Diagonalization of log-integrable operator $\mathbb{R}$-cocycles

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Abstract. The subject of this paper prepared as the report for NOMA conference (June 2017), is the asymptotic behaviour at large times of scalar (additive) and operator (multiplicative) cocycles in the case of $\mathbb{R}$ – action with a finite invariant measure.

Recall that a scalar additive cocycle associated with 1-parameter group $\{T^s\}_{s \in \mathbb{R}}$ of preserving measure $\mu$ transformations $T^s$ of the probability space $(X, \mu)$ is a measurable mapping $\alpha : \mathbb{R} \times X \to \mathbb{C}$ satisfying the functional equation

$$\alpha(s_1 + s_2, x) = \alpha(s_1, x) + \alpha(s_2, T^{s_1}x).$$

This term came from homology theory of $G$ – modules. In fact, the object defined above, is a 1 – cocycle.

Here is the well known example of a scalar additive cocycle:

$$\alpha(t, x) = \int_0^t h(T^\tau x)d\tau,$$

where $h \in L^1(X, \mu)$. It is clear that in this case $\alpha(t, \cdot) \in L^1(X, \mu)$ for $\mu$ and a. a. $t$.

Our first aim is to discuss the asymptotic behaviour of the general additive scalar 1 – cocycle taking integrable values as a vector-function of $t$, $\alpha(t, \cdot) \in L^1(X, \mu)$. Classic source of this formulation of the problem is the Birkhoff Ergodic Theorem. For averages $t^{-1} \int_0^t h(T^\tau x)d\tau$, $h \in L^1$ this Theorem maintains the existence of their limits as $t \to \infty$.

Not every scalar additive cocycle $\alpha : \mathbb{R} \to L^1(X, \mu)$ is of the form (2) even if $\alpha$ is absolutely continuous on $t$. What type of asymptotic behaviour as $t \to \infty$ can have cocycle $\alpha : \mathbb{R} \to L^1$, $\alpha \to \alpha(t, \cdot)$, in the general case of not necessarily absolute continuous dependence on $t$?

This question is of interest from different points of view and, primarily, in relation with the spectral theory of $\mathbb{R}$ – actions by weighted shift operators and the metric theory of Liapounov exponents for flows with an invariant measure. For example, V. Oseledec in his well known paper "Multiplicative ergodic theorem. Characteristic Liapounov exponents of dynamical systems" writes: It is unknown to me, whether there exists a measurable additive cocycle $\alpha$ with integrable values at every $t$ such that the statement as in the Birkhoff Theorem [concerning convergence of averages $t^{-1}\alpha(t, x)$] fails.

This is why we begin, first of all, with construction of 1 – cocycle $\alpha$ taking values in $L^1(X, \mu)$ considering it as a vector-function of $t$ such that the conclusion of Birkhoff Ergodic Theorem...
fails to be true. It will be done for an arbitrary measure preserving flow acting in the standard probability measure space. To this end I recall the notion of a special flow $T = \{T^s\}_{s \in \mathbb{R}}$ built by a measurable invertible $\nu$–preserving mapping $S$ of the base $(Y, \nu)$ and a $\nu$–measurable function $f : Y \rightarrow \mathbb{R}_+$, $\int f \, d\nu = 1$. The special flow $T$ acts in the space $X$ of pairs $x = (y, \tau), \tau \in [0, f(y))$ endowed with the measure $\mu$ possessing the differential $d\mu = du \cdot dt$. Here $T$ – action is locally vertical with instant switching $(y, f(y)) \rightarrow (Sy, 0)$, i.e. the point $x$ moves vertically up with velocity 1 till reaching graph of $f$ (ceiling function) and so on bearing in mind identification $(y, f(y)) \sim (Sy, 0)$ made. The inverse motion is vertical down with velocity – 1 till reaching base $Y$ with instant switching $(y, 0) \rightarrow (s^{-1}y, f(s^{-1}y))$. Formally $T^s(y, \tau) = (S^ky, \tau + s - \sum_{0}^{k} f(S^iy))$ if $s + \tau \in [\sum_{0}^{k-1} f(S^iy), \sum_{0}^{k} f(S^iy)]$.

