Jordan-like Characterization of Automorphism Groups of Planar Graphs*

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Abstract. In 1975, Babai characterized which abstract groups can be realized as the automorphism groups of planar graphs. In this paper, we give a more detailed and understandable description of these groups. We describe stabilizers of vertices in connected planar graphs as the class of groups closed under the direct product and semidirect products with symmetric, dihedral and cyclic groups. The automorphism group of a connected planar graph is then obtained as a semidirect product of these stabilizers with a spherical group. The formulation of the main result is new and original. Moreover, it gives a deeper in the structure of the groups. As a consequence, automorphism groups of several subclasses of planar graphs, including 2-connected planar, outerplanar, and series-parallel graphs, are characterized. Our approach translates into a quadratic-time algorithm for computing the automorphism group of a planar graph which is the first such algorithm described in detail.

Keywords: planar graphs, automorphism groups, group products, Jordan’s characterization, 3-connected reduction.

Diagram: For a dynamic structural diagram of our results, see the following website (supported Firefox and Google Chrome): [http://pavel.klavik.cz/geom_aut_groups.html](http://pavel.klavik.cz/geom_aut_groups.html)

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1 Introduction

A permutation of the vertices and the edges of a graph is an automorphism if it preserves the incidences. In this paper, we investigate which abstract groups can be realized as the automorphism groups of planar graphs (PLANAR).

Restricted Classes of Graphs. Frucht’s Theorem [15] states that for every finite abstract group \( \Psi \), there exists a graph \( G \) such that \( \text{Aut}(G) \cong \Psi \); so automorphism groups of graphs are universal. We ask which abstract groups can be realized by restricted graph classes:

Definition 1.1. For a graph class \( \mathcal{C} \), let \( \text{Aut}(\mathcal{C}) = \{ \Psi : \exists G \in \mathcal{C}, \text{Aut}(G) \cong \Psi \} \). We call \( \mathcal{C} \) universal if every finite abstract group is in \( \text{Aut}(\mathcal{C}) \), and non-universal otherwise.

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In 1869, Jordan [24] characterized the automorphism groups of trees (TREE):

**Theorem 1.2 (24).** The class $\text{Aut}(\text{TREE})$ is defined inductively as follows:

(a) $\{1\} \in \text{Aut}(\text{TREE})$.
(b) If $\Psi_1, \Psi_2 \in \text{Aut}(\text{TREE})$, then $\Psi_1 \times \Psi_2 \in \text{Aut}(\text{TREE})$.
(c) If $\Psi \in \text{Aut}(\text{TREE})$, then $\Psi \cup S_n \in \text{Aut}(\text{TREE})$.

Recently, the automorphism groups of several classes of graphs were characterized; see Fig. 1. It was proved that interval graphs have the same automorphism groups as trees [29], and unit interval graphs have the same automorphism groups as disjoint unions of caterpillars [29]. For permutation graphs and circle graphs [30], there are similar inductive descriptions as in Theorem 1.2. Comparability and function graphs can realize all abstract groups even for the order dimension at most four [30]. Understanding the structure of $\text{Aut}(C)$ may lead to an effective algorithm for computation of generators of $\text{Aut}(G)$ for a graph $G \in C$. There are two related problems of algorithmic nature which can be considered for a class $C$, in particular if $C$ is the class of planar graphs.

**Graph Isomorphism.** This famous problem asks, whether two given graphs $G$ and $H$ are isomorphic (the same up to a labeling). This problem clearly belongs to $\text{NP}$ and it is a prime candidate for an intermediate problem between $\text{P}$ and $\text{NP}$-complete. The graph isomorphism problem is related to computing automorphism groups. Suppose that $G$ and $H$ are connected. If we know permutation generators of $\text{Aut}(G \cup H)$, then $G \cong H$ if and only if some generator swaps $G$ and $H$. Mathon [33] proved that generators of the automorphism group can be computed using $O(n^3)$ instances of graph isomorphism. For a survey, see [27].

**Regular Covering Testing.** In [11,13], there was introduced a problem called regular graph covering testing which generalizes both graph isomorphism and Cayley graph testing. The input gives two graphs $G$ and $H$. The problem asks whether there exists a semiregular subgroup $\Gamma$ of $\text{Aut}(G)$ such that $G/\Gamma \cong H$. The general complexity of this problem is open, no $\text{NP}$-hardness reduction is known. It is shown in [11,13], that the problem can be solved in $\text{FPT}$ time with respect to the size of $H$ when $G$ is planar. For this algorithm, the structure of all semiregular subgroups of $\text{Aut}(G)$ is described. In this paper, we extend this approach to describe the entire automorphism group $\text{Aut}(G)$.

**Babai’s Characterization.** The automorphism groups of planar graphs were first described by Babai [1, Corollary 8.12] in 1973; see Section 7 for the exact statement.

The automorphism groups of 3-connected planar graphs can be easily described geometrically, they are so called spherical groups, since they correspond to the discrete groups of symmetries of the sphere. More details are given in Section 1 and in [2]. The automorphism groups of $k$-connected planar graphs, where $k < 3$, are constructed by wreath products of the automorphism groups of $(k + 1)$-connected planar graphs and of stabilizers of $k$-connected graphs.

Babai’s characterization has several disadvantages. The statement is very long, separated into multiple cases and subcases. More importantly, is not clear precisely which abstract groups belong to $\text{Aut}(\text{PLANAR})$. The used language is complicated, very difficult for non-experts in permutation group theory and in graph symmetries.

In [2] p. 1457–1459, Babai gives a more understandable overview of two key ideas (automorphism groups of 3-connected planar graphs, and 3-connected reduction), with his characterization only sketched as “a description of the automorphism groups of planar graphs in terms of generalized wreath products of symmetric groups and polyhedral groups.” Babai also states an easy consequence:

**Theorem 1.3 (Babai [2]).** If $G$ is planar, then the group $\text{Aut}(G)$ has a subnormal chain

$$\text{Aut}(G) = \Psi_0 \triangleright \Psi_1 \triangleright \cdots \triangleright \Psi_n = \{1\}$$

Fig. 1. Hasse diagram of graph classes with understood automorphism groups. Characterization for the graph classes depicted in gray is described in this paper.
such that each quotient group $\Psi_{i-1}/\Psi_i$ is either cyclic or symmetric or $A_5$.

This theorem describes $\text{Aut}(G)$ only very roughly, by stating its building blocks. For comparison, an easy consequence of our characterization (Theorem 1.4) describes these building blocks more precisely and also states how they are “put together” in $\text{Aut}(G)$. In particular, the composition of the action of a spherical group $\Sigma$ with a stabilizer of a vertex in a planar graph depends on the action of $\Sigma$ on vertices and edges of the associated 3-connected planar graph. To describe it in detail is not an easy task; see Table 1. Our characterization also describes “geometry” of the automorphism groups in terms of actions on planar graphs and around every 1-cut and 2-cut. See Section 4 for more details.

The class of planar graphs is of great importance, and thus we are convinced that a more detailed and transparent description of their symmetries is of an interest. These reasons led us to write this paper which describes $\text{Aut}(\text{PLANAR})$ more understandably, and in more details.

**Our Characterization.** Let $G$ be a planar graph. If it is disconnected, then $\text{Aut}(G)$ can be constructed from the automorphism groups of its connected components (Theorem 2.1). Therefore, in the rest of the paper, we assume that $G$ is connected.

In Section 3, we describe a reduction process which decomposes $G$ into 3-connected components. It was used in [11,12] to study the behaviour of semiregular subgroups of $\text{Aut}(G)$ with respect to 1-cuts and 2-cuts. This natural idea of the reduction was first introduced in a seminal paper by Trakhtenbrot [38] and further extended in [22,20,6,39]. This reduction can be represented by a tree whose nodes are 3-connected graphs, and this tree is known in the literature mostly under the name $SPQR$ tree (7,8,9,16). The main difference, compared to our approach, is that the reduction applies only to 2-connected graphs, but we use it to reduce simultaneously parts separated by 1-cuts and 2-cuts. This allows to describe the geometry of symmetries, arising from articulations in $G$. Further, the reduction process is done in a way that the essential information on the symmetries is preserved (using colored and directed edges), so that the reconstruction of the automorphism group is possible.

The reduction proceeds as follows. In each step, called reduction, we replace all atoms of the considered graph $G$ by colored (possibly directed) edges, where atoms are certain inclusion minimal subgraphs of $G$. This gives a reduction series consisting of graphs $G = G_0, \ldots, G_r$, where $G_{i+1}$ is created from $G_i$ by replacing all of its atoms with some edges. The final graph $G_r$ contains no proper atoms and is called primitive. It follows that $G_r$ is either 3-connected, or a cycle, or $K_2$, or $K_1$. We can consider $G_r$ as an associated skeleton: the graph $G$ is obtained from $G_r$ by attaching the expanded atoms to its vertices and edges.

In Section 5, we characterize $\text{Aut}(\text{connected PLANAR})$. If $G$ is planar, then all its atoms and $G_r$ are planar graphs. Moreover, they are either very simple, or 3-connected. It is interesting that our characterization describes the automorphism groups of planar graphs without referring to planarity explicitly, as a simple recursive process which builds them from a few basic groups. A short version of our main result reads as follows:

**Theorem 1.4.** Let $G$ be a connected planar graph with the reduction series $G = G_0, \ldots, G_r$. Then $\text{Aut}(G_r)$ is a spherical group and $\text{Aut}(G_i) \cong \Psi_i \rtimes \text{Aut}(G_{i+1})$, where $\Psi_i$ is a direct product of symmetric, cyclic and dihedral groups.

We characterize $\text{Aut}(\text{connected PLANAR})$ in two steps. First, similarly as in Theorem 1.2 we give in Theorem 5.1 an inductive characterization of stabilizers of vertices of planar graphs, denoted $\text{Fix}(\text{PLANAR})$. It is the class of groups closed under the direct product, the wreath product with symmetric and cyclic groups and semidirect products with dihedral groups. In Theorem 5.10 we describe $\text{Aut}(\text{connected PLANAR})$ precisely as the class of groups $(\Psi_1^{m_1} \times \cdots \times \Psi_r^{m_r}) \rtimes \text{Aut}(H)$, where $\Psi_i \in \text{Fix}(\text{PLANAR})$ and $H$ is a 3-connected planar graph with colored vertices and colored, possibly oriented, edges. The group $\text{Aut}(H)$ acts on the factors of the direct product $\Psi_1^{m_1} \times \cdots \times \Psi_r^{m_r}$ in the natural way, permuting the isomorphic factors, following the action of $H$ on the vertices and edges of $H$; for more details, see Section 5.

In Section 6, we apply Jordan-like characterization to describe automorphism groups of 2-connected planar graphs and of the following subclasses of planar graphs. Outerplanar graphs (OUTERPLANAR) are planar graphs having an embedding such that all vertices belong to the outer face. Pseudoforests (PSEUDOFOREST) are planar graphs such that each connected component contains at most one cycle, i.e., it is a pseudotree (PSEUDOTREE). Series-parallel graphs (SERIES-PARALLEL) are planar graphs created by two operations, see Section 6.3 for a detailed definition.

**Quadratic-time Algorithm.** Graph isomorphism of planar graphs was attacked in papers [22,23]. Finally, linear-time algorithms were described by Hopcroft and Wong [21], and by Fontet [14]. As we explain in Section 5 the fundamental difficulty is deciding isomorphism of 3-connected (colored) planar graphs in linear time. The idea in [21] is to modify both graphs by a series of reductions ending with colored platonic solids, cycles, or $K_2$. This is a seminal paper used by many other computer science algorithms as a black box; e.g., [31,34,126,25]. Unfortunately, full versions of [14,21] were never published.

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Colbourn and Booth [4] propose the idea to modify the algorithm of [21] for computing the automorphism groups of planar graphs in linear time. The following is stated in [4]: “Necessarily we will only be able to sketch our procedure. A more complete description and a proof of correctness would require a more thorough analysis of the Hopcroft-Wong algorithm than has yet appeared in the literature.” To the best of our knowledge, no such algorithm was ever described in detail. We note that it is not possible to use the result of [21] as a black box for computing generators of the automorphism group, since one has to check carefully that the applied reductions preserve the automorphism group. (Or that the change of the automorphism group is under control, similarly as in Proposition [17].)

By combining the results of [14,21] and [33], the best previously known polynomial-time algorithm computing generators the automorphism group of a planar graph runs in time $O(n^4)$. In Section 8 we describe a quadratic-time algorithm based on our structural description of the automorphism groups of planar graphs.

**Theorem 1.5.** There exists a quadratic-time algorithm which computes generators of Aut($G$) of an input planar graph $G$ in terms of group products of symmetric and spherical groups and of permutation generators.

**Visualization of Symmetries.** As one of the applications of graph symmetries, drawing of planar graphs maximizing the symmetries of the picture were studied in [10][13][17][15]. Disadvantage of this approach is that even though the automorphism group Aut($G$) of a planar graph $G$ might be huge, it is possible to highlight only a small fraction of its symmetries; usually just a dihedral or cyclic subgroup. Even if we would consider drawing on the sphere, one can only visualize a spherical subgroup of Aut($G$). Based on our structural decomposition of Aut($G$), we propose in Conclusions a different spatial visualization which allows to capture the entire automorphism group, and thus visualize our characterization.

**2 Preliminaries**

In this paper, we work with an extended model of graph which is formally described in [12]. A multigraph $G$ is a pair ($V(G)$, $E(G)$) where $V(G)$ is a set of vertices and $E(G)$ is a multiset of edges. We denote $|V(G)|$ by $v(G) = |G|$ and $|E(G)|$ by $e(G)$. The graph can possibly contain parallel edges and loops, and each loop at $u$ is incident twice with the vertex $u$. Each edge $e = uv$ gives rise to two half-edges, one attached to $u$ and the other to $v$. We denote by $H(G)$ the collection of all half-edges. We denote $|H(G)|$ by $h(G)$, clearly $h(G) = 2e(G)$. In the reductions, we obtain pendant edges, each consisting of two half-edges, lone attached to some vertex $u$, the other attached to no vertex.

Unless the graph is $K_2$, we remove all vertices of degree 1 while keeping both half-edges of the incident edge. A pendant edge attached to $v$ is called a single pendant edge if it is the only pendant edge attached to $v$.

We consider graphs with colored edges and also with both directed and undirected edges. It might seem strange to consider such general graphs. But when we apply reductions, we replace some parts of the graph by edges and the colors are used to encode the isomorphism classes of the replaced parts. This allows the algorithm to work with smaller reduced graphs while preserving important information about the structure of the original large graph. Thus, even if the planar graph is simple, more complicated multigraphs are naturally constructed.

**Equivariance.** Suppose that a group $\Sigma$ acts on two sets $A$ and $B$. We say that the actions are equivariant if there exists an equivariant map $\varphi : A \to B$ which is a bijection and for every $\pi \in \Sigma$, we have $\varphi(\pi(x)) = \pi(\varphi(x))$. Equivariance defines an equivalence relation on orbits of the action of $\Sigma$, consisting of equivalent classes of equivariant orbits.

**Automorphisms.** Let $G$ be a graph. An automorphism $\pi$ of $G$ is fully described by a permutation $\pi_h : H(G) \to H(G)$ of the half-edges, preserving edges and the incidence relation between the half-edges and the vertices. It follows that $\pi_h$ induces two permutations $\pi_v : V(G) \to V(G)$ and $\pi_e : E(G) \to E(G)$ connected together by a natural property: $\pi_v(uv) = \pi_v(u)\pi_v(v)$, for every $uv \in E(G)$. We often omit the subscripts and simply write $\pi(u)$ or $\pi(uv)$. In addition, we require that an automorphism preserves the colors of edges and the orientation of directed edges. Similarly as in the definition of an automorphism, two graphs $G$ and $H$ are isomorphic, denoted $G \cong H$, if there exists an isomorphism from $G$ to $H$ satisfying the aforementioned conditions.

**Automorphism Groups.** We denote the group of all automorphisms of a graph $G$ by Aut($G$). Each element $\pi \in \text{Aut}(G)$ acts on $G$, permutes its vertices and edges while it preserves the incidences. Let $\Psi \subseteq \text{Aut}(G)$. The orbit $[v]$ of a vertex $v \in V(G)$ is the set of all vertices $\{\pi(v) \mid \pi \in \Psi\}$, and the orbit $[e]$ of an edge $e \in E(G)$ is defined similarly as $\{\pi(e) \mid \pi \in \Psi\}$. The stabilizer of $x$ is the subgroup of all automorphisms which fix $x$.

**Groups.** In what follows, we recall some standard notation for particular permutation groups of degree $n$. We use $S_n$ for the symmetric group, $C_n$ for the cyclic group, $D_n$ for the dihedral group ($|D_n| = 2n$), and $A_n$ for the alternating group. We note that $D_1 \cong C_2$ and $D_2 \cong C_2^2$. 

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On the right, the Cayley diagram of \( \text{Aut}(G) \cong \mathbb{C}_2^2 \rtimes \mathbb{C}_2 \cong \mathbb{C}_2 \wr S_2 \). It is not \( \mathbb{C}_2^1 \) since applying the white generator exchanges the roles of the black and gray generators.

Fig. 2. On the left, the graph \( G \) consisting of two copies of \( K_2 \), together with the action of three generators of \( \text{Aut}(G) \). On the right, the Cayley diagram of \( \text{Aut}(G) \cong \mathbb{C}_2^2 \rtimes \mathbb{C}_2 \cong \mathbb{C}_2 \wr S_2 \). We note that when both \( \Psi \) and \( \Sigma \) are given in terms of permutation generators and also the image \( \varphi(\Sigma) \) of \( \Sigma \) is given, we can output permutation generators of \( \Psi \rtimes \varphi \Sigma \).

Jordan described how to compute automorphism groups of graphs from automorphism groups of their connected components. We state it together with a proof since a similar argument is used in the proof of Proposition 3.7.

