A Family of Algorithms for Computing Information-Theoretic Forms of Strong Converse Exponents in Channel Coding and Lossy Source Coding

Yutaka Jitsumatsu Member, IEEE and Yasutada Oohama Member, IEEE

Abstract

This paper studies the computation of error and correct decoding probability exponents in channel coding and lossy source coding. For channel coding, we show that the recently proposed algorithm of Tridenski and Zamir for computing strong converse exponent has the attractive property that the objective function alternately takes the forms appearing in Arimoto’s and Dueck and Körner’s exponents. Because of this property, the convergence of the algorithm directly implies the match of the two exponents. Then, we consider a special case of Tridenski et al.’s generalized algorithm of that has not been examined in depth. We show that the objective function of this algorithm also has the interesting property that it takes the forms of Dueck and Körner’s exponent and another representation of the strong converse exponent derived by Arimoto’s algorithm. This particular case is important because the objective function in this case can be used to prove that the Gallager and Csiszár and Körner error exponents agree. For lossy source coding, we propose two new algorithms for computing the Csiszár and Körner’s strong converse exponent. We also define a function similar to Blahut’s error exponent with a negative slope parameter for source coding. We show that one of the proposed algorithms has the property that the objective function alternately takes the forms of Csiszár and Körner’s exponent and the newly defined exponent function. The convergence of the algorithm proves that the new exponent function coincides with the Csiszár and Körner exponent. We also prove that Arimoto algorithm for computing error exponent of lossy source coding can work for negative $\rho \in [-1, 0)$, and thus can be used to compute the new exponent function. Computation of Arikan and Merhav’s guessing exponent is also discussed.

I. Introduction

The channel coding error exponent is an important quantity expressing the bound of the exponential decay of the decoding error probability of a channel code with coding rate $R$ to zero. The random coding error exponent is the exponent of the decoding error probability averaged over an ensemble of randomly generated codes and two forms of its representation are known. One is the form of Gallager [1] and the other is the form of Csiszár and Körner [2]. The former is expressed by using the maximum value of Gallager’s $E_0$ function with respect to input

Dept. of Information and Communications Engineering, Tokyo Institute of Technology
Dept. of Network and Computer Engineering, The University of Electro-Communications
probability distribution with parameter $0 \leq \rho \leq 1$, while the latter is defined as the maximum with respect to input distribution of the minimum with respect to the conditional probability distribution of a function expressed by using the mutual information and Kullback-Leibler (KL) divergence. They match at all rates of $R \geq 0$, although they look very different. On the other hand, the correct decoding probability exponent, also known as the strong converse exponent, of channel coding expresses the exponential upper bound of the correct decoding for the rate $R$ greater than the channel capacity. Like the error exponent, the channel coding strong converse exponent is also known in two forms. One is Arimoto’s exponent [3] that uses the minimum value with respect to the input probability distribution of the $E_0$ function with parameter $\rho \in [-1, 0]$, and the other is Dueck and Körner’s exponent [4] that is defined as the minimum with respect to joint distribution of a function expressed using the mutual information and the KL divergence. These two strong converse exponents are also known to coincide for any rate $R \geq 0$. Arimoto proposed an algorithm for computing Gallager’s random coding error exponent and Arimoto’s strong converse exponent, or more precisely, an algorithm for finding the input distribution that maximizes the $E_0$ function for parameter $0 < \rho < +\infty$ and minimizes the $E_0$ function for $-1 \leq \rho < 0$.

Oohama and the author proposed an algorithm for computing the Dueck and Körner’s exponent [5] and an extension of this to the case of channel coding under cost constraint [6]. The algorithm of Oohama and the author has a parameter $\lambda \in [0, 1]$, which corresponds to $-\rho$ in Gallager’s exponent. Dueck and Körner’s exponent is defined as the minimum of an objective function with respect to the joint probability distribution. The objective function of Dueck and Körner’s exponent is convex and therefore an algorithm for finding the global minimum can be derived [5], [6]. On the other hand, the Csiszár and Körner’s exponent is defined as the saddle point of a minimax problem and the structure of the problem is different. Thus, the algorithm for computing Dueck and Körner’s exponent cannot be applied to compute Csiszár and Körner’s exponent. To the best of the author’s knowledge, no algorithm for finding the saddle point of the minimax problem for Csiszár and Körner’s function has been found.

In Arimoto’s algorithm as well as Oohama and the author’s algorithm, after computing the optimal probability distribution for a fixed $\rho = -\lambda$, we apply Legendre-Fenchel transformation (LFT) to obtain the exponent for any rate $R \geq 0$. Tridenski and Zamir, on the other hand, proposed two algorithms: One computes directly the strong converse exponent for a fixed $R \geq 0$ without using LFT [7] and the other is an algorithm for fixed parameter $\rho$ [8], [9]. Their algorithm computes the optimal distribution based on a representation of strong converse exponent [10], that differs from Arimoto’s exponent and Dueck and Körner’s exponent. In this paper, we show that the fixed slope version of the Tridenski and Zamir algorithm has a desirable property that has not yet been pointed out.

Table I shows the research results that established the error exponent and the strong converse exponent for channel coding and lossy source coding as well as the research results that proposed algorithms for their computation. Table I also shows the results of this paper. The purpose for this research is to establish all algorithms for computing the optimal distribution for optimization or minimax problems that give error and strong converse exponents in channel coding and lossy source coding. To achieve this goal, we compared the optimization and minimax problems that give the exponents shown in Table I and investigated the relationship between them.

In this paper, we propose Algorithm 2 for computing the strong converse exponent for channel coding and Algorithms 5 and 6 for computing the strong converse exponent for lossy source coding. First, we will explain...
### Table I

**Error and Strong Converse Exponents in Channel and Lossy Source Coding and Their Algorithms**

|                      | Channel Coding | Lossy Source Coding |
|----------------------|----------------|---------------------|
| **Error Exponent**   | Gallager (1965) | Csiszár & Körner (1980) | Blahut (1974) [Remark] Suboptimal in general | Marton* (1974) [Remark] May not be continuous |
| **Algorithm**        | Arimoto (1976)  | Open                | Arimoto (1976) | Open |
| **Strong Converse**  | Arimoto* (1973) | Dueck & Körner* (1980) | Definition 9 | Csiszár & Körner* (1980) |
| **Algorithm**        | Arimoto (1976)  | O. & J. (2015) without cost constraint | Proposition 7 showing that Arimoto (1976) is applicable | J. & O. (2016) |
|                      |                | J. & O. (2020) extension to the channel under cost constraint | Algorithm 2 | Algorithm 6 |
| **Simultaneous**     | Tridenski & Zamir (2018) | Algorithm 5 |
| **Computation**      |                |                     |

The * symbol after the name indicates that the function has been proven optimal.

The algorithm for computing the channel coding strong converse exponent. Fig. 1 shows the relationship between Arimoto’s algorithm [11], Oohama and the author’s algorithm [5], [6]. Tridenski and Zamir’s algorithm with a fixed slope parameter $\rho$ [8], [9], and Algorithm 2 which is studied in detail for the first time.

The box in the leftmost on the bottom row labeled $\min_{p_X} F_{0}^{(\lambda)}(p_X)$ is the optimization problem that appears in Arimoto’s strong converse exponent. The box in the rightmost on the bottom row labeled $\min_{q_{XY}} \Theta^{(\lambda)}(q_{XY})$ is the optimization problem that expresses Dueck and Körner’s exponent. The meaning of each function and variable is explained in Section II. Written directly above these two optimization problems are the double minimization problems in the Arimoto algorithm and Oohama and author’s algorithm, respectively. In a double minimization problem, fixing one variable and minimizing with respect to the other variable yields a single minimization problem. The Arimoto algorithm and Oohama and the author’s algorithm have alternative expressions $\min_{\tilde{p}_{XY}} A^{(-\lambda)}(\tilde{p}_{XY})$ and $\min_{\tilde{q}_{XY}} -\lambda F_{\tilde{A}}^{(-\lambda)}(\tilde{q}_{XY}, \tilde{q}_{X|Y})$, respectively. Tridenski and Zamir’s algorithm, on the other hand, is different from either of these two algorithms. The expression for Tridenski and Zamir’s exponent with a fixed slope $\lambda$ coincides with $\Theta^{(\lambda)}(q_{XY})$. In Tridenski and Zamir’s algorithm, the joint distribution $q_{XY}$ and input distribution $p_X$ are alternately updated. In this paper, we point out that the objective function of Arimoto’s exponent and that of Dueck and Körner’s exponent appear alternately during the update. Thus, we can say that Tridenski and Zamir’s algorithm is an algorithm that can simultaneously compute Arimoto’s exponent and Dueck and Körner’s exponent. Therefore, in Table I, Tridenski and Zamir’s algorithm spans both Arimoto’s exponent and Dueck and Körner’s exponent. This fact is so important that the convergence theorem of Tridenski and Zamir’s algorithm immediately...
The strong converse exponent for channel coding \( 0 < \lambda \leq 1 \)

\[
\begin{align*}
\min_{p_x} \min_{\hat{p}_x|y} & -\lambda J_{\lambda}^{(\lambda)}(p_x, \hat{p}_x|y) \quad \text{Arimoto algorithm} \\
\min_{q_{xy}} \min_{p_x} & J_{\lambda}^{(\lambda)}(q_{xy}, p_x) \quad \text{Tridenski and Zamir’s algorithm} \\
\min_{q_{xy}} \min_{p_x} & J_{\lambda}^{(\lambda)}(q_{xy} | p_x) \quad \text{Algorithm 2} \\
\min_{q_{xy}} \min_{\hat{p}_x|y} & F_{\lambda}^{(\lambda)}(q_{xy}, \hat{p}_x|y) \quad \text{Dueck & Körner’s Exponent}
\end{align*}
\]

Fig. 1. Relation between the four algorithms discussed in this paper and the expressions of the strong converse exponents in channel coding

implies that the two strong converse exponents coincide. Among the four algorithms in Fig. 1 only Tridenski and Zamir’s algorithm can prove the match of the two exponents. After [8], Tridenski et al. [12] proposed a generalized algorithm with four non-negative parameters \( t_1, t_2, t_3, t_4 \geq 0 \) that includes their algorithm [8] and Oohama and the author’s algorithm [5], [6] as special cases. The generalized algorithm is important in the sense that it gives a unified perspective that each algorithm is obtained by setting parameters to specific values. On the other hand, some properties are valid only for some special cases. We focus on a special case that has not been studied in depth, which is called Algorithm 2 in this paper. In this algorithm, the objective function of Dueck and Körner and that of Arimoto’s alternative expression appear alternately by updating the probability distribution. As discussed below, function \( J_{2}^{(\lambda)}(q_{xy}, \hat{p}_x|y) \) used by Algorithm 2 has another important advantage. Relationship between these algorithms is discussed in Section III-C and III-D.

Next, we propose a new algorithm for computing the strong converse exponent of lossy source coding. Fig. 2 shows the relationship between Algorithm [5] [9], the Arimoto algorithm, and the author and Oohama’s algorithm [13]. The structure of the relationship between these algorithms corresponds perfectly to Fig. 1. The rightmost \( \min_{q_{xy}} \Theta_{\lambda}^{(\lambda, \lambda \nu)}(q_{xy}) \) in the bottom row is an optimization problem expressing Csiszár and Körner’s strong converse exponent. Csiszár and Körner’s exponent is the only known expression for the strong converse exponent of lossy source coding. In this paper, we introduce in Definition 9 a new expression of the strong converse exponent that is similar to Blahut exponent, except that the range of \( \rho \) is \([-1, 0]\). This exponent is shown as the leftmost \( \min_{\hat{p}_y} -E_{\nu, \lambda}^{(-\lambda, \lambda \nu)}(\hat{p}_y) \) in the bottom row in Fig. 2. In Algorithm 5, the objective function in Blahut’s exponent, but with a negative \( \rho \), and that of Csiszár and Körner’s exponent appear alternately. The convergence of the algorithm proves that for all rate \( R \geq 0 \), the exponent of Definition 9 coincides with Csiszár and Körner’s exponent. This paper also shows in Proposition 7 that Arimoto algorithm originally proposed for positive \( \rho \) works correctly for the parameter \( \rho \in [-1, 0] \) as well.

For the strong converse exponent, we were able to derive an Algorithm to compute the Dueck and Körner’s exponent for channel coding and the Csiszár and Körner’s exponent for source coding, whereas for the error exponent, we were unable to give an algorithm to compute Csiszár and Körner’s exponent for channel coding and Marton’s exponent for source coding. These remain unresolved. However, by using \( J_{2}^{(\lambda)}(q_{xy}, \hat{p}_x|y) \) used in
The strong converse exponent for lossy source coding 

\[
\max_{\rho} \min_{\nu} \frac{1}{\nu} \left( F^{(\lambda,\nu)}(\hat{q}_{XY}, \hat{q}_{Y\mid X}) - \rho \right)
\]

Algorithm 7

(J. & Oohama 2016)

Algorithm 5

Algorithm 6

Arimoto algorithm

with \( \rho = -\lambda \)

Definition 13

Csiszár & Körner's Exponent

Fig. 2. Relation between the four algorithms discussed in this paper and the expressions of the strong converse exponents in lossy source coding

Algorithm [2] we were able to give a new proof that Gallager’s exponent coincides with that of Csiszár and Körner as a byproduct. This is another result of this paper. The conventional proof method [2 Problem 10.24] evaluates the upper and lower bounds of the saddle point using the KKT conditions satisfied by the saddle point of the minimax problem that determines Csiszár and Körner’s exponent. The advantage of the proof method in this paper is that it is more elementary, since the saddle point can be evaluated by an equality rather than by the KKT condition, i.e. by a set of inequalities.

This paper is organized as follows: Section II reviews the error and the strong converse exponents in channel coding and source coding. Section III and IV give the main results of this paper. We discuss algorithms for computing the strong converse exponent in channel coding in Section III and the one in lossy source coding in Section IV. New iterative algorithms are defined and the convergence of the algorithms to the optimal distribution is proved. Then, we compare the proposed algorithms with Arimoto’s and our previous research. In Section V, error exponent of channel coding is discussed. Using the same functional as in the algorithm proposed in Section III, we give a new proof for the fact that Csiszár and Körner’s error exponent matches with Gallager’s exponent. We also show that Arikan and Merhav’s guessing exponent [14] can be computed by Arimoto’s algorithm for lossy source coding.

II. DEFINITIONS OF EXPONENT FUNCTIONS

As a preliminary to state the main result, this section reviews the definition of the error and the strong converse exponent functions in channel coding and lossy source coding listed in Table I. Each of these exponent functions is defined as an optimization problem with respect to a probability distribution.

A. Exponent functions in channel coding

We consider a DMC with finite input and output alphabet \( \mathcal{X} \) and \( \mathcal{Y} \) that is subject to an input cost constraint. Let \( W(y|x), y \in \mathcal{Y}, x \in \mathcal{X} \), be the transition probability of the DMC. Let \( c(x) \geq 0 \) be a cost function for sending \( x \in \mathcal{X} \) and the average cost of sending codeword \( x_1 x_2 \cdots x_n \) is \( c(x_1, \ldots, c_n) = (1/n) \sum_{i=1}^{n} c(i) \). Let \( R \geq 0 \) be a coding rate and \( \Gamma \geq 0 \) be a maximum allowable cost. We define four functions.
\textbf{Definition 1:} For $R \geq 0$ and $\Gamma \geq 0$, Gallager’s random coding error exponent is defined by
\begin{equation}
E_r(R, \Gamma | W) = \max_{\rho \in [0,1]} \inf_{\nu \geq 0} \max_{p_X \in \mathcal{P}(X)} \{-\rho R + \rho \nu \Gamma + E_0^{(\rho, \nu)}(p_X | W)\},
\end{equation}
where $\mathcal{P}(X)$ is a set of probability distributions on the input alphabet and
\begin{equation}
E_0^{(\rho, \nu)}(p_X | W) = -\log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} p_X(x) W(y|x) e^{q(x)} \right)^{\frac{1}{1+\rho}}
\end{equation}
is called Gallager’s $E_0$ function for cost constraint channels. We use the natural logarithm.

\textbf{Definition 2:} For $R \geq 0$ and $\Gamma \geq 0$, Arimoto’s strong converse exponent is defined by
\begin{equation}
G_{AR}(R, \Gamma | W) = \sup_{\rho \in (-1,0]} \sup_{\nu \geq 0} \min_{p_X \in \mathcal{P}(X)} \{-\rho R + \rho \nu \Gamma + E_0^{(\rho, \nu)}(p_X | W)\}.
\end{equation}

\textbf{Definition 3:} Csiszár and Körner’s error exponent is defined by
\begin{equation}
E_{CK}(R, \Gamma | W) = \max_{q_X \in \mathcal{P}(X)} \min_{\rho \in [-1,0], \nu \geq 0} \{D(q_Y|X | W | q_X) + |I(q_X, q_Y|X) - R|^+\},
\end{equation}
where $\mathcal{P}(Y|X)$ is a set of conditional probability distributions of $X \in \mathcal{X}$ given $Y \in \mathcal{Y}$, $D(q_Y|X | W | q_X)$ is the conditional relative entropy defined by
\begin{equation}
D(q_Y|X | W | q_X) = \sum_{x \in \mathcal{X}} q_X(x) \sum_{y \in \mathcal{Y}} q_Y(y|x) \log \frac{q_Y(y|x)}{W(y|x)}
\end{equation}
and $I(q_X, q_Y|X)$ is the mutual information.

\textbf{Definition 4:} Dueck and Körner’s strong converse exponent is defined as
\begin{equation}
G_{DK}(R, \Gamma | W) = \min_{q_{XY} \in \mathcal{P}(X \times Y)} \{D(q_{Y|X} | W | q_X) + |R - I(q_X, q_Y|X)|^+\},
\end{equation}
where $\mathcal{P}(X \times Y)$ is a set of probability distributions on $\mathcal{X} \times \mathcal{Y}$.

Tridesni and Zamir gave a new expression of the optimal strong converse exponent [10, Eq.(32)].

\textbf{Definition 5:} Tridesni and Zamir’s strong converse exponent is defined by
\begin{equation}
G_{TZ}(R | W) = \min_{p_X \in \mathcal{P}(X)} \min_{q_{XY} \in \mathcal{P}(X \times Y)} \{D(q_{XY} | W | q_X \circ W) + |R - D(q_{XY} | p_X \times q_Y)|^+\},
\end{equation}
where $q_X \circ W$ and $p_X \times q_Y$ are joint distributions defined by $q_X(x)W(y|x)$ and $p_X(x)q_Y(y)$.

This exponent coincides with $G_{DK}(R | W)$ as well as $G_{AR}(R | W)$. We will prove of the match of $G_{TZ}(R | W)$ and $G_{DK}(R | W)$ in the proof of Lemma \footnote{The outline of the proof was given in [15, Footnote 13].}
B. Exponent functions in Lossy source coding

Consider a DMS with source alphabet $\mathcal{X}$ and probability distribution $P_X$ on $\mathcal{X}$. Let $\mathcal{Y}$ be a reproduction alphabet and $d(x, y) \geq 0$ be a distortion measure. We consider the case that the distortion with a block length $n$ is measured by $d(x^n, y^n) = (1/n) \sum_{i=1}^{n} d(x_i, y_i)$. Let $R$ be the coding rate and $\Delta$ is the maximum allowable distortion. We assume that both $\mathcal{X}$ and $\mathcal{Y}$ are finite sets and allow $\mathcal{Y}$ to be different form $\mathcal{X}$ and that for every $x \in \mathcal{X}$ there exist at least one $y \in \mathcal{Y}$ satisfying $d(x, y) = 0$. We define three exponent functions.

**Definition 6:** Blahut’s error exponent [16] is defined by

$$E_B(R, \Delta|P_X) = \sup_{\rho \geq 0} \inf_{\nu \geq 0} \max_{\tilde{p}_Y} \left[ \rho R + \nu \Delta - \log \sum_x P_X(x) \left\{ \sum_y \tilde{p}_Y(y) e^{-\nu d(x, y)} \right\}^{-\rho} \right]. \tag{7}$$

We define the third term in the parenthesis of (7) as

$$E_{0,s}^{(\rho, \nu)}(\tilde{p}_Y|P_X) = \log \sum_x P_X(x) \left\{ \sum_y \tilde{p}_Y(y) e^{-\nu d(x, y)} \right\}^{-\rho}, \tag{8}$$

which plays a role similar to $E_0$-function.

The error exponent of the lossy coding problem was established by Marton [17].

**Definition 7:** Marton’s error exponent for lossy source coding was defined by

$$E_M(R, \Delta|P_X) = \min_{q_X \in P(\mathcal{X})} D(q_X||P_X), \tag{9}$$

where

$$R(\Delta|q_X) = \min_{q_{Y|X} \in P(\mathcal{Y}|\mathcal{X})} I(q_X, q_{Y|X})$$

is a rate distortion function. Marton proved that exponents of upper and lower bounds of the error probability for an optimal pair of encoder and decoder are both given by $E_M(R, \Delta|P_X)$. That is, Marton’s exponent is optimal.

The exponential strong converse theorem for lossy source coding was established by Csiszár and Körner [2].

**Definition 8:** Csiszár and Körner’s strong converse exponent for lossy source coding is defined by

$$G_{CK}(R, \Delta|P_X) = \min_{q_X \in P(\mathcal{X})} \{ D(q_X||P_X) + |R(\Delta|q_X) - R|^+ \}. \tag{10}$$

$G_{CK}(R, \Delta|P_X)$ was proven to be optimal [2]. The strong converse exponent for lossy source coding in Blahut style has not been known. We define

**Definition 9:** For $R \geq 0$ and $\Delta \geq 0$, we define

$$G_{JO}(R, \Delta|P_X) \overset{\text{def}}{=} \sup_{\rho \in [-1, 0]} \sup_{\nu \geq 0} \min_{\tilde{p}_Y \in P(\mathcal{Y})} \{ \rho R + \nu \Delta - E_{0,s}^{(\rho, \nu)}(\tilde{p}_Y|P_X) \}. \tag{11}$$

We will show in Section VI-A that $G_{JO}(R, \Delta|P_X)$ coincides with $G_{CK}(R, \Delta|P_X)$.

Marton mentioned in [17] Section III that the continuity of $E_M(R, \Delta|P)$ with respect to $R$ had not been established. Note that the rate distortion function $R(\Delta|q_X)$ is not convex in $q_X$. Therefore, for a fixed $R$, the feasible set in (9) is not convex. This implies that $E_M(R, \Delta|P_X)$ may jump at some $R$. Then, in 1990, Ahlswede gave an example that Marton’s exponent is not continuous in $R$, using a distortion measure with a special structure [18]. This implies Eq.(9) is a non-convex optimization problem and Blahut’s and Marton’s exponents do not match in
general. The difficulty lies in that we can select any distortion measure \( d(x, y) \geq 0 \) on \( x \in X, y \in Y \). Allswede could construct the example but that distortion function seems not natural. We normally expect \( d(x, y) \) is a function similar to the distance between \( x \) and \( y \). Fortunately, it is guaranteed that if \( d(x, y) \) belongs to an important class of distortion measure that can be expressed as a function that only depends on the difference of \( x \) and \( y \), then \( E_M(R|P_X) \) is a convex function of \( R \) \[14\], which immediately implies that \( E_M(R|P_X) = E_B(R|P_X) \). Hamming distortion is included in this class. See also \[19\] and \[2, Exercise 9.5\].

Hereafter, when it is obvious from the context, we omit the symbol for the set of probability distribution such as \( P(X) \) and \( P(Y|X) \) and write only the probability distribution under \( \max \) and \( \min \) symbols.

III. ALGORITHMS FOR THE CHANNEL-CODING STRONG CONVERSE EXPONENT

The algorithms for computing of the channel coding strong converse exponent are discussed in this section. Arimoto’s algorithm \[11\] was the first for computing the strong converse exponent, which is based on Arimoto’s expression \( \frac{2}{3} \). Usually we are interested not only in the value of the exponent function, i.e. the maximum or the minimum value of the objective function, but also in the probability distribution that attains the optimal value. The optimal distribution is not necessarily unique. In Arimoto’s algorithm as well as other algorithms described in this paper, the probability distribution converges to one of the optimal distributions when the optimal distribution is not unique.

About 40 years later, Oohama and the author \[5\] proposed a new algorithm for computing the strong converse exponent based on Dueck and Körner’s expression \( \frac{2}{3} \). Subsequently, Tridenski and Zamir \[8\] proposed another algorithm that was based on their strong converse exponent expression \( \frac{2}{3} \). Then, Tridenski et al. \[12\] proposed a generalized algorithm which includes \[5\] and \[8\] as special cases.

The main purpose of this section is to illustrate the overall picture of Fig. \[1\] We first explain Tridenski and Zamir’s algorithm in Section III-A. We will show the outstanding property of this algorithm that the objective function takes alternately the forms of the objective functions appeared in Arimoto’s and Dueck and Körner’s exponent. Thus, the convergence of the algorithms to the global minimum directly implies the match of the two exponents. This paper is the first to point out such a desirable property of Tridenski and Zamir’s exponent. Then, in Section III-B we describe the properties of the algorithm shown as Algorithm 2 in Fig. \[1\] This is a special case of the generalized algorithms in \[12\], but the importance of this special case will be clarified for the first time in this paper. The function \( J_2^{(\lambda)}(q_{XY}, \hat{p}_{X|Y}) \) used for Algorithm 2 plays an important role for the error exponent, too. This topic is described in Section IV. We then compare Tridenski and Zamir’s algorithm and Algorithm 2 with Arimoto’s and our previously proposed algorithm.

Before proceeding to Section III-A we review the basic properties of Dueck and Körner’s exponent, which are commonly utilized in all the algorithms in Fig. \[1\] except for Arimoto’s algorithm. Put \( \Gamma_{\min} = \min_{x \in X} c(x) \). Let \( C(\Gamma|W) = \sup_{p_X \in P(X)} I(p_X, W) \) be the channel capacity under input constraint \( (c, \Gamma) \). Dueck and Körner’s exponent satisfies the following property.

Property 1:
(a) For a fixed \( \Gamma \geq \Gamma_{\min} \), \( G_{\text{DK}}(R, \Gamma|W) \) is monotone non-decreasing function of \( R \). For a fixed \( R \geq 0 \), \( G_{\text{DK}}(R, \Gamma|W) \) is monotone non-increasing function of \( \Gamma \).

(b) \( G(R, \Gamma|W) \) is a convex function of \( (R, \Gamma) \).

(c) For \( 0 \leq R \leq C(\Gamma|W) \), \( G_{\text{DK}}(R, \Gamma|W) = 0 \) and for \( R > C(\Gamma|W) \), \( G_{\text{DK}}(R|W) \) is strictly positive.

Proof: See [20] for the proof.

