T-duality invariance, M2-brane bundles and type II classification of gauged supergravities

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Abstract: In this paper we obtain the complete classification of the inequivalent classes of M2-brane symplectic torus bundles with monodromy in $SL(2,\mathbb{Z})$ and their precise T-duality relations among them. There are eight inequivalent classes of bundles which are, at low energies, in correspondence with the eight type II gauged supergravities in nine dimensions. Four of those have been previously found and they correspond to the 'type IIB side'. In this paper we provide the explicit realization of the remaining four classes associated to the 'type IIA side'. The precise T-duality relations between the eight inequivalent classes have allowed to identified the remaining four ones. We find that the gauging groups -classifying the eight types of II gauged supergravity in nine dimensions- are determined by the inequivalent coinvariant classes associated to the base and the fiber of the supermembrane bundles and their duals.

Keywords: M2-brane, supermembrane, U-duality, monodromy, bundles, gauged supergravities.

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1. Introduction

Nowadays it has become evident the relevance of finding T-dual/U-dual invariant theories in String and M-theory in order to have a better insight of the non perturbative structure of these theories. The local description, meaning by this the knowledge of the action or mass operator is not enough to characterize them and a bundle description including topological effects is also needed. Most of the community has focused in the search for those theories, in particular invariant actions, in the context of Effective Field Theories with modified supergravity actions enriched with terms of stringy origin in which T-duality is manifest. In this paper we want to delve into this open question from the Supermembrane theory point of view. Supermembrane theory is a sector of M-theory that has shown to be a natural arena for this search. The supermembrane couples to the $D = 11$ supergravity,
there is a deep relation between both theories: On one hand, Supermembrane theory is conjectured to contain a unique massless groundstate associated to the 11D supergravity multiplet. Several works have been developed in support of this claim, for a recent new approach see [1] and references therein. On the other supermembranes are expected to contain supergravities as their 'low energy limit'. In that respect, the M-theory origin of maximal supergravities in any dimension \( d \leq 11 \) [2, 3], is known to correspond to the 11D supermembrane compactified on a trivial \( T^{11-d} \) torus [4]. Recently [5, 6, 7] it has also been shown that global aspects of the supermembrane theories are the relevant ones to explain the correspondence with gauged supergravities. In particular Supermembrane theories formulated on nontrivial 2-torus bundles correspond to the gauged supergravities in nine dimensions with a 'Type IIB origin' [5]. The relation with the 9D 'type IIA' ones was inferred from the symmetry properties of the Mass operator under T-duality but not explicitly obtained. The main result of this paper is to describe and construct these last ones associated to the 'type IIA sector'. In this way we obtain explicitly all possible torus bundles associated to the supermembrane theory underlying the various type II gauged supergravities in nine dimensions. We establish the precise T-duality relations between those eight inequivalent classes of supermembrane torus bundles. From here on, we refer indistinctly to the 11D supermembrane theory or to the M2-brane theory.

The relation between Gauged Supergravities and bundles appeared in [8, 9, 10] and Scherk-Schwarz (SS) reductions [11] is well-established [12]. They are naturally expressed in terms of bundles as was found in [13, 14, 15]. As discussed in [16, 17], type IIB gauged supergravities in nine dimensions are obtained through SS reduction. These deformations correspond to the gauging of a global \( d \) dimensional isometry \( G \) associated to the compactified manifold in the un-gauged action. In this way the reduced \( d \)-fields with \( d < D \) are allowed to keep a non-trivial phase dependence on the \( (D-d) \) internal spatial coordinates. This phase is associated to a twist given by a monodromy \( M_{\text{sugra}} \), i.e., a topological non-trivial invariant of the compactified manifold when it is described in terms of bundles. In [13, 14] they express those supergravities in terms of principal fiber bundles with twisting. From M-theory viewpoint, nine dimensional SS reduction is related to torus bundles with monodromy in \( SL(2, \mathbb{Z}) \). The invariant functional action is formulated in terms of the local sections of this bundle. In nine dimensions, type II gauged supergravities contain a unique maximal supergravity and eight possible gauged deformations [16, 17], whose origin in 10D are the type IIB supergravity, the maximal type IIA and the two type IIA massive deformations: Romans [18] and Howe-Lambert-West (HLW) [19]. The determination of the M-theory origin of some of its gauged/massive supergravity deformations has been more elusive than the case of maximal supergravities. Indeed gauged supergravities were proposed to be obtained as effective theories of String/M-theory on nontrivial torus bundles [13]. In [5] the authors showed explicitly that the M-theory origin of type II gauged supergravities in 9D are the supermembrane torus bundles with monodromy in \( SL(2, \mathbb{Z}) \).

In addition, a considerable effort has been done trying to obtain the String/M-theory T-duality/U-duality invariant action. These have been attempted mainly by two different effective field approaches: Double Field Theory [20, 21, 22, 23, 24, 25] and Generalized
Geometry [26, 27, 28] focusing either on its stringy action [27, 29] or on the M-theory realization [30, 31, 32]. Since T-duality is a transformation between string theories that interchanges winding modes with Kaluza-Klein (KK) momentum at the same time that the moduli is changed according to Buscher rules [33], any invariant T-duality action must contain the information associated to winding modes. Winding modes are an intrinsic effect of extended objects, so effective field theories described in terms of supergravity must be modified to incorporate this information. In these two approaches the winding have been included by doubling the number of coordinates and then imposing a physical restriction known as Section condition to recover the original theory in a consistent way to incorporate more and more stringy / M-theory aspects. Very recently it has been even possible to incorporate the $\alpha'$ corrections into the analysis [34] as well as winding contributions from the very beginning [35]. See [36, 37, 38] for a review.

In this paper we provide an explicit construction of the four inequivalent classes of M2-brane bundles associated to the 'type IIA side' in nine dimensions. Using the topological information associated to the T-duality transformations at the level of Supermembrane bundle theory we are able to give an explanation for the differences in the structure and gauged groups of type IIA and type IIB gauged supergravities in nine dimensions. The supermembrane theory compactified on an $M_9 \times T^2$ is formulated globally on symplectic torus bundles. These bundles are classified according to two inequivalent topological sectors: they can be principal (i.e. with trivial monodromy) or non trivial with a monodromy group contained in $SL(2,\mathbb{Z})$. The supermembrane theory in the Light Cone Gauge (LCG) has a residual gauge symmetry, the symplectomorphisms on the base manifold which in two dimensions are equivalent to the area preserving diffeomorphisms (APD). When the theory is formulated globally the group of symplectomorphisms corresponds to the structure group of the torus bundle. It is well known that supermembrane theory on a torus when doubled dimensionally reduced corresponds to the type IIB superstring compactified on a circle [39, 40]. The supermembrane compactified on a trivial torus is associated at low energies to a type II maximal supergravity in nine dimensions [39]. When the compactification is non trivial, but associated to a central charge condition the theory is described at low energies by the type IIB gauged supergravities as shown in [5]. Type II maximal supergravity in nine dimensions is obtained as the low energy limit of supermembranes described by torus bundles with trivial monodromy. At supermembrane level the charges are quantized and thus their monodromy groups $\mathcal{M}_G$ are the the arithmetic subgroups of the corresponding monodromies associated to the type II gauged supergravity monodromies $\mathcal{M}_{G-sugra}$.

Since T-duality and S-duality transformations were shown to be symmetries of the mass operator of the Supermembrane theory [5, 6], then the same result was expected to hold for the type IIA sector, hence for the complete classes of 9D type II gauged supergravities. The action of dualities does not act only locally on the Hamiltonian and the mass operator but also globally on the structure of the bundle. Indeed this global action is fundamental to establish the corresponding connection with the type II gauged supergravities.

Global aspects of the T-duality action were studied long time ago, see for example
T-duality maps torus bundles into torus bundles preserving the invariance of the structure of the Hamiltonian but interchanging the cohomological charges of the base manifold with the homological ones of the fiber and the monodromy into a dual monodromy in the same conjugation class. The supermembrane bundle class is specified through the coinvariants of the fiber and the base. T-duality interchanges those invariants, however the map among the coinvariants does not correspond to an equivalence relation. As a consequence, on general grounds, the gauging groups characterizing the inequivalent classes of type II gauged supergravities are not preserved under T-duality [16, 17]. From the viewpoint of supergravity this fact is totally natural since the global $SL(2, R)$ symmetry is not realized on the type IIA side at perturbative level. At supermembrane level however S-duality is realized as a symmetry [40] and a priori one could think that these differences should not be manifest at the level of supermembrane bundles. We will see that the analysis is more subtle. The T-duality action on the global structure of the bundle is the relevant one to explain the difference between the two sectors from an M-theory point of view.

