Persistent currents in mesoscopic rings with a quantum dot

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Using the Anderson model in the Kondo regime, we calculate the persistent current \( j \) in a ring with an embedded quantum dot (QD) as a function of the Aharonov-Bohm flux \( \Phi \) for different ring length \( L \), temperature \( T \) and broadening of the conduction states \( \delta \). For \( T = \delta = 0 \) and \( L \gg \xi \), where \( \xi \) is the Kondo screening length, \( L_j \) tends to the value for a non interacting ideal ring, while it is suppressed for a side coupled QD. For any \( L/\xi \), \( L_j \) is also suppressed when either \( T \) or \( \delta \) increase above a fraction of the level spacing which depends on \( \Phi \).

Electron transport through a quantum dot (QD) has been a subject of great interest in the past few years. The progress in nanofabrication made it possible to use QD’s as ideal realizations of the Kondo effect, which is one of the most exciting and studied problems in condensed matter physics. It consists in the screening of an impurity spin by a cloud of conduction electrons of radius \( \xi \sim h v_F / T_K \) and energies \( \sim 2T_K \) around the Fermi energy \( v_F \), where \( T_K \) is the Kondo temperature and \( v_F \) is the Fermi velocity. The energy scale \( T_K \) is experimentally accessible in different ways, like the width of the peak in linear response conductance through a QD and its temperature dependence \[\bar{\Phi} \]. Instead, a direct measurement of \( \xi \) does not exist so far.

Several interesting experiments were performed recently in an Aharonov-Bohm geometry, in which a ring containing a QD is threaded by a magnetic flux \[\bar{\Phi} \]. Phase coherence along the ring has been demonstrated. The persistent current \( j \) in these rings, and in rings side-coupled to a QD has been studied theoretically \[\bar{\Phi} \]. However, basic results of these works contradict each other and an accurate method to calculate \( j \) for any \( L/\xi \) has not been developed. Exact Bethe ansatz results were known for \( L \gg \xi \) and chiral electrons \[\bar{\Phi} \]. These were extended to electrons moving in both directions \[\bar{\Phi} \]. The precise geometry of these calculations was explained recently \[\bar{\Phi} \]. Perturbative renormalization group (RG) calculations have established the form of \( j(\Phi) \) for \( L \ll \xi \) and \( L \to \infty \). A change in the dependence of \( j \) with magnetic flux is expected between the regimes \( L \ll \xi \) and \( L \gg \xi \). Thus, measurements of \( j \) would provide a way of detecting \( \xi \). Since the average level spacing is \( D = 2\pi h v_F / L \), the condition \( L \sim \xi \) is equivalent to \( D / 2\pi \sim T_K \).

Our purpose is to describe \( j(\Phi) \) accurately through the crossover region, and to consider the effects of temperature \( T \) and finite level width \( \delta \) of the conduction states for the first time. We relate \( j \) with the one-particle Green function at the dot and the latter is calculated using an interpolative perturbative approach (IPA) \[\bar{\Phi} \]. For small \( L \) and \( T = \delta = 0 \), we also calculate \( j \) using numerical exact diagonalization (ED) finding excellent agreement with the IPA.

The Hamiltonian for the embedded dot is:

\[
H = - \sum_{\sigma,j=0}^{L-1} \left( t_j e^{i\varphi_j} c_{j+1\sigma}^\dagger c_{j\sigma} + \text{H.c.} \right) + E_d \sum_\sigma n_{d\sigma} + U n_{d\uparrow} n_{d\downarrow},
\]

with \( c_{d\sigma} \) and \( n_{d\sigma} = c_{d\sigma}^\dagger c_{d\sigma} \). The phases \( \varphi_j \) depend on the choice of gauge and satisfy \( \sum_j \varphi_j = \Phi \), where \( \Phi h c / 2\pi e \) is the magnetic flux threading the ring. The hoppings are all equal except those involving the QD: \( t_j = t_R \) (right) for \( j = 0 \), \( t_j = t_L \) (left) for \( j = L - 1 \), and \( t_j = t \) otherwise. The total current flowing between sites \( l \) and \( l + 1 \) is:

\[
 j_l(\Phi) = \frac{e}{\hbar} \frac{\partial}{\partial \varphi_l} \langle H \rangle = \frac{i e}{\hbar} \sum_\sigma \langle e^{i\varphi_j} c_{j+1\sigma}^\dagger c_{j\sigma} - \text{H.c.} \rangle.
\]

