Newton Type Methods for solving a Hasegawa-Mima Plasma Model

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Abstract

In [1], the non-linear space-time Hasegawa-Mima plasma equation is formulated as a coupled system of two linear PDEs, a solution of which is a pair \((u, w)\), with \(w = (I - \Delta)u\). The first equation is of hyperbolic type and the second of elliptic type. Variational frames for obtaining weak solutions to the initial value Hasegawa-Mima problem with periodic boundary conditions were also derived. In a more recent work [2], a numerical approach consisting of a finite element space-domain combined with an Euler-implicit time scheme was used to discretize the coupled variational Hasegawa-Mima model. A semi-linear version of this implicit nonlinear scheme was tested for several types of initial conditions. This semi-linear scheme proved to lack efficiency for long time, which necessitates imposing a cap on the magnitude of the solution.

To circumvent this difficulty, in this paper, we use Newton-type methods (Newton, Chord and an introduced Modified Newton method) to solve numerically the fully-implicit non-linear scheme. Testing these methods in FreeFEM++ indicates significant improvements as no cap needs to be imposed for long time. In the sequel, we demonstrate the validity of these methods by proving several results, in particular the convergence of the implemented methods.

Keywords: Hasegawa-Mima; Periodic Sobolev Spaces; Petrov-Galerkin Approximations; Finite-Element Method; Implicit Euler; Newton-type methods

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1 Introduction

In this paper, we consider the Hasegawa-Mima equation [3, 4], given by (1)

\[-\Delta u_t + u_t = \{u, \Delta u\} + \{p, u\}\]

where \(\{u, v\} = u_x v_y - u_y v_x\) is the Poisson bracket, \(u(x, y, t)\) describes the electrostatic potential, \(p(x, y) = \ln n_0 / \omega_{ci}\) is a function depending on the background particle density \(n_0\) and the ion cyclotron frequency \(\omega_{ci}\), which in turn depends on the initial magnetic field.

In [1], the Hasegawa-Mima model on a square domain with periodic boundary conditions, is reformulated as a hyperbolic-elliptic coupled system of PDEs, where a new variable \(w = -\Delta u + u\) is introduced, leading to (2)

\[
\begin{cases}
    w_t + \vec{V}(u) \cdot \nabla w = \{p, u\} = \vec{V}(p) \cdot \nabla u & \text{on } \Omega \times (0, T) \\
    -\Delta u + u = w & \text{on } \Omega \times (0, T) \\
    \text{PBC's on } u, u_x, u_y, w & \text{on } \partial\Omega \times [0, T] \\
    u(0) = u_0 \text{ and } w(0) = w_0 & \text{on } \Omega.
\end{cases}
\]

where \(\vec{V}(u) = -u_y \hat{i} + u_x \hat{j}\) is a \textit{divergence-free vector field} \((\text{div}(\vec{V}(u)) = 0)\).

The full discretization of the coupled system were obtained in [2] where starting with the given initial condition at \(t = 0\), the subsequent solutions are approximated for a chosen time interval \(\tau\), to reach the end time \(T\) in a finite number of steps. These can be summarized in the following three sections.

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1.1 Time Integral Variational Formulation

The skew-symmetry property, \( \langle \tilde{V}(u) \cdot \nabla v, w \rangle_2 = -\langle \tilde{V}(u) \cdot \nabla w, v \rangle_2 \), \( \forall u \in H^2 \cap H^1_p, v \in W^{1,\infty} \cap H^1_p, w \in H^1_p \), in addition to the integration of

\[ \langle w, v \rangle_2 = \langle \tilde{V}(u) \cdot \nabla v, w \rangle_2 + \langle \tilde{V}(p) \cdot \nabla u, v \rangle_2, \]
on the interval \([t, t + \tau]\) leads to seeking the pair

\[ \{ u, w \} : [0, T] \rightarrow H^2 \cap H^1_p \times L^2 \]
such that \( v \in W^{1,\infty} \cap H^1_p \), \( 0 \leq t \leq s \leq t + \tau \leq T \), and \( \tau > 0 \)

\[ \begin{align*}
\langle u(t + \tau) - u(t), v \rangle_2 &= \int_{t}^{t+\tau} \langle \tilde{V}(u(s)) \cdot \nabla v, w(s) \rangle_2 + \langle \tilde{V}(p) \cdot \nabla u(s), v \rangle_2 \, ds \\
\langle u(s), v \rangle_{H^1} &= \langle w(s), v \rangle_2,
\end{align*} \tag{1} \]

\[ \begin{align*}
u(0) &= u_0 \in H^2 \cap H^1_p \\
w(0) &= w_0 = u_0 - \Delta u_0
\end{align*} \tag{2} \]

1.2 Full \( \mathbb{P}_1 \) Finite-Element Space, Euler-Implicit Time Discretizations

Let \( \mathcal{P}_x = \{ x_i | i = 1, \ldots, n \} \) be a partition of \((0, L)\): \(0 = x_1 < x_2 < \ldots < x_n = L\) in the \( x \) direction, \( \mathcal{P}_y = \{ y_j | j = 1, \ldots, n \} \). Let now:

\[ \mathcal{N} = \{ P_I(x_i, y_j) | I = 1, 2, \ldots, N = n^2 \} = \mathcal{P}_x \times \mathcal{P}_y, \]

be a structured set of nodes covering \( \bar{\Omega} \). Based on \( \mathcal{N} \), one obtains a conforming (Delaunay) structured triangulation \( \mathcal{T} \) of \( \bar{\Omega} \), i.e., \( \mathcal{T} = \{ E_I | J = 1, 2, \ldots, M \} \), \( \bar{\Omega} = \bigcup_{I} E_I \). The \( \mathbb{P}_1 \) finite element subspace \( X_N \) of \( H^1(\Omega) \) is given by:

\[ X_N = \{ v \in C(\bar{\Omega}) \text{ restricted to } E_I \in \mathbb{P}_1, J = 1, 2, \ldots, M \} \subset W^{1,p}_p, \quad 1 \leq p \leq \infty \]

with \( \bigcup_{N \geq 1} \{ X_N \} \) dense in \( H^1(\Omega) \). For that purpose, we let \( B_N = \{ \varphi_I | I = 1, 2, \ldots, N \} \) be a finite element basis of functions with compact support in \( \Omega \), i.e.,

\[ \forall v_N \in X_N : v_N(x,y) = \sum_{I=1}^{N} V_I \varphi_I(x,y), \quad V_I = v_N(x_I,y_I). \]

To obtain a fully discrete scheme, we start by projecting (3) on \( X_{N,p} \times X_{N,p} \), seeking therefor the pair

\[ \{ u, w \} : [0, T] \rightarrow X_{N,p} \times X_{N,p} \]
such that

\[ \begin{align*}
\langle w_N(t + \tau) - w_N(t), v \rangle_2 &= \int_{t}^{t+\tau} \langle \tilde{V}(u_N(s)) \cdot \nabla v, w_N(s) \rangle_2 + \langle \tilde{V}(p) \cdot \nabla u_N(s), v \rangle_2 \, ds \\
\langle u_N(s), v \rangle_{H^1} &= \langle w_N(s), v \rangle_2,
\end{align*} \tag{1} \]

\[ \begin{align*}
u_N(0) &= u_N(0) = \pi_N[w_0] \\
w_N(0) &= \pi_N[w_0] = \langle w_N(0), v \rangle_2
\end{align*} \tag{2} \]

\( \forall v \in X_{N,p}, \quad 0 \leq t \leq s \leq t + \tau \leq T \), and \( \tau > 0 \), where \( \pi_N(v) := \sum_{I=1}^{N} \langle v, \varphi_I \rangle_2 \varphi_I(x,y) \in X_N \) is the L2 projection of \( v \) on \( X_N \). In addition, to obtain the Euler-Implicit formulation, we replace the term \( \int_{t}^{t+\tau} \langle \tilde{V}(u(s)) \cdot \nabla v, w(s) \rangle_2 \) with \( \tau \langle \tilde{V}(u(t + \tau)) \cdot \nabla v, w(t + \tau) \rangle_2 \), thus yielding the following fully implicit Computational Model (5).

Given \( \langle u_N(t), w_N(t) \rangle \in X_{N,p} \times X_{N,p} \), one seeks \( \langle u_N(t + \tau), w_N(t + \tau) \rangle \in X_{N,p} \times X_{N,p} \), such that:

\[ \begin{align*}
\langle w_N(t + \tau) - w_N(t), v \rangle_2 &= \tau \langle \tilde{V}(u_N(t + \tau)) \cdot \nabla v, w_N(t + \tau) \rangle_2 \\
\langle u_N(s), v \rangle_{H^1} &= \langle w_N(s), v \rangle_2, \quad \forall s \in \{ t, t + \tau \},
\end{align*} \tag{1} \]

\[ \begin{align*}
\langle u_N(s), v \rangle_{H^1} &= \langle w_N(s), v \rangle_2, \quad \forall s \in \{ t, t + \tau \}, \quad \forall v \in X_{N,p}, \text{ and } \forall t \in [0, T].
\end{align*} \tag{2} \]
1.3 The Non-Linear Algebraic system

When implementing system (5) one takes periodicity into account, reducing the degrees of freedom from $N = n^2$ to $N = (n - 1)^2$. Thus, in matrix notations and using the expressions:

$$w_N(t) = \sum_{l=1}^{N} W_l(t) \varphi_l(x, y), \quad \text{and} \quad u_N(x, y, t) = \sum_{J=1}^{N} U_J(t) \varphi_J(x, y),$$

where $W_l(t) = w_N(x_l, y_l, t)$ and $U_J(t) = w_N(x_J, y_J, t)$, then (5) can be rewritten as follows:

Given $(U(t), W(t)) \in \mathbb{R}^N \times \mathbb{R}^N$, seek $(U(t + \tau), W(t + \tau)) \in \mathbb{R}^N \times \mathbb{R}^N$, such that:

$$\begin{align*}
(M + \tau S(U(t + \tau))) W(t + \tau) - \tau RU(t + \tau) &= MW(t) \quad (1) \\
KU(s) &= MW(s), \quad \forall s \in \{t, t + \tau\} \quad (2)
\end{align*}$$

with $M$, $K$, $S(U)$ and $R$, $N \times N$ matrices, whose entries are defined as follows for $1 \leq I, J \leq N$:

- $M_{I,J} = \langle \varphi_I, \varphi_J \rangle_{L^2}$, $M$ is the well-known Mass matrix for periodic boundary conditions.
- $K_{I,J} = \langle \varphi_I, \varphi_J \rangle_{H^1}$, $K = M + A$, where $A$ is the stiffness matrix for periodic boundary conditions.
- $R_{I,J} = \langle \bar{V}(p) \nabla \varphi_J, \varphi_I \rangle$.
- $S_{I,J}(U) = - \langle \bar{V}(u_N) \cdot \nabla \varphi_I, \varphi_J \rangle_2 = \langle \bar{V}(u_N) \cdot \nabla \varphi_J, \varphi_I \rangle_2$.

In [2], we prove the existence of a solution to system (6) for $\tau \leq \frac{1}{2\|p\|_{1,\infty} \|W(t)\|_M}$, with a stronger restriction for uniqueness,

$$\tau \leq \min \left\{ \frac{1}{2\|p\|_{1,\infty}}, \frac{h^2}{16c_{0,inv} \|W(t)\|_M} \right\} \quad (7)$$

where $c_{0,inv}$ is an inverse inequality constant, as obtained in Ciarlet [5] Theorem 3.2.6

Remark 1.1. However, note that in our computations we did not use the restrictive condition (7), as we took $\tau = O(h)$.

The nonlinearity of the problem originates from $S(U)$, that must be computed at each iteration. The derivation of $S(U)$ and $R$ is detailed in [6], where they have the same block sparsity patterns as that of $M$ and $K$.

First results to solve (6) were obtained in [2] using a simple semi-linear practical approach:

$$\begin{align*}
(M + \tau S(U(t))) W(t + \tau) &= MW(t) + \tau RU(t) \quad (1) \\
KU(t + \tau) &= MW(t + \tau) \quad (2)
\end{align*}$$

However, this approach fails to simulate accurately the wave phenomena that is supposed to remain bounded. As a matter of fact, in the semilinear approach, one has to put a cap on the amplitude of the wave that stops the algorithms once this cap value is reached.

1.4 Results

To remedy the ill-behaved semi-linear approach (8), we propose in this paper Newton-type algorithms that are based on the well-known Newton’s method for solving the full discrete system (6). System (6) is equivalent to finding $(U, W) \in \mathbb{R}^N \times \mathbb{R}^N$ such that:

$$\begin{align*}
F_1(U, W) &= (M + \tau S(U))W - \tau RU - Z = 0 \\
F_2(U, W) &= KU - MW = 0
\end{align*}$$

(9)
where $U = U(t + \tau), W = W(t + \tau),$ and $Z = MW(t)$ which is given.

In vector form, (9) is equivalent to

$$F(U, W) = \begin{bmatrix} F_1(U, W) \\ F_2(U, W) \end{bmatrix} = \begin{bmatrix} (M + \tau S(U))W - \tau RU - Z \\ KU - MW \end{bmatrix} = \begin{bmatrix} -\tau R & M + \tau S(U) \\ K & -M \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} - \begin{bmatrix} Z \\ 0 \end{bmatrix} = 0 \quad (10)$$

which can be solved using Newton-type methods that require the computation of the Jacobian Matrix of $F(U, W)$, given by

$$J_F(U, W) = \begin{bmatrix} \tau B(W) - \tau R & M + \tau S(U) \\ K & -M \end{bmatrix}, \quad (11)$$

as derived in section 3.1. Note that $K, M,$ and $R$ are fixed matrices that are computed once, whereas $S(U)$ and $B(W)$ have to be computed for each $U$ and $W$.

