On Langlands reciprocity for $C^*$-algebras

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Abstract

We introduce a motivic $C^*$-algebra $A_V$ of the arithmetic variety $V$ and an automorphic $C^*$-algebra $A_G$ of the reductive group $G$ over the ring of adeles of a number field $K$. It is proved that if $V$ is a $G$-coherent variety (e.g. the Shimura variety of $G$), then there exists an embedding $A_V \rightarrow A_G$. As a corollary, the motivic $L$-function of $V$ is shown to be a product of the automorphic $L$-functions corresponding to the irreducible representations of $G$.

Key words and phrases: Langlands program, Serre $C^*$-algebra

MSC: 11F70 (automorphic representations); 46L85 (noncommutative topology)

1 Introduction

The Langlands conjectures say that all zeta functions are automorphic [Langlands 1978] [8]; the aim of our note is to recast and understand (one of) the conjectures in terms of the non-commutative algebra [Dixmier 1977] [4]. Let us outline the main ideas.

Denote by $G(A_K)$ a reductive group over the ring of adeles $A_K$ of a number field $K$ and by $G(K)$ its discrete subgroup over $K$. The Banach algebra $L^1(G(K)\setminus G(A_K))$ consists of the integrable complex-valued functions endowed with the operator norm and pointwise addition and multiplication given by the convolution product:

$$(f_1 * f_2)(g) = \int_{G(A_K)} f_1(gh^{-1})f_2(h)dh. \quad (1)$$
Consider an enveloping $C^*$-algebra, $\mathcal{A}_G$, of the algebra $L^1(G(K)\backslash G(\mathcal{A}_K))$, see [Dixmier 1977, Section 13.9] for the details; the $\mathcal{A}_G$ encodes the unitary irreducible representations of the locally compact group $G(\mathcal{A}_K)$ induced by $G(K)$. (The interested reader is referred to [Gelbart 1984] for a link of such representations with the automorphic cusp forms and the non-abelian class field theory.) We shall call the $\mathcal{A}_G$ an automorphic $C^*$-algebra.

Let $V$ be a projective variety over the complex numbers $\mathbb{C}$. For an automorphism $\sigma : V \to V$ and the invertible sheaf $\mathcal{L}$ of linear forms on $V$, one can construct a twisted homogeneous coordinate ring $B(V, \mathcal{L}, \sigma)$ of the variety $V$, i.e. a non-commutative ring satisfying the Serre isomorphism:

$$\text{Mod} \left( B(V, \mathcal{L}, \sigma) \right) / \text{Tors} \cong \text{Coh} (V),$$

where $\text{Mod}$ is the category of graded left modules over the graded ring $B(V, \mathcal{L}, \sigma)$, $\text{Tors}$ the full subcategory of $\text{Mod}$ of the torsion modules and $\text{Coh}$ the category of quasi-coherent sheaves on the variety $V$, see [Stafford & van den Bergh 2001] p. 180. The norm-closure of a self-adjoint representation of the ring $B(V, \mathcal{L}, \sigma)$ by the linear operators on a Hilbert space is called the Serre $C^*$-algebra of $V$. In what follows, we shall focus on the case $V$ being an arithmetic variety; the Serre $C^*$-algebra of $V$ is denoted by $\mathcal{A}_V$. Since $V$ is an arithmetic variety, it is related to the Weil conjectures and the Grothendieck theory of motives; hence the $\mathcal{A}_V$ is a motivic $C^*$-algebra.

Roughly speaking, a philosophy of [Langlands 1978] does not distinguish the motivic and automorphic objects; if so, a regular map between the algebras $\mathcal{A}_V$ and $\mathcal{A}_G$ must exist. We prove that such a map is in fact an embedding $\mathcal{A}_V \hookrightarrow \mathcal{A}_G$. (The variety $V$ must be $G$-coherent in this case; for instance, the Shimura variety of the group $G(\mathcal{A}_K)$ is always $G$-coherent [Deligne 1971].) To give an exact statement of our results, we shall need the following definitions.

The $C^*$-algebra $\mathcal{A}$ is called an AF-algebra (Approximately Finite $C^*$-algebra), if $\mathcal{A}$ is the inductive limit of the finite-dimensional $C^*$-algebras [Bratteli 1972]; the inductive limit is described by an infinite sequence of non-negative integer matrices $B_i = (b_{rs})$ and if $B_i = \text{Const}$, then the AF-algebra is called stationary. The AF-algebras $\mathcal{A}$ are classified (up to a Morita equivalence) by their dimension groups $(K_0(\mathcal{A}), K_0^+(\mathcal{A}))$, where $K_0(\mathcal{A})$ and $K_0^+(\mathcal{A}) \subset K_0(\mathcal{A})$ are the $K_0$-group and the Grothendieck semigroup of algebra $\mathcal{A}$, respectively; see [Elliott 1976]. In our case the $(K_0(\mathcal{A}), K_0^+(\mathcal{A}))$
can be identified with an additive abelian subgroup $K_0(\mathbb{A})$ of the real line $\mathbb{R}$, such that $K_0^+(\mathbb{A})$ corresponds to the positive reals [Effros 1981] [5].

