String inspired solution to the sign problem and overlapping problem

Masanori Hanada, Yoshinori Mastuo
KEK Theory Center, High Energy Accelerator Research Organization (KEK), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
E-mail: hanada@post.kek.jp, ymatsuo@post.kek.jp

Naoki Yamamoto
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA
Maryland Center for Fundamental Physics, Department of Physics, University of Maryland, College Park, MD 20742-4111, USA
E-mail: nyama@umd.edu

Abstract. Because of the sign problem, it is difficult to study finite-density QCD. In order to circumvent the sign problem, we prove that the phase-quenched approximation is exact to $O(N_f/N_c)$ for any physical observables in a certain region of the phase diagram. We also find a quantitative evidence for the validity of the phase quenching from existing lattice QCD results at $N_c=3$. Our results show that the phase-quenched approximation is good already at $N_c=3$, and the $1/N_c$ correction can be incorporated by the phase reweighting method without suffering from a severe overlap problem. The same equivalence holds in effective models and holographic models. It gives us a theoretical understanding of the empirical facts (exactness of the phase quenching in the mean-field approximation) found in model calculations.

1. Introduction

QCD at a finite baryon chemical potential and/or finite temperature is an important subject. Although the lattice QCD simulations should play an important role for studying the strongly coupled parameter region of this theory, the notorious *sign problem* prevents us from a direct application of the simulation at finite baryon chemical potential. A possible option is the phase-quenched simulation. In principle, the effect of the phase can be taken into account by the phase reweighting. However whether it is practical or not is not clear a priori; when the number of the flavors $N_f$ is two, the phase-quenched theory is the QCD with the isospin chemical potential\(^1\), whose phase diagram is different from the original theory (especially the existence of the pion condensation), and hence a severe overlap problem can appear.

Actually the phase quenching and the phase reweighting are practically useful techniques. The first to emphasized this fact, albeit empirically, are probably Kogut and Sinclair [1]. They pointed out various model calculations give the same answer for certain observables in the full

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\(^1\) As we will see in Sec. 2.2, although this statement is correct for the partition function, there is a difference when one considers the physical observables.
and phase-quenched theories, as long as the pion condensation does not take place in the latter. They also pointed out that known lattice data give very similar results. Independently, Cohen [2] and Toublan [3] pointed out the similarity in the large-$N_c$ limit. Recently these facts have been understood in theoretically unified manner [4, 5, 6]: there is an exact equivalence between the full and phase-quenched QCD at large-$N_c$, which provides a good approximation at $N_c = 3$.\(^2\) The equivalence is a version of the large-$N_c$ orbifold equivalence [7, 8], which was discovered through the study of the string theory. (For other interesting applications of the orbifold equivalence see e.g. [9].) Below we briefly summarize the equivalence, show the lattice data which verify that the equivalence can be seen already at $N_c = 3$, and point out the same equivalence holds for various effective models in the mean field approximation (MFA).

2. The (partial) equivalence between QCD\(_B\) and QCD\(_I\) in the large-$N_c$ limit

2.1. The equivalence between QCD\(_B\) and QCD\(_I\)

Let us start with the orbifold equivalence. First we choose the discrete symmetry \(P\) (subgroup of gauge, flavor, or spacetime symmetry) of the parent theory, which is the SO(2\(N_c\)) or Sp(2\(N_c\)) theory with the baryon chemical potential (SO\(_B\) or Sp\(_B\)) in the present case. We then throw away all the degrees of freedom not invariant under \(P\). This procedure is called the orbifold projection. After the projection, we obtain a new theory called the daughter. We consider two different projections, which give QCD with the baryon and isospin chemical potentials (QCD\(_B\) and QCD\(_I\)) as daughters. The orbifold equivalence states that, in the large-$N_c$ limit, correlation functions of operators \(O^{(p)}(A_\mu, \psi)\) invariant under \(P\) in the parent (called neutral operators) agree with those of the operators \(O^{(d)}(A^{\text{proj}}_\mu, \psi^{\text{proj}})\) that consist of projected fields in the daughter:

\[
\langle O_1^{(p)} O_2^{(p)} \cdots \rangle_p = \langle O_1^{(d)} O_2^{(d)} \cdots \rangle_d. \tag{1}
\]

Here the coupling constants are related as \(g_{SU}^2 = g_{SO}^2 = g_{Sp}^2\), where the ’t Hooft coupling \(g_{SU}^2 N_c\) is kept finite. The field theoretic proof was given in [8] for a class of theories, which can be generalized to various cases. For QCD\(_B\), QCD\(_I\), SO\(_B\) and Sp\(_B\), evidence for nonperturbative equivalence were also provided by the weak-coupling analysis in QCD and QCD-like theories at high density limit [5], low-energy effective theories [10], chiral random matrix models [5] and holographic models [11].

