Reservoir engineering of bosonic lattices using chiral symmetry and localized dissipation

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We show how a generalized kind of chiral symmetry can be used to construct highly-efficient reservoir engineering protocols for bosonic lattices. These protocols exploit only a single squeezed reservoir coupled to a single lattice site; this is enough to stabilize the entire system in a pure, entangled steady state. Our approach is applicable to lattices in any dimension, and does not rely on translational invariance. We show how the relevant symmetry operation directly determines the real space correlation structure in the steady state, and give several examples that are within reach in several one and two dimensional quantum photonic platforms.

Pure quantum states with non-classical properties such as entanglement or squeezing play an important role in quantum computing and communication, and robust methods for their preparation are an important resource. One powerful general approach is reservoir engineering [1, 2], where carefully tailored dissipation is used to prepare and stabilize non-trivial quantum states. Reservoir engineering of a few degrees of freedom is by now a well-established technique, and has been implemented experimentally in a range of systems spanning atomic physics, quantum optics, superconducting circuits and optomechanics (see e.g. Refs. 3–8).

Dissipative state stabilization methods can also be formulated for lattice systems having many degrees of freedom, potentially allowing the preparation of correlated quantum states. Reservoir engineering of a few degrees of freedom is by now a well-established technique, and has been implemented experimentally in a range of systems spanning atomic physics, quantum optics, superconducting circuits and optomechanics (see e.g. Refs. 3–8).

Model–We start by considering bosons hopping on an arbitrary d-dimensional lattice of sites (see Fig. 1), as described by a generic tight-binding Hamiltonian:

\[
\hat{H} = \sum_{m,n} H_{m,n} \hat{a}_m^\dagger \hat{a}_n + \sum_{n} V_n \hat{a}_n^\dagger \hat{a}_n + \sum_{m \neq n} J_{m,n} \hat{a}_m^\dagger \hat{a}_n
\]

related to the nature of the symmetry, and not dependent on further details of the system’s eigenmodes. The resulting entanglement structure is particularly rich when the system has broken time-reversal symmetry.

The understanding of the steady state in terms of the symmetry of the non-dissipative lattice Hamiltonian paves the way for custom design of lattice systems to be used in the preparation of a variety of many-mode steady states, as we demonstrate in several examples. The results we present are directly applicable to a number of experimental platforms. In particular, experiments in superconducting circuits have recently demonstrated all the required elements for our approach, including the construction of non-trivial lattice structures [20–22] and the ability to couple strongly to a squeezed vacuum reservoir [23–26].
Here $\hat{a}_n$ ($\hat{a}_n^\dagger$) is the annihilation (creation) operator for a boson on site $n$, $J_{m,n} = J^*_{n,m}$ is the hopping strength between sites $m,n$, and $V_n$ is the potential on site $n$. Summations are over all $N$ sites in the lattice. Note that we do not assume translational invariance.

We take a single “drain” site, $n_0$, to be linearly coupled to a squeezed zero-temperature Markovian reservoir. In a photonic realization of our system, where each site is a cavity, this simply corresponds to driving site $n_0$ with broadband squeezed vacuum noise. We use the standard input-output treatment of the resulting dissipation [27, 28], yielding the Heisenberg-Langevin equations of motion

$$\hat{a}_n = -i\left[\hat{a}_n, \hat{H}\right] - \delta_{n,n_0} \left( \frac{\Gamma}{2} \hat{a}_{n_0} - \sqrt{\Gamma} \hat{\zeta} \right).$$

(2)

The rate $\Gamma$ parameterizes the strength of the coupling to the reservoir and the operator $\hat{\zeta}$ describes the squeezed vacuum fluctuations associated with it. This is operator-valued Gaussian white noise, with correlators

$$\langle \hat{\zeta}^\dagger(t) \hat{\zeta}(t') \rangle = \delta(t-t') \mathcal{N}, \quad \langle \hat{\zeta}(t) \hat{\zeta}(t') \rangle = \delta(t-t') \mathcal{M},$$

$$\mathcal{N} = \sinh^2 r, \quad \mathcal{M} = e^{i\phi} \cosh r \sinh r,$$

(3)

where $r$ ($\phi$) is the squeezing parameter (angle).

Diagonalizing our tight-binding Hamiltonian yields

$$\hat{H} = \sum_i \varepsilon_i \hat{b}_i^\dagger \hat{b}_i, \quad \hat{b}_i^\dagger = \sum_n \psi_i[n] \hat{a}_n,$$

(4)

where $\hat{b}_i$ annihilates an energy eigenmode with mode energy $\varepsilon_i$ and wavefunction $\psi_i[n]$. Without loss of generality, we label the modes so that $\varepsilon_{i+1} \geq \varepsilon_i$. Including the coupling to the reservoir, the equations of motion in the energy eigenmode basis take the form

$$\dot{\hat{b}}_i = -i \sum_j A_{i,j} \hat{b}_j + e^{-i\varphi_i} \sqrt{\Gamma_i} \hat{\zeta},$$

