Clifford (Geometric) Algebra Wavelet Transform

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May 7, 2014

Abstract

While the Clifford (geometric) algebra Fourier Transform (CFT) is global, we introduce here the local Clifford (geometric) algebra (GA) wavelet concept. We show how for \( n = 2, 3 \mod 4 \) continuous \( Cl_n \)-valued admissible wavelets can be constructed using the similitude group \( SIM(n) \). We strictly aim for real geometric interpretation, and replace the imaginary unit \( i \in \mathbb{C} \) therefore with a GA blade squaring to \(-1\). Consequences due to non-commutativity arise. We express the admissibility condition in terms of a \( Cl_n \) CFT and then derive a set of important properties such as dilation, translation and rotation covariance, a reproducing kernel, and show how to invert the Clifford wavelet transform. As an explicit example, we introduce Clifford Gabor wavelets. We further invent a generalized Clifford wavelet uncertainty principle. Extensions of CFTs and Clifford wavelets to \( Cl_{0,n'}, n' = 1, 2 \mod 4 \) appear straight forward.

Keywords: Clifford geometric algebra, Clifford wavelet transform, multidimensional wavelets, continuous wavelets, similitude group.

AMS Subj. Class.: 15A66, 42C40, 94A12.

1 Introduction

The meaning and importance of wavelets is clearly seen in a biographical note on J. P. Morlet: Following in the footsteps of Denis Gabor (father of holography), Morlet was disconcerted by the poor results he [Gabor] obtained; but, being inquisitive and persistent, he asked himself, "Why?" and immediately provided the answer. Gabor paved the time-frequency plane in uniform cells and associated each cell with a wave shape of invariant envelope with a carrier of variable frequency. Morlet kept the constraint resulting from the uncertainty principle applied to time and frequency, but he perceived that it was the wave shape that must be invariant to give uniform resolution in the entire plane. For this he adapted the sampling rate to the frequency, thereby creating, in effect, a changing time scale producing a stretching of the wave shape. Today the wavelet transform is also called the "time-scale analysis" approach, which is comparable to the conventional time-frequency analysis. . . . It has been rediscovered as a very useful tool, particularly in data compression where it can produce significant savings in storage and transmission costs but also in mathematics, data processing, communications, image analysis, and many other engineering problems.\[1\]

In order to favorably combine wavelet techniques with Clifford (geometric) algebra, which provides a complete algebra of a vector space and all its subspaces, several efforts have been undertaken. They include Clifford multi resolution analysis (MRA) \[2\], quaternion MRA \[4\], Clifford wavelet networks, quaternion wavelet transforms (QWT) applied to image analysis (using the QWT phase concept), image processing and motion estimation \[5\], quaternion-valued admissible wavelets, Clifford algebra-valued admissible (continuous) wavelets using complex Fourier transforms for the spectral representation \[6\], monogenic wavelets over the unit ball \[7\], Clifford continuous wavelet transforms (ContWT) in \( L_{0,2}, L_{0,3} \), wavelets on the 3D sphere...
with Cauchy kernel in Clifford analysis (2009), diffusion wavelets [8], ContWT in Clifford analysis, wavelet frames on the sphere, benchmarking of 3D Clifford wavelet functions, metric dependent Clifford analysis, new multivariable polynomials and associated ContWT: Clifford versions of Hermite, Hermitean Clifford-Hermite, bi-axial Clifford-Hermite, Jacobi, Gegenbauer, Laguerre, and Bessel polynomials [3].

Fourier transformations have been successfully developed in the framework of real Clifford (geometric) algebra (GA), replacing the imaginary unit \(i\) by a geometric (GA) square root of \(-1\) [9]. These Clifford Fourier transformations (CFT) [10–12] have already found interesting applications in vector field analysis and pattern matching [17]. A special case are the so-called quaternion Fourier transforms (QFT) [13–15].

