Fast forward approach to stochastic heat engine

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The fast-forward (FF) scheme proposed by Masuda and Nakamura (Proc. R. Soc. A 466, 1135 (2010)) in the context of conservative quantum dynamics can reproduce a quasi-static dynamics in an arbitrarily short time. We apply the FF scheme to the classical stochastic Carnot-like heat engine which is driven by a Brownian particle coupled with a time-dependent harmonic potential and working between the high ($T_h$)- and low ($T_c$)-temperature heat reservoirs. Concentrating on the underdamped case where momentum degree of freedom is included, we find the explicit expressions for the FF protocols necessary to accelerate both the isothermal and thermally-adiabatic processes, and obtain the reversible and irreversible works. The irreversible work is shown to consist of two terms with one proportional to and the other inversely proportional to the friction coefficient. The optimal value of efficiency $\eta$ at the maximum power of this engine is found to be universal and given by $\eta^* = \frac{1}{2} \left( 1 - \frac{1}{2} \left( \frac{T_c}{T_h} \right)^{1/2} + O \left( \frac{T_c}{T_h} \right)^{1/2} \right)$ and $\eta^* = 1 - \left( \frac{T_c}{T_h} \right)^{1/2}$, respectively in the cases of strong and weak dissipation.

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I. INTRODUCTION

Carnot’s concept of heat engines belongs to a classical subject of thermodynamics. To achieve the highest efficiency, a heat engine needs to operate a reversible thermodynamic cycle which requires a quasi-static process and results in a vanishing power. The power means the work per one-cycle time. The quasi-static thermodynamic cycle should be speeded up so as to produce a finite power of realistic heat engines. It is desirable to investigate how large the efficiency of a heat engine can be reached when the engine operates in the region of the maximum power. This issue has led to the birth of finite-time thermodynamics which has attracted much attention for many years. The most notable result in finite-time thermodynamics is the Curzon-Ahlborn (CA)’s efficiency, $\eta_{CA} = 1 - \sqrt{T_c/T_h}$, which is the efficiency at maximum power for a macroscopic endo-reversible heat engine [1] operating between a cold bath at temperature $T_c$ and hot bath at temperature $T_h$ ($T_c > T_h$). CA noted the finite temperature difference between the heat bath and the working substance and took into consideration the finite time needed for the heat transfer between them.

In contrast to the macroscopic heat engines considered in endo-reversible thermodynamics, thermal fluctuations play a crucial role in nano-scale systems, where dynamics cannot be described on a deterministic (macroscopic) level. Sekimoto’s stochastic energetics [2–4] is a key to thermodynamic description of Langevin systems driven far from equilibrium, which can define thermodynamic quantities on a single stochastic trajectory [3, 5, 6] and yield the ensemble quantities after averaging.

In the context of nano-scale motors, Brownian heat engines have received a wide attention, which mimic a simple system of a stochastic heat engine whose degrees of freedom are subject to a time-dependent potential and working between hot and cold heat baths. The efficiency of the engines of this kind at maximum power was investigated by [7–11], which assumed the time dependence of the effective temperature (e.g., variance of the particle position) during the isothermal process. More recent works [12] and [13] proposed the engineering swift equilibration and the shortcut to isothermality, respectively, which kept the effective temperature during the isothermal process, but provided neither kinetics corresponding to the thermally-adiabatic process nor investigation on the power and efficiency of the heat engine.

On the other hand, independently from the research activities in Brownian heat engines, Masuda and Nakamura [14–16] proposed a way to accelerate quantum dynamics with use of a characteristic driving potential determined by the underlying adiabatic wave function.
This kind of acceleration is called the fast forward, which means to reproduce a series of events or a history of matters on a shortened time scale, like a rapid projection of movie films on the screen. The fast forward theory constitutes one of the promising ways of shortcuts to adiabaticity (STA) devoted to tailor excitations in nonadiabatic processes\textsuperscript{[17–22]}. This theory revealed the non-equilibrium equation of states for the quantum gas under a rapid piston \textsuperscript{[23]} and provided a simple protocol to accelerate the adiabatic quantum dynamics of spin clusters\textsuperscript{[24]}. It is fascinating to investigate the fast forward of the heat engine which is classical and stochastic, find the fast-forward protocols, and investigate the power and efficiency of the engine.

In this paper we shall develop the fast-forward theory of the stochastic Carnot-like heat engine driven by a Brownian particle coupled with a time-dependent harmonic potential and working between the high ($T_h$)- and low ($T_c$)-temperature heat reservoirs. The momentum degree of freedom is taken into consideration throughout the paper, since energetic interaction between the particle and heat reservoir is also carried by the momentum exchange between them. In Section \textbf{III} we are concerned with the isothermal process, apply the fast forward theory to Fokker-Planck-Kramers or simply the Kramers equation, obtain the fast-forward protocols, and calculate both the reversible and irreversible works. In Section \textbf{IV} we treat the thermally-adiabatic process where there is no averaged heat transfer between the system and heat reservoir, find a fast-forward protocol which shows a crucial role of momentum degree of freedom, and obtain the reversible work. In Section \textbf{V} the efficiency at maximum power is calculated and compared with existing references. Section \textbf{VI} is devoted to summary and discussions. Appendices \textbf{A} and \textbf{B} are devoted to some theorems associated with the irreversible works during the fast-forward protocols.

\textbf{II. FAST-FORWARD OF ISOTHERMAL PROCESS}

\subsection{A. Derivation of driving potential}

We shall develop the probabilistic theory of the stochastic heat engine using a Brownian particle confined by the harmonic potential which has a time-dependent stiffness coefficient.

