Quasi-set-theoretical foundations of statistical mechanics: a research program

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Abstract

Quasi-set theory provides us a mathematical background for dealing with collections of indistinguishable elementary particles. In this paper, we show how to obtain the usual statistics (Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac) into the scope of quasi-set theory. We also show that, in order to derive Maxwell-Boltzmann statistics, it is not necessary to assume that the particles are distinguishable. In other words, Maxwell-Boltzmann statistics is possible even in an ensemble of indistinguishable particles, at least from the theoretical point of view. The main goal of this paper is to provide the mathematical grounds of a quasi-set-theoretical framework for statistical mechanics.

1 Introduction

According to any textbook about statistical mechanics, we know that Maxwell-Boltzmann (MB) statistics gives us the most probable distribution of \( N \) distinguishable objects into, say, boxes with a specified number of objects in each box. In this paper we show that the hypothesis concerning distinguishable objects is unnecessary. Usually, classical and quantum distribution
functions are mathematically derived in a naive fashion; but in our case an axiomatic framework is necessary if we want to show that individuality is not a necessary assumption in classical statistical mechanics. Classical logic and mathematics are committed with a conception of identity which does not make any distinction between identity and indistinguishability: indistinguishable things are the very same thing and conversely. In several standard textbooks on quantum mechanics, for example, there is no clear distinction between indistinguishability and identity. So, it is necessary to settle some philosophical terms in order to avoid confusions. When we say that \(a\) and \(b\) are identicals, we mean that they are the very same individual, that is, there are no ‘two’ individuals at all, but only one which can be named indifferently by either \(a\) or \(b\). By indistinguishability we simply mean agreement with respect to attributes. We recognize that this is not a rigorous definition. Nevertheless such an intuition is better clarified in the next Sections.

Our proposed axiomatic framework for dealing with quantum and classical statistics is quasi-set theory \[\mathbb{Q}\]. Quasi-set theory \(\mathbb{Q}\) allows us the presence of two sorts of atoms (atoms in the mathematical sense, that is, Urelemente), termed \(m\)-atoms and \(M\)-atoms, identified by two unary predicates \(m(x)\) and \(M(x)\), respectively. Concerning the \(m\)-atoms, a weaker ‘relation of indistinguishability’ (denoted by the symbol \(\equiv\)), is used instead of identity, and it is postulated that \(\equiv\) has the properties of an equivalence relation. The predicate of equality cannot be applied to the \(m\)-atoms, since no expression of the form \(x = y\) is a well formed formula if \(x\) or \(y\) denote \(m\)-atoms. Hence, there is a precise sense in saying that \(m\)-atoms can be indistinguishable without being identical. In standard mathematics, when we say that \(x = y\) (\(x\) is identical to \(y\)) we are talking about the very same object, with two different names: \(x\) and \(y\). The axioms of quasi-set theory are a very natural extension of the axioms of Zermelo-Fraenkel (ZF) set theory.

This work is part of a research program concerning the problems of non-individuality in quantum mechanics and related topics. In previous works it was presented a manner to cope with collections of ‘physically’ indistinguishable particles in a set-theoretical framework (that is, standard set theory) by using hidden variables \[\mathbb{H}, \mathbb{F}\]. Quasi-set theory was first proposed as a mathematical framework for quantum distributions in \[\mathbb{Q}\], where a quasi-set-theoretical predicate for collections of indiscernibles was presented and Bose-Einstein (BE) and Fermi-Dirac (FD) statistics were derived. Here we show a simpler manner to derive quantum statistics based on quasi-set theory.
and we also discuss about MB statistics. We propose a generalization $Q'$ of quasi-set theory and show that MB statistics is possible even in an ensemble of indistinguishable particles.

By using quasi-set theory instead of standard set theory, our paper provides a way of obtaining the statistics from the assumption that the ‘non-individuality’ of quantum objects should be ascribed right at the start. In the set-theoretical picture presented in [3] and [4], such an assumption is not (and cannot be) stated.

In the next Section we present quasi-set theory. In Section 3 we show how to derive MB statistics in a collection of indistinguishable objects. In Section 4 we make a generalization of quasi-set theory, which we call $Q'$, and show that quantum statistics may be seen as a special case of MB statistics. In Section 5 we prove that $Q'$ is consistent iff ZF set theory is consistent. In Section 6 we present the main conclusions and make a conjecture in the context of this quasi-set theoretical approach to statistical mechanics.