In Section 2 Clifford (geometric) algebra is introduced including multivector signal functions, the Clifford Fourier transform, and the similitude group of dilations, rotations and translations. Section 3 defines Clifford mother and daughter wavelets, spectral representation, discusses admissibility, the Clifford wavelet transformation and its spectral CFT representation. This is followed by a detailed discussion of Clifford wavelet properties, i.e. linearity, covariance w.r.t. dilation, rotation and translation, inner product and norm relations, the inverse Clifford wavelet transform, a reproducing kernel and a Clifford wavelet uncertainty principle. Finally the example of Clifford Gabor wavelets is given.

2 Clifford (geometric) algebra and multivector signals

2.1 Clifford (geometric) algebra

Clifford (geometric) algebra is based on the geometric product of vectors \(a, b \in \mathbb{R}^{pq}, p + q = n\)

\[
a b = a \cdot b + a \wedge b, \tag{1}
\]

and the associative algebra \(Cl_{pq}\) thus generated with \(\mathbb{R}\) and \(\mathbb{R}^{pq}\) as subspaces of \(Cl_{pq}\). \(a \cdot b\) is the symmetric inner product of vectors and \(a \wedge b\) is Grassmann’s outer product of vectors representing the oriented parallelogram area spanned by \(a, b\).

As an example we take the Clifford geometric algebra \(Cl_3 = Cl_{3,0}\) of three-dimensional (3D) Euclidean space \(\mathbb{R}^3 = \mathbb{R}^{3,0}\). \(\mathbb{R}^3\) has an orthonormal basis \(\{e_1, e_2, e_3\}\). \(Cl_3\) then has an eight-dimensional basis of

\[
\{1, e_1, e_2, e_3, e_2 e_3, e_3 e_1, e_1 e_2, i = e_1 e_2 e_3\}. \tag{2}
\]

Here \(i\) denotes the unit trivector, i.e. the oriented volume of a unit cube, with \(i^2 = -1\). The even grade subalgebra \(Cl^+_3\) is isomorphic to Hamilton’s quaternions \(\mathbb{H}\). Therefore elements of \(Cl^+_3\) are also called rotors (rotation operators), rotating vectors and multivectors of \(Cl_3\).

In general \(Cl_{p,q}, p + q = n\) is composed of so-called \(r\)-vector subspaces spanned by the induced bases

\[
\{e_{k_1} e_{k_2} \ldots e_{k_r} \mid 1 \leq k_1 < k_2 < \ldots < k_r \leq n\}, \tag{3}
\]

each with dimension \(\binom{n}{r}\). The total dimension of the \(Cl_{p,q}\) therefore becomes \(\sum_{r=0}^{n} \binom{n}{r} = 2^n\).

General elements called multivectors \(M \in Cl_{p,q}\) have \(k\)-vector parts \((0 \leq k \leq n)\): scalar part \(Sc(M) = \langle M \rangle = \langle M \rangle_0 = M_0 \in \mathbb{R}\), vector part \(\langle M \rangle_1 \in \mathbb{R}^{pq}\), bi-vector part \(\langle M \rangle_2\), ..., and pseudoscalar part \(\langle M \rangle_n \in \Lambda^n \mathbb{R}^{pq}\)

\[
M = \sum_{A=1}^{2^n} M_A e_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n. \tag{4}
\]

The reverse of \(M \in Cl_{p,q}\) defined as

\[
\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k, \tag{5}
\]

often replaces complex conjugation and quaternion conjugation. Taking the reverse is equivalent to reversing the order of products of basis vectors in the basis blades of [3]. The scalar product of two multivectors \(M, \widetilde{N} \in Cl_{p,q}\) is defined as

\[
M \ast \widetilde{N} = \langle M \widetilde{N} \rangle = \langle M \widetilde{N} \rangle_0. \tag{6}
\]

