The dissipative linear Boltzmann equation for hard spheres

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Abstract

We prove the existence and uniqueness of an equilibrium state with unit mass to the dissipative linear Boltzmann equation with hard–spheres collision kernel describing inelastic interactions of a gas particles with a fixed background. The equilibrium state is a universal Maxwellian distribution function with the same velocity as field particles and with a non–zero temperature lower than the background one, which depends on the details of the binary collision. Thanks to the $H$-theorem we then prove strong convergence of the solution to the Boltzmann equation towards the equilibrium.

Key words. Granular gases, equilibrium state, linear Fokker–Planck equation, trend to equilibrium.

1 Introduction

The present paper deals with the linear dissipative Boltzmann equation for hard spheres interactions, and solves some questions left open in the very recent work by Toscani and Spiga [26], devoted to the investigation of a linear dissipative Boltzmann equation for Maxwell molecules.

In the last years, the study of kinetic models for granular flows received a significant interest. The largest part of this work deals with kinetic nonlinear models based upon generalizations of the Boltzmann–Enskog equation. We refer the interested reader to the review articles [11, 18, 19]. However, most of the studies refer to inelastic Maxwell particles, both for the driven case [4, 9] or for the free case [3, 5, 6]. Such (pseudo–)Maxwellian models enjoy nice mathematical simplifications and lead to exact analytical results [14, 15] (see also the recent developments on the inelastic Kac model [24, 25]). At present, only a few papers consider real interactions among grains, which are well described by the inelastic hard–spheres models [7, 17].

Despite their importance for practical applications, linear equations for dissipative models have been much less addressed. To our knowledge the only progresses on the matter are those of the afore–mentioned paper [26] and of R. Pettersson [23]. Linear models describe the time evolution of the distribution function $f(x, v, t)$ of particles of masses $m$ (representing the granular gas) colliding inelastically with particles with masses $m_1$ of a fixed background. Throughout this paper, the subscript $(1)$ will be addressed to the fixed field particles whose distribution function is known and is assumed to be a normalized Maxwellian $M_1$ with given mass velocity and temperature. Note that, the grains being cohesionless, long–range interactions of any kind are irrelevant. Thus, the only model with real physical interest is the hard–spheres model. As introduced in [26], the evolution of $f(x, v, t)$ is given by

$$
\frac{\partial f}{\partial t}(v, t) + v \cdot \nabla_x f(x, v, t) = \frac{1}{2\pi \lambda} \int_{\mathbb{R}^3} |q \cdot n| \left[ \frac{1}{\epsilon} f(v_*) M_1(w_*) - f(v) M_1(w) \right] dw dn.
$$

Here $\lambda$ denotes the constant mean free path, $q$ is the relative velocity, $q = v - w$. The velocities $(v_*, w_*)$ are the pre–collisional velocities of the so–called inverse collision, which results in $(v, w)$ as post-collisional velocities. In the granular setting, the most important feature of the collision
mechanism is its inelastic character which induces that (generally) the total kinetic energy is dissipated. The constant parameter \(0 < \epsilon < 1\) is called the restitution coefficient and measures the inelasticity of the collisions. Whenever \(\epsilon = 1\) we recover the usual linear Boltzmann equation (see Section 2 for details).

One of the main features of this paper is to prove the existence and uniqueness of the (homogeneous) equilibrium state of equation (1.1). Precisely, we exhibit a (non-trivial) distribution function \(M\) (depending on the velocity only) such that

\[
Q(M) = 0
\]

where \(Q\) denotes the right-hand side operator of (1.1). When \(\epsilon = 1\), this question is trivial since the conservation of momentum and energy implies

\[
\tilde{M}(v)M_1(w) = \tilde{M}(v_*)M_1(w_*)
\]

where \(\tilde{M}\) stands for the Maxwellian distribution with same drift velocity and temperature as \(M_1\) but corresponding to particles of mass \(m\) (see Section 2 for more details). Then, one sees immediately that the integrand of \(Q\) vanishes for \(f = \tilde{M}\). Clearly, for \(0 < \epsilon < 1\) this is no more the case and it appears difficult to determine explicitly the (eventual) equilibrium state. Actually, the two following questions are far from being trivial:

- Does an equilibrium state exist?
- If it does, is it given by some suitable Maxwellian distribution?

In this paper, we answer positively to both questions showing that the equilibrium state is a Maxwellian distribution with the same mean velocity as \(M_1\) and with a universal non-zero temperature lower than the given background temperature. Moreover, this equilibrium state is unique, provided its mass is prescribed. This Maxwellian distribution coincide with the one derived in the pseudo–Maxwellian case [20]. Actually, it is also possible to show that, as in the non–dissipative case, the equilibrium state is universal in the sense that its temperature does not depend on the collision kernel, but only on the inelasticity parameter \(\epsilon\) which determines the details of the binary interaction. Let us mention here that our results answer to some open questions from [26] and complete the study of [23] where the existence of an equilibrium state was used as an assumption for some of the results.

The two previous questions, as well as the problem of the rate of convergence towards equilibrium, have been recently addressed in [26] for the pseudo-Maxwellian approximation. This pseudo-Maxwellian approximation consists in replacing the relative velocity \(q\) appearing in the collision kernel \(|q \cdot n|\) of \(Q\) by the unit vector in the direction of \(q\). The pseudo-Maxwellian model enjoys in particular two fundamental properties. First, the associated moment equations are closed with respect to the moments of the distribution function. Hence, it is possible to derive the time evolution of the drift velocity \(u(t)\) and the temperature \(T(t)\) of \(f(v, t)\) and to predict the mass velocity and temperature of the eventual equilibrium state. Moreover, as pointed out by A. Bobylev [2], Maxwell models lend themselves to a convenient Fourier analysis. These two important properties enabled to determine the Maxwellian equilibrium state for the pseudo-Maxwellian model and to prove also exponential convergence of the solution to (1.1) towards the equilibrium (in the homogeneous setting).

Unfortunately, these two tools do not apply to the hard–spheres model (1.1) and one has to proceed in a different way. The main problem is actually to predict what should be the eventual steady-state. To this aim, we derive formally a linear Fokker–Planck equation which is naturally associated to the dissipative Boltzmann model (Section 3) through the asymptotic of the grazing collisions (see e.g. [10, Chapter II.9]). There is a noticeable amount of results on the matter for the elastic (nonlinear) Boltzmann equation. We mention here the papers [28, 29, 12, 13] which enlighten the connection between the nonlinear Boltzmann equation and the Landau–Fokker–Planck equation and the works [16, 21] which describe the same procedure for linear problems. We emphasize the fact that the main goal of this analysis is not to prove rigorously any kind of asymptotic procedure, even if it is possible to make our results rigorous (see Appendix). The method of Section 3 must only be viewed as a formal (but efficient) way to find a suitable approximation of the
collision operator $Q$ which maintains the equilibrium distribution. To our knowledge, it is the first time (in kinetic theory) that such a limiting process is performed within this scope. The utility of this procedure comes out from the fact that the equilibrium state of the Fokker–Planck operator is immediate to obtain. It is a Maxwellian distribution which appears then as the candidate to be the stationary solution of (1.1). The main problem is then to show that this Maxwellian is effectively a steady state for $Q$.

This will be done by means of Fourier transform arguments.

A second important task addressed in this paper is the large–time behavior of the solution to the space homogeneous version of (1.1). Actually, once the existence and uniqueness of an equilibrium state established, the question of trend towards this equilibrium is of primary importance in kinetic theory. In this paper, we are able to show a strong $L^1$–convergence result to the Maxwellian steady–state. As in [23], our result is based upon weak–compactness arguments by means of the so–called $H$–theorem and some estimates on the second moment of the solution to (1.1).