(5)

where $\varphi_i = \arg(\psi_i[n_0]), \Gamma_i = |\psi_i[n_0]|^2 \Gamma$ is the magnitude of the coupling between mode $i$ and the reservoir, and the dynamical matrix $A_{i,j}$ is given by

$$A_{i,j} = \delta_{i,j} \varepsilon_i - ie^{i(\varphi_j - \varphi_i)} \frac{1}{2} \sqrt{\Gamma_i \Gamma_j},$$

(6)

Ensuring a unique steady state—The dynamics corresponding to the linear equations in Eq. (5) are controlled by the complex eigenvalues $\lambda$ of the matrix $A$. Note first that energy eigenmodes having a node at the drain site are completely unaffected by the dissipation, and thus yield $\lambda = \varepsilon_i$. Such dark modes are a generic feature of Hamiltonians possessing degenerate spectra, as one can construct a basis for each $M$-fold degenerate subspace consisting of a single “bright” mode which couples to the bath, and $M-1$ uncoupled “dark” modes.

The coupling to the reservoir will both mix and cause decay of the bright energy eigenmodes. The resulting dynamical matrix eigenvalues can be written as $\lambda = \nu - i \frac{\gamma}{2}$, where $\nu, \gamma$ are real solutions of the equation [29]

$$\sum_j \frac{\Gamma_j}{\frac{\gamma}{2} + i(\nu - \varepsilon_j)} = 1.$$  

(7)

Examining the real part of this condition, we immediately find that all bright eigenmodes have relaxation rates $\gamma > 0$. Thus the “bright” portion of the Hilbert space will relax to a unique steady state. The final state of the system is dictated the bright mode steady state and the initial state of any dark modes.

A unique steady state is desirable for most (but not all [7]) applications of reservoir engineering. Here, we see that this can be accomplished by designing a Hamiltonian that does not have symmetries leading to degeneracies, and by placing the drain site so that it does not coincide with nodes of the remaining modes. In what follows, we assume that these conditions are met and there are no dark modes. We will provide several concrete examples showing that these conditions are indeed achievable in realistic models [30].

Chiral symmetry and the steady state—While the absence of dark modes ensures a unique steady state, it does not ensure that this state will be pure [29]. We find, however, that we can guarantee a pure steady state by imposing a simple symmetry requirement on our system: the existence of a generalized chiral symmetry which leaves the drain site invariant. More explicitly, we require the eigenmodes of $\hat{H}$ to come in pairs of opposite energy and equal wavefunction amplitude at the drain,

$$\varepsilon_{-i} = -\varepsilon_i, \quad |\psi_{-i}[n_0]| = |\psi_{i}[n_0]|.$$  

(8)

Here we have indexed the modes $i \in \{-\lfloor \frac{N}{2} \rfloor, ..., \lfloor \frac{N}{2} \rfloor \}$, skipping $i = 0$ if there is an even number.

The above spectral structure arises in a large variety of tight-binding models, including disordered systems, and is often associated with a sublattice symmetry. For example, in the absence of any on-site potential, it is present in a 1D lattice with (arbitrary, possibly random) nearest neighbour hopping, and more generally in any system with a bipartite hopping structure [31]. Other well known examples of tight-binding models with chiral symmetry include the SSH model in 1D [32], and in two dimensions the graphene bandstructure [33] and the well-known Hofstadter model [34].

To understand how the chiral structure in Eq. (8) constrains the steady state, we first note that this structure ensures that $\hat{H}$ is invariant under any two-mode squeezing (or Bogoliubov) transformation that mixes eigenmode operators $\hat{b}_i$ and $\hat{b}_{-i}^\dagger$. We thus define a new set of canonical annihilation operators $\beta_i = \cosh r \hat{b}_i - e^{i(\varphi_i - \varphi_{-i})} \sinh r \hat{b}_{-i}^\dagger$. Note that the
definition of these modes depends both on the properties of the squeezed reservoir (through r and ϕ), and on the position of the drain (through ϕi and ϕ−i).

Using Eq. (5), we find these new quasiparticle operators obey the equations of motion
\[
\dot{\hat{\beta}}_i = -i \sum_j A_{ij} \hat{\beta}_j + e^{-i\epsilon_i} \sqrt{\Gamma_i} \xi_i + (|\psi_i[n_0]| - |\psi_{-i}[n_0]|) [\{\cdots\} \hat{\beta}_j + \{\cdots\} \hat{\xi}_j + \cdots].
\] (9)

Here, \( \xi = \cosh r \xi - e^{i\phi} \sinh r \xi\) is a noise operator with correlation functions corresponding to simple (unsqueezed) vacuum noise: \( \langle \xi(t) \xi(t') \rangle = \delta(t - t'), \) \( \langle \xi_\dagger(t) \xi(t') \rangle = \langle \xi(t) \xi(t') \rangle = 0. \)

