TOPICAL REVIEW

Nonequilibrium mesoscopic transport: a genealogy

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Abstract
Models of nonequilibrium quantum transport underpin all modern electronic devices, from the largest scales to the smallest. Past simplifications such as coarse graining and bulk self-averaging served well to understand electronic materials. Such particular notions become inapplicable at mesoscopic dimensions, edging towards the truly quantum regime. Nevertheless a unifying thread continues to run through transport physics, animating the design of small-scale electronic technology: microscopic conservation and nonequilibrium dissipation. These fundamentals are inherent in quantum transport and gain even greater and more explicit experimental meaning in the passage to atomic-sized devices. We review their genesis, their theoretical context, and their governing role in the electronic response of meso- and nanoscopic systems.

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1. Introduction
As soon as it became possible to fabricate conducting devices at scales comparable to the scattering mean free paths of the underlying material, experiments began to reveal some remarkable departures from bulk electrical-response behaviour. Among the earliest notable phenomena were universal conductance fluctuations [1], followed by the first measurements of conductance quantization itself, probed as a function of carrier density in the device [2, 3].

These novel effects highlighted the inadequacy of standard theoretical techniques, which had proved themselves extraordinarily effective in the rapid development of solid-state electronics right up to the 1980s. Typical of these older procedures is impurity-site averaging for conductance calculations in the bulk; relying as it does on an assumption of homogeneous randomness, spatial averaging breaks down when the conducting channel of an actual sample is too small to hold more than a few impurities, if any at all.

A second, more basic, instance is the limit to the applicability of Bloch’s theorem. Standard band-structure analysis fails in the absence of lattice regularity involving, again, an assumption of large-scale uniformity. Few of the new types of structure are defined by this level of periodicity.

Of prime significance to mesoscopic transport is the openness and intimacy of contact between the nominally active region and its macroscopic environment. These complements—the mesoscopic structure and its surroundings—are no longer subject to conventional approx-
formulated by Landau and Silin [23].

The subject of the quantum Boltzmann equation (QBE) is the mean distribution of carriers as it changes over time $t$ within the single-particle configuration space for position and wavevector $(r, k)$:

$$
\left[ \frac{\partial}{\partial t} + v_k \cdot \frac{\partial}{\partial r} + \frac{F(r, t)}{\hbar} \cdot \frac{\partial}{\partial k} \right] f_k(r, t)
= - \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}}(k; r, t). \tag{1}
$$

With group velocity $v_k$ in the carrier band and the force field $F(r, t)$ acting on each carrier, the left-hand side of the equation gives the convective derivative: the intrinsic or proper rate of change of the mean particle distribution $f_k(r, t)$ in a frame co-moving with the semiclassical flow. The right-hand quantity in equation (1) accounts for the loss (net outflow from the state labelled $k$) via the microscopic scattering processes active in the system.

The quantum Boltzmann equation provided the kinetic template for successive, more general nonequilibrium theories. The key to its enduring success lies in Boltzmann’s original formulation of the collision term and its method of construction. But, before anything else can be said about collision physics, conservation must be set in place as the prime, overarching physical constraint.

The unconditional requirement for conservation emerges when the momentum dependence of the one-body distribution is integrated over, to obtain the local number density and flux; respectively,

$$
\rho(r, t) = \Omega^{-1} \sum_k f_k(r, t) \quad \text{and} \quad j(r, t) = \Omega^{-1} \sum_k v_k f_k(r, t), \tag{2}
$$

with $\Omega$ a volume cell, centred at $r$, over which the band structure is (locally) defined. Applying the trace operation of equation (2) to each side of the QBE produces

$$
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} \cdot j = -\Omega^{-1} \sum_k \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}}(k; r, t). \tag{3}
$$

In a system closed to particle exchange with external reservoirs (as for a working electrical circuit) the continuity equation, equation (3) above, must be source-free. The carrier number density is microscopically conserved if, and only if, the trace sum on the right-hand side of the relation vanishes identically to lead to the equation of continuity.

Why is this the crucial fact, and why even recall such an ‘elementary’ given? First, it is crucial to each and every model because, otherwise, charge could appear and disappear arbitrarily and beyond the model’s control. Nor does conservation come cheaply, in terms of intellectual effort. Second, no matter how cosmetically attractive its other features might be, a scheme that was unable to account for its microscopic conservation properties would face a serious credibility problem, practically as well as formally.

The primacy of conservation and continuity leaves two options for progress. To begin with, microscopic analysis of the Boltzmann collision term will guarantee its vanishing...
trace. Let us apply the Uehling–Uhlenbeck construction of the scattering rate to Fermi particles [19, 21]. The term corresponding to collisions of, say, individual electrons with a set of external scattering agents (the particle–particle collision integral is more complex, but analogously structured) is

\[
\frac{df}{dt}_{\text{coll}} \equiv \sum_k \left[ (1 - f_k(r, t)) Q_k^{-1} \cdot \left( k' - k \right)(r, t) - (1 - f_k(r, t)) Q_k^{-1} \cdot \left( k - k' \right)(r, t) \right].
\]

where the form of the transition rate \( Q_k^{-1} \cdot \left( k' - k \right) \) carries the collision physics.

Equation (4) differs from the classical through its explicit allowance for Pauli blocking of the outgoing scattering states, a quintessentially nonclassical effect and the most overt of the quantum phenomena entering into normal metallic transport. The leading contribution to the right-hand side sum sends a particle out of state \( k \) to state \( k' \). Its counter-term accounts for the inverse process whereby a new carrier enters state \( k \) out of \( k' \).

The scattering rate is local in space and time, consistent with the long-wavelength time-local (Markov) structure of the QBE’s left-hand side. Other than Pauli exclusion, all the short-range correlations or coherences in time and space, on the scale of the Fermi frequency and wavelength, are subsumed in \( Q_k^{-1} \cdot \left( k' - k \right) \). These ‘intra-collisional’ quantum effects do not directly influence the QBE’s kinetic shape.

The collision term on the right of equation (4) is traceless. It satisfies conservation automatically but, in full form, it can pose technical difficulties to an effective solution. The level of practical effort worth spending on it naturally depends on the fidelity we require in the answer. Even with large computer resources the competing demands of inhomogeneity, realistic and complex band structures, and so on may justify a simplified approach to the scattering properties of the QBE as such.

At this point the alternative option offers itself. We can try for a much simpler Ansatz for the collision integral. If the new structure fails to vanish identically, however, conservation may have to be imposed as an additional constraint, no longer given for free.

