Random primes in arithmetic progressions

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Abstract
We describe a straightforward method to generate a random prime $q$ such that the multiplicative group $F_q^*$ also has a random large prime-order subgroup. The described algorithm also yields this order $p$ as well as a $p$'th primitive root of unity $\omega$. The methods here are efficient asymptotically, but due to large constants may not be very useful in practical settings.

1 Introduction

In various contexts, for example in sparse polynomial evaluation and interpolation algorithms, it is necessary to have a finite field $F_q$ that admits an order-$p$ multiplicative subgroup with generator $\omega$. There are typically some non-divisibility properties both on the field size $q$ and the subgroup order $p$.

In this note, we briefly sketch efficient algorithms to probabilistically generate such $q, p, \omega$ tuples. The results are neither surprising to practitioners in this area, nor are they particularly original. However, we have found them useful, and so decided to publish in this short note with complete proofs.

2 Statement of results

Our approach to produce triples $(p, q, \omega)$ such that $\omega$ generates an order-$p$ multiplicative subgroup of $F_q^*$ is straightforward. We first sample integers $p$ until a prime number is found. Then, we need to find a prime number $q$ such that $p \mid (q - 1)$, that is $q$ is in the arithmetic progression $\{ap + 1 : a \geq 1\}$, and such that $q = \text{poly}(p)$. Again, we sample integers $a$ until $ap + 1$ is prime. Finally, we sample elements $\zeta \in F_q^*$ until $\zeta^{(q-1)/p} \neq 1$ and return $(p, q, \zeta^{(q-1)/p})$.

Each step is justified by the abundance of good integers or elements of $F_q^*$. The computation of $p$ relies on an effective version of the prime number theorem.
of Rosser and Schoenfeld (1962). The computation of \( q \) relies on effective versions of Dirichlet’s theorem (or more precisely Bombieri-Vinogradov theorem) of Akbary and Hambrook (2015) and Sedunova (2018). The computation of \( \omega \) relies on the fact that there are at most \( \frac{(q-1)}{p} \) values \( \zeta \) such that \( \zeta^{(q-1)/p} = 1 \).

Our approach is closely related to similar results in Arnold’s Ph.D. thesis (2016). We replace a constant probability of success by an arbitrary high probability of success, and we use better bounds.

**Theorem 2.1.** There exists an explicit Monte Carlo algorithm which, given a bound \( \lambda \geq \frac{256}{\mu^2} \), produces a triple \( (p, q, \omega) \) that has the following properties with probability at least \( 1 - \epsilon \), and return FAIL otherwise:

- \( p \) is uniformly distributed amongst the primes of \( (\lambda, 2\lambda) \);
- \( q \leq \lambda^6 \) is a prime such that \( p \mid (q - 1) \);
- \( \omega \) is a \( p \)-primitive root of unity in \( \mathbb{F}_q \);

Its worst-case bit complexity is \( \text{polylog}(\lambda) \).

An additional requirement in some situations is that the prime \( q \) does not divide an (unknown!) large integer. This is achieved by taking \( \lambda \) sufficiently large.

**Theorem 2.2.** Let \( K \) be an unknown integer, and let \( (p, q, \omega) \) a triple produced by the algorithm of Theorem 2.1 on some input \( \lambda \). If \( \lambda \geq \max \left( \frac{256}{\mu^2}, \sqrt[48]{\mu \ln K} \right) \), the probability that \( q \) divides \( K \) is at most \( \mu \).

The large constant in the statement of Theorem 2.1 is required in order to get rigorous unconditional complexity bounds. Yet, it makes the algorithm not very practical because of the bit-length of the primes produced. Section 4 presents experimental results indicating the results actually hold for much smaller values of \( \lambda \).

## 3 Proofs

To construct a field \( \mathbb{F}_q \) with a \( p \)-PRU \( \omega \), we first need to generate random prime numbers. The well-known technique for this is to sample random integers and test them for primality. In order to get Las Vegas algorithm, we rely on the celebrated AKS algorithm.

**Fact 3.1 (Agrawal, Kayal, and Saxena (2004)).** There is a deterministic algorithm that, given any integer \( n \), determines whether \( n \) is prime or composite and has bit complexity \( \text{polylog}(n) \).

While the original bit complexity was \( \tilde{O}(\log^{10.5} n) \), this has been subsequently improved to \( \tilde{O}(\log^6 n) \) in a revised version by Lenstra and Pomerance (2011). In practice, a better option is to use the Monte Carlo Miller-Rabin
primality test which has a worst-case bit complexity of \( \tilde{O}(\log^2 n) \) but a low probability of incorrectly reporting that a composite number is prime (Rabin, 1980).

