ON RADICALLY GRADED FINITE DIMENSIONAL QUASI-HOPF ALGEBRAS

PAVEL ETINGOF AND SHLOMO GELAKI

Abstract. In this paper we continue the structure theory of finite dimensional quasi-Hopf algebras started in [EG] and [G]. First, we completely describe the class of radically graded finite dimensional quasi-Hopf algebras over C, whose radical has prime codimension. As a corollary we obtain that if p > 2 is a prime then any finite tensor category over C with exactly p simple objects which are all invertible must have Frobenius-Perron dimension $p^N$, $N = 1, 2, 3, 4, 5$ or 7.

Second, we construct new examples of finite dimensional quasi-Hopf algebras which are not twist equivalent to a Hopf algebra. For instance, to every finite dimensional simple Lie algebra $g$ and an odd integer $n$, coprime to 3 if $g = G_2$, we attach a quasi-Hopf algebra of dimension $n^{\dim g}$.

1. Introduction

In [EO] it is proved that any finite tensor category over C with integer Frobenius-Perron dimensions of objects is equivalent to a representation category of a finite dimensional quasi-Hopf algebra (the Frobenius-Perron dimension of a representation coincides with its dimension as a vector space). Therefore the classification of finite tensor categories with integer Frobenius-Perron dimensions of objects is equivalent to the classification of complex finite dimensional quasi-Hopf algebras. The simplest finite tensor categories to try to understand are those which have only 1-dimensional simple objects which form a cyclic group of prime order under tensor product. Equivalently, one is led to the problem of classifying finite dimensional quasi-Hopf algebras with (Jacobson) radical of prime codimension.

Let $p$ be a prime, and let $RG(p)$ denote the class of radically graded finite dimensional quasi-Hopf algebras over C, whose radical has codimension $p$. It was shown in [EG] that any $H \in RG(2)$ is equivalent to a Nichols Hopf algebra $H_{2^n}$, $n \geq 1$, or to a lifting of one of the four special quasi-Hopf algebras $H(2)$, $H(8)$, $H_{-}(8)$, $H(32)$ of dimensions 2, 8, 8, and 32 (the algebra $H(2)$ is the group algebra of $\mathbb{Z}_2$ with a nontrivial associator).

Later, it was shown in [G] that if $H \in RG(p)$, $p > 2$, has a nontrivial associator and if the rank of $H[1]$ over $H[0]$ is 1, then $H$ is equivalent to one of the quasi-Hopf algebras $A(q)$ of dimension $p^3$, introduced in [G]. More precisely, the result of [G] is formulated under the assumption that $H$ is basic (i.e., $H/\text{Rad}(H) = \mathbb{C}[\mathbb{Z}_p]$ with some associator), but by [ENO], Corollary 8.31, this is automatic.

The purpose of this paper is to continue the structure theory of finite dimensional quasi-Hopf algebras started in [EG] and [G]. More specifically, we completely describe the class $RG(p)$, and construct new examples of finite dimensional quasi-Hopf algebras which are not twist equivalent to a Hopf algebra.

Date: January 13, 2005.

Key words and phrases. Quasi-Hopf algebras, finite tensor categories.
The structure of the paper is as follows. In Section 2 we recall the definition of the quasi-Hopf algebras $A(q)$ and $H_+(p)$.

In Section 3 we show that if $H \in RG(p)$ has a nontrivial associator, then the rank of $H[1]$ over $H[0]$ is $\leq 1$. This yields the following classification of $H \in RG(p)$, $p > 2$, up to twist equivalence.

(a) Duals of pointed Hopf algebras with $p$ grouplike elements, classified in [AS], Theorem 1.3.

(b) Group algebra of $\mathbb{Z}_p$ with associator defined by a 3-cocycle.

(c) The algebras $A(q)$.