**Theorem 1**  The $\mathbb{A}_G$ is a product of the stationary AF-algebras $\{\mathbb{A}_G^p \subseteq \mathbb{A}_G^\infty | p \text{ prime}\}$ of the form

$$\mathbb{A}_G \cong \prod_{p \leq \infty} ' \mathbb{A}_G^p,$$

where $\prod'$ is restricted to all but a finite set of primes.

Recall that the $i$-th trace cohomology $\{H^i_{tr}(V) | 0 \leq i \leq 2 \dim_C V\}$ of an arithmetic variety $V$ is an additive abelian subgroup of $\mathbb{R}$ obtained from a canonical trace on the Serre $C^*$-algebra of $V$ [9]; on the other hand, the dimension group $K_0(\mathbb{A}_G)$ of the AF-algebra $\mathbb{A}_G^\infty$ is also an additive abelian subgroup of $\mathbb{R}$ [Effros 1981, Chapter 6] [5]. The following definition is natural.

**Definition 1**  The arithmetic variety $V$ is called G-coherent, if

$$H^i_{tr}(V) \subseteq K_0(\mathbb{A}_G^\infty) \quad \text{for all} \quad 0 \leq i \leq 2 \dim_C V. \quad (4)$$

**Remark 1**  If $V$ is the Shimura variety of $G(A_K)$ [Deligne 1971] [3], then $V$ is a G-coherent variety; in particular, a set of the G-coherent varieties is non-empty.

**Theorem 2**  If $V$ is a G-coherent variety, there exists a canonical embedding $\mathcal{A}_V \hookrightarrow \mathbb{A}_G$ of the motivic $C^*$-algebra $\mathcal{A}_V$ into an automorphic $C^*$-algebra $\mathbb{A}_G$.

**Remark 2**  To keep it simple, theorem 2 says the non-commutative coordinate ring $\mathcal{A}_V$ of a G-coherent variety $V$ is a subalgebra of the algebra $\mathbb{A}_G$; such a result can be viewed as an analog of the Langlands reciprocity in the category of operator algebras.

An application of theorem 2 is as follows. To each arithmetic variety $V$ one can attach a motivic (Hasse-Weil) $L$-function [Langlands 1978] [8]; likewise, to each irreducible representation of the locally compact group $G(A_K)$ one can attach an automorphic (standard) $L$-function [Gelbart 1984] [7]. Theorem 2 implies one of the conjectures formulated in [Langlands 1978] [8].

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Corollary 1 The motivic L-function of a G-coherent variety V is a product of the automorphic L-functions.

Our note is organized as follows. Section 2 is preliminary; theorems 1, 2 and corollary 1 are proved in Section 3. An example is constructed in Section 4.

2 Preliminaries

This section is a brief account of preliminary facts involved in our paper; we refer the reader to [Bratteli 1972] [2], [Dixmier 1977] [4], [Langlands 1978] [8] and [Stafford & van den Bergh 2001] [17].

2.1 AF-algebras

A $C^*$-algebra is an algebra over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| ||b||$ and $||a^*a|| = ||a^2||$ for all $a, b \in A$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$; otherwise, $A$ represents a noncommutative topological space.

An AF-algebra (Approximately Finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots,$$

where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. If $\varphi_i = Const$, then the AF-algebra $A$ is called stationary; such an algebra defines and is defined by a shift automorphism $\sigma_{\varphi} : A \to A$ corresponding to a map $i \mapsto i + 1$ on $\varphi_i$ [Effros 1981, p. 37] [5]. The homomorphisms $\varphi_i$ can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \cdots \oplus M_{i_k}$ and $M_{i'} = M_{i'_1} \oplus \cdots \oplus M_{i'_{k'}}$ be the semi-simple $C^*$-algebras and $\varphi_i : M_i \to M_{i'}$ the homomorphism. One has two sets of vertices $V_{i_1}, \ldots, V_{i_k}$ and $V_{i'_1}, \ldots, V_{i'_{k'}}$ joined by $b_{rs}$ edges whenever the summand $M_{i_r}$ contains $b_{rs}$ copies of the summand $M_{i'_s}$ under the embedding $\varphi_i$. As $i$ varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra.
The matrix $B = (b_{rs})$ is known as a partial multiplicity matrix; an infinite sequence of $B_i$ defines a unique AF-algebra.

For a unital $C^*$-algebra $A$, let $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$; projections $p, q \in V(A)$ are equivalent if there exists a partial isometry $u$ such that $p = uu^*$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$; the equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called the $K_0$-group of the algebra $A$. The functor $A \rightarrow K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+ \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0(A)$. The ordered abelian group $(K_0(A), K_0^+, u)$ is called a dimension group; an order-isomorphism class of the latter we denote by $(G, G^+)$. If $A$ is an AF-algebra, then its dimension group $(K_0(A), K_0^+, u)$ is a complete isomorphism invariant of algebra $A$ [Elliott 1976] [6]. The order-isomorphism class $(K_0(A), K_0^+, u)$ is an invariant of the Morita equivalence of algebra $A$, i.e. an isomorphism class in the category of finitely generated projective modules over $A$.

