More on Jacobi metric:
Lifts, Frame dragging, and Jacobi metric sharing

Sumanto Chanda

Indian Institute of Astrophysics
Block 2, 100 Feet Road, Koramangala, Bengaluru 560034, India.
sumanto.chanda@iiap.res.in

15th May 2024

Abstract

In this article I discuss the Jacobi metric in various contexts such as momentum constraint, Eisenhart lift, frame dragging effect, and Randers-Finsler metrics that share a common Jacobi metric. First, I introduce the constraint with relativistic momentum, demonstrate how it can act as a generator of equations of motion, followed by two applications: the derivation of harmonic oscillator, and analysis of Schwarzschild Randers-Finsler metric. Then I present derivation of the Jacobi metric using my constraint formulation, comparing it for null curves to optical metrics, and discuss a Hamiltonian mechanics approach to frame dragging, demonstrating how Jacobi metric proves the constraint more suitable when the Hamiltonian is unavailable. I next demonstrate the limitations of the Eisenhart lift for Randers-Finsler metrics, and discuss an alternative via Jacobi metric. Finally, I discuss metrics sharing a common Jacobi metric considering examples of the Schwarzschild-Painlevé metric and the Kerr metric.

Contents

1 Introduction 1
2 Preliminaries 2
3 Constraint for Randers-Finsler metric 3
  3.1 Simple Harmonic Oscillator on curved space 5
  3.2 Schwarzschild Randers-Finsler metric 5
4 The Jacobi-Maupertuis metric 7
  4.1 Jacobi metric for Null curves 9
  4.2 Frame dragging effect 10
5 Eisenhart lift for the RF metric 12
6 Sharing the JMRF 14
  6.1 Schwarzschild Gullstrand-Painlevé metric 16
  6.2 Kerr metric 17
7 Conclusion and Discussion 18
1 Introduction

The Jacobi metric is a topic of significant interest that has been studied for many years \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\]. Ong described an interesting application of the theory to gravity \[1\] to study the curvature of Jacobi metric of Newtonian N-body problem, while Bera, Ghosh, and Majhi used it to study Hawking Radiation \[9\]. It is a projection of a geodesic in spacetime onto a hypersurface characterized by its energy. The basic concept originates in the Maupertuis principle from which it is formulated for Hamiltonian systems \[2, 3, 4, 5, 6, 7, 8\], and it has found many applications in gravity \[1, 9, 10, 11, 12\]. More recently, alongside Gibbons and Guha, I have discussed Jacobi metric in the study of geodesic flows \[4\], an application to the study of Kepler systems \[5\], and the gravitational magnetoelectric effect for stationary spacetimes in \[7\] alongside Maraner and Werner.

Riemannian geometry plays a significant role in the study of gravity in general relativity \[13, 14\], by measuring the length of worldline curves with the square root of the norm on the local tangent space \(T_x M\). However, in 1941 Randers \[15\] took it a step further by modifying it into a Finsler metric known as Randers-Finsler (RF) metrics by adding a linear term \(A_i(x)dx^i\), where \(A_i(x)\) are potentials, thus simultaneously accounting for curvature and potential functions.

\[F_{RF}(x, y) = \sqrt{g_{ij}(x)y^iy^j + A_i(x)y^i}, \quad y = y^i\partial_i \in T_x M.\]

The equations of motion derived for the above metric are best described as Newton’s equations of motion involving Lorentz force, instead of the geodesic equation associated with a Riemannian metric. Recently Stavrinos, Basalikos, Triantafyllopoulos, and others have studied the example of the Schwarzschild Randers-Finsler metric \[16, 17, 18\]. The main reason for discussing the RF metric was seen in \[7\] where we saw that stationary metrics leads to Jacobi metrics with a RF form defined as Jacobi-Maupertuis-Randers-Finsler (JMRF) metric. This is because cross-terms between space and time suggest a moving inertial frame or frame-dragging effect \[19\], which can also be described as gauge fields of gravitomagnetism.

Eisenhart In 1929 \[20\] compared dynamical system trajectories in classical configuration space in \(n\) co-ordinates to geodesics in \(n+2\)-co-ordinates. Based on his work, Gibbons, Duval, Horvathy, Minguzzi and many others developed the Eisenhart lift for natural Hamiltonian systems \[21, 22, 23, 24, 25, 26, 27, 28, 29\], and it has found many applications in the study of gravity and integrable systems \[23, 24, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39\]. More recently, the Eisenhart lift was generalised for application to field theory \[40, 41\]. It is a method for geometrising potentials of non-relativistic systems by adding extra degrees of freedom while preserving Hamilton’s equations of motion, thus converting them into relativistic systems, allowing us to use geometric approach to study them.

The RF metric describes a relativistic system with potentials comparable to a magnetic gauge field, which leads one to ask if these potentials can be geometrised via Eisenhart lift the usual non-relativistic systems it is applied to. However, when attempted directly, the Eisenhart lift faces limitations when dealing with RF metrics, so we must seek an alternative to extend its utility beyond natural Hamiltonian systems. The Jacobi metric provides a route to one such alternative, since for stationary metrics, it is a procedure for the reverse, i.e.- converting geometry into potentials.

In this article, I shall return to the topic of Jacobi metrics and discuss more on it and aspects related to stationary and RF metrics. I will begin by reviewing preliminaries of basic classical mechanics associated with RF spacetime. Then I will develop upon mechanics centered around the momentum constraint that was first introduced in \[7\], and further developed upon in \[41\] as a generator of Hamilton’s equations with a modification to allow consideration of null curves, followed by two applications as exercises: derivation of relativistic simple harmonic oscillator, and analysis of Schwarzschild Randers-Finsler metric. Next, I will describe formulation of JMRF centered around the constraint and for null curves, followed by discussion of derivation of frame dragging for the Jacobi metric. Afterwards, I shall demonstrate why the limitations
faced by the Eisenhart lift during application to RF metrics, and use the theory of the Jacobi metric to overcome the problem. Finally, I will discuss the nature of RF metrics and Riemannian metrics that share a common JMRF, discussing two examples: the Schwarzschild Painlevé metric and the Kerr metric.

2 Preliminaries

General Relativity [13, 14] can be regarded as an advanced version of classical mechanics, borrowing and applying many principles from the latter to a more sophisticated level in curved space. In this section, I shall review the associated mechanical preliminaries relevant later on in this article.

Although Maupertuis is credited for the principle of least action that applies to all physical systems, he had originally applied it only to light [12], evidence pointing to Euler [43] for intuitively connecting it to mechanics. In its modern form, Maupertuis principle has proven very relevant in mechanics, providing a linear form of action applied in path-integral formulation in quantum mechanics.

If the worldline length $s$ of a curve between two points given by integration of the metric $ds$ is parametrised by $\tau$, it can be written in terms of a Lagrangian $L$ such that:

$$s = \int_1^2 ds = \int_1^2 d\tau L(x, \dot{x}), \quad \text{where} \quad L = \frac{ds}{d\tau}, \quad \dot{x} = \frac{dx}{d\tau}. \quad (2.1)$$

Varying the Lagrangian in (2.1) gives us:

$$\delta L = \left\{ \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) \right\} \delta x^\mu + \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right). \quad (2.2)$$

Motion will occur along the curve described by the solution of Euler-Lagrange equation, called the geodesic:

$$\frac{\partial L}{\partial x^\mu} - \frac{dp^\mu}{d\tau} = 0, \quad p^\mu = \frac{\partial L}{\partial \dot{x}^\mu}. \quad (2.3)$$

The variation of the curve length (2.1) close to the geodesic (2.3) is given by the variation at the ends of the curve, shown by applying (2.3) to (2.2):

$$\delta s = \int_1^2 d\tau \delta L = [p^\mu \delta x^\mu]_1^2 \equiv \left[ \frac{\partial s}{\partial x^\mu} \delta x^\mu \right]_1^2 \Rightarrow p^\mu = \frac{\partial s}{\partial x^\mu},$$

from which we can write the Maupertuis principle:

$$ds = \frac{\partial s}{\partial x^\mu} dx^\mu = p^\mu dx^\mu. \quad (2.4)$$

Thus, from (2.4), we can deduce the Maupertuis form of the Lagrangian, for which the overall Hamiltonian $H$ according to Legendre’s principle vanishes

$$L = p^\mu \dot{x}^\mu \quad \Rightarrow \quad H = p^\mu \dot{x}^\mu - L = 0. \quad (2.5)$$

Upon parametrization wrt any one of the co-ordinates $x^0$, the velocity is concealed ($\dot{x}^0 = 1$), and the Legendre principle leads to the associated momentum $p_0$:

$$L = p_i \dot{x}^i + p_0 \quad \Rightarrow \quad H = -p_0 = p_i \dot{x}^i - L. \quad (2.6)$$
Since the velocity component for time $t$ is lost upon being given the status of a parameter, its conjugate momentum defined as the Hamiltonian $H$ is provided by the Legendre theorem (2.6). Since $H$ is a function of $x$ and $p$, the variation of $H$ gives Hamilton’s equation of motion:

\[
\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}.
\] (2.7)

In case of regular Lagrangian systems, the Lagrangian does not match the Maupertuis form (2.5), which allows us to formulate the Hamiltonian function in the phase space using (2.6). Under the circumstances that the system is independent of a co-ordinate $x^0$ referred to as a cyclic co-ordinate, the corresponding momentum will be a conserved quantity according to (2.3) and (2.7), the existence of which is essential for the Eisenhart lift and its alternatives for Randers-Finsler metrics.

\[
\frac{dp_0}{d\tau} = \frac{\partial L}{\partial x^0} = -\frac{\partial H}{\partial x^0} = 0 \quad \Rightarrow \quad p_0 = q(\text{const}).
\] (2.8)

This results in the associated term in the Maupertuis Lagrangian (2.5) becoming a total time derivative that can be dismissed from the Lagrangian:

\[
L = p_i \dot{x}^i + q \dot{x}^0 \quad \Rightarrow \quad L_{\text{eff}} = p_i \dot{x}^i.
\] (2.9)

Thus, according to (2.9), extra degrees of freedom can in principle be removed or inserted into the description of a particles mechanics. It is here that the Hamiltonian proves essential to formulating Eisenhart and Jacobi metrics for Hamiltonian systems without altering the equations of motion [3, 4]. However, one has to consider circumstances where a proper Hamiltonian cannot be deduced.

