SOME PROPERTIES OF UNIVALENT LOG-HARMONIC MAPPINGS

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Abstract. We determine the representation theorem, distortion theorem, coefficients estimate and Bohr’s radius for log-harmonic starlike mappings of order $\alpha$, which are generalization of some earlier results. In addition, the inner mapping radius of log-harmonic mappings is also established by constructing a family of 1-slit log-harmonic mappings. Finally, we introduce pre-Schwarzian, Schwarzian derivatives and Bloch’s norm for non-vanishing log-harmonic mappings, several properties related to these are also obtained.

1. Introduction

Let $B$ denote the set of all bounded analytic functions defined on the unit disk $D = \{z : |z| < 1\}$ satisfying $|\omega(z)| < 1$ for all $z \in D$. Then the differential operators
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
show that the Laplacian is given by
\[
\Delta = 4 \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]
Thus a $C^2$-function $f$ defined on the unit disk $D$ is said to be harmonic in $D$ if $\Delta f = 0$ therein. Analogously, a log-harmonic mapping defined on $D$ is a solution of the nonlinear elliptic partial differential equation
\[
\frac{f_z}{f} = \mu f_{\bar{z}},
\]
for some $\mu \in B$, where $\mu$ is called the second complex-dilatation of $f$. It follows that the Jacobian
\[
J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2(1 - |\mu|^2)
\]
is positive and all non-constant log-harmonic mappings are therefore sense-preserving and open in $D$. If $f$ does not vanish in $D$, then $f$ can be expressed as
\[
f(z) = h(z)g(z),
\]
where $h$ and $g$ are analytic in $D$. On the other hand, if $f$ is a non-constant log-harmonic mapping that vanishes only at $z = 0$, then $f$ admits the representation
\[
f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)}, \tag{1.1}
\]

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where $m$ is a non-negative integer, $\Re \beta > -1/2$, $h$ and $g$ are analytic in $\mathbb{D}$ satisfying $h(0) \neq 0$ and $g(0) = 1$ (see [4]). We see that $\beta$ in (1.1) depends only on $\mu(0)$ and can be expressed as

$$\beta = \mu(0) \frac{1 + \mu(0)}{1 - |\mu(0)|^2}.$$ 

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent log-harmonic mapping in $\mathbb{D}$ vanishes at the origin if and only if $m = 1$. In other words, every univalent log-harmonic mapping in $\mathbb{D}$ which vanishes at the origin has the form

$$f(z) = z |z|^{2\beta} h(z) g(z),$$

where $\Re \beta > -1/2$ and $0 \not\in (h g)(\mathbb{D})$. The class of such functions has been widely studied. See for example [4, 5, 7, 8].

In this paper, our emphasis is primarily on sense-preserving univalent log-harmonic mappings in $\mathbb{D}$ with $\mu(0) = 0$. These mappings have the form

$$f(z) = z h(z) g(z),$$

where $h$ and $g$ are analytic in $\mathbb{D}$ such that

$$h(z) = \exp \left( \sum_{n=1}^{\infty} a_n z^n \right) \quad \text{and} \quad g(z) = \exp \left( \sum_{n=1}^{\infty} b_n z^n \right).$$

(1.3)

Here $h(z)$ and $g(z)$ may be called as analytic and co-analytic factors of $f(z)$. Denote by $S_{Lh}$ the class which consists of all such mappings.

It follows from (1.2) that the functions $h, g$ and the dilatation $\mu$ satisfy

$$\mu(z) = \frac{z g'(z) / g(z)}{1 + z h'(z) / h(z)} = \frac{z (\log g)'(z)}{1 + z (\log h)'(z)}.$$ (1.4)

We say that a univalent log-harmonic mapping $f$ of the form (1.2) is log-harmonic starlike mapping of order $\alpha$, denoted by $f \in S_{Lh}^*(\alpha)$, if

$$\frac{\partial}{\partial \theta} \left( \arg f(re^{i\theta}) \right) = \Re \left( \frac{D f(z)}{f(z)} \right) = \Re \left( \frac{z f_z(z) - z \overline{f_z(z)}}{f(z)} \right) > \alpha,$$

for all $z = re^{i\theta} \in \mathbb{D} \setminus \{0\}$ and for some $0 \leq \alpha < 1$. If $\alpha = 0$, then we get the class of log-harmonic starlike mappings, $S_{Lh}^*(0) =: S_{Lh}^*$. If $f$ is analytic in $\mathbb{D}$, then denote by $S^*(\alpha)$ the class of analytic starlike function of order $\alpha$, and $S^*(0) =: S^*$.

The following theorem establishes a link between the classes $S_{Lh}^*(\alpha)$ and $S^*(\alpha)$.

**Theorem A.** ([5, Lemma 2.4] and [1, Theorem 2.1]) Let $f(z) = z h(z) g(z)$ be a log-harmonic mapping on $\mathbb{D}$, $0 \not\in (h g)(\mathbb{D})$. Then $f \in S_{Lh}^*(\alpha)$ if and only if $\varphi \in S^*(\alpha)$, where $\varphi(z) = z h(z) / g(z)$.

In [1], Abdulhadi and Abumuhanna obtained the following representation theorem and distortion theorem for functions in $S_{Lh}^*(\alpha)$.

