TENSOR PRODUCT DECOMPOSITIONS FOR COHOMOLOGIES OF BOTT-SAMELSON VARIETIES

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Abstract. Let $T$ be a maximal torus of a semisimple complex algebraic group, $BS(s)$ be the Bott-Samelson variety for a sequence of simple reflections $s$ and $BS(s)^T$ be the set of $T$-fixed points of $BS(s)$. We prove the tensor product decompositions for the image of the restriction $H^*_T(BS(s), \mathbb{k}) \to H^*_T(X, \mathbb{k})$, where $X \subset BS(s)^T$ is defined by some special not overlapping equations $\gamma_i \gamma_{i+1} \cdots \gamma_j = w_{i,j}$ with right-hand sides belonging to the Weyl group.

1. Introduction

Let $G$ be a semisimple complex algebraic group. We fix a Borel subgroup $B \leq G$ and a maximal torus $T \leq B$ and denote by $W$ the corresponding Weyl group. For a sequence of simple reflections $s = (s_1, \ldots, s_n)$, we consider the Bott-Samelson variety

$$BS(s) = P_{s_1} \times \cdots \times P_{s_n}/B^n$$

where $P_s = B \cup BsB$ is the minimal parabolic subgroup. The torus $T$ acts on $BS(s)$ via the first component. So we can consider the set of $T$-fixed points $BS(s)^T$, which can be identified with the set of combinatorial galleries

$$\Gamma(s) = \{(\gamma_1, \ldots, \gamma_n) \in W^n \mid \forall i : \gamma_i = 1 \text{ or } \gamma_i = s_i\}$$

Let $\mathbb{k}$ be a principal ideal domain with invertible 2. Then any restriction $H^*_T(BS(s), \mathbb{k}) \to H^*_T(\Gamma(s), \mathbb{k})$ is injective.

Suppose that $X$ is a subset of $\Gamma(s)$. We would like to know what is the image of the restriction

$$H^*_T(BS(s), \mathbb{k}) \to H^*_T(X, \mathbb{k}). \quad (1)$$

This question stated in this generality does not have an answer. However, the answer is know in the following two cases: $X = \Gamma(s)$ and $X = \{(\gamma_1, \ldots, \gamma_n) \in \Gamma(s) \mid \gamma_1 \cdots \gamma_n = w\}$ for some $w \in W$. We denote the image of (1) by $\mathcal{X}(s)$ in the first case and by $\mathcal{X}(s, w)$ in the second case. Note that $\mathcal{X}(s) \cong H^*_T(BS(s), \mathbb{k})$ and $\mathcal{X}(s, w) \cong H^*_T(\pi(s)^{-1}(wB), \mathbb{k})$, where $\pi(s) : BS(s) \to G/B$ is the map $(g_1, \ldots, g_n)B^n \mapsto g_1 \cdots g_nB$. The description of $\mathcal{X}(s)$ and $\mathcal{X}(s, w)$ by congruences is given by M. Härterich [1]. See also [2] for the coefficients different from fields of characteristic zero.

Instead of a single restriction $\gamma_1 \cdots \gamma_n = w$ in the definition of $X$, we can try to impose multiple restrictions of this form: we take some indexing set of pairs $R \subset \{(r_1, r_2) \subset \mathbb{Z}^2 \mid 1 \leq r_1 \leq r_2 \leq n\}$ and a map $v : R \to W$ and consider the set

$$X = \{(\gamma_1, \ldots, \gamma_n) \in \Gamma(s) \mid \forall r \in R \gamma_{r_1} \cdots \gamma_{r_2} = v_r\}.$$  

Let $\mathcal{X}(s, v)$ denote the image of the restriction $H^*_T(BS(s), \mathbb{k}) \to H^*_T(X, \mathbb{k})$. In this paper, we consider only the case where the imposed restrictions do not overlap, that is, the case $r_1 \leq r_1' \leq r_2 \leq r_2'$ is impossible for distinct $r, r' \in R$.

The main results of this paper are Theorems [16] and [17]. They assert that under some restrictions on $(s, v)$, the module $\mathcal{X}(s, v)$ can be decomposed into a tensor product of some modules $\mathcal{X}(t)$ and $\mathcal{X}(t, w)$ for the corresponding sequences $t$ and elements $w \in W$. Moreover,
this decomposition is given by the exact formula, see (44). The restrictions imposed on 
\((s, v)\) claim that certain sequences of reflections associated to this pair be of gallery type, 
see Definitions 1 and 3.

Perhaps the main application of this result is the possibility to construct elements of 
the dual module of \(X(s, v)\) from the elements of the dual modules of the factors \(X(t)\) and 
\(X(t, w)\) entering into the tensor product decomposition, see Corollaries 18 and 19.

The paper is organized as follows. In Section 2 we fix the notation for simisimple com-
plex algebraic groups, their compact subgroups, Bott-Samelson varieties \(\text{BS}(s)\) and their 
compactly defined variants \(\text{BS}_c(s)\). To avoid renumeration, we consider the sequences as 
maps defined on totally ordered sets, see Section 2.2.

In Section 3 we recall the main constructions from [S3] concerning nested structures, 
the subspaces \(\text{BS}_c(s, v)\) of \(\text{BS}_c(s)\) and the corresponding fibre bundles. In Section 4.1 we 
consider equivariant cohomologies. The main technical tool here is the Stiefel manifolds, 
see Section 4.2 for the definitions. In Section 4.3 we also consider the twisted actions of the 
equivariant cohomology of the point on \(H_k^*(\text{BS}_c(s), \mathbb{k})\), where \(K\) is a maximal compact torus.

These actions become important later in Section 5.6 where we consider tensor products of 
bimodules.

Section 5 is devoted to the proof of Theorem 11 about the tensor product decomposition 
for \(K\)-equivariant cohomologies of the spaces \(\text{BS}_c(s, v)\). Here we use the same approach 
as in [S3], that is, first we embed Borel constructions into products of Borel constructions 
(Lemma 7) and then compute the cohomology of the difference to assure the surjectivity of 
restriction (Lemma 8). In this case, we also use fibre bundles for differences (Lemma 9), as 
we did in [S3] Lemma 22]. However in the present case, we need to consider the additional 
fibre bundle for the composition, see part (2) of Lemma 8.

Finally, in Section 6 we come back to the \(T\)-equivariant cohomologies of \(\text{BS}(s)\). We 
consider the images \(X(s, v)\) of restriction (1). Theorems 16 and 17 provide their decompo-
sitions into tensor products. Based on these results, we construct a map for dual modules in 
Section 6.5.

In Section 6.6 we give an example. It actually states that

\[H_T^*(\pi(t)^{-1}(\omega_3\omega_4\omega_3 B), \mathbb{k}) \otimes \mathbb{k} \cong H_T^*(\pi(s)^{-1}(\omega_1\omega_3\omega_4 B), \mathbb{k}),\]

where \(\omega_1, \omega_2, \omega_3, \omega_4\) are the simple roots for \(G = \text{GL}_3(\mathbb{C})\), \(t = (\omega_4, \omega_3, \omega_4, \omega_3, \omega_4)\), \(r = (\omega_2, \omega_1, \omega_2, \omega_1, \omega_2)\) and \(s = (\omega_4, \omega_3, \omega_4, \omega_3, \omega_4, \omega_3, \omega_4)\). The minimal degree element 
of the dual module for the right hand-side is a nontrivially twisted product of the minimal 
degree elements of the dual modules for the cohomologies of the left-hand side.

As in our previous paper [S3], we consider here only the sheaf cohomology. We use some 
standard notation like \(\mathbb{k}_X\) for the constant sheaf on a topological space \(X\), \(\text{GL}(V)\) for the 
group of linear isomorphisms \(V \rightarrow V\), \(M_{m,n}(\mathbb{k})\) for the set of \(m \times n\)-matrices with entries 
in \(\mathbb{k}\) and \(U(n)\) for the group of unitary \(n \times n\)-matrices. We also apply spectral sequences 
associated with fibre bundles. All of them are first quadrant sequences.

Also note that although principal ideal domains with invertible 2 are our main rings of 
coefficients, we impose on them as little restrictions as we can in order to prove each separate 
result about cohomologies.

## 2. Bott-Samelson varieties

### 2.1. Compact subgroups of complex algebraic groups

Let \(G\) be a semisimple complex group with root system \(\Phi\). We assume that \(\Phi\) is defined with respect to the Euclidian space \(E\) with the scalar product \((\cdot, \cdot)\). This scalar product determines a metric on \(E\) and therefore a topology. We will denote by \(\overline{A}\) the closure of a subset \(A \subset E\).
We consider $G$ as a Chevalley group generated by the root elements $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathbb{C}$ (see [St]). We also fix the following elements of $G$:

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \quad \omega_\alpha = w_\alpha(1), \quad h_\alpha(t) = w_\alpha(t)\omega_\alpha^{-1}.$$ 

For any $\alpha \in \Phi$, we denote by $s_\alpha$ the reflection of $E$ through the hyperplane $L_\alpha$ of the vectors perpendicular to $\alpha$. We call $L_\alpha$ a wall perpendicular to $\alpha$. These reflections generate the Weyl group $W$. We choose a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots and write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$). Let $\Pi \subset \Phi^+$ be the set of simple roots corresponding to this decomposition. We denote

$$T(W) = \{ s_\alpha \ | \ \alpha \in \Phi \}, \quad S(W) = \{ s_\alpha \ | \ \alpha \in \Pi \}.$$ 

and call these sets the set of reflections and the set of simple reflections respectively. We use the standard notation $(\alpha, \beta) = 2(\alpha, \beta)/(\beta, \beta)$. There is the analytic automorphism $\sigma$ of $G$ such that $\sigma(x_\alpha(t)) = x_{-\alpha}(-t)$ [St, Theorem 16]. Here and in what follows $\overline{t}$ denotes the complex conjugate of $t$. We denote by $C = G_\sigma$ the set of fixed points of this automorphism. We also consider the compact torus $K = T_\sigma = B_\sigma$, where $T$ is the subgroup of $G$ generated by all $h_\alpha(t)$ and $B$ is the subgroup generated by $T$ and all root elements $x_\alpha(t)$ with $\alpha > 0$.

For $\alpha \in \Phi$, we denote by $G_\alpha$ the subgroup of $G$ generated by all root elements $x_\alpha(t)$ and $x_{-\alpha}(t)$ with $t \in \mathbb{C}$. There exists the homomorphism $\varphi_\alpha : \text{SL}_2(\mathbb{C}) \to G$ such that

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t).$$

Hence, we get

$$C \cap G_\alpha = \varphi_\alpha(\text{SU}_2).$$

Let $N$ be the subgroup of $G$ generated by the elements $\omega_\alpha$ and the torus $K$. Clearly $N \subset C$. The arguments of [St, Lemma 22] prove that there exists an isomorphism $\varphi : W \cong \text{SU}_2/K$ given by $s_\alpha \mapsto \omega_\alpha K$. We choose once and for all, a representative $\hat{w} \in \varphi(w)$ for any $w \in W$. Abusing notation, we denote $wK = \hat{w}K = \varphi(w)$.

Now consider the product $C_\alpha = \varphi_\alpha(\text{SU}_2)K$. This is a compact and closed subgroup of $G$. Moreover, $C_\alpha = C_{-\alpha}$. So we can denote $C_{s_\alpha} = C_\alpha$. For any $w \in W$, we have $\hat{w}C_{s_\alpha}\hat{w}^{-1} = C_{ws_\alpha w^{-1}} = C_{w\alpha}$.

\section*{2.2. Sequences and products.} Let $I$ be a finite totally ordered set. We denote by $\leq (, > , etc.)$ the order on it. We also add two additional elements $-\infty$ and $+\infty$ with the natural properties: $-\infty < i$ and $i < +\infty$ for any $i \in I$ and $-\infty < +\infty$. For any $i \in I$, we denote by $i + 1$ (resp. $i - 1$) the next (resp. the previous) element of $I \cup \{-\infty, +\infty\}$. We write $\text{min} I$ and $\text{max} I$ for the minimal and the maximal elements of $I$ respectively and denote $I' = I \setminus \{ \text{max} I \}$ if $I \neq \emptyset$. Note that $\text{min} \emptyset = -\infty$ and $\text{max} \emptyset = +\infty$. For $i, j \in I$, we set $[i, j] = \{ k \in I \mid i \leq k \leq j \}$.

Any map $s$ from $I$ to an arbitrary set is called a sequence on $I$. We denote by $|s|$ the cardinality $|I|$ of $I$ (we will use this notation for any finite set) and call this number the length of $s$. For any $i \in I$, we denote by $s_i$ the value of $s$ on $i$. Finite sequences in the usual sense are sequences on the initial intervals of the natural numbers, that is, on the sets $\{1, 2, \ldots, n\}$. For any sequence $s$ on a nonempty set $I$, we denote by $s' = s|_{I'}$ its truncation.

We often use the following notation for Cartesian products:

$$\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \cdots \times X_n.$$
and denote by $p_{i_1i_2...i_m}$ the projection of this product to $X_{i_1} \times X_{i_2} \times \cdots \times X_{i_m}$. Similar notation is used for tensor products over a commutative ring $\mathbb{k}$:

$$\bigotimes_{k} M_k = M_{i_1} \otimes_\mathbb{k} \cdots \otimes_\mathbb{k} M_{i_n},$$

where $I = \{i_1, \ldots, i_n\}$, the elements $i_1, \ldots, i_n$ are distinct and $M_{i_1}, \ldots, M_{i_n}$ are $\mathbb{k}$-modules either left of right, which is not important as we consider all such modules automatically as $\mathbb{k}$-$\mathbb{k}$-bimodules.

If we have maps $f_i: Y \to X_i$ for all $i = 1, \ldots, n$, then we denote by $f_1 \boxtimes \cdots \boxtimes f_n$ the map that takes $y \in Y$ to the $n$-tuple $(f_1(y), \ldots, f_n(y))$. To avoid too many superscripts, we denote

$$X(I) = \prod_{i \in I} X.$$

We consider all such products with respect to the product topology if all factors are topological spaces. Suppose that $X$ is a group. Then $X(I)$ is a topological group with respect to the componentwise multiplication. In that case, for any $i \in I$ and $x \in X$, we consider the indicator sequence $\delta_i(x) \in X(I)$ defined by

$$\delta_i(x)_j = \begin{cases} x & \text{if } j = i; \\ 1 & \text{otherwise} \end{cases}$$

Let us additionally assume that $I$ is embedded into a totally ordered set $J$. Then for any $j \in J \cup \{-\infty\}$ and a sequence $\gamma$ on $I$, we will use the notation $\gamma^j = \gamma_{\min}^{\min(I)} \gamma_{\min(I+1)} \cdots \gamma_{\min}^{I}$, where $i$ is the maximal element of $I$ less than or equal to $j$ or $-\infty$ if there is no such element. Obviously $\gamma^{-\infty} = 1$. We set $\gamma^{\max} = \gamma^{\max(I)}$.

We will also use the following notation. Let $G$ be a group. If $G$ acts on the set $X$, then we denote by $X/G$ the set of $G$-orbits of elements of $X$. Suppose that $\varphi : X \to Y$ is a $G$-equivariant map. Then $\varphi$ maps any $G$-orbit to a $G$-orbit. We denote the corresponding map from $X/G$ to $Y/G$ by $\varphi/G$.

2.3. Galleries. As $L_{-\alpha} = L_\alpha$ for any root $\alpha \in \Phi$, we can denote $L_{s_\alpha} = L_\alpha$. For any $w \in W$, we get $wL_{s_\alpha} = wL_\alpha = L_{w_\alpha} = L_{s_{w_\alpha}} = L_{w_{s_\alpha}w^{-1}}$.

A chamber is a connected component of the space $E \setminus \bigcup_{\alpha \in \Phi} L_\alpha$. We denote by $\text{Ch}$ the set of chambers and by $L$ the set of walls $L_{s_\alpha}$. We say that a chamber $\Delta$ is attached to a wall $L$ if the intersection $\Delta \cap L$ has dimension $\dim E - 1$. The fundamental chamber is defined by

$$\Delta_+ = \{ v \in E \mid (v, \alpha) > 0 \text{ for any } \alpha \in \Pi \}.$$ 

A labelled gallery on a finite totally ordered set $I$ is a pair of maps $(\Delta, \mathcal{L})$, where $\Delta : I \cup \{-\infty\} \to \text{Ch}$ and $\mathcal{L} : I \to L$ are such that for any $i \in I$, both chambers $\Delta_{i-1}$ and $\Delta_i$ are attached to $\mathcal{L}_i$. The Weyl group $W$ acts on the set of labelled galleries by the rule $w(\Delta, \mathcal{L}) = (w\Delta, w\mathcal{L})$, where $(w\Delta)_i = w\Delta_i$ and $(w\mathcal{L})_i = w\mathcal{L}_i$.

