INVERSE SCATTERING PROBLEMS FOR THE HARTREE EQUATION WHOSE INTERACTION POTENTIAL DECAYS RAPIDLY.

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Abstract. We consider inverse scattering problems for the three-dimensional Hartree equation. We prove that if the unknown interaction potential $V(x)$ of the equation satisfies some rapid decay condition, then we can uniquely determine the exact value of $\partial^\alpha \hat{V}(0)$ for any multi-index $\alpha$ by the knowledge of the scattering operator for the equation. Furthermore, we show some stability estimate for identifying $\partial^\alpha \hat{V}(0)$.

1. Introductions

This paper is concerned with inverse scattering problems for the three-dimensional Hartree equation

\[ i\partial_t u + \Delta u = (V \ast |u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \tag{1.1} \]

Here, $u = u(t, x)$ is a complex-valued unknown function, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\Delta = \partial^2_1 + \partial^2_2 + \partial^2_3$, $\partial_j = \partial/\partial x_j \ (j = 1, 2, 3)$, $V = V(x)$ is a real-valued measurable function, the symbol $\ast$ denotes the convolution in $\mathbb{R}^3$. The equation (1.1) is approximately derived by the time-dependent multi-body Schrödinger equation and the function $V(x)$ means the interaction potential for particles.

Our aim of this paper is to identify the unknown interaction potential $V(x)$ by the knowledge of the scattering operator. In order to mention the existence of the scattering operator for the equation (1.1), we introduce some notation. For $1 \leq p \leq \infty$, we denote the Lebesgue space $L^p(\mathbb{R}^3)$ and its norm by $L^p$ and $\| \cdot \|_p$, respectively. For $1 \leq p \leq \infty$, we denote the usual Sobolev space $W^1_p(\mathbb{R}^3)$ by $W^1_p$ and we put $H^1 = W^1_2$. For a Banach space $Y$ and for $1 \leq q \leq \infty$, we put $C_b Y = C(\mathbb{R}; Y) \cap L^\infty(\mathbb{R}; Y)$ and $L^q Y = L^q(\mathbb{R}; Y)$. For a Banach space $Y$ equipped with the norm $\| \cdot \|_Y$ and

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for $\delta > 0$, we define the closed ball $B(\delta, Y)$ by $B(\delta, Y) = \{ f \in Y; \| f \|_Y \leq \delta \}$. For $t \in \mathbb{R}$, the linear operator $U(t)$ is defined by $U(t) = e^{it\Delta}$. Then $v(t) = U(t)\phi$ ($\phi \in L^2$) solves the free Schrödinger equation $i\partial_t v + \Delta v = 0$ with the initial condition $v(0) = \phi$. From Strauss [8], we see the existence of the scattering operator for (1.1):

**Theorem 1.1.** Let $M > 0$. Assume that $V \in B(M, L^1)$. Then there exist some positive numbers $\delta_0(M)$ and $C_0$ depending only on $M$ such that the following properties hold:

1. For any $\phi_- \in B(\delta_0(M), H^1)$, there uniquely exists a time-global solution $u \in Z := C_bH^1 \cap L^3 W^{1,5}_{18/5}$ to (1.1) such that $u(0) \in B(C_0\delta_0(M), H^1)$ and
   $$\lim_{t \to -\infty} \| u(t) - U(t)\phi_- \|_{H^1} = 0.$$  
   Thus, the wave operator
   $$W_+: B(\delta_0(M); H^1) \ni \phi_- \mapsto u(0) \in B(C_0\delta_0(M); H^1)$$
   is well-defined.

2. For any $\phi_0 \in B(C_0\delta_0(M); H^1)$, there uniquely exist $v \in Z$ and a datum $\phi_+ \in B(C_0^2\delta_0(M); H^1)$ such that $v$ solves (1.1) with the initial condition $v(0) = \phi_0$ and satisfies
   $$\lim_{t \to +\infty} \| v(t) - U(t)\phi_+ \|_{H^1} = 0.$$  
   Thus, the inverse wave operator
   $$\mathcal{V}_+: B(C_0\delta_0(M); H^1) \ni \phi_0 \mapsto \phi_+ \in B(C_0^2\delta_0(M); H^1)$$
   and the scattering operator
   $$S = \mathcal{V}_+ \circ W_- : B(\delta_0(M); H^1) \to B(C_0^2\delta_0(M); H^1)$$
   are well-defined.

3. We have for any $\phi_- \in B(\delta_0(M), H^1)$,
   $$S(\phi_-) = \phi_- - i \int_{\mathbb{R}} U(-t)(V * |u(t)|^2)u(t)dt$$
   and
   $$\| u(t) - U(t)\phi_- \|_Z \leq C_0 \| \phi_- \|_{H^1}^3,$$
   where $u(t)$ is the time-global solution mentioned in (1).

**Remark 1.1.** For the equation (1.1), the scattering operator is well-defined on some neighborhood of 0 in some suitable Hilbert space if the interaction potential $V$ satisfies either $\sup_{x \in \mathbb{R}^3} |x|^{2\gamma} |V(x)| < \infty$ for some $\gamma \in (1, 3)$ or $V \in L^r$ for some $r \in [1, 3)$. For a proof, see [1, 2, 8].
The inverse scattering problem for the perturbed Schrödinger equation is to recover the perturbed term by applying the knowledge of scattering states. Before we introduce our main results, we first review some known results of inverse scattering problems for $n$-dimensional nonlinear Schrödinger equations briefly. Strauss [7] considered the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = W|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$  

Here, $p$ is a given number and $W = W(x)$ is unknown. It was proved that if $p$ and $W$ satisfy suitable conditions, then the unknown $W$ is uniquely reconstructed by

$$W(x_0) = \lim_{R \to \infty} R^{n+2} K_p[\phi_{R,x_0}] = \int_{\mathbb{R} \times \mathbb{R}^n} |U(t)\phi(x)|^{p+1} d(t,x),$$

where $\phi_{R,x_0}(x) = \phi(R(x - x_0))$ ($R > 0, x \in \mathbb{R}^n$) and

$$K_p[\phi] = \lim_{\varepsilon \to 0} i\varepsilon^{-p} \langle (S - \text{id})(\varepsilon \phi), \phi \rangle_{L^2(\mathbb{R}^n)}, \quad (1.4)$$

which is called the small amplitude limit, the notation “id” is the identity mapping. Later, Weder [14, 15, 16, 17, 18, 19] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover, a method is given for the unique reconstruction of the potential that acts as a linear operator.

Unfortunately, the above methods to obtain the reconstruction formula are not applicable to the case of the Hartree term $(V * |u|^2)u$ (for details, see, e.g. Section 1 of [5]). Watanabe [9, 10, 11, 12] studied the Hartree equation with a potential

$$i\partial_t u + \Delta u = W_1 u + (W_2 * |u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$  

Here, $W_1(x)$ satisfies some suitable condition, $W_2(x) = Q|x|^{-\gamma}$ ($Q > 0$ and $\gamma \in [2, 4]$ with $\gamma < n$). It was proved that $W_1, Q$ and $\gamma$ are uniquely determined (see also [13, 4]). Sasaki [5] considered the three-dimensional Hartree equation with a potential

$$i\partial_t u + \Delta u = W_3 u + (W_4 * |u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3,$$

where $W_j = Q_j \exp(-\gamma_j|x|)/|x|$ ($Q_j \in \mathbb{R}$ and $\gamma_j > 0, j = 3, 4$). It was shown that if $|Q_j| < \gamma_3$, then unknown parameters $Q_j$ and $\gamma_j$ ($j = 3, 4$) are uniquely determined. In the above inverse scattering problems for the Hartree equation, we assume that the interaction potential is a known function with unknown parameters. In other words, we already know what kind of shape the interaction potential has.

We next consider inverse scattering problems for the Hartree equation and the case where we do not know what kind of shape the interaction potential $V$ has. Sasaki–Watanabe [6] proved that if $2 \leq n \leq 6$, $V \in L^1(\mathbb{R}^n)$ and $\phi \in H^1(\mathbb{R}^n) \setminus \{0\}$,
then the following formula holds:

\[ \mathcal{F}V(0) = \lim_{\lambda \to 0} \lambda^{n+2} K_3[\phi_{\lambda,0}] / \left(2\pi\right)^{n/2} \|U(t)\phi\|_{L^1(\mathbb{R};L^1)} \]

where \( \mathcal{F}V \) is the Fourier transform of \( V \) and \( K_3[\phi] \) is the small amplitude limit defined by (1.4) with \( p = 3 \). Later, Sasaki [3] proved that if \( n \geq 3 \) and \( V \) is radial and satisfies some decay condition, then we have for any non-negative integer \( m \),

\[ \nu_m := \left. \frac{d^m}{d\rho^m} \mathcal{F}V(\rho,0,\cdots,0) \right|_{\rho=0} \frac{\partial^m}{\partial \lambda^m} \left( \lambda^{n+2} K_3[\phi_{\lambda,0}] \right) / \int_{\mathbb{R}} \|t^m/2 \mathcal{F} |U(t)\phi|^2\|^2_{L^2(\mathbb{R}^n)} dt, \quad \phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}. \]

Furthermore, if we assume that \( \mathcal{F}V \) is an entire function, then we have

\[ \mathcal{F}V(\xi) = \sum_{m=0}^{\infty} \frac{\nu_m}{m!} |\xi|^m, \quad \xi \in \mathbb{R}^n \quad (1.5) \]

and we can hence reconstruct \( V \).

