Global $W^{1,p}$ regularity for elliptic problem with measure source and Leray-Hardy potential

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Abstract

In this paper, we develop the Littman-Stampacchia-Weinberger duality approach to obtain global $W^{1,p}$ estimates for a class of elliptic problems involving Leray-Hardy operators and measure sources in a distributional framework associated to a dual formulation with a specific weight function.

Key Words: Leray-Hardy potential; Duality approach; Radon measure.
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1 Introduction

Our concern in this paper is to derive global $W^{1,p}$ estimates for weak solutions of non-homogeneous Hardy problem

$$\begin{cases}
\mathcal{L}_\mu u = \nu & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

where $N \geq 3$, $\Omega$ is a bounded $C^2$ domain containing the origin, $\mathcal{L}_\mu$ is the Leray-Hardy operator defined by $\mathcal{L}_\mu = -\Delta + \frac{\mu}{|x|^2}$, $\mu \geq \mu_0 := -\frac{(N-2)^2}{4}$ and $\nu$ is a non-homogeneous source.

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Note that the inverse square potential in $L_\mu$ makes this operator degenerate at the origin. Due to this fact, challenging problems appear in the related Hardy semilinear elliptic equations. The numerous underlying problems were a source of motivation for many researchers. In order to solve (1.1) in the standard case, $\nu$ is a function, many aspects have been developed: Hardy inequality, see the references \[5, 15, 19, 21, 28, 29\], variational methods \[16, 20\] for example, the singularities of semilinear Hardy elliptic equation by \[7, 9, 14, 22, 23, 24, 26, 30\] and the references therein. When $N \geq 3$, $\mu_0 \leq \mu < 0$, Dupaigne in \[15\] (also see \[4, 5\]) studied the weak solutions for the equation $L_\mu u = u^p + tf$ in $\Omega$, subject to the zero Dirichlet boundary condition, with $t > 0$, $p > 1$ and $f \in L^1(\Omega)$, in the distributional sense that $u \in L^1(\Omega), u^p \in L^1(\Omega, \rho dx)$ and

$$\int_\Omega u L_\mu \xi \, dx = \int_\Omega u^p \xi \, dx + t \int_\Omega f \xi \, dx, \quad \forall \xi \in C^{1,1}_0(\Omega),$$

where $\rho(x) = \text{dist}(x, \partial \Omega)$.

However, the normal distributional identity (1.2) does not seem to extend to address other types of isolated singular solutions, such as the solutions of

$$L_\mu u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$ (1.3)

Indeed, when $\mu \geq \mu_0$, problem (1.3) has two radially symmetric solutions with the explicit formulas given by

$$\Phi_\mu(x) = \begin{cases} |x|^{\tau_-(\mu)} & \text{if } \mu > \mu_0, \\ -|x|^{\tau_-(\mu)} \ln |x| & \text{if } \mu = \mu_0 \end{cases} \quad \text{and} \quad \Gamma_\mu(x) = |x|^{\tau_+(\mu)},$$

where

$$\tau_-(\mu) = -\frac{N - 2}{2} - \sqrt{\mu - \mu_0} \quad \text{and} \quad \tau_+(\mu) = -\frac{N - 2}{2} + \sqrt{\mu - \mu_0},$$

$\tau_-(\mu)$ and $\tau_+(\mu)$ are the zero points of $\mu - \tau(\tau + N - 2) = 0$. To simplify the notations, we write $\tau_+$ and $\tau_-$ in what follows. Obviously, $\Phi_\mu$ has a too strong singularity at the origin to hope that $\Phi_\mu \in L^1_{\text{loc}}(\mathbb{R}^N)$ when $\mu \geq \frac{N^2 + 4}{2}$. Due to its singularity, $\Phi_\mu$ can’t be viewed as a fundamental solution expressed by Dirac mass at origin in the normal distributional identity for $\mu \neq 0$. 


To overcome this difficulty, \cite{11} introduces a new distributional identity for $\Phi_{\mu}$ as
\[
\int_{\mathbb{R}^N} \Phi_{\mu} \mathcal{L}^*_{\mu} \xi \, d\gamma_{\mu} = c_{\mu} \xi(0), \quad \forall \xi \in C_c^1(\mathbb{R}^N),
\] (1.5)
where $\delta_0$ is the Dirac mass at the origin, here and in what follows,
\[
d\gamma_{\mu}(x) = \Gamma_{\mu}(x) \, dx, \quad \mathcal{L}^*_{\mu} = -\Delta - \frac{2\tau}{|x|^2} x \cdot \nabla, \quad c_{\mu} = \begin{cases} 2\sqrt{\mu - \mu_0} |\mathbb{S}^{N-1}| & \text{if } \mu > \mu_0, \\ |\mathbb{S}^{N-1}| & \text{if } \mu = \mu_0 \end{cases}
\] (1.6)
and $\mathbb{S}^{N-1}$ is the sphere of the unit ball in $\mathbb{R}^N$. The distributional identity (1.5) could be seen as
\[
\mathcal{L}_{\mu} \Phi_{\mu} = c_{\mu} \delta_0 \quad \text{in } \mathbb{R}^N.
\] (1.7)
From above distributional identity, $\Phi_{\mu}$ is a fundamental solution of Hardy operator $\mathcal{L}_{\mu}$.

Inspired by the new dual identity (1.5), semilinear Hardy problems involving Radon measure
\[
\begin{cases} \mathcal{L}_{\mu} u + g(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega 
\end{cases}
\] (1.8)
have been considered in \cite{12} (also \cite{13} when the origin is at boundary) in a measure framework for $\nu = \nu|_{\Omega^*} + k\delta_0$, where $\Omega^* = \Omega \setminus \{0\}$, $\nu|_{\Omega^*} \in \mathfrak{M}(\Omega^*; \Gamma_{\mu})$, here $\mathfrak{M}(\Omega^*; \Gamma_{\mu})$ is denoted by the set of Radon measures $\nu$ in $\Omega^*$ such that
\[
\|\nu\|_{\mathfrak{M}(\Omega^*; \Gamma_{\mu})} = \int_{\Omega^*} \Gamma_{\mu} d|\nu| := \sup \left\{ \int_{\Omega^*} \zeta d|\nu| : \zeta \in C_c(\Omega^*), \ 0 \leq \zeta \leq \Gamma_{\mu} \right\} < \infty.
\] (1.9)
We denote by $\overline{\mathfrak{M}}(\Omega; \Gamma_{\mu})$ the set of measures which can be written under the form
\[
\nu = \nu|_{\Omega^*} + k\delta_0,
\] (1.10)
where $\nu|_{\Omega^*} \in \mathfrak{M}(\Omega; \Gamma_{\mu})$ and $k \in \mathbb{R}$. Let $\overline{\Omega} := \overline{\Omega} \setminus \{0\}$ and
\[
\mathcal{X}_{\mu}(\Omega) = \left\{ \xi \in C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}^+) : |x|\mathcal{L}^*_{\mu} \xi \in L^\infty(\Omega) \right\}.
\] (1.11)
We note that $C_0^{1,1}(\overline{\Omega}) \subset \mathcal{X}_{\mu}(\Omega)$. 
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**Definition 1.1** We say that $u$ is a weak solution (or a very weak solution) of (1.8) with $\nu = \nu|_{\Omega^*} + k\delta_0 \in \mathfrak{M}(\Omega; \Gamma_{\mu})$, if $u \in L^1(\Omega, |x|^{-1}d\gamma_{\mu})$, $g(u) \in L^1(\Omega, d\gamma_{\mu})$ and

$$
\int_{\Omega} u \mathcal{L}_{\mu} \xi d\gamma_{\mu} + \int_{\Omega} g(u) \xi d\gamma_{\mu} = \int_{\Omega} \xi d(\Gamma_{\mu} \nu) + c_{\mu} k \xi(0) \quad \text{for all } \xi \in \mathcal{X}_{\mu}(\Omega).
$$

(1.12)

Similarly, a function $u$ is called a weak solution (or a very weak solution) of (1.1) with $\nu = \nu|_{\Omega^*} + k\delta_0$, if $u$ is in $L^1(\Omega, |x|^{-1}d\gamma_{\mu})$ and satisfies (1.12) with $g \equiv 0$.

