Perestroikas of vertex sets at umbilic points

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Abstract

Mark all vertices on a curve evolving under a family of curves obtained by intersecting a smooth surface $M$ with the 1-parameter family of planes parallel to the tangent plane of $M$ at a point $p$. Those vertices trace out a set, called the vertex set through $p$. We take $p$ to be a generic umbilic point on $M$ and describe the perestroikas of the vertex set under generic $n$-parameter small deformations of the surface.

Beyond the Mathematical interest on vertices of families of curves, this work was primarily motivated by the medial representation of shapes in Computer Vision and Image Analysis, where the behaviour of vertices plays a crucial role in the qualitative changes of the skeleton or Blum medial axis of curves.

1 Introduction

Curves on surfaces play an important role in applications such as Computer Vision, Shape Analysis, etc. In their former work, P.J. Giblin and second author have proposed to represent image information as a collection of medial representations\textsuperscript{1} for the level sets of intensity (isophote curves). They have investigated the geometry of a class of parameter families of curves arising as a generalisation of isophote curves on surfaces. These curves are obtained as sections of a surface by a continuous family of planes parallel to and near the tangent plane of the surface at a point $p$. Such families contain singular members corresponding to the tangent planes themselves. Hence standard results from Singularity Theory, as in [6], do not apply to

\textsuperscript{1}The symmetry set of a curve (resp. surface) $S$ is the closure of the loci of centers of all circles (spheres) which are tangent to $S$ at more than one place. The medial axis is obtained when we only consider the circles (resp. sphere) whose radii equal the distance from their centers to the curve (surface). In medial representation, one studies the properties and structure of medial axes and seeks to get information on curves (surfaces, shapes) from them.
them. In [8], near any of the (elliptic, umbilic, hyperbolic, parabolic, cusp of Gauss) points \( p \) of a generic smooth surface in 3-space, they carry out an extensive study of the behaviour of vertices and inflexions for curves evolving near the singular member (the curve through \( p \)), as well as the limits of curvatures at vertices as the curves collapse to the singular one. They also classify all possible arrangements of the branches of the vertex set (set of all local patterns of vertices) and the inflexion set.

This has been motivated, on the one hand, by the fact that, for a curve, centers of circles of curvature at vertices are endpoints of the so-called symmetry set of the curve and inflexions correspond to where a local branch of the symmetry set recedes to infinity. Hence, from the way vertices and inflexions behave, one can deduce a great deal of information about the local number of branches of the symmetry set and their qualitative changes, as the isophotes evolve. The qualitative topological changes (or transitions) on symmetry sets carry the information about the so-called medial axis (or skeleton) used in medial representations of shapes.

On the other hand, the study of vertices of curves has raised up a great interest in particular in Geometry and Singularity Theory, in line with several problems such as the 4-vertex Theorem, the local geometry of surfaces and Geometry of Caustics. These classical subjects have received a new impulsion due to the development of Symplectic and Contact Geometry, especially in the works of V. Arnold on Lagrangian and Legendrian Collapse and Legendrian Sturm Theory (see [2], [3] and [4]; see also [10]), showing the relation between vertices of plane curves and Lagrangian and Legendrian singularities and Sturm-Hurwitz Theorem (see [3], [9]). In this context, Uribe-Vargas has devoted several of his work to the same subject. Namely, proving a conjecture of V. Arnold, he showed that the surface of changing four vertices for six in general 2-parameter families of level curves near an umbilic point is a hypocycloidal cup (see Problem 1993-3 of [5] and [11]). Uribe-Vargas also proved that the bifurcation diagram has at most one modulus.

In this paper, we describe the perestroikas of the vertex sets at generic umbilic points of surfaces undergoing an evolution in an \( n \)-parameter family of surfaces. The corresponding discriminants of the vertex set are also studied. One of the main consequences of our results is that, in some sense (see Theorem [2]), generically the study of the discriminants of small \( n \)--deformations, with \( n \geq 2 \), simplifies to that of 2--deformations. The case of generic 1-parameter families of surfaces is also considered in details, we draw the singular surface of the deforming vertex set in \((2 + 1)\)-space where the parameter is an additionnal variable.