Then, we give a parametric expression of the exponent function. To this aim, we define the following functions:

**Definition 10:** For any fixed \( \lambda \in [0, 1] \), we define

\[
G_{\text{DK}}^{(\lambda)}(\Gamma|W) = \min_{q_{XY} \cdot q_X[c(X)] \leq \Gamma} \{ D(q_{Y|X}||W|q_X) - \lambda I(q_X, q_{Y|X}) \}.
\]  

(12)

**Definition 11:** For fixed \( \lambda \in [0, 1] \) and \( \mu \geq 0 \), we define

\[
\Theta^{(\lambda, \mu)}(q_{XY}|W) = D(q_{Y|X}||W|q_X) - \lambda I(q_X, q_{Y|X}) + \mu E_{q_X}[c(X)]
\]

\[
= E_{q_{XY}} \left[ \log \frac{q_{Y|X}(Y|X)q_{X}(X)}{W(Y|X)e^{-\mu c(X)}} \right],
\]  

(13)

\[
\Theta^{(\lambda, \mu)}(W) \triangleq \min_{q_{XY}} \Theta^{(\lambda, \mu)}(q_{XY}|W).
\]  

(14)

It was shown [6] Appendix C that, for a fixed \( \lambda \in [0, 1] \) and \( \mu \geq 0 \), the function \( \Theta^{(\lambda, \mu)}(q_{XY}|W) \) is convex in \( q_{XY} \). We define the following function.

The following lemma states that the Dueck and Körner’s exponent is obtained by evaluating the minimum of \( \Theta^{(\lambda, \nu)}(q_{XY}|W) \).

**Lemma 1 ([9]):** For any fixed \( \lambda \in [0, 1] \), \( \Gamma \geq 0 \), and \( W \in \mathcal{P}(Y|X) \), we have

\[
G_{\text{DK}}^{(\lambda)}(\Gamma|W) = \sup_{\mu \geq 0} \{-\mu \Gamma + \Theta^{(\lambda, \mu)}\}.
\]  

(15)

For any \( R \geq 0 \), \( \Gamma \geq 0 \) and \( W \in \mathcal{P}(Y|X) \), we have

\[
G_{\text{DK}}(R, \Gamma|W) = \max_{0 \leq \lambda \leq 1} \{ \lambda R + G_{\text{DK}}^{(\lambda)}(\Gamma|W) \}
\]

(16)

\[
= \max_{0 \leq \lambda \leq 1} \sup_{\mu \geq 0} \left\{ \lambda R - \mu \Gamma + \Theta^{(\lambda, \mu)}(W) \right\}.
\]  

(17)

Computation of the exponent functions involves the Legendre-Fenchel transformation (LFT) [21]. The LFT of a function \( F(x) \) of an \( n \)-dimensional vector \( x \) is defined by

\[
F^*(y) = \sup_{x \in \mathbb{R}^n} \{ x^T y - F(x) \},
\]  

(18)

where \( x^T \) is the transpose of \( x \). We use the LFT to derive a parametric expression of the exponent functions, by considering supporting lines to the curve of exponent functions. This is the same as the computation of the rate distortion function and the channel capacity under cost-constraint, which is obtained by considering the supporting lines of the rate-distortion function and the capacity-cost function [2 Chapter 8].

By comparing Eq.(15) with (18), we see that for fixed \( \lambda \in [0, 1] \), \( G_{\text{DK}}^{(\lambda)}(\Gamma|W) \) as a function of \( y = \Gamma \) is the one-dimensional LFT of \( -\Theta^{(\lambda, \mu)}(W) \) as a function of \( x = -\mu \in (-\infty, 0] \). Then, by comparing Eq.(16) with (18), we see that for a fixed \( \Gamma \), \( G_{\text{DK}}(R, \Gamma|W) \) is the one-dimensional LFT of \( -G_{\text{DK}}^{(\lambda)}(\Gamma|W) \) as a function of
Strong converse exponent functions and generalized cut-off rate in channel coding

![Diagram showing the strong converse exponent function](image)

Fig. 3. The strong converse exponent $G_{DK}(R, \Gamma|W)$ for a fixed $\Gamma$ and the generalized cutoff rate

$\lambda \in [0, 1]$. This implies that $G_{DK}(R, \Gamma|W)$ is computed by the two-dimensional LFT of $-\Theta^{(\lambda, \mu)}(W)$ as a function of $x = (\lambda, -\mu) \in [0, 1] \times (-\infty, 0]$. The numerical computation of the LFT can be performed efficiently. See, for example, [22]. Thus, the target for the algorithm is to compute an optimal $q_{XY} = q_{XY}^{\ast}(\lambda, \mu)$ that attains $\Theta^{(\lambda, \mu)}(W)$ for any given $\lambda \in [0, 1]$ and $\mu \geq 0$. Fig. 3 shows a rough sketch of the error and the strong converse exponents in channel coding. In Fig. 3, $\Gamma \geq 0$ is fixed, $G_{DK}(R, \Gamma|W)$ as a function of $R$ is depicted by a solid curve, and its supporting line of slope $\lambda$, expressed by $\lambda R + G_{DK}^{(\lambda)}(\Gamma|W)$, is by dotted line. The $R$-axis intercept of this line is called the generalized cutoff rate [23], denoted by $C^{(\lambda)}(\Gamma|W)$ in this figure. If $R \leq C(\Gamma|W)$, $G_{DK}(R, \Gamma|W) = 0$, while if $R \geq R^{\ast}$, $G_{DK}(R, \Gamma|W) = R + G_{DK}^{(\lambda)}(\Gamma|W)$, where $R^{\ast} = \left[ \frac{\partial}{\partial \lambda} \{G_{DK}^{(\lambda)}(\Gamma|W)\} \right]_{\lambda \rightarrow 1^{-}}$. See [6, Lemma 10] for detail.

For given $\lambda \in (0, 1)$, $\mu \geq 0$ and $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$, we do not have general formula for the explicit expression of the optimal joint distribution that attains [14]. For $\lambda = 0$ and $1$, one can easily see that $\Theta^{(0, \mu)}(W) = \min_{q_{XY}} \{D(q_{Y|X}|W|q_{X}) + \mu E_{q_{X}}[c(X)]\} = \mu \ln\max_x c(x) = \mu \Gamma_{\min}$ and $\Theta^{(1, \mu)}(W)$ is evaluated as follows:

$$
\begin{align*}
\Theta^{(1, \mu)}(W) &= \min_{q_{XY}} \Theta^{(1, \mu)}(q_{XY}|W) = \min_{q_{XY}} E_{q_{XY}} \left[ \log \frac{q_{Y}(Y)}{W(Y|X)e^{-\mu c(X)}} \right] \\
&= \min_{q_{XY}} \sum_y q_{Y}(y) \left\{ \log q_{Y}(y) - \sum_x q_{X|Y}(x|y) \log W(y|x)e^{-\mu c(x)} \right\} \\
&= \min_{q_{Y}} \sum_y q_{Y}(y) \left[ \log q_{Y}(y) - \log \max_x W(y|x)e^{-\mu c(x)} \right] \\
&= -\log \max_y W(y|x)e^{-\mu c(x)}, \quad (19)
\end{align*}
$$

where (a) holds with equality if and only if $q_{X|Y}(x'|y) = 1$ for $(x', y)$ such that $x' = \arg \max_x W(y|x)$ for all $y \in \mathcal{Y}$ and (b) holds with equality if and only if $q_{Y}(y) = \max_y W(y|x)e^{-\mu c(x)}$.

We now perform a change of variables: let $\mu = \lambda \nu$, and instead of the parameter pair $(\lambda, \mu)$, let $(\lambda, \nu)$ be the variable parameter pair. By doing so, in the limit of $\lambda \rightarrow 0$, one of the new algorithms is reduced to the
Arimoto-Blahut algorithm that computes the channel capacity under cost constraints.

In the following subsections, we describe algorithms shown in Fig. 1. In Section III-A, Tridenski and Zamir’s algorithm is explained. Then, Algorithm 2 is explained in Section III-B. Comparison with Arimoto’s and Oohama and the author’s algorithms are given in Sections III-C and III-D.

A. Tridenski and Zamir’s algorithm

Tridenski and Zamir proposed two types of algorithms. One is the algorithm for computing Tridenski and Zamir’s exponent (6) for fixed $R$ without using the slope parameter [7], [8] and the other is for fixed slope [8, Section VII]. This subsection describes the latter and shows its attractive properties, which have not been discussed so far.

For describing Tridenski and Zamir’s algorithm, we need to show that $G_{TZ}(R, \Gamma | W) = G_{DK}(R, \Gamma | W)$.

Tridenski and Zamir introduced the following function [8, Section VII]:

$$F^{(\lambda)}_{TZ}(q_{XY}, p_X | W) = D(q_{XY} || p_X \circ W) - \lambda D(q_{XY} || p_X \times q_Y)$$  (20)

The following lemma holds.

**Lemma 2:** For fixed $\lambda \in [0, 1)$ and $q_{XY}$, $F^{(\lambda)}_{TZ}(q_{XY}, p_X | W)$ is minimized by $p_X = q_X$ and its minimum value is

$$F^{(\lambda)}_{TZ}(q_{XY}, q_X | W) = D(q_{Y|X} || W|q_X) - \lambda I(q_X, q_{Y|X}).$$  (21)

**Proof:** By the definition of the ordinary and the conditional divergences, we have

$$F^{(\lambda)}_{TZ}(q_{XY}, p_X | W) = E_{q_{XY}} \left[ \frac{q_{XY} (X, Y)}{p_X (X) W(Y|X)} - \lambda \frac{q_X^\lambda (X)}{p_X^\lambda (X)} \right]$$

$$= E_{q_{XY}} \left[ \frac{q_X^1 - \lambda}{p_X^1 - \lambda} \frac{q_Y^\lambda (Y|X) q_X^\lambda (X)}{W(Y|X)} \frac{q_Y^1 - \lambda (Y)}{p_Y^1 - \lambda (Y)} \right]$$

$$= D(q_{Y|X} || W|q_X) - \lambda I(q_X, q_{Y|X}) + (1 - \lambda) D(q_X || p_X)$$  (22)

Because $D(q_X || p_X)$ is non-negative and takes zero if and only if $p_X = q_X$, we have $\min_{p_X} F^{(\lambda)}_{TZ}(q_{XY}, p_X | W) = D(q_{Y|X} || W|q_X) - \lambda I(q_X, q_{Y|X})$, which completes the proof.

Then we have the following lemma.

**Lemma 3:** For any $R \geq 0$ and $\Gamma \geq 0$, we have

$$G_{DK}(R, \Gamma | W) = G_{TZ}(R, \Gamma | W)$$  (23)

In [8], [10], the proof of the the match of $G_{TZ}(R, \Gamma | W)$ and $G_{AR}(R, \Gamma | W)$ for a DMC without input-cost was given. Because the match of $G_{AR}(R, \Gamma | W)$ and $G_{DK}(R, \Gamma | W)$ is a known fact [24], this suggests $G_{TZ}(R, \Gamma | W) = G_{DK}(R, \Gamma | W)$, too. However, it should be better if we can show this match directly.
Lemma 2: Because the expression in the parenthesis is convex in $x$, Step (a) and (d) follows from the identity

The Tridenski and Zamir’s original algorithm in [8] used the function $F_{ij}(\lambda, \nu)$, Eq. (26) shows that the function $J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W)$ is the extension of $F_{ij}^{(\lambda, \nu)}(q_{XY}, p_{X} | W)$ to channels under input cost.

**Property 2:** For any fixed $-1 < \lambda < 0$, $\nu > 0$ and any fixed $p_{X}$, $J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W)$ is convex in $q_{XY}$. For any fixed $-1 < \lambda < 0$, $\nu > 0$ and any fixed $q_{XY}$, $J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W)$ is convex in $p_{X}$.

See Appendix A for the proof.

We have the following lemma.

**Lemma 4:** For any fixed $\lambda \in [0, 1]$, $\nu > 0$, and any fixed $q_{XY} \in P(\mathcal{X} \times \mathcal{Y})$, $J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W)$ is minimized by $p_{X} = q_{X}$ and its minimum value is

$$J_1^{(\lambda, \nu)}(q_{XY}, q_{X} | W) = \Theta^{(\lambda, \nu)}(q_{XY} | W).$$  

Proof: We have the following chain of equalities:

$$G_{TZ}(R, \Gamma | W)$$

$$= \min_{q_{XY}: \mathbb{E}[X] \leq \Gamma} \min_{p_{X}} \left\{ D(q_{XY} \parallel p_{X} \circ W) + |R - D(q_{XY} | p_{X})|^{+} \right\}$$

(a) $\min_{q_{XY}: \mathbb{E}[X] \leq \Gamma} \min_{p_{X}} \left\{ D(q_{XY} \parallel p_{X} \circ W) + \max_{0 \leq \lambda \leq 1} \lambda (|R - D(q_{XY} \parallel p_{X} \times q_{Y})|) \right\}$

(b) $\min_{q_{XY}: \mathbb{E}[X] \leq \Gamma} \max_{0 \leq \lambda \leq 1} \min_{p_{X}} \left\{ \lambda R + D(q_{XY} \parallel p_{X} \circ W) - \lambda D(q_{XY} \parallel p_{X} \times q_{Y}) \right\}$

(c) $\min_{q_{XY}: \mathbb{E}[X] \leq \Gamma} \max_{0 \leq \lambda \leq 1} \left\{ \lambda R + D(q_{XY} \parallel W | q_{X}) - \lambda I(q_{X}, q_{Y} | X) \right\}$

(d) $\min_{q_{XY}: \mathbb{E}[X] \leq \Gamma} \left\{ D(q_{XY} \parallel W | q_{X}) + |R - I(q_{X}, q_{Y} | X)|^{+} \right\}$

$$= G_{DK}(R, \Gamma | W)$$

(24)

Step (a) and (d) follows from the identity $[x]^{+} = \max_{0 \leq \lambda \leq 1} \lambda x$. Step (b) follows from the mini-max theorem because the expression in the parenthesis is convex in $p_{X}$ and linear in $\lambda$ for a fixed $q_{XY}$. Step (c) follows from Lemma 2. 

Now, we describe Tridenski and Zamir’s algorithm for fixed slope parameter $\lambda$. In this paper, the algorithm for the channel under input cost $\mathcal{W}$ is discussed. To this aim, we define

$$J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W) = \Theta^{(\lambda, \nu)}(q_{XY} | W) + (1 - \lambda) D(q_{X} \parallel p_{X}).$$  

(25)

Tridenski and Zamir’s algorithm is based on the double minimization of $J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W)$ with respect to $q_{XY}$ and $p_{X}$. From Lemma 2 and the definition of $\Theta^{(\lambda, \nu)}(q_{XY} | W)$, for any $\lambda \in [0, 1]$ and $\nu \geq 0$, we have

$$J_1^{(\lambda, \nu)}(q_{XY}, p_{X} | W) = F_{ij}^{(\lambda)}(q_{XY}, p_{X} | W) + \lambda \nu \mathbb{E}_{q_{X}}[c(X)].$$  

(26)
This implies that
\[
\min_{q_{XY}} \min_{p_X} J_1^{(\lambda, \nu)}(q_{XY}, p_X | W) = \min_{q_{XY}} J_1^{(\lambda, \nu)}(q_{XY}, q_X | W) = \min_{q_{XY}} \Theta^{(\lambda, \lambda \nu)}(q_{XY}) - D(q_{XY} || p_X) = \Theta^{(\lambda, \lambda \nu)}(W).
\]  

**Proof:** This lemma follows from Eq. (25) and the condition for \( D(q_X || p_X) = 0 \).

We also have the following lemma. The case of DMC without cost constraint was [8, Lemma 5].

**Lemma 5:** For any fixed \( \lambda \in [0, 1) \), \( \nu \geq 0 \) and any fixed \( p_X \), \( F_{TZ}^{(\lambda, \nu)}(q_{XY}, p_X | W) \) is minimized by

\[
q_{X|Y}(x|y) = \frac{p_X(x)\{W(y|x)e^{-\lambda \nu c(x)}\}^{\frac{1}{1-\lambda}}}{\sum_{x'} p_X(x')\{W(y|x')e^{-\lambda \nu c(x')}\}^{\frac{1}{1-\lambda}}},
\]

\[
q_Y(y) = \frac{\sum_x p_X(x)\{W(y|x)e^{-\lambda \nu c(x)}\}^{\frac{1}{1-\lambda}}}{\sum_{y'} \sum_x p_X(x)\{W(y'|x)e^{-\lambda \nu c(x)}\}^{\frac{1}{1-\lambda}}}. \tag{29}
\]

Denote the joint distribution computed from the above \( q_{X|Y} \) and \( q_Y \) by \( q_{XY}^* \). The minimum value of \( J_1^{(\lambda, \nu)}(q_{XY}, p_X | W) \) for a fixed \( p_X \) is

\[
J_1^{(\lambda, \nu)}(q_{XY}^*, p_X | W) = -\log \sum_y \left[ \sum_x p_X(x)\{W(y|x)e^{-\lambda \nu c(x)}\}^{\frac{1}{1-\lambda}} \right]^{1-\lambda}.
\]

\[
= E_0^{(-\lambda, \nu)}(p_X | W). \tag{30}
\]

This implies that

\[
\min_{q_{XY}} \min_{p_X} J_1^{(\lambda, \nu)}(q_{XY}, p_X | W) = \min_{p_X} J_1^{(\lambda, \nu)}(q_{XY}^*, p_X | W) = \min_{p_X} E_0^{(-\lambda, \nu)}(p_X | W). \tag{31}
\]

See [8] for the proof.

Tridenski and Zamir’s algorithm is shown in Algorithm [1] Here Eq. (35) is a computation of the marginal distribution from \( (q_0^{[t]}, q_{X|Y}^{[t]}) \).

The convergence theorem of this algorithm is stated as follows:

**Theorem 1** (Theorem 8 in [8]): \( J_1^{(\lambda, \nu)}(q_{XY}^{[t]}, p_X^{[t]} | W) \) converges to

\[
\min_{q_X : \text{supp}(q_X) \subseteq \text{supp}(q_0^{[0]})} E_0^{(-\lambda, \nu)}(p_X | W)
\]

from above. Here \( \text{supp}(q_0^{[0]}) \) denotes the support of \( q_0^{[0]} \in \mathcal{P}(X) \).

Now, we can state the following attractive property of Tridenski and Zamir’s algorithm:
Lemma 4 and (32) in Lemma 5. This completes the proof.

\[ \geq \Theta \] and the

Proposition 1: For \( t = 1, 2, \ldots \), we have

\[
\begin{align*}
J_1^{(\lambda, \nu)}(q_{X|Y}^{[0]}, p_X^{[0]}|W) &\geq J_1^{(\lambda, \nu)}(q_{X|Y}^{[0]}, p_X^{[1]}|W) \geq J_1^{(\lambda, \nu)}(q_{X|Y}^{[1]}, p_X^{[1]}|W) \ldots \\
&\geq J_1^{(\lambda, \nu)}(q_{X|Y}^{[t]}, p_X^{[t]}|W) = E_0^{(-\lambda, \nu)}(p_X^{[t]}|W) \\
&\geq J_1^{(\lambda, \nu)}(q_{X|Y}^{[t+1]}, p_X^{[t+1]}|W) = \Theta^{(\lambda, \nu)}(q_{X|Y}^{[t]}|W) \\
&\geq J_1^{(\lambda, \nu)}(q_{X|Y}^{[t+1]}, p_X^{[t+1]}|W) = E_0^{(-\lambda, \nu)}(p_X^{[t+1]}|W) \geq \ldots \\
&\geq \min_{q_{X|Y}, p_X} \Theta^{(\lambda, \nu)}(q_{X|Y}|W) = \min_{p_X} E_0^{(-\lambda, \nu)}(p_X|W).
\end{align*}
\]

Proof: Step (a) follows from Lemma 4 and step (b) follows from Lemma 5 Step (c) follows from (27) in Lemma 4 and (32) in Lemma 5 This completes the proof.

A feature of Algorithm 1 indicated by Proposition 1 is that the \( E_0 \)-function for Arimoto’s strong converse exponent and the \( \Theta^{(\lambda)}(q_{XY}|W) \) for the Csiszár and Kőrner’s exponent appear alternately. The convergence of Algorithm 1 immediately gives the proof of the fact that Dueck and Kőrner’s and Arimoto’s exponents coincide, stated in the following proposition.

\[ G_{DK}(R, \Gamma|W) = G_{AR}(R, \Gamma|W). \]
Proof: We have the following chain of equalities:
\[
G_{AR}(R, \Gamma|W) \equiv (a) \sup_{\lambda \in [0, 1]} \sup_{\nu \geq 0} \min_{p_X} E_0^{(-\lambda, \nu)}(p_X|W) + \lambda R - \lambda \nu \Gamma \\
\equiv (b) \sup_{\lambda \in [0, 1]} \sup_{\nu \geq 0} \min_{q_{XY}} \Theta^{(\lambda, \nu)}(q_{XY}|W) + \lambda R - \lambda \nu \Gamma \\
\equiv (c) \max_{\lambda \in [0, 1]} \sup_{\nu \geq 0} \min_{q_{XY}} \Theta^{(\lambda, \nu)}(q_{XY}|W) + \lambda R - \lambda \nu \Gamma \\
\equiv (d) G_{DK}(R, \Gamma|W).
\]

Step (a) follows from definition [3] Step (b) follows from the equation (c) in Proposition [1] Step (c) holds because we have \( \lim_{\lambda \to 1} \Theta^{(\lambda, \nu)}(q_{XY}, p_X|W) = \Theta^{(1, \nu)}(q_{XY}, p_X|W) \) for a fixed \( q_{XY} \), and Step (d) follows from Definition 5.

The match of Dueck and Körner’s and Arimoto’s strong converse exponents is well-known but no proof was written in [4]. An explicit proof for the channel with cost constraint is found in [24]. In the above proof, the convergence of Tridenski and Zamir’s algorithm plays a critical role. This proof is interesting because it differs significantly from previously known proofs.

In Section III-C, we will compare Tridenski and Zamir’s algorithm with Arimoto’s algorithm, where update rule (33) is the same as in Arimoto algorithm. Another update rule (34) is not explicitly used in the Arimoto algorithm, but we will give the view that (34) is a hidden update rule of the Arimoto algorithm.

B. Algorithm 2

In [12], Tridenski et al. generalized the computation algorithm so that it includes the algorithms [5], [6] and [8] as special cases. As in the case of [7], [8], there are two types of the generalized algorithms. One is for a fixed rate \( R \) and the other is for a fixed slope parameter \( \lambda \). The latter is discussed here. The objective function is [12, Eq.(24)]
\[
F_{TSZ}^{(\lambda, \nu, t)}(q_{XY}, p_{XY}|W) = \Theta^{(\lambda, \nu)}(q_{XY}|W) + (1 - \lambda)D^t(q_{XY}, p_{XY}),
\]
where \( t = (t_1, t_2, t_3, t_4) \) is a vector of four non-negative coefficients and
\[
D^t(q_{XY}, p_{XY}) = t_1 D(q_X||p_X) + t_2 D(q_{Y|X}||p_{Y|X}|q_X) + t_3 D(q_Y||p_Y) + t_4 D(q_{X|Y}||p_{X|Y}|q_Y).
\]
It follows from the definition that \( \min_{p_{XY}} F_{TSZ}^{(\lambda, \nu, t)}(q_{XY}, p_{XY}|W) = \Theta^{(\lambda, \nu)}(q_{XY}|W) \) holds for \( 0 \leq \lambda < 1 \) irrespective of \( t.i.s \). On the other hand, \( \min_{q_{XY}} F_{TSZ}^{(\lambda, \nu, t)}(q_{XY}, p_{XY}|W) \) is parameterized by the four parameters \( t_1, t_2, t_3, t_4 \). It was shown [12] that explicit update rules are available in two cases\(^2\). One is the case of \( t_1 = t_2 + 1 \) with \( t_2, t_3, t_4 \geq 0 \), and the other is the case of \( t_4 = t_3 + \lambda/(1 - \lambda) \) with \( t_1, t_2, t_3 \geq 0 \). They are denoted as Case 1 and Case 2 in Fig. [4].

The probability update rules shown in [12, Lemma 3] appear to be erroneous. We denote the probability update rules for Cases 1 and 2 in Appendix I.

\(^2\)The probability update rules shown in [12, Lemma 3] appear to be erroneous. We denote the probability update rules for Cases 1 and 2 in Appendix I.
Dueck & Körner’s Exponent
Parameter $t = t_1, t_2, t_3, t_4$
Case 1 Case 2
Explicit update is possible

Fig. 4. The double minimization of the generalized algorithm [12] with parameters $t_1, t_2, t_3, t_4 \geq 0$ and the expressions of the strong converse exponent derived from it. For the case 1 with $t = 1 = t_2 + 1$ and the case 2 with $t_4 = t_3 + \lambda/(1 - \lambda)$, the distribution update is explicit.

will be described in Section III-D corresponds to the case $t = (1, 0, 0, \lambda/(1 - \lambda))$, while Tridenski and Zamir’s algorithm [8] corresponds to the case $t = (1, 0, 0, 0)$.

In this subsection, we consider another special case, $t = (0, 0, 0, \lambda/(1 - \lambda))$. This case is important in the sense that during the alternate updates of the probability distribution, the corresponding algorithm, which is Algorithm 2, alternates between the form of the Dueck and Körner’s exponents and the form of the objective function of the alternative representation of the Arimoto’s strong converse exponent.

We use the following objective function of $q_{XY}$ and $p_{X|Y}$:

$$J_2^{(\lambda, \nu)}(q_{XY}, \hat{p}_{X|Y}|W) = \Theta^{(\lambda, \nu)}(q_{XY}|W) + \lambda D(q_{X|Y}||\hat{p}_{X|Y}|q_Y)$$

$$= E_{q_{XY}} \left[ \log \frac{q_{Y|X}(Y|X)q_X^\lambda(X)}{\hat{p}_{X|Y}^\lambda(X|Y)W(Y|X)e^{-\lambda \nu c(X)}} \right] \quad (40)$$

The reason why the conditional distribution is denoted as $\hat{p}_{X|Y}$ instead of $p_{X|Y}$ in (40) is to unify the notation of probability distributions in Fig 5. The first thing to check is the convexity of this function. Here, we check it for $\lambda \in [-1, 1]$. Positive $\lambda$ is for strong converse exponent and negative $\lambda$ is for error exponent which is the topic of Section IV.

Property 3:

a) For any fixed $\lambda \in [0, 1]$ (resp. $\lambda \in [-1, 0]$), $\nu \geq 0$, and any fixed $q_{XY}$, $J_2^{(\lambda, \nu)}(q_{XY}, \hat{p}_{X|Y}|W)$ is convex (resp. concave) in $\hat{p}_{X|Y}$.

b) For any fixed $\lambda \in [0, 1]$ (resp. $\lambda \in [-1, 0]$), $\nu \geq 0$, any fixed $\hat{p}_{X|Y}$, and any fixed $q_{Y|X}$, $J_2^{(\lambda, \nu)}((q_{X}, q_{Y|X}), \hat{p}_{X|Y}|W)$ is convex (resp. concave) in $q_{X}$.

c) For any fixed $-1 \leq \lambda \leq 1$, $\nu \geq 0$, any fixed $q_{X}$ and $p_{X|Y}$, $J_2^{(\lambda, \nu)}((q_{X}, q_{Y|X}), \hat{p}_{X|Y}|W)$ is convex in $q_{Y|X}$.

