A general strong law of large numbers for additive arithmetic functions

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August 5, 2009

Abstract

Let \( f(n) \) be a strongly additive complex valued arithmetic function. Under mild conditions on \( f \), we prove the following weighted strong law of large numbers: if \( X, X_1, X_2, \ldots \) is any sequence of integrable i.i.d. random variables, then

\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} f(n)X_n}{\sum_{n=1}^{N} f(n)} \overset{a.s.}{=} E.X.
\]

1 Introduction and main result

Consider a strongly additive complex valued arithmetic function \( f(n), n = 1, 2, \ldots \). Thus \( f \) satisfies

\[
\begin{align*}
    f(mn) &= f(m) + f(n) & (m, n) = 1, \\
    f(p^n) &= f(p), & p \text{ a prime, } \alpha = 1, 2, \ldots 
\end{align*}
\]

and it follows that

\[
f(n) = \sum_{p\mid n} f(p),
\]

so that \( f \) is completely determined by its values taken over the prime numbers. We put

\[
F(n) = \sum_{m \leq n} f(m) \quad G(n) = \sum_{m \leq n} |f(m)|^2.
\]

Note that

\[
F(n) = \sum_{p \leq n} f(p) \frac{n}{p}, \quad G(n) = \sum_{p \leq n} |f(p)|^2 \frac{n}{p} + 2 \Re \left\{ \sum_{2 \leq p \leq q \leq n} f(p)f(q) \frac{n}{pq} \right\}.
\]

The general problem of determining the order of magnitude of additive arithmetic functions is a difficult task, and we refer to the books Elliott [4] and Kubilius [8] for a thorough treatment.

In this work we are interested in the validity of the weighted strong law of large numbers, when the weights are given by \( f \). More precisely, let \( X = \{X, X_m, m \geq 1\} \) be i.i.d. random variables with basic probability space \((\Omega, \mathcal{A}, P)\) and such that \( E|X| < \infty \). We look for criteria for the weighted SLLN, i.e. the relation

\[
\lim_{n \to \infty} \frac{\sum_{m=1}^{n} f(m)X_m}{F(n)} \overset{a.s.}{=} E.X.
\]

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Such an SLLN is a delicate refinement of the usual SLLN for i.i.d. random variables. Indeed, by rewriting the sum in (5) in the form

\[ \sum_{m \leq n} f(m)X_m = \sum_{m \leq n} X_m \sum_{p|m} f(p) = \sum_{p \leq n} f(p) \sum_{p \leq m \leq n} X_m \]

we see that (5) relies upon all averages of the type

\[ \frac{1}{m} \sum_{k=1}^{m} X_{kp}, \quad p \text{ prime and } m \geq 2, \quad (7) \]

and in fact it means that

\[ \lim_{n \to \infty} \frac{1}{m} \sum_{p \leq n} f(p) \left( \frac{n}{p} \right) \left( \frac{1}{\phi(p)} \sum_{k=1}^{\phi(p)} X_{kp} \right) \rightharpoonup E \cdot X. \]

Thus the validity of (5) is intimately connected with uniformity in the SLLN for the averages in (7). Put

\[ A_n = \left| \sum_{p \leq n} f(p) \right|, \quad B_n = \sum_{p \leq n} \frac{|f(p)|^2}{p}. \]

If \( f \) is real valued and nonnegative, then by (4) for any \( 0 < c \leq 1/2 \)

\[ c \frac{n A_{2n}}{A_n} \leq F(n) \leq n A_n \quad \text{and} \quad G(n) \leq n(B_n + A_n^2). \quad (8) \]

So if \( A_{2n} \asymp A_n \), it follows that \( F(n) \asymp n A_n \). (Here, and in the sequel, \( x_n \asymp y_n \) means \( 0 < \lim \inf_{n \to \infty} |x_n/y_n| \leq \lim \sup_{n \to \infty} |x_n/y_n| < \infty \).)

