Localization for Linearly Edge Reinforced Random Walks

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Abstract

We prove that the linearly edge reinforced random walk (LRRW) on any graph with bounded degrees is recurrent for sufficiently small initial weights. In contrast, we show that for non-amenable graphs the LRRW is transient for sufficiently large initial weights, thereby establishing a phase transition for the LRRW on non-amenable graphs. While we rely on the description of the LRRW as a mixture of Markov chains, the proof does not use the magic formula. We also derive analogous results for the vertex reinforced jump process.

1 Introduction

The linearly edge reinforced random walk (LRRW) is a model of a self-interacting (and hence non-Markovian) random walk, proposed by Coppersmith and Diaconis, and defined as follows. Each edge $e$ of a graph $G = (V, E)$ has an initial weight $a_e > 0$. A starting vertex $v_0$ is given. The walker starts at $v_0$. It examines the weights on the edges around it, normalizes them to be probabilities, and then chooses an edge with these probabilities. The weight of the edge traversed is then increased by 1 (the edge is “reinforced”). The process then repeats with the new weights.

The process is called linearly reinforced because the reinforcement is linear in the number of steps the edge was crossed. Of course one can imagine many other reinforcement schemes, and those have been studied (see e.g. [17] for a survey). Linear reinforcement is special because the resulting process is partially exchangeable. This means that if $\alpha$ and $\beta$ are two finite paths such that every edge is crossed exactly the same number of times by $\alpha$ and $\beta$, then they have the same probability (to be the beginning of an LRRW). Only linear reinforcement has this property.

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Diaconis & Freedman [2, Theorem 7] showed that a recurrent partially exchangeable process is a mixture of Markov chains. Today the name random walk in random environment is more popular than mixture of Markov chains, but they mean the same thing: that there is some measure $\mu$ on the space of Markov chains (known as the “mixing measure”) such that the process first picks a Markov chain using $\mu$ and then walks according to this Markov chain. In particular, this result applies to the LRRW whenever it is recurrent. There “recurrent” means that it returns to its starting vertex infinitely often. We find this result, despite its simple proof (it follows from de Finetti’s theorem for exchangeable processes, itself not a very difficult theorem) to be quite deep. Even for LRRW on a graph with three vertices it gives non-trivial information. For general exchangeable processes recurrence is necessary; see [2, Example 19c] for an example of a partially exchangeable process which is not a mixture of Markov chains. For LRRW this cannot happen, it is a mixture of Markov chains even when it is not recurrent (see Theorem 4 below).

On finite graphs, the mixing measure $\mu$ has an explicit expression, known fondly as the “magic formula”. See [11] for a survey of the formula and the history of its discovery. During the last decade significant effort was invested to understand the magic formula, with the main target the recurrence of the process in two dimensions, a conjecture dating back to the 80s (see e.g. [16, §6]). Notably, Merkl and Rolles [14] showed, for any fixed $a$, that LRRW on certain “dilute” two dimensional graphs is recurrent, though the amount of dilution needed increases with $a$. Their approach did not work for $\mathbb{Z}^2$, but required stretching each edge of the lattice to a path of length 130 (or more). The proof uses the explicit form of the mixing measure, which turns out to be amenable to entropy arguments. These methods involve relative entropy arguments which also lead to the Mermin-Wagner theorem [10]. This connection suggests that the methods should not apply in higher dimension.

An interesting variation on this theme is when each directed edge has a weight. When one crosses an edge one increases only the weight in the direction one has crossed. This process is also partially exchangeable, and is also described by a random walk in a random environment. On the plus side, the environment is now i.i.d., unlike the magic formula which introduces dependencies between the weights of different edges at all distances. On the minus side, the Markov chain is not reversible, while the Markov chains that come out of the undirected case are reversible. These models are quite different. One of the key (expected) features of LRRW — the existence of a phase transition on $\mathbb{Z}^d$ for $d \geq 3$ from recurrence to transience as $a$ varies — is absent in the directed model [18, 8]. In this paper we deal only with the reversible one.

Around 2007 it was noted that the magic formula is remarkably similar to the formulæ that appear in supersymmetric hyperbolic $\sigma$ models which appear in the study of quantum disordered systems, see Disertori, Spencer and Zirnbauer [6] and further see Efetov’s book [7] for an account of the utility of supersymmetry in disordered systems. Very recently, Sabot and Tarrès [19] managed to make this connection explicit. Since
recurrence at high temperature was known for the hyperbolic $\sigma$ model [5], this led to a proof that LRRW is recurrent in any dimension, when $a$ is sufficiently small (high temperature in the $\sigma$ model corresponds to small $a$). We will return to Sabot and Tarrèse’s work later in the introduction and describe it in more details. However, our approach is very different and does not rely on the magic formula in any way.

Our first result is a general recurrence result:

**Theorem 1** For any $K$ there exists some $a_0 > 0$ such that if $G$ is a graph with all degrees bounded by $K$, then the linearly edge reinforced random walk on $G$ with initial weights $a \in (0, a_0)$ is a.s. positive recurrent.

Positive recurrence here means that the walk asymptotically spends a positive fraction of time at any given vertex, and has a stationary distribution. In fact, the LRRW is equivalent to a random walk in a certain reversible, dependent random environment (RWRE) as discussed below. We show that this random environment is a.s. positively recurrent. We formulate and prove this theorem for constant initial weights. However, our proof also works if the initial weights are unequal as long as for each edge $e$, the initial weight $a_e$ is at most $a_0$. With minor modifications, our argument can be adapted to the case of random independent $a_e$’s with sufficient control of their moments. See the discussion after the proof of Lemma 8 for details.

Let us stress again that we do not use the explicit form of the magic formula in the proof. We do use that the process has a reversible RWRE description, but we do not need any details of the measure. The main results in [19] are formulated for the graphs $\mathbb{Z}^d$ but Remark 5 of that paper explains how to extend their proof to all bounded degree graphs. Further, even though [19] claims only recurrence, positive recurrence follows from their methods. Thus the main innovation here is the proof technique.

It goes back to [2] that in the recurrent case the weights are uniquely defined after normalizing, say by $W_{v_0} = 1$ (see there also an example of a transient partially exchangeable process where the weights are not uniquely defined). Hence with Theorem 1 the weights are well defined and we may investigate them. Our next result is that the weights decay exponentially in the graph distance from the starting point. Denote graph distance by $\text{dist}(\cdot, \cdot)$. Also, let $\text{dist}(e, v_0)$ denote the minimal number of edges in any path from $v_0$ to either endpoint of $e$.

**Theorem 2** Let $G$ be a graph with all degrees bounded by $K$ and let $s \in (0, 1/4)$. Then there exists $a_0 = a_0(s, K) > 0$ such that for all $a \in (0, a_0)$, if $e_1$ is the first edge crossed by $X$,

$$\mathbb{E}(W_e)^s \leq \mathbb{E}\left(\frac{W_e}{W_{e_1}}\right)^s \leq 2K\left(C(s, K)\sqrt{a}\right)^{\text{dist}(e,v_0)}. \quad (1)$$

Note that $a_0$ does not depend on the graph except through the maximal degree. The factor $2K$ on the right-hand side is only relevant, of course, when $\text{dist}(e,v_0) = 0$, other-
wise it can be incorporated into the constant inside the power.

The parameter \( s \) deserves some explanation. It is interesting to note that despite the exponential spatial decay, each \( W_i \) does not have arbitrary moments. Examine, for example, the graph with three vertices and two edges. The case when the initial edge weights are 2 and the process starts from the center vertex is particularly famous as it is equivalent to the standard Pólya urn. In this case the weights are distributed uniformly on \([0,1]\). This means that the ratio of the weights of the two edges does not even have a first moment. Of course, this is not quite the quantity we are interested in, as we are interested in the ratio \( W_e / W_f \) where \( W_f \) is the first edge crossed. This, though, is the same as an LRRW with initial weights \( a, a + 1 \) and starting from the side. Applying the magic formula can show that the ratio has \( 1 + a/2 \) moments, but not more. It is also known directly that in this generalized Pólya urn the weights have a Beta distribution (we will not do these calculations here, but they are straightforward). Hence care is needed with the moments of these ratios.

Our methods can be easily improved to give a similar bound for \( s < \frac{1}{3} \), with \( \sqrt{a} \) replaced by a suitably smaller power. See the discussion after Lemma 8 for details. Our proof can probably be modified to give \( 1/2 \) a moment, and depending on \( a \), a bit more. Going beyond that seems to require a new idea.

The most interesting part of the proof of Theorems 1 and 2 is to show Theorem 2 given that the process is already known to be recurrent, and we will present this proof first in § 2. Theorem 1 then follows easily by approximating the graph with finite subgraphs where the LRRW is of course recurrent. This is done in § 3.

### 1.1 Transience

An exciting aspect of LRRW is that on some graphs it undergoes a phase transition in the parameter \( a \). LRRW on trees was analyzed by Pemantle [16]. He showed that there is a critical \( a_c \) such that for initial weights \( a < a_c \), the process is positively recurrent, while for \( a > a_c \) it is transient. We are not aware of another example where a phase transition was proved, nor even of another case where the process was shown to be transient for any initial weights. The proof of Pemantle relies critically on the tree structure, as in that case, when you know that you have exited a vertex through a certain edge, you know that you will return through the very same edge, if you return at all. This decomposes the process into a series of i.i.d. Pólya urns, one for each vertex. Clearly this kind of analysis can only work on a tree.

The next result will show the existence of a transient regime in the case of non-amenable graphs. Recall that a graph \( G \) is non-amenable if for some \( \ell > 0 \) and any finite set \( A \subset G \),

\[ |\partial A| \geq \ell |A| \]

where \( \partial A \) is the external vertex boundary of \( A \) i.e. \( \partial A = \{ x : \text{dist}(x, A) = 1 \} \). The largest
constant \( \iota \) for which this holds is called the **Cheeger constant** of the graph \( G \).

**Theorem 3**  For any \( K, c_0 > 0 \) there exists \( a_0 \) so that the following holds. Let \( G \) be a graph with Cheeger constant \( \iota \geq c_0 \) and degree bound \( K \). Then for \( a > a_0 \) the LRRW on \( G \) with all initial weights \( a \) on all edges is transient.