In this paper, we consider inverse scattering problems for (1.1), supposing some decay condition for \( V \). We show that for any multi-index \( \alpha \), we can uniquely determine the exact value of \( \partial^\alpha \mathcal{F}V(0) \) and we can reconstruct \( V \) even if we do not suppose any symmetric conditions for \( V \).

1.1. Notation. To state our main results precisely, we introduce some notation. Let \( \mathbb{N} \) be the set of all positive integers. Put \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). We denote the Schwartz class \( \mathcal{S}(\mathbb{R}^3) \) by \( \mathcal{S} \). The Fourier transform \( \mathcal{F} \) on \( L^1 \) is defined by

\[ \mathcal{F}f(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx, \quad f \in L^1, \ \xi \in \mathbb{R}^3. \]

We define 3 \times 3 matrices \( I_m \) (\( m = 1, 2, 3 \)) by

\[ I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]

For multi-index \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{0,1\}^3 \), we put

\[ m(\epsilon) = \begin{cases} 1 & \text{if } |\epsilon| = 0, \\ \min\{m; \epsilon_m = 1\} & \text{if } |\epsilon| \neq 0 \end{cases} \]

and \( I(\epsilon) = I_{m(\epsilon)} \). For \( N \in \mathbb{N}_0 \) and \( \epsilon \in \{0,1\}^3 \), we define 3 \times 3 matrices \( D_{N,\epsilon}^\lambda \mu \) (\( \lambda, \mu > 0 \), non-negative integers \( P_{N,\epsilon}(\alpha) \) (\( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3 \)), linear operators
For $\epsilon > 0, t \in \mathbb{R}$ by

$$D_{N,\epsilon}^{\lambda \mu} = \lambda I(\epsilon) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^{N+1} \end{pmatrix} I(\epsilon), \quad P_{N,\epsilon}(\alpha) = (0, 1, N+1) I(\epsilon) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

$$U_{N,\epsilon}^{\mu}(t) = \mathcal{F}^{-1} \exp \left( -it |D_{N,\epsilon}^{1,\mu} \xi|^2 \right) \mathcal{F}, \quad U_{\epsilon}(t) = \mathcal{F}^{-1} \exp \left( -it \epsilon_m^2 \right) \mathcal{F},$$

respectively. We now enumerate two examples. If $\epsilon = (1, 0, 1)$, then

$$I(\epsilon) = I_1, \quad D_{N,\epsilon}^{\lambda \mu} = \text{diag} (\lambda, \lambda \mu, \lambda^{N+1}), \quad P_{N,\epsilon}(\alpha) = \alpha_2 + (N+1) \alpha_3,$$

$$U_{N,\epsilon}^{\mu}(t) = \exp \left( it \left\{ \partial_1^2 + \mu^2 \partial_2^2 + \mu^{2N+2} \partial_3^2 \right\} \right), \quad U_{\epsilon}(t) = \exp \left( it \partial_1^2 \right).$$

If $\epsilon = (0, 0, 1)$, then

$$I(\epsilon) = I_3, \quad D_{N,\epsilon}^{\lambda \mu} = \text{diag} (\lambda^{N+1}, \lambda \mu, \lambda), \quad P_{N,\epsilon}(\alpha) = \alpha_2 + (N+1) \alpha_1,$$

$$U_{N,\epsilon}^{\mu}(t) = \exp \left( it \left\{ \mu^{2N+2} \partial_1^2 + \mu^2 \partial_2^2 + \partial_3^2 \right\} \right), \quad U_{\epsilon}(t) = \exp \left( it \partial_2^2 \right).$$

For $N \in \mathbb{N}_0$, we put $N^* = \# \{ \alpha \in \mathbb{N}_{3,1}^3 \mid |\alpha| = N \} = (N+1)(N+2)/2$. For any $N \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}^3$, there uniquely exists a sequence of multi-indices $\{\alpha(k)\}_{k=1}^{N^*} \subset \mathbb{N}_{3,1}^3$ satisfying the following two properties:

- It follows that $|\alpha(k)| = N$ for any $k = 1, \cdots, N^*$.
- If $1 \leq k < l \leq N^*$, then $P_{N,\epsilon}(\alpha(k)) < P_{N,\epsilon}(\alpha(l))$.

Let us denote such a sequence by $\{\alpha(N, \epsilon; k)\}_{k=1}^{N^*}$. For a $3 \times 3$ matrix $D$ and for a function $\varphi : \mathbb{R}^3 \to \mathbb{C}$, we put $\varphi(Dx) = \varphi(Dx) (x \in \mathbb{R}^3)$. For $L \in \mathbb{N}_0$, $\lambda > 0$ and for a function $h : (0, \infty) \to \mathbb{C}$, we define $\Delta_L^h(\lambda)\mu$ by

$$\Delta_L^h(\lambda)\mu = \sum_{l=0}^{L} \frac{(-1)^{L-l} L!}{l!(L-l)!} h((l+1)\lambda).$$

For $\phi, \tilde{\phi} \in \mathcal{S}, \mu, \lambda > 0, N \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}^3$, we set

$$J_{N,\epsilon}^{\lambda \mu} \left[ \phi, \tilde{\phi}, \mu \right] = \frac{i \mu^{N+2} \lambda^{-N-|\epsilon|}}{(2\pi)^{3/2}} \Delta_{N,\epsilon}^{2N+|\epsilon|} \left\{ \lambda^{-3N-7} \left( S - \text{id} \right) \left( \lambda^{N+4} \phi \circ D_{N,\epsilon}^{\lambda \mu} \right), \tilde{\phi} \circ D_{N,\epsilon}^{\lambda \mu} \right\},$$

$$\Phi_{N,\epsilon} \left( \phi, \tilde{\phi}, \mu; t, \xi \right) = \mathcal{F} \left| U_{N,\epsilon}^{\mu}(t)\phi \right|^2 (\xi) \times \mathcal{F} \left( \overline{U_{N,\epsilon}^{\mu}(t)\phi} \times U_{N,\epsilon}^{\mu}(t)\tilde{\phi} \right) (\xi),$$

$$\Phi_{\epsilon} \left( \phi, \tilde{\phi}; t, \xi \right) = \mathcal{F} \left| U_{\epsilon}(t)\phi \right|^2 (\xi) \times \mathcal{F} \left( \overline{U_{\epsilon}(t)\phi} \times U_{\epsilon}(t)\tilde{\phi} \right) (\xi).$$

Remark that if we fix $N, \epsilon$ and $\mu$, then $\lambda^{N+4} \phi \circ D_{N,\epsilon}^{\lambda \mu}$ belongs to the domain of the scattering operator $S$ for sufficiently small $\lambda > 0$. Furthermore, we define $N^*$-th
column vectors $\mathcal{J}_{N,\epsilon}^\lambda[\phi, \tilde{\phi}, \mu]$, $a_{N,\epsilon}$ and the $N^* \times N^*$ matrix $M_{N,\epsilon}[\phi, \tilde{\phi}, \mu]$ by
\[
\mathcal{J}_{N,\epsilon}^\lambda[\phi, \tilde{\phi}, \mu] = \left( J_{N,\epsilon}^\lambda \left[ \phi, \tilde{\phi}, \mu^{(N+1)2(j-1)} \right] \right)_{1 \leq j \leq N^*},
\]
\[
a_{N,\epsilon} = \left( \partial_\xi^{2\alpha(N,\epsilon;k)+\epsilon} FV(0) \right)_{1 \leq k \leq N^*},
\]
\[
M_{N,\epsilon}[\phi, \tilde{\phi}, \mu] = \left( \frac{(2N + |\epsilon|)!}{(2\alpha(N,\epsilon;k) + |\epsilon|)!} P_{N,\epsilon}(2\alpha(N,\epsilon;k) + |\epsilon|) \times N^{(N+1)2(j-1)} \right.
\]
\[\times \int_{\mathbb{R}^{1+3}} \xi^{2\alpha(N,\epsilon;k) + \epsilon} \Phi_{N,\epsilon} \left( \phi, \tilde{\phi}, \mu^{(N+1)2(j-1)} ; t, \xi \right) d(t, \xi) \right)_{1 \leq j, k \leq N^*}.
\]
We remark that $\mathcal{J}_{N,\epsilon}^\mu$ and $M_{N,\epsilon}$ are given and that $a_{N,\epsilon}$ is unknown. Let $E_1 = \text{diag}(-1,1,1)$, $E_2 = \text{diag}(1,-1,1)$ and $E_3 = \text{diag}(1,1,-1)$. For $N \in \mathbb{N}_0$ and $\epsilon \in \{0,1\}^3$, we set the three following conditions for $\phi, \tilde{\phi} \in S$:

(C1) For any $x \in \mathbb{R}^3$,
\[
\phi \circ E_j(x) = \phi(x), \quad j = 1, 2, 3.
\]

(C2) For any $x \in \mathbb{R}^3$,
\[
\tilde{\phi} \circ E_j(x) = (-1)^{j+1} \tilde{\phi}(x), \quad j = 1, 2, 3.
\]

(C3) For any $k \in \{1, \ldots, N^*\}$,
\[
\int_{\mathbb{R}^{1+3}} \xi^{2\alpha(N,\epsilon;k) + \epsilon} \Phi_{N,\epsilon} \left( \phi, \tilde{\phi}, t, \xi \right) d(t, \xi) \neq 0.
\]

1.2. Main Results. We are ready to state our main results.

**Theorem 1.2.** Assume that the interaction potential $V$ of the equation (1.1) satisfies the following condition:

(V1) There exists some positive number $A$ such that $e^{A|x|} |V(x)| \in L^1(\mathbb{R}_+^3)$.