At each time step, it is assumed that $(U(t), W(t))$ was already computed/approximated, then $(U(t + \tau), W(t + \tau))$, the solution of (6), is approximated using Full Newton’s method by solving system (12) iteratively till convergence up to some given tolerance

$$J_F(U^{(k)}, W^{(k)}) \begin{bmatrix} U^{(k+1)} - U^{(k)} \\ W^{(k+1)} - W^{(k)} \end{bmatrix} = -F(U^{(k)}, W^{(k)}) \quad (12)$$

which is equivalent to solving

$$\begin{bmatrix} \tau B(W^{(k)}) - \tau R & M + \tau S(U^{(k)}) \\ K & -M \end{bmatrix} \begin{bmatrix} U^{(k+1)} \\ W^{(k+1)} \end{bmatrix} = \begin{bmatrix} \tau S(U^{(k)})W^{(k)} + MW(t) \\ 0 \end{bmatrix} \quad (13)$$

where $(U^{(0)}, W^{(0)}) = (U(t), W(t))$.

This paper is divided as follows.

In section 2, we prove an apriori error estimate for the solution $\{u_N(t), w_N(t)\}$ to (5), specifically Theorem 2.1.

In section 3, we derive Newton’s method and prove the existence of a unique solution to (12) as a consequence of Theorem 3.1, for $\tau = O(h^{2.5})$. Moreover, we prove the local convergence of Newton’s method in Theorem 3.4.

However, the Jacobian of $F(U, W), J_F(U^{(k)}, W^{(k)})$, has to be recomputed at every Newton iteration, which is computationally intense. Thus, in section 4 we discuss two variants of Newton’s method:

- Chord’s method (section 4.1) which differs from Newton’s method in the sense that the Jacobian matrix, $J_F(U^{(0)}, W^{(0)})$, is fixed throughout all the iterations within one time step. Thus, the existence of a solution is also a corollary of Theorem 3.1, for $\tau = O(h^{2.5})$. Hence, we prove the local convergence of Chord’s method in Theorem 4.2.

- Modified Newton’s method (section 4.2) which avoids computing $B(W^{(k)})$, leading to a modified Jacobian matrix, $J_F(U, W)$. Thus, we prove the existence of a unique solution to the system solved at each iteration of the Modified Newton’s method (section 4.2.1) for $\tau = O(h^2)$, as a corollary of Theorem (4.3). Moreover, we prove the global convergence of the method in Theorem 4.6 (section 4.2.2).

In section 5, numerical testing on the three Newton-type methods are performed where we compare the number of iterations, runtime and behavior of solution with respect to time.

In section 6 we give concluding remarks.

## 2 Apriori Error Estimates for Solutions to (5)

In this section, we prove an apriori error estimate on $w_N$ and $u_N$ solutions to (5), stated as follows.

**Theorem 2.1.** For $t = m\tau \leq T,$ $m \in \mathbb{N},$ and $\tau < \frac{1}{2 \|p\|_{1,\infty}},$ every solution $\{u_N(t), w_N(t)\}$ to (5) satisfies:

$$\|u_N(t)\|_2 \leq e^{3T\|p\|_{1,\infty}} \|u_N(0)\|_2 \leq e^{3T\|p\|_{1,\infty}} \|u_0\|_2 \quad (14)$$

$$\|w_N(t)\|_2 \leq e^{3T\|p\|_{1,\infty}} \|w_N(0)\|_2 \leq e^{3T\|p\|_{1,\infty}} \|w_0\|_2 \quad (15)$$
Proof. Let \( v = u_N(s) \) in the second equation of (5), then

\[
\langle u_N(s), u_N(s) \rangle_{H^1} = \| u_N(s) \|_{H^1}^2 = \| w_N(s) \|_2 \| u_N(s) \|_2 \leq \| w_N(s) \|_2 \| u_N(s) \|_{H^1} \\
\implies \| u_N(s) \|_2 \leq \| u_N(s) \|_{H^1} \leq \| w_N(s) \|_2, \quad \forall s \in \{ t, t + \tau \}
\] (16)

Let \( v = w_N(t + \tau) \) in the first equation of (5), and assuming \( p \in C^\infty \), then

\[
\langle w_N(t + \tau) - w_N(t), w_N(t + \tau) \rangle_2 = \tau \langle \tilde{V}(u_N(t + \tau)) \cdot \nabla w_N(t + \tau), w_N(t + \tau) \rangle_2 + \tau \langle \tilde{V}(p) \cdot \nabla u_N(t + \tau), w_N(t + \tau) \rangle_2 \\
\implies \| w_N(t + \tau) \|_2^2 = \| w_N(t) \|_2^2 + \| w_N(t + \tau) \|_2 + \| \tilde{V}(p) \cdot \nabla u_N(t + \tau) \|_2 \cdot \| w_N(t + \tau) \|_2
\]

and therefore \( \| w_N(t + \tau) \|_2 \leq \| w_N(t) \|_2 + \tau \| \tilde{V}(p) \cdot \nabla u_N(t + \tau) \|_2 \) \) (17)

Note that

\[
\| \tilde{V}(p) \cdot \nabla u_N(t + \tau) \|_2 \leq \| u_N(t + \tau)p_x - u_N(t + \tau)p_y \|_1 \leq \| p \|_{1,\infty} (u_N(t + \tau) - u_N(t))
\]

\[
\implies \| \tilde{V}(p) \cdot \nabla u_N \|_2 \leq \| p \|_{1,\infty} \| u_N(t + \tau) \|_2 + \| u_N(t) \|_2 \leq 2 \| p \|_{1,\infty} \| u_N(t + \tau) \|_{H^1} \leq 2 \| p \|_{1,\infty} \| w_N(t + \tau) \|_2 \quad \text{by (16)} \] (18)

Replacing (18) in (17), and \( t + \tau \) by \( t \), we get for \( \tau < \frac{1}{2 \| p \|_{1,\infty}} \)

\[
\| w_N(t) \|_2 \leq \| w_N(t - \tau) \|_2 + \tau \| p \|_{1,\infty} \| w_N(t) \|_2
\]

\[
\implies \| w_N(t) \|_2 \leq \frac{1}{1 - \tau \| p \|_{1,\infty}} \| w_N(t - \tau) \|_2 \] (19)

Note that for each \( \frac{1}{\alpha} = 2 + \delta > 2 \), there exists \( \delta_\alpha > 0 \), such that if \( \| p \|_{1,\infty} < \delta_\alpha \), then

\[
\frac{1}{1 - \tau \| p \|_{1,\infty}} < 1 + \alpha \| p \|_{1,\infty}
\]

Specifically,

\[
(1 - \tau \| p \|_{1,\infty})(1 + \alpha \| p \|_{1,\infty}) = 1 + (\alpha - 2)\tau \| p \|_{1,\infty} - 2\alpha \tau^2 \| p \|_{1,\infty}^2 > 1
\] (20)

\[
\iff 2\alpha \tau^2 \| p \|_{1,\infty}^2 < (\alpha - 2)\tau \| p \|_{1,\infty}
\] (21)

and therefore \( \tau < \frac{\alpha - 2}{2\alpha \| p \|_{1,\infty}} = \frac{\delta_\alpha}{2(2 + \delta) \| p \|_{1,\infty}} \) \) (22)

Let \( \delta = 1 \), i.e. \( \alpha = 3 \), if \( \tau < \frac{1}{6 \| p \|_{1,\infty}} \) then

\[
\frac{1}{1 - \tau \| p \|_{1,\infty}} \leq 1 + 3\tau \| p \|_{1,\infty}
\]

\[
\implies \| w_N(t) \|_2 \leq (1 + 3\tau \| p \|_{1,\infty}) \| w_N(t - \tau) \|_2
\] (23)

and therefore: \( \| w_N(t) \|_2 \leq (1 + 3\tau \| p \|_{1,\infty})^m \| w_N(0) \|_2 = e^{m \ln(1 + 3\tau \| p \|_{1,\infty})} \| w_N(0) \|_2 \leq e^{3\tau \| p \|_{1,\infty}} \| w_N(0) \|_2 \) \) (24)

where \( t = m\tau \leq T \) for \( m \in \mathbb{N} \). Moreover, since \( \| w_N(0) \|_2 \leq \| w_0 \|_2 \), then the result is obtained. \( \square \)


3 Newton’ Method

In this section, we discuss in details Newton’s method (section 3.2), for solving the full discrete system (6), by first deriving the corresponding Jacobian matrix (section 3.1). Then, in sections 3.3 and 3.4, we prove respectively the existence of a unique solution to (30), and the convergence of Newton’s method.

3.1 Jacobian Matrix

The Jacobian matrix of $F(U, W)$ defined in (10), is a $2N \times 2N$ block matrix given by:

$$J_F(U, W) = \begin{bmatrix} F_{1,U} & F_{1,W} \\ F_{2,U} & F_{2,W} \end{bmatrix} = \begin{bmatrix} \tau(S(U)W)_U - \tau R & M + \tau S(U) \\ K & -M \end{bmatrix}$$

Let $B(W) = (S(U)W)_U$. Then, one way to obtain the matrix $B(W)$ is based on the observation that the matrix $S(U)$ is linear in $U$, i.e.

$$U = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ \vdots \\ U_{N,1} \end{bmatrix} = \sum_{j=1}^{N} U_{j,1} e_j \implies S(U) = S \left( \sum_{j=1}^{N} U_{j,1} e_j \right) = \sum_{j=1}^{N} U_{j,1} S(e_j)$$

and therefore $B(W) = (S(U)W)_U = [S(e_1)W \ S(e_2)W \cdots S(e_N)W]$.

Hence,

$$J_F(U, W) = \begin{bmatrix} \tau B(W) - \tau R & M + \tau S(U) \\ K & -M \end{bmatrix},$$

where $K$, $M$, and $R$ are fixed matrices that are computed once.

However, $S(U)$ and $B(W)$ have to be computed for each $U$ and $W$. $S(e_1), S(e_2), \cdots, S(e_N)$ can be computed once and stored. Yet for large $N$ values storing this set of $N$ matrices might not be feasible. In this case, they can be recomputed once needed.

Note that $S(U)$ is a block tridiagonal matrix with 2 additional blocks in the upper right and lower left corner. Moreover, it is a skew-symmetric matrix $(S(U)^T = -S(U))$ that is linear in $U$, with 6 nonzero entries per row, 6 nonzero entries per column, and zeros on the diagonal assuming the meshing of $\Omega$ is uniform.

$$S(U) = \frac{1}{6} \begin{bmatrix} S_{1,1} & S_{1,2} & 0 & \cdots & 0 & S_{1,k} \\ S_{2,1} & S_{2,2} & S_{2,3} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{j,l} & S_{j,j} & S_{j,i} & 0 \\ 0 & \cdots & 0 & S_{i,j} & S_{i,i} & S_{i,k} \\ S_{k,1} & 0 & \cdots & 0 & S_{k,i} & S_{k,k} \end{bmatrix}$$

with $S_{i,j} \equiv S_{i,j}(U)$

where $i = n - 2, j = n - 3, k = n - 1, l = n - 4$, and the $3(n - 1)$ nonzero block matrices $S_{i,j}$ are of size $(n - 1) \times (n - 1)$ with $2(n - 1)$ nonzero entries each, and the following sparsity patterns:

- $S_{i,i}$ for $i = 1, \ldots, n - 1$ are tridiagonal matrices with zero diagonal entries, and nonzero $S_{i,i}(1, n - 1)$, and $S_{i,i}(n - 1, 1)$.

- $S_{1,n-1}$ and $S_{i+1,i}$ for $i = 1, 2, 3, \ldots, n - 2$ are lower bidiagonal matrices, with nonzero entry in first row and column $n - 1$.

- $S_{n-1,1}$ and $S_{i,i+1}$ for $i = 1, 2, \ldots, n - 2$ are upper bidiagonal matrices with nonzero entry in first column and row $n - 1$.

