Local models of isolated singularities for affine special Kähler structures in dimension two

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Abstract

We construct local models of isolated singularities for special Kähler structures in real dimension two assuming that the associated holomorphic cubic form does not have essential singularities. As an application we compute the holonomy of the flat symplectic connection, which is a part of the special Kähler structure.

1 Introduction

The notion of a special Kähler structure appeared for the first time in physics [Gat84, dWVP84] and was formalized by Freed [Fre99]. For reader’s convenience, let us recall the definition of the affine special Kähler structure.

Definition 1. An (affine) special Kähler structure on a manifold \( M \) is a quadruple \((g, I, \omega, \nabla)\), where \((M, g, I, \omega)\) is a Kähler manifold with Riemannian metric \( g \), complex structure \( I \), and symplectic form \( \omega(\cdot, \cdot) = g(I \cdot, \cdot) \), and \( \nabla \) is a flat symplectic torsion-free connection on the tangent bundle \( TM \) such that

\[
(\nabla_X I)Y = (\nabla_Y I)X
\]

holds for all vector fields \( X \) and \( Y \).

If \( I \) is fixed, which is always assumed to be the case below, we say for simplicity that \((g, \nabla)\) is a special Kähler structure.

The importance of special Kähler structures stems from the so called c-map construction, which associates to each special Kähler manifold a hyperKähler one, which is equipped with the structure of a holomorphic Lagrangian fibration [CFG89, Fre99, MS15]. Moreover, under suitable hypotheses the converse construction also exists [Fre99, Sect. 3]. Typically, a holomorphic Lagrangian fibration contains singular fibers and in this case the corresponding special Kähler structure has singularities. The importance of singular special Kähler structures can be also seen from the following fact [Lu99, BC01]: A complete special Kähler metric is necessarily flat.

Isolated singularities of affine special Kähler structures in the simplest case of two real dimensions are in the focus of this article, which is closely related to [Hay15]. However, here we focus on the properties of \( \nabla \) rather than \( g \) near an isolated singularity. Properties of \( \nabla \), in particular its holonomy, contain important information about behavior of the corresponding holomorphic Lagrangian fibration near a singular fiber.
A number of special Kähler structures with isolated singularities can be found [Hay15, CH17]. We now describe a family of examples, whose significance will be clear below. Thus, let $B^*_1 := \{0 < |z| < 1\}$ be the punctured disc; Denote by $(r, \theta)$ the polar coordinates on $\mathbb{C}^*$, where $\theta \in (0, 2\pi)$, and put $\rho := \log r$.

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

We show that for any $k \in \mathbb{Z}$, $C \in \mathbb{R}_{>0}$, and $b \in \mathbb{C}$, $|b| = 1$, the following

$$g_k = -C r^k \log r \, |dz|^2,$$

$$\omega_{k,\nabla} = \frac{1}{2} \left( k I_2 + \begin{pmatrix} \text{Im}(be^{ik\theta}) & -1 + \text{Re}(be^{ik\theta}) \\ 1 + \text{Re}(be^{ik\theta}) & -\text{Im}(be^{ik\theta}) \end{pmatrix} \rho^{-1} \right) \, d\theta$$

$$+ \frac{1}{2} \left( k I_2 + \begin{pmatrix} 1 - \text{Re}(be^{ik\theta}) & \text{Im}(be^{ik\theta}) \\ \text{Im}(be^{ik\theta}) & 1 + \text{Re}(be^{ik\theta}) \end{pmatrix} \rho^{-1} \right) \, d\rho$$

is a special Kähler structure on $B^*_1$, where the dependence on $C$ and $b$ is suppressed in the notations. Here $\omega_{k,\nabla}$ is the connection one-form of $\nabla$ with respect to the trivialization $(\partial_x, \partial_y)$, where $z = x + yi$; See Section 3 for further details. Even though (3) may look incomprehensible at first glance, we show that many objects of interest, such as special holomorphic coordinates, the associated holomorphic cubic form, and the holonomy around the origin for these special Kähler structures can be computed explicitly.

While for some values of $k$ the metric $g_k$ was previously known to be special Kähler, see for instance [CH17, Ex. 23, 24], we believe for most values of $k$ these examples are new.

Furthermore, we show that (3) together with flat cones

$$g_\beta^c = r^\beta |dz|^2, \quad \omega^c_{\beta,\nabla} = \omega_{LC} = \frac{\beta}{2} (I_2 \, d\rho + I_2 \, d\theta),$$

where $\beta \in \mathbb{R}$, are local models of isolated singularities of affine special Kähler structures in real dimension two. To explain, let $(M, I)$ be a smooth Riemann surface and let $(g, \nabla)$ be a special Kähler structure on $M$ possibly singular at some $m_0 \in M$. Assume that the associated holomorphic cubic form $\Xi$ has a finite order $n \in \mathbb{Z}$ at $m_0$. Choosing a local holomorphic coordinate $z$ near $m_0$, we can write $g = w |dz|^2$ and $\nabla = d + \omega_{\nabla}$. By the main theorem of [Hay15] there are two possibilities: $w = -|z|^{n+1} \log |z| (C + o(1))$ or there is $\beta < n + 1$ such that $w = |z|^\beta (C + o(1))$. In this article we prove the following result.

**Theorem 5.** Let $(g, \nabla)$ be a special Kähler structure on $M$ possibly singular at $m_0$, where $(M, I)$ is a smooth Riemann surface. Assume that $\Xi$ has a finite order $n$ at $m_0$. Let $z$ be a local holomorphic coordinate near $m_0$.

1. If $w = r^\beta (C + o(1))$, where $\beta < n + 1$, then there exists $\varepsilon > 0$ such that

$$\omega_{\nabla} = \omega^c_{\beta,\nabla} + o(1) \, d\theta + o(\varepsilon^\rho) \, d\rho \quad \text{for} \quad \rho \to -\infty.$$  

2. If $w = -r^{n+1} \log r (C + o(1))$, then there exists $b \in \mathbb{C}$, $|b| = 1$, such that

$$\omega_{\nabla} = \omega^c_{n+1,\nabla} + o(\rho^{-1}) \, d\theta + o(\rho^{-1}) \, d\rho \quad \text{for} \quad \rho \to -\infty.$$  

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Motivated by Theorem 5 we adopt the following terminology.

**Definition 6.** We say that a special Kähler structure \((g, \nabla)\) on \(B^*_1\) has a conical singularity of order \(\frac{1}{2} \beta\) at the origin, if \((g, \omega_{\nabla})\) is asymptotic to \((g_{\beta}, \omega_{\beta}, \nabla)\). We say that \((g, \nabla)\) has a logarithmic singularity of order \(\frac{1}{2} k, \ k \in \mathbb{Z}\), at the origin, if \((g, \omega_{\nabla})\) is asymptotic to \((g_k, \omega_k, \nabla)\).

