THE MODULI SPACE OF $S^1$-TYPE ZERO LOCI FOR $\mathbb{Z}/2$-HARMONIC SPINORS IN DIMENSION 3

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ABSTRACT. Let $M$ be a compact oriented 3-dimensional smooth manifold. In this paper, we will construct a moduli space consisting of the following data $\{(\Sigma, \psi)\}$ where $\Sigma$ is a $C^1$-embedding $S^1$ curve in $M$, $\psi$ is a $\mathbb{Z}/2$-harmonic spinor vanishing only on $\Sigma$ and $\|\psi\|_{L^2} = 1$. We will prove that this moduli space can be parametrized by the space of Riemannian metrics on $M$ locally as the kernel of a Fredholm operator.

CONTENTS

1. Introduction 1
2. Basic setting and results 5
3. Harmonic sections defined on the tubular neighborhood with the Euclidean metric 9
4. Variational formula and perturbation of curves 18
5. The general $\Sigma$ embedding in $M$ 29
6. Fredholm property 35
7. Proof of the main theorem: Part I 52
8. Proof of the main theorem: Part II 71
9. Appendix 74
References 75

1. Introduction

1.1. Main theorem and its background and motivations. In this paper, I will give a local parametrization of the set of triples of the form $(g, \Sigma, \psi)$ with $g$ being a Riemannian metric, $\Sigma$ being a $C^1$ embedded circle and $\psi$ being a $\mathbb{Z}/2$-harmonic spinor defined on the complement of $\Sigma$ whose norm extends across as zero as to give a Hölder continuous function on $M$. Here the $\mathbb{Z}/2$-harmonic spinor can be defined as the follows. The $\mathbb{Z}/2$-spinor is a smooth section defined on the twisted spinor bundle $\mathcal{S} \otimes \mathcal{I}$ where $\mathcal{S}$ is the spinor bundle with respect to the metric $g$ defined on $M$ and $\mathcal{I}$ is a real line bundle defined on $M - \Sigma$ which has $\mathbb{Z}/2$-monodromy over $\Sigma$. Secondly, we call a $\mathbb{Z}/2$-spinor harmonic if and only if it satisfies the Dirac equation $D\psi = 0$.

To say more about this, let

$$\mathcal{A} = \{\Sigma \subset M| \Sigma \text{ is the image of a } C^1 \text{ embedding of the circle}\}.$$
For each $\Sigma \in A$, define $H$ be the subset of $H^1(M - \Sigma; \mathbb{Z}/2)$ with non-zero monodromy around $\Sigma$. Each $e \in H$ corresponds to a real line bundle $\mathcal{I}_{\Sigma,e}$ on $M - \Sigma$. So as $\Sigma$ varies, the set $H$ varies continuously to define a finite sheeted covering space of $A$. This is denoted by $A_H$. Denote by $X$ the space of Riemannian metrics on $M$. Each metric $g \in X$ has a corresponding spinor bundle $\mathcal{S}_g \to M$. Denote by $\mathcal{S}_g,\Sigma,e$ the bundle $\mathcal{S}_g \otimes \mathcal{I}_{\Sigma,e}$; this is a spinor bundle over $M - \Sigma$. This is called the $\mathbb{Z}/2$ spinor bundle. Define $Y$ to be $X \times A_H$.

Let $E \to Y$ denote the infinite dimensional vector bundle defined as follows: Supposing that $y = (g, \Sigma, e) \in X \times A_H$, then the fiber of $E$ over $y$ is the infinite dimensional vector space of $L^2$ sections over $M - \Sigma$ of the $\mathbb{Z}/2$ spinor bundle $\mathcal{S}_g,\Sigma,e$. This vector space is denoted by $E_y$. Let $D(y)$ denote the Dirac operator defined on $E_y$ by the metric $g$. This operator gives a bounded, linear map from $E_y$ to the space of square integrable sections of $\mathcal{S}_g,\Sigma,e$.

The set $M$ inherits a topology from $E$. The goal is to give it some additional structure. To say more about $M$, we can consider the vector bundle $\mathcal{F}$ over $Y$ whose fiber $\mathcal{F}_y$ is the $L^2$ sections of $\mathcal{S}_y$. Then $M$ will be contained in the kernel of $D : E \to \mathcal{F}$ where $D|_{\mathcal{S}_y} = D(y)$.

I will prove the following:

**Theorem 1.1.** Let $(y = (g, \Sigma, e), \psi)$ denote a given element in $M$. There are finite dimensional vector spaces $K_1$ and $K_0$, a ball $B \subset K_1$ centered at the origin, a set $B \subset X$ with $B = p_1(N)$ being the projection of $N$, a neighborhood of $y$, from $Y$ to $X$ and a $C^1$ map to be denoted by $f$ from $B \times B$ to $K_0$ such that $M$ near $(y, \psi)$ is homeomorphic to $f^{-1}(0)$.

The vector spaces $K_1$ and $K_0$ in this theorem can be generated by the kernel and cokernel of a Fredholm operator respectively. This theorem shows us several facts. First of all, the $C^1$-curve component $\Sigma$ in $M$ can only be perturbed in finite dimensional directions. Secondly, when $\dim(K_0) = 0$, then $M$ near $(y, \psi)$ is homeomorphic to $B \times B$.

The operator that leads to $K_0$ and $K_1$ comes from a formal linearization of the equations that are obtained by deforming the metric and the curve and the spinor so as to stay in $M$. This operator seems to be novel and the fact that it is Fredholm does not appear to follow from the usual considerations. By the same token, the proof of Theorem 1.1 is not a standard application of the inverse function theorem as it required a delicate iteration to integrate the formal tangent space given by the kernel of $df$ at $(g, 0) \in B \times B$ to obtain the given parametrization of $M$. 
The study of these data \((g, \Sigma, \psi)\) started from the work of \(PSL(2; \mathbb{C})\) compactness theorem proved by Clifford Taubes. In [1], Clifford Taubes proved a generalized version of Uhlenbeck’s compactness theorem [2]. Let \(M\) be a 3-dimensional manifold. The Uhlenbeck’s compactness theorem [3] can be stated in the following way:

**Theorem 1.2.** Suppose \(P\) is a principal \(G\) bundle over \(M\) for some compact Lie group \(G\) and \(\{A_i\}\) be a sequence of connections on \(P\) satisfying

\[
\|F(A_i)\|_{L^2} \leq C
\]

for some constant \(C\) which is independent of \(i\). Then there exists a subsequence of \(\{A_i\}\) converging (up to gauge transformations) weakly in \(L^2\) to an \(L^2\) connection.

To state the theorem proved in [1], I need to introduce some notations. Firstly, Taubes used the fact that \(\mathfrak{sl}(2; \mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)\) and \(P\) can be regarded as one of its \(SO(3)\)-reductions associated with \(PSL(2; \mathbb{C})\). So he can fix one reduction and denote \(P\) by \(P \times SO(3)_{PSL(2; \mathbb{C})}\). Therefore, he can always decompose a connection \(A = A + ia\) where \(A\) is the connection one form on the \(SO(3)\)-reduction of \(P\) and \(a\) is a \(\mathfrak{su}(2)\)-valued one form. Secondly, if we denote the group of gauge transformations (the automorphism group of \(P\)) by \(G\), then the Lie algebra \(\mathfrak{sl}(2; \mathbb{C})\) does not have norms which are invariant under the action of \(G\). So we should refine the \(L^2\) boundedness condition (1.1) as follows:

**Definition 1.3.** Let

\[
F(A) = \inf_{A + ia \in \mathcal{G}_A} \int |F(A) - a \wedge a|^2 + |dAa|^2 + |dA * a|^2
\]

where \(\mathcal{G}_A\) is the \(G\)-orbit of \(A\).

Now, the generalized Uhlenbeck’s compactness theorem proved in [1] can be stated as follows:

**Theorem 1.4.** For any sequence of connections \(\{A_i = A_i + ia_i\}\) defined on \(P \times SO(3)_{PSL(2; \mathbb{C})}\), which has \(\{F(A_i)\}\) being bounded, we have

- If \(\{\|a_i\|_{L^2}\}\) is bounded, then we can find a subsequence of \(\{A_i\}\) which is weakly \(L^2\) convergent up to automorphisms of \(P\).
- If \(\|a_i\|_{L^2} \to \infty\), we can find a closed, Hausdorff dimension at most 1 subset \(\Sigma\) and a subsequence of \(\{A_i = A_i + ia_i\}\) such that
  1. \(\{A_i\}\) converges weakly in \(L^2_{1, \text{loc}}\)-sense on \(M - \Sigma\) up to automorphisms of \(P\)
  2. \(\{\frac{1}{\|a_i\|_{L^2}^2}a_i\}\) also converges weakly in \(L^2_{1, \text{loc}}\)-sense on \(M - \Sigma\) up to automorphisms of \(P\).

Moreover, the data \(\Sigma\) can be formulated as the zero locus of a \(\mathbb{Z}/2\)-harmonic spinor. In [1], Taubes showed the set \(\Sigma\) will always have a corresponding \(\mathbb{Z}/2\)-spinor \(\psi\) which satisfies the Dirac equation \(D\psi = 0\) and \(\|\psi\|\) can be extended Hölder continuously to zero on \(\Sigma\).

The \(PSL(2; \mathbb{C})\) compactness theorem suggests that data sets consisting of pairs \((\Sigma, \psi)\) with \(\Sigma\) being a closed Hausdorff dimension 1 set and \(\psi\) a \(\mathbb{Z}/2\) harmonic spinor with norm zero on \(\Sigma\) have a role to play in 3 dimensional differential topology. So a natural question we can ask is the following: Can we find a way to parametrize
the data \((\Sigma, \psi)\)?

Meanwhile, in [2], Taubes showed more properties for this data set \(\Sigma\). It is a conjecture that \(\Sigma\) is a \(C^1\) curve for the metric \(g\) suitably generic, this conjecture is also mentioned in [10]. So we consider the case that \(\Sigma\) is a 1-dimensional submanifold in this paper.

1.2. The outline of the proof and the structure of this paper. In the first part of this paper, we shall study model solutions of Dirac equation with \(\Sigma\) fixed. We parametrize a tubular neighborhood of \(\Sigma\), \(\mathcal{N}_{R}\), by \((t, z) \in [0, 2\pi] \times \{z \in \mathbb{C}||z| < R\}\). Then, we will show that any harmonic spinor \(\psi\) which vanishes along \(\Sigma\) is in \(\ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})\) and vice versa. For any \(\psi \in \ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})\), it can be written locally as

\[
\psi = \left( \begin{array}{c} d^+(t)\sqrt{z} \\ d^-(t)\sqrt{z} \end{array} \right) + \text{higher order term}
\]

on \(\mathcal{N}_{R}\). Here the "higher order term" is a smooth section with order \(|z|^p\) for some \(p > \frac{1}{2}\). In addition, by Lichnerowicz-Weitzenböck formula, we will see \(\dim(\ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})) < \infty\). All these basic analysis results for \(L^2_2\)-harmonic spinors will be shown in section 2 and 3. Also, the analysis of \(L^2\)-harmonic spinors will be derived in section 2 and 3 for later use, too.

According to these observations, one can consider the linear perturbation for any given \(p = (g_0, \Sigma_0, \psi_0) \in \mathcal{M}\) (Note that the element \(e \in H\) will be omitted in the rest of this paper because this discrete data wouldn’t change in any local perturbation). This perturbation can be written as \((g_0 + s\delta, \Sigma_s, \psi_s)\) for small \(s \in \mathbb{R}\). Here \(\delta\) is a smooth 2-form with \(\text{supp}(\delta) \cap \Sigma_0 = \emptyset\); \(\Sigma_s = \{(t, z - s\eta)\} \in C^1(\mathbb{S}^1; \mathbb{C})\); \(\psi_s(t, z) = \psi_0(t, z - s\eta) + s\phi(t, z - s\eta)\) for some \(\phi \in L^2_2(\mathcal{S}_{0,0})\). Let us denote by \(D^{(s)}\) the Dirac operator with respect to \(g_0 + s\delta\) and define \(\Sigma_p(\delta, \eta, \phi) := \frac{d}{ds}(D^{(s)}\psi_s)|_{s=0}\).

By (1.2) and some basic analysis results derived in section 2, 3 and 4, we will prove that there exists \(\Phi(\delta, \eta, \phi) \in \ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})\) determined by \(\delta\) such that \(\Sigma_p(\delta, \eta, \phi) - \Phi(\delta) \in \text{range}(D|_{L^2(\mathcal{S}_{0,0},^e)})\). The space \(\mathbb{K}_1\) is therefore defined to be the kernel of \(\Sigma_p|_{\delta=0}\), so any element in \(\ker(\Sigma_p|_{\delta=0})\) corresponds to an element in \(\ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})\) of the form

\[
\left( \begin{array}{c} 2^\eta \\ 2^\eta \end{array} \right) + \text{higher order term} + \phi.
\]

Notice that the condition \(\frac{|\psi_0(p)|}{\text{dist}(p, \Sigma_0)^2} > 0\) near \(\Sigma_0\) implies \(|d^+|^2 + |d^-|^2 \neq 0\). The pair \((d^+, d^-)\) is called the leading coefficient for \(\psi_0\), which plays an important role in this paper.

Now, since \(\dim(\ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})) = \infty\) in general, one cannot show directly that \(\mathbb{K}_1\) is finite dimensional. To deal with this problem, we prove in section 6.1 that for any \(u \in \ker(D|_{L^2_2(\mathcal{S}_{0,0},^e)})\), it always can be written as

\[
u = \left( \begin{array}{c} u^+ \\ u^- \end{array} \right) + \text{higher order term}
\]
with \( u \rightarrow u^+ \) being a Fredholm operator from \( \ker(D|_{\mathcal{I}_2}) \) to \( L^2(S^1; \mathbb{C}) \). We call \((u^+, u^-) \in L^2(S^1; \mathbb{C}^2)\) the leading term of \( u \). So the vector space of leading terms determined by elements in \( \ker(D|_{L^2(S^1; \mathbb{C})}) \) will be isomorphic to a copy of \( L^2(S^1; \mathbb{C}) \) sitting in \( L^2(S^1; \mathbb{C}^2) \), up to quotients of finite dimensional subspaces determined by the Fredholm operator. Meanwhile, by \( (1.3) \) and the fact \(|d^+|^2 - |d^-|^2 \neq 0\), we will expect that the vector space of leading coefficients of \((1.3)\) is also isomorphic to another copy of \( L^2(S^1; \mathbb{C}) \) in \( L^2(S^1; \mathbb{C}^2) \), up to finite dimensional quotients. We have to prove that these two images of isomorphisms intersect only on a finite dimensional subspace in \( L^2(S^1; \mathbb{C}^2) \), which is \( \mathbb{K}_1 \). That will be the main result in section 6.

Here I shall mention that the computation for the linearization \( \Sigma_p \) will be shown in section 6.2. We will construct \( \mathbb{K}_1 \) to be a space isomorphic to \( \ker(\Sigma_p|_{\delta=0}) \) and \( \mathbb{K}_0 \) to be a space contains \( \text{coker}(\Sigma_p|_{\delta=0}) \).

Finally, in section 7 and section 8, we will derive a particular kind of implicit function theorem to prove our main theorem. Unfortunately, this part is very tedious because there is no standard notation for Kuranishi problems perturbing both the domain \((M - \Sigma)\) and the section \((\psi)\) simultaneously in our way.

## 2. Basic setting and results

### 2.1. Functional spaces.

Let \((M, g)\) be a compact 3-dimensional Riemannian manifold and \( \Sigma \in \mathcal{A} \) be a \( C^1 \)-embedding circle in \( M \). Moreover, we suppose that \( g \) is a product type metric near \( \Sigma \). Namely, there exists \( N_R, \) a small tubular neighborhood of \( \Sigma \) which is parametrized by coordinates \((t, r, \theta, t) \in [0, 2\pi] \times [0, 2\pi] \times [0, R], \) such that \( g|_{N_R} = dt^2 + dr^2 + r^2 d\theta^2 \). We can parametrize \( \Sigma \) by \( t \in [0, 2\pi] \). Also, we use the following notation for cut-off functions: For any \( a, b \) with \( a < b \leq R \), we define a nonnegative smooth function

\[
\chi_{a,b} = \begin{cases} 
0 & \text{on } N_a \\
1 & \text{on } M - N_b 
\end{cases}
\]

with \( |\nabla(\chi_{a,b})| \leq \frac{C}{\pi - a} \) for a universal constant \( C \). We will use this notation many times in our paper.

Let \( \mathcal{S} \) be a spinor bundle over \( M \) with respect to \( g \) and \( \mathcal{I} \) be a real line bundle defined on \( M - \Sigma \). We suppose that \( \mathcal{I} \) cannot be extended to the entire manifold \( M \), which means \( \mathcal{I}|_{t=a, r=b} \simeq [0, 2\pi] \times \mathbb{R}/\{(0, x) \sim (2\pi, -x) \} \) for all \( a \in [0, 2\pi] \) and \( 0 < b < R \). We also fix an inner product on \( \mathcal{I} \). So we define \(|v \otimes w| = |v||w|\) for any \((v, w) \in \mathcal{S} \otimes \mathcal{I}\).

\( \mathcal{S} \) itself is equipped with the standard connection \( \nabla^\mathcal{S} \), see [4]. By the inner product defined on \( \mathcal{I} \), there exists a unique connection \( \nabla^\mathcal{I} \) defined on \( \mathcal{I} \) which is compatible with this inner product; i.e. \( X(s_1, s_2) = \langle \nabla^\mathcal{I}_X s_1, s_2 \rangle + \langle s_1, \nabla^\mathcal{I}_X s_2 \rangle \) for any vector field \( X \) on \( M \) and any smooth section \( s_1, s_2 \) on \( \mathcal{I} \). We define the connection \( \nabla^{\mathcal{S} \otimes \mathcal{I}} = \nabla^\mathcal{S} \otimes id_\mathcal{I} + id_\mathcal{S} \otimes \nabla^\mathcal{I} \) on the bundle \( \mathcal{S} \otimes \mathcal{I} \).

With the norm and the connection defined, one can define the following functional spaces.
Definition 2.1. Let \( u \in C^\infty(M - \Sigma, \mathcal{S} \otimes \mathcal{I}) \) be a smooth section of \( \mathcal{S} \otimes \mathcal{I} \). We define the following norms and corresponding spaces:

1. \( \|u\|_{L^2_i} = (\int_{M - \Sigma} |u|^2)^{1/2} \);
2. \( \|u\|_{L^2_i} = (\int_{M - \Sigma} |u|^2 + |
abla u|^2)^{1/2} \);
3. \( \|u\|_{L^2_{i,\text{cpt}}} = \sup \{ \int_{M - \Sigma} \langle \nabla u, v \rangle | v |_{L^2_i} \leq 1 \} \).

Moreover, the spaces of sections bounded with respect to these norms will be denoted by

\[ L^2_i(M - \Sigma; \mathcal{S} \otimes \mathcal{I}) = \text{closure of } \{ u \in C^\infty(M - \Sigma, \mathcal{S} \otimes \mathcal{I}) \mid \|u\|_{L^2_i} \leq \infty \} \]

for \( i = 1, 0, -1 \). In the following paragraphs, we simply use the notation \( L^2_i \) to denote \( L^2_i(M - \Sigma; \mathcal{S} \otimes \mathcal{I}) \) and usually omit the subscript \( i \) when it is zero.

Similarly, we can define the space of compactly supported sections, \( L^2_{i,\text{cpt}} \), by taking the closure of the set of smooth, compactly supported sections with respect to the norm \( \| \cdot \|_{L^2_i} \).

Remark 2.2. We should always remember that the space \( L^2_{-1} \) is the dual space of \( L^2_1 \) in our case. In a general open domain \( \Omega \) on \( \mathbb{R}^n \), the notation \( L^2_{-1}(\Omega) \) usually denotes the dual space of \( L^2_{1,\text{cpt}}(\Omega) \). The advantage of taking dual of \( L^2_{1,\text{cpt}}(\Omega) \) is the following: We can ”differentiate” an \( L^2(\Omega) \) function formally by coupling it with \( L^2_{1,\text{cpt}} \) sections. This gives us a functional defined on \( L^2_{1,\text{cpt}} \). Then compactly supported inputs of this functional allow us doing integration by parts formally without having the boundary term. However, we will see that the dual spaces of \( L^2_1 \) and \( L^2_{1,\text{cpt}} \) are the same by Lemma 2.6 below. Therefore, our definition is consist with the usual one.

The space \( L^2_{-1} \) has the following property. This is an analog version of Theorem 1 in section 5.9 of [7].

Proposition 2.3. Let \( f \in L^2_{-1} \). Then there exists a pair

\[ (f_0, f_1) \in L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I}) \times L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I} \otimes T^*M) \]

such that

\[ \int_{M - \Sigma} \langle \nabla f, v \rangle = \int_{M - \Sigma} \langle \nabla f_0, v \rangle + \langle \nabla v, f_1 \rangle \]

for all \( v \in L^2_1 \). Furthermore, we have

\[ \|f\|_{L^2_{-1}} = \left( \int_{M - \Sigma} |f_0|^2 + |f_1|^2 \right)^{1/2}. \]

Proof. Let \( T_f : L^2_1 \to \mathbb{C} \) be a bounded functional sending each \( v \) to \( \int_{M - \Sigma} \langle v, f \rangle \). By Riesz Representation Theorem, there exists \( u \in L^2_1 \) such that

\[ T_f(v) = \int_{M - \Sigma} \langle \nabla v, \nabla u \rangle. \]

So we can simply take \( f_0 = u \) and \( f_1 = \nabla u \).

To prove the second part, by taking \( v = u \) in (2.3), we have

\[ \|u\|_{L^2_1}^2 = T_f(u) \leq \|u\|_{L^2_i} \|f\|_{L^2_{-1}}, \]

for \( i = 1, 0, -1 \).
This inequality implies that \((\int_{M-\Sigma} |f_0|^2 + |f_1|^2)^{\frac{1}{2}} = \|u\|_{L^2_1} \leq \|f\|_{L^2_1}\).

Meanwhile, from (2.3) we have
\[
|T_i(v)| \leq \left( \int_{M-\Sigma} |f_0|^2 + |f_1|^2 \right)^{\frac{1}{2}} \|
if \|v\|_{L^2_1} \leq 1. So by Definition 2.1, we have
\[
\|f\|_{L^2_1} \leq \left( \int_{M-\Sigma} |f_0|^2 + |f_1|^2 \right)^{\frac{1}{2}}.
\]
\(\square\)

2.2. Some analytical properties of Dirac operators on \(M - \Sigma\).

We prove the following basic properties in this section. These are very similar to some well-known results in [4].

**Proposition 2.4.** Let \(D|_{L^2_1} : L^2_1 \to L^2\) be the Dirac operator. Then we have the following properties:
1. \(\ker(D|_{L^2_1})\) is finite dimensional.
2. \(\text{range}(D|_{L^2_1})\) is closed.
3. Suppose we write the adjoint of \(D|_{L^2_1}\) to be \(D|_{L^2}\), then we have
\[
L^2 = \text{range}(D|_{L^2_1}) \oplus \ker(D|_{L^2_1}).
\]

**Remark 2.5.** \(\ker(D|_{L^2})\) is not finite dimensional in general.

To prove this proposition, we need the following lemma, which is also very useful in the rest of this article.

**Lemma 2.6.** For any \(u \in L^2_1\), we have
\[
\int_{N_r} |u|^2 \leq 4\pi^2 r^2 \int_{N_r} |\nabla u|^2
\]
for all \(r \leq R\).

**Proof.** Let \(u \in L^2_1\) and \(\{u_n\}\) be a sequence of smooth sections such that
\[
u_n \to u
\]
in \(L^2_1\) sense. Since \(I\) is nontrivial along \(\theta\) direction, we have
\[
|u_n(r,s,t)| \leq \left| \int_0^{2\pi} \partial_\theta |u_n(r,\theta,t)|d\theta \right|
\leq \int_0^{2\pi} |\nabla e_2 u_n(r,\theta,t)|rd\theta
\leq \sqrt{2\pi r^2 \left( \int_0^{2\pi} |\nabla e_2 u_n(r,\theta,t)|^2 rd\theta \right)^{\frac{1}{2}}}
\]
for any \(s,t \in [0,2\pi]\), \(0 < r \leq R\), where \(e_2 = \frac{1}{r} \partial_\theta\). So we have
\[
\int_{N_r} |u_n|^2 \leq \int_0^{2\pi} \int_0^{2\pi} |u_n(r,s,t)|^2 rds dt dr
\leq 4\pi^2 r^2 \int_{N_r} |\nabla e_2 u_n|^2.
\]
By taking \(n \to \infty\), we prove this lemma. \(\square\)
Proof. (of Proposition 2.4)
First of all, for any \( u \in L^2_1 \), one can write the Lichnerowicz-Weitzenböck formula
\[ D^2 u = \Delta u + \frac{\mathcal{R}}{4} u \]
in the following sense:
\[
\int \langle D\zeta, D u \rangle = \int \langle \nabla \zeta, \nabla u \rangle + \int \frac{\mathcal{R}}{4} \langle \zeta, u \rangle
\]  
for all \( \zeta \in L^2_1, \text{cpt} \). Here \( \mathcal{R} \) is the scalar curvature of \( M \). We should prove that (2.4) is true for all \( \zeta \in L^2_1 \).

By Lemma 2.6, we have
\[
\int_{N_r} |\zeta|^2 \leq 4\pi^2 r^2 \int_{N_r} |\nabla \zeta|^2
\]
for all \( \zeta \in L^2_1 \). Let us denote \( \int_{N_r} |\nabla \zeta|^2 = f(\zeta) \). We have \( f(\zeta) \to 0 \) as \( r \to 0 \).

We now take the family of cut-off functions \( \chi_\delta := \chi_{\delta, \delta} \) with \( \|\nabla (\chi_\delta)\| \leq \frac{C}{\delta} \) for \( \delta > 0 \) (Recall the definition (2.1)). So by (2.4), we have
\[
\int \langle D(\chi_\delta \zeta), D u \rangle = \int \langle \nabla (\chi_\delta \zeta), \nabla u \rangle + \int \frac{\mathcal{R}}{4} \langle \chi_\delta \zeta, u \rangle
\]
for all \( \zeta \in L^2_1 \). Clearly the second terms on the right-hand side of (2.6) converges to \( \int \frac{\mathcal{R}}{4} \langle \zeta, u \rangle \) as \( \delta \to 0 \) by Cauchy’s inequality.

For the left-hand side of (2.6), we have
\[
\int \langle D(\chi_\delta \zeta), D u \rangle = \int \chi_\delta \langle D\zeta, D u \rangle + \epsilon.
\]
Because of the inequality (2.5), \( \epsilon \) can be bounded as follows.
\[
|\epsilon| \leq \frac{C}{\delta} \int_{N_\delta} |\zeta| \|\nabla u\| \leq \frac{C}{\delta} \left( \int_{N_\delta} |\zeta|^2 \right)^{\frac{1}{2}} \|D u\|_{L^2} \leq C f(\delta) \|D u\|_{L^2}.
\]
So we have
\[
\int \langle D(\chi_\delta \zeta), D u \rangle \to \int \langle D\zeta, D u \rangle
\]
as \( \delta \to 0 \).

Similarly, we have
\[
\int \langle \nabla (\chi_\delta \zeta), \nabla u \rangle \to \int \langle \nabla \zeta, \nabla u \rangle
\]
as \( \delta \to 0 \). So
\[
\int \langle D\zeta, D u \rangle = \int \langle \nabla \zeta, \nabla u \rangle + \int \frac{\mathcal{R}}{4} \langle \zeta, u \rangle
\]
for all \( \zeta \in L^2_1 \).
Once we have \((2.7)\) for all \(\zeta \in L^2_1\), the proof of Proposition 2.4 will be obtained immediately from the standard argument. Readers can see Chapter 3 and 4 in [4] for details.

So far we prove that \(D|_{L^2_1}\) has closed range and finite dimensional kernel. However the cokernel of \(D|_{L^2_1}\), which is also the kernel of \(D|_{L^2} : L^2 \to L^2_{-1}\), is infinite dimensional in general. In section 3, we will formulate elements in \(\ker(D|_{L^2})\) explicitly in terms of Bessel functions on a tubular neighborhood of \(\Sigma\).

3. Harmonic sections defined on the tubular neighborhood with the Euclidean metric

3.1. \(L^2\) and \(L^2_1\) harmonic sections expressed by modified Bessel functions.

Let us consider the space \(N = \mathbb{R}^2 \times S^1\), which can be regarded as a local model for the tubular neighborhood of \(\Sigma\). The Dirac operator on \(N\) can be written as

\[
D = e_1 \cdot \frac{\partial}{\partial t} + e_2 \cdot \frac{\partial}{\partial x} + e_3 \cdot \frac{\partial}{\partial z}
\]

where

\[
e_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\]

and \(z = x + iy\).

Under the cylindrical coordinate, \(r := \sqrt{x^2 + y^2}\) and \(\theta = \arctan(\frac{y}{x})\), we can write down the Fourier expansion of \(u\) as

\[
u(t, r, \theta) = \sum_{l,k} e^{ilt} \left( e^{i(k-\frac{1}{2})\theta} U^+_{k,l} \quad e^{i(k+\frac{1}{2})\theta} U^-_{k,l} \right)
\]

for any \(C^\infty\)-section \(u\) of the twisted spinor bundle \(S \otimes I\). Here \(k\) runs over all integers and \(l\) can be either in \(\mathbb{Z}\) or \(\mathbb{Z} + \frac{1}{2}\) according to the spin structure we chose (see Chapter 2 in [6]). The Dirac operator can be written in terms of \(\theta, r\) by changing of coordinates:

\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = e^{i\theta} \left( \frac{\partial}{\partial r} - \frac{id}{r \partial \theta} \right); \\
\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} = e^{-i\theta} \left( \frac{\partial}{\partial r} + \frac{id}{r \partial \theta} \right).
\]

Suppose \(u\) is a harmonic section. Then we have

\[
Du = \sum_{l,k} e^{ilt} \left( e^{i(k-\frac{1}{2})\theta} \left( \frac{d}{dr} U^+ + \frac{(k+\frac{1}{2})}{r} U^- \right)_{k,l} \quad e^{i(k+\frac{1}{2})\theta} \left( -\frac{d}{dr} U^- + \frac{(k-\frac{1}{2})}{r} U^+ \right)_{k,l} \right) = 0
\]

which gives us the following system of equations:

\[
\frac{d}{dr} \begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l} = \begin{pmatrix} \frac{1}{r} & -1 \\ -l & -\frac{1}{r} \end{pmatrix} \begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l}.
\]

For \(l \neq 0\), these equations have standard solutions of the form

\[
\begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l} = \begin{pmatrix} u^+_{k,l} (\frac{l}{r}) \frac{1}{k+\frac{1}{2}} I_{k+\frac{1}{2}} (lr) - u^-_{k,l} (\frac{l}{r}) \frac{1}{k-\frac{1}{2}} I_{k-\frac{1}{2}} (lr) \\ -u^+_{k,l} (\frac{l}{r}) \frac{1}{k-\frac{1}{2}} I_{k+\frac{1}{2}} (lr) + u^-_{k,l} (\frac{l}{r}) \frac{1}{k+\frac{1}{2}} I_{k-\frac{1}{2}} (lr) \end{pmatrix}
\]
where
\[ I_p(r) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + p + 1)} \left( \frac{r}{2} \right)^{2m+p} \]
is the modified Bessel function. For the properties of Bessel functions, readers can see [14] for more details.

For \( l = 0 \) we have
\[
\begin{pmatrix}
U^+ \\
U^-
\end{pmatrix}_{k,l} = \begin{pmatrix}
u_{k,0}^+ r^{k-\frac{1}{2}} \\
u_{k,0}^- r^{k-\frac{1}{2}}
\end{pmatrix}.
\]
Clearly we have \( I_p(r) = O(r^p) \). To normalize the leading coefficient of \( I_p(lr) \), we define \( \mathcal{I}_{p,l}(r) = l^{-p} I_p(lr) \).

We now apply these results to sections of \( S \otimes \mathcal{I} \) over \( N \). Fix \( R > 0 \), we simply write \( N_R := N \cap \{ r < R \} \). Suppose \( u \in L^2(N; S \otimes \mathcal{I}) \) and \( Du|_{N_R} = 0 \), then
\[
u = \sum_{k \geq 0; l \neq 0} u_{k,l}^+ e^{ilt} + \sum_{k \geq 0; l \neq 0} u_{k,l}^- e^{ilt} + \sum_{k \leq 0; l \neq 0} u_{k,l}^+ e^{ilt} + \sum_{k \leq 0; l \neq 0} u_{k,l}^- e^{ilt}
\]
which has the leading term of order \( r^{-\frac{1}{2}} \), i.e.
\[
u = \begin{pmatrix}
u_{0,0}^+ e^{-i\frac{\theta}{2} r^{-\frac{1}{2}}} \\
u_{0,0}^- e^{i\frac{\theta}{2} r^{-\frac{1}{2}}}
\end{pmatrix} + \sum_{l \neq 0} e^{ilt} \begin{pmatrix}
u_{0,l}^+ e^{-i\frac{\theta}{2} r^{-\frac{1}{2}}} \\
u_{0,l}^- e^{i\frac{\theta}{2} r^{-\frac{1}{2}}}
\end{pmatrix} + \sum_{l \neq 0} e^{ilt} \begin{pmatrix}-le^{-i\frac{\theta}{2} r^{-\frac{1}{2}}} \\
le^{i\frac{\theta}{2} r^{-\frac{1}{2}}}
\end{pmatrix}
\]
+ higher order terms.

The Bessel functions \( I_{\frac{\theta}{2}}(x) \) and \( L_{\frac{\theta}{2}}(x) \) can be written explicitly as \( \sqrt{\frac{2}{\pi x}} \sinh(x) \) and \( \sqrt{\frac{2}{\pi x}} \cosh(x) \). So the leading term can be expressed in terms of \( \frac{e^{-ir\frac{\theta}{2}}}{\sqrt{\nu}} \) and \( \frac{e^{ir\frac{\theta}{2}}}{\sqrt{\nu}} \).

Let us use this expression, then
\[
u = \sum_{l} e^{ilt} \begin{pmatrix}
u_{0,l}^+ e^{-ir\frac{\theta}{2}} \\
u_{0,l}^- e^{ir\frac{\theta}{2}}
\end{pmatrix} + \sum_{l} e^{ilt} \begin{pmatrix}-le^{-ir\frac{\theta}{2}} \\
le^{ir\frac{\theta}{2}}
\end{pmatrix}
\]
+ higher order terms
where \( \tilde{u}_{0,l}^+ = (u_{0,l}^+ - \text{sign}(l)u_{0,l}^-) \) and \( \tilde{u}_{0,l}^- = (u_{0,l}^+ + \text{sign}(l)u_{0,l}^-) \).

