Finite size scaling of random XORSAT

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Abstract

We consider a “configuration model” for random XORSAT which is a random system of $n$ equations over $m$ variables in $\mathbb{F}_2$. Each equation is of the form $y_1 + y_2 + \cdots + y_k = b$ where $k \geq 3$ is fixed, $y_1, y_2, \cdots$ are variables (not necessarily distinct) and $b \in \mathbb{F}_2$. The equations are chosen independently and uniformly at random with replacement. It is known [5, 4, 8] that there exists $\rho_k$ such that $m/n = \rho_k$ is a sharp threshold for the satisfiability of this system. In this note we show that for the configuration model, the width of SAT-UNSAT transition window for random $k$-XORSAT is $\Theta(n^{-1/2})$ and also derive the exact scaling function.

Key words and phrases. Random $k$-XORSAT, Random constraint satisfaction problems, Finite size scaling.

1 Introduction

Consider a random instance of a system of $n$ equations over $m$ variables in $\mathbb{F}_2$ as follows. The $j$-th equation ($j \in [n]$) is of the form $y^1_j + y^2_j + \cdots + y^k_j = b^j$, where $y^1_j, y^2_j, \cdots, y^k_j$ are chosen independently and uniformly from the variables $x_1, x_2, \cdots, x_m$; $b^j$ is uniform in $\{0,1\}$ and is independent with $y^1_j, y^2_j, \cdots, y^k_j$. The tuples $(y^1_j, y^2_j, \cdots, y^k_j; b^j)$’s are also independent. We refer to this system as $E_{k,m,n}$. A natural object of interest related to $E_{k,m,n}$ is:

$$P_k(m, n) = \mathbb{P}(E_{k,m,n} \text{ is solvable in } \mathbb{F}_2).$$

It is known that $P_k(m, n)$ exhibits a sharp phase transition around a critical value $\rho_k$ of the ratio $m/n$ when $k \geq 3$. More precisely, $\lim_{m,n \to \infty, m/n \to \rho} P_k(m, n) = 1$ or 0 accordingly as $\rho > \rho_k$ or $< \rho_k$ respectively. This was first shown by Dubois and Mandler [5] for $k = 3$ and independently by Dietzfelbinger et. al. [4] and Pittel and Sorkin [8] for all $k \geq 3$. In this paper we determine the finite size scaling behavior of $P_k(m, n)$ around the threshold $\rho_k$. Our main result is the following theorem.

**Theorem 1.1.** Let $k \geq 3$ and $m = \lfloor n \rho_k + r n^{1/2} \rfloor$ for some $r \in \mathbb{R}$. There exist positive numbers $s_k, C^*_k$ depending only on $k$ and a positive constant $c^*$ such that for all large enough $n$

$$|P_k(m, n) - \Phi(rs_k)| \leq C^*_k n^{-c^*},$$

where $\Phi(.)$ is the standard Gaussian distribution function.

1.1 Backgrounds and related works

The existence of a sharp threshold for satisfiability of random $k$-XORSAT for general $k \geq 3$ was established separately by Dietzfelbinger et. al. [4] and Pittel and Sorkin [8]. Their approaches are somewhat different and for the purpose of this paper we will discuss the latter work in particular. In their paper Pittel and Sorkin first derived a similar threshold for what they called a “constrained” $k$-XORSAT model (introduced by Dubois and Mandler in [5]) where the system of equations is uniformly random over the subclass of
systems in which each variable appears at least twice. They found the critical ratio of the number of variables to number of equations for this constrained setup as 1. The threshold \( \rho_k \) for unconstrained model was then derived as the variables-equations ratio such that the same in the core of the associated hypergraph approaches 1 in probability as \( n \) becomes large (see section 2, first paragraph for a detailed discussion). They refined their result for the constrained setup (see \[8, Theorem 2\]) to include cases when \( m - n \) wanders off to \( \infty \) or \( -\infty \) arbitrarily slowly. This sharp transition implies the scaling factor for the transition window should be \( n^{-1/\nu} \) if the typical fluctuation in the difference between number of variables and equations in the core is \( \Theta(n^{1-1/\nu}) \) for some \( \nu > 1 \). As a candidate for the scaling exponent \( \nu \), numerical simulations [7] seem to suggest the value \( \nu = 2 \) for \( k = 3 \). This is also the lower bound (assuming existence) proved in [12] for a class of problems including random XORSAT. Among other problems in the class perhaps the most relevant for us is the appearance of a non-empty core in random \( k \)-uniform hypergraphs. For this problem the lower bound was indeed found to be the true scaling exponent (see [2, 3]). A refined scaling law was established by Dembo and Montanari in [3] where they analyzed a naive algorithm for obtaining the core. In the same paper they remarked (see [3, Remark 2.6]) that their techniques were applicable to a wide variety of properties of the core “in the scaling regime” that their techniques were applicable to a wide variety of properties of the core “in the scaling regime” \( \rho = \rho_c + \tau n^{-1/2} \) where \( \rho_c \) is the threshold for the appearance of core. In this note we use the tools developed in [3] to determine the asymptotic distribution of the difference between number of variables and equations in the core (see Proposition 2.6) in a different regime namely when \( \rho_n \) approaches the satisfiability threshold \( \rho_k \). Combined with the result of [8] this then yields the scaling law of random \( k \)-XORSAT.

**Remark 1.2.** The usual random model for \( k \)-XORSAT is where each of the \( n \)-equations’ \( k \) variables are drawn uniformly without replacement from the set of all \( m \) variables. In contrast we work with a configuration model in this paper which allows same variable to appear multiple times in an equation. A similar model was considered in [8] for the constrained set up (see section 3). In fact the bounds in [8, Theorem 2] were first derived for this configuration model which has the critical threshold 1 and the uniformly random set up was subsequently tackled using them (see Lemma 7 and Corollary 8). As a result the critical threshold for our model is same as \( \rho_k \). The main difficulty with the uniformly random model is the analysis of the core and is purely technical in nature (see [3, Remark 2.7] for more discussions on this). However we expect the finite size scaling behavior to be identical for these two models up to a renormalization of the scaling function.

### 1.2 A word on the organization

In section 2 we discuss the results from [3] and [8] that are relevant for the current paper and provide a sketch of our proof. In section 3 we discuss the properties of some ODE’s which describe the evolution of a Markov process associated with the systems of equations leading to the core. Finally in section 4 we derive the asymptotic distribution of the difference between number of variables and equations in the core.

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### 2 The discussion of previous results and the sketch of proof

Central to our analysis is the so called peeling algorithm which removes equations from the system one at a time as long as there is a variable appearing exactly once. Before we make it precise, let us link the system to a \( k \)-uniform directed hypergraph with \( m \) vertices and \( n \) hyperedges. We do this by identifying the variables \( x_1, x_2, \cdots, x_m \) with vertices \( v_{x_1}, v_{x_2}, \cdots, v_{x_m} \) of a hypergraph \( H_{k,m,n} \) and then including, for each equation \( y_1^1 + y_2^1 + \cdots + y_k^1 = b_1 \) in the system, the ordered list of vertices \( (v_{y_1}, v_{y_2}, \cdots, v_{y_k}) \) as
We denote the transition kernel by \( \hat{p}(\tau(z + 1)) \) after \( \tau \) steps. As proved in \( \text{[3, Lemma 3.1]} \), this process is a time inhomogeneous Markov Process. Let us introduce some notations and definitions. Let \( H \) be a hyperedge. The degree of a vertex in \( H_{k,m,n} \) (or the variable it corresponds to) is the total number of times it occurs in all the hyperedges of \( H_{k,m,n} \) counting repetitions. The 2-core (or simply the core) of \( H_{k,m,n} \) is the maximal subhypergraph such that minimum degree of vertices in it is at least 2. In terms of \( E_{k,m,n} \) it is the largest subsystem of equations such that any variable appearing in it appears at least twice. We will also refer to this subsystem as the core and the particular usage should be clear from the context.

At each step the peeling algorithm removes an equation from the system which it picks uniformly from the set of all equations containing at least one variable of degree 1. It is easy to see that this algorithm stops at the core. We can naturally associate a \( \mathbb{Z}^2 \)-valued process \( \{ z^\tau \} = \{(z_1^\tau, z_2^\tau), n \geq \tau \geq 0 \} \) with this algorithm, where \( z_1^\tau \) and \( z_2^\tau \) are respectively the number of variables of degree 1 and \( \geq 2 \) after \( \tau \) steps. As proved in \( \text{[3, Lemma 3.1]} \), this process is a time inhomogeneous Markov Process. Let us denote its transition probabilities as \( W^+_\tau(\Delta z^\tau \mid z^\tau) = P(\Delta z^\tau \mid z^\tau \tau = \tau^\tau) \) for large \( n \) and “suitable” range of \( \tau \) this transition kernel is well approximated \( \text{(\text{[3, Lemma 4.5]})} \) by a simpler transition kernel given by,

\[
\hat{p}(\Delta z^\tau (\tau + 1) = \Delta z^\tau + (q_1 - q_0, q_1) \mid z^\tau (\tau) = \Delta z^\tau) = (k - 1) p_{0,1}^0 q_{1,1}^1 q_{0,2}^2, \tag{2.1}
\]

where \( q_0 + q_1 + q_2 = k \) and \( p_0, p_1, p_2 \) are defined as follows. For \( \Delta z^\tau = z^\tau/n, \theta = \tau/n, \)

\[
p_0 = \frac{\max(x_1, 0)}{k(1 - \theta)} \quad, \quad p_1 = \frac{x_2 \lambda^2}{k(1 - \theta)(e^\lambda - 1 - \lambda)} \quad, \quad p_2 = \frac{x_2 \lambda}{k(1 - \theta)}, \tag{2.2}
\]

