ITERATIVE METHODS FOR SOLVING LARGE SPARSE LYAPUNOV EQUATIONS AND APPLICATION TO MODEL REDUCTION OF INDEX 1 DIFFERENTIAL-ALGEBRAIC-EQUATIONS

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Abstract. To implement the balancing based model reduction of large-scale dynamical systems we need to compute the low-rank (controllability and observability) Gramian factors by solving Lyapunov equations. In recent time, Rational Krylov Subspace Method (RKSM) is considered as one of the efficient methods for solving the Lyapunov equations of large-scale sparse dynamical systems. The method is well established for solving the Lyapunov equations of the standard or generalized state space systems. In this paper, we develop algorithms for solving the Lyapunov equations for large-sparse structured descriptor system of index-1. The resulting algorithm is applied for the balancing based model reduction of large sparse power system model. Numerical results are presented to show the efficiency and capability of the proposed algorithm.

1. Introduction. Lyapunov or Lyapunov-like equations play important roles in various disciplines of science and engineering such as system and control theory, optimization, linear algebra, differential equations, boundary value problem, signal processing, power system, structural dynamics and so on (see, e.g., [29, 15, 27, 20]). In control theory besides the stability analysis and stabilization of systems, Lyapunov equations are also used in computing balancing transformation, Gramian based model reduction, $H_2$ optimal control and Riccati based optimal control. Due to broad applications, solution of large-scale Lyapunov matrix equations has currently been the center of attention for research in control theory, dynamical system, and other fields.

This paper discusses the low-rank approximate solutions of continuous-time algebraic Lyapunov equations of the form

\begin{align}
APE^T + EPA^T + BB^T &= 0, \\
A^TQE + E^TQA + C^TC &= 0,
\end{align}

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where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$ and $p, m \ll n$. The solutions $P$ and $Q$ are respectively known as the controllability and observability Gramians of the linear time-invariant (LTI) continuous-time system

$$
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \quad t \geq t_0 \\
y(t) &= Cx(t) + Du(t),
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the vector of states, $u(t) \in \mathbb{R}^p$ is the input vector and $y(t) \in \mathbb{R}^m$ is the output vector. Dynamical system (3) is extracted from the differential-algebraic equations (DAEs)

$$
\begin{align*}
E_1 \dot{x}(t) &= J_1 x(t) + J_2 z(t) + B_1 u(t) \\
0 &= J_3 x(t) + J_4 z(t) + B_2 u(t) \\
y(t) &= C_1 x(t) + C_2 z(t) + D_z u(t),
\end{align*}
$$

by eliminating the algebraic variables i.e., $z(t) \in \mathbb{R}^l$ of the system. The coefficient matrices in (3) are defined as follows:

$$
E := E_1, \quad A := J_1 - J_2 J_4^{-1} J_3, \quad B := B_1 - J_2 J_4^{-1} B_2, \\
C := C_1 - C_2 J_4^{-1} J_3, \quad D := D_z - C_2 J_4^{-1} B_2.
$$

The dynamical system (3) and (4) are equivalent since the responses of the systems are same and their finite eigenvalues coincide. Although we are particularly interested in power system model [12] here, such a descriptor system also appears in RLC circuit for modified nodal analysis [13]. If the mathematical models are very large, their simulation within a limited time-frame is infeasible. Therefore, it is suggested to replace the higher dimensional model by a substantially lower dimensional model. Model reduction is a process of converting the large-scale model (4) into small-scale model:

$$
\begin{align*}
\hat{E} \dot{\hat{x}}(t) &= \hat{A} \hat{x}(t) + \hat{B} u(t), \\
\hat{y}(t) &= \hat{C} \hat{x}(t) + \hat{D} u(t),
\end{align*}
$$

where $\hat{E}, \hat{A} \in \mathbb{R}^{k \times k}$, $\hat{B} \in \mathbb{R}^{k \times p}$, $\hat{C} \in \mathbb{R}^{m \times k}$ and $\hat{D} \in \mathbb{R}^{m \times p}$ ($k \ll n$).

