ON MULTIPLICITIES OF POINTS ON SCHUBERT VARIETIES IN GRASSMANNIANS II

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ABSTRACT. We prove a conjecture by Kreiman and Lakshmibai on a combinatorial description of multiplicities of points on Schubert varieties in Graßmannians in terms of certain sets of reflections in the corresponding Weyl group. The proof is accomplished by setting up a bijection between these sets of reflections and the author’s previous combinatorial interpretation of these multiplicities in terms of nonintersecting lattice paths (Séminaire Lotharingien Combin. 45 (2001), Article B45c).

1. Introduction. The multiplicity of a point on an algebraic variety is an important invariant that “measures” singularity of the point. It was an important recent advance in Schubert calculus when Rosenthal and Zelevinsky [13] gave a determinantal formula for the multiplicity of a point on a Schubert variety in a Graßmannian. It paved the way to a combinatorial understanding of this multiplicity. More precisely, it was shown in [6] that it counts certain families of nonintersecting lattice paths (and also certain tableaux). An alternative, conjectural combinatorial interpretation was proposed by Kreiman and Lakshmibai in [9, Conjecture 2], in terms of certain sets of reflections. The purpose of this paper is to prove that this latter combinatorial interpretation is indeed valid.

The reason for the proposition of this alternative combinatorial interpretation of the multiplicity of a point on a Schubert variety in a Graßmannian in terms of sets of reflections is that it appears that these sets of reflections also allow the computation of the Hilbert series of the tangent cone at this point (see [9, Conjecture 1]). While we are not able to prove this more general conjecture, we provide an equivalent form of the conjecture in which the Hilbert series is essentially given in terms of a generating
function for certain families of nonintersecting lattice paths which are counted with respect to turns. This equivalent form of the conjecture has the advantage over the original form that it reduces the computation of the Hilbert series to a finite problem. Moreover, it is analogous to similar formulas for the Hilbert series associated to related determinantal varieties (see [2, Eq. (1)], [7, p. 1021, line 11] or [8, Theorem 1]).

Our paper is organised as follows. In the next section we fix notation and formulate the multiplicity conjecture by Kreiman and Lakshmibai. There we also recall the author’s combinatorial interpretation of the multiplicity in terms of nonintersecting lattice paths. Section 3 contains the proof of the conjecture, which is accomplished by setting up a bijection between these nonintersecting lattice paths and the sets of reflections of Kreiman and Lakshmibai. Finally, in Theorem 2 in Section 4, we show that the results from Section 3 allow in fact the above mentioned reformulation, in terms of nonintersecting lattice paths, of Conjecture 1 in [9] on the Hilbert series of the tangent cone at a point on a Schubert variety in the Graßmannian. It is an open problem to find a compact formula for the generating function of nonintersecting lattice paths that appears in this formulation (see Remark (2) after Theorem 2).

2. Combinatorial interpretations of multiplicities of points on Schubert varieties in Graßmannians. We recall some basic notions from the Schubert calculus in the Graßmannian, and fix the notation that we are going to use. We refer the reader to [1, Sec. 3.1] and [3, Sec. 9.4] for in-depth introductions into the subject.

Let \( d \) and \( n \) be positive integers with \( 0 \leq d \leq n \). The Graßmannian \( \text{Gr}_{d}(V) \) is the variety of all \( d \)-dimensional subspaces in an \( n \)-dimensional vector space \( V \) (over some algebraically closed field of arbitrary characteristic). Schubert varieties in the Graßmannian \( \text{Gr}_{d}(V) \) are indexed by elements in \( S_{n}/(S_{d} \times S_{n-d}) \), where \( S_{m} \) denotes the symmetric group of order \( m \). Any coset \( C \) in \( S_{n}/(S_{d} \times S_{n-d}) \) has a minimal representative, which is the unique permutation \( w = i_{1}i_{2}\ldots i_{n} \) in \( C \) such that \( i_{1} < i_{2} < \cdots < i_{d} \) and \( i_{d+1} < \cdots < i_{n-1} < i_{n} \). We will often identify such a minimal representative \( w \) with the vector \( i = (i_{1}, i_{2}, \ldots, i_{d}) \) of its first \( d \) elements. The usual Ehresmann–Bruhat order on \( S_{n} \) induces an order on the cosets of \( S_{n}/(S_{d} \times S_{n-d}) \).

