ANALYSIS OF STRATEGIC CUSTOMER BEHAVIOR IN FUZZY QUEUEING SYSTEMS

GANG CHEN, ZAIMING LIU AND JINGCHUAN ZHANG*

School of Mathematics and Statistics
Central South University
Changsha 410083, Hunan, China

(Communicated by Wuyi Yue)

Abstract. This paper analyzes the optimal and equilibrium strategies in fuzzy Markovian queues where the system parameters are all fuzzy numbers. In this work, tools from both fuzzy logic and queuing theory have been used to investigate the membership functions of the optimal and equilibrium strategies in both observable and unobservable cases. By Zadeh’s extension principle and α-cut approach, we formulate a pair of parametric nonlinear programs to describe the family of crisp strategy. Then the membership functions of the strategies in single and multi-server models are derived. Furthermore, the grated mean integration method is applied to find estimate of the equilibrium strategy in the fuzzy sense. Finally, numerical examples are solved successfully to illustrate the validity of the proposed approach and a sensitivity analysis is performed, which show the relationship of these strategies and social benefits. Our finding reveals that the value of equilibrium and optimal strategies have no deterministic relationship, which are different from the results in the corresponding crisp queues. Since the performance measures of such queues are expressed by fuzzy numbers rather than by crisp values, the system managers could get more precise information.

1. Introduction. In recent years, the management of the high-technology inventory system or service system is particularly significant for the enterprises in the real-world applications, which attracting the attention of both academics and practitioners. In the classical inventory queueing models([18]), the terms produced or received are implicitly assumed to be with perfect quality. However, operations in manufacturing and service processes are often imperfect due to natural disasters, human factors and also many other reasons. Thus, that basic models may have several shortcomings. One of the areas that the classical models do not fit well with real applications is the case where the information for a queueing system is uncertain. Many researchers are seeking for solutions to the problems by using

2010 Mathematics Subject Classification. Primary: 90B22; Secondary: 60K25.
Key words and phrases. Fuzzy sets, queueing game, equilibrium strategy, social benefit, membership functions.

This work is partially supported by the National Natural Science Foundation of China (11671404), and the Fundamental Research Funds for the Central Universities of Central South University (2017zzts061, 2017zzts386).
* Corresponding author. Email address: math@csu.edu.cn (Jingchuan Zhang).
different mathematical theories. In economics, production and operations management literature, fuzzy set theory and queue models are recognized as an appropriate methodology to deal quantitatively with the uncertain decision-making problems.

In many real-world applications, the statistical information may be obtained subjectively. In other words, these system parameters are more possibilistic as well as probabilistic. In order to face the fuzzy environment factor, fuzzy queues are potentially useful and are much more realistic than the commonly used crisp queues [13, 24]. In fuzzy logic literature, fuzzy queues are largely studied. A comprehensive discussion on fuzzy queueing systems can be found in the book of Zhang et al. [26]. Ke and Lin [11] analyze a fuzzy Markovian queue with an unreliable server by the nonlinear programming approach. Ke et al. [8] use the parametric programming approach to study the membership functions of the system characteristics of a batch arrival queuing model with vacation policies and fuzzy parameters. And then they investigate the retrial queueing model and the heterogeneous-server queueing model with fuzzy parameters by the same method [9, 10]. For the fuzzy optimization problems with complex multi-objective and constraints in fuzzy queue models, some new methods are proposed recently. Vahdani et al. [22] consider a bi-objective model for designing a reliable network of bi-directional facilities in logistics network under uncertainties. To solve the model, a new solution approach is proposed by combining queuing theory, fuzzy possibilistic programming and fuzzy multi-objective programming. Jolai et al. [7] study a multi-objective priority assignment problem in a fuzzy queue by the fuzzy data envelopment analysis (FDEA) method. The advantages of FDEA include handling fuzzy uncertainty, its simplicity, efficiency, and its multivariate nature.

Focusing on the problems of economic analysis in queueing systems, strategic customers behavior and optimal social benefits have gained a considerable amount of interest since the work of Naor [16]. Hassin and Haviv [6] and Stidham [21] summarize the main approaches and several results about various models with extensive bibliographical references ([1, 14, 27]). Most of them are forced on the ideas in three directions. One such direction is the consideration of fluid queues and non-Markovian models ([4, 25]). The second direction is the study of two-dimensional Markovian models and the third direction concerns the effect of the levels of information on the customers’ behavior ([5, 20, 23]). In the existing literature of queueing game, the inter-arrival times and service times of customers are required to follow certain probability distributions, that is the system parameters are all assumed to be crisp numbers. For the management of the queueing systems it is more important to consider the fuzzy environment, which can provide more precise and comprehensive information about the decision policy for the system manager [8]. When the equilibrium analysis of customer behaviors in the usual crisp queues can be extended to fuzzy queues, these queueing models will have wider applications.

The model is motivated by a situation that occurs in an automated car wash operation with a machine [2]. Concretely, a strategic customer is typically the driver who wants to wash the car in an automated car wash system. The driver arrives at such a system can make a decision whether to join or not immediately. And the machine can be regarded as a server or service station where managers wish to set the processing rate to minimize the total cost per unit time of the machine. Therefore, managers should determine how to minimize the total cost in such a system with several random and fuzzy parameters, such as customer arrival rate, service rate, holding cost and so on, and the results of the fuzzy queueing system
can be applied to improve the management of the system, which can help managers take the optimal control policy to minimize the average cost per unit time. This research is a step toward gaining fundamental insights into the behaviors of the strategic customers under fuzzy environment.

A closer look at the literature reveals that no previous study has considered to solve problems of customers’ equilibrium strategy and social optimization in fuzzy queues. This paper construct an approach to find the membership functions of the equilibrium strategy and optimal social benefit. Based on the results of corresponding crisp queue, we analyze the membership functions of the equilibrium strategy and optimal social benefit in fuzzy queues through the Zadeh’s extension principle and $\alpha$-cuts (see [28]). The benefit and significance of such fuzzy strategies lie in the fact that it completely maintains the fuzziness of the input information. As the value of $\alpha$ varies, the family of crisp strategy and social benefit are then described and solved by parametric nonlinear programming (NLP). For the fuzzy queueing model, a method of defuzzification, called the graded mean integration is developed to get the estimate of the equilibrium strategy in the fuzzy sense. Moreover, we present some numerical examples to demonstrate how the proposed approach can be applied to this model.

Our main work is to analyze the optimal and equilibrium balking strategies in fuzzy queues under two different levels of information. The contributions can be summarized into three aspects. To begin with, our paper is the first to apply the fuzzy set theory to study the equilibrium strategy and optimal social benefits in fuzzy queues, which fills a gap in the rich literatures of queueing games. The important feature of our work is to consider both fuzziness and randomness for the customers’ strategy. Second, a parametric nonlinear programming analysis method is proposed to derive the membership functions of the optimal and equilibrium strategies and social benefits in both single and multi-server models. It is worthwhile to note that it can be applied to the classes of queueing models, which fit with more complex situations in practical applications. Finally, the study of our model with fuzzy parameters may provide more precise information, which is more accurately and useful for system manager. Our finding reveals that different from the strategy in corresponding crisp queueing system, the optimal strategies are not always larger than the equilibrium strategies (threshold and arrival rate) in fuzzy queues.