The invariance of the drain site under the generalized chiral symmetry (c.f. Eq. (8)) ensures that the second line of Eq. (9) vanishes. We thus find that the new \( \hat{\beta}_i \) modes evolve with the same dynamical matrix as the original \( \hat{b}_i \) energy eigenmodes, but the noise term now corresponds to simple vacuum noise. As the dynamical matrix is unchanged, we again have no dark modes, and thus a unique steady state. Further, as the dynamical matrix corresponds to simple hopping and local damping, the unique steady state is the joint vacuum of all \( \hat{\beta}_i \) modes. This steady state yields non-trivial correlations between the original energy eigenmodes:
\[
\langle \hat{b}_i^\dagger \hat{b}_j \rangle = \delta_{i,j} N, \quad \langle \hat{b}_i^\dagger \hat{b}_j \rangle = \delta_{i,-j} e^{-i(\phi_i + \phi_{-j})} M.
\] (10)

Thus, the single, locally-coupled squeezed reservoir leads to a pure steady state where each pair of \( \varepsilon, -\varepsilon \) energy eigenmodes is in a pure two-mode squeezed state with squeezing parameter \( r. \)

It is even more interesting to consider the correlation structure in real-space. The state has a uniform average photon number on each site, an absence of any beam-splitter correlations, and non-trivial pattern of anomalous correlators:
\[
\langle \hat{a}_m^\dagger \hat{a}_n \rangle = \delta_{m,n} N, \quad \langle \hat{a}_m^\dagger \hat{a}_n \rangle = \delta_{m,n} M,
\]
\[
\sigma_{m,n} = \sum_j e^{-i(\phi_j + \phi_{-j})} \psi_j[n]\psi_{-j}[m].
\] (11)

The pattern of correlations depends explicitly on the position of the drain site via the phases \( \phi_j. \) One always finds that \( \sigma_{m,n_0} = \delta_{m,n_0}, \) implying that the drain site is in a pure squeezed state and unentangled with the rest of the lattice.

**Connection to symmetry operations**—We see that all the non-trivial correlation structure of the pure steady state is contained in the \( N \times N \) matrix \( \sigma \) in Eq. (11). While its formal definition in terms of eigenmodes may seem opaque, it has a simple physical meaning: it directly defines a symmetry operation on \( \mathcal{H}. \) More explicitly, the real-space tight-binding Hamiltonian matrix \( \mathcal{H} \) (c.f. Eq. (1)) satisfies (see EPAPS [29]):
\[
\sigma^\dagger \cdot \mathcal{H} \cdot \sigma = -\mathcal{H}^*, \quad \sigma_{m,n_0} = \delta_{m,n_0}.
\] (12)

The unitary (and symmetric) matrix \( \sigma \) thus represents a symmetry operation which maps \( \mathcal{H} \) to \( -\mathcal{H}^*. \) At the level of operators, this equation can be expressed as
\[
\hat{U} \mathcal{H} + \mathcal{H} \hat{U} = 0.
\] (13)

The operator \( \hat{U} \) is most easily understood in the case where \( \mathcal{H} \) has time reversal symmetry, such that we can work in a gauge where \( \mathcal{H} = \mathcal{H}^* \), and where the eigenwavefunctions are real. In this case, \( \hat{U} \) is a unitary symmetry operator associated with the chiral symmetry of the Hamiltonian. Explicitly, we have:
\[
\hat{U} \to \hat{S} : \quad \hat{S} \hat{a}_m \hat{S}^{-1} = \sum_n \sigma_{m,n} \hat{a}_n.
\] (14)

In this case, we see that the anomalous correlators in real space are simply the matrix elements of the chiral symmetry operator.

In the more general case where time-reversal symmetry is broken, the relevant symmetry is a particle-hole transformation:
\[
\hat{U} \to \hat{C} : \quad \hat{C} \hat{a}_m \hat{C}^{-1} = \sum_n \sigma_{m,n} \hat{a}_n^\dagger.
\] (15)

One can confirm that this definition satisfies Eq. (13) [29].

We again have that the pattern of anomalous correlations in the steady state are directly set by the real-space matrix elements of the particle-hole symmetry operator. While it might seem surprising to focus on the particle-hole symmetry of a single particle Hamiltonian, it is the relevant symmetry operation here as we are ultimately interested in pairing correlations induced by the squeezed reservoir.

The upshot of our analysis is that the generalized chiral structure of \( \mathcal{H} \) does more than ensure a pure steady state: the corresponding symmetry operation directly determines its pattern of correlations. Thus, one does not need knowledge of all the eigenmodes to understand the steady state, and simply identifying the relevant symmetry operation is enough. This provides a powerful and very general principle for engineering steady states with the desired correlation patterns. We now go on to provide several examples of systems with a generalized chiral symmetry.

**Chiral symmetry from bipartite hopping**—Consider first a system with time-reversal symmetry, vanishing on-site energies and hopping terms that connect two distinct sub-lattices, with Hamiltonian
\[
\mathcal{H} = \sum_{a \in A, b \in B} J_{a,b} (\hat{a}_a^\dagger \hat{a}_b + \hat{a}_b^\dagger \hat{a}_a).
\] (16)

for a real \( J_{a,b} \) and some partition of the sites, \( A \cap B = \emptyset. \) The system has a unitary chiral symmetry (c.f. Eq. (14)) defined by
\[
\sigma_{m,n} = (-1)^n \delta_{m,n},
\] (17)
where \( s_n = 0,1 \) for \( n \in A,B \) respectively. A variety of tight-binding models have this form, including all bipartite lattices with nearest neighbor hopping (e.g. a one-dimensional chain or two-dimensional square lattice); the symmetry holds with arbitrary (possibly random) nearest neighbor hopping matrix elements.

