

SOME GEOMETRIC RELATIONS FOR EQUIPOTENTIAL CURVES

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ABSTRACT. Let \( U(r), r \in \Omega \subset \mathbb{R}^2 \) be a harmonic function that solves an exterior Dirichlet problem. If all the level sets of \( U(r), r \in \Omega \) are smooth Jordan curves, then there are several geometric inequalities that correlate the curvature \( \kappa(r) \) with the magnitude of gradient \( |\nabla U(r)| \) on each level set (“equipotential curve”). One of such inequalities is \( \langle |\kappa(r) - \langle \kappa(r) \rangle| |\nabla U(r)| - \langle |\nabla U(r)| \rangle \rangle \geq 0 \), where \( \langle \cdot \rangle \) denotes average over a level set, weighted by the arc length of the Jordan curve. We prove such a geometric inequality by constructing an entropy for each level set \( U(r) = \varphi \), and showing that such an entropy is convex in \( \varphi \). The geometric inequality for \( \kappa(r) \) and \( |\nabla U(r)| \) then follows from convexity and monotonicity of our entropy formula. A few other geometric relations for equipotential curves are also built on a convexity argument.

1. INTRODUCTION

1.1. Background and motivations. Consider a non-constant harmonic function \( U(r) \) that satisfies the Laplace equation

\[
\nabla^2 U(r) = 0, \quad r \in \Omega \subset \mathbb{R}^2
\]

in an unbounded domain \( \Omega \) whose boundary \( \partial \Omega \) is a smooth Jordan curve. We may further impose a Dirichlet boundary condition that \( U(r), r \in \partial \Omega \) remains a constant. (By the Riemann mapping theorem in complex analysis, we can deduce from this boundary condition that all the level sets of \( U(r), r \in \Omega \subset \mathbb{R}^2 \) are smooth Jordan curves, and \( |\nabla U(r)| \neq 0, r \in \Omega \cup \partial \Omega \).) The total flux across the level set \( \partial \Omega \) is prescribed as

\[
-\int_{\partial \Omega} \mathbf{n} \cdot \nabla U(r) \, ds = \Phi > 0,
\]

where \( \mathbf{n} \) denotes outward unit normal vector. (This is also the total flux across every level set of \( U(r), r \in \Omega \subset \mathbb{R}^2 \), according to the Laplace equation and Green’s theorem.) As |\( r \)| goes to infinity, we have the following asymptotic behavior:

\[
U(r) \sim -\frac{\Phi}{2\pi} \log \frac{2\pi |r|}{L_{\partial \Omega}},
\]

where \( L_{\partial \Omega} = \int_{\partial \Omega} ds \) denotes circumference of the boundary.

Such a 2-dimensional exterior Dirichlet problem (“2-exD” hereafter) is found in at least three different physical contexts, in which there are interesting questions regarding the geometry of a level set and the distribution of \( |\nabla U(r)| \) thereupon.

The first 2-exD example is the electrostatic equilibrium of metallic conductors with boundary \( \partial \Omega \). Let \( U(r) \) be the electric potential, then \( \partial \Omega \) is a level set (“equipotential curve” in physical terms). The static electric field \( \mathbf{E}(r) = -\nabla U(r) \), \( r \in \Omega \subset \mathbb{R}^2 \) must be divergence-free in a domain \( \Omega \) devoid of electric charges, according to one of the Maxwell equations \( \nabla \cdot \mathbf{E}(r) = 0 \), also known as the Gauß law of electrostatics [7, §1.4, §1.7]. The total surface flux \( \Phi \) is proportional to the total electric charge carried by the metallic conductor [7, §1.3]. According to electricians’ folklore (Fig. 1), sharp tips (which “stick out”) on the conductor surface tend to give off more electrostatic sparks than the depressed pits (which “cave

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in”), and higher curvatures tend to be associated with higher surface charge densities, the latter of which are pointwise proportional to $|\nabla U(r)|, r \in \partial \Omega$.

The second 2-exD case is given by the Hele-Shaw flow, which involves an inviscid fluid slowly spreading in a highly viscous fluid [6, 16]. Let $U(r) = p(r), r \in \Omega$ be the pressure field in the unbounded region $\Omega$ occupied by the highly viscous phase, then the velocity field $v(r) \propto \nabla p(r)$ is proportional to the pressure gradient. The incompressibility condition $\nabla \cdot v(r) = 0$ guarantees that the harmonicity of the pressure field $p(r), r \in \Omega$. Inside the inviscid phase $\mathbb{R}^2 \setminus (\Omega \cup \partial \Omega)$, the gradient of the pressure vanishes $\nabla p(r) = 0, r \in \mathbb{R}^2 \setminus (\Omega \cup \partial \Omega)$, hence the pressure $p(r)$ remains constant on the fluid interface $\partial \Omega$, bringing us again an exterior Dirichlet problem for $U(r) = p(r)$. As the inviscid phase expands, the fluid interface moves at a velocity $v(r) \propto \nabla p(r)$, the total flux $\Phi$ is proportional to the volume growth rate of the bounded inviscid phase. Empirically speaking, as the volume of $\mathbb{R}^2 \setminus (\Omega \cup \partial \Omega)$ increases, the “tips” have a higher tendency than the “pits” to stretch out further. This analog of electricians’ folklore then gives rise to a tree-like pattern (technically speaking, “dendritic growth”). Needless to say, quantitative relations between the shape of $\partial \Omega$ and the distribution of $|\nabla p(r)|, r \in \partial \Omega$ will help us understand the evolution of interface morphology under the Hele-Shaw flow.

Our third 2-exD example concerns the diffusion limited aggregation [19, 18], a dendritic growth process that is more or less a stochastic version of the Hele-Shaw flow. Consider a suspended colloidal particle executing a random walk in the unbounded region $\Omega$, and assume that the random walker will stick to a pre-existing aggregate occupying $\mathbb{R}^2 \setminus (\Omega \cup \partial \Omega)$ as soon as it hits the boundary $\partial \Omega$ of the cluster. By the harmonic theory of Brownian motion [9], the “probability density” $U(r) = \rho(r), r \in \Omega$ of finding a colloidal particle is a harmonic function in $\Omega$ that assumes constant value at the boundary $\partial \Omega$, and the magnitude of the boundary gradient $|\nabla U(r)| = |\nabla \rho(r)|$ at the level set $\partial \Omega$ is proportional to the probability of growth. There is experimental evidence that the fjord regions are generally less frequently visited by the random walker than the sharp tips in this stochastic growth process, so that the high probability of growth (i.e. large values of $|\nabla U(r)|$) and the outstanding edges seem to concur, giving rise to a branched, tree-like pattern. In all these 2-exD problems, the overall shape of a level set affects the gradient $\nabla U(r)$ thereupon in a non-local manner: it is technically incorrect to say that a large local curvature causes a large local value of $|\nabla U(r)|$. Instead of seeking a pointwise causation that ties curvature to gradient, we will look for statistical correlations between the curvature of the level set and the magnitude of $\nabla U(r)$, such as the geometric inequality stated in the abstract.

1We need regularization at large distances to make this density function integrable.

2While introducing diffusion-limited aggregation, Witten and Sander [18] stated that “... it is well known that electric fields are large near sharp points on a conductor; thus sharp points grow unstably ...” without providing further quantitative arguments to substantiate this “well-known” folklore.
where \( \kappa \) (1.6)–(1.11). Such convexity arguments are morally similar to entropy monotonicity relations for level \( \Sigma \) is the average over \( \Omega \cup \partial \Omega \subset \mathbb{R}^2 \). We show that all these expressions are convex functions in \( \phi \). Partial differential equations that arise from classical and quantum physics.