Given two representatives, identified with \( i = (i_{1}, i_{2}, \ldots, i_{d}) \) and \( j = (j_{1}, j_{2}, \ldots, j_{d}) \), respectively, \( j \) is less or equal than \( i \) in this induced Bruhat order if and only if \( j_{\ell} \leq i_{\ell} \) for all \( \ell = 1, 2, \ldots, d \).

Given a minimal representative \( w \), we denote the corresponding Schubert variety in the Graßmannian \( \text{Gr}_{d}(V) \) by \( X(w) \). It is well-known that \( X(w) \) decomposes into the disjoint union of Schubert cells which are indexed by elements \( \tau \in S_{n}/(S_{d} \times S_{n-d}) \) with \( \tau \leq w \). The multiplicity of a point \( x \) in \( X(w) \) is constant on each Schubert cell. Following [9] we denote the multiplicity of a point \( x \) in the Schubert cell indexed by \( \tau \) by \( \text{mult}_{\tau} X(w) \). In slight abuse of terminology we will often call it the “multiplicity of the point \( \tau \) on the Schubert variety \( X(w) \).”

Let us now recall the multiplicity formula conjectured in [9]. We are given two elements \( w \) and \( \tau \) in \( S_{n}/(S_{d} \times S_{n-d}) \). In Conjecture 2 of [9], sets \( S \) of reflections \( s = (x, y), 1 \leq x \leq d, d + 1 \leq y \leq n \) (here we use standard transposition notation), are considered with the property that

(S1) Any chain \( s_{1} > s_{2} > \cdots > s_{t} \) of commuting reflections, all of them contained
in $S$, satisfies $w \geq \tau s_1 \cdots s_t$ (in the induced Bruhat order on $S_n/(S_d \times S_{n-d})$); (S2) $S$ is maximal with respect to property (S1).

Now we are in the position to formulate Conjecture 2 from [9], which becomes a theorem by our proof in Section 3.

**Theorem 1.** The multiplicity of the point $\tau$ on the Schubert variety $X(w)$ is given by

$$\text{mult}_\tau X(w) = |\{S : S \text{ satisfies (S1) and (S2)}\}|.$$  \hspace{1cm} (2.1)

Next we recall the combinatorial interpretation of multiplicities in terms of non-intersecting lattice paths from [6], which is a more or less straight-forward combinatorial translation of the Rosenthal–Zelevinsky formula [13] using the Lindström–Gessel–Viennot theorem [11, Lemma 1], [4, Theorem 1]. As before, let $w$ and $\tau$ be two elements from $S_n/(S_d \times S_{n-d})$, and identify them with $i = (i_1, i_2, \ldots, i_d)$,
$i_1 < i_2 < \cdots < i_d$, and $j = (j_1, j_2, \ldots, j_d)$, $j_1 < j_2 < \cdots < j_d$, respectively. Furthermore, we define the numbers $\kappa_q := |\{\ell : i_q < j_\ell\}|$. Then

$$\text{mult}_\tau X(w) = \#(\text{families } (P_1, P_2, \ldots, P_d) \text{ of nonintersecting lattice paths,}
\text{where the path } P_\ell \text{ runs from } (d+1-\ell, d) \text{ to}
(d-\kappa_\sigma(\ell), \kappa_\sigma(\ell) + i_\sigma(\ell)), \ell = 1, 2, \ldots, d),$$

(2.2)

where $\sigma$ is some permutation in $S_d$. See Figure 1. There, $d = 9$, $i = (4, 6, 7, 13, 14, 17, 19, 20, 21)$ and $j = (1, 2, 4, 7, 10, 12, 13, 15, 16)$. For this choice the vector of the $\kappa_q$’s is $(6, 6, 5, 2, 2, 0, 0, 0, 0)$. Figure 1 shows a typical family of paths as described in (2.2) for this choice of $i$ and $j$. The permutation $\sigma$ is 674583129 in this example.

