Analytically continued physical states in the path-integral: a sign-problem-free Quantum Monte Carlo simulation of Bell states dynamics.

Evgeny A. Polyakov
Russian Quantum Center, Novaya 100, 143025 Skolkovo, Moscow Region, Russia and Faculty of Physics, Saint Petersburg State University, 7/9 Universitetskaya Naberezhnaya, Saint Petersburg 199034, Russia

Alexey N. Rubtsov
Russian Quantum Center, Novaya 100, 143025 Skolkovo, Moscow Region, Russia and Department of Physics, Lomonosov Moscow State University, Leninskie gory 1, 119991 Moscow, Russia

The derivation of path integrals is reconsidered. It is shown that the expression for the discretized action is not unique, and the path integration domain can be deformed so that at least Gaussian path integrals become probabilistic. This leads to a practical algorithm of sign-problem-free Monte Carlo sampling from the Gaussian path integrals. Moreover, the dynamical influence of Gaussian quantum system (the bath) on any other quantum system can be exactly represented as interaction with classical non-Markovian noise. We discuss the relation of these findings to the Bell’s theorem and the Feynman’s conjecture on the exponential complexity of the classical simulation of quantum systems. In Feynman’s path integral we have quasiprobability distributions for trajectories, and in analytically continued path integrals we have probability distributions for quasitrajectories.

I. INTRODUCTION

One of the most characteristic features of quantum theory is that it cannot be interpreted as a local causal (possibly stochastic) real-time evolution in a certain space of “hidden parameters” [1]. This is the content of Bell’s theorem: we cannot have simultaneously locality, causality, physical probabilities, and physical states. Something should be sacrificed. Feynman’s path-integral interpretation of quantum mechanics succeeds in representing evolution as statistics of diffusive trajectories (field configurations) $x$. To satisfy the Bell’s theorem, this interpretation sacrifices the physicality of probabilities: they are no longer real positive but arbitrary complex weights $\exp(-iS(x)/\hbar)$, where $S(x)$ is the action for configuration $x$. This is not so important for analytic calculations, but becomes crucial for the numerics.

The physicality of trajectory probabilities is crucially important in quantum Monte Carlo (QMC) simulations. The QMC is the only choice in cases when it is difficult to identify small or large parameter, and the dimension of phase space is high. A number of efficient QMC algorithms are known for the imaginary time domain, so that contemporary QMC can efficiently solve equilibrium quantum problems. However in real time the trajectory weights are complex and oscillating. Currently there is no understanding of how to efficiently arrange the sampling procedure. This is the so-called sign problem.

For more than half a century, it has been made a lot of attempts to solve this problem. The most notable and oldest are: to enforce the controlled cancellation of oscillating trajectory weights (filtering [2–4] and multilevel blocking [5–6] techniques); analytic continuation with respect to the time variable $t$ [7–9]. Nevertheless, none of these methods enables us to simulate large correlated systems for long enough times.

Another group of approaches which is currently under active investigations is based on the idea to perform analytic continuation of probability amplitudes with respect to the (field) configurations $x$. This approach takes its origin from the works of Parisi [10] and of Klauder and Petersen [11] where first time the idea was proposed of how to sample the complex probabilities with the aid of complex Langevin equation. While having limited success for certain model systems, the method of complex Langevin equations suffer from instabilities when applied to more realistic systems and for longer simulation times [12–16]. Later a modification of the idea was proposed: to introduce complexified configurations $x$ in path integrals and shift the integration contour (domain) in order to make the oscillations of trajectory weights milder [17]. This is achieved by locating the stationary points of complexified (analytically continued) action $S(x)$ and find the paths (hypersurfaces) of steepest descent. These paths are called Lefschetz thimbles (LT). They have the property that the phase (real part of $S(x)$) is constant on them. The idea is to perform Monte Carlo sampling on Lefschetz thimbles [18]. The limitations of this approach follow from the fact that Lefschetz thimble is a complicated curved hypersurface for which there is no explicit relations. Nevertheless, there are promising results and developments [19–24]. In a recently published work an idea was proposed to conduct Monte Carlo simulations in the vicinity of Lefschetz thimble so that the oscillations of trajectory weights are greatly supressed [25].