### 3 Constraint for Randers-Finsler metric

In RF metrics [15] the first part with the norm under the square root accounts for the influence of curvature, while the linear term outside accounts for gauge field interaction. Sometimes the second part is geometric in origin.

\[
ds = \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu} + A_\lambda(x)dx^\lambda.
\] (3.1)

Using the Lagrangian $L$ defined according to (2.1), the canonical momenta $p$ according to (2.3) leads us to the gauge-covariant momenta $\pi$ given by:

\[
L = \sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} + A_\lambda(x)\dot{x}^\lambda \quad \Rightarrow \quad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{g_{\mu\nu}(x)\dot{x}^\nu}{\sqrt{g_{\alpha\beta}(x)dx^\alpha dx^\beta}} + A_\mu(x),
\]

\[
\pi_\mu = p_\mu - A_\mu(x) = g_{\mu\nu}(x)\frac{dx^\nu}{d\sigma}, \quad \text{where} \quad d\sigma := \sqrt{g_{\alpha\beta}(x)dx^\alpha dx^\beta}.
\] (3.2)

We can verify that the form of the Lagrangian $L$ will match its Maupertuis form (2.5), which means that the Hamiltonian cannot be formulated via Legendre’s theorem (2.6). Thus, an alternative generator of equations in phase space must be found. To this end, I show that the gauge-covariant momenta $\pi$ (3.2) obey the constraint:

\[
\phi(x, p) = \sqrt{g^{\mu\nu}(x)\pi_\mu \pi_\nu} = \sqrt{g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} = 1,
\] (3.3)

which acts as a generator of equations of motion [11], demonstrated by taking a derivative of the constraint:

\[
\frac{d\phi}{d\sigma} = \frac{\partial \phi}{\partial x^\mu} \frac{dx^\mu}{d\sigma} + \frac{\partial \phi}{\partial p_\mu} \frac{dp_\mu}{d\sigma} = 0,
\] (3.4)
then I can show by applying (3.2) and (3.3) into (3.4) that one will have:

\[
\frac{\partial \phi}{\partial p_\mu} = g^{\mu\nu}(x)\pi_\nu = \frac{dx^\mu}{d\sigma} \quad \Rightarrow \quad \frac{\partial \phi}{\partial x^\mu} = -\frac{dp_\mu}{d\sigma}.
\]

Thus, we have the constraint equivalent of Hamilton’s equations of motion:

\[
\begin{align*}
\frac{dx^\mu}{d\sigma} &= \frac{\partial \phi}{\partial p_\mu}, \\
\frac{dp_\mu}{d\sigma} &= -\frac{\partial \phi}{\partial x^\mu}.
\end{align*}
\tag{3.5}
\]

Under the circumstances that one is dealing with spacetime metrics that possess Minkowskian signature, there is entirely a possibility that one may deal with null curves which lead to the constraint becoming undefined according to (3.3). To overcome this obstacle, we insert an extra auxiliary co-ordinate \(y\) without disturbing the mechanics of the system, by writing the RF metric (3.1) into:

\[
\tilde{d}s = \sqrt{\kappa dy^2 + d\sigma^2} + A_\mu(x)dx^\mu,
\]

where

\[
\kappa = \begin{cases} 
0 & \text{if } d\sigma^2 \neq 0 \\
1 & \text{if } d\sigma^2 = 0
\end{cases}
\].
\tag{3.6}

This way, when we deduce the momenta according to (2.3), then we can write under the limit (3.6):

\[
p_y = \kappa \frac{dy}{d\sigma} = \frac{\kappa dy}{\sqrt{\kappa dy^2 + d\sigma^2}} = \begin{cases} 
0 & \text{if } d\sigma^2 \neq 0 \\
1 & \text{if } d\sigma^2 = 0
\end{cases}, \quad \text{where } d\tilde{\sigma}^2 = dy^2 + d\sigma^2.
\tag{3.7}
\]

the constraint (3.3) will become:

\[
\phi(x, p) = \sqrt{p_y^2 + g^{ij}(x)\pi_i\pi_j} = 1, \quad \text{where } p_y = \begin{cases} 
0 & \text{if } d\sigma^2 \neq 0 \\
1 & \text{if } d\sigma^2 = 0
\end{cases}
\].
\tag{3.8}

Although the RF Lagrangian exactly matching the Maupertuis form prevents us from deducing a Hamiltonian function, it does allow us to determine the metric using the Maupertuis principle. When starting from the constraint, this is done by applying (3.2) and the first equation of (3.5) to Maupertuis principle (2.4):

\[
ds = p_\mu dx^\mu = g_{\mu\nu}(x)\frac{\partial \phi}{\partial p_\mu} dx^\mu + A_\mu(x)dx^\mu = d\sigma + A_\mu(x)dx^\mu.
\tag{3.9}
\]

From (3.3), we can write \(d\sigma = \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu}\), which upon application to (3.9), will give us the original RF metric (3.1).

One can modify the action to parametrization wrt observed time \(x^0 = t\), by writing the (00) component for time as a perturbation around the flat space as \(g_{00}(x) = 1 - 2\Phi(x)\), setting \(\dot{t} = 1 \Rightarrow d\sigma = dt\). Upon binomial expansion of the part under square root up to the first order, as shown in [44], we get:

\[
L = \frac{1}{2} g_{ij}(x,t)\dot{x}^i\dot{x}^j + \Phi_k(x,t)\dot{x}^k - V(x,t),
\tag{3.10}
\]

where \(\Phi_k(x,t) = A_k(x,t) + g_{0i}(x,t)\), \(V(x,t) = \Phi(x,t) - A_0(x,t)\).

Furthermore, under the circumstances of a time-dependent or non-autonomous system, parametrising the action wrt observed time by setting \(\dot{t} = 1\) hides the velocity \(\dot{t}\), making the Lagrangian appear dependent on time \(t\), but independent of \(\dot{t}\).

Conventionally one solves the Euler-Lagrange equations when dealing with problems related to classical dynamics. In case of Randers-Finsler metrics, there is a simpler alternative that can prove quicker, i.e., to use the constraint of the system as a substitute for the Hamiltonian. Here, I will demonstrate two exercises to demonstrate mechanics formulated around the constraint which are the Harmonic Oscillator and the Schwarzschild-Randers-Finsler (SRF) metric discussed in [18].
3.1 Simple Harmonic Oscillator on curved space

The SHO is a simple, but very interesting system studied because its Hamilton’s equations are identical and dual to each other. This is why they appear invariant under specific canonical transformations \[15\].

Suppose we have the following constraint equations (3.5) for a simple harmonic oscillator on curved space:

\[
\frac{dx^i}{ds} = \frac{\partial \Phi}{\partial p_i} = \delta_{ij} p_j , \quad \frac{dp_i}{ds} = -\frac{\partial \Phi}{\partial x^i} = -\omega^2 \delta_{ij} x^j .
\]  \hspace{1cm} (3.1.1)

Using (3.1.1), we can solve the following equation to write:

\[
d\Phi = \frac{\partial \Phi}{\partial x^i} dx^i + \frac{\partial \Phi}{\partial p_i} dp_i = \omega^2 \delta_{ij} x^j dx^i + \delta_{ij} p_j dp_i = 0
\]

\[\big(A + |\vec{p}|^2\big) + \big(B + \omega^2 x^2\big) = D, \quad \text{where} \quad x^2 = |\vec{x}|^2 = \delta_{ij} x^i x^j , \quad |\vec{p}|^2 = \delta_{ij} p_i p_j \]  \hspace{1cm} (3.1.2)

where \(A, B, D\) are all constants. To fit the solution (3.1.2) into the form of the constraint (3.3) and keep a Lorentzian signature, we can take its square root and write:

\[
\Phi(x, p) = \sqrt{|\vec{p}|^2 - (1 - k^2 x^2)} Q^2 = 1 \quad \text{(3.1.3)}
\]

where we can assume that \(Q\) is a conserved momentum conjugate to a cyclic co-ordinate \(t\), which we can suppose to be the time co-ordinate. Thus, we can further deduce from (3.1.3)

\[
\frac{dt}{ds} = \frac{\partial \Phi}{\partial Q} = -(1 - k^2 x^2) Q .
\]  \hspace{1cm} (3.1.4)

and derive a Riemannian metric for the oscillator by applying (3.1.1) and (3.1.4) into (3.1.3):

\[
ds^2 = |d\vec{x}|^2 - \frac{1}{1 - k^2 x^2} dt^2 ,
\]  \hspace{1cm} (3.1.5)

which has Lorentzian signature and is flat at \(\vec{x} = 0\). This metric is not applicable at large distances since the metric diverges towards infinity.

