Cone types and geodesic languages for lamplighter groups and Thompson’s group $F$

Sean Cleary$^1$

Department of Mathematics, The City College of New York, New York

Murray Elder$^2$

School of Mathematics and Statistics, University of St Andrews, Scotland

Jennifer Taback$^3$

Department of Mathematics, Bowdoin College, Brunswick, Maine

Abstract

We study languages of geodesics in lamplighter groups and Thompson’s group $F$. We show that the lamplighter groups $L_n$ have infinitely many cone types, have no regular geodesic languages, and have 1-counter, context-free and counter geodesic languages with respect to certain generating sets. We show that the full language of geodesics with respect to one generating set for the lamplighter group is not counter but is context-free, while with respect to another generating set the full language of geodesics is counter and context-free. In Thompson’s group $F$ with respect to the standard finite generating set, we show there are infinitely many cone types and that there is no regular language of geodesics. We show that the existence of families of “seesaw” elements with respect to a given generating set in a finitely generated infinite group precludes a regular language of geodesics and guarantees infinitely many cone types with respect to that generating set.

Key words: Regular language, rational growth, cone type, context-free grammar, counter automata, lamplighter groups, Thompson’s group $F$

Email addresses: cleary@sci.ccny.cuny.edu (Sean Cleary), murray@mcs.st-and.ac.uk (Murray Elder), jtaback@bowdoin.edu (Jennifer Taback).

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1 Introduction

In this article we prove some language-theoretic consequences of work by Cleary and Taback describing geodesic words in the lamplighter groups [9,11] and Thompson’s group $F$ [8,10]. We consider Thompson’s group $F$ with its standard finite generating set,

$$F = \langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$  

We consider the lamplighter groups $L_m$ in the standard wreath product presentation,

$$L_m = \langle a, t | [t^i a t^{-i}, t^j a t^{-j}], a^m, i, j \in \mathbb{Z} \rangle.$$  

We also consider an alternate generating set for $L_2$ which arises from considering this group as an automata group, in which case the natural generators are $t$ and $(ta)$.

We prove the following results for $L_m$ with $m \geq 2$ with respect to generating set $\{a, t\}$, for $L_2$ with respect to the automata group generating set $\{t, (ta)\}$, and for Thompson’s group $F$ with respect to the standard finite generating set $\{x_0, x_1\}$.

- There are infinitely many cone types; that is, there are infinitely many families of possible geodesic extensions to group elements.
- There is no regular language of geodesics which includes at least one representative of each group element.

We prove that there are 1-counter languages of geodesics with a unique representative for each element for $L_m$ with $m \geq 2$ with respect to generating set $\{a, t\}$ and for $L_2$ with respect to the automata group generating set $\{t, (ta)\}$.

However, the formal language class of the full language of geodesics in $L_2$ depends on the choice of generating set. For the automaton generating set $\{t, ta\}$ the set of all geodesics is 1-counter, which implies it is both context-free and counter, yet for the wreath generating set $\{a, t\}$ the set of all geodesics for $L_m$ with $m \geq 2$ is shown to be context-free but not counter.

In addition, we show that if a finitely generated group contains a family of “seesaw elements” of arbitrary swing with respect to a finite generating set, then the group has infinitely many cone types with respect to that generating set and cannot have a regular language of geodesics with respect to that generating set.

The concepts of cone types, regular, context-free, counter and 1-counter geodesic languages are intimately connected, and tell us much about the structure of geodesics in a given group presentation. If a group has finitely many cone
types, then the full language of geodesics is regular. The converse is true when all relators in the presentation have even length [21], and is conjectured to be true in general. In addition, Grigorchuk and Smirnova-Nagnibeda [17] showed that if a group has finitely many cone types then it has a rational complete growth function. While the growth of the lamplighter groups is much studied, the rationality of the growth function for Thompson’s group is an open question.

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2 Languages, Grammars and Cone types

We now present the necessary definitions for the different types of languages and finite state automata which we consider below. If $X$ is a finite set of symbols then we let $X^*$ be the set of all finite strings in the symbols of $X$, including the empty string $\varepsilon$ which has no letters. A language over $X$ is a subset of $X^*$. A regular language is one accepted by some finite state automaton. See Epstein et al [14] and Sipser [23] for an introduction to regular languages.

A key tool in the theory of regular languages is the Pumping Lemma:

**Lemma 1 (Pumping Lemma for regular languages)** Let $A$ be a regular language. There is a number $p$ (the pumping length) so that if $s$ is any word in $A$ of length at least $p$, then $s$ may be divided into three pieces $s = xyz$ where we can require either that $|xy| \leq p$ or $|yz| \leq p$ and have both

- For each $i \geq 0$, $xy^iz \in A$
- $|y| > 0$

See Sipser [23] for a proof of the Pumping Lemma. A standard method of showing that a given language fails to be regular is via this lemma.

Let $G$ be a group with a word metric defined with respect to a finite generating set. Cannon [6] defined the cone type of an element $w \in G$ to be the set of geodesic extensions of $w$ in the Cayley graph.

**Definition 2 (Cone type)** A path $p$ is outbound if $d(1, p(t))$ is a strictly increasing function of $t$. For a given $g \in G$, the cone at $g$, denoted $C'(g)$ is the set of all outbound paths starting at $g$. The cone type of $g$, denoted $C(g)$, is $g^{-1}C'(g)$.

This definition applies both in the discrete setting of the group and in the one-dimensional metric space which is the Cayley graph. In the Cayley graph,
a cone type may include paths that end at the middle of an edge. If the presentation of $G$ consists of only even-length relators, then all cone types will consist solely of full edge paths. Neumann and Shapiro [22] show that whether a group has finite or infinitely many cone types may depend on the choice of generating set and they describe a number of properties in of cone types in [21].

In all that follows, when we consider the set of cone types of a group, we fix a particular generating set. Knowing the set of all possible cone types of elements of a particular group provides information about the full language of geodesics of that group with respect to that particular generating set. We may also get information and about the growth function of the group as follows.

**Lemma 3** [21] If $G$ has finitely many cone types with respect to a finite generating set, then the full language of geodesics with respect to that generating set is regular.

It is not known whether the converse is true in general, but the converse is true for presentations having only even-length relators.

**Lemma 4** [21] Let $G$ be a finitely-presented group with a generating set in which all relators have even length. If the full language of geodesics with respect to this generating set is regular, then $G$ has finitely many cone types with respect to this generating set.

We consider not only regular languages below, but context free and counter languages as well.

**Definition 5 (Pushdown automaton)** A pushdown automaton is a machine consisting of

- a stack,
- a finite set of stack symbols, including $\$ \, \text{which is a marker symbol for the beginning and end of the stack}$, and
- a finite state automaton with possible additional edge labels to affect the stack

where each transition may, in addition to reading a letter from the input word, potentially also push or pop a stack symbol on or off the stack. A word is accepted by the pushdown automaton if it represents a sequence of transitions from the start state to an accept state so that the stack is empty at the final state.

**Definition 6 (Context-free)** A language is context-free if it is the set of all strings recognized by some pushdown automaton.
For example, the language \( \{ a^n b^n \mid n \in \mathbb{N} \} \) is accepted by the pushdown automaton in Figure 1 with alphabet \( a, b \) and stack symbols \( \$, 1 \), and this language is not regular.

![Pushdown automaton accepting \( a^n b^n \) with start state \( q_0 \).

The following result is proved in [23].

**Lemma 7** Context-free languages are closed under union.

There is a pumping lemma for context-free languages as well.

