The influence of magnetic field on the pion superfluidity and phase structure in the NJL model

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The influence of the magnetic field on the pion superfluidity and the phase structure is analyzed in the framework of the two-flavor Nambu–Jona-Lasinio (NJL) model. To do this, we first derive the thermodynamic potential from the Lagrangian density of the NJL model in the mean field approximation. Using this thermodynamic potential, we get the gap equation of the chiral condensate and the pion condensate. The effect of external magnetic field on the pion condensate is not simple promotion or suppression, which we will discuss in detail in the paper. It is shown that the tricritical point on the pion superfluidity phase transition line moves to the space with smaller isospin chemical potential and higher temperature when the external magnetic field becomes stronger. The influence of external magnetic field on the chiral condensate is also studied.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is the gauge theory of strong interaction. Recently, the QCD phase diagram including chiral symmetry breaking (restoration) [1, 2], quark confinement (deconfinement) [10, 11], color superconductivity [12, 17], were investigated with finite temperature and finite chemical potential. Furthermore, the studies are extended to finite isospin chemical potential [18–22]. It is found that at a critical isospin chemical potential, which is about the pion mass in vacuum, the phase transition from normal phase to pion superfluidity phase will happen. The understanding of the properties of pion superfluidity and phase structure under the extreme conditions are important in conducting research of the physics of the compact objects and relativistic heavy ion collisions.

It is reported that very strong magnetic field may be generated in the heavy-ion collision [24, 26]. The magnetic field magnitudes were estimated to be $5.3m^2/e$ at the Relativistic Heavy-Ion Collision (RHIC) and $6m^2/e$ at the Large Hadron Collision (LHC), and even higher [26]. Such magnetic field can also exist in magnets [30] and in the early universe [31, 33]. Despite the origin of such strong field is not clear yet. Above phenomena lead us to think deeply about the influence of the magnetic field on the QCD phase diagram [34]. Recently, much work has been done about the influence of the magnetic field on the properties of quark matter [35], chiral transition [36] and color superconductivity [37, 39].

Normally, QCD is widely accepted as the correct theory describing strongly interacting matter at high temperature and high density. The low energy QCD vacuum is hard to be fully understood by using of perturbative methods, because the characteristics of chiral symmetry breaking and color confinement have a non-perturbative origin. To solve the problem, two approaches are introduced, one is lattice QCD simulations [40] and the other is the effective model method.

The Nambu–Jona-Lasinio (NJL) model [1, 42, 43] is used as a simple and practical chiral model, which satisfies the basic mechanism of spontaneous breaking of chiral symmetry and key features of QCD at finite temperature and chemical potential. One of the basic properties of this model is that it includes a gap equation which connect the chiral condensate to the dynamical quark mass. It is well known that the hadronic mass spectra and the static properties of meson, especially the chiral symmetry spontaneous breaking, can be obtained through the mean field approximation (RPA) of meson in the NJL model. This model and its extended version (PNJL, EPNJL, etc.) are also widely used to study the properties of deconfinement phase, the color superconductivity phase, and the pion superfluidity phase and the related phase transitions under extreme conditions. Recently, the strong magnetic field effect on the properties of quark matter [46, 48], and the phase transitions, including chiral restoration transition, deconfinement transition [49, 50], and color superconductivity transition [52, 53], has been investigated in the NJL-type models by many groups.

In this paper, we will mainly focus on the effect of external magnetic fields on the pion condensate and the pion superfluidity phase structure at finite temperature and finite isospin chemical potential. The influence of the magnetic field on the chiral condensate is also studied.

This paper is organized as follows. In Sec. II, we will give a simple introduction of our model and calculate the thermodynamics potential within mean field approximation. Sec. III is devoted to the numerical results of the
quark pair condensation and the phase diagram in the $T - \mu_I - eB$ space. We summarize and conclude our job in Sec. IV.

