Guaranteed Cost Consensus for Fuzzy Multi-Agent Systems With Switching Topologies

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Abstract This paper concentrates on guaranteed cost consensus problem for nonlinear multi-agent systems (MASs) under directed switching topologies. First, a novel polynomial fuzzy model is constructed for describing the nonlinear error MASs that are formulated by follower agents and a leader agent. Then, a consensus tracking protocol is devised to ensure that the MASs with changing topologies can achieve agreement. For the established fuzzy MASs, guaranteed cost function is introduced based on error states and control inputs. By the multiple polynomial Lyapunov function, we present relaxed sufficient condition to reach consensus with a guaranteed cost for the fuzzy MASs. Moreover, the designed consensus protocol can realize guaranteed cost by the upper bound for any initial error state. The obtained condition is converted to sum of squares and solved numerically. Finally, an example is given to illustrate the validity of the proposed consensus technique.

Index Terms Switching topology, multi-agent system, guaranteed cost, fuzzy modeling.

I. INTRODUCTION

The Takagi-Sugeno (T-S) fuzzy models can efficiently approximate any smooth nonlinear function with arbitrary precision, which have gained widespread attention in the past decades. Fruitful results have been achieved, for example, see [4]–[6] and the references therein.

Recently, [7] presents a polynomial fuzzy model for modeling a nonlinear system by polynomial expression, which has attracted more and more attention in recent years. Rich results have been gained. One may refer to [8], [9].

On the other hand, consensus problems of multi-agent systems (MASs) have attracted extensive attention due to their wide application and remarkable scalability, such as formation control and synchronization of dynamical networks. Numerical results have been achieved for MASs [10]–[14]. Various control schemes are utilized to realize consensus, such as adaptive control [15]–[19], $\mathcal{H}_\infty$ control [20], [21], and event-triggering control [1]. For example, the $\mathcal{H}_\infty$ consensus tracking problem is investigated in [21] for first-order MASs with a fixed undirected graph in a polynomial fuzzy framework.

In practical application, the communication link failures and the establishment of new link lead to frequent changes in the communication topology among agents due to communication equipment faults and disturbances. To name a few, in [22], the consensus tracking problems are considered for one-sided Lipschitz MASs under switching topologies. The consensus tracking is investigated for Lipschitz MASs under switching directed topologies in [23]. Therefore, Switching topology is one of the problems worth studying in the current work.

On another research front line, guaranteed cost control is an efficient control strategy to handle multi-objective problems and can make sure that control systems are not only asymptotically stable, but also maintaining a certain level of energy consumption. It should be emphasized that guaranteed cost function is usually established by states and control inputs for an isolated system. For example, see [24]–[28] and the references therein. Different from the isolated systems,
the guaranteed cost function for MASs is concerned with error states rather than states. Since MASs achieve consensus, the states of each agent may be nonzero. Recently, guaranteed cost consensus control for MASs has received considerable attention. Fruitful results have been yielded. To mention a few, guaranteed cost synchronization is discussed for linear singularly perturbed systems in [29]. Control problems of guaranteed cost consensus are investigated for linear MASs under switching undirected topologies in [30]. The problems of guaranteed cost consensus are considered for singular linear MASs under switching undirected topologies in [31]. Guaranteed-based output consensus issues are investigated for descriptor MASs by considering jointly connected switching topologies in [32]. The work in [33] and [34] investigates guaranteed cost consensus for linear and nonlinear MASs with jointly connected topologies, respectively. For all we know, few result is reported on guaranteed cost consensus for polynomial fuzzy MASs under changing directed communication topologies.

Motivated by the aforementioned discussions, we aim to study the guaranteed cost consensus issue of polynomial fuzzy MASs with switching topologies in this paper. The main contributions are provided as follows:

(i) A novel polynomial fuzzy model is constructed for describing the nonlinear error MASs that are generated by followers and one leader.

(ii) Consensus protocols are designed to ensure that polynomial fuzzy MASs can achieve a guaranteed cost consensus for a given cost function established by the error states and control inputs. Moreover, for any initial error state, the cost function’s upper bound is gained.

(iii) Based on multiple Lyapunov analysis, new relaxed sufficient condition is proposed to ensure the guaranteed cost consensus for polynomial fuzzy MASs with switching directed topologies, respectively. The presented condition is converted to sum of squares (SOS) and can be numerically solved [35].

The rest of this paper is organized as follows: In Section II, the polynomial fuzzy model is constructed. In Section III, the main results are given. Section IV provides an example. Finally, concluding remarks are given in Section V.

Notation: The symbol $\otimes$ refers to the Kronecker product. $I$ denotes the identity matrices with appropriate dimensions. $\text{diag}(...)$ represents a block-diagonal matrix. The superscript $T$ denotes matrix transpose. In matrix expressions, asterisk * stands for symmetric elements in the symmetrical matrix. $Y > 0$ means that $Y$ is a symmetric and positive definite matrix. $\text{Sym}(M)$ denotes $M + M^T$.



