On the perturbed Schwarzschild geometry for determination of particle motion

Alessandro D.A.M. Spallicci

Gravitation Research Group, Salerno Univ. at Benevento
Kamerlingh Onnes Lab., Dept. Phys.& Astr., Fac. Math.& Nat. Sc., Leiden Univ.

Abstract

A novel method for calculation of the motion and radiation reaction for the two-body problem (body plus particle, the small parameter $m/M$ being the ratio of the masses) is presented. In the background curvature given by the Schwarzschild geometry rippled by gravitational waves, the geodesic equations insure the presence of radiation reaction also for high velocities and strong field. The method is generally applicable to any orbit, but radial fall is of interest due to the non-adiabatic regime (equality of radiation reaction and fall time scales), in which the particle locally and immediately reacts to the emitted radiation. The energy balance hypothesis is only used (emitted radiation equal to the variation in the kinetic energy) for determination of the 4-velocity via the Lagrangian and normalization of divergencies. The solution in time domain of the Regge-Wheeler-Zerilli-Moncrief radial wave equation determines the metric tensor expressing the polar perturbations, in terms of which the geodesic equations are written and shown herein.
ON THE PERTURBED SCHWARZSCHILD GEOMETRY FOR DETERMINATION OF PARTICLE MOTION

ALESSANDRO D.A.M. SPALLICCI
Gravitation Research Group, Salerno Univ. at Benevento
Kamerlingh Onnes Lab., Dept. Phys.& Astr., Fac. Math.& Nat. Sc., Leiden Univ.

A novel method for calculation of the motion and radiation reaction for the two-body problem (body plus particle, the small parameter $m/M$ being the ratio of the masses) is presented. In the background curvature given by the Schwarzschild geometry rippled by gravitational waves, the geodesic equations insure the presence of radiation reaction also for high velocities and strong field. The method is generally applicable to any orbit, but radial fall is of interest due to the non-adiabatic regime (equality of radiation reaction and fall time scales), in which the particle locally and immediately reacts to the emitted radiation. The energy balance hypothesis is only used (emitted radiation equal to the variation in the kinetic energy) for determination of the 4-velocity via the Lagrangian and normalization of divergencies. The solution in time domain of the Regge-Wheeler-Zerilli-Moncrief radial wave equation determines the metric tensor expressing the polar perturbations, in terms of which the geodesic equations are written and shown herein.

1 Introduction

Regge and Wheeler proved the vacuum stability of the Schwarzschild black hole, while Zerilli studied the emitted radiation via the radial wave equation for polar perturbations which source is a freely falling test mass $m$ towards the black hole of large mass $M$. Moncrief showed the gauge invariant significance of the wave equations. Past work was concerned on the Fourier analysis of the emitted radiation (for a review see Ruffini) and not on the motion of the particle. Radiation reaction for particles around black holes has been tackled by Cutler and co-workers whose analysis (via Teukolsky formula) is based on the adiabatic approximation, the postulate of energy-balance, and is limited to $6M$; for the Dirac-Galits'ov method antitransformation of all frequencies is required, while the axiomatic approach is conceived within the one-body problem. The aim of this work is the identification of radiation reaction, without the assumption of adiabacity, with minimal use of the energy balance postulate, and identification of the trajectory up to the horizon. Future developments may include a general method for motion of small objects in any orbit; the solution of 2nd-order equation, which energy-momentum tensor is based on the geodesic equations calculated herein, and a post-Schwarzschildian formalism.
2 Approach

Past work was confined to a particle falling in an unperturbed Schwarzschild metric, but radiation reaction requires a geodesic path through the emission of radiation, in a perturbed Schwarzschild metric. To this end it appears necessary: a numeric or analytic solution of Regge-Wheeler-Zerilli-Moncrief (RWZM) equation in time domain; determination of the perturbation components and of 1st-order Lagrangian; finally identification of radiation reaction by subtraction of the 0th-order terms in the geodesic equations. The RWZM equation for polar perturbations is:

\[
\frac{d^2 \Psi_l(r, t)}{dr^2} - \frac{d^2 \Psi_l(r, t)}{dt^2} - V_l(r) \Psi_l(r, t) = S_l(r, t) \tag{1}
\]

where \( r^* = r + 2M \ln \left( \frac{r}{2M} - 1 \right) \) is the tortoise coordinate and the potential \( V_l(r) \) is:

\[
V_l(r) = \left( 1 - \frac{2M}{r} \right) \frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2r + 18M^3}{r^3(\lambda r + 3M)^2} \tag{2}
\]