2.2 Serre $C^*$-algebras and trace cohomology

Let $V$ be an $n$-dimensional complex projective variety endowed with an automorphism $\sigma : V \rightarrow V$ and denote by $B(V, L, \sigma)$ its twisted homogeneous coordinate ring, see [Stafford & van den Bergh 2001] [17]. Let $R$ be a commutative graded ring, such that $V = Spec (R)$. Denote by $R[t, t^{-1}; \sigma]$ the ring of skew Laurent polynomials defined by the commutation relation $b^\sigma t = t b$ for all $b \in R$, where $b^\sigma$ is the image of $b$ under automorphism $\sigma$. It is known, that $R[t, t^{-1}; \sigma] \cong B(V, L, \sigma)$.

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a ring of skew Laurent polynomials $R[t, t^{-1}; \sigma]$, consider a homomorphism:

$$\rho : R[t, t^{-1}; \sigma] \rightarrow B(\mathcal{H}).$$

Recall that $B(\mathcal{H})$ is endowed with a $*$-involution; the involution comes from the scalar product on the Hilbert space $\mathcal{H}$. We shall call representation $\rho$ $*$-coherent, if (i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(t) = \rho(t^{-1})$ and (ii) for all $b \in R$ it holds $(\rho^*(b))^\sigma(\rho) = \rho^*(b^\sigma)$, where $\sigma(\rho)$ is an
automorphism of $\rho(R)$ induced by $\sigma$. Whenever $B = R[t, t^{-1}; \sigma]$ admits a $\ast$-coherent representation, $\rho(B)$ is a $\ast$-algebra; the norm closure of $\rho(B)$ is a $C^\ast$-algebra [Dixmier 1977] \[4\]. We shall denote it by $A_V$ and refer to $A_V$ as the Serre $C^\ast$-algebra of variety $V$.

Let $K$ be the $C^\ast$-algebra of all compact operators on $H$. We shall write $\tau : A_V \otimes K \to \mathbb{R}$ to denote the canonical normalized trace on $A_V \otimes K$, i.e. a positive linear functional of norm 1 such that $\tau(xy) = \tau(yx)$ for all $x, y \in A_V \otimes K$, see [Blackadar 1986] \[1\], p. 31. Because $A_V$ is a crossed product $C^\ast$-algebra of the form $A_V \cong C(V) \rtimes \mathbb{Z}$, one can use the Pimsner-Voiculescu six term exact sequence for the crossed products, see e.g. [Blackadar 1986] \[1\], p. 83 for the details. Thus one gets the short exact sequence of the algebraic $K$-groups: $0 \to K_0(C(V)) \xrightarrow{i_*} K_0(A_V) \to K_1(C(V)) \to 0$, where map $i_*$ is induced by the natural embedding of $C(V)$ into $A_V$. We have $K_0(C(V)) \cong K^0(V)$ and $K_1(C(V)) \cong K^{-1}(V)$, where $K^0$ and $K^{-1}$ are the topological $K$-groups of $V$, see [Blackadar 1986] \[1\], p. 80. By the Chern character formula, one gets $K^0(V) \otimes \mathbb{Q} \cong H^{even}(V; \mathbb{Q})$ and $K^{-1}(V) \otimes \mathbb{Q} \cong H^{odd}(V; \mathbb{Q})$, where $H^{even}$ ($H^{odd}$) is the direct sum of even (odd, resp.) cohomology groups of $V$. Notice that $K_0(A_V \otimes K) \cong K_0(A_V)$ because of a stability of the $K_0$-group with respect to tensor products by the algebra $K$, see e.g. [Blackadar 1986] \[1\], p. 32. One gets the commutative diagram in Fig. 1, where $\tau_*$ denotes a homomorphism induced on $K_0$ by the canonical trace $\tau$ on the $C^\ast$-algebra $A_V \otimes K$. Since $H^{even}(V) := \bigoplus_{i=0}^{n} H^{2i}(V)$ and $H^{odd}(V) := \bigoplus_{i=1}^{n} H^{2i-1}(V)$, one gets for each $0 \leq i \leq 2n$ an injective homomorphism $\tau_* : H^i(V) \to \mathbb{R}$.

\[
\begin{array}{ccc}
H^{even}(V) \otimes \mathbb{Q} & \xrightarrow{i_*} & K_0(A_V \otimes K) \otimes \mathbb{Q} \\
& & \xrightarrow{\tau_*} \\
& & \mathbb{R}
\end{array}
\]

**Figure 1:** The trace cohomology.

**Definition 2** By an $i$-th trace cohomology group $H^i_{tr}(V)$ of variety $V$ one understands the abelian subgroup of $\mathbb{R}$ defined by the map $\tau_*$. 