As for the explicit expressions/values of the entries, refer to appendix A.2 of [6].
Moreover the matrices $B(W)$ and $S(U)$ satisfy the following relation:

$$B(W)U = S(U)W$$

(27)

since by the linearity of the $S$ matrix we get

$$B(W)U = [S(e_1)W \cdots S(e_N)W] \begin{bmatrix} U_{t1} \\
\vdots \\
U_{tN} \end{bmatrix} = \sum_{j=1}^{N} U_{tj} S(e_j)W = S \left( \sum_{j=1}^{N} U_{tj} e_j \right) W = S(U)W$$

3.2 Newton’s Method

At each time step, it is assumed that $(U(t), W(t))$ was already computed/approximated, then $(U(t + \tau), W(t + \tau))$, the solution of (6), is approximated using Newton’s method by solving system (28) iteratively till convergence up to some given tolerance

$$J_F(U^{(k)}, W^{(k)}) \begin{bmatrix} U^{(k+1)} - U^{(k)} \\
W^{(k+1)} - W^{(k)} \end{bmatrix} = -F(U^{(k)}, W^{(k)})$$

(28)

where $(U^{(0)}, W^{(0)}) = (U(t), W(t))$, and $J_F(U, W)$ is the Jacobian of $F(U, W)$ that has to be recomputed at every Newton iteration. It is possible to solve system (28) directly and obtain the vector

$$\begin{bmatrix} \Delta U \\
\Delta W \end{bmatrix} = \begin{bmatrix} U^{(k+1)} - U^{(k)} \\
W^{(k+1)} - W^{(k)} \end{bmatrix}$$

then $U^{(k+1)} = U^{(k)} + \Delta U$ and $W^{(k+1)} = W^{(k)} + \Delta W$. However, computing $F(U^{(k)}, W^{(k)})$ requires 4 matrix-vector multiplications, $MW^{(k)}$, $S(U^{(k)})W^{(k)}$, $RU^{(k)}$, and $KU^{(k)}$. But by replacing $F(U^{(k)}, W^{(k)})$ and $J_F(U^{(k)}, W^{(k)})$ by their expressions, (10) and (26) respectively, and using property (27), system (28) is reduced to the linear system (29) where the right-hand side vector requires the computation of just one matrix-vector multiplication $S(U^{(k)})W^{(k)}$.

$$\begin{bmatrix} \tau B(W^{(k)}) - \tau R & M + \tau S(U^{(k)}) \\
K & -M \end{bmatrix} \begin{bmatrix} U^{(k+1)} \\
W^{(k+1)} \end{bmatrix} = \begin{bmatrix} \tau B(W^{(k)}) - \tau R & M + \tau S(U^{(k)}) \\
K & -M \end{bmatrix} \begin{bmatrix} U^{(k)} \\
W^{(k)} \end{bmatrix} - F(U^{(k)}, W^{(k)})$$

(29)

Thus at the $(k + 1)^{th}$ Newton iteration, the Jacobian matrix $J_F(U^{(k)}, W^{(k)})$ has to be recomputed by computing $B(W^{(k)})$ and $S(U^{(k)})$. Similarly, the right-hand side vector is computed. Then, system (29) is solved. At each timestep, the above procedure is repeated until convergence, i.e. the relative error $\frac{||U^{(k+1)} - U^{(k)}||_2}{||U^{(k)}||_2}$ is less than some given tolerance. This procedure is summarized in Algorithm (1) where the vectors $U(t), W(t), U^{(k)}(t), W^{(k)}(t)$ are denoted by $U_t, W_t, U_{t,k}, W_{t,k}$ respectively.

At every iteration of Newton’s method, there is a need to solve some linear system of form (30), where $[\alpha, \beta]^T \in \mathbb{R}^{2N}$. System (30) is equivalent to the linear system (31) using property (27).

$$J_F(U, W) \begin{bmatrix} \alpha \\
\beta \end{bmatrix} = \begin{bmatrix} \tau B(W) - \tau R & M + \tau S(U) \\
K & -M \end{bmatrix} \begin{bmatrix} \alpha \\
\beta \end{bmatrix} = \begin{bmatrix} \tau S(U)W + Z \\
0 \end{bmatrix}$$

(30)

$$\begin{cases} 
\tau S(U)\alpha - \tau R\alpha + M\beta + \tau S(U)\beta = \tau S(U)W + Z \\
K\alpha = M\beta
\end{cases}$$

(31)
Specifically, at iteration $k + 1$ of Newton’s method, $\alpha = U^{(k+1)}$, $\beta = W^{(k+1)}$, $W = W^{(k)}$, $U = U^{(k)}$ and $Z = MW^{(0)} = MW^{(t)}$ based on (29).

To prove the convergence of Newton’s methods, we prove first the existence of a unique solution of linear system (30) in section 3.3, then we conclude by the convergence proof in section 3.4.

### 3.3 Existence of a Unique Solution to (30)

Let $M\gamma = \tau S(U)W + Z$. Then, to prove the existence of a unique solution to linear system (30), we show that the Jacobian matrix is invertible. We start by showing that there exists some $C \in \mathbb{R}$ independent of $\tau$ and $h$, such that

$$\|\alpha\|^2_M + \|\beta\|^2_M \leq C \|\gamma\|^2_M$$

where $\|\alpha\|^2_M = \alpha^T M \alpha$ and $M$ is the Mass matrix. For that purpose, we use variational formulation.

Let $\phi_N(x, y) = \sum_{I=1}^N \alpha_I \varphi_I(x, y)$, $\psi_N(x, y) = \sum_{I=1}^N \beta_I \varphi_I(x, y)$, and $\xi_N(x, y) = \sum_{I=1}^N \gamma_I \varphi_I(x, y)$, then system (31) can be expressed in variational form elementwise (for $1 \leq I \leq N$) as (32)

$$\begin{align*}
-\tau \langle \tilde{V}(\phi_N) \cdot \nabla w_N, \varphi_I \rangle_2 + \tau \langle \tilde{V}(p) \nabla \phi_N, \varphi_I \rangle_2 + \langle \psi_N, \varphi_I \rangle_2 - \tau \langle \tilde{V}(u_N) \cdot \nabla \psi_N, \varphi_I \rangle_2 &= \langle \xi_N, \varphi_I \rangle_2 \\
\langle \phi_N, \varphi_I \rangle_{H^1} &= \langle \psi_N, \varphi_I \rangle_2
\end{align*}$$

based on (33)-(37).

$$\begin{align*}
(M\beta)_I &= \sum_{J=1}^N M_{I,J} \beta_J = \sum_{J=1}^N \langle \varphi_I, \varphi_J \rangle_2 \beta_J = \langle \varphi_I, \sum_{J=1}^N \varphi_J \beta_J \rangle_2 = \langle \varphi_I, \psi \rangle_2 = \langle \psi, \varphi_I \rangle_2 \tag{33} \\
(K\alpha)_I &= \sum_{J=1}^N K_{I,J} \alpha_J = \sum_{J=1}^N \langle \varphi_I, \varphi_J \rangle_{H^1} \alpha_J = \langle \varphi_I, \sum_{J=1}^N \varphi_J \alpha_J \rangle_{H^1} = \langle \varphi_I, \phi \rangle_{H^1} = \langle \phi, \varphi_I \rangle_{H^1} \tag{34} \\
(R\alpha)_I &= \sum_{J=1}^N R_{I,J} \alpha_J = \sum_{J=1}^N \langle \tilde{V}(p) \cdot \nabla \varphi_I, \psi \rangle_2 \alpha_J = \langle \tilde{V}(p), \nabla \varphi_I, \sum_{J=1}^N \varphi_J \alpha_J \rangle_2 = \langle \tilde{V}(p), \nabla \varphi_I, \phi \rangle_2 \\
&= -\langle \tilde{V}(p), \nabla \phi, \varphi_I \rangle_2 \quad \text{by skew symmetry} \tag{35} \\
(S(U)\beta)_I &= \sum_{J=1}^N S_{I,J} (U) \beta_J = \sum_{J=1}^N \langle \tilde{V}(u_N) \cdot \nabla \varphi_I, \varphi_J \rangle_2 \beta_J = \langle \tilde{V}(u_N) \cdot \nabla \varphi_I, \sum_{J=1}^N \varphi_J \beta_J \rangle_2 \tag{36} \\
&= \langle \tilde{V}(u_N) \cdot \nabla \varphi_I, \psi \rangle_2 = -\langle \tilde{V}(u_N) \cdot \nabla \psi, \varphi_I \rangle_2 \\
(S(\alpha)W)_I &= \langle \tilde{V}(\phi) \cdot \nabla \varphi_I, w_N \rangle_2 = -\langle \tilde{V}(\phi) \cdot \nabla w_N, \varphi_I \rangle_2 \tag{37}
\end{align*}$$

Moreover, $\|\beta\|^2_M = \|\psi_N\|^2_2$ by (33). Similarly $\|\alpha\|^2_M = \|\phi_N\|^2_2$ and $\|\gamma\|^2_M = \|\xi_N\|^2_2$. Thus, we need to show that

$$\|\phi_N\|^2_2 + \|\psi_N\|^2_2 \leq C \|\xi_N\|^2_2$$

(38)

For any $v = \sum_{I=1}^N v_I \varphi_I(x, y) \in X_{N,p}$, system (32) can be written as

$$\begin{align*}
\left\{\begin{array}{l}
\langle \psi_N - \tau \tilde{V}(\phi_N) \cdot \nabla w_N - \tau \tilde{V}(u_N) \cdot \nabla \psi_N + \tau \tilde{V}(p) \cdot \nabla \phi_N, v \rangle_2 = \langle \xi_N, v \rangle_2 \\
\langle \phi_N, v \rangle_{H^1} = \langle \psi_N, v \rangle_2
\end{array}\right. \tag{39}
\end{align*}$$

**Theorem 3.1.** Let $D := D(\Omega, p, T, w_0) = c_{inv}^2 ||p||_{1,\infty} ||w_0||_2 + 2 ||p||_{1,\infty}$ where $c_{inv}$ is an inverse inequality constant as provided in Ciarlet ([5], Theorem 3.2.6):

$$\forall v \in X_N, \quad ||v||_{1,4} \leq c_{inv} h^{-5/4} ||v||_2 \tag{40}$$
Then, for $h < 1$ and $\tau \leq \min \left\{ \frac{1}{6||p||_{1,\infty}}, \frac{h^2}{16c_{0,\text{inv}} ||W(t)||_{L^\infty}}, \frac{h^{5/2}}{2D} \right\} = O(h^{2.5})$.
\[
\left\| \phi_N \right\|_{2}^2 + \left\| \psi_N \right\|_{2}^2 \leq 8 \left\| \xi_N \right\|_{2}^2.
\] (41)

**Proof.** By setting $v = \phi_N$ in the second equation of system (39) and using Cauchy-Schwarz we get (42).
\[
\left\| \phi_N \right\|_{2}^2 \leq \left\| \phi_N \right\|_{H^1}^2 = \left\langle \psi_N, \phi_N \right\rangle_2 \leq \left\| \psi_N \right\|_2 \left\| \phi_N \right\|_2.
\] (42)
\[
\therefore \left\| \phi_N \right\|_{2}^2 + \left\| \psi_N \right\|_{2}^2 \leq 2 \left\| \psi_N \right\|_{2}^2.
\] (43)

Thus, to obtain our result, we seek an upper bound on $\left\| \psi_N \right\|_{2}^2$ in terms of $\left\| \xi_N \right\|_{2}^2$. Let $v = \psi_N$ in the first equation of system (39) we get (44). Then, using Cauchy-Schwarz we get (45).
\[
\left\langle \xi_N, \psi_N \right\rangle_2 = \left\langle \psi_N - \tau \mathbf{V}(\phi_N) \cdot \nabla w_N - \tau \mathbf{V}(u_N) \cdot \nabla \psi_N + \tau \mathbf{V}(p), \nabla \phi_N, \psi_N \right\rangle_2
\] (44)
\[
\left\| \psi_N \right\|_{2}^2 = \left\langle \xi_N, \psi_N \right\rangle_2 + \left\langle \tau \mathbf{V}(\phi_N) \cdot \nabla w_N, \psi_N \right\rangle_2 + \left\langle \tau \mathbf{V}(u_N) \cdot \nabla \psi_N, \psi_N \right\rangle_2 - \left\langle \tau \mathbf{V}(p), \nabla \phi_N, \psi_N \right\rangle_2
\] (45)

We upper bound the last two terms of (45) in terms of $\left\| \psi_N \right\|_{2}$.
\[
\mathbf{V}(\phi_N) \cdot \nabla w_N = -\phi_{Ny} w_{N,x} + \phi_{N,x} w_{N,y} = -\mathbf{V}(w_N) \cdot \nabla \phi_N
\]
\[
\left\| \mathbf{V}(\phi_N) \cdot \nabla w_N \right\|_{2} \leq \left\| \phi_{Ny} w_{N,x} \right\|_{2} + \left\| \phi_{N,x} w_{N,y} \right\|_{2}
\] (46)
\[
\leq \left\| \phi_{Ny} \right\|_4 \left\| w_{N,x} \right\|_4 + \left\| \phi_{N,x} \right\|_4 \left\| w_{N,y} \right\|_4 \leq \left\| w_{N} \right\|_{1,4} \cdot \left\| \phi_{N} \right\|_{1,4}
\] (47)

Assuming $p \in C^\infty$, then
\[
\mathbf{V}(p) \cdot \nabla \phi_N = \phi_{Ny} p_x - \phi_{N,x} p_y \leq \left\| p \right\|_{1,\infty} \left\| \phi_{Ny} - \phi_{N,x} \right\|_2
\] (48)
\[
\therefore \left\| \mathbf{V}(p) \cdot \nabla \phi_N \right\|_{2} \leq \left\| p \right\|_{1,\infty} \left( \left\| \phi_{Ny} \right\|_2 + \left\| \phi_{N,x} \right\|_2 \right) \leq 2 \left\| p \right\|_{1,\infty} \left\| \phi_{N} \right\|_{H^1}
\] (49)

Replacing (47) and (49) in (45) we get
\[
\left\| \psi_N \right\|_{2} \leq \left\| \xi_N \right\|_{2} + \tau c_{inv}^2 h^{-5/2} \left\| w_{N} \right\|_{2} \left\| \psi_{N} \right\|_{2} + 2\tau \left\| p \right\|_{1,\infty} \left\| \psi_{N} \right\|_{2}
\] (50)
\[
\therefore \left\| \psi_{N} \right\|_{2} \leq \frac{1}{\tau} \left\| \xi_{N} \right\|_{2}
\]
and therefore
\[
\left\| \phi_{N} \right\|_{2}^2 + \left\| \psi_{N} \right\|_{2}^2 \leq 2 \left\| \psi_{N} \right\|_{2}^2 \leq \frac{2}{\tau c_{inv}^2 h^{-5/2} \left\| w_{N} \right\|_{2} + 2\tau \left\| p \right\|_{1,\infty}} \frac{\left\| \xi_{N} \right\|_{2}^2}{2}
\] (51)

where $c = \frac{2}{\tau c_{inv}^2 h^{-5/2} \left\| w_{N} \right\|_{2} + 2\tau \left\| p \right\|_{1,\infty}}$ in (38) and $\tilde{c} = 1 - \tau c_{inv}^2 h^{-5/2} \left\| w_{N} \right\|_{2} - 2\tau \left\| p \right\|_{1,\infty}$.