Away from the origin, the operator $\mathcal{L}_{\mu}$ is uniformly elliptic, therefore local regularity properties for second elliptic equation could be applied directly locally in $\Omega \setminus \{0\}$. The main goal of this paper is to study the global regularity of solutions of (1.1) and to apply the global regularity to semilinear elliptic equation (1.8). Our main result reads as follows.

**Theorem 1.1** Assume that $N \geq 3$, $\mu > 2\mu_0$, $\nu \in \mathfrak{M}(\Omega; \Gamma_{\mu})$ with the form $\nu = \nu|_{\Omega^*} + k\delta_0$ for $k \in \mathbb{R}$, and $u$ is a very weak solution of (1.1). Let

$$
p_{\mu}^* = \frac{N + \tau_+}{N - 1 + \tau_+}.
$$

(1.13)

Then (i) $u \Gamma_{\mu} \in W^{1,p}_0(\Omega)$ with $p \in [1, \frac{N}{N-1})$ and there exists $C_1 > 0$ depending on $N$, $p$, $\mu$ and $\Omega$ such that

$$
\|u \Gamma_{\mu}\|_{W^{1,p}(\Omega)} \leq C_1(\|\nu\|_{\mathfrak{M}(\Omega^*; \Gamma_{\mu})} + k).
$$

(1.14)

(ii) $u \in W^{1,p}_0(\Omega, d\gamma_{\mu})$ with $p \in [1, \min\{p_{\mu}^*, \frac{N}{N-1}\})$ and there exists $C_2 > 0$ depending on $N$, $p$, $\mu$ and $\Omega$ such that

$$
\|u\|_{W^{1,p}(\Omega, d\gamma_{\mu})} \leq C_2(\|\nu\|_{\mathfrak{M}(\Omega^*; \Gamma_{\mu})} + k).
$$

(1.15)

**Remark 1.1** (i) $p_{\mu}^* \leq \frac{N}{N-1}$ for $\mu \geq 0$ and $p_{\mu}^* > \frac{N}{N-1}$ for $\mu \in [\mu_0, 0)$;

(ii) An equivalent form of $p_{\mu}^*$ is $p_{\mu}^* = 1 - \frac{2}{\tau_-}$ by definition of $\tau_-$, and $\min\{p_{\mu}^*, \frac{N}{N-1}\}$ is the critical exponent for the existence of (1.8) for $\nu \in \mathfrak{M}(\Omega; \Gamma_{\mu})$ in [12], is the critical exponent for isolated singular solutions of $\mathcal{L}_{\mu} u = u^p$ in $\Omega \setminus \{0\}$, $u = 0$ on $\partial \Omega$ in [10].
It is well-known that the classical method for global $W^{1,p}$ estimates for second order elliptic equations named as Littman-Stampacchia-Weinberger duality approach (LSW approach for short) in [25, 27], with the coefficients being in $L^\infty$. LSW approach has been used to deal with various elliptic equations, see references [6, 7, 8, 22, 28]. Global $W^{1,p}$ estimates in Theorem 1.1 are derived by LSW approach. Due to the inverse-square potential, the Hardy operator is degenerated. Our dual operator for $L_\mu$ is no longer $L_\mu$, but is $L^*_\mu$ in our distributional sense, see Definition 1.1. Observe that the coefficient of dual operator $L^*_\mu$ in gradient term is singular. That is the main reason for assumptions that $N \geq 3$ and $\mu > \frac{3}{4} \mu_0$, and under these assumptions, the solutions for nonhomogeneous Dirichlet problem

$$L^*_\mu u = f_n + \text{div} F_n \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega$$

(1.16)

keep bounded in the approximation of $f$ and $F$ in $L^r$ nonhomogeneous terms with $r > N$ by smooth functions $\{f_n\}_n$ and $\{F_n\}_n$. These uniform estimates enable us to make use of the LSW duality approach.

While if $\mu > 0$ for $N = 2$ or if $\mu \in [\mu_0, \frac{3}{4} \mu_0]$ for $N \geq 3$, dual problem (1.16) has a unique classical solution by constructing super and sub solutions when $f$ and $F$ are regular. However, it is very challenging to obtain weak solutions and $L^\infty$ estimates in the approximation of $f$ and $F$ in $L^2$ spaces. It seems that one has to come up with a new and self-contained approach to solve this problem.

For our semilinear elliptic problem (1.8), a direct corollary for a global regularity could be obtained as follows.

**Corollary 1.1** Assume that $N \geq 3$, $\mu > \frac{3}{4} \mu_0$ and $\nu \in \mathcal{M}(\Omega; \Gamma_\mu)$. If problem (1.8) admits a weak solution $u_g$, then $u_g \Gamma_\mu \in W^{1,p}(\Omega)$ for $p \in [1, \frac{N}{N-1})$ and $u \in W_0^{1,p}(\Omega, d\gamma_\mu)$ with $p \in [1, \min\{p^*_\mu, \frac{N}{N-1}\})$.

The global regularity $u_g \Gamma_\mu \in W^{1,p}_0(\Omega)$ obtained in Corollary 1.1 indicates that the zero Dirichlet boundary restriction of (1.8) could be understood in the trace sense.

The rest of the paper is organized as follows. In Section 2, we approximate the solution of $L^*_\mu v = f \Gamma_\mu + \text{div}(F \Gamma_\mu)$, subject to zero Dirichlet boundary condition and build $L^\infty$ estimate for functions $f, F$ in suitable spaces. Section 3 is devoted to prove Theorem 1.1 by developing LSW duality approach.
2 Weak solution of Dual Hardy problem

2.1 Existence

The essential point to obtain $W^{1,p}$ estimates by applying the Littman-Stampacchia-Weinberger duality method is to get $L^\infty$ estimates of the related dual nonhomogeneous problem. In what follows, we denote by $c_i$ a generic positive constant.

Let us consider the solution $w$ of the problem

$$\begin{cases}
L_\mu^* w = f + \text{div} F & \text{in } \Omega, \\
        w = 0 & \text{on } \partial \Omega,
\end{cases} \tag{2.1}$$

where $f : \Omega \rightarrow \mathbb{R}$ and $F : \Omega \rightarrow \mathbb{R}^N$. Let $\mathbb{H}_0^1(\Omega)$ be the Hilbert space defined as the closure of functions in $C_c^\infty(\Omega)$ under the norm

$$
\|u\|_{\mathbb{H}_0^1(\Omega)} = \sqrt{<u, u>_{\mathbb{H}_0^1(\Omega)}}
$$

with the inner product

$$
<u, v>_{\mathbb{H}_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx.
$$

A function $w$ is called a weak solution of (2.1) if $w \in \mathbb{H}_0^1(\Omega)$ and

$$
\int_\Omega \left( \nabla w \cdot \nabla v - \frac{2\tau+1}{|x|^2} (x \cdot \nabla w) v \right) \, dx = \int_\Omega (fv - F \cdot \nabla v) \, dx \quad \text{for any } v \in \mathbb{H}_0^1(\Omega).
$$

A function $w$ is called a classical solution of (2.1) if $w \in \mathbb{H}_0^1(\Omega) \cap C^2(\Omega \setminus \{0\})$ and $w$ verifies equation (2.1) pointwisely in $\Omega \setminus \{0\}$.

Our existence results state as follows.

**Proposition 2.1** Assume that $N \geq 3$ and $\mu > \frac{3}{4} \mu_0$. Then

(i) for $\theta \in (0, 1)$, for any $f \in C^\theta(\bar{\Omega})$ and $F \in C^{1,\theta}(\bar{\Omega}; \mathbb{R}^N)$, problem (2.1) has a bounded, unique classical solution.

(ii) for $f \in L^2(\Omega)$ and $F \in L^2(\Omega; \mathbb{R}^N)$, problem (2.1) has a unique weak solution $w \in \mathbb{H}_0^1(\Omega)$ satisfying that

$$
\|w\|_{\mathbb{H}_0^1(\Omega)} \leq c_1 \left( \|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} \right). \tag{2.2}
$$
Due to the singularity of the coefficient of $L^{*}_{\mu}$, we use the following sequence of operator to approximate it. Given $\epsilon \in (0, 1)$, we denote

$$L^{*}_{\mu, \epsilon} = -\Delta - 2\tau_{+}\frac{1}{|x|^2} x \cdot \nabla$$

and we consider the approximated problem

$$\begin{cases}
L^{*}_{\mu, \epsilon} w = f + \text{div} F & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases} \quad (2.3)$$

where $f : \Omega \to \mathbb{R}$, $F : \Omega \to \mathbb{R}^N$ and $1_{B^{c}_{\epsilon}}$ is the characteristic function in $\mathbb{R}^N \setminus B_{\epsilon}(0)$.