**Theorem 2.1 (Jordan [24]).** If \( G_1, \ldots, G_n \) are pairwise non-isomorphic connected graphs and \( G \) is the disjoint union of \( k_i \) copies of each \( G_i \), then

\[
\text{Aut}(G) \cong \text{Aut}(G_1) \wr \cdots \wr \text{Aut}(G_n) \wr S_{k_n}.
\]

**Proof.** Since the action of \( \text{Aut}(G) \) is independent on non-isomorphic components, it is clearly the direct product of factors, each corresponding to the automorphism group of one isomorphism class of components. It remains to show that if \( G \) consists of \( k \) isomorphic components \( H_1, \ldots, H_k \) of a connected graph \( H \), then \( \text{Aut}(G) \cong \text{Aut}(H) \wr S_k \). An example is given in Fig. 2.

For \( i > 1 \), let \( \sigma_{i,1} \) be an arbitrarily chosen isomorphism from \( H_1 \) to \( H_i \), and we put \( \sigma_{1,1} = \text{id} \) and \( \sigma_{i,j} = \sigma_{i,j}^{-1} \). Observe that each isomorphism from \( H_i \) to \( H_j \) can be decomposed into \( \sigma_{i,j} \) and some automorphism of \( H_j \). Let \( \pi \in \text{Aut}(G) \). It can be decomposed into a composition \( \mu \cdot \sigma \) of two automorphisms. The automorphism \( \sigma \) permutes the components as in \( \pi \), so when \( \pi(H_i) = H_j \), then \( \sigma|H_i = \sigma_{i,j}. \) The automorphism \( \mu \) maps each component \( H_i \) to itself, so \( \mu|H_i = \pi|H_i \cdot \sigma_{i,j}^{-1}. \) We have \( \pi = \mu \cdot \sigma \) since

\[
\mu|H_i \cdot \sigma|H_i = \pi|H_i \cdot \sigma_{i,j}^{-1} \sigma_{i,j} = \pi|H_i.
\]

The automorphisms \( \mu \) can be bijectively identified with the elements of \( \text{Aut}(H)^k \) and the automorphisms \( \sigma \) with the elements of \( S_k \).

Let \( \pi, \pi' \in \text{Aut}(G) \). Consider the composition \( \mu \cdot \sigma \cdot \mu' \cdot \sigma' \), we want to swap \( \sigma \) with \( \mu' \) and rewrite this as a composition \( \mu \cdot \tilde{\mu} \cdot \sigma \cdot \sigma' \). Clearly the components are permuted in \( \pi \cdot \pi' \) exactly as in \( \sigma \cdot \sigma' \), so \( \tilde{\sigma} = \sigma \). On the other hand, \( \tilde{\mu} \) is not necessarily equal \( \mu' \). Let \( \mu' \) be identified with the vector \((\mu'_1, \ldots, \mu'_k) \in \text{Aut}(H)^k \). Since \( \mu' \) is applied after \( \sigma \), it acts on the components permuted according to \( \sigma \). Therefore \( \tilde{\mu} \) is constructed from \( \mu' \) by permuting the coordinates of its vector by \( \sigma \):

\[
\tilde{\mu} = (\mu'_{\sigma(1)}, \ldots, \mu'_{\sigma(k)}).
\]

This is precisely the definition of the wreath product, so \( \text{Aut}(G) \cong \text{Aut}(H) \wr S_k \).

**3 Reduction to 3-connected Components**

In this section, we describe a modified reduction procedure, introduced in [11][12]. We show that under certain conditions, the reduction procedure allows to reconstruct \( \text{Aut}(G) \) from simpler graphs.
Block-tree and Central Block. The block-tree \( T \) of \( G \) is a tree defined as follows. Consider all articulations in \( G \) and all maximal 2-connected subgraphs which we call blocks (with bridge-edges also counted as blocks). The block-tree \( T \) is the incidence graph between the articulations and the blocks.

It is well-known that the automorphisms of \( \text{Aut}(G) \) induces automorphisms of \( \text{Aut}(T) \). Recall that for a tree, its center is either the central vertex or the central edge of a longest path, depending on the parity of its length. Every automorphism of a tree preserves its center. For a block-tree, there is always a central vertex. Therefore, every automorphism \( \pi \in \text{Aut}(G) \) preserves the central block or the central articulation.

We orient the edges of the block-tree \( T \) towards the central vertex; so the block-tree becomes rooted. A (rooted) subtree of the block-tree is defined by any vertex different from the centrum acting as root and by all its descendants.

**Definition of Atoms.** Let \( B \) be one block of \( G \), so \( B \) is a 2-connected graph. Two vertices \( u \) and \( v \) form a 2-cut \( U = \{u, v\} \) if \( B \setminus U \) is disconnected. We say that a 2-cut \( U \) is non-trivial if \( \text{deg}(u) \geq 3 \) and \( \text{deg}(v) \geq 3 \).

We first define a set \( P \) of subgraphs of \( G \) called parts which are candidates for atoms:

- A **block part** is a subgraph non-isomorphic to a pendant edge induced by the blocks of a subtree of the block-tree.
- A **proper part** is a subgraph \( S \) of \( G \) defined by a non-trivial 2-cut \( U \) of a block \( B \). The subgraph \( S \) consists of a connected component \( C \) of \( G \setminus U \) together with \( u \) and \( v \) and all edges between \( \{u, v\} \) and \( C \). In addition, we require that \( S \) does not contain the central block/articulation; so it only contains some block of the subtree of the block-tree given by \( B \).
- A **dipole part** is any dipole which is defined as follows. Let \( \{u, v\} \) be two distinct vertices of degree at least three joined at least two parallel edges. Then the subgraph induced by \( u \) and \( v \) is called a dipole.

The inclusion-minimal elements of \( P \) are called atoms. We distinguish block atoms, proper atoms and dipoles according to the type of the defining part. Block atoms are either pendant stars called star block atoms, or pendant blocks possibly with single pendant edges attached to them called non-star block atoms. Also each proper atom is a subgraph of a block, together with some single pendant edges attached to it. Notice that a dipole part is by definition always inclusion-minimal, and therefore it is an atom. For an example, see Fig. 3.

We use the topological notation to denote the boundary \( \partial A \) and the interior \( A \) of an atom \( A \). If \( A \) is a dipole, we set \( \partial A = V(A) \). If \( A \) is a proper or block atom, we put \( \partial A \) equal to the set of vertices of \( A \) which are incident with an edge not contained in \( A \). For the interior, we use the standard topological definition \( A = A \setminus \partial A \) where we only remove the vertices \( \partial A \), the edges adjacent to \( \partial A \) are kept in \( A \).

**Lemma 3.1** ([12], Lemma 3.3). For atoms \( A \neq A' \), we have \( A \cap A' = \partial A \cap \partial A' \).

Note that \(|\partial A| = 1\) for a block atom \( A \), and \(|\partial A| = 2\) for a proper atom or dipole \( A \). The interior of a dipole is a set of free edges. We note that dipoles are exactly the atoms with no vertices in their interiors.

Observe for a proper atom \( A \) that the vertices of \( \partial A \) are exactly the vertices \( \{u, v\} \) of the non-trivial 2-cut used in the definition of proper parts. Also the vertices of \( \partial A \) of a proper atom are never adjacent in \( A \), but may be adjacent in \( G \). Further, no block or proper atom contains parallel edges; otherwise a dipole would be its subgraph, so it would not be inclusion minimal.

**Lemma 3.2** ([12], Lemma 3.8). Let \( A \) be an atom and let \( \pi \in \text{Aut}(G) \). Then the image \( \pi(A) \) is an atom isomorphic to \( A \), \( \pi(\partial A) = \partial \pi(A) \), \( \pi(\hat{A}) = \hat{\pi}(A) \).
Fig. 4. A symmetric proper atom $A$ and an asymmetric proper atom $A'$. In the reductions, explained later, they are replaced by undirected and directed edges, respectively.

**Structure of Atoms.** We call a graph essentially 3-connected if it is a 3-connected graph with possibly some single pendant edges attached to it. Similarly, a graph is called essentially a cycle if it is a cycle with possibly some single pendant edges attached to it. For a proper atom $A$ with $\partial A = \{u, v\}$, we define $A^+$ as $A$ with the additional edge $uv$.

**Lemma 3.3 ([12], Lemma 3.5).** Every non-star block atom $A$ is either $K_2$ with an attached single pendant edge, essentially a cycle, or essentially 3-connected.

**Lemma 3.4 ([12], Lemma 3.6).** For every proper atoms $A$, the extended proper atom $A^+$ is either essentially a cycle, or essentially 3-connected.

Also, single pendant edges are always attached to $\tilde{A}$.

**Symmetry Types of Atoms.** For an atom $A$, we denote by $\text{Aut}_{\partial A}(A)$ the set-wise stabilizer of $\partial A$ in $\text{Aut}(A)$, and by $\text{Fix}(\partial A)$ the point-wise stabilizer of $\partial A$ in $\text{Aut}(A)$. Let $A$ be a proper atom or dipole with $\partial A = \{u, v\}$. We distinguish the following two symmetry types, see Fig. 4:

- **The symmetric atom.** There exits an automorphism $\tau \in \text{Aut}_{\partial A}(A)$ which exchanges $u$ and $v$.
- **The asymmetric atom.** There is no such automorphism in $\text{Aut}_{\partial A}(A)$.

In Fig. 5 the dipoles are symmetric but the proper atoms are asymmetric. If $A$ is a block atom, then it is by definition symmetric. For instance a dipole $A$ with $\partial A = \{u, v\}$ is symmetric, if and only if it has the same number of directed edges going from $u$ to $v$ as from $v$ to $u$. For a block atom or an asymmetric atom, we have $\text{Fix}(\partial A) = \text{Aut}_{\partial A}(A)$, but not for a symmetric proper atom or dipole.

**Reduction.** The reduction produces a series of graphs $G = G_0, \ldots, G_r$. To construct $G_{i+1}$ from $G_i$, we find the collection of all atoms $A$. Two atoms $A$ and $A'$ are isomorphic if there exists an isomorphism which maps $\partial A$ to $\partial A'$. We obtain isomorphism classes for $A$, and to each isomorphism class, we assign one new color not yet used in the graphs $G_0, \ldots, G_i$. We replace a block atom $A$ by a pendant edge of the assigned color based at $u$ where $\partial A = \{u\}$, and a proper atom or a dipole $A$ with $\partial A = \{u, v\}$ by a new edge $uv$ of the assigned color. If $A$ is symmetric, the edge $uv$ is undirected. If $A$ is asymmetric, the edge $uv$ is directed and we consistently choose one orientation for the entire isomorphism class. According to Lemma 3.1, the replaced interiors of the atoms of $A$ are pairwise disjoint, so the reduction is well defined. We repeatedly apply reductions and we stop in the step $r$ when $G_r$ contains no atoms, and we call such $G_r$ a primitive graph. For an example of the reduction, see Fig. 5.

We note the following detail. Replacing proper atoms and dipoles by edges preserves the block structure. Replacing block atoms with pendant edges removes leaves from the block tree. The central block/vertex is preserved by the reductions, so we define the atoms $A$ of $G_i$ with respect to the central block/vertex of $G_0$ (which may not be central in $G_i$).

Fig. 5. On the left, the graph $G_0$ has three isomorphism classes of atoms, one of each type. We reduce $G_0$ to $G_1$, which is an eight cycle with single pendant edges, with four black halvable edges replacing the dipoles, four gray undirected edges replacing the block atoms, and four white directed edges replacing the proper atoms. The reduction series ends with $G_1$ since it is primitive.
Lemma 3.5 ([12], Lemma 3.4). Let \( G \) be a primitive graph. If \( G \) has a central block, then it is a 3-connected graph, a cycle \( C_n \) for \( n \geq 2 \), or \( K_2 \), or can be obtained from these graphs by attaching single pendant edges to at least two vertices. If \( G \) has a central articulation, then it is \( K_1 \), possible with a single pendant edge attached.

Reduction Tree. For every graph \( G \), the reduction series determines the reduction tree which is a rooted tree defined as follows. The root is the primitive graph \( G_0 \), and the other nodes are the atoms obtained during the reductions. If a node contains a colored edge, it has the corresponding atom as a child. Therefore, the leaves are the atoms of \( G_0 \), after removing them, the new leaves are the atoms of \( G_1 \), and so on. For an example, see Fig. 6.

Reduction Epimorphism. The algebraic properties of the reductions, in particular how the groups \( \text{Aut}(G_i) \) and \( \text{Aut}(G_{i+1}) \) are related, are captured by a natural mapping \( \Phi_i : \text{Aut}(G_i) \to \text{Aut}(G_{i+1}) \) called the reduction epimorphism which we define as follows. Let \( \pi \in \text{Aut}(G_i) \). For the common vertices and edges of \( G_i \) and \( G_{i+1} \), we define \( \Phi_i(\pi) \) to be the same automorphism. If \( A \) is an atom of \( G_i \), then according to Lemma 3.3, \( \pi(A) \) is an atom isomorphic to \( A \). In \( G_{i+1} \), we replace the interiors of both \( A \) and \( \pi(A) \) by the edges \( e_A \) and \( e_{\pi(A)} \) of the same type and color. We define \( \Phi_i(\pi)(e_A) = e_{\pi(A)} \). In the same way, the action of \( \Phi_i(\pi) \) is defined for half-edges. Since \( \pi(\partial A) = \partial(\pi(A)) \), we have \( \Phi_i(\pi) \in \text{Aut}(G_{i+1}) \). It is proved in [12] Proposition 4.2 that the mapping \( \Phi_i \) is a group epimorphism, in particular, it is surjective. By Homomorphism Theorem, we know that \( \text{Aut}(G_i) \) is an extension of \( \text{Aut}(G_{i+1}) \) by \( \text{Ker}(\Phi_i) \)

\[
\text{Aut}(G_{i+1}) \cong \text{Aut}(G_i)/\text{Ker}(\Phi_i).
\]

Lemma 3.6 ([12], Lemma 4.4). \( \text{Ker}(\Phi_i) \cong \prod_{A \in \mathcal{A}} \text{Fix}(\partial A). \)

Our aim is to investigate when

\[
\text{Aut}(G_i) \cong \text{Ker}(\Phi_i) \rtimes \text{Aut}(G_{i+1}).
\] (1)

Let \( A \) be an atom with \( \partial A = \{ u, v \} \). If \( A \) is symmetric, there exists some automorphism of \( A \) exchanging \( u \) and \( v \). If \( A \) is a symmetric dipole, one can always find an involution exchanging \( u \) and \( v \). This is not true when

![Fig. 6. The reduction tree for the reduction series in Fig. 5. The root is the primitive graph \( G_1 \) and leaves are atoms of \( G_0 \).](image)

![Fig. 7. (a) An example of a symmetric proper atom \( A \) with no involution exchanging \( u \) and \( v \). There are two automorphisms which exchange \( u \) and \( v \), one rotates the four-cycle formed white directed edges by one clockwise, the other one counterclockwise. The set-wise stabilizer of \( u \) and \( v \) is \( C_4 \).](image)

(b) On the left, the graph \( G_{i+1} \) having colored edges \( e_1, \ldots, e_6 \) corresponding to copies \( A_1, \ldots, A_6 \) of \( A \). On the right, the groups \( \text{Aut}(G_i) \) and \( \text{Aut}(G_{i+1}) \cong \text{D}_6 \) with the homomorphism \( \Phi_i \). The rotations in \( \text{Aut}(G_{i+1}) \) can be easily extended, consider the depicted reflection \( \pi' \). Let \( \pi \in \text{Aut}(G_i) \) such that \( \Phi_i(\pi) = \pi' \). The automorphism \( \pi|A_1 \) is one of the two automorphisms of \( A \) exchanging \( u \) and \( v \) described in (a), and similarly \( \pi|A_4 \). Therefore, \( \pi^2 \neq \text{id} \) (since \( \pi^2|A_1 \neq \text{id} \) and \( \pi^2|A_4 \neq \text{id} \)) while \( (\pi')^2 = \text{id} \), and only \( \pi^4 = \text{id} \). Therefore, no complementary subgroup \( \Psi < \text{Aut}(G_i) \) exists and \( \text{Aut}(G_i) \) cannot be constructed using the semidirect product \( \Pi \).
A is a symmetric proper atom. Figure 4 gives an example of a symmetric proper atom with no involution exchanging the two vertices of the boundary. When all symmetric proper atoms have such involutions, we derive 5. Figure 5 explains that this assumption is necessary.

**Proposition 3.7.** Suppose that every symmetric proper atom $A$ of $G_1$ with $\partial A = \{u, v\}$ has an involutory automorphism $\tau$ exchanging $u$ and $v$. Then the following holds:

(a) There exists $\Psi \leq \text{Aut}(G_1)$ such that $\Phi_1(\Psi) = \text{Aut}(G_{i+1})$ and $\Phi_1|_{\Psi}$ is an isomorphism.

(b) $\text{Aut}(G_1) \cong \text{Ker}(\Phi_1) \times \text{Aut}(G_{i+1})$.

**Proof.** (a) Let $\pi' \in \text{Aut}(G_{i+1})$, we want to extend $\pi'$ to $\pi \in \text{Aut}(G_i)$ such that $\Phi_1(\pi) = \pi'$. We just describe this extension on a single edge $e = uv$. If $e$ is an original edge of $G$, there is nothing to extend. Suppose that $e$ was created in $G_{i+1}$ from an atom $A$ in $G_i$. Then $\hat{e} = \pi'(e)$ is an edge of the same color and the same type as $e$, and therefore $\hat{e}$ is constructed from an isomorphic atom $\hat{A}$ of the same symmetry type. The automorphism $\pi'$ preserves the action on the boundary $\partial A$. We need to show that it is possible to define an action on $A$ consistently:

- $A$ is a block atom: The edges $e$ and $\hat{e}$ are pendant, attached by articulations $u$ and $u'$. We define $\pi$ by an isomorphism $\sigma$ from $A$ to $\hat{A}$ which takes $\partial A$ to $\partial \hat{A}$.

- $A$ is an asymmetric proper atom or dipole: By the definition, the orientation of $e$ and $\hat{e}$ is consistent with respect to $\pi'$. Since $A$ is isomorphic to the interior of $\hat{A}$, we define $\pi$ on $A$ according to one such isomorphism $\sigma$.