For a sequence $s : I \to \mathcal{T}(W)$, we define the set of generalized combinatorial galleries

$$\Gamma(s) = \{ \gamma : I \to W \mid \gamma_i = s_i \text{ or } \gamma_i = 1 \text{ for any } i \in I \}.$$ 

If $s$ maps $I$ to $\mathcal{S}(W)$, then the elements of $\Gamma(s)$ are called combinatorial galleries. To any combinatorial gallery $\gamma \in \Gamma(s)$ there corresponds the following labelled gallery:

$$i \mapsto \gamma^i \Delta_+, \quad i \mapsto L_{\gamma_i(s_i(\gamma))^{-1}}.$$ 

Moreover, any labelled gallery $(\Delta, \mathcal{L})$ on $I$ such that $\Delta_{-\infty} = \Delta_+$ can be obtained this way (for the corresponding $s$ and $\gamma$).
Moreover, this composition is associative: for any δ ∈ Γ(s^{(γ)}) be another generalized combinatorial gallery. Then we define γ ◦ λ ∈ Γ(s) by

\[(γ ◦ λ)^i = (γ)^i \lambda_i^{-1}.\]

It is easy to check that \((γ ◦ λ)^i = λ^i γ^i.\) Hence

\[s^{(γ)} \circ λ = s^{(λ ◦ γ)}.\]

Moreover, this composition is associative: for any δ ∈ Γ(s^{(λ ◦ γ)}), we have

\[γ ◦ λ ◦ δ = γ ◦ (λ ◦ δ).\]

We also get γ ◦ ε = γ and ε ◦ λ = λ, where ε is the map \(I \rightarrow \{1\}.\) The inverse of γ is γ⁻¹ ∈ Γ(s^{(γ)}) given by

\[(γ⁻¹)^i = γ^i⁻¹.\]

We have \((γ⁻¹)^i = (γ^i⁻¹).\) Applying this fact, it is easy to prove that γ ◦ γ⁻¹ = γ⁻¹ ◦ γ = ε and \((γ⁻¹)^{-1} = γ.\) Finally, note that \(s^{(ε)} = s.\)

Let \(w ∈ W\) and \(s : I \rightarrow T(W).\) Then we define the sequence \(s^w : I \rightarrow T(W)\) by \((s^w)^i = ws_i w⁻¹.\) For any γ ∈ Γ(s), we denote by γ^w the generalized combinatorial gallery of Γ(s^w) defined by \(γ^w = wγ_i w⁻¹.\)

**Definition 1.** A sequence \(s : I \rightarrow T(W)\) is of gallery type if there exists a labelled gallery \((Δ, L)\) on \(I\) such that \(L_i = L_{si}\) for any \(i ∈ I.\)

In this case, there exist an element \(x ∈ W,\) a sequence of simple reflections \(t : I \rightarrow S(W)\) and a combinatorial gallery γ ∈ Γ(t) such that \(xΔ_{−∞} = Δ_{−}\) and \(x(Δ, L)\) corresponds to γ by (2). We call \((x, t, γ)\) a gallerification of \(s.\) This fact is equivalent to \(t^{(γ)} = s^x.\)

Note that for any \(w ∈ W\) a sequence \(s\) is of gallery type if and only if the sequence \(s^w\) is of gallery type.

**Remark.** If \(Φ\) is of type \(A_1\) or \(A_2,\) then every sequence \(s : I \rightarrow T(W)\) is of gallery type \([S3\) Example].

### 2.4. Definition via the Borel subgroup

For any simple reflection \(t ∈ S(W),\) we consider the minimal parabolic subgroup \(P_t = B \cup BtB.\) Note that \(C_t\) is a maximal compact subgroups of \(P_t\) and \(C_t = P_t \cap C.\) Let \(s : I \rightarrow S(W)\) be a sequence of simple reflections. We consider the space

\[P(s) = \prod_{i ∈ I} P_{si}\]

with respect to the product topology. The group \(B(I)\) acts on \(P(s)\) on the right by \((pb)^i = b_i⁻¹ p_i b_i.\) Here and in what follows, we assume \(b_{−∞} = 1.\) Let

\[BS(s) = P(s)/B(I).\]

This space is compact and Hausdorff. The Borel subgroup \(B\) acts continuously on \(P(s)\) by

\[(bp)^i = \begin{cases} bp_i & \text{if } i = \min I; \\ p_i & \text{otherwise}. \end{cases}\]

As this action commutes with the right action of \(B(I)\) on \(P(s)\) described above, we get the left action of \(B\) on \(BS(s).\)
We also have the map \( \pi(s) : \text{BS}(s) \to G/B \) that maps \( pB(I) \) to \( p^{\text{max}}B \). This map is invariant under the left action of \( B \). However, we need here only the action of the torus \( T \). We have \( \text{BS}(s)^T = \{ \gamma B(I) \mid \gamma \in \Gamma(s) \} \), where \( \gamma \) is identified with the sequence \( i \mapsto \gamma_i \) of \( P(s) \). We will identify \( \text{BS}(s)^T \) with \( \Gamma(s) \). For any \( w \in \Gamma(s) \), we set \( \text{BS}(s,w) = \pi(s)^{-1}(wB) \).

2.5. **Definition via the compact torus.** Let \( s : I \to T(W) \) be a sequence of reflections. We consider the space

\[
C(s) = \prod_{i \in I} C_{s_i}
\]

with respect to the product topology. The group \( K(I) \) acts on \( C(s) \) on the right by \( (ck)_i = k_i^{-1}c_ik_i \). Here and in what follows, we assume \( k_{-\infty} = 1 \). Let

\[
\text{BS}_c(s) = C(s)/K(I).
\]

We denote by \([c]\) the right orbit \( cK(I) \) of \( c \in C(s) \). The space \( \text{BS}_c(s) \) is compact and Hausdorff.

The action of \( K(I) \) on \( C(s) \) described above commutes with the following left action of \( K \):

\[
(ck)_i = \begin{cases} 
  kc_i & \text{if } i = \min I; \\
  c_i & \text{otherwise}.
\end{cases}
\]

Therefore, \( K \) acts continuously on \( \text{BS}_c(s) \) on the left: \( k[c] = [kc] \).

We also have the map \( \pi_c(s) : \text{BS}_c(s) \to C/K \) that maps \([c]\) to \( c^{\max} K \). This map is obviously invariant under the left action of \( K \). We have \( \text{BS}_c(s)^K = \{ [\gamma] \mid \gamma \in \Gamma(s) \} \), where \( \gamma \) is identified with the sequence \( i \mapsto \gamma_i \) of \( C(s) \). We also identify \( \text{BS}_c(s)^K \) with \( \Gamma(s) \). For any \( w \in W \), we set \( \text{BS}_c(s,w) = \pi_c(s)^{-1}(wK) \). The set of \( K \)-fixed points of this space is

\[
\text{BS}_c(s,w)^K = \{ \gamma \in \Gamma(s) \mid \gamma^{max} = w \} = \Gamma(s,w).
\]

For any \( w \in W \), let \( l_w : C/K \to C/K \) and \( r_w : C/K \to C/K \) be the maps given by \( l_w(cK) = \dot{w}cK \) and \( r_w(cK) = cwK \). They are obviously well-defined and \( l_w^{-1} = l_{w^{-1}} \). Moreover, \( l_w \) and \( r_w \) commute for any \( w,w' \in W \). Let \( d_w : \text{BS}_c(s) \to \text{BS}_c(s^w) \) be the map given by \( d_w([c]) = [c^w] \), where \( (c^w)_i = \dot{w}c_i\dot{w}^{-1} \). This map is well-defined and a homeomorphism. Moreover, \( d_w(ka) = \dot{w}k\dot{w}^{-1}a \) for any \( k \in K \) and \( a \in \text{BS}_c(s) \). We have the commutative diagram

\[
\begin{array}{ccc}
\text{BS}_c(s) & \xrightarrow{d_w} & \text{BS}_c(s^w) \\
\pi_c(s) \downarrow & & \downarrow \pi_c(s^w) \\
C/K & \xrightarrow{l_w r_w^{-1}} & C/K \\
\end{array}
\]

Hence for any \( x \in W \), the map \( d_w \) yields the isomorphism

\[
\text{BS}_c(s,x) \cong \text{BS}(s^w,wxw^{-1}). \tag{3}
\]

There is a similar construction for a generalized combinatorial gallery \( \gamma \in \Gamma(s) \). We define the map \( D_\gamma : \text{BS}_c(s) \to \text{BS}_c(s^{[\gamma]}) \) by

\[
D_\gamma([c]) = [c^{[\gamma]}], \quad \text{where } c^{[\gamma]}_i = (\dot{\gamma}_{i-1}\dot{\gamma}_{i-2}\cdots\dot{\gamma}_{\min I}c\dot{\gamma}_{i-1}\cdots\dot{\gamma}_{\min I})^{-1}c_ig_{\dot{\gamma}_{i-1}}\cdots\dot{\gamma}_{\min I}.
\]

This map is well-defined. Indeed let \( [c] = [\tilde{c}] \) for some \( c,\tilde{c} \in C(s) \). Then \( \tilde{c} = ck \) for some \( k \in K(I) \). Then \( \tilde{c}^{[\gamma]} = c^{[\gamma]}k' \), where \( k'_i = (\dot{\gamma}_{i-1}\cdots\dot{\gamma}_{\min I})^{-1}k_i\dot{\gamma}_{i-1}\cdots\dot{\gamma}_{\min I} \). The map
Note that BS is commutative diagram:

\[
\begin{array}{ccc}
BS_c(s) & \overset{\sim}{\longrightarrow} & BS_c(s^{(y)}) \\
\pi_c(s) \downarrow & & \downarrow \pi_c(s^{(y)}) \\
C/K & \overset{r_{\gamma}^{-1}\max I}{\sim} & C/K
\end{array}
\]

Hence for any \( x \in W \), the map \( D_x \) yields the isomorphism

\[
BS_c(s, x) \cong BS_c(s^{(y)}, x(\gamma^{\max})^{-1}).
\]  

2.6. Isomorphism of two constructions. Consider the Iwasawa decomposition \( G = CB \) (see, for example, [St Theorem 16]). Hence for any simple reflection \( s : I \rightarrow S(W) \), the natural embedding \( C(s) \subset P(s) \) induces the homeomorphism \( BS_c(s) \sim \rightarrow BS_c(s) \). This homeomorphism is clearly \( K \)-equivariant.

Moreover, the following diagram is commutative:

\[
\begin{array}{ccc}
BS_c(s) & \overset{\sim}{\longrightarrow} & BS(s) \\
\downarrow & & \downarrow \\
C/K & \overset{\sim}{\longrightarrow} & G/B
\end{array}
\]

Note that \( BS_c(s)K \) is mapped to \( BS(s)K = BS(s)^T \). These sets are identified with \( \Gamma(s) \).

Any isomorphism of finite totally ordered set \( \iota : J \rightarrow I \) induces the obvious homeomorphisms \( BS(s) \sim \rightarrow BS(s\iota) \) and \( BS_c(s) \sim \rightarrow BS_c(s\iota) \). They obviously respect the isomorphisms described above.

3. Nested fibre bundles

3.1. Nested structures. Here we recall the main constructions of [St]. Let \( I \) be a finite totally ordered set. We denote the order on it by \( \leq \) (respectively, \( \geq, < \), etc.). Let \( R \) be a subset of \( I^2 \). For any \( r \in R \), we denote by \( r_1 \) and \( r_2 \) the first and the second component of \( r \) respectively. So we have \( r = (r_1, r_2) \). We will also use the notation \( [r] = [r_1, r_2] \) for the intervals. We say that \( R \) is a nested structure on \( I \) if

- \( r_1 \leq r_2 \) for any \( r \in R \);
- \( \{r_1, r_2\} \cap \{r'_1, r'_2\} = \emptyset \) for any \( r, r' \in R \);
- for any \( r, r' \in R \), the intervals \([r]\) and \([r']\) are either disjoint or one of them is contained in the other.

Abusing notation, we will write

- \( r \subseteq r' \) (resp. \( r \not\subseteq r' \)) to say that \([r] \subset [r'] \) (resp. \([r] \not\subset [r'] \));
- \( r < r' \) to say that \( r_2 < r'_2 \);

Let \( s : I \rightarrow T(W) \) and \( v : R \rightarrow W \) be arbitrary maps. We define

\[
BS_c(s, v) = \{ [c] \in BS_c(s) \mid \forall r \in R : c_{r_1} c_{r_1+1} \cdots c_{r_2} \in v_r K \}.
\]  

This set is well-defined, as \( \hat{v}_r K = K \hat{v}_r \) and is closed in \( BS_c(s) \). Consider the maps \( \xi_r : C(s) \rightarrow C/K \) defined by \( \xi_r(c) = c_{r_1} c_{r_1+1} \cdots c_{r_2} K \). Then \( BS_c(s, v) \) is the image of the intersections

\[
C(s, v) = \bigcap_{r \in R} \xi_r^{-1}(v_r K)
\]
under the quotient map $C(s) \to \textrm{BS}_c(s)$.

Let $F \subset R$ be a nonempty subset such that the intervals $[f]$, where $f \in F$, are pairwise disjoint. In what follows, we use the notation

$$F = \{f^1, \ldots, f^n\}$$

with elements written in the increasing order: $f^1 < f^2 < \cdots < f^n$. We set

$$I^F = I \setminus \bigcup_{f \in F} [f], \quad R^F = R \setminus \{r \in R \mid \exists f \in F : r \subset f\}.$$

Obviously, $R^F$ is a nested structure on $I^F$.

We also introduce the following auxiliary notation. Let $i \in I^F$. We choose $m = 0, \ldots, n$ so that $f^2_m < i < f^m_{m+1}$. In these inequalities and in what follows, we assume that $f^0_0 = -\infty$ and $f^n_{n+1} = +\infty$. We set

$$v^i = v_{f^1}v_{f^2} \cdots v_{f^m}, \quad \dot{v}^i = \dot{v}_{f^1}\dot{v}_{f^2} \cdots \dot{v}_{f^m}.$$

Clearly, $v^i$ is image of $\dot{v}^i$ under the quotient homomorphism $\mathcal{N} \to W$. We define the sequences $s^F : I^F \to \mathcal{T}(W)$ and $v^F : R^F \to W$ by

$$s^F_i = v^i s_i(v^i)^{-1}, \quad v^F_i = v^{r^1} v_{r^2}(v^{r^2})^{-1}.$$

**Definition 2.** A sequence $c \in C(s, v)$ is called $F$-balanced if

$$c_{f^1}c_{f^1+1} \cdots c_{f^2} = \dot{v}^i$$

for any $f \in F$.

It is easy to prove that for any $a \in \textrm{BS}_c(s, v)$, there exists some $F$-balanced $c \in C(s, v)$ such that $a = [c]$. For an $F$-balanced $c \in C(s, v)$, we define the sequence $c^F \in C(s^F)$ by

$$c^F_i = \dot{v}^i c_i(\dot{v}^i)^{-1}.$$

Then we set $p^F([c]) = [c^F]$. By [53, Lemma 4], the map $p^F$ is a well-defined, $K$-equivariant and continuous map from $\textrm{BS}_c(s, v)$ to $\textrm{BS}_c(s^F, v^F)$. We call it the projection along $F$.

A nested structure $R$ on $I$ is called closed if $I \neq \emptyset$ and it contains the pair $(\min I, \max I)$, which we denote by span $I$.

Let $f \in F$. For any sequence $\gamma$ on $I$, we denote by $\gamma_f$ the restriction of $\gamma$ to $[f]$. We also set

$$R_f = \{r \in R \mid r \subset f\}, \quad v_f = v|_{R_f}.$$  

From this construction, it is obvious that $R_f$ is a closed nested structure on $[f]$. By [53, Theorem 7], the map $p^F : \textrm{BS}_c(s, v) \to \textrm{BS}_c(s^F, v^F)$ is a fibre bundle with fibre $\textrm{BS}_c(s^f, v^f) \times \cdots \times \textrm{BS}_c(s^m, v^m)$.

### 3.2. Affine pavings

We say that a topological space $X$ has an affine paving if there exists a filtration

$$\emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-1} \subset X_n = X$$

such that all $X_i$ are closed and each difference $X_i \setminus X_{i-1}$ is homeomorphic to $\mathbb{C}^{m_i}$ for some integer $m_i$.

Let $r \in R$ and $r^1, \ldots, r^q$ be the maximal (with respect to the inclusion) elements of those elements of $R$ that are strictly contained in $r$. We write them in the increasing order $r^1 < \cdots < r^q$. We set

$$I(r, R) = [r] \setminus \bigcup_{r' \in R \setminus r} [r'] = [r] \setminus \bigcup_{m=1}^q [r^m].$$
We define the sequence \( s^{(r,v)} : I(r, R) \to T(W) \) by

\[
s^{(r,v)}_i = v_{r_1} \cdots v_{r_m} s_i(v_{r_1} \cdots v_{r_m})^{-1},
\]

where \( r_2^m < i < r_1^{m+1} \). Here we suppose as usually that \( r_2^0 = -\infty \) and \( r_1^{q+1} = +\infty \).