Note that, assuming $h < 1$ and using the apriori error estimates (15) on $\left\| w_{N} \right\|_{2}$ for $\tau < \frac{1}{6 ||p||_{1,\infty}}$ we get
\[
\tilde{c} = \tilde{c}(\tau, h) = 1 - \tau c_{inv}^2 h^{-5/2} \left\| w_{N}(t) \right\|_{2} - 2\tau \left\| p \right\|_{1,\infty} = 1 - \tau h^{-5/2} (c_{inv} \left\| w_{N}(t) \right\|_{2} + 2h^{5/2} \left\| p \right\|_{1,\infty})
\] (51)
where \( D = c_\text{inv}^2 e^{3T\|p\|_1} \|w_0\|_2 + 2 \|p\|_1 \), is a constant independent of \( \tau \) and \( h \).

Thus, if \( \tau h^{-5/2} \leq \frac{1}{2D}, \) and \( h < 1 \), then \( \varepsilon \geq \frac{1}{2} \) for \( \tau \leq \frac{h^{5/2}}{2D} \), and therefore \( c = \frac{2}{c\varepsilon^2} \leq 8 \). \( \square \)

A consequence of Theorem (3.1) is the existence of a unique solution to the linear system (30).

**Theorem 3.2.** Let \( D := D(\Omega, p, T, w_0) = c_\text{inv}^2 e^{3T\|p\|_1} \|w_0\|_2 + 2 \|p\|_1 \), then system (30) has a unique solution for \( h < 1 \) and \( \tau \leq \min \left\{ \frac{1}{6\|p\|_1}, \frac{h^2}{16c_\text{inv}\|W(t)\|_M}, \frac{h^{5/2}}{2D} \right\} = O(h^{2.5}) \).

**Proof.** Let \( \gamma = 0 \), then \( \xi_N = 0 \) and by theorem 3.1

\[
\|\phi_N\|_2^2 + \|\psi_N\|_2^2 \leq 0
\]

for \( h < 1 \) and \( \tau \leq \frac{h^{5/2}}{2D} \).

Thus, \( \|\phi_N\|_2^2 = \|\alpha\|_M^2 = 0 \) and \( \|\psi_N\|_2^2 = \|\beta\|_M^2 = 0 \), implying that \( \alpha = \beta = 0 \). Thus, \( \text{Null}\{J_F(U, W)\} = \{0\} \) implying that \( J_F(U, W) \) is invertible and the linear system (30) has a unique solution. \( \square \)

### 3.4 Convergence

We are approximating the solution of the nonlinear system (6) by using Newton’s method (29). Let

\[
\begin{align*}
\alpha &= U(t + \tau) \\
\alpha^{(k)} &= U^{(k)}(t) \\
\alpha^{(0)} &= U^{(0)}(t) = U(t) \\
\phi_N(x, y) &= \sum_{i=1}^{N} \alpha_i \varphi_i(x, y) \\
\phi_N^{(k)}(x, y) &= \sum_{i=1}^{N} \alpha_i^{(k)} \varphi_i(x, y)
\end{align*}
\]

Then system (6) can be expressed in variational form for any \( v = \sum_{i=1}^{N} v_i \varphi_i(x, y) \in X_{N,p} \) using (33)-(37) as

\[
\begin{align*}
\left\langle \psi_N - \nabla \phi_N \cdot \psi_N + \tau \nabla \psi(p) \cdot \nabla \phi_N, v \right\rangle_2 &= \left\langle \psi_N^{(0)}, v \right\rangle_2 \\
\{\psi_N, v\}_{H^1} &= \{\psi_N^{(0)}, v\}_2
\end{align*}
\]

Similarly, the iterative Newton’s method, (29) or equivalently (31), can be expressed in variational form as

\[
\begin{align*}
\left\langle \psi_N^{(k+1)} - \nabla \phi_N^{(k+1)} \cdot \psi_N - \tau \nabla \psi(p) \cdot \nabla \phi_N^{(k+1)}, v \right\rangle_2 &= \left\langle \psi_N^{(0)}, v \right\rangle_2 - \tau \left\langle \nabla \phi_N^{(k+1)} \cdot \nabla \phi_N^{(k)}, v \right\rangle_2 \\
\{\phi_N^{(k)}, v\}_{H^1} &= \{\psi_N^{(0)}, v\}_2
\end{align*}
\]

for \( i = \{k, k+1\} \).

Let \( \epsilon^{(k)} = \psi_N - \psi_N^{(k)} \) and \( g^{(k)} = \phi_N - \phi_N^{(k)} \), then to prove the convergence of Newton’s method to the unique solution of (6), we prove that there exists some constant \( c < 1 \) such that

\[
\left\|\epsilon^{(k+1)}\right\|_2^2 + \left\|g^{(k+1)}\right\|_2^2 \leq \left\|\epsilon^{(0)}\right\|_2^2 c^{2(k+1)}
\]

**Theorem 3.3.** Assume that \( \phi_N^{(0)} \) is chosen such that \( \forall \kappa \geq k_0 \geq 0, \left\|\phi_N^{(k+1)} - \phi_N^{(k)}\right\|_2 < \epsilon_{\text{tol}}. \)

Then, for \( \tau \leq \min \left\{ \frac{1}{6\|p\|_1}, \frac{h^2}{16c_\text{inv}\|W(t)\|_M}, \frac{h^{5/2}}{2D} \right\} = O(h^{2.5}) \) there exists a constant \( c < 1 \) such that

\[
\left\|\epsilon^{(k+1)}\right\|_2^2 \leq c^2 \left\|\epsilon^{(k)}\right\|_2^2
\]

where \( D_1 := D_1(\Omega, T, p, w_0) = c_\text{inv}^2 (\epsilon_{\text{tol}} + e^{3T\|p\|_1} \|w_0\|_2 + 2 \|p\|_1) \), and \( h < 1. \)
Proof. By theorem 3.2, (55) has a unique solution \( \{ \phi_N^{(k+1)}, \psi_N^{(k+1)} \} \) for \( \tau \leq \min \left\{ \frac{1}{6||p||_{1,\infty}}, \frac{h^2}{16c_{inv} ||W(t)||_{MT}}, \frac{h^{5/2}}{2D} \right\} \)

and \( h < 1 \), where \( D := D(\Omega, p, T, w_0) = c_{inv}^2 e^{3T||p||_{1,\infty} ||w_0||_2 + 2 ||p||_{1,\infty}} \).

Then, by subtracting the second equation of (55) from that of (54), we get (57) for \( i = \{ k, k + 1 \} \). Letting \( v = g^{(i)} \) we get (58)

\[
\langle g^{(i)}, v \rangle_{H^1} = \langle e^{(i)}, v \rangle_2
\]

\[
\|g^{(i)}\|_{H^1}^2 = \|e^{(i)}\|_2 \leq \|e^{(i)}\|_2 \cdot \|g^{(i)}\|_{H^1}
\]

\[
\therefore \|g^{(i)}\|_2 \leq \|g^{(i)}\|_{H^1} \leq \|e^{(i)}\|_2
\]

By subtracting the first equations of (55) from that of (54), we get (59) by linearity of \( \bar{V} \) operator.

\[
\langle e^{(k+1)}, v \rangle_2 = \tau \left( \bar{V}(\phi_N) \cdot \nabla \psi_N - \bar{V}(\phi_N^{(k+1)}) \cdot \nabla \psi_N^{(k+1)} + \bar{V}(\phi_N^{(k)}) \cdot \nabla \psi_N^{(k)} - \bar{V}(p) \cdot \nabla (g^{(k+1)}), v \right)_2
\]

\[
= \tau \left( \bar{V}(g^{(k+1)}) \cdot \nabla \psi_N + \bar{V}(g^{(k+1)}) \cdot \nabla (e^{(k)}) + \bar{V}(\phi_N^{(k)}) \cdot \nabla (e^{(k+1)} - e^{(k)}) - \bar{V}(p) \cdot \nabla (g^{(k+1)}), v \right)_2
\]

\[
= \tau \left( \bar{V}(g^{(k+1)}) \cdot \nabla \psi_N + \bar{V}(g^{(k+1)} - g^{(k+1)}) \cdot \nabla (e^{(k)}) + \bar{V}(\phi_N^{(k)}) \cdot \nabla (e^{(k+1)} - e^{(k)}) - \bar{V}(p) \cdot \nabla (g^{(k+1)}), v \right)_2
\]

Let \( v = e^{(k+1)} \) in (60), then

\[
\|e^{(k+1)}\|_2^2 = \tau \left( \bar{V}(g^{(k+1)}) \cdot \nabla \psi_N + \bar{V}(g^{(k+1)} - g^{(k+1)}) \cdot \nabla (e^{(k)}) - \bar{V}(p) \cdot \nabla (g^{(k+1)}), e^{(k+1)} \right)_2
\]

\[
\leq \tau \|\bar{V}(g^{(k+1)}) \cdot \nabla \psi_N + \bar{V}(g^{(k+1)} - g^{(k+1)}) \cdot \nabla (e^{(k)}) - \bar{V}(p) \cdot \nabla (g^{(k+1)})\|_2 \|e^{(k+1)}\|_2
\]

\[
\therefore \|e^{(k+1)}\|_2 \leq \tau \|\bar{V}(g^{(k+1)}) \cdot \nabla \psi_N\|_2 + \tau \|\bar{V}(g^{(k+1)} - g^{(k+1)}) \cdot \nabla (e^{(k)})\|_2 + \tau \|\bar{V}(p) \cdot \nabla (g^{(k+1)})\|_2
\]

Similarly to (46), by Ciarlet ([5], Theorem 3.2.6), we have (62)-(63)

\[
\|\bar{V}(g^{(k+1)}) \cdot \nabla \psi_N\|_2 \leq c_{inv}^2 h^{-5/2} \|\psi_N\|_2 \cdot \|g^{(k+1)}\|_2 \leq c_{inv}^2 h^{-5/2} \|\psi_N\|_2 \cdot \|e^{(k+1)}\|_2
\]

\[
\|\bar{V}(g^{(k+1)} - g^{(k+1)}) \cdot \nabla (e^{(k)})\|_2 \leq c_{inv}^2 h^{-5/2} \|e^{(k)}\|_2 \cdot \|g^{(k+1)} - g^{(k+1)}\|_2 = c_{inv}^2 h^{-5/2} \|e^{(k)}\|_2 \cdot \|\phi_N^{(k+1)} - \phi_N^{(k)}\|_2
\]

\[
\|\bar{V}(p) \cdot \nabla (g^{(k+1)})\|_2 \leq 2 \|p||_{1,\infty} \cdot \|g^{(k+1)}\|_{H^1} \leq 2 \|p||_{1,\infty} \cdot \|e^{(k+1)}\|_2
\]

Replace equations (62), (63) and (64) in (61) and using the apriori estimate on \( \|\psi_N\|_2 \) (Theorem 2.1), we get

\[
\|e^{(k+1)}\|_2 \leq \tau \|\bar{V}(g^{(k+1)}) \cdot \nabla \psi_N\|_2 + \tau \|\bar{V}(g^{(k+1)} - g^{(k+1)}) \cdot \nabla (e^{(k)})\|_2 + \tau \|\bar{V}(p) \cdot \nabla (g^{(k+1)})\|_2
\]

\[
\leq \tau c_{inv}^2 h^{-5/2} a_0 \cdot \|e^{(k+1)}\|_2 + \tau c_{inv}^2 \epsilon_{tot} h^{-5/2} \|e^{(k)}\|_2 + 2 \tau \|p||_{1,\infty} \cdot \|e^{(k+1)}\|_2
\]

\[
\therefore \|e^{(k+1)}\|_2 \leq \frac{\tau c_{inv}^2 \epsilon_{tot} h^{-5/2}}{1 - \tau c_{inv}^2 \epsilon_{tot} h^{-5/2} a_0 - 2 \tau \|p||_{1,\infty}} \|e^{(k)}\|_2 = c \|e^{(k)}\|_2
\]

where \( a_0 = e^{3T||p||_{1,\infty} ||w_0||_2} \), and \( \tau \leq \frac{1}{c_{inv}^2 h^{-5/2} a_0 + 2 \|p||_{1,\infty}} = \frac{h^{5/2}}{c_{inv}^2 a_0 + 2 h^{5/2} \|p||_{1,\infty}} \).

In addition, \( c = \frac{\tau c_{inv}^2 \epsilon_{tot} h^{-5/2}}{1 - \tau c_{inv}^2 \epsilon_{tot} h^{-5/2} a_0 - 2 \tau \|p||_{1,\infty}} < 1 \) if and only if

\[
\tau < \frac{1}{c_{inv}^2 h^{-5/2}(\epsilon_{tot} + a_0) + 2 \|p||_{1,\infty}} = \frac{h^{5/2}}{c_{inv}^2 (\epsilon_{tot} + a_0) + 2 h^{5/2} \|p||_{1,\infty}} < \frac{h^{5/2}}{c_{inv}^2 a_0 + 2 h^{5/2} \|p||_{1,\infty}}.
\]
Moreover, since $h < 1$, then 

$$\frac{h^{5/2}}{2D_1} < \frac{h^{5/2}}{D_1} < \frac{h^{5/2}}{c_{inv}(\epsilon_{tol} + a_0) + 2h^{5/2}||p||_{1,\infty}}.$$ 

Thus, let $\tau < \frac{h^{5/2}}{2D_1} < \frac{h^{5/2}}{2D}$ where $D_1 = c_{inv}^2(\epsilon_{tol} + a_0) + 2||p||_{1,\infty} = c_{inv}^2\epsilon_{tol} + D$, which ends the proof. \hfill \blacksquare

A consequence of Theorem (3.3) is the local convergence of Newton’s method.