Let us explain now in some details the organization of the paper. In Section 2, we describe briefly the dissipative Boltzmann linear model and its properties. In Section 3 we deal with the asymptotics of the grazing collision, from which we derive the Fokker–Planck approximation of $Q$ which help us to identify the (possible) equilibrium state. Then in Section 4 we show that the Maxwellian obtained through the grazing collisions procedure is really a stationary solution to (1.1). Thanks to the Boltzmann $H$–theorem we then prove in Section 5 that the equilibrium state is unique (provided its mass is prescribed) and that the solution to (1.1) converges (in the strong $L^1$–sense) towards the equilibrium. Finally, we end this paper by presenting some open questions and perspectives.

2 The dissipative linear Boltzmann equation

As briefly described in the introduction, in this paper we are concerned with the evolution of the distribution function $f(v, t)$ of granular gas particles with masses $m$ which undergo inelastic collisions with the field particles (of masses $m_1$) of a fixed background. The background is supposed to be at thermodynamical equilibrium with given temperature $T_1$ and given mass velocity $u_1$, i. e. its distribution function is the normalized Maxwellian

$$M_1(v) = \left( \frac{m_1}{2\pi T_1} \right)^{3/2} \exp\left\{ -\frac{m_1(v - u_1)^2}{2T_1} \right\} \quad v \in \mathbb{R}^3.$$ 

The main feature of the binary dissipative collisions is that part of the normal relative velocity is lost in the interaction, so that

$$(v^* - w^*) \cdot n = -\epsilon (v - w) \cdot n,$$

where $n \in S^2$ is the unit vector in the direction of impact, $(v, w)$ stand for the velocities before impact whereas $(v^*, w^*)$ denote the post–collisional velocities. The so-called (constant) restitution coefficient $\epsilon$ is such that $0 < \epsilon < 1$, the case $\epsilon = 1$ corresponding to elastic collisions. Thanks to (2.1) and assuming the conservation of momentum

$$m v^* + m_1 w^* = m v + m_1 w$$

one then finds the collision mechanism

$$\begin{cases} v^* = v - 2\alpha (1 - \beta) [(v - w) \cdot n] n \\ w^* = w + 2(1 - \alpha)(1 - \beta) [(v - w) \cdot n] n, \end{cases}
$$

where $\alpha$ is the mass ratio and $\beta$ denotes the inelasticity parameter

$$\alpha = \frac{m_1}{m + m_1}, \quad \beta = \frac{1 - \epsilon}{2}.$$ 

The parameter $\alpha$ is such that $0 < \alpha < 1$ (we exclude the limiting cases of Lorentz and Rayleigh gases), while the inelasticity parameter satisfies $0 < \beta < 1/2$. We refer to [17] for a detailed
description of the geometry of the collisions. It is easy to see that system (2.2) is invertible and provides the pre-collisional velocities of the so-called inverse collisions, resulting in \((v, w)\) as post-collisional velocities
\[
\begin{align*}
\mathbf{v}_* &= \mathbf{v} - 2\alpha \frac{1 - \beta}{1 - 2\beta} (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n} \mathbf{n} \\
\mathbf{w}_* &= \mathbf{w} + 2(1 - \alpha) \frac{1 - \beta}{1 - 2\beta} (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n} \mathbf{n}.
\end{align*}
\]
In contrast to the elastic case \((\epsilon = 1)\), the binary collision dissipates part of the kinetic energy
\[
m|v^*|^2 + m_1|w^*|^2 - (m|v|^2 + m_1|w|^2) = -4\frac{mm_1}{m + m_1} \beta (1 - \beta) |q \cdot n|^2 \leq 0.
\]
In space homogeneous conditions, upon using \(\lambda\) as a time scale, equation (1.1) can be re-written in the dimensionless form
\[
\frac{\partial f}{\partial t}(v, t) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times S^2} |q \cdot n| \left[ \frac{1}{\epsilon^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] d\mathbf{w} d\mathbf{n}. \tag{2.3}
\]
The factor \(\epsilon^{-2}\) in the gain term above appears respectively from the Jacobian of the transformation \(d\mathbf{v}, d\mathbf{w}\) into \(d\mathbf{v} d\mathbf{w}\) and from the length of the cylinders \(|q \cdot n| = \epsilon |q \cdot n|\) (see [11] for details). Let \(\Omega\) be the (dissipative) linear Boltzmann collision operator (acting only on the velocity space)
\[
\Omega(f) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times S^2} |q \cdot n| \left[ \frac{1}{\epsilon^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] d\mathbf{w} d\mathbf{n} \tag{2.4}
\]
The change of variables \(n \rightarrow -n\) leads to the equivalent expression
\[
\Omega(f) = \frac{1}{\pi} \int_{\mathbb{R}^3 \times S^2} H(|q\cdot n|) q \cdot n \left[ \frac{1}{\epsilon^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] d\mathbf{w} d\mathbf{n},
\]
where \(H(\cdot)\) is the Heavyside step function. We can also define the collision operator by its action on the observables. Precisely, for any regular test–function \(\psi(v)\)
\[
\langle \psi, \Omega(f) \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |q \cdot n| f(v) M_1(w) [\psi(v^*) - \psi(v)] d\mathbf{v} d\mathbf{w} d\mathbf{n}. \tag{2.5}
\]
Clearly, \(\psi(v) \equiv 1\) is a collision invariant (mass conservation) whereas, in contrast to the elastic case, \(\psi(v) = v\) and \(\psi(v) = v^2\) are not (dispersion of kinetic energy). Note that an important feature of the hard–spheres model is that (even in the elastic case) the moments equations of \(\Omega(f)\) are not closed with respect to the ones of \(f\).

## 3 The Fokker–Planck approximation

The main goal of this section is the formal derivation of a linear Fokker–Planck equation, obtained from (2.3) through a kind of grazing collisions asymptotics. We point out that, our aim in this paper, is not to prove rigorously the convergence of the (re-scaled) dissipative Boltzmann operator \(\Omega\) towards the Fokker–Planck operator \(\mathcal{K}\) below as the collisions become grazing. The limiting process we perform here must only be seen as an efficient tool to predict the nature of the equilibrium state of \(\Omega\) (if it exists). Nevertheless, this approximation result can be made rigorous and this shall be done in the Appendix. Here we proceed only at a formal level. Let us assume that all the collisions concentrate around
\[
|q \cdot n|/|q| \sim 0. \tag{3.1}
\]
Consequently, according to (2.2) one has \(|v^* - v| \sim 0\) and, for any smooth function \(\psi\), one can perform a Taylor expansion of \(\psi(v^*)\) around \(v\) leading, at the second order, to:
\[
\psi(v^*) = \psi(v) + \nabla_v \psi(v) \cdot (v^* - v) + \frac{1}{2} \nabla^2 \psi(v)(v^* - v) \otimes (v^* - v) + o(|v^* - v|^2)
\]
\[
= \psi(v) - 2\alpha(1 - \beta)(q \cdot n) \nabla_v \psi(v) \cdot n + \frac{2\alpha(1 - \beta)q \cdot n}{2} \nabla^2 \psi(v) \cdot n \otimes n + o(|q \cdot n|/|q|^2). \tag{3.2}
\]
In \( D^2 \psi \) is the Hessian matrix of \( \psi \). The \( o(|v^* - v|^2) \) term will be neglected in the sequel. One clearly observes that the expansion \( D^2 \psi \) is similar to that obtained in the study of elastic collisions between particles of same masses \( \text{[21]} \). This property obviously comes from the fact that \( D^2 \psi \) implies also \( v' \sim v \) where \( v' \) is the post-collisional velocity in the elastic case. The only difference is that, in the classical elastic theory, the multiplicative constant \( 2\alpha(1-\beta) \) is taken to be equal to \( 1/2 \). Thus, staying at a formal level, \textit{dissipative collisions between particles of unequal masses does not lead to supplementary difficulties.} 