At this point, there are two remarks to be made: First, in [6] the starting points of the paths are $(-\ell+1, \ell-1)$ and the end points are $(-\kappa_\ell, \kappa_\ell + i_\ell)$, $\ell = 1, 2, \ldots, d$ (the latter in some order, determined by the permutation $\sigma$). If we shift everything by $d$ units to the right then we obtain the points $(d+1-\ell, \ell-1)$ and $(d-\kappa_\ell, \kappa_\ell + i_\ell)$. Whereas now the end points are in agreement, the starting points still differ slightly. However, the arguments in Section 4 of [6] (and, in fact, figures such as Figure 3 in [6]) show that portions of paths below the horizontal line $y = d$ are forced and can therefore be omitted. This means that we may replace the starting points $(d+1-\ell, \ell-1)$ by the points $(d+1-\ell, d)$. (In fact, Figure 1 shows exactly the result when the forced portions of the paths in Figure 3 of [6] are cut off.) Second, the order in which starting and end points are connected by the nonintersecting paths is always the same, i.e., for fixed $i$ and $j$ the permutation $\sigma$ in (2.2) is always the same.

3. Proof of the theorem. We will prove that the multiplicity formulas in Theorem 1 and (2.2) are equivalent. First we claim:

Claim 1. If a reflection $s = (x, y)$ is identified with the point $(x, y)$ in the plane, then a set of reflections as described in (2.1) is the set of all the lattice points with $y$-coordinates $> d$ on the paths of a family of paths as described in (2.2). In turn, given a family of paths as described in (2.2), the set of lattice points on the paths with $y$-coordinate $> d$ form a set of reflections as described in (2.1), under the above identification of reflections and points in the plane.

To return to our example in Figure 1: the set of lattice points on the paths with $y$-coordinate $> 9$, i.e., the set $\{(1, 10), (1, 11), (1, 12), (1, 13), (2, 13), (2, 14), \ldots, (9, 21), (2, 10), \ldots, (3, 12), \ldots\}$, is a set of reflections with the properties (S1) and (S2).

In fact, we are going to prove a more general claim. In order to be able to formulate it, we have to explain the “light-and-shadow procedure with the sun in the south-east.” We will do this by considering an example.

Suppose that we are given a multiset $S$ of reflections, identified with points in the plane as in Claim 1. For example, Figure 2.a shows the multiset of reflections (points)

$$(2, 13), (3, 10), (3, 10), (3, 11), (3, 11), (4, 10), (4, 16),
(5, 18), (5, 18), (5, 18), (6, 17), (6, 18), (7, 11), (7, 16), (7, 16),
(7, 19), (8, 21), (8, 21), (8, 21), (8, 21), (9, 13), (9, 18))$$.
Next we suppose that there is a light source being located in the bottom-right corner. The shadow of a point \((x, y)\) is defined to be the set of points \((x', y')\) \(\in \mathbb{R}^2\) (\(\mathbb{R}\) denoting the set of real numbers) with \(x' \leq x\) and \(y' \geq y\). We consider the (bottom-right) border of the union of the shadows of all the points of the multiset \(S\). We also include the shadows of the starting points \(A_\ell = (d + 1 - \ell, d)\) and the end points \(E_\ell = (d - \kappa_\ell, \kappa_\ell + i_\ell)\), \(\ell = 1, 2, \ldots, d\). This border is a lattice path. We restrict our attention to the portion of this lattice path between \(A_1\) and \(E_{\sigma(1)}\). (Here, as before, \(\sigma\) is the permutation as in (2.2) which describes how starting and end points are connected in the case of nonintersecting lattice paths. In our example in Figure 2, \(A_1\) and \(E_{\sigma(1)}\) are the points \((9, 9)\) and \((9, 17)\), respectively.) We remove all the points of the multiset that lie on this path, including \(A_1\) and \(E_{\sigma(1)}\). (In our example, we would remove \((9, 9)\), \((9, 13)\) and \((9, 17)\).) Then the light and shadow procedure is repeated.
with the remaining points. (That is, in the next step the roles of \(A_1\) and \(E_{\sigma(1)}\) are played by \(A_2\) and \(E_{\sigma(2)}\), respectively, etc.) We stop after a total of \(d\) iterations. (The result of applying this procedure to the multiset in Figure 2.a is shown in Figure 2.b.) It is obvious that at this point we will have obtained \(d\) nonintersecting lattice paths, the \(\ell\)-th path connecting \(A_\ell\) and \(E_{\sigma(\ell)}\). We are now ready to state:

**Claim 2.** If light-and-shadow with the sun in the south-east is applied to a multiset \(S\) of reflections satisfying (S1) (where we again identify a reflection \(s = (x, y)\) with the point \((x, y)\) in the plane), then one obtains a family of paths as described in (2.2) which in addition cover all the points of \(S\). In turn, given a family of paths as described in (2.2), any submultiset of the lattice points on the paths with \(y\)-coordinate \(> d\) forms a multiset of reflections which satisfies (S1), under the above identification of reflections and points in the plane.

Claim 2 does indeed imply Claim 1: For suppose that we are given a set \(S\) of reflections (viewed as set of points) as described in (2.1). Then the first assertion of Claim 2 says that this set of points lies on a family of paths as described in (2.2). Moreover, if \(S\) were not the complete set of lattice points on the paths with \(y\)-coordinate \(> d\), then we may add such a missing point, \((x, y)\) say, to \(S\). The second assertion of Claim 2 then says that \(S \cup \{(x, y)\}\) is a set of reflections satisfying (S1). Thus \(S\) was not maximal, a contradiction. On the other hand, if we are given a family of paths as described in (2.2) and consider the set \(S\) of all lattice points on the paths with \(y\)-coordinate \(> d\), then the second assertion of Claim 2 says that this is a set of reflections satisfying (S1). In addition, it is maximal with respect to (S1). For suppose that it is not. Then we may add another reflection, \((x, y)\) say, to \(S\), thus obtaining \(S' = S \cup \{(x, y)\}\). Clearly, if we apply light-and-shadow to \(S'\) then we will not have exhausted all elements of \(S'\) after these \(d\) iterations (i.e., the \(d\) paths obtained will not cover all elements of \(S'\)). However, this is a contradiction to the first assertion of Claim 2.

Claim 2 will be fully exploited in Section 4.

Let us call a set \(\{(x_1, y_1), (x_2, y_2), \ldots, (x_t, y_t)\}\) of points with \(x_1 < x_2 < \cdots < x_t\) and \(y_1 > y_2 > \cdots > y_t\) a *chain*. Furthermore, given a point \(A = (a_1, a_2)\), let us define regions \(R(A)\) by

\[
R(A) := \{(x, y) : x \leq a_1, \ y > a_2\}.
\]

(This is the region in the plane weakly to the left and strictly above the point \(A\).)

Claim 2 follows immediately from Claim 3 below. There, and in the following, we assume tacitly that any occurring set (multiset) of points is a subset (submultiset) of the rectangle \(\{(x, y) : 1 \leq x \leq d, \ d + 1 \leq y \leq i_d\}\).

**Claim 3.** Let \(i\) and \(j\) be as before. Then both the point multisets that satisfy (S1) and submultisets of lattice points with \(y\)-coordinate \(> d\) taken from a family of paths as described in (2.2) can be characterized as follows: For any \(q\) with \(1 \leq q \leq d\), the maximal number of points that can be chosen from such a multiset such that all of them are located inside \(R(E_q)\) and in addition form a chain is at most \(d - \kappa_q - q\). Here, as before, \(E_q = (d - \kappa_q, \kappa_q + i_q)\).
In the sequel, we will call the condition spelled out in the next-to-last sentence of Claim 3 the \textit{chain condition}.