**Theorem 3.4.** Assume that $\phi_N^{(0)}$ is chosen such that $\forall k \geq k_0 \geq 0$, $||\phi_N^{(k+1)} - \phi_N^{(k)}||_2 < \epsilon_{tol}$. Then, for

$$\tau \leq \min \left\{ \frac{1}{6||p||_{1,\infty}}, \frac{k^2}{16c_{inv} \|W(t)\|_M}, \frac{h^{5/2}}{2D_1} \right\},$$

Newton’s method converges to the unique solution of (6),

$$\lim_{k \to \infty} \phi_N^{(k)} = \phi_N \quad \text{and} \quad \lim_{k \to \infty} \psi_N^{(k)} = \psi_N$$

where $D_1 := D_1(\Omega, T, p, w_0) = c_{inv}^2(\epsilon_{tol} + e^{3T||p||_{1,\infty}} \|w_0\|_2) + 2||p||_{1,\infty}$ and $h < 1$.

**Remark 3.5.** This additional assumption $(\forall k \geq k_0 \geq 0, ||\phi_N^{(k+1)} - \phi_N^{(k)}||_2 < \epsilon_{tol})$ is computational in nature, since it could be used as a stopping criteria for any iterative method solving a nonlinear problem.

**Proof.** By theorem 3.3, there exists $c < 1$ such that

$$\frac{\left|\epsilon^{(k+1)}\right|_2^2}{\left|\epsilon^{(k)}\right|_2^2} \leq c^2 \frac{\left|\epsilon^{(k)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2} \quad \Rightarrow \quad \frac{\left|\epsilon^{(k+1)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2} \leq c^2 \frac{\left|\epsilon^{(k)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2}$$

and

$$\lim_{k \to \infty} \frac{\left|\epsilon^{(k+1)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2} = 0 \quad \Leftrightarrow \quad \lim_{k \to \infty} \frac{\left|\epsilon^{(k)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2} = 0$$

Similarly,

$$\lim_{k \to \infty} \frac{\left|\epsilon^{(k+1)}\right|_2^2 + \left|g^{(k+1)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2} = 0 \quad \Leftrightarrow \quad \lim_{k \to \infty} \frac{\left|g^{(k)}\right|_2^2}{\left|\epsilon^{(0)}\right|_2^2} = 0$$

\[ \therefore \lim_{k \to \infty} \psi_N^{(k+1)} = \psi_N \quad \text{and} \quad \lim_{k \to \infty} \phi_N^{(k+1)} = \phi_N \] \hfill \blacksquare

## 4 Variants of Newton’s method

In this section we discuss two variants of Newton’s method, Chord’s method (section 4.1) and Modified Newton’s method (section 4.2), that are less computationally intensive than Newton’s method.

### 4.1 Chord’s Method

To avoid recomputing the Jacobian matrix at each Newton iteration, Chord’s method approximates the solution of (6), by solving system (66) iteratively till convergence up to some given tolerance

$$J_F(U^{(0)}, W^{(0)}) \begin{bmatrix} U^{(k+1)} - U^{(k)} \\ W^{(k+1)} - W^{(k)} \end{bmatrix} = -F(U^{(k)}, W^{(k)})$$

(66)

where $(U^{(0)}, W^{(0)}) = (U(t), W(t))$, and $J_F(U^{(0)}, W^{(0)})$ is the Jacobian matrix that is computed once per time iteration.

By replacing $F(U^{(k)}, W^{(k)})$ and $J_F(U^{(0)}, W^{(0)})$ by their expressions, (26) and (10) respectively, system (66) is reduced to the linear system (67), where computing the right-hand side vector requires 3 matrix-vector multiplications. However, using property (27), this can be reduced to just 2 matrix-vector multiplications, $S(U^{(k)})(W^{(0)} - W^{(k)})$ and
by subtracting the second equation of (69) from that of (54), we get (70) for $S$ where the right hand side is replaced by $M\gamma$

Assume that

Theorem 4.1.

Let

We are approximating the solution of the nonlinear system (6) by using Chord’s method (66) or equivalently (68),

4.1.1 Convergence

Note that (68) is equivalent to system (30) and (31) where $\alpha = U^{(k+1)}, \beta = W^{(k+1)}, W = W^{(0)}, U = U^{(0)}$, and the right hand side is replaced by $M\gamma = \tau S(U^{(k)})W^{(0)} + \tau S(U^{(0)})W^{(k)} - \tau S(U^{(k)})W^{(k)} + Z$. Thus, the existence of a unique solution to (68) is a corollary of theorem 3.1 proven in section 3.3.

4.1.1 Convergence

We are approximating the solution of the nonlinear system (6) by using Chord’s method (66) or equivalently (68), which can be expressed in variational form for any $v = \sum_{I=1}^{N} v_{I}\varphi_{I}(x, y) \in X_{N,p}$ using (33)-(37) and (52)-(53) as

Let $e^{(k)} = \psi_{N} - \psi_{N}^{(k)}$ and $g^{(k)} = \phi_{N} - \phi_{N}^{(k)}$, then we prove the local convergence of Chord’s method.

Theorem 4.1. Assume that $\psi_{N}^{(0)}$ and $\phi_{N}^{(0)}$ are chosen such that $\|\phi_{N} - \phi_{N}^{(0)}\|_{2} < c_{1}$ and $\forall k \geq k_{0} \geq 0$,

$$\|\phi_{N}^{(k+1)} - \phi_{N}^{(k)}\|_{2} < \epsilon_{tol1} \quad \text{and} \quad \|\psi_{N}^{(k+1)} - \psi_{N}^{(k)}\|_{2} < \epsilon_{tol2}. \tag{69}$$

Then, for $\tau \leq \min \left\{ \frac{1}{6}[p]_{1,\infty}, \frac{h^{2}}{16c_{0,inv} \|W(t)\|_{M}}, \frac{h^{5/2}}{D_{2}} \right\} = O(h^{2.5})$ there exists a constant $c_{1} < 1$ such that

$$\|e^{(k+1)}\|_{2} \leq c_{1}^{2} \|e^{(k)}\|_{2}^{2} \tag{70}$$

where $D_{2} := D_{2}(\Omega, T, p, w_{0}) = c_{2,inv}(4c_{1} + \epsilon_{sol1} + \epsilon_{tol2} + e^{3T}[p]_{1,\infty} \|w_{0}\|_{2}) + 2[p]_{1,\infty}$ and $h < 1$.

Proof. Similarly to theorem 3.2, and as a consequence of theorem 3.1, (69) has a unique solution $\{\phi_{N}^{(k+1)}, \psi_{N}^{(k+1)}\}$ for $\tau \leq \min \left\{ \frac{1}{6}[p]_{1,\infty}, \frac{h^{2}}{16c_{0,inv} \|W(t)\|_{M}}, \frac{h^{5/2}}{2D} \right\}$, where $D = c_{2,inv}e^{3T}[p]_{1,\infty} \|w_{0}\|_{2} + 2[p]_{1,\infty}$ and $h < 1$. Then, by subtracting the second equation of (69) from that of (54), we get (70) for $i = \{k, k+1\}$. Letting $v = g^{(i)}$ we get (71)

$$\langle g^{(i)}, v \rangle_{H^{1}} = \langle e^{(i)}, v \rangle_{2} \tag{70}$$

$$\|g^{(i)}\|_{H^{1}} = \|g^{(i)}\|_{2} \leq \|e^{(i)}\|_{2} \|g^{(i)}\|_{2} \leq \|e^{(i)}\|_{2} \|g^{(i)}\|_{H^{1}} \tag{71}$$
By subtracting the first equations of (69) from that of (54), we get (72) by linearity of $\tilde{V}$ operator.

$$\langle e^{(k+1)}, v \rangle_2 = \tau \langle \tilde{V}(\phi_N) \cdot \nabla \psi_N - \tilde{V}(\phi^{(k+1)}_N) \cdot \nabla \psi^{(0)}_N - e^{(k+1)} \cdot \nabla e^{(k)} - \tilde{V}(p), \nabla (g^{(k+1)}), v \rangle_2$$
$$+ \tau \langle \tilde{V}(\phi^{(k)}_N) \cdot \nabla \psi^{(k)}_N + \tilde{V}(\phi^{(k)}_N) \cdot \nabla \psi^{(0)}_N, v \rangle_2$$
$$= \tau \langle \tilde{V}(g^{(k+1)}) \cdot \nabla \psi_N + \tilde{V}(\phi^{(k+1)}_N) \cdot \nabla e^{(0)} + \tilde{V}(\phi^{(0)}_N) \cdot \nabla (e^{(k+1)} - e^{(k)}) - \tilde{V}(p), \nabla (g^{(k+1)}), v \rangle_2$$
$$+ \tau \langle \tilde{V}(\phi^{(k)}_N) \cdot \nabla (e^{(k)} - e^{(0)}), v \rangle_2$$
$$= \tau \langle \tilde{V}(g^{(k+1)}) \cdot \nabla \psi_N + \tilde{V}(g^{(k)} - g^{(k+1)}) \cdot \nabla e^{(0)} + \tilde{V}(\phi^{(0)}_N) \cdot \nabla (e^{(k+1)} - e^{(k)}) - \tilde{V}(p), \nabla (g^{(k+1)}), v \rangle_2$$
$$+ \tau \langle \tilde{V}(\phi^{(k)}_N) \cdot \nabla (e^{(k)} - e^{(0)}), v \rangle_2$$

(72)

$$\langle \tilde{V}(e^{(0)}), \cdot \nabla (e^{(k)} - e^{(k+1)}), v \rangle_2$$

Note that

$$- \langle \tilde{V}(g^{(k)}) \cdot \nabla e^{(k)}, v \rangle_2 = \langle \tilde{V}(g^{(k+1)} - g^{(k)}) \cdot \nabla e^{(k)}, v \rangle_2 + \langle \tilde{V}(g^{(k+1)}) \cdot \nabla (e^{(k+1)} - e^{(k)}), v \rangle_2$$
$$- \langle \tilde{V}(g^{(k)}) \cdot \nabla e^{(k)}, v \rangle_2$$

(74)

$$\langle \tilde{V}(\phi^{(0)}_N) \cdot \nabla (e^{(k+1)} - e^{(k)}), v \rangle_2 = \langle \tilde{V}(g^{(0)}), \cdot \nabla (e^{(k)} - e^{(k+1)}), v \rangle_2 + \langle \tilde{V}(\phi^{(0)}_N) \cdot \nabla (e^{(k+1)} - e^{(k)}), v \rangle_2$$

(75)

Replacing (74) and (75) in (73) we get

$$\langle e^{(k+1)}, v \rangle_2 = \tau \langle \tilde{V}(g^{(k+1)}) \cdot \nabla \psi_N + \tilde{V}(g^{(k)} - g^{(k+1)}) \cdot \nabla e^{(0)} + \tilde{V}(g^{(0)}), \cdot \nabla (e^{(k+1)} - e^{(k)}), v \rangle_2$$
$$+ \tau \langle \tilde{V}(g^{(k+1)} - g^{(k)}) \cdot \nabla e^{(k)} + \tilde{V}(g^{(k+1)}) \cdot \nabla (e^{(k+1)} - e^{(k)}), v \rangle_2$$
$$- \tau \langle \tilde{V}(g^{(k+1)}) \cdot \nabla e^{(k)}, v \rangle_2 + \tau \langle \tilde{V}(\phi^{(0)}_N) \cdot \nabla (e^{(k+1)} - e^{(k)}), v \rangle_2$$

(76)

Let $v = e^{(k+1)}$ in (76), then

$$\| e^{(k+1)} \|_2 \leq \tau \| \tilde{V}(g^{(k+1)}) \cdot \nabla \psi_N + \tilde{V}(g^{(k)} - g^{(k+1)}) \cdot \nabla e^{(0)} + \tilde{V}(g^{(0)}), \cdot \nabla (e^{(k+1)} - e^{(k)}), v \|_2$$
$$+ \tau \| \tilde{V}(g^{(k+1)} - g^{(k)}) \cdot \nabla e^{(k)} + \tilde{V}(g^{(k+1)}) \cdot \nabla (e^{(k+1)} - e^{(k)}) \|_2$$
$$\leq \tau \| \tilde{V}(g^{(k+1)}) \cdot \nabla \psi_N \|_2 + \tau \| \tilde{V}(g^{(k)} - g^{(k+1)}) \cdot \nabla e^{(0)} \|_2 + \tau \| \tilde{V}(g^{(0)}), \cdot \nabla (e^{(k+1)} - e^{(k)}) \|_2$$
$$+ \tau \| \tilde{V}(g^{(k+1)} - g^{(k)}) \cdot \nabla e^{(k)} \|_2 + \tau \| \tilde{V}(g^{(k+1)}) \cdot \nabla (e^{(k+1)} - e^{(k)}) \|_2$$

(77)