In the slightly different way but almost similarly the special action can be written for $t < 0$.

In the construction of the required cocycle we use the following fact: every ergodic flow acting on the standard probability measure space possesses the special representation, moreover, one can assume that its ceiling function $f$ satisfies the condition $a < f < 2a$ for some $a > 0$.

To define $1$–cocycle $\alpha$ for the special flow $\{T^s\}_{s \in \mathbb{R}}$ it is sufficient to specify the values $\alpha(t, y)$ for points $y$ of its base $Y$. Here we identify the base space $Y$ of the special flow with the subset $\{(y, 0), y \in Y\} \subset X, y \leftrightarrow (y, 0)$. Moreover, the restriction of $\alpha$ to the base $Y$ can be restored by its values $\alpha(t, y)$ on the intervals $.[0, f(y))$, $y \in Y$. Bearing it in mind we consider associated with the special flow $\{T^s\}$ cocycle $\alpha$ satisfying

$$\alpha(t, y) = g(f(y) - t) - g(f(y))$$

as $t \in [0, f(y))$, where $g$ is an integrable function with an unbounded singularity at $f(y)$, say $g(\xi) = \xi^{-1/2}$. Besides, we set $\alpha(f(y), y) = 0$. Then

$$\alpha(s, (y, \tau)) = g(\sum_{0}^{n-1} f(S^k y) - s - \tau) - g(f(S^n y)) - g(f(y) - \tau) - g(f(y))$$

as $\sum_{0}^{n-1} f(S^k y) \leq \tau + s < \sum_{0}^{n} f(S^k y)$.

As it follows from this formula, cocycle $\alpha$ is $\mu$–integrable at every $t$. Setting $\tau = 0$ we get that $\alpha(s, y)$ as a function of $s \in (\sum_{0}^{n} f(S^k y), \sum_{0}^{n+1} f(S^k y))$ monotonically increases to $\infty$ as $s \rightarrow \sum_{0}^{n+1} f(S^k y)$. Hence, $\alpha(s, y)$ is (essentially) unbounded with respect to $s$ on every interval of length $2a$. This is why there are no limit of the averages $t^{-1}\alpha(t, y)$ as $t \rightarrow \infty$ for $\nu$ – a. a. $y \in Y$. From here and cohomological relation one concludes that averages $t^{-1}\alpha(t, x)$ do not converge as $t \rightarrow \infty$ for $\mu$ – a. a. $x \in X$.

Nevertheless, for $\mu$ – a. a. $x \in X$ there is convergence of the averages $t^{-1}\alpha(t, x)$ as $t \rightarrow \infty$ along the set of density 1.

**Definition.** Borel set $\tau \subset \mathbb{R}^+$ has density 1 if $\lim_{t \rightarrow \infty} \lambda(\tau \cap [0, t]) = 1$, where $\lambda$ is the Lebesgue measure (for subsets of $\mathbb{R}$ the Definition is similar).

As an acceptable formulation of ergodic theorem for $1$–cocycles $\alpha(t, x)$ having integrable values as vector-functions on $t$ one can accept the following statement giving satisfactory description of asymptotic ($t$–large) behaviour of the averages $t^{-1}\alpha(t, x)$.

**Theorem 1.** Let $(X, \mu)$ be the standard probability measure space, $T = \{T^s\}_{s \in \mathbb{R}}$ be $\mu$–preserving flow on $(X, \mu)$ and $\alpha$ be additive scalar $1$–cocycle associated with $T$ such that $\alpha(t, \cdot) \in L^1(X, \mu)$ for all $t$.

Then there exists a measurable mapping $x \mapsto \tau(x)$ taking values in the space of Borel subsets $\mathbb{R}$ of density 1 and a measurable $T$–invariant function $\beta : X \rightarrow \mathbb{C}$ such that

$$\lim_{\tau(x) \nu t \rightarrow \infty} t^{-1}\alpha(t, x) = \beta(x) \text{ for } \mu - \text{a.e.}.$$
moreover, \( \int \beta \, d\mu = t^{-1} \int \alpha(t, \cdot) \, d\mu \).

