terms proportional to \( \eta^{\mu
u} \) contribute),

\[
-\eta_{\mu \nu} \int d^4x' [\mu P^\nu] \chi_e (x, x') |_{\text{de Sitter}} A_\nu (x') = \frac{\alpha H^2}{2\pi k^3} \alpha e^{i \vec{k} \cdot \vec{x}} (\partial_0^2 + k^2) \int_{\eta_0}^\eta d\eta' a' (k \Delta \eta, \frac{k}{H'}) \epsilon_i (k, \eta') \\
- \frac{\alpha H^4}{2\pi k^3} \alpha e^{i \vec{k} \cdot \vec{x}} (\partial_0^2 + k^2) \int_{\eta_0}^\eta d\eta' a'^2 (1 - \Delta \eta \partial_0) \Xi_1 (k \Delta \eta, \frac{k}{H'}) \epsilon_i (k, \eta').
\]

(45)

Here \( \eta_0 = -1/H \) denotes the conformal time at the beginning of inflation, at which \( a(\eta_0) = 1 \). We define \( k \equiv \| \vec{k} \|, \Delta \eta = \eta - \eta' \). We also made use of the following elementary integrals,

\[
\int_0^{\Delta \eta} d^3 \vec{x} e^{-i \vec{k} \cdot \vec{x}} \left\{ \ln \left[ H^2 (\Delta \eta^2 - \| \Delta \vec{x} \|^2) + 1 \right] \right\} = 4\pi \int_0^{\Delta \eta} r^2 dr \frac{\sin(kr)}{kr} \left\{ \ln \left[ H^2 (\Delta \eta^2 - \| \Delta \vec{x} \|^2) + 1 \right] \right\} = 4\pi \int_0^{\Delta \eta} \frac{k}{k^2} \Xi_1 (k \Delta \eta, k/H').,
\]

(46)

where \( \Delta \vec{x} = \vec{x} - \vec{x}' \),

\[
\Xi_n (z, \zeta) \equiv [\sin(z) - x \cos(z)] [2 \ln(z/\zeta) + n] + z^2 \xi(z)
\]

\[
z^2 \xi(z) \equiv z^2 \int_0^1 z' d' \sin(z') \ln(1 - z'^2)
\]

\[
= 2 \sin(z) - [\cos(z) + z \sin(z)] [\sin(2z) + \pi/2] + [\sin(z) - z \cos(z)] [\cos(2z) - \gamma_E - \ln(z/2)]
\]

\[
\text{si}(z) \equiv - \int_z^\infty \frac{\sin(t)}{t} dt = \int_0^z \frac{\sin(t)}{t} dt - \frac{\pi}{2}
\]

\[
\text{ci}(z) \equiv - \int_z^\infty \frac{\cos(t)}{t} dt = \int_0^z \frac{\cos(t) - 1}{t} dt + \gamma_E + \ln(z),
\]

(47)

and \( \gamma_E \equiv - \psi(1) = -(d/dz) \ln[\Gamma(z)] \mid_{z=1} = 0.577 215 664... \) is Euler’s constant. Now making use of,

\[
\Xi_1 (z, \zeta) \equiv \frac{1}{2} (\partial_z^2 + 1) \Xi_1 (z, \zeta) = \sin(z) [2 \ln(2z/\zeta) + 2 + \text{ci}(2z) - \gamma_E - \ln(2z)] - \cos(z) \text{si}(2z) + \pi/2
\]

\[
(z \partial_z - 1)(\partial_z^2 + 1) \Xi_1 (z, \zeta) = 4 \sin(z) - 2 \Xi_2 (z, \zeta),
\]

(49)

the integral \( [15] \) reduces to

\[
-\eta_{\mu \nu} \int d^4x' [\mu P^\nu] \chi_e (x, x') |_{\text{de Sitter}} A_\nu (x') = \frac{\alpha H^2}{\pi k} \alpha e^{i \vec{k} \cdot \vec{x}} (\partial_0^2 + k^2) \int_{\eta_0}^\eta d\eta' a' (k \Delta \eta, \frac{k}{H'}) \epsilon_i (k, \eta') \\
+ \frac{\alpha H^4}{\pi k} \alpha^2 e^{i \vec{k} \cdot \vec{x}} \int_{\eta_0}^\eta d\eta' a'^2 \left[ 2 \sin(k \Delta \eta) - \Xi_2 (k \Delta \eta, \frac{k}{H'}) \right] \epsilon_i (k, \eta').
\]

