USING QUATERNION-VALUED LINEAR ALGEBRA

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Abstract. Linear algebra is usually defined over a field such as the reals or complex numbers. It is possible to extend this to skew fields such as the quaternions. However, to the authors’ knowledge there is no commonly accepted notation of linear algebra over skew fields. To this end, we discuss ways of notation that account for the non-commutativity of the quaternion multiplication.

1. Introduction

The use of quaternion linear algebra is emerging among researchers. However, defining the properties of quaternionic matrices is still subject to research. Nevertheless, advances have been made especially in analyzing eigenvalues ([9], [3], [10], [5], [1]) of quaternion matrices.

Albeit the analytical analyses, a practical framework is needed to use quaternion matrices. A basic problem arises due to the non-commutativity of the quaternion algebra. That is, for two matrices $A \in H^{M \times K}$ and $B \in H^{K \times N}$ the matrix products $\sum_{k=1}^{K} [A]_{m,k} [B]_{k,n}$ and $\sum_{k=1}^{K} [B]_{k,n} [A]_{m,k}$ are in general not the same.

To this end, we propose a new notation for quaternion matrix multiplication. In this notation we take care of the multiplication order.

Matrices are denoted using capital letters in bold face. Vectors are denoted by lower case characters in bold face. The $T$ operator indicates the transpose of a matrix or a vector. Additionally, $*$ and $H$ stand for the conjugation and the Hermitian transpose (conjugate transpose) respectively. The $\text{diag}(\cdot)$ operator transforms a vector into a square matrix having the elements of the vectors on the main diagonal.

2. Review on Quaternions

The current section will introduce the quaternions and the notation used in the subsequent sections.

2.1. Quaternions and Complex Numbers. Quaternions have been first discovered by W. R. Hamilton [4]. The quaternions are one of several possible extensions of complex numbers.

The set of quaternions $\mathbb{H}$ can be constructed from the set of complex numbers $\mathbb{C}$. For that, let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ be two complex numbers. Additionally, we introduce another imaginary unit $j$ with $j^2 = -1$. Then, a quaternion $q \in \mathbb{H}$ is a pair of complex numbers $(z_1, z_2) \equiv z_1 + z_2j$. Hence, we have

\begin{equation}
q = z_1 + z_2j = a_1 + b_1i + a_1j + b_2j
\end{equation}

By introducing a third imaginary unit $k := ij$, with $k^2 = -1$, we obtain the common form of a quaternion.

\[ q = a_1 + b_1i + a_1j + b_2k \]

From now on, we will denote the set of quaternions as $\mathbb{H}$. The associated algebra is expressed in terms of pairs of complex numbers. To this end, let $x_1$, $x_2$, $z_1$, and $z_2$ be two pairs of complex numbers that form two quaternions $q_1 = x_1 + x_2j$ and $q_2 = z_1 + z_2j$. In this case, addition, multiplication and conjugation are defined as follows.

\begin{align*}
(x_1, x_2) + (z_1, z_2) &= (x_1 + z_1, x_2 + z_2) \\
(x_1, x_2) \cdot (z_1, z_2) &= (x_1z_1 - x_2z_2^*, x_1z_2^* + x_2z_1) \\
(x_1, x_2)^* &= (x_1^*, -x_2)
\end{align*}

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2.2. Symplectic Decomposition. Let \( q = a_0 + a_1i + a_2j + a_3k \in \mathbb{H} \) be a quaternion. The value \( s(q) = a_0 \) is called the **scalar part** (or real part) and the value \( v(q) = a_1i + a_2j + a_3k \) is called the **vector part**.

A **pure quaternion** is a quaternion having a vanishing scalar part. Similar to the complex numbers, the modulus of a quaternion \( q \) can be expressed as \( |q| := \sqrt{q^*q} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2} \).

