Finite determinacy of matrices over local rings II.

Genrich Belitskii and Dmitry Kerner

Abstract. Consider matrices with entries in a local ring, Mat(m, n; R). Fix a group action, G ⊙ Mat(m, n; R), and a subset of allowed deformations, Σ ⊆ Mat(m, n; R). We study the finite-(Σ, G)-determinacy of matrices. In Part I the determinacy question was translated into the study of the tangent spaces T_{Σ, A}, T_{G, A}, more precisely to the annihilator ideal of their quotient, ann T_{Σ, A}/T_{G, A}.

In this Part II of the work we develop the method to compute/approximate this annihilator ideal. We illustrate the method for the basic group actions, e.g., GL(m, R) × GL(n, R) and GL(m, R) × GL(n, R) × K. The method gives explicit, ready-to-use criteria.

In the simplest cases this generalizes/strengthens the classical finite determinacy conditions for maps, ideals, embedded modules etc.

1. Introduction

This is the continuation of our work on the determinacy of matrices. See [BK-1] for the definitions, motivation, details.

1.1. Setup. Let R be a (commutative, associative) local ring over some base field k of zero characteristic. Let m ⊂ R be the maximal ideal. (As the simplest case, one can consider regular rings, O_{k,p,0}, e.g., the rational functions regular at the origin, k[x_1, ..., x_p]_k or the formal power series, k[[x_1, ..., x_p]]. For k ⊆ C, or any other normed field, one can consider the convergent power series, k{...} or the smooth functions C^∞(R^p).) Geometrically, R is the ring of regular functions on the (algebraic/formal/analytic etc.) germ Spec(R).

Let Mat(m, n; R) be the space of m × n matrices with entries in R. We always assume m ≤ n, otherwise one can transpose the matrix. Various groups act on Mat(m, n; R), cf. [BK-1, Example 1.1].

Example 1.1. • The left multiplications G_l := GL(m, R), the right and two-sided multiplications G_r := GL(n, R), G_{lr} := G_l × G_r.

• The ”local changes of coordinates”, K (see [BK-1, §2] for the possible differences of the ring automorphisms, Aut_k(R), and the coordinate changes). The corresponding semi-direct products, G_{lJ} := G_l × K, G_{lrJ} := G_{lr} × K. Sometimes one considers only those coordinate changes that preserve some ideal I, i.e. the locus V(I) ⊆ Spec(R).

• Transformations that are trivial modulo some (prescribed) ideal. More precisely, for a proper ideal J ⊆ R and a group action G ⊙ Mat(m, n; R) consider the subgroup

\begin{equation}
G(J) := \{ g ∈ G | g · Mat(m, n; J) = Mat(m, n; J) \} \cap Mat(m, n; R)/\Mat(m, n; J).
\end{equation}

For example, for J = m^k the group G(m^k) consists of elements that are identities up to the order (k - 1). Similarly, Aut(J)(R) = \{ φ ∈ K | φ = Id ⊖ K/J \} Note that G(J) is ”unipotent” in the sense of [BK-1, §2]. (In the notations of Bruce.du-Plessis, Wall, for J = m, this corresponds to the groups $K_k, A_k$ etc.)

In the applications one often deforms a matrix not inside the whole Mat(m, n; R) but only inside a subset ”allowed deformations”, A ⊆ A + B, A + B ⊆ Σ_A ⊆ Mat(m, n; R), cf. [BK-1, Example 1.2]. Note that in general Σ ⊆ Mat(m, n; R) is not a linear subspace.

Example 1.2. • Consider the congruence of (anti-)symmetric matrices, A → G^{sym} U Aut, U ∈ GL(m, R). As the congruence preserves the (anti-)symmetry, it is natural to deform the matrix by the (anti-)symmetric matrices only,

Σ = Mat^{sym} or Σ = Mat^{anti-sym}.

• The G_lJ-equivalence preserves the ideal of maximal minors, in fact all the Fitting ideals I_{j}(A) (i.e. the ideals of j × j minors, for 1 ≤ j ≤ m). Thus it is sometimes natural to restrict the deformations to Σ := Σ_A = \{ A + B | I_{j}(A + B) = I_{j}(A), j = 1, \ldots, m \} ⊆ Mat(m, n; R).

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The conjugation, $G_{\text{conj}}$, preserves the whole characteristic polynomial. Thus it is sometimes natural to restrict the deformations to $\Sigma := \Sigma_A = \{ A + B \mid \det(A + B - \lambda I) = \det(A - \lambda I) \}$.

1.2. (In)finite determinacy. Given a prescribed deformation space and a group action, $G \circ \Sigma \subset \text{Mat}(m, n; R)$, one asks about the $(\Sigma, G)$-stability/infinite determinacy of $A$.

Definition 1.3. The (classical) order of determinacy is $\text{ord}_G^2(A) := \min \{ k : (\Sigma - \{ A \}) \cap \text{Mat}(m, n; m^{k+1}) \subseteq GA - \{ A \} \}$.

If $\text{ord}_G^2(A) < \infty$ then $A$ is called finitely-$(\Sigma, G)$-determined. For non-Noetherian rings in general $m^\infty := \cap_{n>0} m^n \neq \{0\}$. If $(\Sigma - \{ A \}) \cap \text{Mat}(m, n; m^\infty) \subseteq GA - \{ A \}$ then $A$ is called infinitely-$(\Sigma, G)$-determined. (In this case $A$ is determined by its $m$-adic completion $\bar{A}$, i.e. its Taylor expansion.)

As was explained in Part I the finite determinacy depends on a more general (and more manageable) question:

(2) \quad what is the biggest $R$-module $\Lambda \subseteq \Sigma$ satisfying: $\{ A \} + \Lambda \subseteq GA$?

(When $\Sigma \subseteq \text{Mat}(m, n; R)$ is a linear subspace this gives the classical notion.)

1.3. Part I of the work. In [BK-1] we began to study this (”local” or ”linear”) determinacy for a broad class of scenarios $(A, \Sigma, G)$. The classical approach to (dis)prove the finite determinacy is to compare the relevant tangent spaces, $T_{(\Sigma, A)}$ and $T_{(GA, A)}$. Accordingly, the main object of the current paper is the conductor of $\Sigma$.

Theorem 1.4. [BK-1] Fix a local ring $R$ and a scenario $A, \Sigma, G, J$.

1. If $\{ A \} + J^k \cdot T_{(\Sigma, A)} \subseteq G^J(A)$ then $J^k \subseteq \text{ann}(T_{(\Sigma, A)}/T_{(GA, A)}) + J^\infty$, where $J^\infty = \cap_{k>1} J^k$.

2. If $R$ has the relevant approximation property and $J^k \subseteq \text{ann}(T_{(\Sigma, A)}/T_{(GA, A)})$ then $\{ A \} + J^{k+1} \cdot T_{(\Sigma, A)} \subseteq G^J(A)$.

2'. If moreover $J^k T_{(\Sigma, A)} \subseteq T_{(G^J(A), A)}$ then $\{ A \} + J^{k+1} \cdot T_{(\Sigma, A)} \subseteq G^J(A)$.

(Here the ”relevant approximation property” depends on the type of equations, cf. §3.2, e.g. the Artin approximation property suffices when all the equations are polynomial.)

This theorem is a linearization result, it transforms the initial question (highly non-linear in general) to the comparison of modules and the annihilator of their quotient, an object of Commutative Algebra.

Note that $\Sigma$ participates in Theorem 1.4 only through its tangent space. In fact, being an answer to question 2, the theorem was formulated with an $R$-module $\Lambda$ instead of $T_{(\Sigma, A)}$. However, when looking for the biggest possible $\Lambda$, one usually starts from $T_{(\Sigma, A)}$.

1.4. The method to compute/approximate the annihilator ideal. Both $T_{(\Sigma, A)}$ and $T_{(GA, A)}$ are (in general non-free) $R$-modules of high rank. Their quotient is usually complicated, even in the case of a regular ring $R = O_{(k_p, 0)}$, $p > 1$.

In §4 we develop the general method to compute/approximate $\text{ann}(T_{(\Sigma, A)}/T_{(GA, A)})$. For good enough scenarios the tangent space is naturally embedded into a free $R$-module of the same rank, $T_{(\Sigma, A)} \subseteq F$. Accordingly we consider the embedding $T_{(GA, A)} \subseteq F$ and construct the generating matrix $A_{(A, \Sigma, G)}$ of $T_{(GA, A)} \subseteq F$. Then we relate $\text{ann}(T_{(\Sigma, A)}/T_{(GA, A)})$ to $\text{ann}(F/T_{(\Sigma, A)})$ and $\text{ann.coker} A_{(A, \Sigma, G)}$.

While the ideal $\text{ann}(F/T_{(\Sigma, A)})$ is often rather simple, the computation of $\text{ann.coker} A_{(A, \Sigma, G)}$ is in general difficult. Therefore, in most cases it is very difficult to write down the ideal $\text{ann}(T_{(\Sigma, A)}/T_{(GA, A)})$ explicitly. However in many cases we can approximate this annihilator by some (tight) lower/upper bounds.

As the first approximation, one finds the (set-theoretic) support of the module $T_{(\Sigma, A)}/T_{(GA, A)}$, §4.5.2. This is done by checking ("set-theoretically") the degeneracy of $A_{(A, \Sigma, G)}$ at the points of $\text{Spec}(R)$. The points of degeneracy define a locus in $\text{Spec}(R)$ whose ideal is precisely the radical $\sqrt{\text{ann}(T_{(\Sigma, A)}/T_{(GA, A)})}$.

To obtain a better approximation than the radical we change the base ring, corresponding to the maps of smooth curve-germs, $(C, 0) \to \text{Spec}(R)$. Algebraically, we go over all the homomorphisms $R \xrightarrow{\phi} S$ where $S$ is a discrete
valuation domain (DVR). The image $\phi T_{(\Sigma,A)}/T_{(G,A,A)}$ is a module over a DVR, usually it is manageable. (This step involves the canonical forms of matrices over DVR, well known for many group actions.) By going over all such maps we bound the integral closure of the ideal, $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)})$, or the integral closure of modules, $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)})^\ell$, cf. §4.2.

We remark that the bound by integral closure, $I \subseteq \hat{I}$, is rather tight. And for most applications just the knowledge of $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)})$ or $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)})^\ell$ suffices.

This criterion with maps to DVR assumes that $R$ is a Noetherian ring. To extend the so obtained bounds to the non-Noetherian case (e.g. to $C^\infty(\mathbb{R}^p,0)$) one uses the completion/base-change properties which will be developed in the subsequent work, cf. §2.6.

In §5 we illustrate the method in the "simplest" cases, the actions $G_l, G_r, G_{lr}, \mathcal{R}, G_l, G_r, G_{lr} \circ \text{Mat}(m, n; R)$, their unipotent versions $G_l^{(J)}$ and the deformations spaces $\Sigma = \text{Mat}(m, n; J)$. These come from the classical equivalences of Singularity Theory and Commutative Algebra.

Various further scenarios and questions are sketched in §2.6 and treated in [BK-3], [BK-4].

2. The results

2.1. Notations. Denote by $I_j(A)$ the ideal generated by all the $j \times j$ minors of $A$, this is also known as the $j$'th Fitting ideal. Denote by $\text{ann.coker}(A)$ the annihilator-of-cokernel ideal of the homomorphism $R^\oplus n \to R^\oplus m$.

The annihilator-of-cokernel is a rather delicate invariant but it is approximated by the ideal of maximal minors (the minimal Fitting ideal), $I_m(A)$, which is much simpler, stable and easy to compute (cf. §3.3):

\begin{equation}
\text{ann.coker}(A)^m \subseteq I_m(A) \subseteq \text{ann.coker}(A) \subseteq \sqrt{I_m(A)}.
\end{equation}

In particular, for one-row matrices, $m = 1$, or when $I_m(A)$ is a radical ideal, $I_m(A) = \text{ann.coker}(A)$. Further, for square matrices $\text{ann.coker}(A) = (\det(A)) : I_{m-1}(A)$, while for non-square matrices $\text{ann.coker}(A) = I_m(A)$, provided $I_m(A)$ is of expected depth $(n - m + 1)$, [Eisenbud, pg.511].

We often use the quotient of ideals, it is defined by $I : J = \{f \in R \mid fJ \subseteq I\}$.

For any ideal $J \subseteq R$ denote by $lR(J) \leq \infty$ the Loewy length, i.e. the minimal $N \leq \infty$ such that $I \supseteq m^N$. (If $m^N \not\subseteq I$ then put $lR(I) = -1$.)

2.2. The criteria for $G_l, G_l, G_{lr}$ actions.

**Theorem 2.1.** Fix some ideals $b \subseteq a \subseteq R$, let $\Sigma = \text{Mat}(m, n; a)$ for $m \leq n$. For $G \in G_l, G_r, G_{lr}$ consider the subgroup $G^{(b)}_l \subseteq G$, cf. example 1.1.

1. $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)}) = b \cdot \text{ann.coker}(A) : a$.

1'. If $m < n$ then $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)}) = 0$. If $m = n$ then $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)}) = b \cdot \text{ann.coker}(A) : a$.

2. If $m = 1$ then $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)}) = b \cdot I_1(A) : a$.

2'. If $m > 1$ then $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)}) = b \cdot \text{ann.coker}(A) : a$.

2''. If $m > 1$ and $R$ is Noetherian then $\text{ann}(T_{(\Sigma,A)}/T_{(G,A,A)}) = \text{Ext}^1(\text{coker}(A), \text{coker}(A)) \subseteq b(\text{ann}(R^\oplus m / I_m(A))) : a$.

In the simplest case, $a = b = R$, this bounds the order-of-determinacy as follows.

**Corollary 2.2.** Suppose either $R$ is Noetherian or $R \to \hat{R}$, where the $m$-adic completion $\hat{R}$ is Noetherian. Fix $\Sigma = \text{Mat}(m, n; R)$. Let $a \subseteq R$.

1. If $m < n$ then no matrix $A \in \text{Mat}(m, n; R)$ is finitely-$G_l$-determined.

2. Suppose $J^y \subseteq \text{ann.coker}(A)$ and $B \in \text{Mat}(m, n; J^y + J)$. Then $A + B$ is finitely-$G_l$-determined.