**Definition 3.1.** For any \( R > 0 \) given. Let \( K_R \) be a subspace of \( L^2(N_R; S \otimes \mathcal{I}) \) defined by
\[
K_R = \left\{ u \in L^2(N_R; S \otimes \mathcal{I})| Du = 0 \text{ and } \tilde{u}_{l} = 0 \text{ for all } || > \frac{1}{2R} \right\}.
\]

**Definition 3.2.** Let \( u \) be a harmonic section in \( L^2(N_R; S \otimes \mathcal{I}) \). Then there are the corresponding Fourier coefficients \( \{ u_{l}^\pm \} \). We define the following terminologies:

- We call \( \{ (\tilde{u}_{0,l}^+, \tilde{u}_{0,l}^-), -\text{sign}(l)\tilde{u}_{0,l}^+, +\text{sign}(l)\tilde{u}_{0,l}^-) \} \in \ell^2 \times \ell^2 \) to be the sequence of
leading coefficients of $u$.

- Define $\{(u_+^I, u_-^I)\} \in \ell^2 \times \ell^2$ to be

$$
(u_+^I, u_-^I) = (\hat{u}_{0, l}^+, -\text{sign}(l)\hat{u}_{0, l}^-) \quad \text{for } |l| > \frac{1}{2R}.
$$

$$(u_+^I, u_-^I) = (\hat{u}_{0, l}^+, -\text{sign}(l)\hat{u}_{0, l}^+) + (\hat{u}_{0, l}^-, \text{sign}(l)\hat{u}_{0, l}^-) \quad \text{for } |l| \leq \frac{1}{2R}.
$$

We call $\{(u_+^I, u_-^I)\}$ the sequence of $K_R$-leading coefficients of $u$.

- We call $(\sum_l (\hat{u}_{0, l}^+ + \hat{u}_{0, l}^-)e^{ilt}, \sum_l (-\text{sign}(l)\hat{u}_{0, l}^+ + \text{sign}(l)\hat{u}_{0, l}^-)e^{ilt})$ to be the leading term of $u$.

- Define $u^+(t) = \sum_l u_+^I e^{ilt}$ and $u^-(t) = \sum_l u_-^I e^{ilt}$ where $\{u_+^I\}$ is the sequence of $K_R$-leading coefficients of $u$. We call $u^+(t)$ to be the $K_R$-leading term of $u$.

- We call $(u^+(t)\sqrt{2}, u^-(t)\sqrt{2})$ the $K_R$-dominant term of $u$, where $u^+(t)$ is the $K_R$-leading term of $u$.

Moreover, we can see that if $u \in K_R$, then the (sequence of) $K_R$-leading term (coefficients) will be the (sequence of) leading term (coefficients) of $u$. For $u \in \ker(D_{L^2})$, the sequence of leading coefficients of $u$ can also be regarded as the sequence of $K_0$-leading coefficients of $u$.

When we perturb some $(g, \Sigma, \psi) \in \mathcal{M}$ later, the leading term of $\psi$ plays a crucial rule in the linearization of $\mathcal{M}$. So readers should be familiar with these definitions.

Now if we consider $v \in L^2_1(N_R; \mathcal{S} \otimes \mathcal{I})$ satisfying $Dv = 0$, we will have

$$
v = \sum_{k \geq 1, l \neq 0} v_{k, l}^+ e^{ilt} \left( e^{i(k - \frac{i}{2})\theta} f_{k - \frac{i}{2}, l}(r) - e^{i(k + \frac{i}{2})\theta} f_{k + \frac{i}{2}, l}(r) \right) + \sum_{k \leq -1, l \neq 0} v_{k, l}^- e^{ilt} \left( -e^{i(k - \frac{i}{2})\theta} f_{-k - \frac{i}{2}, l}(r) e^{i(k + \frac{i}{2})\theta} f_{-k + \frac{i}{2}, l}(r) \right) + \sum_{k \geq 1} \left( v_{k, 0}^+ e^{i(k - \frac{i}{2})\theta} r_{k - \frac{i}{2}} \right) + \sum_{k \leq -1} \left( v_{k, 0}^- e^{i(k + \frac{i}{2})\theta} r_{-k - \frac{i}{2}} \right).
$$

So we can write

$$
v = \left( v_{+1,0}^+ e^{\frac{i}{2} \theta} r^\frac{i}{2}, v_{0,0}^- e^{\frac{i}{2} \theta} r^\frac{i}{2} \right) + \sum_{l \neq 0} e^{ilt} \left( v_{+1, l}^+ e^{\frac{i}{2} \theta} f_{\frac{i}{2}, l}(r), v_{+1, l}^- e^{\frac{i}{2} \theta} f_{\frac{i}{2}, l}(r) \right) + \text{higher order terms}.
$$

Again, we define leading coefficients and the leading term for $v$.

**Definition 3.3.** Let $v$ be a harmonic section in $L^2_1(N_R; \mathcal{S} \otimes \mathcal{I})$.

- We call the Fourier coefficients, $\{(v_{-1, l}^+, v_{1, l}^-)\}$, denoted by $\{v_{l}^\pm\} \in (\mathbb{C}^2)^2$, to be the sequence of leading coefficients of $v$.

- We define $v^+(t) = \sum_l v_{l}^+ e^{ilt}$ and $v^-(t) = \sum_l v_{l}^- e^{ilt}$, to be the leading term of $v$.

- We call $v^+(t)\sqrt{2}, v^-(t)\sqrt{2}$ the dominant term of $v$.

In the rest of this paper, we always use letters of Fraktur script, $u, v, h, c$, etc., to denote the sections defined on $L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$ or $L^2_1(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$. If they satisfy the Dirac equation on $N_R$ for some $R > 0$, their corresponding sequences of $(K_R)$-leading coefficients will be denoted by letters of normal script $\{u_l^\pm\}, \{v_l^\pm\}, \{h_l^\pm\}, \{c_l^\pm\}$, etc. which are in $\ell^2 \times \ell^2$. Meanwhile, the corresponding $(K_R)$-leading terms will be denoted by $u^\pm = \sum u_l^\pm e^{ilt}$, $v^\pm, h^\pm, c^\pm$ which
are in $L^2(S^1; \mathbb{C})$. Therefore, we have the $L^2$-norm for $u^\pm$ will be the same as $(\|u_+^1\|_{L^2}^2 + \|u_+^1\|_{L^2}^2)^{1/2}$.

By Definition 3.2 (3.3), any $L^2(L^2_1)$-harmonic spinor $u(v)$ can be decomposed as a sum of a dominant term and a remainder term. In the following proposition, we take care of the regularity estimate for these remainder terms.

**Proposition 3.4.** We have the following two properties.

a. Let $u \in \mathcal{K}_R$, then we can decompose

$$u = \begin{pmatrix} u^+(t) \frac{1}{\sqrt{2}} \\ u^-(t) \frac{1}{\sqrt{2}} \end{pmatrix} + u_R$$

for some $u_R \in L^2(N \mathbb{R}; S \otimes T)$ where $u^\pm(t) = \sum u^\pm_t e^{ilt}$ and

(3.1) $\|u_R\|_{L^2(N \mathbb{R})} \leq CR^{-1}\|u\|_{L^2(N \mathbb{R})}$

for some constant $C$. In the following paragraphs, we call $(u - u_R)$ the $\mathcal{K}_R$-dominant term of $u$ and call $u_R$ the remainder term of $u$.

b. Let $v \in L^2(N \mathbb{R}; S \otimes T)$ and $Dv = 0$, then we can decompose

$$v = \begin{pmatrix} v^+(t) \frac{1}{\sqrt{2}} \\ v^-(t) \frac{1}{\sqrt{2}} \end{pmatrix} + v_R$$

for some $v_R \in L^2(N \mathbb{R}; S \otimes T)$ where $v^\pm(t) = \sum v^\pm_t e^{ilt}$ and

(3.2) $\|v_R\|_{L^2(N \mathbb{R})} \leq CR^{-2}\|v\|_{L^2(N \mathbb{R})}$

for some constant $C$. Similarly, in the following paragraphs, we call $(v - v_R)$ the dominant term of $v$ and call $v_R$ the remainder term of $v$.

**Proof.** (proof of part a). We claim the following two inequalities:

Firstly, we have $D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{2}} \\ u^-(t) \frac{1}{\sqrt{2}} \end{pmatrix} \in L^2(N \mathbb{R})$ and

(3.3) $\left\| D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{2}} \\ u^-(t) \frac{1}{\sqrt{2}} \end{pmatrix} \right\|_{L^2(N \mathbb{R})}^2 \leq CR^{-2}\|u\|_{L^2(N \mathbb{R})}^2$

for some $C > 0$. Secondly,

(3.4) $\left\| \begin{pmatrix} u^+(t) \frac{1}{\sqrt{2}} \\ u^-(t) \frac{1}{\sqrt{2}} \end{pmatrix} \right\|_{L^2(N \mathbb{R})}^2 \leq C\|u\|_{L^2(N \mathbb{R})}^2$

We will prove these inequalities in Corollary 3.6.

We now fix $K > 0$ and define

$$u_{R,K} = \sum_{k \neq 0} \sum_{|\ell| \leq K} e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} U^+_{k,l} \\ e^{i(k+\frac{1}{2})\theta} U^-_{k,l} \end{pmatrix} - \sum_{l \neq 0} \sum_{|\ell| \leq K} e^{ilt} \begin{pmatrix} u^+_l \frac{1}{\sqrt{2}} \\ u^-_l \frac{1}{\sqrt{2}} \end{pmatrix}.$$  

We can easily see that $|u_{R,K}| \leq C_K \sqrt{T}$ and $|\nabla u_{R,K}| \leq C_K \frac{1}{\sqrt{T}}$, which means there will be no boundary term when we do the integration by part for the Lichnerowicz-Weitzenböck formula. Let $\chi = 1 - \chi_{\frac{2}{5}R,R}$ be a cut-off function. By applying
Lichnerowicz-Weitzenböck formula on \( \chi_{\Omega_1} \) and using (3.3), (3.4) above, we have
\[
(3.5) \quad \|u_{\Omega_1}K\|^2_{L^2_t(N_{\frac{4R}{\varepsilon}})} \leq \|D u_{\Omega_1}K\|^2_{L^2_t(N_{\varepsilon})} + C \frac{1}{R^2} \|u_{\Omega_1}K\|^2_{L^2_t(N_{\varepsilon})}
\]
\[
\leq \left\| D \left( \frac{u^+(t) \sqrt{L}}{v^{-}(t) \sqrt{L}} \right) \right\|^2_{L^2_t(N_{\varepsilon})} + C \frac{1}{R^2} \|u_{\Omega_1}K\|^2_{L^2_t(N_{\varepsilon})}
\]
\[
\leq CR^{-2} \|v\|^2_{L^2_t(N_{\varepsilon})}
\]
for some \( C > 0 \).

By taking \( K \to \infty \) in (3.5), we have
\[
\|u_{\Omega_1}\|^2_{L^2_t(N_{\frac{4R}{\varepsilon}})} \leq CR^{-2} \|v\|^2_{L^2_t(N_{\varepsilon})}.
\]

(proof of part b). Similar to the proof of part a, we claim the following two inequalities which will be proved in Corollary 3.8.
\[
(3.6) \quad \|D \left( \frac{u^+(t) \sqrt{L}}{v^{-}(t) \sqrt{L}} \right) \right\|^2_{L^2_t(N_{\varepsilon})} \leq CR^{-2} \|v\|^2_{L^2_t(N_{\varepsilon})},
\]
\[
(3.7) \quad \| \left( \frac{u^+(t) \sqrt{L}}{v^{-}(t) \sqrt{L}} \right) \right\|^2_{L^2_t(N_{\varepsilon})} \leq C \|v\|^2_{L^2_t(N_{\varepsilon})}.
\]

Fix \( K > 0 \), define
\[
v_{\Omega_1}K = \sum_{k \neq 0} \sum_{|l| \leq K} e^{ilt} \left( \frac{L_{k,l} v^+_k}{L_{k,l} v^-_k} \right) - \sum_{t \neq 0, |l| \leq K} e^{ilt} \left( \frac{v^+_\xi}{v^-_\xi} \right).
\]
We have \( |v_{\Omega_1}K| \leq C_K \sqrt{t} \) and \( |\nabla v_{\Omega_1}K| \leq C_K \sqrt{t} \) and \( |\nabla \nabla v_{\Omega_1}K| \leq C_K \sqrt{t} \). So by applying Lichnerowicz-Weitzenböck formula on \( \chi v_{\Omega_1}K \), we have
\[
(3.8) \quad \|v_{\Omega_1}K\|^2_{L^2_t(N_{\frac{4R}{\varepsilon}})} \leq \|D v_{\Omega_1}K\|^2_{L^2_t(N_{\varepsilon})} + C \frac{1}{R^2} \|v_{\Omega_1}K\|^2_{L^2_t(N_{\varepsilon})}
\]
\[
\leq \left\| D \left( \frac{u^+(t) \sqrt{L}}{v^{-}(t) \sqrt{L}} \right) \right\|^2_{L^2_t(N_{\varepsilon})} + C \frac{1}{R^2} \|v_{\Omega_1}K\|^2_{L^2_t(N_{\varepsilon})}
\]
\[
\leq CR^{-2} \|v\|^2_{L^2_t(N_{\varepsilon})}
\]
for some \( C > 0 \). By taking the limit \( K \to \infty \), we have
\[
\|v_{\Omega_1}\|^2_{L^2_t(N_{\frac{4R}{\varepsilon}})} \leq CR^{-2} \|v\|^2_{L^2_t(N_{\varepsilon})}.
\]

Notice that \( [\nabla, D] = 0 \), so we can use the same argument on \( \nabla v \). Here we need the following inequalities which are also proved in Corollary 3.8.
\[
(3.9) \quad \left\| D \left( \frac{u^+(t) \sqrt{L}}{v^{-}(t) \sqrt{L}} \right) \right\|^2_{L^2_t(N_{\varepsilon})} \leq CR^{-4} \|v\|^2_{L^2_t(N_{\varepsilon})}
\]
and
\[
(3.10) \quad \left\| \left( \frac{u^+(t) \sqrt{L}}{v^{-}(t) \sqrt{L}} \right) \right\|^2_{L^2_t(N_{\varepsilon})} \leq CR^{-2} \|v\|^2_{L^2_t(N_{\varepsilon})}.
\]
So we have
\[
\|v_{r,K}\|_{L^2(N_{2R})}^2 \leq \|D(\nabla v_{r,K})\|_{L^2(N_R)}^2 + C \frac{1}{R^2} \|v_{r,K}\|_{L^2(N_R)}^2
\]
\[
\leq \left\| D\left( \frac{v^+(t) \sqrt{z}}{v^-(t) \sqrt{z}} \right) \right\|_{L^2(N_R)}^2 + C \frac{1}{R^2} \|v_{r,K}\|_{L^2(N_R)}^2
\]
\[
\leq CR^{-4} \|v\|_{L^2(N_R)}^2
\]
for some $C > 0$. By taking the limit $K \to \infty$, we prove this proposition. \qed

3.2. Regularity properties and the asymptotic behavior of $L^2$-harmonic sections on the tubular neighborhood. In this section, we will derive some regularity theorems for harmonic spinors $u \in L^2(N_R; S \otimes \mathcal{I})$. These estimates are similar to the doubling estimate appearing in [12]. Recall that, by standard interior regularity theorem, $u$ is a smooth section on any compact subset of $N_R$. We write
\[
u = \sum_{l,k} e^{ilt} \left( \begin{array}{c} e^{\frac{i}{2} \frac{l}{R^2} \theta} U_{k,l}^+ \\ e^{\frac{i}{2} \frac{l}{R^2} \theta} U_{k,l}^- \end{array} \right)
\]
where
\[
\left( \begin{array}{c} U^+ \\ U^- \end{array} \right)_{k,l} = \left( \begin{array}{c} u_{k,l}^+ \mathcal{J}_{k-l}^+(r) - u_{k,l}^- \mathcal{J}_{k-l}^-(r) \\ -u_{k,l}^+ \mathcal{J}_{k-l}^+(r) + u_{k,l}^- \mathcal{J}_{k-l}^-(r) \end{array} \right)
\]
for $l \neq 0$ and
\[
\left( \begin{array}{c} U^+ \\ U^- \end{array} \right)_{k,0} = \left( \begin{array}{c} u_{k,0}^+ r^{k-rac{1}{2}} \\ -u_{k,0}^- r^{k-rac{1}{2}} \end{array} \right).
\]

Since $u \in L^2$, so we have
\[
u_{k,l}^+ = 0 \text{ for } k \leq -1;
\]
\[
u_{k,l}^- = 0 \text{ for } k \geq 1.
\]
Moreover, let us define
\[
E_{k,l} = \left\{ e^{ilt} \left( \begin{array}{c} u_{k,l}^+ e^{\frac{i}{2} \frac{l}{R^2} \theta} \mathcal{J}_{k-l}^+(r) - u_{k,l}^- e^{\frac{i}{2} \frac{l}{R^2} \theta} \mathcal{J}_{k-l}^-(r) \\ -u_{k,l}^+ e^{-\frac{i}{2} \frac{l}{R^2} \theta} \mathcal{J}_{k+l}^+(r) + u_{k,l}^- e^{-\frac{i}{2} \frac{l}{R^2} \theta} \mathcal{J}_{k+l}^-(r) \end{array} \right) \in L^2 \right\},
\]
then $E_{k,l} \perp E_{k',l'}$ for any two couples $(k, l) \neq (k', l')$.

By using these observations, we can prove the following proposition.

**Proposition 3.5.** Let $u \in L^2(N_R; S \otimes \mathcal{I}) \cap \ker(D)$ with the corresponding Fourier coefficients $\{u_{k,l}^\pm\}$. Then the sequence of $\mathcal{K}_R$-leading coefficients $\{u_{k,l}^\pm\}$ is in $L^2_{2k}$ for all $k \in \mathbb{N}$. Moreover, we have
\[
\|(l^k u_{k,l}^\pm)_{l \in \mathbb{Z}}\|^2_{L^2} \leq \frac{3(2k+1)!}{R^{2k+1}} \|u\|^2_{L^2}.
\]

**Proof.** First of all, let $P_{k,l}: \ker(D)_{L^2} \to E_{k,l}$ be the orthonormal projection. We have
\[
P_{0,l}(u) = e^{ilt} \left( \begin{array}{c} \hat{u}_{0,l}^+ \frac{e^{ilr}}{\sqrt{z}} + \hat{u}_{0,l}^- \frac{e^{-ilr}}{\sqrt{z}} \\ -\text{sign}(l) \hat{u}_{0,l}^+ \frac{e^{ilr}}{\sqrt{z}} + \text{sign}(l) \hat{u}_{0,l}^- \frac{e^{-ilr}}{\sqrt{z}} \end{array} \right)
\]
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for any $l$.

Recall that $(u^+_l, u^-_l) = (\hat{u}^+_0, -\text{sign}(l)\hat{u}^+_0)$ for $|l| > \frac{1}{2R}$ and $(u^+_l, u^-_l) = (\hat{u}^+_0, -\text{sign}(l)\hat{u}^+_0) + (\hat{u}^-_0, \text{sign}(l)\hat{u}^-_0)$ for $|l| \leq \frac{1}{2R}$. We can compute directly to get

$$\|u\|^2_{L^2(N_R)} \geq \sum_l |P_{0,l}(u)|^2$$

$$\geq \sum_l |\hat{u}^+_0|^2 \int_0^R e^{2|l|r} \, dr + \sum_l |\hat{u}^-_0|^2 \int_0^R e^{-2|l|r} \, dr$$

$$\geq \sum_l |\hat{u}^+_0|^2 \int_0^R e^{2|l|r} \, dr$$

$$\geq \sum_k \sum_l |\hat{u}^+_0|^2 \frac{(2l)^{2k+1}}{(2k+1)!}.$$ 

Meanwhile, the second line of this inequality also tells us that

$$\|u\|^2_{L^2(N_R)} \geq \sum_l |\hat{u}^-_0|^2 \int_0^R e^{-2|l|r} \, dr$$

$$\geq \sum_{|l| \leq \frac{1}{2R}} e^{-1}|\hat{u}^-_0|^2 R$$

$$\geq \sum_{|l| \leq \frac{1}{2R}} e^{-1}|\hat{u}^-_0|^2 |l|^{2k} R^{2k+1}.$$ 

So we prove (3.12).

By using this proposition, we can prove (3.3) and (3.4) in the following way.

**Corollary 3.6.** Suppose that \( \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{z}} \end{pmatrix} \) is the $K_R$-dominant term of an $L^2$-harmonic section $u$ as we showed in Proposition 3.4, then

$$\left\| D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{z}} \end{pmatrix} \right\|_{L^2(N_R)}^2 \leq CR^{-2} \|u\|^2_{L^2(N_R)}$$

and

$$\left\| \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{z}} \end{pmatrix} \right\|_{L^2(N_R)}^2 \leq C\|u\|^2_{L^2(N_R)}$$

for some constant $C > 0$.

**Proof.** We can compute directly that

$$D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{z}} \end{pmatrix} = \begin{pmatrix} \dot{u}^+(t) \frac{1}{\sqrt{z}} \\ \dot{u}^-(t) \frac{1}{\sqrt{z}} \end{pmatrix}.$$ 

Then by Proposition 3.5, we can prove this corollary immediately. \qed
3.3. Regularity properties and the asymptotic behavior of $L^2$-harmonic sections on the tubular neighborhood. Suppose that $v$ is an $L^2$-harmonic section, then we can write

$$v = \sum_{l,k} e^{ilt} \left( \frac{e^{(k+\frac{1}{2})\theta} P^+_{k,l}}{e^{(k+\frac{1}{2})\theta} P^-_{k,l}} \right)$$

where

$$\begin{pmatrix} V^+ \\ V^- \end{pmatrix}_{k,l} = \begin{pmatrix} v^+_{k,l} \mathcal{J}_{k-\frac{1}{2},l}(r) - v^-_{k,l} \mathcal{I}_{k+\frac{1}{2},l}(r) \\ -v^+_{k,l} \mathcal{I}_{k+\frac{1}{2},l}(r) + v^-_{k,l} \mathcal{J}_{k-\frac{1}{2},l}(r) \end{pmatrix}$$

for $l \neq 0$ and

$$\begin{pmatrix} V^+ \\ V^- \end{pmatrix}_{k,0} = \begin{pmatrix} v^+_{k,0} r^{k-\frac{1}{2}} \\ v^-_{k,0} r^{k-\frac{1}{2}} \end{pmatrix}.$$

Since $v \in L^2$, so we have

$$v^+_l = 0 \text{ for } k \leq 0;$$
$$v^-_l = 0 \text{ for } k \geq 0.$$

**Proposition 3.7.** Let $v \in L^2(N_R; S \otimes I) \cap \ker(D)$ with the corresponding coefficients $\{v^+_l\}$. Then the sequence of leading coefficients $\{(v^+_l)\}$ defined in Definition 3.3 is in $l^2_k$ for all $k \in \mathbb{N} \cup \{0\}$. Moreover, we have

$$(3.13) \| (l^k v^+_l)_{l \in \mathbb{N}} \|_{l^2_k}^2 \leq \frac{2k+3}{R^{2k+3}} \|v\|_{L^2}^2$$

**Proof.** We use the same notations defined in Proposition 3.5.

$$P_{-1,l} = \begin{pmatrix} -v^-_{1,l} \mathcal{J}_{l,\frac{1}{2}}(r) \\ v^-_{1,l} \mathcal{I}_{l,\frac{1}{2}}(r) \end{pmatrix}, \quad P_{1,l} = \begin{pmatrix} v^+_{1,l} \mathcal{J}_{l,\frac{1}{2}}(r) \\ -v^+_{1,l} \mathcal{I}_{l,\frac{1}{2}}(r) \end{pmatrix}.$$

for $l \neq 0$ and

$$P_{-1,0} = \begin{pmatrix} 0 \\ v^-_{1,0} r^{\frac{1}{2}} \end{pmatrix}, \quad P_{1,0} = \begin{pmatrix} v^+_{1,0} r^{\frac{1}{2}} \\ 0 \end{pmatrix}.$$

Since $\mathcal{J}_{l,\frac{1}{2}} = \frac{\sin klr}{l^{1/4} r}$, we have

$$\|v\|_{L^2}^2 \geq \sum_{l \neq 0} \{(v^+_l)^2 + |v^-_l|^2\} \int_0^R \frac{\sin^2 klr}{l^2} dr + \left( |v^+_{1,0}|^2 + |v^-_{1,0}|^2 \right) \int_0^R r^2 dr$$

$$\geq \sum_{l \neq 0} \{(v^+_l)^2 + |v^-_l|^2\} \sum_{k=0}^{\infty} \frac{\ell_k^2 R^{2k+3}}{(k+3)!}$$

$$= \sum_{l} |v^+_l|^2 \sum_{k=0}^{\infty} \frac{\ell_k^2 R^{2k+3}}{(2k+3)!}$$

Therefore, we prove this proposition. \(\square\)

The proof of the following corollary is similar to the proof of Corollary 3.6. So we omit the proof for this corollary.
Corollary 3.8. Suppose \( \left( \frac{v^+(t)\sqrt{z}}{v^-(t)\sqrt{z}} \right) \) is the dominant term of an \( L^2 \)-harmonic section \( v \) as we showed in Proposition 3.4, then we have

a. \[
\left\| D \left( \frac{v^+(t)\sqrt{z}}{v^-(t)\sqrt{z}} \right) \right\|_{L^2(N_R)}^2 \leq CR^{-2} \|v\|_{L^2(N_R)}^2
\]
and \[
\left\| \left( \frac{v^+(t)\sqrt{z}}{v^-(t)\sqrt{z}} \right) \right\|_{L^2(N_R)}^2 \leq C \|v\|_{L^2(N_R)}^2
\]
for some constant \( C > 0 \).

b. \[
\left\| D \left( \nabla \left( \frac{v^+(t)\sqrt{z}}{v^-(t)\sqrt{z}} \right) \right) \right\|_{L^2(N_R)}^2 \leq CR^{-4} \|v\|_{L^2(N_R)}^2
\]
and \[
\left\| \left( \frac{v^+(t)\sqrt{z}}{v^-(t)\sqrt{z}} \right) \right\|_{L^2(N_R)}^2 \leq CR^{-1} \|v\|_{L^2(N_R)}^2
\]
for some constant \( C > 0 \).

Finally, we can prove the following theorem by using Proposition 3.7 now.

Theorem 3.9. For any \( v \in L^2_1(N_R) \cap \ker(D) \), we have
\[
\|v\|_{L^2(N_R)}^2 \leq r^3 \frac{C}{R^5} \|v\|_{L^2(N_R)}^2.
\]
Moreover, we can also prove that
\[
\|\nabla v\|_{L^2(N_R)}^2 \leq r^3 \frac{C}{R^5} \|v\|_{L^2(N_R)}^2.
\]
for some constant \( C > 0 \) and all \( r \leq \frac{R}{2} \), where \( v_t = \partial_t v \).

Proof. To prove the first statement, we use Lemma 2.6 to get
\[
\|v\|_{L^2(N_r)}^2 \leq C r^2 \|\nabla v\|_{L^2(N_r)}^2
\]
for all \( v \in L^2_1(N_R) \) and \( r < R \).

By Lemma 2.6, Proposition 3.7 and Proposition 3.4 b, we have
\[
\int_{N_r} |v|^2 \leq C r^2 \int_{N_r} |\nabla v|^2 \leq 2C r^2 \int_{N_r} \left( \frac{v^+(t)\sqrt{z}}{v^-(t)\sqrt{z}} \right)^2 + |\nabla v|_{L^2}^2
\]
\[
\leq 2C \frac{r^3}{R^3} \|v\|_{L^2(N_R)}^2 + 2C r^4 \|v\|_{L^2(N_R)}^2
\]
\[
\leq 4C \frac{r^3}{R^3} \|v\|_{L^2(N_R)}^2
\]
for some \( C > 0 \).

To prove the second statement, we notice that by applying Lemma 2.6 on \( v_t \),
\[
\int_{N_r} |v_t|^2 \leq r^2 \int_{N_r} |\nabla v_t|^2 \leq 2r^2 \int_{N_r} \left( \frac{v^+_t(t)\sqrt{z}}{v^-_t(t)\sqrt{z}} \right)^2 + |\nabla (v_t)|_{L^2}^2.
\]
By using Proposition 3.7, we have
\[ r^2 \int_{N_r} \left| \nabla \left( \frac{v^+(t) \sqrt{z}}{v^-(t) \sqrt{z}} \right) \right|^2 \leq 2 \frac{r^3}{R^5} \|v\|_{L^2(N_R)}^2. \]

So we have
\[ \int_{N_r} |v|^2 \leq 2 \frac{r^3}{R^5} \|v\|_{L^2(N_R)}^2 + 2r^2 \|v_{2N}\|_{L^2(N_r)}^2. \]

Then by the first statement proved above and Proposition 3.4 b,
\[ \|v_{2N}\|_{L^2(N_r)}^2 \leq \frac{C}{R^3} \|v\|_{L^2(N_R)}^2 \leq C \frac{r^3}{R^5} \|v\|_{L^2(N_R)}^2. \]

So we prove the second statement.

\[\square\]

Remark 3.10. a. By using this theorem, Proposition 2.3 and Lemma 2.6, we can prove that for any \( v \in L^2_1(N_R) \cap \ker(D) \), we have
\[ \|v\|_{L^2_1(N_r)}^2 \leq \frac{C}{R^5} \|v\|_{L^2(N_R)}^2 \]
for some constant \( C > 0 \).

b. By Proposition 3.7 and the definition of modified Bessel functions, one can prove directly that the remainder term \( v_{2N} \) is bounded by the order \( r^p \) for any \( p \leq \frac{3}{2} \). Similarly, for an \( L^2 \)-harmonic spinor \( u \) with \( \mathcal{K}_R \)-leading coefficients, the remainder term \( u_{2N} \) is bounded by the order \( r^p \) for any \( p \leq \frac{1}{2} \). This can be obtained by Proposition 3.5. This result is also true for \( L^2 \)-harmonic spinors with its sequence of leading coefficients has \( L^2_1 \) bound.

4. Variational formula and perturbation of curves

The previous two sections (section 3.2 and 3.3) give us some analysis tools to handle the perturbation of \( \psi \) later. This section will give us some important analysis tools to deal with the perturbation of the metric \( g \) and \( \Sigma \).

4.1. Variational formula. We should review the following fact about the Sobolev inequality and introduce a modified Poincaré inequality first.

Let \( u \in L^2(M - \Sigma; S \otimes T) \). We have \( |u| \in L^2(M - \Sigma; \mathbb{R}) \). Since \( \Sigma \) is a measure zero subset of \( M \), \( |u| \) can be extended as an \( L^2 \) section over \( M \). Moreover, suppose \( u \) is in \( L^2_1(M - \Sigma; S \otimes T) \), then we will have \( |u| \in L^2_1(M; \mathbb{R}) \).

Now, by Sobolev inequality, we have
\[ \|u\|_{L^6(M; \mathbb{R})} \leq C \|u\|_{L^2_1(M; \mathbb{R})} \]
for some constant \( C > 0 \).

Another important tool we need is the following modified Poincaré inequality.
Lemma 4.1. Let \( u \in L^2_1 \) and \( u \perp \ker(D) \), then we have
\[
\|u\|_{L^2_1} \leq C\|Du\|_{L^2}
\]
for some \( C \) depending only on the volume of \( M \).

Proof. The inequality,
\[
\|u\|_{L^2} \leq C\|Du\|_{L^2},
\]
can be obtained immediately by proving Dirac operator has empty residual spectrum, empty continuous spectrum and has nonnegative first eigenvalue. See Chapter 4 in [4] for the proof. Then, (4.2) can be obtained by (4.3) and (2.7). \( \square \)

Definition 4.2. Let \( \bar{f} \in L^2_{-1} \), we define the functional
\[
E_\bar{f}(u) = \int_{M-\Sigma} |Du|^2 + \langle u, \bar{f} \rangle
\]
for all \( u \in L^2_1 \).

Since \( D \) is self-adjoint, the Euler-Lagrange equation of \( E_\bar{f} \) will be
\[
D^2u = \bar{f}.
\]

Proposition 4.3. Let \( \bar{f} \in L^2_{-1} \) be given. For any \( u \in L^2_1 \cap \ker(D|_{L^2_1})^\perp \), we have
\[
E_\bar{f}(u) \geq \alpha\|Du\|_{L^2}^2 - \beta
\]
for some \( \alpha > 0, \beta \in \mathbb{R} \) (This property is usually called coercive). Moreover, if we consider the admissible set of \( E_\bar{f} \) to be all sections in \( L^2_1 \cap \ker(D|_{L^2_1})^\perp \), then \( E_\bar{f} \) has a unique minimizer.

Proof. The inequality (4.5) can be obtained directly from Proposition 2.3 and Lemma 4.1. So we should only prove that \( E_\bar{f} \) has a unique minimizer in \( L^2_1 \cap \ker(D|_{L^2_1})^\perp \) by using (4.5). Suppose we have a sequence \( \{u_n\} \subset L^2_1 \cap \ker(D|_{L^2_1})^\perp \) such that
\[
\lim_{n \to \infty} E_\bar{f}(u_n) = \inf_{u \in L^2_1 \cap \ker(D|_{L^2_1})^\perp} E_\bar{f}(u).
\]
Let us call \( \inf_{u \in L^2_1 \cap \ker(D|_{L^2_1})^\perp} E_\bar{f}(u) = m \). Then there exists \( n_0 \in \mathbb{N} \) such that
\[
E_\bar{f}(u_n) \leq m + 1
\]
for all \( n > n_0 \). So
\[
\alpha\|Du_n\|_{L^2}^2 - \beta \leq E_\bar{f}(u_n) \leq m + 1
\]
for all \( n > n_0 \). This inequality implies that the sequence \( \{\|Du_n\|_{L^2}\}_{n>n_0} \) is bounded. By Lemma 4.1, \( \{\|u_n\|_{L^2}\} \) is bounded. So a subsequence of \( \{u_n\} \) has a weak limit, say \( u \), which is a minimizer of \( E_\bar{f} \).

