Iterated Brownian motion ad libitum is not the pseudo-arc

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Abstract

We show that the construction of a random continuum $C$ from independent two-sided Brownian motions as considered in [11] almost surely yields a non-degenerate indecomposable but not-hereditary indecomposable continuum. In particular $C$ is (unfortunately) not the pseudo-arc.

1 Introduction

Iterated Brownian motions ad libitum. Let $(B_i)_{i \geq 1}$ be a sequence of i.i.d. two-sided Brownian motions (BM), i.e. $(B_i(t))_{t \geq 0}$ and $(B_i(-t))_{t \geq 0}$ are independent standard linear Brownian motions started from 0. The $n$th iterated BM is

$$I^{(n)} = \circled{B_1 \cdots B_n}. \quad (1)$$

The doubly iterated Brownian motion $I^{(2)}$ has been deeply studied in the 90’s. It permits to construct solutions to partial differential equations [9] and lots of results about its probabilistic and analytic properties can be found in [1, 4, 5, 8, 10, 16, 17] and references therein. Of course $I^{(n)}$ is wilder and wilder as $n$ increases (see Figure 1) but in [7], second author and Konstantopoulos proved that the occupation measure of $I^{(n)}$ over $[0, 1]$ converges as $n \to \infty$ towards a random probability measure $\Xi$ which can be though of as iterated Brownian motions ad libitum. This object has then been studied in [6] by the first author and Marckert, and they gave a description of $\Xi$ using invariant measure of an iterated functions system (IFS). However, many distributional properties of $\Xi$ remain open.

Continuum and pseudo-arc. In a recent work, Kiss and Solecki used iterated Brownian motions to define a random continuum. Recall that a continuum is a nonempty, compact, connected metric space. They were interested by the so-called pseudo-arc. The pseudo-arc is a homogeneous continuum which is similar to an arc, so similar, that its existence was unclear in the beginning of the last century. A continuum $C$ is

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Figure 1: Simulations of $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$, the first three iteration of independent two-
sided Brownian motions. The article studies random continuum build out the sequence
of $(I^{(n)} : n \geq 1)$.

- **chainable** (also called arc-like, see [15, Theorem 12.11]), if for each $\varepsilon > 0$, there exists a con-
tinuous function $f : C \to [0, 1]$ such that the pre-images of points under $f$ have diameter less than $\varepsilon$.

- **decomposable**, if there exist $A$ and $B$ two subcontinua of $C$ such that $A, B \neq C$ and $C = A \cup B$. A non decomposable continuum is called indecomposable.

- **hereditarily indecomposable** if any of its subcontinuum (non reduced to a singleton) is inde-
composable.

By [3], the pseudo-arc is the unique (up to homeomorphisms) chainable and hereditarily in-
decomposable continuum non reduced to a singleton. In particular, any subcontinuum (non reduced to a singleton) of a pseudo-arc is a pseudo-arc. Its name “pseudo-arc” comes from this property because arcs have the same property, in the sense that any subcontinuum (non reduced to a singleton) of an arc is an arc. For more information on pseudo-arc, we refer the interested reader to the second paragraph of [15, Chapter XII] and to [2, 3, 12, 13]. Sadly, it is very complicated to get a “drawing” of the pseudo-arc due to its complicated crocked structure, see [15, Exercise 1.23]. Following the works of Bing, one can wonder whether the pseudo-arc is typical among arc-like continua and ask whether there is a natural probabilistic construction of the pseudo-arc.

Let us recall the construction of continua from inverse limits used in [11], see [15, Section II.2] for details. Suppose we are given a sequence

$$\ldots \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$$

where for any $i \geq 1$, the metric space $(X_i, d_i)$ is compact and $f_i : X_{i+1} \to X_i$ is a continuous surjective function. Then the inverse limit of $(\{X_i, f_i\})_{i \geq 1}$ is the subspace of $\prod_{i \geq 1} X_i$ defined by

$$\lim \lim_{i \to \infty} (f_i, X_i : i \geq 1) = \left\{ (x_i)_{i \geq 1} \in \prod_{i \geq 1} X_i : f_i(x_{i+1}) = x_i \right\}.$$  \hspace{1cm} (2)