3. If $\text{ann.coker}(A) \supseteq m^\infty$, then the order of determinacy satisfies: $lR(\text{ann.coker}(A)) - 1 \leq \text{ord}^\ell_{G_l}(A) \leq lR(\text{ann.coker}(A))$.

3'. If $R$ is Noetherian then $lR(\text{ann}(R^\oplus m / I_m(A))) = \text{ord}^\ell_{G_l}(A) \leq lR(\text{ann.coker}(A))$.

4. If $m^\infty \neq \{0\}$ and $\text{ann.coker}(A) \supseteq m^\infty$ then $A$ is not infinitely-$G_l$-determined.

(Recall that by Borel’s lemma, [Rudin, pg. 284, exercise 12], $C^\infty(\mathbb{R}^p,0) \to \mathbb{R}[[x]]$.)
The condition \( \text{ann.coker}(A) \supseteq m^N \), for \( N < \infty \) can be stated also as: "the module \( \text{coker}(A) \) is supported only at one point, the origin". The proof and various further corollaries/examples are given in §5.1.

Example 2.3. Part 2 of the corollary can be restated in the following form.

(5) \[ J^q \subseteq \text{ann.coker}(A_1). \] If \( A_1 = A_2 \mod(J^{q+1}) \) then \( A_1 \xrightarrow{G_{(J^q)}} A_2 \).

This both strengthens and generalizes [Cutkosky-Srinivasan, Theorem 5.2]: "For \( R \) a domain and complete with respect to \( I_m(A_1) \)-filtration, there exist \( k_1, k_2 \) such that if \( A_1 = A_2 \mod(I_m(A_1))^j \) for \( j \geq k_1 \) then \( A_1 = U A_2 V^{-1} \), with \( U = \mathbb{I} \mod(I_m(A_1))^{j-k_2}, V = \mathbb{I} \mod(I_m(A_1))^{j-k_2}. \)"

Example 2.4. Let \( R = \mathbb{k}[x, y] \), suppose the matrix \( A \in \text{Mat}(2, 3; m) \) is generic enough, such that the ideal of the maximal minors, \( I_2(A) \), has 3 generators whose quadratic parts are \( \mathbb{k} \)-linearly independent. Then \( \text{jet}_2(I_2(A)) = \text{jet}_2(m^2) \), hence by Nakayama \( I_2(A) = m^2. \) Then by corollary 2.2: \( \text{ord}_{G_r}^\Sigma \leq 2. \)

Note that the condition "\( \text{ann.coker}(A) \supseteq m^N \) for some \( N < \infty \)" places severe restrictions on the ring. For example (corollary 5.2): If \( \text{dim}(R) > n - m + 1 \) then no matrix \( A \in \text{Mat}(m, n; m) \) is finitely determined with respect to \( G_r. \) This negative result is not a surprise, as the \( G_r \)-equivalence preserves all the Fitting ideals. Thus it is natural to allow only those deformations that preserve these ideals, cf. §2.6.

2.3. The criteria for \( \mathfrak{R} \). Finite determinacy for the right equivalence. The group-action \( \mathfrak{R} \circ \text{Mat}(m, n; R) \) the group does not use any matrix structure, thus we can put \( m = 1. \) Such "one-row matrices" are just maps and we have the classical right equivalence, \( \mathfrak{R} \circ \text{Maps}(\text{Spec}(R), (k^n, 0)). \) In particular, our method generalizes various classical statements.

Denote by \( \text{Der}(R) \) the \( R \)-module of all \( k \)-linear derivations of the ring. Denote by \( \text{Der}(R, m) \) the module of those derivations that send \( R \) to the maximal ideal. Denote by \( \text{rank}(\text{Der}(R)) \) the rank of this module. Note that \( \text{rank}(\text{Der}(R)) = \text{rank}(\text{Der}(R, m)) \).

For the classical regular rings, e.g. \( k[x], k\{x\}, \) etc., \( \text{Der}(R) \) is generated by the partial derivatives, \( \text{Der}(R) = \langle \partial_1, \ldots, \partial_p \rangle. \) Further, \( \text{Der}(R, m) = m \text{Der}(R) \) and \( \text{rank}(\text{Der}(R)) = \text{dim}(R) \).

Fix some set of generators \( \{\mathcal{D}_\alpha\} \) of \( \text{Der}(R, m) \), write the entries of \( A \) as a column and consider the associated "Jacobian matrix" \( \text{Jac}_{c(m)}(A) := \{\mathcal{D}_\alpha a_i\}_{\mathcal{D}_\alpha \in \text{Der}(R, m)} \). It has \( n \) rows, while the number of columns depends on \( \text{Der}(R, m) \) (and can possibly be infinite).

Theorem 2.5. Let \( \Sigma = \text{Mat}(m, n; R). \)
1. \( \text{ann}(T_{[\Sigma, A]}(T_{[\Sigma, A]}(A))) = \text{ann.coker}(\text{Jac}_{c(m)}(A)). \)
2. Suppose \( R \) has the relevant approximation property (cf. §3.3), e.g. \( R \) is the quotient of a Weierstrass system. (Examples of Weierstrass system are: \( k[x], k\{x\}, \) Grevy functions.) Let \( J^q \subseteq \text{ann.coker}(\text{Jac}_{c(m)}(A)) \) and \( B \in \text{Mat}(m, n, J^{q+1}). \) Then \( A + B \xrightarrow{\text{Aut}^0(R)} A. \)
3. In particular, \( \text{ll}_R\left(\text{ann.coker}(\text{Jac}_{c(m)}(A))\right) - 1 \leq \text{ord}_{\mathfrak{R}}(A) \leq \text{ll}_R\left(\text{ann.coker}(\text{Jac}_{c(m)}(A))\right). \)

(The group \( \text{Aut}^0(R) \) is defined in 1.1.)

Example 2.6. For \( n = 1 \) we have the right-determinacy of functions (or hypersurface singularities). Accordingly replace \( A \) by \( f \). The matrix \( \text{Jac}_{c(m)}(f) \) has one row, thus the ideal \( \text{ann.coker}(\text{Jac}_{c(m)}(f)) \) is generated by the entries of \( \text{Jac}_{c(m)}(f) \). Suppose \( R \) is regular, then \( \text{ann.coker}(\text{Jac}_{c(m)}(f)) = m \cdot (\partial_1 f, \ldots, \partial_p f) \) and Part 3 reads:

(6) \[ \text{ll}_R\left(m^2 \cdot (\partial_1 f, \ldots, \partial_p f)\right) \leq \text{ord}_{\mathfrak{R}}(f) \leq \text{ll}_R\left(m \cdot (\partial_1 f, \ldots, \partial_p f)\right). \]

For \( R = k[x] \) or \( k\{x\} \), when these numbers are finite this gives e.g. [GLS, Part 1 of Theorem 2.23], [Wall81, Theorem 1.2]. For \( R = \mathbb{C}^\infty(R^p, 0) \) and \( \text{I}_1(\text{Jac}_{c(m)}(f)) \supseteq m^\infty \) this gives [Wall81, Theorem 6.1].

Example 2.7. Suppose \( \text{ann.coker}(\text{Jac}_{c(m)}(f)) \) contains no power of \( m \). Take its saturation \( \sum_j \text{ann.coker}(\text{Jac}_{c(m)}(f)) : m^j. \) Suppose \( J^q \subseteq \sum_j \text{ann.coker}(\text{Jac}_{c(m)}(f)) : m^j. \) Then \( \Sigma = J^{q+1} \) is a module of admissible deformations. Consider the coordinate changes that preserve \( J \): \( \mathfrak{R}_J = \{ \phi \in \mathfrak{R} | \phi(J) = J \}. \) The tangent space of this group is \( T_{\mathfrak{R}_J} = \{ \mathcal{D} \in \text{Der}(R, m) | D(J) \subseteq J \}. \) We get: \( \text{ann}(T_{[\Sigma, A]}(T_{[\mathfrak{R}_J]}(f))) = \text{I}_1(\text{Jac}_{c(m)}(f)) : J^{q+1}, \) where \( \text{I}_1(\text{Jac}_{c(m)}(f)) = \{D(f) \mathcal{D}\}_{\mathcal{D} \in T_{[\mathfrak{R}_J]}(R)} \). Thus the order of determinacy for these deformations satisfies:

(7) \[ \text{ll}_R\left(m^2 \cdot \text{I}_1(\text{Jac}_{c(m)}(f)) : J^{q+1}\right) \leq \text{ord}_{\mathfrak{R}}(f) \leq \text{ll}_R\left(m \cdot \text{I}_1(\text{Jac}_{c(m)}(f)) : J^{q+1}\right). \]

In the simplest case, let \( R = \mathcal{O}_{k[x]}(0) \), suppose the singular locus itself is a smooth germ (as a set) defined by the ideal \( J = \langle x_1, \ldots, x_h \rangle \subset R. \) Suppose the generic multiplicity of \( f \) along its singular locus is \( p \), so that \( f \in J^p. \) For \( p = 2 \)
we get: \( J = \bigoplus I_j(\text{jac}(m)) : m^2 \), so the space of admissible deformations is \( J^2 \). Then \( h_R(I) = 1 \leq \operatorname{ord}_A^e(\text{jac}(m)) (I) \leq h_R(I) \), where \( I = m(J\partial_1 f, \ldots, J\partial_k f, \partial_{k+1} f, \ldots, \partial_{p_f}) : J^2 \). Compare to [Siersma83], [Pellikaan90], [Jiang], see also [Sun-Wilson, Theorem 4], [Grandjean00, Theorem 3.5], [Thilliez06, Theorem 2.1].

In the case \( n > 1 \) the finite \( R \)-determinacy of maps is a very restrictive condition, cf. §5.2. It can be thought of as the simultaneous \( R \)-determinacy of a tuple of functions.

2.4. Criteria for \( G_r \), \( G_t \), \( G_tr \) actions. For any Fitting ideal \( I_f(A) \) choose some set of generators, \( \{f_\beta\} \). Write them as a column and consider the "associated Jacobian matrix" \( \text{jac}(m)\) of \( I_f(A) := \{D_\alpha f_\beta\} \) \( d_\alpha \in \text{Der}(R, m) \).

Geometrically the ideal \( I_f(A) \) defines the degeneracy locus of \( A \), all the points where \( \text{rank}(A) < j \). The expected co-dimension of this locus, i.e. the expected height of \( I_f(A) \), is \( \min((m + 1 - j)(n + 1 - j), \dim(R)) \). The "singular part" of this locus is defined by the ideal \( I_{(m+1-j)(n+1-j)}(\text{jac}(m)(I_f(A))) + I_f(A) \). This ideal does not depend on the choice of generators of \( I_f(A) \) or of \( \text{Der}(R, m) \).

**Theorem 2.8.** Let \( \Sigma = \text{Mat}(m, n; R) \).

1. \( \text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)}) \supseteq \text{ann.coker}(A) + \text{ann.coker}(\text{jac}(m)(A)) \).

If \( m = 1 \) then \( \text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)}) = I_1(A) + \text{ann.coker}(\text{jac}(m)(A)) \).

2. Let \( R \) be Noetherian.

i. \( \sqrt{\text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)})} = \bigcap_{j=1}^m \sqrt{T_{(j-1)(A)} \bigcup I_{(m+1-j)(n+1-j)}(\text{jac}(m)(I_f(A))) : I_{j-1}(A)} \).

ii. If \( \text{rank}_R(\text{Der}(R)) < (n-j)(m-j) \) then \( \text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)}) \subseteq \sqrt{T_{(j+1)(A)}} \).

iii. If \( \text{rank}_R(\text{Der}(R)) < n(m-j) \) then \( \text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)}) \subseteq \sqrt{T_{j+1}(A)} \).

iv. If \( \text{rank}_R(\text{Der}(R)) < m(n-j) \) then \( \text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)}) \subseteq \sqrt{T_{j+1}(A)} \).

(Here for \( j = 0 \) we put \( I_0(A) = R \). The expressions for \( \sqrt{\text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)})} \), \( \sqrt{\text{ann}(T_{(\Sigma,A)}/T_{(\bar{G}_t,A)})} \) are more involved and can be obtained using the proof in §5.3.) In particular (provided \( R \) has the relevant approximation property), \( A \) is finitely-\( G_t \)-determined iff the radical ideal in part 2 equals \( m \).

**Corollary 2.9.** 1. Suppose \( \text{rank}(\text{Der}(R)) < (n-j)(m-j) < \dim(R) \) for some \( 1 \leq j < m \). Then if \( A \in \text{Mat}(m, n; m) \) is \( (m) \)-finitely-\( G_t \)-determined.

i‘. Suppose \( (n-j)(m-j) < \dim(R) \) and \( \text{rank}(\text{Der}(R)) < n(m-j) \) for some \( 1 \leq j < m \). Then \( A \in \text{Mat}(m, n; m) \) is \( (m) \)-finitely-\( G_t \)-determined.

i‘‘. Suppose \( (n-j)(m-j) < \dim(R) \) and \( \text{rank}(\text{Der}(R)) < m(n-j) \) for some \( 1 \leq j < m \). Then \( A \in \text{Mat}(m, n; m) \) is \( (m) \)-finitely-\( G_t \)-determined.

2. Suppose \( R \) has the relevant approximation property (cf. §3.2). If \( \dim(R) \leq n - m + 1 \) then \( A \in \text{Mat}(m, n; m) \) is finitely-\( G_t \)-determined if it is finitely-\( G_t \)-determined.

3. Let \( A, B \in \text{Mat}(1, n; R) \). If the entries of \( B \) lie in \( m \cdot (I_1(A)) + m \cdot (\text{ann.coker}(\text{jac}(m)(A))) \) then \( A + B \cong (m \cdot 1 \cdot m \cdot n) \).

The proof and some immediate applications are given in §5.3.

**Example 2.10.** For \( m = 1 \leq n \) and \( G = G_r \), part 3 of the corollary strengthens (and extends) [Cutkosy-Srinivasan, Theorem 2.1], which was proved for \( R = k(2) \).