In this Section we develop the fast forward theory for the isothermal process in the Carnot-like cycle. Here the Brownian particle is in touch with a reservoir at temperature $k_B T (= \frac{1}{T})$ and working under the expanding or compressing trapping potential. In the stochastic energetics\textsuperscript{[21]} on which the present article is based, the inertial effect or momentum degree of freedom plays an essential role. So we shall investigate the underdamped region of a Brownian particle, where the Kramers equation for its distribution function $\rho_0 (x, p, t)$ is derived through the continuity equation\textsuperscript{[25]},

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_p}{\partial p} = 0$$ \hspace{1cm} (2.1)

with the vector flux $(J_x, J_p)$ given by

$$J_x = \rho_0 \frac{\partial H_0}{\partial p},$$

$$J_p = -\rho_0 \left( \frac{\partial H_0}{\partial x} + \gamma \frac{\partial H_0}{\partial p} + \frac{\gamma}{\beta \rho_0} \frac{\partial \rho_0}{\partial p} \right).$$ \hspace{1cm} (2.2)

where $H_0 = \frac{x^2}{2} + \frac{1}{2} \lambda x^2$ is Hamiltonian for a particle with unit mass trapped by the harmonic potential with stiffness coefficient $\lambda$. $\gamma$ stands for the friction coefficient responsible to dissipation. The last term of $J_p$ in Eq. (2.2) is traced back to the Gaussian white noise in the underlying Langevin equation. Using Eq. (2.2), Eq. (2.1) can be rewritten as

$$\frac{\partial \rho_0}{\partial t} = \{H_0, \rho_0\}$$

$$+ \gamma \partial_p (\rho_0 p + \frac{1}{\beta} \partial_p \rho_0),$$ \hspace{1cm} (2.3)

where $\{\cdots, \cdots\}$ means the Poisson bracket.

If $\lambda = \text{const.}$, we have the equilibrium Gaussian distribution function $\rho_0^e$ at $t \to \infty$. Assuming $\partial_t \rho_0 = 0$ in Eq. (2.3), we see:

$$\rho_0^e = \frac{\beta \sqrt{\lambda}}{2 \pi} \exp (-\beta H_0 (\lambda)), \hspace{1cm} (2.4)$$

which fulfills the normalization, $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_0^e (x, p) dx dp = 1$.

If $\lambda$ will be time dependent, the solution in Eq. (2.4) becomes meaningless. But the idea of fast forward can guarantee the form in Eq. (2.4), even when $\lambda$ is time dependent.

The first half of the fast forward scheme is the regularization procedure. Let $\lambda$ vary in time very slowly, namely in a quasi-static way:

$$\lambda(t) \equiv \lambda_0 + \epsilon t$$ \hspace{1cm} (2.5)
with the growth rate $|\epsilon| \ll 1$, which means that it requires a very long time $T = O(\frac{1}{|\epsilon|})$, to see a recognizable change of $\lambda(t)$.

We take the regularized distribution function $\rho_0^{\text{reg}}(x,p;\lambda(t))$ which has the same functional form as $\rho_0^{\text{eq}}$ in Eq. (2.4):

$$\rho_0^{\text{reg}} = \exp \left[ -\beta H_0(\lambda(t)) + \Gamma(\lambda(t)) \right] \quad \text{(2.6)}$$

where

$$H_0(\lambda(t)) = \frac{p^2}{2} + \frac{\lambda(t)}{2} x^2,$$

$$\exp(-\Gamma(\lambda(t))) = \frac{\beta \sqrt{\lambda(t)}}{2\pi}. \quad \text{(2.7)}$$

Then, adding a potential $\epsilon h$ to $H_0$ in Eq. (2.2), we regularize the Kramers equation as

$$\frac{\partial \rho_0^{\text{reg}}}{\partial t} = \{ H_0 + \epsilon h, \rho_0^{\text{reg}} \} + \gamma \partial_p (\rho_0^{\text{reg}} \partial_p) + \epsilon \gamma \partial_p (\partial_t \rho_0^{\text{reg}} \partial_p). \quad \text{(2.8)}$$

$h = h(x,p;\lambda)$ will be determined so that $\rho_0^{\text{reg}}$ in Eq. (2.6) should satisfy Eq. (2.8).

Noting

$$\partial_t \rho_0^{\text{reg}} = \frac{\partial \rho_0^{\text{reg}}}{\partial \lambda} \frac{d\lambda}{dt} = \epsilon \left[ \frac{\beta}{2} x^2 + \frac{1}{2\lambda} \right] \rho_0^{\text{reg}}, \quad \text{(2.9)}$$

let’s compare both sides of Eq. (2.8) in each order of $\epsilon$. Firstly we obtain the equality of $O(1)$:

$$\{ H_0, \rho_0^{\text{reg}} \} + \gamma \partial_p (\rho_0^{\text{reg}} \partial_p + \frac{1}{\beta} \partial_p \rho_0^{\text{reg}}) = 0. \quad \text{(2.10)}$$

It is evident that Eq. (2.10) is satisfied. By using the expression for $\rho_0^{\text{reg}}$ in Eq. (2.6), each of the first and second terms on the left-hand side of Eq. (2.10) can be shown to vanish.

Then the equality of $O(\epsilon)$ from Eq. (2.8) is

$$\left[ \frac{\beta}{2} x^2 + \frac{1}{2\lambda} \right] \rho_0^{\text{reg}} = \{ h, \rho_0^{\text{reg}} \} + \gamma \partial_p (\rho_0^{\text{reg}} \partial_p), \quad \text{(2.11)}$$

which will determine the function $h$. Noting that $\partial_p \rho_0^{\text{reg}} = -\beta \rho_0^{\text{reg}}$ and $\partial_x \rho_0^{\text{reg}} = -\beta \lambda x \rho_0^{\text{reg}}$, Eq. (2.11) can be rewritten as

$$-\frac{\beta}{2} x^2 + \frac{1}{2\lambda} = +\beta [\lambda x \partial_x h - p \partial_p h] - \gamma \partial_p h + \gamma \partial_p^2 h. \quad \text{(2.12)}$$

Equation (2.12) for $h$ can be solved by assuming

$$h = ap^2 + bpx + cx^2. \quad \text{(2.13)}$$

In fact, using Eq. (2.13) in Eq. (2.12) and equating the constant term and each coefficient of $p^2$, $x^2$ and $px$ to be zero, we have 4 linear algebraic equations (with rank 3):

$$b + 2\gamma a = 0, \quad \lambda b = -\frac{1}{2}, \quad 2\lambda a - 2c - \beta b = 0, \quad \frac{1}{2\lambda} - 2\gamma a = 0. \quad \text{(2.14)}$$

The solution of Eq. (2.14) is $a = \frac{1}{4\lambda}$, $b = -\frac{1}{4\lambda}$, and $c = \frac{1}{4}(\frac{1}{\gamma} + \frac{1}{\lambda})$. Hence Eq. (2.13) reduces to

$$h = \frac{1}{4\gamma \lambda} p^2 - \frac{1}{2\lambda} px + \left( \frac{1}{4\gamma} + \frac{\gamma}{4\lambda} \right) x^2. \quad \text{(2.15)}$$