**Lemma 8** (Pumping Lemma for context-free languages) Let \( A \) be a context-free language. There is a number \( p \) (the pumping length) so that if \( s \) is any word in \( A \) of length at least \( p \), then \( s \) may be divided into five pieces \( s = uvxyz \) such that

- For each \( i \geq 0 \), \( uv^i xy^i z \in A \)
- \( |v|, |y| > 0 \)
- \( |vxy| \leq p \)

See [23] for a proof of Lemma 8.

**Definition 9** (G-automaton) Let \( G \) be a group and \( \Sigma \) a finite set. A (non-deterministic) \( G \)-automaton \( A_G \) over \( \Sigma \) is a finite directed graph with a distinguished start vertex \( q_0 \), some distinguished accept vertices, and with edges labelled by elements of \((\Sigma^{\pm 1} \cup \{\epsilon\}) \times G\). If \( p \) is a path in \( A_G \), then the element of \((\Sigma^{\pm 1})\) which is the first component of the label of \( p \) is denoted by \( w(p) \), and the element of \( G \) which is the second component of the label of \( p \) is denoted \( g(p) \). If \( p \) is the empty path, \( g(p) \) is the identity element of \( G \) and \( w(p) \) is the empty word. \( A_G \) is said to accept a word \( w \in (\Sigma^{\pm 1}) \) if there is a path \( p \) from the start vertex to some accept vertex such that \( w(p) = w \) and \( g(p) =_G 1 \).

If \( G \) is the trivial group, then a \( G \)-automaton is just a finite state automaton.

**Definition 10** (Counter) A language is \( k \)-counter if it is accepted by some \( \mathbb{Z}^k \)-automaton. We call the (standard) generators of \( \mathbb{Z}^k \) and their inverses counters. A language is counter if it is \( k \)-counter for some \( k \geq 1 \).
For example, the language \( \{a^n b^n a^n \mid n \in \mathbb{N} \} \) is accepted by the \( \mathbb{Z}^2 \)-automaton in Figure 2, with alphabet \( a, b \) and counters \( x_1, x_2 \), and this language is not context-free [23].

In the case of \( \mathbb{Z} \)-automata, we assume that the generator is 1 and the binary operation is addition, and we may insist without loss of generality that each transition changes the counter by either 0, 1 or \(-1\). We can do this by adding states and transitions to the automaton appropriately. That is, if some edge changes the counter by \( k \neq 0, \pm 1 \) then divide the edge into \( |k| \) edges using more states. The symbols +, − indicate a change of 1, −1 respectively on a transition.

Note that with this definition, a \( \mathbb{Z} \)-automaton cannot “see” the value of the counter until it reaches an accept state, so is not equivalent to a pushdown automaton with one stack symbol, which can determine if the stack is empty at any time. In the literature these counter automata are sometimes called “partially blind” to emphasize this difference.

Elder [12] shows that counter languages have the following properties:

**Lemma 11** 1-counter languages are context-free.

**Lemma 12 (Closure properties)** 1-counter languages are not closed under concatenation or intersection. If \( C \) is \( k \)-counter for \( k \geq 1 \) and \( L \) is regular, then \( C \cap L, CL \) and \( LC \) are all \( k \)-counter. The union of a finite number of \( k \)-counter languages is \( k \)-counter.

In order to show that a language is not counter we make use of the following lemma from [12].

**Lemma 13 (Swapping Lemma)** If \( L \) is counter then there is a constant \( s > 0 \), the “swapping length”, such that if \( w \in L \) with length at least \( 2s + 1 \) then \( w \) can be divided into four pieces \( w = uxyz \) such that \( |uxy| \leq 2s + 1, |x|, |y| > 0 \) and \( uxyz \in L \).

**Proof** Let \( p \) be the number of states in the counter automaton. If a path visits each state at most twice then it cannot have length more than \( 2p \), so \( w \) visits
some state $s$ at least three times. Let $u$ be the prefix of $w$ ending when $w$ reaches state $s$ for the first time. Let $x$ be the continuation of $u$, ending when $w$ reaches state $s$ for the second time. Let $y$ be the continuation of $x$, ending when $w$ reaches state $s$ for the third time. Finally, let $z$ be the remainder of $w$. So $w = uxyz$ ends at an accept state, with all the counters balanced correctly. If we switch the orders of the strings $x$ and $y$, then we will also have a path leading to the same accept state with the correct counters, so $uyxz \in L$. \hfill \Box

The Swapping Lemma is similar to the Pumping Lemmas for regular and context-free languages. It is only useful if the word $w$ has no repeated concurrent subwords; if there are repeated subwords, we can take $x = y$ in the statement of the lemma, and the resulting word will be identical to the initial word so the conclusion is vacuously true.

For more background on counter languages see Mitrana and Stiebe [20], Elston and Ostheimer [13], Elder [12], and Gilman [16].

3 Geodesic languages for the lamplighter groups in the wreath product generating set

The lamplighter group $L_m$, with presentation

$$L_m = \langle a, t \mid [t^i at^{-i}, t^j at^{-j}], a^m \rangle$$

is the wreath product $\mathbb{Z}_m \wr \mathbb{Z}$, where the generator $a$ in the above presentation generates $\mathbb{Z}_m$ and $t$ generates $\mathbb{Z}$. We will refer to these generators as the wreath product generators to distinguish them from the generators which arise naturally when the group $L_2$ is considered as an automata group.

In [9], Cleary and Taback studied metric properties of this wreath product presentation for $L_m$. In this paper we consider some language theoretic consequences of that work. We begin with a geometric interpretation for elements of $L_m$, first in the case $m = 2$, and then for $m > 2$.

An element of $L_2$ is best understood via the following geometric picture. We visualize each $\mathbb{Z}_2$ factor as a light bulb which is either on or off, and the wreath product with $\mathbb{Z}$ thus creates an infinite string of light bulbs, one at each integer. An element $w \in L_2$ is represented by a configuration of a finite number of illuminated bulbs, and a position of the “lamplighter” or cursor. A word $\gamma$ in the generators $\{a, t\}$ denoting the element $w$ can be thought of as a sequence of instructions for creating the configuration representing $w$, in the following way.

The position of the cursor indicates the current bulb under consideration by
Fig. 3. The element \( w = a_4a_5a_6a_{-1}a_{-6}t^{-2} \) of \( L \), expressed as a configuration of illuminated bulbs and a cursor in the position \(-2\). Shaded bulbs represent bulbs which are illuminated, clear bulbs represent bulbs which are off, the vertical bar denotes the origin in \( \mathbb{Z} \), and the arrow denotes the position of the cursor.

the “lamplighter”. The generator \( a \) changes the state of the bulb at the cursor position, and the generator \( t \) moves the cursor one unit to the right. Thus prefixes of the word \( \gamma \) appear to “move” the cursor to different integral positions, possibly changing the state of bulbs as well, ending with the configuration representing \( w \).

The identity word is represented by the configuration of bulbs which are all in the off state, and the cursor at the origin. Figure 3 gives an example of an element of \( L_2 \) represented in this way. For a given word \( w \), we consider the successive prefixes of \( w \) as steps involved in the creation of \( w \).

Elements of \( L_m \) for \( m > 2 \) can be understood via the analogous picture in which the bulbs have \( m \) states. Again, the generator \( t \) moves the cursor one unit to the right, and \( a \) increments the state of the current bulb under consideration.

### 3.1 Normal forms for elements of \( L_2 \)

We first give normal forms for elements of \( L_2 \) in detail, then generalize these forms to \( L_m \) for \( m > 2 \).

Normal forms for elements of \( L_2 \) with respect to the wreath product generators \( a \) and \( t \) are given in terms of the conjugates \( a_k = t^ka^{-k} \), which move the cursor to the \( k \)-th bulb, turn it on, and return the cursor to the origin.