1.2. Statement of results and plan of proof. Let \( \kappa(r) \) be the signed curvature of the level set at a point \( r \in \Omega \cup \partial \Omega \subset \mathbb{R}^2 \). In this work, we follow the convention that the unit circle has positive curvature: \( \kappa = +1 \). For simplicity, we will write \( \kappa(r) \). These techniques not only inject geometric insights into level sets of harmonic functions, but also add to our geometric understanding of many other types of (linear or non-linear) partial differential equations that correlate \( \kappa(r) \) with \( E(r) \) on each level set \( \Sigma \) in 2-exD problems:

\[
\text{cov}_\Sigma \left( \frac{\kappa(r)}{E(r)}, E(r) \right) \geq 0,
\]

where \( \text{cov}_\Sigma \left( \kappa(r), E(r) \right) \geq 0 \), \( \text{cov}_\Sigma \left( \kappa(r), E(r) \right) \geq 0 \), \( \text{cov}_\Sigma \left( \frac{\kappa(r)}{2\pi \log \frac{E(r) L_\Sigma}{\Phi}}, \frac{E(r) L_\Sigma}{\Phi} \right) \geq 0 \), \( \text{cov}_\Sigma \left( \frac{\kappa(r)}{E(r)} \right) \geq \frac{2\pi}{\Phi} \frac{1}{E(r)} \),

\[
\left\langle \frac{\kappa(r)}{E(r)} \right\rangle \leq \frac{2\pi}{\Phi} \left\langle \frac{1}{E(r)} \right\rangle,
\]

where \( \nabla f(r) \) denotes tangential gradient of a scalar \( f(r) \) (up to a 90° rotation). Furthermore, these inequalities are strict, unless \( \partial \Omega \) is a circle.

Some of these geometric inequalities provide quantitative support for the aforementioned electricians’ folklore, and also explain the statistical trend in dendritic growth governed by Hele-Shaw flow or diffusion limited aggregation.

In §2 we study four functionals of \( \kappa(r) \) and \( E(r) \), which are integrals over the level set \( \Sigma_\varphi := \{ r \in \Omega \cup \partial \Omega | U(r) = \varphi \} \):

\[
\mathcal{H}(\varphi) := \int_{\Sigma_\varphi} \frac{E(r) L_\Sigma}{\Phi} \log \frac{E(r) L_\Sigma}{\Phi} \, ds,
\]

\[
\mathcal{E}(\varphi) := \int_{\Sigma_\varphi} \frac{\kappa^2(r)}{E(r)} \, ds - \int_{\Sigma_\varphi} \frac{\left| \nabla \log E(r) \right|^2}{E(r)} \, ds,
\]

\[
\mathcal{F}(\varphi) := \int_{\Sigma_\varphi} \frac{\left| \nabla \log E(r) \right|^2}{E(r)} \, ds,
\]

\[
\mathcal{L}(\varphi) := \log \int_{\Sigma_\varphi} \frac{ds}{E(r)}.
\]

We show that all these expressions are convex functions in \( \varphi \), and their convexity eventually entails (1.6)–(1.11). Such convexity arguments are morally similar to entropy monotonicity relations for level...
sets of Green’s functions in manifolds of dimension greater than 2, as developed by Colding [1] and Colding–Minicozzi [2, 3].

In §3 we generalize our analysis to the Green’s functions for 2-dimensional interior Dirichlet problems (“2-inD” hereafter). For an unbounded domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial \Omega$ is a smooth Jordan curve, the 2-inD problem can be formulated as

$$\begin{cases} \nabla^2 G(r) = 0, & r \in \mathbb{R}^2 \setminus (\Omega \cup \partial \Omega \cup \{0\}), \\ G(r) = \text{const.}, & r \in \partial \Omega, \\ -\lim_{t \to 0^+} \int_{|r|=\varepsilon} n \cdot \nabla G(r) \, d\sigma = \Phi, \end{cases} \tag{1.16}$$

where $\Phi \neq 0$. (As before, we automatically have $|\nabla G(r)| \neq 0$ for $r \in \mathbb{R}^2 \setminus (\Omega \cup \{0\})$, by the Riemann mapping theorem. If we set $\Phi = 1$ in the 2-inD problem, then $G(r) = G_{\partial \Omega}^D(0, r)$ is called a Dirichlet Green’s function in electrostatics.) Let $\Sigma$ be a level set of $G(r)$ in $\mathbb{R}^2 \setminus (\Omega \cup \{0\})$, then we will show that

$$\text{cov}_{\Sigma} \left( \frac{\kappa(r)}{|\nabla G(r)|}, |\nabla G(r)| \right) \leq 0, \tag{1.6}$$

$$\text{cov}_{\Sigma} \left( \frac{\kappa(r)}{|\nabla G(r)|}, \log \frac{|\nabla G(r)|L_{\Sigma}}{\Phi} \right) \leq \text{cov}_{\Sigma} \left( \frac{|\nabla G(r)|}{\Phi}, \log \frac{|\nabla G(r)|L_{\Sigma}}{\Phi} \right), \tag{1.9}$$

$$\left\langle \frac{\kappa(r)}{|\nabla G(r)|^2} \right\rangle_\Sigma \geq \frac{2\pi}{\Phi} \left\langle \frac{1}{|\nabla G(r)|} \right\rangle_\Sigma, \tag{1.10}$$

$$\left\langle \frac{\kappa(r)}{|\nabla G(r)|^2} \right\rangle_\Sigma \geq \frac{2\pi}{\Phi} \left\langle \frac{1}{|\nabla G(r)|} \right\rangle_\Sigma, \tag{1.11}$$

where the inequalities are strict except when $\partial \Omega$ is a circle centered at the origin $0$. There is a notable reversal of inequality signs [as compared to (1.6), (1.9), (1.10) and (1.11) for 2-exD], whose physical significance will be explained in §3.2.

In §4 we show that our derivations are sensitive to level set topology and space dimension. First, we discuss how our arguments are affected when the equipotential curve is no longer a smooth Jordan curve. Second, we mention some technical difficulties that one may encounter while extending the present results to Euclidean spaces in higher dimensions.

2. Entropy, Conservation Law and Geometric Inequalities

2.1. Geometric entropy and curvature correlations. The function $\mathcal{H}(\varphi)$ defined in (1.12) is the relative entropy between two mutually non-singular probability measures: the flux density $d\mu = E(r)\, d\, s/\Phi$ and the line density $d\nu = d\, s/L_{\Sigma}$. By Jensen’s inequality, we have

$$\mathcal{H}(\varphi) = -\int_{\Sigma^c} \log \frac{d\nu}{d\mu} \, d\mu \geq -\log \int_{\Sigma^c} \frac{d\nu}{d\mu} \, d\mu = 0. \tag{2.1}$$

Before evaluating the derivatives $\mathcal{H}'(\varphi)$ and $\mathcal{H}''(\varphi)$, we need some geometric preparations.