For \(M, \widetilde{N} \in Cl_n = Cl_{n,0}\) we get \(M \ast \widetilde{N} = \sum_A M_A N_A\). The modulus \(|M|\) of a multivector \(M \in Cl_n\) is defined as

\[
|M|^2 = M \ast \widetilde{M} = \sum_A M_A^2. \tag{7}
\]
For $n\equiv 2(\text{mod } 4)$ and $n\equiv 3(\text{mod } 4)$ the pseudoscalar is $i_n = e_1 e_2 \ldots e_n$ with (also valid for $Cl_{0,n'}$, $n' = 1, 2, 3(\text{mod } 4)$)

$$i_n^2 = -1. \quad (8)$$

A blade $B_k = b_1 \wedge b_2 \ldots \wedge b_k, b_i \in \mathbb{R}^{p,q}, 1 \leq l \leq k \leq n = p + q$ describes a $k$-dimensional vector subspace

$$V_B = \{x \in \mathbb{R}^{p,q}|x \wedge B = 0\}. \quad (9)$$

Its dual blade

$$B^* = B_i n^{-1} \quad (10)$$
describes the complimentary ($n-k$)-dimensional vector subspace $V_B^\perp$. The pseudoscalar $i_n \in Cl_n$ is central for $n\equiv 3(\text{mod } 4)$

$$i_n M = M i_n, \quad \forall M \in Cl_n. \quad (11)$$

But for even $n$ we get due to non-commutativity [11] of the pseudoscalar $i_n \in Cl_n$ for all $M \in Cl_n, \lambda \in \mathbb{R}$

$$i_n M = M_{\text{even}} i_n - M_{\text{odd}} i_n, \quad (12)$$

$$e^{i_n \lambda} M = M_{\text{even}} e^{i_n \lambda} + M_{\text{odd}} e^{-i_n \lambda}. \quad (13)$$

### 2.2 Multivector signal functions

A multivector valued function $f : \mathbb{R}^{p,q} \rightarrow Cl_{p,q}, p + q = n$, has $2^n$ blade components ($f_A : \mathbb{R}^{p,q} \rightarrow \mathbb{R}$)

$$f(x) = \sum_A f_A(x) e_A. \quad (14)$$

We define the inner product of $\mathbb{R}^n \rightarrow Cl_n$ functions $f, g$ by

$$(f, g) = \int_{\mathbb{R}^n} f(x) g(x) d^n x \quad (15)$$

$$= \sum_{A,B} e_A e_B \int_{\mathbb{R}^n} f_A(x) g_B(x) d^n x, \quad (16)$$

and the $L^2(\mathbb{R}^n; Cl_n)$-norm

$$\|f\|^2 = \langle (f, f) \rangle = \int_{\mathbb{R}^n} |f(x)|^2 d^n x$$

$$= \sum_A \int_{\mathbb{R}^n} f_A^2(x) d^n x, \quad (17)$$

$$L^2(\mathbb{R}^n; Cl_n) = \{f : \mathbb{R}^n \rightarrow Cl_n | \|f\| < \infty\}. \quad (18)$$

For the Clifford geometric algebra Fourier transformation (CFT) [11] the complex unit $i \in \mathbb{C}$ is replaced by some geometric (square) root of $-1$, e.g. pseudoscalars $i_n, n = 2, 3(\text{mod } 4)$. Complex functions $f$ are replaced by multivector functions $f \in L^2(\mathbb{R}^n; Cl_n)$.

**Definition 1** (Clifford geometric algebra Fourier transformation (CFT)). The Clifford GA Fourier transform $\mathcal{F}\{f\} : \mathbb{R}^n \rightarrow Cl_n, n = 2, 3(\text{mod } 4)$ is given by

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^n} f(x) e^{-i_n \omega \cdot x} d^n x, \quad (19)$$

for multivector functions $f : \mathbb{R}^n \rightarrow Cl_n$.