If \( q' \) is an arbitrary quaternion with non-vanishing vector part, then

\[
\mu(q') = \frac{v(q')}{|q'|}
\]

is a pure unit quaternion (PUQ). Throughout the subsequent sections we will use \( \mathbb{H}_{pu} \) to indicate the set of all pure unit quaternions.

\[
\mathbb{H}_{pu} := \left\{ a_1i + a_2j + a_3k : \sum_{n=1}^{3} a_n^2 = 1, a_n \in \mathbb{R} \right\}
\]

It can easily be proven that for any PUQ \( \mu \in \mathbb{H}_{pu} \) the relation \( \mu^2 = -1 \) holds. Based on this observation, it is possible to build subsets of quaternions that each are isomorphic to complex numbers. To this end, let \( \mu \in \mathbb{H}_{pu} \) be an arbitrary but fixed pure unit quaternion. Consequently, the set

\[
\mathbb{C}_\mu := \{ a_0 + a_1\mu : a_n \in \mathbb{R} \}
\]

is isomorphic to the set of complex numbers \( \mathbb{C} \). One special consequence is the following fact: If \( q \) and \( p \in \mathbb{H} \) there exists a PUQ \( \mu \in \mathbb{H}_{pu} \) such that \( q \in \mathbb{C}_\mu \) and \( p \in \mathbb{C}_\mu \), the product of both becomes commutative (i.e., \( qp = pq \)).

Before continuing, let us first define the notion of orthogonality of PUQs. Let \( \mu = a_1i + a_2j + a_3k \in \mathbb{H}_{pu} \) and \( \mu_{\perp} = b_1i + b_2j + b_3k \in \mathbb{H}_{pu} \) be two pure unit quaternions. \( \mu \) and \( \mu_{\perp} \) are said to be orthogonal iff \( \sum_{n=1}^{3} a_nb_n = 0 \).

For notational convenience, in all following analyses we will assume that the Greek letter \( \mu \) always refers to a PUQ. Additionally, \( \mu_{\perp} \) will always refer to some PUQ which is orthogonal to \( \mu \).

Finally, the **symplectic decomposition** as presented in [2] can be obtained in the following way: Given a quaternion \( q \in \mathbb{H} \) choose two PUQs \( \mu \) and \( \mu_{\perp} \). Then, \( q \) can be decomposed into the following form:

\[
q = q_0 + q_1\mu + (q_2 + q_3\mu)\mu_{\perp}, \quad q_n \in \mathbb{R}
\]

\[
q = q'_0 + q'_1\mu_{\perp}, \quad q'_n \in \mathbb{C}_\mu
\]

It turns out that [3] is as generalized version of [1]. In order to remain most general all subsequent equations will be given in terms of pure unit quaternions rather than in terms of \( i, j, \) and \( k \).

2.3. Euler’s Formula. In the four-dimensional quaternionic space each pure unit quaternion can be seen as an axis. In conjunction with the real axis infinitely many planes exist that contain complex-isomorphic numbers. A rotation within one of such planes is similar to a rotation in the ordinary complex plane. Hence, Euler’s formula obtains the following form in the quaternion domain.

Let \( q \in \mathbb{H} \) be some non-zero quaternion. Then, up to a sign ambiguity there exists a unique axis \( \mu \in \mathbb{H}_{pu} \) and an angle \( \alpha \in (-\pi, \pi] \) such that

\[
q = |q|\exp(\mu\alpha) = |q|(\cos(\alpha) + \mu\sin(\alpha)).
\]

In [3] \( \exp(\cdot) \) denotes the exponential function. It may be noted that [3] also implies the fact that for any quaternion \( q \) there exists a \( \mu \) such that \( q \in \mathbb{C}_\mu \).

In general, for two quaternions \( q \) and \( p \) the value \( \exp(q + p) \) does not equal \( \exp(q)\exp(p) \). However, if \( q \) and \( p \) commute, then it is true that \( \exp(q + p) = \exp(q)\exp(p) \).

2.4. Similar Quaternions. Due to the lack of commutativity in the quaternion algebra the term \( p = s^{-1}qs \) for two quaternions \( q, s \in \mathbb{H} \) is in general not the same as \( q \). Nevertheless, \( p \) is said to be **similar** to \( q \). This is an equivalence relation, where the equivalence class \( S(q) \) is defined as follows:

\[
S(q) := \{ p \in \mathbb{H} : p = s^{-1}qs, s \in \mathbb{H}, |s| = 1 \}
\]
3. Matrix Products

3.1. Left and Right Matrix Multiplication. The quaternion multiplication is not commutative. That is, in general $ab$ is not the same as $ba$ for two quaternions $a \in \mathbb{H}$ and $b \in \mathbb{H}$. A problem arises when computing the product of two quaternionic matrices $A \in \mathbb{H}^{M \times K}$ and $B \in \mathbb{H}^{K \times N}$.