- $A$ is a symmetric or a halvable proper atom or a dipole: Let $\sigma$ be an isomorphism of $A$ and $\hat{A}$. Either $\sigma$ maps $\partial A$ exactly as $\pi'$, and then we can use $\sigma$ for defining $\pi$. Or we compose $\sigma$ with an automorphism of $A$ exchanging the two vertices of $\partial A$. (We know that such an automorphism exists since $A$ is not antisymmetric.)

To establish (a), we need to do this consistently, in such a way that these extensions form a subgroup $\Psi$ which is isomorphic to $\text{Aut}(G_{i+1})$.

Let $e_1, \ldots, e_t$ be colored edges of one orbit of the action of $\text{Aut}(G_{i+1})$ such that these edges replace isomorphic atoms $A_1, \ldots, A_t$ in $G_i$; see Fig. 3 for an overview. We divide the argument into three cases:

**Case 1:** The atom $A_1$ is a block atom: Let $u_1, \ldots, u_t$ be the articulations such that $\partial A_i = \{u_i\}$. Choose arbitrarily isomorphisms $\sigma_{i,j}$ from $A_1$ to $A_j$ such that $\sigma_{1,i}(u_1) = u_i$, and put $\sigma_{i,1} = \text{id}$ and $\sigma_{i,j} = \sigma_{i,j}^{-1}$. If $\pi'(e_i) = e_j$, we set $\pi|_{A_i} = \sigma_{i,j}|_{A_i}$. Since

$$\sigma_{i,k} = \sigma_{j,k}\sigma_{i,j}, \quad \forall i,j,k,$$

the composition of the extensions $\pi_1$ and $\pi_2$ of $\pi_1$ is defined on the interiors of $A_1, \ldots, A_t$ exactly as the extension of $\pi_1 \pi_2$. Also, by (2), an identity $\pi_1 \pi_2 \cdots \pi_t = \text{id}$ is extended to an identity.

**Case 2:** The atom $A_1$ is an asymmetric proper atom or dipole: Let $e_i = u_i v_i$. We approach it exactly as in Case 1, just we require that $\sigma_{1,i}(u_1) = u_i$ and $\sigma_{1,i}(v_1) = v_i$.

**Case 3:** The atom $A_1$ is a symmetric proper atom or a dipole: For each $e_i$, we arbitrarily choose one endpoint as $u_i$ and one as $v_i$. Again, we arbitrarily choose isomorphisms $\sigma_{1,j}$ from $A_1$ to $A_i$ such that $\sigma_{1,i}(u_1) = u_i$ and $\sigma_{1,i}(v_1) = v_i$, and define $\sigma_{1,j} = \sigma_{1,j}^{-1} \sigma_{1,j}$. We further consider an involution $\tau_1$ of $A_1$ which exchanges $u_1$ and $v_1$. (Such an involution exists for symmetric proper atoms by the assumptions, and for symmetric dipoles by the definition.) Then $\tau_1$ defines an involution of $A_i$ by conjugation as $\tau_i = \tau_1 \sigma_i^{-1} \tau_1$. It follows that

$$\tau_j = \sigma_{i,j} \tau_i \sigma_{i,j}^{-1}, \quad \text{and consequently} \quad \sigma_{i,j} \tau_i = \tau_j \sigma_{i,j}, \quad \forall i,j.$$

**Fig. 8.** Case 1 is demonstrated on the left for $\ell = 3$, the respective block atoms are $A_1$, $A_2$, and $A_3$. Case 3 is demonstrated on the right for $\ell = 2$. The additional semiregular involution $\tau_1 \in \text{Fix}(\partial A_1)$ transposes $u_1$ and $v_1$. 

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We put $\hat{\sigma}_{i,j} = \sigma_{i,j} \tau_i = \tau_j \sigma_{i,j}$ which is an isomorphism mapping $A_i$ to $A_j$ such that $\hat{\sigma}_{i,j}(u_i) = v_j$ and $\hat{\sigma}_{i,j}(u_j) = u_i$. In the extension, we put $\pi|_{A_i} = \sigma_{i,j}|_{A_i}$ if $\pi'(u_i) = u_j$, and $\pi|_{A_i} = \sigma'_{i,j}|_{A_i}$ if $\pi'(u_i) = v_j$.

Aside from this, we get the following additional identities:

$$\hat{\sigma}_{i,k} = \sigma_{j,k} \hat{\sigma}_{i,j}, \quad \hat{\sigma}_{i,k} = \hat{\sigma}_{j,k} \sigma_{i,j}, \quad \text{and} \quad \sigma_{i,k} = \hat{\sigma}_{j,k} \hat{\sigma}_{i,j}, \quad \forall i, j, k. \quad (3)$$

We just argue the last identity:

$$\hat{\sigma}_{j,k} \hat{\sigma}_{i,j} = \tau_k (\sigma_{j,k} \sigma_{i,j}) \tau_i = \tau_k \sigma_{i,k} \tau_i = \tau_k \tau_k \sigma_{i,k} = \sigma_{i,k},$$

where the last equality holds since $\tau_k$ is an involution. It follows that the composition $\pi_2 \pi_1$ is correctly defined as above, and it maps identities to identities.

We have described how to extend the elements of $\text{Aut}(G_{i+1})$ on one edge-orbit, and apply this process repeatedly to all edge-orbits. The set $\Psi \leq \text{Aut}(G_i)$ consists of all these extensions $\pi$ from every $\pi' \in \text{Aut}(G_{i+1})$. It is a subgroup by (2) and (3), and since the extension $\pi' \mapsto \pi$ is injective, $\Psi \cong \text{Aut}(G_{i+1})$.

(b) By (a), we know that $\text{Ker}(\Phi_i) \leq \text{Aut}(G_i)$ has a complement $\Psi$ isomorphic to $\text{Aut}(G_{i+1})$. Actually, this already proves that $\text{Aut}(G_i)$ has the structure of the internal semidirect product.

We give more insight into its structure by describing it as an external semidirect product. Each element of $\text{Aut}(G_i)$ can be written as a pair $(\pi', \sigma)$ where $\pi' \in \text{Aut}(G_{i+1})$, and $\sigma \in \text{Ker}(\Phi_i)$. We first apply the extension $\pi \in \Psi$ of $\pi'$ and permute $G_i$, mapping interiors of the atoms as blocks. Then $\sigma$ permutes the interiors of the atoms, preserving the remainder of $G_i$.

It remains to understand how composition of two automorphisms $(\pi', \sigma)$ and $(\hat{\pi}', \hat{\sigma})$ works. We get this as a composition of four automorphisms $\hat{\sigma} \circ \hat{\pi} \circ \sigma \circ \pi$, which we want to write as a pair $(\tau, \rho)$. Therefore, we need to swap $\hat{\pi}$ with $\sigma$. This clearly preserves $\hat{\pi}$, since the action $\hat{\sigma}$ on the interiors does not influence it; so we get $\tau = \hat{\pi} \circ \sigma$.

But $\sigma$ is changed by this swapping. According to Lemma 3.6, we get $\sigma = (\sigma_1, \ldots, \sigma_s)$ where each $\sigma_i \in \text{Fix}(\partial A_i)^{m_i}$. Since $\pi$ preserves the isomorphism classes of atoms, it acts on each $\sigma_i$ independently and permutes the isomorphic copies of $A_i$. Suppose that $A$ and $A'$ are two isomorphic copies of $A_i$ and $\pi(A) = A'$. Then the action of $\sigma_i$ on the interior of $A$ corresponds after the swapping to the same action on the interior of $A' = \pi(A)$. This can be described using the semidirect product, since each $\pi$ defines an automorphism of $\text{Ker}(\Phi_i)$ which permutes the coordinates of each $\text{Fix}(\partial A_i)^{m_i}$, following the action of $\pi'$ on the colored edges of $G_{i+1}$.

\[ \square \]

4 Automorphism Groups of Polyhedral Graphs

In this section, we review geometric properties of automorphism groups of 3-connected planar graphs. They are based on Whitney’s Theorem [10] stating that 3-connected planar graphs have unique embeddings onto the sphere. Using these properties, we describe possible automorphism groups of planar atoms and primitive graphs.

**Spherical Groups.** A group is spherical if it is the group of the isometries of a tiling of the sphere. The first class of spherical groups are the subgroups of the automorphism groups of the platonic solids. Their automorphism groups are $S_4$ for the tetrahedron, $S_4 \times C_2$ for the cube and the octahedron, and $A_5 \times C_2$ for the dodecahedron and the icosahedron; see Fig. 9. The second class of spherical groups is formed by four infinite families, namely $C_n$, $D_n$, $C_n \times C_2$, and $D_n \times C_2$, $n \geq 2$. They act as groups of automorphism of n-sided prisms.

**Maps.** A (spherical) map $M$ is a 2-cell decomposition of the sphere $S$. A map is usually defined by a 2-cell embedding of a connected graph $i : G \hookrightarrow S$. The connected components of $S \setminus i(G)$ are called faces of $M$. An automorphism of a map is an automorphism of the graph preserving the incidences between vertices, edges, and faces. Clearly, $\text{Aut}(M)$ is one of the spherical groups and with the exception of paths and cycles, it is a subgroup of $\text{Aut}(G)$. As a consequence of Whitney’s theorem [10] we have the following.

![Fig. 9. The five platonic solids together with their automorphism groups.](image-url)
Theorem 4.1. Let \( \mathcal{M} \) be the map given by the unique 2-cell embedding of a 3-connected graph into the sphere. Then \( \text{Aut}(G) \cong \text{Aut}(\mathcal{M}) \). \( \square \)

Geometry of Automorphisms. Mani [32] gives the following insight into geometry of \( \text{Aut}(G) \) of a 3-connected planar graph \( G \). There exists a polyhedron \( P \) such that \( \text{Aut}(G) \) coincides with the group of isometries of \( P \). Figure 6 gives examples of such polyhedra associated to the graphs of platonic solids. Also, the polyhedron \( P \) can be placed in the center of a sphere and projected onto it, so that each isometry of \( P \) corresponds to some isometry of the sphere. Therefore, every automorphism in \( \text{Aut}(G) \) can be geometrically viewed as an isometry of the sphere with \( G \) drawn onto it, and this is essential for the Jordan-like characterization in Section 5. In particular, we have:

Theorem 4.2. Let \( G \) be a 3-connected graph. Then \( \text{Aut}(G) \) is isomorphic to one of the spherical groups.

We recall some basic definitions from geometry [37,35]. An automorphism of a 3-connected planar graph \( G \) is called orientation preserving if the respective isometry preserves the global orientation of the sphere. It is called orientation reversing if it changes the global orientation of the sphere. A subgroup of \( \text{Aut}(G) \) is called orientation preserving if all its automorphisms are orientation preserving, and orientation reversing otherwise. We note that every orientation reversing subgroup contains an orientation preserving subgroup of index two. (The reason is that the composition of two orientation reversing automorphisms is an orientation preserving automorphism.)

Stabilizers. Let \( u \in V(G) \). The stabilizer of \( u \) in \( \text{Aut}(G) \) is a subgroup of a dihedral group and it has the following description in the language of isometries. If \( \text{Stab}(u) \cong \mathbb{C}_n \), for \( n \geq 3 \), it is generated by a rotation of order \( n \) that fixes \( u \) and the opposite point of the sphere, and fixing no other point of the sphere. The opposite point of the sphere may be another vertex or a center of a face. If \( \text{Stab}(u) \cong \mathbb{D}_n \), it consists of rotations and reflections fixing a circle passing through \( u \) and the opposite point of the sphere. Each reflection always fixes either a center of some edge, or another vertex. When \( \text{Stab}(u) \cong \mathbb{D}_1 \cong \mathbb{C}_2 \), it is generated either by a 180° rotation or by a reflection.

Let \( e \in E(G) \). The stabilizer of \( e \) in \( \text{Aut}(G) \) is a subgroup of \( \mathbb{C}_2 \). When \( \text{Stab}(e) \cong \mathbb{C}_2 \), it contains the following three non-trivial isometries. First, the 180° rotation around the center of \( e \) and the opposite point of the sphere that is a vertex, center of an edge, or center of an even face. Next, two reflections orthogonal to each other which fix circles through \( u \) and the opposite point of the sphere. When \( \text{Aut}(G) \cong \mathbb{C}_2 \), it is generated by only one of these three isometries.

4.1 Automorphism Groups of Planar Primitive Graphs and Atoms

Theorem 4.2 allows us to describe possible automorphism groups of planar atoms and primitive graphs which appear in the reduction tree for a planar graph \( G \). First, we describe the automorphism groups of planar primitive graphs.

Lemma 4.3. The automorphism group \( \text{Aut}(G) \) of a planar primitive graph \( G \) is a spherical group.

Proof. Recall that a graph is essentially 3-connected if it is a 3-connected graph with attached single pendant edges to some of its vertices. If \( G \) is essentially 3-connected, then \( \text{Aut}(G) \) is a spherical group from Theorem 4.2. Since the family of spherical groups is closed under taking subgroups, the subgroup of color- and orientation-preserving automorphisms is spherical as well. If \( G \) is \( K_1 \), \( K_2 \) or \( C_n \) with attached single pendant edges, then \( \text{Aut}(G) \) is a subgroup of \( \mathbb{C}_2 \) or \( \mathbb{D}_n \). \( \square \)

Next, we deal with the automorphism groups of planar atoms. Let \( A \) be a planar atom. Recall that \( \text{Aut}_{\partial A}(A) \) is the set-wise stabilizer of \( \partial A \), and \( \text{Fix}(\partial A) \) is the point-wise stabilizer of \( \partial A \). The following lemma determines \( \text{Aut}_{\partial A}(A) \); see Fig. 10 for examples.

Lemma 4.4 ([12], Lemma 5.3). Let \( A \) be a planar atom.
(a) If \( A \) is a star block atom, then \( \text{Aut}_{\partial A}(A) = \text{Fix}(\partial A) \) which is a direct product of symmetric groups.
(b) If \( A \) is a non-star block atom, then \( \text{Aut}_{\partial A}(A) = \text{Fix}(\partial A) \) and it is a subgroup of a dihedral group.
(c) If \( A \) is a proper atom, then \( \text{Aut}_{\partial A}(A) \) is a subgroup of \( \mathbb{C}_2 \) and \( \text{Fix}(\partial A) \) is a subgroup of \( \mathbb{C}_2 \).
(d) If \( A \) is a dipole, then \( \text{Fix}(\partial A) \) is a direct product of symmetric groups. If \( A \) is symmetric, then \( \text{Aut}_{\partial A}(A) = \text{Fix}(\partial A) \times \mathbb{C}_2 \). If \( A \) is asymmetric, then \( \text{Aut}_{\partial A}(A) = \text{Fix}(\partial A) \).

Proof. (a) The edges of each color class of the star block atom \( A \) can be arbitrarily permuted, so \( \text{Aut}_{\partial A}(A) = \text{Fix}(\partial A) \) which is a direct product of symmetric groups.

(b) For the non-star block atom \( A \), the boundary \( \partial A = \{u\} \) is stabilized. We have one vertex in both \( \text{Aut}_{\partial A}(A) \) and \( \text{Fix}(\partial A) \) fixed, thus the groups are the same. By Lemma 3.3, we have that \( A \) is either a
essentially a cycle, $K_2$ with attached single pendant edges, or essentially 3-connected, so $\text{Aut}_{\partial A}(A)$ is a subgroup of $D_n$ where $n$ is the degree of $u$.

(c) Let $A$ be a proper atom with $\partial A = \{u,v\}$. By Lemma 3.4, $A^+$ is either essentially a cycle, or essentially 3-connected. The first case is trivial, so we deal with the latter case. Since $\text{Aut}_{\partial A}(A)$ preserves $\partial A$, we have $\text{Aut}_{\partial A}(A) = \text{Aut}_{\partial A^+}(A^+)$, and $\text{Aut}_{\partial A^+}(A^+)$ fixes in addition the edge $uv$. Because $A^+$ is essentially 3-connected, $\text{Aut}_{\partial A^+}(A^+)$ corresponds to the stabilizer of $uv$ in $\text{Aut}(\mathcal{M})$ for a map $\mathcal{M}$ of $A^+$. But such a stabilizer is a subgroup of $C_2$. Since $\text{Fix}(\partial A)$ stabilizes the vertices of $\partial A$, it is a subgroup of $C_2$.

(d) For an asymmetric dipole, we have $\text{Aut}_{\partial A}(A) = \text{Fix}(\partial A)$ which is a direct product of symmetric groups. For a symmetric dipole, we can permute the vertices in $\partial A$, so we get the semidirect product with $C_2$.

Last, we argue that every planar symmetric atom $A$ has an involutory automorphism exchanging $\partial A$, so the assumptions of Proposition 3.7 are always satisfied for planar graphs.

**Lemma 4.5.** For every planar symmetric proper atom $A$ with $\partial A = \{u,v\}$, there exists an involutory automorphism exchanging $u$ and $v$.

**Proof.** Since $\text{Aut}_{\partial A}(A)$ is a subgroup of $C_2^r$, all elements are involutions. $\square$

## 5 The Jordan-like Characterization

The automorphism groups of planar graphs are constructed using Theorem 2.1 from the automorphism groups of its connected components. It remains to deal with the automorphism groups of connected planar graphs. We describe them in this section, using the results of Sections 3 and 4, thus proving the main result of this paper. We show that $\text{Aut}(\text{connected PLANAR})$ can be described by a semidirect product series composed from few basic groups. In Subsection 5.1 we prove Theorems 1.3 and 1.4 as an easy consequence of our previous results. In Subsection 5.2, we give a Jordan-like characterization of all point-wise stabilizers of a vertex, or of a pair of vertices, in the automorphism groups of connected planar graphs. In Subsection 5.3, we describe all possible compositions of actions of spherical groups with the stabilizers described in Subsection 5.2.

### 5.1 Characterization by Semidirect Product Series

First, we prove Theorems 1.3 and 1.4 which can be viewed as a rough approximation of the main result.

