Vertices of Schubitopes

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Abstract. Schubitopes were introduced by Monical, Tokcan and Yong as a specific family of generalized permutohedra. Important cases of Schubitopes include the Newton polytopes of Schubert polynomials and key polynomials. We develop a combinatorial rule to generate the vertices of Schubitopes. As an application, we show that the vertices of the Newton polytope of a key polynomial are generated by permutations in a lower interval in the Bruhat order, settling a conjecture of Monical, Tokcan and Yong.

1 Introduction

The objective of this paper is to investigate the vertices of Schubitopes introduced by Monical, Tokcan and Yong \cite{Monical2019} during their study of Newton polytopes in algebraic combinatorics. Schubitopes are a family of generalized permutohedra initially studied by Postnikov \cite{Postnikov2006}. In particular, it was conjectured by Monical, Tokcan and Yong \cite{Monical2019} and shown by Fink, Mészáros and St. Dizier \cite{Fink2015} that the Newton polytopes of Schubert polynomials and key polynomials are Schubitopes.

We provide a combinatorial algorithm to generate the vertices of Schubitopes. As an application, we prove that the vertices of the Newton polytope of a key polynomial can be generated by permutations in a lower interval in the Bruhat order. This confirms a conjecture proposed by Monical, Tokcan and Yong \cite[Conjecture 3.13]{Monical2019}.

Schubitopes are polytopes associated to diagrams in an $n \times n$ grid. A diagram $D$ is a collection of boxes in an $n \times n$ grid. The Schubitone $S_D$ associated to $D$ can be defined as follows. For $1 \leq j \leq n$ and a subset $S$ of $[n] = \{1,2,\ldots,n\}$, define a string word$_{j,S}(D)$ by reading the $j$-th column of the $n \times n$ grid from top to bottom and recording:

- $(i,j) \notin D$ and $i \in S$;
- $(i,j) \in D$ and $i \notin S$;
- $\star$ if $(i,j) \in D$ and $i \in S$,

where $(i,j)$ denotes the box of the $n \times n$ grid in row $i$ and column $j$. Here the rows (resp., columns) are labeled $1,2,\ldots,n$ from top to bottom (resp., from left to right). Let

$$\theta_D^j(S) = \#\{\text{paired } ()'s \text{ in word}_{j,S}(D)\} + \#\{\star \text{ 's in word}_{j,S}(D)\},$$

where \( \theta_D^j(S) \) is the number of paired parentheses in the word $\text{word}_{j,S}(D)$.
where the pairing is by the standard “inside-out” convention. Set $\theta_D(S) = \sum_{j=1}^n \theta^j_D(S)$. The Schubitope $S_D$ is defined by

$$S_D = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in [n]} x_i = \#D \text{ and } \sum_{i \in S} x_i \leq \theta_D(S) \text{ for } S \subset [n] \right\}.$$ 

In this paper, we characterize the vertices of the Schubitopes $S_D$ in terms of certain fillings of $D$. This leads to simple descriptions of the vertices of the Newton polytopes of Schubert polynomials and key polynomials. Let $S_n$ denote the set of permutations of $[n]$. Given a permutation $w = w_1 w_2 \cdots w_n \in S_n$, define $F_w(D)$ to be the filling of $D$ with the entries of $w$ as follows. Fill the columns of $D$ from left to right one by one. For the $j$-th (1 $\leq j \leq n$) column $D_j$, fill the integers $w_1, \ldots, w_n$ in turn into the empty boxes of $D_j$ as below. For 1 $\leq k \leq n$, put $w_k$ into the first (from top to bottom) empty box whose row index is larger than or equal to $w_k$. If there are no such empty boxes, then $w_k$ does not appear in the filling and skip to $w_{k+1}$. For example, Figure 1.1 illustrates the filling $F_w(D)$ for $w = 315624$.

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & & & \\
& & & & & & \\
3 & & & & & & \\
2 & 3 & 3 & 3 & 1 & & \\
& 5 & & & & & \\
5 & 6 & 1 & 3 & & & \\
\end{array}$$

Figure 1.1: The filling $F_w(D)$ for $w = 315624$.

**Theorem 1.1.** Let $D$ be a diagram of $[n]^2$. Then the vertex set of the Schubitope $S_D$ is

$$\{x(w) : w \in S_n\},$$

where $x(w) = (x_1, x_2, \ldots, x_n)$ is the vector such that $x_k$ ($1 \leq k \leq n$) is the number of appearances of $k$ in $F_w(D)$.

For the running example as displayed in Figure 1.1 we have $x(w) = (6, 2, 6, 0, 3, 1)$.

Given a polynomial $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_{\alpha} x^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$, the Newton polytope of $f$ is the convex hull of the exponent vectors of $f$, namely,

$$\text{Newton}(f) = \text{conv}(\{\alpha : c_{\alpha} \neq 0\}).$$

Specifying $D$ to the Rothe diagram $D(w)$ of a permutation $w$, the Schubitope $S_{D(w)}$ is the Newton polytope $\text{Newton}(\mathcal{S}_w)$ of the Schubert polynomial $\mathcal{S}_w(x)$ \[3, 8\]. The Rothe diagram $D(w)$ of $w \in S_n$ is the diagram obtained from the $n \times n$ grid by deleting the box $(i, w_i)$ as well as the boxes to the right of $(i, w_i)$ or below $(i, w_i)$. Figure 1.2(a) illustrates the Rothe diagram of $w = 1432$. So, when $D$ is the Rothe diagram $D(w)$, Theorem 1.1 gives a characterization of the vertices of Newton($\mathcal{S}_w$).

