SPACE-TIME KERNEL BASED NUMERICAL METHOD FOR GENERALIZED BLACK-SCHOLES EQUATION

MARJAN UDDIN* AND HAZRAT ALI

Department of Basic Sciences
University of Engineering and Technology Peshawar, Pakistan

Abstract. In approximating time-dependent partial differential equations, major error always occurs in the time derivatives as compared to the spatial derivatives. In the present work the time and the spatial derivatives are both approximated using time-space radial kernels. The proposed numerical scheme avoids the time stepping procedures and produced sparse differentiation matrices. The stability and accuracy of the proposed numerical scheme is tested for the generalized Black-Scholes equation.

1. Introduction. In the financial market a right given to an owner to buy or sell the underlying asset on a certain date for a certain price is known as call or put option. The specified date is known a maturity date and specified price is known a strike price. There are two types of options, the one which can be exercised at any time till the maturity date is known as American option, while the option which can be exercised at the expiration date only is known as European option. These two types of option prices can be identified by a second-order partial differential equation (Black-Scholes equation) with respect to the underlying asset price \( x \), and time variable \( t \) respectively. The most generalized form of Black-Scholes equation [20] is given by

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2(x,t)x^2 \frac{\partial^2 u}{\partial x^2} - [D(x,t) - R(x,t)] x \frac{\partial u}{\partial x} - R(x,t)u = 0, \tag{1}
\]

where \( (x,t) \in \mathbb{R}^+ \times (0,T) \) and \( u \) is price of option.

The Black-Scholes equation can be solved analytically with constant or space-independent coefficients. However, when the coefficient in the Black-Scholes equation are generalized functions or discontinuous functions, then computing the exact solution is always a difficult task. For example in (1) the dividend \( D \), the interest rate \( R \), and the volatility \( \sigma \) are functions of \( t, x \), then we can not transform the Black-Scholes equation to the standard heat equation. In fact in many cases it is not possible to obtain the exact solution of the generalized Black-Scholes equation. Therefore the numerical methods are very essential for accurate approximation of many time-dependent partial differential equations. As for as the numerical solution of Black-Scholes equation is concerned, few results are available in the literature. The finite difference as well as the finite elements methods are successfully applied...
for solving the Black-Scholes equations [10, 2]. A lattice technique developed for
the numerical solution of Black-Scholes equation by Cox [6], later on the numerical
scheme due to Cox was improved to an explicit time-stepping scheme [12]. Most of
the numerical methods in the literature are based on time-stepping schemes. These
numerical techniques include the finite difference methods, the finite elements meth-
ods, the boundary elements methods, the meshless methods, the wavelets methods,
and the spectral methods [18, 21, 7, 5, 4, 3, 24, 25, 28, 26]. In most of these methods
the time integration is carried out with various explicit and implicit methods, Lie
splitting methods and Runge-Kutta methods. Although few methods are available
to approximate time and space derivative at the same time. For example, space-
time numerical methods were developed for the solution of inverse problems (see
[19, 16]). Li and Mao work was extended further in [17, 15] for approximating in
[15] for the estimate river pollution. Some other efficient space-time methods de-
veloped can be found in [23, 14, 22, 1, 30]. A recent work [8, 11] was carried out
to develop time-space methods for numerically approximating the time-dependent
partial differential equations. In the present work, a time-space numerical technique
is constructed which is based on time-space radial kernels for solving the general-
ized Black-Scholes equation (1). The rest of the paper is organized as follows. In
next section we give a brief discussion of space-time kernel based time. Section 2 is
devoted to the stability analysis of the proposed method. Section 3 is dedicated to
numerical examples and finally a conclusion on results is indicated.