(50)
The contribution from $\delta n^2$ to the spatial components of Eq. (8) can be obtained quite easily by the methods analogous to those employed for the contribution from $\chi_c|_{\text{de Sitter}}$; the result can be written as,

$$\eta_{\mu} \int d^4x'[\mu\Phi^\nu]\delta n^2(x,x') A_\nu(x') = \frac{\alpha H^4}{\pi k}a^2e^{i\vec{k}\cdot\vec{x}} \int_{\eta_0}^{\eta} d\eta' a^2\Xi(z/(k H)) \epsilon_i(k,\eta').$$  \hspace{1cm} (51)

Remarkably, this contribution is fully canceled by the analogous one in (50). Keeping the terms that contribute at order $a^2$, we have,

$$\eta_{\mu} \int d^4x'[\mu\Pi^\nu_{\text{rel}}](x,x') A_\nu(x') = \eta_{\mu} \int d^4x'(-[\mu\Phi^\nu]\chi_c(x,x')|_{\text{de Sitter}} + [\mu\Phi^\nu]\delta n^2(x,x')) A_\nu(x') + O(a) \hspace{1cm}$$

$$= \frac{\alpha H^2}{\pi k}a^2e^{i\vec{k}\cdot\vec{x}}(\partial_0^2 + k^2) \int_{\eta_0}^{\eta} d\eta' a^2 Y_1(k\Delta\eta, k/\eta') \epsilon_i(k,\eta')$$

$$+ \frac{2\alpha H^4}{\pi k}a^2e^{i\vec{k}\cdot\vec{x}} \int_{\eta_0}^{\eta} d\eta' a^2 \sin(k\Delta\eta) \epsilon_i(k,\eta') + O(a),$$

where $O(a)$ indicates that the neglected terms can contribute to the vacuum polarization at most linearly in $a$.

Now from the small argument expansions of the sine and cosine integral (cf. Eq. [8.232] in Ref. [13]),

$$\text{si}(z) = -\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}z^{2n-1}}{(2n-1)(2n-1)!}, \hspace{1cm} |z| \ll 1$$

$$\text{ci}(z) = \gamma_E + \ln(z) + \sum_{n=1}^{\infty} \frac{(-1)^nz^{2n}}{(2n)(2n)!}, \hspace{1cm} |z| \ll 1$$

we obtain easily the following small argument expansion of $Y_1$,

$$Y_1(z,\zeta) = 2z \ln\left(\frac{2z}{\zeta}\right) - \frac{1}{3}z^3\left[\ln\left(\frac{2z}{\zeta}\right) - \frac{31}{12}\right] + O(z^5 \ln(z), z^5).$$  \hspace{1cm} (54)

This implies that, attempting to act with the derivative $\partial_0^2$ on the integral in (52), would result in a divergent contribution at the upper limit of integration, $\eta' = \eta$. In order to overcome this difficulty, observe first that one time derivative may be taken,

$$\frac{\partial}{\partial \eta} \int_{\eta_0}^{\eta} d\eta' a' Y_1(\Delta\eta, k/\eta') \epsilon_i(k,\eta') = \int_{\eta_0}^{\eta} d\eta' a'[\partial Z Y_1(\Delta\eta, k/\eta')] \bigg|_{z=k\Delta\eta} \epsilon_i(k,\eta').$$

Note we can write,

$$\partial Z Y_1(z,\zeta) = \cos(z)[2 \ln(2z/\zeta) + 2 + \text{ci}(2z) - \gamma_E - \ln(2z)] + \sin(z)[\text{si}(2z) + \pi/2]$$

$$= \left[\partial Z Y_1(z,\zeta) - 2 \ln(2z/\zeta) - 2\right] + 2 \ln(2z/\zeta) + 2,$$  \hspace{1cm} (56)
where in the last line we have added and subtracted the troublesome term which is logarithmically divergent at the upper limit of integration, \( \eta' \to \eta \). We can now act with the second time derivative on the first term, to arrive at,

\[
\frac{\partial^2}{\partial \eta^2} + k^2 \int_{\eta_0}^{\eta} d\eta' a' Y_1 \left( k \Delta \eta, \frac{k}{H} \right) \epsilon_i(k, \eta') = -2 \int_{\eta_0}^{\eta} d\eta' a' \frac{1 - \cos(k \Delta \eta)}{\Delta \eta} \epsilon_i(k, \eta') + 2 \partial_0 \int_{\eta_0}^{\eta} d\eta' a' \left[ \ln(2H' \Delta \eta) + 1 \right] \epsilon_i(k, \eta').
\]

