The Rich Are Different!
Pareto Law from asymmetric interactions in asset exchange models

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Fitzgerald: The rich are different from you and me
Hemingway: Yes, they have more money

It is known that asset exchange models with symmetric interaction between agents show either a Gibbs/log-normal distribution of assets among the agents or condensation of the entire wealth in the hands of a single agent, depending upon the rules of exchange. Here we explore the effects of introducing asymmetry in the interaction between agents with different amounts of wealth (i.e., the rich behave differently from the poor). This can be implemented in several ways: e.g., (1) in the net amount of wealth that is transferred from one agent to another during an exchange interaction, or (2) the probability of gaining vs. losing a net amount of wealth from an exchange interaction. We propose that, in general, the introduction of asymmetry leads to Pareto-like power law distribution of wealth.

1 Introduction

“The history of all hitherto existing society is a history of social hierarchy” – Joseph Persky

As is evident from the above quotation, the inequality of wealth (and income) distribution in society has long been common knowledge. However, it was not until the 1890s that the nature of this inequality was sought to be quantitatively established. Vilfredo Pareto collected data about the distribution of income across several European countries, and stated that, for the high-income range, the probability that a given individual has income greater than or equal to \( x \) is \( P_x(x) \sim x^{-\alpha} \), \( \alpha \) being known as the Pareto exponent. Pareto had observed \( \alpha \) to vary around 1.5 for the data available to him and believed \( \alpha \approx 1.5 \) to be universal (i.e., valid across different societies). However, it is now known that \( \alpha \) can vary over a very wide range; furthermore,
Wealth W, Income X (Indian Rupees)

Fig. 1. Wealth and income distribution in India: (Left) Rank ordered wealth distribution during the period 2002-2004 plotted on a double-logarithmic scale, showing the wealth of the $k$-th ranked richest person (or household) in India against the rank $k$ (with rank 1 corresponding to the wealthiest person) as per surveys conducted by Business Standard [7] in Dec 31, 2002 (squares), Aug 31, 2003 (triangles) and Aug 31, 2004 (circles). The broken line having a slope of $-1.23$ is shown for visual reference. (Right) Cumulative income distribution during the period 1929-30 as per information obtained from Income Tax and Super Tax data given in Ref. [8]. The plot has Gibbs/log-normal form at the lower income range, and a power law tail with Pareto exponent $\alpha \approx 1.15$ for the highest income range.

for the low-income end, the distribution follows either a log-normal [4] or exponential distribution [5]. Similar power law tails have been observed for the wealth distribution in different societies. While wealth and income are obviously not independent of each other, the exact relation between the two is not very clear. While wealth is analogous to voltage, being the net value of assets owned at a given point of time, income is analogous to current, as it is the net flow of wages, dividends, interest payments, etc. over a period of time. In general, it has been observed that wealth is more unequally distributed than income. Therefore, the Pareto exponent for wealth distribution is smaller than that for income distribution.

Most of the empirical studies on income and wealth distribution have been done for advanced capitalist economies, such as, Japan and USA. It is interesting to note that similar distributions can be observed even for India [6], which until recently had followed a planned economy. As income tax and other records about individual holdings are not publicly available in India, we had to resort to indirect methods. As explained in detail in Ref. [6], the Pareto exponent for the power-law tail of the wealth distribution was determined from the rank-ordered plot of wealth of the richest Indians [Fig. 1 (left)]. This procedure yielded an average Pareto exponent of $\alpha \approx 1/1.23 = 0.82$. A similar exercise carried out for the income distribution in the highest income range produced a Pareto exponent $\alpha \approx 1.51$. Surprisingly, this is identical to what Pareto had thought to be the universal value of $\alpha$. Comparing this with historical data of income distribution in India [8], we again observe the power-law
tail although with a different exponent [Fig. 1 (right)]. In addition, we note that the low-income range has a log-normal or Gibbs form very similar to what has been observed for advanced capitalist economies [3]. In the subsequent sections, we will try to reproduce these observed features of wealth & income distributions through models belonging to the general class of asset exchange models.