**Theorem B.** ([1, Theorem 2.2]) $f(z) = z h(z) g(z)$ $\in S_{Lh}^*(\alpha)$ with $\mu(0) = 0$ if and only if there are two probability measures $\delta$ and $\kappa$ such that

$$f(z) = z \exp \left( \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} K(z, \eta, \xi) d\delta(\eta) d\kappa(\xi) \right),$$
where

\[ K(z, \eta, \xi) = (1 - \alpha) \log \left( \frac{1 + \xi z}{1 - \eta z} \right) + T(z, \eta, \xi). \]

Here

\[ T(z, \eta, \xi) = \begin{cases} 
-2(1 - \alpha) \text{Im} \left( \frac{\eta + \xi}{\eta - \xi} \right) \arg \left( \frac{1 - \xi z}{1 - \eta z} \right) - 2\alpha \log |1 - \xi z| & \text{if } |\eta| = |\xi| = 1, \eta \neq \xi, \\
(1 - \alpha) \text{Re} \left( \frac{4\eta z}{1 - \eta z} \right) - 2\alpha \log |1 - \eta z| & \text{if } |\eta| = |\xi| = 1, \eta = \xi.
\end{cases} \]

**Theorem C.** ([1, Theorem 3.1]) Let \( f(z) = zh(z)g(z) \in S_{Lh}^*(\alpha) \) with \( \mu(0) = 0 \). Then for \( z \in \mathbb{D} \) we have

\[
\frac{|z|}{(1 + |z|)^{2\alpha}} \exp \left( (1 - \alpha) \frac{-4|z|}{1 + |z|} \right) \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^{2\alpha}} \exp \left( (1 - \alpha) \frac{4|z|}{1 - |z|} \right).
\]

The equalities occur if and only if \( f(z) \) is one of the functions of the form \( \overline{\eta} f_\alpha(\eta z), |\eta| = 1 \), where \( f_\alpha(z) \) is given by

\[
f_\alpha(z) = \frac{z}{1 - z} \frac{1}{(1 - \overline{\eta} z)^{2\alpha - 1}} \exp \left( (1 - \alpha) \text{Re} \left( \frac{4z}{1 - z} \right) \right). \tag{1.5}
\]

The paper is organized as follows. In Section 2, we present the representation theorem for the analytic and the co-analytic products of \( S_{Lh}^*(\alpha) \) and use them to derive distortion theorems, coefficients estimates and Bohr’s radius. In Section 4, the inner mapping radius of log-harmonic mappings is established by constructing a family of 1-slit log-harmonic mappings and propose a problem of the inner mapping radius for log-harmonic mappings. In Section 5, we introduce pre-Schwarzian, Schwarzian derivatives and log-harmonic Bloch mappings and in Section 6, we continue to discuss the log-harmonic Bloch space \( \mathcal{B}_{Lh} \) of non-vanishing log-harmonic mappings.

### 2. Coefficients estimate

In order to prove our main results, we shall need the following lemmas.

**Lemma 2.1.** ([22, Corollary 3.6]) Let \( p(z) \) be analytic in \( \mathbb{D} \) with \( p(0) = 1 \). Then \( \text{Re} p(z) > 0 \) in \( \mathbb{D} \) if and only if there is a probability measure \( \delta \) on \( \partial \mathbb{D} \) such that

\[ p(z) = \int_{\partial \mathbb{D}} \frac{1 + \eta z}{1 - \eta z} d\delta(\eta), \quad z \in \mathbb{D}. \]

Since each \( p \) has the form

\[ p(z) = \frac{1 + \mu(z)}{1 - \mu(z)} = 1 + \frac{2\mu(z)}{1 - \mu(z)} \]

for some \( \mu \in \mathcal{B} \), we have the following equivalent version of Lemma 2.1.

**Lemma 2.2.** If \( \mu \in \mathcal{B} \) with \( \mu(0) = 0 \), then

\[ \frac{\mu(z)}{1 - \mu(z)} = \int_{\partial \mathbb{D}} \frac{\xi z}{1 - \xi z} d\kappa(\xi), \quad z \in \mathbb{D}, \]

for some probability measure \( \kappa \) on \( \partial \mathbb{D} \).
Theorem 2.3. A log-harmonic mapping \( f(z) = zh(z)g(z) \in S^*_Lh(\alpha) \) if and only if there are two probability measures \( \delta \) and \( \kappa \) on \( \partial \mathbb{D} \) such that

\[
h(z) = \exp \left( \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} K_1(z, \eta, \xi) \, d\delta(\eta) \, d\kappa(\xi) \right),
\]

where

\[
K_1(z, \eta, \xi) = \begin{cases} 
\frac{(1 - 2\alpha)\eta + \xi}{\eta - \xi} \log \left( \frac{1 - \xi z}{1 - \eta z} \right) - \log(1 - \eta z) & \text{if } |\eta| = |\xi| = 1, \eta \neq \xi, \\
\frac{2(1 - \alpha)\eta z}{1 - \eta z} - \log(1 - \eta z) & \text{if } |\eta| = |\xi| = 1, \eta = \xi,
\end{cases}
\]

and

\[
g(z) = \exp \left( \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} K_2(z, \eta, \xi) \, d\delta(\eta) \, d\kappa(\xi) \right),
\]

where

\[
K_2(z, \eta, \xi) = \begin{cases} 
\frac{(1 - 2\alpha)\eta + \xi}{\eta - \xi} \log \left( \frac{1 - \xi z}{1 - \eta z} \right) + (1 - 2\alpha) \log(1 - \eta z) & \text{if } \eta \neq \xi, \\
\frac{2(1 - \alpha)\eta z}{1 - \eta z} + (1 - 2\alpha) \log(1 - \eta z) & \text{if } \eta = \xi.
\end{cases}
\]

Proof. The proof could be extracted from Theorem B after some computation. Because of it is independent interest and use in our investigation, we need explicit representation for the analytic and co-analytic factors \( h(z) \) and \( g(z) \) of \( f(z) \), and thus we include the proof.