**Definition 3.** Let \( s \) be a sequence of reflections on \( I \) and \( v \) be a map from a nested structure \( R \) on \( I \) to \( W \). We say that the pair \( (s, v) \) is of gallery type if for any \( r \in R \), the sequence \( s^{(r,v)} \) is of gallery type.

The following results are Lemma 11 and Corollary 12 from [S3].

**Proposition 4.** If \( (s, v) \) is of gallery type, then \( (s^F, v^F) \) and \( (s^{f_1}, v^{f_1}), \ldots, (s^{f_q}, v^{f_q}) \) are also of gallery type. Therefore, \( BS_c(s, v) \) has an affine paving.

### 4. Equivariant cohomology

**4.1. Definitions.** Let \( p_T : E_T \to B_T \) be a universal principal \( T \)-bundle. For any \( T \)-space \( X \), we consider the Borel construction \( X \times_T E_T = (X \times E_T)/T \), where \( T \) acts on the Cartesian product diagonally: \( t(x, e) = (tx, te) \). Then we define the \( T \)-equivariant cohomology of \( X \) with coefficients \( k \) by

\[
H^\bullet_T(X, k) = H^\bullet(X \times_T E_T, k).
\]

This definition priory depends on the choice of a universal principal \( T \)-bundle. However, all these cohomologies are isomorphic. Indeed let \( p'_T : E'_T \to B'_T \) be another universal principal \( T \)-bundle. We consider the following diagram (see [J, 1.4]):

\[
\begin{array}{ccc}
(X \times E_T \times E'_T)/T & \xrightarrow{p_{12}/T} & X \times_T E_T \\
& \searrow & \nearrow \\
& \searrow & \nearrow \\
& X \times_T E'_T & \xrightarrow{p_{13}/T} & (X \times E_T \times E'_T)/T \\
\end{array}
\]

where the \( T \) acts diagonally on \( X \times E_T \times E'_T \). As \( p_{12}/T \) and \( p_{13}/T \) are fibre bundles with fibres \( E'_T \) and \( E_T \) respectively, we get by the Vietoris-Begle mapping theorem the following diagram for cohomologies:

\[
\begin{array}{ccc}
H^\bullet(X \times_T E_T, k) & \xrightarrow{(p_{12}/T)^\ast} & H^\bullet((X \times E_T \times E'_T)/T, k) \\
\sim & \sim & \sim \\
H^\bullet((X \times E_T \times E'_T)/T, k) & \xleftarrow{(p_{13}/T)^\ast} & H^\bullet(X \times_T E'_T, k) \\
\end{array}
\]

It allows us to identify the cohomologies \( H^\bullet(X \times_T E_T, k) \) and \( H^\bullet((X \times E_T \times E'_T)/T, k) \). The reader can easily check that this identification respects composition and is identical if both universal principal bundles are equal.

Similar constructions are possible for the compact torus \( K \): for a \( K \)-space \( X \), we define

\[
H^\bullet_K(X, k) = H^\bullet(X \times_K E_K, k),
\]

where \( p_K : E_K \to B_K \) is a universal principal \( K \)-bundle. For any \( T \)-space, we have \( H^\bullet_T(X, k) \cong H^\bullet_K(X, k) \), see for example [J, 1.6].
4.2. Stiefel manifolds. Being a Chevalley group, the group $G$ admits an embedding $G \leq GL(V)$ for some faithful representation $V$ of the Lie algebra of $G$. Let $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition into a direct sum of irreducible $G$-modules. By [St, Chapter 12], there exist for each $i$ a positive definite Hermitian form $\langle \cdot, \cdot \rangle_{V_i}$ on $V_i$ such that $\langle gu, v \rangle_{V_i} = \langle u, \sigma(g^{-1})v \rangle_{V_i}$ for any $x \in G$ and $u, v \in V_i$, where $\sigma$ is the automorphism of $G$ defined in Section 2.1. Their direct sum $\langle \cdot, \cdot \rangle_V$ is a positive definite Hermitian form on $V$ satisfying the same property. We consider the unitary group

$$U(V) = \{ g \in GL(V) \mid \langle gu, gv \rangle_V = \langle u, v \rangle_V \forall u, v \in V \},$$

for which we have $U(V) \cap G = C$. Choosing an orthonormal basis $(v_1, \ldots, v_t)$ of $V$, we get an isomorphism $U(V) \xrightarrow{\sim} U(t)$ taking an operator of $U(V)$ to its matrix in this basis.

For any natural number $N \geq t$, we consider the Stiefel manifold $E^N = \{ A \in M_{t,N}(\mathbb{C}) \mid A A^T = I_t \}$, where $M_{t,N}(\mathbb{C})$ is the space of $t \times N$ matrices with respect to metric topology and $I_t$ is the identity matrix. The group $U(t)$ of unitary $t \times t$ matrices acts on $E^N$ on the left by multiplication. Similarly, $U(N)$ acts on $E^N$ on the right. The last action is transitive and both actions commute. The quotient space $Gr^N = E^N/U(t)$ is called a Grassmanian and the corresponding quotient map $E^N \rightarrow Gr^N$ is a principal $U(t)$-bundle. Note that the group $U(N)$ also acts on $Gr^N$ by the right multiplication. For $N' > N$, we get the embedding $E^N \rightarrow E^{N'}$ by adding $N' - N$ zero columns to the right.

Taking the direct limits

$$E^\infty = \lim_{\rightarrow} E^N, \quad Gr = \lim_{\rightarrow} Gr^N,$$

we get a universal principal $U(t)$-bundle $E^\infty \rightarrow Gr$.

We need the spaces $E^N$ to get the principal $K$-bundles $E^N \rightarrow E^N/K$. It is easy to note that this bundle for $N = \infty$ is the direct limit of the bundles for $N < \infty$.

Note that for any $K$-space $X$, the group $U(N)$ acts on $X \times_K E^N$ on the right by $K(x, e)g = K(x, eg)$, where $x \in X$, $e \in E^N$ and $g \in U(N)$.

4.3. Equivariant cohomology of a point. We denote by $S = H^*_T(pt, \mathbb{C})$ the equivariant cohomology of a point. It is well known that $S$ is a polynomial ring with zero odd degree component. More exactly, let $\mathfrak{X}(T)$ be the group of all continuous homomorphisms $T \rightarrow \mathbb{C}^\times$. For each $\lambda \in \mathfrak{X}(T)$, let $\mathbb{C}_\lambda$ be the $\mathbb{C}T$-module that is equal to $\mathbb{C}$ as a vector space and has the following $T$-action: $tc = \lambda(t)c$. Then we have the line bundle $\mathbb{C}_\lambda \times_T E_T \rightarrow B_T$ denoted by $\mathcal{L}_T(\lambda)$, where $E_T \rightarrow B_T$ is a universal principal $T$-bundle. We get the map $\mathfrak{X}(T) \rightarrow H^2_T(pt) = H^2(B_T, \mathbb{C})$ given by $\lambda \mapsto c_1(\mathcal{L}_T(\lambda))$, where $c_1$ denotes the first Chern class, which extends to the isomorphism with the symmetric algebra:

$$\text{Sym}(\mathfrak{X}(T) \otimes_\mathbb{C} \mathbb{C}) \xrightarrow{\sim} S. \quad (9)$$

Similarly, let $\mathfrak{X}(K)$ be the group of continuous homomorphisms $K \rightarrow \mathbb{C}^\times$. For each $\lambda \in \mathfrak{X}(K)$, we have the bundle $\mathcal{L}_K(\lambda)$ similar to $\mathcal{L}_T(\lambda)$. Therefore, we have the isomorphism

$$\text{Sym}(\mathfrak{X}(K) \otimes_\mathbb{C} \mathbb{C}) \xrightarrow{\sim} S \quad (10)$$

induced by $\lambda \mapsto c_1(\mathcal{L}_K(\lambda))$. In what follows, we identify $\mathfrak{X}(T)$ with $\mathfrak{X}(K)$ via the restriction. Then both isomorphisms $\langle 9 \rangle$ and $\langle 10 \rangle$ become equal. Note that the Weyl group $W$ acts on $\mathfrak{X}(K)$ and $\mathfrak{X}(T)$ by $(w\lambda)(t) = \lambda(\tilde{w}^{-1}tw)$.

---

This space is usually denoted by $V_\tau(\mathbb{C}^N)$ or $\mathbb{C}V_{N,\tau}$. We also transpose matrices, as we want to have a left action of $U(\tau)$.
We are free to choose a universal principal $K$-bundle $E_K \to B_K$ to compute $S = H^*(B_K, \mathbb{k})$. We assume that the $K$-action on $E_K$ can be extended to a continuous $C$-action. The quotient map $E^\infty \to E^\infty/K$ is an example of such a bundle. The map $\rho \circ \pi : E_K \to E_K$ defined by $\rho \circ \pi(e) = we$ factors through the action of $K$ and we get the map $\rho \circ \pi : E_K/K \to E_K/K$. This map induces the ring homomorphism $(\rho \circ \pi)^*: S \to S$. It is easy to check that the pullback of $\mathcal{L}_K(\lambda)$ along $\rho \circ \pi/K$ is $\mathcal{L}(w^{-1}\lambda)$. Therefore, under identification (10), we get

$$\rho \circ \pi(K) = w^{-1}u$$

for any $u \in S$.

For a finite space $X$ with the discrete topology and trivial action of $K$, we identify $H^\bullet_{\mathbb{k}}(X, \mathbb{k})$ with $S(X)$. More exactly, let $x \in X$ be an arbitrary point. Consider the map $j_x : E_K/K \to X \times_K E_K$ given by $j_x(Ke) = K(x, e)$. Then any element $h \in H^n_K(X, \mathbb{k})$ is identified with the function $x \mapsto j_x^* h$. A similar identification is possible for a finite discrete space $X$ with the trivial action of $T$.

\subsection{4.4. Twisted actions of $S$}

Suppose that $E_K \to B_K$ is a universal principal $K$-bundle such that the $K$-action on $E_K$ can be extended to a continuous $C$-action. An example of such a bundle is the quotient map $E^\infty \to E^\infty/K$ (see, Section 4.2). Let $s : I \to T(W)$ be a sequence. We assume that $I$ is embedded into another totally ordered set $J$ (this assumption will be used in Section 5.4). Let $j$ be an element of $J \cup \{-\infty\}$. We define the map $\Sigma(s, j, w) : BS_{\mathbb{k}}(s) \times_K E_K \to E_K/K$ by

$$K([c], e) \mapsto K(c^jw)^{-1}e. \quad (12)$$

The reader can easily check that this map is well-defined and continuous. We call the map $\Sigma(s, -\infty, 1)$ the canonical projection from $BS_{\mathbb{k}}(s) \times_K E_K$ to $E_K/K$.

Taking cohomologies, we get the map

$$\Sigma(s, j, w)^* : S \to H^\bullet_{\mathbb{k}}(BS_{\mathbb{k}}(s), \mathbb{k}).$$

This map induces the action of $S$ on $H^\bullet(\mathbb{k})$ by $u \cdot h = \Sigma(s, j, w)^*(u) \cup h$. For $w = 1$ and $j = -\infty$, we get the canonical action of $S$. Note that these actions are independent of the choice of the universal principal $K$-bundle and of the action of $C$.

We also consider the finite dimensional version of these maps. Let $\Sigma^N(s, j, w) : BS_{\mathbb{k}}(s) \times_K E_K^N \to E_K^N/K$ be the map given by (12). Here $N$ may be an integer greater than or equal to $\tau$ or $\infty$. Note that $\Sigma^\infty(s, j, w)$ is a representative of $\Sigma(s, j, w)$.

\section{5. Tensor products}

From now on, we assume that $\mathbb{k}$ is a commutative ring having a finite global dimension. We denote by $D_{\mathbb{k}}$ the derived category of the category of $\mathbb{k}$-modules and by $D_{\mathbb{k}}(X)$ the derived category of the category of sheaves of $\mathbb{k}$-modules on a topological space $X$. We consider the bounded-below, bounded-above and bounded subcategories $D^+_{\mathbb{k}}$, $D^-_{\mathbb{k}}$, $D^b_{\mathbb{k}}$ of $D_{\mathbb{k}}$, respectively. Moreover, we set $D_{\mathbb{k}}^\circ = D_{\mathbb{k}}$. The similar notation is used for $D_{\mathbb{k}}(X)$.

\subsection{5.1. Tensor products of complexes of modules}

The tensor product $N \otimes_{\mathbb{k}} M$ of two left $\mathbb{k}$-modules is again a left $\mathbb{k}$-module. Similarly the tensor product $C^\bullet \otimes_{\mathbb{k}} D^\bullet$ of two complexes of left $\mathbb{k}$-modules is again a complex of left $\mathbb{k}$-modules. If the ring $\mathbb{k}$ is obvious from the context, then we write $\otimes$ instead of $\otimes_{\mathbb{k}}$. For any integer $n$, there is the natural homomorphism

$$\bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(D^\bullet) \to H^n(C^\bullet \otimes D^\bullet). \quad (13)$$
We would like to know if this homomorphism is an isomorphism for \( C^\bullet \) and \( D^\bullet \) satisfying certain conditions.

**Lemma 5.** Let \( D^\bullet \) be a complex of projective \( \mathbb{k} \)-modules bounded above such that \( H^j(D^\bullet) \) is projective for any \( j \). Then \( (13) \) is an isomorphism.

**Proof.** The proof is the same as that of [H, Theorem 3B.5] with the exception of the case where the differentials of \( C^\bullet \) are all zero. In that case, the complex \( C^\bullet \otimes D^\bullet \) is a direct sum of complexes \( C^i[-i] \otimes D^\bullet \). So it suffices to consider the case where \( C^\bullet \) is concentrated in degree zero. We note that the complex \( D^\bullet \) is isomorphic to the following one:

\[
\cdots \to B^i \oplus H^i(D^\bullet) \oplus B^{i+1} \overset{\partial^i}{\to} B^{i+1} \oplus H^{i+1}(D^\bullet) \oplus B^{i+2} \to \cdots ,
\]

where \( B^i \) is the \( i \)th coboundary and \( \partial^i \) maps the first to summands to zero and the third one identically. As the tensor product is distributive over the direct sum, the required result follows. \( \square \)

For every complex \( X \in \text{Ob}(D^*_k) \) where \( * = -, +, b, \emptyset \), there exist a complex of projective modules \( P \in \text{Ob}(D^*_k) \) and a quasi-isomorphism \( P \to X \). This allows us to define the derived tensor product:

\[
L^{} \otimes: D^*_k \times D^*_k \to D^*_k.
\]

Thus we get the map

\[
\bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(D^\bullet) \to H^n(C^\bullet \otimes D^\bullet).
\]

(14)

for any complexes \( C^\bullet, D^\bullet \in \text{Ob}(D^*_k) \). We will need the following result.

**Lemma 6.** Let \( C^\bullet, D^\bullet \in \text{Ob}(D^*_k) \) be such that \( H^i(C^\bullet) = 0 \) for \( i < 0 \) and \( H^j(D^\bullet) \) is projective for \( j \leq N \). Then \( (14) \) is an isomorphism for \( n \leq N - \text{gld}(k) \).