The rest of the paper is structured as follows. Section 2 introduces the basic model and preliminary concepts in details. In Section 3, main results for both single and multi-server models are presented, and we solve equilibrium and optimal strategies under two different levels of information by using a mathematical programming approach. Section 4 investigates a solution procedure to numerically construct the membership function of the equilibrium balking strategies and a sensitivity analysis is also performed by numerical examples. Finally, we conclude paper in Section 5 and suggest directions for future work.

2. Model description and preliminary concepts.

2.1. Basic model. We consider a Markovian queueing system with first come first served discipline. Customers arrive according to a Poisson process with rate $\Lambda$, and they are allowed to decide whether to join or balk upon arrival. There is a single server, and service times of all customers are independent and exponentially distributed with parameter $\mu$. A waiting cost of $C$ per unit time incurs when customers stay in the system. Every customer receives a reward of $R$ units after
service. This reward quantifies his satisfaction and/or the added value of being served. We assume that customers are risk-neutral, and choose to join the queue if and only if the expected cost of waiting is less than or equal to \( R \). The offered load for the system is denoted by \( \rho = \frac{\lambda}{\mu} \).

We represent the state of the system at time \( t \) by \( N(t) \), which denotes the number of customers in the system (including the one being served). It is clear that the process \( \{ N(t), t \geq 0 \} \) is a continuous time Markov chain with the state space \( E = \{ 0, 1, 2, \ldots \} \). In this paper, we consider two levels of system information (queue length \( N(t) \)) available to customers before the decision is made: observable case and unobservable case.

In observable model, it can be shown that customers who wish to maximize their individual welfare will follow a pure threshold strategy. This means that there exists an integer \( n_c \) such that newly-arrived customers will join the queue if and only if the number of other customers already present in the system is smaller than \( n_c \). For the social optimization, one aims to find the integer \( n_s \) which maximizes the overall social welfare \( S_{so}(n) = \frac{1 - \rho^n}{1 - \rho^{n+1}} - \frac{1}{\nu_s} \left( \frac{1}{1 - \rho} - \frac{(n+1)}{1 - \rho^{n+1}} \right) \). Furthermore, we have \( n_c = [\nu_s], \ n_s = [\nu] \) and \( n_c \leq n_s \), where \( \nu_s = \frac{R \mu}{C} \), and \( \nu \) satisfies \( \nu = \frac{v(1-\rho)-\rho(1-\rho^2)}{1-\rho} \).

In unobservable model, every customer has no information upon entering the system. It is therefore reasonable to assume that all customers adopt the same randomized strategy for deciding whether to join or balk. The common strategy of customers can be represented by a value \( p \in [0, 1] \) such that a customer will choose to join the queue with probability \( p \). It follows that the rate at which customers join the queue is \( \lambda = p \Lambda \). For the social optimization, one aims to control the effective arrival rate \( \lambda_s \) which maximizes the overall social welfare \( S_{so}(\lambda) = RL - \frac{C \lambda}{\mu - \lambda} \), \( \lambda \in [0, \Lambda] \). [19] shows that \( \lambda_c = \min\{ C/R, \Lambda \} \), \( \lambda_s = \min\{ 1 - \sqrt{C/R}, \Lambda \} \) and \( \lambda_c \leq \lambda_s \).

2.2. Preliminary concepts. In order to investigate the optimal and equilibrium strategies in the Markovian queues with fuzzy environment, we firstly introduce the fuzzy arithmetic operations and graded mean integration representation method. In the fuzzy queues, we assume that customer arrival rate \( \Lambda \), service rate \( \mu \), reward \( R \) and holding cost \( C \) are fuzzy numbers which are approximately known. Then, these fuzzy numbers could be represented as follows:

\[
\hat{\Lambda} = \{(x, \eta_{\hat{\Lambda}}(x)) | x \in X \}, \quad \hat{\mu} = \{(y, \eta_{\hat{\mu}}(y)) | y \in Y \},
\]

\[
\hat{R} = \{(r, \eta_{\hat{R}}(r)) | r \in G \}, \quad \hat{C} = \{(c, \eta_{\hat{C}}(c)) | c \in H \},
\]

where \( X, Y, G, H \) are the crisp universal sets of \( x, y, r, c \) respectively, and \( \eta_{\hat{\Lambda}}(x), \eta_{\hat{\mu}}(y), \eta_{\hat{R}}(r) \) and \( \eta_{\hat{C}}(c) \) are the corresponding membership functions. Because the arrival rate \( \hat{\Lambda} \), service rate \( \hat{\mu} \), reward \( \hat{R} \) and holding cost \( \hat{C} \) are fuzzy, the steady-state probabilities and the performance measures for this system in the steady state must also be fuzzy.

Assume that \( \mathbb{R} \) is the set of real numbers, the mapping \( \eta_{\hat{A}}: \mathbb{R} \rightarrow [0, 1] \), which determines the fuzzy subset \( \hat{A} \) on \( \mathbb{R} \). The mapping \( \eta_{\hat{A}} \) is called the membership function of \( \hat{A} \). Any convex normalized fuzzy subset \( \hat{A} \) on \( \mathbb{R} \) with membership function \( \eta_{\hat{A}} \) is called a fuzzy number (Dubois and Prade [3]). The fuzzy number \( \hat{A} \) is said to be a trapezoidal fuzzy number if it is determined by the crisp numbers \( [a_1, a_2, a_3, a_4] \), where \( a_1 < a_2 < a_3 < a_4 \), with membership function of the form
Furthermore, we assume that \( \tilde{\eta} \), we assume that \( \tilde{\eta} \) and equilibrium balking strategies in the queues with fuzzy parameters. From now

3. Main results.

Assume that \( \tilde{A} = [a_1, a_2, a_3, a_4] \) is a trapezoidal fuzzy number. It is clearly to see that the functions \( L \) and \( R \) have inverse functions denoted by \( L^{-1} \), \( R^{-1} \) respectively.

Define the graded mean \( h \)-level value of \( \tilde{A} \) as \( \frac{h(L^{-1}(h)+R^{-1}(h))}{2} \). The graded mean integration representation of fuzzy number \( \tilde{A} \) can be computed as

\[
P(\tilde{A}) = \frac{\int_0^1 h(L^{-1}(h)+R^{-1}(h)) \, dh}{\int_0^1 \, dh} = \int_0^1 h(L^{-1}(h)+R^{-1}(h)) \, dh.
\]

For trapezoidal fuzzy number \( \tilde{A} = [a_1, a_2, a_3, a_4] \), \( L^{-1} = a_1 + (a_2 - a_1)h \), \( R^{-1} = a_4 - (a_4 - a_3)h \), the graded mean integration representation of trapezoidal fuzzy number \( \tilde{A} = [a_1, a_2, a_3, a_4] \) is given by

\[
P(\tilde{A}) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}.
\]

3. Main results.

3.1. Optimal and equilibrium strategies. In this section, we study the optimal and equilibrium balking strategies in the queues with fuzzy parameters. From now on, we assume that \( \tilde{R} - \tilde{C}/\tilde{\mu} \geq 0 \), otherwise no customers will ever enter the system. Furthermore, we assume that \( \tilde{\rho} = \tilde{\Lambda}/\tilde{\mu} \) and \( 1 - \tilde{\rho} > 0 \) in the unobservable case.