The CFT [11] is inverted by

$$f(x) = \mathcal{F}^{-1}\{\mathcal{F}\{f\}(\omega)\}$$

$$= \left(\frac{-1}{2\pi n}\right) \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\omega) e^{i_n \omega \cdot x} d^n x. \quad (20)$$

The similitude group $\mathcal{G} = SIM(n)$ of dilations, rotations and translations is a subgroup of the affine group of $\mathbb{R}^n$

$$\mathcal{G} = \mathbb{R}^+ \times SO(n(n) \times \mathbb{R}^n$$

$$= \{(a, r_\theta, b) | a \in \mathbb{R}^+, r_\theta \in SO(n), b \in \mathbb{R}^n\}. \quad (21)$$

The left Haar measure on $\mathcal{G}$ is given by

$$d\lambda = d\lambda(a, \theta, b) = d\mu(a, \theta) d^m b, \quad (22)$$

$$d\mu = d\mu(a, \theta) = \frac{d\theta}{a^{n+1}}. \quad (23)$$

where $d\theta$ is the Haar measure on $SO(n)$. For example

$$d\theta = \begin{cases} \frac{d\theta}{2\pi}, & n = 2 \\ \frac{1}{8\pi} \sin \theta_1 d\theta_1 d\theta_2 d\theta_3, & n = 3 \end{cases}. \quad (24)$$

We define the inner product of $f, g : \mathcal{G} \rightarrow Cl_n$ by

$$(f, g) = \int_{\mathcal{G}} f(a, \theta, b) g(a, \bar{\theta}, b) d\lambda(a, \theta, b), \quad (25)$$

and the $L^2(\mathcal{G}; Cl_n)$-norm

$$\|f\|^2 = \langle (f, f) \rangle = \int_{\mathcal{G}} |f(a, \theta, b)|^2 d\lambda, \quad (26)$$

$$L^2(\mathcal{G}; Cl_n) = \{f : \mathcal{G} \rightarrow Cl_n | \|f\| < \infty\}. \quad (27)$$

The CFT can be defined analogously for $Cl_{0,n'}$, $n' = 1, 2(\text{mod } 4)$.
The variations of a multivector signal $f \in L^2(\mathbb{R}^n; Cl_n)$ in position $x \in \mathbb{R}^n$ and frequency $\omega \in \mathbb{R}^n$ are related by the CFT uncertainty principle \cite{1111921}
\[
\|xf\|^2_{L^2(\mathbb{R}^n; Cl_n)} \|\omega f\|^2_{L^2(\mathbb{R}^n; Cl_n)} \geq n \left(\frac{2\pi}{4}\right)^n \|f\|^4_{L^2(\mathbb{R}^n; Cl_n)}.
\] (28)

3 Clifford GA wavelets

3.1 Real admissible continuous Clifford GA wavelets

We represent the transformation group $G = SIM(n)$ by applying translations, scaling and rotations to a so-called Clifford mother wavelet $\psi : \mathbb{R}^n \rightarrow Cl_n$
\[
\psi(x) \mapsto \psi_{a, \theta, b}(x) = \frac{1}{a^{n/2}} \psi(r_{\theta}^{-1}(x - b)) \left(\frac{a}{b}\right).
\] (29)

The family of wavelets $\psi_{a, \theta, b}$ are so-called Clifford daughter wavelets.

Lemma 1 (Norm identity). The factor $a^{-n/2}$ in $\psi_{a, \theta, b}$ ensures (independent of $a, \theta, b$) that
\[
\|\psi_{a, \theta, b}\|_{L^2(\mathbb{R}^n; Cl_n)} = \|\psi\|_{L^2(\mathbb{R}^n; Cl_n)}.
\] (30)

Proof.
\[
\|\psi_{a, \theta, b}\|^2_{L^2(\mathbb{R}^n; Cl_n)} = \int_{\mathbb{R}^n} \sum_A \frac{1}{a^n} \psi_A(r_{\theta}^{-1}(x - b)) d^n x = \int_{\mathbb{R}^n} \sum_A \psi_A^2(z) z^n d^n z = \int_{\mathbb{R}^n} \psi^2(z) d^n z = \|\psi\|^2_{L^2(\mathbb{R}^n; Cl_n)}.
\] (31)

The spectral CFT representation of Clifford daughter wavelets is
\[
\mathcal{F}\{\psi_{a, \theta, b}\}(\omega) = a^n \hat{\psi}(ar_{\theta}^{-1}(\omega)) e^{-i a b \omega}.
\] (32)