A common feature of complex Langevin and Lefschetz thimble methods is that they start from a given discretized path integral on a given integration domain and then just try to complexity it (to continuously deform

\footnote{e.a.polyakov@gmail.com}
the integration domain into the complex plane).

In this work we reconsider the standard derivation of path integrals and construct a discretized action where the path integration domain can be deformed so that at least Gaussian integrals become probabilistic. It turns out that there is rich freedom when constructing discretized actions. Here we explore only one component of this freedom, the integration contour (domain) independence, which turns out to be richer than in Lefschetz thimble methods. We discuss the relation of the analytical continuation approach with the Feynman’s arguments that it is impossible to simulate quantum dynamics on classical local probabilistic computer [1]. Our approach is based on works [28–32]. As a particular application of this general methodology, we consider a quantum system interacting with Gaussian heat bath and show that probabilistic path integral for the heat bath leads to a stochastic evolution of reduced system state, analogous to what was derived from different considerations in works [25, 32].

We believe that the rich freedom implicitly present in the discretized path integral actions will help us to devise novel simulation algorithms for real-time quantum dynamics, and to improve efficiency of the existing complex Langevin and Lefschetz thimble methods.

II. GENERALIZED PATH INTEGRALS

In this section we demonstrate that the expression for discretized path integral action is not unique, and we can exploit this fact in order to make the Gaussian path integral real positive.

The propagation of bosonic quantum many-particle system is described by evolution operator

$$\hat{U} = e^{-i(\hat{H}t - t_i)\hat{\delta}(\hat{a}^\dagger, \hat{a})}. \quad (1)$$

The matrix element of \( \hat{U} \) between given initial \( \Psi_i \) and final \( \Psi_f \) wavefunctional can be represented formally as path integral

$$\langle f | \hat{U} | i \rangle = \int D[\alpha, \alpha^*] \times \exp\left\{ \int_{t_i}^{t_f} dt \int d\alpha \left[ \frac{1}{2} (\alpha \partial_\alpha \alpha^* - \alpha^* \partial_\alpha \alpha) - iH (\alpha, \alpha^*) \right] \right\} \times \exp\left\{ \frac{1}{2} \int d\alpha \left[ |\alpha(x, t_i)|^2 + |\alpha(x, t_f)|^2 \right] \right\} \times \Psi_f^* (\alpha (\cdot, t_f)) \Psi_i (\alpha (\cdot, t_i)). \quad (2)$$

Here, the second line can be interpreted as defining the canonical structure of the evolution; the third line as convergence factor of the integral; the last line specifies boundary conditions for the matrix element.

We recall that the standard derivation of path integral begins with introduction of the holomorphic representation of quantum states. Here, each quantum state with occupied modes \( i_1 \ldots i_N \) is mapped onto analytic function of variables \( \alpha_{i_1} \ldots \alpha_{i_N} \) according to the rule

$$\Psi (\hat{\alpha}) = \hat{a}_{i_1}^\dagger \ldots \hat{a}_{i_N}^\dagger \ket{0} \rightarrow \Psi (\alpha) = \alpha_{i_1} \ldots \alpha_{i_N}, \quad (3)$$

and superpositions of states are represented by superpositions of analytic functions. It can be shown that in holomorphic picture the inner product between any pair of quantum states is given by

$$\langle \Psi_1^* (\alpha) | \Psi_2 (\alpha) \rangle = \int \prod_{k=1}^{M} d\alpha_k d\alpha^*_k \times e^{-\sum_k \alpha_k \alpha^*_k [\Psi_1 (\alpha)]^* \Psi_2 (\alpha)}. \quad (4)$$

The action of creation and annihilation operators is represented by

$$\hat{a}_k^\dagger \Psi (\alpha) \rightarrow \alpha_k \Psi (\alpha), \quad \hat{a}_k \Psi (\alpha) \rightarrow \partial_{\alpha_k} \Psi (\alpha). \quad (5)$$

