Non-abelian statistics of half-quantum vortices in $p$-wave superconductors

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Excitation spectrum of a half-quantum vortex in a $p$-wave superconductor contains a zero-energy Majorana fermion. This results in a degeneracy of the ground state of the system of several vortices. From the properties of the solutions to Bogoliubov-de-Gennes equations in the vortex core we derive the non-abelian statistics of vortices identical to that for the Moore-Read (Pfaffian) quantum Hall state.

Certain types of superconductors with triplet pairing allow half-quantum vortices. Such vortices appear if the multi-component order parameter has extra degrees of freedom besides the overall phase, and the vortex involves both a rotation of the phase by $\pi$ and a rotation of the "direction" of the order parameter by $\pi$, so that the order parameter maps to itself on going around the vortex. The magnetic flux through such a vortex is one half of the superconducting flux quantum $\Phi_0$.

Another signature of this unusual flux quantization is a Majorana fermion level at zero energy inside the vortex core. This energy level has a topological nature and from the continuity considerations must be stable to any local perturbations. In terms of the energy levels, the Majorana fermions in vortex cores imply a $2n$-fold degeneracy of the ground state of a system with $2n$ isolated vortices. If we let vortices adiabatically move around each other, this motion may result in a unitary transformation in the space of ground states (non-abelian statistics). We shall see that it is indeed the case.

The non-abelian statistics for half-quantum vortices has been originally derived for the Pfaffian quantum Hall state proposed by Moore and Read. The Pfaffian state is of Laughlin type and may be possibly realized for filling fractions with even denominator. The excitations in the Pfaffian state are half-quantum vortices, and their non-abelian statistics has been derived in the field-theoretical framework.

On the other hand, recently Read and Green suggested that the Pfaffian state belongs to the same topological class as the BCS pairing state and thus the latter must have the same non-abelian statistics. In our paper we verify this directly in the BCS framework as the property of solutions to Bogoliubov-de-Gennes equations. Our derivation provides an alternative (and possibly more transparent) point of view on the non-abelian statistics of half-quantum vortices as well as an additional verification of topological equivalence between Pfaffian and BCS states.

Let us begin our discussion with reviewing the properties of a half-quantum vortex. To be specific, we consider a chiral two-dimensional superconductor with the order parameter of $A$ phase of $^3$He. The order parameter is characterized by the direction $\hat{d}$ of the spin triplet (the projection on which of the spin of the Cooper pair is zero) and by the overall phase $\varphi$. The wave function of the condensate is

$$\Psi_{\pm} = e^{i\varphi} \left[ d_x (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) + id_y (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) + d_z (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right] (k_x \pm i k_y).$$

(1)

The $\pm$ signs denote the two possible chiralities of the condensate. The chirality breaks the time-reversal symmetry and means a non-zero angular momentum of the Cooper pairs. In a physical chiral superconductor there must exist domain walls separating domains of opposite chirality. Experimentally, domain walls may possibly be expelled from the sample by an external field which makes one of the chiralities energetically favorable. In our discussion we do not consider interaction of vortices with domain walls, but assume that the chirality is fixed over the region where the vortex braiding occurs (and takes positive sign in eq. (1)).

For the half-quantum vortex to exist, the vector $\hat{d}$ must be able to rotate (either in a plane or in all three dimensions). An important observation is that the order parameter maps to itself under simultaneous change of sign of the vector $\hat{d}$ and shift of the phase $\varphi$ by $\pi$: $(\varphi, \hat{d}) \mapsto (\varphi + \pi, -\hat{d})$. The half-quantum vortex then combines rotations of the vector $\hat{d}$ by $\pi$ and of the phase $\varphi$ by $\pi$ on going around the vortex core (Fig. 1). This vortex is topologically stable, i.e. it cannot be removed by a continuous (homotopic) deformation of the order parameter.

Without loss of generality, we consider the vector $\hat{d}$ rotating in the $x$-$y$ plane. The direction of the rotation of the phase $\varphi$ may either coincide or be opposite to the chirality of the condensate, which defines either a positive ($\Phi = 1/2$) or a negative ($\Phi = -1/2$) vortex respectively.