II. PRELIMINARIES AND PROBLEM FORMULATION

A. GRAPH THEORY

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph, in which $\mathcal{V} = \{1, 2, \ldots, N\}$, $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\}$, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denote the node set, the edge set and the weighted adjacency matrix, respectively. The Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $L_{ij} = \sum_{k=1}^{N} a_{ik}$, if $i = j$, otherwise $L_{ij} = -a_{ij}$.

Finally, concluding remarks are given in Section V.

Define $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ as the neighboring set of the $i$th agent. $\bar{j} \in \mathcal{N}_i$ means that agent $i$ can gain access to the information of agent $j$. Suppose that $\mathcal{G}$ denotes a fixed topology generated by all followers. Denote $\bar{G}$ as the interaction topology containing the leader, $\mathcal{G}$, and the directed edges between appropriate followers and the leader.

For switching topologies, there exists a time interval $[t_k, t_{k+1})$ with $k \in \mathbb{N}$ and $t_0 = 0$, $t_1 \geq t_{k+1} - t_k \geq t_0 > 0$. $t_0$ is called the dwell time, during which the interaction topology is time invariant, while $t_k$ with $k \in \mathbb{N}$ is said to be the switching sequence, at which the topology is time variant. Denote $\Omega = \{G_1, G_2, \ldots, G_m\}$ as a topology set for all possible interaction topologies. Let $\sigma(t) : [0, +\infty) \rightarrow \mathcal{I}_m = \{1, 2, \ldots, m\}$ be a switching signal whose value at time $t$ is the index of possible interaction topology.

B. POLYNOMIAL FUZZY MODEL

Here, we establish a new polynomial fuzzy model for first-order nonlinear error MASs with one leader and $N$ followers. The system of each follower is described by

$$\dot{x}_i = f(x_i) + u_i, \quad i = 1, 2, \ldots, N, \quad (1)$$

where $x_i \in \mathbb{R}^n$ denotes the state of agent $i$. $u_i \in \mathbb{R}^p$ is the control input to be determined later. $f(x_i) \in \mathbb{R}^n$ denotes the polynomial vector in $x_i$.

The leader’s dynamics is

$$\dot{x}_0 = f(x_0), \quad (2)$$

where $x_0 \in \mathbb{R}^n$ denote the leader’s state.

Let $\rho_i = x_i - x_0$ be the error state. Then, the error dynamics between (1) and (2) is generated as

$$\dot{\rho}_i = f(x_i) - f(x_0) + u_i. \quad (3)$$

Construct a polynomial fuzzy model as

$$\mathbb{R}^p : \text{IF } \theta_{i,1} \text{ is } M_{i,1}^p \text{ and } \ldots \text{ and } \theta_{i,q} \text{ is } M_{i,q}^p,$$

Then $\hat{\rho}_i = a_p(\rho_i)\rho_i + u_i, \quad (4)$

where $\theta_i = [\theta_{i,1}^T, \ldots, \theta_{i,q}^T]^T$ is the premise variable vector. $M_{i,j}^p$ for $p = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, q$ stands for the fuzzy sets. Constant $r$ denotes the number of fuzzy rules. $a_p(\rho_i)$ represents the polynomial matrix in $\rho_i$, which indicates that $a_p(\rho_i)\rho_i + u_i$ is a polynomial vector. Hence, (4) is other than the conventional T-S fuzzy model.

The mode (4) is written in a compact form

$$\dot{\hat{\rho}}_i = \sum_{p=1}^{r} h_p(\theta_i)(a_p(\rho_i)\rho_i + u_i), \quad (5)$$

where

$$h_p(\theta_i) = \frac{\omega_p(\theta_i)}{\sum_{p=1}^{r} \omega_p(\theta_i)}, \quad \omega_p(\theta_i) = \prod_{j=1}^{q} M_{i,j}^p(\theta_i).$$
The function $h_p(\theta(t))$ has the properties of
\[ \sum_{p=1}^{r} h_p(\theta_i) = 1, \quad h_p(\theta_i) \geq 0. \]
Then, system (5) for all agents can be expressed as
\[ \dot{\rho} = \sum_{p=1}^{r} h_p(\theta) \left[ A_p(\rho) \rho + u \right], \quad (6) \]
where
\[ h_p(\theta) \triangleq \text{diag}(h_p(\theta_1), h_p(\theta_2), \ldots, h_p(\theta_N)), \]
\[ A_p(\rho) \triangleq \text{diag}(a_p(\rho_1), a_p(\rho_2), \ldots, a_p(\rho_N)), \]
\[ \rho \triangleq [\rho_1^T \rho_2^T \ldots \rho_N^T]^T, \quad u \triangleq [u_1^T u_2^T \ldots u_N^T]^T. \]