This result implies, in particular, that if $p > 2$ is a prime then any finite tensor category over $\mathbb{C}$ with exactly $p$ simple objects which are all invertible must have Frobenius-Perron dimension $p^N$, $N = 1, 2, 3, 4, 5$ or 7.

In Section 4 we construct new examples of finite dimensional quasi-Hopf algebras $H$, which are not twist equivalent to a Hopf algebra. They are radically graded, and $H/\text{Rad}(H) = \mathbb{C}[\mathbb{Z}_m^+]$, with a nontrivial associator. For instance, to every finite dimensional simple Lie algebra $g$ and an odd integer $n$, coprime to 3 if $g = G_2$, we attach a quasi-Hopf algebra of dimension $n^{\dim g}$.

Acknowledgments. The research of the first author was partially supported by the NSF grant DMS-9988796. The second author was supported by Technion V.P.R. Fund - Dent Charitable Trust - Non Military Research Fund, and by The Israel Science Foundation (grant No. 70/02-1). He also thanks MIT for its warm hospitality. Both authors were supported by BSF grant No. 2002040.

2. Preliminaries

All constructions in this paper are done over the field of complex numbers $\mathbb{C}$.

We refer the reader to [D] for the definition of a quasi-Hopf algebra and a twist of a quasi-Hopf algebra.

2.1. We recall the theory of the radical filtration for finite dimensional quasi-Hopf algebras, discussed in [EG]. It is completely parallel to the classical theory of such filtration in finite dimensional Hopf algebras.

Let $H$ be a finite dimensional quasi-Hopf algebra, and $I$ be the radical of $H$. Assume that $I$ is a quasi-Hopf ideal, i.e., $\Delta(I) \subseteq H \otimes I + I \otimes H$. In categorical terms, this means that the category of representations $\text{Rep}(H)$ has Chevalley property, i.e., the tensor product of irreducible $H$-modules is completely reducible. This is satisfied, for example, if $H$ is basic, i.e., every irreducible $H$-module is 1-dimensional.

In this situation, the filtration of $H$ by powers of $I$ is a quasi-Hopf algebra filtration. Thus the associated graded algebra $\text{gr}(H)$ of $H$ under this filtration has a natural structure of a quasi-Hopf algebra.

Let now $\overline{H}$ be a finite dimensional quasi-Hopf algebra with a $\mathbb{Z}_+$-grading, i.e., $\overline{H} = \oplus_{m \geq 0} \overline{H}[m]$, with all structure maps being of degree zero. In this case, $\overline{H}[0]$ is a quasi-Hopf algebra, $\overline{H}[i]$ is a free module over $\overline{H}[0]$ for all $i$ (by Schauenburg’s theorem [S]), and the radical $I$ of $\overline{H}$ is a quasi-Hopf ideal.

One says that $\overline{H}$ is radically graded if $\overline{H}^k = \oplus_{m \geq k} \overline{H}[m]$, for $k \geq 1$. In this case, $\overline{H}[0]$ is semisimple, and $\overline{H}$ is generated by $\overline{H}[0]$ and $\overline{H}[1]$.

An example of a radically graded quasi-Hopf algebra is the algebra $\text{gr}(H)$ defined above. Moreover, $H$ is radically graded if and only if $\text{gr}(H) = H$. 
Finally, we observe that if $H$ is radically graded and basic, then $H[0] = \text{Fun}(G)$ for a finite group $G$, and the associator (being of degree zero) corresponds to a class in $H^3(G, \mathbb{C}^*)$.

2.2. The following are the simplest examples of quasi-Hopf algebras not twist equivalent to a Hopf algebra.

Let $p > 2$ be a prime, and $\varepsilon = e^{2\pi i/p}$. If $z \in \mathbb{Z}$, we denote by $z'$ the projection of $z$ to $\mathbb{Z}_p$.