2. Local time-space radial kernel method. The partial differential equations
on a spatial domain $\Omega_d \in \mathbb{R}^d$, $d \geq 1$, $(0, T_f)$ denote time domain, satisfies
\[
(\frac{\partial}{\partial t} - L_1)u(x, t) = f(x, t), \ (x, t) \in \Omega = \Omega_d \times (0, T_f),
\]
subject to well defined boundary and initial conditions, where $L_1$ is spatial operator.
This time-dependent partial differential equation is reduced to a problem in $\mathbb{R}^{d+1}$.
The problem (2) subject to the boundary and initial conditions is read as
\[
L[u(\vec{x})] = f(\vec{x}), \text{where } \vec{x} = (x, t) \in \Omega, \ t > 0,
\]
\[
B[u(\vec{x})] = g(\vec{x}), \text{where } \vec{x} = (x, t) \in \partial \Omega, \ t > 0,
\]
\[
u(\vec{x}) = u_0, \text{where } \vec{x} = (x, t) \in \Omega, \ t = 0.
\]

This reduced problem is well-posed (being a square system ), hence leads to
approximate the solution $u(\vec{x})$ of the reduced problem. The transformed problem
(3)-(5) can be approximated with a local kernel based method. Consider the nodal
points $\{u(x_i), i = 1, 2, ..., m\}$ corresponding to smooth function $u(x)$, such that
$\{x_1, ..., x_m\} \subset \Omega \subset \mathbb{R}^d, d \geq 1$. The function $u(x)$ is approximated by local kernel
method at $x_i \in \Omega$,
\[
u(x_i) = \sum_{x_j \in \Omega} a_i^j \psi^j(\|x_i - x_j\|),
\]
where, $a_i = [a_i^1, a_i^2, ..., a_i^n]$ is vector of unknown coefficients, and $r_{ij} = ||x_i - x_j||$
is the norm between notes $x_i$ and $x_j$, $\psi(r), r \geq 0$ is a radial kernel (radial basis
function) and $\Omega_i \subset \Omega$ is a local domain for around each $x_i$, contains $n$ neighboring
nodes around the node $x_i$. So we have $m$ small size linear systems each of order
\( n \times n \) given by
\[
\begin{pmatrix}
u_1^i & \psi_{12}^i & \cdots & \psi_{1n}^i \\
u_2^i & \psi_{22}^i & \cdots & \psi_{2n}^i \\
\vdots & \vdots & \ddots & \vdots \\
u_n^i & \psi_{n1}^i & \cdots & \psi_{nn}^i \\
\end{pmatrix}
\begin{pmatrix}
a_1^i \\a_2^i \\
\vdots \\
a_n^i \\
\end{pmatrix}
\]

, \( i = 1, 2, \ldots, m \),

which can be denoted by
\[
u^i = \Psi^i a^i, i = 1, 2, \ldots, m,
\]

where \( \psi_{kj}^i = \psi^i(||x_k - x_j||), x_k, x_j \in \Omega_i \), the matrix \( \Psi^i \) is called system matrix.

Similarly apply the operator \( L \) we get
\[
Lu(x_i) = \sum_{x_j \in \Omega_i} a_{ij}^i L \psi^i(||x_i - x_j||),
\]

In vector form we have,
\[
Lu(x_i) = \mathbf{v}^i \cdot \mathbf{a}^i,
\]
where \( \mathbf{v}^i \) is given by
\[
\mathbf{v}^i = L \psi^i(||x_i - x_j||), x_j \in \Omega_i,
\]

the unknown coefficients can be eliminated from equation (7)
\[
a^i = (\Psi^i)^{-1} \mathbf{u}^i,
\]

putting the values of \( a^i \) in (9) to have,
\[
Lu(x_i) = \mathbf{v}^i(\Psi^i)^{-1} \mathbf{u}^i = \mathbf{w}^i \mathbf{u}^i
\]

where,
\[
\mathbf{w}^i = \mathbf{v}^i(\Psi^i)^{-1}.
\]

This gives the localized discretized form of the linear operator
\[
Lu \equiv \mathbf{L}u,
\]

here \( \mathbf{L} \) is \( m \times m \) differentiation matrix, with \( n \) non-zeros values while \( m - n \) zeros values in each row, where \( n \) denote the nodes local sub-domain \( \Omega_i \) for each \( i \).