Naturally if \( f \) is complex valued, the above bound for \( G(n) \) ceases to be true, because the sum

\[ \Re \left\{ \sum_{2 \leq p < q \leq n} f(p) \overline{f(q)} \frac{n}{pq} \right\}, \]

is in general no longer comparable to \( A_n^2 \).

In a recent work [2], we studied the weighted SLLN when \( f \geq 0 \) and proved the following result (see Theorem 1.1 in [2]).

**Theorem 1** Assume that \( f \geq 0 \) and

\[ B_p \to \infty, \quad f(p) = o(B_p^{1/2}) \quad \text{as } p \to \infty, \quad (9) \]

Then (5) holds.

Condition (9) plays an important role in probabilistic number theory as a nearly optimal sufficient condition for the central limit theorem

\[ \lim_{N \to \infty} \frac{1}{N} \#\{n \leq N : \frac{f(n) - A_N}{B_N^{1/2}} \leq x\} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-t^2/2} dt \quad (10) \]
(see e.g. Elliott [4], Kubilius [8].) Halberstam [5] proved that replacing the $o$ by $O$ in (9) the CLT (10) becomes generally false. Note that relation (11) implies the Lindeberg condition
\[
\lim_{n \to \infty} \frac{1}{B_n} \sum_{p \leq n, |f(p)| \geq \varepsilon B_n} \frac{f^2(p)}{p} = 0 \quad \text{for any } \varepsilon > 0,
\]
and, under mild technical assumptions on $f$, condition (11) is necessary and sufficient for the CLT (10), see again Elliott [4], Kubilius [8].

In [2] we also proved that (9) implies the law of the iterated logarithm corresponding to (5) (see Theorem 1.2 in [2]). We further indicated that if $f(p)$ does not fluctuate too wildly, for instance if
\[
\sup_{n \leq p, q \leq n^2} \frac{f(p)}{f(q)} = O(1),
\]
then Theorem 1 remains valid under condition (11). We raised the question of the validity of Theorem 1 under the sole Lindeberg condition. Recently, Fukuyama and Komatsu [3] answered this question affirmatively.

**Theorem 2** Assume that $f \geq 0$ and the Lindeberg condition (11) is satisfied. Then (4) holds.

Their approach is simple and elegant and is based on Abel summation, and moreover it shows the interesting fact that the Lindeberg condition implies
\[
\sum_{p > t} \frac{f(p)}{p^2 A_p} = O(1/t).
\]
The estimates $F(n) \geq C_1 n A_n$, $G(n) \leq C_2 n A_n^2$, which are implied by (11) (see for instance Lemma 2.1 in [2]), are crucial in their proof, and their result remains valid under these sole conditions. In particular, this is the case if $A_{2n} \approx A_n$ and
\[
B_n^{1/2} = O(A_n).
\]
Actually, the condition
\[
F(n) \geq C_1 n \sup(A_n, B_n^{1/2})
\]
would also suffice, since by (8) it implies (14) and thus $G(n) \leq C_2 n A_n^2$. Condition (15) seems to be the relevant assumption in this problem. Note the interesting implication
\[
(15) \implies (13).
\]

A typical example of application is the well-known von Mangoldt arithmetical function $\Lambda$, which is neither additive nor multiplicative. Recall that $\Lambda$ is defined by
\[
\Lambda(n) = \begin{cases} 
\log p, & \text{if } n = p^k, \\
0, & \text{otherwise}.
\end{cases}
\]
It is elementary to see that
\[
A_n = \sum_{p < n} \frac{\Lambda(p)}{p} \sim \log n, \quad B_n = \sum_{p < n} \frac{\Lambda^2(p)}{p} \sim \log^2 n.
\]
Since $A_n$ is slowly varying, on using (8) it follows that (15) is valid. Therefore (5) holds when the weights are given by von Mangoldt’s function. Note also that in this case (13) reduces to the trivial estimate $\sum_{p>t} p^{-2} = O(1/t)$. By a result of Wierdl [11], the result extends to the case when $X$ is a stationary ergodic sequence in $L^p$, $p > 1$ (and not if $p = 1$, see [1]), which is a quite remarkable fact.