3.2 Schwarzschild Randers-Finsler

When calculating the deflection angle for either a massive particle or a photon travelling along a null curve, normally one derives and solves the geodesic equations. However, it is entirely possible to bypass this step and calculate the deflection angle faster by using the constraint.

The Schwarzschild-Randers-Finsler metric is a modification of the existing Schwarzschild metric. Here, we shall consider a modification that effectively inserts a scalar potential into the system.

This Schwarzschild-Randers-Finsler metric discussed in \[18\] is given by:

\[
ds = \sqrt{f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + A_0 \sqrt{f} dt ,
\]  \hspace{1cm} (3.2.1)

where \(f(r) = 1 - \frac{r_0}{r}\), \(r_0\) being the Schwarzschild radius. If we modify it according to (3.6) then we will have:

\[
d\tilde{s} = \sqrt{dy^2 + f(r) dt^2 - \frac{1}{f(r)} dr^2 - (d\theta^2 + r^2 \sin^2 \theta d\phi^2)} + \tilde{A}_0 \sqrt{f} dt . \quad (3.2.2)
\]
Setting $\theta = \frac{\pi}{2}$, we will have the constraint according to (3.8):

$$
\phi(x, p) = \sqrt{p_y^2 + \left(\frac{E - \tilde{A}_0 \sqrt{f}}{f(r)}\right)^2 - f(r) p_r^2 - \frac{l^2}{r^2}} = 1, \quad (3.2.3)
$$

into which we can substitute $p_r = \frac{1}{f(r)} \frac{dr}{d\sigma}$ given by (3.2) to get according to (3.7):

$$
\left(\frac{dr}{d\sigma}\right)^2 - \left(E - \tilde{A}_0 \sqrt{f}\right)^2 = -f(r) \left[\frac{l^2}{r^2} + (1 - p_y^2)\right] = \begin{cases} -f(r) \left(\frac{l^2}{r^2} + 1\right) & \text{massive} \\ -f(r) \frac{l^2}{r^2} & \text{null}. \end{cases} \quad (3.2.4)
$$

Finally substituting the angular momentum $l = r^2 \left(\frac{d\varphi}{d\sigma}\right)$, we can finally write that:

$$
\left(\frac{dr}{d\varphi}\right)^2 - \frac{r^4}{l^2} \left(E - \tilde{A}_0 \sqrt{f}\right)^2 = -r^2 f(r) \left[1 + \frac{r^2}{l^2} (1 - p_y^2)\right] = \begin{cases} -r^2 f(r) \left(1 + \frac{r^2}{l^2}\right) & \text{massive} \\ -r^2 f(r) & \text{null}, \end{cases} \quad (3.2.5)
$$

from which we can proceed to compute deflection angle via integration. This result was achieved faster without solving the geodesic equations.

The photon sphere is a zone where photons are captured and trapped. Naturally, this means that they inhabit a spherical surface of fixed radius. Using the null part of (3.2.4) we can write:

$$
\frac{dr}{d\sigma} = 0 \quad \Rightarrow \quad \frac{E - \tilde{A}_0 \sqrt{f(r)}}{l} = \pm \sqrt{f(r)} \frac{r}{r}, \quad (3.2.6)
$$

while from the constraint (3.2.3), using constraint equations (3.5) we have:

$$
\frac{dr}{d\sigma} = \frac{\partial \varphi}{\partial p_r} = f(r) p_r = 0 \quad \Rightarrow \quad p_r = 0,
$$

$$
\frac{d^2r}{d\sigma^2} = \frac{df(r)}{d\sigma} p_r + f(r) \frac{dp_r}{d\sigma} = f(r) \frac{dp_r}{d\sigma} = -f(r) \frac{\partial \phi}{\partial r} = 0,
$$

$$
\Rightarrow \quad \left(\frac{E - \tilde{A}_0 \sqrt{f(r)}}{l}\right)^2 = \frac{\tilde{A}_0 \sqrt{f(r)}}{l} \left(\frac{E - \tilde{A}_0 \sqrt{f(r)}}{l}\right) - 2 \left(\frac{f(r)}{r}\right)^2 = 0,
$$

$$
\Rightarrow \quad \frac{E - \tilde{A}_0 \sqrt{f(r)}}{l} = -\frac{\tilde{A}_0 \sqrt{f(r)}}{2l} \pm \sqrt{\left(\frac{\tilde{A}_0 \sqrt{f(r)}}{2l}\right)^2 + 2 \left(\frac{f(r)}{r}\right)^2}, \quad (3.2.7)
$$

Thus, comparing (3.2.6) and (3.2.7) we shall have the radius of the photon sphere is given by:

$$
\sqrt{\frac{f(r)}{r}} = \pm \frac{\tilde{A}_0 \sqrt{f(r)}}{2l} \pm \sqrt{\left(\frac{\tilde{A}_0 \sqrt{f(r)}}{2l}\right)^2 + 2 \left(\frac{f(r)}{r}\right)^2}
$$

$$
\Rightarrow \quad r = 3 \frac{\tilde{A}_0 r_s}{2l} \left(1 \pm \frac{\tilde{A}_0 r_s}{2l}\right), \quad (3.2.8)
$$

indicating that there are two different concentric spheres that act as traps for photons.
4 The Jacobi-Maupertuis metric

While the Eisenhart lift is a procedure to insert an additional direction of symmetry, formulating the Jacobi metric involves removing an existing one. Moreover, the latter can be directly formulated in RF geometry via the constraint, unlike the former as I will also describe in this section. Thus, it is simple to realise that reversing the formulation of the Jacobi metric is a suitable alternative to the Eisenhart lift. We must note that when dealing with a time-independent space time metric, we use time to formulate the Jacobi metric to leave the spatial description of the geodesic undisturbed.

First, I shall discuss how to project a RF geodesic in \( n + 1 \) co-ordinates into the Jacobi-Maupertuis-Randers-Finsler (JMRF) geodesic in \( n \) co-ordinates via a new constraint with a conformal factor from the original constraint. Consider the following RF metric with a cyclic co-ordinate \( x^0 = t \):

\[
ds_{\text{RF}} = \sqrt{\gamma_{ij}(x)dx^i dx^j + g_{00}(x) \left( \frac{dt}{d\sigma} + \frac{g_{0i}(x)}{g_{00}(x)} dx^i \right)^2} + A_i(x)dx^i + A_0(x)dt,
\]

where \( \gamma_{ij}(x) = g_{ij}(x) - \frac{g_{0i}(x)g_{0j}(x)}{g_{00}(x)} \). The canonical momenta are deduced according to (2.3), where the momentum conjugate to \( x^0 \) is a constant of motion \( p_0 = q \). The gauge-covariant momenta (3.2) for \( d\sigma = \sqrt{g_{\alpha\beta}(x)dx^\alpha dx^\beta} \) are:

\[
\pi_0 = q - A_0(x) = g_{0\alpha}(x) \frac{dx^\alpha}{d\sigma} + g_{00}(x) \frac{dt}{d\sigma} = Q(x),
\]

\[
\pi_i = p_i - A_i(x) = \gamma_{ij}(x) \frac{dx^j}{d\sigma} + \frac{g_{0i}(x)}{g_{00}(x)} Q(x).
\]

Here I will introduce a new gauge-covariant momentum \( \Pi \) from (4.2):

\[
\Pi_i = p_i - \alpha_i(x) = \gamma_{ij}(x) \frac{dx^j}{d\sigma}, \quad \text{where} \; \alpha_i(x) = A_i(x) + \frac{g_{0i}(x)}{g_{00}(x)} Q(x)
\]

with which the constraint (3.3) for (4.1) is written as:

\[
\phi(x, p) = \sqrt{g^{ij}(x)\Pi_i \Pi_j + \frac{(Q(x))^2}{g_{00}(x)}} = 1.
\]

To formulate the Jacobi metric, I shall rewrite the constraint (4.1) discussed in Subsection 5 for the Randers-Finsler metric with a cyclic co-ordinate into a different constraint of the same form as (3.3)

\[
\Gamma(x, p) = \sqrt{1 - \frac{(Q(x))^2}{g_{00}(x)}} g^{ij}(x)\Pi_i \Pi_j = 1.
\]

We can therefore define the Jacobi metric as:

\[
J^{ij}(x) := \frac{g^{ij}(x)}{\mathcal{C}(x)} \Rightarrow J_{ij}(x) = \mathcal{C}(x) \gamma_{ij}(x), \quad \text{where} \; \mathcal{C}(x) = 1 - \frac{(Q(x))^2}{g_{00}(x)}.
\]