Let $s$ be an integer such that $1 \leq s \leq p - 1$. Let $Q = \varepsilon^{-s}$. The $p$–dimensional quasi-Hopf algebra $H(p, s)$, is generated by a grouplike element $a$ such that $a^p = 1$, with non-trivial associator

$$\Phi_a := \sum_{i,j,k=0}^{p-1} Q^{-i(j+k)-(j+k)'_p} 1_i \otimes 1_j \otimes 1_k$$

where $\{1_i|0 \leq i \leq p-1\}$ is the set of primitive idempotents of $\mathbb{Z}_p$ (i.e., $1_i a = Q^i 1_i$), distinguished elements $\alpha = a$, $\beta = 1$, and antipode $S(a) = a^{-1}$.

Let $s_0 \in \mathbb{Z}_p$ be a quadratic nonresidue. It can be shown (by considering automorphisms $a \mapsto a^{s_0}$) that for any $s$, $H(p, s)$ is isomorphic to $H_+(p) := H(p, 1)$ if $s$ is a quadratic residue, and to $H_-(p) := H(p, s_0)$ if $s$ is a non-quadratic residue. On the other hand, $H_+(p)$ and $H_-(p)$ are not equivalent.

Thus it follows from [ENO], Corollary 8.31, that any $p$–dimensional semisimple quasi-Hopf algebra is twist equivalent either to $\mathbb{C}[\mathbb{Z}_p]$ or to $H_{\pm}(p)$.

2.3. The following are examples of $p^3$–dimensional basic quasi-Hopf algebras with radical of codimension $p$, which are not twist equivalent to a Hopf algebra.

**Theorem 2.1.** [C] Let $p$ be a prime number.

(i) There exist $p^3$–dimensional quasi-Hopf algebras $A(q)$, parametrized by primitive roots of unity $q$ of order $p^2$, which have the following structure. As algebras $A(q)$ are generated by $a, x$ with the relations $ax = q^p xa$, $a^p = 1$, $x^{p^2} = 0$. The element $a$ is grouplike, while the coproduct of $x$ is given by the formula

$$\Delta(x) = x \otimes \sum_{y=0}^{p-1} q^y 1_y + 1 \otimes (1 - 1_0)x + a^{-1} \otimes 1_0 x,$$

where $\{1_i|0 \leq i \leq p-1\}$ is the set of primitive idempotents of $\mathbb{C}[a]$ defined by the condition $a 1_i = q^p 1_i$, the associator is $\Phi_a$ (where $s$ is defined by the equation $\varepsilon^{-s} = q^p$), the distinguished elements are $\alpha = a$, $\beta = 1$, and the antipode is $S(a) = a^{-1}$, $S(x) = -x \sum_{z=0}^{p-1} q^{p^2} 1_z$.

(ii) The quasi-Hopf algebras $A(q)$ are pairwise non-equivalent. Any finite dimensional radically graded basic quasi-Hopf algebra $H$ with radical of codimension $p$ and nontrivial associator, such that $H[1]$ is a free module of rank 1 over $H[0]$, is equivalent to $A(q)$ for some $q$.

3. QUASI-HOPF ALGEBRAS WITH RADICAL OF PRIME CODIMENSION

3.1. The main result. Let $p > 2$ be a prime number. Our main result in this section is the following theorem.

**Theorem 3.1.** Let $H$ be a radically graded basic quasi-Hopf algebra with radical of codimension $p$. If the associator of $H$ is nontrivial, then the rank of $H[1]$ over $H[0]$ is $\leq 1$. 

Theorem 3.1 is proved in the next subsection.

Theorem 3.1 and the results cited above imply the following classification result.

**Theorem 3.2.** Let $H$ be a radically graded finite dimensional quasi-Hopf algebra with radical of codimension $p$. Then $H$ is one of the following quasi-Hopf algebras, up to twist equivalence:

(a) Duals of pointed Hopf algebras with $p$ grouplike elements, classified in [AS], Theorem 1.3 (including the group algebra $\mathbb{C}[\mathbb{Z}_p]$).

(b) The algebras $H_+(p)$ and $H_-(p)$.