It can also be pointed out here that Abel summation alone suffices to prove a result valid for general weights. In fact, the proof in [3] yields the following

**Proposition 3** Let $f \geq 0$ be an arbitrary function and assume that

\[
\sum_{n>t} \frac{G(n)|F(n+1) - F(n)|}{F^2(n)F(n+1)} = O(1/t).
\] (18)

Then (5) holds.

**Remark 4** (a) If there exists a nondecreasing function $H(n)$ such that

\[
F(n) \geq C_1 n H(n), \quad G(n) \leq C_2 n H^2(n), \quad |F(n+1) - F(n)| \leq C_3 H(n),
\] (19)

then condition (18) is satisfied and thus (5) holds.

(b) Under the Lindeberg condition we have

\[
\sum_{n>t} \frac{G(n)|F(n+1) - F(n)|}{F^2(n)F(n+1)} \leq C \left(1 + \sum_{p>t} \frac{\mathbb{1}}{p^2 A_p}\right) = O(1/t).
\]

**Proof.** We have

\[
\#\{n > t : F(n) \leq tf(n)\} \leq t^2 \sum_{n>t} \frac{f^2(n)}{F^2(n)} = t^2 \sum_{n>t} \left(\sum_{t<k \leq n} f^2(k)\right) \left(\frac{1}{F^2(n)} - \frac{1}{F^2(n+1)}\right) \leq 2t^2 \sum_{n>t} \frac{G(n)|F(n+1) - F(n)|}{F^2(n)F(n+1)} \leq Ct.
\] (20)

Thus

\[
\sup_{t>0} \frac{1}{t} \#\{n : F(n) \leq tf(n)\} < \infty,
\]

which, by Lemma 2.1 (see Section 2) suffices to ensure (5).

It is natural to ask about extensions of these results for complex valued additive arithmetic functions. As we mentioned after (8), the complex valued case requires a different treatment. We do not know how to use Abel summation in this case. Also, no estimate of type $F(n) \geq C_1 n A_n, G(n) \leq C_2 n A^2_n$ or even $G(n) \leq C_3 n (B_n + A^2_n)$ is available. Further, the use of Abel summation leads to series involving $|f(n)|$, which are not related to

\[ A_n = \left|\sum_{2 \leq p \leq n} \frac{f(p)}{p}\right|. \]
We will show, however, that a slight modification in the use of the randomization argument introduced in the proof of Theorem 1.1 in [2] allows in turn to prove a rather general SLLN in this context.

Let us first introduce some notation. Let \( \varepsilon_i, i \geq 1 \) denote a Bernoulli sequence defined on a probability space \( (\tilde{\Omega}, \tilde{A}, \tilde{P}) \), with partial sums \( S_n = \varepsilon_1 + \ldots + \varepsilon_n \). Let \( \tilde{E} \) denote the corresponding expectation symbol. Put 

\[
\eta = \sup\{ \rho > 0 : \tilde{P}\{ \inf_{n \geq 1} S_n / n \geq \rho \} > 0 \}.
\]

It is immediate to see that \( \eta > 0 \). By the SLLN we have \( S_n / n \to 1/2 \) almost surely, so there is an integer \( N \geq 3 \) for which 

\[
\tilde{P}\{ \inf_{n \geq N} S_n / n \geq \eta \} \geq 2/3.
\]

Now 

\[
\frac{1}{2} = \tilde{P}\{ \varepsilon_1 = 1 \} \leq \tilde{P}\{ \inf_{n \leq N} S_n / n \geq \inf_{n \leq N} \varepsilon_1 / n \geq \frac{1}{N} \}.
\]

Hence \( \tilde{P}\{ \inf_{n \geq 1} S_n / n \geq \frac{1}{N} \} \geq 1/6 \), which yields that \( \eta \geq 1/N \).

Let \( f \) be a complex valued strongly additive arithmetic function. We will prove the following result.

