$\mathcal{N} = 2$ supersymmetric gauge theories with massive hypermultiplets and the Whitham hierarchy

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Abstract

We embed the Seiberg–Witten solution for the low energy dynamics of $\mathcal{N} = 2$ super Yang–Mills theory with an even number of massive hypermultiplets into the Whitham hierarchy. Expressions for the first and second derivatives of the prepotential in terms of the Riemann theta function are provided which extend previous results obtained by Gorsky, Marshakov, Mironov and Morozov. Checks in favour of the new equations involve both their behaviour under duality transformations and the consistency of their semiclassical expansions.
1 Introduction

Soon after the paper of Seiberg and Witten [1] appeared, the coincidence between spectral curves of soliton equations of the Toda type and the Seiberg–Witten hyperelliptic curves for the low energy effective action of $\mathcal{N} = 2$ SUSY Yang–Mills was established [2, 3]. The averaged dynamics of these integrable systems, that goes under the generic name of Whitham hierarchy [4], allows for an interpretation of the effective prepotential as the logarithm of a quasiclassical tau function [5, 6]. Later on, some non-perturbative results concerning the derivatives of the prepotential as functions over the moduli space [7, 8] were rephrased neatly in this language [9, 10]. However, it was not until very recently that the richness of this approach showed its true thrust. The rôle of the Whitham times as Wilsonian couplings, or infrared counterpart of microscopic deformations by higher polynomial interactions, was well settled in the context of two-dimensional topological conformal field theory and the Kontsevich model [11]. In four dimensions, this conjectured link was put on a firm basis thanks to the work carried out by Gorsky, Marshakov, Mironov and Morozov [12], where first and second derivatives of the prepotential with respect to these times were computed as functions over the moduli space. The appearance of a logarithmic derivative of the Riemann theta function confirmed and extended analogous formulas for the contact terms in topological $\mathcal{N} = 2$ twisted theory obtained from the so-called $u$-plane integral [13, 14, 15, 16]. In Ref. [17], the analytical results of [12] were put to work. A first calculational goal was an efficient algorithm for recursive evaluation of the semiclassical expansion of the prepotential. Also, the relation between Whitham parameters and microscopic deformations of pure $SU(N)$ was analysed (see also [18]) and, finally, the Whitham times were seen to provide generalized spurionic sources for breaking supersymmetry softly down to $\mathcal{N} = 0$. In [19], this formalism was used to extract (or test) non trivial strong coupling information that is difficult to obtain from other methods as, for example, the off-diagonal couplings at the maximal singularities of the moduli space. For a review on the latest developments in these subjects, see Refs. [18, 20].

In the present paper, we shall extend this formalism to the case of $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with any classical gauge group and massive matter hypermultiplets in the fundamental representation. As it is well known, the Seiberg–Witten ansatz also holds in this situation and one still has a geometrical picture in terms of an auxiliary punctured Riemann surface $\Gamma$. Also, the connection with integrable models has been observed in this case [14, 21]. The masses of the hypermultiplets are (linearly related to) the residues of the Seiberg–Witten differential at the poles. From the point of view of the theory of Riemann surfaces (and of the Whitham hierarchy), this implies the incorporation
to the game of differentials with simple poles on $\Gamma$ (third kind meromorphic differentials).

The paper is organized as follows. We start, in Section 2, by reviewing the construction of the universal Whitham hierarchy along the lines of Refs.\[4, 6\]. In Section 3, we particularize this setup to the case of interest by introducing the Seiberg–Witten hyperelliptic curve. Section 4 is devoted to the construction of the generating meromorphic differential implied by the Whitham hierarchy, in such a way that the Seiberg–Witten differential naturally fits into the framework.

In Section 5, general formulas for the first and second derivatives of the prepotential are obtained within this enlarged framework. They usually involve the Riemann theta function and extend the pure gauge results of Ref.\[12\]. In Section 6, we obtain the whole set of duality transformations of the previously computed couplings by analyzing their behaviour under symplectic transformations of the homology basis and deformations of integration contours. Of course, the particular Whitham hierarchy we are considering is strongly motivated by the Seiberg–Witten solution for the effective $\mathcal{N} = 2$ super Yang–Mills theory with massive hypermultiplets whose moduli space, ultimately, should be recovered as a submanifold. This is done in Section 7, where we end up by giving several formulas for the first and second order derivatives of the effective prepotential of $\mathcal{N} = 2$ supersymmetric gauge theories. In Section 8, we give two types of arguments supporting the expressions found in this way. First, we show that these expressions exhibit the required duality covariance. Second, we see by explicit computation that they are consistent in a highly non-trivial way with the semiclassical expansion of the prepotential. In fact, following the same lines of Ref.\[17\], one can use this formalism to develop a recursive procedure to obtain the instanton expansion of the effective prepotential up to arbitrary order \[22\]. We extend our results to any classical gauge group in Section 9. Finally, in the last section, we include some further remarks and present some avenues for future research.

2 Whitham equations and the prepotential

It is well known from the general theory of Riemann surfaces that there are three basic types of Abelian differentials. They can be characterized by their Laurent expansion about selected points called punctures. Let $P$ or $Q$ denote two such points on a Riemann surface $\Gamma$ of genus $g$, with local coordinates $\xi_P$ and $\xi_Q$ about them.

(i) Holomorphic differentials, $d\omega_i$. In any open set $U \subset \Gamma$, with complex coordinate $\xi$, they are of the form $d\omega = f(\xi)d\xi$ with $f$ an holomorphic function. The vector
space of holomorphic differentials has complex dimension $g$. If the complex curve is hyperelliptic,
\[ y^2 = \prod_{i=1}^{2(g+1)} (\lambda - e_i) , \quad (2.1) \]
a suitable basis for these differentials is given by the following set of $g$ holomorphic 1-forms
\[ dv_k = \frac{\lambda^{g-k} d\lambda}{y}, \quad k = 1, 2, \ldots, g. \quad (2.2) \]
Given a symplectic basis of homology cycles $A^i, B_i \in H_1(\Gamma, \mathbb{Z})$ one may compute their period integrals
\[ A^i_k = \oint_{A^i} dv_k \quad B_{ik} = \oint_{B_i} dv_k. \quad (2.3) \]
A canonical basis $\{d\omega_j\}$ can be defined by the $g$ linear equations
\[ \oint_{A^i} d\omega_j = \delta^i_j, \quad (2.4) \]
and clearly both bases are related as $d\omega_j = A^{-1} \omega_j dv_k$. Now the matrix of $B_i$–periods yield moduli of the curve
\[ \oint_{B_i} d\omega_j = \tau_{ij}. \quad (2.5) \]

(ii) **Meromorphic differentials of the second kind, $d\Omega_n^P$.** These have a single pole of order $n + 1$ at point $P \in \Gamma$, and zero residue. In local coordinates $\xi_P$ about $P$, ($\xi_P(P) = 0$), the normalization
\[ d\Omega_n^P = (\xi_P^{n-1} + O(1)) d\xi_P, \quad (2.6) \]
determines $d\Omega_n^P$ up to an arbitrary combination of holomorphic differentials $d\omega_i$. To fix the regular part we shall require that it has vanishing $A^i$–periods. Altogether, $d\Omega_n^P$ is uniquely defined by
\[ \text{res}_P \xi_P^m d\Omega_n^P = \delta_{mn}, \quad \forall m > 0, \quad (2.7) \]
\[ \oint_{A^i} d\Omega_n^P = 0, \quad \forall i. \quad (2.8) \]

\[ ^2\text{We absorb a factor of $(2\pi i)^{-1}$ into the definition of every integral that runs around a closed contour so that, for example, } \oint_0 \frac{d\xi}{\xi} = \text{res}_0 \frac{d\xi}{\xi} = 1. \]
(iii) Meromorphic differentials of the third kind, $d\Omega^{P,Q}_0$. These have first order poles at $P$ and $Q$ with opposite residues taking values $+1$ and $-1$ respectively. In local coordinates $\xi_P$ ($\xi_Q$) about $P$ ($Q$)

$$d\Omega^{P,Q}_0 = (\xi_P^{-1} + O(1))\,d\xi_P = -(\xi_Q^{-1} + O(1))\,d\xi_Q.$$  \hspace{1cm} (2.9)

We shall also normalize the regular part of these differentials by demanding that their $A^i$-periods vanish, $\oint_{A^i} d\Omega^{P,Q}_0 = 0$, $\forall i$.

Following Krichever’s construction \[4\], the moduli space of the universal Whitham hierarchy $\hat{\mathcal{M}}_{g,p}$ is given by the set of algebraic-geometrical data

$$\hat{\mathcal{M}}_{g,p} \equiv \{ \Gamma_g, P_a, \xi_a(P), a = 1, ..., p \} \hspace{1cm} (2.10)$$

in which

1. $\Gamma_g$ denotes a smooth algebraic curve of genus $g$, and $P$ a point on it.
2. $P_a$ are a set of $p$ points (punctures) on $\Gamma_g$ in generic positions.
3. $\xi_a$ are local coordinates in the neighbourhood of the $p$ points, \textit{i.e.} $\xi_a(P_a) = 0$.

Fix a basis point $P_0$. For each given puncture $P_a, a = 1, 2, ..., p$, a set of \textit{slow times} $T^0_{P_a,P_0}$ and $T^n_{P_a}$, $n = 1, 2, ...$ are assigned in correspondence with the meromorphic forms $d\Omega^{P_a,P_0}_0$ and $d\Omega^{P_a}_n$ respectively. Defining the collective index $A = (P_a; n), B = (P_b; m)$, etc. (also including the possibility $A = (P_a, P_0; 0)$), we shall write $T^A, T^B, ...$ and $d\Omega_A, d\Omega_B, ...$. In its original form, the Whitham hierarchy can be defined by the following set of differential equations

$$\frac{\partial d\Omega_A}{\partial T^B} = \frac{\partial d\Omega_B}{\partial T^A}.$$  \hspace{1cm} (2.11)

The set of data (2.10) specify the quasi-periodic integrable model involved. For example, a Riemann surface with a single puncture provides solutions for the KdV equation. With two singularities, solutions for the Toda lattice can be obtained, etc.