where for \( x_2 > 0, \lambda \) is the unique positive solution of

\[
f_1(\lambda) \equiv \frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = \frac{k(1 - \theta) - \max(x_1, 0)}{x_2}, \tag{2.3}
\]

so that \( p_0 + p_1 + p_2 = 1 \) whereas for \( x_2 = 0, p_1 \) and \( p_2 \) are set at 0 and \( 1 - p_0 \) respectively by continuity. We denote this transition kernel by \( W^\theta(\Delta z^\tau \mid \Delta z^\tau) \). In fact the definition can be extended to include all vectors in \( \Delta z^\tau \in \mathbb{R}^2 \) by defining

\[
\hat{W}_\theta(\Delta z^\tau \mid \Delta z^\tau) = \hat{p}(\Delta z^\tau (\tau + 1) = \Delta z^\tau + (q_1 - q_0, q_1) \mid z^\tau (\tau) = \Delta z^\tau), \tag{2.4}
\]

where for each \( \theta \in [0, 1) \) \( K_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denotes the projection onto the convex set \( K_\theta \equiv \{ z^\tau \in \mathbb{R}^2 : x_1 + 2x_2 \leq k(1 - \theta) \} \). A consequence of this approximation is that we can couple the Markov processes with kernels \( W^+_\tau(\Delta z^\tau \mid z^\tau) \) and \( W^\theta(\Delta z^\tau \mid \Delta z^\tau) \) (and the same initial conditions) while keeping their “graphs” close to each other with high probability. In order to state this result precisely we need to introduce some notations and definitions. Let \( G_k(n,m) \) denote the collection of all possible instances of \( H_{k,m,n} \) and \( P_{G_k(n,m)}(\cdot) \) denote the uniform distribution on the set. Define two \( \mathbb{Z}^2 \)-valued Markov chains with distributions \( P_{n,\rho}(\cdot) \) and \( \hat{P}_{n,\rho}(\cdot) \) respectively as follows:

\[
P_{n,\rho}(\Delta z^\tau (0) = \Delta z^\tau) = \hat{P}_{n,\rho}(\Delta z^\tau (0) = \Delta z^\tau) = P_{G_k(n,m)}(\Delta z^\tau (G) = \Delta z^\tau),
\]

if \( \Delta z^\tau \in \mathbb{Z}^2_+ \) such that \( z_1 + 2z_2 \leq nk \), and 0 otherwise. Coming to the transition kernels define

\[
W^\theta(\Delta z^\tau \mid \Delta z^\tau) = \begin{cases} W^+_\tau(\Delta z^\tau \mid \Delta z^\tau) & \text{if } z_1 \geq 1, n^{-1} \Delta z^\tau \in K_{\tau/n}, \\ \hat{W}_{\tau/n}(\Delta z^\tau \mid n^{-1} \Delta z^\tau) & \text{otherwise} \end{cases}
\]

Now evolve the two Markov chains according to:

\[
P_{n,\rho}(\Delta z^\tau (\tau + 1) = \Delta z^\tau + \Delta \Delta z^\tau (\tau) = \Delta z^\tau) = W^\theta_{\tau/n}(\Delta z^\tau \mid \Delta z^\tau), \quad \text{and} \quad \hat{P}_{n,\rho}(\Delta z^\tau (\tau + 1) = \Delta z^\tau + \Delta \Delta z^\tau (\tau) = \Delta z^\tau) = \hat{W}_{\tau/n}(\Delta z^\tau \mid n^{-1} \Delta z^\tau)
\]
for \( \tau = 0, 1, \ldots, n - 1 \). Our original Markov process is not quite same as the one associated with \( \mathbb{P}_{n, \rho}(\cdot) \). However, they coincide until the first time \( \tau \) such that \( z_1(\tau) = 0 \), i.e., when the peeling algorithm terminates at the core. For this reason whenever we mention the process \( \{\mathbf{z}(\tau)\} \) without any reference to its distribution, the latter is implicitly understood to be \( \mathbb{P}_{n, \rho} \). Now we can state the lemma we were looking for.

**Lemma 2.1.** [3, Lemma 5.1] There exist finite \( C_\ast = C_\ast(k, \epsilon) \) and positive \( \lambda_\ast = \lambda_\ast(k, \epsilon) \), and a coupling between \( \{\mathbf{z}(\tau)\} \) and \( \{\mathbf{z}'(\tau)\} \) defined \( \mathbb{P}_{n, \rho}(\cdot) \), such that for any \( n, \rho \in [\epsilon, 1/\epsilon] \) and \( r > 0 \),

\[
\mathbb{P}\left( \sup_{\tau \leq \tau_s} ||\mathbf{z}(\tau) - \mathbf{z}'(\tau)|| > r \right) \leq C_\ast e^{-\lambda_\ast r},
\]

where \( \tau_s \leq n \) denotes the first time such that \( (\mathbf{z}(\tau_s), \tau_s) \notin \mathcal{Q}(\epsilon) \) and for each \( \epsilon > 0 \),

\[
\mathcal{Q}(\epsilon) \equiv \{(\mathbf{z}, \tau) : -nk + ne \leq z_1; ne < z_2; 0 \leq \tau \leq n(1 - \epsilon); ne \leq (n - \tau)k - \max(z_1, 0) - 2z_2 \}.
\]

Another important ingredient is the asymptotic joint distribution of number of degree 1 and 2 vertices in \( \mathcal{H}_{k, m, n} \). For \( \mathbf{\mu} \in \mathbb{R}^d \) and a positive definite \( d \)-dimensional matrix \( \Sigma \), denote the \( d \)-dimensional Gaussian density of mean \( \mathbf{\mu} \) and covariance \( \Sigma \) by \( \phi_d(\cdot; \mathbf{\mu}; \Sigma) \). Entries of the vector \( \mathbf{z} = (z_1, z_2) \) denote the number of vertices of degree 1 and at least 2 respectively in a random graph drawn uniformly from \( \mathcal{G}_k(n, [n\rho]) \).

Define

\[
\mathbf{\bar{y}} = \mathbf{\bar{y}}(\rho) = (ke^{-k/\rho}, \rho(1 - e^{-k/\rho}) - ke^{-k/\rho}),
\]

and denote by \( \mathcal{Q} = \mathcal{Q}(\rho) \) some positive definite matrix which we will specify later. Then,

**Lemma 2.2.** [3, Lemma 4.4] For any \( \epsilon > 0 \) there exist positive constants \( \kappa_0, \kappa_1, \kappa_2, \kappa_3 \), such that for all \( n, r, \rho \in [\epsilon, 1/\epsilon] \),

\[
||\mathbb{E}\mathbf{\bar{z}} - n\mathbf{\bar{y}}|| \leq \kappa_0,
\]

\[
\mathbb{P}(||\mathbf{\bar{z}} - \mathbb{E}\mathbf{\bar{z}}|| \geq r) \leq \kappa_1 e^{-r^2/\kappa_2 n},
\]

and

\[
\sup_{\mathbf{u} \in \mathbb{R}^2} \sup_{x \in \mathbb{R}} \mathbb{P}(\mathbf{u} \cdot \mathbf{\bar{z}} \leq x) - \int_{\mathbf{u} \cdot \mathbf{\bar{z}} \leq x} \phi_2(\mathbf{\bar{z}}; n\mathbf{\bar{y}}; n\mathcal{Q}) d\mathbf{\bar{z}} \leq \kappa_3 n^{-1/2}.
\]

One immediate consequence of Lemma 2.2 is that \( n^{-1}\mathbf{\bar{z}}(0) \) converges in probability to \( \mathbf{\bar{y}}(0) \). A natural question then is if we can say something similar about \( n^{-1}\mathbf{\bar{z}}(n\theta) \). If the convergence still holds, one and perhaps the only reasonable candidate for the limit would be \( \lim n^{-1}\mathbb{E}\mathbf{\bar{z}}(n\theta) = \mathbf{\bar{y}}(\theta) \) (say). Since the transition kernel in (2.1) depends on \( \mathbf{\bar{z}} \) and \( \tau \) only through the scaled variables \( \mathbf{\bar{z}} \) and \( \theta \), the gradient of \( \mathbf{\bar{y}}(\theta) \) should roughly equal \( n^{-1}\mathbb{E}(\Delta \mathbf{\bar{z}}(n\theta))/n^{-1} = \mathbb{E}(\Delta \mathbf{\bar{z}}(n\theta)) \) where the expectations are with respect to the same kernel. We further get from (2.1) that this expectation is \((-1 + (k - 1)(p_1 - p_0), -(k - 1)p_1)\) which we denote by \( \mathbf{\bar{F}}(\mathbf{\bar{y}}(\theta), \theta) \). Thus the process \( \{\mathbf{\bar{z}}(n\theta)/n\}_{0 \leq \theta < 1} \) can be hoped to concentrate around the solution of the ODE,

\[
\frac{d\mathbf{\bar{y}}}{d\theta}(\theta) = \mathbf{\bar{F}}(\mathbf{\bar{y}}(\theta), \theta),
\]

with the initial condition \( \mathbf{\bar{y}}(\theta) = \mathbf{\bar{y}} \) from (2.4). We denote the solution to (2.5) subject to the initial conditions (2.4) as \( \mathbf{\bar{y}}(\theta, \rho) \) although the dependence on \( \rho \) will often be suppressed. We discuss the analytical properties of \( \mathbf{\bar{y}}(\theta, \rho) \) in section [3], but for the time being let us precisely formulate the concentration of \( \mathbf{\bar{z}} \) around it. First we present a similar result for \( \mathbb{P}_{n, \rho} \).