We assign local orthogonal curvilinear coordinates $r(\varphi, u)$ to points $r \in \Omega \cup \partial \Omega$, so that $\varphi$ coincides with $U(r)$, and a pair of points on different level sets share the same $u$ coordinate if and only if they are joined by an integral curve of $\nabla U(r)$ (known as “electric field line” in electrostatics). In such a curvilinear coordinate system, we can rewrite the Euclidean metric $d\, r \cdot d\, r = (dx)^2 + (dy)^2$ as

$$d\, r \cdot d\, r = \frac{(d\varphi)^2}{E^2} + g(du)^2, \quad \text{where} \quad \frac{1}{E} := \frac{1}{|\nabla U(r)|} = \left| \frac{\partial r}{\partial \varphi} \right|, \quad g := \left| \frac{\partial r}{\partial u} \right|^2. \tag{2.2}$$

Unfortunately, due to the reversed inequality in (1.6), we are not yet able to produce an analog of (1.7) or (1.8), for the 2-inD problems.
Accordingly, the Laplacian in the Euclidean space \( \Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) can be decomposed into [20, Proposition 1.2]

\[
\Delta = \Delta_{\Sigma_\varphi} + E^2 \frac{\partial^2}{\partial \varphi^2} - \frac{1}{gE} \frac{\partial E}{\partial u} \frac{\partial}{\partial u}.
\]  

(2.3)

Here,

\[
\Delta_{\Sigma_\varphi} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u} \left( \sqrt{g} \frac{\partial}{\partial u} \right) = \frac{\partial^2}{\partial s^2}
\]  

(2.4)

is the Laplacian on the level set \( \Sigma_\varphi \), whose arc length parameter \( s \) satisfies \((d s)^2 = g(d u)^2\). It is then an elementary exercise in differential geometry to show that [20, (5), (6) and Proposition 1.3]

\[
\frac{\partial (E \sqrt{g})}{\partial \varphi} = 0, \quad \frac{\partial E}{\partial \varphi} = \kappa, \quad \frac{\partial \kappa}{\partial \varphi} = \frac{\kappa^2}{E} + \Delta_{\Sigma_\varphi} \frac{1}{E}.
\]  

(2.5)

Combining the tools above with the Umlaufsatz \( \oint_{\Sigma_\varphi} \kappa \, d s = 2\pi \), we can readily compute

\[
\mathcal{H}'(\varphi) = \frac{1}{\Phi} \oint_{\Sigma_\varphi} E \frac{\partial}{\partial \varphi} \left( \log \frac{E L_{\Sigma_\varphi}}{\Phi} \right) \, d s = \oint_{\Sigma_\varphi} \kappa \frac{d s}{\Phi} - \oint_{\Sigma_\varphi} \frac{\kappa d s}{E L_{\Sigma_\varphi}} = \frac{2\pi}{\Phi} - \oint_{\Sigma_\varphi} \kappa \frac{d s}{E L_{\Sigma_\varphi}},
\]  

(2.6)

and

\[
\mathcal{H}''(\varphi) = -\oint_{\Sigma_\varphi} \frac{1}{E} \Delta_{\Sigma_\varphi} \frac{1}{E} \frac{d s}{L_{\Sigma_\varphi}} - \frac{1}{L_{\Sigma_\varphi}} \oint_{\Sigma_\varphi} \kappa \frac{\partial}{\partial \varphi} \left( \frac{1}{E} \right) \, d s - \left( \oint_{\Sigma_\varphi} \frac{\kappa d s}{E L_{\Sigma_\varphi}} \right) \frac{\partial}{\partial \varphi} \left( \frac{1}{L_{\Sigma_\varphi}} \right)
\]

\[
= \left( \left| \mathbf{n} \times \nabla \frac{1}{E} \right|^2 - \frac{\kappa}{E} - \left( \frac{\kappa}{E / \Sigma_\varphi} \right)^2 \right) \geq 0.
\]  

(2.7)

Here, the term \( \left| \mathbf{n} \times \nabla \frac{1}{E} \right|^2 \, d s = \left( \frac{\partial}{\partial s} \frac{1}{E} \right)^2 \, d s \) arises from integrating the differential form \( -\frac{1}{E} \Delta_{\Sigma_\varphi} \frac{1}{E} \, d s = -\frac{1}{E} \frac{\partial^2}{\partial s^2} \frac{1}{E} \, d s \) by parts. This proves the convexity of our entropy function \( \mathcal{H}(\varphi) \). Furthermore, it is clear that we have \( \mathcal{H}''(\varphi) > 0 \) unless \( E \) and \( \kappa \) both remain constant on \( \Sigma_\varphi \).

Since we can rewrite the computations above as

\[
\mathcal{H}'(\varphi) = \frac{2\pi}{\Phi} + \frac{d}{d \varphi} \log L_{\Sigma_\varphi}, \quad \mathcal{H}''(\varphi) = \frac{d^2}{d \varphi^2} \log L_{\Sigma_\varphi} \geq 0,
\]  

(2.8)

we also have a by-product: the circumference \( L_{\Sigma_\varphi} \) of the equipotential curve \( \Sigma_\varphi \) is logarithmically convex in \( \varphi \). This strengthens a previous result of Longinetti [13, Theorem 3.1], which demonstrated logarithmic convexity for convex ring configurations, without using an entropy argument.

As \( |r| \to +\infty \), we have \( \varphi \to -\infty \) according to (1.3), and

\[
\kappa(r) \sim \frac{1}{|r|}, \quad E(r) \sim \frac{\Phi}{2\pi|r|},
\]  

(2.9)

so

\[
\lim_{\varphi \to -\infty} \mathcal{H}'(\varphi) = \frac{2\pi}{\Phi} \frac{2\pi}{\Phi} \lim_{\varphi \to -\infty} \oint_{\Sigma_\varphi} \frac{d s}{L_{\Sigma_\varphi}} = 0,
\]  

(2.10)

and

\[
\mathcal{H}'(\varphi) = \int_{-\infty}^{\varphi} \mathcal{H}''(\phi) \, d \phi \geq 0.
\]  

(2.11)
The last inequality can be recast into
\[
\frac{2\pi}{\Phi} - \int_{\Sigma_\phi} \frac{\kappa \, d s}{E L_{\Sigma_\phi}} = \frac{L_{\Sigma_\phi}}{\Phi} \left( \int_{\Sigma_\phi} \frac{\kappa E \, d s}{E L_{\Sigma_\phi}} - \int_{\Sigma_\phi} \frac{\kappa d s}{L_{\Sigma_\phi}} \int_{\Sigma_\phi} \frac{E \, d s}{E L_{\Sigma_\phi}} \right) = \frac{L_{\Sigma_\phi}}{\Phi} \text{cov}_{\Sigma_\phi} \left( \frac{\kappa}{E} \right) \geq 0, \tag{2.12}
\]
which proves our first geometric inequality stated in (1.6). Such an inequality becomes an equality only if \(\mathcal{H}''(\phi) = 0\) for all \(\phi < \phi\), that is, both \(E\) and \(\kappa\) remain constant on every level set enclosing \(\Sigma_\phi\)—a scenario that happens only when \(\partial \Omega\) is a circle.

We note that \(\frac{1}{2} \int_{\Sigma_\phi} \nabla \cdot \mathbf{r} \, d s = \int_{\Sigma_\phi} d^2 \mathbf{r} = A_{\Sigma_\phi}\) equals the area of the region \(\Sigma_\phi\) bounded by the equipotential curve \(\Sigma_\phi\). To prove our second geometric inequality (1.7) for \(\Sigma = \partial \Sigma\), we can rearrange (1.7) into
\[
\frac{4\pi A_{\Sigma_\phi}}{L_{\partial \Sigma_\phi}^2} \left( 2 \left( \frac{\kappa}{E} \right)_{\Sigma_\phi} - \frac{L_{\Sigma_\phi}}{A_{\Sigma_\phi}} \right) = \frac{d}{d \phi} A_{\Sigma_\phi} \geq 0. \tag{2.13}
\]
Here, in the first step, we have exploited the Cauchy–Schwarz inequality and the correlation inequality in (1.6). In the last step, we have used the isoperimetric inequality. In view of the coarea formula [13 (4.1)]
\[
\frac{d}{d \phi} A_{\Sigma_\phi} = \int_{\Sigma_\phi} \frac{d s}{E}, \tag{2.14}
\]
we can rearrange (1.7) into
\[
\frac{4\pi A_{\Sigma_\phi}}{L_{\partial \Sigma_\phi}^2} \left( 2 \left( \frac{\kappa}{E} \right)_{\Sigma_\phi} - \frac{L_{\Sigma_\phi}}{A_{\Sigma_\phi}} \right) = \frac{d}{d \phi} \frac{4\pi A_{\Sigma_\phi}}{L_{\partial \Sigma_\phi}^2} \leq 0. \tag{2.15}
\]
Therefore, the left-hand side of the isoperimetric inequality \(1 - 4\pi A_{\Sigma_\phi}/L_{\partial \Sigma_\phi}^2 \geq 0\) decays monotonically to zero, as \(|r| \to +\infty, \phi \to -\infty\). In other words, as one tracks down the electrostatic potential \(\phi\), the equipotential curve \(\Sigma_\phi\) becomes rounder and rounder. This trend is strictly monotone, unless both \(E\) and \(\kappa\) remain constant on every level set, that is, unless \(\partial \Omega\) is a circle.