**Proof (Proof of Theorem 1.3).** We define an epimorphism $\Theta_i: \text{Aut}(G) \to \text{Aut}(G_i)$ by $\Theta_i = \Phi_0 \circ \cdots \circ \Phi_{i-1}$, for $i = 1, \ldots, r-1$. We have $\ker(\Theta_{r-i}) > \ker(\Theta_{r-i-1})$ and since all the $\ker(\Theta_{r-i})$ are normal in $\ker(\Theta_r)$, we can write $\ker(\Theta_{r-i}) \supset \ker(\Theta_{r-i-1})$. By definition, $\Theta_{r-i} = \Theta_{r-i-1} \circ \Phi_{r-i}$. Therefore, $\ker(\Theta_{r-i})/\ker(\Theta_{r-i-1}) \cong \ker(\Phi_{r-i})$, for $i = 1, \ldots, r-1$. By Lemmas 3.3 and 4.4, $\ker(\Phi_{r-i})$ is isomorphic to a direct product of symmetric, cyclic, and dihedral groups. Moreover, $\text{Aut}(G_i)/\ker(\Theta_{r-i}) \cong \text{Aut}(G_i)$. By Lemma 4.3, $\text{Aut}(G_r)$ is isomorphic to a spherical group. We have a subnormal chain $\text{Aut}(G) = \Psi_0 > \Psi_1 > \cdots > \Psi_{r-i} = 1$, where $\Psi_i = \ker(\Theta_{r-i})$ such that $\Psi_i/\Psi_{i+1}$ is a product of the required groups. By Jordan-Hölder theorem there exists a refinement satisfying the statement of the above subnormal chain. $\square$

**Proof (Proof of Theorem 1.4).** The primitive graph $G_r$ has $\text{Aut}(G_r)$ isomorphic to a spherical group by Lemma 4.3. By Lemma 4.5, we can apply Proposition 3.7 and $\text{Aut}(G_i) \cong \ker(\Phi_i) \rtimes \text{Aut}(G_{i+1})$. By Lemma 3.6, the kernel $\ker(\Phi_i)$ is the direct product of the groups $\text{Fix}(\partial A)$ for all atoms $A$ in $G_r$. Each of these groups is isomorphic to either to a cyclic, or to a dihedral group, or to a direct product of symmetric groups. $\square$
Theorems [13] and [14] impose some necessary conditions fulfilled by the automorphism groups of a planar graph. On the other hand, not every abstract group satisfying these conditions is isomorphic to the automorphism group of some planar graph. First, $\text{Aut}(G_{i+1})$ admits an induced action on the groups $\text{Fix}(\partial A)$, where $A$ ranges through all atoms of $G_i$. In particular, the sizes of orbits of $\text{Aut}(G_{i+1})$ are reflected in $\text{Aut}(G_i)$, since an orbit of length $m$ gives rise to $m$ copies of $\text{Fix}(\partial A)$, for some atom $A$. For instance, if $\text{Aut}(G_r) \cong C_n$, then every orbit is of size 1, or $n$. Therefore, the possible powers of $\text{Fix}(\partial A)$ in $\text{Ker}(\Phi_{r+1})$ are restricted. Applying Theorem 1.4 repeatedly, we can construct $\text{Aut}(G)$ recursively, starting in the root of the reduction tree and terminating its leaves. In the remainder of this section, we revert the approach and construct a Jordan-like characterization of the automorphism groups of planar graphs from the leaves to the root of reduction trees.

### 5.2 Fixer of the Boundary of an Expanded Atom

Consider the reduction tree of a planar connected graph $G$. For an atom $A$ in $G$, let $A^*$ denote the subgraph of $G$ corresponding to the node $A$ and all its descendants in the reduction tree. In other words, $A^*$ is the fully expanded atom $A$. Let $\text{Fix}(\partial A^*)$ be the point-wise stabilizer of $\partial A^* = \partial A$ in $\text{Aut}_{\partial A^*}(A^*)$.

Equivalently, $\text{Fix}(\partial A^*) = \{ \text{Fix}(\partial A^*) : A \text{ is an atom of the reduction tree of a planar graph} \}$.

**Theorem 5.1.** The class $\text{Fix}(\text{connected PLANAR})$ is defined inductively as follows:

(a) $\{1\} \in \text{Fix}(\text{connected PLANAR})$.

(b) If $\Psi_1, \Psi_2 \in \text{Fix}(\text{connected PLANAR})$, then $\Psi_1 \times \Psi_2 \in \text{Fix}(\text{connected PLANAR})$.

(c) If $\Psi \in \text{Fix}(\text{connected PLANAR})$, then $\Psi : S_n, \Psi : C_n \in \text{Fix}(\text{connected PLANAR})$.

(d) If $\Psi_1, \Psi_2, \Psi_3 \in \text{Fix}(\text{connected PLANAR})$, then $\Psi_1^n \times \Psi_2^n \times \Psi_3^n \times \Psi_0^n \times D_n \in \text{Fix}(\text{connected PLANAR})$, for every $n$ even.

(e) If $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5 \in \text{Fix}(\text{connected PLANAR})$, then $\left(\Psi_1^{2^n} \times \Psi_2^{2^n} \times \Psi_3^{2^n} \times \Psi_4^{2^n} \times \Psi_5^{2^n}\right) \times D_n \in \text{Fix}(\text{connected PLANAR})$, for every $n \geq 4$, even.

(f) If $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6 \in \text{Fix}(\text{connected PLANAR})$, then $\left(\Psi_1^4 \times \Psi_2^4 \times \Psi_3^4 \times \Psi_4^4 \times \Psi_5^4 \times \Psi_6^4\right) \times C_2 \in \text{Fix}(\text{connected PLANAR})$.

We prove this theorem in a series of lemmas below. We note that the homomorphisms defining the semidirect products in the operations (d), (e), and (f) are specified in the proofs of Lemmas 5.4, 5.7, 5.8, and 5.9.

**Lemma 5.2.** The class $\text{Fix}(\text{connected PLANAR})$ is closed under operations (a)–(f). Further, every such group can be realized by a block atom, by a proper atom, or by a dipole, in arbitrarily many non-isomorphic ways.

**Proof.** It is clear for (a) and Fig. [11] shows the constructions for the operations (b)–(f). Concerning the second part, for every group $\Psi$, arbitrarily many non-isomorphic atoms $A$ such that $\Psi \cong \text{Fix}(\partial A)$ can be constructed. For instance, we can do it by replacing edges of a realization of $\Psi$ by suitable rigid planar graphs (having no non-trivial automorphisms) consistently with the action of $\Psi$. Similarly, if some group can be realized by, say block atom, we can attach the corresponding pendant edge to some, say, rigid proper atom which preserves the group. 

In the rest of this section, we prove that each group in $\text{Fix}(\text{connected PLANAR})$ arises by using operations (b)–(f) repeatedly. We prove this by induction according to the depth of the reduction tree. Let $A$ be an atom in $G_{i+1}$, with each colored edge corresponding to some atom $A$ in $G_i$ which is expanded to $A^*$. The expanded atom $A^*$ is constructed from $A$ by replacing all colored edges with expanded atoms $\hat{A}$. By induction hypothesis, we assume that the groups $\text{Fix}(\partial A^*)$ can be constructed using (a)–(f).

We relate $\text{Fix}(\partial A)$ and $\text{Fix}(\partial A^*)$ similarly as in Proposition 3.7. Let $\Phi : \text{Fix}(\partial A^*) \to \text{Fix}(\partial A)$ be the reduction epimorphism $\Phi : \pi^* \mapsto \pi$ defined as follows. For $\pi^* \in \text{Fix}(\partial A^*)$, the automorphism $\pi = \Phi(\pi^*)$ maps the common parts of $A$ and $A^*$, the same while $\pi$ maps the colored edges of $A$ in $\pi^*$ maps the expanded edges in $A^*$. Note that $\Phi$ is $\Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_i$, restricted to $\text{Fix}(\partial A^*)$. Similarly as in the proof of Lemma 3.6, we have $\text{Ker}(\Phi) \cong \text{Fix}(\partial A^*)$. Lemma 4.4 describes the group $\text{Fix}(\partial A)$, depending on the type of the atom $A$. Also, Lemma 4.5 generalizes to planar symmetric expanded atoms $A^*$. Therefore, every planar symmetric atom $A$ has an involution $\tau^* \in \text{Aut}_{\partial A^*}(A^*)$ swapping the boundary $\partial A^*$. Exactly as in Proposition 3.7, we can prove that

$$\text{Fix}(\partial A^*) \cong \text{Ker}(\Phi) \rtimes \text{Fix}(\partial A).$$

(4)
We describe the semidirect product in [4] in more detail. The group $\text{Ker}(\Phi)$ consists of all automorphisms which fix $A$ and only act non-trivially on interiors of all expanded atoms $\hat{A}$. Each automorphism $\pi^* \in \text{Fix}(A^*)$ can be written as a composition $\sigma \cdot \pi'$ of two automorphisms. First, the automorphism $\sigma \in \text{Ker}(\Phi)$ acts on interiors of all $\hat{A}$. Then, the automorphism $\pi'$ acts on $\hat{A}$ as $\pi \in \text{Fix}(A)$ acts on $A$, and $\pi'$ maps interiors of $\hat{A}$ exactly as $\pi$ maps the corresponding colored edges. Then we can form a composition $\pi = \sigma \cdot \pi'$. We want to swap $\sigma_1$ with $\sigma_2$ to write the resulting automorphism in the form $\sigma \cdot \pi'$, which is done by the semidirect product. In the proof of Proposition 3.7a, we explain how to define the correspondence $\pi \mapsto \pi'$ consistently.

Below, we divide the proof into several lemmas, according to the type of $A$, and further simplify (4) to get the operations (b) to (f). Suppose that two edge-orbits of $A$, corresponding to expanded atoms $\hat{A}$ and $\hat{A}$, respectively, are equivalent in $\text{Fix}(\partial A)$. Then using (b), we can construct $\text{Fix}(\partial A^*) \times \text{Fix}(\partial A^*)$ and work with it, using distributivity, as with one group in [4]. Therefore, we need to identity all equivariance classes of edge-orbits in $\text{Fix}(\partial A)$.

The following two types of edge-orbits are considered. An edge-orbit of size $k$ is called fixed, denoted $k$, if the corresponding half-edges form two orbits of size $k$. An edge-orbit of size $k$ is called reflected, denoted $k_+$, if the corresponding half-edges form one orbit of size $2k$. We distinguish different geometric actions on the set of half-edges $H(A)$, so a fixed edge-orbit of size $k$ is non-equivariant with a reflected orbit of size $k$.

### Dipoles and Star Block Atoms

**Lemma 5.3.** Let $A$ be a star block atom or a dipole. Then $\text{Fix}(\partial A^*)$ can be constructed from the groups $\text{Fix}(\partial A)$, using operations (b) and operations (c) for symmetric groups, where $\hat{A}$ ranges through all atoms corresponding to colored edges in $A$.

**Proof.** The edges of the same type (for dipoles, we have undirected, directed in one way, directed in the other way) and color can be arbitrarily permuted. By Lemma 4.4, the action of $\text{Fix}(\partial A)$ has $\ell$ orbits, each consisting of all colored edges of one color and of the same type and orientation. These orbits have sizes $m_1, \ldots, m_\ell$, so $\text{Fix}(\partial A) \cong S_{m_1} \times \cdots \times S_{m_\ell}$. Colored edges in these orbits correspond to atoms $\hat{A}_1, \ldots, \hat{A}_\ell$.

Since $\text{Fix}(\partial A^*)$ acts independently on the atoms corresponding to each orbit of colored edges in $\text{Fix}(\partial A)$, each orbit contributes by one factor and $\text{Fix}(\partial A^*)$ is the direct product of these factors. The atoms corresponding to each orbit can be arbitrarily permuted, thus each factor is isomorphic to $\text{Fix}(\partial A^*) \wr S_{m_i}$. □

---

**Fig. 11.** Constructions for the operations (b)-(f), every colored edge corresponds to an atom $\hat{A}$ with $\text{Fix}(\partial \hat{A})$ isomorphic to the denoted group. In (d), we have three equivariant classes of edge-orbits in the action of $D_n$ when $n$ is odd. In (e), we have two further equivariant classes of edge-orbits in the action of $D_n$ when $n \geq 4$ is even. In (f), there is one extra equivariant class consisting of one edge-orbit in the action of $D_2 = C_2$ generated by two reflections.
Proper Atoms.

Lemma 5.4. Let $A$ be a proper atom. Then $\text{Fix}(\partial A^*)$ can be constructed from the groups $\text{Fix}(\partial A^*)$, using operations (b) and operations (d) for $\mathbb{D}_1 \cong C_2$, where $A$ ranges through all atoms corresponding to colored edges in $A$.

Proof. By Lemma 4.4, we know that $\text{Fix}(\partial A)$ is a subgroup of $C_2$. If $\text{Fix}(\partial A) \cong C_1$, then $\text{Fix}(\partial A^*)$ can be easily constructed only using (b). Otherwise, $\text{Fix}(\partial A) \cong C_2$. Then the non-trivial automorphism $\pi \in \text{Fix}(\partial A)$ corresponds to a reflection through $\partial A$. Therefore, $\text{Fix}(\partial A)$ has some edge-orbits of colored edges of size two, and at most two types of edge-orbits of colored edges of size one, as depicted in Fig. 12:

- **Edge-orbits of type 2.** We have $\ell_2$ equivariant edge-orbits of size two, whose edges are reflected to each other by $\pi$. The colored edges in these orbits correspond to atoms $A_1, \ldots, A_{\ell_2}$.

- **Edge-orbits of type 1.** We have $\ell_2$ equivariant edge-orbits of size one, in which both half-edges forming each edge are fixed by $\pi$, together with the incident vertices. The colored edges in these orbits correspond to atoms $B_1, \ldots, B_{\ell_2}$.

- **Edge-orbits of type 1,4.** We have $\ell_3$ equivariant edge-orbits of size one, in which the half-edges forming each edge are exchanged by $\pi$. Therefore, these half-edges belong to one orbit, the incident vertices also belong to one orbit and the corresponding edges are reflected by $\pi$. The colored edges in these orbits correspond to (necessarily) symmetric atoms $C_1, \ldots, C_{\ell_3}$. Let $\tau \in \text{Aut}_{\partial A^*}(C_1) \times \cdots \times \text{Aut}_{\partial A^*}(C_{\ell_3})$ be an involution which exchanges the boundaries of each of these atoms (ensured by Lemma 4.5), and $\tau^*$ be a corresponding involution in $\text{Aut}_{\partial A^*}(C_1^*) \times \cdots \times \text{Aut}_{\partial A^*}(C_{\ell_3}^*)$.

We need to distinguish two equivariant classes of edge-orbits of size one since the reflection $\pi$ behaves differently with respect to them. For size one, fixed, two half-edges also form orbits of size one. On the other hand, for size one, reflected, both half-edges belong to the same orbit of size one. In $\pi^*$, the boundaries of $B_1^*$ are fixed, but the boundaries of $C_1^*$ are swapped, by applying $\tau^*$ on $C_1^*$. To be able to distinguish these two cases in $A$, it is important to consider automorphisms on half-edges instead of edges.

To construct $\text{Fix}(\partial A^*)$, we put

$$\Psi_1 = \prod_{i=1}^{\ell_1} \text{Fix}(\partial A_i^*), \quad \Psi_2 = \prod_{i=1}^{\ell_2} \text{Fix}(\partial B_i^*), \quad \Psi_3 = \prod_{i=1}^{\ell_3} \text{Fix}(\partial C_i^*)$$

using (b). Then it easily follows that

$$\text{Fix}(\partial A^*) \cong (\Psi_1^2 \times \Psi_2 \times \Psi_3) \rtimes_\varphi C_2,$$

(5)

where $\varphi$ is the homomorphism defined as

$$\varphi(0) = \text{id}, \quad \varphi(1) = (\alpha_1, \alpha_1^*, \alpha_2, \alpha_3) \mapsto (\alpha_1^*, \alpha_1, \alpha_2, \tau^* \cdot \alpha_3),$$

$\alpha_1 \in \Psi_1$, $\alpha_1^* \in \Psi_1^*$, $\Psi_2 \cong \Psi_1$, $\alpha_2 \in \Psi_2$, and $\alpha_3 \in \Psi_3$. So $\text{Fix}(\partial A^*)$ can be constructed using (b) and (d) (since $\mathbb{D}_1 \cong C_2$).

We note that semidirect product in (5) can be further simplified into $(\Psi_1^2 \times \Psi_3) \rtimes_\varphi C_2 \times \Psi_2$ since $\varphi$ acts trivially on the coordinate corresponding to $\Psi_2$. So the operation (d) for $\mathbb{D}_1 \cong C_2$ could be simplified. We use this simplification in Section 6.

![Diagram](image)

Fig. 12. On the left, the action of $\text{Fix}(\partial A)$ is generated by the reflection $\pi$. Observe that $\pi$ acts differently on the edges corresponding to $B_1$ and $B_2$ ($\pi$ fixes them) than on the edges corresponding to $C_1$ and $C_2$ ($\pi$ reflects them). Therefore, in $\text{Fix}(\partial A^*)$, we compose $\pi$ with an involution $\tau^*$ reflecting $C_1^*$ and $C_2^*$, depicted on the right.
Fig. 13. (a) An example of a non-star block atom $A$ with $\text{Fix}(\partial A) \cong C_6$ generated by the $60^\circ$ rotation through $u$ and $v$. We have two orbits of colored edges of size 6, whose edges correspond to atoms $A_1$ and $A_2$. Further, the vertex $v$ has attached a single pendant edge, corresponding to a block atom $B$.

(b) On the left, an example of a non-star block atom $A$ with $\text{Fix}(\partial A) \cong D_5$ consisting of five rotations by multiples of $72^\circ$ and five depicted reflections. On the right, the view from above, with colored edges labeled by the corresponding atoms. Different types of edge-orbits are depicted with different types of edges. We have $\ell_1 = \ell_2 = 1$ and $\ell_3 = 2$.