The time-space kernels method need to satisfy partial differential equation and corresponding initial-boundary conditions at every node in space-time domain \( \Omega \).
If we denote \( N_I \) inner nodes, \( N_B \) boundary, and \( N_T \) the initial time \( (t = 0) \) nodes respectively.

\[
Lu(\mathbf{x}_i) = f(\mathbf{x}_i), i = 1, \ldots, N_I,
\]

\[
Bu(\mathbf{x}_i) = g(\mathbf{x}_i), i = N_I + 1, \ldots, N_I + N_B,
\]

\[
u(\mathbf{x}_i) = u_0, i = N_I + N_B + 1, \ldots, N.
\]

The equations (15)-(17) imply the linear system of equations of order \( N \times N \), equivalently we get,
\[
\begin{pmatrix}
L_C \\
L_B \\
L_I
\end{pmatrix}
\begin{pmatrix}
a \\
f \\
g \\
u_0
\end{pmatrix}
\]

or
\[
\mathbf{L}a = \mathbf{b}.
\]

The linear system of equations (19) may be solved for unknown vector \( a \) using some equations solver.
3. Stability analysis of the time-space scheme. In this section, we define the stability of scheme (19) in the following

\[ \mathbf{La} = \mathbf{b}, \]  

(20)

where \( L \) is the sparse differentiation matrix of order \( N \times N \), so stability constant of system (19) is defined by

\[ c_s = \sup_{a \neq 0} \frac{\|a\|}{\|La\|}. \]  

(21)

The value \( c_s \) is bounded over any discrete norms \( \|\cdot\| \) on \( \mathbb{R}^N \). Hence we have

\[ \|L\|^{-1} \leq \frac{\|a\|}{\|La\|} \leq c_s. \]  

(22)

Equivalently in case of pseudoinverse \( L^\dagger \) of \( L \), we get

\[ \|L^\dagger\| = \sup_{v \neq 0} \frac{\|L^\dagger v\|}{\|v\|}, \]  

(23)

writing

\[ \|L^\dagger\| \geq \sup_{v=La \neq 0} \frac{\|L^\dagger La\|}{\|La\|} = \sup_{a \neq 0} \frac{\|a\|}{\|La\|} = c_s. \]  

(24)

Hence equations (22) and (24) gives the bounds for stability constant \( c_s \).

4. Application of the time-space kernels method. In this section, the proposed numerical space-time scheme is applied for solving generalized Black-Scholes partial differential equation. In this work, we used the multiquadric space-time kernel [27] defined by

\[ \phi = \sqrt{1 + (\varepsilon r)^2}, \]

\[ \phi_t = \frac{\varepsilon^2 t}{\sqrt{1 + (\varepsilon r)^2}}, \]

\[ \phi_x = \frac{\varepsilon^2 x}{\sqrt{1 + (\varepsilon r)^2}}, \]

\[ \phi_{xx} = \frac{\varepsilon^2}{\sqrt{1 + (\varepsilon r)^2}} - \frac{x^2 \varepsilon^4}{(1 + (\varepsilon r)^2)^{1.5}}, \]

where \( r = \sqrt{x^2 + t^2} \), and \( \varepsilon \) is a shape parameter. The accuracy of the numerical scheme is greatly depends on arbitrary parameter \( \varepsilon \) which can be optimize using the procedure developed in [9]. The performance and accuracy of present time-space method is validated in term of error \( L_\infty \) defined by

\[ L_\infty = \max_{1 \leq j \leq N} \| \hat{u}(\tau_j) - u(\tau_j) \|, \]  

(25)

where \( \hat{u} \) and \( u \) denote the numerical and exact solutions respectively.
4.1. Problem 1. In this problem we consider the most generalized form of the Black-Scholes equation

\[ \frac{\partial u}{\partial t} = A(x,t) \frac{\partial^2 u}{\partial x^2} + B(x,t) \frac{\partial u}{\partial x} + C(x,t)u + f(x,t). \]  

