Thermal Hall effect, spin Nernst effect, and spin density induced by thermal gradient in collinear ferrimagnets from magnon-phonon interaction

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We theoretically study the intrinsic thermal Hall and spin Nernst effect in collinear ferrimagnets on a honeycomb lattice with broken inversion symmetry. The broken inversion symmetry allows in-plane Dzyaloshinskii-Moriya interaction between the nearest neighbors, which does not affect the magnon spectrum in the linear spin wave theory. However, the Dzyaloshinskii-Moriya interaction can induce large Berry curvature in the magnetoelastic excitation spectrum through the magnon-phonon interaction to produce thermal Hall current. Furthermore, we find that the magnetoelastic excitations transport spin, which is inherited from the magnon bands. Therefore, the thermal Hall current is accompanied by spin Nernst current. Because the magnon-phonon interaction does not conserve the spin, we also study the spin density induced by thermal gradient in the presence of magnon-phonon interaction. We find that the intrinsic part of the spin density shows no asymmetric spin accumulation near the boundary of the system having a stripe geometry. However, because of the magnon-phonon interaction, we find nonzero total spin density in the system having armchair edges. The extrinsic part of the spin density, on the other hand, shows asymmetric spin accumulation near the boundary for both armchair and zigzag edges because of the magnon-phonon interaction. In addition, we find nonzero total spin density in the system having zigzag edge.

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ticle lifetime, reflects the spin Nernst current through asymmetric spin accumulation [50]. Moreover, nonzero total spin expectation value is induced by the intrinsic (extrinsic) mechanism in the system having armchair (zigzag) edges, but not for zigzag (armchair) edges. Finally, we discuss the relevance of our model to the thermal Hall conductivity measured in Fe$_2$Mo$_3$O$_8$ [31].

Model.— Our model is motivated by Fe$_2$Mo$_3$O$_8$, which consists of Fe-O layers separated by sheets of Mo$_4$O$_7$ [32] [33]. The Fe$^{2+}$ ions in the Fe-O layer form a honeycomb-like lattice as in Fig.1(a), where the Fe$^{2+}$ ions at $A$ and $B$ sites are displaced vertically with respect to each other. Because the $A$ ($B$) sites are surrounded by octahedral (tetrahedral) cage of oxygen atoms, the inversion symmetry between the nearest neighbors is broken. However, the mirror symmetry $\mathcal{M}_z: (x, y) \rightarrow (-x, y)$ about the center of a hexagon is retained.

For our model, we simplify the lattice as shown in Fig.1(b), where we keep only the magnetic ions forming a two-dimensional honeycomb lattice. We consider the spin Hamiltonian given by

$$\mathcal{H}_m = \mathcal{H}_J + \mathcal{H}_D + \mathcal{H}_\alpha + \mathcal{H}_H.$$  \hspace{1cm} (1)

Here, $\mathcal{H}_J = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j$ with $J_1 > 0$ is the antiferromagnetic nearest-neighbor Heisenberg interaction, and $\mathcal{H}_D = J_2^A \sum_{\langle ij \rangle} S_i \cdot S_j + J_2^B \sum_{\langle ij \rangle} S_i \cdot S_j$ is the ferromagnetic next-nearest-neighbor Heisenberg interaction between the $A$ sites ($J_2^A < 0$) and $B$ sites ($J_2^B < 0$). To reflect the broken inversion symmetry that relates $A$ and $B$ sites, we put $J_2^A \neq J_2^B$ and include the nearest-neighbor DM interaction $\mathcal{H}_D = \sum_{\langle ij \rangle} D_{ij} \cdot [S_i \times S_j]$, where the direction of $D_{ij}$ is indicated in Fig.1. We note, however, that the DM interaction does not contribute to the magnon spectrum because the magnetic ordering direction, normal to the honeycomb plane, is perpendicular to the DM vector. Finally, $\mathcal{H}_\alpha = \sum_i \alpha_i (S_i^z)^2$ with $\alpha_i < 0$ is the easy-axis anisotropy ($\alpha_A \neq \alpha_B$) and $\mathcal{H}_H = \sum_i \mu_i S_i \cdot \mathbf{H}$ is the Zeeman coupling to the external magnetic field applied parallel to the magnetic ordering direction ($\mu_A \neq \mu_B$).

The magnon Hamiltonian is obtained by writing $S_i = \hat{x}_i S_i^x + \hat{y}_i S_i^y + \hat{z}_i S_i^z$ where $\hat{x}_i$, $\hat{y}_i$, and $\hat{z}_i$ are local orthogonal coordinates, and introducing the Holstein-Primakoff operators $a_i$ and $a_i^\dagger$ as detailed in the Supplemental Material (SM) [34]. The magnon spectrum with $\mathbf{H} = 0$ is shown in Fig.2(a). We find that the upper (lower) magnon band carries spin $-1 (+1)$ by using the definition of the magnon spin operator $\mathbf{S}^\alpha = \sum_i \text{sgn}(i) a_i^\dagger a_i$, where $\text{sgn}(i) = -1 (+1)$ when $i$ is one of the $A$ ($B$) sites. Note that although the DM interaction breaks the SO(2) symmetry about the $z$ axis, it does not appear in the linear spin wave theory, and the magnon spin is conserved in this limit.

For the phonon Hamiltonian, we consider a simple harmonic oscillator model of the form

$$\mathcal{H}_p = \frac{1}{2} \sum_{\langle ij \rangle} \left[ \mathbf{p}_i^2 \delta_{ij} + \mathbf{u}_i K_{ij} (\mathbf{R}_i - \mathbf{R}_j) \mathbf{u}_j \right],$$  \hspace{1cm} (2)

where $\mathbf{R}_i$ is the position of the atom at site $i$ and $K_{ij} (\mathbf{R}_i - \mathbf{R}_j)$ are the spring constant matrices between the atoms at sites $i$ and $j$. Note that we have absorbed the atomic mass $M_i$ by defining the rescaled displacement and momentum operators $\hat{u}_i = \sqrt{M_i} u_i$ and $\hat{p}_i = \frac{p_i}{\sqrt{M_i}}$, where $u_i$ and $p_i$ are the displacement and momentum operators. Let $K_L$ and $K_T$ be the nearest-neighbor longitudinal and transverse spring constants, respectively, and let $k_L^A$ and $k_T^A$ ($k_L^B$ and $k_T^B$) be the next-nearest-neighbor longitudinal and transverse spring constants between AA (BB) sites. These can be organized into spring constant matrices as discussed in the SM [34]. The resulting phonon spectrum is shown in Fig.2(a).

Finally, we consider the MPI arising from the fluctuation of $D_{ij}$ when the atoms deviate from their equilibrium positions by Taylor expanding $D_{ij}$ in terms of $u_i - \bar{u}_i$, where $l$ is the distance between nearest neighbors (the full expression of MPI is given in the SM). Note that because MPI mixes magnon and phonon, $\delta^\alpha$ is not conserved. We show the magnetoelastic excitation spectrum with MPI in Fig.2(b), where we also turn on the out-of-plane magnetic field. The energy bands with up and down spins respond oppositely to the magnetic field, so that with our choice of parameters (see Fig.2), the energy of the spin up and down bands are lowered and raised, respectively. This produces two overlapping regions near the $\Gamma$ point between the magnon and phonon bands, which hybridize because of the DM interaction. This is clarified in Fig.2(c), where the two anticrossing regions between energy bands 4 and 5 near the $\Gamma$ point are indicated with dotted circles. It is important to note that these two anticrossing regions correspond to the Berry curvature hotspots encircling the $\Gamma$ point in Fig.2(d), which are crucial for the thermal Hall and spin Nernst effect.

Thermal Hall conductivity.— The thermal Hall conductivity $\kappa_{xy}$ is defined by the expression $j_y^Q = -\kappa_{xy} \nabla_y T$, where $j_y^Q$ is the heat current and $T$ is the temperature. The semiclassical and linear response theories both yield $\kappa_{xy} = \kappa_{xy}^{\text{class}}$ and $\kappa_{xy}^{\text{lin}}$.
Energy breaks the κdom. axis, which acts on both spin and position degrees of freedom. The parameters for the magnon Hamiltonian are \( K_L = 120 \) (meV)\(^2\), \( K_T = 25 \) (meV)\(^2\), \( k_1 = 10 \) (meV)\(^2\), \( k_2 = 5 \) (meV)\(^2\). The highest to lowest energy bands are labeled from 1 to 6. The color line indicates the magnon and phonon content of the magnetoelectronic modes. (c) Close-up view of the two anticrossing regions \((\alpha, \beta)\) near the Γ point, with a tiny gap at each crossing point. (d) Berry curvature density for energy bands 4 and 5. The boundary of the first Brillouin zone is indicated with dotted line.

\[
-k^2 T \sum_k \sum_{n=1}^{N} \left[ c_2(g(E_{n,k})) - \frac{2}{\pi^2} \right] \Omega_{n,k},
\]

where the summation is over only the particle bands, \( \Omega_{n,k} \) is the Berry curvature, \( c_2(x) = (1 + x) \ln(1 + x) - (\ln x)^2 - 2Liz(-x) \), and \( Liz \) is the polylogarithm function \( Li_n \) for \( n = 2 \).

Without MPI, the individual magnon and phonon bands satisfy the \( \mathcal{M}_x \mathcal{C}_{2z}^S \) symmetry, which forces \( \kappa_{xy} = 0 \) [34]. Here, \( \mathcal{M}_x \) is the mirror symmetry about the plane normal to the \( x \) axis, which acts on both spin and position degrees of freedom. \( \mathcal{C}_{2z}^S \) acts only on the spin degrees of freedom, and it rotates all of the spin about the \( x \) axis by \( \pi \) without changing their position.

Because DM interaction requires spin-orbit coupling, MPI breaks the \( \mathcal{M}_x \mathcal{C}_{2z}^S \) symmetry, which decouples spin and orbital degrees of freedom. We therefore obtain nonzero \( \kappa_{xy} \) as shown in Fig. 3 (a). Because the lowest three magnetoelectronic bands in our model do not carry Chern numbers [34], the sign change in \( \kappa_{xy} \) around 15K cannot be explained by sign alternation of Chern numbers [38, 39] between the magnetoelectronic bands. Instead, we notice that the two Berry curvature hotspots \( \alpha, \beta \) near the Γ point have opposite signs. Therefore, at low temperature, the smaller region \((\alpha)\) with energy approximately 1 meV and negative Berry curvature is the main contributor, so that \( \kappa_{xy} < 0 \) (note that \( k_B T \approx 0.86 \) meV). On the other hand, the larger region \((\beta)\) with positive Berry curvature is located around 5 meV, and therefore starts to contribute significantly at higher temperature to flips the sign of \( \kappa_{xy} \).

Spin Nernst effect.— The spin Nernst coefficient \( \alpha_{xy}^S \) is defined from the expression for the spin current density \( j_{xy}^S = -\alpha_{xy}^S \nabla_y T \) when the spin is conserved, the semiclassical [19] and the linear response theory both \[4, 5, 20, 29, 40\]. Writing the linear response theory in Fig. 3 (b) (blue curve). As can be seen, the behavior of the \( \alpha_{xy}^S \) closely follows \( \kappa_{xy} \). This is because the low-energy magnetoelectronic excitations with large Berry curvature are mixtures of phonon and magnon with spin +1, so that thermal Hall current is accompanied by spin Nernst current. However, in contrast to \( \kappa_{xy} \), \( \alpha_{xy}^S \) does not change sign near \( T = 15K \) because of the subtle distribution of \( \langle \delta^S \rangle_{n,k} \Omega_{n,k} \), which is analyzed in the SM [34].