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2.3 Langlands reciprocity

Let \( V \) be an \( n \)-dimensional complex projective variety over a number field \( K \); consider its reduction \( V(\mathbb{F}_p) \) modulo the prime ideal \( \mathfrak{p} \subset K \) over a good prime \( p \). Recall that the Weil zeta function is defined as:

\[
Z_p(t) = \exp \left( \sum_{r=1}^{\infty} \frac{|V(\mathbb{F}_{p^r})| t^r}{r} \right), \quad r \in \mathbb{C},
\]

where \( |V(\mathbb{F}_{p^r})| \) is the number of points of variety \( V(\mathbb{F}_{p^r}) \) defined over the field with \( p^r \) elements. It is known that:

\[
Z_p(t) = P_1(t) \ldots P_{2n-1}(t)
\]

\[
P_i(t) = 1 - p^i t, \quad P_0(t) = 1 - t, \quad P_{2n}(t) = 1 - p^{2n} t
\]

(8)

Consider an infinite product:

\[
L(s, V) := \prod_p Z_p(p^{-s}) = \frac{L^1(s, V) \ldots L^{2n-1}(s, V)}{L^0(s, V) \ldots L^{2n}(s, V)},
\]

where \( L^i(s, V) = \prod_p P_i(p^{-s}) \); since the above \( L \)-function is related to the Weil conjectures and the Grothendieck motives, the \( L(s, V) \) is called a motivic (or Hasse-Weil) \( L \)-function of \( V \).

On the other hand, if \( K \) is a number field then the adele ring \( \mathbb{A}_K \) of \( K \) is a locally compact subring of the direct product \( \prod_v K_v \) taken over all places \( v \) of \( K \); the \( \mathbb{A}_K \) is endowed with a canonical topology. Consider a reductive group \( G(\mathbb{A}_K) \) over \( \mathbb{A}_K \); the latter is a topological group with a canonical discrete subgroup \( G(K) \). Denote by \( L^2(G(K) \backslash G(\mathbb{A}_K)) \) the Hilbert space of all square-integrable complex-valued functions on the homogeneous space \( G(K) \backslash G(\mathbb{A}_K) \) and consider the right regular representation \( \mathcal{R} \) of the locally compact group \( G(\mathbb{A}_K) \) by linear operators on the space \( L^2(G(K) \backslash G(\mathbb{A}_K)) \) given by formula (1). It is well known, that each irreducible component \( \pi \) of the unitary representation \( \mathcal{R} \) can be written in the form \( \pi = \otimes \pi_v \), where \( v \) are all unramified places of \( K \). Using the spherical functions, one gets an injection \( \pi_v \mapsto [A_v] \), where \([A_v]\) is a conjugacy class of matrices in the group \( GL_n(\mathbb{C}) \). The automorphic (standard) \( L \)-function is given by the formula:

\[
L(s, \pi) = \prod_v \left( \det \left[ I_n - [A_v](Nv)^{-s} \right] \right)^{-1}, \quad s \in \mathbb{C},
\]

(10)
where \( Nv \) is the norm of place \( v \); we refer the reader to [Langlands 1978, p. 170] [8] and [Gelbart 1984, p. 201] [7] for details of this construction.

The following conjecture relates the motivic and automorphic \( L \)-functions.

**Conjecture 1 ([Langlands 1978] [8])** For each \( 0 \leq i \leq 2n \) there exists an irreducible representation \( \pi_i \) of the group \( G(A_K) \), such that \( L^i(s, V) \equiv L(s, \pi_i) \).

### 3 Proofs

#### 3.1 Proof of theorem [1]

We shall split the proof in a series of lemmas.

**Lemma 1** The \( G(Z_p) \) is a profinite group.

*Proof.* Recall that a *profinite group* is a topological group which can be represented as inverse limit \( \lim \leftarrow G_i \) of finite groups \( G_i \). The group of \( p \)-adic integers \( Z_p \) is profinite, since \( Z_p = \lim \leftarrow Z/p^iZ \), where \( Z/p^iZ \) are finite groups. If \( G \) is a reductive group, then

\[
G(Z_p) = \lim \leftarrow G(Z/p^iZ). \tag{11}
\]

But the groups \( G_i = G(Z/p^iZ) \) are finite for all \( 1 \leq i \leq \infty \). Therefore \( G(Z_p) \) is a profinite group. Lemma [1] follows.

**Lemma 2** The \( C^* \)-algebra \( \mathcal{A}_G^{Z_p} := \mathcal{A}_{G(Z_p)} \) is a stationary \( AF \)-algebra.

*Proof.* (i) Let us prove that the \( \mathcal{A}_G^{Z_p} \) is an \( AF \)-algebra. Recall that if \( G \) is a finite group, then the *group algebra* \( C[G] \) over the field \( C \) has the form

\[
C[G] \cong M_{n_1}(C) \oplus \cdots \oplus M_{n_h}(C), \tag{12}
\]

where \( n_i \) are degrees of the irreducible representations of \( G \) and \( h \) is the total number of such representations; we refer the reader to [Serre 1967, Proposition 10] [15] for the proof of this fact.
By Lemma 1, the $G(\mathbb{Z}_p)$ is a profinite group of the form $G(\mathbb{Z}_p) = \lim_{\leftarrow} G(\mathbb{Z}/p^i\mathbb{Z})$. Consider a group algebra

$$\mathbb{C}[G_i] \cong M_{m_1}^{(i)}(\mathbb{C}) \oplus \cdots \oplus M_{m_h}^{(i)}(\mathbb{C})$$

(13)
corresponding to the finite group $G_i := G(\mathbb{Z}/p^i\mathbb{Z})$. Notice that the $\mathbb{C}[G_i]$ is a finite-dimensional $\mathbb{C}^*$-algebra.