For the projection from SO\(_B\) to QCD\(_B\), we use the \(Z_4\) discrete symmetries of SO\(_B\) generated by \(J_c = -i\sigma_2 \otimes 1_{N_c}\) (\(1_N\) is an \(N \times N\) identity matrix) and \(\omega = e^{i\pi/2} \in U(1)_B\). We require the gauge field \(A^{SO}_{\mu,ab}\) and the fermion \(\psi^{SO}_{\alpha,a}\) to be invariant under the following \(Z_2\) transformation embedded in the gauge and U(1)\(_B\) transformation [4],

\[
A^{SO}_{\mu,ab} = (J_c)_{aad'} A^{SO}_{\mu,d'b'} (J_c^{-1})_{b'b}, \tag{2}
\]
\[
\psi^{SO}_{\alpha,a} = \omega (J_c)_{aad'} \psi^{SO}_{\alpha,d'}. \tag{3}
\]

The projection symmetry breaks down in the BEC/BCS crossover region (diquark condensation region) of SO\(_B\), because the U(1)\(_B\) symmetry is broken to \(Z_2\) there.

One can also construct the projection from SO\(_B\) to QCD\(_I\) for even \(N_f\) by choosing another \(Z_2\) symmetry [4, 5],

\[
A^{SO}_{\mu,ab} = (J_c)_{aad'} A^{SO}_{\mu,d'b'} (J_c^{-1})_{b'b}, \tag{4}
\]
\[
\psi^{SO}_{\alpha,a} = (J_c)_{aad'} \psi^{SO}_{\alpha,af'} (J_c^{-1})_{ff'}. \tag{5}
\]

\(^2\) Previously we argued the equivalence is restricted to a class of observables. As we will see, however, the equivalence holds for any observables. We thank F. Karsch for a valuable critical comment, which made us revisit the issue and led to more precise statement.
where $J_I = -i\sigma \otimes 1_{N_f/2}$ generates $\mathbb{Z}_4$ subgroup of $\text{SU}(2)$ isospin symmetry and the projection condition for the gauge field is the same as (2). In this case, the isospin symmetry used for the projection is unbroken everywhere, and so the orbifold equivalence holds also in the BEC/BCS region of the phase diagram. Therefore, through the equivalence with $SO_B$, we obtain the equivalence between $\text{QCD}_B$ and $\text{QCD}_I$ outside the BEC/BCS region (pion condensation region) of the latter; the phase quenching is exact for neutral sectors in this region.

The $1/N_c$ corrections to the equivalence can be estimated as follows. In the ’t Hooft large-$N_c$ limit, expectation values of gluonic operators trivially agree because the fermions are not dynamical. Now consider finite-$N_c$, say $N_c = 3$ and $N_f = 2$. Then the largest correction to the ’t Hooft limit comes from one-fermion-loop planar diagrams, which, as we have seen, do not distinguish $\mu_B$ and $\mu_I$. Therefore the difference of expectation values of gluonic operators is at most $(N_f/N_c)^3$ (two-fermion-loop planar diagrams). In particular, the deconfinement temperatures, which are determined by the Polyakov loop, agree up to corrections of this order. A similar observation was made in [3].

2.2. The equivalence between $\text{QCD}_B$ and phase-quenched $\text{QCD}$ in the large-$N_c$ limit

It is often said that $\text{QCD}_I$ and the phase-quenched $\text{QCD}$ are the same. However, although this statement is correct for the partition function, there is a difference when one studies the physical observables. In $\text{QCD}_I$, the propagators of up and down quarks are $D^{-1}(\mu)$ and $D^{-1}(-\mu)$. On the other hand, in the phase-quenched $\text{QCD}$, both are taken as $D^{-1}(+\mu)$. (In the terms of the lattice $\text{QCD}$ simulation, the configuration are generated by using $\text{QCD}_I$, while the same operators as $\text{QCD}_B$ are used for the measurement.) Therefore the expectation values of the chiral condensate, the baryon density, and the isospin density are:

| Chiral Condensate | $\text{QCD}_B$ | $\text{QCD}_I$ | Phase-quenched $\text{QCD}$ |
|-------------------|----------------|----------------|-----------------------------|
| Chiral Condensate | $2\langle \text{Tr} D^{-1}(\mu) \rangle_B$ | $\langle \text{Tr} D^{-1}(\mu) \rangle_I + \langle \text{Tr} D^{-1}(-\mu) \rangle_I$ | $2\langle \text{Tr} D^{-1}(\mu) \rangle_I$ |
| Baryon Density    | $2\langle \text{Tr} \gamma^0 D^{-1}(\mu) \rangle_B$ | $\langle \text{Tr} \gamma^0 D^{-1}(\mu) \rangle_I + \langle \text{Tr} \gamma^0 D^{-1}(-\mu) \rangle_I$ | $2\langle \text{Tr} \gamma^0 D^{-1}(\mu) \rangle_I$ |
| Isospin Density   | 0              | $\langle \text{Tr} \gamma^0 D^{-1}(\mu) \rangle_I - \langle \text{Tr} \gamma^0 D^{-1}(-\mu) \rangle_I$ | 0                                       |