Because it is not clear what approximations are made in the semiclassical approach, we also derive \( \alpha_{xy}^S \) using the semiclassical theory in Fig. 3 (b) (blue curve). As can be seen, the behavior of the \( \alpha_{xy}^S \) closely follows \( \kappa_{xy} \). This is because the low-energy magnetoelectronic excitations with large Berry curvature are mixtures of phonon and magnon with spin +1, so that thermal Hall current is accompanied by spin Nernst current. However, in contrast to \( \kappa_{xy} \), \( \alpha_{xy}^S \) does not change sign near \( T = 15K \).

FIG. 3: (a) Thermal Hall conductivity and (b) spin Nernst coefficient arising from MPI computed with the parameters used in Fig. 2 (b).
where \( v = \frac{1}{\hbar} \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \), \( w = S^z \tau_z v \), and \( u = v \tau_z S^z \).

We show \( \alpha_{xy}^{\text{SP}} \) calculated using this approximation in Fig. 5 (b) (red curve). We find that the behavior does not differ significantly from spin Nernst coefficient calculated using the semiclassical theory (blue curve).

**Spin density.**—Because spin is not conserved by MPI and edge spin accumulation is the experimentally measurable consequence of the spin current [43], we study the spatial distribution of spin density as in electronic systems [34, 41–44]. The spin density \( \zeta_{xy} \) induced by the thermal gradient at position \( r \) is given by \( \langle \delta S^x(r) \rangle = \langle S^x(r) \rangle_{\text{eq}} - \langle S^x(r) \rangle_0 = -\zeta_{x}(r) \nabla \nu T \). Let us divide \( \zeta_{xy}(r) = \zeta_{xy}^{\text{in}}(r) + \zeta_{xy}^{\text{ext}}(r) \) and study the two parts separately. Here, the intrinsic part \( \zeta_{xy}^{\text{in}}(r) \) is independent of quasiparticle lifetime, while the extrinsic part \( \zeta_{xy}^{\text{ext}}(r) \) is approximately proportional to the quasiparticle lifetime.

Let us first examine \( \zeta_{xy}^{\text{in}}(r) \) using the Kubo’s formula [34]. We find that when the system has zigzag edge, spin density uniformly vanishes whether or not there is MPI. Similarly, for the armchair edge, we find a symmetric distribution of spin, regardless of the presence of MPI. On the other hand, \( \sum_x \zeta_{xy}^{\text{in}}(x)\big|_{D=0} = 0 \) for the armchair edge when there is MPI while \( \sum_x \zeta_{xy}^{\text{in}}(x)\big|_{D=0} = 0 \) when there is no MPI. Thus, although MPI does not induce asymmetric edge spin accumulation through the intrinsic mechanism, it can change the total spin density of the system under thermal gradient. In Fig. 4 (a), we show the spin density caused by MPI for the armchair edge, i.e., \( \zeta_{xy}^{\text{in}}(x)\big|_{D=0} = \zeta_{xy}^{\text{in}}(x)\big|_{D=0} \).

To observe the spin accumulation induced by spin Nernst current, we need to consider the finite quasiparticle lifetime [30]. We study this extrinsic effect using the Boltzmann transport theory within the constant relaxation time approximation [30]. The spin density \( \zeta_{xy}^{\text{ext}}(r) \) is approximately proportional to the quasiparticle lifetime.

Whenever the spin is not conserved and the system has sufficiently low symmetry, such as broken inversion symmetry. The spatial distribution and the total sum of spin also depend strongly on the direction of thermal gradient. These behaviors in our model can be explained using symmetry arguments, which is given in the SM.

**Material realization.**—We suggest that the thermal Hall current arising from MPI may be relevant to Fe\(_2\)Mo\(_3\)O\(_8\). In Ref. [31], giant thermal Hall conductivity of the order \( 10^{-2} \text{Wm}^{-1}\text{K}^{-1} \) was observed in Fe\(_2\)Mo\(_3\)O\(_8\), which was attributed to skew-scattering of phonon. According to the phonon scattering mechanism, \( \kappa_{xx} \propto \kappa_{xy} \propto \tau_1 \), where \( \tau_1 \) is the phonon lifetime. However, the data in Ref. [31] suggests that this relation does not hold at large magnetic field and high temperature. We suggest that this may be due to the intrinsic contribution to the thermal Hall conductivity originating from the MPI. Although the parameters for magnon and phonon are not available to us except for the magnetic moment [46], from Fig. 4 and the interlayer distance of Fe\(_2\)Mo\(_3\)O\(_8\), we can estimate the order of magnitude of the thermal Hall conductivity arising from the band crossing between one magnon and one phonon bands to be around \( 0.3 \times 10^{-3}\text{Wm}^{-1}\text{K}^{-1} \) when \( D = 0.94\text{meV} \). Therefore, depending on the material parameters, the MPI can potentially generate a thermal Hall response order of \( 10^{-3}\text{Wm}^{-1}\text{K}^{-1} \) to throw off the relation \( \kappa_{xx} \propto \kappa_{xy} \propto \tau_1 \).

**Discussion.**—In this work, we examined the heat and spin responses arising from MPI in a noncentrosymmetric collinear ferromagnet with anisotropy. The unique feature of ferromagnets is that either of the spin \( \pm 1 \) magnon band always decreases in energy to hybridize with phonon band when magnetic field is applied parallel to the magnetic order, which is important for thermal Hall and spin Nernst effect. This is distinct from ferromagnets, in which all of the magnon bands increase or decrease in energy. Moreover, the intrinsic spin Nernst current originating from MPI induces edge spin accumulation through the extrinsic contribution to the spin density.
This can serve as an indicator of MPI contribution to the thermal Hall conductivity, since magnon does not, by itself, show thermal Hall or spin Nernst effect. However, since MPI does not conserve the spin, revealing the correspondence between the spin accumulation and the spin current is quite a subtle issue, which we leave for future study. Finally, we showed that nonzero total spin density induced by thermal gradient can serve as an additional evidence of MPI.

Note added—During the preparation of our manuscript, a related work appeared in which the thermal Hall effect in antiferromagnetic insulators is discussed [47]. Although their work also mentions the spin Nernst effect, it is more focused on the SU(3) topology of magnetoelastic excitations and the ensuing thermal Hall conductivity. We focus more on the correspondence between the spin Nernst current and the spin density.

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should be a correction to the definition of magnetization resulting from the gravitational potential as noted in Refs. [48, 49].
Supplemental Material (SM)

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SM A: Magnon

In this section, we study the magnon Hamiltonian given in Eq. (1) of the main text. To carry out the spin wave expansion, we introduce the local axes

\[ \hat{x}_A = (1, 0, 0), \quad \hat{y}_A = (0, 1, 0), \quad \hat{z}_A = (0, 0, 1) \]
\[ \hat{x}_B = (1, 0, 0), \quad \hat{y}_B = (0, -1, 0), \quad \hat{z}_B = (0, 0, -1), \] (A1)

so that \( S_i = S_i^x \hat{x}_i + S_i^y \hat{y}_i + S_i^z \hat{z}_i \) where \( i = A, B \). To obtain the linear spin wave theory, it suffices to put

\[ S_i^x = \frac{\sqrt{2} \gamma}{2} (a_i + a_i^\dagger), \quad S_i^y = \frac{\sqrt{2} \gamma}{2i} (a_i - a_i^\dagger), \quad S_i^z = S - a_i^\dagger a_i. \] (A2)

We note any orthonormal system of local axes is valid so long as \( \hat{z}_i \) points along the magnetic order.

To the quadratic order in the Holstein-Primakoff (HP) operators, the nearest-neighbor interaction is \( (J_1 > 0) \)

\[ \mathcal{H}_{J_1} = \sum_{(ij)} J_1 S_i \cdot S_j \]
\[ = -3 J_1 N (S^2 + S) + \frac{1}{2} \sum_k \gamma_k H_{J_1} (k) \phi_k, \] (A3)

where

\[ H_{J_1} (k) = J_1 S \begin{bmatrix} 3 & 0 & 0 & \gamma_k^* \\ 0 & 3 & \gamma_k & 0 \\ 0 & \gamma_k^* & 3 & 0 \\ \gamma_k & 0 & 0 & 3 \end{bmatrix}. \] (A4)

and

\[ \phi_k = \begin{bmatrix} a_{A,k} \\ a_{B,k} \\ a_{N,k}^{\dagger} \\ a_{N,k} \end{bmatrix}. \] (A5)

We have defined the function \( \gamma_k = \sum_\delta e^{i \delta k} \), where \( \delta_1 = (0, 1)l, \delta_2 = (-\sqrt{3}, -\frac{1}{2})l \) and \( \delta_3 = (\sqrt{3}, -\frac{1}{2})l \) are the relative positions of the nearest neighbors, and \( l \) is the distance between \( A \) and \( B \).

Similarly, the next-nearest-neighbor Heisenberg interaction is \( (J_A, J_B < 0) \)

\[ \mathcal{H}_{J_2} = J_2^A \sum_{(ij)} S_i \cdot S_j + J_2^B \sum_{(ij)} S_i \cdot S_j \]
\[ = 3 (J_2^A + J_2^B) N (S^2 + S) + \frac{1}{2} \sum_k \phi_k^2 H_{J_2} (k) \phi_k, \] (A6)

where

\[ H_{J_2} (k) = S (-6 + \tilde{\gamma}_k) \begin{bmatrix} J_2^A & 0 & 0 & 0 \\ 0 & J_2^B & 0 & 0 \\ 0 & 0 & J_2^A & 0 \\ 0 & 0 & 0 & J_2^B \end{bmatrix}. \] (A7)

Here, we defined \( \tilde{\gamma}_k = \sum_\delta e^{i \delta k} \), where \( \delta \) are the relative positions of the six next-nearest neighbors.

Let us note that the easy axis anisotropy must be along the \( z \) direction because of the three-fold rotations about \( A \) and \( B \) sites. We have \( (\alpha_A, \alpha_B < 0) \)

\[ \mathcal{H}_{\alpha} = \sum_i \alpha_i (S_i^z)^2 \]
\[ = (\alpha_A + \alpha_B) N (S^2 + S) + \frac{1}{2} \sum_k \phi_k^2 H_{\alpha} (k) \phi_k, \] (A8)

where

\[ H_{\alpha} (k) = -2 S \begin{bmatrix} \alpha_A & 0 & 0 & 0 \\ 0 & \alpha_B & 0 & 0 \\ 0 & 0 & \alpha_A & 0 \\ 0 & 0 & 0 & \alpha_B \end{bmatrix}. \] (A9)

Applying the magnetic field \( \mathbf{H} = H \hat{z} \), we have

\[ \mathcal{H}_{H} = \sum_i \mu_i S_i \cdot \mathbf{H} \]
\[ = H (\mu_A - \mu_B) N (S + \frac{1}{2}) + \frac{1}{2} \sum_k \phi_k^2 H_{H} (k) \phi_k, \] (A10)

where

\[ H_{H} (k) = H \begin{bmatrix} -\mu_A & 0 & 0 & 0 \\ 0 & \mu_B & 0 & 0 \\ 0 & 0 & -\mu_A & 0 \\ 0 & 0 & 0 & \mu_B \end{bmatrix}. \] (A11)
Let us briefly discuss the $M_x \mathbb{Z}_2$ symmetry mentioned in the main text. In the spin sector, both $M_x$ and $\mathbb{Z}_2$ rotates the spin direction about the $x$ axis by $180^\circ$, so that when combined, it does not change the spin direction. However, $M_x$ sends the spin at $(x,y)$ to $(-x,y)$. Thus, the action of the $M_x \mathbb{Z}_2$ symmetry on the HP operators is $a_i(k_x,k_y) \rightarrow a_i(-k_x,k_y)$ for $i = A, B$. It is easy to see that the magnon Hamiltonian satisfies this symmetry from the expressions above.