(c) The algebras $A(q)$.

**Proof.** By Corollary 8.31 of [ENO], $H$ is necessarily basic.

If the associator of $H$ is trivial, then we may assume that $H$ is a Hopf algebra. Thus $H^*$ is a coradically graded pointed Hopf algebra with $G(H^*) = \mathbb{Z}_p$. Such algebras are classified in [AS], Theorem 1.3, so we are in case (a).

If the associator is nontrivial, then by Theorem 3.1 the rank of $H[1]$ over $H[0]$ is at most 1. If the rank is 0, we are in case (b). If the rank is 1, we are in case (c) by Theorem 2.1.

We refer the reader to [EO], for the definition of a finite tensor category and the notion of its Frobenius-Perron dimension.

**Corollary 3.3.** Let $p > 2$ be a prime. Let $\mathcal{C}$ be a finite tensor category, which has exactly $p$ simple objects which are all invertible. Then the possible values of the Frobenius-Perron dimension of $\mathcal{C}$ are $p^N$, $N = 1, 2, 3$ (for all $p$), 4 (for $p = 3$ and $p = 3k + 1$), 5 (for $p = 3$ and $p = 4k + 1$) and 7 (for $p = 3$ and $p = 3k + 1$).

**Proof.** It is clear that the Frobenius-Perron dimension of objects in $\mathcal{C}$ are integers. Hence by [EO], there exists a quasi-Hopf algebra $A$ such that $\mathcal{C} = \text{Rep}(A)$. This quasi-Hopf algebra is basic, so its radical is a quasi-Hopf ideal and hence $A$ admits a radical filtration. Let $H := \text{gr}(A)$ (with respect to this filtration). Then Theorem 3.2 applies to $H$, hence the result.

3.2. **Proof of Theorem 3.1** Let us assume that $H[1]$ has rank $> 1$ over $H[0]$. From this we will derive a contradiction. We may assume that $H$ has the minimal possible dimension.

Let $a$ be a generator of $\mathbb{Z}_p$. We have $H[0] = \mathbb{C}[\mathbb{Z}_p]$ with associator $\Phi_s$ for some $s$.

Let us decompose $H[1]$ into a direct sum of eigenspaces of $a$: $H[1] = \bigoplus_{r=0}^{p-1} H_r[1]$, where $H_r[1]$ is the space of $x \in H[1]$ such that $axa^{-1} = Q^r x$ (we recall that $Q := e^{-s}$). Note that $1_i x = x 1_{i-r}$ for $x \in H_r[1]$. Also, by Theorem 2.17 in [EO], $H_0[1] = 0$.

Let $\tilde{H}$ be the free algebra generated by $H[1]$ as a bimodule over $H[0]$; i.e., $\tilde{H}$ is the tensor algebra of $H[1]$ over $H[0]$. Then $\tilde{H}$ is (an infinite dimensional) quasi-Hopf algebra, and we have a surjective homomorphism $\varphi : \tilde{H} \to H$ (it is surjective since $H$ is radically graded and hence generated by $H[0]$ and $H[1]$).

Let $q$ be a number such that $q^p = Q$. Define an automorphism $\gamma$ of $\tilde{H}$ by the formula $\gamma|_{H[0]} = 1$ and $\gamma|_{H[1]} = q^r$. (It is well defined since $\tilde{H}$ is free.)

Let $L$ be the sum of all quasi-Hopf ideals in $\tilde{H}$ contained in $\bigoplus_{d \geq 2} \tilde{H}[d]$. Clearly, $\text{Ker}\varphi \subseteq L$, so $H$ projects onto $\tilde{H}/L$. However, since $H$ has the smallest dimension, it follows that $\tilde{H}/L = H$.
Now, \( \gamma(L) = L \), so \( \gamma \) acts on \( H \). Let us define a new quasi-Hopf algebra \( \hat{H} \) generated by \( H \) and a grouplike element \( g \) with relations \( g^p = a \), \( g g^{-1} = \gamma(z) \) for \( z \in H \). Clearly, \( \text{Ad}(g) = \gamma^p \), and \( g \) generates a group isomorphic to \( \mathbb{Z}_p \).