Non-star Block Atoms. Now, we deal with non-star block atoms $A$ which are the most involved. By Lemma 4.4, we know that $\text{Fix}(\partial A)$ is a subgroup of a dihedral group, so it is isomorphic to $C_n$, or to $D_n$. By Lemma 5.3, $A$ is either $K_2$ with an attached pendant edge, essentially a cycle, or essentially 3-connected. In the first two cases, $\text{Fix}(\partial A)$ is a subgroup of $C_2$. If $\text{Fix}(\partial A)$ is not isomorphic to a subgroup of $C_2$, then $A$ is necessarily an essentially 3-connected graph.

Lemma 5.5. Let $A$ be a non-star block atom with $\text{Fix}(\partial A) \cong C_n$. Then $\text{Fix}(\partial A^*)$ can be constructed from the groups $\text{Fix}(\partial A^*)$, using operations (b), operations (c) for $C_n$, and operations (d) for $D_1 \cong C_2$, where $A$ ranges through all atoms corresponding to colored edges in $A$.

Proof. When $\text{Fix}(\partial A) \cong C_1$, it has no non-trivial automorphism and $\text{Fix}(\partial A^*)$ can be constructed using only (b). When $\text{Fix}(\partial A) \cong C_2$, it has a single non-trivial automorphism $\pi$ which is either a reflection or a $180^\circ$ rotation around $\partial A$. In the case of the $180^\circ$ rotation, there is at most one edge-orbit of size 1 which is either fixed (for a pendant edge), or reflected (for a normal edge). Further, we proceed similarly as in the case of a proper atom in Lemma 5.4 to prove that $\text{Fix}(\partial A^*)$ can be constructed using (b) and (d).

Suppose that $\text{Fix}(\partial A) \cong C_n$ for some $n \geq 3$. Recall that $A$ is essentially 3-connected. The situation is depicted in Fig. 13a. The group $\text{Fix}(\partial A)$ is the stabilizer of the unique vertex $u$ in $\partial A$. Recall from Section 4 that in the language of isometries its action is generated by a rotation around $u$ and the opposite point of the sphere. Therefore, every edge-orbit of $\text{Fix}(\partial A)$ is of size one or $n$. All the edge-orbits of size $n$ are equivariant. Suppose that the action of $\text{Fix}(\partial A)$ consists of $\ell$ equivariant edge-orbits of colored edges of size $n$. The colored edges in these edge-orbits correspond to atoms $A_1, \ldots, A_{\ell}$.

The opposite point of the sphere is a vertex $v$ or the center of a face. (Since $n \geq 3$, it cannot contain the center of an edge.) In the former case, there might be an edge-orbit of size one consisting of a single pendant edge attached to $v$, and suppose that this pendant edge corresponds to a block atom $B$. In the later case, there is no edge-orbit of size one.

Let $\Psi = \text{Fix}(\partial A_1^*) \times \cdots \times \text{Fix}(\partial A_{\ell}^*)$, we can construct it using (b). Then we get

$$\text{Fix}(\partial A^*) \cong \Psi \wr C_n \times \text{Fix}(\partial B^*),$$

where $\text{Fix}(\partial B^*) = \{1\}$ if no edge-orbit of size one exists. So we construct $\text{Fix}(\partial A^*)$ using (b) and (c). \qed

It remains to deal with dihedral groups. First, we determine the possible counts of equivariant classes of edge-orbits.

Lemma 5.6. Let $A$ be a non-star block atom with $\text{Fix}(\partial A) \cong D_n$.

- If $n$ is odd, then all edge-orbits of type $n$ are equivariant and all edge-orbits of type $n^{*+}$ are equivariant.
- If $n$ is even, then there are at most two equivariant classes of edge-orbits of types $n$ and $n^{*+}$.
Proof. The statement clearly holds for \( n = 1 \), so in what follows, we assume \( n \geq 2 \). Recall from Section 4 that in the language of isometries the action of \( \text{Fix}(\partial A) \) consists of \( n \) rotations and \( n \) reflections. Each rotation fixes only \( \partial A \) and the opposite point of the sphere, and each reflection fixes a circle containing \( \partial A \) and the opposite point. Let \( r \) be the rotation by \( 360°/n \), then all rotations are \( \text{id}, r, r^2, \ldots, r^{n-1} \). Let \( f_1, f_2, \ldots, f_n \) be the reflections as their planes are ordered cyclically, perpendicular to the axis of the rotations; see Fig. 14.

These reflections are cyclically linked by the conjugation \( f_{i+2} = r^{-1}f_i r \). The key distinction is that for \( n \) odd, all reflections are conjugate of each other, but for \( n \) even, we get two conjugacy classes \( f_1, f_3, \ldots, f_{n-1} \) and \( f_2, f_4, \ldots, f_n \).

Let \( e \) be an edge belonging to a fixed/reflected edge-orbit \([e]\) of size \( n \). The rotation \( r \) does not stabilize any edge in \([e]\), so each edge is stabilized by some reflection. From the geometry, \([e] = (e, r \cdot e, r^2 \cdot e, \ldots, r^{n-1}e)\). Suppose that \( e \) is stabilized by \( f_i \). Then \( r \cdot e \) is stabilized by \( f_{i+2}, r^2 \cdot e \) by \( f_{i+4} \), and so on.

When \( n \) is odd, each reflection stabilizes exactly one edge in \([e]\); see Fig. 14 on the left. Therefore, two edge-orbits \([e]\) and \([e']\) of size \( n \), both fixed or reflected, are equivariant: the edges in \([e]\) and \([e']\) having the same stabilizer can be matched.

When \( n \) is even, only one conjugacy class of reflections stabilizes edges in \([e]\), each stabilizing \( r^i \cdot e \) and \( r^{i+n/2} \cdot e \). When two edge-orbits \([e]\) and \([e']\) of size \( n \), both fixed or reflected, are stabilized by the same conjugacy class of reflections, they are equivariant. Therefore, we get at most two equivariant classes of both fixed and reflected edge-orbits of size \( n \).

To specify the semidirect products in (d) and (e), we describe the action of \( D_n \) on an edge-orbit \([e]\) of size \( n \). The rotation \( r \) maps \( r^k \cdot e \) to \( r^{k+1} \cdot e \). When the reflection \( f_1 \) stabilizes \( e' \in [e] \), then it swaps \( r \cdot e' \) with \( r^{-1} \cdot e', r^2 \cdot e' \) with \( r^{-2} \cdot e' \), and so on; it fixes \( e' \) and for \( n \) even also \( r^{n/2} \cdot e' \). The reflection \( f_{i+1} \) swaps \( r \cdot e' \) with \( e', r^2 \cdot e' \) with \( r^{-1} \cdot e', r^3 \cdot e' \) with \( r^{-2} \cdot e' \), and so on. As stated, the reflection \( f_{i+2} \) stabilizes \( r^2 \cdot e' \), and so on.

Let \( h \) and \( h' \) be the half-edges corresponding to \( e \). Consider \( 2n \) half-edges corresponding to edges in \([e]\). In the action of \( \{r\} \), they form two orbits \( \{h, r \cdot h, r^2 \cdot h, \ldots, r^{n-1} \cdot h\} \) and \( \{h', r \cdot h', r^2 \cdot h', \ldots, r^{n-1} \cdot h'\} \) of size \( n \). When \([e]\) is fixed, these two orbits are preserved in the action of \( D_n \). When \([e]\) is reflected, each reflection \( f_1 \) swapping \( r \cdot e \) with \( r^i \cdot e \) swaps \( r^i \cdot h \) with \( r^{i+1} \cdot h' \) and \( r^i \cdot h' \) with \( r^{i+1} \cdot h \) so we get one orbit of half-edges of size \( 2n \) in the action of \( D_n \). In \( D_n \), we note the action of \( D_n \) on an edge-orbit of size \( 2n \) is the same as the action on the half-edges corresponding to \( n \).

When \( \text{Fix}(\partial A) \cong D_1 \cong C_2 \), we use Lemma 5.5. We start with an easier case of \( \text{Fix}(\partial A) \cong D_n \), for \( n \geq 3 \), odd.

**Lemma 5.7.** Let \( A \) be a non-star block atom with \( \text{Fix}(\partial A) \cong D_n \) for \( n \geq 3 \) and odd. Then \( \text{Fix}(\partial A^*) \) can be constructed from the groups \( \text{Fix}(\partial A) \), using operations (b), and operations (d), where \( \partial A \) ranges through all atoms corresponding to colored edges in \( A \).

**Proof.** As it is described in the proof of Lemma 5.6 the group \( \text{Fix}(\partial A) \cong D_n \) consists of \( n \) rotations and \( n \) reflections; see Fig. 13. It acts semiregularly on the angles of the map and all edge-orbits are of size one, \( n \), or \( 2n \). By Lemma 5.6 all fixed/reflected edge-orbits of size \( n \) are equivariant and \( \text{Fix}(\partial A) \) acts on them as described below Lemma 5.6.

– **Edge-orbits of type 1.** The opposite point of the sphere either contains a vertex \( v \), or the center of a face.

In the former case, there might be at most one edge-orbit of size one, consisting of a single pendant edge attached to \( v \) corresponding to a block atom \( D \). In the latter case, no edge-orbit of size one exists.

![Fig. 14. Two block atoms \( A \) with \( \text{Fix}(\partial A) \cong D_n \) in the view from above, with \( n = 5 \) on the left and \( n = 6 \) on the right. The rotation \( r \) and the reflections \( f_1, \ldots, f_n \) are denoted. On the left, each edge of an edge-orbit of size \( n \) is stabilized by exactly one reflection. On the right, each pair of opposite edges of an edge-orbit of size \( n \) is stabilized by exactly one reflection from one of the conjugacy classes \( f_1, f_3, \ldots, f_{n-1} \) and \( f_2, f_4, \ldots, f_n \).](image-url)
Fig. 15. On the left, an example of a non-star block atom \(A\) with \(\text{Fix}(\partial A) \cong \mathbb{D}_6\) consisting of 6 rotations by multiples of 60° and six depicted reflections. On the right, the view from above, with colored edges labeled by the corresponding atoms. Different types of edge-orbits are depicted with different types of edges. We have \(\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell_5 = 1\).

- **Edge-orbits of type \(n\).** We have \(\ell_2\) equivariant fixed edge-orbits of colored edges of size \(n\), corresponding to atoms \(B_1,\ldots,B_{\ell_2}\).
- **Edge-orbits of type \(n,\ell_3\).** We have \(\ell_3\) equivariant reflected edge-orbits of colored edges of size \(n\), corresponding to necessarily symmetric atoms \(C_1,\ldots,C_{\ell_3}\). Let \(\tau^*_i \in \text{Aut}_{\partial C^*_i}(C^*_i)\) be an involution exchanging \(\partial C^*_i\), ensured by Lemma 4.5.
- **Edge-orbits of type \(2n\).** We have \(\ell_1\) equivariant edge-orbits of colored edges of size \(2n\), corresponding to atoms \(A_1,\ldots,A_{\ell_1}\).

We put

\[
\psi_1 = \prod_{i=1}^{\ell_1} \text{Fix}(\partial A^*_i), \quad \psi_2 = \prod_{i=1}^{\ell_2} \text{Fix}(\partial B^*_i), \quad \psi_3 = \prod_{i=1}^{\ell_3} \text{Fix}(\partial C^*_i).
\]

It follows that

\[
\text{Fix}(\partial A^*) \cong (\psi_1^{2n} \times \psi_2^{n} \times \psi_3^{n}) \times_{\varphi} \mathbb{D}_{2n} \times \text{Fix}(\partial D^*),
\]

where \(\text{Fix}(\partial D^*) = \{1\}\) if there is no edge-orbit of size one. The homomorphism \(\varphi\) is defined based on the description of the action of \(\mathbb{D}_{2n}\) below Lemma 5.6. It permutes the coordinates of \(\Psi_1^{2n}\) regularly as \(\mathbb{D}_{2n}\) acts on half-edges of a reflected edge-orbit of size \(n\). It permutes the coordinates in \(\Psi_2^{n}\) and \(\Psi_3^{n}\) following the action on the edges of fixed and reflected edge-orbits of size \(n\), respectively. For the edges of reflected edge-orbits corresponding to \(C^*_i\), when half-edges are swapped by an element \(\pi \in \text{Fix}(\partial A)\), the involution \(\tau^*_i\) is used in the action of \(\pi^* \in \text{Fix}(\partial A^*)\) on the corresponding atoms \(C^*_i\). Therefore, \(\text{Fix}(\partial A^*)\) can be constructed using (b) and (d).

\(\square\)

Next, we deal with the case \(\text{Fix}(\partial A) \cong \mathbb{D}_n\), for \(n \geq 4\) and even.

**Lemma 5.8.** Let \(A\) be a non-star block atom with \(\text{Fix}(\partial A) \cong \mathbb{D}_n\) for \(n \geq 4\), even. Then \(\text{Fix}(\partial A^*)\) can be constructed from the groups \(\text{Fix}(\partial A^*)\), using operations (b), and operations (c), where \(A\) ranges through all atoms corresponding to colored edges in \(A\).

**Proof.** As it is described in the proof of Lemma 5.6 the group \(\text{Fix}(\partial A) \cong \mathbb{D}_n\) consists of \(n\) rotations and \(n\) reflections; see Fig. 15. It acts semiregularly on the angles of the map and all edge-orbits are of size one, \(n\), or \(2n\). By Lemma 5.6 there are at most two equivariant classes of fixed/reflected edge-orbits of size \(n\) and \(\text{Fix}(\partial A)\) acts on them as described below Lemma 5.6.

- **Edge-orbits of type \(1\).** Exactly as in the proof of Lemma 5.7 there is at most one edge-orbit of size one, consisting of a single pendant edge corresponding to a block atom \(F\).
- **Edge-orbits of type \(n\).** We have two equivariant classes of \(\ell_2\) and \(\ell_4\) fixed edge-orbits of colored edges of size \(n\), corresponding to atoms \(B_1,\ldots,B_{\ell_2}\) and \(D_1,\ldots,D_{\ell_4}\), respectively.
Fig. 16. On the left, an example of a non-star block atom \( A \) with \( \text{Fix}(\partial A) \cong \mathbb{C}_2^2 \) generated by two depicted reflections. On the right, the view from above, with colored edges labeled by the corresponding atoms, the central edge corresponds to an atom \( F \). Different types of edge-orbits are depicted with different types of edges. We have \( \ell_1 = 3, \ell_2 = \ell_4 = 1 \) and \( \ell_3 = \ell_5 = 2 \).

- **Edge-orbits of type \( n_u \).** We have two equivariance classes of \( \ell_4 \) and \( \ell_5 \) reflected edge-orbits of colored edges of size \( n \), corresponding to necessarily symmetric atoms \( C_1, \ldots, C_{\ell_1}, E_1, \ldots, E_{\ell_5} \). Let \( \tau^*_i \in \text{Aut}_\partial C_i(C_i^*) \) be an involution exchanging \( \partial C_i^* \) and let \( \hat{\tau}^*_i \in \text{Aut}_\partial E_i(E_i^*) \) be an involution exchanging \( \partial E_i^* \), ensured by Lemma 5.6.

- **Edge-orbits of type \( 2n \).** We have \( \ell_1 \) equivariant edge-orbits of colored edges of size \( 2n \), corresponding to atoms \( A_1, \ldots, A_{\ell_1} \).

We put

\[
\Psi_1 = \prod_{i=1}^{\ell_1} \text{Fix}(\partial A_i^*), \quad \Psi_2 = \prod_{i=1}^{\ell_2} \text{Fix}(\partial B_i^*), \quad \Psi_3 = \prod_{i=1}^{\ell_3} \text{Fix}(\partial C_i^*), \quad \Psi_4 = \prod_{i=1}^{\ell_4} \text{Fix}(\partial D_i^*), \quad \Psi_5 = \prod_{i=1}^{\ell_5} \text{Fix}(\partial E_i^*).
\]

It follows that

\[
\text{Fix}(\partial A^*) \cong (\Psi_1^2 \times \Psi_2^2 \times \Psi_3^2 \times \Psi_4^2 \times \Psi_5^2) \times \mathbb{D}_{2\ell} \times \text{Fix}(\partial F^*),
\]

where \( \text{Fix}(\partial F^*) = \{1\} \) if there is no edge-orbit of size one. The homomorphism \( \varphi \) is defined based on the description of the action of \( \mathbb{D}_{2\ell} \) below Lemma 5.6. It permutes the coordinates of \( \Psi_1^2, \Psi_2^2, \Psi_3^2, \Psi_4^2, \Psi_5^2 \) regularly following the action of \( \mathbb{D}_{2\ell} \), on half-edges of a reflected edge-orbit of size \( n \). It permutes the coordinates in \( \Psi_2^2, \Psi_3^2, \Psi_4^2, \Psi_5^2 \) in the same way as the edges of two equivariance classes of fixed and reflected edge-orbits of size \( n \), respectively. For the edges of reflected edge-orbits corresponding to \( C_i^* \) and \( E_i^* \), when half-edges are swapped by an element \( \pi \in \text{Fix}(\partial A) \), the involutions \( \tau^*_i \) and \( \hat{\tau}^*_i \) are used in the action of \( \pi^* \in \text{Fix}(\partial A^*) \) on the corresponding atoms \( C_i^* \) and \( E_i^* \), respectively. Therefore, \( \text{Fix}(\partial A^*) \) can be constructed using (b) and (e). \( \Box \)

It remains to deal with the last case of \( \text{Fix}(\partial A) \cong \mathbb{D}_2 \cong \mathbb{C}_2^2 \) which may have the most involved structure of edge-orbits.

**Lemma 5.9.** Let \( A \) be a non-star block atom with \( \text{Fix}(\partial A) \cong \mathbb{D}_2 \cong \mathbb{C}_2^2 \). Then \( \text{Fix}(\partial A^*) \) can be constructed from the groups \( \text{Fix}(\partial A^*) \), using operations (b) and (f), where \( A \) ranges through all atoms corresponding to colored edges in \( A \).