Finally, we prove the uniqueness. Suppose we have \( u_a, u_b \) are two minimizers in \( L^2_1 \cap \ker(D|_{L^2_1})^\perp \), then
\[
E_\bar{f}(\frac{u_a + u_b}{2}) = \int \frac{1}{4}(|Du_a + Du_b|^2) + \frac{1}{2}\langle u_a, \bar{f} \rangle + \frac{1}{2}\langle u_b, \bar{f} \rangle
\]
\[
\leq \int \frac{1}{2}|Du_a|^2 + \frac{1}{2}|Du_b|^2 + \frac{1}{2}\langle u_a, \bar{f} \rangle + \frac{1}{2}\langle u_b, \bar{f} \rangle
\]
\[
= m
\]
by Cauchy’s inequality. The equality holds if and only if \( Du_a = Du_b \), which implies \( u_a = u_b \) by Lemma 4.1.

\[\square\]

4.2. Perturbation of \( \Sigma \): local trivialization. In this section, we define some notations and explain the local trivialization of \( \mathcal{E} \) (We follow the notation in section 1). First of all, let \( N_R \) be the tubular neighborhood of \( \Sigma \in \mathcal{A} \). There exists a neighborhood of \( \Sigma \) in \( \mathcal{A} \), say \( \mathcal{V}_{\Sigma} \), such that \( \Sigma' \subset N_{R} \) for all \( \Sigma' \in \mathcal{V}_{\Sigma} \). Therefore, we can parametrize elements in \( \mathcal{V}_{\Sigma} \) by \( \{ \eta : S^1 \to \mathbb{C} \eta \in C^1 \text{ and } \| \eta \|_{C^1} \leq C_R \} \) for some \( C_R \) depending on \( \mathcal{V} \). We map \( \eta \) to \( \{ (t, \eta(t)) \} \equiv \Sigma' \subset N_R \).

Here we choose a variable \( r < \frac{1}{2} \). This variable will also be used in the rest of this paper. Also, we fix a \( T > 1 \) which will be specified in the following sections. We use the notation \( \chi^{(t)} \) for the cut-off function \( 1 - \chi_{\frac{1}{T}r} \) (We will omit the superscript \( (t) \) later, but keep in mind that this function depends on \( t \)).

For each \( (\eta, t) \), we now define the following map

\[
\phi^{(t)} : M - \Sigma \to M - \Sigma';
\]

\[(t, z) \mapsto (t, z + \chi^{(t)}(z)\eta(t))\]

with \( \Sigma' = \{ (\eta(t), t) \} \). This map is a diffeomorphism if \( \| \eta \|_{C^1} \leq C_t \) for some constant \( C_t \) depending on \( t \).

We fix \( g \) for a moment. Recall that the fiber of \( \mathcal{E} \) over \( (g, \Sigma', e) \in \mathcal{X} \times \mathcal{A}_H \) is the space \( L^2_{\Sigma'} (M - \Sigma'; S_{g, \Sigma'}, e) \), which can be identified with \( L^2_{\Sigma'} (M - \Sigma; S_{\phi^{(t)}, g, \Sigma, e}) \). Therefore, for any element \( (g, \Sigma) \in \mathcal{X} \times \mathcal{A}_H \), there exists \( \mathcal{N} \subset \mathcal{X} \times \mathcal{A}_H \), a neighborhood of \( (g, \Sigma) \), such that the bundle \( \mathcal{E}_{|\mathcal{N}} \simeq \pi_1(\mathcal{N}) \times \mathbb{B}_e \times L^2_{\Sigma'} \) where \( L^2_{\Sigma'} \simeq L^2_{\Sigma'} (M - \Sigma; S_{g, \Sigma, e}) \) and \( \mathbb{B}_e = \{ \eta : S^1 \to \mathbb{C} \eta \in C^1 \text{ and } \| \eta \|_{C^1} \leq \varepsilon \} \) for some small \( \varepsilon > 0 \).

By the same token, we have the local trivialization of \( \mathcal{F} \) near \( (g, \Sigma, e) \) to be \( \pi_1(\mathcal{N}) \times \mathbb{B}_e \times L^2 \). The Dirac operator \( D : \mathcal{E} \to \mathcal{F} \) will be a family of first order differential operator mapping from \( \mathbb{B}_e \times L^2_{\Sigma'} \) to \( \mathbb{B}_e \times L^2 \). Therefore, the kernel of the linearization map of \( \mathcal{M} \) (when \( g \) is fixed), \( \mathcal{K}_1 \), will be contained in \( \mathcal{V} \times L^2_{\Sigma'} \) where

\[ \mathcal{V} = \{ \eta : S^1 \to \mathbb{C} \eta \in C^1 \}. \]

By Proposition 2.4, we know the projection of \( \mathcal{K}_1 \) on the second factor, \( L^2_{\Sigma'} \), is finite dimensional. We will prove that the projection of \( \mathcal{K}_1 \) on \( \mathcal{V} \) is also finite dimensional in section 6.

4.3. Perturbation of \( \Sigma \): estimates. Recall that we assume the product metric being defined on \( N_R \), which is \( g_{N_R} = dt^2 + dv^2 + r^2 d\theta \). In the following sections, we choose a positive constant \( r < \frac{1}{4} \). The precise value of \( r \) can be assumed to decrease between each successive appearance. Also, we fix a \( T > 1 \) which will be specified in the following sections.
Consider a pair \((\chi, \eta)\) where \(\eta \in C^\infty(S^1; \mathbb{C})\) (here \(\chi = \chi^{(r)}\)). We can define the corresponding one-parameter family of diffeomorphisms
\[
\phi_s : M - \Sigma \rightarrow M - \Sigma_s;
\]
(4.7)
\[
(t, z) \mapsto (t, z + s\chi(z)\eta(t)) \quad \text{on} \quad N_R,
\]
\[
\phi_s(p) = p \quad \text{for all} \quad p \in M - N_R
\]
with \(0 \leq s \leq t_0\) for some small \(t_0\) and \(\Sigma_s = \{(t, s\eta(t))\}\). We fix a positive \(s \leq t_0\) and use \((\tau, u)\) to denote the coordinate on \(\phi_s(N_R)\) in the following paragraphs.

If we write down the relationship of \(\partial_\tau, \partial_z\) and \(\partial_{\bar{z}}\) and the pull-back tangent vectors \((\phi_s)^* (\partial_\tau), (\phi_s)^* (\partial_z)\) and \((\phi_s)^* (\partial_{\bar{z}})\),
\[
\begin{aligned}
\partial_\tau &= (\phi_s)^* (\partial_\tau) + \frac{\partial u}{\partial t} \partial_u + \frac{\partial \bar{u}}{\partial t} \partial_{\bar{u}} \\
\partial_z &= (\phi_s)^* (\partial_z) + \frac{\partial u}{\partial z} \partial_u + \frac{\partial \bar{u}}{\partial z} \partial_{\bar{u}} \\
\partial_{\bar{z}} &= (\phi_s)^* (\partial_{\bar{z}}) + \frac{\partial u}{\partial \bar{z}} \partial_u + \frac{\partial \bar{u}}{\partial \bar{z}} \partial_{\bar{u}}
\end{aligned}
\]
we will have
\[
(\phi_s)^* \begin{pmatrix} \partial_\tau \\ \partial_u \\ \partial_{\bar{u}} \end{pmatrix} = M \begin{pmatrix} \partial_\tau \\ \partial_z \\ \partial_{\bar{z}} \end{pmatrix}
\]
where
\[
M = \frac{1}{1 + s(\chi \eta + \chi \bar{\eta})} \begin{pmatrix} 1 + s(\chi \eta + \chi \bar{\eta}) & 0 & 0 \\ -s\chi \eta - s^2 \chi \bar{\eta}(\eta \bar{\eta} - \bar{\eta} \eta) & 1 + s\chi \bar{\eta} & -s\chi \eta \\ -s\chi \bar{\eta} - s^2 \chi \eta(\eta \bar{\eta} - \bar{\eta} \eta) & -s\chi \eta & 1 + s\chi \eta \end{pmatrix}
\]
(We use \(\bar{\eta}\) to denote \(\eta_t\) when an equation is complicated).

Since the metric and spinor bundle are fixed over \(M\) here, so the Clifford multiplication \(\kappa : TM \rightarrow Cl(TM)\) will always send \(\partial_\tau, \partial_u, \partial_{\bar{u}}\) to 
\[
e_1 = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
respectively. Therefore, the Dirac operator \(D_s\) defined on \(\phi_s(N_R)\) will be
\[
D_s = e_1 \cdot \partial_\tau + e_2 \cdot \partial_u + e_3 \cdot \partial_{\bar{u}} + \frac{1}{2} \sum_{i=1}^{3} \sum_{k,l} \omega_{kl}(e_i) e_k e_l
\]
where \(\omega_{kl}\) is the forms defining the Levi-Civita connection.

In the following sections, all these perturbed curves will be identified with \(\Sigma\) by using the pull-back operator \((\phi_s)^*\). So we have to write down the corresponding Dirac operator explicitly
\[
D_{s\chi \eta} = (\phi_s)^* \circ D_s = e_1 \cdot (\phi_s)^* (\partial_\tau) + e_2 \cdot (\phi_s)^* (\partial_u) + e_3 \cdot (\phi_s)^* (\partial_{\bar{u}})
\]
\[
+ \frac{1}{2} \sum_{i=1}^{3} \sum_{k,l} (\phi_s)^* (\omega_{kl}(e_i)) e_k e_l.
\]
We can see that, after some standard computation,
\[(4.8) \quad (\phi_s)^* (\omega_{kl}(e_i)) = \mathcal{M}(\omega(e_i))\mathcal{M}^{-1} + (d\mathcal{M})\mathcal{M}^{-1} = (d\mathcal{M})\mathcal{M}^{-1}. \]
Here we write down precisely the $O(s)$ order term of $\sum_{i=1}^{3} e_i \sum_{k,l} (\phi_s)^* (\omega_{kl}(e_i))e_k e_l$, which is
\[- [(d\mathcal{M})_{11}(e_1)Id + (d\mathcal{M})_{11}(e_2)e_2 + (d\mathcal{M})_{11}(e_3)e_3] \]
\[+ [-(d\mathcal{M})_{12}(e_1)e_2 - (d\mathcal{M})_{13}(e_1)e_3 + (d\mathcal{M})_{23}(e_1)e_1 e_2 e_3 + (d\mathcal{M})_{32}(e_1)e_3 e_2] \]
\[= D(s(\chi_\eta + \chi_\bar{\eta})Id) + D\left(\begin{pmatrix} 0 & s\chi_\eta \\ -s\chi_\eta & 0 \end{pmatrix}\right) := \mathcal{F}_s. \]
So the term \( \frac{1}{2} \sum_{i=1}^{3} e_i \sum_{k,l} (\phi_s)^* (\omega_{kl}(e_i))e_k e_l \) can be expressed as
\[(4.9) \quad \frac{1}{2} \sum_{i=1}^{3} e_i \sum_{k,l} (\phi_s)^* (\omega_{kl}(e_i))e_k e_l = \mathcal{F}_s + \mathcal{A}_s \]
where $\mathcal{F}_s$ is the $O(s)$-zero order differential operator described as above and $\mathcal{A}_s$ is an $O(s^2)$-zero order differential operator.

Meanwhile, suppose that we have the following assumptions: There exist $\kappa_0$ such that
\[(4.10) \quad \|\eta\|_{L^2(S^1)} \leq \kappa_0 t^2, \]
\[(4.11) \quad \|\eta_t\|_{L^2(S^1)} \leq \kappa_0 t, \]
\[(4.12) \quad \|\eta_{tt}\|_{L^2(S^1)} \leq \kappa_0. \]
We will see that these inequalities will imply that there exists $\kappa_1 = O(\kappa_0)$ such that
\[(4.13) \quad \max\{|\chi_\eta|, |\chi_\bar{\eta}|, |\eta_t|\} \leq \gamma_T \kappa_1 t^\frac{1}{2}, \]
\[(4.14) \quad \|\chi_\eta \eta_t\|_{L^2}, \|\chi_\bar{\eta} \eta_t\|_{L^2} \leq \gamma_T \kappa_1 \]
\[(4.15) \quad \|\chi_\eta \eta\|_{L^2}, \|\chi_\bar{\eta} \eta\|_{L^2} \leq \gamma_T^2 \kappa_1, \]
where we denote \( \frac{T}{T-1} \) by $\gamma_T$.

Here we prove (4.13), (4.14) and (4.15). Firstly, notice that by Sobolev inequality, we have $\eta$ is continuous. So
\[
|\eta|^2(t) \leq \frac{1}{2\pi} \int_0^{2\pi} |\eta|^2 + \int_0^{2\pi} |\eta_t|^2 d\theta \]
\[\leq \frac{1}{2\pi} \|\eta\|_{L^2}^2 + 2\|\eta\|_{L^2} \|\eta_t\|_{L^2} \]
\[\leq \frac{1}{2\pi} \kappa_0^2 t^4 + 2\kappa_0^{2} t^3 \]
\[\leq (\frac{1}{2\pi} + 2)\kappa_0^2 t^3. \]
Meanwhile, we have $|\chi_\eta|, |\chi_\bar{\eta}| \leq C\frac{T}{T-1}$. Therefore,
\[
|(|\eta)|_{\eta}\|_{\eta}, |(|\eta)|_{\bar{\eta}}\|_{\bar{\eta}} \leq C\kappa_0 t^\frac{1}{2}. \]
This implies (4.13). The inequality (4.14) can be proved by the fact $|\chi_z|, |\chi_{\bar{z}}| \leq C\gamma_1 \frac{1}{\tau}$; (4.15) can be proved by the fact $|\chi_{zz}|, |\chi_{\bar{z}z}|, |\chi_{\bar{z}\bar{z}}| \leq C\gamma_1^2 \frac{1}{\tau}$ and (4.10).

Under these assumptions, for any $s$ small, we have
\[
\left| \frac{1}{1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})} - 1 \right| \leq 2s\gamma_1 \kappa_1 t \frac{1}{\tau^2}.
\]

We can write $\frac{1}{1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})} = 1 + \varrho_{s\chi \eta}$. Then
\[
(4.16) \quad |\varrho_{s\chi \eta}| \leq 2s\gamma_1 \kappa_1 t \frac{1}{\tau^2}.
\]

For the perturbed Dirac operator $D_{s\chi \eta}$, we have the following proposition.

**Proposition 4.4.** There exists $\kappa_1 = O(\kappa_0)$ depending on $\kappa_0$ with the following significance. The perturbed Dirac operator $D_{s\chi \eta}$ with $\eta$ satisfying (4.10) - (4.12) can be written as follows:
\[
(4.17) \quad D_{s\chi \eta} = (1 + \varrho_{s\chi \eta})D + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})(e_1 \partial_t) + \Theta_s + \mathcal{R}_s + A_s + \mathcal{F}_s
\]

where
- $\Theta_s = [e_1(s\chi \eta \partial_z + s\chi_{\bar{z}} \partial_{\bar{z}}) + e_2(s\chi_z \eta \partial_{\bar{z}} - s\chi_{\bar{z}} \eta \partial_z) + e_3(-s\chi_z \eta \partial_{\bar{z}} + s\chi_{\bar{z}} \eta \partial_z)]$ is a first order differential operator.
- $\mathcal{R}_s : L^2 \to L^2$ is an $O(s^2)$-first order differential operator supported on $N_t - N_\perp$ with its operator norm $\|\mathcal{R}_s\| \leq \gamma_1^2 s^2$.
- $A_s$ is an $O(s^2)$-zero order differential operator supported on $N_t - N_\perp$. Moreover, let us denote $\eta_\nu$ by $\tilde{\eta}$, the vector field defined on $N_\nu$, then
\[
(4.18) \quad \int_{\{t = r_0\}} |A_s|^2 i dV \leq \gamma_1^2 \kappa_1^4 t s^4
\]

for all $r_0 \leq \tau$.
- $\mathcal{F}_s$ is an $O(s)$-zero order differential operator where
\[
(4.19) \quad \mathcal{F}_s = D(s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})Id) + D\left( \begin{pmatrix} 0 & s i \chi \eta \bar{\eta} \\ -s i \chi \eta \bar{\eta} & 0 \end{pmatrix} \right).
\]

**Proof.** By using the conventions defined above, we have
\[
(4.20) \quad \mathcal{M} = (1 + \varrho_{s\chi \eta}) \left[ \begin{array}{ccc} 1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -s \chi \eta \partial_{\bar{z}} & s \chi_{\bar{z}} \eta & -s \chi_{\bar{z}} \eta \\ -s \chi \eta \partial_z & -s \chi_z \eta & s \chi_z \eta \end{array} \right] + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -s^2 \chi \eta (\bar{\eta} + \eta) & 0 & 0 \\ -s^2 \chi_{\bar{z}} (\eta \bar{\eta} - \bar{\eta} \eta) & 0 & 0 \end{array} \right].
\]

Therefore, by (4.9), we can rewrite
\[
(4.21) \quad D_{s\chi \eta} = (1 + \varrho_{s\chi \eta})(D + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})(e_1 \partial_t))
\]
\[
+ (1 + \varrho_{s\chi \eta})[e_1(s\chi \eta \partial_z + s\chi_{\bar{z}} \partial_{\bar{z}}) + e_2(s\chi_z \eta \partial_{\bar{z}} - s\chi_{\bar{z}} \eta \partial_z) + e_3(-s\chi_z \eta \partial_{\bar{z}} + s\chi_{\bar{z}} \eta \partial_z)]
\]
\[
+ \mathcal{R}_s + A_s + \mathcal{F}_s
\]

where
(4.22) \( \tilde{\mathcal{R}}_s := \frac{-1}{1 + s(\chi \eta + \chi \tilde{\eta})} [e_2(s^2 \chi \chi \tilde{z}(\tilde{\eta} \tilde{\eta} - \tilde{\eta} \eta)\partial_t) + e_3(s^2 \chi \chi \tilde{z}(\eta \tilde{\eta} - \eta \tilde{\eta})\partial_t)] \)

is a first order differential operator satisfying \( \|\tilde{\mathcal{R}}_s\| \leq \gamma^2 \kappa^2 s^2 \).

Finally, we define the following two terms for the second term on the right-hand side of (4.21):

\[
\Theta_s = [e_1(s \chi \tilde{\eta} \partial_z + s \chi \tilde{\eta} \partial_z) \\
+ e_2(s \chi \eta \partial_z - s \chi z \tilde{\eta} \partial_z) \\
+ e_3(-s \chi \eta \partial_z + s \chi z \eta \partial_z)]
\]

and

\[
\delta \mathcal{R}^{(1)}_s = \theta_{s \chi \eta} [e_1(s \chi \tilde{\eta} \partial_z + s \chi \tilde{\eta} \partial_z) \\
+ e_2(s \chi \eta \partial_z - s \chi z \tilde{\eta} \partial_z) \\
+ e_3(-s \chi \eta \partial_z + s \chi z \eta \partial_z)]
\]

where \( \delta \mathcal{R}^{(1)}_s \) is an \( O(s^2) \)-first order differential operator. We can also simplify the first term on the right-hand side of (4.21) by writing \( (1 + \theta_{s \chi \eta}) (s \chi \eta + \chi z \tilde{\eta}) (e_1 \partial_t) = s (\chi \eta + \chi z \tilde{\eta}) (e_1 \partial_t) + \delta \mathcal{R}^{(2)}_s \) where \( \delta \mathcal{R}^{(2)}_s \) is also an \( O(s^2) \)-first order differential operator. So we can rewrite (4.21) as the following.

\[
D_{s \chi \eta} = (1 + \theta_{s \chi \eta}) D + s (\chi \eta + \chi z \tilde{\eta})(e_1 \partial_t) + \Theta_s + \mathcal{R}_s + \mathcal{A}_s + \mathcal{F}_s
\]

where \( \mathcal{R}_s = \tilde{\mathcal{R}}_s + \delta \mathcal{R}^{(1)}_s + \delta \mathcal{R}^{(2)}_s \).

To prove the estimate (4.18) for \( \mathcal{A}_s \), we notice that the term \( (dM) M^{-1} \) involves at most the second derivative of \( \chi \) and \( \eta \), which can be estimated by (4.12), (4.14) and (4.15). So we get (4.18) immediately. \( \Box \)

By using conventions of this proposition, we have the following proposition.

**Proposition 4.5.** Let \( \psi \in L^2_\gamma \) be a harmonic section. Then

\[
\|\mathcal{R}_s(\psi)\|_{L^2} \leq C \gamma^2 \kappa^2 s^2
\]

for some constant \( C \) depending on the \( \|\psi\|_{L^2} \). In fact, this estimate is true for any \( \psi \in L^2_\gamma \) which can be expressed as \( \psi = \sqrt{v} \psi(t, \theta, r) \) with \( v \) being a \( C^1 \)-bounded section.

**Proof.** By Proposition 3.4 b, we have \( \psi = \sqrt{v} \psi(t, \theta, r) \) where \( v \) is a \( C^1 \)-bounded section. We write down by the definition:

\[
\mathcal{R}_s = \frac{-1}{1 + s(\chi \eta + \chi z \tilde{\eta})} [e_2(s^2 \chi \chi \tilde{z}(\tilde{\eta} \tilde{\eta} - \tilde{\eta} \eta)\partial_t) + e_3(s^2 \chi \chi \tilde{z}(\eta \tilde{\eta} - \eta \tilde{\eta})\partial_t)] \\
+ \theta_{s \chi \eta} [e_1(s \chi \tilde{\eta} \partial_z + s \chi \tilde{\eta} \partial_z) + e_2(s \chi \eta \partial_z - s \chi z \tilde{\eta} \partial_z) + e_3(-s \chi \eta \partial_z + s \chi z \eta \partial_z)] \\
+ \theta_{s \chi \eta} (s \chi \eta + \chi z \tilde{\eta})(e_1 \partial_t).
\]

By (4.16), we can bound \( \frac{-1}{1 + s(\chi \eta + \chi z \tilde{\eta})} \) by \( 1 + 2s \gamma \kappa \kappa^2 \). Then by using (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15) we notice that every term in \( \mathcal{R}_s \) can be
written as the type \( \sum_{i=1}^{3} s^2 \alpha_i \beta_i \partial_i \) with \((\partial_1, \partial_2, \partial_3) = (\partial_t, \partial_x, \partial_y)\), \(\|\alpha_i\|_{L^\infty} \leq \gamma_T \kappa_1 r^2\)
and
\[
\int_{r=r_0} |\beta_i|^2 i_dV ol(M) \leq \gamma_T \kappa_1 r^2.
\]

So we have
\[
\|R_s(\psi)\|_{L^2} \leq s^2 \|v\|_{C^1} \gamma_T \kappa_2 r^2.
\]

\[\square\]

4.4. Compositions of perturbations: estimates. In this section, we discuss the composition of perturbations and its corresponding Dirac operator. These results will be used in the following sections.

Let \( r < \frac{4}{T}, T > P > 1 \) be fixed for a moment. We consider a sequence \( \{(\chi_i, \eta_i)\} \) satisfying the following conditions:

1. \( \chi_i := 1 - \chi \frac{r}{T}, \frac{r}{T} \) is a cut-off function (Recall the definition (2.1)).
2. There exists \( \kappa_2 > 0 \) such that \( \|\eta_i\|_{L^2(S^1)} \leq \kappa_2 r^2 \)
(4.23)
\[
\|((\eta_i)_t)\|_{L^2(S^1)} \leq \kappa_2 r^2
\]
(4.24)
\[
\|((\eta_i)_{tt})\|_{L^2(S^1)} \leq \kappa_2 r^2
\]
(4.25)
for all \( i \in \mathbb{N} \).

Similar to the argument of (4.13), (4.14) and (4.15), we have the following results
\[
\max\{ |(\chi_i) \eta_i|, |(\chi_i)_t \eta_i|, |(\chi_i)_{tt} \eta_i|, |(\chi_i)_{ttt} \eta_i| \} \leq \gamma_T \kappa_3 r^2
\]
(4.26)
\[
\|((\chi_i) \eta_i)\|_{L^2}, \|((\chi_i)_{tt} \eta_i)\|_{L^2} \leq \gamma_T \kappa_3 r^2
\]
(4.27)
\[
\|((\chi_i)_{ttt} \eta_i)\|_{L^2} \leq \gamma_T \kappa_3 r^2
\]
(4.28)
for some \( \kappa_3 = O(\kappa_2) \). We denote \( \sum_{j=0}^{i} \chi_j \eta_i \) by \( \eta^i \).

As we have shown in previous section, we define the following family of diffeomorphisms
\[
\phi^i_s : M - \Sigma \rightarrow M - \Sigma_s; \quad (t, z) \mapsto (t, z + s \eta^i(t)) \text{ on } N_R,
\]
(4.29)
\[
\phi^i_s(p) = p \text{ for all } p \in M - N_R
\]
with \( 0 \leq s \leq t_0 \) for some small \( t_0 \) and \( \Sigma_s = \{(t, \eta^i(t))\} \). Now fix \( s \), we use \((u, \tau)\) to denote the coordinate on \( \phi^i_s(N_R) \).
The Dirac operator $D_{s\eta}$ on $M - \Sigma$ will be

$$D_{s\eta} = (\phi_s)^i \circ D_s = e_1 \cdot (\phi_s)^i(\partial_r) + e_2 \cdot (\phi_s)^i(\partial_u) + e_3 \cdot (\phi_s)^i(\partial_{\bar{u}}) + \frac{1}{2} \sum_{i=1}^{3} e_i \sum_{k<l} (\phi_s)^i(\omega_{kl}(e_i))e_ke_l.$$  

**Proposition 4.6.** There exists $\kappa_3 = O(\kappa_2)$ depending on $\kappa_2$ with the following significance. The perturbed Dirac operator $D_{s\eta}$ with $\eta$ satisfying (4.23) - (4.25) can be written as follows:

$$D_{s\eta}^{i+1} = (1 + g^{i+1})D_{s\eta} + s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_1\partial_t + \Theta_s^{i+1} + R_s^{i+1} + \hat{A}_s^{i+1} + F_s^{i+1}$$

where

1. $\Theta_s^{i+1}$, the $(\chi, \eta) = (\chi_{i+1}, \eta_{i+1})$ version of $\Theta_s$, is a first order differential operator of order $O(s)$. 
2. $R_s^{i+1} : L^2 \to L^2$ is an $O(s^2)$-first order differential operator supported on $N_{\frac{r}{T}} - N_{\frac{r}{T}+1}$ with its operator norm bounded in the following way:

$$\|R_s^{i+1}\| \leq \gamma_2^2 \kappa_2^2 s^2.$$ 

3. $\hat{A}_s^{i+1}$ is an $O(s^2)$-zero order differential operator. Moreover, let us denote by $\vec{a} = \partial_r$ the vector field defined on $N_R$, then

$$\int_{\{r=r_0\}} |\hat{A}_s^{i+1}|^2 \vec{a} d\text{Vol}(M) \leq \gamma_3^4 \kappa_3 \frac{(i+1)r}{T+1} s^4.$$ 

for all $r_0 \leq \frac{r}{T}$. 
4. $F_s^{i+1}$ is an $O(s)$-zero order differential operator where

$$F_s^{i+1} = D(s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})Id) + D\left( \begin{pmatrix} 0 & s\chi_{i+1}\eta_{i+1} \\ -s\chi_{i+1}\bar{\eta}_{i+1} & 0 \end{pmatrix} \right).$$

**Proof.** We can define the matrix $\mathcal{M}^i$ to be

$$(\phi_s^i)^* \begin{pmatrix} \partial_r \\ \partial_u \\ \partial_{\bar{u}} \end{pmatrix} = \mathcal{M}^i \begin{pmatrix} \partial_r \\ \partial_u \\ \partial_{\bar{u}} \end{pmatrix}.$$ 

Notice that the support of $(\chi_i)z$ and $(\chi_j)\bar{z}$ are disjoint for all $i \neq j$. Therefore, we can write $\mathcal{M}^{i+1}$ as follows

$$\mathcal{M}^{i+1} = \frac{1}{1 + s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})} \mathcal{M}^i + \Lambda^{i+1}.$$
where $\mathcal{N}^{t+1}$ is a $(\chi_{i+1}, \eta_{i+1})$ version of $M$:

$$
\mathcal{N}^{t+1} = \frac{1}{1 + s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})} \begin{pmatrix}
- s\chi_{i+1}\bar{\eta}_{i+1}
- s^2\chi_{i+1}(\chi_{i+1})z(\bar{\eta}_{i+1}\bar{\eta}_{i+1} - \eta_{i+1}\bar{\eta}_{i+1}) - s(\chi_{i+1})z\bar{\eta}_{i+1}
\end{pmatrix}.
$$

Let us define $D_s^{i+1} = (1 + \rho^{i+1})(D_{stp} - \delta_s^i) + s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_i\partial_t$

$$
+ \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + \delta_s^{i+1}
= (1 + \rho^{i+1})D_{st} + s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_i\partial_t
+ \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + [\delta_s^{i+1} - (1 + \rho^{i+1})\delta_s^i].
$$

where $\delta_s^{i+1} = \sum_{j=1}^{3} e_j \sum_{k<j} (\omega_{kl}^{i+1}(e_j))e_k e_l$ with $\omega_{kl}^{i+1}$ being the pull back of Levi-Civita connection $(\phi_s^{i+1})^*(\omega)$. Using these conventions, we have

$$
|\delta_s^{i+1} - (1 + \rho^{i+1})\delta_s^i| = |(dM^{i+1})(M^{i+1})^{-1} - (dM)(M)^{-1} - \rho^{i+1}(dM^{i+1})(M^{i+1})^{-1}|.
$$

Now by using (4.32) and (4.33), we can see that $F_s^{i+1}$ is the $O(s)$ order term of $(dM^{i+1})(M^{i+1})^{-1} - (dM)(M)^{-1}$. Therefore, by using (4.23), (4.24) and (4.26), we have

$$
\int_{r=r_0}^r |(dM^{i+1})(M^{i+1})^{-1} - (dM)(M)^{-1} - F_s^{i+1}|^2 i\bar{r}dVol(M) \leq C\gamma^4 r^2 (\frac{r}{T+1})\kappa^3 s^4
$$

for some universal constant $C$. Therefore, we can choose $\kappa_3 = O(\kappa_2)$ large enough such that the right-hand side of this equation is smaller than $\frac{1}{2}\gamma^4 r^2 (\frac{r}{T+1})\kappa^3 s^4$.

Meanwhile,

$$
\int_{r=r_0}^r |(dM^{i+1})(M^{i+1})^{-1} - (dM)(M)^{-1}|^2 i\bar{r}dVol(M) \leq C\gamma^2 \kappa^2 s^2
$$

for all $j$, so we have

$$
\int_{r=r_0}^r |(dM)(M)^{-1}|^2 i\bar{r}dVol(M) \leq \gamma^2 r^2 (i + 1)\kappa^2 s^2.
$$

Now recall that $|\rho^{i+1}| \leq \gamma s\kappa_3 (\frac{r}{T+1})^{\frac{1}{2}}$. So we have

$$
\int_{r=r_0}^r |\rho^{i+1}(dM^{i+1})(M^{i+1})^{-1}|^2 i\bar{r}dVol(M) \leq \gamma^4 r^2 (i + 1)\kappa_3 s^4.
$$

Therefore, by taking

$$
\delta_s^{i+1} = \delta_s^{i+1} - (1 + \rho^{i+1})\delta_s^i - F_s^{i+1},
$$

we prove this proposition. $\Box$
We have a similar version of Proposition 4.5 as follows.

**Proposition 4.7.** Let \( \psi \in L^2_1 \) be a harmonic section. Then
\[
\|R_s^{1+1}(\psi)\|_{L^2} \leq C \gamma_2^2 \kappa_3^2 (\frac{r}{T^{t+1}})^2 s^2.
\]
for some constant \( C \) depending on the \( \|\psi\|_{L^2} \). In fact, this estimate is true for any \( \psi \in L^2_1 \) which can be expressed as \( \psi = \sqrt{r} v(t, \theta, r) \) where \( v \) is a \( C^1 \)-bounded section.

4.5. **Variational formula for perturbed Dirac operators.** In section 4.1, we prove that there exists a unique minimizer of \( E_j \) in \( L^2_1 \cap \ker (D|_{L^2_1}) \) when the metric \( g \) is Euclidean on a tubular neighborhood of \( \Sigma \). The argument in section 4.1 works not only for \( D \) but also for the perturbed Dirac operator \( D_{\Delta^j} \), \( D_{\Delta^j} \) appearing in section 4.3 and 4.4. However, using the variational method to find the solution \( D_{\Delta^j} u_s = f \) wouldn’t give us enough information about \( u_s \) changing by varying \( s \). Therefore, we will prove the following proposition to clarify this part.

In addition, the minimizer for \( E_j \) will satisfy the equation \( Du = f \) only if \( f \in L^2_{-1} \setminus \ker (D|_{L^2_1}) \). Namely, Proposition 4.3 gives us the following statement: For any \( f \in L^2_{-1} \), there exists \( u \in D(L^2_1 \cap \ker (D|_{L^2_1})) \) such that
\[
Du = f + \text{some elements in ker}(D|_{L^2_1}).
\]

We will use \( \text{mod}(D|_{L^2}) \) to denote ”some elements in \( \ker(D|_{L^2}) \)” in the rest of our paper.

**Proposition 4.8.** For any \( j > 0 \) fixed. Suppose \( f \in L^2_{-1} \) and \( u_0 \in L^2 \) is a section which satisfies
\[
Du_0 = f + \text{mod}(D|_{L^2_1}),
\]
then there exist \( u = u_0 + u^s \) and \( t_0 > 0 \) such that
\[
D_{\Delta^j} u = f + \text{mod}(D|_{L^2_1})
\]
and \( \|u^s\|_{L^2} \leq C(\|u_0\|_{L^2} + \|f\|_{L^2_1})s \) for \( s \in [0, t_0] \) and \( C \) being a universal constant. Furthermore, the existence of \( u_0 \) can be given by Proposition 4.3 or Proposition 6.2 which appears later.

**Proof.** We can assume \( \ker(D|_{L^2_1}) = 0 \) for a moment. The general case follows the same argument as below.

Suppose \( D_{\Delta^j} \) is the perturbed Dirac operator and \( f \in L^2_{-1} \). We want to solve \( u \in L^2 \) satisfying
\[
D_{\Delta^j} u = f.
\]

We solve this equation iteratively. Firstly, we know that the perturbed Dirac operator \( D_{\Delta^j} \) can be written as \( D + \delta^j \) where \( \delta^j : L^2 \to L^2_{-1} \) is a first order differential operator with its operator norm \( \|\delta^j\| \leq Cs \) for some \( C > 0 \). Meanwhile, by Proposition 4.3, there exists \( u_0 \in L^2 \) such that
\[
Du_0 = f.
\]
So we have
\[
D_{\Delta^j} u_0 = f - \delta^j(u_0).
\]
Since \( \|u_0\|_{L^2} \leq C\|f\|_{L^2_{-1}} \), we have \( \|\delta^i_s(u_0)\|_{L^2_{-1}} \leq C_s\|f\|_{L^2_{-1}} \). By taking \( s \) small enough, we have \( \|\delta^i_s(u_0)\|_{L^2_{-1}} \leq \frac{1}{2}\|f\|_{L^2_{-1}} \).

Now we solve \( v_1 \in L^2 \) such that
\[
Dv_1 = \delta_s^i(u_0)
\]
by using Proposition 4.3. Then we have
\[
D_{s\eta^j} (u_0 + v_1) = f + \delta_s^i(v_1)
\]
where \( \|\delta_s^i(v_1)\|_{L^2_{-1}} \leq \frac{1}{2}\|\delta_s^i(u_0)\| \leq \frac{1}{2}\|f\|_{L^2_{-1}} \).