As defined above, the vertex set of a surface at a point \( p \) is the locus of
the curvature extrema on sections of the surface by planes parallel to the tangent one at \( p \), and these extrema are euclidean invariants. In the conclusion (Section 4) we discuss very briefly the projective differential geometry version of this study, as well as its possible extension to higher degeneration.

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2 Presentation of main results

Consider a smooth (embedded) curve in the Euclidian plane \( \mathbb{R}^2 \). The osculating circle of the curve at a point \( p \) is the circle tangent to the curve at \( p \) with order at least 2, that is, the circle passing through 3 infinitesimally close points of the curve. The curvature of the curve at this point is the inverse of the radius of the osculating circle. A point of the curve is a vertex if its osculating circle has a tangency of order at least 3. A vertex of a curve is a critical point of its curvature. A vertex is non-degenerate if the corresponding osculating circle has tangency of order 3 with the curve. The vertex is \( n \)-degenerate \( (n \in \mathbb{N} \cup \{\infty\}) \) if the curvature of the curve has there an \( A_{n+1} \) critical point.

Let us consider a smooth surface \( M \) in the Euclidean 3-space \( \mathbb{R}^3 = \{x, y, z\} \) and a point \( p \) in \( M \) considered as the origin of \( \mathbb{R}^3 \). Without loss of generality, we assume that the tangent plane of the surface at \( p \) is \( z = 0 \). Then the surface is, at least locally, the graph \( z = f_0(x, y) \) of a smooth function \( f_0 : \mathbb{R}^2 \rightarrow \mathbb{R} \).

In this setting, the intersection of \( M \) with planes parallel to the tangent plane at \( p \), are the level curves \( f_0 = k \), where \( k = 0 \) corresponds to the tangent plane itself. The patterns of vertices of \( f = k \), when \( k \) varies, trace out a set called the vertex set through \( p \)(see [8]).

We summarise some of the results in [8] as follows.

**Theorem 1.** [8] The germ of the vertex set at a point \( p \) of a generic smooth surface in 3-space has:

- four smooth transverse branches, tangent to the principal directions and the asymptotic directions at \( p \), if \( p \) is a hyperbolic point of the surface;
- two smooth transverse branches, tangent to the principal directions at \( p \), if \( p \) is a non-umbilic elliptic point;
- three smooth transverse branches, if \( p \) is a generic umbilic point;
- three branches tangent to the zero-curvature principal direction, one of which is smooth and the other two ones have an ordinary cusp, if \( p \) is an ordinary parabolic point;
- two smooth transverse branches, one of which is tangent to the parabolic curve, if \( p \) is an elliptic cusp of Gauss;
- six smooth branches, five of which are tangent to the parabolic curve and the other one is transverse, if \( p \) is a hyperbolic cusp of Gauss.

Throughout this paper, we suppose that \( p \) is a generic umbilic of \( M \), that is, the quadratic part \( q_0 \) of \( f_0 \) is proportional to \( x^2 + y^2 \). The genericity condition is that \( q_0 \) does not divide the cubic part of \( f_0 \). We may assume without loss of generality that the cubic part of \( f_0 \) equals \( ax^3 + bx^2y + axy^2 + cy^3 \) (see Lemma 1 of [7]). The genericity condition is then \( b \neq c \).

Let us consider an \( n \)-parameter deformation
\[
z = f(x, y; \tau) = f_0(x, y) + R(x, y; \tau)
\]
of our surface \( z = f_0(x, y) \), where \( R : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth mapping identically vanishing at \( \tau = 0 \). Up to affine coordinate changes depending on the deformation parameters, we may assume that the origin \( p \) is not moved by the deformation and that at this point the tangent plane to the surface is always \( z = 0 \). Thus, we assume that the linear part of \( R \) vanishes for all \( \tau \).

**Definition 1.** The bifurcation diagram of the vertex sets of the family of surfaces \( z = f \) is the germ at the origin of the closure of the set in the \((n + 1)\)-space \( \mathbb{R}^n \times \mathbb{R} = \{\tau, k\} \) formed by the elements \((\tau, k)\) such that the level curve \( f(\cdot; \tau) = k \) has (at least) a degenerate vertex.

