Dynamics of one-dimensional supersolids

Masaya Kunimi, Michikazu Kobayashi, and Yusuke Kato
Department of Basic Science, The University of Tokyo, Japan
E-mail: kunimi@vortex.c.u-tokyo.ac.jp

Abstract.
We study dynamical properties of one-dimensional supersolids in the framework of the Gross-Pitaevskii equation with a finite-range two-body interaction in the presence of an obstacle. Above a critical velocity, superfluidity breaks down and solitons are emitted periodically as in the case of superfluids. However, near the critical velocity, the period of soliton emissions obeys a scaling law which is different from that in superfluids.

1. Introduction
Since the first theoretical prediction of a supersolid was proposed by Andreev and Lifshitz[1], the study of the supersolid is gathering many scientist’s interests even now. Recently, Kim and Chan[2, 3, 4] have measured a nonclassical rotational inertia (NCRI)[5] of solid $^4$He. Many interesting properties of solid $^4$He were also observed in these experiments[6, 7]. The possibility for supersolids is also discussed in ultra-cold atomic gases: Recently, there are many predictions that the long-range interactions such as magnetic dipole interaction between $^{52}$Cr[8], van der Waals interaction between Rydberg atoms[9, 10], and electric dipole interaction between polar molecules, can stabilize supersolid phase.

One of the nature of superfluidity is the existence of a critical velocity. In a uniform system, the critical velocity is determined as the Landau critical velocity[11]. A stable superflow can exist if the velocity of the flow is lower than the Landau critical velocity. If an obstacle preventing a flow exists, the critical velocity depends on the shape of the obstacle. Above the critical velocity, superfluidity breaks down by generating solitons in one-dimensional(1D) system [12] or quantized vortices(2D or 3D system)[13, 14]. The emission rates of solitons or vortices obey a scaling relation near the critical velocity. This scaling law is predicted by a saddle-node bifurcation[15, 16].

A theoretical model for the supersolid in the framework of the Gross-Pitaevskii(GP) equation[17, 18] has been proposed by Pomeau and Rica[19]. The supersolid phase in this framework is defined as the state in which $U(1)$ and translational symmetry are spontaneously broken. Using not a contact-type interaction but a finite-range interaction, they have shown that the ground state exhibits crystalline order if the strength of the interaction or the particle density is large enough. In our previous work[20], 1D supersolids exhibit superfluidity even in the presence of an obstacle using the 1D version of the Pomeau and Rica model[21]. This work shows that static properties of 1D supersolids are similar to those in superfluids.

In this paper, we focus on a breakdown dynamics of supersolid above a critical velocity in the presence of an obstacle. We show that solitons are emitted periodically in the supersolid phase above the critical velocity and the period of emissions obeys a scaling relation.
2. Model and method
We consider a 1D system with system size $L$. The GP equation with an external potential $U$ for an obstacle moving at constant velocity $v (v > 0)$ and a finite-range two-body interaction $V$ is given by[12]

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x + vt) + \int_0^L dy V(x - y)|\Psi(y, t)|^2 \right] \Psi(x, t), \quad (1)$$

where $\Psi(x, t)$ is the condensate wave function, $U(x)$ is of a Gaussian form

$$U(x) = \frac{U_0}{\sqrt{\pi d}} \exp \left[ - \left( \frac{x - L/2}{d} \right)^2 \right], \quad (2)$$

and $V(x)$ is a two-body repulsive soft-core interaction with a constant strength $V_0$ and an interaction range $a$[19]

$$V(x) \equiv V_0 \theta(a - |x|). \quad (3)$$

Equation (1) can be rewritten in the moving frame with the obstacle:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - i\hbar v \frac{\partial}{\partial x} + U(x) + \int_0^L dy V(x - y)|\Psi(y, t)|^2 \right] \Psi(x, t) \equiv \mathcal{H}\Psi(x, t). \quad (4)$$

We impose the periodic boundary condition $\Psi(x + L, t) = \Psi(x, t)$ on the condensate wavefunction.

In the following, we measure the length, energy, time, velocity, and $\Psi$ in units of $a$, $\epsilon_0 \equiv \hbar^2/(ma^2)$, $\tau \equiv ma^2/\hbar$, $v_0 \equiv \hbar/(ma)$, and $\sqrt{\nu_0} \equiv \sqrt{N/L}$, respectively, where $N$ is the number of particles. We define a dimensionless parameter $g$ representing the strength of interaction[21, 20]:

$$g \equiv \frac{2n_0ma^3V_0}{\hbar^2}. \quad (5)$$

When $g \geq g_c \simeq 21.05$ and $v = 0$ in the absence of obstacles, the ground state is supersolid phase[21].

Time evolution of the condensate wave function is obtained by Crank-Nicholson method and the interaction term is calculated by the fast Fourier transformation. The initial condition is a stationary solution of eq. (4) at $v = 0$ obtained by the imaginary time evolution. We set the parameters as follows: $L = 512a, d = 0.1a, U_0 = 5\epsilon_0a, 7.5\epsilon_0a, 10\epsilon_0a$, and $g = 5, 25, 30, 40$. In the following, we will mainly present the results for $g = 25$.

3. Results
We first prepare the initial condition $\Psi(x, t = 0) \equiv \Psi(x)$ by solving time-independent GP equation: $\mathcal{H}\Psi(x) = \mu \Psi(x)$, where $\mu$ is the chemical potential which is determined to satisfy the condition $N = \int_0^L dx |\Psi(x)|^2$. Figure 1 shows the steady solution of time-independent GP equation with $g = 25$ and $v = 0$.