Similarly to (46), by (40) ([5], Theorem 3.2.6), we have (78)-(82)

$$\| \tilde{V}(g^{(k+1)}) \cdot \nabla \psi_N \|_2 \leq c_{inv}^2 h^{-5/2} \| \psi_N \|_2 \cdot \| e^{(k+1)} \|_2$$

(78)

$$\| \tilde{V}(g^{(k)} - g^{(k+1)}) \cdot \nabla e^{(0)} \|_2 \leq c_{inv}^2 h^{-5/2} \| e^{(0)} \|_2 \cdot \| g^{(k)} - g^{(k+1)} \|_2 \leq c_{inv}^2 h^{-5/2} c_{1} \left( \| e^{(k)} \|_2 + \| e^{(k+1)} \|_2 \right)$$

(79)

$$\| \tilde{V}(g^{(0)}), \cdot \nabla (e^{(k)} - e^{(k+1)}) \|_2 \leq c_{inv}^2 h^{-5/2} \| g^{(0)} \|_2 \cdot \| e^{(k)} - e^{(k+1)} \|_2 \leq c_{inv}^2 h^{-5/2} c_{21} \left( \| e^{(k)} \|_2 + \| e^{(k+1)} \|_2 \right)$$

(80)

$$\| \tilde{V}(g^{(k+1)} - g^{(k)}) \cdot \nabla e^{(k)} \|_2 \leq c_{inv}^2 h^{-5/2} \| g^{(k+1)} - g^{(k)} \|_2 \cdot \| e^{(k)} \|_2 \leq c_{inv}^2 h^{-5/2} c_{2} \| e^{(k)} \|_2$$

(81)

$$\| \tilde{V}(g^{(k+1)}) \cdot \nabla (e^{(k+1)} - e^{(k)}) \|_2 \leq c_{inv}^2 h^{-5/2} \| g^{(k+1)} \|_2 \cdot \| e^{(k+1)} - e^{(k)} \|_2 \leq c_{inv}^2 h^{-5/2} c_{22} \| g^{(k+1)} \|_2$$

(82)

$$\| \tilde{V}(p) \cdot \nabla g^{(k+1)} \|_2 \leq 2 \| p \|_{1,\infty} \cdot \| g^{(k+1)} \|_{H^1}$$

Similarly to (48)

(83)

$$\| e^{(k+1)} \|_2$$

By (71)
In addition, a corollary of Theorem (4.1) is the local convergence of Chord’s method. A corollary of Theorem (4.1) is the local convergence of Chord’s method.

Replace equations (78) - (83) in (77), then

\[
\left\| e^{(k+1)} \right\|_2 \leq 2\tau \| p \|_{1,\infty} \cdot \left( \left\| e^{(k+1)} \right\|_2 + \tau c^2_{in} h^{-5/2} \left[ a_0 \left\| e^{(k+1)} \right\|_2 + 2c_1 \left( \left\| e^{(k)} \right\|_2 + \left\| e^{(k+1)} \right\|_2 \right) \right] \right)
\]

\[
\vdots \left\| e^{(k+1)} \right\|_2 \leq \left( 1 - 2\tau \| p \|_{1,\infty} - \tau c^2_{in} h^{-5/2}(a_0 + 2c_1 + \varepsilon_{tol2}) \right) \left\| e^{(k)} \right\|_2 = c \left\| e^{(k)} \right\|_2
\]

where \( a_0 = e^{3T} \| p \|_{1,\infty} \| u_0 \|_2 \). \( \tau \leq \frac{1}{c^2_{in} h^{-5/2}(a_0 + 2c_1 + \varepsilon_{tol2}) + 2 \| p \|_{1,\infty}} = \frac{c^2_{in}(a_0 + 2c_1 + \varepsilon_{tol2}) + 2h^{5/2} \| p \|_{1,\infty}}{h^{5/2}}. \)

In addition, \( c = \frac{1}{1 - 2\tau \| p \|_{1,\infty} - \tau c^2_{in} h^{-5/2}(2c_1 + \varepsilon_{tol1})} < 1, \) if and only if

\[
\tau < \frac{h^{5/2}}{D_2 + 2h^{5/2} \| p \|_{1,\infty}} = \frac{h^{5/2}}{D_2 + 2h^{5/2} \| p \|_{1,\infty}} \leq \frac{h^{5/2}}{c^2_{in}(a_0 + 2c_1 + \varepsilon_{tol2}) + 2h^{5/2} \| p \|_{1,\infty}}
\]

where \( D_2 = \tilde{D}_2 + 2 \| p \|_{1,\infty} \). Thus, let \( \tau < \frac{h^{5/2}}{2D_2} = \frac{h^{5/2}}{D_2} \) where \( D = c^2_{in}a_0 + 2 \| p \|_{1,\infty} \), which ends the proof.

A corollary of Theorem (4.1) is the local convergence of Chord’s method.

Theorem 4.2. Assume that \( \psi_N^{(0)} \) and \( \phi_N^{(0)} \) are chosen such that \( \left\| \phi_N - \phi_N^{(0)} \right\|_2 < \epsilon_{tol1} \) and \( \forall k \geq k_0 \geq 0 \),

\[
\left\| \phi_N^{(k+1)} - \phi_N^{(k)} \right\|_2 < \epsilon_{tol1} \quad \text{and} \quad \left\| \psi_N^{(k+1)} - \psi_N^{(k)} \right\|_2 < \epsilon_{tol2}.
\]

Then, for \( \tau \leq \min \left\{ \frac{1}{6\| p \|_{1,\infty}}, \frac{h^2}{16c_0 \| W(t) \|_M}, \frac{h^{5/2}}{D_2} \right\} = O(h^{2.5}) \) Chord’s method converges to the unique solution of (6),

\[
\lim_{k \to \infty} \psi_N^{(k)} = \psi_N \quad \text{and} \quad \lim_{k \to \infty} \phi_N^{(k)} = \phi_N
\]

where \( D_2 := D_2(\Omega, T, p, u_0) = c^2_{in}(4c_1 + \epsilon_{tol1} + \epsilon_{tol2} + e^{3T} \| p \|_{1,\infty} \| u_0 \|_2 + 2 \| p \|_{1,\infty} \), and \( h < 1 \).

Proof. By theorem 4.1, there exists \( \epsilon < 1 \) such that

\[
\left\| e^{(k+1)} \right\|_2 \leq e^2 \left\| e^{(k)} \right\|_2 \Rightarrow \left\| e^{(k+1)} \right\|_2 \leq e^{2(k+1)} \left\| e^{(0)} \right\|_2
\]

\[
\lim_{k \to \infty} \left\| e^{(k+1)} \right\|_2 \leq 0 \quad \Rightarrow \quad \lim_{k \to \infty} \left\| e^{(k)} \right\|_2 = 0
\]

\[
\left\| e^{(k+1)} \right\|_2 + \left\| g^{(k+1)} \right\|_2 \leq 2 \left\| e^{(k+1)} \right\|_2 \leq 2e^2 \left\| e^{(k)} \right\|_2 \Rightarrow \left\| e^{(k+1)} \right\|_2 + \left\| g^{(k+1)} \right\|_2 \leq 2e^2 \left\| e^{(k)} \right\|_2
\]

\[
\lim_{k \to \infty} \left\| g^{(k+1)} \right\|_2 = 0 \quad \Rightarrow \quad \lim_{k \to \infty} \left\| g^{(k)} \right\|_2 = 0
\]

\[
\therefore \lim_{k \to \infty} \psi_N^{(k+1)} = \psi_N \quad \text{and} \quad \lim_{k \to \infty} \phi_N^{(k+1)} = \phi_N
\]

4.2 Modified Newton’s Method

In both Newton and Chord’s methods, the computation of the Jacobian matrix requires the computation of the \( B(W) \) matrix that consists of \( N \) matrix-vector multiplications. To avoid the computation of the \( B(W) \) matrix, we introduce a Modified Newton’s Method. Starting with Newton’s equation (29) or equivalently (85)

\[
\leftrightarrow \left\{ \begin{array}{l}
\tau B(W)U^{(k+1)} - \tau R(U^{(k+1)} + MW^{(k+1)} + \tau S(U^{(k)})W^{(k+1)} = \tau S(U^{(k)})W^{(k)} + Z \\
KU^{(k+1)} - MW^{(k+1)} = 0
\end{array} \right. 
\]

(85)
and using property (27), we approximate \( B(W^{(k)})U^{(k+1)} \) by \( S(U^{(k)})W^{(k)} \),

\[
B(W^{(k)})U^{(k+1)} = S(U^{(k+1)})W^{(k)} \approx S(U^{(k)})W^{(k)}
\]

leading to the Modified Newton system (86) whose right-hand side is fixed throughout the Modified Newton iterations. Moreover, the modified Jacobian matrix \( \tilde{J}_F(U, W) \) (86) requires updating one of its blocks by computing \( S(U^{(k)}) \) at each Modified Newton iteration. The procedure is summarized in Algorithm (3).

\[
\begin{align*}
\tau S(U^{(k)})W^k - \tau RU^{(k+1)} + MW^{(k+1)} + \tau S(U^{(k)})W^{(k+1)} = \tau S(U^{(k)})W^{(k)} + Z \\
KU^{(k+1)} - MW^{(k+1)} = 0
\end{align*}
\]

\[
\Leftrightarrow \begin{cases} 
\tau S(U^{(k)})W^k - \tau RU^{(k+1)} + MW^{(k+1)} + \tau S(U^{(k)})W^{(k+1)} = \tau S(U^{(k)})W^{(k)} + Z \\
KU^{(k+1)} - MW^{(k+1)} = 0
\end{cases}
\]

\[
\Leftrightarrow \begin{bmatrix} -\tau R & M + \tau S(U) \\ K & -M \end{bmatrix} \begin{bmatrix} U^{(k+1)} \\ W^{(k+1)} \end{bmatrix} = \begin{bmatrix} Z \\ 0 \end{bmatrix} \quad (86)
\]

Note that the Modified Newton method, defined by (86), is equivalent to the iterative solution \( x^{(k+1)} = G(x^{(k)}) \) of the fixed point problem of (6) or equivalently (9), i.e.

\[
\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} -\tau R & M + \tau S(U) \\ K & -M \end{bmatrix}^{-1} \begin{bmatrix} Z \\ 0 \end{bmatrix} = \tilde{J}_F(U, W)^{-1} \begin{bmatrix} Z \\ 0 \end{bmatrix} = G([U, W]^T)
\]

where we prove in section (4.2.1) that the Modified Jacobian matrix \( \tilde{J}_F(U, W) \) is invertible.

At every iteration of the Modified Newton’s method, there is a need to solve a system of form (87), where \([\alpha, \beta]^T \in \mathbb{R}^{2N} \). System (87) is equivalent to system (88).

\[
\begin{align*}
\tilde{J}_F(U, W) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\tau R & M + \tau S(U) \\ K & -M \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} M\gamma \\ 0 \end{bmatrix} \\
\Leftrightarrow \begin{cases} 
-\tau R\alpha + M\beta + \tau S(U)\beta = M\gamma \\
K\alpha = M\beta
\end{cases}
\end{align*}
\]

(88)

where at the \( k + 1 \)th iteration \( \alpha = U^{(k+1)}, \beta = W^{(k+1)}, \gamma = W^{(0)}, W = W^{(k)}, \) and \( U = U^{(k)} \) based on (86).

To prove the convergence of this method (section 4.2.2), we prove first the existence of a unique solution of system (87) in section 4.2.1.

### 4.2.1 Existence of a Unique Solution to (87)

To prove the existence of a unique solution to (87), we start by showing that there exists some \( C \in \mathbb{R} \) independent of \( \tau \) and \( h \), such that \( \|\alpha\|_M^2 + \|\beta\|_M^2 \leq C \|\gamma\|_M^2 \) using variational formulation.

Let \( \phi_N(x, y) = \sum_{I=1}^N \alpha_I \varphi_I(x, y), \psi_N(x, y) = \sum_{I=1}^N \beta_I \varphi_I(x, y), \) and \( \xi_N(x, y) = \sum_{I=1}^N \gamma_I \varphi_I(x, y), \) then system (88) can be expressed in variational form elementwise (for \( 1 \leq I \leq N \)) as (89) based on (33)-(37).

\[
\begin{align*}
\left\{ \begin{array}{l}
\tau \langle \tilde{V}(p), \nabla \phi_N, \varphi_I \rangle_2 + \langle \psi_N, \varphi_I \rangle_2 - \tau \langle \tilde{V}(u_N), \nabla \psi_N, \varphi_I \rangle_2 = \langle \xi_N, \varphi_I \rangle_2 \\
\langle \phi_N, \varphi_I \rangle_{H_1} = \langle \psi_N, \varphi_I \rangle_2
\end{array} \right.
\end{align*}
\]

(89)

Moreover, \( \|\beta\|^2_M = \|\psi_N\|^2_2 \) by (33). Similarly \( \|\alpha\|^2_M = \|\phi_N\|^2_2 \) and \( \|\gamma\|^2_M = \|\xi_N\|^2_2 \). Thus, we need to show that

\[
\|\phi_N\|^2_2 + \|\psi_N\|^2_2 \leq C \|\xi_N\|^2_2
\]

(90)

For any \( v = \sum_{I=1}^N v_I \varphi_I(x, y) \in X_{N,p} \), system (89) can be written as

\[
\begin{align*}
\left\{ \begin{array}{l}
\langle \psi_N - \tau \tilde{V}(u_N), \nabla \psi_N + \tau \tilde{V}(p), \nabla \phi_N, v \rangle_2 = \langle \xi_N, v \rangle_2 \\
\langle \phi_N, v \rangle_{H_1} = \langle \psi_N, v \rangle_2
\end{array} \right.
\end{align*}
\]

(91)
Theorem 4.3. Let $\tau \leq \min \left\{ \frac{1}{4\|p\|_{1,\infty}} \cdot \frac{h^2}{16\epsilon_0 \text{inv} \|W(t)\|_M} \right\} = O(h^2)$ then
\[ \|\phi_N\|_2^2 + \|\psi_N\|_2^2 \leq 8\|\xi_N\|_2^2. \] (92)

Proof. Similarly to (42), By setting $v = \phi_N$ in the second equation of system (91) and using Cauchy-Schwarz we get
\[ \implies \|\phi_N\|_2 \leq \|\psi_N\|_2 \] (93)
\[ \therefore \|\phi_N\|_2^2 + \|\psi_N\|_2^2 \leq 2\|\psi_N\|_2^2 \] (94)

Thus, to obtain our result we must upper bound $\|\psi_N\|_2^2$ in terms of $\|\xi_N\|_2^2$.