SM B: Phonon

In this section, we list the spring constant matrices. Using the expression for the spring constant matrix between the nearest neighbor $A$ and $B$,

$$K(\delta_1) = \begin{bmatrix} -K_T & 0 \\ 0 & -K_L \end{bmatrix} \quad \text{(B1)}$$

and imposing the $\mathbb{Z}_2$ symmetry, we have $K(\delta_2) = C_{3z} K(\delta_1) C_{3z}$ and $K(\delta_3) = C_{3z} K(\delta_1) C_{3z}$, where

$$C_{3z} = \frac{-1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{bmatrix} \quad \text{(B2)}$$

For the next-nearest neighbors between the $A$ sites, we have

$$K(\delta_3 - \delta_2) = \begin{bmatrix} -k_{T}^{A} & 0 \\ 0 & -k_{T}^{A} \end{bmatrix} \quad \text{(B3)}$$

$K(\delta_1 - \delta_3) = C_{3z} K(\delta_1 - \delta_2) C_{3z}$ and $K(\delta_2 - \delta_1) = C_{3z} K(\delta_1 - \delta_2) C_{3z}$, while the other spring constants follow from the identity $K(\Delta \mathbf{R}) = K(-\Delta \mathbf{R})$. The spring constant matrices between the $B$ sites is obtained by making the replacement $k_{T}^{A} \rightarrow k_{T}^{B}, k_{L}^{A} \rightarrow k_{L}^{B}$. Finally, the onsite potentials follow from the condition that there is no change in Hamiltonian from uniform shift of the lattice. The contributions from nearest-neighbor interactions to the $A$ and $B$ sites are respectively given by

$$K_{nn}(0) = \frac{1}{2} \begin{bmatrix} \frac{3}{2}(K_T + K_L) & 0 \\ 0 & \frac{3}{2}(K_T + K_L) \end{bmatrix}$$

$$K_{nn}'(0) = \frac{1}{2} \begin{bmatrix} \frac{9}{2}(K_T + K_L) & 0 \\ 0 & \frac{3}{2}(K_T + K_L) \end{bmatrix} \quad \text{(B4)}$$

For the next-nearest neighbors, we have

$$K_{nnn}(0) = \begin{bmatrix} \frac{3}{2}(k_{T}^{A} + k_{L}^{A}) & 0 \\ 0 & \frac{3}{2}(k_{T}^{A} + k_{L}^{A}) \end{bmatrix}$$

$$K_{nnn}'(0) = \begin{bmatrix} \frac{3}{2}(k_{T}^{B} + k_{L}^{B}) & 0 \\ 0 & \frac{3}{2}(k_{T}^{B} + k_{L}^{B}) \end{bmatrix} \quad \text{(B5)}$$

SM C: Magnon-Phonon Interaction

As explained in the main text, we consider the magnon-phonon interaction arising from the fluctuation of DM vector direction when the atoms fluctuate from their equilibrium position. To do this, note that when the atoms are at their equilibrium positions, we can write the DM vector as (see Fig. 1 (b) in the main text) $D_{ij} = D \hat{r}_{ij} \times \hat{z}$, where $\hat{r}_{ij} = \frac{r_i - r_j}{|r_i - r_j|}$. Assuming for simplicity that $D$ is independent of $u_i$ and that $M_A = M_B = M$, the part of $D_{ij}$ that is linear in $u_{ij} = u_i - u_j$ is $-\frac{M_A}{M} \hat{r}_{ij} \left( \frac{u_i}{\sqrt{M}} + \frac{u_j}{\sqrt{M}} \right) \times \hat{z} \frac{2}{T}$, where $l$ is the distance between the nearest neighboring atoms, $R_{ij} = R_i - R_j$, and $u_{ij} = u_i - u_j$. Similarly, the terms linear in the Holstein-Primakoff operators in $S_i \times S_j$ are given by $S_i^z \hat{x}_i \times \hat{z}_j + S_i^y \hat{y}_i \times \hat{z}_j + S_j^z \hat{x}_j \times \hat{y}_j$, where $\hat{x}_i, \hat{y}_i, \hat{z}_i$ are the local axes introduced in Eq. (A1). The MPI at the quadratic level is obtained by multiplying these two terms.

Introducing the HP operators as in Eq. (A2) and taking the Fourier transformation, the magnon-phonon interaction is

$$\lambda \sum_{\delta_i} \left\{ [u_{A,B}^i(a_{A,B} - k + a_{A,B}^\dagger e^{ik\cdot\delta_i} + a_{A,B}^\dagger e^{ik\cdot\delta_i})] + u_{B,A}^i(a_{A,B} - k + a_{A,B}^\dagger e^{ik\cdot\delta_i} + a_{A,B}^\dagger e^{ik\cdot\delta_i}) \right\} \quad \text{(C1)}$$

where we have defined $u_{A,B}^i = \tilde{u}_{A,B} \cdot \delta_i$ and $\lambda = \frac{DS}{2\pi} \sqrt{\frac{2\pi}{M}}$.

SM D: BdG Hamiltonian and Berry curvature

In this work, we have computed all of the response functions by transforming the magnetoelastic Hamiltonian into bosonic BdG form as explained below. Another popular method is to write the Hamiltonian in terms of the phonon operators so that the phonon sector of the Hamiltonian is diagonalized from the outset. However, this may not result in the correct response functions such as the thermal Hall conductivity, as pointed out in Ref. [25].

To make the connection between the magnetoelastic Hamiltonian and the bosonic BdG Hamiltonian, let us first clarify the relation between the phonon Hamiltonian and bosonic BdG Hamiltonian. Below, we work in the system of units where $\hbar = 1$ and energy is measured in meV. If we define

$$v_{A/B,k} = \frac{\sqrt{2}}{2} (p_{A/B,k} - i\tilde{u}_{A/B,k}) \quad \text{(D1)}$$

and

$$y_{k} = (v_{A,k}, v_{B,k}, v_{A,-k}^\dagger, v_{B,-k}^\dagger) \quad \text{(D2)}$$
we have

\[ [y_{mk}, y_{nk}^\dagger] = (\rho_z)_{mn} \delta_{k,k'}, \quad y_k^\dagger = \rho_z y_{-k} \quad \text{(D3)} \]

where \( m \) and \( n \) runs over all the components of \( y_k \), and \( \rho_i \) are the Pauli matrices relating the particle and hole sectors of the field operator \( y_k \). Note that Eq. (D3) is the defining relation of a bosonic BdG field operator.

Next, let us review the basic properties of BdG Hamiltonian. Let \( \mathcal{H} = \frac{1}{2} \sum_k \Phi_{k}^\dagger H_k \Phi_k \) be the bosonic BdG Hamiltonian where \( \Phi_k \) satisfy

\[ [\Psi_{mk}, \Psi_{nk}^\dagger] = (\tau_z)_{mn} \delta_{k,k'}, \quad \Psi_k^\dagger = \rho_z \Psi_{-k}, \quad \text{(D4)} \]

where \( m, n = -N, ..., -1, 1, ..., N \). The transformation to the energy eigenbasis is given by \( \Phi_k = T_k \Psi_k \), so that

\[ \mathcal{H} = \frac{1}{2} \sum_k \Phi_{k}^\dagger T_k^\dagger H_k T_k \Phi_k = \frac{1}{2} \sum_k \Phi_{k}^\dagger E_k \Phi_k, \quad \text{where } E_k \text{ is diagonal, and } \langle \Phi_k, m_n | \Phi_{k'}, n_m \rangle = (\tau_z)_{mn} \delta_{k,k'}. \]

This requires that \( T_k^\dagger H_k T_k = E_k \) and \( T_k^\dagger \tau_z T_k = \tau_z \), where \( \tau_z \) are the Pauli matrices in the particle and hole sectors. For example, \( (\tau_z)_{mn} = \delta_{mn} \) for \( m, n > 0 \), \( (\tau_z)_{mn} = -\delta_{mn} \) for \( m, n < 0 \), and \( (\tau_z)_{mn} = 0 \) otherwise. Denoting by \( |n,k\> \) the column vectors of \( T_k \) and noticing that \( T_k^{-1} = \tau_z T_k^\dagger \tau_z \), these conditions translate to \( \tau_z H_k |n,k\> = E_k |n,k\> \) and \( \langle m,k \| \tau_z |n,k\> = (\tau_z)_{mn} \).

The Berry curvature for this Hamiltonian is defined to be \( B_{k,n} = i(\tau_z)_{nn} \nabla \times |n,k\> |\tau_z \nabla |n,k\> \). In particular, we denote the \( z \) component of the Berry curvature by \( \Omega_k, n \). In Fig. (S1) we plot the full Berry curvature density for magnon, phonon, and magnetoelastic bands and the corresponding energy spectrums. Let us also calculate the Chern numbers, which is defined as \( C_n = \frac{1}{\pi} \int_{BZ} dk \Omega_k, n \). We find that the individual magnon and phonon bands are topologically trivial. Specifically, using the labels for the energy bands in Fig. (S1) we have \( C_a = C_b = 0, C_c + C_d = C_e + C_f = 0 \). Note that we have grouped together the bands which are not gapped.
find that the magnetoelastic bands are topologically nontrivial. Specifically, $C_g + C_h = -1$, $C_i = 1$, $C_j = C_k + C_l = 0$.

SM E: Symmetry analysis of thermal Hall and spin Nernst effect

In the main text, we have stated that the $M_x E_{2x}^S$ symmetry forces $\kappa_{xy} = \alpha_{Sxy}^z = 0$. To show this, recall that $M_x E_{2x}^S$ is the mirror symmetry about the plane normal to the $x$ axis, which does not change the spin direction. It therefore imposes $E_n(k_x,k_y) = E_{n,(-k_x,k_y)}$. Because Berry curvature transforms like magnetic field in the momentum space, we also have $\Omega_{n,(k_x,k_y)} = -\Omega_{n,(-k_x,k_y)}$. Using this, the contribution to $\kappa_{xy} = -k_B T \sum_k \sum_{n=1}^{N} \left| c_2(g(E_{n,k}) - \frac{\pi}{2}) \right| \Omega_{n,k}$ from $(k_x, k_y)$ and $(-k_x, k_y)$ cancel pairwise, so that $\kappa_{xy} = 0$. In addition, because the $M_x E_{2x}^S$ symmetry does not change the spin direction, it imposes $(\delta^z)_{n,(k_x,k_y)} = (\delta^z)_{n,(-k_x,k_y)}$. Using this, the contribution to $\alpha_{Sxy}^z = -\frac{k_B}{h} \sum_k \sum_{n=1}^{N} (\delta^z)_{n,k} \Omega_{n,k} c_1(E_{n,k}/k_B T)$ from $(k_x, k_y)$ and $(-k_x, k_y)$ cancel pairwise, so that $\alpha_{Sxy}^z = 0$.

SM F: Absence of Sign Change in Spin Nernst coefficient

In this section, we explain why the spin Nernst coefficient computed using the semiclassical theory $\alpha_{Sxy}^z = -\frac{k_B}{h} \sum_k \sum_{n=1}^{N} (\delta^z)_{n,k} \Omega_{n,k} c_1(E_{n,k}/k_B T)$ does not change sign as temperature is varied, contrary to the case of thermal Hall conductivity. To this end, it is useful to examine the momentum space distribution of $\Omega_{n,k}$, which was defined as the spin operator along the direction of the magnetic order in Eq. (A2). In matrix form, we have $\Psi_k = \begin{bmatrix} a_{A,k} & a_{B,k}^\dagger \end{bmatrix}$, $\psi_k$, where the explicit expression for $S^z$ is given in Eq. (G1).