Theorem 5 yields in particular the following result, which has been announced in [CH17].

**Corollary 7.** Let \((g, \nabla)\) be as in Theorem 5. If \((g, \nabla)\) has a logarithmic singularity of order \(\frac{1}{2} (n + 1)\), we put by definition \(\beta = n + 1\). Denote by \(\text{Hol}(S^1, \nabla)\) the holonomy of \(\nabla\) along a circle centered at the origin and oriented in the counterclockwise direction. Then the following holds:

- If \(\beta \notin \mathbb{Z}\), then \(\text{Hol}(S^1, \nabla)\) is conjugate to \(\begin{pmatrix} \cos \pi \beta & -\sin \pi \beta \\ \sin \pi \beta & \cos \pi \beta \end{pmatrix}\);
- If \(\beta \in 2\mathbb{Z}\), then \(\text{Hol}(S^1, \nabla)\) is trivial or conjugate to \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\);
- If \(\beta \in 2\mathbb{Z} + 1\), then \(\text{Hol}(S^1, \nabla)\) is \(-1\) or conjugate to \(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\).

Moreover, if \((g, \nabla)\) has a conical singularity of order \(\frac{1}{2} \beta\), where \(\beta \in \mathbb{Z}\), then \(\text{Hol}(S^1, \nabla) = 1\) if \(\beta\) is even and \(\text{Hol}(S^1, \nabla) = -1\) if \(\beta\) is odd.

Proofs of these statements as well as some corollaries can be found in Section 4. Numerous local examples of isolated singularities of affine special Kähler structures are contained in [Hay15, CH17]. Continuous families of special Kähler structures on \(\mathbb{CP}^1\) with isolated singularities can be found in [CH17, Sect. 3.2].

The reader may wonder why the holonomy of the flat symplectic connection around the singularity depends on \(\beta\) only, i.e., on the asymptotic of the metric. This seems particularly strange in view of the following: Since \(\Xi\) can be seen as measuring the difference between \(\nabla\) and the Levi-Civita connection \(\nabla^{LC}\) [Fre99] and \(\Xi\) may have a pole at the origin, one might expect that \(\nabla\) and \(\nabla^{LC}\) are unrelated in general. However, it follows from Theorem 5 that this is not the case: The leading terms of \(\omega_{\nabla}\) and \(\omega_{LC}\) coincide. Indeed, this is immediate in the conical case and in the logarithmic one this follows from the observation that the Levi-Civita connection of \(g_k\) is \(\frac{1}{2}(k + \rho^{-1})(I_2 \, \rho^2 + I_2 \, d\theta)\). In fact, in both cases we have \(\omega_{\nabla} - \omega_{LC} = o(r^{-1})\) with respect to the flat reference metric on \(B^*_1\). Thus, the singularity of the cubic holomorphic form reflects the singularity of \(g\) rather than the deviation of \(\nabla\) and \(\nabla^{LC}\) near an isolated singularity.

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## 2 Preliminaries

The main purpose of this section is to fix notations. Details can be found for instance in [Fre99, Hay15]; See also [CH17] for an elementary exposition.
Conventions. Throughout this paper we adopt the convention
\[ \alpha \wedge \beta = \frac{1}{2} (\alpha \otimes \beta - \beta \otimes \alpha), \]
where \( \alpha, \beta \) are 1-forms, i.e., we think of differential forms as tensors. This convention implies \( 2 \, dx \wedge dy (\partial_x, \partial_y) = 1 \); in particular, the Kähler form of the flat metric \( |dz|^2 = dx^2 + dy^2 \) on \( \mathbb{R}^2 \) is \( |dz|^2 (I, \cdot , \cdot ) = 2 \, dx \wedge dy \). This also means that \( (p, q) \) are Darboux coordinates for a symplectic form \( \omega \in \Omega^1(\mathbb{R}^2) \), if \( \omega = 2 \, dp \wedge dq \).

This convention coincides with the one of [Hay15], but differs from others, which can be found in the literature.

A special Kähler structure can be conveniently described locally in terms of special holomorphic coordinates. The notion of special holomorphic coordinates makes sense in any dimension, but here we restrict ourselves to the case of real dimension two.

Following [Fre99], we say that a holomorphic coordinate \( Z \) is special, if \( p = \text{Re} \, Z \) is flat, i.e., \( \nabla dp = 0 \). Two special holomorphic coordinates \( Z \) and \( W \) are said to be conjugate, if \( p \) and \( q := - \text{Re} \, W \) are Darboux coordinates for \( \omega \), i.e., \( \omega = 2 \, dp \wedge dq \). Such coordinates always exist in a neighborhood of any regular point [Fre99]. This does not need to be the case if the point is singular [CH17, Sect. 2.4].

One way to construct a pair of conjugate holomorphic coordinates in a neighborhood of some point \( m \in M \) is as follows. Choose local coordinates so as to identify a neighborhood of \( M \) with the disc in \( \mathbb{R}^2 \) of radius 1. Choose a pair of covectors \( (\alpha_1, \alpha_2) \) from \( T_m^* M \) such that \( \omega_m = 2 \, \alpha_1 \wedge \alpha_2 \). Use parallel transport along radial lines to obtain a pair of one-forms, which are closed, since \( \nabla \) is torsion-free. Hence, these forms are in fact exact, say \( (dp, dq) \), so that \( (p, q) \) are Darboux coordinates. Then \( Z \) and \( W \) can be found from the relations: \( \text{Re} \, Z = p \) and \( \text{Re} \, W = -q \).

A useful object, which can be attached to a special Kähler structure, is the so called holomorphic cubic form, which is defined as follows. Consider the fiberwise projection \( \pi^{(1,0)} \) onto the \( T^{1,0} M \subset T_{0} \mathbb{C} \mathbb{P}^1 \) as a 1-form with values in \( T_{0} \mathbb{C} \mathbb{P}^1 \). Since this form vanishes on vectors of type \( (0,1) \), we can think of \( \pi^{(1,0)} \) as an element of \( \Omega^{1,0}(M; T_{0} \mathbb{C} \mathbb{P}^1) \). Then, the holomorphic cubic form is
\[ \Xi := -\omega(\pi^{(1,0)}, \nabla \pi^{(1,0)}) \in H^0(M; \text{Sym}^3 T^* M). \]

In terms of conjugate special holomorphic coordinates \( (Z, W) \), \( \Xi \) can be expressed as follows:
\[ \Xi = \frac{1}{4} \frac{\partial^2 W}{\partial Z^2} \, dZ^3. \]

Let us now provide a description of special Kähler structures in terms of solutions to a system of nonlinear PDEs, which were first obtained in [Hay15]. The analysis of these PDEs is a crucial ingredient of the proof of Theorem 5.