The Whitham hierarchy may be further enhanced to incorporate also holomorphic differentials, $d\omega_i$, with associated parameters $\alpha^i$, such that the system (2.11) is enlarged as follows

$$\frac{\partial d\omega_i}{\partial \alpha^j} = \frac{\partial d\omega_i}{\partial \alpha^j} ; \quad \frac{\partial d\omega_i}{\partial T^A} = \frac{\partial d\Omega_A}{\partial \alpha^j} ; \quad \frac{\partial d\Omega_A}{\partial T^B} = \frac{\partial d\Omega_B}{\partial T^A}.$$  \hspace{1cm} (2.12)

Equations (2.12) are nothing but the integrability conditions implying the existence of a \textit{generating meromorphic differential} $dS(\alpha^i, T^A)$ satisfying

$$\frac{\partial}{\partial \alpha^i} dS = d\omega_i ; \quad \frac{\partial}{\partial T^A} dS = d\Omega_A.$$  \hspace{1cm} (2.13)
Moreover, the Whitham equations implicitly define a certain function, the prepotential, \( F(\alpha^i, T^A) \), through the following set of equations

\[
\frac{\partial F}{\partial \alpha^j} = \oint_{B_j} dS , \quad (2.14)
\]

\[
\frac{\partial F}{\partial T^A_{Pa}} = \frac{1}{2\pi i} \text{res}_{P_a} \xi^{-n} dS = \frac{1}{2\pi i} \oint_{P_a} \xi^{-n} dS , \quad (2.15)
\]

\[
\frac{\partial F}{\partial T^0_{P_a,P_b}} = \frac{1}{2\pi i} \int_{P_a} dS . \quad (2.16)
\]

The consistency of (2.13)–(2.16) follows from a direct computation, relying solely on Riemann bilinear relations. For completeness, the relevant information is given in Appendix A.

Due to (2.13) and the definitions in (2.14)–(2.16), the local behaviour of \( dS \) near each puncture \( P_a \) is given by

\[
dS \sim \left\{ \sum_{n \geq 1} T^n_{Pa} \xi_a^{-n-1} + T^0_{Pa,P_b} \xi_a^{-1} + 2\pi i \sum_{n \geq 1} n \frac{\partial F}{\partial T^n_{Pa}} \xi_a^{n-1} \right\} d\xi_a . \quad (2.17)
\]

An interesting class of solutions, and certainly that which is relevant in connection with \( \mathcal{N} = 2 \) super Yang–Mills theories, is given by those prepotentials that are homogeneous of degree two:

\[
\sum_{i=1}^g \alpha^i \frac{\partial F}{\partial \alpha^i} + \sum_{a=1}^p T^0_{Pa,P_b} \frac{\partial F}{\partial T^0_{Pa,P_b}} + \sum_{a=1}^p \sum_{n \geq 1} T^n_{Pa} \frac{\partial F}{\partial T^n_{Pa}} = 2F . \quad (2.18)
\]

For this kind of solutions, it is easy to see that the generating differential \( dS \) admits the following decomposition:

\[
dS = \sum_{i=1}^g \alpha^i d\omega_i + \sum_{a=1}^p T^0_{Pa,P_b} d\Omega^n_{Pa,P_b} + \sum_{a=1}^p \sum_{n \geq 1} T^n_{Pa} d\Omega^n_{Pa} . \quad (2.19)
\]

Indeed, it suffices to show that near each puncture \( P_a \), \( dS \) can be expanded as in (2.17). To this end, one finds from (2.13)–(2.16) that the local expansion of \( d\omega_i \) and \( d\Omega^n_{Pa} \) around \( P_a \) involve the second derivatives of \( F \) as follows:

\[
d\omega_j = 2\pi i \sum_{m \geq 1} m \frac{\partial^2 F}{\partial \alpha^j \partial T^m_{Pa}} \xi_a^{m-1} d\xi_a , \quad (2.20)
\]

\[
d\Omega^n_{Pa} = \left( \xi_a^{-n-1} + 2\pi i \sum_{m \geq 1} m \frac{\partial^2 F}{\partial T^n_{Pa} \partial T^m_{Pa}} \xi_a^{m-1} \right) d\xi_a , \quad (2.21)
\]

\[
d\Omega^0_{Pa,P_b} = \left( \xi_a^{-1} + 2\pi i \sum_{m \geq 1} m \frac{\partial^2 F}{\partial T^0_{Pa} \partial T^m_{Pa,P_b}} \xi_a^{m-1} \right) d\xi_a . \quad (2.22)
\]
Inserting these expansions in (2.19) and using (2.18) one arrives at (2.17) as desired.

Given (2.19) and the normalization (2.7)–(2.8), we recognize that the parameters, \( \alpha^i \), and the slow times, \( T^n_{P_a} \), can also be recovered from \( dS \),

\[
\alpha^i = \oint_{A^i} dS \quad , \quad T^n_{P_a,P_0} = \text{res}_{P_a} dS = -\text{res}_{P_0} dS \quad , \quad T^n_{P_a} = \text{res}_{P_a} \xi^n dS .
\]

Finally, inserting (2.14)–(2.16) and (2.23) into (2.18), a formal expression for \( F \) in terms of \( dS \) can be found,

\[
F = \frac{1}{2} \sum_{i=1}^g \oint_{A^i} dS \oint_{B^i} dS + \frac{1}{4 \pi i} \sum_{a=1}^p \left( \oint_{P_a} dS \oint_{P_0} dS + \sum_{n \geq 1} \frac{1}{n} \oint_{P_a} \xi^n dS \oint_{P_a} \xi^{-n} dS \right) .
\]

(2.24)

3 The Whitham hierarchy

In this section we adapt the above formalism to the situation that will lead naturally to a connection with the low-energy dynamics of asymptotically free \( \mathcal{N} = 2 \) super Yang–Mills theories with matter hypermultiplets. To this end, we shall specify the particular set of algebraic-geometrical data that corresponds to the Whitham hierarchy of our interest, i.e. the complex curve \( \Gamma_g \), the set of punctures and the local coordinates in their vicinities.

3.1 The hyperelliptic curve

Inspired by the Seiberg–Witten solution to the low-energy dynamics of \( \mathcal{N} = 2 \) super Yang–Mills theory with gauge group \( SU(N_c) \) and \( N_f < 2N_c \) massive hypermultiplets \[23\] we shall consider the following algebraic curve of genus \( g = N_c - 1 \),

\[
y^2 = \left( P(\lambda, u_k) + T(\lambda, m_f) \right)^2 - 4F(\lambda, m_f) ,
\]

(3.1)

where \( P \) is the characteristic polynomial,

\[
P(\lambda; u_k) = \lambda^{N_c} - \sum_{k=2}^{N_c} u_k \lambda^{N_c-k} ,
\]

(3.2)

and \( \beta = 2N_c - N_f \) is the coefficient of the one-loop \( \mathcal{N} = 2 \) beta function. Concerning \( T \) and \( F \), they are polynomials that do not depend on the moduli \( u_k \):

\[
F(\lambda, m_f) = \prod_{r=1}^{N_f} (\lambda - m_f) = \lambda^{N_f} + \sum_{j=1}^{N_f} t_j \lambda^{N_f-j} ,
\]

(3.3)
and $T$ is a homogeneous polynomial in $\lambda$ and $m_f$ of degree $N_f - N_c$, which is different from zero only when $N_f > N_c$. All dependence on $T$ can be absorbed in a redefinition of the classical order parameters so that the effective prepotential, the basic object of interest, does not depend on it \[24\]. Thus, we can set $T = 0$, and write the hyperelliptic curve as follows:

$$y^2 = P^2(\lambda, u_k) - 4F(\lambda, m_f) .$$

This curve represents a double cover of the Riemann sphere branched over $2N_c$ points. The moduli space of this genus $g = N_c - 1$ curve is parametrized by the complex numbers $u_k, k = 2, ..., N_c$. In the Seiberg–Witten model, these complex numbers are homogeneous combinations of the vacuum expectation values of the Casimirs of the gauge group $SU(N_c)$ and they parametrize the quantum moduli space of vacua, while the $m_f$ are constant parameters related to the bare masses of the hypermultiplets. As explained in the previous section, a precise choice of the local coordinates $\xi_a$ around punctures is in order. These coordinates are kept fixed while coordinates of moduli space are varied. The following functions $w$ and $w^{-1}$,

$$w^{\pm 1}(\lambda) = \frac{P \pm y}{2\sqrt{F}} .$$

provide the natural candidates to construct such well-behaved local coordinates. In terms of them,

$$P = \sqrt{F}\left(w + \frac{1}{w}\right), \quad y = \sqrt{F}\left(w - \frac{1}{w}\right) .$$

From (3.3)–(3.6) a relation between the variation of the different parameters of the curve follows. Define $W \equiv P/\sqrt{F}$, then

$$(\partial_{u_k}W)\delta u_k + W'\delta \lambda + (\partial_{m_f}W)\delta m_f = \frac{y}{\sqrt{F}} \delta \log w ,$$

where ( )’ stands for $\partial_\lambda( )$ and repeated indices are summed over. For a given curve, that is, for fixed $u_k$ and $m_f$,

$$\frac{dw}{w} = \frac{\sqrt{F}}{y} \frac{W'}{\sqrt{W'^2 - 4}} d\lambda .$$

Note that these formulas are the same as for $SU(N_c)$ without matter ($N_f = 0$) \[12\], upon replacing

$$P \rightarrow W \equiv P/\sqrt{F} , \quad y \rightarrow \tilde{y} \equiv y/\sqrt{F} = \sqrt{W'^2 - 4} ,$$

$$N_c \rightarrow N \equiv N_c - 1/2N_f ,$$

where now $W$ and $\tilde{y}$ are polynomials of order $N$ in $\lambda$. 7
Notice the appearance of the square root of $F$. As explained in Appendix C, at some stage of the computation of the derivatives of the prepotential, this square root will be asked to be a rational function of $\lambda$. Unavoidably, $F$ must be a square, $F = R^2$. This implies that, in the present framework, $N_f$ must be an even integer and, moreover, massive hypermultiplets must come up in degenerated pairs $m_f = m_{f+N_f/2}$. In principle one could think of more perverse possibilities for $w^\pm$ that generalize the analogous formulas for $SU(N_c)$. Namely, let

$$w = \frac{1}{2R}(P + y) \quad ; \quad w^{-1} = \frac{1}{2R}(P - y)$$

with $R = \prod_{f=1}^{n_f} (\lambda - m_f'), R' = \prod_{f=1}^{n_f'} (\lambda - m_f)$, $n_f + n_f' = N_f$ and $F = RR'$. The ansatz in (3.3) corresponds to a “symmetric scenario” where $R = R' = \sqrt{F}$ i.e. $n_f = n_f' = N_f/2$, which is the only one that preserves the involution symmetry $(\lambda, y) \leftrightarrow (\lambda, -y) \leftrightarrow w \leftrightarrow w^{-1}$ between the two branches of the Riemann surface.

