A new method for measuring the CMB temperature quadrupole with an accuracy better than cosmic variance

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We present a new method for measuring the CMB temperature quadrupole, using large scale CMB polarization. The method exploits the fact that CMB polarization is partially sourced by the local temperature quadrupole. We link the temperature with the polarization spectrum directly by relating the local quadrupole at the onset of reionization to both of them. The dominant contribution is at \( l < 30 \) and since we use many \( l \) values, we can reduce the error significantly below cosmic variance. In particular, for our fiducial model, the error on the temperature quadrupole is reduced to 24%. This has the potential of reducing the probability of a low quadrupole by two orders of magnitude.

How precisely can \( C_2^T \), the Cosmic Microwave Background (CMB) temperature quadrupole, be measured? At first sight, the question seems trivial to answer: the measurement is limited by cosmic variance, which at the quadrupole is equal to \( \sqrt{\frac{2}{5} C_2^T} \). The first release of Wilkinson Microwave Anisotropy Probe (WMAP) data \(^1\) has regenerates interest in the value of the CMB quadrupole, as WMAP measured a lower quadrupole than expected, based on a \( \Lambda \)CDM cosmology. This confirmed the Cosmic Background Explorer (COBE) \(^2\) measurement but more cleanly, as the lower detector noise and wider frequency range to pin down galactic foreground emission \(^3\) renders the measurement more robust.

Let us consider the value of the quadrupole \( \Delta T_2^T \) where \( \Delta T_2^T = \frac{1}{3} (l+1) C_l \). Consider also the best fit adiabatic model of \(^4\) as a fiducial model for the rest of the paper, a six parameter model with physical baryon and cold dark matter densities of \( \omega_0 = 0.023 \) and \( \omega_c = 0.117 \) respectively, relative cosmological constant density \( \Omega_\Lambda = 0.715 \), optical depth to reionization \( \tau = 0.137 \) and scalar spectral index \( n = 0.974 \). Using frequentist statistics given the fiducial model and assuming full sky coverage with cosmic variance as the only source of error, the probability of measuring a quadrupole as low as or lower than the quoted WMAP value of \( \Delta T_2^T = 123.4 \) (\( \mu K \))^2 is 0.01. Using a sky cut would actually raise the probability. Efstathiou \(^5\) has argued that statements such as the one above should not be taken too seriously and do not favor the \( \Lambda \)CDM concordance model. If for example, one uses \( \Delta T_2^T = 201.6 \) (\( \mu K \))^2 quoted by Tegmark \textit{et al.} \(^6\) using their different method of foreground subtraction, the probability rises to 0.03. Efstathiou also showed that the value of the quadrupole is sensitive to the estimator used \(^7\). In particular, the use of the quadratic estimator \(^8\) rather than the pseudo-\( C_l \) estimator used by WMAP \(^9\) is better suited for low \( l \) values and could further increase the quadrupole to \( \Delta T_2^T = 250 \) (\( \mu K \))^2. The corresponding probability for the fiducial model would then increase to 0.05, five times larger than the originally quoted value. Other apparent coincidences such as the alignment between the quadrupole and the octopole \(^10\), the north-south asymmetry \(^11\) or possible non-Gaussianity/global anisotropy \(^12\) do not concern us here even though a normal quadrupole would certainly weaken these findings.

The probability arguments discussed above depend strongly on the cosmic variance limit. If there were a way to measure \( C_2^T \) with precision better than cosmic variance, the underlying probability distribution of the quadrupole would be different and the probabilities could change drastically. Is there a different measurement one might make, that could measure the CMB temperature quadrupole and has an error smaller than cosmic variance? This question has been raised and partially answered by Kamionkowski and Loeb \(^13\). They considered the polarization spectrum produced by clusters through the Sunyaev-Zel’dovich effect \(^14\). Since the polarization spectrum depends on the local quadrupole at the cluster, one can get information on the quadrupole \( C_2^T(z) \) at redshift \( z \), by taking large samples of clusters and taking an average. One hopes that the averaging method will recover a quadrupole close to the true cosmological value. The authors mention another way of getting a measure of \( C_2^T \), namely large angle CMB polarization, generated by reionization. They argue however that this would depend on the reionization details and will therefore be very difficult to handle. Further calculations involving clusters have been carried out in \(^15\) and more recently in \(^16\).

The connection between the temperature quadrupole and polarization has been exploited further by Doré \textit{et al.} \(^17\). They considered the consistency of the observed temperature spectrum with the polarization-temperature cross correlation, given the best fit model of WMAP. In particular, if the observed temperature quadrupole was anomalously low, then one would expect low large angle polarization power as well. Their method consists of Monte-Carlo realizations of their fiducial model combined with the well known statistical correlations between the two above spectra \(^18\). Using a frequentist approach, they find that the two are inconsistent at the 98.5% level.
This hints that the low temperature quadrupole might not be just due to an unlucky throw of dice.