2 Quasi-set theory

2.1 The Language

This Section is essentially based on [3], which is variant of the formulation presented in [4].

The language of quasi-set theory $Q$ is that of the first order predicate calculus without identity. The intuitive idea is to allow the existence of Urelemente of two kinds, which are called $m$-atoms and $M$-atoms. The latter act as atoms of ZFU (Zermelo-Fraenkel with Urelemente), while the former are supposed to be objects to which the concept of identity cannot be applied in a sense.

The specific symbols of $Q$ are three unary predicates $m$, $M$ and $Z$, two binary predicates $\equiv$ and $\in$ and an unary functional symbol $qc$. Terms and well-formed formulas are defined in the standard way, as are the concepts of free and bound variables, etc.. We use $x$, $y$, $z$, $u$, $v$, $w$ and $t$ to denote individual variables, which range over quasi-sets (henceforth, qsets) and Urelemente. Intuitively, $m(x)$ says that ‘$x$ is a microobject’ ($m$-atom), $M(x)$ says that ‘$x$ is a macroobject’ ($M$-atom) while $Z(x)$ says that ‘$x$ is a set’. The term $qc(x)$ stands for ‘$x$ is a cardinal of (the qset) $x$’. The sets are supposed to be
exact copies of the sets in ZFU.

The formulas \( \forall P \cdot x(\ldots) \) and \( \exists P \cdot x(\ldots) \) abbreviate \( \forall x(P(x) \rightarrow (\ldots)) \) and \( \exists x(P(x) \land (\ldots)) \) respectively, where \( P \) is a predicate of the language, \( \rightarrow \) is the conditional of propositional calculus, and \( \forall \) and \( \exists \) are, respectively, the universal and the existential quantifiers of predicate calculus. We use further standard logical notation: \( \neg \) is negation, \( \land \) is conjunction, \( \lor \) is disjunction, and \( \leftrightarrow \) is biconditional.

**Definition 1**

1. \( Q(x) := \neg(m(x) \lor M(x)) \) (\( x \) is a quasi-set)

2. \( P(x) := Q(x) \land \forall y(y \in x \rightarrow m(y)) \) (\( x \) is a ‘pure’ quasi-set, that is, a quasi-set whose elements are \( m \)-atoms only).

3. \( D(x) := M(x) \lor Z(x) \) (\( x \) is a classical object, or Ding, in Zermelo’s original sense, that is, \( x \) is a (classical) Urelement or a set).

4. \( E(x) := Q(x) \land \forall y(y \in x \rightarrow Q(y)) \)

5. \([\text{Extensional Equality}]\) For all \( x \) and \( y \), if they are not \( m \)-atoms, then:

\[
\begin{align*}
\exists_E x &= y := \forall z(z \in x \leftrightarrow z \in y) \lor (M(x) \land M(y) \land x \equiv y)
\end{align*}
\]

6. \([\text{Subquasi-set}]\) For all \( x \) and \( y \), if they are not atoms, then:

\[
x \subseteq y := \forall z(z \in x \rightarrow z \in y)
\]

If \( x \not\equiv_E y \), that is, \( \neg(x =_E y) \), we say that \( x \) and \( y \) are extensionally distinct. As is usual, \( x \subset y \), means \( x \subseteq y \land x \not\equiv y \). It is immediate that \( x \subseteq y \land y \subseteq x \rightarrow x =_E y \).

### 2.2 The First Axioms

The first four axioms of \( Q \) are The Axioms of Indistinguishability:

- **Q1** \( \forall x(x \equiv x) \)
- **Q2** \( \forall x \forall y(x \equiv y \rightarrow y \equiv x) \)
- **Q3** \( \forall x \forall y \forall z(x \equiv y \land y \equiv z \rightarrow x \equiv z) \)
Axiom **Q4** excludes $m$-atoms from the substitutivity law since if substitutivity is postulated to include them as well, then **Q1–Q4** turn to be exactly the axioms usually used for the predicate of identity \([5]\) and no syntactical difference between identity and indistinguishability could be achieved. By using **Q4** as above, we preserve Leibniz Law of Identity of Indiscernibles for the ‘macroscopic’ (that it, those which are not $m$-atoms) indistinguishable objects (including qsets).