Let $N \in \mathbb{N}_0$ and $\epsilon \in \{0,1\}^3$. Fix $\phi, \tilde{\phi} \in S \setminus \{0\}$. We assume that $\phi = \tilde{\phi}$ and (C1) holds if $|\epsilon| = 0$, and assume that (C1)–(C3) hold if $|\epsilon| \neq 0$.

Then there exists some positive number $\tilde{\mu}$ such that the matrix $M_{N,\epsilon}[\phi, \tilde{\phi}, \mu]$ is invertible for any $\mu \in (0, \tilde{\mu})$. Furthermore, for any $\mu \in (0, \tilde{\mu})$, there exist some positive numbers $\lambda_\mu$ and $C_\mu$ depending only on $N, A, \|V\|_1, \mu, \phi$ and $\tilde{\phi}$ such that
\[
|a_{N,\epsilon} - M_{N,\epsilon}[\phi, \tilde{\phi}, \mu]^{-1} \mathcal{J}_{N,\epsilon}^\lambda[\phi, \tilde{\phi}, \mu]| \leq C_\mu \lambda, \quad \lambda \in (0, \lambda_\mu).
\]

In particular, the unknown vector $a_{N,\epsilon}$ is determined by
\[
a_{N,\epsilon} = M_{N,\epsilon}[\phi, \tilde{\phi}, \mu]^{-1} \lim_{\lambda \to 0} \mathcal{J}_{N,\epsilon}^\lambda[\phi, \tilde{\phi}, \mu], \quad \mu \in (0, \tilde{\mu}).
\]
We next mention some uniqueness and stability for identifying $\partial_\xi^2 \mathcal{F}V(0)$.

**Corollary 1.3.** Let $j = 1, 2$. Suppose that $V_j$ satisfies (V1) with $V = V_j$ and that $V_1, V_2 \in B(M, L^1)$ for some $M > 0$. Let $S_j : B(\delta_0(M), H^1) \to B(C_0^0 \delta_0(M), H^1)$ be the scattering operator for the equation (1.1) with $V = V_j$. Here, $C_0$ and $\delta_0(M)$ are positive numbers mentioned in Theorem 1.1. Then the following properties hold:

1. Define $\|S_1 - S_2\|$ by

$$\|S_1 - S_2\| = \sup \left\{ \|S_1 - S_2\|_1, S \in B(\delta_0(M); H^1) \setminus \{0\} \right\}.$$

Then for any $\alpha \in \mathbb{N}_0^3$, we have

$$|\partial_\xi^\alpha \mathcal{F}V_1(0) - \partial_\xi^\alpha \mathcal{F}V_2(0)| \leq C \left( \|S_1 - S_2\|_1 + \|S_1 - S_2\| \right),$$

where the constant $C$ is dependent on $|\alpha|$, $A$ and $M$.

2. If $S_1 = S_2$, then $V_1 = V_2$.

We enumerate some remarks for Theorem 1.2 and Corollary 1.3.

**Remark 1.2.** Let $N \in \mathbb{N}_0$ and let $\epsilon \in \{0, 1\}^3$ with $|\epsilon| \neq 0$. We now introduce an example of $\phi, \tilde{\phi} \in S \setminus \{0\}$ satisfying (C1)–(C3). Fix $\varphi_j \in S(\mathbb{R}) \setminus \{0\}$ ($j = 1, 2, 3$) which are even. We put

$$\phi(x_1, x_2, x_3) = \varphi_1(x_1) \varphi_2(x_2) \varphi_3(x_3), \quad \tilde{\phi}(x_1, x_2, x_3) = \partial_\xi^3 \varphi(x_1, x_2, x_3).$$

Then we immediately see that $\phi$ and $\tilde{\phi}$ satisfy (C1) and (C2), respectively. It follows from Proposition A.2 below that (C3) holds.

**Remark 1.3.** We immediately see that for any $\beta \in \mathbb{N}_0^3$, there uniquely exist $N \in \mathbb{N}_0$, $\epsilon \in \{0, 1\}^3$ and $k \in \{1, \cdots, N^*\}$ such that $\beta = 2\alpha(N, \epsilon, k) + \epsilon$. Therefore, using our main results, we can uniquely determine the exact value of $\partial_\xi^2 \mathcal{F}V(0)$ for any $\beta \in \mathbb{N}_0^3$. Furthermore, it follows from Proposition A.1 that

$$\mathcal{F}V(\xi) = \sum_{|\beta| \geq 0} \frac{\beta!}{\partial_\xi^\beta \mathcal{F}V(0)} \xi^\beta, \quad |\xi| < A/3.$$

Using Proposition A.1 again and again, we see the exact value of $\mathcal{F}V(\xi)$ ($\xi \in \mathbb{R}^3$) and we can hence reconstruct $V$.

Introducing the contents of the rest of this paper, we close this section. In Section 2, we show that the matrix $\mathcal{M}_{N, \epsilon}[\phi, \tilde{\phi}, \mu]$ is invertible for some $\phi, \tilde{\phi} \in S \setminus \{0\}$ and $\mu > 0$. In order to prove the invertibility, we first show some propositions for functions $\Phi_{N, \epsilon}$ and $\Phi_\epsilon$. Section 3 is devoted to the proof of main results. In Appendix A, we prove some supplementary propositions.
2. Preliminaries

In this section, we show propositions used in Section 3 below. In particular, we show the following lemma:

Lemma 2.1. Let $N \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}^3$. Fix $\phi, \tilde{\phi} \in \mathcal{S} \setminus \{0\}$. Assume that $\phi = \tilde{\phi}$ if $|\epsilon| = 0$ and that (C3) holds if $|\epsilon| \neq 0$. Then there exists some positive number $\mu$ such that for any $\mu \in (0, \tilde{\mu})$, the matrix $\mathcal{M}_{N,\epsilon}[\phi, \tilde{\phi}, \mu]$ is invertible.

To show Lemma 2.1, we first prove some properties for functions $\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)$ and $\Phi_{\epsilon}(\phi, \tilde{\phi}; t, \xi)$.

Proposition 2.2. Let $N \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}^3$. For any $\mu > 0$ and $L > 3$, there exists some positive number $C_L$ such that

$$
|\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)| \leq \begin{cases} 
\frac{C_L (\xi)^{-2L}}{1 + \mu^2 (\xi_1^2 + \mu^4 \xi_2^2)} & \text{if } \epsilon = (0, 0, 0), \\
\frac{C_L (\xi)^{-2L}}{1 + \mu^2 \xi_3^2} & \text{if } \epsilon \neq (0, 0, 0)
\end{cases}
$$

for any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^3$.

Proof. It suffices to show the case $\epsilon = (0, 0, 0)$ because the other case $\epsilon \neq (0, 0, 0)$ can be proved more easily. It follows that

$$
\mathcal{F} \left( \frac{U_{N,\epsilon}(t) \phi \times U_{N,\epsilon}(t) \tilde{\phi}}{1 + \mu^2 (\xi_1^2 + \mu^4 \xi_2^2)} \right) (\xi) = (2\pi)^{-3/2} \left( \mathcal{F} \left( \frac{U_{N,\epsilon}(t) \phi}{1 + \mu^2 (\xi_1^2 + \mu^4 \xi_2^2)} \right) \ast \mathcal{F} \left( \frac{U_{N,\epsilon}(t) \tilde{\phi}}{1 + \mu^2 (\xi_1^2 + \mu^4 \xi_2^2)} \right) \right) (\xi)
$$

$$
= (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp \left( -it \left\{ (\xi_1 - \eta_1)^2 + \mu^2 (\xi_2 - \eta_2)^2 + \mu^{2N+2} (\xi_3 - \eta_3)^2 \right\} \right) \\
\times \exp \left( -it \left\{ \eta_1^2 + \mu^2 \eta_2^2 + \mu^{2N+2} \eta_3^2 \right\} \right) \mathcal{F}^{-1} \phi(\xi - \eta) \mathcal{F} \tilde{\phi}(\eta) d\eta
$$

$$
= (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp \left( -it \left\{ \eta_1^2 + \mu^2 \eta_2^2 + \mu^{2N+2} \eta_3^2 \right\} \right) \mathcal{F}^{-1} \phi(\xi - \eta) \mathcal{F} \tilde{\phi}(\eta) d\eta.
$$