In this paper, we exploit the same connection between temperature and polarization as above and demonstrate, using a different method, the possibility of getting a measure on $C^T_2$ from large angle polarization generated by reionization. We show that contrary to the argument in [13], the method does not depend on the details of reionization but rather on the initial amplitude of the local quadrupole at the onset of reionization. Any reionization history dependence can be accommodated by using extra parameters but this does not appear to be an obstacle to our method. That the details of reionization can be observed in the CMB has been studied before [21]. Even in the worst case scenario, the information about reionization contained in the CMB boils down to at most five parameters [21] but this is something that would have to be taken into account for standard CMB parameter extraction as well. Therefore in what follows, we use a sharp reionization transition at (conformal) time $t_r$, the time where the electron ionization fraction first rises above its residual recombination value.

We first review how the relevant spectra are related. Let $\Delta^T_2(k,t)$ and $\Delta^P_2(k,t)$ denote the l-multipole of the temperature and polarization transfer functions respectively. Let also t be conformal time with $t_0$ being the time today, $k$ the wavenumber and $\tau(t) = \int^{t_0} dt \phi(t)$ the optical depth to time t where a dot indicates differentiation with respect to conformal time.

The temperature quadrupole is then given in terms of an initial power spectrum $P_\Phi(k)$ and transfer function $\Delta^T_2(k)$ as

$$C^T_2 = \frac{2}{\pi} \int_0^\infty dk \, k^2 \, P_\Phi(k) \, |\Delta^T_2(k)|^2 . \tag{1}$$

The quadrupole's transfer function obeys the differential equation

$$\dot{\Delta}^T_2 + \frac{9}{10} \Delta^T_2 = \frac{k}{5} (2\Delta^T_2 - 3\Delta^T_3) + \frac{1}{10} (\Delta^P_0 + \Delta^P_2) . \tag{2}$$

The general solution is given in terms of an initial condition (the initial quadrupole) multiplied by the relevant Green’s function and a particular integral $\Delta^{Pf}_2(k)$ over the source $S_2$ which is given by the RHS of (3). For reasons to become clearer below, we choose the initial time to be the reionization time $t_r$, so that the initial condition is the local quadrupole at the onset of reionization. The local quadrupole today is then given by

$$\Delta^T_2(k,t_0) = e^{\frac{3}{5}\tau_r} \Delta^T_2(k,t_r) + \int_{t_r}^{t_0} dt \, e^{\frac{3}{5}\tau(t)} \, S_2(k,t). \tag{3}$$

Similarly the E-mode polarization spectrum is given in terms of the initial power spectrum and E-mode transfer function $E_1(k)$ as

$$C^E_l = \frac{2}{\pi} \int_0^\infty dk \, k^2 \, P_\Phi(k) \, |E_1(k)|^2 . \tag{4}$$

The E-mode transfer function is given by the line-of-sight integral along the past light cone as

$$E_1(k) = \frac{3}{4} \sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^{t_0} \frac{dt}{x^2} \, \dot{\tau} e^{\tau} \Pi(k,t) , \tag{5}$$

where $x \equiv k(t_0 - t)$, $\Pi(k,t)$ is the polarization source and $j_l(x)$ is the spherical Bessel function. The polarization source is given in terms of temperature quadrupole and polarization monopole and quadrupole as

$$\Pi(k,t) = \Delta^T_2(k,t) + \Delta^P_0(k,t) + \Delta^P_2(k,t) \tag{6}$$

and obeys the differential equation

$$\dot{\Pi} + \frac{3}{10} \dot{\tau} \Pi = \frac{k}{5} (2\Delta^T_2 - 3 (\Delta^T_2 + \Delta^P_0 + \Delta^P_2) . \tag{7}$$

The general solution of the above equation for any time $t > t_r$ is

$$\Pi(k,t) = e^{\frac{3}{5}\tau(t)} \Delta^T_2(k,t_r) + e^{-\frac{3}{5}\tau(t)} \int_{t_r}^t dt' \, e^{\frac{3}{5}\tau(t')} \, S_{\Pi}(k,t') . \tag{8}$$

where again we have chosen the initial time to be $t_r$ and where $S_{\Pi}$ is given by the RHS of (4). Note that in the above equation we have replaced $\Pi(k,t_r)$ which should have been the true initial condition with $\Delta^T_2(k,t_r)$. It turns out that due to free-streaming from recombination to reionization, the two are equal to one part in $10^6$. During reionization, however, this is no longer true as the three terms comprising $\Pi(k,t)$ become comparable because of rescattering. We are therefore forced to put the initial condition at the onset of reionization. As one can see, the temperature quadrupole produced by reionization is fully connected with the polarization spectrum through the initial condition $\Delta^T_2(k,t_r)$. This forms the basis of our method: given a model and the polarization
autocorrelation spectrum $C_T^E$ one can get information about the local quadrupole at reionization from which the quadrupole today can be inferred.