### 3.2 Punctures and local coordinates

Besides the curve, the set of algebraic-geometrical data demands the specification of a set of punctures and local coordinates around them. Again, since we are trying to embed the Seiberg–Witten solution, the natural choice is given by the two points at infinity $(\lambda = \infty, y = \pm \infty)$ plus the $2N_f$ points $(\lambda = m_f, y = \pm P(m_f, u_k))$, that will be denoted respectively $\infty^\pm$ and $m_f^\pm$. Following [12] the local coordinates will be chosen to be the appropriate powers of $w$ that uniformize the curve around them. More specifically, near the points $\infty^\pm$, we have $y \sim \pm P(\lambda) (1 + O(W^{-2}))$ so that $w^\pm \sim \lambda^N$. Then, in the vicinities of these punctures, $\xi \equiv w^{-1/N} \sim \lambda^{-1}$ near $\infty^+$ and $\xi \equiv w^{+1/N} \sim \lambda^{-1}$ close to $\infty^-$. Also, near “mass” punctures $m_f^\pm$, the local coordinates are $\xi \equiv w^{\pm 1} \sim (\lambda - m_f)$ for the symmetric scenario.

The general framework introduced in the previous section would allow us to consider a Whitham system (2.12) given by the whole set of meromorphic differentials corresponding to these punctures. This is out of the scope of the present paper and we will restrict ourselves to a smaller system which is enough to accomodate the Seiberg–Witten solution of $\mathcal{N} = 2$ super Yang–Mills theories with massive hypermultiplets. Namely: we are not going to include in our discussion meromorphic differentials of the second kind with higher poles at $m_f^\pm$. According to the general prescriptions (2.6) and (2.9), the canonical meromorphic differentials $d\Omega_n^{\infty^\pm}$, associated with times $T_n^{\infty^\pm}$, are expanded as follows

$$d\Omega_n^{\infty^\pm} \sim ((w^{\pm 1/N})^{-n-1} + O(1)) d(w^{\pm 1/N}) = \pm \frac{1}{N} (w^{\pm n/N} + O(w^{\mp 1/N})) \frac{dw}{w}, \quad (3.11)$$
whereas for the differentials of the third kind we have, after choosing
\[ P_0 = \infty, \]
\[ d\Omega_{m^+}^{\infty, \infty} \xrightarrow{\infty} - (w^{1/N})^{-1} + O(1) \] \( dw \),
\[ m^+ \quad (\infty^{+1} + O(1)) d(w^{+1}) = \mp(w^{-1} + O(w^{+1})) dw. \] (3.12)
They are associated with times \( T_{m^+, \infty}^{0, m^+} \).

### 4 The generating meromorphic 1-form, \( dS \)

In the previous sections we have reviewed the general framework of the Whitham hierarchy and introduced the auxiliary curve. The next ingredient is the generating 1-form \( dS \). At the end of the day we will manage to identify it, on a certain submanifold of the full Whitham moduli space, with the Seiberg–Witten differential \( dS_{SW} \). Let us start by considering the following set of meromorphic 1-forms
\[ d\hat{\Omega}_n \equiv \left( [W_{nN}]_{+} - \frac{1}{2N}\sum_{f=1}^{N_f} [W_{nN}]_{\lambda=m_f} \right) \frac{dw}{w}, \] (4.1)
were \([f]_+\) stands for the Laurent part of \( f(\lambda) \) at \( \lambda = \infty \), i.e. \([f]_+ = \sum_{k \geq 0} c_k \lambda^k\), \([f]_- = \sum_{k < 0} c_k \lambda^k = f - [f]_+\). For example,
\[ [W_{1N}]_+ = \lambda - \frac{t_1}{2N}, \]
\[ [W_{2N}]_+ = \lambda^2 - \frac{t_1}{2N} \lambda + \frac{1 + N}{2N^2} t_1^2 - \frac{2}{N} t_2 - \frac{2}{N} u_2, \] (4.2)
\[ \vdots \]

Let us define for convenience the quantities \( \kappa^f_n \) and \( \kappa_n \),
\[ \kappa^f_n(u_k, m_s) \equiv [W_{nN}]_{\lambda=m_f} \quad , \quad \kappa_n(u_k, m_s) \equiv \frac{1}{2N} \sum_{f=1}^{N_f} \kappa^f_n, \] (4.3)
in terms of which, after (3.8), the differentials \( d\hat{\Omega}_n \) can be casted in the form
\[ d\hat{\Omega}_n = \sqrt{F} \left( [W_{nN}]_+ - \kappa_n \right) W_n \frac{d\lambda}{y}. \] (4.4)
In contrast to \( d\Omega_n \), \( d\hat{\Omega}_n \) do have non-vanishing \( A^i \)-periods,
\[ c^i_n(u_k, m_f) \equiv \oint_{A^i} d\hat{\Omega}_n, \] (4.5)
as well as a non-vanishing residue at $\lambda = m_f$,

$$b^f_n(u_k, m_s) \equiv \pm \text{res}_{m_f^+} d\hat{\Omega}_n = \kappa_n - \kappa_n^f . \tag{4.6}$$

Notice that we are already working at the symmetric scenario, $\sum_{f=1}^{N_f} b^f_n = -N \kappa_n$ and, in particular, $b^1_n = -m_f$.

The meromorphic differentials $d\hat{\Omega}_n$ are the natural generalization of the analogous objects in pure $SU(N_c)$ \cite{12}, which can be recovered after setting, $F \to 1$ and $\kappa_n \to 0$ (as long as $N_f \to 0$). Using (4.4), we furthermore see that $d\Omega_1$ takes the familiar form $d\hat{\Omega}_1 = \lambda \frac{dw}{w}$ which is the Seiberg–Witten differential \cite{1, 21}. Naively, one could be tempted to consider a different set of meromorphic differentials $d\hat{\Omega}_n' = \left[W^{n/N}\right]_+ \frac{dw}{w}$, which also generalizes the pure $SU(N_c)$ case \cite{26}. However, the Seiberg–Witten differential would be in this case a combination of $d\Omega_1'$ and $d\Omega_0' = \frac{dw}{w}$, which in turn forces the introduction of a new variable $T^0_{\infty^+, \infty^-}$. This extra time \cite{3} has to be treated as an independent variable and introduces a number of unnecessary complications. In particular, the derivative of $F$ with respect to it (2.16) is hard to compute on general grounds. With our definition, instead, every differential $d\hat{\Omega}_n$ has residue balanced between $\infty$ and all masses $m_f$,

$$\text{res}_{\infty^\pm} d\hat{\Omega}_n = \text{res}_{\infty^\pm} \sqrt{F} \left(W^\pm \frac{n}{W} - \left[W^\pm \frac{n}{W}\right]_+ \right) W' \frac{d\lambda}{y} - \kappa_n \text{res}_{\infty^\pm} \sqrt{F} W' \frac{d\lambda}{y} = \text{res}_{\infty^\pm} (W^\pm \frac{n}{W}) W' \frac{d\lambda}{W} \pm N \kappa_n = \pm N \kappa_n = -\sum_{f=1}^{N_f} \text{res}_{m_f^\pm} d\hat{\Omega}_n .$$

In view of (2.4) and (2.7), the above set of residues and periods given in (4.5)–(4.6) and (4.7) can be taken into account by means of the following decomposition

$$d\hat{\Omega}_n = d\Omega_n + \sum_i c^i_n d\omega_i + \sum_f b^f_n d\Omega_{0,f} , \tag{4.7}$$

where $d\Omega_n = -N(d\Omega_n^{\infty^+} - d\Omega_n^{\infty^-})$, and $d\Omega_{0,f} = d\Omega_0^{m_f^+, \infty^+} - d\Omega_0^{m_f^-, \infty^-}$. Accordingly, $T^n$ and $T_0^f$ will stand for $T^n_{\infty^+} = -T^n_{\infty^-}$ and $T_0^{m_f^+, \infty^+} = -T_0^{m_f^-, \infty^-}$ respectively. We can now proceed to evaluate the derivatives of these differentials with respect to $u_k$ and $m_f$ (holding $w$ fixed).

**Lemma A:** The following equations hold

$$\left. \frac{\partial d\hat{\Omega}_n}{\partial u_k} \right|_w = \sum_i \left( \frac{\partial c^i_n}{\partial u_k} \right) d\omega_i + \sum_f \left( \frac{\partial b^f_n}{\partial u_k} \right) d\Omega_{0,f} , \tag{4.8}$$

\cite{3}The possibility of including an extra parameter like this was earlier considered in \cite{8}. It should be absent both in pure gauge and massless theories.
\[
\frac{\partial \Omega_n}{\partial m_s} \bigg|_w = \sum_i \left( \frac{\partial c_i^n}{\partial m_s} \right) d\omega_i + \sum_f \left( \frac{\partial b_f^n}{\partial m_s} \right) d\Omega_{0,f} .
\] (4.9)

The proof of this lemma is given in Appendix B.