We plot this quantity in Fig. S2 for $n = 4$ and $5$, as they are relevant for the spin Nernst coefficient at low temperatures. Comparing $\Omega_{4,k}$ and $\Omega_{5,k}$, we see that $\Omega_{4,k}$ shows a circular region where it takes positive values and has smaller radius. The positive region extends towards the inner regions of the circle and has lower energy. $\Omega_{5,k}$, on the other hand, shows a circular region where it takes negative and has larger radius. The negative region extends towards the outer regions of the circle and has higher energy. Because $c_1(E/k_B T)$ for $E > 0$ is positive and decays quickly to zero as a function of energy at low temperature, the positive contribution to $\alpha_{Sxy}^z$ from the negative $\Omega_{4,k}$ in band 4 around the red circular region is larger than the negative contribution to $\alpha_{Sxy}^z$ from the positive $\Omega_{5,k}$ in band 5 around the blue circular region at low temperature, which explains why $\alpha_{Sxy}^z > 0$ at low temperature (the temperature scale is set by the energy at which these circular regions occur, which is about 1 meV). Because the region outside of the blue circle in band 5 quickly becomes red as the radius is increased while the corresponding region in band 4 stays white, we should expect $\alpha_{Sxy}^z$ to decrease at larger temperature, which is consistent with the numerical calculation in Fig. (3) (b) in the main text.

SM G: Spin Nernst Effect

Much of the linear response theory used here was already developed in Refs. [5][20], but we repeat them for completeness. Let

$$\Psi_k = \begin{bmatrix} a_{A,k} & a_{B,k} \end{bmatrix}, \psi_k, \begin{bmatrix} a_{A,k}^\dagger & a_{B,k}^\dagger \end{bmatrix}, \begin{bmatrix} \psi_k^\dagger & S^z \psi_k \end{bmatrix}$$

be the bosonic BdG basis, where $a_{A/B,k}$ are the Holstein-Primakoff operators and $\psi_{A/B,k}$ are the bosonic BdG fields for phonons defined in Eq. (D1). Then,

$$\mathcal{H}_0 = \frac{1}{2} \sum_k H_k \Psi_k, \quad S^z = \frac{1}{2} \sum_k \Psi_k^\dagger S^z \Psi_k,$$

where $\mathcal{H}_0$ is the magnetoelastic Hamiltonian and $S^z$ is the spin excitation operator, which should not be confused with $S^z_f$, which was defined as the spin operator along the direction of the magnetic order in Eq. (A2). In matrix form, we have

$$S^z = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where $0_4$ is the $4 \times 4$ zero-matrix. We find

$$[\mathcal{H}_0, S^z] = \frac{1}{2} \sum_k \Psi_k^\dagger \left( H_k \tau_z S^z - S^z \tau_z H_k \right) \Psi_k.$$
Explicitly evaluating \( H_k \tau_z S^z - S^z \tau_z H_k \), one finds that only terms that couple magnon and phonon survive. Namely, \( \mathcal{H}_0 \) and \( S^z \) commute when the magnon-phonon interaction is neglected.

While the spin Nernst effect in Refs. [19, 20] was analyzed for the case when spin is conserved, here we extend the theory to the case when the spin is not conserved.

1. Method of pseudo-gravitational potential

Let us study the spin current for a general BdG Hamiltonian in response to thermal gradient. Below, repeated Roman indices implies summation over the BdG field operator indices taking the values \(-N, -N-1, \ldots, -1, 1, \ldots, N-1, N\). Calligraphic letters are reserved for operators containing BdG fields. Finally, we put \( \hbar = 1 \) and restore it at the end.

Let

\[
\mathcal{H}_0 = \frac{1}{2} \sum_{\delta} \int dr \Psi_{\delta m}^\dagger(r) H_{\delta mn} \Psi_n(r + \delta)
\]

\[
= \frac{1}{2} \sum_{\delta} \int dr \Psi_{\delta}^\dagger(r) H_{\delta} e^{i p \cdot \delta} \Psi(r)
\]

\[
= \frac{1}{2} \int dr \Psi_{\delta}^\dagger(r) h_0 \Psi(r)
\]

\[
= \frac{1}{2} \int dr \hat{h}_0(r).
\]

Here, \( p \) is the momentum operator. Taking the Fourier transformation of the field operators

\[
\Psi(r) = \frac{1}{\sqrt{V}} \sum_k \Psi_k e^{i k \cdot r},
\]

we find that

\[
H_k = \sum_{\delta} e^{i k \cdot \delta} H_{\delta},
\]

where \( H_k \) was defined in Eq. (G2).

According to Luttinger [29], one can compute the response to thermal gradient by introducing the gravitational scalar potential \( \chi(r) \) that couples to the Hamiltonian. Assuming that the potential is linear in position, i.e. \( \chi(r) = r \cdot \nabla \chi \), this interaction is given by

\[
\mathcal{Y} = \frac{1}{4} \int dr \Psi^\dagger(r)(h_0 \nabla r + r h_0) \Psi(r) \cdot \nabla \chi.
\]

The total Hamiltonian,

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{Y},
\]

is equivalent up to linear order in \( \chi(r) \) to

\[
\mathcal{H} = \frac{1}{2} \sum_{\delta} \int dr \hat{\Psi}_{\delta m}^\dagger(r) H_{\delta mn} \hat{\Psi}_n(r + \delta).
\]

Because of the relation [29]

\[
\langle j^S_{\mu} \rangle = L^S_{\mu
u} \left( T \nabla_{\nu} \frac{1}{T} - \nabla_{\nu} \chi \right)
\]

we have \( \alpha^S_{\mu \nu} = L^S_{\mu \nu}/T \).

2. Current operator

Let

\[
S^z(r) = \frac{1}{2} \Psi_m^\dagger(r) S^z\Psi_n(r),
\]

be the spin excitation operator. Its time evolution is given by

\[
-i \frac{\partial S^z(r)}{\partial t} = [\mathcal{H}, S^z(r)]
\]

\[
= \frac{1}{4} \sum_{\delta} \int dr \left( [\Psi_{\delta m}^\dagger(r') H_{\delta mn} \Psi_n(r + \delta), \Psi_{\delta}^\dagger(r) S^{z l}_{\delta kl} \Psi_l(r) - \Psi_{\delta}^\dagger(r) S^{z l}_{\delta kl} \Psi_l(r), \Psi_{\delta}^\dagger(r') H_{\delta mn} \Psi_n(r + \delta) - \Psi_{\delta m}^\dagger(r') H_{\delta mn} \Psi_n(r + \delta)] \right)
\]

\[
= \frac{1}{4} \sum_{\delta} \left( (-i \tau_y)_{mk} \xi(r) H_{\delta mn} \xi(r + \delta) \Psi_{\delta mn}(r + \delta) + \Psi_{\delta m}^\dagger(r - \delta) \xi(r - \delta) H_{\delta mn} \xi(r)(\tau_z)_{nk} S^{z l}_{kl} \Psi_l(r) \right)
\]

\[
- \frac{1}{4} \sum_{\delta} \Psi_{\delta}^\dagger(r) S^{z l}_{\delta kl} \xi(r) H_{\delta mn} \xi(r + \delta) \Psi_{\delta mn}(r + \delta) + \Psi_{\delta m}^\dagger(r - \delta) \xi(r - \delta) H_{\delta mn} \xi(r)(\tau_y)_{mn} \xi(r - \delta),
\]

where we have used

\[
[\Psi_m(r), \Psi_n^\dagger(r')] = (\tau_z)_{mn} \delta(r - r'), \quad [\Psi_m^\dagger(r), \Psi_n^\dagger(r')] = -(i \tau_y)_{mn} \delta(r - r'), \quad [\Psi_m(r), \Psi_n(r')] = i(\tau_y)_{mn} \delta(r - r').
\]

The third line of Eq. (G14) containing \( \tau_y \) can be manipulated by using

\[
\Psi_m^\dagger(r) = (\tau_x)_{mn} \Psi_n^\dagger(r), \quad \tau_x S^z \tau_x = (S^z)^T, \quad \tau_x H^\delta \tau_x = (H^{-\delta})^T.
\]
Note that the first equality follows from definition of BdG field. The second and the third follow from the first equality. The third line of Eq. \((G14)\) becomes

\[
\frac{1}{4} \sum_{\delta} \tilde{\Psi}^{\dagger}(r + \delta) H^{-\delta} \tau_z S^z \tilde{\Psi}(r) - \tilde{\Psi}^{\dagger}(r) S^z \tau_z H^{-\delta} \tilde{\Psi}(r - \delta).
\]  

(G17)

In total, we obtain

\[
\frac{1}{2} \sum_{\delta} \Psi^{\dagger}(r - \delta) \xi(r - \delta) H^\delta \tau_z S^z \Psi(r) - \Psi^{\dagger}(r) S^z \tau_z \xi(r) H^\delta \tau_z \xi(r + \delta) \Psi(r + \delta).
\]  

(G18)

Thus,

\[
-\frac{i}{\hbar} \frac{\partial \delta S^z}{\partial t} = \frac{1}{2} \sum_{\delta} \Psi^{\dagger}(r - \delta) H^\delta \tau_z S^z \Psi(r) - \Psi^{\dagger}(r) S^z \tau_z H^\delta \tilde{\Psi}(r + \delta).
\]  

(G19)

Now, note that

\[
v = i[H, r] = i \sum_{\delta} \delta H^\delta e^{ip\delta}.
\]  

(G20)

Thus,

\[
\frac{\partial \delta S^z}{\partial t} = -\frac{i}{2} \sum_{\delta} \left( \tilde{\Psi}^{\dagger}(r) S^z \tau_z H^\delta \tilde{\Psi}(r) - \tilde{\Psi}^{\dagger}(r - \delta) S^z \tau_z H^\delta \tilde{\Psi}(r) + \tilde{\Psi}^{\dagger}(r) S^z \tau_z H^\delta \tilde{\Psi}(r + \delta) - \tilde{\Psi}^{\dagger}(r - \delta) S^z \tau_z H^\delta \tilde{\Psi}(r + \delta) \right)
\]  

\[
= -\frac{i}{2} \sum_{\delta} \left( \nabla \cdot \left( \tilde{\Psi}^{\dagger}(r) S^z \tau_z H^\delta \tilde{\Psi}(r) + \tilde{\Psi}^{\dagger}(r) H^\delta \tau_z S^z \tilde{\Psi}(r + \delta) \right) + \tilde{\Psi}^{\dagger}(r - \delta) S^z \tau_z H^\delta \tilde{\Psi}(r) - \tilde{\Psi}^{\dagger}(r - \delta) H^\delta \tau_z S^z \tilde{\Psi}(r) \right)
\]  

\[
= -\nabla \cdot \tilde{\Psi}^{\dagger}(r) \frac{S^z \tau_z v + v \tau_z S^z}{2} \tilde{\Psi}(r) - \frac{i}{2} \sum_{\delta} \tilde{\Psi}^{\dagger}(r) (S^z \tau_z H^\delta e^{ip\delta} - H^\delta e^{ip\delta} \tau_z S^z) \tilde{\Psi}(r).
\]  

(G21)

Therefore, we define the spin current operator as

\[
\tilde{j}_S = \tilde{\Psi}^{\dagger}(r) \frac{v \tau_z S^z + S^z \tau_z v}{2} \tilde{\Psi}(r).
\]  

(G22)

3. Linear response to temperature gradient

To linear order in temperature gradient, we have

\[
j_S(r) = \frac{1}{2} \Psi^{\dagger}(r) (S^z \tau_z v + v \tau_z S^z) \Psi(r) + \frac{\nabla \cdot \lambda}{4} \left( S^z \tau_z v \mu + S^z \tau_z \tau_z \mu v + v \mu \tau_z S^z + r \mu v \tau_z S^z \right)
\]  

\[
= j_S^{(0)} + j_S^{(1)},
\]  

(G23)

respectively. Define \(J_S = \int dr j_S(r)\). To linear order in temperature gradient,

\[
\langle J_S \rangle = \langle J_S^{(0)} \rangle_{\text{neq}} + \langle J_S^{(1)} \rangle_{\text{eq}}.
\]  

(G24)

Notice that \(J_S^{(0)}\) should be averaged over the non-equilibrium distribution, while \(J_S^{(1)}\), which is already linear in the temperature gradient, should be averaged over the equilibrium distribution.