Further \( \lambda = \frac{1}{2}(l - 1)(l + 2) \) and \( S_l(r, t) \) is the \( 2^l \)-pole source component and, for a radially falling particle, is:

\[
S_l = \left( 1 - \frac{2M}{r} \right) k \left\{ r \left( 1 - \frac{2M}{r} \right) \delta'[r - r_0(t)] + \left[ \frac{6M \left( 1 - \frac{2M}{r} \right)^2}{\lambda r + 3M} + \frac{M}{r} - \lambda - 1 \right] \delta[r - r_0(t)] \right\} \tag{3}
\]

where \( k = 4M \sqrt{(2l + 1)\pi} \) and \( r_0(t) \) is the inverse of:

\[
t = -4M \left( \frac{r}{2M} \right)^{1/2} - 4 \left( \frac{r}{2M} \right)^{3/2} - 2M \ln \left[ \left( \sqrt{\frac{r}{2M}} - 1 \right) \left( \sqrt{\frac{r}{2M}} + 1 \right)^{-1} \right] \tag{4}
\]

Eq.(4) can be rewritten in terms of \( t, r^* \) with constant coefficients of the 2nd derivatives but solely via an approximate inverse function \( r(r^*) \) and thus resulting into an approximate p.d.e. (further, the solution is most interesting at \( r^* = \infty \)). In the \((t, r)\) domain instead, eq.(4) becomes:

\[
\frac{1}{A^2(r)} \frac{d^2 \Psi_l}{dr^2} - \frac{1 - A(r)}{rA^2(r)} \frac{d \Psi_l}{dr} - \frac{d^2 \Psi_l}{dt^2} - V_l(r) \Psi_l = S_l(r, t) \tag{5}
\]

where \( A(r) = \frac{dr^*}{dr} = \frac{r}{r - 2M} \). The polar perturbations \( h_{\mu \nu} \) are determined by:
\[
\begin{pmatrix}
(1 - \frac{2M}{r}) H_0 & H_1 & h_0 \frac{\partial}{\partial \theta} & h_0 \frac{\partial}{\partial \phi} \\
\text{sym} & \left(1 - \frac{2M}{r}\right)^{-1} H_2 & h_1 \frac{\partial}{\partial \theta} & h_1 \frac{\partial}{\partial \phi} \\
\text{sym} & \text{sym} & r^2 \left[K + G \frac{\partial^2}{\partial \theta^2} \right] & \text{sym} \\
\text{sym} & \text{sym} & r^2 G \left(\frac{\partial^2}{\partial \phi \partial \theta} - \cot \theta \frac{\partial}{\partial \phi}\right) & r^2 \left[K \sin^2 \theta + G \left(\frac{\partial^2}{\partial \theta^2} + \sin \theta \cos \theta \frac{\partial}{\partial \phi}\right)\right]
\end{pmatrix}
\]

where \(H_0, H_1, H_2, h_0, h_1, K, G\) are functions of \(t, r\). The Regge-Wheeler gauge specifies \(G = h_0 = h_1 = 0\). There are relations between \(\Psi\) and derivatives and the perturbation components. Analytically the initial value problem is well defined (\(\Psi\) and \(\dot{\Psi}\) are zero, i.e. particle at rest at infinity). Numerically, Lousto and Price have determined intermediate conditions, solving eq. (1) with a finite difference scheme and direct integration of the source term (thus an interpolation should be performed for finding \(\Psi\) and \(\dot{\Psi}\)). Alternatively for an analytic solution of eq.(5), the asymptotic solutions for \(r \to \infty\) and \(r \to 0\) would be instrumental for qualitative methods as differential inequalities or Lagrange identities techniques.

3 The equations of motion

The Schwarzschild metric up to 1st-order perturbations in the Regge-Wheeler gauge is:

\[
d s^2 = d s_0^2 + \left(1 - \frac{2M}{r}\right) H_0 Y d t^2 + \left(1 - \frac{2M}{r}\right)^{-1} H_2 Y d r^2 + r^2 K Y d \theta^2 + r^2 \sin^2 \theta K Y d \phi^2 + 2 H_1 Y d t d r
\]