Theorem 3 will be proved in §4. As with Theorem 1, our proof works with non-equal initial weights, provided that \( a_e > a_0 \) for all \( e \). It is tempting to push for stronger results by considering graphs \( G \) with intermediate expansion properties such that the simple random walk on \( G \) is transient. To put this in perspective, let us give some examples where LRRW has no transient regime.

1. The **canopy graph** is the graph \( \mathbb{Z}^+ \) with a finite binary tree of depth \( n \) attached at vertex \( n \). It is often poetically described as “an infinite binary tree seen from the leaves”. Since the process must leave each finite tree eventually, the process on the “backbone” is identical to an LRRW on \( \mathbb{Z}^+ \), which is recurrent for all \( a \) (say from [1] or from the techniques of [16]).

2. Let \( T \) be the infinite binary tree. Replace each edge on the \( n \)th level by a path of length \( n^2 \). The random walk on the resulting graph is transient (by a simple resistance calculation, see e.g. [4]). Nevertheless, LRRW is recurrent for any value of \( a \). This is because LRRW on \( \mathbb{Z}^+ \) has the expected weights decaying exponentially (again from the techniques of [16]) and this decay wins the fight with the exponential growth of the levels.

3. These two example can be combined (a stretched binary tree with finite decorations) to give a transient graph with exponential growth on which LRRW is recurrent for any \( a \).

We will not give more details on any of these examples as that will take us off-topic, but they are all more-or-less easy to do. The proof of Theorem 3 again uses that the process has a dual representation as both an self-interacting random walk and as an RWRE. This might be a good point to reiterate that the LRRW is a mixture of Markov chains even in the transient case (the counterexample of Diaconis and Freedman is for a partially exchangeable process, but that process is not a LRRW). This was proved by Merkl and Rolles [13], who developed a tightness result for this purposes which will also be useful for us. Let us state a version of their result in the form we will need here along with the original result of Diaconis and Freedman:

**Theorem 4** ([2, 13])  Fix a finite graph \( G = (V, E) \) and initial weights \( a = (a_e)_{e \in E} \) with \( a_e > 0 \). Then \( X \) is a mixture of Markov chains in the following sense: There exists a unique probability measure \( \mu \) on \( \Omega \) so that

\[
P = \int P^W d\mu(W)
\]
is an identity of measures on infinite paths in $G$. All $W(e) > 0$ for all $e$.

Moreover, the $W_e$'s form a tight family of random variables: if we set $W_v = \sum_{e \ni v} W_e$ and $a_v = \sum_{e \ni v} a_e$, there are contestants $c_1, c_2$ depending only on $a_v, a_e$ so that

$$\mu(W_e/W_v \leq \varepsilon) \leq c_1 \varepsilon^{a_e/2} \quad \text{and} \quad \mu(W_e/W_v \geq 1 - \varepsilon) \leq c_2 \varepsilon^{(a_v-a_e)/2}.$$ 

As noted, we mainly need the existence of such a representation of the LRRW, as well as the tightness of the $W$’s, but not the explicit bounds.

It is natural to conjecture that in fact a phase transition exists on $\mathbb{Z}^d$ for all $d \geq 3$, analogously to the phase transition for the supersymmetric $\sigma$ model. What happens in $d = 2$? We believe it is recurrent for all $a$, but dare not guess whether it enjoys a Kosterlitz-Thouless transition or not.

1.2 Back to Sabot and Tarrèses

The main object of study for Sabot and Tarrèses [19] are vertex reinforced jump processes (VRJP). Unlike LRRW, this is defined in continuous time process with reinforcement acting through local times of vertices. One begins with a positive function $J = (J_e)_{e \in E}$ on the edges of $G$. These are the initial rates for the VRJP process $Y_t$, and are analogous to the initial weights $a$ of LRRW. Let $(L_x(t))_{x \in V}$ be the local times for $Y$ at time $t$ and vertex $x$. If $Y_t = x$ then $Y$ jumps to a neighbor $y$ with rate $J_{xy}(1 + L_y(t))$. See [19] for history and additional reference for this process.

The VRJP shares a key property with the LRRW: a certain form of partial exchangeability after applying a certain time change. This suggests that it too has a RWRE description (Diaconis and Freedman [2] only consider discrete time processes, but their ideas should carry over). Such a RWRE description exists was found by Sabot and Tarrèses by other methods. The existence (though not the formula) for such a form is fundamental for our proof. Their main results are the following: First, the law of LRRW with initial weight $a$ is identical to the time-discretization of $Y_t$ when $J$ is i.i.d. with marginal distribution $\Gamma(a, 1)$. Secondly, after a time change, $Y_t$ is identical to a mixture of continuous-time reversible Markov chains. Moreover, the mixing measure is exactly the marginal of the supersymmetric hyperbolic $\sigma$ models studied in [5, 6]. This is analogous to the magic formula for the LRRW. Finally, as already mentioned, this allowed them to harness the techniques of [5, 6] to prove that LRRW is recurrent for small $a$ in all dimensions.

Since the VRJP has both a dynamic and an RWRE representation, our methods apply to this model too. Thus we show:

**Theorem 5** Let $G$ be a fixed graph with degree bound $K$. Let $J = (J_e)_{e \in E}$ be a family of independent initial rates with

$$\mathbb{E}J^{1/5} < c(K),$$

where $c(K)$ is a constant depending only on $K$. Then (a.s. with respect to $J$), $Y_t$ is recurrent.
In particular this holds for any fixed, sufficiently small $\mathbf{J}$. We formulate this result for random $\mathbf{J}$ because of the relation to LRRW explained above — we could have also proved the LRRW results for random $\mathbf{a}$ but we do not see a clear motivation to do so. We will prove this in § 5, where we will also give more precise information about the dependency of $c$ on $K$.

Next we wish to present a result on exponential decay of the weights, an analogue of Theorem 2. To make the statement more meaningful, let us describe the RWRE (which, we remind, is not $Y_t$ but a time change of it). We are given a random function $W$ on the vertices of our graph $G$. The process is then a standard continuous-time random walk which moves from vertex $i$ to vertex $j$ with rate $\frac{1}{2}I_{ij}W_j/W_i$. The result is now:

**Theorem 6**  
If the $J_e$ are independent, with $\mathbb{E}[J_e^{1/5}] < c(K)$ then for a.e. $\mathbf{J}$, the infinite volume mixing measure exists and under the joint measure, for any vertex $v \in G$,

$$\mathbb{E}W_v^{1/5} < 2K^{-4\text{dist}(v_0,v)}.$$  

In particular, this implies that the time-discretization of $Y$ is positively recurrent, and not just recurrent. For any $s < 1/5$, bounds on $\mathbb{E}[J_e^s]$ can yield the an estimate on $\mathbb{E}W_v^s$ and Theorem 5.

It is interesting to note that the proof of Theorems 5 and 6 are simpler than that of Theorems 1 and 2, even though the techniques were developed to tackle LRRW. The reader will note that for VRJP, each edge can be handled without interference from adjoining edges on the same vertex, halving the length of the proof. Is there some inherent reason? Is VRJP (or the supersymmetric $\sigma$ model) more basic in some sense? We are not sure.

### 1.3 Notations

In this paper, $G$ will always denote a graph with bounded degrees, and $K$ a bound on these degrees. The set of edges of $G$ will be denoted by $E$. $a_e$ will always denote the initial weights, and when the notation $a$ is used, it is implicitly assumed that $a_e = a$ for all edges $e \in E$.

Let us define the LRRW again, this time using the notations we will use for the proofs. Suppose we have constructed the first $k$ steps of the walk, $x_0, \ldots, x_k$. For each edge $e \in E$, let

$$N_k(e) = |\{j < k : e = \langle X_j, X_{j+1}\rangle\}|$$

be the number of times that the undirected edge $e$ has been traversed up to time $k$. Then each edge $e$ incident on $X_k$ is used for the next step with probability proportional to $a + N_k(e)$, that is, if $X_k = v$ then

$$\mathbb{P}(X_{k+1} = w | X_0, \ldots, X_k) = \frac{a + N_k((v, w))}{d_v a + N_k(v)} 1\{v \sim w\},$$
where \( d_v \) is the degree of \( v \); where \( N_k(v) \) denotes the sum of \( N_k(e) \) over edges incident to \( v \); and where \( \sim \) is the neighborhood relation i.e. \( v \sim w \iff \langle v, w \rangle \in E \). It is crucial to remember that \( N_k \) counts traversals of the undirected edges. We stress this now because at some points in the proof we will count oriented traversals of certain edges.

While all graphs we use are undirected, it is sometimes convenient to think of edges as being directed. Each edge \( e \) has a head \( e^- \) and tail \( e^+ \). The reverse of the edge is denoted \( e^{-1} \). A path of length \( n \) in \( G \) may then be defined as a sequence of edges \( e_1, \ldots, e_n \) such that \( e^+_i = e^-_{i+1} \) for all \( i \). Vice versa, if \( v \) and \( w \) are two vertices, \( \langle v, w \rangle \) will denote the edge whose two vertices are \( v \) and \( w \).

By \( W \) we will denote a function from \( E \) to \([0, \infty)\) (“the weights”). We will denote \( W(e) \) instead of \( W(e) \) and for a vertex \( v \) we will denote \( W_v := \sum_{e \ni v} W(e) \).

The space of all such \( W \) will be denoted by \( \Omega \) and measures on it typically by \( \mu \). The \( \mu \) which describes our process (whether unique or not) will be called “the mixing measure”.

Given \( W \) we define a Markov process with law \( P^W \) on the vertices of \( G \) as follows. The probability to transition from \( v \) to \( w \), denoted \( P^W(v, w) \), is \( \frac{W_{\langle v, w \rangle}}{W_v} \). For a given \( \mu \) on \( \Omega \), the RWRE corresponding to \( \mu \) is a process on the vertices of \( G \) given by

\[
P(X_0 = v_0, \ldots, X_n = v_n) = \int \prod_{i=1}^{n} P^W(X_{i-1}, X_i) \, d\mu(W).
\]

This process will always be denoted by \( X \).