Therefore, arbitrary normally ordered operator \( O (\hat{a}^\dagger, \hat{a}) \) is represented as a differential operator \( O (\alpha, \partial_\alpha) \). However, in order to arrive at the path integral representation [29], we need to represent the action of \( O (\hat{a}^\dagger, \hat{a}) \) as integral kernel. We achieve this by employing integral automorphisms of holomorphic representation:

$$\Psi (\alpha) = \int \prod_{k=1}^{M} \frac{d\alpha_k d\alpha^*_k}{2\pi i} A (\alpha, \alpha') \Psi (\alpha'). \quad (6)$$

The integral kernels \( A (\alpha, \alpha') \) contain considerable degree of freedom. Restricting ourselves to a from which is Gaussian in \( \alpha', \alpha'^* \), we have the following family of \( A (\alpha, \alpha') \):

$$A (\alpha, \alpha') = \det A \exp \left\{ -\sum_{k,l} (\alpha^*_k - \alpha_k) A_{kl} (\alpha^*_l - b_l) \right\}, \quad (7)$$

where in general \( A_{kl} \) and \( b_l \) may depend on \( \alpha \) and \( \alpha^* \). In the standard derivation of path integral one usually sets \( A_{kl} = \delta_{kl} \) and \( b_l = 0 \) [29]. The action of \( A \) is based on Wick theorem: \( \alpha'_{k} \) are replaced by their “mean values” \( \alpha_k \); if \( \alpha^*_l \) were present then they would be replaced by \( b_l \); if the pairs \( \alpha^*_k \alpha'_l \) were present then they would be contracted by \( (A^{-1})_{kl} \). For the Wick theorem to hold the integration contour (hypersurface) is irrelevant. The only requirement is that all the Gaussian integrals coming from all \( A \)s should be convergent. The standard choice is that the real \( \alpha_{xz} \) and imaginary \( \alpha_{yk} \) parts of \( \alpha_k \) run along the real axis from \(-\infty\) to \(+\infty\). However we will assume that the contour is arbitrary and \( \alpha_{xz} \) and \( \alpha_{yk} \) are independent complex numbers. Therefore, instead of \( \alpha_k \) we will employ \( \alpha_k^x = \alpha_{xz} - i\alpha_{yk} \). This way the normally ordered operator \( O (\hat{a}^\dagger, \hat{a}) \) is represented as a kernel of integral operator

$$O (\hat{a}^\dagger, \hat{a}) \rightarrow O (\alpha, \alpha') = O (\alpha, \partial_\alpha) A (\alpha, \alpha'). \quad (8)$$
Now we introduce the grid of time moments \( t_j, j = 0 \ldots P \), with time step \( \Delta t \) and boundary conditions \( t_0 = t_i \) and \( t_P = t_f \). The holomorphic representation of initial and final quantum states are

\[
|\Psi_0\rangle = \Psi_0 (\alpha_0), \quad |\Psi_P\rangle = \Psi_P (\alpha_P). \tag{9}
\]

The propagation during time interval \([t_j, t_{j+1}]\),

\[
|\Psi_{j+1}\rangle = \left(1 - i\Delta t\hat{H} (\hat{a}^\dagger, \hat{a}) + O (\Delta t^2)\right) |\Psi_j\rangle, \tag{10}
\]

in holomorphic picture assumes the form

\[
\Psi_{j+1} (\alpha (j + 1)) = \int \prod_k \frac{d\alpha_k (j) d\alpha_{yk} (j)}{2\pi i} \times \exp \left[- \sum_k \alpha_k^2 (j) (\alpha_k (j + 1) - \alpha_k (j)) \right.
\]

\[
- i\Delta tH (\alpha (j + 1), \alpha^* (j)) + O (\Delta t^2) \left. \right] \Psi_j (\alpha (j)). \tag{11}
\]

Here we have employed the standard form of automorphism [20]. Using this recurrent relation between the neighboring \( \Psi_j (\alpha (j)) \) and the inner product formula [14], we construct the discretized path integral. For a general Gaussian problem we have

\[
\langle f | \hat{U} | i \rangle = \int \prod_{k,j} \frac{d\alpha_{yk} (j) d\alpha_{yk} (j)}{2\pi i} \times \exp \left[- \sum_{kjlp} \alpha_k (j) (G^{-1})_{kj,lp} \alpha_l^* (p) \right]. \tag{12}
\]