In the fuzzy queues, since the system parameters \( \tilde{\Lambda}, \tilde{\mu}, \tilde{R} \) and \( \tilde{C} \) are all fuzzy parameters, it is clear that the optimal and equilibrium strategies which include the equilibrium and optimal thresholds for the observable case, optimal and equilibrium arrival rates for the unobservable case must also be fuzzy. They are denoted by \( \tilde{n}_e, \tilde{\lambda}_e, \tilde{\lambda}_e, \tilde{S}_{ob}, \tilde{S}_{un} \) respectively. For the optimal social strategies, the objective functions are designed to determine the optimal threshold and arrival rate that maximize the fuzzy expected social benefits in both observable and unobservable cases as follows:

\[
\tilde{S}_{ob}(n) = \frac{1 - \tilde{\rho}^n}{1 - \tilde{\rho}^{n+1}} - \frac{\tilde{C}}{\tilde{R} \tilde{\mu}} \left[ \frac{1}{1 - \tilde{\rho}} - \frac{(n + 1) \tilde{\rho}^{n+1}}{1 - \tilde{\rho}^{n+1}} \right],
\]

\[
\tilde{S}_{un}(\lambda) = \tilde{R} \lambda - \frac{\tilde{C} \lambda}{\tilde{\mu} - \tilde{\lambda}}.
\]

In the fuzzy queues the maximal expected social benefit per unit time is a fuzzy number, not a crisp number. We can not directly derive the optimum so that we should analyze its membership function. We denote \( \Omega = \{ \nu | \frac{\nu y}{\tau} = \frac{\nu(1-x/y)-(x/y)(1-(x/y))}{(1-x/y)^2} \} \).
By Zadeh’s extension principle, the membership functions of the objective values \( \tilde{n}_e, \tilde{n}_s, \bar{\lambda}_e, \bar{\lambda}_s, \tilde{S}_{ob}, \tilde{S}_{un} \) are defined as

\[
\eta_{\tilde{n}_e}(z) = \sup_{x \in X, y \in Y, r \in G, c \in H} \min \{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{R}}(r), \eta_{\tilde{C}}(c) | \ z = n_e(x, y, r, c) \},
\]

\[
\eta_{\tilde{n}_s}(z) = \sup_{x \in X, y \in Y, r \in G, c \in H} \min \{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{R}}(r), \eta_{\tilde{C}}(c) | \ z = n_s(x, y, r, c) \},
\]

\[
\eta_{\tilde{\lambda}_e}(z) = \sup_{x \in X, y \in Y, r \in G, c \in H} \min \{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{R}}(r), \eta_{\tilde{C}}(c) | \ z = \lambda_e(x, y, r, c) \},
\]

\[
\eta_{\tilde{\lambda}_s}(z) = \sup_{x \in X, y \in Y, r \in G, c \in H} \min \{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{R}}(r), \eta_{\tilde{C}}(c) | \ z = \lambda_s(x, y, r, c) \},
\]

\[
\eta_{\tilde{S}_{ob}}(z) = \sup_{x \in X, y \in Y, r \in G, c \in H} \min \{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{R}}(r), \eta_{\tilde{C}}(c) | \ z = \max S_{ob}(n) \},
\]

\[
\eta_{\tilde{S}_{un}}(z) = \sup_{x \in X, y \in Y, r \in G, c \in H} \min \{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{R}}(r), \eta_{\tilde{C}}(c) | \ z = \max_{0 \leq \lambda \leq \bar{\lambda}} S_{un}(\lambda) \}.
\]

(7)

From the results of the corresponding crisp queues in [19], we know that

\[
n_e(x, y, r, c) = \left\lfloor \frac{ry}{c} \right\rfloor,
\]

\[
n_s(x, y, r, c) = \left\lceil \nu \right\rceil, \ \nu \in \Omega,
\]

\[
\lambda_e(x, y, r, c) = \min \{ y - c/r, x \},
\]

\[
\lambda_s(x, y, r, c) = \min \{ y - \sqrt{yc}/r, x \},
\]

\[
S_{ob}(n) = \frac{1 - (x/y)^n}{1 - (x/y)^{n+1}} - \frac{c}{ry} \left( \frac{1}{1 - x/y} - \frac{(n + 1)(x/y)^{n+1}}{1 - (x/y)^{n+1}} \right),
\]

\[
S_{un}(\lambda) = r\lambda - \frac{c\lambda}{y - \bar{\lambda}}, \ \ 0 \leq \lambda \leq x.
\]

Although the membership functions defined in (2)-(7) are theoretically correct, the function is not written in the typical form for practical use, and it is difficult to visualize its shape. In fact, the membership functions can be defined by determining the left and right shape functions of \( \tilde{n}_e, \tilde{n}_s, \bar{\lambda}_e, \bar{\lambda}_s, \tilde{S}_{ob}, \tilde{S}_{un} \), which can be achieved using the mathematical programming technique.

3.2. Parametric nonlinear programming approach. In most instances, the values of the membership functions of the optimal strategies are difficult to be solved analytically, therefore, it is hard to obtain a closed-form membership function for these strategies and social benefits. While the numerical solutions for \( \eta_{\tilde{n}_e}(z) \), \( \eta_{\tilde{n}_s}(z) \), \( \eta_{\tilde{\lambda}_e}(z) \), \( \eta_{\tilde{\lambda}_s}(z) \), \( \eta_{\tilde{S}_{ob}}(z) \), \( \eta_{\tilde{S}_{un}}(z) \) at different possibility levels can be collected to approximate the shapes of the membership functions. That is, the set of crisp intervals reveals the shape of the fuzzy optimal strategies. In this section we construct the membership function on \( \eta_{\tilde{f}}(z) \) by deriving the a-cuts. The a-cuts of the fuzzy parameters \( \tilde{\Lambda}, \tilde{\mu}, \tilde{R} \) and \( \tilde{C} \) are defined as follows.

\[
\Lambda(\alpha) = \{ x \in X \mid \eta_{\tilde{\Lambda}}(x) \geq \alpha \}, \quad \mu(\alpha) = \{ y \in Y \mid \eta_{\tilde{\mu}}(y) \geq \alpha \},
\]

\[
R(\alpha) = \{ r \in G \mid \eta_{\tilde{R}}(r) \geq \alpha \}, \quad C(\alpha) = \{ c \in H \mid \eta_{\tilde{C}}(c) \geq \alpha \}.
\]