**Definition** A process is recurrent if it returns to every vertex infinitely many times.

This is the most convenient definition of recurrence for us. It is formally different from the definition of [2] we quoted earlier, but by the results of [13] quoted above they are in fact equivalent for LRRW.

We also use the following standard notation: for two vertices \( v \) and \( w \) we define \( \text{dist}(v, w) \) as the length of the shortest path connecting them (or 0 if \( v = w \)). For an edge \( e \) we define \( \text{dist}(v, e) = \min\{\text{dist}(v, e^-), \text{dist}(v, e^+)\} \). For a set of vertices \( A \) we define \( \partial A = \{x : \text{dist}(x, A) = 1\} \) i.e. the external vertex boundary. By \( X \overset{D}{=} Y \) we denote that the variables \( X \) and \( Y \) have the same distribution. \( \text{Bern}(p) \) will denote a Bernoulli variable with probability \( p \) to take the value 1 and \( 1 - p \) to take the value 0 and \( \text{Exp}(\lambda) \) will denote an exponential random variable with expectation \( 1/\lambda \). We will denote constants whose precise value is not so important by \( c \) and \( C \), where \( c \) will be used for constants sufficiently small, and \( C \) for constants sufficiently large. The value of \( c \) and \( C \) (even if they are not absolute but depend on parameters, such as \( C(K) \)) might change from one
place to another. By $x \sim y$ we denote $cx \leq y \leq Cx$.

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2 Proof of Theorem 2

In this section we give the most interesting part of the proof of Theorem 2, showing exponential decay assuming a priori that the process is recurrent. We give an upper bound which only depends on the maximal degree $K$. In the next section use apply this result to a sequence of finite approximations to $G$ to prove recurrence for the whole graph and complete the proof.

Before we begin, we need to introduce a few notations. For any edge $e = (e^-, e^+)$ that is traversed by the walk, let $e'$ be the edge through which $e^-$ is first reached. In particular, $e'$ is traversed before $e$. If $e$ is traversed before its inverse $e^{-1}$, then $e'$ is distinct from $e$ as an undirected edge. Iterating this construction to get $e''$, $e'''$, etc. yields a path $\gamma = \{\ldots, e'', e', e\}$ starting with the first edge used from $v_0$, and terminating with $e$. We call $\gamma$ the path of domination of $e$. This path is either a simple path, or a simple loop in the case that $e^+$ is the starting vertex $v_0$. In the former case $\gamma$ is the backwards loop erasure of the LRRW. All edges in the path are traversed before their corresponding inverses. Let $\mathcal{D}_\gamma$ be the event that the deterministic path $\gamma$ is the path of domination corresponding the final edge of $\gamma$.

For an edge $e$ with $e' \neq e^{-1}$, let $Q(e)$ be an estimate for $W_e/W_{e'}$ defined as follows: If $e$ is crossed before $f := (e')^{-1}$ then set $M_e$ to be the number of times $e$ is crossed before $f$, and set $M_f = 1$. If $f$ is crossed before $e$ then set $M_f$ to be the number of times $f$ is crossed before $e$, and set $M_e = 1$. In both cases we count crossing as directed edges. In other words, we only count crossings that start from $e^-$, the common vertex of $e$ and $f$ (in which case the walker chooses between them). Then

$$Q(e) := \frac{M_e}{M_f}$$

is our estimate for $W_e/W_{e'}$. Thus to find $Q(e)$ we wait until the LRRW has left $e^-$ along both $e$ and $f$, and take the ratio of the number of departures along $e$ and along $f$ at that time. Note again that we do not include transitions along $e^{-1}$ or $f^{-1} = e'$, so that by definition one of the two numbers $M_e$ and $M_f$ must be 1.

With $Q$ defined we can start the calculation. Recall that $e_1$ is the first edge crossed by the walk. Suppose $x$ is some edge of $G$ and we want to estimate $W_x/W_{e_1}$. We fix $x$
for the reminder of the proof. Let $\Gamma = \Gamma_x$ denote the set of possible values for the path of domination i.e. the set of simple paths or loops whose first edge is one of the edges coming out of $v_0$ and whose last edge is either $x$ or $x^{-1}$ (depending on which is crossed first).

Split the probability space according to the value of the path of domination:

$$
E \left[ \left( \frac{W_x}{W_{e_1}} \right)^s \right] = \sum_{\gamma \in \Gamma} E \left[ \left( \frac{W_x}{W_{e_1}} \right)^s 1\{D_\gamma\} \right].
$$

Naturally, under $D_{\gamma}$, $e_1$ must be the first edge of $\gamma$. We remind the reader that we assume a priori that our process is recurrent. This has two implications: first, $x$ will a.s. be visited eventually, and so the path of domination is well-defined. Second, the weights are unique, so $W_x/W_{e_1}$ is a well-defined variable on our probability space.

Given the weights $W$, let $R$ be the actual ratios along the path of domination, so for $e \in \gamma \setminus \{e_1\}$,

$$
R(e) = \frac{W_e}{W_{e'}}.
$$

On the event $D_\gamma$ for $\gamma$ fixed, we may telescope $W_x/W_{e_1}$ via

$$
\frac{W_x}{W_{e_1}} = \prod_{e \in \gamma \setminus e_1} R(e) = \prod_{e \in \gamma \setminus e_1} \frac{R(e)}{Q(e)} \prod_{e \in \gamma \setminus e_1} Q(e).
$$

An application of the Cauchy-Schwarz inequality then gives

$$
E \left[ \left( \frac{W_x}{W_{e_1}} \right)^s 1\{D_\gamma\} \right] \leq \mathbb{E} \left[ \prod_{e \in \gamma \setminus e_1} \left( \frac{R(e)}{Q(e)} \right)^{2s} 1\{D_\gamma\} \right] \cdot \mathbb{E} \left[ \prod_{e \in \gamma \setminus e_1} Q(e)^{2s} 1\{D_\gamma\} \right]^{\frac{1}{2}}. \quad (2)
$$

Theorem 2 then essentially boils down to the following two lemmas.

**Lemma 7** For any graph $G$, any starting vertex $v_0$ and any $a \in (0, \infty)$ such that the LRRW on $G$ starting from $v_0$ with initial weights $a$ is recurrent, for any edge $x \in E$, any $\gamma \in \Gamma_x$ and any $s \in (0,1)$,

$$
\mathbb{E} \left[ \prod_{e \in \gamma \setminus e_1} \left( \frac{R(e)}{Q(e)} \right)^s 1\{D_\gamma\} \right] \leq C(s)^{\gamma-1}
$$

where $C(s)$ is some constant that depends on $s$ but not on $G$, $v_0$, $a$ or anything else.

**Lemma 8** For any graph $G$ with degrees bounded by $K$, any starting vertex $v_0$ and any $a \in (0, \infty)$ such that LRRW on $G$ starting from $v_0$ with initial weights $a$ is recurrent, for any edge
\[ x \in G, \text{ any } \gamma \in \Gamma_x \text{ and any } s \in (0,1/2), \]

\[ \mathbb{E} \left[ \prod_{e \in \gamma \setminus e_1} Q(e)^s 1\{\mathcal{D}_\gamma\} \right] \leq \left[ C(s,K) a \right] |\gamma|^{-1}, \]

where \( C(s,K) \) is a constant depending only on \( s,K \).

In these two lemmas, it is not difficult to make \( C(s) \) and \( C(s,K) \) explicit. Following the proof gives
\[ C(s) = O\left( \frac{1}{1-s} \right) \text{ and } C(s,K) = O\left( \frac{K}{1-2s} + \frac{1}{s} \right). \]

However, there seems to be little reason to be interested in the \( s \)-dependency. We will apply the lemmas with some fixed \( s \), say \( 1/4 \). We do not have particularly strong feelings about the \( K \)-dependency either.

It is worth noting that Lemma 7 is proved by using the random environment point of view on the LRRW, while Lemma 8 is proved by considering the reinforcements. Thus both views are central to our proof. We note that Theorem 2 can be extended to \( s < 1/3 \) by simply using Hölder’s inequality instead of Cauchy-Schwartz. Relaxing the limit on \( s \) in Lemma 7 may allow any \( s < 1/2 \), but that is the limit of our approach, since the \( 1/2 \) in Lemma 8 is best possible.

**Proof of Lemma 7.** For this lemma the RWRE point of view is used, as it must, since the weights \( W \) appear in the statement, via \( R \). Our first step is to throw away the event \( 1\{\mathcal{D}_\gamma\} \) i.e. to write
\[
\mathbb{E} \left[ \prod_{e \in \gamma \setminus e_1} \left( \frac{R(e)}{Q(e)} \right)^s 1\{\mathcal{D}_\gamma\} \right] \leq \mathbb{E} \left[ \prod_{e \in \gamma \setminus e_1} \left( \frac{R_{\gamma}(e)}{Q_{\gamma}(e)} \right)^s \right] \tag{3}
\]
where the terms on the right-hand side are as follows: \( R_{\gamma}(e) \) is the ratio between \( W_e \) and \( W_f \) where \( f \) is the predecessor of \( e \) in \( \gamma \); and \( Q_{\gamma}(e) \) is defined by following the process until both \( e \) and \( f \) are crossed at least once from \( e^- \) and then define \( M_e, M_f \) and \( Q \) according to these crossings. Clearly, under \( \mathcal{D}_\gamma \) both definitions are the same so (3) is justified.

This step seems rather wasteful, as heuristically one can expect to lose a factor of \( K^{\lvert \gamma \rvert} \) from simply ignoring a condition like \( \mathcal{D}_\gamma \). But because our eventual result (Theorem 2) has a \( C(s,K)^{\lvert \gamma \rvert} \) term, this will not matter. Since \( \gamma \) is fixed, from this point until the end of the proof of the lemma we will denote \( R = R_{\gamma} \) and \( Q = Q_{\gamma} \).