When $D$ is restricted to the skyline diagram $D(\alpha)$ of a composition $\alpha$, the Schubitope $S_{D(\alpha)}$ is the Newton polytope $\text{Newton}(\kappa_\alpha)$ of the key polynomial $\kappa_\alpha(x)$ \[3, 8\]. The skyline
diagram \(D(\alpha)\) of a composition \(\alpha\) is the diagram consisting of the first \(\alpha_i\) boxes in row \(i\), see Figure 1.2(b) for the skyline diagram of \(\alpha = (1, 2, 0, 1)\). In this case, Theorem 1.1 can be used to generate the vertices of Newton(\(\kappa_\alpha\)).

Monical, Tokcan and Yong [8, Conjecture 3.13] conjectured an alternative characterization of the vertices of Newton(\(\kappa_\alpha\)) in terms of the Bruhat order on permutations.

Let \(\alpha\) be a composition, and \(\lambda(\alpha)\) be the partition obtained by resorting the parts of \(\alpha\) decreasingly. Write \(w(\alpha)\) for the (unique) permutation of shortest length that sends \(\lambda(\alpha)\) to \(\alpha\). For two compositions \(\alpha, \beta\), define

\[
\beta \leq \alpha \quad \text{if} \quad \lambda(\beta) = \lambda(\alpha) \quad \text{and} \quad w(\beta) \leq w(\alpha) \quad \text{in the Bruhat order}.
\]

Based on the decomposition of a key polynomial into Demazure atoms, Monical, Tokcan and Yong [8, Theorem 3.12] showed that if \(\beta \leq \alpha\), then \(\beta\) is a vertex of Newton(\(\kappa_\alpha\)). They [8, Conjecture 3.13] conjectured that the converse is still true, that is, if \(\beta\) is a vertex of Newton(\(\kappa_\alpha\)), then \(\beta \leq \alpha\). Invoking Theorem 1.1, we confirm this conjecture.

**Theorem 1.2.** Let \(\alpha\) be a composition. Then the vertex set of the Newton polytope Newton(\(\kappa_\alpha\)) is \(\{\beta: \beta \leq \alpha\}\).

When the parts of \(\alpha\) are weakly increasing, \(\kappa_\alpha(x)\) is the Schur function \(s_{\lambda(\alpha)}(x)\) [11]. In this case, Theorem 1.2 implies the classical result that the Newton polytope of a Schur function \(s_\lambda(x)\) is \(P_\lambda\), the permutohedron whose vertices are rearrangements of \(\lambda\).

Theorem 1.2 also establishes a connection between the Newton polytopes of certain key polynomials and Bruhat interval polytopes. For two permutations \(u \leq v\) in the Bruhat order, the Bruhat interval polytope \(Q_{u,v}\) is the convex hull of the permutations in the Bruhat interval \([u, v]\). Bruhat interval polytopes were introduced by Kodama and Williams [6] in the context of the Toda lattice and the moment map on the flag variety, and their combinatorial properties were studied by Tsukerman and Williams [12]. The following corollary is a direct consequence of Theorem 1.2.

**Corollary 1.3.** Let \(w = w_1 \cdots w_n \in S_n\) be a permutation. View \(w\) as a composition \((w_1, \ldots, w_n)\). Then the Newton polytope Newton(\(\kappa_w\)) of \(\kappa_w(x)\) is the Bruhat interval polytope \(Q_{w_0, w}\), where \(w_0 = n \cdots 21\) is the largest permutation of \(S_n\) in the Bruhat order.

We end this section with an overview of the definitions of Schubert and key polynomials. Schubert polynomials \(\mathcal{S}_w(x)\) for permutations \(w \in S_n\) were introduced by Lascoux and Schützenberger [7], which represent the cohomology classes of Schubert cycles in flag varieties. To define \(\mathcal{S}_w(x)\), let \(\partial_i\) be the divided difference operator which sends a
polynomial $f$ to $\partial_i f = (f - s_i f)/(x_i - x_{i+1})$, where $s_i f$ is obtained from $f$ by exchanging $x_i$ and $x_{i+1}$. For the permutation $w_0 = n (n - 1) \cdots 1$, set $\mathcal{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$.

For $w \neq w_0$, choose a position $1 \leq i < n$ such that $w_i < w_{i+1}$. Let $w'$ be the permutation obtained from $w$ by interchanging $w_i$ and $w_{i+1}$. Set $\mathcal{S}_w(x) = \partial_i \mathcal{S}_{w'}(x)$.

Key polynomials $\kappa_\alpha(x)$ (also called Demazure characters) indexed by compositions $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ were introduced by Demazure [1], which are defined using the Demazure operator $\pi_i = \partial_i x_i$. If $\alpha$ is a partition, then set $\kappa_\alpha(x) = x^\alpha$. Otherwise, choose $i$ such that $\alpha_i < \alpha_{i+1}$. Let $\alpha'$ be the composition obtained from $\alpha$ by interchanging $\alpha_i$ and $\alpha_{i+1}$. Set

$$\kappa_\alpha(x) = \pi_i \kappa_{\alpha'}(x) = \partial_i(x_1 \kappa_{\alpha'}(x)).$$

This paper is structured as follows. In Section 2, we review a result shown in [3] that Schubertopes are Minkowski sums of Schubert matroid polytopes. This implies that the Schubertope $S_D$ is a base polytope associated to a certain submodular function. Edmonds [2] found a characterization of vertices of base polytopes for submodular functions. Based on Edmonds’s characterization, we prove Theorem 1.1 in Section 3. In the final section, we present a proof of Theorem 1.2.

## 2 Schubert matroid polytopes

A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$, called independent sets, such that

(i) $\emptyset \in \mathcal{I}$;

(ii) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$;

(iii) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

By (ii), a matroid $M$ is determined by the collection $\mathcal{B}$ of maximal independent sets, called the bases of $M$. So we can write $M = (E, \mathcal{B})$. Moreover, it follows from (iii) that the bases have the same size.

Let $S$ be a subset of $[n]$. The Schubert matroid $SM_n(S)$ is the matroid with basis

$$\{T \subseteq [n] : T \leq S\}.$$

The notation $T \leq S$ means that

1. $\#T = \#S$;

2. If we write $T = \{a_1 < a_2 < \cdots < a_k\}$ and $S = \{b_1 < b_2 < \cdots < b_k\}$, then $a_i \leq b_i$ for $1 \leq i \leq k$.