**Theorem 5** Assume that there exists a nondecreasing function \( U : \mathbb{N} \to \mathbb{R}^+ \) with \( \lim_{n \to \infty} U(n) = \infty \) such that

\[
c_1 U(n) \leq |F(n)| \leq c_2 U(n)
\]

for some positive constants \( c_1, c_2 \). Assume further that for some \( 0 < h < 1/4 \)

\[
\sup_{n^h < p \leq n} |f(p)| \ll |F(nh)|/n, \quad A_{n^h} \ll |F(nh)|/n, \quad B_{n^h}^{1/2} \ll |F(nh)|/n. \tag{21}
\]

Then (5) holds.

**Remark 6 (a)** Note that in condition (21) we have

\[
A_{n^h} = \left| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right| \quad \text{instead of} \quad \sum_{2 \leq p \leq n^h} \frac{|f(p)|}{p}.
\]

This was made possible by using a stronger estimate for divisors of Bernoulli sums than the one used in [2].

**Remark 6 (b)** If \( f \geq 0 \), one can take \( U = F \). If \( F(n) \geq Cn \max(A_n, B_n^{1/2}) \), then a sufficient condition for (21) is

\[
\sup_{n^h < p \leq n} |f(p)|^2 \ll \max(A_{n^h}^2, B_{n^h})
\]

This is satisfied e.g. if \( f \) is bounded.
(c) Condition (21) can be replaced by a slightly weaker condition of type (18):
\[
\sum_{n \geq t} \frac{1}{|F(\eta)|^2} \left( \sup_{n^h < p \leq n} |f(p)|^2 + A_n^2 + B_n^h \right) = O(1/t).
\]
However, since the two conditions are close to each other and both are probably far from being necessary and sufficient, it is preferable to use the simpler assumption (21).

2 Preliminaries

In this section we formulate some lemmas needed for the proof of Theorem 5.

Let \( X = \{X_k, k \geq 1\} \) be i.i.d. random variables and let \( w = \{w_k, k \geq 1\} \) be complex numbers with partial sums \( W_n = \sum_{k=1}^{n} w_k, n \geq 1 \). We assume that
\[
|W_n| \uparrow \infty, \quad n \to \infty. \tag{22}
\]
Consider the weighted averages
\[
M_n(w, X) = \frac{1}{W_n} \sum_{k=1}^{n} w_k X_k, \quad n = 1, \ldots
\]

Lemma 7 We have \( \lim_{n \to \infty} M_n(w, X) = 0 \) almost surely for every i.i.d. sequence \( X \) of nondegenerate, centered, integrable random variables if and only if
\[
\lim sup_{t \to \infty} \frac{1}{t} \# \{ n : \left| \frac{W_n}{w_n} \right| \leq t \} < \infty.
\]
Note that the last condition implies
\[
\lim_{n \to \infty} \left| \frac{w_n}{W_n} \right| = 0,
\]
since, for any \( \rho > 0 \), the number of integers \( n \) such that \( |w_n/W_n| > \rho \) is finite.

The characterization above is due to Jamison, Orey and Pruitt (see Theorems 1 and 3 in [7]) under the additional fact that the weights \( w_k \) are positive reals, in which case condition (22) is trivially satisfied. As a matter of fact, the same proof allows to work with complex weights.

It would be natural to verify the conditions of Lemma 7 in order to prove Theorem 5 but for technical reasons we were not able to do this. Instead, we will use the following sufficient criterion for the weighted SLLN, also proved (in the case of positive weights) in Jamison et al. [7].

Lemma 8 Put
\[
N(x) = \begin{cases} 
\# \{ k : |W_k/w_k| \leq x \} & \text{if } x \geq 1, \\
0 & \text{if } 0 \leq x < 1
\end{cases} \tag{23}
\]
and assume that \( E |X| < \infty \) and
\[
E |X|^2 \left( \int_{|y| \geq |X|} \frac{N(y)}{|y|^3} dy \right) < \infty. \tag{24}
\]
Then we have \( \lim_{n \to \infty} M_n(w, X) = 0 \) a.s.
Again, the proof given in [7] works in the complex case with trivial changes. Let $\Psi$ denote the distribution function of $X$ and let

$$Y_k = X_k \cdot \chi\{|X_k| < |W_k/w_k|\}, \quad \zeta_k = |w_k/W_k| (Y_k - EY_k).$$