We can prove our third geometric inequality (1.8) by differentiating
\[
L_{\Sigma_\phi} \text{cov}_{\Sigma_\phi} (\kappa, E) = \int_{\Sigma_\phi} \kappa E \, d s - \frac{2\pi \Phi}{L_{\Sigma_\phi}} \tag{2.16}
\]
as follows:
\[
\frac{d}{d \phi} \left[ L_{\Sigma_\phi} \text{cov}_{\Sigma_\phi} (\kappa, E) \right] = \int_{\Sigma_\phi} E A_{\Sigma_\phi} \frac{1}{E} \, d s + \int_{\Sigma_\phi} \kappa^2 \, d s - \frac{2\pi \Phi}{L_{\Sigma_\phi}^2} \int_{\Sigma_\phi} \kappa \, d s
\]
\[
= \int_{\Sigma_\phi} |n \times \nabla \log E|^2 \, d s + \int_{\Sigma_\phi} \kappa^2 \, d s + \frac{2\pi \Phi}{L_{\Sigma_\phi}} \left[ \mathcal{H}'' (\phi) - \frac{2\pi}{\Phi} \right]
\]
\[
\geq L_{\Sigma_\phi} \left[ |n \times \nabla \log E|^2 + (\kappa - \langle \kappa \rangle_{\Sigma_\phi})^2 \right]_{\Sigma_\phi} \geq 0, \tag{2.17}
\]
and noting that \(\lim_{\phi \to -\infty} L_{\Sigma_\phi} \text{cov}_{\Sigma_\phi} (\kappa, E) = 0\). Here, we have exploited (2.11) in the penultimate step of (2.17). Again, we have a strict inequality \(\text{cov}_{\Sigma_\phi} (\kappa, E) > 0\) unless \(\kappa\) and \(E\) remain constant on each level set enclosing \(\Sigma_\phi\), which only occurs when \(\partial \Omega\) is a circle.

The inequality (1.8) has a simple application in the Hele-Shaw flow, where the boundary surface \(\partial \Omega_t\) moves according to \(\partial r/\partial t = v(r, t) n\) for \(r \in \partial \Omega_t\). Here, we have \(v(r, t) = |\nabla U(r, t)|\), where \(U(r, t)\) solves 2-exD in the unbounded region \(\Omega_t\). Direct computation reveals that
\[
\frac{d}{dt} \left( L_{\partial \Omega_t}^2 - 4\pi A_{\mathbb{R}^2 \setminus \Omega_t} \right) = 2L_{\partial \Omega_t}^2 \text{cov}_{\partial \Omega_t} (\kappa(r, t), v(r, t)) \geq 0. \tag{2.18}
\]
Therefore, the Hele-Shaw flow drives the boundary away from roundness, as measured by the monotonically non-decreasing quantity \(L_{\partial \Omega_t}^2 - 4\pi A_{\mathbb{R}^2 \setminus \Omega_t}\).
We pause to offer a concrete example that supports the inequalities in (1.6) and (1.8). Let \( \partial \Omega \) be an ellipse in a 2-exD problem. One can show, by using Joukowsky’s conformal mapping or the like, that all the level sets \( \Sigma_\varphi \) are ellipses confocal with \( \partial \Omega \), and that the expression \( \kappa/E^3 \) remains constant on each equipotential curve \( \Sigma_\varphi \). In this specific example, the inequalities in (1.6) and (1.8) become \( \text{cov}_{\Sigma_\varphi}(E^2,E) \geq 0 \) and \( \text{cov}_{\Sigma_\varphi}(E^3,E) \geq 0 \). This is not unexpected, as a positive random variable \( X \) and its own positive power \( X^\alpha \) (\( \alpha > 0 \)) always have non-negative covariance:

\[
\frac{\text{cov}(X,X^\alpha)}{\langle X \rangle^{1+\alpha}} := \frac{\langle X^{1+\alpha} \rangle - \langle X \rangle \langle X^\alpha \rangle}{\langle X \rangle^{1+\alpha}} = \int_0^\alpha \left( \frac{X^\beta}{\langle X \rangle^\beta} \right) \log \frac{X}{\langle X \rangle} \, d\beta \geq 0.
\]  
(2.19)

2.2. Geometric conservation law and correlation comparison. Our next goal is to show that the quantity \( \mathcal{E}(\varphi) \) defined in (1.13) is in fact a conservation law:

\[
\mathcal{E}(\varphi) := \int_{\Sigma_\varphi} \frac{\kappa^2}{E} \, d\mu = \frac{\|n \times \nabla \log E \|^2}{E} \equiv 4\pi^2/\Phi.
\]  
(2.20)

In other words, there is an exact identity with respect to the probability measure \( d\mu = E \, ds/\Phi \):

\[
\int_{\Sigma_\varphi} \left( \frac{\kappa}{E} - \int_{\Sigma_\varphi} \frac{\kappa}{E} \, d\mu \right)^2 \, d\mu = \int_{\Sigma_\varphi} \left| n \times \nabla \frac{1}{E} \right|^2 \, d\mu.
\]  
(2.21)

From this identity, it is also clear that \( E(r) \equiv \text{const.}, r \in \Sigma_\varphi \) implies that \( \Sigma_\varphi \) is a circle.

To demonstrate the conservation law, we enlist the help from (2.5) to compute

\[
\frac{d}{d\varphi} \left[ \int_{\Sigma_\varphi} \frac{\kappa^2}{E} \, ds - \int_{\Sigma_\varphi} \left| n \times \nabla \log E \right|^2 \, ds \right]
= \int_{\Sigma_\varphi} \frac{2\kappa}{E} \left( \Delta_1 \frac{1}{E} + \kappa^2 \right) \, ds - \int_{\Sigma_\varphi} \frac{\kappa^2}{E} \, ds - 2 \int_{\Sigma_\varphi} \frac{1}{E} \, ds - 2 \int_{\Sigma_\varphi} \frac{\partial (\log E) \, \partial (\kappa/E)}{\partial u} \, du
= \int_{\Sigma_\varphi} \frac{2\kappa}{E} \, ds + \int_{\Sigma_\varphi} \frac{\partial (1/E) \, \partial (2\kappa/E)}{\partial s} \, ds
= \int_{\Sigma_\varphi} \frac{\partial}{\partial s} \left( \frac{2\kappa \, \partial (1/E) \, \partial (2\kappa/E)}{\partial s} \right) \, ds = 0,
\]  
(2.22)

and then equate \( \mathcal{E}(\varphi) \) with \( \lim_{\varphi \to -\infty} \mathcal{E}(\varphi) \). To evaluate the last limit, we need two observations. First, the asymptotic behavior in (2.9) immediately leads us to \( \lim_{\varphi \to -\infty} \int_{\Sigma_\varphi} \frac{\kappa^2}{E} \, ds = 4\pi^2/\Phi \). Second, we have the following multipole expansion for a harmonic function:\footnote{The expression \( \log|\nabla U(xe_1 + ye_2)| \) is the real part of a complex-analytic function, hence harmonic. Here, one can construct \( f(x + iy) = U(xe_1 + ye_2) + iV(xe_1 + ye_2) \) from \( U \) and its conjugate harmonic function \( V \).}

\[
\log E(r)(e_x \cos \theta + e_y \sin \theta) = c_0 - \log |r| + \frac{c_1 \cos \theta + s_1 \sin \theta}{|r|} + O \left( \frac{1}{|r|^2} \right)
\]  
(2.23)

for constants \( c_0, c_1, s_1 \), which enables us to estimate

\[
|n \times \nabla \log E(r)| = O \left( \frac{1}{|r|^2} \right), \quad \text{as } |r| \to +\infty,
\]  
(2.24)

so \( \lim_{\varphi \to -\infty} \int_{\Sigma_\varphi} \left| n \times \nabla \log E(r) \right| \, ds = 0 \).

