REGULARITY AND STABILITY
OF A WAVE EQUATION WITH A STRONG DAMPING
AND DYNAMIC BOUNDARY CONDITIONS

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ABSTRACT. We present an analysis of regularity and stability of solutions corresponding to wave equation with dynamic boundary conditions. It has been known since the pioneering work by [26, 27, 30] that addition of dynamics to the boundary may change drastically both regularity and stability properties of the underlying system. We shall investigate these properties in the context of wave equation with the damping affecting either the interior dynamics or the boundary dynamics or both.

This leads to a consideration of a wave equation acting on a bounded 3-d domain coupled with another second order dynamics acting on the boundary. The wave equation is equipped with a viscoelastic damping, zero Dirichlet boundary conditions on a portion of the boundary and dynamic boundary conditions. These are general Wentzell type of boundary conditions which describe wave equation oscillating on a tangent manifold of a lower dimension. We shall examine regularity and stability properties of the resulting system –as a function of strength and location of the dissipation. Properties such as well-posedness of finite energy solutions, analyticity of the associated semigroup, strong and uniform stability will be discussed.

The results obtained analytically are illustrated by numerical analysis. The latter shows the impact of various types of dissipation on the spectrum of the generator as well as the dynamic behavior of the solution on a rectangular domain.

1. Introduction. We consider the following damped wave equation with dynamic boundary conditions:

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\[
\begin{align*}
\begin{cases}
  u_{tt} - k_\Omega \Delta u_t - \Delta u &= 0 & \text{in } \Omega \\
  u_{tt} - \frac{\partial}{\partial \nu}(u + k_\Omega u_t) - k_\Gamma \Delta_\Gamma (\alpha u_t + u) &= 0 & \text{in } \Gamma_1 \\
  u &= 0 & \text{in } \Gamma_0 \\
  u(0, x) &= u_0, \ u_t(0, x) = u_1 & \text{on } \Omega
\end{cases}
\end{align*}
\]

where \( u = u(x, t), t \geq 0, x \in \Omega \) is a bounded three dimensional domain, \( \Delta \) denotes the Laplacian operator with respect to the \( x \) variable, \( \delta \Omega = \Gamma_0 \cup \Gamma_1, \Gamma_0 \cap \Gamma_1 = \emptyset, \) \( \frac{\partial}{\partial \nu} \) denotes the outer normal derivative, \( \Delta_\Gamma \) denotes the Laplace-Beltrami operator on the boundary \( \Gamma_1 \) with respect to the variable \( x \), \( k_\Omega, k_\Gamma \) and \( \alpha \) are non-negative constant. In order to ensure the presence of damping and thus energy dissipation in this system, we impose the following condition:

\[
\max\{k_\Omega, k_\Gamma \alpha\} > 0
\]

The term \( \Delta u_t \), which we will call the viscoelastic damping in the interior indicates that the stress is proportional not only to the strain but also the strain rate.

This equation corresponds to a physical model describing gas dynamics evolving in a bounded three dimensional domain \( \Omega \) with a boundary denoted by \( \Gamma \), and it was first described by Morse-Ingrad in \cite{34}. More precisely, we consider gas undergoing small irrotational perturbations away from \( \Omega \) which are concentrated on the portion of the boundary denoted by \( \Gamma_1 \). In general, when a small-amplitude plane wave hits a plane surface such as a wall, some of the energy is absorbed by the surface while some is reflected. The pressure generated, during this phenomenon, will tend to make the surface move, or else tends to force more fluid into the pores of the surface, creating, in the tangential direction motion, which will be characterized by the surface properties. When the motion at one point is independent of the motion of any other part of the surface; thus each point acts like a spring in response to the excess pressure in the gas, such a surface is called one of local reaction. However, if the surface behavior at one point depends on the behavior at neighboring points, so that the reaction is different for different incident waves, the surface can be called one of extended reaction. It means that the boundary condition are more complex that a simple harmonic oscillator, i.e., the acceleration is not proportional to the displacement anymore but to the relative displacement compared to its neighbours. Such surfaces are of many sorts: one which behaves like membranes and one with laminated structure, in which waves are propagated parallel to the surface; one in which waves penetrate into the material of the wall. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such type of boundary conditions are usually called dynamic boundary conditions and find numerous of applications in the bio-medical domain \cite{10, 43}, as well as in applications related to stabilization and active control of large elastic structures. See \cite{33} for some applications.

While undamped wave equations are well-posed in an \( L^p \)-setting if and only if \( p = 2 \) or the space dimension is 1 (\cite{25}), it is well-known that the wave equation with viscoelastic damping - strongly damped wave equation -, i.e., \( k_\Omega > 0 \), under classical boundary conditions such as Dirichlet or Robin generates an exponentially stable analytic \( C_0 \)-semigroup \cite{11}. In fact, this result is known for a larger class of scalar problems where viscoelastic damping is given in a form of fractional powers of Laplacian.
The purpose of this paper is to study how the dynamic boundary conditions on \( \Gamma \) affect the properties of the dynamics. We are particularly interested in qualitative properties of the resulting semigroup that include regularity and long time behavior with the analysis of various types of stability. It is well known that the presence of a damping impacts these two properties. Sufficient amount of dissipation in the system may provide not only strong decays of the energy when time \( t \to \infty \), but also may regularize semigroup by providing more smoothness to the solutions. It is also known that “more damping” does not mean more decays. The phenomenon of overdamping is well known both in practice (engineering) and in theory (mathematics). This motivates our interest in studying the balance between the competing damping mechanisms in the interior \( \Omega \) and on the boundary \( \Gamma \). This general problem is, of course, not new and goes back to the fundamental work by Littman-Markus [26, 27, 29] where “hybrid” systems of elasticity were studied. It was then discovered that an addition of a lower dimensional dynamics may destabilize the system. In a similar vein it was shown in [30] that the hybrid system with \( k_\Omega > 0 \)

\[
\begin{align*}
\text{u}_{tt} - \Delta u &= 0 & x \in \Omega, t > 0 \\
u &= 0 & x \in \Gamma_0, t > 0 \\
mv_{tt} + \frac{\partial}{\partial v}(u + u_t) &= 0 & x \in \Gamma_1, t > 0 \\
u(0, x) &\in H^1(\Omega), u_t(0, x) \in L^2(\Omega) & x \in \Omega
\end{align*}
\]

with \( m > 0 \) is not uniformly stable, while the case when \( m = 0 \) is exponentially stable assuming suitable geometric conditions imposed on \( \Omega \) [24].

The first mathematical model for surface of local reaction was introduced by Beale-Rosencrans, in [5], where he described the acoustic/structure interaction by a wave equation with acoustic boundary conditions:

\[
\begin{align*}
\text{u}_{tt} - \Delta u &= 0 & x \in \Omega, t > 0 \\
u(x, t) &= 0 & x \in \Gamma_0, t > 0 \\
m(x)d_t + d(x)d_t + k(x)d = -\rho\frac{\partial}{\partial v}u_t & x \in \Gamma_1, t > 0 \\
\delta_t &= \frac{\partial}{\partial v}u & x \in \Gamma_1 \\
u(0, x) &\in H^1(\Omega), u_t(0, x) \in L^2(\Omega) & x \in \Omega \\
z(0, x) &\in H^1(\Omega), z_t(0, x) \in L^2(\Omega) & x \in \Omega
\end{align*}
\]

Beale, in [7, 6] demonstrated that the problem was governed by a \( C_0 \)-semigroup of contractions. Later, similar models were studied ([20] and references therein) pointing out long time behavior and continuous dependence of solutions on the mass of the structure. Also, an extensive study of this model within an abstract framework is given by Mugnolo in [35].

Not only acoustic boundary conditions and the present study model similar physical phenomenon but they also coincide with the so-called general Wentzell boundary conditions (GWBC) given conditions on the parameters ([16]). Such boundary conditions involve second derivatives as well as lower order terms (Robin type) and Laplace-Beltrami terms and were intensively studied by Favini, Goldstein, Gal et al., in [12, 14] and references therein, in the context of hybrid problems. The Wentzell-type boundary conditions are intrinsically related to the one in our model (1), as we will see in section 7. Other papers used this framework in the study of one-dimensional wave equation without internal damping, with Wentzell boundary conditions, see [1, 13, 44].
Wave equations with viscoelastic damping and dynamic boundary conditions acting on a surface of local reaction have been studied within the framework of the following general model:

\[
\begin{cases}
  u_{tt} - k_0 \Delta u_t - \Delta u = f_1(u) & x \in \Omega, t > 0 \\
  u = 0 & x \in \Gamma_0, t > 0 \\
  u_{tt} + \frac{\partial}{\partial \nu}(u + k_2 u_t) + \rho(u_t) + f_2(u) = 0 & x \in \Gamma_1, t > 0 \\
  u(0,x) = u_0, \ u_t(0,x) = u_1 & x \in \Omega
\end{cases}
\]

Gerbi and Said-Houari in [18, 17] studied the problem (3) with \( f_2 = 0 \), \( f_1(u) = |u|^{p-2}u \) and a nonlinear boundary damping term of the form \( \rho(u_t) = |u_t|^{m-2} \).

A local existence result was obtained by combining the Faedo-Galerkin method with the contraction mapping theorem. The authors also showed that under some restrictions on the exponents \( m \) and \( p \), there exists initial data such that the solution is global in time and decay exponentially. Graber and Said-Houari in [19] extended some of these results to more general function \( f_1 \) and \( f_2 \). They also demonstrate that for \( \rho = f_2 = 0 \), the model (3) was governed by an analytic semigroup on \( H^{1,0}_0(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \).

**Remark 1.** We note that the model under consideration (1) can be recast in the abstract form as

\[
\begin{cases}
  u_{tt} = Au(t) + Cu_t(t) \\
  w_{tt} = B_1 u(t) + B_2 u_t(t) + B_3 u(t) + B_4 u_t(t)
\end{cases}
\]

where the operators \( A, C, B_i, i = 1 - 4 \) satisfy suitable conditions and variables \( u \) and \( w \) are connected via “trace” type operator \( L \) so that \( w = Lu \). In fact, this is the framework pursued in by Mugnolo in [36] where the second order abstract system (4) is reformulated as an abstract second order Cauchy problem on the product space \( X \times \partial X \) in the variable \( U(t) \equiv (u(t), Lu(t)) \) satisfying

\[
U_{tt} = AU(t) + CU_t(t), \ t > 0
\]

on a product space \( X = X \times \partial X \) where

\[
A = \begin{pmatrix} A & 0 \\ B_1 & B_3 \end{pmatrix}, \ C = \begin{pmatrix} C & 0 \\ B_2 & B_4 \end{pmatrix}
\]

are operator matrices on \( X \) with suitably defined domains. We also mentioned that a similar approach was used by Xiao et al. in [46, 45], where boundary conditions were reformulated as differential inclusions in suitable equivalence classes. However, the authors only discussed smoothing properties, without explicit generation result of analytic semigroups.

The above formulation can be further reduced to first order Cauchy problem by introducing variable \( u \equiv (U, U_t) \) and writing

\[
u_t(t) = Au(t)
\]

\[
A = \begin{pmatrix} 0 & I \\ A & C \end{pmatrix}
\]

A comprehensive study of well-posedness and regularity of second order evolutions (5) is given in [36] under the following standing set of general assumptions:

1. \( Y, X, \partial Y, \partial X \) are Banach spaces such that \( Y \hookrightarrow X, \partial Y \hookrightarrow \partial X \).
2. \( A : D(A) \subset X \rightarrow X, C : D(C) \subset X \rightarrow X \) are linear operators.
3. \( L : D(A) \cap D(C) \rightarrow \partial X \) is linear and surjective.
4. \( B_1 : D(A) \to \partial X, \ B_2 : D(C) \to \partial X \) are linear operators.

5. \( B_3 : D(B_3) \subset \partial Y \to \partial X, \ B_4 : D(B_4) \subset \partial X \to \partial X \), are linear closed operators.

Thus, in order to compare our results with those obtained in [36] we shall recast our problem within this more general framework. This is easily accomplished by setting:

\[
A = \Delta, \ C = k_\Omega \Delta, \ Lu = \gamma \equiv u|_{\Gamma_1},
\]

\[
B_1 = -\frac{\partial}{\partial \nu}, \ B_2 = -k_\Omega \frac{\partial}{\partial \nu}, \ B_3 = k_\Gamma \Delta_\Gamma, \ B_4 = k_\Gamma \alpha \Delta_\Gamma
\]

with the corresponding spaces (assuming \( k_\Gamma > 0 \)):

\( X = L^2(\Omega), \ Y = H^1_{0,0}(\Omega), \ \partial X = L^2(\Gamma_1), \ \partial Y = H^1(\Gamma_1) \)

The basic framework presented in [36] aims at proving generation of \( C_0 \) semigroups (resp. analyticity) as being equivalent having the same properties for the blocks of operators

\[
\begin{pmatrix}
0 & I \\
A & C
\end{pmatrix}, \ \text{and} \ \begin{pmatrix}
0 & I \\
B_3 & B_4
\end{pmatrix}
\]

considered on \( Y \times X \) and \( \partial Y \times \partial X \) (see Thm 3.3, 3.8, 4.5, 4.12 [36]).

Clearly the role of the “coupling” operator \( L \) is critical. In fact, the results in [36] are categorized with respect to the properties of \( L \) as

- unbounded \( Y \to \partial X \)
- bounded from \( Y \to \partial X \) but unbounded \( X \to \partial X \)

It is this second scenario that is relevant in our situation. The corresponding results are in section 4 [36]. As we shall see below the results of both Theorem 4.5 and Theorem 4.12 are not applicable -due to severity of assumptions imposed either on the operators \( B_i, i = 1 - 4 \). To wit, Part I of Thm 4.5 assumes that \( B_1 \in L(Y, \partial X), B_2 \in L(X, \partial X). \) The above is never satisfied with \( B_1 = -\frac{\partial}{\partial \nu}, B_2 = -k_\Omega \frac{\partial}{\partial \nu} \) and the choices of spaces \( X, Y, \partial X \). In addition, operators \( B_3, B_4 \) do not comply with regularity requirements postulated in Thm 4.5-unless they are bounded. Similar conclusion applies to Part II of Thm 4.5. Part II of that Theorem pertains to generation of analytic semigroups. However, here the operators \( B_1, B_2 \) do not comply with regularity requirements unless the Dirichlet map is sufficiently smooth-as in one dimensional case. (note that \( D(A)_L \) corresponds to \( H^{1/2}(\Omega) \) , unless the dimension of \( \Omega = 1 \).