Following [7], we get

$$\sum_{k=1}^{\infty} P\{X_k \neq Y_k\} = \sum_{k=1}^{\infty} \int_{|v| \geq |w_k/w_k|} \Psi(dv) = E N(|X|)$$

and

$$\sum_{k=1}^{\infty} E|\zeta_k|^2 \leq 4 \int x^2 \left( \int_{y \geq |x|} \frac{N(y)}{y^3} dy \right) \Psi(dx).$$

As noted in [7], relation (24) implies $EN(|X|) < \infty$ and thus the lemma follows from the Borel-Cantelli lemma and the Kolmogorov two series criterion.

Next we need a lemma on divisors of Bernoulli sums. Let $d(n) = \#\{y : y|n\}$ be the divisor function. Consider the elliptic Theta function

$$\Theta(d, m) = \sum_{\ell \in \mathbb{Z}} e^{im\pi \frac{\ell^2}{2d^2}}.$$  \hfill (25)

The following lemma is Theorem II from [10] which we recall for convenience.

**Lemma 9** We have the following uniform estimate:

$$\sup_{2 \leq d \leq n} \left| \frac{N}{d} \right| \leq C \left( (\log n)^{5/2} n^{-3/2} + \frac{\pi}{\log n} \right) \quad \text{if } d \leq \sqrt{n},$$

$$\frac{C}{\sqrt{n}} \quad \text{if } \sqrt{n} \leq d \leq n.$$ \hfill (26)

Further, for any $\alpha > 0$

$$\sup_{d < \pi \sqrt{n}} \left| \frac{N}{d} \right| = O\left( n^{-\alpha + \varepsilon} \right) \quad \text{for all } \varepsilon > 0.$$ \hfill (27)

and for any $0 < \rho < 1$,

$$\sup_{d < \pi \sqrt{n}} \left| \frac{N}{d} \right| = O\left( e^{-(1-\varepsilon)n^\rho} \right), \quad \text{for all } 0 < \varepsilon < 1.$$ \hfill (28)

**Remark 10** By using the Poisson summation formula (see e.g. [6], p. 42)

$$\sum_{\ell \in \mathbb{Z}} e^{-i(\ell+\delta)^2 \pi x} = x^{1/2} \sum_{\ell \in \mathbb{Z}} e^{2i\pi \ell \delta - \ell^2 \pi x},$$ \hfill (30)

where $x$ is any real number and $0 \leq \delta \leq 1$, with the choices $x = \pi n/(2d^2)$, $\delta = n/(2d)$, we get

$$\frac{\Theta(d, n)}{d} = \frac{1}{d} \sum_{\ell \in \mathbb{Z}} e^{i\pi n \frac{\ell^2}{2d^2}} = \sqrt{\frac{2}{\pi n}} \sum_{\ell \in \mathbb{Z}} e^{-2i(\frac{\pi n}{2d})^2 \frac{\ell^2}{n}}.$$
Thus
\[
\sup_{2 \leq d \leq n} \left| \tilde{P} \{ d | S_n \} - \sqrt{\frac{2}{\pi n}} \sum_{t \in \mathbb{Z}} e^{-2(\frac{t}{n} + 1)^2 \frac{d^2}{n}} \right| = O \left( \frac{(\log n)^{5/2}}{n^{3/2}} \right). \tag{31}
\]

Lemma 9 was recently improved by the second named author for the range of values \( d \geq \sqrt{n} \), one the basis of these estimates.