One conceptual difficulty with the method is the following. Since the quadrupole today $C_T^E(t_0)$ probes scales larger than the quadrupole at reionization $C_T^E(t_r)$ it seems at first that there should not be any way to make the method work as, strictly speaking, we are measuring $C_T^E(t_r)$ which is definitely not equal to $C_T^E(t_0)$. Examining the issue more carefully however, one sees that there is significant overlap between the scales spanned by $\Delta_T^E(k, t_0)$ and $\Delta_T^E(k, t_r)$ as shown in Fig. 1. Since the $l = 2$ moment is mapped into $k$-space by $j_2(x)$, we expect to get a wide range of scales contributing to $C_T^E(t_0)$ as $j_2(x)$ is broadly distributed. Moreover $\Delta_T^E(k, t_r)$ is also convolved with $G_1(k) = \frac{2}{3} \sqrt{\frac{\Omega}{\Omega - 2}} \tilde{\Psi}^{+ \tau} \int_{t_0}^{\infty} dt \tilde{\dot{r}}(x) \tilde{\xi}^{+ \tau}$ as implied by combining (5) and (8), which further increases the overlap. The first five $G_1$’s as well as their multiplication with $\Delta_T^E(k, t_r)$ are also shown in Fig. 1. Therefore even though what we really measure is not $C_T^E$ but rather the contribution to $C_T^E$ coming from the quadrupole at reionization, this is sufficient to reduce the error on $C_T^E$ significantly below the cosmic variance limit.

One may also wonder why we could not choose some very early time prior to recombination to set the initial condition, as during tight coupling we have $\Pi(k, t) = \frac{2}{3} \Delta_T^E(k, t)$ and so we can also relate the quadrupole with in fact the whole of the polarization spectrum, not just the part produced by reionization. The problem however in this case is that the overlap between the local quadrupole at the early time and $\Delta_T^E(k, t_0)$ would be minuscule. The quadrupole today will be dominated by the particular integral instead, and its error would be effectively cosmic variance again.

To get an estimate for the error on $C_T^E$ we let the amplitude of $\Delta_T^E(k, t_r)$ vary as a free parameter. This can be modeled by multiplying it with a parameter $q$ by hand, i.e. $\Delta_T^E(k, t_r) \rightarrow q \Delta_T^E(k, t_r)$ in the two relevant equations $5$ and $8$. The variance $V_r$ of $\Delta_T^E(k, t_r)$ can then be estimated as $V_r = V_q C_T^E$ where $V_q = \text{Var}[q]$ can be obtained using the Fisher information matrix and where $C_T^E = \frac{1}{5} \sum_m |\hat{\sigma}_m|^2 = \frac{2}{5} \int_0^{\infty} dk k^2 P_0 |\Delta_T^E(k, t_r)|^2$. Assuming that the only error in the polarization spectrum comes from cosmic variance, the Fisher matrix is a scalar given by

$$F = \sum_l (l + 1) \frac{1}{2} \frac{\partial}{\partial q} \ln C_T^E |^2.$$  

The derivative in the expression above can be calculated numerically by double-sided finite difference which we have taken to be $\delta q = 0.02$ around the fiducial value of $q = 1$. This is shown in Fig. 2. The transfer functions were calculated using a modified version of DASH [24]. For the model considered we get a variance of $q$ of $V_q = \frac{1}{5} = 0.002$.

Since we have used the Fisher matrix to get our estimate, we can assume that the posterior pdf of $\hat{\sigma}_{2m}$ is Gaussian, with variance $V_r$. Therefore the posterior pdf of the total $\hat{\sigma}_{2m} = e^{+ \tau} \hat{\sigma}_{2m} + \hat{\sigma}_{2m}^{\prime}$ will be Gaussian with variance $V_2 = e^{+ \tau} V_r + C_N^{\prime} + 2 e^{+ \tau} C_N$, where $C_N^{\prime} = \frac{1}{5} \sum_m |\hat{\sigma}_{2m}^{\prime}|^2 = \frac{2}{5} \int dk k^2 P_0 |\Delta_T^E(k)|^2$ is the variance of the particular integral and $C_N = \frac{1}{5} \sum_m \hat{\sigma}_{2m}^{\prime} \hat{\sigma}_{2m}^{\prime} + c.c.$ = $\frac{2}{5} \int dk k^2 P_0 \Delta_T^E(k, t_r) \Delta_T^E(k, t_r)$ the variance of the correlation. Given that we have no extra information about the particular integral we can assume that its variance will not change from the cosmic variance value. The variance of $C_T^E = \frac{1}{5} \sum_m |\hat{\sigma}_{2m}|^2$ can then be estimated as $\text{Var} [C_T^E] = \frac{2}{5} (V_2)^2$, since under the above assumptions, $C_T^E$ will obey a $\chi^2$ distribution with five degrees of freedom. Our fiducial model gives $\text{Var} [C_T^E] / (C_T^E)^2 = 0.06$, which gives an error on $C_T^E$ of 24%.