No fast deterministic algorithm is known to construct a prime number with a given bit length \( b \). However, sampling random \( b \)-bit integers and testing their primality using AKS algorithm results in a Las Vegas randomized algorithm. The expected running time relies on the fact that there are at least \( \Omega(2^b/b) \) primes with \( b \) bits. We recall some more precise bounds.

**Fact 3.2 (Rosser and Schoenfeld (1962)).** For \( \lambda \geq 21 \), there exist at least \( \frac{2}{3}\lambda \ln \lambda \) prime numbers between \( \lambda \) and \( 2\lambda \).

Once we have a prime number \( p \), we want to find a prime number \( q \) in the arithmetic progression \( p + 1, 2p + 1, 3p + 1, \ldots \). Dirichlet’s theorem says that, asymptotically, the distribution of primes in this arithmetic progression is the same as the distribution of primes in \( \mathbb{Z} \). The Bombieri-Vinogradov theorem refines it with bounds on the error terms. This indicates that a good strategy to generate \( q \) is simply to pick a random (even) positive integer \( k \) and test whether \( pk + 1 \) is prime, repeating until a prime of that form is found.

The question is, how large should \( k \) be in the strategy above in order to guarantee a reasonable chance of success? Recent results of Akbary and Hambrook (2015) and Sedunova (2018) give explicit bounds for the Bombieri-Vinogradov theorem.

**Fact 3.3 (Sedunova (2018, Corollary 1.5)).** Let \( \pi(x) \) denote the number of prime numbers \( \leq x \), \( \pi(x; m, a) \) the number of prime numbers \( \leq x \) that are congruent to a modulo \( m \), and \( \ell(x) \) the smaller prime divisor of \( x \). Then for any \( \gamma \geq 4 \) and \( \lambda_1 \leq \lambda_2 \leq \gamma^{1/2} \),

\[
\sum_{m \leq \lambda_2} \max_{\ell(m) > \lambda_1} \max_{\ell \leq \gamma, \gcd(a, m) = 1} \left| \pi(y; m, a) - \frac{\pi(y)}{\phi(m)} \right| \\
\leq 122.77 \left( 14 \frac{\gamma}{\lambda_1} + 4\gamma^{1/2}\lambda_2 + 15\gamma^{2/3}\lambda_2^{1/2} + 4\gamma^{5/6} \ln\left(\frac{\lambda_2}{\lambda_1}\right) \right) (\ln \gamma)^{7/2}.
\]

From this fact, we obtain the following probabilistic result on the number of primes in an arithmetic progression.

**Corollary 3.4.** Let \( 0 < \epsilon < \frac{1}{2} \) and \( \lambda \geq \frac{2e^4}{\epsilon^2} \), and \( p \) be a random prime from \((\lambda, 2\lambda)\). Then with probability at least \( 1 - \epsilon \), the number of prime numbers \( q \leq \lambda^6 \) of the form \( q = ap + 1 \) is \( \geq \lambda^5/(24 \ln \lambda) \).

**Proof.** We apply Fact 3.3 with \( \lambda_1 = \lambda \), \( \lambda_2 = 2\lambda \) and \( \gamma = \lambda^6 \). We note that the sum is over the prime numbers (since \( \ell(m) > \lambda_1 \geq m/2 \)). We then simplify it by choosing \( y = \gamma \) and \( a = 1 \) in the formula, which can only make the sum smaller. Then

\[
\sum_{\lambda < p < 2\lambda \atop p \text{ prime}} \left| \pi(\lambda^6; p, 1) - \frac{\pi(\lambda^6)}{p - 1} \right| \\
\leq 1.09 \cdot 10^6 \left( \lambda^5 + 1.27\lambda^{4.5} + 0.48\lambda^4 \right) (\ln \lambda)^{7/2}.
\]
For $\lambda \geq 2^8$, the sum is bounded by $1.2 \cdot 10^6 \lambda^5 (\ln \lambda)^{7/2}$. Now we wish to count the bad primes in $(\lambda, 2\lambda)$ such that $\pi(\lambda^6; p, 1) \leq \lambda^5/(24\ln \lambda)$. Since $\pi(\lambda^6) \geq \lambda^6/(6\ln \lambda)$, if $p$ is a bad prime, then $\pi(\lambda^6)/(p - 1) \geq \pi(\lambda^6; p, 1)$ and since $p - 1 \leq 2\lambda$,

$$\left| \frac{\pi(\lambda^6; p, 1) - \pi(\lambda^6)}{p - 1} \right| \geq \frac{\lambda^6/(6\ln \lambda)}{p - 1} - \frac{\lambda^5}{24\ln \lambda} \geq \frac{\lambda^5}{24\ln \lambda}.$$ 