Since $2\tau_{+}1_{B^{c}_{\epsilon}}|x|^{-2}x$ is bounded, smooth in $\Omega \setminus \partial B_{\epsilon}(0)$ and $L^{*}_{\mu, \epsilon}$ is a uniformly elliptic operator, then for any $f \in C^{1}(\bar{\Omega})$ and $F \in C^{2}(\bar{\Omega}, \mathbb{R}^N)$, problem (2.3) admits a unique classical solution $u_{\epsilon}$ by the Perron’s super and sub solutions’ method and comparison principle. On the other hand, by the fact that $0 \in \Omega$, there exist $\epsilon_0 \in (0, 1)$ and $R_0 > \epsilon_0$ such that

$$B_{\epsilon_0}(0) \subset \Omega \subset B_{R_0}(0).$$

**Lemma 2.1** Assume that $N \geq 2$, $\epsilon \in [0, \epsilon_0)$, $\mu > \frac{3}{4}\mu_0$, $f \in C^{1}(\bar{\Omega})$, $F \in C^{2}(\bar{\Omega}, \mathbb{R}^N)$ and $u_{\epsilon}$ is the unique classical solution of problem (2.3). Then there exists $c_2 > 0$ independent of $f, F$ and $\epsilon$ such that

$$\|\nabla u_{\epsilon}\|_{L^2(\Omega)} \leq c_2 \left( \|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} \right). \quad (2.4)$$

**Proof.** It is obvious that $u_{\epsilon} \in H^{1}_0(\Omega)$ for $\epsilon \in (0, 1)$ and we observe that

$$(N - 2)\tau_{+} > -\frac{(N - 2)^2}{4} \text{ if } \mu > -\frac{3}{4}\frac{(N - 2)^2}{4}, \quad (2.5)$$

that is,

$$\frac{4\tau_{+}}{N - 2} > -1.$$
By Hölder inequality and Poincaré inequality, we have that
\[
\int_{\Omega} (f u_\epsilon - F \cdot \nabla u_\epsilon) \, dx \leq (\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega; \mathbb{R}^N)}) (\|u_\epsilon\|_{L^2(\Omega)} + \|\nabla u_\epsilon\|_{L^2(\Omega)}) \\
\leq c_3 (\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega; \mathbb{R}^N)}) \|\nabla u_\epsilon\|_{L^2(\Omega)},
\]
where \(c_3 > 0\) is independent of \(f, F\) and \(\epsilon\). Observe that \(\frac{x}{|x|^2} u_\epsilon^2 \in W^{1,1}(\Omega \setminus B_{\epsilon}(0) : \mathbb{R}^N)\) and the divergence theorem (see e.g. [31, Theorem 6.3.4]), imply that
\[
\int_{\Omega \setminus B_{\epsilon}(0)} \nabla (u_\epsilon^2) \frac{x}{|x|^2} \, dx + \int_{\Omega \setminus B_{\epsilon}(0)} u_\epsilon^2 \div \left( \frac{x}{|x|^2} u_\epsilon \right) \, dx = \int_{\Omega \setminus B_{\epsilon}(0)} \div \left( \frac{x}{|x|^2} u_\epsilon^2 \right) \, dx \\
= \int_{\partial B_{\epsilon}(0)} u_\epsilon^2 \frac{x}{|x|^2} \cdot \frac{\nabla \omega(x)}{|x|} \, d\omega(x),
\]
thus,
\[
\int_{\Omega} \left( |\nabla u_\epsilon|^2 - \frac{\tau + 1}{|x|^2} x \cdot \nabla (u_\epsilon^2) \right) \, dx \geq \int_{\Omega} |\nabla u_\epsilon|^2 \, dx + \tau \int_{\Omega} u_\epsilon^2 \div \left( \frac{x}{|x|^2} 1_{B_{\epsilon}} \right) \, dx \\
= \|\nabla u_\epsilon\|_{L^2(\Omega)}^2 + (N - 2) \tau \int_{\Omega \setminus B_{\epsilon}(0)} \frac{u_\epsilon^2}{|x|^2} \, dx.
\]
When \(\tau \geq 0\) i.e. \(\mu \geq 0\),
\[
\int_{\Omega} \left( |\nabla u_\epsilon|^2 - \frac{\tau + 1}{|x|^2} x \cdot \nabla (u_\epsilon^2) \right) \, dx \geq \|\nabla u_\epsilon\|_{L^2(\Omega)}^2.
\]
When \(N \geq 3\) and \(0 \leq \mu < -\frac{3}{4} \left( \frac{N-2}{N} \right)^2\), by Hardy inequality, we have that
\[
\int_{\Omega} \left( |\nabla u_\epsilon|^2 - \frac{\tau + 1}{|x|^2} x \cdot \nabla (u_\epsilon^2) \right) \, dx \geq \|\nabla u_\epsilon\|_{L^2(\Omega)}^2 + (N - 2) \tau \int_{\Omega} \frac{u_\epsilon^2}{|x|^2} \, dx \\
\geq (1 + \frac{4\tau}{N-2}) \|\nabla u_\epsilon\|_{L^2(\Omega)}^2.
\]
Therefore, there exists \(c_4 > 0\) independent of \(\epsilon, f\) and \(F\) such that
\[
\|\nabla u_\epsilon\|_{L^2(\Omega)} \leq c_4 (\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega; \mathbb{R}^N)}),
\]
which completes the proof. \(\square\)
Lemma 2.2 Assume that \( N \geq 3 \), \( 0 < \epsilon_1 < \epsilon_2 < \epsilon_0 \), \( \mu > \frac{3}{4} \mu_0 \) and \( f \in C^1(\Omega) \) and \( F \in C^2(\Omega, \mathbb{R}^N) \) and \( u_{\epsilon_1}, u_{\epsilon_2} \) are the unique classical solutions of problem \((2.3)\) with \( \epsilon = \epsilon_1 \) and \( \epsilon = \epsilon_2 \) respectively.

Then there exists \( c_5 > 0 \) independent of \( \epsilon_1 \) and \( \epsilon_2 \) such that

\[
\| \nabla (u_{\epsilon_1} - u_{\epsilon_2}) \|_{L^2(\Omega)} \leq c_5 \left( \| f \|_{L^2(\Omega)} + \| F \|_{L^2(\Omega)} \right) \sqrt{\epsilon_2^{N-2} - \epsilon_1^{N-2}}. \tag{2.7}
\]

Proof. Due to the boundedness of \( \Omega \), there exists \( R_0 \geq \epsilon_0 \) such that \( \Omega \subset B_{R_0}(0) \). Let \( a_0 = \| f \|_{L^\infty(\Omega)} + \| \text{div} F \|_{L^\infty(\Omega)} \) and

\[
W_0(x) = \frac{a_0}{N+2} (R_0^2 - |x|^2), \quad \forall x \in \Omega. \tag{2.8}
\]

For \( \mu > \frac{3}{4} \mu_0 \), we have that \( 2N + 2\tau_+ > N + 2 \) and

\[
\mathcal{L}^\ast_{\mu, \epsilon} W_0 = \frac{a_0}{N+2} (2N + 2\tau_+ 1_{B_1}) \geq a_0.
\]

In addition, Comparison Principle implies that for any \( \epsilon \in (0, \epsilon_0) \),

\[
|u_\epsilon| \leq W_0 \quad \text{in} \quad \Omega.
\]