Once the basic set of meromorphic differentials has been described, we can go one step further and set
\[
dS = \sum_{n \geq 1} T^n d\Omega_n .
\] (4.10)

In principle this expression defines \(dS(T^n, u_k, m_f)\). The idea now is to trade the \(N_c - 1 + N_f\) moduli \((u_k, m_f)\), for the equal number of Whitham coordinates \((\alpha^i, T^{0,f})\). In order to do so, we notice that the structure of poles and periods of \(dS\) can be taken into account in the “integrabilistic” basis of 1-forms, as in (2.19):
\[
dS = \sum_{n \geq 1} T^n d\Omega_n + \sum_{i=1}^{N_c-1} \alpha^i d\omega_i + \sum_{f=1}^{N_f} T^{0,f} d\Omega_{0,f} .
\] (4.11)

Now, \(T^n\) are independent variables given by
\[
T^n = \pm \frac{1}{N} \text{res}_{\infty} w^{-n/N} dS .
\] (4.12)

Using (4.7) and (4.10), we can compute \(\alpha^i\) and \(T^{0,f}\) as functions of \((T^n, u_k, m_f)\):
\[
\alpha^i = \sum_{n \geq 1} T^n c_n^i(u_k, m_f) = \oint_{A^i} dS ,
\] (4.13)
\[
T^{0,f} = \sum_{n \geq 1} T^n b_n^f(u_k, m_s) = \pm \text{res}_{m_f} dS .
\] (4.14)

We can solve these equations for \(u_k\) and \(m_f\) as functions of \((\alpha^i, T^n, T^{0,f})\), and this leads to the Whitham equations. These equations just emphasize the role of \(\alpha^i, T^{0,f}\) and \(T^n\) as independent coordinates. More explicitly, making use of (4.13) and (4.14), we demand
\[
\frac{d\alpha^i}{dT^n} = 0 = c^i_n + \sum_{m \geq 1} T^m \left( \frac{\partial u_k}{\partial T^n} \frac{\partial c^i_m}{\partial u_k} + \frac{\partial m_g}{\partial T^n} \frac{\partial c^i_m}{\partial m_g} \right) .
\] (4.15)

and
\[
\frac{dT^{0,f}}{dT^n} = 0 = b^f_n + \sum_{m \geq 1} T^m \left( \frac{\partial u_k}{\partial T^n} \frac{\partial b^f_m}{\partial u_k} + \frac{\partial m_g}{\partial T^n} \frac{\partial b^f_m}{\partial m_g} \right) .
\] (4.16)

In other words, calling \(\rho_l = \{u_k, m_f\}, (l = 1, \ldots, N_c + N_f - 1)\) the “old” moduli, and \(\gamma_a = \{\alpha^i, T^{0,f}\}, (a = 1, \ldots, N_c + N_f - 1)\) the “new” ones, the Whitham equations assert that \(\gamma_a\) and \(T^n\) form a set of independent coordinates:
\[
\frac{d\gamma_a}{dT^n} = \frac{\partial \gamma_a}{\partial T^n} + \frac{\partial \rho_l}{\partial T^n} \frac{\partial \gamma_a}{\partial \rho_l} = 0 ,
\] (4.17)
which can be solved for $\partial u_k / \partial T^n$ and $\partial m_f / \partial T^n$ by inverting $\partial \gamma_a / \partial \rho_l$,

$$
\frac{\partial \rho_l}{\partial T^n} = - \frac{\partial \gamma_a}{\partial T^n} \frac{\partial \rho_l}{\partial \gamma_a}, \quad \text{where} \quad \frac{\partial \rho_l}{\partial \gamma_a} \equiv \left. \left( \frac{\partial \gamma_a}{\partial \rho_l} \right)^{-1} \right|_{T^n}. \quad (4.18)
$$

The solution of these equations embodies functions $u_k$ and $m_f$ homogeneous of degree zero in $T^n, T^{0,f}$ and $\alpha^i$. This fact follows automatically after multiplying (4.18) by $T^n$ and summing up in $n$. Indeed, we see from (4.13) and (4.14) that $\gamma_a$ are linear in $T^m$, thus

$$
\sum_{n \geq 1} T^n \frac{\partial \rho_l}{\partial T^n} + \sum_a \gamma_a \frac{\partial \rho_l}{\partial \gamma_a} = 0. \quad (4.19)
$$

**Lemma B:** The generating differential $dS$ satisfies

$$
\frac{\partial dS}{\partial \alpha^i} = d\omega_i, \quad \frac{\partial dS}{\partial T^{0,f}} = d\Omega_{0,f}, \quad \frac{\partial dS}{\partial T^n} = d\Omega_n. \quad (4.20)
$$

The proof of this lemma is left to Appendix B.

## 5 Prepotential Derivatives

In the present section, a set of expressions is given for the dependence of the first and second derivatives of the prepotential with respect to the independent parameters $(\alpha^i, T^n, T^{0,f})$ as functions over the moduli space, i.e. of $u_k$ and $m_f$. In contrast to the situation in pure super Yang–Mills, the parameter $N = N_c - \frac{1}{2} N_f$ can become as small as one. For this reason, whenever possible, we have tried to push the range of validity of the formulas to higher times $T^n$ than those in [12]. In each case we will clearly specify the allowed range.

### 5.1 First Derivatives

The formal expressions for these functions are

$$
\alpha_i^D \equiv \frac{\partial F}{\partial \alpha^i} = \oint_{B_i} dS, \quad (5.1)
$$

$$
T_n^D \equiv \frac{\partial F}{\partial T^n} = - \frac{N}{\pi i} \text{res}_{\gamma^+} w^{n/N} dS, \quad (5.2)
$$

$$
T_{0,f}^D \equiv \frac{\partial F}{\partial T^{0,f}} = \frac{1}{\pi i} \int_{\gamma^+}^m dS. \quad (5.3)
$$
From these, only (5.2) can be worked out to yield some polynomial function of \( u_k \) and \( m_f \):

\[
\frac{\partial F}{\partial T_n} = -\frac{N}{\pi i n} \text{res}_\infty w^{n/N} dS
= -\frac{N^2}{\pi i n^2} \sum_{m \geq 1} T^m \text{res}_\infty \left( [W^{\mp}]_+ - \kappa_m \right) dw^{n/N}.
\]

Since \( w^{n/N} = W^{n/N} \left( 1 - \frac{n}{N} W^{-2} + \mathcal{O}(W^{-4}) \right) \), this derivative takes the form

\[
\frac{\partial F}{\partial T_n} = \frac{N^2}{\pi i n} \sum_{m \geq 1} T^m \text{res}_\infty^+ W^{n/N} \left( 1 - \frac{n}{N} W^{-2} + \mathcal{O}(W^{-4}) \right) d [W^{\mp}]_+
= \frac{N}{\pi i n} \sum_{m \geq 1} m T^m \left( \mathcal{H}_{n+1,m+1} \left[ W^{\mp} \right]_+ - \frac{n-2N}{N} \mathcal{H}_{n+1-2N,m+1} + \ldots \right) . \tag{5.4}
\]

This expression is valid for \( n + m < 4N \), and the second term gives non-vanishing contributions only for \( n + m \geq 2N \). In Eq.(5.4), \( \mathcal{H}_{p+1,q+1} \) stand for polynomials in \( (u_k, m_f) \) defined by

\[
\mathcal{H}_{p+1,q+1} = \frac{N}{pq} \text{res}_\infty W^p d [W^q]_+ = \mathcal{H}_{q+1,p+1} . \tag{5.5}
\]

The first few examples of these polynomials are

\[
\mathcal{H}_{2,2} = u_2 - \frac{1+2N}{8N} t^2_1 + \frac{t_2}{2}, \tag{5.6}
\]

\[
\mathcal{H}_{2,3} = u_3 - \frac{t_1}{N} u_2 + \frac{t_3}{2} - \frac{1}{2N} t_1 t_2 + \frac{1}{12N^2} t^3_1, \]

\[
\mathcal{H}_{3,3} = u_4 + \frac{N-2}{2N} u^2_2 + \frac{t^2_3 - 2t_2}{2N} u_2 + \frac{N+1}{2N} t^2_1 t_2
- \frac{1}{4N} t^2_2 - \frac{1}{4N} t_1 t_3 + \frac{t_4}{2} + \frac{1}{48N^3} t^4_1.
\]

Concerning Eq.(5.3), let us point out that, since \( dS \) has first order poles at \( m_f^\pm \), it is actually logarithmically divergent and needs a regularization.

### 5.2 Second Derivatives

The formal expresions for these functions are obtained directly from the general Whitham setup adapted to the present context

\[
\frac{\partial^2 F}{\partial \alpha^i \partial \alpha^j} = \oint_{B_j} d\omega_i , \tag{5.7}
\]
\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^n} = -\frac{N}{\pi in} \text{res}_{\infty^+} \left( w_i^m d\omega_i \right) = \oint_{B_i} d\Omega_n , \\
(5.8)
\]

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^m} = -\frac{N}{\pi in} \text{res}_{\infty^+} \left( w_i^m d\Omega_m \right) ,
(5.9)
\]

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^0, f} = \frac{1}{\pi i} \int_{m_f^+} \omega_i = \oint_{B_i} d\Omega_{0, f} ,
(5.10)
\]

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^0, g} = \frac{1}{\pi i} \int_{m_g^+} \omega_i = \oint_{B_i} d\Omega_{0, g} ,
(5.11)
\]

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^0} = -\frac{N}{\pi in} \text{res}_{\infty^+} \left( w_i^m d\Omega_{0, f} \right) = \frac{1}{\pi i} \int_{m_f^+} d\Omega_n . 
(5.12)
\]

For the first equation above we simply have

\[
\frac{\partial^2 F}{\partial \alpha^i \partial \alpha^j} = \tau_{ij} .
(5.13)
\]

We shall obtain in what follows closed expressions, as functions over the moduli space, for those derivatives involving local (residue) calculations, such as (5.8)–(5.9) and (5.12).

### 5.2.1 Mixed derivatives with respect to \( T^n \) and \( \alpha^i \)

The mixed derivatives with respect to \( T^n \) and \( \alpha^i \) (5.8) are given by

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^n} = -\frac{N}{\pi in} \text{res}_{\infty^+} w_i^m d\omega_i = -\frac{N}{\pi in} \text{res}_{\infty^+} W_i^m \left( 1 - \frac{n}{N} W^{-2} + O(W^{-4}) \right) d\omega_i .
(5.14)
\]

To obtain these derivatives we still have to expand \( d\omega_i \) near \( \infty^+ \) as follows

\[
d\omega_i = \sum_k \frac{\partial u_k}{\partial \alpha^i} \, dv_k = \sum_k \frac{\partial u_k}{\partial \alpha^i} \, \lambda^{Nc-1-k} P \left( 1 + 2W^{-2} + O(W^{-4}) \right) \, d\lambda \\
= -\sum_k \frac{\partial u_k}{\partial \alpha^i} \frac{\partial}{\partial u_k} \log W \left( 1 + 2W^{-2} + O(W^{-4}) \right) \, d\lambda ,
(5.15)
\]

so that, finally, the residue in (5.14) can be written as

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^n} = \frac{N^2}{\pi i n^2} \frac{\partial}{\partial \alpha_i} \text{res}_{\infty^+} \left( W_i^m \left( \frac{n - 2N}{N} W_i^{m-2} + O(W_i^{m-4}) \right) \right) d\lambda .
(5.16)
\]

We can better write this result in terms of the polynomials \( \mathcal{H}_{p+1} \equiv \mathcal{H}_{p+1, 2} = \mathcal{H}_{2, p+1} \) as follows

\[
\frac{\partial^2 F}{\partial \alpha^i \partial T^n} = \frac{N}{\pi in} \frac{\partial}{\partial \alpha_i} \left( \mathcal{H}_{n+1} - \frac{n - 2N}{N} \mathcal{H}_{n-2N+1} + \ldots \right) ,
(5.17)
\]

where the dots denote terms that contribute only for \( n \geq 4N - 1 \), and the derivative \( \partial/\partial a_i \) should be taken at constant \( m_f \).
5.2.2 Second derivatives with respect to $T^n$ and $T^m$