The main reason is that the treatment given in [36] treats the coupling operator \( B_1, B_2 \) like a perturbation -rather than a carrier of regularity. It is this second approach that is used in our paper where the matrix operators is not a perturbation of two blocks of operator matrices -but rather perturbation of a new system which is related to Wentzell problem.

What distinguishes our model from other systems studied is the presence of Laplace Beltrami operator in the dynamic boundary condition. The latter transforms the local reaction surface into an extended reaction surface.

In the case of viscously elastic dampers located in \( \Omega \), our aim is to provide a comprehensive study of (i) well-posedness, (ii) regularity and (iii) stability of solutions under the influence of competing both interior and boundary damping. In order to quantize the analysis we introduce the energy function.
With respect to this energy we are interested in the well-posedness and stability of solutions \((u|_{\Omega}, u_t|_{\Omega}, u|_{\Gamma_1}, u_t|_{\Gamma_1})^T\) to (1), along with the decay of the energy \(E(t)\) of (1) at \(t \to \infty\). We begin by defining energy functions representing both internal and boundary energy of the system as well as the damping term \(D(t)\):

\[
E(t) = E_\Omega(t) + E_{\Gamma_1}(t)
\]

\[
E_\Omega(t) = \int_{\Omega} \left( |\nabla u(t)|^2 + |u_t(t)|^2 \right) d\Omega
\]

\[
E_{\Gamma_1}(t) = \int_{\Gamma_1} \left[ k_{\Gamma_1} |\nabla u|_{\Gamma_1}(t)|^2 + |u_t|_{\Gamma_1}(t)|^2 \right] d\Gamma_1
\]

\[
D(t) = \int_{\Omega} k_{\Omega} |\nabla u_t(t)|^2 d\Omega + \int_{\Gamma_1} k_{\Gamma_1} \alpha |\nabla u_t|_{\Gamma_1}(t)|^2 d\Gamma_1
\]

We have the following (formal) energy identity:

\[
E(0) = E_\Omega(t) + E_{\Gamma_1}(t) + 2 \int_0^t D(s)ds
\]

\[
E(0) = |\nabla u(t)|_{\Omega}^2 + |u_t(t)|_{\Omega}^2 + k_{\Gamma_1} |\nabla u|_{\Gamma_1}(t)|^2_{\Gamma_1} + |u_t|_{\Gamma_1}(t)|^2_{\Gamma_1}
\]

\[
+ 2 \int_0^t k_{\Omega} |\nabla u_t(s)|^2_{\Gamma_1} + k_{\Gamma_1} \alpha |\nabla u_t|_{\Gamma_1}(s)|^2_{\Gamma_1} ds
\]

The above energy identity suggests that the energy is decreasing. However, how fast, it needs to be determined. The energy balance also suggests that there is an extra regularity in the damping. How this regularity is propagated onto the entire system is a question we aim to resolve.

After establishing a preliminary well-posedness of finite energy solution (generation of a semigroup of contractions), we shall proceed with a study of:

- Additional regularity (analyticity) properties of solutions, both in \(L_2\) and \(L_p\) framework.
- Stability analysis of the resulting semigroups with particular emphasis on differentiating between strong and uniform stability. The latter will depend on the values of damping parameters.
- Numerical results illustrating sharpness of the estimates obtained.

The paper is organized into eight sections. In section 2, we give an appropriate abstract description of the problem. The results are presented in section 3, while the following sections (4-8) each prove one theorem. We show in the theorem 3.1 our model (1) is governed by a \(C_0\)-semigroup of contractions. Then, we assume, in theorem 3.2 that the semigroup is exponentially stable, in the presence of a structural damping in the interior. Then, theorems 3.3 and 3.4 both show the analyticity of the semigroup under certain conditions for the dampings. We also show that the spectrum is contained in the left-half plane (see theorem 3.5). Finally, section 9 presents numerical results obtained with finite element methods, as well as an exhaustive study of the different possible scenarios for this model

2. Definitions and preliminaries. We first introduce a couple of operators from [42] which we will use in this study to pose the abstract problem and prove the theorems.
\textbf{Definition 2.1 (Neumann map).} Define the map $N$ by
\begin{align*}
z = N g &\iff \begin{cases}
\Delta z = 0 &\text{on } \Omega \\
\frac{\partial}{\partial \nu} z|_{\Gamma_1} = g &\text{on } \Gamma \\
z|_{\Gamma_0} = 0 &\text{on } \Gamma_0
\end{cases}
\end{align*}
Then, elliptic theory gives $N \in \mathcal{L}(L^2(\Gamma_1), H^2(\Omega) \subset \mathcal{D}(A^{1/2-e})) \forall e > 0$

\textbf{Definition 2.2 (The Laplacian in $\Omega$).} Let the operator $A : L^2(\Omega) \supset \mathcal{D}(A) \to L^2(\Omega)$ be defined by:
\[
Au = -\Delta u, \quad cD(A) = \{u \in L^2(\Omega), Au \in L^2(\Omega), u|_{\Gamma_0} = 0, \frac{\partial}{\partial \nu} u|_{\Gamma_1} = 0\}
\]
Then $A$ is self-adjoint, positive definite, and therefore the fractional powers of $A$ are well-defined. In particular, we have the following characterization:
\[
\mathcal{D}(A^{1/2}) = H^1_{\Gamma_0}(\Omega) = \{z \in H^1(\Omega), z = 0 \text{ on } \Gamma_0\}
\]
with $\|z\|^2_{\mathcal{D}(A^{1/2})} = \|A^{1/2}z\|^2_{L^2(\Omega)} = \int_{\Omega} |\nabla z|^2 = \|z\|^2_{H^1_{\Gamma_0}(\Omega)}, \forall z \in \mathcal{D}(A^{1/2})$
where the last equality follows from Poincaré’s inequality. In order to simplify the notation, we will denote the $L^2$-norm in the following way: $\|u\|^2_{L^2(\Omega)} = |u|^2_{\Omega}$ and $\|u\|^2_{L^2(\Gamma_1)} = |u|^2_{\Gamma_1}$

\textbf{Definition 2.3 (Trace map $\gamma$.)} Let $\gamma : H^1(\Omega) \to L^2(\Gamma_1) \subset H^{1/2}(\Gamma_1)$ be the restriction to $\Gamma_1$:
\[
\forall z \in H^1_{\Gamma_0}(\Omega), \quad \gamma(z) = z|_{\Gamma_1}
\]
Then, by [42, Lemma 2.0]:
\[
N^* A = \gamma(z) \forall z \in \mathcal{D}(A^{1/2})
\]

\textbf{Definition 2.4 (Laplace-Beltrami on the boundary).} For all $k_\Gamma \geq 0$, set $B : L^2(\Gamma_1) \supset \mathcal{D}(B) \to L^2(\Gamma_1)$ to be:
\[
Bz = -k_\Gamma \Delta z, \quad \mathcal{D}(B) = \{z \in L^2(\Gamma_1), k_\Gamma \Delta z \in L^2(\Gamma_1)\}
\]
with the associated norm:
\[
\|z\|^2_{\mathcal{D}(B^{1/2})} = \|B^{1/2}z\|^2_{L^2(\Gamma_1)} = \int_{\Gamma_1} k_\Gamma |\nabla z|^2 d\Gamma_1 = k_\Gamma \|u\|^2_{H^1(\Gamma_1)}
\]
Let $\gamma(u, u, u|_{\Gamma_1}, u|_{\Gamma_1}) = (u_1, u_2, u_3, u_4)$. The third coordinate $u_3$ is defined as the trace of $u_1$:
\[
uz = u_1|_{\Gamma_1}
\]

\textbf{Definition 2.5 (Energy spaces).} The associated energy space is:
\[
\mathcal{H} = \{(u_1, u_2, u_3, u_4) \in \mathcal{D}(A^{1/2}) \times L^2(\Omega) \times \mathcal{D}(B^{1/2}) \times L^2(\Gamma_1), u_1|_{\Gamma_1} = u_3\}
\]
with associated norm: $\|u\|^2_{\mathcal{H}} = |\nabla u_1|^2_{\Omega} + |u_2|^2 + k_\Gamma |\nabla u_3|^2_{\Gamma_1} + |u_4|^2_{\Gamma_1}$.
We note that $\mathcal{D}(B^{1/2})$ is equipped with the graph norm
\[
\|u\|^2_{\mathcal{D}(B^{1/2})} \equiv |u|^2_{\Gamma_1} + |B^{1/2}u|^2_{\Gamma_1} = |u|^2_{\Gamma_1} + k_\Gamma |\nabla u|^2_{\Gamma_1}
\]
With the definitions of $A, B$ and $N$, we set:

$$
A = \begin{pmatrix}
0 & I & 0 & 0 \\
\Delta & k_\Omega \Delta & 0 & 0 \\
0 & 0 & 0 & I \\
-\frac{\partial}{\partial \nu} & -k_\Omega \frac{\partial}{\partial \nu} & -B & -\alpha B
\end{pmatrix}
$$

(8)

$$
D(A) = \{ [u_1, u_2, u_3, u_4]^T \in D(A) \times D(A) \times D(B) \times D(B), \\
\text{such that } \Delta (u_1 + k_\Omega u_2) \in L^2(\Omega), \\
\frac{\partial}{\partial \nu} (u_1 + k_\Omega u_2) + B^\frac{1}{2} (B^\frac{1}{2} u_3 + \alpha B^\frac{1}{2} u_4) \in L^2(\Gamma_1), \\
u_1|_{\Gamma_1} = N^* Au_1 = u_3, \ u_2|_{\Gamma_1} = N^* Au_2 = u_4
\}$$

which is densely defined in $\mathcal{H}$.

**Remark 2.** The following representation will be frequently used.

$$
-\Delta u = A(I - N \frac{\partial}{\partial \nu}) u
$$

3. Statement of the main results.

**Theorem 3.1.** The operator $A$, given in (8), generates a $C_0$-semigroup of contractions $\{e^{At}\}_{t \geq 0}$ on $\mathcal{H}$. In addition, under the condition (2), i.e. $\max\{k_\Omega, k_\Gamma \alpha\} > 0$, the said semigroup is strictly contractive.

**Theorem 3.2.** Let $k_\Omega > 0$. Consider the operator $A$ given in (8), then the $C_0$-semigroup $\{e^{At}\}_{t \geq 0}$ is exponentially stable, i.e.,

$$
\exists C, \omega \geq 0 \text{ such that } \|e^{At}u\|_\mathcal{H} \leq Ce^{-\omega t} \|u\|_\mathcal{H}
$$

**Remark 3.** When $k_\Omega = 0$ the system is no longer exponentially stable. This has been first discovered by Littman and Markus in [27] for the Scele model [26], the only form of stability retained for finite energy initial data is strong stability (see theorem 3.6). If one considers more regular initial data (with a membership in the domain of the generator) then there may be a possibility of proving algebraic decay rates for the energy function. However these rates, by necessity, can not be uniform with respect to the underlined topology of finite energy space. For this type of results, applicable to some coupled models with interface conditions, we refer to recent work of Avalos (e.g [4]) which exploits sharp characterization [9] of algebraic decay rates in terms of the estimates of the resolvent along the imaginary axis. Whether similar technique could be applied in the present context is not known at the present time.

**Theorem 3.3.** Let $k_\Omega, k_\Gamma, \alpha > 0$. Suppose $|\beta| \geq 1$ so that $(i \beta - A)^{-1} \in \mathcal{L}(\mathcal{H})$, then with $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$

$$
\|R(i \beta; A)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|\beta|}
$$

It follows that $\{e^{At}\}_{t \geq 0}$ is an analytic semigroup on $\mathcal{H}$.

**Theorem 3.4.** Let $k_\Omega, \alpha > 0$.

First Part: $L^2$ theory:

Suppose $k_\Gamma \geq 0$, and consider $A_{k_\Gamma \geq 0}$ given by $A$ in (8), then $A_{k_\Gamma \geq 0}$ generates an analytic $C_0$-semigroup $\{e^{A_{k_\Gamma \geq 0}t}\}_{t \geq 0}$ on $\mathcal{H}$. 

Second Part: $L_p$ theory:

Suppose $k_\Gamma = 0$, and consider $A_{k_\Gamma=0}$ given by $A$ in (8), then $A_{k_\Gamma=0}$ generates an analytic $C_0$-semigroup $\{e^{A_{k_\Gamma=0}t}\}_{t \geq 0}$ on

$$\mathcal{H}_p = \{(u_1, u_2, u_3) \in W^{1,p}_0(\Omega) \times L^p(\Omega) \times L^p(\Gamma_1), u_1|_{\Gamma_1} = u_3\}$$

for all $p \in [1, \infty]$.

**Theorem 3.5.** Suppose that the damping condition (2) holds, i.e. $\max\{k_\Omega, k_\Gamma \alpha\} > 0$. With $A$ defined in (8), $\sigma(A) \cap i\mathbb{R} = \emptyset$.