**Example 2.11.** Consider two (finitely generated) ideals, \( J_1, J_2 \subset R \). Fix some sets of generators, \( J_i = \{f_1^{(i)}, \ldots, f_{n_i}^{(i)}\} \). Then \( J_1 \oplus J_2 \subset R \) iff the matrices \( A_i := (f_1^{(i)}, \ldots, f_{n_i}^{(i)}) \) are stably-\( G_r \)-equivalent. Namely, (assuming \( n_2 \geq n_1 \)), \( A_2 \cong (A_1, 0, \ldots, 0) \). Similarly, \( J_1 \cong J_2 \) iff \( A_1, A_2 \) are stably-\( G_r \)-equivalent. Using part 3 we get:

\[
(8) \quad \text{if } J_1 \equiv J_2 \bmod m \cdot \left( J_1 + \text{ann.coker}(\text{jac}(m)(J_1)) \right) \text{ then } J_1 \equiv J_2 \bmod m \cdot \left( J_1 + \text{ann.coker}(\text{jac}(m)(J_1)) \right)
\]

More generally, if \( J_i \equiv J_i \bmod I \cdot \left( J_1 + \text{ann.coker}(\text{jac}(m)(J_1)) \right) \), for some \( I \subset m \), then \( J_i \cong J_i \bmod \left( J_1 + \text{ann.coker}(\text{jac}(m)(J_1)) \right) \). This strengthens and generalizes [Cutkosy-Srinivasan, Theorem 1.6]. One can reformulate this scenario as the conditions on the isomorphism of quotients \( R/J_1 \cong R/J_2 \). The isomorphism conditions were obtained in a different form in [Möring-van-Straten, Theorem 1.1].
Example 2.12. We get that if $A \in Mat(m, n; \mathfrak{m})$ is finitely-$G_{tr}$-determined then all the Fitting ideals are of expected height, i.e. (for $k = k_1$) the loci $V(I_j(A)) \subset \text{Spec}(R)$ are of the expected codimensions. In particular:

- either $\dim(R) > mn$ and $I_1(A)$ is a complete intersection ideal
- or $\dim(R) \leq mn$ and $I_1(A)$ contains a power of the maximal ideal.

Example 2.13. If the ring $R$ is regular then $A$ is finitely determined iff all the complements $V(I_j(A)) \setminus V(I_{j-1}(A))$ are of expected codimension and smooth. For square matrices over $\mathbb{C} \{z\}$ this is [Bruce-Tari04, proposition 3.2].

Parts 1,1’ show the analogues of the classical "bad dimensions" in Singularity Theory. On the other hand, for $G_{tr}$ the theorem implies: for regular rings the finitely determined matrices are dense. This extends the classical Tougeron/Mather criteria, cf. §2.8.1. (Again, for square matrices over $\mathbb{C} \{z\}$ this is [Bruce-Tari04, proposition 3.3].)

Corollary 2.14. Suppose $k = \mathbb{k}$, $R$ is regular, local, Noetherian with the "strongest" approximation property cf. §3.2.

1. If $\dim(R) \leq 2(n-m+2)$, then $A$ is finitely-$G_{tr}$-determined iff $V(I_m(A))$ is of expected codimension and $\text{Sing}(V(I_m(A))) = V(I_{m-1}(A)) = \{0\} \subset \text{Spec}(R)$.

2. In particular, if $\dim(R) = n-m+2$, then $A$ is finitely-$G_{tr}$-determined iff $I_m(A)$ defines an isolated curve singularity (necessarily reduced).

3. If $A$ is finitely-$G_{tr}$-determined then $I_1(A)$ defines a sub germ of $\text{Spec}(R)$ with at most an isolated singularity.

Example 2.15. Let $R = \mathbb{k}[x_1, \ldots, x_p]$, let $A = \left(\begin{array}{cc} x_2 & x^k_1 \\ x^l_1 & x_2 \end{array}\right)$. We illustrate part 2 of the theorem by checking the $G_{tr}$-determinacy of $A$. For $j = 2$ we have:

\[
I_2(A) = (\det(A)) = (x_2^2 - k l_1), \quad \text{Jac}(\mathfrak{m})(I_2(A)) = \mathfrak{m} \times (k+1, x_2, 0, \ldots), \quad \sqrt{I_2(A) + I_1(\text{Jac}(\mathfrak{m})(I_2(A)))} : I_1(A) = R.
\]

(Here the matrix $\text{Jac}(\mathfrak{m})(I_2(A))$ is of size $1 \times p^2$, it is obtained by multiplying all the columns of the matrix of size $1 \times p$ by the generators of $\mathfrak{m}$.) Similarly, for $j = 1$:

\[
I_1(A) = (x_2) + (x^l_1) + (x^k_1), \quad \text{Jac}(\mathfrak{m})(I_1(A)) = \mathfrak{m} \times \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ x^{-l_1} & 0 & 0 & \cdots \\ x^{-k_1} & 0 & 0 & \cdots \end{pmatrix}, \quad I_2(\text{Jac}(\mathfrak{m})(I_1(A))) = \{0\},
\]

(Here $\text{Jac}(\mathfrak{m})(I_1(A))$ is the matrix of size $4 \times p^2$, it is obtained by multiplying all the columns of the matrix of size $4 \times p$ by the generators of $\mathfrak{m}$. As there are only two $R$-linearly independent columns, every maximal minor is zero.) Thus

\[
\sqrt{I_1(A) + I_2(\text{Jac}(\mathfrak{m})(I_1(A)))} : I_0(A) = (x_1) + (x_2).\]

In total:

\[
\text{ann}(T(\mathfrak{c}, A)/T(\mathfrak{c}_{tr}, A, A_0)) = \left(\sqrt{I_2(A) + I_1(\text{Jac}(\mathfrak{m})(I_2(A)))} : I_1(A)\right) \cap \left(\sqrt{I_1(A) + I_2(\text{Jac}(\mathfrak{m})(I_1(A)))} : I_0(A)\right) = (x_1) + (x_2).
\]

Example 2.16. Let $R = \mathbb{k}[x_1, \ldots, x_p]$. Suppose $I_1(A) \subseteq (x_1, \ldots, x_{p-1})^2$. Then theorem 2.8 gives (for $j = 1$):

\[
\text{ann}(T(\mathfrak{c}, A)/T(\mathfrak{c}_{tr}, A, A_0)) \subseteq (x_1, \ldots, x_{p-1}) \subset R.
\]

In particular this ideal cannot contain any power of $\mathfrak{m}$. Therefore $A$ is not finitely-$G_{tr}$-determined.

Corollary 2.17. Let $\mathbb{R} \subseteq \mathbb{k} \subseteq \mathbb{C}$ and $R = \mathbb{k}(\mathbb{C})/I$. $A \in \text{Mat}(m, n; R)$ is finitely-$G_{tr}$-determined iff for each point $0 \neq pt \in \text{Spec}(R)$ the matrix $A$ is $G_{tr}$-stable at $pt$.

(In this statement by $\text{Spec}(R)$ we mean a small enough neighborhood of the origin, so that $A$ is defined at each point of $\text{Spec}(R)$. By the group action of $G_{tr}$ at $pt$ we mean $GL(m, \mathbb{O}) \times GL(n, \mathbb{O}) \rtimes \mathfrak{R}_{pt}$, where $\mathbb{O}$ is the local ring, while $\mathfrak{R}_{pt}$ is the group of local coordinate changes that do not necessarily preserve $pt$, i.e. $\mathfrak{R}$ includes the translations.)

2.5. Applications to the finite determinacy of maps/complete intersections under the contact equivalence. For matrices with just one row, $\text{Mat}(1, n; R)$ the $G_{tr}$-action/equivalence coincides with the contact equivalence of maps/complete intersections, $\text{Maps}(\text{Spec}(R), (\mathbb{k}^n, 0))$, cf. §2.8.1. In this case the criteria are especially simple, §5.4. In particular, if $R$ is a regular ring, i.e. $\text{Spec}(R)$ is a smooth germ, our results imply the classical determinacy criteria.

Example 2.18. In the simplest case, $m = 1 = n$, this gives the contact-determinacy for hypersurface singularities. Suppose $R$ is regular, then $I_1(f) + \text{ann.coker}(\text{Jac}(\mathfrak{m})(f)) = (f) + \mathfrak{m}(\partial f, \ldots, \partial f)$.

i. If this contains a power of the maximal ideal then we get Part 2 of [GLS, Theorem 2.23]:

\[
\text{ord}_k(f) \leq \text{ord}_R \left( (f) + \mathfrak{m} \partial f, \ldots, \partial f \right).
\]

If this contains only $\mathfrak{m} \infty$ then we get [Wall81, theorem 6.1].
ii. When the saturation contains no \( m \) the singularity is non-isolated. Suppose \( R \) is Noetherian, define
\[
J = \sqrt{(f) + \text{ann.coker(Jac}(m)(f))}.
\]
This ideal defines (set-theoretically) the singular locus of the hypersurface \( \{ f = 0 \} \subset \text{Spec}(R) \). As the singularity is non-isolated \( J \subseteq m \). We want to understand the space of admissible deformations, i.e. the biggest ideal \( I \subset R \) with finite determinacy for deformations inside \( \Sigma = I \). By theorem 1.4
\[
J \sum_{j \geq 0} \left((f) + \text{ann.coker(Jac}(m)(f))\right)^j : m^j
\]

As is stated above, we often forget the matrix structure and consider matrices as maps, Maps(Spec(R), (k^m, 0)). In the converse direction, start from the space Maps(Spec(R), (k^N, 0)), with \( N = mn \), \( 1 \leq m \leq n \), and identify \( k^m \approx \text{Mat}(m, n; k) \). We can associate to any map the corresponding matrix \( A^{m,n}(m, n; m) \). The group \( G^{m,n}_lr := \text{GL}(m; R) \times \text{GL}(n; R) \times K \cap \text{Mat}(m, n; k) \) is obviously a subgroup of \( K \cap \text{Maps}(\text{Spec}(R), (k^N, 0)) \). And the orbits of \( G^{m,n}_lr \) are much smaller than those of \( K \). (The \( G^{m,n}_lr \)-orbits are of infinite codimension inside the \( K \)-orbits.) Thus, the finite \( G^{m,n}_lr \)-determinacy is a much stronger property than the finite \( K \)-determinacy.

In view of theorem 2.8 we get: even for the group \( G^{m,n}_lr \) the finite determinacy is often the generic property. (At least this holds when \( R \) is a regular ring). This both strengthens Mather’s/Tougeron’s results and extends them to the broader category.

2.6. Further work. In [BK-3],[BK-4] we treat many other questions and compute the ideals \( \text{ann}(T_{(\Sigma, \emptyset)}(G, A, A)) \) for the following scenarios.

- As is mentioned above, the \( G_{lr} \)-equivalence preserves all the Fitting ideals. Thus, to get a reasonable notion of determinacy for \( G \subseteq G_{lr} \) we must consider only those deformations that preserve the Fitting ideals. Note that the corresponding subset \( \Sigma_{I_{mn,1}, 1} \subset \text{Mat}(m, n; m) \) is not a linear space.
- In many areas (e.g. representation theory) the relevant equivalence is the conjugation, \( A \rightarrow UAU^{-1} \), where \( U \in G \subseteq \text{GL}(n, R) \). The conjugation preserves not only the Fitting ideals, but also characteristic polynomial \( \text{det}(A - \lambda \mathbb{I}) \). Thus it is natural to deform only by the corresponding subsets \( \Sigma_{\det(A - \lambda \mathbb{I})}, \Sigma_{\det(A - \lambda \mathbb{I}) I_{mn,1}, 1} \subset \text{Mat}(m, n; m) \).
- In various applications (e.g. bilinear/quadratic forms) one considers the congruences \( G_{\text{congr}} G_{\text{congr}} \cap \text{Mat}(m, n; m) \). Recall that the congruence preserves the (anti-)symmetry. Thus it is natural to deform the matrix only by the (anti-)symmetric matrices.
- One can consider various subgroups \( G \subseteq \text{GL}(n, R) \), e.g. orthogonal matrices, upper triangular matrices etc.
- The change of the ring. A morphism \( R \xrightarrow{S} S \) induces \( \text{Mat}(m, n; R) \xrightarrow{S} \text{Mat}(m, n; S) \). Accordingly one compares the annihilator ideals (over \( R, S \)) and the orders of determinacy. For example, for some computations of \( \text{ann}(T_{(\Sigma, \emptyset)}(G, A, A)) \) one assumes that \( R \) is Noetherian, thus excluding the important ring of smooth functions, \( R = C^\infty(\mathbb{R}^p, 0) \). Then we use the completion map \( R \rightarrow \hat{R} = \mathbb{R}[x_1, \ldots, x_p] \) to extend the determinacy results to \( C^\infty(\mathbb{R}^p, 0) \).
- Algebrization of the matrices (for various group actions) and the corresponding Weierstrass preparation theorems for matrices.

2.7. Remarks and further results. Theorems 2.1, 2.5 and 2.8 are absolutely explicit, ready-to-use criteria. This is analogous to (for \( k = \mathbb{k} \))

* "the hypersurface singularity is finitely determined iff its uniminimal deformation is finite dimensional".
* "the hypersurface singularity is finitely determined iff it is isolated",

Aside from the particular applications, we emphasize one "theoretical" consequence of our method (Remark 4.6): the study of determinacy for a given scenario \((A, \Sigma, G)\), is reduced to the study of the \( G_{lr} \)-determinacy of some associated matrix \( A_{(A, \Sigma, G)} \). In other words, the problem for an arbitrary \((A, \Sigma, G)\) is "embedded" into the problem for \((A_{(A, \Sigma, G)}, \text{Mat}(\tilde{m}, \tilde{n}, m), G_{lr})\).