Let $v = \psi_N$ in the first equation of system (91) we get (95). Then, using Cauchy-Schwarz we get (96).
\[ \langle \xi_N, \psi_N \rangle_2 = \left\langle \psi_N - \tau \tilde{V}(u_N) \cdot \nabla \psi_N + \tau \tilde{V}(p) \cdot \nabla \phi_N, \psi_N \right\rangle_2 \] (95)
\[ \|\psi_N\|_2^2 = \langle \xi_N, \psi_N \rangle_2 + \left\langle \tau \tilde{V}(u_N) \cdot \nabla \psi_N, \psi_N \right\rangle_2 - \left\langle \tau \tilde{V}(p) \cdot \nabla \phi_N, \psi_N \right\rangle_2 \]
\[ = \langle \xi_N, \psi_N \rangle_2 - \left\langle \tau \tilde{V}(p) \cdot \nabla \phi_N, \psi_N \right\rangle_2 \] using skew-symmetry
\[ \leq \|\xi_N\|_2 \|\psi_N\|_2 + \tau \left\| \tilde{V}(p) \cdot \nabla \phi_N \right\|_2 \|\psi_N\|_2 \]
\[ \therefore \|\psi_N\|_2 \leq \|\xi_N\|_2 + \tau \left\| \tilde{V}(p) \cdot \nabla \phi_N \right\|_2 \] (96)

Assuming $p \in C^\infty$, we upper bound the last terms of (96) in terms of $\|\psi_N\|_2$ similarly to (49).
\[ \implies \left\| \tilde{V}(p) \cdot \nabla \phi_N \right\|_2 \leq 2\|p\|_{1,\infty} \|\phi_N\|_{H^1} \leq 2\|p\|_{1,\infty} \|\psi_N\|_2 \] (97)

Replacing (97) in (96) we get
\[ \|\psi_N\|_2 \leq \|\xi_N\|_2 + 2\|p\|_{1,\infty} \|\psi_N\|_2 \]
\[ \implies \|\psi_N\|_2 \leq \frac{1}{C} \|\xi_N\|_2 \]
\[ \therefore \|\phi_N\|_2^2 + \|\psi_N\|_2^2 \leq 2\|\psi_N\|_2^2 \leq \frac{2}{C^2} \|\xi_N\|_2^2 \] (98)

where $C = \frac{2}{C^2}$ in (90), $\hat{C} = 1 - 2\tau \|p\|_{1,\infty}$. Thus, if $\tau \leq \frac{1}{4\|p\|_{1,\infty}}$, then $\hat{C} \geq \frac{1}{2}$, and therefore $C = \frac{2}{C^2} \leq 8$. \[ \square \]

A corollary of Theorem (4.3) is the existence of a unique solution to system (87).

Theorem 4.4. Let $\tau \leq \min \left\{ \frac{1}{4\|p\|_{1,\infty}} \cdot \frac{h^2}{16\epsilon_0 \text{inv} \|W(t)\|_M} \right\} = O(h^2)$ then system (87) has a unique solution.

Proof. Let $\gamma = 0$, then $\xi_N = 0$ and by theorem 4.3
\[ \|\phi_N\|_2^2 + \|\psi_N\|_2^2 \leq 0 \]
for $\tau \leq \frac{1}{4\|p\|_{1,\infty}}$. Thus, $\|\phi_N\|_2^2 = \|\alpha\|_M^2 = 0$ and $\|\psi_N\|_2^2 = \|\beta\|_M^2 = 0$, implying that $\alpha = \beta = 0$.

Thus, $\text{Null}(\tilde{J}_F(U, W)) = \{0\}$, implying that $\tilde{J}_F(U, W)$ is invertible and system (87) has a unique solution. \[ \square \]

4.2.2 Convergence

We are approximating the solution of the nonlinear system (6) by using Modified Newton’s method (86), which can be expressed in variational form for any $v = \sum_{j=1}^{N} v_j \varphi_j(x, y) \in X_{N,p}$ using (33)-(37) and (52)-(53) as
\[ \left\{ \begin{array}{l}
\left\langle \psi_N^{(k+1)} - \tau \tilde{V}(\phi_N^{(k)}) \cdot \nabla \psi_N^{(k+1)} + \tau \tilde{V}(p) \cdot \nabla \phi_N^{(k+1)}, v \right\rangle_2 = \left\langle \psi_N^{(0)}, v \right\rangle_2 \\
\left\langle \phi_N^{(i)}, v \right\rangle_{H^1} = \left\langle \psi_N^{(i)}, v \right\rangle_2 \\
\end{array} \right. \quad \text{for } i = \{k, k+1\} \] (99)
Let \( e^{(k)} = \psi_N - \psi_N^{(k)} \) and \( g^{(k)} = \phi_N - \phi_N^{(k)} \), then we prove the convergence of Modified Newton’s method.

**Theorem 4.5.** Let \( \tau \leq \min \left\{ \frac{1}{6|p|_{1,\infty}}, \frac{h^2}{16c_0, inv \|W(t)\|_M}, \frac{h^{5/2}}{D_3} \right\} = O(h^{2.5}) \), then there exists a constant \( c < 1 \) such that

\[
\left\| e^{(k+1)} \right\|_2^2 + \left\| g^{(k+1)} \right\|_2^2 \leq c^2 \left( \left\| e^{(k)} \right\|_2^2 + \left\| g^{(k)} \right\|_2^2 \right)
\]

(100)

where \( D_3 := D_3(\Omega, p, T, w_0) = c^2_{inv} e^{3T|p|_{1,\infty} \|W(t)\|_M} + 2 \|p\|_{1,\infty} \) and \( h < 1 \).

**Proof.** By theorem 4.3, (99) has a unique solution \( \{\phi_N^{(k+1)}, \psi_N^{(k+1)}\} \) for \( \tau \leq \min \left\{ \frac{1}{4|p|_{1,\infty}}, \frac{h^2}{16c_0, inv \|W(t)\|_M} \right\} \). Then, by subtracting the second equation of (99) from that of (54), we get (101) for \( i = \{k, k+1\} \). Letting \( v = g^{(i)} \) we get (102)

\[
\left\langle g^{(i)}, v \right\rangle_{H^1} = \left\langle e^{(i)}, v \right\rangle_2
\]

(101)

\[
\left\| g^{(i)} \right\|_{H^1} \leq \left\| e^{(i)} \right\|_2 \left\| g^{(i)} \right\|_2 \leq \left\| e^{(i)} \right\|_2 \left\| g^{(i)} \right\|_{H^1}
\]

(102)

By subtracting the first equations of (99) from that of (54), we get (103) by linearity of \( \overline{V}(. \cdot) \) operator.

\[
\left\langle e^{(k+1)}, v \right\rangle_2 = \tau \left\langle \overline{V}(\phi_N) \cdot \nabla \psi_N - \overline{V}(\phi_N^{(k)}) \cdot \nabla \psi_N^{(k+1)} - \overline{V}(\psi_N) \cdot \nabla (g^{(k+1)}), v \right\rangle_2
\]

(103)

Let \( v = e^{(k+1)} \) in (103), then

\[
\left\| e^{(k+1)} \right\|_2^2 = \tau \left\langle \overline{V}(g^{(k)}) \cdot \nabla \psi_N, e^{(k+1)} \right\rangle_2 + \tau \left\langle \overline{V}(\phi_N^{(k)}) \cdot \nabla (e^{(k+1)}), e^{(k+1)} \right\rangle_2 - \tau \left\langle \overline{V}(\psi_N) \cdot \nabla (g^{(k+1)}), e^{(k+1)} \right\rangle_2
\]

(104)

\[
\leq \tau \left\| \overline{V}(g^{(k)}) \cdot \nabla \psi_N \right\|_2 \left\| e^{(k+1)} \right\|_2 + \tau \left\| \overline{V}(\psi_N) \cdot \nabla (g^{(k+1)}) \right\|_2 \left\| e^{(k+1)} \right\|_2
\]

(105)

Similarly to (46), by Ciarlet ([5], Theorem 3.2.6), we have (106), then for \( \tau \leq \frac{1}{6|p|_{1,\infty}} \) we get (107)

\[
\left\| \overline{V}(g^{(k)}) \cdot \nabla \psi_N \right\|_2 \leq c^2_{inv} h^{-5/2} \psi_N \left\| e^{(k)} \right\|_2 \frac{\|g^{(k)}\|_2}{\|e^{(k)}\|_2}
\]

(106)

\[
\leq c^2_{inv} h^{-5/2} e^{3T|p|_{1,\infty}} \|w_0\|_2 \left\| g^{(k)} \right\|_2 = h^{-5/2} \tilde{D}_3 \left\| g^{(k)} \right\|_2 \quad \text{By (15)}
\]

(107)

\[
\left\| \overline{V}(\psi_N) \cdot \nabla (g^{(k+1)}) \right\|_2 \leq 2 \|p\|_{1,\infty} \cdot \left\| g^{(k+1)} \right\|_{H^1} \quad \text{Similarly to (48)}
\]

(108)

where \( \tilde{D}_3 := \tilde{D}_3(\Omega, p, T, w_0) = c^2_{inv} e^{3T|p|_{1,\infty}} \|w_0\|_2 \). Replacing (107) and (108) in (105), we get (109)

\[
\left\| e^{(k+1)} \right\|_2 \leq c^2 \left\| g^{(k)} \right\|_2
\]

(109)

\[
\implies \left\| e^{(k+1)} \right\|_2^2 \leq c^2 \left\| g^{(k)} \right\|_2^2
\]

(110)
where $c = \frac{\tau h^{-5/2} \tilde{D}_3}{1 - 2\tau ||p||_1,\infty}$. If $\tau < \frac{1}{h^{-5/2} D_3 + 2 ||p||_1,\infty} = \frac{h^{5/2}}{D_3 + 2 h^{5/2} ||p||_1,\infty}$, then $c < 1$. Moreover, assuming $h < 1$, then $\tau < \frac{h^{5/2}}{D_3} < \frac{h^{5/2}}{D_3 + 2 h^{5/2} ||p||_1,\infty}$ where $D_3 = \tilde{D}_3 + 2 ||p||_1,\infty$, which ends the proof. \hfill \Box

A corollary of Theorem (4.5) is the global convergence of Modified Newton’s method.

**Theorem 4.6.** Let $\tau \leq \min \left\{ \frac{1}{6 ||p||_1,\infty}, \frac{h^2}{16 c_0 ||W(t)||_\infty}, \frac{h^{5/2}}{D_3} \right\} = O(h^{2.5})$, then, for any choice of initial guesses \{\phi^{(0)}_N, \psi^{(0)}_N\}, Modified Newton’s method converges to the unique solution of (6).

\[
\lim_{k \to \infty} \phi^{(k)}_N = \phi_N \quad \text{and} \quad \lim_{k \to \infty} \psi^{(k)}_N = \psi_N,
\]

where $D_3 := D_3(\Omega, p, w_0) = c_{inv} e^{3T} ||w_0||_2 + 2 ||p||_1,\infty$ and $h < 1$.

**Proof.** By theorem 4.5, there exists $c < 1$ such that

\[
\left\| e^{(k+1)} \right\|_2^2 + \left\| g^{(k+1)} \right\|_2^2 \leq c^2 \left( \left\| e^{(k)} \right\|_2^2 + \left\| g^{(k)} \right\|_2^2 \right) \leq c^{2(k+1)} \left( \left\| e^{(0)} \right\|_2^2 + \left\| g^{(0)} \right\|_2^2 \right) = 0
\]

\[
\Rightarrow \lim_{k \to \infty} \left\| e^{(k+1)} \right\|_2 = 0 \quad \text{and} \quad \lim_{k \to \infty} \left\| g^{(k+1)} \right\|_2 = 0
\]

\[
\therefore \lim_{k \to \infty} \psi^{(k+1)}_N = \psi_N \quad \text{and} \quad \lim_{k \to \infty} \phi^{(k+1)}_N = \phi_N.
\]

5 **Computer Simulations and Testings**

We implement the three discussed methods, Newton (Algorithm 1), Chord (Algorithm 2), Modified Newton (Algorithm 3) for $k_{max} = 20$ and $tol = 10^{-14}$ using Freefem++ [7], a programming language and software focused on solving partial differential equations using the finite element method.

**Algorithm 3 Solving HM using Modified Newton Method**

**Input:** $A$: stiffness matrix; $M$: Mass matrix; $K = M + A$; $S(U)$; $R$: As defined in [6]; $U_0$; $W_0$: the discrete initial condition vectors; $T$: end time; $\tau$: time step; $N$: total number of mesh nodes; $k_{max}$: maximum Modified Newton iterations; $tol$: Modified Newton’s relative error stopping tolerance.