We call \( \delta_s^i(u_0) = z_0, -\delta_s^i(v_1) = z_1 \) and \( u_0 + v_1 = u_1 \). Suppose that we have \((u_i, z_i)\) satisfying
\[
D_{s\eta^j} u_i = f - z_i
\]
with \( \|z_i\|_{L^2_{-1}} \leq \frac{1}{2^{i+2}}\|f\|_{L^2_{-1}} \) for some \( i \in \mathbb{N} \), then we can solve \( v_{i+1} \in L^2 \) which satisfies
\[
Dv_{i+1} = \delta_s^i
\]
by Proposition 4.3. So we have
\[
D_{s\eta^j} (u_i + v_{i+1}) = f + \delta_s^i(v_{i+1})
\]
where \( \|\delta_s^i(v_{i+1})\|_{L^2_{-1}} \leq \frac{1}{2}\|z_i\| \leq \frac{1}{2^{i+2}}\|f\|_{L^2_{-1}} \). By taking \( u_i + v_{i+1} = u_{i+1} \) and \(-\delta_s^i(v_{i+1}) = z_{i+1} \), we can repeat this argument inductively.

Finally, by taking the limit \( i \to \infty \), then we have \( u_{i+1} \to u \) in \( L^2 \)-sense which satisfies
\[
D_{s\eta^j} u = f.
\]
Moreover, since \( u - u_0 = \sum_{i=1}^{\infty} v_i \) and \( Dv_i = (-1)^{(i-1)}\delta_s^i(u_{i-1}) \), we have \( \sum_{i=1}^{\infty} v_i \) is an \( O(s) \)-order \( L^2 \) section. We call \( \sum_{i} v_i = u^s \).

Therefore, \( u_0 + u^s \) satisfies
\[
(4.36) \quad D_{s\eta^j} (u_0 + u^s) = f.
\]

Remark 4.9. In our proof, since we can always write \( \delta_s^i = \sum_{i=1}^{\infty} s^i \delta_s^i \) where the operator norm of \( \delta_s^i \) is bounded uniformly, \( u \) can be written as \( \sum_{i=0}^{\infty} s^i u^{(i)} \).
\[
\| \sum_{i=m}^{\infty} s^i u^{(i)} \|_2 \to 0 \quad \text{as} \quad m \to \infty.
\]

5. The General \( \Sigma \) Embedding in \( M \)

We now try to derive same results as we did in previous section without assuming that \( \Sigma \) has a product type metric on the tubular neighborhood.
5.1. Asymptotic behavior of the $L^2_1$-harmonic section. Let $g$ be a smooth metric and $\Sigma \subset M$ be a $C^1$ curve embedded in $M$. We use the exponential map to send elements in the normal bundle $\{v \in \nu_{\Sigma}| |v| \leq R\}$ to the tubular neighborhood of $\Sigma$ in $M$. We can parametrize this neighborhood by a cylindrical coordinate $(t, r, \theta)$ and $g = dt^2 + dr^2 + r^2d\theta^2 + O(r^2)$. In the following section, we will use $D_{\text{prod}}$ to denote the Dirac operator of the product type metric.

Lemma 5.1. For any $R > 0$ fixed. Let $v \in L^2_1(\mathcal{N}_r; S \otimes \mathcal{I})$ such that $D(v) = 0$, then there exists $v^* \in L^2_1(\mathcal{N}_r; S \otimes \mathcal{I})$ such that $D_{\text{prod}}v^* = 0$ and

\[ v_{r,0} := v - v^* \]

satisfying the following estimate:

\[ \|v_{r,0}\|_{L^2_1(\mathcal{N}_r)} \leq O(r^\frac{1}{2}). \]

Proof. We divide our proof into two parts. 

**Step 1.** Here we set up the strategy of the proof. Firstly, it is clear that we can write $D = D_{\text{prod}} + O(r^2)L_1 + O(r)L_0$ where $L_1$ is a bounded first order differential operator and $L_0$ is a zero order operator, composed by Clifford multiplications.

Secondly, the argument in Lemma 2.6 still works for elements in $L^2_1(\mathcal{N}_r; S \otimes \mathcal{I})$. So by using the equation

\[ D_{\text{prod}}v = O(r^2)L_1(v) + O(r)L_0(v), \]

we have $\|O(r^2)L_1(v) + O(r)L_0(v)\|_{L^2_1(\mathcal{N}_r)} \leq O(a^2)$ for all $a < R$. So we have

\[ D_{\text{prod}}v = f \]

for some $f$ satisfying $\|f\|_{L^2_1(\mathcal{N}_r)} \leq O(a^2)$ for all $a < R$.

We also have the following regularity theorem [11].

Theorem 5.2. Let $R$ be the Atiyah-Patodi-Singer boundary condition for the spinor bundle $S$ on the manifold $X$ with boundary $Y$, then we have for any $v \in L^2_1(X; S)$ we have

\[ \|v\|_{L^2_1(X)} \leq C(\|R(v)\|_{L^2_1(Y)} + \|v\|_{L^2_1(X)} + \|D_{\text{prod}}v\|_{L^2_1(X)}) \]

for some constant $C$.

In our case, $\Sigma$ can be regarded as a degenerated boundary. We take $X = M - N_r$ and $Y = \partial N_r$, then we have

\[ \|v\|_{L^2_1(M - N_r)} \leq C(\|R(v)\|_{L^2_1(\partial N_r)} + \|v\|_{L^2_1(M - N_r)} + \|D_{\text{prod}}v\|_{L^2_1(M - N_r)}). \]

If we take $r$ goes to 0, the boundary term $\|R(v)\|_{L^2_1(\partial N_r)}$ will vanish by Lemma 2.6. So we have

\[ \|v\|_{L^2_1(M - \Sigma)} \leq C(\|v\|_{L^2_1(M - \Sigma)} + \|D_{\text{prod}}v\|_{L^2_1(M - \Sigma)}). \]

Therefore, if we can prove that there exists $v^* \in L^2_1$ such that $D_{\text{prod}}v^* = 0$ and

\[ \left( \int_{r=r_0} |v - v^*|^2 i\eta dVol(M) \right)^\frac{1}{2} = o(r_0^{\frac{1}{2}}), \]
we can prove Lemma 5.1 by using (5.3).

**Step 2.** Here we prove the existence of $v^* \in L^2_r$. To prove this part, we write down the Fourier expression of $v$ on $N_R$ as we have done in section 3.

$$v(t, r, \theta) = \sum_{l, k} e^{ilt} \left( e^{i(k - \frac{l}{2})\theta} V_{k, l}^+ - e^{i(k + \frac{l}{2})\theta} V_{k, l}^- \right).$$

The equation $Dv = 0$ will give us the equation

$$\frac{d}{dr} V_{k, l}^- + \alpha V_{k, l}^+ + \frac{l}{(1 + O(r^2))} V^+ + P_{k, l}^+(f) = 0;$$

$$\frac{d}{dr} V_{k, l}^+ - \beta V_{k, l}^- + \frac{l}{(1 + O(r^2))} V^- + P_{k, l}^-(f) = 0$$

where $P^*$ is the projection mapping to the first component of the Fourier expansion. $\alpha, \beta$ have the form $(k + \frac{1}{2} + O(r^2)) r(1 + O(r^2))$.

Now we find the two nonzero functions $\rho_k^\pm(r)$ by solving the following ODE:

$$\frac{d}{dr} \rho_{k, l}^+ = \alpha \rho_{k, l}^+;$$

$$\frac{d}{dr} \rho_{k, l}^- = -\beta \rho_{k, l}^-.$$

So we have $C_1 r^{(k + \frac{1}{2})} < \rho_{k, l}^+ < C_2 r^{(k + \frac{1}{2})}$ and $C_1 r^{-(k + \frac{1}{2})} < \rho_{k, l}^- < C_2 r^{-(k + \frac{1}{2})}$ for some $C_2 > C_1 > 0$.

Therefore, we have

$$\frac{d}{dr}(\rho_{k, l}^+ V_{k, l}^-) = -\rho_{k, l}^+ \frac{l}{(1 + O(r^2))} V_{k, l}^+ - \rho_{k, l}^+ P_{k, l}^+(v);$$

$$\frac{d}{dr}(\rho_{k, l}^- V_{k, l}^+) = -\rho_{k, l}^- \frac{l}{(1 + O(r^2))} V_{k, l}^- - \rho_{k, l}^- P_{k, l}^-(v);$$

for all $k, l$.

Suppose $k \geq 0$, the integral of (5.4) shows that

$$|\rho_{k, l}^+ V_{k, l}^-(b) - \rho_{k, l}^+ V_{k, l}^-(a)| \leq \int_a^b \rho_{k, l}^+ (|V_{k, l}^+| + |P_{k, l}^+(v)|) \leq (b^{2k+2} - a^{2k+2})^{\frac{1}{2}} \left( \int_a^b O(1) \right)^\frac{1}{2}$$

$$\leq C(b^{2k+2} - a^{2k+2})^{\frac{1}{2}} (b - a)^{\frac{1}{2}}.$$

By using this inequality we have

$$\lim_{r \to 0} \rho_{k, l}^+ V_{k, l}^-(r) = c$$

for some $c \in \mathbb{C}$. $|V_{k, l}^+| > |c| r^{-k - \frac{1}{2}} \geq \frac{|c|}{2} r^{-\frac{1}{2}}$ which is contradictory to Lemma 2.6 if $c \neq 0$. So we have $\lim_{r \to 0} \rho_{k, l}^+ V_{k, l}(r) = 0$.

By taking $a \to 0$ in (5.6), we have

$$C_1 b^{k + \frac{1}{2}} |V_{k, l}^-(b)| \leq |\rho_{k, l}^+ V_{k, l}^-(b)| \leq b^{k + \frac{1}{2}}.$$
So we have
\[ |V_{k,l}^-| (r) \leq n_{k,l} O(r) \tag{5.7} \]
for all \( k \geq 0 \) with \( \sum_{k,l} |n_{k,l}|^2 < \infty \). Similarly, by using the same argument, we can also prove that
\[ |V_{k,l}^+| (r) \leq n_{k,l} O(r) . \tag{5.8} \]
for all \( k \leq 0 \).

For the case \( k = -1 \), by (5.7) we have
\[ \lim_{r \to 0} \rho_{k,l}^+ V_{k,l}^- = c \text{ for some } c \in \mathbb{C}. \]
So we have
\[ V_{-1,l}^- (r) = v_{-1,l} r^{\frac{1}{2}} + o(r^{\frac{1}{2}}). \]
Similarly, we have
\[ V_{1,l}^+ (r) = v_{1,l} r^{\frac{1}{2}} + o(r^{\frac{1}{2}}). \]

For the case \( k < -1 \), if we have
\[ \limsup_{r \to 0} |\rho_{k,l}^+ V_{k,l}^-| (r) = c < \infty, \]
then \( |V_{k,l}^-| (r) \leq cr^{-k-\frac{1}{2}} \leq cr^{\frac{3}{2}} \). On the other hand, if we have
\[ \limsup_{r \to 0} |\rho_{k,l}^+ V_{k,l}^-| (r) = \infty, \]
k < -2 by (5.6) and (5.8). Moreover, (5.6) implies that
\[ |\rho_{k,l}^+ V_{k,l}^- (b) - \rho_{k,l}^+ V_{k,l}^- (a)| \leq C \rho_{k,l}^+ (a) a^2. \]
So
\[ \left| \frac{\rho_{k,l}^+ (b)}{a \rho_{k,l}^+ (a)} V_{k,l}^- (b) - a^{-2} V_{k,l}^- (a) \right| \leq n_{k,l} O(1). \]
Therefore, we have
\[ \limsup_{a \to 0} |a^2 V_{k,l}^- (a)| \leq n_{k,l} O(1) \]
which implies
\[ |V_{k,l}^-| (r) \leq n_{k,l} O(r^2). \]
So we can conclude that
\[ |V_{k,l}^-| (r) \leq n_{k,l} O(r^{\frac{3}{2}}) \]
for all \( k < -1 \). We finish our proof.

Remark 5.3. By the same token, we can also show that elements in \( \ker(D|_{L^2}) \) have a similar decomposition. To be more precisely, for any \( u \in \ker(D|_{L^2}) \), there is a decomposition \( u = u^* + u_{\alpha,0}^* \) such that \( D_{prod}(u^*) = 0 \) and \( |u_{\alpha,0}^*| = o(R^{-\frac{1}{2}}) \).
5.2. **Modify propositions in section 5.** Now we modify results in section 4 without assuming a Euclidean metric on the tubular neighborhood.

First of all, we should set up several notations. Let $N_R$ to be the tubular neighborhood of $\Sigma$, and $D_{\text{prod}}$ to be the Dirac operator with respect to Euclidean metric on $N_R$. We define $D^{(n)} = \chi_n D_{\text{prod}} + (1 - \chi_n)D$, where $\chi_n = 1 - \chi_{\frac{r}{r_1 + 1}}$ is defined in section 4.4. So we have

$$D^{(n)} = D_{\text{prod}}$$

Moreover, we have the following proposition (Here we take $\partial_1 = \partial_r$, $\partial_2 = \partial_\theta$ and $\partial_3 = \partial_t$).

**Proposition 5.4.** Let $(D^{(n)} - D) = \delta^{(n)}$, we have

$$\delta^{(n)} = \delta_1^{(n)} + \delta_0^{(n)}$$

where

- $\delta_1^{(n)}$ is a first order differential operator supported on $N_{\frac{r}{r_1}}$ such that

  $$\delta_1^{(n)} = \sum_{i=1}^{3} a_i \partial_i \text{ with } |a_1| \leq O(r^2) \text{ and } |a_2|, |a_3| \leq O(r).$$

- $\delta_0^{(n)}$ is a zero order differential operator supported on $N_{\frac{r}{r_1}}$ such that

  $$|\delta_0^{(n)}| = O(r).$$

We follow the setting in section 4. Suppose $(\eta_1, \chi_1)$ satisfies (4.10), (4.11), (4.12). We also define

$$\phi_s(t, z) = (t, z + s\eta_1(t)) \text{ on } N_R,$$

$$\phi_s(p) = p \text{ on } M - N_R$$

and

$$D_{s\eta_1} = \sum_{i=1}^{3} e_i \cdot (\phi_s)^* (e_i) + \sum_{i=1}^{3} e_i \sum_{j,k=1}^{3} (\phi_s)^* (w_{jk}) e_j e_k.$$

Then we have the following proposition.

**Proposition 5.5.** The perturbed Dirac operator can be written as

$$D_{s\eta_1} = (1 + \rho^1)D^{(1)} + s((\chi_1)_{\bar{z}}\eta_1 + (\chi_1)_{\bar{\eta}})(e_1 \partial_t) + \Theta_s + \mathcal{R}_s + \mathcal{A}_s + \mathcal{F}_s + \delta^{(1)}.$$

where

- $\Theta_s = [e_1(s\chi\eta\partial_z + s\chi\bar{\eta}\partial_{\bar{z}}) + e_2(s\chi_{\bar{z}}\bar{\eta}\partial_z - s\chi_{\bar{z}}\bar{\eta}\partial_{\bar{z}}) + e_3(-s\chi_{\bar{z}}\eta\partial_z + s\chi_{\bar{z}}\eta\partial_{\bar{z}})]$ is a first order differential operator.

- $\mathcal{R}_s : L^2 \to L^2$ is an $O(s^2)$-first order differential operator supported on $N_{\frac{r}{r_1}} - N_\perp$ with its operator norm $\|\mathcal{R}_s\| \leq \gamma_1^2 \kappa_1^2 s^2$. Moreover, for any $\psi \in L^2 \cap \ker(D)$ we have

  $$\|\mathcal{R}_s(\psi)\|_{L^2} \leq C\gamma_1^2 \kappa_1^2 s^2.$$
for some constant $C$ depending on $\|\psi\|_{L^2_0}$.

- $A_s$ is an $O(s^2)$-zero order differential operator supported on $N_{T} - N_{\tau}$. Moreover, let us denote $\partial_r$ by $\bar{n}$, the vector field defined on $N_R$, then

\begin{equation}
\int_{\{r=r_0\}} |A_s|^2 i_\bar{n}dVol(M) \leq \gamma_s^4 \kappa_1^4 s^4
\end{equation}

for all $r_0 \leq \tau$.

- $F_s$ is an $O(s)$-zero order differential operator where

\begin{equation}
F_s = D(\begin{pmatrix} s(\chi z_\eta + \chi_\bar{z}\bar{\eta})Id & D(\begin{pmatrix} 0 & s\chi_\bar{\eta} \\ -s\chi\bar{n} & 0 \end{pmatrix}) \end{pmatrix}).
\end{equation}

- $\delta^{(1)}$ can be written as $\delta^{(1)} = \delta_0^{(1)} + \delta_1^{(1)}$ where $\delta_1^{(1)}$ is a first order operator with

\begin{equation}
\delta_1^{(1)} = \sum a_i \partial_i \text{ with } |a_i| \leq O(r^2) \text{ and } |a_1|, |a_2| \leq O(r)
\end{equation}

and $\delta_0^{(1)}$ is a zero order operator with $|\delta_0^{(1)}| = O(r)$.

Moreover, $\delta^{(1)}$ is supported on $N_R$.

Similarly, we have a new version of Proposition 4.6. Suppose that we have a sequence of pairs, $\{(\chi_i, \eta_i)\}$, which is defined in section 4.4. Moreover, we suppose that $\eta_i$ satisfies \[\begin{pmatrix} 4.23 \end{pmatrix}, \begin{pmatrix} 4.24 \end{pmatrix}, \begin{pmatrix} 4.25 \end{pmatrix}\] and we write $\eta^i = \sum_{j=0}^{i} \chi_j \eta_j$. Then we have

**Proposition 5.6.** There exists $\kappa_3 = O(\kappa_2)$ depending on $\kappa_2$ with the following significance. The perturbed Dirac operator $D_{s\eta^i}$ which satisfies \[\begin{pmatrix} 4.25 \end{pmatrix}-\begin{pmatrix} 4.27 \end{pmatrix}\] can be written as follows:

\begin{equation}
D_{s\eta^{i+1}} = (1 + q^{i+1}) D_{s\eta^i} + s((\chi_i+1)z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_1 \partial_t
\end{equation}

\begin{equation}
+ \Theta_{s}^{i+1} + \mathcal{R}_{s}^{i+1} + \mathcal{A}_{s}^{i+1} + F_{s}^{i+1} + \delta^{(i+1)}
\end{equation}

where

- $\Theta_{s}^{i+1}$, the $(\chi, \eta) = (\chi_{i+1}, \eta_{i+1})$ version of $\Theta_{s}^{0}$, is a first order differential operator of order $O(s)$.

- $\mathcal{R}_{s}^{i+1} : L^2_{T} \rightarrow L^2_{T}$ is an $O(s^2)$-first order differential operator supported on $N_{T} - N_{\tau}$ with its operator norm

\begin{equation}
\|\mathcal{R}_{s}^{i+1}\| \leq \gamma_s^2 \kappa_3^2 s^2.
\end{equation}

- $\mathcal{A}_{s}^{i+1}$ is an $O(s^2)$-zero order differential operator. Moreover, let us denote $\bar{n} = \partial_r$ be the vector field defined on $N_R$, then

\begin{equation}
\int_{\{r=r_0\}} |\mathcal{A}_{s}^{i+1}|^2 i_\bar{n}dVol(M) \leq \gamma_s^4 \kappa_3^4 (\frac{(i+1)r_{\tau}}{T r_{\tau+1}}) s^4.
\end{equation}

for all $r_0 \leq \frac{T}{r_{\tau}}$.

- $F_{s}^{i+1}$ is an $O(s)$-zero order differential operator where

\begin{equation}
F_{s}^{i+1} = D(s((\chi_{i+1}z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})Id) + D(\begin{pmatrix} 0 & s\chi_{i+1}\bar{\eta}_{i+1} \\ -s\chi_{i+1}\eta_{i+1} & 0 \end{pmatrix}).
\end{equation}
• \( \delta^{(i+1)} \) can be written as \( \delta^{(i+1)} = \delta_0^{(i+1)} + \delta_1^{(i+1)} \) where \( \delta_1^{(i+1)} \) is a first order operator with
  \[
  \delta_1^{(i+1)} = \sum a_i \partial_i \text{ with } |a_i| \leq O(r^2) \text{ and } |a_2|, |a_3| \leq O(r)
  \]
  and \( \delta_0^{(i+1)} \) is a zero order operator with
  \[
  |\delta_0^{(i+1)}| = O(r).
  \]
  Moreover, \( \delta^{(i+1)} \) is supported on \( N_{\mathbb{R}}^r \).

6. Fredholm property

6.1. Basic setting. In this section, we develop an important theorem which indicates that the perturbation along \( V \) is finite dimensional as I mentioned in section 4.2. The operator, \( T_{d, d^-} \), we construct in this section is an important part in the linear approximation of the moduli space \( M \) we defined in our main theorem. We explain the idea of construction this operator in the following sections first.

The idea comes from [8]. Let \( N \) be a tubular neighborhood of \( \Sigma \) equipped with the Euclidean metric. By the computation in section 3.1, we know that for any \( u \) in the \( \ker(D|_{L^2(N; S \otimes I^2)}) \) can be written as

\[
  u = \sum_l e^{ilr} \left[ \hat{u}_{0, l}^+ \left( \frac{e^{ilr}}{\sqrt{\pi}} \right) - \text{sign}(l) e^{ilr} \right] + \text{higher order terms}.
\]

We define

\[
  (6.1) \quad B : \ker(D|_{L^2(N; S \otimes I^2)}) \to L^2(S^1; \mathbb{C}^2),
  \quad u \mapsto \left( \sum_l \hat{u}_{0, l}^+ e^{ilr}, \sum_l \text{sign}(l) \hat{u}_{0, l}^+ e^{ilr} \right).
\]

Secondly, we define the following spaces

\[
  \text{Exp}^+ = \left\{ \left( \sum_l u_l e^{i|l|t}, \sum_l -\text{sign}(l) u_l e^{i|l|t} \right)(u) \in \ell_2 \right\} \quad \text{and}
\]

\[
  \text{Exp}^- = \left\{ \left( \sum_l u_l e^{-i|l|t}, \sum_l \text{sign}(l) u_l e^{-i|l|t} \right)(u) \in \ell_2 \right\},
\]

then we have the corresponding projections \( \pi^\pm : L^2(S^1; \mathbb{C}^2) \to \text{Exp}^\pm \).

**Proposition 6.1.** Define the maps \( p^\pm = \pi^\pm \circ B \) in the following diagram.

\[
  \begin{array}{ccc}
  \text{Exp}^+ & \xrightarrow{p^+} & \ker(D|_{L^2(M - \Sigma; S \otimes I^2)}) \\
  \downarrow{\pi^+} & & \downarrow{B} \\
  \text{Exp}^- & \xleftarrow{p^-} & L^2(S^1; \mathbb{C}^2)
  \end{array}
\]
We will have

a. \( p^- : \ker(p^+) \to \text{Exp}^- \) is a Fredholm operator.
b. \( p^+ : \ker(p^-) \to \text{Exp}^+ \) is a compact operator.

**Proof.** a. First of all, for any \( r > 0 \) small enough, we have

\[
\int_{M-N_r} |Du|^2 = \int_{M-N_r} |\nabla u|^2 + \int_{\partial N_r} (u, \partial_{\nu} u) d\nu + \int_{M-N_r} \langle \mathcal{A}u, u \rangle
\]

by Lichnerowicz-Weitzenböck formula. Now by taking the limit \( r \to 0 \) and \( u \in \ker(D|_{L^2(M-\Sigma;\mathbb{C} \otimes \mathbb{C})}) \), we have

\[
0 = \int_{M-\Sigma} |\nabla u|^2 - \sum_l |l| \langle \hat{u}_{0,l}^-, \hat{u}_{0,l}^- \rangle + \sum_l |l| \langle \hat{u}_{0,l}^+, \hat{u}_{0,l}^+ \rangle + \int_{M-\Sigma} \mathcal{B}|u|^2.
\]

So

\[
\|u\|^2_{L^2(M-\Sigma)} \leq \sum_l |l| \|\hat{u}_{0,l}^-\|^2 + C\|u\|^2_{L^2(M-\Sigma)}
\]

for some constant \( C = \sup |\mathcal{B}| + 1 \). If \( u \in \ker(p^-) \), \( \sum_l |l| \|\hat{u}_{0,l}^-\|^2 = 0 \), which implies that

\[
\|u\|^2_{L^2(M-\Sigma)} \leq C\|u\|^2_{L^2(M-\Sigma)}.
\]

So the kernel of \( p^- \) is finite dimensional.

To prove that \( p^- \) has finite dimensional cokernel, we can prove the following statement instead: There exists \( n > 0 \) with the following significance. For any \( \sum_l e^{it \hat{u}_{0,l}^+} \in \text{Exp}^- \) such that \( \hat{u}_{0,l}^- = 0 \) for all \( l \) satisfying \( |l| \leq n \), there exists \( u \in \ker(p^+) \) such that \( \|B(u) - \langle \hat{u}_{0,l}^+ \rangle\|_{L^2} \leq \frac{1}{2\delta}\|\langle \hat{u}_{0,l}^+ \rangle\|_{L^2} \).

Suppose this claim is true. Let \( \mathcal{W} := \{ \sum_l e^{it \hat{u}_{0,l}^+} |\hat{u}_{0,l}^- = 0 \text{ for all } |l| > n \} \). We prove that \( \text{range}(p^-) + \mathcal{W} = \text{Exp}^- \) as the follows. Suppose not; there exists \( v \in L^2(S^1;\mathbb{C}) \) such that \( v \notin \text{range}(p^-) + \mathcal{W} \). Then we can assume that \( v \perp (\text{range}(p^-) + \mathcal{W}) \). So by using the claim in previous paragraph, for any \( \sum_l e^{it \hat{u}_{0,l}^-} \in \text{Exp}^- \) with \( \|\sum_l e^{it \hat{u}_{0,l}^-}\|_{L^2} = 1 \), we have

\[
\langle v, \sum_l e^{it \hat{u}_{0,l}^-} \rangle = \langle v, \sum_{|l| \leq n} e^{it \hat{u}_{0,l}^-} \rangle + \langle v, \sum_{|l| > n} e^{it \hat{u}_{0,l}^-} \rangle
\]

\[
= \langle v, \sum_{|l| \leq n} e^{it \hat{u}_{0,l}^-} \rangle + \langle v, B(u) \rangle + X
\]

\[
= X
\]

for some \( |X| \leq \frac{1}{2\delta}\|v\|_{L^2} \), which is a contradiction. Therefore, we have

\[
dim(\text{coker}(p^-)) \leq 2n + 1.
\]
Here we suppose $\|\hat{u}_{0,l}\|_{l^2} = 1$ without loss of generality. To prove this claim, we can consider the following section
\[
    u_0 = \chi \sum_{|l| \geq n} e^{it} u_{0,l} \left( e^{-i\frac{\theta}{2} \frac{1}{2} \partial_r (r)} + \text{sign}(l) \left( -le^{-i\frac{\theta}{2} \frac{1}{2} \partial_r (r)} e^{i\frac{\theta}{2} \frac{1}{2} \partial_r (r)} \right) \right)
\]
\[
    = \chi \sum_{|l| \geq n} e^{it} \hat{u}_{0,l} \left( \frac{e^{-|l|r}}{\sqrt{2}} \text{sign}(l) \frac{e^{-|l|r}}{\sqrt{2}} \right)
\]
with $\chi = 1 - \chi \frac{2}{R}$. So by this setting, we have
\[
    \|D(u_0)\|_{L^2} \leq C \frac{e^{-nR}}{R}.
\]
By using the argument in Proposition 4.3, we minimize the functional $E_{D(u_0)}$ among $L^2_1 \cap \ker(D)^\perp$. We can find $u^*$ such that $D(u^*) = D(u_0)$. Moreover, we have $\|B(u^*)\|_{L^2} \leq C \frac{e^{-nR}}{R}$. So by taking $u = u_0 - u^*$, we finish the proof of this claim.

b. Notice that the coefficients of $u$ in $\text{Exp}^+$ are corresponding to exponential increasing Fourier modes. Therefore, we have
\[
    \sum_l |l| |\hat{u}_{0,l}|^2 \leq C \|u\|_{L^2(M - \Sigma)}^2.
\]
So any bounded sequence $\{u^{(n)}\}$ such that $\{p^+(u^{(n)}) = (\hat{u}^{(n)}_{0,l})^+\}$ converges, we have
\[
    \sum_l |\hat{u}_{0,l}|^2 + \sum_l |\hat{u}_{0,l}^+|^2 \leq C.
\]
This implies that there exists a convergent subsequence of $\{u^{(n)}\}$ which converges to some $u$ and $\lim_{n \to \infty} p^+(u^{(n)}) = p^+(u)$. Therefore, $p^+$ is compact. \qed

We should remember that under a small perturbation of the metric and $\Sigma$, the dimension of cokernel of $p^+$ will be an upper semi-continuous function. I will leave this proof in Appendix 9.2.

By Proposition 6.1 and Proposition 4.3, we have the following result.

**Proposition 6.2.** Suppose that $f \in L^2(M - \Sigma; S \otimes \mathbb{I})$ and $f|_{N_{\alpha}} = 0$ for some $r \leq \alpha$. Then there exists $h \in L^2(M - \Sigma; S \otimes \mathbb{I})$ such that $Dh = f \mod \ker(D|_{L^2_1})$

a. $\|h\|_{L^2} \leq C \|f\|_{L^2_{-1}}$, for some universal constant $C > 0$;

b. The leading term of $h$, $h^\pm$, will satisfy
\[
    r_0 \|h^\pm\|_{L^2}^2, r_0^3 \|h^\pm\|_{L^2}^2, r_0^5 \|(h^\pm)_{tt}\|_{L^2}^2 \leq C \|f\|_{L^2_{-1}}^2
\]
for some universal constant $C > 0$.

**Proof.** First of all, we claim that, for any $l > 0$, there exists $u_l \in L^2(M - \Sigma; S \otimes \mathbb{I})$ with
\[
    u_l = e^{it} \left( \frac{e^{-|l|r}}{\sqrt{2}} \text{sign}(l) \frac{e^{-|l|r}}{\sqrt{2}} \right)
\]
on \( N_R \) such that \( Du = 0 \) on \( M - \Sigma \). This assumption is not true in general. In fact, by Proposition 6.1, it is only true when \( p^+ = 0 \) and \( p^- \) invertible. However, Proposition 6.1 tells us that \( p^+ \) is a compact operator, limit of finite dimensional operator, and \( p^- \) is a Fredholm operator, so we can modify this claim and get the proof for the general case.

We have

\[
||u||_{L^2} \leq \frac{2C}{|l|^2}.
\]

Meanwhile, by using Proposition 4.3, there exists \( \hat{h} \in L^2_1 \) such that \( D^2\hat{h} = f \). Taking \( \tilde{h} = D\hat{h} \), we have \( D\hat{h} = f \). Now, since \( \hat{h} \in \text{range}(D|_{L^2_1}) \), it is perpendicular to \( \ker(D|_{L^2}) \) by Proposition 2.4. So it is perpendicular to \( u \). Suppose that the Fourier coefficients of \( \hat{h} \) are \( h_{k,l}^\pm \).

Therefore, we can define

\[
\hat{u}_l := \frac{\hat{h}_{0,l}^- u_l}{|\hat{h}_{0,l}|},
\]

where \( \hat{h}_{0,l}^+ = (h_{0,l}^+ - \text{sign}(l)h_{0,l}^-) \) and \( \hat{h}_{0,l}^- = (h_{0,l}^- + \text{sign}(l)h_{0,l}^+) \). We also have

\[
||\hat{u}_l||_{L^2} \leq \frac{2C}{|l|^2}.
\]

Meanwhile, we have

\[
\int_{M - \Sigma} \langle \tilde{h}, \hat{u}_l \rangle = 0 = |\hat{h}_{0,l}^-| \int_0^{r_\theta} e^{-2|l| r} dr + \int_{M - N_r} \langle \hat{h}, \hat{u}_l \rangle.
\]

So we have

\[
|\hat{h}_{0,l}^-| \leq \frac{4C|l|^2}{1 - e^{-2|l| r_\theta}} ||P_l(\hat{h})||_{L^2}
\]

where \( P_l \) is the orthogonal projection from \( L^2(M - N_{r_\theta}) \) to \( \text{span}\{u_l\} \). Now define

\[
\eta = \sum_{|l| > \frac{1}{r_\theta}} \hat{h}_{0,l}^- u_l.
\]

Then we have

\[
||\eta||_{L^2} \leq C||r_\theta^{\frac{1}{2}}\eta||_{\partial N_{r_\theta}} ||L^2_{-1/2} = \sum_{|l| > \frac{1}{r_\theta}} \frac{|\hat{h}_{0,l}^-|^2}{|l|} \leq \sum_{|l| > \frac{1}{r_\theta}} \frac{4C}{(1 - e^{-2|l| r_\theta})^2} ||P_l(\hat{h})||_{L^2}^2 \leq \frac{4C}{(1 - e^{-2})^2} \sum ||P_l(\hat{h})||_{L^2}^2 \leq C||\hat{h}||_{L^2}^2.
\]

Let \( \hat{h} = \tilde{h} - \eta \), which satisfies \( D\hat{h} = 0 \) and \( \hat{h} \in K_{r_\theta} \). Moreover, we have

\[
||\hat{h}||_{L^2} \leq C||\hat{h}||_{L^2}.
\]

Notice that by Lemma 4.1, we have by Cauchy inequality

\[
||\hat{h}||_{L^2}^2 \leq C||\hat{h}||_{L^2}^2 \leq C||\hat{h}||_{L^2_1} f||_{L^2_{-1}} \leq \varepsilon ||\hat{h}||_{L^2_1}^2 + \frac{C}{4\varepsilon} ||f||_{L^2_{-1}}.
\]
So by choosing \( \epsilon \) small enough, Lemma 4.1 tells us
\[
\| \tilde{b} \|_{L^2} \leq C \| f \|_{L^2_{-1}}.
\]
Therefore, we prove \( a \). For \( b \), we can get it immediately by using Proposition 3.5.

For the general case (\( p^+ \) is nonzero and \( p^- \) is Fredholm), we have similar argument by modifying \( u_t \) to be 
\[
e^{\text{ill}t} \left( \frac{e^{-|l|r}}{\sqrt{2}} \sign(l) \frac{e^{-|l|r}}{\sqrt{2}} \right) + O_{1},
\]
where \( \| O_1 \|_{L^2} \leq C e^{-\| l \|_{L^2}} \)
(We can therefore choose \( r \) small such that \( |l| > \frac{1}{2} \) is very large). The existence of these \( u_t \) can be proved by using Proposition 4.3. The term 
\[
e^{\text{ill}t} \left( \frac{e^{-|l|r}}{\sqrt{2}} \sign(l) \frac{e^{-|l|r}}{\sqrt{2}} \right)
\]
will dominate \( O_1 \). So we can check that the argument above works for these \( u_t \) and the argument in Proposition 3.5 works for them, too. Therefore, we prove this proposition.