Among several methods the system theoretic method balanced truncation (BT) has recently become prominent for the model reduction of large-scale sparse dynamical system. Besides the stability preservation, the method also guarantees the global error bound. These essentially make the method superior to some other existing method. To implement this method one has to solve two Lyapunov equations which is an expensive computational task. Two efficient methods, namely, the LR-ADI (low-rank-alternating direction implicit) method and the Krylov subspace method are frequently used for the solution of Lyapunov equations of large-scale sparse systems. Both methods can compute low-rank factors of the approximate solutions of the Lyapunov equations.

Freitas et. al. in [12] discusses a balancing based method for the model reduction of index-1 descriptor system (4). They apply the LR-ADI to solve the Lyapunov equations. We know that LR-ADI can only be applied to asymptotically stable systems. Note that the model that we use for the numerical test (which is also used in [12]) is not asymptotically stable. To circumvent the problem, authors in [12] used the so-called $\alpha$-shift approach to stabilize the system. In this paper, to solve the Lyapunov equations, we apply the Rational Krylov Subspace Method (RKSM) introduced by V. Simoncini (for standard system) in [11]. In this method, we are not required to stabilize the system using the $\alpha$-shift. In addition, unlike
here we use an efficient technique to compute the shift parameters for better convergence of the RKSM. The computed low-rank Gramian factors are then applied to the balancing based model reduction. The proposed theories are applied to find the model reduction of several sized systems of Brazilian power system models. Numerical results are discussed to show the efficiency and capability of the proposed method.

2. Balanced truncation and low-rank RKSM for generalized systems. In this section, we briefly review some basic concepts and results including BT and RKSM for the solutions of Lyapunov equations that will be used throughout this paper.

Consider the linear time-invariant (LTI) continuous-time state space system
\[\Sigma : \begin{cases} \dot{\eta}(t) = A\eta(t) + BU(t), \\ y(t) = C\eta(t) + Du(t), \end{cases} \tag{7}\]
where \(E, A \in \mathbb{R}^{n_g \times n_g}, B \in \mathbb{R}^{n_g \times p}, C \in \mathbb{R}^{m \times n_g}, D \in \mathbb{R}^{m \times p}\), state \(\eta(t) \in \mathbb{R}^{n_g}\), input \(u(t) \in \mathbb{R}^p\) and output \(y(t) \in \mathbb{R}^m\). Assume that the matrix \(E\) is non-singular.

To perform the BT method of system (7) we need to carry out the following steps.

At first we compute the low-rank controllability Gramian factor
\[R := P \approx RR^T \quad \text{and} \quad L := Q \approx LL^T \]
by solving the two Lyapunov equations
\[A^TQ + EQA = -C^TC, \tag{10}\]
\[AP \Sigma^T + \Sigma^TAP = -BB^T \quad \text{and} \quad A^TQ \Sigma + \Sigma^TA^T = -C^TC. \tag{11}\]
respectively. For stable \(A\), they have a unique positive definite solutions \(P\) and \(Q\).

In BT method, the system is transformed to the balanced structure in which the controllability and observability Gramians are both equal and diagonal.

The singular value decomposition (SVD) of Gramian factors can be computed as:
\[R^T\Sigma L = \Phi \Sigma \Gamma^T = [\Phi_1 \quad \Phi_2]\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} \begin{bmatrix} \Gamma_1^T \\ \Gamma_2 \end{bmatrix}, \tag{12}\]
where block matrices \(\Phi_1\) and \(\Gamma_1\) are assumed to be constructed by corresponding leading size \(k(< n)\) of the orthogonal matrices \(\Phi\) and \(\Gamma\) respectively. Furthermore, \(\Sigma\) is a diagonal matrix with Hankel singular values (HSVs) \([1]\) in decreasing order, i.e., \(\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, \ldots, \sigma_n)\) and \(\sigma_{i+1} \leq \sigma_i\) where \(\sigma_i, i = 1, 2, \ldots, n\) are HSVs. Here \(\Sigma_1\) preserves the first \(k \times k\) block. The reduced order model can be easily obtained by truncating the states which are associated with the set of the least HSVs.