Direct computation shows that

\[
0 = \int_\Omega (\mathcal{L}^\ast_{\mu, \epsilon_1} u_{\epsilon_1} - \mathcal{L}^\ast_{\mu, \epsilon_2} u_{\epsilon_2})(u_{\epsilon_1} - u_{\epsilon_2}) \, dx
\]

\[
\geq \int_\Omega |\nabla (u_{\epsilon_1} - u_{\epsilon_2})|^2 \, dx + \tau_+ \int_\Omega (u_{\epsilon_1} - u_{\epsilon_2})^2 \text{div} \left( \frac{x}{|x|^2} 1_{B_{\epsilon_1}} \right) \, dx
\]

\[
-2\tau_+ \int_{B_{\epsilon_2}(0) \setminus B_{\epsilon_1}(0)} (u_{\epsilon_1} - u_{\epsilon_2}) x \cdot \nabla u_{\epsilon_2} \, dx
\]

\[
\geq (1 + \frac{4\tau_+}{N-2}) \| \nabla (u_{\epsilon_1} - u_{\epsilon_2}) \|^2_{L^2(\Omega)} - 2|\tau_+| \left( \int_{B_{\epsilon_2}(0) \setminus B_{\epsilon_1}(0)} \frac{|u_{\epsilon_1} - u_{\epsilon_2}|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \| \nabla u_{\epsilon_2} \|_{L^2(\Omega)}
\]

\[
\geq (1 + \frac{4\tau_+}{N-2}) \| \nabla (u_{\epsilon_1} - u_{\epsilon_2}) \|^2_{L^2(\Omega)} - 4a_0 |\tau_+| \left( \int_{B_{\epsilon_2}(0) \setminus B_{\epsilon_1}(0)} \frac{1}{|x|^2} \, dx \right)^{\frac{1}{2}} \| \nabla u_{\epsilon_2} \|_{L^2(\Omega)},
\]

thus, by Lemma 2.1, we have that

\[
\| \nabla (u_{\epsilon_1} - u_{\epsilon_2}) \|^2_{L^2(\Omega)} \leq c_5 \left( \| f \|^2_{L^2(\Omega)} + \| F \|^2_{L^2(\Omega)} \right) \sqrt{\epsilon_2^{N-2} - \epsilon_1^{N-2}},
\]

which completes the proof. \( \square \)
Remark 2.1. In the above Lemma, the dimension \( N \geq 3 \) plays an essential role in the calculation of \( \int_{B_{\varepsilon_2}(0) \setminus B_1(0)} \frac{1}{|x|} \, dx \).

Proof of Proposition 2.1. (i) Take \( \varepsilon_n = \frac{1}{n} \) with \( n \in \mathbb{N} \) and \( n \geq \frac{1}{c_0} \) and let \( u_n \) be the unique solution of (2.3), then by Lemma 2.2, we have that \( \{u_n\} \) is a Cauchy sequence in \( H^1_0(\Omega) \). Therefore, there exists \( u_0 \in H^1_0(\Omega) \) such that

\[
 u_n \to u_0 \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad n \to +\infty.
\]

Multiplying \( v \in H^1_0(\Omega) \) in (2.3) and integrating over \( \Omega \), we obtain that

\[
 \int_{\Omega} \left( \nabla u_n \cdot \nabla v - \frac{2\tau_+}{|x|^2} (x \cdot \nabla u_n)v \right) \, dx = \int_{\Omega} (fv - F \cdot \nabla v) \, dx. \tag{2.9}
\]

Passing to the limit in the identity (2.9) as \( n \to +\infty \), we have that

\[
 \int_{\Omega} \left( \nabla u_0 \cdot \nabla v - \frac{2\tau_+}{|x|^2} (x \cdot \nabla u_0)v \right) \, dx = \int_{\Omega} (fv - F \cdot \nabla v) \, dx \quad \text{for any} \quad v \in H^1_0(\Omega). \tag{2.10}
\]

From the proof of Lemma 2.2, the sequence \( \{u_n\} \) is uniformly bounded by barrier \( W_0 \) defined by (2.8), so \( |u_0| \leq W_0 \) a.e. in \( \Omega \). Standard interior regularity shows that \( u_0 \) is a classical solution of (2.1).

(ii) For \( f \in L^2(\Omega) \) and \( F \in L^2(\Omega; \mathbb{R}^N) \), let \( \{f_m\} \) and \( \{F_m\} \) be two sequence in \( C^1(\Omega) \) and \( C^2(\Omega; \mathbb{R}^N) \) converging to \( f \) and \( F \) in \( L^2(\mathbb{R}^N) \) and \( L^2(\Omega; \mathbb{R}^N) \) respectively.

By Lemma 2.1 with \( \varepsilon = 0 \),

\[
 \|\nabla u_m\|_{L^2(\Omega)} \leq c_5 \left( \|f_m\|_{L^2(\Omega)} + \|F_m\|_{L^2(\Omega)} \right) \tag{2.11}
\]

and

\[
 \|\nabla (u_{m_1} - u_{m_2})\|_{L^2(\Omega)} \leq c_5 \left( \|f_{m_1} - f_{m_2}\|_{L^2(\Omega)} + \|F_{m_1} - F_{m_2}\|_{L^2(\Omega)} \right). \tag{2.12}
\]

Therefore, we have that \( \{u_m\} \) is a Cauchy sequence in \( H^1_0(\Omega) \) and there exists \( u \in H^1_0(\Omega) \) such that \( u_m \to u \) in \( H^1_0(\Omega) \) as \( m \to +\infty \). Additionally, we have that

\[
 \int_{\Omega} \left( \nabla u_m \cdot \nabla v - \frac{2\tau_+}{|x|^2} (x \cdot \nabla u_m)v \right) \, dx = \int_{\Omega} (fv - F \cdot \nabla v) \, dx \quad \text{for any} \quad v \in H^1_0(\Omega). \tag{2.13}
\]
Passing to the limit as $m \to +\infty$, we have that
\[
\int_{\Omega} \left( \nabla u \cdot \nabla v - \frac{2\tau}{|x|^2} (x \cdot \nabla u) v \right) \, dx = \int_{\Omega} (fv - F \cdot \nabla v) \, dx \quad \text{for any } v \in H^1_0(\Omega). \tag{2.14}
\]
Therefore, $u$ is a weak solution of (2.1) and (2.2) follows by (2.11) directly for $\mu > -\frac{3}{4} (N-2)^2$.

The uniqueness follows by Lemma 2.1. In fact, if there exists two (classical or weak) solutions $u_1, u_2 \in H^1_0(\Omega)$ of (2.1), then it follows by Lemma 2.1 that $\|
abla (u_1 - u_2)\|_{L^2(\Omega)} \leq 0$ and then $u_1 = u_2$ a.e. in $\Omega$. \qed

### 2.2 $L^\infty$ estimates for dual problems

Given $k > 0$, let $S_k$ be the function defined for $t \in \mathbb{R}$ by
\[
S_k(t) = \begin{cases} 
  t + k & \text{if } t < -k, \\
  0 & \text{if } -k \leq t \leq k, \\
  t - k & \text{if } t > k.
\end{cases} \tag{2.15}
\]

**Lemma 2.3** Assume that $w$ is a measurable function in $L^1(\Omega)$ and there exist $\alpha > 1$ and $A > 0$ such that for every $k > 0$,
\[
\|S_k(w)\|_{L^1(\Omega)} \leq A\{|w| > k\}^\alpha, \tag{2.16}
\]
where $\{|w| > k\} = \{x \in \Omega : |w(x)| > k\}$. Then $w \in L^\infty(\Omega)$ and
\[
\|w\|_{L^\infty(\Omega)} \leq c_6 A^{\frac{1}{\alpha}} \|w\|_{L^1(\Omega)}^{1 - \frac{1}{\alpha}}. \tag{2.17}
\]

**Proof.** We will follow the same idea of the proof Lemma 5.2 in [28]. For the convenience of the reader, we provide all the details of the proof.

By Cavalieri’s principle, we have that
\[
\|S_k(w)\|_{L^1(\Omega)} = \int_0^\infty |\{|S_k(w)| > s\}| \, ds = \int_k^\infty |\{|w| > s\}| \, ds
\]
Using (2.16), it follows that
\[
\int_k^\infty |\{|w| > s\}| \, ds \leq A|\{|w| > k\}|^\alpha.
\]
Let \( H : [0, \infty) \to \mathbb{R} \) be the function defined for \( t \in 0 \) by
\[
H(t) = \int_t^\infty |\{ |w| > s \}| \, ds.
\]
Since \( s \to |\{ |w| > s \}| \) is non-increasing, it is then continuous except for countable many points. Thus, for almost every \( t \geq 0 \), we have that
\[
-H'(t) = |\{ |w| > t \}| \geq \left[ \frac{H(t)}{A} \right]^{\frac{1}{\alpha}}.
\]
Integrating this inequality, we conclude that if \( \alpha > 1 \), then
\[
H(k_0) = 0 \quad \text{for some} \quad k_0 \geq 0 \quad \text{such that} \quad k_0 \leq c_7 A^{\frac{1}{\alpha}} H(0)^{1-\frac{1}{\alpha}},
\]
that is, \( \| S_k(w) \|_{L^\infty(\Omega)} \leq k_0 \) and \( H(0) = \| w \|_{L^1(\Omega)} \). This ends the proof. \( \square \)

**Proposition 2.2** Assume that \( N \geq 3 \), \( \mu > \frac{3}{4} \mu_0 \), \( f \in L^r(\Omega) \), \( F \in L^{r'}(\Omega; \mathbb{R}^N) \) with \( r > N \) and \( u_0 \in H_0^1(\Omega) \) is the unique solution of problem (2.1).