Write a special Kähler metric \( g \) on \( B_1^* \) in the form
\[ g = e^{-u} |dz|^2. \]

With respect to the trivialization \( (\partial_x, \partial_y) \) of \( T B_1^* \) the connection \( \nabla \) is described by its connection 1-form \( \omega_\nabla \in \Omega^1(\Omega; \mathfrak{gl}(2, \mathbb{R})) \). A computation shows that \( \nabla \) is torsion-free and satisfies (2) if and only if \( \omega_\nabla \) can be written in the form
\[
\omega_\nabla = \begin{pmatrix}
\omega_{11} & -* \omega_{11} \\
* \omega_{22} & \omega_{22}
\end{pmatrix},
\]

(8)
where $\ast$ denotes the Hodge operator with respect to the flat metric $|dz|^2 = dx^2 + dy^2$. Furthermore, one can show that there is a pair $(h, a) \in C^\infty(B^*_4) \times \mathbb{R}$ such that

$$2\omega_{11} = e^u (dh + a \varphi) - du \quad \text{and} \quad 2\omega_{22} = -e^u (dh + a \varphi) - du. \quad (9)$$

Here $\varphi = -d\theta$ is a generator of the first de Rham cohomology group of $B^*_4$. A computation ([Hay15, Cor 2.3]) shows that $\nabla$ is flat if and only if $(h, u, a)$ satisfies

$$\Delta h = 0 \quad \text{and} \quad \Delta u = |dh + a \varphi|^2 e^{2u}. \quad (10)$$

Here $\Delta = \partial_{xx}^2 + \partial_{yy}^2$ is the (non-positive) Laplace operator with respect to the flat metric of $B^*_4$.

In terms of solutions of (10) the associated holomorphic cubic form can be expressed as follows

$$\Xi = \frac{1}{2} \left( \frac{a}{2z^i} - \frac{\partial h}{\partial z^i} \right) dz^3. \quad (11)$$

**Remark 12.** Note that if $(u, h, a)$ determines an affine special Kähler structure with $g = e^{-u} |dz|^2$, then, for $C > 0$, the triple $(u - \log C, Ch, Ca)$ determines a new special Kähler structure with the rescaled metric $Cg = Ce^{-u} |dz|^2$. In particular, the connection 1-forms agree and the cubic holomorphic form scales by the same factor $C$ as the metric.

**Remark 13.** Observe that the triple $(h, u, a)$ depends on the choice of the local coordinate. We wish to find the triple $(\tilde{h}, \tilde{u}, \tilde{a})$ corresponding to the new local coordinate $\tilde{z} = \lambda \cdot z = e^{i\theta_0} \cdot z$, where $\lambda$ is a complex number of unit length. Write $h = h_0 + b \log r$, where $h_0$ is the real part of some holomorphic function $f$, which may have a pole at the origin, and $b \in \mathbb{R}$. Then a computation shows that

$$\tilde{h}(\tilde{z}, \bar{z}) = \text{Re}(\lambda^2 f(\lambda \tilde{z})) + \text{Re}(\lambda^2 (b + ai)) \log |\tilde{z}|, \quad \tilde{a} = \text{Im}(\lambda^2 (b + ai)),$$

and $\tilde{u} = u$.

Notice also that the rescaling $\tilde{z} = \lambda \cdot z$ with $\lambda \in \mathbb{R}_{>0}$ leads to $(\tilde{h}, \tilde{u}, \tilde{a}) = (h, u + 2 \log \lambda, a)$.

## 3 Models of singular special Kähler structures

### 3.1 Flat cones

Consider $\mathbb{C}^*$ as a special Kähler manifold equipped with the flat conical structure (4). Despite its simplicity, it is instructive to compute special holomorphic coordinates and the holonomy of this structure.

For $r \in (0, 1)$ let $\sigma : [r, 1] \to \mathbb{C}^*$ be the straight line segment, $\sigma(s) = s$. A straightforward computation yields that the parallel transport $P_\sigma$ along $\sigma$ is given by

$$P_\sigma(dx) = r^{-\beta/2} dx \quad \text{and} \quad P_\sigma(dy) = r^{-\beta/2} dy.$$

Similarly, if $\gamma_r : [0, \psi] \to \mathbb{C}^*$ is the arc of angle $\psi$ starting at the point $r$, then for the parallel transport along $\gamma_r$ we have

$$P_{\gamma_r}(dx) = \cos(\frac{\beta}{2} \psi) dx - \sin(\frac{\beta}{2} \psi) dy \quad \text{and} \quad P_{\gamma_r}(dy) = \sin(\frac{\beta}{2} \psi) dx + \cos(\frac{\beta}{2} \psi) dy,$$
i.e., $P_{\gamma}$ is the rotation by the angle $\frac{\beta}{2}\psi$. In particular,

$$\text{Hol}(S^1; \nabla) = \begin{pmatrix} \cos \pi \beta & -\sin \pi \beta \\ \sin \pi \beta & \cos \pi \beta \end{pmatrix}. $$

Denote by $P_{\sigma}$ the parallel transport from $1$ to $r$ along the segment $[r, 1]$. The 1-forms $(dp, dq) = (P_{\gamma} \circ P_{\sigma}(dx), P_{\gamma} \circ P_{\sigma}(dy))$ are well-defined on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Since $\nabla$ is torsion-free, these forms are closed and therefore exact (hence the notation). Explicitly, we have

$$ dp = \text{Re}(z^{\beta/2}dz) \quad \text{and} \quad dq = \text{Im}(z^{\beta/2}dz).$$

Hence, if $\beta \neq -2$ we obtain that

$$ (Z, W) = \frac{2}{\beta+2} \left( z^{\beta/2}, iz^{\beta/2} \right) $$

are conjugate special holomorphic coordinates. For $\beta = -2$, we obtain the following pair of conjugate coordinates:

$$ (Z, W) = (\log z, i \log z), $$

where $\log z = \log r + i \theta$.

In both cases we have the relation $W = iZ$, which implies that the holomorphic cubic form $\Xi$ vanishes, which is clear anyway, since $\nabla = \nabla^L C$.

Notice that flat cones do not satisfy the hypotheses of Theorem 5 since $\Xi$ does not have a finite order at the origin. Nevertheless, as we show below they serve as models for special Kähler structures, whose associated holomorphic cubic forms have finite orders at isolated singularities.