2.8. Relations to other fields and motivation.

2.8.1. Singularity Theory. For a short discussion of the results on determinacy cf. [BK-1, §2]

- The case of square matrices (for \( k = \mathbb{R} \) or \( k = \mathbb{C} \), \( R = k\{x_1, \ldots, x_p\} \) and \( G = G_{lr} \)) was considered in [Bruce-Tari04], and further studied in [Bruce-Goryunov-Zakalyukin02], [Bruce03], [Goryunov-Mond05], [Goryunov-Zakalyukin03]. In particular, the generic finite determinacy was established and the simple types were classified.
- The notion of admissible deformations for complex non-isolated hypersurface singularities was studied in [Pellikaan90], [Siersma00], [de Jong-van Straten90], cf. also [de Jong90], [de Jong-de Jong90].
- Sometimes one considers the coordinate changes that preserve a locus/subscheme in \( \text{Spec}(R) \), e.g. Arnol’d has initiated the study of functions on manifolds with boundaries. For the \( C^\infty(\mathbb{R}^p, 0) \) version see [Grandjean00].
• Finite determinacy is equivalent to the finite dimensionality of the miniversal deformation. In particular, the genericity of finite determinacy, for a fixed scenario \((A, \Sigma, G)\), means: the stratum of matrices whose Tjurina number is \(\infty\), \(\{A\} + \Sigma^G_{r=\infty} \subset \{A\} + \Sigma\), is of infinite codimension.

2.8.2. Commutative Algebra. • Any matrix is the presentation matrix of its cokernel module, \(R^{\oplus m} \xrightarrow{A} R^{\oplus m} \to \text{coker}(A) \to 0\). The \(G_t\)-transformations of \(A\) preserve the module \(\text{coker}(A)\) (up to isomorphism). The \(G_t\)-transformations preserve this module ”up to base change”, \(\text{coker}(A) \otimes_{\phi(R)} R\), for \(\phi \in \mathfrak{R}\).

From the commutative algebra point of view, the classical stability means the rigidity of the module, while the finite determinacy means that the miniversal deformation of a module is of finite dimension.

Another relation is via the embedded modules. Let \(M \subset R^{\oplus m}\) be a finitely generated \(R\)-submodule. Fix some set of generators of \(M\), combine them into a matrix \(A\).

• The finite-\(G_t\)-determinacy of \(A\) means: ”if \(M, N \subset R^{\oplus m}\) and \(jet_kM = jet_kN\) for \(R \xrightarrow{jet_k} R^{\oplus m}_{k+1}\), then \(M = N \subset R^{\oplus m}\) for some \(L \in GL(m, R)\)”.

• The projection \(R \xrightarrow{jet_k} R^{\oplus m}_{k+1}\) induces the map of categories \(\text{Mod}_R \to \text{Mod}_{R^{\oplus m}_{k+1}}\), defined by \(M \mapsto M^{\oplus m}_{k+1}\). This map is surjective, an \(R^{\oplus m}_{k+1}\) module \(M\) is also an \(R\) module. (For \(r \in R\) define \(rM := \text{jet}_{k-1}(r)M\).) But the map is usually not injective, e.g. if the presentation matrix of \(M\) has all its entries in \(m\) then \(M^{\oplus m}_{k+1}\) is a free \(R^{\oplus m}_{k+1}\) module. Our results give the regions of parameters (the size of presentation matrices, \(\text{dim}(R)\), \(\text{rank}(\text{Der}(R))\)) for which the projection map is ”generically injective”.

• A particularly important (and well studied) case is: \(m = n\) and \(\det(A) \in S\) is not a zero divisor. Then \(\text{coker}(A)\) is a maximally Cohen-Macaulay module over \(R := S/(\det(A))\), [Yoshino], [Leuschke-Wiegand]. In this case there are ”Maranda type results”. Let \((R, m)\) be a Cohen-Macaulay ring admitting a faithful system of parameters, \(x_1, \ldots, x_{\text{dim}(R)}\).

[Leuschke-Wiegand, §15.2] Let \(\{x_i\}\) be a faithful system of parameters for \(R\), let \(M, N\) be two mCM \(R\)-modules. If \(M^{\oplus m}_{\{x_i\}} \xrightarrow{\phi} N^{\oplus m}_{\{x_i\}}\), then there exists an isomorphism \(M \xrightarrow{\hat{\phi}} N\) such that \(\hat{\phi} \otimes R^{\oplus m}_{\{x_i\}} = \phi \otimes R^{\oplus m}_{\{x_i\}}\).

(The faithful system of parameters exists e.g. for a complete Cohen-Macaulay ring with at most an isolated singularity.)

In terms of the presentation matrices, \(A_M, A_N\), this implies: if \(\det(A_M) = \det(A_N) = f \in S\) and \(A_M \otimes S^{\{x_i\}}\rightarrow A_M^{G_t}, A_N \otimes S^{\{x_i\}}\rightarrow A_N^{G_t}\), then \(A_M^{G_t} \cong A_N^{G_t}\).

3. Preparations

We denote the zero matrix by \(\emptyset\), the identity matrix by \(I\). For the basics of commutative algebra cf. [Eisenbud].

3.1. Smith normal form. Suppose \(R\) is a DVR, in particular a regular local ring of Krull dimension one (over a field), e.g. \(R = \mathbb{Z}[\mathfrak{p}]\) or \(\mathbb{Z}\{x\}\). Let \(A \in \text{Mat}(m, n; R)\). Then \(A\) is \(G_t\)-equivalent to a matrix with one diagonal, \(A \sim \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & a_{mm} \end{pmatrix}\). Considering \(A\) as a presentation matrix of the \(R\)-module \(\text{coker}(A)\), this can be stated as: ”every module over a DVR is the direct sum of cyclic modules”.

3.2. The relevant approximation properties. Let \(G \subset G_t\). Consider the completion \(R \to \hat{R}\), denote by \(\hat{G}\) the corresponding completion of the group. We often need the following approximation property:

\[
(14) \quad \text{Given the equation } gA = A + B \text{ for } g \in G. \text{ If there exists a formal solution, } \hat{g}A = \hat{A} + \hat{B}, \text{ then there exists an ordinary solution, } gA = A + B. 
\]

The needed condition on the ring depends on the type of the equations, cf. [BK-1, §2] for the details/proofs.

• If \(G \subset G_t\) is defined by \(R\)-linear equations and the condition \(gA = A + B\) can be written as a system of linear equations on \(g = (U, V)\) then the property (14) holds for \(G\) and arbitrary Noetherian local ring \(R\). In the non-Noetherian case the property holds if \(R \to \hat{R}\) and \(\text{ann.coker}(A) \supseteq m^\infty\). Note that by Borel’s lemma: \(R = C^\infty(\mathbb{R}^n, 0) \to \mathbb{R}[\mathfrak{p}]\), [Rudin, pg. 284, exercise 12]. Further, the condition \(\text{ann.coker}(A) \supseteq m^\infty\) can checked by Lojasiewicz inequality.

• If \(G \subset G_t\) is defined by polynomial/analytic equations and the condition \(gA = A + B\) can be written as a system of polynomial/analytic equations on \(g = (U, V, \phi)\), then the property (14) holds over any Henselian Noetherian ring. When the group involves the coordinate changes and \(\phi \neq Id\) then the assumption implies that \(A\) is a matrix of polynomials/analytic functions.
3.3. Fitting ideals and annihilator of cokernel. [Eisenbud, §20] For $0 \leq j \leq m$ and $A \in Mat(m,n;R)$ denote by $I_j(A) \subseteq R$ the Fitting ideal, generated by all the $j \times j$ minors of $A$. By definition $I_0(A) = R$. When the size of $A$ is not stated explicitly we denote the ideal of maximal minors by $I_{\text{max}}(A)$. Note that the chain of ideals $R = I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_m(A)$ is invariant under the $G_{rt}$-action and transforms under the action of $R$. The height of $I_j(A)$ is at most $\dim \left( (m+1-j)(n+1-j), \dim(R) \right)$ and this bound is achieved for $A \in Mat(m,n;m)$ generic enough.

A matrix $A \in Mat(m,n;m)$ can be considered as a map of free modules, its cokernel is an $R$-module as well:

\[(15) \quad R^{\oplus n} \xrightarrow{A} R^{\oplus m} \to \text{coker}(A) \to 0.\]

Then one takes the annihilator-of-cokernel ideal: $\text{ann.coker}(A) = \{ f \in R : f \text{coker}(A) = 0 \} = \text{ann}R^{\oplus m}/fR^{\oplus m} \subseteq R$. This ideal is a refinement of the Fitting ideal $I_m(A)$. We use the following properties:

**Lemma 3.1.**

1. $\text{ann.coker}(A)$ is $G_{rt}$-invariant. $\text{ann.coker}(A)^m \subseteq I_m(A) \subseteq \text{ann.coker}(A) \subseteq \sqrt{I_m(A)}$.
2. $\text{ann.coker}(A \oplus B) = \text{ann.coker}(A) \cap \text{ann.coker}(B)$.
3. If $A$ is a square matrix then $\text{ann.coker}(A) = \text{ann.coker}(A^T)$.
3’. If moreover $R$ is a unique factorization domain (UFD) then $\text{ann.coker}(A)$ is a principal ideal.
4. Suppose $A \in Mat(m,m;R)$ is (upper or lower) triangular. Denote by $A_{i \times i}$ the block of entries sitting in the first $i$ rows and columns. Then $\text{ann.coker}(A_{1 \times 1}) \supseteq \cdots \supseteq \text{ann.coker}(A_{i \times i}) \supseteq \text{ann.coker}(A_{(i+1) \times (i+1)}) \supseteq \cdots \supseteq \text{ann.coker}(A_{m \times m})$.

**Proof.** 1. The first two inclusions are proved e.g. in [Eisenbud, proposition 20.7]. The inclusion $\text{ann.coker}(A) \subseteq \sqrt{I_m(A)}$ follows.

Statements 2 and 4 are immediate.

3. Suppose $f \in \text{ann.coker}(A)$ then for some matrix $B$: $AB = f \mathbb{1}$, thus $B^TA^T = f \mathbb{1}$. Which means $\text{Im}(A^T) \supseteq \text{Im}(B^TA^T) \supseteq fR^{\oplus m}$, i.e. $f \in \text{ann.coker}(A^T)$. Thus $\text{ann.coker}(A) \subseteq \text{ann.coker}(A^T)$. In the same way $\text{ann.coker}(A^T) \subseteq \text{ann.coker}(A)$.

3’. By part 1 the height of $\text{ann.coker}(A) \subseteq R$ is one. If $R$ is UFD then $\text{ann.coker}(A)$ is generated by just one element.

**Example 3.2.** If $A$ has a row of zeros then $\text{ann.coker}(A) = 0$. If a square matrix $A$ has a column of zeros then $\text{ann.coker}(A) = 0$.

Further properties of $\text{ann.coker}(A)$ and its generalization, the counterparts of all $I_j(A)$, are given in [BK-3].

3.4. Integral closure of ideals and modules. The integral closure of ideal $I \subseteq R$ is defined as

\[(16) \quad \bar{I} = \left\{ f \in R, \ f^d + \sum_{j=1}^{d} a_j f^{d-j} = 0, \text{ for some } d \in \mathbb{N} \text{ and some } \{ a_j \in I \}_{j=1,...,d} \right\}.\]

This subset is an ideal and is itself integrally closed, i.e. $\bar{I} = \bar{I}$.

**Example 3.3.** Consider some monomial ideal $I \subseteq R = k[x_1, \ldots, x_p]$. Consider its diagram of exponents: $\Gamma_I := \{ \underline{d} : \underline{d}^\mathsf{T} \in I \}$. Then $\bar{I} = \{ \underline{d}^\mathsf{T} : \underline{d} \in \text{Conv}(\Gamma_I + R^p_{\geq 0}) \},$ cf.[Huneke-Swanson, §1.4].

In general the computation of $\bar{I}$ is rather involved. Things simplify due to the following "geometric" criterion. [Huneke-Swanson, Theorem 6.8.3]

\[(17) \quad \text{Suppose } R \text{ is Noetherian, then } \bar{I} = \bigcap_{R^\mathsf{T} \subseteq S} \phi^{-1}S\phi(I)\]

(Here the intersection is over all the homomorphisms to discrete valuation domains.) In words: $f \in \mathcal{I}$ iff for any homomorphism to a DVR, $R \xrightarrow{\phi} S$, $\phi(f) \in S\phi(I)$.

Geometrically this criterion (initially for $R = k\{ \underline{x} \}$, [Teissier, 1.3.4]) reads:

\[(18) \quad f \in \mathcal{I} \text{ iff for any map of the smooth curve-germ, } (C,0) \xrightarrow{\nu} \text{Spec}(R) \text{ the pullbacks satisfy: } \nu^*(f) \in \nu^*(I)\mathcal{O}_{(C,0)}.\]
For the general definition of the integral closure of modules cf. [Huneke-Swanson, definition 16.1.1]. In our case all the modules are embedded into $R^\oplus m$, the ring is Noetherian over a field, $\text{char}(k) = 0$, thus the definition simplifies. Note that any ring homomorphism $R \xrightarrow{\phi} S$ induces $R^\oplus m \supseteq M \xrightarrow{\phi} \phi(M) \subseteq S^\oplus m$.

**Definition 3.4.** $\overline{M} := \bigcap_{R \xrightarrow{\phi} S_{DVR}} \phi^{-1}S\phi(M)$.

In words, $z \in \overline{M}$ iff for any projection onto a discrete valuation domain, $R \xrightarrow{\phi} S_{DVR}$: $\phi(z) \in S\phi(M)$.

**Example 3.5.**

1. $\overline{\text{Im}(A)} = \bigcap_{R \xrightarrow{\phi} S_{DVR}} \phi^{-1}\text{Im}(\phi(A))$. (Follows from $S \cdot \text{Im}(\phi(A)) = \text{Im}(\phi(A))$.)

2. $f \in \text{ann} R^\oplus m \setminus \text{Im}(A)$ iff for any $R \xrightarrow{\phi} S_{DVR}$: $\phi(f) \in \text{ann}\text{coker}(A)$.

We use these notions in §4.2 to obtain good upper bounds on the annihilator.

### 3.5. Points in the neighborhood of the origin

Frequently $R$ is the ring of "genuine" functions, i.e. for any element $f \in R$ the germ $\text{Spec}(R)$ has a representative that contains other closed points besides the origin and $f$ can be actually computed at those points "off the origin". Thus, for any $A \in \text{Mat}(m, n; R)$ we can take a small enough representative of $\text{Spec}(R)$ and for any point of it we can evaluate the matrix, $A|_{\text{pt}} \in \text{Mat}(m, n; k)$.