The system in the balanced state space form is obtained by applying transformation matrix. Therefore, left and right balancing and truncating transformations \([21]\) are formed by
\[T_L := R\Phi_1\Sigma_1^{-\frac{1}{2}}, \quad T_R := L\Gamma_1\Sigma_1^{-\frac{1}{2}}. \tag{13}\]

Applying transformations \(T_L\) and \(T_R\) in (7), we obtain \(k\) dimensional reduced-order model
\[\hat{\Sigma} : \begin{cases} \hat{\eta}(t) = \hat{A}\hat{\eta}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{\eta}(t) + \hat{D}u(t), \end{cases} \tag{14}\]
Algorithm 1: Generalized square-root balanced truncation

Input : \( E, A, B, C, D \)

Output: Matrices \( \hat{E}, \hat{A}, \hat{B}, \hat{C} \) and \( \hat{D} \) of stable reduced-order model.

1. Compute low-rank solution factors \( R, L \) of the system Gramians;
2. Compute and partition a (thin) singular value decomposition 
\[
R^T EL = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Sigma_1 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}^T;
\]
3. Construct \( T_L := R \Phi_1 \Sigma_1^{-\frac{1}{2}} \) and \( T_R := L \Gamma_1 \Sigma_1^{-\frac{1}{2}} \);
4. Compute the reduced-order system 
\[
\hat{E} := T_L^T E T_R, \quad \hat{A} := T_L^T A T_R, \quad \hat{B} := T_L^T B, \quad \hat{C} := CT_R, \quad \hat{D} := D.
\]

where the reduced matrices are formed as follows:
\[
\hat{E} = T_L^T E T_R, \quad \hat{A} = T_L^T A T_R, \quad \hat{B} = T_L^T B, \quad \hat{C} = CT_R, \quad \hat{D} = D.
\]

The method ensures that it satisfies the global error bound \[1, 14\]
\[
\|G(s) - \hat{G}(s)\|_{H_\infty} \leq 2 \sum_{i=k+1}^{n} \sigma_i.
\] (13)

Here, \( G(s) = C(sE - A)^{-1}B \) and \( \hat{G}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \) are the transfer function of the full and reduced models respectively, and \( \|\cdot\|_{H_\infty} \) denotes the \( H_\infty \) norm. Given a error tolerance, one can easily truncate the required HSVs and determine the reduced-order model. This means, for a given system, the method can generate a best approximate system. The above procedure is often referred to as square-root method (SR) for balanced truncation. The resulting algorithm is summarized in Algorithm 1.

The only disadvantage of the BT algorithm is to solve two continuous-time Lyapunov equations. Over the last few decades, several iterative methods were proposed to solve large-scale Lyapunov equations e.g. LRCF-ADI (low-rank Cholesky factor alternating direction implicit) iterations \[5, 18, 22, 23\], cyclic low-rank Smith methods \[16, 24\], projection methods \[10, 17, 19, 28, 26, 31\] and sign function methods \[3, 6, 7, 9\]. Although most of the methods are shown to be applicable for large-scale sparse dynamical systems, the LRCF-ADI iteration \[5, 30\] and Krylov subspace based projection method \[28, 32\] are particularly advantageous in the context of Gramian based model reduction. These approaches have been developed so as to allow exploiting the sparsity of the matrices. Moreover, both methods have low-rank version for computing the low-rank Gramian factors. This is important since Gramians can be approximated by there low-rank factors if the number of inputs and outputs are significantly less than the number of DoFs (Degree of Freedoms). Therefore, instead of computing the full Gramian, the low-rank Gramian factor is computed which is much cheaper. In computing low-rank factors \( R \) and \( L \), we concentrate on rational Krylov subspace method which was originally introduced in \[11\] and the authors in \[11\] proposed the method for standard systems. This technique is favorable since it is more efficient than LR-ADI. In the following we briefly discuss RKSM for generalized system. Since the Lyapunov equations \( (8) \) and \( (9) \) are the dual of each other we focus only on the Lyapunov equation \( (8) \) elaborately.
Algorithm 2: Rational Krylov subspace method for Gramian.