At $t = 0$, we start to move the obstacle with the velocity $-v$. In the frame of moving obstacle, this corresponds to injecting a flow in the positive direction. Figures 2 and 3, and 4 show that snapshots of density and phase profiles above the critical velocity. Figures 2 and 3 display one and two depressions of the density modulation propagating downstream, respectively. These depressions are emitted from the obstacle periodically. At the moment of emissions, the
condensate wave function vanishes and we can not define the phase at the location of the obstacle. This corresponds to $2\pi$ phase slip\cite{22} occurrence as shown in Figure 4. An analogous phenomenon was observed in numerical calculations of a superfluid\cite{12}. Therefore, we identify these depressions as gray solitons of the supersolid phase.

In order to study the phase slip quantitatively, we define a winding number

$$n \equiv \frac{1}{2\pi} \int_0^L dx \frac{\partial \varphi(x,t)}{\partial x},$$

(6)

where $\varphi(x,t)$ is the phase of the condensate wave function. This quantity is an integer due to the periodic boundary condition. The change of the winding number by $\pm 1$ corresponds to the occurrence of one phase slip. We can measure the period of the phase slip from the change of the winding number. Figure 5 shows that the time dependence of the winding number.

From the time dependence of the winding number, we obtain the period of the phase slip for various values of $v$. Figure 6 shows $v$-dependence of the period $T$ for three different potential heights of the obstacle. Straight lines are determined by fitting function $T/\tau = A(v - v_c)^c$, where $A$, $v_c$, and $c$ are fitting parameters and $v_c$ denotes the critical velocity. These values are shown in Table 1. The data over one or two digits can be fitted with a single exponent. For $g = 30$ and 40, we also obtain similar results(Data are not shown.). We compare these results to those in the model with the contact-interaction, where the value of the exponent is $-0.5$\cite{15, 16}, reflecting from the characteristic time-scale $t^* \sim (v - v_c)^{-0.5}$ near the saddle-node bifurcation.

On the other hand, our result of the exponent in the supersolid phase is different from $-0.5$. These results suggest that supersolid phase has a different bifurcation structure.

Table 1. The values of fitting parameters $A$, $v_c$, and $c$.

| $g$    | $U_0$     | $A$      | $v_c/v_0$ | $c$   |
|--------|-----------|----------|-----------|-------|
| 25     | $5\epsilon_0a$ | 0.81     | 0.22824   | -0.578|
| 7.5    | $5\epsilon_0a$  | 0.99     | 0.15654   | -0.571|
| 10     | $\epsilon_0a$   | 1.06     | 0.11256   | -0.586|
Figure 4. Snapshots of the phase scaled by $2\pi$ at \( t = 25\tau, 40\tau, \) and \( 60\tau \) respectively. We fix \( \varphi(x = 0, t) = 0 \).

Figure 5. Time dependence of the winding number. The parameters are the same as Figure 2.

Figure 6. Log-log plot of the \( v \)-dependence of the period of phase slip for \( g = 25 \). Each straight line represents fitting function \( T/\tau = A(v - v_c)^{\varepsilon} \).

4. Conclusion

In the present work, we have investigated dynamics of one-dimensional supersolids above the critical velocity using the Gross-Pitaevskii equation with a finite-range two-body interaction. Periodic soliton-emissions have been found in the supersolid phase above the critical velocity. The period of these emissions was found to obey a scaling law.

A remaining problem is identification of the bifurcation type. To extend our results to higher dimensional systems is another future work.

Acknowledgements

We thank to D. A. Takahashi, S. Watabe, and Y. Nagai for useful discussions. This work is supported by Grant-in-Aid for Scientific Research (Grant No. 21540352) from JSPS. M. Kunimi acknowledges support by Grant-in-Aid for JSPS Fellows (239376).

References

[1] Andreev A F and Lifshitz I M 1969 Sov. Phys. JETP 29 1107
[2] Kim E and Chan M H W 2004 Nature (London) 427 225
[3] Kim E and Chan M H W 2005 Science 305 1941
[4] Leggett A J 1970 Phys. Rev. Lett. 25 1543
[5] Prokof’ev N 2007 Adv. Phys. 56 381
[6] Balibar S 2010 Nature, 464 176
[7] Griesmaier A, Werner J, Hensler S, Stuhler J, and Pfau T 2005 Phys. Rev. Lett. 94 160401
[8] Henkel N, Nath R, and Pohl T 2010 Phys. Rev. Lett. 104 195302
[9] Cinti F, Jain P, Boninsegni M, Micheli A, Zoller P, and Pupillo G 2010 Phys. Rev. Lett. 105 135301
[10] Landau L D 1941 J. Phys. (USSR) 5 71
[11] Hakim V 1997 Phys. Rev. E 55 2835
[12] Frisch T, Pomeau, Y, and Rica S 1992 Phys. Rev. Lett. 69 1644
[13] Aftalion A, Du Q, and Pomeau Y 2003 Phys. Rev. Lett. 91 090407
[14] Pham C -T and Brachet M, 2002 Physica D 163 127
[15] Gross E P, 1961 Nuovo Cimento 20 454
[16] Huepe C and Brachet M -E, 2000 Physica D 140 126
[17] Anderson P W 1966 Rev. Mod. Phys. 38 298