**Proof.** Represent the flow \( T \) as the special flow built by a base automorphism \( S \) of the base space \( (Y, \nu) \) and a ceiling function \( f : Y \to \mathbb{R}_+ \). According to the functional relation for a cocycle one has

\[
t^{-1} \alpha(t, T^s y) = t^{-1} \alpha(t + s, y) - t^{-1} \alpha(s, y).
\]

Hence, \( \mu - \text{a. e.} \) convergence of \( t^{-1} \alpha(t, x) \) as \( t \to \infty \) is equivalent to \( \nu - \text{a. e.} \) convergence \( t^{-1} \alpha(t, y) \) and, moreover, these limits agree.

To express the average \( t^{-1} \alpha(t, y) \) in terms of the ceiling function \( f \) and corresponding restriction \( \alpha(f(\cdot), \cdot) \) we consider how to recover a cocycle by his restriction on the base of the special representation. Restriction on the base \( Y \) uniquely determines cocycle:

\[
\alpha(t, y) + \alpha(s, T^t y) = \alpha(t + s, y).
\]

The function \( \alpha(t, y) \) can not be arbitrary. For correctness of the recovery procedure the following relation should be fulfilled:

\[
\alpha(f(y), y) + \alpha(t, Sy) = \alpha(f(y) + \tau, y). \tag{3}
\]

By values of the cocycle on the set \( \{y \in Y, 0 \leq t < f(y)\} \) one can restore the cocycle in the following way. Relation (3) gives the values \( \alpha(t, y) \) for \( t \in [f(y), f(y) + f(Sy)) \), namely,

\[
\alpha(t, y) = \alpha(f(y), y) + \alpha(t - f(y), Sy).
\]

Inductively applying (3) one gets

\[
\alpha(t, y) = \alpha(f(y), y) + \cdots + \alpha(f(S^k y), S^k y) + \alpha(t - f(y) - \cdots - f(S^k y), S^{k+1} y)
\]

if \( t \in [f(y) + \cdots + f(S^k y), f(y) + \cdots + f(S^{k+1} y)) \). Thus, \( \alpha(t, y) \) can be expressed by \( \alpha(\tau, S^{k+1} y) \) for \( 0 \leq \tau < f(S^{k+1} y) \) and \( S \) - shifts \( \alpha(f(S^k y), S^k y) \) of the function \( \alpha(f(\cdot), \cdot) := g(\cdot) \). Varying the base of the special representation, the time scale and applying Fubini Theorem we can assume that \( g \in L^1(Y, \nu) \). It is also useful to apply here the stronger version of the special Representation Theorem due to Rudolph.

By the Ergodic Theorem (applied to \( S \)) one gets

\[
k^{-1} \sum_{0}^{k} g(S^{-i} y) \to \int g \, d\nu, \quad k^{-1} \sum_{0}^{k} f(S^i y) \to 1
\]

as \( k \to \infty \). Hence,

\[
2(f(y) + \cdots + f(S^k y), y)/f(y) + \cdots + f(S^k y) \to \int g \, d\nu
\]

for \( \nu - \text{a. a.} \) \( y \)'s. In other words, if \( t \to \infty \) taking values \( \sum_{0}^{k} f(S^i y) \), then

\[
t^{-1} \alpha(t, y) \to \int g \, d\nu.
\]

We show now that for \( \nu - \text{a. a.} \) \( y \in Y \) there exists subset \( M_y \subset \mathbb{R}_+ \) of density 1 such that

\[
t^{-1} \alpha(t, y) \to \int g \, d\nu
\]
as $M_y \ni t \to \infty$.

If $f(y) + \cdots + f(S^k y) \leq t < f(y) + \cdots + f(S^{k+1} y)$, where $\tau = t - \sum_{0}^{k} f(S^i y)$, and

$$\alpha(t, y) = \frac{g(y) + \cdots + g(S^k y) + \alpha(\tau, S^{k+1} y)}{f(y) + \cdots + f(S^k y) + \tau},$$

It implies that the equivalence

$$t^{-1} \alpha(t, y) \sim \int g d\nu + k^{-1} \alpha(t - \sum_{0}^{k} f(S^i y))$$

(4)

holds $\nu - a. a.$ as $t \to \infty$.