The expectation value entering Eq. (2) is given in terms of Green functions \( \langle [c_{l\sigma}^\dagger, c_{j\sigma}^\dagger] \rangle \). These in turn can be expressed in terms of the QD Green function \( G_{d\sigma}(\omega) = \langle [c_{d\sigma}^\dagger, c_{d\sigma}^\dagger] \rangle_\omega \) using equations of motion. Choosing \( \varphi_0 = 0 \) for \( 0 < j < L - 1 \) and \( \varphi_0 = \varphi_{L-1} = \pi / 2 \) for \( j = L - 1 \), we obtain, after some algebra for \( 0 \neq \varphi \neq L - 1 \):

\[
 j_l(\Phi) = \frac{16 e}{\pi \hbar L^2} H_{RR} n L \sin \Phi \sum_n (-1)^n \sum_{n'} p(l, n) p(l + 1, n') \times \sum \int d\omega f(\omega) \frac{G_{d\sigma}(\omega + i\eta)}{(\omega + i\eta - \epsilon_n)(\omega + i\eta - \epsilon_{n'})};
\]

\[
p(l, n) = \sin \frac{\pi n l}{L}; \quad \sin \frac{\pi n}{L}; \quad \epsilon_n = -2t \cos \frac{\pi n}{L} - i\delta.
\]

Here \( \sum_n \) runs over all integers \( 1 \leq n \leq L - 1 \), while \( \sum_{n'} \) is restricted to integers of opposite parity to that of \( n \), \( f(\omega) \) is the Fermi function, \( \eta \) is a positive infinitesimal, and \( \epsilon_n \) are the eigenenergies in absence of the QD, allowing for a finite broadening \( \delta \). Eq. (2) is exact. However, we use an approximate \( G_{d\sigma} \). Within the precision of the numerical integration, we have verified that the resulting \( j_l \) is independent of \( \delta \), as it should be because of conservation of the current. Therefore we drop the subscript in what follows.
We calculate $G_{ds}$ using a self-consistent IPA based on perturbation theory in $U$ up to second order \[13,14\], generalized to allow spin dependence:

$$G_{ds}^{-1}(\omega) = [G_{ds}^0(\omega)]^{-1} - Un_\sigma - \Sigma_\sigma(\omega),$$

where $G_{ds}^0$ is the Green function for $U = 0$ and $E_d$ replaced by an effective energy $\varepsilon_{eff}$ determined selfconsistently

$$[G_{ds}^0(\omega)]^{-1} = \omega - \varepsilon_{eff} - \sum_n \frac{|V_n|^2}{\omega - \varepsilon_n};$$

$$V_n = \sqrt{\frac{2}{L}} \sin \frac{\pi n}{L} \left[ t_R e^{i\Phi/2} - (-1)^n t_R e^{-i\Phi/2} \right],$$

$n_\sigma = \langle n_{d\sigma} \rangle$, and:

$$\Sigma_\sigma(\omega) = \frac{n_\sigma(1-n_\sigma)\Sigma_\sigma^{(2)}}{n_\sigma^2(1-n_\sigma) - [(1-n_\sigma)U + E_d - \varepsilon_{eff}]^2 \Sigma_\sigma^{(2)}},$$

where $n_\sigma^0$ is the expectation value of $n_{d\sigma}$ calculated with $G_{ds}^0$ and $\Sigma_\sigma^{(2)}$ is the ordinary second order correction to the self energy, calculated from a Feynmann diagram involving the analytical extension of $G_{ds}^0(\omega)$ to Matsubara frequencies $\[12\]$:

$$\Sigma_\sigma^{(2)}(i\omega_n, T) = U^2 T \sum_m G_{ds}^0(i\omega_n - i\nu_m)\chi(i\nu_m);$$

$$\chi(i\nu_m) = -T \sum_n G_{ds}^0(i\omega_n)G_{ds}^0(i\omega_n + i\nu_m).$$

The resulting $G_{ds}$ is not only valid up to $U^2$, but it is also exact for a decoupled dot ($t_L = t_R = 0$), and reproduces the leading term for $\omega \rightarrow \infty$.\[14\] We determine $\varepsilon_{eff}$ by imposing that $n_\sigma^0 = n_\sigma$ for both spins.