**Proof.** Consider the following distinguished triangle:

\[
\tau_{\leq N} D^\bullet \to D^\bullet \to \tau_{> N} D^\bullet \xrightarrow{\sim},
\]

where \( \tau_{\leq N} \) and \( \tau_{> N} \) are the standard truncation functors. Tensoring with \( C^\bullet \), we get another distinguished triangle:

\[
C^\bullet \otimes \tau_{\leq N} D^\bullet \to C^\bullet \otimes D^\bullet \to C^\bullet \otimes \tau_{> N} D^\bullet \xrightarrow{\sim}
\]

Hence we get an exact sequence

\[
H^{n-1}(C^\bullet \otimes \tau_{> N} D^\bullet) \to H^n(C^\bullet \otimes \tau_{\leq N} D^\bullet) \to H^n(C^\bullet \otimes D^\bullet) \to H^n(C^\bullet \otimes \tau_{> N} D^\bullet).
\]

(15)

Without loss of generality, we can assume that \( C^i = 0 \) for \( i < 0 \). Then there exists a complex \( P^\bullet \) of projective modules and a quasi-isomorphism \( P^\bullet \to C^\bullet \) such that \( P^i = 0 \) for \( i < - \text{gld}(k) \). This follows, for example, from the Cartan-Eilenberg resolution. We get

\[
(C^\bullet \otimes \tau_{> N} D^\bullet)^n = P^\bullet \otimes \tau_{> N} D^\bullet \to \bigoplus_{i+j=n, i \geq - \text{gld}(k), j > N} P^i \otimes D^j.
\]
The last sum is obviously zero if \( n \leq N - \text{gld}(\mathfrak{k}) \). In this case, we get from (15) the commutative diagram

\[
\begin{array}{ccc}
H^n(C^\bullet \otimes \tau_{\leq N} D^\bullet) & \xrightarrow{\sim} & H^n(C^\bullet \otimes D^\bullet) \\
\uparrow & & \uparrow \\
\bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(\tau_{\leq N} D^\bullet) & \xrightarrow{\sim} & \bigoplus_{i+j=n} H^i(C^\bullet) \otimes H^j(D^\bullet)
\end{array}
\]

The left vertical arrow is an isomorphism by Lemma 5 and the bottom horizontal arrow is an isomorphism as the summation in both formulas can be restricted to \( j \leq n \leq N \). Hence the right vertical arrow is also an isomorphism. \( \square \)

5.2. Projection formula and base change for proper maps. Let \( f : X \to Y \) be a proper map between locally compact topological spaces, \( \mathcal{F}^\bullet \in \text{Ob}(D_k(X)^+) \) and \( \mathcal{G}^\bullet \in \text{Ob}(D_k(Y)^+) \). Then there exists an isomorphism

\[
Rf_* \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet \xrightarrow{\sim} Rf_* (\mathcal{F}^\bullet \otimes f^* \mathcal{G}^\bullet),
\]

which is called the projection formula. It follows from the proof of this formula, for example [KS, Proposition 2.6.6], that (in this special case where \( f \) is proper) it is actually conjugate to the natural map \( f^* Rf_* \mathcal{F}^\bullet \otimes f^* \mathcal{G}^\bullet \to \mathcal{F}^\bullet \otimes f^* \mathcal{G}^\bullet \) (see the proof of [KS, Proposition 2.5.13] and the proof of formula (2.3.21) from the same book).

We have a similar description for the base change morphism. Let that \( X \) and \( Y \) be compact topological spaces and \( \mathcal{F}^\bullet \in \text{Ob}(D_k(X)^+) \). We consider the Cartesian product \( X \times Y \) and the following natural maps:

\[
\begin{array}{ccc}
\times Y & \xrightarrow{p_Y} & Y \\
\downarrow & & \downarrow \\
X & \xleftarrow{p_X} & X
\end{array}
\]

Let

\[
a_Y^* Ra_Y \mathcal{F}^\bullet \to Rp_Y^* p_X^* \mathcal{F}^\bullet \tag{17}
\]

be the morphism conjugate to the morphism \( p_Y^* a_Y^* Ra_Y \mathcal{F}^\bullet = p_Y^* a_X^* Ra_X \mathcal{F}^\bullet \to p_X^* \mathcal{F}^\bullet \) induced by the counit \( a_Y^* Ra_Y \to \text{id} \). To prove that (17) is an isomorphism, we can argue locally and reduce the problem to the case \( Y = \text{pt} \). In that case, the result follows from the zigzag identity for the unit and counit.

5.3. Cross product and the K"unneth formula. Let \( X \) be a topological space and \( \mathcal{F}^\bullet, \mathcal{G}^\bullet \in \text{Ob}(D_k(X)^+) \). Consider the morphism

\[
\bigcup_X : Ra_X \mathcal{F}^\bullet \otimes Ra_X \mathcal{G}^\bullet \to Ra_X (\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)
\]

that is conjugate to the product of counits

\[
a_X^* (a_X \mathcal{F}^\bullet \otimes a_X \mathcal{G}^\bullet) = a_X^* a_X^* \mathcal{F}^\bullet \otimes a_X^* a_X^* \mathcal{G}^\bullet \to \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet.
\]

Taking cohomologies and applying morphism (14), we get the map

\[
\bigoplus_{i+j=n} \mathbb{H}^i(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^j(X, \mathcal{G}^\bullet) \to \mathbb{H}^n(X, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet),
\]

which is called the cup product [KS, Exercise II.17].
Now let that $X$ and $Y$ be topological spaces and $\mathcal{F}^\bullet \in \text{Ob}(D_k(X)^+)$ and $\mathcal{G}^\bullet \in \text{Ob}(D_k(Y)^+)$. Then we have the following diagram:

$$
Ra_* p_X^* \mathcal{F}^\bullet \otimes Ra_* p_Y^* \mathcal{G}^\bullet \xrightarrow{\cup_{X \times Y}} Ra_* (p_X^* \mathcal{F}^\bullet \otimes p_Y^* \mathcal{G}^\bullet) \sim \Ra_{X*} \mathcal{F}^\bullet \otimes \Ra_{Y*} \mathcal{G}^\bullet
$$

(18)

The left arrow comes from the units of adjunction $\text{id} \rightarrow p_X^* p_Y^*$ and $\text{id} \rightarrow p_Y^* p_X^*$ and the right arrow is the Künneth isomorphism. As we want to prove the commutativity of the above diagram, we need to look more closely at this isomorphism. Actually it is the following composition of projection formulas and the base change:

$$
\Ra_{X*} \mathcal{F}^\bullet \otimes \Ra_{Y*} \mathcal{G}^\bullet \xrightarrow{\text{projection formula}} \Ra_{Y*} (a_Y^* \Ra_{X*} \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \xrightarrow{\text{base change}} \Ra_{Y*} (p_Y^* \mathcal{F}^\bullet \otimes \Ra_Y p_Y^* \mathcal{G}^\bullet).
$$

Now it remains to apply the constructions of Section 5.2.

Let $X$ and $Y$ be compact topological spaces such that $H^j(X, k)$ is a free $k$-module for $j \leq N$. For $n \leq N - \text{gld}(k)$, we get the following commutative diagram:

$$
\bigoplus_{i+j=n} H^i(X \times Y, k) \otimes H^j(X \times Y, k) \xrightarrow{\text{cup product}} H^n(X \times Y, k) \xrightarrow{\sim} \bigoplus_{i+j=n} H^i(X, k) \otimes H^j(Y, k)
$$

This construction can be easily iterated as follows. Let $X_1, \ldots, X_m$ be topological spaces such that $H^n(X_j, k)$ are free for all $n \leq N$ and $j = 2, \ldots, m$. Then for any $n \leq N - \text{gld}(k)$ we have the isomorphism

$$
\bigoplus_{i_1+\cdots+i_m=n} H^{i_1}(X_1, k) \otimes \cdots \otimes H^{i_m}(X_m, k) \sim \bigoplus_{i+j=n} H^i(X_1, k) \otimes H^j(Y, k)
$$

(19)

that is given by the cross product $a_1 \otimes \cdots \otimes a_m \mapsto p_1^*(a_1) \cup \cdots \cup p_m^*(a_m)$, where $p_i : X_1 \times \cdots \times X_m \rightarrow X_1$ is the projection to the $i$th coordinate.

5.4. **Embeddings of the Borel constructions.** We return here to the notation of Section 3. For any $m = 1, \ldots, n$ and $N \geq r$, we define the map

$$
q^{m,N} : \text{BS}_c(s, v) \times_K E^N \rightarrow \text{BS}_c(s_{f^m}, v_{f^m}) \times_K E^N
$$

by

$$
K([c], e) \rightarrow K([c_{f^m}], (c_{f^m}^{-1})^{-1} e).
$$

(20)

It is easy to check that this map is well-defined and continuous. Remember also from Section 4.4 that we have the maps

$$
\Sigma^N(s^F, f_{1}^{m}, v_{f^1} \cdots v_{f^m-1}) : \text{BS}_c(s^F) \times_K E^N \rightarrow E^N/K.
$$
Let $\sigma^N_m : BS_c(s^F, v^F) \times_K E^N \to E^N/K$ be the restriction of this map. More precisely, we have

$$K([d], e) \xrightarrow{\sigma^N_m} K((d_{min} \cdots d_{j_1-1}d_{j_1+1} \cdots d_{j_1-1} \cdots d_{j_m-1} \cdots d_{j_m-1} \cdots v_{j_1} \cdots v_{j_m-1})^{-1} e.$$  

We consider the map

$$\mu^N = (p^F \times_K \text{id}) \boxtimes q^1, \ldots, \boxtimes q^n, N$$

from $BS_c(s, v) \times_K E^N$ to

$$M^N = (BS_c(s^F, v^F) \times_K E^N) \times \prod_{m=1}^n BS_c(s_{fm}, v_{fm}) \times_K E^N.$$  

We abbreviate $M = M^\infty$, $\mu = \mu^\infty$ and $\sigma_m = \sigma^\infty_m$.

The following result allows us to embed Borel constructions to Cartesian product of Borel constructions in a way similar to [S3, Lemma 21].

**Lemma 7.** For $N < \infty$, the map $\mu^N$ is a topological embedding. Its image consists of all $n+1$-tuples $(K([d], e), K([c^{(1)}], e_1), \ldots, K([c^{(n)}], e_n))$ such that

$$Ke_m = \sigma^N_m(K([d], e))$$  

for any $m = 1, \ldots, n$.

**Proof.** Part 1: $\mu^N$ is a topological embedding. As $\mu^N$ maps a compact space to a Hausdorff space, it suffices to prove that it is injective. Let $K([c], e)$ and $K([d], h)$ be two orbits of $BS_c(s, v) \times_K E^N$ mapped to the same $n+1$-tuple by $\mu^N$. We can assume that $c$ and $d$ are $F$-balanced. It follows that there exist some elements $k_0, \tilde{k}_1, \ldots, \tilde{k}_n \in K$ such that

$$([d^F], h) = \tilde{k}_0^{-1}([c^F], e), \quad ([d_{fm}], (d_{fm} - 1) h) = \tilde{k}_m^{-1}(c_{fm}, (c_{fm} - 1) e)$$  

for $m = 1, \ldots, n$. Hence in its turn, it follows that

$$h = \tilde{k}_0^{-1} e, \quad (d_{fm} - 1) h = \tilde{k}_m^{-1}(c_{fm}, (c_{fm} - 1) e)$$  

and that there exist some sequences $k^{(0)} \in K([f^1]), k^{(1)} \in K([f^2]), \ldots, k^{(n)} \in K([f^n])$ such that

$$d^F = \tilde{k}_0^{-1} c^F k^{(0)}, \quad d_{fm} = \tilde{k}_m^{-1} c_{fm} k^{(m)}$$  

for any $m = 1, \ldots, n$.

The fact that $c$ and $d$ are $F$-balanced and the second equality of (23) immediately prove that

$$v_{fm} = \tilde{k}_m^{-1} v_{fm} k^{(m)}$$  

for any $m = 1, \ldots, n$. For our calculations below, it is convenient to define first $k^{(0)}_{-\infty} = \tilde{k}_0$ and then for any $m = 1, \ldots, n$ to define $k^{(0)}_{f_m - 1} = k_{j-1}^{(0)}$, where $j$ is the maximal element of $I^F \cup \{- \infty\}$ less than or equal to $f_{m-1}$.

We will prove by induction on $m = 1, \ldots, n$ that

$$k_m = (v_{f_1} \cdots v_{f_{m-1}})^{-1} k^{(0)}_{f_{m-1}} v_{f_1} \cdots v_{f_{m-1}}.$$  

First consider the case $m = 1$ and $f_1 = \min I$. The second equality of (22) now takes the form $h = \tilde{k}_1^{-1} e$. Comparing it with the first equality of (22), we get

$$\tilde{k}_1 = \tilde{k}_0 = k^{(0)}_{-\infty} = k^{(0)}_{f_1-1}.$$  

Now let $m = 1$ and $f_m > \min I$. From the first equality of (23), we get

$$d_{fm}^{-1} = \tilde{k}_m^{-1} c_{fm}^{-1} k^{(m)}_{f_{m-1}}.$$  


Substituting this to the second equality of (22) for \( m = 1 \), we get
\[
(k^{(0)}_{f_1} - 1)(c^{f_1 - 1} - 1)k_0 h = \tilde{k}_1^{-1}(c^{f_1 - 1} - 1)e
\]
Hence and from the first equality of (22), we get \( k^{(0)}_{f_1} = \tilde{k}_1 \).

Now suppose that \( m > 1 \) and for smaller indices (25) is true. Applying the second equation of (23), we get
\[
d^{f_{m-1}} = d^{f_1 - 1}(\tilde{k}_1^{-1}c^{f_1} \cdots c^{f_2}k^{(1)}_{f_2})
\]
\[
\times d_{f_2 + 1} \cdots d_{f_1 - 1}(\tilde{k}_2^{-1}c^{f_1} \cdots c^{f_2}k^{(2)}_{f_2}) \cdots d_{f_m - 2 + 1} \cdots d_{f_m - 1 - 1} \times
\]
\[
\times (\tilde{k}_{m-1}^{-1}c^{f_{m-1}} \cdots c^{f_m - 1}k^{(m-1)}_{f_m - 1})d_{f_m - 1 + 1} \cdots d_{f_m - 1}.
\]
Here we replaced elements of \( d \) with the corresponding elements of \( c \) in the brackets. To replace the remaining elements of \( d \) with elements of \( c \), we apply the first equation of (23):
\[
d^{f_{m-1}} = (d^f)^{f_{m-1}} = \tilde{k}_1^{-1}(c^{f_1} - 1)k^{(0)}_{f_1} = \tilde{k}_1^{-1}c^{f_1 - 1}k_1
\]
and
\[
d_{f_2 + 1} \cdots d_{f_1 + 1 - 1} = (k^{(0)}_{f_1} - 1)c_{f_2 + 1} \cdots c_{f_1 + 1 - 1}k^{(0)}_{f_1 + 1 - 1}
\]
for any \( h = 1, \ldots, m - 1 \). Conjugating both sides of the last equality by \((\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}\), we get
\[
d_{f_2 + 1} \cdots d_{f_1 + 1 - 1} = (\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}(k^{(0)}_{f_1} - 1)\hat{v} \hat{f} \cdots \hat{v} \hat{p} \times
\]
\[
\times c_{f_2 + 1} \cdots c_{f_1 + 1 - 1}(\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}k^{(0)}_{f_1 + 1 - 1} \hat{v} \hat{f} \cdots \hat{v} \hat{p}.
\]
We claim that
\[
(\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}(k^{(0)}_{f_1} - 1)\hat{v} \hat{f} \cdots \hat{v} \hat{p} = k^{(h)}_{f_1}
\]
for \( 1 \leq h \leq m - 1 \) and that
\[
(\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}k^{(0)}_{f_1 + 1 - 1} \hat{v} \hat{f} \cdots \hat{v} \hat{p} = \tilde{k}_{h+1}.
\]
1 \leq h < m - 1. The second equality follows from the inductive hypothesis. Conjugating the first equality by \( \hat{v} \hat{p} \), we get an equivalent equality
\[
(\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}(k^{(0)}_{f_1} - 1)\hat{v} \hat{f} \cdots \hat{v} \hat{p} = \hat{v} \hat{p}k^{(h)}_{f_2}(\hat{v} \hat{p})^{-1},
\]
which is equivalent to
\[
\tilde{k}_h = \hat{v} \hat{p}k^{(h)}_{f_2}(\hat{v} \hat{p})^{-1}
\]
by the inductive hypothesis. This equality follows from (24). Substituting (30) and (31) to (29), we get
\[
d_{f_2 + 1} \cdots d_{f_1 + 1 - 1} = (k^{(h)}_{f_1} - 1)c_{f_2 + 1} \cdots c_{f_1 + 1 - 1}k_{h+1}
\]
for \( 1 \leq h < m - 1 \)
\[
d_{f_m - 1 + 1} \cdots d_{f_m - 1} = (k^{(m-1)}_{f_m - 1} - 1)c_{f_m - 1 + 1} \cdots c_{f_m - 1}k^{(0)}_{f_m - 1} \hat{v} \hat{f} \cdots \hat{v} \hat{p}.
\]
Substituting these two equations and (23) to (27), we get
\[
d^{f_{m-1}} = \tilde{k}_0^{-1}c^{f_{m-1}}(\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}k^{(0)}_{f_m - 1} \hat{v} \hat{f} \cdots \hat{v} \hat{p}.
\]
Multiplying \( h \) by the inverse of both sides, we get by the first equation of (22) that
\[
(d^{f_{m-1}})^{-1}h = (\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}(k^{(0)}_{f_m - 1} - 1)\hat{v} \hat{f} \cdots \hat{v} \hat{p} - (c^{f_{m-1}})^{-1} \tilde{k}_0 h
\]
\[
= (\hat{v} \hat{f} \cdots \hat{v} \hat{p})^{-1}(k^{(0)}_{f_m - 1} - 1)\hat{v} \hat{f} \cdots \hat{v} \hat{p} - (c^{f_{m-1}})^{-1} e.
\]
Comparing it with the second equation of (22), we get (25).

Now let us define the sequence \( k' \in K(I) \) by

\[
k'_i = \begin{cases} 
(\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}k_i^{(0)} \hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}} & \text{if } f_2^{m-1} < i < f_1^m; \\
k_i^{(m)} & \text{if } i \in [f_1^m].
\end{cases}
\]

It is convenient to define \( k'_{-\infty} = \hat{k}_0 \).