For example, this happens for the ring of rational functions on $(\mathbb{k}[z])$. Complete rings, e.g. $(\mathbb{k}[z])$, are not of this type; their elements in general cannot be computed "off the origin".

Still, the geometric description is frequently used as the guiding tool to formulate criteria. Usually the geometric conditions are of the type a property $P$ is satisfied "generically" on $\text{Spec}(R)$; or satisfied on $\text{Spec}(R) \setminus \{0\}$; or on some locus in $\text{Spec}(R)$.

When $R$ is the ring of "genuine" functions this means: for a small enough representative $U$ of $\text{Spec}(S)$, there exists a locus $V \subset U$ (open or locally closed) such that the condition $P$ is satisfied at each point of $V$. Thus, in many places in the paper, we formulate the relevant condition both "algebraically" (in terms of $R, A$ and associated ideals) and "geometrically" (in terms of $A$ computed at some points near the origin).

Finally, we emphasize that when speaking (geometrically) about the points near the origin we always take the algebraic closure, $\overline{k}$ of the field $k$, so $\text{pt} \in \text{Spec}(R \otimes \overline{k})$.

### 3.6. $\text{Mat}(m, n; R)$ as an affine space

Sometimes we consider $\text{Mat}(m, n; R)$ as an affine space over the base ring $R$. As $\text{Mat}(m, n; R)$ is an $R$-module, it is isomorphic to the Zariski tangent space at each point: $T(\text{Mat}(m, n; R), A) \approx \text{Mat}(m, n; R)$.

Sometimes we consider $\text{Mat}(m, n; R)$ as an infinite dimensional affine space over $k$. When saying that the generic matrix satisfies some property, the genericity is taken in the sense of [Tougeron68]: the subset of matrices that do not satisfy this property is of "essentially" infinite codimension in $\text{Mat}(m, n; R)$. More precisely: the subsets of $\text{Mat}(m, n; R/m^k)$ that do not satisfy this property are algebraic subspaces, and their codimension tends to infinity with $k$.

### 3.7. The tangent spaces

We consider only those subsets $G, \Sigma$ that satisfy the additional restriction, cf. [BK-1, §2.2]:

(19) \[ \text{the tangent spaces } T_{(G, A), A}, T_{(\Sigma, A)} \text{ are } R\text{-modules (and not just the } k\text{-linear spaces).} \]

**Example 3.6.** The group-actions from the introduction satisfy these assumptions. Their tangent spaces are computed in [BK-1, §2]. We need the following cases:

1. $G_l: A \rightarrow AU$. Here $T_{(G_l, A, A)} = \text{Span}_R \{uA, Av\}_{(u,v)\in \text{Mat}(m, m; R) \times \text{Mat}(n, n; R)}$. Similarly for $G_l$ and $G_r$.
2. $\Psi: A \rightarrow A(\phi(z))$. (Note that the automorphisms of a local ring preserve the origin of $\text{Spec}(R)$.). Here $T_{(\Psi, A, A)} = \text{Span}_R \{D(A)\}_{D \in \text{Der}(R, m)}$, where $\text{Der}(R, m)$ is the module of those derivations of $R$ that send $m$ into itself. For a regular local ring $D(\text{Der}(R, m)) = m \text{Der}(R)$, where the module $\text{Der}(R)$ is generated by the first order partial derivatives $\{\partial_i\}$.
3. $G_r: A \rightarrow UA(\phi(z))V$. Here $T_{(G_r, A, A)} = \text{Span}_R \{uA, Av, D(A)\}_{(u,v,D)\in \text{Mat}(m, m; R) \times \text{Mat}(n, n; R) \times \text{Der}(R, m)}$. Similarly for $G_l$ and $G_r$.

Recall that the classical group of left-right equivalence of maps, $A$, does not satisfy this assumption, $T_{(A, e)}$ is not an $R$-module, cf. [BK-1, §].
4. How to compute/approximate the annihilator

4.1. Invariance of the annihilator. The element \( h = (U, V, \phi) \in \mathcal{G}_r \) acts on \( R \) by \( f \mapsto \phi^*(f) \). We use the sloppy notation \( h^*(f) \) and \( h^*(J) \) for an ideal \( J \subset R \). Suppose \( h \in \mathcal{G} \) acts on \( \Sigma \), i.e. it sends the germ \( \Sigma(A) \) to the germ \( \Sigma(hA) \).

Proposition 4.1. Let \( R \) be a local ring. Suppose the tangent spaces \( T_{(G,A),A} \), \( T_{\gamma(A)} \) are \( R \)-modules. Suppose \( h \in \mathcal{G}_r \) acts on \( \Sigma \) and also commutes with the \( G \)-action, i.e. \( hGA = GhA \). Then \( h^* \left( \text{ann}T_{\gamma(A),A}/T_{(G,A),A} \right) = \text{ann}T_{(G,hA),A}/T_{(G,A),A} \). In particular \( \text{ord}_{\mathcal{G}}(A) = \text{ord}_{\mathcal{G}}(hA) \).

Proof. Consider \( h \) as a \( k \)-linear automorphism of \( \text{Mat}(m,n;R) \). It induces the isomorphism of the tangent spaces:
\[
T_{\text{Mat}(m,n;R),A} \xrightarrow{h^*} T_{\text{Mat}(m,n;R),hA}.
\]
As \( h \) acts on \( \Sigma \), the tangent isomorphism restricts to \( T_{\gamma(A)} \). Further, the restriction \( (GA,A) \xrightarrow{h^*} (hGA,hA) \) induces the isomorphism \( T_{(G,A),A} \rightarrow T_{(G,hA),A} \).

If \( h \in \mathcal{G}_r \) then this map is \( R \)-linear. If \( h \in \mathcal{G}_r \) then the map is compatible with \( R \)-action:
\[
h^*(f \cdot T_{\gamma(A),A}/T_{(G,A),A}) = h^*(f) \cdot h^*(T_{\gamma(A),A}/T_{(G,A),A}).
\]
Altogether:
\[
h^* \left( \text{ann}T_{\gamma(A),A}/T_{(G,A),A} \right) = \text{ann}T_{(G,hA),A}/T_{(G,A),A}.
\]
As \( h \) is invertible, we get the inverse inclusion as well.

Example 4.2. • In the trivial case, \( h \in \mathcal{G} \), we get that the trivial property: the \( G \)-determinant is constant along the \( G \)-orbit.
• In many cases choosing \( h \in \mathcal{G} \) is not enough, e.g. \( A \) has no nice canonical form under the \( G \)-action. Then one extends \( G \) by its normalizer, as in the proposition. For example we use the following normal extensions: \( G_l, G_r \triangleleft G \) is usually complicated, even in the case of regular ring \( R = \mathcal{O}(k^p,0) \).
• Note that \( h \in \mathcal{G}_r \) does not in general normalize the \( G \)-action. Similarly for \( h \in \mathcal{G}_r \) does not in general normalize the \( G \)-action.

4.2. The integral closure of the annihilator. Matrices over a discrete valuation ring (DVR) often have nice canonical forms, e.g. §3.1. This simplifies the computation of an upper bound on the annihilator as follows.

Lemma 4.3. Suppose \( R \) is local, Noetherian. Fix a scenario \((A, \Sigma, G)\). Then:
\[
\frac{\text{ann}T_{\Sigma(A)}/T_{(G,A)},A} {\text{ann}T_{\gamma(A),A}/T_{(G,A),A}} \subseteq \text{ann}T_{\Sigma(A),A}/T_{(G,A),A} = \bigcap_{R \in S_{DVR}} S^{-1} \text{ann} \left( \frac{S \cdot \phi(\Sigma(A),A)} {S \cdot \phi(T_{\gamma(A)}/T_{(G,A),A})) \right).
\]

Here the projections are to all the possible discrete valuation rings. Each such projection induces \( \text{Mat}(m,n;R) \) which restricts to \( \phi(T_{\gamma(A)}/T_{(G,A),A}) \) and \( \phi(T_{\gamma(A),A}/T_{(G,A),A}) \).

Proof. The second inclusion. Let \( f \in \text{ann}T_{\Sigma(A),A}/T_{(G,A),A} \), then by equation (17) \( \phi(f) \in S \cdot \phi(\text{ann}(..)) \) for any \( R \in S_{DVR} \). This means:
\[
\phi(f) = \sum_s s_i \phi(g_i) \text{ where } s_i \in S \text{ and } \phi(g_i) \in \phi(\text{ann}(..)).
\]
Then \( g_i \in \text{ann}(..) + \ker(\phi) \), i.e. \( g_i \cdot T_{\Sigma(A)} \subseteq T_{(G,A),A} \) and \( \ker(\phi) \). Therefore \( \phi(f)S \cdot \phi(T_{\Sigma(A)}) \subseteq S \cdot \phi(T_{(G,A)},{A}) \).

The (last) equality. \( f \in \text{ann}T_{\Sigma(A),A}/T_{(G,A),A} \) iff \( f \cdot T_{\Sigma(A),A} \subseteq T_{(G,A),A} \). The later means that for any \( \phi \):
\[
\phi(f) \phi(T_{\Sigma(A)}) = \phi(f) \phi(T_{\Sigma(A)}) = \phi(f) \phi(T_{(G,A)},{A}) = \phi(T_{(G,A),A}).
\]
But the later means: \( \phi(f) \in \text{ann} \left( \frac{S \cdot \phi(T_{\Sigma(A)})} {S \cdot \phi(T_{(G,A)},{A})} \right) \).

For many scenarios \( S \cdot \phi(T_{\Sigma(A)}) = T_{\phi(0)} \), for example this holds when \( \Sigma \subseteq \text{Mat}(m,n;R) \) is defined by linear equations with coefficients in \( k \). Similarly, very often \( S \cdot \phi(T_{(G,A)},{A}) = T_{G\phi(A),A} \), this holds e.g. for all the subgroups of \( G_r \), defined by linear equations with coefficients in \( k \).

Finally, for every restriction \( \frac{T_{\phi(0)}} {T_{(G\phi(A),A)}} \) we can use the invariance of the annihilator, §4.1, to compute it efficiently.

4.3. The generating matrix \( A_{(A,\Sigma,G)} \) of the tangent space. Both \( T_{\Sigma(A)} \) and \( T_{(G,A),A} \) are (non-free) \( R \)-modules of high rank. Their quotient, \( T_{\Sigma(A)}/T_{(G,A),A} \) is usually complicated, even in the case of regular ring \( R = \mathcal{O}(k^p,0), p > 1 \).

The first complication occurs because \( T_{\Sigma(A)} \) is usually not a free \( R \)-module. But in most cases it is naturally embedded into a (finitely generated) free module, \( T_{\Sigma(A)} \subseteq F \). Moreover, the embedding often has non-zero conductor, i.e. \( JF \subseteq T_{\Sigma(A)} \subseteq F \), for some ideal \( J \subset R \) that contains non-zero divisors. In this case \( \text{rank}(T_{\Sigma(A)}) = \text{rank}(F) \).
Example 4.4. • In the simplest case $\Sigma = \text{Mat}(m, n; R)$, thus $T_{(\Sigma, A)} \cong \text{Mat}(m, n; m^3)$. The natural embedding is $\text{Mat}(m, n; R) \subset \text{Mat}(m, n; R^m)$, giving $\text{rank}(T_{(\Sigma, A)}) = mn$
• More generally, start from some vector subspace $V_k \subset \text{Mat}(m, n; V)$. Identify $k$ with its image inside $R$, this embeds $\text{Mat}(m, n; k) \hookrightarrow \text{Mat}(m, n; R)$. Then $RV_k \subset \text{Mat}(m, n; R)$ is a free module. For the groups $G_{m^3}$ of example 1.1 we often need the module $RV_k \cap \text{Mat}(m, n; m^3) = m^3 V_k$. It is not free but the embedding $m^3 V_k \subset RV_k$ has conductor $m^3$.
• More generally, in many cases $T_{(\Sigma, A)}$ can be identified with $JF$, where $F$ is a free $R$-module while $J \subset R$ is some ideal.

For a given scenario $(A, \Sigma, G)$ embed the tangent space into a free module, $T_{(\Sigma, A)} \subset F$, as above. Choose some generators $\{ v_i \}_{i \in I}$ of the module $T_{(GA,A)}$, these are some elements of $F$. Fix some basis of $F$ over $R$, then $\{ v_i \}$ are some column vectors. Combine them into the matrix, $A_{(A, \Sigma, G)}$, so that $T_{(GA,A)}$ is the image of $R^{|I|}A_{(A, \Sigma, G)}$. This generating matrix of $T_{(GA,A)}$ is $R$-valued and is defined up to $\text{GL}(R^{|I|}) \times \text{GL}(F)$ transformations. We use $A_{(A, \Sigma, G)}$ to compute/approximate $\text{ann} T_{(\Sigma, A)} / T_{(GA,A)}$.

Proposition 4.5. 1. If $T_{(\Sigma, A)} \subseteq JF$ then

\be\label{22}
\text{ann} T_{(\Sigma, A)} / T_{(GA,A)} \supseteq \text{ann} \text{coker} (A_{(A,F,G)}) : J \supseteq \text{ann} \left( T_{(\Sigma, A)} / T_{(GA,A)} \right) \left( \text{ann} F / T_{(\Sigma, A)} : J \right).
\ee

2. In particular, if $T_{(\Sigma, A)} = JF$ then $\text{ann} T_{(\Sigma, A)} / T_{(GA,A)} = \text{ann} \text{coker} (A_{(A,F,G)}) : J$.

3. If $T_{(G^{(j)},e)} : J = T_{(G,e)}$ then $\text{ann} T_{(\Sigma, A)} / T_{(G^{(j)}A,A)} \supseteq J \cdot \text{ann} T_{(\Sigma, A)} / T_{(GA,A)}$.