From now on, we drop the subscript \(S\) on the spin current.
4. Evaluation of Kubo formula

The Kubo formula is

$$
\langle J_{\mu}^{(0)} \rangle_{\text{eq}} = -\lim_{\omega \rightarrow 0} \frac{\partial}{\partial \omega} \int_0^\beta d\tau e^{i\omega \tau} \langle T_{\tau} J_{\mu}^{(0)}(\tau) J_{\nu}^{(0)}(0) \rangle \nabla_{\nu} \chi \equiv -S_{\mu\nu} \nabla_{\nu} \chi, \hspace{1cm} \text{(G25)}
$$

where $\omega = 2\pi n / \beta$ and $n$ is an integer. Here, $J_{\mu}^{(0)}$ satisfies $\frac{\partial J_{\mu}}{\partial t} = J_{\nu} \cdot \nabla \chi$. Since

$$
\frac{\partial J_{\mu}}{\partial t} = i[H_0, J_{\mu}] = \frac{i}{4} \nabla \chi \cdot \sum_{\delta, \delta'} \int dr \Psi^\dagger (r + \delta') H^{-\delta}(-\tau_z)(2r + \delta') H^\delta \Psi(r + \delta) + \Psi^\dagger (r) H^\delta \tau_z H^{\delta'}(2r + \delta + \delta') \Psi(r + \delta + \delta')
$$

we have

$$
J_{\mu}^{(0)} = \frac{1}{4} \int dr \Psi^\dagger (r) (h_0 \tau_z v + v \tau_z h_0) \Psi(r).
$$

To obtain the second line, we used Eqs. (G15) and (G16), and to obtain the third line, we shifted the integration variable. Finally, to obtain Eq. (G27), we use Eqs. (G5) and (G20).

Taking the Fourier transform, we have

$$
J^{(0)} = \frac{1}{2} \sum_{k, \delta} \Psi^\dagger_k [S^\tau \tau_z \delta H^\delta e^{ik \cdot \delta} + \delta H^\delta e^{ik \cdot \delta} \tau_z S^\tau] \Psi_k
$$

and

$$
J^{Q} = \frac{1}{4} \sum_{k, \delta, \delta'} \Psi^\dagger_k [H^\delta e^{i k \cdot \delta} \tau_z \delta'H^\delta e^{i k ' \cdot \delta'} + \delta H^\delta e^{i k \cdot \delta} \tau_z H^\delta e^{i k ' \cdot \delta'}] \Psi_k
$$

Let us introduce the field operators for the energy eigenstates (see below Eq. (D4)),

$$
\Phi_k = T_k \Psi_k.
$$

We then have

$$
S_{\mu\nu} = \frac{1}{8} \lim_{\omega \rightarrow 0} \frac{\partial}{\partial \omega} \int_0^\beta e^{i\omega \tau} \sum_{k, k'} \langle \Phi^\dagger_k(\tau) T_k \sum \Phi^\dagger_{k'}(0) T_{k'} \Psi_{k'}(\tau) \rangle \times \Phi^\dagger_{k'}(0) T_{k'} \sum \Phi_{k'}(0).
$$

Note the identity

$$
\langle \Phi_{k', m}(\tau) \Phi_{k, n}(\tau) \Phi_{k, q}(0) \Phi_{k', q}(0) \rangle = \langle \Phi_{k, m}(\tau) \Phi_{k, n}(\tau) \rangle \langle \Phi_{k', q}(0) \Phi_{k', q}(0) \rangle + \langle \Phi_{k, m}(\tau) \Phi_{k', q}(0) \rangle \langle \Phi_{k, n}(\tau) \Phi_{k', q}(0) \rangle
$$

and

$$
\langle \Phi_{k, m}(\tau) \Phi_{k', q}(0) \rangle \langle \Phi_{k, n}(\tau) \Phi_{k', q}(0) \rangle
$$
The integral \( \int_0^\beta e^{i\omega\tau} \langle \Phi_{k,\mu}(\tau)\Phi_{k'}(\tau) \rangle \langle \Phi_{k,\nu}(0)\Phi_{k'}(0) \rangle \) vanishes (because \( \omega = 2\pi n/\beta \)), so we only need the correlation functions between different times. We have

\[
\langle \Phi_{k,m}(\tau)\Phi_{k',n}(0) \rangle = \delta_{k,-k',(i\tau_y)_{m,n}} g[(\tau_z E_k)_{m,m}] e^{(i\tau_z E_k)_{m,m}\tau}
\]

(G33)

\[
\langle \Phi_{k,m}(0)\Phi_{k',n}(\tau) \rangle = \delta_{k,-k'}(-i\tau_y)_{m,n} g[(-\tau_z E_k)_{m,m}] e^{-(i\tau_z E_k)_{m,m}\tau}
\]

(G34)

\[
\langle \Phi_{k,m}(\tau)\Phi_{k',n}(0) \rangle = \delta_{k,k'} \delta_{k,-k'} g[(\tau_z E_k)_{m,m}] e^{(i\tau_z E_k)_{m,m}\tau}
\]

(G35)

\[
\langle \Phi_{k,m}(0)\Phi_{k',n}(\tau) \rangle = \delta_{k,k'} \delta_{k,-k'} g[(-\tau_z E_k)_{m,m}] e^{-(i\tau_z E_k)_{m,m}\tau}
\]

(G36)

where \( g[x] = \frac{1}{e^{x} - 1} \) is the Bose-Einstein distribution. If we define \( V^S_k = T^k_1[S^z \tau_z \v_k + \v_k \tau_z S^z] T_k \) and \( V_k = T^k_1 \v_k T_k \), we have

\[
S_{\mu\nu} = \frac{1}{8} \lim_{\omega \to 0} \frac{\partial}{\partial \omega} \int_0^\beta \frac{e^{i\omega\tau}}{\omega} \sum_{k,k'} \left[ V^S_{k,\mu} \right]_{mn} \left[ E_{k',\tau} \v_{k',\nu} + \v_{k',\nu} \tau_z E_{k'} \right]_{pq} \left[ (\tau_y)_{mp} (\tau_y)_{nq} \delta_{k,-k'} - (\tau_z)_{mq} (\tau_z)_{np} \delta_{k,k'} \right] g[(\tau_z E_k)_{m,m}] e^{(i\tau_z E_k)_{m,m}\tau}
\]

\[
\times g[(-\tau_z E_k)_{m,m}] e^{-(i\tau_z E_k)_{m,m}\tau}.
\]

(G37)

The integral is

\[
\lim_{\omega \to 0} \frac{\partial}{\partial \omega} \int_0^\beta \frac{e^{i\omega+\tau_z E_k)_{m,m} - (\tau_z E_k)_{m,m})\tau}}{\omega} = \lim_{\omega \to 0} \frac{\partial}{\partial \omega} \frac{e^{i\omega+\tau_z E_k)_{m,m} - \beta (\tau_z E_k)_{m,m} - 1}}{\omega} - i \frac{e^{i\beta (\tau_z E_k)_{m,m} - (\tau_z E_k)_{m,m})\tau}}{\omega}.
\]

(G38)

Using the identity

\[
[g(x) - g(y)] = -g(x)g(-y)[e^{\beta x - \beta y} - 1],
\]

(G39)

we have

\[
S_{\mu\nu} = \frac{i}{8} \sum_{k,k'} \left[ V^S_{k,\mu} \right]_{mn} \left[ E_{k',\tau} \v_{k',\nu} + \v_{k',\nu} \tau_z E_{k'} \right]_{pq} \left[ (\tau_y)_{mp} (\tau_y)_{nq} \delta_{k,-k'} - (\tau_z)_{mq} (\tau_z)_{np} \delta_{k,k'} \right] g[(\tau_z E_k)_{m,m}] - g[(\tau_z E_k)_{m,m}]
\]

\[
\times \left[ (\tau_z E_k)_{m,m} - (\tau_z E_k)_{m,m} \right].
\]

(G40)

Now, note the identities

\[
E_{-k} = \tau_x E_k \tau_x, \quad T_{-k} = \tau_x T_{k} \tau_x, \quad \v_{-k} = -\tau_x \v_{k} \tau_x.
\]

(G41)

The first two follow from \( H_{-k} = \tau_x H_k \tau_x \). More generally, \( T_{-k} = (T_k P_k)^* \), where \( P_k \) is such that \( E_{-k} = \tau_x P_k E_k \tau_x \). However, the results do not depend on the gauge choice, so for simplicity, we put \( P_k = 1 \). Finally, the third identity is obtained as follows:

\[
\v_{-k} = \sum_{\delta} \delta H_{H,\delta} e^{-i k \delta} = \sum_{\delta} \delta \tau_x \delta H_{H,\delta} e^{-i k \delta} = -\sum_{\delta} \delta \tau_x \delta H_{H,\delta} e^{-i k \delta} \tau_x = -\tau_x \v_{k} \tau_x.
\]

(G42)

Using these identities, we can manipulate the term containing \( \tau_y \) as follows:

\[
\frac{i}{8} \sum_{k,k'} \left[ V^S_{k,\mu} \right]_{mn} \left[ E_{k',\tau} \v_{k',\nu} + \v_{k',\nu} \tau_z E_{k'} \right]_{pq} \left[ (\tau_y)_{mp} (\tau_y)_{nq} \delta_{k,-k'} - (\tau_z)_{mq} (\tau_z)_{np} \delta_{k,k'} \right] g[(\tau_z E_k)_{m,m}] - g[(\tau_z E_k)_{m,m}]
\]

\[
\times \left[ (\tau_z E_k)_{m,m} - (\tau_z E_k)_{m,m} \right] = \frac{i}{8} \sum_{k} \left[ V^S_{k,\mu} \right]_{mn} \left[ (\tau_y) (E_{-k} T^k_{k} \v_{k,v} \tau_z E_{k} + E_{z} T^k_{k} \v_{k,v} \tau_z E_{k}) \right]_{mn} g[(\tau_z E_k)_{m,m}] - g[(\tau_z E_k)_{m,m}]
\]

\[
\times \left[ (\tau_z E_k)_{m,m} - (\tau_z E_k)_{m,m} \right].
\]

(G43)

Therefore, the term containing \( \tau_y \) is identical to the term containing \( \tau_z \). Thus,

\[
S_{\mu\nu} = \frac{i}{8} \sum_{k} \left[ V^S_{k,\mu} \right]_{mn} \left[ (\tau_y) (E_{-k} T^k_{k} \v_{k,v} \tau_z E_{k} + E_{z} T^k_{k} \v_{k,v} \tau_z E_{k}) \right]_{mn} g[(\tau_z E_k)_{m,m}] - g[(\tau_z E_k)_{m,m}]
\]

\[
\times \left[ (\tau_z E_k)_{m,m} - (\tau_z E_k)_{m,m} \right].
\]
If we define
\[ w_k = S^z \tau_z v_k, \quad u_k = v_k \tau_z S^z, \quad (G45) \]
this is equal to
\[ S_{\mu \nu} = -\frac{i}{4} \sum_k [(n|w_\mu|m) + (n|u_\mu|m)][E_{\nu n}(m|v_\nu|n)(\tau_z)_{nn} + E_{\nu m}(m|v_\nu|n)(\tau_z)_{nm}] g[(\tau_z E)_{nn} - g[(\tau_z E)_{nm}]][((\tau_z E)_{nm} - (\tau_z E)_{mm})^2, \quad (G46) \]
where we omit \( k \) dependence.