In the proof of (32) the CFT properties of scaling, $x$-shift and rotation are applied. A Clifford mother wavelet $\psi \in L^2(\mathbb{R}^n; Cl_n)$ is admissible if
\[
C_\psi = \int_{\mathbb{R}^n} \int_{SO(n)} a^n \{\hat{\psi}(ar_{\theta}^{-1}(\omega))\} \hat{\psi}(ar_{\theta}^{-1}(\omega)) d\mu
= \int_{\mathbb{R}^n} \hat{\psi}(\omega) \hat{\psi}(\omega) \|\omega\|^n d^n \omega,
\] (33)
is an invertible multivector constant and finite at a.e. $\omega \in \mathbb{R}^n$. We must therefore have $\hat{\psi}(\omega = 0) = 0$
\[
\hat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x) e^{i a b} \hat{\psi}(a^2 \omega) d^n x = \int_{\mathbb{R}^n} \psi(x) d^n x
= \sum_A \int_{\mathbb{R}^n} \psi_A(x) d^n x e_A = 0,
\] (34)
and therefore for all $2^n$ Clifford mother wavelet components
\[
\int_{\mathbb{R}^n} \psi_A(x) d^n x = 0.
\] (35)

By construction $C_\psi = C_\psi$. Hence for $n = 2, 3(\text{mod} 4)$
\[
C_\psi = \langle C_\psi \rangle_0 + \langle C_\psi \rangle_1 + \langle C_\psi \rangle_4 + \langle C_\psi \rangle_5 + \ldots
= \sum_{k=0}^{[n/4]} \langle C_\psi \rangle_{4k} + \langle C_\psi \rangle_{4k+1},
\] (36)
and
\[
\langle C_\psi \rangle_0 = \int_{\mathbb{R}^n} \|\hat{\psi}(\xi)\|^2 \frac{1}{|\xi|^n} d\xi^n > 0.
\] (37)

The invertibility of $C_\psi$ depends on its grade content, e.g. for $n = 2, 3, C_\psi$ is invertible, if and only if $\langle C_\psi \rangle_1^2 \neq \langle C_\psi \rangle_0^2$
\[
C_\psi^{-1} = \frac{\langle C_\psi \rangle_0 - \langle C_\psi \rangle_1}{\langle C_\psi \rangle_0 - \langle C_\psi \rangle_1}
\] (38)

Definition 2 (Clifford GA wavelet transformation (CWT)). For an admissible GA mother wavelet $\psi \in L^2(\mathbb{R}^n; Cl_n)$ and a multivector signal function $f \in L^2(\mathbb{R}^n; Cl_n)$
\[
T_\psi : L^2(\mathbb{R}^n; Cl_n) \rightarrow L^2(\mathbb{R}^n; Cl_n),
\] (39)
\[
f \mapsto T_\psi f(a, \theta, b) = \int_{\mathbb{R}^n} f(x) \psi_{a, \theta, b}(x) d^n x.
\] (40)

- Because of (12) we need to restrict the mother wavelet $\psi$ for $n = 2(\text{mod} 4)$ to even or odd grades: Either we have a spinor wavelet $\psi \in L^2(\mathbb{R}^n; Cl_n)$ with $\varepsilon = 1$, or we have an odd parity wavelet $\psi \in L^2(\mathbb{R}^n; Cl_n)$ with $\varepsilon = -1$.

- For $n = 3(\text{mod} 4)$, no grade restrictions exist. We then always have $\varepsilon = 1$. 