**Lemma 2.3.** [3, Lemma 5.2] For any \( k \geq 3 \) and \( \epsilon > 0 \) there exist positive \( \eta \leq \epsilon \), and \( C_0, C_1, C_2, C_3 \), such that, for any \( n, \rho \in [\epsilon, 1/\epsilon] \) and \( \tau \in \{0, \ldots, [n(1 - \epsilon)]\} \),
(a) $\bar{z}(\tau)$ is exponentially concentrated around its mean
$$\mathbb{P}_{n,\rho}(\|\bar{z}(\tau) - \mathbb{E}\bar{z}(\tau)\| \geq r) \leq 4e^{-r^2/C_0 n}.$$

(b) $\bar{z}(\tau)$ is close to the solution of the ODE \((2.5)\),
$$\mathbb{E}\|\bar{z}(\tau) - n\bar{y}(\tau/n)\| \leq C_1 \sqrt{n\log n}.$$

(c) $(\bar{z}(\tau), \tau) \in Q(\eta)$ with high probability; more precisely,
$$\mathbb{P}_{n,\rho}(\bar{z}(\tau), \tau) \notin Q(\eta)) \leq C_2 e^{-C_3 n}.$$

Part (c) and Lemma 2.3 should give us analogous results for $\mathbb{P}_{n,\rho}$. This is confirmed by the following two corollaries.

**Corollary 2.4 (Corollary 5.4., [3]).** For any $\epsilon > 0$, there exists $0 < \eta < \epsilon$ and positive, finite constants $C_4, C_5$ such that if $\rho \in [\epsilon, 1/\epsilon]$, then
$$\mathbb{P}_{n,\rho}(\bar{z}(\tau), \tau) \notin Q(\eta) \quad \forall \ 0 \leq \tau \leq n(1 - \epsilon) \geq 1 - C_4 e^{-C_5 n}.$$

**Corollary 2.5.** For any $\epsilon > 0$, there exist finite, positive $A = A(k, \epsilon)$, $C = C(k, \epsilon)$, and a coupling between $\{\bar{z}(\tau)\} \overset{d}{=} \mathbb{P}_{n,\rho}(\cdot)$ and $\{\bar{z}(\tau)\} \overset{d}{=} \mathbb{P}_{n,\rho}(\cdot)$, such that for any $n, \rho \in [\epsilon, 1/\epsilon]$,
$$\mathbb{P}\left( \sup_{\tau \leq n(1 - \epsilon)} \|\bar{z}(\tau) - \bar{z}(\tau)\| \geq A \log n \right) \leq C n^{-1}.$$

**Proof.** This follows immediately from the previous corollary and Lemma 2.3 \qed

In section 3 we show that there exists $\theta_k \in (0, 1)$ such that (1) $\theta_k = \min_{\theta \in [0, 1]} \{\theta \in [0, 1] : y_1(\theta, \rho_k) = 0\}$ and (2) $y_2(\theta_k, \rho_k) = 1 - \theta_k$. Furthermore in a small neighborhood of $\theta_k$,
$$y_1(\theta, \rho_k) \approx \frac{\partial y_1}{\partial \theta}(\theta_k, \rho_k)(\theta - \theta_k), \text{ and } y_2(\theta) \approx 1 - \theta_k + \frac{\partial y_2}{\partial \theta}(\theta_k, \rho_k)(\theta - \theta_k),$$
where both of these partial derivatives are negative. Fluctuations of $\bar{z}(n\theta_k)$ around $\bar{y}(n\theta_k)$ are gained in $n\theta_k$ stochastic steps, and are therefore should be of order $\sqrt{n}$. We show in section 4 that the rescaled variable $(\bar{z}(n\theta_k) - \bar{y}(n\theta_k))/\sqrt{n}$ converges to a Gaussian random vector. Its covariance matrix $\mathbb{Q}(\theta, \rho) = \{Q_{ab}(\theta, \rho); 1 \leq a, b \leq 2\}$ is the symmetric positive definite solution of the ODE:
$$\frac{d\mathbb{Q}(\theta)}{d\theta} = \mathbb{G}(\bar{y}(\theta), \theta) + \mathbb{A}(\bar{y}(\theta), \theta)\mathbb{Q}(\theta) + \mathbb{Q}(\theta)\mathbb{A}(\bar{y}(\theta), \theta)^T,$$
where $\mathbb{A}(\bar{y}(\theta), \theta) = \{A_{ab}(\bar{y}(\theta), \theta); 1 \leq a, b \leq 2\}$ for $A_{ab}(\bar{y}(\theta), \theta) = \partial_{x_a} F_a(\bar{y}(\theta), \theta)$, and $\mathbb{G}(\bar{y}(\theta), \theta)$ is the covariance of $\bar{z}(\tau + 1) - \bar{z}(\tau)$ w.r.t the transition kernel \((2.1)\), i.e., the nonnegative definite symmetric matrix with entries
\[
\begin{align*}
\mathbb{G}_{11}(\bar{y}(\theta), \theta) &= (k - 1)[p_0 + p_1 - (p_0 - p_1)^2], \\
\mathbb{G}_{12}(\bar{y}(\theta), \theta) &= -(k - 1)[p_0 p_1 + p_1 (1 - p_1)], \\
\mathbb{G}_{22}(\bar{y}(\theta), \theta) &= (k - 1)p_1 (1 - p_1).
\end{align*}
\]
Like before we will omit $\rho$ from the notation when its values is fixed in a context. The positive definite initial condition $Q(0)$ for \((2.5)\) is computed on the original graph ensemble, and is given by
\[
\begin{align*}
Q_{11}(0) &= \frac{k}{\gamma} e^{-2\gamma} [(e\gamma - 1) + \gamma - \gamma^2], \\
Q_{12}(0) &= -\frac{k}{\gamma} e^{-2\gamma} [(e\gamma - 1) - \gamma^2], \\
Q_{22}(0) &= \frac{k}{\gamma} e^{-2\gamma} [(e\gamma - 1) + \gamma (e\gamma - 2) - \gamma^2 (1 + \gamma)],
\end{align*}
\]
where $\gamma = \frac{k}{\rho}$. We can now state the following proposition whose proof will be the main focus of this paper.
Proposition 2.6. Let \( \tilde{\xi}(r) \) is a bivariate Gaussian random vector of mean \( \frac{\partial \tilde{\eta}}{\partial r} \) and variance \( \tilde{Q} \) (both evaluated at \( \theta = \theta_k \) and \( \rho = \rho_k \)). Denote by \( n_{\text{core}}, m_{\text{core}} \), the number of equations and variables in the core of \( \mathcal{E}_{k,m,n} \) and by \( \mathbb{P}_{\text{core}}(.) \) their joint law. Then there exists a positive constant \( \eta \) such that for all \( A > 0, r \in \mathbb{R} \), and \( n \) large enough, if \( \rho_n = \rho_k + rn^{-1/2} \), we have

\[
|\mathbb{P}_{\text{core}}(m_{\text{core}} < A \log n) - \mathbb{P}(\xi_1(r) + \xi_2(r) \geq 0)| \leq n^{-\eta},
\]

and

\[
|\mathbb{P}_{\text{core}}(m_{\text{core}} \geq -A \log n) - \mathbb{P}(\xi_1(r) + \xi_2(r) \leq 0)| \leq n^{-\eta}.
\]

The key to Proposition 2.6 is the construction of another Markov chain as in [3, equation 2.12] which, within the critical time window \( J_n \equiv [n\theta_k - n^\beta, n\theta_k - n^\beta] \) (for some \( \beta \in (\frac{1}{2}, 1) \)), serves as a good approximation for the chain with transition kernel (2.1). This Markov chain is evolved as:

\[
\tilde{z}'(\tau + 1) = \tilde{z}'(\tau) + \tilde{\alpha}_r(n-1\tilde{z}'(\tau) - \tilde{y}(\tau/n)) + \Delta_r,
\]

where \( \tau_n \equiv [n\theta_k - n^\beta], \tilde{\alpha}_r \equiv \mathbb{I}_{\tau_n} \tilde{\alpha}(\tilde{y}(\tau/n, \rho), \tau/n) \) and \( \Delta_r \)'s are independent random variables with mean \( \tilde{F}((\tilde{y}(\tau/n, \rho), \tau/n)) \) and covariance \( \tilde{G}((\tilde{y}(\tau/n, \rho), \tau/n)) \). Denote by \( \tilde{\mathbb{P}}_{n,\rho}(\cdot) \) the law of the \( \mathbb{R}^2 \)-valued Markov chain \( \{\tilde{z}'(\tau)\} \), where \( \tilde{z}'(0) \) has the uniform distribution on the graph ensemble \( \mathcal{G}_n(m,n) \) for \( m \equiv [n\rho] \), and

\[
\tilde{\mathbb{P}}_{n,\rho}(\tilde{z}'(\tau + 1) = \tilde{z}'(\tau) + \Delta_r + \tilde{\alpha}_r(n-1\tilde{z}'(\tau) - \tilde{y}(\tau/n))|\tilde{z}'(\tau) = \tilde{z}')
\]

\[
= \tilde{W}_{\tau/n}(\Delta_r|\tilde{y}(\tau/n)).
\]

Then we have the following proposition regarding approximation of \( \tilde{\mathbb{P}}_{n,\rho}(\cdot) \) by \( \tilde{\mathbb{P}}_{n,\rho}(\cdot) \) which is a slightly modified version of [3, Proposition 5.5] and is central to proving Proposition 2.6.

Proposition 2.7. Fixing \( \beta \in (\frac{1}{2}, 1) \) and \( \beta' < \beta \), for any \( \delta > \frac{1}{2} \lor (\beta - \frac{1}{4}) \), there exist constants \( \alpha, c \) and a coupling of the processes \( \{\tilde{z}(\cdot)\} \) of distribution \( \tilde{\mathbb{P}}_{n,\rho}(\cdot) \) and \( \{\tilde{z}'(\cdot)\} \) of distribution \( \tilde{\mathbb{P}}_{n,\rho}(\cdot) \) such that for all \( n \) and \( |\rho - \rho_k| \leq n^{\beta'-1} \),

\[
\mathbb{P}\left( \sup_{\tau \in J_n \equiv [n\theta_k - n^\beta, n\theta_k + n^\beta]} \|\tilde{z}(\tau) - \tilde{z}'(\tau)\| \geq cn^\delta \right) \leq \alpha n^{-1}
\]

\[
(2.12)
\]

Proof. Very similar to the proof of [3, Proposition 5.5]. All the arguments are valid for general \( \rho \) bounded away from 0 and \( \infty \) except for [3, Corollary 5.3] which corresponds to a different threshold than ours (denoted as \( \rho_c \) in the paper). But this poses no problem as we have an analogue of this corollary (see Lemma 3.2) for \( \rho_k \).