It follows from Eq. (11) that if we now locally couple this system to squeezed dissipation (with arbitrary choice of drain site), it will relax into a product state where each site is in a pure squeezed state with parameters \( r, \phi \). We thus have a robust method for preparing a lattice of squeezed states, using a single squeezing source. Moreover, given the nature of the symmetry being exploited, the steady state is robust against any amount of disorder in the tight-binding matrix elements.

**Generalized chiral symmetry from spatial inversion**—Consider next a system where spatial inversion about the origin takes \( H \to -H^* \). With an appropriate labelling of lattice sites, where inversion corresponds to \( n \to -n \), this symmetry corresponds to

\[
J_{-m,-n} = -J^*_{m,n} \quad V_{-n} = -V_n.
\]

Such systems formally have a particle-hole symmetry (as defined in Eq. (15)), with a symmetry operator in position space described by

\[
\sigma_{m,n} = \delta_{m,-n}.
\]

Note that inversion necessarily leaves the origin \( n = 0 \) invariant. Thus, if we couple the origin to our squeezed reservoir, \( n_0 = 0 \), the steady state is described by Eq. (11) with \( \sigma_{m,n} \) given above. The state thus factorizes into a product of two-mode squeezed states, with each site \( n \) entangled with the site \( -n \).

An analogous result can be achieved with a slightly different symmetry under inversion. Consider a bipartite lattice with nearest neighbour hopping, where the onsite energies \( V_n \) are odd under inversion, while the hoppings satisfy \( J_{-m,-n} = J^*_{m,n} \). Such systems have a particle-hole symmetry described by the matrix \( \sigma_{m,n} = (-1)^n \delta_{m,-n} \). If we couple the drain at the original, local dissipation again leads to pure two-mode squeezing between sites \( n \) and \( -n \). In the specific case of a 1D model with uniform hopping, this result corresponds to that found by Zippilli et al in Ref. [17].

**Particle-hole symmetries in the Hofstadter model**—Consider a two-dimensional square lattice with a (synthetic) flux \( \Phi \) through each plaquette. Labelling the sites via \( n = (x,y) \) with \( x,y \in (-M,M) \), we have:

\[
\hat{H} = -J \sum_{x,y} \left( \hat{a}_{(x+1,y)}^\dagger \hat{a}_{(x,y)} + e^{i\Phi x} \hat{a}_{(x,y+1)}^\dagger \hat{a}_{(x,y)} + \text{h.c.} \right).
\]

Such a system has recently been realized both with coupled optical cavities [35], and with coupled superconducting microwave cavities [22, 36]. One can in general choose \( \Phi \) and \( n \) to ensure the absence of dark modes (e.g. all results in Fig. 2 correspond to a situation with no dark modes).

For all values of the flux, the system in Eq. (20) has a simple chiral symmetry \( \hat{S} \), which in real-space is described by \( \sigma_{(x,y),(x',y')} = (-1)^{x+y} \delta_{x,x'} \delta_{y,y'} \). In contrast, the particle-hole symmetry \( \hat{C} \) that is relevant to the steady-state entanglement can be far more complex. As always, the relevant symmetry depends on the choice of drain site \( n_0 \). If one places the drain site at certain positions with high spatial symmetry, the relevant particle-hole symmetry has a simple form:

\[
\sigma_{(x,y),(x',y')} = \begin{cases} 
(-1)^{x+y} \delta_{x,-x} \delta_{y,y'} & \text{if } n_0 = (z,0) \\
(-1)^{x+y} \delta_{x,x'} \delta_{y,-y'} & \text{if } n_0 = (0,z) \\
(-1)^{x+y} \delta_{x,y'} \delta_{y,x'} e^{i\Phi xy} & \text{if } n_0 = (z,z) 
\end{cases}
\]

where \( z \neq 0 \) is arbitrary [37]. For these positions, we obtain a pure steady state which is a product of two-mode
squeezed states in real space; the nature of the pairing depends on the choice of drain site. We stress that all of the transformations $\sigma$ above are symmetries of the system, and they are closely related: in terms of energy eigenstates, they all have the form $\hat{C} \hat{b}_m \hat{C}^{-1} = e^{-i(\varphi_m + \varphi_{-m})} \hat{b}_m^\dagger$.

The choice of $n_0$ that determines which is relevant for calculating steady state correlations. These relatively simple steady states are shown in Fig. 2 (for $\Phi = \pi/2$ and a $9 \times 9$ lattice).

If we instead couple to the lattice at a position $n_0$ not described above, we still obtain a pure steady state. However, the resulting steady state is an entangled state that does not factor simply in real space. An example of the complex real-space correlations in such a state is shown in Figs. 2d and 2h. These states have a general structure reminiscent of complex multi-mode Gaussian entangled states known as cluster states [29].