There are also two possible directions of rotating the vector $\hat{d}$. If the vector $\hat{d}$ is confined to a plane (i.e. takes values on a one-dimensional circle) by an anisotropy interaction, this gives two possible winding numbers of the vector...
\( \hat{d} \) \((m = \pm 1/2)\). If the vector \( \hat{d} \) is not confined to a plane, but takes values on a two-dimensional sphere, the two directions of winding vector \( \hat{d} \) are topologically equivalent, and there is no additional quantum number characterizing the vortex.

We further restrict our discussion to \( \Phi = 1/2 \) vortices (\( \Phi = -1/2 \) vortices have a slightly different structure of the quasiparticle eigenfunctions, but their spectrum and braiding statistics are the same as for \( \Phi = 1/2 \) vortices). At such a vortex, the condensate wave function \( \Psi \) takes the form:

\[
\Psi(r, \theta) = \Delta(r) \left[ e^{i\theta} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right] (k_x + ik_y). \tag{2}
\]

Here \( r \) and \( \theta \) are the polar coordinates in the vortex, the windings of the phase \( \phi \) and of the vector \( \hat{d} \) have been taken into account.

The Bogoliubov-de-Gennes Hamiltonian in this case decouples into the two Hamiltonians for spin-up and spin-down electrons. The spin-down sector contains no vortex and no subgap states. The Hamiltonian of the spin-up sector is

\[
H = \int d^2r \left[ \Psi_\uparrow \left( \frac{p^2}{2m} - \varepsilon_F \right) \Psi_\uparrow + e^{i\theta} \Delta(r) \Psi_\uparrow^\dagger (\nabla_x + i\nabla_y) \Psi_\uparrow^\dagger + h.c. \right]. \tag{3}
\]

Thus it may be considered as a single-quantum vortex in a superconductor of spinless fermions. The quasiparticles do not have a definite spin projection, but are mixtures of spin-up electrons and spin-down holes:

\[
\gamma^\dagger = u \Psi_\uparrow^\dagger + v \Psi_\uparrow. \tag{4}
\]

Bogoliubov-de-Gennes equations for \((u, v)\) are obtained as \([H, \gamma^\dagger] = E \gamma^\dagger\). They are identical to those for a single-quantum vortex (with the vector \( \hat{d} \) constant in space) and were solved by Kopnin and Salomaa in the context of superfluid \(^3\)He vortices \([3]\). The low-energy spectrum is

\[
E_n = n\omega_0, \tag{5}
\]

where \( \omega_0 \propto \Delta^2/\varepsilon_F \) is the level spacing. The quantum number \( n \) takes integer values (which distinguishes \( p \)-wave vortex states from Caroli-de-Gennes-Matricon states in \( s \)-wave vortices \([10]\) and has the meaning of the angular momentum of the quasiparticle.

In contrast to the single-quantum vortex considered by Kopnin and Salomaa, in the half-quantum vortex there is an additional relation between positive- and negative-energy eigenstates, namely \( \gamma^\dagger(E) = \gamma(-E) \). In other words, the solutions with positive and negative energies are the creation and annihilation operators for the same fermionic level. Therefore the number of degrees of freedom in a half-quantum vortex is one half of that in a single-quantum vortex. The zero-energy level becomes a Majorana fermion:

\[
\gamma^\dagger(E = 0) = \gamma(E = 0). \tag{6}
\]

It is worth mentioning that this Majorana fermion is stable with respect to any local perturbation including external potential, electromagnetic vector potential, local deformations of the order parameter, spin-orbit interaction.
and Zeeman splitting (in a single-quantum vortex, only the first three of those perturbations preserve the zero-energy level \([11]\)). We can easily prove it with continuity considerations. Indeed, suppose that we gradually increase perturbation to the vortex Hamiltonian (which includes both the spin-up and spin-down sectors). The levels will shift and mix, but they must do it continuously, and therefore the number of levels is preserved. Since it is half-integer without perturbation, it must remain half-integer for the perturbed Hamiltonian, i.e. the Majorana fermion survives the perturbation. This argument is valid as long as the perturbation is sufficiently small so that the low-lying states remain localized in the vortex.