6.2. Linearization. Here we derive the linearization of \( \mathfrak{M} \). Suppose that \( \psi \) is an \( L^2_1 \)-harmonic spinor with respect to metric \( g \), which is locally written as
\[
\psi = \left( \begin{array}{c} d^+(t) \sqrt{z} \\ d^-(t) \sqrt{z} \end{array} \right) + \text{higher order terms.}
\]
\( \Sigma \) is its zero locus. Denote by \( p \) the triple \((g, \Sigma, \psi) \in \mathfrak{M}\):
\[
\mathcal{B} = \{ C^\infty - \text{real valued 2-form} \delta \text{ with } \text{supp}(\delta) \cap \Sigma = \emptyset \};
\]
\[
\mathcal{V} = \{ \eta : S^1 \to \mathbb{C} | \eta \in C^1 \}.
\]
Now suppose that we have a differentiable one-parameter perturbation \((g_s, \Sigma_s, \psi_s) \) with \((g_0, \Sigma_0, \psi_0) = (g, \Sigma, \psi) \) which can be written as
\[
g_s = g_0 + s\delta_s, \\
\Sigma_s = \{ (t, s\eta(t) + O(s^2)) \},
\]
\[
\psi_s(t, z) = \psi(t, z - s\eta + O(s^2)) + s\phi_s = \left( \begin{array}{c} d^+(t) \sqrt{z - s\eta} \\ d^-(t) \sqrt{z - s\eta} \end{array} \right) + O_L^2(s) + s\phi_s
\]
for some \( \delta_s \in \mathcal{B}, \eta \in \mathcal{V} \) and \( \phi_s \in L^2_1(M - \Sigma; S_{g, \Sigma}) \cong L^2_1(M - \Sigma; S_{g, \Sigma}) \). We use \( O_L^2(s) \) to denote a one-parameter family of sections \( f_s \) satisfying \( \| f_s \|_{L^2} \leq C s \) for some constant \( C \). Let \( D^{(s)} \) be the Dirac operator with respect to \( g_s \), then we have
\[
D^{(s)} = D + sT + O(s^2)
\]
for some first order differential operator \( T \). Notice that the support of \( T \) is disjoint from a tubular neighbourhood of \( \Sigma \).

Therefore, the linearization of \( \mathfrak{M} \) at \( p \) can be written as
\[
(6.2) \quad \mathcal{L}_p(\delta_0, \eta, \phi_0) := \left. \frac{\partial}{\partial s} (D^{(s)} \psi_s) \right|_{s=0} = T(\psi) + D \left( \left. \frac{\partial}{\partial s} \left( \psi(t, z - s\eta + O(s^2)) \right) \right|_{s=0} + O_L^2(1) + \phi_0 \right).
\]
\( \mathcal{L}_p \) is a map from \( \mathcal{B} \times \mathcal{V} \times L^2_1(M - \Sigma; S_{g, \Sigma}) \) to \( L^2(M - \Sigma; S_{g, \Sigma}) \).
Notice that $T(\psi) \in L^2$ is compactly supported away from $\Sigma$. By Proposition 4.3, there exists $\eta \in L^2$ such that $D\eta = -T(\psi) \ mod(\ker(D|_{L^2}))$. We can write
\[
\eta = \left( \frac{h^+}{\sqrt{z}} \right) + \text{higher order terms}.
\]
Therefore, the right-hand side of (6.2) can be rewritten as
\[
D \left( \left( \frac{d^+ \eta}{2\sqrt{z}} \right) + \left( \frac{h^+}{\sqrt{z}} \right) + O_{L^2}(1) + \phi_0 \right).
\]
This implies that if $(\delta_0, \eta, \phi_0) \in \ker(\mathcal{L}_p)$, the element
\[
(6.3) \quad \left( \left( \frac{d^+ \eta}{2\sqrt{z}} \right) + \left( \frac{h^+}{\sqrt{z}} \right) + O_{L^2}(1) + \phi_0 \right)
\]
is an $L^2$-harmonic spinor. By using notations of Proposition 6.1, we can rewrite this condition as follows:
\[
(d^+ \eta + 2h^+, d^- \bar{\eta} + 2\bar{h}^-) = B(u)
\]
for some $u \in \ker(D|_{L^2})$. In particular, this defines a map $\Psi : \mathcal{B} \times \mathcal{V} \times L^2 \to \ker(D|_{L^2})$ with $\Psi(\delta, \eta, \phi_0) = u$. Our goal is to prove that for any $h^\pm$ given, there are only finite dimensional solutions $\eta$ satisfying $(d^+ \eta + 2h^+, d^- \bar{\eta} + 2\bar{h}^-) \in B(\ker(D|_{L^2}))$. Namely, we have to show that the equations
\[
\begin{align*}
d^+ \eta + c^+ &= -2h^+, \\
d^- \bar{\eta} + c^- &= -2\bar{h}^-.
\end{align*}
\]
for $(c^\pm) \in B(\ker(D|_{L^2}))$ have a finite dimensional solution space. These equations have the following constraint:
\[
|d^+|^2 + |d^-|^2 \neq 0.
\]
which comes from the assumption that $\frac{|\psi(p)|}{\text{dist}(p, \Sigma)^2} > 0$ for all $p$. By some basic computation, these equations imply
\[
(6.5) \quad d^- c^+ - d^+ \bar{c}^- = -2d^- h^+ + 2d^+ \bar{h}^-.
\]
Therefore, we can define the following operator
\[
(6.6) \quad T_{d^+, d^-} : L^2(S^1; \mathbb{C}^2) \to L^2(S^1; \mathbb{C});
\]
\[
T_{d^+, d^-}(c^\pm) = d^- c^+ - d^+ \bar{c}^-.
\]
One can check that $\ker(\mathcal{L}_p|_{d=0}) = ker(T_{d^+, d^-} \circ B)$, we leave the proof of this part in Appendix 9.3. One can also show that $\text{coker}(\mathcal{L}_p|_{d=0}) \subset \text{coker}(T_{d^+, d^-} \circ B) \oplus \ker(D|_{L^2})$ (also proved in Appendix 9.3). We therefore define
\[
(6.7) \quad \mathbb{K}_1 = ker(T_{d^+, d^-} \circ B);
\]
\[
(6.8) \quad \mathbb{K}_0 = \text{coker}(T_{d^+, d^-} \circ B) \oplus \ker(D|_{L^2}).
\]
So our goal in this section is to show that $T_{d^+, d^-} \circ B$ is Fredholm. That will imply $\mathbb{K}_0$ and $\mathbb{K}_1$ are finite dimensional.
In addition, recall that \( p^+ \) is a compact operator. So

\[
T_{d^+,d^-} \circ B = T_{d^+,d^-}|_{Exp^+} \circ p^- + \text{a compact operator.}
\]

We have that \( T_{d^+,d^-} \circ B \) is Fredholm if and only if \( T_{d^+,d^-}|_{Exp^+} \) is Fredholm (also recall that \( p^- \) is a Fredholm operator).

### 6.3. Fredholmness of finite Fourier mode case

We now consider the equation

\[
d^+ \eta + c^+ = -2h^+,
d^- \bar{\eta} + c^- = -2h^-.
\]

with constraint (6.4) and \( c^\pm \in \text{Exp}^- \). So there is the following relationship between \( c^+ \) and \( c^- \): if we write \( c^+ = \sum p_l e^{ilt} \), we have \( c^- = \sum \text{sign}(l)p_l e^{ilt} \). Namely, the \( c^- \) is determined by \( c^+ \).

In this section, we will use the following notation.

**Definition 6.3.** Let \( g = \sum_l g_l e^{ilt} \in L^2 \), we write \( g^{\text{aps}} = \sum_l \text{sign}(l)g_l e^{ilt} \).

So we can rewrite our operator in the following way:

\[
T_{d^+,d^-}(c) := \tilde{d}^-c - d^+ c^{\text{aps}}
\]

with \( T_{d^+,d^-} : L^2(S^1; \mathbb{C}) \to L^2(S^1; \mathbb{C}) \). Here we should explain the meaning of this \( L^2(S^1; \mathbb{C}) \) space. We can easily see that, \( T_{d^+,d^-} \) is not a \( \mathbb{C} \)-linear operator, since the conjugate term \( c^{\text{aps}} \) involved. However, it is still an \( \mathbb{R} \)-linear operator. Therefore, we define our index under the real vector space.

So in our case, we should define the inner product to be

\[
(f, g) := \text{Re}\left( \int_{S^1} f \cdot \bar{g} dt \right)
\]

for all \( f, g \in C^\infty(S^1; \mathbb{C}) \). We can see that, under this definition, the \( L^2 \)-bounded space will be coincident with the one equipped with the usual inner product over \( \mathbb{C} \).

We will prove the following property:

**Proposition 6.4.** \( T_{d^+,d^-} \) is a Fredholm operator and index\( (T_{d^+,d^-}) = 0 \) when both \( d^+ \) and \( d^- \) have only finite many Fourier modes:

\[
d^+ = \sum_{-M}^{M} d_i^+ e^{ilt}, \quad d^- = \sum_{-M}^{M} d_i^- e^{ilt}
\]

for some \( M \in \mathbb{N} \).

In this section, we will assume that \( d^\pm \) have only finite many Fourier modes and prove Proposition 6.4. Then we will prove the general case in the next section.

**Definition 6.5.** Let \( a = (x, y) \in \mathbb{C} \times \mathbb{C} \), we define the spouse of \( a \), denoted by \( \check{a} \), to be \( (\bar{y}, -x) \in \mathbb{C} \times \mathbb{C} \). We can easily see that \( \check{\check{a}} = -a \).

Similarly, for any \( p \)-tuple of complex pairs, we have the following definition.

**Definition 6.6.** Let \( A = (a_1, a_2, ..., a_{p-1}, a_p) \in (\mathbb{C} \times \mathbb{C})^p \) for some \( p \in \mathbb{N} \). We define the spouse of \( A \), denoted by \( \check{A} \), to be \( (\check{a}_p, \check{a}_{p-1}, ..., \check{a}_2, \check{a}_1) \in (\mathbb{C} \times \mathbb{C})^p \).
Now we write our proof of Proposition 6.4 in the following 8 steps.

**Step 1.** In this and the next step, we will prove that $T_{d^+,d^-}$ has finite dimensional kernel. Firstly, we notice that the $n$-th Fourier coefficient of $(\bar{d}^--d^+c^{aps})$ can be written as

$$(\bar{d}^-c-d^+c^{aps})_n = \sum_{l=-M}^{M} \bar{d}^-_{-l}p_{n-l} + \text{sign}(l-n)d^+_l \bar{p}_{l-n}.$$ 

When $n > M$, $\text{sign}(l-n) = -1$ for all $l = -M, ..., M$. So we have

$$(\bar{d}^-c-d^+c^{aps})_n = \sum_{l=-M}^{M} \bar{d}^-_{-l}p_{n-l} - d^+_l \bar{p}_{l-n}$$

for $n > M$.

Similarly

$$(\bar{d}^-c-d^+c^{aps})_n = \sum_{l=-M}^{M} \bar{d}^-_{-l}p_{n-l} + d^+_l \bar{p}_{l-n}$$

for $n < -M$.

If we take $n = -n'$ and then take the conjugation on both side of the equation above, we will have the following equation:

$$(\bar{d}^-c-d^+c^{aps})_{n'} = \sum_{l=-M}^{M} \bar{d}^-_{-l}p_{n'-l} - d^+_l \bar{p}_{l-n'}$$

for $n' > M$.

**Step 2.** To show that the kernel of $T_{d^+,d^-}$ is finite dimensional, here is the idea: we claim that every element in $\ker(T_{d^+,d^-})$ can be determined by their Fourier coefficients from $-2M$ to $2M$. Therefore, the dimension of $\ker(T_{d^+,d^-})$ cannot exceed $4M + 2$. To prove this claim, suppose there are two elements $c_1$ and $c_2$ in $\ker(T_{d^+,d^-})$ which have same Fourier coefficients from $-2M$ to $2M$. Then $c_1 - c_2$ is also in $\ker(T_{d^+,d^-})$. Therefore, our claim is true iff any $c \in \ker(T_{d^+,d^-})$ which has zero Fourier coefficients from $-2M$ to $2M$ is identically zero.

Now we prove this claim. Suppose that $c \in \ker(T_{d^+,d^-})$ has zero Fourier coefficients from $-2M$ to $2M$. Because $c \in \ker(T_{d^+,d^-})$, we have

$$\sum_{l=-M}^{M} \bar{d}^-_{-l}p_{n-l} - d^+_l \bar{p}_{l-n} = 0$$

$$\sum_{l=-M}^{M} \bar{d}^+_{-l}p_{n-l} + d^-_l \bar{p}_{l-n} = 0$$
for $n > M$, we can rewrite this equation by pairing $(p_j, \bar{p}_{-j}) := v_j$ and $(\bar{d}_{-j}, -d^+_j) := d_j$ for all $j \in \mathbb{Z}$. Now we have

$$
\sum_{l=-M}^{M} \langle d_l, \bar{v}_{n-l} \rangle = 0 \\
\sum_{l=-M}^{M} \langle \bar{d}_{-l}, \bar{v}_{n-l} \rangle = 0
$$

with the bracket $\langle \cdot, \cdot \rangle$ denoting the usual inner product over $\mathbb{C}$. Here we can use the following convention: Let $U = (u_i), W = (w_i) \in (\mathbb{C} \times \mathbb{C})^\mathbb{Z}$. Define a new bracket $\langle\langle \cdot, \cdot \rangle \rangle$ to be

$$
\langle\langle U, W \rangle \rangle_n = \sum_{i \in \mathbb{Z}} \langle u_i, w_{n-i} \rangle.
$$

So our equation can be written as

$$
\langle\langle D, \bar{V} \rangle \rangle_n = 0 \\
\langle\langle \hat{D}, \bar{V} \rangle \rangle_n = 0
$$

where $D = (d_l), V = (v_l)$ and $n > M$.

Now we apply the following squeezing lemma.

**Lemma 6.7.** Given $A = (a_j)_{j=1,2,\ldots,p} \in (\mathbb{C} \times \mathbb{C})^p$. If $V = (v_j)_{j \in \mathbb{Z}} \in (\mathbb{C} \times \mathbb{C})^\mathbb{Z}$ satisfies

$$
\langle\langle A, \bar{V} \rangle \rangle_m = 0; \langle\langle \hat{A}, \bar{V} \rangle \rangle_m = 0
$$

for all $m > 0$, then there is $B = (0, \ldots, 0, b_1, \ldots, b_q) \in (\mathbb{C} \times \mathbb{C})^p$ with and det $\left( \begin{smallmatrix} b_q \\ b_1 \end{smallmatrix} \right) \neq 0$ such that

$$
\langle\langle B, \bar{V} \rangle \rangle_m = 0; \langle\langle B^*, \bar{V} \rangle \rangle_m = 0,
$$

where $B^* = (0, \ldots, 0, \hat{b}_q, \ldots, \hat{b}_1), \text{ for all } m > 0$.

**Proof.** If $\text{det} \left( \begin{smallmatrix} a_p \\ a_1 \end{smallmatrix} \right) \neq 0$, then we can just take $A = B$. The lemma holds trivially.

Suppose now $\text{det} \left( \begin{smallmatrix} a_p \\ a_1 \end{smallmatrix} \right) = 0$. Then we have $\alpha a_p = \hat{a}_1$ for some $\alpha \in \mathbb{C} - \{0\}$. So we have

$$
\langle\langle \hat{A}, \bar{V} \rangle \rangle_m - \alpha \langle\langle A, \bar{V} \rangle \rangle_m = \langle\langle \hat{A} - \alpha A, \bar{V} \rangle \rangle_m = 0.
$$

We also have

$$
\langle\langle A, \bar{V} \rangle \rangle_m + \bar{\alpha} \langle\langle \hat{A}, \bar{V} \rangle \rangle_m = \langle\langle A + \bar{\alpha} \hat{A}, \bar{V} \rangle \rangle_m = 0.
$$

Denote

$$
B'_1 = (\hat{A} - \alpha A) = (\hat{a}_p - \alpha a_1, \hat{a}_{p-1} - \alpha a_2, \ldots, \hat{a}_2 - \alpha a_{p-1}, 0).
$$

Notice that: Since $\alpha a_p = \hat{a}_1$, we have $\hat{a}_p - \alpha a_1 = \hat{a}_p + |\alpha|^2 \hat{a}_p = (1 + |\alpha|^2) \hat{a}_p \neq 0$. This implies $B'_1 \neq 0.$
Now let \( B_1 = (0, \hat{a}_p - \alpha a_1, \hat{a}_{p-1} - \alpha a_2, \ldots, \hat{a}_2 - \alpha a_{p-1}) \). We can easily verify that
\[
\langle \langle \hat{A} - \alpha A, V \rangle \rangle_{m+1} = \langle \langle B_1, V \rangle \rangle_m = 0
\]
for all \( m > 0 \).

Since \((\hat{A} - \alpha A) = (A + \hat{A})^\ast \), the second equation gives us
\[
\langle \langle A - \alpha \hat{A}, V \rangle \rangle_m = \langle \langle B_1^\ast, V \rangle \rangle_m = 0
\]
for all \( m > 0 \).

Now by repeating this process inductively, we prove this lemma. \( \square \)

Back to our problem, we have the equations
\[
\langle \langle D, V \rangle \rangle_n = 0
\]
\[
\langle \langle \hat{D}, V \rangle \rangle_n = 0
\]
for \( n > M \). Now we can apply Lemma 6.7 on \( A = (d_{-M}, d_{-(M-1)} \ldots, d_M) \), \( m = n - M \). So there exists \( B \in (\mathbb{C} \times \mathbb{C})^p \) such that \( \det \begin{pmatrix} b_q \\ b_1 \end{pmatrix} \neq 0 \) and
\[
\langle \langle B, V \rangle \rangle_n = 0
\]
\[
\langle \langle B^\ast, V \rangle \rangle_n = 0
\]
for all \( n > M \). Together with the condition \( v_l = 0 \) for \( l = 0, 1, \ldots, 2M \), we have
\[
\langle \langle B, V \rangle \rangle_{M+1} = \langle b_q, v_{2M+1} \rangle = b_q^+ p_{(2M+1)} + b_q^- p_{-(2M+1)} = 0
\]
\[
\langle \langle B^\ast, V \rangle \rangle_{M+1} = \langle \hat{b}_1, v_{2M+1} \rangle = \hat{b}_1^- p_{(2M+1)} + \hat{b}_1^+ p_{-(2M+1)} = 0,
\]
which implies \( v_{2M+1} = 0 \) because \( \det \begin{pmatrix} b_q \\ b_1 \end{pmatrix} \neq 0 \). Now we can solve \( v_k \) inductively:
Suppose \( v_1, v_2, \ldots, v_{M+k} \) are all zero for some \( k > M + 1 \). Then the equation tells us that
\[
\langle \langle B, V \rangle \rangle_{k+1} = \langle b_q, v_{M+k+1} \rangle = b_q^+ p_{(M+k+1)} + b_q^- p_{-(M+k+1)} = 0
\]
\[
\langle \langle B^\ast, V \rangle \rangle_{k+1} = \langle \hat{b}_1, v_{M+k+1} \rangle = \hat{b}_1^- p_{(M+k+1)} + \hat{b}_1^+ p_{-(M+k+1)} = 0.
\]
So we have \( v_{M+k+1} = 0 \). Therefore, we have \( v_l = 0 \) for all \( l \) which implies \( c \equiv 0 \).

**Step 3.** To show that \( T_{d^+, d^-} \) is a Fredholm operator, we can either prove \( T_{d^+, d^-}^\ast \) has finite dimensional cokernel, or we can prove the following properties instead:
1. \( \ker(T_{d^+, d^-}^\ast) \) is finite dimensional,
2. \( \text{range}(T_{d^+, d^-}^\ast) \) is closed,
3. \( \text{range}(T_{d^+, d^-}) \) is closed.

We prove these properties step by step.

**Step 4.** We prove \( \ker(T_{d^+, d^-}^\ast) \) is finite dimensional in this step. Here \( T_{d^+, d^-}^\ast \) is the adjoint operator of \( T_{d^+, d^-} \). We can get the following computation by definition:
Let $c = \sum p_i e^{ilt}$, $k = \sum q_i e^{ilt} \in L^2$

\[
(T_{d^+,d^-}(c), k) = \text{Re} \left( \int_{\mathbb{S}^1} T_{d^+,d^-}(c) \cdot \bar{k} dt \right).
\]

\[
= \frac{1}{2} \left( \int_{\mathbb{S}^1} T_{d^+,d^-}(c) \cdot \bar{k} dt + \int_{\mathbb{S}^1} k \cdot \bar{T}_{d^+,d^-}(c) dt \right)
\]

\[
= \sum_{n \in \mathbb{Z}} \sum_{l=-M}^{M} (\bar{d}_-(p_{n-l} + \text{sign}(l-n) d_l^+ \bar{p}_{l-n}) \bar{q}_n \\
+ \sum_{n \in \mathbb{Z}} \sum_{l=-M}^{M} q_n (d_{-l}^+ \bar{p}_{n-l} - \text{sign}(l-n) d_l^+ p_{l-n}) \\
= \sum_{n \in \mathbb{Z}} \sum_{l=-M}^{M} (d_{-l}^- q_{n+l} + \text{sign}(n) d_l^+ \bar{q}_{n+l}) \bar{q}_n \\
+ \sum_{n \in \mathbb{Z}} p_n \sum_{l=-M}^{M} (d_{-l}^- q_{n+l} + \text{sign}(n) d_l^+ \bar{q}_{n+l}) \\
= (c, T_{d^+,d^-}(k)).
\]

We get the last equality by taking

\[
T_{d^+,d^-}(k) = \sum_{n \in \mathbb{Z}} \sum_{l=-M}^{M} (d_{-l}^- q_{n+l} + \text{sign}(n) d_l^+ \bar{q}_{n+l}) e^{int}.
\]

Now we can repeat the argument in step 1 and 2 on $T_{d^+,d^-}$, then we will get $\dim(\ker(T_{d^+,d^-})) < \infty$.

**Step 5.** Property 2 and 3 in step 3 are similar. Here we only prove property 2. Readers can prove Property 3 by applying the same argument again.

Before we prove Property 2, we need the following lemma.

**Lemma 6.8.** Let $P_k : L^2 \rightarrow L^2$ is the projection defined by

\[
P_k : \sum f_n e^{int} \mapsto \sum_{|n| \leq k} f_n e^{int}.
\]

Then we have

\[
T_{d^+,d^-}|_{(I-P_{2M})(L^2)} : (I - P_{2M})(L^2) \rightarrow (I - P_M)(L^2)
\]

is injective.

**Proof.** Let $f \in (I - P_{2M})(L^2)$. Clearly $T(f) \in (I - P_M)(L^2)$, so we should prove this map is one to one. We prove this by induction.

Suppose $f = \sum f_ke^{ikt} \in (I - P_M)(L^2)$, by the equation given by Lemma 6.7,

\[
\langle (B, V) \rangle_{M+1} = \langle b_1, v_{2M+1} \rangle = b_1^+ p_{2(M+1)} + b_1^- \bar{p}_{-(2M+1)} = f_{M+1},
\]

\[
\langle (B^*, V) \rangle_{M+1} = \langle b_1, v_{2M+1} \rangle = b_1^+ p_{2(M+1)} + b_1^+ \bar{p}_{-(2M+1)} = \bar{f}_{-(M+1)}.
\]
we can solve $v_{(2M+1)} = (p_{2M+1}, \bar{p}_{-(2M+1)})$, which is unique.

Now suppose $v_{(2M+1)}, \ldots, v_{M+k}$ are uniquely determined (where $k > M + 1$), we consider the equation

\[
\langle (B, \bar{V}) \rangle_{k+1} = \langle b_q, v_{M+k+1} \rangle
\]

\[
= b^+_q p_{(M+k+1)} + b^-_q \bar{p}_{-(M+k+1)} + F_k(v_{(2M+1)}, \ldots, v_{M+k}) = f_{k+1}
\]

\[
\langle (B^*, \bar{V}) \rangle_{k+1} = \langle b_1, v_{M+k+1} \rangle
\]

\[
= \hat{b}^+_1 p_{(M+k+1)} + \hat{b}^-_1 \bar{p}_{-(M+k+1)} + G_k(v_{(2M+1)}, \ldots, v_{M+k}) = \hat{f}_{-(k+1)}.
\]

where $F_k(v_{(2M+1)}, \ldots, v_{M+k}) = f_{k+1}$ and $G_k(v_{(2M+1)}, \ldots, v_{M+k})$ are determined by \{v_{(2M+1)}, \ldots, v_{M+k}\}. So we can solve $v_{(M+k+1)}$ uniquely.

Therefore, the map $T_{d^+, d^-} |_{(I - P_{2M})(L^2)}$ is an injective map from $(I - P_{2M})(L^2)$ to $(I - P_M)(L^2)$. \(\square\)

If we decompose $L^2 = P_{2M}(L^2) \oplus (I - P_{2M})(L^2)$, we have $T_{d^+, d^-}(P_{2M}(L^2)) \subset P_{3M}(L^2)$ and $T_{d^+, d^-}((I - P_{2M})(L^2)) \subset (I - P_M)(L^2)$.

**Step 6.** We will prove Property 2 in Step 3 in the following 2 steps. Suppose now we have $c^k \in L^2$, $k \in \mathbb{N}$ such that $\{T_{d^+, d^-}(c^k)\}$ converges to some $f \in L^2$. Let \{v^k_p\} be the corresponding pairing $\ell^2$-coefficients of $c^k$. Here we can assume that $c^k \perp \ker(T_{d^+, d^-})$ without loss of generality. We have to show that there exist $c$ such that $T_{d^+, d^-}(c) = f$.

First of all, suppose that $c^k$ is bounded by some constant $K$. We choose a subsequence, which is denoted by $c^k$ again, such that $\{v^k_p\}_{k \in \mathbb{N}}$ converges for any $p \leq 3J$ with some $J > M$. Let us say

\[
v^k_p \to v_p
\]

for $p \leq 3J$ and choose $J$ large enough such that $v_p \neq 0$. Now by Lemma 6.8, there is a unique solution $c$ such that

\[
T_{d^+, d^-}(c) = f
\]

where the corresponding $\ell^2$-coefficients of $c$ are $v_p$ for $p \leq 3M$. So we only need to show that $c$ is in $L^2$.

Now for any $r \in \mathbb{N}$, we have

\[
\sum_{i \leq r} \|v_i\|_{l^2}^2 \leq \sum_{i \leq r} \|v^k_i - v_i\|_{l^2}^2 + \sum_{i \leq r} \|v^k_i\|_{l^2}^2
\]

\[
\leq \sum_{i \leq r} \|v^k_i - v_i\|_{l^2}^2 + K.
\]

We notice that the first term converges to 0 as $k \to \infty$. Therefore, we have

\[
\sum_{i \leq r} \|v_i\|_{l^2}^2 \leq 1 + K
\]

for any $r > 0$. So $c \in L^2$. 

Step 7. Suppose that $c^k$ is unbounded, say $\|c_k\|_{L^2} \to \infty$ (by taking subsequence if it is necessary), we can take $\hat{c}^k = \frac{c^k}{\|c_k\|_{L^2}}$ which satisfies $T_{d^+,d^-}(\hat{c}^k) \to 0$. We should prove that this case will lead a contradiction. This is the part that condition (6.4) involved.

To begin with, we should define the following notations.

**Definition 6.9.** Let $\varepsilon > 0$. We define the number $\tau = \inf \{\sqrt{|d^+|^2 + |d^-|^2}\}$ and the following sets:

1. $\Omega_1 = \{|d^+| = |d^-|\} \subset S^1$,
2. $\Omega_{1,\varepsilon} = \{|\varepsilon \tau| - |d^+| \leq \varepsilon \tau\}$,
3. $\Omega_\varepsilon^+ = \{|d^+| > |d^-| + \varepsilon \tau\}$,
4. $\Omega_\varepsilon^- = \{|d^-| > |d^+| + \varepsilon \tau\}$.

So we have $S^1 = \Omega_{1,\varepsilon} \cup \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. 

Now we fix a $\varepsilon \leq \frac{1}{3}$ which will be specified later. We define $\chi_{1,\varepsilon}$ to be a nonnegative real valued function defined on $S^1$ which has value 1 in $\Omega_{1,\varepsilon}$ and 0 in $\Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. Also, define $\chi_{2,\varepsilon}$ to be 1 in $\Omega_\varepsilon^+$ and 0 on $\{|d^+| \leq |d^-| + \frac{3}{5} \tau\}$. Define $\chi_{3,\varepsilon}$ to be 1 in $\Omega_\varepsilon^-$ and 0 on $\{|d^+| \geq |d^-| + \frac{3}{5} \tau\}$. Moreover, suppose that

$$\chi_{1,\varepsilon} + \chi_{2,\varepsilon} + \chi_{3,\varepsilon} \equiv 1.$$

**Step 8.** In this Step, we will modify the statement in the first paragraph of step 7 by some observation and define some notations which will be used later: Suppose we have a sequence $\{c^k\}$ with their $L^2$ norms equaling 1 and $T_{d^+,d^-}(c^k)$ converging to 0 in $L^2$ sense. For any $i \in \mathbb{Z}$ fixed, suppose that the limit sup of $\{\|c_i^k\|^2\}$ is nonzero, than we can use the argument in step 6 by taking $J > i$ to achieve a contradiction.

Now, for any $L \in \mathbb{N}$ let $P_L : L^2 \to L^2$ be the projection which maps $\sum_{l \in \mathbb{Z}} q_l e^{ilt}$ to $\sum_{|l| \leq L} q_l e^{ilt}$. By using the observation in the previous paragraph, for any $L \in \mathbb{N}$ given, we can add the additional assumption into our statement: $P_L(c^k) = 0$ for some $k$. This number $L$ will be specified later which is determined by $\varepsilon$ and $\chi_{1,\varepsilon}$.

Here we should restate the statement as follows.

**Lemma 6.10.** There exists $L \in \mathbb{N}$ depending only on $d^\pm$, such that for any sequence $\{c^k\}_{k \in \mathbb{N}} \subset L^2$ satisfying

$$\|c^k\|_{L^2} = 1, \ P_L(c^k) = 0 \text{ for all } k \in \mathbb{N},$$

we have $\inf_{k \in \mathbb{N}} \{\|T_{d^+,d^-}(c^k)\|_{L^2}\} > C_0$, where $C_0$ depending only on the $C^1$-norm of $d^\pm$ and $\tau$.

We still have several constants to define. We consider the function $Q = \frac{d^+}{d^-}$ defined on $\Omega_{1,\varepsilon}$. Extend this function as a $C^1$ function defined on $S^1$. Then we can approximate it by its first $N_2$ Fourier modes, $S$, such that the $L^2$-norm and $L^\infty$-norm of $|Q - S|$ are $O(\varepsilon)$.

Since $\chi_{1,\varepsilon} + \chi_{2,\varepsilon} + \chi_{3,\varepsilon} \equiv 1$, we have

$$1 = \|c^k\|_{L^2} \leq \|\chi_{1,\varepsilon} c^k\|_{L^2} + \|\chi_{2,\varepsilon} c^k\|_{L^2} + \|\chi_{3,\varepsilon} c^k\|_{L^2}.$$

Therefore, there exists $i \in \{1, 2, 3\}$ such that $\|\chi_{i,\varepsilon} c^k\|_{L^2} \geq \frac{1}{3}$ infinite many times. We take this subsequence and renumber them consecutively from 1. Since $\chi_{1,\varepsilon}$ is
a smooth function, we approximate $\chi_{1, \varepsilon}$ by its first $N_1$ Fourier mode, denoted by $\zeta_{1, \varepsilon}$, such that $\|\chi_{1, \varepsilon} - \zeta_{1, \varepsilon}\|_{L^2} \leq \varepsilon < \frac{\delta}{6}$ and $\sup|\chi_{1, \varepsilon} - \zeta_{1, \varepsilon}| \leq \varepsilon$, so by Cauchy’s inequality, we have $\|\zeta_{i, \varepsilon} c^k\|_{L^2} \geq \frac{1}{6}$. Now we shall start the proof of Lemma 6.10 case by case.

**Proof. Case 1.** If $i = 1$, we have

$$\zeta_{1, \varepsilon} T_{d^+, \varepsilon}^{(c^k)} = \zeta_{1, \varepsilon} f^k,$$

where $\limsup \|\zeta_{1, \varepsilon} f^k\|_{L^2} \leq \varepsilon$. Now we can write

$$\zeta_{1, \varepsilon} T_{d^+, \varepsilon}^{(c^k)} = T_{d^+, \varepsilon}^{(c_{1, \varepsilon}^k)} + (\zeta_{1, \varepsilon} T_{d^+, \varepsilon}^{(c^k)} - T_{d^+, \varepsilon}^{(c_{1, \varepsilon}^k)}).$$

The second term on the right can be written as $[T_{d^+, \varepsilon}, \zeta_{1, \varepsilon}](f^k)$. Let $\zeta_{1, \varepsilon} = \sum_{l \in \mathbb{Z}} \zeta_l e^{ilt}$, then we can get

$$[T_{d^+, \varepsilon}, \zeta_{1, \varepsilon}](f^k) = \zeta_{1, \varepsilon}((c^k)_{ap)} - (\zeta_{1, \varepsilon} c^k)^{ap}$$

$$= \sum_{n \in \mathbb{Z}} \left( \sum_{|j| \leq N_1} \zeta_j \text{sign}(n - j)p_n^k \right) - \sum_{|j| \leq N_1} \text{sign}(n)\zeta_j p_n^k)^{ap}$$

$$= \sum_{|n| \leq N_1} \pm 2\left( \sum_{|j| \leq N_1} \zeta_j p_n^k \right)^{ap}.$$

So this term will be 0 by taking $L > 2N_1$.

Therefore, we have

$$T_{d^+, \varepsilon}^{(c_{1, \varepsilon}^k)} = \zeta_{1, \varepsilon} f^k = \bar{d} - \zeta_{1, \varepsilon} c^k - d^+ (\zeta_{1, \varepsilon} c^k)^{ap}.$$

Dived both side by $\bar{d}^-$, then we have

$$\zeta_{1, \varepsilon} c^k - \frac{d^+ (\zeta_{1, \varepsilon} c^k)^{ap}}{\bar{d}^-} = \frac{\zeta_{1, \varepsilon} f^k}{\bar{d}^-}.$$
We notice that

\[ P^\pm(A^k) = \overline{P^\pm(B^k)}, \]

and

\[ [P^+, S]B^k = (P^+SB^k - SP^+B^k) \]

\[ = \sum_{n>0} \left( \sum_{|j| \leq N_2} S_j B_{n-j} e^{int} \right) - \sum_{n \geq j} \left( \sum_{|j| \leq N_2} S_j B_{n-j} e^{int} \right) \]

\[ = \sum_{|n| \leq N_2} \sum_{|j| \leq N_2} S_j B_{n-j} e^{int} \]

the last term will be 0 when we take \( L > 2N_1 + 2N_2 \).