### 3.2 Logarithmic singularities

In this section we describe the model special Kähler structures on $B^*_1$ with logarithmic singularities mentioned in the introduction. We first present a more computational approach much in the spirit of the previous subsection. A somewhat less computational approach, which also establishes links between $(g_k, \omega_k, \nabla)$ as $k$ varies, is given afterwards.

For simplicity of exposition, in the main body of this section we consider only the case $C = 1 = b$ in (3). In this case the connection 1-form is given by

$$ \omega_{k, \nabla} = \frac{1}{2} \left( k I_2 + \begin{pmatrix} \sin k \theta & -1 + \cos k \theta \\ 1 + \cos k \theta & -\sin k \theta \end{pmatrix} \rho^{-1} \right) d\theta + \frac{1}{2} \left( k I_2 + \begin{pmatrix} -\cos k \theta & \sin k \theta \\ \sin k \theta & 1 + \cos k \theta \end{pmatrix} \rho^{-1} \right) d\rho. $$

The general case is easy to obtain by a suitable choice of $h$ and $a$ below, see Remark 18 for explicit formulae.

**First approach: Direct computation.** Pick an integer $k \neq 0$ and consider the functions

$$ h(z, \bar{z}) := \text{Re} \left( \frac{z^k}{\bar{z}} \right) \quad \text{and} \quad u(z, \bar{z}) := -k \log |z| - \log |\log |z||,$$
which are smooth on $B^*_1$. Clearly, $h$ is harmonic whereas for $u$ we have

$$\Delta u = -\Delta (\log | \log |z||) = \frac{1}{|z|^2 (\log |z|)^2}. $$

Using $dh = r^{k-1} \cos(k\theta) \, dr - ir^k \sin(k\theta) \, d\theta$, we obtain $| dh |^2 = r^{2(k-1)} = |z|^{2(k-1)}$, which yields in turn

$$\Delta u = \frac{1}{|z|^2 (\log |z|)^2} = \frac{1}{|z|^{2k} (\log |z|)^2} = | dh |^2 e^{2u}. $$

Therefore $(h, u, 0)$ satisfies (10); Hence, the metric $g = e^{-u} |dz|^2 = -|z|^k \log |z||dz|^2$ is special Kähler.

Using (9), we compute the components of the connection 1-form:

$$2 \omega_{11} = - (\log r)^{-1} (r^{-1} \cos(k\theta) dr - \sin(k\theta) d\theta) + \left( \frac{k}{r} + \frac{1}{r \log r} \right) dr, $$

$$2 \omega_{22} = (\log r)^{-1} (r^{-1} \cos(k\theta) dr - \sin(k\theta) d\theta) + \left( \frac{k}{r} + \frac{1}{r \log r} \right) dr. $$

This shows that $g = -r^k \log r |dz|^2$ and (14) is indeed a special Kähler structure.

Similarly, for $k = 0$ one checks that $h(z, \bar{z}) := \log |z|$, $u(z) := -\log |\log |z||$, and $a = 0$ determine the special Kähler structure with the metric $g_0 = -\log |z||dz|^2$ and connection 1-form $\omega_{0, \gamma}$ from (14).

Furthermore, the parallel transport $P_\sigma$ of a 1-form $\eta_1 \, dx + \eta_2 \, dy$ along the segment $\sigma = [r_0, r_1] \subset \mathbb{C}^*$, where $0 < r_0 < r_1 < 1$, is described by the system of ODEs

$$\dot{\eta}_1(r) = k \frac{r_1}{r_0} \eta_1(r) \quad \text{and} \quad \dot{\eta}_2(r) = \frac{1}{2r} \left( k + \frac{2}{\log r} \right) \eta_2(r),$$

which can be easily solved explicitly. This yields:

$$P_\sigma(dx) = \left( \frac{r_1}{r_0} \right)^{\frac{k}{2}} dx \quad \text{and} \quad P_\sigma(dy) = \left( \frac{r_1}{r_0} \right)^{\frac{k}{2}} \frac{\log r_1}{\log r_0} dy. $$

For the parallel transport along the arc $\gamma_r : [0, \theta] \to \mathbb{C}^*$, $\gamma_r(s) = re^{is}$, we obtain the system

$$\left( \frac{\dot{\eta}_1(s)}{\dot{\eta}_2(s)} \right) = \frac{1}{2 \log r} \left( \begin{array}{cc} \sin(k s) & \cos(k s) \\ \cos(k s) & -\sin(k s) \end{array} \right) - \left( k \log r + 1 \right) I_2 \left( \begin{array}{c} \eta_1(s) \\ \eta_2(s) \end{array} \right),$$

One can check that the solution subject to the initial condition $\eta_1(0) = 1, \eta_2(0) = 0$ is given by:

$$\eta_1(s) = \cos \left( \frac{k}{2} s \right), \quad \eta_2(s) = -\sin \left( \frac{k}{2} s \right).$$

For the initial condition $\eta_1(0) = 0, \eta_2(0) = 1$, the corresponding solution is

$$\eta_1(s) = \sin \left( \frac{k}{2} s \right) + \frac{1}{\log(r)} \cos \left( \frac{k}{2} s \right), \quad \eta_2(s) = \cos \left( \frac{k}{2} s \right) - \frac{1}{\log(r)} \sin \left( \frac{k}{2} s \right).$$

Hence, the holonomy around the origin with a basepoint $r_1$ is given by

$$\text{Hol}(S_1^1, \nabla) = (-1)^k \left( \begin{array}{cc} 1 & \frac{2\pi}{\log(r_1)} \\ 0 & 1 \end{array} \right),$$

(15)
**Remark 16.** At this point the reader may wonder why the holonomy of $\nabla$ depends on $r_1$. While $S^1_{r_1}$ and $S^1_{r_2}$ are of course homotopic as free loops in $B^*_1$ for any $r_1, r_2 \in (0, 1)$, these loops are based at different points if $r_1 \neq r_2$. Therefore, $\text{Hol}(S^1_{r_1}, \nabla)$ and $\text{Hol}(S^1_{r_2}, \nabla)$ are conjugate, but do not need to be equal.