Let us consider a referential frame with the \( x \)-axis directed along \( q \). Then,

\[
\begin{align*}
\left\{ \begin{array}{l}
n = (\cos \theta, \sin \theta \cos \xi, \sin \theta \sin \xi) \quad 0 \leq \theta \leq \pi/2, \ 0 \leq \xi \leq 2\pi, \\
\cos \theta = \frac{|q \cdot n|}{|q|} \quad \text{and} \quad dn = \sin \theta d\theta d\xi.
\end{array} \right.
\end{align*}
\] (3.3)

Assuming that the collisions concentrate around \( \theta \sim \pi/2 \), we define

\[
b_\delta(\theta) = \frac{2}{\pi} \chi_{[\pi/2-\delta,\pi/2]}(\theta) \quad (\delta > 0)
\]

and

\[
I_\delta = \int_0^{\pi/2} b_\delta(\theta) \cos^3 \theta \sin \theta \, d\theta.
\]

One sees that

\[
I_\delta \sim \frac{\delta^4}{2\pi} \quad \text{as} \quad \delta \sim 0.
\] (3.4)

Consequently, let us define the associated collision kernel

\[
B_\delta(q, n) = b_\delta(\theta) |q \cdot n|,
\]

and denote by \( Q_\delta \) the collision operator obtained by replacing \( |q \cdot n| \) by \( \delta^{-4} B_\delta(q, n) \) in \( \text{[26]} \).

**Remark 3.1.** Note that the introduction of the multiplicative factor \( \delta^{-4} \) can be seen as a suitable \textit{time-scaling} in \text{[26]} (see Appendix \textit{6} for further details).

Our aim is to show that for small values of \( \delta \), the (re-scaled) operator \( Q_\delta \) is closed to be a Fokker–Planck collision operator. Precisely, let us fix \( f \in L^1(\mathbb{R}^3) \) and a smooth test–function \( \psi(v) \). Using \( Q_\delta \) into \text{[26]} leads to

\[
\langle \psi, Q_\delta(f) \rangle = \frac{1}{2\pi \delta^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} B_\delta(q, n) f(v) M_1(w) [\psi(v^*) - \psi(v)] \, dv \, dw \, dn.
\]

Now, inserting the expansion \text{[26]} in the above expression gives the second order approximation

\[
\langle \psi, Q_\delta(f) \rangle = J_1^\delta + J_2^\delta = \\
= -\frac{2\alpha(1-\beta)}{2\pi \delta^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} B_\delta(q, n)(q \cdot n) f(v) M_1(w) \nabla \psi(v) \cdot n \, dv \, dw \, dn \\
+ \frac{(2\alpha(1-\beta))^2}{4\pi \delta^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} B_\delta(q, n) (q \cdot n)^2 f(v) M_1(w) D^2 \psi(v) \cdot n \otimes n \, dv \, dw \, dn. \quad (3.5)
\]

To estimate \( J_1^\delta \), we first compute the integral with respect to \( dn \). According to \text{[26]}

\[
\int_{\mathbb{S}^2} B_\delta(q, n)(q \cdot n)dn = 2\pi |q| \int_0^{\pi/2} B_\delta(q, n)(\cos^2 \theta, 0, 0) \sin \theta d\theta
= 2\pi |q|^2 \int_0^{\pi/2} b_\delta(\theta)(\cos^3 \theta, 0, 0) \sin \theta d\theta
= 2\pi |q|^2 (I_\delta, 0, 0) = 2\pi I_\delta |q|.
\]

Therefore

\[
J_1^\delta = -I_\delta \frac{2\alpha(1-\beta)}{\delta^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^2 f(v) M_1(w) \nabla \psi(v, t) \cdot \frac{q}{|q|} \, dv \, dw.
\]
Thanks to (3.4), the coefficient in front of the above integral converges to $-\alpha(1 - \beta)/\pi$ as $\delta$ goes to 0. Consequently
\[ J_3 \simeq -\frac{\alpha(1 - \beta)}{\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q| f(v) M_1(w) \nabla_v \psi(v,t) \frac{q}{|q|} \mathrm{d}v \mathrm{d}w. \quad (3.6) \]

We proceed in the same way for $J_3^1$. One has first
\[ \int_{S^2} B_\delta(q,n) (q \cdot n)^2 n \otimes n \mathrm{d}n = |q|^3 \int_0^{\pi/2} b_\delta(\theta) \cos^3 \theta \sin \theta \mathrm{d}\theta \int_0^{2\pi} n \otimes n \mathrm{d}\xi = 2\pi |q|^3 \int_0^{\pi/2} b_\delta(\theta) \cos^3 \theta \sin \theta \frac{\cos^2 \theta}{\pi} \sin^2 \theta \mathrm{d}\theta, \quad (3.7) \]

where $\text{Diag}[a_1, a_2, a_3]$ is the diagonal matrix in $\mathbb{R}^3 \times \mathbb{R}^3$ whose diagonal entries are $a_i$ ($i = 1, 2, 3$).

Now, defining
\[ K_\delta = \int_0^{\pi/2} b_\delta(\theta) \cos^3 \theta \sin \theta \mathrm{d}\theta, \]

one gets
\[ \int_{S^2} B_\delta(q,n) (q \cdot n)^2 n \otimes n \mathrm{d}n = 2\pi |q|^3 \text{Diag}[K_\delta, \frac{1}{2}(I_\delta - K_\delta), \frac{1}{2}(I_\delta - K_\delta)], \]

and
\[ J_3^1 = \frac{(2\alpha(1 - \beta))^2}{2\beta^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^3 f(v) M_1(w) \mathbb{D}^2 \psi(v,t) \cdot \text{Diag}[K_\delta, \frac{1}{2}(I_\delta - K_\delta), \frac{1}{2}(I_\delta - K_\delta)] \mathrm{d}v \mathrm{d}w. \]

Since $K_\delta$ is negligible with respect to $I_\delta$ one concludes with the following approximation
\[ J_3^1 \simeq \frac{\alpha^2(1 - \beta)^2}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^3 f(v) M_1(w) \mathbb{D}^2 \psi(v) \cdot \text{Diag}[0,1,1] \mathrm{d}v \mathrm{d}w. \quad (3.8) \]

For any $z \in \mathbb{R}^3$ ($z \neq 0$), let $S(z)$ be the symmetric matrix
\[ S(z) = \text{Id} - \frac{z \otimes z}{|z|^2}, \]

i.e. $S(z)$ is the projection on the space orthogonal to $z$. Then, combining (3.5), (3.6) and (3.8), one obtains for $\Omega_\delta$ the approximation
\[ \langle \psi, \Omega_\delta(f) \rangle \simeq -\frac{\alpha(1 - \beta)}{\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^2 f(v) M_1(w) \nabla_v \psi(v) \cdot \frac{q}{|q|} \mathrm{d}v \mathrm{d}w \]
\[ + \frac{\alpha^2(1 - \beta)^2}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^3 f(v) M_1(w) \mathbb{D}^2 \psi(v) \cdot S(v-w) \mathrm{d}v \mathrm{d}w. \]

Now, straightforward computations, using the fact that $2|q|q = \text{Div}_v(|v-w|^3S(v-w))$, yield
\[ \langle \psi, \Omega_\delta(f) \rangle \simeq -\frac{1}{2\pi} \int_{\mathbb{R}^3} \mathrm{d}v \nabla_v \psi(v) \int_{\mathbb{R}^3} |v-w|^3 S(v-w) \{ \kappa M_1(w) \nabla_v f(v) + (\kappa - \mu) f(v) \nabla_w M_1(w) \} \mathrm{d}w \]

where the parameters $\kappa$ and $\mu$ are defined respectively by
\[ \kappa = \alpha^2(1 - \beta)^2; \quad \mu = \alpha(1 - \beta). \]