The second derivatives of the prepotential with respect to the Whitham times are given by

\[
\frac{\partial^2 F}{\partial T^n \partial T^m} = -\frac{N}{i\pi n} \text{res}_\infty w^{\frac{\Phi}{2}} d\Omega_m = \frac{N^2}{i\pi n} \text{res}_\infty w^{\frac{\Phi}{2}} \left( d\Omega_m^+ - d\Omega_m^- \right). \tag{5.18}
\]

To evaluate this residue, it is more convenient to use the canonical differentials in hyper-elliptic coordinates $d\tilde{\Omega}_m^\pm(\lambda)$. The relation between $d\Omega_m^\pm$ and $d\tilde{\Omega}_m^\pm$ is easily obtained by matching the asymptotic behaviour around $\infty^\pm$. Expanding

\[
W_m/N = \sum_{p=-\infty}^m b_{mp}(\lambda)^p
\]

one gets (for $m \leq 2N$)

\[
d\Omega_m^\pm = \frac{1}{m} d\omega_m^\pm/N + \cdots = \frac{1}{m} \sum_{p=1}^m p b_{mp}(\lambda)^p (-\lambda^{p-1} d\lambda + \cdots) = \frac{1}{m} \sum_{p=1}^m p b_{mp} d\tilde{\Omega}_p^\pm,
\]

where the dots denote the regular part of the differentials, which is unambiguously fixed by the condition (2.8). Eq.(5.18) can now be written as

\[
\frac{\partial^2 F}{\partial T^n \partial T^m} = \frac{N^2}{i\pi nm} \sum_{p=1}^m p b_{mp}(\lambda)^p \text{res}_\infty \left( w^{\frac{\Phi}{2}} d\tilde{\Omega}_p^\pm \right). \tag{5.19}
\]

This residue can be computed with the help of expression (3.11), and closely follows the one performed in [12] for the pure gauge theory. It turns out that, as long as $m + n < 2N$, the result found there is still valid (with the obvious replacements given in (3.9))

\[
\frac{\partial^2 F}{\partial T^n \partial T^m} = -\frac{N}{\pi i} \left( \mathcal{H}_{n+1,m+1} + \frac{2N}{mn} \frac{\partial \mathcal{H}_{n+1}}{\partial a_i} \frac{\partial \mathcal{H}_{m+1}}{\partial a_j} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E \right), \tag{5.20}
\]

where $\Theta_E$ is the Riemann theta function with a particular even and half-integer characteristic (see Appendix C). In asymptotically free theories with paired massive hypermultiplets, $N$ can be as small as 1. Thus, we would need to extended the range of validity for this formula to higher times $T^n, T^m$ up to $1 \leq n, m \leq 2N$ with $n + m \leq 2N$, in order to include at least $T^n_1$. In general, when $n + m \geq 2N$, additional contributions must be considered. If $n + m = 2N$, it is just a constant

\[
\frac{N}{i\pi} \left( \frac{m}{n} + \theta(m - N - 1) \frac{4(N - m)N}{mn} \right) \delta_{m+n,2N} = \frac{N}{i\pi} f_{mn} \delta_{m+n,2N}, \tag{5.21}
\]

where $f_{mn} = \min(m,n)/\max(m,n)$. Then, the net result is

\[
\frac{\partial^2 F}{\partial T^n \partial T^m} = -\frac{N}{\pi i} \left( \mathcal{H}_{n+1,m+1} + \frac{2N}{mn} \frac{\partial \mathcal{H}_{n+1}}{\partial a_i} \frac{\partial \mathcal{H}_{m+1}}{\partial a_j} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E - f_{mn} \delta_{m+n,2N} \right). \tag{5.22}
\]
5.2.3 Second derivatives with respect to \( T^m \) and \( T^{0,f} \)

The calculation of the second derivative of the prepotential \( \mathcal{F} \) with respect to \( T^m \) and \( T^{0,f} \) requires the evaluation of the residue

\[
\frac{\partial^2 \mathcal{F}}{\partial T^{0,f} \partial T^m} = -\frac{N}{\pi i n} \text{res}_{\infty^+} w^{n/N} d\Omega_{0,f} = -\frac{N}{\pi i n} \text{res}_{\infty^+} w^{n/N} \left( d\Omega_{0,f}^{m_f^+,\infty^+} - d\Omega_{0,f}^{m_f^-,\infty^-} \right). \tag{5.23}
\]

Like in the preceding derivation, calculations are feasible in hyperelliptic coordinates \( \lambda \). In this case, however, it is clear that \( d\Omega_{0,f}^{m_f^+,\infty^+}(\lambda) = d\Omega_{0,f}^{m_f^-,\infty^-}(w(\lambda)) \) since the coefficient of both singular parts, \( i.e. \) the residue, is fixed to be \( \pm 1 \). Remember also that we are working in the “symmetric scenario” in which the masses come degenerated in pairs; the index \( f \) will then run in the range \( f = 1, \cdots, N_f/2 \). It is possible to obtain, from the general theory of Riemann surfaces [27, 28], a convenient representation for the meromorphic differential of the third kind \( d\Omega_{0,f} \) (the details of this calculation are given in Appendix C)

\[
d\Omega_{0,f}^{m_f^+,\infty^+} - d\Omega_{0,f}^{m_f^-,\infty^-} = \frac{P}{y} \frac{d\lambda}{(\lambda - m_f)} + \frac{1}{\pi i} d\omega_i \partial_i \log \Theta_E(\vec{z}_f | \tau), \tag{5.24}
\]

where the vector \( \vec{z}_f \)

\[
\vec{z}_f = \frac{1}{2\pi i} \int_{\infty^+}^{m_f^+} d\vec{z}, \tag{5.25}
\]

is the image of the divisor \( m_f^+ - \infty^+ \) under the Abel map. Inserting the previous formula in (5.23), we obtain

\[
\frac{\partial^2 \mathcal{F}}{\partial T^{0,f} \partial T^m} = \frac{N}{\pi i n} \kappa_f' + \frac{N}{\pi^2 n} \partial_{H_{n+1}} \partial_i \log \Theta_E(\vec{z}_f | \tau), \tag{5.26}
\]

this result being valid as long as \( n \leq 2N \).

6 Duality transformations

As it is well-known, one of the key properties of the Seiberg–Witten ansatz is the existence of equivalent duality frames for the low-energy theory. In the theories with matter hypermultiplets, as already remarked in [1], the duality transformations (which are usually elements of the symplectic group) pick an inhomogeneous part associated with the masses of the hypermultiplets. We will show in what follows how all this is nicely encoded in the present geometrical framework. The duality symmetry will turn out to be nothing but an ambiguity in the choice of the geometrical data involved in the construction of the prepotential within the Whitham hierarchy. We start by characterizing this ambiguity.
For this purpose it is convenient to recall equation (2.24). As it stands, it is just a formal expression but it nicely exhibits the fact that the different duality frames are associated with different choices of integration contours. We will distinguish two types of operations that can be performed on these contours:

- **changes of the symplectic homology basis** \((A^i, B^j) \rightarrow (\tilde{A}^i, \tilde{B}^j)\). These are performed as usual by means of a matrix \(\Gamma\),

\[
\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2r, \mathbb{Z}),
\]

(6.1)

where \(r\) is the rank of the gauge group and the \(r \times r\) matrices \(A, B, C, D\) satisfy:

\[
A^i D - C^i B = 1, \quad A^i C = C^i A, \quad B^i D = D^i B.
\]

(6.2)

- **deformations of the integration contours**. When we deform a one-cycle across a pole, we pick up the residue of \(dS\). In principle, these deformations can be performed independently on each integration contour.

Bearing in mind the equations (4.12)–(4.14) and (5.1)–(5.3), we see that the previous operations leave \(T^n, T^D_n\) and \(T^{0,f}\) intact

\[
\tilde{T}^n = T^n, \quad \tilde{T}^D_n = T^D_n, \quad \tilde{T}^{0,f} = T^{0,f}.
\]

(6.3)

Therefore, the most general ambiguity yields the ansatz

\[
\tilde{\alpha}^D_i = A^i_j \alpha^D_j + B^i_j \alpha^j + p_{ij} T^{0,f},
\]

(6.4)

\[
\tilde{\alpha}^i = C^{ij} \alpha^D_j + D^{ij} \alpha^j + q^{ij} T^{0,f},
\]

(6.5)

\[
\tilde{T}^{0,f}_{0,f} = T^{0,f}_{0,f} + R_{fi} \alpha^i + S^i_j \alpha^D_i + t_{fg} T^{0,g},
\]

(6.6)

together with (6.3), where \(T^{0,f}\) appears on the right hand side because it is the residue of \(dS\) at the pole \(\lambda = m_f\). \(R_{fi}\) and \(S^i_j\) are matrices of integers that signal the possibility of deforming the contour between \(m_f\) and \(\infty\) by encircling additional cycles. Also, \(p_{ij}\) and \(q^{ij}\) are even (because of the paired masses) integer coefficients that account for poles that are crossed when one-cycle deformations are performed. The extended duality group is, however, not as big as these formulas may suggest. Namely, the deformations in the contours that define \(T^{D}_{0,f}, \alpha^i\) and \(\alpha^D_i\) cannot occur independently. This is not easy to see geometrically, but it is a consequence of the fact that a single function, the prepotential, is behind the whole construction. The representation given in (2.24) is nothing but the
statement of the fact that $F$ is a homogeneous function of degree two. For the present case it can be written as

$$F = \frac{1}{2} (\alpha^i \alpha^D_i + T^m T^n + T^{0,j} T^{0,j}_0) ,$$

(6.7)

where repeated indices $i, n$ and $f$ are summed up. Plugging (6.3)–(6.6) in this formula, and using (6.2), one easily obtains the transformed $\tilde{F}$. This expression can be used to compute $\partial \tilde{F} / \partial \alpha^i$, which should be compared with

$$\partial \tilde{F} / \partial \alpha^i = \partial \tilde{\alpha}_j^D / \partial \alpha^i .$$

(6.8)

Agreement between both expressions enforces the constraint:

$$S_i^f = p_{ij} f C_{ij} - q_{ij} f D_{ij} ; \quad R_{i,f} = p_{ij} A_{ij} - q_{ij} B_{ij} .$$