Now set

$$M_y = \bigcup_{k=0}^{\infty} \{ t : |\alpha(t - \sum_{0}^{k} f(S^i y), S^{k+1} y)| < k^{1/2} \cdot \| \alpha(\cdot, S^{k+1} y) \|_{L^1(0, f(S^{k+1} y))}^{1/2} \}.$$  

By Chebyshev Inequality

$$\text{mes}\{ \tau : \| \alpha(\tau, S^k y) \| \geq k^{1/2} \cdot \| \alpha(\cdot, S^k y) \|_{L^1(0, f(S^k y))}^{1/2} \} \leq k^{-1/2} \cdot \| \alpha(\cdot, S^k y) \|_{L^1(0, f(S^k y))}^{1/2}.$$  

Since the norm $\| \alpha(\cdot, y) \|_{L^1(0, f(S^k y))}$ is $\nu -$ summable (as a function of $y$) then by Borel – Cantelli Lemma or Birkhoff Ergodic Theorem the formula

$$k^{-1} \| \alpha(\cdot, S^k y) \|_{L^1(0, f(S^k y))} \to 0$$

(5)

holds $\nu - a. e.$ as $k \to \infty$. Hence, the limit equality

$$\lim_{k \to \infty} \text{mes}\{ (\mathbb{R}^+ \setminus M_y) \cap \left[ \sum_{0}^{k-1} f(S^i y), \sum_{0}^{k+1} f(S^i y) \right] / f(S^{k+1} y) = 0$$

is valid $\nu - a. e.$ This means that $M_y$ has density 1 for a. a. $y$.

If $t \in M_y \cap \left[ \sum_{0}^{k-1} f(S^i y), \sum_{0}^{k} f(S^i y) \right]$ then

$$|k^{-1} \cdot \alpha(t - \sum_{0}^{k-1} f(S^i y), S^k y)| \leq k^{-1/2} \cdot \| \alpha(\cdot, S^k y) \|_{L^1(0, f(S^k y))}^{1/2}.$$  

Using formula (5) one has $k^{-1} \alpha(t - \sum_{0}^{k} f(S^i y), S^{k+1} y) \to 0$ as $M_y \ni t \to \infty$. Here $k$ depends on $t$ according to inclusion

$$t \in M_y \cap \left[ \sum_{0}^{k} f(S^i y), \sum_{0}^{k+1} f(S^i y) \right].$$

Combining this property and equivalence (4) we get the conclusion of the Theorem 1. Moreover, if $T$ is ergodic then

$$\lim_{M_y \ni t \to \infty} t^{-1} \alpha(t, x) = \int_{Y} \alpha(f(y), y) d\nu.$$
Corollary. For any ergodic flow $T = \{T^s\}_{s \in \mathbb{R}}$ its special representation with the base space $(Y, \nu)$, a ceiling function $f : Y \to \mathbb{R}_+$, $\int f d\nu = 1$ and a scalar additive cocycle $\alpha$ associated with $T$, the quantity $\int \alpha(f(y), y) d\nu$ does not depend on the choice of its special representation.

Remark. 1. Used in the proof of Theorem 1 expression of cocycles associated with flows given in its special representation enables one to obtain necessary and sufficient conditions for convergence of averages $t^{-1}\alpha(t, x)$ almost everywhere as $t \to \infty$.

In the noninvariant form (depending on the choice of a special representation of the flow) the condition is as follows

$$n^{-1} \cdot \sup_{0 \leq t \leq f(S^n y)} |\alpha(\tau, S^n y)| \to 0 \text{ a. e.}$$

In the invariant form the condition requires that

$$n^{-1} \cdot \sup_{0 \leq \tau < 1} |\alpha(\tau, T^n x)| \to 0 \text{ a. e.}$$

2. Divergence of averages $t^{-1}\alpha(t, x)$ as $t \to \infty$ for scalar additive cocycles with values in $L^1$ may even be in the case of cocycles continuously depending on $t$. Smoothing of the construction given before Theorem 1 can be done with the help of the theorem that asserts the existence of topology making weakly mixing flow to possess the property of unique ergodicity.