The geodesic equations encompass three perturbation schemes: i) a perturbative field \(g^{\mu \nu} = \eta_{\mu \nu}^{\text{Schwarzschild}} + h_1^{\mu \nu}\); ii) the particle trajectory \(r_p(t) = r_0(t) + r_1(t) = r_0[1 + \epsilon_r(t)]\) where \(r_0(t)\) is the trajectory crossing the Schwarzschild metric and radiating out but the motion being unperturbed and \(r_1(t)\) is the correction due to radiation reaction, via the small parameter \(\epsilon_r(t)\); iii) the field at \(r_p(t)\) is a McLaurin series at \(r_0, t_0\), and \(t_p = t_0 + t_1 = t_0[1 + \epsilon_t(t)]\):

\[
g^{\mu \nu}(r_p, t_p) \simeq g^{\mu \nu}(r_0, t_0) + \epsilon_r r_0 \frac{\partial g^{\mu \nu}}{\partial r} \bigg|_{(r_0, t_0)} + \epsilon_t t_0 \frac{\partial g^{\mu \nu}}{\partial t} \bigg|_{(r_0, t_0)}
\]

There are two unknown variables \(\epsilon_r(t)\) and \(\epsilon_t(t)\). For the case of radial fall \(d \theta\) and \(d \phi\) vanish, and using the time independence of the Schwarzschild metric, and dropping the \((r_0, t_0)\) notation one equation is:
\[
d\frac{(1 + \varepsilon_r) r_0}{ds^2} + \frac{1}{2} \left( \eta_{rr} + \varepsilon_r r_0 \frac{\partial \eta_{rr}}{\partial r} + h_{rr} + \varepsilon_r r_0 \frac{\partial h_{rr}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{rr}}{\partial t} \right) \times \left\{ \frac{\partial}{\partial r} \left[ \frac{3 \left( \eta_{rr} + \varepsilon_r r_0 \frac{\partial \eta_{rr}}{\partial r} + h_{rr} + \varepsilon_r r_0 \frac{\partial h_{rr}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{rr}}{\partial t} \right)}{\partial r} \right] + 2 \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right\} dr dt \left\{ \frac{\partial}{\partial t} \left[ \frac{2 \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right)}{\partial t} + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right] + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right\} dt dr \left\{ \frac{\partial}{\partial t} \left[ \frac{2 \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right)}{\partial t} + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right] + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right\} dt dt \left\{ \frac{\partial}{\partial s} \left[ \frac{2 \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right)}{\partial s} + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right] + \frac{1}{2} \left( h_{tt} + \varepsilon_r r_0 \frac{\partial h_{tt}}{\partial r} + \varepsilon_t t_0 \frac{\partial h_{tt}}{\partial t} \right) \right\} \right\} = 0 \tag{8}
\]

The quantity \( d^2 r / dt^2 \) is deduced as ratio from eq. (3), and from the same eq. in where \( \mu = t \). Eq. (8) at lowest order (\( h = h' = h'' = \eta' = 0 \)) expresses the known motion of a particle in a unperturbed metric. Otherwise, the particle crosses the perturbed metric and thus its motion is influenced by the emitted radiation. The velocities \( \dot{r} \) and \( \dot{t} \) are derived from the Lagrangian. The field contribution at 1st-order is:
\[ \mathcal{L}_1 = \left(1 - \frac{2M}{r}\right) H_0 Y\dot{r}^2 + \left(1 - \frac{2M}{r}\right)^{-1} H_2 Y\dot{r}^2 + r^2 KY\dot{\theta}^2 + r^2 \sin^2 \theta KY\dot{\phi}^2 + 2H_1 Y\dot{r} \tag{9} \]

where the dot indicates \( s \) derivatives. Differentiating \( \mathcal{L}_1 \) (omitting the index \( l \)) and eliminating angular dependence, the canonical quantities are:

\[ \dot{p}_t = \frac{\partial \mathcal{L}_1}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right) \frac{\partial H_0}{\partial t} Y\dot{r}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \frac{\partial H_2}{\partial t} Y\dot{r}^2 + 2 \frac{\partial H_1}{\partial t} Y\dot{r} \dot{r} \tag{10} \]

\[ p_t = \frac{\partial \mathcal{L}_1}{\partial \dot{t}} = 2 \left(1 - \frac{2M}{r}\right) H_0 Y\dot{r} + 2H_1 Y \dot{r} \tag{11} \]