Then
\[
\| u_0 \|_{L^\infty(\Omega)} \leq c_8 \left( \| f \|_{L^r(\Omega)} + \| F \|_{L^{r'}(\Omega)} \right), \tag{2.18}
\]
where \( c_8 > 0 \) is independent of \( f \) and \( F \).

**Proof.** From (2.14) with \( v = S_k(u_0) \in H_0^1(\Omega) \cap H^1_0(\{ |u_0| > k \}) \), we have that
\[
\int_{\{ |u_0| > k \}} |\nabla u_0|^2 \, dx - 2\tau_+ \int_{\{ |u_0| > k \}} \frac{x \cdot \nabla S_k^2(u_0)}{|x|^2} \, dx = \int_{\{ |u_0| > k \}} \left( f + \text{div} F \right) S_k(u_0) \, dx, \tag{2.19}
\]
where we used the facts that \( \nabla S_k(u_0) = \nabla u_0 \) a.e. in \( \{ |u_0| > k \} \) and \( \nabla S_k(u_0) = 0 \) a.e. in \( \Omega \setminus \{ |u_0| > k \} \).

For \( \mu \geq 0 \), it follows from (2.6) that
\[
\int_{\{ |u_0| > k \}} |\nabla u_0|^2 \, dx - 2\tau_+ \int_{\{ |u_0| > k \}} \frac{x \cdot \nabla S_k^2(u_0)}{|x|^2} \, dx \\
\geq \int_{\{ |u_0| > k \}} |\nabla S_k(u_0)|^2 \, dx + (N - 2) \tau_+ \int_{\{ |u_0| > k \}} \frac{S_k^2(u_0)}{|x|^2} \, dx \\
\geq \int_{\{ |u_0| > k \}} |\nabla S_k(u_0)|^2 \, dx,
\]
where \( \tau_+ \geq 0 \) for \( \mu \geq 0 \).

For \( 0 \geq \mu > -\frac{3(N-2)^2}{N} \), \( \tau_+ < 0 \) and \( 1 + \frac{4\tau_+}{N} > 0 \). It follows by Hardy inequality and the fact that \( S_k(u_0) \in H^1_0(\{ |u_0| > k \}) \), we have that

\[
\int_{\{ |u_0| > k \}} \frac{S^2_k(u_0)}{|x|^2} dx = \int_\Omega \frac{S^2_k(u_0)}{|x|^2} dx \leq \frac{4}{(N-2)^2} \int_\Omega |\nabla S_k(u_0)|^2 dx
\]

and

\[
\int_{\{ |u_0| > k \}} |\nabla u_0|^2 dx - 2\tau_+ \int_{\{ |u_0| > k \}} \frac{x \cdot \nabla S^2_k(u_0)}{|x|^2} dx \geq \int_{\{ |u_0| > k \}} |\nabla S_k(u_0)|^2 dx + (N-2)\tau_+ \int_{\{ |u_0| > k \}} S^2_k(u_0) \frac{|x|^2}{|x|^2} dx
\]

\[
= (1 + \frac{4\tau_+}{N-2}) \int_{\{ |u_0| > k \}} |\nabla S_k(u_0)|^2 dx.
\]

The right hand side of (2.19) has the following estimate:

\[
\left| \int_{\{ |u_0| > k \}} \left( f + \text{div} F \right) S_k(u_0) \right| dx \leq \left( \int_{\{ |u_0| > k \}} f^2 dx \right)^{\frac{1}{2}} \left( \int_{\{ |u_0| > k \}} S_k(u_0)^2 dx \right)^{\frac{1}{2}} + \left( \int_{\{ |u_0| > k \}} |F \cdot \nabla S_k(u_0)| \right) dx \leq \left( \|f\|_{L^2(\{ |u_0| > k \})} + \|F\|_{L^2(\{ |u_0| > k \})} \right) \|\nabla S_k(u_0)\|_{L^2(\{ |u_0| > k \})}.
\]

It then follows from (2.19) that

\[
\|\nabla S_k(u_0)\|_{L^p(\{ |u_0| > k \})} \leq c_7 \left( \|f\|_{L^2(\{ |u_0| > k \})} + \|F\|_{L^2(\{ |u_0| > k \})} \right). \tag{2.20}
\]

Therefore, it follows from the Hölder inequality and Sobolev inequality that

\[
\|S_k(u_0)\|_{L^1(\Omega)} = \|S_k(u_0)\|_{L^1(\{ |u_0| > k \})} \leq \|S_k(u_0)\|_{L^{2p}(\{ |u_0| > k \})} \left| \{ |u_0| > k \} \right|^{\frac{1}{2p} + \frac{1}{r}} \leq \int_{\{ |u_0| > k \}} |\nabla u_0|^2 dx \left| \{ |u_0| > k \} \right|^{\frac{1}{2} + \frac{1}{r}} \tag{2.21}
\]

\[
\leq c_9 \left( \|f\|_{L^r(\{ |u_0| > k \})} + \|F\|_{L^r(\{ |u_0| > k \})} \right) \left| \{ |u_0| > k \} \right|^{\frac{1}{2} + \frac{1}{r} - \frac{1}{2}}.
\]
that is, 
\[ \|S_k(u_0)\|_{L^1(\Omega)} \leq c_{10} \left( \|f\|_{L^r(\{|u_0|>k\})} + \|F\|_{L^r(\{|u_0|>k\})} \right) \{|u_0|>k\}^{1+\frac{1}{r} - \frac{1}{q}}, \] 
where \( r>N \).

We apply Lemma 2.3 with \( \alpha = 1 + \frac{1}{q} - \frac{1}{r} > 1 \) for \( r>N \), then it follows that
\[ \|u_0\|_{L^\infty(\Omega)} \leq c_{11} \left( \|f\|_{L^r(\Omega)} + \|F\|_{L^r(\Omega)} \right), \]
which ends the proof. \( \square \)

**Proposition 2.3** Assume that \( N \geq 3, \mu>0, f \in L^r(\Omega, d\gamma_\mu), F \in L^r(\Omega, d\gamma_\mu; \mathbb{R}^N) \) with \( r>N+\tau_+ \) and \( u_0 \in \mathbb{H}_0^1(\Omega) \) is the unique solution of problem \( (2.1) \).

Then 
\[ \|u_0\|_{L^\infty(\Omega)} \leq c_{12} \left( \|f\|_{L^r(\Omega, d\gamma_\mu)} + \|F\|_{L^r(\Omega, d\gamma_\mu)} \right), \]
where \( c_{12}>0 \) is independent of \( f \) and \( F \).