If there are $k$ bad primes, then the sum is at least $k\lambda^5/24\ln \lambda$. Using the previous bound on the sum, we get the bound

$$k \leq \frac{1.2 \cdot 10^6 \lambda^5 (\ln \gamma)^{7/2}}{\lambda^5/(24\ln \lambda)} = 2.88 \cdot 10^7 (\ln \lambda)^{9/2}.$$

Since there are at least $\frac{\lambda}{2} \lambda / \ln \lambda$ prime numbers between $\lambda$ and $2\lambda$, the probability that a random prime number $p$ chosen in $(\lambda, 2\lambda)$ is bad is at most

$$\frac{2.88 \cdot 10^7 (\ln \lambda)^{9/2}}{\frac{\lambda}{2} \lambda / \ln \lambda} = 4.8 \cdot 10^7 \lambda^{-1} (\ln \lambda)^{11/2}.$$

The probability obviously tends to zero when $\lambda$ tends to infinity. For instance, for $\lambda \geq 2^{55}$ the probability is bounded by $2^{27}\lambda^{-1/2}$. Hence, to get a probability at most $\epsilon$, one can take $\lambda \geq \frac{2^{54}}{\epsilon}$ (which is $> 2^{55}$ as long as $\epsilon \leq \frac{1}{2}$). \hfill \Box

From this effective result, we deduce a Monte Carlo algorithm that produces primes $p, q$ such that $p \mid (q - 1)$, as well as a $p$-PRU modulo $q$.

**Proof of Theorem 2.1.** This is basically Algorithm “GetPrimeAP-5/6” on page 35 of (Arnold, 2016), slightly adapted, where the primality tests are made using AKS algorithm:

1. **sample** $\leq \frac{\lambda}{6} \ln \frac{4}{\epsilon} \ln \lambda$ random odd integers $p \in (\lambda, 2\lambda)$ until $p$ is prime, return **FAIL** if none of them is prime
2. **sample** $\leq 12 \ln \frac{4}{\epsilon} \ln \lambda$ random even integers $a \in [1, \lambda^5]$ until $q = ap + 1$ is prime, return **FAIL** if none of them is prime
3. **sample** $\leq \log_{p, \epsilon} \frac{4}{\epsilon}$ random elements $\zeta \in \mathbb{F}_q$ until $\omega = \zeta^{q - 1}/p \neq 1$, return **FAIL** if $\omega = 1$ for each $\zeta$
4. **return** $(p, q, \omega)$

Since AKS has complexity $\text{polylog}\lambda$ and $\log \frac{1}{\epsilon} = O(\log \lambda)$, the complexity of the whole algorithm is $\text{polylog}(\lambda)$.

There are at least $\frac{\lambda}{2} \lambda / \ln \lambda$ primes in $(\lambda, 2\lambda)$, and $\lambda/2$ odd integers. Therefore, the probability that a random odd integer is prime is at least $6/(5\ln \lambda)$. The probability that no prime is produced after $k$ tries is at most $(1 - 6/(5\ln \lambda))^k \leq e^{-6k/(5\ln \lambda)}$. If $k = \frac{3}{2} \ln \frac{4}{\epsilon} \ln \lambda$, the probability is at most $\frac{\epsilon}{4}$. Hence Step 1 succeeds with probability at least $1 - \frac{\epsilon}{4}$. 

4
Since $\lambda \geq \frac{2^{24}}{\pi^2}$, if the algorithm succeeds in producing $p$, there are at least $\lambda^6/(24\ln\lambda)$ prime numbers $q \leq \lambda^6$ of the form $ap + 1$ with probability at least $1 - \frac{\epsilon}{2}$.

If $p$ satisfies this condition, there are at least $\lambda^6/(24\ln\lambda)$ values of $a$ such that $ap + 1$, amongst the $\frac{1}{2}\lambda^6$ possible values. With the same proof as above, the probability that such an $a$ be found in $\leq 12\ln\frac{1}{\epsilon}\ln\lambda$ tries is at least $1 - \frac{\epsilon}{2}$.

Finally, if $q$ has been found, the third step finds a suitable $\omega$ with probability at least $1 - \frac{\epsilon}{\lambda}$ since there are at most $\frac{2\lambda}{p}$ values of $\zeta$ such that $\zeta^{(q-1)/p} = 1$.