\[ \dot{p}_r = -\frac{\partial \mathcal{L}_1}{\partial r} = -\left[ \frac{2M}{r^2} H_0 + \left(1 - \frac{2M}{r}\right) \frac{\partial H_0}{\partial \dot{r}} \right] Y\dot{r}^2 + \left[ \frac{2M}{(r-2M)^2} H_2 - \left(1 - \frac{2M}{r}\right)^{-1} \frac{\partial H_2}{\partial \dot{r}} \right] Y\dot{r}^2 - 2 \frac{\partial H_1}{\partial \dot{r}} Y\dot{r} \dot{r} \tag{12} \]

\[ p_r = -\frac{\partial \mathcal{L}_1}{\partial \dot{r}} = -2 \left(1 - \frac{2M}{r}\right)^{-1} H_2 Y \dot{r} - 2H_1 \dot{r} \dot{r} \tag{13} \]

The Hamiltonian is equal to the Lagrangian due to the absence of the potential. The Lagrangian is time independent for a conservative system: the power of the emitted radiation \( P_{gw} \) is added to \( \dot{p}_{t,\text{total}} = \dot{p}_t + P_{gw} = 0 \). The integration in time leads to a constant of energy dimensions from which \( i \) is derived:

\[ \dot{p}_{t,\text{total}} = \int \left(1 - \frac{2M}{r}\right) \frac{\partial H_0}{\partial t} Y\dot{r}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \frac{\partial H_2}{\partial t} Y\dot{r}^2 + 2 \frac{\partial H_1}{\partial t} Y\dot{r} \dot{r} dt + \int P_{gw} dt \tag{14} \]

The Lagrangian is unitary for timelike geodesics and substituting (14) in for \( \dot{\theta} = \dot{\phi} = 0, \dot{r}(r) \) and \( \dot{r} \) are calculated. The initial conditions at infinity, \( \dot{r} = 0 \) and \( E_{gw} = 0 \), fix the constant \( E \). The zeroth order terms must be added to the 1st-order ones.

### 3.1 Singularities

The energy going into the black hole is \( E_{in} \approx 0.3927 mc^2 \) while the energy radiated away is \( E_{out} \approx 0.0104 \mu^2 c^2 / M \) where the reduced mass \( \mu \approx m \). The relation between \( E \) and \( \psi \) or:

\[ T^{\mu\nu} = \frac{1}{32\pi} h^\gamma_{\delta\mu\nu} h^{\gamma\delta}_{\nu} \tag{15} \]

should bring to the normalization of divergent expressions. Alternatively a finite size mass, e.g. dust, could be considered.
4 Conclusions

The approach for identification of radiation reaction of masses falling into Schwarzschild black holes via 1st-order polar perturbations has been shown and the equations of motion in symbolic form have been found. Radiation reaction is a fundamental concept in bodies motion theory, but also has relevant implications on detector’s templates since the capture of stars by black holes is a source of gravitational waves.

Acknowledgements

Discussions with V. Pierro and I. Pinto (Salerno), G. Schäfer (Jena) are acknowledged. Encouragement from S. Chandrasekhar was the warmest gift. Financial support for participating at this conference was received from the European Space Research & Technology Centre, Noordwijk.

References

1. T. Regge and J.A. Wheeler, Phys. Rev., 108, 1063 (1957).
2. F.J. Zerilli, Phys. Rev. D 2, 2141 (1970).
3. F.J. Zerilli in Black Holes, Gravitational Waves and Cosmology: an Introduction to Current Research (Gordon & Breach Science Publ., A-7, 1975); errata corriges of Zerilli, 1970.
4. V. Moncrief, Ann. Phys. N.Y., 88, 323 (1974).
5. R. Ruffini, in Black Holes August 1972 Les Houches, C. DeWitt and B. DeWitt eds. (Gordon & Breach Science Publ., 453, 1972).
6. C. Cutler C., D. Kennefick and E. Poisson, Phys. Rev. D 50, 3816 (1994).
7. R.A. Capon, in Mathematics of Gravitation. Part II. Gravitational Wave Detection, 1997 Warszawa (Inst. Mathem., Polish Ac. Sc., Banach Center Publ., 41, 65, 1997).
8. A.D.A.M. Spallicci, in 8th Marcel Grossmann Meeting, 22-28 June 1997 Jerusalem (World Scientific, 1998).
9. Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D 55, 4547 (1997); T.C. Quinn and R.M. Wald, Phys. Rev. D 56, 3381 (1997).
10. C.O. Lousto and R.H. Price, private communication (1997).
11. J.N. Flavin and S. Rionero, Qualitative Estimates for Partial Differential Equations (CRC Press, 1996).