The inverse limit $G(\mathbb{Z}_p) = \lim_{\leftarrow} G_i$ defines an ascending sequence of the group algebras:

$$\lim_{\leftarrow} \mathbb{C}[G_i] \cong \lim_{\leftarrow} M_{m_1}^{(i)}(\mathbb{C}) \oplus \cdots \oplus M_{m_h}^{(i)}(\mathbb{C}).$$

(14)

But the $\lim_{\leftarrow} \mathbb{C}[G_i]$ is nothing but the $\mathbb{C}^*$-algebra $\mathbb{A}_G(\mathbb{Z}_p)$ [Dixmier 1977, Section 13.9] [4]. It follows from (14) that the $\mathbb{A}_G(\mathbb{Z}_p)$ is an AF-algebra, since it is an ascending sequence of the finite-dimensional $\mathbb{C}^*$-algebras $\mathbb{C}[G_i]$. Item (i) follows.

**Remark 3** If $G$ is a profinite group, then the $\mathbb{A}_G$ is an AF-algebra; the proof repeats the argument of item (i) and is left to the reader.

**Example 1** An AF-algebra $\mathbb{A}_{\mathbb{Z}_p}$ corresponding to the ring of $p$-adic integers $\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^i\mathbb{Z}$ is an UHF-algebra $M_{p^n}$ of the form $\mathbb{C} \to M_p(\mathbb{C}) \to M_{p^2}(\mathbb{C}) \to \cdots$ and $K_0(M_{p^n}) \cong \mathbb{Z}[\frac{1}{p}]$ [Rørdam, Larsen & Laustsen 2000, Section 7.4] [14].

(ii) Let us prove that the AF-algebra $\mathbb{A}_G(\mathbb{Z}_p)$ is stationary. Indeed, each element of the finite group $G(p^i\mathbb{Z})$ is fixed under the multiplication by $p$; therefore we have $pG(\mathbb{Z}/p^i\mathbb{Z}) \cong G(\mathbb{Z}/p^i\mathbb{Z})$. Taking the inverse limit, one gets

$$\lim_{\leftarrow} G(\mathbb{Z}/p^i\mathbb{Z}) \cong \lim_{\leftarrow} pG(\mathbb{Z}/p^i\mathbb{Z}) \cong \lim_{\leftarrow} G(p^{-i}\mathbb{Z}).$$

(15)

From (15) we obtain an isomorphism of the corresponding group algebras

$$\lim_{\leftarrow} \mathbb{C}[G(\mathbb{Z}/p^i\mathbb{Z})] \cong \lim_{\leftarrow} \mathbb{C}[G(\mathbb{Z}/p^{-1}\mathbb{Z})].$$

(16)

But $\lim_{\leftarrow} \mathbb{C}[G(\mathbb{Z}/p^i\mathbb{Z})] := \mathbb{A}_G^p$ and (16) gives rise to a shift automorphism $\sigma_p : \mathbb{A}_G^p \to \mathbb{A}_G^p$ of the AF-algebra $\mathbb{A}_G^p$. Such an automorphism exists if and only if the $\mathbb{A}_G^p$ is a stationary AF-algebra; the latter is an ascending sequence of finite-dimensional $\mathbb{C}^*$-algebras of the form

$$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\varphi} \cdots,$$

(17)

Item (ii) and Lemma 2 is proved.
Lemma 3 $A^p_G \subseteq A^\infty_G$.

Proof. Recall that the $AF$-algebra $A^\infty_G$ corresponds to an “infinite prime”. Such a prime relates to the “finite prime” $p$ via the following short exact sequence:

$$0 \to p\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$  \hspace{1cm} (18)

Thus one gets to an extension of the $AF$-algebras of the form:

$$0 \to A^p_G \to A^\infty_G \to \mathbb{Z}/p\mathbb{Z} \to 0,$$  \hspace{1cm} (19)

where $A^p_G \hookrightarrow A^\infty_G$ is an embedding (injective homomorphism). In particular, $A^p_G \subseteq A^\infty_G$, where the non-strict inclusion corresponds to the case $p = \infty$. Lemma 3 follows.

Lemma 4 $A_G \cong \prod'_{p \leq \infty} A^p_G$.