Here $\langle \cdot \rangle_B$ and $\langle \cdot \rangle_I$ are the expectation values with $\text{QCD}_B$ and $\text{QCD}_I$ ensembles, respectively. We can easily see the chiral condensate in $\text{QCD}_I$ and the phase-quenched $\text{QCD}$ take the same value because of the charge-conjugation invariance of the $\text{QCD}_I$ ensemble. Therefore, the orbifold equivalence (the chiral condensate in $\text{QCD}_B =$ the chiral condensate in $\text{QCD}_I$) tells us it is not affected by the phase quenching. For the baryon density, let us remind $\langle \text{Tr} \gamma^0 D^{-1}(-\mu) \rangle_I = -\langle \text{Tr} \gamma^0 D^{-1}(+\mu) \rangle_I$, again because of the charge-conjugation invariance of the $\text{QCD}_I$ ensemble. Therefore, the isospin density in $\text{QCD}_I$ and the baryon density in the phase-quenched $\text{QCD}$ take the same value. (Also the baryon density in $\text{QCD}_B$ becomes zero.) By combining it with the orbifold equivalence (the baryon density in $\text{QCD}_B =$ the isospin density in $\text{QCD}_I$), we conclude that the phase quenching does not affect the expectation value of the baryon density. The same argument holds for other observables too, and the orbifold equivalence leads to the exactness of the phase quenching for any observable to $O(N_f/N_c)$.

3. Evidence from lattice simulations at $N_c = 3$

We have seen that the phase quenching is exact to $O(N_f/N_c)$. But the standard ’t Hooft counting does not tell us the expansion coefficients. That motivates us to look at lattice data of $N_c = 3$ QCD. In the following we summarize lattice studies which compared $\text{QCD}_B$ and the phase-quenched $\text{QCD}$.

- In [12], Nakamura et al. studied two-flavor $\text{QCD}_B$ and phase-quenched $\text{QCD}$ by using staggered fermions. $\text{QCD}_B$ is obtained by the phase reweighting. For the chiral condensate
and the Polyakov loop, they found a perfect agreement between QCD$_B$ and the phase-quenched QCD within numerical errors.

- In the right panel of Fig. 1 and the left panel of Fig. 4 of [13], the free energy at various temperatures between 0.5$T_c$ and 1.1$T_c$ are plotted as functions of the baryon number, which show a nice agreement near the critical temperature.

- Fodor et al. [14] combined the phase reweighting and the density of states methods. In Fig. 4 of [14] they show the critical couplings at $a\mu = 0.3$ both in the phase-quenched and phase-reweighted cases, which take close values.

- The large-$N_c$ equivalence holds for the imaginary baryon and isospin chemical potentials, $(\mu_u, \mu_d) = (i\mu_{img}, i\mu_{img})$ and $(\mu_u, \mu_d) = (i\mu_{img}, -i\mu_{img})$, without any modification. As a result, the chiral condensates $\langle \bar{\psi}\psi \rangle_B$ and $\langle \bar{\psi}\psi \rangle_I$ take the same value at finite imaginary potentials as long as the projection symmetries are unbroken. In [16], the chiral critical temperatures $T_c(\mu)$ in two-flavor QCD were exploited by the extrapolations from the imaginary chemical potential, by using a fitting ansatz $T_c(\mu)/T_c(0) = 1 + a_1 \left( \frac{\mu}{T_c} \right)^2$. They found $a_1 = -0.470(13)$ for $\mu_H$ and $a_1 = -0.522(10)$ for $\mu_B$, which provide a nice quantitative agreement already at $N_c = 3$.

- Let us consider the Taylor expansion method, in which the expectation value of an observable is expanded in powers of $\mu / T$, $\langle \mathcal{O} \rangle_B, I = \sum_{n=0}^{\infty} c_n^{B,I} (\mu / T)^n$. Taylor coefficients $c_n^{B,I}$ and $c_0^{B,I}$, which are functions of the temperature $T$, can be determined by the simulation. The large-$N_c$ equivalence tells that the coefficients agree in the large-$N_c$ limit: $\lim_{N_c \to \infty} c_n^{B,I} = \lim_{N_c \to \infty} c_0^{B,I}$.