Finally, we show in Fig. 3 how both disorder (in the form of random variations in on-site energy) and internal loss degrade our local, symmetry-assisted reservoir engineering protocol. Small amounts of disorder and loss do not completely destroy the entanglement associated with the ideal, reservoir-engineered state.

**Conclusion**— We have developed a simple yet potentially powerful symmetry-based approach for reservoir engineering of entangled steady states of a bosonic lattice. Our approach only employs a single, locally coupled squeezed reservoir, and relies on the existence of chiral symmetry, something that is present in a variety of different tight-binding models. The approach allows the preparation and stabilization of a variety of different kinds of Gaussian entangled states, and is applicable to current state-of-the-art quantum photonic systems. In particular, experiments in superconducting quantum circuits have demonstrated all the required ingredients, including the realization of a Hofstadter model [22], and the ability to strongly couple to a squeezed vacuum reservoir [23–26]. Our work also suggests that symmetry-based approaches could be a powerful way to design lattice reservoir engineering protocols in more complex systems (e.g. where interactions and nonlinearity are also important).

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SUPPLEMENTAL MATERIAL FOR “RESERVOIR ENGINEERING OF BOSONIC LATTICES USING CHIRAL SYMMETRY AND LOCALIZED DISSIPATION”

RELATION OF THE SYMMETRY TO THE CORRELATION MATRIX

The relation between the eigenmodes and the original basis is

\[ \hat{b}_i = \sum_n (\psi_i[n])^* \hat{a}_n \quad \hat{a}_n = \sum_j \psi_j[n] \hat{b}_j \]  

(S.1)

where the wavefunctions are orthonormal,

\[ \sum_n (\psi_i[n])^* \psi_j[n] = \delta_{i,j} \]

(S.2)

The Hamiltonian can be written in the form

\[ \hat{H} = \sum_{m,n} H_{m,n} \hat{a}_m^\dagger \hat{a}_n = \sum_i \varepsilon_i \hat{b}_i^\dagger \hat{b}_i \]

(S.3)

with the matrix relations

\[ H_{m,n} = \sum_i \psi_i[m] \varepsilon_i (\psi_i[n])^*. \]

(S.4)

In the eigenmode basis, we have at the steady state

\[ \langle \hat{b}_i^\dagger \hat{b}_j \rangle \rightarrow \delta_{i,j} N, \quad \langle \hat{b}_i \hat{b}_j \rangle \rightarrow \delta_{i,-j} e^{-i(\varphi_i + \varphi_j)} M. \]

(S.5)

and therefore, in real space

\[ \langle \hat{a}_m^\dagger \hat{a}_n \rangle = \sum_{i,j} (\psi_i[m])^* \psi_j[n] \langle \hat{b}_i^\dagger \hat{b}_j \rangle \rightarrow \sum_i (\psi_i[m])^* \psi_i[n] N = \delta_{m,n} N \]

(S.6)

\[ \langle \hat{a}_m \hat{a}_n \rangle = \sum_{i,j} \psi_i[m] \psi_j[n] \langle \hat{b}_i \hat{b}_j \rangle \rightarrow \sum_i \psi_i[m] \psi_{-i}[n] e^{-i(\varphi_i + \varphi_{-i})} M \equiv \sigma_{m,n} M \]

(S.7)

We then observe

\[ (\sigma^\dagger \cdot H \cdot \sigma)_{m,n} = \sum_{a,b} \sigma_{a,m}^* H_{a,b} \sigma_{b,n} \]

\[ = \sum_{a,b} (\sum_i e^{-i(\varphi_i + \varphi_j)} \psi_i[a] \psi_{-i}[b])^* (\sum_j \varepsilon_j \psi_j[a]) \varepsilon_j (\sum_i e^{-i(\varphi_{i} + \varphi_{-i})} \psi_i[b] \psi_{-i}[n]) \]

\[ = \sum_{i,j,l} e^{i(\varphi_i + \varphi_{-i} - \varphi_j - \varphi_{-j})} (\psi_{-i}[m])^* (\sum_a (\psi_i[a])^* \psi_j[a]) \varepsilon_j (\sum_b (\psi_j[b])^* \psi_l[b]) \psi_{-l}[n] \]

\[ = \sum_{i,j,l} e^{i(\varphi_i + \varphi_{-i} - \varphi_j - \varphi_{-j})} (\psi_{-i}[m])^* \delta_{i,j} \delta_{j,l} \psi_{-l}[n] \]

\[ = \sum_i (\psi_{-i}[m])^* \varepsilon_i \psi_{-i}[n] = \sum_i (\psi_{-i}[m])^* (-\varepsilon_{-i}) \psi_{-i}[n] = -H^*_{m,n}. \]
OPERATOR FORMULATION OF SYMMETRY

We show that the transformations of Eqs. (14) and (15) satisfy Eq. (12). The Hamiltonian is, again,

\[ \hat{\mathcal{H}} = \sum_{m,n} H_{m,n} \hat{a}_m^\dagger \hat{a}_n. \]  

(S.8)

In the real case, we have

\[ \hat{U} \rightarrow \hat{S} : \quad \hat{S} \hat{a}_m \hat{S}^{-1} = \sum_n \sigma_{m,n} \hat{a}_n, \]  