The main usefulness of this approximation lies in the following observation. Taking

\[
\tilde{\mathbb{B}}_\sigma \equiv \left( \mathbb{I} + \frac{1}{n} \tilde{\alpha}_\sigma \right) \ldots \left( \mathbb{I} + \frac{1}{n} \tilde{\alpha}_\sigma \right),
\]

\[
(2.13)
\]

for integers \( 0 \leq \sigma \leq \tau \) (while \( \tilde{\mathbb{B}}_0 \equiv \mathbb{I} \) in case \( \tau < \sigma \), we see that

\[
\tilde{z}'(\tau) = \tilde{\mathbb{B}}_0^{-1}\tilde{z}(0) + \sum_{\sigma=0}^{\tau-1} \tilde{\mathbb{B}}_{\sigma+1}^{-1}(\Delta_\sigma - \tilde{\alpha}_\sigma \tilde{y}(\tau/n, \rho))
\]

\[
(2.14)
\]

is a sum of (bounded) independent random variables, hence of approximately Gaussian distribution which prepares the ground for Proposition 2.6.

We are just steps away from proving Theorem 1.1. [3, Lemma 3.1] tells us that conditional on \( n_{\text{core}} \) and \( m_{\text{core}} \) i.e. the number of equations and variables in the core, the system is uniformly distributed on all possible instances of \( \mathcal{E}_{k,m_{\text{core}},n_{\text{core}}} \) that has no variable with degree 1. Such a system of equations was called constrained k-XORSAT in [8]. Our very last ingredient is the following theorem proved from [8] (see the discussions in Remark 1.2).
**Theorem 2.8.** Let $Ax = b$ be a uniformly random constrained $k$-XORSAT instance with $n$ equations and $m$ variables, with $k \geq 3$ and $m, n \to \infty$ with $\lim \inf n/m > 2/k$. Then, for any $w(n) \to +\infty$, if $n + w(n) \leq m$ then $Ax = b$ is almost surely satisfiable, with satisfiability probability $1 - O((n^{-k})^2 + \exp(-c^*\omega(n)))$, while if $n \geq m + \omega(n)$ then $Ax = b$ is almost surely unsatisfiable, with satisfiability probability $O(2^{-\omega(n)})$. Here $c^*$ is a positive constant.

**Proof of Theorem 1.1:** Theorem 1.1 now follows immediately from Proposition 2.6 and Theorem 2.8

### 3 Solutions to ODE’s (2.5) and (2.6)

In this section we will discuss the properties of the solutions to (2.5) and (2.6) which will be used repeatedly throughout our analysis. The results are based on the continuity of $(\overrightarrow{x}, \theta) \mapsto p_a(\overrightarrow{x}, \theta)$, $a = 0, 1, 2$ on the following compact subsets of $\mathbb{R}^2 \times \mathbb{R}_+$:

$$
\overrightarrow{q}(\epsilon) \equiv \{(\overrightarrow{x}, \theta) : -k \leq x_1; 0 \leq x_2; \theta \in [0, 1 - \epsilon]; 0 \leq (1 - \theta)k - \max(x_1, 0) - 2x_2 \},
$$

and

$$
\overrightarrow{q}_+(\epsilon) = \overrightarrow{q}(\epsilon) \cap \{x_1 \geq 0\}.
$$

**Lemma 3.1.** [3, Lemma 4.1] For any $\epsilon > 0$, the functions $(\overrightarrow{x}, \theta) \mapsto p_a(\overrightarrow{x}, \theta)$, $a = 0, 1, 2$ are $[0, 1]$-valued, Lipschitz continuous on $\overrightarrow{q}(\epsilon)$. Further, on $\overrightarrow{q}_+(\epsilon)$ the functions $(\overrightarrow{x}, \theta) \mapsto p_a(\overrightarrow{x}, \theta)$ have Lipschitz continuous partial derivatives.

The next proposition provides some important properties of the solutions for general $\rho$. Define for $\rho > 0$, $h_{\rho, 1}(u) \equiv u - 1 + \exp(-ku^{1-1}/\rho)$ and $h_{\rho, 2}(u) \equiv 1 - (1 + ku\rho^{-1}/\rho)\exp(-ku\rho^{-1}/\rho)$.

**Proposition 3.2.** [3, Proposition 4.2] For any $\epsilon > 0$, $\theta < 1 - \epsilon$, the ODE $\overrightarrow{\gamma}(\theta)$ admits a unique solution $y(\theta)$ subject to the initial conditions (2.4), and the ODE $\overrightarrow{\gamma}(\theta)$ admits a unique, positive definite, solution $\overrightarrow{Q}$ subject to the initial conditions (2.8), such that:

(a) For any $\epsilon > 0$, $\theta < 1 - \epsilon$, we have that $(\overrightarrow{y}(\theta, \rho), \theta)$ is in the interior of $\overrightarrow{q}(\epsilon)$, with both functions $(\theta, \rho) \mapsto \overrightarrow{y}$ and $\theta \mapsto \overrightarrow{Q}$ Lipschitz continuous on $(\theta, \rho) \in [0, 1 - \epsilon] \times [\epsilon, 1/\epsilon]$.

(b) Let $u(\theta) \equiv (1 - \theta)^{1/k}$ and $\theta_-(\rho) \equiv \inf\{\theta \geq 0 : h_{\rho, 1}(u(\theta)) < 0\} \wedge 1$. Then, for $\theta \in [0, \theta_-(\rho)]$,

$$
y_1(\theta, \rho) = ku(\theta)^{k-1}[u(\theta) - 1 + e^{-\gamma u(\theta)^{k-1}}], \quad (3.1)
$$

$$
y_2(\theta, \rho) = \frac{k}{\gamma}[1 - e^{-\gamma u(\theta)^{k-1}} - \gamma u(\theta)^{k-1}e^{-\gamma u(\theta)^{k-1}}], \quad (3.2)
$$

(where $\gamma = k/\rho$). In particular, $(\theta, \rho) \mapsto \overrightarrow{y}$ is infinitely continuously differentiable and $(\theta, \rho) \mapsto \overrightarrow{Q}$ is Lipschitz continuous on $\{(\theta, \rho) : \theta \leq \min(\theta_-(\rho), 1 - \epsilon), \epsilon \leq \rho \leq 1/\epsilon\}$.

Using the previous two results we can now derive several important properties of the solution to (2.5) when $\rho = \rho_k$.

**Proposition 3.3.** Define $\theta_*(\rho) \equiv \inf\{\theta \geq 0 : h_{\rho, 1}(u(\theta)) \leq 0\}$. Then,

(a) $\theta_*(\rho) = 1$ for $\rho > \max_{x \geq 0} \frac{k(1-e^{-x})^{k-1}}{x}$. For $\rho \leq \max_{x > 0} \frac{k(1-e^{-x})^{k-1}}{x}$, $\theta_*(\rho) = 1 - (1 - e^{-\lambda_\rho})^k$ where $\lambda_\rho$ is the maximum positive solution to $\rho = \frac{k(1-e^{-\lambda})^{k-1}}{\lambda}$, Furthermore for these values of $\rho$, $y_2(\theta_*(\rho), \rho) = \frac{k}{f_1(\lambda_\rho)}(1 - \theta_*(\rho))$ where $f_1$ is the same function as in (2.3).

(b) $\rho_k$ satisfies the equation $y_2(\theta_*(\rho_k), \rho_k) = 1 - \theta_*(\rho_k) > 0$. 

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(II) Denoting $\theta^*_k(p_k)$ by $\theta_k$, we have for $p = p_k$, $y'_1(\theta_k) < 0$ and $y'_2(\theta_k) < 0$.

(III) Although $\bar{y}(\theta)$ (for $p = p_k$) may not be twice continuously differentiable at $\theta = \theta_k$, $\bar{y}(\theta)$ is nonetheless twice continuously differentiable when considered on $[0, \theta_k]$ and $[\theta_k, \theta_k + \epsilon]$ separately for some $\epsilon > 0$.

Proof. (a) Since $h_{\rho,1}(u(0)) > 0$ and $h_{\rho,1}(u(1)) = 0$ we have $\theta_*(\rho) \in (0, 1]$ for all $\rho > 0$. Notice that

$$h_{\rho,1}(u) > 0 \Leftrightarrow \exp\left(-\frac{k u^{k-1}}{\rho}\right) > 1 - u,$$

for $u \in (0, 1)$. Writing $-\log(1 - u) = x \in (0, \infty)$ we find that,

$$\theta_*(\rho) = 1 \Leftrightarrow h_{\rho,1}(u) > 0 \ \text{for} \ u \in (0, 1) \ \Leftrightarrow \ \theta \geq \frac{k(1 - e^{-x})^{k-1}}{x}.$$ 

Hence $\theta_*(\rho) = 1$ for $\rho > \max_{x>0} \frac{k(1 - e^{-x})^{k-1}}{x}$. This chain of equivalences also give us that

$$\theta_*(\rho) = 1 - (1 - e^{-\lambda_\rho})^k$$

for $\rho \leq \max_{x>0} \frac{k(1 - e^{-x})^{k-1}}{x}$ and $\lambda_\rho$ being the maximum positive solution to $\rho = \frac{k(1 - e^{-\lambda})^{k-1}}{\lambda}$. The expression for $y_2(\theta_*(\rho), \rho)$ now follows from a routine algebra:

$$y_2(\theta_*(\rho), \rho) = \rho h_{\rho,2}(u(\theta_*(\rho))) = \frac{k(1 - e^{-\lambda_\rho})^k}{\lambda_\rho} - k(1 - e^{-\lambda_\rho})^{k-1} + (1 - e^{-\lambda_\rho})^k$$

$$= k \frac{1 - e^{-\lambda_\rho}(1 + \lambda_\rho)}{\lambda_\rho(1 - e^{-\lambda_\rho})}(1 - e^{-\lambda_\rho})^k = k \frac{\lambda_\rho}{f_1(\lambda_\rho)}(1 - \theta_*(\rho)).$$

(b) (I) [8 Theorem 16] tells us that

$$\rho_k = \frac{k(1 - e^{-\lambda_k})^{k-1}}{\lambda_k},$$

where $\lambda_k$ is the unique positive solution to $f_1(\lambda_k) = k$. $\lambda_k > 0$ since $\lim_{\lambda \to 0^+} f_1(\lambda) = 2$ and $k \geq 3$. Thus from part (a) we get $y_2(\theta_*(\rho_k), \rho_k) = 1 - \theta_*(\rho_k) > 0$.