Summarizing what we have done in the last paragraph, we see that \( \mathcal{E}(\varphi) \equiv 4\pi^2/\Phi \) holds.

Such a conservation law is useful in our subsequent investigation of

\[
L_{\Sigma_\varphi} \left[ \text{cov}_{\Sigma_\varphi} \left( \frac{\kappa}{2\pi}, \log \frac{E_{\Sigma_\varphi}}{\Phi} \right) - \text{cov}_{\Sigma_\varphi} \left( \frac{E}{\Phi}, \log \frac{E_{\Sigma_\varphi}}{\Phi} \right) \right]
= \int_{\Sigma_\varphi} \kappa \log \frac{E_{\Sigma_\varphi}}{\Phi} \, ds - \int_{\Sigma_\varphi} E \log \frac{E_{\Sigma_\varphi}}{\Phi} \, ds,
\]  
(2.25)
whose derivative evaluates to

\[
\frac{1}{2\pi} \oint_{\Sigma_{\varphi}} \left( \Delta_{\Sigma} \frac{1}{E} \right) \log \frac{E L_{\Sigma_{\varphi}}}{\Phi} \, ds + \frac{1}{2\pi} \oint_{\Sigma_{\varphi}} \frac{\kappa^2}{E} \, ds - \frac{1}{\Phi} \oint \kappa \, ds \\
= \frac{1}{2\pi} \left( \oint_{\Sigma_{\varphi}} \frac{\kappa^2}{E} \, ds + \oint_{\Sigma_{\varphi}} \frac{|n \times \nabla \log E|^2}{E} \, ds - 4\pi^2 \right) + \frac{1}{\pi} \oint_{\Sigma_{\varphi}} |n \times \nabla \log E|^2 \, ds \geq 0.
\]  

(2.26)

Now that the expression in (2.25) vanishes as \( \varphi \to -\infty \), by virtue of (2.9), we have the correlation comparison inequality stated in (1.9). As before, we can argue that (1.9) becomes an equality only when \( \partial\Omega \) is a circle.

In particular, from (1.9) and (2.1), we know that

\[
\oint_{\Sigma_{\varphi}} \kappa \log \frac{E L_{\Sigma_{\varphi}}}{\Phi} \, ds \geq \oint_{\Sigma_{\varphi}} E \log \frac{E L_{\Sigma_{\varphi}}}{\Phi} \, ds \geq 0. \tag{2.27}
\]

This inequality provides semi-quantitative insights into electricians’ folklore when \( \kappa(r), r \in \partial\Omega \) changes sign \( \kappa(r) > 0 \) for “tips” and \( \kappa(r) < 0 \) for “pits”). Concretely speaking, if \( E(r), r \in \partial\Omega \) in this 2-exD problem went against electricians’ folklore point by point, with weaker field \( E(r) L_{\partial\Omega} < \Phi \) at “tips” and stronger field \( E(r) L_{\partial\Omega} > \Phi \) at “pits”, then the left-most integral in (2.27) would become negative—a contradiction.

However, it is worth pointing out that (2.27) is not strong enough to bring us non-negativity of the integrand \( \kappa(r) \log \frac{E L_{\Sigma_{\varphi}}}{\Phi} \) at every point \( r \in \Sigma_{\varphi} \). Like other inequalities presented in this work, it provides only a statistical correlation averaged over a level set \( \Sigma_{\varphi} \), rather than a pointwise causation between \( \kappa(r) \) and \( E(r) \) on \( \Sigma_{\varphi} \).

2.3. Statistical (mis)alignment of tangential gradients. We say that the tangential gradients of two scalar functions \( f_1(r), r \in \Sigma_{\varphi} \) and \( f_2(r), r \in \Sigma_{\varphi} \) are aligned (resp. misaligned) at a point \( r \) if

\[
[n \times \nabla f_1(r)] \cdot [n \times \nabla f_2(r)] \geq 0 \quad \text{(resp. } [n \times \nabla f_1(r)] \cdot [n \times \nabla f_2(r)] \leq 0). \tag{2.28}
\]

We say that there are statistical alignment (resp. misalignment) of tangential gradients if

\[
\oint_{\Sigma_{\varphi}} [n \times \nabla f_1(r)] \cdot [n \times \nabla f_2(r)] \, ds \geq 0 \quad \text{(resp. } \oint_{\Sigma_{\varphi}} [n \times \nabla f_1(r)] \cdot [n \times \nabla f_2(r)] \, ds \leq 0). \tag{2.29}
\]

Statistical (mis)alignment of tangential gradients tells us how the monotonicity of two functions, on average, are tied to each other.

Before moving onto the convexity proof for

\[
F(\varphi) := \oint_{\Sigma_{\varphi}} \left| n \times \nabla \log E \right|^2 \, ds = \oint_{\Sigma_{\varphi}} \frac{\kappa^2}{E} \, ds - \frac{4\pi^2}{\Phi}, \tag{2.30}
\]

we exploit the decomposition of Laplacian in (2.3) to show that

\[
\frac{d^2}{d\varphi^2} \oint_{\Sigma_{\varphi}} E(r) f(r) \, ds = \oint_{\Sigma_{\varphi}} \frac{1}{E} (\Delta f - \Delta_{\Sigma} f) \, ds - \oint_{\Sigma_{\varphi}} \frac{1}{g} \frac{\partial (1/E)}{\partial u} \frac{\partial f}{\partial u} \, ds = \oint_{\Sigma_{\varphi}} \frac{1}{E} \Delta f \, ds, \tag{2.31}
\]

holds for any suitably regular \( f(r) \) defined in a neighborhood of \( \Sigma_{\varphi} \). Here, in the last step, we have integrated by parts:

\[
- \oint_{\Sigma_{\varphi}} \frac{1}{E} \Delta_{\Sigma} f \, ds - \oint_{\Sigma_{\varphi}} \frac{1}{g} \frac{\partial (1/E)}{\partial u} \frac{\partial f}{\partial u} \, ds = - \oint_{\Sigma_{\varphi}} \frac{1}{E} \frac{\partial^2 f}{\partial s^2} \, ds - \oint_{\Sigma_{\varphi}} \frac{\partial (1/E)}{\partial s} \frac{\partial f}{\partial s} \, ds = 0. \tag{2.32}
\]
By virtue of (2.31) and Talenti’s relation $\Delta \frac{\kappa}{E} = 0$ [17], we can quickly compute
\[
\mathcal{F}''(\varphi) = \frac{d^2}{d\varphi^2} \int_{\Sigma_{\varphi}} \frac{\kappa^2}{E} d\sigma - \int_{\Sigma_{\varphi}} \frac{1}{E} \Delta \frac{\kappa}{E} d\sigma
\]
\[
= 2 \int_{\Sigma_{\varphi}} \left| \nabla \frac{\kappa}{E} \right|^2 d\sigma + 2 \int_{\Sigma_{\varphi}} \frac{1}{E} \Delta \frac{\kappa}{E} d\sigma = 2 \int_{\Sigma_{\varphi}} \frac{1}{E} \left| \nabla \frac{\kappa}{E} \right|^2 d\sigma \geq 0. \tag{2.33}
\]
From this convexity we can deduce the monotonicity of $\mathcal{F}(\varphi)$:
\[
0 \geq -\mathcal{F}'(\varphi) = - \frac{d}{d\varphi} \int_{\Sigma_{\varphi}} \frac{\kappa^2}{E} d\sigma = - 2 \int_{\Sigma_{\varphi}} \frac{\kappa}{E} \frac{\partial}{\partial \varphi} \left( \frac{\kappa}{E} \right) E d\sigma
\]
\[
= - 2 \int_{\Sigma_{\varphi}} \frac{\kappa}{E} \frac{\partial^2}{\partial \varphi^2} \frac{1}{E} d\sigma = 2 \int_{\Sigma_{\varphi}} \left[ n \times \nabla \frac{\kappa}{E} \right] \cdot \left[ n \times \nabla \frac{1}{E} \right] d\sigma, \tag{2.34}
\]
after we establish $\lim_{\varphi \to -\infty} \mathcal{F}'(\varphi) = 0$ on the asymptotic behavior:
\[
\left| n \times \nabla \frac{\kappa(r)}{E(r)} \right| = O \left( \frac{1}{|r|^2} \right), \quad \left| n \times \nabla \frac{1}{E(r)} \right| = O(1), \tag{2.35}
\]
as $|r| \to +\infty$. Here, the tangential derivative of the harmonic function $\frac{\kappa}{E}$ can be estimated in a similar fashion as (2.24).