Since the above approximation is valid for any arbitrary smooth function $\psi$, one sees that, as $\delta$ goes to 0, $\Omega_\delta$ can be approximated (up to the constant $1/2\pi$) by the Fokker–Planck operator
\[ \Omega_{FP}(g)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} |v-w|^3 S(v-w) \cdot \{ \kappa M_1(w) \nabla_v g(v) + (\kappa - \mu) g(v) \nabla_w M_1(w) \} \mathrm{d}w. \]
Let us write $Q_{FP}$ in a nicer way. Using the fact that $S(v - w) \cdot (v - w) = 0$ and

$$\nabla_w M_1(w) = -\frac{m_1(w - u_1)}{T_1} M_1(w),$$

one has

$$Q_{FP}(g)(v) = \kappa \nabla_v \cdot \int_{\mathbb{R}^3} |v - w|^3 M_1(w) S(v - w) \cdot \left\{ \nabla_v g(v) - \frac{m_1(\kappa - \mu)}{\kappa T_1} (v - u_1) g(v) \right\} \, dw$$

$$= \kappa \nabla_v \cdot \left[ A(v) \cdot \left\{ \nabla_v g(v) + \frac{m_1(\mu - \kappa)}{\kappa T_1} (v - u_1) g(v) \right\} \right],$$

where $A(v)$ denotes the invertible matrix

$$A(v) = \int_{\mathbb{R}^3} |v - w|^3 M_1(w) S(v - w) \, dw.$$

**Remark 3.2.** In a different spirit, J. J. Brey et al. [8] derived a linear Fokker–Planck equation from (2.3) in the limit of small mass ratio ($\alpha \to 0$). We also point out that it is possible to consider (see e.g. [27]) the quasi–elastic approximation of $Q$ assuming that $\beta \ll 1$. We adopt here the grazing collisions asymptotic since it preserves the parameters $\alpha$ and $\beta$ (and therefore the elasticity) modifying only the geometry of the collisions. Note however that the existence of an equilibrium state for the linear quasi–elastic approximation of $Q$ is an open problem to our knowledge.

Now, one recognizes immediately that the above procedure preserves the (eventual) equilibrium state. Precisely, if $F$ is an equilibrium state of $Q_1$, then $Q_{FP}(F) = 0$. This strongly suggests that one has to select the candidate for being an equilibrium state of $Q$ as the one of $Q_{FP}$. Of course, the interest of the above procedure lies in the fact that this latter is easy to exhibit. Indeed, it is obvious from [26] that the unique solution with unit mass to $Q_{FP}(g) = 0$ is given by a Maxwellian distribution with drift velocity $u_1$ and temperature

$$T^{\#} = \left( \frac{m_1(\mu - \kappa)}{\kappa T_1} \right)^{-1}.$$

This suggests the following dichotomy:

- Either $Q(f) = 0$ has no non-trivial solution.
- Either the unique solution to $Q(f) = 0$ with unit mass is the Maxwellian:

$$M(v) = \left( \frac{m}{2\pi T^{\#}} \right)^{3/2} \exp\left\{ -\frac{m(v - u_1)^2}{2 T^{\#}} \right\} \quad v \in \mathbb{R}^3 \quad (3.10)$$

where

$$T^{\#} = \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha(1 - \beta)} T_1. \quad (3.11)$$

At this point, we remark that the above Maxwellian distribution is exactly the equilibrium state found in [26] in the study of the pseudo-Maxwellian approximation of (2.3). This supports our belief that $M$ is indeed the steady state of $Q$ and that, moreover, it is also a universal stationary solution (independent of the collision kernel) as it occurs in the elastic case. This will be proved rigorously in the following section.

### 4 Is the Maxwellian the equilibrium state?

The problem of finding the equilibrium state of the linear Boltzmann equation (23) has now been reduced to determine whether $Q(M) \equiv 0$ or not, where

$$M(v) = \left( \frac{m}{2\pi T^{\#}} \right)^{3/2} \exp\left\{ -\frac{m(v - u_1)^2}{2 T^{\#}} \right\}, \quad v \in \mathbb{R}^3;$$
with $T^\# = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)}T_1$. Surprisingly, apart for the peculiar case of the 1D-model, we have not been able to prove by direct computation that

$$\Omega(M)(\mathbf{v}) = 0, \quad \mathbf{v} \in \mathbb{R}^3.$$ 

Actually, we prove this result through Fourier analysis, i.e., we show that

$$\widehat{\Omega(M)}(\xi) = 0, \quad \xi \in \mathbb{R}^3,$$

where $\widehat{\Omega(M)}(\xi)$ denotes the Fourier transform of $\Omega(M)$. By (2.5), it is given by

$$\widehat{\Omega(M)}(\xi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\mathbf{q} \cdot \mathbf{n}| M(\mathbf{v})M_1(\mathbf{w}) \exp \{-i\xi \cdot \mathbf{v}\} - \exp \{-i\xi \cdot \mathbf{v}\} d\mathbf{w} d\mathbf{v} d\mathbf{n}.$$ 

It is immediate to show that, up to a translation of the referential frame, one can assume

$$\mathbf{u}_1 = 0.$$ 

For the sake of simplicity, we introduce the parameter

$$C = \left( \frac{m m_1}{2 T_1 T^\#} \right)^{3/2}$$

and we recall that $\mu = \alpha(1-\beta)$. Then,

$$\widehat{\Omega(M)}(\xi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{q}| M(\mathbf{v})M_1(\mathbf{w}) \exp \{-i\xi \cdot \mathbf{v}\} d\mathbf{w} \times$$

$$\times \int_{S^2} \frac{\mathbf{q} \cdot \mathbf{n}}{|\mathbf{q}|} (\exp \{2i\mu (\mathbf{q} \cdot (\xi \cdot \mathbf{n})\} - 1) d\mathbf{n}.$$ 

Now, the key point of our computations is the identity

$$M(\mathbf{v})M_1(\mathbf{w}) = C \exp \left\{ -\frac{m_1}{2\mu T_1} \left[ \mu \mathbf{w}^2 + (1-\mu)\mathbf{v}^2 \right] \right\}$$

$$= C \exp \left\{ -\frac{m_1}{2\mu T_1} \left[ \mu(1-\mu)\mathbf{q}^2 + (\mathbf{v} - \mu\mathbf{q})^2 \right] \right\},$$

that follows from the equality $\frac{m}{T^\#} = -(\mu - 1)\frac{m_1}{\mu T_1}$. Then,

$$\widehat{\Omega(M)}(\xi) = C \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{q}| \exp \left\{ -\frac{m_1}{2\mu T_1} \left[ \mu(1-\mu)\mathbf{q}^2 + (\mathbf{v} - \mu\mathbf{q})^2 \right] \right\} \exp \{-i\xi \cdot \mathbf{v}\} d\mathbf{w} \times$$

$$\times \int_{S^2} \frac{\mathbf{q} \cdot \mathbf{n}}{|\mathbf{q}|} (\exp \{2i\mu (\mathbf{q} \cdot (\xi \cdot \mathbf{n})\} - 1) d\mathbf{n}.$$ 