(6.9)

Hence (6.6) reduces to

$$\tilde{T}^{D}_0, f = T^{D}_0, f + p_{ij} f (C^{ij} \alpha^D_i + D^{ij} \alpha^j) - q_{ij} f (A_{ij} \alpha^D_i + B_{ij} \alpha^j) + t_{f} T^{0,g} ,$$

(6.10)

and, from Eq.(6.7), we obtain the generalized duality transformation rule for $F$ (see also [29]):

$$\tilde{F}(\tilde{\alpha}^i, T^n, T^{0,f}) = F(\alpha^i, T^n, T^{0,f}) + \frac{1}{2} \alpha^i (D^T B)_{ij} \alpha^j + \frac{1}{2} \alpha^D_i (C^T A)_{ij} \alpha^D_j + \alpha^D_i (B^T C)_{ij} \alpha^j$$

$$+ p_{ij} T^{0,j} (C^{ij} \alpha^D_j + D^{ij} \alpha^j) + \frac{1}{2} (p_{ij} q_{ij} + t_{f} g) T^{0,f} T^{0,g} .$$

(6.11)

Using this expression, one can easily compute the transformation properties of the first and second derivatives of the prepotential. Reserving the indices $i, j, k$ for the cycles $\alpha^i$ and $\alpha^D_i$, $f, g$ for the times associated to the masses $T^{0,f}, T^{0,f}_0$, and $n, m$ for the higher Whitham times $T^n, T^n_0$, we define the generalized couplings

$$\tau_{ij} = \frac{\partial^2 F}{\partial \alpha^i \partial \alpha^j} \quad \tau_{ij} = \frac{\partial^2 F}{\partial \alpha^i \partial T^{0,f}} \quad \tau_{in} = \frac{\partial^2 F}{\partial \alpha^i \partial T^n}$$

$$\tau_{fg} = \frac{\partial^2 F}{\partial T^{0,f} \partial T^{0,g}} \quad \tau_{fn} = \frac{\partial^2 F}{\partial T^{0,f} \partial T^n} \quad \tau_{nm} = \frac{\partial^2 F}{\partial T^n \partial T^m} .$$

(6.12)

To unravel the transformation rules for these couplings the most efficient way is to make use of their geometrical definition (5.7)–(5.12). The geometrical data involved are contours, residues and differentials. After the previous discussions, the contours change as follows

$$\oint B_i = A_{ij} \oint B_j + B_{ij} \oint A_i + \sum_f p_{ij} \text{res}_{m_f} ,$$

(6.13)
\[
\int_{A^f} = C^{ij} \int_{B^j} + D^i_j \int_{A^f} + \sum_f q^i_f \text{res}_{m_f},
\]
\[
\int_{m_f}^{\infty} = \int_{m_f}^{\infty} + p_{ij}(C^{ij} \int_{B^j} + D^i_j \int_{A^f}) - q^i_f (A^i_j \int_{B^j} + B_{ij} \int_{A^f}) + \sum_g t_{fg} \text{res}_{m_g}.
\]

The change in the symplectic homology basis can be easily pulled back to the canonically normalized basis of meromorphic differentials
\[
d\tilde{\omega}_i = d\omega_j (C\tau + D)^{-1} j_i,
\]
\[
d\tilde{\Omega}_\tau = d\Omega_n - d\omega_i ((C\tau + D)^{-1} C)^{ij} \int_{B^j} d\Omega_n,
\]
\[
d\tilde{\Omega}_{0,f} = d\Omega_{0,f} - d\omega_i ((C\tau + D)^{-1} C)^{ij} \int_{B^j} d\Omega_{0,f} - (C\tau + D)^{-1} j_i q^j_f.
\]

Inserting (6.13)–(6.15) and (6.16)–(6.18) in (5.7)–(5.12), the transformation rules for the couplings (6.12) come out straightforwardly
\[
\tilde{\tau}_{ij} = [(A\tau + B)(C\tau + D)^{-1}]_{ij},
\]
\[
\tilde{\tau}_{im} = [(C\tau + D)^{-1}]^j_i \tau_{jm},
\]
\[
\tilde{\tau}_{mn} = \tau_{mn} - \tau_{im} [(C\tau + D)^{-1} C]^{ij} \tau_{jn},
\]
\[
\tilde{\tau}_{if} = [(C\tau + D)^{-1}]^j_i \tau_{jf} - [(A\tau + B)(C\tau + D)^{-1}]_{ij} q^j_f + p_{if},
\]
\[
\tilde{\tau}_{fn} = \tau_{fn} - q^i_f [(C\tau + D)^{-1}]^j_i \tau_{jn} - \tau_{if} [(C\tau + D)^{-1} C]^{ij} \tau_{jn},
\]
\[
\tilde{\tau}_{fg} = \tau_{fg} - \tau_{if} [(C\tau + D)^{-1} C]^{ij} \tau_{jg} + q^i_f [(A\tau + B)(C\tau + D)^{-1}]_{ij} q^j_g
\]
\[
- q^i_f [(C\tau + D)^{-1}]^j_i \tau_{jg} - q^i_g [(C\tau + D)^{-1}]^j_i \tau_{jf} - p_{if} q^j_g + t_{fg}.
\]

Eventually, we find another constraint on \(t_{fg}\) from the requirement of symmetry under \(f \leftrightarrow g\) in the last expressions. This is solved in general by taking \(t_{fg} = p_{ij} q^i_g + s_{fg}\) with \(s_{fg}\) an arbitrary integer valued symmetric matrix. It is reassuring to find that (6.11) and (6.19)–(6.24) fully coincide and generalize the results presented in [17] for pure \(SU(N_c)\) and [29], to which they reduce when there is only one higher Whitham time \(T^n \sim \Lambda \delta_{n1}\).

7 The Seiberg–Witten hyperplane

In this section we shall identify the Seiberg–Witten solution as a submanifold of the Whitham configuration space. In the former, the \(a^i\) variables of the prepotential, for the duality frame associated to the \(A^i\)-cycles, are given by the integrals over these cycles of
a certain meromorphic one-form, $dS_{SW}$, that can be written as

$$a^i(u_k, m_f; \Lambda) = \oint_{A^i} dS_{SW} \equiv \oint_{A^i} \frac{\lambda W'(u_k, m_f)}{\sqrt{W^2(u_k, m_f) - 4\Lambda^2}} d\lambda ,$$

(7.1)

and the same expression holds for the dual variables $a^D_i$ with $B_i$ replacing $A_i$. Here, $\Lambda$ stands for $\Lambda_{N_f}$, the quantum generated dynamical scale. On the Whitham side, correspondingly, we have the $\alpha^i$ variables given by

$$\alpha^i(u_k, m_f; T_1, T_2, ...) = T_1 a^i(u_k, m_f; 1) + \mathcal{O}(T_n > 1) = a^i(\bar{u}_k, \bar{m}_f, \Lambda = T_1) + \mathcal{O}(T_n > 1) ,$$

(7.3)

where we have introduced the set of rescaled variables

$$\bar{u}_k = (T_1)^k u_k , \quad \bar{m}_f = T_1 m_f .$$

(7.4)

Summarizing, the Seiberg–Witten differential of [23] can be exactly recovered after performing dimensional analysis in units of the scale set by $T_1$, and tuning $T_n > 1 = 0$. Using (4.14), we also see that the Whitham times $T^{0,f}$ become (up to a sign) the bare masses:

$$T^{0,f} = -\bar{m}_f \quad f = 1, ..., N_f/2 .$$

(7.5)

In view of the previous considerations, we shall define the following change of variables

$$(\alpha, T^{0,f}, T^{n>1}, T_1) \to (\alpha, T^{0,f}, T^{n>1}, \log \Lambda) ,$$

where

$$\log \Lambda = \log T_1 , \quad T^n = (T_1)^{-n} T^n \quad (n > 1) ,$$

(7.6)

and, consequently,

$$\frac{\partial}{\partial \log \Lambda} = \sum_{m \geq 1} m T^m \frac{\partial}{\partial T^m} , \quad \frac{\partial}{\partial T^n} = (T_1)^n \frac{\partial}{\partial T^n} \quad (n > 1) .$$

(7.7)

With the help of these expressions, one can rewrite all the formulas given in the last section for the derivatives of $\mathcal{F}$, as derivatives with respect to $\alpha^i, T^{0,f}, T^{n>1}$ and $\log \Lambda$. Most of

\footnote{Notice that the variable $a^i$ in [1.17], [5.22] and [5.26] stands precisely for $a^i(u_k, m_f; 1)$.}
the factors $T^1$ are used to promote $u_k$ to $\bar{u}_k$ or, rather, the homogeneous combinations thereof

$$\bar{H}_{m+1,n+1} = (T^1)^{m+n}H_{m+1,n+1} \quad \Rightarrow \quad \bar{H}_{m+1} = (T^1)^{m+1}H_{m+1}, \quad (7.8)$$

the remaining ones are absorbed in making up $\bar{a}^i \equiv T^1a^i(u_k, m_f; 1) = a^i(\bar{u}_k, \bar{m}_f; T^1)$ and $\bar{m}_f \equiv T^1m_f$ (see [17] for this explicit intermediate step).