$[AB]_{m,n} := \sum_{k=1}^{K} [A]_{m,k} [B]_{k,n}$

The entries of $A$ are multiplied from the left. Likewise, the entries of $B$ are multiplied from the right. If the order needs to be swapped, the following expression gives the correct result.

$$\left( (B^T A^T)^T \right)_{m,n} := \sum_{k=1}^{K} [B]_{k,n} [A]_{m,k} \quad (4)$$

However, the left-hand side of (4) is somewhat less intuitive. Hence, in [8] we introduced a notation for left multiplication and right multiplication.

$$[A \cdot_L B]_{m,n} := \sum_{k=1}^{K} [A]_{m,k} [B]_{k,n}$$

$$[A \cdot_R B]_{m,n} := \sum_{k=1}^{K} [B]_{k,n} [A]_{m,k}$$

Consider the case of conformant matrices, i.e. where $M = N$. It follows that four different matrix products, $A \cdot_L B$ and $B \cdot_R A$, are possible. This highlights the fact that we deal with two different kinds of ordering: The first kind of ordering denotes inner products of either the rows of $A$ and the columns of $B$ or inner products of the rows of $B$ and the columns of $A$. Commonly, this is represented by the the ordering in which the matrices $A$ and $B$ appear in an equation. The second kind of ordering refers to the ordering of the scalars within each inner product. This ordering is specified by the proposed operators $\cdot_L$ and $\cdot_R$.

Next, we observe that the usage of the left and right multiplication operators allows for a convenient description of how the transpose, conjugation and Hermitian transpose (conjugate transposition) act on a product of two quaternion matrices.

$$\left( A \cdot_L B \right)^T = B^T \cdot_R A^T \quad (5)$$

$$\left( A \cdot_L B \right)^* = A^* \cdot_R B^* \quad (6)$$

Note that the second relation follows from the fact that for two quaternion scalars $q$ and $p$ we have $(pq)^* = q^* p^*$. Combining (5) and (6) yields the following.

$$\left( A \cdot_L B \right)^H = B^H \cdot_L A^H$$

$$\left( A \cdot_R B \right)^H = B^H \cdot_R A^H$$

Hence, the Hermitian conjugate behaves the same as in the complex case since the type of multiplication (left or right) is not altered.

3.2. Matrix Product of Three Matrices. Additionally, let us look at the Product of three matrices. There are six different ways of ordering the product of the matrix elements which are all covered by the
operators introduced above.

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} a_{m,k} b_{k,\ell} c_{\ell,n} = [A \cdot \_L B \cdot \_L C]_{m,n}
\]

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} c_{\ell,n} b_{k,\ell} a_{m,k} = [A \cdot \_R B \cdot \_R C]_{m,n}
\]

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} a_{m,k} c_{\ell,n} b_{k,\ell} = [A \cdot \_L (B \cdot \_R C)]_{m,n}
\]

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} c_{\ell,n} a_{m,k} b_{k,\ell} = [(A \cdot \_L B) \cdot \_R C]_{m,n}
\]

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} a_{m,k} c_{\ell,n} b_{k,\ell} = [(A \cdot \_R B) \cdot \_L C]_{m,n}
\]

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} b_{k,\ell} c_{\ell,n} a_{m,k} = [(A \cdot \_R (B \cdot \_L C)]_{m,n}
\]

\[
\sum_{k=1}^{K} \sum_{\ell=1}^{L} b_{k,\ell} a_{m,k} c_{\ell,n} = (A \cdot \_R B) \cdot \_L (B \cdot \_R C)
\]

Note that the first two identities do not contain inner brackets due to the following associativity property.

(7) \((A \cdot \_L B) \cdot \_C = A \cdot \_L (B \cdot \_L C)\)

(8) \((A \cdot \_R B) \cdot \_C = A \cdot \_R (B \cdot \_R C)\)

Similarly, iff \([A]_{m,n} \in \mathbb{C}_\mu\) and \([B]_{u,v} \in \mathbb{C}_\mu\) are in the same complex-isomorphic set \(\mathbb{C}_\mu\), then associativity does hold for the remaining equations as well.