This proves the statistical (mis)alignment of tangential gradients stated in (1.10). This is a strict inequality unless $\nabla \frac{\kappa}{E} = 0$ on all the level sets enclosing $\Sigma_{\varphi}$, in view of (2.33). Since $0 = \frac{\partial}{\partial \sigma} \frac{\kappa}{E} = \frac{1}{E} \frac{\partial^2}{\partial \varphi^2} \frac{1}{E}$ implies constant $E(r)$ on each level set, and $n \times \nabla \frac{\kappa}{E} = 0$ further entails constant $\kappa(r)$ on each level set, the situation $\mathcal{F}'(\varphi) = 0$ happens only if $\partial \Omega$ is a circle.

Admittedly, at this point, we have not yet exhausted all the possible geometric integrals that are convex in $\varphi$. By (2.31) and the Kong–Xu equation $\Delta \frac{n \times \nabla \kappa}{E} = 0$ [10], one can also show that $\frac{d^2}{d\sigma^2} \int_{\Sigma_{\varphi}} \frac{1}{E} \left| n \times \nabla \kappa \right|^2 d\sigma \geq 0$. However, we are unable to reinterpret the first-order derivative $\frac{d}{d\varphi} \int_{\Sigma_{\varphi}} \frac{1}{E} \left| n \times \nabla \kappa \right|^2 d\sigma$ as statistical (mis)alignment of geometrically/physically interesting quantities, as in the case of $\mathcal{F}'(\varphi)$.

2.4. Longinetti functional and weighted correlation. In [13, Theorem 4.1], Longinetti has shown that $\mathcal{L}(\varphi) := \log \frac{f_{\varphi}}{f_{E}}$ is convex in $\varphi$, using support functions in convex domains. Now, we will prove $\mathcal{L}''(\varphi) \geq 0$, namely
\[
\int_{\Sigma_{\varphi}} \frac{d\sigma}{E} \frac{d^2}{d\varphi^2} \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} \geq \left( \frac{d}{d\varphi} \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} \right)^2 \tag{2.36}
\]
without requiring geometric convexity.

Since the differential formulae in (2.35) bring us
\[
\frac{d}{d\varphi} \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} = -2 \int_{\Sigma_{\varphi}} \frac{\kappa}{E^2} d\sigma, \tag{2.37}
\]
and
\[
\frac{d}{d\varphi} \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} = -2 \int_{\Sigma_{\varphi}} \frac{1}{E^2} \frac{\partial^2}{\partial \varphi^2} \frac{1}{E} d\sigma + 4 \int_{\Sigma_{\varphi}} \frac{\kappa^2}{E^3} d\sigma = 4 \int_{\Sigma_{\varphi}} \left( \left| n \times \nabla \frac{1}{E} \right|^2 + \frac{\kappa^2}{E^2} \right) d\sigma, \tag{2.38}
\]
we may proceed with the computation
\[
\int_{\Sigma_{\varphi}} \frac{d\sigma}{E} \frac{d^2}{d\varphi^2} \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} - \left( \frac{d}{d\varphi} \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} \right)^2
\]
\[
= 4 \left( \int_{\Sigma_{\varphi}} \frac{d\sigma}{E} \right)^2 \int_{\Sigma_{\varphi}} \left[ \left| n \times \nabla \frac{1}{E} \right|^2 + \left( \frac{\kappa}{E} - \int_{\Sigma_{\varphi}} \frac{\kappa}{E} d\lambda \right)^2 \right] d\lambda \geq 0, \tag{2.39}
\]
for a probability measure \( d \lambda = \frac{dx}{E} / \int f_s \frac{dx}{E} \). So far, we have proved Longinetti’s inequality in (2.36), for all 2-exD problems whose boundary \( \partial \Omega \) is a smooth Jordan curve. Arguing as before, we know that \( \mathcal{L}''(\varphi) = 0 \) happens only when \( \partial \Omega \) is a circle.

From \( \mathcal{L}''(\varphi) \geq 0 \), we may deduce the inequality stated in (1.11):

\[
\mathcal{L}'(\varphi) = -\frac{2 \int f_s \frac{kds}{E}}{\int f_s \frac{dx}{E}} \geq \lim_{\phi \to -\infty} \mathcal{L}'(\phi) = -\frac{4\pi}{\Phi}.
\]  

(2.40)

This may also be rewritten as a non-negative correlation between \( E^2 \) and \( \kappa/E \), weighted by the probability measure \( d \lambda = \frac{dx}{E} / \int f_s \frac{dx}{E} \):

\[
\frac{2\pi}{\int f_s \frac{dx}{E}} = \int \frac{E^2 \kappa}{E} d\lambda \geq \int \frac{E^2 k \sigma}{E} d\lambda = \frac{\Phi \int f_s \frac{kds}{E}}{(\int f_s \frac{dx}{E})^2}.
\]  

(2.41)

3. GEOMETRIC RELATIONS FOR GREEN’S FUNCTIONS

3.1. Differences between interior and exterior Dirichlet problems. From the statement of (1.6), (1.9) and (1.10), we see that the level sets for Green’s functions in 2-inD do not behave the same as the level sets for harmonic functions in 2-exD. Once (1.6), (1.9) and (1.10) are proven, we will know that a Dirichlet Green’s function \( G_D^{\beta}(0,r) \), \( r \in \mathbb{R}^2 \setminus (\Omega \cup \{0\}) \) in 2-inD cannot be analytically extended to \( \mathbb{R}^2 \setminus \{0\} \) without encountering singularities of \( \kappa(r) \), unless \( \partial \Omega \) is a circle centered at \( \{0\} \).

To get a concrete idea about the obstacle in such analytic continuations, we consider a special Dirichlet Green’s function \( G_D^{\beta}(0,r) \), \( r \in \mathbb{R}^2 \setminus (\Omega \cup \{0\}) \) in 2-inD, where the unbounded region \( \Omega = \{(x,y) \in \mathbb{R}^2 | x(x + 2) + y^2 > 1\} \) has a circular boundary \( \partial \Omega \) that is not centered at the origin \( \{0\} \). Such a Green’s function can be expressed in closed form:

\[
G_D^{\beta}(0,r) = -\frac{1}{\pi} \log \frac{x^2 + y^2}{(x-1)^2 + y^2} , \quad r \in \mathbb{R}^2 \setminus (\Omega \cup \{0\}).
\]  

(3.1)

One cannot avoid the singularity at \( x = 1, y = 0 \) while analytically extending the domain of definition for this harmonic function.