See Appendix A.
Definition 12: For a given transition probability \( W(y|x) \) and \( 0 < |\rho| \leq 1, \nu \geq 0 \), we define

\[
A^{(\rho,\nu)}(\hat{p}_X|Y|W) = \rho \log \sum_x \left[ \sum_y \hat{p}_{X|Y}(x|y)W(y|x)e^{\nu c(x)} \right]^{-1/\rho}.
\]

This implies that

\[
\min_{q_{XY}} \min_{\hat{p}_X|Y} J_2^{(\lambda,\nu)}(q_{XY}, \hat{p}_X|Y|W) = \min_{q_{XY}} J_2^{(\lambda,\nu)}(q_{XY}, q_{X|Y}|W) = \min_{q_{XY}} \Theta^{(\lambda,\nu)}(q_{XY}|W).
\]

Proof: It is obvious from the property of \( D(q_{X|Y}||\hat{p}_X|Y|q_{Y}) \).

Lemma 6: For any fixed \( \lambda \in (0, 1], \nu \geq 0 \) and any fixed \( q_{XY} \), \( J_2^{(\lambda,\nu)}(q_{XY}, \hat{p}_X|Y|W) \) is minimized by \( \hat{p}_X|Y = q_{X|Y} \) and the minimum value is

\[
J_2^{(\lambda,\nu)}(q_{XY}, q_{X|Y}|W) = \Theta^{(\lambda,\nu)}(q_{XY}|W).
\]

Lemma 7: For any fixed \( \lambda \in (0, 1], \nu \geq 0 \), and any fixed \( p_{X|Y} \), \( J_2^{(\lambda,\nu)}(q_{XY}, \hat{p}_X|Y|W) \) is minimized by

\[
q_{Y|X}(y|x) = \frac{\hat{p}_X^\lambda(x|y)W(y|x)e^{-\lambda \nu c(x)}}{\sum_{y'} p_{X|Y}(x|y')W(y'|x)e^{-\lambda \nu c(x')}},
\]

\[
q_X(x) = \frac{\left\{ \sum_y \hat{p}_X^\lambda(x|y)W(y|x)e^{-\lambda \nu c(x)} \right\}^{1/\lambda}}{\sum_{x'} \left\{ \sum_y \hat{p}_X^\lambda(x'|y)W(y|x')e^{-\lambda \nu c(x')} \right\}^{1/\lambda}}.
\]

Denote the joint distribution calculated from the above \( q_{Y|X} \) and \( q_X \) by \( \tilde{q}_{XY}(p_{X|Y}) \). Then, the minimum value of \( J_2^{(\lambda,\nu)}(q_{XY}, \hat{p}_X|Y|W) \) for fixed \( p_{X|Y} \) is

\[
J_2^{(\lambda,\nu)}(\tilde{q}_{XY}(p_{X|Y}), \hat{p}_X|Y|W) = -\lambda \log \sum_x \left[ \sum_y \hat{p}_X^\lambda(x|y)W(y|x)e^{-\lambda \nu c(x)} \right]^{1/\lambda}
\]

\[
= A^{(-\lambda,\nu)}(\hat{p}_X|Y|W).
\]

This implies that

\[
\min_{q_{XY}} \min_{\hat{p}_X|Y} J_2^{(\lambda,\nu)}(q_{XY}, \hat{p}_X|Y|W) = \min_{\tilde{p}_{X|Y}} J_2^{(\lambda,\nu)}(\tilde{q}_{XY}(\tilde{p}_{X|Y}), \hat{p}_X|Y|W) = \min_{\tilde{p}_{X|Y}} A^{(-\lambda,\nu)}(\tilde{p}_X|Y|W).
\]

See Appendix B for the proof.
Lemma 6 and (47) in Lemma 7. This completes the proof. 

Input: The conditional probability of the channel \( W, \lambda \in (0, 1) \), and \( \nu \geq 0 \). Choose any initial joint probability distribution \( p_{X|Y}^{[0]} \), such that all \( p_{X|Y}^{[0]}(x|y) \) are positive.

\[
\text{for } t = 0, 1, 2, \ldots, \text{ do}
\]

\[
q_{Y|X}^{[t]}(y|x) = \frac{\hat{p}_{X|Y}^{[t]}(x|y) \lambda W(y|x)e^{-\lambda \nu c(x)}}{\sum_{y'} \hat{p}_{X|Y}^{[t]}(x|y') \lambda W(y'|x)e^{-\lambda \nu c(x)}},
\]

\[
q_{X}^{[t]}(x) = \frac{e^{-\nu c(x)} \left\{ \sum_{y} \hat{p}_{X|Y}^{[t]}(x|y) \lambda W(y|x) \right\}^{1/\lambda}}{\sum_{x'} e^{-\nu c(x')} \left\{ \sum_{y} \hat{p}_{X|Y}^{[t]}(x'|y) \lambda W(y|x') \right\}^{1/\lambda}},
\]

\[
\hat{p}_{X|Y}^{[t+1]}(x|y) = \frac{q_{Y|X}^{[t]}(y|x) q_{X}^{[t]}(x)}{\sum_{x'} q_{Y|X}^{[t]}(y|x') q_{X}^{[t]}(x')}.
\]

end for

Proposition 3: For \( t = 1, 2, \ldots, \) we have

\[
J_2^{(\lambda,\nu)}(q_{XY}^{[0]}, p_{X|Y}^{[0]}|W) \geq J_2^{(\lambda,\nu)}(q_{XY}^{[1]}, \hat{p}_{X|Y}^{[1]}|W) \geq J_2^{(\lambda,\nu)}(q_{XY}^{[2]}, \hat{p}_{X|Y}^{[2]}|W) \ldots
\]

(proof follows from Lemma 6 and step (b) follows from Lemma 7). This completes the proof.

Proposition 2 shows that the objective function takes the forms of the alternative expression of Arimoto’s exponent and the parametric expression of Dueck and Körner’s exponent alternately and that \( J_2^{(\lambda,\nu)}(q_{XY}^{[t]}, \hat{p}_{X|Y}^{[t]}|W) = \Theta^{(\lambda,\lambda \nu)}(q_{XY}^{[t]}|W) \) monotonically decreases. Next theorem shows that \( \Theta^{(\lambda,\lambda \nu)}(q_{XY}^{[t]}|W) \) converges to \( \Theta^{(\lambda,\lambda \nu)}(W) \).

Theorem 2: For any \( \lambda \in (0, 1], \nu \geq 0 \) and \( W \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \), the series of distributions \( q_{XY}^{[t]} \) defined by (48)–(50) converges to an optimal distribution \( q_{XY}^* \) that minimizes \( \Theta^{(\lambda,\lambda \nu)}(q_{XY}|W) \). The approximation error \( \Theta^{(\lambda,\lambda \nu)}(q_{XY}^{[t]}|W) - \Theta^{(\lambda,\lambda \nu)}(q_{XY}^*|W) \) is inversely proportional to the number of iterations.

The proof appears in Appendix C.
opposite cases. In the limit of $\lambda \to 0$, (49) becomes

$$q_X^{[t]}(x) = \frac{e^{-\nu \rho(x)} \prod_y p_X^{[t]}(x|y) W(y|x)}{\sum_{x'} e^{-\nu \rho(x')} \prod_y p_X^{[t]}(x|y) W(y|x)}$$

and Algorithm 2 reduces to the Arimoto-Blahut algorithm for computing the channel capacity.

C. Comparison with Arimoto algorithm for channel coding exponents

This subsection gives comparisons between Tridenski and Zamir’s algorithm, Algorithm 2, and Arimoto’s algorithms for channel coding [11]. Arimoto’s algorithm was derived from the optimization problem appeared in [1] and (5). Unlike Tridenski and Zamir’s algorithm and Algorithm 2, Arimoto’s algorithm can be applied to the computation of both the error and the strong converse exponents.

For fixed $\rho$ and $\nu \geq 0$, the function $E^{(\rho,\nu)}_0(p_X|W)$ is concave in $p_X$ when $\rho$ is positive and is convex in $p_X$ when $\rho$ is negative. Arimoto chose $(1/\rho) E^{(\rho,\nu)}_0(p_X|W)$ as an objective function which is concave for positive and negative $\rho$. Arimoto [11] introduced the following function.

Definition 13: For a given transition probability of a channel $W(y|x)$, $-1 < \rho \leq 1$, and $\nu$, we define

$$E^{(\rho,\nu)}_{AR}(p_X, \hat{p}_X|Y|W) = -\frac{1}{\rho} \log \sum_{x \in X} \sum_{y \in Y} p_X^{1+\rho}(x) \hat{p}_X^\rho (x|y) W(y|x) e^{-\lambda \nu c(x)}.$$ (51)

Arimoto’s algorithm is an algorithm that finds

$$\max_{p_X} \max_{\hat{p}_X} E^{(\rho,\nu)}_{AR}(p_X, \hat{p}_X|Y|W)$$ (52)

by optimizing $p_X$ and $\hat{p}_X|Y$ alternately. Arimoto proved the following theorem.

Theorem 3 (Arimoto [11]): For a fixed $-1 < \rho \leq 1$, $\nu \geq 0$ and any fixed $p_X$, we have

$$\frac{1}{\rho} E^{(\rho,\nu)}_0(p_X|W) = \max_{p_X|Y} E^{(\rho,\nu)}_{AR}(p_X, \hat{p}_X|Y|W)$$ (53)

and the maximum value is attained by

$$\hat{p}_X|Y(x|y) = \frac{p_X(x) \{W(y|x) e^{-\lambda \nu c(x)}\}^{1/\rho}}{\sum_{x'} p_X(x') \{W(y|x') e^{-\lambda \nu c(x')}\}^{1/\rho}}.$$ (54)

On the other hand, for a fixed $\rho \in [-1,0) \cup (0,1]$, $\nu \geq 0$ and any fixed $\hat{p}_X|Y$, we have

$$\frac{1}{\rho} A^{(\rho,\nu)}(\hat{p}_X|Y|W) = \max_{p_X} E^{(\rho,\nu)}_{AR}(p_X, \hat{p}_X|Y|W).$$ (55)

The maximum value is attained by

$$p_X(x) = \left( \sum_{y} \hat{p}_X^\rho (x|y) W(y|x) e^{-\lambda \nu c(x)} \right)^{-1/\rho} \left( \sum_{x'} \{ \sum_{y} \hat{p}_X^\rho (x'|y) W(y|x') e^{-\lambda \nu c(x')} \}^{-1/\rho} \right)^{-1/\rho}.$$ (56)

See [11] for the proof. A quick proof will be shown later. Based on Theorem 3, Arimoto presented an iterative algorithm shown in Algorithm 3.
Algorithm 3 Arimoto’s algorithm for error and strong converse exponents in channel coding [11]

Input: The conditional probability of the channel $W$, $\rho \in (-1, 0) \cup (0, \infty)$ and $\nu \geq 0$. Choose initial $\hat{p}_X^{[0]}$ such that all components are nonzero.

for $t = 1, 2, 3, \ldots$ do

\[
\hat{p}_X^{[t]}(x|y) = \frac{p_X^{[t]}(x)\{W(y|x)e^{\rho c(x)}\}^{1/\rho}}{\sum_{x'} p_X^{[t]}(x')\{W(y|\nu e^{\rho c(x')})\}^{1/\rho}},
\]

(57)

\[
\hat{p}_X^{[t+1]}(x) = \frac{\left\{\sum_y \hat{p}_X^{[t]}(y|\nu e^{\rho c(x)})\right\}^{-1/\rho}}{\sum_{x'} \left\{\sum_y \hat{p}_X^{[t]}(y|\nu e^{\rho c(x')})\right\}^{-1/\rho}}.
\]

(58)

end for

When $\rho \to 0$, Arimoto’s extended algorithm reduces to the ordinary Arimoto-Blahut algorithm for computing the channel capacity. For the case of $\rho = 0$, $(1/\rho)E_0^{(\rho, \nu)}(p_X|W)$ is interpreted as its limiting function, i.e.,

\[
\lim_{\rho \to 0} \frac{1}{\rho} E_0^{(\rho, \nu)}(p_X|W) = I(p_X, W) - \nu E_{p_X}[c(X)].
\]

(59)

When $\rho = 0$, $F_{AR}^{(\rho, \nu)}(p_X, \hat{p}_X|Y|W)$ is interpreted as

\[
\lim_{\rho \to 0} F_{AR}^{(\rho, \nu)}(p_X, \hat{p}_X|Y|W)
\]

\[
= \sum_{x \in X} p_X(x)W(y|x) \log \frac{\hat{p}_X|(y|x)e^{-\nu c(x)}}{p_X(x)}.
\]

(60)

The rhs of Eq. (60) is the function that appears in Arimoto-Blahut algorithm for finding the channel capacity. Thus we can regard $F_{AR}^{(\rho, \nu)}(p_X, \hat{p}_X|Y|W)$ as a generalization of (60) to the case $\rho \neq 0$.

From Eqs. (53) and (55) in Theorem 3 we have the following corollaries.

Corollary 1: For any fixed $\rho \in (-1, 0) \cup (0, 1]$ and $\nu \geq 0$, we have

\[
\max_{p_X} \frac{1}{\rho} E_0^{(\rho, \nu)}(p_X|W) = \max_{p_X \in \mathcal{P}_X|Y} F_{AR}^{(\rho, \nu)}(p_X, \hat{p}_X|Y|W)
\]

\[
= \max_{p_X \in \mathcal{P}_X|Y} A^{(\rho, \nu)}(\hat{p}_X|Y|W).
\]

(61)

Corollary 2: Define the following functions.

\[
\tilde{E}_t(R|W) = \max_{\rho \in [0, 1]} \inf_{\nu \geq 0} \left\{ \max_{\tilde{p}_{X,Y} \in \mathcal{P}(X|Y)} A^{(\rho, \nu)}(\tilde{p}_{X,Y}|W) - \rho R + \nu \Gamma \right\},
\]

(62)

\[
\tilde{G}_{AR}(R|W) = \sup_{\rho \in (-1, 0)} \sup_{\nu \geq 0} \left\{ \min_{\tilde{p}_{X,Y} \in \mathcal{P}(X|Y)} A^{(\rho, \nu)}(\tilde{p}_{X,Y}|W) - \rho R + \nu \Gamma \right\}.
\]

(63)

Then, we have

\[
\tilde{E}_t(R, \Gamma|W) = E_t(R, \Gamma|W), \quad \tilde{G}_{AR}(R, \Gamma|W) = G_{AR}(R, \Gamma|W).
\]

(64)

We call $\tilde{E}_t(R, \Gamma|W)$ and $\tilde{G}_{AR}(R, \Gamma|W)$ alternate error exponent and alternate strong converse exponent.

There should be deep insight into how Arimoto came up with $E_0$ function to Eq. (51). Verdú [25], [26] and Cai and Verdú [27] describes many attractive properties of Rényi entropy, Rényi divergence, and exponent function.
One of the important properties discussed in [25]–[27] is that the Gallager’s error exponent is expressed by using Arimoto’s conditional Rényi entropy [28] as well as Sibson’s mutual information of $\alpha$. Denote the Rényi divergence and the conditional Rényi divergence of order $\alpha$ by

$$
D_\alpha(p_X \| q_X) = \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} p_X^\alpha(x)q_X^{1-\alpha}(x),
$$

(65)

$$
D_\alpha(p_{X|Y} \| q_{X|Y}|p_Y) = \frac{1}{\alpha - 1} \log \sum_{y \in \mathcal{Y}} p_Y(y) \sum_{x \in \mathcal{X}} p_{X|Y}^\alpha(x|y)q_{X|Y}^{1-\alpha}(x|y).
$$

(66)

Rényi divergence and conditional Rényi divergence is nonnegative and vanish if and only if $p_X = q_X$ and $p_{X|Y} = q_{X|Y}$. To the best of our knowledge, the following relations have not been pointed out so far.

Lemma 8: For a given $W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$, $p_X \in \mathcal{P}(\mathcal{X})$, and $\hat{p}_{X|Y} \in \mathcal{P}(\mathcal{X}|\mathcal{Y})$, set

$$
\hat{p}_{X|Y}(p_X)(x|y) = \frac{p_X(x) \{ W(y|x)e^{-\rho c(x)} \}^{1/(1+\rho)}}{\sum_{x' \in \mathcal{X}} p_X(x') \{ W(y|x')e^{\rho c(x')} \}^{1/(1+\rho)}},
$$

(67)

$$
\hat{p}_Y(p_X)(y) = \left\{ \frac{1}{\sum_{x \in \mathcal{X}} p_X(x) \{ W(y|x)e^{\rho c(x)} \}^{1/(1+\rho)}} \right\}^{1+\rho},
$$

(68)

$$
\hat{p}_X^\rho(\hat{p}_{X|Y})(x) = \left( \frac{1}{\sum_{x' \in \mathcal{X}} \hat{p}_{X|Y}^\rho(x'|y) \{ W(y|x')e^{\rho c(x')} \}^{1/(1+\rho)}} \right)^{-1/(1+\rho)}.
$$

(69)

Then we have

$$
F_{AR}^{(\rho, \nu)}(p_X, \hat{p}_{X|Y}|W) = \frac{1}{\rho} \rho E_{0}^{(\rho, \nu)}(p_X|W) - D_{1+\rho}(\hat{p}_{X|Y}(p_X) \| \hat{p}_{X|Y}(p_X))
$$

(70)

$$
= \frac{1}{\rho} A^{(\rho, \nu)}(\hat{p}_{X|Y}|W) - D_{1+\rho}(p_X \| \hat{p}_X(\hat{p}_{X|Y})).
$$

(71)

See Appendix 3 for the proof. From this lemma and the property of Rényi divergence, Theorem 3 immediately follows.

It should be noted that although Eq. (33) in Algorithm [1] and Eq. (49) in Algorithm [2] were derived from Dueck and Körner’s exponent function, their functional structures are the same as Eqs. (57) and (58) in Arimoto’s algorithm. Note also that the structure of (68) in Lemma 8 is the same as (34). Arimoto algorithm does not use (68), but it was involved when we analyze the objective function for double maximization in Arimoto algorithm.

D. Comparison with our previous algorithm

This subsection gives a comparison between Tridenski and Zamir’s algorithm, Algorithms 2, and our previous algorithm in [5]. Before [5], no algorithm was known for computing the optimal distribution for Dueck and Körner’s strong converse exponent. The algorithm for DMCs without input cost constraint was extended to the algorithm for DMCs under input cost [6]. We introduced the following function to obtain the minimum of $\Theta^{(\lambda, \nu)}(q_{XY}|W)$.

$$
F_{JO}^{(\lambda, \nu)}(q_{XY}, \hat{q}_{XY}|W) = E_q \left[ \log \frac{q_{Y|X}^{1-\lambda}(Y|X)\hat{q}_{Y}^{\lambda}(Y)}{W(Y|X)e^{-\nu c(x)}} \right] + D(q_{XY} \| \hat{q}_{XY})
$$

(72)

Regarding this function, the following two lemmas hold.
Lemma 9 ([6]): For a fixed \( \lambda \in [0, 1], \nu \geq 0 \) and \( q_{XY} \), \( F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}, \hat{q}_{XY}|W) \) is minimized by \( \hat{q}_{XY} = q_{XY} \) and its minimum value is

\[
F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}, q_{XY}|W) = \Theta^{(\lambda, \nu)}(q_{XY}|W). \tag{73}
\]

This implies that

\[
\min \min_{q_{XY}} F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}, \hat{q}_{XY}|W) = \min_{q_{XY}} F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}, q_{XY}|W) = \min_{q_{XY}} \Theta^{(\lambda, \nu)}(q_{XY}|W). \tag{74}
\]

Lemma 10 ([6]): For a fixed \( \lambda \in [0, 1], \nu \geq 0 \) and \( \hat{q}_{XY} \), \( F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}, \hat{q}_{XY}|W) \) is minimized by

\[
q_{XY}(x, y) = \sum_{x'} \sum_{y'} q_{X}^{1-\lambda}(x) q_{Y}^{\lambda}(y|x) \{ W(y|x) e^{-\lambda \nu c(x)} \}
\]

and its minimum value is

\[
F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}(\hat{q}_{XY}), \hat{q}_{XY}|W)
= -\log E_{\hat{q}_{XY}} \left[ W(Y|X)e^{-\lambda \nu c(x)} \right]
\]

This implies that

\[
\min \min_{q_{XY}} F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}, \hat{q}_{XY}|W) = \min_{q_{XY}} F_{\lambda, \nu}^{(\lambda, \nu)}(q_{XY}(\hat{q}_{XY}), \hat{q}_{XY}|W)
= \min_{q_{XY}} -\log \sum_{x} \sum_{y} q_{X}^{1-\lambda}(x) q_{Y}^{\lambda}(y|x) W(y|x) e^{-\lambda \nu c(x)}. \tag{77}
\]

See [6] for the proof of Lemmas 1, 2, and 3.

The algorithm for computing \( \min_{q_{XY}} \Theta^{(\lambda, \lambda \nu)}(q_{XY}|W) \) is shown in Algorithm 4. In [6], the condition for the termination of the iterative updating is mentioned but is omitted here.

**Algorithm 4** Computation of \( \min_{q_{XY}} \Theta^{(\lambda, \lambda \nu)}(q_{XY}|W) \) [6]

**Input:** The conditional probability of the channel \( W, \lambda \in (0,1) \) and \( \nu \geq 0 \) Choose initial joint probability distribution \( q_{XY}^{[0]} \) such that \( q_{XY}^{[0]}(x, y) = 0 \) if \( W(y|x) = 0 \) and \( q_{XY}^{[0]}(x, y) > 0 \) if \( W(y|x) > 0 \).

**for** \( t = 0, 1, 2, \ldots, \) **do**

\[
q_{XY}^{[t+1]}(x, y) = \frac{q_{X}^{[t]}(x) 1 - \lambda q_{Y}^{[t]}(x|y) W(y|x) e^{-\lambda \nu c(x)}}{\sum_{x'} \sum_{y'} q_{X}^{[t]}(x') 1 - \lambda q_{Y}^{[t]}(x'|y') W(y'|x') e^{-\lambda \nu c(x')}} \tag{78}
\]

**end for**
The important observation here is that the function in the rhs of (76) is exactly the same as the function in the rhs of Eq. (51). Then, it can be seen that the function \( F_{JO}^{(\lambda,\nu)}(q_{XY},\hat{q}_{XY}|W) \) in (72) satisfies

\[
F_{JO}^{(\lambda,\nu)}(q_{XY},\hat{q}_{XY}|W) = E_{\hat{q}_{XY}} \left[ \log \frac{q_{XY}(X,Y)}{q_X^\lambda(X)q_Y^\lambda(Y|X)W(Y|X)e^{-\lambda\nu c(x)}} \right]
\]

\[
= \Theta^{(\lambda,\nu)}(q_{XY}|W) + (1-\lambda)D(q_X || \hat{q}_X) + \lambda D(q_X|Y \parallel \hat{q}_X|Y|q_Y). 
\]

(79)

Here, \( \hat{q}_X \) and \( \hat{q}_{XY|Y} \) are the marginal and the conditional distribution derived from the joint distribution \( \hat{q}_{XY} \). However, what if we can replace \( \hat{q}_{XY|Y} \) with another conditional distribution function \( \hat{p}_{XY|Y} \) that is not related to \( \hat{q}_X \)? By slightly modifying \( F_{JO}^{(\lambda,\nu)}(q_{XY},\hat{q}_{XY}|W) \), we have introduced the following function (\( \hat{q}_X \) is also replaced with \( p_X \)) \([6]\).

\[
\hat{F}_{JO}^{(\lambda,\nu)}(q_{XY},p_X,\hat{p}_{XY|Y}|W) = \Theta^{(\lambda,\nu)}(q_{XY}|W) + (1-\lambda)D(q_X || p_X) + \lambda D(q_X|Y \parallel \hat{p}_{XY|Y}|q_Y). 
\]

(80)

Then, we can easily prove the following lemmas.

**Lemma 11** \([6]\): For a fixed \( \lambda \in [0,1], \nu \geq 0 \) and \( q_{XY} \), we have

\[
\min_{p_X} \min_{\hat{p}_{XY|Y}} \hat{F}_{JO}^{(\lambda,\nu)}(q_{XY},p_X,\hat{p}_{XY|Y}|W) = \Theta^{(\lambda,\nu)}(q_{XY}|W). 
\]

(81)

The minimum is attained if and only if \( p_X = q_X \) and \( \hat{p}_{XY|Y} = q_{XY|Y} \) holds.

**Lemma 12** \([6]\): For a fixed \( \lambda \in [0,1], \nu \geq 0, p_X, \) and \( \hat{p}_{XY|Y} \), we have

\[
\min_{q_{XY}} \hat{F}_{JO}^{(\lambda,\nu)}(q_{XY},p_X,\hat{p}_{XY|Y}|W) = -\lambda F_{AR}^{(-\lambda,\nu)}(p_X,\hat{p}_{XY|Y}|W). 
\]

(82)

The minimum is attained if and only if

\[
q_{XY}(x,y) = \frac{p_X^{1-\lambda}(x)p_{XY}^\lambda(x|y)W(y|x)e^{-\lambda\nu c(x)}}{\sum_{x'} \sum_{y'} p_X^{1-\lambda}(x')p_{XY}^\lambda(x'|y')W(y'|x')e^{-\lambda\nu c(x')}}.
\]

The above is what we discussed in \([6\] Section III-C]. Then, it is obvious that the following two lemmas hold.

**Lemma 13:** For fixed \( \lambda \in [0,1], \nu \geq 0, p_X, \) and \( q_{XY} \), we have

\[
\min_{\hat{p}_{XY|Y}} \hat{F}_{JO}^{(\lambda,\nu)}(q_{XY},p_X,\hat{p}_{XY|Y}|W) = \Theta^{(\lambda,\nu)}(q_{XY}|W) + (1-\lambda)D(q_X || p_X) \]

(83)

**Lemma 14:** For a fixed \( \lambda \in [0,1], \hat{p}_{XY|Y}, \) and \( q_{XY} \), we have

\[
\min_{p_X} \hat{F}_{JO}^{(\lambda,\nu)}(q_{XY},p_X,\hat{p}_{XY|Y}|W) = \Theta^{(\lambda,\nu)}(q_{XY}|W) + \lambda D(q_X|Y \parallel \hat{p}_{XY|Y}|q_Y) \]

(84)

It is seen that the rhs of (83) is \( J_1^{(\lambda,\nu)}(q_{XY},p_X|W) \) and the rhs of (84) is \( J_2^{(\lambda,\nu)}(q_{XY},\hat{p}_{XY|Y}|W) \). This relationship is illustrated in Fig. 5. The triple minimization \( \min_{q_{XY}} \min_{p_X} \min_{\hat{p}_{XY|Y}} \hat{F}_{JO}^{(\lambda,\nu)}(q_{XY},p_X,\hat{p}_{XY|Y}|W) \), shown in the top level, is connected to three double minimization forms. Therefore, these three algorithms are related each
The strong converse exponent for channel coding \((0 < \lambda \leq 1, \nu \geq 0)\)

![Diagram](image_url)

Fig. 5. Relation between several expressions for the strong converse exponent for the channel coding of DMCs

other. We have shown that Tridenski and Zamir’s objective function \(F_{TZ}(q_{XY}, p_X | W)\) defined (20) is equal to \(J_1(\lambda, \nu)(q_{XY}, p_X | W)\). We have derived Algorithm 2 by the double minimization of \(J_2(\lambda, \nu)(q_{XY}, \hat{p}_X | Y)\) with respect to \(q_{XY}\) and \(\hat{p}_X | Y\).