\((A \cdot \_L B) \cdot \_C = A \cdot \_L (B \cdot \_L C)\)

\((A \cdot \_R B) \cdot \_C = A \cdot \_R (B \cdot \_R C)\)

4. THE FUNDAMENTAL SUBSPACES

In the quaternion domain we may define eight fundamental subspaces using the proposed left and right matrix multiplication. This is due to the fact that for the row spaces, column spaces and null spaces we have to define the order of multiplication. Hence, these are the fundamental subspaces of a matrix \(A \in \mathbb{H}^{M \times N}\).

Left row space (\(\mathcal{LR}\)) and right row space (\(\mathcal{RR}\)).

\(\mathcal{LR}(A) := \{ y \in \mathbb{H}^{M \times 1} : y = A \cdot \_L x, \forall x \in \mathbb{H}^{N \times 1} \}\)

\(\mathcal{RR}(A) := \{ y \in \mathbb{H}^{M \times 1} : y = A \cdot \_R x, \forall x \in \mathbb{H}^{N \times 1} \}\)

Left column space (\(\mathcal{LC}\)) and right column space (\(\mathcal{RC}\)).

\(\mathcal{LC}(A) := \{ y \in \mathbb{H}^{N \times 1} : y^T = x^T \cdot \_L A, \forall x \in \mathbb{H}^{M \times 1} \}\)

\(\mathcal{RC}(A) := \{ y \in \mathbb{H}^{N \times 1} : y^T = x^T \cdot \_R A, \forall x \in \mathbb{H}^{M \times 1} \}\)

Left row null space (\(\mathcal{LRN}\)) and right row null space (\(\mathcal{RRN}\)).

\(\mathcal{LRN}(A) := \{ x \in \mathbb{H}^{M \times 1} : A \cdot \_L x = 0 \}\)

\(\mathcal{RRN}(A) := \{ x \in \mathbb{H}^{M \times 1} : A \cdot \_R x = 0 \}\)

Left column null space (\(\mathcal{LCN}\)) and right column null space (\(\mathcal{RCN}\)).

\(\mathcal{LCN}(A) := \{ x \in \mathbb{H}^{N \times 1} : x^T \cdot \_L A = 0^T \}\)

\(\mathcal{RCN}(A) := \{ x \in \mathbb{H}^{N \times 1} : x^T \cdot \_R A = 0^T \}\)
Relations between subspaces. Similar to the complex case, the transpose operation relates some of the subspaces to each other.

\[
\begin{align*}
\mathcal{LR}(A) &= \mathcal{RC}(A^T) \\
\mathcal{RR}(A) &= \mathcal{LC}(A^T) \\
\mathcal{LRN}(A) &= \mathcal{RCN}(A^T) \\
\mathcal{RRN}(A) &= \mathcal{LCN}(A^T)
\end{align*}
\]

5. The Matrix Inverse

We now turn to investigate matrix inverses based on the proposed left and right multiplication.

Let \( X \) be some matrix \( X \in \mathbb{H}^{M \times N} \) and let \( A \in \mathbb{H}^{M \times M} \) be a square matrix. The left inverse \( A^\triangleleft \) satisfies the following condition.

\[
A^\triangleleft \cdot_L (A \cdot_L X) = X \tag{9}
\]

Similarly, the right inverse \( A^\triangleright \) satisfies this condition:

\[
A^\triangleright \cdot_R (A \cdot_R X) = X \tag{10}
\]

By exploiting the associativity properties \( [11] \) and \( [8] \) one can observe the following identities.

\[
A^\triangleleft \cdot_L (A \cdot_L X) = (A^\triangleleft \cdot_L A) \cdot_L X = X
\]

\[
A^\triangleright \cdot_R (A \cdot_R X) = (A^\triangleright \cdot_R A) \cdot_R X = X
\]

Hence, we have:

\[
A^\triangleleft \cdot_L A = I_M \tag{11}
\]

\[
A^\triangleright \cdot_R A = I_M \tag{12}
\]

In order to define the left inverse \( A^\triangleleft \) and right inverse \( A^\triangleright \) we use the symplectic decomposition of \( A \).

\[
A := A_0 + A_1 \mu_L, \quad A_i \in \mathbb{C}^{M \times M}
\]