Given a matroid $M = (E, \mathcal{B})$ with $E = [n]$, the associated matroid polytope of $M$ is constructed as follows. For $1 \leq i \leq n$, let $e_i$ be the standard basis of $\mathbb{R}^n$. For a subset $B = \{b_1, \ldots, b_k\}$ of $[n]$, write

$$e_B = e_{b_1} + \cdots + e_{b_k}.$$
The matroid polytope $P(M)$ is defined by
\[
P(M) = \text{conv}\{e_B : B \in \mathcal{B}\}.
\]
The matroid polytope is a generalized permutohedron parametrized by its rank function $\{r_M(S)\}$, see [3] for a reference. To be specific,
\[
P(M) = \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} x_i = r_M([n]) \quad \text{and} \quad \sum_{i \in S} x_i \leq r_M(S) \quad \text{for} \ S \subseteq [n] \right\},
\]
where the rank function $r_M$ of $M$ is a map from the subsets of $E$ to $\mathbb{Z}_{\geq 0}$ defined by
\[
r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}, \quad \text{for} \ S \subseteq E.
\]

It turns out that the Schubitope $S_D$ is the Minkowski sum of Schubert matroid polytopes associated to the columns of $D$. Let $D$ be a diagram of $[n]^2$. Write $D = (D_1, \ldots, D_n)$, where, for $1 \leq j \leq n$, $D_j$ is the $j$-th column of $D$. The column $D_j$ can be viewed as a subset of $[n]$:
\[
D_j = \{1 \leq i \leq n : (i, j) \in D\}.
\]

Then the column $D_j$ defines a Schubert matroid $SM_n(D_j)$. For two polytopes $P$ and $Q$, the Minkowski sum of $P$ and $Q$ is defined by
\[
P + Q = \{u + v : u \in P, v \in Q\}.
\]

**Theorem 2.1** (Fink-Mészáros-St. Dizier [3]). Let $D = (D_1, \ldots, D_n)$ be a diagram of $[n]^2$, and let $r_j$ denote the rank function of $SM_n(D_j)$. Then
\[
S_D = P(SM_n(D_1)) + \cdots + P(SM_n(D_n))
\]
\[
= \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} x_i = \#D \quad \text{and} \quad \sum_{i \in S} x_i \leq r_D(S) \quad \text{for} \ S \subseteq [n] \right\},
\]
where
\[
r_D(S) = r_1(S) + \cdots + r_n(S).
\]

3 Proof of Theorem 1.1

In this section, we present a proof of Theorem 1.1. A crucial observation is that the Schubitope $S_D$ is the base polytope associated to the function $r_D$. Edmonds [2] obtained a characterization of the vertices of any given base polytope. Based on Edmonds's characterization, we arrive at a proof of Theorem 1.1.
3.1 Schubitopes are base polytopes

Base polytopes are polytopes associated to submodular functions. A function \( f \) from subsets of \([n]\) to \( \mathbb{R} \) is called a submodular function, if, for any subsets \( S, T \subseteq [n] \),

\[
f(S) + f(T) \geq f(S \cup T) + f(S \cap T).
\]

To a submodular function \( f \), the associated base polytope \( B_f \) is defined by

\[
B_f = \left\{ x \in \mathbb{R}^n : x_1 + \cdots + x_n = f([n]), \sum_{i \in S} x_i \leq f(S) \text{ for } S \subseteq [n] \right\}.
\]

Edmonds [2] used the greedy algorithm to obtain a description of the vertices of base polytopes for submodular functions, see also [4].

**Theorem 3.1** (\([2,4]\)). Let \( f : 2^{[n]} \to \mathbb{R} \) be a submodular function. Then the vertex set of the base polytope \( B_f \) is precisely

\[
\{ x(w) : w \in S_n \},
\]

where \( x(w) = (x_1, \ldots, x_n) \) is the vector in \( \mathbb{R}^n \) defined by

\[
x_{wk} = f(\{w_1, \ldots, w_k\}) - f(\{w_1, \ldots, w_{k-1}\}).
\]

It is well known that the rank function \( r_M \) of a matroid \( M \) is submodular [9]. Hence the function \( r_D \) defined in \([2,3]\) is submodular. By Theorem 3.1, we obtain the following characterization of the vertex set of a Schubitope.

**Theorem 3.2.** Let \( D \) be a diagram of \([n]^2\). Then the vertex set of the Schubitope \( S_D \) is

\[
\{ x(w) : w \in S_n \},
\]

where \( x(w) = (x_1, \ldots, x_n) \) is the vector in \( \mathbb{R}^n \) defined by

\[
x_{wk} = r_D(\{w_1, \ldots, w_k\}) - r_D(\{w_1, \ldots, w_{k-1}\}).
\]

3.2 Rank function of a Schubert matroid

Throughout this subsection, we let \( C \) be a column of a diagram of \([n]^2\). Of course, we can regard \( C \) itself as a diagram of \([n]^2\) such that the boxes lie in exactly one column. Let \( SM_n(C) \) be the Schubert matroid associated to \( C \). We show that the filling \( F_w(C) \) generated by the algorithm in Introduction can be used to compute the rank function \( r_C \) of \( SM_n(C) \). This, together with Theorem 3.2, leads to a proof of Theorem 1.1.

A filling \( F \) of \( C \) is an assignment of positive integers into some of the boxes of \( C \). A box of \( F \) is called empty if it is not assigned any number. A filling \( F \) is called column-strict if the numbers appearing in \( F \) are distinct, and \( F \) is called flagged if for any nonempty box in row \( i \), the number assigned in it does not exceed \( i \). For a subset \( S \)
of $[n]$, we denote by $F(C, S)$ the set of column-strict flagged fillings $F$ of $C$ such that all the numbers appearing in $F$ belong to $S$. We also denote $F_{\leq}(C, S)$ to be the subset consisting of the fillings $F \in F(C, S)$ such that the numbers in $F$ are increasing from top to bottom. Let $|F|$ denote the number of non-empty boxes of $F$.