The change of variables $(\mathbf{v}, \mathbf{w}) \rightarrow (\mathbf{v}, \mathbf{q})$ yields

$$\widehat{\Omega(M)}(\xi) = C \int_{\mathbb{R}^3} |\mathbf{q}| \exp \{ -\frac{m_1}{2\mu T_1} \mu(1-\mu)\mathbf{q}^2 \} d\mathbf{q} \times$$

$$\times \int_{\mathbb{R}^3} \exp \{ -\frac{m_1}{2\mu T_1} (\mathbf{v} - \mu\mathbf{q})^2 \} \exp \{-i\xi \cdot \mathbf{v}\} d\mathbf{v} \times$$

$$\times \int_{S^2} \frac{\mathbf{q} \cdot \mathbf{n}}{|\mathbf{q}|} (\exp \{2i\mu (\mathbf{q} \cdot (\xi \cdot \mathbf{n})\} - 1) d\mathbf{n}.$$ 

Performing first the second integral leads to

$$\widehat{\Omega(M)}(\xi) = C \exp \left\{ \frac{\mu T_1}{2m_1} \xi^2 \right\} \int_{\mathbb{R}^3} |\mathbf{q}| \exp \{ -\frac{m_1}{2T_1} (1-\mu)\mathbf{q}^2 \} \exp \{-i\mu \mathbf{q} \cdot \xi\} d\mathbf{q} \times$$

$$\times \int_{S^2} \frac{\mathbf{q} \cdot \mathbf{n}}{|\mathbf{q}|} (\exp \{2i\mu (\mathbf{q} \cdot (\xi \cdot \mathbf{n})\} - 1) d\mathbf{n},$$
where we used the fact that the Fourier transform of the Gaussian \(\exp\left\{-\frac{(v - u)^2}{2\Theta}\right\}\) is equal to \(C_{\Theta} \exp\left\{-i\bf{u} \cdot \bf{w} - \Theta/2\omega^2\right\}\) for any \(\Theta > 0\) and \(\bf{u} \in \mathbb{R}^3\) (here \(\bf{u} = \mu \bf{q}\) and \(\Theta = m_1/\mu T_1\)) where \(C_{\Theta}\) is a multiplicative constant depending only on \(\Theta\). Now, as pointed out first by A. Bobylev \[^2\], the inner integral on the unit sphere is an isotropic function of the vectors \(\xi\) and \(\bf{q}\) and is therefore equal to

\[
\int_{S^2} |\xi \cdot \bf{n}|/|\xi| \left\{ \exp\{2i\mu (\bf{q} \cdot \bf{n}) (\xi \cdot \bf{n})\} - 1 \right\} \, d\bf{n}.
\]

Consequently,

\[
\widehat{Q(M)}(\xi) = C \exp\left\{-\frac{\mu T_1}{2m_1} \xi^2\right\} \int_{\mathbb{R}^3} |\bf{q}| \exp\left\{-\frac{m_1}{2T_1} (1 - \mu) q^2\right\} d\bf{q} \times
\int_{S^2} |\xi \cdot \bf{n}|/|\xi| \left\{ \exp\{-i\mu (\bf{q} \cdot \xi^+\} - \exp\{-i\mu (\bf{q} \cdot \xi^-)\}\right\} \, d\bf{n},
\]

where

\[
\xi^+ = \xi - 2(\xi \cdot \bf{n})\bf{n}.
\]

Since the last integral on the unit sphere only depends on \(|\xi|\) and \(\xi \cdot \bf{q}\), and \(|\xi^+| = |\xi|\), we conclude that

\[
\widehat{Q(M)}(\xi) = 0 \quad \text{for any} \quad \xi \in \mathbb{R}^3.
\]

We proved

**Theorem 4.1.** The Maxwellian distribution

\[
M(v) = \left(\frac{m}{2\pi T^*}\right)^{3/2} \exp\left\{-\frac{m(v - u_1)^2}{2T^*}\right\} \quad v \in \mathbb{R}^3,
\]

with \(T^* = \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha(1 - \beta)} T_1\) is an equilibrium state for \(Q\).

**Remark 4.2 (Universality of the Maxwellian).** One can easily generalize the above computations to show that the Maxwellian \[^3,10\] is an equilibrium state of any collision operator \(Q_B\) taking the weak form

\[
\langle \psi, Q_B(f) \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B(q, n) f(v) M_1(w) [\psi(v^*) - \psi(v)] d\bf{v} d\bf{w} d\bf{n},
\]

for any smooth function \(\psi\). In \[^11\] the collision kernel \(B(\cdot, \cdot)\) is assumed to be given by

\[
B(q, n) = |q|^\gamma b(q \cdot n/|q|)
\]

with \(-1 \leq \gamma \leq 1\) and \(b(\cdot)\) nonnegative. This shows that, as it happens in the elastic case, the equilibrium state of the dissipative (linear) Boltzmann equation is universal in the sense that it does not depend on the collision kernel.

**Remark 4.3.** Note that for a general collision kernel \(B(q, n)\) it is convenient to use \[^4,11\] as a definition for \(Q_B\) instead of its strong form:

\[
Q_B(f) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times S^2} B(q, n) \left\{ J(q, n) f(v_*) M_1(w_*) - f(v) M_1(w) \right\} d\bf{v} d\bf{w} d\bf{n}
\]

where the factor \(J\) depends of \(B, \alpha\) and \(\beta\) in a complicated way.

## 5 On the trend to equilibrium

In this section we investigate the large-time behavior of the solution to the linear dissipative Boltzmann equation \[^2,3\]. Precisely, let \(f_0\) be a given (nonnegative) distribution function and consider the Cauchy problem

\[
\begin{aligned}
\frac{\partial}{\partial t} f(v, t) &= Q(f)(v, t), \quad v \in \mathbb{R}^3, t \geq 0 \\
f(v, t = 0) &= f_0(v).
\end{aligned}
\]

(5.1)
Since the above problem is linear (and homogeneous), it is not difficult to construct a (nonnegative) mild solution to (5.1) by a simple iterative method. Moreover, this solution is unique and preserves the mass
\[
\int_{\mathbb{R}^3} f(v, t) dv = \int_{\mathbb{R}^3} f_0(v) dv \quad \text{for any } t \geq 0.
\]
For further details, we refer to [23] where the problem is studied in high generality, including the spatially inhomogeneous equation with suitable boundary conditions.

A fundamental task in kinetic theory is to determine whether the solution to (5.1) converges toward the equilibrium state of \( \mathcal{Q} \) or not. Assuming the existence of a unique equilibrium distribution, this result has been proved by R. Petterson [23] for collision kernels of the form \( 4.2 \) with \(-1 < \gamma < 1 \) (corresponding to hard or soft interactions).

Actually, the proof of [23] requires to prove that the equilibrium state we exhibited is unique. This can be done by means of the \( H \)-theorem.

The \( H \)-theorem. Let us recall here that the linear \( H \)-theorem for the dissipative Boltzmann equation \( 2.3 \) has been first established by R. Pettersson [23], assuming the existence of an equilibrium state for \( \mathcal{Q} \). We point out that, in contrast to what happens in the nonlinear setting, the existence of such a steady-state is necessary in order to prove the \( H \)-theorem.

Once the existence of a steady state has been established, one can state the corresponding \( H \)-theorem and to prove the uniqueness of the stationary solution as a corollary.

We give here an elementary formal proof of the \( H \)-theorem (see Pettersson [23] for a more general result for the linear inhomogeneous equation with suitable boundary conditions). Let \( \Phi : \mathbb{R}^+ \rightarrow \mathbb{R} \) be a \textit{convex} \( C^1 \)-function. The associated entropy functional reads
\[
H_\Phi(f|M) = \int_{\mathbb{R}^3} M(v) \Phi \left( \frac{f(v)}{M(v)} \right) dv, \tag{5.2}
\]
where \( M(v) \) is the Maxwellian \( 3.10 \). The \( H \)-theorem asserts that \( H_\Phi(\cdot|M) \) is a Lyapunov functional for the solution of the linear Boltzmann equation.