**IV. GALLAGER’S AND Csiszár AND Körner’s ERROR EXPONENTS**

Arimoto’s algorithm is based on the double maximization expression in Eq.(52) for positive and negative \(\rho\) and therefore can be applied to both the error exponent and the strong converse exponent. Algorithms 1, 2, and 4 were developed based on the fact that \(\Theta(\lambda, \mu)(q_{XY} | W)\) is a convex function of \(q_{XY}\) for \(0 \leq \lambda \leq 1\). Comparing Eq.(4) with (5), Eq. (5) is a minimum value of a functional with respect to a joint distributions, whereas Eq.(4) is a saddle point of a functional. At the saddle point, the functional is maximized with respect to the input probability distribution \(q_X\) and it is minimized with respect to a conditional distribution \(q_Y | X\). In order to derive Arimoto-Blahut type algorithm, the exponent function must be defined as a joint maximization problem or a joint minimization problem. Therefore, Algorithms 1, 2, and 4 can only be applied to the strong converse exponent.

Csiszár and Körner’s exponent is expressed also using the function \(\Theta(\lambda, \mu)(q_{XY} | W)\) but with negative \(\lambda\) (we put \(\rho = -\lambda\)). When \(0 \leq \rho \leq 1\), \(\Theta(\rho, \mu)(q_{XY} | W)\) satisfies the following property.

**Property 4:** Suppose \(\rho \in [0, 1]\) is fixed. Then \(\Theta(\rho, \mu)(q_{XY} | W)\) is concave in \(q_X\) for a fixed \(q_Y | X\) and convex in \(q_Y | X\) for a fixed \(q_X\).

**Proof:** By definition, we have \(\Theta(\rho, \mu)(q_{XY} | W) = I(q_X, q_{Y | X}) + \rho D(q_{Y | X} | W | q_X) + \mu E_{q_X}[c(X)]\). The first term is concave in \(q_X\) for a fixed \(q_Y | X\) and is convex in \(q_Y | X\) for a fixed \(q_X\). The second term is convex in \(q_Y | X\) for a fixed \(q_X\) and is a linear function of \(q_X\) for a fixed \(q_Y | X\). The last term is linear in \(q_{XY}\). Therefore, the property holds.

This section shows that we can prove that Csiszár and Körner’s exponent matches with the alternative error exponent expression, defined in (63). Because the alternative expression matches with Gallager’s error exponent,
the proof immediately implies that Csiszár and Körner’s exponent matches with Gallager’s exponent.

The following proposition shows that Csiszár and Körner’s exponent matches with Gallager’s exponent.

**Proposition 4:** For any \( R \geq 0 \) and \( \Gamma \geq 0 \), we have

\[
E_{\text{CK}}(R, \Gamma|W) = E_t(R, \Gamma|W). \tag{85}
\]

In this section, we first review the standard method to proof of Proposition 4. Then, we give the new proof.

A. A proof shown in Exercise 10.24 in [2]

The following steps are suggested in [2, Exercise 10.24] to prove Proposition 4 which states that Csiszár and Körner’s exponent and Gallager’s exponent match. In order to explicitly discuss the difference between this steps and our proof, we describe the steps in detail. Cost constraints essentially have no effect on the proof step. Thus, as in [2, Exercise 10.24], we consider a DMC without cost constraint in this subsection. See also Lapidoth and Miliou [29], showing that these two exponents are Lagrange dual each other.

1) Prove that

\[
E_{\text{CK}}(R|W) = \max_{q_X} \max_{\rho \in [0,1]} \{ \min_{q_Y|X} \Theta^{(-\rho)}(q_{XY}|W) - \rho R \}. \tag{86}
\]

2) Define

\[
J^{(\rho)}_{\text{CK}}(q_{XY}, \hat{p}_Y|W) = \Theta^{(-\rho)}(q_{XY}|W) + \rho D(q_Y||\hat{p}_Y) \tag{87}
\]

and prove that

\[
\min_{q_Y|X} \Theta^{(-\rho)}(q_{XY}|W) = \min_{\hat{p}_Y} J^{(\rho)}_{\text{CK}}(q_X, \hat{p}_Y|W), \tag{88}
\]

where

\[
\hat{J}^{(\rho)}_{\text{CK}}(q_X, \hat{p}_Y|W) = \min_{q_Y|X} J^{(\rho)}_{\text{CK}}(q_{XY}, \hat{p}_Y|W)
= -(1 + \rho) q_X(x) \log \sum_y W_{X \rightarrow Y}^{\rho} (y|x) \hat{p}_Y^{\rho} (y). \tag{89}
\]

3) In the following, we give an upper and a lower bounds on \( \max_{q_X} \min_{\hat{p}_Y} \hat{J}^{(\rho)}_{\text{CK}}(q_X, \hat{p}_Y|W) \). If the upper and the lower bounds are equal, the bound is found the desired saddle point. First, by the concavity of log function and Jensen’s inequality, we have

\[
\hat{J}^{(\rho)}_{\text{CK}}(q_X, \hat{p}_Y|W) \geq -(1 + \rho) q_X(x) \sum_y W_{X \rightarrow Y}^{\rho} (y|x) \hat{p}_Y^{\rho} (y). \tag{90}
\]

To be precise, it shows the proof that the two forms of the sphere-packing exponent function match.
Then, show that the right-hand side of (90) is minimized by

\[ \hat{p}_Y(y) = \Lambda^{-1} \left[ \sum_{x} q_X(x) W^{\frac{1}{1+\rho}}(y|x) \right]^{1+\rho}, \tag{91} \]

where \( \Lambda \) is a normalization factor. Denote this \( \hat{p}_Y \) by \( \hat{p}_Y^*(q_X) \).

4) From Step 3, we have

\[
\min_{\hat{p}_Y} J_{\text{CK}}^{(\rho)}(q_X, \hat{p}_Y | W) \\
\geq (a) \min_{\hat{p}_Y} - (1 + \rho) \log \sum_{x} q_X(x) \sum_y W^{\frac{1}{1+\rho}}(y|x) \hat{p}_Y^{\frac{\rho}{1+\rho}} (y) \\
= (b) - \log \sum_y \left[ \sum_{x} q_X(x) W^{\frac{1}{1+\rho}}(y|x) \right]^{1+\rho}. \tag{92} \]

Step (a) follows from Eq.(90). In Step (b), \( \hat{p}_Y = \hat{p}_Y^*(q_X) \) is substituted. Using Lagrange’s undetermined multiplier method, show that the \( q_X \) that maximizes (92) satisfies

\[
\sum_y W^{\frac{1}{1+\rho}}(y|x) \left[ \sum_{x'} q_X(x') W^{\frac{1}{1+\rho}}(y|x') \right]^{\rho} \\
\geq (a) \sum_y \left[ \sum_{x'} q_X(x') W^{\frac{1}{1+\rho}}(y|x') \right]^{1+\rho} = \Lambda. \tag{93} \]

Inequality (a) holds with equality for every \( x \) such that \( q_X(x) > 0 \).

5) Fix \( \hat{p}_Y \) to some distribution to give an upper bound of \( \max_{q_X} \min_{\hat{p}_Y} J_{\text{CK}}^{(\rho)}(\hat{p}_Y, q_X | W) \). For this purpose, assume \( \hat{p}_Y = \hat{p}_Y^* \). Then,

\[
\max_{q_X} \min_{\hat{p}_Y} J_{\text{CK}}^{(\rho)}(q_X, \hat{p}_Y | W) \\
\leq \max_{q_X} J_{\text{CK}}^{(\rho)}(q_X, \hat{p}_Y^* | W) \\
= \max_{q_X} -(1 + \rho) \sum_{x} q_X(x) \log \Lambda^{-\frac{\rho}{1+\rho}} \\
\cdot \sum_y W^{\frac{1}{1+\rho}}(y|x) \left[ \sum_{x'} q_X(x') W^{\frac{1}{1+\rho}}(y|x') \right]^{\rho} \\
\leq (a) \max_{q_X} -(1 + \rho) \sum_{x} q_X(x) \log \Lambda^{-\frac{\rho}{1+\rho} + 1} \\
= \max_{q_X} - \log \sum_y \left[ \sum_{x'} q_X(x') W^{\frac{1}{1+\rho}}(y|x') \right]^{1+\rho}. \tag{94} \]

Step (a) follows from Eq.(93). Hence, \( \max_{q_X} E_{\text{CK}}^{(\rho)}(p_X | W) \) is found to be an upper as well as lower bound of
Random coding error probability exponent \(0 < \rho < 1\)

\[
\max_{q_X} \min_{q_Y} J_{\text{CK}}^{(\rho)}(q_X Y, \hat{p}_Y | W) \quad \text{Therefore}
\]

\[
E_{\text{CK}}(R|W) = \max_{q_X} \max_{\rho \in [0, 1]} \left\{ \min_{q_Y} \Theta^{(-\rho)}(q_{XY}|W) - \rho R \right\}
\]

\[
= \max_{\rho \in [0, 1]} \left\{ \min_{q_Y} J_{\text{CK}}^{(\rho)}(\hat{p}_Y, q_X | W) - \rho R \right\}
\]

\[
= \max_{\rho \in [0, 1]} \left\{ \max_{q_Y} E_0^{(\rho)}(q_X | W) - \rho R \right\}
\]

\[
= E_r(R|W). \quad (95)
\]

As shown above, Step 3 uses Jensen’s inequality and steps 4 and 5 evaluate the upper and lower bounds. In the next subsection, we show another procedure to show the match of the two error exponents. In this new procedure, the evaluation of each step is explicit and thus understanding each step of the proof is easy. The interesting point is that we use the function \(J_{\text{CK}}^{(\lambda)}\) that was used to derive Algorithm 2.

B. New proof for the match of random coding error exponents

Here, we give a new proof for the match of Gallager’s and Csiszár and Körner’s error exponents. In the previous section, we removed the cost constraint for simplicity and to match the description in Exercise 10.24. In this section, we consider DMCs with cost constraints. The proof steps are the same with or without cost constraints. We use \(J_{2}^{(-\rho, \nu)}(q_X, \hat{p}_X|Y|W)\) defined in (40) for \(\lambda = -\rho\) with \(0 < \rho < 1\). Define

\[
J_{2}^{(-\rho, \nu)}(q_X, \hat{p}_X|Y|W) = \sum_x q_X(x) \log \frac{q_X^{-\rho}(x)}{W(y|x)e^{-\lambda^{-\nu}(x)} \hat{p}_X^{-\nu}(x|y)}. \quad (96)
\]

We have the following lemmas.
Lemma 15: For any fixed $0 < \rho \leq 1$ and any fixed $q_{XY}$, $J_2^{(-\rho,\nu)}(q_{XY}, \hat{p}_X|Y) = \Theta(-\rho,-\nu)(q_{XY}|W)$ is maximized by $\hat{p}_X|Y = q_X|Y$ and the maximum value is

$$J_2^{(-\rho,\nu)}(q_{XY}, q_X|Y|W) = \Theta(-\rho,\nu)(q_{XY}|W)$$

(97)

This implies that

$$\max_{q_X} \min_{q_Y|X} \max_{\hat{p}_X|Y} J_2^{(-\rho)}(q_{XY}, \hat{p}_X|Y|W) = \max_{q_X} \min_{q_Y|X} \max_{\hat{p}_X|Y} J_2^{(-\rho,\nu)}(q_{XY}, q_X|Y|W)$$

(98)

Proof: It is obvious from the definition of $J_2^{(\lambda,\nu)}(q_{XY}, \hat{p}_X|Y|W)$.

Lemma 16: For any fixed $0 < \rho \leq 1$, $\nu \geq 0$, $\hat{p}_X|Y$ and $q_X$, $J_2^{(-\rho,\nu)}(q_{XY}, \hat{p}_X|Y|W)$ is minimized by

$$q_Y|X(y|x) = \frac{W(y|x)e^{-\lambda(x)\lambda(x)\rho}p_X|Y(x|y)}}{\sum_{y'} W(y'|x)e^{-\lambda(x')\lambda(x')\rho}p_X|Y(x'|y')}} \triangleq q_Y^*|X(\hat{p}_X|Y)(y|x)$$

(99)

and the minimum value is

$$J_2^{(-\rho,\nu)}((q_X, q_Y^*|X(\hat{p}_X|Y)), \hat{p}_X|Y|W) = \tilde{J}_2^{(-\rho,\nu)}(q_X, \hat{p}_X|Y|W).$$

(100)

This implies that

$$\max_{q_X} \min_{\hat{p}_X|Y} \max_{q_Y|X} J_2^{(-\rho)}(q_{XY}, \hat{p}_X|Y|W) = \max_{q_X} \min_{\hat{p}_X|Y} \max_{q_Y|X} J_2^{(-\rho,\nu)}((q_X, q_Y^*|X(\hat{p}_X|Y)), \hat{p}_X|Y|W)$$

(101)

$$= \max_{q_X} \max_{\hat{p}_X|Y} J_2^{(-\rho,\nu)}(q_X, \hat{p}_X|Y|W).$$

Lemma 17: For any fixed $0 < \rho \leq 1$ and any fixed $\hat{p}_X|Y$, $\tilde{J}_2^{(-\rho,\nu)}(q_X, \hat{p}_X|Y|W)$ is maximized by

$$q_X(x) = \frac{\left\{\sum_{y} W(y|x)e^{-\lambda(x)\lambda(x)\rho}p_X|Y(x|y)}\right\}^{-1/\rho}}{\sum_{x' \neq x} \left\{\sum_{y} W(y|x')e^{-\lambda(x')\lambda(x')\rho}p_X|Y(x'|y)}\right\}^{-1/\rho}} \triangleq q_X^*(\hat{p}_X|Y)(x)$$

(102)

and the maximum value is

$$\tilde{J}_2^{(-\rho,\nu)}(q_X^*(\hat{p}_X|Y), \hat{p}_X|Y|W) = A^{(\rho,\nu)}(\hat{p}_X|Y).$$

(103)

This implies that

$$\max_{q_X} \max_{\hat{p}_X|Y} \tilde{J}_2^{(-\rho,\nu)}(q_X, \hat{p}_X|Y|W)$$

$$= \max_{\hat{p}_X|Y} \tilde{J}_2^{(-\rho,\nu)}(q_X^*(\hat{p}_X|Y), \hat{p}_X|Y|W)$$

$$= \max_{\hat{p}_X|Y} A^{(\rho,\nu)}(\hat{p}_X|Y).$$

(104)

See Appendix B for the proof of Lemma 16 and 17.
Proof of Proposition 2: For any fixed \( \rho \in (0, 1], \nu \geq 0 \), we have

\[
\max \min_{q_X, q_Y|X} \Theta^{(-\rho, \nu)}(q_{XY}|W) \\
= \max \min_{q_X} \max_{q_Y|X \bar{p}_X|Y} J_2^{(-\rho, \nu)}(q_{XY}, \bar{p}_X|Y|W) \\
= \max_{q_X} \max \min_{q_Y|X} \bar{p}_X|Y J_2^{(-\rho, \nu)}(q_{XY}, \bar{p}_X|Y|W) \\
= \max \max_{q_X} \bar{p}_X|Y J_1^{(-\rho, \nu)}(q_X, \bar{p}_X|Y|W) \\
= \max_{q_X} A^{(\rho, \nu)}(\bar{p}_X|Y|W). \tag{105}
\]

Step (a) follows from Lemma 15. In step (b), the interchange of the order of min and max is justified because of Property 3. Step (c) follows from Lemma 16 and step (d) follows from Lemma 17. Therefore we have

\[
E_{CK}(R, \Gamma|W) \overset{(e)}{=} \sup_{\rho \in (0, 1]} \inf_{\nu \geq 0} \left\{ \max \min_{q_X, q_Y|X} \Theta^{(-\rho, \nu)}(q_{XY}|W) - \rho R + \rho \nu \Gamma \right\} \\
\overset{(f)}{=} \sup_{\rho \in (0, 1]} \inf_{\nu \geq 0} \left\{ \max_{p_X|Y} A^{(\rho, \nu)}(p_X|Y|W) - \rho R + \rho \nu \Gamma \right\} \\
= \tilde{E}_r(R, \Gamma|W) \overset{(g)}{=} E_r(R, \Gamma|W). \tag{106}
\]

Step (e) follows from (86), step (f) follows from (105), and step (g) follows from Corollary 2. This completes the proof.

The guideline shown in [2] Exercise 10.24 is a standard method to prove the match of Csiszár and Körner’s exponent and Gallager’s exponent. It is interesting that our procedure uses Corollary 2 i.e., the match of \( \bar{E}_r \) and \( E_r \), which was derived by Arimoto’s algorithm. It should be noted that due to the structural difference with \( J_2^{(\lambda)}(q_{XY}, \bar{p}_X|Y|W) \), the function \( J_1^{(\lambda)}(q_{XY}, p_X|W) \) cannot be used to prove the match between the two exponents. The difference is that for \( J_2^{(\lambda)}(q_{XY}, \bar{p}_X|Y|W) \), the joint distribution \( q_{XY} \) can be separated \( q_X \) and \( q_{Y|X} \) as shown above, while it is separated as \( q_Y \) and \( q_X|Y \) for \( J_1^{(\lambda)}(q_{XY}, p_X|W) \).

If we think the double maximization in the third line of (105) will lead to an algorithm, that’s a misunderstanding. In fact, if the optimum solution when one of the variables is fixed cannot be written explicitly, it will not lead to a computation algorithm for alternating optimization like the Arimoto algorithm. The reason why it is difficult to derive the computation algorithm for the Csiszár and Körner’s error exponent is that the the exponent is defined as the minimax problem, unlike the Dueck and Körner’s strong converse exponent. Even if the form of the double maximization problem can be derived, it is still necessary to be able to explicitly write the solution when one of the variables is fixed. This condition makes it difficult to derive the computation algorithm for Csiszár and Körner’s error exponent.

V. Algorithms for the Lossy Source-Coding Strong Converse Exponent

So far, we have discussed computation algorithms for the strong converse exponents in channel coding. This section describes algorithms for computing exponents in lossy source coding. The first algorithm for computing Csiszár and Körner’s strong converse exponent was due to the author and Oohama [13]. In the following two
The strong converse exponent for lossy source coding \((0 < \lambda \leq 1, \nu \geq 0)\):

\[
\min_{q_{XY}, \hat{p}_Y, p_Y} \min_{p_X} \min_{\lambda} F^{(\lambda, \nu)}_{\text{Jü,s}}(q_{XY}, \hat{p}_Y, p_Y|X)
\]

Algorithm 7 (J. & Oohama 2016)

Algorithm 5

Algorithm 6

Arimoto algorithm with \(\rho = -\lambda\)

Definition 13

Csiszár & Körner’s Exponent

Fig. 7. Relation between seven expressions for the strong converse exponent for lossy source coding of DMSs

subsections, new algorithms are proposed for the lossy source coding strong converse exponent. Fig. 7 shows the relation between the algorithms. The structure of the diagram of Fig. 7 is quite similar to Fig. 5. In Fig. 7, nine quantities defined by minimization problems are shown. They are the quantities used to express the strong converse exponent with parameter \(\lambda \in [0, 1]\) and \(\nu \geq 0\). The function \(F^{(\lambda, \nu)}_{\text{Jü,s}}(q_{XY}, \hat{p}_Y, p_Y|X)\) will be defined in Section V-C.

In Arimoto’s paper [11], an algorithm for computing Blahut’s error exponent was given. The validity of computing the strong converse exponent with Arimoto algorithm has not been clarified. The issue is discussed in this section and Section VI-A and its validity is clarified.

The rate-distortion function \(R(\Delta|q_X)\) is generally not convex in \(q_X\) and therefore it is difficult to develop an algorithm directly from the expression of [5]. This motivated us to define the following function:

**Definition 14:** For any fixed \(R \geq 0, \Delta \geq 0\), we have

\[
\tilde{G}_{\text{CK}}(R,\Delta|P_X) = \min_{q_{XY} \in \mathcal{P}(X,Y): E[d(X,Y)] \leq \Delta} \{D(q_X||P_X) + |I(q_X,q_Y|X) - R|^+\}.
\]

Then, we have the following lemma:

**Lemma 18:** For any fixed \(R \geq 0, \Delta \geq 0\), and any fixed \(P_X \in \mathcal{P}(X)\), we have

\[
G_{\text{CK}}(R,\Delta|P_X) = \tilde{G}_{\text{CK}}(R,\Delta|P_X).
\]

See Appendix D for the proof of Lemma 18.

Lemma 18 assures that \(G_{\text{CK}}(R,\Delta|P_X)\) can be computed by evaluating \(\tilde{G}_{\text{CK}}(R,\Delta|P_X)\), the form of which is quite similar to \(G_{\text{CK}}(R,\Delta|P_X)\). In the representation of \(\tilde{G}_{\text{CK}}(R,\Delta|P_X)\) we could eliminate the rate-distortion function. Based on this representation, we can derive an algorithm.

The function \(\tilde{G}_{\text{CK}}(R,\Delta|P_X)\) satisfies the following property, which is required to derive its parametric expression:

**Property 5:**
a) $\tilde{G}_{\text{CK}}(R, \Delta|P_X)$ is a monotone non-increasing function of $R \geq 0$ for a fixed $\Delta \geq 0$ and is a monotone non-increasing function of $\Delta \geq 0$ for a fixed $R \geq 0$.

b) $\tilde{G}_{\text{CK}}(R, \Delta|P_X)$ is a convex function of $(R, \Delta)$.

c) $\tilde{G}_{\text{CK}}(R, \Delta|P_X)$ takes positive value for $0 \leq R < R(\Delta|P_X)$. For $R \geq R(\Delta|P_X)$, $\tilde{G}_{\text{CK}}(R, \Delta|P_X) = 0$.

d) For $R' \geq R \geq 0$, we have $\tilde{G}_{\text{CK}}(R, \Delta|P_X) - \tilde{G}_{\text{CK}}(R', \Delta|P_X) \leq R' - R$.

See Appendix F for the proof.

Then, we define the following function that will be used as an objective function of the algorithm.

**Definition 15:** For any fixed $\lambda \in [0, 1]$, we define

$$G_{\text{CK}}^{(\lambda)}(\Delta|P_X) = \min_{D(q_X||P_X) + \lambda I(q_X, q_Y|X) \leq \Delta} \{D(q_X||P_X) + \lambda I(q_X, q_Y|X)\}. \quad (109)$$

**Definition 16:** For any fixed $q_{XY} \in \mathcal{P}(X \times Y)$, $\lambda \in [0, 1]$, and $\mu \geq 0$, we define

$$\Theta_{s}^{(\lambda, \mu)}(q_{XY}|P_X) = D(q_X||P_X) + \lambda I(q_X, q_Y|X) + \mu E_{q_{XY}}[d(X, Y)]$$

$$= E_{q_{XY}} \left[ \log \frac{q_X^{-\lambda}(X)q_Y^{\lambda}(X|Y)}{P_X(X)e^{\mu d(X, Y)}} \right], \quad (110)$$

$$\Theta_{s}^{(\lambda, \mu)}(P_X) = \min_{q_{XY}} \Theta_{s}^{(\lambda, \mu)}(q_{XY}|P_X). \quad (111)$$

The subscript ‘s’ means that this function is the source coding counterpart of the corresponding function in channel coding. One can observe that Eq. (110) for $\Theta_{s}^{(\lambda, \mu)}(q_{XY}|P_X)$ is similar to Eq. (13) for $\Theta^{(\lambda)}(q_{XY}|W)$ for the channel coding.

**Lemma 19:** For any fixed $\lambda \in [0, 1]$, $\Delta \geq 0$, and $P_X \in \mathcal{P}(X)$, we have

$$G_{\text{CK}}^{(\lambda)}(\Delta|P_X) = \sup_{\mu \geq 0} \left\{ \Theta_{s}^{(\lambda, \mu)}(P_X) - \mu \Delta \right\}. \quad (112)$$

For any fixed $R \geq 0$, $\Delta \geq 0$, and $P_X \in \mathcal{P}(X)$, we have

$$\tilde{G}_{\text{CK}}(R, \Delta|P_X) = \max_{0 \leq \lambda \leq 1} \left\{ G_{\text{CK}}^{(\lambda)}(\Delta|P_X) - \lambda R \right\} \quad (113)$$

$$= \max_{0 \leq \lambda \leq 1, \mu \geq 0} \left\{ \Theta_{s}^{(\lambda, \mu)}(P_X) - \mu \Delta - \lambda R \right\}. \quad (114)$$

See Appendix G for the proof of Lemma 19.

Eq. (112) shows that $G_{\text{CK}}^{(\lambda)}(\Delta|P_X)$ is computed by the Legendre transform of $-\Theta_{s}^{(\lambda, \mu)}(P_X)$ with respect to $-\mu \leq 0$. Then, Eq. (113) shows that $\tilde{G}_{\text{CK}}(R, \Delta|P_X)$ is computed by the Legendre transform of $-G_{\text{CK}}^{(\lambda)}(\Delta|P_X)$ with respect to $-\lambda \in [-1, 0]$. This immediately implies that $\tilde{G}_{\text{CK}}(R, \Delta|P_X)$ is computed by the two-dimensional Legendre transform of $\Theta_{s}^{(\lambda, \mu)}(P_X)$ whose domain is $(\lambda, \mu) \in [0, 1] \times [0, \infty)$, as shown in Eq. (114). An image of the shape of function $G_{\text{CK}}(R, \Delta|P_X)$ is shown in Fig. 8. The supporting line to this function of slope $-\lambda$ is expressed by $-\lambda R + G_{\text{CK}}^{(\lambda)}(\Delta|P_X)$. We can easily see that $G_{\text{CK}}^{(1)}(\Delta|P_X) = \min_{d(X, Y) \leq \Delta} D(q_X||P_X) = 0$ by choosing $q_X = P_X$ and $q_Y|X(y|x) = 1$ for every $x$ and $y$ with $d(x, y) = 0$. For $\lambda = 1$, we have

$$\Theta_{s}^{(1, \mu)}(P_X) = -\log \max_y \sum_{x} P_X(x)e^{-\mu d(x, y)}. \quad (115)$$
Strong converse exponent functions and generalized cut-off rate in lossy channel coding

\[ G_{CK}^{(\lambda)}(\Delta | P_X) \]

slope = -1  

supporting line of slope \(-\lambda\)

Fig. 8. Csiszár and Körner’s strong converse exponent and the generalized cutoff rate in lossy source coding.