The first result (theorem 3.1) asserts that problem (1) is well-posed and generates a $C_0$-semigroup of contractions with respect to the finite energy space $\mathcal{H}$. Then, theorem 3.2 provides us the exponential stability of the semigroup $\{e^{At}\}_{t \geq 0}$, provided interior damping ($k_\Omega > 0$). This result is obtained using energy estimates. In presence of viscoelastic damping both in the interior and on the boundary, theorem 3.3 reveals the $C_0$-semigroup of contractions from theorem 3.1 is also analytic, this result obtained with the resolvent equation and relies on the exponential stability property. Another way to show analyticity - without the exponential stability property - is presented in 3.4 (first part of the theorem) where we proceed to a change of variable transforming the problem into a heat equation with General Wentzell Boundary Conditions (GWBC). Such an approach is related to the work made during the last ten years by Favini and Goldstein’s group: [14] and [12]. This approach also presents the advantage to treat the case $k_\Gamma = 0$, in which case, we have a stronger result (see second part of the theorem 3.4). We also mention that analyticity should be achieved using energy method and following Haraux’s approach in [21].

Finally, theorem 3.5 gives an important spectral property providing stability results for the model (1). For instance, we will be able to recover the exponential stability for the analytic semigroups $\{e^{At}\}_{t \geq 0}$ described in theorem 3.4 using the following:

**Theorem A** (Theorem 4.4.3 - [37]). Let $A$ be the generator of an analytic semigroup $T(t)$. If

$$\sup\{\text{Re}\lambda : \lambda \in \sigma(A)\} < 0$$

then there are constants $M \geq 1$ and $\mu > 0$ such that $\|T(t)\| \leq Me^{-\mu t}$, i.e. $T(t)$ is exponentially stable.

This same theorem 3.5 will also give strong stability for the semigroup $\{e^{At}\}_{t \geq 0}$ (for any $\alpha > 0$ described in theorem 3.1 after application of the following classical result quoted:

**Theorem B** ([2]). Let $T(t)$ be a bounded $C_0$-semigroup with generator $A$. Assume that $\sigma_r(A) \cap i\mathbb{R} = \emptyset$, where $\sigma_r(.)$ denotes the residual spectrum. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $T(t)$ is strongly stable. This is to say; for every $u \in \mathcal{H}$

$$\|T(t)u\|_{\mathcal{H}} \to 0, \ t \to \infty$$

The above theorem will immediately provide the proof of the following result.

**Theorem 3.6.** Let $k_\Omega = 0$. Then the semigroup is strongly stable.

The illustration of strong stability is given on Figure 5 where it is shown that one branch of the eigenvalues approaches imaginary axis. While boundary stabilization
leads to exponential stability, this is so for models without inertial terms on the boundary. This phenomenon was first discovered by Markus and Littman in the context of the Scole model.

4. $C_0$-semigroup of contractions - proof of theorem 3.1. Let $k_{\Omega}, k_{\Gamma} \alpha \geq 0$, and consider the problem (1):

$$
\begin{align*}
&u_{tt} - k_{\Omega} \Delta u_t - \Delta u = 0 \quad \text{in } \Omega \\
u_{tt} - \frac{\partial}{\partial \nu}(u + k_{\Omega} u_t) - k_{\Gamma} \Delta_{\Gamma}(\alpha u_t + u) = 0 \quad \text{in } \Gamma_1 \\
u = 0 \quad \text{in } \Gamma_0 \\
u(0, x) = u_0, \; u_t(0, x) = u_1 \quad \text{on } \Omega
\end{align*}
$$

and recall the definition of (8) using:

$$
-\Delta u = A(I - N \frac{\partial}{\partial \nu})u.
$$

The following regularity properties of the elements in the domain result from the definition of $D(A)$

$$
D(A) = \{[u_1, u_2, u_3, u_4]^T \in D(A^{1/2}) \times D(A^{1/2}) \times D(B^{1/2}) \times D(B^{1/2}),
\text{ such that } A(I - N \frac{\partial}{\partial \nu})(u_1 + k_{\Omega} u_2) \in L^2(\Omega),
\frac{\partial}{\partial \nu}(u_1 + k_{\Omega} u_2) + B^{1/2}(B^{1/2} u_3 + \alpha B^{1/2} u_4) \in L^2(\Gamma_1),
u|_{\Gamma_1} = N^* Au_1 = u_3, \; u_2|_{\Gamma_1} = N^* Au_2 = u_4\}
$$

which is closed and densely defined in $H$.

The following regularity properties of the elements in the domain result from the definition of $D(A)$

$$
\frac{\partial}{\partial \nu}(u_1 + k_{\Omega} u_2) \in [D(B^{1/2})]'\tag{9}
$$

this in particular implies $\frac{\partial}{\partial \nu}(u_1 + k_{\Omega} u_2) \in L^2(\Gamma_1), k_{\Gamma} = 0$. The above regularity result is stronger than classical elliptic theory implies $\frac{\partial}{\partial \nu}(u_1 + k_{\Omega} u_2) \in H^{-1/2}(\Gamma_1)$, where the latter is due to the fact that $\Delta(u_1 + k_{\Omega} u_2) \in H^1(\Omega)$ along with $u_1, u_2 \in H^1(\Omega)$. As a consequence of (9) one has well defined duality pairing

$$
< \frac{\partial}{\partial \nu}(u_1 + k_{\Omega} u_2), v >, \forall v \in D(B^{1/2}), \forall U \in D(A) \tag{10}
$$

In order to show the semigroup generation for the dynamics $A$, we wish to use the Lumer-Phillips theorem, [32] and hence we must show the maximal dissipativity of $A$.

**Step 1.** $A$ is dissipative.
∀U = (u_1, u_2, u_3, u_4) ∈ D(A), take the inner product with U:

\[ <AU, U> = \left( \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right) \in D(A), \text{ take the inner product with } U:\]

\[ <AU, U>_H = \left( \begin{array}{c} u_2 \\ u_1 \end{array} \right) \in D(A_1^2) + \left( \begin{array}{c} u_4 \\ u_3 \end{array} \right) \in D(B_1^2) \]

\[ = \left( A^2 u_2, A^2 u_1 \right)_\Omega - \left( A^2 u_1 + k_\Omega A^2 u_2, A^2 u_2 \right)_\Omega \]

\[ + \left( \frac{\partial}{\partial \nu} (u_1 + k_\Omega u_2), u_4 \right)_\Gamma_1 + \left( B^2 u_4, B^2 u_3 \right)_\Gamma_1 \]

\[ - \left( \frac{\partial}{\partial \nu} (u_1 + k_\Omega u_2), u_4 \right)_\Gamma_1 - \left( B^2 u_3, B^2 u_4 \right)_\Gamma_1 \]

\[ - \alpha \left( B^2 u_4, B^2 u_4 \right)_\Gamma_1 \]

\[ = -k_\Omega \left| A^2 u_2 \right|^2_\Omega - \alpha \left| B^2 u_4 \right|^2_\Gamma_1 \leq 0 \]

Therefore, \( A \) is dissipative.

**Step 2.** Range condition. Our aim is to show \( R(\lambda I - A) = H \).

To show that \( A \) is a contraction semigroup, it remains to show that the range condition is satisfied, i.e., if \( \lambda \in \mathbb{C}, \ Re \lambda > 0 \) and \( \left( \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \right) \in H \) is given, then the stationary equation:

\[ (\lambda I - A) \left( \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right) = \left( \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \right) \]  

is satisfied for some \( \left( u_1, u_2, u_3, u_4 \right) \in D(A) \).

Then (11) becomes:

\[
\begin{cases}
\lambda u_2 + A(I - N \frac{\partial}{\partial \nu})u_1 + k_\Omega A(I - N \frac{\partial}{\partial \nu})u_2 = f_1 \\
\lambda u_3 - u_4 = f_2 \\
\lambda u_4 + \frac{\partial}{\partial \nu} (u_1 + k_\Omega u_2) + Bu_3 + \alpha Bu_4 = f_4
\end{cases}
\]

In the first equation, note that \( f_1, u_1 \in D(A^2) \), so \( u_2 \in D(A^2) \). Similarly, from the third equation we have \( u_4 \in D(B^2) \).

From the second equation \( f_2, u_2 \in L^2(\Omega) \) implies \( A(I - N \frac{\partial}{\partial \nu})(u_1 + k_\Omega u_2) \in L^2(\Omega) \).

The fourth equation, with \( u_4, f_4 \in L^2(\Gamma_1) \) implies that \( \frac{\partial}{\partial \nu} (u_1 + k_\Omega u_2) \in [D(B^{1/2})'] \).
The above leads to the following representation:

\[
\begin{align*}
\begin{cases}
  u_2 &= \lambda u_1 - f_1 \\
  u_4 &= \lambda u_3 - f_3 \\
  \lambda^2 u_1 + A(I - N \frac{\partial}{\partial n}) u_1 + \lambda k_0 A(I - \frac{\partial}{\partial n}) u_1 &= f_2 + \lambda f_1 + k_0 A(I - \frac{\partial}{\partial n}) f_1 \\
  \lambda^2 u_3 + \frac{\partial}{\partial n} (u_1 + k_0 \lambda u_1) + B u_3 + \lambda \alpha B u_3 &= f_4 + \lambda f_3 + k_0 \frac{\partial}{\partial n} f_1 + \alpha B f_3
\end{cases}
\end{align*}
\]  

(12)

To solve the stationary problem (12), we shall use a weak formulation and Lax-Milgram theorem. Let \((v_1, v_2, v_3, v_4) \in \mathcal{D}(A)\), and for the time being we also take \(F = (f_1, f_2, f_3, f_4) \in \mathcal{D}(A)\). Later we shall extend the argument by density to all \(F \in \mathcal{H}\). We consider the two last equations, multiply them by \((v_1, v_3)\) and integrate in space over \(\Omega\) and \(\Gamma_1\), respectively:

\[
\begin{align*}
\begin{cases}
  (\lambda^2 u_1, v_1)_{\Omega} + (A(I - N \frac{\partial}{\partial n}) u_1, v_1)_{\Omega} + (k_0 \lambda A(I - \frac{\partial}{\partial n}) u_1, v_1)_{\Omega} &= (f_2 + \lambda f_1 + k_0 A(I - \frac{\partial}{\partial n}) f_1, v_1)_{\Omega} \\
  \langle \lambda^2 u_3, v_3 \rangle_{\Gamma_1} + \langle \frac{\partial}{\partial n} (u_1 + k_0 \lambda u_1), v_3 \rangle_{\Gamma_1} + (1 + \lambda \alpha) \langle B u_3, v_3 \rangle_{\Gamma_1} &= \langle f_4 + \lambda f_3 + k_0 \frac{\partial}{\partial n} f_1 + \alpha B f_3, v_3 \rangle_{\Gamma_1}
\end{cases}
\end{align*}
\]

Then combining these two equations, we get:

\[
\begin{align*}
  (\lambda^2 u_1, v_1)_{\Omega} + (1 + \lambda k_0) \langle A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_1 \rangle_{\Omega} + (\langle \lambda^2 u_3, v_3 \rangle_{\Gamma_1} + (1 + \lambda \alpha) \langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} v_3 \rangle_{\Gamma_1} &= (f_2 + \lambda f_1, v_1)_{\Omega} + (k_0 A^{\frac{1}{2}} f_1, A^{\frac{1}{2}} v_1)_{\Omega} + (f_4 + \lambda f_3, v_3)_{\Gamma_1} + (\alpha B^{\frac{1}{2}} f_3, B^{\frac{1}{2}} v_3)_{\Gamma_1}
\end{align*}
\]

This leads us to consideration of a bilinear form

\[
a(u_1, u_3, v_1, v_3) \equiv (\lambda^2 u_1, v_1)_{\Omega} + (1 + \lambda k_0) \langle A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_1 \rangle_{\Omega} + (\langle \lambda^2 u_3, v_3 \rangle_{\Gamma_1} + (1 + \lambda \alpha) \langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} v_3 \rangle_{\Gamma_1} \equiv \langle (v_1, v_3) \rangle \in \mathcal{V}, \text{ where } \mathcal{V} \equiv \{(v_1, v_3) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(B^{1/2}), v_3 = v_1|_{\Gamma_1}\}.\text{We are solving for the variable } u \text{ the variational equation:}
\]

\[
a(u, v) = F(v), \quad \forall v \in \mathcal{V} \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(B^{1/2})
\]
Recall that exponential stability of $u$ since there exists a unique solution $u$ such that $V \times V$ we have continuity of bilinear form on $V \times V$:

$$|a(u, v)| \leq \max \{\lambda^2 + k_\Omega |\lambda| + \alpha |\lambda|, 1\} \|u\|_V \|v\|_V$$

$$|F(v)| \leq \max \{\|\lambda k_\Omega + \alpha, 1\} \|F\|_H \|v\|_V$$

The bilinear is also coercive:

$$Re a(u, u) = (\lambda^2 u_1, u_1)_\Omega + (A u_1, A u_1)_\Omega + (k_\Omega \lambda A u_1, A u_1)_\Omega + (A u_3, A u_3)_\Gamma + (\lambda \alpha B u_3, B u_3)_\Gamma \geq C \left[ Re \lambda^2 |u_1|^2_\Omega + |A u_3|^2_\Gamma + Re \lambda^2 |u_3|^2_\Gamma + |B u_3|^2_\Gamma \right] \geq C \|u\|^4_V$$

Therefore $a(u, v)$ is both bounded and coercive, so by Lax Milgram for every $F \in \mathcal{H}$ there exists a unique solution $u \in \mathcal{V}$. Moreover $u = (u_1, u_3)$ satisfies the last two equations in (12).

Next we reconstruct the remaining part of the vector $U$. From (12)

$$u_2 = \lambda u_1 - f_1 \in D(A^{1/2}), \quad u_4 = \lambda u_3 - f_3 \in D(B^{1/2}), \quad \forall F \in \mathcal{H}$$

Since $u_3 = N^* A u_1$ and $f_3 = N^* f_1$ we conclude that $u_4 = N^* A u_2$ as required by the membership in the $D(A)$. The remaining regularity requirements simply follow from the structure of equations in (12).