We present two normal forms for an element \( w \in L_2 \), the right-first normal form given by

\[
rf(w) = a_{i_1}a_{i_2} \cdots a_{i_m}a_{-j_1}a_{-j_2} \cdots a_{-j_l}t^r
\]

and the left-first normal form given by

\[
lf(w) = a_{-j_1}a_{-j_2} \cdots a_{-j_l}a_{i_1}a_{i_2} \cdots a_{i_m}t^r
\]

with \( i_m > \ldots > i_2 > i_1 \geq 0 \) and \( j_l > \ldots > j_2 > j_1 > 0 \). In the successive prefixes of the right-first normal form, the position of the cursor moves first toward the right, and appropriate bulbs are illuminated in nonnegative positions.
In later prefixes, the position of the cursor moves to the left, and once its
position is to the left of the origin, the appropriate bulbs in negative positions
are illuminated as well. The left-first normal form follows this procedure in
reverse.

One or possibly both of these normal forms will lead to a minimal length
representative for $w \in L_2$ with respect to the wreath product generating set,
depending upon the final location of the cursor relative to the origin. That
position is easy to detect from the sign of the exponent sum of $t$, given as $r$
above. If $r > 0$, then the left-first normal form will produce a minimal length
representative for $w$, and if $r < 0$, the right-first normal form will produce a
minimal length representative for $w$. If the exponent sum of $t$ is zero, then
both normal forms lead to minimal length representatives for $w$.

Cleary and Taback [9] used these normal forms to compute the word length
of $w \in L_2$ with respect to the wreath product generators as follows.

**Proposition 14 ([9], Proposition 3.2)** Let $w \in L_2$ be in either normal
form given above, and define

$$D(w) = m + l + \min\{2j_l + i_m + |r - i_m|, 2i_m + j_l + |r + j_l|\}.$$ 

The word length of $w$ with respect to the generating set $\{a, t\}$ is exactly $D(w)$.

The geodesic representatives for $w \in L_2$ arising from either the left-first or
right-first normal forms are not necessarily unique. Suppose that $\gamma$ is a geodesic
word in $\{t, a\}$ representing $w \in L_2$, and $\gamma$ has two prefixes with the same
exponent sum. This corresponds to a bulb at some position $k$ which is “visited
twice” by the cursor during the construction of $w$. If this bulb is illuminated
in the word $w$, then there is a choice as to whether it is illuminated during the
first prefix which leaves the cursor in this position, or the second. Similarly, if
the bulb in position $k$ is not illuminated in $w$, then either it remains off during
both prefixes, or is turned on in the first, and off in the second.

If $w \in L_2$ has illuminated bulbs which are visited more than once by the cursor
during the construction of $w$, then there will be multiple possible geodesics
representing $w$. A typical element $w$ where the cursor is not left at the origin
may have $k$ illuminated bulbs which are visited exactly twice, giving $2^k$ possible
geodesics which represent $w$. For elements where the final position of the cursor
is at the origin, there are such families of geodesics from both the right-first
and left-first methods of construction. If bulb zero is illuminated and the cursor
remains at the origin with illuminated bulbs on both sides of the origin, there
will be 3 visits of the cursor to the origin, giving $3 \cdot 2^k$ possible geodesic
representatives for each of right-first and left-first manners of constructing the
element and thus $6 \cdot 2^k$ total possible geodesic representatives. It is not hard
to see using Proposition 14 that all geodesic representatives for $w$ must be of
this form.

3.2 Normal forms for elements of $L_m$

The normal forms given above for elements of $L_2$ have obvious extensions to $L_m$ for $m > 2$. Occurrences of $a$ in the normal forms above must now be replaced by $a^k$, for $k \in \{-h, -h + 1, \cdots, -1, 0, 1, 2, \cdots, h\}$ where $h$ is the integer part of $\frac{m}{2}$. When $m$ is even, we may omit $a^{-h}$ to ensure uniqueness, since $a^h = a^{-h}$ in $\mathbb{Z}_{2h}$.

There is an analogous definition for $D(w)$ in [9] when $w \in L_m$ which again determines the word length of $w$ with respect to the wreath product generating set.

As with $L_2$, the left- and right-first normal forms for elements of $L_m$ do not necessarily give all minimal length representatives for group elements. If a geodesic representative for $w \in L_m$ has two prefixes with exponent sum $k$, and the bulb in position $k$ in $w$ is illuminated to state $l < m$, then there are many choices as to what state the bulb is left in at the first prefix, with the second prefix allowing the bulb to be illuminated to the final state.

3.3 Geodesic languages for $L_m$

We now prove that there is no collection of geodesic paths representing elements of $L_m$ which is accepted by a finite state automaton. In this section we always consider $L_m$ with respect to the wreath product generators $\{a, t\}$.

**Theorem 15** The lamplighter groups $L_m$ with respect to the wreath product generating set $\{a, t\}$ have no regular languages of geodesics.

**Proof:** Let $g_n = a_n a_{-n} \in L_m$ be the group element corresponding to a configuration of bulbs in which the two bulbs at distance $n$ from the origin are turned on to the first state, and the cursor is at the origin. This element has exactly two geodesic representatives: $t^n a t^{-2n} a t^n$ and $t^{-n} a t^{2n} a t^{-n}$.

Suppose that there is a regular language of geodesics for $L_m$ with respect to this generating set. Then the Pumping Lemma for regular languages guarantees a pumping length $p$ for this language. Choose $n > p$ and consider $g_n$ as defined above. Suppose that the first geodesic representative for $g_n$ is an element of the regular language.
Since \( n > p \), we can write \( g_n = xyz \) such that \( |xy| < p \), so \( x = t^i, y = t^j \) with \( j > 0 \) and \( j < p < n \), and \( z = t^{n-i-j}at^{-2n}at^n \). Then by the Pumping Lemma, \( xy^2z \) must also be in the language. So \( t^{i+2j}t^{n-i-j}at^{-2n}at^n = t^{n+j}at^{-2n}at^n \) would be geodesic. This word has length \( 4n + j + 2 \) and corresponds to the configuration of one lamp on at \( n + j \), one lamp on at \(-n + j \) and the cursor at position \( j \), to the right of the origin. A shorter word for this configuration is \( t^{-n+j}at^{2n}at^{-n} \) which has length \( 4n - j + 2 \), yielding a contradiction.

Similarly, if the second geodesic representative for \( g_n \) was part of the regular language, we would obtain an analogous contradiction. Thus neither geodesic representative for \( g_n \) can be part of any regular language of geodesics for \( L_m \).

\[ \blacksquare \]

**Corollary 16** The lamplighter group \( L_m \) for \( m \geq 2 \) has infinitely many cone types with respect to the wreath product generating set.

**Proof**: Theorem 15 states that the full language of geodesics is not regular, so the contrapositive of Lemma 3 implies that the number of distinct cone types is not finite. \( \blacksquare \)

One can observe this directly as well by considering the elements \( g_n = a_n a_{-n} = t^n at^{-2n}at^n \) used in the proof of Theorem 15. The cone type of \( t^n at^{-2n}a \) contains \( t^n \) but not \( t^{n+1} \), so for each \( n \) we have a distinct cone type with respect to \( \{a, t\} \).

**Theorem 17** There is a language of geodesics for \( (L_m, \{a, t\}) \) with a unique representative for each element that is accepted by a 1-counter automaton.

**Proof**: We first describe the 1-counter automaton accepting a language of geodesics for \( L_2 \), and then give the generalization to \( L_m \) for \( m > 2 \). In \( L_m \), for \( m \geq 2 \), each group element corresponds to a configuration of bulbs in some states and a cursor position. Such a configuration can always be obtained in the following manner.