Input : $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$, $\mu_1$ (initial shift) and tolerance $\epsilon$.
Output: $\mathcal{R}$ such that $\mathcal{P} \approx \mathcal{R}\mathcal{R}^*$.  

1. Compute $v_1 = (\mathcal{A} - \mu_1 \mathcal{E})^{-1}\mathcal{B}$, $V_1 = \frac{v_1}{\|v_1\|}$;

2. while $m \leq i_{\text{max}}$ do
   3. Find $v_{m+1} = (\mathcal{A} - \mu_{m+1} \mathcal{E})^{-1}v_m$ and compute new shifts if store is empty;
   4. Orthogonalize $v_{m+1}$ against $V_m$ to obtain $V_{m+1}$;
   5. $V_{m+1} = [V_m, v_{m+1}]$;
   6. Solve the small Lyapunov equation
      \[ \tilde{\mathcal{A}}\tilde{\mathcal{P}}\tilde{\mathcal{E}}^T + \tilde{\mathcal{E}}\tilde{\mathcal{P}}\tilde{\mathcal{A}}^T = -\tilde{\mathcal{B}}\tilde{\mathcal{B}}^T, \]
      for $\tilde{\mathcal{P}}$ where $\tilde{\mathcal{E}} = V_{m+1}^*\mathcal{E}V_{m+1}$, $\tilde{\mathcal{A}} = V_{m+1}^*AV_{m+1}$ and $\tilde{\mathcal{B}} = V_{m+1}^*\mathcal{B}$;
   7. Compute the norm of the residual $\delta$ as defined in Proposition 1;
   8. if $\delta \leq \epsilon$ then
      9. Stop
   10. Compute eigenvalue decomposition $\mathcal{P} = \mathcal{T}\Pi\mathcal{T}^* = [\mathcal{T}_1, \mathcal{T}_2] \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} [\mathcal{T}_1^*, \mathcal{T}_2^*]$;
   11. Construct full Gramian factor $\mathcal{R} = V_{m+1}\mathcal{T}_1\Pi_1^{-1}$ by truncating small eigenvalues $\Pi_2$.

The RKSM finds the low-rank Gramian factor $\mathcal{R}$ to the approximate solution $(\mathcal{P}_a)$ of the Lyapunov equation (8), in other words -
\[ \mathcal{P} \approx \mathcal{P}_a = \mathcal{R}\mathcal{R}^*. \]

The method is carried out by projecting the system onto a lower-dimensional rational Krylov subspace which can be generated iteratively. The $m$ dimensional rational Krylov subspace can be constructed for a set of given shift parameters $\mu_i \in \mathbb{C}; i = 1, 2, \cdots, m$ as
\[ \mathcal{K}_m := \text{span}\left\{ (\mathcal{A} - \mu_1 \mathcal{E})^{-1}\mathcal{B}, \cdots, \prod_{i=1}^{m} (\mathcal{A} - \mu_i \mathcal{E})^{-1}\mathcal{B} \right\}. \]

Applying the modified Gram-Schmidt orthogonalization procedure, one can generate an orthogonal basis for the rational Krylov subspace $\mathcal{K}_m$. Let $\mathcal{V}$ be a matrix whose orthonormal basis span $\mathcal{K}_m$ in such a way that
\[ \text{Range}(\mathcal{V}) = \text{span}\left\{ \prod_{i=1}^{m} (A - \mu_i \mathcal{E})^{-1}\mathcal{B} \right\}. \]

By imposing the Galerkin condition (i.e., $\mathcal{V}\mathcal{V}^T = I$), we can find an approximate solution of the projected Lyapunov equation
\[ \mathcal{V}^*(\mathcal{A}\mathcal{P}\mathcal{E}^T + \mathcal{E}\mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T)\mathcal{V} = 0. \] (14)