An early and still popular collision approximation is the Drude model, shown here in its most basic form:

\[
\frac{df}{dt}_{\text{coll}} \equiv \frac{1}{\tau_{\text{el}}} (f_k(r, t) - f_{k, f}(r, t))_{E_k=E_k}\]

\[+ \frac{1}{\tau_{\text{in}}} (f_k(r, t) - f_{k, f}(r, t)).\]

The elastic and inelastic collision times \( \tau_{\text{el}} \) and \( \tau_{\text{in}} \) reflect, in a coarse-grained way, the more complex physics of the original scattering prescription, equation (4). The Drude collision times can depend on the single-particle energy \( E_k \); here we choose \( \tau_{\text{el}}, \tau_{\text{in}} \) as constants, purely to streamline the discussion. The elastic and inelastic collision times \( \tau_{\text{el}} \) and \( \tau_{\text{in}} \) reflect, in a coarse-grained way, the more complex physics of the original scattering prescription, equation (4). The Drude collision times can depend on the single-particle energy \( E_k \); here we choose \( \tau_{\text{el}}, \tau_{\text{in}} \) as constants, purely to streamline the discussion.

The equilibrium distribution \( f_{k, f}^{\text{eq}} \) within the inelastic term, second on the right of equation (5), is the usual Fermi–Dirac function in terms of the thermal energy \( k_B T \) and global chemical potential \( \mu \), as fixed by the environment:

\[
f_{k, f}^{\text{eq}}(r, t) \equiv (1 + \exp((E_k + u(r, t) - \mu)/(k_B T)))^{-1},
\]

where \( u(r, t) \) is an effective local potential, to be determined. In the elastic contribution, first on the right of equation (5), the expectation \( \langle f_k^{\text{el}} \rangle_{E_k=E_k} \) averages \( f \) over all possible wavevector orientations, on the assumption that an elastic collision randomizes the exit wavevector completely, instantaneously, and locally but without energy exchange between carrier and scatterer.

As with the elastic component, the assumption behind the inelastic Drude term is that the collision instantaneously and totally randomizes the perturbed distribution, resetting it to (local) equilibrium. In the process the mean energy density of the carriers is locally changed, but the particle density cannot change. This is precisely because all scattering processes are conceived as strictly localized in real space. The need for conservation forecloses any option for a discontinuous change in local carrier number.

A conserving procedure for the inelastic Drude term was originally proposed by Greene et al [24] and independently by Mermin [25]. The local potential \( u(r, t) \) is determined by demanding that the trace of the inelastic term should be zero for all \((r, t)\). For the actual system density \( \rho(r, t) \) this requires that

\[
\Omega^{-1} \sum_k f_{k}^{\text{eq}} = \rho(r, t).
\]

Continuity adds a consistency condition for \( u(r, t) \) as, from equation (7), it further demands that

\[
\frac{\partial \rho}{\partial t} = \Omega^{-1} \sum_k \left( -\frac{\partial f_{k}^{\text{eq}}}{\partial \mu} \frac{\partial \mu}{\partial t} \right) = -\frac{\partial u}{\partial t} = -\frac{\partial}{\partial t} \cdot \mathbf{j};
\]

so

\[
\frac{\partial u}{\partial t} = \frac{\partial \mu}{\partial \rho} \Omega^{-1} \sum_k \mathbf{v}_k \cdot \frac{\partial}{\partial \mathbf{r}} f_{k}(r, t). \tag{8}
\]

Equation (8) defines the local time dependence of the effective equilibrium, subject to the model’s assumption that all relaxation events are instantaneous.

At global equilibrium we have \( \mathbf{F}(r) = -\partial u/\partial r \) and the Fermi–Dirac distribution of equation (6) exactly satisfies the stationary, collisionless QBE. Away from equilibrium, the local structure of \( f_{k}^{\text{eq}}(r, t) \) forces the solution to be conserving.

This is a generic requirement on \( f^{\text{eq}} \) and brings in the principle of detailed balance for equation (4). For all state labels \( k, k' \), the identity

\[
\langle f_{k}^{\text{eq}} Q_{k' \leftarrow k}(1 - f_{k}^{\text{eq}}) \rangle = \langle 1 - f_{k}^{\text{eq}} Q_{k \leftarrow k' f_{k}^{\text{eq}}} \rangle
\]

holds. Equivalently,

\[
Q_{k' \leftarrow k} = e^{-(E_{k' f_{k}^{\text{eq}}}) (k_B T)} Q_{k \leftarrow k' f_{k}^{\text{eq}}}
\]

The requirement of detailed equilibrium between individual collision contributions reveals that the Boltzmann rates involve more than simply knowing the set of elementary scattering cross-sections. Their explicit dependence on the thermal energy tells us that, unless the scattering is strictly elastic, many-body interactions in the system—at minimum, those coupling to the huge numbers of excitable modes in the heat bath—must enter at the microscopic level. It is a deeper manifestation of the same physics that underlies Brownian motion, intimately associated with both fluctuations and dissipation.
By a continuous and local resetting of the effective (though not the global) chemical potential $\mu = u(r, t)$, the revised inelastic Drude Ansatz obeys the zero-trace requirement at every step of the model system’s evolution. (Physically it also conforms to a thermodynamic rule of thumb: a disturbed system’s eventual global relaxation tends first to pass rapidly to a temporary, locally defined equilibrium.)

With continuity now well controlled, the local equilibrium function forms the physical input to the otherwise purely formal solution of the full nonequilibrium distribution function. However, there is a cost. The procedure introduces a self-consistent loop into the calculation since, through $\rho$ and therefore $u$, the reference equilibrium distribution is itself a functional of the actual distribution to be solved.

Here a further consideration appears: regardless of its inner detail, the QBE standing alone does not contain enough structure to close the physical transport problem. The computational loop requires a connection between the one-body nonequilibrium density $\rho$ and the locally induced force field $F$. In the context of electron transport, closure comes through the constitutive relation, that is the Poisson equation, tying $u$ to $\rho$ and generating the local induced electrostatic field, adjacent to the externally supplied driving field.

Conservation has to be taken seriously. If so, it is true that simplifying the collisional part of a transport problem is offset by the extra effort to obtain the auxiliary distribution. Nevertheless, as a practical matter, self-consistency cannot be avoided. For systems with nonuniform charge distributions (including almost every realistic mesoscopic case) kinetics and electrostatics become inextricable. A familiar example is the ‘pinch-off’ phenomenon at the heart of a field-effect transistor’s action.

2.2. Kubo formula

Kubo [26, 28, 29] established a very general transport framework complementary to earlier kinetic methods. To appreciate the difference we recapitulate the goals of each perspective.

In both classical and quantum kinetics one starts from the abstract but complete multi-particle description in the system’s phase space, and takes expectation values in the dynamical (Liouville) equation. Statistical averages are first performed over the general evolution, before any particular observable is identified for solution, and then subsets of the internal variables are further removed by systematic integration; a simple illustration is the reduction of equations (1)–(3).

The result is an enormous nesting of reduced but strongly coupled equations of motion for all the expectation values. To terminate the otherwise unmanageable hierarchy, higher-order many-body expectations must be factorized, in some physically guided way, and modelled as functionals of the lower-order ones. The classic instance of this is the ‘Stosszahlansatz’ of Boltzmann: the stochastic argument leading to the characteristic shape of his collision terms.

As section 2.1 shows, it is essential for any such averaging to preserve continuity, first and foremost at the single-carrier level (analogous higher-order identities, or sum rules [23], also exist). That this is a nontrivial matter is evident in the careful steps needed to make even the ‘simple’ Drude model microscopically conserving.