2 Asset exchange models

Asset exchange models belong to a class of simple models of a closed economic system, where the total wealth available for exchange, $W$, and the total number of agents, $N$, trading among each other, are fixed [9, 10, 11, 12, 13]. Each agent $i$ has some wealth $W_i(t)$ associated with it at time step $t$. Starting from an arbitrary initial distribution of wealth $(W_i(0), i = 1, 2, 3, \ldots)$, during each time step two randomly chosen agents $i$ and $j$ exchange wealth, subject to the constraint that the combined wealth of the two agents is conserved by the trade, and that neither of the two has negative wealth after the trade (i.e., debt is not allowed). In general, one of the players will gain and the other player will lose as a result of the trade. If we consider an arbitrarily chosen pair of agents $(i, j)$ who trade at a time step $t$, resulting in a net gain of wealth by agent $i$, then the change in their wealth as a result of trading is:

$$ W_i(t + 1) = W_i(t) + \Delta W; W_j(t + 1) = W_j(t) - \Delta W; $$

where, $\Delta W$ is the net wealth exchanged between the two agents.

Different exchange models are defined based on how $\Delta W$ is related to $[W_i(t), W_j(t)]$. For the random exchange model, the wealth exchanged is a random fraction of the combined wealth $[W_i(t) + W_j(t)]$, while for the minimum exchange model, it is a random fraction of the wealth of the poorer agent, i.e., $\min[W_i(t), W_j(t)]$. The asymptotic distribution for the former is exponential, while the latter shows a condensation of the entire wealth $W$ into the hands of a single agent [Fig. 2 (left)]. Neither of these reflect the empirically observed distributions of wealth in society, discussed in the previous section.

Introducing savings propensity in the exchange mechanism, whereby agents don't put at stake (and are therefore liable to lose) their entire wealth, but put in reserve a fraction of their current holdings, does not significantly change the steady state distribution [11]. By increasing the savings fraction (i.e., the fraction of wealth of an agent that is not being put at stake during a trade), one observes that the steady-state distribution becomes non-monotonic, although the tail still decays exponentially. However, randomly assigning different savings fractions (between $[0,1]$) to agents lead to a power-law tail in the asymptotic distribution [13].

This result raises the question of whether it is the differential ability of agents to save that gives rise to the Pareto distribution. Or, turning the question around, we may ask whether the rich save more. This question has been the
subject of much controversy, but recent work seems to have answered this in the affirmative [14]. As mentioned in a leading economics textbook, savings is the greatest luxury of all [15] and the amount of savings in a household rises with income. In terms of the asset exchange models, one can say that an agent with more wealth is more likely to save (or saves a higher fraction of its wealth). Implementing this principle appropriately in the exchange rules, one arrives at the asymmetric exchange model.

3 Asymmetric exchange model

The model is defined by the following exchange rules specifying the change in wealth, $W_A(t+1) - W_A(t)$, of agent $A$ who wins a net amount of wealth after trading with agent $B$ [$W_B(t+1) - W_B(t) = W_A(t) - W_A(t+1)$]:

$$W_A(t+1) = W_A(t) + \epsilon(1 - \tau[1 - \frac{W_A(t)}{W_B(t)}])W_B(t), \text{ if } W_A(t) \leq W_B(t),$$

$$= W_A(t) + \epsilon W_B(t), \text{ otherwise},$$

where $\epsilon$ is a random number between 0 and 1, specifying the fraction of wealth that has been exchanged. For $\tau = 0$, this is the random exchange model, while for $\tau = 1$, it is identical to the minimum exchange model [Fig. 2 (left)]. In the general case, $0 < \tau < 1$, the relation between the agents trading with each other is asymmetric, the richer agent having more power to dictate terms of trade than the poorer agent. The parameter $\tau$ (thrift) measures the degree to which the richer agent is able to use this power.

As $\tau$ is increased from 0 to 1, the asymptotic distribution of wealth is observed to change from exponential to a condensate (all wealth belonging to a single agent). However, at the transition between these two very different types
of distribution (τ → 1) one observes a power-law distribution! As seen in Fig. 2 (right), the power-law extends for almost the entire range of wealth and has a Pareto exponent ≃ 0.5. This is possibly the simplest asset exchange model that can give rise to a power-law distribution. Note that, unlike other models [13], here one does not need to assume the distribution of a parameter among agents.