C. PROBLEM FORMULATION

Here, we consider a first-order nonlinear multi-agent system with changing topologies. The consensus protocol for the $i$th agent is designed as:
\[ u_i = -c \sum_{j \in \mathcal{N}_i^{(t)}} d_i^{(t)}(\Gamma(\rho))(x_i - x_j), \]
\[ -cd_i^{(t)}(\Gamma(\rho))(x_i - x_0), \quad i = 1, 2, \ldots, N, \quad (7) \]
where $\mathcal{N}_i^{(t)}$ denotes the neighboring set of the $i$th agent at $\sigma(t)$, $d_i^{(t)}$ stands for the information flow weight. If the $i$th agent can gain the leader’s state at $\sigma(t)$, then $d_i^{(t)} > 0$, otherwise $d_i^{(t)} = 0$. Coefficient $c$ stands for the coupling strength. $\Gamma(\rho) \in \mathbb{R}^{n \times n} > 0$ denotes control gain matrix, to be determined. The consensus protocol $u_i$ aims to assure that the trajectories of each follower converge to those of the leader. $\mathcal{G}^{(t)}$ and $\mathcal{L}^{(t)}$ denote the communication topology and associated Laplacian matrix at $\sigma(t)$, respectively. Laplacian matrix $\mathcal{L}^{(t)}$ characterizes the topological structure of a communication network constructed by followers in this paper.

By substituting (7) into (6), the fuzzy closed-loop error system can be characterized by
\[ \dot{\rho} = \sum_{p=1}^{r} h_p(\theta(t))[A_p(\rho)\rho - \mathcal{W}_p^{\rho} \rho], \quad (8) \]
where
\[ \mathcal{W}_p^{\rho} \triangleq c(\mathcal{F}_p^{\sigma} \otimes \Gamma(\rho)), \quad \mathcal{F}_p^{\sigma} \triangleq \mathcal{L}^{(t)} + \mathcal{D}_p^{(t)}, \]
\[ \mathcal{D}_p^{(t)} \triangleq \text{diag}(d_1^{(t)}, \ldots, d_N^{(t)}). \]

Then, we give the definition of a guaranteed cost function that needs to be optimized, which is constructed by error states and control inputs as
\[ \mathcal{J} \triangleq \int_{0}^{\infty} \begin{bmatrix} \rho^T & 0 \\ u & 0 \\ \end{bmatrix} \begin{bmatrix} (I_N \otimes \mathcal{Q}) & 0 \\ * & (I_N \otimes \mathcal{R}) \\ \end{bmatrix} \begin{bmatrix} \rho \\ u \\ \end{bmatrix} dt = \int_{0}^{\infty} \xi^T \begin{bmatrix} (I_N \otimes \mathcal{Q}) & 0 \\ * & (I_N \otimes \mathcal{R}) \\ \end{bmatrix} \xi dt, \quad (9) \]
where $\xi \triangleq \sum_{p=1}^{r} h_p(\theta(t))[I - \mathcal{W}_p^{\rho}] \rho$, $\mathcal{Q} \in \mathbb{R}^{n \times n}$ and $\mathcal{R} \in \mathbb{R}^{n \times n}$ are given symmetric positive definite matrices.

Remark 1: It should be mentioned that guaranteed cost function is usually defined by states and control inputs for an isolated system, see [24], [26]. Different from the previous works, instead of utilizing state information, a guaranteed cost function based on error states and control inputs is employed for polynomial fuzzy MASs in this paper, because convergence states of agents may be nonzero when MASs achieve consensus. Our aim is to transform the consensus problem with guaranteed cost into a stabilization problem.

Before proceeding further, the following assumption and concepts are given.

Assumption 1: For switching topologies and fixed topology, $\mathcal{G}^{(t)}$ and $\mathcal{G}$ contain one directed spanning tree with the leader being a root node, respectively.

Definition 1 ([35]): If there exist polynomials $\zeta_1(\rho), \ldots, \zeta_N(\rho)$, such that
\[ G(\rho) = \sum_{i=1}^{N} \zeta_i^2(\rho), \quad (10) \]
then the polynomial $G(\rho)$ in $\rho \in \mathbb{R}^n$ is called an SOS.

Lemma 1 ([36]): The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has one simple zero eigenvalue and the other eigenvalues have positive real parts if and only if $\mathcal{G}$ has one directed spanning tree.

Definition 2 ([23]): A matrix $T = [t_{pq}] \in \mathbb{R}^{n \times n}$ is said to be a nonsingular M-matrix if $t_{pq} \leq 0, \forall p \neq q, p, q = 1, \ldots, n$, and all the leading principal minors of $T$ are positive.

Lemma 2 ([23]): Suppose that matrix $P = [p_{km}] \in \mathbb{R}^{n \times n}$ has $p_{km} \leq 0$, for $k \neq m, k, m = 1, \ldots, n$. Then, the following conditions are equivalent:
1) $P$ is a nonsingular M-matrix;
2) $P^T J + HP > 0$, where $H > 0$ is a diagonal matrix;
3) Each eigenvalue of $P$ has positive real parts.