By contrast, although in Kubo’s method one also starts with the system’s full quantum dynamical equation—given the Hamiltonian—the quantum commutation relations are first applied to it systematically. This has the effect of projecting the single-particle current operator, for example, uniquely into an equivalent, two-particle quantum correlation operator. Only after this mathematical procedure are physical averages taken on each side of the exact relation. This prescribes the specific relation between the mean, measurable current response and the mean (in principle also measurable) current–current correlation.

In summary, a kinetic equation works directly by functionally linking expectation values of particle correlations, operating at distinct orders in powers of the carrier density. This is possible only by first making a (conserving) approximation for the higher-level distributions in terms of the lower ones. For a Kubo formula, every such correlation stays on an equal footing. Expectations strictly come after the dynamical relationship between an observable and its next-higher-order analogue has been singled out. Ensemble averaging then projects both objects out of the original, complete Hamiltonian evolution.

We now consider the eigenstates of the Hamiltonian $H_0$ for a many-particle system of interest. An external time-dependent perturbation $H'(t) \equiv \Theta(t)U(t)A$ begins to act at $t = 0$—hence the step function $\Theta(t)$—such that an external field $U(t)$ couples to a system observable $A$. In charge transport, $A$ could be the density or the flux-density operator.

A single assumption is made in the Kubo analysis: the characteristic energy $\langle H'(t) \rangle$ must be small enough (compared, say, to the free energy) to justify a finite, and in practice linear, perturbation expansion in powers of $H'(t)$. For $t > 0$ the perturbation preserves a one-to-one mapping between the basis states of $H_0$ and those of the complete Hamiltonian $H(t) = H_0 + H'(t)$. The dynamical states remain orthonormal but, to linear order, changes in their mutual orientations in Hilbert space (relative to the initial basis) can be shown to cancel exactly in the time-dependent expectation of any observable $B$ in the Heisenberg picture:

$$\langle B(t) \rangle = \left( \sum_n e^{-E_n(N)/k_B T} \right)^{-1} \times \sum_n \langle \psi_n | B(t) | \psi_n \rangle e^{-E_n(N)/k_B T},$$

with

$$B(t) \equiv \exp \left( \frac{i}{\hbar} \int_0^t H(t') \, dt' \right) B \exp \left( -\frac{i}{\hbar} \int_0^t H(t'') \, dt'' \right).$$

The trace in equation (9) runs over the equilibrium distribution for the whole $N$-particle system. Only those terms directly involving the time development of $B(t)$ contribute, since the expectations themselves are simply traces over the unperturbed eigenstates, populated at thermal equilibrium.
The convolution expressed in equation (10) simplifies on expanding its pair of unitary evolution operators to linear order to arrive at

$$B(t) \to e^{iH_0 t/\hbar} \left( 1 + \frac{i}{\hbar} \int_0^t e^{-iH_0 t'/\hbar} H'(t') \, dt' e^{iH_0 t'/\hbar} \right) \times B \left( 1 - \frac{i}{\hbar} \int_0^t e^{-iH_0 t'/\hbar} H'(t') \, dt' e^{iH_0 t'/\hbar} \right) e^{-iH_0 t/\hbar}$$

$$\equiv B_0(t) + \frac{i}{\hbar} e^{iH_0 t/\hbar} \times \int_0^t dt' \langle \hat{H}(t') B - B \hat{H}(t') \rangle e^{-iH_0 t'/\hbar}, \quad (11)$$

where $B_0(t) \equiv e^{iH_0 t/\hbar} B e^{-iH_0 t/\hbar}$, $\hat{H}(t) \equiv e^{-iH_0 t/\hbar} H(t) e^{iH_0 t/\hbar}$.

Expectations are taken on both sides. Since $\langle B_0(t) \rangle$ keeps its original equilibrium value (normally, this vanishes) we focus on the linear departure of the expectation

$$\langle \delta B(t) \rangle \equiv \langle B(t) - B_0(t) \rangle.$$

The cyclic trace property $(XY) = (YX)$ allows the exponentials in $\hat{H}(t)$ to be relocated within the right-hand side of equation (11) to yield the Kubo formula

$$\langle \delta B(t) \rangle = \frac{i}{\hbar} \int_0^t dt' U(t')[\{A, B_0(t')\}]. \quad (12)$$

The formula reveals a direct and deep equivalence between the one-body average response, such as current, on its left-hand side and a corresponding two-body correlation function within the integral on the right.

The significance of equation (12) is that it is universal for normal systems (those with a stable free-energy minimum). To make it concrete, consider a simple, circular one-dimensional (1D) metallic loop of length $L$, threaded by an even magnetic flux changing linearly in time. The flux induces an electric field $E(t)$ everywhere on the perimeter. The electromagnetic Hamiltonian couples the vector potential $-\hat{E}(t)$, unique up to a conservative gauge term, to the current operator $I(x) = q\dot{q}(x)$. The expectation of the current itself responds according to

$$\langle \delta I(x, t) \rangle = -\frac{q^2 E}{\hbar} \int_0^t dt' \int_0^L dx' \langle \{j_0(x', t'), j_0(x, t)\} \rangle \quad (13)$$

from which the system conductance can be defined (uniformity means that $\langle I(x, t) \rangle$ in equation (13) does not depend on $x$) as

$$G(t) = \frac{\langle \delta I(t) \rangle}{E L} \equiv \frac{q^2}{\hbar} \int_0^t dt' \langle C(t') \rangle \quad \text{for}$$

$$C(t') \equiv -i \int_0^L dx \langle \{j_0(x, 0), j_0(0, t')\} \rangle. \quad (14)$$

Equation (14) is much more general that its restricted derivation above suggests. Moreover it carries special implications for one-dimensional mesoscopic transport. The time integral involving the correlator $C$ is dimensionless, and behaves in a way closely similar to the dimensionless transmission factor of the Landauer model for quantization of $G$ in one-dimensional ballistic conductors (see below). For further insight, we cite the Kubo analysis of Kameyev and Kohn [30] for closed driven mesoscopic circuits, including the dominant Coulomb-screening effects expected in realistic nonuniform cases.

Another perspective on equation (14) comes from its long-time limit. Empirically, a driven normal conductor will always reach steady state, usually very rapidly. Therefore $\int_0^\infty \langle C(t) \rangle \, dt$ should be finite; all transients will have fully contributed to the time integral.

There is a critical proviso. The Hamiltonian must include one or more scattering mechanisms of a many-body nature: carrier–phonon terms, or carrier–carrier, or both. If $H_0$ is strictly a single-particle object with no dynamical coupling to collective excitations, it is straightforward to show that equation (14) will not have a well-defined asymptote unless an infinitesimally damped, but nevertheless phenomenological, factor $e^{-\eta t}$ is adjoined to $C(t)$.