At this point, and until the end of the lemma, we fix one realization of the weights \( W \) and condition on it being chosen. This conditioning makes the \( R(e) \) just numbers, while the \( Q(e) \) become independent. Indeed, given \( W \), the random walk in the random environment can be constructed by associating with each vertex \( v \) a sequence \( Z_v \) of i.i.d. edges incident to \( v \) with law \( W_v / \sum_{e \ni v} W_e \). If the walk is recurrent, then \( v \) is reached infinitely often, and the entire sequence is used in the construction of the walk. If we fix an edge \( e \) and let \( f = e^{-1} \), then \( M_e \) (resp. \( M_f \)) is the number of appearances of \( e \)
(resp. \( f \)) in the sequence \( \{Z^v\} \) up to the first time that \( e \) and \( f \) have both appeared. As a consequence, since the sequences for different vertices are independent, we get that conditioned on the environment \( W \), the estimates \( Q(e) \) for \( W_e/W_f \) for pairs incident to different vertices are all independent.

Thus to prove our lemma it suffices to show that for any two edges \( e, f \) leaving some vertex \( v \), with \( M_e \) and \( M_f \) defined as above we have

\[
\mathbb{E} \left[ \left( \frac{W_e M_f}{W_f M_e} \right)^s \bigg| W \right] \leq C(s).
\]

We now show that this holds uniformly in the environment.

First, observe that entries other than \( e, f \) in the sequence of i.i.d. edges at \( v \) have no effect on the law of \( M_e \) and \( M_f \), so we may assume w.l.o.g. that \( e \) and \( f \) are the only edges coming out of \( e^- \). Denote the probability of \( e \) by \( p \) and of \( f \) by \( q = 1 - p \) (for some \( p \in (0,1) \)). Now we have for \( n \geq 1 \), that the probability that \( e \) appears \( n \) times before \( f \) is \( p^n q \), and similarly for \( f \) before \( e \) with the roles of \( p \) and \( q \) reversed. Thus

\[
\mathbb{E} \left[ \left( \frac{W_e M_f}{W_f M_e} \right)^s \bigg| W \right] = \left( \frac{p}{q} \right)^s \left[ \sum_{k \geq 1} k^s p q^k + \sum_{k \geq 1} k^{-s} q p^k \right].
\]

It is a straightforward calculation to see that this is bounded for \( |s| < 1 \). The first term is the \( s \)-moment of a \( \text{Geom}(p) \) random variable, which is of order \( p^{-s} q \), and with the pre-factor comes to \( q^{1-s} \). The second term is the \((-s)\)-moment of a \( \text{Geom}(q) \) random variable, which is of order \( p q^s \), and with the pre-factor gives \( p^{1+s} \). Thus

\[
\mathbb{E} \left[ \left( \frac{W_e M_f}{W_f M_e} \right)^s \bigg| W \right] \asymp q^{1-s} + p^{1+s} \leq C(s)
\]

(recall that we assumed \( s \in (0,1) \)). This finishes the lemma.

Now we move on to the proof of Lemma 8.

**Proof of Lemma 8.** Fix a path \( \gamma \). We shall construct a coupling of the LRRW together with a collection of i.i.d. random variables \( \overline{Q}(e) \), associated with the edges of \( \gamma \) (except \( e_1 \)) such that on the event \( \mathcal{E}_\gamma \), for every edge \( e \in \gamma \setminus e_1 \) we have \( Q(e) \leq \overline{Q}(e) \), and such that for \( s < 1/2 \),

\[
\mathbb{E} \overline{Q}(e)^s \leq C(s, K)a.
\]
The claim would follows immediately, because

\[
\mathbb{E} \prod Q(e)^s 1\{\mathcal{D}_\gamma\} \leq \mathbb{E} \prod Q(e)^s 1\{\mathcal{P}_\gamma\} \leq \mathbb{E} \prod Q(e)^s = \prod \mathbb{E} Q(e)^s \\
\leq \prod C(s, K)a = (C(s, K)a)^{|\gamma|-1}.
\]

The remarks in the previous proof about “waste” are just as applicable here, since we also, in the second inequality, threw away the event \(\mathcal{D}_\gamma\). Note that we cannot start by eliminating the restriction, since we only prove \(Q \leq \overline{Q}\) on the event \(\mathcal{D}_\gamma\).

Let us first describe the random variables \(\overline{Q}\). Estimating their moments is then a straightforward exercise. Next we will construct the coupling, and finally we shall verify that \(Q(e) \leq \overline{Q}(e)\). For an edge \(e = (e^-, e^+)\) of \(\gamma\), we construct two sequences of Bernoulli random variables (both implicitly depending on \(e\)). For \(j \geq 0\), consider Bernoulli random variables

\[
Y_j = \text{Bern} \left( \frac{a}{j+1+2a} \right), \quad Y'_j = \text{Bern} \left( \frac{1+a}{2j+1+Ka} \right).
\]

where Bern(\(p\)) is a random variable that takes the value 1 with probability \(p\) and 0 with probability \(1-p\). All \(Y\) and \(Y'\) variables are independent of each other and of those associated with other edges in \(\gamma\). In the context of the event \(\mathcal{D}_\gamma\), we think of \(Y'_0\) as the event that decides which of \(e\) and \(f\) is crossed first. For \(j \geq 1\), think about \(Y_j\) as telling us whether on the \(j^{th}\) visit to \(e^-\) we depart along \(e\) and \(Y'_j\) telling us whether we depart along \(f = e'^-\). This leads to the definition

\[
\overline{Q} = \overline{M}_e / \overline{M}_f,
\]

where

\[
\overline{M}_e = \min\{j \geq 1 : Y'_j = 1\} \quad \text{and} \quad \overline{M}_f = 1, \quad \text{if } Y'_0 = 0, \\
\overline{M}_f = \min\{j \geq 1 : Y_j = 1\} \quad \text{and} \quad \overline{M}_e = 1, \quad \text{if } Y'_0 = 1.
\]

**Moment estimation.** To estimate \(\mathbb{E} \overline{Q}^s\) we note

\[
\mathbb{P}(Y'_0 = 0, \overline{M}_e = n) = \frac{(K-1)a}{1+Ka} \frac{1+a}{2n+1+Ka} \prod_{j=1}^{n-1} \left( 1 - \frac{1+a}{2j+1+Ka} \right).
\]

The first two terms we estimate by

\[
\frac{(K-1)a}{1+Ka} \frac{1+a}{2n+1+Ka} \leq \frac{Ka}{1+Ka} \frac{1+Ka}{2n} = \frac{Ka}{2n}
\]
while for the product we note that for any $a > 0$,

$$\frac{1 + a}{2j + 1 + Ka} \geq \min \left\{ \frac{1}{2j + 1}, \frac{1}{K} \right\}.$$ 

Putting these together we get

$$\mathbb{P}(Y'_0 = 0, M_e = n) \leq \frac{Ka}{2n} \prod_{j=1}^{n-1} \exp \left( -\frac{1}{2j} + O(j^{-2}) \right) = \frac{Ka}{2n} \exp \left( -\frac{1}{2} \log(n) + O(1) \right) \leq C(K)an^{-3/2}.$$ 

Thus for $s < 1/2$,

$$\mathbb{E} \left[ \mathcal{Q}(e)^s (\{Y'_0 = 0\} \right] \leq \sum_{n \geq 1} n^s \mathbb{P}(Y'_0 = 0, M_e = n) \leq \sum_{n \geq 1} \mathcal{C}(K) n^{-3/2} \leq C(s, K)a.$$

(This is the main place where the assumption $s < 1/2$ is used.)

For the case $Y'_0 = 1$ we write

$$\mathbb{P}(Y'_0 = 1, M(f) = n) \leq \mathbb{P}(Y'_0 = 1) \leq \frac{a}{n},$$

and so

$$\mathbb{E} \left[ \mathcal{Q}(e)^s (\{Y'_0 = 1\} \right] = \sum_{n \geq 1} an^{-1-s} < C(s)a.$$

Together we find $\mathbb{E} \mathcal{Q}(e)^s \leq C(s, K) a$ as claimed.

**The coupling.** Here we use the linear reinforcement point of view of the walk. We consider the Bernoulli variables as already given, and construct the LRRW as a function of them (and some additional randomness). Suppose we have already constructed the first $t$ steps of the LRRW, are at some vertex $v$, and need to select the next edge of the walk. There are several cases to consider.

**Case 0.** $v \notin \gamma$: In this case we choose the next edge according to the reinforced edge weights, independently of all the $Y$ and $Y'$ variables.

**Case 1.** $v \in \gamma$ and the LRRW so far is not consistent with $\mathcal{D}_\gamma$: We may safely disregard the $Y$ variables, as nothing is claimed in this case. This case occurs if for some edge $e \in \gamma$ is traversed only after its inverse is, or if the first arrival to $e^+$ was not through the edge preceding $e$ in $\gamma$. 

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Case 2. \( v \in \gamma \), and \( Q \) is already determined: If \( v = e^- \) for \( e \in \gamma \), and both \( e \) and \( (e')^{-1} \) have both already been traversed, then \( Q(e) \) is determined by the path so far. Again, we disregard the \( Y \) variables.

For the remaining cases, we may assume that \( \mathcal{D}_\gamma \) is consistent with the LRRW path so far, and that \( v = e^- \) for \( e \in \gamma \setminus e_1 \). As before, let \( f = (e')^{-1} \). If we are not in Case 2, then one of \( e, f \) has not yet been traversed.

Case 3. This is the first arrival to \( v \). In this case the weights of all edges incident to \( v \) are still \( a \), except for \( f \) which has weight \( 1 + a \). Thus the probability of exiting along \( f \) is

\[
\frac{1 + a}{1 + d_v a} \geq \frac{1 + a}{1 + Ka}.
\]

where as usual \( d_v \) is the degree of the vertex \( v \). Thus we can couple the LRRW step so that if \( Y_0' = 1 \), then the walk exits along \( f \) (and occasionally also if \( Y_0' = 0 \)).

