Representing Lie algebras
using approximations with nilpotent ideals

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Abstract

We prove a refinement of Ado’s theorem for Lie algebras over an algebraically-closed field of characteristic zero. We first define what it means for a Lie algebra \( \mathfrak{g} \) to be approximated with a nilpotent ideal, and we then use such an approximation to construct a faithful representation of \( \mathfrak{g} \). The better the approximation, the smaller the degree of the representation will be. We obtain, in particular, explicit and combinatorial upper bounds for the minimal degree of a faithful \( \mathfrak{g} \)-representation. The proofs use the universal enveloping algebra of Poincaré-Birkhoff-Witt and the almost-algebraic hulls of Auslander and Brezin.

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1 Introduction

The classical theorem of Ado and Iwasawa states that every finite-dimensional Lie algebra \( \mathfrak{g} \) over a field \( F \) admits a finite-dimensional, faithful representation. That is: there exists a natural number \( n \) and a homomorphism \( \varphi : \mathfrak{g} \rightarrow \mathfrak{gl}_n(F) \) of Lie algebras with \( \ker(\varphi) = \{0\} \).

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But it is in general quite difficult to determine whether a given Lie algebra $\mathfrak{g}$ admits a faithful representation of a given degree $n$.

The problem is easily seen to be equivalent to the computation of the minimal degree $\mu(\mathfrak{g})$ of a faithful representation of $\mathfrak{g}$. The study of this invariant was crucial in finding (filiform) counter-examples to a conjecture by Milnor on the existence of affine structures on manifolds, [19], [4] and [9]. Lower and upper bounds for $\mu(\mathfrak{g})$ in terms of other natural invariants of $\mathfrak{g}$ were given for various families of Lie algebras by Benoist, Birkhoff, Burde, de Graaf, Eick, Grunewald and the author, [5], [6], [8], [10], [11], and [13]. This paper aims to refine the current upper bounds.

We recall two major techniques that have been used historically to construct faithful representations. A nilpotent Lie algebra $\mathfrak{n}$ is known to act faithfully on its universal enveloping algebra $\mathcal{U}(\mathfrak{n})$, and Birkhoff observed that the subspace $S$ of all sufficiently large elements (made precise in the next sections), yields a faithful quotient module $\mathcal{U}(\mathfrak{n})/S$ of dimension $d^{c-1} - d - 1$, where $d$ is the dimension of $\mathfrak{n}$ and $c$ is its class. Mal’cev later observed that Ado’s theorem is a natural consequence of the existence of almost-algebraic hulls (also called splittings) and Birkhoff’s construction. A constructive approach to Mal’cev’s theorem by Neretin made use of elementary expansions and it allowed Burde and the author to find explicit upper bounds for $\mu(\mathfrak{g})$ for all finite-dimensional complex Lie algebras $\mathfrak{g}$: $\mu(\mathfrak{g}) = O(2^d)$.

Several examples in the literature then suggested that Lie algebras which have an abelian ideal of small codimension in the solvable radical also have a small, faithful representation (see for example [10] and propositions 2.12, 2.15, 4.5 and remark 4.8 of [11]). This was made precise and proven by Burde and the author to obtain bounds of the form $\mu(\mathfrak{n}) = O(d^{r+1})$, where $\mathfrak{n}$ is a $d$-dimensional nilpotent Lie algebra and $\gamma$ is the minimal codimension of an abelian ideal.

We will prove that these results can be generalised to nilpotent ideals of arbitrary class:

**Theorem 1.0.1.** Consider a Lie algebra $\mathfrak{g}$ over an algebraically-closed field of characteristic zero. Let $d$ be its dimension, let $r$ be the dimension of the solvable radical and let $n$ be the dimension of the nilradical. Suppose $\mathfrak{g}$ has a nilpotent ideal of class $\varepsilon_1$ and codimension $\varepsilon_2$ in $\text{rad}(\mathfrak{g})$. Then

$$\mu(\mathfrak{g}) \leq d - n + \left(\frac{r + \varepsilon_1}{\varepsilon_1}\right) \cdot \left(\frac{r + \varepsilon_2}{\varepsilon_2}\right).$$

We note that de Graaf’s theorem, [13], corresponds with the special case where the Lie algebra is itself nilpotent and the ideal is chosen to be the whole Lie algebra: $d := n$, $\varepsilon_1 := c(\mathfrak{g})$ and $\varepsilon_2 := 0$. We define the nil-defect $\varepsilon = \varepsilon(\mathfrak{g})$ of $\mathfrak{g}$ to be the minimal value $\varepsilon_1 + \varepsilon_2$ as $h$ runs over all nilpotent ideals of $\mathfrak{g}$.