Upon applying (4.6) to the constraint (4.5), the 1st of the constraint equations (3.5) allows us to write:

\[
\frac{dx^i}{d\lambda} = \frac{\partial \Gamma}{\partial p_i} = J^{ij}(x) \Pi_j \Rightarrow \Pi_i = p_i - \alpha_i(x) = J_{ij}(x) \frac{dx^j}{d\lambda},
\]

\[
J^{ij}(x) \Pi_i \Pi_j = J_{ij}(x) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 1 \Rightarrow d\lambda^2 = J_{ij}(x) \; dx^i \; dx^j.
\]
According to the Maupertuis principle, using (4.3), (4.7), and (4.8), one can write the JMRF metric:

\[ ds_J = \sqrt{1 - \frac{(Q(x))^2}{g_{00}(x)}} \gamma_{ij}(x) dx^i dx^j + \left( A_i(x) + \frac{g_{0i}(x)}{g_{00}(x)} Q(x) \right) dx^i. \]  

(4.9)

Furthermore, we can see from (4.7) that:

\[ \frac{dx^i}{d\lambda} = (C(x))^{-1} g^{ij}(x) \Pi_j = (C(x))^{-1} \left[ \frac{\partial \phi}{\partial p_i} \right] = (C(x))^{-1} \frac{dx^i}{d\sigma}, \]

showing that the direction ratios along the geodesic of the Jacobi metric will be the same as with the original geodesic. If the RF Lagrangian is parametrized wrt the cyclic co-ordinate and expanded binomially as shown in Subsection 3, but without restriction to first order for low energy approximation, the Jacobi metric may be formulated as described by Maraner in [8] for general Lagrangian systems. If more than one cyclic co-ordinate is available, the procedure can be repeated, until none are left.

Having formulated the Jacobi metric for the Randers-Finsler metric, I shall now discuss the Jacobi metric for different settings of the original RF metric.

**Riemannian metric**

If we start with a Riemannian metric \( A_{\mu}(x) = 0 \) in (5.1), then we will have the gauge fields according to (4.2) and (4.3):

\[ Q(x) = q, \quad \alpha_i(x) = -\gamma_{ik}(x) q^k(\gamma) q, \]

and thus, we will have the Jacobi metric:

\[ ds_J = \sqrt{1 - \frac{q^2}{g_{00}(x)}} \gamma_{ij}(x) dx^i dx^j + q \frac{g_{0i}(x)}{g_{00}(x)} dx^i, \]  

(4.10)

which is the result presented in [7]. This setting shows how the Jacobi metric creates new potentials from the metric.

**Static space flat along cyclic co-ordinate**

In this case where \( g_{00}(x) = 1, g_{0i}(x) = 0, A_{\mu}(x) \neq 0 \), the Jacobi metric is:

\[ ds_J = \sqrt{1 - (Q(x))^2} g_{ij}(x) dx^i dx^j + A_i(x) dx^i. \]

If we can say that

\[ Q(x) = q - A_0(x) = 1 + \varepsilon - A_0(x), \]

where \( \varepsilon - A_0(x) \ll 1 \), then I can write:

\[ (W(x))^2 \approx 1 + 2 (\varepsilon - A_0(x)). \]

\[ \therefore \quad ds_J = \sqrt{-2 (\varepsilon - A_0(x))} g_{ij}(x) dx^i dx^j + A_i(x) dx^i, \]

which is the non-relativistic limit discussed in [4] when \( A_i(x) = 0 \), and in [7]. Under the circumstances that \( A_i(x) = 0 \forall i \), this example describes a case where the Jacobi metric geometrizes the potential \( A_0(x) \) absorbing it into the metric.
4.1 Jacobi metric for Null curves

In 1662, Fermat speculated in his principle of least time [46] that light travels along paths requiring the shortest time interval, defined by null-geodesics. This makes Fermat’s principle the optical version of the Brachistochrone problem [47, 48], as discussed by Erlichson [49] and Broer [50]. Null-geodesics are unique since the speed of a particle (photon) travelling along them remains unchanged under local Lorentz transformations. In special relativity, in flat spaces this leads to Einstein’s postulate on the universality of the speed of light in all inertial frames, which holds true locally, even in refracting media.

Since the length of a null curve vanishes, one may introduce a metric based on Fermat’s principle, called the Optical metric. We shall demonstrate this starting with the stationary spacetime metric given below:

$$ds^2 = g_{00}(x)dt^2 + 2g_{i0}(x) dt dx^i + g_{ij}(x)dx^i dx^j. \quad (4.1.1)$$

Viewing the null version of (4.1.1) as a quadratic equation

$$g_{00}(x)dt^2 + 2g_{i0}(x) dt dx^i + g_{ij}(x)dx^i dx^j = 0,$$

we can write the Optical metric $ds_O$ as a solution of the quadratic equation for $dt$

$$dt = \pm \sqrt{-\frac{\gamma_{ij}(x)}{g_{00}(x)} dx^i dx^j - \frac{g_{i0}(x)}{g_{00}(x)} dx^i},$$

where we will take $+$ solution since $dt > 0$

$$ds_O = dt = \sqrt{-\frac{\gamma_{ij}(x)}{g_{00}(x)} dx^i dx^j - \frac{g_{i0}(x)}{g_{00}(x)} dx^i}, \quad (4.1.2)$$

which we can see is a Randers type of Finsler metric [15]. Maupertuis speculated in [42] that light passing through a medium was refracted due to gravitational effects, implying that from an optical perspective, one can interpret gravitational fields as transparent media and vice versa.

Under the circumstances that one is dealing with a null curve, we will have to resort to the modification described in (3.6). In this case according to (3.8), the constraint (4.4) would be rewritten as:

$$\phi(x, p) = \sqrt{1 + g^{ij}(x)\Pi_i \Pi_j + \frac{(Q(x))^2}{g_{00}(x)}} = 1. \quad (4.1.3)$$

Thus, the Jacobi metric constraint would become:

$$\Gamma(x, p) = \sqrt{-\left(\frac{(Q(x))^2}{g_{00}(x)}\right)^{-1} g^{ij}(x)\Pi_i \Pi_j} = 1. \quad (4.1.4)$$

Thus, the Jacobi metric this time will be:

$$ds_J = p_i dx^i = \sqrt{-\frac{(Q(x))^2}{g_{00}(x)} \gamma_{ij}(x) dx^i dx^j + \left(A_i(x) + \frac{g_{i0}(x)}{g_{00}(x)} Q(x)\right) dx^i}. \quad (4.1.5)$$

Now, if we consider only Riemannian metrics ($A_i(x) = A_0(x) = 0$) then we will have:

$$d\tilde{s}_J = \frac{ds_J}{q} = \sqrt{-\frac{\gamma_{ij}(x)}{g_{00}(x)} dx^i dx^j + \frac{g_{i0}(x)}{g_{00}(x)} dx^i}. \quad (4.1.6)$$
Under the circumstances that one is dealing with a static metric \( g_{00}(x) = 0 \) we can write:

\[
\frac{d\tilde{s}_{ij}}{q} = \frac{ds_{ij}}{q} = \sqrt{\frac{\gamma_{ij}(x)}{g_{00}(x)}}dx^idx^j.
\] (4.1.7)

which happens to be the form of the Optical metric formulated for static metric according to Fermat’s principle. However, we can see from \((4.1.6)\) that for stationary metrics the form of the Jacobi metric deviates from the optical metric \((4.1.2)\). Thus, we can say that optical metrics are not Jacobi metrics for null curves, and their similarity for Riemannian static metrics as shown by \((4.1.7)\) is merely coincidence.

### 4.2 Frame dragging effect

So far, we have seen that since there is no Hamiltonian described for the Jacobi metric, it is not always possible to describe mechanics using the Hamiltonian \((2.6)\) and Hamilton’s equations \((2.7)\). Thus, the constraint \((3.3)\) and its equations \((3.5)\) are more reliable alternatives to Hamilton’s equations. This is furthermore evident when one considers the question of frame dragging in the Jacobi metric. In [19], Epstein discusses a Hamiltonian approach to studying frame dragging. Such frame dragging effects should also exist in a Jacobi metric based description of dynamics around a black hole. However, since a Hamiltonian is unavailable, it should also be possible to describe frame dragging using the constraint.