**Q5** No *Urelemente* is at the same time an $m$-atom and an $M$-atom:
\[
\forall x (m(x) \lor M(x) \rightarrow \neg (m(x) \land M(x)))
\]

**Q6** If $x$ has an element, then $x$ is a qset. In other words, the atoms are empty:
\[
\forall x \forall y (x \in y \rightarrow Q(y))
\]

**Q7** Every set is a qset:
\[
\forall x (Z(x) \rightarrow Q(x))
\]

**Q8** No set contains $m$-atoms as elements:
\[
\forall x (\exists \exists m y (y \in x) \rightarrow \neg Z(x))
\]

**Q9** Qsets whose elements are ‘classical objects’ are sets and conversely:
\[
\forall x (\forall y (y \in x \rightarrow D(y)) \leftrightarrow Z(x))
\]

**Theorem 1** If $x$ is an $M$-atom (respectively, a qset) and $x \equiv y$, then $y$ is also an $M$-atom (respect., a qset).

**Proof:** (See \([3]\))

If $x$ is an $m$-atom, the analogous case of the above theorem cannot be proven. For details see \([3]\). So, we need the following postulate:
Q10 Objects which are indistinguishable from \( m \)-atoms are also \( m \)-atoms:

\[
\forall x (m(x) \land x \equiv y \rightarrow m(y))
\]

From the above axioms, it follows that sets cannot have \( m \)-atoms as elements and, in order its elements also be ‘classical’, they also cannot have \( m \)-atoms as elements, and so on. Hence this idea pervades the ‘interior’ of the elements of a qset, and this implies that a qset is a set iff its transitive closure (this concept can be defined in the standard way) does not contain \( m \)-atoms.

Q11 There exists a qset (denoted ‘\( \emptyset \)’) which is a set and which does not have elements:

\[
\exists Z \forall y (\neg (y \in x))
\]

**Definition 2** [Similar quasi-sets] For all quasi-sets \( x \) and \( y \),

\[
Sim(x, y) := \forall z \forall t (z \in x \land t \in y \rightarrow z \equiv t)
\]

Intuitively, similar qsets have as elements objects ‘of the same sort’. The idea of ‘objects of the same sort’ can be realized by passing the quotient by the relation of indistinguishability. This procedure defines equivalence classes of indistinguishable objects and, if they are ‘classical’, the classes turn to be unitary sets, since the indistinguishability relation coincides with equality in this case.

Q12 Indistinguishable sets are extensionally identicals:

\[
\forall_Z \forall_Z x (x \equiv y \rightarrow x =_E y)
\]

Q12 imposes the requeriment that the usual extensional properties of the sets of ZFU are valid for the sets of \( Q \). Further explanations regarding this axiom are presented after the axiom Q27.

Q13 [‘Weak-Pair’] For all \( x \equiv y \), there exists a qset whose elements are the indistinguishable objects from either \( x \) or \( y \):

\[
\forall x \forall y \exists Q \forall t (t \in z \iff t \equiv x \lor t \equiv y)
\]
The weak-pair of $x$ and $y$ is denoted $[x, y]$ and in the case when $x$ and $y$ are both classical objects, we may use the standard notation $\{x, y\}$, since in this case the only things indistinguishable from $x$ and $y$ will be respectively $x$ and $y$ themselves. If $x \equiv y$, we denote the weak-pair by $[x]$, called the weak-singleton of $x$, which is the qset of that which is indistinguishable from $x$. It is important to realize, as it will be clear below, that it is consistent with the theory to admit that the weak-singleton of $x$ may have quasi-cardinal greater than one. In this sense, $\mathcal{Q}$ allows the existence of indistinguishable objects which cannot be said to be identical.

**Q14 [The Separation Schema]** By considering the usual syntactical restrictions on the formula $A(t)$, we have:

$$\forall \mathcal{Q}x \exists \mathcal{Q}y \forall t (t \in y \leftrightarrow t \in x \land A(t))$$

This qset will be written $[t \in x : A(t)]$. The separation axiom allows us to form subquasi-sets of a quasi-set $x$ by considering those elements of $x$ that satisfy a certain property expressed (in the language of $\mathcal{Q}$) by a formula $A(t)$. This idea conforms itself with the intended interpretation of the $m$-atoms as elementary particles, since in ordinary physics it is possible to ‘select’, from a certain collection of elementary particles, a certain number of them that satisfy a particular condition.