As can be seen, the a-cuts of these fuzzy system parameters \( \Lambda(\alpha), \mu(\alpha), R(\alpha) \) and \( C(\alpha) \) are crisp sets rather than fuzzy sets. The fuzzy system parameters \( \tilde{\Lambda}, \tilde{\mu}, \tilde{R} \) and \( \tilde{C} \) can be represented by different levels of confidence intervals (Klir and Yuan [12]). By the definition of the fuzzy number, their a-cuts are crisp intervals that
can be expressed in the following alternate forms:

\[ \Lambda(\alpha) = \min_x \{ x \in X \mid \eta_\alpha(x) \geq \alpha \}, \quad \max_x \{ x \in X \mid \eta_\alpha(x) \geq \alpha \} = [X^L_\alpha, X^U_\alpha], \]

\[ \mu(\alpha) = \min_y \{ y \in Y \mid \eta_\alpha(y) \geq \alpha \}, \quad \max_y \{ y \in Y \mid \eta_\alpha(y) \geq \alpha \} = [Y^L_\alpha, Y^U_\alpha], \]

\[ R(\alpha) = \min_r \{ r \in G \mid \eta_\alpha(r) \geq \alpha \}, \quad \max_r \{ r \in G \mid \eta_\alpha(r) \geq \alpha \} = [G^L_\alpha, G^U_\alpha], \]

\[ C(\alpha) = \min_c \{ c \in H \mid \eta_\alpha(c) \geq \alpha \}, \quad \max_c \{ c \in H \mid \eta_\alpha(c) \geq \alpha \} = [H^L_\alpha, H^U_\alpha]. \]

For the simplicity of the representation, we denote \( f = n_e, \lambda_e, n_s, \lambda_s, S_{ob}, S_{un} \) and the set as follows

\[ \Phi = \{ (x, y, r, c) \mid X^L_\alpha \leq x \leq X^U_\alpha, Y^L_\alpha \leq y \leq Y^U_\alpha, G^L_\alpha \leq r \leq G^U_\alpha, H^L_\alpha \leq c \leq H^U_\alpha \} \]

It is clearly to see that the membership functions defined in (2)-(7) are also parameterized by \( \alpha \). Thus, we use the \( \alpha \)-cut of \( \tilde{f} \) to construct these membership functions.

Our goal is to determine the lower bound \( f^L(\alpha) \) and the upper bound \( f^U(\alpha) \) on the \( \alpha \)-cut of \( \tilde{f} \). According to (2)-(7), \( \eta_\alpha(z) \) is the minimum of several other membership functions \( \eta_\alpha(x), \eta_\alpha(y), \eta_\alpha(r) \) and \( \eta_\alpha(c) \). To ensure that the obtained interval for the membership value is indeed the \( \alpha \)-cut of \( \tilde{f} \), at least one of \( x, y, r, c \) must lie on the boundary of its \( \alpha \)-cut to satisfy \( \eta_\alpha(z) = \alpha \). In other words, to satisfy \( \eta_\alpha(z) = \alpha \), it is required that \( \eta_\alpha(x) \geq \alpha, \eta_\alpha(y) \geq \alpha, \eta_\alpha(r) \geq \alpha \) and \( \eta_\alpha(c) \geq \alpha \), and at least one of these four inequalities should be active such that \( z = n_e, \lambda_e, n_s, \lambda_s, S_{ob}, S_{un} \). This outcome can be accomplished via parametric programming. By the definitions of \( f^L(\alpha) \) and \( f^U(\alpha) \), these terms can be obtained as follows.

(S1) For the membership function \( \eta_{\tilde{n}_e}(z) \) of fuzzy number \( \tilde{n}_e \), we have

\[ n^L_e(\alpha) = \min\{ r | y/c | Y^L_\alpha \leq y \leq Y^U_\alpha, G^L_\alpha \leq r \leq G^U_\alpha, H^L_\alpha \leq c \leq H^U_\alpha \} \]
\[ n^U_e(\alpha) = \max\{ r | y/c | Y^L_\alpha \leq y \leq Y^U_\alpha, G^L_\alpha \leq r \leq G^U_\alpha, H^L_\alpha \leq c \leq H^U_\alpha \} \]

which can be reformulated as follows:

\[ n^L_e(\alpha) = [G^L_\alpha Y^L_\alpha / H^U_\alpha], \quad n^U_e(\alpha) = [G^U_\alpha Y^U_\alpha / H^L_\alpha]. \]  

(S2) For the membership function \( \eta_{\tilde{n}_s}(z) \) of fuzzy number \( \tilde{n}_s \), we have

\[ n^L_s(\alpha) = \min\{ |\nu| \mid \nu \in \Omega \}, \quad n^U_s(\alpha) = \max\{ |\nu| \mid \nu \in \Omega \}, \]
\[ s.t. \quad X^L_\alpha \leq x \leq X^U_\alpha, \quad Y^L_\alpha \leq y \leq Y^U_\alpha, \quad G^L_\alpha \leq r \leq G^U_\alpha, \quad H^L_\alpha \leq c \leq H^U_\alpha. \]

(S3) For the membership function \( \eta_{\tilde{\lambda}_e}(z) \) of fuzzy number \( \tilde{\lambda}_e \), we have

\[ \lambda^L_e(\alpha) = \min\{ n - c/r, x \mid (x, y, r, c) \in \Phi \}, \]
\[ \lambda^U_e(\alpha) = \max\{ n - c/r, x \mid (x, y, r, c) \in \Phi \}, \]

which can be reformulated as follows:

\[ \lambda^L_e(\alpha) = \min\{ Y^L_\alpha - H^U_\alpha / G^L_\alpha, X^L_\alpha \}, \quad \lambda^U_e(\alpha) = \min\{ Y^U_\alpha - H^L_\alpha / G^U_\alpha, X^U_\alpha \}. \]

(S4) For the membership function \( \eta_{\tilde{\lambda}_s}(z) \) of fuzzy number \( \tilde{\lambda}_s \), we have

\[ \lambda^L_s(\alpha) = \min\{ y - \sqrt{c}/r, x \mid (x, y, r, c) \in \Phi \}, \]
\[ \lambda^U_s(\alpha) = \max\{ y - \sqrt{c}/r, x \mid (x, y, r, c) \in \Phi \}, \]

which can be reformulated as follows:

\[ \lambda^L_s(\alpha) = \min\{ Y^L_\alpha - \sqrt{Y^L_\alpha H^U_\alpha} / G^L_\alpha, X^L_\alpha \}, \quad \lambda^U_s(\alpha) = \min\{ Y^U_\alpha - \sqrt{Y^U_\alpha H^L_\alpha} / G^U_\alpha, X^U_\alpha \}. \]
For the membership function \( \eta_{S_{ob}} \) of fuzzy number \( S_{ob} \), we have
\[
S_{ob}^L(\alpha) = \min \{ S_{ob}(n, x, y, r, c) \mid (x, y, r, c) \in \Phi \}, \quad S_{ob}^U(\alpha) = \max \{ S_{ob}(n, x, y, r, c) \mid (x, y, r, c) \in \Phi \}.
\]
which can be reformulated as follows:
\[
\begin{align*}
S_{ob}^L(\alpha) &= \min_{n \in \Omega} \{ S_{ob}(n, x, y) \mid r = G^L_{\alpha}, c = H^U_{\alpha} \}, \\
S_{ob}^U(\alpha) &= \max_{n \geq 0} \{ S_{ob}(n, x, y) \mid r = G^U_{\alpha}, c = H^L_{\alpha} \}, \\
\text{s.t.} \quad X^L_{\alpha} \leq x \leq X^U_{\alpha}, \quad Y^L_{\alpha} \leq y \leq Y^U_{\alpha}.
\end{align*}
\]
(S6) For the membership function \( \eta_{S_{un}} \) of fuzzy number \( S_{un} \), we have
\[
S_{un}^L(\alpha) = \min \{ S_{un}^*(\lambda) \mid (x, y, r, c) \in \Phi \}, \quad S_{un}^U(\alpha) = \max \{ S_{un}^*(\lambda) \mid (x, y, r, c) \in \Phi \},
\]
which can be reformulated as follows:
\[
\begin{align*}
S_{un}^L(\alpha) &= \min_{n \geq 0} \{ S_{un}(n, x, y) \mid \lambda = \min \{ y - \sqrt{cy/r} \mid r = G^U_{\alpha}, c = H^L_{\alpha} \}, x \}, \\
S_{un}^U(\alpha) &= \max_{0 \leq \lambda \leq \lambda} \{ S_{un}(\lambda, x, y) \mid y = G^L_{\alpha}, r = H^U_{\alpha} \}, \\
\text{s.t.} \quad X^L_{\alpha} \leq x \leq X^U_{\alpha}, \quad Y^L_{\alpha} \leq y \leq Y^U_{\alpha}.
\end{align*}
\]
where \( S_{un}^* = \max \{ S_{un}(n) \} \) and \( S_{un}^* = \max \{ S_{un}(\lambda) \} \). This can be accomplished by the parametric NLP techniques. If both \( f^L(\alpha) \) and \( f^U(\alpha) \) of \( \tilde{f} \) are invertible with respect to \( \alpha \), then a left shape function \( L(z) \) and a right shape function \( R(z) \) can be obtained, from which the membership function \( \eta_{\tilde{f}}(z) \) is constructed.
\[
\eta_{\tilde{f}}(\alpha) = \begin{cases} 
L(z) & z_1 \leq z \leq z_2, \\
1 & z_2 \leq z \leq z_3, \\
R(z) & z_3 \leq z \leq z_4, \\
0 & \text{otherwise}.
\end{cases}
\]
In most cases, the values of \( z_i \) can not be solved analytically. Consequently, a closed-form membership function for \( \tilde{f} \) cannot be obtained. However, the numerical solutions for \( z_i \) at different possibility level \( \alpha \) can be collected to approximate the shapes of \( L(z) \) and \( R(z) \). In Section 4, we will present an efficient solution algorithm to compute the membership values of \( f \) at different possibility level \( \alpha \).

3.3. Extension. The optimal and equilibrium balking strategies in a single server Markovian queue with fuzzy parameters are studied. We provide the analysis method for deriving the membership function of these strategies. As we know, multiserver queueing systems can be applied for modelling various real life systems. In this subsection we illustrate that the proposed method applies to settings with a single server as well as to the multiserver Markovian model. Here we consider the model same with the description in Section 3.1 except that there are \( K \) (\( K \geq 1 \)) servers who provide service to customers. We first show the optimal and equilibrium balking strategies of the crisp \( M/M/K \) queueing system. Interestingly, limit results of the multiserver queue problem have been treated in the literature. In this case, [17] consider an \( M/M/K/n \) queue for the optimal and equilibrium baking strategies in the observable case and the stationary probability distribution of having \( i \)
customers in the system is given by

\[ \pi_i = d_i / \sum_{i=0}^{K} d_i, \quad 0 \leq i \leq n, \]

where

\[ d_i = \begin{cases} \frac{(K\rho_1)^i}{i!}, & 0 \leq i \leq K - 1, \\ \frac{(K\rho_1)^K}{\rho_1^{i-K}}, & K \leq i \leq n, \end{cases} \]

and \( \rho_1 = \frac{\lambda}{K\mu} \). Assume that \( \beta = \frac{(K\rho_1)^K}{\rho_1 + (K\rho_1)^K} \) and \( U(\lambda) = R - C[\frac{\beta}{K\mu - \lambda} + \frac{1}{\mu}] \). Applying the method and partial results in [17], we investigate the optimal and equilibrium strategies in crisp \( M/M/K \) queueing system as follows.

1. For the observable case:
   \[ n_c = \lfloor RK\mu / C \rfloor, \quad n_s = \max_{n \geq 0} \{ S_{ob}(n) \}, \]
   \[ S_{ob}(n) = \Lambda \sum_{i=0}^{n-1} \pi_i [R - \frac{(i + 1)C}{K\mu}] \]
2. For the unobservable case:
   \[ \lambda_e = \min \{ \lambda' \}, \quad \lambda_s = \max_{\lambda \geq 0} \{ \lambda \}, \]
   \[ S_{un}(\lambda) = \lambda R - C[\frac{\beta}{K\mu - \lambda} + \frac{1}{\mu}] \]

where \( \lambda' \) is the root of equation \( U(\lambda) = 0 \). In Section 3.2 we propose a parametric nonlinear programming method for calculating the membership function of the optimal and equilibrium strategies in the single server fuzzy queue. As the technique works, it can be applied in many models, even though it is not possible to derive the closed-form expressions for the optimal and equilibrium strategies in the crisp queues. Here we consider the optimal and equilibrium strategies in a multiserver Markovian fuzzy queue by the method proposed in Section 3.2. We directly give the membership functions of the optimal and equilibrium strategies as follows.

M1. For the membership function \( n_{\tilde{c}}(\alpha) \) of fuzzy number \( \tilde{c} \), we have
   \[ n_{\tilde{c}}(\alpha) = \lfloor KG_a^{L} \alpha / L^U / H_a^{L} \rfloor, \quad n_{\tilde{c}}(\alpha) = \lfloor KG_a^{U} \alpha / L^U / H_a^{U} \rfloor. \]

M2. For the membership function \( n_{\tilde{c}}(\alpha) \) of fuzzy number \( \tilde{n} \), we have
   \[ n_{\tilde{n}}(\alpha) = \min \{ \arg \max \{ S_{ob}(n) \} \}, \quad n_{\tilde{n}}(\alpha) = \max \{ \arg \max \{ S_{ob}(n) \} \}, \]
   \[ s.t. \quad X_a^{L} \leq x \leq X_a^{U}, \quad Y_a^{L} \leq y \leq Y_a^{U}, \quad G_a^{L} \leq r \leq G_a^{U}, \quad H_a^{L} \leq c \leq H_a^{U}. \]

M3. For the membership function \( \lambda_{\tilde{c}}(\alpha) \) of fuzzy number \( \tilde{\lambda} \), we have
   \[ \lambda_{\tilde{c}}(\alpha) = \min \{ \lambda' \}, \quad \lambda_{\tilde{c}}(\alpha) = \max \{ \lambda' \}, \]
   \[ s.t. \quad X_a^{L} \leq x \leq X_a^{U}, \quad Y_a^{L} \leq y \leq Y_a^{U}, \quad G_a^{L} \leq r \leq G_a^{U}, \quad H_a^{L} \leq c \leq H_a^{U}. \]

M4. For the membership function \( \lambda_{\tilde{c}}(\alpha) \) of fuzzy number \( \tilde{\lambda} \), we have
   \[ \lambda_{\tilde{c}}(\alpha) = \min \{ \arg \max \{ S_{un}(\lambda) \} \}, \quad X_a^{L} \]
\[ \lambda^U_\alpha (\alpha) = \min \{ \max \{ \arg \max \{ S_{uu}(\lambda) \} \}, X^U_\alpha \} , \]

\[ s.t. \ X^L_\alpha \leq x \leq X^U_\alpha , \ Y^L_\alpha \leq y \leq Y^U_\alpha , \ G^L_\alpha \leq r \leq G^U_\alpha , \ H^L_\alpha \leq c \leq H^U_\alpha . \]

The multiserver Markovian model is the natural generalization of its single server counterpart. As can be seen, the results for the multiserver model are more complex than that of single server model. While, the membership functions of the optimal and equilibrium strategies can be derived in the same analysis method proposed in this paper. In order to illustrate the effectiveness of the proposed methods and solution procedure, we present some numerical examples in the next section.