Now consider $\hat{\mathcal{P}} = \mathcal{V}^*\mathcal{P}\mathcal{V}$, $\hat{\mathcal{E}} = \mathcal{V}^*\mathcal{E}\mathcal{V}$, $\hat{\mathcal{A}} = \mathcal{V}^*\mathcal{A}\mathcal{V}$ and $\hat{\mathcal{B}} = \mathcal{V}^*\mathcal{B}$; then equation (14) yields a small-scale Lyapunov equation
\[ \hat{\mathcal{A}}\hat{\mathcal{P}}\hat{\mathcal{E}}^T + \hat{\mathcal{E}}\hat{\mathcal{P}}\hat{\mathcal{A}}^T = -\hat{\mathcal{B}}\hat{\mathcal{B}}^T, \] (15)

which can be solved easily by any exact solver e.g. Bartels-Stewart method [2]. Since $\mathcal{P}$ is symmetric and positive (semi-)definite matrix, it can be factorized as
\( \tilde{P} = SS^* \). Then, the approximate solution is \( P_a = V \tilde{P} V^* \). Hence, \( P_a = RR^* \) implies
\[
P_a = RR^* = (VS)(VS)^*.
\]
Consequently, we can write
\[
R = VS.
\]
Applying the eigenvalue decomposition to the matrix \( \tilde{P} \) and truncating the negligible eigenvalues, which ensure that the computed Gramian factor \( R \) has the smallest possible number of columns.

The method can be stopped at the \( i \)-th iteration if
\[
\frac{\| R_e(P_a) \|_F}{\| BB^T \|_F} \leq \text{tol},
\]
where \( \| . \|_F \) denotes Frobenius norm and the residual at \( m \)-th iteration step is
\[
R_e(P_a) = AP_aE^T + EP_aA^T + BB^T.
\]
(16)
The residual norm can be computed cheaply using the following observation.

**Proposition 1.** Let \( V_m \) be the orthogonal basis of \( m \) dimensional rational Krylov subspace \( K_m \) and \( P_a = V_m \tilde{P} V_m^* \) be the approximate solution of the Lyapunov equation (8). Then the residual \( R_e \) of (16) can be computed by the Frobenius norm
\[
\| R_e \|_F = \| UJU^T \|_F, \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]
where \( U \) is the 3 \( \times \) 3 upper triangular matrix in the QR factorization of
\[
Z = [v_m + \mu_{m+1}, \quad EV_m \tilde{P} H_m e_m h_{m+1,m}, \quad -(I - V_m V_m^*)A v_{m+1}].
\]
Here, \( H_m \) is the \( m \times m \) block upper Hessenberg matrix, \( h_{m+1,m} \) denotes the last row of \( H_{m+1,m} \) and \( e_m \) is the matrix of the last \( p \) columns of the \( mp \times mp \) identity matrix i.e. \( H_{m+1,m} = [H_m; h_{m+1} e_m^T] \).

**Proof.** Following Arnoldi algorithm in [25], we get
\[
(A - \mu_j + 1 E)^{-1} v_j = V_{j+1} H_{j+1,j}, \quad j = 1, \cdots, m.
\]
After some derivation (see, [25]), we can write
\[
AV_m = V_m T_m + v_{m+1} h_{m+1,m} e_m^T D m H_m^{-1} - f h_{m+1,m} e_m^T H_m^{-1},
\]
where \( f = (I - V_m V_m^*) A v_{m+1} \) and \( D_m = \text{diag}(\mu_2, \cdots, \mu_{m+1}) \).

Using results, the residual in (16) can be written as
\[
R_e = AV_m \tilde{P} V_m^* E^T + EV_m \tilde{P} V_m^* A^T + BB^T.
\]
current subspace have been used, the next set of shifts can be preserved by following the range of $V$ projecting onto the subspace $B$ and $C$ solving the Lyapunov equation (8) is summarized in Algorithm 2. On the other task for fast convergence of the RKSM. Unlike [11], we propose an efficient technique to compute shift parameters. In [8] a simple and efficient shift computation strategy has been proposed which was a modified version of the strategy in [4].

A set of optimal shift parameters $\{\mu_i\}_{i=1}^k$ can be computed by solving the rational $\min - \max$ problem [24]. Therein, shift parameters are generated automatically by the algorithm itself. The eigenvalues of the matrix pencil $(\mu E - A)$ is used by projecting onto the subspace $V_m$ span $K_m$. Whenever all the current set of shifts have been used, the next set of shifts can be preserved by following the range of current subspace $V_m$. If any eigenvalues are in $C^+$, we project them on the negative real part of the complex plane.