Let \( J := \sum_{i,j} c(i,j)1_i \otimes 1_j \), \( c(i,j) := q^{-(j-j')} \), where \( j' \) denotes the remainder of division of \( j \) by \( p \), be the twist in \( \mathbb{C}[\mathbb{Z}_p^2] \otimes \mathbb{Z}_p \) defined in \( \mathbb{G} \). Define \( H \) to be the twist of \( \hat{H} \) by \( J^{-1} : \hat{H} := \hat{H} J^{-1} \). Since by \( \mathbb{G} \), \( dJ = \Phi \), \( \hat{H} \) is a finite dimensional radically graded Hopf algebra. Since the rank of \( \hat{H}[1] \) over \( \mathbb{H}[0] \) is \( > 1 \), we have at least 2 independent over \( \mathbb{H}[0] \) skew primitive elements \( x_1, x_2 \in \hat{H}[1] \) which are eigenvectors for \( \text{Ad}(g) \):

\[
g x_1 g^{-1} = q^{d_1} x_1, \Delta(x_1) = x_1 \otimes g_{b_1} + 1 \otimes x_1
\]

and

\[
g x_2 g^{-1} = q^{d_2} x_2, \Delta(x_2) = x_2 \otimes g_{b_2} + 1 \otimes x_2.
\]

Since \( H_0[1] = 0 \), \( d_1, d_2 \) must be relatively prime to \( p \). Also, since \( H \) has minimal dimension, the algebra \( H \) is generated by \( g, x_1, x_2 \).

By \( \mathbb{G} \), the function \( \frac{c(i,j)}{c(1,1)} q^j \) is \( p \)-periodic in each variable. Moreover, the coproduct of \( \hat{H} \) maps \( x_i \) into \( \hat{H} \otimes \hat{H} \); thus, similarly to \( \mathbb{G} \), the function \( \frac{c(i,j)}{c(1,1)} q^b_{i,j} \) is \( p \)-periodic in each variable for \( k = 1, 2 \). Hence the function \( \frac{c(i,j)}{c(1,1)} q^{b_{i,j}(d/d_k)} \) is \( p \)-periodic in each variable for \( k = 1, 2 \) (here \( b_{i,j} \) is the ratio taken in \( \mathbb{Z}_p^2 \)). We thus conclude that \( b_{i,j} = d \) modulo \( p \), for \( k = 1, 2 \).

Now set \( \bar{g} := g^{b_1}, \bar{q} := q^{d_1 b_1}, b := b_2/b_1 \) and \( d := d_2/d_1 \). We obtain

\[
\bar{g} x_1 \bar{g}^{-1} = \bar{q} x_1, \Delta(x_1) = x_1 \otimes \bar{g} + 1 \otimes x_1
\]

and

\[
\bar{g} x_2 \bar{g}^{-1} = \bar{q} x_2, \Delta(x_2) = x_2 \otimes \bar{q} + 1 \otimes x_2,
\]

where \( b, d \in \mathbb{Z}_p \) and \( b = d \) modulo \( p \).

Extend \( H \) to a Hopf algebra \( H' \) generated by \( \hat{H} \) and two commuting grouplike elements \( g_1, g_2 \), with relations \( g_1 x_i g_2^{-1} = \bar{q}^{b_{i,j}} \), \( g_i^2 = 1 \) for \( i, j = 1, 2 \), and \( \bar{q} = g_1 g_2^\lambda \).

(Proof that this is possible is the same as the proof given above of the fact that \( H \) can be extended by adjoining \( g \).)