Additionally, let us define the left adjoint matrix\(^1\)

\[
\chi_\mu \{ A \} := \begin{bmatrix} A_0 & A_1 \\ -A_1^* & A_0^* \end{bmatrix}
\]

as well as the right adjoint matrix

\[
\chi'_\mu \{ A \} := \begin{bmatrix} A_0 & -A_1^* \\ A_1 & A_0^* \end{bmatrix}
\]

These matrices render the direct complex-valued representation of a quaternion matrix. The left adjoint matrix is connected to the left matrix multiplication whereas the right adjoint matrix inherently represents the right matrix multiplication. Therefore, the left and right inverses can be computed with complex arithmetics by noting the following two identities (see also \([10]\)).

\[
\chi_\mu \{ A^\triangleleft \} = \chi_\mu^{-1} \{ A \}
\]

\[
\chi'_\mu \{ A^\triangleright \} = \chi'^{-1}_\mu \{ A \}
\]

Note that if \( A = A_0 \) and therefore \( A \in \mathbb{C}_\mu^{M \times M} \) both adjoint matrices are equal. That is, for fields which are isomorphic to the complex numbers both, \( A^\triangleleft \) and \( A^\triangleright \), reduce to the complex inverse \( A^{-1} \).

Without giving the proof we state that the right and left inverse defined above remain the same when the order of the matrix products is changed.

\[
(X \cdot_L A) \cdot_L A^\triangleleft = X
\]

\[
(X \cdot_R A) \cdot_R A^\triangleright = X
\]

\(^1\)In \([2]\) and \([3]\) this matrix is simply called adjoint matrix since only left matrix multiplication has been considered.
Hence, in addition to \( (11) \) and \( (12) \) we have the following identities:

\[
A^l \cdot A^\circ = I_M \\
A^r \cdot A^\triangledown = I_M
\]

We complete the investigation of the quaternion matrix inverses by pointing how they are connected.

\[
(A^\circ)^T = (A^T)^\triangledown \iff (A^\triangledown)^T = (A^T)^\circ
\]

This becomes clear by applying \( (10) \).

\[
I_M = (A^\circ)^l \cdot A^T \\
= (A^\circ)^T \cdot_r A^T
\]

Hence, \( A^T \) must be the right inverse of \( (A^\circ)^T \) and vice versa. Moreover, if all entries of \( A \) are located in the same set \( \mathbb{C}^{M \times M}_r \), \( (13) \) reduces to the well known relation \( (A^{-1})^T = (A^T)^{-1} \).

6. **The Kronecker Product And The Khatri-Rao Product**

6.1. **The Left and Right Kronecker Product.** In addition to the matrix multiplication, it is also desirable to examine Kronecker products of two quaternion matrices \( A \in \mathbb{H}^{M \times N} \) and \( B \in \mathbb{H}^{N \times K} \). In this case, one must also distinguish between the left Kronecker product

\[
A \otimes_l B := \begin{bmatrix}
[A]_{1,1} \cdot B & \cdots & [A]_{1,N} \cdot B \\
\vdots & \ddots & \vdots \\
[A]_{M,1} \cdot B & \cdots & [A]_{M,N} \cdot B
\end{bmatrix}
\]

and the right Kronecker product

\[
A \otimes_r B := \begin{bmatrix}
B \cdot [A]_{1,1} & \cdots & B \cdot [A]_{1,N} \\
\vdots & \ddots & \vdots \\
B \cdot [A]_{M,1} & \cdots & B \cdot [A]_{M,N}
\end{bmatrix}
\]

In contrast to the matrix product, transposing the Kronecker product does not change its type.

\[
(A \otimes_l B)^T = A^T \otimes_l B^T \\
(A \otimes_r B)^T = A^T \otimes_r B^T
\]

6.2. **Kronecker Product and Vectorization.** It is possible to reformulate the vectorization of a product of three matrices \( A \in \mathbb{H}^{M \times K} \), \( B \in \mathbb{H}^{K \times L} \), and \( C \in \mathbb{H}^{L \times N} \) in the quaternion domain.

\[
\text{vec} \left( A \cdot_l [B \cdot_r C] \right) = (C^T \otimes_r A) \cdot_r \text{vec} (B) \\
\text{vec} \left( A \cdot_r [B \cdot_l C] \right) = (C^T \otimes_l A) \cdot_l \text{vec} (B) \\
\text{vec} \left( [A \cdot_l B] \cdot_r C \right) = (C^T \otimes_l A) \cdot_l \text{vec} (B) \\
\text{vec} \left( [A \cdot_r B] \cdot_l C \right) = (C^T \otimes_r A) \cdot_r \text{vec} (B)
\]

However, there is no such expression for \( \text{vec} (A \cdot_l B \cdot_l C) \) and \( \text{vec} (A \cdot_r B \cdot_r C) \) using the operators proposed in this work. The reason is that this would involve multiplying the components of \( B \) between \( A \) and \( C \).