According to Theorem A we see that \( f(z) = zh(z)g(z) \in S^*_Lh(\alpha) \) if and only if \( \varphi(z) = zh(z)/g(z) \in \mathcal{S}^*(\alpha) \), i.e.,

\[
\frac{z\varphi'(z)}{\varphi(z)} = (1 - \alpha)p(z) + \alpha,
\]

where \( p \) is analytic in \( \mathbb{D} \) such that \( p(0) = 1 \) and \( \text{Re} \, p(z) > 0 \) in \( \mathbb{D} \). Thus, by Lemma 2.1 it follows that

\[
\frac{z\varphi'(z)}{\varphi(z)} = (1 - \alpha) \int_{\partial \mathbb{D}} \frac{1 + \eta z}{1 - \eta z} \, d\delta(\eta) + \alpha,
\]

and therefore, we have the following well-known representation for \( \varphi(z) \in \mathcal{S}^*(\alpha) \):

\[
\varphi(z) = z \exp \left( -2(1 - \alpha) \int_{\partial \mathbb{D}} \log(1 - \eta z) \, d\delta(\eta) \right).
\]

From (1.4), (2.4) and Lemma 2.2 it follows that

\[
g(z) = \exp \left( \int_{0}^{z} \left( \frac{\mu(s)}{1 - \mu(s)} \cdot \frac{\varphi'(s)}{\varphi(s)} \right) \, ds \right)
\]

so that

\[
g(z) = \exp \left( \int_{0}^{z} \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} \frac{\xi}{1 - \xi s} \left( (1 - \alpha) \frac{1 + \eta s}{1 - \eta s} + \alpha \right) \, d\delta(\eta) \, d\kappa(\xi) \, ds \right)
\]

for some probability measures \( \delta \) and \( \kappa \) on \( \partial \mathbb{D} \).
Moreover, if \( \eta \neq \xi \), we may reduce \( g \) in the form
\[
g(z) = \exp \left( \int_{\partial D} \int_{\partial D} \int_0^z \frac{\xi}{1 - \xi s} \left( (1 - \alpha) \frac{1 + \eta s}{1 - \eta s} + \alpha \right) \, ds \, d\delta(\eta) \, d\kappa(\xi) \right)
\]
\[
= \exp \left( \int_{\partial D} \int_{\partial D} \left( \frac{2(1 - \alpha)\eta + \xi}{\eta - \xi} \log(1 - \xi z) - \frac{2(1 - \alpha)\eta}{\eta - \xi} \log(1 - \eta z) \right) \, d\delta(\eta) \, d\kappa(\xi) \right)
\]
\[
= \exp \left( \int_{\partial D} \int_{\partial D} \left( \frac{1 - \eta s}{1 - \eta z} + (1 - 2\alpha) \log(1 - \eta z) \right) \, d\delta(\eta) \, d\kappa(\xi) \right).
\]
On the other hand, if \( \eta = \xi \), then from (2.7) we see that
\[
g(z) = \exp \left( \int_{\partial D} \int_{\partial D} \int_0^z \frac{\eta}{1 - \eta s} \left( (1 - \alpha) \frac{1 + \eta s}{1 - \eta s} + \alpha \right) \, ds \, d\delta(\eta) \right)
\]
\[
= \exp \left( \int_{\partial D} \int_{\partial D} \left( \frac{2(1 - \alpha)\eta}{1 - \eta z} \log(1 - \eta z) \right) \, d\delta(\eta) \right).
\]
Finally, by writing
\[
h(z) = \frac{\varphi(z)}{z} g(z)
\]
the representation for \( h \) can easily be obtained from the last expression for \( g \) and (2.5). □

**Theorem 2.4.** Let \( f(z) = z h(z) g(z) \in S^*_L(\alpha) \) with \( \mu(0) = 0 \). Then for \( z \in D \),
\[
(1) \quad \frac{1}{1 + |z|} \exp \left( (1 - \alpha) \frac{2|z|}{1 + |z|} \right) \leq |h(z)| \leq \frac{1}{1 - |z|} \exp \left( (1 - \alpha) \frac{2|z|}{1 - |z|} \right);
\]
\[
(2) \quad \frac{1}{(1 + |z|)^{2\alpha - r}} \exp \left( (1 - \alpha) \frac{2|z|}{1 + |z|} \right) \leq |g(z)| \leq \frac{1}{(1 - |z|)^{2\alpha - r}} \exp \left( (1 - \alpha) \frac{2|z|}{1 - |z|} \right).
\]
The equalities occur if and only if \( f(z) \) is one of the functions of the form \( \overline{\Pi} f_\alpha(\eta z), |\eta| = 1 \), where \( f_\alpha(z) \) is given by (1.3).

**Proof.** Let \( \varphi(z) = z h(z)/g(z) \in S^*(\alpha) \) so that
\[
h(z) = \frac{\varphi(z)}{z} g(z) \quad \text{and} \quad f(z) = \varphi(z) |g(z)|^2.
\]
For \( |z| = r < 1 \), by Theorem A we know that
\[
\left| \frac{z \varphi'(z)}{\varphi(z)} \right| \leq (1 - \alpha) \frac{1 + r}{1 - r} + \alpha.
\]
Because \( \mu \in B \) with \( \mu(0) = 0 \), we have
\[
\left| \frac{\mu(z)}{z(1 - \mu(z))} \right| \leq \frac{1}{1 - r} \quad \text{and} \quad |\varphi(z)| \leq \frac{r}{(1 - r)^{2(1 - \alpha)}},
\]
which by (2.4) and (2.6) imply that
\[
|g(z)| \leq \exp \left( \int_0^r \frac{1}{1 - s} \left( (1 - \alpha) \frac{1 + s}{1 - s} + \alpha \right) \, ds \right)
\]
\[
= \exp \left( (1 - \alpha) \frac{2r}{1 - r} - (2\alpha - 1) \log(1 - r) \right)
\]
\[
= \frac{1}{(1 - r)^{2\alpha - 1}} \exp \left( (1 - \alpha) \frac{2r}{1 - r} \right).
\]
The known estimate for $\varphi \in S^*(\alpha)$ and the last inequality give

$$|h(z)| = \left| \frac{\varphi(z)}{z} \right| |g(z)| \leq \frac{1}{(1-r)^{2(1-\alpha)}} \cdot \frac{1}{(1-r)^{2\alpha-1}} \exp \left( (1-\alpha) \frac{2r}{1-r} \right) = \frac{1}{1-r} \exp \left( (1-\alpha) \frac{2r}{1-r} \right).$$

Equality occurs if and only if $\mu(z) = \eta z$ and $\varphi(z) = \frac{\eta z}{1-z^{(1-\alpha)}}$, $|\eta| = 1$, which leads to $f(z) = \eta f_\alpha(\eta z)$, where $f_\alpha(z)$ is given by \([1.3]\).