Just like in the previous subsection, we compute the parallel transport of $\alpha_1 = r_1^{\frac{k}{2}} dx \in T^*_1 B^*_1$ and $\alpha_2 = -r_1^{\frac{k}{2}} \log r_1 dy \in T^*_1 B^*_1$ and obtain

$$(dp, dq) = \left( \text{Re}(z^{\frac{k}{2}} dz), -\text{Im}(\log(z) z^{\frac{k}{2}} dz) \right).$$

Hence we obtain that for $k \neq -2$ the pair

$$(Z, W) = \left( \frac{2}{k+2} z^{\frac{k+2}{2}}, -\frac{2}{k+2} z^{\frac{k+2}{2}} \left( \log z - \frac{2}{k+2} \right) \right) \quad (17)$$

consists of special conjugate coordinates on $B^*_1 \setminus \mathbb{R}_{>0}$.

Similarly, for $k = -2$, we obtain conjugate coordinates

$$(Z, W) = \left( \log(z), \frac{1}{2} \log(z)^2 \right).$$

Furthermore, we have

$$\Xi = -\frac{i}{2} \frac{\partial h}{\partial z} dz^3 = -\frac{i}{4k} \frac{\partial z^k}{\partial z} dz^3 = -\frac{i}{4} z^{k-1} dz^3.$$  

In particular, $\Xi$ has order $n = k - 1$ at the origin.

**Remark 18.** If $b$ and/or $C$ in (3) is not necessarily 1, the triple $(h, u, a)$ is given by

$$h(z, \bar{z}) := C \text{Re} \left( b \frac{z^k}{k} \right), \quad u(z, \bar{z}) := -k \log |z| - \log |\log |z|| - \log C, \quad \text{and} \quad a := 0$$

for $k \neq 0$ and by

$$h(z, \bar{z}) := C \text{Re}(b) \log |z|, \quad u(z, \bar{z}) := -\log |\log |z|| - \log C, \quad \text{and} \quad a := C \text{Im}(b)$$

for $k = 0$.

**Second approach: A fundamental example and its pull-backs.** Consider the special Kähler structure on $B^*_1$ determined by the triple $(u, h, a)$, where

$$u = \log |z| - \log |\log |z||, \quad h = -\text{Re} z^{-1} = -\frac{x}{x^2 + y^2}, \quad (19)$$

and $a = 0$. In other words, the corresponding metric and holomorphic cubic form are given by

$$g = -|z|^{-1} \log |z| |dz|^2, \quad (20)$$

$$\Xi = -\frac{i}{4} z^{-2} dz^3. \quad (21)$$

This is a special case of the setup of the previous approach corresponding to $k = -1$. 

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We claim that a pair of conjugate special holomorphic coordinates on $B^*_1 \setminus \mathbb{R}_{>0}$ is given by

$$(Z,W) = \left(2\sqrt{z}, -2i\sqrt{z}(\log z - 2)\right).$$

Here $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ in polar coordinates.

Let us check that $(Z,W)$ are indeed special coordinates. Denoting $p := 2 \Re \sqrt{z} = 2\sqrt{r}\cos \frac{\theta}{2}$ and $q := -\Re W = -2\sqrt{r}(\theta \cos \frac{\theta}{2} + \log r \sin \frac{\theta}{2} - 2 \sin \frac{\theta}{2})$,

we obtain

$$dp = \frac{1}{\sqrt{r}}(\cos \frac{\theta}{2} dr - r \sin \frac{\theta}{2} d\theta),$$

$$dq = -\frac{1}{\sqrt{r}}((\theta \cos \frac{\theta}{2} + \log r \sin \frac{\theta}{2}) dr + r(\log r \cos \frac{\theta}{2} - \theta \sin \frac{\theta}{2}) d\theta).$$

A straightforward computation yields

$$dp \wedge dq = -\log r \, dr \wedge d\theta = -(r^{-1} \log r) \, r \, dr \wedge d\theta,$$

which shows that $(p,q)$ are Darboux coordinates for the Kähler form of $(20)$. We leave to the reader to check that $(p,q)$ are flat.

It is clear from $(22)$ that the holonomy of the flat symplectic connection $\nabla$ as we go once around a circle centered at the origin is conjugate to

$$\begin{pmatrix} -1 & 2\pi \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Here “$\sim$” means that the above two matrices are conjugate.

Let $f : B^*_1 \to B^*_1$, $f(z) = z^{k+2}$, where $k \in \mathbb{Z}$, $k \neq -2$. For the pull-back of $(20), (21)$ we obtain:

$$f^*g = -(k + 2)^3|z|^k \log |z| |dz|^2$$

and

$$f^*\Xi = -\frac{(k + 2)^3i}{4} z^{k-1} \, dz^3.$$

Then clearly $(f^*g, f^*\nabla)$ is again a special Kähler structure. Thus, we immediately obtain that special holomorphic coordinates are $(f^*Z, f^*W)$, which yields (17) up to the multiplication by $(k + 2)^3$; This factor corresponds to different normalizations of $f^*g$ and $g_k$.

A straightforward albeit laborious computation shows that the connection 1-form of $f^*\nabla$ is given by (14). In this way one can obtain all structures $(g_k, \omega_k, \nabla)$ except for $k = -2$, which must be considered separately just like in the previous approach. We omit the details.

Remark 23. It is not true that the connection form of $f^*\nabla$ is represented by $f^*\omega_{\nabla}$ in the sense of [Hay15]. The reason is simple: The pair of real coordinates $^1(a, b)$, where $a + bi = z^{k+2} = f(z)$, also gives rise to a trivialization of the tangent bundle and $\omega_{\nabla}$ depends on this trivialization, cf. Remark 13. Rather, the representation of $f^*\nabla$ can be obtained from $f^*\omega_{\nabla}$ by applying a gauge transformation. It is also not true that $(f^*g, f^*\nabla)$ is represented by the triple $(f^*h, f^*u, 0)$, where $(h, u)$ are as in (19).

^1(a, b) can be viewed as local coordinates on the $k$-sheeted covering of $B^*_1$. 


4 Proofs of the main results and some corollaries

Write $\Xi = \Xi_0 \, dz^3$, where $\Xi_0$ is a holomorphic function on $B_1^*$, and denote $\tilde{\Xi}_0(z) := z^{-n} \Xi_0(z)$. Observe that the order of $\tilde{\Xi}_0$ at the origin vanishes, i.e., $\tilde{\Xi}_0(0)$ is well defined and does not vanish.

**Lemma 24.** Assume that the holomorphic cubic form $\Xi$ does not have an essential singularity at the origin and denote $n := \text{ord}_0 \Xi \in \mathbb{Z}$. Then there exists $\beta \leq n + 1$ such that

\[
\begin{align*}
\frac{\partial}{\partial x} v & \quad \text{are continuous at } 0, \\
\frac{\partial}{\partial y} v & = O(1), \\
\frac{\partial}{\partial x} v & = O(|z|^{-2(\beta - n)}), \\
\frac{\partial}{\partial y} v & = O(|z|^{-1}(\log |z|)^{-2}),
\end{align*}
\]

where the remainder functions $v$ and $\tilde{v}$ are continuous on $B_1$ and $\tilde{v}(0) = -2 \log 2 - \log |\tilde{\Xi}_0(0)|$.