In conclusion, for all $F \in \mathcal{H}$ we obtain $U = (u_1, u_2, u_3, u_4)$ in $D(\mathcal{A})$ such that $(\lambda I - \mathcal{A}) U = F \in \mathcal{H}$. Thus $U$ is our desired solution. As a consequence, $\mathcal{A}$ generates a strongly continuous semigroup of contraction $\{e^{\mathcal{A}t}\}_{t \geq 0}$. Note that the result on generation holds without any dissipation active, i.e., the range condition still holds if $k_\Omega = 0$ or $k_\Gamma \alpha = 0$

5. Exponential stability - proof of theorem 3.2. Let $k_\Omega > 0$ and recall the model (1):

$$\begin{cases}
  u_{tt} - k_\Omega \Delta u_t - \Delta u = 0 \quad &\text{in } \Omega \\
  u_{tt} - \frac{\partial}{\partial \nu}(u + k_\Omega u_t) - k_\Gamma \Delta \Gamma (\alpha u_t + u) = 0 \quad &\text{in } \Gamma_1 \\
  u = 0 \quad &\text{in } \Gamma_0 \\
  u(0, x) = u_0, \ u_t(0, x) = u_1 \quad &\text{on } \Omega
\end{cases}$$

Recall that $B = -k_\Gamma \Delta \Gamma$ (see definition 2.4). By [37, Theorem 2.4.1], to show the exponential stability of $\{e^{\mathcal{A}t}\}_{t \geq 0}$ it is enough to show that

$$\int_0^T \|e^{\mathcal{A}t}U\|_H^2 \, dt < \infty$$
Multiply by $u_t$ the first equation of (1):
\[
(u_{tt}, u_t)_\Omega + k_\Omega |\nabla u_t|_\Omega^2 + (\nabla u, \nabla u_t)_\Omega \\
+ (u_{tt}, u_t)_{\Gamma_1} + k_{\Gamma_1} |\nabla u_t|_{\Gamma_1}^2 + k_{\Gamma} (\nabla u, \nabla u_t)_\Omega = 0
\]
(14)
\[
\frac{1}{2} \frac{d}{dt} (|u_t|_\Omega^2 + |\nabla u_t|_\Omega^2 + |u_t|_{\Gamma_1}^2 + k_{\Gamma} |\nabla u_t|_{\Gamma_1}^2) = -k_\Omega |\nabla u_t|_\Omega^2 - k_{\Gamma_1} |\nabla u_t|_{\Gamma_1}^2
\]
Let $E_p(t) = |u_t|_\Omega^2 + |u_t|_{\Gamma_1}^2$, $E_k(t) = |\nabla u_t|_\Omega^2 + k_{\Gamma} |\nabla u_t|_{\Gamma_1}^2$ be the potential and kinetic energy respectively, then define the energy of this system as the summation of the potential and kinetic energies: $E(t) = E_p(t) + E_k(t)$.
Using these definitions, we obtain the following energy equality after integrating in time (14):
\[
E(0) = E(t) + \int_0^t k_\Omega |\nabla u_t|_\Omega^2 + k_{\Gamma_1} |\nabla u_t|_{\Gamma_1}^2 \, ds
\]
(15)
Now multiply by $u$ the first equation of (1) and again integrate in space:
\[
(u_{tt}, u)_\Omega + k_\Omega (\nabla u_t, \nabla u)_\Omega + |\nabla u_t|_\Omega^2 \\
+ (u_{tt}, u)_{\Gamma_1} + k_{\Gamma_1} (\nabla u_t, \nabla u)_{\Gamma_1} + k_{\Gamma} |\nabla u_t|_\Omega^2 = 0
\]
(16)
Integrate in time, and use integration by parts:
\[
\int_0^t - |u_t|_\Omega^2 - |u_t|_{\Gamma_1}^2 + |\nabla u_t|_\Omega^2 + k_{\Gamma} |\nabla u_t|_{\Gamma_1}^2 \, ds \\
= \int_0^t \frac{1}{2} \frac{d}{dt} (k_\Omega |\nabla u_t|_\Omega^2 + k_{\Gamma_1} |\nabla u_t|_{\Gamma_1}^2) \, ds + (u_t, u)_\Omega |_t^0 + (u_t, u)_{\Gamma_1} |_t^0
\]
(17)
In (17), we identify the potential energy on the left hand side and we bound the right hand side using the energy at $t = 0$:
\[
\int_0^t E_p(s) ds \leq \int_0^t |u_t|_\Omega^2 + |u_t|_{\Gamma_1}^2 \, ds + C_0 E(0)
\]
\[
\int_0^t E_k(s) ds \leq 2 \int_0^t |u_t|_\Omega^2 + |u_t|_{\Gamma_1}^2 \, ds + C_0 E(0)
\]
(18)
5.1. Case 1: $\alpha = 0$ or $k_{\Gamma} = 0$. By Poincaré’s inequality that $|u_t|_{\Omega}^2 \leq C_2 |\nabla u_t|_{\Omega}^2$, then by the trace moment inequality (see [8])
\[
|u_t|_{\Gamma_1}^2 \leq C |\nabla u_t|_{\Omega}^2 |u_t|_{\Omega}^2 \\
\leq C |\nabla u_t|_{\Omega}^2
\]
(19)
Also by Poincaré’s inequality, the equation (17) becomes:
\[
\int_0^t E(s) ds \leq C_3 (\int_0^t |\nabla u_t|_{\Omega}^2 ds + E(0))
\]
(20)
where $C_3 = \text{max}\{C_0, C_1, C_2\}$ By the equality (15), we get:
\[
\int_0^t E(s) ds \leq CE(0)
\]
where $C$ is a constant. Therefore, by [37, Theorem 4.4.1], the semigroup $\{e^{At}\}_{t\geq0}$ is exponentially stable, if $\alpha = 0$ or $k_{\Gamma} = 0$. 
5.2. Case 2: $\alpha > 0$. This case is straightforward, indeed: by Poincaré’s inequality we have:

$$|u_t|^2_{\Omega} \leq C_1 |\nabla u_t|^2_{\Omega} \text{ by Poincaré’s inequality.}$$

$$|u_t|^2_{\Gamma_1} \leq C_2 |\nabla u_t|^2_{\Omega} \text{ by (19).}$$

(21)

Therefore, using these two inequalities from above in (16), we obtain:

$$\int_0^t E(s)ds \leq C_3 (\int_0^t |\nabla u_t|^2_{\Omega} + |\nabla u_t|^2_{\Gamma_1} ds + E(0)) \leq CE(0) \text{ by (15).}$$

6. Analyticity of the semigroup - proof of theorem 3.3. Assume that $k_\Omega$, $k_\Gamma$, $\alpha > 0$ and recall that $A$ is exponentially stable (see theorem 3.2).

Let $\beta \in \mathbb{R}$ so that $(i\beta - A)^{-1} \in \mathcal{L}(\mathcal{H})$, then with $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ and pre-image $U = (u_1, u_2, u_3, u_4)^T \in D(A)$, consider the resolvent equation $(i\beta - A)U = F$:

$$(i\beta - A) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = (i\beta - A)^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = (i\beta - A)^{-1} F$$

We want to show $|\beta| \|U\|_\mathcal{H} \leq C \|F\|_\mathcal{H}$.

Since $\{e^{At}\}_{t \geq 0}$ is exponentially stable, then:

$$\|U\|_\mathcal{H} \leq C \|F\|_\mathcal{H}$$

$$\begin{cases}
    i\beta u_1 - u_2 = f_1 \\
    i\beta u_3 - u_4 = f_3 \\
    i\beta u_2 - \Delta u_1 - k_\Omega \Delta u_2 = f_2 \\
    i\beta u_4 + \frac{\partial}{\partial \nu}(u_1 + k_\Omega u_2) + Bu_3 + \alpha Bu_4 = f_4
\end{cases}$$

(22)

Multiply the third equation by $\overline{u_2}$ and make use of the fourth equation:

$$i\beta |u_2|^2_{\Omega} + (\nabla u_1, \nabla u_2)_{\Omega} + k_\Omega |\nabla u_2|^2_{\Omega} + \langle i\beta u_4 + Bu_3 + \alpha Bu_4 - f_4, u_2 \rangle_{\Gamma_1} = (f_2, u_2)_{\Omega}$$

(23)

Using the first equation, observe that:

$$\langle \nabla u_1, \nabla u_2 \rangle_{\Omega} = (\nabla u_1, i\beta \nabla u_1 - \nabla f_1)_{\Omega}$$

$$\langle Bu_3, u_2 \rangle_{\Gamma_1} = \langle B^\frac{1}{2} u_3, B^\frac{1}{2} u_4 \rangle_{\Gamma_1}$$

$$= \langle B^\frac{1}{2} u_3, i\beta B^\frac{1}{2} u_3 - \beta^\frac{1}{2} f_1 \rangle_{\Gamma_1}$$

$$= -i\beta \left| B^\frac{1}{2} u_3 \right|_{\Gamma_1}^2 - \langle B^\frac{1}{2} u_3, B^\frac{1}{2} f_3 \rangle_{\Gamma_1}$$

(24)
Then, combine (23) and (24):

\[
iβ \left[ |u_2|^2_Ω + |u_4|^2_{Γ_r} - |∇u_1|^2_Ω - |B^\frac{1}{2}u_3|^2_{Γ_r} \right] + k_Ω |∇u_2|^2_Ω + α |B^\frac{1}{2}u_4|^2_{Γ_r} = (f_2, u_2)_Ω + (f_4, u_4)_Ω + (∇u_1, ∇f_1)_Ω + \left< B^\frac{1}{2}u_3, B^\frac{1}{2}f_3 \right>_{Γ_r}
\]

(25)

Taking the real part, one gets:

\[
k_Ω |∇u_2|^2_Ω + α |B^\frac{1}{2}u_4|^2_{Γ_r}
\]

\[
\leq |(f_2, u_2)_Ω + (f_4, u_4)_Ω + (∇u_1, ∇f_1)_Ω + \left< B^\frac{1}{2}u_3, B^\frac{1}{2}f_3 \right>_{Γ_r}| ^2
\]

\[
\leq C \|F\|_H \|U\|_H
\]

(26)

\[
\leq C \|F\|^2_H
\]

Using the first two equations in (22) and applying estimate (26):

\[
\begin{cases}
β |∇u_1|_Ω \leq |∇u_2|_Ω + |∇f_1|_Ω \leq \frac{C}{k_Ω} \|F\|_H \\
β |B^\frac{1}{2}u_3|_{Γ_r} \leq |B^\frac{1}{2}u_4|_{Γ_r} + |B^\frac{1}{2}f_3|_{Γ_r} \leq \frac{C}{α} \|F\|_H
\end{cases}
\]

(27)

Let \( K = min\{k_Ω, α\} \), then, with estimate (26), (27) becomes:

\[
β \left[ |∇u_1|_Ω + |B^\frac{1}{2}u_3|_{Γ_r} \right] \leq \frac{C}{K} \|F\|_H
\]

(28)

Note that the presence of \( K \) on the denominator forces both damping coefficients to be non-zero.

Back to (25), multiply by \( β \) and take the imaginary part:

\[
β^2 \left[ |u_2|^2_Ω + |u_4|^2_{Γ_r} \right] \leq β^2 \left[ |∇u_1|^2_Ω + |B^\frac{1}{2}u_3|^2_{Γ_r} \right]
\]

\[
+ |f_2|_Ω |βu_2|_Ω + |∇f_1|_Ω |β∇u_3|_Ω + |B^\frac{1}{2}f_3|_{Γ_r} |βB^\frac{1}{2}u_3|_{Γ_r} + |f_4|_{Γ_r} |βu_4|_{Γ_r}
\]

(29)

Splitting:

\[
\begin{cases}
|f_2|_Ω |βu_2|_Ω \leq C_ε |f_2|_Ω^2 + ε |βu_2|_Ω^2 \\
|f_4|_{Γ_r} |βu_4|_{Γ_r} \leq C_ε |f_4|_{Γ_r}^2 + ε |βu_4|_{Γ_r}^2
\end{cases}
\]

(30)

Using (27) to estimate the first terms on the right-hand side of (29):

\[
β^2 \left[ |u_2|^2_Ω + |u_4|^2_{Γ_r} \right] \leq C_ε \left( \frac{1}{K} + 1 \right) \|F\|^2_H
\]

(31)

Combine (28) with (31)

\[
|β|^2 \left[ |∇u_1|^2_Ω + |u_2|^2_Ω + |B^\frac{1}{2}u_3|^2_{Γ_r} + |u_4|^2_{Γ_r} \right] = |β|^2 \|R(β - A)|^2_Ω \leq C_ε \left( \frac{1}{K} + 1 \right) \|F\|^2_H
\]

It follows that \( \{e^{At}\}_{t≥0} \) is analytic on \( H \), provided \( k_Ω, k_Γ, α > 0 \).
7. Analyticity of $C_0$-semigroup - proof of theorem 3.4. In the following, we first show (theorem 3.4 - first part) that $A_{k\Gamma \geq 0}$ generates an analytic semigroup on $\mathcal{H}$, by using the General Wentzell Boundary Conditions since this approach offers a more complete study of this problem. Indeed, if one adds the additional condition: $k\Gamma = 0$, i.e., in the absence of Laplace-Beltrami term on the boundary, the semigroup generated by $A_{k\Gamma = 0}$ is analytic not only in $\mathcal{H} \subset H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times D(B^{1/2}) \times L^2(\Gamma_1)$ but also in

$$\mathcal{H}_p = \{(u_1, u_2, u_3) \in \mathcal{W}^{1,p}_{\Gamma_0}(\Omega) \times L^p(\Omega) \times L^p(\Gamma_1), u_1|_{\Gamma_1} = u_3\}$$

for $1 \leq p \leq \infty$ (theorem 3.4 - second part).