We can deform the integration hypersurface for \( \alpha_{yk} \) and \( \alpha_{yk} \) such that the integrand becomes real positive. Indeed, suppose we have some factorization of \( G \),

\[
G = UV^*. \tag{13}
\]

Then we can choose such integration hypersurface that the following variables become complex conjugate:

\[
\gamma_k = \sum_{ij} \alpha_i (j) (V^{-1})^*_{ij,k}, \quad \gamma_k^* = \sum_{ij} (U^{-1})_{k,ij} \alpha_j^* (j). \tag{14}
\]

This way we obtain the following probabilistic path integral:

\[
\langle f | \hat{U} | i \rangle = \int \prod_{k,j} \frac{d\alpha_{yk} (j) d\alpha_{yk} (j)}{2\pi i} \times \exp \left[- \sum_k \gamma_k (\alpha_x, \alpha_y) \gamma_k^* (\alpha_x, \alpha_y) \right]. \tag{15}
\]

The main result of this section is that such properties of holomorphic representation as contour-independence and the existence of a rich set of automorphisms allows one to solve the sign problem at least for Gaussian systems. The case of interacting systems (e.g., the presence of quartic terms in the action) deserves further study.

If we change the variables from \( \alpha, \alpha^2 \) to \( \gamma, \gamma^* \), we consider them as new independent classical random quantities with the following statistics:

\[
\overline{\gamma_k \gamma_l} = \delta_{kl}. \tag{16}
\]

From Eq. (14) we have:

\[
\alpha = V^1 \gamma, \quad \alpha^2 = U \gamma^*. \tag{17}
\]

Therefore, the covariance of \( \alpha, \alpha^2 \) is

\[
\alpha^2 \otimes \alpha^2 = U \gamma^* \otimes \gamma^* V^* = UV^* = G \tag{18}
\]
as it should be for the original path integral [12].

### III. MONTE CARLO SIMULATION OF BELL STATES

In the preceding section we have shown that it is possible to represent arbitrary quantum Gaussian evolution as a classical stochastic process. This may seem somewhat surprising, especially in light of classical work of Feynman [11] where he argues that it is impossible to efficiently simulate quantum evolution on a classical local probabilistic computer. Since Feynman in his work considers the two-photon correlation experiment, here we provide computations for this case.

Let us consider evolution under the parametric down-conversion (PDC) process, which has the following effective Hamiltonian [32]

\[
\hat{H} = ik \left( \hat{a}_1^\dagger \hat{b}_1^\dagger - \hat{a}_1 \hat{b}_1 \right) + i\kappa \left( \hat{a}_2^\dagger \hat{b}_2^\dagger - \hat{a}_2 \hat{b}_2 \right), \tag{19}
\]

where 1 and 2 denote the two orthogonal polarizations; the photonic modes \( \hat{a}_k \) propagate “to the left” wing of the experiment, and the photonic modes \( \hat{b}_k \) propagate “to the right” wing of the experiment. At each wing of the experiment there are polarizers \( A \) and \( B \) at the angles \( \theta \) and \( \phi \) correspondingly. We measure the transmitted \( \hat{c}_+ \) and reflected \( \hat{c}_- \) modes at polarizer \( A \)

\[
\hat{c}_+ = \hat{a}_1 \cos \theta + \hat{a}_2 \sin \theta, \tag{20}
\]

\[
\hat{c}_- = -\hat{a}_1 \sin \theta + \hat{a}_2 \cos \theta, \tag{21}
\]

and the transmitted \( \hat{d}_+ \) and reflected \( \hat{d}_- \) modes at polarizer \( B \),

\[
\hat{d}_+ = \hat{b}_1 \cos \phi + \hat{b}_2 \sin \phi, \tag{22}
\]

\[
\hat{d}_- = -\hat{b}_1 \sin \phi + \hat{b}_2 \cos \phi. \tag{23}
\]
We use the Clauser-Horne-Bell inequality with intensity moments [33]:

\[
S_{CH} = \frac{I_{AB}^A (\theta, \phi) - I_{AB}^A (\theta, \phi') + I_{AB}^B (\theta', \phi) + I_{AB}^B (\theta', \phi')}{I_{AB}^A (\theta') + I_{AB}^B (\phi)} \leq 1. \tag{24}
\]