Moreover, we have

\[
\begin{align*}
\frac{\partial}{\partial x} v & = O(1), \\
\frac{\partial}{\partial y} v & = O(1), \\
\frac{\partial}{\partial x} v & = O(|z|^{-2(\beta - n)}), \\
\frac{\partial}{\partial y} v & = O(|z|^{-1}(\log |z|)^{-2}).
\end{align*}
\]

**Proof.** Write $\Xi = \Xi_0 \, dz^3$, where $\Xi_0$ is a holomorphic function on $B_1^*$. By the proof of Theorem 1.1 of [Hay15] $u$ satisfies $\Delta u = 16|\Xi_0|^2 e^{2u}$.

Observe that since $\Xi_0$ is holomorphic, $\log |\Xi_0|$ is harmonic. Using this, a straightforward computation shows that $u_1 := 2u + 2 \log |\Xi_0| + 5 \log 2$ is a solution of Liouville’s equation $\Delta u_1 = e^{u_1}$. With this at hand, the statement of this lemma is immediately obtained from [Nit57, Thm. 1.1].

We say that the order of $h$ at the origin is $\hat{N} = N + 1$ if

\[
h = \begin{cases} 
\sum_{j=N+1}^{\infty} p_j(x, y), & \text{if } N + 1 > 0, \\
q_0 \log |z| + h_0, & \text{if } N + 1 = 0, \\
\sum_{j=1}^{N+1} q_j(x, y) - |z|^{2j} + q_0 \log |z| + h_0, & \text{if } N + 1 < 0,
\end{cases}
\]

where $p_j, q_j$ are homogeneous harmonic polynomials of degree $j$ and $h_0$ is a smooth harmonic function; cf. [ABR01, p. 219].

By (11), the order of $\Xi$ at the origin is

\[
\text{ord}_0 \Xi = \begin{cases} 
N & \text{if } a = 0, \\
\min \{-1, N\} & \text{if } a \neq 0.
\end{cases}
\]

Notice also that the harmonicity of $h$ implies that $\frac{\partial h}{\partial z}$ is holomorphic.

**Proof of Theorem 5.** Let $(h, u, a)$ be a triple representing the special Kähler structure $(g, \nabla)$ as in (10). As before, denote by $\hat{N} = N + 1$ the order of $h$ at 0. From (25), we have the following asymptotics:

\[
h = \begin{cases} 
\frac{r^{N+1}}{N+1} \text{Re}(a_N e^{i(N+1)\theta}) + O(r^{N+2}) & \text{if } N \notin \{-2, -1\}, \\
- \text{Re}(a_{-2} r^{-1} e^{-\theta}) + O(\log(r)) & \text{if } N = -2, \\
\text{Re}(a_{-1} \log(r e^{\theta})) + O(1) & \text{if } N = -1.
\end{cases}
\]
Note that, while \( a_N \in \mathbb{C} \) in general, we have \( a_{-1} \in \mathbb{R} \), since \( h \) is not allowed to be multivalued. Furthermore,

\[
\partial_r h = r^N \Re(a_N e^{i(N+1)\theta}) + O(r^{N+1}) \quad \text{and} \quad \partial_\theta h = -r^{N+1} \Im(a_N e^{i(N+1)\theta}) + O(r^{N+2}).
\]

Combining (27), (28), and (11), we also obtain

\[
\Xi_0(z) = \begin{cases} 
\frac{1}{4}a_N z^N + O(r^{N+1}) & \text{if } N \neq -1, \\
\frac{1}{4}(a + a_{-1}) z^{-1} + O(1) & \text{if } N = -1.
\end{cases}
\]

**Case 1: The conical singularity.** Writing \( u = -\beta \log |z| + v \), with the help of Lemma 24 we obtain

\[
|\partial_\theta h| \lesssim r^{n+1}, \quad |e^u \partial_\theta h| \lesssim r^{n+1-\beta},
\]

\[
|\partial_v u| = |\partial_\theta v| \lesssim r(|\partial_v v| + |\partial_\theta v|) = o(r^\epsilon),
\]

\[
|r \partial_v u + \beta| = |r \partial_v v| = o(r^\epsilon),
\]

for a suitable \( \epsilon > 0 \).

Notice also that if \( a \neq 0 \), then \( n = \text{ord}_0 \Xi \leq -1 \). Therefore, \( \beta < n + 1 \leq 0 \), and we obtain

\[
e^u = e^{-\beta \log |z| + v} = |z|^{-\beta} e^v = o(r^\epsilon).
\]

Furthermore, by (9) we have

\[
2\omega_{11} = (e^u \partial_r h - \partial_r u) \, dr + (e^u \partial_\theta h - \partial_\theta u - e^v a) \, d\theta,
\]

\[
2\omega_{22} = -(e^u \partial_r h + \partial_r u) \, dr - (e^u \partial_\theta h + \partial_\theta u - e^v a) \, d\theta.
\]

Substituting (30) and (31) into (32), we obtain

\[
\lim_{|z| \to 0} \left( \omega_{\nabla} \left( \frac{\partial}{\partial \theta} \right) \right) = \frac{1}{2} \begin{pmatrix} 0 & -\beta \\ -\beta & 0 \end{pmatrix} \quad \text{and} \quad \omega_{\nabla} \left( \frac{\partial}{\partial r} \right) = \begin{pmatrix} \frac{\beta}{2r} & 0 \\ 0 & \frac{\beta}{2r} \end{pmatrix} + o(r^{-1+\epsilon}).
\]