Proof. The spectral (CFT) representation of the Clifford wavelet transform is
\[
T_{\psi} f(a, \theta, b) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \hat{f}(\omega) e^{\frac{i}{2} \omega^T \alpha a} \left( \hat{\psi}(ar_\theta^{-1}(\omega)) \right) \sim e^{i \alpha x} b \omega \, d^n \omega. \tag{41}
\]

Proof. By definition
\[
T_{\psi} f(a, \theta, b) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \hat{f}(\omega) \left( \hat{\psi}(ar_\theta^{-1}(\omega)) \right) \sim e^{i \alpha x} b \omega \, d^n \omega. \tag{42}
\]

Proof. By definition
\[
T_{\psi} f(a, \theta, b) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \hat{f}(\omega) \left( \hat{\psi}(ar_\theta^{-1}(\omega)) \right) \sim e^{i \alpha x} b \omega \, d^n \omega. \tag{43}
\]

Proof. By definition
\[
T_{\psi} f(a, \theta, b) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \hat{f}(\omega) \left( \hat{\psi}(ar_\theta^{-1}(\omega)) \right) \sim e^{i \alpha x} b \omega \, d^n \omega. \tag{44}
\]

Dilation covariance: If \(0 < c \in \mathbb{R}\) then
\[
[T_{\psi} f(c \cdot)](a, \theta, b) = \frac{1}{c^n} T_{\psi} f(ca, \theta, cb). \tag{45}
\]

Rotation covariance: If \(r = r_0, \theta_0 = \theta_0\) and \(r' = r_0 r = \theta_0 r \theta_0\) are rotations, then
\[
[T_{\psi} f(r_0 \cdot)](a, \theta, b) = T_{\psi} f(a, \theta', r_0 b). \tag{46}
\]

3.2 Properties of real Clifford GA wavelets

We immediately see from Definition\[2\] that the Clifford GA wavelet transform is left linear with respect to multivector constants \(\lambda_1, \lambda_2 \in \mathcal{C}_n\).

We further have the following set of properties.

Translation covariance: If the argument of \(T_{\psi} f(x)\) is translated by a constant \(x_0 \in \mathbb{R}^n\) then
\[
[T_{\psi} f(\cdot - x_0)](a, \theta, b) = T_{\psi} f(a, \theta, b - x_0). \tag{47}
\]

Proof. By definition
\[
[T_{\psi} f(\cdot - x_0)](a, \theta, b) = \int_{\mathbb{R}^n} f(x - x_0) \psi_{a, \theta, b}(x) \, d^n x
\]

Now we will see some differences from the classical wavelet transforms.

The next property is an inner product relation: Let \(f, g \in \mathcal{L}^2(\mathbb{R}^n; \mathcal{C}_n)\) arbitrary. Then we have
\[
(T_{\psi} f, T_{\psi} g)_{\mathcal{L}^2(\mathbb{R}^n; \mathcal{C}_n)} = (fC_{\psi}, g)_{\mathcal{L}^2(\mathbb{R}^n; \mathcal{C}_n)} \tag{49}
\]

In the following proof of \(49\) we will use the abbreviations
\[
F_{a, \theta}(\omega) = a^{\frac{n}{2}} \hat{f}(\omega) \{\hat{\psi}(ar_\theta^{-1}(\omega))\} \sim, \tag{50}
\]
\[
G_{a, \theta}(\omega') = a^{\frac{n}{2}} \hat{g}(\omega') \{\hat{\psi}(ar_\theta^{-1}(\omega))\} \sim. \tag{51}
\]
and the spectral representations  

\[
T_\psi f(a, \theta, b) = \frac{\mathcal{F}\{F_{a, \theta}\}(-\varepsilon b)}{(2\pi)^n}, 
\]

(52)

\[
T_\psi g(a, \theta, b) = \frac{\mathcal{F}\{G_{a, \theta}\}(-\varepsilon b)}{(2\pi)^n}. 
\]

(53)

**Proof.** Using the abbreviations [51], [52] and spectral representations [52], [53] we get

\[
(T_\psi f, T_\psi g)_{L^2(\mathbb{G}; C_\mathbb{G}^0)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{S_0(n)} \left( \int_{\mathbb{R}^n} F_{a, \theta}(\xi) d\mu \right) d\xi = 1
\]

\[
\]  

\[
[\text{norm relation}] 
\]

\[
= 1 \quad \text{for any } \psi \in L^2(\mathbb{R}^n; C_\mathbb{G}^0) 
\]

The integral converging in the weak sense.