In the application below $X_i$ are compact intervals of $\mathbb{R}$ and in this case, by [15, Theorems 2.4
and 12.19], the inverse limit is a chainable continuum. In [11], Kiss and Solecki constructed a
system as above using two-sided independent Brownian motions \((\mathcal{B}_i : i \geq 1)\). More precisely, they proved that for any interval \(J\) of \(\mathbb{R}\) with \(0 \in J\) and \(J \neq \{0\}\), the following limit exists almost surely
\[
\mathcal{I}_i = \lim_{m \to \infty} \mathcal{B}_i (\mathcal{B}_{i+1} (\ldots (\mathcal{B}_{i+m} (J)) \ldots)), \quad (3)
\]
and does not depend on \(J\), so that we can consider the random chainable continuum \(C\) obtained as the inverse limit of the system
\[
\ldots \xrightarrow{\mathcal{B}_3} \mathcal{I}_3 \xrightarrow{\mathcal{B}_2} \mathcal{I}_2 \xrightarrow{\mathcal{B}_1} \mathcal{I}_1.
\]

Kiss and Solecki proved \([11, \text{Theorem 2.1}]\) that the random chainable continuum \(C\) is almost surely non-degenerate and indecomposable. This note answers negatively the obvious question the preceding result triggers:

**Theorem 1.** Almost surely, the random continuum \(C\) is not hereditary indecomposable (hence is not the pseudo-arc).

The proof below could be adapted to prove that a random continuum constructed similarly from a sequence of i.i.d. reflected Brownian motions is neither a pseudo-arc, answering a question in \([11, \text{Section 3.1.1}]\). Although almost surely not homeomorphic to the pseudo-arc, the random continuum \(C\) is interesting in itself and one could ask about its topological property, e.g. we wonder whether the topology of \(C\) is almost surely constant and if it is easy to characterise.

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2 Finding good intervals

In the rest of the article the Brownian motions \(\mathcal{B}_i\) are fixed and we recall the definition of \(\mathcal{I}_i\) in \((3)\) and of the continuum \(C\). We will show that Theorem 1 follows from the proposition below stated in terms of images of intervals under the flow of independent Brownian motions whose proof occupy the remaining of the article:

**Proposition 2.** For any \(\varepsilon > 0\) small enough, with probability at least
\[
p_\varepsilon = \prod_{i=1}^{\infty} 1 - 2 \left( \varepsilon^{(5/4)^{i-1}} \right)^{1/8} > 0,
\]
there exists two sequences \((U_i)_{i \geq 1}\) and \((V_i)_{i \geq 1}\) of subintervals of \(\mathbb{R}\) such that, for any \(i \geq 1\), the five following conditions are satisfied

1. \(U_i, V_i \subset \mathcal{I}_i\) where \(\mathcal{I}_i\) is defined in \((3)\).
2. \(U_i \nsubseteq V_i\) and \(V_i \nsubseteq U_i\),
3. \( U_i \cap V_i \neq \emptyset \),
4. \( U_i = \mathcal{B}_i(U_{i+1}) \) and \( V_i = \mathcal{B}_i(V_{i+1}) \),
5. \( |U_i|, |V_i| \leq \varepsilon^{(5/4)^{i-1}} \).

Proof of Theorem \([7]\) Given Proposition \(2\). In the proof, since we are always working with the functions \( \mathcal{B}_i \), we write \( \lim (W_i : i \geq 1) \) for the inverse limit previously denoted by \( \lim (\mathcal{B}_i, W_i : i \geq 1) \) for any sequence of intervals \( W_1, W_2, ... \) such that \( W_{i+1} \xrightarrow{\mathcal{B}_i} W_i \). On the event described in the above proposition we have with probability at least \( p_\varepsilon > 0 \):