Lemma 3 ([23]): Suppose that Assumption 1 is satisfied. If there exists a positive vector $\varphi^{(t)}(\rho) = (\varphi_1^{(t)}, \ldots, \varphi_N^{(t)})^T \in \mathbb{R}^N$, such that $\mathcal{F}^{\sigma} \varphi^{(t)}(\rho) = 1_N$ and
\[ \Phi^{\sigma}(\mathcal{F})^{\sigma} + (\mathcal{F}^{\sigma})^T \Phi^{\sigma}(\mathcal{F})^{\sigma} > 0, \quad (11) \]
where $\Phi^{\sigma}(\rho) = \text{diag}\left\{ \frac{1}{\varphi_1^{(t)}}, \ldots, \frac{1}{\varphi_N^{(t)}} \right\}$, and $\mathcal{F}^{\sigma}$ is defined in (8).

Clearly, the eigenvalues of $\mathcal{F}^{\sigma}$ have positive real part, it follows from Lemma 2 that $\mathcal{F}^{\sigma}$ is a nonsingular M-matrix.

Definition 3 (Guaranteed Cost Consensualization): Polynomial fuzzy error dynamic system (6) is called guaranteed cost consensualization with the cost function (9) by consensus protocol (7), if there exists a gain matrix $\Gamma(\rho)$ such that polynomial fuzzy error dynamic system (8) achieves the guaranteed cost consensus with a minimum upper bound of the given cost function.
III. GUARANTEED COST CONSENSUS CONTROL DESIGN

Here, we first address the guaranteed cost consensus issues of polynomial fuzzy MASs under switching topologies. Then, the obtained results are applied to the fixed topology.

A. CONSENSUS CONDITION UNDER SWITCHING TOPOLOGIES

Now, we give our first main result for system (8).

Theorem 1: Suppose that Assumption 1 holds. If there exist positive definite matrices $\gamma(\sigma) \in \mathbb{R}^{n \times n}$ and $\Gamma(\sigma) \in \mathbb{R}^{n \times n}$, arbitrary vectors $\eta_1, \eta_2,$ and $\eta_3$, nonnegative polynomials $\epsilon_1, \epsilon_2,$ and $\epsilon_3$, such that

\[
\eta_1^T(e_1I - \epsilon_1 I)e_1 = \text{SOS}, \quad (12)
\]
\[
\eta_2^T(e_2I - \epsilon_2 I)e_2 = \text{SOS}, \quad (13)
\]
\[
-\eta_3^T (e_3 I + \epsilon_3 I) = \text{SOS}, \quad (14)
\]

then system (8) is asymptotically stable; that is, all the agents can achieve the guaranteed cost consensus with a minimum upper bound, where

\[
\Xi = \begin{bmatrix}
\Omega_p & \Psi_1 & \Psi_2 \\
* & \Psi_3 & 0 \\
* & * & \Psi_4
\end{bmatrix},
\]
\[
\Omega_p = \text{Sym}(\Phi(\sigma) \otimes p(\rho)\sigma(\sigma)),
\]
\[
-\lambda_0 \Phi(\sigma) \otimes \Gamma(\sigma),
\]
\[
+ \lambda \Phi(\sigma) \otimes \gamma(\sigma),
\]
\[
\Psi_1 = Y(\sigma),
\]
\[
\Psi_2 = -(F^T \otimes \Gamma(\sigma))^T,
\]
\[
\Psi_3 = -(I_N \otimes \mathcal{Q})^{-1},
\]
\[
\Psi_4 = -(I_N \otimes \mathcal{R})^{-1},
\]
\[
\Gamma(\sigma) = \begin{bmatrix}
\gamma(\sigma)
\end{bmatrix},
\]
\[
Y(\sigma) = c\begin{bmatrix}
\gamma(\sigma)
\end{bmatrix},
\]
\[
\mathcal{Q} = \sum_{i=1}^{m} \chi_i \mathcal{Q}_i,
\]

where $\lambda_0 = \min_{i=1,\ldots,m}(\Phi F^T + (F^T \Phi)^T), \mathcal{Q}_i \psi_i = 1_N, \psi_i = (\psi_i, \psi_i, \ldots, \psi_i)^T \in \mathbb{R}^n$, in which $i = \{1, \ldots, m\}, \psi_0 = \min_{i,j}(\psi_i, \psi_j)$, in which $j = \{1, \ldots, N\}$.

Adding $\lambda V(t)$ into both side of (17), one has

\[
\dot{V}(t) + \lambda V(t) = \sum_{p=1}^{r} h_p(\theta) \rho^T \left\{ \begin{array}{l}
\Phi(\sigma) \otimes P(\sigma) p(\rho) \\
-\lambda_0 \Phi(\sigma) \otimes \gamma(\sigma) \\
+ \lambda \Phi(\sigma) \otimes \gamma(\sigma)
\end{array} \right\} \rho
\]

\[
= \sum_{p=1}^{r} h_p(\theta) \rho^T \left\{ \begin{array}{l}
\Phi(\sigma) \otimes P(\sigma) p(\rho) \\
-\lambda_0 \Phi(\sigma) \otimes \gamma(\sigma) \\
+ \lambda \Phi(\sigma) \otimes \gamma(\sigma)
\end{array} \right\} \rho
\]

\[
\leq \sum_{p=1}^{r} h_p(\theta) \rho^T \left\{ \begin{array}{l}
\Phi(\sigma) \otimes P(\sigma) p(\rho) \\
-\lambda_0 \Phi(\sigma) \otimes \gamma(\sigma) \\
+ \lambda \Phi(\sigma) \otimes \gamma(\sigma)
\end{array} \right\} \rho.
\]