Moreover, all the level sets \( \Sigma_\varphi, \varphi \in [\log_2 \pi^2, +\infty) \) for \( G_D^{\beta}(0,r) \), \( r \in \mathbb{R}^2 \setminus (\Omega \cup \{0\}) \) defined in (3.1) are circles. On each such circle, the curvature \( \kappa(r), r \in \Sigma_\varphi \) is a positive constant \( \kappa \), and we have

\[
\frac{1}{\kappa} \text{cov}_{\Sigma_\varphi} \left( \frac{\kappa(r)}{|\nabla G_D^{\beta}(0,r)|}, |\nabla G_D^{\beta}(0,r)| \right) = 1 - \left< \frac{1}{|\nabla G_D^{\beta}(0,r)|} \right>_{\Sigma_\varphi} < 0
\]  

(3.2)

given the Cauchy–Schwarz inequality, as well as a trivially positive integral:

\[
\left< \left[ n \times \nabla \frac{\kappa(r)}{|\nabla G(r)|} \right] \cdot \left[ n \times \nabla \frac{1}{|\nabla G(r)|} \right] \right>_{\Sigma_\varphi} = \kappa \int_{\Sigma_\varphi} \left| n \times \nabla \frac{1}{|\nabla G(r)|} \right|^2 d\sigma > 0.
\]  

(3.3)

This is a brute-force verification of (1.6) and (1.10) for a special 2-inD problem. A general proof will follow soon.

3.2. Geometric relations for interior problems. For the level sets \( \Sigma_\varphi \) of Green’s functions in 2-inD, our arguments in \( \S \) still bring us \( \mathcal{H}''(\varphi) \geq 0, \mathcal{E}'(\varphi) = 0 \) and \( \mathcal{F}''(\varphi) \geq 0 \). However, the asymptotic analysis in 2-inD (\( |r| \to 0, \varphi \to +\infty \)) is critically different from that in 2-exD (\( |r| \to +\infty, \varphi \to -\infty \)).

To prove that \( \lim_{\varphi \to +\infty} \mathcal{E}(\varphi) = \frac{4\pi^2}{\Phi} \), we need two steps. First, we note that the asymptotic behavior in (2.9) also applies to \( |r| \to 0 \), so \( \lim_{\varphi \to +\infty} \int_{\Sigma_\varphi} \frac{kds}{E} = \frac{4\pi^2}{\Phi} \). Second, in view of the multipole expansion

\[
\log E(|r|/e_x \cos \theta + e_y, \sin \theta)) = -\log |r| + |r|(e_\theta' \cos \theta + s_\theta' \sin \theta) + O(|r|^2)
\]  

(3.4)

as \( |r| \to 0 \), we have \( |n \times \nabla \log E(r)| = O(1) \) and \( \frac{dx}{E(r)} = O(|r|^2 d\theta) \), so \( \lim_{\varphi \to +\infty} \int_{\Sigma_\varphi} |n \times \nabla \log E|^2 d\sigma = 0 \).
To prove (1.9′), one simply checks that (2.25) and (2.26) remain valid in 2-inD:
\[
\frac{d}{d\varphi} \left\{ L_{\Sigma'} \left[ \text{cov}_{\Sigma'} \left( \frac{\kappa}{\pi^2}, \log \frac{EL_{\Sigma'}}{\Phi} \right) - \text{cov}_{\Sigma'} \left( \frac{E}{\Phi}, \log \frac{EL_{\Sigma'}}{\Phi} \right) \right] \right\} = \frac{1}{\pi} \int_{\Sigma'} |n \times \nabla \log E|^2 ds E \geq 0, \tag{3.5}
\]
and that
\[
\lim_{\varphi \to +\infty} L_{\Sigma'} \left[ \text{cov}_{\Sigma'} \left( \frac{\kappa}{\pi^2}, \log \frac{EL_{\Sigma'}}{\Phi} \right) - \text{cov}_{\Sigma'} \left( \frac{E}{\Phi}, \log \frac{EL_{\Sigma'}}{\Phi} \right) \right] = 0. \tag{3.6}
\]

One can then compute directly that \( \lim_{\varphi \to +\infty} \mathcal{H}'(\varphi) = 0 \) and \( \lim_{\varphi \to +\infty} \mathcal{F}'(\varphi) = 0 \), so we have \( \mathcal{H}'(\varphi) \leq 0 \) and \( \mathcal{F}'(\varphi) \leq 0 \) for 2-inD, thus confirming (1.6) and (1.10). To prove (1.11), it would suffice to write down \( \mathcal{L}'(\varphi) \leq \lim_{\varphi \to +\infty} \mathcal{L}'(\phi) = \frac{4\pi}{\kappa} \) by analogy to (2.40).

Furthermore, the equalities in (1.6), (1.9), (1.10) and (1.11) hold only when \( E \) and \( \kappa \) both remain constant on each level set, which means that \( \partial \Omega \) is a circle centered at \( 0 \).

Before closing this section, we comment on the physical significance of (1.6), (1.9) and (1.10).

On one hand, the reversal in inequality signs (as compared to (1.6), (1.9) and (1.10) for 2-exD) is compatible with electricians’ folklore for interior problems: in 2-inD cases, “tips” are marked by \( \kappa < 0 \) while “pits” are marked by \( \kappa > 0 \), which reverses the interpretation of curvature signs in 2-exD scenarios. In particular, the inequalities (1.6) and (1.10) are statistically consistent with the empirical rule of “stronger electric field at sharper points”.

On the other hand, we may consider an interface \( \partial \Omega \) that is close to a circle (say, a curve \( \rho = 1 + \varepsilon \cos m\theta \) in polar coordinates, with \( |\varepsilon| \ll 1 \) and \( m \in \mathbb{Z} \)), whose outward growth speed is proportional to \( -\nabla G(\rho), \rho \in \partial \Omega \). This is (close to) the setting in internal diffusion limited aggregation \( [15, 4] \), or the corresponding Hele-Shaw flow as its deterministic limit \([12]\). Here, the inequalities (1.6) and (1.10) tell us that on average, the indentations (“tips” in 2-inD) have a higher outward growth rate than the protrusions (“pits” in 2-inD) on the near-circular interface \( \partial \Omega \). Such a feedback mechanism will make the interface more and more circular \([11, 8]\), rather than producing branched structures in the Witten–Sander process \([19, 18]\).

4. Discussions and Outlook

4.1. Dirichlet problems with non-Jordan boundaries. In both our treatments of 2-exD in §2 and 2-inD in §3 we have assumed that \( \partial \Omega \) is a smooth Jordan curve. This is sometimes known as the “isolated conductor condition” in electrostatics, where electricians’ folklore attributes boundary charge distribution to the curvature effects (rather than the presence of “other” metallic conductors in the vicinity). In such “isolated conductor cases”, all the level sets \( \Sigma_\varphi \) in the region of interest are smooth Jordan curves, and \( |\nabla U(\rho)| \) never vanishes. In fact, the integral curves of \( \nabla U(\rho) \) induce diffeomorphisms among all these level sets \( \Sigma_\varphi \), allowing us to compare certain integrals on \( \Sigma_\varphi \) to the limit case of \( \Sigma_{-\infty} \) (resp. \( \Sigma_{+\infty} \)) in 2-exD (resp. 2-inD).

When we consider an exterior Dirichlet boundary problem for a harmonic function where \( \partial \Omega \) is not a Jordan curve, or an interior Dirichlet boundary problem with multiple point sources, the gradient \( \nabla U(\rho) \) may vanish somewhere in our region of interest. The interruption of the integral curves of \( \nabla U(\rho) \) at these critical points of gradient may cause phase transitions in the topology of \( \Sigma_\varphi \). Such phase transitions can make our integrals on \( \Sigma_\varphi \) ill-defined for countably many values of \( \varphi \), rendering some results of §§2–3 inapplicable to non-Jordan boundaries.