The entire process of finding the low-rank controllability Gramian factor $R$ by solving the Lyapunov equation (8) is summarized in Algorithm 2. On the other hand, equation (8) can be converted into (9) if we consider $(E, A, B) = (E^T, A^T, C^T)$. Therefore, Algorithm 2 can be applied to solve the Lyapunov equation (9) for computing the low-rank observability Gramian factor $L$ replacing the inputs $E$, $A$ and $B$ by $E^T$, $A^T$ and $C^T$ respectively. Proper shift computation is another important task for fast convergence of the RKSM. Unlike [11], we propose an efficient technique to compute shift parameters. In [8] a simple and efficient shift computation strategy has been proposed which was a modified version of the strategy in [4].

3. Balanced truncation and low-rank RKSM for index-1 DAEs. In this section we return back to the index-1 descriptor system as in (4). In Section 1, we showed that the index-1 descriptor system (4) can be converted into the generalized state space system (3). Hence, we can apply the procedure of balanced truncation by following Section 2. However, due to the computational complexity we cannot form the ODE system explicitly. In this section we will show the implementation of the whole balanced truncation procedure of index-1 system (4) without forming the ODE system (3) explicitly.

3.1. Computation of low-rank Gramian factors. To implement the balanced truncation of the underlying system, we need to compute the low-rank Gramian factors by solving the Lyapunov equations (1) and (2). At first we consider the Lyapunov equation (1). In the following, we discuss how to solve it efficiently by applying Algorithm 2.
Algorithm 3: RKSM for index-1 descriptor system.

**Input:** $E_1, J_1, J_2, J_3, J_4, B_1, B_2, \mu_1$ (initial shift) and tolerance $\epsilon$.

**Output:** $R$ such that $P \approx RR^T$.

1. Compute $v_1$ by solving (17) and form $V_1 = \frac{v_1}{\|v_1\|}$.
2. while $m \leq i_{\text{max}}$ do
3. 
4. Find $v_{m+1}$ by solving (18) and compute new set of shifts if store is empty as Section 2;
5. Compute $\tilde{E} = V_{m+1}^*E_1V_{m+1}$, $\tilde{A} = V_{m+1}^*J_1V_{m+1} - (V_{m+1}^*J_2)J_4^{-1}(J_3V_{m+1})$ and $\tilde{B} = V_{m+1}^*B_1 - (V_{m+1}^*J_2)J_4^{-1}B_2$;
6. Solve the small Lyapunov equation, $\tilde{A}\tilde{P}\tilde{E} + \tilde{E}\tilde{P}\tilde{A}^* = -\tilde{B}\tilde{B}^*$, for $\tilde{P}$;
7. Compute the norm of the residual $\delta$ as defined in Proposition 1;
8. if $\delta \leq \epsilon$ then
9. Stop
10. Compute eigenvalue decomposition $\tilde{P} = T\Pi T^*$ = $[T_1 \quad T_2]$ $[\Pi_1 \quad 0 \quad \Pi_2]$ $[T_1^* \quad T_2^*]$;
11. Truncate small eigenvalues $\Pi_2$ and construct the full Gramian factor $R = V_{m+1}T_1\Pi_1^2$.

Following Step 1 of Algorithm 2, in our case $v_1 = (A - \mu_1E)^{-1}B$. This can be computed by solving the linear system

$$(A - \mu_1E)v_1 = B.$$ 

Plugging $E$, $A$ and $B$ from (5) and algebraic manipulation will turn to solve the linear system

$$\begin{bmatrix} J_1 - \mu_1E_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} v_1 \\ * \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{17}$$

Similarly, to follow Step 3 of Algorithm 2, we can compute $V_{m+1}$ by solving the linear system

$$\begin{bmatrix} J_1 - \mu_{m+1}E_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} v_{m+1} \\ * \end{bmatrix} = \begin{bmatrix} v_m \\ 0 \end{bmatrix}. \tag{18}$$

Applying these identities, a complete procedure for computing the low-rank controllability Gramian factor $R$ such that $P \approx RR^T$ by solving Lyapunov equation (1) is presented in Algorithm 3.