**Corollary 1.0.2.** Consider a Lie algebra $\mathfrak{g}$ over an algebraically-closed field of characteristic zero. Let $d$ be its dimension and let $\varepsilon$ be its nil-defect. Then $\mathfrak{g}$ has a faithful representation of degree

$$P_{\varepsilon}(d) := d + \frac{(d + \varepsilon) \cdots (d + 1)}{\frac{d}{2}! \cdot \frac{d+1}{2}!}.$$

We can also apply the construction to graded Lie algebras; there exists a function $E : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds.
Corollary 1.0.3. Consider a Lie algebra \( g = \oplus_{a \in A} g_a \) graded by an abelian, finitely-generated, torsion-free group \((A,+))\). Let \( \sigma := |\{a \in A| g_a \neq \{0\}\}| \) be the cardinality of the support and let \( \delta := \dim(g_0) \) be the dimension of the homogeneous component corresponding with the neutral element. Then \( g \) admits a faithful representation of degree \( P_{E(\sigma,\delta)}(d) \).

Convention: we will only consider finite-dimensional Lie algebras over an algebraically-closed field of characteristic zero.

2 Preliminaries

In this section we introduce some of the concepts used in our construction of faithful representations. We first define the nil-defect \( \varepsilon(g) \) of a Lie algebra \( g \). We then show that \( g \) can (almost) be embedded into an almost-algebraic Lie algebra \( \widehat{g} \) for which the nil-defect \( \varepsilon(\widehat{g}) \) is at most \( \varepsilon(g) \).

2.1 The nil-defect of a Lie algebra

The following definition is justified by the existence of a (unique) solvable radical, that is: the theorem of Levi-Malcev.

Definition 2.1.1 (Nil-defect). Let \( r \) be a solvable Lie algebra and \( n \) be a nilpotent ideal of \( r \). The nil-defect \( \varepsilon(r,n) \) of \( n \) in \( r \) is \( \dim(r/n) + c(n) \). The nil-defect \( \varepsilon(r) \) of \( r \) is

\[
\min_{n} \{\varepsilon(r,n)\},
\]

where \( n \) runs over all nilpotent ideals of \( r \). The nil-defect of an arbitrary Lie algebra is the nil-defect of its solvable radical.

Let us consider a few special cases.

Example 2.1.2. The nil-defect of a semisimple Lie algebra is 0. The nil-defect of a nilpotent Lie algebra \( n \) is bounded by the nilpotency class: \( \varepsilon(n) \leq \varepsilon(n,n) = c(n) \). In particular: the family of all Lie algebras of nil-defect at most \( \varepsilon \) contains the family of all nilpotent Lie algebras of class at most \( \varepsilon \).

We note that Lie algebras of a given nil-defect can have arbitrarily high nilpotency class.

Example 2.1.3. A standard filiform Lie algebra has a nil-defect of 2, while its class can be chosen arbitrarily high. More generally: for filiform Lie algebras \( f \), we have \( \varepsilon(f) \leq 2\sqrt{\dim(f)} + 1 \). This is a direct consequence of the fact that any ideal of \( f \) with codimension \( a > 1 \) is nilpotent of class at most \( \lceil (c(f) + 1)/a \rceil - 1 \), cf. [22].

The following result, not strictly necessary for the rest of the paper, is due to B. Kostant and allows us to approximate solvable Lie algebras with nilpotent subalgebras, rather than ideals (personal communication with G. Glauberman and N. Wallach; see for example [16] and [2]. See also [7].)

Theorem 2.1.4. Consider a finite-dimensional complex, solvable Lie algebra \( r \) and a nilpotent subalgebra \( n \). Then \( r \) has an ideal \( h \) of class at most \( c(n) \) and dimension equal to \( \dim(n) \).