Below we prove Claim 3, which in fact means to prove four assertions, labelled A1–A4.

A1. \textit{Any submultiset of lattice points with \(y\)-coordinate > \(d\) taken from a family of paths as described in (2.2) satisfies the chain condition.} This is obvious once one observes that \(d - \kappa_q - q\) is the number of lattice paths in the family that start strictly to the left of \(E_q\) and terminate weakly to the right of \(E_q\) (and, enforcedly, pass above \(E_q\); cf. Figure 1).

A2. \textit{If a multiset of lattice points satisfies the chain condition then it is a submultiset of the lattice points with \(y\)-coordinate > \(d\) of a family of paths as described in (2.2).} This is also more or less “obvious.” The only matter is notation. Probably the most convenient way to prove this rigorously is by induction on \(d\).
For $d = 1$ the assertion is obvious (the quantity $d - \kappa_q - q$ being 0 for $d = q = 1$). Let us now assume that we have already proved the assertion for $d$. Given $i = (i_1, i_2, \ldots, i_d)$ and $j = (j_1, j_2, \ldots, j_d)$ and a multiset of lattice points satisfying the chain condition, we apply light-and-shadow (with respect to the starting and end points determined by $i$ and $j$). We restrict our attention to the rightmost strip of the picture, i.e., the region of points with $x$-coordinate between $d$ and $d + 1$, see Figure 3. There, we have chosen $d = 8$, $i = (4, 6, 7, 13, 14, 17, 19, 20, 21)$ and $j = (1, 2, 4, 7, 10, 12, 13, 15, 16)$. The starting and end points determined by $i$ and $j$ are indicated by circles. The multiset of points is indicated by bold dots, multiplicities being indicated by the numbers in parentheses. (This is in fact the same example as in Figure 2. The path pieces should be ignored for the moment.)

Let $k$ be minimal such that $i_k \geq j_{d+1}$. (In our example we have $k = 6$.) Then the end points with $x$-coordinate $d + 1$ are $E_k, E_{k+1}, \ldots, E_{d+1}$. Clearly, under light-and-shadow, $E_k$ is connected with $A_1$. The path portions leading to the other end points $E_{k+1}, \ldots, E_{d+1}$ hit the vertical line $y = d$ the last time in the points $E_{k+1}', \ldots, E_{d+1}'$, say (see Figure 3). It is easy to see that for any $q$ with $k + 1 \leq q \leq d + 1$ the maximal number of points that can be chosen from such a multiset such that they form a chain and all of them are located inside $R(E_q')$ is at most $d + 1 - q$. Now we apply the induction hypothesis to $i' = (i_1, \ldots, i_{k-1}, i_{k+1}', \ldots, i_{d+1})$ and $j' = (j_1, j_2, \ldots, j_d)$, where $i_{k}'$ denotes the $y$-coordinate of $E_{k}'$, $\ell = k + 1, \ldots, d + 1$. It should be observed that, up to a vertical shift of 1 unit, the starting points determined by $i'$ and $j'$ are $A_2, A_3, \ldots, A_{d+1}$, whereas the corresponding end points are $E_1, \ldots, E_{k-1}, E_{k+1}', \ldots, E_{d+1}'$. By the above consideration, the multiset satisfies the chain condition with respect to these new starting and end points. The induction hypothesis then guarantees that light-and-shadow yields a family of paths connecting the (new) starting points with the (new) end points, thereby covering all (remaining) elements of the multiset. This family of paths is finally concatenated with the path portions that we already obtained in the strip between the vertical lines $y = d$ and $y = d + 1$.