Next we shall enter the second half of the fast forward scheme. $\rho_0^{\text{reg}}$ in Eq. (2.6) and the regularized Kramers equation in Eq. (2.8) are meaningful for long time, but only on slow time scale. They can be made effective also on a rapid time scale, however, by introducing the fast-forward time scale $T$, i.e.,

$$\Lambda(t) = \int_0^t \alpha(t')dt'. \quad \text{(2.16)}$$

with $\alpha(t) \gg 1$. The simplest expression for $\alpha(t)$ in the fast-forward range ($0 \leq t \leq T_{FF}$) is given by:

$$\alpha(t) = \tilde{\alpha} - (\tilde{\alpha} - 1) \cos \left( \frac{2\pi}{T_{FF}} t \right), \quad \text{(2.17)}$$

where $\tilde{\alpha}$ is the mean value of $\alpha(t)$ and is given by $\tilde{\alpha} = T/T_{FF}$. Here $T = O(\frac{1}{|\epsilon|})$ is a long time interval of the quasi-static isothermal process and $T_{FF}$ is an arbitrarily short time to reproduce this process. After the above time scaling, the stiffness coefficient $\lambda$ now varies in time rapidly as

$$\lambda(\Lambda(t)) = \lambda_0 + c\Lambda(t). \quad \text{(2.18)}$$

Let’s define the fast-forwarded distribution $\rho_{FF}$ as

$$\rho_{FF}(x,p,t) = \rho_0^{\text{reg}}(x,p;\lambda(\Lambda(t))) = \exp \left[ -\beta H_0(\lambda(\Lambda(t))) - \Gamma(\lambda(\Lambda(t))) \right]. \quad \text{(2.19)}$$
Then $\rho_{FF}$ satisfies the same Kramers equation as Eq. (2.8) with $\epsilon$ prior to $h$ and $\lambda(t)$ included in $H_0, h$ being replaced by $\epsilon\alpha$ and $\lambda(\Lambda(t))$, respectively. In fact, the time derivative of $\rho_{FF}$ becomes

$$
\frac{\partial \rho_{FF}}{\partial t} = \frac{\partial \lambda}{\partial t} \frac{\partial \rho_{FF}}{\partial \lambda} = \epsilon \alpha \frac{\partial \rho_{FF}}{\partial \lambda}
= \epsilon \alpha \{ h, \rho_{0}^{\text{reg}} \} + \gamma \epsilon \alpha \frac{\partial h}{\partial \rho_{0}^{\text{reg}}},
$$

where the 3rd equality comes from the fast-forward variant of Eq. (2.11). The remaining terms on the right-hand side of Eq. (2.8) proves vanishing, which is the fast-forward version ($t \to \Lambda(t)$) of Eq. (2.10).

Taking the asymptotic limit $\lim_{\epsilon \to 0, \tilde{\alpha} \to \infty} \epsilon \tilde{\alpha} = \tilde{v}$ with $\tilde{v} > 0$ ($\tilde{v} < 0$) for $\epsilon \to +0$ ($\epsilon \to -0$), we obtain the Kramers equation working for the rapid-time scale region:

$$
\frac{\partial \rho_{FF}}{\partial t} = \{ H_0 + v(t)h, \rho_{FF} \}
+ \gamma \frac{\partial \rho_{FF}}{\partial \rho} \frac{\partial (v(t)h)}{\partial \rho},
$$

(2.21)

Here $v(t)$ is a velocity function available from $\alpha(t)$ in the asymptotic limit [15]:

$$
v(t) = \lim_{\epsilon \to 0, \tilde{\alpha} \to \infty} \epsilon \alpha(t) = \tilde{v} \left( 1 - \cos \left( \frac{2\pi}{T_{FF}} t \right) \right).
$$

(2.22)

Consequently, for $0 \leq t \leq T_{FF}$,

$$
\lambda(\Lambda(t)) = \lambda_0 + \lim_{\epsilon \to 0, \tilde{\alpha} \to \infty} \epsilon \Lambda(t) = \lambda_0 + \int_0^t v(t') dt'
= \lambda_0 + \tilde{v} \left[ t - \frac{T_{FF}}{2\pi} \sin \left( \frac{2\pi}{T_{FF}} t \right) \right].
$$

(2.23)

From now on we take the following prescription:

$$
\lambda(t) \equiv \lambda(\Lambda(t)),
\dot{\lambda} \equiv \frac{d\lambda(\Lambda(t))}{dt} = v(t).
$$

(2.24)

Then we see the values of $\lambda$ at the initial and final stages of the FF dynamics:

$$
\lambda(0) = \lambda_0, \quad \lambda(T_{FF}) = \lambda_0 + \tilde{v} T_{FF}
$$

(2.25)

and

$$
\dot{\lambda}(0) = \dot{\lambda}(T_{FF}) = 0.
$$

(2.26)

Figure 1 shows schematic curves of $\lambda(t)$ in each of fast-forwarded isothermal and adiabatic processes. The Curve from $\lambda_0(\Lambda_2)$ to $\lambda_1(\Lambda_3)$ with time interval $T_{FF} = t_1$ ($T_{FF} = t_3$) corresponds to isothermal expansion (compression) in contact with the high- (low-) temperature reservoir.

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**FIG. 1**: Schematic curves for stiffness coefficient ($\lambda$) as a function of time ($t$) in the Carnot-like cycle. $t_1, t_2, t_3$ and $t_4$ are the fast-forward time ($T_{FF}$) in each of sub-processes.

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**B. Work and heat**

Now, with use of $\rho_{FF}$ in Eq. (2.19) we shall evaluate the thermodynamic quantities, i.e., work $W$, heat $Q$, and internal energy $E$. The fast forwarded Hamiltonian for the Brownian particle is given by:

$$
H_{FF}(x, p, t) = H_0 + \lambda h,
$$

(2.27)

where $H_0$ and $h$ are defined respectively in Eqs. (2.27) and (2.15), with $\lambda$ being replaced by its FF version in Eq. (2.22). $H_{FF}(x, p, t)$ is varied during the time interval $0 \leq t \leq T_{FF}$ at a constant temperature $k_B T = \frac{1}{\beta}$.