6.3. **The Khatri-Rao Product.** The last pair of operators we introduce is the left Khatri-Rao product

\[
C \diamond_l D := [c_1 \otimes_l d_1 \cdots c_N \otimes_l d_N]
\]

as well as the right Kathri-Rao product

\[
C \diamond_r D := [c_1 \otimes_r d_1 \cdots c_N \otimes_r d_N]
\]

of two matrices \( C = [c_1 \cdots c_N] \) and \( D = [d_1 \cdots d_N] \).
6.4. The Khatri-Rao Product and Vectorization. Let $A \in \mathbb{H}^{M \times K}$ and $C \in \mathbb{H}^{K \times N}$ be two matrices. Additionally, let $B = \text{diag}(b)$ be a diagonal matrix having the entries of a vector $b \in \mathbb{H}^{K \times 1}$ on its main diagonal. In this case, the following identities hold.

$$\text{vec}(A \cdot L [B \cdot C]) = (C^T \circ_r A) \cdot_r b$$
$$\text{vec}(A \cdot_r [B \cdot C]) = (C^T \circ_l A) \cdot_l b$$
$$\text{vec}([A \cdot L B] \cdot_r C) = (C^T \circ_l A) \cdot_r b$$
$$\text{vec}([A \cdot_r B] \cdot L C) = (C^T \circ_r A) \cdot_l b$$

However, there is no such expression for $\text{vec}(A \cdot L B \cdot L C)$ and $\text{vec}(A \cdot_r B \cdot r C)$.

7. Examples

In this section we will give some examples showing the benefit of using the proposed ways of notation.

7.1. Systems of Complex Widely Linear Equations. Let us consider a widely linear system of equations in the complex domain with $A, B \in \mathbb{C}^{M \times M}$, $X \in \mathbb{C}^{M \times P}$, and $C \in \mathbb{C}^{M \times P}$,

$$AX + BX^* = C$$

(14)

The complex conjugate prevents factoring out $X$. A possible solution would be to look at the real part and imaginary part of the above equation and to compute the result in the real domain.

Nevertheless, we may also consider (14) as a quaternion equation with $A, B \in \mathbb{C}_\mu^{M \times M}$, $X \in \mathbb{C}_\mu^{M \times P}$, and $C \in \mathbb{C}_\mu^{M \times P}$, where $\mu \in \mathbb{H}_\text{pu}$ is an arbitrary pure unit quaternion (PUQ). Then, let $\mu_\perp \in \mathbb{H}_\text{pu}$ be another PUQ that is orthogonal to $\mu$. It can readily be verified that $X^* = -\mu_\perp X \mu_\perp$. Therefore we may rewrite (14) as follows.

$$A \cdot L X - B \cdot L (\mu_\perp X \mu_\perp) = X$$

(15)

Using the left matrix product (15) becomes:

$$[A \quad -B \mu_\perp] \cdot_l \begin{bmatrix} X \\ X \mu_\perp \end{bmatrix} = C$$

Next, by using the right matrix product we may factor out $X$.

$$[A \quad -B \mu_\perp] \cdot_r \left( \begin{bmatrix} I_M \\ \mu_\perp I_M \end{bmatrix} \cdot_r X \right) = C$$

Moreover, let us define the matrices $F_1$ and $G$.

$$F_1 := [A \quad -B \mu_\perp] \quad G := \begin{bmatrix} I_M \\ \mu_\perp I_M \end{bmatrix}$$

Using these matrices we arrive at a compact expression.

$$F_1 \cdot_l (G \cdot_r X) = C$$

(16)

The same derivation can be done for (14) having taken the conjugate on both sides of the equation.