A3. Any multiset of reflections satisfying (S1) satisfies the chain condition. Suppose we are given a multiset of reflections which satisfies (S1) but does not satisfy the chain condition. Then for some $q$ there is a chain of $d - \kappa_q - q + 1$ reflections from the multiset which, when viewed as points in the plane, are all located inside $R(E_q)$. Let the reflections in the chain be $s_1, s_2, \ldots, s_{d-\kappa_q-1}$. Let us consider a reflection in the chain, $(x, y)$ say. Since $(x, y) \in R(E_q)$ we have $x \leq d - \kappa_q$. Furthermore, we have

$$\tau s_1 \cdots s_{d-\kappa_q-1}(x) = \tau(y),$$

where as before $\tau$ is identified with $j$, i.e., $\tau = j_1 j_2 \cdots j_n$ with $j_1 < j_2 < \cdots < j_d$ and $j_1 < \cdots < j_{n-1} < j_n$. Since $(x, y)$ is contained in $R(E_q)$ we have $y > \kappa_q + i_q \geq d$. Because of $j_{d+1} \cdots < j_{n-1} < j_n$, this implies $\tau(y) \geq \tau(\kappa_q + i_q + 1)$.

We claim that $\tau(\kappa_q + i_q + 1) = i_q + 1$. This is seen as follows. Taking into account the trivial fact that the set of values $\{j_{d+1}, j_{d+2}, \ldots, j_n\}$ is equal to the complement of $\{j_1, j_2, \ldots, j_d\}$ in $\{1, 2, \ldots, n\}$, a value $\tau(y) = j_y$ for $y > d$ is characterized by

$$j_y = (y - d) + |\{\ell : j_\ell < j_y\}|.$$
Thus we may verify our claim by setting \( y = \kappa_q + i_q + 1 \) and substituting \( i_q + 1 \) for \( j\kappa_q + i_q + 1 \) in this equation. Indeed, we have \( i_1 + 1 = (\kappa_q + i_q + 1 - d) + (d - \kappa_q) \). Hence, we have \( \tau(y) > i_q \).

In summary, we have found \( d - \kappa_q - q + 1 \) values of \( x \) such that

\[
(\tau s_1 \cdots s_{d-\kappa_q-q+1})(x) > i_q,
\]

all of which are \( \leq d - \kappa_q \). Moreover, if \( d - \kappa_q < x \leq d \), then we have

\[
(\tau s_1 \cdots s_{d-\kappa_q-q+1})(x) = \tau(x) = j_x > i_q.
\]

Hence, in total we found \( (d - \kappa_q - q + 1) + \kappa_q = d - q + 1 \) values \( x \) for which (3.1) holds. If we recall that we also always identify \( w \), then this is a contradiction to (S1).

4. A formula for the Hilbert series of the tangent cone at a point. Now the full significance of Claim 2 can be revealed. Briefly, it allows the formulation of a version of Conjecture 1 in [9] which has the advantage of being efficient, as it reduces the computation of the Hilbert function to a finite problem. More precisely, we can express the Hilbert series in form of a finite summation. This form of the conjecture is the analogue of, say, formulas for the Hilbert series as in [2, Eq. (1)], [7, p. 1021, line 11] or in [8, Theorem 1].

In order to formulate this equivalent form, we need to introduce some notation. A point in a lattice path \( P \) which is the end point of a horizontal step and at the same time the starting point of a vertical step will be called an east-north turn (EN-turn for short) of the lattice path \( P \). For example, the EN-turns of the leftmost lattice paths in Figure 1 are \((2,13), (4,16) \text{ and } (5,18)\). We write \( \text{EN}(P) \) for the number of NE-turns of \( P \). Also, given a family \( \mathbf{P} = (P_1, P_2, \ldots, P_d) \) of paths \( P_i \), we write \( \text{EN}(\mathbf{P}) \) for the number \( \sum_{i=1}^d \text{EN}(P_i) \) of all EN-turns in the family. By \( \mathbb{P}_L^+(A \to E) \) we denote the set of all families \( (P_1, P_2, \ldots, P_d) \) of nonintersecting lattice paths, where \( P_i \) runs from \( A_i \) to \( E_i \). Finally, given any weight function \( \mu \) defined on a set \( \mathcal{M} \), by the generating function \( \text{GF}(\mathcal{M}; \mu) \) we mean \( \sum_{x \in \mathcal{M}} \mu(x) \).
Theorem 2. Conjecture 1 from [9] is equivalent to saying that the Hilbert series of the tangent cone to $X(w)$ at $\tau$ is equal to

$$\frac{\text{GF}(\mathbb{P}^+(A \to E); z^{\text{EN}.})}{(1 - z)^{\sum_{\ell=1}^{d} i_\ell - \binom{d+1}{2}}}, \quad (4.1)$$

with $A_\ell = (d + 1 - \ell, d)$ and $E_\ell = (d - \kappa_\ell, \kappa_\ell + i_\ell)$, $\ell = 1, 2, \ldots, d$, as before.