For the left side estimates of (2), by (2.1), we obtain that

$$\log|h(z)| = \text{Re} \left( \int_{\partial D} \int_{\partial D} K_1(z, \xi, \eta) d\delta(\eta) d\kappa(\xi) \right),$$

where $K_1(z, \xi, \eta) d\delta(\eta)$ is defined by (2.2) and may be rewritten as

$$K_1(z, \xi, \eta) = \begin{cases} 
(1-\alpha) \frac{\eta + \xi}{\eta - \xi} \log \left( \frac{1 - \xi z}{1 - \eta z} \right) - \alpha \log(1 - \xi z) - (1-\alpha) \log(1 - \eta z) & \text{if } \eta \neq \xi, \\
\frac{2(1-\alpha)\eta z}{1 - \eta z} \log(1 - \eta z) & \text{if } \eta = \xi,
\end{cases}$$

and $|\eta| = |\xi| = 1$. Then for $|z| = r$, we have

$$\log|h(z)| = \text{Re} \left( \int_{\partial D} \int_{\partial D} K_1(z, \xi, \eta) d\delta(\eta) d\kappa(\xi) \right) \geq \min_{\delta, \kappa} \left\{ \min_{|z|=r} \text{Re} \left( \int_{\partial D} \int_{\partial D} K_1(z, \xi, \eta) d\delta(\eta) d\kappa(\xi) \right) \right\}$$

$$= \min \left\{ \min_{|z|=r} \inf_{0 < |l| \leq \pi/2} \left[ -(1-\alpha) \text{Im} \left( \frac{1 + e^{2il} \eta z}{1 - e^{2il} \eta z} \right) \right] - \log(1 + r), \right\}$$

$$\left(1-\alpha\right) \frac{-2r}{1 + r} - \log(1 + r),$$

where $e^{2il} = \overline{\eta} \xi$. Now, let

$$\Phi_r(l) = \begin{cases} 
\min_{|z|=r} \left[ -(1-\alpha) \text{Im} \left( \frac{1 + e^{2il} \eta z}{1 - e^{2il} \eta z} \right) \right] - \log(1 + r) & \text{if } 0 < |l| < \pi/2, \\
(1-\alpha) \frac{-2r}{1 + r} - \log(1 + r) & \text{if } l = 0.
\end{cases}$$

In a manner similar in the proof of \([11]\) Theorem 2], we see that the function $\Phi_r(l)$ is continuous and is even in the interval $|l| \leq \pi/2$. Hence

$$\log|h(z)| \geq \inf_{0 \leq |l| \leq \pi/2} \Phi_r(l) = (1-\alpha) \frac{-2r}{1 + r} - \log(1 + r).$$

For the lower bound of $|g(z)|$ in Theorem 2.4[2], a similar discussion applied to (2.3) yields

$$\log |g(z)| \geq \inf_{0 \leq |l| \leq \pi/2} \left( \Phi_r(l) + 2(1-\alpha) \log(1 + r) \right) = (1-\alpha) \frac{-2r}{1 + r} - (2\alpha - 1) \log(1 + r).$$

The proof is complete. □
Corollary 2.5. Let \( f(z) = zh(z) \bar{g}(z) \in S^{*}_{Lh}(\alpha) \). Also, let \( H(z) = zh(z) \) and \( G(z) = zg(z) \). Then

\[
\begin{align*}
(1) \quad \frac{1}{2 \epsilon^{1-\alpha}} &\leq d(0, \partial H(\mathbb{D})) \leq 1; \\
(2) \quad \frac{1}{2 \alpha - 1 + \alpha} &\leq d(0, \partial G(\mathbb{D})) \leq 1; \\
(3) \quad \frac{1}{2 \alpha - 1 + \alpha} &\leq d(0, \partial f(\mathbb{D})) \leq 1.
\end{align*}
\]

The equalities occur if and only if \( f(z) \) is one of the functions of the form \( \eta f_{\alpha}(\eta z), |\eta| = 1 \), where \( f_{\alpha}(z) \) is given by (1.5).

Proof. By Theorem 2.4,

\[
d(0, \partial H(\mathbb{D})) = \lim \inf_{|z| \to 1} |H(z) - H(0)| = \lim \inf_{|z| \to 1} \frac{|H(z) - H(0)|}{|z|} = \lim \inf_{|z| \to 1} |h(z)| \geq \frac{1}{2 \epsilon^{1-\alpha}}.
\]

On the other hand, the minimum modulus principle shows that

\[
d(0, \partial H(\mathbb{D})) = \lim \inf_{|z| \to 1} |h(z)| \leq 1,
\]

since \( |h(0)| = 1 \). The same approach may be applied to \( G(z) \) and \( f(z) \) to find proofs of the remaining inequalities.

Now, we give a sharp upper bound for the coefficients of \( h(z) \) and \( g(z) \).

Theorem 2.6. Let \( f(z) = zh(z) \bar{g}(z) \in S^{*}_{Lh}(\alpha) \). Then

\[
|a_n| \leq 2(1 - \alpha) + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2(1 - \alpha) + \frac{2 \alpha - 1}{n}
\]

for all \( n \geq 1 \). The equalities occur if and only if \( f(z) \) is one of the functions of the form \( \eta f_{\alpha}(\eta z), |\eta| = 1 \), where \( f_{\alpha}(z) \) is given by (1.5).