For a permutation $\pi$ of a subset $S$ of $[n]$, we can generate a filling $F_{\pi}(C)$ of $C$ by the algorithm given in Introduction. Notice that there may exist empty boxes in $F_{\pi}(C)$.

**Theorem 3.3.** Let $C$ be a column of a diagram of $[n]^2$, and $r_C$ be the rank function of $SM_n(C)$. For a $k$-subset $S$ of $[n]$, let $\pi = \pi_1 \pi_2 \cdots \pi_k$ be any given permutation of elements of $S$. Then

$$r_C(S) = |F_{\pi}(C)|.$$  \hspace{1cm} (3.1)

To prove Theorem 3.3 we need the following characterization of the rank function $r_C$.

**Theorem 3.4.** For any subset $S$ of $[n]$, we have

$$r_C(S) = \max\{|F| : F \in F(C, S)\}.$$  \hspace{1cm} (3.2)

To prove Theorem 3.4 we define two operations acting on $F(C, S)$ and $F_{\leq}(C, S)$, respectively. Let $F \in F(C, S)$. The first one is the sorting operation, which transforms $F$ to a filling sort($F$) by keeping the empty boxes of $F$ unchanged and rearranging the numbers of $F$ increasingly from top to bottom. An example to illustrate the sorting operation is given in Figure 3.3.

![Figure 3.3: The sorting operation and standardization operation.](image)

**Proposition 3.5.** For $F \in F(C, S)$, the filling sort($F$) belongs to $F_{\leq}(C, S)$.

**Proof.** Obviously, sort($F$) is column-strict. We need to verify that sort($F$) is flagged. Let $a_1a_2\cdots a_k$ be the word by reading the numbers of $F$ from top to bottom. Define the inversion number $\text{inv}(F)$ of $F$ to be the number of pairs $(i, j)$ such that $a_i > a_j$.

The proof is by induction on $\text{inv}(F)$. If $\text{inv}(F) = 0$, then sort($F$) = $F \in F_{\leq}(C, S)$. We now consider the case $\text{inv}(F) > 0$. Choose $i < j$ such that $a_i > a_j$. Let $F'$ be the filling obtained from $F$ by interchanging $a_i$ and $a_j$. Clearly, $\text{inv}(F') < \text{inv}(F)$. We claim that $F'$ belongs to $F(C, S)$. This can be seen as follows. Suppose that $a_i$ lies in the $p$-th row of $F$, and $a_j$ lies in the $q$-th row of $F$, where $p < q$. Since $F$ is flagged, we have
\[ a_i \leq p \text{ and } a_j \leq q. \] Combining the facts that \( a_i > a_j \) and \( p < q \), we reach that \( a_i \leq q \) and \( a_j \leq p \). This implies that \( \mathcal{F}' \) is flagged, concluding the claim. By induction, \( \text{sort}(\mathcal{F}') \) belongs to \( \mathcal{F}_\leq(C,S) \). Since \( \text{sort}(\mathcal{F}) = \text{sort}(\mathcal{F}') \), we complete the proof.

The second operation is the standardization operation acting on \( \mathcal{F}_\leq(C,S) \). Let \( \mathcal{F} \in \mathcal{F}_\leq(C,S) \). The standardization of \( \mathcal{F} \) is the filling \( \text{standard}(\mathcal{F}) \) obtained by moving upwards the numbers in \( \mathcal{F} \) as high as possible subject to the flag condition. More precisely, let \( a_1 < a_2 < \cdots < a_k \) be the numbers in \( \mathcal{F} \) from top to bottom. Construct a sequence of fillings \( \mathcal{F} = \mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(k)} \) as follows. For \( 1 \leq t \leq k \), \( \mathcal{F}^{(t)} \) is generated from \( \mathcal{F}^{(t-1)} \) according to the following two cases:

(1) The indices of empty boxes in \( \mathcal{F}^{(t-1)} \) above \( a_t \) are all strictly smaller than \( a_t \). In this case, let \( \mathcal{F}^{(t)} = \mathcal{F}^{(t-1)} \);

(2) There exist empty boxes in \( \mathcal{F}^{(t-1)} \) above \( a_t \) with row indices greater than or equal to \( a_k \). Let \( i_t \) be the smallest such row index. Then \( \mathcal{F}^{(t)} \) is obtained from \( \mathcal{F}^{(t-1)} \) by moving \( a_t \) up to the box in row \( i_t \).

Define \( \text{standard}(\mathcal{F}) = \mathcal{F}^{(k)} \). By construction, it is easily seen that \( \text{standard}(\mathcal{F}) \) belongs to \( \mathcal{F}_\leq(C,S) \). Figure 3.3 is an illustration of the standardization operation.

We can now give a proof of Theorem 3.4.

**Proof of Theorem 3.4** Let

\[ \tau_C(S) = \max\{|\mathcal{F}| : \mathcal{F} \in \mathcal{F}(C,S)\}. \tag{3.3} \]

We first show that \( \tau_C(S) \leq r_C(S) \). Suppose that \( \mathcal{F}_0 \in \mathcal{F}(C,S) \) attains the maximal cardinality among all \( \mathcal{F} \in \mathcal{F}(C,S) \), namely, \( \tau_C(S) = |\mathcal{F}_0| \). Set

\[ \mathcal{F}_0' = \text{standard}(\text{sort}(\mathcal{F}_0)). \]

