Periodic orbits in dynamic systems with Wien Bridge

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Abstract. The following document showed the state space system that governs the behavior of the Wien Bridge, which is an electronic oscillator that generates sinusoidal waves, through an amplifier fed back by a bridge composed of four resistors and two capacitors. Through the simulations carried out, the existence of periodic orbits was verified when this system operates in a stable state. For this, the limit cycles were studied by being able to determine the origin of the periodic oscillation in the circuit in areas very close to the point of stability by means of the focus-center-limit cycle bifurcation presented in the Poincaré-Bendixon Theorem.

1. Introduction
Analytically solving the problem of nonlinear Cauchy (differential equation) can be very difficult and in many cases have no solution. This is why analytical, geometric and numerical techniques have been developed to approximate the solution, which allows us to understand the qualitative behavior of the solutions. One of the most important aspects in qualitative analysis is to know the asymptotic behavior of the solutions and this is achieved by studying the dynamics of the critical points of the system and the existence of periodic orbits. A point $x_0 \in \mathbb{R}$ is a critical point if it is fulfilled that for the Cauchy problem $\dot{x} = f(x)$, the function becomes null when evaluated at that point, that is $f(x_0) = 0$ [1]. The Poincaré-Bendixon theorem offers the conditions to solve the nonlinear Cauchy theorem which will be applied to verify through limit cycles (closed trajectories in a space for which at least one other trajectory converges) the existence of periodic orbits when the Wien bridge oscillator (object of study of this document), operates in a stable state.

2. Content
2.1. Wien oscillator
The Wien Bridge oscillator has two main parts, a band pass filter that is responsible for generating the oscillation at a frequency equal to its center frequency, and a non-inverting amplifier that maintains the oscillator gain in one [2]. The band pass filter has the input of the operational amplifier $v_0$ as input, and its output $v_+$ is fed back to the non-inverting pin of the same amplifier [3]. Since the filter output is attenuated since it is a passive filter, with the RF and $R_1$ resistors the adjustment is made so that the gain $A$ is the unit [4]. Then first the band pass filter analysis will be performed Figure 1. The goal of adding the bandpass filter is to allow only some frequencies to be viewed over the domain.
Applying Kirchhoff's laws to the previous circuit (Wien Bridge) the following is taken Equations (1)

\[ R_1 C_1 \dot{V}_{c1} = -V_{c1} - V_{c2} + V_0, \]  

\[ C_1 \dot{V}_{c1} - C_2 \dot{V}_{c2} = \frac{V_{c2} - E}{R_2}, \]  

where Equation (3).

\[ V_0 = \frac{2E}{\pi} \tan^{-1}\left(\frac{\pi\alpha V_{c2}}{2E}\right), \]  

where E is the saturation voltage of the operational amplifier and \( \alpha = 1 + \frac{R_1}{R_2} \) is the gain of the circuit equivalent to the operational amplifier. The parameters are determined so that the circuit generates a 60 Hz frequency oscillatory signal. The presence of the filter will act to be able to leave only some frequencies on the signal window but what is desired is that the frequency of the input signal is not disturbed, Figure 2.

\[
\begin{bmatrix}
\dot{V}_{c2} \\
\dot{V}_{c1}
\end{bmatrix} = 
\begin{bmatrix}
\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} & \frac{1}{R_1 C_2} \\
\frac{1}{R_1 C_1} & \frac{1}{R_1 C_1}
\end{bmatrix}
\begin{bmatrix}
V_{c2} \\
V_{c1}
\end{bmatrix} + 
\begin{bmatrix}
\frac{1}{R_1 C_2} \\
\frac{1}{R_1 C_1}
\end{bmatrix} V_0 + 
\begin{bmatrix}
E_b \\
0
\end{bmatrix}.
\]

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![Figure 1. Wien Bridge circuit.](image1)

**Figure 1.** Wien Bridge circuit.

![Vol](image2)