Note that, same algorithm (i.e. Algorithm 3) can also be applied for solving equation (2) to compute the low-rank observability Gramian factor $L$ such that $Q \approx LL^T$. For this purpose, the inputs $E_1, J_1, J_2, J_3, J_4, B_1$ and $B_2$ are changed to $E_1^T, J_1^T, J_2^T, J_3^T, J_4^T, C_1^T$ and $C_2^T$ respectively.

1 can be solved by MATLAB lyap command.
Algorithm 4: Balanced truncation for index-1 descriptor system

**Input**: $E_1$, $J_1$, $J_2$, $J_3$, $J_4$, $B_1$, $B_2$, $C_1$, $C_2$ and $D_a$.

**Output**: Matrices $\hat{E}$, $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$.

1. Compute low-rank Gramian factors $R$ and $L$ by solving the Lyapunov equations (1) and (2), respectively.

2. Compute and partition a (thin) singular value decomposition
   \[ R^T E_1 L = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}^T; \]

3. Construct matrices $T_L := R \Phi_1 \Sigma_1^{-\frac{1}{2}}$ and $T_R := L \Gamma_1 \Sigma_1^{-\frac{1}{2}}$;

4. Form reduced-order matrices as follows:
   \[
   \begin{align*}
   \hat{E} & := T_L^T E_1 T_R, \quad \hat{A} := \hat{J}_1 - \hat{J}_2 J_4^{-1} \hat{J}_3, \quad \hat{B} := \hat{B}_1 - \hat{J}_2 J_4^{-1} B_2, \\
   \hat{C} & := \hat{C}_1 - C_2 J_4^{-1} \hat{J}_3, \quad \hat{D} := D_a - C_2 J_4^{-1} B_2,
   \end{align*}
   \]
   where $\hat{J}_1 = T_L^T J_1 T_R$, $\hat{J}_2 = T_L^T J_2$, $\hat{J}_3 = J_3 T_R$, $\hat{B}_1 = T_L^T B_1$ and $\hat{C}_1 = C_1 T_R$.

3.2. Balanced truncation for index-1 descriptor system. We solve two Lyapunov equations (1) and (2) for computing the low-rank controllability Gramian factor and observability Gramian factor efficiently, by following the above procedure. Then, following Step 2 to Step 3 of Algorithm 1, construct the balancing and truncating transformations and construct the reduced model as in (6). The complete procedure is summarized in Algorithm 4.

4. Numerical experiments. This section presents some results of numerical experiments to demonstrate the effectiveness of our proposed methods for index-1 descriptor systems. We apply the technique to several practical test systems derived from Brazilian Interconnected Power System (BIPS) models introduced in [12].

All of the following results were carried out in MATLAB 8.1 (R2013b) on an Intel Core i3 @ 1.70 GHz clock, RAM 8GB with relative machine precision $\epsilon = 1.7 \times 10^{-16}$.

The authors in [12] generate different sizes of large-scale descriptor power system models. Table 1 shows some information for the considered data i.e. different sizes of the models. It is shown that all models have 4 inputs and 4 outputs.

The proposed method (i.e. Algorithm 3) is applied to all the models for computing their low-rank Gramian factors separately. For all models, 6 proper shift parameters are selected in each cycle, following the shift computation strategy. For initial shift, consider the largest eigenvalue of pair $(-A, E)$ using MATLAB eigs. Each shifted linear system like $(A - \mu E)$ is solved using a sparse direct solver (backslash in MATLAB). For stopping criterion, the relative residual Frobenius norm is considered as Proposition 1. In all experiments, $10^{-11}$ is used as a stopping tolerance.

Applying the balanced truncation procedure (i.e. Algorithm 4), the corresponding reduced-order models are computed for each original model examples. Table 2 reports the comparison of the original and reduced models by showing their absolute and relative deviations with truncation tolerance $10^{-2}$.