For the purpose of comparison with the methods developed in the work [20, 13], we select the space time domain \((x,t) \in (-2,2) \times (0,1)\), where the coefficients functions \(A, B\) and \(C\) are given by

\[ A(x,t) = 0.08[2 + (1 - t) \sin(e^x)]^2, \]  
\[ B(x,t) = 0.06[1 + te^{-e^x} - 0.02e^{(-t-e^x)} - A(x,t)], \]  
\[ C(x,t) = -0.06(1 + te^{-e^x}). \]

The analytic solution of problem 1 is given by \(u(x,t) = e^{x-t}\) and \(f(x,t)\) can be calculated from the analytic solution see for example [20, 13]. The boundary and initial conditions can be used from the analytic solution. To apply the space-time kernel method, we used a model space-time domain as shown in Figure 1. The local nature of the present method has demonstrated in term of sparsity of the matrix of corresponding discretized time-space operators shown in Figure 1. Despite the fact that the dimension of the problem is increased by 1, yet this kernel method is efficient as it works in multi-dimensions domains because of solving small system matrices of order of stencil sizes as shown in Figure 1.

The other advantage of the present time-space kernel method is to avoid the time integration classical approach. Some time we need very small time step \(\delta t\) to get better accuracy at the cost of large computations. The present time-space method approximate the time-dependent PDEs just like the time-independent PDEs, which require less computer time. The above problem is solved with time-space as well as time-stepping methods and the results are shown in Table 1.

The results of space-time kernel method for problem 1 are obtained for various number of points \(m\) in the global domain and points \(n\) in local sub-domain. These results have demonstrated that space-time (ST) method takes less computer time as compared to time-integration (TI) method. The results of various other methods [20, 13] are shown in Table 2. It can been seen that the numerical solution of the present space-time method is well comparable with methods in comparison at the cost of less amount of computer time.

| Table 2. Observed maximum absolute error for example 1 in Reza [20] and Kadabaju [13] for different \(\theta\) in domain \((x,t) \in (-2,2) \times (0,1)\). |
|---|---|---|---|---|---|
| \(M = N\) | 10 | 20 | 40 | 80 | 160 |
| [20] for \(\theta = 1\) | 7.24E-02 | 3.12E-02 | 1.39E-02 | 6.08E-02 | 2.71E-04 |
| [20] for \(\theta = \frac{1}{2}\) | 1.12E-03 | 2.08E-02 | 3.91E-05 | 7.19E-06 | 1.31E-06 |
| [13] for \(\theta = 1\) | 7.44E-02 | 3.94E-02 | 2.02E-02 | 1.02E-02 | 5.16E-03 |
| [13] for \(\theta = \frac{1}{2}\) | 5.89E-03 | 1.46E-03 | 3.64E-04 | 9.10E-05 | 2.27E-05 |
FIGURE 1. A typical centers arrangements in global space-time domain as well as in a local sub-domain, and sparsity of descretized operator of problem 1, where \( m = 100, n = 10 \).

TABLE 1. Space-time (ST) solution of problem 1 for different total collocation points \( m \) and stencil size \( n \), and time integration (TI) solution in domain \( (x, t) \in (-2, 2) \times (0, 1) \).

| \( m \) | \( n \) | \( L_\infty \) | ST method (C.time) | TI method (C.time) |
|-------|-------|-------------|------------------|------------------|
| 100   | 10    | 1.23E-02    | 0.6783           | 3.2891           |
| 400   |       | 9.49E-02    | 0.7123           | 6.2821           |
| 1600  |       | 1.08E-04    | 0.9129           | 10.2370          |
| 2500  |       | 1.26E-04    | 1.1234           | 15.2950          |
| 100   | 15    | 2.37E-02    | 0.8234           | 4.3491           |
| 400   |       | 1.45E-03    | 10.2356          | 9.2371           |
| 1600  |       | 2.45E-03    | 11.2916          | 13.9820          |
| 2500  |       | 3.45E-04    | 13.1087          | 20.3582          |
| 100   | 20    | 2.22E-02    | 9.1835           | 10.10491         |
| 400   |       | 2.89E-03    | 11.2349          | 12.7146          |
| 1600  |       | 9.88E-03    | 14.8679          | 21.3812          |
| 2500  |       | 7.23E-04    | 16.1955          | 25.0492          |