**Proof.** For any \( g \in L^2(\mathbb{R}^n; C_\mathbb{G}^0) \)

\[
(T_\psi f, T_\psi g)_{L^2(\mathbb{G}; C_\mathbb{G}^0)} = \int_\mathbb{G} \int_{\mathbb{R}^n} T_\psi f(a, \theta, b) \psi_{a, \theta, b}(x) g(x) d^m b d\mu d^m b 
\]

\[
= \int_\mathbb{G} \int_{\mathbb{R}^n} T_\psi f(a, \theta, b) \psi_{a, \theta, b}(x) d^m b g(x) d^m x 
\]

\[
= \left( \int_\mathbb{G} T_\psi f(a, \theta, b) \psi_{a, \theta, b}(x) d^m b, g \right)_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)} 
\]

\[
= \left( fC_\psi, g \right)_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)}, 
\]

PT denotes the inner product relation [59].

\[
\]  

\[
\]  

\[
\text{where } PT \text{ denotes the CFT Plancherel theorem.}
\]

For the second equality we have also used the fact, that a substitution \( b' = -\varepsilon b, \varepsilon = \pm 1 \), as in \( \int_{\mathbb{R}^n} h(\varepsilon b) d^m b = \int_{\mathbb{R}^n} h(b') d^m b' \), does not change the overall sign.  

As a corollary we get the following norm relation:

\[
\|T_\psi f\|_{L^2(\mathbb{G}; C_\mathbb{G}^0)} = S c(fC_\psi, f)_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)} = C_\psi (f, f)_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)}, \quad (55)
\]

\[
= C_\psi \ast (f, f)_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)}, 
\]

\[
(56)
\]

We can further derive the

**Theorem 1** (Inverse Clifford \( C_\mathbb{G}^0 \) wavelet transform). Any \( f \in L^2(\mathbb{R}^n; C_\mathbb{G}^0) \) can be decomposed with respect to an admissible Clifford GA wavelet as

\[
f(x) = \int_{\mathbb{G}} T_\psi f(a, \theta, b) \psi_{a, \theta, b} C^{-1}_\psi d^m b \quad (57)
\]

\[
= \int_{\mathbb{G}} (f, \psi_{a, \theta, b})_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)} \psi_{a, \theta, b} C^{-1}_\psi d^m b, 
\]

Next is the reproducing kernel: We define for an admissible Clifford mother wavelet \( \psi \in L^2(\mathbb{R}^n; C_\mathbb{G}^0) \)

\[
\mathcal{K}_\psi(a, \theta, b; a', \theta', b') = \left( \psi_{a, \theta, b} C^{-1}_\psi, \psi_{a', \theta', b'} \right)_{L^2(\mathbb{R}^n; C_\mathbb{G}^0)} \quad (61)
\]

Then \( \mathcal{K}_\psi(a, \theta, b; a', \theta', b') \) is a reproducing kernel in \( L^2(\mathbb{G}, d\lambda) \), i.e.

\[
T_\psi f(a', \theta', b') = \int_{\mathbb{G}} T_\psi f(a, \theta, b) \mathcal{K}_\psi(a, \theta, b; a', \theta', b') d\lambda. \quad (62)
\]

**Proof.** By inserting for \( f(x) \) the inverse CWT [57] into the definition of the CWT we obtain

\[
T_\psi f(a', \theta', b') = \int_{\mathbb{R}^n} \left( \int_{\mathbb{G}} T_\psi f(a, \theta, b) \psi_{a', \theta', b'}(x) d\lambda C^{-1}_\psi \right) d^m x
\]