The discriminant of the vertex sets is the projection of the singular locus of the bifurcation diagram to the \( n \)-parameter space \( \{\tau\} \) by the mapping “forgetting \( k \”).

The forms \( x^2 + y^2 \), \( x^2 - y^2 \) and \( 2xy \) form a basis of the 3-dimensional space \( Q \) of the quadratic forms on \( \mathbb{R}^2 = \{x, y\} \). Therefore, the umbilic forms (i.e., proportional to \( x^2 + y^2 \)) span a codimension 2 subspace of this space \( Q \) (according to the fact that a generic surface has only isolated umbilics).

Let us denote by \( T \) the affine plane \( q_0 + \langle x^2 - y^2, 2xy \rangle_{\mathbb{R}} \subset Q \), which is transversal at \( q_0 \) to the line \( \langle x^2 + y^2 \rangle_{\mathbb{R}} \) of the umbilic forms in the space of the quadratic forms \( Q \).
Consider the mapping $F$ which associates to a parameter $\tau$ the natural projection on $T \approx \mathbb{R}^2 = \{\lambda, \mu\}$ of the corresponding quadratic part of $f(\cdot; \tau)$.

**Definition 2.** The rank of the deformation $f$ of $f_0$ is the rank of the derivative of $F : \mathbb{R}^n \rightarrow \mathbb{R}^2$, at the parameters’ origin.

**Remark 1.** The small deformations of rank 0 perturb the surface $z = f_0$ among the surfaces having a generic umbilic at the origin. Therefore they do not change the vertex set of the surface (up to diffeomorphisms).

Therefore, we will discuss only deformations of rank 1 and 2. Notice that a deformation has generically the maximum rank possible (i.e., 1 for 1-parameter deformations and 2 for $(n \geq 2)$-parameter deformations).

**Theorem 2.** For every $n \geq 2$, the germ at the origin of the discriminant of the vertex set of any $n$-parameter rank 2 deformation $z = f$ of the surface $z = f_0$ is diffeomorphic to the germ at the origin of an $(n - 2)$-cylinder over the union of three transverse smooth curves on the plane $T$, tangent to the lines $\mu = 0, \pm \sqrt{3} \lambda$.

The vertex set at a generic umbilic point of a surface is the union of three transverse smooth branches $C_1, C_2$ and $C_3$ (Theorem [1]). Each of these branches can be seen as the union $C_i = C_i^+ \cup C_i^-$ of two half-branches issuing from the origin.

**Theorem 3.** Let $z = f$ be an $n$-parameter rank 2 deformation of $z = f_0$. Fix a ball centered at the origin $(x = 0, y = 0)$ of radius arbitrarily small. For any $\tau$ small enough outside the deformation’s discriminant, the vertex set of the surface $z = f(\cdot; \tau)$ in this ball is the union of a smooth branch not passing through the origin and two smooth branches passing through the origin. The first of these branches is a smooth $C^0$-small deformation of two consecutive half-branches, say $C_1^+ \cup C_2^+$, of the unperturbed vertex set; it is disjoint to the other two, which are $C^0$-small deformations of the pairs of non consecutive remaining half-branches $C_1^- \cup C_3^-$ and $C_2^- \cup C_3^+$ (see Figure [1]).

The projection on $T$ of the discriminant of any 2-parameter rank 2 deformation of a surface near a generic umbilic point is shown in Figure [2] together with the corresponding vertex sets.

**Remark 2.** Since generic umbilics are stable, the perturbed surface has a generic umbilic close to that of the unperturbed surface. Whenever the coordinates of the projection of the deformation $R$ on the plane $T$ are not
both vanishing, the umbilic slightly moves away from the origin. For small perturbations, the tangent plane to the perturbed surface at the umbilic point is transversal to the planes $z = k$, so we cannot see the umbilic in the diagram above as intersection of vertex set branches.