The paper is structured as follows: In section 2 we review the formulation of the Supermembrane theory on torus bundles with monodromy in $SL(2, Z)$. In section 3 and in the Appendices B and C we present our new results. We describe in detail the T-duality for the supermembrane bundle. We specify its local and global action on the torus bundles according to its coinvariant classification. We obtain explicitly, by analyzing the action of T-duality on the coinvariant structures, the four inequivalent classes of bundles of M2-brane bundles associated at low energies with the gaugings in the type IIA supergravity sector in nine dimensions. We present the discussion on the role of T-duality to explain the differences in the deformations allowed at supergravity level from the supermembrane point of view, and finally we present our conclusions. In the Appendix A we review the well-known results of the complete characterization of type II gauged supergravities in nine dimensions [42, 16]. In the Appendix B we describe some of the properties of the coinvariants of bundles with monodromy and in the Appendix C we deduce the most general supermembrane T-duality transformation.

2. Supermembrane theory on a symplectic torus bundle

The hamiltonian of a supermembrane theory with central charges formulated in the Light Cone Gauge (LCG) on a target space $M_9 \times T^2$, the supermembrane subject to a central charge condition, is the following one [43, 44, 45, 46, 47, 48, 49]:

$$H = \int_{\Sigma} \sqrt{\rho} \left[ \frac{1}{2} \left( \frac{P_\alpha}{\sqrt{\rho}} \right)^2 + \frac{1}{2} \frac{P_\alpha P_\beta}{\sqrt{\rho}} + \frac{T_{\alpha\beta}}{4} \{X^\alpha, X^\beta\}^2 + \frac{T_{\alpha\beta}}{2} (D X^\alpha)(\overline{D} X^\beta) \right] +$$

$$+ \int_{\Sigma} \frac{T_{\alpha\beta}^2}{\sqrt{\rho}} \left[ \frac{1}{2} (\frac{P_\alpha}{\sqrt{\rho}})^2 + \frac{1}{2} \frac{P_\alpha P_\beta}{\sqrt{\rho}} + \frac{T_{\alpha\beta}}{4} \{X^\alpha, X^\beta\}^2 + \frac{T_{\alpha\beta}}{2} (D X^\alpha)(\overline{D} X^\beta) \right] +$$

$$+ \int_{\Sigma} \frac{T_{\alpha\beta}^2}{\sqrt{\rho}} \left[ - \overline{\Gamma}_{\alpha\beta} \{X^\alpha, \Psi\} - \frac{1}{2} \overline{\Gamma}_{\alpha\beta} \Gamma^\mu \partial_\mu \Psi - \frac{1}{2} \overline{\Gamma}_{\alpha\beta} \Gamma \overline{D} \Psi \right]$$

$$+ \int_{\Sigma} \sqrt{\rho} L \left[ \frac{1}{2} \overline{D} \left( \frac{P}{\sqrt{\rho}} \right) + \frac{1}{2} \overline{D} \left( \frac{\overline{P}}{\sqrt{\rho}} \right) + \{X^\alpha, \frac{P_\alpha}{\sqrt{\rho}}\} - \{\overline{\Gamma}_{\alpha\beta}, \Psi\} \right],$$

(2.1)
where $L$ is a Lagrange multiplier and $T_{M2}$ is the 11D tension of the supermembrane, $\rho$ is the determinant of non-flat two torus $\Sigma$ that corresponds to the spatial part of the worldvolume metric. The symplectic bracket is defined as $\{A, B\} = \omega_{ab} \partial_a A \partial_b B$ whose symplectic 2-form is $\omega = \omega^{ab} d\sigma_a \otimes d\sigma_b$ with $\omega^{ab} = \frac{i\epsilon^{ab}}{2\sqrt{\rho}}$ with $a, b = z, \bar{z}$ the local complex coordinates and respectively its complex conjugate $\bar{z}$, defined on the base manifold $\Sigma$. $X^n$ are the embedding maps $\Sigma \rightarrow M_9$ where $n = 3, \ldots, 9$ and $X = X^1 + iX^2$ are the embedding ones from $\Sigma \rightarrow T^2$. They are scalars parametrizing the transverse coordinates of the supermembrane in the target space. $P_n$ are densities and they are the canonical momenta associated to the $X_n$, and respectively $P$ that of the field $X$. $\Psi$ are scalars on the worldvolume but an $SO(7)$ spinor on the target space, $\Gamma_n$ are seven Gamma matrices and $\Gamma = \Gamma_1 + i\Gamma_2$, denoting by $\Gamma$ its complex conjugate. The 2-torus $T^2$ of the target space is characterized by the moduli $R$ the radius, and $\tau$ the complex Teichmüller parameter. The winding numbers are $l_s, m_s$ with $r, s = 1, 2$ associated to the wrapping of the supermembrane on the $T^2$. They define a matrix $W = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$. When the wrapping is irreducible its determinant $\mathfrak{n}$ is different from zero [47] and the theory has discrete spectrum [48] in distinction with the wrapped supermembrane without this topological condition. This sector defines a topological condition associated to the existence of worldvolume monopoles that algebraically imply the existence of a non-vanishing central charge in the supersymmetric algebra. For this reason this sector was called supermembrane with central charges. On this sector there is a symplectic curvature defined on the base manifold is $\mathcal{F} = \nabla A - D A + \{A, \bar{A}\}$, with $A$ a connection under the infinitesimal symplectomorphism transformation $\delta \epsilon = D \epsilon$. See [43, 50] for a detailed analysis. The symplectic covariant derivative is defined as $D\bullet = D\bullet + \{A, \bullet\}$, with $D\bullet = e_t^a \partial_a$ a rotated covariant derivative [51, 5] defined in terms of a zwei-bein $e_t^a$ as

$$e_t^a := -2\pi R(l_r + m_r \tau) \Theta_{sr} \Theta^a_b \partial_b \hat{X}^s,$$

with $r, s = 1, 2$. $d\hat{X}$ are the harmonic one-form basis defined on $\Sigma$. The Hamiltonian is invariant the residual symmetry under Area Preserving Diffeomorphisms (APD) connected and not connected to the identity.

The hamiltonian is subject to the APD group residual constraints (connected to the identity $\phi_1$, but also to the large APD $\phi_2$)

$$\phi_1 : d\left( \frac{1}{2}(Pd\hat{X} + \bar{P}dX) + P_m dX^m - \bar{\Psi} \Gamma_{-} \Psi \right) = 0;$$

$$\phi_2 : \oint_{C_s} \left[ \frac{1}{2}(Pd\hat{X} + \bar{P}dX) + P_m dX^m - \bar{\Psi} \Gamma_{-} d\Psi \right] = 0,$$

(2.3)

where $C_s$ is the canonical 1-homology basis on $T^2$.

The theory is invariant under two different $SL(2, \mathbb{Z})$ discrete symmetries: The first one is associated to the invariance under the change of the basis of the harmonic one forms defined on the Riemann worldvolume [40] and the windings,

$$d\hat{X} \rightarrow S d\hat{X}, \quad \mathcal{W} \rightarrow S^{-1} \mathcal{W},$$

(2.4)
with \( S \in SL(2, \mathbb{Z}) \). This dependence is encoded in the matrix \( \Theta \in SL(2, \mathbb{Z}) \) [51] in (2.2).

The second one is associated to an invariance of the mass operator involving \( SL(2, \mathbb{Z}) \) symmetry related to the target 2-torus \( T^2 \), so it is invariant under S-duality transformations,

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad R \rightarrow R|c\tau + d|, \quad A \rightarrow Ae^{i\varphi}, \quad \mathbb{W} \rightarrow \left( \begin{array}{cc} a & -b \\ -c & d \end{array} \right) \mathbb{W}, \quad Q \rightarrow \left( \begin{array}{c} a \\ b \end{array} \right) Q, \quad (2.5)
\]

where \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) \) and \( c\tau + d = |c\tau + d|e^{-i\varphi} \) and \( Q = \left( \begin{array}{c} p \\ q \end{array} \right) \) is the KK charge of the supermembrane propagating on the target 2-torus considered. The homological charges of the target torus \( H_1(T^2) \) are interpreted as the quantized KK charges of the compactified supermembrane [5].

Now we formulate the previous embedding description in terms of a symplectic torus bundle with monodromy in \( SL(2, \mathbb{Z}) \). This global formulation is going to make manifest some topological invariants that carry physical information. The total space \( E \) is defined in terms of a fiber \( F = M_9 \times T^2 \) and \( \Sigma \) as the base manifold. The structure group \( G \) is the symplectomorphisms leaving invariant the canonical symplectic structure in \( T^2 \). The action of \( G \) on \( F \) produces a \( \pi_0(G) \)-action on the homology and cohomology of \( F \). The monodromy \( M_G \) is defined as \( M_G : \pi_1(\Sigma) \rightarrow \pi_0(G) \), with \( G = \text{Symp}(T^2) \) and \( \pi_0(G) = SL(2, \mathbb{Z}) \).