\textbf{Theorem 5.1 (\( H \)-theorem).} Let \( \Phi : \mathbb{R}^+ \rightarrow \mathbb{R} \) be a \textit{convex} \( C^1 \)-function and let \( f_0 \) be a distribution function with unit mass such that \( H_\Phi(f_0|M) < \infty \). Then,
\[
\frac{d}{dt} H_\Phi(f(t)|M) \leq 0 \quad (t \geq 0) \tag{5.3}
\]
where \( f(t) \) stands for the (unique) solution to (5.1).

\textbf{Proof :} It is clear that
\[
\frac{d}{dt} H_\Phi(f(t)|M) = \int_{\mathbb{R}^3} \frac{\partial f}{\partial t}(v, t) \Phi \left( \frac{f(v, t)}{M(v)} \right) dv
\]
\[
= \int_{\mathbb{R}^3} \mathcal{Q}(f)(v, t) \Phi \left( \frac{f(v, t)}{M(v)} \right) dv
\]
and the proof of (5.3) amounts to show that
\[
\int_{\mathbb{R}^3} \mathcal{Q}(f)(v) \Phi \left( \frac{f(v)}{M(v)} \right) dv \leq 0 \tag{5.4}
\]
for any distribution function \( f \) with unit mass for which the above integral is meaningful. From (2.6)
\[
\int_{\mathbb{R}^3} \mathcal{Q}(f)(v) \Phi \left( \frac{f(v)}{M(v)} \right) dv = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} |\mathbf{q} \cdot \mathbf{n}| f(v) M_1(w) \times
\]
\[
\times \left\{ \Phi \left( \frac{f(v^*)}{M(v^*)} \right) - \Phi \left( \frac{f(v)}{M(v)} \right) \right\} dv dw dn
\]
and this last integral is also equal to
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\mathbf{q} \cdot \mathbf{n}| M(v) M_1(w) \left\{ \left[ \frac{f(v)}{M(v)} - \frac{f(v^*)}{M(v^*)} \right] \Phi' \left( \frac{f(v^*)}{M(v^*)} \right) + \frac{f(v^*)}{M(v^*)} \Phi' \left( \frac{f(v^*)}{M(v^*)} \right) - \frac{f(v)}{M(v)} \Phi' \left( \frac{f(v)}{M(v)} \right) \right\} \, dv \, dw \, dn.
\]

Now, since \( \left\langle \frac{f}{M} \Phi' \left( \frac{f}{M} \right), Q(M) \right\rangle = 0 \),
\[
\int_{\mathbb{R}^3} Q(f)(v) \Phi' \left( \frac{f(v)}{M(v)} \right) \, dv = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\mathbf{q} \cdot \mathbf{n}| M(v) M_1(w) \times \frac{f(v)}{M(v)} \Phi' \left( \frac{f(v)}{M(v)} \right) \, dv \, dw \, dn.
\]

The conclusion follows since, \( \Phi \) being convex,
\[
\Phi'(a)(b-a) \leq \Phi(b) - \Phi(a) \quad (a, b \in \mathbb{R})
\]
and \( \left\langle \Phi \left( \frac{f}{M} \right), Q(M) \right\rangle = 0 \). \( \square \)

Direct consequence of the above result is the following uniqueness result (due to Pettersson \cite{Pettersson} in a more general setting).

**Corollary 5.2.** The Maxwellian distribution \( M \) given by (5.10) is the unique stationary solution to (2.3) with unit mass.

**Proof :** Let \( F \) be another equilibrium state with unit mass. Then, according to (5.6) with \( \Phi(z) = \frac{(z - 1)^2}{2} \) one sees that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\mathbf{q} \cdot \mathbf{n}| M(v) M_1(w) \left\{ \frac{F(v)}{M(v)} - \frac{F(v^*)}{M(v^*)} \right\} \frac{F(v^*)}{M(v^*)} \, dv \, dw \, dn = 0.
\]
This implies that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\mathbf{q} \cdot \mathbf{n}| M(v) M_1(w) \left\{ \frac{F(v)}{M(v)} - \frac{F(v^*)}{M(v^*)} \right\} ^2 \, dv \, dw \, dn = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\mathbf{q} \cdot \mathbf{n}| F(v) M_1(w) \left\{ \frac{F(v)}{M(v)} - \frac{F(v^*)}{M(v^*)} \right\} \, dv \, dw \, dn = 0
\]
where this last integral is null since \( Q(F) = 0 \). Consequently, one gets that
\[
\frac{F(v)}{M(v)} = \frac{F(v^*)}{M(v^*)} \quad \text{for any} \quad (v, w) \in \mathbb{R}^3
\]
and this last identity leads, as in the elastic case, to \( F = M \). \( \square \)

**The evolution of the second moment.** To prove that the solution to the Cauchy problem (5.1) converges towards the (unique) equilibrium state, one has to establish some \textit{a priori} estimates on moments. Actually, in contrast to the elastic case and because of the lack of collision invariants, it is not trivial to estimate the evolution of the moments of \( f(v,t) \). This difficulty is peculiar to the hard–spheres model and does not occurs for pseudo–Maxwellian molecules \cite{Pettersson}, since in this case the equations for moments are in close form. For long–range interactions forces, R. Pettersson succeeded in proving uniform estimates on the higher moments of \( f(v,t) \) (see \cite{Pettersson} Theorem 4.1) for various types of kernels. Unlikely, his arguments do not apply to the hard–sphere model. Here we show that the second moment of the solution to (5.1) remains uniformly bounded in time.
Precisely, let the initial distribution function $f_0 \in L^1(\mathbb{R}^3)$ have \textit{unit mass} and let $f(t)$ be the solution to the Cauchy problem (5.1). Recall that for any $t \geq 0$, $f(v, t)$ has also unit mass. One considers the following second moment of $f(t)$:

$$T(t) = \frac{m}{3} \int_{\mathbb{R}^3} f(v, t)(v - u_1)^2 \, dv.$$ 

Note that $T(t)$ is not \textit{stricto sensu} the temperature of $f(t)$ which is defined by replacing the velocity $u_1$ by the drift velocity of $f(t)$. Define also

$$F(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 f(v, t)M_1(w) \, dv \, dw.$$ 

One has

$$F(t) = \int_{\mathbb{R}^3} (v - u_1)^2 f(v, t) \int_{\mathbb{R}^3} M_1(w) \, dw + \int_{\mathbb{R}^3} f(v, t) \, dv \int_{\mathbb{R}^3} (w - u_1)^2 M_1(w) \, dw$$

$$- 2 \int_{\mathbb{R}^3} (v - u_1)f(v, t) \, dv \cdot \int_{\mathbb{R}^3} (w - u_1)M_1(w) \, dw.$$ 

The last integral is equal to zero by definition of $u_1$. Therefore

$$F(t) = \frac{3}{m} T(t) + \frac{3}{m_1} T_1.$$ 

Now, from (5.3), one has

$$\frac{dT(t)}{dt} = \frac{m}{6\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q \cdot n| f(v, t)M_1(w) \{ (v^* - u_1)^2 - (v - u_1)^2 \} \, dv \, dw \, dn$$

and

$$(v^* - u_1)^2 - (v - u_1)^2 = 4\alpha^2(1 - \beta)^2 |q \cdot n|^2 - 4\alpha(1 - \beta)(q \cdot n)(v - u_1) \cdot n$$

$$= -4\alpha(1 - \beta)[1 - \alpha(1 - \beta)] |q \cdot n|^2 + 4\alpha(1 - \beta)(q \cdot n)(w - u_1) \cdot n.$$ 