**Case 2: The logarithmic singularity.** Just like in Lemma 24, write \( u = -(n + 1) \log r - \log |\log r| + \tilde{v} \). The continuity of \( \tilde{v} \) yields

\[
e^u = -\frac{1}{r^{n+1} \log r} e^{\tilde{v}} = -\frac{C^{-1} + O(1)}{r^{n+1} \log r},
\]

where \( C = e^{\tilde{v}(0)} \) is positive. Moreover, Lemma 24 yields

\[
\partial_r \tilde{v} = O \left( \frac{1}{r \log r} \right) \quad \text{and} \quad \partial_\theta \tilde{v} = O \left( \frac{1}{(\log r)^2} \right),
\]

and, therefore,

\[
du = -\frac{n + 1}{r} \, dr - \frac{1}{r \log(r)} \, dr + \frac{\partial \tilde{v}}{\partial r} \, dr + \frac{\partial \tilde{v}}{\partial \theta} \, d\theta
\]

\[
= -\frac{n + 1}{r} \, dr - \frac{1}{r \log(r)} \, dr + \frac{o(1)}{r \log r} \, dr + \frac{o(1)}{\log r} \, d\theta.
\]
Furthermore, using (28) and (33), we obtain
\[
e^u(dh + a\varphi) = - (C^{-1} + o(1))r^{-(n+1)} \log(r)^{-1} \left( \frac{\partial h}{\partial r} dr + \frac{\partial h}{\partial \theta} d\theta - a d\theta \right)
\]
\[
= - \frac{1}{C \log r} r^{-n} \left( \Re(a_N e^{i(N+1)\theta}) + o(1) \right) dr
\]
\[
+ \frac{1}{C \log r} r^{-n} \left( \Im(a_N e^{i(N+1)\theta}) + o(1) \right) d\theta
\]
\[
+ \frac{1}{C \log r} r^{-(n+1)} a(1 + o(1)) d\theta.
\]

Define \(b \in \mathbb{C}\) as follows:
\[
b := \begin{cases}
C^{-1} a_n & \text{if } n \neq -1, \\
C^{-1} (a_{-1} + ia) & \text{if } n = -1.
\end{cases}
\]
Here, we use \(a_{-1} = 0\) if \(N > -1\).

We claim that \(|b| = 1\) in fact. Indeed, if \(n \neq -1\) this follows from the computation
\[
|b| = |a_n|/C = 4|\tilde{E}_0(0)| e^{\tilde{\epsilon}(0)} = 1,
\]
where the second equality follows by (29) and the last one by Lemma 24. The case \(n = -1\) requires only cosmetic changes in the computation above.

Recalling (26), we obtain from (35):
\[
e^n(dh + a\varphi) = - \frac{1}{r \log r} \left( \Re(be^{i(n+1)\theta}) + o(1) \right) dr
\]
\[
+ \frac{1}{\log r} \left( \Im(be^{i(n+1)\theta}) + o(1) \right) d\theta.
\]

Substituting (34) and (36) into (9), we obtain
\[
2\omega_{11} = \left( n + 1 + \frac{1 - \Re(be^{i(n+1)\theta})}{\log r} + \frac{o(1)}{\log r} \right) \frac{dr}{r} + \left( \frac{\Im(be^{i(n+1)\theta})}{\log r} + \frac{o(1)}{\log r} \right) d\theta,
\]
\[
2\omega_{22} = \left( n + 1 + \frac{1 + \Re(be^{i(n+1)\theta})}{\log r} + \frac{o(1)}{\log r} \right) \frac{dr}{r} + \left( - \frac{\Im(be^{i(n+1)\theta})}{\log r} + \frac{o(1)}{\log r} \right) d\theta.
\]

This immediately yields the statement of this theorem. \qed

Remark 37. In general, the constant \(b\) in Theorem 5 depends on the local coordinate and, in most cases, can be normalized to 1 via suitable choices. For example, if \(n \neq -3\) this can be achieved by the change of the local coordinate as in Remark 13. However, if \(n = -3\), the coefficient of the leading term in (27) does not depend on the choice of the local coordinate and can not be normalized to 1.

It may seem that Theorem 5 implies that the holonomies of \(\nabla\) and its model \((\omega_{ij, \nabla}^{\sigma} \text{ or } \omega_{ij+1, \nabla}^{\sigma})\) coincide. This is not quite accurate, Theorem 5 implies only that the closures of the \(\text{SL}(2, \mathbb{R})\)-conjugacy classes of the holonomies of \(\nabla\) and its local model coincide. To explain, recall that \(\sigma = \sigma_r\) denotes the straight line segment \([r, 1]\) and \(\gamma_r\) denotes the circle of radius \(r\) centered at the origin. Denoting by \(P_r := P_{\sigma_r}\) and \(P_{\gamma_r}\) the parallel transport along \(\sigma_r\) and \(\gamma_r\) respectively, we have
\[
P_{\gamma_1} = P_r \circ P_{\sigma_r} \circ P_r^{-1}.
\]
Notice that $P_{\gamma_r}$ does not need to be constant in $r$, see Remark 16. Moreover, even though $P_{\gamma_r}$ is asymptotic to the parallel transport $P_{\gamma_r}^0$ of the model connection along $\gamma_r$ as $r \to 0$, one can not pass to the limit in (38) since $P_r$ (as well as $P_{\gamma_r}$) may diverge to infinity.

Observe, however, that $\text{tr} P_{\gamma_r}$ converges to $\text{tr} P_{\gamma_r}^0$ as $r \to 0$. Since $\text{tr} P_{\gamma_r}$ does not depend on $r$ indeed, we must have $\text{tr} P_{\gamma_1} = \text{tr} P_{\gamma_r} = \text{tr} P_{\gamma_r}^0 = \text{tr} P_{\gamma_1}^0$. It remains to notice that two matrices in $\text{SL}(2, \mathbb{R})$ have equal traces if and only if the closures of their conjugacy classes are equal. Moreover, the closure of the orbit of $A \in \text{SL}(2, \mathbb{R}) \setminus \{ \pm 1 \}$ is strictly bigger than the orbit of $A$ itself if and only if $\text{tr} A = \pm 2$.

With this understood, we can proceed to the proof of Corollary 7.

**Proof of Corollary 7.** As explained above, the closure of the orbit of $\text{Hol}(S^1, \nabla)$ in $\text{SL}(2, \mathbb{R})$ contains

$$H_\beta := \exp \left( \begin{array}{cc} 0 & \pi \beta \\ -\pi \beta & 0 \end{array} \right) = \left( \begin{array}{cc} \cos(\pi \beta) & \sin(\pi \beta) \\ -\sin(\pi \beta) & \cos(\pi \beta) \end{array} \right), \quad \beta \in (-\infty, n + 1].$$

Hence, if $\text{tr} H_\beta \neq \pm 2$, then $\text{Hol}(S^1, \nabla)$ is conjugate to $H_\beta$. If $\text{tr} H_\beta = \pm 2$, then $\text{Hol}(S^1, \nabla)$ is equal to $\pm 1$ or is conjugate to the elementary Jordan block:

$$\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}.$$

It remains to show, that if $\beta \in \mathbb{Z}$ and $\beta < n + 1$, then $\text{Hol}(S^1, \nabla) = \pm 1$. By Theorem 5, the parallel transport of $\nabla$ along the radial segment $[r, 1] \subset B^*_1$ is described by the system of ODEs

$$\dot{\eta} = -\frac{1}{s} \left( \frac{\beta}{2} + o(s^2) \right) \eta,$$

where $s \in [r, 1]$ and $\eta : [r, 1] \to \mathbb{R}^2$. Notice that the above system becomes singular as $r$ goes to zero.