Consequently \( M_G = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \gamma \in SL(2, \mathbb{Z}) \) and it acts on the homology basis of the \( T^2 \) target torus -where \( \gamma = \gamma_1 + \gamma_2 \) with \( (\gamma_1, \gamma_2) \) are the integers characterizing the element of the homotopic group \( \pi_1(\Sigma) \). The symplectic connection defined on the base manifold transforms with the monodromy, \( dA \rightarrow dAe^{i\varphi M_G} \) where \( \varphi M_G \) is a discrete monodromy phase given by \( \varphi M_G = \frac{c\tau + d}{|c\tau + d|} \) for a given modulus \( \tau \). The inequivalent classes of symplectic torus bundles over \( \Sigma \) are classified by the elements of the second cohomology group, \( H^2(\Sigma, \mathbb{Z}_{M_G}) \) or equivalently by their coinvariants. See [50, 53, 5] for more details. The global symmetries of the theory become restricted by the monodromy.

2.1 The role of fiber \( C_F \) and base \( C_B \) coinvariants on the bundle structure

The supermembrane symplectic torus bundles are characterized by two types of coinvariants relevant for the characterization of the supermembrane bundle. The class of coinvariants associated to the fiber \( C_F \) and the coinvariants associated to the base \( C_B \). The torus bundles with a given monodromy \( M_G \) are classified according to the elements of the twisted second cohomology group \( H^2(\Sigma, \mathbb{Z}_{M_G}) \) of the base manifold \( \Sigma \). Its coefficients are defined on the module generated by the monodromy representation acting on the homology of the target torus [53]. There is a bijective relation with the elements of the coinvariant group \( C_F = \{ C_a \}, a = 1, \ldots, j \) associated with a particular monodromy group \( M_G \), -see appendix A, for more properties of the coinvariant classes-. A coinvariant class in the KK sector is given by

\[
C_F = \{ Q + (M_g - 1)\hat{Q} \}, \quad (2.7)
\]
for any \( g \in \mathcal{M}_G \), and \( \hat{Q} \) is any arbitrary element of the KK sector. That is, two elements belong to the same class if they differ in an element \((g - 1)\hat{Q}\) for some \( g \in \mathcal{M}_G \) and \( \hat{Q} \).

Associated to the monodromy subgroup \( \mathcal{M}_G \) there is an induced action on the cohomology of the base manifold \([6]\), which corresponds to the monodromy group of the winding sector \( \mathcal{M}_G^* \). Since \( \mathcal{M}_G^* = \Omega \mathcal{M}_G \Omega^{-1} \) with \( \Omega = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), it lies in the same conjugacy class of \( \mathcal{M}_G \). \( \mathcal{M}_G^* \) acts on the fields which define the Hamiltonian, that is, on sections of the torus bundle through a matrix \( \Theta = (V^{-1} \mathcal{M}_G^* V)^T \) that appears in the symplectic covariant derivatives \( D_r \) of the Hamiltonian \( H \). \( V \) is associated to the monodromy subgroup \( \mathcal{M}_G \subset SL(2, \mathbb{Z}) \), see \([5]\) for further details. Induced by them there are other two possible invariants \((\mathcal{M}_G^*, C_B)\) characterizing the symplectic bundles. The two monodromies lie in the same conjugation class, however they are not the same and consequently their respective coinvariants are not equivalent.

A coinvariant class in the winding sector is given by

\[
C_B = \{ W + (\mathcal{M}_g^* - I)\hat{W} \},
\]

with \( \mathcal{M}_g^* \in \mathcal{M}_G^* \), the monodromy group acting in the winding sector specified by \( W = \begin{pmatrix} l \\ m \end{pmatrix} \in H^1(\Sigma) \). Due to the \( SL(2, \mathbb{Z}) \) symmetry on the equivalence class of the basis of homology on the base manifold (a torus), it is possible to reduce the problem to work indistinctly with the winding matrix \( W \) or with \( W \) defined by the first row of the matrix \( W \). See appendix A. Then for M2-brane or symplectic torus bundles one needs to specify both types of coinvariants for a given monodromy \( \mathcal{M}_G \), those associated to the fiber \( C_F \) and those associated to the base manifold \( C_B \). Given a symplectic torus bundle the Hamiltonian of the theory and the mass operator are defined on the coinvariant classes.

The dependence of the bundle on the winding charges \( W \) and KK charges \( Q \) is defined in terms of the function \( F \):

\[
F : \quad W \to W, \\
Q \to Q,
\]

which depends on the classes \( C_B \) and \( C_F \). Since these are invariant under the action of the monodromy, the same occur for the Hamiltonian or mass operator. We thus have,

\[
F(C_B) = F(W), \quad F(C_F) = F(Q).
\]

In particular if we consider the action only on the orbits, instead of the coinvariant classes, \( F \) may be defined in terms of a matrix \( \Theta \) acting on the matrix \( W \):

\[
\Theta W
\]

, in a way that under the monodromy \( W \to \mathcal{M}^* W \) and \( \Theta \to \Theta (\mathcal{M}^*)^{-1} \) as we have defined the coinvariant derivative. A way to obtain a function \( F \) with such property is to consider a linear function \( F_L \) defined on the orbits of the elements of the coinvariant class:

\[
F_L(C_B) = F_L(W + (\mathcal{M}^* - I)\hat{W}) = F_L(W) + F_L(\mathcal{M}^* \hat{W}) - F_L(\hat{W}) = F_L(W).
\]
Each coinvariant class is invariant under the action of any \( g \in \mathcal{M}_G \). So the coinvariant class may be considered itself as a class of orbits under the action of \( \mathcal{M}_G \). Given a symplectic torus bundle the Hamiltonian of the theory is defined for any orbit of the coinvariant class.

### 2.2 T-duality for supermembrane theory torus bundles

The T-duality transformation acts on the Hamiltonian \( H \) and the mass operator \( M^2 \) and it has also a action on the topological invariants of the M2-brane torus bundle describing the theory. In what follows, we are going to describe the T-duality action at global level, that is, specifying its action on the supermembrane symplectic torus bundle structure and secondly we will describe its action on the mass operator of the supermembranes.

#### 2.2.1 T-duality action on the coinvariants

The T-duality transformation globally transforms a bundle into a dual one, by interchanging the cohomological charges of the torus base manifold into the homological charges defined on the torus fiber with dual moduli. T-duality also interchanges the coinvariant class of the base and the fiber in the dual T-bundle:

\[
(C_F, C_B) = (\tilde{C}_B, \tilde{C}_F),
\]

where \( \tilde{C} \) denotes the dual coinvariant class. It may occur, however that the transformation becomes non linear. At low energies this fact will be reflected in the change of the gauging group associated to the corresponding dual supergravity. In order to delve in this classification we are going to characterize the action of T-duality over the different classes of M2-brane bundles with monodromies trivial and non trivial:

- **Trivial monodromy**: We first consider the case in which \( \mathcal{M}_G = I \), i.e. when the monodromy group is trivial. In this case the coinvariant classes, which classify the inequivalent torus bundles, have only one element \( Q \) in the KK sector and one element \( W \) in the winding sector. The T-dual transformation is defined in terms of \( T \in SL(2,Z) \) with equal diagonal terms, satisfying

\[
\tilde{Q} = W = TQ, \quad \tilde{W} = Q = T^{-1}W,
\]

where \( T = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} \in SL(2,Z) \) is defined as in the appendix C. Given \( Q \) and an associated winding matrix \( W \), there always exists a winding matrix on the equivalence class of \( W \) defined by the action from the right by \( S \in SL(2,Z) \) such that

\[
WS = TQ,
\]

\( Q \) is the matrix whose first row is \( Q \) and it also has determinant \( n \). Given \( Q \) there always exists \( \tilde{Q} \), though it is not unique. The most general one is obtained by multiplying from the right by a parabolic matrix with integer coefficients and equal diagonal elements:

\[
QK, \quad K = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.
\]
The matrix $K$ can always be absorbed into $S$. The symplectic torus bundles are classified in this case, i.e. $\mathcal{M}_G = I$, by two integers, the elements $Z \otimes Z$. The symplectic torus bundles are in one to one correspondence with the $U(1) \times U(1)$ principle bundle over the base manifold. Since the monodromy is trivial, the structure group may reduce to the group of symplectomorphism homotopic to the identity. The dual transformation is then completed by the transformation of the moduli as given in (2.21).