**Q15 [Union]** $\forall \mathcal{Q}x(E(x) \rightarrow \exists \mathcal{Q}y(\forall z(z \in y) \leftrightarrow \exists t(z \in t \land t \in x)))$

As usual, this qset is written

$$\bigcup_{t \in x} t$$

and we still write $x \cup y$ in the same sense as in the standard set theories.

**Q16 [Power-qset]** $\forall \mathcal{Q}x \exists \mathcal{Q}y \forall t (t \in y \leftrightarrow t \subseteq x)$

The power qset of $x$ is denoted by $\mathcal{P}(x)$.

**Definition 3**

1. $\pi := [y \in x : m(y)]$
2. $\langle x, y \rangle := [[x], [x, y]]$ (the generalized ordered pair)
3. For every quasi-sets \( x \) and \( y \), \( x \times y := \{ (z, u) \in \mathcal{P}\mathcal{P}(x \cup y) : z \in x \wedge u \in y \} \)

4. The intersection \( x \cap y \) of two quasi-sets can be defined do that \( t \in x \cap y \) iff \( t \in x \wedge t \in y \) as usual.

Q17 [Infinity] \( \exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup [y] \in x)) \)

Q18 [Regularity] Quasi-sets are well-founded, that is, for every qset \( x \), there are no infinite chains \( \ldots \in x_2 \in x_1 \in x \):
\[
\forall Q x (E(x) \wedge x \neq \emptyset \rightarrow \exists Q (y \in x \wedge y \cap x = \emptyset))
\]

2.3 Quasi-Relations

The concept of relation and in particular that of equivalence relation is like the standard one: \( w \) is a relation between two quasi-sets \( x \) and \( y \) if \( w \) satisfies the following predicate \( R \):
\[
R(w) := Q(w) \wedge \forall z (z \in w \rightarrow \exists u \exists v(u \in x \wedge v \in y \wedge z =_E \langle u, v \rangle))
\]

As in the classical case, \( R \in \mathcal{P}\mathcal{P}\mathcal{P}(x \cup y) \). Furthermore, as usual, if \( x =_E y \), we say that \( R \) is a relation on \( x \). We denote by \( \text{Dom}(R) \) (the domain of \( R \)) the qset \( [u \in x : \langle u, v \rangle \in R] \) and by \( \text{Rang}(R) \) (the range of \( R \)) the qset \( [v \in y : \langle u, v \rangle \in R] \).

A particular interesting case of an equivalence relation on a qset \( x \) is the indistinguishability relation, which satisfies the predicate \( R \) above and, due to the axioms \( Q1 - Q3 \), has the required properties. In this case, if \( x \) is a pure qset, then the ‘quotient qset’ \( x/\equiv \) stands for a collection of equivalence classes of indistinguishable objects.

**Theorem 2** No partial, total or strict order relation can be defined on a pure qset whose elements are indistinguishable from one another.

**Proof:** (see [3]).
2.4 Axioms of Quasi-Cardinals

Q19 Every object which is not a qset (that is, every \textit{Urelement}) has quasi-cardinal zero:

\[ \forall x (\neg Q(x) \rightarrow qc(x) = E\ 0) \]

Q20 Every qset has an unique quasi-cardinal which is a cardinal (as defined in the ‘copy’ of ZFU) and, if the qset is in particular a set, then this quasi-cardinal is its cardinal stricto sensu:

\[ \forall Q \exists! y (Cd(y) \land y = E\ qc(x) \land (Z(x) \rightarrow y = E\ card(x))) \]

Q21 Every non-empty qset has a non null quasi-cardinal:

\[ \forall Q x (x \neq E\ \emptyset \rightarrow qc(x) \neq E\ 0) \]

The next axiom says that if the quasi-cardinal of a qset \( x \) is \( \alpha \), then for every quasi-cardinal \( \beta \leq \alpha \), there is a a subquasi-set of \( x \) whose quasi-cardinal is \( \beta \).