- For any \( i \geq 1 \), \( \mathcal{B}_i(U_{i+1} \cup V_{i+1}) = U_i \cup V_i \) (point 4) and \( U_i \cup V_i \subseteq \mathcal{I}_i \) (point 1) and \( U_i \cup V_i \) is an interval (point 3), so by Lemma 2.6 of \([15]\), \( \lim (U_i \cup V_i : i \geq 1) \) is a subcontinuum of \( \mathcal{C} \).
- By Lemma 2.6 of \([15]\), both \( \lim (U_i : i \geq 1) \) and \( \lim (V_i : i \geq 1) \) are also subcontinua of \( \lim (U_i \cup V_i : i \geq 1) \).
- Let \( x = (x_i)_{i \geq 1} \in \lim (U_i \cup V_i : i \geq 1) \), then
  - either, for any \( i \), we have \( x_i \in U_i \cap V_i \), and so \( x \in \lim (U_i : i \geq 1) \) and \( x \in \lim (V_i : i \geq 1) \),
  - or there exists \( j \geq 1 \) such that \( x_j \in U_j \) and \( x_j \notin V_j \), but then by point 4 we have \( x_j \in U_j \) for all \( i \geq j \) and so \( x \in \lim (U_i : i \geq 1) \),
  - or there exists \( j \geq 1 \) such that \( x_j \notin U_j \) and \( x_j \in V_j \) and similarly we deduce that \( x \in \lim (V_i : i \geq 1) \).

Hence, \( \lim (U_i \cup V_i : i \geq 1) \subseteq \lim (U_i : i \geq 1) \cup \lim (V_i : i \geq 1) \) and the reverse inclusion is obvious.

- \( \lim (U_i \cup V_i : i \geq 1) \neq \lim (U_i : i \geq 1) \) nor \( \lim (U_i \cup V_i : i \geq 1) \neq \lim (V_i : i \geq 1) \) by combining point 2 and point 4.

All of these points imply that \( \lim (U_i \cup V_i : i \geq 1) \) is a decomposable subcontinuum of \( \mathcal{C} = \lim (\mathcal{I}_i : i \geq 1) \). That implies that \( \mathcal{C} \) is not a pseudo-arc with probability at least \( p_\varepsilon \) for any \( \varepsilon > 0 \). As \( p_\varepsilon \to 1 \) when \( \varepsilon \to 0 \), it is not a pseudo-arc with probability one.

2.1 Construction of a decomposable subcontinuum using good shape excursions

Let us now explain the idea behind the construction of the intervals of Proposition \(2\). This relies on the concept of excursions with a good shape. Imagine that we have a sequence of non trivial intervals \( [u_i, v_i] \subset [0, 1] \) such that \( \mathcal{B}_i([u_{i+1}, v_{i+1}]) = [u_i, v_i] \) and furthermore that \( \mathcal{B}_i(u_{i+1}) = u_i \) and \( \mathcal{B}_i(v_{i+1}) = v_i \) and \( \mathcal{B}_i(t) \in (u_i, v_i) \) for \( t \in (u_{i+1}, v_{i+1}) \). In words, over the time interval \( [u_{i+1}, v_{i+1}] \), the Brownian motion \( \mathcal{B}_i \) makes an excursion from \( u_i \) to \( v_i \). We say that this excursion has a good shape if it stays in the pentomino of Figure \(2\).
Figure 2: An excursion from $u_i$ to $v_i$ over the time interval $[u_{i+1}, v_{i+1}]$ has a good shape if it stays in the light grey region.

If we have such a sequence of intervals and excursions, then one can define a sequence of intervals $U_i, V_i$ by setting for any $i \geq 1$,

$$U_i = \lim_{n \to \infty} \left( \mathcal{B}_i \circ \mathcal{B}_{i+1} \circ \cdots \circ \mathcal{B}_{i+n-1} \right) \left( \left[ u_{i+n}, \frac{u_{i+n} + 2v_{i+n}}{3} \right] \right)$$

and

$$V_i = \lim_{n \to \infty} \left( \mathcal{B}_i \circ \mathcal{B}_{i+1} \circ \cdots \circ \mathcal{B}_{i+n-1} \right) \left( \left[ \frac{2u_{i+n} + v_{i+n}}{3}, v_{i+n} \right] \right).$$