- **Non trivial monodromy.** We now consider the case where the monodromy group $\mathcal{M}_G$ is non trivial. It is an abelian subgroup of $SL(2, Z)$. The T-dual transformation maps as before coinvariant classes on the KK sector onto coinvariant class in the winding sector. In order to define the T-duality map we take any element $Q$ of $C_F$ and any element $W$ of $C_B$ and map them as in (2.14). The map is given by the following transformation,

$$
\begin{align*}
Q & \mapsto W, \\
\mathcal{M}_G & \xrightarrow{\Omega} \Omega \mathcal{M}_G \Omega^{-1}, \\
\hat{Q} & \mapsto \hat{W}, \\
C_F & \mapsto C_B,
\end{align*}
$$

(2.17)

where $f$ is a general linear map from the $\hat{Q}$ sector onto the $\hat{W}$ sector. In particular the map $f$ can be defined as:

$$
\hat{W} = T \hat{Q}.
$$

(2.18)

However, in general it is not necessary to relate $f$ to $T$.

Suppose now that instead of mapping $Q$ to $W$ we map it to another member of the coinvariant class to which $W$ belongs:

$$
Q \rightarrow W + (\mathcal{M}_G^* - I)\hat{W}_1,
$$

(2.19)

then,

$$
C_F = \{Q + (\mathcal{M}_G - I)\hat{Q}\} \rightarrow \{W + (\mathcal{M}_G^* - I)(\hat{W} + \hat{W}_1)\} = C_B.
$$

(2.20)

That is, the new map is only a translation on the $\hat{W}$ sector, the map $f$ includes a translation by $\hat{W}_1$. Equation (2.20) shows that changing the map for $Q \rightarrow W$ to $Q \rightarrow (W + (\mathcal{M}_G^* - I)\hat{W})$ is equivalent to leave the map $Q \rightarrow W$ and change $f$ by a translation. The translation by $\hat{W}_1$ is irrelevant from the point of view of the coinvariant class since $\hat{W} + \hat{W}_1$ is a general element of the winding sector. Hence the map between the coinvariant classes is only determined by $T$ which is constructed from one element of each class $Q$ and $W$ respectively. The generator $\mathcal{M}_G$ can be parabolic, elliptic or hyperbolic.
2.2.2 T-duality action on the mass operator

The duality transformation on the symplectic torus bundle has an action on the charges but also on the geometrical moduli. We define dimensionless variables\(^1\) \( Z \), where \( Z = (T_{M^2}A)Y^{1/3} \) with \( A = (2\pi R)^2 Im\tau \), the area of the target torus and \( Y = \frac{R_{\text{torr}}}{q\tau-p} \) a variable proportional to the \( R \) radius of the complex torus. \( Y \) is invariant under the monodromy group if we consider \( Q \) the components of \( F(Q) \). The T-duality transformation is given by:

\[
\tilde{Z} \tilde{Z} = 1, \quad \frac{\tilde{\tau}}{\tau} = \frac{\alpha\tau + \beta}{\gamma\tau + \alpha},
\]

(2.21)

The moduli:

\[
\tilde{Z} \tilde{Z} = 1, \quad \tilde{\tau} = \frac{\alpha\tau + \beta}{\gamma\tau + \alpha}.
\]

The charges:

\[
\tilde{Q} = TQ, \quad \tilde{W} = T^{-1}W,
\]

with \( \alpha, \beta, \gamma \), the integer entries of the \( T \) matrix in (2.14). The charges \( Q, W \) transform depending on the type of bundle considered, i.e. with trivial or non trivial monodromy. We notice that \( Z, Y \) and their duals are invariant on an orbit generated by \( M_G \) contained in the respective coinvariant class, provided that \( \tau \) and \( Q \) transform as in (2.5). Moreover, they are independent of the coinvariant class when we define \( Y \) in terms of the components of \( F(Q) \) instead of \( Q \) and leave \( \tau \) as a invariant moduli under monodromy. The symmetry of the Hamiltonian related to the basis of harmonic one-forms of the Riemann worldvolume [40] allows to define the class of orbits associated to the winding matrices \([W]\). Following (2.14) there always exists \( T \) such that

\[
TQ = [W], \quad T^{-1}W = [Q],
\]

(2.22)

such that T-duality maps classes into classes

\[
[W] \to [\tilde{Q}] = [W], \quad [W] \to [\tilde{W}] = [Q].
\]

(2.23)

Let us recall [5, 6], the relation between the radius modulus and its dual follows from (2.21), was obtained in:

\[
\tilde{R} = \left| \frac{q\tau-p}{R^2(Im\tau)^{4/3}(2\pi)^{4/3}R} \right|^{3/2}.
\]

(2.24)

T-duality defines a nonlinear transformation on the charges of the supermembrane since \( T \) is constructed from them, in distinction with the usual \( SL(2, Z) \) action on the moduli which is a linear one. This property will be very relevant for understanding the dual multiplet structure. The KK modes are mapped onto the winding modes and viceversa as expected. This property together with the condition \( Z\tilde{Z} = 1 \) ensure that \( (\text{T-duality})^2 = I \). This transformation becomes a symmetry for \( Z = \tilde{Z} = 1 \) which imposes a relation between the tension, the moduli and the KK charges of the wrapped supermembrane,

\[
T_{M^2}^0 = \frac{|q\tau-p|}{R^3(Im\tau)^2}.
\]

(2.25)

Given the values of the moduli it fixes the allowed tension \( T_{M^2}^0 \) or on the other way around, for a fixed tension \( T_{M^2}^0 \), the radius, the Teichmüller parameter of the 2-torus, and the

\(^1\)In [40, 5] the dimensionless variables were defined as \( Z := T_{M^2}A\tilde{Y} \) and \( \tilde{Z} := T_{M^2}\tilde{A}Y \). Following (2.21) one can verify that both definitions are equivalent.
KK charges satisfy (2.25). For $Z = 1$ the Hamiltonian and the mass operator of the supermembrane with central charges are invariant under T-duality:

$$M^2 = (T_{n2}^0)^2 n^2 \tilde{A}^2 + \frac{k^2}{Y^2} + (T_{n2}^0)^2/3 H = \frac{n^2}{Y^2} + (T_{n2}^0)^2 k^2 \tilde{A}^2 + (T_{n2}^0)^2/3 \tilde{H},$$

(2.26)

with $H = \tilde{H}$. See [5] for further details.

In [6] the authors showed that there always exists a $\mathcal{T}$ a parabolic matrix transformation of T-duality given for any arbitrary value of the KK and winding charges. This parabolic transformation depends on the winding and KK momenta of the supermembrane bundle. In this work we extend this analysis to characterize in detail the most general T-duality transformation $\mathcal{T}$, see the appendix C. In principle the T-duality transformations are a subset of the parabolic, elliptic and hyperbolic transformations with equal diagonal terms. We show there that the parabolic one plays a distinguished role since it is the only class able to map any kind of winding and KK charges for a given supermembrane torus bundle with arbitrary monodromy and general central charge into its dual. If we restrict the central charge to $n = 1$, then all the different types of T-dual transformations $\mathcal{T}$, are allowed, i.e. elliptic $\mathcal{T}_e$, parabolic $\mathcal{T}_p$ and hyperbolic ones $\mathcal{T}_h$ however, since the supermembrane can wrapped any arbitrary times a 2-torus so there is no justification to restrict to $n = 1$. For $n \neq 1$ the situation is different: the $\mathcal{T}_{Z4}$ elliptic case of T-duality transformations fail to map the T-bundles except for very specific KK and W charges. In distinction, the hyperbolic T-duality matrices $\mathcal{T}_h$, always exist for arbitrary $Q$, $W$, and $n$. The difference with respect to the parabolic case $\mathcal{T}_p$, relies in the fact that for each set of charges there is needed a different hyperbolic realization with different trace. For a general transformation, the parabolic T-duality one $\mathcal{T}_p$ is going to be the one responsible to characterize the different M2-torus bundles dual which at low energies are associated with the different type II gauged supergravities in nine dimensions, for that reason we will denote $\mathcal{T}_p \equiv \mathcal{T}$.

3. Classification of T-dual supermembrane bundles

In this section we are going to establish the precise correspondence between the type IIA side of the supermembrane bundle with parabolic, elliptic, hyperbolic and trombone monodromies. At low energies they correspond respectively to the 9D type II gauged supergravities: type IIB origin and their T-duals of type IIA origin. These type IIA 9D gauged supergravities are obtained either through KK reduction of the HLW and Romans type IIA supergravity in 10D or the SS reduction of the 10D maximal type IIA supergravity. Schematically we can summarize our results in Figure 1.

The gauging groups of the gauged supergravities associated to the monodromies differs in the two sectors (type IIA and type IIB) of 9D gauged supergravities. The M2-brane bundle analysis explains this fact since the T-dual transformation in general does not commute with the monodromy group (except for the parabolic monodromy case) and consequently its associated T-dual coinvariant class of the bundle does not lie in same
Figure 1: These are the precise relations between the M2-brane bundle with monodromy in $SL(2, \mathbb{Z})$ inequivalent classes when a M2-brane T-duality is performed. The formers are in correspondence with the gauged type IIB supergravities and the latter with those of the type IIA in nine dimensions.