**Example 1.** The vertex sets and some level lines of the surface $z = f$, with function

$$f(x, y; \lambda, \mu) = x^2 + y^2 + x^3 - y^3 + \lambda(x^2 - y^2) + 2\mu xy,$$

are drawn in Figure 3 for the parameter values $(\lambda, \mu) = (0, 0), (1/10, 0)$ and $(0, 1/10)$.

Proving a conjecture by V. Arnold, R. Uribe-Vargas has shown that the bifurcation diagram of the vertex sets of a generic 2-parameter deformation of a surface at a generic umbilic point is diffeomorphic to the **hypocycloidal cup** in the 3-space $\mathbb{C} \times \mathbb{R}$, parametrized by the mapping

$$(\varphi, k) \mapsto (-2\sqrt{k}(5e^{-i\varphi} + e^{5i\varphi}), k).$$

The curves above the hypocycloidal cup have 6 vertices while the curves below it have 4 vertices. The curves on the regular part of the hypocycloidal cup have a 1-degenerate vertex, while those on the semicubic cuspidal edge have a 2-degenerate vertex.

The proof, similar to that of the analogous result on the nearby problem on vanishing flattenings of spatial curves (see [1]), is based on Arnold’s Lagrangian Collapse and Sturm-Hurwitz Theorem (see [2], [9]). The relation between these two problems is explained in Uribe-Vargas’ comment to problem 1993-3 of [5].

The discriminant of a rank 2 perturbation is hence the projection on the parameter plane of the semicubical cuspidal edges of the hypocycloidal cup (see figure 4).
Figure 2: Vertex set discriminant of 2-parameter rank 2 deformations of surfaces at generic umbilic points (middle picture) together with all corresponding perestroikas.

**Remark 3.** Consider an \( n \)-parameter rank 2 deformation of \( z = f_0 \). If \( \tau \) is small enough and belongs to the discriminant’s complement, then there exists a level curve \( f(\cdot; \tau) = k^* \) having a 1-degenerate vertex. This level separates the level curves having 4 vertices from those having 6 vertices. The value \( k^* \) is arbitrarily small, provided that \( \tau \) is small enough: it is actually of the order of \( \lambda^2 + \mu^2 \), as follows from the parameterisation of the hypocycloidal cup (where \( \lambda \) and \( \mu \) are the coordinates of the projection of the deformation \( R \) on the plane \( T \)).

Since rank 1 deformations are obviously induced from rank 2 deformations, they are completely described by the above theorems. In the most interesting case of generic 1-parameter deformations we obtain immediately the following result.
Figure 3: Vertex sets of the surface $z = f$ for different values of the deforming parameters.

Figure 4: Arnold–Uribe-Vargas’ hypocycloidal cup.

**Theorem 4.** Let us consider a deformation of a surface $z = f_0$, depending on one parameter $t \in \mathbb{R}$. Assume that the projection on $T$ of the deformation is a curve transversal at $\lambda = \mu = 0$ to the lines $\mu = 0, \pm \sqrt{3}\lambda$. Label the branches in such a way that for $t < 0$ the vertex set origin-avoiding branch is a smooth $C^0$-small deformation of the two consecutive half-branches $C^+_1 \cup C^+_2$ of the unperturbed vertex set. Then for $t > 0$ the vertex set origin-avoiding branch is a smooth $C^0$-small deformation of the two opposite consecutive half-branches $C^-_1 \cup C^-_2$.

The perestroika of the vertex set under a generic 1-parameter deformation of the surface $z = f_0$ is illustrated in Figure 5.

The germ at the origin of the surfaces formed in the 3-space $\{x, y, t\}$ by the vertex sets along generic 1-parameter deformations of the surface is depicted in figure 6.
Figure 5: Perestroika of the vertex set under a generic 1-parameter deformation of the surface at a generic umbilic.

Figure 6: Surface spanned by the deforming vertex sets on a generic 1-parameter deformation. The slices of this surface by planes $t = \text{const}$ give the corresponding perestroika of the vertex set (compare Theorem 4 and Fig. 5.)