**Figure 2.** Sinusoidal signal generated by Wien Bridge oscillator.

It should be borne in mind that the model of a system is not unique and for this reason the analysis could change depending on the perspective. In order to carry out studies on the stability of a non-linear system, it is very convenient and efficient to analyze the qualitative behavior of said system, since the analysis starts from the representation of the system in state space, and by means of linearization around the equilibrium points [1] They can perform analyzes of various factors such as fork, types of breakpoints, phase portrait, etc.; the state space system for the Wien circuit is given by Equation (4).
2.2. Limit cycle
A periodic orbit $\gamma$ in a plane (or in a two-dimensional variety) is called the limit cycle if it is the set of limits $\alpha$ or the set of limits $\omega$ of some point $z$ that is not in the periodic orbit, that is, the set of accumulation points of the path forward or backward through $z$, respectively, are exactly $\gamma$. Asymptotically stable and unstable periodic orbits are examples of limit cycles [5,6]. Figure 3 shows the classification of limit cycles [6-8], the qualitative theory allows us the characterization of this type of cycles as well as the analysis of local stability, something that would not be possible through the Laplace transform or another analytical method.

![Figure 3. Classification of limit cycles [6].](image)

2.3. Poincaré-Bendixon theorem
Let $M \subset \mathbb{R}^2$ be open and $v = (v_1, v_2): M \rightarrow \mathbb{R}^2$, a vector field of class $r (r \geq 1)$. Let $F \subset M$ be compact and $x_0 \in F$. Is also positively invariant, that is, for all $y_0 \in F$ the solution $\varphi(t)$ de $\dot{x} = v(x), x(0) = y_0$ meets $\varphi(t) \in F$ for all $t \geq 0$. Then, at least one of the following statements occurs Figure 4 [9]. The following figure allows to see in a didactic way the existence of a periodic point on the system, something that could be more difficult to explain through quantitative analysis:

- $x_0$ is an equilibrium point.
- The solution $\varphi(t)$ de $\dot{x} = v(x), x(0) = x_0$ complies that $\lim_{t \rightarrow \infty} \varphi(t) = x^* \in F$ being $x^*$ equilibrium point.
- The solution $\varphi(t)$ de $\dot{x} = v(x), x(0) = x_0$ is periodic and the closed orbit is contained in $F$.
- The orbit that passes through $x_0$ approaches asymptotically to a closed orbit contained in $F$.

![Figure 4. Existence of periodic orbit.](image)

The set that is selected in this case is $F \equiv 1 \leq (x_1 + 2)^2 + x_2^2 \leq 2$, said set is compact (closed and bounded as can be seen in Figure 4. To verify that it is positively invariant, the scalar product $\langle x, v(x) \rangle$ is performed and evaluated at the boundaries of field $F$ to see where the field lines are directed [10].

3. Results
Using the previously established equations for the Vienna bridge oscillator [11], the simulation of the oscillator behavior, Figure 5 and Figure 6, was performed in the MATLAB software. The concentration points of the system were identified (Figure 7) to verify the existence of periodic orbits.
In the graph of Figure 5 you can see specifically the voltage behavior in the Wien Bridge Oscillator. After 0.01311 seconds the voltage stabilizes. In the graph of Figure 6 you can see how as time passes the signal begins to branch; but this signal reaches a stable point. In the graph of Figure 7 you can see the concentration points of the system and in the middle the periodic points associated with the critical point of the system for which Figure 7 is stabilized [12]. In it you can see the phase diagram, clearly in which the critical points and the stability of the system are presented.

![Figure 5. Voltage behavior in Wien Bridge oscillator.](image1)

![Figure 6. Current behavior in Wien Bridge oscillator.](image2)

![Figure 7. Behavior of critical points.](image3)

4. Conclusions
Through the simulations carried out, the existence of periodic orbits was verified when this system operates in a stable state. The focus-center-limit cycle bifurcation presented in the Poincaré-Bendixon theorem justifies the birth of the periodic oscillation in the circuit in areas very close to the point of stability; when analyzing the system bifurcation for the $R_f$ branch parameter, it is presented that for values that exceed a threshold in said parameter the periodic orbits of the system disappear, this phenomenon is recognized as bifurcation equilibrium boundaries.

Limit cycles are of great importance since they actually have many applications, especially in biological and physical models; since the existence of a limit cycle in a model guarantees the coexistence of two or more curves or trajectories that approach it. This behavior was what was found when examining the critical points in the Wien Bridge oscillator exposed in this work.
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