Therefore, we have

\[ P^+ A^k + SP^+ B^k = O_{L^2}(\varepsilon) + P^+(\frac{\zeta_{1,\varepsilon} f^k}{d^-}), \]

\[ P^+ B^k - SP^+ A^k = O_{L^2}(\varepsilon) + P^-(\frac{\zeta_{1,\varepsilon} f^k}{d^-}) \]

Since \( \|A^k\|_{L^2} \geq \frac{1}{6} \), we can suppose that either \( \|P^+ A^k\|_{L^2} > \frac{1}{12} \) or \( \|P^- A^k\|_{L^2} > \frac{1}{12} \). Suppose that \( \|P^+ A^k\|_{L^2} > \frac{1}{12} \) then we will have

\[ P^+ A^k + SP^+ B^k - S(P^+ B^k - SP^+ A^k) = (1 + |S|^2)P^+ A^k \]

\[ = O_{L^2}(\varepsilon) + P^+(\frac{\zeta_{1,\varepsilon} f^k}{d^-}) + SP^-(\frac{\zeta_{1,\varepsilon} f^k}{d^-}) \]

Therefore, we have

\[ \frac{1}{12} \leq \|P^+ A^k\|_{L^2} \leq \|(1 + |S|^2)P^+ A^k\|_{L^2} \leq O(\varepsilon) + \|P^+(\frac{\zeta_{1,\varepsilon} f^k}{d^-})\|_{L^2} + \|SP^-(\frac{\zeta_{1,\varepsilon} f^k}{d^-})\|_{L^2} \]

\[ \leq O(\varepsilon) + \frac{1}{4}\|f^k\|_{L^2} \]

for \( \varepsilon \) arbitrary. so we have

\[ \|f^k\|_{L^2} \geq \frac{\tau}{13} \]

for all \( k \).

**Case 2.** If \( i = 2 \), we have

\[ \zeta_{2,\varepsilon} T_{d^+,d^-}(c^k) = T_{d^+,d^-}(\zeta_{2,\varepsilon} c^k) = \zeta_{2,\varepsilon} f^k \]

\[ = d^- (\zeta_{2,\varepsilon} c^k) + d^+ (\zeta_{2,\varepsilon} c^k)^- = \zeta_{2,\varepsilon} f^k \]

Dividing both side by \( d^+ \) and notice that \( |\frac{d^-}{d^+}| \leq 1 - \frac{\tau}{2}\varepsilon \) on \( \Omega_1^+ \), so we have

\[ \|\frac{\zeta_{2,\varepsilon} f^k}{d^+}\|_{L^2(\Omega_1^+)} = \|\frac{d^-}{d^+}(\zeta_{2,\varepsilon} c^k) + (\zeta_{2,\varepsilon} c^k)^-\|_{L^2(\Omega_1^+)} \]

\[ \geq \|\zeta_{2,\varepsilon} c^k\|_{L^2(\Omega_1^+)} (1 - \frac{\tau}{2}\varepsilon)\|\zeta_{2,\varepsilon} c^k\|_{L^2(\Omega_1^+)} \]

\[ = \frac{\tau}{2}\varepsilon\|\zeta_{2,\varepsilon} c^k\|_{L^2(\Omega_1^+)} \].
Therefore, we have
\[ \frac{\tau}{2} \varepsilon \left( \frac{1}{3} - O(\varepsilon) \right) \leq \frac{\tau}{2} \varepsilon \| \zeta_{2 \varepsilon, \varepsilon} e^k \|_{L^2(\Omega^+)} \]
\[ \leq \frac{\varepsilon}{d^+} \left( \frac{\zeta_{2 \varepsilon, \varepsilon} e^k}{d^+} \right) \|_{L^2(\Omega^+)} \leq \frac{2}{\tau} \| f^k \|_{L^2}. \]
Then fix a small \( \varepsilon \) such that the left end is a positive constant. We get
\[ \| f^k \|_{L^2} \geq C \tau^2 \]
where \( C \) is a constant depending only on \( C^1 \)-norm of \( d^\pm \).

\[ \square \]

Remark 6.11. We should notice that this lower bounded \( C_0 \) can be chosen as a continuous function \( C_0(\tau, \| d^+ \|_{C^1}, \| d^- \|_{C^1}) \). Moreover, if we have a sequence of \( \{d^\pm(k)\} \) such that the corresponding \( \tau(k), \| d^\pm(k) \|_{C^1} \) are bounded and do not accumulate at 0, then \( \inf \{C_0(\tau(k), \| d^\pm(k) \|_{C^1}, \| d^\pm(k) \|_{C^1}) \} > 0 \).

6.4. General cases. Now we turn to the proof of the general case: \( d^\pm \) have infinite many Fourier modes. We will prove the following theorem.

Theorem 6.12. Let
\[ T_{d^+, d^-} = d^- c - \frac{d^+ c}{\| d^+ \|_{C^1}} \]
be the operator from \( L^2 \) to \( L^2 \), with the following constraint:
\[ |d^+|^2 + |d^-|^2 \neq 0. \]
Moreover, suppose that
\[ \| d^+ \|_{C^1}, \| d^- \|_{C^1} < \infty. \]
Then we have \( T_{d^+, d^-} \) is a Fredholm operator and the index will be 0.

Proof. Step 1. To prove this theorem, notice that we can approximate the operator \( T_{d^+, d^-} \) by a sequence of Fredholm operators \( \{T_{d^+, d^-}(k)\}_{k \in \mathbb{N}} \), where \( d^\pm(k) \) are summations of the first \( k \) Fourier modes of \( d^\pm \). Since the Fredholm operators form an open set in \( \text{Hom}(L^2) \), this is insufficient to say that \( T_{d^+, d^-} \) itself is a Fredholm operator. However, recall that we have the following well-known equivalent statement for the Fredholm operators [9].

Lemma 6.13. Let \( X \) be a Hilbert space and \( F \in \text{Hom}(X) \). Then \( F \) is a Fredholm operator iff there is \( S \in \text{Hom}(X) \) such that
\[ SF = FS = I \text{ mod}(\text{Com}(X)) \]
where \( \text{Com}(X) \) is the subspace(ideal) consisted by all compact operators mapping from \( X \) to itself.

Now since \( T_{d^+, d^-}(k) \) is a Fredholm operator for all \( k \in \mathbb{N} \) by Proposition 6.3, there exists a sequence of right inverse \( \{S^k\} \) such that
\[ T_{d^+, d^-}(k) S^k = I \text{ mod}(\text{Com}(X)). \]
Suppose that \( \| S^k \| \) is bounded uniformly by a number \( K \). For any \( \varepsilon > 0 \), there exists a constant \( N > 0 \) such that \( \| T_{d^+, d^-}(k) - T_{d^+, d^-}(N) \| \leq \varepsilon \) for all \( k \geq N \). So we have
\[ T_{d^+, d^-} S^N = T_{d^+, (N), d^-, (N)} S^N + O(\varepsilon) S^N = I + O(\varepsilon) S^N \text{ mod}(\text{Com}(X)). \]
Since $\|O(\varepsilon)S^N\| \leq O(\varepsilon)K$, we can choose $\varepsilon$ small enough such that $\|O(\varepsilon)S^N\| \leq \frac{1}{2}$. Therefore, we have $I + O(\varepsilon)S^N$ invertible. Let $V$ be the inverse of $I + O(\varepsilon)S^N$, we have

$$\mathcal{T}_{d^+,d^-}S^NV = I \mod(\text{Com}(X)).$$

So $\mathcal{T}_{d^+,d^-}$ has the right inverse $S^NV$ modulo the ideal of compact operators. Similar result is true for the existence of the left inverse. Therefore, it is a Fredholm operator.

**Step 2.** In step 1, we prove that if there is a uniform bound for $\{\|S^k\|\}$, then Theorem 6.12 will be immediately true. To prove this claim, we should know how to construct inverse $S^k$ for each $k$. In the following paragraphs, we use $\mathcal{T}^k$ to denote the operator $\mathcal{T}_{d^+,k,d^-,k}$ and $\mathcal{T}$ to denote $\mathcal{T}_{d^+,d^-}$.

A standard way to construct $S^k$ is to use the decomposition $L^2 = N(\mathcal{T}^k) \oplus N(\mathcal{T}^k)^\perp = R(\mathcal{T}^k) \oplus N(\mathcal{T}^k)$, where $\mathcal{T}^k$ is the operator $S^k$ to denote. By standard Fredholm alternative, we have $N(\mathcal{T}^k)^\perp \rightarrow R(\mathcal{T}^k)$ is a bijection. Therefore, by open mapping theorem (see [15]), there is a bounded inverse map $\mathcal{S}^k : R(\mathcal{T}^k) \rightarrow N(\mathcal{T}^k)^\perp$.

Here we should imitate this idea to construct $S^k$. Here we know that $\mathcal{T}^k : (I - P_L)(L^2) \rightarrow \mathcal{T}^k((I - P_L)(L^2)) \subset (I - P_L)(L^2)$ is a bijection, where $L$ is the number given by Lemma 6.10. Moreover, we can prove that $\mathcal{T}^k((I - P_{2k})(L^2))$ is a closed subspace by using the argument in step 6.7.8 in section 6.3. Therefore, we have a bounded inverse $\mathcal{R}^k : \mathcal{T}^k((I - P_L)(L^2)) \rightarrow (I - P_L)(L^2)$. Meanwhile, Remark 6.11 tells us that $\mathcal{R}^k$ have a uniform bounded norm. Now we set our $S^k$ to be $\mathcal{S}^k \circ \mathcal{R}(\mathcal{T}^k)$.

**Step 3.** Finally, we should prove that $S^k$ is actually an inverse of $\mathcal{T}^k$, modulo the ideal of compact operators. To prove this, just recall that both $(I - P_L)(L^2)$ and $\mathcal{T}^k((I - P_L)(L^2))$ are finite codimensional. We denote $(I - P_L)(L^2) = A$ and $\mathcal{T}^k((I - P_L)(L^2)) = B$ for a while, so $L^2 = A \oplus A^\perp = B \oplus B^\perp$ and

$$(\mathcal{T}^kS^k - I)(v) = 0 \text{ for any } v \in B.$$ 

So for any bounded sequence $\{u^k = (u_1^k, u_2^k) \in B \oplus B^\perp = L^2\}$, we have

$$(\mathcal{T}^kS^k - I)(u^k) = (\mathcal{T}^kS^k - I)(u_2^k)$$

where $\{u_2^k\}$ lies in a finite dimensional space $B^\perp$. We can get a convergence subsequence of $\{u^k_2\}$ easily. This implies

$$(\mathcal{T}^kS^k - I) = 0 \mod(\text{Com}(X)).$$

Similarly, we have $(S^k\mathcal{T}^k - I) = 0 \mod(\text{Com}(X))$, too. Therefore, we finish our proof for the Fredholmness.

The computation of the index is simple: One can choose $d^+$ be nonzero everywhere and $d^- = 0$. In this case, $\mathcal{T}_{d^+,d^-}$ is invertible. So $\text{index}(\mathcal{T}_{d^+,d^-}) = 0$. \hfill \Box

Remember that $(d^+, d^-)$ is the leading term of an $L^2_1$ harmonic section, so by Proposition 3.7, it is smooth. Meanwhile, notice that $\mathcal{T}_{d^+,d^-}$ maps from $L^2_k$ to $L^2_k$.
for any \( k \in \mathbb{N} \). We can easily show that all these maps are Fredholm by using the same argument.

### 6.5. Relations between \( T \) and the original equation.

Recall that by the argument in section 6.2, we want to solve the equation

\[
\begin{align*}
  d^+ \eta + c &= -2h^+, \\
  d^- \tilde{\eta} + c_{\text{ps}} &= -2h^- 
\end{align*}
\]

which will give us the equation \( T_{d^+, d^-}(c) = -2( \tilde{d}^- h^+ - d^+ \tilde{h}^- ) \). Here we define the map \( \mathcal{J} \) by \( \mathcal{J}(h^+, h^-) = -2( \tilde{d}^- h^+ - d^+ \tilde{h}^- ) \) and the map \( \mathcal{O} \), which maps from \( \ker(T) \) to \( L^2(S^1; \mathbb{C}) \), by

\[
\mathcal{O}(c) = \frac{\tilde{d}^+ c + d^- c_{\text{ps}}}{|d^+|^2 + |d^-|^2}.
\]

This map will give us \( \eta \) when \( h^\pm = 0 \).

Now by using the notations in section 6.1, we can always be decomposed the pair \( (u^+, u^-) \in L^2(S^1; \mathbb{C}) \times L^2(S^1; \mathbb{C}) \) as \( \pi^+(u^+, u^-) + \pi^-(u^+, u^-) \). By using this proposition and the Fredholmness of \( T_{d^+, d^-} \), we can find a finite dimensional vector space \( U \subset \text{Exp}^+ \) such that \( \text{range}(T_{d^+, d^-}) \oplus \mathcal{J}(U) = L^2 \).

### 7. Proof of the main theorem: Part I

In this section, we will prove Theorem 1.1 in the version without showing \( f \) is \( C^1 \). In the next section, we will prove that \( f \) is a \( C^1 \) map.

In the rest of this paper, we suppose \( p^+ = 0 \), since the general case can be obtained by the same argument and this assumption can simplify our notations effectively. Meanwhile, the argument in this section assumes that the metric \( g \) defined on a tubular neighborhood is Euclidean. The case with a general metric is more complicated but follows the similar argument, see Appendix 9.1 for details.

### 7.1. Reformulate \( \mathbb{K}_1 \) and \( \mathbb{K}_0 \).

The definition of \( \mathbb{K}_0 \) and \( \mathbb{K}_1 \) are given by (6.7) and (6.8). We also notice that \( \ker(D|_{L^2}) = B(\ker(D|_{L^2}) \oplus \ker(D|_{L^1})) \). Use \( \mathbb{H}_1 \) to denote the space \( \mathcal{O}[B(\ker(T_{d^+, d^-} \circ B))] \). In addition, we define \( \mathbb{H}_0 = \text{coker}(T_{d^+, d^-} \circ B) \). Then \( \mathbb{K}_1 \) and \( \mathbb{K}_0 \) can be written as follows

\[
\begin{align*}
  \mathbb{K}_1 &\cong \mathbb{H}_1 \times \ker(D|_{L^2}); \\
  \mathbb{K}_0 &\cong \mathbb{H}_0 \times \ker(D|_{L^1}).
\end{align*}
\]

To prove this, we notice that the map \( \mathcal{O} \) is injective on \( \ker(T_{d^+, d^-}) \) since the equation

\[
\begin{align*}
  \tilde{d}^+ c + d^- c_{\text{ps}} &= 0, \\
  d^- c - d^+ c_{\text{ps}} &= 0
\end{align*}
\]

implies \( c = 0 \). So

\[
\mathbb{H}_1 \times \ker(D|_{L^2}) \cong B(\ker(T_{d^+, d^-} \circ B)) \oplus \ker(D|_{L^1}) \cong \ker(\ker(T_{d^+, d^-} \circ B)).
\]
7.2. Basic setting. Before we start our argument, we define some notations.

Firstly, in the following paragraphs, we fix $r < \frac{R}{T}$, $T > 1$ for a moment. The precise values of $r$ and $T$ will be specified later. Moreover, let us assume that $\|T_{d_1,d_2}|_{\text{range}(\tau_{s,1}^{a_0,d_2})}\| \leq 1$.

Secondly, we suppose that there exists $t_0 > 0$ which is the upper bound for $s$. The precise value of $t_0$ can be assumed to decrease between each successive appearance. We also define the following notations.

**Definition 7.1.** For any $A \subset M$, we call a section $u : [0,t_0] \times A \rightarrow S \otimes I$ is in $C^\omega([0,t_0]; L^2_1(A; S \otimes I))$ if and only if $\|u(s, \cdot)|_{L^2_1(A; S \otimes I)} < \infty$ for all $s \in [0,t_0]$ and $u(\cdot, x) : [0,t_0] \rightarrow (S \otimes I)_x$ varies analytically on $[0,t_0]$ (The remainder of Taylor series will converge to zero in $L^2$-sense).

**Definition 7.2.** For any $i \in \mathbb{N}$, $k > 0$, we define

\begin{equation}
\mathfrak{C}^i_{k+1} = \{ f \in C^\omega([0,t_0]; L^2_1(M-N_2; S \otimes I)) : \|f(s, \cdot)|_{L^2_1} \leq \frac{k}{T^i} \};
\end{equation}

\begin{equation}
\mathfrak{B}^i_{k+1} = \{ f \in C^\omega([0,t_0]; L^2(N_{M-N_2}; S \otimes I)) \} \|f(s, \cdot)|_{L^2_1} \leq \frac{k}{T^i} \};
\end{equation}

\begin{equation}
\mathfrak{C}^i_{r+1} = \{ f \in C^\omega([0,t_0]; L^2(N_{\frac{1}{r_2}}; S \otimes I)) \} \|f(s, \cdot)|^2_{L^2(N_{M-N_2})} \leq \kappa(r_1 - r_2)^{(\frac{r}{T^i})^2} \text{ for all } r_2 < r_1 \leq \frac{r}{T^i} \}.
\end{equation}

Thirdly, suppose that we perturb the metrics $g$ on the region $M - N_2$ analytically with the parameter $s$. Let us call this family of perturbed metric $g^*$. We use the notation $D_{\text{pert}} = D + T^*$ to denote the Dirac operator perturb by metric. The operator $T^* : L^2 \rightarrow L^2_1$ will be a first order differential operator with its operator norm $\|T^*\| \leq C$s.

Therefore, we have

\[ D_{\text{pert}} \psi = s\psi \]

for some $\psi = T^*(\psi) \in C^\omega([0,t_0]; L^2)$.

To prove Theorem 1.1, we need to prove the following claim: There exists $\varepsilon > 0$ with the following significance. For any $\xi \in \mathbb{H}_1$ with $\|\xi\|_{L^2_2} = \varepsilon$ there exist $\eta_s \in C^\omega([0,t_0]; C^1(S^1; \mathbb{C}))$ and $t_s \in \{ u \in L^2|B(u) \in \mathbb{H}_0 \}$ such that

\begin{equation}
D_{\text{pert},\eta_s}(\psi + s\psi) = 0
\end{equation}

for all $s \in [0,t_0]$ with the constraint $\eta_s = s\xi + \eta_s^\perp, \eta_s^\perp \perp \mathbb{H}_1$. Moreover, we have to show these data $(\eta_s, t_s)$ will be homeomorphic to an open set in $\mathbb{R}^k$ with $k = \text{dim}(\ker(D|_L^2))$. By using this claim, we can define the map $f$ by $f(g^*, s\xi, \psi) = B(s\psi)$ for any $\psi \in \text{dim}((\ker(D|_L^2)))$ with $\|\psi\|_2$ small. Finally, we shall prove that $f$ is $C^1$. 

So I separate my proof into two parts. In this section, I will prove that there exists \((\eta_*, t_*)\) satisfying (7.4). In the next section, I will prove the set of data \((\eta_*, t_*)\) satisfying (7.4) will be homeomorphic to an open set in \(\mathbb{R}^k\) with \(k = dim(ker(D|_{L^2}))\) and \(f\) is a \(C^1\)-map.

**Remark 7.3.** In fact, the \(\xi\) we choose in our claim can be a smooth map \(\xi : [0, t_0] \to \mathbb{H}_1\) with \(\|\xi\|_{L^2} = \varepsilon\) and \(\psi\) can be replaced by a smooth family \(\psi(s) \in ker(D|_{L^2})\). The argument in the rest of this section will still hold under this setting.

7.3. **Part I of the proof:** First order approximation of \(\eta_*\) and \(t_*\). Now we are ready to prove our claim. I separate this part into 10 steps.

**Step 1.** In this and the next step, we will denote by \(\kappa_0\) an \(O(1)\) constant. The precise value of \(\kappa_0\) can be assumed to increase between each successive appearance.

By using Proposition 6.2, there exists \(h_0 \in L^2\) such that
\[
Dh_0 = f_0 \mod(ker(D|_{L^2}))
\]
So we have
\[
D_{pert}(\psi - sh_0) = -sT^*(h_0) \mod(ker(D|_{L^2})).
\]
Since \(T^*\) is a first order differential operator, we have
\[
\|T^*(h_0)\|_{L^2} \leq C|s|h_0\|L^2 \leq C|s|f_0\|L^2.
\]
This implies
\[
sT^*(h_0) \in s^2\mathbb{A}^1
\]
by taking \(\kappa_0 \geq 2C|f_0|L^2\) large enough.

**Step 2.** In this step we construct the data of perturbation \(\eta_0\) and prove \(\eta_0\) will satisfy the condition \((4.10), (4.11), (4.12)\).

Since \(f_0 = 0 \mod(ker(D|_{L^2}))\) on \(N_r\), we have \(Dh_0 = 0\) on \(N_r\). So by Proposition 3.4, we can write
\[
h_0 = \begin{pmatrix} h_0^+ \\ \frac{\sqrt{\varepsilon}}{h_0^+} \end{pmatrix} + h_{2r,0}.
\]
on \(N_r\). By Theorem 6.12, there exists \((\eta_0, c_0)\) such that
\[
\begin{cases} 2h_0^+ + d^+\eta_0 + c_0 = k_0^+ \\ 2h_0^- + d^-\eta_0 + c_{aps} = k_0^-
\end{cases}
\]
where \((k_0^+, k_0^-) \in \mathbb{H}_0\) and \((k_0^+, k_0^-) \perp (2h_0^+ - k_0^+, 2h_0^- - k_0^-)\) in \(L^2\)-sense. So there is a corresponding \(c_0\) which satisfies \(Dc_0 = 0\) on \(M - \Sigma\) and
\[
c_0 = \begin{pmatrix} \frac{c_0^+}{\sqrt{\varepsilon}} \\ \frac{c_0^-}{\sqrt{\varepsilon}} \end{pmatrix} + c_{2r,0}.
\]
Since we have \(h_0\) satisfies \(Dh_0 = f_0 \mod(ker(D|_{L^2}))\) which is given by Proposition 6.2, so
\[
\|h_0^+\|_{L^2}^2, \|h_0^+\|_{L^2}^2, \|h_0^+\|_{L^2}^2, \|h_0^+\|_{L^2}^2 \leq C |h_0|_{L^2}^2 \leq C |f_0|_{L^2}^2
\]
Furthermore, since (7.5)

\[ \|h^\pm_0\|_{L^2}^2 \leq \frac{\kappa_0}{2} r^2, \quad \|(h^\pm_0)_{tL}\|_{L^2}^2 \leq \frac{\kappa_0}{2} r, \]

\[ \|(h^\pm_0)_{tL}\|_{L^2}^2 \leq \frac{\kappa_0}{2}, \quad \|b_0\|_{L^2}^2 \leq \kappa_0. \]

Moreover, since \( T_{d^+,d^-}(c_0) = d^-h^+ - d^+h^- \ mod(\mathcal{J}(\mathbb{H}_0)) \), we can choose \( c_0 \) such that

\[ \|c_0\|_{L^2}^2 \leq \frac{\kappa_0}{2} r^2, \quad \|(c_0)_{tL}\|_{L^2}^2 \leq \frac{\kappa_0}{2} r, \]

\[ \|(c_0)_{tL}\|_{L^2}^2 \leq \frac{\kappa_0}{2}, \quad \|c_0\|_{L^2}^2 \leq \kappa_0. \]

\( \eta_0 = \frac{d^+}{(d^+)^2 + (d^-)^2} (h^+_0 - 2h^+_0 - c_0) + \frac{d^-}{(d^+)^2 + (d^-)^2} (h^-_0 - 2h^-_0 - c_0^\text{op}) \) will satisfy (4.10), (4.11) and (4.12), so it satisfies (4.13), (4.14) and (4.15).

We should notice that the condition \( \kappa_0 \geq 2 \frac{C}{r^2} \|f_0\|_{L^2}^2 \) will give us a constraint for \( q^\ast \). In the following paragraphs, we should always assume \( \|f_0\|_{L^2}^2 \leq r^2 \). This assumption will give us some restriction to define \( N \) in Theorem 1.1. We will discuss this part in section 7.5.

By this setting, we will also have

\[ \|k^\pm_0\|_{L^2}^2 \leq \frac{\kappa_0}{2} r^2, \quad \|(k^\pm_0)_{tL}\|_{L^2}^2 \leq \frac{\kappa_0}{2} r, \quad \|(k^\pm_0)_{tL}\|_{L^2}^2 \leq \frac{\kappa_0}{2}. \]

Furthermore, since

\[ \|T^\ast(c_0)\|_{L^2} \leq C s \|c_0\|_{L^2} \leq s \frac{\kappa_0}{2} \]

so we have \( sT^\ast(c_0) \in s^2 \mathfrak{A}_{1,t}^\text{op} \).

Finally, notice that we still have some options for the choice of \( c_0 \). We can choose another \( c_0 \) by adding an element in \( B[\ker(T \circ B)] \). So we choose \( c_0 \) such that the corresponding \( \eta_0 = \xi + \eta_0^\perp \) with \( \eta_0^\perp \perp \mathbb{H}_1 \) and \( \xi \) satisfying (4.10), (4.11) and (4.12) (replacing \( \eta \) by \( \xi \)). So (7.6) still holds in this case.

**Remark 7.4.** We know that \( \eta_0 \) satisfies (4.10), (4.11) and (4.12). By using the same argument in the proof of (4.13), we have

\[ \|\eta_0\|_{L^2}^2 \leq C(\|\eta_0\|_{L^2}^2 + \|\eta_0\|_{L^2} + \|\eta_0\|_{L^2} + \|\eta_0\|_{L^2} + \|\eta_0\|_{L^2} \leq C \kappa_0^2 r. \]

Meanwhile, we can estimate the following H"older seminorm (1 follow the standard way to estimate the H"older norms, readers can see [13] for details):

\[ [\eta_k]_{a, \frac{1}{4}} = \sup_{a \neq b} \frac{|\eta_k(a) - \eta_k(b)|}{|a - b|^{1/4}}. \]

When \( |a - b| \leq r \), we have

\[ [\eta_k]_{a, \frac{1}{4}} \leq \frac{1}{|a - b|^{1/4}} \int_a^b \delta_t |\eta_k|(s) ds \leq \sup \|\eta_k\|_{L^2} |a - b|^{1/4} \leq C \kappa_0 r^{1/4}; \]
when $|a - b| > r$, we have
\[
\|\eta_t\|_{0, \frac{1}{t^+}} \leq C \sup |\eta_t| \frac{1}{t^+} \leq C\kappa_0 \frac{1}{t^+}.
\]
So we have the Hölder estimate
\[
(7.8) \quad \|\eta_0\|_{C^1, \frac{1}{t^+}} \leq C\kappa_0 \frac{1}{t^+}.
\]

**Remark 7.5.** We should also notice that the choice of $(\eta_0, \kappa_0^\perp)$ is unique. More precisely, for any $\xi \in \mathbb{H}_1$, the choice of $\eta_0^\perp$ is unique.

**Step 3.** Now we can fix $\kappa_0$ forever. In this and the next steps, we will determine another constant $\kappa_1 = O(\kappa_0)$. The precise value of $\kappa_1$ can be assumed to increase between each successive appearance. First of all, since $\eta_0$ satisfies (4.13), (4.14) and (4.15), we should assume that $\kappa_1$ is the constant appearing in these estimates in the beginning.

On $N_R$, we can define
\[
h_0^b = \chi_0 \left( \frac{\kappa_0^+}{\sqrt{z}} \right); \quad \psi_0^b = \psi_0 \left( \frac{\kappa_0^+}{2\sqrt{z}} \right); \quad \xi_0^b = \chi_0 \left( \frac{-i\kappa_0^+}{\sqrt{z}} \right).
\]
We also define $h_0^\perp = h_0 - h_0^b$ and $\psi_0^\perp = \psi_0 - \psi_0^b$.

So we have
\[
D_{\text{pert}}(\psi + s\psi_0 - s\psi_0) = sT^s(\psi_0 - h_0)
\]
\[
= D_{\text{pert}}(\psi + s\psi_0^\perp - s\psi_0^\perp) + D_{\text{pert}}(s\psi_0^b - s\psi_0^b)
\]
\[
= D_{\text{pert}}(\psi + s\psi_0^\perp - s\psi_0^\perp) + D_{N_R}(s\psi_0^b - s\psi_0^b) \mod(\ker(D_{L_1})).
\]

Notice that
\[
D_{|N_R}(s\psi_0^b - s\psi_0^b) = sD_{|N_R}(\chi_0 \left( \begin{array}{c}
-\frac{c_0 - 2h_0^+}{2\sqrt{z}} \\
-\frac{c_0 - 2h_0^+}{2\sqrt{z}}
\end{array} \right))
\]
\[
= s\chi_0 \left( \begin{array}{c}
\frac{i\psi_0 + \frac{2h_0^+}{2\sqrt{z}}} {2\sqrt{z}} \\
\frac{i\psi_0 + \frac{2h_0^+}{2\sqrt{z}}} {2\sqrt{z}}
\end{array} \right) + s\psi_0(\chi_0)\psi_0^b - s\psi_0(\chi_0)h_0^b
\]
\[
= s\chi_0 \left( \begin{array}{c}
\frac{-i\psi_0 + \psi_0^b}{\sqrt{z}} \\
\frac{-i\psi_0 + \psi_0^b}{\sqrt{z}}
\end{array} \right) + s\psi_0(\chi_0)\psi_0^b - s\psi_0(\chi_0)h_0^b
\]
\[
= s\chi_0 \left( \begin{array}{c}
\frac{-i\psi_0}{\sqrt{z}} \\
\frac{-i\psi_0}{\sqrt{z}}
\end{array} \right) + s\chi_0 \left( \begin{array}{c}
\frac{-i\psi_0}{\sqrt{z}} \\
\frac{-i\psi_0}{\sqrt{z}}
\end{array} \right) + s\psi_0(\chi_0)\psi_0^b - s\psi_0(\chi_0)h_0^b
\]
\[
- s\psi_0(\chi_0)\psi_0^b - sD_{|N_R}(\xi_0^b).}
\]
So we have

\[
s\sigma(\chi_0)\chi_0^{b} - s\sigma(\chi_0)\eta_0^{b} - s\sigma(\chi_0)\theta_0^{b} = s \left( \frac{\chi_z \epsilon_0^{a \pm} - \eta_0^{b}}{\sqrt{\chi}} - \frac{\chi_z \epsilon_0^{a \pm} - \eta_0^{b}}{\sqrt{\chi}} \right) = s \left( -\chi_z \eta_0^{d \pm} \frac{\eta_0^{d \pm}}{\sqrt{\chi}} \right).
\]

We can check that

\[
s\chi_0 \left( \frac{-i d^{+} \eta_0 \sqrt{\chi}}{i d^{+} \eta_0 \sqrt{\chi}} \right) + s\sigma(\chi_0)\epsilon_0^{b} - s\sigma(\chi_0)\theta_0^{b} - s\sigma(\chi_0)\eta_0^{b} = \Theta_0^0(\psi).
\]

So

\[(7.9) \quad D|_{N_R}(-s\epsilon_0^{b} - s\eta_0^{b}) = s\chi_0 \left( \frac{-i d^{+} \eta_0 \sqrt{\chi}}{i d^{+} \eta_0 \sqrt{\chi}} \right) + \Theta_0^0(\psi) - D_{pert}(s\epsilon_0^{b}).\]

Meanwhile, we define

\[
\epsilon_0 = \chi_0 \left( \frac{-i d^{-} \eta_0 \sqrt{\chi}}{i d^{-} \eta_0 \sqrt{\chi}} \right)
\]

which satisfies \(D(s\epsilon_0) = \chi_0 \left( \frac{-i d^{-} \eta_0 \sqrt{\chi}}{i d^{+} \eta_0 \sqrt{\chi}} \right) + se_1 \partial_1 \epsilon_0 + sD(\chi_0)(\frac{\epsilon_0}{\chi_0}).\) So we can simplify \((7.10)\) as follows:

\[(7.10) \quad D|_{N_R}(-s\epsilon_0^{b} - s\eta_0^{b}) = D(s\epsilon_0) - se_1 \partial_1 \epsilon_0 + \Theta_0^0(\psi) - sD(\chi_0)(\frac{\epsilon_0}{\chi_0}) - D_{pert}(s\epsilon_0^{b}).\]

Recall that the Dirac operator \(D_{s\chi_0\eta_0}\) can be written as

\[
D_{s\chi_0\eta_0} = (1 + \Theta^0)D + s((\chi_0)z \eta_0 + (\chi_0)z \eta_0) e_1 \partial_1 + \Theta_s^0 + \mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0
\]

\[
= (1 + \Theta^0)D + s((\chi_0)z \eta_0 + (\chi_0)z \eta_0) e_1 \partial_1 + \Theta_s^0 + \mathcal{W}_s^0 + \mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0
\]

where

\[
\Theta_s^0 = e_1(s \chi_0 \eta_0 \partial_z + s \chi_0 \eta_0 \partial_z),
\]

\[
\mathcal{W}_s^0 = e_2(s \chi_0 z \eta_0 \partial_z - s \chi_0 z \eta_0 \partial_z) + e_3(-s \chi_0 z \eta_0 \partial_z + s \chi_0 z \eta_0 \partial_z).
\]
We use the following notations to simplify the upcoming equation:

\begin{align}
\mathcal{W}_x^0(s\xi_0) + \varrho^0(e_2\partial_z + e_3\partial\bar{z})(s\xi_0) &= s^2\mathcal{B}_1; \\
-\mathcal{W}_x^0(s\xi_0^b - sh_0^b) - \varrho^0(e_2\partial_z + e_3\partial\bar{z})(s\xi_0^b - sh_0^b) &= s^2\mathcal{B}_2; \\
\mathcal{W}_x^0(s\xi_0') + \varrho^0(e_2\partial_z + e_3\partial\bar{z})(s\xi_0') &= s^2\mathcal{B}_3; \\
\eta(s((\chi_0)z\bar{\eta}_0 + (\chi_0)z\bar{\eta}_0)e_1\partial_1\psi = s\mathcal{C}_0; \\
-sD(\chi_0)(\xi_0) + (\varrho^0 - 1)e_1\partial_z(s\xi_0) + s((\chi_0)z)\bar{\eta}_0 + (\chi_0)z\bar{\eta}_0)e_1\partial_t(s\xi_0) &= s\mathcal{C}_1; \\
-\varrho^0e_1\partial_t(s\xi_0^b - sh_0^b) - s((\chi_0)z)\bar{\eta}_0 + (\chi_0)z\bar{\eta}_0)e_1\partial_t(s\xi_0^b - sh_0^b) &= s\mathcal{C}_2; \\
\varrho^0e_1\partial_t(s\xi_0') + s((\chi_0)z)\bar{\eta}_0 + (\chi_0)z\bar{\eta}_0)e_1\partial_t(s\xi_0') &= s\mathcal{C}_3; \\
\phi(s\xi_0') &= s^2\mathcal{Q}_1; \\
-\phi(s\xi_0^b - sh_0^b) &= s^2\mathcal{Q}_2 \\
\phi(s\xi_0') &= s^2\mathcal{Q}_3.
\end{align}

where

\[
\xi_0' = -(\chi_0z + \chi\bar{\xi}\bar{\eta})\psi - \begin{pmatrix} 0 & i\chi\dot{\eta} \\ -i\chi\dot{\eta} & 0 \end{pmatrix} \psi
\]

which satisfies the equation

\[
D(s\xi_0') = -\mathcal{F}_x^0(\psi).
\]