The generalized cutoff rate, denoted by \( R^{(-\lambda)}(\Delta | P_X) \) is the \( R \)-axis intercept of this supporting line. Therefore

\[ R^{(-\lambda)}(\Delta | P_X) = \frac{1}{\lambda} G_{CK}^{(\lambda)}(\Delta | P_X). \]

We now perform a change of variables: let \( \mu = \lambda \nu \), and instead of the parameter pair \((\lambda, \mu)\), let \((\lambda, \nu)\) be the variable parameter pair. By doing so, in the limit of \( \lambda \to 0 \), one of the new algorithms is reduced to the Arimoto-Blahut algorithm that computes the rate distortion function. From (112), we have

\[ R^{(-\lambda)}(\Delta | P_X) = \frac{1}{\lambda} \sup_{\mu \geq 0} \left\{ \Theta_s^{(\lambda,\mu)}(P_X) - \mu \Delta \right\} \]

\[ = \sup_{\mu \geq 0} \left\{ \frac{1}{\lambda} \Theta_s^{(\lambda,\mu)}(P_X) - \frac{\mu}{\lambda} \Delta \right\} \]

\[ = \sup_{\nu \geq 0} \left\{ \frac{1}{\lambda} \Theta_s^{(\lambda,\lambda \nu)}(P_X) - \nu \Delta \right\}. \] (116)

It will be shown that the generalized cutoff rate in lossy source coding is expressed by using a variant of \( d \)-tilted information.

The following discussions for source coding are parallel with the discussions for channel coding. We define the following functions:

\[ J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y | P_X) = \Theta_s^{(\lambda,\lambda \nu)}(q_{XY} | P_X) + \lambda D(q_{Y} | \hat{p}_Y) \]

\[ = E_{q_{XY}} \left[ \log \frac{q_X(X) q_Y^\lambda(Y | X)}{P_X(X) e^{-\lambda d(X,Y) \hat{p}_Y(Y)}} \right] \] (117)

\[ J_{2,s}^{(\lambda,\nu)}(q_{XY}, p_{Y | X} | P_X) = \Theta_s^{(\lambda,\lambda \nu)}(q_{XY} | P_X) + (1 - \lambda) D(q_{Y | X} | p_{Y | X} | q_X) \]

\[ = E_{q_{XY}} \left[ \log \frac{q_Y^{1-\lambda}(Y) q_X(X) q_Y^\lambda(Y | X)}{P_X(X) e^{-\lambda d(X,Y) p_{Y | X}^{1-\lambda}(Y | X)}} \right] \] (118)

In Section V-A, we give Algorithm 5 that is derived from \( J_{1,s}^{(\lambda,\nu)} \) and in Section V-B, we give Algorithm 6 that is derived from \( J_{2,s}^{(\lambda,\nu)} \).
A. Algorithm 5

This subsection gives Algorithm 5. We will define a new exponent function that is basically the same as Blahut exponent but the parameter $\rho$ takes negative values. The match of Csiszár and Körner’s exponent and this new exponent function is proved by the convergence of Algorithm 5, as the match of Arimoto’s exponent and Dueck and Körner’s exponent is proved by the convergence of Algorithm 1.

The function $J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y|P_X)$ satisfies the following two lemmas.

**Lemma 20:** For any fixed $\lambda \in [0, 1]$, $\mu \geq 0$, and any fixed $q_{XY}$, $J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y|P_X)$ is minimized by $\hat{p}_Y = q_Y$ and the minimum value is
\[
J_{1,s}^{(\lambda,\nu)}(q_{XY}, q_Y|P_X) = \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X).
\] (119)

This implies that
\[
\min_{q_{XY}} \min_{\hat{p}_Y} J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y|P_X) = \min_{q_{XY}} J_{1,s}^{(\lambda,\nu)}(q_{XY}, q_Y|P_X) = \min_{q_{XY}} \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X).\] (120)

**Lemma 21:** For any fixed $\lambda \in [0, 1]$, $\nu \geq 0$, and any fixed $\hat{p}_Y$, $J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y|P_X)$ is minimized by
\[
q_{Y|X}(y|x) = \frac{\hat{p}_Y(y)e^{-\nu d(x,y)}}{\sum_{y'} \hat{p}_Y(y')e^{-\nu d(x,y')}} \quad \text{(121)}
\]
\[
q_X(x) = \frac{P(x)\{\sum_y \hat{p}_Y(y)e^{-\nu d(x,y)}\}^\lambda}{\sum_x P(x')\{\sum_y \hat{p}_Y(y)e^{-\nu d(x',y)}\}^\lambda}. \quad \text{(122)}
\]

Denote the joint distribution calculated from the above $q_{Y|X}$ and $q_X$ by $q_{XY}^*(\hat{p}_Y)$. Then, the minimum value of $J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y|P_X)$ for fixed $\hat{p}_Y$ is
\[
J_{1,s}^{(\lambda,\nu)}(q_{XY}^*(\hat{p}_Y), \hat{p}_Y|P_X) = -\log \sum_x P_X(x) \left\{ \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} \right\}^\lambda
= -E_{0,s}^{(-\lambda,\nu)}(\hat{p}_Y|P_X). \quad \text{(123)}
\]

This implies that
\[
\min_{\hat{p}_Y} \min_{q_{XY}} J_{1,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_Y|P_X) = \min_{\hat{p}_Y} J_{1,s}^{(\lambda,\nu)}(q_{XY}^*(\hat{p}_Y), \hat{p}_Y|P_X) = \min_{\hat{p}_Y} -E_{0,s}^{(-\lambda,\nu)}(\hat{p}_Y|P_X). \quad \text{(124)}
\]
See Appendix B for the proof.

An algorithm derived from Lemmas 20 and 21 is shown in Algorithm 5.

We have the following proposition:
Algorithm 5: Algorithm for computing $\min_{q_{XY}} \min_{p_Y} J_{1,s}^{(\lambda,\nu)} (q_{XY}, \hat{p}_Y | P_X)$

**Input:** The probability distribution $P_X$ on $\mathcal{X}$, the distortion measure $d(x,y)$, $\lambda \in (0,1]$ and $\nu \geq 0$.

Choose any initial distribution $\hat{p}_Y^{[0]}$ such that $\hat{p}_Y^{[0]}(y) > 0$ for all $y \in \mathcal{Y}$.

for $t = 0, 1, 2, \ldots$ do

$$q_Y^{[t]}(x) = \frac{\hat{p}_Y^{[t]}(y) e^{-\nu d(x,y)}}{\sum_y \hat{p}_Y^{[t]}(y') e^{-\nu d(x,y')}}$$  \hspace{1cm} (125)

$$q_Y^{[t]}(y) = \frac{P(x) \{ \sum_y \hat{p}_Y^{[t]}(y) e^{-\nu d(x,y)} \}^{\lambda}}{\sum_{x'} P(x') \{ \sum_y \hat{p}_Y^{[t]}(y) e^{-\nu d(x',y)} \}^{\lambda}},$$  \hspace{1cm} (126)

$$\hat{p}_Y^{[t+1]}(y) = \sum_x q_X^{[t]}(x) q_Y^{[t]}(y|x).$$  \hspace{1cm} (127)

end for

**Proposition 5:** For $t = 0, 1, 2, \ldots$, we have

$$J_{1,s}^{(\lambda,\nu)} (q_{XY}, \hat{p}_Y^{[0]} | P_X) \geq J_{1,s}^{(\lambda,\nu)} (q_{XY}, \hat{p}_Y^{[1]} | P_X) \geq J_{1,s}^{(\lambda,\nu)} (q_{XY}, \hat{p}_Y^{[t]} | P_X) \geq \cdots$$

$$\geq J_{1,s}^{(\lambda,\nu)} (q_{XY}, \hat{p}_Y^{[t+1]} | P_X) = \Theta_s^{(\lambda,\nu)} (q_{XY} | P_X)$$

**Proof:** Step (a) follows from Lemma 22 and step (b) follows from Lemma 23. Step (c) follows from (136) in Lemma 23. This completes the proof.

When $\lambda = 0$, Algorithm 5 reduces to Blahut’s algorithm for computing the rate distortion function $\mathbb{R}$. The next theorem shows that $J_{1,s}^{(\lambda,\nu)} (q_{XY}, \hat{p}_Y^{[t+1]} | P_X) = \Theta_s^{(\lambda,\nu)} (q_{XY} | P_X)$ converges to $\Theta_s^{(\lambda,\nu)} (P_X)$.

**Theorem 4:** For any $\lambda \in (0,1]$ and for any $\nu \geq 0$, the series of distributions $q_{XY}^{[t]}$ defined by (125)-(127) converges to an optimal distribution $q_{XY}^*$ that minimizes $\Theta_s^{(\lambda,\nu)} (q_{XY} | P_X)$. The approximation error $\Theta_s^{(\lambda,\nu)} (q_{XY}^{[t]} | P_X) - \Theta_s^{(\lambda,\nu)} (q_{XY}^* | P_X)$ is inversely proportional to the number of iterations.

See Appendix C for the proof.

We have the following proposition.

**Proposition 6:** A necessary and sufficient condition for $\hat{p}_Y^*$ to minimize $-E_{0,s}^{(-\lambda,\nu)} (\hat{p}_Y | P_X)$ is that

$$- \log \sum_x P_X(x) e^{-\nu d(x,y)} \left\{ \sum_{y'} \hat{p}_Y^*(y') e^{-\nu d(x,y')} \right\}^\lambda$$

$$\begin{cases} = \Theta_s^{(\lambda,\nu)} (P_X) \text{ if } \hat{p}_Y^*(y) > 0, \\ > \Theta_s^{(\lambda,\nu)} (P_X) \text{ if } \hat{p}_Y^*(y) = 0. \end{cases}$$  \hspace{1cm} (128)

**Proof:** This is exactly the KKT condition for $\hat{p}_Y^*$. Its derivation is straightforward and thus is omitted.
As shown in Proposition 5, the minimum value of \( G_{s}^{(\lambda,\lambda \nu)}(q_{XY}|P_{X}) \) equals to the minimum value of \(-E_{0,s}^{(-\lambda,\nu)}(\hat{p}_{Y}|P_{X})\). Using the latter expression, we can show that Csiszár and Körner’s strong converse exponent is equal to (11) which is similar to Blahut’s error exponent, except that it has a negative \( \rho \). We have the following proposition.

**Proposition 7:** For any \( R \geq 0 \), and \( \Delta \geq 0 \), we have

\[
G_{CK}(R, \Delta|P_{X}) = G_{JO}(R, \Delta|P_{X}).
\]

**Proof:** We have the following equations.

\[
G_{CK}(R, \Delta|P_{X}) \equiv \hat{G}_{CK}(R, \Delta|P_{X}) = \max_{\lambda \in [0,1]} \sup_{\mu \geq 0} \{ \Theta_{s}^{(\lambda,\mu)}(P_{X}) - \lambda R - \mu \Delta \}
\]

\[
= \sup_{\lambda \in [0,1]} \sup_{\nu \geq 0} \{ \min_{\hat{p}_{Y}} (-E_{0,s}^{(-\lambda,\nu)}(\hat{p}_{Y}|P_{X})) - \lambda R - \nu \Delta \}
\]

\[
= \sup_{\rho \in [-1,0]} \sup_{\nu \geq 0} \{ -E_{0,s}^{(\rho,\nu)}(\hat{p}_{Y}|P_{X}) + \rho R + \nu \Delta \}
\]

\[
= G_{JO}(R, \Delta|W)
\]

Step (a) follows from Lemma 18. Step (b) follows from Lemma 19. Step (c) follows from the equation (c) in Proposition 5 setting \( \mu = -\rho \nu \) and we put \( \lambda = -\rho \) in Step (d).

Proposition 7 says that Csiszár and Körner’s strong converse exponent matches with \( G_{JO}(R, \Delta|P_{X}) \), which is written in a similar form as Blahut’s error exponent.

For any fixed \( \lambda \in (0, 1] \), the generalized cut-off rate satisfies the following property. Let \( \nu^{*} = \nu^{*}(\lambda) \) be an optimal \( \nu \) that attains the rhs of (116) and let \( \hat{p}_{Y}^{*} = \hat{p}_{Y}^{*}(\lambda) \) be an optimal distribution that attains \( \min_{\hat{p}_{Y}} -E_{0,s}^{(\lambda,\nu^{*}(\lambda))}(\hat{p}_{Y}|P_{X}) \). Then we have the following.

**Property 6:** For \( \lambda \in (0, 1] \), and \( \Delta \geq 0 \), we have

\[
R^{(-\lambda)}(\Delta|P_{X}) = -\frac{1}{\lambda} \log \sum_{x} P_{X}(x) \left\{ \sum_{y} \hat{p}_{Y}^{*}(y) e^{-\nu^{*}[d(x,y)-\Delta]} \right\}^{\lambda}.
\]

**Proof:** We have the following chain of equalities:

\[
R^{(-\lambda)}(\Delta|P_{X}) \equiv \frac{1}{\lambda} G_{CK}^{(\lambda)}(\Delta|P_{X}) \equiv \frac{1}{\lambda} \left\{ \Theta_{s}^{(\lambda,\lambda \nu^{*})}(P_{X}) - \lambda \nu^{*} \Delta \right\}
\]

\[
= \frac{1}{\lambda} \left\{ -E_{0,s}^{(-\lambda,\nu^{*})}(\hat{p}_{Y}^{*}|P_{X}) - \lambda \nu^{*} \Delta \right\}
\]

\[
= -\frac{1}{\lambda} \log \sum_{x} P_{X}(x) \left\{ \sum_{y} \hat{p}_{Y}^{*}(y) e^{-\nu^{*}[d(x,y)-\Delta]} \right\}^{\lambda}
\]

Step (a) follows from Lemma 19. Step (b) follows from (116) and the assumption of \( \nu^{*} \). Step (c) follows from the assumption of \( \hat{p}_{Y}^{*} \), and Step (d) follows from the definition of \( E_{0,s}^{(\rho,\nu)}(\hat{p}_{Y}|P_{X}) \). This completes the proof.
B. Algorithm 6

Next, we derive another algorithm from $J_{2, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X)$. In order to describe it, we use the following function that appears in Arimoto's algorithm.

**Definition 17:** For a given source distribution $P_X$, transition probability of a test channel $p_{Y|X}$, and parameters $\rho$ and $t$, define

$$A_s^{(\rho, \nu)}(p_{Y|X}|P_X) = (1 + \rho) \log \sum_y \sum_x P_X(x) e^{\rho \nu d(x, y)} p_{Y|X}^{1+\rho}(y|x).$$  \hfill (134)

**Lemma 22:** For any fixed $\lambda \in [0, 1]$ and $\nu \geq 0$, $J_{2, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X)$ is minimized if and only if $p_{Y|X} = q_{Y|X}$ and the minimum value is

$$J_{2, s}^{(\lambda, \nu)}(q_{XY}, q_{Y|X}|P_X) = \Theta_s^{(\lambda, \nu)}(q_{XY}|P_X).$$  \hfill (135)

This implies

$$\min_{q_{XY}} \min_{p_{Y|X}} J_{2, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X) = \min_{q_{XY}} J_{2, s}^{(\lambda, \nu)}(q_{XY}, q_{Y|X}|P_X) = \min_{q_{XY}} \Theta_s^{(\lambda, \nu)}(q_{XY}|P_X).$$  \hfill (136)

**Proof:** It is obvious from the nonnegativity of the relative entropy $D(q_{Y|X}||p_{Y|X}|q_X)$.  

**Lemma 23:** For any fixed $\lambda \in [0, 1]$ and $\nu \geq 0$, $J_{1, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X)$ is minimized by

$$q_{Y|X}(x|y) = \frac{P_X(x) e^{-\lambda \nu d(x, y)} p_{Y|X}^{1-\lambda}(y|x)}{\sum_{x'} P_X(x') e^{-\lambda \nu d(x', y)} p_{Y|X}^{1-\lambda}(y|x')}.$$  \hfill (137)

$$q_Y(y) = \frac{\sum_{x} P_X(x) e^{-\lambda \nu d(x, y)} p_{Y|X}^{1-\lambda}(y|x)}{\sum_{y'} \sum_{x} P_X(x) e^{-\lambda \nu d(x, y')} p_{Y|X}^{1-\lambda}(y'|x')}.$$  \hfill (138)

Denote the joint distribution computed from the above $q_{Y|X}$ and $q_Y$ by $q_{XY}^*(p_X)$. The minimum value of $J_{1, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X)$ is

$$J_{2, s}^{(\lambda, \nu)}(q_{XY}^*(p_X), p_{Y|X}|P_X) = -(1 - \lambda) \log \sum_{y} \sum_{x \in X} P_X(x) e^{-\lambda \nu d(x, y)} p_{Y|X}^{1-\lambda}(y|x)$$

$$= -A_s^{(-\lambda, \nu)}(p_{Y|X}|P_X).$$  \hfill (139)

This implies that

$$\min_{q_{XY}} \min_{p_{Y|X}} J_{2, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X) = \min_{p_{Y|X}} J_{2, s}^{(\lambda, \nu)}(q_{XY}^*(p_X), p_{Y|X}|P_X) = \min_{p_{Y|X}} -A_s^{(-\lambda, \nu)}(p_{Y|X}|P_X).$$  \hfill (140)

See Appendix B for the proof.

Note that the rhs of Eq. (139) has the same form of the rhs of Eq. (134). We have the following.

$$\min_{q_{XY}} \Theta_s^{(\lambda, \nu)}(q_{XY}|P_X)$$

$$= \min_{q_{XY}} \min_{p_{Y|X}} J_{2, s}^{(\lambda, \nu)}(q_{XY}, p_{Y|X}|P_X)$$

$$= \min_{p_{Y|X}} \left\{ -A_s^{(-\lambda, \nu)}(p_{Y|X}|P_X) \right\}.$$  \hfill (141)
For a fixed $\nu \geq 0$, we can show that the following proposition holds.

**Proposition 8:** We have

$$
limit_{\rho \to 0} \frac{1}{\rho} A_s^{(\rho,\nu)}(p_{Y|X}|P_X) = I(P_X, P_{Y|X}) + \nu E_{(P_X, P_{Y|X})}[d(X,Y)].$$

(142)

See Appendix B for the proof. Thus, the function $(1/\rho)A_s^{(\rho,\nu)}(p_{Y|X}|P_X)$, which we call an alternative form, has a natural connection to information theoretic quantities.

An algorithm derived from Lemmas 22 and 23 is shown in Algorithm 6.

**Algorithm 6** Algorithm for computing $\min_{q_{XY}} \min_{q_{Y|X}} J_{2,s}^{(\lambda,\nu)}(q_{XY}, p_{Y|X}|P_X)$

**Input:** The probability distribution $P_X$ on $\mathcal{X}$, the distortion measure $d(x,y)$, $\lambda \in (0,1)$ and $\nu \geq 0$.

Choose any initial distribution $p_{Y|X}^0$ such that $p_{Y|X}^0(y|x) > 0$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$.

for $t = 0, 1, 2, \ldots$ do

$$q_{X|Y}^t(x|y) = \frac{P_X(x)e^{-\lambda d(x,y)}p_{Y|X}^t(y|x)^{1-\lambda}}{\sum_{x'} P_X(x')e^{-\lambda d(x',y)}p_{Y|X}^t(y|x')^{1-\lambda}},$$

(143)

$$q_{Y}^t(y) = \frac{\left[\sum_{x} P_X(x)e^{-\lambda d(x,y)}p_{Y|X}^t(y|x)^{1-\lambda}\right]^{1/(1-\lambda)}}{\sum_{y'} \left[\sum_{x} P_X(x)e^{-\lambda d(x,y')}p_{Y|X}^t(y'|x)^{1-\lambda}\right]^{1/(1-\lambda)}},$$

(144)

$$p_{Y|X}^{[t+1]}(y|x) = \frac{q_{Y}^t(y)q_{X|Y}^t(x|y)}{\sum_{y'} q_{Y}^t(y')q_{X|Y}^t(x|y')}.$$  

(145)

end for

We have the following proposition:

**Proposition 9:** For $t = 0, 1, 2 \ldots$, we have

$$J_{2,s}^{(\lambda,\nu)}(q_{XY}^0, p_{Y|X}^0|P_X) \geq J_{2,s}^{(\lambda,\nu)}(q_{XY}^1, p_{Y|X}^1|P_X) \geq J_{2,s}^{(\lambda,\nu)}(q_{XY}^2, p_{Y|X}^2|P_X) \cdots$$

$$\geq J_{2,s}^{(\lambda,\nu)}(q_{XY}^t, p_{Y|X}^t|P_X) = -A_s^{-(-\lambda,\nu)}(p_{Y|X}|P_X).$$

(143)

$$(a) \geq J_{2,s}^{(\lambda,\nu)}(q_{XY}^{[t+1]} p_{Y|X}^{[t+1]}|P_X) = \Theta_{s}^{(\lambda,\nu)}(q_{XY}|P_X)$$

$$(b) \geq J_{2,s}^{(\lambda,\nu)}(q_{XY}^{[t+1]} p_{Y|X}^{[t+1]}|P_X) = -A_s^{-(-\lambda,\nu)}(p_{Y|X}|P_X) \geq \ldots$$

$$\geq \min_{q_{XY}} \Theta_{s}^{(\lambda,\nu)}(q_{XY}|P_X) \geq \min_{P_X} A_s^{-(-\lambda,\nu)}(p_{Y|X}|P_X).$$

**Proof:** Step (a) follows from Lemma 22 and step (b) follows from Lemma 23. Step (c) follows from (136) in Lemma 22 and (140) in Lemma 23. This completes the proof.

The next theorem shows that $J_{2,s}^{(\lambda,\nu)}(q_{XY}^{[t]} p_{Y|X}^{[t+1]}|P_X) = \Theta_{s}^{(\lambda,\nu)}(q_{XY}|P_X)$ converges to $\Theta_{s}^{(\lambda,\nu)}(P_X)$.

**Theorem 5:** For any $\lambda \in (0,1]$ and for any $\nu \geq 0$, the series of distributions $q_{XY}^{[t]}$ defined by (143)-(145) converges to an optimal distribution $q_{XY}^{*}$ that minimizes $\Theta_{s}^{(\lambda,\nu)}(q_{XY}|P_X)$. The approximation error $\Theta_{s}^{(\lambda,\nu)}(q_{XY}|P_X) - \Theta_{s}^{(\lambda,\nu)}(q_{XY}^{*}|P_X)$ is inversely proportional to the number of iterations.

See Appendix C for the proof.
C. Comparison with our previous algorithm

Next, we describe the algorithm proposed in [31] for computing Csiszár and Körner’s strong converse exponent. Proofs for the theorems and lemmas omitted in [31] are provided in Appendix of this paper.

Then, in order to derive an algorithm for computing $\min_{q_{XY}} \Theta_s^{(\lambda,\lambda\nu)}(q_{XY}|P_X)$, we define the following function.

$$
F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}, \hat{q}_{XY}|P_X) = \Theta_s^{(\lambda,\lambda\nu)}(q_{XY}|P_X) + D(q_{XY}||\hat{q}_{XY})
$$

$$
= E_{X,Y} \left[ \log \frac{q_{XY}(X,Y)}{P_X(X)e^{-\lambda d(X,Y)}\hat{q}_Y^\lambda(Y)\hat{q}_{Y|X}^{1-\lambda}(Y|X)} \right] \tag{146}
$$

We have the following Lemmas.

**Lemma 24:** For any fixed $\lambda \in [0, 1]$, $\nu \geq 0$, and any fixed $P_X \in \mathcal{P}(X)$, $q_{XY} \in \mathcal{P}(X \times Y)$, $F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}, \hat{q}_{XY}|P_X)$ is minimized if and only if $\hat{q}_{XY} = q_{XY}$ and its minimum value is

$$
F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}, q_{XY}|P_X) = \Theta_s^{(\lambda,\lambda\nu)}(q_{XY}|P_X). \tag{147}
$$

This implies that

$$
\min_{q_{XY}} \min_{\hat{q}_{XY}} F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}, \hat{q}_{XY}|P_X) = \min_{\hat{q}_{XY}} F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}, \hat{q}_{XY}|P_X) = \min_{q_{XY}} \Theta_s^{(\lambda,\lambda\nu)}(q_{XY}|P_X).
$$

**Lemma 25:** For any fixed $\lambda \in [0, 1]$, $\mu \geq 0$, $P_X \in \mathcal{P}(X)$, and any fixed $\hat{q}_{XY} \in \mathcal{P}(X \times Y)$, $F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}, \hat{q}_{XY}|P_X)$ is minimized if and only if

$$
q_{XY}(x, y) = \frac{P_X(x)e^{-\lambda d(x,y)}\hat{q}_Y^\lambda(y)\hat{q}_{Y|X}^{1-\lambda}(y|x)}{\sum_{x',y'} P_X(x')e^{-\lambda d(x',y')}\hat{q}_Y^\lambda(y')\hat{q}_{Y|X}^{1-\lambda}(y'|x')}
$$

$$
= \hat{q}_{XY}(\hat{q}_{XY})(x, y)
$$

and its minimum value is

$$
F_{\lambda\nu}^{(\lambda,\lambda\nu)}(q_{XY}(\hat{q}_{XY}), \hat{q}_{XY}|P_X) = -\log E_{\hat{q}_{XY}} \left[ \frac{P_X(X)e^{-\lambda d(X,Y)}}{\hat{q}_Y^\lambda(X)\hat{q}_{Y|X}^{1-\lambda}(Y)} \right]
$$

$$
= -\log \sum_{x,y} P_X(x)e^{-\lambda d(x,y)}\hat{q}_Y^\lambda(y)\hat{q}_{Y|X}^{1-\lambda}(y|x). \tag{148}
$$

This implies that

$$
\min_{\hat{q}_{XY}} \min_{q_{XY}} F_{\lambda\nu}^{(\lambda,\mu)}(q_{XY}, \hat{q}_{XY}|P_X) = \min_{\hat{q}_{XY}} F_{\lambda\nu}^{(\lambda,\mu)}(q_{XY}(\hat{q}_{XY}), \hat{q}_{XY}|P_X) = \min_{\hat{q}_{XY}} \left( -\log \sum_{x,y} P_X(x)e^{-\lambda d(x,y)}\hat{q}_Y^\lambda(y)\hat{q}_{Y|X}^{1-\lambda}(y|x) \right).
$$

See Appendix [F] for the proof.

The algorithm based on Lemmas 24 and 25 is shown in Algorithm [7]
Proof: It is obvious from the definition of Algorithm 7 for computing $\min_{q_{XY}} \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X)$.

Input: The probability distribution $P_X$ of the information source, $\lambda \in (0,1)$, and $\nu \geq 0$. The distortion measure $d(x,y)$ is also given.

Choose any initial joint distribution $q_{XY}^0$ such that every element is strictly positive.

for $t = 0, 1, 2, \ldots$ do

Update the joint probability distribution as

$$q_{XY}^{t+1}(x,y) = \frac{P_X(x)e^{-\lambda d(x,y)}q_Y^t(y)^\lambda q_{Y|X}^t(y|x)^{1-\lambda}}{\sum_{x'}\sum_{y'} P_X(x')e^{-\lambda d(x',y')}q_Y^t(y')^\lambda q_{Y|X}^t(y'|x')^{1-\lambda}}. \quad (149)$$

end for

Algorithm 7 for $G_{CK}(R, \Delta|P)$ is analogous to Algorithm 4 for $G_{DK}(R, \Gamma|W)$. As is discussed below, this observation motivated the development of Algorithms 5 and 6 which are similar to Algorithms 4 and 2.