II. THE MODEL

The Lagrangian density of two-flavor Nambu-Jona-Lasinio (NJL) model is defined as:

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu D^\mu - m_0)\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5 \tau^5 \psi)^2], \quad (1)$$

where $\psi = (\psi_u, \psi_d)^T$ is the quark field, $m_0 = diag (m_u, m_d)$ is the current quark mass matrix with $m_u = m_d \equiv \bar{m}_0$ (the isospin symmetry). $D^\mu = \partial^\mu + ieA^\mu$ is the covariant derivative, $A^\mu = \delta^\mu_0 A^0$ and $A^0 = -iA_4$ represents the gauge field. $G$ is the four-quark coupling constant with dimension $GeV^{-2}$. The Pauli matrices $\tau_i$ $(i = 1, 2, 3)$ are defined in isospin space.

With scalar and pseudoscalar interactions corresponding to $\sigma$ and $\pi$ excitation, the lagrangian density has the symmetry of $SU(1) \otimes SU(2) \otimes SU_A(2)$, corresponding to baryon number symmetry, isospin symmetry and chiral symmetry, respectively. The chiral symmetry $SU_A(2)$ breaks down to $U_A(1)$ global symmetry which is associated with the chiral condensation of the sigma meson.

$$\sigma = \langle \bar{\psi}\psi \rangle = \sigma_u + \sigma_d, \quad (2)$$
$$\sigma_u = \langle \bar{u}u \rangle, \sigma_d = \langle \bar{d}d \rangle. \quad (3)$$

The isospin symmetry $SU_f(2)$ breaks down to $U_f(1)$ global symmetry with the generator $f_3$ which is related to the condensate of charged pions, $\pi^+$ and $\pi^-$,

$$\pi^+ = \langle \bar{u}\gamma_5 \tau^3 \psi \rangle = \sqrt{2} \langle \bar{u}\tau^3 u \rangle, \quad (4)$$
$$\pi^- = \langle \bar{d}\tau^3 \psi \rangle = \sqrt{2} \langle \bar{d}\tau^3 d \rangle. \quad (5)$$

With $\tau_3 = (\tau_1 \pm \tau_2)/\sqrt{2}$. At extremely high $\mu_I > 0$ the condensate of $u$ and anti-$d$ quark is favored. At extremely high $\mu_I < 0$ the condense of $d$ and anti-$u$ quark is favored.

The system is in the global thermal equilibrium, $\pi^+ = \pi^-$, as is assumed in this paper, the whole superfluidity is electric charge neutral.

By means of the mean field approximation, the thermodynamic potential of two flavor NJL model at finite isospin chemical potential, finite temperature and strong magnetic field is given as

$$\Omega = G(\sigma^2 + \pi^2) - 2N_c \sum_{f=u,d} \sum_\kappa \alpha_\kappa \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} \frac{|Q_f eB|}{2\pi} \left[ \omega_f^\pi + 2Tln(1 + e^{-\beta \omega_f^\pi}) \right], \quad (6)$$

with $\omega_f^\pi$ on behalf of $\omega_u^\pi$ and $\omega_d^\pi$, the quasi-particle energy of $u$ and $d$ quark. $Q_f$ means the electric charge of $Q_u$ and $Q_d$. $\kappa$ is non-negative integer which denotes the Landau level and $\alpha_\kappa = 2 - \delta_{\kappa,0}$ is the corresponding degeneracy. In the equation $(1)$,

$$\omega_f^\pi = \sqrt{(\omega_f + \mu_I)^2 + 4G^2\pi^2},$$
$$\omega_f = \sqrt{p_z^2 + 2|Q_f eB|\kappa + M^2}, \quad (7)$$

and

$$M = m_0 - 2G \sigma. \quad (8)$$

If we use the replacement for the momentum integral and the quark energy,

$$2 \int \frac{d^3p}{(2\pi)^3} \leftrightarrow \frac{|Q_f eB|}{2\pi} \sum_\kappa \alpha_\kappa \int \frac{dp_z}{2\pi}, \quad (9)$$

$$\sqrt{p_z^2 + M^2} \leftrightarrow \sqrt{p_z^2 + 2|Q_f eB|\kappa + M^2}, \quad (10)$$

the thermodynamic potential $\Omega$ above will coincide with that in the zero magnetic field case [24]. For the upper limit of $p_z$ integral, we use the hard cut $\sqrt{A^2 - 2|eB|}$. Then the upper limit of $\kappa$ sum is $\kappa_{\text{max}} = int[\sqrt{A^2 - 2|eB|}]$, for $Q_u = +\frac{2}{3}e$ and $Q_d = -\frac{1}{3}e$.