3 Proof of Theorem 5

We put
\[
L(t) = \# \{ n : |F(n)| \leq t|f(n)| \}. \tag{32}
\]

Since \( E |X| < \infty \), according to Lemma 8 in order to prove Theorem 5, it suffices to prove
\[
E |X|^2 \int_{y \geq |X|} \frac{L(y)}{y^3} dy < \infty. \tag{33}
\]

We use the same probabilistic trick as in [2]. We assume that the Bernoulli sequence \( \{ \varepsilon_i, i \geq 1 \} \) is defined on a probability space \((\Omega, \mathcal{A}, \tilde{P})\), and denote by \( \tilde{E} \) the corresponding expectation symbol. Then, letting \( F_\eta(n) = \inf_{m \geq \eta n} |F(m)| \) we get
\[
L(t) \leq \# \{ n : F_\eta(n) \leq t|f(S_n)| \} \leq \# \{ n : \eta|S_n| \leq t|f(S_n)| \}, \tag{34}
\]

and this is true for any \( t > 0 \), simply because the graph of the random walk \( \{ S_n, n \geq 1 \} \) replicates all positive integers with possible multiplicities. If \( \Omega_\eta = \{ S_n \geq \eta n \text{ for all } n \geq 1 \} \) then \( \tilde{P}(\Omega_\eta) > 0 \). Reading (34) on \( \Omega_\eta \) gives:
\[
L(t) \leq \# \{ n : F_\eta(n) \leq t|f(S_n)| \} \quad \text{on } \Omega_\eta \text{ for all } t > 0. \tag{35}
\]

But for all \( t > 0 \)
\[
\frac{1}{t} \# \{ n : F_\eta(n) \leq t|f(S_n)| \} \leq 1 + \frac{1}{t} \# \{ n \geq t : F_\eta(n) \leq t|f(S_n)| \} = 1 + \frac{1}{t} \sum_{n \geq t} \chi \{ \frac{F_\eta^2(n)}{t} \leq |f(S_n)|^2 \} \leq 1 + t \sum_{n \geq t} \frac{|f(S_n)|^2}{F_\eta^2(n)}. \tag{36}
\]

We now prove the following lemma.

Lemma 11 There exists a constant \( C_h \) depending on \( h \) only such that for any sufficiently large \( n \)
\[
\| f(S_n) \|_{2, \tilde{P}} \leq C_h \sup_{n^h < p \leq n} |f(p)| + \left| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right| + C \varepsilon \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2}.
\]

Proof. We have
\[
f(S_n) = \sum_{2 \leq p \leq S_n} f(p) \chi(p | S_n) = \sum_{2 \leq p \leq n} f(p) \chi(p | S_n),
\]
and given any real $h$ with $0 < h < 1/4$

$$
\left| f(S_n) - \sum_{2 \leq p \leq n^h} f(p)\chi(p|S_n) \right| = \left| \sum_{n^h < p \leq n} f(p) \right| \leq \sum_{n^h < p \leq n} |f(p)| \leq C_h \sup_{n^h < p \leq n} |f(p)|. \tag{37}
$$

The last bound is justified by the fact that if $S_n$ admits $K$ different prime factors greater than $n^h$, then we have the inequalities $n^{Kh} \leq S_n \leq n$; whence $Kh \leq 1$. And so

$$
\left| \left\| f(S_n) \right\|_{2,\bar{P}} - \left\| \sum_{2 \leq p \leq n^h} f(p)\chi(p|S_n) \right\|_{2,\bar{P}} \right| \leq C_h \sup_{n^h < p \leq n} |f(p)|. \tag{38}
$$

Now denote by $1$ the function equal to 1 everywhere on $\Omega$. Then

$$
\left| \left\| \sum_{2 \leq p \leq n^h} f(p)\bar{P}(p|S_n) \right\|_{2,\bar{P}} - \left\| \sum_{2 \leq p \leq n^h} f(p)\chi(p|S_n) \right\|_{2,\bar{P}} \right| \leq \left\| \sum_{2 \leq p \leq n^h} f(p)\left(\chi(p|S_n) - \bar{P}(p|S_n)\right) \right\|_{2,\bar{P}}. \tag{39}
$$