5. Method of Smrcka and Streda

Our goal here is to evaluate
\[ \langle J^{(1)}_\mu \rangle_{\text{eq}} = -M_{\mu \nu} \nabla_\nu \chi \quad (G47) \]
First, we note that
\[ J^{(1)} = \int d\tau \Phi^\dagger (r) \text{Sym}[S^z \tau_z v_\chi(r)] \Psi(r) = \sum_k \Phi^\dagger_k \text{Sym}[S^z \tau_z v_k \chi(r)] \Psi_k = \sum_k \Phi^\dagger_k T_k \text{Sym}[S^z \tau_z v \chi(r)] T_k \Phi_k, \quad (G48) \]
where \( \text{Sym}[S^z \tau_z v \chi(r)] = \frac{1}{2} \{ S^z \tau_z, \{ v, \chi(\tau) \} \} \) is the symmetrization. Note that the \( r \) in \( \chi(r) \) becomes \( i \frac{\partial}{\partial \tau} \). Taking the expectation value, we obtain
\[ \langle J^{(1)}_\mu \rangle_{\text{eq}} = \sum_k \text{Tr} [\tau_z T_k \text{Sym}[S^z \tau_z v_k \chi(r)] T_k g[\tau_z E_k]]. \quad (G49) \]
Using the identity
\[ e^{\tau_z E_k} = \tau_z T_k e^{\tau_z H_k} T_k, \quad (G50) \]
we have
\[ \langle J^{(1)}_\mu \rangle_{\text{eq}} = \sum_k \nabla_\nu \chi \text{Tr} [\tau_z T_k \text{Sym}[S^z \tau_z v_k \mu \nu v_k] g[\tau_z H_k]] = M_{\mu \nu} \nabla_\nu \chi. \quad (G51) \]
Recalling the definition in Eq. \( (G45) \), we have
\[ M_{\mu \nu} = -\frac{1}{4} \sum_k \int d\eta g(\eta) \text{Tr}[\tau_z (w_k \mu \nu + r_\nu r_\mu \nu + u_k \mu \nu + u_\nu u_k \mu \nu) \delta(\eta - \tau_z H_k)]. \quad (G52) \]
Below, we will often omit the \( k \) dependence for notational simplicity.

Define
\[ A_{\mu \nu} = \frac{i}{2} \text{Tr} [\tau_z w_\mu \dfrac{dG^+}{d\eta} \tau_z v_\nu \delta(\eta - \tau_z H) - \tau_z w_\mu \delta(\eta - \tau_z H) \tau_z v_\nu \dfrac{dG^-}{d\eta}] \quad (G53) \]
\[ B_{\mu \nu} = \frac{i}{2} \text{Tr} [\tau_z w_\mu G^+ \tau_z v_\nu \delta(\eta - \tau_z H) - \tau_z w_\mu \delta(\eta - \tau_z H) \tau_z v_\nu G^-], \quad (G54) \]
where
\[ G^\pm = \frac{1}{\eta \pm i \epsilon - \tau_z H} \quad (G55) \]
satisfies
\[ i\delta(\eta - \tau_z H) = -\frac{1}{2\pi} (G^+ - G^-), \quad \dfrac{dG^\pm}{d\eta} = -(G^\pm)^2, \quad i \dfrac{d}{d\eta} \delta(\eta - \tau_z H) = \frac{1}{2\pi} ((G^+)^2 - (G^-)^2). \quad (G56) \]
Using these, we find
\[
A_{\mu\nu} - \frac{1}{2} \frac{dB_{\mu\nu}}{d\eta} = \frac{1}{8\pi} \text{Tr} \left[ -\tau_\mu \frac{dG^+}{d\eta} \tau_\nu G^+ - \tau_\mu \frac{dG^-}{d\eta} \tau_\nu G^- + \tau_\mu \frac{dG^+}{d\eta} \tau_\nu G^+ + \tau_\mu \frac{dG^-}{d\eta} \tau_\nu G^- \right]
\]
Adding the last term causes the problem because of the non-commutativity of the Hamiltonian and the spin. Defining \( C_S = \tau_z [H, S^z \tau_z] \), the second term becomes
\[
- \frac{1}{4\pi} \text{Tr} [S^z \tau_\mu \tau_\nu (G^- - G^+) - S^z \tau_\mu (G^- - G^+) \tau_\nu + C_S \tau_\mu G^+ \tau_\nu G^- + C_S \tau_\mu G^- \tau_\nu G^+].
\]
Because \([r_\mu, r_\nu] = 0\), the first two terms cancel. Using Eq. (G56), we obtain
\[
A_{\mu\nu} - \frac{1}{2} \frac{dB_{\mu\nu}}{d\eta} = - \frac{1}{4} \text{Tr}[\tau_\mu \frac{d}{d\eta} \delta(\eta - \tau_z H) r_\nu + \tau_\mu \frac{d}{d\eta} \delta(\eta - \tau_z H)]
\]
Let us call the first line \( m_{\mu\nu}^{(0)} \) and the second line \( m_{\mu\nu}^{(1)} \). Replacing \( w \) by \( u \), we have
\[
\frac{i}{2} \text{Tr}[C_S r_\mu \delta(\eta - \tau_z H) r_\nu G^+] = \frac{i}{2} \langle \tau_z \rangle_{nn} \langle n | \tau_z C_S r_\mu | m \rangle \langle m | \tau_z \delta(\eta - \tau_z H) r_\nu G^+ | n \rangle
\]
and similarly,
\[
\frac{i}{2} \text{Tr}[C_S r_\mu G^- r_\nu \delta(\eta - \tau_z H)] = \frac{i}{2} \langle \tau_z \rangle_{nn} \langle n | \tau_z C_S r_\mu G^- | m \rangle \langle m | \tau_z r_\nu \delta(\eta - \tau_z H) | n \rangle
\]
and similarly

\[
\tilde{m}^{(1)}_{\mu\nu} = \sum_{n \neq m} i \frac{1}{2} (\tau_z)_{nn}(\tau_z)_{mm} \langle n | \tau_z r_\mu C_S | m \rangle \langle m | \tau_z r_\nu | n \rangle \left[ \frac{\delta(\eta - \tau_z E)_{mm}}{(\tau_z E)_{mm} - (\tau_z E)_{nn}} + \frac{\delta(\eta - \tau_z E)_{nn}}{(\tau_z E)_{nn} - (\tau_z E)_{mm}} \right]. 
\]  

(G67)

Note that \((\tilde{m}^{(1)}_{\mu\nu})* = m^{(1)}_{\mu\nu}\) because \(C_S^\dagger = -\tau_z C_S \tau_z\). Using these expressions, we can see that for a bounded spectrum,

\[
\int_{-\infty}^{\infty} d\eta \left[ A_{\mu\nu}(\eta) - \frac{1}{2} \frac{d}{d\eta} B_{\mu\nu}(\eta) \right] = 0, \quad \int_{-\infty}^{\infty} d\eta \left[ \dot{A}_{\mu\nu}(\eta) - \frac{1}{2} \frac{d}{d\eta} \dot{B}_{\mu\nu}(\eta) \right] = 0. 
\]  

(G68)

Thus,

\[
M_{\mu\nu} = -\sum_k \int_{-\infty}^{\infty} d\eta g(\eta) \int_{\eta}^{\infty} d\eta' \left[ A_{\mu\nu}(\eta') - \frac{1}{2} \frac{d}{d\eta'} B_{\mu\nu}(\eta') + \dot{A}_{\mu\nu}(\eta') - \frac{1}{2} \frac{d}{d\eta'} \dot{B}_{\mu\nu}(\eta') \right] \int_{\eta}^{\infty} d\eta''(-g(\eta'')). 
\]  

(G69)

Let us call the first four terms \(M_{\mu\nu}^{(0)}\) and last two terms \(M_{\mu\nu}^{(1)}\). Using Eq. (G68), we have

\[
M_{\mu\nu}^{(0)} = -\sum_k \int_{-\infty}^{\infty} d\eta g(\eta) \int_{\eta}^{\infty} d\eta' \left[ A_{\mu\nu}(\eta') - \frac{1}{2} \frac{d}{d\eta'} B_{\mu\nu}(\eta') + \dot{A}_{\mu\nu}(\eta') - \frac{1}{2} \frac{d}{d\eta'} \dot{B}_{\mu\nu}(\eta') \right] \int_{\eta}^{\infty} d\eta''(-g(\eta'')). 
\]  

(G70)

Similarly,

\[
M_{\mu\nu}^{(1)} = \sum_k \mathrm{Re} \left[ i \sum_{n \neq m} (\tau_z)_{nn}(\tau_z)_{mm} \langle n | \tau_z r_\mu C_S | m \rangle \langle m | \tau_z r_\nu | n \rangle \int_{-(\tau_z E)_{mm}}^{(\tau_z E)_{nn}} d\eta(-g(\eta)) \right]. 
\]  

(G71)

To evaluate Eq. (G70), it is convenient to express \(A_{\mu\nu} - \frac{1}{2} \frac{d}{d\eta} B_{\mu\nu}\) in terms of Green’s function and delta functions

\[
A_{\mu\nu} - \frac{1}{2} \frac{d}{d\eta} B_{\mu\nu} = -\frac{i}{4} \mathrm{Tr} \left[ \tau_z w_\mu \frac{dG^+}{d\eta} \tau_z v_\nu \delta(\eta - \tau_z H) - \tau_z v_\nu \frac{dG^-}{d\eta} \tau_z w_\mu \delta(\eta - \tau_z H) \right. \\
+ \tau_z v_\nu G^- \tau_z w_\mu \frac{d}{d\eta} \delta(\eta - \tau_z H) - \tau_z w_\mu G^+ \tau_z v_\nu \frac{d}{d\eta} \delta(\eta - \tau_z H) \left. \right]. 
\]  

(G72)

Using Eq. (G63), we find that its contribution to \(M_{\mu\nu}^{(0)}\) is

\[
M_{\mu\nu}^{(0)} = -\frac{i}{2} \sum_k \int_{-\infty}^{\infty} d\eta \int_{\eta}^{\infty} (-g(\eta)) d\eta' \left[ (\tau_z)_{nn} \langle n | v_\mu | m \rangle \langle m | w_\nu | n \rangle \delta(\eta - (\tau_z E)_{nn}) \right. \\
- (\tau_z)_{nn} \langle n | w_\mu | m \rangle \langle m | v_\nu | n \rangle \delta(\eta - (\tau_z E)_{nn}) \\
- \frac{i}{4} \sum_k \int_{-\infty}^{\infty} d\eta \left[ (\tau_z)_{nn} \langle n | w_\mu | m \rangle \frac{(\tau_z)_{mm}}{\eta - (\tau_z E)_{mm} + i\epsilon} \langle m | v_\nu | n \rangle g(\eta) \delta(\eta - (\tau_z E)_{nn}) \right. \\
- (\tau_z)_{nn} \langle n | v_\nu | m \rangle \frac{(\tau_z)_{mm}}{\eta - (\tau_z E)_{mm} - i\epsilon} \langle m | w_\mu | n \rangle g(\eta) \delta(\eta - (\tau_z E)_{nn}) \right] \\
- \frac{i}{4} \sum_{k, m \neq n} \langle n | w_\mu | m \rangle \langle m | v_\nu | n \rangle \frac{(\tau_z)_{mm}(\tau_z)_{nn}}{(\tau_z E_{mm})^2 - (\tau_z E_{nn})^2} \left[ \int_{-(\tau_z E)_{mm}}^{(\tau_z E)_{nn}} d\eta g(\eta) - \int_{-(\tau_z E)_{mm}}^{(\tau_z E)_{nn}} d\eta g(\eta) \right] \\
- \frac{i}{4} \sum_{k, m \neq n} (\tau_z)_{nn}(\tau_z)_{mm} \langle n | w_\mu | m \rangle \langle m | v_\nu | n \rangle g(\tau_z E_{nn}) + g(\tau_z E_{mm}) \frac{g(\tau_z E_{nn}) + g(\tau_z E_{mm})}{(\tau_z E_{nn})^2 - (\tau_z E_{mm})^2}. 
\]  