By Proposition 3.5, \( \mathcal{F}_0' \) belongs to \( \mathcal{F}_\leq(C,S) \). Let \( \mathcal{F}_0'' \) be the filling of \( C \) obtained from \( \mathcal{F}_0' \) by assigning each empty box with its row index. By the construction of the standardization operator, it is easily checked that \( \mathcal{F}_0'' \) is a column-strict flagged filling of \( C \) such that the numbers in \( \mathcal{F}_0'' \) are increasing from top to bottom. Hence the set of numbers in \( \mathcal{F}_0'' \) forms a base \( B_0 \) of the Schubert matroid \( SM_n(C) \). Moreover,

\[ #(S \cap B_0) \geq |\mathcal{F}_0|, \]

which implies that

\[ \tau_C(S) = |\mathcal{F}_0| \leq #(S \cap B_0) \leq r_C(S). \]

We now verify the reverse direction \( \tau_C(S) \geq r_C(S) \). Let \( B_0 \) be a base of the Schubert matroid \( SM_n(C) \) such that \( S \cap B_0 \) has the maximal cardinality, that is, \( r_C(S) = #(S \cap B_0) \). Define a filling \( \mathcal{F}_{B_0} \) of \( C \) as follows: Assign the elements of \( B_0 \) into the boxes of \( C \) such that the numbers are increasing from top to bottom, and then delete the numbers not belonging to \( S \). Since \( B_0 \leq C \), it is clear that \( \mathcal{F}_{B_0} \) is a filling in \( \mathcal{F}_\leq(C,S) \). As \( |\mathcal{F}_{B_0}| = #(S \cap B_0) \), we see that

\[ \tau_C(S) \geq |\mathcal{F}_{B_0}| = r_C(S). \]
This completes the proof.

Using Theorem 3.4, we can finish the proof of Theorem 3.3.

**Proof of Theorem 3.3.** We make induction on the cardinality of $S = \{\pi_1, \ldots, \pi_k\}$.

Consider the initial case $k = 1$. It is obvious that $r_C(S) = 1$ or 0, depending on whether $C$ has a box with row index greater than or equal to $\pi_1$. So the equality (3.1) holds.

Assume now that $k \geq 2$ and (3.1) is true for $k - 1$. Let

$$S' = S \setminus \{\pi_k\} = \{\pi_1, \ldots, \pi_{k-1}\}$$

and $\pi' = \pi_1 \pi_2 \cdots \pi_{k-1}$. Recall that

$$r_C(S') = \max\{\#(S' \cap B) : B \in \mathcal{B}\}$$

and

$$r_C(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\},$$

where $\mathcal{B}$ is the basis of the Schubert matroid $SM_n(C)$. So we see that

$$r_C(S) = r_C(S') \quad \text{or} \quad r_C(S) = r_C(S') + 1. \quad \text{(3.4)}$$

Keep in mind that $\mathcal{F}_\pi(C)$ is obtained from $\mathcal{F}_{\pi'}(C)$ by putting $\pi_k$ into the topmost empty box of $\mathcal{F}_{\pi'}(C)$ subject to the flag condition. We conclude the proof by considering the following cases.

Case 1. $\mathcal{F}_\pi(C) \neq \mathcal{F}_{\pi'}(C)$. In this case, $|\mathcal{F}_\pi(C)| = |\mathcal{F}_{\pi'}(C)| + 1$. Since $\mathcal{F}_\pi(C) \in \mathcal{F}(C, S)$, it follows from Theorem 3.4 that

$$r_C(S) \geq |\mathcal{F}_\pi(C)| = |\mathcal{F}_{\pi'}(C)| + 1.$$

By induction, $r_C(S') = |\mathcal{F}_{\pi'}(C)|$. So $r_C(S) \geq r_C(S') + 1$. In view of (3.4), we have

$$r_C(S) = r_C(S') + 1 = |\mathcal{F}_\pi(C)|,$$

as desired.

Case 2. $\mathcal{F}_\pi(C) = \mathcal{F}_{\pi'}(C)$. In this case, there are no allowable empty boxes in $\mathcal{F}_{\pi'}(C)$ to place $\pi_k$. There are two subcases.

Case I. There are no empty boxes in $\mathcal{F}_{\pi'}(C)$. By induction,

$$r_C(S') = |\mathcal{F}_{\pi'}(C)| = \#C.$$

Since $r_C(S) \leq \#C$, by 3.4, we obtain

$$r_C(S) = r_C(S') = \#C,$$

and hence $r_C(S) = |\mathcal{F}_\pi(C)|$.

Case II. There exist empty boxes in $\mathcal{F}_{\pi'}(C)$, but we cannot put $\pi_k$ into any of these empty boxes. Suppose that $l$ is the largest row index of the empty boxes. Assume that there are $b$ boxes of $C$ lying strictly below row $l$. By the construction of $\mathcal{F}_{\pi'}(C)$, it is
obvious that each number filled in those \( b \) boxes below row \( l \) is strictly larger than \( l \). As the box in row \( l \) is empty, by the construction of \( F_\pi(C) \), we have \( \pi_k > l \).

Assume that \( r_C(S') = m \). Let \( \pi_{i_1}, \ldots, \pi_{i_m} \) be the elements of \( S' \) that are filled in \( F_\pi'(C) \). Again, as the box in row \( l \) is empty, by the construction of \( F_\pi'(C) \), it is easy to check that each integer in the set

\[
S' \setminus \{\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}\},
\]
is larger than \( l \).

We aim to show that \( r_C(S) = m \). Suppose to the contrary that \( r_C(S) \neq m \). By (3.4), we have \( r_C(S) = m + 1 \). By Theorem 3.3 there is a filling \( F \in F(C, S) \) such that \( |F| = m + 1 \). Notice that \( \pi_k \) must belong to \( F \), since otherwise \( F \) is a filling in \( F(C, S') \) which, together with Theorem 3.4 would imply that \( r_C(S') \geq m + 1 \), leading to a contradiction.