3'. Suppose moreover $T_{(\Sigma, A)}$ is a free $R$-submodule of $\text{Mat}(m, n; R)$, which is a free summand. Then $\text{ann} T_{(\Sigma, A)} / T_{(G^{(j)}A,A)} : J = \text{ann} T_{(\Sigma, A)} / T_{(GA,A)}$.

Proof. 1. If $f \in \text{ann} \text{coker} (A_{(A,\Sigma, G)}) : J$ then $f \cdot J(F/T_{(GA,A)}) = 0$. As $T_{(\Sigma, A)} \subseteq JF$, we have $f T_{(\Sigma, A)} / T_{(GA,A)} = 0$, i.e. $f \in \text{ann} T_{(\Sigma, A)} / T_{(GA,A)}$. Similarly for the second inclusion.

2. If $T_{(\Sigma, A)} = JF$ then $\text{ann} F / T_{(\Sigma, A)} : J = R$, now use part 1.

3. Is obvious.

3'. We should prove $\text{ann} T_{(\Sigma, A)} / T_{(G^{(j)}A,A)} : J \supseteq \text{ann} T_{(\Sigma, A)} / T_{(GA,A)}$. Suppose $f \in \text{ann} T_{(\Sigma, A)} / T_{(G^{(j)}A,A)} : J$, then $f J \cdot T_{(\Sigma, A)} \subseteq J \cdot T_{(G,A)}$. As $T_{(\Sigma, A)} \subset \text{Mat}(m, n; R)$ is a free summand we get $f \cdot T_{(\Sigma, A)} \subseteq T_{(GA,A)}$, hence the statement.  

Thus the computation of the annihilator is translated into the study of the matrix $A_{(A, \Sigma, G)}$. For example, if $J = m$ then $\text{ord}^G(A)$ is the minimal $k$ such that $m \left( \text{ann} \text{coker} (A_{(A, \Sigma, G)}) : m \right) \supseteq m^k$.

Remark 4.6. As theorem 2.1 reads, the condition "$\text{ann} \text{coker} (A_{(A, \Sigma, G)})$ contains some power of $m$" is equivalent to: "$A_{(A, \Sigma, G)}$ is finitely $G_r$-determined". Therefore proposition 4.5 implies the general statement:

\be\label{23}
\text{The determinacy problem for the scenario } (A, \Sigma, G) \text{ is reduced to that for the scenario } (A_* , \text{Mat}(**), G_r).
\ee

In other words, the problem for an arbitrary action $G \circ \Sigma$ is "embedded" into the problem for $G_r$.

4.4. Example: the matrices $A_{G_r}$, $A_{G_l}$, and $A_{0R}$. Consider the (most inclusive) case $G_{lr} \circ \text{Mat}(m, n; R)$. Using example 3.6 we get that the $R$-submodule $T_{(G,A,A)} \subset \text{Mat}(m, n; R)$ is generated by

\be\label{24}
\left\{ \sum_j A_{ij} E_{kj} \right\}_{i,k}, \quad \left\{ \sum_i A_{ij} E_{ik} \right\}_{j,k}, \quad \{ \bar{D}A \}
\ee

Here $E_{ij}$ are elementary matrices, $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$. In the first brackets we have the matrices with only one non-zero row, an arbitrary row of $A$. In the second brackets - matrices with only one non-zero column, an arbitrary column of $A$. In the third brackets, $\bar{D}$ is the row of the generators of the $R$-module $\text{Der}(R, m)$, see example 3.6.

Identify $\text{Mat}(m, n; R)$ with the space of column vectors, $R^{m \times n}$. We present the matrix $A_{(A, \Sigma, G_{lr})}$ in three blocks. The first block (with $mn$ rows) corresponds to the change of variables. It generates $T_{(\Sigma A, A)}$. The second $mn \times n^2$ block corresponds to $G_r$, the third $mn \times m^2$ block corresponds to $G_l$. To present the matrices in the compact form
we use the notations: \( \vec{a}_{ki} := (a_{k1}, \ldots, a_{kn}) \) and \( \vec{a}_{jk} := (a_{1k}, \ldots, a_{nk}) \). Then, in the basis \((\vec{a}_{1i}, \vec{a}_{2i}, \ldots, \vec{a}_{mi})\) we have:

\[
\begin{pmatrix}
\vec{E}_{i1} & \vec{E}_{i2} & \ldots & \vec{E}_{im} \\
\vec{B}_{11} & \vec{B}_{12} & \ldots & \vec{B}_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
\vec{B}_{ni} & \vec{B}_{n2} & \ldots & \vec{B}_{nm}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\vec{E}_{i1} & \vec{E}_{i2} & \ldots & \vec{E}_{im} \\
A^T & 0 & \ldots & 0 \\
0 & A^T & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

The over-the-matrix symbols \( \vec{E}_{ki} = (E_{k1}, \ldots, E_{km}) \) and \( \vec{E}_{jk} = (E_{1k}, \ldots, E_{nk}) \) denote the particular generators of the tangent space, as they appear in equation (24).

Sometimes we need this matrix in the basis \((\vec{a}_{j1}, \vec{a}_{j2}, \ldots, \vec{a}_{jn})\):

\[
\begin{pmatrix}
\vec{B}_{11} & \vec{B}_{12} & \ldots & \vec{B}_{1m} \\
\vec{B}_{n1} & \vec{B}_{n2} & \ldots & \vec{B}_{nm}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{pmatrix} A & 0 & \ldots & 0 \\
0 & A & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix} & \vec{a}_{j1} & \vec{a}_{j2} & \ldots & \vec{a}_{jn}
\end{pmatrix}
\]

4.5. **The properties of** \( T_{(\Sigma,A)}/T_{(G,A,A)} \) **and of** \( A_{(A,\Sigma,G)} \). Proposition 4.5 reduces the problem to the computation of \( ann.coker(A_{*+}) \). Even this computation is in most cases complicated. However, in many cases we can obtain useful bounds on \( ann.coker(A_{*+}) \).

4.5.1. **The criterion in terms of the Fitting ideal.** Often instead of computing \( ann.coker(A_{*+}) \) it is enough to study the weaker version, the ideal of maximal minors, \( I_{max}(A_{*+}) \), cf.\S3.3. We get:

**Proposition 4.7.** Given a fixed embedding into a free module, \( m^+F \subseteq T_{(\Sigma,A)} \subseteq F \). Then \( A \in \Sigma \) is finitely-(\( \Sigma, G \))-determined iff \( I_{max}(A_{(A,\Sigma,G)}) \) contains a power of the maximal ideal of \( R \).

This follows immediately from \( I_{max}(A_{(A,\Sigma,G)}) \subseteq ann.coker(A_{(A,\Sigma,G)}) \subseteq \sqrt{I_{max}(A_{(A,\Sigma,G)})} \), cf. lemma 3.1.

4.5.2. **Set-theoretic support of** \( T_{(\Sigma,A)}/T_{(G,A,A)} \) **and the radical of** \( ann(T_{(\Sigma,A)}/T_{(G,A,A)}) \). Here we assume the ring \( R \) to be Noetherian. The most basic invariant of the module \( T_{(\Sigma,A)}/T_{(G,A,A)} \) is its "pointwise" support, i.e. all the points of \( Spec(R) \) where the localization of \( T_{(\Sigma,A)}/T_{(G,A,A)} \) does not vanish. The locus of such points defines the radical \( \sqrt{ann(T_{(\Sigma,A)}/T_{(G,A,A)})} \).

Suppose \( A \) is the matrix of "genuine functions", in the sense of \( \S3.5 \). Then the support can be obtained by going over the closed points of \( Spec(R) \). For example, the condition, \( f \in I_{max}(A_{(A,\Sigma,G)}) \) contains a power of the maximal ideal", means: for any point of the punctured neighborhood of the origin, \( pt \in Spec(R) \setminus \{0\} \), there is a function \( f \in I_{max}(A_{(A,\Sigma,G)}) \) with \( f(pt) \neq 0 \). Alternatively, the left kernel of the numerical matrix \( A_{(A,\Sigma,G)}|pt \) is trivial.

For complete rings, we cannot compute \( A_{(A,\Sigma,G)} \) at a point off the origin. The statement of the lemma below is just the reformulation of the condition "\( ker^{(l)}(A_{(A,\Sigma,G)}|pt) = \{0\} \)" to the case of such rings.

For any given ring \( S \), the left kernel of a matrix \( B \in Mat(m, n; S) \) is defined by

\[
ker^{(l)}(B) := \{ v \in S^{\oplus m} | vB = 0 \in S^{\oplus n} \}.
\]
By construction, it is an $S$ submodule of the free module $S^\oplus m$, thus it is torsion-free (or zero).

Given the triple $(A, \Sigma, G)$ and the embedding $T_{(\Sigma, A)} \subseteq F$ into a free module. Suppose $N < \infty$. Let $A_{(A, \Sigma, G)}$ be the generating matrix of $T_{(G, A, \Sigma)} \subseteq F$.

**Lemma 4.8.** Let $R$ be a local Noetherian ring.

1. If $\ker^{(1)}(A_{(A, \Sigma, G)}) \neq \{0\}$ then $ann T_{(\Sigma, A)}/T_{(G, A)} = \{0\}$.
2. Suppose $\ker^{(1)}(A_{(A, \Sigma, G)}) = \{0\}$. Denote by $P$ the set of all the prime ideals $0 \neq p \subseteq R$ satisfying $\ker^{(1)}(\phi(A)) \neq \{0\}$ for the projection $R \onto R_p$. Then $\sqrt{ann T_{(\Sigma, A)}T_{(G, A)}} = \bigcap_{p \in P} p$. In particular if $P = \emptyset$ then $T_{(G, A)} = T_{(\Sigma, A)}$.
3. $ord_2^n(A) < \infty$ iff $\ker^{(1)}(A_{(A, \Sigma, G)}|_{pt}) = \{0\}$ for all the points of the punctured neighborhood, $pt \in Spec(R) \setminus \{0\}$. (Algebraically: for any non-maximal prime ideal, $p \not\subseteq m$ the matrix $\phi(A_{(A, \Sigma, G)})$ has no non-trivial left kernel.)

**Proof.** As $m^N F \subseteq T_{(\Sigma, A)}$, we have: $\sqrt{ann T_{(\Sigma, A)}T_{(G, A)}} = \sqrt{ann F}T_{(G, A)} = \sqrt{T_{max}(A_{(A, \Sigma, G)})}$. 1. If $A_{(A, \Sigma, G)}$ has a non-trivial left kernel then $I_{max}(A_{(A, \Sigma, G)}) = \{0\}$.

2. Note that $\cap P$ corresponds precisely to the primary decomposition of $I_{max}(A_{(A, \Sigma, G)})$, [Eisenbud, §20]. More precisely, $\phi(A_{(A, \Sigma, G)})$ has a non-trivial left kernel iff $p \supseteq \sqrt{I_{max}(A_{(A, \Sigma, G)})}$. And $I_{max}(A_{(A, \Sigma, G)})$ contains a power of the maximal ideal iff for any non-maximal prime ideal: $m \supseteq p \nsubseteq I_{max}(A_{(A, \Sigma, G)})$. Thus $\phi(A_{(A, \Sigma, G)})$ has no left kernel. Suppose $I_{max}(A_{(A, \Sigma, G)})$ does not contain any power of the maximal ideal, then $I_{max}(A_{(A, \Sigma, G)}) \subseteq p \not\subseteq m$ for some non-maximal prime ideal. Then $\phi(I_{max}(A_{(A, \Sigma, G)}) = \{0\}) \subseteq R$. Thus $\phi(A_{(A, \Sigma, G)})$ has the non-zero left kernel.

3. $ord_2^n(A) < \infty$ iff $ann T_{(\Sigma, A)}T_{(G, A)}$. As $R$ is Noetherian, the later holds iff $A_{(A, \Sigma, G)}$ is non-degenerate off the origin.

**Remark 4.9.** If we want to check the left kernel only "pointwise", i.e. to compute $\sqrt{ann(\cdot)}$, then in the block $A_{ir}$ in equation (26) it is enough to take just the columns of the "ordinary" derivations, $\{D_{a_{ir}}\}_{D \in Der(R)}$, instead of the more complicated $\{D_{a_{ir}}\}_{D \in Der(R)}$.

4.5.3. **Example of $A_{Gi_A, A_{Gi_r, A_{ir}}, A_{ir}}$, continued.** Denote by $A_{Gi_A, A_{Gi_r}, A_{ir}, A_{ir}}$ the corresponding blocks of $A_{ir}$, §4.4.

**Proposition 4.10.** 1. $\text{ann. coker}(A_{Gi_A}) = \text{ann. coker}(A)$. If $m < n$ then $\text{ann. coker}(A_{Gi_A}) = 0$.

2. $\text{ann. coker}(A) \subseteq \text{ann. coker}(A_{Gi_r}) \subseteq \text{ann. coker}(A_{Gi_A}) \subseteq \text{ann. coker}(A)$. 2'. If $m = 1$ or $A$ is $G_{ir}$ equivalent to a matrix with a column of zeros then $\text{ann. coker}(A_{Gi_r}) = \text{ann. coker}(A)$. 3. $\text{ann. coker}(A) + \text{ann. coker}(Jac_{(m)}(A)) \subseteq \text{ann. coker}(A_{Gi_r}) \subseteq \text{ann. coker}(A_{Gi_A})$, where $I_{mn}(Jac_{(m)}(A))$ is the ideal of the critical locus, cf. §4.2. 3'. If $m = 1$ then $\text{ann. coker}(A_{Gi_A}) = \text{ann. coker}(A_{Gi_r}) = I_{1}(A) + \text{ann. coker}(Jac_{(m)}(A)).$

4. For an ideal $J \subseteq R$ consider the quotient $R \onto R/J$.

$\ker^{(1)}(\phi(A_{Gi_A})) = \left\{ B \in Mat(n, m, R/J) \mid \text{trace}(B\phi(D_A)) = 0 \right\}$.

$\ker^{(1)}(\phi(A_{Gi_r})) = \left\{ B \in Mat(n, m, R/J) \mid \phi(A)B = B\phi(D_A) = 0 \right\}$.

$\ker^{(1)}(\phi(A_{Gi_A})) = \left\{ B \in Mat(n, m, R/J) \mid B\phi(A) = B\phi(D_A) = 0 \right\}$.