(II) From (2.5) we get at $\rho = \rho_k$,

$$y'_1(\theta_k) = -1 + (k - 1)(p_1(\bar{y}(\theta_k, \rho_k), \theta_k) - p_0(\bar{y}(\theta_k, \rho_k), \theta_k)).$$

Since $y_1(\theta_k, \rho_k) = 0$, we have

$$y'_1(\theta_k) = (k - 1)\frac{y_2(\theta_k, \rho_k) \lambda_k^2}{k(1 - \theta_k)(e^{\lambda_k} - 1 - \lambda_k)} = (k - 1)\left(\frac{\lambda_k^2}{k(e^{\lambda_k} - 1 - \lambda_k)} - \frac{1}{k - 1}\right)$$

where $f_1(\lambda_c) = \frac{k(1 - \theta_k)}{y_2(\theta_k, \rho_k)} = k$ i.e. $\lambda_c = \lambda_k$ (see the discussion around (2.3)). Hence we can write

$$y'_1(\theta_k) = (k - 1)\left(\frac{\lambda_k}{e^{\lambda_k} - 1} - \frac{1}{k - 1}\right).$$

Thus in order to prove $y'_1(\theta_k) < 0$, we just need to show $\frac{\lambda_k - 1}{e^{\lambda_k} - \lambda_k - 1} > k - 1$. Now notice that

$$f_1(\lambda_k) = \frac{\lambda_k(e^{\lambda_k} - 1)}{e^{\lambda_k} - \lambda_k - 1} = k > k - 1.$$
So it suffices to show $e^{\lambda_k} \geq 1 + \lambda_k + \lambda_k^2$. But for $x > 0$, 
\[ e^x - 1 - x - x^2 > \frac{-x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} = \frac{x^2}{24}(x^2 + 4x - 12) > 0 \]
whenever $x \geq 2$. Since $f_1(\lambda_k) = k \geq 3$, $f_1(2) < 3$ and $f_1(x)$ is strictly increasing on $[0, \infty)$, it follows that $\lambda_k > 2$. On the other hand 
\[ y_2'(\theta_k) = -(k-1)p_1(\tilde{y}(\theta_k, \rho_k), \theta_k) = -(k-1)\frac{(1-\theta_k)^2}{k(1-\theta_k)} < 0. \]

(III) Part (b) of Proposition 3.2 already settles the $\theta \leq \theta_k$ part. For $\theta \geq \theta_k$ part, notice that from Lemma 3.1, Proposition 3.2 and the previous parts of the current proposition we get an $\epsilon \in (0, 1 - \theta_k)$ such that 
\[ F_1(y_1(\theta), \theta) < 0, \text{ and } k - \frac{1}{4} < \frac{k(1-\theta)}{y_2(\theta)} < k + \frac{1}{4} \]
on $(\theta_k, \theta_k + \epsilon]$. The former implies $y_1(\theta) < 0$ on $(\theta_k, \theta_k + \epsilon]$ and consequently $p_0(\tilde{y}(\theta), \theta) = 0$. Hence for $\theta \in [\theta_k, \theta_k + \epsilon]$ we have from (2.2) and (2.3) that 
\[ 1 - p_1(\tilde{y}(\theta), \theta) = p_2(\tilde{y}(\theta), \theta) = f_1^{-1}\left(\frac{k(1-\theta)}{y_2(\theta)}\right) \frac{y_2(\theta)}{k(1-\theta)}. \]

Since $F_2(y_2(\theta), \theta)$ is Lipschitz continuous on $[\theta_k, \theta_k + \epsilon]$ by Lemma 3.1, it suffices to prove that $f_1^{-1}\left(\frac{k(1-\theta)}{y_2(\theta)}\right)$ is continuously differentiable on $[\theta_k, \theta_k + \epsilon]$. But we have already shown $\frac{k(1-\theta)}{y_2(\theta)}$ lies between $k - \frac{1}{4}$ and $k + \frac{1}{4}$ on the same set. The proof is now complete with the observation that $f_1^{-1}$ is continuously differentiable on $[k - \frac{1}{4}, k + \frac{1}{4}]$. \[ \square \]

Consider a new sequence of vectors defined as 
\[ \tilde{y}^*(\tau + 1) = \tilde{y}^*(\tau) + n^{-1}\tilde{A}_\tau(\tilde{y}^*(\tau) - \tilde{y}(\tau/n)) + n^{-1}F(\tilde{y}(\tau/n), \tau/n), \] (3.3)
where $\tilde{y}^*(0) \equiv \tilde{y}(0, \rho)$ and $\tilde{A}_\tau \equiv \mathbb{1}_{r < \tau, k}(\tilde{y}(\tau/n), \rho, \tau/n)$. Also define the positive definite matrices 
\[ Q_\tau = \tilde{B}^{-1}_{\tau}(Q(0, \rho) + Q(0, \rho)') + \frac{1}{n} \sum_{\sigma = 0}^{\tau-1} \tilde{B}^{-1}_{\sigma+1}\mathcal{G}(\tilde{y}(\sigma/n), \sigma/n)', \]
(3.4)
where $\tilde{B}_{\tau}$ is same as in (2.13). (3.3) and (3.4) are discrete recursions corresponding to the mean and covariance of the process $\tilde{z}(\cdot)$ of (2.11). The lemma below shows that $\tilde{y}^*(\tau)$ and $Q_\tau$ are near the solutions of appropriate ODE’s up to time $\tau_n \equiv [n\theta_k - n^\beta]$ when $\rho$ is near $\rho_k$. The lemma and its proof are very similar to [3]. Lemma 4.3] except for a few modifications.

**Lemma 3.4.** Fixing $\beta \in (\frac{1}{2}, 1)$ and $\beta' < \beta$, we have for all sufficiently large $n$ and $|\rho - \rho_k| \leq n^{\beta'-1}$, 
\[ |y_1^*(\tau_n) + \frac{\partial y_1}{\partial \rho}(\theta_k, \rho_k)n^{\beta-1} - (\rho - \rho_k)\frac{\partial y_1}{\partial \rho}(\theta_k, \rho_k)| \leq Cn^{2(\beta-1)}, \] (3.5)
\[ |y_2^*(\tau_n) - (1 - \theta_k) + \frac{\partial y_2}{\partial \theta}(\theta_k, \rho_k)n^{\beta-1} - (\rho - \rho_k)\frac{\partial y_2}{\partial \rho}(\theta_k, \rho_k)| \leq Cn^{2(\beta-1)}, \] (3.6)
for some positive $C = C(\beta, \beta')$. Furthermore, the matrices $\{\tilde{B}_{\sigma} : \sigma, \tau \leq n\}$ and their inverses are uniformly bounded with respect to the $L_2$-operator norm (denoted $||.||$) and 
\[ ||Q_\tau - Q(\theta_k, \rho_k)|| \leq Cn^{\beta-1} \] (3.7)
for all $n$. 9
Proof. From part (a) of Proposition 3.2 we get that \( \bar{y}(\theta, \rho) \in \tilde{q}(\epsilon) \) for \( \theta \leq 1 - 2\epsilon \) and \( \rho \in [\epsilon, 1/\epsilon] \). Now choose \( \epsilon < (1 - \theta_k)/2 \). Part (b) of Proposition 3.3 says \( y_1(\theta, \rho_k) > 0 \) for \( 0 \leq \theta < \theta_k \) while \( y_1(\theta_k, \rho_k) = 0 \). From the same proposition we also have that \( \frac{\partial y_1}{\partial \theta}(\theta_k, \rho_k) < 0 \). Hence from Lipschitz continuity of \( p_\alpha(\bar{\mathcal{T}}, \theta) \), \( (a = 0, 1, 2) \) on \( \tilde{q}(\epsilon) \), as given by Lemma 3.1 we get \( y_1(\theta, \rho_k) \geq c_0 n^{\beta - 1} \) for some positive \( c_0 = c_0(\epsilon) \) whenever \( 0 \leq \theta \leq \theta_n \). Combined with the fact that \( (\theta, \rho) \mapsto \bar{y}(\theta, \rho) \) is Lipschitz continuous on \( (\theta, \rho) \in [0, 1 - \epsilon] \times [\epsilon, 1/\epsilon] \) (part (a), Proposition 3.2), this further implies \( y_1(\theta, \rho) \geq c_1 n^{\beta - 1} \) for some positive \( c_1 = c_1(\epsilon, \beta, \beta') \) whenever \( 0 \leq \theta \leq \theta_n \), \( \rho - \rho_k \leq n^{\beta - 1} \) and \( n \geq n_0(\epsilon, \beta, \beta') \). Consequently \( \bar{y} \in \tilde{q}_+(\epsilon) \) for \( 0 \leq \theta \leq \theta_n \), \( \rho - \rho_k \leq n^{\beta - 1} \) and all such \( n \). Hence by Lemma 3.1 we obtain that the entries of the matrices \( \mathbb{A}_\tau \) are bounded uniformly in \( \tau \leq \tau_n \), \( \| \rho - \rho_k \| \leq n^{\beta - 1} \) and \( n \) as before. From the expression of \( \mathbb{B}_n \) in (2.13) we thus conclude that the matrices \{\( \mathbb{B}_n^\tau : \sigma, \tau \leq n \)\} and their inverses are bounded w.r.t. the \( L_2 \) operator norm uniformly in \( |\rho - \rho_k| \leq n^{\beta - 1} \) and \( n \). The remaining part of the proof can be completed by mimicking the same in [3]. \( \square \)