If $\dot{V}(t) + \lambda V(t) \leq 0$, then the error state $\rho$ will converge to zero. Therefore, we need

\[
\dot{V}(t) + \lambda V(t) \leq 0
\]

which holds if

\[
\Theta_p \leq 0.
\]

From the SOS condition (14), we have

\[
\Omega_p = \text{Sym}(\Phi(\sigma) \otimes p(\rho)\gamma(\sigma)),
\]
\[
-\lambda_0 \Phi(\sigma) \otimes \gamma(\sigma),
\]
\[
+ \lambda \Phi(\sigma) \otimes \gamma(\sigma),
\]
\[
\gamma(\sigma) \Theta_p \gamma(\sigma)
\]

Multiplying both sides of (20) by $\gamma(\sigma)$ yields

\[
\Omega_p \leq 0.
\]

From (18) and Schur complement, one gets

\[
\dot{V}(t) \leq -\lambda V(t)
\]

\[
\leq -\lambda \rho^T \gamma(\sigma) \rho.
\]
where $t \in [t_0, t_1)$. In addition, the communication topology changes at time $t = t_1$. Then, one has
\[
V(t_1^-) < e^{-\lambda(t_1-t_0)}V(t_0) < e^{-\lambda t_0}V(t_0).
\]
According to (16), one obtains
\[
V(t_1) < r_0 V(t_1^-),
\]
where $r_0 = \bar{\varphi}/\varphi$, and
\[
\bar{\varphi} = (\max_{i,j} \psi_i^j)_{\lambda_{\max}(P^*)}, \varphi = (\min_{i,j} \psi_i^j)_{\lambda_{\min}(P^*)}, i \in \{1, \ldots, m\}, j \in \{1, \ldots, N\}.
\]
Thus, one has
\[
V(t_1) < r_0 e^{-\lambda t_0}V(t_0),
\]
that is
\[
V(t_1) < e^{-(\lambda t_0 + \ln r_0)}V(t_0),
\]
From the fact that $\lambda > (\ln r_0)/\lambda$, one has that $\lambda - \ln r_0 t_0 > 0$. Therefore, one gets
\[
V(t_1) < e^{-\lambda t_0}V(t_0),
\]
where $\alpha = \lambda - \ln r_0 t_0 > 0$.

For any given $t > t_1$, there exists a positive integer $m \geq 1$ such that $t_m < t \leq t_{m+1}$. By the recursion, for $k \in \mathbb{N}$, one obtains
\[
V(t_{k+1}) = e^{-\lambda t_0}V(t_k) < \ldots < e^{-\lambda t_0}V(t_0) = e^{-\lambda t_0}V(0).
\]
when $t \in (t_m, t_{m+1})$, one gets
\[
V(t) < e^{-\lambda (t-t_m)}V(t_m) < \ldots < e^{-\lambda t_0}V(0) = e^{-\lambda t_0}V(0).
\]
Since $m \geq 1$, one has
\[
V(t) < e^{-\lambda t_0}V(0), t \in (t_m, t_{m+1}).
\]
For $t = t_{m+1}$, it follows from (27) that
\[
V(t) < e^{-\lambda t_0}V(0).
\]
By (28) and (29), one concludes that $\rho \to 0$ as $t \to +\infty$. Thus, (8) can reach the consensus under the protocol (29). Here, suppose that there exists $P^{(i)}$ satisfying
\[
\dot{V} + \rho \left( \sum_{p=1}^{r} h_p(\theta) \left[ -I \otimes \mathcal{W}^{(i)} \right] \right) \times \left[ (I_N \otimes Q) 0 \right] \times \left( \sum_{p=1}^{r} h_p(\theta) \left[ I \otimes \mathcal{W}^{(i)} \right] \right) \rho < 0.
\]
Since
\[
\rho \left( \sum_{p=1}^{r} h_p(\theta) \left[ -I \otimes \mathcal{W}^{(i)} \right] \right) \times \left[ (I_N \otimes Q) 0 \right] \times \left( \sum_{p=1}^{r} h_p(\theta) \left[ I \otimes \mathcal{W}^{(i)} \right] \right) \rho > 0,
\]
we obtain $\dot{V} < 0$ at $\rho \neq 0$. Therefore, system (8) is asymptotically stable if (12) and (30) hold. That is, all agents can reach guaranteed cost agreement.