According to Epstein [19], the frame dragging effect describes motion independent of momentum. However, if we use the constraint instead of the Hamiltonian, then the constraint for stationary metric \((4.1.1)\) according to \((3.3)\) is given by:

\[
\phi(x, p) = \sqrt{g^{ij}(x)p_ip_j + 2g^{0j}(x)p_ip_0 + g^{00}(x)(p_0)^2} = \sqrt{f^{ij}(x)p_ip_j + g^{00}(x)}\left(p_0 + \frac{g^{0m}(x)}{g^{00}(x)}p_m\right)^2 = 1,
\] (4.2.1)

where \(f^{ij}(x) = g^{ij}(x) - \frac{g^{0j}(x)p_0}{g^{00}(x)}\). We shall have the following constraint equations according to \((3.5)\):

\[
\begin{align*}
\frac{dx^i}{ds} &= \frac{\partial\phi}{\partial p_i} = g^{ij}(x)p_j + g^{0j}(x)p_0 = f^{ij}(x)p_j + \frac{g^{0j}(x)}{g^{00}(x)}\left(g^{00}(x)p_0 + g^{0j}(x)p_j\right), \\
\frac{dt}{ds} &= \frac{\partial\phi}{\partial p_0} = g^{0j}(x)p_j + g^{00}(x)p_0 = \sqrt{g^{00}(x)}\left(1 - f^{ij}(x)p_ip_j\right)
\end{align*}
\] (4.2.2)

from which we can see that

\[
\frac{dx^i}{dt} = \frac{f^{ij}(x)}{1 - \sqrt{g^{0j}(x)\left(1 - f^{ij}(x)p_ip_j\right)}}p_j + \frac{g^{0j}(x)}{g^{00}(x)}
\] (4.2.3)

which matches what Epstein discussed in [19]. Ultimately, frame dragging is manifested as motion or velocity that exists in the absence of momentum. This can also be seen from \((4.2.3)\) or directly from the constraint equations \((4.2.2)\):

\[
p_j = 0 \quad \forall \ j \quad \Rightarrow \quad \left\{ \begin{array}{l}
\left(\frac{dx^i}{ds}\right)_{p_j=0} = g^{0j}(x)p_0 \\
\left(\frac{dt}{ds}\right)_{p_j=0} = g^{0j}(x)p_0
\end{array} \right. \quad \Rightarrow \quad \left(\frac{dx^i}{dt}\right)_{p_j=0} = g^{0j}(x)\frac{g^{0j}(x)}{g^{00}(x)}.
\] (4.2.4)
From the constraint (4.2.4), we can say that:

$$p_i = 0 \quad \Rightarrow \quad g^{00}(x)(p_0)^2 = 1,$$

which allows us to write the proper velocities of (4.2.4) as:

$$\frac{dx^i}{ds}_{p_j=0} = g^{0i}(x)\sqrt{g^{00}(x)} \quad , \quad \frac{dt}{ds}_{p_j=0} = \sqrt{g^{00}(x)}.$$

(4.2.6)

Most importantly, we have the frame dragging proper velocity given by the first equation of (4.2.6), which is completely independent of momentum. Now, consider the constraint (3.3) for a general RF metric (3.1).

$$\phi(x, p) = \sqrt{g^{\mu\nu}(x)(p_\mu - A_\mu(x))(p_\nu - A_\nu(x))} = 1.$$

In this case, the constraint equations (3.5) are:

$$\frac{dx^\mu}{d\sigma} = \frac{\partial \phi}{\partial p_\mu} = g^{\mu\nu}(x)(p_\nu - A_\nu(x)) \quad \Rightarrow \quad \left(\frac{dx^\mu}{d\sigma}\right)_{p_\nu=0} = W^\mu(x) = -g^{\mu\nu}(x)A_\nu(x),$$

(4.2.7)

in which case the frame dragging is truly independent of any aspects of the particle’s motion. Consider the Jacobi metric for a stationary Riemannian metric given by (4.10). This metric has the constraint given according to (3.5):

$$\Gamma(x, p) = \sqrt{\left(1 - \frac{q^2}{(V(x))^2}\right)^{-1} g^{ij}(x)(p_i - qW_i(x))(p_j - qW_j(x))} = 1,$$

(4.2.8)

where we have:

$$W_i(x) = \frac{g_{0i}(x)}{(V(x))^2}, \quad (V(x))^2 = g_{00}(x),$$

(4.2.9)

from which according to the constraint equations for Jacobi metric (4.7), we will have:

$$\frac{dx^i}{d\lambda} = \frac{\partial \Gamma}{\partial p_i} = \left(1 - \frac{q^2}{(V(x))^2}\right)^{-1} g^{ij}(x)(p_j - qW_j(x)).$$

(4.2.10)

Thus, as with (4.2.4), we can describe the frame dragging velocity to be:

$$\left(\frac{dx^i}{d\lambda}\right)_{p_j=0} = \left(1 - \frac{q^2}{(V(x))^2}\right)^{-1} qW^i(x),$$

(4.2.11)

where $W^i(x) = -g^{ij}(x)W_j(x) = g^{00}(x)$. Furthermore, from the constraint (4.2.8), we will have:

$$\left(\Gamma(x, p)\right)_{p_j=0} = \sqrt{\left(1 - \frac{q^2}{(V(x))^2}\right)^{-1} q^2 |W(x)|^2} = 1$$

$$\Rightarrow \quad q^2 \left(|W(x)|^2 + \frac{1}{(V(x))^2}\right) = 1,$$

(4.2.12)

where $|W(x)|^2 = g^{ij}(x)W_i(x)W_j(x) = \gamma_{ij}(x)W^i(x)W^j(x) = -g^{0i}(x)\frac{g_{00}(x)}{g_{00}(x)}$, which allows us to write (4.2.11) as:

$$\left(\frac{dx^i}{d\lambda}\right)_{p_j=0} = \frac{1}{q} \frac{W^i(x)}{|W(x)|^2} = \sqrt{|W(x)|^2 + \frac{1}{(V(x))^2} \frac{W^i(x)}{|W(x)|^2}}.$$

(4.2.13)
Upon substituting the functions with (4.2.9), we have:

\[
\left( \frac{dx^i}{d\lambda} \right)_{p_j=0} = \left( 1 - \frac{1}{g^{00}(x)} \right) \frac{g^{0i}(x)}{\sqrt{g^{00}(x)}}. \tag{4.2.14}
\]

Thus, when the Hamiltonian is absent for cases such as the Jacobi metric, the constraint proves much more suitable for dynamical analysis.

5 Eisenhart lift for the RF metric

Eisenhart’s comparison of geodesics in \( n + 2 \) co-ordinates to trajectories of regular Lagrangian systems in configuration space with \( n \) co-ordinates \[20\] hints at their equivalence, evident in projection of geodesics into regular Lagrangian systems. The Eisenhart lift reverses the projection, converting a regular Lagrangian into a geodesic one \[25, 26, 27, 28, 32\].

A Lagrangian parametrized wrt time \( t \) derived from a Randers-Finsler metric under non-relativistic approximation becomes a regular Lagrangian \(3.10\). Starting with the metric \( \tilde{ds}^2 = dt^2 + ds^2 \), where \( ds^2 \) involves a cyclic co-ordinate \( v \), and expanding it binomially to 1st order, we have:

\[
S = \int_1^2 d\tilde{s} = \int_1^2 \sqrt{dt^2 + ds^2} \approx \int_1^2 dt \left[ 1 + \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \right] = \int_1^2 dt (1 + \mathcal{L}), \tag{5.1}
\]

with the geodesic Lagrangian \( \mathcal{L} \), from which I will derive the regular Hamiltonian as shown in \[27\] and the regular Lagrangian \(3.10\), which can be lifted in 2 ways depending on its time-dependence.

So far, the Eisenhart lift was performed for natural Hamiltonian systems. Here, I shall attempt to do the same by using the constraint to project a curve described by a Riemannian metric onto a fixed hypersurface as a RF metric. Consider the Riemannian metric given below with cyclic co-ordinate \( x^0 = T \):

\[
ds_R = G_{ij}(x)dx^i dx^j + 2G_{i0}(x)dx^i dT + G_{00}(x)dT^2. \tag{5.2}
\]

The canonical momenta from \(5.2\) for \( d\lambda = \sqrt{G_{\mu\nu}(x)dx^\mu dx^\nu} \) and the conserved energy \( k = \text{const.} \) are:

\[
p_T = G_{i0}(x)\frac{dx^i}{d\lambda} + G_{00}(x)\frac{dT}{d\lambda}, \quad p_i = \nabla_i(x)\frac{dx^i}{d\lambda} + kG_{i0}(x)G_{00}(x),
\]

using which I can define a gauge covariant momentum \( \Pi^* \):

\[
\Pi^*_i = p_i - kG_{i0}(x)G_{00}(x),
\]

and use it to write the constraint according to \(3.3\) as:

\[
\psi(x, p) = \sqrt{G^{ij}(x)\Pi^*_i \Pi^*_j + \frac{k^2}{G_{00}(x)}} = 1, \quad \text{where} \quad G^{ik}(x)\nabla_k(x) = \delta^i_j. \tag{5.3}
\]

If we want the constraint \(5.3\) to match the form of \(3.3\), the last term of \(5.3\) must vanish. In simple words, we require that

\[
\frac{k^2}{G_{00}(x)} = 0. \tag{5.4}
\]
Given a RF metric if we can identify a conformal factor in the metric such that
\[ ds = \sqrt{1 - \frac{(k - U(x))^2}{\beta(x)}} G_{ij}(x) dx^i dx^j + \left( A_i(x) + \frac{\alpha_i(x)}{\beta(x)} (k - U(x)) \right) dx^i, \] (5.5)
or in the constraint (4.5) such that
\[ \Gamma(x, p) = \sqrt{1 - \frac{(k - U(x))^2}{\beta(x)}}^{-1} G^{ij}(x) \pi_i \pi_j, \] (5.6)
where \( \pi_i = p_i - \left[ A_i(x) + \frac{\alpha_i(x)}{\beta(x)} (k - U(x)) \right], \)
then by reversing the steps to derive JMRF metric (4.6), I can deduce the ER metric by writing the constraint \( \phi(x, p) \), lifting it (replacing \( k = p_0 \)) and writing the first of constraint equations (3.5).