7.1. Proof of theorem 3.4 - first part. Consider (1) with $k_\Omega, \alpha > 0$:

$$\begin{cases}
    u_{tt} - k_\Omega \Delta u_t - \Delta u = 0 & \text{in } \Omega \\
    u_{tt} - \frac{\partial}{\partial t}(u + k_\Omega u_t) - k_\Gamma \Delta(\alpha u_t - u) = 0 & \text{in } \Gamma_1 \\
    u = 0 & \text{in } \Gamma_0 \\
    u(0, x) = u_0, \ u_t(0, x) = u_1 & \text{on } \Omega
\end{cases}
$$

with the associated Cauchy problem operator $A_{k\Gamma \geq 0}$ given in (8).

The above model contains two possible scenarios:

**Scenario 1:** $k_\Omega > 0, k\Gamma = 0$ which corresponds to the absence of Laplace Beltrami operator. In that case, the result has been known, however our more general proof will also yield the expected conclusion.

**Scenario 2.** $k_\Omega, k\Gamma, \alpha > 0$ corresponds to the presence of Laplace Beltrami operator with additional strong damping on the boundary $\alpha > 0$. It an alternate proof to the previous theorem.

Note that there is a third related scenario:

**Scenario 3:** $k_\Omega > 0, k\Gamma > 0, \alpha = 0$ corresponds to the presence of Laplace Beltrami operator without the additional strong damping on the boundary $\alpha = 0$. In this case, the analyticity does not hold but we expect some smoothing behavior in line with Gevrey class behavior. See figure 6 where the eigenvalues are aligned along the parabola $Im\lambda = a - b|Re\lambda|^2$.

Our proof is based on connecting the problem under consideration to Wentzell semigroups studied in [12, 14]. To this end we introduce the new variable:

$$z = u + k_\Omega u_t \text{ on } \Omega \text{ and } \Gamma_1$$

We recall from the definition 2.4 that $B = -k_\Gamma \Delta$, then the two first equations of (1) are equivalent to:

$$\begin{cases}
    u_t + \frac{u}{k_\Omega} = \frac{z}{k_\Omega} & \text{in } \Omega \\
    \gamma(u_t) + \gamma\left(\frac{u}{k_\Omega}\right) = \gamma\left(\frac{z}{k_\Omega}\right) & \text{in } \Gamma_1 \\
    \frac{\partial}{\partial t}z + \frac{k_\Omega}{k_\Gamma} \frac{\partial}{\partial t} z - \frac{k_\Gamma}{k_\Omega} \Delta z - k_\Gamma (1 - \frac{\alpha}{k_\Omega}) \Delta z = 0 & \text{in } \Gamma_1
\end{cases} \quad (32)$$

Note that the first two equations are simply an ordinary differential equation. By the initial condition of (1), we are looking for

$$U \equiv (u, u|_{\Gamma_1}) \in X_u = \{(u|_{\Omega}, u|_{\Gamma_1}) \in H^1_{\Gamma_0}(\Omega) \times D(B^{1/2}), \gamma(u) = u|_{\Gamma_1}\}$$

with the associated norm:

$$\|U\|_{X_u}^2 = \|\nabla u\|_{\Omega}^2 + \|B^{1/2} u|_{\Gamma_1}\|_{\Gamma_1}^2$$
Thus $X_u$ is a closed subspace of $H^1(\Omega) \times \mathcal{D}(B^{1/2})$.

Consider the abstract ODE

$$V_t + \frac{1}{k_\Omega} V = 0, \quad V(0) \in X_u$$

and denote by $T(t)$ the governing semigroup. This semigroup is obviously analytic on $X_u$ and given by $T(t)V = e^{-\frac{t}{k_\Omega}} V$.

$$\|T(t)U_0\|_{X_u} = \|T(t)u_0|\Omega\|_{H^1(\Omega)} + \|T(t)|r_1\|_{\mathcal{D}(B^{1/2})}$$

$$\leq e^{-\frac{1}{k_\Omega}t} \left[\|u_0|\Omega\|_{H^1(\Omega)} + \|u_0|\Gamma_1\|_{\mathcal{D}(B^{1/2})} \right]$$

$$\leq e^{-\frac{1}{k_\Omega}t} \|u_0\|_{X_u}$$

Since $\|V(t)\|_{X_u} \leq C \|V_0\|_{X_u}$, then $T(t)$ is an analytic $C_0$-semigroup on $X_u$.

Note that the solution to the first two equations of (32) can be written as a perturbation

$$U_t + \frac{1}{k_\Omega} U = \frac{1}{k_\Omega} Z, \quad U(0) = U_0$$

where $Z \equiv (z|\Omega, \gamma(z))$. Thus, by the variation of parameter formula we obtain:

$$U(t) = T(t)U_0 + \int_0^t T(t-s)Z(s)ds$$

The caveat is that the above equation is considered on $X_u$ which means that with $z \in L^2(\Omega)$ one obtains unbounded forcing in ((34)). However, the analyticity of the semigroup will help in handling this part.

Before we move to the other two equations, we need to do some preliminary work on the $z$-dynamics.

**Definition 7.1 ($X_z$).** Identify every $z \in C(\bar{\Omega})$ with $Z = (z|\Omega, z|\Gamma_1)$ and define $X_z$ to be the completion of $C(\bar{\Omega})$ in the norm:

$$\|z\|_X := (\|z\|_\Omega^2 + \|z\|_{\Gamma_1}^2)^{1/2}$$

In general, a member of $X_z$ is $H = (f, g)$, where $f \in L^2(\Omega)$, $g \in L^p(\Gamma_1)$. Note that $f$ may not have a trace on $\Gamma_1$, and even if $f$ does, this trace needs not equal $g$.

Also, define the formal Laplacian operator $A$ with General Wentzell Boundary Conditions (GWBC) by:

$$Au = \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) \quad \text{in } \Omega$$

$$A z + \frac{\partial}{\partial \nu} z + \gamma z - \zeta \beta \Delta_{\Gamma} z = 0 \quad \text{on } \delta \Omega$$

In [12, Theorem 3.1] and then in [14, Theorem 3.2], Favini has shown that the heat equation with GWBC was governed by an analytic semigroup on $X_z$.

**Theorem C** (Theorem 3.2 - [14]). The realization $A_2$ of $A$ in $X_z$ with domain

$$\mathcal{D}(A_2) = \{ Z = (z|\Omega, z|\delta \Omega) \in C^2(\bar{\Omega}), \ z|\delta \Omega \text{ satisfies (36)} \}$$

is symmetric and bounded above on $X_z$.

Also, the closure $G_2$ of $A_2$ is selfadjoint and generates a cosine function and a quasi-contraction (contraction, if $\gamma \geq 0$) semigroup $\{e^{G_2t}\}_{t \geq 0}$ on $X_2$ which is analytic in the right half plane.
Set $\beta = k_\Omega$, $\gamma = 0$, $\zeta = k_\Gamma \alpha$ and rewrite (32) using the operator $A = k_\Omega \Delta$:

$$\begin{align*}
\begin{aligned}
&u_t + \frac{\alpha}{k_\Omega} z = \frac{z}{k_\Omega} & &\text{in } \Omega \\
&\gamma(u_t) + \gamma\left(\frac{z}{k_\Omega}\right) = \gamma\left(\frac{z}{k_\Omega}\right) & &\text{in } \Gamma_1 \\
z_t = Az + \frac{\alpha}{k_\Omega} z - k_\Gamma \alpha \Delta_{\Gamma} z = k_\Gamma (k_\Omega - \alpha) \Delta_{\Gamma} u & &\text{in } \Gamma_1
\end{aligned}
\end{align*}$$

(37)

Observe that the last equation in (37) corresponds to the heat equation (35) with General Wentzell Boundary Conditions (36) perturbed on $\Omega$ by $\frac{z}{k_\Omega} - \frac{u}{k_\Omega}$ and on $\Gamma_1$ by $-k_\Gamma (k_\Omega - \alpha) \Delta_{\Gamma} u|_{\Gamma_1}$.

**Remark 4.** Note that the fourth equation can be rewritten as an evolution equation in $\gamma(z)$:

$$z_t = -k_\Omega \frac{\partial}{\partial \nu} z + k_\Gamma \alpha \Delta_{\Gamma} z + \frac{z}{k_\Omega} - \frac{u}{k_\Omega} + k_\Gamma (k_\Omega - \alpha) \Delta_{\Gamma} u$$

(38)

This way of writing makes a connection with evolution equations governed by dynamic boundary conditions.

To show the analyticity of (37) on $\mathcal{H}_{u,z} = X_u \times X_z$, we will proceed in two steps: first, we show that the system without perturbation on $\Gamma_1$ is analytic (Lemma 7.2). Then, we will show that the analyticity with the perturbation on $\Gamma_1$ is preserved (Lemma 7.3)

The system given in (37) can be written in a compact form as

$$\begin{align*}
\begin{aligned}
&U_t + \frac{1}{k_\Omega} U = \frac{1}{k_\Omega} Z & &\text{in } \Omega \\
&Z_t + G_2 Z = \frac{1}{k_\Omega} (Z - U) + P(\gamma(u)) & &\text{in } \Gamma_1
\end{aligned}
\end{align*}$$

where $P(\gamma(u)) \equiv [0, k_\Gamma (k_\Omega - \alpha) \Delta_{\Gamma} \gamma(u)]^T$. We begin the analysis with unperturbed system. We shall show that the associated semigroup inherits analyticity properties from Wentzell semigroup.

**Lemma 7.2.** Given the system:

$$\begin{align*}
\begin{aligned}
&u_t + \frac{\alpha}{k_\Omega} z = \frac{z}{k_\Omega} & &\text{in } \Omega \\
&\gamma(u_t) + \gamma\left(\frac{z}{k_\Omega}\right) = \gamma\left(\frac{z}{k_\Omega}\right) & &\text{in } \Gamma_1 \\
z_t = Az + \frac{\alpha}{k_\Omega} z - k_\Gamma \alpha \Delta_{\Gamma} z = 0 & &\text{in } \Gamma_1
\end{aligned}
\end{align*}$$

(39)

where $(u(\Omega, u|_{\Gamma_1}, z|_{\Omega}, z|_{\Gamma_1})^T \in H^1_{\Gamma_1}(\Omega) \times \mathcal{D}(B^{1/2}) \times \mathcal{D}(G_2^{1/2})$. Then the following inequality holds,

$$\|U_t(t)\|_{X_u} + \|Z_t(t)\|_{X_z} \leq C_0 \|U_0\|_{X_u} + \frac{C_1}{t} \|Z_0\|_{X_z}$$

(40)

It follows that the semigroup associated generates an analytic $C_0$-semigroup on $\mathcal{H}_{u,z}$.

Note that lemma 7.2 corresponds to the case $\alpha = k_\Omega$.

**Proof.** First, recall the action of $G_2$ the closure of $A_2$ the realizaion of $A$ in $X_z$, for some $Z = (z|_{\Omega}, z|_{\Gamma_1}) \in \mathcal{D}(G_2^{1/2})$:

$$\begin{align*}
\langle G_2 Z, Z \rangle_{X_z} &= k_\Omega \langle \Delta z, z \rangle_{\Omega} - k_\Omega \left\langle \frac{\partial}{\partial \nu} z, z \right\rangle_{\Gamma_1} + k_\Gamma \langle \Delta_{\Gamma} z, z \rangle_{\Gamma_1} \\
&= -k_\Omega |\nabla z|_{\Omega}^2 - k_\Gamma \alpha |\nabla_{\Gamma} z|_{\Gamma_1}^2 \leq 0
\end{align*}$$

where $\nabla_{\Gamma} z|_{\Gamma_1}^2$ is the gradient of $z$ on $\Gamma_1$.
Note that \( G_2 \) is dissipative, and now observe that by Favini’s result quoted in theorem C about analyticity:
\[
\|G_2^\theta Z(t)\|_{X_z} \leq \frac{1}{t^\theta} \|Z_0\|_{X_z} \quad \text{for } \theta \in [0,1]
\]

Therefore we have the following inequality for all \( t > 0 \) and taking into account that \( Z(t) \in \mathcal{D}(G_2^{\frac{3}{2}}) \):
\[
\|Z(t)\|_{X_z}^2 = |\nabla z(t)|_{\Omega}^2 + k_\Gamma |\nabla z(t)|_{\Gamma_1}^2 \leq C \|G_2^{\frac{3}{2}} Z(t)\|_{X_z}^2 \leq \frac{C}{t} \|Z_0\|_{X_z}^2 \quad \text{(41)}
\]

By the variation of parameters, the first two equations of (39) have a solution in \( X_u \):
\[
\|U(t)\|_{X_u} \leq e^{-\frac{\alpha}{t} t} \|U_0\|_{X_u} + \int_0^t e^{-\frac{\alpha}{t} (t-s)} \|Z(s)\|_{X_u} \, ds \\
\leq C \|U_0\|_{X_u} + C \int_0^t e^{-\frac{\alpha}{t} (t-s)} \|G_2^{\frac{3}{2}} Z(s)\|_{X_z} \, ds \\
\leq C \|U_0\|_{X_u} + C \int_0^t e^{-\frac{\alpha}{t} (t-s)} \frac{1}{\sqrt{s}} \|Z_0\|_{X_z} \, ds \\
\leq C \left( \|U_0\|_{X_u} + \|Z_0\|_{X_z} \right) \text{ since } e^{-\frac{\alpha}{t} (t-s)} \frac{1}{\sqrt{s}} \in L^1(0,T) \quad \text{(42)}
\]

Finally, the desired estimate is obtained by combining (41) and (42):
\[
\|U(t)\|_{X_u} + \|Z(t)\|_{X_z} \leq \frac{1}{k_\Omega} \left[ \|Z(t) - U(t)\|_{X_u} + \|Z(t) - U(t)\|_{X_z} \right] + \|G_2 Z(t)\|_{X_z} \\
\leq C \left[ \|U(t)\|_{X_u} + \|Z(t)\|_{X_z} \right] + \|G_2 Z(t)\|_{X_z} \\
\leq C \left( \|U_0\|_{X_u} + (1 + \frac{1}{\sqrt{t}}) \|Z_0\|_{X_z} \right) + \frac{1}{t} \|Z_0\|_{X_z}
\]

which completes the proof of equation (40), implying analyticity of the corresponding semigroup. \( \square \)