$\ker^{(1)}(\phi(A_{Gi_A})) = \left\{ B \in Mat(n, m, R/J) \mid B\phi(A) = B\phi(D_A) = 0 \right\}$.

**Proof.** 1. The part $\text{ann. coker}(A_{Gi_A}) = \text{ann. coker}(A)$ is obvious from equation (26), as $Im(A_{Gi_A}) = \oplus Im(A)$. Similarly, $I_{mn}(A_{Gi_A}) = \oplus Im(A^T)$, where $R^\oplus m \overset{A^T}{\rightarrow} R^\oplus m$. Thus for $m < n$ the module $Im(A^T)$ cannot be of the rank $n$, giving $\text{ann. coker}(A_{Gi_A}) = 0$.

But for $m = n$: $\text{ann. coker}(A_{Gi_A}) = \text{ann. coker}(A^T) = \text{ann. coker}(A)$, cf. §3.3.

2. As $Im(A_{Gi_A}) \supseteq Im(A_{Gi_r})$ one has $\text{ann. coker}(A) \subseteq \text{ann. coker}(A_{Gi_A})$. To establish the other inclusion consider the images $\phi(A_{Gi_A})$ for all the possible homomorphisms $R \overset{\phi}{\rightarrow} S_{DR}$, see §4.2. For $\phi(A_{Gi_A})$, a matrix over a DVR, we use invariance under the $G_{ir}$, §4.1, and the Smith normal form, §3.1. Thus we assume that the only possibly non-zero entries of the matrix lie on the diagonal,

\[
\phi(A) = \begin{pmatrix}
{a_{11}} & 0 & \cdots & 0 \\
0 & {a_{22}} & \cdots & 0 \\
0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & {a_{mm}}
\end{pmatrix}
\]
and \((a_{11}) \supset (a_{22}) \supset \cdots\). Then by the column operation on the right block of \(\nu^* A_{Gir}\) one can bring it to the form:

\[
\phi(A_{Gir}) \mapsto \left( \begin{array}{ccccccc}
\nu^* A & 0 & \cdots & 0 & a_{11} & 0 & 0 & \cdots \\
0 & \nu^* A & 0 & \cdots & 0 & a_{22} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & \nu^* A & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots \\
\end{array} \right)
\]

Thus, for any map of the smooth curve-germ we get \(ann.coker(\phi A_{Gir}) = (a_{mm}) = ann.coker(\phi(A))\).

Therefore \(ann.coker(A_{Gir}) \subset ann \frac{R^0(mn)}{Im A_{Gir}} = ann \frac{R^0(m)}{Im(A)}\), cf. §4.2.

If \(m = 1\) then \(Im(A_{Gir}) = \frac{n}{n} Im(A)\), thus \(ann.coker(A_{Gir}) = ann.coker(A)\).

Suppose \(A\) is \(Gir\) equivalent to a matrix with at least one column of zeroes. We assume that \(a_{*n} = 0\). Then, by equation 26, \(A_{Gir}\) is of the form: \(\left( \begin{array}{cc}
** & 0 \\
0 & A \end{array} \right)\). Thus \(ann.coker(A_{Gir}) = ann.coker(A)\).

3. The inclusion \(ann.coker(A_{G\ell}) \subset ann.coker(A_{\ell \ell})\) follows from \(Im(A_{G\ell}) \subset Im(A_{\ell \ell})\). Similarly \(Im(A_{G\ell}) + Im(A_{\ell \ell}) \subset Im(A_{G\ell})\), giving the first inclusion.

3'. For \(m = 1\) by the direct check: \(Im(A_{G\ell}) = I_1(A)R^0 + Im(A_{\ell \ell})R^0\).

4. To understand the left kernel we just write down \(ker(1)(A_{G\ell})\) in the matrix form, using the identification

\[
(w_1, \ldots, w_{mn}) \rightarrow \left( \begin{array}{cccc}
w_1 & \cdots & w_n \\
w_{n+1} & \cdots & w_{2n} \\
\cdots & \cdots & \cdots \\
0 & \cdots & w_{mn} \\
\end{array} \right)
\]

Sometimes one needs to check the equivariance of the left kernel explicitly.

**Example 4.11.** Consider the simplest case, \(G = G_{ir}\), \(\Sigma = Mat(m, n; R)\). If \((U, V) \in G_{ir}\) then

\[
ker(1)(A_{(UAV, G_{ir})}) = \{ B | \text{trace}(BUAV)_{|pt} = 0, UAVA|_{|pt} = 0, \text{trace}(BBD(UAV)_{|pt}) = 0 \} = \{ B | \text{trace}(BUAV)_{|pt} = 0, UAVA|_{|pt} = 0, \text{trace}(VBUAV)_{|pt} + \text{trace}(AVA|_{|pt}) = 0 \} = \{ V_{|pt} | ker(1)(A_{(G_{ir})}) U_{|pt} = 0 \}
\]

Here for the last equality one uses \(AVA|_{|pt} = 0, UAVA|_{|pt} = 0\) to get:

\[
\text{trace}(BUAV)_{|pt} + \text{trace}(AVA|_{|pt}) = \text{trace}(AVAU^{-1}BUAV)_{|pt} = 0 + 0,
\]

This equivariance preserves the (non-)triviality of \(ker(1)(A_{G_{ir}})\). Thus, we can assume \(A\) in some particularly nice form. For example, if \(rank(A)_{|pt} = j\) and \(G \supset G_{ir}\) then we can assume (near \(pt\)): \(A = I_{j \times j} + A_{(m-j) \times (m-j)}\), where \(A|_{|pt} = 0\).

4.5.4. How to check the (non-)triviality of \(ker(1)(A_{(G_{ir})})\). Lemma 4.8 translates the understanding of the ideal \(\sqrt{ann(\Sigma, A)/\mathcal{I}(G, A, A)}\) into the (non-)triviality of \(ker(1)(A_{(G, \Sigma, G)})\) for points \(0 \neq pt \in \text{Spec}(R)\). Proposition 4.10 gives the simplest examples of these left-kernel spaces. We have some systems of matrix equations, often it is complicated. By proposition 4.1, to check the (non-)triviality of the left kernel we can use the group action and assume \(A\) in some particularly nice form.

Assuming \(A\) in a suitable form, one first treats the conditions on \(B\) that arise from the \(G \cap G_{ir}\) part of the group. \(\Sigma = Mat(m, n; R)\). If \((U, V) \in G_{ir}\) then \(A|_{|pt} = 0\) and/or \(B|_{|pt} = 0\). This forces some sub-blocks of \(B\) to vanish. Then one has to check the condition of the form \(trace(BD\bar{A})_{|pt} = 0\). Here \(\bar{A}\) is constructed from some blocks of \(A\) and \(\bar{A}|_{|pt} = 0\), while \(\bar{B}\) is some numerical matrix. This condition is reinterpreted as follows. Write all the entries of \(\bar{A}\) as a column, write all the entries of \(\bar{B}\) as a row. Then \(\bar{D}\bar{A}\) gives the generating matrix \(\bar{A}_{\bar{A}, \bar{D}}\), e.g. as in equation (26). And the condition \(trace(BD\bar{A}) = 0\) means that (the row) \(\bar{B}\) is in the left kernel of \(A_{(\Sigma)}|_{|pt}\). This has non-trivial solutions (for \(\bar{B}\)) iff the rows of \(\bar{A}_{(\Sigma)}|_{|pt}\) are \(k\)-linearly dependent. The later means that some associated degeneracy locus is either not smooth or not of expected dimension.

5. Examples: Determinacy for the Actions \(G_{ir}, G_{r}, G_{G_{ir}}, R, G_{ir}, G_{r}, G_{tir} \odot Mat(m, n; R)\)

5.1. The case of \(G_{ir}\).

**Proof of theorem 2.1.**
By proposition 4.5:

\[(33) \quad \text{ann} \frac{T_{(M(m,n;\Omega))}}{T_{(G^*)_{A,A}}} = \text{ann} \frac{T_{(M(m,n;\Omega))}}{T_{(G^*)_{A,A}}} : a = b \cdot \text{ann} \frac{T_{(M(m,n;\Omega))}}{T_{(G^*)_{A,A}}} : a.\]

Finally \(\text{ann} \frac{T_{(M(m,n;\Omega))}}{T_{(G^*)_{A,A}}} = \text{ann.coker}(A).\) The latter is computed/studied in proposition 4.10, parts 1, 2 and 2’. This proves all the statements. ■

**Proof of corollary 2.2.** Note that for \(G_t\)-action all the equations are linear, \((A + B) = AV.\) Thus the relevant approximation property to use theorem 1.4 is, cf. §3.2: either \(R\) is Noetherian or \(R \to \hat{R}.\)

Thus all the statements follows from theorem and 2.1 theorem 1.4. ■

**Example 5.1.** Consider the trivial case: \(A\) is a ”numerical” matrix (with entries in \(k\), \(m \leq n\). Then \(A_G\) is a constant matrix and \(\text{ann.coker}(A_G)\) is either \(R\) or \(\{0\}\). Hence \(A\) is finitely \(G_t\)-determined iff at least one of its maximal minors is a non-zero constant, i.e. \(A\) is of the full rank. (In other words, \(A\) is invertible from the left, i.e. it has no left-kernel.)

In this case, for \(m \leq n\), \(A\) is 0-determined with respect to \(G_r\), in fact \(A\) is \(G_r\)-stable.

Theorem 2.1 bounds the annihilator in terms of \(\text{ann.coker}(A)\) and \(\text{ann.coker}(A)\). This ideal is usually rather small, thus finite-\(G_t\)-determinacy places severe restrictions on the ring \(R\).

**Corollary 5.2.** Suppose either \(R\) is Noetherian or \(R \to \hat{R}\), where \(\hat{R}\) is Noetherian. Let \(\Sigma = M(m,n;R)\), with \(\dim(R) > 0\).

1. If \(m = n\) then \(A \in M(m,m,m)\) is finitely-\(G_t\)-determined iff \(\dim(R) = 1\), \(\det(A) \in R\) is not a zero divisor and \(\det(A) \notin m^\infty.\)

2. If \(\dim(R) > |n - m| + 1\) then no matrix in \(M(m,n; m)\) is finitely-\(G_t\)-determined.

3. Suppose \(\dim(R) \leq |n - m| + 1\). Given \(A \in M(m,n; R)\) and \(N > 0\), for any generic enough \(B \in M(m,m; m^N)\), the matrix \(A + B\) is finitely \(G_r\)-determined in particular, the set of matrices that are not finitely determined is of infinite codimension in \(M(m,m; R)\).

4. Suppose \(R\) is Noetherian and \(\dim(R) = 2\). Suppose \(A\) has at least two \(m \times m\) blocks whose determinants are relatively prime, (i.e. if \(\Delta_i = a_i h \in R\) then \(h \in R\) is invertible), not zero divisors, neither belong to \(m^\infty.\) Then \(A\) is finitely \(G_r\)-determined.

Here Part 1 generalizes Part 1 of [Bruce-Tari04, Theorem 1.1]. Parts 2.3 imply: either the finite determinacy is the generic property (in the sense of §3.6) or there are no finitely determined matrices. This is the analogue of the ”bad dimensions” for the determinacy of maps \(\text{Maps}\left((k^n,0),(k^m,0)\right), [AGLV-1, III.1].\)

**Proof.** (1) For square matrices \(\text{height}(\text{ann.coker}(A)) = \text{height}(\det(A)) \leq 1.\) Thus it can contain a power of the maximal ideal only when \(\dim(R) \leq 1.\) Now invoke corollary 2.2.

(2) Assume \(m \leq n\). If \(I_m(A) \supseteq m^N\) for some \(N > 0\) then the germ defined by this ideal, over \(k = \bar{k}\), \(V(I_m) \subset \text{Spec}(R)\), is supported at the origin only, i.e. the dimension of \(V(I_m)\) is zero. But the height of the ideal of maximal minors (the codimension of the corresponding germ if \(k = \bar{k}\)) is at most \((n - m + 1)\). So \(\text{dim}(V(I_m)) \geq \text{dim}(R) - (n - m + 1) > 0\), contradicting \(I_m \supseteq m^N.\)

(3) Follows by observation that for generic enough \(B\) the height of \(I_m(A + B)\).

(4) Let \(\Delta_1, \Delta_2\) be two such minors then the height of the ideal \((\Delta_1) + (\Delta_2)\) is two, i.e. the scheme \((\Delta_1 = 0 = \Delta_2) \in \bar{k} \otimes \text{Spec}(R)\) is supported at the origin only. Thus the local ring contains a power of \(m.\) ■

5.2. **The case of \(\mathbb{R}\).** Finite determinacy of maps under the right equivalence. Theorem 2.5 follows directly from the form of \(\mathcal{A}_\mathbb{R}\), equation 26 and theorem 1.4.

**Corollary 5.3.** Let \(R\) be a regular local ring with the relevant approximation, cf. §3.2, and suppose \(n > 1\). Then the map \(A \in \text{Maps}(\text{Spec}(R), (k^n,0))\) is finitely-\(\mathbb{R}\)-determined iff \(\dim(R) \geq n\) and the entries of \(A\) form a sequence of generators of \(\mathbb{R}\) (over \(R\)). In the later case \(A\) is \(\mathbb{R}\)-stable.

**Proof.** By proposition 4.7 it is enough to check whether the ideal \(I_{\max}(\mathcal{A}_\mathbb{R})\) contains some power of the maximal ideal. According to remark 4.9 instead of \(\mathcal{A}_\mathbb{R}\) we can consider the matrix whose rows are \(\{\partial_{a_{ij}}\}_{\partial \in \text{Der}(R)}\).

As \(R\) is regular, \(\text{Der}(R) = R < (\partial_1, \ldots, \partial_p)\) is a free \(R\)-module of rank \(p\). Thus \(\{\partial_{a_{ij}}\}_{\partial \in \text{Der}(R)}\) is a \(n \times p\) matrix with the rows \((\partial_{a_{ij}}, \ldots, \partial_{a_{ij}}).\)

If \(n > \dim(R)\) then \(I_{\text{max}}(\{\partial_{a_{ij}}\}_{\partial \in \text{Der}(R)}\) = 0, as there are more rows than columns.