At the end of the day, the restriction to the submanifold $T^{n>1} = 0$ and $T^1 = \Lambda$, yields formulas which are ready for use in the Seiberg–Witten analysis. Notice that in this subspace $\alpha^i(u_k, m_f; T^1 = \Lambda, T^{n>1} = 0) = a^i(\bar{u}_k, \bar{m}_f, \Lambda = T^1) \equiv \bar{a}^i$; hence (after omitting all bars for clarity) one can write

$$\frac{\partial F}{\partial \log \Lambda} = \frac{N}{\pi i} (H_2 + \Lambda^2 \delta_{N,1}) \quad , \quad \frac{\partial^2 F}{\partial a^i \partial \log \Lambda} = \frac{N}{\pi i} \frac{\partial H_2}{\partial a^i}, \quad (7.9)$$

$$\frac{\partial^2 F}{\partial (\log \Lambda)^2} = -\frac{2N^2}{\pi i} \frac{\partial H_2}{\partial a^j} \frac{1}{\pi i} \partial_{r_j} \log \Theta_E(0 | \tau) + \frac{2N^2}{\pi i} \Lambda^2 \delta_{N,1}, \quad (7.10)$$

$$\frac{\partial^2 F}{\partial m_f \partial \log \Lambda} = -\frac{N}{\pi i} \left( m_f - \frac{t_1}{2N} \right) - \frac{N}{\pi^2} \frac{\partial H_2}{\partial a^i} \partial_{r_i} \log \Theta_E(\bar{z}_f | \tau), \quad (7.11)$$

as well as, for derivatives with respect to higher Whitham times, we obtain

$$\frac{\partial F}{\partial T^n} = \frac{N}{\pi in} \left( H_{n+1} + \frac{1}{N} \Lambda^{2n} \delta_{n,2N-1} \right) \quad , \quad \frac{\partial^2 F}{\partial a^i \partial T^n} = \frac{N}{\pi in} \frac{\partial H_{n+1}}{\partial a^i}, \quad (7.12)$$

$$\frac{\partial^2 F}{\partial \log \Lambda \partial T^n} = -\frac{2N^2}{\pi in} \frac{\partial H_{n+1}}{\partial a^j} \frac{1}{\pi i} \partial_{r_j} \log \Theta_E(0 | \tau) + \frac{n + N}{\pi in} \Lambda^{2N} \delta_{n,2N-1}, \quad (7.13)$$

$$\frac{\partial^2 F}{\partial m_f \partial T^n} = -\frac{N}{\pi in} \kappa_n^f - \frac{N}{\pi^2 n} \frac{\partial H_{n+1}}{\partial a^i} \partial_{r_i} \log \Theta_E(\bar{z}_f | \tau), \quad (7.14)$$

$$\frac{\partial^2 F}{\partial T^m \partial T^n} = -\frac{N}{\pi i} \left( H_{n+1,m+1} + \frac{2N}{mn} \frac{\partial H_{n+1}}{\partial a^j} \frac{\partial H_{m+1}}{\partial a^i} \frac{1}{\pi i} \partial_{r_j} \log \Theta_E(0 | \tau) \right.$$

$$\left. - \frac{\min(m, n)}{\max(m, n)} \Lambda^{2N} \delta_{n+m,2N} \right), \quad (7.15)$$

where $t_1 = -2\sum_{j=1}^{N_f/2} m_f$ (cf. Eq. (7.3)) and $m, n \geq 2$, whereas $n \leq 2N$ and $m + n \leq 2N$ in (7.13). It is worth to remark that, whereas the latter set of equations (7.12)–(7.13) involve deformations of the effective prepotential parametrized by higher Whitham times, the former one (7.9)–(7.11) is entirely written in terms of the original Seiberg–Witten variables.
Finally, one can combine Eqs.\((7.12)-(7.14)\) to write the following interesting expressions for the derivatives of (homogeneous combinations of) higher Casimir operators with respect to \(\Lambda\) and \(m_f\),

\[
\frac{\partial H_{n+1}}{\partial \log \Lambda} = -2N \frac{\partial H_{n+1}}{\partial a^i} \frac{\partial H_2}{\partial a^j} \frac{1}{\pi i} \partial_{ij} \log \Theta_E(0|\tau) + \frac{N - 1}{N} \Lambda^{2N} \delta_{n,2N-1}, \tag{7.16}
\]

\[
\frac{\partial H_{n+1}}{\partial m_f} = -\kappa_n^f + \frac{1}{\pi i} \frac{\partial H_{n+1}}{\partial a^i} \partial_i \log \Theta_E(\vec{z}_f | \tau). \tag{7.17}
\]

Let us provide in the following section some non-trivial checks supporting these results.

8 Some checks

One of the main results in this paper is given by the whole set of equations \((7.9)-(7.17)\) for the derivatives of the effective prepotential. The equation \((7.10)\) is by now a well settled result. In Ref.\([13]\), an independent derivation coming from topological field theory was obtained prior to the work \([12]\). For the pure gauge SU(2) case, it was checked by using the Picard–Fuchs equations in \([12]\). In Refs.\([17, 22]\), it was put in the test bench and two additional proofs were passed. First of all, the right hand side was shown to reproduce correctly the appropriate duality transformation rules. In addition, this equation was used to obtain the semiclassical expansion of the effective prepotential up to arbitrary instanton corrections with remarkable success. In this section, we will see that equation \((7.13)\) also enjoys the generalized duality properties and is consistent with the instanton expansion.

Let us first analyze the duality transformations \((6.3)-(6.6)\) and \((6.10)\) where the new ingredient, as we have already remarked, is the inhomogeneous piece associated to the presence of masses. This inhomogeneous piece stems from the deformation of contours across simple poles \((6.13)-(6.15)\). We can reinterpret this ambiguity in the context of the formulas involving the Riemann Theta function. First of all, the vector \(\vec{z}_f\) \((5.25)\) lives in the Jacobian of the hyperelliptic curve, as the image of the divisor \(m^+_f - \infty^+\) under the Abel map, thus being defined up to transformations of the form:

\[
z_{f,i} \to z_{f,i} + n_{f,i} + \tau_{ij} \ell_{f}^j \tag{8.1}
\]

with integers \(n_{f,i}\) and \(\ell_{f}^j\). Taking into account that \(z_{f,i} = 1/2 \tau_{ij}\) (cf. \(5.10)\), we see that \((8.1)\) reproduces the formula \((5.22)\), when the symplectic rotation is the identity, with \(p_{i,f} = 2n_{i,f}, q_{f}^j = -2\ell_{f}^j\). Now we can check that the formula \((5.26)\) is consistent with the
transformation law given in equation (6.23). Using the transformation property of the theta function under shifts

$$
\Theta \left[ \vec{\alpha} \beta \right] (z_{f,i} + n_{f,i} + \tau_{ij} \ell_j | \tau) = \exp(-\pi i \ell_i \tau_{ij} \ell_j - 2\pi i \ell_f (z_{f,i} + \beta_i) + 2\pi i \alpha^i n_{f,i}) \Theta \left[ \vec{\alpha} \beta \right] (z_{f,i} | \tau),
$$

we easily obtain

$$
\frac{1}{\pi i} \frac{\partial}{\partial z_i} \log \Theta_E(z_{f,i} | \tau) \rightarrow -2 \ell_i + \frac{1}{\pi i} \frac{\partial}{\partial z_i} \log \Theta(z_{f,i} | \tau),
$$

which induces precisely the transformation law (6.23) with $q_i^f = -2 \ell_i^f$. Next, for the behavior of the Theta function under homogeneous symplectic transformations we find

$$
\frac{1}{2\pi i} \frac{\partial}{\partial z_i} \log \Theta_E(z_{f,i} | \tau) \rightarrow \frac{1}{2\pi i} \frac{\partial}{\partial z_i} \log \Theta(z_{f,i} | \tau) + \left[(C \tau + D)^{-1} C\right]_{ij} z_{f,j},
$$

and taking into account (5.17), we find again the right transformation properties in the whole set of expressions (6.19)–(6.24).

A second, and much more stringent check of equation (7.11) is provided by the semi-classical expansion of the prepotential in powers of $\Lambda$. One can write the ansatz

$$
\mathcal{F} = \frac{3N}{4Nc\pi i} \sum_{\alpha_+} Z_{\alpha_+} - \frac{i}{4\pi} \sum_{\alpha_+} Z_{\alpha_+} \log \frac{Z_{\alpha_+}}{\Lambda^2} - \frac{i}{4\pi} \sum_{f=1}^{N_f} \sum_{p=1}^{N_c} (e_p + m_f)^2 \log \frac{(e_p + m_f)^2}{\Lambda^2}
$$

$$
- \frac{N_f}{8\pi i} \sum_{f=1}^{N_f/2} m_f^2 \log \frac{m_f^2}{\Lambda^2} + \frac{t_1^2}{16\pi i} \log \frac{t_1^2}{\Lambda^2} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_k (Z) \Lambda^{2kN}. \tag{8.4}
$$

The set $\{\alpha_i\}_{i=1,\ldots,r}$ stands for the simple roots of the corresponding classical Lie algebra. Also in (8.4), $\alpha_+$ denotes a positive root and $\sum_{\alpha_+}$ is the sum over all positive roots. The dot product ($\cdot$) of two simple roots $\alpha_i$ and $\alpha_j$ gives an element of the Cartan matrix, $A_{ij} = \alpha_i \cdot \alpha_j$ and extends bilinearly to arbitrary linear combinations of simple roots. For any root $\alpha = n^i \alpha_j \in \Delta$, the quantities $Z_{\alpha}$ are defined by $Z_{\alpha} = a \cdot \alpha \equiv a^i A_{ij} n^j$ where $a = a^i \alpha_i$. Simple roots can be written in terms of the orthogonal set of unit vectors $\{e_p\}_{p=1,\ldots,N_c}$ and the order parameters $a^i$ and $e_p$ are related by $e_p = a \cdot e_p$. Also $\mathcal{F}_k (a^i, m_f)$ are homogeneous functions of degree $2 - 2kN$ that represent the instanton corrections to the perturbative 1-loop effective action.

In Refs. [17, 22], it was shown that inserting (8.4) into (7.10), and expanding both members of the equation in powers of $\Lambda^{2N}$, the instanton corrections $\mathcal{F}_k$ could be fixed completely in a recursive way. As an example, for $SU(2)$ with two degenerated flavors...
\( (N_f = 2) \) one readily obtains

\[
\mathcal{F}_1 = \frac{u_2 + m^2}{2u_2}, \quad \text{(8.5)}
\]

\[
\mathcal{F}_2 = \frac{u_2^2 - 6u_2m^2 + 5m^4}{64u_2^3}, \quad \text{(8.6)}
\]

\[
\mathcal{F}_3 = \frac{5u_2^2m^2 - 14u_2m^4 + 9m^6}{192u_2^5} \ldots \quad \text{(8.7)}
\]

where \( u_2 \) stands for the quadratic polynomial \( u_2 = a^2 \). Once the prepotential expansion has been solved up to a certain power of \( \Lambda \), it can be inserted in expression (7.11). Matching both sides is highly non-trivial since this equation involves simultaneously three different types of couplings, namely \( \tau_{f\Lambda} \) on the left hand side, and \( \tau_{ij} \) and \( \tau_{ij} \) as arguments of the Riemann theta function on the right hand side. We have checked on the computer that, indeed, this equation is satisfied order by order for \( SU(2) \) with \( N_f = 2 \) up to \( \Lambda^6 \), for \( SU(3) \) with \( N_f = 2 \) and 4 up to \( \Lambda^8 \) and \( \Lambda^4 \) respectively, and for \( SU(4) \) with \( N_f = 2 \) and 4 up to order \( \Lambda^{12} \) and \( \Lambda^8 \). We believe that this test gives a strong support to the expression (7.11).