However, the Pareto exponent for this model is quite small compared to those empirically observed in real economies. This situation is remedied if instead of considering a fixed value of τ for all agents, we consider the heterogeneous case where τ is distributed randomly among agents according to a quenched distribution. For an uniform distribution of τ, the steady-state distribution of wealth has a power-law tail with α = 1.5 [Fig. 3 (left)], which is the value predicted by Pareto, while at the region corresponding to low wealth, the distribution is exponential. By changing the nature of the random distribution, one observes power-law tails with different exponents. For example, for $P(τ) \sim τ$, the resulting distribution has a Pareto exponent $α \sim 1.3$, while for $P(τ) \sim τ^{-2/3}$, one obtains $α \sim 2.1$. A non-monotonic U-shaped distribution of τ yields $α \sim 0.73$. However, the fact that even with these extremely different distributions of τ one always obtains a power-law tail for the wealth distribution, underlines the robustness of our result.

4 Asymmetric Winning Probability Model

Asymmetry in the interaction between agents (as a function of their wealth) can also be introduced through the probability that an agent will gain net wealth from an exchange. Consider a variant of the minimum exchange model

Fig. 3. (Left) Asymptotic wealth distribution (inset shows the cumulative distribution) with a power-law tail having Pareto exponent $α ≃ 1.5$, for the asymmetric exchange model with $τ$ distributed uniformly over the unit interval [0, 1] among $N$ agents ($N = 1000$, $t = 1 \times 10^7$ iterations, averaged over $10^4$ realizations). (Right) Asymptotic wealth distribution for model having asymmetric winning probability with $β = 0.1$ [pluses] (slope of the power-law curve is $1.30 \pm 0.05$) and $β = 0.01$ [crosses] (slope of the power-law curve is $1.27 \pm 0.05$). ($N = 1000$, $t = 1.5 \times 10^7$ iterations, averaged over 5000 realizations).
where the probability that agent $A$ (wealth $W_A$) will win a net amount in an exchange with $B$ (wealth $W_B$) is

$$p(A|A, B) = \frac{1}{1 + e^{\beta \left[ W_A(t) W_B(t) - 1 \right]}}$$

where $\frac{1}{\beta}$ is the indifference to relative wealth (for details see Ref. [12]). For $\beta = 0$, i.e., $p(A|A, B) = \frac{1}{2}$, the minimum exchange model is retrieved, where, in the steady state, the entire wealth belongs to a single agent (condensation). However, for a finite value of $\beta$, the poorer agent has a higher probability of winning. For large $\beta$, the asymptotic distribution is exponential, similar to the random exchange model. At the transition between these two very different types of distributions (condensate and exponential) we observe a power-law distribution of wealth [Fig. 3 (right)].

5 Discussion

The two models discussed here for generating Pareto-like distribution of wealth are both instances of the “Rich Are Different” principle, implemented in the formalism of asset exchange models. It is interesting to note that other recently proposed models for generating Pareto law also use this principle, whether this is in terms of kinetic theory as in the present paper [16, 17] or in a network context [18, 19]. This leads us to conclude that asymmetry in agent-agent interactions is a crucial feature of models for generating distributions having power-law tails.

To conclude, we have presented two models illustrating the general principle of how Pareto-like distribution of wealth (as observed in empirical observations in society) can be reproduced by implementing asymmetric interactions between agents in asset exchange models. In the models presented here the asymmetry is based on wealth of agents, with the rich agents behaving differently from the poor, either in terms of net wealth changing hands, or the probability of gaining net wealth out of a trade. One of the models is possibly the simplest asset exchange model that gives a power-law distribution. The results are also very robust, the power law being observed for a wide variety of parameter distributions. The different values of $\alpha$ obtained for different parameter distributions is a possible explanation of why different Pareto exponents have been measured in different societies, as well as in the same society at different times.

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