Let

$$
\Psi(t, \xi) = \int_{\mathbb{R}^3} \exp \left( -2it \left( \xi_1 \eta_1 + \mu^2 \xi_2 \eta_2 + \mu^{2N+2} \xi_3 \eta_3 \right) \right) \mathcal{F}^{-1} \phi(\xi - \eta) \mathcal{F} \tilde{\phi}(\eta) d\eta.
$$

Since $\phi, \tilde{\phi} \in \mathcal{S}$, we obtain for any $L > 3$,

$$
\int_{\mathbb{R}^3} \left| \mathcal{F}^{-1} \phi(\xi - \eta) \mathcal{F} \tilde{\phi}(\eta) \right| d\eta \leq C \int_{\mathbb{R}^3} \langle \xi - \eta \rangle^{-L} \langle \eta \rangle^{-L} d\eta \leq C \langle \xi \rangle^{-L}
$$

and we hence see that

$$
|\Psi(t, \xi)| \leq C \langle \xi \rangle^{-L}.
$$

(2.2)
Furthermore, by using integration by parts with respect to $\eta_1$, we have
\[
\Psi(t, \xi) = \frac{1}{2it\xi_1} \int_{\mathbb{R}^3} \exp \left( -2it \left( \xi_1 \eta_1 + \mu^2 \xi_2 \eta_2 + \mu^{2N+2} \xi_3 \eta_3 \right) \right) 
\times \left\{ -\partial_t \mathcal{F}^{-1} \phi(\xi - \eta) \mathcal{F} \tilde{\phi}(\eta) + \overline{\mathcal{F}^{-1} \phi(\xi - \eta)} \partial_t \mathcal{F} \tilde{\phi}(\eta) \right\} d\eta.
\]
From (2.1), we see that
\[
|\Psi(t, \xi)| \leq \frac{C \langle \xi \rangle^{-L}}{|t\xi_1|}.
\] (2.3)
Using integration by parts with respect to $\eta_2$, we have
\[
|\Psi(t, \xi)| \leq \frac{C \langle \xi \rangle^{-L}}{|t\mu^2\xi_2|}.
\] (2.4)
By (2.2)–(2.4), we complete the proof.

**Proposition 2.3.** Let $N \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}^3$. For any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^3$, we have
\[
\lim_{\mu \to 0} \Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu ; t, \xi) = \Phi_\epsilon(\phi, \tilde{\phi}; t, \xi).
\] (2.5)

**Proof.** For any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^3$, we obtain
\[
\left| \mathcal{F} \left( \overline{U_{N,\epsilon}^\mu(t) \phi \times U_{N,\epsilon}^\mu(t) \tilde{\phi}} \right) (\xi) - \mathcal{F} \left( \overline{U_\epsilon(t) \phi \times U_\epsilon(t) \tilde{\phi}} \right) (\xi) \right|
\leq \left| \mathcal{F} \left( \left( \overline{U_{N,\epsilon}^\mu(t) \phi} \times U_{N,\epsilon}^\mu(t) \tilde{\phi} \right) - \overline{U_\epsilon(t) \phi} \times U_\epsilon(t) \tilde{\phi} \right) (\xi) \right|
+ \left| \mathcal{F} \left( \overline{U_\epsilon(t) \phi} \times \left( U_{N,\epsilon}^\mu(t) \tilde{\phi} - U_\epsilon(t) \tilde{\phi} \right) \right) (\xi) \right|
\leq C \left\| U_{N,\epsilon}^\mu(t) \phi - U_\epsilon(t) \phi \right\|_2 \left\| U_{N,\epsilon}^\mu(t) \tilde{\phi} \right\|_2 + C \left\| U_\epsilon(t) \phi \right\|_2 \left\| U_{N,\epsilon}^\mu(t) \tilde{\phi} - U_\epsilon(t) \tilde{\phi} \right\|_2
\leq C \left\| U_{N,\epsilon}^\mu(t) \phi - U_\epsilon(t) \phi \right\|_2 \left\| \tilde{\phi} \right\|_2 + C \left\| \phi \right\|_2 \left\| U_{N,\epsilon}^\mu(t) \tilde{\phi} - U_\epsilon(t) \tilde{\phi} \right\|_2.
\]
Since we have for any $\varphi \in \mathcal{S}$ and $t \in \mathbb{R}$,
\[
\exp \left( -it \left| D_{N,\epsilon}^{1,\mu} \xi \right|^2 \right) \mathcal{F} \varphi(\xi) \to \exp \left( -it \xi^{2 \alpha_{m(\epsilon)}} \right) \mathcal{F} \varphi(\xi) \quad \text{in} \quad L^2(\mathbb{R}^3_\xi) \text{ as } \mu \to 0,
\]
we see that (2.5) holds.

**Proposition 2.4.** Let $N \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}^3$. For any $\alpha \in \mathbb{N}_0^3$ with $\alpha_{m(\epsilon)} \neq 0$, functions $\xi^\alpha \Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu ; t, \xi) \ (\mu > 0)$ and $\xi^\alpha \Phi_\epsilon(\phi, \tilde{\phi}; t, \xi)$ are integrable on $\mathbb{R}_t \times \mathbb{R}^3_\xi$. Furthermore, we have
\[
\lim_{\mu \to 0} \int_{\mathbb{R}^{1+3}} \xi^\alpha \Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu ; t, \xi) d(t, \xi) = \int_{\mathbb{R}^{1+3}} \xi^\alpha \Phi_\epsilon(\phi, \tilde{\phi}; t, \xi) d(t, \xi).
\] (2.6)
Proof. Let $L > |\alpha| + 3$. It follows from Proposition 2.2 that
\[
|\xi^\alpha \Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)| \leq \frac{C \langle \xi \rangle^{-2L} |\xi^\alpha|}{1 + t^2 \xi_{m(\epsilon)}^2}.
\]
Since $\alpha_{m(\epsilon)} \neq 0$, we have
\[
\int_{\mathbb{R}^{1+3}} \frac{\langle \xi \rangle^{-2L} |\xi^\alpha|}{1 + t^2 \xi_{m(\epsilon)}^2} dt d\xi \leq \int_{\mathbb{R}^{1+3}} \langle \xi \rangle^{-|\alpha| - 7} d\xi < \infty.
\]
Thus, for any $\mu > 0$, the function $\xi^\alpha \Phi_{N,\epsilon}(\phi, \tilde{\phi}; t, \xi)$ is integrable on $\mathbb{R} \times \mathbb{R}^3$. The integrability of $\xi^\alpha \Phi_{\epsilon}(\phi, \tilde{\phi}; t, \xi)$ is shown analogously. We see from Proposition 2.3 and the Lebesgue dominated convergence theorem that (2.6) holds. □

Proposition 2.5. Let $N \in \mathbb{N}_0$ and $\epsilon = (0, 0, 0)$. Assume that $\phi \in S \setminus \{0\}$. Let $m, l \in \mathbb{N}_0$. Then there exist positive numbers $\mu_1$ and $C_2$ such that for any $\mu \in (0, \mu_1)$,
\[
C_1 \leq \int_{\mathbb{R}^{1+3}} \xi_2^{2m} \xi_3^{2l} \Phi_{N,\epsilon}(\phi, \phi, \mu; t, \xi) dt d\xi \leq C_2 (1 + |\log \mu|). \tag{2.7}
\]