Case 4. Later visits to \( v \) when \( Y_0' = 0 \). Suppose this is the \( n \)th visit to \( v \). The weight of \( f \) is at least \( 1 + a \), since we first entered \( v \) through \( f^{-1} \). The total weight of edges at \( v \) is \( 2n - 1 + d_v a \). Thus the probability of the LRRW exiting through \( f \) is

\[
\frac{N_t(f) + a}{2n - 1 + d_v a} \geq \frac{1 + a}{2n - 1 + Ka}
\]

where \( t \) is the time of this visit to \( v \), and \( N_t(f) \) is, as usual, the number of crossings of the edge \( f \) up to time \( t \). Thus we can couple the LRRW step so that if \( Y_{n-1}' = 1 \), then the walk exits along \( f \) (and occasionally also if \( Y_{n-1}' = 0 \)).

Case 5. Later visits to \( v \) with \( Y_0' = 1 \). Here, \( f \) was traversed on the first visit to \( v \). Since we are not in Case 2, \( e \) has not yet been traversed. Since we are not in Case 1, neither has \( e^{-1} \), and so \( e \) still has weight \( a \). In this case, we first decide with appropriate probabilities, and independent of the \( Y \), whether to use one of \( \{e, f\} \), or another edge from \( v \). If we decide to use another edge, we ignore the \( Y \) variables. If we decide to use one of \( \{e, f\} \), and this is the \( n \)th time this occurs, then \( N_t(f) \geq n \) (since we had only chosen \( f \) so far in these cases, and also used \( f^{-1} \) at least once). Thus the probability that we select \( e \) from \( \{e, f\} \) is

\[
\frac{a}{N_t(f) + 2a} \leq \frac{a}{n + 2a}.
\]

Thus we can couple the LRRW step so that if \( Y_{n-1}' = 0 \), then we select \( f \).

Domination of \( Q(e) \). We now check that on \( \mathcal{D}_\gamma \) we have \( Q(e) \leq \overline{Q}(e) \). Assume \( \mathcal{D}_\gamma \) occurs. When \( Y_0' = 0 \) the coupling considers only the \( Y' \) variables, one at each visit to
v until f is used. If n is minimal s.t. \( Y'_n = 1 \) then f is used on the \((n+1)\)st visit to v or earlier. Thus

\[ Q(e) \leq M_f \leq n = \overline{Q}(e). \]

Note that if \( Y'_0 = 0 \) it is possible that the walk uses f before e, and then \( Q(e) \leq 1 \).

If \( Y'_0 = 1 \) then the coupling considers only the \( Y_s \), one at each time that either e or f is traversed (and not at every visit to v). If n is the minimal \( n \geq 1 \) s.t. \( Y_n = 1 \), then f is used at least n times before e. Thus

\[ Q(e) = \frac{1}{M_f} \leq \frac{1}{n} = \overline{Q}(e), \]

and in both cases \( Q(e) \leq \overline{Q}(e) \), completing the proof.

Let us remark briefly on how this argument changes if the weights \( a \) are not equal, or possibly random. In order to get the domination \( Q(e) \leq \overline{Q}(e) \) the variables \( Y_j \) and \( Y'_j \) are defined differently. Setting \( a_v = \sum_{e \ni v} a_e \), we use

\[ Y_j = \text{Bern} \left( \frac{a_e}{j + 1 + a_e + a_f} \right), \quad Y'_j = \text{Bern} \left( \frac{1 + a_f}{2j + 1 + a_v} \right). \]

This changes the moment estimation, and instead of \( Ca \) we get \( C (a_v + a^{1+s}_v) \). If the \( a_s \) are all sufficiently small there is no further difficulty. If the \( a \)'s are random, this introduces dependencies between edges, and some higher moments must be controlled.

**Corollary** Theorem 2 holds on graphs where the LRRW is recurrent.

**Proof.** This is just an aggregation of the results of this section:

\[
E \left( \frac{W_x}{W_{e_1}} \right)^s = \sum_{\gamma} E \left[ \left( \frac{W_x}{W_{e_1}} \right)^s \mathbf{1}_{\{D_{\gamma} \}} \right]
\]

by (2)

\[
\leq \sum_{\gamma} E \left[ \prod_{e \in \gamma \setminus e_1} \left( \frac{R(e)}{Q(e)} \right)^{2s} \mathbf{1}_{\{D_{\gamma} \}} \right] \frac{1}{2} E \left[ \prod_{e \in \gamma \setminus e_1} Q(e)^{2s} \mathbf{1}_{\{D_{\gamma} \}} \right] \frac{1}{2}
\]

By Lemmas 7 and 8

\[
\leq \sum_{\gamma} \left[ C(2s)^{|\gamma|-1} \right] \frac{1}{2} \left[ (C(2s,K)a)^{|\gamma|-1} \right] \frac{1}{2} = \sum_{\gamma} (C_0 \sqrt{a})^{|\gamma|-1},
\]

for \( C_0 = \sqrt{C(2s)C(2s,K)} \). Now, the number of paths of length \( \ell \) is at most \( K^\ell \) and the shortest path to e has length \( \text{dist}(e,v_0) + 1 \). If we take \( a_0 \) such that \( KC_0 \sqrt{a_0} = \frac{1}{2} \) then for
$a < a_0$ longer paths give at most a factor of 2 and so

$$\mathbb{E} \left( \frac{W_e}{W_{e'}} \right)^s \leq 2K(KC_0\sqrt{a})^{\text{dist}(e,v_0)} \quad \square$$

3 Recurrence on Bounded Degree Graphs for Small Values of $a$

We are now ready to prove Theorem 1. As noted above, the main idea is to approximate the LRRW on $G$ by LRRW on finite balls in $G$. Let $R > 0$ and let $X^{(R)}$ be the LRRW in the finite volume ball $B_R(v_0)$. By Theorem 4, for each $R$ the full distribution of $X^{(R)}$ is given by a mixture of random conductance models with edge weights denoted by $(W_e^{(R)})_{e \in B_R(v_0)}$, and mixing measure denoted by $\mu^{(R)}$. Recall $\Omega = \mathbb{R}_+^E$ is the configuration space of edge weights on the entire graph $G$. For fixed $a$, the measures $\mu^{(R)}$ form a sequence of measures on $\Omega$, equipped with the product Borel $\sigma$-algebra.

**Proof of Theorem 1.** Theorem 2 (or 4 if you prefer) implies that the measures $\mu^{(R)}$ are tight, and so have subsequential limits. Let $\mu$ be one such subsequence limit. Clearly the law of the random walk in environment $W$ is continuous in the weights. However, the first $R$ steps of the LRRW on $B_R(v_0)$ have the same law as the first $R$ steps of the LRRW on $G$. Thus the LRRW on $G$ is the mixture of random walks on weighted graphs with the mixing measure being $\mu$.

Fix some $s < 1/4$ for the remainder of the proof. For any edge $e \in B_R(v_0)$ we have by Markov’s inequality that

$$\mu^{(R)}(W_e^{(R)} > Q) \leq \frac{2K(C\sqrt{a})^{\text{dist}(e,v_0)}}{Q^s},$$

where $C = C(s,K)$ comes from Theorem 2, which we already proved for recurrent graphs. Taking $Q = (2K)^{-\text{dist}(e,v_0)}$, and since the number of edges at distance $\ell$ from $v_0$ is at most $K^{\ell+1}$, we find that the probability of having an edge at distance $\ell$ with $W_e > (2K)^{-\ell}$ is at most

$$K^{\ell+1}(2K)^s2K(C\sqrt{a})^\ell = 2K^2 \left(2^sK^{1+s}C\sqrt{a} \right)^\ell.$$

Let $a_0$ be such that $2^sK^{1+s}\sqrt{a_0} = \frac{1}{2}$. Then for $a < a_0$ this last probability is at most $K^22^{1-\ell}$. Crucially, this bound holds uniformly for all $\mu^{(R)}$, and hence also for $\mu$. By Borel-Cantelli, it follows that $\mu$-a.s., for all but a finite number of edges we have $W_e \leq (2K)^{-\text{dist}(e,v_0)}$. On this event the random network $(G,W)$ has finite total edge weight $W = \sum_e W_e$, and therefore the random walk on the network is positive recurrent, with stationary measure $\pi(v) = W_v/2W$. \quad \square
3.1 Trapping and Return Times on Graphs

With the main recurrence results, Theorems 1 and 2 proved, we wish to make a remark about the notions of “localization” and “exponential decay”. We would like to point out that one should be careful when translating them from disordered quantum systems into the language of hyperbolic non-linear $\sigma$-models and the LRRW. To illustrate this point, let $G$ be a graph with degrees bounded by $K$, and consider the behavior of the tail of the return time to the initial vertex $v_0$. Let

$$\tau = \min \{ t > 0 : X_t = v_0 \}.$$ 

**Proposition 9** Let $K$ be fixed. Suppose that $G$ is a connected graph with degree bound $K$ and some edge not incident to $v_0$. Then there is a constant $C > 0$ depending on $K$ but not on $a$ so that

$$\mathbb{P}(\tau > M) \geq c(a, K) M^{-(K-1)a}.$$ 

Thus despite the exponential decay of the weights, the return time has at most $(K-1)a - \frac{1}{2}$ finite moments.