3 Proofs and further discussions

Let us consider as above a smooth $n$-parameter family of functions

$$f = f_0 + R : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R},$$

with quadratic part equal to $x^2 + y^2$ and cubic part equal to $ax^3 + bx^2y + axy^2 + cy^3$, where $b \neq c$ by the genericity assumption.

To any fixed value $\tau$ of the deforming parameters corresponds the vertex set of the perturbed surface at the origin. This vertex set is the zero level of a smooth function $V_f(\cdot; \tau)$, depending smoothly on the parameter value $\tau$. Therefore we obtain a smooth $n$-parameter family of vertex set functions $V_f : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$. These functions are computed from $f$ by the following
explicit formula, given in [8]:

\[
V_f = (f^2_x + f^2_y)(f^3_y f_{yyy} - 3f^2_x f_y f_{xyy} + 3f_x f^2_y f_{xxy} - f^3_x f_{xxx}) + \\
+ 3f_x f_y (f^2_x f^2_y - f^2_x f_{yy} + (f^2_y - f^2_x)(f_{xx} f_{yy} + 2f^2_y)) + \\
+ 3f_{xy} (f_{xx} f_y^4 - 3f^2_x f^2_y (f_{xx} - f_{yy}) - f_{yy} f_x^4). 
\]

(1)

In order to describe geometrically these vertex sets, we consider the natural equivalence relation acting on these functions. Since we are interested on the zero levels of these functions, the relevant equivalence relation is a version of the usual \(V\)-equivalence, preserving the distinguished role of the deformation parameters.

Consider the natural structure of trivial fiber bundle on \(\mathbb{R}^2 \times \mathbb{R}^n\), defined by the natural projection \(\pi(x,y; \tau) = \tau\) onto the parameter space.

**Definition 3.** Two functions \(F, G : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}\) are \(V^*\)-equivalent (or fibered \(V\)-equivalent) if there exist diffeomorphisms

\[
\Phi = (\Phi_1, \Phi_2) : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}^2 \times \mathbb{R}^n, \quad \psi : \mathbb{R} \rightarrow \mathbb{R},
\]

with \(\psi(0) = 0\), making commutative the following diagram:

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\pi} & \mathbb{R}^2 \times \mathbb{R}^n \\
\Phi_2 \downarrow & & \Phi \downarrow \psi \\
\mathbb{R}^n & \xrightarrow{\pi} & \mathbb{R}^2 \times \mathbb{R}^n \\
& & G \\
\end{array}
\]

Notice that the left part of the diagram just means that \(\Phi\) is a fibered diffeomorphism of the total space of the fiber bundle \(\mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}^n\). In particular, each \(\Phi_\tau(\cdot) := \Phi_1(\cdot; \tau)\) is a diffeomorphism of each fiber \(\mathbb{R}^2\).

A similar definition of \(V^*\)-equivalence holds for germs.

**Proposition 1.** Suppose that \(F, G : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}\) are \(V^*\)-equivalent. Then the zero level sets of the functions \(F(\cdot; \tau)\) and \(G(\cdot, \Phi_2(\tau))\) are diffeomorphic.

**Proof.** Let \(X_\tau\) be the zero level set of \(F(\cdot; \tau)\). Hence

\[
X_\tau = \{(x,y) : F(x,y; \tau) = 0\} = \{(x,y) : \psi^{-1} \circ G \circ \Phi(x,y; \tau) = 0\}. 
\]

Since \(\psi^{-1}(t) = 0\) if and only if \(t = 0\), we have

\[
X_\tau = \{(x,y) : G(\Phi_\tau(x,y); \Phi_2(\tau)) = 0\}.
\]

Therefore the zero level set of \(G(\cdot; \Phi_2(\tau))\) is \(\Phi_\tau(X_\tau)\). \(\square\)
$V^*$-equivalence and Proposition 1 allow us to replace the qualitative study of the vertex set by the corresponding study of a diffeomorphic curve $\tilde{V}_f = 0$ (which is not necessarily the vertex set of some surface).