First, these two limits exist a.s. and are closed intervals a.s. because they are limits of a sequence of decreasing closed intervals. Indeed, because $\mathcal{B}_{i+n}$ performs a good shape excursion from $u_{i+n}$ to $v_{i+n}$ over $[u_{i+n+1}, v_{i+n+1}]$ we have

$$\mathcal{B}_{i+n} \left( \left[ u_{i+n+1}, \frac{u_{i+n+1} + 2v_{i+n+1}}{3} \right] \right) \subset \left[ u_{i+n}, \frac{u_{i+n} + 2v_{i+n}}{3} \right],$$

and so $U_{i,n+1} \subset U_{i,n}$, and $U_{i,n}$ are intervals because the BM is continuous a.s. It is then an easy matter to check that the interval constructed above satisfies points 2-4 of Proposition 2. Our task is thus to construct the sequence $u_i, v_i$ so that $\mathcal{B}_i$ performs a good shape excursion from $u_i$ to $v_i$ over $[u_{i+1}, v_{i+1}]$ and to ensure points 1 and 5 of Proposition 2. The key idea is to look for these intervals in the vicinity of 0 because any given small interval close to 0 has MANY pre-images close to 0 by a Brownian motion. These many pre-images enable us to select one with a good shape.
2.2 Pre-images of a small interval by a Brownian motion

In the following lemma the dependence in $i$ is superfluous but we keep it to make the connection with the preceding discussion easier to understand.

**Lemma 3.** Let $a_i$ be any real positive number small enough. Fix $[u_i, v_i] \subset [0, a_i]$. Then with probability at least

$$1 - 2a_i^{1/8}$$

we can find $[u_{i+1}, v_{i+1}] \subset [0, a_i^{5/4}]$ so that $B_i$ performs an excursion with a good shape from $u_i$ to $v_i$ over the time interval $[u_{i+1}, v_{i+1}]$.

**Proof.** Fix $0 < u_i < v_i$ and consider the successive excursions $E_1, E_2, \ldots$ that the Brownian motion $B_i$ performs from $u_i$ to $v_i$ over the respective time intervals $[u_{i+1}^{(1)}, v_{i+1}^{(1)}], [u_{i+1}^{(2)}, v_{i+1}^{(2)}], \ldots$. By the Markov property of Brownian motion and standard argument in excursion theory, these excursions are i.i.d. We claim that

$$r = \mathbb{P}(E \text{ has a good shape}) > 0.$$  

Indeed, since the law of Brownian motion has full support in the space of continuous functions (with the topology of uniform convergence over all compacts of $\mathbb{R}_+$), the first excursion from $u_i$ to $v_i$ might be close to any prescribed continuous function and in particular, the probability to have a good shape is strictly positive. See Figure 3.

**Figure 3:** For any given continuous function $f$ starting from 0 and any $\epsilon > 0$, the Brownian motion may stay within distance $\epsilon > 0$ of $f$ up to time 1 with a positive probability. Choosing $f$ carefully, we deduce that the first excursion from $u_i$ to $v_i$ has a good shape with positive probability.

Hence, the probability that at least one of the $k$ first excursions has a good shape is at least

$$1 - (1 - r)^k.$$
Hence \( w_{i+1}^{(1)} < v_{i+1}^{(1)} < w_{i+1}^{(2)} < v_{i+1}^{(2)} < \cdots \) are the successive hitting times of \( u_i, v_i, u_i, v_i \) by \( \mathcal{B}_i \), see Figure 4. For \( a \geq 0 \), we let \( \mathcal{T}_a = \inf \{ t \geq 0 : \mathcal{B}_i(t) = a \} \) the hitting time of \( a \) by a standard linear Brownian motion. It is classic (see e.g. [14, Theorem 2.35]) that for \( a > 0 \) we have \( \mathcal{T}_a = a^2 \cdot \mathcal{T}_1 \) in law where \( \mathcal{T}_1 \) is distributed according to the Lévy law

\[
\mathcal{T}_1 = \frac{dt}{(d) \sqrt{2\pi t^3}} \exp \left(-\frac{1}{2t}\right) 1_{t>0}.
\]