Proof. Since the function $\Phi_{\epsilon}(\phi, \phi; t, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^3$, non-negative and does not equal to a zero function, we see that
\[
C_3 := \int_{\mathbb{R}^{1+3}} \xi_2^{2m} \xi_3^{2l} \Phi_{\epsilon}(\phi, \phi; t, \xi) dt d\xi \in (0, \infty].
\]
It follows from Proposition 2.3 and Fatou’s lemma that
\[
C_3 = \int_{\mathbb{R}^{1+3}} \xi_2^{2m} \xi_3^{2l} \liminf_{\mu \to 0} \Phi_{N,\epsilon}(\phi, \phi, \mu; t, \xi) dt d\xi \leq \liminf_{\mu \to 0} \int_{\mathbb{R}^{1+3}} \xi_2^{2m} \xi_3^{2l} \Phi_{N,\epsilon}(\phi, \phi, \mu; t, \xi) dt d\xi.
\]
Therefore, there exists some $\mu_1 \in (0, 1)$ such that for any $\mu \in (0, \mu_1)$,
\[
\min \left\{ \frac{C_3}{2}, 1 \right\} \leq \int_{\mathbb{R}^{1+3}} \xi_2^{2m} \xi_3^{2l} \Phi_{N,\epsilon}(\phi, \phi; t, \xi) dt d\xi. \tag{2.8}
\]
Henceforth, we suppose that \( \mu \in (0, \mu_1) \). Let \( L \) be a sufficiently large positive number. By Proposition \( 2.2 \), we have
\[
\int_{\mathbb{R}^{1+2}} \xi_2^{2m} \xi_3^{2l} \Phi_{N, \xi}(\phi, \phi, \mu; t, \xi) d(t, \xi)
\leq C \int_{\mathbb{R}^2} \xi_2^{2m} \xi_3^{2l} \left( \int_{\mathbb{R}^{1+1}} \langle \xi \rangle^{-2L} d(t, \xi) \right) d(\xi_2, \xi_3)
\leq C \int_{\mathbb{R}^2} \frac{\xi_2^{2m} \xi_3^{2l}}{(1 + \xi_2^2 + \xi_3^2)^{L/2}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{-L}}{1 + t^2 (\xi_1^2 + \mu^4 \xi_2^2)} dt \right) d\xi_1 \right) d(\xi_2, \xi_3).
\]
Furthermore, we obtain
\[
\int_{|\xi_1| \geq 1} \left( \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{-L}}{1 + t^2 (\xi_1^2 + \mu^4 \xi_2^2)} dt \right) d\xi_1 \leq \int_{|\xi_1| \geq 1} \langle \xi_1 \rangle^{-L} |\xi_1|^{-1} d\xi_1 \cdot \int_{\mathbb{R}} \frac{dt}{1 + t^2}
\leq \pi \int \langle \xi_1 \rangle^{-L} d\xi_1
\]
and
\[
\int_{|\xi_1| < 1} \left( \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{-L}}{1 + t^2 (\xi_1^2 + \mu^4 \xi_2^2)} dt \right) d\xi_1
\leq \int_{-1}^{1} \left( \int_{\mathbb{R}} \frac{dt}{1 + t^2 (\xi_1^2 + \mu^4 \xi_2^2)} \right) d\xi_1 = \int_{-1}^{1} \frac{d\xi_1}{\sqrt{\xi_1^2 + \mu^4 \xi_2^2}} \int_{\mathbb{R}} \frac{dt}{1 + t^2}
= \pi \log \left( \frac{1 + \sqrt{1 + \mu^4 \xi_2^2}}{1 + \mu^4 \xi_2^2} \right) - 2\pi \log \mu - \pi \log |\xi_2| - 2\pi \log \mu - \pi \log |\xi_2|.
\]
Thus, we see that
\[
\int_{\mathbb{R}^{1+2}} \xi_2^{2m} \xi_3^{2l} \Phi_{N, \xi}(\phi, \phi, \mu; t, \xi) d(t, \xi)
\leq C \int_{\mathbb{R}^2} \frac{\xi_2^{2m} \xi_3^{2l}}{(1 + \xi_2^2 + \xi_3^2)^{L/2}} \left\{ 1 + \log \left( 1 + \sqrt{1 + \xi_2^2} \right) + |\log |\xi_2|| - \log \mu \right\} d(\xi_2, \xi_3)
\leq C \left( 1 + |\log \mu| \right).
\] (2.9)
From (2.8) and (2.9), we have (2.7). \( \square \)

We next prove Lemma \( 2.1 \). If \( N = 0 \), then we easily see that the lemma holds. For other cases, we show dividing three steps.
(Step I) Let \( N \in \mathbb{N} \) and \( \epsilon \in \{0, 1\}^3 \). Then for any \( 2 \leq M \leq N^* \), we have

\[
\sum_{j=1}^{M} (N+1)^{2(j-1)} P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 1 - j)) \\
\leq (N+1)^{2(M-1)} P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 1 - M)) \\
\quad + \sum_{j=1}^{M-1} (N+1)^{2(j-1)} P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 1 - j)) \\
\leq (N+1)^{2(M-1)} P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 1 - M)) \\
\quad + \frac{(N+1)^{2(M-1)} - 1}{(N+1)^2 - 1} \times N(N+1) \\
\leq (N+1)^{2(M-1)} \left\{ P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 1 - M)) + \frac{N(N+1)}{N(N+2)} \right\} \\
\leq (N+1)^{2(M-1)} P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 2 - M)).
\]

(Step II) Fix \( N \in \mathbb{N} \). Let \( \mathcal{S} \) be the symmetric group of degree \( N^* \). We define \( \sigma_0 \in \mathcal{S} \) by

\[
\sigma_0(j) = N^* + 1 - j, \quad j = 1, \ldots, N^*.
\]

Furthermore, we define

\[
Q_{N,\epsilon}(j, k) = 2(N+1)^{2(j-1)} P_{N,\epsilon} (\alpha(N, \epsilon; k)),
\]

\[
Q_{N,\epsilon} = \sum_{j=1}^{N^*} Q_{N,\epsilon}(j, \sigma_0(j)),
\]

\[
\tilde{Q}_{N,\epsilon} = \inf_{\sigma \in \mathcal{S} \setminus \{\sigma_0\}} \sum_{j=1}^{N^*} Q_{N,\epsilon}(j, \sigma(j)) \quad \text{if} \ N \geq 1.
\]

Let \( \sigma \in \mathcal{S} \setminus \{\sigma_0\} \). Then there exists some \( M \in \{1, \ldots, N^*\} \) such that \( \sigma(M) > \sigma_0(M) = N^* + 1 - M \), and such that if \( M < N^* \) then \( \sigma(j) = \sigma_0(j) \) \((M+1 \leq j \leq N^*)\). Therefore, we obtain

\[
Q_{N,\epsilon} = \sum_{j=1}^{M} Q_{N,\epsilon}(j, \sigma_0(j)) + \sum_{j=M+1}^{N^*} Q_{N,\epsilon}(j, \sigma_0(j)) \\
\leq 2 \sum_{j=1}^{M} (N+1)^{2(j-1)} P_{N,\epsilon} (\alpha(N, \epsilon; N^* + 1 - j)) + \sum_{j=M+1}^{N^*} Q_{N,\epsilon}(j, \sigma(j)).
\]
where a term \( \sum_{j=N^*+1}^{N^*} a_j \) is understood as 0. We see from Step I and \( \sigma(M) \geq N^* + 2 - M \) that

\[
Q_{N,\varepsilon} < 2(N + 1)^{2(M-1)} P_{N,\varepsilon} \left( \alpha(N, \varepsilon; N^* + 2 - M) \right) + \sum_{j=M+1}^{N^*} Q_{N,\varepsilon}(j, \sigma(j))
\]

\[
\leq 2 \sum_{j=1}^{M} (N + 1)^{2(j-1)} P_{N,\varepsilon} \left( \alpha(N, \varepsilon; \sigma(j)) \right) + \sum_{j=M+1}^{N^*} Q_{N,\varepsilon}(j, \sigma(j))
\]

\[
= \sum_{j=1}^{N^*} Q_{N,\varepsilon}(j, \sigma(j)).
\]

Therefore, we have \( Q_{N,\varepsilon} < \tilde{Q}_{N,\varepsilon} \).

(Step III) Fix \( N \in \mathbb{N} \). For \( \mu > 0 \), we set

\[
C_{N,\varepsilon}(j, k; \mu) = \int_{\mathbb{R}^{1+3}} \xi^{2\alpha(N,\varepsilon,k) + \varepsilon} \Phi_{N,\varepsilon} \left( \phi, \tilde{\phi}, \mu^{(N+1)(2(j-1))}; t, \xi \right) d(t, \xi), \quad 1 \leq j, k \leq N^*,
\]

\[
A(\mu) = (\mu^{Q_{N,\varepsilon}(j,k)} C_{N,\varepsilon}(j,k; \mu))_{1 \leq j,k \leq N^*}.
\]

Then it suffices to show that for some \( \tilde{\mu} > 0 \), we have \( \det A(\mu) \neq 0 \ (\mu \in (0, \tilde{\mu})) \).

We see from the assumptions of \( \phi \) and \( \tilde{\phi} \), Propositions 2.4 and 2.5 that there exist some \( \mu_1 \in (0, 1) \) and \( C_1, C_2 > 0 \) such that for any \( \mu \in (0, \mu_1) \),

\[
C_1 \leq |C_{N,\varepsilon}(j, k; \mu)| \leq C_2 (1 + |\log \mu|).
\]

Therefore, for any \( \mu \in (0, \mu_1) \), it follows from Step II that

\[
|\det A(\mu)| \geq \prod_{j=1}^{N^*} \mu^{Q_{N,\varepsilon}(j, \sigma_0(j))} |C_{N,\varepsilon}(j, \sigma_0(j); \mu)| - \sum_{\sigma \in \Sigma \setminus \{\sigma_0\}} \prod_{j=1}^{N^*} \mu^{Q_{N,\varepsilon}(j, \sigma(j))} |C_{N,\varepsilon}(j, \sigma(j); \mu)|
\]

\[
\geq C_1^{N^*} - N^! C_2^{N^*} (1 + |\log \mu|)^{N^*} \mu^{\tilde{Q}_{N,\varepsilon}}
\]

\[
\geq \left( C_1^{N^*} - N^! C_2^{N^*} (1 + |\log \mu|)^{N^*} \right) \mu^{\tilde{Q}_{N,\varepsilon}}.
\]

Since

\[
\lim_{\mu \to 0} (1 + |\log \mu|)^{N^*} \mu = 0,
\]

there exists \( \tilde{\mu} > 0 \) such that

\[
C_1^{N^*} - N^! C_2^{N^*} (1 + |\log \mu|)^{N^*} \mu > 0, \quad \mu \in (0, \tilde{\mu}),
\]

which completes the proof.
3. Proof of Main Theorems

In this section, we prove main results. Throughout this section, we fix \( N \in \mathbb{N}_0 \), \( \epsilon \in \{0, 1\}^3 \) and \( \phi, \tilde{\phi} \in S \setminus \{0\} \). Furthermore, we assume that \( \phi = \tilde{\phi} \) and (C1) holds if \( |\epsilon| = 0 \), and assume that (C1)–(C3) hold if \( |\epsilon| \neq 0 \). Fix \( \mu \in (0, \tilde{\mu}) \), where \( \tilde{\mu} \) is the positive number mentioned in Lemma 2.1. Then the matrix \( M_{N,\epsilon}[\phi, \tilde{\phi}, \mu] \) is invertible. We remark that all positive constants \( C \) which appear in this section are independent of the positive parameter \( \lambda \).