We claim that \( d = \hat{k}_0^{-1}c \).

Conjugating this equality by \((\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}\) and applying (25) and (24), we get

\[
d_i = (\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}(k_i^{(0)})^{-1}\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}}c_i(\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}k_i^{(0)} \hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}}c_i = \hat{v}_{j_1}^{-1}\hat{v}_{j_{m-1}}^{-1} \hat{v}_{j_{m-1}}c_i k_i' = (k_i^{(m)})^{-1}c_i k_i' = (k_{i-1})^{-1}c_i k_i'.
\]

Now suppose that \( i = \min I \). From the first formula of (23), we get

\[
d_i = d_i^F = \hat{k}_0^{-1}c_i k_i'^{(0)} = \hat{k}_0^{-1}c_i k_i^{(0)} = (k'_{i-1})^{-1}c_i k_i'.
\]

Now consider the case \( i \in [f_1^m] \) but \( i-1 \notin [f_1^m] \). Then we have \( i = f_1^m \). First suppose that \( i-1 \in I^F \). Then \( f_2^{m-1} < i-1 < f_1^m \). From the second equality of (23) and by (25), we get

\[
d_i = \hat{k}_m^{-1}c_i k_i^{(m)} = (\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}(k_i^{(0)})^{-1}\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}}c_i k_i^{(m)} = (k_{i-1})^{-1}c_i k_i'.
\]

If \( i = \min I \), then \( m = 1 \) and \( f_1^1 = \min I \). From the second equality of (23) and (26), we get

\[
d_i = \hat{k}_1^{-1}c_i k_i^{(1)} = \hat{k}_0^{-1}c_i k_i' = (k'_{i-1})^{-1}c_i k_i'.
\]

Finally suppose that \( m > 1 \) and \( i-1 = f_2^{m-1} \). Then we have \( k_i^{(m-1)} = k_i^{(m-1)} \). From the second equality of (23), by (25) applied twice and by (24), we get

\[
d_i = \hat{k}_m^{-1}c_i k_i^{(m)} = (\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}(k_i^{(0)})^{-1}\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}}c_i k_i^{(m)} = (\hat{v}_{j_1} \cdots \hat{v}_{j_{m-1}})^{-1}(k_i^{(m-1)})^{-1}c_i k_i^{(m)} = (k_{i-1})^{-1}c_i k_i'.
\]

We have therefore proved that \([d] = \hat{k}_0^{-1}[c]\). Hence and from the first equation of (22), we get

\[
K([d], h) = K(\hat{k}_0^{-1}[c], \hat{k}_0^{-1}e) = K([c], c).
\]

**Part 2: the description of the image.** All elements of the image satisfy (21), as for any \( F \)-balanced \( c \in \mathcal{C}(s, v) \) we have

\[
(c_{f_1^m})^{-1}c = (c_{f_1^m}^{-1}c_f^{m-1}c_{f_1^m+1} \cdots c_{f_2^m}^{-1}c_{f_{f_2^m+1}} \cdots c_{f_{f_1^m}}^{-1}c)^{-1}c = (c_{f_1^m}^{-1}c_f^{m-1}c_{f_1^m+1} \cdots c_{f_{f_2^m+1}}^{-1}c_{f_{f_2^m+1}} \cdots c_{f_{f_1^m}}^{-1}c)^{-1}.
\]
generality assume that $d = u^F$ (see the remark before this proof). Let us write (21) in the form

$$k_m e_m = (u^F_{(m)} \cdots u^F_{(1)} \cdot \hat{v}^F_{(1)} \cdots \hat{v}^F_{(m)})^{-1} e$$

for some $k_m \in K$. By \cite{S3} Lemma 5, we can write $k_m[z^{(m)}] = [\tilde{z}^{(m)}]$ for some $\tilde{z}^{(m)} \in \tilde{C}(s_f^m, v_f^m)$. We define

$$c_i = \begin{cases} u_i & \text{if } i \in I^F \\ z_i^{(m)} & \text{if } i \in [f^m]. \end{cases}$$

It is easy to see that $c$ is $F$-balanced and $c^F = u^F$. We get

$$(p^F \times_K \id)(K([c], e)) = K([c^F], e) = K([u^F], e) = K([d], e).$$

On the other hand, for any $m = 1, \ldots, n$, applying (32), we get

$$q^{m, n}(K([c], e)) = K([z^{(m)}], (c^F_{(m)})^{-1} e) = K([z^{(n)}], \sigma^{F_{(n)}}_{(n)} \cdot \cdots \cdot \sigma^{F_{(2)}}_{(2)} \cdot \sigma^{F_{(1)}}_{(1)}) = K(k_m[c^{(m)}], k_m e_m) = K([c^{(m)}], e_m).$$

Hence we get

$$\mu^N(K([c], e)) = (K([d], e), K([c^{(1)}], e_1), \ldots, K([c^{(n)}], e_n)).$$

\[\square\]

5.5. The difference. We now would like to study the cohomology with compact support of the space $M^N \setminus \im \mu^N$ for $N < \infty$ with the help of sequence formulas. To do it, we need the corresponding fibre bundles. They are given by the following lemma.

Lemma 8. Let $r \leq N < \infty$ and $\varphi: M^N \setminus \im \mu^N \to \BS_c(s^F, v^F) \times_K E^N$ be the projection to the first component and $\eta: \BS_c(s^F, v^F) \times_K E^N \to E^N/K$ be the canonical projection.

(1) $\varphi$ is a fibre bundle.

(2) the composition $\eta \varphi$ is a fibre bundle.

Proof. Let $b$ be an arbitrary element (orbit) of $\BS_c(s^F, v^F) \times_K E^N$. The right action of the unitary group $U(N)$ on $E^N$ induces a right action of $U(N)$ on $E^N/K$. For any $m = 1, \ldots, n$, let $t_m: U(N) \to E^N/K$ be the map defined by $t_m(g) = \sigma^N_m(b)g$. As $U(N)$ acts transitively on $E^N$, it acts transitively on $E^N/K$ and $t_m$ is a fibre bundle. Therefore, there exists an open neighbourhood of $V_m$ of $\sigma_m(b)$ and a continuous section $g_m: V_m \to U(N)$ of $t_m$. Hence for any $u \in V_m$, we get

$$u = t_m(g_m(u)) = \sigma^N_m(b)g_m(u).$$

(33)

We define $H = \bigcap_{m=1}^{n} (\sigma^N_m)^{-1}(V_m)$. It is an open subset of $\BS_c(s^F, v^F) \times_K E^N$ containing $b$.

We construct the map $\varphi: H \times \varphi^N(1) \to M^N$ by

$$(h, (b, a_1, \ldots, a_n)) \mapsto \varphi^N(h), (a_1 g_1 (\sigma^N_1(h)), \ldots, a_n g_n (\sigma^N_n(h)))$$

Here we used the right action of $U(N)$ on Borel constructions mentioned in Section 4.2.

Suppose that the right-hand side of the above formula belongs to $\im \mu^N$. Let us write $a_m = K(b_m, e_m)$ for some $b_m \in BS_c(s^F, v^F)$ and $e_m \in E^N$. By Lemma 7 we get that

$$K e_m g_m (\sigma^N_m(h)) = \sigma^N_m(h)$$

Suppose that the right-hand side of the above formula belongs to $\im \mu^N$. Let us write $a_m = K(b_m, e_m)$ for some $b_m \in BS_c(s^F, v^F)$ and $e_m \in E^N$. By Lemma 7 we get that

$$K e_m g_m (\sigma^N_m(h)) = \sigma^N_m(h)$$
for any \( m = 1, \ldots, n \). By (53), this equality is equivalent to
\[
K e_m = \sigma_m^N(h) g_m(\sigma_m^N(h))^{-1} = \sigma_m^N(b).
\]
Hence by Lemma \([4]\) we get a contradiction \((b, a_1, \ldots, a_n) \in \text{im} \mu^N\). Thus we have proved that \( \text{im} \varphi \cap \text{im} \mu^N = \emptyset \). On the other hand, it is obvious that \( \text{im} \varphi \subseteq \mathfrak{x}^{-1}(H) \). Therefore, \( \varphi \) is actually a continuous map from \( H \times \mathfrak{x}^{-1}(b) \) to \( \mathfrak{x}^{-1}(H) \).

It is easy to prove that \( \varphi \) is a homeomorphism. Indeed the inverse map \( \mathfrak{x}^{-1}(H) \to H \times \mathfrak{x}^{-1}(b) \) is given by
\[
(h, a_1, \ldots, a_n) \mapsto (h, (b, a_1 g_1(\sigma_1^N(h))^{-1}, \ldots, a_n g_n(\sigma_n^N(h))^{-1})).
\]

We get the following commutative diagram:
\[
\begin{array}{ccc}
H \times \mathfrak{x}^{-1}(b) & \xrightarrow{\varphi} & \mathfrak{x}^{-1}(H) \\
\downarrow{\cong} & & \downarrow{\cong} \\
H & \xrightarrow{\mathfrak{x}} & H
\end{array}
\]

Finally note that \( \mathfrak{x}^{-1}(b) \) are homeomorphic for different \( b \), as the space \( BS_c(s^F, v^F) \times_K E^N \) is connected and compact.

[2] Let \( \bar{e} \) be an arbitrary point of \( E^N/K \). Let \( \nu_N : E^N \to E^N/K \) denote the quotient map. As it is a principal \( K \)-bundle, there exists an open neighbourhood \( U \subset E^N/K \) of \( \bar{e} \) and a homeomorphism \( \varphi \) such that the following diagram is commutative
\[
\begin{array}{ccc}
U \times K & \xrightarrow{\varphi} & \nu_N^{-1}(U) \\
\downarrow{\cong} & & \downarrow{\cong} \\
U & \xrightarrow{\nu_N} & U
\end{array}
\]

and \( \varphi(u, k_1 k_2) = k_1 \varphi(u, k_2) \). We set \( \mathbf{k} = p_2 \varphi^{-1} \). It is a continuous map from \( \nu_N^{-1}(U) \) to \( K \) such that \( \varphi(\nu_N(e), \mathbf{k}(e)) = e \) for any \( e \in \nu_N^{-1}(U) \). It satisfies the following property:
\( \mathbf{k}(ke) = \mathbf{k}k(e) \) for any \( e \in \nu_N^{-1}(U) \) and \( k \in K \).

We also consider the section \( t : U \to E^N \) of \( \nu_N \) given by \( t(u) = \varphi(u, 1) \). By definition, we get \( \mathbf{k}(t(u)) = 1 \).

Let \( z : U(N) \to E^N \) be the map defined by \( z(g) = t(\bar{e}) g \). As \( U(N) \) acts transitively on \( E^N \), we get that \( z \) is a fibre bundle. Therefore, there exists an open neighbourhood \( V \) of \( t(\bar{e}) \) and a continuous section \( y : V \to U(N) \) of \( z \). Hence for any \( w \in V \), we get
\[
w = z(y(w)) = t(\bar{e}) y(w).
\]

Consider the set \( U' = t^{-1}(V) \). It is an open subset of \( U \) and hence is an open neighbourhood of \( \bar{e} \) in \( E^N/K \). Considering the restriction \( \varphi' = \varphi|_{U' \times K} \), we get the following commutative diagram
\[
\begin{array}{ccc}
U' \times K & \xrightarrow{\varphi'} & \nu_N^{-1}(U') \\
\downarrow{\cong} & & \downarrow{\cong} \\
U' & \xrightarrow{\nu_N} & U'
\end{array}
\]

We define the map \( \psi : U' \times (\eta \mathfrak{x})^{-1}(\bar{e}) \to M_N \) by
\[
\left( u, (K([d], t(\bar{e})), a_1, \ldots, a_n) \right) \mapsto \left( K([d], t(u)), a_1 y(t(u)), \ldots, a_n y(t(u)) \right).
\]
Suppose that the right-hand side of the above formula belongs to $\text{im} \mu^N$. Let us write $a_m = K(b_m, e_m)$ for some $b_m \in \text{BS}_c(s_{f_m}, v_{f_m})$ and $e_m \in E^N$. By Lemma 7 we get that

$$K e_m y(t(u)) = \sigma_m^N(K([d], t(u))) = K(d_{m, 1} \cdots d_{f_{2, 1} - 1} d_{f_{2, 1} + 1} \cdots d_{f_1 - 1} \cdots d_{f_{m-1, 1} + 1} \cdots d_{f_{m, 1} - 1} \cdots \bar{v_{f_1}} \cdots \bar{v_{f_m-1}})^{-1} t(u),$$

for any $m = 1, \ldots, n$. By (54), this equality is equivalent to

$$K e_m = K(d_{min, 1} \cdots d_{f_{1, 1} - 1} d_{f_{1, 1} + 1} \cdots d_{f_{1, 1} - 1} \cdots d_{f_{2, 1} - 1} d_{f_{2, 1} + 1} \cdots d_{f_{2, 1} - 1} \cdots d_{f_{m-1, 1} + 1} \cdots d_{f_{m, 1} - 1} \cdots \bar{v_{f_1}} \cdots \bar{v_{f_m-1}})^{-1} t(u) y(t(u))^{-1}$$

$$= \sigma_m^N(K([d], t(\bar{e}))).$$

Hence we get a contradiction $(K([d], t(\bar{e})), a_1, \ldots, a_n) \in \text{im} \mu^N$. Therefore, the image of $\psi$ is a subset of $M^N \setminus \text{im} \mu^N$. It is easy to see that $\psi$ is actually a map to $(\eta \varphi)^{-1}(U')$.

In order to prove that $\psi$ is a homeomorphism, we need to construct the inverse map $\xi : (\eta \varphi)^{-1}(U') \to U' \times (\eta \varphi)^{-1}(\bar{e})$. It is easy to see that it is given by

$$(K([d], e), a_1, \ldots, a_n) \mapsto \left(\nu_N(e), (K(k(e)^{-1}[d], t(\bar{e})), a_1 y(t(\nu_N(e)))^{-1}, \ldots$$

$$\ldots, a_n y(t(\nu_N(e)))^{-1}\right).$$

We get the following commutative diagram:

$$\begin{array}{ccc}
U' \times (\eta \varphi)^{-1}(\bar{e}) & \xrightarrow{\psi} & (\eta \varphi)^{-1}(U') \\
\downarrow{\text{pr}_1} & & \downarrow{\eta \varphi} \\
U' & \xleftarrow{\eta \varphi} & \text{induced by } \mu \text{ is surjective } \end{array}$$

Finally, note that $(\eta \varphi)^{-1}(\bar{e})$ are homeomorphic for different $\bar{e}$, as $E^N/K$ is connected and compact.

\textbf{Lemma 9.} Suppose that $(s, v)$ is of gallery type.

1. If $r < N < \infty$, then $H^c_n(M^N \setminus \text{im} \mu^N, k) = 0$ for odd $n < 2(N - r) - \text{gld } k$.

2. If $r < N < \infty$, then the restriction map $H^n(M^N, k) \to H^n(\text{BS}_c(s, v) \times E^N, k)$ induced by $\mu^N$ is surjective for $n < 2(N - r) - \text{gld } k - 1$.

3. For any $n$, the restriction map $H^n(M, k) \to H^n_k(\text{BS}_c(s, v), k)$ induced by $\mu$ is surjective.

\textbf{Proof.} We write the Leray spectral sequence for cohomologies with compact support for $\eta \varphi$, where $\eta$ and $\varphi$ are as in Lemma 8. As by part (2) of this lemma $\eta \varphi$ is a fibre bundle and $E^N/K$ is simply connected, it has the following second page:

$$E^{p,q}_2 = H^p(E^N/K, H^q_c((\eta \varphi)^{-1}(\bar{e}), k)), $$

where $\bar{e}$ is an arbitrary point of $E^N/K$. We claim that

\begin{itemize}
  \item[(*)] $H^q((\eta \varphi)^{-1}(\bar{e}), k)$ is free of finite rank and equals zero for odd $q$ if $q < 2(N - r) - \text{gld } k$;
\end{itemize}
In order to prove it, we consider the map \( \tilde{\zeta} : (\eta \zeta)^{-1}(\tilde{e}) \to \eta^{-1}(\tilde{e}) \) that is the restriction of \( \zeta \). Note that \( \tilde{\zeta} \) is a fibre bundle as a restriction of \( \zeta \), which is a fibre bundle by Lemma 8(1).