**Proof.** As it is described in the proof of Lemma 5.6, the group \( \text{Fix}(\partial A) \cong \mathbb{D}_2 \cong \mathbb{C}_2^2 \) is generated by two reflections \((1,0)\) and \((0,1)\) through \( \partial A \), orthogonal to each other. Composition of these two reflections forms the \( 180^\circ \) rotation \((1,1)\) through \( \partial A \) and the opposite point of the sphere. (We identified the geometric transformations with the elements of the elementary abelian group of order 4.) Therefore, every edge-orbit of \( \text{Fix}(\partial A) \) is of size one, two, or four, and we describe them below; see Fig. 16 for an example.

- **Edge-orbits of type 1 and of type \( 1_{us} \).** The rotation \((1,1)\) and the reflections \((1,0)\) and \((0,1)\) stabilize, aside \( \partial A \), the opposite point of the sphere which contains either a vertex, or the center of an edge, or the center of a face.
• If they stabilize the center of a face, there is no edge-orbit of size 1.
• If they stabilize a vertex \( e \), there might be a fixed edge-orbit of size 1 consisting of a single pendant edge attached at \( e \). We deal with it using (b) as in the proofs of Lemmas 5.3, 5.7, and 5.8 and we put \( \psi_0 = \{1\} \) and \( \tau^*_0 = \text{id} \).

• If they stabilize the center of an edge \( e \), then \( e \) is a reflected edge-orbit of size 1. Let \( F \) be a symmetric atom corresponding to the colored edge \( e \) and we put \( \psi_0 = \text{Fix}(\partial F^*) \). By Lemma 4.5, there exists an involution \( \tau_0 \) exchanging \( \partial F \) and let \( \tau^*_0 \) be a corresponding involution in \( \text{Aut}(\partial F^*) \).

Edge-orbits of type \( 2 \) and of type \( 2^* \). By Lemma 5.3, there are at most two equivariant classes of fixed/reflected edge-orbits of size 2, one class stabilized by \((1,0)\) and the other one by \((0,1)\).

• There are \( \ell_2 \) equivariant edge-orbits of colored edges reflected by the reflection \((1,0)\). These colored edges correspond to atoms \( B_1, \ldots, B_{\ell_2} \).

• There are \( \ell_3 \) equivariant edge-orbits of colored edges reflected by \((1,0)\). These colored edges correspond to symmetric atoms \( \mathcal{C}_1, \ldots, \mathcal{C}_{\ell_3} \). Let \( \tau^*_3 \in \text{Fix}(\partial \mathcal{C}_1^*) \times \cdots \times \text{Fix}(\partial \mathcal{C}_{\ell_3}^*) \) be an involution exchanging their boundaries.

• There are \( \ell_4 \) equivariant edge-orbits of colored edges reflected by \((0,1)\). These colored edges correspond to atoms \( D_1, \ldots, D_{\ell_4} \).

• There are \( \ell_5 \) equivariant edge-orbits of colored edges reflected by \((0,1)\). These colored edges correspond to symmetric atoms \( \mathcal{E}_1, \ldots, \mathcal{E}_{\ell_5} \). Let \( \tau^*_5 \in \text{Fix}(\partial \mathcal{E}_1^*) \times \cdots \times \text{Fix}(\partial \mathcal{E}_{\ell_5}^*) \) be an involution exchanging their boundaries.

Edge-orbits of type \( 4 \). The group \( \text{Fix}(\partial A) \) acts regularly on edge-orbits of size four. Suppose we have \( \ell_4 \) equivariant edge-orbits of colored edges of size four, and these colored edges correspond to atoms \( A_1, \ldots, A_{\ell_4} \).

We put

\[
\psi_1 = \prod_{i=1}^{\ell_4} \text{Fix}(\partial A_i^*), \quad \psi_2 = \prod_{i=1}^{\ell_4} \text{Fix}(\partial B_i^*), \quad \psi_3 = \prod_{i=1}^{\ell_3} \text{Fix}(\partial C_i^*), \quad \psi_4 = \prod_{i=1}^{\ell_4} \text{Fix}(\partial D_i^*), \quad \psi_5 = \prod_{i=1}^{\ell_5} \text{Fix}(\partial E_i^*).
\]

It easily follows that

\[
\text{Fix}(\partial A^*) \cong (\psi_1^4 \times \psi_2^2 \times \psi_3^2 \times \psi_4^2 \times \psi_5^5) \rtimes \varphi \mathcal{C}_2^2.
\]

Assuming that \((1,0)\) reverses the edge \( e_6 \), the homomorphism \( \varphi \) is defined by

\[
\begin{align*}
\varphi(1,0) &= (\pi_1, \pi_1', \pi_1'', \pi_2, \pi_2', \pi_3, \pi_4, \pi_5, \pi_5', \pi_6) \\
&\mapsto (\pi_1, \pi_1', \pi_1'', \pi_2, \pi_2', \pi_3, \pi_3', \pi_4, \pi_5, \pi_5', \pi_6),
\end{align*}
\]

\[
\varphi(0,1) = (\pi_1, \pi_1', \pi_1'', \pi_2, \pi_2', \pi_3, \pi_3', \pi_4, \pi_4', \pi_5, \pi_5', \pi_6) \\
\mapsto (\pi_1, \pi_1', \pi_1'', \pi_1', \pi_2, \pi_2', \pi_3, \pi_3', \pi_4, \pi_4', \pi_5, \pi_5', \pi_6).
\]

where \( \tau^*_0 = \text{id} \) if \( e_6 \) does not exist. Therefore, \( \text{Fix}(\partial A^*) \) can be constructed using (b) and (f).

Now, we are ready to prove the Jordan-like characterization of \( \text{Fix}(\text{connected PLANAR}) \).

Proof (Theorem 5.1). Lemma 5.2 describes constructions. We prove the opposite implication by induction according to the depth of the subtree of a reduction tree. Let \( A \) be an atom and suppose that the subtrees rooted at all its children can be realized by (b) to (f). By Lemmas 5.3, 5.4, 5.5, 5.7, 5.8, and 5.9 also \( \text{Fix}(\partial A^*) \) can be realized by (b) to (f). \( \square \)

5.3 Composition of Spherical groups with Fixers

It remains to deal with the root of the reduction tree, corresponding to the primitive graph \( G_r \). In comparison with atoms, \( \text{Aut}(G_r) \) does not have to stabilize any vertex or edge, unlike \( \text{Fix}(\partial A) \) which stabilizes \( \partial A \). Therefore, all spherical groups are available for \( \text{Aut}(G_r) \). Now, we are ready to prove the main result of this paper, classifying \( \text{Aut}(\text{connected PLANAR}) \).

Theorem 5.10. Let \( H \) be a planar graph with colored vertices and colored (possibly oriented) edges, which is either 3-connected, or \( K_1 \), or \( K_2 \), or a cycle \( C_n \). Let \( m_1, \ldots, m_{\ell} \) be the sizes of the vertex- and edge-orbits of the action of \( \text{Aut}(H) \). Then for all choices \( \psi_1, \ldots, \psi_{\ell} \in \text{Fix}(\text{connected PLANAR}) \), we have

\[
(\psi_1^{m_1} \times \cdots \times \psi_{\ell}^{m_{\ell}}) \rtimes \text{Aut}(H) \in \text{Aut}(\text{connected PLANAR}),
\]

where \( \text{Aut}(H) \) permutes the factors of \( \psi_1^{m_1} \times \cdots \times \psi_{\ell}^{m_{\ell}} \) following the action on the vertices and edges of \( H \).

On the other hand, every group of \( \text{Aut}(\text{connected PLANAR}) \) can be constructed in the above way as

\[
(\psi_1^{m_1} \times \cdots \times \psi_{\ell}^{m_{\ell}}) \rtimes \Sigma,
\]

where \( \psi_1, \ldots, \psi_{\ell} \in \text{Fix}(\text{connected PLANAR}) \) and \( \Sigma \) is a spherical group.
Fig. 17. On the left, a 3-connected planar graph \( H \) obtained from the cube, with only three front faces depicted. We have \( \text{Aut}(H) \cong C_2 \times S_3 \) and its action \(*432\) in Table 1. Different orbits are shown in different colors: there are two vertex-orbits (of sizes 8 and 6) and two edge-orbits (of sizes 24 and 12).

On the right, \( H \) is modified by attaching single pendant edges of different colors for each vertex-orbit. For arbitrary choices of \( \Psi_1, \Psi_2, \Psi_3, \Psi_4 \in \text{Fix}(\text{connected PLANAR}) \), we can expand \( H \) to \( H^* \) with \( \text{Aut}(H^*) \cong (\Psi_1^4 \times \Psi_2^2 \times \Psi_3^3 \times \Psi_4^2) \times (C_2 \times S_3) \). Notice that some automorphisms of \( \text{Aut}(H) \) reflect some white edges and the corresponding automorphism in \( \text{Aut}(H^*) \) reflect the expanded atoms corresponding to these edges by \( \tau_i^* \).

Proof. Let \( H \) be a graph satisfying the assumptions; for an example, see Fig. 17. First, we replace colors of the vertices of \( H \) with colored single pendant edges attached to them. Using Lemma 5.2, we choose arbitrary pairwise non-isomorphic extended atoms \( A_1^*, A_2^*, \ldots, A_l^* \) such that \( \text{Fix}(\partial A_i^*) \cong \Psi_i \), and we replace the corresponding colored edges with them. If the edge-orbit replaced by \( A_i^* \) consists of undirected edges, we assume that \( A_i^* \) are symmetric atoms, and let \( \tau_i^* \in \text{Aut}(\partial A_i^*) \) be an involution exchanging \( \partial A_i^* \). If it consists of directed edges, we assume that \( A_i^* \) are asymmetric atoms placed consistently with the orientation. We denote this modified planar graph by \( H^* \).

Exactly as in the proof of Proposition 3.7, we get that

\[
\text{Aut}(H^*) \cong (\Psi_1^{m_1} \times \cdots \times \Psi_l^{m_l}) \rtimes_{\varphi} \text{Aut}(H).
\]

An automorphism \( \pi^* \in \text{Aut}(H^*) \) permutes the extended atoms exactly as \( \pi \in \text{Aut}(H) \) permutes the colored edges. If \( \pi \) reflects an edge representing a symmetric atom \( A_i^* \), then \( \pi^* \) reflects \( A_i^* \) using \( \tau_i^* \).

For the other implication, let \( G \) be a planar graph. We apply the reduction series and obtain a primitive graph \( G_r \). By Lemma 4.3, we know that \( \text{Aut}(G_r) \) is a spherical group. Suppose that we have \( l \) edge-orbits of colored edges in the action of \( \text{Aut}(G_r) \). Suppose that their sizes are \( m_1, \ldots, m_l \) and their colored edges correspond to expanded atoms \( A_1^*, \ldots, A_l^* \). By Theorem 5.1, we know that \( \text{Fix}(\partial A_i^*) \in \text{Fix}(\text{connected PLANAR}) \). Further, for symmetric expanded atoms \( A_i^* \), by Lemma 4.5, there exists an involution \( \tau_i^* \) exchanging \( \partial A_i^* \). We proceed exactly as in the proof of Proposition 3.7, and we obtain

\[
\text{Aut}(G) \cong (\text{Fix}(\partial A_i^*)^{m_1} \times \cdots \times \text{Fix}(\partial A_l^*)^{m_l}) \rtimes_{\varphi} \text{Aut}(G_r).
\]

Possible Lengths of Orbits. To describe the groups realizable as automorphism groups of connected planar graphs, we need to understand what are the possible restrictions on sizes \( m_i \) of the orbits of \( \text{Aut}(G) \) in a 3-connected planar graph \( G \). When \( G \) is \( K_4 \), \( K_6 \) or a cycle \( C_n \), we get more restricted orbits than in the case of 3-connected planar graphs \( G \). For instance, the wheel \( W_6 \) is 3-connected and contains all orbits of \( C_n \). We investigate possible actions of spherical groups \( \Sigma \) realized as groups of isometries of polytopes projected onto the sphere.

In [28], the following characterization of possible equivariance classes of orbits given in Table 1 is proved. Table 1 is organized as follows. Each row of the table corresponds to a distinguished (parametrized) spherical group \( \Sigma \) described using the notation of Conway and Thurston [5]. There are fourteen types of actions, several small special cases are discussed separately. The second column describes \( \Sigma \) as an abstract group and the third column gives the order of \( \Sigma \).

The fourth column gives the numbers of equivariance classes of point-orbits of \( \Sigma \). By \( \cdot \cdot \cdot a^b \), we denote that there are \( c \) equivariance classes of point-orbits of size \( a \), each class of size \( b \). For instance, in the second row, the fourth entry contains 48, 3, \( 24 \cdot 4 \), 12, 8, 6. This means that there are infinitely many mutually equivariant point-orbits of size 48, three infinite equivariant classes of point-orbits of size 24, and single point-orbits of sizes 12, 8, and 6.
| Action | \( \Sigma \) | \(|\Sigma|\) | Point-orbits | Vertex-orbits | Edge-orbits |
|--------|---------|------|-------------|--------------|-------------|
| *532   | \( A_5 \times C_2 \) | 120  | \( 120 \infty \), \( 3 \cdot 60 \infty \), \( 30^1 \), \( 20^1 \), \( 12^1 \) | \( 120 \), \( 3 \cdot 60 \), \( 30 \), \( 20 \), \( 12 \) | \( 120 \), \( 3 \cdot 60 \), \( 3 \cdot 60^\infty \), \( -\infty \) |
| 432    | \( S_4 \times C_2 \) | 48   | \( 48 \infty \), \( 3 \cdot 24 \infty \), \( 12^1 \), \( 6^1 \), \( 6^2 \) | \( 48 \), \( 3 \cdot 24 \), \( 12 \), \( 8 \), \( 6 \) | \( 48 \), \( 3 \cdot 24 \), \( 24 \), \( 8 \), \( -\infty \) |
| 332    | \( S_4 \) | 24   | \( 24 \infty \), \( 2 \cdot 12 \), \( 6^1 \), \( 4^1 \) | \( 24 \), \( 2 \cdot 12 \), \( 6 \), \( 4 \) | \( 24 \), \( 2 \cdot 12 \), \( -\infty \) |
| *22n   | \( D_n \times C_2 \) \( n \geq 3 \), odd | 4n   | \( (4n) \infty \), \( 2 \cdot (2n) \), \( n^2 \), \( 2^1 \) | \( 4n \), \( 2 \cdot 2n \), \( n \), \( 2 \) | \( 4n \), \( 2 \cdot 2n \), \( 2 \cdot n^\infty \), \( n \) |
| *22n   | \( D_n \times C_2 \) \( n \geq 4 \), even | 4n   | \( (4n) \infty \), \( 3 \cdot (2n) \), \( 2 \cdot n^1 \), \( 2^1 \) | \( 4n \), \( 3 \cdot 2n \), \( 2 \cdot n \), \( 2 \) | \( 4n \), \( 3 \cdot 2n \), \( 2 \cdot n^\infty \), \( n \) |
| *222   | \( D_2 \times C_2 \) \( = C_2 \times C_2 \) | 8    | \( 8 \infty \), \( 3 \cdot 4 \), \( 3 \cdot 2^1 \) | \( 8 \), \( 3 \cdot 4 \), \( 3 \cdot 2 \) | \( 8 \), \( 3 \cdot 4 \), \( 3 \cdot 4 \), \( 2 \) |
| 532    | \( A_5 \) | 60   | \( 60 \infty \), \( 30^1 \), \( 20^1 \), \( 12^1 \) | \( 60 \), \( 30 \), \( 20 \), \( 12 \) | \( 60 \), \( -\infty \) |
| 432    | \( S_4 \) | 24   | \( 24 \infty \), \( 12^1 \), \( 8^1 \), \( 6^1 \) | \( 24 \), \( 12 \), \( 8 \), \( 6 \) | \( 24 \), \( -\infty \) |
| 332    | \( A_4 \) | 12   | \( 12 \infty \), \( 6^1 \), \( 4^2 \) | \( 12 \), \( 6 \), \( 4 \) | \( 12 \), \( -\infty \) |
| 22n    | \( D_n \) \( n \geq 3 \), odd | \( 2n \) | \( (2n) \infty \), \( n^2 \), \( 2^1 \) | \( 2n \), \( n \), \( 2 \) | \( 2n \), \( -\infty \) |
| 22n    | \( D_n \) \( n \geq 4 \), even | \( 2n \) | \( (2n) \infty \), \( 2 \cdot n^1 \), \( 2^1 \) | \( 2n \), \( 2 \cdot n \), \( 2 \) | \( 2n \), \( n \) |
| 222    | \( D_2 = C_2 \) | 4    | \( 4 \infty \), \( 3 \cdot 2^1 \) | \( 4 \cdot 2 \), \( 4 \cdot 2 \) | \( 4 \cdot 2 \), \( -\infty \) |
| 3*2    | \( A_4 \times C_2 \) | 24   | \( 24 \infty \), \( 12 \), \( 8^1 \), \( 6^1 \) | \( 24 \), \( 12 \), \( 8 \), \( 6 \) | \( 24 \), \( 12 \), \( -\infty \) |
| 2*n    | \( D_{2n} \), \( n \geq 3 \) | \( 4n \) | \( (4n) \infty \), \( 2 \cdot (2n) \), \( 1^1 \) | \( 4n \), \( 2 \cdot 2n \), \( 2 \) | \( 4n \), \( 2 \cdot 2n \) |
| 2*2    | \( D_4 \) | 8    | \( 8 \infty \), \( 4 \), \( 4 \), \( 4^2 \) | \( 8 \), \( 4 \), \( 4 \), \( 2 \) | \( 8 \), \( 4 \), \( -\infty \) |
| *nn    | \( D_n \) \( n \geq 3 \), odd | \( 2n \) | \( (2n) \infty \), \( n^\infty \), \( 1^2 \) | \( 2n \), \( n \), \( 1 \) | \( 2n \), \( n \) |
| *nn    | \( D_n \) \( n \geq 4 \), even | \( 2n \) | \( (2n) \infty \), \( 2 \cdot n^1 \), \( 1^2 \) | \( 2n \), \( 2 \cdot n \), \( 1 \) | \( 2n \), \( -\infty \) |
| *22    | \( D_2 = C_2 \) | 4    | \( 4 \infty \), \( 2 \), \( 2 \), \( 1 \) | \( 4 \), \( 2 \), \( 2 \) | \( 4 \), \( -\infty \) |
| n      | \( C_n \), \( n \geq 3 \) | \( n \) | \( n \), \( 1^2 \) | \( n \), \( 1 \) | \( n \) |
| 22    | \( C_2 \) | \( n \) | \( 2^2 \), \( 4^2 \) | \( 2 \), \( 1 \) | \( 2 \), \( -\infty \) |
| 2n    | \( C_{2n} \), \( n \geq 3 \) | \( 2n \) | \( (2n) \infty \), \( 2^4 \) | \( 2n \), \( 2n \) | \( 2n \) |
| 2s    | \( C_4 \) | 8    | \( 4 \infty \), \( 4 \), \( 2 \) | \( 4 \), \( -\infty \) | \( 4 \), \( 2 \) |
| 2n*   | \( C_n \times C_2 \) \( n \geq 3 \) | \( 2n \) | \( (2n) \infty \), \( n^\infty \), \( 2^1 \) | \( 2n \), \( n \), \( 2 \) | \( 2n \), \( n \) |
| 2*    | \( C_2 \times C_2 \) | 4    | \( 4 \infty \), \( 2 \), \( 2^1 \) | \( 4 \), \( 2 \) | \( 4 \), \( 2 \) |

Table 1. The list of all possible types of lengths of orbits in Theorem 5.10.
The fifth and sixth columns describe equivariance classes of vertex- and edge-orbits of \( \Sigma \), respectively. In the second row, there are two options for sequences of possible vertex- and edge-orbits:

- either \( 48, 3 \cdot 24, 12, 8, 6 \) and \( 48, 3 \cdot 24, 3 \cdot 24_{es}, -e_{es} \),
- or \( 48, 3 \cdot 24, -, 8, 6 \) and \( 48, 3 \cdot 24, 3 \cdot 24_{es}, 12_{es} \).