If the exponent sum of the generator \( t \) in either normal form for \( w \in L_2 \) is negative, so the cursor’s final position is given by \( i < 0 \), then we construct a minimal length representative for the element in a “right-first manner” as follows. Suppose that the rightmost illuminated bulb in \( w \) is in position \( l \). We begin the geodesic representative with \( t^i a \), which illuminates this bulb. We then add suffixes of the form \( t^{-k}a \); each suffix puts the cursor in the position of another bulb which must be turned on, and this is accomplished via the generator \( a \). This is done until the leftmost illuminated bulb is turned on. We add a final suffix of the form \( t^n \) which brings the cursor to its position in \( w \).

The 1-counter automaton in Figure 4 accepts this set of words. The counter keeps track of the current position of the cursor.
If the exponent sum of the generator \( t \) in either normal form given above is nonnegative, so that the cursor ends at a position \( i \geq 0 \), then we construct a minimal length representative for the element in a “left-first manner”. This is done by interchanging the generators \( t \) and \( t^{-1} \) in the above method, and first turns on the leftmost illuminated bulb.

The 1-counter automaton in Figure 5 accepts this set of words. Again the counter keeps track of the current position of the cursor.

This process gives a unique representative for each group element, and in [9] it is shown that these words are geodesic.

To construct a 1-counter automaton which accepts these representatives of elements of \( L_m \), we modify the automata given in Figures 4 and 5 as follows. Additional loops must be added to states \( q_1 \) and \( q_3 \), allowing for the bulbs to be turned to any possible state. Namely, there must be loops at state \( q_1 \) with labels \((t^{-1}a^k, -)\) for \( k \in \{-h, -h + 1, \ldots, -1, 0, 1, 2, \ldots h\}\), where \( h \) is the integer part of \( \frac{m}{2} \). When \( m \) is even, we omit \( a^{-h} \) to ensure uniqueness, since \( a^h = a^{-h} \) in \( \mathbb{Z}_{2h} \). We also need to add edges from \( q_1 \) to \( q_2 \) with labels \((t^{-1}a^k, -)\) for \( k \in \{\pm 1, \pm 2, \ldots \pm h\}\).

The loops (and edges) added to state \( q_3 \) are of the form \((ta^k, +)\), with the same restrictions on \( k \).

Now we consider the full language of geodesics. As described above, there may be many possible geodesic representatives for an element. Bulbs may be visited
twice or even three times in the extreme case, when the cursor ends back at the origin, with bulbs illuminated to the left and right. There may be geodesic representatives which turn on such a bulb at any of those opportunities, but no geodesic representative can change the state of a particular bulb more than once in $L_2$.

Clearly, remembering which bulbs have already been turned on is not a job for a finite state automaton, as proved above, but it is easy to avoid switching on and off the same bulb by keeping track of previous switchings using a stack. This is the key to the proof of the next theorem.

**Theorem 18 (Context free full language for $(L_2, \{a, t\})$)** The full language of geodesics for the lamplighter group with the wreath product generating set is context-free.

*Proof:* We describe the complete set of geodesic words which represent each group element $w \in L_2$, which we view as a configuration of illuminated bulbs along with a position of the cursor.

If the cursor position in $w$ is $i \leq 0$ with no bulbs to the right of position 0 illuminated, then geodesic representatives for this element are of the form $g_0g_{-1} \ldots g_{-m}(v)$ where $g_i$ is either $t^{-1}$ or $at^{-1}$ (which illuminates the bulb at position $i$), $v$ is either empty or $ag_{-m}g_{-1-m} \ldots g_r$, where $-m \leq r \leq 0$ and

\[
 g_i' = \begin{cases} 
 t & \text{if } g_i = at^{-1} \\
 at \text{ or } t & \text{otherwise.}
\end{cases}
\]

We push a 1 on the stack to indicate a bulb is switched on, and a 0 if the bulb is not switched on in that position. Then when we return to a position we can only switch if a 0 is on the top of the stack. The pushdown automaton in Figure 6 accepts these words.

If the cursor position in $w$ is $i \geq 0$ with no bulbs to the left of position 0 illuminated, then geodesic representatives for this element are of the form $g_0g_1 \ldots g_{n-1}(v)$ where $g_i$ is either $t$ or $at$ (if the lamp at position $i$ is illuminated), and $v$ is either $\epsilon$ or $ag_{n-1}g_{n-2} \ldots g_r$, where $0 \leq r \leq n-1$ and

\[
 g_i' = \begin{cases} 
 t^{-1} & \text{if } g_i = at \\
 t^{-1}a \text{ or } t^{-1} & \text{otherwise.}
\end{cases}
\]

The pushdown automaton in the right of Figure 7 accepts these words.
Fig. 6. The pushdown automaton accepting the set of geodesics for elements of $L_2$ in which the cursor is in position $i \leq 0$ and there are no illuminated bulbs to the right of the origin. The start state is labelled $q_0$.

Fig. 7. The pushdown automaton accepting geodesics in which the cursor is in position $i \geq 0$ and there are no illuminated bulbs to the left of the origin. The start state is labelled $q_0$.

If the cursor position in $w$ is $i \leq 0$ with the rightmost illuminated bulb at $n > 0$, then geodesic representatives for this element are of the form

$$(u)f_1 \ldots f_{n-1}tat^{-1}g_{n-1} \ldots g_1(v)g'_{n-1} \ldots g'_1(w)$$

where $u$ is either $\epsilon$ or $a$ (turns bulb at 0 on), $f_i$ is either $t$ or $ta$ (turns bulb at $i$ on),

$$g_i = \begin{cases} t^{-1} & \text{if } f_i = ta \\ at^{-1} & \text{or } t^{-1} \text{ otherwise (}at^{-1} \text{ turns bulb at } i \text{ on)} \end{cases}$$

$$v = \begin{cases} a & \text{if } u = \epsilon \text{ (turns bulb at } i \text{ on)} \\ \epsilon & \text{otherwise,} \end{cases}$$

$$g'_i = t^{-1} \text{ or } t^{-1}a \text{ (turns the bulb at } i \text{ on)}, \quad w = \epsilon \text{ if the cursor ends at or to the left of the leftmost illuminated bulb, or } t^{-1}atk_{-m} \ldots k_r \text{ with } r \leq 0, \text{ for } -m \leq i < 0$$
\( k_i = \begin{cases} 
  t & \text{if } g_i = t^{-1}a \\
  t a \text{ or } t & \text{otherwise (at turns bulb at } i \text{ on),}
\end{cases} \)

and if \( r = 0 \) then

\( k_0 = \begin{cases} 
  a & \text{if } u = \epsilon \text{ and } v = \epsilon \text{ (turns bulb at } i \text{ on),} \\
  \epsilon & \text{otherwise.}
\end{cases} \)

As mentioned above, we can keep track of whether or not a bulb has been illuminated as we move the cursor right then left then right using a stack. In the pushdown automaton in Figure 8 we push a 1 on the stack to indicate a bulb is on, and a 0 if the bulb in that position is off.

We need to take care with the bulb at position 0, since there may be three possible opportunities to illuminate it. We use different “bottom of stack” markers \$ and \# depending on whether this bulb has been turned on or not.

Fig. 8. The pushdown automaton accepting the set of geodesics for elements of \( L_2 \) in which the cursor is in position \( i \leq 0 \) and some bulb is illuminated at position \( n > 0 \). The state labelled \( q_0 \) is the start state.