For \(n \leq \dim(R)\) the ideal \(I_{\text{max}}(\{\partial_{a_{ij}}\}_{\partial \in \text{Der}(R)}\) is either non-proper, i.e. \(R\), or of height at most \((\dim(R) + 1 - n)\), which is less than \(\dim(R).\) Thus this ideal contains a power of the maximal ideal iff \(I_{\text{max}}(\ldots) = R\). Which means: at
least one of the maximal minors is invertible. But the later means precisely that the entries of $A$ form a sequence of generators of $m$ (over $R$). Then the stability follows. ■

We remark that if $R$ is not regular then $A^{-1}(0)$ is not necessarily reduced or has an isolated singularity, cf. example 5.4 and remark 5.6.

5.3. The case of $G_{l^r}, G_r, G_l$.

Proof of theorem 2.8.

1. i. We study the set-theoretic support of the module $T(\Sigma,A)/\mathcal{T}(G_{l^r},A,A)$. Thus we check the triviality of the kernel module at each point of $\text{Spec}(R)$, cf. proposition 4.10:

$\text{(34)} \quad \ker(l^t)(A_{G_{l^r}},|_{pt}) = \{ B \in \text{Mat}(n,m,\mathbb{k}) | BA|_{pt} = 0, A|_{pt} B = 0, \text{ trace}(B \overline{\partial} A)|_{pt} = 0 \} \not= \{0\}$

Suppose $\text{rank}(A|_{pt}) = j$, we work locally near $pt$. As is explained in §4.5.4, we can assume $A = \begin{pmatrix} I_{j \times j} & 0 \\ 0 & \bar{A} \end{pmatrix}$, where $\bar{A}|_{pt} = 0$. Present $B$ accordingly, \( \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \). Then $BA|_{pt} = 0$ and $A|_{pt} B = 0$ mean $B_1 = 0$, $B_2 = 0$, $B_3 = 0$. If $j = m$ then the equations force $B = 0$, i.e. $\ker(l^t)(A_{G_{l^r}},|_{pt}) = \{0\}$.

Assume $j < m$. The remaining condition reads: $\text{trace}(B \overline{\partial} A)|_{pt} = 0$. If we write down all the entries of $A$ as a column and all the entries of $B_4$ as a row, then $\text{trace}(B_4 \overline{\partial} \bar{A})|_{pt} = 0$ means that the row $B_4$ belongs to the left kernel of the matrix $\text{Jac}(\bar{A})|_{pt}$ that has $(m - j)(n - j)$ rows. Which means: the rows of $\text{Jac}(\bar{A})|_{pt}$ are linearly dependent. Geometrically this happens when the locus $\{ \bar{A} = 0 \} \subset \text{Spec}(R) \otimes \mathbb{k}$ is singular or not of expected dimension.

Algebraically the degeneracy can be written as $I_{(m-j)(n-j)}(\text{Jac}(\bar{A}))|_{pt} = 0$. Note that near $pt \in \text{Spec}(R)$ the entries of $\bar{A}$ generate the ideal $I_{j+1}(A)$. Therefore in terms of $A$ the degeneracy condition is written as

$\text{(35)} \quad I_{(m-j)(n-j)}(\text{Jac}(I_{j+1}(A)))|_{pt} = 0.$

Here $\text{Jac}(I_{j+1}(A))$ is defined in §2.4, it is a matrix with $(m - j)(n - j)$ rows. Thus, set theoretically, the locus where $\text{rank}(A) = j$ and $\ker(l^t)(A_{G_{l^r}},|_{pt}) \not= \{0\}$ is defined by \( \left\{ I_{j+1}(A) + I_{(m-j)(n-j)}(\text{Jac}(I_{j+1}(A))) \right\} : I_{j}(A) \). Here one divides by $I_{j}(A)$ to exclude the points where $\text{rank}(A) < j$. As we work set-theoretically we take the radical of this ideal.

By going over all the ranks we get the union of the loci corresponding to the intersection of ideals:

$\text{(36)} \quad \bigcap_{j=0}^{m-1} \left( \sqrt{I_{j+1}(A) + I_{(m-j)(n-j)}(\text{Jac}(I_{j+1}(A)))} : I_{j}(A) \right).$

ii. If $\text{rank}(\text{Der}(R)) < (n - j)(m - j)$ then $I_{(m-j)(n-j)}(\text{Jac}(I_{j+1}(A))) = \{0\}$. Therefore $\text{ann} \frac{T(\Sigma,A)}{\mathcal{T}(G_{l^r},A,A)} \subseteq \sqrt{I_{j+1}(A)}$.

iii. We check the triviality of the kernel module at each point, cf. proposition 4.10:

$\text{(37)} \quad \ker(l^t)(A_{G_r},|_{pt}) = \{ B \in \text{Mat}(n,m,\mathbb{k}) | BA|_{pt} = 0, \text{ trace}(B \overline{\partial} A)|_{pt} = 0 \} \not= \{0\}$

Suppose $\text{rank}(A|_{pt}) = j$, such points always exist if $\text{dim}(R) > (m - j)(n - j)$. As is explained in §4.1, §4.5.4, we can use the $GL(m,\mathbb{k}) \times GL(n,R)$-action. Thus we assume $A = \begin{pmatrix} I_{j \times j} & 0 \\ 0 & A_3 \end{pmatrix}$, where $A_3|_{pt} = 0$ and $A_4|_{pt} = 0$.

Present $B$ accordingly, \( \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \). Then $BA|_{pt} = 0$ means $B_1 = 0$ and $B_2 = 0$. The only remaining condition is:

$\text{trace}(B_3 \overline{\partial} A_3) + \text{trace}(B_4 \overline{\partial} A_4) = 0$. Here $A_3 \in \text{Mat}(m - j,j;\mathbb{m})$, $A_4 \in \text{Mat}(m - j,n - j;\mathbb{m})$. As is explained in §4.5.4, if we write all the entries of $A_3$ and $A_4$ in one column, we get the matrix $\text{Jac}(A_3, A_4)$ with $(m - j)$ rows. The trace condition means: $\text{rank}(\text{Jac}(A_3, A_4))|_{pt} < n(m - j)$, but if $\text{rank}(\text{Der}(R)) < n(m - j)$ then this condition is empty! And then $\ker(l^t)(A_{G_r},|_{pt}) \not= \{0\}$ if $\text{rank}(A|_{pt}) < j$. Thus $\text{ann} \frac{T(\Sigma,A)}{\mathcal{T}(G_r,A,A)} \subseteq \sqrt{I_{j+1}(A)}$.

iv. This case is done similarly to $G_r$. ■

Proof of corollary 2.9.1. If $(n - j)(m - j) < \text{rank}(\text{Der}(R)) < \text{dim}(R)$ then, by theorem 2.8, $\text{ann} \frac{T(\Sigma,A)}{\mathcal{T}(G_r,A,A)} \subseteq \sqrt{I_{j+1}(A)}$. And $\text{height}(I_{j+1}(A)) \leq (n - j)(m - j)$, thus $\text{ann} \frac{T(\Sigma,A)}{\mathcal{T}(G_r,A,A)}$ cannot contain any power of $m$. Thus $A$ is not finitely-$G_{l^r}$-determined.

1'. and 1". are proved similarly.
2. As $G_r \subset \mathcal{G}_r$, the only non-trivial direction is that $\mathcal{G}_r$-determinacy implies the $G_r$-determinacy. If $A$ is $\mathcal{G}_r$-finitely determined then $\text{height}(I_m(A)) = \min\left(n - m + 1, \text{dim}(R)\right)$. As $\text{dim}(R) \leq n - m + 1$, $\text{height}(I_m(A)) = \text{dim}(R)$, thus $I_m(A)$ contains a power of the maximal ideal. But then, by theorem 2.1, $A$ is finitely $G_r$-determined.

3. Follows straight from theorem 2.8, the case of $m = 1$. ■

Proof of corollary 2.14

Proof. 1. If $\text{dim}(R) \leq 2(n - m + 2)$ then the expected dimension of $\Sigma_r(A)$ for $r < m - 1$ is zero. Thus among the conditions of part 3 of theorem 2.8 only the case $j = m - 1$ is relevant.

2. Immediate. We should only add: as $\text{Spec}(R)$ is smooth, and the determinantal ideals $I_j(A)$ are of expected height, their zero loci $V(I_j(A))$ are Cohen-Macaulay. In particular they have no embedded components.

3. Apply theorem 2.8, the $j = 1$ case. ■

Proof of corollary 2.17. The linear independence of the vectors $\{\tilde{\partial}(A_{ik})\}_{i=1, \ldots, m-1}^{k=1, \ldots, (n-j)}$ means the $k$-linear independence of the forms $\{\text{jet}_1(A_{ik})\}_{i=1, \ldots, m-1}^{k=1, \ldots, (n-j)}$ near pt. But then, over $\mathcal{O}_{\mathcal{U}(pt)}$, all the higher order terms of $\tilde{A}$ can be killed just by a change of coordinates. And then, by proposition 5.3, $\tilde{A}$ is $\text{Aut}(\mathcal{O}_{\mathcal{U}(pt)})$-stable. Thus $A$ is $\mathcal{G}_r$-stable. ■

Example 5.4. • If $R$ is not a regular ring then being finitely determined does not imply the ideal $I_1(A)$ is radical (i.e. the zero locus $A^{-1}(0)$ is reduced). For example, let $R = \mathbb{k}[x, y, z]/(xz, yz)$ and $A = x + y + z \in \text{Mat}(1, 1, R)$. Then $A$ is obviously $\mathfrak{R}$-finitely-determined (even stable), but $I_1(A)$ defines a non-reduced scheme, whose local ring is $\mathbb{k}[x, y, z]/(x, z + x + y, z) \cong \mathbb{k}[x, y]/(x^2, xy)$.

• Further, cf. remark 5.6, even if $R$ is a complete intersection and $A$ is finitely-$\mathcal{G}_r$-determined, the Fitting ideal $I_1(A)$ can define a scheme with multiple components.

5.4. An application to the contact equivalence of maps. Suppose $m = 1$, then $\text{Mat}(1, n; \mathfrak{m})$ can be considered as $\text{Maps}(\text{Spec}(R), (\mathbb{k}^n, 0))$. Note that in this case the $\mathcal{G}_c$ and $\mathcal{G}_r$ equivalences coincide. Moreover, they coincide with the classical contact equivalence $\mathcal{K}$, [BK-1, §2]. Our results extend the well known results to the case of non-regular rings.

Take the Jacobian matrix $\text{Jac}(A)$ of $A \in \text{Mat}(1, n; R)$, cf. §2.3. The corresponding degeneracy ideal is the ideal of the maximal minors $I_n(\text{Jac}(A))$.

Corollary 5.5. Consider the map $\text{Spec}(R) \xrightarrow{A} (\mathbb{k}^n, 0)$, where $A = (a_1, \ldots, a_n)$.

1. $A$ is finitely-$\mathcal{G}_r$-determined iff $A^{-1}(0) \subset \text{Spec}(R)$ is a (possibly non-reduced) point. In particular, if $n < \text{dim}(R)$ then no such map is finitely-$\mathcal{G}_r$-determined.

2. If $\text{dim}(R) \leq n$ then $A$ is finitely-$\mathcal{K}$-determined iff it is finitely-$\mathcal{G}_r$-determined. For $\text{dim}(R) \leq n$ the generic matrix is finitely determined.

3. $A$ is finitely-$\mathcal{K}$-determined iff the ideal $I_1(A) + I_n(\text{Jac}(A))$ contains $\mathfrak{m}^N$, for some $N$.

3'. If $R$ is regular and $\text{dim}(R) > n$ then $A$ is finitely-$\mathcal{K}$-determined iff $A^{-1}(0)$ is a complete intersection (of codimension $n$) with at most an isolated singularity. In particular, the generic matrix is finitely-$\mathcal{K}$-determined.

Proof. 1. and 2. This is just the $\mathcal{G}_r$-criterion from theorem 2.1 and corollary 5.2.

3. Note that in this case the generating matrix of $T_{(\mathfrak{m}, A, A)}$ has $n$-rows:

$$
\mathcal{A}_K = \text{Jac}(A) \begin{pmatrix}
\bar{a} & \bar{0} & \ldots & \ldots & \bar{0} \\
\bar{0} & \bar{a} & \bar{0} & \ldots & \bar{0} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\bar{0} & \ldots & \ldots & \bar{0} & \bar{a}
\end{pmatrix}.
$$

(38)

So, if $\text{dim}(R) < n$ then $\sqrt{I_{\text{max}(A)}} = \sqrt{I_1(A)}$, while if $\text{dim}(R) \geq n$ then $\sqrt{I_n(A)} = \sqrt{I_n(\text{Jac}_{(\mathfrak{m})}(A)) + I_1(A)}$. So, the first assertion is just the statement of proposition 4.7 for the case $\text{Mat}(1, n; R)$.

The rest follows because $\text{height}(I_1(A)) = n$. ■

Remark 5.6. The statement (3') of the corollary is the classical criterion, e.g. [Wall81]. For non-regular rings the locus $A^{-1}(0)$ can have non-isolated singularity but $A$ can still be finitely determined. For example, let $R = \mathbb{k}[x_1, \ldots, x_n]/(x_1^n)$, then the module of derivations is generated by $(x_1 \partial_1, \partial_2, \ldots, \partial_n)$. Consider $A \in \text{Mat}(1, 1; \mathfrak{m})$ whose only entry is $x_1 + x_n$. Note that $\mathcal{A}_{(A, \mathfrak{g}_r)} = (x_1, 0, 0, \ldots, 1, x_1 + x_n)$, thus $I_{\text{max}}(A) = R$, hence finite-$\mathcal{K}$-determinacy.

(In fact, $A$ is even $\mathfrak{R}$-finitely determined.) But the zero locus is $\{x_1 + x_n = 0\} \approx \text{Spec}(\mathbb{k}[x_1, x_2, \ldots, x_{n-1}]/(x_1^n))$, i.e. is a multiple hyperplane.