Before we turn to discussing the non-abelian statistics of vortices, let us see how the Majorana fermion \(\gamma(E = 0)\) transforms under \(U(1)\) gauge transformations. If the overall phase of the superconducting gap shifts by \(\phi\), it is equivalent to rotating electronic creation and annihilation operators by \(\phi/2\): \(\Psi_\alpha \mapsto e^{i\phi/2}\Psi_\alpha\), \(\Psi_\alpha^\dagger \mapsto e^{-i\phi/2}\Psi_\alpha^\dagger\). The solution \((u, v)\) transforms accordingly: \((u, v) \mapsto (ue^{i\phi/2}, ve^{-i\phi/2})\). The important consequence of this transformation rule is that under change of the phase of the order parameter by \(2\pi\) the Majorana fermion in the vortex changes sign: \(\gamma \mapsto -\gamma\). This is an obvious consequence of the fact that the quasiparticle is a linear combination of fermionic creation and annihilation operators carrying charge \(\pm 1\).

Now consider a system of \(2n\) vortices, far from each other (at distances much larger than \(\xi_0 \sim v_F/\Delta\)). To each vortex there corresponds one Majorana fermion (further we shall denote them by \(c_i, i = 1, \ldots, 2n\)) commuting with the Hamiltonian. They can be combined into \(n\) complex fermionic operators and therefore give rise to the degeneracy of the ground state equal to \(2^n\) (each fermionic level may be either filled or empty). If the vortices move adiabatically slowly so that we can neglect transitions between subgap levels, the only possible effect of such vortex motion is a unitary evolution in the space of ground states.

Let us fix the initial positions of vortices. Consider now a permutation (braiding) of vortices which returns vortices to their original positions (possibly in a different order). Such braid operations form a braid group \(B_{2n}\) (multiplication in this group corresponds to the sequential application of the two braid operations) \([12]\). This group may be described formally in the following way.

Suppose we order all particles along a fixed non-self-intersecting path. Then braid operations are generated by elementary interchanges of two neighboring particles. Denote such an elementary operation interchanging particles \(i\) and \(i + 1\) by \(T_i\) \((i = 1, \ldots, 2n - 1)\). Then the group \(B_{2n}\) is generated by operators \(T_i\) modulo the following relations (see Fig. 2):

\[
T_i T_j = T_j T_i, \quad |i - j| > 1, \\
T_i T_j T_i = T_j T_i T_j, \quad |i - j| = 1. 
\]  

(7)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Defining relation for the braid group: \(T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}\).}
\end{figure}

The braiding statistics is defined by the unitary operators in the space of ground states representing the braid operations from \(B_{2n}\). Here an important reservation has to be made. When a single vortex moves along a closed loop, the multi-particle state acquires a phase proportional to the area inside the loop (every electron inside the loop effectively moves around the vortex). We shall disregard this effect and, as a consequence, lose information about the overall phase of the wave function. In other words, we shall speak about only a projective representation of the braid group \(B_{2n}\). However, since the representation is multi-dimensional, the resulting projective representation is still nontrivial and transforms different states into each other — which implies the non-abelian statistics of vortices.

Since the Majorana fermions \(c_i\) change sign under a shift of the superconducting phase by \(2\pi\), we introduce cuts connecting vortices to the left boundary of the system (Fig. 3). We take the superconducting phase single-valued away from the cuts and jumping by \(2\pi\) across the cuts. From examining Fig. 3 one easily obtains that the transformation exchanging the two vortices \(i\) and \(i + 1\) (with no vortices between them) changes the phase of the order parameter at one of the vortices by \(2\pi\), which results in the following transformation rule:
This defines the action of $T_i$ on Majorana fermions. One easily checks that this action obeys the commutation relations of the braid group.

Now the action of operators $T_i$ may be extended from operators to the Hilbert space. Since the whole Hilbert space can be constructed from the vacuum state by fermionic creation operators, and the mapping of the vacuum state by $T_i$ may be determined uniquely up to a phase factor, the action of $T_i$ on operators uniquely defines a projective representation of $B_{2n}$ in the space of ground states.

The explicit formulas for this representation may be written in terms of fermionic operators. Namely, we need to construct operators $\tau(T_i)$ obeying $\tau(T_i)c_j|\tau(T_i)|^{-1} = T_i(c_j)$, where $T_i(c_j)$ is defined by (8). If we normalize the Majorana fermions by

$$\{c_i, c_j\} = 2\delta_{ij},$$

then the expression for $\tau(T_i)$ is

$$\tau(T_i) = \exp\left(\frac{\pi}{4}c_{i+1}c_i\right) = \frac{1}{\sqrt{2}}(1 + c_{i+1}c_i)$$

(up to a phase factor).