$$B^* X + A^* X^* = C^*$$

The result is similar to (16)

$$F_2 \cdot_l (G \cdot_r X) = C^*,$$

(17)

with

$$F_2 := \begin{bmatrix} B^* & -A^* \mu_\perp \end{bmatrix}.$$
By noting that $G$ has orthogonal columns

$$G^H \cdot_L G = G^H \cdot_R G = 2I_M,$$

the following solution to (14) is obtained

$$X = 0.5 \, G^H \cdot_R (F^s \cdot_L C_s)$$

7.2. The Eigendecomposition. As mentioned in Section 1 there has already been some research concerning the eigenvalues of a quaternion matrix $A \in \mathbb{H}^{M \times M}$. We will now review the definition of quaternionic eigenvectors and eigenvalues. Based on this, we examine how the proposed notation renders useful when defining Eigenvalue decompositions.

The Left Eigendecomposition. If for a scalar $\lambda_L \in \mathbb{H}$ and a vector $q_L \in \mathbb{H}^{M \times 1}$ the following condition holds, $\lambda_L \in \mathbb{H}$ is called a left eigenvalue and $q_L$ is said to be a left eigenvector of $A$.

$$AQ_L = \lambda_L q_L$$

Hence, we call

$$A = (Q_L \cdot_R A_L) \cdot_L Q_L^\dagger$$

the left eigendecomposition of $A$, where the matrix $Q_L := [q_{L,1}, \ldots, q_{L,M}]$ stores the left eigenvectors corresponding to the eigenvalues stored in the diagonal matrix $A_L := \text{diag}(\lambda_{L,1}, \ldots, \lambda_{L,M})$.

The Right Eigendecomposition. If for a scalar $\lambda_R \in \mathbb{H}$ and a vector $q_R \in \mathbb{H}^{M \times 1}$ the following condition holds, $\lambda_R \in \mathbb{H}$ is called a right eigenvalue and $q_R$ is said to be a right eigenvector of $A$.

$$AQ_R = q_R \lambda_R$$

However, in general there might exist infinitely many right eigenvalues. This can be seen by multiplying (18) from the right with a non-zero quaternion $s \in \mathbb{H}$.

$$AQ_R s = q_R \lambda_R s$$

Hence, if $\lambda_R$ is a right eigenvalue of $A$, then all similar quaternions $\lambda'_R \in S(\lambda_R)$ are right eigenvalues as well (see [9, p. 36]).

To define a right eigendecomposition, let $A_R := \text{diag}(\lambda_{R,1}, \ldots, \lambda_{R,M})$ be a diagonal matrix of mutually non-similar right eigenvalues. Moreover, let $Q_R := [q_{R,1}, \ldots, q_{R,M}]$ be the the matrix of corresponding right eigenvectors. The resulting eigendecomposition obtains the following form.

$$A = Q_R \cdot_L A_R \cdot_R Q_R^\dagger$$

Note that this decomposition is not unique. However, according to [9, theorem 5.4] any matrix $A \in \mathbb{H}^{M \times M}$ has exactly $M$ complex right eigenvalues. Additionally, these eigenvalues all have non-negative imaginary parts. These eigenvalues are called standard eigenvalues and may be used to resolve the uniqueness problem.

7.3. The Quaternion Discrete Fourier Transform. The Quaternion Discrete Fourier Transform (QDFT) has already been used in applications such as image processing (see [2]). However, it still lacks a convenient notation.

Let us first have a look at the hitherto QDFT notation of a matrix $A \in \mathbb{H}^{M \times N}$. To this end, let $f_{1,m,u}^{(\mu_1)}$ and $f_{2,n,v}^{(\mu_2)}$ denote the Fourier basis function of the QDFT, where $\mu_i \in \mathbb{H}_{pu}$.

$$f_{1,m,u}^{(\mu_1)} := \frac{1}{\sqrt{M}} \exp(-\mu_1 2\pi u M)$$

$$f_{2,n,v}^{(\mu_2)} := \frac{1}{\sqrt{N}} \exp(-\mu_2 2\pi v N)$$
The two-side, left-side and right-side DQFT arise by multiplying the basis functions from different directions with respect to the entries of \( A \) (see also [7]).