Now by using the fact that \(D\psi = 0\), (7.10) yields

\[
D|_{\mathcal{N}_\alpha}(-s\xi_0^b - sh_0^b) = D(s\xi_0) - se_1\partial_t\xi_0 + D_{\chi\phi\eta}(\psi) - (A_0^0 + \mathcal{F}_x^0 + \mathcal{R}_x^0)(\psi) + s\mathcal{C}_1 - s\xi_0^b
\]

\[
= D_{\chi\phi\eta}(s\xi_0) + D_{\chi\phi\eta}(\psi) - (A_0^0 + \mathcal{F}_x^0 + \mathcal{R}_x^0)(\psi + s\xi_0)
\]

\[
+ s^2\mathcal{B}_1 + s\mathcal{C}_0 + s\mathcal{C}_1 - s^2\mathcal{Q}_1 - D_{\text{pert}}(s\xi_0^b).
\]

Therefore, we have

\[
D_{\text{pert}}(\psi + s\xi_0 - sh_0) = sT^\alpha(\xi_0 - h_0)
\]

\[
= D_{\text{pert}}(\psi + s\xi_0^b - sh_0^b + s\xi_0') + D_{\chi\phi\eta}(\psi + s\xi_0) - \mathcal{F}_x^0(s\xi_0)
\]

\[
- (A_0^0 + \mathcal{R}_x^0)(\psi + s\xi_0) + s^2\mathcal{B}_1 + s(\mathcal{C}_0 + \mathcal{C}_1) - s^2\mathcal{Q}_1 - D_{\text{pert}}(s\xi_0^b)
\]

\[\text{mod}(\ker(D|_{\mathcal{L}_z^3})).\]

So

\[
D_{\chi\phi\eta,\text{pert}}(\psi + s\xi_0^b - sh_0^b + s\xi_0) = (A_0^0 + \mathcal{R}_x^0 + \mathcal{F}_x^0)(s\xi_0 + s\xi_0' + s\xi_0^b - sh_0^b)
\]

\[
+ (A_0^0 + \mathcal{R}_x^0)(\psi) + sT^\alpha(\xi_0 - h_0)
\]

\[
- s^2(\sum_{i=1}^{3}B_i) - s(\sum_{i=0}^{3}C_i) + s^2(\sum_{i=0}^{3}Q_i)
\]

\begin{align}
(7.21) \\
- D_{\text{pert}}(s\xi_0^b) \mod(\ker(D|_{\mathcal{L}_z^3})).
\end{align}
Here we will show that

\begin{equation}
(7.22) \quad \mathcal{A}_s^0(\psi) + (\mathcal{A}_s^0 + \mathcal{F}_s^0)(s\varepsilon_0 + s\varepsilon_0' + s\bar{\kappa}^2_0 - s\bar{h}_0) - s\left(\sum_{i=0}^{3} C_i\right) \in s\mathfrak{C}_{i+1}^1,
\end{equation}

\begin{equation}
(7.23) \quad \mathcal{R}_s^0(\psi + s\varepsilon_0 + s\varepsilon_0' + s\bar{\kappa}^2_0 - s\bar{h}_0) - s^2\left(\sum_{i=1}^{3} B_i\right) \in s^2\mathfrak{B}_{i+1}^1,
\end{equation}

\begin{equation}
(7.24) \quad sT^s(\varepsilon_0 - b_0) \in s^2\mathfrak{A}_{i+1}^0.
\end{equation}

We already show that \(sT^s(b_0) \in \mathfrak{A}_{i+1}^{2p}\) in step 1 and \(sT^s(\varepsilon_0) \in \mathfrak{A}_{i+1}^{2p}\) in step 2, so we only need to prove (7.22) and (7.23). We should also notice that \(\eta_0\) satisfies use (4.13), (4.14) and (4.15). So we have control for those terms in \(C_0, C_1, C_2, C_3\) which involve \(\eta_0\). Meanwhile, terms in \(C_2\) involve \(c_0^2 - \bar{b}_0^0\) can be taken care by Remark 3.10 b. Therefore, we can see that \(s(\sum_{j=0}^{3} C_j) \in s\mathfrak{C}_{i+1}^1\) for some \(\kappa_1\). Meanwhile, by Proposition 4.4, we have \(\mathcal{A}_s^0(\psi + s\varepsilon_0 + s\varepsilon_0' + s\bar{\kappa}^2_0 - s\bar{h}_0) \in s\mathfrak{C}_{i+1}^1\). So we prove (7.22).

Finally, by Proposition 4.4, we have

\[
\|\mathcal{R}_s^0(s\varepsilon_0 + s\varepsilon_0' + s\bar{\kappa}^2_0 - s\bar{h}_0)\|_{L^2} \leq \gamma_2 s^2 \|\psi\|_{L^2} + s\varepsilon_0 + s\varepsilon_0' - s\bar{h}_0 \|_{L^2}
\]

\[
\leq \gamma_2 s^2 \bar{\kappa}_1^2 s^2 \|\psi\|_{L^2} \leq \kappa_1 t^2 s^2
\]

for any \(s \leq \frac{1}{\gamma_2^2 \kappa_1^2}\). Meanwhile, by Proposition 4.6, we have

\[
\|\mathcal{R}_s^0(\psi)\|_{L^2} \leq C\gamma_2^2 \kappa_1^2 s^2 \leq \kappa_1 t^2 s^2
\]

for any \(t \leq \frac{1}{C\gamma_2^2 \kappa_1^2}\). So we prove (7.23).

**Step 4.** In this step we prove that there exists \(e' \in L^2_1\) such that \(D_{\chi\varepsilon_0}\left(s^2 e'_i\right) = s^2 Q_i + s^3 B + s^2 C\) for some \(B \in \mathfrak{B}_{i+1}^1\) and \(C \in \mathfrak{C}_{i+1}^1\) where \(i = 1, 2, 3\).

**Lemma 7.6.** Let \(Q\) be either of the type \(s^2\chi_0\left(\begin{array}{c} \frac{q^+(t)}{\sqrt{2}} \\ \frac{q^-(t)}{\sqrt{2}} \end{array}\right)\) or of the type \(s^2\chi_0\left(\begin{array}{c} \frac{q^+(t)}{\sqrt{2}} \\ \frac{q^-(t)}{\sqrt{2}} \end{array}\right)\)

with \(\|q^+\|_{L^2} \leq \kappa_1\), \(\|(q^-)\|_{L^2} \leq \kappa_1\). Then there exists an \(L^2_1\) section \(e'\) which can be written as

\[
e' = \sum_{i \geq 2} s^i \chi_0^i \left(\begin{array}{c} e_i^+(t) \sqrt{2} \\ e_i^-(t) \sqrt{2} \end{array}\right)
\]

for the first type and

\[
e' = \sum_{i \geq 0} s^i \chi_0^i \left(\begin{array}{c} e_i^+(t) \sqrt{2} \\ e_i^-(t) \sqrt{2} \end{array}\right)
\]

for the second such that \(D_{\chi\varepsilon_0}\left(s^2 e'\right) = s^2 Q + s^3 B + s^2 C\) for some \(B \in \mathfrak{B}_{i+1}^1\) and \(C \in \mathfrak{C}_{i+1}^1\) for all \(s \leq \frac{1}{2\gamma_2^2 \kappa_1^2}\). Furthermore, we have \(\|e'\|_{L^2_1} \leq 2\kappa_1\).

**Proof.** First of all, let \(Q\) is of the first type. We start with the element

\[
e'_0 = \chi_0 \left(\begin{array}{c} q^- \sqrt{2} \\ q^+ \sqrt{2} \end{array}\right).
\]
Under a straightforward direct computation, we have
\[ D(s^2\epsilon'_0) = s^2Q + s^2B + s^2C \]
with \( B \in \mathfrak{B}_1^{\gamma^+_2\alpha^2} \) and \( C \in \mathcal{C}_1^{\kappa^+1} \).
Recall that by Proposition 4.4, we have
\[ D_{s\chi_{0}\eta_0} = (1 + \varepsilon^0)D + s((\chi_0)x\eta_0 + (\chi_0)x\bar{\eta}_0)e_1\partial_1 + \Theta_0^0 + A_0^0 + F_0^0 + R_s^0. \]
By the argument proving the results (7.22) and (7.23), we have
\[ s((\chi_0)x\eta_0 + (\chi_0)x\bar{\eta}_0)e_1\partial_1 + (A_0^0 + F_0^0)(s^2\epsilon'_0) \in s^2\mathfrak{B}_1^{\gamma^+_2\nu^2}. \]
\[ \varepsilon^0D(s^2\epsilon'_0) + R_s^0(s^2\epsilon'_0) \in s^2\mathfrak{B}_1^{\gamma^+_2\nu^2}. \]
Meanwhile, recall that \( \Theta_0^0 = [e_1(s\chi\dot{\eta}\partial_z + s\chi\dot{\eta}\partial_z) + e_2(s\chi\bar{\eta}\partial_z - s\chi\bar{\eta}\partial_z) + e_3(-s\chi\bar{\eta}\partial_z + s\chi\eta\partial_z)] \). Here we recall the decomposition
\[ \Theta_0^0 = \hat{\Theta}_0^0 + \mathcal{W}_s^0. \]
Notice that \( \mathcal{W}_s^0 \) is an \( O(\kappa_1) \)-first order differential operator with its support on \( N_r - N_{\mathfrak{B}} \), which implies \( \mathcal{W}_s^0(s^2\epsilon'_0) \in s^2\mathfrak{B}_1^{\gamma^+_2\nu^2}. \) So we have
\[ \Theta_0^0(s^2\epsilon'_0) = \hat{\Theta}_0^0(s^2\epsilon'_0) + s^2B \]
for some \( B \in \mathfrak{B}_1^{\gamma^+_2\nu^2}. \) Moreover, since
\[ \Theta_0^0(s^2\epsilon'_0) = \chi_0\Theta_0^0(s^2\epsilon'_0) + \Theta_0^0(\chi_0)s^2\epsilon'_0 \]
with the second term in \( s^2\mathfrak{B}_1^{\gamma^+_2\nu^2} \), so we have
\[ \Theta_0^0(s^2\epsilon'_0) = \chi_0\Theta_0^0(s^2\epsilon'_0) + s^2B \]
for some \( B \in \mathfrak{B}_1^{\gamma^+_2\nu^2}. \)

Now we call \( Q_1 = \chi_0\Theta_0^0(s^2\epsilon'_0) \), which can be simplified as
\[ Q_1 = s\chi_0^2 \left( \begin{array}{c} q_1^+ (t) \\ \frac{q_1^+(t)}{\sqrt{s}} \\ q_1^-(t) \end{array} \right) \]
where
\[ q_1^+ = -i(\chi_0\dot{\eta}_0)q^+ \]
\[ q_1^- = -i(\chi_0\dot{\eta}_0)q^- . \]
By using the fact \( \|q^+\|_{L^2} \leq \kappa_1t \), \( \|q^+\|_{L^2} \leq \kappa_1t \), fundamental theorem of calculus and H"older’s inequality, we have \( \|q^+\|_{L^2} \leq \kappa_1t^2 \). Therefore, by using (4.13), we have \( \|q_1^\pm\|_{L^2} \leq \kappa_1^2t^2 \), \( \|q_1^\pm\|_{L^2} \leq \kappa_1^2t^2 \). So we have
\[ D_{s\chi_{0}\eta_0}(s^2\epsilon'_0) = s^2Q_1 + s^2B^0 + s^2C^0 \]
for some \( B^0 \in \mathfrak{B}_1^{\gamma^+_2\nu^2} \) and \( C^0 \in \mathcal{C}_1^{\kappa^+1} \).
Here we define an $L^2(S^1; \mathbb{C})$-module $V$ which is generalized by
\[
\{ \begin{pmatrix} z^a z^b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z^a z^b \end{pmatrix} \} | (a,b) \in (\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z} \text{ or } (a,b) \in \mathbb{Z} \times (\mathbb{Z} + \frac{1}{2}), a + b = \frac{1}{2} \}.
\]

Now, we define a linear map $J$ by the following rule:
\[
J \left( \frac{q^+ z^a z^b}{q^- z^b z^a} \right) = \left( -i\bar{\eta}_0 q^+ z^b z^a \right) + \frac{b}{a + 1} \left( -i\bar{\eta}_0 q^+ z^{b-1} z^{a+1} \right)
\]

This map is not well defined on the entire $V$ since it makes no sense when $a = -1$. However, if we start with $x = \left( \frac{q^+ z^a z^b}{q^- z^b z^a} \right)$ with $(a,b) = (\frac{1}{2}, 0)$ or $(a,b) = (0, \frac{1}{2})$, we can always define $J^n(x)$ for any $n$. Here we call the term $x = \left( \frac{q^+ z^a z^b}{q^- z^b z^a} \right)$ is of the type $(a,b)$. To prove that $J^n(x)$ is well-defined for all $n$ when $x$ is of the type $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$, we should prove that there is no term in $J^n(x)$ which is of the type $(\frac{1}{2}, \frac{3}{2})$ or $(\frac{3}{2}, -1)$. For the first case that the component appearing in $J^n(x)$ is of the type $(\frac{3}{2}, -1)$, it must be generated from a component in $J^{n-1}(x)$ of the type $(\frac{1}{2}, -1)$, which is a contradiction ($n$ is the smallest). For the second case that the component appearing in $J^n(x)$ is of the type $(\frac{3}{2}, -1)$, either this component comes from a component in $J^{n-1}(x)$ of the type $(\frac{1}{2}, -1)$, which is a contradiction again, or it comes from a component in $J^{n-1}(x)$ of the type $(\frac{1}{2}, -2)$. The later case is also impossible because we start from the term of the type $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. At each time we apply $J$ on it, it will only change $(a,b)$ by adding $(\pm 1, \pm 1)$. So there must be a number $m < n - 1$ such that $J^m(x)$ contains a component of the type $(\frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{2}, -1)$, which leads a contradiction. Therefore, all the components in $J^n(x)$ are not of the type $(\frac{3}{2}, -1)$, which means $J^n(x)$ is well-defined for all $n$.

Now we define $\zeta'_k$ inductively by
\[
\zeta'_k = x^{k+1} J \left( \frac{\zeta'_{k-1} - \zeta'_{k-2}}{\lambda_0^k} \right) + \zeta'_{k-1}.
\]

By induction hypothesis, we suppose that $\zeta_k \in L^2_1$ satisfying
\[
D_{s\lambda_0 s^0} (s^2 \zeta'_k) = s^2 Q^{k+1} + s^2 B^k + s^2 C^k
\]
where $B^k \in \mathbb{B}_{\lambda_0}^1 \sum_{j=0}^k s^{k+1} (k+1) \gamma_j^2 \kappa_j^2$, $C^k \in \mathbb{C}_{\lambda_0}^1 \sum_{j=0}^k s^{k+1} (k+1) \gamma_j^2 \kappa_j^2$ and
\[
Q^{k+1} = \lambda_0^k \hat{\Theta}_s^0 \frac{\zeta'_{k+1} - \zeta'_{k}}{\lambda_0^{k+1}}.
\]

By taking $s < \frac{1}{\kappa_1 \kappa_2}$, we can see that the sequence $\{\zeta_k\}$ will converge in $L^2_1$ sense to some $\zeta'$. Meanwhile, we can see that
\[
D_{s\lambda_0 s^0} (s^2 \zeta'_{k+1}) = \lambda_0^{k+2} \hat{\Theta}_s^0 (s^2 \frac{\zeta'_{k+1} - \zeta'_{k}}{\lambda_0^{k+2}}) + s^2 \delta B^{k+1} + s^2 \delta C^{k+1} + s^2 B^k + s^2 C^k.
\]
where \( \delta B^{k+1} \in \mathcal{B}_1^{k+2} \) and \( \delta C^{k+1} \in \mathcal{C}_1^{k+2} \). We define inductively that \( B^{k+1} = \delta B^{k+1} + B^k \), \( C^{k+1} = \delta C^{k+1} + C^k \) and

\[
\lambda_0^{k+2} \tilde{\Theta}_0^0 \left( \frac{\epsilon_{k+1}^0 - \epsilon_k^0}{\lambda_0^{k+2}} \right) = Q^{k+2}.
\]

Furthermore, if we take \( s \) small enough such that \( \sum_{j=0}^{\infty} s^{k+1}(k+1) = \frac{s}{(1-s)^2} \leq \frac{1}{\gamma^2 \kappa_1} \), we then have \( B^k \to B \in \mathcal{B}_1^{\kappa_1} \) and \( C^k \to C \in \mathcal{C}_1^{\kappa_1} \).

Therefore, by taking \( k \to \infty \), we finish our proof by induction.

To get the \( L_2^2 \)-estimate of \( \epsilon' \), we have

\[
\|\epsilon'_{k+1} - \epsilon_k'\|_{L_2^2} = \|s^{k+1} \chi_0^{k+2} J_k \left( \frac{\epsilon_k'}{\lambda_0^k} \right) \|_{L_2^2} \leq \frac{1}{2^{k} \kappa_1}
\]

by using the fact \( \|q_{k+1}^b\|_{L^2} \leq \frac{\kappa_1^{k+1}}{4} \) and \( \|\langle q_{k+1}^b \rangle_t\| \leq \kappa_1^{k+2} \). So we have \( \|\epsilon'\|_{L_2^2} \leq \frac{1}{2^{k_1}} \).

Now we apply this lemma to the \( Q_1, Q_2 \) and \( Q_3 \) in (7.18), (7.19) and (7.20), we can find \( \epsilon'_1 \) and \( \epsilon'_2 \) such that

\[
D_{s \chi_0 t_0}(s^2 \epsilon'_1) = s^2 Q_1 + s^2 B + sC \text{ for } i = 1, 2.
\]

For \( Q_3 \), we notice that

\[
s^2 Q_3 = \hat{\Theta}_s^0(s \chi_2 \eta + s \chi_2 \tilde{\eta}) \psi + \hat{\Theta}_s^0 \left( \begin{array}{cc} 0 & i \chi \hat{\eta} \\ -i \chi \hat{\eta} & 0 \end{array} \right) \psi.
\]

the first term is in \( s \mathcal{C}_1^{\kappa_1} \) and the second term is the first type of Lemma 7.6. So there exists \( \epsilon'_3 \) such that

\[
D_{s \chi_0 t_0}(s^2 \epsilon'_3) = s^2 Q_3 + s^2 B + sC.
\]

Finally, we can prove that \( D_{pert}(s t_{0,s}^b) = D_{s \chi_0 t_0.pert}(s t_{0,s}^b) \) for some \( t_{0,s}^b \in L_2^2 \) by Proposition 4.8. Furthermore, we can decompose \( t_{0,s}^b = t_{0,s}^b + s t_{0,s}^b \), where \( B(t_{0,s}^b) \in \mathcal{H}_0 \) and \( B(t_{0,s}^b) \in \mathcal{H}_0^+ \). Again, by Proposition 6.2, we have the following estimates for \( B(t_{0,s}^b) \):

\[
\|B(t_{0,s}^b)\|_{L_2^2}^2 \leq \frac{\kappa_0}{2} \epsilon^2, \|B(t_{0,s}^b)\|_{L_2^2}^2 \leq \frac{\kappa_0}{2} \epsilon^2 \|B(t_{0,s}^b)\|_{L_2^2}^2 \leq \frac{\kappa_0}{2}.
\]

Therefore, we can rewrite (7.21) as

\[
D_{s \chi_0 t_0.pert}(\psi - s \epsilon_0^0 - s h_0^b - s c_0^0 + s t_{0,s}^b + s t_{0,s}^b) = s^2 A + s^2 B + sC \mod(\ker(D_{L_2^2}))
\]

where \( c_0^0 = c_0 + c'_0 + s \sum_{j=1}^3 \epsilon'_j, A \in \mathcal{A}_1^{\kappa_0}, B \in \mathcal{B}_1^{\kappa_1} \) and \( C \in \mathcal{C}_1^{\kappa_1} \). We give \( -s c_0^0 - s h_0^b - s c_0^0 + s t_{0,s}^b \) a name \( t^0 \).

Now we can fix \( \kappa_1 \) forever.
7.4. Part I of the proof: Iteration of \((\eta_i, (\epsilon_i^0, \epsilon_i^g, \epsilon_i^b, \epsilon_i^h, t_{i,s}^\perp, f_i,))\). In this section we will construct an iterative process by determining the following two constants, \(1 > r \) and \( P \in (T^4 + 1, T^4) \) where \( T > 512 \) is any fix number. We will also use another constant \( \varepsilon > 0 \) which depends only on \( r \). In addition, we will also give the upper bound for \( t_0 \). We divide our argument into the following 6 steps.

Step 1. Suppose we have \( \psi_i = \psi + st^i \in L^2 \) satisfies

\[
D_{sy^{i, \text{pert}}} (\psi_i + s^2 t_{i,s}^\perp) = sf_i \mod (\ker(D|_{L^2}))
\]

where \( \eta^i = \sum_{j=0}^i \chi_j \eta_j \). Moreover, we assume the following conditions:

\[
\text{Inductive Assumptions:}
\]

1. \( sf_i \) can be decomposed as

\[
sf_i = s^2 f_{i,A} + s^2 t_{i,B} + sf_{i,C}
\]

where \( f_{i,A} \in \mathfrak{A}_{i}^{P^i \kappa_0}, f_{i,B} \in \mathfrak{B}_{i}^{P^i \kappa_1} \) and \( f_{i,C} \in \mathfrak{C}_{i}^{P^i \kappa_1} \).

2. The sequence \( \{(\chi_j, \eta_j)\}_{1 \leq j \leq i} \) satisfies \((4.23), (4.24), (4.25)\) with \( \kappa_2 = \varepsilon P^i \kappa_0 \).

3. We have \( t^i = \sum_{j=0}^i (-s\epsilon_j^g - s\epsilon_j^b - s\epsilon_j^h + t_{j,s}^b) \) and \( \{t^i\} \) converges in \( L^2 \) sense.

In fact, \( \sum_{j=0}^i (-s\epsilon_j^g - s\epsilon_j^b - s\epsilon_j^h) \) converges in \( L^2 \) sense.

To do the iteration, we need to construct the following data

\[
(\eta_{i+1}, (\epsilon_{i+1}^0, b_{i+1}^g, \epsilon_{i+1}^b, t_{i+1,s}^b, f_{i+1}), f_{i+1})
\]

in \( L^2(S^1; C) \times (L^2)^3 \) \times \( \times (s^2 \mathfrak{A}_{i+1}^{P^i \kappa_0} + s^2 \mathfrak{B}_{i+1}^{P^i \kappa_1} + s^2 \mathfrak{C}_{i+1}^{P^i \kappa_1}) \)

form all previous data \( \{(\eta_{j}, (\epsilon_{j}^0, b_{j}^g, \epsilon_{j}^b, t_{j,s}^b, f_{j}))\}_{j \leq i} \). We will show that all conditions in \((7.29)\) will be satisfied inductively.

Step 2. In this step, we will construct \( b_{i+1} \) and determine the constant \( t_0 \) in terms of \( \varepsilon, r \) and \( T \). First of all, since \( f_{i,C} \in \mathfrak{C}_{i}^{P^i \kappa_1} \) so we have

\[
\chi_{i+1} f_{i,C} \in \mathfrak{C}_{i+1}^{P^i \kappa_1}
\]

and

\[
(1 - \chi_{i+1}) f_{i,C} \in \mathfrak{B}_{i}^{P^i \kappa_1}.
\]

Now we can rewrite

\[
(7.30) \quad sf_i = s^2 f_{i,A} + sf_{i,B} + s\epsilon_i
\]

where \( f_{i,A} \in \mathfrak{A}_{i}^{P^i \kappa_0}, f_{i,B} = s^2 f_{i,B} + (1 - \chi_{i+1}) f_{i,C} \) with \( \epsilon_i := \chi_{i+1} f_{i,C} \in \mathfrak{C}_{i+1}^{P^i \kappa_1} \).

Before we start to solve \( b_{i+1} \), we need to show that \( f_{i,B} \in \frac{s\epsilon}{4T^2} \mathfrak{A}_{i}^{P^i \kappa_0} \).
Firstly, by taking $s$ small enough, we will have $s^T_i.B \in \frac{\varepsilon^2}{8T^2} \mathcal{B}_{i,t}^{P^iK_0}$. This fact can be achieved if we assume $t_0 \leq \frac{\varepsilon^2}{8T^2} \frac{(\varepsilon_0)}{K_1}$.

Secondly, by Lemma 2.6, for any $\zeta \in L^2_1$ and $\|\zeta\|_{L^2_1} = 1$, we have

$$| \int \langle \zeta, (1 - \chi_{i+1})f_i, c \rangle | = | \int \langle (1 - \chi_{i+1})\zeta, f_i, c \rangle |$$

$$\leq C \frac{r}{T^2} \|f_i, c\|_{L^2}$$

$$\leq C(\frac{r}{T^2})^\frac{1}{2} P^iK_1(\frac{r}{T^2})^\frac{1}{2}$$

$$\leq \frac{\varepsilon}{8} P^iK_0(\frac{r}{T^2})^\frac{1}{2}$$

by taking $r$ small enough. Therefore, we have $\|(1 - \chi_{i+1})f_i, c\|_{L^2_1} \leq \frac{\varepsilon P^iK_0}{8T^2}$, which implies that $f_i, B \in \frac{\varepsilon P^iK_0}{4T^2} \mathcal{B}_{i,t}^{P^iK_0}$.

Suppose (7.28) and (7.30) are true for $i$. We can solve

$$D_{s^T_i,A} h_{i+1,A} = s^T_i,A \text{ mod}(ker(D|_{L^2_1}))$$

$$D_{s^T_i,B} h_{i+1,B} = s^T_i,B \text{ mod}(ker(D|_{L^2_1}))$$

by using Proposition 4.9. Since $f_i.A |_{N - \frac{r}{T^{i+1}}} = f_i,B |_{N - \frac{r}{T^{i+1}}} = 0$, we have

$$\left(\frac{r}{T^{i+1}}\right)^5 \|h_{i+1,A}^{\pm}\|_{L^2} \leq \|h_{i+1,A}\|_{L^2} \leq \frac{s^2}{4} \|f_i,A\|_{L^2_1} \leq \frac{s^2 P^iK_0}{T^{5(i+1)}}.$$ 

This implies that

(7.31) \[ \|h_{i+1,A}\|_{L^2} \leq \frac{\varepsilon P^iK_0}{4T^{2(i+1)}}, \] \[ \|h_{i+1,A}^\pm\|_{L^2} \leq \frac{\varepsilon P^iK_0}{4T^{2(i+1)}}, \] \[ \|h_{i+1,A}\|_{L^2} \leq \frac{\varepsilon P^iK_0}{4T^{5(i+1)}}, \]

by taking $t_0 \leq \frac{\varepsilon}{4} (\frac{r}{T})^2$.

Meanwhile, we have

(7.32) \[ \|h_{i+1,B}\|_{L^2} \leq \frac{\varepsilon P^iK_0}{4T^{2(i+1)}}, \] \[ \|h_{i+1,B}^\pm\|_{L^2} \leq \frac{\varepsilon P^iK_0}{4T^{2(i+1)}}, \] \[ \|h_{i+1,B}\|_{L^2} \leq \frac{\varepsilon P^iK_0}{4T^{5(i+1)}}. \]

So we put these data together. Denote $h_{i+1}$ by $h_{i+1,A} + h_{i+1,B} - s^T_i,c$, then we have

$$D_{s^T_i, \text{pert}}(\psi_i - sh_{i+1}) = sT^i(h_{i+1}) \text{ mod}(ker(D|_{L^2_1}))$$

$sT^i(h_{i+1})$ is an order $O(s^2)$ term in $\mathcal{A}_{i+1,t}^{P^iK_0}$. 
Step 3. By Theorem 6.12, we can find \((\eta_{i+1}, c_{i+1})\) such that
\[
\begin{align*}
2h^+_{i+1} + d^+ \eta_{i+1} + c_{i+1} &= k^+_{i+1} \\
2h^-_{i+1} + d^- \eta_{i+1} + c_{i+1}^\pm &= k^\pm_{i+1}
\end{align*}
\]
for some \((k^+_{i+1}, k^-_{i+1}) \in \mathbb{H}_0\) where \((k^+_{i+1}, k^-_{i+1}) \perp (sh^+_{i+1} - k^+_{i+1}, 2h^-_{i+1} - k^-_{i+1})\) in \(L^2\)-sense. So
\[
\begin{align*}
\|k^\pm_{i+1}\|_2^2 &\leq \frac{\varepsilon P^i \kappa_0}{2T^{2i+1}}, \quad \|(k^\pm_{i+1})_t\|_2^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{i+1}}.
\end{align*}
\]
By using Proposition 4.8, there exists \(c_{i+1}\) such that \(D_{n^i}c_{i+1} = 0\)
\[
c_{i+1} = \left( \frac{c_{i+1}}{\varepsilon P^i \kappa_0} \right) + c_{\eta,i+1} + c_{\eta,i+1}.
\]
Moreover, since \(c_{i+1}\) satisfies \(T_{d^+,d^-}(c_{i+1}) = d^- (k^+_{i+1} - 2h^+_{i+1}) - d^+ (k^-_{i+1} - 2h^-_{i+1})\), we have
\[
\begin{align*}
\|c_{i+1}\|_2^2 &\leq \frac{\varepsilon P^i \kappa_0}{2T^{2i+1}}, \quad \|(c_{i+1})_t\|_2^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{i+1}}, \\
\|(c_{i+1})_t\|_2^2 &\leq \frac{\varepsilon P^i \kappa_0}{2}, \quad \|c_{i+1}\|_2^2 \leq \varepsilon P^i \kappa_0.
\end{align*}
\]
According to these estimates, we can show that \(sT^a(c_{i+1}) \in \Omega_{i+1}^{P^i \kappa_0}\).

Meanwhile, we can easily check that \(\eta_{i+1}\) satisfies \(i + 1\)-th version of (4.23), (4.24) and (4.25) with \((\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)\) and so it satisfies the condition (4.26), (4.27), and (4.28). Therefore, the inductive assumption 2 in (7.29) holds. Also, we have the \(\kappa_3 = \varepsilon P^i \kappa_1\) version of Proposition 4.6 and Proposition 4.7. Therefore,
\[
\begin{align*}
\int_{\{r = r_0\}} |\tilde{A}^i_{i+1}|^2 |dV| = \epsilon \gamma^4 \varepsilon^4 P^{i+1} \kappa_4^4 \left( \frac{r}{T^{i+1}} \right)^{i+\frac{i}{2}} \bar{s}^4 \leq \varepsilon^2 P^{2i} \kappa_4^2 \left( \frac{r}{T^{i+1}} \right)^{i+\frac{i}{2}} \bar{s}^2
\end{align*}
\]
by taking \(P \leq T^{\frac{i}{2}}\) and \(s\) small enough.

Remark 7.7. Here we show the estimate of the Hölder norm of \(\eta_i^+ \in \mathcal{H}_i^+\). By the argument similar to Remark 7.4, we have the following Hölder estimate
\[
\|\eta_i^+\|_{C^1, \frac{i}{2}} \leq C \kappa_0 P^i \left( \frac{r}{T_i} \right)^{i+\frac{i}{2}} \leq C \kappa_0 T^\frac{i}{2} \left( \frac{r}{T_i} \right)^{i+\frac{i}{2}} \leq C \kappa_0 \left( \frac{r}{T_i} \right)^{i+\frac{i}{2}}
\]
for all \(i\).