This formula presents the main result of our calculation. On inspection, this representation coincides with that described by Nayak and Wilczek for the statistics of the Pfaffian state (our Majorana fermions correspond to the Majorana fermions in the space of ground states. The two simplest examples of the representation are the cases of two and four vortices. These examples were previously discussed to some extent in the Pfaffian framework in refs. [5,6], and we review them here for illustration purposes.

In the case of two vortices, the two Majorana fermions may be combined into a single complex fermion as $\Psi = (c_1 + ic_2)/2, \Psi^\dagger = (c_1 - ic_2)/2$. The ground state is doubly degenerate, and the only generator of the braid group $T$ is represented by

$$\tau(T) = \exp\left(\frac{\pi}{4}c_2c_1\right) = \exp\left[i\frac{\pi}{4}(2\Psi^\dagger\Psi - 1)\right] = \exp\left(i\frac{\pi}{4}\sigma_z\right),$$

where $\sigma_z$ is a Pauli matrix in the basis $|0\rangle, \Psi^\dagger |0\rangle$).

In the case of four vortices, the four Majorana fermions combine into two complex fermions $\Psi_1$ and $\Psi_2$ by $\Psi_1 = (c_1 + ic_2)/2, \Psi_2 = (c_3 + ic_4)/2$ (and similarly for $\Psi_1^\dagger$ and $\Psi_2^\dagger$). The ground state has degeneracy four, and the three generators $T_1, T_2,$ and $T_3$ of the braid group are represented by

$$\tau(T_1) = \exp\left(i\frac{\pi}{4}\sigma_z(1)\right) = \begin{pmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{pmatrix},$$

$$\tau(T_2) = \exp\left(i\frac{\pi}{4}\sigma_z(2)\right) = \begin{pmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix},$$

$$\tau(T_3) = \exp\left(i\frac{\pi}{4}c_3c_2\right) = \frac{1}{\sqrt{2}}(1 + c_3c_2) = \frac{1}{\sqrt{2}}\left[1 + i(\Psi_2^\dagger + \Psi_2)(\Psi_1^\dagger - \Psi_1)\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix},$$

FIG. 3. Elementary braid interchange of two vortices.
where the matrices are written in the basis \((|0\rangle, \Psi^\dagger_1 |0\rangle, \Psi^\dagger_2 |0\rangle, \Psi^\dagger_1 \Psi^\dagger_2 |0\rangle)\).

There are two important properties of the representation \((10)\). The first one is that \(\tau(T)\) are even in fermionic operators and therefore preserve the parity of the number of fermions (physically, this simply means that the superconducting Hamiltonian creates and destroys electrons only in pairs). Therefore the representation may be restricted to odd or even sector of the space of ground states, each of them containing \(2^{n-1}\) states (this degeneracy was also found for the Pfaffian state in refs. \([5\, 6]\)). Still, in each of these subspaces the representation operators are non-trivial and non-commuting.

The second property of the representation \((10)\) is that \(T^4_i\) is represented by a scalar matrix (projectively equivalent to the unity matrix, since we disregard the overall phase). That is, an elementary interchange of two vortices repeated four times produces an identity operator (up to an overall phase).

Quite remarkably, our derivation of the non-abelian statistics only relies on the two facts: first, the flux quantization (half-quantum for spin-1/2 electrons or, equivalently, single-quantum for spinless fermions) and, second, that the Majorana fermions carry odd charge with respect to the vortex gauge field, i.e. they transform as \(c_i \mapsto -c_i\) when the phase of the order parameter changes by \(2\pi\). But these are quantization properties that depend only on the presence of the Majorana fermion in the vortex spectrum, but not on the exact form of the Hamiltonian. Therefore, if we introduce disorder or other local perturbation in the BCS Hamiltonian (such as electromagnetic vector potential, spin-orbit scattering or local deformation of the order parameter), then not only the Majorana fermions survive, but also the braiding statistics \((10)\) remains unchanged (provided the Majorana fermions stay localized in vortices). Thus we may speak of the topological stability of the non-abelian statistics \((10)\).

Finally, we mention that the operators \(\tau(T_i)\) have also been discussed in the context of quantum computation as part of a universal set of operators \([13]\). Also, non-abelian anyons provide a topologically stable realization of unitary operators for quantum computing \([13]\). Thus, should \(p\)-wave superconductors with sufficiently large \(T_c\) (or, equivalently, large \(\omega_0\)) be discovered, they may provide a promising hardware solution for quantum computation.

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