Therefore, the algorithm returns a triple $(p, q, \omega)$ satisfying the three properties with probability at least $1 - \epsilon$. \hfill \Box

Proof of Theorem 2.2. Since $\lambda \geq \frac{2^{24}}{(p/2)^2}$, the prime $p$, if produced, satisfies that there are at least $\lambda^6/(24\ln\lambda)$ primes $q \leq \lambda^6$ of the form $ap + 1$ with probability at least $1 - \frac{\epsilon}{2}$. The number of those primes than can divide $K$ is at most $\log\lambda K$ since all of them are $\leq \lambda^6$. Therefore, the probability that one of them chosen at random divides $K$ is at most $24\log_p K \ln\lambda/\lambda^6 \leq \frac{\epsilon}{2}$. \hfill \Box

4 Experiments

Theorems 2.1 and 2.2 are only valid for large values of $\lambda$. This is only an artefact due to the known explicit constants known for Bombieri-Vinogradov theorem. Actually, very similar results hold with smaller values. As an experimental justification of this, we perform the following computations.

A strong form of Dirichlet’s theorem due to de la Vallée Poussin states that asymptotically, the proportion of primes in the arithmetic progression $\{2kp + 1 : 1 \leq k \leq p/2\}$ is $O\left(\frac{1}{\ln(p)}\right)$. For each prime number $p$ of bitsize between 10 and 20 (there are 81,928 of them), we estimate this proportion. For bitsizes 10 to 14, we actually compute the proportion exactly, testing the primality for each value $k$. For larger bitsizes, we estimate the proportion by sampling: We sample $N \geq 1000$ random elements in the set and test their primality.

The SAGEMath code used for the computations is given in Listing 1. Table 1 provides, for each bitsize, the smallest and largest proportions found, as well as the average proportion. Figure 1 plots the proportion for each prime.

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Listing 1: Code for the proportion of primes in arithmetic progressions

def primes_in_arith_prog(p, bound, samples=1000, random=True):
    """
    Estimate the number of primes <= bound in the arithmetic progression A = \{ 2pk+1 : k \geq 1 \}, by sampling samples random values k. The primality test is randomized if random = True.
    """
    count = 0
    size = bound // (2* p)
    if size > samples:
        for _ in range(samples):
            k = ZZ(randint(1, size))
            if (2*k*p+1).is_prime(proof=random):
                count += 1
        return RR(count / samples * size)
    for k in range(size):
        if ZZ(2*k*p+1).is_prime(proof=random):
            count += 1
    return count

def distrib_primes(bitsize, bound, samples=1000, random=True):
    """
    Estimate the proportion of primes <= bound in each arithmetic progression A = \{2 kp+1 : k \geq 1\} for primes p of the given bitsize.
    """
    L = []
    for p in prime_range(2**(bitsize-1),2**bitsize):
        n = primes_in_arith_prog(p, bound, samples, random)
        L.append(float(n/(bound//((2*p)))))
    return L

def test(bmin, bmax):
    """
    This function generates the whole data set as a dictionary and prints the table that summarizes the result.
    """
    D = {}
    for b in range(bmin,bmax):
        L = distrib_primes(b,2**(2*b),samples=1000*(b-10+1))
        D[b] = L
    print("\nbitsize\tmin\taverage\tmax\n")
    for b in D:
        print(f"\{{{b}\n\t\{min(D[b]):.3f}\n\t\{mean(D[b]):.3f}\n\t\{max(D[b]):.3f}\}"")
    return D

test(10, 21)
Table 1: Proportion of primes in \(\{2kp + 1 : 1 \leq k \leq p/2\}\).

| Bitsize | Minimum | Average | Maximal | Theory |
|---------|---------|---------|---------|--------|
| 10      | 13.36%  | 15.63%  | 18.49%  | 16.03% |
| 11      | 12.50%  | 14.12%  | 15.80%  | 14.43% |
| 12      | 11.00%  | 12.79%  | 14.33%  | 13.12% |
| 13      | 10.13%  | 11.79%  | 13.55%  | 12.02% |
| 14      | 9.50%   | 10.93%  | 12.54%  | 11.10% |
| 15      | 8.83%   | 10.14%  | 11.65%  | 10.30% |
| 16      | 8.24%   | 9.46%   | 10.74%  | 9.62%  |
| 17      | 7.70%   | 8.89%   | 10.00%  | 9.02%  |
| 18      | 7.19%   | 8.36%   | 9.41%   | 8.49%  |
| 19      | 6.80%   | 7.91%   | 9.11%   | 8.01%  |
| 20      | 6.53%   | 7.49%   | 8.52%   | 7.59%  |

Figure 1: Proportion of primes in \(\{2kp + 1 : 1 \leq k \leq p/2\}\).
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