We take $t = 1$ as the unit of energy and keep $E_d = -U/2$ and $t_L = t_R$. This allows us to exploit electron-hole and reflection symmetry in some cases. As a basis for our study we choose $U = 2$ and $t_L = 0.4$. For $L \rightarrow \infty$ and $\varepsilon_F = 0$, this leads to a resonant level width $\Delta = 0.32$ \[13\] (neglecting its energy dependence). The ratio $U/\Delta = 6.25$ is large enough for the system to be in the Kondo regime of the model, but low enough to ensure the validity of the IPA \[13\]. The impurity spectral density $\rho_{ds}(\omega)$ shows three peaks at $E_d$, $E_d + U$ and $\varepsilon_F$ characteristic of the Kondo regime. For $\delta = 0$ and finite $L$, $\rho_{ds}(\omega)$ consists in a set of delta functions. Using \[16\]:

$$T_K = \sqrt{U/\Delta}e^{\pi(E_d-\varepsilon_F)(E_d-\varepsilon_F+U)/2U\Delta},$$

the above parameters lead to $T_K \sim 0.05$ and $\xi \sim 40$. We begin showing the results for $T = \delta = 0$. They depend drastically on the parity of the number of particles $N$. For even $N$, the results for odd $N/2$ are essentially the same as those for $N \pm 2$ shifting $\Phi$ by $\pi$. Then we restrict to either odd $N$ or $N/4$ integer. Using reflection symmetry around the QD ($c_{j\sigma} \rightarrow c_{L-j\sigma}$), one realizes that $j(\Phi) = -j(-\Phi)$. Thus, it is sufficient to represent $j$ in the interval $0 \leq \Phi \leq \pi$. Fig. 1 displays the evolution of $j(\Phi)$ as a function of ring size for even $N$. For $L = 8$ and $N = 7, 8$ we also compare the IPA results with those obtained using $j(\Phi) = -(eL/h)\partial E(\Phi)/\partial \Phi$, with the ground state energy $E$ calculated by ED. The maximum deviation between both results takes place around $\Phi = 0.2$ and is below 0.05et/h. For $L \rightarrow \infty$ and $N$ even, $j(\Phi)$ converges to the non-interacting ideal result, as predicted by an analysis of the strong-coupling fixed point of RG \[3\]. This supports the validity of the IPA results for all $L$. Our results also agree qualitatively with those of RG for $L \ll \xi$, but disagree with those of Ref. \[3\].

![FIG. 1. Current in units of $et/h$ as a function of magnetic flux for $U = 2$, $t_L = 0.4$ and different values of $L$. Unless otherwise indicated $N = L$.](image1.png)

![FIG. 2. Current as a function of $L$ for two sets of parameters.](image2.png)

To study the scaling properties in the dependence with $L$, we have calculated $j(\pi/2)$ for $8 \leq L \leq 1000$ and
two sets of parameters: $U = 2$, $t_L = 0.4$ as before, and $U = 0.5$, $t_L = 0.2$. The new choice should reduce $\Delta$ by a factor $\sim 4$, keeping then nearly the same value of $U/\Delta$, and resulting in $\sim 4$ times smaller $T_K$ (see Eq. (3)), and $\sim 4$ times larger $\xi$. In fact, as shown in Fig. 2, scaling $L$ by a factor $3.79$, both functions $Lj(\pi/2, L)$ practically coincide, confirming that $Lj$ is a universal function of $L/\xi$, as expected for finite $\xi$. Thus for $U = 0.5$, using $\xi = 151.6$ as derived from Eq. (3), the results for $8 \leq L \leq 240$ fit with negligible errors on the curve:

$$Lj(\pi/2)\hbar/te = 1.53 + 0.24\ln(L/\xi). \quad (5)$$

**FIG. 3.** Same as Fig. 1 for odd $N$ and $U = 1.5$.

Using ED, we find that in general for given $\varepsilon_F$, the states with odd $N$ are less stable than those of even $N$. In particular, it is not possible to find a unique value of $\varepsilon_F$ for which some odd $N$ are stable for all $\Phi$. Even $N$ with even (odd) $N/2$ are favored near $\Phi = 0$ ($\Phi = \pi$), for which $j(\Phi)$ has a small amplitude. For example, for $L = 12$ and $\varepsilon_F = -0.7$, $N$ decreases from 10 for $\Phi = 0$ to 8 for $\Phi = \pi$, jumping from 10 to 9 near $\Phi = 0.16\pi$, and from 9 to 8 near $\Phi = 0.82\pi$. As a consequence $j(\Phi)$ is discontinuous and small in magnitude ($|j(\Phi)| < 0.5\varepsilon_F/\hbar$).

The IPA for odd $N$ was applied imposing a given $N$ (although it might correspond to a metastable state) and adjusting $\varepsilon_F$ as a function of $\Phi$. A technical difficulty is that for large $U$ and $L$ we could not find the self-consistent solution. This is related to the fact that for finite $L$, $n^0_\sigma$ and $n_\sigma$ are discontinuous functions of the $\varepsilon^{\text{eff}}_\sigma$ and it is not always possible to satisfy $n^0_\sigma = n_\sigma$. As seen in Fig. 1, and the above mentioned case for $L = 12$, $j(\Phi)$ is strongly suppressed for odd $N$ and small $L$. This is related to a partial suppression of the Kondo effect. For $L = 8$, the expectation value of the impurity spin $s_z = (n_\uparrow - n_\downarrow)/2 \sim 0.4$, indicating only a small partial screening. Also, $j(\Phi)$ displays positive and negative values in the interval $0 \leq \Phi \leq \pi$, suggesting a tendency to periodicity in $\pi$ instead of $2\pi$. This periodicity is exact for $N = L$ odd, $E_\sigma = -U/2$ and $t_L = t_R$: the electron-hole transformation $c^\dagger_{j\sigma} \rightarrow c_{j\sigma}$, $\varphi_j \rightarrow \varphi_j + \pi$, maps $H(-\Phi)$ onto $H(\Phi + \pi)$ and $j(\Phi)$ onto $-j(\Phi + \pi)$ (see Eqs. (4) and (6)). Combining this with $j(\Phi) = -j(-\Phi)$, one has $j(\Phi) = j(\Phi + \pi)$, and $j(\pi/2) = 0$. In Fig. 3 we show $j(\Phi)$ in two of these cases with $U$ reduced to 1.5 to be able to obtain self consistent solutions up to $L = 37$. We estimate $\xi \sim 25$. According to RG arguments, as $L$ increases, $j(\Phi)$ tends to the result of a fictitious non-interacting system with spin dependent $\varepsilon_F$ to allow for odd $N$. Our results are consistent with this. $s_z$ decreases from $\sim 0.37$ to $\sim 0.15$ as $L$ increase from 9 to 37.

**FIG. 4.** Same as Fig. 1 for different temperatures and (a) $U = 2$, $t_L = 0.4$, $L = 800$, (b) $U = 0.5$, $t_L = 0.2$, $L = 8$.

We have calculated the $T$ dependence of $j(\Phi)$ for even $N$ and several $L/\xi$. In Fig. 4, we show two cases: (a) $T_K \sim 0.05$, $L/\xi \sim 20$ and (b) $T_K \sim 0.012$, $L/\xi \sim 1/20$. From the known results for the conductance through a QD [15] one would expect $T_K$ to be the relevant scale for the $T$ dependence. However, it is a fraction of the level spacing $D = 2\pi\hbar e/\ell$ in both cases. This is easy to understand in case (a): for $T \ll T_K$ and $L \gg \xi$, the physics is still dominated by the strong coupling fixed point of RG for which the model reduces to a non-interacting one [15]. In turn, the conductance of the latter is strongly reduced for $T \sim D \ll T_K$. In contrast to the infinite system, for which the Kondo peak in $\rho_{\ell \ell} (\omega)$ looses approximately half its intensity for $T \sim T_K$ [15], when $T_K \ll D$.
\( L \ll \xi \), \( G_{ds} \) is practically not modified with increasing \( T \), until \( T \) reaches a sizeable fraction of \( D \). As a consequence for all \( L \) the scale for the \( T \) dependence is \( \sim D/5 \), but depends on \( \Phi \). Note that these arguments are independent of the parity of \( N \). The last argument is in agreement with recent RG calculations, which show that for even \( N \) and \( T_K \ll D \), the screening of the localized spin takes place at \( T \sim D \), while as discussed above, the screening is only partial for odd \( N \) \[12\]. In other words decreasing \( L \), the Kondo effect is enhanced for \( N \) even and inhibited for \( N \) odd.