4. Numerical experiment. In this section, we focus on the solution algorithm and sensitivity analysis of the optimal and equilibrium strategies in the single server fuzzy queues by some numerical examples. Assume that the trapezoidal fuzzy numbers \( \Lambda \), \( \mu \), \( \bar{R} \) and \( \bar{C} \) are represented by \([x_1, x_2, x_3, x_4], [y_1, y_2, y_3, y_4], [r_1, r_2, r_3, r_4] \) and \([c_1, c_2, c_3, c_4] \) respectively, where \( x_4 > x_3 > x_2 > x_1, y_4 > y_3 > y_2 > y_1, r_4 > r_3 > r_2 > r_1 \) and \( c_4 > c_3 > c_2 > c_1 \). Using the proposed approach stated in Section 3, it is easy to obtain the \( \alpha \)-cut sets of \( \hat{n}_c, \hat{n}_s, \lambda_c, \lambda_s, S_{ob} \) and \( S_{un} \). In order to illustrate the proposed method, we outline a simple step-by-step procedure for constructing the membership function \( \eta_f(z) \) as follows, where \( f = n_c, n_s, \lambda_c, \lambda_s, S_{ob}, S_{un} \).

**Step 1.** Input the arrival rate, service rate, reward and holding cost, which are trapezoidal fuzzy number represented by \([x_1, x_2, x_3, x_4], [y_1, y_2, y_3, y_4], [r_1, r_2, r_3, r_4] \) and \([c_1, c_2, c_3, c_4] \).

**Step 2.** For \( \alpha \) at a possibility level from 0 to 1 with step size of 0.01.

**Step 3.** Compute the upper and lower bounds of \( \hat{\Lambda}, \hat{\mu}, \hat{R} \) and \( \hat{C} \) as follows:

\[ X^L_\alpha = (x_2 - x_1)\alpha + x_1; \ X^U_\alpha = x_4 - (x_4 - x_3)\alpha; \]
\[ Y^L_\alpha = (y_2 - y_1)\alpha + y_1; \ Y^U_\alpha = y_4 - (y_4 - y_3)\alpha; \]
\[ G^L_\alpha = (r_2 - r_1)\alpha + r_1; \ G^U_\alpha = r_4 - (r_4 - r_3)\alpha; \]
\[ H^L_\alpha = (c_2 - c_1)\alpha + c_1; \ H^U_\alpha = c_4 - (c_4 - c_3)\alpha; \]

**Step 4.** For \( x \) from \( X^L_\alpha \) to \( X^U_\alpha \) with a step size of \( \Delta_1 = (X^U_\alpha - X^L_\alpha)/100. \)

**Step 5.** For \( y \) from \( Y^L_\alpha \) to \( Y^U_\alpha \) with a step size of \( \Delta_2 = (Y^U_\alpha - Y^L_\alpha)/100. \)

**Step 6.** For \( r \) from \( G^L_\alpha \) to \( G^U_\alpha \) with a step size of \( \Delta_3 = (G^U_\alpha - G^L_\alpha)/100. \)

**Step 7.** For \( c \) from \( H^L_\alpha \) to \( H^U_\alpha \) with a step size of \( \Delta_4 = (H^U_\alpha - H^L_\alpha)/100. \)

**Step 8.** Calculate \( \alpha \)-cut of the fuzzy number \( f, [f^L(\alpha), f^U(\alpha)] \) by applying (8)-(13), where \( f = n_c, n_s, \lambda_c, \lambda_s, S_{ob} \) and \( S_{un} \).

**Step 9.** Using \( f^L(\alpha) \) and \( f^U(\alpha) \) to approximate the membership functions \( \eta_f(z) \) by (14) at different levels of \( \alpha \).

4.1. Membership functions of the equilibrium strategies. To illustrate how the proposed approach is applied to analyze the optimal and equilibrium strategies in fuzzy Markovian queues, in this section we investigate some numerical examples. A software Matlab version 6.0 for windows is used to solve the mathematical programs and then the shape of membership functions of the optimal and equilibrium strategies can be found. By enumerating different \( \alpha \) values, the lower and upper bounds of the \( \alpha \)-cuts of the performance measure are calculated to approximate
the membership function. Here we enumerate 101 values of $\alpha$: 0, 0.01, 0.02, ..., 1.00. The figures depict the rough shape from these values. In the following examples, we assume that the arrival rate $\Lambda$, service rate $\mu$, reward $R$ and holding cost $C$ are trapezoidal fuzzy numbers represented by $[1, 2, 3, 4]$, $[5, 6, 7, 8]$, $[10, 15, 20, 25]$, $[2, 4, 6, 8]$, respectively. As is shown in the figures, we can make the following observations.

![Figure 1](image_url)

**Figure 1.** The approximate membership function of optimal and equilibrium threshold $n$.

In Figures 1-3, we describe the membership functions of the equilibrium and optimal strategies in the single server fuzzy queues under both observable and unobservable cases. For the optimal and equilibrium strategies in observable case, as we can see from Figure 1, their membership functions are approximately trapezoidal. At one extreme end for possibility level $\alpha = 1$, the range of the fuzzy optimal and equilibrium thresholds are approximately $[8, 25]$ and $[15, 35]$ respectively, which indicates that it is definitely possible for $n_e$ and $n_*$ to fall between these intervals, although it is imprecise. At the other extreme end for possibility level $\alpha = 0$, it shows that $n_e$ and $n_*$ will never exceed 3 or fall below 100. The above results will certainly be useful and significant for system designers. Moreover, we find that there are some differences and relationship between the membership functions of optimal and equilibrium strategies. For example, for the case $\alpha = 0.6$, it is easy to see that $n_e \in [3, 87]$ and $n_* \in [6, 100]$, which imply $n_e \cap n_* \neq \emptyset$. It is an interesting phenomenon that different from the results in the corresponding crisp queues, the value of equilibrium and optimal strategies $n_e$ and $n_*$ have no deterministic relationship in fuzzy Markovian queues. For the optimal and equilibrium strategies in unobservable case, Figure 2 presents the membership functions of the equilibrium and optimal strategies in the unobservable case. We find that $\lambda_e \in [1.0, 4.0]$ and $\lambda_* \in [1.0, 7.2]$, which imply $\lambda_e \cap \lambda_* \neq \emptyset$. It is evident that the result $\lambda_e \geq \lambda_*$ in the crisp queues does not hold in the corresponding fuzzy queues.
For the optimal social benefits in both observable and unobservable cases, the membership functions of optimal social benefits are given in Figure 3. It shows the impact of the system information on the optimal social benefits. For the special case with $\alpha = 1$, we have $S_{ob} > S_{un}$, while this inequality does not always hold for
varying values of $\alpha \in [0,1]$. Moreover, as can be seen in Figure 3, the range of the $S_{ob}$ and $S_{un}$ are $[47.1, 92.0]$ and $[24.7, 98.0]$ respectively, which shows that the factor on the fuzziness has larger impact of the social benefit in unobservable case than that in observable case. In a word, the above results are not always same with the corresponding results in the crisp queues. Therefore, the fuzzy factors are important for the system manager to control the system. Since the performance measures are expressed by membership functions rather than by crisp values, the derived results can be used to represent the real time systems as fuzzy system more accurately. As can be seen in these figures, the membership functions of the strategies and social benefits are approximately trapezoidal, which are result from the assumption that the four fuzzy parameters in this model are trapezoidal fuzzy numbers.