3.1. Proof of Theorem 1.2 We assume that the interaction potential \( V \) of the equation (1.1) satisfies (V1). We see from Theorem 1.1(3) that for any \( \psi \in S \), the scattering operator \( S \) is expressed by

\[
S(\epsilon\psi) = \epsilon\psi - i \int_{\mathbb{R}} U(-t)F(u_\epsilon(t))dt,
\]

where \( \epsilon \) is sufficiently small and \( u_\epsilon \) is the time-global solution to (1.1) satisfying

\[
\lim_{t \to -\infty} \|u(t) - U(t)(\epsilon\psi)\|_{H^1} = 0.
\]

By the inequality (1.3), we have

\[
\left| i \epsilon^{-3} \left( (S - id)(\epsilon\psi), \tilde{\psi} \right) - (2\pi)^{3/2} \int_{\mathbb{R}^{1+3}} \mathcal{F}V(\xi)\Phi_{N,\epsilon}(\psi, \tilde{\psi}, 1; t, \xi)d(t, \xi) \right| \leq C\epsilon^2 \|\psi\|_{H^1}^5 \left\|\tilde{\psi}\right\|_{H^1}.
\]

Substituting \( \psi = \phi \circ D_{N,\epsilon}^\lambda, \tilde{\psi} = \tilde{\phi} \circ D_{N,\epsilon}^\lambda \) and \( \epsilon = \lambda^{N+4} \) for the above inequality, we obtain for sufficiently small \( \lambda > 0 \),

\[
\left| i \lambda^{-3N-12} \left( (S - id)(\lambda^{N+4}\phi \circ D_{N,\epsilon}^\lambda, \tilde{\phi} \circ D_{N,\epsilon}^\lambda) \right) \right|
\]

\[
- (2\pi)^{3/2} \int_{\mathbb{R}^{1+3}} \mathcal{F}V(\xi)\Phi_{N,\epsilon}(\phi \circ D_{N,\epsilon}^\lambda, \tilde{\phi} \circ D_{N,\epsilon}^\lambda, 1; t, \xi)d(t, \xi)
\]

\[
\leq C\lambda^{2N+8} \left\|\phi \circ D_{N,\epsilon}^\lambda\right\|_{H^1}^5 \left\|\tilde{\phi} \circ D_{N,\epsilon}^\lambda\right\|_{H^1}
\]

\[
\leq C\lambda^{2N+8}\lambda^{-\frac{3}{2}6} \left\|\phi\right\|_{H^1}^5 \left\|\tilde{\phi}\right\|_{H^1}.
\]

Since

\[
\int_{\mathbb{R}^{1+3}} \mathcal{F}V(\xi)\Phi_{N,\epsilon}(\phi \circ D_{N,\epsilon}^\lambda, \tilde{\phi} \circ D_{N,\epsilon}^\lambda, 1; t, \xi)d(t, \xi)
\]

\[
= \lambda^{-5}\mu^{-N-2} \int_{\mathbb{R}^{1+3}} \mathcal{F}V(D_{N,\epsilon}^\lambda\xi)\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)d(t, \xi),
\]

we have
we have
\[
\left\| \frac{i\lambda^{-3N-7}\mu^{N+2}}{(2\pi)^{3/2}} \left( (S - \text{id})(\lambda^{N+4}\phi \circ D_{N,\epsilon}^{\lambda,\mu}), \tilde{\phi} \circ D_{N,\epsilon}^{\lambda,\mu} \right) \right. \\
- \int_{\mathbb{R}^{1+3}} \mathcal{F}V(D_{N,\epsilon}^{\lambda,\mu}\xi)\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)d(t, \xi) \left\| \right. \\
\leq C\lambda^{2N+4} ||\tilde{\phi}||_{H^1}^5 ||\phi||_{H^1}.
\]
Therefore, it follows that
\[
\left| \frac{i\mu^{N+2}}{(2\pi)^{3/2}} \lambda^{-2N-|\epsilon|} \Delta_\lambda^{2N+|\epsilon|} \left\{ \lambda^{-3N-7} \left( (S - \text{id})(\lambda^{N+4}\phi \circ D_{N,\epsilon}^{\lambda,\mu}), \tilde{\phi} \circ D_{N,\epsilon}^{\lambda,\mu} \right) \right. \\
- \lambda^{-2N-|\epsilon|} \Delta_\lambda^{2N+|\epsilon|} \int_{\mathbb{R}^{1+3}} \mathcal{F}V(D_{N,\epsilon}^{\lambda,\mu}\xi)\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)d(t, \xi) \left\| \right. \\
\leq C\lambda
\]
for sufficiently small \(\lambda > 0\).

We now define
\[
f(\lambda) = \int_{\mathbb{R}^{1+3}} \mathcal{F}V(D_{N,\epsilon}^{\lambda,\mu}\xi)\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)d(t, \xi).
\]
Since \(V\) satisfies the condition (V1), we see from Proposition 2.2 that \(f(\lambda)\) is smooth on \((0, \lambda_0)\) for some \(\lambda_0 > 0\). Using Proposition A.3 below, we obtain
\[
\left| \lambda^{-2N-|\epsilon|} \Delta_\lambda^{2N+|\epsilon|} f(\lambda) - \frac{\partial^{2N+|\epsilon|}}{\partial\lambda^{2N+|\epsilon|}} f(\lambda) \right| \leq C\lambda
\]
for any \(\lambda \in (0, \lambda_0/(2N + 4))\). Furthermore, we see that
\[
\frac{\partial^{2N+|\epsilon|}}{\partial\lambda^{2N+|\epsilon|}} f(\lambda) = \sum_{|\beta| = 2N+|\epsilon|} |\beta|! \mu_{\lambda,\epsilon}^{p,\epsilon(\beta)} \int_{\mathbb{R}^{1+3}} \xi^\beta \partial^\beta \mathcal{F}V(D_{N,\epsilon}^{\lambda,\mu}\xi)\Phi_{N,\epsilon}(\phi, \tilde{\phi}, \mu; t, \xi)d(t, \xi).
\]
For any \(\lambda > 0\), \(\beta \in \mathbb{N}_0^3\) and \(\xi \in \mathbb{R}^3\), we see from the condition (V1) that
\[
\left| \partial^\beta \mathcal{F}V(D_{N,\epsilon}^{\lambda,\mu}\xi) - \partial^\beta \mathcal{F}V(0) \right| \\
\leq (2\pi)^{-3/2} \left\| x^\beta V(x) \left( \exp \left( -iD_{N,\epsilon}^{\lambda,\mu}(\xi \cdot x) \right) - 1 \right) \right\|_{L^1(\mathbb{R}^3)} \\
\leq (2\pi)^{-3/2} \left\| e^{-A|x|/2} \left( \exp \left( -iD_{N,\epsilon}^{\lambda,\mu}(\xi \cdot x) \right) - 1 \right) \right\|_{L^\infty(\mathbb{R}^3)} \left\| x^\beta e^{A|x|/2} V(x) \right\|_{L^1(\mathbb{R}^3)} \\
\leq C|\lambda\xi|.
\]
Hence we see that
\[
\left| \frac{\partial^{2N+|\alpha|}}{\partial \lambda^{2N+|\alpha|}} f(\lambda) - \sum_{|\beta|=2N+|\alpha|} \frac{\beta!}{\beta!} \mu^{P_{N,\epsilon}(\beta)} \partial_\xi^3 F_V(0) \int_{\mathbb{R}^{1+3}} \xi^\alpha \Phi_N(\phi, \tilde{\phi}, \mu; t, \xi) d(t, \xi) \right| \leq C\lambda
\]
for any \( \lambda \in (0, \lambda_0/(2N+4)) \). By conditions for \( \phi \) and \( \tilde{\phi} \), we have
\[
\int_{\mathbb{R}^{1+3}} \xi^{2\alpha+7} \Phi_N(\phi, \tilde{\phi}, \mu; t, \xi) d(t, \xi) = 0
\]
for any \( \alpha \in \mathbb{N}_0^3 \) and for any \( \tilde{\epsilon} \in \{0, 1\}^3 \setminus \{\epsilon\} \). Therefore, it follows that
\[
\left| \frac{\partial^{2N+|\alpha|}}{\partial \lambda^{2N+|\alpha|}} f(\lambda) - \left\{ \sum_{k=1}^{N^*} \frac{(2N+|\alpha|)!}{(2\alpha(N, \epsilon; k)+\epsilon)!} \mu^{P_{N,\epsilon}(2\alpha(N, \epsilon; k)+\epsilon)} \right. \right.
\]
\[
\times \left. \partial_\xi^{2\alpha(N, \epsilon; k)+\epsilon} F_V(0) \int_{\mathbb{R}^{1+3}} \xi^{2\alpha(N, \epsilon; k)+\epsilon} \Phi_N(\phi, \tilde{\phi}, \mu; t, \xi) d(t, \xi) \right\} \right| \leq C\lambda
\]
for any \( \lambda \in (0, \lambda_0/(2N+4)) \).