In particular:

\[
\varepsilon(r) = \min_{m} \{c(m) + \text{codim}_r(m)\},
\]

where \( m \) runs over the nilpotent subalgebras of \( r \).
2.2 Almost-algebraic Lie algebras

Definition 2.2.1 (Almost-algebraic). A Lie algebra $g$ is almost-algebraic if it admits a decomposition of the form $g = p \ltimes m$, where $m$ is the nilradical of $g$ and $p$ is a subalgebra of $g$ that acts fully reducibly on $g$ (by the adjoint representation).

A theorem by Mal’cev, later generalized by Auslander and Brezin, states that every finite-dimensional Lie algebra over an algebraically closed field of characteristic zero can be embedded into an almost-algebraic Lie algebra, and that there is a minimal such algebra: the almost-algebraic hull, [18], [3], [21]. Neretin later gave an explicit construction, as a succession of finitely many elementary expansions, of such an embedding, [20]. This construction was used by Burde and the author to obtain explicit upper bounds for $\mu(g)$, [11].

Auslander and Brezin observed that ideals are compatible with elementary expansions:

Lemma 2.2.2. Let $\iota_1 : g_1 \to g_2$ be an elementary expansion of $g_1$. Then every ideal $i$ of $g_1$ maps onto an ideal $\iota_1(i)$ of $g_2$.

In particular: if $(\iota_j : g_j \to g_{j+1})_{1 \leq j \leq u}$ is a finite sequence of elementary expansions, and $i := \iota_u \circ \iota_{u-1} \circ \cdots \circ \iota_1$, then every ideal $i$ of $g_1$ maps onto an ideal $\iota(i)$ of $g_{u+1}$. We may thus combine the lemma with proposition 4.1 of [11] to obtain the following theorem.

Proposition 2.2.3 (Good embedding). Let $g$ be a finite-dimensional Lie algebra over the complex numbers. Let $r$ be its solvable radical and let $n$ be its nilradical. Then there exists an embedding $\iota : g \to \hat{g}$ of $g$ into the Lie algebra $\hat{g}$ such that:

1. (Decomposition): $\hat{g}$ decomposes as $p \ltimes m$, where $m$ is nilpotent and $p$ acts fully reducibly on $\hat{g}$.
2. (Control of dimensions): $\dim(m) = \dim(r)$ and $\dim(p) = \dim(g/n)$,
3. (Preservation of ideals): if $h$ is a nilpotent ideal of $g$, then $\iota(h)$ is a nilpotent ideal of $p \ltimes m$ contained in $m$.

Proof. Points (1) and (2) can be obtained by expanding $\dim(r/n)$ times with respect to the nilradical, cf. 4.1 of [11]. Point (3) follows from the lemma.

Remark 2.2.4. In the above decomposition $p \ltimes m$, the nilpotent ideal $m$ need not be the whole nilradical. However, if we let $p_0$ be the kernel of the action of $p$ on $m$, then $p \ltimes m \cong p_0 \oplus (p/p_0 \ltimes m)$ and $m$ will be the nilradical of the almost-algebraic algebra $(p/p_0 \ltimes m)$.

In order to construct faithful representations of $g$ it therefore suffices to construct (sufficiently small) faithful representations of $\hat{g}$.

3 Quotients of the universal enveloping algebra

In this section we will construct faithful representations of almost-algebraic Lie algebras $p \ltimes m$. In order to do this we first introduce weight functions $\omega : U(m) \to \mathbb{N} \cup \{\infty\}$ on the universal enveloping algebra $U(m)$ of the nilpotent Lie algebra $m$. We shall then see that the elements of $U(m)$ that are sufficiently large with respect to such a weight function form a $(p \ltimes m)$-submodule $S'$ of $U(m)$. A good choice of weight functions will allow us to construct a quotient $U(m)/S'$ that is faithful and finite-dimensional.
3.1 Filtrations

In the following sections we will be working with pairs of filtrations of a given nilpotent Lie algebra. It will be convenient to have a basis that is compatible with those filtrations.

Definition 3.1.1. A filtration of a Lie algebra \( g \) is a flag \((g(t))_{t \in \mathbb{N}}\) of subspaces of \( g \) of the form, \( g = g(0) \supseteq g(1) \supseteq g(2) \supseteq \cdots \) such that for all \( g(i), g(j) \) we have \([g(i), g(j)] \subseteq g(i + j)\).