The multiplicity of orbits in equivariance classes is not displayed. The difference between the two cases comes from the fact that the unique point-orbit of size 12 is either a vertex-orbit or an edge-orbit. The subscript \( \leftrightarrow \) means that the edge-orbit is reflexive. Similarly as in the proofs in Section 5.2, we distinguish edge-orbits which are fixed (the corresponding half-edges form two orbits of the same size), depicted as \( e \cdot a \), and which are reflected, denoted as \( e \cdot a_{es} \) (the corresponding half-edges form one orbit of the double size).

6 Applications of Jordan-like Characterization

In this section, we apply the Jordan-like characterization of Theorems 5.1 and 5.10 to describe automorphism groups of several important subclasses of planar graphs. First, we determine possible atoms, and primitive graphs and their automorphism groups. Then we determine possible stabilizers similarly as in the proof of Theorem 5.1, however, only some of the group products appear. Lastly, we combine these stabilizers together with spherical groups which are representable by primitive graphs, again only some of the (possibly restricted) cases of Table 1 may happen. In what follows, for a subclass \( C \) of planar graphs, we set

\[
\text{Fix}(C) = \{ \text{Fix}(\partial A^+): A \text{ is an atom of the reduction tree of a graph in } C \}.
\]

For instance, consider the class of all trees (TREE). The only primitive graph is \( K_1 \) (with a single pendant edge attached), so its automorphism group is trivial. All atoms are block atoms (either star block atoms or \( K_2 \) with a single pendant edge attached). Therefore, the stabilizers are determined by Lemma 5.3. The class is closed under the direct product and the wreath product with symmetric groups. Since \( K_1 \) has the trivial automorphism group, we get that the automorphism groups of trees are same as are the vertex-stabilizers of trees, so we get the Jordan’s characterization; see Theorem 1.2.

6.1 Automorphism Groups of 2-connected Planar Graphs

Denote the class of 2-connected planar graphs by 2-connected PLANAR. Consider the reduction tree of a 2-connected planar graph. There are no block atoms since all the atoms are proper or dipoles. Note that Fix(2-connected PLANAR) consists of all point-wise stabilizers of edges in 2-connected planar graphs. The reason is that for a proper atom/dipole \( A \) with \( \partial A = \{u, v\} \), we may consider the extended atom \( A^+ \) constructed from \( A \) by adding the edge \( uv \), and the corresponding extended extended atom \( (A^+)^* \). Then Fix(\( \partial A^+ \)) is the point-wise stabilizer of the edge \( uv \) in Aut(\( \partial A^+ \))\( \big((A^+)^*\big) \).

Lemma 6.1. The class Fix(2-connected PLANAR) is defined inductively as follows:

(a) \( \{1\} \in 2\text{-Fix}(2\text{-connected PLANAR}) \).
(b) If \( \Psi_1, \Psi_2 \in 2\text{-Fix}(2\text{-connected PLANAR}) \), then \( \Psi_1 \times \Psi_2 \in 2\text{-Fix}(2\text{-connected PLANAR}) \).
(c) If \( \Psi \in 2\text{-Fix}(2\text{-connected PLANAR}) \), then \( \Psi \times S_n \in 2\text{-Fix}(2\text{-connected PLANAR}) \).
(d) If \( \Psi_1, \Psi_2 \in 2\text{-Fix}(2\text{-connected PLANAR}) \), then \( \Psi_1^2 \times \Psi_2 \times C_2 \in 2\text{-Fix}(2\text{-connected PLANAR}) \).

Proof. The constructions are explained in Fig. 11b, c, d. For the other implication, we argue exactly as in the proof of Theorem 5.1. We apply induction according to the depth of the reduction tree. Let \( A \) be an atom with colored edges corresponding to proper atoms/dipoles \( \hat{A} \). We assume that Fix(\( \partial A^+ \)) can be constructed using the operations (a)–(d). Since \( A \) is a proper atom or a dipole, only Lemmas 5.3 and 5.4 apply, so the operations (b), (c), (d) are sufficient. For (d), we use the simplification described below the proof of Lemma 5.4.\( \square \)

Notice that the operations (c) and (d) are restrictions of (c) and (d) from Theorem 5.1 to \( S_n \) and \( D_1 \cong C_2 \), respectively. Also, the class Fix(2-connected PLANAR) is more rich than the class Aut(TREE), characterized by Jordan (Theorem 1.2) employing the operations (a)–(c). Therefore,

\[
\text{Aut}(\text{TREE}) \subset \text{Fix}(2\text{-connected PLANAR}) \subset \text{Fix}(\text{connected PLANAR}).
\]

Finally, we deal with a primitive graph in the root of the reduction tree. We easily modify the characterization in Theorem 5.10. There are two key differences. First, we use the class Fix(2-connected PLANAR) instead of Fix(connected PLANAR). Second, we only consider edge-orbits since there are no single pendant edges in primitive graphs, i.e., no expanded block atoms attached to their vertices.
Theorem 6.2. The class $\text{Aut}(2\text{-connected PLANAR})$ consists of the following groups. Let $H$ be a planar graph with colored (possibly oriented) edges, which is either 3-connected, or $K_2$, or a cycle $C_n$. Let $m_1, \ldots, m_e$ be the sizes of the edge-orbits of the action of $\text{Aut}(H)$. Then for all choices $\Psi_1, \ldots, \Psi_e \in \text{Fix}(2\text{-connected PLANAR})$, we have

$$(\Psi_1^{m_1} \times \cdots \times \Psi_e^{m_e}) \rtimes \text{Aut}(H) \in \text{Aut}(2\text{-connected PLANAR}),$$

where $\text{Aut}(H)$ permutes the factors of $\Psi_1^{m_1} \times \cdots \times \Psi_e^{m_e}$ following the action on the edges of $H$.

On the other hand, every group of $\text{Aut}(2\text{-connected PLANAR})$ can be constructed in the above way as

$$(\Psi_1^{m_1} \times \cdots \times \Psi_e^{m_e}) \rtimes \Sigma,$$

where $\Psi_1, \ldots, \Psi_e \in \text{Fix}(2\text{-connected PLANAR})$ and $\Sigma$ is a spherical group.

Proof. The reduction tree of a 2-connected planar graph contains only proper atoms and dipoles, and the primitive graph cannot be $K_1$. The proof proceeds as in Theorem 5.10 but the only difference is that we use Lemma 6.1 instead of Theorem 5.1. □

6.2 Automorphism Groups of Outerplanar Graphs

Let $G$ be a connected outerplanar graph with the reduction series $G = G_0, \ldots, G_r$. All graphs $G_i$ are outerplanar. Since no 3-connected planar graph is outerplanar, $G_r$ is by Lemma 3.3 either $K_1$, $K_2$, or a cycle $C_n$ (possibly with a single pendant edge attached). So, $\text{Aut}(G_r)$ is a subgroup of a dihedral group.

Next, we describe possible atoms encountered in the reduction:

- **Star block atoms.** We have arbitrary star block atoms.
- **Non-star block atoms.** Each non-star block atom $A$ is outerplanar. By Lemma 3.3 $A$ is either $K_2$ or $C_n$ with single pendant edges attached. Therefore, $\text{Fix}(\partial A)$ is a subgroup of $C_2$.
- **Proper atoms.** For a proper atom $A$ with $\partial A = \{u, v\}$, the extended proper atom $A^+$ is an outerplanar graph having an embedding with the edge $uv$ in the outer face. Therefore, by Lemma 3.4 $A$ is a non-trivial path, so $\text{Fix}(\partial A) \cong C_2$.
- **Dipoles.** For a dipole $A$ with $\partial A = \{u, v\}$, the extended dipole $A^+$ (with the edge $uv$) has an embedding such that the edge $uv$ belong to the outer face. Therefore, assuming that $G$ contains no parallel edges, $A$ consists of exactly two edges, one corresponding to an edge of $G$, and the other to a proper atom. Again, $\text{Fix}(\partial A) \cong C_2$.

Lemma 6.3. $\text{Fix}(\text{connected OUTERPLANAR}) = \text{Aut}($TREE$)$.

Proof. By induction, when $A$ is a proper atom or a dipole, we get that $\text{Fix}(\partial A^*)$ is the direct product of $\text{Fix}(\partial A^*)$ of the attached extended block atoms. Alternatively, it can be argued that each 2-connected outerplanar graph $G$ has $\text{Aut}(G)$ a subgroup of $\mathbb{D}_n$. Unless $G$ is a cycle, $\text{Aut}(G)$ stabilizes the outer face.

We use the same approach as in the proof of Theorem 5.11. Only Lemmas 3.3 and 3.4 for $C_1$, and 3.5 for subgroups of $C_2$ apply. For a block atom $A$ with $\text{Fix}(\partial A) \cong C_2$, there might be either an edge-orbit of type 1 consisting of a pendant edge corresponding to a block-atom $B$, or an edge-orbit of type 1, consisting of an edge corresponding to a proper atom or a dipole $C$. In the latter case, $\text{Fix}(\partial C^*)$ is just the direct product of extended block atoms attached in $C^*$, so the reflection in $\text{Fix}(\partial A)$ just swaps them, while at most one is fixed. Therefore, $\text{Fix}(\text{connected OUTERPLANAR})$ is defined inductively by the operations (a)–(c) from Theorem 1.2. □

By adapting the proof of Theorem 5.10 we get the following:

Theorem 6.4. The class $\text{Aut}($connected OUTERPLANAR$)$ consists of the following groups:

(i) If $\Psi \in \text{Fix}($connected OUTERPLANAR$)$, then $\Psi \circ C_n \in \text{Aut}($connected OUTERPLANAR$)$.

(ii) If $\Psi_1, \Psi_2 \in \text{Fix}($connected OUTERPLANAR$)$, then

$$(\Psi_1^{2n} \times \Psi_2^{2n}) \rtimes \mathbb{D}_n \in \text{Aut}($connected OUTERPLANAR$), \quad \forall n \text{ odd}.$$  

(iii) If $\Psi_1, \Psi_2, \Psi_3 \in \text{Fix}($connected OUTERPLANAR$)$, then

$$(\Psi_1^{2n} \times \Psi_2^{2n} \times \Psi_3^{2n}) \rtimes \mathbb{D}_n \in \text{Aut}($connected OUTERPLANAR$), \quad \forall n \text{ even}.$$  

Moreover, $\text{Aut}($connected OUTERPLANAR$) = \text{Aut}($PSEUDOTREE$)$.  

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6.3 Automorphism Groups of Series-Parallel Graphs

Series-parallel graphs are defined inductively as follows. Each series-parallel graph contains a pair of terminal vertices \((s, t)\). The graph \(K_2\) with the vertices \((s, t)\) is series-parallel. For two series-parallel graphs \(G_1(s_1, t_1)\) and \(G_2(s_2, t_2)\), we can construct a series-parallel graph on the vertices \(V(G_1) \cup V(G_2)\) by using two operations. The **parallel operation** identifies \(s_1 = s_2\) and \(t_1 = t_2\) and has the terminal vertices \((s_1, t_1)\). The **series operation** identifies \(t_1 = s_2\) and has the terminal vertices \((s_1, t_2)\). The class of all generalized series-parallel graphs (or just series-parallel graphs) consists of all graphs having each block a series-parallel graph, we denote this class by **SERIES-PARALLEL**. Clearly, **OUTERPLANAR** \(\subseteq** **SERIES-PARALLEL**.

Let \(G\) be a connected series-parallel graph with the reduction series \(G = G_0, \ldots, G_r\). All graphs the graphs \(G\) remain series-parallel since each 1-cut and 2-cut is introduced in the composition of the graph using one or the other operation. The only exception is \(G_r\), where we allow \(G_r = K_1\). Since no 3-connected planar graph is series-parallel, \(G_r\) is by **Lemma 3.5** again either \(K_1, K_2\), or a cycle \(C_n\), with attached single pendant edges.

So, \(\text{Aut}(G_r)\) is a subgroup of a dihedral group.

Next, we describe possible atoms encountered in the reduction:

- **Star block atoms.** Star block atoms may be arbitrary.
- **Non-star block atoms.** Each non-star block atom \(A\) is a series-parallel graph. By **Lemma 3.3** we get that \(A\) is either \(K_2\) or a cycle with attached single pendant edges, so \(\text{Fix}(\partial A)\) is again a subgroup of \(C_2\).
- **Proper atoms.** For a proper atom \(A\), the extended proper atom \(A^+\) is a series-parallel graph. Therefore, by **Lemma 3.3**, \(A\) is a path, so \(\text{Fix}(\partial A) \cong C_1\).
- **Dipoles.** Dipoles may be arbitrary.

**Lemma 6.5.** \(\text{Fix}(**CONNECTED SERIES-PARALLEL**) = \text{Fix}(2-connected PLANAR)\).

**Proof.** We use the same approach as in the proof of **Theorem 5.1**. Since the groups Fix(\(\partial A\)) of encountered atoms \(A\) are restricted, by **Lemmas 5.3, 5.4** and **5.5** the groups Fix(\(\partial A^+\)) can be constructed using (a)–(d) of **Lemma 6.1**. We note that each non-star block atom \(A\) with \(\text{Fix}(\partial A) \cong C_2\) has at most one edge-orbit of size 1, which is either of type 1, or of type 1*.

By adapting the proof of **Theorem 5.10** we get the following:
Theorem 6.6. The class \( \text{Aut}(\text{connected SERIES-PARALLEL}) \) consists of the following groups:

(i) If \( \Psi \in \text{Fix}(\text{connected SERIES-PARALLEL}) \), then \( \Psi \wr C_n \in \text{Aut}(\text{connected SERIES-PARALLEL}) \).

(ii) If \( \Psi_1, \Psi_2, \Psi_3 \in \text{Fix}(\text{connected SERIES-PARALLEL}) \), then
\[
(\Psi_1^{2n} \times \Psi_2^n \times \Psi_3^n) \rtimes D_n \in \text{Aut}(\text{connected SERIES-PARALLEL}), \quad \forall n \text{ odd}.
\]

(iii) If \( \Psi_1, \Psi_2, \Psi_3 \in \text{Fix}(\text{connected SERIES-PARALLEL}) \), then
\[
(\Psi_1^{2n} \times \Psi_2^n \times \Psi_3^n) \rtimes D_n \in \text{Aut}(\text{connected SERIES-PARALLEL}), \quad \forall n \text{ even}.
\]

We note that the semidirect products in (ii) and (iii) are different, see the proof for details.

Proof. Fig. 19 depicts the constructions. For the other direction, we deal with the case that the primitive graph \( G_r \) is a cycle \( C_r \) with attached single pendant edges, otherwise it is trivial. As in the proof of Theorem 6.4 we get three cases leading to different automorphism groups from the statement: (i) \( \text{Aut}(G_r) \cong C_n \), for \( n \neq 2 \),
(ii) \( \text{Aut}(G_r) \cong D_n \), for \( n \) odd, (iii) \( \text{Aut}(G_r) \cong D_n \), for \( n \) even.

The case (i) is exactly the same as in Theorem 6.4. In the cases (ii) and (iii), we have edge-orbits of types \( 2n, n, \) and \( n_{++} \). The group \( \text{Aut}(G_r) \) acts on the edge-orbits of type \( 2n \) regularly, exactly as in the proofs of Lemmas 5.7 and 5.8. We argue that the edge-orbits of types \( n \) and \( n_{++} \) are more restricted.

In the case (ii), we have exactly one conjugacy class of reflections \( f_i \), recall the notation from the proof of Lemma 5.6. Each of the reflections either stabilizes two edges of the cycle or two vertices if \( n \) is even; or a vertex and an edge if \( n \) is odd. In the first case, we get two equivariant edge-orbits of type \( n_{++} \). In the second case, we get at most two equivalent edge-orbits of type \( n \) consisting of pendant-edges attached to stabilized vertices. In the last case, we get an edge-orbit of type \( n_{++} \) and at most one edge-orbit of type \( n \) consisting of pendant-edges. The group \( \text{Aut}(G) \) can be constructed by the operation (ii) in the same way as in the case (d) in the proof of Lemma 5.7.