For convenience, we illustrate only one test system mod-2 ($n = 9735$) since the results are similar for all BIPS models. Applying Algorithm 3, we compute the controllability Gramian factor $R \in \mathbb{C}^{1142 \times 198}$ and the observability Gramian factor
Table 1. Number of differential & algebraic variables and largest eigenvalue of \((-A, E)\) for different models.

| Model | differential | algebraic | eigs\((-A, E)\) | inputs/outputs |
|-------|--------------|-----------|----------------|----------------|
| Mod-1 | 606          | 6529      | 10727          | 4/4            |
| Mod-2 | 1142         | 8593      | 10727          | 4/4            |
| Mod-3 | 3078         | 18050     | 10669          | 4/4            |

Table 2. Comparisons between full systems and their reduced models.

| Model | Dimension | Error       |
|-------|-----------|-------------|
|       | full      | reduced     | absolute error | relative error |
| Mod-1 | 7135      | 87          | 3.1 \times 10^{-3} | 1.5 \times 10^{-4} |
| Mod-2 | 9735      | 86          | 5.3 \times 10^{-2} | 4.7 \times 10^{-4} |
| Mod-3 | 21128     | 77          | 5.6 \times 10^{-1} | 4.3 \times 10^{-2} |

Table 3. Balanced truncation tolerances and dimensions of reduced-order model.

| Model | tolerance | dimension of ROM |
|-------|-----------|------------------|
|       | \(10^{-4}\) | 118              |
|       | \(10^{-3}\) | 104              |
| Mod-2 | \(10^{-2}\) | 86               |
|       | \(10^{-1}\) | 70               |

Figure 1. Convergence histories of both Gramians by RKSM for mod-2.

\(L \in \mathbb{C}^{1142 \times 153}\) for mod-2. The stopping criterion was monitored at every iteration. The convergence history for controllability Gramian and observability Gramian are shown in Figure 1 with 40 iterations and ensure good performance.
Later following Algorithm 4, we truncate the Hankel singular values using SVD (MATLAB: \texttt{svd}) with inner truncation tolerance $10^{-2}$, and preserve 86 Hankel singular values. Using these HSVs, construct the transformation matrices $T_L \in \mathbb{C}^{1142 \times 86}$ and $T_R \in \mathbb{C}^{1142 \times 86}$. Finally, we approximate the original system by 86 dimensional reduced-order system matrices ($\hat{E}, \hat{A}, \hat{B}, \hat{C}$). Table 3 shows that the dimension of the reduced-order model can be decreased or increased by changing (increasing or decreasing) the truncation tolerance if desired or required.

Figure 2a depicts the maximum singular values of the transfer function matrix $G(j\omega) \in \mathbb{C}^{4 \times 4}$ for the full model and $\hat{G}(j\omega) \in \mathbb{C}^{4 \times 4}$ for reduced-order model in frequency domain $\omega \in (10^{-2}, 10^2)$ respectively. Good matching ensures that the 86 state reduced-order model is highly accurate. The absolute and relative deviations of the original and reduced models are depicted in Figure 2b and 2c respectively and these are below $10^{-4}$.

To compute time responses, we use an implicit Euler method with fixed time step size of $10^{-2}$. Figure 3a shows the time response of the original model and the reduced-order model; their absolute deviations are shown in Figure 3b. From these figures, we can conclude that the generated reduced-order model ensures good
quality and hence they can be used for controller design, simulation and optimization. Figure 4 shows that Hankel singular values of full (i.e. 1142 state) system and computed 86 state reduced system are approximately same.
5. **Conclusions.** We have introduced a method to compute the low-rank Gramian factors by solving the Lyapunov equations that arise from the large-scale sparse index-1 descriptor systems. The method was based on the rational Krylov subspace method which originated in [11] for standard state space system. We have also discussed shift computation strategy for better convergence of the RKSM. The computed low-rank Gramian factors have been applied to the balancing based model reduction of the underlying descriptor system. We have applied the proposed technique to several test systems derived from Brazilian Interconnected Power System (BIPS) models found in [12]. The efficiency and accuracy of technique have been discussed through numerical results.

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