The mean work $W$ done from outside is

$$
W = \int_0^{T_{FF}} dt \langle \frac{\partial H_{FF}}{\partial t} \rangle.
$$

(2.28)

Noting

$$
\langle \frac{\partial H_{FF}}{\partial t} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial H_{FF}}{\partial t} \rho_{FF} dx dp
= \left( \frac{\lambda}{2} - \gamma \frac{\lambda^2}{4\lambda^2} + \frac{1}{\beta} \right) \frac{1}{\beta} 
+ \left( \frac{\lambda}{4\gamma \lambda} \right) \frac{1}{\beta},
$$

(2.29)

the work proves to be a sum of the reversible ($W_{\text{rev}}$) and irreversible ($W_{\text{irr}}$) parts as:

$$
W = W_{\text{rev}} + W_{\text{irr}}
$$

(2.30)
where

\[ W_{rev} = \frac{1}{2\beta} \ln \left| T_{FF} \right|_0 = \frac{1}{2\beta} \ln \frac{\lambda(T_{FF})}{\lambda_0} \] (2.31)

and

\[ W_{irr} = \left( \frac{\gamma}{8} \int_0^{T_{FF}} \lambda \lambda^2 dt + \frac{1}{4\gamma} \int_0^{T_{FF}} \lambda \lambda dt \right) \frac{1}{\beta}. \] (2.32)

In obtaining the compact expression Eq.(2.32) from Eqs.(2.28) and (2.29), we used the equality

\[ \int_0^{T_{FF}} \lambda^m dt = \left\{ \begin{array}{ll}
\frac{1}{m+1} T_{FF}^{m+1} & \text{with } m > 1, \\
0 & \text{with } m = 1.
\end{array} \right. \] (2.33)

with \( m > 1 \), which can be verified with use of the boundary characteristics in Eq.(2.29).

The irreversible work \( W_{irr} \) in Eq.(2.32) consists of the integral of the type, \( \int_0^{T_{FF}} \lambda^m dt \), which can be expressed in terms of the initial value \( \lambda(0) \) and the relative growth rate of \( \lambda \) defined by

\[ \xi = \frac{\lambda(T_{FF}) - \lambda(0)}{\lambda(0)}, \] (2.34)

during the fast-forwarding time from \( t = 0 \) through \( t = T_{FF} \). Using the definition of \( \lambda \) in Eqs.(2.23) and (2.24) and making a variable change from \( t \) to \( s = \frac{t}{T_{FF}} \), we can rewrite \( \int_0^{T_{FF}} \lambda^{m-1} dt \) as

\[ \int_0^{T_{FF}} \frac{\lambda}{\lambda^0} dt = \frac{2\pi n}{T_{FF}} \lambda^0 \int_0^{1} \frac{\sin(2\pi s)}{\lambda_0 + \bar{v}T_{FF} \left( s - \frac{1}{2\pi} \sin(2\pi s) \right)} ds, \] (2.35)

where

\[ Z_n(\xi) \equiv \int_0^{1} \frac{\sin(2\pi s)}{\left[ 1 + \xi \left( s - \frac{1}{2\pi} \sin(2\pi s) \right) \right]^n} ds. \] (2.36)

The expression for \( W_{irr} \) is thus given by:

\[ W_{irr} = \frac{\pi k_B T}{4\lambda(0) T_{FF}} Z_2(\xi) + \frac{\pi k_B T}{2\gamma T_{FF}} Z_1(\xi). \] (2.37)

As shown in Fig. 2 and in Appendix A, \( Z_1(\xi), Z_2(\xi) \) and thereby \( W_{irr} \) are always nonnegative. The irreversible work in Eq.(2.37) is inversely proportional to \( T_{FF} \), which is consistent with the results of the engineering swift equilibration (overdamped case) [12] and of the shortcut to isothermality (underdamped case) [13]. Our new discovery here is that the irreversible work consists of the term proportional to the friction coefficient \( \gamma \) and one inversely proportional to \( \gamma \). Note: \( \frac{1}{\lambda(0)} \) and \( \frac{1}{\gamma} \) has the same dimension under the prescription of unit mass.

Alternative derivation of Eq.(2.37) is given in Appendix B.

In a similar way of calculating the mean work, we obtain the internal energy \( E(t) \)

\[ E(t) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_F F \rho_F F dxdp = \left( 1 + \bar{v} \left( \frac{1}{2\gamma} + \frac{\gamma}{4\lambda^2} \right) \right) \frac{1}{\beta}. \] (2.38)

from which we see the increment \( \Delta E \) during the fast forward of the isothermal process is given by \( \Delta E = E(T_{FF}) - E(0) = 0 \) because of the boundary values of \( \bar{v} \) in Eq.(2.26).

In the isothermal process, the first law of thermodynamics

\[ \Delta E = Q + W \] (2.39)

together with \( \Delta E = 0 \) determines the heat \( Q \) taken from the heat bath at temperature \( T \) as

\[ Q = -W = -(W_{rev} + W_{irr}). \] (2.40)

III. FAST FORWARD OF THERMALLY-ADIABATIC PROCESS

In this Section, we shall embark upon the fast forward of the thermally-adiabatic process, where the particle is
isolated from a reservoir and working under the expanding or compressing trapping potential. In the case of the stochastic microscopic heat engine, however, the unambiguous treatment of the thermally-adiabatic process is controversial and has not yet been settled [4-6]. In the thermally-adiabatic process, therefore, we shall choose a strategy of the system and its surrounding [2-4]. The contribution of the thermally-adiabatic process is stochastic microscopic heat engine, however, the unambiguous treatment of the thermally-adiabatic process is controversial and has not yet been settled [4, 9, 26, 27].

According to the stochastic energetics, both frictions and random force contribute to the heat transfer between the system and its surroundings [2-4]. In the thermally-adiabatic process, therefore, we shall choose a strategy of using the same Kramers equation as in Eq.(2.3) with the inverse temperature \((\beta)\) changing smoothly in a way that guarantees the vanishing heat transfer between a system and the reservoir.