Q22 \[ \forall Q x (qc(x) = E\ \alpha \rightarrow \forall \beta (\beta \leq E\ \alpha \rightarrow \exists Q y (y \subseteq x \land qc(y) = E\ \beta)) \]

Q23 The quasi-cardinal of a subquasi-set of \( x \) is not greater than the quasi-cardinal of \( x \):

\[ \forall Q x \forall Q y (y \subseteq x \rightarrow qc(y) \leq E\ qc(x)) \]

Q24 \[ \forall Q x \forall Q y (Fin(x) \land x \subset y \rightarrow qc(x) < qc(y)), \text{ where } Fin(x) \text{ corresponds to say that } x \text{ is finite.} \]

Q25 \[ \forall Q x \forall Q y (\forall w -(w \in x \land w \in y) \rightarrow qc(x \cup y) = E\ qc(x) + qc(y)) \]

In the next axiom, \( 2^{qc(x)} \) denotes (intuitively) the quantity of subquasi-sets of \( x \). Then,

Q26 \[ \forall Q x (qc(P(x)) = E\ 2^{qc(x)}) \]

This last axiom is one of our central interests in this paper, as we see below.
2.5 ‘Weak’ Extensionality

We begin by recalling that the quasi-sets $x$ and $y$ are similar, $(\text{Sim}(x, y))$ – cf. Definition (2) – if their elements are indistinguishable. Then, we define:

**Definition 4** The quasi-sets $x$ and $y$ are Q-Similar if they are similar and have the same quasi-cardinality.

By observing that the quotient quasi-set $x/\equiv$ may be regarded as a collection of equivalence classes of indistinguishable objects, the weak axiom of extensionality is stated as:

**Q27** [Weak Extensionality]

$$
\forall_Q x \forall_Q y (\forall z (z \in x/\equiv \rightarrow \exists t (t \in y/\equiv \land Q\text{Sim}(z, t) \land \forall t (t \in y/\equiv \rightarrow \exists z (z \in x/\equiv \land Q\text{Sim}(t, z) \rightarrow x \equiv y)))
$$

This axiom simply says that those quasi-sets that have the ‘the same quantity of elements of the same sort’ are indistinguishable.

**Theorem 3** $\forall_Q x \forall_Q y (\text{Sim}(x, y) \land \text{qc}(x) =_E \text{qc}(y) \rightarrow x \equiv y)$

**Proof:** (see [3])

As a corollary, it follows that $x =_E y \rightarrow x \equiv y$.

**Theorem 4** $\forall_Q x \forall_Q y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x \equiv y)$

**Proof:** (See [3])

**Theorem 5** $x \equiv y \land \text{qc}([x]) =_E \text{qc}([y]) \leftrightarrow [x] \equiv [y]$

**Proof:** (see [3])

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2.6 Quasi-Functions

With respect to the concept of function, we note that functions, as usu-
ally conceived, cannot distinguish between its arguments and values if there
were \( m \)-atoms involved. So, a more general concept of a \( q \)-function
(quasi-function) as a relation which maps indistinguishable objects into indistin-
guishable objects is introduced:

**Definition 5** Let \( x \) and \( y \) be quasi-sets. Then we say that \( f \) is a \( q \)-function
from \( x \) to \( y \) if \( f \) is such that (\( R \) is the predicate for ‘relation’ defined previ-
ously):

\[
R(f) \land \forall u (u \in x \rightarrow \exists v (v \in y \land \langle u, v \rangle \in f)) \land \\
\forall u \forall u' \forall v \forall v' (\langle u, v \rangle \in f \land \langle u', v' \rangle \in f \land u \equiv u' \rightarrow v \equiv v')
\]

If \( f \) is a \( q \)-function from \( x \) to \( y \) and satisfies the additional condition:

\[
\forall u \forall u' \forall v \forall v' (\langle u, v \rangle \in f \land \langle u', v' \rangle \in f \land v \equiv v' \rightarrow u \equiv u')
\]

\[ \land \text{qc}(\text{Dom}(f)) \leq E \text{qc}(\text{Range}(f)) \]

then \( f \) is a \( q \)-injection, and \( f \) is a \( q \)-surjection if it is a function from \( x \) to \( y \)
such that

\[
\forall v (v \in y \rightarrow \exists u (u \in x \land \langle u, v \rangle \in f)) \land \text{qc}(\text{Dem}(f)) \geq E \text{qc}(\text{Range}(f)).
\]

An \( f \) which is both a \( q \)-injection and a \( q \)-surjection is said to be a \( q \)-bijection.
In this case, \( \text{qc}(\text{Dom}(f)) = E \text{qc}(\text{Range}(f)) \).

In the general case there is no criterion to check if two quasi-sets have
the same quasi-cardinal or not, since there is no ‘counting process’ if they
have \( m \)-atoms as elements. This means, for instance, that if (say) \( x \) has five
elements (formally: its quasi-cardinal is 5), then we cannot define a bijection
from 5 = \{0, 1, 2, 3, 4\} to \( x \), since we would not be able to define without
ambiguity the images of \( f(0) \ldots f(4) \).