Note that (30) can be expressed as
\[
\xi^T \left[ (I_N \otimes Q) 0 \right] < -\dot{V}.
\]
Furthermore, by integrating both sides of (31) from 0 to $\infty$, one has
\[
\mathcal{J} = \int_0^{\infty} \xi^T \left[ (I_N \otimes Q) 0 \right] \eta dt < -\dot{\mathcal{J}}^{\infty}.
\]
Since system (8) is asymptotically stable if both (12) and (30) hold, we have $\lim_{t \to \infty} \|\rho(t)\| = 0$. Thus, (32) is represented as
\[
\mathcal{J} < \rho(0)^T (\Phi^{(i)} \otimes P^{(i)}) \rho(0).
\]
An upper bound of the cost function (9) is gained as
\[
\mathcal{J}^{*} = \rho(0)^T (\Phi^{(i)} \otimes P^{(i)}) \rho(0).
\]
Next, we will show that (30) holds if the SOS condition (14) is satisfied.
If the SOS condition (14) holds, one has
\[
\sum_{p=1}^{r} h_p(\theta) \left[ \Omega_p \right. \left. -Y^{(i)}(i) \right] < 0.
\]
Applying Schur complement, (34) can be expressed as
\[
\sum_{p=1}^{r} h_p(\theta) \left( \sum_{p=1}^{r} h_p(\theta) \left[ -F^{(i)} \otimes \mathcal{G}^{(i)} \right] \right)^T \times \left[ (I_N \otimes Q) 0 \right] \times \left( \sum_{p=1}^{r} h_p(\theta) \left[ I \otimes \mathcal{W}^{(i)} \right] \right) \rho < 0.
\]
By multiplying both sides of (35) by $Y^{(i)}(i)^{-1}$, one obtains
\[
\sum_{p=1}^{r} h_p(\theta) \left( \sum_{p=1}^{r} h_p(\theta) \left[ -F^{(i)} \otimes \mathcal{G}^{(i)} \right] \right)^T \times \left[ (I_N \otimes Q) 0 \right] \times \left( \sum_{p=1}^{r} h_p(\theta) \left[ I \otimes \mathcal{W}^{(i)} \right] \right) \rho < 0.
\]
Hence, (30) holds if the SOS condition (14) is satisfied. System (8) is asymptotically stable; that is, all the agents can achieve guaranteed cost consensus. By Definition 3, the conclusion is obtained.

Remark 2: In this work, the stability of multi-agent system (8) with switching directed topologies is shown based on multiple Lyapunov analysis. SOS-based relaxed guaranteed cost consensus criterion is proposed for fuzzy MASs under switching directed topologies. It should be mentioned that the derived conditions are converted to the SOS form. From (12)-(14), we gain the solutions of \( \hat{\Gamma}^{\sigma(i)} \) and \( y^{\sigma(i)} \). Then, the control gain matrices are calculated by \( \Gamma(\rho) = \hat{\Gamma}^{\sigma(i)}(\sigma(i))^{-1} \). Moreover, an upper bound for a given cost function is obtained.

**B. Consensus Condition Under Fixed Topology**

Here, we give the guaranteed cost consensus conditions for polynomial fuzzy MASs under fixed directed topology.

For the fixed topology, the consensus protocol is devised as:

\[
\dot{u}_i = -c \sum_{j \in \mathcal{N}_i} a_{ij} \Gamma(\rho)(x_i - x_j) - c d_i \Gamma(\rho)(x_i - x_0),
\]

\( i = 1, 2, \ldots, N, \) \( (37) \)

where \( \Gamma(\rho) \in \mathbb{R}^{n \times n} > 0 \) represents a polynomial matrix in \( \rho \).

By substituting (37) into (6), the overall error system is represented by

\[
\dot{\rho} = \sum_{p=1}^{r} h_p(\theta) [A_p(\rho) - \mathcal{W}] \rho, \quad (38)
\]

where \( \mathcal{W} \triangleq c(\mathcal{F} \otimes \Gamma(\rho)), \quad \mathcal{F} \triangleq \mathcal{L} + \mathcal{D}, \)

\( \mathcal{D} \triangleq \text{diag}(d_1, d_2, \ldots, d_N). \)

Using the polynomial Lyapunov function, we present the following result for system (38).

**Theorem 2:** Suppose that Assumption 1 holds. If there exist positive definite polynomial matrices \( Y(\rho) \in \mathbb{R}^{n \times n} \) and \( \hat{\Gamma}(\rho) \in \mathbb{R}^{n \times n} \), arbitrary vectors \( \eta_1 \) and \( \eta_2 \) and nonnegative polynomials \( \epsilon_1(\rho) , \epsilon_2(\rho) , \) and \( \epsilon_3(\rho) , \) such that

\[
\eta_1^T(Y(\rho) - \epsilon_1(\rho) I) \eta_1 \quad \text{is SOS}, \quad (39)
\]

\[
\eta_2^T(\hat{\Gamma}(\rho) - \epsilon_2(\rho) I) \eta_2 \quad \text{is SOS}, \quad (40)
\]

\[
-\eta_3^T\left( \hat{\Gamma} + \epsilon_3(\rho) I \right) \eta_3 \quad \text{is SOS}, \quad (41)
\]

then all agents can achieve the guaranteed cost consensus, where

\[
\Omega_p \triangleq \begin{bmatrix}
\Omega_{1p} & \Psi_1 & \Psi_2 \\
* & \Psi_3 & 0 \\
* & * & \Psi_4
\end{bmatrix},
\]

\[
\Omega_p = \text{Sym}(A_p(\rho)Y(\rho)) - \text{Sym}(\mathcal{F} \otimes \hat{\Gamma}(\rho))
\]

\[
-\sum_{k=1}^{n} \frac{\partial Y(\rho)}{\partial \rho_k} (A_p^k(\rho) - (\mathcal{W}^k) \rho
\]