**A. Derivation of driving potential**

We shall again apply the fast-forward scheme which consists of the regularization of Kramers equation and the fast forward time-rescaling. The regularized Kramers equation takes the same form as Eq.(2.8), but here the inverse temperature is time dependent through the time-dependent stiffness coefficient \(\lambda(t)\), i.e., \(\beta = \beta(\lambda(t))\). The regularized distribution function is defined by

\[
\rho_0^{reg} = \exp \left[ -\beta(\lambda(t))H_0(\lambda(t)) - \Gamma(\lambda(t)) \right],
\]  

(3.1)

where \(H_0(\lambda(t))\) and \(\Gamma(\lambda(t))\) are given below Eq.(2.6). The definition of \(\lambda(t)\) is traced back to Eq. (2.5).

The left-hand side of Eq. (3.5) is of \(O(\epsilon)\) and is given by

\[
\partial_t \rho_0^{reg} = \frac{\partial \rho_0^{reg}}{\partial \lambda} \beta + \epsilon \left[ -\frac{\beta p^2}{2} - \frac{\beta}{2} x^2 - \lambda \frac{\partial \beta}{\partial \lambda} x^2 + \frac{1}{\beta} \frac{\partial \beta}{\partial \lambda} + \frac{1}{2\lambda} \right] \rho_0^{reg}.
\]  

(3.2)

The right-hand side of Eq. (3.5) consists of \(O(1)\) and \(O(\epsilon)\). The contribution of \(O(1)\) vanishes due to Eq. (2.10). The contribution of \(O(\epsilon)\) is the same as in the isothermal process:

\[
\{ \epsilon h, \rho_0^{reg} \} = \gamma \epsilon \partial_0^{reg} \partial p h = \epsilon \beta \{ x \partial_0 h - p \partial_x h \} \rho_0^{reg} + \gamma \epsilon \{ -\beta p \partial_0 h + \partial_p^2 h \} \rho_0^{reg}.
\]  

(3.3)

We obtain the equation to solve \(h\) by equating the right-hand sides of Eqs. (3.2) and (3.3). Using the expansion of \(h\) as in Eq. (2.13) and equating the constant term and each coefficient of \(p, x^2\) and \(px\) to be zero, we have the following equations.

\[
\begin{align*}
\frac{1}{2} \frac{\partial \beta}{\partial \lambda} &= \beta b + 2\gamma a \beta, \\
\beta \lambda b &= -\frac{\lambda}{2} \frac{\partial \beta}{\partial \lambda} - \frac{\beta}{2}, \\
2\lambda a - 2\beta c - \gamma \beta &= 0, \\
\frac{1}{\beta} \frac{\partial \beta}{\partial \lambda} + \frac{1}{2\lambda} - 2\gamma a &= 0.
\end{align*}
\]  

(3.4)

There are 4 unknowns \((a, b, c \text{ and } \beta)\). Among 4 equations above, however, the independent ones are 3, and we need one more independent equation, which will be available by assuming vanishing heat transfer in the dynamics of regularized equation.

In the case that the stiffness coefficient \(\lambda\) is constant, the time derivative of the mean heat absorbed from the reservoir is:

\[
\frac{dQ}{dt} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp (J_x \partial H_0 / \partial x + J_p \partial H_0 / \partial p),
\]  

(3.5)

where \((J_x, J_p)\) is given in Eq. (2.2). When \(\lambda\) changes very slowly as in Eq. (2.5), the regularization in the fast-forward scheme requires \(H_0\) in Eq. (3.5) to be replaced by \(H_0 + \epsilon h(x, p)\) as in Eq. (2.8), while using the distribution \(\rho_0^{reg}\) in Eq. (3.1).

On the slow time scale, Eq. (3.5) becomes:

\[
\frac{dQ}{dt} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp (\gamma \rho_0^{reg} (p + \epsilon h) + \frac{\partial h}{\partial p})
\times (p + \epsilon \frac{\partial h}{\partial p} + \frac{1}{\beta} \frac{\partial \rho_0^{reg}}{\partial p}),
\]  

(3.6)

where \(h\) is expanded again as Eq. (2.13).

Noting the equality

\[
p + \frac{1}{\beta \rho_0^{reg}} \frac{\partial \rho_0^{reg}}{\partial p} = p - \frac{1}{\beta \rho_0^{reg}} \beta \rho_0^{reg} p = 0,
\]

and using the expression \(\frac{\partial h}{\partial p} = 2ap + bx\) due to Eq. (2.13), Eq. (3.6) reduces to

\[
\frac{dQ}{dt} = \epsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp (\gamma \rho_0^{reg} (2ap^2 + bpx)) = \epsilon \frac{2\gamma a}{\beta},
\]  

(3.7)

up to the leading order of \(\epsilon\).
Then the vanishing heat transfer ($\frac{\partial Q}{\partial T} = 0$) during the quasi-static thermally-adiabatic process is satisfied by

$$a = 0.$$  \hfill (3.8)

Using Eq. (3.8) in Eq. (3.4), we obtain:

$$b = \frac{1}{4\lambda},$$

$$c = \frac{\gamma}{8\lambda},$$

$$\beta \sqrt{\lambda} = \text{constant},$$  \hfill (3.9)

and consequently, the driving potential proves to be

$$h = -\frac{1}{4\lambda} px + \frac{\gamma}{8\lambda} x^2.$$  \hfill (3.10)

The second half of the fast-forward scheme in the thermally-adiabatic process is exactly parallel to the description from Eqs. (2.16) through Eq. (2.26) in the isothermal process, except for the difference in the inverse temperature ($\beta$) which is time dependent through $\beta = \frac{\text{constant}}{\sqrt{\lambda}}$. Now we have the fast-forwarded Hamiltonian $H_{FF} = H_0 + \lambda h$ with $H_0 = \frac{p^2}{2} + \frac{1}{2} \lambda(t)x^2$ and $h$ given in Eq. (3.10). (t) is the same as in Eqs. (2.24) and (2.24). The fast-forwarded distribution is $\rho_{FF}(x, p, t) \equiv \rho_{reg}(x, p; \lambda(\lambda(t)))$.

**B. Work**

During the fast forward of the thermally-adiabatic process, the work $W$ done from outside is given by

$$W = \Delta E = E(T_{FF}) - E(0),$$  \hfill (3.11)

where $E(t) = \langle H_{FF} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{FF} dxdp$. Here $H_{FF} = H_0 + \lambda h$ with $h$ obtained in Eq. (3.10). Noting $E(t) = \frac{1}{\beta} + \frac{2\lambda}{8\lambda} \frac{1}{\beta} t$ together with the boundary condition $\lambda(T_{FF}) = \lambda(0) = 0$, we find

$$W = W_{rev} = \frac{1}{\beta(T_{FF})} - \frac{1}{\beta(0)} = k_B T_{final} - k_B T_{initial}.$$  \hfill (3.12)

In the thermally-adiabatic process we have no irreversible work.