If the cursor position in \( w \) is \( i \geq 0 \) with the leftmost illuminated bulb at position \( -m < 0 \), then geodesics for this element are of the form 
\[(u)f_{-1} \ldots f_{1-m} t^{-1} at g_{1-m} \ldots g_{-1}(v)g'_{1} \ldots g'_n(w)\]
where \( u \) is either \( \epsilon \) or \( a \) (turns bulb at 0 on), \( f_i \) is either \( t^{-1} \) or \( t^{-1}a \) (turns bulb at \( i \) on),

\[ g_i = \begin{cases} 
  t & \text{if } f_i = t^{-1}a \\
  a \text{ or } t & \text{otherwise,}
\end{cases} \]

\[ v = \begin{cases} 
  a & \text{if } u = \epsilon \\
  \epsilon & \text{otherwise,}
\end{cases} \]

\( g'_i = t \text{ or } ta \) (turns the bulb at \( i \) on), \( w = \epsilon \) if the cursor ends at or to the right of the rightmost illuminated bulb, or \( tat^{-1}k_n \ldots k_r \) with \( r \leq 0 \), for \( 0 < i \leq n \).
\[ k_i = \begin{cases} 
  t^{-1} & \text{if } g_i' = ta \\
  at^{-1} \text{ or } t^{-1} & \text{otherwise}, 
\end{cases} \]
and if \( r=0 \) then
\[ k_0 = \begin{cases} 
  a & \text{if } u = \epsilon \text{ and } v = \epsilon \\
  \epsilon & \text{otherwise}. 
\end{cases} \]

Fig. 9. The pushdown automaton accepting the set of geodesics for elements of \( L_2 \) in which the cursor is in position \( i \geq 0 \) and some bulb is illuminated at position \( n < 0 \). The state labelled \( q_0 \) is the start state.

It follows from [9] that all possible geodesics for elements of \( L_2 \) in this generating set have one of these forms, and is accepted by one of the pushdown automata in Figures 6 - 9. By Lemma 7 the union of these four languages is context-free.

Next, we show that the full language of geodesics is not counter. The proof mimics Elder’s proofs in [12] which show that there are context-free languages that are not counter, and is derived from the fact that one can write out a string on three letters of arbitrary length that has no concurrent repeating subwords, due to Thue and Morse [12]. We call such a string of letters with no repeating subwords a Thue-Morse word.

**Theorem 19** The language of all geodesics for the lamplighter group \( L_m \) with the wreath product generating set \( \{a, t\} \) is not counter.

**Proof:** Suppose that \( C \) is the full language of geodesics for \( L_m \), and that it is counter. As earlier, we let \( h \) be the integer part of \( \frac{m}{2} \), and consider the regular language \( L \) of all strings consisting of \( a^h \) and \( t^{-1} \), beginning with \( t \) and ending with \( a \), with at most 3 consecutive \( t^{-1} \) symbols. The intersection of \( C \) with \( L \) is counter by Lemma 12.

We form a language \( E \) on the letters \( \pm 1, \pm 2, \pm 3 \) to encode words from \( C \cap L \) as follows. Encode a word \( t^{w_1}a^ht^{w_2}a^h \ldots t^{w_n}a \) by its \( t \)-exponents, to obtain \( w_1w_2\ldots w_n \). So for example the word \( w = t^2a^h t^3a^h t^a t^2a^h \) is encoded as \( e = 2312 \).
An encoded word \( e = e_1 e_2 \cdots e_k \) with all \( e_i \) positive represents a configuration of bulbs with some subset of the bulbs in positions from 1 and \( \sum e_i \) illuminated to state \( h \). We consider a Thue-Morse word \( w \) in \( L \cap C \), which has by definition no repeated subwords, and encode it to form a string in \( E \) which also has no repeated subwords. This word and its encoding both represent the same configuration of bulbs, where all of the illuminated bulbs are in positions between 1 and \( \sum e_i \).

We append a suffix to \( w \) as follows, so that the new word has all bulbs in positions between 1 and \( \sum e_i \) illuminated to state \( h \). Namely, we attach strings of the form \( t^{-s}a^h \) to the end of \( w \), for \( s \in \{1, 2, 3\} \), until all bulbs are illuminated to state \( h \).

For example, if \( m = 2 \) and \( e = 2312 \) represents a configuration of illuminated bulbs in positions from 1 to 8, then \( 2312(-1)(-3)(-1)(-2) \) is the encoding for the word with all bulbs illuminated between these positions.

When \( e \) is a Thue-Morse word, there is a nice rewriting of the positive part of the word to obtain the negative part, but for this argument we merely need to observe the following. If \( e \) is a Thue-Morse word, then we let \( f \) be the encoding suffix described above corresponding to the group element \( v \), so that \( wv \) represents a group element with all bulbs illuminated to state \( h \) from positions 1 to \( \sum e_i \) with the cursor at the origin. Suppose that \( w' \) represents a different configuration of bulbs illuminated to state \( h \) at positions between 1 and \( \sum e_i \). Then \( w'v \) represents a word in which some bulb is turned to state \( 2h \). If \( 2h = m \), then this bulb has been turned on and then off, thus the word is not geodesic. If \( 2h = m - 1 \), then there is a bulb which is turned to state \( 2h \); a geodesic representative for this element would use a single generator, \( a^{-1} \), to achieve this state. The point is that the pairs of positive and negative words defined here are carefully chosen to interleave illumination of the bulbs, thus ensuring that changing either of the parts alone will render the word non-geodesic.

If the language \( C \cap L \) is counter, then it is accepted by some counter automaton. We construct a counter automaton accepting the encoded language \( E \) as follows. For every path in the finite state automaton labelled by \( t^i a^h \) plus some counters \( u \), where \( i \in \{ \pm 1, \pm 2, \pm 3 \} \), replace the path by an edge labelled \((i, u)\). This gives a new counter automaton with the same counters, accept states and start state, accepting the language of encoded words. Let \( p \) be the swapping length for this language, guaranteed by the Swapping Lemma (Lemma 13).

Take a Thue-Morse word in \( 1, 2, 3 \) of length greater than \( 2p + 1 \), then append a word in \(-1, -2, -3\) so that the full word represents a group element with all lamps illuminated in positions from 1 to some large positive integer. Call
this element $w$.

By the Swapping Lemma, we must be able to swap two adjacent subwords in the initial positive segment of $w$ and obtain another word $w'$ in the language. But if we do this the negative part of $w'$ is the same as the negative part of $w$ and is now no longer compatible with the changed positive part. Thus we have an encoding of a word that is not geodesic, so $w'$ cannot be in $C \cap L$, a contradiction. Thus, the full language of geodesics is not counter.

\[ \square \]

4 Geodesic languages for the lamplighter groups with the automata generating set

Grigorchuk and Zuk [18] prove that the lamplighter group $L_2$ is an example of an automata group. The natural generating set which arises from this interpretation is \{t, ta\} which we will call the automata generating set for $L_2$. They compute the spectral radius of $L_2$ with respect to this generating set and find remarkably that it is a discrete measure. Bartholdi and Sunik [1] prove that all lamplighter groups $L_m$ are automata groups.

We now compare the language theoretic properties of $L_2$ with respect to the automata generating set \{t, ta\} to the analogous properties described in the previous sections. We still view elements of $L_2$ as a configuration of light bulbs, with a cursor pointing to an integral position. However, the generator $ta$ combines the two basic “motions” of the wreath product generators, namely multiplication by $ta$ both moves the cursor and turns on a bulb. We show here that, as with respect to the wreath product generating set, there is no regular language of geodesics for $L_2$ and there is a counter language of unique geodesic representatives. In contrast to the wreath product generating set, however, the full language of geodesics with respect to the automata generating set is 1-counter.

4.1 Geodesic paths with respect to the automata generators

We can construct minimal length representatives with respect to the automata generating set using the same normal forms for elements of $L_2$ described above when considering the wreath product generating set, as described by Cleary and Taback [11].

We state an analogue of Proposition 14 which will allow us to recognize which of these normal forms are geodesic with respect to the automata generating set. As described in [11], the length of an element with respect to the automata
generating set depends only upon the positions of the leftmost- and rightmost-illuminated bulbs and the final location of the cursor, since it is possible to turn on or off intermediate bulbs with no additional usage of generators by choosing to use $ta$ instead of $t$ for moving the cursor.