(57)

Now from,

\[
a' \ln(2H' \Delta \eta) = -\frac{1}{H\eta'} \left[ \ln(-2H' \eta') - \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{n}{\eta'} \right)^n \right]
\]

we can rewrite the second integral in (57) as follows,

\[
\int_{\eta_0}^{\eta} d\eta' a' \ln(2H' \Delta \eta) \epsilon_i(k, \eta') = -\frac{1}{H} \left\{ \frac{1}{2} \ln^2(-2H' \eta') + \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{\eta}{\eta'} \right)^n \right\} \bigg|_{\eta' = \eta_0} + \int_{\eta_0}^{\eta} d\eta' \left( \frac{1}{2} \ln(-2H' \eta') + \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{n}{\eta'} \right)^n \right) \frac{d}{d\eta'} \epsilon_i(k, \eta').
\]

(58)

Its time derivative equals,

\[
\partial_0 \int_{\eta_0}^{\eta} d\eta' a' \left[ \ln(2H' \Delta \eta) + 1 \right] \epsilon_i(k, \eta') = a \left[ \ln(-2H' \eta) + 1 \right] \epsilon_i(k, \eta) + a \ln \left( 1 - \frac{\eta}{\eta_0} \right) \epsilon_i(k, \eta_0)
\]

\[
+ a \int_{\eta_0}^{\eta} d\eta' \ln \left( 1 - \frac{\eta}{\eta'} \right) \frac{d}{d\eta'} \epsilon_i(k, \eta').
\]

(59)

Upon collecting all of the terms, Eq. (52) reduces to,

\[
\eta_{ijkl} \int d^4 x' \Pi_{rel}^l(x, x') A_{ij}(x') = -a^2 \frac{2 \alpha H^2}{\pi} e^{i \vec{k} \cdot \vec{x}} \left[ - \left( \ln(-2H' \eta) + 1 \right) \epsilon_i(k, \eta)
\]

\[
- \frac{H^2}{\alpha} \int_{\eta_0}^{\eta} d\eta' a' a'' \sin(k \Delta \eta) \epsilon_i(k, \eta') + \frac{1}{a} \int_{\eta_0}^{\eta} d\eta' a' \frac{1 - \cos(k \Delta \eta)}{\Delta \eta} \epsilon_i(k, \eta')
\]

\[
- \int_{\eta_0}^{\eta} d\eta' \ln \left( 1 - \frac{\eta}{\eta'} \right) \frac{d}{d\eta'} \epsilon_i(k, \eta')
\]

\[= \ln \left( 1 - \frac{1}{a} \right) \epsilon_i(k, \eta_0) \right],
\]

(60)

where we took account of \( \eta_0 = -1/H \). When \( a \gg 1 \), the last local term, which depends on the initial photon amplitude \( \epsilon_i(k, \eta_0) \), contributes as \( O(a^{-1}) \), such that, in the limit when \( a \gg 1 \), it
can be consistently neglected.

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[14] We thank Ewald Puchwein for pointing out the improvement.

[15] The asymptotic expansion for the Airy functions,

$$\text{Ai}(-z) \sim \frac{1}{\pi^{1/2} z^{1/4}} \left[ \cos \left( \frac{2}{3} z^{3/2} - \frac{\pi}{4} \right) + O(z^{-3/2}) \right], \quad |z| \gg 1$$

$$\text{Bi}(-z) \sim \frac{1}{\pi^{1/2} z^{1/4}} \left[ \sin \left( \frac{2}{3} z^{3/2} - \frac{\pi}{4} \right) + O(z^{-3/2}) \right], \quad |z| \gg 1.$$