**Proof.** There is some edge $e$ at distance 1 from $v_0$. Consider the event $E_M$ that the LRRW moves towards $e$ and then crosses $e$ back and forth $2M$ times. On this event, $\tau > 2M$. The probability of this event is at least

$$\mathbb{P}(E_M) \geq \frac{1}{K} \left[ \prod_{0 \leq j < M} \frac{2j + a}{2j + 1 + Ka} \right] \left[ \prod_{0 \leq j < M} \frac{2j + 1 + a}{2j + 1 + Ka} \right]$$

$$= \frac{1}{K} \prod_{0 \leq j < M} \left( 1 - \frac{(K-1)a}{2j + 1} + O\left( \frac{1}{j^2} \right) \right)^2$$

$$\geq c(a, K) M^{-(K-1)a}.$$ 

One might claim that we only showed this for the first return time. But in fact, moving to the RWRE point of view shows that this is a phenomenon that can occur at any time. Indeed, if $\mathbb{P}(\tau > M) \geq \varepsilon$ then this means that with probability $\geq \frac{1}{2}\varepsilon$, the environment $W$ satisfies that $\mathbb{P}(\tau > M \mid W) \geq \frac{1}{2}\varepsilon$. But of course, once one conditions on the environment, one has stationarity, so the probability that the $k$th excursion length (call it $\tau_k$) is bigger than $M$ is also $\geq \frac{1}{2}\varepsilon$. Returning to unconditioned results give that

$$\mathbb{P}(\tau_k > M) \geq \frac{1}{4}\varepsilon^2 = c(a, K) M^{-2(K-1)a} \forall k$$

This is quite crude, of course, but shows that the phenomenon of polynomial decay of the return times is preserved for all times.
3.2 Reminiscences

Let us remarks on how we got the proof of these results. We started with a very simple heuristic picture, explained to us by Tom Spencer: when the initial weights are very small, the process gets stuck on the first edge for very long. A simple calculation shows that this does not hold forever but that at some point it breaks to a new edge. It then runs over these two edges, roughly preserving the weight ratio, until breaking into a third edge, roughly at uniform and so on.

Thus our initial approach to the problem was to try and show using only the LRRW picture that weights stabilize. This is really the point about the linearity — only linear reinforcement causes the ratios of weights of different edges to converge to some value. One direction that failed was to show that it is really independent of the rest of the graph. We tried to prove that “whatever the rest of the graph does, the ratio of the weights of two edges is unlikely to move far, when both have already been visited enough times”. So we emulated the rest of the graph by an adversary (which, essentially, decides through which edge to return to the common vertex), and tried to prove that any adversary cannot change the weight ratio. This, however, is not true. A simple calculation that shows that an adversary which simply always returns the walker through edge $e$ will beat any initial advantage the other edge, $f$ might have had. Trying to somehow move away from a general adversary, we had the idea that the RWRE picture allows to restrict the abilities of the adversary, which finally led to the proof above (which has no adversary, of course). We still think it would be interesting to construct a pure LRRW proof, as it might be relevant to phase transitions for other self-interacting random walks, which are currently wide open.

4 Transience on Non-amenable Graphs

In this section we prove Theorem 3. The proof is ideologically similar to that of Theorem 2, and in particular we prove the main lemmas under the assumption that the process is recurrent — there it seemed natural, here it might have seemed strange, but after reading §3 maybe not so much: in the end we will apply these lemmas for finite subgraphs of our graph. We will then use compactness, i.e. Theorem 4, to find a subsequential limit on $\mathcal{G}$ which also describes the law of LRRW on $G$.

Let us therefore begin with these lemmas, which lead to a kind of ”local stability” of the environment $W$ when the initial weights $a$ are uniformly large.

As in the case of small $a$, our stability argument has two parts. We use the dynamic description to argue that the walk typically uses all edges leaving a vertex roughly the same number of times (assuming it enters the vertex sufficiently many times). We then use the random conductance viewpoint to argue that the random weights are typically close to each other. Finally, we use a percolation argument based on the geometry of the graph to deduce a.s. transience.
4.1 Local Stability if LRRW is Recurrent

Let us assume throughout this subsection that the graph $G$ and the weights $a$ are given and that they satisfy that LRRW is recurrent.

Let $L$ be some parameter which we will fix later. The main difference between the proofs here and in §2 is that here we will examine the process until it leaves a given vertex $L$ times through each edge rather than once. Let therefore $\tau = \tau(L,v)$ denote the number of visits to $v$ until the LRRW has exited $v$ at least $L$ times along each edge $e \ni v$. Note that since we assume that the LRRW is recurrent, these stopping times are a.s. finite. Let $M(e) = M(L,v,e)$ denote the total number of outgoing crossings from $v$ along $e$ up to and including the $\tau^\text{th}$ exit. Given the environment $W$, call a vertex $v$ $\varepsilon$-faithful if $M(e)/\tau$ is close to its asymptotic value of $W_e/W_v$, i.e.

$$\frac{M(e)/\tau}{W_e/W_v} \in [1 - \varepsilon, 1 + \varepsilon] \quad \text{for all } e \ni v.$$

**Lemma 10** Let the degree bound $K$ be fixed. Then for any $\varepsilon, \delta > 0$ there exists $L = L(K,\varepsilon,\delta)$ so that

$$\mathbb{P}^W(v \text{ is not } \varepsilon\text{-faithful}) < \delta.$$

Moreover, these events are independent under $\mathbb{P}^W$.

A crucial aspect of this lemma is that $L$ does not depend on the environment $W$. A consequence is that for this $L$, the $\varepsilon$-faithful vertices stochastically dominate $1 - \delta$ site percolation on $G$ for any $\mathbb{P}^W$, and hence also for the mixture $\mathbb{P}$.

**Proof.** Given $W$, the exits from $v$ are a sequence of i.i.d. random variables taking $d_v \leq K$ different values with some probabilities $p(e) = W_e/W_v$. Let $A_n(e)$ be the location of the $n^\text{th}$ appearance of $e$ in this sequence, so that $A_n$ is the cumulative sum of Geom($p$) random variables. By the strong law of large numbers, there is some $L$ so that with arbitrarily high probability (say, $1 - \delta$), we have

$$\left| \frac{pA_n}{n} - 1 \right| < \varepsilon \quad \text{for all } n \geq L. \quad (4)$$

We now claim that $L$ can be chosen uniformly in $p$ (for $\varepsilon$ and $\delta$ fixed). This can be proved either by replacing the law of large numbers by a quantitative analog, say a second moment estimate, or by using continuity in $p$: Indeed, the only possible discontinuity is at zero, and $pA_n$ converges as $p \to 0$ to a sum of i.i.d. exponentials, and so with this interpretation the same claim holds also for $p = 0$. Since the probability of a large deviation at some large time for a sum of i.i.d. random variables is upper semi-continuous in the distribution of the steps, and since upper semi-continuous functions on $[0,1]$ are bounded from above, we get the necessary uniform bound.
We now move back from the language of $A_n$ to counting exits at specific time $t$. Denote $M_t(e)$ the number of exits from $v$ through $e$ in the first $t$ visits to $v$. If $t \in [A_n, A_{n+1})$ then $M_t = n$ and hence, for $t \geq A_t$, and with probability $> 1 - \delta$,

$$\frac{M_t(e)}{pt} \in \left( \frac{n}{pA_{n+1}}, \frac{n}{pA_n} \right) \subset [1 - \epsilon', 1 + \epsilon']$$

(where $\epsilon'$ is some function of $\epsilon$ with $\epsilon' \to 0$ when $\epsilon \to 0$).

Since $\tau \geq A_t(e)$ for all edges $e$ coming out of $v$, we get $M(e)/p\tau \in [1 - \epsilon', 1 + \epsilon']$ for all $e$, with probability $> 1 - K\delta$. Replacing $\epsilon'$ with $\epsilon$ and $K\delta$ with $\delta$ proves the lemma.

The second step is to use the dynamical description of the LRRW to show that if $a$ is large then the $M(L, v, e)$ are likely to be roughly equal. Let $S(L, v) = \max_{e \in v} M(L, v, e)$ be the number of exits from $v$ along the most commonly used edge. By the definition, $M(L, v, e) \geq L$ for all $e, v$. We will therefore call a vertex $\epsilon$-balanced if $S(L, v) \leq (1 + \epsilon)L$.

**Lemma 11** For any $K, \epsilon, \delta$ there is an $L_0 = L_0(K, \epsilon, \delta)$ such that for any $L > L_0$ there is some $a_0$ so that for any $a > a_0$ such that LRRW on $G$ with initial weight $a$ is recurrent we have

$$P(v \text{ is not } \epsilon\text{-balanced}) < \delta.$$  

Moreover, for such $a$, the $\epsilon$-balanced vertices stochastically dominate $1 - \delta$ site percolation.

**Proof.** We prove directly the stochastic domination. To this end, we prove that any $v$ is likely to be $\epsilon$-balanced even if we condition on arbitrary events occurring at other vertices. At the $i$th visit to $v$, the probability of exiting via any edge $e$ is at least $a/(d_v a + 2i - 1)$. Throughout the first $T$ visits to $v$, this is at least $1/(d_v - 2T/a)$. Since this bound is uniform in the trajectory of the walk, we find that the number of exits along an edge $e$ stochastically dominates a $\text{Bin}(T, 1/(d_v - 2T/a))$ random variable, even if we condition on any event outside of $v$.

We take $T = (1 + \epsilon)d_v L$. If $a$ is large enough ($CK^2L/\epsilon$ suffices) then the binomial has expectation at least $L - 1/2\epsilon L$. Since it has variance at most $T = O(L)$, if $L$ is sufficiently large then the binomial is very likely to be at least $L$. In summary, given $\delta$ and $\epsilon$ we can find some large $L$ so that for any large enough $a$, with probability at least $1 - \delta$ there are at least $L$ exits along any edge $e$ up to time $T$. This occurs for all edges $e \ni v$ with probability at least $1 - K\delta$.

Finally notice that if all edges are exited at least $L$ times, then this accounts for $d_v L$ exits, and only $T - d_v L$ exits remain unaccounted for. Even if they all happen at the same edge, that edge would still have only $L + \epsilon d_v L$ exits, hence $S \leq L + \epsilon d_v L$. Therefore $v$ is $\epsilon d_v$-balanced with probability at least $1 - K\delta$. Redefining $\epsilon$ and $\delta$ gives the lemma.

Call a vertex $\epsilon$-good (or just good) if it is both $\epsilon$-faithful and $\epsilon$-balanced. Otherwise, call the vertex bad. Note that if $v$ is good, then weights of edges leaving $v$ differ by a
factor of at most $\frac{1+\epsilon}{1-\epsilon} \leq (1-\epsilon)^{-2}$.

**Corollary 12** For any $K, \epsilon, \delta$, for any large enough $L$, for any large enough $a$, the set of $\epsilon$-good vertices stochastically dominates the intersection of two Bernoulli site percolation configurations with parameter $1-\delta$.

Unfortunately, these two percolation processes are not independent, so we cannot merge them into one $(1-\delta)^2$ percolation.