Hence we can consider the Leray spectral sequence for cohomologies with compact support for \( \tilde{\zeta} \). It has the following second page:

\[
E_2^{t,t} = H^t_c\left( \eta^{-1}(\tilde{e}), \mathcal{F}^t \right),
\]

where \( \mathcal{F}^t = \mathcal{H}^t(\tilde{\zeta}_0)_{(\eta \zeta)^{-1}(\tilde{e})} \). This is a local (not necessarily constant) system on \( \eta^{-1}(\tilde{e}) \cong \text{BS}_c(s^F, v^F) \). We claim that

(***) the stalks of \( \mathcal{F}^t \) are free of finite rank and equal zero for odd \( t \) if \( t < 2(N - r) - \text{gld} \ k \);

To prove it, let us take an arbitrary point \( b \in \eta^{-1}(\tilde{e}) \) and consider the following Cartesian diagram:

\[
x^{-1}(b) \overset{i}{\hookrightarrow} (\eta \zeta)^{-1}(\tilde{e}) \xrightarrow{\tilde{\zeta}} \eta^{-1}(\tilde{e})
\]

Hence by the proper base change, we get

\[
i^* \mathcal{F}^t = \mathcal{H}^t(\tilde{\zeta}_0)_{(\eta \zeta)^{-1}(\tilde{e})} = \mathcal{H}^t(\tilde{\zeta}_0)_{(\eta \zeta)^{-1}(\tilde{e})} = \mathcal{H}^t(\tilde{\zeta}_0)_{(\eta \zeta)^{-1}(\tilde{e})} = H^t_c(\eta^{-1}(b), \mathbb{k}).
\]

To compute the last module, consider the canonical projections \( \eta_m : \text{BS}_c(s_{2m}, v_{2m}) \times_K E^N \to E^N/K \) for \( m = 1, \ldots, n \). Their direct product

\[
\eta_1 \times \cdots \times \eta_n : (\text{BS}_c(s_{2}, v_{2}) \times_K E^N) \times \cdots \times (\text{BS}_c(s_{2n}, v_{2n}) \times_K E^N) \to (E^N/K)^n
\]

is a fibre bundle with fibre \( \text{BS}_c(s_{2}, v_{2}) \times \cdots \times \text{BS}_c(s_{2n}, v_{2n}) \). It has an affine paving by Proposition 2. By Lemma 7 (the claim about the image), we get that \( \zeta^{-1}(b) \) is homeomorphic to

\[
(\text{BS}_c(s_{2}, v_{2}) \times_K E^N) \times \cdots \times (\text{BS}_c(s_{2n}, v_{2n}) \times_K E^N) \setminus (\eta_1 \times \cdots \times \eta_n)^{-1}(\sigma_1^{-1}(b), \ldots, \sigma_n^{-1}(b)),
\]

By [S3, Lemma 19(2)] and the Künneth formula in form (19), we get that \( H^t((E^N/K)^n, \mathbb{k}) \) is free of finite rank and equals zero for odd \( t \) if \( t < 2(N - r) + 1 - \text{gld} \ k \). Therefore, we can apply [S3, Lemma 23] to the fibre bundle \( \eta_1 \times \cdots \times \eta_n \). We thus get that \( H^t_\ast(\zeta^{-1}(b), \mathbb{k}) \) is free of finite rank and equals zero for odd \( t \) if \( t < 2(N - r) - \text{gld} \ k \). Thus we have proved (***).

By Proposition 2 the space \( \text{BS}_c(s^F, v^F) \) has an affine paving. Therefore, by (***), we get that \( E_2^{t,t} \) is zero except the following cases both \( r \) and \( t \) are even; \( t \geq 2(N - r) - \text{gld} \ k \). Moreover \( E_2^{r,r} \) is free of finite rank for \( t < 2(N - r) - \text{gld} \ k \).

The differentials coming to and starting from \( E_2^{r,t} \) are zero for \( a \geq 2 \) if \( r + t < 2(N - r) - \text{gld} \ k \). Thus \( E_2^{a,t} = E_2^{a,t} \) for \( r + t < 2(N - r) - \text{gld} \ k \). Therefore, \( H^0_\ast(\eta \zeta^{-1}(\tilde{e}), \mathbb{k}) \) is free of finite rank and equals zero for odd \( q \) if \( q < 2(N - r) - \text{gld} \ k \). Thus (**) is proved.

Finally, let us look at \( E_2^{p,q} \). By (**) and [S3, Lemma 19(2)], we get that \( E_2^{p,q} \) is zero except the following cases: \( p \) and \( q \) are both even; \( p \geq 2(N - r) + 1 \); \( q \geq 2(N - r) - \text{gld} \ k \).

That the differentials coming to and starting from \( E_2^{p,q} \) are zero for \( a \geq 2 \) if \( p + q < 2(N - r) - \text{gld} \ k \). Thus \( E_\infty^{p,q} = E_2^{p,q} \) for \( p + q < 2(N - r) - \text{gld} \ k \). It follows now from the Leray spectral sequence that \( H^n(\mu^N \setminus \text{im} \mu^N, \mathbb{k}) = 0 \) for odd \( n < 2(N - r) - \text{gld} \ k \).

[2] Let \( n < 2(N - r) - \text{gld} \ k - 1 \). If \( n \) is even, then by the first part of this lemma, we get an exact sequence

\[
H^n(M^N, \mathbb{k}) \to H^n(\text{BS}_c(s, v) \times_K E^N, \mathbb{k}) \to H_{even}^{n+1}(M^N \setminus \text{im} \mu^N, \mathbb{k}) = 0.
\]

If \( n \) is odd, then the restriction under consideration is surjective as \( H^n(\text{BS}_c(s, v) \times_K E^N, \mathbb{k}) = 0 \) by [S3, Lemma 20].
This result follows from the previous part and [3] Lemma 18] and the following commutative diagram:

\[
\begin{array}{ccc}
H^n(M^N, \mathbb{k}) & \xrightarrow{(\mu^N)^*} & H^n(BS_c(s) \times_K E^N, \mathbb{k}) \\
\| & & \| \\
H^n(M, \mathbb{k}) & \xrightarrow{\mu^*} & H^n_K(BS_c(s), \mathbb{k})
\end{array}
\]

which holds for \( n < 2(N - t) + 1 \).

\[\square\]

**Remark.** It easily follows from this proof that the kernels of the restrictions \( H^n(M, \mathbb{k}) \rightarrow H^n_K(BS_c(s, v), \mathbb{k}) \) are free of finite rank. We do not need this fact in the sequel.

### 5.6. Tensor product decomposition.

Note that for a pair \((s, v)\) of gallery type, we get by the Künneth formula and [3] Lemma 18] the map

\[
\mu^* : H^\bullet(M, \mathbb{k}) \cong H^\bullet_K(BS_c(s^F, v^F), \mathbb{k}) \otimes_k \bigotimes_{m=1}^{n} k H^\bullet_K(BS_c(s_{f_m}, v_{f_m}), \mathbb{k}) \rightarrow H^\bullet_K(BS_c(s, v), \mathbb{k}).
\]

By Lemma [3], this map is surjective. We would like to study its kernel. Let \( Q = S \otimes_k \cdots \otimes_k S \) be the tensor product of \( n \) copies of \( S \). Clearly \( Q \) is a commutative ring isomorphic to the polynomial ring \( \mathbb{k}[x_1, \ldots, x_n] \). We consider \( H^\bullet_K(BS_c(s^F, v^F), \mathbb{k}) \) as an \( S-Q \)-bimodule with the canonical left action of \( S \) and the following right action of \( Q \):

\[
h(c_1 \otimes \cdots \otimes c_n) = h \cup \sigma_1^*(c_1) \cup \cdots \cup \sigma_n^*(c_n)
\]

for \( h \in H^\bullet_K(BS_c(s^F, v^F), \mathbb{k}) \) and \( c_m \in S \). Here \( \sigma_m \) is the map as in Section 5.4. On the other hand, we consider the tensor product

\[
\bigotimes_{m=1}^{n} k H^\bullet_K(BS_c(s_{f_m}, v_{f_m}), \mathbb{k})
\]

as the left \( Q \)-module such that the \( m \)th copy of \( S \) acts on \( H^\bullet_K(BS_c(s_{f_m}, v_{f_m}), \mathbb{k}) \) canonically:

\[
(c_1 \otimes \cdots \otimes c_n)(h_1 \otimes \cdots \otimes h_n) = (\eta_1^*(c_1) \cup h_1) \otimes \cdots \otimes (\eta_n^*(c_n) \cup h_n),
\]

where \( \eta_m : BS_c(s_{f_m}, v_{f_m}) \times_K E^\infty \rightarrow E^\infty / K \) is the canonical projection.

As the space \( E^\infty / K \) is connected, we get \( \sigma_m^*(c) = (\eta^F)^*(c) = c \), for any \( c \in S^0 = H^0(\text{pt}, \mathbb{k}) \), where \( \eta^F : BS_c(s^F, v^F) \times_K E^\infty \rightarrow E^\infty / K \) is the canonical projection. Thus considering \( Q \) as a \( Q-k \)-bimodule, we get that

\[
H^\bullet_K(BS_c(s^F, v^F), \mathbb{k}) \otimes_Q Q \cong H^\bullet_K(BS_c(s^F, v^F), \mathbb{k}) \quad (35)
\]
as a \( k-k \)-bimodule with respect to the canonical actions of \( k \) on both sides.

In the proof of Lemma 7 it was shown that the image of \( \mu^N \) satisfies (21). This argument applies equally well to the case \( N = \infty \). Therefore, we get the following commutative diagram:

\[
\begin{array}{ccc}
BS_c(s, v) \times_K E^\infty & \xrightarrow{p_1 \mu} & BS_c(s^F, v^F) \times_K E^\infty \\
\downarrow_{p_m + 1 \mu} & & \downarrow_{\sigma_m} \\
BS_c(s_{f_m}, v_{f_m}) \times_K E^\infty & \xrightarrow{\eta_m} & E^\infty / K
\end{array}
\]

Suppose that we have homogeneous elements \( h \in H^\bullet_K(BS_c(s^F, v^F), \mathbb{k}) \) and \( c_m \in S \), \( h_m \in H^\bullet_K(BS_c(s_{f_m}, v_{f_m}), \mathbb{k}) \) for \( m = 1, \ldots, n \). Taking cohomologies in the above diagram and
using identification (19), we get

\[ \mu^* \left( h \otimes ((c_1 \otimes \cdots \otimes c_n)(h_1 \otimes \cdots \otimes h_n)) \right) = \mu^* \left( h \otimes (\eta_1^*(c_1) \cup h_1) \otimes \cdots \otimes (\eta_n^*(c_n) \cup h_n) \right) \]

As graded \( k \)-modules. Hence

\[ \mu^* \left( p_1^*(h) \cup p_2^*(\eta_1^*(c_1) \cup h_1) \cup \cdots \cup p_{n+1}^*(\eta_n^*(c_n) \cup h_n) \right) \]

On the other hand, by \([S3, \text{Theorem 7}]\), applying the Leray spectral sequence, we get

\[ H^s_c(s^F, v^F, k) \otimes_{Q} k \]

By the following result.

\[ (36) \]

Hence

\[ h \otimes ((c_1 \otimes \cdots \otimes c_n)(h_1 \otimes \cdots \otimes h_n)) - h(c_1 \otimes \cdots \otimes c_n) \otimes h_1 \otimes \cdots \otimes h_n \in \ker \mu^* \]

Let \( A \) be the \( k \)-submodule of \( H^*(M, k) \) generated by all the above differences. Clearly \( A \subset \ker \mu^* \). Therefore, applying (35), we get that the quotient \( H^*(M, k)/A \) is the following tensor product:

\[ H^s_c(s^F, v^F, k) \otimes_{Q} k \]

On the other hand, by \([S3, \text{Theorem 7}]\), applying the Leray spectral sequence, we get

\[ H^s_c(s^F, v^F, k) \otimes_{Q} k \]

as graded \( k \)-modules. Hence \( H^s_c(s, v, k) \cong S \otimes_k H^s_c(s, v, k) \cong H^s(M, k)/A \). Hence we get \( A = \ker \mu^* \) by the following result.

**Proposition 10.** Let \( L \subset L' \subset M \) be \( k \)-modules such that

\[ M/L \cong k^n, \quad M/L' \cong k^n \]

for some integer \( n \). Then \( L = L' \).

**Proof.** It follows from \([R, \text{Theorem 3.6}]\) applied to the quotient homomorphism

\[ M/L \to (M/L)/(L'/L) \cong M/L'. \]

**Theorem 11.** For a pair \( (s, v) \) is of gallery type, the embedding \( \mu \) induces an isomorphism of left \( S \)-modules

\[ \mu^o : H^s_c(s^F, v^F, k) \otimes_{Q} k \]

for some integer \( n \). Then \( L = L' \).

**Proof.** It follows from \([R, \text{Theorem 3.6}]\) applied to the quotient homomorphism

\[ M/L \to (M/L)/(L'/L) \cong M/L'. \]
Proof. It remains to prove that this map is an isomorphism of \( S \)-modules. It suffices to prove that \( \mu^* \) is a morphism of \( S \)-modules. We have the following commutative diagram:

\[
\begin{array}{ccc}
BS_c(s, v) \times_K E^\infty & \xrightarrow{\eta} & E^\infty/K \\
\downarrow_{\eta_1} & & \downarrow_{\eta_F} \\
BS_c(s^F, v^F) \times_K E^\infty
\end{array}
\]

where \( \eta \) is the canonical projection.

Suppose that we have homogeneous elements \( h \in H^*_K(\text{BS}_c(s^F, v^F), c \in S \) and \( h_m \in H^*_K(\text{BS}_c(s^F, v^F), k) \) for \( m = 1, \ldots, n \). Then we get

\[
\mu^*(c(h \otimes h_1 \otimes \cdots \otimes h_n)) = \mu^*(c(h_1 \otimes h_2 \otimes \cdots \otimes h_n)) = \mu^*(p_1^*(c) \cup \cdots \cup p_n^*(c)) = (\eta^F_{s,v})^*(c) \cup \cdots \cup (\eta^F_{s,v})^*(c) = c \mu^*(h \otimes h_1 \otimes \cdots \otimes h_n).
\]

6. Results for the big torus

6.1. Fixed points. In Section 2.3 we identified the set of \( K \)-fixed points \( \text{BS}_c(s)^K \) with the set of generalized combinatorial galleries \( \Gamma(s) \). We would like to know what the constructions of Section 3 look like when restricted to the fixed points under this identification. We set

\[
\Gamma(s, v) = \{ \gamma \in \Gamma(s) \mid \forall r \in R : \gamma_r \gamma_{r+1} \cdots \gamma_{r+2} = v_r \}.
\]

We also define the map \( p^F_K : \Gamma(s, v) \to \Gamma(s^F, v^F) \) by

\[
p^F_K(\gamma)_i = v^{i} \gamma_i(v^{i})^{-1}.
\]

We get the following obvious result.

Lemma 12. We have \( \text{BS}_c(s, v)^K = \Gamma(s, v) \). Moreover, the projection \( p^F \) restricted to the \( K \)-fixed points is \( p^F_K \).

Note that \( \mu \) maps \( \Gamma(s, v) \times_K E^\infty \) to the following subspace of \( M \):

\[
M_K = (\Gamma(s^F, v^F) \times_K E^\infty) \times \prod_{m=1}^n \Gamma(s^F, v^F) \times_K E^\infty.
\]

We denote by \( \mu_K : \Gamma(s, v) \times_K E^\infty \to M_K \) the restriction of \( \mu \). Thus we get the map

\[
\mu_K^* : H^*_K(M_K, k) \cong H^*_K(\Gamma(s^F, v^F), k) \otimes_k \bigotimes_{m=1}^n H^*_K(\Gamma(s^F, v^F), k) \to H^*_K(\Gamma(s, v), k)
\]

similarly to Section 5.6. This map is automatically surjective as \( \mu_K \) is an embedding and all spaces are discrete. We also make \( H^*_K(\Gamma(s^F, v^F), k) \) into an \( S-Q \)-bimodule by letting \( S \) act on the left canonically and defining the right \( Q \)-action by

\[
h(c_1 \otimes \cdots \otimes c_n) = h \cup (\sigma^K_m(c_1) \cup \cdots \cup (\sigma^K_m(c_n)),
\]

where \( \sigma^K_m : \Gamma(s^F, v^F) \times_K E^\infty \to E^\infty/K \) is the restriction of \( \sigma^K_m \). Repeating the arguments of Section 5.6 we get the following result.

Lemma 13. The embedding \( \mu_K \) induces the isomorphism of left \( S \)-modules

\[
\mu^*_K : H^*_K(\Gamma(s^F, v^F), k) \otimes_Q \bigotimes_{m=1}^n H^*_K(\Gamma(s^F, v^F), k) \cong H^*_K(\Gamma(s, v), k).
\]
If \((s, v)\) is of gallery type, then the following diagram is commutative:

\[
\begin{array}{c}
\xymatrix{
H^*_{K}(\text{BS}_c(s^F, v^F), \mathbb{k}) \ar[r]_{\mu^\gamma_{K}} & H^*_{K}(\text{BS}_c(s_{f_{m}}, v_{f_{m}}), \mathbb{k}) \\
H^*_{K}(\Gamma(s^F, v^F), \mathbb{k}) \ar[u] \ar[r]_{\mu^\gamma_{K}} & H^*_{K}(\Gamma(s_{f_{m}}, v_{f_{m}}), \mathbb{k}) \\
\end{array}
\]

where the vertical arrows are restriction and tensor product of restrictions.