If \( A(x, y) \) is a formula in which \( x \) and \( y \) are free variables, we say that
\( A(x, y) \) defines a \( x \)-functional condition on the quasi-set \( t \) if \( \forall w (w \in t \rightarrow \\
\exists s A(w, s) \land \forall w \forall w' (w \in t \land w' \in t \rightarrow \forall s \forall s' (A(w, s) \land A(w', s') \land w \equiv w' \rightarrow \\
s \equiv s')) \) (this is abbreviated by \( \forall x \exists y A(x, y) \)). Then, we have:
Q28 [Replacement]

∀x∃!yA(x, y) → ∀Q∀u∃Qv(∀z(z ∈ v → ∃w(w ∈ u ∧ A(w, z))))

Intuitively, the replacement schema says that the images of qsets by q-functions are also qsets. It is easy to see that if there are no m-atoms involved, that is, if the qsets are sets, then the above axiom is exactly that of ZFC – Zermelo-Fraenkel with Axiom of Choice – (or of ZFU – Zermelo-Fraenkel with Urelemente).

Definition 6 A strong singleton of x is a quasi-set x' which satisfies the following predicate St:

\[ St(x') ↔ x' ⊆ [x] ∧ qc(x') = 1 \]

That is, x' is a subquasi-set of [x] that has just ‘one element’ which is indistinguishable from x.

Theorem 6 For all x, there exists a strong singleton of x.

Proof: (see [3])

Q29 [The Axiom of Choice]

∀x(E(x) ∧ ∀y∀z(y ∈ x ∧ z ∈ x → y ∩ z =E 0 ∧ y ≠E 0) →
∃u∀y∀v(y ∈ x ∧ v ∈ y → ∃w(w ⊆ [v] ∧ qc(w) =E 1 ∧ w ∩ y =E w ∩ u)))

Theorem 7 The extensional equality has all the properties of the usual equality.

Proof: (see [3])

Theorem 8 [Unobservability of Permutations] Let x be a qset and z an m-atom such that z ∈ x. If w ≡ z, then

\[(x - z') ∪ w' ≡ x\]
The operation of difference between qsets is defined as in standard set-theories. This theorem is an immediate consequence of Q27.

We recall that \( z' \) (respect. \( w' \)) denotes the strong singleton of \( z \) (respect., of \( w \)). Furthermore, it may be the case that \( w \notin x \), and this motivates the interpretation according to which the theorem is saying that we have ‘exchanged’ an element of \( x \) by an indistinguishable one, and the resulting fact is that ‘nothing has occurred at all’. In other words, the resulting qset is indistinguishable from the original one. This theorem is the quasi-set theoretical version of the quantum mechanical fact which expresses that permutations of indistinguishable particles are not regarded as observable, as expressed by the so called Indistinguishability Postulate in quantum mechanics.

3 Maxwell-Boltzmann Statistics

3.1 Some Standard Results in ZF

It is a well known theorem in Zermelo-Fraenkel set theory the following:

**Lemma 1** If \( x \) is a finite ZF-set, then

\[
\#\mathcal{P}(x) = 2^{\#x},
\]

where \( \#x \) denotes the cardinal of the set \( x \).

**Theorem 9** Let \( x \) be a non-empty and finite ZF-set. If we define \( x_2 \) as a set of ordered pairs \((y_1, y_2)\) such that \( y_1, y_2 \in \mathcal{P}(x) \), \( x_1 \cup x_2 = x \), and \( x_1 \cap x_2 = \emptyset \) then \( \#x_2 = 2^{\#x} \).

**Proof:** Straightforward from Lemma (1).

**Theorem 10** Let \( x \) be a finite ZF-set such that \( \#x = N \). If we define \( x_n \) as a set of ordered \( n \)-tuples \((y_1, \cdots, y_n)\) such that for all \( i = 1, \cdots, n \) we have \( y_i \in \mathcal{P}(x) \), \( \bigcup_i y_i = x \), and \( i \neq j \rightarrow y_i \cap y_j = \emptyset \), then \( \#x_n = n^N \).