Let \( \lambda \in \mathbb{Z}_p^2 \). Let

\[
T = T_\lambda := \sum_{\gamma, \beta} q^{\lambda b_1 \gamma_1} 1_\beta \otimes 1_\gamma \in \mathbb{C}[\mathbb{Z}_p^2 \times \mathbb{Z}_p^2],
\]

where \( \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \) and \( \{1_\beta | \beta \in \mathbb{Z}_p^2 \times \mathbb{Z}_p^2\} \) is the set of primitive idempotents of \( \mathbb{Z}_p^2 \times \mathbb{Z}_p^2 \). This is a Hopf twist. Consider the new coproduct \( \Delta_T \), obtained by twisting \( \Delta \). That is, \( \Delta_T(z) = T \Delta(z) T^{-1} \).

Using the facts that \( 1_\beta g_1 = \bar{q}^{b_{1,i}} 1_\beta \) and \( 1_\beta x_i = x_i 1_{\beta^{-\epsilon_i}}, i = 1, 2 \), where \( \epsilon_1 := (1, 0) \) and \( \epsilon_2 := (0, 1) \), it is straightforward to verify that

\[
\Delta_T(x_1) = x_1 \otimes g_1 g_2^{\lambda d} + 1 \otimes x_1 \text{ and } \Delta_T(x_2) = x_2 \otimes g_1^{b_1} g_2^{bd} + g_1^\lambda \otimes x_2.
\]

Therefore, if we set

\[
z_1 := x_1, z_2 := g_1^{-\lambda} x_2, \text{ } h_1 := g_1 g_2^{\lambda + d} \text{ and } h_2 := g_1^{b_1 - \lambda} g_2^{bd}
\]

we get

\[
\Delta_T(z_1) = z_1 \otimes h_1 + 1 \otimes z_1 \text{ and } \Delta_T(z_2) = z_2 \otimes h_2 + 1 \otimes z_2.
\]
Now, the relations
\[ h_1z_1h_1^{-1} = q z_1, \ h_1z_2h_1^{-1} = q^{\lambda+d}z_2, \ h_2z_1h_2^{-1} = q^{b-\lambda}z_1 \text{ and } h_2z_2h_2^{-1} = q^{bd}z_2, \]
imply that the braiding matrix \( B \) of \((H')^T\) (in the sense of AS) is given by \( b_{11} = q, \ b_{12} = q^{\lambda+d}, \ b_{21} = q^{b-\lambda} \text{ and } b_{22} = q^{bd}. \)

Now set \( \lambda = (b-d)/2. \) In this case \( b_{12} = b_{21} = q^{(b+d)/2}, \) so the braiding matrix is symmetric, and the corresponding Nichols algebra is of FL type in the sense of AS.

According to AS, the Cartan matrix \( A \) corresponding to \( B \) has \( a_{12} = b + d \) and \( a_{21} = (b + d)/bd \) (modulo \( p^2 \)). Since in our situation \( b = d \) modulo \( p \), we get that modulo \( p \), \( a_{12} = 2b \) and \( a_{21} = 2/b \), and hence that \( a_{12}a_{21} = 4 \) modulo \( p \). We claim that this implies that the Cartan matrix \( A \) cannot be of finite type.

Indeed, in the finite type case \((A_1 \times A_1, A_2, B_2 \text{ and } G_2), a_{12}a_{21} = 0, 1, 2, 3. \) Therefore if \( p > 3 \), \( A \) cannot be of finite type. For \( p = 3 \), we get that \( a_{12} = a_{21} = -1 \) (A_2 case) and \( b = 1 \) modulo 3. But this implies that \( b^2 + b + 1 = 0 \) modulo 9, which leads to a contradiction.

Now by Theorem 1.1 (ii) in AS, the algebra \((H')^T\) (and hence \( H' \)) is infinite dimensional. This gives a contradiction and completes the proof of Theorem 3.1.

4. Construction of finite dimensional basic quasi-Hopf algebras

In this section we generalize the construction of \( A(q) \) from G, and construct finite dimensional basic quasi-Hopf algebras which are not twist equivalent to a Hopf algebra.