It is a positive filtration if \( g(0) = g(1) \).

Note that each element of a filtration is an ideal of the Lie algebra. We will consider the following example in the next paragraphs.

Example 3.1.2. Let \( h \) be an ideal of a Lie algebra \( g \). Then
\( g(0) := g, h(1) := h \) and \( g(i) := [h, h_{i-1}] \) for \( i \geq 2 \) defines a filtration of \( g \). Let us call this the \((g, h)\)-filtration.

The \((g, h)\)-filtration is clearly positive if \( g = h \) is nilpotent.

Definition 3.1.3. Let \( V \) be a vector space and consider a flag \((V_j)_{j \in \mathbb{N}}\) of \( V \). We say that a basis \( B \) of \( V \) is weakly adapted to the flag, iff for each \( V_j \) there exists a subset \( B_j \) of \( B \) that is a basis of \( V_j \).

Some elementary observations in linear algebra lead to the following.

Lemma 3.1.4. Consider a Lie algebra \( g \) and a pair of \( g \)-filtrations. Then \( g \) has a basis that is weakly adapted to both filtrations.

The corresponding statement for triples of filtrations fails trivially.

3.2 Notation

Let us fix some notation. We let \( g \) be a fixed almost-algebraic Lie algebra with corresponding decomposition \( p \triangleleft m \) and a nilpotent ideal \( h \) of \( g \), necessarily contained in \( m \). Let us also fix the following pair of filtrations of \( m \): the \((m, m)\)-filtration and the \((m, h)\)-filtration of \( m \). The lemma above then allows us to choose a basis for \( m \) that is weakly adapted to both filtrations. Let \( \{x_1, \ldots, x_d\} \) be such a basis.

The Poincaré-Birkhoff-Witt theorem states that the standard (non-commutative, ordered) monomials in the \( x_i \) form a basis for the universal enveloping algebra \( U(m) \) of \( m \).

We also recall that \( p \triangleleft m \) acts naturally on \( U(m) \): \( p \) acts by derivations and \( m \) acts by left multiplication. To be precise:

\[
\delta \ast (x_{i_1} \cdots x_{i_t}) := \sum_{1 \leq j \leq t} x_{i_1} \cdots x_{i_{j-1}} \cdot [\delta, x_{i_j}] \cdot x_{i_{j+1}} \cdots x_{i_t}
\]

and \( x \ast (x_{i_1} \cdots x_{i_t}) := x \cdot x_{i_1} \cdots x_{i_t} \) for all \( x \in m, \delta \in p \), and monomials \( x_{i_1} \cdots x_{i_t} \). The \( p \triangleleft m \)-module \( U(m) \) is faithful but infinite-dimensional.

3.3 From filtrations to weights and submodules

Let us first define weight functions on \( U(m) \) and then show how they can be constructed from positive filtrations on \( m \). Set \( \overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\} \).

Definition 3.3.1. A map \( \omega : U(m) \rightarrow \overline{\mathbb{N}} \) is a weight on \( U(m) \) iff it satisfies the following conditions:

1. \( \omega(X) = +\infty \Leftrightarrow X = 0 \)
Lemma 3.3.4. If 

Proof. Property (1) holds since 

Example 3.3.3. Suppose \( n \) is a nilpotent ideal of \( m \), for example \( m \) itself or \( h \). We then let \( \omega_{(m,n)} \) be the weight on \( \mathcal{U}(m) \) obtained from the \((m,n)\)-filtration. We let \( \lambda \) be the weight on \( \mathcal{U}(m) \) obtained from the trivial, positive filtration \( m := m(0) := m(1) \) and \( m(i) := \{0\} \) for \( i \geq 2 \).

This \( \lambda \) can be considered a length function (on the standard monomials).

Lemma 3.3.4. If \( n \) is a nilpotent ideal of \( p \ltimes m \) contained in \( m \), then \( \omega_{(m,n)} \) is a weight on \( \mathcal{U}(m) \) that is compatible with the action of \( p \ltimes m \).