Proof. From (2.1) and (2.3), we get the following expressions

\[
a_n = \frac{1}{n} \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} \left( \eta^n + \frac{(1 - 2\alpha)\eta + \xi}{\eta - \xi} (\eta^n - \xi^n) \right) d\delta(\eta) d\kappa(\xi)
\]

\[
= \frac{1}{n} \int_{\partial \mathbb{D}} \eta^n + \int_{\partial \mathbb{D}} \left( ((1 - 2\alpha)\eta + \xi) \sum_{k=0}^{n-1} \eta^{n-k-1}\xi^k \right) d\kappa(\xi) d\delta(\eta)
\]

and

\[
b_n = \frac{1}{n} \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} \left( (2\alpha - 1)\eta^n + \frac{(1 - 2\alpha)\eta + \xi}{\eta - \xi} (\eta^n - \xi^n) \right) d\delta(\eta) d\kappa(\xi)
\]

\[
= \frac{1}{n} \int_{\partial \mathbb{D}} \left( (2\alpha - 1)\eta^n + \int_{\partial \mathbb{D}} \left( (1 - 2\alpha)\eta + \xi) \sum_{k=0}^{n-1} \eta^{n-k-1}\xi^k \right) d\kappa(\xi) \right) d\delta(\eta).
\]

The maximum of \( |a_n| \) (resp. \( |b_n| \)) is attained when \( \delta \) and \( \kappa \) are Dirac measures. Therefore, we have

\[
|a_n| \leq \max \left\{ \frac{1}{n} \left| \eta^n + \left( (1 - 2\alpha)\eta + \xi \right) \sum_{k=0}^{n-1} \eta^{n-k-1}\xi^k \right| : |\eta| = |\xi| = 1 \right\}
\]

\[
\leq 2(1 - \alpha) + \frac{1}{n}
\]
and
\[ |b_n| \leq \max \left\{ \frac{1}{n} \left| (2\alpha - 1)\eta^n + ((1 - 2\alpha)\eta + \xi) \sum_{k=0}^{n-1} \eta^{n-k-1}\xi^k \right| : |\eta| = |\xi| = 1 \right\} \]
\[ \leq 2(1 - \alpha) + \frac{2\alpha - 1}{n}. \]

The equalities occur if and only if \( f(z) \) is one of the functions of the form \( \eta f_\alpha(\eta z), |\eta| = 1 \), where \( f_\alpha(z) \) is given by (1.5), which may be rewritten as
\[ f_\alpha(z) = z \exp \left( \sum_{n=1}^{\infty} \left( 2(1 - \alpha) + \frac{1}{n} \right) z^n \right) \exp \left( \sum_{n=1}^{\infty} \left( 2(1 - \alpha) + \frac{2\alpha - 1}{n} \right) z^n \right). \]

This completes the proof. \( \square \)

3. Bohr’s radius for \( S^*_{Lh}(\alpha) \)

The classical Bohr inequality states that if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \), then
\[ M_r(f) = \sum_{n=0}^{\infty} |a_n|r^n \leq 1 \]
for all \( |z| = r \leq 1/3 \) (see Bohr [13]). Bohr actually obtained the inequality only for \( |z| \leq 1/6 \), Wiener, Riesz and Schur independently established the sharp inequality for \( |z| \leq 1/3 \) and showed that the bound 1/3 was sharp. See the recent survey on this topic [12] and the references therein. In recent years, the space of subordinations and the space of complex-valued bounded harmonic mappings are considered in the study of Bohr’s inequality, for example in [9, 10, 24].

The following results concern Bohr’s radius of log-harmonic starlike mappings of order \( \alpha \).

**Theorem 3.1.** Let \( f(z) = zh(z)g(z) \in S^*_{Lh}(\alpha), H(z) = zh(z) \) and \( G(z) = zg(z) \). Then

1. \( |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right) \leq d(0, \partial H(\mathbb{D})) \quad \text{for } |z| \leq r_H, \text{ where } r_H \text{ is the unique root in } (0, 1) \) of the equation
\[ \frac{r}{1 - r} \exp \left( (1 - \alpha) \frac{2r}{1 - r} \right) = \frac{1}{2e^{1-\alpha}}, \]

2. \( |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) \leq d(0, \partial G(\mathbb{D})) \quad \text{for } |z| \leq r_G, \text{ where } r_G \text{ is the unique root in } (0, 1) \) of the equation
\[ \frac{r}{(1 - r)^{2\alpha - 1}} \exp \left( (1 - \alpha) \frac{2r}{1 - r} \right) = \frac{1}{2^{2\alpha - 1}e^{1-\alpha}}, \]

3. \( |z| \exp \left( \sum_{n=1}^{\infty} (|a_n| + |b_n|)|z|^n \right) \leq d(0, \partial f(\mathbb{D})) \quad \text{for } |z| \leq r_f, \text{ where } r_f \text{ is the unique root in } (0, 1) \) of the equation
\[ \frac{r}{(1 - r)^{2\alpha}} \exp \left( (1 - \alpha) \frac{4r}{1 - r} \right) = \frac{1}{2^{2\alpha}e^{2(1-\alpha)}}. \]
All the radius are sharp and attained by a suitable rotation of the log-harmonic right-half plane mapping \( f_\alpha(z) \), where \( f_\alpha(z) \) is given by (1.5).