4 Gaussian approximation and the proof of Proposition 2.6

This section is devoted to proving Proposition 2.6 Notice that \( n_{\text{core}} = n - \tau_c \) where \( \tau_c \) is the first time the process \{\( \tilde{z}(\tau) \)\} hits the \( z_1 = 0 \) line and \( m_{\text{core}} = z_2(\tau_c) \). Our goal is to estimate the distribution of \( z_2(\tau_c) - (n - \tau_c) \). We will begin with an approximation of the process \( \tilde{z}'(\tau) \) of (2.11) by a suitable Gaussian process. To this end let \( \mathcal{G}_\theta(\cdot, \mathbb{A}) \) denote the \( d \)-dimensional Gaussian distribution with mean \( \tilde{z} \) and covariance \( \mathbb{A} \). We denote by \( \mathcal{P}_{n,\rho}(\cdot) \) the law of a \( \mathbb{R}^2 \)-valued process \{\( \tilde{z}''(\tau) \)\} of (2.11) where \( \tilde{z}''(0) \) has the uniform distribution \( \mathcal{P}_{\tilde{z}(n,\rho)}(\cdot) \) on the graph ensemble \( \mathcal{G}_{k(n,\rho)} \) and

\[
\begin{align*}
\mathcal{P}_{n,\rho}^G(\tilde{z}''(\tau + 1) &= \tilde{z}''(\tau) + \Delta'_\tau + \tilde{z}'_\tau(n^{-1}\tilde{z}' - \bar{y}(\tau/n))|\tilde{z}''(\tau) = \tilde{z}'') \\
&= \mathcal{G}_2(\Delta'_\tau|\mathbb{F}(\tilde{y}(\tau/n), \tau/n), \mathbb{G}(\tilde{y}(\tau/n), \tau/n)).
\end{align*}
\]

(4.1)

for \( \tau < \tau_n \). For \( \tau \in J_n \equiv [n\theta_k - n^\beta, n\theta_k + n^\beta] \) and \( |\rho - \rho_k| \leq n^{\beta - 1} \), we expect \( \Delta'_\tau \) to be distributed like \( \Delta'_{{\lceil n\theta_k \rceil}} \) which has mean \( ((k - 1)p_1(\bar{y}(\theta_k, \rho_k), \theta_k) - 1, -((k - 1)p_1(\bar{y}(\theta_k, \rho_k), \theta_k)) \) and a positive semidefinite covariance matrix \( \mathbb{G}'(\theta_k, \rho_k) \) whose both diagonal entries are \( \mathbb{G}_{11}(\bar{y}(\theta_k, \rho_k), \theta_k) \). For \( \tau \geq \tau_n \), we modify the transition kernel as follows:

\[
\begin{align*}
\mathcal{P}_{n,\rho}^G(\tilde{z}''(\tau + 1) &= \tilde{z}''(\tau) + \Delta'_\tau + \tilde{z}'_\tau(n^{-1}\tilde{z}' - \bar{y}(\tau/n))|\tilde{z}''(\tau) = \tilde{z}'') \\
&= \mathcal{G}_2(\Delta'_\tau|\mathbb{F}(\tilde{y}(\theta_k, \rho_k), \theta_k), \mathbb{G}'(\theta_k, \rho_k)).
\end{align*}
\]

(4.2)

Lemma 4.1. Fixing \( \beta \in \left( \frac{1}{2}, \frac{5}{4} \right) \), set \( 0 < \beta' < \beta \) and \( 2\beta - 1 < \beta'' < 2(1 - \beta) \). Then there exist \( \alpha = \alpha(\beta, \beta'', k) > 0 \), \( 0 < \delta_1 = \delta_1(\beta, \beta'') < \frac{1}{2} \) and a coupling between the processes \{\( \tilde{z}'(\tau) \)\} of distribution \( \mathcal{P}_{n,\rho}(\cdot) \) and \{\( \tilde{z}''(\tau) \)\} of distribution \( \mathcal{P}_{n,\rho}^G(\cdot) \) such that for all large \( n \) and \( |\rho - \rho_k| \leq n^{\beta' - 1} \),

\[
\mathbb{P}(\sup_{\tau \in J_n} \| \tilde{z}'(\tau) - \tilde{z}''(\tau) \| \geq n^{\delta_1}) \leq \alpha n^{-\delta''}.
\]

(4.3)

Proof. The key to this lemma is the fact that under certain conditions we can couple of partial sums of independent random vectors with the same for independent gaussian random vectors having similar moments. For the \( \tau < \tau_n \) case we recall from (2.11) that \( \tilde{z}'(\tau_n) - \tilde{z}'(0) \) can be written as a sum of independent random vectors. Then by a multidimensional version of a strong approximation result of Sakhanenko (see [13] Theorem 1.2), there exist a sequence of independent gaussian vectors \( \Delta'_\tau \)'s distributed like in (4.1), \( c_2 = c_2(k) > 0 \) and \( \alpha_0 = \alpha_0(k) \) such that:

\[
\mathbb{P}(\| \sum_{\tau < \tau_n} (\Delta_\tau - \Delta'_\tau) \| \geq c_2 \log n) \leq \alpha_0 n^{-1}
\]

(4.4)
for $|\rho - \rho_k| \leq n^{\beta'-1}$ and all $n$. Furthermore $\Delta_{\tau}'$s are independent with $\mathcal{Z}'(0)$. \cite{13} Theorem 1.2 requires some ellipticity conditions on the covariance matrices of the summands which follow in this case from the (easily verifiable) fact that $p_\alpha(\mathcal{Y}(\theta, \rho), \theta)$’s are uniformly bounded away from 0 when $0 \leq \theta \leq \theta_k/2$ and $|\rho - \rho_k| \leq n^{\beta'-1}$.

For the $\tau \geq \tau_n$ part, first notice that

$$\text{Var}(\Delta_{\tau,1} + \Delta_{\tau,2}) = (k - 1)p_0(\tau/n, \rho)(1 - p_0(\tau/n, \rho)),$$

where $\Delta_{\tau} = (\Delta_{\tau,1}, \Delta_{\tau,2})$. From part (a), Proposition 3.2 and Lemma 3.1, we get for $|\rho - \rho_k| \leq n^{\beta'-1}$, $\tau \in \mathcal{J}_n$ and all large $n$,

$$p_0(\tau/n, \rho) = p_0(\theta_k, \rho_k) \leq C_6(|\theta_k - \tau/n| + |\rho - \rho_k|) \leq 2C_6n^{\beta'-1},$$

where $C$ is a positive constant. Hence

$$\sum_{\tau \in \mathcal{J}_n} \text{Var}(\Delta_{\tau,1} + \Delta_{\tau,2}) \leq 2(k - 1)C_6n^{2\beta'-1}.$$

Now by Kolmogorov’s maximal inequality

$$\Pr\left(\max_{\tau \in \mathcal{J}_n} \left| \sum_{\tau_n \leq t < \tau} ((\Delta_{t,1} + \Delta_{t,2}) - \mathbb{E}(\Delta_{t,1} + \Delta_{t,2})) \right| \geq n^\delta \right) \leq 2(k - 1)C_6n^{2\beta'-1 - 2\delta} .$$

(4.5)

So choosing $\delta_1 = \beta + \frac{\beta''}{2} - \frac{1}{2}$ we incur an error of at most $n^{\delta_1}$ with probability at least $1 - 2C_6(k - 1)n^{-\beta''}$ if we replace $\Delta_{\tau,2}$ with $\mathbb{E}(\Delta_{\tau,1} + \Delta_{\tau,2}) - \Delta_{\tau,1}$. Since $\Delta_{\tau_n + i,1}$’s are independent and uniformly bounded by $4k$, we can apply Sakhanenko’s refinement of the Hungarian construction (see [9, 10]) to deduce the existence of a sequence of independent gaussian variables $\{\Delta_{\tau,1}''\}_{\tau \geq \tau_n}$ such that

$$E\Delta_{\tau,1}'' = E\Delta_{\tau,1} = F_1(\overline{y}(\tau/n, \rho), \tau/n), \quad \text{Var}\Delta_{\tau,1}'' = \text{Var}\Delta_{\tau,1} = \mathbb{G}_{11}(\overline{y}(\tau/n, \rho), \tau/n)$$

and

$$\Pr\left(\sup_{\tau \in \mathcal{J}_n} \left| \sum_{\tau_n \leq t < \tau} (\Delta_{t,1}'' - \Delta_{t,1}') \right| \geq c_3 \log n \right) \leq c_1 n^{-1}$$

(4.6)

for some $c_3 = c_3(\beta, k) > 0$ and $c_1 = c_1(\beta, k)$.

$$\Delta_{\tau,1}' = \frac{\sqrt{\mathbb{G}_{11}(\overline{y}(\theta_k, \rho_k), \theta_k)}}{\sqrt{\mathbb{G}_{11}(\overline{y}(\tau/n, \rho), \tau/n)}} (\Delta_{\tau,1}' - F_1(\overline{y}(\tau/n, \rho), \tau/n)) + F_1(\overline{y}(\theta_k, \rho_k), \theta_k),$$

and

$$\Delta_{\tau,2}' = \mathbb{E}(\Delta_{\tau,1} + \Delta_{\tau,2}) - \Delta_{\tau,1}'.$$