Here, assume that there exists \( P(\rho) \) satisfying

\[
\dot{V} + \rho^T \left( \sum_{p=1}^{r} h_p(\theta) \left[ \begin{array}{c} I \\ -\mathcal{W} \end{array} \right] \right)^T \left( [I_N \otimes \mathcal{Q}] 0 \right) \left( [I_N \otimes \mathcal{R}] \right) \rho < 0. \quad (44)
\]
Since
\[
\rho^T \left( \sum_{p=1}^{r} h_p(\theta) \begin{bmatrix} I & -W \end{bmatrix} \right)^T \left( \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} \right) \rho \geq 0,
\]

one has \( \dot{V} < 0 \) at \( \rho \neq 0 \). Therefore, all agents can reach guaranteed cost agreement.

Note that (44) can be expressed as
\[
\zeta^T \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} \zeta < -\dot{V}.
\]

Furthermore, integrating both sides of (45) from 0 to \( \infty \), one obtains
\[
\mathcal{J} = \int_{0}^{\infty} \zeta^T \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} \zeta \, dt < -\dot{V}|_{t=0}^{\infty}.
\]

Since the polynomial fuzzy system (38) is asymptotically stable if both (39) and (44) hold, we have \( \lim_{t \to \infty} \| \rho(t) \| = 0 \). Thus, (46) can be expressed as
\[
\mathcal{J} < \rho(0)^T P(\rho(0)) \rho(0).
\]

An upper bound of (9) is obtained as
\[
\mathcal{J}^* = \rho(0)^T P(\rho(0)) \rho(0).
\]

Next, we will show that (44) holds if the SOS condition (41) is satisfied.

If the SOS condition (41) holds, one gets
\[
\sum_{p=1}^{r} h_p(\theta) \begin{bmatrix} \hat{\Omega}_p & Y(\rho) \\ -Y(\rho) & -F(\rho) \end{bmatrix} < 0.
\]

By employing Schur complement, (48) can be represented as
\[
\sum_{p=1}^{r} h_p(\theta) \hat{\Omega}_p + \left( \sum_{p=1}^{r} h_p(\theta) \begin{bmatrix} -Y(\rho) & -F(\rho) \\ -F(\rho) & -F(\rho) \end{bmatrix} \right)^T 
\times \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} \times \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} \sum_{p=1}^{r} h_p(\theta) \begin{bmatrix} Y(\rho) \\ -F(\rho) \end{bmatrix} < 0.
\]

From the SOS condition (41), one obtains
\[
\hat{\Omega}_p = \text{Sym}(A_p(\rho)Y(\rho)) - \text{Sym}(\hat{F} \otimes \hat{F}(\rho))
- \sum_{k=1}^{n} \frac{\partial Y(\rho)}{\partial \rho_k} (A^k_p(\rho) - (\hat{W}^k)\rho).
\]

Since \( P(\rho)Y(\rho) = I \), partially differentiating it with respect to \( \rho_k \) gives
\[
- \frac{\partial Y(\rho)}{\partial \rho_k} = Y(\rho) \frac{\partial P(\rho)}{\partial \rho_k} Y(\rho).
\]

Therefore, substituting (51) into (50) yields
\[
\hat{\Omega}_p = \text{Sym}(A_p(\rho)Y(\rho)) - \text{Sym}(\hat{F} \otimes \hat{F}(\rho))
+ \sum_{k=1}^{n} Y(\rho) \frac{\partial P(\rho)}{\partial \rho_k} Y(\rho)(A^k_p(\rho)
- (\hat{W}^k)\rho)
= Y(\rho)\hat{\Omega}_p Y(\rho).
\]

By multiplying both sides of (49) by \( Y(\rho)^{-1} \), one obtains
\[
\sum_{p=1}^{r} h_p(\theta) \hat{\Omega}_p + \left( \sum_{p=1}^{r} h_p(\theta) \begin{bmatrix} I & -W \end{bmatrix} \right)^T
\times \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} \times \begin{bmatrix} I_N \otimes Q & 0 \\ I_N \otimes R \end{bmatrix} < 0.
\]

Hence, (44) holds if the SOS condition (41) is satisfied. All agents can achieve guaranteed cost consensus. By Definition 3, the conclusion is obtained.