Let \( S \) be the generator of the analytic semigroup governing model (39) on the space \( \mathcal{H}_{u,z} = X_u \times X_z \) with the domain:
\[
\mathcal{D}(S) = \{ V = (u|\Omega, u|\Gamma_1, z|\Omega, z|\Gamma_1) \} \subseteq H_{1_0}^1(\Omega) \times \mathcal{D}(B^{\frac{1}{2}}) \times \mathcal{D}(G_2^{\frac{3}{2}})
\]

Then the original system (37) with the fourth equation replaced by (38) is equivalent to:
\[
V_t = (S + P) V \quad \text{(43)}
\]

where
\[
S = \begin{pmatrix} -J & J \\ -J & K \end{pmatrix} \quad \text{with } J = \frac{1}{k_\Omega} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} G_2 + \frac{L}{2\nu} & 0 \\ -k_\Omega \frac{L}{2\nu} & k_\Gamma \alpha \Delta_\Gamma + \frac{L}{k_\Omega} \end{pmatrix}
\]
\[
P = \begin{pmatrix} 0_{2x2} & 0_{2x2} \\ 0_{2x2} & L \end{pmatrix} \quad \text{with } L = \begin{pmatrix} 0 & 0 \\ 0 & k_\Gamma (k_\Omega - \alpha) \Delta_\Gamma \end{pmatrix}
\]

Lemma 7.3. The semigroup generated by \( S + P \) is analytic on \( \mathcal{H}_{u,z} \).
Proof. Let \( V = (U, Z) = (u_{t_0}, u_{t_1}, z_{t_0}, z_{t_1}) \in H^{1}_{t_0}(\Omega) \times D(B^{1/2}) \times X_z, \) with \( V_0 = (u_{t_0}, u_{t_1}, z_{t_0}, z_{t_1}) \) be solution of (43), then
\[
V(t) = e^{St}V_0 + \int_0^t e^{S(t-\tau)} PV(\tau) d\tau
\]
Taking the Laplace Transform
\[
V(\lambda) = R(\lambda, S)V_0 + R(\lambda, S)PV(\lambda)
\]
leads to the following relation:
\[
[I - R(\lambda, S)P] V(\lambda) = R(\lambda, S)V_0
\]
Thus, \( S + P \) generates an analytic \( C_0 \)-semigroup if and only if \( [I - R(\lambda, S)P] \) is invertible. This last statement follows from
\[
V(\lambda) = [I - R(\lambda, S)P]^{-1} R(\lambda, S)V(0)
\]
and the estimate
\[
|R(\lambda, S)|_{\mathcal{L}(\mathcal{H}_{u,z})} \leq \frac{C|\lambda|^{-1}}{1 - \frac{1}{\lambda^2}}
\]
Thus it is enough to check \( \|R(\lambda, S)PV\|_{\mathcal{H}_{u,z}} \leq \frac{1}{2} \):}

\[
\|R(\lambda, S)PV\|_{\mathcal{H}_{u,z}} = \|R(\lambda, S)S^{\frac{1}{2}}S^{-\frac{1}{2}} PV\|_{\mathcal{H}_{u,z}}
\]
\[
\leq \|S^{\frac{1}{2}} R(\lambda, S)\|_{\mathcal{L}(\mathcal{H}_{u,z})} \|S^{-\frac{1}{2}} PV\|_{\mathcal{H}_{u,z}}
\]
\[
\leq \frac{1}{\sqrt{\lambda}} \|S^{-\frac{1}{2}} PV\|_{\mathcal{H}_{u,z}}
\]
since \( S \) generates an analytic semigroup on \( \mathcal{H}_{u,z} \). It remains to bound the second term. For all \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in D(S^{\frac{1}{2}}) \subset H^{1}_{t_0}(\Omega) \times D(B^{\frac{1}{2}}) \times D(G^{\frac{1}{2}}), \) define \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) = [S^{-\frac{1}{2}}]^* \phi \in D(S^{\frac{1}{2}}) \subset H^{1}_{t_0}(\Omega) \times D(B^{1/2}) \times D(G^{\frac{1}{2}}), \) thus \( \|S^{\frac{1}{2}} \varphi\|_{\mathcal{H}_{u,z}} \leq C \|\phi\|_{\mathcal{H}_{u,z}}. \) Then, we have:
\[
\left( S^{-\frac{1}{2}} PV, \varphi \right)_{\mathcal{H}_{u,z}} = (PV, \varphi_4)_{\mathcal{H}_{u,z}}
\]
\[
\leq (k_{\Gamma} (k_{\Omega} - \alpha) \nabla_{\Gamma} u|_{\Gamma_1}, \nabla_{\Gamma} \varphi_4)_{L^2(\Gamma)}
\leq C \|u|_{\Gamma_1}\|H^1(\Gamma_1)\| \|\varphi_4\|_{H^1(\Gamma_1)}
\leq C \|U\|_{\mathcal{H}_{u,z}} \|\phi\|_{\mathcal{H}_{u,z}}
\]
where we have assumed that \( D(B^{1/2}) \sim H^1(\Gamma_1) \) (since otherwise \( k_{\Gamma} = 0 \), by choosing \( \lambda > R^2 \) with \( R \) sufficiently large:
\[
\|R(\lambda, S)PV\|_{\mathcal{H}_{u,z}} \leq \frac{C}{R} \|V\|_{\mathcal{H}_{u,z}} \Rightarrow \|I - R(\lambda, S)P\|_{\mathcal{L}(\mathcal{H}_{u,z})} \geq \frac{1}{2}
\]
\( \square \)

Then \( \mathcal{A} \) generates an analytic semigroup \( \{e^{\mathcal{A} t_{\geq 0}}\}_{t \geq 0} \) on \( \mathcal{H} \), for \( k_{\Omega}, \alpha > 0. \)
7.2. Proof of theorem 3.4 - second part. Consider (1) with \( k_Ω > 0, \ k_Γ = 0 \):

\[
\begin{cases}
  u_{tt} - k_Ω \Delta u_t - \Delta u = 0 & \text{in } Ω \\
  u_{tt} - \frac{∂}{∂ν}(u + k_Ω u_t) - k_Γ \Delta_Γ(αu_t + u) = 0 & \text{in } Γ_1 \\
  u = 0 & \text{in } Γ_0 \\
  u(0, x) = u_0, \ u_t(0, x) = u_1 & \text{on } Ω
\end{cases}
\]

To prove that \( A_{k_Γ=0} \) generates an analytic semigroup on \( \mathcal{H}_p \), we proceed as in the previous proof. It is enough to consider the first two equations of (32), which are ordinary differential equations can be defined in \( X_u = \{ u ∈ W^{1,p}_Γ(Ω), γ(u) = u|Γ_1 \} \), for \( 1 \leq p ≤ ∞ \). Then trace theory provides the following estimate for \( u|Γ_1 \):

\[ ||u||_{X_u} \leq C ||u||_{W^{1,p}(Ω)} \], \( ∀u ∈ W^{1,p}_Γ(Ω) \)

For the \( z \)-dynamics, we first generalize the space \( X_2 \) by \( X_p \):

**Definition 7.4 \( (X_p) \).** Identify every \( z ∈ C(Ω) \) with \( z = (z|Ω, z|Γ_1) \) and redefine \( X_p \) to be the completion of \( C(Ω) \) in the norm:

\[ ||z||_p := \left( \int_Ω |u|_Ω^p \, dx + \int_{Γ_1} |u|_{Γ_1}^p \, dS \right)^{\frac{1}{p}} \]

for \( 1 ≤ p ≤ ∞ \).

Then, an analogy of theorem C was established, which we quote:

**Theorem D** (Theorem 3.3 - [14]). In the absence of Laplace-Beltrami term on the boundary, the closure \( G_p \) of the realization \( A_p \) of \( A \) in \( X_p \) with domain

\[ \mathcal{D}(A_p) = \{ z = (z|Ω, z|Γ_1) ∈ \mathcal{D}(A_2) ∩ X_p \mid z|_{δΩ} \text{ satisfies (36)} \} \]

is analytic on \( X_p \) for \( p ∈ [1, ∞) \).

By the above theorem, the domain \( \mathcal{D}(G_p^\frac{1}{2}) \) is contained in \( W^{1,p}_Γ(Ω) × W^{1−\frac{1}{p}}(Γ_1) \) and we have the following relationship, using interpolation theorem:

\[ ||Z(t)||_{X_p}^p = ||z(t)||_{W^{1,p}(Ω)}^p ≤ ||G_p^\frac{1}{2} Z(t)||_{X_p}^p ≤ \left( \frac{1}{\sqrt{t}} ||Z_0||_{X_p} \right)^p \]

Following the proof of lemma 7.2, we use the variation of parameters and proceed as in (42):

\[ ||U(t)||_{X_u} ≤ e^{-\frac{1}{k_Ω}t} ||U_0||_{X_u} + \int_0^t e^{-\frac{1}{k_Ω}(t-s)} ||Z(s)||_{X_u} \, ds \]

\[ ≤ C ||U_0||_{X_u} + C \int_0^t e^{-\frac{1}{k_Ω}(t-s)} ||G_p^\frac{1}{2} Z(s)||_{X_p} \, ds \]

\[ ≤ C \left[ ||U_0||_{X_u} + ||Z_0||_{X_p} \right] \]

We deduce:

\[ ||U(t)||_{X_u} + ||Z(t)||_{X_p} ≤ \frac{1}{k_Ω} \left[ ||Z(t) - U(t)||_{X_u} + ||Z(t) - U(t)||_{X_p} \right] + ||G_p Z(t)||_{X_p} \]

\[ ≤ C \left[ ||U_0||_{X_u} + (1 + \frac{1}{\sqrt{t}}) ||Z_0||_{X_p} \right] + \frac{1}{T} ||Z_0||_{X_p} \]

It follows that the semigroup generated by \( A_{k_Γ=0} \) is analytic on \( \mathcal{H}_p \).
8. Spectral property - proof of the theorem 3.5. Suppose that the condition (2) holds, i.e. $\max\{k_1, k_2\} > 0$. In order to show that $\sigma(A) \cap i\mathbb{R} = \emptyset$, we shall follow the general line of arguments presented in [3]. To wit, we begin by computing the adjoint of $A$:

**Lemma 8.1 (Adjoint of $A$).** With $A$ as defined in (8) and using the notation $-\Delta u = A(I - N \frac{\partial}{\partial v})u$, the adjoint $A^*$ is given to be

$$
A^* = \begin{pmatrix}
0 & -I & 0 & 0 \\
A(I - N \frac{\partial}{\partial v}) & -k_1 A(I - N \frac{\partial}{\partial v}) & 0 & 0 \\
0 & 0 & -I & 0 \\
\frac{\partial}{\partial v} & -k_1 \frac{\partial}{\partial v} & B & -\alpha B
\end{pmatrix}
$$

(44)

$$
\mathcal{D}(A^*) = \{[u_1, u_2, u_3, u_4]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}),
$$

such that $A(I - N \frac{\partial}{\partial v})(u_1 - k_1 u_2) \in L^2(\Omega)$,

$$
\frac{\partial}{\partial v}(u_1 - k_1 u_2) + B^{\frac{1}{2}}(B^{\frac{1}{2}} u_3 - \alpha B^{\frac{1}{2}} u_4) \in L^2(\Gamma_1),
$$

$$
u_1|_{\Gamma_1} = N^* A u_1 = u_3 \text{ and } u_2|_{\Gamma_1} = N^* A u_2 = u_4\}
$$

**Proof.** Let

$$
S = \{[u_1, u_2, u_3, u_4]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}),
$$

such that $A(I - N \frac{\partial}{\partial v})(u_1 - k_1 u_2) \in L^2(\Omega)$,

$$
\frac{\partial}{\partial v}(u_1 - k_1 u_2) + B^{\frac{1}{2}}(B^{\frac{1}{2}} u_3 - \alpha B^{\frac{1}{2}} u_4) \in L^2(\Gamma_1),
$$

$$
u_1|_{\Gamma_1} = N^* A u_1 = u_3 \text{ and } u_2|_{\Gamma_1} = N^* A u_2 = u_4\}
$$

Then $\forall u \in \mathcal{D}(A)$, $y = [y_1, y_2, y_3, y_4]^T \in S$:

$$
(A u, y)_{H^{-\frac{1}{2}} \times L^2(\Omega) \times L^2(\Gamma_1)} =
$$

$$
\left( A^{\frac{1}{2}} u_2, A^{\frac{1}{2}} y_1 \right)_{L^2(\Omega)} - \left( A(u_1 - N \frac{\partial}{\partial v} u_1), y_2 \right)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})}
$$

$$
- k_1 \left( A(u_2 - N \frac{\partial}{\partial v} u_2), y_2 \right)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})} + \alpha \left( B^{\frac{1}{2}} u_4, B^{\frac{1}{2}} y_3 \right)_{L^2(\Gamma_1)}
$$

$$
- \left( \frac{\partial}{\partial v} u_1, y_4 \right)_{L^2(\Gamma_1)} - k_1 \left( \frac{\partial}{\partial v} u_2, y_4 \right)_{L^2(\Gamma_1)}
$$

$$
- (B u_3, y_4)_{[\mathcal{D}(B^{\frac{1}{2}})]^* \times \mathcal{D}(B^{\frac{1}{2}})} - \alpha (B u_4, y_4)_{[\mathcal{D}(B^{\frac{1}{2}})]^* \times \mathcal{D}(B^{\frac{1}{2}})}
$$
where Λ =

\[
\begin{pmatrix}
0 & -I & 0 & 0 \\
-\frac{\partial}{\partial \nu} & -k_0A(I - N\frac{\partial}{\partial \nu}) & 0 & 0 \\
0 & 0 & 0 & -I \\
\frac{\partial}{\partial \nu} & -k_0\frac{\partial}{\partial \nu} & B & -B
\end{pmatrix}
\]

Therefore, \( S \subseteq \mathcal{D}(A^*) \) and \( A^*|_S = \Lambda \).