(G73)
To obtain the first equality, we cyclically permute the delta functions to the right and then integrate by parts. Thus,

\[ S_{\mu \nu} + M^{(0)}_{\mu \nu} = -\frac{i}{2} \sum_{k,m \neq n} \langle n|w_{\mu} + u_{\mu}|m|v_{\nu}|n\rangle \frac{(\tau_{z})_{mm}(\tau_{z})_{nn}}{((\tau_{z})_{mn} - (\tau_{z})_{nm})^2} \left( \int_{(\tau_{z})_{mn}}^{(\tau_{z})_{nm}} d\eta \frac{d^2}{d\eta^2} \right) \]

Changing the variable to \( \eta = k_B T x \) and restoring the \( h \), the spin Nernst conductivity \( \alpha_{\mu \nu}^S = \frac{S_{\mu \nu} + M_{\mu \nu}}{VT} \) is given by

\[ \alpha_{\mu \nu}^S = -k_B \frac{1}{V} \sum_{k,m \neq n} \langle n|w_{\mu} + u_{\mu}|m|v_{\nu}|n\rangle \frac{(\tau_{z})_{mn}(\tau_{z})_{nn}}{((\tau_{z})_{mn} - (\tau_{z})_{nm})^2} \int_{(\tau_{z})_{mn}}^{(\tau_{z})_{nm}} d\eta \frac{d^2}{d\eta^2} \]

where \( \rho(x) = \frac{1}{e^x - 1} \). The second term is difficult to evaluate because we need to find (locally) differentiable wave function throughout the Brillouin zone. As explained in more detail below, we expect this term to be small in comparison to the first because it is proportional to \( C_S \) which is proportional to the magnon-phonon interaction strength, which is about 0.05 for \( D = 0.94 \) (in units where we measure energy in meV and \( h = 1 \)).

Now, a note on the integrals. Using

\[ \rho(-x) = -1 - \rho(x), \]

we have

\[ \int_{0}^{E} \rho(x) dx = \int_{0}^{E} (1 + \rho(x)) dx, \quad \int_{0}^{E} x \rho(x) dx = \int_{0}^{E} x (1 + \rho(x)) dx. \]

It follows that

\[ \int_{a}^{b} dx \rho(x) = \tilde{c}(b) - \tilde{c}(a), \quad \int_{a}^{b} dx \frac{d^2}{dx^2} \rho(x) = c_1(|b|) - c_1(|a|), \]

where \( \tilde{c}(x) = \log |1 - e^{-|x|}| + |x| \theta(-x) \) and \( c_1(x) = \int_{x}^{\infty} dx' x' \left( -\frac{d^2}{dx'^2} \right) = (1 + \rho(x)) \log(1 + \rho(x)) - \rho(x) \log \rho(x) \).

We also have

\[ v_{\mu} = \frac{1}{h} \frac{\partial H}{\partial k_{\mu}}, \quad w_{\mu} = \frac{1}{h} S^z \tau_{z} \frac{\partial H}{\partial k_{\mu}}, \quad u_{\mu} = \frac{1}{h} \frac{\partial H}{\partial k_{\mu}} \tau_{z} S^z, \]

so that

\[ \alpha_{\mu \nu}^S \approx \frac{k_B}{h} \frac{i}{2V} \sum_{k,m \neq n} \langle n|S^z \tau_{z} \frac{\partial H}{\partial k_{\mu}} + \frac{\partial H}{\partial k_{\mu}} \tau_{z} S^z|m\rangle \langle m|\frac{\partial H}{\partial k_{\nu}}|n\rangle \frac{(\tau_{z})_{mm}(\tau_{z})_{nn}}{((\tau_{z})_{mn} - (\tau_{z})_{nm})^2} \]

\[ \times \left[ c_1\left( \frac{E_{mn}}{k_B T} \right) - c_1\left( \frac{E_{nm}}{k_B T} \right) \right], \]

where we have neglected the term containing \( C_S \).

To justify this, let us note that

\[ \langle n|\frac{\partial H}{\partial k_{\mu}}|m\rangle = \frac{\partial}{\partial k_{\mu}} \langle n|H|m\rangle - \left( \frac{\partial}{\partial k_{\mu}} \langle n|H|m\rangle \right) \frac{\partial H}{\partial k_{\mu}} - \langle n|H \left( \frac{\partial}{\partial k_{\mu}} \right)|m\rangle \]

\[ = \frac{\partial}{\partial k_{\mu}} E_{nm} + \langle n|\tau_{z} \frac{\partial}{\partial k_{\mu}}|m\rangle (\tau_{z})_{mn} - (\tau_{z})_{nm} \],

(G81)
where we have used \( \langle n | H | m \rangle = E_{nm} \) and \( \frac{\partial}{\partial \tau_n} \langle n | \tau_z | m \rangle = 0 \). Then, Eq. (G80) can be rewritten as
\[
\frac{k_B}{h} \frac{i}{2V} \sum_{k,m \neq n} \left\langle n | S^z \tau_z \frac{\partial H}{\partial k_{\mu}} + \frac{\partial H}{\partial k_{\mu}} S^z \tau_z | m \right\rangle \left\langle m | \tau_z \frac{\partial}{\partial k_{\nu}} | n \right\rangle (\tau_z)_{nm} (\tau_z)_{mm} c_1 \left( E_{nm} k_B \right) - c_1 \left( E_{mm} k_B \right) (\tau_z E_{nm} - (\tau_z E)_{mm}.
\]
\[(G82)\]

Since \( r_{\mu} = i \frac{\partial}{\partial k_{\mu}} \), this expression can be compared with the expression containing \( C_S \). Using the completeness relation in Eq. (G63), we have
\[
H = \sum_p \tau_z |p \rangle E_p \langle p | \tau_z, \quad C_S = \sum_{p,q} \tau_z |p \rangle \langle q | C_S |q \rangle \langle \tau_z | q \rangle q \langle \tau_z \]
\[(G83)\]

If we consider the energy scale in which the anticrossing between magnon and phonon bands occur, we expect \( \langle p | C_S |q \rangle \) to smaller by a factor of 10 for the band crossing with the lowest energy, which occurs around 1 meV, and we expect it to be much smaller for the band crossing which occurs at higher energies.

**SM H: Spin Density**

We assume periodicity along the direction of temperature gradient, but assume finite size along the edge perpendicular to it. Let us first examine the intrinsic contribution to the spin density. From the Kubo formula, the intrinsic contribution to the spin density is
\[
\langle \delta S^z (r) \rangle = \langle \delta S^z (r) \rangle_{neq} - \langle \delta S^z (r) \rangle_{eq} = - \lim_{\omega \to 0} \frac{\partial}{\partial \omega} \int_0^\beta d\tau e^{i\omega \tau} \langle T_\tau \delta S^z (r, \tau) J^\alpha (0) \rangle_{eq} \nabla_{\nu} \chi.
\]
\[(H1)\]

Here, \( \delta S^z (r) \) is the spin density of the \( r \)th strip, where \( r \) is the index for the unit cells along the finite size, which is for the armchair edge and \( y \) for the zigzag edge. From the Kubo formula, the intrinsic contribution to the spin density is
\[
\langle \delta S^z (r) \rangle = \langle \delta S^z (r) \rangle_{eq} - \langle \delta S^z (r) \rangle_{neq} = - \lim_{\omega \to 0} \frac{\partial}{\partial \omega} \int_0^\beta d\tau e^{i\omega \tau} \langle T_\tau \delta S^z (r, \tau) J^\alpha (0) \rangle_{eq} \nabla_{\nu} \chi.
\]
\[(H1)\]

Here, \( \delta S^z (r) \) is the spin density operator for the \( r \)th strip. Proceeding as in Sec. [4], we have \( \langle \delta S^z (r) \rangle = -Z_{\nu} (r) \nabla_{\nu} \chi \), where
\[
Z_{\nu} (r) = -i \frac{1}{2} \sum_{k,m \neq n} [T_k |S^z T_k|_{mn}] [\tau_z (T_k |v_{k,\nu} T_k \tau_z E_k + E_k \tau_z T_k |v_{k,\nu} T_k \tau_z)_{nm} - g_1 (\tau_z E_{km}) - g_1 (\tau_z E_{mn}) - 2 (\tau_z E_{mm})^2]
\]
\[(H2)\]

where \( \delta S^z (r) \) is the spin density operator for the \( r \)th strip. Proceeding as in Sec. [4], we have \( \langle \delta S^z (r) \rangle = -Z_{\nu} (r) \nabla_{\nu} \chi \), where
\[
Z_{\nu} (r) = -i \frac{1}{2} \sum_{k,m \neq n} [T_k |S^z T_k|_{mn}] [\tau_z (T_k |v_{k,\nu} T_k \tau_z E_k + E_k \tau_z T_k |v_{k,\nu} T_k \tau_z)_{nm} - g_1 (\tau_z E_{km}) - g_1 (\tau_z E_{mn}) - 2 (\tau_z E_{mm})^2]
\]
\[(H3)\]

and the superscript ‘in’ indicates the intrinsic contribution. Then, \( \langle \delta S^z (r) \rangle = -\zeta_{\nu} (r) \nabla_{\nu} \chi \), where \( \zeta_{\nu} \) is \( Z_{\nu} (r) / T \).

In Fig. [S3] we plot \( \zeta_{\nu} (r) \) for the zigzag and armchair edges, with and without magnon-phonon interaction. Here, we note that we have put \( l = 1 \), where \( l \) is the distance between \( A \) and \( B \) sites. Thus, the distance between the unit cells along the \( x \) (\( y \)) direction for the zigzag (armchair) edge is \( \sqrt{3} \). As can be seen in Fig. [S3] in the absence of magnon-phonon interaction, the spin density for the zigzag edge vanishes while the spin density for the armchair edge is almost symmetric. Such distribution of spin density can be explained by examining the symmetry transformation property of \( \zeta_{\nu} \). Even in the presence of the magnon-phonon interaction, the spin density for the zigzag edge remains zero, while the spin density for the armchair edge remains (nearly) symmetric. To explain such a distribution, we examine some of the properties of the spin density induced by thermal gradient.