Here

\[
I_{AB}^A (\theta') = \left\langle \hat{c}_+^\dagger \hat{c}_+ \left( \hat{d}_+^\dagger \hat{d}_+ + \hat{d}_-^\dagger \hat{d}_- \right) \right\rangle, \tag{25}
\]

is the correlation of the intensity of photons which are transmitted through the polarizer \(A\) at the angle \(\theta'\) with the full intensity of photons which are incident on the polarizer \(B\):

\[
I_{AB}^B (\phi) = \left\langle \hat{d}_+^\dagger \hat{d}_+ \left( \hat{c}_+^\dagger \hat{c}_+ + \hat{c}_-^\dagger \hat{c}_- \right) \right\rangle \tag{26}
\]

is the symmetrically mirrored quantity;

\[
I_{AB}^A (\theta, \phi) = \left\langle \hat{c}_+^\dagger \hat{c}_+ \hat{d}_+^\dagger \hat{d}_+ \right\rangle \tag{27}
\]

is the correlation between the intensities of the photons which are transmitted through the polarizers \(A\) and \(B\) simultaneously, at the angles \(\theta\) and \(\phi\) correspondingly. The maximal violation of (24) should occur at \(\theta = 0^\circ\), \(\theta' = 45^\circ\), \(\phi = 22.5^\circ\), \(\phi' = 67.5^\circ\) when \(S_{CH} \approx 1.2\). The Keldysh Green functions \(G\) for Hamiltonian Eq. (19) can be computed analytically by switching to the normal modes of this Hamiltonian. Factorizing \(G\) according to [15] and introducing the classical stochastic processes \((c_+, c_-, d_+, d_-)\) according to [17], we compute the intensity moments Eqs. (25)-(27) as classical expectations \(E_{\beta, \beta^*} \left[ c_+ (t) c_+^\dagger (t) \left( d_+ (t) d_+^\dagger (t) + d_- (t) d_-^\dagger (t) \right) \right]\),

\[
E_{\beta, \beta^*} \left[ c_+ (t) d_+^\dagger (t) \left( c_+^\dagger (t) c_+ (t) + c_- (t) c_-^\dagger (t) \right) \right],
\]

and

\[
E_{\beta, \beta^*} \left[ c_+ (t) c_-^\dagger (t) d_+ (t) d_-^\dagger (t) \right] \text{ correspondingly.}
\]

In order to give an idea of the calculated intensity correlations, on Fig. 1 we present the results for \(I_{AB}^A (\theta')\). The dynamical evolution of the left hand side of the Clauser-Horne-Bell inequality Eq. (24) is presented on Fig. 2. The large variance of computed \(S_{CH}\) near \(\kappa t = 0\) is because all the intensity correlations become exponentially small when \(\kappa t \to 0\) (see e.g. Fig. 1). In fact, this large variance reflects the fact that such an experimental setup becomes inefficient for the Bell-type tests when \(\kappa t \ll 1\), since the detection of photons becomes exponentially-seldom event. The computation is carried at the angles of maximal violation of \(S_{CH}\) (dash-dotted line on Fig. 2). Nevertheless, the value of maximal violation is not reached and \(S_{CH}\) is decreased with time because during the evolution under the PDC Hamiltonian [19] the increasing multiphoton contribution to the Bell state makes it non-ideal.