\[ \phi(x, p) = \sqrt{G^{ij}(x) \pi_i \pi_j + \frac{(p_v - U(x))^2}{\beta(x)}} = \sqrt{\Omega_{\mu\nu}(x) \frac{dx^\mu}{d\theta} \frac{dx^\nu}{d\theta}} = 1, \] (5.7)

\[ \frac{dv}{d\theta} = \frac{\partial \phi}{\partial p_v} = \frac{p_v - U(x)}{\beta(x)} - \frac{\alpha_i(x) dx^i}{\beta(x)} \implies p_v = \beta(x) \frac{dv}{d\theta} + \frac{\alpha_j(x) dx^j}{d\theta} + U(x), \]

\[ \frac{dx^i}{d\theta} = \frac{\partial \phi}{\partial p_i} = G^{ij}(x) \pi_j \implies p_i = g_{ij}(x) \frac{dx^j}{d\theta} + \frac{\alpha_i(x) dx^i}{d\theta} + A_i(x), \] (5.8)

where \( g_{ij}(x) = G_{ij}(x) + \frac{\alpha_i(x) \alpha_j(x)}{\beta(x)} \) Thus, by applying (5.7), (5.8), and the Maupertuis principle (2.4), I complete the Jacobi lift by writing:

\[ ds = p_i dx^i + p_v dv = \Omega_{\mu\nu}(x) \frac{dx^\mu}{d\theta} \frac{dx^\nu}{d\theta} + A_\mu(x) dx^\mu, \]

\[ ds = \sqrt{g_{ij}(x) dx^i dx^j + 2 \alpha_i(x) dx^i dv + \beta(x)(dv)^2 + A_i(x) dx^i + U(x) dv}. \] (5.9)

To lift a RF metric in \( n \) co-ordinates to a Riemannian metric in \( n + 1 \) co-ordinates, we simply identify the conformal factor and gauge fields in (5.5) and (5.6) such that \( A_i(x) = U(x) = 0 \)

\[ \frac{\bar{\beta}(x)}{q^2} = \frac{\beta(x)}{(k - U(x))^2}, \quad q \frac{\bar{\alpha}_i(x)}{\bar{\beta}(x)} = A_i(x) + \frac{\alpha_i(x)}{\beta(x)} (k - U(x)) , \]

\[ ds = \sqrt{1 - \frac{q^2}{\beta(x)}} G_{ij}(x) dx^i dx^j + q \frac{\bar{\alpha}_i(x)}{\bar{\beta}(x)} dx^i, \] (5.10)
such that we get the Riemannian metric:

\[ ds^2 = \left( G_{ij}(x) + \frac{\tilde{a}_i(x)\tilde{a}_j(x)}{\beta(x)} \right) dx^i dx^j + 2\tilde{a}_i(x) dx^i dv + \tilde{\beta}(x)(dv)^2. \quad (5.11) \]

We will next discuss the nature of RF metrics that share the same JMRF.

6 Sharing the JMRF

While we are unable to directly lift a RF metric, we can absorb the gauge potential \( A(x) \) in (3.1) into \( g_{\mu\nu}(x) \) to convert it into a Riemannian metric with the same number of coordinates. However, instead of inserting a new direction of symmetry, this procedure requires identifying a pre-existing one and either replacing or rescaling it. In effect, we will be describing how to formulate all the RF metrics sharing a common JMRF.

Now let us suppose that the RF metric (4.1) and the Riemannian metric (5.2) share the same JMRF described below:

\[ ds_J = \sqrt{1 - \frac{(Q(x))^2}{g_{00}(x)}} \gamma_{ij}(x) dx^i dx^j + \left( A_i(x) + \frac{g_{0i}(x)}{g_{00}(x)} Q(x) \right) dx^i \]

\[ = \sqrt{1 - \frac{k^2}{G_{00}(x)}} \gamma_{ij}(x) dx^i dx^j + k G_{0i}(x) dx^i. \quad (6.1) \]

It is possible to equate the two constraints (4.1) and (5.2), showing that a RF metric can be equated to a Riemannian metric so long as both have at least one cyclic co-ordinate, by writing:

\[ G_{00}(x) = \left( \frac{k}{Q(x)} \right)^2 g_{00}(x), \]

\[ G_{i0}(x) = \frac{k}{Q(x)} \left( g_{i0}(x) + \frac{A_i(x)}{Q(x)} g_{00}(x) \right), \quad (6.2) \]

\[ G_{ij}(x) = g_{ij}(x) + \frac{A_j(x)}{Q(x)} g_{i0}(x) + \frac{A_i(x)}{Q(x)} g_{j0}(x) + \frac{A_i(x)A_j(x)}{(Q(x))^2} g_{00}(x). \]

So according to (6.2) we have the Riemannian metric:

\[ ds_R^2 = (g_{ij}(x) + \Sigma_{ij}(x)) dx^i dx^j \]

\[ + 2 \frac{k}{Q(x)} \left( g_{i0}(x) + \frac{A_i(x)}{Q(x)} g_{00}(x) \right) dx^i dT + \left( \frac{k}{Q(x)} \right)^2 g_{00}(x) dT^2, \quad (6.3) \]

where \( \Sigma_{ij}(x) = \frac{A_j(x)}{Q(x)} g_{i0}(x) + \frac{A_i(x)}{Q(x)} g_{j0}(x) + \frac{A_i(x)A_j(x)}{(Q(x))^2} g_{00}(x). \)

The shared constraint for the two metrics (4.1) and (5.2) is given by:

\[ \phi(x, p) = \sqrt{g^{ij}(x) \Pi_i \Pi_j + \frac{(Q(x))^2}{g_{00}(x)}} = \sqrt{G^{ij}(x) \Pi_i \Pi_j + \frac{k^2}{G_{00}(x)}} = 1. \quad (6.4) \]
As we can see from (6.3), the signature of the metric is preserved, meaning that if \( t \) is time, then \( T \) can be treated as a rescaled time. Furthermore, since \( \psi(x, p, p_T) = \phi(x, p, p_t) \), according to the first equation of (3.5), we can write
\[
\frac{\partial \phi}{\partial p_i} = \frac{\partial \psi}{\partial p_i} \quad \Rightarrow \quad \frac{dx^i}{d\sigma} = \frac{dx^i}{d\lambda} \quad \Rightarrow \quad d\sigma = d\lambda.
\] (6.5)

Applying (3.9) to both, RF (4.1) and Riemannian (6.3) metrics, we can say that according to (6.5)
\[
\begin{align*}
ds_{RF} &= p_i dx^i + q \, dt = d\sigma + A_{\mu}(x)dx^\mu, \\
ds_R &= p_i dx^i + k \, dT = d\lambda,
\end{align*}
\] \( \Rightarrow \quad \frac{dT}{dt} = \omega(x) - \alpha_i(x) \frac{dx^i}{dt} \). (6.6)

where
\[Q(x) = k \omega(x), \quad A_i(x) = k \alpha_i(x),\]

If \( A_i(x) = 0 \), then the time rescaling is position dependent
\[
A_i(x) = 0 \quad \Rightarrow \quad \frac{dT}{dt} = \omega(x). \] (6.7)

While the metric may not have been lifted by increasing the number of canonical pairs, I have converted the action from the Randers-Finsler form into the Riemannian action form free of gauge fields.

On the other hand, under the circumstances that one deals with a static RF metric
\[
ds_{RF} = \sqrt{g_{ij}(x)dx^i dx^j + g_{00}(x)dt^2 + A_i(x)dx^i + A_0(x)dt},
\] (6.8)
this setting (6.3) produces a stationary spactime metric.
\[
ds^2_R = \left( g_{ij}(x) + \frac{g_{00}(x)}{(Q(x))^2} A_i(x)A_j(x) \right) dx^i dx^j + 2k \frac{g_{00}(x)}{(Q(x))^2} A_i(x)dx^i dT + k^2 \frac{g_{00}(x)}{(Q(x))^2} dT^2,
\] (6.9)
thus, supporting the interpretation that the linear terms of a RF metric are comparable to the potential terms of a vector potential, and that motion in a spacetime described by a stationary metric is comparable to motion in the presence of a magnetic field. Furthermore, if we set that \( q = k \) and \( A_0(x) = 0 \), then (6.9) will become:
\[
ds^2_R = (g_{ij}(x) + g_{00}(x)\alpha_i(x)\alpha_j(x)) \, dx^i dx^j + 2g_{00}(x)\alpha_i(x)dx^i dT + g_{00}(x)dt^2,
\] (6.10)
where the time according to (6.6) is given by:
\[
\frac{dT}{dt} = 1 - \alpha_i(x) \frac{dx^i}{dt}.
\] (6.11)

Now I shall analyze the application of this procedure upon some spacetimes as examples.
6.1 Schwarzschild Gullstrand-Painlevé metric

An interesting example to consider is the Schwarzschild metric described in Gullstrand-Painlevé (GP) co-ordinates \([51, 52, 53]\). This is an example where a static metric appears in stationary form due to co-ordinate transformation of the time, and as a result has an apparent magnetic field influencing motion. Let us start by considering the Schwarzschild metric in its regular form:

\[
\begin{align*}
\text{ds}_R^2 &= f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \quad \text{where} \quad f(r) = 1 - \frac{r_0}{r}, \\
&= \left(1 - \frac{r_0}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{r_0}{r}\right)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \quad (6.1.1)
\end{align*}
\]

for which the conserved momentum associated with time \(t\) according to (3.2) is:

\[
\begin{align*}
p_0 &= f(r)\frac{dt}{ds} = k(\text{const}). \\
&= f(r)\frac{dt}{ds} = k(\text{const}). \quad (6.1.2)
\end{align*}
\]