Let us consider an $n$-parameter rank 2 deformation $z = f_0 + R$ of the surface $z = f_0$. Then $n \geq 2$ and, by the implicit function theorem, the perturbing term is (up to a coordinate change in the parameter space) of the form
\[
R(\tau) = \lambda (x^2 - y^2) + 2\mu xy + R_3(\tau),
\]
where $\lambda$ and $\mu$ are two distinguished parameters among the $\tau$'s and, as a function of $x, y$, $R_3(\tau)$ has a 2-jet which is identically zero.

**Lemma 1.** The vertex set function $V_f$ is $V^*$-equivalent to a function $\tilde{V}_f$ such that
\[
\tilde{V}_4 = (1 - \lambda^2 - \mu^2)^2 (x^2 + y^2 + \lambda (x^2 - y^2) + 2\mu xy) (2\lambda xy + \mu (y^2 - x^2)),
\]
\[
\tilde{V}_5 \big|_{\tau = 0} = x(x^2 + y^2)(x^2 - 3y^2), \quad \tilde{V}_6 \big|_{\tau = 0} = 0,
\]
where $\tilde{V}_f = \tilde{V}_4 + \tilde{V}_5 + \tilde{V}_6 + \ldots$ is the homogeneous expansion of $\tilde{V}$ (each $\tilde{V}_i$ is a homogeneous polynomial of degree $i$ in the variables $x$ and $y$, whose coefficients depend on the parameters $\tau$).

**Proof.** Compute $V_f$ in terms of $f$ by the explicit expression (1), and consider the $V^*$-equivalent function
\[
\tilde{V}_f := \frac{(c - b)^4}{192} V_f + q_1 xV_f + q_2 yV_f,
\]
for some coefficients $q_i$ (depending on the parameters). Consider a coordinate change of the form
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{c-b} x + p_0 x^2 + p_1 xy + p_2 y^2 \\ \frac{1}{c-b} y + p_3 x^2 \end{pmatrix},
\]
where the coefficients $p_i$ depend smoothly on the parameters. One directly check that for a suitable choice of these coefficients $p, q$ (depending also on the quartic part of $f$), this function fulfills the required conditions. Notice that $b \neq c$, since the umbilic is generic. 

**Remark 4.** $\tilde{V}_4$ depends only on the parameters $\lambda$ and $\mu$. In particular, the coefficients of $x^4$ and $y^4$ of $\tilde{V}_4$ are respectively
\[
-\mu (1 + \lambda) (1 - \lambda^2 - \mu^2)^2 \quad \text{and} \quad \mu (1 - \lambda) (1 - \lambda^2 - \mu^2)^2,
\]
so near $\tau = 0$ they both vanish if and only if $\mu = 0$. 

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Lemma 2. For every small enough value of the parameter deformation \( \tau \), such that \((\lambda, \mu) \neq (0, 0)\), the vertex set of the perturbed surface \( z = f \) has two real smooth branches passing through the origin. They are tangent to the lines
\[
\mu y = (-\lambda \pm \sqrt{\lambda^2 + \mu^2}) x
\]
for \( \mu \neq 0 \) and to the lines \( x = 0, y = 0 \) for \( \mu = 0 \) and \( \lambda \neq 0 \). For \( \lambda = \mu = 0 \), the vertex set has three real smooth branches passing through the origin, tangent to the lines \( x = 0 \) and \( x = \pm \sqrt{3} y \).

Proof. We shall use the Newton diagrams of the vertex set functions for the different fixed values of the parameters. The relevant diagrams are shown in figure 7.

Figure 7: Newton diagrams of \( \tilde{V}_f \).

Suppose first \( \mu \neq 0 \). Then \( \tilde{V}_f \) is quasi-homogeneous, with principal part \( \tilde{V}_4 \) (given in Lemma 2). Solving \( \tilde{V}_4 = 0 \) we get:
\[
y = \frac{-\lambda \pm \sqrt{\lambda^2 + \mu^2}}{\mu} x \quad y = \frac{\mu \pm \sqrt{\lambda^2 + \mu^2 - 1}}{\lambda - 1} x .
\]
Hence, for \( \lambda \) and \( \mu \) small enough, \( \tilde{V}_f = 0 \) has two real smooth branches passing through the origin, whose tangent directions are given by the first two lines above. The vertex set has also two smooth complex conjugate branches (tangent to the second two lines above), whose only real point is the origin (they appear also in the forthcoming cases, but we do not insist about that).