**Proof.** Let \( D \) be a nonempty open domain, then by Hölder inequality, we have that
\[ \|g\|_{L^2(D)}^2 \leq \|g\|_{L^r(D, d\gamma_\mu)}^2 \left( \int_D \Gamma_\mu^{-\frac{2\tau_+}{q_1}} dx \right)^\frac{1}{q_1} |D|^\frac{1}{q_2}, \]
where our purpose is to find \( r>2 \) such that there exist \( q_1, q_2>1 \) satisfying that
\[ \left\{ \begin{array}{l}
\frac{2}{r} + \frac{1}{q_1} + \frac{1}{q_2} = 1, \\
-\frac{2\tau_+}{r} q_1 > -N, \\
\frac{1}{2q_2} > \frac{1}{2} - \frac{1}{N}.
\end{array} \right. \]  
(2.24)

Thanks to the fact \( \tau_+>0 \), when \( r>N+\tau_+ \), (2.21) holds for some \( q_1, q_2>1 \). Additionally, we note that for \( r>N+\tau_+ \),
\[ L^r(D, d\gamma_\mu) \subset L^2(D). \]

Thus, from (2.21), we have that
\[ \|S_k(u_0)\|_{L^1(\Omega)} \leq c_{13} \left( \|f\|_{L^2(\{|u_0|>k\})} + \|F\|_{L^2(\{|u_0|>k\})} \right) \{|u_0|>k\}^{1+\frac{1}{r} - \frac{1}{q}}, \]
\[ \leq c_{14} \left( \|f\|_{L^r(\Omega, d\gamma_\mu)} + \|F\|_{L^r(\Omega, d\gamma_\mu)} \right) \{|u_0|>k\}^{1+\frac{1}{r} + \frac{1}{q}}, \]
where $\frac{1}{2} + \frac{1}{N} + \frac{1}{q_2} > 1$. Then by Lemma 2.3 with $\alpha = \frac{1}{2} + \frac{1}{N} + \frac{1}{q_2} > 1$ for $r > N$, then it follows that
\[
\|u_0\|_{L^\infty(\Omega)} \leq c_{15} \left( \|f\|_{L^r(\Omega, d\gamma_\mu)} + \|F\|_{L^r(\Omega, d\gamma_\mu)} \right),
\]
which ends the proof. \hfill \Box

3 Proof of our main results

From the form of $\nu = \nu|_{\Omega^*} + k\delta_0$ and linearity, we have to consider the regularity of the solutions of (1.1) with $\nu = \nu|_{\Omega^*}$ and $\nu = k\delta_0$ respectively.

Let $v_0$ be a weak solution of
\[
\begin{cases}
    \mathcal{L}_\mu u = \delta_0 & \text{in } \Omega, \\
    u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
in the distributional sense that
\[
\int_\Omega v_0 \mathcal{L}_\mu^* \xi \, d\gamma_\mu(x) = c_\mu \xi(0) \quad \text{for all } \xi \in X_\mu(\Omega),
\]
where $\mathcal{L}_\mu^*$ is given by (1.6) and $c_\mu$ is the normalized constant.

**Proposition 3.1** Let $N \geq 2$, $\mu > \mu_0$ $p_\mu^*$ is given in (1.13) and $v_0$ be the very weak solution of (3.1).

Then (i) $v_0 \Gamma_\mu \in W^{1,q}_0(\Omega)$ with $q \in \left[1, \frac{N}{N-1}\right)$,
\[
\|v_0 \Gamma_\mu\|_{W^{1,q}(\Omega)} \leq c_{16};
\]
(ii) $v_0 \in W^{1,q}_0(\Omega, d\gamma_\mu)$ with $q \in \left[1, p_\mu^*\right)$,
\[
\|v_0\|_{W^{1,q}(\Omega, d\gamma_\mu)} \leq c_{17}.
\]

**Proof.** The existence of solution $v_0$ could see [11, Theorem 1.2].

Let $\eta_0 : [0, +\infty) \to [0, 1]$ be a decreasing $C^\infty$ function such that
\[
\eta_0 = 1 \quad \text{in } [0, 1] \quad \text{and} \quad \eta_0 = 0 \quad \text{in } [2, +\infty).
\]

Take $n_0 \geq 1$ such that
\[
\frac{1}{n_0} \sup\{r > 0 : B_r(0) \subset \Omega\} \leq \frac{1}{2}.
\]
Denote $\eta_{n_0}(r) = \eta_0(n_0 r)$ for $r \geq 0$, $w_1 = \Phi^\mu \eta_{n_0}$, then Direct computation shows that $w_1$ satisfies (3.3) and (3.4).

Note that
\[ \mathcal{L}_\mu w_1 = -\nabla \eta_0 \cdot \nabla \Phi^\mu - \Phi^\mu \Delta \eta_0 \quad \text{in } \Omega \setminus \{0\}, \quad (3.6) \]
where $-\nabla \eta_0 \cdot \nabla \Phi^\mu - \Phi^\mu \Delta \eta_0$ is smooth and has compact support in $B_{\frac{2}{n_0}}(0) \setminus B_{\frac{1}{n_0}}(0)$.

Let $w_2$ be a solution of
\[
\begin{align*}
\mathcal{L}_\mu w_2 &= -\nabla \eta_0 \cdot \nabla \Phi^\mu - \Phi^\mu \Delta \eta_0 \quad \text{in } \Omega \setminus \{0\}, \\
w_2 &= 0 \quad \text{on } \partial \Omega, \\
\lim_{x \to 0} w_i(x) \Phi^{-1}(x) &= 0.
\end{align*}
\]
then $w_2 \in H^{1, q}(\Omega) \cap C^2(\Omega \setminus \{0\})$ and
\[ \int_\Omega |\nabla w_2|^2 dx < c_{19}. \]
By comparison principle, we have that $|w_2| \leq c_{18} \Gamma_\mu$. As a consequence, $w_2$ satisfies (3.3) and (3.4). Note that $v_0 = w_1 - w_2$ and the proof is complete. □

**Lemma 3.1** Assume that $N \geq 3$, $\mu > \frac{2}{N} \mu_0$ and $\nu \in \mathcal{M}_+(\Omega^*; \Gamma_\mu)$. Let $v_1$ be the unique weak solution of $\mathcal{L}_\mu^* u = \nu$ in $\Omega$ under the zero Dirichlet boundary condition.

Then $v_1 \Gamma_\mu \in W^{1, q}(\Omega)$ with $q \in [1, \frac{N}{N-1})$ such that
\[ \|v_1 \Gamma_\mu\|_{W^{1, q}(\Omega)} \leq c_{20} \|\nu\|_{\mathcal{M}(\Omega; \Gamma_\mu)}, \quad (3.8) \]
where $c_{20} > 0$ is independent of $\nu$.

**Proof.** For every $\xi \in C_0^{1, 1}(\Omega)$, the $d\mu$-very weak solution $u$ of (1.4) satisfies the inequality
\[ \left|\int_\Omega u \mathcal{L}_\mu^* \xi \, d\gamma_\mu\right| = \left|\int_\Omega \xi \, d(\Gamma_\mu \nu)\right| \leq \|\nu\|_{\mathcal{M}(\Omega, \Gamma_\mu)} \|\xi\|_{L^\infty(\Omega)}, \quad (3.9) \]
where $d\gamma_\mu = \Gamma_\mu dx$ and $d(\Gamma_\mu \nu) = \Gamma d\nu$. 


For any $f \in C^1(\bar{\Omega})$, let $\xi$ be the solution of $(2.1)$ with $F = 0$. For $q \in (1, \frac{N}{N-1})$, $q' = \frac{q}{q-1} > N$ and by Proposition 2.2 it follows that

$$\|\xi\|_{L^\infty(\Omega)} \leq c_8 \|f\|_{L^{q'}(\Omega)}.$$ 

Therefore, we have that

$$\left| \int_{\Omega} u\Gamma_\mu f \, dx \right| \leq c_8 \|\nu\|_{M(\Omega, \Gamma_\mu)} \|f\|_{L^{q'}(\Omega)}.$$ 

Since $C^1(\bar{\Omega})$ is dense in $L^{q'}(\Omega)$, then Riesz representation theorem implies that $u\Gamma_\mu \in L^q(\Omega)$ and

$$\|u\Gamma_\mu\|_{L^q(\Omega)} \leq c_{21} \|\nu\|_{M(\Omega, \Gamma)}.$$ 

For $F \in C^2(\bar{\Omega}; \mathbb{R}^N)$, we may let $\xi$ be the solution of $(2.1)$ with $f = 0$. By Proposition 2.2 it follows that for $q \in (1, \frac{N}{N-1})$,

$$\|\xi\|_{L^\infty(\Omega)} \leq c_8 \|F\|_{L^{q'}(\Omega)}.$$ 

Henceforth, we have that

$$\left| \int_{\Omega} u\Gamma_\mu \text{div} F \, dx \right| \leq c_8 \|\nu\|_{M(\Omega, \Gamma_\mu)} \|F\|_{L^{q'}(\Omega)}.$$ 