The GP co-ordinate system \(T = t - a(r)\), \(a(r)\) being some function is meant to describe the metric as observed by a radially infalling observer.

\[
\begin{align*}
\text{ds}_R^2 &= f(r)\left( dT^2 + 2a'(r)dT\,dr \right) - \left[ \frac{1}{f(r)} - f(r)\left(a'(r)\right)^2 \right]dr^2 - r^2\left( d\theta^2 + \sin^2\theta\,d\phi^2 \right). \\
&= f(r)\left( dT^2 + 2a'(r)dT\,dr \right) - \left[ \frac{1}{f(r)} - f(r)\left(a'(r)\right)^2 \right]dr^2 - r^2\left( d\theta^2 + \sin^2\theta\,d\phi^2 \right). \quad (6.1.3)
\end{align*}
\]

The momentum associated with \(T\) according to (3.2) is given by:

\[
\begin{align*}
P_0 &= f(r)\left( \frac{dT}{ds} + a'(r)\frac{dr}{ds} \right) = f(r)\frac{dt}{ds} = k(\text{const}). \\
&= f(r)\frac{dT}{ds} + a'(r)\frac{dr}{ds} = f(r)\frac{dt}{ds} = k(\text{const}). \quad (6.1.4)
\end{align*}
\]

which according to (6.1.2) is the same value of constant conserved momentum, implying that Symmetry Replacement procedure can be applied to Schwarzschild metric in GP co-ordinates. If we compare (6.1.3) to (6.9) and choose to set \(q = k\) and \(A_0(x) = 0\), then we can write (6.8) as:

\[
\begin{align*}
\text{ds}_{RF} &= \sqrt{f(r)d\tilde{t}^2 - \frac{1}{f(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) + ka'(r)dr}, \\
&= \sqrt{f(r)d\tilde{t}^2 - \frac{1}{f(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) + ka'(r)dr} \quad (6.1.5)
\end{align*}
\]

where we can see that the additive linear term at the end of (6.1.5) is a gradient of the function \(a(r)\), and is dismissable according to Lagrangian mechanics. We can also see that since we have deduced a RF metric from the Riemannian Schwarzschild GP metric, we have according to (6.6):

\[
\frac{dT}{d\tilde{t}} = 1 - a'(r)\frac{dr}{d\tilde{t}} \quad \Rightarrow \quad \tilde{t} = T + a(r) = t.
\]

thus showing that the Schwarzschild metric in GP co-ordinates is essentially no different from the default Schwarzschild metric (6.1.1) with a linear gradient term added. If one were to deduce the Jacobi metric for the Schwarzschild metric in GP co-ordinates (6.1.3), then we will have according to (4.9):

\[
\begin{align*}
\text{ds}_J &= \sqrt{- \left( 1 - \frac{1}{f(r)} \right)\left[ \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \right] + ka'(r)dr}, \\
&= \sqrt{- \left( 1 - \frac{1}{f(r)} \right)\left[ \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \right] + ka'(r)dr} \quad (6.1.6)
\end{align*}
\]

where again, the linear additive term outside square root is a dismissable gradient term, showing that the final Jacobi metric is the same as that of the familiar Schwarzschild metric.
6.2 Kerr metric

Here, I shall return to exploring the Kerr metric discussed previously in [7]. The Kerr metric describes a rotating uncharged black hole that is a generalisation of the Schwarzschild black hole to include rotation, the exact solution of which was discovered by Kerr in 1963 [54]. The Kerr black hole is readily used as a basic example when discussing the theory of frame dragging effect that occurs around rotating masses. Here, I shall use the equivalent RF form of the Kerr black hole metric to explore the gravitoelectromagnetic interpretation to frame dragging.

The Kerr metric in Boyer-Lindquist co-ordinates is given by:

\[
ds^2_R = \left(1 - \frac{2Mr}{\rho^2}\right)dT^2 + \frac{4Mar\sin^2 \theta}{\rho^2}d\varphi dT - \rho^2 \left[\frac{dr^2}{\Delta} + d\theta^2 + \frac{\sin^2 \theta}{\rho^4} \left\{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta\right\} d\varphi^2\right],
\]

(6.2.1)

where \(\Delta(r) = r^2 - 2Mr + a^2\), \(\rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta\). If we compare (6.2.1) to the form of (6.10) for \(q = k, A_0(x) = 0\), then we shall have:

\[g_{00}(x) = 1 - \frac{2Mr}{\rho^2}, \quad A_\varphi = k \frac{2Mar \sin^2 \theta}{\Delta - a^2 \sin^2 \theta}\]

and its corresponding symmetry replaced RF form according to (6.8) will be:

\[ds_{RF} = \sqrt{\left(1 - \frac{2Mr}{\rho^2}\right)} dt^2 - \rho^2 \left[\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\varphi^2\right] + k \frac{2Mar \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\varphi,\]

(6.2.2)

where \(\frac{dT}{dt} = 1 - A_\varphi \frac{d\varphi}{dt}\),

showing that the Kerr spacetime is comparable to a static spacetime with a magnetic field generated by a magnetic dipole. Naturally, upon setting \(a = 0\), we recover the Schwarzschild metric (6.1.1).

Now, since there is no explicit dependence on the azimuthal angle \(\varphi\) in the metric (6.2.2), we can say that the angular momentum \(p_\varphi\) is preserved, suggesting that the plane of revolution can remain constant. Thus, we can set \(\theta = \frac{\pi}{2}\). If one were to write the geodesic equations of the metric (6.2.2), then we would see that the angular momentum for revolution in the equatorial plane is:

\[p_\varphi = (r^2 + a^2) \frac{\Delta}{\Delta - a^2} \frac{d\varphi}{ds} + \frac{2Mar}{\Delta - a^2} = l(const.),\]

(6.2.3)

from which the azimuthal acceleration is given by:

\[
\frac{d^2 \varphi}{ds^2} = \frac{\partial}{\partial r} \left[\frac{(\Delta - a^2) l - 2Mar}{(r^2 + a^2)\Delta}\right] \frac{dr}{ds}
\]

(6.2.4)

Thus, when there is no radial movement during circular motion, there will be no azimuthal acceleration, just a stable drift at constant radius. If we consider only the external gauge field only, then the Lorentz force contribution in (6.2.4) is given by the 2nd term:

\[F_{Lorentz} = -\frac{\partial}{\partial r} \left[\frac{2Mar}{(r^2 + a^2)\Delta}\right] \frac{dr}{ds},\]

which vanishes when the black hole stops rotating for \(a = 0\). This Lorentz force is another way of interpreting the frame dragging effect of the rotating Kerr black hole.
7 Conclusion and Discussion

I showed that when dealing with RF metrics the constraint can act as a suitable and more general alternative to the Hamiltonian as a generator of Hamilton’s equations of motion. A simple modification of the metric that adds an auxiliary co-ordinate makes the constraint suitable for studying light-like curves as well. Furthermore, we can see that it can be used to solve problems in relativistic mechanics more efficiently, such as formulating a model of relativistic harmonic oscillator, and studying the Schwarzschild RF metric, describing its massive particle and null curves, and the radii of the photon spheres.

Next, I deduced the Jacobi metric for a given RF metric for an autonomous relativistic system using the constraint, and discussing two settings as examples. I also discuss the modification of the metric for optical curves, showing that it allows us to deduce the Jacobi metric, and showed that it is distinct from the optical metric deduced according to Fermat’s principle of path of least time for stationary spacetimes. I also discussed the frame dragging effect from a Hamiltonian mechanics approach using constraint mechanics. Since frame dragging manifests from the cross terms of stationary spacetimes, I derived it for RF metrics and the JMRF metric. Here, we can see that mechanics with the constraint is more suitable than with the Hamiltonian since cases like the Jacobi metric has no Hamiltonian.

I then showed that the Eisenhart lift cannot be directly applied to an RF metric in the manner it is for non-relativistic problems. The only way out is to identify the RF metric as a JMRF metric and reverse the process of its derivation to lift it into a Riemannian metric, thus geometrizing the gauge potentials. Thus, the process of Eisenhart lift can also be applied to relativistic systems.

Finally, I discussed autonomous pairs of RF and Riemannian metrics that share a common JMRF. The case of stationary Riemannian metric and a static RF metric shows that the cross terms are dynamically comparable to magnetic gauge fields. When applied to Schwarzschild Painlevé metric to derive a static RF metric, we see that the cross term introduced via co-ordinate transformation is comparable to a total function derivative, which is dismissible from any Lagrangian. I also deduced a RF metric sharing the same JMRF with the Kerr metric as an exercise to show that the cross terms act as a magnetic vector potential.

Acknowledgements I wish to acknowledge P. Guha, G. W. Gibbons, P. Maraner, M. Werner, M. Cariglia and Sanved Kolekar for various discussions related to this topic and their support that was essential in the preparation of this article, and thank P. Horvathy, K. Morand, A. Galajinsky, E. Minguzzi, M.F. Ranada, and Joydeep Chakravarty for supportive comments that helped improve and develop its content.