Proof. Property (1) holds since \( n \) is nilpotent (and only if the ideal is nilpotent). Since \( \mathcal{U}^k(m,\omega_{(m,n)}) \) is spanned by the standard monomials of degree at least \( k \), we also have the second property. If \( x_i \) and \( x_j \) are basis vectors with \( j < i \), then we have \( \omega(x_i \cdot x_j) = \omega(x_j \cdot x_i + [x_i,x_j]) \geq \min(\omega(x_i \cdot x_j),\omega([x_i,x_j])) \geq \omega(x_i) + \omega(x_j) \), by using the definition of \( \omega \) on standard monomials and the property of the grading on \( m \). By using induction on the length of a standard monomials (and (2)), we then obtain property (3). Property (4) follows from (3) and the fact that \( \delta \) stabilises the flag (since \( n \) is an ideal of \( p \ltimes m \)).

We note that if a weight \( \omega \) is compatible with the action of \( p \ltimes m \), then each such subspace \( \mathcal{U}^k(m,\omega) \) is a \((p \ltimes m)\)-submodule of \( \mathcal{U}(m) \). In particular: we conclude that for each \( k_1, k_2 \) in \( \mathbb{N} \),

\[ S(m,\omega_{(m,m)}, k_1, \omega_{(m,h)}, k_2) := \mathcal{U}^{k_1}(m,\omega_{(m,m)}) + \mathcal{U}^{k_2}(m,\omega_{(m,h)}) \]

is a \((p \ltimes m)\)-stable subspace of \( \mathcal{U}(m) \) and we may consider the quotient module.
3.4 Properties of the quotient module

We now need to determine two things: the dimension of the quotient and for which choices the quotient will be faithful.

**Proposition 3.4.1** (Faithful quotient). Suppose \( p \) acts faithfully on \( m \). If \( k_1 > c(m) \) and \( k_2 > c(h) \), then the quotient of \( \mathcal{U}(m) \) by \( S(m, \omega_{(m,m)}, k_1, \omega_{(m,h)}, k_2) \) is a faithful \( (p \ltimes m) \)-module.

**Proof.** Note that the submodule \( S \) is contained in the subspace \( \Lambda_{\geq 2} := \mathcal{U}^2(m, \lambda) \) of \( \mathcal{U}(m) \) that is spanned by all standard monomials of length at least two. Now suppose that \( (\delta, x) \in p \ltimes m \) maps \( \mathcal{U}(m) \) into \( S \). Then \( x = (\delta, x) * 1 = \delta(1) + x * 1 = 0 + x = x \in S \subseteq \Lambda_{\geq 2} \). Note that \( m \) is the vector space spanned by all the standard monomials of length one. We conclude that \( x \in \Lambda_{\geq 2} \cap m = \{0\} \). Similarly, \( p \) maps \( m \) into \( \Lambda_{\geq 2} \cap m = \{0\} \). Since \( p \) is assumed to act faithfully on \( m \), we have \( (\delta, x) = (0,0) \). \qed

We recall the following well-known result about Sylvester denumerants.

**Lemma 3.4.2.** Consider a finite multiset \( M = \{m_1, \ldots, m_p\} \) of positive integers and \( t \in \mathbb{N} \). Then the number \( \Delta(t; M) \) of \( M \)-partitions of \( t \) is bounded from above by \( \binom{p T}{t-1} \).

In particular: the number of \( M \)-partitions of \( 0 \leq t \leq T \) is at most \( \binom{p T}{t-1} \). The proposition now suggests the choice \( k_1 := c(m) + 1 \) and \( k_2 := c(h) + 1 \). We then get:

**Proposition 3.4.3** (Upper bound). The \( (p \ltimes m) \)-module

\[
\mathcal{Q} := \mathcal{U}(m)/S(m, \omega_{(m,m)}, c(m) + 1, \omega_{(m,h)}, c(h) + 1)
\]

has dimension at most

\[
\left( \frac{\dim(m) + \dim(m/h)}{\dim(m/h)} \right) \cdot \left( \frac{\dim(m) + c(h)}{c(h)} \right).
\]