FIGURE 2. The exact solution versus the approximate solution in space-time domain corresponding to problem 1, when \( m = 1600 \) and \( n = 10 \) in domain \( (x, t) \in (-2, 2) \times (0, 1) \).
Problem 2. To validate the present space-time numerical scheme, we consider the generalized Black-Scholes model with different coefficients $A$, $B$, $C$ and $f$ and the initial and boundary conditions given by

$$\frac{\partial u(x,t)}{\partial t} = A \frac{\partial^2 u(x,t)}{\partial x^2} + B \frac{\partial u(x,t)}{\partial x} - Cu(x,t) + f(x,t),$$

(30)

$$u(x,0) = x^2(1-x), \quad u(0,t) = 0, \quad u(1,t) = 0,$$

(31)

where and the source term

$$f = (1 - x)x^2(2 + 2t) - \left[A(2 - 6x) + B(2x - 3x^2) - Cx^2(1 - x)\right] (t + 1)^2,$$

is selected in such a way that exact solution of (30) reduced to the form $u(x,t) = x^2(1-x)(t+1)^2$. The particular coefficients values used in this computations, are $\sigma = 0.25$, $R = 0.05$, $A = \frac{1}{2}\sigma^2$, $C = R$ $B = R - A$. The space time method is applied for solving problem 2 in the space time domain $(x,t) \in (0,1) \times (0,1)$. Initially we used $m = 400$, the total points in the global domain, where as in the local sub-domain $n = 10$. The shape parameter $\varepsilon$ involved depends both on $m$ and $n$. Also the accuracy depends how to optimally select $m$ and $n$ and $\varepsilon$ which is an open problem. We used uncertainty principal to select shape parameter $\varepsilon$ for fixing $m$ and varies $n$. The numerical solution as well as error plots for this problem are shown Figure 3, which are well comparable. We used only one shape parameter only in space dimension. If another shape parameter in time dimension be used, then the accuracy can be increased up to much higher order.

![Figure 3](image-url)

Figure 3. The numerical solution and error in space-time domain, corresponding to problem 2 when $m = 400$ and $n = 10$ in the domain $(x,t) \in (0,1) \times (0,1)$.

Problem 3. In the last problem we consider the particular case of the generalized Black-Scholes model which govern the European option defined by

$$\frac{\partial u(S,t)}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u(S,t)}{\partial S^2} + (R - D) S \frac{\partial u(S,t)}{\partial S} - Ru(S,t),$$

(32)

where, $(S,t) \in (0,\infty) \times (0,T_e)$, subject to the conditions

$$u(b_1,t) = P(t), \quad u(b_2,t) = Q(t), \quad u(s,T_e) = v(s),$$

(33)

We applied the proposed space-time kernel method to solve the BS equation governing the European model. We used the conditions defined by $v(s) = \max\{0, S -$
and boundary values \( P(t) = 0 \) and \( Q(t) = 0 \) of the BS model (32). This BS model governing European double barrier knock-out call option.

The solution curves corresponding to double barrier option price at different values of fractional order \( \alpha \) are shown in Figure 4. The parameters \( \sigma = 0.45, R = 0.03, T_e = 1 \) (year), \( K = 10 \), \( b_1 = 3 \), \( b_2 = 15 \) and the dividend is \( D = 0.01 \). The plots in Figure 4 are well consistent with the results in [31]. It is shown in Figure 4 that when \( S \) is less than a critical value \( K \) ie strike price, lower prices values are obtained. while for fat tails, the higher prices are obtained when \( S > K \). It is concluded that the present numerical scheme successfully capture jump or large movement in the process.