Suppose now \( \mu = 0, \lambda \neq 0 \). In this case the vertex set has two real smooth branches with equation
\[
2\lambda (1 + o(\lambda)) xy + \text{hot}(x, y) = 0 ,
\]
which are therefore tangent at the origin to the lines \( x = 0 \) and \( y = 0 \).

Finally, assume \( \lambda = \mu = 0 \). Then the vertex set has three smooth real branches, according to the fact that under such a deformation the origin is still an umbilic of our surface. These branches are provided by an equation of the form
\[
x(x^2 + y^2)(x^2 - 3y^2) + \text{hot}(x, y) = 0,
\]
so they are tangent to the lines \( x = 0, \pm \sqrt{3}y \).

In order to describe the perestroikas of the vertex sets of the family of surfaces \( z = f(x, y) \), we first focalize on the case \( \mu = 0 \).

Let us fix a ball centered at the origin, of radius arbitrary small. For \( \tau = 0 \), the real smooth branches of the vertex set meet the ball’s boundary at 6 points, close to the vertex of the regular hexagon inscribed in the ball and symmetric with respect to \( x = 0 \). When \( \tau \) varies, these six points slightly move along the ball’s boundary, provided that the variation is small enough.

**Proposition 2.** Assume \( \mu = 0, \lambda \neq 0 \). The branch of the vertex set, which is tangent to the line \( y = 0 \), is parabolic near the origin, namely it has a second order tangency with a parabola of the form
\[
2\lambda y = (-1 + \text{hot}(\tau)) x^2, \quad (\text{for } \tau \to 0).
\]
The second vertex set branch has a second order tangency with its tangent line \( x = 0 \) at the origin.

**Proof.** The vertex set branch which is tangent to \( y = 0 \) at the origin is of the form \( y = h(x) = C x^2 + o(x^2) \), for some “constant” \( C \) depending smoothly on the value of the parameter \( \tau \). Putting that into the expression of \( \tilde{V}_f \), we get (according to figure 7)
\[
\tilde{V}_f(x, h(x); \lambda, \mu = 0) = (1 + 2\lambda(1 + \text{hot}(\tau)) C + \text{hot}(\tau)) x^5 + o(x^5) = 0.
\]
Hence, \( C = (-1 + \text{hot}(\tau))/2\lambda \).

Similarly, we get the expression \( x = o(y^2) \) for the other vertex set smooth branch passing through the origin.

This proposition allows us to describe the perestroika of the vertex set along this deformation. In particular, the convexity of this branch is toward the positive direction (resp. negative direction) of the \( y \)-axis whenever \( \lambda \) is negative (resp. positive). The result is shown in figure 8.

Notice that the pattern of the third vertex set branch depicted in figure 8 is the only possible (provided that the deformation is small enough, as
Figure 8: Perestroika of the vertex set along the deformation \( \mu = 0 \) when \( \lambda \) crosses 0.

well as the radius of the ball inside which we are looking the vertex set). Indeed, for \( \tau, k \) small enough, \( k > 0 \), the level curve \( f = k \) has at most one 2-degenerate vertex by Uribe-Vargas’ theorem.

In particular, the vertex corresponding to the self-intersection of the vertex set under our deformations is exactly 2-degenerate.

The 6-jet of the vertex set equation has an hexagonal symmetry which simplifies our computations. Indeed, one easily check the following fact.

**Proposition 3.** For every \( n \in \mathbb{Z} \), the terms \( \tilde{V}_4, \tilde{V}_5 |_{\tau=0} \) and \( \tilde{V}_6 |_{\tau=0} \) are invariant under the simultaneous rotations of angles \( 2n\pi/3 \) and \( -2n\pi/3 \) in the \( \{x,y\} \) plane and in the \( \{\lambda,\mu\} \) plane.