4.2. Sensitivity analysis of fuzzy parameters. In this subsection, we set four trapezoidal fuzzy numbers of the input parameters to represent the components of fuzzy models. Tables 1-4 show the results of the sensitivity analysis for each input parameter of the problem in both observable case and unobservable case. In the calculations, for these parameters, their defuzzified values are determined by applying the graded mean integration method (1) and each of the parameters is increased by 10% in each step separately.

Table 1. Fuzzy trapezoidal value for the input parameters $\tilde{\Lambda}$

| $\tilde{\Lambda}$ | $P(\tilde{\Lambda})$ | $n_c$ | $n_e$ | $\lambda_c$ | $\lambda_e$ |
|-------------------|----------------------|-------|-------|-------------|-------------|
| (1.0, 2.0, 3.0, 4.0) | 2.50 | 22 | 14 | 2.50 | 2.50 |
| (1.1, 2.2, 3.3, 4.4) | 2.75 | 22 | 13 | 2.75 | 2.75 |
| (1.2, 2.4, 3.6, 4.8) | 3.00 | 22 | 13 | 3.00 | 3.00 |
| (1.3, 2.6, 3.9, 5.2) | 3.25 | 22 | 12 | 3.25 | 3.25 |
| (1.4, 2.8, 4.2, 5.6) | 3.50 | 22 | 11 | 3.50 | 3.50 |
| (1.5, 3.0, 4.5, 6.0) | 3.75 | 22 | 10 | 3.75 | 3.75 |

Table 2. Fuzzy trapezoidal value for the input parameters $\tilde{\mu}$

| $\tilde{\mu}$ | $P(\tilde{\mu})$ | $n_c$ | $n_e$ | $\lambda_c$ | $\lambda_e$ |
|----------------|------------------|-------|-------|-------------|-------------|
| (5.0, 6.0, 7.0, 8.0) | 6.50 | 22.75 | 14 | 2.5 | 2.5 |
| (5.5, 6.6, 7.7, 8.8) | 7.15 | 25.03 | 16 | 2.5 | 2.5 |
| (6.0, 7.2, 8.4, 9.6) | 7.80 | 27.30 | 19 | 2.5 | 2.5 |
| (6.5, 7.8, 9.1, 10.4) | 8.45 | 29.58 | 21 | 2.5 | 2.5 |
| (7.0, 8.4, 9.8, 11.2) | 9.10 | 31.85 | 23 | 2.5 | 2.5 |
| (7.5, 9.0, 10.5, 12.0) | 9.75 | 34.13 | 25 | 2.5 | 2.5 |

Table 1 indicates the results of the sensitivity analysis for the parameter $\tilde{\Lambda}$. From the table we can see, the change of $\tilde{\Lambda}$ affect optimal threshold $n_e$, optimal arrival rate $\lambda_e$, and equilibrium arrival rate $\lambda_e$, but not equilibrium threshold $n_c$. As the parameter $\tilde{\Lambda}$ monotonically increases, the optimal threshold monotonically decreases, and optimal arrival rate $\lambda_e$ and equilibrium arrival rate $\lambda_e$ all monotonically increases. Therefore, this parameter is a sensitive parameter of $n_e$, $\lambda_e$ and $\lambda_e$ in
Table 3. Fuzzy trapezoidal value for the input parameters $\tilde{R}$

| $\tilde{R}$          | $P(\tilde{R})$ | $n_e$ | $n_*$ | $\lambda_e$ | $\lambda_*$ |
|----------------------|-----------------|-------|-------|--------------|--------------|
| (10.0, 15.0, 20.0, 25.0) | 17.50          | 22    | 14    | 2.5          | 2.5          |
| (11.0, 16.5, 22.0, 27.5) | 19.25          | 25    | 16    | 2.5          | 2.5          |
| (12.0, 18.0, 24.0, 30.0) | 21.00          | 27    | 17    | 2.5          | 2.5          |
| (13.0, 19.5, 26.0, 32.5) | 22.75          | 29    | 18    | 2.5          | 2.5          |
| (14.0, 21.0, 28.0, 35.0) | 24.50          | 31    | 20    | 2.5          | 2.5          |
| (15.0, 22.5, 30.0, 37.5) | 26.25          | 34    | 21    | 2.5          | 2.5          |

Table 4. Fuzzy trapezoidal value for the input parameters $\tilde{C}$

| $\tilde{C}$          | $P(\tilde{C})$ | $n_e$ | $n_*$ | $\lambda_e$ | $\lambda_*$ |
|----------------------|-----------------|-------|-------|--------------|--------------|
| (2.0, 4.0, 6.0, 8.0) | 5.00           | 22    | 14    | 2.5          | 2.5          |
| (2.2, 4.4, 6.6, 8.8) | 5.50           | 20    | 13    | 2.5          | 2.5          |
| (2.4, 4.8, 7.2, 9.6) | 6.00           | 18    | 12    | 2.5          | 2.5          |
| (2.6, 5.2, 7.8, 10.4) | 6.50           | 17    | 11    | 2.5          | 2.5          |
| (2.8, 5.6, 8.4, 11.2) | 7.00           | 16    | 10    | 2.5          | 2.5          |
| (3.0, 6.0, 7.0, 12.0) | 6.83           | 16    | 10    | 2.5          | 2.5          |

this model. Table 2 shows the results of the sensitivity analysis for the parameter $\tilde{\mu}$. The change of $\tilde{\mu}$ are fairly sensitive for both equilibrium and optimal strategies in the observable and unobservable cases, and both monotonically increase as $\tilde{\mu}$ monotonically increases. Similarly, Tables 3 and Table 4 indicate the results of the sensitivity analysis for the parameters $\tilde{R}$ and $\tilde{C}$. Take the same analytical method, $\tilde{R}$ and $\tilde{C}$ have significant effect on the equilibrium and optimal strategies. As $\tilde{R}$ monotonically increases, equilibrium and optimal strategies both increase, and both decrease with $\tilde{C}$ monotonically increasing. As could be expected, $\tilde{\mu}$, $\tilde{R}$, $\tilde{C}$ are associated with equilibrium and optimal strategies in the observable and unobservable cases, while $\Lambda$ is related to the optimal threshold in the observable case, and optimal arrival rate and equilibrium arrival rate in unobservable case.