**Proof.** Since \( S \) is spanned by all standard monomials \( X \) satisfying \( \omega_{(m,m)}(X) \geq c(m) + 1 \) or \( \omega_{(m,h)}(X) \geq c(h) + 1 \), the dimension of \( \mathcal{Q} \) is bounded from above by the number of standard monomials \( Y \) satisfying \( \omega_{(m,m)}(Y) \leq c(m) \) and \( \omega_{(m,h)}(Y) \leq c(h) \). The lemma above then gives the crude upper bound

\[
\dim(\mathcal{Q}) \leq \left( \frac{\dim(m/h) + c(m)}{c(m)} \right) \cdot \left( \frac{\dim(h) + c(h)}{c(h)} \right)
\]

and the identity \( \binom{a+b}{b} = \binom{a+b}{a} \) finishes the proof. \qed

We note that if a weight \( \omega \) is given explicitly, it makes sense to compute the corresponding Sylvester denumerator directly.

**Example 3.4.4.** If \( \mathfrak{f} \) is filiform, then we can find a decomposition \( p \ltimes m \) of \( \mathfrak{f} \) (cf. [22] and [11]) such that the number of standard monomials \( X \) of \( \mathcal{U}(m) \) satisfying \( \omega_{(m,m)}(X) = t \) is given by the usual partition function

\[
p(t) \sim \frac{e^{\pi \sqrt{t}}}{4t^{3/2}}
\]
4 Proof of the main results

Recall that $\mu(g)$ is the minimal degree of a faithful representation of $g$ and that we wish to prove the inequality

$$\mu(g) \leq d - n + \left(\frac{r + \varepsilon_1}{\varepsilon_1}\right) \cdot \left(\frac{r + \varepsilon_2}{\varepsilon_2}\right).$$

Proof. (Theorem 1.0.1) Let $\iota : g \to p \ltimes m$ be the embedding of proposition 2.2.3. We may decompose $p \ltimes m$ as in remark 2.2.4: $p_0 \oplus (p/p_0 \ltimes m)$. Since the $\mu$-invariant is monotone and sub-additive (cf. [11]), we obtain the upper bound

$$\mu(g) \leq \mu(p \ltimes m) \leq \mu(p_0) + \mu(p/p_0 \ltimes m).$$

Since $p$ acts reductively on itself, it is itself reductive and its reductive ideal $p_0$ satisfies $\mu(p_0) \leq \dim(p_0) \leq \dim(p)$, [11]. Proposition 2.2.3 gives $\dim(p) \leq \dim(g/n)$, so that $\mu(p_0) \leq d - n$.

Proposition 2.2.3 guarantees that $\iota(h)$ is an ideal of $p/p_0 \ltimes m$ of codimension $\dim(r/h)$ in $m$. The proposition also gives $\dim(m) = \dim(r)$. Since $p/p_0$ acts faithfully on $m$, we may apply propositions 3.4.1 and 3.4.3 to conclude that $\mu(p/p_0 \ltimes m) \leq \left(\frac{r + \varepsilon_1}{\varepsilon_1}\right) \cdot \left(\frac{r + \varepsilon_2}{\varepsilon_2}\right)$. This finishes the proof.

Note that we obtain an upper bound for $\mu(g)$ that is a polynomial in $d$ of degree $\varepsilon_1 + \varepsilon_2$.

5 Application: representations of graded Lie algebras

A well-known theorem by Jacobson, on weakly closed sets of nilpotent operators, states that a Lie algebra $g$ is nilpotent if it admits a regular derivation, [14]. (For Lie algebras admitting such a transformation, it is known that $\mu(g) = O(\dim(g))$.) The theorem was later generalized and refined in many different ways, see for example [17], [12], and [15].

Theorem 5.0.1 (Khukhro-Makarenko-Shumyatsky – [15]). There exist functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ for which the following is true. Consider Lie algebra $g = \bigoplus_{u \in A} g_u$ graded by a finitely-generated, torsion-free group $(A, +)$. Let $\sigma$ be the cardinality of the support and let $\delta$ be the dimension of the trivial component $g_0$. Then $g$ admits a nilpotent ideal $i$ satisfying $\dim(g/i) \leq f(\sigma)$ and $c(i) \leq g(\sigma, \delta)$.