In our case, applying the strong Markov property at time \( w_{i+1}^{(1)} < v_{i+1}^{(1)} < w_{i+1}^{(2)} < v_{i+1}^{(2)} < \cdots \) and using invariance by symmetry we deduce that we have the equalities in distribution

\[
w_{i+1}^{(1)} \overset{(d)}{=} \mathcal{T}_{u_i}, \quad v_{i+1}^{(1)} \overset{(d)}{=} \mathcal{T}_{u_i + |v_i - u_i|}, \quad w_{i+1}^{(2)} \overset{(d)}{=} \mathcal{T}_{u_i + 2|v_i - u_i|}, \quad \cdots, \quad v_{i+1}^{(k)} \overset{(d)}{=} \mathcal{T}_{u_i + (2k-1)|v_i - u_i|},
\]

for \( k \geq 2 \). Since \( \mathcal{T}_{u_i + (2k-1)|v_i - u_i|} \leq \mathcal{T}_{2ka_i} \), the probability that the first \( k \) excursions of \( \mathcal{B}_i \) occurs before \( a_i^{5/4} \) is at least

\[
P\left( \mathcal{T}_{2ka_i} < a_i^{5/4} \right) = P\left( \mathcal{T}_1 < \left( \frac{1}{2k a_i^{3/8}} \right) \right) \geq 1 - \sqrt{2 \pi} 2ka_i^{3/8} \text{ (for } ka_i^{3/8} \text{ small enough)}.
\]

Gathering-up the above remarks and taking \( k = \lfloor a_i^{-1/4} \rfloor \), we deduce that the probability to do not find an excursion from \( u_i \) to \( v_i \) with a good shape in \([0, a_i^{5/4}]\) is bounded above by

\[
(1 - r) a_i^{-1/4} + 2 \sqrt{2 \pi} |a_i^{-1/4}| a_i^{3/8} \leq 2a_i^{-1/8} \text{ (for } a_i \text{ small enough)}.
\]
Let \((B_i)_{i \geq 1}\) be a sequence of i.i.d. two-sided Brownian motions, and \(\varepsilon\) be any real positive number small enough. For any \(i \geq 1\), take \(a_i = \varepsilon^{(5/4)^{i-1}}\).

Firstly, we put \([u_1, v_1] = [0, \varepsilon] = [0, a_1]\), by Lemma 3 with probability at least \(1 - 2^{a_1/8}\), there exists an interval \([u_2, v_2] \subset [0, a_1^{5/4}] = [0, a_2]\) such that \(B_1\) performs a good shape excursion from \(u_1\) to \(v_1\) over the time interval \([u_2, v_2]\). Now, we apply Lemma 3 to \([u_2, v_2] \subset [0, a_2]\), etc.

At the end, with probability at least \(\prod_{i=1}^{\infty} 1 - 2 \left(\varepsilon^{(5/4)^{i-1}}\right)^{1/8}\), we obtain a sequence of non trivial intervals \(([u_i, v_i])_{i \geq 1}\) such that for any \(i\), \(B_i\) makes a good shape excursion from \(u_i\) to \(v_i\) over \([u_i+1, v_i+1]\). By Section 2.1, we can then construct two sequences of intervals \(U_i, V_i\) that satisfy points 2-4 of Proposition 2. Moreover, by construction, \(U_i, V_i \subset [u_i, v_i] \subset [0, a_i]\), hence point 5 is also satisfied.

Finally, to obtain point 1, just remark that, for any \(i, n \geq 1\), \([u_{i+n}, v_{i+n}] \subset [0, a_{i+n}] \subset [0, 1]\), so

\[
U_i = \lim_{n \to \infty} (B_i \circ B_{i+1} \circ \cdots \circ B_{i+n}) \left(\left[u_{i+n+1}, \frac{u_{i+n+1} + 2v_{i+n+1}}{3}\right]\right)
\subseteq \lim_{n \to \infty} (B_i \circ B_{i+1} \circ \cdots \circ B_{i+n}) ([0, 1]) = I_i \quad \text{(by (3)).}
\]

Similarly, \(V_i \subset I_i\).

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