### 4.2 Application to Infinite Graphs

Let $G_n$ denote the ball of radius $n$ in $G = (V, E)$ with initial vertex $v_0$. Let $\mu_n$ denote the mixing measure guaranteed by the first half of Theorem 4. Further let $\mathcal{P}_n$ be the sequence of coupling measures guaranteed by Corollary 12.

According to the second half of Theorem 4 the measures $\mu_n$ are tight. Quite obviously the remaining marginals of $\mathcal{P}_n$ are also tight. Therefore $\mathcal{P}_n$ is tight. Thus we can always pass to a subsequential limit $\mathcal{P}$ so that the first marginal is a mixing measure for $\mu$ and the conclusion of Corollary 12 holds. We record this in a proposition:

**Proposition 13** For any $K, \epsilon, \delta$, for any large enough $L$, for any large enough $a$ the following holds. For any weak limit $\mu$ of finite volume mixing measures there is a coupling so that the set of $\epsilon$-good vertices (with respect to $\mu$) stochastically dominates the intersection of two Bernoulli site percolation configurations with parameter $1-\delta$.

We now use a Peierls’ argument to deduce transience for large enough $a$. We shall use two results concerning non-amenable graphs. The standard literature uses edge boundaries so let us give the necessary definitions: we define the edge boundary of a set by $\partial_E(A) = \{(x, y) \in E : x \in A, y \notin A\}$, $\text{Vol } A = \sum_{v \in A} d_v$ and $\iota = \inf_{A \subset G, \text{Vol } A < \infty} \frac{\partial_E(A)}{\text{Vol } A}$.

Clearly

$$\iota = \inf \frac{\partial A}{\text{Vol } A} \leq \inf \frac{\partial_E A}{K \text{Vol } A} = \frac{1}{K^2 \iota_E}.$$  

and similarly $\iota \leq K^2 \iota_E$.

The first result that we will use is Cheeger’s inequality:

**Theorem 14** If $G$ is non-amenable then the random walk on $G$ has return probabilities $p_n(0, 0) \leq Ce^{-\beta n}$, where $\beta > 0$ depends only on the Cheeger constant $\iota_E(G)$.

Cheeger proved this for manifolds. See e.g. [3] for a proof in the case of graphs. A second result we use is due to Virág [20, Proposition 3.3]. Recall that the anchored
expansion constant is defined as
\[
\alpha(G) = \lim_{n \to \infty} \inf_{|A| \geq n} \frac{|\partial_E A|}{\text{Vol}(A)}
\]
where the infimum ranges over connected sets containing a fixed vertex \(v\). It is easy to see that \(\alpha\) is independent of the choice of \(v\).

**Proposition 15** ([20]) Every infinite graph \(G\) with \(\alpha\)-anchored expansion contains for any \(\varepsilon > 0\) an infinite subgraph with edge Cheeger constant at least \(\alpha - \varepsilon\).

**Proof of Theorem 3.** Let \(\varepsilon\) be some small number to be determined later, and let \(G_\varepsilon\) denote the set of \(\varepsilon\)-good vertices (and also the induced subgraph). For any set \(F\) with boundary \(\partial F\), the \(\varepsilon\)-bad vertices in \(\partial F\) are a union of two Bernoulli percolations, each of which is exponentially unlikely to have size greater than \(\frac{1}{4}|\partial F|\), provided that \(\delta < \frac{1}{4}\).

Specifically,
\[
P\left(|G_\varepsilon \cap \partial F| \leq \frac{1}{2}|\partial F|\right) \leq e^{-c|\partial F|} \leq e^{-\iota |F|},
\]
where \(c = c(\delta)\) tends to \(\infty\) as \(\delta \to 0\), and \(\iota > 0\) is the Cheeger constant of the graph. The number of connected sets \(F\) of size \(n\) containing the fixed origin \(o\), is at most \(2^{Kn}\) (this is true in any graph with degree bound \(K\) — the maximum is easily seen to happen on a \(K\)-regular tree, on which the set can be identified by its boundary which is just \(Kn\) choices of whether to go up or down). Thus if \(\delta\) is small, so that \(c(\delta) > K(\log 2)/\iota\), then a.s. only finitely many such sets \(F\) have bad vertices for half their boundary.

Taking such a \(\delta\), we find that \(G_\varepsilon\) contains an infinite cluster \(\mathcal{C}\) which has “vertex anchored expansion” at least \(\iota(G)/2\). Moving to edge anchored expansion loses a \(K^2\) and we get \(\alpha \geq \iota/2K^2\). By proposition 15 we find that \(\mathcal{C}\) contains a subgraph \(H\) with \(\iota_E(H) > \iota(G)/3K^2\).

By Theorem 14, the simple random walk on \(H\) has exponentially decaying return probabilities. However, since all vertices of \(H\) are \(\varepsilon\)-good, edges incident on vertices of \(H\) have weights within \((1 - \varepsilon)^{-2}\) of equal. Thus the random walk on the weighted graph \((H, W)\) is close to the simple random walk on \(H\). Specifically, letting \(p^W\) denote the heat kernel for the \(W\)-weighted random walk restricted to \(H\), then we find
\[
p^W_n(0, 0) \leq (1 - \varepsilon)^{-2n} p_n(0, 0) \leq C(1 - \varepsilon)^{-2n} e^{-\beta n},
\]
where \(\beta\) is some constant depending only on \(G\). In particular for \(\varepsilon\) small enough, these return probabilities are summable and the walk is transient.

Finally by Rayleigh monotonicity (see e.g. [4]), \((G, W)\) is transient as well, completing the proof.

Since there are several parameters that need to be set in the proof, let us summarize
the final order in which they must be chosen. The graph $G$ determines $K$ and $\iota(G)$. Then $\delta$ is chosen to get large enough anchored expansion. This determines via Theorem 14 the value of $\beta$ for the subgraph $H$, which determines how small $\epsilon$ needs to be. Finally, given $\epsilon$ and $\delta$ we take $L$ large enough to satisfy Lemmas 10 and 11 and the minimal $\alpha$ is determined from Lemma 11.

5 Vertex Reinforced Jump Process

In this section we apply our basic methods to the VRJP models mentioned in the introduction. This demonstrates the flexibility of the approach and gives a second proof of recurrence of LRRW based on the embedding of LRRW in VRJP with initial rates $J$ i.i.d. with marginal distribution $\Gamma(a,1)$.

5.1 Times they are a changin’

As explained in the introduction, the VRJP has a dynamic and an RWRE descriptions, related by a time change. Let us give the details. The dynamic version we will denote by $Y_t$, and its local time at a vertex $x$ by $L_x(t)$, so that $t = \sum_x L_x(t)$. Recall that $Y_t$ moves from $x$ to $y$ with rate $J_{xy}(1 + L_y(t))$.

The RWRE picture is defined in terms of a positive function $W = (W_v)_{v \in V}$ on the vertices. Given $W$ we will denote by $Z_s = Z_s^W$ the random walk in continuous time that jumps from $x$ to $y$ with rate $\frac{1}{2}J_{xy}W_y/W_x$. We will denote the local time of $Z$ by $M_x(s)$, and again $s = \sum_x M_x(s)$. When some random choice of $W$ is clear from the context we will denote by $Z_s$ the mixed process.

The time change relating $s$ and $t$ is then given by the relation that $M_x = L_x^2 + 2L_x$, or equivalently that $L_x = \sqrt{1 + M_x} - 1$. Summing over all vertices gives a relation between $s$ and $t$. Since the local times are only increasing at the presently occupied vertex, this gives the equivalent relations $ds = 2(1 + L_{Y_t}(t))dt$ and $dt = ds/2\sqrt{1 + M_{Z_s}(s)}$.

**Theorem 16 ([19])** On any finite graph there exists a random environment $W$ so that $(Z_s)$ is the time change of $(Y_t)$ given above.

For the convenience of the reader, here is a little table comparing our notation with those of [19]:

| Here | $J$ | $W$ | $L$ |
|------|-----|-----|-----|
| [19] | $W$ | $e^{L}$ | $L - 1$ |

In fact, Sabot and Tarrès also give an explicit formula for the law of the environment $W$. However, as with the LRRW, we do not require the formula for this law, but only that it exists.
One more way to think of all this is that the process has two clocks measuring the occupation time at each vertex. From the VRJP viewpoint, the process jumps from \(i\) to \(j\) at rate \(J_{ij}(1 + L_j(t))dt\) (i.e. with respect to the \(L\)'s). From the RWRE viewpoint, the process jumps at rate \(\frac{1}{2}J_{ij}W_j/W_i ds\). Theorem 16 states that the above relation between \(ds\) and \(dt\), these two descriptions give the same law for the trajectories.

It is interesting to also describe the process in terms of both the local times and reinforcement. Using the time of the \(L\)'s, and given the environment, we find that the process jumps from \(i\) to \(j\) at rate \(J_{ij}(W_j/W_i)(1 + L_i)dt\). This gives a description of \(Z\) with no random environment, but using reinforcement. Nevertheless, it will be most convenient to use \(Z\) for the RWRE side of the proof and \(Y\) for the dynamical part.

5.2 Guessing the environment

As with the LRRW, the main idea is to extract from the processes some estimate for the environment, and show that it is reasonably close to the actual environment on the one hand, and behaves well on the other.

For neighboring vertices \(i, j\), let \(S_{ij}\) be the first time at which \(Z\) jumps from \(i\) to \(j\), and let \(\tau_{ij} = M_i(S_{ij})\) be the local time for \(Z\) at \(i\) up to that time. Given the environment \(W\), we have that \(\tau_{ij} \overset{D}{=} \text{Exp}(\frac{1}{2}J_{ij}W_j/W_i)\). Thus we can use \(Q_{ij} := \sqrt{\frac{\tau_{ji}}{\tau_{ij}}}\) as an estimator for \(R_{ij} = W_j/W_i\).