Consequently,

$$\frac{dT(t)}{dt} = -\frac{2m}{3\pi} \mu(1 - \alpha(1 - \beta)) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q \cdot n|^3 f(v, t)M_1(w) \, dv \, dw \, dn$$

$$+ \frac{2m}{3\pi} \mu \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q \cdot n|(q \cdot n)((w - u_1) \cdot n)f(v, t)M_1(w) \, dv \, dw \, dn.$$ 

Recall that $0 < \mu = \alpha(1 - \beta) < 1$. Since

$$\int_{\mathbb{S}^2} |q \cdot n|^3 \, dn = \pi |q|^3 \quad \text{and} \quad \int_{\mathbb{S}^2} |q \cdot n|(q \cdot n)(w - u_1) \cdot n \, dn = \pi |q|q \cdot (w - u_1),$$

one gets

$$\frac{dT(t)}{dt} \leq -\frac{2m}{3} \mu(1 - \mu) \int_{\mathbb{R}^3} |q|^3 f(v, t)M_1(w) \, dv \, dw$$

$$+ \frac{2m}{3} \mu \int_{\mathbb{R}^3} |q|^2 |w - u_1| f(v, t)M_1(w) \, dv \, dw.$$ 

Let us first look for a upper bound to the second integral in terms of $F(t)$. Using the fact that

$$\int_{\mathbb{R}^3} (w - u_1)|w - u_1| M_1(w) \, dw = 0,$$ 

we obtain

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} |q|^2 |w - u_1| f(v, t)M_1(w) \, dv \, dw = \int_{\mathbb{R}^3} (v - u_1)^2 f(v, t) \int_{\mathbb{R}^3} |w - u_1| M_1(w) \, dw$$

$$+ \int_{\mathbb{R}^3} f(v, t) \, dv \int_{\mathbb{R}^3} |w - u_1|^3 M_1(w) \, dw.$$
Thus,
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^2 |w - u_1| f(v, t) M_1(w) \text{d}v \text{d}w \leq C_1 F(t), \tag{5.7}
\]
where
\[
C_1 = \max \left\{ \int_{\mathbb{R}^3} |w - u_1| M_1(w) \text{d}w, \frac{\int_{\mathbb{R}^3} |w - u_1|^3 M_1(w) \text{d}w}{\int_{\mathbb{R}^3} |w - u_1|^2 M_1(w) \text{d}w} \right\}
\]
is a positive (explicit) constant depending only on $M_1$. Moreover, Jensen’s inequality gives
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^3 f(v, t) M_1(w) \text{d}v \text{d}w \geq \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^2 f(v, t) M_1(w) \text{d}v \text{d}w \right)^{3/2}. \tag{5.8}
\]
Now, combining (5.7) and (5.8) and (5.6) one gets the differential inequality
\[
\frac{d F(t)}{dt} \leq -2\mu(1 - \mu) F(t)^{3/2} + 2\mu C_1 F(t). \tag{5.9}
\]
A direct inspection then shows that the solution to (5.9) satisfies the bound
\[
F(t) \leq \max \left\{ \frac{C_1^2}{(1 - \mu)^2}, F(0) \right\} \quad t \geq 0.
\]
Turning back to $T(t)$ one obtains that
\[
\sup_{t \geq 0} \int_{\mathbb{R}^3} (v - u_1)^2 f(v, t) \text{d}v < \infty \tag{5.10}
\]
provided \(\int_{\mathbb{R}^3} \nabla^2 f_0(v) \text{d}v < \infty\). Note that the above bound for $T(t)$ is explicitly computable in terms of $f_0$, $C_1$, $\alpha$ and $\beta$.

Now, mass conservation and the $H$-theorem, together with estimate (5.10) imply that, provided
\[
\int_{\mathbb{R}^3} (1 + v^2 + |\log f_0(v)|) f_0(v) \text{d}v \leq K < \infty \tag{5.11}
\]
at any subsequent time $t > 0$
\[
\int_{\mathbb{R}^3} (1 + (v - u_1)^2 + |\log f(v, t)|) f(v, t) \text{d}v \leq K_1 < \infty,
\]
and this implies the weak–compactness in $L^1(\mathbb{R}^3)$ of the family $\{f(v, t)\}_{t \geq 0}$. Now, following the strategy of [23], one obtains first the weak–convergence, and using translation continuity, the strong convergence towards the equilibrium of $f(v, t)$.

**Theorem 5.3.** Let $f_0 \in L^1(\mathbb{R}^3)$ be a distribution function with unit mass satisfying (5.11) and let $f(v, t)$ be the solution to the Cauchy problem (5.1). Then
\[
\lim_{t \to \infty} \|f(t) - f_0\|_{L^1(\mathbb{R}^3)} = 0.
\]

**Remark 5.4.** We conjecture that, as it occurs for the pseudo-Maxwellian approximation [20], the decay of $\|f(t) - f_0\|_{L^1(\mathbb{R}^3)}$ towards 0 is exponential (with an explicit rate).

### 6 Appendix

Let us make rigorous the derivation of the Fokker–Planck equation we formally obtained in Section 3. This can be done a posteriori using the results of the previous Section. We maintain here the notations of Section 3. The only difference is that, hereafter $Q_{\delta}$ denotes the collision operator with kernel
\[
B_{\delta}(q, n) = b_{\delta}(\theta)|q \cdot n|,
\]
whereas in Section 5 we considered a re-scaled kernel (see Remark 5.1). Actually, in Section 5 we were concerned with an approximation of the Boltzmann collision operator whereas in this Appendix, we approximate the solution to the Boltzmann equation as collisions become grazing. This difference will appear clearly in the sequel. Let $f_0$ be a nonnegative distribution function fulfilling estimate 5.11 and consider the following Cauchy problem

$$
\begin{cases}
\frac{\partial f_\delta}{\partial t}(v, t) = Q_\delta(f_\delta)(v, t) & t > 0, v \in \mathbb{R}^3 \\
f_\delta(v, 0) = f_0(v).
\end{cases}
$$

(6.1)

As we outlined in Section 5, problem (6.1) admits a (unique) weak-solution which satisfies

$$
\int_0^\infty \int_{\mathbb{R}^3} f_\delta(v, t) \partial_t \psi(v, t) dv - \int_{\mathbb{R}^3} f_0(v) \psi(v, 0) dv
= \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_\delta(q, n)f_\delta(v, t)M_1(w)[\psi(v^+, t) - \psi(v, t)] dv dwdn,
$$

(6.2)

for any $\psi \in C^1_{2, c}(0, +\infty|\mathbb{R}^3)$. In usual notations, $C^1_{2, c}(0, +\infty|\mathbb{R}^3)$ denotes the space of all functions $\psi$ which are continuously differentiable with compact support in $[0, +\infty[$ and twice continuously differentiable in $\mathbb{R}^3$. We recall that

$$
I_\delta = \int_0^{\pi/2} b_\delta(\theta) \cos^3 \theta \sin \theta d\theta \sim \frac{\delta^4}{2\pi} \quad \text{as} \quad \delta \sim 0.
$$

(6.3)

Let us introduce the time-scaling

$$
g_\delta(v, t) = f_\delta(v, \delta^{-4} t).
$$

Then, considering a test-function of the form $\psi_\delta(v, t) = \psi(v, \delta^{-4} t)$ into (6.2) leads to

$$
\int_0^\infty \int_{\mathbb{R}^3} g_\delta(v, t) \partial_t \psi(v, t) dv - \int_{\mathbb{R}^3} f_0(v) \psi(v, 0) dv
= \frac{1}{2\pi \delta^4} \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_\delta(q, n)g_\delta(v, t)M_1(w)[\psi(v^+, t) - \psi(v, t)] dv dwdn.
$$

(6.4)

The key point of the approximation procedure is the following

**Proposition 6.1.** There exists a nonnegative function $g : [0, +\infty[ \rightarrow L^1(\mathbb{R}^3)$ and a subsequence, still denoted $(g_\delta)_{\delta \geq 0}$ such that $g_\delta$ converges weakly in $L^1_{\text{loc}}([0, +\infty[, L^1(\mathbb{R}^3))$ towards $g$ as $\delta$ goes to zero.