Proof. Let $K$ be a number field. The ring of adeles $A_K$ of $K$ can be written in the form

$$A_K \cong K \otimes \prod_{p \leq \infty} \mathbb{Z}_p \cong \prod'_{p \leq \infty} K_p,$$  \hspace{1cm} (20)

where $K_p$ denotes the $p$-adic completion of $K$ at a prime ideal $p$ and $\prod'$ is a restricted product of all but a finite number of the (ramified) ideals. If $G$ is a reductive group, one obtains from (20)

$$G(A_K) \cong \prod'_{p \leq \infty} G(K_p).$$  \hspace{1cm} (21)

The group algebras corresponding to (21) can be written in the form

$$\mathbb{C}[G(A_K)] \cong \prod'_{p \leq \infty} \mathbb{C}[G(K_p)].$$  \hspace{1cm} (22)

But $\mathbb{C}[G(A_K)] \cong A_G$ and $\mathbb{C}[G(K_p)] \cong A^p_G$; thus one gets an isomorphism

$$A_G \cong \prod'_{p \leq \infty} A^p_G.$$  \hspace{1cm} (23)

Lemma 4 is proved.

Remark 4 The $A_G$ is an $AF$-algebra, because the product of (finite or infinite) number of the $AF$-algebras is also an $AF$-algebra. However, an elegant description of the $A_G$ is an open problem.

Theorem 1 follows from lemmas 2-4.
3.2 Proof of theorem

Notice that by lemma 3 we have an inclusion \( \mathcal{A}_G^p \subseteq \mathcal{A}_G^\infty \) and by lemma 4 there exists an isomorphism \( \mathcal{A}_G \cong \prod_{p \leq \infty} \mathcal{A}_G^p \). Therefore, it is sufficient to construct an embedding

\[
\mathcal{A}_V \hookrightarrow \mathcal{A}_G^\infty.
\] (24)

To prove (24) we shall use the Pimsner Embedding Theorem of the crossed product \( \mathcal{A}_V = C(V) \rtimes \mathbb{Z} \) into an AF-algebra \( \mathcal{A} \) [Pimsner 1983, Theorem 7]. The \( G \)-coherence of variety \( V \) will imply the Morita equivalence between the AF-algebras \( \mathcal{A} \) and \( \mathcal{A}_G^\infty \). Let us pass to a detailed argument.

Let \( V \) be a \( G \)-coherent variety defined over \( \mathbb{C} \). Following [Pimsner 1983] we shall think of \( V \) as a compact metrizable topological space \( X \). Recall that for a homeomorphism \( \varphi : X \to X \) the point \( x \in X \) is called non-wandering if for each neighborhood \( U \) of \( x \) and every \( N > 0 \) there exists \( n > N \) such that

\[
\varphi^n(U) \cap U \neq \emptyset.
\] (25)

(In other words, the point \( x \) does not “wander” too far from its initial position in the space \( X \).) If each point \( x \in X \) is a non-wandering point, then the homeomorphism \( \varphi \) is called non-wandering.

Let \( \sigma : V \to V \) be an automorphism of finite order of the \( G \)-coherent variety \( V \), such that the representation (6) is \( * \)-coherent. Then the crossed product

\[
\mathcal{A}_V = C(V) \rtimes_{\sigma} \mathbb{Z}
\] (26)

is the Serre \( C^* \)-algebra of \( V \). Since \( \sigma \) is of a finite order, it is a non-wandering homeomorphism of \( X \). In particular, the \( \sigma \) is a pseudo-non-wandering homeomorphism [Pimsner 1983, Definition 2]. By a result of Pimsner, there exists a unital (dense) embedding

\[
\mathcal{A}_V \hookrightarrow \mathcal{A},
\] (27)

where \( \mathcal{A} \) is an AF-algebra defined by the homeomorphism \( \varphi \) [Pimsner 1983, Theorem 7].

Let us show that the algebra \( \mathcal{A} \) is Morita equivalent to the AF-algebra \( \mathcal{A}_G^\infty \). Indeed, the embedding (27) induces an injective homomorphism of the \( K_0 \)-groups

\[
K_0(\mathcal{A}_V) \hookrightarrow K_0(\mathcal{A}).
\] (28)
As explained in Section 2.2, the map (28) defines an inclusion

\[ H^i_{tr}(V) \subseteq K_0(\mathbb{A}). \] (29)

On the other hand, the trace cohomology of a \( G \)-coherent variety \( V \) must satisfy an inclusion

\[ H^i_{tr}(V) \subseteq K_0(\mathbb{A}_G^{\infty}). \] (30)

Let \( b^* = \max_{0 \leq i \leq 2n} b_i \) be the maximal Betti number of variety \( V \); then in formulas (29) and (30) the inclusion is an isomorphism, i.e. \( H^i_{tr}(V) \cong K_0(\mathbb{A}) \) and \( H^i_{tr}(V) \cong K_0(\mathbb{A}_G^{\infty}) \). One concludes that

\[ K_0(\mathbb{A}) \cong K_0(\mathbb{A}_G^{\infty}). \] (31)

In other words, the \( AF \)-algebras \( \mathbb{A} \) and \( \mathbb{A}_G^{\infty} \) are Morita equivalent. Formula (27) implies a dense embedding \( \mathcal{A}_V \hookrightarrow \mathbb{A}_G^{\infty} \) and therefore an embedding \( \mathcal{A}_V \hookrightarrow \mathbb{A}_G \). Theorem 2 is proved.