Remark 3: In Theorem 2, the stability of system (38) under fixed topology is shown based on polynomial Lyapunov analysis. SOS-based relaxed guaranteed cost consensus criterion is presented for polynomial fuzzy MASs under the fixed topological structure. From (39)-(41), we give the control gain matrix \( \Gamma(\rho) = \frac{1}{\epsilon^2} Y(\rho)^{-1}(\rho) \). Moreover, an upper bound (42) for a given cost function is obtained.

IV. ILLUSTRATIVE EXAMPLE
Here, we will provide an example to verify the effectiveness of obtained results.

Considering a nonlinear multi-agent system containing one leader and four followers, switching directed topologies in the switching set are shown in Figure 1.

The dynamics of each follower is described by chaotic equation [20]:
\[
\dot{x}_i(t) = f(x_i) + u_i,
\]

FIGURE 1. Switching topologies.
where

\[ f(x_i) = \begin{bmatrix} -10x_1 + 10x_2 \\ 28x_1 - x_2 - x_1x_3 \\ x_1x_2 - \frac{8}{3}x_3 \end{bmatrix}, \quad (55) \]

The dynamics of leader is same as (2). The error dynamic system is

\[ \dot{\rho}_i = \begin{bmatrix} -10 & 10 & 0 \\ 28 - x_3 & -1 & \rho_i - x_1 \\ x_1 - \rho_i & -\frac{8}{3} \end{bmatrix} \rho + u_i. \quad (56) \]

The fuzzy model of (56) is constructed by

\[ R^{l,m q} : \quad \text{IF } x_1 \text{ is } M^l_1, x_2 \text{ is } M^m_2, \text{ and } x_3 \text{ is } M^q_3, \]

Then \( \dot{\rho}_i = a^{l,m q} \rho_i + u_i, \)

where \( l, m, q = 1, 2. \) Suppose \( \theta_1(t) \in [M^l_1, M^l_2], \theta_2(t) \in [M^m_2, M^m_2], \) and \( \theta_3(t) \in [M^q_3, M^q_3], \) where \( M^l_1 = -50, M^m_2 = 50, M^m_2 = -46, M^q_3 = 46, M^q_3 = -65, M^q_3 = 65. \)

Under any initial condition, consider the consensus problem with guaranteed cost for fuzzy MASs. If the \( i \)th agent can get the leader’s state at \( \sigma(t), \) then \( d_i = 1. \) We assume that the weight value of each edge for network is 1. Thus, we have

\[ \mathcal{F} = \begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, & \mathcal{F}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \\ \mathcal{F}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{cases} \]

Choose positive definite matrices as

\[ Q = \begin{bmatrix} 4 & 0 & 0 \\ * & 4 & 0 \\ * & * & 4 \end{bmatrix}, \quad R = \begin{bmatrix} 5 & 0 & 0 \\ * & 5 & 0 \\ * & * & 5 \end{bmatrix}. \]

From \( \varphi_0 = \min_{i=1,\ldots,3} \varphi_j, j = \{1, 2, 3, 4\}, \) we obtain \( \varphi_0 = 1, \lambda_0 = \min_{i=1,\ldots,3} \lambda_{\min}(\Phi^T \mathcal{F}^j + (\mathcal{F}^j)^T \Phi^j) = 0.3900. \) Take \( \alpha = 0.3, \beta = 5. \) We give \( 2\alpha/\lambda_0 = 1.5385. \) Thus, we take the coupling strength \( c = 2. \) By (24), \( r_0 = \bar{\varphi}/\bar{\varphi} = 21.1816, \)

\[ \tau_0 > \ln r_0/\beta = 0.6106. \]

The initial error state value is given by \( \rho_i(0) = [4, -5, 8]^T (i = 1, 2, 3, 4). \) Using the SOSTOOLS [38] for solving SOS, from Theorem 1, we obtain \( J^1 = 3030.6, J^2 = 1755.6, \) and \( J^3 = 2746.8, \) and the control gains

\[ \Gamma^1 = \begin{bmatrix} 12.6438 & 0 & 0 \\ * & 2.5105 & 0 \\ * & * & 2.4908 \end{bmatrix}, \]

\[ \Gamma^2 = \begin{bmatrix} 10.8389 & 0 & 0 \\ * & 2.1773 & 0 \\ * & * & 2.1082 \end{bmatrix}, \]

\[ \Gamma^3 = \begin{bmatrix} 12.1037 & 0 & 0 \\ * & 2.4128 & 0 \\ * & * & 2.4027 \end{bmatrix}. \]

Considering switching directed topologies case, the switching signal is depicted in Figure 2. The interaction topologies are given by \( G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3, \) with \( \tau_0 = 1s. \) Figure 3 depicts the state trajectories of all agents. The error states are shown in Figure 4. It can be seen from Figures 3 and 4 that all followers track the leader and error states converge to zero.
V. CONCLUSION
In this paper, we construct a new polynomial fuzzy model for describing the nonlinear error MASs. By multiple Lyapunov function, SOS-based relaxed sufficient conditions are proposed to guarantee the consensus with guaranteed cost for MASs under switching topologies. Simulation results are provided to verify the validity of presented design techniques.

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