**Definition 20** Let $w = a_{i_1}a_{i_2} \ldots a_{i_m}a_{-j_1}a_{-j_2} \ldots a_{-j_l}t^r \in L_2$, with $0 < i_1 < i_2 \cdots < i_m$ and $0 \leq j_1 < j_2 \cdots < j_l$. If $l = 0$, there are no bulbs illuminated at or to the left of the origin and we set $D'(w) = i_m + |r - i_m|$. Otherwise, we set

$$D'(w) = \min\{2(j_l + 1) + i_m + |r - i_m|, 2i_m + j_l + 1 + |r + j_l + 1|\}.$$ 

With respect to the automata generating set it is more convenient to group the bulb in position 0 with the bulbs in negative positions in the normal form given above than with the bulbs in positive positions, as in [11]. This is done because the element $a \in L_2$ has length two with respect to this generating set, and is explained fully in [11].

**Proposition 21** ([11], Proposition 2.4) The word length of $w \in L_2$ with respect to the automata generating set $\{t, ta\}$ is given by $D'(w)$.

We use Proposition 21 to show that $L_2$ has no regular language of geodesics with respect to the automata generating set.

**Theorem 22** The lamplighter group with the automata generating set has no regular language of geodesics.

*Proof:* The argument is similar to that for the wreath product generating set. Again, we consider words of the form $g_n = a_n a_{-n}$. This element has length $4n + 2$ with respect to the automata generating set. There are again two families of geodesic representatives for $g_n$, those arising from the right-first normal forms, such as $t^n(ta)^{-1}t^{-2n}(ta)t^n$, and those arising from the left-first normal forms, such as $t^{-n-1}(ta)t^{2n}(ta)^{-1}t^{-n+1}$. Applying the Pumping Lemma as in the proof of Theorem 15 yields the desired contradiction.

\[\square\]

As before, it follows from Lemma 3 that there are infinitely many cone types.

**Corollary 23** The lamplighter group has infinitely many cone types with respect to the automata generating set.

Again, we can observe this directly by considering the elements $g_n = a_n a_{-n} = t^n(ta)^{-1}t^{-2n-1}(ta)t^n$. The cone type of $g_n t^{-k}$ contains $t^k$ but not $t^{k+1}$ for $0 \leq k \leq n$, so for each $k$ we have a distinct cone type and as $n$ increases, we have infinitely many cone types.
Fig. 10. A 1-counter automaton accepting geodesics representing elements $w$ with illuminated bulbs at $n > 0$ and $-m$ with $m \leq 0$ and cursor at $k \leq 0$.

Though the languages of geodesics with respect to the automata generating set cannot be regular, there are geodesic languages with respect to this generating set which are 1-counter. In contrast to the wreath product generating set, it is now possible to have geodesic words which change the state of a bulb several times, so it is not essential to remember which bulbs have already been illuminated by using a stack.

**Theorem 24** There is a language of geodesics for $(L_2, \{t, (ta)\})$ with a unique representative for each element that is accepted by a 1-counter automaton.

**Proof**: Let $w \in L_2$. We choose the unique representative for $w$ in our language to be of one of the following forms.

Suppose that $w$ has illuminated bulbs to the right of the origin and also at or to the left of the origin and the position of the cursor is at or to the left of the origin. Let the position of the rightmost illuminated bulb be $n > 0$, and the position of the leftmost illuminated bulb be $-m$ with $m \leq 0$. We choose a representative of $w$ of the form $t^n(ta)^{-1}g_{n-1}g_{n-2}\ldots g_{1-m}(ta)^{-1}t^l$ where $l \leq m$ and

$$g_i = \begin{cases} (ta)^{-1} & \text{if bulb } i \text{ is on} \\ t^{-1} & \text{if bulb } i \text{ is off}. \end{cases}$$

The 1-counter automaton must check that the number of $t$ letters at the start and end of a word is at most the number of $(ta)^{-1}, t^{-1}$ letters, and the word starts with at least one $t$ letter. See Figure 10.

If there are illuminated bulbs only at or to the left of the origin, then we choose a representative for $w$ of the form $g_0g_1\ldots g_m(v)$ where $g_i$ is either $t^{-1}$ or $(ta)^{-1}$ as defined above, and $v$ is either empty (if the cursor is to the left of the leftmost illuminated bulb), or $(ta)^{-1}t^l$ where $1 \leq l \leq m$. These words are obtained in the 1-counter automaton in Figure 10, by following the $\epsilon$ edge from the start state.
Suppose that \( w \) has illuminated bulbs to the right of the origin and also at or to the left of the origin and the position of the cursor is to the right of the origin. Let \( n \) and \( m \) be as above, and choose a representative of the form 

\[
(t^{m-1}ta)g_1g_2 \ldots g_{n-1}(ta)t^l
\]

where and \( l > -n \) and

\[
g_i = \begin{cases} 
(ta) & \text{if bulb } i \text{ is on} \\
t & \text{if bulb } i \text{ is off}
\end{cases}
\]

The 1-counter automaton for these words must check that the number of \( t^{-1} \) letters at the start and end of a word is at most the total number of \( (ta), t \) letters, and the word starts with at least one \( t^{-1} \) letter. See Figure 11.

If there are illuminated bulbs only to the right of the origin, then we choose a representative for \( w \) of the form \((u)g_1g_2 \ldots g_{n-1}(v)\) where \( u \) is either empty (if the bulb at the origin is off) or \( t^{m-1}(ta) \) (if the bulb at the origin if on), \( g_i \) is either \( t \) or \( (ta) \) as defined above, and \( v \) is either empty (if the cursor is at or to the right of the rightmost illuminated bulb), or \( (ta)t^{-l} \) where \( 1 \leq l \leq n \). These words are obtained in the 1-counter automaton in Figure 11, by following the \( \epsilon \) edge from the start state.

Again, in contrast to the wreath product generating set, the full language of geodesics with respect to the automata generating set can be recognized by a simpler machine— it does not require a stack to keep track of which bulbs have been illuminated.

**Theorem 25 (1-counter full language for \( (L_2, \{t, (ta)\}) \))** The full language of geodesics for the lamplighter group with the automata generating set is accepted by a 1-counter automaton.

**Proof:** We exhibit four 1-counter automata, each recognizing a type of geodesic word. There are several variations within each family of geodesics which must be recognized as well.
We first consider geodesic representatives for elements $w \in L_2$ in which all illuminated bulbs lie to the right of the origin, with the rightmost illuminated bulb in position $n > 0$.

In this case, we build an automaton which recognizes geodesics of the following forms.

- If the cursor position in $w$ is at or to the right of $n$, then geodesics are words of the form $g_1 \ldots g_n$ with $g_i = t$ or $(ta)$.
- If the cursor is to the left of $n$, then geodesics are words of the form $g_1 \ldots g_{n-1}uv$ with $g_i = t$ or $(ta)$, $u$ is $(ta)t^{-1}$ or $t(ta)^{-1}$, $v$ is $\epsilon$, $g'_{n-1} \ldots g'_r$ with $r \geq 0$, or $g'_{n-1} \ldots g'_0t^{-r}$ with $r > 0$, and $g'_i = t^{-1}$ or $(ta)^{-1}$.

These forms are the language of the counter automaton in Figure 12. The first type are obtained by following the $t$ or $(ta)$ edge to the accept state on the left, and no counters are needed.

Second, we consider geodesic representatives for elements $w$ in which all illuminated bulbs lie at or to the left of the origin, with the leftmost illuminated bulb in position $-m$ with $m \geq 0$.

In this case, we build an automaton which recognizes geodesics of the following forms.