Proof. (Corollary 1.0.3) The nil-defect of $g$ is bounded from above by $\varepsilon(\text{rad}(g), i) \leq E(\sigma, \delta) := f(\sigma, \delta) + g(\sigma)$. Note that the functions $f$ and $g$ may grow very quickly as $\sigma$ and $\delta$ increase.

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References

[1] I. Ado, *The representation of Lie algebras by matrices*. Amer. Math. Soc. Translation 1949, no. 2, 21 pp.

[2] J. L. Alperin and G. Glauberman: *Limits of abelian subgroups of finite p-groups*. J. Algebra 203 (1998), no. 2, 533-566.

[3] L. Auslander and J. Brezin: *Almost algebraic Lie algebras*. J. Algebra 8 1968 295-313.

[4] Y. Benoist: *Une nilvarié eté non affine*. J. Differential Geom. 41 (1995), no. 1, 215-22.

[5] G. Birkhoff, *Representability of Lie algebras and Lie groups by matrices*. Ann. Math. 38 (1937), 562-532.

[6] D. Burde: *On a refinement of Ado’s theorem*. Arch. Math. (Basel) 70 (1998), no. 2, 118-127.

[7] D. Burde, M. Ceballos: *Abelian ideals of maximal dimension for solvable Lie algebras*. J. Lie Theory 22 (2012), no. 3, 741-756.

[8] D. Burde, B. Eick and W. de Graaf, *Computing faithful representations for nilpotent Lie algebras*. J. Algebra 322 (2009), no. 3, 602-612.

[9] D. Burde and F. Grunewald, *Modules for Lie algebras of maximal class*. J. Pure Appl. Algebra 99 (1995), 239-254.

[10] D. Burde and W. Moens: *Minimal faithful representations of reductive Lie algebras*. Arch. Math. (Basel) 89 (2007), no. 6, 513-523.

[11] D. Burde and W. A. Moens: *Faithful Lie algebra modules and quotients of the universal enveloping algebra*. J. Algebra 325 (2011), 440-460.

[12] D. Burde and W. A. Moens: *Periodic derivations and prederivations of Lie algebras*. J. Algebra 357 (2012), 208-221.

[13] W. A. de Graaf: *Constructing faithful matrix representations of Lie algebras*. Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI), 54-59 (electronic), ACM, New York, (1997).

[14] N. Jacobson: *A note on automorphisms of Lie algebras*. Pacific J. Math. 12 1962 303-315.

[15] E. I. Khukhro, N. Makarenko and P. Shumyatsky: *Nilpotent ideals in graded Lie algebras and almost constant-free derivations*. Comm. Algebra 36 (2008), no. 5, 1869-1882.

[16] B. Kostant: *Eigenvalues of the Laplacian and commutative Lie subalgebras*. Topology 3 1965 suppl. 2, 147-159.

[17] A. I. Kostrikin and M. I. Kuznetsov: *Two remarks on Lie algebras with non-degenerate derivation*. Trudy Mat. Inst. Steklov. 208 (1995), Teor. Chisel, Algebra i Geom., 186-192.

[18] A. I. Malcev: *Solvable Lie algebras*. Amer. Math. Soc. Translation 1950, (1950). no. 27, 36 pp.

[19] J. Milnor: *On fundamental groups of complete affinely flat manifolds*. Advances in Math. 25 (1977), no. 2, 178-187.
[20] Y. Neretin A construction of finite-dimensional faithful representation of Lie algebra. Proceedings of the 22nd Winter School "Geometry and Physics" (Srni, 2002). Rend. Circ. Mat. Palermo (2) Suppl. No. 71 (2003), 159161.

[21] L. Onishchik and E. B. Vinberg: Lie groups and Lie algebras, III. Structure of Lie groups and Lie algebras. A translation of Current problems in mathematics. Fundamental directions. Vol. 41 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR1056485 (91b:22001)]. Translation by V. Minachin [V. V. Minakhin], Translation edited by A. L. Onishchik and E. B. Vinberg. Encyclopaedia of Mathematical Sciences, 41. Springer-Verlag, Berlin, 1994. iv+248 pp. ISBN: 3-540-54683-9

[22] M. Vergne Réductibilité de la variété des algèbres de Lie nilpotentes. (French) C. R. Acad. Sci. Paris Sér. A-B 263 1966 A4A6.