Therefore, we see from (3.1)–(3.3) that there exists some positive number \( \tilde{\lambda} \) such that
\[
\left| \tilde{\lambda}^\alpha \Phi_N(\phi, \tilde{\phi}, \mu) - M_{N,\epsilon}[\phi, \tilde{\phi}, \mu] a_{N,\epsilon} \right| \leq C\lambda, \quad \lambda \in (0, \tilde{\lambda}).
\]
Since the matrix \( M_{N,\epsilon}[\phi, \tilde{\phi}, \mu] \) is invertible, the proof is complete.

3.2. **Proof of Corollary 1.3** Let \( j = 1, 2 \). Suppose that \( V_j \) satisfies (V1) with \( V = V_j \). Let \( S_j \) be the scattering operator for the equation (1.1) with \( V = V_j \). We denote \( a_{N,\epsilon} \) (resp. \( \tilde{\lambda}^\alpha \Phi_N(\phi, \tilde{\phi}, \mu) \)) with \( V = V_j \) by \( a_{N,\epsilon}^j \) (resp. \( \tilde{\lambda}^\alpha \Phi_N(\phi, \tilde{\phi}, \mu) \)). We see from Theorem 1.2 that for some \( \lambda_0 > 0 \) and \( \phi, \tilde{\phi} \in S \),
\[
|a_{N,\epsilon}^j - a_{N,\epsilon}^2| \leq \left| M_{N,\epsilon}[\phi, \tilde{\phi}, \mu]^{-1} \left( \tilde{\lambda}^\alpha \Phi_N(\phi, \tilde{\phi}, \mu) - \tilde{\lambda}^\alpha \Phi_N(\phi, \tilde{\phi}, \mu) \right) \right| + C\lambda, \quad \lambda \in (0, \lambda_0).
\]
Since
\[
\left| \lambda^{-2N-|\alpha|} A_{\lambda}^{2N+|\alpha|} \left\{ \lambda^{-3N-7} \langle (S_1 - S_2)(\lambda^{N+4}\phi \circ D_{N,\epsilon}^{\lambda\mu}, \tilde{\phi} \circ D_{N,\epsilon}^{\lambda\mu}) \rangle \right\} \right| \leq \left| \lambda^{-2N-|\alpha|} \lambda^{-3N-7} \|S_1 - S_2\| \left\| \lambda^{N+4}\phi \circ D_{N,\epsilon}^{\lambda\mu} \right\|_{H^1}^3 \left\| \tilde{\phi} \circ D_{N,\epsilon}^{\lambda\mu} \right\|_{H^1} \right| \leq C\lambda^{-2N-|\alpha|} \lambda^{-3N-7} \lambda^{3N+12} \lambda^{-\frac{7}{2}} \|S_1 - S_2\| \leq C\lambda^{-2N-|\alpha|-1} \|S_1 - S_2\|,
\]
we have

$$|a_{N,\epsilon}^1 - a_{N,\epsilon}^2| \leq C \left\{ \lambda^{-2N-|\epsilon|-1} \| S_1 - S_2 \| + \lambda \right\}.$$ 

Let $a = (2N + |\epsilon| + 2)^{-1}$ and put

$$\lambda = \min \{ \lambda_0, \| S_1 - S_2 \|^a \}.$$ 

Then (1.8) holds. By (1.8) and Proposition A.1 below, we see that the property (2) holds. We hence complete the proof.

**Appendix A. Supplementary Propositions**

In this section, we prove supplementary propositions used above. For $n \in \mathbb{N}$, we define the Fourier transform $\mathcal{F}_n$ on $L^1(\mathbb{R}^n)$ by

$$\mathcal{F}_n f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad f \in L^1(\mathbb{R}^n), \ \xi \in \mathbb{R}^n.$$ 

We first show the analyticity of the Fourier transform of a function $f(x)$ which exponentially decreases.

**Proposition A.1.** Let $n \in \mathbb{N}$ and let $A > 0$. Assume that a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ satisfies $f(x) \exp(A|x|) \in L^1(\mathbb{R}^n)$. Then $\mathcal{F}_n f(\xi)$ is analytic on $\mathbb{R}^n$. Furthermore, for any $\xi_0 \in \mathbb{R}^n$, if $\xi \in \mathbb{R}^n$ satisfies $|\xi - \xi_0| < A/n$, then we have

$$\mathcal{F}_n f(\xi) = \sum_{|\alpha| \geq 0} \frac{\partial^\alpha \mathcal{F}_n f(\xi_0)}{\alpha!} (\xi - \xi_0)^{\alpha}.$$ 

**Proof.** Fix $n \in \mathbb{N}$, $\xi_0 \in \mathbb{R}^n$ and $\beta \in \mathbb{N}_0^n$. Then we see that $\mathcal{F}_n f$ is smooth and that

$$\left| \partial^\beta \mathcal{F}_n f(\xi_0) \right| = \left| \mathcal{F}_n (x^\beta f)(\xi_0) \right| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |x^\beta f(x)| dx$$

$$\leq (2\pi)^{-n/2} \left\| f e^{A|x|} \right\|_{L^1(\mathbb{R}^n)} \left\| x^\beta e^{-A|x|} \right\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\left\| x^\beta e^{-A|x|} \right\|_{L^\infty(\mathbb{R}^n)} \leq \prod_{j=1}^n \sup_{\beta_j \geq 0} (\beta_j!) e^{-A\beta_j/n} = \prod_{j=1}^n \beta_j \frac{A^{-\beta_j} \beta_j e^{-\beta_j}}{\beta_j} = n^{\beta_j} A^{-|\beta|} \prod_{j=1}^n \beta_j e^{-\beta_j}.$$ 

It follows from Stirling’s formula that

$$\beta! = \prod_{j=1}^n \beta_j! \geq \prod_{j=1}^n \beta_j \beta_j e^{-\beta_j},$$

and we hence obtain

$$\left| \frac{\partial^\beta \mathcal{F}_n f(\xi_0)}{\beta!} \right| \leq C_A \left( \frac{n}{A} \right)^{|\beta|},$$

where

$$C_A = \frac{1}{\pi} \prod_{j=1}^n \frac{1}{\beta_j}.$$
where $C_A = (2\pi)^{-n/2} \| f e^{A|x|} \|_{L^1(\mathbb{R}^n)}$ is a positive constant independent of $\beta$. Therefore, for any $\xi \in \mathbb{R}^n$ with $|\xi - \xi_0| < A/n$, we have

$$\sum_{|\alpha| \geq 0} \left| \frac{\partial^{\alpha}_\xi F_n f(\xi_0)}{\alpha!} (\xi - \xi_0)^\alpha \right| \leq C_A \sum_{|\alpha| \geq 0} \left\{ \left( \frac{n}{A} \right) |\xi - \xi_0| \right\}^{|\alpha|}$$

$$= C_A \sum_{m=0}^\infty \sum_{|\alpha| = m} \left\{ \left( \frac{n}{A} \right) |\xi - \xi_0| \right\}^m$$

$$= C_A \sum_{m=0}^\infty \# \{ \alpha \in \mathbb{N}_0^n; |\alpha| = m \} \times \left\{ \left( \frac{n}{A} \right) |\xi - \xi_0| \right\}^m$$

$$\leq C_A \sum_{m=0}^\infty (m + n - 1)^{n-1} \times \left\{ \left( \frac{n}{A} \right) |\xi - \xi_0| \right\}^m < \infty.$$  

Let $N \in \mathbb{N}$. Then we obtain

$$\left| \sum_{0 \leq |\alpha| \leq N} \frac{\partial^{\alpha}_\xi F_n f(\xi_0)}{\alpha!} (\xi - \xi_0)^\alpha - F_n f(\xi) \right|$$

$$\leq \sum_{|\alpha| = N+1} \sup_{\theta \in [0,1]} \left| \frac{\partial^{\alpha}_\xi F_n f((1-\theta)\xi + \theta \xi_0)}{\alpha!} (\xi - \xi_0)^\alpha \right|$$

$$\leq C_A (N + n)^{n-1} \left\{ \left( \frac{n}{A} \right) |\xi - \xi_0| \right\}^{N+1} \to 0 \quad \text{as } N \to \infty.$$  

Therefore, we see that $F_n f$ is analytic. We hence complete the proof. \hfill \Box

In Remark 1.2 above, we introduce an example of data $\phi$ and $\tilde{\phi}$ satisfying conditions (C1)–(C3). We now prove that the example $(\phi, \tilde{\phi})$ actually satisfies (C3).