![Figure 1. Correlation \(I_{AB}^A (\theta')\) of the intensity of transmitted photons through polarizer \(A\) with the full intensity of photons through polarizer \(B\). The central black line is the calculation result. The lower and the upper color lines denote the standard deviation of the result.](image1)

**IV. OPEN SYSTEMS AND PROBABILISTIC PATH INTEGRAL FOR THE BATH**

The Monte Carlo computation of polynomial averages for a Gaussian system with known Green function represents purely methodological interest since in this case we can apply Wick theorem. However the problem of the dynamics of an open quantum system interacting with a Gaussian bath is already nontrivial. Such a problem emerges in different fields of research, ranging from open systems and decoherence theory [34] to computation methods for correlated systems (Anderson impurity model and the dynamical mean field theory [35]). In this section we discuss how the probabilistic path integral leads to computational recipes for this kind of prob-
lems. This problem was already treated from a different point of view in works \cite{28-32}, however our consideration provides us with a different methodological perspective.

Suppose we have a system

$$\hat{H} = \hat{H}_q + h \left( \hat{b}^\dagger \hat{a} + \hat{b} \hat{a}^\dagger \right) + \hat{H}_b,$$

(28)

where \( \hat{H}_b \) is a Gaussian bath with known contour Green function

$$G_b(\tau - \tau') = \langle 0_b | T \hat{a}(\tau) \hat{a}^\dagger(\tau') | 0_b \rangle$$

(29)

for its coupled degree of freedom \( \hat{a} \). We assume that the initial state of the system and of the bath is factorized as \(|\Psi_{\text{ini}}\rangle |0_b\rangle\), where \(|0_b\rangle\) is the vacuum of the bath. We also suppose that we are dealing with a certain dynamical problem up to a real time \( T \) for which appropriate contour (e.g. Keldysh) was chosen with forward \( C_+ \) and backward \( C_- \) branches. The system \( \hat{H}_q \) may be arbitrary yet numerically tractable (that is whose Hilbert space dimension is not very large).

Let us describe the evolution of the reduced state of the system. Since in a reduced description we average over the bath, we represent it as a path integral. This way each occurrence of the operators \( \hat{a} \) and \( \hat{a}^\dagger \) is replaced by \( c \)-number stochastic fields \( \alpha^\dagger(\tau) \) and \( \alpha(\tau) \). Now, we deform the integration domain to make the bath path integral probabilistic. The price paid is that instead of complex conjugated fields \( \alpha^\dagger(\tau) \) and \( \alpha(\tau) \) we now deal with non-conjugated fields \( \alpha^2(\tau) \) and \( \alpha(\tau) \), and the corresponding evolution of the system is not unitary. The obtained Hamiltonian in the interaction picture with respect to the bath has the following stochastic form

$$\hat{H}_{\text{stoch}}\left( \alpha_\pm(\tau), \alpha^2_\pm(\tau) \right) = \hat{H}_q + h \left( \alpha_\pm(\tau) \hat{b} + \alpha^2_\pm(\tau) \hat{b}^\dagger \right),$$

(30)

which governs the evolution of system’s states \(|\Psi_+(\tau)\rangle\) on the forward branch and \(|\Psi_-(\tau)\rangle\) on the backward branch according to the following stochastic differential equations:

$$d\langle |\Psi_+(\tau)\rangle \rangle = -i dt \left[ \hat{H}_q + h \left( \alpha_+(\tau) \hat{b} + \alpha^2_+(\tau) \hat{b}^\dagger \right) \right]$$

$$\times |\Psi_+(\tau)\rangle\rangle,$$

(31)

$$d \langle |\Psi_-(\tau)\rangle \rangle = \langle |\Psi_-(\tau)\rangle \rangle \times i dt \left[ \hat{H}_q + h \left( \alpha_-(\tau) \hat{b} + \alpha^2_-(\tau) \hat{b}^\dagger \right) \right].$$

(32)

The average value of system’s observable is computed as

$$\langle \hat{O}(\tau) \rangle = \frac{\langle |\Psi_-(\tau)\rangle \langle \hat{O} |\Psi_+(\tau)\rangle \rangle}{\langle |\Psi_+(\tau)\rangle \rangle},$$

(33)

If the system is initially in the state \(|\Psi_{\text{ini}}\rangle\), the equations \ref{eq:31} and \ref{eq:32} have the initial conditions \( |\Psi_{\pm}(0)\rangle = |\Psi_{\text{ini}}\rangle\).