What does this mean? If $H_0$ does not explicitly include many-body contributions of some kind, the long-time behaviour of the product $\langle \{j_0(x, 0), j_0(0, t')\} \rangle$ cannot guarantee convergence to a well-defined steady state. Stabilizing it with a formal ad hoc damping factor (in fact this is the Fermi golden rule in disguise) is tantamount to mimicking damping effects that could not otherwise emerge except by multi-particle excitations; compare the analysis given by Hershfield [31].

How does natural many-body damping appear in the Kubo formula? The time dependence of the current operator $j_0(0, t)$ leads to the spreading out, among more and more collective modes, of its self-overlap in the correlator. In fashionable parlance, the injected single-carrier states become progressively ‘entangled’ with the far more abundant set of multi-particle states within the environment.

At the quantum level, therefore, propagation is not by single carriers but by entangled clusters. Furthermore this entanglement is as much kinematic (via Pauli exclusion and microscopic conservation) as it is dynamic (via interactions). It follows that many-body arguments are necessarily implicated and that any analysis ought not to pass them by.

The act of taking the trace average destroys the coherences that remain encoded (albeit inexorably entangled) in the commutator. The observable result is the actual decay of $C(t)$; and by far the most important idea to keep in mind for this mechanism, as it is manifested in $C(t)$, is that it always represents the evolution of dynamically correlated particle-pair states and not of autonomous single-particle entities. This is how conservation is maintained, while the indispensable processes of energy loss are quantified in terms of the rate of energy redistribution from the relatively few externally excited modes to the exponentially more numerous multi-pair excitations of the system. In a real scenario, the injected energy should be first spread out among the collective...
modes will be further dispersed (dissipated) by passage to the even larger thermal environment.

For a system with effective mass $m^*$, dimensional analysis of the correlator shows that its integral scales as

$$\int_0^\infty t C(t) \, dt = \frac{\hbar}{m^* L^2} n \tau$$

(15)

in which $n = \langle \rho \rangle$ is the mean carrier density and $\tau$ is a time constant that characterizes the magnitude of the integral. Clearly, while $\tau$ is the outcome of generally very complex kinetic events deep in the system, it quantifies the observable attrition rate due to averaging over the ‘landscape’ of all physically possible many-body excitations. The Kubo expression comes down to

$$G = \frac{q^2 n \tau}{m^* L^2},$$

recognizable as the classic Drude formula [32].

A clarification is in order. It is sometimes claimed that the Kubo prescription is an artifice, a somewhat academic result restricted to idealized systems where global charge transfer poses no issue [4]. The implication is that, to push it into useful form, extra assumptions foreign to the spirit of the problem are necessary.

That is not so. Passing over the fact that practical working circuits are electrically closed by definition, the Kubo formula extends with equal rigour to any sub-system of a normal conductor (having an absolutely stable ground state), no matter whether the structure under study has closed or open boundary conditions, is macroscopic or mesoscopic, homogeneous or inhomogeneous.

It can be argued that Kubo’s result is all the more powerful, applied to open structures with charge transfer to external reservoirs. This is because of the formula’s inbuilt and particularly stringent constraints on global, over and above simply local, charge conservation. Interested readers are referred to the electrodynamic analyses by Magnus and Schoenmaker [33] and Sols [34] in addition to [30].

2.3. Field-theoretical (Green-function) methods

The analysis of response and transport reaches its most complete form in many-body quantum kinetics, nowadays set in the formalism of quantum fields [23, 28, 29, 35, 36]. This is not the place to give more than a verbal sketch but at least it can be said that semiclassical kinetics (post-Boltzmann), the Kubo formalism, and much more are subsumed in the field-theoretical point of view, commonly also referred to as the Green-function approach.

As the single-particle distribution $f_\rho(r, t)$ is the fundamental quantity of the QBE, so the one-body density matrix

$$\rho(r, t; r', t') = -\Theta(t - t') \langle \psi(r, t) \psi^\dagger(r', t') \rangle$$

$$+ \Theta(t' - t) \langle \psi^\dagger(r', t') \psi(r, t) \rangle$$

(17)

becomes the fundamental one for the dynamical equations of quantum kinetic analysis. The expression above has the following interpretation. The Hilbert space of many-electron states is now extended to arbitrary numbers of electrons, and the statistical ensemble average $\langle \cdots \rangle$ traces over all of them.

Acting on this greatly augmented state space, the operator $\psi^\dagger(r', t')$ ‘creates’ a particle at the space–time point $(r', t')$ by projecting any $N$-particle state to another with $N + 1$ particles, one of which is initially at $r'$. In the same vein its conjugate $\psi(r, t)$ ‘annihilates’ an electron at $(r, t)$ and sends any $N$-electron state to some $N - 1$ state with an electron initially absent at $r$. The operators are cast in the Heisenberg picture, as with equation (10) above. Fermion antisymmetry means that, at equal times, they satisfy the anticommutation relation

$$\psi(r, t) \psi^\dagger(r', t) + \psi^\dagger(r', t) \psi(r, t) = 6(r - r').$$

In equation (17), the time ordering reverses the role and sign of its fermion operator pair: for $t > t'$ we add a single-electron excitation, later to be removed. This probes the overlap, or correlation, within the evolving one-body distribution and is the simplest measure (though theoretically insufficient for conductance) of the system’s relaxation during the interval $t - t'$. Conversely, for $t' > t$ the operation removes an electron—adds a hole—and restores it later to measure relaxation of the hole in the interacting-electron background. (The choice of time ordering is for convenience since, mathematically, a hole that evolves forwards in time looks just like an electron evolving ‘backwards’ in time.)

Given a system’s interacting Hamiltonian, its many-body Schrödinger equation leads to a conjugate pair of coupled equations of motion for $\rho(r, t; r', t')$. The objects $\psi$ and $\psi^\dagger$ act like quantum-field operators and $\rho$ has all the characteristics of the Green function, or ‘propagator’, for the coupled equations. This makes the language highly abstract; in compensation it also makes accessible, to practical condensed-matter transport calculations, some sophisticated techniques of field theory such as Feynman-diagram analysis [28] and Keldysh time-evolution contours [36].

As a two-point double-time quantity, the complexity of the density matrix exceeds that of the solution to the QBE. On the other hand, it necessarily encodes far more information: its evolution admits every possible quantum coherence effect, at every scale. Nevertheless, despite the quantum subtleties, the way in which the evolutionary character of $\rho$ is interpreted stays conceptually quite close to the older understanding that motivates the Boltzmann equation.

The quantum many-body equations’ kinematic complexity is only the beginning. The creation–annihilation pair within the density matrix couples to the Hamiltonian such that two-pair, three-pair and, in general, multi-pair excitations all enter and contribute to its space–time development. Their interaction within the density matrix appears as a hierarchy.
of convolutions which represent the full generalization of Boltzmann’s collision terms: there is a ‘scattering-out’ component, or self-energy, that removes particle strength from \( \rho \) and a ‘scattering-in’ component, or vertex, that tends to restore particle strength.