In [15], the coefficient $c_2^B$ and $c_2^I$ for the chiral condensate and the pressure of the quark-gluon gas have been calculated in two-flavor QCD. The agreement is good, especially for the chiral condensate.

4. The equivalence in the effective models

Let us consider the Nambu–Jona-Lasinio (NJL) model as an example of the effective models. We concentrate on the chiral limit in order to simplify the discussion. The starting point is the Lagrangian with the U($N_c$) color current interaction with $N_f$ flavors,

$$\mathcal{L}_{NJL} = \bar{\psi}_f \left( \gamma^\mu \partial_\mu + \mu_f \gamma^A \right) \psi_f + \frac{G}{N_c} J_{\mu A}^{(U)} J_{\mu A}^{(U)} ,$$

where $J_{\mu A}^{(U)} = \bar{\psi}_f \gamma^\mu T_A \psi_f$ and $T_A$ are the U($N_c$) color generators and summation is taken over repeated indices. The coupling constant $G$ is taken to be of order $N_f^0$. One rewrites it keeping only the interactions in the scalar and pseudoscalar channels after Fierz transformations:

$$\mathcal{L}_{NJL} = \bar{\psi}_f \left( \gamma^\mu \partial_\mu + \mu_f \gamma^A \right) \psi_f + \mathcal{L}_{\text{int}},$$

$$\mathcal{L}_{\text{int}} = \frac{G}{N_c} \left[ (\bar{\psi}_f \gamma^5 \psi_f)(\bar{\psi}_f \gamma^5 \psi_f) + (\bar{\psi}_f \gamma^5 \psi_f)(\bar{\psi}_f \gamma^5 \psi_f) \right].$$

In the Lagrangians (6) and (7), the invariance under U($N_c$) gauge symmetry and U($N_f$)$_L$ × U($N_f$)$_R$ flavor symmetry are manifest. Here we ignore the effect of instantons or the U(1)$_A$ anomaly which explicitly breaks the U(1)$_A$ symmetry, because it is subleading in $1/N_c$. (From the viewpoint of the orbifold equivalence, there is no reason for the exactness of the phase quenching at the level of MFA if we take into account the 1/$N_c$-suppressed instanton effects.

3 For odd $n$, $c_n^B$ and $c_n^I$ vanish, and the first nontrivial $\mu$-dependences appear in $c_4^B$ and $c_4^I$. Although $c_n^B$ ($n \geq 4$) have been calculated, $c_n^I$ ($n \geq 4$) have not been calculated in [15]. (Note that, for $n \geq 4$, they use the same symbol $c_n^I$ for another quantity.)
However, even if we incorporate them, the phase quenching for the chiral condensate turns out to be exact within the NJL model \[6\]. For $\text{SO}(2N_c)$ theory, we can construct the corresponding NJL model in the same manner, by using the $\text{SO}(2N_c)$ current. The proof of the large-$N_c$ orbifold equivalence applies to the NJL model, by starting with the NJL model for $\text{SO}_B$ and by using similar projection conditions as the previous section \[6\].

In order to consider the large-$N_c$ limit, the correct $1/N_c$-counting scheme which reproduces the correct $1/N_c$-scaling in the large-$N_c$ QCD is needed. The quark $\psi$ has $N_c$ colors so that a closed color loop gives a factor of $N_c$. The coupling constant of the four-fermi interaction should be taken as $O(N_c^{-1})$, and furthermore, the form of possible four-fermi interactions are restricted; in other words only the interactions which have origins in QCD are allowed. Then the right $1/N_c$-counting follows and we can use the same proof of the orbifold equivalence as the large-$N_c$ QCD \[6\]. A rather standard argument relates this large-$N_c$ equivalence to the equivalence at the level of MFA. Similar arguments hold also for various other theories, such as linear sigma model \[6\], Polyakov-Nambu-Jona-Lasinio model \[6\], Polyakov-quark-meson model \[6\], chiral random matrix model \[5\], Sakai-Sugimoto model \[6\] and D3/D7 model \[11\].

5. Conclusion
We have seen the exactness of the phase quenching to $O(N_f/N_c)$ outside the BEC/BCS crossover region. In other words, the effect of the phase is $1/N_c$-suppressed, and hence the reweighting method works without the overlap problem. Previous lattice studies confirm the effect of the phase is small already at $N_c = 3$. We have also shown the exactness of the phase quenching in effective models, which had been realized by explicit calculations and used to justify the reweighting method. Because the phase quenching and phase reweighting methods have been theoretically justified, it is important to study the QCD phase diagram by using them.

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