Suppose that \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > 0 \), to show the opposite containment, it is enough to verify that \( \forall F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H} \), there exists \( Y = (y_1, y_2, y_3, y_4)^T \in \mathcal{D}(A^*) \) such that

\[
\lambda Y - A^*Y = F \in \mathcal{H}
\]

\[
\begin{aligned}
y_1 + y_2 &= f_1 \\
y_2 - A(I - N\frac{\partial}{\partial \nu})y_1 - k_0A(I - N\frac{\partial}{\partial \nu})y_2 &= f_2 \\
y_3 + y_4 &= f_3 \\
y_4 - \frac{\partial}{\partial \nu}y_1 - k_0\frac{\partial}{\partial \nu}y_2 + By_3 - \alpha BY_4 &= f_4 \\
y_2 &= f_1 - \lambda y_1 \\
y_4 &= f_3 - \lambda y_3 \\
y_3 &= f_2 - \lambda y_1 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})y_1 \\
y_3 &= f_2 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})f_1 \\
y_3 &= f_2 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})f_1 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})y_3 \\
y_3 &= f_2 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})f_1 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})y_3 - \lambda k_0A(I - N\frac{\partial}{\partial \nu})y_3
\end{aligned}
\]

(45)

To solve the stationary problem (45), we shall use a weak formulation and Lax-Milgram theorem. To this end, introduce \((y_1, y_2, y_3, y_4) \in \mathcal{D}(A^*)\) and \((y_1, y_2, y_3, y_4) \in \mathcal{H}\). Later we shall extend the argument by density to all \( F \in \mathcal{H} \). We consider the two last equations of (45), multiply them by \((v_1, v_3)\) and integrate in space over \( \Omega \) and \( \Gamma_1 \), respectively:
where $y \in Y$ such that $Y = \mathcal{D}\left(\mathcal{A}^*\right)$ such that $\lambda Y - \mathcal{A}^* Y = F \in \mathcal{H}$. Thus $Y$ is our desired solution and we conclude that $Y \in S$. Therefore, $\mathcal{D}(\mathcal{A}^*) = S$ and $\mathcal{A}^* = \Gamma$. \hfill \square
Now consider the spectrum of $A$: $\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A)$ where $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ denotes respectively the point spectrum, the residual spectrum and the continuous spectrum of $A$; and show that it does not contain the imaginary axis: $i\mathbb{R}$.

**Step 1.** $\sigma_c(A) \cap i\mathbb{R} = \emptyset$

Let’s first cite a theorem [15, Problem 2.54 p. 128]: If $\lambda \in \sigma_c(A)$, then $A - \lambda$ does not have a closed range.

With $A$ as given in (8), assume that $\lambda = ir \in \sigma_c(A)$ with $r \neq 0$.

$\forall f = [f_1, f_2, f_3, f_4]^T \in H$, suppose that $u = [u_1, u_2, u_3, u_4] \in D(A)$ such that:

$$ (ir - A) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} $$

which is equivalent to:

$$ \begin{cases} 
ir u_1 - u_2 = f_1 \\
ir u_3 - u_4 = f_3 \\
ir u_2 + A(u_1 - N \frac{\partial}{\partial \nu} u_1) + k_\Omega A(u_2 - N \frac{\partial}{\partial \nu} u_2) = f_2 \\
ir u_4 + \frac{\partial}{\partial \nu} u_1 + k_\Omega \frac{\partial}{\partial \nu} u_2 + Bu_3 + \alpha Bu_4 = f_4 
\end{cases} \quad (47) $$

The two last equations can be rewritten, using the first two and $u_3 = N^*Au_1$ as follows:

$$ \begin{cases} 
-r^2 u_1 + (1 + k_\Omega ir)Au_1 - AN(\frac{\partial}{\partial \nu} u_1 + k_\Omega \frac{\partial}{\partial \nu} u_2) = ir f_1 + k_\Omega Af_1 + f_2 \\
\frac{\partial}{\partial \nu} u_1 + k_\Omega \frac{\partial}{\partial \nu} u_2 = f_4 + r^2 N^*Au_1 + ir f_3 - BN^*Au_1 - AIRBN^*Au_1 + \alpha BN^*Af_1 
\end{cases} \quad (47) $$

Substitute the second equation into the first one:

$$ -r^2 u_1 + (1 + k_\Omega ir)Au_1 - AN(r^2 N^*Au_1 - BN^*Au_1 - AIRBN^*Au_1) = ir f_1 + k_\Omega Af_1 + f_2 + AN(f_4 + ir f_3 + \alpha Bf_3) \quad (48) $$

Let $V = D(A^{\frac{3}{2}}) \cap D(B^{\frac{3}{2}})$ with associated norm:

$$ \|u\|_V = \left|A^{\frac{3}{2}} u\right|^2_\Omega + \left|B^{\frac{3}{2}} u\right|^2_{\Gamma_1} \quad (49) $$

Consider the left hand side of (48) and define the operators $T, M, K : V \rightarrow V^*$, by

$$ \begin{cases} 
T = M + K \\
M = A + ANBN^*A + ir(k_\Omega A + \alpha ANBN^*A) \\
K = -r^2(I + ANN^*A) 
\end{cases} $$

Also, using the right hand side of (48), define $F : \mathcal{H} \subset D(A^{\frac{3}{2}}) \times L^2(\Omega) \times D(B^{\frac{3}{2}}) \times L^2(\Gamma_1) \rightarrow V^*$ by:

$$ F = ((ir + k_\Omega A) I AN(ir + \alpha B) AN) $$

Observe that $F$ is well-defined: let $v \in V^*$. First, the term $\alpha ANBf_3$:

$$ \langle \alpha ANBf_3, v \rangle_{[D(A^{\frac{3}{2}})]^* \times D(A^{\frac{3}{2}})} = \alpha \left(B^{\frac{3}{2}} f_3, N^*AB^{\frac{3}{2}} v\right)_{L^2(\Gamma_1)} \leq \alpha \left|B^{\frac{3}{2}} f_3\right|^2_{\Gamma_1} \left|B^{\frac{3}{2}} v\right|^2_{\Gamma_1} $$

Since $f_3 \in D(B^{\frac{3}{2}})$, then $\alpha ANBf_3 \in [D(A^{\frac{3}{2}})]^*$. 
Therefore, we obtain:

$$|F(V)| \leq C \|F\|_H \|v\|_V.$$

It remains to show that $T$ is invertible, we will proceed in three steps.

**Step 1-a.** Compactness of $K$

By definition of $A$, $N$ and $N^*A$, $K$ is compact from $D(A^{\frac{1}{2}})$ into its dual.

**Step 1-b.** $M$ is boundedly invertible.

Given $M$, $u \in V$ and $v \in V^*$, define its bilinear form $M(,)$, by:

$$M(u,v) = (1 + k_r i r) \left( A^{\frac{1}{2}} u, A^{\frac{1}{2}} v \right)_\Omega + (1 + \alpha r) \left( B^{\frac{1}{2}} N^* A u, B^{\frac{1}{2}} N^* A v \right)_{\Gamma_1}$$

Then $M(,)$ is a coercive and bounded bilinear form, setting $K_M = \max\{1, k_r r + \alpha r\}$:

$$|M(u,v)| \leq K_M \left[ \|A^{\frac{1}{2}} u\|^2_\Omega + \|A^{\frac{1}{2}} v\|^2_\Omega + \|B^{\frac{1}{2}} N^* A u\|^2_{\Gamma_1} + \|B^{\frac{1}{2}} N^* A v\|^2_{\Gamma_1} \right]$$

$$\leq C \|u\|_V \|v\|_V \quad \text{by (49)}$$

$$\Re|M(u,v)| = \|A^{\frac{1}{2}} u\|^2_\Omega + \|B^{\frac{1}{2}} N^* A u\|^2_{\Gamma_1} + k_r r \|A^{\frac{1}{2}} u\|^2_\Omega + \alpha r \|B^{\frac{1}{2}} N^* A u\|^2_{\Gamma_1}$$

$$\geq C\left(\|A^{\frac{1}{2}} u\|^2_\Omega + \|B^{\frac{1}{2}} N^* A u\|^2_{\Gamma_1}\right) = C \|u\|_V$$

Therefore by Lax-Milgram, the operator $M$ is boundedly invertible. By the Fredholm’s alternative, we deduce the desired invertibility of $T$ provided that $T$ is injective.

**Step 1-c.** $T$ is injective.

Suppose that $Tu_1 = 0$, assume that $r \neq 0$ and take the duality pairing with respect to $v = u_1 \in V^*$, then:

$$((1 + k_r i r)Au_1, v)_{D(A^{\frac{1}{2}})^* \times D(A^{\frac{1}{2}})} - (r^2(1 + AN^* Au_1), v)_{D(A^{\frac{1}{2}})^* \times D(A^{\frac{1}{2}})}$$

$$+ ((1 + \alpha r)ANBN^* Au_1, v)_{D(A^{\frac{1}{2}})^* \times D(A^{\frac{1}{2}})}$$

$$= (1 - r^2) \left( A^{\frac{1}{2}} u_1 \right)_\Omega - r^2 \left( N^* Au_1 \right)_{\Gamma_1} + \left( B^{\frac{1}{2}} N^* Au_1 \right)_{\Gamma_1}^2$$

$$+ i r \left( k_r \left( A^{\frac{1}{2}} u_1 \right)_\Omega + \alpha \left( B^{\frac{1}{2}} N^* Au_1 \right)_{\Gamma_1}^2 \right) = 0$$

Taking the imaginary part in (50) implies $u_1 = 0$. Thus $T$ is injective. Note that the injectivity does not necessarily hold if we do not impose the damping condition.

Thus, $T$ is invertible which achieves the proof of the first step: $\sigma_c(A) \cap i\mathbb{R} = 0$.

**Step 2.** $\sigma_p(A) \cap i\mathbb{R} = 0$

With $A$ as given in (8), if for $r \in \mathbb{R}$ and $r \neq 0$, there exists $u = [u_1, u_2, u_3, u_4]^T \in D(A)$ such that:

$$A \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = i r \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$
which is equivalent to:

\[
\begin{align*}
& \begin{cases} 
    u_2 = i r u_1 \\
    u_4 = i r u_3 \\
    i r u_2 + A (u_1 - N \frac{\partial}{\partial \nu} u_1) + k_\Omega A (u_2 - N \frac{\partial}{\partial \nu} u_2) = 0 \\
    i r u_4 + \frac{\partial}{\partial \nu} u_1 + k_\Omega \frac{\partial}{\partial \nu} u_2 + B u_3 + q B u_4 = 0
  \end{cases} \\
& \begin{cases} 
    \frac{\partial}{\partial \nu} u_1 + r B u_3 - \alpha B u_4 = i r u_4
  \end{cases}
\end{align*}
\]  

(51)

Observe that (51) is equivalent to \( T u_1 = 0 \). Then, in step 1, we have already shown that \( u_1 = 0 \) which achieves the proof for step 2.

**Step 3.** \( \sigma_r(\mathcal{A}) \cap i \mathbb{R} = \emptyset \)

Again, let’s cite a theorem [15, p. 127]: *If the eigenvalue \( \lambda \in \mathbb{C} \) is in the residual spectrum of \( \mathcal{A} \), then \( \lambda \in \sigma_p(\mathcal{A}^*) \)*

With \( \mathcal{A}^* \) as given in (44), if for \( r \in \mathbb{R} \) and \( r \neq 0 \), there exists \( u = [u_1, u_2, u_3, u_4]^T \in D(\mathcal{A}^*) \) such that:

\[
\mathcal{A}^* \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = i r \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}
\]

which is equivalent to:

\[
\begin{align*}
& \begin{cases} 
    - u_2 = i r u_1 \\
    - u_4 = i r u_3 \\
    A u_1 - A N \frac{\partial}{\partial \nu} u_1 - k_\Omega A u_2 + k_\Omega A N \frac{\partial}{\partial \nu} u_2 = i r u_2 \\
    \frac{\partial}{\partial \nu} u_1 - k_\Omega \frac{\partial}{\partial \nu} u_2 + B u_3 - \alpha B u_4 = i r u_4
  \end{cases}
\end{align*}
\]  

(52)

Proceeding as in (47) and (48), then (52) can also be written as \( T u_1 = 0 \), which again implies that \( u_1 = 0 \), by step 1. Therefore, the residual does not intersect the imaginary axis.

**Remark 5.** Since \( i \mathbb{R} \notin \sigma(\mathcal{A}) \), the semigroup \( \{e^{\mathcal{A} t}\}_{t \geq 0} \) generated in theorem 3.1 is strongly stable by the stability theorem from Arendt-Batty theorem [2, Theorem 2.4] cited in theorem B. With the additional assumption that \( k_\Omega, \alpha > 0 \), the semigroup \( \{e^{\mathcal{A} t + \alpha t}\}_{t \geq 0} \) is analytic by theorem 3.4 in \( \mathcal{H} \), therefore it is exponentially stable by the stability theorem A cite from [37, Theorem 4.4.3].