The gap equation of the mean field $\sigma$ and $\pi$ are derived from

$$\frac{\partial \Omega}{\partial \sigma} = 0, \frac{\partial \Omega}{\partial \pi} = 0. \quad (11)$$

When there exist multi-roots, only the solution which satisfies the minimum condition is physical. The gap equation of $\sigma$ is

$$\sigma + 2N_c \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} \sum_\kappa \alpha_\kappa \frac{M|eB|}{2\pi} \left[ Q_u[1 - 2f(\omega_u^\pi)] \right. \left. \frac{\omega_u + \mu_I}{\omega_u^\pi \omega_u} + Q_d[1 - 2f(\omega_d^\pi)] \frac{\omega_d - \mu_I}{\omega_d^\pi \omega_d} = 0 \right]. \quad (12)$$

The gap equation of $\pi$ is

$$1 - 4N_c G \sum_{f=u,d} \sum_\kappa \alpha_\kappa \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} \left[ \frac{|eB|}{2\pi} \frac{Q_f[1 - 2f(\omega_f^\pi)]}{\omega_f^\pi} = 0 \right]. \quad (13)$$

$\pi = 0$ is the trivial solution of this equation, which has been ignored here. The Fermi function is $f(x) = \frac{1}{(1 + e^{ax})}$. We choose the following values as numerical analysis parameters: $m_0 = 5MeV$, a three-dimensional momentum cut-off $\Lambda = 650.7MeV(T = 0)$, $G = 5.01GeV^{-2}$, $m_\pi = 139MeV$ and $|<\psi_u\psi_d>|^{1/2} = -250MeV$.

III. NUMERICAL RESULTS

A. the pion and chiral condensate

In Fig. 1 we plot the behavior of the chiral condensate $\sigma$ and the pion condensate $\pi$, measured in units of the chi-
pion condensate in vacuum $\sigma_0$, with varying isospin chemical potential $\mu_I$ at fixed temperature $T = 100\, \text{MeV}$ for various magnitude of the magnetic field, $eB = 0$, $20m_\pi^2$, and $30m_\pi^2$.

In the upper panel, for $eB = 0$, we reproduced the result of Ref. [24]. The $\pi$ superfluidity begin at critical isospin chemical potential $\mu_I^c = m_\pi/2 \approx 70\, \text{MeV}$, then the pion condensate increases when $\mu_I$ becomes larger, which says that the isospin chemical potential enhance the pion condensate $[54]$. When the external magnetic field is included, such as $eB = 20m_\pi^2$ and $30m_\pi^2$, the critical $\mu_I^c$ for occurrence of pion condensate increases with the increasing magnetic fields, indicating the magnetic field suppresses the formation of the pion superfluidity. For different magnetic fields, the pion condensate gaps remain the similar shape as a function of $\mu_I$, the difference is that the maximum of gap is bigger with larger $eB$. For example, the maximum of $\pi$ gap is $1.1\sigma_0$ when $eB = 0$, but it reaches to $1.5\sigma_0$ when $eB = 30m_\pi^2$. For $\mu_I > 550\, \text{MeV}$, there exist three solutions for the $\pi$ condensation corresponding to a given isospin chemical potential (one of them is zero). Such feature of the order parameter normally indicates that the phase transition is of first order.

In the lower panel, for $eB = 0$, the chiral condensation also coincides with the result in Ref. [24]. The chiral condensate decreases with increasing $\mu_I$, which reflects the restoration of the chiral symmetry. When the external magnetic fields are taken into account, such as $eB = 20m_\pi^2$ and $30m_\pi^2$, the $\sigma$ condensate is enhanced with increasing $B$, this is the so called chiral catalytic effect of the magnetic field $[52, 53]$. The restoration of chiral symmetry still happen with the increasing $\mu_I$. From the change tendency of the $\sigma$ condensate we can roughly extract the critical isospin chemical potential $\mu_I^c$. The bigger the magnetic field, the bigger the $\mu_I^c$. This implies that the external magnetic field hinders the restoration of chiral symmetry.

In Fig. 1 we show the behavior of the pion condensate $\pi$ and chiral condensate $\sigma$, with varying magnetic field strength at fixed isospin chemical potential $\mu_I = 100\, \text{MeV}$ for several values of temperature, $T = 0$, $100\, \text{MeV}$ and $130\, \text{MeV}$.