To flesh out, we consider a harmonic function
\[
U_C(\rho) = \log \sqrt{(x+1)^2 + y^2} + \log \sqrt{(x-1)^2 + y^2}, \tag{4.1}
\]
5By constructing a conjugate harmonic function \( V(\rho) \) for \( U(\rho) \), one has \( |\nabla U(\rho)| = |f'(x+iy)| \) for a locally defined complex-analytic function \( f(x+iy) = U(xe_x + ye_y) + iV(xe_x + ye_y) \). According to the open mapping theorem in complex analysis, we know that the set \( \{ \rho \in \mathbb{C} \mid \nabla U(\rho) = 0 \} \) has no accumulation points.
which is defined wherever \([(x + 1)^2 + y^2][(x - 1)^2 + y^2] \neq 0\). Its level sets \(\Sigma_\varphi := \{(x, y) \in \mathbb{R}^2 | [(x + 1)^2 + y^2][(x - 1)^2 + y^2] = e^{2\varphi}\}, \varphi \in \mathbb{R}\) are Cassini ovals. The only critical point \(x = 0, y = 0\) sits on \(\Sigma_0\). The functions \(\mathcal{H}(\varphi), \mathcal{E}(\varphi)\) and \(\mathcal{F}(\varphi)\) are well-defined for \(\varphi \in (-\infty, 0) \cup (0, +\infty)\). Whenever \(\varphi > 0\), the level set \(\Sigma_\varphi\) is the union of two disjoint smooth Jordan curves. The computation in (2.22) remains valid, so \(\mathcal{E}'(\varphi) = 0, \varphi > 0\), and

\[
\mathcal{E}(\varphi) = \lim_{\varphi \to +\infty} \mathcal{E}(\varphi) = 4\pi, \quad \varphi > 0.
\]

Meanwhile, whenever \(\varphi < 0\), the level set \(\Sigma_\varphi\) is a smooth Jordan curve, and we have

\[
\mathcal{E}(\varphi) = \lim_{\varphi \to -\infty} \mathcal{E}(\varphi) = \pi, \quad \varphi < 0.
\]

Thus, the function \(\mathcal{E}(\varphi), \varphi \in (-\infty, 0) \cup (0, +\infty)\) is piecewise constant, rather than a global conservation law. The behavior of \(\mathcal{H}'(\varphi)\) and \(\mathcal{F}'(\varphi)\) on a connected (resp. disconnected) level set \(\Sigma_\varphi\) enclosing (resp. enclosed by) \(\Sigma_0\) resembles the situations in 2-exD (resp. 2-inD):

\[
\mathcal{H}'(\varphi) = \frac{L_{\Sigma_\varphi}}{4\pi} \text{cov}_{\Sigma_\varphi} \left( \frac{k}{E}, E \right) \in \begin{cases} (0, +\infty), & \varphi < 0, \\ (-\infty, 0), & \varphi > 0; \end{cases}
\]

\[
\mathcal{F}'(\varphi) = -2 \oint_{\Sigma_\varphi} \left[ n \times \nabla \frac{k}{E} \right] \cdot \left[ n \times \nabla \frac{1}{E} \right] \, ds \in \begin{cases} (0, +\infty), & \varphi < 0, \\ (-\infty, 0), & \varphi > 0. \end{cases}
\]

Moreover, neither \(\mathcal{H}'(\varphi)\) nor \(\mathcal{F}'(\varphi)\) admits a continuous extension to \(\varphi = 0\).

We leave the detailed proof of the statements in the last paragraph to diligent readers.

4.2. **Equipotential (hyper)surfaces in higher dimensions.** It is appropriate to consider Dirichlet problems for harmonic functions in higher dimensional Euclidean spaces \(\mathbb{R}^d\) (\(d > 2\)). One can formulate \(d\)-exD as a Laplace equation

\[
\nabla^2 U(r) = 0, \quad r \in \Omega \subset \mathbb{R}^d
\]

in an unbounded domain \(\Omega\), whose boundary \(\partial \Omega\) is a smooth and connected (hyper)surface, on which \(U(r)\) remains constant. Such a boundary is orientable, with an outward unit normal vector \(n\) defined everywhere on \(\partial \Omega\). The flux condition

\[
-\oint_{\partial \Omega} n \cdot \nabla U(r) \, d\Sigma = \Phi > 0
\]

(with \(d\Sigma\) being the induced Lebesgue measure on \(\partial \Omega\)) translates into the following asymptotic behavior as \(|r|\) goes to infinity:

\[
U(r) \sim \frac{\Phi}{4\pi^{d/2}|r|^{d-2}} \int_0^{\infty} t^{(d-4)/2} e^{-t} \, dt.
\]

If \(0 \notin \Omega \cup \partial \Omega\), then one can define the Green’s function in \(d\)-inD as a solution to

\[
\begin{cases} 
\nabla^2 G^\Omega_D(0, r) = 0, & r \in \mathbb{R}^d \setminus (\Omega \cup \partial \Omega \cup \{0\}), \\
G^\Omega_D(0, r) = \text{const.}, & r \in \partial \Omega, \\
-\lim_{\varepsilon \to 0^+} \oint_{|r| = \varepsilon} n \cdot \nabla G^\Omega_D(0, r) \, d\Sigma = 1.
\end{cases}
\]

In both \(d\)-exD and \(d\)-inD, we will be interested in the geometric relations on the level sets \(\Sigma_\varphi\) [equipotential (hyper)surfaces in physical parlance], in a similar vein as §2.3.

Akin to §4.1, the possible existence of critical points (where the gradient of a harmonic function vanishes) forms a major obstacle to finding higher dimensional analogs of the monotonicity results in 2-exD (§2) and 2-inD (§3). Lacking a generalization of the Riemann mapping theorem to \(\mathbb{R}^d\) (\(d > 2\)), we usually cannot rule out the critical points, by relying on the smoothness and connectedness of the boundary \(\partial \Omega\) alone. A recent result of Ma–Zhang \[14, Proposition 3.2\] ensures the non-existence of
critical points in \( d\)–exD and \( d\)–inD if \( \partial \Omega \) is both smooth and convex. Thus, it is sensible to limit our scope to \( d\)–exDc and \( d\)–inDc, which are problems with the additional constraint on geometric convexity of the boundary \( \partial \Omega \). (For the special case of 3–inDc, one can also deduce the non-existence of critical points from Gergen’s answer \([5, (1.1)]\) to a question of Morse.)

Yet another obstacle to obtaining higher dimensional generalizations of the geometric inequalities in \S\S2–3 is the unavailability of auxiliary harmonic functions, like those studied by Talenti \([17]\) and Kong–Xu \([10]\). Without such “extra harmonicity” in higher dimensional spaces, we may lose control of the sign in certain derivatives with respect to \( \varphi \).

Despite these difficulties, we can still construct some (partial and conditional) generalizations to the current work in \( \mathbb{R}^d \) (\( d > 2 \)). For example, instead of a conservation law in \( \S\S2.2 \), our closest analogs in \( \mathbb{R}^3 \) \([21]\) will be the following inequality for every level set \( \Sigma_\varphi \) in 3–exDc (strict unless \( \partial \Omega \) is a sphere):

\[
\oint_{\Sigma_\varphi} \frac{4[H^2(r) - K(r)] - |n \times \nabla \log |\nabla U(r)||^2}{|\nabla U(r)|^2} \, dS \geq 0, \tag{4.6}
\]

and following inequality for every level set \( \Sigma_\varphi \) in 3–inDc (strict unless \( \partial \Omega \) is a sphere centered at the origin):

\[
\oint_{\Sigma_\varphi} \frac{4[H^2(r) - K(r)] - |n \times \nabla \log |\nabla G_{\partial \Omega}^{\partial D}(0, r)||^2}{|\nabla G_{\partial \Omega}^{\partial D}(0, r)|^2} \, dS \leq 0. \tag{4.7}
\]

Here, the mean curvature and the Gaussian curvature are denoted by \( H \) and \( K \) respectively, while \( dS \) is the surface measure.

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