Let us compute the map \((\sigma^K_m)^* : S \to H^*_{K}(\Gamma(s^F, v^F), \mathbb{k}) = S(\Gamma(s^F, v^F))\) exactly. Note that the map \(\sigma^K_m\) is actually a map from the disjoint union \(\Gamma(s^F, v^F) \times K E^\infty \cong \Gamma(s^F, v^F) \times (E^\infty/K)\) of several copies \(E^\infty/K\) to \(E^\infty/K\). For each \(\gamma \in \Gamma(s^F, v^F)\), the restriction of this map to \(\{\gamma\} \times (E^\infty/K)\) equals \(\rho_{(\gamma f^m_1 \cdots v_{f_{m-1}}^-1)/K}\) in the notation of Section 1.3 Hence by (11), we get

\[(\sigma^K_m)^*(c)(\gamma) = \gamma f^m_1 \cdots v_{f_{m-1}} c.
\]

From this formula, we get the right action of \(Q\) on \(H^*_{K}(\Gamma(s^F, v^F), \mathbb{k})\):

\[h(c_1 \otimes \cdots \otimes c_n)(\gamma) = h(\gamma) \prod_{m=1}^{n} \gamma f^m_1 \cdots v_{f_{m-1}} c_m.
\]

Finally, let us compute the map \(\mu^K_{K}\). Let \(\gamma\) be an arbitrary point of \(\Gamma(s, v)\). Let us denote \(\delta = p^K_{K}(\gamma)\) for brevity. We have the following commutative diagram

\[
\begin{array}{c}
E^\infty/K \xrightarrow{j_\gamma} \Gamma(s, v) \times K E^\infty \\
\downarrow \quad \quad \quad \downarrow \rho_{1\mu_{K}}
\Gamma(s^F, v^F) \times K E^\infty
\end{array}
\]

Here \(j_\gamma\) and \(j_{\delta}\) are the maps defined in Section 1.3 Moreover, for each \(m = 1, \ldots, n\), we denote \(\delta^m = \gamma f_{m}\) and \(w_m = \gamma f_{m-1}^-\) for brevity. We get the following commutative diagram:

\[
\begin{array}{c}
E^\infty/K \xrightarrow{j_\gamma} \Gamma(s, v) \times K E^\infty \\
\downarrow \quad \quad \quad \downarrow \rho_{w_{m-1}/K} \\
E^\infty/K \xrightarrow{j_{\delta}} \Gamma(s_{f_{m}}, v_{f_{m}}) \times K E^\infty
\end{array}
\]

Suppose that we have \(h \in H^*_{K}(\text{BS}_c(s^F, v^F), \mathbb{k})\) and \(h_m \in H^*_{K}(\Gamma(s_{f_{m}}, v_{f_{m}}), \mathbb{k})\) for \(m = 1, \ldots, n\). Using the commutative diagrams above, we get

\[
\left(\mu^K_{K}(h \otimes h_1 \otimes \cdots \otimes h_n)\right)(\gamma) = j_{\gamma}^* \mu^K_{K}(p^K_{1}(h) \cup p^K_{2}(h_{1}) \cup \cdots \cup p^K_{n+1}(h_{n}))
= (p_{1\mu_{K}j_{\gamma}})^*(h) \cup (p_{2\mu_{K}j_{\gamma}})^*(h_{1}) \cup \cdots \cup (p_{n+1\mu_{K}j_{\gamma}})^*(h_{n})
= j_{\delta}^*(h) \cup (j_{1\delta}^*(\rho_{w_{n-1}/K})^*(h_{1}) \cup \cdots \cup (j_{\delta^{(n)}}^*(\rho_{w_{n-1}/K})^*(h_{n})
= h(\delta) \cup (\rho_{w_{n-1}/K})^*(h_{1}(\delta^{(1)})) \cup \cdots \cup (\rho_{w_{n-1}/K})^*(h_{n}(\delta^{(n)})).
\]

Applying (11) and omitting the sign of the cup product, we get

\[
\mu^K_{K}(h \otimes h_1 \otimes \cdots \otimes h_n)(\gamma) = h(p^K_{K}(\gamma)) \left(\gamma f_{1}^-h_{1}(\gamma f_{1})\right) \cdots \left(\gamma f_{n-1}^-h_{n}(\gamma f_{n})\right).
\]
6.2. Images, localization and good rings. We denote by $\mathcal{X}_c(s)$ the image of the restriction

$$H^*_K(\text{BS}_c(s), \mathbb{k}) \to H^*_K(\Gamma(s), \mathbb{k}).$$

Similarly, let $\mathcal{X}_c(s, x)$ and $\mathcal{X}_c(s, v)$, where $x \in W$ and $v : R \to W$ is a map as in Section 3 denote the images of the restrictions

$$H^*_K(\text{BS}_c(s, x), \mathbb{k}) \to H^*_K(\Gamma(s, x), \mathbb{k}), \quad H^*_K(\text{BS}_c(s, v), \mathbb{k}) \to H^*_K(\Gamma(s, v), \mathbb{k})$$

respectively.

Let recall the homeomorphism $d_w : \text{BS}_c(s) \to \text{BS}_c(s^w)$ from Section 2.3. We get the homeomorphism $d_w \times_K \rho_w : \text{BS}_c(s) \times_K E^\infty \to \text{BS}_c(s^w) \times_K E^\infty$. Hence we get the isomorphism $d^{\text{im}}_w : \mathcal{X}_c(s^w) \to \mathcal{X}(s)$ such that the diagram

$$\begin{array}{ccc}
H^*_K(\text{BS}_c(s^w), \mathbb{k}) & \xrightarrow{(d_w \times_K \rho_w)^*} & H^*_K(\text{BS}_c(s), \mathbb{k}) \\
\downarrow & & \downarrow \\
\mathcal{X}_c(s^w) & \xrightarrow{d^{\text{im}}_w} & \mathcal{X}(s)
\end{array}$$

is commutative. Similarly, considering the spaces $\text{BS}_c(s, x)$ and $\text{BS}(s^w, wxw^{-1})$, we get the isomorphism $d^{\text{im}}_{w, x} : \mathcal{X}_c(s^w, wxw^{-1}) \to \mathcal{X}(s, x)$. Applying (11), the reader can easily check that

$$d^{\text{im}}_w(f)(\lambda) = w^{-1}f(\lambda^w), \quad d^{\text{im}}_{w, x}(f)(\lambda) = w^{-1}f(\lambda^w). \quad (40)$$

In a similar way, we can consider the homeomorphism $D_\gamma : \text{BS}_c(s) \to \text{BS}_c(s^{(\gamma)})$ defined in Section 3. It is $K$-equivariant. Therefore, it induces the isomorphisms $D^{\text{im}}_\gamma : \mathcal{X}_c(s^{(\gamma)}) \to \mathcal{X}_c(s)$ and $D^{\text{im}}_{\gamma, x} : \mathcal{X}(s^{(\gamma)}, x^{(\gamma, \text{max})^{-1}}) \to \mathcal{X}(s, x)$. Applying (11), we get

$$D^{\text{im}}_\gamma(f)(\lambda) = f(\gamma^{-1} \circ \lambda), \quad D^{\text{im}}_{\gamma, x}(f)(\lambda) = f(\gamma^{-1} \circ \lambda), \quad (41)$$

where we used the operations on the generalized combinatorial galleries defined in Section 2.3.

To obtain the tensor product theorem for images similar to Theorem 11 we need to impose some restrictions on the ring of coefficients $\mathbb{k}$.

**Definition 14.** We say that a commutative ring $\mathbb{k}$ is good if it has finite global dimension, 2 is invertible in $\mathbb{k}$ and for any sequence of simple reflections $s$ and any $x \in W$, the restrictions

$$H^*_T(\text{BS}(s), \mathbb{k}) \to H^*_K(\Gamma(s), \mathbb{k}), \quad H^*_T(\text{BS}(s, x), \mathbb{k}) \to H^*_K(\Gamma(s, x), \mathbb{k})$$

are injective and the restriction $H^*_T(\text{BS}(s), \mathbb{k}) \to H^*_T(\text{BS}(s, x), \mathbb{k})$ is surjective.

The results of [SI1] imply that any principal ideal domain with invertible 2 is a good ring.
Suppose that \((s, v)\) is of gallery type and \(k\) is a good ring. Then we have the following commutative diagram:

\[
\begin{array}{c}
H^*_K(\text{BS}_c(s^F, v^F), k) \otimes Q \bigotimes_{m=1}^{n} H^*_K(\text{BS}_c(s_{f,m}, v_{f,m}), k) \xrightarrow{\mu^c} H^*_K(\text{BS}_c(s, v), k) \\
\downarrow \hspace{1cm} \downarrow \\
\Lambda^*_c(s^F, v^F) \otimes Q \bigotimes_{m=1}^{n} \Lambda^*_c(s_{f,m}, v_{f,m}) \xrightarrow{\mu^c_{f,m}} \Lambda^*_c(s, v) \\
\downarrow \hspace{1cm} \downarrow \\
H^*_K(\Gamma(s^F, v^F), k) \otimes Q \bigotimes_{m=1}^{n} H^*_K(\Gamma(s_{f,m}, v_{f,m}), k) \xrightarrow{\mu^c_k} H^*_K(\Gamma(s, v), k)
\end{array}
\]

This diagram can actually be obtained from Diagram (37) by inserting the middle row.

6.3. \textbf{Surjectivity of restriction.} We introduce the following parameter that we plan to use for induction. Let \(v : R \to W\) be a map, where \(R\) is a nested structure on a finite totally ordered set \(I\) (see Section 3.1). Its \textit{index} \(\text{ind}_I v\) is a pair \((|R|, \eta), \) where \(\eta = 0\) if \(R\) is closed and \(n = 1\) otherwise. These indices are compared lexicographically: \((x, n) < (x', n')\) if and only if \(x < x'\) or \(x = x'\) and \(n < n'\).

\textbf{Theorem 15.} Suppose that \(k\) is a good ring. Then for any pair \((s, v)\) of gallery type the restriction \(H^*_K(\text{BS}_c(s), k) \to H^*_K(\text{BS}_c(s, v), k)\) is surjective.

\textbf{Proof.} We denote by \(\iota\) the natural embedding \(\text{BS}_c(s, v) \hookrightarrow \text{BS}_c(s)\). So we have to prove the surjectivity of \(\iota^* : H^*_K(\text{BS}_c(s), k) \to H^*_K(\text{BS}_c(s, v), k)\).

We will prove the lemma by induction on \(\text{ind}_I v\). It is easy to see that two minimal possible values are \((0, 1)\) and \((1, 0)\). In the first case, the restriction is an identical map. So we consider the second case. We get \(R = \{\text{span} I\}\) and \(s\) is itself of gallery type. We have \(\text{BS}_c(s, v) = \text{BS}_c(s, w)\), where \(w = v_{\text{span} R}\). Let \((x, t, \gamma)\) be a gallerification of \(s\) (see Section 2.3). We get the following commutative diagram

\[
\begin{array}{c}
\text{BS}_c(s, w) \xrightarrow{\varphi_w} \text{BS}_c(s) \\
\varphi \downarrow \hspace{1cm} \varphi \downarrow \\
\text{BS}(t, xwx^{-1}\gamma_{\text{max}}) \xleftarrow{\iota} \text{BS}(t)
\end{array}
\]

(see the calculations from the proof of [3, Lemma 9]). The isomorphisms \(\varphi\) and \(\varphi_w\) have the property \(\varphi(ka) = \hat{\chi} k^{-1} \varphi(a)\) and \(\varphi_w(ka) = \hat{\chi} k^{-1} \varphi_w(a)\) for \(k \in K\) and \(a \in \text{BS}_c(s)\) or \(a \in \text{BS}_c(s, w)\) respectively. Multiplying by \(E^\infty\) and taking taking quotients, we get the commutative diagram

\[
\begin{array}{c}
\text{BS}_c(s, w) \times_K E^\infty \xrightarrow{\varphi_w \times_K \rho_z} \text{BS}_c(s) \times_K E^\infty \\
\varphi_{\text{BS}_c(s,w)} \downarrow \hspace{1cm} \varphi_{\text{BS}_c(s)} \downarrow \\
\text{BS}(t, xwx^{-1}\gamma_{\text{max}}) \times_K E^\infty \xleftarrow{\iota} \text{BS}(t) \times_K E^\infty
\end{array}
\]
Taking cohomologies, we get the commutative diagram

\[
\begin{array}{ccc}
H_K(BS_c(s, w), k) & \xleftarrow{i} & H_K(BS_c(s), k) \\
H^*_T(\text{BS}(t, xw^{-1}t^{\max}), k) & \xleftarrow{i'} & H^*_T(\text{BS}(t), k)
\end{array}
\]

The bottom arrow is surjective by the surjectivity condition. Hence the upper one is also surjective.

Now suppose that \(\text{ind}_I v\) is neither \((0, 1)\) nor \((1, 0)\). Let us consider the case where the second component of \(\text{ind}_I v\) equals 1, that is, \(\text{span } I \notin R\). Let \(F\) be the set of all maximal elements of \(R\). Our assumption implies that \(F \neq \emptyset\). By Lemma [3] and the K"unneth formula, we get that \(H^*_K(BS_c(s, v), k)\) is generated by elements \(\mu^* p^*_1(h)\) and \(\mu^* p^*_{m+1}(h_m)\), where \(h \in H^*_K(BS_c(s^F), k)\) and \(h_m \in H^*_K(BS_c(s_{f_m}, v_{f_m}), k)\). We need to prove that all these elements can be lifted to some elements of \(H^*_K(BS_c(s), k)\).

First consider the element \(\mu^* p^*_1(h)\). By Lemma 5.4, it suffices to consider the case \(h = \Sigma(s^F, i, 1)^* (a)\) for some \(i \in \mathbb{F} \cup \{-\infty\}\) and \(a \in S\). We get

\[
\mu^* p^*_1(h) = (p_1 \mu)^*(h) = (p^F \times K \text{id})^*(h)
\]

\[
= (\Sigma(s^F, i, 1)(p^F \times K \text{id}))^*(a) = (\Sigma(s, i, (v^i)^{-1})\mu)^*(a) = v^* \Sigma(s, i, (v^i)^{-1})^*(a).
\]

Hence \(\Sigma(s, i, (v^i)^{-1})^*(a)\) delivers the required lifting of \(\mu^* p^*_1(h)\).

Now let us lift elements \(\mu^* p^*_{m+1}(h_m) = (p_{m+1} \mu)^*(h_m) = (q^{m, \infty})^*(h)\). To do it notice that we can extend the map \(q^{m, \infty} : BS_c(s, v) \times_K E^\infty \rightarrow BS_c(s_{f_m}, v_{f_m}) \times_K E^\infty\) defined in Section 5.4 to the map \(\tilde{q}^{m, \infty} : BS_c(s) \times_K E^\infty \rightarrow BS_c(s_{f_m}, v_{f_m}) \times_K E^\infty\) given by (20) as well as \(q^{m, \infty}\). We get the commutative diagram

\[
\begin{array}{ccc}
BS_c(s) \times_K E^\infty & \xrightarrow{\tilde{q}^{m, \infty}} & BS_c(s_{f_m}) \times_K E^\infty \\
\times \text{id} & \xleftarrow{i} & \times \text{id} \\
BS_c(s, v) \times_K E^\infty & \xrightarrow{q^{m, \infty}} & BS_c(s_{f_m}, v_{f_m}) \times_K E^\infty
\end{array}
\]

Taking cohomologies, we get the commutative diagram

\[
\begin{array}{ccc}
H^*_K(BS_c(s), k) & \xleftarrow{(q^{m, \infty})^*} & H^*_K(BS_c(s_{f_m}), k) \\
\downarrow i^* & & \downarrow i^* \\
H^*_K(BS_c(s, v), k) & \xleftarrow{(q^{m, \infty})^*} & H^*_K(BS_c(s_{f_m}, v_{f_m}), k)
\end{array}
\]

Noting that \(\text{ind}_{[f_m]} v_{f_m} < \text{ind}_I v\), we get by the inductive hypothesis that there exists some \(\tilde{h} \in H^*_K(BS_c(s_{f_m}), k)\) whose restriction to \(BS_c(s_{f_m}, v_{f_m})\) is \(h\). From the above diagram, we get \((q^{m, \infty})^*(h) = i^*(q^{m, \infty})^*(\tilde{h})\). Hence \((q^{m, \infty})^*(\tilde{h})\) delivers the required lifting.