Self-energy and vertex parts are intricate functionals of the higher-order multi-pair excitations coupled to the original electron–hole pair (\( \psi \psi^\dagger \) or \( \psi^\dagger \psi \)) represented by \( \rho \). To satisfy particle conservation it is not only imperative for both parts to be retained in any approximate treatment of the dynamics but that they should be inter-related in a highly specific way, through the Ward identities [29]. Strategies for securing the essential conserving properties within approximate models of quantum transport are fully explored by Kadanoff and Baym [35]. Exactly the same considerations apply to the problem when posed in the language of Keldysh [36].

The Ward identities generalize the optical theorem of scattering [29], which asserts that the loss of energy and momentum from an initial freely propagating state must be made up by the total gain of all the other states in the Schrödinger problem. Since unitary scattering couples all such modes, their evolution must be solved jointly. In transport, such modes are physically occupied, so energy–momentum conservation becomes inherently a many-body problem. The Ward identities quantify this collective coupling subject to conservation, in a way similar to the optical theorem’s requirement on the underlying basis states. But while knowledge of the single-particle modes is a prerequisite to set up the many-body transport solution, it can never supplant it.

If the intrinsic particle–particle interactions are strong and/or long-ranged, normal perturbation theory will not work. The literature describes countless ingenious ways to deal with the internal correlations. It has a long pedigree, it is vast and it is still growing. For this discussion, the general references [23, 28, 29, 35, 36] must suffice.

Although internal interactions may be strong, one can still use linear response for the conductance of a metallic electron system, if driven by a weak enough external field. Under this condition, the Kubo formalism is replicated within the Green-function picture [29].

A completely quantum treatment of \( \rho \) is needed whenever the range of its quantum correlations begins to be reached by the physical size of the system or by the timescale for effects of interest; in less extreme applications the full description can be simplified. For example, to study processes whose rates are slow compared to the inverse Fermi frequency we can restrict the analysis to the density-matrix form

\[
\phi(r, r'; t) \equiv \lim_{\ell \to \ell^+} \rho(r, \ell; r', \ell') = \langle \psi^\dagger (r', \ell) \psi (r, \ell) \rangle.
\]

Electron correlations are regarded as instantaneous, while their spatial nonlocality, or off-diagonal structure, remains an essential feature. The overall time evolution of the system, analysed quasi-statically (relative to the electrons), recalls the Born–Oppenheimer decoupling of slow and fast motions within electronic systems. The dynamics of this limit has been studied in depth by Mahan [29].

The equal-time matrix \( \phi(r, r'; t) \) is governed by a so-called master equation, one version of which is the Lindblad equation [37]. Its form can be distilled from its original quantum dynamical setting, though not without additional assumptions extending the arguments of Uehling and Uhlenbeck to deal with the off-diagonal nature of the density matrix.

As a final step the quantum Boltzmann equation re-emerges on restricting attention to \( \phi \) in the neighbourhood of its diagonal elements \( \phi(R, R; t) \). From \( \phi \), Wigner’s distribution function can be defined as

\[
f_k(R, t) \equiv \int d^3r e^{-i k \cdot r} \phi(R + r, R - r, t).
\]

If \( \phi \) decreases sharply away from the central coordinate \( R \), the Wigner function behaves as a localized slowly varying quantity. Any remnant off-diagonal nonlocality is taken up into the dependence on wavevector \( k \). The latter spans a background electronic band structure, defined over some cell \( \Omega \) centred on \( R \), whose size is small on the typical scale of \( |R| \) yet still large compared to the Fermi wavelength of the electrons; if not, the band-structure picture would not make good physical sense. As derived by Kadanoff and Baym [35] (with some assumptions for the \( k \)-dependent self-energy and vertex structures destined to become the locally defined Boltzmann collision term) this limit of the Wigner function \( f \) becomes the semiclassical fermion occupancy, obeying a reduced equation identical to the QBE.

Distinct from the long-wavelength recovery of semiclassical kinetics, the original Wigner distribution can be studied and solved as a fully quantum distribution in its own right, equivalent to the equal-time density matrix and with its master equation transcribed to the Fourier domain. Wigner-function methods form an extensive body of work, particularly in high-field problems where Kubo and linear-response approaches lose their immediate utility. An instructive reference to this different transport methodology is Frensley [38].

2.4. Landauer formula

Anticipating the eventual creation of structures with feature sizes much below any bulk mean free path, Landauer [39] and subsequent developers of his approach [4–6] were motivated to understand conductance as a mesoscopic phenomenon with less emphasis on kinetic relaxation and more on a hydrodynamic (in the quantum limit, wave-like) concept of the interaction with scattering sources. The sources are treated not so much as mediators of dynamical many-body excitations but as modifiers of single-particle ballistic ‘flow’ which remains, at all events, microscopically free and undisturbed.

The analogy shifts from the staccato motion of colliding billiard balls towards the smooth geometry of a working aerofoil. One might say—with a bit of poetic licence—that the...
Landauer approach to mesoscopics aims to be holistic while standard ones are bound to remain thoroughly reductionist. In stressing a continuum-like rather than particulate formal description of transport, one feature of this viewpoint is a more diffuse theoretical link between the average response of a system and fluctuations about the average.

Landauer characterized his own viewpoint in part as ascribing conceptual parity to the roles of current and voltage [39]. Either should equally well be able to act as the causative agent of transport, with its partner representing the effect. This is radically different from the Kubo analysis, for example, where an external field has absolute meaning as the applied stimulus to conduction while the current is invariably the system’s causal response. Causality turns out to be intimately tied to the notion of dissipative relaxation.

It may appear ironic that the Landauer formula applied to 1D conductance does not actually require any novel assumptions, as introduced originally by Landauer, to establish it. To show that it does not, we derive it straightforwardly from canonical transport theory. The proof has several stages.

- **First**: we invoke the Kubo formula, equation (14), for the current traversing an open one-dimensional mesoscopic channel [34]: the working channel is a uniform wire smoothly embedded within a sub-system itself forming a segment of a macroscopically closed driven loop [30]. The sub-system encompasses the mesoscopic interface regions, since these are comparable in size to the channel’s conductive core. Even if the core is ballistic, scattering in the boundaries is a real and dominant effect. Equation (15) therefore applies.

- **Second**: existence of a steady state implies that the time integral of the correlator in equation (15) is finite. Therefore the conductance takes the form of equation (16):

\[
G = \frac{e^2 n}{m^* L} \frac{\bar{T}_{in} \bar{T}_{el}}{\bar{T}_{in} + \bar{T}_{el}},
\]

where, by Matthiessen’s rule, the stochastically independent collision rates \( \tau_{in}^{-1} \) and \( \tau_{el}^{-1} \) add together and lead to the composite collision time \( \tau = (\tau_{in}^{-1} + \tau_{el}^{-1})^{-1} \).

- **Third**: the sub-system’s operative length \( L \) must be determined by the physical environment of the open configuration for the core plus boundary regions. In a ballistic structure, the effective mean free paths are no longer uniquely set by the bulk material, but by the dynamics at the interfaces. This implies that \( L \) becomes, in effect, the maximum of the mean free paths \( \lambda_{in} \) and \( \lambda_{el} \), or any others that can be theoretically (and experimentally) distinguished over the span of the device.