Consequently, the functional

$$F \in C^2(\bar{\Omega}; \mathbb{R}^N) \mapsto \int_{\Omega} u\Gamma_\mu \text{div} F \, dx$$

admits a unique continuous linear extension in $L^{q'}(\Omega; \mathbb{R}^N)$. By the Riesz representation theorem, there exists a unique function $\tilde{u} \in L^q(\Omega; \mathbb{R}^N)$ such that for any $F \in L^{q'}(\Omega; \mathbb{R}^N)$,

$$\int_{\Omega} u\Gamma_\mu \text{div} F \, dx = \int_{\Omega} \tilde{u} \cdot F \, dx,$$

which implies that $\tilde{u} = \nabla(u\Gamma_\mu) \in L^q(\Omega; \mathbb{R}^N)$. Thus, $u\Gamma_\mu \in W^{1,q}_0(\Omega)$ and

$$\|u\Gamma_\mu\|_{W^{1,q}(\Omega)} \leq c_{22} \|\nu\|_{M(\Omega, \Gamma_\mu)}$$

for $q \in (1, \frac{N}{N-1})$. Theorem 1.1 with $q = 1$ follows by Hölder inequality. □
Lemma 3.2 Assume that \( N \geq 3, \mu > \frac{3}{4} \mu_0 \mu \neq 0, \nu \in \mathcal{R}_+ (\Omega^*; \Gamma_\mu) \) and \( p_\mu^* \) is given by (1.13). Let \( v_1 \) be the unique weak solution of (1.1)

Then \( v_1 \in W^{1,q}_0 (\Omega, d\gamma_\mu) \) with \( q \in [1, \min \{ p_\mu^*, \frac{N}{N-1} \}) \) such that

\[
\| v_1 \|_{W^{1,q}_0 (\Omega, d\gamma_\mu)} \leq c_{23} \| \nu \|_{\mathcal{R}_+ (\Omega^*; \Gamma_\mu)},
\]

(3.10)

where \( c_{23} > 0 \) is dependent of \( \nu \).

**Proof.** The case of \( \mu > 0 \). For any \( f \in C^1 (\bar{\Omega}) \), let \( \xi \) be the solution of (2.1) with \( F = 0 \). For \( q \in (1, p_\mu^*), q' = \frac{q}{q-1} > N + \tau_+ \) and by Proposition 2.3, it follows that

\[
\| \xi \|_{L^\infty (\Omega)} \leq c_8 \| f \|_{L^{q'} (\Omega, d\gamma_\mu)}.
\]

Therefore, (3.9) implies that

\[
\left| \int_{\Omega} u\Gamma_\mu f \, dx \right| \leq c_8 \| \nu \|_{\mathcal{M}(\Omega, \Gamma_\mu)} \| f \|_{L^{q'} (\Omega, d\gamma_\mu)}.
\]

Since \( C^1 (\bar{\Omega}) \) is dense in \( L^{q'} (\Omega) \), then Riesz representation theorem implies that \( u \in L^q (\Omega, d\gamma_\mu) \) and

\[
\| u \|_{L^q (\Omega, d\gamma_\mu)} \leq c_8 \| \nu \|_{\mathcal{M}(\Omega, \Gamma_\mu)}.
\]

For \( F \in C^2 (\bar{\Omega}; \mathbb{R}^N) \), we may let \( \xi \) be the solution of (2.1) with \( f = 0 \). By Proposition 2.3, it follows that for \( q \in (1, p_\mu^*) \),

\[
\| \xi \|_{L^\infty (\Omega)} \leq c_{17} \| F \|_{L^{q'} (\Omega, d\gamma_\mu)}.
\]

Henceforth, we have that

\[
\left| \int_{\Omega} u\Gamma_\mu \text{div} F \, dx \right| \leq c_{18} \| \nu \|_{\mathcal{M}(\Omega, \Gamma_\mu)} \| F \|_{L^{q'} (\Omega, d\gamma_\mu)}.
\]

Consequently, the functional

\[
F \in C^2 (\bar{\Omega}, d\gamma_\mu; \mathbb{R}^N) \mapsto \int_{\Omega} u\Gamma_\mu \text{div} F \, dx
\]

admits a unique continuous linear extension in \( L^{q'} (\Omega, d\gamma_\mu; \mathbb{R}^N) \). By the Riesz representation theorem, there exists a unique function \( \tilde{u} \in L^q (\Omega, d\gamma_\mu; \mathbb{R}^N) \) such that for any \( F \in L^{q'} (\Omega, d\gamma_\mu; \mathbb{R}^N) \),

\[
\int_{\Omega} u\Gamma_\mu \text{div} F \, dx = \int_{\Omega} \tilde{u} \cdot F \, dx = \int_{\Omega} (\gamma_\mu^{-\frac{1}{q}} \tilde{u}) \cdot (\gamma_\mu^{\frac{1}{q}} F) \, dx,
\]
which implies that $\Gamma_\mu \frac{1}{p} \nabla (\nu \Gamma_\mu) = \Gamma_\mu \frac{1}{p} \nabla u + \Gamma_\mu \frac{1}{p} u \frac{x}{|x|^2} \in L^q(\Omega; \mathbb{R}^N)$ and
\[
\|((\Gamma_\mu \frac{1}{p} \nabla u + \Gamma_\mu \frac{1}{p} u \frac{x}{|x|^2})\|_{L^q(\Omega)} \leq c_{23}\|\nu\|_{M(\Omega; \Gamma_\mu)}.
\]

**Claim 1:** For $1 \leq q < p_\mu^*$, we have that
\[
\|\Gamma_\mu \frac{1}{p} u \frac{x}{|x|^2}\|_{L^q(\Omega)} \leq c_{24}\|\nu\|_{M(\Omega; \Gamma_\mu)}.
\]

Thus, $\Gamma_\mu \nabla u = [(\Gamma_\mu \frac{1}{p} \nabla u + \Gamma_\mu \frac{1}{p} u \frac{x}{|x|^2}) - \Gamma_\mu \frac{1}{p} u \frac{x}{|x|^2}] \in L^q(\Omega; \mathbb{R}^N)$, i.e. $|\nabla u| \in L^q(\Omega, d\gamma_\mu)$.

We end the proof by showing Claim 1. In fact, by Lemma 3.1, we have that $u \Gamma_\mu \in W^{1,p}_0(\Omega)$ for any $p \in [1, N - 1)$, then by Sobolev Embedding theorem, for any $\sigma \in [1, N - 2)$, there holds that
\[
\|u \Gamma_\mu\|_{L^\sigma(\Omega)} \leq c_{25}\|\nu\|_{M(\Omega; \Gamma_\mu)}.
\]

We observe that for $1 \leq q < p_\mu^*$,
\[
\int_{\Omega} \left| \Gamma_\mu \frac{1}{p} u \frac{x}{|x|^2} \right|^q dx = \int_{\Omega} |x|^\tau q (1 - q) - q |x|^\tau q u dx \\
\leq (\int_{\Omega} |x|^{\tau q (1 - q) - q} dx)^{\frac{q}{q_1}} (\int_{\Omega} |x|^{\tau q (1 - q) - q} dx)^{\frac{1}{q_2}},
\]
where $q_2 = \frac{q_1}{q_1 - q}$ and we require that
\[
q_1 < \frac{N}{N - 2}
\]
and
\[
[\tau q (1 - q) - q] q_2 > -N.
\]

(3.12) could be equivalent to $Nq < [N - q - \tau q (q - 1)]q_1$, then by (3.11), we have that
\[
(N - 2)q < N - q - \tau q (q - 1),
\]
which holds for any $1 \leq q < p_\mu^*$. 