We leave the proof to Proposition 6.1 to the end of this Appendix and explain now how to derive the Fokker–Planck equation from it. Inserting the expansion 5.2 into 6.4 leads to the approximation

$$
- \int_0^\infty \int_{\mathbb{R}^3} g_\delta(v, t) \partial_t \psi(v, t) dv - \int_{\mathbb{R}^3} f_0(v) \psi(v, 0) dv = \int_0^\infty J_\delta^3(t) dt + \int_0^\infty J_\delta^4(t) dt + R_\delta
= \frac{-2\alpha(1 - \beta)}{2\pi \delta^4} \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_\delta(q, n)(q \cdot n)g_\delta(v, t)M_1(w) \nabla \psi(v, t) \cdot n dv dwdn
+ \frac{(2\alpha(1 - \beta))^2}{4\pi \delta^4} \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_\delta(q, n)(q \cdot n)^2 g_\delta(v, t)M_1(w) \nabla^2 \psi(v, t) \cdot n \otimes n dv dwdn + R_\delta.
$$

In the above expression, $R_\delta = R_\delta(\psi)$ is a remainder term obtained from the Taylor expansion of $\psi$ 5.2 (see 20 for details). One can prove as in the elastic case 20 that

$$
\lim_{\delta \to 0} R_\delta(\psi) = 0
$$

(6.5)
for any test–function $\psi \in C^1_{\text{loc}}([0, +\infty]\times \mathbb{R}^3)$. As in Section 5 one can show that

$$
\int_0^\infty J^3_\delta(t)dt = -I_\delta \frac{2\alpha(1-\beta)}{4\pi} \int_0^\infty dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^2 g_\delta(v, t)M_1(w) \nabla_v \psi(v, t) \cdot \frac{q}{|q|} \, dv \, dw.
$$

Consequently, thanks to Proposition 6.1

$$
\lim_{\delta \to 0} \int_0^\infty J^3_\delta(t)dt = -\frac{\alpha(1-\beta)}{\pi} \int_0^\infty dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} |q|^2 g(v, t)M_1(w) \nabla_v \psi(v, t) \cdot \frac{q}{|q|} \, dv \, dw. \tag{6.6}
$$

We proceed in the same way for $J^3$ to obtain

$$
\int_0^\infty dt \int_{\mathbb{R}^3} g(v, t)\partial_t \psi(v, t)dv - \int_{\mathbb{R}^3} f_0(v)\psi(v, 0)dv = -\frac{1}{2\pi} \int_0^\infty dt \int_{\mathbb{R}^3} dv \nabla_v \psi(v, t) \cdot \int_{\mathbb{R}^3} |v-w|^2 \Sigma(v-w) \{\kappa M_1(w) \nabla_v g(v, t) + (\kappa - \mu)g(v, t) \nabla_w M_1(w)\} \, dw, \tag{6.7}
$$

where $\kappa$ and $\mu$ are defined in Section 3. At this point it is not difficult to recognize in (6.8) the weak formulation of the Cauchy problem

$$
\begin{aligned}
\frac{\partial g}{\partial t}(v, t) &= \frac{1}{2\pi} Q_{\text{FP}}(g)(v, t) \quad t \geq 0, v \in \mathbb{R}^3 \\
g(v, 0) &= f_0(v)
\end{aligned} \tag{6.9}
$$

where the Fokker–Planck collision operator is given by (3.9). It is well–known that problem (6.9) admits a (unique) nonnegative weak solution $g$. We proved

**Theorem 6.2.** There exists a subsequence, still denoted $g_\delta$, such that $g_\delta \rightharpoonup g$ weakly in $L^1_{\text{loc}}([0, +\infty], L^1(\mathbb{R}^3))$ where $g$ is a weak solution to the Fokker–Planck equation (6.9) with initial datum $f_0$ satisfying (6.11).

It remains now to prove Proposition 6.1. Clearly, it is enough to prove that the sequence $g_\delta$ satisfies the uniform estimate

$$
\sup_{\delta \geq 0, t \geq 0} \int_{\mathbb{R}^3} (1 + |v-u|^2 + |\log g_\delta(v, t)|)g_\delta(v, t)dv < \infty. \tag{6.10}
$$

Now the estimate

$$
\sup_{\delta \geq 0, t \geq 0} \int_{\mathbb{R}^3} (1 + |\log g_\delta(v, t)|)g_\delta(v, t)dv < \infty
$$

follows from the H–theorem applied to $f_\delta(t)$. Note that the H-theorem turns to be valid for the collision kernel $B_\delta(\cdot)$ since the equilibrium state is universal (see Remark 6.2). Now to prove the remaining estimate, we proceed as we did in Section 5 to derive formula (6.10). We only sketch here the main changes. Let

$$
T_\delta(t) = \frac{m}{3} \int_{\mathbb{R}^3} g_\delta(v, t)(v-u_1)^2dv = \frac{m}{3} \int_{\mathbb{R}^3} f_\delta(v, \delta^{-4}t)(v-u_1)^2dv.
$$

Then,

$$
\frac{dT_\delta(t)}{dt} = \frac{m\delta^4}{6\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_\delta(q \cdot n)f_\delta(v, \delta^{-4}t)M_1(w) \{(v^* - u_1)^2 - (v - u_1)^2\} \, dv \, dw \, dn.
$$

Since

$$
\int_{\mathbb{S}^2} |q \cdot n|^2 B_\delta(q \cdot n)dn = 2\pi I_\delta |q|^3,
$$

$$
\int_{\mathbb{S}^2} B_\delta(q \cdot n)(q \cdot n)(w-u_1) \cdot n \, dn \leq 2\pi I_\delta |q||w-u_1|
$$

and $\lim_{\delta \to 0} 2\pi \delta^4 I_\delta = 1$, we can draw the same conclusion of Section 5.
7 Conclusions

We proved existence and uniqueness of a collision equilibrium for the dissipative linear Boltzmann equation with general collision kernel. This equilibrium state is a universal Maxwellian with the same mass velocity as the field particles background and with a (non-zero) temperature always lower than the one of the background (depending on mass ratio and inelasticity). As early noticed in [26], this follows from the combined effects of momentum and energy exchange with fields particles on the one side and, on the other side, of energy dissipation in the binary collisions.

We point out that the existence of a Maxwellian equilibrium at non-zero temperature is of primary importance to reckon the hydrodynamic equations for the considered granular flow. This can be done through a Chapman–Enskog procedure (see the conclusions of [26]).

In space homogeneous conditions, we showed that the solution to the linear dissipative Boltzmann equation converges towards the equilibrium state as time goes to infinity for any initial datum with finite entropy and temperature. Unlikely, our convergence result is based upon compactness arguments and we have not been able to determine the decay rate towards the equilibrium. We may hope that, as it occurs for the pseudo–Maxwellian approximation [26], the relaxation to equilibrium is exponential. The results of Appendix show that the Fokker–Planck equation (6.9) is a good approximation of (2.3) when collisions become grazing. It is well–known (see [1]) that the solution to (6.9) relaxes to $M$ exponentially with an explicit rate related to $T^\#$. This supports us in the belief that the same occurs for the dissipative Boltzmann equation (2.3). We may infer that dissipation–dissipation entropy methods should lead to such a result. Work is in progress in this direction.

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