Proof. By assumption

\[
H(z) = z \exp \left( \sum_{n=1}^{\infty} a_n z^n \right) \quad \text{and} \quad G(z) = z \exp \left( \sum_{n=1}^{\infty} b_n z^n \right).
\]

Firstly, we have

\[
r \exp \left( \sum_{n=1}^{\infty} |a_n|r^n \right) \leq r \exp \left( \sum_{n=1}^{\infty} \left( 2(1 - \alpha) + \frac{1}{n} \right) r^n \right) \quad \text{(by Theorem 2.6)}
\]

\[
= \frac{r}{1-r} \exp \left( 2(1 - \alpha) \frac{r}{1-r} \right)
\]

\[
= \frac{1}{2e^{1-\alpha}} \leq d(0, \partial H(\mathbb{D})) \quad \text{(by (3.1) and Corollary 2.5)}.
\]

Similarly, using Theorem 2.6, (3.2) and Corollary 2.5 we have

\[
r \exp \left( \sum_{n=1}^{\infty} |b_n|r^n \right) \leq r \exp \left( \sum_{n=1}^{\infty} \left( 2(1 - \alpha) + \frac{2\alpha - 1}{n} \right) r^n \right)
\]

\[
= \frac{r}{(1-r)^{2\alpha-1}} \exp \left( 2(1 - \alpha) \frac{r}{1-r} \right)
\]

\[
\leq d(0, \partial G(\mathbb{D})).
\]

Furthermore, using Theorem 2.6, (3.3) and Corollary 2.5 we have

\[
r \exp \left( \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \right) \leq r \exp \left( \sum_{n=1}^{\infty} \left( 4(1 - \alpha) + \frac{4\alpha}{n} \right) r^n \right)
\]

\[
= \frac{r}{(1-r)^{2\alpha}} \exp \left( (1 - \alpha) \frac{4r}{1-r} \right)
\]

\[
\leq d(0, \partial f(\mathbb{D})).
\]

Finally, it is evident that all radius are attained by suitable rotations of the log-harmonic right half plane mapping \( f_\alpha(z) \), where \( f_\alpha(z) \) is given by (1.5). \( \Box \)

If \( \alpha = 0 \), then Theorem 3.1 reduces to Theorem 3 in [11]. If \( \alpha \to 1 \), then \( r_H = r_G = 1/3 \), and \( r_f = 3 - 2\sqrt{2} \) which is same as Bohr’s radius of the subordinating family of univalent functions (see [9, Theorem 1]) which we recall for a ready reference below.

Theorem D. Suppose that \( f, g \) are analytic in \( \mathbb{D} \) such that \( f \) is univalent in \( \mathbb{D} \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) belongs to \( S(f) = \{ \varphi : \varphi \prec f \} \), where \( \prec \) denotes the usual subordination. Then inequality

\[
\sum_{n=1}^{\infty} |b_n|r^n \leq \text{dist}(f(0), \partial f(\mathbb{D}))
\]

holds with \( r_f = 3 - 2\sqrt{2} \approx 0.17157 \). The sharpness of \( r_f \) is shown by the Koebe function \( f(z) = z/(1 - z)^2 \).
4. The inner mapping radius of log-harmonic mappings

In [25], the authors have proposed the following two conjectures.

**Conjecture A.** Let \( f(z) = zh(z)g(z) \in S_{Lh} \), where the representation of \( h(z) \) and \( g(z) \) are given by (1.3). Then for all \( n \geq 1 \),

1. \(|a_n| \leq 2 + \frac{1}{n^2} \); 
2. \(|b_n| \leq 2 - \frac{1}{n^2} \); 
3. \(|a_n - b_n| \leq \frac{2}{n^2} \);

This conjecture has been verified for starlike log-harmonic mappings, see [6, Theorem 3.3] and [25, Theorem 3.3]. The log-harmonic Koebe function \( f_0(z) \) given by (1.3) gives the sharpness. In [25, Theorem 3.3], it was also proposed that \( \{w : |w| < 1/e^2\} \subseteq f(\mathbb{D}) \) if \( f(z) = zh(z)g(z) \in S_{Lh} \).

**Definition 1.** For \( f(z) = zh(z)g(z) \in S_{Lh} \), the inner mapping radius \( \rho_0(f) \) of the domain \( f(\mathbb{D}) \) is to be the real number \( \psi'(0) \), where \( \psi \) is the analytic function that maps \( \mathbb{D} \) onto \( f(\mathbb{D}) \) such that \( \psi(0) = 0 \) and \( \psi'(0) > 0 \).

Recall that the tip of the slit of the log-harmonic Koebe function \( f_0(z) \) is at \(-1/e^2\) while the tip of the slit for the analytic Koebe function \( k(z) = \frac{z}{1-z} \) is at \(-1/4\). Obviously, the images of the unit disk under \( \frac{4}{e^2}k(z) \) and under \( f_0(z) \) are the same, i.e.,

\[
\frac{4}{e^2}k(\mathbb{D}) = f_0(\mathbb{D}).
\]

This multiplier factor of \( 4/e^2 \) is the inner mapping radius for \( f_0(\mathbb{D}) \). For other log-harmonic functions in \( S_{Lh} \), the inner mapping radius may be different. For example, consider log-harmonic right half-plane mapping \( LR(z) \) and log-harmonic two-slits mapping \( LS(z) \) (see from Examples 2 and 3 in [25]) given by

\[
LR(z) = \frac{z}{1-z} \exp \left( \Re \left( \frac{2z}{1-z} \right) \right)
\]

and

\[
LS(z) = \frac{z}{1-z^2} |1-z^2| \exp \left( \Re \left( \frac{2z^2}{1-z^2} \right) \right),
\]

respectively. The inner mapping radius for \( LR(\mathbb{D}) \) and \( LS(\mathbb{D}) \) is \( 1/e \), since

\[
\frac{1}{e}R(\mathbb{D}) = LR(\mathbb{D}) \quad \text{and} \quad \frac{1}{e}S(\mathbb{D}) = LS(\mathbb{D}),
\]

where \( R(z) = z/(1-z) \) and \( S(z) = z/(1-z^2) \) denote the analytic right half-plane mapping and two-slits mapping, respectively.

In the example above \( \psi(z) = \frac{4}{e^2}k(z) \), and the inner mapping radius \( \rho_0(f_0) = \frac{4}{e^2} \). In the following example, we show that \( \frac{4}{e^2} \leq \rho_0(f) \leq 4 \) for one slit log-harmonic mappings \( f \in S_{Lh} \).