Next we consider the European call option, with boundary and initial conditions defined by \( P(t) = 0 \), and \( Q(t) = b_2 K \exp (-R T_e t) \) and \( v(S) = \max \{0, S - K\} \). The parameters \( \sigma = 0.25, R = 0.05, b_1 = 0.1, b_2 = 100, T_e = 1 \) (year) and \( K = 50 \).

Once again for European put option the initial and boundary values \( v(S) = \max \{0, K - S\}, P(t) = K \exp (-R T_e t) \) and \( Q(t) = 0 \) and used with other parameters \( \sigma = 0.25, R = 0.05, b_1 = 0.1, b_2 = 100, T_e = 1 \) (year) and \( K = 50 \) as considered in [31]. The solution curves corresponding to call option price as well as put option price are shown in Figures 5 and 6, respectively.

![Figure 4. Double barrier option prices obtained by space-time local kernel method.](image)

4.4. Conclusions. In this work, we constructed a space-time numerical scheme based on radial kernels for generalized Black-Scholes equation. The time and space are both collocated using space-time radial kernel. As a result the dimension is increased by 1, yet it works well, as it is designed for multi-dimensions and irregular domain. The advantages of the proposed space-time numerical scheme is to avoid the time-stepping procedures such as the implicit and explicit numerical schemes, and the \( \theta \)-weighted schemes etc. Such types of methods require very small step size for better accuracy and stability. The proposed numerical scheme reduced cost of computer time as it avoids to compute the differentiation matrix at each time step as compared to the methods like RK4 method for time-depending partial differential equations.
Acknowledgments. We would like to thank HEC Pakistan and UET Peshawar for financial support and the reviewers for their constructive comments.

REFERENCES

[1] V. R. Ambati and O. Bokhove, Space-time discontinuous Galerkin finite element method for shallow water flows, J. Comput. Appl. Math., 204 (2007), 452–462.
[2] C. Canuto, M. Y. Hussaini, A. Z. Quarteroni and T. Zang, Option Pricing: Mathematical Models and Computation, Springer, 1993.
[3] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods, Evolution to Complex Geometries and Applications to Fluid Dynamics, Scientific Computation, Springer, Berlin, 2007.
[4] C. Chen, A. Karageorghis and Y. Smyrlis, *The Method of Fundamental Solutions: A Meshless Method*, Dynamic Publishers Atlanta, 2008.

[5] A. Cohen, *Numerical Analysis of Wavelet Methods*, Studies in Mathematics and its Applications, 32. North-Holland Publishing Co., Amsterdam, 2003.

[6] J. C. Cox, S. A. Ross and M. Rubinstein, Option pricing: A simplified approach, *J. Financ. Econ.*, 7 (1979), 229–263.

[7] G. E. Fasshauer, *Meshfree Approximation Methods with MATLAB*, With 1 CD-ROM (Windows, Macintosh and UNIX), Interdisciplinary Mathematical Sciences, 6. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[8] G. E. Fasshauer and M. McCourt, *Kernel-Based Approximation Methods Using Matlab*, World Scientific pub. Co, 2015.

[9] G. E. Fasshauer and J. G. Zhang, On choosing optimal shape parameters for RBF approximation, *Numerical Algorithms*, 45 (2007), 345–368.

[10] R. Geske and K. Shastri, Valuation by approximation: A comparison of alternative option valuation techniques, *Journal of Financial and Quantitative Analysis*, 20 (1985), 45–71.

[11] M. Hamaidi, A. Naji and A. Charaf, Space-time localized radial basis function collocation method for solving parabolic and hyperbolic equations, *Eng. Anal. Bound. Elem.*, 67 (2016), 152–163.

[12] J. Hull and A. White, The use of the control variate technique in option pricing, *Journal of Financial and Quantitative analysis*, 23 (1988), 237–251.

[13] M. K. Kadalbajoo, L. P. Tripathi and A. Kumar, A cubic B-spline collocation method for a numerical solution of the generalized Black-Scholes equation, *Math. Comput. Modelling*, 55 (2012), 1483–1505.