**Proof:** It is straightforward from combinatorial in ZF set theory.

We could rewrite theorem (10) as:
Theorem 11 Let $x$ be a finite ZF-set such that $\#x = N$. If we define $x_n$ as a set of ordered $n$-tuples $(y_1, \ldots, y_n)$ such that for all $i = 1, \ldots, n$ we have $y_i \in \mathcal{P}(x)$, $\bigcup_i y_i = x$, and $\sum_i \#y_i = \#x$, then $\#x_n = n^N$.

Proof: Analogous to the proof of theorem (10), since $\bigcup_i y_i = x$, and $i \neq j \rightarrow y_i \cap y_j = \emptyset$ iff $\bigcup_i y_i = x$, and $\sum_i \#y_i = \#x$.

3.2 Our Proposal

We propose to replace axiom Q26 in quasi-set theory $\mathcal{Q}$ by the following assumption (which is a generalization of Q26 as well as a quasi-set theoretical version of theorem (11)):

Q26’ Let $x$ be a finite quasi-set such that $qc(x) = N$. If we define $z_n$ as the quasi-set whose elements are ordered $n$-tuples $\langle y_1, \ldots, y_n \rangle$, where, for all $i = 1, \ldots, n$, we have $y_i \in \mathcal{P}(x)$, $\bigcup_i y_i = x$, and $\sum_i qc(y_i) = qc(x)$, then we have the following:

$$qc(z_n) = n^N.$$  \hspace{1cm} (1)

In the case where $n = 2$, we have a sentence which is equivalent to axiom Q26.

The main role of axiom Q26’ is to allow us a quasi-set theoretical combinatorics which can be useful to cope with distribution functions. From the mathematical point of view, it is important to show that the replacement of axiom Q26 by axiom Q26’ does not entail any inconsistency in quasi-set theory. This is proved in the Section 5. The point, at this moment, is that Q26 is very ‘poor’ if we are interested on a quasi-set-theoretical combinatorics with more than two physical states or ‘boxes’, as exemplified in the Introduction. Besides, axiom Q26’ is our quasi-set theoretical version of theorem (11).

If we recall the polynomial of Leibniz, we can rewrite equation (1) as:

$$qc(z_n) = n^N = \sum \frac{N!}{\prod_{i=1,\ldots,n} n_i!},$$  \hspace{1cm} (2)

where the sum is over all possible combinations of nonnegative integers $n_i$ such that $\sum_{i=1,\ldots,n} n_i = N$. 

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If we interpret $n$ as the number of physical states, $N$ as the total number of particles and $n_i$ as the number of particles associated to each physical state $i$, then it is easy to see that each parcel of the summation in equation (2) is a possible MB distribution of $N$ particles among $n$ states. The most probable among all these parcels is the MB distribution. So, we can add equation (2), with its respective interpretation, as another extra-assumption in quasi-set theory. In other words, we are generalizing theory $Q$, by replacing axiom $Q26$ by axiom $Q26'$. We refer to this generalized quasi-set theory as $Q'$. 

It is easy to see that, for all $i$ we have $n_i = qc(y_i)$. Axiom $Q26'$ is just another manner to say that the number of ways we can distribute $N$ objects (distinguishable or not) among $n$ boxes is $n^N$. The condition that $\bigcup_i y_i = x$, and $\sum_i qc(y_i) = qc(x)$ is simply a manner to guarantee that there will be no ‘repeated occurrence’ of the same object in two boxes. Nevertheless, it is obvious that the expression ‘repeated occurrence’, in this quasi-set-theoretical context, is just an intuitive approach for didactical purposes, since there is no sense in saying that the ‘same’ object cannot occupy two boxes.

The reader could ask: what are the so-called boxes? Each $y_i$ corresponds to a given box or physical state. There can be, of course, two indistinguishable boxes $y_i$ and $y_j$. In this case, the labels $i$ and $j$ cannot individualize each box. They are just different names, or labels, attributed to two indistinguishable objects (qsets, in this case).

### 3.3 One Simple Example

Now, let us exhibit an example in order to illustrate our ideas. Consider a collection of three indistinguishable particles to be distributed between two possible states or ‘boxes’. According to standard textbooks on statistical mechanics there are only four possibilities of distribution. On the other hand, according to our axiomatic framework – axiom $Q26'$ – there are eight possibilities. If we impose that the occupation number of each box is constant, the number of possibilities corresponds to one parcel of the sum in equation (3).