**Example 1.** Consider the family of functions \( F_\lambda(z) = f_1^\lambda(z)f_2^{1-\lambda}(z) \) (\( 0 \leq \lambda \leq 1 \)), where

\[
f_1(z) = \frac{z}{(1-z)^2} |1-z|^2 \quad \text{and} \quad f_2(z) = \frac{z}{(1-z)^2} |1-z|^2 \exp \left( \Re \left( \frac{4z}{1-z} \right) \right).
\]
Simple calculations show that $f_1$ and $f_2$ are starlike log-harmonic with dilatations $\mu_1(z) = -z$ and $\mu_2(z) = z$. Also $F_\lambda$ is log-harmonic with the dilatation

$$
\mu(z) = \frac{z[(1 - 2\lambda) + z]}{1 + (1 - 2\lambda)z}.
$$

It is clear that $|\mu(z)| < 1$ for $0 \leq \lambda \leq 1$, and therefore $F_\lambda$ is sense-preserving in $D$. Since the conditions of Theorem 3 in [2] are satisfied (or see the details in Example 3 in [2]), we thus have that $F_\lambda$ is univalent and starlike in $D$. Moreover,

$$
F_\lambda(z) = f_1^\lambda(z)f_2^{1-\lambda}(z) = \frac{z}{(1 - z)^2}|1 - z|^2 \exp \left( (1 - \lambda) \Re \left( \frac{4z}{1 - z} \right) \right).
$$

Because $k(D)$ is $\mathbb{C}$ minus the slit on the negative real axis from $-1/4$ to $\infty$, and $F_\lambda(D)$ is $\mathbb{C}$ minus the slit on the negative real axis from $-e^{-2(1-\lambda)}$ to $\infty$, we obtain that for $0 < \lambda < 1$,

$$
\frac{4}{e^2} \leq \rho_0(F_\lambda) \leq 4.
$$

**Problem 1.** Show that the inner mapping radius of log-harmonic mapping $f \in S_{Lh}$ satisfy

$$
\frac{4}{e^2} \leq \rho_0(f) \leq 4;
$$

or else find a class of log-harmonic mappings $f \in S_{Lh}$ such that either $\rho_0(f) > 4$ or $\rho_0(f) < \frac{4}{e^2}$.

### 5. Pre-Schwarzian derivatives and log-harmonic mappings

In this section, we introduce pre-Schwarzian, Schwarzian derivatives and log-harmonic Bloch function for non-vanishing log-harmonic mappings analogous to analytic and harmonic mappings.

The pre-Schwarzian and Schwarzian derivatives of a locally univalent analytic function $h$ are given (cf. [21]) by

$$
Ph(z) = \frac{h''(z)}{h'(z)} \quad \text{and} \quad Sh(z) = \left( \frac{h''(z)}{h'(z)} \right)' - \frac{1}{2} \left( \frac{h''(z)}{h'(z)} \right)^2,
$$

respectively. These notions for complex valued harmonic mappings was presented by Chuaqui et al. [17] and investigated by a number of authors. See [16, 18, 19, 23] and the references therein. In [26], Mao and Ponnusamy investigated the Schwarzian derivative of log-harmonic mappings, and they obtained several necessary and sufficient conditions for Schwarzian derivative $S_f$ to be analytic. In this paper, we modify the definitions of pre-Schwarzian $P_f$ and Schwarzian $S_f$ derivatives for the sense-preserving univalent log-harmonic mappings and notice that the new definitions preserve the standard properties of the classical Schwarzian.
derivative and they are given in the following way:

\[ P_f(z) = (\log J_f)_z = \left( \frac{h''(z)}{h'(z)} - \frac{h'(z)}{h(z)} \right) - \frac{\mu(z)\mu'(z)}{1 - |\mu(z)|^2}, \]

\[ S_f(z) = (P_f(z))' - \frac{1}{2} (P_f(z))^2 = \left( \frac{h''(z)}{h'(z)} - \frac{h'(z)}{h(z)} \right)' - \frac{1}{2} \left( \frac{h''(z)}{h'(z)} - \frac{h'(z)}{h(z)} \right)^2 + \left( \frac{h''(z)}{h'(z)} - \frac{h'(z)}{h(z)} \right) \frac{\mu(z)\mu'(z)}{1 - |\mu(z)|^2} \]

\[ - \frac{\mu(z)\mu''(z)}{1 - |\mu(z)|^2} - \frac{3}{2} \left( \frac{\mu(z)\mu'(z)}{1 - |\mu(z)|^2} \right)^2, \]

where

\[ J_f(z) = \left| \frac{h'(z)}{h(z)} \right|^2 (1 - |\mu(z)|^2) \quad \text{and} \quad \mu(z) = \frac{|g'(z)/g(z)|}{|h'(z)/h(z)|} \]

are the Jacobian of log-harmonic mapping \( f \) and the dilatation of \( f \), respectively. The pre-Schwarzian and Schwarzian derivatives of log-harmonic mappings have the chain rule property exactly in the same form as in the analytic case: if \( f \) is a sense-preserving log-harmonic mapping and \( \varphi \) is a locally univalent analytic function for which the composition \( f \circ \varphi \) is defined, then a straightforward calculation shows that

\[ P_{f \circ \varphi}(z) = (P_f \circ \varphi(z)) \cdot \varphi'(z) + P_{\varphi}(z) \quad \text{and} \quad S_{f \circ \varphi}(z) = (S_f \circ \varphi(z)) \cdot (\varphi'(z))^2 + S_{\varphi}(z). \]

If we assume that the pre-Schwarzian derivative \( P_f \) of a log-harmonic mapping \( f = h \overline{g} \) with dilatation \( \mu(z) \) is analytic, then we get that

\[ \frac{\partial P_f}{\partial \overline{z}} = \frac{|\mu'(z)|^2}{(1 - |\mu(z)|^2)^2} = 0 \quad (z \in \mathbb{D}), \]

which implies that \( \mu(z) \) is constant. In other words, \( P_f \) is analytic if and only if the dilatation of \( f \) is constant. Actually, we get the following more general result.