(S.9)

and

\[
\begin{align*}
\hat{U} \hat{\mathcal{H}} \hat{U}^{-1} &= \sum_{m,n} H_{m,n} \hat{S} \hat{a}_m^\dagger \hat{a}_n \hat{S}^{-1} = \sum_{m,n} H_{m,n} \hat{S} \hat{a}_m^\dagger \hat{S}^{-1} \hat{S} \hat{a}_n \hat{S}^{-1} \\
&= \sum_{m,n} H_{m,n} \left( \sum_{m'} \sigma_{m',m} \hat{a}_{m'}^\dagger \right) \left( \sum_{n'} \sigma_{n,n'} \hat{a}_{n'} \right) = \sum_{m',n'} (\sigma^\dagger \cdot H \cdot \sigma)_{m',n'} \hat{a}_{m'}^\dagger \hat{a}_{n'} \\
&= \sum_{m',n'} -H^*_{n',m'} \hat{a}_{m'}^\dagger \hat{a}_{n'} = - \sum_{m',n'} H_{m',n'} \hat{a}_{m'}^\dagger \hat{a}_{n'} = -\hat{\mathcal{H}}.
\end{align*}
\]

More generally,

\[ \hat{U} \rightarrow \hat{C} : \quad \hat{C} \hat{a}_m \hat{C}^{-1} = \sum_n \sigma_{m,n} \hat{a}_n^\dagger, \]  

(S.10)

and

\[
\begin{align*}
\hat{U} \hat{\mathcal{H}} \hat{U}^{-1} &= \sum_{m,n} H_{m,n} \hat{C} \hat{a}_m^\dagger \hat{a}_n \hat{C}^{-1} = \sum_{m,n} H_{m,n} \hat{C} \hat{a}_m^\dagger \hat{C}^{-1} \hat{C} \hat{a}_n \hat{C}^{-1} \\
&= \sum_{m,n} H_{m,n} \left( \sum_{m'} \sigma_{m',m} \hat{a}_{m'} \right) \left( \sum_{n'} \sigma_{n,n'} \hat{a}_{n'}^\dagger \right) = \sum_{m',n'} (\sigma^\dagger \cdot H \cdot \sigma)_{m',n'} \hat{a}_{m'} \hat{a}_{n'}^\dagger \\
&= \text{Tr} H + \sum_{m',n'} -H^*_{n',m'} \hat{a}_{m'}^\dagger \hat{a}_{n'} = - \sum_{m',n'} H_{m',n'} \hat{a}_{m'}^\dagger \hat{a}_{n'} = -\hat{\mathcal{H}}.
\end{align*}
\]

where \( \text{Tr} H = \text{Tr}(-H) = 0. \)
the \( \Phi = 0 \) model is highly-degenerate and has many dark modes. We note that in the absence of a flux, \( \Phi \), the bright modes behave in much the same way as in its presence. However, eigenmodes. To stress this, we show the effect of these transformations at \( \Phi = 0 \), where the model is exactly solvable.

They both take each mode, up to a phase, to the antiparticle of its negative energy mode.

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SYMMETRY TRANSFORMATIONS IN A 2D LATTICE

In the text, we discuss the the Hofstadter Hamiltonian,

\[
\hat{H} = -J \sum_{x,y} \left( \hat{a}_{(x+1,y)}^\dagger \hat{a}_{(x,y)} + e^{i\Phi x} \hat{a}_{(x,y+1)}^\dagger \hat{a}_{(x,y)} \right) + \text{h. c.}
\]  

(S.11)

and the three real-space transformations that should be used depending on the position of the drain site,

\[
\hat{C}_{z,0} \Rightarrow \sigma(x,y),(x',y') = (-1)^{x+y} \delta_{x,-x'} \delta_{y,y'}
\]

\[
\hat{C}_{z,0} \Rightarrow \sigma(x,y),(x',y') = (-1)^{x+y} \delta_{x,x'} \delta_{y,-y'}
\]

\[
\hat{C}_{z,z} \Rightarrow \sigma(x,y),(x',y') = (-1)^{x+y} \delta_{x,y} \delta_{y,x} e^{i\Phi xy}
\]

It’s important to stress that these transformations are different in real space, but very similar in their effects on the eigenmodes. To stress this, we show the effect of these transformations at \( \Phi = 0 \), where the model is exactly solvable. We note that in the absence of a flux, \( \Phi \), the bright modes behave in much the same way as in its presence. However, the \( \Phi = 0 \) model is highly-degenerate and has many dark modes.

We take a 2D lattice of size \( N = 2M + 1 \times 2M + 1 \). Its eigenmodes are given by

\[
\hat{a}_{(k,q)} = \frac{1}{M+1} \sum_{x,y} \sin[k(x + M + 1)] \sin[q(y + M + 1)] \hat{a}_{(x,y)}
\]

(S.12)

for \( k, q \in \frac{\pi}{2(M+1)} \times \{1, \ldots, 2M + 1\} \), with energies

\[
\left[ \hat{a}_{(k,q)}, \hat{H} \right] = -2J(\cos k + \cos q) \hat{a}_{(k,q)}.
\]

(S.13)

Note that \( \varepsilon_{(\pi-k,\pi-q)} = -\varepsilon_{(k,q)} \) and that \( \varepsilon_{(q,k)} = \varepsilon_{(k,q)} \).