First, let us assume that there is no magnon-phonon interaction. In such a case, the total spin density of the system vanishes. This occurs for a bipartite antiferromagnetic system whenever the spin is conserved, so that the Hamiltonian can be written in a block form for each spin sector. For our system (applicable for both the finite size and the bulk system), the magnon Hamiltonian can be written in the following block form,
\[
\mathcal{H} = \frac{1}{2} \sum_k |\phi_k^+ H_k | \phi_k^0 \phi_k, \quad \phi_k = \left( \begin{array}{c} \phi_{Ak} \\ \phi_{B-k} \\ \phi_{Bk} \\ \phi_{A-k} \end{array} \right), \quad H_k = \left( \begin{array}{cc} H_{1k} & 0 \\ 0 & H_{1k} \end{array} \right).
\]
\[(H4)\]
Thus, the symmetry constrains the distribution of spin density to be symmetric along the \( x \) axis, and which are cancelled by the bulk. In the presence of magnon-phonon interaction (\( D = 0.94 \) meV), the spin density still vanishes for the zigzag edge, while a relatively large bulk spin density is induced for the armchair edge. The difference \( \zeta^{\text{inh}}_\nu(r)|_{D=0.94} - \zeta^{\text{inh}}_\nu(r)|_{D=0} \) represents the spin density induced by magnon-phonon interaction.

where \( \phi_{A_k} (\phi_{B_k}) \) contains only the magnon annihilation operators on \( A (B) \) sites. In this basis, the total spin density operator is

\[
\sigma^{z}_{\text{tot}}(\mathbf{r}) = \frac{1}{2} \sum_{\mathbf{k}} \Psi_{\mathbf{k}} \sigma^{z}_{\mathbf{k}} \Psi_{\mathbf{k}}, \quad S^{z}_{\text{tot}} = \frac{\hbar}{V} \begin{pmatrix} -12N & 0 & 0 & 0 \\ 0 & 12N & 0 & 0 \\ 0 & 0 & 0 & 12N \\ 0 & 0 & 0 & -12N \end{pmatrix},
\]

(H5)

where \( 12N \) is the \( 2N \) by \( 2N \) identity matrix. Because each of the blocks can be diagonalized using paraunitary transformation, we find that in the case of zigzag edge,

\[
(M^S_{\text{zz}})^{\dagger} M^S_{\tau \tau} = H_{-k_x}, \quad (M^S_{\text{zz}})^{\dagger} M^S_{\tau \tau} = S_{y}, \quad \text{and} \quad M^S_{\tau \tau} |n, k_x \rangle = |n, -k_x \rangle,
\]

from which it follows that \( \zeta^{\text{m}}_\nu(y) = -\zeta^{\text{m}}_\nu(y) = 0 \). Specifically, if we define

\[
f((E_{k_x})_{mm}, (E_{k_x})_{nn}) = \frac{g([\tau_{\nu} E_{k_x}]_{mm}) - g([\tau_{\nu} E_{k_x}]_{nn})}{[\tau_{\nu} E_{k_x}]_{mm} - [\tau_{\nu} E_{k_x}]_{nn}},
\]

we have

\[
Z^{\text{m}}_{x}(y) = -\frac{i}{2} \sum_{k_x, m \neq n} \langle m, k_x | S^z_{y} | n, k_x \rangle \langle n, k_x | \partial_y H_{k_x} | m, k_x \rangle f((E_{k_x})_{mm}, (E_{k_x})_{nn})
\]

\[
= -\frac{i}{2} \sum_{k_x, m \neq n} \langle m, -k_x | S^z_{y} | n, -k_x \rangle \langle n, -k_x | \partial_y H_{-k_x} | m, -k_x \rangle f((E_{-k_x})_{mm}, (E_{-k_x})_{nn})
\]

\[
= Z^{\text{m}}_{x}(y).
\]

(H6)

For the armchair edge, if we neglect the slight asymmetry from the edge configuration, the lattice is symmetric under \( \mathcal{M}_x \) passing through the center of the lattice. Letting \( M_{x}x \) denote position of the position \( x \) under the action of this symmetry, we have

\[
(M^S_{\text{zz}})^{\dagger} H_{k_x} (M^S_{\text{zz}}) = H_{k_x}, \quad (M^S_{\text{zz}})^{\dagger} S^z_{\tau \tau} M^S_{\text{zz}} = S^z_{\text{M}_x x}, \quad \text{and} \quad M^S_{\tau \tau} |n, k_y \rangle = |n, k_y \rangle,
\]

from which it follows that \( \zeta^{\text{m}}_y(x) = \zeta^{\text{m}}_y(M_{x}x) \). Thus, the symmetry constrains the distribution of spin density to be symmetric along the \( x \) axis, as can be checked in Fig. S3.
Next, let us show that the $C_{3z}$ symmetry, which is present even when $D \neq 0$, constrains $\zeta_{\nu}^{in} = 0$ for periodic system (i.e. system without edges). To derive this, let $C_{3z}$ be the representation of the symmetry, i.e. $C_{3z} H_k C_{3z} = H_{C_{3z} k}$ and $|n, k\rangle = C_{3z} |n C_{3z} k\rangle$. We find that $C^{in} = C_{3z} C^{in} = 0$. That is, threefold rotation symmetry disallows bulk spin density induced by thermal gradient. Indeed, we find that there is no bulk spin density in our system regardless of the presence of magnon-phonon interaction. On the other hand, this does not require vanishing spin density in the presence of zigzag and armchair edges since the threefold symmetry is broken by the presence of edges. Before moving on, let us note that the inversion symmetry also constrains $C^{in} = 0$.

Next, let us discuss the zigzag and armchair edges when $D \neq 0$. We find that the spin density for the zigzag edge continues to be identically zero even though the $M_x \mathcal{C}_{2x}^2$ is broken, and the spin density for the armchair continues to be symmetric along the $x$ axis. This can be explained using a rather complicated antisymmetricity, which we denote by $\mathcal{Y}$. Let us first discuss its constraints on the bulk properties. Let $\mathcal{K}$ be the complex conjugation operator and let $m_{u_y}$ be the operator that sends $u_y(r) \rightarrow -u_y(r)$ while keeping everything else fixed, where $u_y(r)$ is the displacement operator in the $y$ direction located at position $r$. Let $m_x$ be the operator that sends $\Psi(r) \rightarrow \Psi(M_x r)$, where $\Psi(r)$ is the field operator that contains both the magnon and phonon fields located position $r$. Then, we define $\mathcal{Y} \equiv \mathcal{K} m_{u_y} m_x$. Roughly, $\mathcal{Y}$ can be understood as a combination of pseudo time reversal symmetry $\mathcal{T}$ which does not flip the spin direction, with the pseudo mirror symmetry $m_{u_y} m_x$ about the plane normal to the $x$ axis which also does not flip the spin direction. Let $Y \mathcal{K}$ be the matrix representation of $\mathcal{Y}$ in the $k$ space. We note that $Y$ is the matrix that sends $u_x^{A/B} \rightarrow -u_x^{A/B}$ and act as an identity on other field operators. Under this symmetry, we find that $\mathcal{Y} \mathcal{K} H^{\nu}_n(k_x, k_y) \mathcal{Y} \mathcal{K} = H(k_x, -k_y)$ and $|n, k_x, -k_y\rangle = \mathcal{Y} \mathcal{K} |n, k_x, k_y\rangle$. From this, it follows that $\Omega_n(k_x, k_y) \rightarrow \Omega_n(k_x, -k_y)$ and $(\delta^z)^n(k_x, k_y) \rightarrow (\delta^z)^n(k_x, -k_y)$, the point being that the thermal Hall and spin Nernst currents are not forbidden. On the other hand, proceeding similarly as before, we find that $\zeta_{\nu y}^{in} \rightarrow -\zeta_{\nu y}^{in}$ for the zigzag edge and $\zeta_{\nu x}^{in} \rightarrow \zeta_{\nu x}^{in}(M_x x)$ for the armchair edge (if we ignore the slight asymmetry caused by the edge configuration). We therefore conclude that because of the $\mathcal{Y}$ symmetry, the intrinsic contribution to the spin density does not cause asymmetric spin distribution.

Next, let us evaluate the extrinsic contribution to the spin density. The Boltzmann transport theory within the constant relaxation-time approximation gives \cite{22} $g_{eq(E)} = g_{eq(E)} - \tau v_y \nabla T_y E \frac{E}{k_B T} \frac{e^{E/k_BT} - 1}{e^{E/k_BT}}$, so that

$$\zeta_{\nu}^{ext}(r) = \frac{\tau}{2 k_B T^2} \frac{1}{V} \sum_{k} \sum_{n=-N}^{N} \langle n, k | S_z^r | n, k \rangle \langle n, k | v_{k,\nu} | n, k \rangle (\tau z E_k)_{n n} \frac{e^{(\tau z E_k)_{nn}/k_BT}}{(e^{(\tau z E_k)_{nn}/k_BT} - 1)^2}. \quad (H7)$$

We note that the same equation can be derived using the Kubo formula by making a constant lifetime approximation and restricting to intraband contribution, as in Ref. \cite{23}. As discussed in the case for intrinsic contribution to the spin density, when the system is symmetric with respect to $M_x \mathcal{C}_{2x}$, this symmetry constrains the spin density to appear symmetrically for the armchair edge, and constrains the spin density to vanish for the zigzag edge. On the other hand, the $\mathcal{Y}$ symmetry constrains the spin density to appear antisymmetrically for the armchair edge, but it does not constrain the spin density for the zigzag edge. Thus, in the absence of magnon-phonon coupling, in which case both symmetries are present, the spin density vanishes for both the armchair and zigzag edge. However, in the presence of magnon-phonon coupling, only the $\mathcal{Y}$ symmetry is present, so that the spin density appears antisymmetrically for the armchair edge, while no constraint is imposed on the zigzag edge. From this, we see that total spin density is forced to vanish in the armchair edge while nonzero total spin density can be induced in the presence of zigzag edge. Let us note that this behavior of the extrinsic contribution to the spin density is opposite to that of the intrinsic contribution, and that the origin of this behavior lies in the different behavior of intrinsic and extrinsic contributions under the action of antiunitary symmetry, such as $\mathcal{Y}$. We confirm the above statements numerically as shown in Fig. S4.
we show the extrinsic contribution to the spin density in the presence of magnon-phonon interaction. We conclude that the intrinsic spin Nernst current induces asymmetric spin density through the extrinsic contribution to the spin density.

Finally, let us comment on the magnetization induced by temperature gradient. According to Refs. \[48, 49\], the magnetization susceptibility $\mu_\nu$ which we define from the relation $\langle \delta M_z \rangle = -\mu_\nu \nabla_\nu T$, is given by

$$\mu_\nu = \frac{1}{T} \sum_{n,k} \text{Re} \left[ \sum_{m \neq n} \frac{i \langle n, k | \partial_\nu H_k | m, k \rangle \langle m, k | M_z^{\text{eq}} | n, k \rangle \tau_{mn} \tau_{nn}}{(\tau_z E_k)_{nn} - (\tau_z E_k)_{mm}} \right] \left[ (\tau_z E_k)_{nn} g((\tau_z E_k)_{nn}) + \int_{(\tau_z E_k)_{nn}}^\infty dx g(x) \right]. \tag{H8}$$

Here, $\langle \delta M_z \rangle = \langle \delta \mathcal{M}^z \rangle_{\text{neq}} - \langle \delta \mathcal{M}^z \rangle_{\text{eq}}$ is the change in the magnetization density arising from the temperature gradient, and the quantities $\mathcal{M}^z$ and $\tilde{\mathcal{M}}^z$ are the equilibrium and non-equilibrium magnetization density, respectively, and are defined as

$$\mathcal{M}^z = \frac{1}{2} \sum_k \Psi_k^\dag M^z \Psi_k, \quad \tilde{\mathcal{M}}^z = \frac{1}{2} \sum_k \tilde{\Psi}_k^\dag M^z \tilde{\Psi}_r, \quad M_z^{\text{tot}} = \frac{1}{V} \begin{pmatrix} -\mu_A \times 12N & 0 & 0 & 0 \\ 0 & \mu_B \times 12N & 0 & 0 \\ 0 & 0 & \mu_B \times 12N & 0 \\ 0 & 0 & 0 & -\mu_A \times 12N \end{pmatrix}, \tag{H9}$$

where $\mu_A$ and $\mu_B$ are the magnetic moment of the spins at $A$ and $B$ sites (for the definition of $\tilde{\Psi}$, see Eq. (G11)). We note that this quantity is not derived from the $\zeta_\nu$ by simply replacing $S_z$ with $M_z^{\text{tot}}$ because the definition of magnetization is modified from $\mathcal{M}^z$ to $\tilde{\mathcal{M}}^z$ in the presence of temperature gradient \[48, 49\]. We note that the properties of the spin density operator we have discussed can be straightforwardly applied to this quantity as well. In particular, the symmetry constraints are the same when $\mu_A = \mu_B$. 

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