**IV. EFFICIENCY AT THE MAXIMUM POWER OF FAST-FORWARDED STOCHASTIC HEAT ENGINE**

The stochastic engine works between the hot ($T_h$) and cold ($T_c$) reservoirs. Using the results of Sections [II] and [III] we shall evaluate the efficiency of the fast-forwarded Carnot-like cycle at the maximum power. The cycle consists of following 4 steps as shown in Table 1. See also Fig. [I]. We take $\lambda_j$ with $j = 0, 1, 2, 3$ as the stiffness coefficients at the nodes of the cycle. Among a variety of choices $\{\lambda_j\}$, we concentrate on the symmetric case that the ratio of the initial and final stiffness coefficients is the same in both of the expanding and contracting isothermal processes: $\frac{\lambda_1}{\lambda_0} = \frac{\lambda_2}{\lambda_3} = q$, and $\frac{\lambda_2}{\lambda_3} = \frac{\lambda_3}{\lambda_0} = \bar{q}$. Table 1 also includes the relative increment of the stiffness coefficient $\xi$, the time interval $T_{FF}$ (arbitrary) and the mean velocity $\bar{v}$ in each of 4 steps.

| sub-processes          | $\lambda(0)$ | $\lambda(T_{FF})$ | $\xi \equiv \frac{\lambda(T_{FF}) - \lambda(0)}{\lambda(0)}$ | $T_{FF}$ | $\bar{v} \equiv \frac{\lambda(T_{FF}) - \lambda(0)}{T_{FF}}$ |
|------------------------|---------------|-------------------|-----------------------------------------------|----------|------------------------------------------------------------------|
| isothermal expansion at $T_h$ | $\lambda_0$  | $\lambda_1$  | $q < 1$) | $q - 1 < 0$) | $t_1$ | $\frac{(q - 1)\lambda_0}{t_1}$ |
| thermally-adiabatic expansion | $\lambda_1$ | $\lambda_2$ | $q < 1$) | $q - 1 < 0$) | $t_2$ | $\frac{(q - 1)\lambda_1}{t_2}$ |
| isothermal compression at $T_c$ | $\lambda_2$ | $\lambda_3$ | $q > 1$) | $q - 1 > 0$) | $t_3$ | $\frac{(q - 1)\lambda_2}{t_3}$ |
| thermally-adiabatic compression | $\lambda_3$ | $\lambda_0$ | $q > 1$) | $q - 1 > 0$) | $t_4$ | $\frac{(q - 1)\lambda_3}{t_4}$ |

We now calculate the reversible and irreversible parts of work which the heat engine does on the outside during its one cycle. In the isothermal processes consisting of the steps 1 and 3, the reversible part gives a nonvanishing contribution:

$$-\frac{k_B}{2} (T_h - T_c) \ln q,$$  \hfill (4.1)
while the contribution due to the irreversible part is

\[ - \frac{\pi k_B}{4} \left( \frac{Z_2(q - 1)}{\lambda_0} + \frac{2}{\gamma}Z_1(q - 1) \right) \frac{T_h}{t_1} \]

\[ - \frac{\pi k_B}{4} \left( \frac{Z_2(\frac{q}{2} - 1)}{\lambda_2} + \frac{2}{\gamma}Z_1(\frac{1}{q} - 1) \right) \frac{T_c}{t_3}. \]

(4.2)

In the thermally-adiabatic processes consisting of the steps 2 and 4, we have no irreversible work and the net reversible part gives no contribution:

\[ - k_B(T_c - T_h) - k_B(T_h - T_c) = 0. \]

(4.3)

The total work for one cycle is a sum of contributions from the isothermal and adiabatic processes and is given by:

\[ W_{total} = -\frac{k_B}{2}(T_h - T_c) \ln q \]

\[ - \frac{\pi k_B}{4} \left( \frac{Z_2(q - 1)}{\lambda_0} + \frac{2}{\gamma}Z_1(q - 1) \right) \frac{T_h}{t_1} \]

\[ - \frac{\pi k_B}{4} \left( \frac{Z_2(\frac{q}{2} - 1)}{\lambda_2} + \frac{2}{\gamma}Z_1(\frac{1}{q} - 1) \right) \frac{T_c}{t_3}. \]

(4.4)

On the 3rd line, we employed the theorem of Appendix A

\[ Z_n \left( \frac{1}{q} - 1 \right) = q^{n-1}Z_n(q - 1), \]

(4.5)

together with \( \lambda_2 = \bar{q}\lambda_1 = q\bar{q}\lambda_0 \) available from Table 1. Concerning the factor \( \bar{q} \), we can see:

\[ \bar{q} = \frac{\lambda_2}{\lambda_1} = \left( \frac{T_c}{T_h} \right)^2 \]

(4.6)

with use of the constant of motion in Eq. (4.9) during the thermally-adiabatic process.

Below we shall first concentrate on the case of a large dissipation with \( \gamma \gg \sqrt{\lambda_0} \). The one-cycle work is expressed as

\[ W_{total} = -\frac{k_B}{2}(T_h - T_c) \ln q \]

\[ - \frac{\pi k_B}{4} Z_2(q - 1) T^2_c \left( \frac{T_h}{t_1} - \frac{T_c}{t_1} \right) \]

\[ - \frac{\pi k_B}{4} Z_2(\frac{q}{2} - 1) T^2_c \left( \frac{T_h}{t_3} - \frac{T_c}{t_3} \right) \]

(4.7)

where the contribution of the term proportional to \( \frac{1}{\gamma} \) in Eq.(4.4) is suppressed. The heat transfer to the particle from the hot heat bath at temperature \( T_h \) is given by

\[ Q_{in} = -\frac{1}{2} k_B T_h \ln q - \frac{\pi k_B}{4\lambda_0} Z_2(q - 1) T^2_h \frac{T_h}{t_1} \]

(4.8)