- If the cursor position in $w$ is to the left of $-m$, then geodesics are words of the form $g_0 \ldots g_{-m}$ with $g_i = t^{-1}$ or $(ta)^{-1}$.
- If the cursor is at or to the right of $-m$, then geodesics are words of the form $g_0 \ldots g_{-m}uv$ with $g_i = t^{-1}$ or $(ta)^{-1}$, $u$ is $(ta)^{-1}t$ or $t^{-1}(ta)$, $v$ is $\epsilon$, $g'_{-m} \ldots g'_r$ with $r \leq 0$, or $g'_{-m} \ldots g'_0t^r$ with $r > 0$, and $g'_i = t$ or $(ta)$.

These forms are the language of the counter automaton in Figure 12. The first type are obtained by following the $t^{-1}$ or $(ta)^{-1}$ edge to the accept state on the left, and no counters are needed.

Finally, we consider geodesic representatives for elements $w$ in which illu-
In this case, we build an automaton which recognizes geodesics of the following forms.

- If the cursor position in \( w \) is at or to the left of the origin, then geodesics are words of the form \( g_1 \ldots g_{n-1} u g'_{n-1} \ldots g'_0 v \) with \( g_i = t \) or \((ta)\), \( u \) is \((ta)t^{-1} \) or \( t(ta)^{-1} \), \( g'_i = t^{-1} \) or \((ta)^{-1} \), \( v \) is \( \epsilon \) or \( h_1 \ldots h_r x y \) with \( h_i = t^{-1} \) or \((ta)^{-1} \), \( x \) is \((ta)^{-1} t \) or \( t^{-1}(ta) \), and \( y \) is \( h'_1 \ldots h'_m \) with \( h'_i = t \) or \((ta) \) and \( m \geq 0 \).

- If the cursor is at or to the right of the origin, then geodesics are words of the form \( g_1 \ldots g_{n-1} u g'_{n-1} \ldots g'_0 v \) with \( g_i = t^{-1} \) or \((ta)^{-1} \), \( u \) is \((ta)t^{-1} \) or \( t^{-1}(ta) \), \( g'_i = t \) or \((ta) \), \( v \) is \( \epsilon \) or \( h_1 \ldots h_r x y \) with \( h_i = t \) or \((ta) \), \( x \) is \((ta)t^{-1} \) or \( t(ta)^{-1} \), and \( y \) is \( h'_1 \ldots h'_m \) with \( h'_i = t^{-1} \) or \((ta)^{-1} \) and \( m \geq 0 \).

Each of these forms are accepted by the counter automata in Figures 14 and 15.

Theorems 19 and 25 together show that \( L_2 \) has at least one generating set which yields a full language of geodesics that is counter, and one generating set whose full language of geodesics is not counter.
Fig. 15. 1-counter automaton for geodesics with bulbs to the right and at or to the left of the origin and cursor at or to the right of the origin.

5 Geodesic languages for Thompson’s group $F$

Thompson’s group $F$ is a fascinating group which is studied from many different perspectives. Analytically, $F$ is understood as a group of piecewise linear homeomorphisms of the interval, whose finitely-many discontinuities of slope have dyadic rational coordinates, and whose linear pieces have slopes which are powers of 2. See Cannon, Floyd and Parry [5] for an excellent introduction to $F$.

Algebraically, $F$ has two standard presentations: an infinite one with a convenient set of normal forms, and a finite one, with respect to which we study the metric. These presentations are as follows:

$$F = \langle x_0, x_1 | [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2] \rangle = \langle x_i, i \geq 0 | x_i^{-1}x_jx_i = x_{j+1}, \text{for } i < j \rangle.$$

Geometrically, $F$ is studied as a group of pairs of finite binary rooted trees, where group multiplication is analogous to function composition. Fordham [15] developed a remarkable way of measuring the word length of an element of $F$, with respect to the finite generating set $\{x_0, x_1\}$, directly from the tree pair diagram representing the element. Belk and Brown [2] and Guba [19] also have geometric methods for computing the word length of an element of $F$ in this generating set.

All of these methods for computing word length in the generating set $\{x_0, x_1\}$ have led to greater understanding of the the geometry of the Cayley graph of $F$ with respect to this generating set. For example, Cleary and Taback [7] show that this Cayley graph is not almost convex. Belk and Bux [3] show that this Cayley graph is additionally not minimally almost convex. Burillo [4], Guba [19], and Belk and Brown [2] have studied the growth of the group by trying to compute the size of balls in this Cayley graph. These volumes have
been estimated, but the exact growth function of $F$ is not known and it is not even known if it is rational.

In this section, we prove directly that $F$ contains infinitely many cone types with respect to the standard finite generating set $\{x_0, x_1\}$. This fact also follows as a corollary to Corollary 31 below. Our explicit example uses a family of "seesaw elements", described by Cleary and Taback [10], which are words for which there are two different possible suffixes for geodesic representatives.

We extend the results of Section 5 to general groups containing seesaw elements in Section 6.

**Definition 26** An element $w$ in a finitely generated group $G$ with finite generating set $X$ is a seesaw element of swing $k$ with respect to a generator $g$ if the following conditions hold.

- Right multiplication by both $g$ and $g^{-1}$ reduces the word length of $w$; that is, $|wg^\pm 1| = |w| - 1$, and for all $h \in X \setminus \{g^\pm 1\}$, we have $|wh^\pm 1| \geq |w|$.
- Additionally, $|wg^l| = |wg^{l+1}| - 1$ for integral $l \in [1, k]$, and $|wg^m h^\pm 1| \geq |wg^m|$ for all $h \in X \setminus \{g\}$ and integral $m \in [1, k - 1]$.
- Similarly, $|wg^{-l}| = |wg^{l+1}| - 1$ for integral $l \in [1, k]$, and $|wg^{-m} h^\pm 1| \geq |wg^{-m}|$ for all $h \in X \setminus \{g^{-1}\}$ for integral $m \in [1, k - 1]$.

Seesaw elements pose difficulty for finding unique geodesic representatives consistently for group elements, and are used to show that $F$ is not combable by geodesics in [10]. We notice from the definition that if $w$ is a seesaw element of swing $k$ then geodesic representatives for $w$ can have exactly two suffixes of length $k$—either $g^k$ or $g^{-k}$.

Examples of seesaw elements in $F$ are given explicitly in the following theorem, and illustrated in Figure 5. We note that the description given below is in terms of the infinite generating set $\{x_0, x_1, x_2, \ldots\}$ of $F$ for convenience but these could be expressed in terms of $x_0$ and $x_1$ by substituting $x_n = x_1^n x_0^{-n}$ for $n \geq 2$.

**Theorem 27** ([10], Theorem 4.1) The elements

$$x_0^k x_1 x_3 x_3^{-1} x_5 x_5^{-1} \cdots x_k x_k^{-1} x_0 x_0^{-1}$$

are seesaw elements of swing $k$ with respect to the generator $x_0$ in the standard generating set $\{x_0, x_1\}$.

We use the seesaw elements defined in Theorem 27 to find an infinite number of distinct cone types in $F$.

**Theorem 28** Thompson’s group $F$ contains infinitely many cone types with respect to the generating set $\{x_0, x_1\}$.
Fig. 16. The seesaw element $x_0^k x_1 x_{3k+3} x_{3k+2} x_{3k+1}^{-1} x_{3k}^{-1} x_{k+4}^{-1} x_{k+2}^{-1} x_{k}^{-1}$ of swing $k$ in $F$.