Now let us discuss how to generate the field configurations \( \alpha^2(\tau) \) and \( \alpha(\tau) \). The most straightforward way would be to apply the relations Eq. \ref{eq:17} and to sample \( \gamma_\ell \) from the standard complex normal distribution. The singular value decomposition of Green function

$$G = U_s \Sigma V_s^*$$

(34)

provides us with \( U \equiv U_s \sqrt{\Sigma} \) and \( V^* \equiv \sqrt{\Sigma} V_s^* \). As preliminary calculations show, this approach is exact and converging when the simulation sample size is increased. Nevertheless, the statistical variance of Monte Carlo results grows exponentially with time. The reason of this behaviour becomes clear if we look at the time dependence of system’s state norm (overlap). Since the norm is conserved and the simulation is exact, we have

$$\langle |\Psi_-(\tau)\rangle |\Psi_+(\tau)\rangle \rangle = 1.$$  

(35)

Nevertheless, for a given realization of field trajectories \( \alpha^2(\tau) \) and \( \alpha(\tau) \) we have

$$\langle |\Psi_-(\tau)\rangle |\Psi_+(\tau)\rangle \rangle = \langle |\Psi_-(0)\rangle |\Psi_+(0)\rangle \rangle \times \exp \left\{ i \int_0^\tau d\tau' \left[ \alpha_-(\tau') - \alpha_+(\tau') \right] \right\}$$

$$+ i \int_0^\tau d\tau' \left[ \alpha^2_-(\tau') - \alpha^2_+(\tau') \right],$$

(36)

where \( \bar{b}(\tau) \) and \( \bar{b}^\dagger(\tau) \) are the “instantaneous” mean values of system annihilation and creation operators:

$$\bar{b}(\tau) = \frac{\langle \Psi_-(\tau) | \hat{b} | \Psi_+(\tau) \rangle \rangle}{\langle \Psi_-(\tau) | \Psi_+(\tau) \rangle \rangle},$$

(37)

$$\bar{b}^\dagger(\tau) = \frac{\langle \Psi_-(\tau) | \hat{b}^\dagger | \Psi_+(\tau) \rangle \rangle}{\langle \Psi_-(\tau) | \Psi_+(\tau) \rangle \rangle},$$

(38)

From Eq. \ref{eq:36} we see that the normalization of system’s state fluctuates exponentially with time and this is the reason for the growth of the variance of Monte Carlo results. At least for harmonic system \( \hat{H}_q = \varepsilon \hat{b}^\dagger \hat{b} \) it is possible to overcome this problem in the following way. Firstly, we employ the trick analogous to \cite{29}: we change the variables in the bath path integral according to

$$\alpha(\tau) \to \alpha(\tau) + f(\tau),$$

(39)

$$\alpha^2(\tau) \to \alpha^2(\tau) + f^2(\tau),$$

(40)

where \( f(\tau) \) and \( f^2(\tau) \) are retarded functionals of the fields \( \alpha^2(\tau) \) and \( \alpha(\tau) \):

$$f(\tau) = -i \int \frac{d\tau'}{c} \bar{b}(\tau') G(\tau, \tau'),$$

(41)
Here according to Eqs. (37) and (38), $\bar{b}^\dagger (\tau')$ and $\bar{b} (\tau')$ take the same values on $C_ \pm$ and $C_ \mp$. The functional Jacobian matrices of the transformation (39)-(40) are lower triangular with unit diagonal, thus their determinant is unity, and the integration measure is not changed. Then it can be shown that the overlap factor Eq. (48) is compensated. The modified stochastic fields Eqs. (37)-(38) are sampled as

$$\alpha (\tau) = \sum_k V^\dagger_{lk} \gamma_k + f (\tau),$$

$$\alpha^2 (\tau) = \sum_k U_{lk} \gamma_k^2 + f^2 (\tau),$$

where as previously $\gamma_k$ are sampled from the standard normal distribution. Now in principle we could solve the stochastic equations (31)-(32) with this modified noise and evaluate the observables as