**Theorem 5.1.** Suppose that \( f(z) = h(z)\overline{g(z)} \) is a sense-preserving log-harmonic mapping in \( \mathbb{D} \). Then pre-Schwarzian derivative \( P_f \) of \( f(z) \) is harmonic if and only if the dilatation \( \mu(z) \) of \( f(z) \) is constant.

**Proof.** By a straightforward calculation, we obtain

\[ \frac{\partial^2 P_f}{\partial z \partial \overline{z}} = \frac{\mu' \left( \mu'' (1 - |\mu|^2) + 2\mu'^2 \overline{\mu} \right)}{(1 - |\mu|^2)^3}. \quad (5.1) \]

If \( \mu \) is constant, then it is clear that \( \Delta P_f \equiv 0 \) and so \( P_f(z) \) is harmonic in \( \mathbb{D} \). Now we assume that \( P_f(z) \) is harmonic. By (5.1), we get

\[ \mu' \left( \mu'' (1 - |\mu|^2) + 2\mu'^2 \overline{\mu} \right) = 0. \]

If \( \mu \) is not constant, then the last relation reduced to

\[ \frac{\mu''}{\mu'^2} = -\frac{2\overline{\mu}}{1 - |\mu|^2}, \]

which is analytic in \( \mathbb{D} \). Thus, we see that \( \mu \) is a constant which contradicts our assumption. The proof is complete. \( \square \)
6. Log-harmonic Bloch space

**Definition 2.** A non-vanishing log-harmonic mapping \( f(z) = h(z)g(z) \) in \( \mathbb{D} \) is said to be a **log-harmonic Bloch function** if

\[
\beta(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \left| \frac{h'(z)}{h(z)} \right| + \left| \frac{g'(z)}{g(z)} \right| \right) < +\infty,
\]

where \( h \) and \( g \) are analytic in \( \mathbb{D} \),

\[
h(z) = \exp \left( \sum_{n=0}^{\infty} a_n z^n \right) \quad \text{and} \quad g(z) = \exp \left( \sum_{n=1}^{\infty} b_n z^n \right).
\]

The space of all log-harmonic Bloch functions is denoted by \( \mathcal{B}_{Lh} \).

The space \( \mathcal{B}_{Lh} \) forms a complex Banach space with the norm \( \| \cdot \|_{\mathcal{B}_{Lh}} \) given by (see \[20\])

\[
\|f\|_{\mathcal{B}_{Lh}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \left| \frac{h'(z)}{h(z)} \right| + \left| \frac{g'(z)}{g(z)} \right| \right)
\]

\[
= |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{h'(z)}{h(z)} \right| (1 + |\mu(z)|).
\]

We refer it as the **log-harmonic Bloch norm** and the elements of the log-harmonic Bloch space are called log-harmonic Bloch functions.

Now we will show that \( \mathcal{B}_{Lh} \) has the affine and linear invariance. To do this, we let

\[
\phi_\alpha(z) = \frac{z + \alpha}{1 + \bar{\alpha}z}, \quad z \in \mathbb{D},
\]

where \( |\alpha| < 1 \).

**Proposition 6.1.** If \( f(z) = h(z)g(z) \in \mathcal{B}_{Lh} \), then

(i) \( f^a f^b \in \mathcal{B}_{Lh} \) for any \( a, b \in \mathbb{C} \) (affine invariance)

(ii) \( f \circ \phi_\alpha \in \mathcal{B}_{Lh} \) for any \( \alpha \in \mathbb{D} \) (linear invariance).

**Proof.** For the proof of (i), we let \( f = h\overline{g} \), and consider

\[
F = f^a f^b = h^a \overline{g^b} \overline{h^b} \overline{g^a}.
\]

Elementary computations give

\[
\beta(F) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \left| \frac{h^a(z)g^b(z)}{h^a(z)g^b(z)} \right| + \left| \frac{\overline{h^b(z)g^a(z)}}{\overline{h^b(z)g^a(z)}} \right| \right)
\]

\[
= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \left| \frac{h'(z)}{h(z)} \right| + b \left| \frac{g'(z)}{g(z)} \right| + a \left| \frac{h'(z)}{h(z)} + \overline{a} \frac{g'(z)}{g(z)} \right| \right)
\]

\[
\leq (|a| + |b|) \beta(f) < +\infty.
\]

By Definition \[2\] the desired assertion follows.

For the proof of (ii), we write \( F = f \circ \phi_\alpha = H \overline{G} \) so that

\[
\frac{H'(z)}{H(z)} = \frac{h' \left( \phi_\alpha(z) \right)}{h \left( \phi_\alpha(z) \right)} \cdot \frac{1 - |\alpha|^2}{(1 + \bar{\alpha}z)^2} \quad \text{and} \quad \frac{G'(z)}{G(z)} = \frac{g' \left( \phi_\alpha(z) \right)}{g \left( \phi_\alpha(z) \right)} \cdot \frac{1 - |\alpha|^2}{(1 + \bar{\alpha}z)^2}.
\]
Consequently,
\[
\beta(F) = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)(1 - |\alpha|^2)}{|1 + \bar{\alpha}z|^2} \left( \left| \frac{h'(\phi_\alpha(z))}{h(\phi_\alpha(z))} \right| + \left| \frac{g'(\phi_\alpha(z))}{g(\phi_\alpha(z))} \right| \right)
\]
\[
= \sup_{z \in \mathbb{D}} (1 - |\phi_\alpha(z)|^2) \left( \left| \frac{h'(\phi_\alpha(z))}{h(\phi_\alpha(z))} \right| + \left| \frac{g'(\phi_\alpha(z))}{g(\phi_\alpha(z))} \right| \right),
\]
which gives that \( \beta(F) = \beta(f) \). The proof is complete. \( \square \)

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