\[
\mathcal{F}^{(1)}_{\mu_1, \mu_2} \{ A \}_{u,v} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{1,m,u}^{(\mu_1)} \cdot f_{2,n,v}^{(\mu_2)} \cdot [A]_{m,n}
\]

\[
\mathcal{F}^{(2)}_{\mu_1, \mu_2} \{ A \}_{u,v} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{1,m,u}^{(\mu_1)} \cdot f_{2,n,v}^{(\mu_2)} \cdot [A]_{m,n}
\]

\[
\mathcal{F}^{(3)}_{\mu_1, \mu_2} \{ A \}_{u,v} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} [A]_{m,n} \cdot f_{1,m,u}^{(\mu_1)} \cdot f_{2,n,v}^{(\mu_2)}
\]

Notice that for each DQFT it would be possible to swap the order of the Fourier basis functions. However, we will not consider these DQFTs in the following.

The authors of [2] presented the special case where \( \mu_1 = \mu_2 \) for the left-side and right-side DQFT. Additionally, the authors did not take advantage of a left matrix multiplication and a right matrix multiplication. Hence, we present a new notation of the DQFT using the notation proposed in Section 3. To that end, we define the quaternion Fourier matrices \( F_{1}^{(\mu_1)} \in \mathbb{C}^{M \times M} \) and \( F_{2}^{(\mu_2)} \in \mathbb{C}^{N \times N} \) based on (19) as follows.

\[
F_{1}^{(\mu_1)} := \frac{1}{\sqrt{M}} \exp \left( -\mu_1 2\pi mn^T M^{-1} \right)
\]

\[
F_{2}^{(\mu_2)} := \frac{1}{\sqrt{N}} \exp \left( -\mu_2 2\pi mn^T N^{-1} \right)
\]

The index vectors \( m \) and \( n \) are defined as \( m = [0 \ldots M-1]^T \), \( n = [0 \ldots N-1]^T \). Now, the DQFTs given in (21) – (23) of a matrix \( A \in \mathbb{H}^{M \times N} \) may conveniently be written as follows.

\[
\mathcal{F}^{(1)}_{\mu_1, \mu_2} \{ A \} = F_{1}^{(\mu_1)} \cdot L \ A \cdot L \ F_{2}^{(\mu_2)}
\]

\[
\mathcal{F}^{(2)}_{\mu_1, \mu_2} \{ A \} = F_{1}^{(\mu_1)} \cdot L \ (A \cdot R \ F_{2}^{(\mu_2)})
\]

\[
\mathcal{F}^{(3)}_{\mu_1, \mu_2} \{ A \} = (F_{1}^{(\mu_1)} \cdot R \ A) \cdot R \ F_{2}^{(\mu_2)}
\]

Moreover, the inverse discrete quaternion Fourier transform (IDQFT) of a matrix \( A \in \mathbb{H}^{M \times N} \) obtains the following form.

\[
\mathcal{F}^{(1)}_{\mu_1, \mu_2} \{ A \} = F_{1}^{(\mu_1)}^H \cdot L \ A \cdot L \ F_{2}^{(\mu_2)^H}
\]

\[
\mathcal{F}^{(2)}_{\mu_1, \mu_2} \{ A \} = (F_{1}^{(\mu_1)}^H \cdot L \ A)^H \cdot R \ F_{2}^{(\mu_2)^H}
\]

\[
\mathcal{F}^{(3)}_{\mu_1, \mu_2} \{ A \} = F_{1}^{(\mu_1)}^H \cdot R \ (A \cdot L \ F_{2}^{(\mu_2)^H})
\]

In general, the matrix product defined by the left-side (I)DQFT and the right-side (I)DQFT is not associative. This changes when \( \mu_1 \) and \( \mu_2 \) are chosen to be equal. In this case, the parenthesis can be set arbitrarily.

8. Conclusions

Using linear algebra over a skew-field such as the quaternions often lacks a convenient notation. To this end, we propose the left and right matrix product. These products inherently cover the problem of defining the order in which the matrix components are being multiplied.

Further on, it turns out that this notation is capable of defining the set of quaternionic fundamental subspaces in a convenient manner. Additionally, the feasibility of this notation is shown by applying it to the Eigenvalue decomposition as well as to the Quaternion Discrete Fourier transform. In both cases a more compact and simple form is achieved.

In complex linear algebra the Kronecker product as well as the Khatri-Rao product often prove beneficial. Hence, these products are shortly investigated as well.
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