The following holds for the function $F_{JO,s}^{(\lambda,\nu)}(q_{XY}, \hat{q}_{XY}|P_X)$.

$$F_{JO,s}^{(\lambda,\nu)}(q_{XY}, \hat{q}_{XY}|P_X) = \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X) + \lambda D(q_{Y}||\hat{q}_{Y}) + (1-\lambda)D(q_{Y|X}||\hat{q}_{Y|X}|q_{X}) \quad (150)$$

Then, as before, we replace $\hat{q}_{Y}$ and $\hat{q}_{Y|X}$ in (150) with $\hat{p}_{Y}$ and $p_{Y|X}$ to define

$$\hat{F}_{JO,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_{Y}, p_{Y|X}|P_X) = \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X) + \lambda D(q_{Y}||\hat{p}_{Y}) + (1-\lambda)D(q_{Y|X}||p_{Y|X}|q_{X}) \quad (151)$$

Here, we use the following function introduced by Arimoto [11] to derive his algorithm.

Definition 18: For a given source probability distribution $P_X$, an output distribution of the test channel $p_Y$, and parameters $\rho$ and $t$, define*:

$$F_{AR,s}^{(\rho,\nu)}(\hat{p}_{Y}, p_{Y|X}|P_X) = \frac{1}{\rho} \log \sum_x \sum_y P_X(x)e^{\rho d(x,y)}\hat{p}_{Y}^{-\rho}(y)p_{Y|X}^{1+\rho}(y|x). \quad (152)$$

We observe the similarity in (151) and (152).

Then, we have the following lemma.

Lemma 26: For a fixed $\lambda \in [0,1]$ and $q_{XY}$, we have

$$\min_{\hat{p}_{Y} \in P_X | Y} \hat{F}_{s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_{Y}, p_{X|Y}|P_X) = \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X). \quad (153)$$

Proof: It is obvious from the definition of $\hat{F}_{s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_{Y}, p_{X|Y}|P_X)$ and non-negativity of the relative entropy.

Lemma 27: For a fixed $\lambda \in [0,1]$, $\nu \geq 0$, $\hat{p}_{Y}$, and $p_{Y|X}$, we have

$$\min_{q_{XY}} \hat{F}_{JO,s}^{(\lambda,\nu)}(q_{XY}, \hat{p}_{Y}, p_{X|Y}|P_X) = -\log \sum_{x,y} P_X(x)e^{-\lambda d(x,y)}\hat{p}_{Y}(y)p_{Y|X}^{1-\lambda}(y|x)$$

$$= \lambda F_{AR,s}^{(\lambda,\nu)}(\hat{p}_{Y}, p_{Y|X}|P_X). \quad (154)$$

*More precisely, Arimoto defined (152) multiplied $-1$ as the objective function and its maximization was discussed.
See Appendix H for the proof.

The above-mentioned relationship is illustrated in Fig. 6. The triple minimization $\min_{q_{XY}} \min_{\hat{p}_Y} \min_{p_{X|Y}} \hat{F}_{JO,s}(q_{XY}, \hat{p}_Y, p_{X|Y})$ is shown in the top level and is connected to three double minimization, as described in the following proposition.

**Proposition 10:** For any fixed $\lambda \in [0, 1]$, $\nu \geq 0$, and $P_X \in \mathcal{P}(X)$, we have

$$\min \min \min \hat{F}_{JO,s}^{(\lambda, \nu)}(q_{XY}, \hat{p}_Y, p_{X|Y}|P_X)$$

$$= \min \min J_{1,s}(q_{XY}, \hat{p}_Y|P_X),$$

$$= \min \min J_{2,s}(q_{XY}, \hat{p}_Y|P_X),$$

$$= \min \min J_{AR,s}(\hat{p}_Y, p_{Y|X}|P_X).$$

We have derived Algorithms 5 and 6 from the double minimization of (155) and (156). The property of the double minimization of (157) will be discussed in Section VI-A. We will show that the algorithm derived from (157) is the same as Arimoto algorithm except that $\rho$ is negative. We then show that Arimoto algorithm works correctly also for negative $\rho$.

### VI. Computation of the Error Exponents in Lossy Source Coding

In this section, Arimoto’s algorithm for computing the error exponents in lossy source coding is discussed and is compared with Algorithms 5, 6, and 7. As we have shown in Section V-A, Csiszar and Körner’s strong converse is equal to a function that is similar to Blahut’s error exponent, except that the slope parameter $\rho$ takes negative value. Hence, it is natural to expect that the Arimoto algorithm can be used to compute the strong converse exponent in lossy source coding. To the best our knowledge, this issue has not been studied. In Section VI-A, we have proved that Csiszar and Körner’s strong converse exponent matches with a Blahut-style exponent function with parameter $\rho \in [-1, 0]$. Thus, the answer to the natural question above is yes. Note that the strong converse exponent of lossy source coding [2] was established after the publication of [11], and Arimoto was unable to mention the computation of this exponent.

We will also show that Arimoto’s algorithm can be used to compute Arikan and Merhav’s guessing exponent [14].

#### A. Arimoto algorithm for lossy source coding exponents

The target of Arimoto’s algorithm for lossy source coding is Blahut’s error exponent for lossy source coding, defined by

$$E_B(R, \Delta|P_X) = \max_{\rho \geq 0} \min_{\nu \geq 0} \left[ \rho R + \rho \nu \Delta - \min_{\hat{p}_Y} E_{0,s}^{(\rho, \nu)}(\hat{p}_Y|P_X) \right].$$

The exponent functions satisfy the following property.

**Property 7 (Theorem 21 and Theorem 22 in [16]):**

(a) For a fixed $\Delta \geq 0$, $E_B(R, \Delta|P_X)$ is monotone non-decreasing function of $R$. For a fixed $R \geq 0$, $E_B(R, \Delta|P_X)$ is monotone non-decreasing $\Delta$.

(b) $E_B(R, \Delta|P_X)$ is convex in $(R, \Delta)$. 

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Fig. 9. Blahut’s error exponent and the generalized cutoff rate in lossy source coding.

(c) For $R \leq R(\Delta |P_X)$, $E_B(R|P_X) = 0$ and for $R > R(\Delta |P_X)$, $E_B(R|P_X)$ is strictly positive.

Graphing of the exponent function $E_B(R, \Delta |P_X)$ is similar to that of $G_CK(R, \Delta |P_X)$. For the convenience, let us denote the minimum value of $E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X)$ by

$$E_{0,s}^{(\rho,\nu)}(P_X) \triangleq \min_{\hat{p}_Y} E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X).$$

A rough sketch of $E_B(R, \Delta|P)$ is shown in Fig. 9 as a solid curve. In this figure, we define

$$E_B^{(\rho)}(\Delta|P_X) \triangleq \rho \max_{\nu \geq 0} \left[ \nu \Delta - \frac{1}{\rho} E_{0,s}^{(\rho,\nu)}(P_X) \right], \quad \rho > 0. \tag{160}$$

For $\rho = 0$, $(1/\rho)E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X)$ is interpreted as

$$\lim_{\rho \to 0} \frac{1}{\rho} E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X) = - \sum_x P_X(x) \log \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)}. \tag{161}$$

Then, we have

$$E_B(R, \Delta|P_X) = \max_{\rho \geq 0} \left[ \rho R + E_B^{(\rho)}(\Delta|P_X) \right]. \tag{162}$$

Eq.(162) shows that $E_B(R, \Delta|P_X)$ is obtained by the Lagrange transformation of $-E_B^{(\rho)}(\Delta|P_X)$ as a function of $\rho \geq 0$. Moreover, Eq.(159) shows that for a fixed $\rho \geq 0$, $E_B^{(\rho)}(\Delta|P_X)$ is obtained by the Lagrange transformation of $(1/\rho)E_{0,s}^{(\rho,\nu)}(P_X)$ as a function of $\nu \geq 0$, multiplied by $\rho$. In Fig. 8, $R^{(\rho)}(\Delta|P_X)$ is the source-coding generalized cutoff rate defined as the $R$-axis intercept of the supporting line of slope $\rho > 0$ to the curve $E_B(R, \Delta|P_X)$.

Now we describe the Arimoto’s algorithm for computing (159). The following theorem in the case of $\rho > 0$ was suggested in [11], although the details of the proof were omitted. We give a complete proof for $\rho > 0$ as well as $\rho \in [-1,0)$. The theorem for the case of negative $\rho$ suggests that Arimoto algorithm successfully works for computing the exponent function defined in Definition 9.
Similarly, from (168), we also have

\[
\min_{p_{Y|X}} F_{\mathcal{AR},s}^{(\rho,\nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \frac{1}{\rho} \log \sum_x P_X(x) \left\{ \sum_y \hat{p}_Y(y) e^{-\nu d(x,y)} \right\}^{-\rho} = \frac{1}{\rho} E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X).
\]

(163)

The maximum value is achieved if and only if

\[
p_{Y|X}(y|x) = \frac{\hat{p}_Y(y)e^{-\nu d(x,y)}}{\sum_{y'} \hat{p}_Y(y')e^{-\nu d(x,y')}} \triangleq p_{Y|X}^*(\hat{p}_Y)(y|x).
\]

(164)

On the other hand, for any fixed \( \rho \in [-1, 0) \cup (0, \infty), \nu \geq 0 \), and any fixed \( p_{Y|X} \), we have

\[
\min_{\hat{p}_Y} F_{\mathcal{AR},s}^{(\rho,\nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \frac{1}{\rho} A_s^{(\rho,\nu)}(p_{Y|X}|P_X),
\]

(165)

where \( A_s^{(\rho,\nu)}(p_{Y|X}|P_X) \) is the function defined in (134). The maximum value is achieved if and only if

\[
\hat{p}_Y(y) = \left\{ \sum_x P_X(x)e^{\rho d(x,y)} \right\}^{1/(1+\rho)} \cdot \frac{\sum_y \left\{ \sum_x P_X(x)e^{\rho d(x,y')} \right\}^{1/(1+\rho)}}{\sum_y \left( \sum_x P_X(x)e^{\rho d(x,y')} \right)^{1/(1+\rho)}} \triangleq \hat{p}_Y^*(p_{Y|X})(y).
\]

(166)

The proof for the case \( \rho > 0 \) was omitted in [11] because it is analogous to the proof for the similar theorem for channel coding. Since we have extended the range of \( \rho \) to \([-1, \infty)\), we give the proof of Theorem 6 to make the paper self-contained. To this aim, we give the following lemma.

**Lemma 28**: We have

\[
F_{\mathcal{AR},s}^{(\rho)}(\hat{p}_Y, p_{Y|X}|P_X) = \frac{1}{\rho} E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X) + D_{1+\rho}(p_{Y|X}||p_{Y|X}^*(\hat{p}_Y)|P_X)
\]

(167)

and the minimum value is achieved if and only if \( p_{Y|X} = p_{Y|X}^*(\hat{p}_Y) \) because of the property of Rényi divergence.

Similarly, from (168), we also have

\[
\min_{p_Y} F_{\mathcal{AR},s}^{(\rho,\nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \frac{1}{\rho} A_s^{(\rho,\nu)}(p_{Y|X}|P_X).
\]

(168)

See Appendix C for the proof.

**Proof of Theorem 6**: From (167), we have

\[
\min_{p_{Y|X}} F_{\mathcal{AR},s}^{(\rho,\nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \frac{1}{\rho} E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X).
\]

(169)

and the minimum value is achieved if and only if \( p_{Y|X} = p_{Y|X}^*(\hat{p}_Y) \).

Similarly, from (168), we also have

\[
\min_{p_Y} F_{\mathcal{AR},s}^{(\rho,\nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \frac{1}{\rho} A_s^{(\rho,\nu)}(p_{Y|X}|P_X).
\]

(170)

and the minimum value is achieved if and only if \( \hat{p}_Y = \hat{p}_Y^*(p_{Y|X}) \). This completes the proof.

**Corollary 3**: For any fixed \( \rho \in [-1, 0) \cup (0, \infty) \) and \( \nu \geq 0 \), we have

\[
\min_{\hat{p}_Y} \frac{1}{\rho} E_{0,s}^{(\rho,\nu)}(\hat{p}_Y|P_X) = \min_{p_{Y|X}} \min_{\hat{p}_Y} F_{\mathcal{AR},s}^{(\rho,\nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \min_{p_{Y|X}} \min_{\rho \geq 0} \frac{1}{\rho} A_s^{(\rho,\nu)}(p_{Y|X}|P_X).
\]

(171)

**Corollary 4**: Define the following function

\[
\hat{E}_B(R, \Delta|P_X) = \max \min_{\rho \geq 0} \left\{ \rho R + \rho \nu \Delta - \min_{p_{Y|X}} A_s^{(\rho,\nu)}(p_{Y|X}|P_X) \right\}
\]

(172)
Then, for any $R \geq 0$, $\Delta \geq 0$, and $P_X \in \mathcal{X}$, we have

$$\tilde{E}_B(R, \Delta|P_X) = E_B(R, \Delta|P_X).$$  \hfill (173)

We call $\tilde{E}_B(R, \Delta|P_X)$ as the alternative expression for the Blahut’s error exponent.

Arimoto algorithm for computing the Blahut’s error exponent is shown in Algorithm 8 which updates $\hat{p}_Y$ and $p_{Y|X}$ alternately.

**Algorithm 8** Arimoto’s algorithm [11]

**Input:** A probability distribution of information source $P_X$, $\rho > -1$, and $\nu \geq 0$. A distortion measure $d(x, y)$ is also given.

Choose initial $\hat{p}_Y^{[1]}$ such that all components are nonzero.

for $t = 1, 2, 3, \ldots$ do

$$p_{Y|X}^{[t]}(y|x) = \frac{\hat{p}_Y^{[t]}(y)e^{-\nu d(x, y)}}{\sum_{y'} \hat{p}_Y^{[t]}(y')e^{-\nu d(x, y')}},$$  \hfill (174)

$$\hat{p}_Y^{[t+1]}(y) = \left\{\frac{\sum_x P_X(x)e^{\rho \nu d(x, y)} p_{Y|X}^{[t]}(y|x)^{1+\rho}}{\sum_{y'} \left\{\sum_x P_X(x)e^{\rho \nu d(x, y')} p_{Y|X}^{[t]}(y'|x)^{1+\rho}\right\}^{1/(1+\rho)}}\right\}^{1/(1+\rho)}.$$  \hfill (175)

end for

For the case of $\rho = 0$, the update rules (174) and (175) reduces to the rule of the Arimoto-Blahut algorithm for the rate-distortion function. $F_{AR,s}^{(0, \nu)}(\hat{p}_Y, p_{Y|X}|P_X)$ for $\rho = 0$ is interpreted as

$$\lim_{\rho \to 0} F_{AR,s}^{(\rho, \nu)}(\hat{p}_Y, p_{Y|X}|P_X) = \sum_{x,y} P_X(x)p_{Y|X}(y|x) \left(\log \frac{p_{Y|X}(y|x)}{p_Y(y)} + \nu d(x, y)\right).$$  \hfill (176)

At the end of this subsection, we give a remark regarding the function $(1/\rho)A_s^{(\rho, t)}(\hat{p}_{Y|X}|P_X)$. When $\nu = 0$, the function is equal to Sibson’s mutual information of order $\alpha = 1 + \rho$, i.e., we have

$$\frac{1}{\rho} A_s^{(\rho, 0)}(p_Y|P_X) = I_{1+\rho}^s(P_X, p_Y|X).$$  \hfill (177)

One can easily check this equality holds. Hence $(1/\rho)A_s^{(\rho, \nu)}(p_{Y|X}|P_X)$ is considered as an extension of Sibson’s mutual information for lossy source coding. Note that, for Gallager’s $E_0$-function, $\alpha$ is $1/(1 + \rho)$, i.e.,

$$\frac{1}{\rho} E_0^{(\rho, 0)}(P_X|W) = I_{1+\rho}^{s_0}(p_X, W).$$

**B. Arikan and Merhav’s guessing exponent**

In this subsection, we show that Arimoto’s algorithm can be used to compute Arikan and Merhav’s guessing exponent [14]. The optimal error exponent in guessing problem under distortion was established by Arikan and Merhav [14]. Their exponent is defined as follows:


**Definition 19 ([14]):** For $\Delta \geq 0$, $\rho \geq 0$, and $P_X \in \mathcal{P}(\mathcal{X})$, $\rho$-th order guessing exponent at distortion level $\Delta$ is defined by

$$E_{AM}^{(\rho)}(\Delta|P_X) = \sup_{q_X \in \mathcal{P}(\mathcal{X})} \{ \rho R(\Delta|q_X) - D(q_X||P_X) \}. \quad (178)$$

We have the following lemma.

**Lemma 29:** For any $\Delta \geq 0$, $\rho \geq 0$, $P_X \in \mathcal{P}(\mathcal{X})$, we have

$$E_{AM}^{(\rho)}(\Delta|P_X) = -E_B^{(\rho)}(\Delta|P_X). \quad (179)$$

**Proof:** The rate distortion function is expressed as [30], [2] Chapter 8.

$$R(\Delta|P_X) = \max_{\nu \geq 0} \left\{ -\sum_x P_X \log \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} - \nu \Delta \right\}. \quad (180)$$

Then, substituting (180) into (178), we have

$$E_{AM}^{(\rho)}(\Delta|P_X) = \begin{align*}
&= \sup_{q_X \in \mathcal{P}(\mathcal{X})} \max_{\nu \geq 0} \min_{\rho} \left\{ -\rho \sum_x q_X(x) \log \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} - \rho \nu \Delta - D(q_X||P_X) \right\} \\
&= \max_{\nu \geq 0} \min_{\rho} \sup_{q_X \in \mathcal{P}(\mathcal{X})} \left\{ -\sum_x q_X(x) \log \frac{q_X(x)}{P_X(x) \left( \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} \right)^{-\rho}} - \rho \nu \Delta \right\} \\
&= \max_{\nu \geq 0} \min_{\rho} \left\{ \log \sum_x P_X(x) \left[ \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} \right]^{-\rho} - \rho \nu \Delta \right\} \\
&= -E_B^{(\rho)}(\Delta|P_X). \quad (181)
\end{align*}$$

Step (a) follows from (180), Step (b) holds because the objective function is concave in $q_X$ for a fixed $\hat{p}_Y$ and convex in $\hat{p}_Y$ for a fixed $q_X$, Step (c) holds because the maximum value of the objective function over $q_X$ is attained if

$$q_X(x) = \frac{P_X(x) \left[ \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} \right]^{-\rho}}{\sum_x P_X(x) \left[ \sum_y \hat{p}_Y(y)e^{-\nu d(x,y)} \right]^{-\rho}},$$

and Step (d) follows from (160).

Arikan and Merhav showed that for fixed $\Delta \geq 0$, $E_{AM}^{(\rho)}(\Delta|P_X)$ as a function of $\rho$ is the one-sided LFT of Marton’s error exponent $E_M(R,\Delta|P_X)$ as a function of $R$ and its inverse relation that the one-sided LFT of $E_{AM}^{(\rho)}(\Delta|P_X)$ as a function of $\rho$ is the lower convex hull of $E_M(R,\Delta|P_X)$ as a function of $R$ [14, Theorem 2]. Thus from Lemma 29 we have the following proposition.

**Proposition 11:** Blahut’s error exponent is equal to a lower convex full of Marton’s error exponent that is expressed by

$$E_B(R, \Delta|P_X) = \tilde{E}_M(R, \Delta|P_X) \triangleq \max_{\rho \geq 0} \min_{q_X} \{ D(q_X||P_X) + \rho [R - R(\Delta|q_X)] \}. \quad (182)$$

Expression (182) is found in [14, Eq.(30)], but the explicit relationship with Blahut’s exponent is shown here for the first time. Arikan and Merhav described how to calculate $E_{AM}^{(\rho)}(\Delta|P_X)$ using Gallager’s formula [32, Theorem 9.4.1] for the rate distortion function, but did not mention its relationship to Arimoto’s algorithm. Lemma 29 and 160 ensure that we can employ Arimoto’s algorithm for computing Arikan and Merhav’s guessing exponent.
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APPENDIX A

Proofs of Properties 2 and 3

In this section, we give proof of Properties 2 and 3.

**Proof of Property 2** We can decompose $J_1^{(\lambda, \mu)}(q_{XY}, p_X|W)$ as

$$J_1^{(\lambda, \mu)}(q_{XY}, p_X|W) = -(1 - \lambda)H(q_{XY}) - \lambda H(q_Y) - E_{q_{XY}}[\log W(Y|X)] - (1 - \lambda)E_{q_{XY}}[\log p_X(X)] + \mu E_{q_X}[c(X)].$$

The first term is a joint entropy of $X$ and $Y$ multiplied by $-(1 - \lambda) < 0$ and thus is a convex function of $q_{XY}$. The marginal distribution $q_Y$ is a linear function of $q_{XY}$ and thus the second term is a convex function of $q_{XY}$. The third term is a linear function of $q_{XY}$. The fourth term is a linear function of $p_X$ for a fixed $q_{XY}$ and a convex function of $p_X$ for a fixed $q_{XY}$ because $-\log t$ is a convex function of $t$. The fifth term is linear in $q_X$. This completes the proof.

Next, we prove Property 3.

**Proof of Property 3** We can expand $J_2^{(\lambda, \mu)}(q_{XY}, \hat{p}_X|Y|W)$ as

$$J_2^{(\lambda, \mu)}(q_{XY}, \hat{p}_X|Y|W) = D(q_Y|X|W||q_X) - \lambda H(q_X) - \lambda E_{q_{XY}}[\log \hat{p}_X|Y (Y|X)] + \mu E_{q_X}[c(X)].$$

Part a) holds because the first, the second and the fourth terms are constant in $\hat{p}_X|Y$, and the third term is convex (resp. concave) because of the concavity of $\log$ and linearity of the expectation operation. Part b) holds because of the convexity (resp. concavity) of the second term, while the first, the third and the fourth terms are constant in $q_X$. Part c) holds because of the convexity of $D(q_Y|X|W||q_X)$, which completes the proof.
APPENDIX

PROOF OF LEMMAS 7, 21, 23, 16, AND 17

Proof of Lemma 7 For $\lambda \in [0, 1]$, we have the following chain of inequalities.

\[
J_{2}^{(\lambda, \mu)}(q_{XY}, \tilde{P}_{X|Y}|W) = \mathbb{E}_{q} \left[ \log \frac{q_{Y|X}(Y|X)q_{X}^{\lambda}(Y)}{W(Y|X)e^{-\mu c(X)}P_{X|Y}^{\lambda}(X|Y)} \right]
\]

\[= -\lambda H(q_{X}) + \mathbb{E}_{q_{X|Y}} \left[ \log \frac{q_{Y|X}(Y|X)q_{X}^{\lambda}(Y)}{W(Y|X)e^{-\mu c(X)}P_{X|Y}^{\lambda}(X|Y)} \right]
\]

\[= -\lambda H(q_{X}) + D(q_{Y|X}|\tilde{q}_{Y|X}(\tilde{P}_{X|Y}|)q_{X}) - \sum_{x} q_{X}(x) \log \sum_{y} W(y|x)e^{-\mu c(x)}P_{X|Y}^{\lambda}(x|y)
\]

\[\geq -\lambda H(q_{X}) - \sum_{x} q_{X}(x) \log \frac{q_{X}(x)}{\left\{ \sum_{y} W(y|x)e^{-\mu c(x)}P_{X|Y}^{\lambda}(x|y) \right\}^{1/\lambda}}
\]

\[= \lambda D(q_{X}|\tilde{q}_{X}(\tilde{P}_{X|Y})) - \lambda \log \sum_{x} \left\{ \sum_{y} W(y|x)e^{-\mu c(x)}P_{X|Y}^{\lambda}(x|y) \right\}^{1/\lambda}
\]

\[\geq A^{(-\lambda, \mu)}(\tilde{P}_{X|Y}|W).
\] (185)

Steps (a) and (b) follow from the nonnegativity conditional relative entropy and ordinary relative entropy. Their equality condition show that the minimum is attained when $q_{Y|X} = \tilde{q}_{X|Y}(\tilde{P}_{X|Y})$ and $q_{X} = \tilde{q}_{X}(\tilde{P}_{X|Y})$ are satisfied. Thus we have $\min_{q_{X}} \min_{q_{Y|X}} J_{2}^{(\lambda, \mu)}(q_{XY}, \tilde{P}_{X|Y}|W) = A^{(-\lambda, \mu)}(\tilde{P}_{X|Y}|W)$. This completes the proof.

The sign of $\lambda$ is changed from positive for Lemma 5 to negative in Lemmas 16 and 17. We need to carefully check the convexity of $J_{2}^{(\lambda)}(q_{XY}, \tilde{P}_{X|Y}|W)$.

Proof of Lemma 21 Let the rhs’s of (121) and (122) be $\tilde{q}_{Y|X}(\tilde{P}_{Y})(y|x)$ and $\tilde{q}_{X}(\tilde{P}_{Y})(x)$. Then, we have

\[
J_{1, s}^{(\lambda, \nu)}(q_{XY}, \tilde{P}_{Y}|W) = \mathbb{E}_{q_{XY}} \left[ \log \frac{q_{X}(X)q_{X|Y}(X|Y)}{P_{X}(X)e^{-\nu d(X,Y)}\tilde{P}_{Y}^{\nu}(Y)} \right]
\]

\[= D(q_{X}|P_{X}) + \lambda \mathbb{E}_{q_{XY}} \left[ \log \frac{q_{Y|X}(Y|X)}{\tilde{P}_{Y}(Y)e^{-\nu d(X,Y)}} \right]
\]

\[= D(q_{X}|P_{X}) + \lambda D(q_{Y|X}|\tilde{q}_{Y|X}(\tilde{P}_{Y})|q_{X}) - \lambda \sum_{x} q_{X}(x) \log \sum_{y} \tilde{P}_{Y}(y)e^{-\nu d(x,y)}
\]

\[\geq D(q_{X}|P_{X}) - \lambda \sum_{x} q_{X}(x) \log \sum_{y} \tilde{P}_{Y}(y)e^{-\nu d(x,y)}
\]

\[= \sum_{x} q_{X}(x) \log \frac{q_{X}(x)}{P_{X}(x) \left\{ \sum_{y} \tilde{P}_{Y}(y)e^{-\nu d(x,y)} \right\}^{\lambda}}
\]

\[= D(q_{X}|\tilde{q}_{X}(\tilde{P}_{Y})) - \log \sum_{x} P_{X}(x) \left\{ \sum_{y} \tilde{P}_{Y}(y)e^{-\nu d(x,y)} \right\}^{\lambda}
\]

\[\geq - \log \sum_{x} P_{X}(x) \left\{ \sum_{y} \tilde{P}_{Y}(y)e^{-\nu d(x,y)} \right\}^{\lambda}
\]

\[= E_{0, s}^{(-\lambda, \nu)}(\tilde{P}_{Y}||P_{X}).
\]
Proof of Lemma 23: Let the rhs’s of (137) and (138) be \( \bar{q}_{X|Y}(p_{Y|X}) \) and \( \bar{q}_Y(p_{Y|X}) \). Then we have

\[
J_{1,s}^{(\lambda,\nu)}(q_{XY}, p_{Y|X}|P_X) = E_{q_{XY}} \left[ \log \frac{q_{Y}(Y)q_{X|Y}(X|Y)}{P_X(X)e^{-\lambda \nu d(X,Y)p_{Y|X}(Y|X)}} \right]
\]

\[
= - (1 - \lambda)H(q_Y) + E_{q_{XY}} \left[ \log \frac{q_{X|Y}(X|Y)}{P_X(X)e^{-\lambda \nu d(X,Y)p_{Y|X}(Y|X)}} \right]
\]

\[
= -(1 - \lambda)H(q_Y) + D(q_{X|Y}||\bar{q}_{X|Y}(p_{Y|X}))
\]

\[
- \sum_y q_Y(y) \log \sum_x P_X(x)e^{-\lambda \nu d(x,y)p_{Y|X}(y|x)} \geq (1 - \lambda) \sum_y q_Y(y) \log \frac{q_Y(y)}{\sum_x P_X(x)e^{-\lambda \nu d(x,y)p_{Y|X}(y|x)}^{1/(1-\lambda)}}
\]

\[
= (1 - \lambda)D(q_Y||\bar{q}_Y(p_{Y|X}))
\]

\[
- (1 - \lambda) \sum_y \left\{ \sum_x P_X(x)e^{-\lambda \nu d(x,y)p_{Y|X}(y|x)} \right\}^{1/(1-\lambda)}
\]

\[
\geq -(1 - \lambda) \sum_y \left\{ \sum_x P_X(x)e^{-\lambda \nu d(x,y)p_{Y|X}(y|x)} \right\}^{1/(1-\lambda)}
\]

\[
= A_s^{(\lambda,\nu)}(p_{Y|X}|P_X).
\]

As before, inequalities (a) and (b) follow from the nonnegativity of conditional and ordinary relative entropies. The equality conditions implies that minimum of \( J_{1,s}^{(\lambda,\nu)}(q_{XY}, p_{Y|X}|P_X) \) is attained if both \( q_Y = \bar{q}_Y(p_{Y|X}) \) and \( q_{X|Y} = \bar{q}_{X|Y}(p_{Y|X}) \) are satisfied.