Suppose the mean speed of the carriers within the system is \( v_0 \). Then

\[
L = \max(\lambda_{in}, \lambda_{el}) = v_0 \max(\tau_{in}, \tau_{el})
\]

so

\[
G = \frac{e^2 n}{m^* L} \frac{\lambda_{in} \lambda_{el}}{v_0 (\lambda_{in} + \lambda_{el})} = \frac{q^2 n}{m^* v_0 (\lambda_{in} + \lambda_{el})} \frac{\lambda_{in} \lambda_{el}}{\lambda_{in} + \lambda_{el}} = \frac{q^2 n}{m^* v_0} \left( 1 - \frac{\lambda_{in} - \lambda_{el}}{2(\lambda_{in} + \lambda_{el})} \right), \tag{19}
\]

- **Fourth**: in the degenerate limit, the characteristic speed of the carriers becomes the Fermi velocity; \( v_0 = v_F \). At the same time the density is given by \( n = 2k_F/\pi \), where \( k_F \) is the Fermi wavevector, related to \( v_F \) through the momentum: \( h k_F = m^* v_F \). Substitution for \( n \) and \( v_0 \) in equation (19) gives

\[
G = G_0 \left( 1 - \frac{\lambda_{in} - \lambda_{el}}{\lambda_{in} + \lambda_{el}} \right); \quad G_0 = \frac{q^2}{\pi h} \tag{20}
\]

The expression in parentheses on the right-hand side of equations (20) operates in every way identically to the transmission factor \( T \) in the standard form of the Landauer conductance,

\[
G = G_0 T \quad \text{with } 0 \leq T \leq 1. \tag{21}
\]

What is the experimental distinction between equations (20) derived directly from the standard quantum Kubo formula and equation (21) obtained by rather different physical arguments [4, 5]? There is no empirical distinction at the level of so-called ‘two-probe’ current–voltage measurements (both quantities being read off the same pair of probe terminals).

However, the underlying differences in origin and interpretation may be substantial. Bringing these out could be a matter of some interest, calling for additional measurements designed around, say, well constructed ‘four-probe’ configurations to obtain current and voltage reliably and independently across the inner core. Also necessary would be improved experimental control of inelastic, with respect to elastic, scattering effects.

Another central quantity for testing issues of principle is current noise, providing knowledge of the two-body correlator \( C \); recall equation (14). Noise properties lie beyond the capacity of two- and four-probe methods, which test only the single-carrier response, but high-quality nonequilibrium noise measurements are difficult to carry through. The basic implications of existing ballistic-noise experiments, analysed quantum kinetically, have been addressed elsewhere [40].

In the Landauer equation (21), the ideal scenario plays out when \( G \) reaches the full conductance quantum \( G_0 \). This occurs if, and only if, \( T = 1 \). The channel is then considered to be perfectly transparent to the carriers.

The Kubo-derived equations (20) attain ideality if, and only if, the mean free paths are equally matched, both consequently being equal to the ballistic length of the structure as a whole: interfaces as well as interposed conductor. This does not entail ‘perfect’—totally collisionless—transport since both inelastic and elastic scattering act unfettered at the scale of the overall structure.

It is because the mesoscopic Kubo description necessarily incorporates the interface physics into that of the ‘real’
device, sandwiched between, that collision events retain their explicit and pre-eminent role even in the case of ideally quantized ballistic conductance. This differs from the Landauer description in that (a) elastic scattering is present and robust, and (b) the physics of dissipation does not devolve to the asymptotic equilibrium regions in the leads [4] but acts co-operatively with the elastic effects. In Kubo, dissipation is the inalienable signature of dynamics in the active, nonequilibrium region.

The boundaries are always with us. The mesoscopic boundaries are the physical loci of all scattering, elastic and inelastic together. Yet it is inelastic scattering alone that stabilizes the steady state for every transport problem, irrespective of the device scale. The deeper reason why that is so concerns us in section 3.

3. Dissipation and conductance

3.1. Dissipation

As long as a conducting structure is Ohmic, it dissipates its externally supplied power at the usual rate

$$P = IV = GV^2.$$  (22)

A circuit in which such dissipation occurs is insensitive to the microscopic details of conduction; it is essential only that there exist a path—any path at all—for energy loss. Should it happen that \( G = G_0 = 77.4809 \) μS, it makes no difference to equation (22) whether this conductance belongs to a common-stock resistor, or is measured for a uniquely hand-crafted and very costly ballistic research device.

The key question is straightforward. In terms of its energetics, what distinguishes the process of conduction in a conventional channel from a ballistic one? A crude answer might be 'nothing'.

Unsatisfactory as it seems, such an answer does not fundamentally require too much elaboration. The universality of electrical energy dissipation is plain; what is at times less obvious is how to resolve the apparent incompatibility between ideally ballistic transport on the one hand and, on the other hand, the absolute necessity for accelerated electrical charges to shed their energy gain inelastically.

In section 2.2 we intimated that the physical answer to the origin of energy dissipation rests in the correlation structure of the Kubo formula, and in section 2.4 we made direct use of that relation to recover, in a standard way, quantization of \( G \) in a one-dimensional metallic mesoscopic conductor. Commensurability of the nondissipative (elastic) scattering length and the dissipative (inelastic) length, and of each of them with the operational size of the ballistic conductor, is the essential element in the emergence of quantization within the Landauer formula, as a corollary to the general Kubo analysis.

Dissipation is thus a given in normal transport and the necessity for its presence is widely appreciated. Nevertheless, discussion sometimes still surrounds not the reality of dissipation, but its actual centre of operation. Is energy loss an active process, immediately tied to the dynamics of the structure under test and thus demanding explicit modelling [27]? Or is it instead a remote and passive effect, buried deep within the equilibrium background and hence free of all need to represent it within the physics of a viable transport model [4]?

Kubo’s formula reveals the first of the alternatives above to be the one that faithfully reflects the physics of dissipation. Conditioned as it is by the many-body correlation properties, the Kubo formula demands that inelastic relaxation be present and accounted for quantitatively within the correlator. Were elastic scattering to be the only process allowed, the system could not relax; it would have no mechanism for shedding the energy pumped into its carrier population by the driving field. The system would become grossly disequilibrated and steady state would be impossible.

3.2. Fluctuations

A concrete example illustrates the drastic thermodynamic consequences if mesoscopic transport were exclusively determined by elastic (nondissipative) scattering. We preface the example by introducing an essential idea, the fluctuations in a mesoscopic conductor. To that end we recall the Kubo relation in its alternative form [28, 29] (we will specialize to 1D)

$$G = \frac{1}{k_B T} \langle \mathcal{I}_0 \mathcal{I}_0(0) \rangle; \quad \mathcal{I}_0 = \int_0^\infty dt \mathcal{I}_0(t).$$  (23)

Equation (23) is the zero-frequency—that is, steady-state—version of the fluctuation–dissipation theorem (FDT). It asserts that the low-field conductance is defined by the equilibrium fluctuations, or statistical self-correlation, of the current about its zero mean; the very heart of the Kubo formula. In Ziman’s words [28]: ‘conductivity is a property inherent in the quantum-mechanical description of the unperturbed system; the application of a weak electric field merely exposes the time correlations of the fluctuating components of the electric current in the equilibrium state’. This is a general reformulation of Einstein’s early conclusion that random (Brownian) motion determines the measurable mobility of a test particle.