The T-dual transformation maps a given charge $Q$ in the KK sector into a winding $W = TQ$ and the coinvariant class of $Q$, to the coinvariant class of $W$. That is, the coinvariant class

$$C_F \equiv \{ Q + (M_G - \mathbb{I}) \hat{Q} \} ,$$

is mapped into

$$\tilde{C}_F = \{ W + (M_G^2 - \mathbb{I}) \hat{W} \} = C_B ,$$

on the winding sector, which may or may not coincide with

$$T C_F = \{ TQ + T(M_G - \mathbb{I})T^{-1}(T \hat{Q}) \} ,$$

because generically

$$M_G^C := TM_G T^{-1} \neq M_G^* = \Omega M_G \Omega^{-1} .$$

If both classes define the same coinvariant class, i.e. $\tilde{C}_F = T C_F$, then for any element of the matrix group monodromy $M_{g_1}$ and $\hat{W}$

$$(M_G - \mathbb{I})T \hat{Q} - (M_{g_1} - \mathbb{I})\hat{W} = (M_{g_2} - \mathbb{I})\hat{W} ,$$

because $\tilde{C}_F = T C_F$. This happens even though the monodromy and its dual are in the same conjugation class as originally signalled in [54].
Figure 2: This diagram shows the precise T-duality correspondence between the eight M2-brane theories resulting from the 11D M2-brane theory formulated over trivial torus bundles and torus bundles with non-trivial monodromies. On top of it we also show their connections with the maximal and the gauged supergravities theories in nine dimensions found in [42, 55].

for some $\mathcal{M}^*_9$, $\mathcal{W}$ denotes an arbitrary element in the winding sector. $\mathcal{M}^*_9 \mu, \mathcal{M}^*_2 \in \mathcal{M}^*_9$. In this case, the T-duality map transforms the symplectic torus bundles with a given monodromy group onto themselves. Generically that is not the case, i.e., $\tilde{C}_F \neq T C_F$, so we have to study in detail what occurs for each type of bundle of monodromy: parabolic, elliptic, hyperbolic and trombone.

3.1 T-dual of M2-brane parabolic torus bundles

The M2-brane torus bundles with parabolic monodromies has two inequivalent nontrivial
monodromies classes
\[ M_p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad M_{Z_2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.6) \]
satisfying that \(|Tr(M_p)| = 2\). While associated to the first type of monodromy the coinvariants are generically torsionless (the only exception is the \((0,0)\) class associated to torsion one), in the second type of parabolic monodromy all coinvariants have torsion and the KK charges are restricted to be \((0,0), (0,1), (1,0), (1,1)\). A coinvariant with torsion determines a symplectic torus bundle whose local symplectic structure on the fibers is the restriction of the global symplectic one. For the parabolic case the dual monodromy group is given by
\[ M^C_p = TM_p T^{-1} = M_p \text{ since } T \text{ commute with } M_p. \]
In addition the group \(M^*_p = \Omega M_p \Omega^{-1}\) coincides with the group \(M_p\) since \(\Omega g \Omega^{-1} = g^{-1}\) for any \(g \in M_p\). The condition \((3.5)\) is then satisfied. The T-duality map acts linearly and it transforms the class of the M2-brane symplectic torus bundles with parabolic monodromy group onto itself. At low energies they correspond to a parabolic gauged supergravity on the type IIA side. This 9D gauged supergravity is associated to the KK reduction of the 10D Romans supergravity \([42]\).

### 3.2 T-duals of M2-brane elliptic and hyperbolic torus bundles

We will analyze the two cases separately and we will see how under T-duality they generate a unique class of supermembrane torus bundles.

The elliptic monodromy group \(M_e\) \([56]\) is finitely generated by the following matrices
\[ M_{Z_3} = \gamma, \quad M_{Z_4} = \gamma, \quad M_{Z_6} = \gamma, \quad (3.7) \]
satisfying that \(|Tr(M_e)| < 2\) with \(\gamma \in Z\). Their associated coinvariant of the fiber equivalence classes can be computed explicitly using \((2.7)\). For the \(Z_3\) case there are three classes
\[ C^F_{Z_3} = \{Q_i + (M_{Z_3} - I)\hat{Q}\}, \quad i = 1, 2, 3, \quad (3.8) \]
with
\[ Q_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad (3.9) \]

It can be shown that \((3.5)\) cannot be satisfied. In fact \((3.5)\) means that for any \(\hat{g} \in M^C_{Z_3}\) and \(\hat{Q}\), there exists \(g \in M^*_Z\) such that
\[ (\hat{M}_g - I) T \hat{Q} = (g - I)\hat{W}. \quad (3.10) \]
One can show that there does not exist \(\hat{W}\) satisfying the equality \((3.10)\) for suitable \(\hat{g} \in M^C_{Z_3}\) and \(\hat{Q}\). In the same way it can be shown that \((3.5)\) is not satisfied for the monodromy groups \(M_{Z_4}, M^*_Z\). In distinction \((3.5)\) is satisfied for \(M_{Z_6}\) and \(M^*_Z\). This means that given any element \(\hat{g} \in M^C_{Z_6}\) and any \(\hat{Q}\) there exists \(g \in M^*_Z\) and \(\hat{W}\) satisfying \((3.5)\) and viceversa. This
follows because one element of the group is $g_{Z_6} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, hence $(g_{Z_6}^* - \mathbb{I})$ has determinant equal to 1, then there always exists $\hat{W}$ satisfying (3.5):

$$(g_{Z_6}^* - \mathbb{I})^{-1}(\hat{g}_{Z_6}^* - \mathbb{I}) T \hat{Q} = \hat{W}. \quad (3.11)$$

The other way round follows in the same way, given any $g \in \mathcal{M}_{Z_6}^*$ and $\hat{W}$ there always exists $\hat{Q}$ satisfying (3.10). In fact

$$det(T g_{Z_6} T^{-1} - \mathbb{I}) = 1, \quad (3.12)$$

and we proceed as before. The coinvariant abelian group associated to the monodromy $\mathcal{M}_{Z_6}$ has only one element in the KK sector given by the class

$$C_{Z_6}^F = \{q_0 + (\mathcal{M}_{Z_6} - \mathbb{I})\hat{Q}\}, \quad (3.13)$$

where $q_0$ is any particular charge and respectively the only one element in the winding sector.

Let us now analyze the case of hyperbolic monodromy. There are infinite abelian monodromy groups of hyperbolic matrices constructed in terms of

$$\mathcal{M}_h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\gamma,$$

such that $|Tr(\mathcal{M}_h)| > 2$ with $\mathcal{M}_h \in SL(2, \mathbb{Z})$ [56]. It can also be explicitly shown that generically the coinvariant structure of the dual bundle is not equivalent to the original one.

Let us now compare the dual coinvariants of elliptic and hyperbolic monodromy bundles with respect to their conjugate coinvariants. We consider the generic monodromy case $g \in \mathcal{M}_G$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, then

$$T g T^{-1} = \begin{pmatrix} a + tc & -t(a + tc) + b + td \\ c & -ct + d \end{pmatrix}. \quad (3.15)$$

By defining with $u = a - d$, we can always express

$$(T g T^{-1} - \mathbb{I}) - ((g^*)^{-1} - \mathbb{I}) = (u + tc)[B - tA], \quad (3.16)$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Notice that $A, B$, satisfy the algebra:

$$[A, B] = 2A, \quad (3.17)$$

which corresponds to a non abelian group $A(1)$ associated to the collinear transformations in one dimensions (translation and scaling). This algebra was already identified at the level of nine type II gauged supergravity [42].
Notice that for the parabolic case \((a = 1, b = p, c = 0, d = 1)\) both coefficients in (3.16) vanish, and the coinvariants and their duals lie in the same equivalence class.

Clearly the elliptic and hyperbolic coinvariant classes are mapped under T-duality into an inequivalent coinvariant equivalence class. Naively one could have expected that at low energies, this would not imply a difference between the gauged group and structure of the multiplets of 9D type IIA and type IIB gauged supergravities. However as it has been emphasized all along this work the structure of the coinvariants is the one characterizing the bundle and consequently it is related to the gauging group at the level of supergravity. For the elliptic and hyperbolic supermembrane bundles, T-duality acts nonlinearly on the supermembrane dual bundle and this dual realization is associated to a nonabelian algebra of the \(A(1)\) group. We expect it to be the M-theory origin of 9D type IIA gauged supergravities corresponding to a nonabelian \(A(1)\) scaling of \(R\) type.