5. Conclusions. Our main goal in this paper was to investigate the optimal and equilibrium strategies in the fuzzy queues, where the arrival rate, service rate, as well as the reward and holding cost are all trapezoidal fuzzy numbers. Based on the results of the optimal and equilibrium strategies in the crisp Markovian queues, We derived the membership functions of the these strategies and optimal social benefits in both single and multi-server fuzzy queues. Then, an efficient algorithm is proposed to construct the membership functions of the strategies and social benefits. We calculated numerical solutions for different $\alpha$ values to approximate the membership functions by NLP. The graded mean integration method is developed to get the estimate of the equilibrium strategy in the fuzzy sense. Moreover, a sensitivity analysis are provided by numerical experiments. The main implication of our results is that the equilibrium and optimal strategies have no deterministic relationship, which are different from the results in the corresponding crisp queues. Since the optimal and equilibrium strategies are expressed by membership functions rather
than by crisp values, they completely conserve the fuzziness of input information when some data of queueing systems are ambiguous.

We expect that the proposed model and method can also be extended in different directions. One possible change is to consider systems in which the customer arrival time and service time distributions follow the general distribution by embedded Markov chain method. In addition, further extensions would be researched about the equilibrium strategy and dynamic pricing problem in the queueing system with fuzzy parameters or fuzzy states. In the real-world, impatience and vacation policy are often encountered. Another way to generalize the model is to study the models with impatient customers and vacation policy. It will be interesting to see whether they can be handled by modifying the present model and future research along these lines is of great interest and challenge.

**Acknowledgments.** This work is partially supported by the National Natural Science Foundation of China (11671404), and the Fundamental Research Funds for the Central Universities of Central South University (2017zzts061, 2017zzts386). The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers.

**REFERENCES**

[1] O. Bountali and A. Economou, *Equilibrium joining strategies in batch service queueing systems*, European Journal of Operational Research, **260** (2017), 1142–1151.

[2] S.-P. Chen, *Time value of delays in unreliable production systems with mixed uncertainties of fuzziness and randomness*, European Journal of Operational Research, **255** (2016), 834–844.

[3] D. J. Dubois, *Fuzzy Sets and Systems: Theory and Applications*, vol. 144, Academic Press, 1980.

[4] A. Economou and A. Manou, *Strategic behavior in an observable fluid queue with an alternating service process*, European Journal of Operational Research, **254** (2016), 148–160.

[5] P. Guo and R. Hassin, *Strategic behavior and social optimization in markovian vacation queues: The case of heterogeneous customers*, European Journal of Operational Research, **222** (2012), 278–286.

[6] R. Hassin and M. Haviv, *To queue or not to queue: Equilibrium behavior in queueing systems*, vol. 59, Springer Science & Business Media, 2003.

[7] F. Jolai, S. M. Asadzadeh, R. Ghodsi and S. Bagheri-Marani, *A multi-objective fuzzy queueing priority assignment model*, Applied Mathematical Modelling, **40** (2016), 9500–9513.

[8] J.-C. Ke, H.-I. Huang and C.-H. Lin, *Parametric programming approach for batch arrival queues with vacation policies and fuzzy parameters*, Applied Mathematics and Computation, **180** (2006), 217–232.

[9] J.-C. Ke, H.-I. Huang and C.-H. Lin, *On retrial queueing model with fuzzy parameters*, Physica A: Statistical Mechanics and its Applications, **374** (2007), 272–280.

[10] J.-C. Ke, H.-I. Huang and C.-H. Lin, *Analysis on a queue system with heterogeneous servers and uncertain patterns*, Journal of Industrial & Management Optimization, **6** (2010), 57–71.

[11] J.-C. Ke and C.-H. Lin, *Fuzzy analysis of queueing systems with an unreliable server: A nonlinear programming approach*, Applied Mathematics and Computation, **175** (2006), 330–346.

[12] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, vol. 4, Prentice Hall New Jersey, 1995.

[13] R.-J. Li and E. Lee, *Analysis of fuzzy queues*, Computers & Mathematics with Applications, **17** (1989), 1143–1147.

[14] Y. Ma, Z. Liu and Z. G. Zhang, *Equilibrium in vacation queueing system with complementary services*, Quality Technology & Quantitative Management, **14** (2017), 114–127.

[15] G. C. Mahata and A. Goswami, *Fuzzy inventory models for items with imperfect quality and shortage backordering under crisp and fuzzy decision variables*, Computers & Industrial Engineering, **64** (2013), 190–199.

[16] P. Naor, *The regulation of queue size by levying tolls*, Econometrica: Journal of the Econometric Society, **37** (1969), 15–24.
[17] L. J. Ratliff, C. Dowling, E. Mazumdar and B. Zhang, To observe or not to observe: Queuing game framework for urban parking, in Decision and Control (CDC), 2016 IEEE 55th Conference on, IEEE, 2016, 5286–5291.

[18] M. Saffari, S. Asmussen and R. Haji, The M/M/1 queue with inventory, lost sale, and general lead times, Queueing Systems, 75 (2013), 65–77.

[19] R. Shone, V. A. Knight and J. E. Williams, Comparisons between observable and unobservable M/M/1 queues with respect to optimal customer behavior, European Journal of Operational Research, 227 (2013), 133–141.

[20] E. Simhon, Y. Hayel, D. Starobinski and Q. Zhu, Optimal information disclosure policies in strategic queueing games, Operations Research Letters, 44 (2016), 109–113.

[21] S. Stidham Jr, Optimal Design of Queueing Systems, CRC Press, 2009.

[22] B. Vahdani, R. Tavakkoli-Moghaddam and F. Jolai, Reliable design of a logistics network under uncertainty: A fuzzy possibilistic-queueing model, Applied Mathematical Modelling, 37 (2013), 3254–3268.

[23] J. Wang and F. Zhang, Strategic joining in M/M/1 retrial queues, European Journal of Operational Research, 230 (2013), 76–87.

[24] Y. C. Wang, J. S. Wang and F. H. Tsai, Analysis of discrete-time space priority queue with fuzzy threshold, Journal of Industrial and Management Optimization, 5 (2009), 467–479.

[25] F. Zhang, J. Wang and B. Liu, Equilibrium joining probabilities in observable queues with general service and setup times, Journal of Industrial and Management Optimization, 9 (2013), 901–917.

[26] R. Zhang, Y. A. Phillis and V. S. Kouikoglou, Fuzzy Control of Queuing Systems, Springer Science & Business Media, 2005.

[27] S. Zhu and J. Wang, Strategic behavior and optimal strategies in an M/G/1 queue with bernoulli vacations, Journal of Industrial & Management Optimization, (2008), 56–64.

[28] H.-J. Zimmermann, Fuzzy Set Theory and Its Applications, Second edition, Kluwer Academic Publishers, Boston, MA, 1992.

Received for publication June 2018.

E-mail address: chenmathcsu@163.com
E-mail address: math_lzm@csu.edu.cn
E-mail address: math_zjc@csu.edu.cn