The fluctuation–dissipation theorem’s significance is that every microscopically conserving transport model must imply it. This covers every model that can be consistently formulated within quantum kinetics (section 2.3) including, apart from the Kubo formula, the entire class of quantum Boltzmann theories (section 2.1).

We emphasize that the FDT is a necessary consequence of conservation. A transport model, if forced to assume the fluctuation–dissipation relation as an additional external hypothesis, could not have incorporated the physically required level of conservation in the first place. As with the Drude model of equation (5) the remedy would be to restructure the model’s equation of motion to make it microscopically conserving; the theorem would then follow. It is not enough simply to add the FDT phenomenologically, without reconsidering the model’s underlying kinetics.

The scope of the FDT omits nonequilibrium fluctuations, however. While the latter must conform to the FDT in the
linear weak-field limit, the relation by itself cannot reveal anything of the system’s behaviour as soon as it departs from linear response. A genuinely nonlinear kinetic framework is needed.

To compare the wider consequences of active dissipation with those of a dissipation-free picture of ballistic conductance, we revisit a conserving Boltzmann–Drude description of uniform 1D mesoscopic transport [41]. The model recovers the FDT in the weak-field regime and reproduces the Landauer conductance from equations (20). Away from that limit it gives a quantitative account of nonequilibrium current fluctuations for a ballistic conductor at high driving fields [40] and thus offers a test-bed for such effects.

We return to the zero-frequency conductance given by that model for metallic (degenerate) carriers, but with a new working premise: the mesoscopic sample length $L$ is to be set independently of the scattering mechanisms at the sample boundaries (as just one instance, it might be visualized to be the wire’s dimension in fabrication). Combining equations (18) and (23), the low-field conductance is accompanied by the current-fluctuation strength

$$S(0) \equiv \langle I_0 I_0(0) \rangle = k_B T G_0 \frac{2 \lambda_{in} \lambda_{el}}{L (\lambda_{in} + \lambda_{el})}$$

in the zero-voltage limit.

The same theory also yields the corresponding nonlinear current–current fluctuation at higher applied voltages. For finite $V$ this is [41]

$$S(V) = S(0) \left[ 1 + \left( \frac{q V}{m^* v_F^2} \right)^2 \left( \frac{\lambda_{in}}{L} \right)^2 \left( 2 - \frac{\lambda_{in}^2}{(\lambda_{in} + \lambda_{el})^2} \right) \right]$$

while the power dissipation in the model remains as

$$P = V^2 G_0 \frac{2 \lambda_{in} \lambda_{el}}{L (\lambda_{in} + \lambda_{el})}$$

so the energy loss rate stays directly proportional to $S(0)$.

The zero-field fluctuation strength $S(0)$ scales with the typical energy $k_B T$ of a carrier at thermal equilibrium. What, then, does the difference of the current fluctuations, $S(V) - S(0)$, represent? It can be characterized in terms of an effective ‘hot-carrier’ excess temperature $\Delta T \equiv T(S(V)/S(0) - 1)$; that is, it measures the overhead of energy converted into—and retained as—the mean kinetic energy of highly excited carriers accelerated by the driving field.

Thus the nature of $S(V) - S(0)$ is nondissipative; excess fluctuations are not associated with the energy transfer that is manifested as Joule heating in the structure and its surroundings, since this is keyed to $S(0)$. Rather, it reflects the average inventory of energy built up and stored in the carrier population by the underlying dynamical processes that sustain the nonequilibrium carriers in steady state.

Consider next what would happen in the limit of asymptotically weak inelastic scattering; all along, $L$ stays fixed by hypothesis. Then $\lambda_{in} \gg L, \lambda_{el}$ and leads to

$$S(0) \to k_B T G_0 \frac{2 \lambda_{el}}{L}, \quad \text{while}$$

$$S(V) \to S(0) \left[ 1 + \frac{q V}{m^* v_F^2} \frac{\lambda_{in}^2}{(\lambda_{in} + \lambda_{el})^2} \right]$$

so (24)

This example highlights the consequence of taking inelastic scattering completely out of the mesoscopic transport picture. Even though the conductance is not qualitatively affected, the nonequilibrium current fluctuations diverge; they do so, in fact, for any nontrivial applied voltage. As a result, while the low-field current response $I = GV$ may well remain finite—in a strictly formal sense—its deviation from that average blows out uncontrollably. This makes the very possibility of measuring the current’s mean value empirically void.

A mesoscopic calculation carried out strictly in linear response but excluding from consideration all inelastic collisions will, notwithstanding, yield a plausible value for the low-field conductance—and fail to reveal the uncontrollably wild behaviour of the nonequilibrium current fluctuations, which is inevitable except at $V = 0$. Deprived of any mechanism for inelastic energy loss, a prediction that appears quite reasonable at linear response must, in reality, hide the seriously unphysical nature of its associated fluctuations.

One can contrast the nonequilibrium implications of the calculation for a sample length that is in fact physically delimited by the scattering physics in the interface regions, as described previously. With this understanding of $L$, and assuming now that the inelastic mean free path is the largest one in the device configuration, we have $L = \lambda_{in} \gg \lambda_{el}$. Substitution into equations (24) shows that $S(0)$ and $G$ are diffusively dominated by elastic scattering. Meanwhile, the excess fluctuations, no longer divergent, are well behaved in all circumstances.

The example above entails four considerations.

(i) A description of transport in a normal, thermodynamically stable system should incorporate a specific model for inelastic scattering; otherwise, the existence of a meaningful steady state is not guaranteed. This constraint applies just as much to mesoscopic charge transport as to any other setting for electrical conduction. Inelasticity is intimately linked to the dissipative energetics of transport and is essential to the stability of the steady state.

(ii) In the mesoscopic context, the scales for both elastic and inelastic scattering become comparable to each other and to the operative length of a conductor. Conventional bulk notions of a purely geometric device ‘length’, that is, one conceptually divorced from the particular mechanisms for real scattering, are problematic when the device dimensions shrink dramatically and its boundaries are wide open to direct energy exchange.

(iii) The current-fluctuation structure of a conductor is inseparable from the current response. Transport models that are conserving always lead to a fluctuation–dissipation relation, which will automatically take care of the essential link between response and fluctuations.