**Proposition A.2.** Let $\epsilon \in \{0, 1\}^3$ with $|\epsilon| \neq 0$. Fix $\varphi_j \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$ ($j = 1, 2, 3$) which are even. Define $\phi(x)$ and $\tilde{\phi}(x)$ ($x = (x_1, x_2, x_3) \in \mathbb{R}^3$) by

$$\phi(x_1, x_2, x_3) = \varphi_1(x_1) \varphi_2(x_2) \varphi_3(x_3) \quad \text{and} \quad \tilde{\phi}(x) = \partial_\xi \phi(x),$$

respectively. Then we have for any $\alpha \in \mathbb{N}_0^3$,

$$\int_{\mathbb{R}^{1+3}} \xi^{2\alpha+\epsilon} \Phi_\epsilon \left( \phi, \tilde{\phi}; t, \xi \right) d(t, \xi) \neq 0. \quad (A.1)$$

**Proof.** We consider only the case $\epsilon = (1, 0, 1)$ because other cases are proved similarly. Let $\rho, \tilde{\rho} \in \mathcal{S}(\mathbb{R})$. Then we have the following properties:

(i) For any $\zeta \in \mathbb{R}$, it follows that $\zeta (F_1 \rho)(\zeta) = -i F_1 (\rho')(\zeta)$.

(ii) If $\rho$ is odd, then $-i F_1 \text{Re} \rho = \text{Im} F_1 \rho$.

(iii) If $\rho$ is even, then $F_1 |\rho|^2$ is a real-valued function.
We define the function $\Psi(\rho, \tilde{\rho}; t, \zeta) \ (t, \zeta \in \mathbb{R})$ by

$$
\Psi(\rho, \tilde{\rho}; t, \zeta) = \mathcal{F}_1 (|U_1(t)\rho|^2) (\zeta) \times \mathcal{F}_1 \left( \frac{U_1(t)\rho \times U_1(t)\tilde{\rho}}{} \right)(\zeta),
$$

where $U_1(t) = \mathcal{F}_1^{-1} e^{-it\zeta^2} \mathcal{F}_1$. If $\rho$ is even, then it follows from (i)–(iii) that

$$
\zeta \Psi(\rho, \rho; t, \zeta) = \mathcal{F}_1 (|U_1(t)\rho|^2) (\zeta) \times \zeta \mathcal{F}_1 \left( \frac{U_1(t)\rho \times U_1(t)\rho}{U_1(t)\rho \times U_1(t)\rho} \right)(\zeta)
$$

$$
= 2 \mathcal{F}_1 (|U_1(t)\rho|^2) (\zeta) \times (-i) \mathcal{F}_1 \text{Re} \left( \frac{U_1(t)\rho \times U_1(t)\rho}{U_1(t)\rho \times U_1(t)\rho} \right)(\zeta)
$$

$$
= 2 \mathcal{F}_1 (|U_1(t)\rho|^2) (\zeta) \times \text{Im} \mathcal{F}_1 \left( \frac{U_1(t)\rho \times U_1(t)\rho}{U_1(t)\rho \times U_1(t)\rho} \right)(\zeta)
$$

$$
= -2 \text{Im} \left\{ \mathcal{F}_1 (|U_1(t)\rho|^2) (\zeta) \times \mathcal{F}_1 \left( \frac{U_1(t)\rho \times U_1(t)\rho}{U_1(t)\rho \times U_1(t)\rho} \right)(\zeta) \right\}
$$

$$
= -2 \text{Im} \Psi(\rho, \rho; t, \zeta), \ t, \zeta \in \mathbb{R}.
$$

Hence we have

$$
\text{Im} \Psi(\rho, \rho; t, \zeta) = \frac{1}{2} \zeta \Psi(\rho, \rho; t, \zeta), \ t, \zeta \in \mathbb{R}. \quad (A.2)
$$

Let $m \in \mathbb{N}_0$. As in the proof of Proposition 2.2, we obtain

$$
\Psi(\rho, \rho; t, \zeta) \leq C \langle \zeta \rangle^{-m-2} \frac{1}{1 + t^2 \zeta^2}, \ (t, \zeta) \in \mathbb{R} \times \mathbb{R}.
$$

Therefore, we see that $\zeta^{2m+1} \Psi(\rho, \rho'; t, \zeta)$ and $\zeta^{2m+1} \Psi(\rho, \rho'; 0, \zeta)$ are integrable on $\mathbb{R}_t \times \mathbb{R}_\zeta$ and $\mathbb{R}_\zeta$, respectively. Furthermore, if $\rho$ is nonzero, then we have

$$
\int_{\mathbb{R}} \zeta^{2m+2} \Psi(\rho, \rho; 0, \zeta) \, d\zeta > 0. \quad (A.3)
$$

Therefore, if $\rho$ is even and nonzero, then we see from (A.2) and (A.3) that

$$
\text{Im} \int_{\mathbb{R}^{1+1}} \zeta^{2m+1} \Psi(\rho, \rho'; t, \zeta) \, d(t, \zeta) = -\frac{1}{2} \int_{\mathbb{R}_t^{1+1}} \zeta^{2m+2} \Psi(\rho, \rho; t, \zeta) \, d(t, \zeta) \neq 0, \quad (A.4)
$$

$$
\text{Im} \int_{\mathbb{R}} \zeta^{2m+1} \Psi(\rho, \rho'; 0, \zeta) \, d\zeta = -\frac{1}{2} \int_{\mathbb{R}} \zeta^{2m+2} \Psi(\rho, \rho; 0, \zeta) \, d\zeta \neq 0. \quad (A.5)
$$
Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3 \). Then it follows from the equality
\[
\int_{\mathbb{R}^{1+3}} \xi^{2\alpha+\epsilon} \Phi_\epsilon \left( \phi, \tilde{\phi}; t, \xi \right) d(t, \xi)
= \int_{\mathbb{R}^{1+3}} \xi^{2\alpha_1+1} \Psi (\varphi_1, \varphi'_1; t, \xi_1) d(t, \xi_1)
\times \int_{\mathbb{R}} \xi^{2\alpha_2} \Psi (\varphi_2, \varphi'_2; 0, \xi_2) d\xi_2 \times \int_{\mathbb{R}} \xi^{2\alpha_3+1} \Psi (\varphi_3, \varphi'_3; 0, \xi_3) d\xi_3
\]
and (A.3)–(A.5) that (A.1) holds. \( \square \)

Finally, we show the following inequality used in the proof of Theorem 1.2:

**Proposition A.3.** Fix \( \lambda_0 > 0 \) and \( L \in \mathbb{N}_0 \). Let \( h \in C^{L+1}((0, \lambda_0)) \). If
\[
\max_{l=0, \ldots, L+1} \sup_{\lambda \in (0, \lambda_0)} \left| \frac{d^l}{d\lambda^l} h(\lambda) \right| < \infty, \quad (A.6)
\]
then we have
\[
\left| \frac{d^L}{d\lambda^L} h(\lambda) - \lambda^{-L} \Delta^L h(\lambda) \right| \leq C\lambda, \quad \lambda \in \left( 0, \frac{\lambda_0}{L+1} \right)
\]
for some positive number \( C \) independent of \( \lambda \).

**Proof.** Fix \( \lambda \in (0, \lambda_0/(L+1)) \). Then we see from Taylor’s theorem and (A.6) that for any \( l = 0, 1, \ldots, L \),
\[
\left| h((l+1)\lambda) - \sum_{k=0}^{L} \frac{(l\lambda)^k}{k!} h^{(k)}(\lambda) \right| \leq C\lambda^{L+1}.
\]
Since
\[
\sum_{l=0}^{L} \frac{(-1)^{L-l} L!}{l!(L-l)!} \sum_{k=0}^{L} \frac{(l\lambda)^k}{k!} h^{(k)}(\lambda) = L! \sum_{k=0}^{L} \left( \sum_{l=0}^{L} \frac{(-1)^{L-l} k^l}{l!(L-l)!} \right) \frac{\lambda^k}{k!} h^{(k)}(\lambda),
\]
we obtain
\[
\left| \Delta^L h(\lambda) - L! \sum_{k=0}^{L} \left( \sum_{l=0}^{L} \frac{(-1)^{L-l} k^l}{l!(L-l)!} \right) \frac{\lambda^k}{k!} h^{(k)}(\lambda) \right| \leq C\lambda^{L+1}.
\]
Therefore, it suffices to prove the following equality:
\[
\sum_{l=0}^{L} \frac{(-1)^{L-l} k^l}{l!(L-l)!} = \begin{cases} 
0 & \text{if } k = 0, 1, \ldots, L-1, \\
1 & \text{if } k = L. \end{cases} \quad (A.7)
\]
Let $g(y) = y^L$ ($y \in \mathbb{R}$). Then the $L$-th forward difference of $g$ satisfies

$$
\sum_{l=0}^{L} \frac{(-1)^{L-l}L!}{l!(L-l)!} g(y + l) = L!
$$

and

$$
\sum_{l=0}^{L} \frac{(-1)^{L-l}L!}{l!(L-l)!} g(y + l) = L! \sum_{k=0}^{L} \frac{L!}{k!(L-k)!} \left( \sum_{l=0}^{L} \frac{(-1)^{l-k}L!}{l!(L-l)!} \right) y^{L-k}.
$$

Since $y$ is arbitrary, we have \eqref{A.7}, which completes the proof.

\[\square\]

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