For the case $\mu \in \left(\frac{3}{4}\mu_0, 0\right)$, we note that
\[
\int_{\Omega} |\nabla u|^p \, d\gamma_{\mu} = \int_{\Omega} \Gamma_{\mu} |\nabla (\Gamma_{\mu}^{-1}(\Gamma_{\mu} u))|^p \, dx 
\leq 2^p \left( |\tau_+|^p \int_{\Omega} |x|^{(1-p)|\tau_+|} |\Gamma_{\mu} u|^p \, dx + \int_{\Omega} |x|^{(1-p)|\tau_+|} |\nabla (\Gamma_{\mu} u)|^p \right) 
\leq c_{26} \left( \int_{\Omega} |\Gamma_{\mu} u|^p \, dx + \int_{\Omega} |\nabla (\Gamma_{\mu} u)|^p \, dx \right)
\leq c_{27} \|\nu\|_{M^p(\Omega^{*}; \Gamma_{\mu})}^p,
\]
where we used $p \geq 1$ and $\tau_+ < 0$ for $\tau < 0$. We complete the proof. \hfill \Box

**Proof of Theorem 1.1** Part (i) follows directly by Proposition 3.1 and Lemma 3.1. Part (ii) does Proposition 3.1 and Lemma 3.2. \hfill \Box

**Proof of Corollary 1.1** Note that from definition 1.1, any weak solution $u$ of (1.8) has the property that $g(u) \in L^1(\Omega, d\gamma_{\mu})$. In fact, for any $w \in L^1(\Omega, d\gamma_{\mu})$, we have that
\[
\int_{\Omega^*} |w| \, d\gamma_{\mu} = \int_{\Omega} |w| \, d\gamma_{\mu} < +\infty,
\]
then $L^1(\Omega, d\gamma_{\mu}) \subset M(\Omega^*; \Gamma_{\mu})$ and then $\nu - g(u) \in \overline{M}(\Omega; \Gamma_{\mu})$. Apply Theorem 1.1 we obtain the global regularity. \hfill \Box

## 4 More global regularity: the Marcinkiewicz estimates

In this section, we append a bit discussion of global regularity for non-homogeneous Hardy problem (1.1). To this end, we recall the definition and basic properties of the Marcinkiewicz spaces.

**Definition 4.1** Let $\Omega \subset \mathbb{R}^N$ be a domain and $\nu$ be a positive Borel measure in $\Omega$. For $\kappa > 1$, $\kappa' = \kappa/(\kappa - 1)$ and $u \in L^1_{\text{loc}}(\Omega, d\nu)$, we set
\[
\|u\|_{M^\kappa(\Omega, d\nu)} = \inf \{c \in [0, \infty) : \int_E |u| \, d\nu \leq c \left( \int_E |u|^{\kappa'/(\kappa - 1)} \right)^{1/(\kappa - 1)} , \forall E \subset \Omega \text{ Borel set} \}
\]
and
\[
M^\kappa(\Omega, d\nu) = \{ u \in L^1_{\text{loc}}(\Omega, d\nu) : \|u\|_{M^\kappa(\Omega, d\nu)} < +\infty \}.
\]
Global $W^{1,p}$ regularity for Hardy problem

$M^\kappa(\Omega, d\nu)$ is called the Marcinkiewicz space with exponent $\kappa$ or weak $L^\kappa$ space and $\| \cdot \|_{M^\kappa(\Omega, d\nu)}$ is a quasi-norm. The following property holds.

**Proposition 4.1** Assume that $1 \leq q < \kappa < +\infty$ and $u \in L^1_{\text{loc}}(\Omega, d\nu)$. Then there exists $C(q, \kappa) > 0$ such that

$$\int_E |u|^q d\nu \leq C(q, \kappa) \| u \|_{M^\kappa(\Omega, d\nu)} \left( \int_E d\nu \right)^{1-q/\kappa}$$

for any Borel set $E$ of $\Omega$.

**Proposition 4.2** Let $\mu > 0$, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded $C^2$ domain and $G_{\mu}[\cdot](\cdot) = \int_\Omega G_{\mu}(\cdot, y) d\sigma(y)$ for $\sigma \in \mathcal{M}(\Omega, \Gamma_{\mu})$. Then there exists $c_{28} > 0$ such that

$$\| G_{\mu}[\sigma] \|_{M^q(\Omega, d\gamma_{\mu})} \leq c_{28} \| \sigma \|_{\mathcal{M}(\Omega, \Gamma_{\mu})},$$

where $1 < q \leq \frac{N+\tau_+\mu}{N-2+\tau_+\mu}$.

**Proof.** For $\lambda > 0$ and $y \in \Omega$, we denote

$$A_\lambda(y) = \{x \in \Omega \setminus \{y\} : G_{\mu}(x, y) > \lambda\} \quad \text{and} \quad m_\lambda(y) = \int_{A_\lambda(y)} d\gamma_{\mu}.$$

From [11, Lemma 4.1] it follows that for any $(x, y) \in (\Omega \setminus \{0\}) \times (\Omega \setminus \{0\})$,

$$G_{\mu}(x, y) \leq c_{29} \frac{\Gamma_{\mu}(y)}{|x-y|^{N-2+\tau_+\mu}}.$$

Observe that

$$A_\lambda(y) \subset \left\{x \in \Omega \setminus \{y\} : \frac{c_{29} \Gamma_{\mu}(y)}{|x-y|^{N-2+\tau_+\mu}} > \lambda\right\} = D_\lambda(y),$$

where $D_\lambda(y) = \left\{x \in \Omega : |x-y| < \left(\frac{c_{29} \Gamma_{\mu}(y)}{\lambda}\right)^{\frac{1}{N-2+\tau_+\mu}}\right\}$. Then we have that

$$m_\lambda(y) \leq \int_{D_\lambda(y)} d\gamma_{\mu} \leq \int_{D_\lambda(0)} d\gamma_{\mu} = c_{29} \left(\frac{c_{29} \Gamma_{\mu}(y)}{\lambda}\right)^{\frac{N+\tau_+\mu}{N-2+\tau_+\mu}}. \quad (4.15)$$
For any Borel set \( E \) of \( \Omega \), we have
\[
\int_E G_\mu(x,y)\Gamma_\mu(x)dx \leq \int_{A_\lambda(y)} G_\mu(x,y)\Gamma_\mu(x)dx + \lambda \int_E d\gamma_\mu.
\]
Using Fubini’s theorem, integration by parts formula and estimate \((4.15)\), we obtain
\[
\int_{A_\lambda(y)} G_\mu(x,y)d\gamma_\mu = -\int_\lambda^\infty sdm_s(y)ds
\]
\[
= \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y)ds
\]
\[
\leq c_30\Gamma_\mu^{N+\tau_+ (\mu)}(y)\lambda^{1-\frac{N+\tau_+ (\mu)}{N-2+\tau_+ (\mu)}}.
\]

Thus,
\[
\int_E G_\mu(x,y)d\gamma_\mu \leq c_30\Gamma_\mu^{N-2+\tau_+ (\mu)}(y)\lambda^{1-\frac{N+\tau_+ (\mu)}{N-2+\tau_+ (\mu)}} + \lambda \int_E d\gamma_\mu.
\]

By choosing \( \lambda = \Gamma_\mu(y)(\int_E d\gamma_\mu)^{\frac{N-2+\tau_+ (\mu)}{N+\tau_+ (\mu)}} \) and \( c_{31} = c_{30} + 1 \), we have
\[
\int_E G_\mu(x,y)d\gamma_\mu \leq c_{31}\Gamma_\mu(y)(\int_E d\gamma_\mu)^{\frac{2+\tau_+ (\mu)}{N+\tau_+ (\mu)}}.
\]

Therefore,
\[
\int_E G_\mu[|\sigma|](x)\Gamma_\mu(x)dx = \int_\Omega \int_E G_\mu(x,y)\Gamma_\mu(x)dxd|\sigma(y)|
\]
\[
\leq c_{31} \int_\Omega \Gamma_\mu(y)d|\sigma(y)| \left( \int_E d\gamma_\mu \right)^{\frac{2+\tau_+ (\mu)}{N+\tau_+ (\mu)}}
\]
\[
\leq c_{31} \|\sigma\|_{\mathcal{W}^2(\Omega,\Gamma_\mu)} \left( \int_E d\gamma_\mu \right)^{\frac{2+\tau_+ (\mu)}{N+\tau_+ (\mu)}}.
\]

As a consequence,
\[
\|G_\mu[\sigma]\|_{ M}^{\frac{N+\tau_+ (\mu)}{N-2+\tau_+ (\mu)}}(\Omega,d\gamma_\mu) \leq c_{31} \|\sigma\|_{\mathcal{W}^2(\Omega,\Gamma_\mu)}.
\]
We complete the proof.

We remark that the biggest defect of this Marcinkiewicz estimate in Proposition 4.2 is the restriction that $\mu > 0$. For $N \geq 3$ and $\mu_0 \leq \mu < 0$, a much weaker type of Marcinkiewicz estimate could be derived due to the Green kernel’s estimates, see [11, Lemma 4.1].

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