This remarkable hexagonal symmetry (related to the Sturm-Hurwitz theorem, cf. [11]) of the vertex set implies that the above perestroika, occurring on the hypersurface \( \mu = 0 \) when \( \lambda \) crosses 0, also occurs along two other smooth hyperplanes, whose projections on the parameter plane \( \{\lambda,\mu\} \) are the lines \( \mu = \pm\sqrt{3}\lambda \).

In particular, these three hypersurfaces are contained in the deformation’s discriminant.

A similar study of the relative positions of the vertex set branches leads to the description of the perestroika of the vertex set along the deforming curves \( \mu = \alpha \lambda \), where \( \alpha \neq 0, \pm\sqrt{3} \).

To describe the behaviour of the vertex set branches (in a small ball centered at the origin), we shall compute the convexity of the branches passing through the origin.
Proposition 4. Suppose $\tau$ small enough and such that $\mu = \alpha \lambda$, $\alpha \neq 0, \pm \sqrt{3}$. Then the two smooth real branches of the vertex set passing through the origin have there a second order tangency with two parabola of the form

$$y = h(x) = \left(\frac{-1 \pm \sqrt{1 + \alpha^2}}{\alpha} + \text{hot}(\tau)\right) x + A_\pm x^2 + o(x^2),$$

(2)

where the coefficients $A_\pm$, depending smoothly on $\tau$ and $\alpha$, are

$$A_\pm = \frac{(\alpha^2 + 6)(\alpha^2 + 1) \mp 2\sqrt{1 + \alpha^2}(2\alpha^2 + 3)}{\lambda \alpha^2(1 + \alpha^2)(-1 \pm \sqrt{1 + \alpha^2})} (1 + \text{hot}(\tau)).$$

In particular, $A_\pm \neq 0$ for every $\lambda \neq 0$ small enough.

Proof. By Lemma 2, the two real smooth branches of the vertex set passing through the origin are of the form (2). Replacing this expression in $\tilde{V}_f$ and using Lemma 1, we get an explicit expression of the form $C_\pm x^5 + o(x^5)$. The equation $C_\pm = 0$ provides the above values of $A_\pm$.

The coefficient $A_-$ vanishes if and only if $\alpha$ is a root of the polynomial

$$(\alpha^2 + 6)(\alpha^2 + 1) - 4(2\alpha^2 + 3)^2 = \alpha^4(\alpha^2 - 3).$$

Thus $A_\pm \neq 0$ for $\lambda$ small enough, under the hypothesis of the proposition.

The resulting arrangement of the vertex set branches is shown in figure 9 in the case $0 < \alpha < \sqrt{3}$ (the other cases can be deduced from this one via the hexagonal symmetry). These arrangements are obtained using the convexity described in the above proposition and the same argument used in the case $\mu = 0$. The key point here is that the tangent lines to the two branches passing through the origin intersect the same sides of the symmetry hexagon for every $0 < \alpha < \sqrt{3}$.

In particular, the degenerate vertex on the origin-avoiding branch is exactly 1-degenerate. These deformations belong to the complement of the discriminant, for every $\tau$ small enough such that $(\lambda, \mu) \neq (0, 0)$. Summing up the descriptions of these deformations, we have proved Theorem 2 and Theorem 3.

4 Conclusion

In this paper, we have discussed the geometry of the vertex sets at isolated umbilic points of a generic surface evolving in an $n$-parameter family of
surfaces. As a follow up, it would be very interesting to extend this investigation to vertex sets at higher degenerate points (in the sense that the tangent plane has a higher order contact with the surface) such as ordinary parabolic points or cusps of Gauss, of \( n \)-parameter small deformations of generic surfaces. For fixed (as opposed to continuous families of) generic surfaces, these degenerations are a part of the work carried out in \[8\]. Recall that the vertices we consider in this paper are Euclidean invariants. A discussion of this subject from the projective differential geometry point of view, is an interesting question. A challenging problem would be, as brought to our attention by V.I. Arnold, the extension of the work within this paper to symplectic and contact geometries, namely the singularities of the corresponding lagrangian map theory, where the vertices are replaced by the singularities of the caustic enveloping the normals. The theory in this context is still awaiting its creation. See [1] for an extensive discussion.

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