Writing $\zeta(s) = s^{\beta/2} \eta(s)$, we obtain

$$\dot{\zeta} = o(s^{s-1}) \zeta. \quad (39)$$

Let us denote $\chi(s) := |\zeta(s)|^2$. Then $\dot{\chi} = 2 \langle \zeta, \dot{\zeta} \rangle = o(s^{s-1}) \chi$. Integrating this equation, we obtain

$$\log \frac{\chi(s_2)}{\chi(s_1)} = \int_{s_1}^{s_2} o(s^{s-1}) ds \geq -C(s_2^{s} - s_1^{s})$$

for some constant $C > 0$ and any $s_1 < s_2$. Exponentiating this inequality we obtain

$$\chi(s_1) \leq \chi(s_2) \exp(C(s_2^{s} - s_1^{s})) \leq \chi(s_2) \exp(Cs_2^s).$$

This implies in particular, that for any solution $\zeta$ of (39) with a fixed initial condition at $s = 1$ the function $\chi = |\zeta|^2$ is bounded on $(0, 1)$.

Think of $P_r$ as a $2 \times 2$-matrix. Then the above consideration shows that the matrix-valued function $r^{-\beta/2} P_r$ is bounded, hence there is a sequence $r_i \to 0$ such that $r_i^{-\beta/2} P_{r_i}$ converges to some $P_0$. Moreover, since $\nabla$ is symplectic, we have $\det(r^{-\beta/2} P_r) = r^{-\beta} \det P_r \to 1$ (recall that the chosen trivialization $(\partial_x, \partial_y)$ is not symplectic). Hence, we can assume that $(r_i^{-\beta/2} P_{r_i})^{-1}$ converges; in particular, $P_0$ is invertible.
Passing to the limit along the sequence \( r_i \) in the equality
\[
P_{\gamma_1} = P_r \circ P_{\gamma_r} \circ P_r^{-1} = r^{-\beta/2} P_r \circ P_{\gamma_r} \circ (r^{-\beta/2} P_r)^{-1},
\]
and using \( P_{\gamma_r} \to \pm 1 \), we obtain \( P_{\gamma_1} = \pm 1 \). □

**Proposition 40.** \( \text{Hol}(S^1, \nabla) \) is conjugate to a matrix lying in \( \text{Sp}(2, \mathbb{Z}) \) if and only if \( \beta \in \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} \).

**Proof.** Since \( \text{Hol}(S^1, \nabla) \in \text{Sp}(2, \mathbb{R}) \), the characteristic polynomial of \( \text{Hol}(S^1, \nabla) \) has integer coefficients if and only if \( \text{tr} \text{Hol}(S^1, \nabla) \in \mathbb{Z} \). This implies that \( \text{Hol}(S^1, \nabla) \) is conjugate to a matrix lying in \( \text{Sp}(2, \mathbb{Z}) \) if and only if \( \cos \pi \beta \in \{0, \pm \frac{1}{2}, \pm 1\} \). □

The context where one meets special Kähler structures, whose flat symplectic connection \( \nabla \) has an integral holonomy, is called algebraic integrable systems. Roughly speaking, an algebraic integrable system is a smooth hyperKähler manifold \( X \) equipped with a holomorphic map \( \pi: X \to M \), whose fibers are smooth compact Lagrangian tori equipped with polarizations. We refer to [Fre99, Def. 3.1] for details. It turns out that an algebraic integrable system induces a special Kähler structure on the base \( M \) [Fre99, Thm. 3.4]. Moreover, if the polarization of the fibers is principal, the holonomy of the corresponding flat symplectic connection is integral. As a refinement of the above proposition, we obtain the following result.

**Proposition 41.** Let \( \pi: X \to B_1 \) be an elliptic fibration such that all fibers \( X_z \), \( z \neq 0 \), are smooth and the central fiber \( X_0 \) is not multiple. Suppose that \( \pi: X \setminus X_0 \to B_1^* \) admits a structure of the algebraic integrable system such that all non-central fibers are principally polarized. Let \( (g, \nabla) \) be the induced special Kähler structure on \( B_1^* \). Assume that the associated holomorphic cubic form \( \Xi \) has a finite order \( n \in \mathbb{Z} \) at the origin. If the Kodaira type of \( X_0 \) is as in the left column of Table 1, then the type and order multiplied by the factor 2 of \( (g, \nabla) \) is given in the middle and right columns of Table 1 respectively. In addition, the order of a conical singularity is always smaller than \( n + 1 \).

| Kodaira type | Type of isolated singularity | Twice the order of isolated singularity |
|--------------|-----------------------------|---------------------------------------|
| I_0          | conical / logarithmic       | \( \beta \) even integer / \( n + 1 \), \( n \) odd |
| I_0*         | conical / logarithmic       | \( \beta \) odd integer / \( n + 1 \), \( n \) even |
| I_b, \( b \neq 0 \) | logarithmic                | \( n + 1 \), \( n \) even |
| I_b* \( b \neq 0 \) | logarithmic                | \( n + 1 \), \( n \) even |
| II or II*    | conical                     | \( \beta = \frac{6k+1}{3} \), \( k \in \mathbb{Z} \) |
| III or III*  | conical                     | \( \beta = \frac{1}{3} + k \), \( k \in \mathbb{Z} \) |
| IV or IV*    | conical                     | \( \beta = \frac{6k+2}{3} \), \( k \in \mathbb{Z} \) |

Table 1: Singular fibers and corresponding singularities of affine special Kähler structures.

**Proof.** Let us justify the first row of Table 1; Other rows can be justified in a similar manner.

Thus, assume that the singular fiber is of type I_0. By the proof of Theorem 11.1 of [BHPV04] and inspection of Table 6 in the op.cit. we conclude that \( \text{tr} \text{Hol}(S^1, \nabla) = 2 \), which yields \( \cos \pi \beta = 1 \). □
Remark 42. It is quite plausible that the statement of the above proposition can be made stronger. For example, it seems likely that in the first two rows of Table 1 only the conical singularity can occur. Also, there should be a relation between $b$ and $n$ in the third and fourth rows. More importantly, the assumption that $\pi: X \setminus X_0 \to B^*_1$ can be extended to an elliptic fibration by adding a suitable central fiber may well follow from the other assumptions. We intend to address these points elsewhere.

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