**Proof**: The seesaw elements given in Theorem 27 are all defined with respect to the generator $x_0$ of $F$. Let $w$ be a seesaw element of swing $k$ of the form given above. The possible geodesic continuations of the word $w x_0^{-l}$ where $l \in [1,k]$ includes $x_0^l$ but not $x_0^{l+1}$. Varying $l$, we have produced a finite set of group elements with distinct cone types. Varying $k$, the “swing” of the element, we can produce larger and larger finite sets of distinct cone types, so the set of cone types is unbounded.

The next theorem follows easily because the relators of the group $F$ in the finite presentation given above all have even length.

**Theorem 29** The full language of geodesics in $F$ with respect to $\{x_0, x_1\}$ is not regular.

**Proof**: The finite presentation of $F$ given above has relators of lengths 10 and 14, and Theorem 28 shows that $F$ has infinitely many cone types. It then follows from Lemma 4 that the full language of geodesics is not regular. 

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6 Groups with infinitely many cone types

The main theorems of this section show that if a group $G$ with finite generating set $S$ contains an infinite family of the seesaw elements of arbitrary swing, as defined in Section 5, then two results follow:
• $G$ has no regular language of geodesics, and
• $G$ has infinitely many cone types with respect to $S$.

These seesaw elements are present in $F$ with the standard generating set \{x_0, x_1\}, in $L_m$ with the wreath product generators, and in a large class of wreath products as described in [9].

Seesaw elements of large swing preclude the possibility of there being a regular language of geodesics, by an argument similar to that used to prove Theorem 29.

**Theorem 30** A group $G$ generated by a finite generating set $X$ with seesaw elements of arbitrary swing with respect to $X$ has no regular language of geodesics.

*Proof:* Suppose there were a regular language of geodesics for $G$ with pumping length $p$, and consider the form of the Pumping Lemma used in Theorem 15. We take $w$ to be a seesaw element of swing $k$ with respect to a generator $t$ with $k > p$, and note that any geodesic representative for $w$ must be written $w = vt^k$ or $w = v't^{-k}$. So a word in one of these two forms must belong to the regular language.

First suppose that $w = vt^k$ belongs to the regular language. Applying the Pumping Lemma to the suffix $t^k$ of $w$, we see that $vt^{k+n}$ must be in the regular language as well. Since $k + n > k$, this path is geodesic only until $vt^k$, and after that, further multiplication by $t$ will decrease word length. Thus $vt^{k+n}$ cannot be geodesic, contradicting the Pumping Lemma.

Similarly, if the regular language contains a geodesic representative of $w$ of the form $v't^{-k}$, we again apply the Pumping Lemma to obtain a contradiction. Thus there can be no regular language of geodesics for $G$. \hfill $\blacksquare$

Since Thompson’s group $F$ contains seesaw elements of arbitrary swing [10], it follows from Theorem 30 that $F$ has no regular language of geodesics. Theorems 15 and 22 also follow from Theorem 30, since $L_m$ in the wreath product generating set is shown to have seesaw elements of arbitrary swing in [9], and $L_2$ is shown to have seesaw elements of arbitrary swing with respect to the automata generating set in [11].

The following corollary follows immediately from Lemma 3.

**Corollary 31** Let $G$ be a finitely generated group with finite generating set $S$. If $G$ has seesaw elements of arbitrary swing then $G$ has infinitely many cone types with respect to the finite generating set $S$. 27
This corollary provides alternative proofs of Corollaries 16 and 23, and Theorem 28.

Seesaw elements are also found in more general wreath products.

**Theorem 32 ([9], Theorem 6.3)** Let $F$ be a finitely generated group containing an isometrically embedded copy of $\mathbb{Z}$, and $G$ any finitely generated non-trivial group. Then $G \wr F$ contains seesaw elements of arbitrary swing with respect to at least one generating set.

Combining Theorems 30 and 32 with Corollary 31, we obtain the following corollary.

**Corollary 33** Let $F$ be a finitely generated group containing an isometrically embedded copy of $\mathbb{Z}$, and $G$ any finitely generated non-trivial group. Then $G \wr F$ contains infinitely many cone types with respect to at least one generating set and there is no regular language of geodesics for $F$ with respect to that generating set.

**References**

[1] Laurent Bartholdi and Zoran Šunič. Some solvable automaton groups. *Contemporary Mathematics*, to appear.

[2] James Belk and Kenneth S. Brown. Forest diagrams for elements of Thompson’s group $F$. *Internat. J. Algebra Comput.*, to appear.

[3] James Belk and Kai-Uwe Bux. Thompson’s group $F$ is not minimally almost convex. In Jose Burillo, Sean Cleary, Murray Elder, Jennifer Taback, and Enric Ventura, editors, *Geometric Methods in Group Theory*. American Mathematical Society, 2005.

[4] José Burillo. Growth of positive words in Thompson’s group $F$. *Comm. Algebra*, 32(8):3087–3094, 2004.

[5] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson’s groups. *L’Ens. Math.*, 42:215–256, 1996.

[6] James W. Cannon. The combinatorial structure of cocompact discrete hyperbolic groups. *Geom. Dedicata*, 16(2):123–148, 1984.

[7] Sean Cleary and Jennifer Taback. Thompson’s group $F$ is not almost convex. *J. Algebra*, 270(1):133–149, 2003.

[8] Sean Cleary and Jennifer Taback. Combinatorial properties of Thompson’s group $F$. *Trans. Amer. Math. Soc.*, 356(7):2825–2849 (electronic), 2004.
Sean Cleary and Jennifer Taback. Dead end words in lamplighter groups and other wreath products. Quarterly Journal of Mathematics, to appear.

Sean Cleary and Jennifer Taback. Seesaw words in Thompson’s group F. In Jose Burillo, Sean Cleary, Murray Elder, Jennifer Taback, and Enric Ventura, editors, Geometric Methods in Group Theory. American Mathematical Society, 2005.

Sean Cleary and Jennifer Taback. Metric properties of the lamplighter group as an automata group. In Jose Burillo, Sean Cleary, Murray Elder, Jennifer Taback, and Enric Ventura, editors, Geometric Methods in Group Theory. American Mathematical Society, 2005.

Murray Elder. A context free and a counter geodesic language for Baumslag-Solitar groups. Theoret. Comput. Sci., to appear.

Gillian Z. Elston and Gretchen Ostheimer. On groups whose word problem is solved by a counter automaton. Theoret. Comput. Sci., 320(2-3):175–185, 2004.

David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. Word processing in groups. Jones and Bartlett Publishers, Boston, MA, 1992.

S. Blake Fordham. Minimal length elements of Thompson’s group F. Geom. Dedicata, 99:179–220, 2003.

Robert H. Gilman. Formal languages and infinite groups. In Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), volume 25 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., pages 27–51. Amer. Math. Soc., Providence, RI, 1996.

Rostislav Grigorchuk and Tatiana Nagnibeda. Complete growth functions of hyperbolic groups. Invent. Math., 130(1):159–188, 1997.

Rostislav I. Grigorchuk and Andrzei Żuk. The lamplighter group as a group generated by a 2-state automaton, and its spectrum. Geom. Dedicata, 87(1-3):209–244, 2001.

V. S. Guba. On the properties of the Cayley graph of Richard Thompson’s group F. Internat. J. Algebra Comput., 14(5-6):677–702, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.

Victor Mitrana and Ralf Stiebe. The accepting power of finite automata over groups. In New trends in formal languages, volume 1218 of Lecture Notes in Comput. Sci., pages 39–48. Springer, Berlin, 1997.

Walter D. Neumann and Michael Shapiro. A short course in geometric group theory. Notes for the ANU Workshop January/February 1996. Topology Atlas Document no. iaai-13.

Walter D. Neumann and Michael Shapiro. Automatic structures, rational growth, and geometrically finite hyperbolic groups. Invent. Math., 120(2):259–287, 1995.
[23] Michael Sipser. *Introduction to the Theory of Computation.* PWS Publishing Co., 1997.