### 3.3 T-dual of M2-brane trombone torus bundles

Trombone symmetry produces supergravities that do not have Lagrangian but are uniquely defined through the equations of motion. The reason is that the trombone symmetry is not a symmetry of the action since it scales the Langrangian but it is a symmetry of the equations of motion. At quantum level however there exists a well defined action since it is possible to define an invariant hamiltonian. In type II supergravity in 9D the global symmetries are \(GL(2, R) = SL(2, R) \times R\), the breaking of the group into its arithmetic subgroup \(GL(2, Z)\) with determinant \(\pm 1\) is not able to capture the effect of the scaling. Hence in order to obtain the scaling symmetries an alternative procedure is needed. This question was in fact solved many years ago by the authors [57], by means of a nonlinear realization of the group \(SL(2, Z)\) that they called active \(SL(2, Z)\). In [5] the authors used this realization to obtain a bundle description of the supermembrane with gauged trombone symmetry. In the following, in order to be self-contained we first summarize those results i.e. the realization of the trombone symmetry and their associated torus bundles, previous to perform the characterization of their duals.

Let us consider a nonlinear representation of the group \(SL(2, Z)\) in terms of the \(2 \times 2\) matrices \(H_{ij}\) as in [5]. Given two different charges \(Q_i, Q_j\) labelled with two different indices \(i, j\) and given \(H_{ij}\) the \(SL(2, Z)\) active transformation, it acts on the charges as follows:

\[
H_{ij} Q_i = Q_j, \quad \text{and} \quad \frac{H_{ij}}{h_{ij}} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix},
\]

(3.18)

the solution for (3.18) is given by

\[
H_{ji} = \begin{pmatrix} \frac{p_j - q_j \tau}{q_j} U + \frac{q_j C}{q_i} \frac{p_j - q_j \tau}{q_i} U - \frac{p_j - q_j \tau}{q_i} C \\ -U \end{pmatrix}
\]

(3.19)

where

\[
h_{ji} = \frac{p_j - q_j \tau}{p_i - q_i \tau}, \quad U = \frac{p_j q_i - p_i q_j}{|p_i - q_i \tau|^2}, \quad C = \frac{|p_j - q_j \tau|^2}{|p_i - q_i \tau|^2}.
\]

(3.20)
For each monodromy group there exists a unique non-linear realization of it $\mathbb{H}_G$. There are three non-linear realizations associated to the elliptic, parabolic and hyperbolic monodromy classes, however they cannot be distinguished among them, so on the type IIB side we only obtain a single M2-brane trombone torus bundle class. This is in agreement with the fact that at low energies on the type IIB side there is a unique 9D trombone supergravity.

This transformation generates the complete lattice of charges for a given vacuum, (that is the asymptotic value of the scalar moduli). $H_{ji} \in GL(2,\mathbb{R})$ is the non-linear representation of $M_{ji}Q_i = Q_j$ with $M_{ji} \in SL(2,\mathbb{Z})$ which acts linearly on the charges but non-linearly on the moduli. The Hamiltonian $H$ is invariant, since

$$\tau \rightarrow \tau, \quad W \rightarrow W, \quad R \rightarrow R.$$  

While the mass operator changes since the KK contribution changes, $KK \rightarrow KK'$ according to (3.18). See [5] for details.

The structure of the supermembrane trombone bundle differs from those in which the monodromy is linearly realized since in the former the moduli is unaltered by the trombone monodromy. Consequently on the 'type IIB side' each bundle only contains a single pair of winding charges $W = \begin{pmatrix} l \\ m \end{pmatrix}$ instead of an orbit, and therefore there is no monodromy associated to the base manifold. The only monodromy of the bundle is associated to the fiber $\mathcal{M}_G$, it is non-linearly realized in terms of $\mathbb{H}_G$. The coinvariant class of charges of the fiber is defined as

$$C_{\text{tromb}}^F = \{ Q + (\mathbb{H}_G - \mathbb{I})\hat{Q} \},$$

and the structure of the bundle in terms of the coinvariant classes is $(C_{\text{tromb}}^F, W)$.

The T-duality transformation for the trombone torus bundle maps the coinvariant class of the fiber of the original bundle into the coinvariant class of the base, and the original winding single charge of the base into a single dual KK charge of the dual fiber. As a result, the dual trombone bundle $(C_{\text{tromb}}^B, Q)$, corresponds to a bundle that has a trivial monodromy on the fiber but a non-trivial monodromy in the base manifold with coinvariant classes given by

$$\tilde{C}_{\text{tromb}}^B = C_{\text{tromb}}^F,$$

the T-dual moduli parameters transform as in (2.21) and (2.24), they are defined by the covariant class. They do not depend on any particular element of the orbit. The geometric structure can be interpreted as a compatible set of fiber bundles with characteristic classes defined by the coinvariant class of winding matrices. Under the T-duality transformation the hamiltonian which is invariant on the orbits of $\mathbb{H}_G$ is transformed to a hamiltonian invariant under the orbits of $\mathbb{H}_G^*$.

We notice that in the expression of the hamiltonian of the trombone torus bundle with coinvariant $C_{\text{tromb}}^F$ there is not a $\Theta$ matrix on the expression of the covariant derivative, since $W, \tau, R$ remain fixed under the action of the non-linear transformation $\mathbb{H}_G$. At low energies they correspond to gauged theories in nine dimensions obtain of the
massive deformation of type IIA supergravity in 10D. However there exists a \( \Theta \) matrix in the covariant derivatives which compensates the transformation of the winding matrix on the dual torus bundle (‘type IIB’ side).

### 3.3.1 Two inequivalent duals of the M2-brane trombone torus bundle

The gauging is obtained by means of the nonlinear representation \( \mathbb{H}^G \) with \( \mathcal{M}_G \subset \mathcal{M}_{ji} \) associated to a particular monodromy equivalence class. To compute it we particularize the monodromy matrix \( \mathcal{M}_G Q_j = Q_j \) and we substitute \( \mathbb{H}_{ji}(\tau, q_i, p_i, q_j, p_j) \) and \( Q_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix} \) and respectively \( Q_j \). The linear representation \( \mathcal{M}_{ji} \) contains the three inequivalent conjugation classes (elliptic, parabolic and hyperbolic) however the nonlinear representation \( \mathbb{H}_{ji} \), does not have any particular property which may distinguish the elliptic to the hyperbolic cases. Under T-duality in distinction with the previous case, there are two inequivalent types of dual principal bundles, one with the parabolic monodromy non linearly realized in the base and a second one associated to the nonlinear realization elliptic and hyperbolic monodromy dual in the base manifold. In the case of parabolic monodromy the duality matrix \( T \) commute with \( \mathbb{H}_{ij} \). In the other two cases the T-duality does not commute with \( \mathbb{H}_{ij} \) but since they cannot be distinguish they form a second inequivalent class of trombone torus bundles on the ‘type IIA side’. Nicely at low energies they are in correspondence with the two trombone gauged supergravities of the type IIA sector.

### 4. Conclusions

We find eight independent inequivalent classes of supermembrane torus bundles with monodromy linearly and non linearly realized in \( SL(2, \mathbb{Z}) \). Four of them had been previously found, but those associated to the type IIA side are new. The M2-brane torus bundle is characterized by the monodromy, the coinvariants of the base manifold \( C_B \) and that of the fiber \( C_F \). At low energies each of these torus bundles match exactly with each of the eight type II gauged supergravities in 9D. We also characterize completely the case of the principal M2-brane torus bundles associated to maximal type II supergravity in nine dimensions. In this paper we explicitly show the role of T-duality action over the M2-brane bundles (what we have called global action) responsible for the relative differences at low energies between the four gauged supergravity description with origin in type IIA sector and those of type IIB one. At low energy, the supermembrane with non trivial central charges compactified on a 2-torus times a 9D space-time corresponds to the 9D type IIB gauged supergravity sector. When it is compactified on its dual, and therefore modeled by what we have called a T-dual M2-brane bundle, it corresponds to 9D type IIA one. Their Hamiltonians and mass operators for the nine different cases (eight inequivalent class of bundles with monodromy and one principal) are all invariant under the duality action we have defined in this work irrespective on the monodromy group. However the structure of the bundle, for arbitrary M2-brane torus bundles with monodromy, is not necessarily
preserved. Only for the cases in which the monodromy is parabolic or trivial, the bundle structure is preserved. We have proved that the T-dual of the parabolic coinvariants remains in the same class of coinvariants, moreover, they seem to be the only ones that preserve the coinvariant class. Consequently the type IIB parabolic bundle is mapped through duality into the parabolic type IIA bundle. At low energies this means that the type IIB parabolic supergravity is mapped onto the type IIA parabolic supergravity, which corresponds to the KK reduction of Romans supergravity in 10D to 9D.