We note first the chiral symmetry,

\[
\hat{S} \hat{a}_{(x,y)} \hat{S}^{-1} = \sum_{x',y'} (-1)^{x+y} \delta_{x,x'} \delta_{y,y'} \hat{a}_{(x',y')}
\]

(S.14)

has

\[
\hat{S} \hat{a}_{(k,q)} \hat{S}^{-1} = \frac{1}{M+1} \sum_{x,y} \sin[k(x + M + 1)] \sin[q(y + M + 1)](-1)^{x+y} \hat{a}_{(x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[(\pi - k)(x + M + 1)] \sin[(\pi - q)(y + M + 1)] \hat{a}_{(x,y)} = \hat{a}_{(\pi-k,\pi-q)}.
\]

This guarantees the existence of a chiral structure and therefore particle-hole symmetries.

Next, we examine the first two particle-hole symmetries,

\[
\hat{C}_{z,0} \hat{a}_{(k,q)} \hat{C}_{z,0}^{-1} = \frac{1}{M+1} \sum_{x,y} \sin[k(x + M + 1)] \sin[q(y + M + 1)](-1)^{x+y} \hat{a}_{(-x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[k(x - M - 1)] \sin[(\pi - q)(y + M + 1)](-1)^{x+M+1} \hat{a}_{(x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[(\pi - k)(x + M + 1)] \sin[(\pi - q)(y + M + 1)] \hat{a}_{(x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[(\pi - k)(x + M + 1)] \sin[(\pi - q)(y + M + 1)] \hat{a}_{(x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[(\pi - k)(x + M + 1)] \sin[(\pi - q)(y + M + 1)] \hat{a}_{(x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[(\pi - k)(x + M + 1)] \sin[(\pi - q)(y + M + 1)] \hat{a}_{(x,y)}
\]

\[
= \frac{1}{M+1} \sum_{x,y} \sin[(\pi - k)(x + M + 1)] \sin[(\pi - q)(y + M + 1)] \hat{a}_{(x,y)}
\]

They both take each mode, up to a phase, to the antiparticle of its negative energy mode.
Finally, the last symmetry operator has

\[ \hat{C}_{z,z} \hat{a}_{(k,q)} \hat{C}_{z,z}^{-1} = \frac{1}{M+1} \sum_{x,y} \sin[k(x + M + 1)] \sin[q(y + M + 1)] (-1)^{x+y} \hat{a}^\dagger_{(y,x)} = \hat{a}^\dagger_{(\pi-q,\pi-k)}. \]

It takes each mode to a \textit{different} mode anti-particle, still corresponding to the same negative energy. This is possible because the zero-flux mode has the degeneracies mentioned above. The addition of flux breaks this degeneracy.
DERIVATION OF THE DISSIPATION SPECTRUM

The evolution equations for the eigenmodes of the system are

\[ \dot{b}_i = -i \sum_j A_{i,j} b_j + e^{-i\varphi} \sqrt{\bar{\Gamma}_i} \zeta, \]

\[ A_{i,j} = \delta_{i,j} \varepsilon_i - ie^{i(\varphi_j - \varphi_i)} \frac{1}{2} \sqrt{\bar{\Gamma}_i \bar{\Gamma}_j}, \quad \bar{\Gamma}_i = |\psi_i[n_0]|^2 \Gamma_i, \quad \varphi_i = \text{arg}(\psi_i[n_0]). \]  

(S.15)

We seek the eigenmodes of \( A \) and their eigenvalues. These have

\[ \dot{b}_\nu = \sum_j u_{\nu,j} b_j, \quad \dot{b}_\nu = -\left( \frac{\gamma}{2} + i\nu \right) b_i + g_\nu \zeta. \]  

(S.16)

Calculating,

\[ \dot{b}_\nu = \sum_j u_{\nu,j} \left[ -i \varepsilon_j b_j + e^{-i\varphi_j} \sqrt{\bar{\Gamma}_j} \left( -\frac{1}{2} \sum_l e^{i\varphi_l} \sqrt{\bar{\Gamma}_l} b_l + \zeta \right) \right] \]

\[ = -\sum_j \left( \frac{1}{2} g_\nu e^{i\varphi_j} \sqrt{\bar{\Gamma}_j} + i\varepsilon_j u_{\nu,j} \right) b_j + g_\nu \zeta \]

\[ = -\sum_j \left( \frac{\gamma}{2} + i\nu \right) b_j + g_\nu \zeta \]

\[ = -\sum_j \left( \frac{\gamma}{2} + i(\nu - \varepsilon_j) \right) u_{\nu,j} \zeta = \frac{1}{2} g_\nu e^{i\varphi_j} \sqrt{\bar{\Gamma}_j} \]

⇒ \[ \left( \frac{\gamma}{2} + i(\nu - \varepsilon_j) \right) u_{\nu,j} = \frac{1}{2} g_\nu e^{i\varphi_j} \sqrt{\bar{\Gamma}_j} \]

• A solution with \( \gamma = 0, \nu = \varepsilon_i \) for some \( i \) is consistent only if \( \bar{\Gamma}_i = 0 \), with \( u_{\nu,j} = \delta_{i,j} \) and \( g_\nu = 0 \). These are the dark modes, which are unaffected by the dissipation.