To obtain a high power heat engine, we take the vanishing time \( t_2 = t_4 \) with \( \bar{q} = \bar{q} \to \infty \) so as to guarantee \( \tilde{Q}_{in} = \bar{q} = 1 = (\frac{T_h}{T_c})^2 - 1 \) and \( \tilde{Q}_{out} = \frac{1}{\gamma} = 1 = (\frac{T_c}{T_h})^2 - 1 \). Introducing

\[ A = -\frac{1}{2} k_B (T_h - T_c) \ln q, \]

\[ B = \frac{\gamma \pi k_B}{4\lambda_0} Z_2(q - 1) T^2_c \]

and assuming \( t_2 = t_4 = 0 \), the power can be defined by

\[ P = \frac{W_{total}}{t_1 + t_3} = A \left( \frac{T_h^{-1}}{t_1} + \frac{T_c^{-1}}{t_3} \right). \]

(4.10)

The time \( t_1^* \) and \( t_3^* \) which maximizes \( P \) is obtained by solving the equations:

\[ \frac{\partial P}{\partial t_1} = 0, \quad \frac{\partial P}{\partial t_3} = 0, \]

(4.11)

which is satisfied by

\[ \frac{t_1^*}{t_3^*} = \left( \frac{T_c}{T_h} \right)^{-\frac{1}{2}}. \]

(4.12)

This issue expresses that \( t_1^* \) and \( t_3^* \) should be different so as to achieve the maximum power. To be explicit, we have

\[ t_1^* = \frac{2B}{A} \left( T_h^{-1} + (T_h T_c)^{-\frac{1}{2}} \right), \]

\[ t_3^* = \frac{2B}{A} \left( T_c^{-1} + (T_h T_c)^{-\frac{1}{2}} \right). \]

(4.13)

The efficiency is written with use of Eqs. (4.9) as

\[ \eta = \frac{W_{total}}{Q_{in}} = \frac{A - B \left( \frac{T_h^{-1}}{t_1} + \frac{T_c^{-1}}{t_3} \right)}{\frac{A T_h^{-1}}{T_h} - B \frac{T_c^{-1}}{t_3}}. \]

(4.14)

We can express the efficiency at maximum power by substituting Eqs. (4.13) into Eq. (4.14) as

\[ \eta^* = \frac{1}{2} \left( 1 - \frac{1}{2} \left( \frac{T_c}{T_h} \right)^{-\frac{1}{2}} - \frac{1}{4} \frac{T_c}{T_h} - 3 \frac{8}{8} \left( \frac{T_c}{T_h} \right)^{-\frac{1}{2}} \right). \]

(4.15)

We see that, for large temperature differences \( (T_h \gg T_c) \), the limiting efficiency \( (\eta^* \rightarrow \frac{1}{2}) \) is less than that \( (\eta^* \rightarrow \frac{2}{3}) \) of analysis of the overdamped case [8].

Then we consider the case of a small dissipation with \( \gamma \ll \sqrt{\lambda_0} \). The one-cycle work and heat transfer from
the hot heat bath are now given by

\[
W_{\text{total}} = -\frac{k_B}{2} (T_h - T_c) \ln q \\
- \frac{\pi k_B}{2\gamma} Z_1(q - 1) \cdot \left( \frac{T_h}{t_1} + \frac{T_c}{t_3} \right). \tag{4.16}
\]

and

\[
Q_{in} = -\frac{1}{2} k_B T_h \ln q - \frac{\pi k_B}{2\gamma} Z_1(q - 1) \cdot \frac{T_h}{t_1}, \tag{4.17}
\]

respectively. Introducing \( B' = \frac{\pi k_B}{2\gamma} Z_1(q - 1) \) instead of \( B \) in Eq. (4.9) and assuming \( t_2 = t_4 = 0 \), the power and efficiency are now given by

\[
P \equiv \frac{W_{\text{total}}}{t_1 + t_3} = \frac{A}{t_1 + t_3} - \frac{B'}{A} \left( \frac{T_h}{t_1} + \frac{T_c}{t_3} \right), \tag{4.18}
\]

and

\[
\eta \equiv \frac{W_{\text{total}}}{Q_{in}} = \frac{A - B' \left( \frac{T_h}{t_1} + \frac{T_c}{t_3} \right)}{A \left( \frac{T_h}{t_1 - T_c} - B' \frac{T_c}{t_1} \right)}, \tag{4.19}
\]

respectively. The time \( t_1^* \) and \( t_3^* \) which maximizes \( P \) in Eq. (4.18) are

\[
t_1^* = \frac{2B'}{A} \left( T_h + \sqrt{T_hT_c} \right), \tag{4.20}
\]

\[
t_3^* = \frac{2B'}{A} \left( T_c + \sqrt{T_hT_c} \right).
\]

Substituting this \( t_1^* \) and \( t_3^* \) into Eq. (4.19), we have

\[
\eta^* = 1 - \sqrt{\frac{T_c}{T_h}}. \tag{4.21}
\]

Interestingly, this result is equal to the Curzon-Ahlborn efficiency for endoreversible heat engines working at maximum power \[1, 28, 29\] although the present stochastic model looks quite different from macroscopic finite-time heat engines. The issue in Eq. (4.20) is also compatible with the assertions of Refs. \[9, 10\] which are concerned with the underdamped case, but are solving the equation of motion for variances of position and momentum of the Brownian particle.

**V. SUMMARY AND DISCUSSIONS**

By extending the idea of the fast forward cultivated in the context of the conservative quantum dynamics, we constructed the fast-forward (FF) theory of the nanoscale stochastic heat engine driven by a Brownian particle coupled with a time-dependent harmonic potential and working between the high-temperature (\( T_h \)) and low temperature (\( T_c \)) heat baths. The FF scheme applied to the Kramers equation for the underdamped case has successfully reproduced the quasi-static dynamics of the stochastic Carnot-like cycle on shortened time scale. We have given the explicit expression for the protocols or the driving potentials in both the isothermal and thermally-adiabatic processes, which guarantee the Gaussian probability distribution function throughout the cycle. The irreversible work is found to consist of two terms with one proportional to and the other inversely proportional to the friction coefficient. With use of the reversible and irreversible works evaluated by the FF protocols, we have found the efficiency of this engine at maximum power is universal, which is \( \eta^* = \frac{1}{2} \left( 1 - \frac{1}{2} \left( \frac{T_h}{T_c} \right)^{\frac{1}{2}} + \left( \frac{T_c}{T_h} \right)^{\frac{1}{2}} \right) \) in the case of strong dissipation and the Curzon-Ahlborn efficiency \( \eta^* = 1 - \left( \frac{T_h}{T_c} \right)^{\frac{1}{2}} \) in the case of weak dissipation. To test the present issue experimentally, a proposal how to prepare the driving potential including the momentum degree of freedom will be necessary, which will constitute our next challenge.