In the case (iii), we have two conjugacy classes of reflections. Each reflection stabilizes either two edges of the cycle, or two vertices, belonging to the same orbit; and this is same for each conjugacy class. Therefore, each conjugacy class defines either an edge-orbit of type \( n \) of attached single pendant edges, or an edge-orbit of type \( n_{++} \), non-equivariant to the edge-orbit defined by the other class. In total, we get three possibilities: two non-equivariant orbits of type \( n \), one edge-orbit of type \( n \) and one edge-orbit of type \( n_{++} \), or two non-equivariant edge-orbits of type \( n_{++} \). These three possibilities lead to different semidirect products in (iii) which are created by restrictions of the operations (e) from the proof of Lemma 5.8.

7 Comparison with Babai’s Characterization

In this section, we compare our characterization of automorphism groups of planar graphs with Babai’s characterization [1].

Statement of Babai’s Characterization. We include the full statement of Babai’s characterization of \( \text{Aut}(\text{PLANAR}) \), copied from [1] with an adapted notation.

Theorem 7.1 (Babai [1], 8.12 The Main Corollary). Let \( \Psi \) be a finite group. All graphs below are assumed to be finite.

(A) \( \Psi \) is representable by a planar graph if and only if
\[
\Psi \cong \Psi_1 \wr S_{n_1} \times \cdots \times \Psi_t \wr S_{n_t},
\]
for some \( t, n_1, \ldots, n_t \) where the groups \( \Psi_i \) are representable by connected planar graphs.
(A') $\Psi$ is representable by a planar graph with a fixed point if and only if $\Psi$ is representable by a planar graph.

(A'') $\Psi$ is representable by a planar fixed-point free graph if and only if a \textit{6} decomposition exists with all groups $\Psi$, possessing planar connected fixed-point-free graph representation.

(B) For $|\Psi| \geq 3$, $\Psi$ is representable by a connected planar graph if and only if

$$\Psi \cong (\Psi_1 \wr S_k) \wr (\Psi_2|_K)$$

for some positive integer $k$, where $\Psi_1$ should be representable by a connected graph having a fixed point; and either $\Psi_2$ is representable by a 2-connected planar graph $G_2$, or $k \geq 2$ and $|\Psi_2| = 1$. In the former case, $\Psi_2|_K$ denotes the not necessarily effective permutation group, acting on some orbit $K$ of $\text{Aut}(G_2)$.

(B') $\Psi$ is representable by a connected planar graph having a fixed point if and only if a \textit{7} decomposition exists as described under (B) with either $|\Psi_2| = 1$ or $G_2$ a 2-connected planar graph with a fixed point.

(B'') $\Psi$ is representable by a connected fixed-point-free planar graph if and only if a \textit{7} decomposition exists with $|\Psi_2| = 1$, $G_2$ fixed-point free (hence $|K| \geq 2$).

(C) $\Psi$ is representable by a 2-connected planar graph if and only if

$$\Psi \cong (\Psi_1 \wr S_k) \wr (\Psi_2|_K)$$

where $|\Psi_1| \leq 2$; $\Psi_2$ is representable by some 2-connected planar graph $G_2$. If $|\Psi_1| = k = 1$, $G_2$ should be 3-connected. $\Psi_2|_K$ denotes the action of $\Psi_2 \cong \text{Aut}(G_2)$, as a not necessarily effective permutation group, on $K$, an orbit of either an ordered pair $(a, b)$ or of an unordered pair $\{a, b\}$ of adjacent vertices $a, b \in V(G_2)$.

(C') $\Psi$ is representable by a 2-connected planar graph with a fixed point or with an invariant edge if and only if a \textit{8} decomposition exists such that $G_2$ has a fixed point or an invariant edge.

(C'') $\Psi$ is representable by a 2-connected planar fixed-point-free graph if and only if a \textit{9} decomposition exists with $G_2$ fixed-point free.

(D) $\Psi$ is representable by a 3-connected planar graph if and only if $\Psi$ is isomorphic to one of the finite symmetry groups of the 3-space:

$$C_n, \ D_n, \ A_4, \ S_4, \ A_5, \ C_n \times C_2, \ D_n \times C_2, \ A_4 \times C_2, \ S_4 \times C_2, \ A_5 \times C_2.$$  \hspace{1cm} (9)

(D') $\Psi$ is representable by a 3-connected planar graph with a fixed point if and only if $\Psi$ is a cyclic or a dihedral group.

(D'') $\Psi$ is representable by a 3-connected planar fixed-point-free graph if and only if $|\Psi| \geq 2$ and $\Psi$ is one of the groups listed under \textit{8}.

The characterization is very long and hard to understand, but it works in a nutshell as follows. The automorphism group of a $k$-connected planar graph ($k \leq 2$) is constructed by combining automorphism groups of smaller $k$-connected planar graphs with stabilizers of $k$-connected planar graphs and automorphism groups of $(k + 1)$-connected graphs.

The part (A) corresponds to Jordan’s Theorem 2.1. The automorphism groups listed in the part (D) are the spherical groups described in Section 3 and are based on the classical results from geometry. Therefore, the novel parts are (B) and (C). Unfortunately, it is not clear which groups are $\Psi_2|_K$, used in (7) and (8).

In principle, it would be possible to derive Theorem 7.1 from the described Jordan-like characterization of Theorems 5.1 and 5.10 and the reader can work out further details. The opposite is not possible because Jordan-like characterizations contains more information about automorphism groups of planar graphs; for instance the one given in Table 1. We note that Jordan-like characterizations for all parts of Theorem 7.1 exist. For example, the characterization of $\text{Aut}(2\text{-connected PLANAR})$ in Section 6.1 corresponds to the part (C) of Theorem 7.1.

**Group Products Instead of Group Extensions.** We explain why the simple version of our characterization given in Theorem 1.4 already describes the structure more accurately than Theorem 1.3.

The following idea was invented by Jordan. If $H$ is a normal subgroup of $G$, then we can understand $G$ by studying two smaller groups: the subgroup $H$ and the quotient group $G/H$. We repeat the same idea on both of these groups, till they cannot be further simplified, and such groups are called simple groups. We obtain a composition series

$$G = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_n = \{1\},$$

such that each quotient group $H_{i-1}/H_i$ is simple. We can imagine these simple quotient groups as building blocks which construct $G$; they play the role of prime numbers for groups. A consequence of Babai’s characterization (Theorem 1.3) describes these building blocks for the automorphism groups of planar graphs.
The celebrated classification of finite simple groups describes all building blocks for finite groups. Therefore, it describes the structure of all finite groups. But this description gives just a part of information, since it is not clear how these building blocks are “put together” to form more complex groups. This is called the group extension problem. The problem to describe all extensions of a group $H$ by $G/H$ is in general a hard problem. In particular, Theorem 1.3 does not describe how the building blocks for automorphism groups of planar graphs are put together.

In certain special cases, the structure of $G$ can be described from $H$ and $G/H$ by semidirect products. This happens exactly when there exists a complement subgroup $K \leq G$ isomorphic to $G/H$ such that $K \cap H = \{1\}$ and $\langle K \cup H \rangle = G$. Then we can build $G$ using the semidirect product:

$$G \cong H \rtimes K \cong H \rtimes G/H.$$ 

A simple version of our characterization (Theorem 1.4) states that the automorphism groups of planar graphs can be built from standard building blocks using a series of semidirect product, so it describes the structure more accurately than Theorem 1.3. In Theorems 5.1 and 5.10, we describe these semidirect products in more detail. As far as we understand Babai’s approach, he uses generalized wreath products instead of semidirect products. More important difference between ours and Babai’s approach consists in the fact that we deal with the 1- and 2-connected case together. This allows us to apply recursion in a more compact way, thus deriving the Jordan-like characterization of stabilizers of 1-cuts and 2-cuts, established in Theorem 5.1.

A group of symmetries of a graph is not fully described just by characterizing it as an abstract group. As far as we understand Babai’s approach, he uses generalized wreath products instead of semidirect products. More important difference between ours and Babai’s approach consists in the fact that we deal with the 1- and 2-connected case together. This allows us to apply recursion in a more compact way, thus deriving the Jordan-like characterization of stabilizers of 1-cuts and 2-cuts, established in Theorem 5.1.

Figure 20 shows two simple graphs with isomorphic abstract automorphism groups, realized by different group actions on the graphs. From Babai’s characterization, the structure of this action is not very clear. On the other hand, Theorems 5.1 and 5.10 reveal the actions of automorphism groups on planar graphs, by describing them with respect to each 1-cut and 2-cut (Theorem 5.1), and with respect to the primitive graph (Theorem 5.10).

8 Quadratic-time Algorithm

In this section, we describe a quadratic-time algorithm which computes the automorphism groups of planar graphs.

Lemma 8.1. Let $G$ be an essentially 3-connected planar graph with colored edges. There exists a quadratic-time algorithm which computes a generating set of $\text{Aut}(G)$ or of the stabilizer of a vertex.

Proof. Consider the unique embedding of $G$ into the sphere. Let $n$ be the number of vertices of $G$. We work with colored pendant edges as with colored vertices. Let $(v, e, e')$ be an arbitrary angle. For every other angle $(\hat{v}, \hat{e}, \hat{e}')$ there exists at most one automorphism $\pi \in \text{Aut}(M)$ which maps $(v, e, e')$ to $(\hat{v}, \hat{e}, \hat{e}')$. We introduce three involutions $\rho, \lambda, \tau$ on the set of oriented angles by setting:

- $\rho(v, e, e') = (v, e', e)$,
- $\lambda(v, e, e') = (v', e', e')$, where $v'$ is the other vertex incident to $e'$ and the angles $(v, e, e')$ and $(v', e', e')$ lie on the same side of $e'$, and
- $\tau(v, e, e') = (v, e, e'')$, where $(v, e, e'')$ is the other angle incident to $v$ and $e$.

The procedure checking whether the mapping $\pi: (v, e, e') \mapsto (\hat{v}, \hat{e}, \hat{e}')$ extends to an automorphism is based on the observation that an automorphism of $G$ commutes with $\rho, \lambda, \tau$. It follows that in time $O(n)$, we can check whether the mapping $\pi: (v, e, e') \mapsto (\hat{v}, \hat{e}, \hat{e}')$ extends to an automorphism or not. In the positive case, we get the automorphism $\pi$ as a byproduct. Moreover, we can easily verify whether the automorphism preserves colors. Therefore, $\text{Aut}(G)$ is computed in time $O(n^2)$ and the algorithm can easily identify which of the abstract spherical groups is isomorphic to $\text{Aut}(G)$. Note that the number of edges of $G$ is $O(n)$. For the stabilizer, we just compute the automorphisms which map $(v, e, e')$ to $(\hat{v}, \hat{e}, \hat{e}')$.

Notice that by the above algorithm, we can compute $\text{Fix}(\partial A)$ of a non-star block atom $A$. 

Fig. 20. Planar graphs $G$ and $G'$ with $\text{Aut}(G) \cong \text{Aut}(G') \cong C_2 \times C_2$, having different actions. For $G$, we get independent reflections/rotations for each of the blocks. For $G'$, the group $\text{Aut}(G')$ is generated by two reflections.
Lemma 8.2. There is a linear-time algorithm which computes a generating set of \( \text{Fix}(\partial A) \) and \( \text{Aut}_{\partial A}(A) \) of a proper atom, dipole, or a star block atom \( A \) with colored edges.

Proof. If \( A \) is a dipole or a star block atom, we get \( \text{Fix}(\partial A) \) as the direct product of symmetric groups, one for each type and color class of edges. Further, if \( A \) is a dipole, it is symmetric if and only if it has the same number of directed edges of each color class in both directions, and then \( \text{Aut}(A) \cong \text{Fix}(\partial A) \rtimes \mathbb{C}_2 \); otherwise \( \text{Aut}_{\partial A}(A) = \text{Fix}(\partial A) \).

Let \( A \) be a proper atom with \( \partial A = \{u, v\} \). Since \( A^+ \) is essentially 3-connected, the reasoning from Lemma 8.1 applies. We know that both \( \text{Aut}_{\partial A}(A) \) and \( \text{Fix}(\partial A) \) are generated by automorphisms which maps the two angles containing \( u \) and \( uv \) and the two angles containing \( v \) and \( uv \) between each other. We can easily test in time \( O(n) \) which of these mappings are automorphisms.

Proof (Proof of Theorem 5.3). Hopcroft and Tarjan [20] give a linear-time algorithm which computes the decomposition to 3-connected components, and it can be easily modified to our definition to output the reduction tree \( \mathcal{T} \) of the graph. In the beginning, all nodes of \( \mathcal{T} \) are unmarked. We process the tree from the leaves to the root, dealing with the nodes which have all children marked, and marking these nodes after. We compute colors and symmetry types of the considered atoms, the groups \( \text{Fix}(\partial A) \) and for symmetric atoms also involutions \( \tau \) exchanging the vertices in the boundaries. Let the colors be integers. Suppose that in some step, we process several atoms whose edges are colored and have computed symmetry types.

**Dipoles and star block atoms.** To each dipole/star block atom with \( n \) edges, we assign the vector \( \mathbf{v} = (t, c_1, \ldots, c_m) \) where \( t \) is the type of the atom and \( c_1, \ldots, c_m \) is the list of colors. By lexicographic sorting of these vectors for all dipole/star block atoms, we can compute isomorphism classes and assign new colors to them. This runs in linear time.

**Non-star block atoms.** Let \( A \) be a non-star block atom with \( \partial A = \{u\} \). We work with single pendant edges as with colors of vertices. Let \( n \) be the number of its vertices and \( m \) the number of its edges. Consider a map of \( A \). For each choice of an angle \( (u, e, e') \), we compute labellings \( 1, \ldots, n \) of the vertices and \( 1, \ldots, m \) of the edges as they appear in BFS of the map. Starting with \( u \), we visit all its neighbors, from the one incident with \( e \) following the rotational scheme. From each neighbor, we visit their unvisited neighbors, and so on.

For a labeling, we compute the vector \( \mathbf{v} = (c_1, \ldots, c_m, (x_1, y_1, c'_1), \ldots, (x_m, y_m, c'_m)) \) where \( c_i \) is the color of the \( i \)-th vertex, and \( x_i < y_i \) are the endpoints and \( c'_i \) the color of the \( j \)-th edge. We compute at most \( 2n \) vectors for all possible choices of \( (u, e, e') \) and we choose the one which is lexicographically smallest.

Notice that two atoms are isomorphic if and only if their associated vectors are identical. So, by sorting the chosen vectors for all non-star block atoms lexicographically, we compute isomorphism classes and assign new colors to them. This runs in quadratic time since we compute \( 2n \) vectors for each atom.

**Proper atoms.** We approach \( A^+ \) similarly as a non-star block atom, but we just need to consider the labellings starting from an angle containing \( uv \) and either \( u \), or \( v \). We have four choices for each vector, so it runs in linear time.

For each considered atom \( A \), we apply one of the algorithms described in Lemmas 8.1 and 8.2, and we compute \( \text{Fix}(\partial A) \). If \( A \) is a dipole or a proper atom, we also compute its type, and if \( A \) is symmetric, we construct an involution \( \tau \in \text{Aut}_{\partial A}(A) \) exchanging the vertices of \( \partial A \).

Following the proof of Theorem 5.1, we can compute in quadratic time also \( \text{Fix}(\partial A^+) \) and \( \tau^* \) for every node which is a child of the root node. By Theorem 5.10, we can compute \( \text{Aut}(G) \) as the semidirect product of these groups \( \text{Fix}(\partial A^+) \) with \( \text{Aut}(H) \) computed by Lemma 8.3. We can output the automorphism groups in terms of permutation generators, or by assigning the computed groups \( \text{Fix}(\partial A) \), \( \tau \) and \( \text{Aut}(H) \) to the corresponding nodes of \( \mathcal{T} \).

\[ \square \]

9 Conclusions

Let \( G \) be a connected planar graph. We propose the following way of imagining the action of \( \text{Aut}(G) \) geometrically, which can be used in a dynamic visualization in 3-space. Suppose that the reduction tree \( \mathcal{T} \) of \( G \) is computed together with the corresponding parts of \( \text{Aut}(G) \), assigned to the nodes. For each 2-connected block, we have some 3-connected colored primitive graph which can be visualized by a symmetric polytope, and these polytopes are connected by articulations as in the block tree; see Fig. 21.

Onto each polytope, we attach a hierarchical structure of colored atoms given by the decomposition. For a dipole \( A \) with \( \partial A = \{u, v\} \), we know that independent color classes can be

![Fig. 21. The bold edges are symmetric proper atoms \( A \) with \( \text{Fix}(\partial A) \cong \mathbb{C}_2 \), generated by \( \pi \). We visualize the automorphism group \( (D_5^4 \times C_2^3) \times S_4 \).](image-url)
arbitrarily permuted, so we assign symmetric groups to them. For a proper atom \( A \) with \( \partial A = \{u, v\} \), the non-trivial element of \( \text{Fix}(\partial A) \) (if it exists) is the reflection through \( u \) and \( v \), and we represent it. These symmetries generate the subgroup of \( \text{Aut}(G) \) which fixes all polytopes. Further, for an edge \( uv \) representing a symmetric atom \( A \), we also add an involution \( \tau \in \text{Aut}_{\partial A}(A) \) exchanging \( u \) and \( v \), which is geometrically a reflection through \( uv \) in \( A^* \), if \( \tau \) is used in \( \text{Aut}(G) \).

The central block is preserved by \( \text{Aut}(G) \), so it is transformed by a spherical group \( \text{Aut}(H) \), permuting also the attached polytopes. Multiple polytopes attached at an articulation correspond to a star block atom. So isomorphic subtrees of blocks can be arbitrarily permuted and we assign symmetric groups to them. Consider a polytope attached by the articulation \( u \) to its parent in the block tree. Since \( \text{Fix}(\partial A) \) of a non-star block atom is either a dihedral or a cyclic group, the polytope can be only rotated/reflected around \( u \).

**Problem 9.1.** For a planar graph \( G \), is it possible to compute a generating set of \( \text{Aut}(G) \) in a linear time?

Our algorithm has two bottlenecks which need to be improved to get a linear-time algorithm: (i) the algorithm of Lemma 8.1 (ii) the procedure of finding lexicographically smallest vectors of non-star block atoms. Both of these can likely be solved by modifying the algorithm of [21].

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