Proof of Lemma 16: From Property 3(c), \( J_2^{(\lambda)}(q_{XY}, \hat{p}_X|Y|W) \) is convex in \( q_{Y|X} \) for a fixed \( q_X \) and \( \hat{p}_X|Y \) for \( \lambda \in [-1, 0) \). Inequality (a) in (185) is valid also. Therefore, we have

\[
\min_{q_{Y|X}} J_2^{(\rho)}(q_{XY}, \hat{p}_X|Y|W) = -\rho \sum_x q_X(x) \log \frac{q_X(x)}{\left\{ \sum_y W(y|x)\hat{p}_{X|Y}(x|y) \right\}^{1/\rho}}
\]

\[
= J_2^{(\rho)}(q_X, \hat{p}_X|Y|W).
\]

The minimum value is attained by \( q_{Y|X} = \bar{q}_{Y|X}(\hat{p}_X|Y) \). This completes the proof.

Proof of Lemma 17: From Property 3(b), \( J_2^{(\rho)}(q_{XY}, \hat{p}_X|Y|W) \) is concave in \( q_X \) for a fixed \( q_{Y|X} \) and \( \hat{p}_X|Y \). For \( \rho \in (0, 1] \). In fact, we have

\[
\max_{q_X} J_2^{(\rho)}(q_X, \hat{p}_X|Y|W)
\]

\[
= \max_{q_X} \left\{ -\rho D(q_X||\bar{q}_X(\hat{p}_X|Y)) + \rho \log \sum_x \left\{ \sum_y W(y|x)\hat{p}_{X|Y}(x|y) \right\}^{1/\rho} \right\}
\]

\[
= \rho \log \sum_x \left\{ \sum_y W(y|x)\hat{p}_{X|Y}(x|y) \right\}^{1/\rho}
\]

\[
= A^{(\rho)}(\hat{p}_X|Y|W).
\]
The maximum is attained by \( q_X = \tilde{q}_X(\hat{p}_{X|Y}) \) because of the equality condition of \( D(q_X||\tilde{q}_X(\hat{p}_{X|Y})) \). This completes the proof.

**APPENDIX C**

**Proofs of Theorems 2, 4, and 5.**

In this section we give proofs of Theorems 2, 4, and 5. Because the proofs are similar each other, we explain only the proof of Theorem 2 in detail. For other proofs, we only give outlines and omit the details.

**Proof of Theorem 2** From (48), we have

\[
\sum_{y'} \hat{p}_{X|Y}(x'|y')W(y'|x) e^{-\mu c(x)} = \frac{\hat{p}_{X|Y}(x|y)A^\lambda W(y|x) e^{-\mu c(x)}}{q_{Y|X}(y|x)}
\]  

(186)

for any \((x, y) \in \mathcal{X} \times \mathcal{Y}\). From (49),

\[
\sum_{x'} \left\{ \sum_{y'} \hat{p}_{X|Y}(x'|y')W(y'|x') e^{-\mu c(x')} \right\}^{1/\lambda} = \left\{ \frac{\sum_{y'} \hat{p}_{X|Y}(x|y')W(y'|x) e^{-\mu c(x)}}{q_{Y|X}(x)} \right\}^{1/\lambda}
\]  

(187)

holds for all \( x \in \mathcal{X} \). Therefore, we have

\[
A(-\lambda, \mu)(\hat{p}_{X|Y}|W)
\]

\[
\overset{(a)}{=} -\lambda \log \sum_{x} \left\{ \sum_{y} \hat{p}_{X|Y}(x|y)W(y|x) e^{-\mu c(x)} \right\}^{1/\lambda}
\]

\[
\overset{(b)}{=} -\lambda \left[ \log \left\{ \sum_{y} \hat{p}_{X|Y}(x|y)W(y|x) e^{-\mu c(x)} \right\}^{1/\lambda} - \log q_{X}(x) \right]
\]

\[
\overset{(c)}{=} -\log \frac{\hat{p}_{X|Y}(x|y)A^\lambda W(y|x) e^{-\mu c(x)}}{q_{Y|X}(y|x)} + \lambda \log q_{X}(x)
\]

\[
= -\log \frac{\hat{p}_{X|Y}(x|y)A^\lambda W(y|x) e^{-\mu c(x)}}{q_{Y|X}(y|x)q_{X}(x)^{\lambda}}
\]  

(188)

for any \((x, y) \in \mathcal{X} \times \mathcal{Y}\). Step (a) follows from the definition. Step (b) follows from (187) and step (c) from (186). Let \( q_{X|Y} \) be a joint distribution that attains \( \min_{q_{XY}} \Theta^{(\lambda, \mu)}(q_{X|Y}|W) \). Then it follow from Proposition 3 and Lemma 6.
that $A^{(-\lambda,\mu)}(\hat{\rho}_{X|Y}|W)$ is minimized by $\hat{\rho}_{X|Y}^* = q_{X|Y}^*$. Then we have

$$0 \leq \Theta^{(\lambda)}(q_{XY}^{[t]}|W) - \Theta^{(\lambda)}(q_{X|Y}^*|W)$$

Step (a) holds because $q_{X|Y}^*$ minimizes $\Theta^{(\lambda)}(P_{X|Y}|W)$. Step (b) follows from Proposition 2. Step (c) follows from (188) holds for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Step (d) follows from (50) and non-negativity of relative entropy.

Let $\xi_t = \Theta^{(\lambda,\mu)}(q_{XY}^{[t]}|W) - \Theta^{(\lambda,\mu)}(q_{X|Y}^*|W)$. Then

$$0 \leq \sum_{t=0}^{T-1} \xi_t$$

$$\leq \lambda \{ D(q_{Y|X}^*||p_{Y|X}^{[0]}|q_{X}^*) - D(q_{Y|X}^*||p_{Y|X}^{[T]}|q_{X}^*) \}$$

$$\leq \lambda D(q_{Y|X}^*||p_{Y|X}^{[0]}|q_{X}^*).$$

(190)

By Proposition 3, $\xi_t$ is a monotone decreasing sequence. Then from (190) we have $0 \leq T \xi_T \leq \lambda D(q_{Y|X}^*||p_{Y|X}^{[0]}|q_{X}^*)$. Thus

$$0 \leq \xi_T \leq \frac{\lambda D(q_{Y|X}^*||p_{Y|X}^{[0]}|q_{X}^*)}{T} \to 0, \quad T \to \infty$$

(191)

Hence, we have

$$\lim_{T \to \infty} \Theta^{(\lambda,\mu)}(q_{XY}^{[t]}|W) = \Theta^{(\lambda,\mu)}(q_{XY}^*|W).$$

(192)

This completes the proof.

Proof of Theorem 2: The line of the proof is the same as Theorem 2. From (125),

$$\sum_{y'} \hat{p}_{Y}^{[t]}(y') e^{-\nu d(x,y')} = \hat{p}_{Y}^{[t]}(y) e^{-\nu d(x,y)}$$

(193)

holds for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and from (126),

$$\sum_{x'} P(x') \left( \sum_{y'} \hat{p}_{Y}^{[t]}(y') e^{-\nu d(x',y')} \right)^{\lambda} = \frac{P(x) \left( \sum_{y'} \hat{p}_{Y}^{[t]}(y') e^{-\nu d(x',y')} \right)^{\lambda}}{q_{X}^{[t+1]}(x)}$$

(194)
holds for every $x \in \mathcal{X}$. Then we have

$$E_{0,s}^{(-\lambda,\nu)}(\hat{p}_Y^{[t]}|P_X)$$

$$= \log \sum_x P(x) \left\{ \sum_y \hat{p}_Y^{[t]}(y)e^{-\nu d(x,y)} \right\}^\lambda$$

$$= \log \frac{P(x)}{q_X^{[t]}(x)} \left\{ \sum_y \hat{p}_Y^{[t]}(y)e^{-\nu d(x,y)} \right\}^\lambda$$

$$= \log \left( \frac{P(x)}{q_X^{[t]}(x)} \right) \left( \frac{\hat{p}_Y^{[t]}(y)e^{-\nu d(x,y)}}{q_{Y|X}^{[t+1]}(y|x)} \right)^\lambda.$$ (195)

Step (a) follows from (194) and step (b) follows from (195).

Let $q_{XY}^*$ be a joint distribution that minimizes $\Theta_s^{(\lambda,\nu)}(q_{XY}|P_X)$. Then, we have

$$0 \leq \Theta_s^{(\lambda,\nu)}(q_{XY}^*|P_X) - \Theta_s^{(\lambda,\nu)}(q_{XY}|P_X)$$

$$\leq -E_{q_{XY}^*}^{(\lambda,\nu)}(\hat{p}_Y^{[t]}|P_X) - \Theta_s^{(-\lambda,\nu)}(q_{XY}^*|P_X)$$

$$= E_{q_{XY}^*} \left[ - \log \frac{P(X, Y)}{q_X^{[t]}(X)} \hat{p}_Y^{[t]}(Y) e^{-\nu d(X,Y)} q_{Y|X}^{[t+1]}(Y|X) \right]$$

$$= E_{q_{XY}^*} \left[ \log \frac{q_Y^{[t]}(Y)}{\hat{p}_Y^{[t]}(Y)} q_{XY}^*(X) q_{Y|X}^*(Y|X) \right] - E_{q_{XY}^*} \left[ \log \frac{q_{XY}^*(X) q_{Y|X}^*(Y|X) \lambda}{P(X) e^{-\lambda d(X,Y)}} \right]$$

$$\leq \lambda E_{q_{XY}^*} \left[ \log \left( \frac{q_Y^{[t]}(Y)}{\hat{p}_Y^{[t]}(Y)} \right) \right] = \lambda \{ D(p_{X|Y}^*||p_{X|Y}) - D(p_{X|Y}^*||p_{X|Y}^{[t+1]}) \}. \quad (196)$$

Step (a) follows from (195) and (124), step (b) holds because (195) holds for every $x$ and $y$, and step (c) follows from (127).

Let $\xi_t = \Theta_s^{(\lambda,\nu)}(q_{XY}^*|P_X) - \Theta_s^{(\lambda,\nu)}(q_{XY}^*|P_X)$. Then by executing the same procedure as in the proof of Theorem 1, we can show that $\Theta_s^{(\lambda,\nu)}(q_{XY}^*|P_X)$ converges to $\Theta_s^{(\lambda,\nu)}(q_{XY}^*|P_X)$. The details are omitted. \qed

**Proof of Theorem 5** From (143) we have

$$\sum_{x'} P_X(x') e^{-\lambda d(x',y)} p_{Y|X}^{[t]}(y|x')^{1-\lambda} = \frac{P_X(x)e^{-\lambda d(x,y)} p_{Y|X}^{[t]}(y|x)^{1-\lambda}}{q_{Y|X}^{[t]}(y)}$$ (197)

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ and from (144).

$$\sum_{y'} \left[ \sum_{x'} P_X(x') e^{-\lambda d(x',y')} p_{Y|X}^{[t]}(y'|x')^{1-\lambda} \right]^{1/(1-\lambda)} = \left[ \sum_{x'} P_X(x') e^{-\lambda d(x',y')} p_{Y|X}^{[t]}(y|x')^{1-\lambda} \right]^{1/(1-\lambda)} \quad (198)$$
for all \( y \in \mathcal{Y} \). Then, we have

\[
-A_s^{(\lambda,\nu)}(p_{Y|X}^{[t]}|P_X)
= -(1 - \lambda) \log \sum_x \left[ \sum_y P_X(x) e^{-\lambda \rho(x,y)} p_{Y|X}^{[t]}(y|x)^{1-\lambda} \right]^{1/(1-\lambda)}
\]

\[
\leq (1 - \lambda) \log \left\{ \sum_x P_X(x) e^{-\lambda \rho(x,y)} p_{Y|X}^{[t]}(y|x) \right\}^{1/(1-\lambda)} - \log q_y^{[t]}(y)
\]

\[
= \log \frac{P_X(x) e^{-\lambda \rho(x,y)} p_{Y|X}^{[t]}(y|x)^{1-\lambda}}{q_y^{[t]}(y)}
\]

Step (a) follows from Proposition \[198\] and Step (b) follows from \[197\]. Let \( q_{XY}^* \) be a joint distribution that minimizes \( \Theta_s^{(\lambda)}(q_{XY}^*|P_X) \). Then, we have

\[
0 \leq \Theta_s^{(\lambda)}(q_{XY}^*|P_X) - \Theta_s^{(\lambda)}(q_{XY}^{[t]}|P_X)
\]

\[
\leq A^{(-\lambda,\nu)}(p_{Y|X}^{[t]}|P_X) - \Theta_s^{(\lambda,\nu)}(q_{XY}^{[t]}|P_X)
\]

\[
= E_{q_{XY}^*} \left[ -\log \frac{P_X(X) e^{-\lambda \rho(X,Y)} p_{Y|X}^{[t]}(Y|X)^{1-\lambda}}{q_{Y|X}^{[t]}(Y|X)^{1-\lambda}} \right] - E_{q_{XY}^{[t]}} \left[ -\log \frac{q_{XY}^*(X) e^{-\lambda \rho(X,Y)}}{P_X(X) e^{-\lambda \rho(x,y)}} \right]
\]

\[
= (1 - \lambda) E_{q_{XY}^{[t]}} \left[ \log \frac{p_{Y|X}^{[t+1]|Y|X}(Y|X)}{p_{Y|X}^{[t]}(Y|X)} \right] - (1 - \lambda) D(q_{XY}^*||q_{Y|X}^{[t]}) - \lambda D(q_{XY}^*||q_{Y|X}^{[t]})
\]

\[
\leq (1 - \lambda) E_{q_{XY}^{[t]}} \left[ \log \frac{p_{Y|X}^{[t+1]|Y|X}(Y|X)}{p_{Y|X}^{[t]}(Y|X)} \right] - D(q_{XY}^*||q_{Y|X}^{[t]}) - D(q_{XY}^*||q_{Y|X}^{[t]})
\]

Step (a) follows from Proposition \[9\], Step (b) holds because \[199\] holds for every \( x, y \). Step (c) follows from \[145\].

Let \( \xi_t = \Theta_s^{(\lambda)}(q_{XY}^{[t]}|P_X) - \Theta_s^{(\lambda)}(q_{XY}^*|P_X) \). Then, by the same procedure as the proof for Theorem \[1\] we can prove \( 0 \leq \xi_t \leq \frac{D(q_{XY}^*||p_{Y|X}^{[t+1]}||q_{XY}^*)}{\lambda} \). The detail is omitted.

\[\text{APPENDIX D}\]

\[\text{PROOFS OF LEMMAS 18}\]

\[\text{Proof of 18}\]

Let \( q_{XY}^* \) be a joint distribution that attains \( G(R, \Delta|P) \). From its formula, we have

\[
R(\Delta|q_X^*) \leq I(q_X^*, q_{Y|X}^*)
\]

\[\text{(201)}\]
Thus,

\[ G(R, \Delta | P) = |I(q_X^*, q_{Y|X}^*) - R|^+ + D(q_X^* || P) \]

\[ \geq |R(\Delta | q_X^*) - R|^+ + D(q_X^* || P) \]

\[ \geq \min_{q_X \in \mathcal{P}(X)} \{ |I(q_X, q_{Y|X}) - R|^+ + D(q_X || P) \} \]

\[ = G_{CK}(R, \Delta | P). \]

Step (a) follows from (201). On the other hand, let \( \tilde{q}_X^* \) be a distribution that attains \( G_{CK}(R, \Delta | P) \) and let \( \tilde{q}_{Y|X}^* \) be a conditional distribution that attains \( R(\Delta | \tilde{q}_X^*) \). Then, we have

\[ G_{CK}(R, \Delta | P) = |I(\tilde{q}_X^*, \tilde{q}_{Y|X}^*) - R|^+ + D(\tilde{q}_X^* || P) \]

\[ \geq \min_{q_X \in \mathcal{P}(X)} \{ |I(q_X, q_{Y|X}) - R|^+ + D(q_X || P) \} \]

\[ = G(R, \Delta | P). \]

Thus, we have \( G_{CK}(R, \Delta | P) = G(R, \Delta | P) \), which completes the proof.

\[ \square \]

**APPENDIX E**

**PROOF OF LEMMATA 8 AND 28**

This section gives the proof of Lemma 8 and Lemma 28.

**Proof of Lemma 8**: We have

\[ D_{1+\rho}(p_{X|Y}^*(p_X)||\hat{p}_{X|Y}^*(p_X)) \]

\[ = \frac{1}{\rho} \log \sum_y \hat{p}_{Y|X}(p_X)^{1+\rho}(x|y) \sum_x \hat{p}_{X|Y}^*(x|y) \]

\[ = \frac{1}{\rho} \log \sum_y \left( \sum_x p_{X}(x) \left\{ W(y|x) e^{\rho \text{ovc}(x)} \right\}^{1/\{1+(1+\rho)\}} \right)^{1+\rho} \sum_x \left\{ p_{X}(x) \left\{ W(y|x) e^{\rho \text{ovc}(x)} \right\}^{1/\{1+(1+\rho)\}} \right\}^{1+\rho} \hat{p}_{X|Y}^*(x|y) \]

\[ \leq 1 \rho \log \sum_y \left( \sum_x p_{X}(x) \{W(y|x) e^{\rho \text{ovc}(x)}\}^{1/\{1+(1+\rho)\}} \right)^{1+\rho} + \frac{1}{\rho} \log \sum_x \sum_y p_{X}(x) \{W(y|x) e^{\rho \text{ovc}(x)}\}^{1/\{1+(1+\rho)\}} \hat{p}_{X|Y}^*(x|y) \]

\[ \leq \frac{1}{\rho} F_{0}^{(\rho, \nu)}(p_{X|W} - F_{AR}^{(\rho, \nu)}(p_{X|\hat{p}_{X|Y}|W}), \]

\[ \geq \min_{\nu \in \mathcal{P}(X)} \{ |I(\tilde{q}_X^*, \tilde{q}_{Y|X}^*) - R|^+ + D(\tilde{q}_X^* || P) \} \]

\[ = G_{CK}(R, \Delta | P). \]
which proves (70) and we also have
\[ D_{1+\rho}(p_X||q_X^* (\hat{p}_X|Y)) \]
\[ = \frac{1}{\rho} \sum_x p_X^{1+\rho}(x) q_X^* (\hat{p}_X|Y)(x)^{-\rho} \]
\[ = \frac{1}{\rho} \log \sum_x p_X^{1+\rho}(x) \left\{ \sum_y W(y|x) e^{\rho d(x,y)} \hat{p}_X^{-\rho}(y) \right\}^{-1/\rho} \]
\[ = \log \sum_y \left[ \sum_{x'} W(y|x') e^{\rho d(x',y)} \hat{p}_X^{-\rho}(y) \right]^{-1/\rho} + \frac{1}{\rho} \log \sum_x p_X^{1+\rho}(x) \sum_y W(y|x) e^{\rho d(x,y)} \hat{p}_X^{-\rho}(x|y) \]
\[ = \frac{1}{\rho} A_s^{(\rho,\nu)} (\hat{p}_X|Y) - F_{AR_s}^{(\rho,\nu)} (p_X, \hat{p}_X|Y|W), \]
which proves (71). This completes the proof.

Proof of Lemma 28 We have
\[ \frac{1}{\rho} E_{0,s}^{(\rho,\nu)} (\hat{p}_Y|P_X) + D_{1+\rho}(p_Y|X||p_Y^*|X)(\hat{p}_Y) \]
\[ = \frac{1}{\rho} \log \sum_x P_X(x) \left\{ \sum_y \hat{p}_Y(y) e^{\rho d(x,y)} \right\}^{-\rho} \]
\[ + \frac{1}{\rho} \log \sum_x P_X(x) \sum_y p_Y^* (y|x) \frac{\hat{p}_Y(y) e^{\rho d(x,y)}}{\sum_{y'} \hat{p}_Y(y') e^{\rho d(x,y')}}^{-\rho} \]
\[ = \frac{1}{\rho} \log \sum_x P_X(x) \sum_y p_Y^* (y|x) \hat{p}_Y^{-\rho}(y) e^{\rho d(x,y)} \]
\[ = F_{AR_s}^{(\rho,\nu)} (\hat{p}_Y, p_Y|X|P_X). \] (202)

This proves (167). We also have
\[ \frac{1}{\rho} A_s^{(\rho,\nu)} (p_Y|X|P_X) + D_{1+\rho}(\hat{p}_Y^*(p_Y|X)||\hat{p}_Y) \]
\[ = \frac{1+\rho}{\rho} \log \sum_y \left[ \sum_x P_X(x) e^{\rho d(x,y)} p_Y^* (y|x) \right]^{1+\rho} \]
\[ + \frac{1}{\rho} \log \sum_y \left\{ \sum_x P_X(x) e^{\rho d(x,y)} p_Y^* (y|x) \right\}^{1+\rho} \hat{p}_Y^{-\rho}(y) \]
\[ = \frac{1}{\rho} \log \sum_x P_X(x) e^{\rho d(x,y)} p_Y^* (y|x) \hat{p}_Y^{-\rho}(y) \]
\[ = F_{AR_s}^{(\rho,\nu)} (\hat{p}_Y^*, p_Y|X|P_X). \] (203)

This proves Eq. (168), which completes the proof.
APPENDIX F
PROOF OF PROPERTY 5

In this appendix, we prove Property 5. By definition, Part a) is obvious. For the proof of Part b), let \( q^{(0)} \) and \( q^{(1)} \) be joint distribution functions that attain \( \hat{G}_{CK}(R_0, \Delta_0|P) \) and \( \hat{G}_{CK}(R_1, \Delta_1|P) \), respectively. Denote

\[
\Theta(R, q|P) \triangleq |I(q_X, q_Y|X) - R| + D(q_X||P) = \max\{D(q_X||P), I(q_X, q_Y|X) - R + D(q_X||P)\}. \tag{204}
\]

By definition, we have

\[
\hat{G}_{CK}(R_i, \Delta_i|P) = \Theta(R_i, q^{(i)}|P) \text{ for } i = 0, 1. \tag{205}
\]

For \( \alpha_1 = \alpha \in [0,1] \) and \( \alpha_0 = 1 - \alpha \), we set \( R_\alpha = \alpha_0 R_0 + \alpha_1 R_1, \Delta_\alpha = \alpha_0 \Delta_0 + \alpha_1 \Delta_1 \), and \( q^{(\alpha)} = \alpha_0 q^{(0)} + \alpha_1 q^{(1)} \).

By linearity of \( E_q[d(X,Y)] \) with respect to \( q \), we have that

\[
E_{q^{(\alpha)}}[d(X,Y)] = \sum_{i=0,1} \alpha_i E_{q^{(i)}}[d(X,Y)] \leq \Delta_\alpha. \tag{206}
\]

Because

\[
I(q_X, q_Y|X) + D(q_X||P) = \sum_{x,y} q_{XY}(x,y) \log \frac{q_{XY}(x,y)}{P(x)}
\]

is convex with respect to \( q_{XY} \) and \( D(q_X||P) \) is convex with respect to \( q_X \), we have

\[
I(q_X^{(\alpha)}, q_Y^{(\alpha)}|X) + D(q_X^{(\alpha)}||P) \leq \sum_{i=0,1} \alpha_i \left\{I(q_X^{(i)}, q_Y^{(i)}|X) + D(q_X^{(i)}||P)\right\}, \tag{207}
\]

\[
D(q_X^{(\alpha)}||P) \leq \sum_{i=0,1} \alpha_i D(q_X^{(i)}||P). \tag{208}
\]

Therefore, we have the following two chains of inequalities:

\[
I(q_X^{(\alpha)}, q_Y^{(\alpha)}|X) + D(q_X^{(\alpha)}||P) - R_\alpha \leq \sum_{i=0,1} \alpha_i \left\{I(q_X^{(i)}, q_Y^{(i)}|X) + D(q_X^{(i)}||P) - R_i\right\}, \tag{209}
\]

\[
D(q_X^{(\alpha)}||P) \leq \sum_{i=0,1} \alpha_i D(q_X^{(i)}||P) \leq \sum_{i=0,1} \alpha_i \Theta(R_i, q^{(i)}|P). \tag{210}
\]

Steps (a) and (c) follow from \( 207 \) and \( 208 \) and Steps (b) and (d) follow from the definition of \( \Theta(R_i, q^{(i)}|P) \) for \( i = 0, 1 \). Then, from \( 204 \) we have

\[
\Theta(R_\alpha, q^{(\alpha)}|P) \leq \sum_{i=0,1} \alpha_i \Theta(R_i, q^{(i)}|P). \tag{211}
\]
Therefore,

\[ \hat{G}_{CK}(R_\alpha, \Delta_\alpha|P) = \min_{q \in \mathbb{P}(X \times Y) \mid d(X,Y) \leq \Delta} \Theta(R_\alpha, q|P) \]

\[ \leq \Theta(R_\alpha, q^{(a)}|P) \leq \sum_{i=0,1} \alpha_i \Theta(R_i, q^{(i)}|P) \]

\[ \leq \sum_{i=0,1} \alpha_i \hat{G}_{CK}(R_i, \Delta_i|P). \]

Step (a) follows from (206), Step (b) follows from (211), and Step (c) follows from (205).