For a vertex \(v\), we consider a random simple path \(\gamma\) from \(v_0\) to \(v\), where each vertex \(x\) is preceded by the vertex from which \(x\) is entered for the first time. This \(\gamma\) is just the backward loop erasure of the process up to the hitting time of \(v\). As for the LRRW, we need two estimates. First an analogue of Lemma 7:

**Lemma 17** For any simple path \(\gamma\) in a finite graph \(G\), any environment \(W\) and any \(s < 1\) we have

\[
\mathbb{E}^W \prod_{e \in \gamma} \left( \frac{R_e}{Q_e} \right)^{2s} = \left( \frac{\pi s}{\sin \pi s} \right)^{|\gamma|}.
\]

Second, we need an analogue of Lemma 8. Recall from § 2 that \(\mathcal{D}_{\gamma}\) denotes the event that the backward loop erasure from \(v\) is a given path \(\gamma\). We use the same notation here.

**Lemma 18** There exists some \(C > 0\) such that for any \(0 < s < 1/4\) the following holds. For any finite graph \(G\), any conductances \(J\), and simple path \(\gamma\) starting at \(v_0\),

\[
\mathbb{E} \prod_{e \in \gamma} Q_e^{2s} 1\{\mathcal{D}_{\gamma}\} \leq C(s)^{|\gamma|} \prod_{e \in \gamma} J_e^{2s}.
\]
Proof of Lemma 17. Given the environment, the time $Z$ spends at $i$ before jumping to $j$ is $\text{Exp}\left(\frac{1}{2}J_{ij}W_i/W_j\right)$ which may be written as $(2W_j/J_{ij}W_i)X_{ij}$, where $X_{ij} \overset{D}{=} \text{Exp}(1)$. Crucially, given the environment the variables $X_{ij}$ are all independent. For an edge $e = (i, j)$ this gives $R_e / Q_e = \sqrt{X_{ij}/X_{ji}}$. Therefore

$$\mathbb{E}^W \prod_{e \in \gamma} \left(\frac{R_e}{Q_e}\right)^{2s} = \left(\Gamma(1 + s)\Gamma(1 - s)\right)^{|\gamma|} = \left(\frac{\pi s}{\sin \pi s}\right)^{|\gamma|},$$

by the reflection identity for the $\Gamma$ function. \hfill \square

Lemma 19. Suppose $0 < s < 1/4$, and let $J > 0$ be fixed. Let $U \overset{D}{=} \text{Exp}(J)$ and conditioned on $U$, let $V \overset{D}{=} \text{Exp}(J(1 + U))$. Then

$$\mathbb{E} \left(\frac{2V + V^2}{2U + U^2}\right)^s \leq \frac{C}{1 - 4s} J^{2s},$$

where $C$ is some universal constant.

Proof. We can reparametrize $U = \frac{X}{\pi}$ and $V = \frac{Y}{\cos \pi}$, where $X$ and $Y$ are independent $\text{Exp}(1)$ random variables. In term of $X, Y$ we have

$$\frac{2V + V^2}{2U + U^2} = \frac{2J^2Y(J + X) + J^2Y^2}{X(J + X)^2(2J + X)} < \frac{2J^2YX^3 + J^2Y^2}{X^4}.$$  

We now calculate, using $(a + b)^s \leq a^s + b^s$ for $0 < s < 1$,

$$\mathbb{E} \left(\frac{2J^2Y}{X^3} + \frac{J^2Y^2}{X^4}\right)^s \leq J^{2s} \left(\mathbb{E}(2YX^{-3})^s + \mathbb{E}(Y^2X^{-4})^s\right)$$

$$= J^{2s} \left(2^s \mathbb{E}X^{3s} \mathbb{E}X^{-3s} + \mathbb{E}Y^{2s} \mathbb{E}X^{-4s}\right) \quad \text{by independence}$$

$$\leq \frac{C}{1 - 4s} J^{2s} \quad \text{since } \mathbb{E}X^a \leq \frac{C}{1 + a}.$$  

(the inequality $\mathbb{E}X^a \leq C / (1 + a)$ holds for $|a| < 1$, the relevant range here). \hfill \square

Proof of Lemma 18. Consider an edge $e = (i, j) \in \gamma$, and let $T_{ij}$ be the first time $t$ at which $Y_t$ jumps from $i$ to $j$ (and similarly define $T_{ji}$). On the event $\mathcal{D}_\gamma$ the process does not visit $j$ before the first jump from $i$ to $j$. Thus $L_i(t) = 0$ for all $t < T_{ij}$. Hence the jump $i \rightarrow j$ occurs at rate $J_{ij}$ whenever $Y$ is at $i$, and so $U := L_i(T_{ij})$ has law $\text{Exp}(J_{ij})$. More precisely, the statement about the jump rate implies that we can couple the process $Y$ with an $\text{Exp}(J_{ij})$ random variable, so that on the event $\mathcal{D}_\gamma$ it equals $U$.

Let $V = L_j(T_{ji})$ be the time spent at $j$ before the first jump back to $i$. Since $L_i(t) \geq U$ from the time we first enter $j$, the rate of such jumps is always at least $J_{ij}(1 + U)$, we find
that $V$ is stochastically dominated by a $\text{Exp}(J_{ij}(1 + U))$ random variable.

All statements above concerning rates of jumps along the edge $e$ hold (on the event $\mathcal{D}_\gamma$), uniformly in anything that the process does anywhere else. Thus it is possible to construct such exponential random variables for every edge $e \in \gamma$, independent of all other edges, so that the $U$ equals the first and $V$ is dominated by the second. The claim then follows by Lemma 19, since $\tau_{ij} = 2U + U^2$ and $\tau_{ji} = 2V + V^2$ by their definitions and the time change formulae. \hfill \square

5.3 Exponential decay and Theorem 6

Let $G_R$ denote the ball of radius $R$ around $v_0$. Denote by $\mu^{(R)}$ the VRJP measure on $G_R$ and the corresponding expectation by $E^{(R)}$. In the proof, $E_J$ denote expectation with respect to $J$. Theorem 6 follows from the following:

**Theorem 20** There is a universal constant $c > 0$ such that the following holds. Let $G$ be a fixed graph with degree bound $K$. Let $J = (J_e)_{e \in E}$ be a family of independent initial rates with

$$E_J^{1/5} < cK^{-4}.$$  

Then (a.s. with respect to $J$) the measures $\mu^{(R)}$ are a tight family and converge to a limit $\mu$ on $\mathbb{R}_+^E$ so that the VRJP is a time change of the process $Z_s$ in the environment given by $\mu$. The limit process is positive recurrent, and the stationary measure decays exponentially.

The moment condition on $J$ is trivially satisfied in the case that all $J_e$’s are bounded by some sufficiently small $J_0$. The particular condition comes from specializing to $s = 1/5$, and can be easily changed by taking other values of $s$ or by using Hölder’s inequality in place of Cauchy-Schwartz in the proof below. The dependence on $K$ may be similarly improved.

**Proof.** Combining Lemmas 17 and 18 with Cauchy-Schwartz, for any $v$, any radius $R > \text{dist}(v_0, v)$ and any path $\gamma : v_0 \to v$ in $G_R$ we have (recall $W_{v_0} = 1$):

$$E^{(R)} W_{v_0}^{s} 1\{\mathcal{D}_\gamma\} \leq \left( E^{(R)} \prod_{e \in \gamma} \left( \frac{R_e}{Q_e} \right)^{2s} \right)^{1/2} \left( E^{(R)} \prod_{e \in \gamma} Q_e^{2s} 1\{\mathcal{D}_\gamma\} \right)^{1/2} \leq C_1^{\gamma} \prod_{e \in \gamma} J_e^s$$

where $C_1$ depends only on $s$. Let the $c$ from the statement of the theorem be $1/C_1(1/5)$. Then with $s = 1/5$ we get

$$E_J E^{(R)} W_{v_0}^{1/5} 1\{\mathcal{D}_\gamma\} \leq K^{-4|\gamma|}.$$  

Since the number of paths of length $n$ is at most $K^n$ and no path to $v$ is shorter than
\( \text{dist}(v_0, v) \) this implies
\[
\mathbb{E}_J \mathbb{E}^{(R)} W_0^{1/5} < 2K^{-3 \text{dist}(v_0, v)}.
\]
Since this bound is uniform in \( R \), the Borel-Cantelli Lemma, applied with respect to \( J \), implies that the measures \( \mu^{(R)} \) are tight and that they have subsequential limits \( J \). Let \( \mu \) be any such subsequential limit (we later deduce that \( \mu \) is unique). It is easy to see that the weak convergence of \( \mu^{(R)} \) to \( \mu \) implies a convergence of \( Z^{(R)}, Y^{(R)} \) and the time change between them to \( Z, Y \) and the time change between them corresponding to the infinite measure (all convergences are along the chosen subsequence). However, from the reinforcement viewpoint, \( Y_t \) has the same law on all \( G_R \) until the first time it reaches the boundary of the ball. Thus \( \mu \) yields the VRJP on the infinite graph \( G \).

As noted above, the discretized \( Z_s \) is just a random walk on \( G \) with conductances \( C_{ij} = J_{ij} W_i W_j \). By Markov’s inequality, \( \mathbb{P}(W_v > K^{-3 \text{dist}(v_0, v)}) \leq 2K^{-2 \text{dist}(v_0, v)} \). By Borel-Cantelli, it follows that a.s. \( W_v \leq K^{-3 \text{dist}(v_0, v)} \) for all but finitely many \( v \), and therefore \( C_e \leq J_{ij} K^{-6 \text{dist}(v_0, v)} \). However, the number of edges at distance \( n \) is at most \( K^n \) and we assumed \( J \) has a finite 1/5 moment so yet another application of the Borel-Cantelli Lemma ensures that \( \sum_e J_e K^{-6 \text{dist}(v_0, v)} < \infty \). Thus \( \sum_e C_e \) is almost surely finite, and the total weight outside \( G_R \) decays exponentially. This implies the positive recurrence.

Finally, since the process is a.s. recurrent, \( Z \) visits each vertex infinitely often, and the environment can be deduced from the observed jump frequencies along edges, the subsequential limit \( \mu \) is in fact unique. With tightness, this implies convergence of the \( \mu_R \). \( \square \)

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