In the upper panel, the $\pi/\sigma_0$ almost remains stable with small $eB$. When $eB$ increases to certain value, the $\pi$ condensation starts to decrease oscillately, and then disappears when the external magnetic field is strong enough. This implies that $\pi$ condensate is suppressed with increasing $eB$. From the figure we can see the higher the temperature, the narrower range of the magnetic field domain of $\pi \neq 0$ and the smaller the critical $eB$ for pion vanishing. That is to say the pion condensate is suppressed by the temperature. An interesting phenomenon is that the pion condensate exhibits an oscillation behav-

![FIG. 1: Upper panel: Pion condensate $\pi$, shown as a function of the isospin chemical potential $\mu_I$ with respect to various magnetic field magnitudes. Lower panel: Chiral condensate $\sigma$, shown as a function of the isospin chemical potential $\mu_I$ with respect to various magnetic field magnitudes.](image1)

![FIG. 2: Upper panel: Pion condensate $\pi$, shown as a function of the magnetic field strength $eB$ with respect to various temperature magnitudes. Lower panel: Chiral condensate $\sigma$, shown as a function of the magnetic field strength $eB$ with respect to various temperature values.](image2)
ior as a function of $eB$. From our calculation, the oscillating behavior is only related to the external magnetic field but not the temperature or the isospin chemical potential, it is also verified in the lower panel.

In the lower panel, with increasing $eB$, the chiral condensate increases oscillately. That is coincide with the chiral catalytic effect. The oscillation is obvious even when the temperature is very high, such phenomenon is the so-called Alfven-de Haas oscillation. The oscillating behavior related to the external magnetic field is similar with the $\pi$. But the oscillating behavior vanishes when $eB > 16.5m_\pi^2$ and the $\sigma$ condensate increases monotonously as increasing $eB$. The reason is that when the $eB$ is higher enough, the Lowest Landau Level (LLL) will chiefly contribute to the properties of the system and then the oscillation disappears. The influence of temperature on chiral condensate is not such apparent as the pion condensation, the curves of $\sigma/\sigma_0$ coincide approximately for the three values of temperature we chose.

We noticed in some $eB$ regions, for example, between $eB = 13m_\pi^2 \sim 17m_\pi^2$, the chiral condensate will decrease as the increasing $eB$. The decrease is a consequence of the oscillating behavior of the $\sigma$ condensate, which is different from the inverse chiral magnetic catalysis effect.

In Fig. 3, we show the pion condensation varies with $eB$ for $\mu_I = 400 MeV$. Now the pion condensation increase oscillately with $eB$, impling the magnetic field can also enhance the pion condensate. This contradicts the behavior for the case of $\mu = 100 MeV$ in Fig. 2. Normally the isospin chemical potential can promote the pion condensate while the temperature depress it. However, the effect of magnetic field is complicated. In some case it depresses the $\pi$ condensation and in the other case it reinforce the $\pi$ condensation. For some given temperature and isospin chemical potential, such as $T > 195 MeV$, $\mu_I = 400 MeV$, the pion condensate does not exist at $eB = 0$, and then it occurs as the $eB$ increases, showing that the external magnetic field can promote the formation of the pion superfluidity. This phenomenon is the result of the coupled influence of $\mu_I$ and $eB$. When we increase $eB$ higher enough, the $\pi$ condensate start to decrease and will disappear at some value of $eB$ (it is not shown in the Figure), that is similar with the pion condensate in Fig.2.

In Fig. 4 we show the behavior of the pion condensate $\pi$ and the chiral condensate $\sigma$, with varying $T$ for isospin chemical potential $\mu_I = 100 MeV$ but for various magnetic field magnitudes, $eB = 0, 20m_\pi^2$ and $23m_\pi^2$, respectively.

In the upper panel, for $eB = 0$ the $\pi$ condensate decreases monotonously as the temperature increases. When $T$ increases to the critical value, the pion condensate disappears. This implies that the temperature depress the pion condensate, as is shown similarly in Fig. 2 and Fig. 3. Taking the magnetic field into consideration, the greater the $eB$, the narrower range of temperature domain of the $\pi \neq 0$ and the lower the critical $T$ for $\pi$ superfluidity disappearing, reflecting that $\pi$ condensate is suppressed by the strong magnetic field for $\mu_I = 100 MeV$. This result is self-consistent with what we have obtained in Fig. 2.