For the proof of Part c), the choice of \(q_X = P\) gives \(\hat{G}_{CK}(R, \Delta|P) = 0\), if \(R \geq R(\Delta|P)\). If \(R < R(\Delta|P)\), the choice of \(q_X = P\) makes the first term of the objective function strictly positive, while any choice of \(q \neq P\), \(D(q||P)\) is strictly positive. This completes the proof of Part c).

For the proof of Part d), let \(q^*\) be a joint distribution that attains \(\hat{G}_{CK}(R', \Delta|P)\). Then,

\[ \hat{G}_{CK}(R, \Delta|P) \leq |I(q_X, q^*_Y|X) - R|^+ + D(q^*_X||P) \]

\[ \leq (R' - R) + |I(q^*_X, q^*_Y|X) - R'|^+ \]

\[ + D(q^*_X||P) \]

\[ = (R' - R) + \hat{G}_{CK}(R', \Delta|P). \]

Step (a) follows from \(|x|^+ \leq |x - q|^+ + c\) for \(c \geq 0\). This completes the proof.

\[ \text{Appendix G} \]

\[ \text{Proofs of Lemmas} [19] \]

In this appendix, we give the proof of Lemmas [19]. To this aim, we define the following functions:

\[ \hat{G}^{(\lambda)}_{CK}(R, \Delta|P_X) \triangleq \min_{q_X Y \mid q_X Y \leq \Delta} \{ D(q_X||P_X) + \lambda[I(q_X, q_Y|X) - R] \}, \]

\[ \hat{G}^{(\lambda, \mu)}_{CK}(R, \Delta|P_X) \triangleq \min_{q_X Y} \{ D(q_X||P_X) + \lambda[I(q_X, q_Y|X) - R] \}

\[ + \mu[E_{q_X Y \mid d(X,Y) - \Delta}]. \]

We first prove

\[ \hat{G}_{CK}(R, \Delta|P_X) = \max_{\lambda \in [0,1]} \hat{G}^{(\lambda)}_{CK}(R, \Delta|P_X) \]

(214)

and then prove

\[ \hat{G}^{(\lambda)}_{CK}(R, \Delta|P_X) = \max_{\mu \geq 0} \hat{G}^{(\lambda, \mu)}_{CK}(R, \Delta|P_X). \]

(215)

Eqs. (214) and (215) imply Eq. (114). The function \(G^{(\lambda)}_{CK}(R, \Delta|P)\) satisfies the following property:

Property 8:

a) \(\hat{G}^{(\lambda)}_{CK}(R, \Delta|P)\) is a monotone decreasing function of \(R \geq 0\) for a fixed \(\Delta \geq 0\) and is a monotone decreasing function of \(\Delta \geq 0\) for a fixed \(R \geq 0\).
b) \( G^{(λ)}_{CK}(R, Δ|P) \) is a convex function of \( (R, Δ) \).

**Proof of Property** \( \overline{R} \) By definition, Part a) is obvious. For the proof of Part b), choose \( R_0, R_1, Δ_0, Δ_1 \geq 0 \) arbitrary. Let \( q^{(0)}_{XY} \) and \( q^{(1)}_{XY} \) be joint distribution functions that attain \( G^{(λ)}_{CK}(R_0, Δ_0|P) \) and \( G^{(λ)}_{CK}(R_1, Δ_1|P) \), respectively. Denote

\[
\Theta^{(λ)}_{s}(q_{XY}|P_X) \triangleq D(q_X||P_X) + λI(q_X, q_{Y|X}).
\]  

By definition, we have

\[
G^{(λ)}_{CK}(R_i, Δ_i|P_X) = \Theta^{(i)}_{s}(q^{(i)}_{XY}|P_X) - λR_i \text{ for } i = 0, 1. \tag{217}
\]

For \( α_1 = α \in [0, 1] \) and \( α_0 = 1 − α \), we set \( R_α = α_0R_0 + α_1R_1, Δ_α = α_0Δ_0 + α_1Δ_1, \) and \( q^{(α)}_{XY} = α_0q^{(0)}_{XY} + α_1q^{(1)}_{XY} \).

By linearity of \( E_{q_{XY}}[d(X, Y)] \) with respect to \( q_{XY} \), we have that

\[
E_{q^{(α)}_{XY}}[d(X, Y)] = \sum_{i=0,1} α_i E_{q^{(i)}_{XY}}[d(X, Y)] \leq Δ_α. \tag{218}
\]

We also have

\[
\Theta^{(λ)}_{s}(q^{(α)}_{XY}|P_X)
= λE_{q^{(α)}_{XY}}[log q_{XY}(X|Y)] - E_{q_{XY}}[log P_X(X)] - (1 - λ)H(q_X). \tag{219}
\]

The second term is a linear function of \( q_{XY} \) and the third term is a convex function of \( q_X \) and thus a convex function of \( q_{XY} \) because \( q_X \) is a linear function of \( q_{XY} \). For the first term, we have

\[
\sum_{x,y} q^{(α)}_{XY}(x, y) log q^{(α)}_{XY}(x|y)
= \sum_{x,y} q^{(α)}_{XY}(x, y) log \frac{q^{(α)}_{XY}(x, y)}{q^{(α)}_{Y}(y)}
= \sum_{x,y} (α_0q^{(0)}_{XY}(x, y) + α_1q^{(1)}_{XY}(x, y)) log \frac{α_0q^{(0)}_{XY}(x, y)}{α_0q^{(0)}_{Y}(y)} + \frac{α_1q^{(1)}_{XY}(x, y)}{α_1q^{(1)}_{Y}(y)}
\overset{(a)}{\leq} \sum_{x,y} q^{(0)}_{XY}(x, y) log \frac{α_0q^{(0)}_{XY}(x, y)}{α_0q^{(0)}_{Y}(y)} + \sum_{x,y} q^{(1)}_{XY}(x, y) log \frac{α_1q^{(1)}_{XY}(x, y)}{α_1q^{(1)}_{Y}(y)}
= \alpha_0q^{(0)}_{XY}(x, y) log q^{(0)}_{XY}(x|y) + α_1q^{(1)}_{XY}(x, y) log q^{(1)}_{XY}(x|y), \tag{220}
\]

where step (a) follows from the log-sum inequality. Inequality \( \overset{(a)}{\leq} \) shows that the first term of \( \overset{(b)}{\leq} \) is convex with respect to \( q_{XY} \) and so is the whole of \( \overset{(b)}{\leq} \). Therefore

\[
G^{(λ)}_{CK}(R_α, Δ_α|P) = \min_{q_{XY} \in P(X \times Y)} \Theta^{(λ)}_{s}(q_{XY}|P) - λR_α
\overset{(a)}{\leq} \Theta^{(λ)}_{s}(q^{(α)}_{XY}|P) - λR_α \overset{(b)}{\leq} \sum_{i=0,1} α_i \{ \Theta^{(λ)}_{s}(q^{(i)}_{XY}|P) - λR_i \}
\overset{(c)}{=} \sum_{i=0,1} α_i G^{(λ)}_{CK}(R_i, Δ_i|P).
\]
Hence, it is sufficient to show that there exists a \( \lambda \) such that
\[
\Theta(\lambda)(q_{XY} \mid P_X) \leq \Delta G_{CK}(R, \Delta | P).
\]
This completes the proof.

This property is used to prove (215). We first prove (214). For any \( \lambda \in [0, 1] \), we have \(|x|^+ \geq \lambda x\). Let \( \hat{q}_{XY} \) be a joint distribution that attains \( G_{CK}(R, \Delta | P) \). Then, we have
\[
\tilde{G}_{CK}(R, \Delta | P) = D(\hat{q}_X \| P) + |I(\hat{q}_X, \hat{q}_{Y|X}) - R|^+
\]
\[
\geq D(\hat{q}_X \| P) + \lambda |I(\hat{q}_X, \hat{q}_{Y|X}) - R|
\]
\[
\geq \min_{q_{XY} \in \mathcal{P}(X \times Y)^{\subseteq \Delta}} \{ D(q_X \| P) + \lambda |I(q_X, q_{Y|X}) - R| \}
\]
\[
= \tilde{G}_{CK}(\lambda)(R, \Delta | P).
\]
Thus,
\[
G_{CK}(R, \Delta | P) \geq \max_{0 \leq \lambda \leq 1} \tilde{G}_{CK}(\lambda)(R, \Delta | P).
\]

Next, we prove (215). From its formula, it is obvious that
\[
\tilde{G}_{CK}(\lambda)(R, \Delta | P) = \tilde{G}_{CK}(\lambda, \mu)(R, \Delta | P).
\]

Fix the above \( \lambda \) and \( q^* \) be a joint distribution that attains \( \tilde{G}_{CK}(\lambda)(R, \Delta | P) \). Then we have
\[
\tilde{G}_{CK}(\lambda)(R, \Delta | P)
\]
\[
\leq \tilde{G}_{CK}(\lambda)(R, \Delta | P) + \lambda (R' - R)
\]
\[
= \min_{q_{XY} \in \mathcal{P}(X \times Y)^{\subseteq \Delta}} \{ D(q_X \| P) + |I(q_X, q_{Y|X}) - R'|^+ \}
\]
\[
+ \lambda (R' - R)
\]
\[
\leq D(q_X \| P) + |I(q_X, q_{Y|X}) - R'|^+ + \lambda (R' - R)
\]
\[
\leq \lambda |I(q_X, q_{Y|X}) - R| + D(q_X \| P)
\]
\[
= \tilde{G}_{CK}(\lambda)(R, \Delta | P).
\]

Step (a) follows from (221) and Step (b) comes from the choice of \( R' = I(q_X^*, q_{Y|X}^*) \). Therefore, there exists a \( 0 \leq \lambda \leq 1 \) such that
\[
\tilde{G}_{CK}(\lambda)(R, \Delta | P) = \tilde{G}_{CK}(\lambda, \mu)(R, \Delta | P).
\]

Next, we prove (215). From its formula, it is obvious that
\[
\tilde{G}_{CK}(\lambda)(R, \Delta | P) \geq \max_{\mu \geq 0} \tilde{G}_{CK}(\lambda, \mu)(R, \Delta | P).
\]

Hence, it is sufficient to show that for any \( R \geq 0 \) and \( \Delta \geq 0 \), there exists \( \mu \geq 0 \) such that
\[
\tilde{G}_{CK}(\lambda)(R, \Delta | P) \leq \tilde{G}_{CK}(\lambda, \mu)(R, \Delta | P).
\]

From Property 5 part a) and b), \( G(\lambda)(R, \Delta | P) \) is a monotone decreasing and convex function of \( \Delta \geq 0 \) for a fixed \( R \). Thus, there exists \( \mu \geq 0 \) such that for any \( \Delta' \geq 0 \), the following inequality holds:
\[
\tilde{G}_{CK}(\lambda)(R, \Delta' | P) \geq \tilde{G}_{CK}(\lambda, \mu)(R, \Delta | P) - \mu (\Delta' - \Delta).
\]
Fix the above $\mu$. Let $q^*$ be a joint distribution that attains $\hat{G}_{\text{CK}}^{(\mu,\lambda)}(R, \Delta|P)$. Set $\Delta' = E_{q^*}[d(X, Y)]$. Then, we have

$$\hat{G}_{\text{CK}}^{(\lambda)}(R, \Delta|P) \overset{(a)}{=} \hat{G}_{\text{CK}}^{(\lambda)}(R, \Delta'|P) - \mu(\Delta - \Delta')$$

$$= \min_{P_{X,Y}[d(X,Y)] \leq \Delta'} \left\{ D(q_X||P) + \lambda [I(q_X, q_Y|X) - R] \right\} - \mu(\Delta - \Delta')$$

$$(b) \leq \lambda [D(q_X^*||P) + I(q_X^*, q_Y^*|X) - R] - \mu(\Delta - E_{q_{XY}}[d(X, Y)])$$

$$= \hat{G}_{\text{CK}}^{(\lambda,\mu)}(R, \Delta|P).$$

Step (a) follows from (224) and Step (b) follows from the definition of $\hat{G}_{\text{CK}}^{(\lambda)}(R, \Delta'|P)$ and the choice of $\Delta' = E_{q^*}[d(X, Y)]$. Thus, for any $\Delta \geq 0$, we have (223) for some $\mu \geq 0$. This completes the proof.

**APPENDIX H**

**PROOFS OF LEMMAS 24 AND 25**

In this appendix, we give proofs of Lemmas 24 and 25.

**Proof of Lemma 24** We have

$$F_{30,s}^{(\lambda,\mu)}(q_{XY}, \hat{q}_{XY}|P_X)$$

$$= E_{q_{XY}} \left[ \log \frac{q_{X}^{1-\lambda}(X)\hat{q}_{X}^{\lambda}(X|Y)}{P_X(X)e^{-\lambda d(X,Y)}} \right] + D(q_{XY}||\hat{q}_{XY})$$

$$= E_{q_{XY}} \left[ \log \frac{q_{X}^{1-\lambda}(X)\hat{q}_{X}^{\lambda}(X|Y)q_{XY}(X,Y)}{P_X(X)e^{-\lambda d(X,Y)}q_{XY}(X,Y)} \right]$$

$$= E_{q_{XY}} \left[ \log \frac{q_{XY}(X,Y)}{P_X(X)e^{-\lambda d(X,Y)}\hat{q}_{X}^{\lambda}(X|Y)q_{Y}^{\lambda|X}} \right]$$

$$= \Theta_{s}^{(\lambda,\mu)}(q_{XY}|P_X) + E_{q_{XY}} \left[ \log \frac{q_{Y}^{1-\lambda}(Y|X)\hat{q}_{Y}^{\lambda}(Y|X)}{q_{Y}^{1-\lambda}(Y|X)\hat{q}_{Y}^{\lambda}(Y|X)} \right]$$

$$= \Theta_{s}^{(\lambda,\mu)}(q_{XY}|P_X) + (1 - \lambda)D(q_{Y}|X|\hat{q}_{Y}|X|q_{X}) + \lambda D(q_{Y}|\hat{q}_{Y}).$$

Hence, by non-negativity of divergence we have

$$F_{30,s}^{(\lambda,\mu)}(q_{XY}, \hat{q}_{XY}|P_X) \geq \Theta_{s}^{(\lambda,\mu)}(q_{XY}|P_X),$$

where equality holds if $\hat{q}_{XY} = q_{XY}$. This completes the proof.

**Proof of Lemma 25** We have

$$F_{30,s}^{(\lambda,\mu)}(q_{XY}, \hat{q}_{XY}|P_X)$$

$$= E_{q_{XY}} \left[ \log \frac{q_{XY}(X,Y)}{P_X(X)e^{-\lambda d(X,Y)}\hat{q}_{X}^{\lambda}(X|Y)q_{Y}^{\lambda|X}} \right]$$

$$= E_{q_{XY}} \left[ \log \frac{q_{XY}(X,Y)}{q_{XY}(\hat{q}_{XY}(X,Y))} \right] - \log \sum_{x,y} P_X(x)e^{-\lambda d(x,y)}\hat{q}_{Y}^{\lambda}(y)q_{Y}^{1-\lambda}(y|x).$$
Hence, by non-negativity of relative entropy, we have

\[ F^{(\lambda, \nu)}_{10}s(q_{XY}, \hat{q}_{XY}|P_X) \geq -\log \sum_{x,y} P_X(x)e^{-\lambda \nu d(x,y)} \hat{q}_Y^\lambda(y)\hat{q}_{Y|X}^{1-\lambda}(y|x), \]

where equality holds if and only if \( q_{XY} = \hat{q}_{XY}(\hat{p}_Y, p_{Y|X}) \) holds. This completes the proof.

Next we give the proof of Lemma 27. The proof is almost the same as the proof of Lemma 25. We have

\[ \tilde{F}^{(\lambda, \nu)}_s(q_{XY}, \hat{p}_Y, p_{Y|X}|P_X) \]

\[ = \Theta^{(\lambda, \lambda \nu)}_s(q_{XY}|P_X) + \lambda D(q_Y||\hat{p}_Y) + (1 - \lambda)D(q_{Y|X}||p_{Y|X}|q_X) \]

\[ = E_{q_{XY}} \left[ \log \frac{q_{XY}(X,Y)}{P_X(X)e^{-\lambda \nu d(X,Y)} \hat{p}_Y^\lambda(Y)p_{Y|X}^{1-\lambda}(Y|X)} \right] \]

\[ = E_{q_{XY}} \left[ \log \frac{q_{XY}(X,Y)}{\hat{q}_{XY}(\hat{p}_Y, p_{Y|X})(X,Y)} \right] - \log \sum_{x,y} P_X(x)e^{-\lambda \nu d(x,y)} \hat{p}_Y^\lambda(y)p_{Y|X}^{1-\lambda}(y|x), \]

where

\[ \hat{q}_{XY}(\hat{p}_Y, p_{Y|X})(x, y) = \frac{P_X(x)e^{-\lambda \nu d(x,y)} \hat{p}_Y^\lambda(y)p_{Y|X}^{1-\lambda}(y|x)}{\sum_{x', y'} P_X(x')e^{-\lambda \nu d(x',y')} \hat{p}_Y^\lambda(y')p_{Y|X}^{1-\lambda}(y'|x')} \]

Hence, by non-negativity of relative entropy, we have

\[ \tilde{F}^{(\lambda, \nu)}_s(q_{XY}, \hat{p}_Y, p_{Y|X}|P_X) \geq -\log \sum_{x,y} P_X(x)e^{-\lambda \nu d(x,y)} \hat{p}_Y^\lambda(y)p_{Y|X}^{1-\lambda}(y|x). \]

Equality holds if and only if \( q_{XY} = \hat{q}_{XY}(\hat{p}_Y, p_{Y|X}) \), which completes the proof.

APPENDIX I

PROOF OF PROPOSITION 8

The limit of \((1/\rho)A^{(\rho, \nu)}_s(\hat{p}_Y, p_{Y|X}|P_X)\) as \( \rho \) approaches to 0 is evaluated as follows.
Proof of Proposition We have

\[
\begin{align*}
&\lim_{\rho \to 0} \frac{1}{\rho} A_s^{(\rho, \nu)}(\hat{p}_{Y|X}|P_X) \\
&\quad \overset{(a)}{=} \lim_{\rho \to 0} \frac{\partial}{\partial \rho} A_s^{(\rho, \nu)}(\hat{p}_{Y|X}|P_X) \\
&\quad = \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \left\{ (1 + \rho) \log \sum_y \left[ \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right]^{1/(1+\rho)} \right\} \\
&\quad = \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \left\{ \sum_y \left[ \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right]^{1/(1+\rho)} \right\} \\
&\quad + \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \left\{ \sum_y \left[ \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right]^{1/(1+\rho)} \right\} \\
&\quad = 0 + \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \left\{ \sum_y \left[ \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right]^{1/(1+\rho)} \right\} \\
&\quad \overset{(b)}{=} \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \left\{ \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right\}^{1/(1+\rho)} \\
&\quad \cdot \left( \frac{-1}{(1 + \rho)^2} \log \left\{ \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right\} + \frac{1}{1 + \rho} \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right) \\
&\quad = \lim_{\rho \to 0} \sum_y \left( \sum_x P_X(x) \hat{p}_{Y|X}(y|x) \right) \left( -\log \sum_x P_X(x) \hat{p}_{Y|X}(y|x) + \frac{\partial}{\partial \rho} \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \right) \\
&\quad = \lim_{\rho \to 0} \sum_y \left( \sum_x P_X(x) \hat{p}_{Y|X}(y|x) \right) \left( -\log \sum_x P_X(x) \hat{p}_{Y|X}(y|x) + \frac{\partial}{\partial \rho} \sum_x P_X(x) e^{\rho \log d(x, y)} \hat{p}_{Y|X}^{1+\rho}(y|x) \log \hat{p}_{Y|X}^{1+\rho}(y|x) \right) \\
&\quad = \sum_y \left\{ -\sum_x P_X(x) \hat{p}_{Y|X}(y|x) \log \sum_{x'} P_X(x') \hat{p}_{Y|X}(y|x') \right. \\
&\quad \quad + \left. \sum_x t_d(x, y) P_X(x) \hat{p}_{Y|X}(y|x) + \sum_x P_X(x) \hat{p}_{Y|X}(y|x) \log \hat{p}_{Y|X}(y|x) \right\} \\
&\quad = I(P_X, \hat{p}_{Y|X}) + \nu E_{q_{x|y}} [d(X, Y)].
\end{align*}
\]

Step (a) follows from the L’Hôpital’s rule. Step (b) holds because we have \((f(x)g(x))' = f(x)g'(x) \left( g'(x) \log f(x) + g(x) \frac{f'(x)}{f(x)} \right)\).
APPENDIX J

EXPLICIT UPDATE RULE FOR THE GENERALIZED ALGORITHM

The distribution updating algorithm of Tridensiki et al.’s generalized algorithm was described in [12, Lemma3] but the parameters $t_2$, $t_3$, and $t_4$ were misspelled as $t_4$, $t_2$, and $t_3$, respectively. It was noted that in both Case 1 and Case 2, the probability update rule is parameterized with two parameters. We argue that in both cases, the rules should be parameterized with three parameters.

Case 1: Set $t_1 = t_2 + 1$. By separating the joint distribution $q_{XY}$ as $(q_Y, q_{X|Y})$, we have

$$\tilde{F}_{TSZ}^{(\lambda,v,t)}(p_{XY}|W) = \min_{q_Y} \min_{q_{X|Y}} F_{TSZ}^{(\lambda,v,t)}(q_{XY}, p_{XY}|W)$$

$$= -\{(1 - \lambda)(t_2 + t_3) + 1\} \log \sum_y p_Y^{-\lambda\lambda(t_2 + t_3)}(y) \left[ \sum_x \left\{ p_X(x)p_{X|Y}^{t_2 + t_4}(x|y) \{W(y|x)e^{-\lambda\lambda(x)}\}^{1/y} \right\}^{1/(1-\lambda)(t_2 + t_3) + 1/2} \right]^{1/(1-\lambda)(t_2 + t_3) + 1/2} \right) \right)$$

(229)

The probability distribution update rule for Case 1 can be obtained from the condition for $(q_Y, q_{X|Y})$ to attain (229), as follows:

$$q_{X|Y}^{[t+1]}(x|y) = K_y^{-1}\left\{ p_X^{[t]}(x)p_{X|Y}^{[t]}(y|x)^{t_2 + t_4}\{W(y|x)e^{-\lambda\lambda(x)}\}^{1/\lambda} \right\}^{1/\lambda/(t_2 + t_3) + 1},$$

(230)

$$q_Y^{[t+1]}(y) = \Lambda_1^{-1}\left\{ p_Y^{[t]}(y) \right\}^{1/(1-\lambda)(t_2 + t_3) + 1} \left[ \sum_x \left\{ p_X(x)p_{X|Y}^{[t]}(x|y)^{t_2 + t_4}\{W(y|x)e^{-\lambda\lambda(x)}\}^{1/\lambda} \right\}^{1/\lambda/(t_2 + t_3) + 1} \right]^{1/(1-\lambda)(t_2 + t_3) + 1},$$

(231)

$$p_{XY}^{[t+1]}(x,y) = q_{XY}^{[t+1]}(x,y),$$

(232)

where $K_y$ and $\Lambda_1$ are normalization factors defined by

$$K_y = \sum_x \left\{ p_X^{[t]}(x)p_{X|Y}^{[t]}(y|x)^{t_2 + t_4}\{W(y|x)e^{-\lambda\lambda(x)}\}^{1/\lambda} \right\}^{1/\lambda/(t_2 + t_3) + 1},$$

(233)

$$\Lambda_1 = \sum_y \left\{ p_Y^{[t]}(y) \right\}^{1/(1-\lambda)(t_2 + t_3) + 1} \left[ \sum_x \left\{ p_X(x)p_{X|Y}^{[t]}(x|y)^{t_2 + t_4}\{W(y|x)e^{-\lambda\lambda(x)}\}^{1/\lambda} \right\}^{1/\lambda/(t_2 + t_3) + 1} \right]^{1/(1-\lambda)(t_2 + t_3) + 1}.$$  

(234)

If $t_2 = t_3 = t_4 = 0$, the updating rule (230) and (231) reduces to (33) and (34) in Algorithm [1].

Case 2: Set $t_4 = t_3 + 1\lambda - 1$. In a similar manner, we have

$$\tilde{F}_{TSZ}^{(\lambda,v,t)}(p_{XY}|W) = \min_{q_{XY}} F_{TSZ}^{(\lambda,v,t)}(q_{XY}, p_{XY}|W)$$

$$= -\{(1 - \lambda)(t_1 + t_3) + 1\} \log \sum_x p_X(x) \left[ \sum_y \left\{ p_Y^{-\lambda\lambda(t_1 + t_3) + 1}\{W(y|x)e^{-\lambda\lambda(x)}\}^{1/\lambda} \right\}^{1/(1-\lambda)(t_1 + t_3) + 1} \right]^{1/(1-\lambda)(t_1 + t_3) + 1} \right) \right)$$

(235)
From the condition for $q_{XY}$ to achieve the above minimum, we see that the probability distribution update rule for Case 2 is the following:

$$q_{Y|X}^{[t+1]}(y|x) = K_x^{-1} \left\{ p_Y^{[t]}(y) - \lambda p_Y^{[t]}(y|x) (1-\lambda) (t_2 + t_3) + \lambda W(y|x) e^{-\lambda \nu c(x)} \right\}^{\frac{1}{1-\lambda (t_2 + t_3) + \lambda}}$$

$$q_X^{[t+1]}(x) = \Lambda_2^{-1} p_X(x) \left[ \sum_y \left\{ p_Y^{[t]}(y) - \lambda p_Y^{[t]}(y|x) (1-\lambda) (t_2 + t_3) + \lambda W(y|x) e^{-\lambda \nu c(x)} \right\}^{\frac{1}{1-\lambda (t_2 + t_3) + \lambda}} \right]^{\frac{(1-\lambda) (t_2 + t_3) + 1}{(1-\lambda) (t_2 + t_3) + \lambda}}$$

$$p_X^{[t+1]}(x,y) = q_{X|Y}^{[t+1]}(x,y)$$

where

$$K_x = \sum_y \left\{ p_Y^{[t]}(y) - \lambda p_Y^{[t]}(y|x) (1-\lambda) (t_2 + t_3) + \lambda W(y|x) e^{-\lambda \nu c(x)} \right\}^{\frac{1}{1-\lambda (t_2 + t_3) + \lambda}}$$

$$\Lambda_2 = \sum_x p_X(x) \left[ \sum_y \left\{ p_Y^{[t]}(y) - \lambda p_Y^{[t]}(y|x) (1-\lambda) (t_2 + t_3) + \lambda W(y|x) e^{-\lambda \nu c(x)} \right\}^{\frac{1}{1-\lambda (t_2 + t_3) + \lambda}} \right]^{\frac{(1-\lambda) (t_2 + t_3) + 1}{(1-\lambda) (t_2 + t_3) + \lambda}}.$$

If $t_1 = t_2 = t_3 = 0$, the updating rule (236) and (237) reduces to (48) and (49) in Algorithm 2.