Step 4. In this step and the next step, we construct \(f_{i+1}\) and prove the inductive assumption 1 in (7.29). Firstly, since \(D_{n^i}c_{i+1} = 0\) we have
\[
D_{n^i, pert}(\psi - sc_{i+1} - s\eta_{i+1}) = sT^a\left(-c_{i+1} - \eta_{i+1} \mod (D|_{L_T^1})\right).
\]
Secondly, recall that we can write
\[
D_{n^i + s\chi_{i+1} \eta_{i+1}} = (1 + g_{\chi_{i+1} \eta_{i+1}}) D_{n^i} + s((\chi_{i+1}) \eta_{i+1} + (\chi_{i+1}) \varepsilon_{\eta_{i+1}}) \varepsilon_{\eta_{i+1}} + \Theta_{\eta_{i+1}} + \mathcal{R}_{\eta_{i+1}} + \mathcal{A}_{\eta_{i+1}} + \mathcal{F}_{\eta_{i+1}}.
\]
Now by Proposition 4.8, we can decompose $b_{t+1} = b_{t+1}^0 + b_{t+1}^s + b_{t+1}^r$ and $c_{t+1} = c_{t+1}^0 + c_{t+1}^s + c_{t+1}^r$ as follows: recall that $b_{t+1} = b_{t+1}^0 + b_{t+1}^s$ and $c_{t+1} = c_{t+1}^0 + c_{t+1}^s$

such that

$$DH_{t+1}^0 = s\bar{f}_{t,A} + f_{t,B} \text{ mod}(ker(D|_{L_i}));$$
$$DH_{t+1}^0 = 0 \text{ mod}(ker(D|_{L_i})).$$

Since $s\bar{f}_{t,A} + f_{t,B} = 0$ on $N_{\chi_{t+1}}$, we have

$$h_{t+1}^0 = \left( \frac{h_{t+1}^0}{\sqrt{z}} \right); \quad c_{t+1}^0 = \left( \frac{c_{t+1}^0}{\sqrt{z}} \right) = \chi_{t+1} + \frac{h_{t+1}^0}{\sqrt{z}}.$$

So we define

$$h_{t+1}^b = \chi_{t+1} \left( \frac{h_{t+1}^b}{\sqrt{z}} \right); \quad c_{t+1}^b = \chi_{t+1} \left( \frac{c_{t+1}^b}{\sqrt{z}} \right); \quad f_{t+1}^b = \chi_{t+1} \left( \frac{f_{t+1}^b}{\sqrt{z}} \right).$$

Now we compute

$$D^b_{x\eta'}|_{\chi_{t+1}} (s(c_{t+1}^b + c_{t+1}^b) + s(h_{t+1}^b + h_{t+1}^b))$$
$$=Ds(c_{t+1}^b + h_{t+1}^b) + (D_{x\eta'} - D)s(c_{t+1}^b + h_{t+1}^b) + D_{x\eta'}s(c_{t+1}^b + h_{t+1}^b)$$
$$=Ds(c_{t+1}^b + h_{t+1}^b) + (D_{x\eta'} - D)s(c_{t+1}^b + h_{t+1}^b).$$

For the first term on the right-hand side of (7.37), we can follow the argument in step 3 in section 7.3 to get

$$\epsilon_{t+1} = \chi_{t+1} \left( \begin{array}{c}
-\text{id} - \bar{\eta}_{t+1} \sqrt{z} \\
-\text{id} + \eta_{t+1} \sqrt{z}
\end{array} \right)$$

such that

$$Ds(-c_{t+1}^b - h_{t+1}^b) = \Theta_{t+1}^s(\psi) + D(s\epsilon_{t+1}) - s\epsilon_{t+1} \partial_i(\epsilon_{t+1}) - sD(\chi_{t+1}) \frac{\epsilon_{t+1}}{\chi_{t+1}} - D(s\bar{f}_{t+1}^b).$$

For the second term on the right-hand side of (7.37), since

$$D_{x\eta'} - D)|_{\chi_{t+1}} = \sum_{j=0}^i \Theta_{s}^j + A_+^i$$
$$= s\left( \sum_{j=0}^i \hat{\eta}_j \partial z + \sum_{j=0}^i \hat{\eta}_j \partial z \right) + A_+^i,$$

we have

$$(D_{x\eta'} - D)s(h_{t+1}^b + c_{t+1}^b) = s \left( \sum_{j=0}^i \Theta_{s}^j (h_{t+1}^g + c_{t+1}^g) + sA_+^i (h_{t+1}^g + c_{t+1}^g) \right).$$
Therefore, we can derive from (7.34) the following equality

\[(7.39)\]

\[
D_{\nu_{n+1}}|_{\nu_{n+1}} (-s(c_i^h + c_i^c) - s(h_i^h + h_i^c)) \\
= \Theta^i_{s+1}(\psi) + D(s\xi_{i+1}) - s\epsilon_1\partial_i(\xi_{i+1}) - sD(\chi_{i+1})(\xi_{i+1}) \\
\quad + s \sum_{j=0}^{i} \Theta^j_s(h_j^g - c_j^g) + sA^i_s(h_i^g - c_i^g) - D_{pert}(s\psi^i_{i+1}).
\]

Recall that the Dirac operator \(D_{\nu_{n+1}}\) can be written as

\[
D_{\nu_{n+1}} = (1 + g^{i+1})D_{\nu_{n}} + s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\eta_0)e_1\partial_i \\
\quad + \Theta^i_{s+1} + \hat{A}^i_{s+1} + R_{s+1}^i + F_{s+1}^i
\]

where

\[
\hat{\Theta}^i_{s+1} = \epsilon_1(s\chi_{i+1}\eta_{i+1}\partial_z + s\chi_{i+1}\hat{\eta}_{i+1}\partial_z), \\
W^i_{s+1} = \epsilon_2(s((\chi_{i+1})\bar{z}\eta_{i+1}\partial_z - s((\chi_{i+1})\bar{z}\eta_{i+1}\partial_z) \\
\quad + \epsilon_3(-s((\chi_{i+1})z\eta_{i+1}\partial_z + s((\chi_{i+1})z\eta_{i+1}\partial_z).
\]

Meanwhile, we use the following notations to simplify the upcoming equation:

\[
(7.40) \quad W^i_{s+1}(\psi_1 - \psi) = s^2B_0;
\]

\[
(7.41) \quad W^i_{s+1}(s\xi_{i+1}) + g^{i+1}(e_2\partial_z + e_3\partial_z)(s\xi_{i+1}) = s^2B_1;
\]

\[
(7.42) \quad -W^i_{s+1}(sc_i^g - sh_i^g) - g^{i+1}(e_2\partial_z + e_3\partial_z)(sc_i^g - sh_i^g) = s^2B_2;
\]

\[
(7.43) \quad \frac{s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_1\partial_i\psi + sA^i_s(sc_i^g - sh_i^g)}{sD(\chi_{i+1})(\xi_{i+1})(\xi_{i+1}) + (g^{i+1} - 1)e_1\partial_i(sc_i^g) = sC_0};
\]

\[
(7.44) \quad +s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_1\partial_i(s\xi_{i+1}) + sA^i_s(s\xi_{i+1}) = sC_1;
\]

\[
(7.45) \quad -g^{i+1}e_1\partial_i(sc_i^g - sh_i^g) - s((\chi_{i+1})z\eta_{i+1} + (\chi_{i+1})z\bar{\eta}_{i+1})e_1\partial_i(s\xi_{i+1}^g - sh_i^g) = sC_2;
\]

\[
(7.46) \quad \hat{\Theta}^i_{s+1}(\psi - \psi) + \sum_{j=0}^{i} \Theta^j_s(sc_i^g - sh_i^g) = s^2Q_0;
\]

\[
(7.47) \quad \hat{\Theta}^i_{s+1}(s\xi_{i+1}) + \sum_{j=0}^{i} \Theta^j_s(s\xi_{i+1}) = s^2Q_1;
\]

\[
(7.48) \quad -\hat{\Theta}^i_{s+1}(sc_i^g - sh_i^g) = s^2Q_2.
\]
Put all these data together, (7.39) yields

\[
D_{SN^i, \text{pert}}(\psi_i - sc_{i+1} - h_{i+1}) = sT^s(-c_{i+1} - h_{i+1}) + s\epsilon_i
\]

\[
= D_{SY^i}(\psi_i - sc_{i+1}^g - sh_{i+1}^g) + (D_{SN^i+1} - D_{SY^i})\psi_i
\]

\[
+ D_{SY^i+1}(sc_{i+1}) - (A_{i+1}^s + R_{i+1}^s)(\psi_i + se_{i+1})
\]

\[
+ s^2(B_0 + B_1) + s(C_0 + C_1) + s(Q_0 + Q_1)
\]

\[
- D_{\text{pert}}(s^b)
\]

\[
= D_{SY^i}(\psi_i - sc_{i+1}^g - sh_{i+1}^g) + (D_{SN^i+1} - D_{SY^i})\psi_i
\]

\[
+ D_{SY^i+1}(sc_{i+1}) + (D_{SN^i+1} - D_{SY^i})(se_{i+1}^g - sh_{i+1}^g)
\]

\[
- (A_{i+1}^s + R_{i+1}^s)(\psi_i + se_{i+1}^g - sh_{i+1}^g + se_{i+1})
\]

\[
+ s^2(\sum_{j=0}^2 B_j) + s(\sum_{j=0}^2 C_j) + s(\sum_{j=0}^2 Q_j)
\]

\[
- D_{\text{pert}}(s^b_{i+1})
\]

\[
\text{mod}(ker(D|_{L^2}))
\]

Therefore, we have

\[
D_{SN^i+1, \text{pert}}(\psi_i - sc_{i+1}^g - sh_{i+1}^g + se_{i+1}) = sT^s(c_{i+1} - h_{i+1}) + s\epsilon_i
\]

\[
- (A_{i+1}^s + R_{i+1}^s)(\psi_i - sc_{i+1}^g - sh_{i+1}^g + se_{i+1})
\]

\[
- s^2(\sum_{j=0}^2 B_j) - s(\sum_{j=0}^2 C_j) - s(\sum_{j=0}^2 Q_j)
\]

\[
- D_{\text{pert}}(s^b_{i+1})
\]

\[
\text{mod}(ker(D|_{L^2}))
\]

Now we prove that

\[
A_{i+1}^s(\psi_i - sc_{i+1}^g - sh_{i+1}^g + se_{i+1}) + s(\sum_{j=0}^2 C_j) + s\epsilon_{i+1} \in sC^{((1+C)P^i)\kappa_0};
\]

\[
R_{i+1}^s(\psi_i - sc_{i+1}^g - sh_{i+1}^g + se_{i+1}) + s^2(\sum_{j=0}^2 B_j) \in s^2B_{i+1}^{C\kappa_0 P^i};
\]

\[
T_0^s(-c_{i+1} - h_{i+1}) \in s2N_{i+1}^{b+1}\kappa_0.
\]

We already prove (7.52) in step 2 and step 3. By using \(\kappa_3 = \varepsilon P^i\kappa_1\) version of

(4.34), we can prove that

\[
\|A_{i+1}^s(\psi_i + sc_{i+1}^g - sh_{i+1}^g + se_{i+1})\|_{L^2(N_i - N_s)} \leq \varepsilon P^i\kappa_1(r^3 - s^3).
\]

Meanwhile, by \((\kappa_2, \kappa_3) = (\varepsilon P^i\kappa_0, \varepsilon P^i\kappa_1)\) version of \([4.23, 4.28]\), we have \(s(\sum_{j=0}^2 C_j) \in sC^{C\kappa_0 P^i}\kappa_1\). So we get (7.50).
Finally, by using \((\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)\) version of Proposition 4.7, we have
\[
\|R^i_{s+1}(\psi_i + s\xi_{s+1} + s\xi_{s+1})\|_2 \leq C_{\gamma_2} \varepsilon P^i \kappa_1 (\frac{r}{T^{i+1}})^{2}
\leq C_{\gamma_2} \varepsilon P^i \kappa_1 (\frac{r}{T^{i+1}})^{2}
\leq \varepsilon P^i \kappa_1 (\frac{r}{T^{i+1}})^{2}
\]
by taking \(P \leq \sqrt{T}\) and \(\varepsilon \leq \frac{1}{C_{\gamma_2} \kappa_1 r^2}\). So we have \(R^i_{s+1}(\psi_i + s\xi_{s+1} + s\xi_{s+1}) \in s^2 B_{i+1}C^2 P^i \kappa_1\). Meanwhile, by \((\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)\) version of (4.23) - (4.28) again, we have \(s(\sum_{j=0}^2 b_j) \in s^2 B_{i+1}C^2 P^i \kappa_1\). So we prove (7.51).

**Step 5.** In this step, we state the following lemma which is the \(i + 1\)-th version of Lemma 7.6. The proof of this lemma can follow from the argument of Lemma 7.6 directly. So we omit the proof.

**Lemma 7.8.** Suppose \(Q\) be either the following 4 types:
\[
s^2 \chi_{i+1} \left( \frac{q^s(t)}{q^s_0(t)} \right), s^2 \chi_{i+1} \left( \frac{q^s(t)}{q^s_0(t)} \right), s^2 \left( \frac{q^s(t)}{q^s_0(t)} \right) \text{ or } s^2 \left( \frac{q^s(t)}{q^s_0(t)} \right)
\]
where \(\|q^s\|_2 \leq \kappa_3 \frac{T}{r^{i+1}}, \|q^s(t)\|_2 \leq \kappa_3\). Then there exists an \(L^i_1\) section \(\epsilon'\) which can be written as
\[
\epsilon' = \sum_{j=0}^2 s^j \chi_{i+1} \left( \frac{e^s_j(t)}{\sqrt{z}} \right), \sum_{j=0}^2 s^j \chi_{i+1} \left( \frac{e^s_j(t)}{\sqrt{z}} \right) \text{ or } \sum_{j=0}^2 s^j \chi_{i+1} \left( \frac{e^s_j(t)}{\sqrt{z}} \right)
\]
for the each type respectively such that \(D_{\text{per}^i} (s^2 \epsilon') = s^2 Q + s^2 B + s C\) where \(B \in B_{i+1}^2\) and \(C \in C_{i+1}^2\) for all \(s \leq \frac{T}{2^{i+1}}\). Furthermore, we have \(\|\epsilon'\|_L^i \leq 2 \kappa_3\).

By using this lemma, we can show that there exist \(\epsilon'_{i+1,j}, j = 0, 1, 2\), such that
\[
D_{\text{per}^i} (s^2 \epsilon'_{i+1,j}) = s^2 Q_j + s^2 B_j + s C_j.
\]

Meanwhile, by Proposition 4.8 and Proposition 6.2, we can show that there exist \(t_{i+1,s}\) and \(t_{i+1,s}^\perp\) satisfying \(D_{\text{per}^i} (t_{i+1,s}) = D_{\text{per}^i} (t_{i+1,s}^\perp), B(t_{i+1,s}) \in H_{i+1}\) and
\[
(7.53)
\|B(t_{i+1,s})\|_L^i \leq \frac{\varepsilon P^i \kappa_0}{2 T^{i+1}}, \|B(t_{i+1,s})t\|_L^i \leq \frac{\varepsilon P^i \kappa_0}{2 T^{i+1}}, \|B(t_{i+1,s})t\|_L^i \leq \frac{\varepsilon P^i \kappa_0}{2}.
\]
Therefore, we can rewrite
\[
(7.54)
D_{\text{per}^i} (\psi_i - s\xi_{s+1} + s\xi_{s+1} + s\xi_{s+1} + s t_{i+1,s} + s^2 t_{i+1,s}) = s^2 A + s^2 B + s C := f_{i+1}
\]
with $c_{i+1}^g = c_{i+1} + \sum_j c_{i+1,j}^g$, $A \in \mathfrak{A}_{i+1}^{(1+C_\varepsilon)P_i K_1}$, $B \in \mathfrak{B}_{i+1}^{C_\varepsilon P_i K_1}$ and $C \in \mathcal{C}_{i+1}^{C_\varepsilon P_i}$. So by taking $\varepsilon \leq \frac{P_i - 1}{C_\varepsilon}$, we prove the inductive assumption 1 in (7.29).

**Step 6.** Finally, we should prove the inductive assumption 3 in (7.29). To prove this part, we notice that both $h_{i+1}^g$ and $c_{i+1}^g$ vanish on $\Sigma$. Therefore, we can do the integration by parts to get

$$\|h_{i+1}^g\|_{L^2}^2 \leq \|D_{\sigma^i} \eta_{i+1}\|_{L^2}^2 + C\|h_{i+1}^g\|_{L^2}^2$$

for some constant $C$ depending on the curvature of $M$. Now by the fact $D_{\sigma^i} h_{i+1} = 0$ on $N_{\frac{1}{i+1}}$ and Corollary 3.6, we have

$$\|D_{\sigma^i} h_{i+1}^g\|_{L^2} \leq |\sigma(\chi_{i+1})|\|h_{i+1}\|_{L^2} + \left\|D_{\sigma^i} \left( \frac{h_{i+1}^g}{\psi_{i+1}} \right) \right\|_{L^2(N_{\frac{1}{i+1}})} \leq C \frac{P_i^{i+1} K_1}{T^{4(i+1)}}$$

and by (7.31) and (7.32) and Corollary 3.6, we have

$$\|h_{i+1}^g\|_{L^2} \leq C\|h_{i+1}\|_{L^2} \leq C \frac{P_i^{i+1} K_1}{T^{4(i+1)}}.$$ 

So we have

$$\|h_{i+1}^g\|_{L^2} \leq C \frac{P_i^{i+1} K_1}{T^{i+1}}.$$ 

Similarly, we have

$$\|c_{i+1}^g\|_{L^2} \leq C \frac{P_i^{i+1} K_1}{T^{i+1}}.$$ 

For $L^2$-bounds, we have

$$\|\tilde{t}_{i+1,s}^g\|_{L^2} \leq C\|\tilde{t}_{i+1}^g\|_{L^2} \leq C \frac{P_i K_0}{T^{2(i+1)}};$$

$$\|\tilde{t}_{i+1,s}^g\|_{L^2} \leq C\|\tilde{t}_{i+1}^g\|_{L^2} \leq C \frac{P_i K_0}{T^{2(i+1)}}.$$ 

So $t_{i+1,s}^g \to 0$ in $L^2$-sense. Therefore, we finish the proof of the inductive assumption 3 in (7.29).

By induction, we get a sequence $\psi_i \in L^2$ and a family of perturbations $\eta^i = \sum_{j=0}^i \chi_j \eta_j$ such that

$$D_{\sigma^i, \text{pert}}(\psi_i + s t_{i+1,s}^g) \to 0 \mod(\ker(D)_{L^2})$$

as $i \to \infty$ in $L^2$-sense. Moreover, since $\|\psi_{i+1} - \psi_i\|_{L^2} \leq C \kappa_3 T^i$ for some $C > 0$, so we have $\psi_i \to \psi_0$ in $L^2$ sense. Meanwhile, since $\|\eta_i\|_{L^2} \leq C \kappa_3 T^i$ for some $C > 0$, we have $\sum \eta_i \to \eta_0$ in $L^2$ sense. In addition, it is easy to see that the element in $\ker(D)_{L^2}$ is also convergent.

To prove that $\eta^i$ converges to a $C^1$ circle, we only need to use the Hölder estimates in Remarks 7.4 and 7.7. We have

$$\|\eta\|_{C^{1, \theta}} \leq C \kappa_0 \left( \frac{T^{\theta}}{T^\theta} \right).$$
for all i. Therefore, by Arzela-Ascoli theorem, there is a subsequence of the partial sum \( \{ \eta^i \} \) converging in \( C^1 \) sense. So the limit, \( \eta \), will be a \( C^1 \) circle.

Because \( B(\psi_s) = 0 \), \( \psi_s \) will vanish on \( \Sigma \) and \( D_{s\eta, \text{pert}}(\psi_s) = 0 \mod(\ker(D|_{L_2})) \), we have \( \psi_s \in L_2^4 \).

\textit{Remark 7.9.} Suppose we consider a smaller neighborhood of \( ((g, \Sigma, e), \psi) \) to parametrize. This means we can take \( r, t_0 \) smaller. In this case, the constant \( \epsilon \) can be chosen smaller, too. We can see that

\[
\frac{1}{r^2} \sum_{j=1}^{\infty} \| \eta_j \|_{C^1} \to 0
\]

as \( r \) goes to 0. Similarly, we have \( t_0 - \theta^0 \) is \( O(\epsilon) \). So all these terms we derived in this iteration process is \( o(s) \)-order.

7.5. \textbf{Part I of the proof: The set} \( \pi_1(\mathcal{N}) \). Here we should say more about the neighborhood \( \mathcal{N} \). We define the topology on \( \mathcal{Y} \) as follows. Let \( ((g, \Sigma, e), \psi) \in \mathfrak{M} \), we refine the notation used in section 4.2 in the following way:

\[
V_{\Sigma, r, C} = \{ \eta : S^1 \to \mathbb{C} \| \eta \|_{C^1} \leq C; (\eta(t), t) \in N_r \}
\]

and define

\[
V_{g, r, C'} = \{ \hat{g} \in \mathcal{N} \| \| \hat{g} - g \|_{C^2} \leq C'; \text{dist}(\Sigma, \text{supp}(\hat{g} - g)) \leq r \}.
\]

So we can generate the topology on \( \mathcal{Y} \) by the family of open sets \( \{ V_{g, r, C'} \times V_{\Sigma, r, C} \} \) for \( r < R \), \( C, C' \in \mathbb{R}_+ \).

Now we define our \( \mathcal{N} = \bigcup_{r>0} V_{g, r, C} \times V_{\Sigma, r, C} \) for some \( C \) small enough. Reader can double check the argument in step 2 of section 7.3: By taking \( \mathcal{N} \) in this way, we have all elements in \( \pi_1(\mathcal{N}) \) will follow the argument in section 7.

\textit{Remark 7.10.} It seems to be impossible to take \( \mathcal{N} \) to be \( \bigcup_{r>0} V_{g, r, C} \times V_{\Sigma, r, C} \) because the map \( f \) is not differentiable on this set. However, the choice of \( r \) can be arbitrarily small.

8. \textbf{Proof of the main theorem: Part II}

In this section, we prove two statements. Firstly, we have to show that the choice of \((\eta_s, \psi_s)\) have dimension equaling \( K_1 \). Secondly, we have to show that the function \( f \) we defined in previous section is \( C^1 \).

8.1. \textbf{Part II of the proof: parametrization of} \((\eta_s, \psi_s)\). First of all, by the argument in the previous section. After we fix a \( \xi \in \mathbb{H}_1 \), we have the choice of \( \eta_s \) is unique. Also, we have \( B(\psi_s) = 0 \).

According to this observation, we can prove the following proposition instead.

\textbf{Proposition 8.1.} For any two solutions \((\eta_s, \psi_s)\) and \((\eta_s, \psi_s^*)\) satisfying \( D(\psi_0 - \psi_0^*) = 0 \), then \( \psi_s - \psi_s^* = 0 \).
Proof. We can write \( D_{s, \text{pert}} = D + P(s) \) where \( P(s) \) is of the order \( O(s) \) and analytic with respect to \( s \). Meanwhile, since we have \( \psi_s - \psi^*_s \in C^\omega([0, t_0]; L^2_1) \), so we have

\[
D_{s, \text{pert}}(\psi_s - \psi^*_s) = D(\psi_s - \psi^*_s) - P(s)(\psi_s - \psi^*_s) = 0.
\]

So inductively, we have \( (\psi_s - \psi^*_s) = O(s^k) \) for all \( k \). This implies \( (\psi_s - \psi^*_s) = 0 \). □

By this proposition, we know that we can parametrize the data \( \psi_s \) by elements in \( \ker(D|_{L^2_1}) \). Therefore, we can define a map \( K: \hat{\psi} \mapsto \psi_s \) with \( \hat{\psi} \in \ker(D|_{L^2_1}) \) and \( \|\hat{\psi}\|_{L^2_1} = 1 \).

8.2. Part II of the proof: \( C^1 \) regularity of \( f \). Since the function \( f \) is defined on an infinity dimensional space, so the definition of \( C^1 \) will be in the sense of Frechet \( C^1 \). Here we recall the definition of Frechet \( C^1 \).

Definition 8.2. Let \( B_1, B_2 \) are two Banach spaces. \( F: B_1 \to B_2 \) be a bounded operator. Then \( F \) is differentiable at \( p \) if and only if there exists a bounded linear operator \( d_p F: B_1 \to B_2 \) such that

\[
\|F(p + x) - d_p F(x) - F(p)\|_{B_2} = o(\|x\|_{B_1}).
\]

In addition, if \( F \) is differentiable everywhere and \( d_p F \) vary continuously. Then we call \( F \) a \( C^1 \) map.

Now let \( F \) maps from \( \mathbb{R}^n \times B \) to \( \mathbb{R}^m \). Suppose we have

(8.1) \( \frac{\partial}{\partial x_i} F(p) := h_i(p) \) is continuous near 0.

(8.2) The family of directional derivatives \( \{D_v F := j_v(p)|v \in B, \|v\| = 1\} \) is equicontinuous near 0,

(8.3) \( \{D_v F = k_p(v)|p \in \mathbb{R}^n \times B\} \) is equicontinuous on \( \{v \in B||v|| = 1\} \).

Then we can define the linear operator as follows:

(8.4) \[
L_p(x, v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(p)x_i + D_{\|v\|} F(p)v
\]

To prove this is the linear approximation, we need to check some other conditions. However, this is the only possible linear operator tangential to \( F \) at 0.

Now, suppose we already show that these linear operators are the differential of \( F \). To show \( F \) is \( C^1 \), it is sufficient to show that \( L_p \) varies continuously. So the condition (8.1) and (8.2) are exactly what we need to show.

Here I divide my proof into two parts. In first part, I will assume that \( f \) is differentiable at every point and then showing that \( f \) is \( C^1 \). In the second part, I will prove that \( f \) is differentiable.

Step 1. Since \( \psi_s \) is analytic, the family of directional derivatives of \( f \) is actually equicontinuous at any point except \( p = 0 \). Therefore, we only need to show conditions (8.1) and (8.2) hold near 0.
Since
\[ s^h_s = \sum_{i=0}^{\infty} s^h_{i,s} = \sum_{i=0}^{\infty} s^h_i + O(s^2), \]
we can further simplify this equation by using the conclusion in Remark 7.9:
\[ s^h_s = s^h_0 + o(s). \]
Now, recall the way we construct \( k^+_0 \) from Step 1 and Step 2 in section 7.3. In
the case that we have no perturbation for \( g \), \( k^+_0 = 0 \). That is to say, \( s^h_s = o(s) \).
Therefore, the directional derivatives of \( f \) along \( \mathbb{H}_1 \) will be 0. Meanwhile, it is
obvious that they are continuous by using (8.6).

To prove (8.2), we use (8.6) again. Here we can check that if we perturb the
metric along the opposite direction, then the corresponding \( s^h_s \) will only change the
sign. So the directional derivatives along \( \pi_1(N) \) also exist and are continuous at 0.
Furthermore, since the estimates shown in section 8 are independent of the choice
of \( g^s \), so it doesn’t depend on \( v \). Therefore, \( \{ j_v(p) \} \) is equicontinuous at 0.

**Step 2.** In this step, we need to show that \( f \) is differentiable. By Definition 8.2,
we need to show that for any \( p = (y, w) \in \mathbb{R}^n \times \mathcal{B} \),
\[ \| f(y + x, w + v) - \mathcal{L}_{(y, w)}(x, v) - f(y, w) \| \leq o(\sqrt{\| x \|^2 + \| v \|^2_{C^2}}) \]
where \( x, y \in \mathbb{K}_1 \) and \( w, v + w \in \pi_1(N) \). All we need to show is the ”small o” in
(8.7) will converge to zero uniformly. Namely, we are going to prove (8.3) here.
Now, since we already prove that the directional derivatives of \( f \) are all continuous,
so we can obtain (8.7) by showing that \( \{ k_p(v) \} \) is equicontinuous.

By using the conclusion in 7.5, we suppose that \( \| \partial_s g^s \|_{C^2} = C \tau^{\frac{5}{2}} \), then the
directional derivative of \( f \) along \( v = \partial_s g^s \) at \( g^{s_0} \) will be 
\[ \frac{1}{C \tau^{\frac{5}{2}}} \partial_s (B(s^h_s)) \mid_{s = s_0}. \]
Now we can prove (8.3) by using the fact that \( s^h_s \) is analytic and the estimates (7.7) and
(7.33).

Therefore, we complete the proof of this part.

**8.3. Summary of the proof.** Let me summarize what we have proved in these
two sections: For any \((g, \Sigma, e, \psi)\), there exist a neighborhood of \( y = (g, \Sigma, e) \),
\( N \subset \mathcal{V} \), finite dimensional ball \( \mathcal{B} \in \mathbb{K}_1 \) and finite dimensional vector space \( \mathbb{K}_0 \) all
defined as above such that \( \mathcal{M} \) will locally homeomorphic to the kernel of \( f \) where
\[ f(g^s, s\xi, s\hat{\psi}) = (B(\mathcal{X}(s\hat{\psi})), P_{ker(D)_{L^2}}(D_{\eta, pert}(\psi_s))). \]
Here \( P_{ker(D)_{L^2}} \) is the orthogonal projection from \( L^2 \) to \( ker(D)_{L^2} \) and \( \mathcal{X} \) is defined
in section 8.1. We can see that \( B(\mathcal{X}(s\hat{\psi})) \in \mathbb{H}_0 \) and \( P_{ker(D)_{L^2}}(D_{\eta, pert}(\psi_s)) \in
ker(D)_{L^2} \). Moreover, we have proved that \( f \) is a \( C^1 \) function.

Here I make one more remark. Recall that we define \( \mathcal{N} = \bigcup_{r > r} \mathcal{V}_{g, r, C^{\gamma/2}} \times \mathcal{Y}_{\Sigma, r, C} \)
in section 7.5. The choice of this open set depends on \( r \), so we can call it \( \mathcal{N}(r) \) for
a while. Now, what we proved in the previous section show us that there exists
$C_r > 0$, which goes to infinity as $r \to 0$, such that $\|df|_{N(r)}\| \leq C_r$ for any $r > 0$. Because of this, there is no uniform control for $\|df\|$ when $r \to 0$. So we cannot choose $N$ to be $\bigcup_{r > 0} \mathcal{V}_{g,r,Cr^{5/2}} \times \mathcal{V}_{\Sigma,r,C}$.

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### 9. Appendix

#### 9.1. Remark of the proof when the metric is not Euclidean around $\Sigma$.

Here I will sketch the proof for the general case that the metric is not Euclidean near $\Sigma$. The idea is to replace Propositions 4.4 and 4.6 by Propositions 5.5 and 5.6 in the argument contained in section 7.

First of all, let me summarize what we have done in section 7. We start with a perturbation $g^s$ which gives us an extra term $f_0$ such that $D_{pert}\psi = f_0$. Then in the next step, we construct a triple $(h_0, c_0, \eta_0)$ such that $Dh_0 = f_0$ (mod a finite dimensional space), $Dc_0 = 0$ and “eliminate” the $\frac{1}{\sqrt{r}}$ part in $h_0$ by $(c_0, \eta_0)$. Then we repeat this process. Each time we will produce a new $f$ which can be decomposed into 3 parts, which belongs to $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ defined in Definition 7.2 (We omit all subscripts here).

Now, we restart the process of producing $(h_0, c_0, \eta_0)$ for the general case, but this time we replace the Dirac operator $D$ by $D^{(1)}$ defined in section 5.2. So $D^{(1)}h_0 = f_0$ (mod a finite dimensional space) and $D^{(1)}c_0 = 0$. By using the same argument, we will still generate $f_1$. The only difference will be an extra term in $\mathfrak{C}$, which is something we can deal with. This part is generated by the operator $\delta^{(1)}$ defined in Proposition 5.3.

Now we do this process step by step. We replace $D$ by $D^{(i)}$ in $i$-th step, then we will get the same result. So the whole argument works for the general case.

#### 9.2. Upper semi-continuity of $\dim(coker(p^-))$.

In this final part, I will answer the question about the upper semi-continuity of $\dim(coker(p^-))$.

Since $p^-$ is a Fredholm operator, we can decompose $\text{Exp}^- = \text{range}(p^+) \oplus \mathbb{W}$ where $\mathbb{W}$ is finite dimensional. Now, for any $c^\pm \in \text{range}(p^-)$, there exists $\epsilon \in \ker(D|_{L^2})$ such that $B(\epsilon) = c^\pm$. Suppose we have a perturbed Dirac operator $D_{pert}$. We can follow the argument in the proof of Proposition 4.8 to get a $\epsilon'$ such that $D_{pert}(\epsilon') = 0$ and $\|B(\epsilon - \epsilon')\| \leq \epsilon \|B(\epsilon)\|$.  

To prove $coker(p^-)$ is upper semi-continuous, we need to show that the dimension of cokernel under a small perturbation will be less or equal than the dimension
of $\mathcal{W}$. We can prove this fact by showing that $\text{range}(p^-_{pert}) + \mathcal{W} = \text{Exp}^-$.

Suppose this is not the case, then we can find $v \in \text{Exp}^-$, $\|v\| = 1$ such that $v \perp \mathcal{W}$ and $v \perp \text{range}(p^-_{pert})$. So we have

$$\langle v, B(c') \rangle = 0 = \langle v, B(c) \rangle + O(\varepsilon).$$

This means that, if we decompose $v = v_0 + v_1$ where $v_0 \in \text{range}(p^-)$ and $v_1 = \mathcal{W}$, then we have $\|v_0\| \leq O(\varepsilon)$ and $v_1 = 0$. Therefore, we have $\|v\| = O(\varepsilon)$, which is a contradiction. Therefore, we prove the upper semi-continuity of $\dim(\text{coker}(p^-))$.

9.3. The bijection from $\ker(\mathcal{L}_p|_{\delta=0})$ and $\mathcal{K}_1$ and the injection from $\text{coker}(\mathcal{L}_p|_{\delta=0})$ and $\mathcal{K}_0$. First of all, we prove that $\ker(\mathcal{L}_p|_{\delta=0})$ is isomorphic to $\mathcal{K}_1 = \ker(\mathcal{T}_{d^+,d^-} \circ B)$. Recall notations in section 6.2, we have the following map:

$$J : \ker(\mathcal{L}_p|_{\delta=0}) \to \ker(\mathcal{T}_{d^+,d^-} \circ B);$$

$$(\eta, \phi_0) \mapsto \left( \begin{array}{c} \frac{d^+ \eta}{\sqrt{2}} \\ \frac{d^- \eta}{\sqrt{2}} \end{array} \right) + \left( \begin{array}{c} \frac{h^+}{\sqrt{2}} \\ \frac{h^-}{\sqrt{2}} \end{array} \right) + O_L^2(1) + \phi_0$$

where the right-hand side is the element defined in (6.3). Notice that the $O_L^2(1)$ term on the right is determined by $\eta$ and $\psi_0$. To prove $J$ is a bijection, we need to find the inverse. Suppose we have $u \in \ker(\mathcal{T}_{d^+,d^-} \circ B)$. $B(u) = (u^+, u^-)$. Then we can solve $\eta = \frac{d^+ u^+}{\sqrt{2}} = \frac{d^- u^-}{\sqrt{2}}$. Once we solve $\eta, \phi_0$ will be solved immediately. So we can construct the inverse map. Therefore, we prove $J$ is a bijection.

Next, we prove that there exists an injection from $\text{coker}(\mathcal{L}_p|_{\delta=0})$ to $\mathcal{K}_0 = \ker(\mathcal{T}_{d^+,d^-} \circ B) \oplus \ker(D|_{L^2\Sigma})$. Notice that $\ker(\mathcal{L}_p|_{\delta=0}) \subset \text{range}(D|_{L^2\Sigma}) \subset \ker(D|_{L^2\Sigma}) = B(\ker(D|_{L^2\Sigma})) \oplus \ker(D|_{L^2\Sigma})$ by (6.2) and Proposition 2.4. So for any $u \in \ker(\mathcal{L}_p|_{\delta=0})$, there is a unique corresponding pair $(B(u), v) = ((u^+, u^-), v) \in B(\ker(D|_{L^2\Sigma})) \oplus \ker(D|_{L^2\Sigma})$. Since $u \perp \text{range}(\mathcal{L}_p|_{\delta=0})$, by integration by parts, we have $d^- u^+ = d^+ u^-$. Therefore, we can define the following map

$$L : \text{coker}(\mathcal{L}_p|_{\delta=0}) \to \text{coker}(\mathcal{T}_{d^+,d^-} \circ B) \oplus \ker(D|_{L^2\Sigma});$$

$$u \mapsto (c, v) \text{ where } c = \frac{\bar{u}^+}{d^+} = \frac{u^-}{d^-}.$$  

To prove this element $c$ is in $\text{coker}(\mathcal{T}_{d^+,d^-} \circ B)$, we use the inner product defined in the section 6.3 and integration by parts:

$$\langle \mathcal{T}_{d^+,d^-} \circ B(v), c \rangle = \text{Re}(\int_{\Sigma} \bar{u}^- w^+ - u^+ w^-) = \text{Re}(\int_{M - \Sigma} \langle Du, w \rangle + \langle u, Dw \rangle) = 0$$

for any $v \in \ker(D|_{L^2\Sigma})$ with $B(v) = (w^+, w^-)$. Finally, it is easy to see from the definition that $L$ is injective. So we finish this proof.

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