Assume that \( \pi_{j_1}, \ldots, \pi_{j_m}, \pi_k \in S \) are the integers filled in \( F \). Notice that each integer in the set

\[
\{\pi_{j_1}, \ldots, \pi_{j_m}\} \setminus \{\pi_{i_1}, \ldots, \pi_{i_m}\}
\]
is strictly larger than \( l \). Recall that the numbers filled in those \( b \) boxes of \( F_\pi'(C) \) below row \( l \) are strictly larger than \( l \). So \( \{\pi_{i_1}, \ldots, \pi_{i_m}\} \) contains at least \( b \) integers larger than \( l \). Thus \( \{\pi_{j_1}, \ldots, \pi_{j_m}\} \) contains at least \( b \) integers larger than \( l \). Combining the fact that \( \pi_k > l \), the set \( \{\pi_{j_1}, \ldots, \pi_{j_m}, \pi_k\} \) contains at least \( b + 1 \) integers larger than \( l \). However, there are exactly \( b \) boxes of \( C \) with row indices strictly larger than \( l \). This means that the \( m + 1 \) integers \( \pi_{j_1}, \ldots, \pi_{j_m}, \pi_k \) cannot be put into the boxes of \( C \) to form a flagged filling, leading to a contradiction. Thus the assumption that \( r_C(S) = m + 1 \) is false. So we have \( r_C(S) = m = |F_\pi(C)| \). This completes the proof.

Using Theorem 3.2 and Theorem 3.3 we can now present a proof of Theorem 1.1.

**Proof of Theorem 1.1** Let \( x(w) = (x_1, \ldots, x_n) \) be the vertex of \( S_n \) labeled by a permutation \( w = w_1 \cdots w_n \in S_n \). By (2.3), Theorem 3.2 and Theorem 3.3 we find that

\[
x_{w_k} = r_D(\{w_1, \ldots, w_k\}) - r_D(\{w_1, \ldots, w_{k-1}\})
\]

\[
= \sum_{j=1}^n r_j(\{w_1, \ldots, w_k\}) - \sum_{j=1}^n r_j(\{w_1, \ldots, w_{k-1}\})
\]

\[
= \sum_{j=1}^n |F_{w_1 \cdots w_k}(D_j)| - \sum_{j=1}^n |F_{w_1 \cdots w_{k-1}}(D_j)|
\]

\[
= |F_{w_1 \cdots w_k}(D)| - |F_{w_1 \cdots w_{k-1}}(D)|.
\]

Thus \( x_{w_k} \) is equal to the number of appearances of \( w_k \) in \( F_{w_1 \cdots w_k}(D) \). It is obvious that the numbers of appearances of \( w_k \) in \( F_{w_1 \cdots w_k}(D) \) and in \( F_w(D) \) are the same, and so \( x_{w_k} \) is equal to the number of appearances of \( w_k \) in \( F_w(D) \). This completes the proof.
4 Proof of Theorem 1.2

Let us begin by reviewing the Bruhat order. We view a permutation \( w = w_1w_2 \cdots w_n \in S_n \) as a bijection on \([n]\), that is, \( w \) maps \( i \) to \( w(i) = w_i \). As usual, for \( 1 \leq i \leq n - 1 \), let \( s_i = (i, i + 1) \) denote the adjacent transposition. So \( ws_i \) is the permutation obtained from \( w \) by interchanging \( w_i \) and \( w_{i+1} \), while \( s_iw \) is obtained by interchanging the values \( i \) and \( i + 1 \). For example, for \( w = 2143 \), we have \( ws_2 = 2413 \) but \( s_2w = 3142 \).

Each permutation can be written as a product of adjacent transpositions. The length \( \ell(w) \) of a permutation \( w \) is the minimum \( k \) such that \( w = s_{i_1}s_{i_2} \cdots s_{i_k} \), and in this case, \( s_{i_1}s_{i_2} \cdots s_{i_k} \) is called a reduced expression of \( w \). The (strong) Bruhat order \( \leq \) on \( S_n \) is the closure of the following covering relation. For \( w, w' \in S_n \), we say that \( w \) covers \( w' \) if there exists a transposition \( t_{ij} = (i, j) \) such that \( w = w't_{ij} \) and \( \ell(w) = \ell(w') + 1 \). The Bruhat order can also be characterized by the Subword Property, see for example [5].

**Theorem 4.1** (Subword Property). Let \( s_{i_1}s_{i_2} \cdots s_{i_k} \) be any given reduced expression of a permutation \( w \). Then \( w' \leq w \) in the Bruhat order if and only if there exists a subexpression of \( s_{i_1}s_{i_2} \cdots s_{i_k} \) that is a reduced expression of \( w' \).

4.1 Properties on vertices of Newton\((\kappa_\alpha)\)

In this subsection, we use Theorem 4.1 to give two relationships on the vertices of Newton\((\kappa_\alpha)\), which will be used in the proof of Theorem 1.2.

Given \( w = w_1 \cdots w_n \in S_n \) and a vector \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), the (right) action of \( w \) on \( v \) is defined as

\[
v \cdot w = (v_{w_1}, \ldots, v_{w_n}).
\]

**Proposition 4.2.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a composition. Assume that there exists \( 1 \leq r \leq n - 1 \) such that \( \alpha_r < \alpha_{r+1} \), and that \( w \) is a permutation in \( S_n \) such that \( r \) appears before \( r+1 \) in \( w \). Then

\[
x(w) = x(s_rw) \cdot s_r. \tag{4.1}
\]

**Proof.** Write \( x(w) = (x_1, \ldots, x_n) \). By Theorem 4.1, \( x_k \) is the number of appearances of \( k \) in \( F_w(D(\alpha)) \). Let \( D_j \) be the \( j \)-th column of \( D(\alpha) \), which can also be viewed as a subset of \([n]\). It suffices to prove the following claim.

Claim. The numbers of appearances of \( r \) and \( r+1 \) in \( F_w(D_j) \) and \( F_{s_rw}(D_j) \) are exchanged, while, for \( k \neq r, r+1 \), the number of appearances of \( k \) in \( F_w(D_j) \) and \( F_{s_rw}(D_j) \) remains the same.