References

[1] O. C. Ong, Curvature and mechanics, Adv. Math. 15 (1975) 269-311.
[2] M. Szydłowski, M. Heller, and W. Sasin, Geometry of spaces with the Jacobi metric, J. Math. Phys. 37 (1996) 346-360.
[3] G. W. Gibbons, The Jacobi metric for timelike geodesics in static spacetimes, Class. Quantum Grav. 33 (2015) 025004.
[4] S. Chanda, G.W. Gibbons and P. Guha, Jacobi-Maupertuis-Eisenhart metric and geodesic flows, J. Math. Phys. 58 (2017), 032503.
[5] S. Chanda, G.W. Gibbons and P. Guha, Jacobi–Maupertuis metric and Kepler equation, Int. J. Geom. Methods Mod. Phys. 14 (2017) 1730002.
[6] A.A. Izquierdo, M.A. Leon, J.M. Guilarte, and M. Mayado, Jacobi metric and Morse theory of dynamical systems, arXiv: math-ph/0212017.
[7] S. Chanda, G. W. Gibbons, P. Guha, P. Maraner, and M.C. Werner, *Jacobi-Maupertuis Randers-Finsler metric for curved spaces and the gravitational magnetoelectric effect*, J. Math. Phys. **60** (2019), 122501.

[8] P. Maraner, *On the Jacobi metric for a general Lagrangian system*, J. Math. Phys. **60** (2019) 112901.

[9] A. Bera, S. Ghosh, B.R. Majhi, *Hawking radiation in a non-covariant frame: the Jacobi metric approach*, Eur. Phys. J. Plus **135** (2020) 670.

[10] M. Arganaraz, and O.L. Andino, *Dynamics in wormhole spacetimes: a Jacobi metric approach*, Class. Quantum Grav. **38** (2020) 045004.

[11] Z. Li, and J. Jia, *The finite-distance gravitational deflection of massive particles in stationary spacetime: a Jacobi metric approach*, Eur. Phys. J. C **80** (2020) 1-13.

[12] Z. Li, and J. Jia, *Kerr-Newman-Jacobi geometry and the deflection of charged massive particles*, Phys. Rev. D **104** (2021) 044061.

[13] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (W. H. Freeman and Company, San Francisco, 1973).

[14] S.M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, (Addison Wesley, San Francisco, CA, USA, 2004).

[15] G. Randers, *On an asymmetrical metric in the four-space of general relativity*, Phys. Rev. **59** (1941) 195.

[16] E. Kapsabelis, A. Triantafyllopoulos, S. Basilakos, and P.C. Stavrinos, *Schwarzschild-like solutions in Finsler-Randers gravity*, Eur. Phys. J. C **80** (2020) 1-14.

[17] E. Kapsabelis, A. Triantafyllopoulos, S. Basilakos, and P.C. Stavrinos, *Application of the Schwarzschild-Finsler-Randers model*, Eur. Phys. J. C **81** (2021) 1-11.

[18] E. Kapsabelis, P. G. Kevrekidis, P. C. Stavrinos, and A. Triantafyllopoulos, *Schwarzschild-Finsler-Randers spacetime: Geodesics, Dynamical Analysis and Deflection Angle*, Eur. Phys. J. C **82** (2022) 1098.

[19] K.J. Epstein, *Hamiltonian approach to frame dragging*, Gen. Relativ. Gravit. **40** (2008) 1367-1378.

[20] L.P. Eisenhart, *Dynamical trajectories and geodesics*, Ann. Math. **30** (1928) 591-606.

[21] A. Lichnerowicz, and T. Teichmann, *Théories relativistes de la gravitation et de l’électromagnétisme*, Physics Today **8** (1955) 24.

[22] C. Duval, G. Burdet, H.P. Künzle, and M. Perrin, *Bargmann structures and Newton-Cartan theory*, Phys. Rev. D **31** (1985) 1841.

[23] E. Minguzzi, *Classical aspects of lightlike dimensional reduction*, Class. Quantum Grav. **23** (2006) 7085.

[24] E. Minguzzi, *Eisenhart’s theorem and the causal simplicity of Eisenhart’s spacetime*, Class. Quantum Grav. **24** (2007) 2781.

[25] M. Cariglia, *Hidden symmetries of dynamics in classical and quantum physics*, Rev. Mod. Phys. **86** (2014) 1283.

[26] M. Cariglia, and G.W. Gibbons, *Generalised Eisenhart lift of the Toda chain*, J. Math. Phys. **55** (2014) 022701.

[27] M. Cariglia, G.W. Gibbons, J.W. van Holten, P.A. Horvathy, and P.M. Zhang, *Conformal Killing tensors and covariant Hamiltonian dynamics*, J. Math. Phys. **55** (2014) 122702.
[28] M. Cariglia, and F.K. Alves, The Eisenhart lift: a didactical introduction of modern geometrical concepts from Hamiltonian dynamics, Eur. J. Phys. 36 (2015) 025018.
[29] A. Galajinsky, and I. Masterov, Eisenhart lift for higher derivative systems, Phys. Lett. B 765 (2017) 86-90.
[30] C. Duval, G.W. Gibbons, and P. Horváthy, Celestial mechanics, conformal structures, and gravitational waves, Phys. Rev. D 43 (1991) 3907.
[31] X. Bekaert, and K. Morand, Embedding nonrelativistic physics inside a gravitational wave, Phys. Rev. D 88 (2013) 063008.
[32] S. Filyukov, and A. Galajinsky, Self-dual metrics with maximally superintegrable geodesic flows, Phys. Rev. D 91 (2015) 104020.
[33] M. Cariglia, and A. Galajinsky, Ricci-flat spacetimes admitting higher rank Killing tensors, Phys. Lett. B 744 (2015) 320-324.
[34] J.F. Carinena, F.J. Herranz, and M.F. Ranada, Superintegrable systems on 3-dimensional curved spaces: Eisenhart formalism and separability, J. Math. Phys. 58 (2017) 022701.
[35] A.P. Fordy, and A. Galajinsky, Eisenhart lift of 2-dimensional mechanics, Eur. Phys. J. C 79 (2018) 301.
[36] A. Galajinsky, Geometry of the isotropic oscillator driven by the conformal mode, Eur. Phys. J. C 78 (2018) 72.
[37] M. Cariglia, A. Galajinsky, G. W. Gibbons, and P. A. Horvathy, Cosmological aspects of the Eisenhart-Duval lift, Eur. Phys. J. C 78 (2018) 314.
[38] K. Morand, Embedding Galilean and Carrollian geometries I. Gravitational waves, J. Math. Phys. 61 (2020) 082502.
[39] A. Sen, B. K. Parida, S. Dhasmana, and Z. K. Silagadze, Eisenhart lift of Koopman-von Neumann mechanics, J. Geom. Phys. 185 (2023) 104732.
[40] K. Finn, S. Karamitsos, and A. Pilaftsis, Eisenhart lift for field theories, Phys. Rev. D 98 (2018) 016015.
[41] S. Chanda, P. Guha, Eisenhart lift and Randers-Finsler formulation for scalar field theory, Eur. Phys. J. Plus 136 (2021) 1-9.
[42] P.L. Maupertuis, (1744). Accord de différentes loix de la nature qui avoient jusqu’ici paru incompatibles, Académie Internationale d’Histoire des Sciences, Paris.
[43] L. Euler, (1744). Methodus inveniendi/Additamentum II.
[44] S. Chanda, and P. Guha, Geometrical formulation of Relativistic Mechanics, Int. J. Geom. Methods Mod. Phys. 15 (2018) 1850062.
[45] H. Goldstein, C. Poole and J. Safko, Classical Mechanics, 3rd Edition, Addison Wesley, Boston, 2002.
[46] M. Born, and E. Wolf, Basic properties of the electromagnetic field, Principles of optics 44 (1980).
[47] J. Bernoulli, Problema novum ad cujus solutionem Mathematici invitantur, Acta Eruditorum 18 (June, 1696) 269.
[48] I. Newton, De ratione temporis quo grave labitur per rectam data puncta conjungentem, ad tempus brevissimum quo, vi gravitatis, transit ab horum uno ad alterum per arcum cycloidis, Philosophical Transactions of the Royal Society of London 19 (January, 1697) 424-425.
[49] H. Erlichson, Johann Bernoulli’s brachistochorone solution using Fermat’s principle of least time, Eur. J. Phys. 20 (1999) 299.
[50] H.W. Broer, *Bernoulli’s light ray solution of the brachistochrone problem through Hamilton’s eyes*, Int. J. Bifurc. Chaos 24 (2014) 1440009.

[51] P. Painlevé, *La mécanique classique et la théorie de la relativité*, C. R. Acad. Sci. (Paris) 173 (1921) 677-680.

[52] A. Gullstrand, *Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie*, Arkiv för Matematik, Astronomi och Fysik. 16 (1922) 1-15.

[53] G. Lemaitre, *L’Univers en expansion*, Annales de la Société Scientifique de Bruxelles. A53 (1933) 51-85.

[54] R. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Phys. Rev. Lett. 11 (1963) 237.