The question now is: what about the extra four possibilities predicted by axiom $Q26'$? The eight possibilities predicted by $Q26'$ and equation (3) come from
So, we have one possibility with 3 particles in the first state and no particle in the second state, plus three *indistinguishable* possibilities with 2 particles in the first state and 1 particle in the second state, plus three *indistinguishable* possibilities with 1 particle in the first state and 2 particles in the second state, plus one single possibility with no particle in the first state and 3 particles in the remaining one. The calculation of the most probable case is made for a large number of particles, following the standard calculations of statistical mechanics.

Following our example, axiom Q26' says that we can distribute 3 objects (indistinguishable or not) among 2 boxes in $2^3$ manners (indistinguishable or not). But this axiom does not say how can we make this distribution. If we do not appeal to equation (2), we have the following: according to Fig. 1, there are, at least, by means of axiom Q16, four possible distributions. But axiom Q26’ says that there are eight possible distributions. One possibility is something like Fig. 2, that is, the four distributions in Fig. 1 plus four distributions which are indistinguishable from the third distribution of Fig. 1. The reader can easily imagine other possibilities. So, axiom Q26’ by itself does not allow us to derive MB statistics. It simply says that MB statistics is a possibility even in a collection of indiscernibles. Axiom Q26’ and equation (2), with its respective interpretation in the context of Q26’, is a manner to say that the only possibility is that one illustrated in Fig. 3.

4 Quantum Statistics

What is the difference between quantum statistics and MB, after all? In Bose-Einstein we take into account only distinguishable possibilities, among all possibilities predicted by axiom Q26’. And Fermi-Dirac is derived in the same manner, but with the extra assumption of Exclusion Principle in its quasi-set-theoretical form: $qc(y_i) \leq 1$ for each $i$ in Q26’. In [4] the usual quantum distribution functions (Bose-Einstein and Fermi-Dirac) are achieved by means of a quasi-set-theoretical framework. In this paragraph we show that this is not necessary. Put it in another way, quantum statistics may be seen as special cases of MB statistics.
5 Consistency of Theory $Q'$

**Theorem 12** $Q'$ is consistent iff ZFC is consistent.

**Proof:** Here we make just a very brief sketch of the proof, which can be made in details by the reader, with no difficulty at all. The translation from the language of ZFU to the language of $Q$ (as well as to the language $Q'$) has shown that if $Q$ (and $Q'$) is consistent, so is ZFU (and, hence, so is ZFC). In [3] the converse result for $Q$ is outlined. A superstructure $Q$ over a given ZF-set is defined, and a proof that $Q$ is a model for quasi-set theory $Q$ is presented. Since our only modification was the replacement of axiom $Q26$ by axiom $Q26'$, we concentrate our attention to $Q26'$. The proof of axiom $Q26$ in the context of the set-theoretical model $Q$ of $Q$ was made by means of a translation of $Q26$ into the language of ZFC. Since such a translation simply states a basic property of cardinals in ZFC – theorem (9) – its proof does not represent any problem. In the case of axiom $Q26'$ we can use the same argument, since its translation into the language of ZFC (as it was made in [3]) simply states a basic property of cardinals in ZFC – theorem (11).

6 Final Remarks

Our main conclusions are:

1. By using quasi-set theory instead of standard set theory, our paper provides a way of obtaining the usual statistics in physics from the assumption that the ‘non-individuality’ of quantum objects should be ascribed right at the start.

2. Maxwell-Boltzmann statistics can be derived even in a collection of indiscernibles.

3. Maxwell-Boltzmann statistics may be seen as a generalization of quantum statistics (BE and FD); or, BE and FD are particular cases of MB.

4. Quasi-set theory $Q'$ is much more ‘powerful’ than $Q$ if we are interested on a quasi-set-theoretical combinatorics.
5. $Q'$ is consistent if and only if ZFC is consistent.

There have been some recent experiments which have demonstrated entangled pairs of atoms [1], which entail the indistinguishability between these atoms. We wonder if it is possible to demonstrate a gas of indistinguishable atoms which preserves the Maxwell-Boltzmann distribution, since it seems that there is no clearly defined frontier between classical and quantum physics.

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Figure 1: The ‘first’ four possible distributions of 3 objects (indistinguishable or not) among 2 boxes. Each line represents one possible distribution and each bullet represents an object.

Figure 2: One possible sequence of the eight possible distributions of 3 objects among 2 boxes according to axiom Q26’.

Figure 3: The only possible distribution of 3 objects among 2 boxes, if we conjugate axiom Q26’ and equation (2).