The factor of two in the projection of the Kondo effect to a re-

mote location (mirage effect) \[12\]. In the case \( D \ll T_K \), it is clear that \( j \) is strongly reduced when \( \delta \) reaches \( D \). In the opposite case \( T_K \ll D \), for \( N \) even (corresponding to some \( \varepsilon_n \) near \( \varepsilon_F \)), \( \rho_{ds}(\omega) \) near \( \varepsilon_F \) evolves with increasing \( \delta \) from two delta functions at both sides of \( \varepsilon_F \) to one peak centered at \( \varepsilon_F \) \[12\]. The corresponding change in \( j(\Phi) \) is shown in Fig. 5. The effect of the broadening of the levels is similar to that of increasing \( T \), but a little bit weaker and more evenly distributed in \( \Phi \).

We discuss briefly the side dot. The Hamiltonian is:

\[
H = - \sum_{\sigma,j = 0}^{L-1} \left( t e^{i \varphi} c_{j+1 \sigma}^\dagger c_{j \sigma} + \text{H.c.} \right) - t_d \sum_{\sigma} (d_{\sigma}^\dagger c_{0 \sigma} + \text{H.c.}) + E_d \sum_{\sigma} n_{d \sigma} + U n_{d \uparrow} n_{d \downarrow},
\]

where now \( \varphi = \Phi / L \) and \( n_{d \sigma} = d_{\sigma}^\dagger d_{\sigma} \). Proceeding as before, for \( \delta = 0 \) the current becomes:

\[
j_s(\Phi) = \frac{2t_e}{\pi \hbar L^2} \left( -2\pi L \sum_k \sin(\varepsilon_k) \right) + t_d^2 \sum_k \sum_{k'} \sin(\varphi + k'(l + 1) - kl) \times \left( \sum_{\sigma} \int d\omega f(\omega) \text{Im} \left[ \frac{G_{ds}(\omega + i\eta)}{(\omega + i\eta - \varepsilon_k)(\omega + i\eta - \varepsilon_{k'})} \right] \right); \quad \varepsilon_k = -2t \cos(\varepsilon + \varphi),
\]

independently of \( l \). The expressions for \( G_{ds} \) are the same as before with \( V_n \) replaced by \( V_k = 1/\sqrt{L} \). In the limit \( t_d \to 0 \), \( L \to \infty \), the resulting resonant level width \( \Delta \) is the same as that of an embedded dot with \( t_L = t_R = t_d / 2 \) \[13\]. Then, we take \( t_d = 0.8 \), \( N = L + 1 \) even and other parameters as in Fig. 1 to study the dependence of \( j_s(\Phi) \) on \( L \). The result is shown in Fig. 6. In agreement with Ref. \[6\] \( j_s(\Phi) \) is suppressed for large \( L \). Note that the first term of Eq. \[6\] is the non-interacting one, which is of the order of 100 times \( j_s(\Phi) \) for \( L = 799 \). The fact that the small \( j_s(\Phi) \) comes from a near cancellation of two terms, and only the second one is approximate, further supports the validity of the IPA.

We have calculated the persistent current \( j(\Phi) \) in a ring with a QD using an interpolative perturbative approach. The method is accurate enough to describe \( j(\Phi) \) for all \( L/\xi \). The universal dependence of \( Lj \) with \( L/\xi \) is displayed. The energy scale for the dependence on temperature and broadening of the levels is a fraction of the level spacing \( D \) instead of \( T_K \). For the side dot \( j_s(\Phi) \) is small and decreases with \( L \). For both systems and odd \( L \), the period of \( j(\Phi) \) is \( \pi \) in half filled symmetric rings. Because of their larger stability and larger currents, states with even number of electrons \( N \) in embedded dots seem the most interesting experimentally.

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