In the lower panel, the $\sigma$ condensate rises slowly with increasing temperature. The above phenomenon only happens in the pion superfluidity phase. In the high $T$ region the $\pi = 0$, the $\sigma$ condensation decreases monotonously as the $T$ increases, meaning the chiral symmetry will restore in this region. When magnetic field is

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**FIG. 3:** The Pion condensate $\pi$, shown as a function of the magnetic field strength $B$ with respect to various temperature magnitude at fixed isospin chemical potential $\mu_I = 400 MeV$.

**FIG. 4:** Upper panel: Pion condensate $\pi$, shown as a function of the temperature with respect to various magnetic field magnitudes. Lower panel: Chiral condensate $\sigma$, shown as a function of the temperature with respect to various magnetic field magnitudes.
included, the varying behaviors of the chiral condensation are similar, but the chiral condensate increases with $eB$. That is to say that the magnetic field can improve the $\sigma$ condensate, which is of course the chiral magnetic catalysis effect again.

The influence of the external magnetic field on the pion condensation is not simple promotion or suppression again the influence of the external magnetic field on the $\pi$ condensation for different $\mu_I$. For fixed $eB$, the $\pi$ condensate decreases monotonously with increasing temperature, that coincides with the case when $\mu_I = 100MeV$. This implies that the temperature always depress the pion condensation. Taking the magnetic field into consideration, the greater the $eB$, the wider range of temperature domain of the $\pi \neq 0$ and the larger the critical temperature for $\pi$ superfluidity transition, reflecting that $\pi$ condensate is promoted with the strong magnetic field for $\mu_I = 400MeV$. Comparing the varying behaviors of the pion condensation for $\mu_I = 100MeV$ with that for $\mu_I = 400MeV$, we prove again the influence of the external magnetic field on the pion condensation is not simple promotion or suppression. The effect of the external magnetic field on the pion superfluidity is related to the value of the isospin chemical potential.

B. the pion superfluidity phase diagram

From the change in the $\pi$ and $\sigma$ order parameter as a function of $T, eB$, and $\mu_I$, we can calculate both the pion superfluidity and the chiral phase transition phase diagram. We confine our analysis to the pion phase diagram in this paper. Here we define the transition happens when the $\pi$ condensation goes to zero, namely the pion superfluidity phase transform to the normal phase.

In the upper panel of Fig. 6 we plot the pion superfluidity phase diagram in $T - \mu_I$ plane at $eB = 0$. The region inside of the curve with $\pi \neq 0$ is superfluid phase, and the region outside of the curve with $\pi = 0$ is normal phase. When $\mu_I$ is small, the phase transition is second order, and then it becomes first order at large $\mu_I$. The point connects the first and the second order phase transition lines is the so-called tricritical point. At $eB = 0$, this point is approximately at $(T, \mu_I) = (456, 130)$. The two thin dashed lines are the metastable lines of the first order phase transition, which can be observed from the upper panel of Fig. 1. The isospin chemical potential on the higher dashed line is the maximal $\mu_I$ when the pion condensation exists, and on the lower dashed line the $\mu_I$ is the point where the pion condensate disappears. The thick dashed line in the middle is the so-called Maxwell line. The points on the Maxwell line meet the condition of phase equilibrium of the first order phase transition with $p_1 = p_2, \mu_{I1} = \mu_{I2}$ and $T_1 = T_2$.

The lower panel of Fig. 6 clearly shows how we determine this Maxwell line. We plot the behavior of the thermodynamic potential $\Omega(= -P)$ with varying $\mu_I$, for example, for $T = 60MeV$ and $eB = 0$. From the cross point on the figure we obtain the value of $\mu_I$ at this temperature. By changing the temperature and doing the same calculation, we then obtain the middle dashed line on the upper panel of Fig. 6.

Fig. 7 show the pion superfluidity phase diagram of our model including the strong magnetic field effect. In the upper panel, we show the pion superfluidity phase diagram in $T - \mu_I$ plane at different magnetic field, $eB = 0, 20m^2_s$, and $30m^2_s$, respectively. The regions outside of the curves are in normal phase. The regions inside the
curves with lower $T$ are the $\pi$ superfluidity phases.