Observe first by using Lemma 9

$$
\left| \left\| \sum_{2 \leq p \leq n^h} f(p)\bar{P}(p|S_n) \right\|_{2,\bar{P}} - \left\| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right\|_{2,\bar{P}} \right| \leq \left| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right| \leq \sum_{2 \leq p \leq n^h} |f(p)|\left(\bar{P}(p|S_n) - \frac{1}{p}\right) \leq \sum_{2 \leq p \leq n^h} |f(p)|\bar{P}(p|S_n) - \frac{1}{p} \leq C_he^{-(1-\varepsilon)n^{1-2h}} \sum_{2 \leq p \leq n^h} |f(p)|.
$$

Next by the Cauchy-Schwarz inequality

$$
\left( \sum_{2 \leq p \leq n^h} |f(p)| \right)^2 = \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|}{\sqrt{p}} \cdot \sqrt{p} \right)^2 \leq \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right) \left( \sum_{2 \leq p \leq n^h} \frac{p}{p} \right) \leq n^{2h} \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right). \tag{40}
$$

Hence

$$
e^{-(1-\varepsilon)n^{1-2h}} \sum_{2 \leq p \leq n^h} |f(p)| \leq e^{-(1-\varepsilon)n^{1-2h}} n^h \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2}.
$$

Consequently,

$$
\left| \left\| \sum_{2 \leq p \leq n^h} f(p)\bar{P}(p|S_n) \right\|_{2,\bar{P}} - \left\| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right\|_{2,\bar{P}} \right|.
$$
Clearly and writing
By (40)
By combining the previous relations it follows that
once n is large enough, which we do assume. Thus (43) and (11) imply
\[
\left\| \sum_{2 \leq p \leq n^h} f(p) \chi(p|S_n) \right\|_{2, \mathcal{P}} - \left\| \sum_{2 \leq p \leq n^h} f(p) \right\| \leq C \varepsilon e^{-(1-\varepsilon)n^{1-2h}} n^h\left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2}.
\]
(41)
\[
\left\| \sum_{2 \leq p \leq n^h} f(p) \chi(p|S_n) \right\|_{2, \mathcal{P}} - \left\| \sum_{2 \leq p \leq n^h} f(p) \right\| \leq C \varepsilon \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2} + \left\| \sum_{2 \leq p \leq n^h} f(p) (\chi(p|S_n) - \mathcal{P}[p|S_n]) \right\|_{2, \mathcal{P}}. 
\]
(42)
Clearly
\[
\mathcal{E} \left| \sum_{2 \leq p \leq n^h} f(p) (\chi(p|S_n) - \mathcal{P}[p|S_n]) \right|^2
= \sum_{2 \leq p \leq n^h} |f(p)|^2 (\mathcal{P}[p|S_n] - \mathcal{P}[p|S_n])^2
+ 2 \mathcal{R} \left\{ \sum_{2 \leq p < q \leq n^h} f(p)\overline{f(q)} (\mathcal{P}[pq|S_n] - \mathcal{P}[pq|S_n]) (\mathcal{P}[q|S_n] - \mathcal{P}[q|S_n]) \right\}. \]
(43)
But by Lemma [9]
\[
\max \left\{ \left| \mathcal{P}[p|S_n] - \frac{1}{p} \right|, \left| \mathcal{P}[pq|S_n] - \frac{1}{pq} \right| \right\} \leq C \varepsilon e^{-(1-\varepsilon)n^{1-4h}},
\]
whence
\[
\left| \mathcal{P}[p|S_n] (1 - \mathcal{P}[p|S_n]) - \frac{1}{p} (1 - \frac{1}{p}) \right| \leq C \varepsilon e^{-(1-\varepsilon)n^{1-4h}},
\]
(44)
and writing
\[
\mathcal{P}[pq|S_n] - \mathcal{P}[p|S_n] \mathcal{P}[q|S_n] = \left( \mathcal{P}[pq|S_n] - \frac{1}{pq} \right) - \left\{ \left( \mathcal{P}[p|S_n] - \frac{1}{p} \right) \mathcal{P}[q|S_n] + \frac{1}{p} (\mathcal{P}[q|S_n] - \frac{1}{q}) \right\}.
\]
we also get
\[
\left| \mathcal{P}[pq|S_n] - \mathcal{P}[p|S_n] \mathcal{P}[q|S_n] \right| \leq C \varepsilon e^{-(1-\varepsilon)n^{1-4h}}.
\]
(45)
By combining the previous relations it follows that
\[
\mathcal{E} \left| \sum_{2 \leq p \leq n^h} f(p) (\chi(p|S_n) - \mathcal{P}[p|S_n]) \right|^2
\leq \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} + C \varepsilon e^{-(1-\varepsilon)n^{1-4h}} \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2}.
\]
(47)
By (40)
\[
e^{-(1-\varepsilon)n^{1-4h}} \left( \sum_{2 \leq p \leq n^h} |f(p)|^2 \right)^2 \leq e^{-(1-\varepsilon)n^{1-4h}} n^{2h} \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right).
\]
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It follows that
\[
\sum_{2 \leq p \leq n^h} |f(p)|^2 \leq \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p}.
\]