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7 This is equivalent to attaching the additional asymptotic convergence factor $e^{-\lambda_{in} L}$ to the Kubo correlation function $C(t)$, securing a stable limit within purely single-particle Hamiltonian models; recall the discussion following equation (4).
(iv) Particularly in the meso- and nanoscopic regimes, strictly linear models of current response may not be sufficient to discriminate among subtle competing dynamical phenomena. For a fuller picture it would be necessary to develop both nonlinear theoretical descriptions and experiments that better revealed those properties lying beyond weak-field measurement [40].

3.3. Many-body aspects

In strongly correlated systems, transport solutions that appear straightforward in the semiclassical regime must be re-examined in the appropriate many-body setting. Of the Green-function techniques, the Keldysh analysis [36] provides the paradigm of current interest [6]. The primary physical issues confronting a many-body approach remain (1) whether it is inherently conserving, and (2) whether the FDT is contained in it rather than having to be phenomenologically assumed. A consistent Keldysh-based analysis will automatically meet both requirements.

Since the first aim is to compute the single-particle current response, one has to look for the particulars by which strong multi-particle interactions filter down to the one-body level and modify the noninteracting form of the conductance. The Kubo formalism already identifies this nexus: it is the architecture of the correlation function \( j(t)j(0) \). In the Keldysh analysis, it is the self-energy and vertex components of the one-body propagator (Green function) that ties the inner details of the many-body correlations to the outward behaviour of the current response.

Briefly stated, a correlated system at rest possesses a self-energy structure precisely attuned to the scattering amplitude (interaction vertex) between carrier pairs. This tuning is essential for microscopic conservation. It is encoded in the Ward identities [29].

Scattering effects that appear in the form of a nonequilibrium correction to the one-body self-energy (the so-called energy relaxation rate) are systematically offset by counter-terms that are nonequilibrium vertex parts, which are not of single-particle form. However, there is a natural quantity that accommodates both types: the two-particle propagator or correlation function, characteristic of the Kubo formula. In the semiclassical limit, the fundamental cancellations between self-energy and vertex terms are manifested as the principle of detailed balance and determine the conserving structure of the Boltzmann collision integral.

The Keldysh formalism allows one to track the modified self-energy structure in a controlled way. It can then be compared with the equilibrium value. Taken in isolation, the self-energy difference defines the loss rate in one-body propagation: the apparent exponential attenuation in the strength of a single injected carrier reflecting its progressive entanglement with innumerable many-body excitations.

Even though the computational accent is on the self-energy we stress that, within the Keldysh expansion, the Ward identities apply and require those contributions not expressible as one-body self-energy parts (namely the vertex corrections) to appear explicitly in the overall relaxation. They do so on fully equal terms with changes that keep the formal shape of a self-energy. The complete form of the correlator \( j(t)j(0) \), and thus (via Kubo and the FDT) the conductance, is the sum of two parts: self-energy and vertex terms. Together, and only together, they determine the momentum relaxation rate [29], whose inverse appears as \( \tau \) in the Kubo formula, equation (15).

A well-known application of Keldysh analysis is the Meir–Wingreen formula [6]

\[
G = -G_0 \int dE \text{Im} \{ \langle \Gamma_0^r \rangle \} \frac{\partial}{\partial \mu} e^{i\delta (E - \mu)}, \quad (25)
\]

where the energy integral is over the asymptotic carrier band of the (noninteracting) current reservoirs attached to the central (interacting) system, and the expression whose imaginary part is taken represents the trace over the internal many-body excitations of the core. The Green function \( \Gamma_0^r \) is in Keldysh retarded form and \( \Gamma \) encodes the inner core’s coupling to the leads.

In the noninteracting case, \( \Gamma(E, \omega) = 2\pi T (\omega) \delta(E - \omega) \) is the one-body transition-matrix operator [28] for the device treated as a barrier to single-particle propagation. The corresponding free single-particle propagator is \( \Gamma^0(\omega, \omega') = 1/(\omega - \omega' + i0^+) \) and the Meir–Wingreen kernel goes to

\[
\Gamma_0^r = \int \frac{d\omega'}{2\pi} \Gamma^0(E, \omega') \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\eta}}{\omega - \omega' + i0^+} = -i\Gamma(E); \quad \eta \to 0^+ . \quad (26)
\]

Note again the need for an asymptotic convergence factor to force this nondissipative analytic structure to simulate a stable steady state, as explained in section 3.2. If \( \langle T(E) \rangle \) varies slowly with energy, equation (25) recovers the Landauer formula (21). In the interacting case, the integral on the right side of equation (25) both formalizes and extends Landauer’s phenomenological transmission coefficient.

For a system thus described—noninteracting or even interacting—the essential functions of dissipation and inelasticity appear to lack clear definition in equation (25). In that respect, three basic many-body aspects still await validation within its expression.

- **Relation to Kubo formula and FDT**: how is the canonical form of the two-particle Kubo correlation function recovered from the integrand on the right-hand side of equation (25)?
- **Tracking the conserving structure of \( \Gamma \)**: how does it conform to the Ward identities, dynamically interconnecting the self-energy and vertex components?
- **Auditing the mechanisms of energy loss**: how do \( \Gamma \) and \( T \) encode energy transfer (inelasticity) from the driving field to the environment, via many-body interaction(s)?

4. Summary

Mesoscopic electrical transport is now a major discipline within condensed matter, and has been so for almost three decades. Yet more than ever it is a rapidly changing,
intellectually stimulating and economically vital research area. In large part, its theoretical initiatives are still fed by the astonishing growth, in sophistication and versatility, of materials technology.

For all of those reasons it is important to hold to the fore, with clarity, the conceptual underpinnings of transport theory. In mesoscopics its points of reference are, and will remain, precisely those that have governed the general understanding of quantum kinetics: chiefly, the conservation laws and the close interplay of electromagnetism with thermodynamics, which leads to the universality of energy dissipation in real mesoscopic systems.

The maturity of kinetic theory has been arrived at by a collective effort going on, without pause, since the formulation of quantum mechanics. Its legacy to mesoscopics is a fruitful, almost extravagant arsenal of well-crafted and tested approaches to solving problems. This applies not just to nonequilibrium mesoscopics but also to the next step down, the quasi-atomic scale.

We have revisited the main, and already classic, kinetic techniques with the goal of teasing out their common unifying strands. These relate directly to the issues that will always confront mesoscopic transport theory: microscopic conservation and, most particularly, the accounting of energy dissipation at small scales. If a theory wants to replicate their explicit consequences, especially in a novel setting, it should first be able to demonstrate that it explicitly respects their agency.

The dissipation issue led us to elucidate how the nonlinear properties of energy loss processes hold the key to establishing thermodynamic stability within a mesoscopic transport description. The task of detailing in full the physics of dissipation outranks even those pragmatic and increasingly pressing questions now driving theoretical initiatives towards high fields.

Far from being remote icons or symbolic figures for lip service, the unifying ideas that we have traced here are the immediate and prime constraints—‘reality checks’—on every mesoscopic model aspiring to serve the novel demands of device design by embodying, as practically and flexibly as possible, deeply established physical rules. We hope that this paper may help both as a reminder of them and as a pointer to the questions that need to be addressed by mesoscopic transport theories, before any answers are ventured.

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