\[
= \int_{\mathbb{G}} \int_{\mathbb{R}^n} (\psi_{a', \theta', b'}(x)) \psi_{a', \theta', b'}(x) d^m x = \mathcal{K}_\psi(a, \theta, b; a', \theta', b') d\lambda. \quad (63)
\]
Theorem 2 (Generalized Clifford GA wavelet uncertainty principle). Let $\psi$ be an admissible Clifford algebra mother wavelet. Then for every $f \in L^2(\mathbb{R}^n; Cl_n)$, the following inequality holds
\[
\|bT_0 f (a, \theta, b)\|_{L^2(\mathbb{R}^n; Cl_n)}^2 C^p_\psi \hat{\varphi}(\vec{\psi}_f) \geq \frac{n(2\pi)^n}{4} \left| C^p_\psi (f, \hat{f})_{L^2(\mathbb{R}^n; Cl_n)} \right|^2. \tag{64}
\]
NB: The integrated variance
\[
\int_{\mathbb{R}^n} \int_{SO(n)} \|\omega \mathcal{F} (T_0 f (a, \theta, \cdot))\|_{L^2(\mathbb{R}^n; Cl_n)}^2 d\mu (\omega) \tag{65}
\]
is independent of the wavelet parity $\varepsilon$. Otherwise the proof is similar to the one for $n = 3$ in [16]. For scalar admissibility constant this reduces to

Corollary 1 (Uncertainty principle for Clifford GA wavelet). Let $\psi$ be a Clifford algebra wavelet with scalar admissibility constant $C^p_\psi \in \mathbb{R}^n$. Then for every $f \in L^2(\mathbb{R}^n; Cl_n)$, the following inequality holds
\[
\|bT_0 f (a, \theta, b)\|_{L^2(\mathbb{R}^n; Cl_n)}^2 \|\omega \hat{f}\|_{L^2(\mathbb{R}^n; Cl_n)}^2 \geq nC^p_\psi \frac{(2\pi)^n}{4} \|f\|_{L^2(\mathbb{R}^n; Cl_n)}. \tag{66}
\]

• This shows indeed, that Theorem 2 represents a multivector generalization of the uncertainty principle for Clifford wavelets with scalar admissibility constant.

• Compare with the (direction independent) uncertainty principle [28] for the CFT.

3.3 Example of Clifford GA Gabor wavelets

Finally Clifford (geometric) algebra Gabor Wavelets are defined as (variances $\sigma_k, 1 \leq k \leq n$, for $n = 2(\text{mod } 4) : A \in Cl_n$ or $A \in Cl^{-1}_n$)
\[
\psi^c (x) = A e^{-\frac{1}{2} \sum_{k=1}^n \sigma_k^2} \phi (x - e^{-\frac{1}{2} \sum_{k=1}^n \sigma_k^2} \omega_{0,k}), \quad x, \omega_0 \in \mathbb{R}^n, \quad \text{constant } A \in Cl_n. \tag{67}
\]

The spectral (CFT) representation of the Clifford Gabor wavelets [27] is
\[
\mathcal{F} \{ \psi^c \} (\omega) = \hat{\psi}^c (\omega) \tag{68}
\]
\[
= A (e^{-\frac{1}{2} \sum_{k=1}^n \sigma_k^2 (\omega_k - \omega_{0,k})^2} - e^{-\frac{1}{2} \sum_{k=1}^n \sigma_k^2 (\omega_k^2 + \omega_{0,k}^2)}). \quad \text{(67)}
\]

It follows with [33] that the Clifford Gabor wavelet admissibility constant
\[
C^p_\psi = \int_{\mathbb{R}^n} \{ \hat{\psi}^c (\xi) \} \frac{\hat{\psi}^c (\xi)}{|\xi|^n} d^n \xi
\]
\[
= \hat{A} \frac{\phi (\xi)^2}{|\xi|^n} d^n \xi. \tag{69}
\]

If e.g. $A$ is a vector or a product of vectors (versor), then $C^p_\psi$ will be scalar.

4 Conclusion

We have introduced real Clifford (geometric) algebra wavelets for multivector signals taking values in $Cl_n$. Real means that we completely avoid to use the field of complex numbers $\mathbb{C}$. This also applies to the use of a real Clifford (geometric) algebra Fourier transform for the spectral representation. An extension to $Cl_{0,n'}, n' = 1, 2(\text{mod } 4)$ appears straightforward.

Acknowledgments

Soli deo gloria. I do thank my dear family, B. Mawardi, G. Schenemann, D. Hildenbrand and V. Skala.

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