3. There exist additive scalar cocycles that are not absolutely continuous in $t$ such that their temporal averages converge as $t \to \infty$. M. Lipatov gave an example of a cocycle such that its averages converge as $t \to \infty$ but the Oseledec’s condition

$$\sup_{|t| \leq 1} \ln |\alpha(t, x)| \in L^1,$$

that is sufficient for such a convergence, fail to be true.

More generally (in comparison with Theorem 1) the following sub-additive ergodic proposition is valid.

Theorem 2. Let $F = \{T^s\}_{s \in \mathbb{R}}$ be a flow acting on the standard probability space $(X, \mu)$, a measurable real-valued function $\alpha$ on $\mathbb{R}_+ \times X$ be such that $\alpha(t, s) \in L^1(X, \mu)$ and $\alpha(t + s, x) \leq \alpha(t, T^sx)$ for every $t, s \in \mathbb{R}_+, x \in X$.

Then there exists a measurable function $\tau$ on $X$ taking values in the space of Borel subsets in $\mathbb{R}_+$ having density 1 and a measurable $F$ – invariant function $\beta : X \to \mathbb{R} \cup \{-\infty\}$ such that $\beta^+ \in L^1(X, \mu)$,

$$\lim_{\tau(x) \geq t \to \infty} t^{-1}\alpha(t, x) = \beta(x) \mu - \text{a.e.},$$

$$\lim_{t \to \infty} t^{-1} \int \alpha(t, \cdot) d\mu = \inf_{t} t^{-1} \int \alpha(t, \cdot) d\mu = \int \beta d\mu.$$

This proposition has two precursors in the discrete time case: well known Fekete Lemma on subadditive sequences and its version for random sequences due to Kingman. Theorem 2 enables one to investigate asymptotic behaviour of operator multiplicative $\mathbb{R}$-cocycles with log-integrable values and to prove their block diagonalization.

To define generalized Liapounov exponents in Theorem 3 below we shall need the notion of $F_A$ – invariant sub-bundle in $X \times V$, where $F = \{T^s\}_{s \in \mathbb{R}}$ is a flow on $(X, \mu)$ and $A$ is an operator – valued cocycle associated with $F$ and taking values in bounded operators on normed space $V$.

For an operator – valued cocycle $A$ define linear extension $F_A = \{T^s_A\}_{s \in \mathbb{R}}$ of $F$ by formulas

$$T^s_A : X \times V \to (T^s x, A(s, x)v).$$
Sub-bundle $L$ of $X \times V$, i.e., the family $\{L(x)\}$ of a subspace is invariant with respect to weighted shift operators $v(x) \to A(s, x)v(T^s x)$ in the space of $V$-valued functions on $X$. For simplicity we confine ourselves to the matrix case.

**Theorem 3.** Let multiplicative cocycle $A : \mathbb{R} \times X \to GL(n, R)$, associated with a $\mu$-preserving flow $F = \{T^s\}_{s \in \mathbb{R}}$ on a standard probability space $(X, \mu)$ satisfy the condition $\log^+ \|A(t, \cdot)\| \in L^1(X, \mu)$ for every $t \in \mathbb{R}$.

Then there exists a measurable mapping $\tau : X \to \{\text{Borel subsets in } \mathbb{R}\}$ of density 1 such that the limits

$$\lim_{\tau(x) \ni t \to \pm \infty} (A^*(t, x)A(t, x))^{1/2|t|} =: \Lambda^\pm(x)$$

exist $\mu$-a.e., and there are $F_A$-invariant linear sub-bundles $L_i = \{L_i(x)\}$, $i = 1, \ldots, k(x)$, $X \times \mathbb{R}^n$ for which uniformly on $L_i(x) \setminus \{0\}$ the limits

$$\lim_{\tau \ni t \to \pm \infty} |t|^{-1}\log \|A(t, x)v\| = \pm \chi_i(x), \ v \in L_i(x)$$

exist $\mu$-a.e., where $\{(\exp^{\pm \chi_i(x)}, \dim L_i(x))\}_{i=1,\ldots,k(x)}$ are the spectra of $\Lambda^\pm(x)$ counted with multiplicities.

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