Let \( n \geq 2 \) be an integer, and \( q \) a primitive root of 1 of order \( n^2 \). Let \( H \) be a finite dimensional Hopf algebra generated by grouplike elements \( g_i \) and skew-primitive elements \( e_i, \ i = 1, \ldots, m, \) such that
\[ g_i^2 = 1, \ g_i g_j = g_j g_i, \ g_i e_j g_i^{-1} = q^{\delta_{ij}} e_j \]
and
\[ \Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \]
where \( K_i := \prod_j g_j^{a_{ij}} \) for some \( a_{ij} \) in \( \mathbb{Z}_{n^2}. \)

Assume that \( H \) has a projection onto \( \mathbb{C}[(\mathbb{Z}_{n^2})^m], g_i \mapsto g_i, e_i \mapsto 0, \) and let \( B \subset H \) be the subalgebra generated by \( \{e_i\} \). Then by Radford’s theorem R, the multiplication map \( \mathbb{C}[(\mathbb{Z}_{n^2})^m] \otimes B \to H \) is an isomorphism of vector spaces. Therefore, \( A := \mathbb{C}[(\mathbb{Z}_n)^m]B \subset H \) is a subalgebra of dimension \( \dim(H)/n^m. \) It is generated by \( q^n \) and \( e_i. \)

Let \( \{1_{\beta}g_i = (\beta_1, \ldots, \beta_m) \in (\mathbb{Z}_n)^m\} \) be the set of primitive idempotents of \( \mathbb{C}[\mathbb{Z}_n]^m, \) and denote by \( e_i \in (\mathbb{Z}_n)^n \) the vector with 1 in the \( i \)th place and 0 elsewhere. Note that
\[ 1_{\beta}g_i = q^{\beta_i} 1_{\beta} \text{ and } 1_{\beta}e_i = e_i 1_{\beta - e_i}. \]

Let \( c(z,y) \) be the coefficients of the twist \( J \) as above introduced in G. Recall from G that \( c(z,y) = q^{-z(y-y')}, \) where \( y' \) denotes the remainder of division of \( y \) by \( n. \)

Let
\[ J := \sum_{\beta, \gamma \in (\mathbb{Z}_n)^m} \prod_{i,j=1}^m c(\beta_i, \gamma_j)^{a_{ij}} 1_{\beta} \otimes 1_{\gamma}. \]
It is clear that it is invertible and \((\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1\). Define a new coproduct \(\Delta_J(z) = J \Delta(z) J^{-1}\).

**Lemma 4.1.** The elements \(\Delta_J(e_i)\) belong to \(A \otimes A\).

*Proof.* This lemma for \(m = 1\) was proved in [G]. The general case follows from the case \(m = 1\) by a straightforward computation. \(\square\)

**Lemma 4.2.** The associator \(\Phi := dJ\) obtained by twisting the trivial associator by \(J\) is given by the formula

\[
\Phi = \sum_{\beta, \gamma, \delta \in \mathbb{Z}_m^n} \left( \prod_{i,j=1}^m \left( a_{ij} ((\gamma_j + \delta_j) - \gamma_i - \delta_i) \right) \right) 1_\beta \otimes 1_\gamma \otimes 1_\delta,
\]

where \(\{1_\beta\}\) are the primitive idempotents of \(\mathbb{Z}_m^n\) \((1_\beta q^n_i = q^{n\beta}_i 1_\beta)\), and we regard the components of \(\beta, \gamma, \delta\) as elements of \(\mathbb{Z}\). Thus \(\Phi\) belongs to \(A \otimes A \otimes A\).