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**Appendix A: Proof of \( Z_n(\frac{1}{q} - 1) = q^{n-1}Z_n(q - 1) \)**

\( Z_n(\xi) \) is defined by

\[
Z_n(\xi) = \xi \int_0^1 \frac{\sin(2\pi s)}{1 + \xi \left( s - \frac{1}{2\pi} \sin(2\pi s) \right)} ds. \tag{A1}
\]

Firstly we shall show its non-negativity. Let’s introduce \( g_\xi(s) = 1 + \xi \left( s - \frac{1}{2\pi} \sin(2\pi s) \right) \), which is positive for \( \xi > -1 \) in the interval \( 0 \leq s \leq 1 \). Then

\[
Z_n(\xi) = \int_0^1 \frac{\dot{g}_\xi(s)}{g_\xi(s)} ds
\]

\[
= \dot{g}_\xi(s)g_\xi^{-n}(s)|_0^1 + n \int_0^1 \frac{\dot{g}_\xi(s)}{g_\xi(s)} g_\xi^{-(n+1)}(s) ds
\]

\[
= n \int_0^1 \frac{\dot{g}_\xi^2(s)}{g_\xi^{-(n+1)}(s)} ds \quad (\geq 0). \quad \tag{A2}
\]

In the last equality, we used \( \dot{g}_\xi(0) = \dot{g}_\xi(1) = 1 \). Therefore \( Z_n(\xi) \geq 0 \) and the equality holds when \( \xi = 0 \).
Secondly we show the relation entitled in this Appendix. \( Z_n\left(\frac{1}{q} - 1\right) \) is explicitly written as

\[
Z_n\left(\frac{1}{q} - 1\right) = \left(\frac{1}{q} - 1\right)q^n\int_0^1 \frac{\sin(2\pi s)}{q + (1 - q)\left(s - \frac{1}{\pi} \sin(2\pi s)\right)} ds.
\]

If we shall make a variable change

\[
s = 1 - s',
\]

then we see the goal:

\[
Z_n\left(\frac{1}{q} - 1\right) = \left(\frac{1}{q} - 1\right)q^n
\times \int_0^1 \frac{\sin(2\pi s')ds'}{q + (1 - q)\left(1 - s' + \frac{1}{\pi} \sin(2\pi s')\right)}
= (1 - q)q^{n-1}
\times \int_0^1 \frac{\sin(2\pi s')ds'}{1 + (q - 1)\left(s' - \frac{1}{\pi} \sin(2\pi s')\right)}
= q^{n-1}Z_n(q - 1).
\]

We used the cases \( n = 1, 2 \) in the text.

Appendix B: Alternative derivation of \( W_{irr} \) in the isothermal process

Let us show another derivation of the irreversible work by using the definition,

\[
W_{irr} = T\Delta S - Q,
\]

where \( \Delta S \) and \( Q \) are respectively increments of entropy and heat during the isothermal (\( \beta = \text{constant} \)) process. We shall apply the definition of \( \frac{d\rho}{dt} \) in Eq. (3.40) to the fast-forwarded isothermal process, by replacing \( \rho_0^{reg}, h, \) and \( \epsilon \) by \( \rho_{FF} \) in Eq. (2.19), \( h \) in Eq. (2.15) and \( \lambda \) in Eq. (2.22), respectively. Then

\[
Q = -\int_{-\infty}^{T_{FF}} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \left[ \gamma \rho_{FF} \left( p + \lambda \frac{\partial h}{\partial p} \right) \right]
\times \left( p + \lambda \frac{\partial h}{\partial p} + \frac{1}{\beta \rho_{FF}} \frac{\partial \rho_{FF}}{\partial p} \right).
\]

Similarly, with use of the definition of ensemble average of trajectory entropy \( S = -\int_{-\infty}^{+\infty} T dt \int_{-\infty}^{+\infty} dp k_B \rho_{FF} \ln \rho_{FF} \), we see:

\[
\Delta S = \int_0^{T_{FF}} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp k_B \left[ \frac{\partial \rho_{FF}}{\partial p} \right]
\times \left( p + \lambda \frac{\partial h}{\partial p} + \frac{1}{\beta \rho_{FF}} \frac{\partial \rho_{FF}}{\partial p} \right).
\]

Noting \( \frac{\partial h}{\partial p} = \frac{1}{2\gamma \lambda} p - \frac{1}{2\lambda} x \) and \( p + \frac{1}{\beta \rho_{FF}} \frac{\partial \rho_{FF}}{\partial p} = p - \frac{1}{\beta \rho_{FF}} \beta \rho_{FF} p = 0 \), we obtain:

\[
Q = -\int_0^{T_{FF}} dt \gamma \left[ \frac{\lambda}{2\gamma \lambda} + \frac{\dot{\lambda}^2}{4\gamma^2 \lambda^2 \beta} + \frac{\dot{\lambda}^2}{4\lambda^3 \beta} \right].
\]

In a similar way, we find:

\[
\frac{\Delta S}{k_B} = \int_0^{T_{FF}} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp k_B \left[ \frac{\partial \rho_{FF}}{\partial p} \right]
\times \left( p + \lambda \frac{\partial h}{\partial p} + \frac{1}{\beta \rho_{FF}} \frac{\partial \rho_{FF}}{\partial p} \right).
\]

Then we can evaluate \( W_{irr} \) as follows:

\[
W_{irr} = -k_B T \int_0^{T_{FF}} dt \frac{\dot{\lambda}}{2\lambda} - \int_0^{T_{FF}} dt \left[ \frac{\dot{\lambda}^2}{2\gamma \lambda} + \frac{\dot{\lambda}^2}{4\gamma^2 \lambda^2 \beta} + \frac{\dot{\lambda}^2}{4\lambda^3 \beta} \right]
\]

With use of Eq. (2.33) in the text, the final issue agrees with Eq. (2.32) and thereby leads to Eq. (2.37).

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