• Otherwise, we solve for \( u_{\nu,j} \) and find the self-consistent equation:

\[ g_\nu = \sum_j u_{\nu,j} e^{-i\varepsilon_j} \sqrt{\bar{\Gamma}_j} = \sum_j \frac{g_\nu \sqrt{\bar{\Gamma}_j}}{\frac{\gamma}{2} + i(\nu - \varepsilon_j)} \sqrt{\bar{\Gamma}_j} = g_\nu \sum_j \frac{\bar{\Gamma}_j}{\frac{\gamma}{2} + i(\nu - \varepsilon_j)} \]  

(S.17)
NON-PURE STEADY STATES FROM ZERO-ENTROPY RESERVOIR

We note in the main text, below Eq. (7), the requirement for a unique steady state is that there are no dark states of the dissipator. We note that this guarantees a unique state but not a unique pure state, even for a zero-entropy reservoir such as the one we use. To demonstrate this, consider a two-mode system,

\[ \hat{H} = \frac{V}{2} (\hat{a}^\dagger_1 \hat{a}_1 - \hat{a}^\dagger_2 \hat{a}_2) - J (\hat{a}^\dagger_1 \hat{a}_2 + \hat{a}^\dagger_2 \hat{a}_1). \]  

(S.18)

Choosing site \( n_0 = 1 \) as the drain, we have the steady state given by

\[ \langle \hat{a}^\dagger_1 \hat{a}_1 \rangle = \langle \hat{a}^\dagger_2 \hat{a}_2 \rangle = \sinh^2 r \quad \langle \hat{a}^\dagger_1 \hat{a}_2 \rangle = 0 \]

\[ \langle \hat{a}_1 \hat{a}_2 \rangle = \frac{4J^2 - \Gamma V}{4J^2 + V^2 - iV} \quad \langle \hat{a}_1 \hat{a}_2 \rangle = \frac{-4J^2}{4J^2 + V^2 - iV} \quad \langle \hat{a}_1 \hat{a}_2 \rangle = \frac{2JV}{4J^2 + V^2 - iV} \]

and we can calculate the purity of the state,

\[ \mu = \sqrt{\frac{(4J^2 + V^2)^2 + \Gamma^2 V^2}{(4J^2 + V^2 \cosh^2 2r)^2 + \Gamma^2 V^2 \cosh^4 2r}}. \]  

(S.19)

As expected, it is a pure state when \( V = 0 \), and we have a chiral system with \( |\psi_{+}[1]| = |\psi_{-}[1]| \), or when \( r = 0 \), and there is no squeezing. Otherwise, the steady state is a mixed state.
A cluster state is a highly-entangled state of a many-qubit [39] or many-oscillator [40] system. Their high degree of entanglement makes them useful resource in a variety of quantum computing and quantum communication applications.

A continuous-variable cluster state is given by the wavefunction [41]

\[ |\psi_A\rangle = \exp \left[ \frac{i}{2} \sum_{m,n} A_{mn} \hat{x}_m \hat{x}_n \right] |\text{vac}\rangle \]  

(S.20)

where the real, symmetric matrix \( A \) is the adjacency matrix representing a graph of connections. It also has a set of nullifiers, given by \( (\hat{p}_m - \sum_n A_{m,n} \hat{x}_n) |\psi_A\rangle = 0 \).

A similarly powerful state, which can be generated by pumping an optical parametric oscillator [42], is known as an \( H \)-graph state and takes the form

\[ |\psi_Z\rangle = \exp \left[ -\frac{i}{2} \sum_{m,n} \alpha G_{m,n} (\hat{x}_m \hat{p}_n + \hat{p}_m \hat{x}_n) \right] |\text{vac}\rangle \]  

(S.21)

where the adjacency matrix is now \( Z = i \exp[-2\alpha G] \) for some real \( G \).

The steady state described by Eqs. (10) and (11) in the main text, which is described by

\[ |\psi_\sigma\rangle = \exp \left[ \frac{1}{2} \sum_{m,n} r (e^{i\phi_\sigma_{m,n}} \hat{a}_m^\dagger \hat{a}_n^\dagger - e^{-i\phi_\sigma_{m,n}} \hat{a}_m \hat{a}_n) \right] |\text{vac}\rangle, \]  

(S.22)

can clearly be taken as an extension of the \( H \)-graph state to a complex adjacency matrix.

We note also that it has a set of nullifier states, given by \( (\hat{p}_m - \sum_n \tilde{A}_{m,n} \hat{x}_n) |\psi_\sigma\rangle \) where

\[ \tilde{A} = (I + \tanh r \text{Re}[e^{i\phi_\sigma}]) \cdot (\tanh r \text{Im}[e^{i\phi_\sigma}]). \]  

(S.23)