For $eB = 0$ case, when $\mu_1$ is larger than $\mu_1^c$ (about $m_\pi/2$), the $\pi$ superfluidity phase forms. The pion superfluidity vanishes with increasing $T$. When the external magnetic fields are included, such as $eB = 20m_\pi^2$ and $30m_\pi^2$, the regions of $\pi \neq 0$ shrink as the $eB$ increases for $\mu_1 < 200MeV$, which can be verified by the upper panel in Fig.4. When $\mu_1$ is very large, for $\mu_1 > 200MeV$, the regions of $\pi \neq 0$ become larger as the $eB$ increases, this phenomenon is coincide with the lower panel in Fig.4. All in all, the region of $\pi \neq 0$ is nonmonotonously affected by the external $eB$. From our calculation, the tricritical point moves to the space with smaller $\mu_1$ and higher $T$ when the external magnetic field becomes stronger.

The middle panel shows us the pion superfluidity phase diagram in $eB - \mu_1$ plane at different values of temperature, $T = 0$, $150MeV$, and $190MeV$, respectively. The regions in the left-side of the curves are the normal phases and the right-side of the curves with higher $\mu_1$ are the pion superfluidity phase. The regions of $\pi \neq 0$ decrease with increasing $T$ because the temperature will suppress the formation of the $\pi$ superfluidity.

In the upper panel, we can roughly conclude that the critical $\mu_1^c$ where the pion superfluidity occurs increases as the $eB$ increases. If we investigate it in detail, shown in the middle panel, the $\mu_1^c$ actually does not rise monotonously but oscillitorily as the $eB$ increases when the magnetic field is not so strong. This oscillation is related to the splitting quark energy level in the surrounding of the magnetic field. For $eB > 40m_\pi^2$, the $\pi$ superfluid phase boundary becomes monotonous and smooth with the increasing $\mu_1$. This is because the Lower Landau Level (LLL) chiefly contributes to the behaviors of phase transition boundaries for large $eB$.

In the lower panel we show the pion superfluidity phase diagram in $T - eB$ plane for different values of the isospin chemical potential, $\mu_1 = 80MeV$, $100MeV$, and $150MeV$, respectively. The regions inside the curves with low $eB$ and low $T$ are the $\pi$ superfluidity phases. The larger the $\mu_1$, the wider the region of $\pi \neq 0$. This means that the $\pi$ superfluid is enhanced with increasing $\mu_1$. Here we still obtain the oscillated phase boundary for the pion superfluid phase and this results to an interesting phenomenon with the increasing $eB$. Taking $\mu_1 = 150MeV$ and $T = 185MeV$ for an example, when increasing the $eB$ the pion superfluidity appears occasionally. This phenomenon of the superfluid phase carry on alternatively. The oscillatory behavior of the pion superfluid phase boundary is also a typical subsequence of the magnetic field effect. This result is similar with the conclusion of the middle panel in this figure.

**IV. CONCLUSION AND DISCUSSION**

The effect of the external magnetic field on the pion condensate and chiral condensate is investigated in the two flavor NJL model at finite temperature and isospin potential. The chiral and pion condensation are calculated in the surrounding of the various magnetic field. The oscillating behavior of the chiral and pion condensation, related to the so-called Alfven-de Haas oscillation, is shown by the numerical result. The phase boundary of the pion superfluid are also investigated with the consideration of the magnetic field. An oscillatory phase boundary is found with increasing the magnetic field. The first order phase transition in the high isospin chemical potential region still exists when the magnetic field effect is taken into account. With the magnetic field considered, the tricritical point on the phase diagram of the $T - \mu_1$ plane moves to the high $T$ and low $\mu_1$ region. Because of the discrete of Landau Level, the influence of the external magnetic field on the pion condensation and the phase boundary is not simple promotion or suppres-
sion, which is different from the effect of the temperature and the isospin chemical potential. The $\sigma$ condensate is enhanced with increasing magnetic field, which is supported by the chiral magnetic catalysis effect. However, because of the oscillation of the chiral condensate, there exist some magnetic field region where the chiral condensation would decrease with the increasing $eB$.

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