### 3.3 Proof of corollary 1

The proof of Conjecture 1 is an observation that for each \( 0 \leq i \leq 2n \), the Frobenius action \( \sigma(Fr_p^i) : H^i_{tr}(V) \to H^i_{tr}(V) \) (i.e. a motivic data) extends to a Hecke operator \( T_p : K_0(\mathbb{A}_G^{\infty}) \to K_0(\mathbb{A}_G^{\infty}) \) (i.e. an automorphic data); such an extension is possible because \( H^i_{tr}(\mathcal{A}_V) \subseteq K_0(\mathbb{A}_G^{\infty}) \) by theorem 2. Let us pass to a detailed argument.

Recall that the Frobenius map on the \( i \)-th trace cohomology of variety \( V \) is given by an integer matrix \( \sigma(Fr_p^i) \in GL_{b_i}(\mathbb{Z}) \), where \( b_i \) is the \( i \)-th Betti number of \( V \); moreover,

\[ |V(\mathbb{F}_p)| = \sum_{i=0}^{2n} (-1)^i \text{tr} \sigma(Fr_p^i), \] (32)

where \( V(\mathbb{F}_p) \) is the reduction of \( V \) modulo a good prime \( p \). (Notice that (32) is sufficient to calculate the Hasse-Weil \( L \)-function \( L(s, V) \) of variety \( V \) via equation (7); hence the map \( \sigma(Fr_p^i) : H^i_{tr}(V) \to H^i_{tr}(V) \) is motivic.)

**Definition 3** Denote by \( T_p^i \) an endomorphism of \( K_0(\mathbb{A}_G^{\infty}) \), such that the diagram in Figure 2 is commutative, where \( \iota \) is the embedding (4). By \( \mathfrak{s}_i \) we understand an algebra over \( \mathbb{Z} \) generated by the \( T_p^i \in \text{End}(K_0(\mathbb{A}_G^{\infty})) \), where \( p \) runs through all but a finite set of primes.
Figure 2: The Hecke operator \( T_p^i \).

**Remark 5** The algebra \( \mathfrak{H}_i \) is commutative. Indeed, the endomorphisms \( T_p^i \) correspond to multiplication of the group \( K_0(\mathfrak{A}_G^\infty) \) by the real numbers; the latter commute with each other. We shall call the \( \{ \mathfrak{H}_i \mid 0 \leq i \leq 2n \} \) an \( i \)-th Hecke algebra.

**Lemma 5** The algebra \( \mathfrak{H}_i \) defines an irreducible representations \( \pi_i \) of the group \( G(\mathbb{A}_K) \).

*Proof.* Let \( f \in L^2(G(K)\backslash G(\mathbb{A}_K)) \) be an eigenfunction of the Hecke operators \( T_p^i \); in other words, the Fourier coefficients \( c_p \) of function \( f \) coincide with the eigenvalues of the Hecke operators \( T_p^i \) up to a scalar multiple. Such an eigenfunction is defined uniquely by the algebra \( \mathfrak{H}_i \).

Let \( \mathcal{L}_f \subset L^2(G(K)\backslash G(\mathbb{A}_K)) \) be a subspace generated by the right translates of \( f \) by the elements of the locally compact group \( G(\mathbb{A}_K) \). It is immediate that the \( \mathcal{L}_f \) is an irreducible subspace of the space \( L^2(G(K)\backslash G(\mathbb{A}_K)) \); therefore it gives rise to an irreducible representation \( \pi_i \) of the locally compact group \( G(\mathbb{A}_K) \). Lemma 5 follows.

**Lemma 6** \( L(s, \pi_i) \equiv L^i(s, V) \).

*Proof.* Recall that the function \( L^i(s, V) \) can be written as

\[
L^i(s, V) = \prod_p \left( \det \left[ I_n - \sigma(Fr_p^i) P^{-s} \right] \right)^{-1},
\]

(33)

where \( \sigma(Fr_p^i) \in GL_{b_i}(\mathbb{Z}) \) is a matrix form of the action of \( Fr_p^i \) on the trace cohomology \( H^i_{tr}(V) \).
On the other hand, from (10) one gets

\[ L(s, \pi_i) = \prod_p \left( \det \left[ I_n - [A_p^i]p^{-s} \right] \right)^{-1}, \]  

(34)

where \([A_p^i] \subset GL_n(C)\) is a conjugacy class of matrices corresponding to the irreducible representation \(\pi_i\) of the group \(G(A_K)\). It is easy to see, that the \([A_p^i]\) is the conjugacy class of the action of the Hecke operator \(T_p^i\) on the group \(K_0(A_G^\infty)\).

But the action of \(T_p^i\) is an extension of the action of \(\sigma(Fr_p^i)\) on \(H_{tr}^i(V)\), see Figure 2. Therefore

\[ \sigma(Fr_p^i) = [A_p^i] \]  

(35)

for all but a finite set of primes \(p\). Comparing formulas (33)-(35), we get that \(L(s, \pi_i) \equiv L^i(s, V)\). Lemma 6 follows.