For ease of description, for any filling \( F \), we use \( i \in F \) to mean that the integer \( i \) appears in \( F \). To verify the Claim, since \( \alpha_r < \alpha_{r+1} \), we have the following three cases.

Case 1. \( r \notin D_j \) and \( r + 1 \notin D_j \). In this case, \( F_{s_rw}(D_j) \) is obtained from \( F_w(D_j) \) by replacing \( r \) (if any) by \( r + 1 \), and replacing \( r + 1 \) (if any) by \( r \).

Case 2. \( r \notin D_j \) and \( r + 1 \in D_j \). This case is essentially the same as Case 1.
Case 3. $r \in D_j$ and $r + 1 \in D_j$. This case is divided into the following subcases.

Subcase I. $r \notin \mathcal{F}_w(D_j)$ and $r + 1 \notin \mathcal{F}_w(D_j)$. It is easy to check that $\mathcal{F}_w(D_j) = \mathcal{F}_{s,w}(D_j)$.

Subcase II. $r \notin \mathcal{F}_w(D_j)$ and $r + 1 \in \mathcal{F}_w(D_j)$. Since $r$ appears before $r + 1$ in $w$, this case is impossible.

Subcase III. $r \in \mathcal{F}_w(D_j)$ and $r + 1 \notin \mathcal{F}_w(D_j)$. In this case, we still have two situations to consider.

(1) $r$ is not filled in the box $(r, j)$. In this case, it easy to check that $\mathcal{F}_{s,w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by replacing $r$ with $r + 1$.

(2) $r$ is filled in the box $(r, j)$. Since $r + 1$ does not appear in $\mathcal{F}_w(D_j)$, the box $(r + 1, j)$ is filled with an integer, say $w_i$, which is smaller than $r$. By the construction of $\mathcal{F}_{s,w}(D_j)$, $w_i$ must appear after $r$, but before $r + 1$. Hence, when we construct $\mathcal{F}_{s,w}(D_j)$, the box $(r + 1, j)$ is occupied by $r + 1$, the box $(r, j)$ is occupied by $w_i$, and each box other than $(r, j)$ and $(r + 1, j)$ is filled with the same integer as $\mathcal{F}_w(D_j)$. This implies that $\mathcal{F}_{s,w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by replacing $r$ with $r + 1$, and then exchanging the values $r + 1$ and $w_i$. The above arguments are best understood by an example as given in Figure 4.4, where $w = 324615$, $r = 4$ and $s_r w = 325614$.

Figure 4.4: An illustration of the proof of Subcase (III)(2).

Subcase IV. $r \in \mathcal{F}_w(D_j)$ and $r + 1 \in \mathcal{F}_w(D_j)$. This case is similar to Subcase III.

(1) $r$ is not filled in the box $(r, j)$. In this case, it easy to check that $\mathcal{F}_{s,w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by interchanging $r$ and $r + 1$.

(2) $r$ is filled in the box $(r, j)$, and $r + 1$ is filled in the box $(r + 1, j)$. In this case, it is easy to check that $\mathcal{F}_{s,w}(D_j) = \mathcal{F}_w(D_j)$.

(3) $r$ is filled in the box $(r, j)$, but $r + 1$ is filled in a box below $(r + 1, j)$. Since $r + 1$ is filled in a box below $(r + 1, j)$, the box $(r + 1, j)$ is filled with an integer, say $w_i$, which is smaller than $r$. With the same arguments as those in Subcase III(2), we obtain that $\mathcal{F}_{s,w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by interchanging $r$ with $r + 1$, and then interchanging $r + 1$ and $w_i$.

The above analysis allows us to conclude the Claim, and so the proof is complete.
**Proposition 4.3.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a composition. Assume that there exists \( 1 \leq r \leq n - 1 \) such that \( \alpha_r < \alpha_{r+1} \), and that \( w \) is a permutation in \( S_n \) such that \( r \) appears before \( r + 1 \) in \( w \). Let \( \alpha' = \alpha \cdot s_r \), and let \( x'(w) \) denote the vertex of Newton(\( \kappa_{\alpha'} \)) labeled by \( w \). Then

\[
(x(w) = x'(w)). \tag{4.2}
\]

**Proof.** Let \( D_j \) be the \( j \)-th column of \( D(\alpha) \). Write \( D' = D(\alpha \cdot s_r) \), and let \( D'_j \) be the \( j \)-th column of \( D(\alpha') \). If \( D_j = D'_j \), it is clear that \( \mathcal{F}_w(D_j) = \mathcal{F}_w(D'_j) \). If \( D_j \neq D'_j \), since \( \alpha_r < \alpha_{r+1} \), we must have \((r,j) \notin D_j \) and \((r,j) \in D'_j \). In this case, it is readily checked that \( \mathcal{F}_w(D'_j) \) is obtained from \( \mathcal{F}_w(D_j) \) by moving the box \((r + 1, j) \) (together with the number filled in the box) up to row \( r \). This, along with Theorem \ref{thm:1.1} completes the proof.

## 4.2 Proof of Theorem \ref{thm:1.2}

Based on Propositions \ref{prop:4.2} and \ref{prop:4.3}, we provide a proof of Theorem \ref{thm:1.2}.