One conceptual objection with the above argument is that in the usual case, one has $\langle \Delta_T^E \rangle = C_T^E$ where as above it is equal to $V_2$ instead. This is not a problem as the true posterior pdf of $C_T^E$ would no longer be $\chi^2$ and therefore its mean and variance would not be related in the usual way.

As argued before, the key point is the overlap of the $G_1(k) \Delta_T^E(k, t_r)$ with $\Delta_T^E(k, t_0)$. This signifies that the lower the reionization redshift, the better the reduction of cosmic variance. Since however for low reionization redshifts, we obtain a smaller signal in the polarization spectrum, we would also get a larger variance for the parameter $q$. We should therefore expect that for low reionization redshifts, the error on $C_T^E$ would not be improved but even be greater that cosmic variance. This means that as one varies the reionization redshift $z_r$ from zero to some large value, the error on $C_T^E$ would become better and better until some $\text{conspiratory}$ value, and then start to become worse and worse until it reaches cosmic variance again. Another way to see this is the following. The correlation $\langle \hat{\sigma}_{2m} \hat{\sigma}_{2m}^{\prime} \rangle$ becomes arbitrarily small with increasing $z_r$, as is implied from Fig. 1 which means

![FIG. 2: The derivative of $\ln C_T^E$ with respect to $q$, the amplitude of the local quadrupole at the onset of reionization.](image-url)
FIG. 3: Top panel: The variation of the quadrupole error \( \frac{\sigma_{q}^{2}}{C_{q}^{2}} \) (solid) and \( \sigma_{q} \) (dash) with reionization redshift. The error on the quadrupole reaches a minimum as expected, around \( z = 3 \). The cosmic variance limit is shown as a gray line.

Bottom panel: Variation of the ratios \( \frac{C_{q}^{Y}}{C_{q}^{T}} \) (solid), \( \frac{C_{q}^{P}}{C_{q}^{T}} \) (dash) and \( \frac{C_{q}^{X}}{C_{q}^{T}} \) (dotted) with reionization redshift.

that we are probing many independent Hubble volumes at that redshift hence the small variance of \( q \). On the other hand (\( \frac{C_{q}^{Q}}{\langle C_{q}^{2} \rangle} \approx \frac{C_{q}^{T}}{C_{q}^{T}} \) is also decreasing (but slowly enough) which means that the propagation of our information on \( C_{q}^{T} \) to \( C_{q}^{T} \) becomes less effective at higher \( z_{r} \). This is shown in Fig. 3.

The role of other cosmological parameters is also very important due to imminent degeneracies, particularly with the optical depth. Still the quoted error above is a lower bound and other parameters can be included at a later stage along with predictions for future polarization experiments.

Finally let us see how a different variance on \( C_{q}^{T} \) could affect the probabilities mentioned in the beginning. Strictly speaking, we need the true posterior pdf of \( C_{q}^{T} \) but based on our assumptions we can assume that it would be approximately Gaussian with mean given by the model’s value and variance the value quoted above. Moreover, the maximum entropy principle, would also give a Gaussian distribution if the only knowledge about the distribution is the mean and the variance. If the fiducial model was the best fit model of a cosmic variance limited polarization experiment then the probability that the quadrupole is as low as or lower than \( 250(\mu K)^{2} \), is reduced to \( 2 \times 10^{-4} \). This should be taken only as an order of magnitude estimate of the actual probability which would have been given by the true pdf. For comparison, using a Gaussian distribution with variance given by cosmic variance instead, we get a probability which is only 8% different than the one quoted in the beginning.

We have shown that it is possible to reduce the error on the CMB temperature quadrupole, to a value better than cosmic variance. The method exploits the connection between the temperature quadrupole and the polarization spectrum generated by a period of reionization. This could reduce the variance of the quadrupole significantly and has the potential to answer with confidence whether the quadrupole is really low or not, compared to a given model. The method is still at its infancy and further treatment is needed before it can be incorporated with parameter estimation techniques. It would be an excellent way to test new physics.

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