Corollary 1 follows from lemma 6 and formula (9).

4 Example

We shall illustrate theorems 1, 2 and corollary 1 for the locally compact group \(G \cong SL_2(A_K)\), where \(K = \mathbb{Q}(\sqrt{D})\) is a real quadratic field and \(D > 0\) is a square-free integer.

**Proposition 1** \(K_0(A_G^\infty) \cong \mathbb{Z} + \mathbb{Z}\omega\), where

\[ \omega = \begin{cases} 
\frac{1 + \sqrt{D}}{2}, & \text{if } D \equiv 1 \text{ mod } 4, \\
\sqrt{D}, & \text{if } D \equiv 2, 3 \text{ mod } 4.
\]  

(36)

*Proof.* If \(\varepsilon\) is the fundamental unit of the field \(K\), then multiplication by \(\varepsilon\) defines an automorphism \(\phi_\varepsilon : K \to K\). Keeping the notation of Section 3.1, one gets a commutative diagram shown in Figure 3, where \(K_0(A_G^\infty) \cong \mathbb{Z} + \mathbb{Z}\theta\) for an irrational number \(\theta\). But \(\tilde{\phi}_\varepsilon\) is a non-trivial automorphism of \(K_0(A_G^\infty)\); this can happen if and only if \(\theta\) is a quadratic irrationality, such that \(\theta = \omega\). Proposition 1 follows.

**Proposition 2** The \(G\)-coherent variety of the group \(G \cong SL_2(A_K)\) is an elliptic curve with complex multiplication \(E_{CM} \cong \mathbb{C}/\mathcal{O}_k\), where \(\mathcal{O}_k\) is the ring of integers of the imaginary quadratic field \(k = \mathbb{Q}(\sqrt{-D})\).
Figure 3: Calculation of the group $K_0(A_K^\infty)$.

Proof. Recall that a non-commutative torus $\mathcal{A}_\theta$ is a $C^*$-algebra generated by the unitary operators $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$ for a constant $\theta \in \mathbb{R}$ [Rieffel 1990] [13]. It is known, that the Serre $C^*$-algebra of an elliptic curve $\mathcal{E}_\tau \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ is isomorphic to $\mathcal{A}_\theta$ for any $\{\tau \mid \text{Im } \tau > 0\}$ and if $\tau \in O_k$ then

$\begin{align*}
H^0_{tr}(\mathcal{E}_{CM}) &= H^2_{tr}(\mathcal{E}_{CM}) \cong \mathbb{Z}, \\
H^1_{tr}(\mathcal{E}_{CM}) &\cong \mathbb{Z} + \mathbb{Z} \omega,
\end{align*}$

(37)

see [10]. Comparing formulas (36) and (37), one concludes that $H^i_{tr}(\mathcal{E}_{CM}) \subseteq K_0(A_K^\infty)$, i.e. the $\mathcal{E}_{CM}$ is a $G$-coherent variety for the locally compact group $G \cong SL_2(A_K)$. Proposition 2 is proved.

Remark 6 Theorem 2 implies an embedding $A_\omega \hookrightarrow A_K^\infty$. Such an embedding was originally proved by [Pimsner & Voiculescu 1980] [12] for any $\theta \in \mathbb{R}$. This observation was a starting point of our study.

Proposition 3 $L(s, \mathcal{E}_{CM}) \equiv \frac{L(s, \pi_1)}{L(s, \pi_0) L(s, \pi_2)}$, where $\pi_i$ are irreducible representations of the locally compact group $SL_2(A_K)$.

Proof. The motivic (Hasse-Weil) $L$-function of the $\mathcal{E}_{CM}$ has the form:

$\begin{align*}
L(s, \mathcal{E}_{CM}) &= \prod_p \left[ \frac{\det (I_2 - \sigma(Fr_p^1)p^{-s})^{-1}}{\zeta(s)\zeta(s-1)} \right], \\
&= \zeta(s), \\
&= \zeta(s-1),
\end{align*}$

(38)

where $\zeta(s)$ is the Riemann zeta function and the product is taken over the set of good primes; we refer the reader to formula (33). It is immediate that

$\begin{align*}
L(s, \pi_0) &= \zeta(s), \\
L(s, \pi_2) &= \zeta(s-1),
\end{align*}$

(39)
where \( L(s, \pi_0) \) and \( L(s, \pi_2) \) are the automorphic \( L \)-functions corresponding to the irreducible representations \( \pi_0 \) and \( \pi_2 \) of the group \( SL_2(\mathbb{A}_K) \). An irreducible representation \( \pi_1 \) gives rise to an automorphic \( L \)-function

\[
L(s, \pi_1) = \prod_p (\det [I_2 - [A_p^1]^p^{-s}])^{-1}.
\]

But formula (35) says that \([A_p^1] = \sigma(Fr_p^1)\) and therefore the numerator of (38) coincides with the \( L(s, \pi_1) \). Proposition 3 is proved.

Remark 7 Proposition 3 can be proved in terms of the Grössencharacters; we refer the reader to [Silverman 1994] [16], Chapter II, §10.

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