$$\langle \hat{O} (\tau) \rangle = \left[ \frac{\langle \Psi_- (\tau) | \hat{O} | \Psi_+ (\tau) \rangle}{\langle \Psi_- (\tau) | \Psi_+ (\tau) \rangle} \right],$$

where the denominator accounts for the fact that we have compensated for the norm. Let us assume that the initial system state $\Psi_{\text{ini}}$ is a coherent one. Under Gaussian evolution the coherent state conserves its form. Therefore, at any time moment $\tau$ we have

$$| \Omega_\pm (\tau), b_\pm (\tau) \rangle = \exp \left[ \Omega_\pm (\tau) + b_\pm (\tau) \hat{b}^\dagger \right] | 0 \rangle,$$

where $\Omega_\pm (\tau)$ is a normalization and $b_\pm (\tau)$ is a displacement. In terms of $\Omega_\pm (\tau)$ and $b_\pm (\tau)$, the stochastic equations Eqs. (31)-(32) take the following form:

$$db_\pm (\tau) = -i \varepsilon \dot{b}_\pm (\tau) d\tau - ih\alpha_\pm (\tau) d\tau,$$

$$d\Omega_\pm (\tau) = -ih b_\pm (\tau) \alpha_\pm (\tau) d\tau,$$

and also the adjoint equations for $\Omega_- (\tau), b_- (\tau)$. The stochastic term in equations for $\Omega_\pm (\tau)$ [Eq. (48)] is multiplicative and this fact could lead to exponential growth of variance. However since we divide by overlap $\langle \Omega_-, b_- (\tau) | \Omega_+, b_+ (\tau) \rangle$, the dependence on $\Omega_\pm (\tau)$ is compensated, and we can drop the equation (48). The “instantaneous” mean values of system annihilation and creation operators Eqs. (37)-(38) assume the following simple form:

$$\bar{b} (\tau) = b_+ (\tau), \quad \bar{b}^\dagger (\tau) = b_-^* (\tau).$$

The resulting simulation procedure is defined by equation of motion Eq. (17), adjoint equation for $b_- (\tau)$, the noise is constructed according to Eq. (49), (11)-(14). Since any state can be represented as a mixture of coherent states, initial conditions of a more general form can be treated by the same procedure. In Fig. 3 we present the results of calculations for a harmonic system with $\varepsilon = 1$ which is coupled to one harmonic mode $\hat{H}_b = \omega \hat{a}^\dagger \hat{a}$ with $\omega = 1$ and $h = 2$. It is seen that the stochastic computation correctly reproduces the beat due to coupling to the harmonic bath mode.

This recipe works only for a harmonic system which is in a Gaussian state. If the system were not harmonic, it would be difficult to introduce coherent states, and it would not be possible to compensate the multiplicative noise terms like those in Eq. (48). If the state was not Gaussian (coherent), then we had to use general formulas Eq. (37)-(38) which are numerically unstable. The case of non-Gaussian evolution deserves further study.

V. DISCUSSION

In his classical paper [1] Feynman argues that the efficient classical simulation of quantum dynamics is possible only on local causal probabilistic computer. The possibility of such simulation would imply the possibility of “hidden parameters” interpretation of quantum mechanics, thus it is impossible. Nevertheless, as we have discussed in this work, we can escape this limitation by noting that Bell’s theorem prohibits only “hidden parameters” interpretation in terms of physical probabilities and physical states. If we make the trajectories of states unphysical by analytic continuation of field variables then at least for Gaussian systems we obtain probabilistic path integrals. The latter can be sampled by Monte Carlo methods. In different terms and context, similar conclusions have been drawn in [27].

The situation when anharmonic interaction terms are present (e.g. quartic in creation/annihilation operators) deserves further study. In principle is is known that the
deformation of contour which eliminates the oscillation of complex phase is possible: this is the main result of the Lefschetz thimbles approach \cite{18}. The open question is to find such a deformed contour that the Monte Carlo sampling along it is not prohibitively computationally complex. It is likely that there is no universal answer to this question for a general nonlinear quantum dynamical problem. Nevertheless, we believe that the freedom in the derivation of discretized path integral action may be useful to increase the efficiency of simulation algorithms for specific classes of dynamical problems.

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