The T-duals of the supermembrane with monodromies elliptic and hyperbolic do not preserve the coinvariant class, even though the dual monodromy is conjugated to the original one. T-duality acting on these M2-brane bundles is a nonlinear transformation. The algebra of the monodromy groups elliptic and hyperbolic with the T-duality transformation form a non-abelian algebra $A(1)$ that acts like an scaling and a translation. The dual M2-brane bundle with these two inequivalent classes of monodromies corresponds to a single class of dual M2-brane bundles that we call $A(1)$. At low energies we relate it to the type IIA $A(1)$ gauged supergravity in 9D.

Trombone symmetry of the supermembrane torus bundle is a non linear realization of the $SL(2, \mathbb{Z})$ group. It acts linearly on the charges but non-linearly on the moduli and the bundle description once it is gauged is completely different to the previous cases considered. The M2-brane bundle with gauged trombone monodromy are classified according to the monodromy, but they contain only one class of coinvariants either of the fiber (‘type IIB side’) or of the base (‘type IIA side’). Under T-duality the nonlinear realization of the three inequivalent classes of M2-brane torus bundles (elliptic, hyperbolic or parabolic subgroups) they form a unique M2-brane trombone bundle on the ‘type IIB’ side. Under T-duality it maps into two inequivalent classes of dual M2-brane trombone bundles on the ‘type IIA’ side. There exists two inequivalent dual trombone symmetries, one associated to the gauging of the non linear realization of the parabolic monodromy and a second one associated to the gauging of the non linear realization of the bundles with elliptic and hyperbolic monodromy. These two inequivalent classes of T-bundles can naturally be associated on the 10D ‘type IIA side’ to the KK reduction of the massive Howe, Lambert and West supergravity and to the SS reduction of type IIA maximal supergravity. The non-preservation of the coinvariant equivalence classes under T-duality is then the reason for the different structure between the type IIB and type IIA gauged supergravities in nine dimensions.

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6. Appendix A: Type II gauged supergravities in nine dimensions

There exist eight gauged supergravities in nine dimensions [42]. Four of them correspond to Kaluza Klein (KK) and Scherk-Schwarz (SS) reductions of type IIA supergravity in ten dimensions. The other four ones to the ten dimensional IIB supergravity. Attending to the name of the parent theory we will name them as 'Type IIA origin' or respectively 'Type IIB origin'. In order to be self-contained in this section we summarize the results found in [42] focusing only on the monodromy aspects of the Scherk-Schwarz reductions since these are the relevant ones to understand their supermembrane bundle description. The reduced $d$-fields are not periodic in the internal variables but keep track of the internal dependence in the SS reduction via a monodromy $M_{sugra}(g)$ defined in terms of elements $g$ of the Lie algebra $G$ of the group $G$, where $G$ is a subgroup of the global symmetry group $G_D$ [13]. Upon reduction it becomes local in the internal variables $y$, i.e. $g(y) \equiv e^{M(y)}$, where $M(y)$ is known as the mass matrix and parametrizes the monodromy group. Generically abelian monodromies are expressed as $M_{sugra} = e^{M(y)}$. The gauged supergravity fields have a modified transformation law due to the presence of a non trivial monodromy group. The Ramond-Ramond gauge vectors, inert under the monodromy (not having weight under the global symmetry that becomes gauged), together with the monodromy group modify the field strength expressions into massive ones [16, 42, 58]. The consistency of the reductions require that the parent theory contains a global symmetry group $G$ which become gauged upon reduction [54].

In nine dimensions the $N = 2$ type II supergravity contains several fields: In the bosonic sector there are three scalars $(\phi, \vartheta, \chi)$, three gauge fields $\{A^{(1)}_\mu, A^{(2)}_\mu\}$, two antisymmetric 2-forms $\{B^{(1)}_{\mu\nu}, B^{(2)}_{\mu\nu}\} = \vec{B}$, a three form $C_{\mu\nu\rho}$ and a vielbein $e^a_\mu$. In the fermionic sector, there are a gravitino $\Phi_\mu$ and two dilatinos $\kappa, \tilde{\kappa}$. For a complete description see [58].

Under a global symmetry $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$ the axion-dilaton modulus defined as $\tau = \chi + ie^{-\phi}$ and the one and two-form vectors transform as:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \vec{A} \rightarrow \Lambda \vec{A}, \quad \vec{B} \rightarrow \Lambda \vec{B}, \quad (6.1)$$

plus fermionic transformations. The lasting scalar $\vartheta$ and the three form $C$ remain invariant. This global symmetry generates a local monodromy subgroup $M_{sugra}$ through the SS reduction. Also, there are gauge transformations with parameter $\lambda$ acting on the fields:

$$\vec{A} \rightarrow \vec{A} - d\lambda, \quad \vec{B} \rightarrow \vec{B} - \vec{A}d\lambda. \quad (6.2)$$

When one of the global symmetries $G$ becomes gauged, the gauge transformations become modified as follows:

$$\vec{A} \rightarrow \vec{A} - d\lambda, \quad \vec{B} \rightarrow M_{sugra}(\vec{B} - \vec{A}d\lambda), \quad (6.3)$$
where \( M_{\text{sugra}} = e^{\mathbf{M} \lambda} \) is the monodromy of the SS reduction, and \( \mathbf{M} \) represents the mass matrix of the corresponding deformation. These deformations modify the fields strengths at the same time that a covariant derivative \( D \) associated to the particular gauging is induced. At quantum level, charges are quantized and consequently the symmetries must break to their invariant arithmetic subgroups, for example \( SL(2,R) \) into \( SL(2,Z) \). The quantum realization of the scaling symmetries \( R^+ \) is much more involved due to the absence of a discrete subgroup. In \([57]\) the authors realize the trombone symmetry in terms of a nonlinear representation of \( SL(2,Z) \) called by the authors active \( SL(2,Z) \).

6.1 'Type IIB origin'

In 10D type IIB supergravity the global symmetry group is \( G_D \equiv GL(2,R) = SL(2,R) \times R^+ \). In 9D according to the symmetry subgroup \( G \subset G_D \) that becomes gauged, there are four inequivalent classes of gauged supergravities in 9D. In the \( SL(2,R) \) sector, there are three inequivalent classes of theories, corresponding to the hyperbolic, elliptic and parabolic \( SL(2,R) \) conjugacy classes \([13]\) which form a triplet. There exists a fourth one whose origin is the gauging of an scaling symmetry \( R^+ \) of the equations of motion called trombone symmetry \([59,60,61]\). The gauging of the trombone symmetry leads to gauged supergravities without Lagrangian description \([60]\), since it is not a symmetry of the action.

The inequivalent \( SL(2,R) \) deformations are classified in conjugacy classes according to their trace as follows:

(a) The parabolic deformations are associated to the gauging of the subgroup \( R \subset SL(2,R) \) with parameter \( u_p \) generated by matrices with \( |TrM_{\text{sugra}_p}| = 2 \),

\[
M_{\text{sugra}_p} = \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix}.
\] (6.4)

(b) The hyperbolic deformations are associated to the gauging of the subgroup \( SO(1,1)^+ \) with parameter \( h \) generated by matrices whose \( |TrM_{\text{sugra}_h}| > 2 \),

\[
M_{\text{sugra}_h} = \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix}.
\] (6.5)

(c) The elliptic deformations are associated to the gauging of the subgroup \( SO(2) \) generated by elements \( M_{\text{sugra}_e} \) of \( SL(2,R) \) with parameter \( \eta \) appearing in matrices with \( |TrM_{\text{sugra}_e}| < 2 \),

\[
M_{\text{sugra}_e} = \begin{pmatrix} \cos \eta & \sin \eta \\ -\sin \eta & \cos \eta \end{pmatrix}.
\] (6.6)

We can group them as,

\[
M_{\text{sugra}_G} = e^{M_i \lambda} \quad \text{with} \quad \mathbf{A} \rightarrow \mathbf{A} - d\lambda, \quad \mathbf{B} \rightarrow M_{\text{sugra}_G}(\mathbf{B} - \mathbf{A}d\lambda).
\] (6.7)

\( M_i, i = 1,2,3 \) are three mass matrices characterizing the \( SL(2,R) \) deformations. Following \([42]\) the \( R^+ \)-symmetry is gauged with parameter \( M_{\text{sugra}_{R^+}} \) such that

\[
M_{\text{sugra}_{R^+}} = e^{-ma_{4\lambda}w_{R^+}} \quad \text{with} \quad \mathbf{A} \rightarrow \mathbf{A} - d\lambda, \quad \mathbf{B} \rightarrow e^{ma_{4\lambda}w_{R^+}}(\mathbf{B} - \mathbf{A}d\lambda),
\] (6.8)
where \( w_{R^+} \) is the weight under \( R^+ \).