[14] C. M. Klaija, J. J. W. van der Vegta and H. van der Venb, Space-time discontinuous Galerkin method for the compressible Navier–Stokes equations, *J. Comput. Phys.*, 217 (2006), 589–611.

[15] M. Li, W. Chen and C. S. Chen, The localized RBFs collocation methods for solving high dimensional PDEs, *Eng. Anal. Bound. Elem.*, 37 (2013), 1300–1304.

[16] Z. Li and X. Z. Mao, Global multiquadric collocation method for groundwater contaminant source identification, *Environmental modelling & software*, 26 (2011), 1611–1621.

[17] Z. Li and X. Z. Mao, Global space-time multiquadric method for inverse heat conduction problem, *Internat. J. Numer. Methods Engrg.*, 85 (2011), 355–379.

[18] F. Moukalled, L. Mangani and M. Darwish, *The Finite Volume Method in Computational Fluid Dynamics*, Fluid Mechanics and its Applications, 113. Springer, Cham, 2016.

[19] H. Netuzhylov, A Space-Time Meshfree Collocation Method for Coupled Problems on Irregularity-Shaped Domains, Ph.D thesis, Zugl., Braunschweig, Univ. in Diss., 2008.

[20] R. Mohammadi, Quintic B-spline collocation approach for solving generalized Black–Scholes equation governing option pricing, *Comput. Math. Appl.*, 69 (2015), 777–797.

[21] S. A. Sauter and C. Schwab, *Boundary Element Methods*, Springer Series in Computational Mathematics, 39. Springer-Verlag, Berlin, 2011.

[22] J. J. Sudirham, J. J. W. van der Vegta and R. M. J. van Damme, Space-time discontinuous Galerkin method for advection-diffusion problems on time-dependent domains, *Appl. Numer. Math.*, 56 (2006), 1491–1518.

[23] T. E. Tezduyar, S. Sathe, R. Keedy and K. Stein, Space-time finite element techniques for computation of fluid-structure interactions, *Comput. Methods Appl. Mech. Engrg.*, 195 (2006), 2002–2027.

[24] C. Turchetti, M. Conti, P. Crippa and S. Orcioni, On the approximation of stochastic processes by approximate identity neural networks, *IEEE Transactions on Neural Networks*, 9 (1998), 1069–1085.

[25] C. Turchetti, P. Crippa, M. Pirani and G. Biagetti, Representation of nonlinear random transformations by non-Gaussian stochastic neural networks, *IEEE transactions on neural networks*, 19 (2008), 1033–1060.

[26] M. Uddin, H. Ali and A. Ali, Kernel-based local meshless method for solving multi-dimensional wave equations in irregular domain, *CMES-Computer Modeling In Engineering & Sciences*, 107 (2015), 463–479.

[27] M. Uddin and H. Ali, The space time kernel based numerical method for Burgers equations, *Mathematics*, 6 (2018), 212–222.
[28] M. Uddin, K. Kamran, M. Usman and A. Ali, On the Laplace-transformed-based local meshless method for fractional-order diffusion equation, *Int. J. Comput. Methods Eng. Sci. Mech.*, 19 (2018), 221–225.

[29] B. M. Vaganan and E. E. Priya, Generalized Cole-Hopf transformations for generalized Burgers equations, *Pramana*, 85 (2015), 861–867.

[30] D. L. Young, C. C. Tsai, K. Murugesana, C. M. Fan and C. W. Chen, Time-dependent fundamental solutions for homogeneous diffusion problems, *Engineering Analysis with Boundary Elements*, 28 (2004), 1463–1473.

[31] H. Zhang, F. Liu, I. Turner and Q. Yang, Numerical solution of the time fractional Black-Scholes model governing European options, *Comput. Math. Appl.*, 71 (2016), 1772–1783.

Received February 2019; 1st revision May 2019; 2nd revision June 2019.

*E-mail address:* marjan@uetpeshawar.edu.pk

*E-mail address:* aliuetpeshawar@gmail.com