Proof. We can simply copy the corresponding proof in [7, first proof of Theorem 2].

According to the conjecture, the dimension of the $m$-th homogeneous component of the tangent cone is equal to the number of multisets of cardinality $m$ which satisfy (S1). Following [9] we denote this dimension by $h_{TC, X(w)}(m)$.

Let $S$ be such a multiset. We apply light-and-shadow to it. By Claim 2, we obtain a family $(P_1, P_2, \ldots, P_d)$ of paths as described in (2.2). Each path $P_\ell$ contains a few (possibly multiple) points of $S$. However, in each EN-turn of $P_\ell$ there has to be at least one element of $S$, $\ell = 1, 2, \ldots, d$. Therefore, and because of the second assertion of Claim 2, given such a family $(P_1, P_2, \ldots, P_d)$ of paths as described in (2.2) with a total number of exactly $t$ EN-turns, there are exactly $\binom{T + m - t - 1}{m - t}$ multisets $S$ of cardinality $m$ that reduce to $(P_1, P_2, \ldots, P_d)$ under light and shadow, where

$$T = \sum_{\ell=1}^{d} ((d - \kappa_\ell) + (\kappa_\ell + i_\ell)) - \sum_{\ell=1}^{d} ((d + 1 - \ell) + d) = \sum_{\ell=1}^{d} i_\ell - \binom{d + 1}{2}$$

is the total number of lattice points with $y$-coordinate $> d$ on the lattice paths $P_1, P_2, \ldots, P_d$. (It is independent of the path family.)

Hence, if we let $h_t$ denote the number of all families $(P_1, P_2, \ldots, P_d)$ of paths as described in (2.2) with a total number of exactly $t$ EN-turns, we obtain for the Hilbert series,

$$\sum_{m=0}^{\infty} h_{TC, X(w)}(m) z^m = \sum_{m=0}^{\infty} \left( \sum_{t=0}^{m} \binom{T + m - t - 1}{m - t} h_t \right) z^m = \sum_{t=0}^{\infty} h_t \sum_{m=0}^{\infty} \binom{T + m - t - 1}{m - t} z^m = \sum_{t=0}^{\infty} h_t z^t \sum_{m=0}^{\infty} \binom{T + m - 1}{m} z^m = \sum_{t=0}^{\infty} h_t z^t \frac{1}{(1 - z)^{T}}.$$

Now, the generating function $\sum_{t=0}^{\infty} h_t z^t$ is exactly the numerator in (4.1). This proves the theorem. □

Remarks. (1) If true, formula (4.1) implies the Rosenthal–Zelevinsky formula. For, the multiplicity $\text{mult}_\tau X(w)$ is equal to the numerator of the Hilbert series of the
tangent cone to $X(w)$ at $\tau$, evaluated at $z = 1$. But, by (4.1), this is exactly the number of all families of nonintersecting lattice paths in $\mathbb{P}^+(A \to E)$, i.e., of all path families as described in (2.2). As we already remarked earlier, the combinatorial interpretation of the multiplicity in terms of nonintersecting lattice paths as given in (2.2) is equivalent to the Rosenthal–Zelevinsky formula.

(2) Unfortunately, all the results that have been found so far on the enumeration of nonintersecting lattice paths with respect to turns (see [5, 7, 8, 10, 12]) do not cover the above case, because the location of the starting and end points is quite unusual. This means that, up to now, there is no compact formula (a determinant, or whatever) for $\text{GF}(\mathbb{P}^+(A \to E); z^{\text{EN}(\cdot)})$.

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