*Proof.* One has

\[
\Phi = \sum_{\beta, \gamma, \delta \in \langle \mathbb{Z}_n, 2 \rangle^m} \left( \prod_{i,j=1}^m \left( c(\beta_i + \gamma_i, \delta_j) c(\beta_i, \gamma_j) \right) a_{ij} \left( c(\beta_i, \gamma_j + \delta_j) c(\gamma_i, \delta_j) \right) \right) 1_\beta \otimes 1_\gamma \otimes 1_\delta.
\]

Substituting the expression of \(c(z, y)\), similarly to [G] we get the statement. \(\square\)

Thus we get our second main result.

**Theorem 4.3.** The algebra \(A\) is a quasi-Hopf subalgebra of \(H^3\), which has coproduct \(\Delta_J\) and associator \(\Phi\).

*Proof.* We have shown that \(\Delta_J : A \to A \otimes A\) and \(\Phi \in A \otimes A \otimes A\). It is also straightforward to show that \(S_J : A \to A\) and \(\alpha \in A\) if \(\beta\) is gauged to be 1 (where \(S_J, \alpha,\) and \(\beta\) are the antipode and the distinguished elements of \(H^3\)). Thus \(A\) is a quasi-Hopf subalgebra of \(H^3\). \(\square\)

This yields many examples of finite dimensional basic quasi-Hopf algebras \(A\). For instance, let \(\mathfrak{g}\) be a finite dimensional simple Lie algebra, and \(\mathfrak{b}\) be a Borel subalgebra of \(\mathfrak{g}\). Assume that \(n\) is odd, coprime to 3 if \(\mathfrak{g} = G_2\). Then we can take \(H\) to be the Frobenius-Lusztig kernel \(u_q(\mathfrak{b})\). In this case, \(A\) is a quasi-Hopf algebra of dimension \(n^{\dim \mathfrak{g}}\). Another example is obtained from \(H = \text{gr}(u_q(\mathfrak{g}))\) (with respect to the coradical filtration).

**Remark 4.4.** If for some \(i\), \(a_{ii} \neq 0\) modulo \(n\), then \(A\) is not twist equivalent to a Hopf algebra. Indeed, the associator \(\Phi\) is non-trivial since the 3–cocycle corresponding to \(\Phi\) restricts to a non-trivial 3–cocycle on the cyclic group \(\mathbb{Z}_n\) consisting of all tuples whose coordinates equal 0, except for the \(i\)th coordinate. Since \(A\) projects onto \((\mathbb{C}[\mathbb{Z}_n^m], \Phi)\) with non-trivial \(\Phi\), \(A\) is not twist equivalent to a Hopf algebra.

For instance, this is the case in the above two examples obtained from \(u_q(\mathfrak{b})\) and \(u_q(\mathfrak{g})\).
References

[AS] N. Andruskiewitsch and H-J. Schneider, Finite quantum groups and Cartan matrices, Adv. Math. 154 (2000), no. 1, 1–45.

[D] V. Drinfeld, Quasi-Hopf algebras. (Russian) Algebra i Analiz 1 (1989), no. 6, 114–148; translation in Leningrad Math. J. 1 (1990), no. 6, 1419–1457.

[EG] P. Etingof and S. Gelaki, Finite-dimensional quasi-Hopf algebras with radical of codimension 2, Mathematical Research Letters 11 (2004), 685–696.

[ENO] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, Annales of Mathematics, math.QA/0203060.

[EO] P. Etingof and V. Ostrik, Finite tensor categories, Moscow Mathematical Journal, math.QA/0301027.

[G] S. Gelaki, Basic quasi-Hopf algebras of dimension $n^3$, Journal of Pure and Applied Algebra, math.QA/0402159.

[N] W. Nichols, Bialgebras of type one. Comm. Algebra 6 (1978), no. 15, 1521–1552.

[R] D. Radford, The structure of Hopf algebras with a projection, J. Algebra 92 (1985), no. 2, 322–347.

[S] P. Schauenburg, A quasi-Hopf algebra freeness theorem, Proc. Amer. Math. Soc. 132 (2004), no.4, 965–972.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: etingof@math.mit.edu

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel
E-mail address: gelaki@math.technion.ac.il