Recall that for a composition \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \lambda(\alpha) \) is the partition obtained by resorting the parts of \( \alpha \) decreasingly, and \( w(\alpha) \) is the shortest length permutation such that

\[
\lambda(\alpha) \cdot w(\alpha) = \alpha, \text{ or equivalently, } \alpha \cdot w(\alpha)^{-1} = \lambda(\alpha).
\]

In fact, \( w(\alpha)^{-1} \) can be read off directly from \( \alpha \) as follows. Let \( t_1 \) be the largest entry of \( \alpha \), and \( t_1 \) appears in \( \alpha \) at positions \( l_1, \ldots, l_{m_1} \) from left to right. Then the first \( m_1 \) entries of \( w(\alpha)^{-1} \) are \( l_1, \ldots, l_{m_1} \). Repeat the same process for the second largest entry of \( \alpha \), etc. For example, let \( \alpha = (2, 0, 1, 3, 2, 0, 1) \). Then \( \lambda(\alpha) = (3, 2, 2, 1, 1, 0, 0) \) and so we have \( w(\alpha)^{-1} = 4153726 \). Thus \( w(\alpha) = 2641375 \). We can also construct \( w(\alpha) \) by a recursive procedure. If \( \alpha \) is a partition, then \( w(\alpha) \) is the identity permutation. Otherwise, choose a position \( r \) such that \( \alpha_r < \alpha_{r+1} \). Let \( \alpha' = \alpha \cdot s_r \). Then

\[
w(\alpha) = w(\alpha') s_r.
\]

The recursive construction eventually leads to a reduced expression of \( w(\alpha) \).

Let \( V(\alpha) \) denote the set

\[
V(\alpha) = \{ \beta : \beta \leq \alpha \} = \{ \lambda(\alpha) \cdot u : u \leq w(\alpha) \}. \tag{4.3}
\]

We need the following decomposition of \( V(\alpha) \).

**Lemma 4.4.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a composition. Assume that there exists \( 1 \leq r \leq n - 1 \) such that \( \alpha_r < \alpha_{r+1} \). Let \( \alpha' = \alpha \cdot s_r \). Then

\[
V(\alpha) = V(\alpha') \cup \{ v \cdot s_r : v \in V(\alpha') \}. \tag{4.4}
\]

**Proof.** It is equivalent to show that

\[
\{ \sigma : \sigma \leq w(\alpha) \} = \{ \tau : \tau \leq w(\alpha') \} \cup \{ \tau s_r : \tau \leq w(\alpha') \}. \tag{4.5}
\]
This can be easily seen from the Subword Property in Theorem 4.1. Since $\alpha_r < \alpha_{r+1}$, we see that $w(\alpha) = w(\alpha')s_r$, and $\ell(w(\alpha)) = \ell(w(\alpha')) + 1$. Let $s_{i_1} \cdots s_{i_k}$ be a reduced expression of $w(\alpha')$. Then $s_{i_1} \cdots s_{i_k} s_r$ is a reduced expression of $w(\alpha)$.

We first verify (4.8) for the case $rinv(\alpha) = 0$. Let $\sigma s_r$ be a reduced expression of $\sigma s_r$. As $\sigma s_r \leq \sigma \leq w(\alpha)$, from Case 1 it follows that $\sigma s_r \leq w(\alpha')$. Since $\sigma = (\sigma s_r)s_r$, we have $\sigma \in \{\tau s_r : \tau \leq w(\alpha')\}$. This verifies (4.8).

The reverse set inclusion can be checked in a similar manner, and thus is omitted.

We are now in a position to finish the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Denote by $U(\alpha)$ the vertex set of Newton($\kappa_\alpha$), that is,

$$U(\alpha) = \{x(w) : w \in S_n\}. \tag{4.7}$$

As mentioned in Introduction, Monical, Tokcan and Yong [8, Theorem 3.12] showed $V(\alpha) \subseteq U(\alpha)$. We finish the proof of Theorem 1.2 by proving

$$U(\alpha) \subseteq V(\alpha). \tag{4.8}$$

The proof is by induction on the “reverse” inversion number of $\alpha$: $rinv(\alpha) = \#\{1 \leq i < j \leq n : \alpha_i < \alpha_j\}$.

We first verify (4.8) for the case $rinv(\alpha) = 0$. In this case, $\alpha$ is a partition. So $w(\alpha)$ is the identity permutation, and thus $V(\alpha) = \{\alpha\}$. On the other hand, it is easy to see that for any permutation $w \in S_n$, $F_w(D)$ is the filling with boxes in row $k$ ($1 \leq k \leq n$) filled with $k$. By Theorem 1.1 we have $U(\alpha) = \{\alpha\}$. This verifies (4.8) for the case $rinv(\alpha) = 0$.

We next consider the case $rinv(\alpha) > 0$. Assume that $r$ is a row index such that $\alpha_r < \alpha_{r+1}$. Let $\alpha' = \alpha \cdot s_r$. It is obvious that $rinv(\alpha') = rinv(\alpha) - 1$. Let $S_n^\leq$ denote the subset consisting of the permutations $w$ of $S_n$ such that $r$ appears before $r + 1$. Let $S_n^\geq$ denote the complement of $S_n^\leq$. It is clear that

$$S_n^\geq = \{s_r w : w \in S_n^\leq\}.$$

Write

$$U^\leq(\alpha) = \{x(w) : w \in S_n^\leq\} \quad \text{and} \quad U^\geq(\alpha) = \{x(w) : w \in S_n^\geq\}.$$ 

Then

$$U(\alpha) = U^\leq(\alpha) \cup U^\geq(\alpha).$$
By Proposition 4.2, we have
\[ U^>(\alpha) = \{ v \cdot s_r : v \in U^<(\alpha) \}. \quad (4.9) \]

By Proposition 4.3, we have
\[ U^< (\alpha) = U^< (\alpha') \subseteq U (\alpha'). \quad (4.10) \]

Therefore,
\[
U (\alpha) = U^< (\alpha) \cup U^> (\alpha)
\]
\[
= U^< (\alpha) \cup \{ v \cdot s_r : v \in U^< (\alpha) \} \quad \text{(by (4.9))}
\]
\[
\subseteq U (\alpha') \cup \{ v \cdot s_r : v \in U (\alpha') \} \quad \text{(by (4.10))}
\]
\[
\subseteq V (\alpha') \cup \{ v \cdot s_r : v \in V (\alpha') \} \quad \text{(by induction)}
\]
\[
= V (\alpha), \quad \text{(by Lemma 4.4)}
\]

which proves (4.8), as desired.

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