As explained in [42] the complete set of deformations \( \{M_i, m_4\} \) for the IIB reductions corresponds to:

\[
M_{\text{sugra}GL(2,R)} = M_{\text{sugra}SL(2,R)}M_{\text{sugra}R^+}.
\]

(6.9)

6.2 'Type IIA origin'

In 10D there is one maximal type IIA supergravity and two massive deformations (Romans and HLW). Nine dimensional type IIA supergravity contains as a global symmetry groups two scaling \( R^+ \) symmetries inherited from the 10D type IIA global symmetries. Closely following the notation of [42] we denote them by \( \alpha, \beta \). While the first one \( \alpha \) is a trombone symmetry, that is, it is only a symmetry of the equations of motion but not of the action, the second one, \( \beta \), is a true symmetry of the Lagrangian.

There are four reductions from the maximal and the two massive (Romans and HLW) IIA supergravity to obtain 9D gauged supergravities. Following [42] we summarize their results: The KK reduction of Romans supergravity corresponds to a parabolic gauged supergravity on the type IIA side. The KK reduction of HLW supergravity in 10D and the SS reduction of the 10D maximal type IIA supergravity lead to two supergravities whose mass parameters are connected through a rotation in \( SL(2,R) \). They correspond -from the 11D perspective- to the alternate reduction in terms of KK, and SS reductions of the 11D supergravity. There is a fourth independent type II gauged supergravity whose origin is the gauging of the 10D scaling symmetry \( \beta \) [58] and it has a nonabelian gauge group \( A(1) \). In [42] they show that this gauge group is enhanced with the one associated to the parabolic subgroup of \( SL(2,R) \) with parameter \( \zeta \). Since these two scaling symmetries do not commute they form the two–dimensional non-Abelian Lie group \( A(1) \), consisting of collinear transformations (scalings and translations) in one dimension with a non-semisimple algebra given by

\[
[T_\zeta, T_\beta] = T_\zeta.
\]

(6.10)

7. Appendix B: Coinvariants

Let us define the map associated to the coinvariant classes on the KK sector to the coinvariant classes on the winding sector. We take one element \( Q \) in a coinvariant class of the KK sector and map it to an element \( W \) of a coinvariant class of the winding sector. There exists an element of \( SL(2,Z) \) with equal diagonal terms \( T \) which maps \( Q \) to \( W \):

\[
W = TQ.
\]

(7.1)

In the case, as we are considering here, where the abelian group \( G \) has only one generator \( J \):

\[
g \in M_G, \quad g = J^a,
\]

(7.2)

\(^2\text{HLW (Howe-Lambert-West) 10D type IIA supergravities. It is a SS reduction of the 11D supergravity promoting to a gauge symmetry the global scaling symmetry of the parent theory.}\)
where \( a \) is an integer, then, 
\[
g - I = J^a - I.
\]
We can distinguish three different cases:

\[
\begin{align*}
\text{if} & \quad a = 0 & g - I &= 0, \\
\text{if} & \quad a > 0 & J^a - I &= (J - I)(J^{a-1} + J^{a-2} + \cdots + I), \\
\text{if} & \quad a < 0 & J^a - I &= (I - J^{-a})J^a.
\end{align*}
\]

Hence if \( a = 0 \) we have one element \( Q \).

If \( a > 0 \) we have
\[
C_F = \{Q + (J - 1)\hat{Q}\}.
\]  
(7.4)

If \( a = 1 \) we get \( \hat{Q} = Q \) and if \( a \geq 2, \hat{Q} \) belongs to a subset of the whole \( \hat{Q} \) sector. We then conclude that the coinvariant class can be expressed as
\[
C_F = \{Q + (J - 1)\hat{Q}\}. \tag{7.5}
\]

The same result holds for \( a < 0 \). Consequently without loose of generality we can take the elements of the coinvariant class as (7.5). Notice that the case \( a = 0 \) is also contained in the class since it corresponds to consider \( \hat{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

8. Appendix C: General T-duality transformation

In this appendix we obtain the most general transformation of T-duality for the supermembrane T-bundles. Given a particular supermembrane wrapping the 2-torus target space it has associated a matrix of windings \( W \) with determinant \( \det(W) = n \neq 0 \), such that applying an \( S \in SL(2,\mathbb{Z}) \) it admits a triangular description,
\[
WS = \begin{pmatrix} n & e \\ 0 & 1 \end{pmatrix}.
\]  
(8.1)

Multiplying on the left hand side by a parabolic matrix \( R \) the winding can be expressed in terms of its canonical form \( RWS = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \). On the other hand, in [7] the authors showed that always exists a map T-duality maps
\[
WS = TQ,
\]  
(8.2)

where \( Q \) is the matrix of charge of determinant \( n \) whose first row is \( Q \). \( T \) for sake of brevity denotes the parabolic T-duality transformation \( T_p \). Then, for arbitrary \( W, Q \) charges we want to obtain the more general \( \hat{S}, \hat{T} \) such that they satisfy \( \hat{W}\hat{S} = \hat{T}\hat{Q} \).

Consequently,
\[
W = \hat{T}\hat{T}^{-1}WS\hat{S}^{-1}.
\]  
(8.3)

We want to find the more general \( u = \hat{T}\hat{T}^{-1} \) and \( v = \hat{S}\hat{S}^{-1} \) such that the above relation is satisfied. This condition is verified for
\[
T = ABA^{-1}(T) = \begin{pmatrix} d - ec & dt - bn + e(a - d - ct - ce) \\ -c & -c(t - e) + a \end{pmatrix},
\]  
(8.4)
with \( A = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} d & -bn \\ -c & a \end{pmatrix} \) with \( A, B \in SL(2, \mathbb{Z}) \).

One can distinguish three different cases according to the value of the trace

\[
Trace(T) = d - ct + a ,
\]

as elliptic, parabolic and hyperbolic. For the case when the central charge is restricted to be \( n = 1 \) all of the three subclasses of T-duality matrices are allowed, however this is not the more general case since for the wrapped supermembrane on a torus arbitrary \( n \) is allowed.

- The parabolic case \( T_p = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \) has already being discussed [6], it corresponds to have \( a = 1, d = 1 \) with \( c = 0 \) and it is satisfied for any configuration of charges, windings and central charge, so in the following we will analyze the two other possibilities.

- The elliptic case corresponds to have \( c = \pm 1 \) and \( bn = \pm 1 - e(e - t) \) since it must verify that the two trace entries are equal. For arbitrary central charge \( n \neq 1 \) the above equation is not satisfied apart for solutions associated to particular values of charges and windings.

- For the hyperbolic case we will analyze separately three different cases restricted to assume that \( n \) is prime:
  - In first place let consider \((2e - t)\) and \( n \) relatively prime integers different from zero, then choose \( a, b \) such that the following relation is satisfied

\[
(2e - t) - b \quad \text{then} \quad n = 1, \quad c = 1 - a^2 .
\]

In this case the trace is equal to \( 2(1 - a^2)(e - t) + a \), then if \( a, b \) are solution then \( a + \lambda n, b + \lambda (2e - t) \) is also a solution. We always can find a solution of reversed sign to \((e - t)\) then \( (1 - a^2)(e - t) \) and \( a \) have the same sign and consequently we only need to choose a \( \lambda \) large enough to guarantee that

\[
|a + \lambda n| > 1 ,
\]

and it corresponds to an hyperbolic solution. It always exists a hyperbolic solution for this first case.

- If \((2e - t)\) and \( n \) different from zero are NOT relatively prime. Then, \((2e - t) = mn\) with \( m \neq 0 \). Substituting in the relation \( ad - bc n = 1 \) one obtains \( a^2 + c(am - b)n = 1 \). There is a solution \( b = am \) with \( a = 1 \) and arbitrary \( c \).

The trace is

\[
Tr(T) = 2(e - t)c + a \geq 2 .
\]

One can choose \( c = e - t \) and then the trace corresponds to a hyperbolic matrix. The case \( e = t \) (which it would not corresponds to an hyperbolic case) cannot happen since in the case we consider \( 2e - t = mn \) then \( e = mn \) but \( e < n \) always, as shown in [40]. Consequently it always exists a hyperbolic matrix for this case.
The last case to consider occurs when \( 2e - k = 0 \) then \( a = n + 1, b = \epsilon = \text{sign}(e - k), c = \epsilon(n + 2) \) with trace

\[
Tr(T) = 2[\epsilon(n + 29(e - k)) + a] > 2(n + 1) > 2.
\]  

(8.9)

Consequently it always exists a hyperbolic matrix for this case.

In conclusion there always exists a parabolic transformation \( T \), and an hyperbolic transformation \( T_h \) inside the set of \( T \)-dualities transformations allowed for any value of the central charge \( n \). Each hyperbolic transformation has a different trace for any different set of charges.

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