Now let us consider the case where the second component of \(\text{ind}_I v\) equals 0, that is, \(\text{span } I \in R\). Let \(F\) be the set of all maximal elements of the set \(\bar{R} = R \setminus \{\text{span } I\}\). As \(\text{ind}_I v \neq (1, 0)\), we get that \(F \neq \emptyset\). The arguments of the previous case work now with the only exception: we need to show that \(\mu^* p^*_1(h)\) can be lifted to an element of \(H^*_K(BS_c(s), k)\) for any \(h \in H^*_K(BS_c(s^F), k)\). We denote \(w = v_{\text{span } I}\). Then we have \(BS_c(s^F, v^F) = BS_c(s^F, w)\). Consider the restriction \(\tilde{v} = v|_{\bar{R}}\). Note that \(\text{ind}_I \tilde{v} < \text{ind}_I v\). Let \(\tilde{p}^F : BS_c(s, \tilde{v}) \rightarrow BS_c(s^F)\) be the projection along \(F\). We get the following commutative
Let $X$ be a nested structure on $I$. Then there is the isomorphism of left $T$-equivariant cohomologies.

Theorem 16. Let $s$ be a sequence of simple reflections on a finite totally ordered set $I$, $R$ be a nested structure on $I$ and $v : R \to W$ be an arbitrary function. Let $F = \{f_1 < \cdots < f_n\}$ be the set of all maximal pairs of $R$. Suppose that $s^F$ and $(s, v)$ are of gallery type and that $k$ is a good ring. Let $(t, \gamma, x)$ be a gallerification of $s^F$. We consider $X(t)$ as an $S$-$Q$-bimodule with the following actions (denoted by $\cdot$):

$$(c \cdot h)(\lambda) = (xc)h(\lambda), \quad (h \cdot (c_1 \otimes \cdots \otimes c_n))(\lambda) = h(\lambda) \prod_{m=1}^n \lambda^{f_m^\tau(\gamma f_m)^{-1}} x v\gamma_1 \cdots v_{f_m-1} c_m. \quad (43)$$

Then there is the isomorphism of left $S$-modules

$$\mu_T^{im} : X(t) \otimes_Q \bigotimes_{m=1}^n X(s_{f_m}, v_{f_m}) \xrightarrow{\sim} X(s, v)$$

given by

$$\mu_T^{im}(h \otimes h_1 \otimes \cdots \otimes h_n)(\lambda) = \left(x^{-1} h(\gamma p_K^F(\lambda)^x) \right) \prod_{m=1}^n \lambda^{f_m^{-1} h_m(\lambda_{f_m})}. \quad (44)$$
Proof. The middle line of Diagram (42) can be written as

$$\mu_{\imath}^{{\imath,m}} : \mathcal{X}_c(s^F) \otimes Q \bigotimes_{m=1}^{n} \mathcal{X}(s_{f_m}, v_{f_m}) \sim \mathcal{X}(s, v).$$

It remains to find an isomorphism for the $S$-$Q$-bimodule $\mathcal{X}_c(s^F)$. We have $s^F = (t^{\gamma})^{x^{-1}}$. So we have the following isomorphisms:

$$\mathcal{X}_c(s^F) \xrightarrow{d_{x^{-1}}} \mathcal{X}_c(t^{\gamma}) \xrightarrow{D_{x^{-1}}} \mathcal{X}_c(t) = \mathcal{X}(t).$$

Let $\varphi = D_{x^{-1}}^{\imath}d_{x^{-1}}^{\imath}$ denote their composition. Applying direct formulas (40) and (41), we get

$$\varphi(h)(\lambda) = \left(D_{x^{-1}}^{\imath}d_{x^{-1}}^{\imath}(h)\right)(\lambda) = d_{x^{-1}}^{\imath}(h)(\gamma^{-1} \circ \lambda) = xh\left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right).$$

We can easily invert $\varphi$ as follows: $\varphi^{-1}(h)(\lambda) = x^{-1}h(\gamma \circ \lambda^x)$. For any $c \in S$ and $h \in \mathcal{X}(t)$, we get

$$c \cdot h(\lambda) = \varphi(c\varphi^{-1}(h))(\lambda) = x(c\varphi^{-1}(h))\left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right) = (xc)(x\varphi^{-1}(h))\left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right) = (xc)h(\lambda).$$

On the other hand, for any $c_1, \ldots, c_n \in S$, applying (38), we get

$$(h \cdot (c_1 \otimes \cdots \otimes c_n))(\lambda) = \varphi(\varphi^{-1}(h)(c_1 \otimes \cdots \otimes c_n))(\lambda) = x(\varphi^{-1}(h)(c_1 \otimes \cdots \otimes c_n))\left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right)$$

$$= x\left(\varphi^{-1}(h)\left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right) \prod_{m=1}^{n} \left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right)^{f_{m}^{x}} v_{f_{1} \cdots v_{f_{m-1}} e_{m}}\right)$$

$$= h(\lambda) \prod_{m=1}^{n} x\left((\gamma^{-1} \circ \lambda)^{x^{-1}}\right)^{f_{m}^{x}} v_{f_{1} \cdots v_{f_{m-1}} e_{m}}$$

$$= h(\lambda) \prod_{m=1}^{n} \left((\gamma^{-1} \circ \lambda)^{f_{m}^{x}} x v_{f_{1} \cdots v_{f_{m-1}} e_{m}}\right)$$

$$= h(\lambda) \prod_{m=1}^{n} \lambda^{f_{m}^{x}}(\gamma^{-1} x)^{-1} v_{f_{1} \cdots v_{f_{m-1}} e_{m}}.$$
The structure of an S-Q-module on the first factor is given by \((43)\) and the isomorphism is given by \((44)\).

**Proof.** We have the isomorphism of the middle line of \((42)\)

\[
\mu_K^{im} : \mathcal{X}_c(s^F, v^F) \otimes_Q \bigotimes_{m=1}^n \mathcal{X}(s_{f_m}, v_{f_m}) \sim \mathcal{X}(s, v).
\]

Now note that \(s^F = s^{(\text{span } I, v)}\) and \(v^F = w(v_{f_1} \cdots v_{f_n})^{-1}\). Hence

\[
\mathcal{X}_c(s^F, v^F) = \mathcal{X}_c(s^{(\text{span } I, v)}, w(v_{f_1} \cdots v_{f_n})^{-1}) \cong \mathcal{X}(t, xw(xv_{f_1} \cdots v_{f_n})^{-1}, \gamma_{\text{max}}).
\]

To describe the structure of an S-Q-bimodule on the right set above and to describe exactly the map \(\mu_T^{im}\), we just repeat the proof of Theorem 17.

\[\square\]

6.5. **Dual modules.** Finally, we describe how the above results induce the maps between the dual modules. In what follows, we assume that \(k\) is a good ring. For a sequence of simple reflections \(s\) on \(I\), a nested structure \(R\) on \(I\) and \(v : R \to W\), we consider the following set of maps:

\[D\mathcal{X}(s, v) = \{\alpha : \Gamma(s, v) \to \text{Frac}(S) \mid (\alpha, g) \in S \ \forall g \in \mathcal{X}(s, v)\},\]

where \(\text{Frac}(S)\) is the field of fractions of \(S\) and \((\cdot, \cdot)\) is the standard Euclidean product:

\[(\alpha, g) = \sum_{\lambda \in \Gamma(s, v)} \alpha(\lambda)g(\lambda).\]

Let \(\tilde{R}\) be a subset of \(R\) and \(\tilde{v} = v|_{\tilde{R}}\). Then for any \(\alpha \in D\mathcal{X}(s, v)\) the continuation by zero \(\tilde{\alpha}\) to \(\Gamma(s, \tilde{v})\) obviously belongs to \(D\mathcal{X}(s, \tilde{v})\). Indeed for any \(g \in \mathcal{X}(s, \tilde{v})\), we get

\[(\tilde{\alpha}, g) = (\alpha, g|_{\Gamma(s, v)}) \in S,\]

as \(g|_{\Gamma(s, v)} \in \mathcal{X}(s, v)\).

**Corollary 18.** It the notation of Theorem 17, there is the map

\[D\mu_T^{im} : D\mathcal{X}(t) \times \prod_{m=1}^n D\mathcal{X}(s_{f_m}, v_{f_m}) \to D\mathcal{X}(s, v)\]

given by

\[D\mu_T^{im}(\alpha, \alpha_1, \ldots, \alpha_n)(\lambda) = \left(x^{-1}a_1\gamma \circ p^F_K(\lambda)^x\right) \prod_{m=1}^n \lambda_{f_m}^{t_m-1} \gamma_{f_m}(\lambda_{f_m}^{t_m}). (45)\]

**Proof.** Let us denote \(\beta = D\mu_T^{im}(\alpha, \alpha_1, \ldots, \alpha_n)\) for brevity. As \(\beta\) is clearly a map from \(\Gamma(s, v)\) to \(\text{Frac}(S)\), we only have to prove that \(\beta \in D\mathcal{X}(s, v)\). Applying the definition of \(D\mathcal{X}(s, v)\) above, we see that we have to sum over \(\lambda \in \Gamma(s, v)\). It is easy to prove that the map \(\lambda \mapsto (p_K^F(\lambda), \lambda_{f_1}, \ldots, \lambda_{f_n})\) is a bijection from \(\Gamma(s, v)\) to \(\Gamma(s^F) \times \Gamma(s_{f_1}, v_{f_1}) \times \cdots \times \Gamma(s_{f_n}, v_{f_n})\).

Indeed the inverse map is given by the map \(\theta\) from \([3] \text{ Section 3.} 4\) to \(\Gamma(t) \times \Gamma(s_{f_1}, v_{f_1}) \times \cdots \times \Gamma(s_{f_n}, v_{f_n})\). Therefore, we sum over \(\delta = \gamma \circ p_K^F(\lambda)^x\) and the segments \(\lambda_{f_1}, \ldots, \lambda_{f_n}\). We get \(p_K^F(\lambda)^{t_m-1} = \lambda_{f_m}^{t_m-1}(v_1 \cdots v_{m-1})^{-1}\) for any \(\lambda \in \Gamma(s, v)\). It follows from this formula that

\[\delta^{t_m-1} = (\gamma \circ p_K^F(\lambda)^x)^{\lambda_{f_m}^{t_m-1}} - (\gamma \circ p_K^F(\lambda)^x)^{\lambda_{f_m}^{t_m-1}} - \gamma^{t_m-1} = x\lambda^{t_m-1}(v_1 \cdots v_{m-1})^{-1} - \gamma^{t_m-1}.
\]

Hence

\[\lambda_{f_m}^{t_m-1} = x^{-1}\delta^{t_m-1}(\gamma^{t_m-1})^{-1}x v_1 \cdots v_{m-1}.
\]
Now we compute the scalar product as follows:

\[
(\beta, \mu_T^{im}(h \otimes h_1 \otimes \cdots \otimes h_n)) = \sum_{\delta \in \Gamma(t)} (x^{-1}(\alpha(\delta)h(\delta))) \prod_{m=1}^{n} x^{-1}\delta f^{m-1}(\gamma f^{m-1})^{-1}xv_1 \cdots v_{m-1}c_m,
\]

where \(c_m = (a_m, h_m)\), which is an element of \(S\).

For each \(m = 1, \ldots, n\), we consider the function \(g_m : \Gamma(t) \to S\) defined by \(g_m(\delta) = \delta f^{m-1}(\gamma f^{m-1})^{-1}xv_1 \cdots v_{m-1}c_m\). Note that \(g_m\) belongs to \(\mathcal{X}(t)\), which we identify with \(\mathcal{X}(t)\), as this map is the restriction of \(\Sigma(t, f_1^m, -1, (\gamma f_1^m)^{-1}xv_1 \cdots v_{m-1})^*(c_m)\) to \(\Gamma(t)\). From (46), we get

\[
(\beta, \mu_T^{im}(h \otimes h_1 \otimes \cdots \otimes h_n)) = x^{-1}(\alpha, hg_1 \cdots g_n).
\]

The right-hand side belongs to \(S\), as we take the (cup) product \(hg_1 \cdots g_n\) of elements of \(\mathcal{X}(t)\).

The same arguments allow us to prove the following result.

**Corollary 19.** It the notation of Theorem 17, there is the map

\[
D\mathcal{X}(t, xv(xv_1 \cdots v^m f^m)_1^{\gamma_{\text{max}}}) \times \prod_{m=1}^{n} D\mathcal{X}(s f^m, v f^m) \to D\mathcal{X}(s, v)
\]

given by (43).

### 6.6. Example of a decomposition.

Suppose that \(G = \text{SL}_3(\mathbb{C})\) with the following Dynkin diagram:

```
α1  α2  α3  α4
```

The simple reflections are \(\omega_i = s_{\alpha_i}\). We consider the following sequence and the element of the Weyl group

\[
s = (\omega_1, \omega_3, \omega_1, \omega_2, \omega_1, \omega_2, \omega_1, \omega_2, \omega_3, \omega_4), \quad w = \omega_1\omega_3\omega_1\omega_3.
\]

We would like to study the module \(\mathcal{X}(s, w)\) under the assumption that the ring of coefficients \(k\) is good. It is easy to note that \(\Gamma(s, w) = \Gamma(s, v)\), where \(v\) is the map from \(R = \{(1, 10), (4, 8)\}\) to \(W\) given by \(v((1, 10)) = w\) and \(v((4, 8)) = \omega_1\). Thus \(\mathcal{X}(s, w) = \mathcal{X}(s, v)\). We can apply Theorem 17 to \(\mathcal{X}(s, v)\) for \(F = \{(4, 8)\}\). Let \(t = (\omega_4, \omega_3, \omega_1, \omega_3, \omega_4)\) and \(\gamma = (1, 1, 1, 1, 1, 1)\). As we have \(s^{(\text{span} F, v)} = s^{F} = t\), the triple \((1, t, \gamma)\) is a gallerification of \(s^{(\text{span} F, v)}\). We obtain the isomorphism of left \(S\)-modules

\[
\mu_T^{im} : \mathcal{X}(t, \omega_2\omega_4\omega_3) \otimes S \mathcal{X}(\omega_2, \omega_1, \omega_2, \omega_1, \omega_2, \omega_1, \omega_2, \omega_3, \omega_4) \cong \mathcal{X}(s, w).
\]

Here all actions of \(S\) are canonical except the right action on \(\mathcal{X}(t, \omega_2\omega_4\omega_3)\). It is given by \((h \cdot c)(\lambda) = h(\lambda)(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\lambda_8)\), where \(h \in \mathcal{X}(t, \omega_2\omega_4\omega_3)\) and \(c \in S\). The above isomorphism is given by the formula

\[
\mu_T^{im}(h \otimes g)(\lambda) = h(\lambda_1, \lambda_2, \lambda_3, \lambda_9, \lambda_{10})(\lambda_1\lambda_2\lambda_3g(\lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)).
\]

We can also represent the minimal degree element \(a \in D\mathcal{X}(s, w)\) as a twisted product of the minimal degree elements \(b \in D\mathcal{X}(t, \omega_2\omega_4\omega_3)\) and \(c \in D\mathcal{X}(\omega_2, \omega_1, \omega_2, \omega_1, \omega_2, \omega_1)\). As all three Bott-Samelson varieties are smooth we get the following values of these functions: \(a\)
maps
\[
(1, 1, \omega_3, 1, 1, 1, \omega_1, 1, \omega_3, \omega_4), (\omega_4, 1, 1, 1, 1, \omega_1, 1, \omega_3, \omega_4), (1, 1, \omega_3, \omega_2, 1, \omega_2, \omega_1, 1, \omega_3, \omega_4),
\]
\[
(1, 1, \omega_4, 1, \omega_1, 1, \omega_2, \omega_3, \omega_4), (\omega_4, 1, 1, 1, \omega_1, 1, 1, 1, \omega_3, \omega_4),
\]
\[
(1, 1, \omega_4, 1, \omega_1, 1, \omega_2, \omega_3, \omega_4), (\omega_4, 1, 1, 1, \omega_1, 1, 1, 1, \omega_3, \omega_4),
\]
\[
(1, 1, \omega_4, 1, \omega_1, 1, \omega_2, \omega_3, \omega_4), (\omega_4, 1, 1, 1, \omega_1, 1, 1, 1, \omega_3, \omega_4),
\]
\[
(\omega_4, \omega_3, 1, 1, 1, 1, \omega_1, 1, \omega_3, \omega_4), (\omega_4, \omega_3, 1, 1, 1, 1, \omega_1, 1, \omega_3, \omega_4),
\]
\[
(1, \omega_3, 1, 1, 1, \omega_1, 1, \omega_3, \omega_4), (\omega_4, \omega_3, 1, 1, 1, 1, \omega_1, 1, \omega_3, \omega_4),
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respectively. One can easily check that $a$ is a twisted product of $b$ and $c$ in the sense of (45), which in our case is

$$c(\lambda) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_9, \lambda_{10})(\lambda_1 \lambda_2 \lambda_3 c(\lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)).$$

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