9. **Numerical results & study of the different cases.** In parallel, this model was studied numerically. It was developed using finite elements in order to get a better intuition of the overall dynamics by observing the solution’s behavior and to analyze its spectrum, thus some stability and regularity properties described before. Let \( \Omega = (0, 10) \times (0, 10) \) be filled with fluid which is at rest except for acoustic wave motion. Assume that the sides are not rigid but subject to small oscillations, which defines the dynamic boundary \( \Gamma_1 \) (like a container of the rectangular shape sealed on the four sides by a resilient material). Assume that a small rectangle inside the domain \( \Omega \) is clamped: \( \Gamma_0 = (3, 4) \times (3, 4) \), then create a mesh of this domain. To illustrate this set-up, we present the two pictures (Figure 1-2) which correspond to the approximate solutions of model (1)

- with internal damping only and boundary damping \( (k_\Omega, k_\Gamma > 0, \alpha = 0) \) - **Figure 1**
- with internal and boundary damping \( (k_\Omega, k_\Gamma, \alpha > 0) \) - **Figure 2**

at the same time \( t > t_0 \) given the same initial conditions \( u_0 \) and \( u_1 \). We first observe the presence of the different waves of the system. The main wave moving in the
interior is feeding waves on each edge of the exterior boundary. We also note the Dirichlet boundary condition where the boundary $\Gamma_0$ is represented by two squares on the left of the highest point. The presence of boundary damping add some smoothness to the solution, not only on the boundary but also in the interior.

First, setting $u(x, t) = u e^{\lambda t}$, we obtain the weak formulation of the characteristic equation associated with (1):

$$
\begin{aligned}
\lambda^2 (u, \chi)_\Omega + k_\Omega \lambda (\nabla u, \nabla \chi)_\Omega + (\nabla u, \nabla \chi)_\Omega \\
+ \lambda^2 \langle u, \chi \rangle_{\Gamma_1} + k_{\Gamma_1} \lambda (\nabla_{\Gamma_1} u, \nabla_{\Gamma_1} \chi)_{\Gamma_1} + k_{\Gamma_1} \langle \nabla_{\Gamma_1} u, \nabla_{\Gamma_1} \chi \rangle_{\Gamma_1} = 0
\end{aligned}
$$

(53)
After an appropriate meshing, with $N$ nodes, of the domain $\Omega$ into rectangles, define \{$S_h$\}$_h$ as the classical family of finite-dimensional subspace of $H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1)$, where $h$ denotes the maximal size of the rectangle’s sides. A typical element of $S_h$ is a piecewise linear function which can be uniquely expressed as a linear combination of pyramid functions \{$\Phi_j$\}$_{j=1}^N \subset S_h$. We refer the reader to Thomée [41], or any classical textbook describing the Galerkin method for more details. After the space discretization, with $u_j^e$ the value of $u$ at the node $j$ we set:

$$u(x,y) \approx u_h(x,y) = \sum_{j=1}^N u_j^e \Phi_j, \text{ where } u^e = \{u_1^e, u_2^e, \ldots, u_N^e\}^T$$

By choosing $\chi = u$, the discretized weak form (53) leads to $N$ algebraic equations, which once gathered can be rewritten in the following matrix form:

$$\begin{bmatrix} \lambda^2 M + k_\Omega \lambda K + K + \lambda^2 M_T + k_T \alpha \lambda K_T + k_T K_T \end{bmatrix} u^e = 0 \tag{54}$$

where $M, K, M_T, K_T$ are the stiffness and mass matrices of size $N \times N$ characterized by:

$$K_{ij} = \int_{\Omega} \Phi_j \Phi_i \, dx \, dy$$

$$M_{ij} = \int_{\Omega} \nabla \Phi_j \nabla \Phi_i \, dx \, dy$$

$$G^M_{ij} = \int_{\Gamma_1} \Phi_i \Phi_j^T \, ds$$

$$G^K_{ij} = \int_{\Gamma_1} \nabla_i \Phi_j \, \nabla_i \Phi_i \, ds$$

The approximated eigenvalues of (1) are obtained by determining numerically the values of $\lambda$ such that the determinant of (54) is null.

Recall our initial model (1):

$$\begin{cases}
  u_{tt} - k_\Omega \Delta u_t - \Delta u = 0 & \text{ in } \Omega \\
  u_{tt} - \frac{\alpha}{\sigma}(u + k_\Omega u_t) - k_T \Delta_\Gamma (\alpha u_t + u) = 0 & \text{ in } \Gamma_1 \\
  u = 0 & \text{ in } \Gamma_0 \\
  u(0,x) = u_0, \ u_t(0,x) = u_1 & \text{ on } \Omega
\end{cases}$$

We will present the results for the different cases provided by this model as a function of the values taken by $k_\Omega$, $k_T$ and $\alpha$. Assume that the damping condition holds (2), it guarantees not only the generation of the strongly continuous semigroup of contraction (Theorem 3.1), but also the spectral property $\sigma(A) \cap i\mathbb{R} = \emptyset$ (Theorem 3.5), which will be confirmed by all the pictures.

9.1. Wave equation with strong damping in the interior and dynamic boundary condition without Laplace-Beltrami term - Figure 3. Consider the following set-up: $k_T = 0$, $k_\Omega > 0$, i.e. we have a wave equation with a strong damping in the interior, with no Laplace-Beltrami term on the boundary. Then, the semigroup is analytic on $\mathcal{H}_p$ for $1 \leq p \leq \infty$ by theorem 3.4 and exponentially stable by the spectral property from theorem 3.5. One could also obtain the exponential stability by theorem 3.2. First of all, the following picture confirms that there is no eigenvalue on the imaginary axis. Also the spectrum is similar to the one described by Chen and Triggiani in [11], i.e., if \{$\lambda_n$\}$_{n \in \mathbb{Z}}$ denote the eigenvalues for a strong damped wave equation with Dirichlet boundary conditions, then the authors explicitly derived the following formula $\lambda_n^{\pm} = \mu_n^{\pm} \left( -k_\Omega \pm \sqrt{k_\Omega^2 - \mu_n^2} \right)$, \{$\mu_n$\}$_{n=1}^\infty$ are the eigenvalues of the laplace operator in 2D as $n \to \infty$. This pattern will come again in some of the following pictures.
9.2. Wave equation with strong damping both in the interior and on the boundary - Figure 4. Consider the following set-up: \( k_\Gamma, k_\Omega, \alpha > 0 \). Then the semigroup associated with this model is analytic on \( \mathcal{H} \) by theorem 3.4 and exponentially stable by the spectral property from theorem 3.5 associated with Pazy’s theorem A. Note that theorem 3.2 also provides the exponential stability. We obtain a similar spectrum to the previous one. On Figure 4, we present another aspect of the study, observing dynamically the impact of the coefficient on the spectrum. It shows that the structural damping \( k_\Omega \Delta u_t \) and \( \alpha \Delta \Gamma u_t \) affects the radius of the circle, it is actually inversely proportional to \( k_\Omega \) and \( k_\Gamma \), i.e., as the coefficient \( k_\Omega \) and \( k_\Gamma \) increases, the circle shrinks. This is the exact pattern described for a wave equation with 0-Dirichlet boundary conditions (see [11]).
9.3. Wave equation with strong damping only on the boundary - Figure 5. The scenario corresponds to Theorem 3.6 \((k_\Omega = 0 \text{ and } k_\Gamma, \alpha > 0)\). Since the semigroup governing this problem is a \(C_0\)-semigroup of contraction on \(\mathcal{H}\) (Theorem 3.1) and the intersection between its spectrum and the imaginary axis is empty \((3.5)\), then, by application of Arendt-Batty theorem (Theorem B), the strong stability immediately follows. **Figure 5** confirms this result. Indeed, we observe one new component of the spectrum spreading asymptotically along the imaginary axis, with \(\text{Re} (\lambda) < 0\), \(\forall \lambda \in \sigma (A)\). We know that boundary dissipation in the absence of inertial terms on the boundary leads to exponential stability of the semigroup. However, inertial boundary terms added to the dynamics destroy exponential stability and boundary damping alone leads to strong stability only. This fact was first observed by Littman and Markus [26].

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - k_\Omega \Delta u - \Delta u &= 0 \quad \text{on } \Omega \\
\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial \eta} (k_\Omega(u_t + u)) - k_\Gamma \Delta \Gamma (\alpha u_t + u) &= 0 \quad \text{on } \Gamma_1 \\
u &= 0 \quad \text{on } \Gamma_0
\end{align*}
\]

**Figure 5.** Spectrum of (1) with \(k_\Gamma, \alpha > 0\) and \(k_\Omega = 0\).

9.4. Wave equation with strong damping only in the interior - Figure 6. Consider the following set-up: \(k_\Gamma, k_\Omega > 0\) and \(\alpha = 0\). Then the semigroup associated with this model is still not analytic but we recover the exponential stability property by theorem 3.2. We observe that the spectrum in the absence of boundary damping (Figure 6) has an additional component: a parabola. We indicated that the parabola is of the form \(|\text{Im } \lambda| = a - b \times |\text{Re } \lambda|^2\), again this confirms that the semigroup can not be analytic, but must have some smoothing property from parabolicity. We expect the semigroup to be in a class between differentiability and analyticity, and related to the theory of Gevrey Semigroups [39]. As for characterization of analyticity, Gevrey’s regularity is described in terms of the bounds on all derivatives of the semigroup:

**Definition 9.1** (Gevrey Semigroup). Let \(T(t)\) be a strongly continuous semigroup on a Banach space \(X\) and let \(\delta > 1\). We say that \(T(t)\) is of Gevrey class \(\delta\) for \(t > t_0\) if \(T(t)\) is infinitely differentiable for \(t \in (t_0, \infty)\) and for every compact \(\mathcal{K} \subset (t_0, \infty)\), and each \(\theta > 0\), there exists a constant \(C = C_{0, \mathcal{K}}\) such that

\[
\left\| T^{(n)} (t) \right\| \leq C \theta^{\delta n} (n!)^\delta, \ \forall t \in \mathcal{K} \text{ and } n \in \{0, 1, 2, \ldots\}
\]
Besides other characterizations, the author provided some sufficient conditions for semigroups to be of Gevrey class.

**Theorem E** ([39]). Let $T(t)$ be a strongly continuous semigroup satisfying $\|T(t)\| \leq Me^{\omega t}$. Assume one of the following holds:

- Suppose that for some $\gamma$ satisfying $0 < \gamma \leq 1$:
  $$\limsup_{\beta \to \infty} |\beta|^{\gamma} \|R(i\beta; A)\| = C < \infty$$
- Suppose that:
  $$\lim_{t \to 0} t^{\delta} \|T'(t)\| = 0$$

Then $T(t)$ is of Gevrey class $\delta$ for $t > 0$ (for every $\delta > \frac{1}{\gamma} \gamma$ for the first item).

The resolvent characterization relies on the contour shape formed by the eigenvalues whose imaginary parts tend to infinity. More precisely, the degree of the described curve is $\delta$. Back to our problem, we identify on Figure 6 such a parabolic component which suggests that the semigroup should of Gevrey class $\delta = 2$. This remark has not yet been proved, but the picture suggests further investigation in this direction should lead to a positive conclusion.

It is also important to note from Theorem E that for $\gamma = 1$ the semigroup is analytic.

This class of semigroup offers an intermediate level between analytic semigroup and differentiable semigroup. Also, while a differentiable semigroup is not stable under bounded perturbation, as it is demonstrated by Renardy in [38], if $B \in \mathcal{H}$ and one of the characterizations from theorem E holds, then not only $A$ is Gevrey but also $A + B$. This can be seen by using a similar argument as for perturbation of analytic semigroup by bounded operator (see [37, Theorem III.3.2.1]).

One could summarize the connection between these different classes of semigroup with the following tree:

\[
\text{Analytic} \rightarrow \text{Theorem E} \rightarrow \text{Gevrey} \rightarrow \text{Differentiable}
\]

S. Taylor and W. Littman used Gevrey’s regularity to study smoothing properties in the context of plate and Schrodinger equations [28, 31, 40]. We also refer the reader to: [22, 23, 30, 31] for other applications of Gevrey regularity.

### 9.5. Wave equation with frictional damping in the interior and strong damping in the interior - Figure 7.

Finally, we conclude by a slight modification of our initial model (1) by replacing the viscoelastic damping $\Delta u_t$ in the interior by a frictional term $u_t$. Thus, we now consider the model:

\[
\begin{align*}
  u_{tt} + c\Omega u_t - \Delta u &= 0 & x \in \Omega, \ t > 0 \\
  u &= 0 & x \in \Gamma_0, \ t > 0 \\
  u_{tt} + \frac{\partial}{\partial n} u - k \Gamma_1 \Delta_\Gamma (\alpha u_t + u) &= 0 & x \in \Gamma_1, \ t > 0 \\
  u(0, x) &= u_0, \ u_t(0, x) = u_1 & x \in \Omega
\end{align*}
\]

The proof for theorem 3.1 would also work to show the generation of the $C_0$-semigroup associated with problem (55). Also, a similar version of the proof of theorem 3.2 is also working by noting that the dissipation term $|\nabla u_t|^2_{\Omega}$ in (14) is replaced by $|u_t|^2_{\Omega}$ and during the reconstruction of the energy with the multiplier $u$, it is not necessary anymore to use Poincaré’ s inequality as in (20) and (21) since we
recover immediately the desired dissipation \(|u_t|^2_{\Omega}\). Therefore, this model generates an exponentially stable \(C_0\)-semigroup of contractions.

9.6. Wave equation with frictional damping in the interior and high order dynamic boundary condition without Laplace-Beltrami term - Figure 8.

Keeping the previous model (55) in the absence of Laplace-Beltrami term on the boundary \((k_\Gamma = 0, c_\Omega > 0)\). Then, we have a \(C_0\)-semigroup of contraction by a similar argument to theorem 3.1 which is only strongly stable by 3.5.

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\[ \begin{aligned}
&c_\Omega = 0.5 \\
c_\Gamma = 0 \\
k_\Omega = 0 \\
\alpha = 0 \\
k_\Gamma = 0
\end{aligned} \]

\[ \begin{aligned}
&k_\Omega = 0 \\
c_\Omega = 0.5 \\
k_\Gamma = 0 \quad \text{on } \Gamma_1 \\
\delta \mu u_t + \delta \nu u_t + \delta \kappa u_t = 0 \quad \text{on } \Gamma_0
\end{aligned} \]

**Figure 8.** Spectrum of (55) with \( k_\Gamma = 0, \ \omega_n > 0. \)

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