We claim that the process $\overline{\mathcal{Z}}''(\tau)$ with increments $\Delta_{\tau}' = (\Delta_{\tau,1}', \Delta_{\tau,2}')$ satisfy \cite{13} for $\delta_1, \beta''$ and a suitable choice of $\alpha$. We will justify this claim in two steps. First notice that, by Lipschitz continuity of $\mathbb{G}_{11}(., .)$, there exists a positive constant $C_7$ such that

$$|\mathbb{G}_{11}(\overline{y}(\tau/n, \rho), \tau/n) - \mathbb{G}_{11}(\overline{y}(\theta_k, \rho_k), \theta_k)| \leq C_7 n^{\beta'-1},$$

for all $\tau \in \mathcal{J}_n$ and $|\rho - \rho_k| \leq n^{\beta'-1}$. Writing

$$\Delta_{\tau,1}'' - E\Delta_{\tau,1}' = \frac{\sqrt{\mathbb{G}_{11}(\overline{y}(\theta_k, \rho_k), \theta_k)}}{\sqrt{\mathbb{G}_{11}(\overline{y}(\tau/n, \rho), \tau/n)}} (\Delta_{\tau,1}' - E\Delta_{\tau,1}') + \left(1 - \frac{\sqrt{\mathbb{G}_{11}(\overline{y}(\theta_k, \rho_k), \theta_k)}}{\sqrt{\mathbb{G}_{11}(\overline{y}(\tau/n, \rho), \tau/n)}}\right) (\Delta_{\tau,1}' - E\Delta_{\tau,1}')$$

$$= \Delta_{\tau,1}' + \Delta_{\tau,1}''.$$
we notice that $\Delta''_{n,1}$’s are independent Gaussian variables with mean 0 and variance $\leq C_8 n^{2\beta - 2}$ for some positive constant $C_8$. Hence by a similar application of Kolmogorov’s inequality as in (4.5) we get

$$\mathbb{P}\left(\max_{\tau \in J_n} \left| \sum_{\tau_n \leq t < \tau} \Delta''_{\tau,1} \right| \geq n^{\delta_1} \right) \leq 2(k - 1)C_8 n^{2\beta - 2 - 2\delta_1} \leq 2(k - 1)C_8 n^{-\beta''}. \tag{4.7}$$

In view of (4.7) and the definition of $\Delta'_n$, it only remains to bound the following:

$$\max_{\tau \in J_n} \left| \sum_{\tau_n \leq t < \tau} \left( F_1(\tilde{\gamma}(t/n, \rho), t/n) - F_1(\tilde{\gamma}(\theta_k, \rho_k), \theta_k) \right) \right|. \tag{4.8}$$

We can do this by applying Lemma 3.1 and Proposition 3.2 part (a), which gives us

$$\max_{\tau \in J_n} \left| \sum_{\tau_n \leq t < \tau} \left( F_1(\tilde{\gamma}(t/n, \rho), t/n) - F_1(\tilde{\gamma}(\theta_k, \rho_k), \theta_k) \right) \right| \leq C_9 n^{2\beta - 1} \leq C_9 n^{\delta_1}, \tag{4.8}$$

for $|\rho - \rho_k| \leq n^{\beta''}$ and some positive constant $C_9$. (4.3) now follows from combining the displays (4.4), (4.5), (4.6), (4.7) and (4.8).

We are interested in the behavior of $\{(\tilde{\tau}''(\tau), \tau)\}_{\tau \geq \tau_n}$. After a little reshuffling we get the following decent expression:

$$\tilde{z}''(\tau) = \tilde{z}^* + (S_\tau, -S_\tau) + \left( \frac{\partial y_1}{\partial \theta}(\tau_n - n\theta_k), \frac{\partial y_2}{\partial \theta}(\tau_n - n\theta_k) \right) + (0, n(1 - \theta_k)) \tag{4.9}$$

where $S_\tau = \sum_{\tau_n=1}^{\tau_n=1} \Delta'_n$ and the process $\{S_\tau\}_{\tau \geq \tau_n}$ is independent of $\tilde{z}^*$ which is defined as

$$\tilde{z}^* = \tilde{z}''(\tau_n) - \left( \frac{\partial y_1}{\partial \theta}(\tau_n - n\theta_k), \frac{\partial y_2}{\partial \theta}(\tau_n - n\theta_k) \right) - (0, n(1 - \theta_k)).$$

Since we are dealing with first hitting times it is convenient to consider processes defined on $\mathbb{R}^+$. Also the presence of an implicit upper bound $n$ on $\tau$ is inconvenient. To get around these issues we first extend the definition of the processes $\{\tilde{z}(\tau)\}$ and $\{\tilde{z}''(\tau)\}$ beyond $n$. For the process $\{\tilde{z}(\tau)\}$ this is achieved by setting $\tilde{z}(\tau) = 0$ for $\tau \geq n$ and for $\{\tilde{z}''(\tau)\}$ we simply carry on (4.2) for all $n$. Now we extend these two processes to all $t \in \mathbb{R}^+$ by linear interpolation. This minor change has some other technical advantages as well. Both the processes now almost surely hit the $x = 0$ line at finite time. Furthermore the minimum $\tau$ such that $z_1(\tau_n) = 0$ is still $\tau_c$. Define $\tau_c^G$ for $\{\tilde{z}''\}$ as the first time $t \geq \tau_n$ such that $z''(\tau_n) = 0$ if $z''(\tau_n) \geq 0$ and $\tau_n$ otherwise. The following lemma is an important intermediate step for our proof of Lemma 4.3 which shows that $(z_2(\tau_c), \tau_c)$ and $(z_2(\tau_c^G), \tau_c^G)$ are pretty close. This lemma is similar in essence to [8 Corollary 5.3].

**Lemma 4.2.** Fix $\beta \in \left(\frac{1}{2}, 1\right)$ and $0 < \beta' < \beta$. Then there exist $C_1(C, \beta') > 0$ and $\eta = \eta(\beta) > 0$ such that

$$\mathbb{P}_{n, \rho}(\{ \min_{\tau \in [0, n\theta_k - n^{\beta}]} z_1(\tau) \leq n^{\beta'} \} \cup \{ z_1([n\theta_k + n^{\beta}]) \geq -n^{\beta'} \}) \leq C e^{-n^\eta},$$

all $n$ and $|\rho - \rho_k| \leq n^{\beta'' - 1}$.

**Proof.** We showed in the proof of Lemma 3.3 that $y_1(\theta, \rho) \geq c_1 n^{\beta - 1}$ for all $0 \leq \theta \leq \theta_n$, $|\rho - \rho_k| \leq n^{\beta'' - 1}$ and large $n$. By part (b) of Lemma 2.3 we then get that for some $C' = C'(\beta') > 0$ and $n$ large enough

$$\mathbb{E}z_1(\tau) \geq ny_1(\tau/n, \rho) - C_1 \sqrt{n \log n} \geq c_1 n^{\beta} - C_1 \sqrt{n \log n} \geq n^{\beta'},$$
whenever \( \tau \in [0, n\theta_k - n^\beta] \) and \( |\rho - \rho_k| \leq n^{\beta - 1} \). Now applying part (a) of Lemma 2.3, we get that for any \( \eta < (2\beta - 1)/2 \), some \( C'' = C''(\beta, \beta', \eta) > 0 \) and \( n \) large enough

\[
\overset{}{\mathbb{P}}_{n, \rho}(z_1(\tau) \leq n^{\beta'}) \leq \overset{}{\mathbb{P}}_{n, \rho}(\|z(\tau) - \mathbb{E}z(\tau)\| \geq C' n^{\beta}/2) \leq C'' e^{-n^{2\eta}},
\]

whenever \( \tau \in [0, n\theta_k - n^\beta] \) and \( |\rho - \rho_k| \leq n^{\beta - 1} \). Similarly we can derive a similar exponential bound on \( \overset{}{\mathbb{P}}_{n, \rho}(z_1([n\theta_k + n^\beta])) \geq -n^{\beta} \}. \) The result now follows by applying a union bound over \( \tau \in [0, n\theta_k - n^\beta] \cup \{[n\theta_k + n^\beta] \} \).

**Lemma 4.3.** Fix \( \beta \in \left(\frac{1}{2}, \frac{3}{4}\right) \) and \( 0 < \beta' < \beta \). There exist \( 0 < \delta_2 < \frac{1}{2} \) and \( \delta_3 > 0 \) depending only on \( \beta, \beta' \), \( B = B(\beta, \beta', k) > 0 \) and a coupling of the processes \( \{\widetilde{z}(\tau)\}_{\tau \in \mathbb{R}^+} \) and \( \{\widetilde{z}''(\tau)\}_{\tau \in \mathbb{R}^+} \), such that for all \( n \) and \( |\rho - \rho_k| \leq n^{\beta - 1} \),

\[
\mathbb{P}(z''_1(\tau^G) = 0, |\tau_c - \tau^G_c| + |z_2(\tau_c) - z''_2(\tau^G_c)| \leq n^{\delta_2}) \geq 1 - Bn^{-\delta_3}.
\]

**Proof.** Since \( z''_1(\tau) - z''_1(\tau_n) \) is a sum of i.i.d random variables with negative mean (see Proposition 3.3), it is clear that \( \tau^G_c \) is almost surely finite. Thus

\[
\mathbb{P}(z''_1(\tau_n) \geq 0, z''_1(\tau^G_n) \neq 0) = 0.
\]