**Output:** $U$: $N \times (T/\tau + 1)$ matrix containing the computed vectors $U_t$ for $t = 0, \tau, 2\tau, \cdots, T$

1: for $t = 0 : \tau : T$ do
2: \hspace{1em} $U_{t,0} = U_t$; $W_{t,0} = W_t$; \text{error} = 1; \hspace{0.5em} $k = 0$
3: \hspace{1em} Let $Z = M + W_t$; Let $r(0 : N - 1) = Z$; and $r(N : 2N - 1) = 0$
4: \hspace{1em} while (\text{error} > tol and $k < k_{max}$ ) do
5: \hspace{2em} let $J_2 = M + \tau S(U_{t,k})$
6: \hspace{2em} Construct $J$ : \hspace{0.5em} $J = [[-\tau R, J_2], [K, -M]]$
7: \hspace{2em} Solve for $V$ : \hspace{0.5em} $J \times V = r$;
8: \hspace{2em} let $U_{t,k+1} = V(0 : N - 1)$ and $W_{t,k+1} = V(n : 2N - 1)$;
9: \hspace{2em} error = $\frac{\left\| U_{t,k+1} - U_{t,k} \right\|}{\left\| U_{t,k} \right\|}$; \hspace{0.5em} $k = k + 1$
10: end while
11: $U_{t+1} = U_{t,k}$; $W_{t+1} = W_{t,k}$
12: end for
the semi-linear algorithm introduced in [2].

We consider the same initial conditions $u_0$ as in [2] (Table 1), and compare the obtained solutions of the three method, the required runtime, and the number of iterations per time step (Table 2). Moreover, we compare them with the semi-linear algorithm introduced in [2].
| Ω          | $u_0(x, y)$                        | $p(x, y) = \ln \frac{n_0}{w_{ci}}$ |
|------------|-----------------------------------|----------------------------------|
| Test 1     | $[0, 1] \times [0, 1]$            | $10^{-5} \sin(10\pi y)$         | 12$x$                           |
| Test 2     | $[0, \pi] \times [0, \pi]$       | $10^{-5} \sin(3y)$              | 12$x$                           |
| Test 3     | $[0, \pi] \times [0, \pi]$       | $10^{-5} \sin(3x)$              | 12$x$                           |
| Test 4     | $[0, \pi] \times [0, \pi]$       | $10^{-10} xy(x - 2) \sin(x)$    | 12$x$                           |
| Test 5     | $[0, 20] \times [0, 20]$          | $-10^{-5}(x - 10)e^{-0.5(x-10)^2} - 0.5(y-10)^2$ | $\ln(10^{13} e^{-(x-10)^2/64-(y-10)^2/64})$ |

Table 1: Considered Test Cases

Similarly to [2], we consider a square domain $[x_0, x_n] \times [y_0, y_n]$ with a uniform mesh in the $x$ and $y$ direction $(x_i - x_{i-1} = \frac{x_n - x_0}{n} = y_i - y_{i-1} = \frac{y_n - y_0}{n}$ for $i = 1, 2, ..., n$ and $n$ intervals in the $x$ and $y$ directions respectively) and the finite element $P_1$ space with periodic boundary conditions, using appropriate Freefem++ functions. Even though theoretically $\tau = O(h^{2.5})$, we use $\tau = O(h)$ in our tests, specifically $\tau = 0.1$. Moreover, the simulation is stopped once the maximum value of $u(t)$ at one of the mesh nodes is 0.3, which corresponds to the maximum value attained physically.

The function $p$ from the initial Hasegawa-Mima PDE and the initial condition $u_0$ are given as input. As for the initial condition $w_0 = u_0 - \Delta u_0$ it could be given as input if $u_0$ is a simple function. However, for any function $u_0$, we compute the vector $W_0 = W(0)$ by solving the linear system

$$M * W_0 = K * U_0$$

where the vector $U_0 = U(0)$. The matrices $M, K, R$ and $S(U^k)$ are generated in Freefem++ using their corresponding variational formulations. As for $B(W^k)$, it requires the generation of $N = (n - 1)^2$ matrices $S(e^j)$ of size $N \times N$, where each is multiplied by $W^k$. The $N$ matrices $S(e^j)$ could be generated once and stored. However, this would require $N^3$ words, which is memory-bound (for $n = 65$, $N^3 = 6.87 \times 10^{10}$). Thus, the matrices $S(e^j)$ are regenerated at every iteration to compute $B(W^k)$.

The algorithmic difference between the three methods is that in Newton $B(W^k)$ is computed at every iteration, in Chord it is computed once per time-step, and in Modified Newton it is not computed at all. The effect of the $B(W^k)$ computation on the runtimes of the three methods is evident in Table 2. We compare the three methods for 16 partition intervals in the $x$ and $y$ directions, since for finer meshes the matrices will be larger, particularly the $B(W^k)$ matrix. We also consider end time $T = 10$ with time-step $\tau = 0.1$.

| Newton (Algorithm 1) | Chord (Algorithm 2) | Modified Newton (Algorithm 3) |
|----------------------|---------------------|-----------------------------|
| Iter | RelErr | Time(s) | Iter | RelErr | Time(s) | Iter | RelErr | Time(s) |
| Test 1 | 2 | $10^{-14}$ | 304.577 | 2 | $10^{-10}$ | 146.755 | 2 | $10^{-14}$ | 1.35253 |
| Test 2 | 2 | $10^{-16}$ | 281.969 | 2 | $10^{-12}$ | 138.803 | 2 | $10^{-16}$ | 1.37822 |
| Test 3 | 1 | $10^{-15}$ | 141.301 | 1 | $10^{-15}$ | 140.674 | 1 | $10^{-16}$ | 0.763779 |
| Test 4 | 2 | $10^{-14}$ | 274.895 | 2 | $10^{-12}$ | 139 | 2 | $10^{-12}$ | 1.32606 |
| Test 5 | 2 | $10^{-14}$ | 277.537 | 2 | $10^{-10}$ | 138.5 | 2 | $10^{-10}$ | 1.38517 |

Table 2: Comparison of the 3 methods for the 5 test cases, with $T = 10$, $n = 17$, $\tau = 0.1$ showing the number of method iterations per time step (Iter), the last relative error per time step (RelErr) and the total runtime of the algorithm (Time).
For all the tests except the third, two method iterations are performed per time-step, where Chord’s method is 2 times faster than Newton’s method, since half the $B(W^k)$ matrices are computed. Moreover, Modified Newton’s method is 200 times faster than Newton’s method since it avoids computing 200 $B(W^k)$ matrices. In the second iteration, the relative error $\frac{|U_{t,k+1} - U_{t,k}|}{|U_{t,k}|}$ varies between $10^{-10}$ and $10^{-16}$, which is much smaller than the tolerance $tol = 10^{-6}$. This implies that the change in the solution in the second iteration is relatively negligible.

For Test 3, one method iteration is performed per time-step. Thus Chord’s method and Newton’s method require the same runtime, since the same number of $B(W^k)$ matrices are computed. Moreover, Modified Newton’s method is 100 times faster than Newton’s method since it avoids computing 100 $B(W^k)$ matrices.

Apart from runtime, it should be noted that the evolution of the solution with respect to time was similar for the three methods. Thus, in Figures 1-5 we show the evolution of the solution for the fastest method, Modified Newton (Algorithm 3), for $n = 33$ or 65 partition points in each direction, and $\tau = 0.1$.

For Tests 1-4, $p(x,y) = 12x$ is of the form $Ax + b$, where $p_x = 12$ and $p_y = 0$. Thus, the solution is expected to be a traveling wave in the $y$-direction for a nonzero $A$. The speed of the motion and its direction depend on the magnitude and sign of $A$ respectively. If $A > 0$, then the motion is in downwards, whereas if $A < 0$ the motion is upwards, and for $A = 0$ no motion. The larger the magnitude of $A$, the faster the motion.

For example in Tests 1 and 2 (Figures 1, and 2) where $u_0$ is a sin function in the $y$-direction, the traveling wave effect in the $y$-direction is clear.

Figure 1: Time evolution of solution $u$ of (6) for Test1 using Algorithm 3, with $\tau = 0.1$, and a $65 \times 65$ grid on $\Omega = [0,1] \times [0,1]$. 

![Time evolution of solution u of (6) for Test1 using Algorithm 3, with \( \tau = 0.1 \), and a 65 \times 65 grid on \( \Omega = [0,1] \times [0,1] \).](image-url)
Whereas in Test 3 (Figure 3), it may seem that the solution is stationary. However, the motion is along the y-direction, and the initial solution which is a \( \sin \) function in the x-direction, is fixed for \( x = a \), \( a \in \mathbb{R} \). Thus, the traveling wave effect is not visible. But if the \( \sin \) function is multiplied by other function so that it is no longer fixed for \( x = a \), like Test 4 (Figure 4), then the motion is visible again. Note that if we set \( p_x = 0 \) and \( p_y = 12 \), then the solution will be moving in the x-direction in a similar manner, where for Tests 1 and 2, the solution will appear stationary and Test 3 and 4 will be traveling in the x-direction.
Figure 4: Time evolution of solution $u$ of (6) for Test 4 using Algorithm 3, with $\tau = 0.1$, and a $33 \times 33$ grid on $\Omega = [0, \pi] \times [0, \pi]$. 
As for Test 5 (Figure 5), the solution is expected to have a circular motion around the center of the domain \((10, 10)\), since \(\nabla p = \left[-\frac{(x - 10)}{32}, -\frac{(y - 10)}{32}\right]\), which is observed. Note that if \(\nabla p = \left[\frac{(x - 10)}{32}, \frac{(y - 10)}{32}\right]\) for the same initial conditions, then the solution will be moving in the opposite direction.

As for the comparison between the Newton-type methods for solving (6) and the semi-linear approach (8) introduced in [2], we note the following. The corresponding expected behavior was observed in all methods with one main difference. In the Newton-type methods that were tested for end time \(T = 300\), the maximum entry in the solution
vector remained \( O(\max(U_0)) \) and the algorithm was never stopped. Thus there was no need to put a cap on the amplitude of the solution. However, this was not the case for the semi-linear approach. For Test1 the solution grows with time to reach \(|u|_\infty = 0.3\) at \( t = 260.4 \) when the algorithm is stopped. For Test2 the solution grows with time at a faster rate to reach \(|u|_\infty = 0.3\) at \( t = 9.6 \). For Test3 the solution remains unchanged up till \( t = 26 \), and after it grows with time to reach \(|u|_\infty = 0.3\) at \( t = 42 \).

Thus, the Newton-type methods are numerically more stable and robust than the semi-linear approach. Moreover, the most competitive one is Modified Newton’s method as it is the fastest with a similar runtime to that of the semi-linear approach.

6 Concluding Remarks

In this paper, we implement Newton-type methods for solving system (6), specifically Newton, Chord and Modified Newton methods. Moreover, we justify the use of these methods by proving several results, in particular the convergence of the implemented methods.

Although the sufficient conditions for proving our theorems restrict the time interval \( \tau \) to be of the order of \( h^{2.5} \) or \( h^2 \), yet in our computational implementations this restriction was lifted as we were able to use \( \tau = O(h) \) without any difficulty. Proving the mathematical validity of such choices remains an open question.

In terms of implementation, given a relative tolerance, all the methods converged in at most \( k = 2 \) iterations per time-step, for all the tested cases. Moreover, the expected runtime behavior is observed, where Modified Newton’s method is \( \frac{T}{10^2} \)-times faster than Chord’s method which is \( k \)-times faster than Newton’s method.

On the other hand, the Newton-type methods are numerically more stable and robust than the semi-linear approach introduced in [2], since there was no need to put a cap on the amplitude of the solution in the algorithm. Yet, the time evolution of the solution using the Newton-type methods followed the expected behavior for the corresponding cases. In addition, Modified Newton’s method has a similar runtime to that of the semi-linear approach.

Thus, Modified Newton’s method appears to be the most competent and robust version to be used for simulations of the Hasegawa-Mima plasma model.

As for future avenues of research, these include principally the following.

1. Proof of convergence of the solution to the nonlinear (5), and (6) schemes as \( \tau \) and \( h \) go to zero, which is currently being investigated.

2. Another interesting problem for which these methods can be applied is the Modon Traveling Waves Solutions to (2). These solutions are obtained by considering the pair of variables \((\xi, \eta)\) given by \( \xi = x, \eta = y - ct \), one looks for solutions to (2) in the form \( u(x, y, t) = \phi(\xi, \eta) = \phi(x, y - ct) \) and \( w(x, y, t) = \psi(\xi, \eta) = \psi(x, y - ct) \). By defining \( \forall t \in (0, T) : \Omega_0 = \{\xi, \eta | 0 < \xi < L, -ct < \eta < L - ct\} \), then in terms of \( \phi \) and \( \psi \), the system (2) reduces to be solved on \( \Omega_0 = \Omega \). Thus, with \( \nabla = \nabla_{\xi, \eta} \), one seeks \( \{\phi, \psi\} : \Omega \to \mathbb{R}^2 \), such that:

\[
\begin{align*}
-c\psi_{\eta} + \nabla(\phi) \cdot \nabla \psi &= k\phi_{\eta} & \text{on } \Omega \\
-\Delta \phi + \phi &= \psi & \text{on } \Omega \\
\text{PBC’s on } \phi, \phi_\xi, \phi_\eta, \psi & \text{on } \partial\Omega 
\end{align*}
\]  

Undergoing research is being carried out on this problem.

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