Thus
\[
\mathbb{E} \left| \sum_{2 \leq p \leq n^h} f(p)(\chi(p|S_n) - \tilde{P}(p|S_n)) \right|^2 \leq \sum_{n^h < p \leq n} \frac{|f(p)|^2}{p}.
\] (48)

By inserting (48) into (42) we arrive at
\[
\left\| \sum_{2 \leq p \leq n^h} f(p)\chi(p|S_n) \right\|_{L^2(P)} - \left\| \sum_{2 \leq p \leq n^h} f(p) \right\| \leq C\varepsilon \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2}.
\] (49)

In view of (37), the last relation implies
\[
\|f(S_n)\|_{L^2(P)} \leq C_h \sup_{n^h < p \leq n} |f(p)| + \left\| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right\| + C\varepsilon \left( \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right)^{1/2}.
\] (50)

This completes the proof of Lemma 11.

We can now finish the proof of Theorem 5. We get from Lemma 11 and (36)
\[
\mathbb{E} \left( \frac{1}{t} \# \left\{ n : F_n(n) \leq tf(S_n) \right\} \right) \leq 1 + t \sum_{n \geq t} \frac{\mathbb{E} \|f(S_n)\|^2}{F^2_n(n)}
\]
\[
\leq 1 + C t \sum_{n \geq t} \frac{1}{F^2_n(n)} \left\{ \sup_{n^h < p \leq n} |f(p)|^2 + \left\| \sum_{2 \leq p \leq n^h} \frac{f(p)}{p} \right\|^2 + \sum_{2 \leq p \leq n^h} \frac{|f(p)|^2}{p} \right\}.
\]

On using assumption (21), we deduce
\[
\sup_{t > 0} \mathbb{E} \left( \frac{1}{t} \# \left\{ n : F_n(n) \leq t|f(S_n)| \right\} \right) \leq C.
\]

It follows that
\[
\mathbb{E} \chi(\Omega_n) \cdot X^2 \int_{y \geq |X|} \frac{\# \left\{ n : F_n(n) \leq y|f(S_n)| \right\}}{y^3} dy \leq C \mathbb{E} X^2 \int_{y \geq |X|} \frac{1}{y^2} dy \leq C \mathbb{E} |X| < \infty.
\]

And in view of (34), (36) and Fubini’s theorem
\[
\mathbb{E} \chi(\Omega_n) \cdot \mathbb{E} X^2 \int_{y \geq |X|} \frac{L(y)}{y^3} dy \leq C \mathbb{E} X^2 \int_{y \geq |X|} \# \left\{ n : F_n(n) \leq y|f(S_n)| \right\} dy
\]
\[
\leq C \mathbb{E} |X| < \infty.
\]

Since
\[
\mathbb{E} \chi(\Omega_n) \cdot \mathbb{E} X^2 \int_{y \geq |X|} \frac{L(y)}{y^3} dy = \mathbb{P}(\chi(\Omega_n)) \mathbb{E} X^2 \int_{y \geq |X|} \frac{L(y)}{y^3} dy,
\]
relation (36) follows, completing the proof.
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