Now we couple the processes \( \{\widetilde{z}(\tau)\}_{\tau \geq 0} \) and \( \{\widetilde{z}''(\tau)\}_{\tau \geq 0} \) by joining the couplings of Lemma 4.1, Proposition 2.7 and Corollary 2.5. We can extend this coupling to all \( t \in \mathbb{R}^+ \) by linear interpolation. From the same results we then get the numbers \( 0 < \delta_2' = \delta_2'(\beta, \beta') < \frac{1}{2}, \delta_3' = \delta_3'(\beta, \beta') > 0 \) and \( B' = B'(\beta, \beta', k) \), such that

\[
\mathbb{P}(\text{sup}_{\tau \in J_n} \|\widetilde{z}(\tau) - \widetilde{z}''(\tau)\| \geq n^{\delta_2'}) \leq B'n^{-\delta_3'},
\]

for all \( n \) and \( |\rho - \rho_k| \leq n^{\beta - 1} \). (4.10) combined with Corollary 2.5 and Lemma 4.2 gives us

\[
\mathbb{P}(z''_1(\tau_n) \geq 0, \tau_c \in J_n, \tau^G_c \in J_n) \geq 1 - B'' n^{-\delta_3'},
\]

for all \( n \), \( |\rho - \rho_k| \leq n^{\beta - 1} \) and some \( B'' = B''(\beta, \beta', k) \). We still need one more ingredient. Since \( \Delta''_n \)’s are i.i.d. Gaussian variables, it follows by Dudley’s entropy bound on the supremum of a Gaussian process (see, e.g., [1] Theorem 4.1) and Gaussian concentration inequality (see e.g., [6] Equation (7.4), Theorem 7.1) that

\[
\mathbb{P}\left(\max_{\tau \in J_n} z''_1(\tau_{\delta_2,+}) - z''_1(\tau) - \mathbb{E}(z''_1(\tau_{\delta_2,+}) - z''_1(\tau)) \geq n^{3\delta_2/4}\right) \leq B'' n^{-1},
\]

for some \( B''' = B'''(\beta, \beta', k) \) and \( \tau_{\delta_2,+} = (\tau + n^{\delta_2}) \wedge (n\theta_k + n^\beta) \). But

\[
\mathbb{E}(z''_1(\tau + n^{\delta_2}) - z''_1(\tau)) = n^{\delta_2} F_1(\widetilde{y}(\theta_k, \rho_k), \theta_k),
\]

where \( F_1(\widetilde{y}(\theta_k, \rho_k), \theta_k) < 0 \). The lemma now follows from this fact together with (4.10), (4.11) and (4.12). \( \square \)

Now let us revisit (4.9). A routine algebra yields that when \( z''_1(\tau^G_c) = 0 \), we have

\[
z''_2(\tau^G_c) - (n - \tau^G_c) = (\xi_1 + \xi_2'),
\]

where \( \widetilde{\xi} = (\xi_1, \xi_2') \). The following is an immediate consequence of Lemma 4.3 and (4.13).
Lemma 4.4. Fix $\beta \in (\frac{1}{3}, \frac{2}{3})$ and $\frac{1}{3} < \beta' < \beta$. There exist $0 < \delta_2 < \frac{1}{3}$ and $\delta_3 > 0$ depending only on $\beta, \beta'$, $B_1 = B_1(\beta, \beta', k) > 0$ such that for all $n, A > 0$ and $|\rho - \rho_k| \leq n^{\beta' - 1}$,

$$
\mathbb{P}_{n,\rho}^G( (\xi_1^* + \xi_2^*) \geq A \log n + n^{\delta_2} ) - B_1 n^{-\delta_3} \leq \mathbb{P}_{n,\rho}( z_2(\tau_c) - (n - \tau_c) \geq A \log n ) \\
\leq \mathbb{P}_{n,\rho}( (\xi_1^* + \xi_2^*) \geq A \log n - n^{\delta_2} ) + B_1 n^{-\delta_3},
$$

and

$$
\mathbb{P}_{n,\rho}^G( (\xi_1^* + \xi_2^*) \leq -A \log n - n^{\delta_2} ) - B_1 n^{-\delta_3} \leq \mathbb{P}_{n,\rho}( z_2(\tau_c) - (n - \tau_c) \leq -A \log n ) \\
\leq \mathbb{P}_{n,\rho}( (\xi_1^* + \xi_2^*) \leq -A \log n + n^{\delta_2} ) + B_1 n^{-\delta_3}.
$$

Proof of Proposition 2.6. Notice that,

$$
\bar{\xi}^* = \left( \bar{\xi}''(\tau_n) - \mathbb{E}_0^\tau_n \rho - \mathbb{E}_0^\tau_n \rho(0) \right) - \left( \frac{\partial y_1}{\partial \theta}(\tau_n - n\theta_k), \frac{\partial y_2}{\partial \theta}(\tau_n - n\theta_k) \right) - (0, n(1 - \theta_k)) + \mathbb{E}_0^\tau_n \rho(0).
$$

From (2.14) and the coupling defined in Lemma 4.1 we can see that $\bar{\xi}''(\tau_n) - \mathbb{E}_0^\tau_n \rho - \mathbb{E}_0^\tau_n \rho(0)$ and $\bar{\xi}(0)$ are independent. Also from (4.1), (3.3) and (3.4) we have

$$
\bar{\xi}''(\tau_n) - \mathbb{E}_0^\tau_n \rho - \mathbb{E}_0^\tau_n \rho(0) \overset{\mathcal{D}}{=} \mathcal{G}_2 \left( n(\sqrt{y}''(\tau_n) - \mathbb{E}_0^\tau_n \rho - \mathbb{E}_0^\tau_n \rho(0), \rho), n(Q_{\tau_n} - \mathbb{E}_0^\tau_n \rho - \mathbb{E}_0^\tau_n \rho(0)^T) \right).
$$

Hence by Lemma 2.2 we get for some $\kappa_3 > 0$,

$$
\sup_{x \in \mathbb{R}} \mathbb{P}(\xi_1^* + \xi_2^* \leq x) - \int_{z_1 + z_2 \leq x} \phi_2(\bar{\xi}^* ; nQ(\tau_n))d\bar{\xi} \leq \kappa_3 n^{-1/2}.
$$

On the other hand Lemma 3.4 tells us

$$
\| nQ_{\tau_n} - nQ_{\theta_k, \rho_k} \| \leq Cn^\beta,
$$

and

$$
\| n\sqrt{y}''(\tau_n) - n((\rho - \rho_k)\frac{\partial y_1}{\partial \rho}(\theta_k, \rho_k), (\rho - \rho_k)\frac{\partial y_2}{\partial \rho}(\theta_k, \rho_k)) - (\frac{\partial y_1}{\partial \theta}(\tau_n - n\theta_k), \frac{\partial y_2}{\partial \theta}(\tau_n - n\theta_k)) \right)
$$

$$
- (0, n(1 - \theta_k)) \| \leq 2Cn^{2\beta - 1}
$$

for some $C = C(\beta, \beta')$. Now using formula for the Kullback-Leibler divergence between two multivariate Gaussian distributions and Pinsker’s inequality (see, e.g., [11] p. 132) we get for $\rho = \rho_k + rn^{-1/2}$,

$$
\| \mathcal{L}(\sqrt{n}(\xi_1(r) + \xi_2(r))) - \mathcal{L}(\xi_1^* + \xi_2^*) \|_{TV} \leq \max(n^{(4\beta - 3)/2}, n^{(\beta - 1)/2}),
$$

(4.14)

where $\bar{\xi}(r)$ is same as in Proposition 2.6 and $\mathcal{L}(X)$ denotes the law of the random variable $X$. The proof now follows from (4.14) and Lemma 4.4.

References

[1] R. J. Adler. An introduction to continuity, extrema and related topics for general gaussian processes. 1990. Lecture Notes - Monograph Series. Institute Mathematical Statistics, Hayward, CA.

[2] A. Amraoui, A. Montanari, T. Richardson, and R. Urbanke. Finite-length scaling for iteratively decoded LDPC ensembles. IEEE Trans. Inform. Theory, 55(2):473–498, 2009.
[3] A. Dembo and A. Montanari. Finite size scaling for the core of large random hypergraphs. *Ann. Appl. Probab.*, 18(5):1993–2040, 2008.

[4] M. Dietzfelbinger, A. Goerdt, M. Mitzenmacher, A. Montanari, R. Pagh, and M. Rink. Tight Thresholds for Cuckoo Hashing via XORSAT, pages 213–225. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[5] O. Dubois and J. Mandler. The 3-XORSAT threshold. *C. R. Math. Acad. Sci. Paris*, 335(11):963–966, 2002.

[6] M. Ledoux. The concentration of measure phenomenon, volume 89 of mathematical surveys and monographs. 2001. American Mathematical Society, Providence, RI.

[7] M. Leone, F. Ricci-Tersenghi, and R. Zecchina. Phase coexistence and finite-size scaling in random combinatorial problems. *Journal of Physics A: Mathematical and General*, 34(22):4615, 2001.

[8] B. Pittel and G. B. Sorkin. The satisfiability threshold for k-xorsat. *Combinatorics, Probability and Computing*, 25(02):236–268, 2016.

[9] A. Sakhanenko. Rate of convergence in the invariance principle for variables with exponential moments that are not identically distributed. *Trudy Inst. Mat. SO AN SSSR*, 3:4–49, 1984.

[10] Q. M. Shao. Strong approximation theorems for independent random variables and their applications. *J. Multivariate Anal.*, 52(1):107–130, 1995.

[11] A. B. Tsybakov. Introduction to nonparametric estimation. revised and extended from the 2004 french original. translated by vladimir zaiats, 2009.

[12] D. B. Wilson. On the critical exponents of random k-SAT. *Random Structures Algorithms*, 21(2):182–195, 2002.

[13] A. Y. Zaitsev. Multidimensional version of a result of sakhanenko in the invariance principle for vectors with finite exponential moments. i. *Theory of Probability & Its Applications*, 45(4):624–641, 2001.