Note in passing that, as compared to the usual ansatz for \( \mathcal{F} \), the fourth and fifth terms in (8.4) have been added for consistency of all the equations. These terms do not depend on \( a_i \) and, being linear in \( \log \Lambda \), they only contribute to the derivatives \( \frac{\partial \mathcal{F}}{\partial \log \Lambda} \), \( \frac{\partial^2 \mathcal{F}}{\partial T^0 \partial \log \Lambda} \) and \( \frac{\partial^2 \mathcal{F}}{\partial T^0 \partial \sigma} \); neither to the couplings nor to the instanton expansion. So, they correspond to a freedom of the prepotential that is fixed by the embedding into the Whitham hierarchy. A similar feature was observed before in the uses of this framework to study the strong coupling regime of \( \mathcal{N} = 2 \) pure gauge theories near the maximal singularities of the quantum moduli space [19].

### 9 Other Gauge Groups

We can extend the results of previous sections to all classical gauge groups \( SO(2r) \), \( SO(2r + 1) \) and \( Sp(2r) \) with even \( N_f \) matter hypermultiplets degenerated in pairs. For these groups, the characteristic polynomial is

\[
P(\lambda, u_{2k}) = \lambda^{2r} - \sum_{k=1}^{r} u_{2k} \lambda^{2r-2k}, \quad \text{(9.1)}
\]
and the low-energy dynamics of the corresponding $\mathcal{N} = 2$ super Yang–Mills theory is described by the hyperelliptic curves \cite{30}

\begin{align*}
  y^2 &= P^2(\lambda, u_{2k}) - 4\Lambda^{4r-2N_f}\lambda^2 \prod_{j=1}^{N_f} (\lambda^2 - m_j^2) && SO(2r) , \quad (9.2) \\
  y^2 &= P^2(\lambda, u_{2k}) - 4\Lambda^{4r-4-2N_f}\lambda^4 \prod_{j=1}^{N_f} (\lambda^2 - m_j^2) && SO(2r+1) , \quad (9.3) \\
  y^2 &= (\lambda^2 P(\lambda, u_{2k}) + A_0)^2 - 4\Lambda^{4r+4-2N_f} \prod_{j=1}^{N_f} (\lambda^2 - m_j^2) && Sp(2r) , \quad (9.4)
\end{align*}

with $A_0 = \Lambda^{2r-N_f+2} \prod_{j=1}^{N_f} m_j$. In order to treat $Sp(2r)$ on equal footing to the other gauge groups, it is convenient to restrict to the case where two hypermultiplets are massless, which we denote by $Sp(2r)''$. We can then write the hyperelliptic curve for all these cases as

\begin{align*}
  y^2 &= P^2(\lambda, u_{2k}) - 4\Lambda^{4r-2q-2N_f}\lambda^{2q} \prod_{j=1}^{N_f} (\lambda^2 - m_j^2) , \quad (9.5)
\end{align*}

where $q = 1$ for $SO(2r)$, $q = 2$ for $SO(2r+1)$ and $q = 0$ for $Sp(2r)''$ (in this last case, $N_f$ accounts for matter hypermultiplets other than the two mentioned above). These curves have genus $g = 2r - 1$, where $r$ is the rank of the gauge group. Then, if we now adjust masses to come in pairs, the curves take the form

\begin{align*}
  y^2 &= P^2(\lambda, u_{2k}) - 4\Lambda^{4r-2q-2N_f} (\lambda^q R(\lambda))^2 , \quad (9.6)
\end{align*}

where

\begin{align*}
  R(\lambda) = \prod_{j=1}^{N_f/2} (\lambda^2 - m_j^2) . \quad (9.7)
\end{align*}

Now, similarly to the $SU(N_c)$ case, we define $W \equiv P/(\lambda^q R)$ and the description of the theory is the same as before, with $N = 2r - q - N_f$. The Seiberg–Witten differential is also given by $dS_{SW} = \lambda^{2w} w$, and its variation with respect to the moduli $u_{2k}$ is

\begin{align*}
  \left. \frac{\partial dS_{SW}}{\partial u_{2k}} \right|_w = \frac{\lambda^{2r-2k-q}}{RW'} \frac{dw}{w} = \lambda^{2r-2k} \lambda \frac{d\lambda}{y} = dv^{2k} , \quad k = 1, \ldots, r . \quad (9.8)
\end{align*}

From the original space of holomorphic differentials corresponding to the hyperelliptic curve of genus $g = 2r - 1$, one really deals with a subspace of dimension $r$, which is the complex dimension of the quantum moduli space, generated by those invariant under the reflection $\lambda \rightarrow -\lambda$ \cite{30}. This symmetry, of course, also has to be taken into account in
the definition of $d\hat{\Omega}_n$. Among the meromorphic differentials (4.1), only those with odd $n$ are invariant under this reflection. That is, the differentials we have to consider are

$$d\hat{\Omega}_{2n-1} \equiv \left( W^{2n-1}_N \right) + \frac{2}{2r-q} \sum_{f=1}^{N/2} \left( W^{2n-1}_N \right)\bigg|_{\lambda=m_f} \int \frac{dw}{w} .$$

With these remarks in mind, one can proceed along the calculations of the preceding sections obtaining analogous results. Clearly, these theories have only odd Whitham times. The recovery of the Seiberg–Witten solution also goes as in the $SU(N_c)$ case. That is, one must rescale the times $\bar{T}^{2n-1} = (T^1)^{(2n-1)}T^{2n+1}$, the moduli $\bar{u}_{2k} = (T^1)^{2k} u_{2k}$, and the masses $\bar{m}_f = T^1 m_f$ so, for example, $\alpha^i$ reads

$$\alpha^i(u_{2k}, T^1, \bar{T}^3, \bar{T}^5, \ldots) = \sum_{n \geq 1} \bar{T}^{2n-1} \frac{1}{2\pi i} \oint_A \frac{W^{2n-1}_N(\bar{u}_{2k})W'(\bar{u}_{2k})}{\sqrt{W^2(\bar{u}_{2k}) - 4(T^1)^2 N}} \, d\lambda .$$

Then, $\alpha^i(u_{2k}, T^1, \bar{T}^{2n-1}>3 = 0) = T^1 a^i(u_{2k}, 1) = a^i(\bar{u}_{2k}, \Lambda = T^1)$. In summary, it is clear at this point that the same formulas (7.9)–(7.17) are obtained for the first and second derivatives of the effective prepotential, provided the appropriate value of $W$ and $N$ is considered, and changing $n, m, \ldots$ by $2n - 1, 2m - 1, \ldots$. Similar checks to the ones discussed in the previous section were carried out in these cases. Moreover, the semiclassical expansion of the prepotential up to arbitrary instantonic corrections can be recursively obtained from these equations in a remarkably simple way.

10 Concluding Remarks

In this paper we have undertaken the embedding of the Seiberg–Witten ansatz for the low-energy effective dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories with an even number of massive fundamental hypermultiplets within a Whitham hierarchy. Aside from its mathematical beauty, this formalism leads to new differential equations for the effective prepotential that can be easily applied to obtain powerful results as its whole semiclassical expansion. The expressions obtained by these means are also consistent with the duality properties of the effective couplings, including those resulting from the derivation of the prepotential with respect to Whitham times.

This work opens, or suggests, several interesting avenues for further research. The most immediate one seems to be its generalization to any number of matter hypermultiplets non-degenerated in mass. This is quite problematic within our present approach. A possible derivation of the corresponding equations for arbitrary masses is to use the $u$-plane integral.
of the topological theory \cite{13}. The second derivatives of the \prepotential with respect to higher Whitham times can be understood in that context as contact terms \cite{14}. At the same time, these contact terms can be obtained from the behavior under blowup of the twisted low-energy effective action \cite{15}. One should be able to generalize the arguments explained in Ref.\cite{18} to the case of theories with massive hypermultiplets. The first step in this direction would be to generalize the $u$-plane integral of \cite{14} in order to extract the corresponding blowup formula, and from it one could read the appropriate theta function. The possible ambiguities in this derivation can be fixed in principle by looking at the behavior at infinity, as explained in \cite{18}.

The expressions that we have provided extend in most cases the ones in \cite{12} to higher times, $T^n, n < 2N_c$. There is an obvious question about the microscopical origin of these deformations. In two dimensional topological conformal field theory they correspond to marginal deformations by gravitational descendants. It would be very interesting to have a clear understanding of the corresponding operators here.

In Ref.\cite{17}, it was shown that the Whitham times provide generalized spurionic sources for soft breaking of supersymmetry down to $\mathcal{N} = 0$. In this spirit, the additional times $T^{0,i}$ would also admit an interpretation as spurion superfields. The formulas we have obtained in the present paper are ready for use in a study of such softly broken theories along the lines of \cite{17,29}.

Another interesting possibility is the use of the new equations (7.9)-(7.17) to study the strong coupling expansion of the \prepotential near the singularities of the quantum moduli space, as it was done in Ref.\cite{19} for the case of pure $SU(N_c)$. Finally, another avenue for future research is, certainly, the connection of this formalism with the string theory and D-brane approach to supersymmetric gauge theories, where some steps has already been given in the last few years \cite{31}. We believe that all these matters deserve further study.

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A  Riemann bilinear relations.

We denote by $\tilde{\Gamma}$ the cut-Riemann surface, that is the surface with boundary obtained by removing all $A^i$- and $B_i$-cycles from $\Gamma$. Let $A^\pm_i$ and $B^\pm_j$ denote the left and right edges of the appropriate cuts,

$$\partial \tilde{\Gamma} = \sum_{j=1}^{g} (A^+_j + B^+_j - A^-_j - B^-_j) .$$  \hspace{1cm} (A.1)

Any abelian differential of the first or second kind is single-valued on $\tilde{\Gamma}$. It is sufficient to require that the integration path should not intersect any $A_i$- or $B_i$-cycles. At the boundary $\partial \tilde{\Gamma}$, the abelian integral $\Omega(P)$ satisfies

$$\Omega(P)|_{A^+_i} - \Omega(P)|_{A^-_i} = -\oint_{B_j} d\Omega ,$$

$$\Omega(P)|_{B^+_j} - \Omega(P)|_{B^-_j} = \oint_{A^i} d\Omega .$$  \hspace{1cm} (A.2)

To distinguish a single-valued branch on a third-kind differential, $d\Omega^a_{P_0}$, it is necessary to draw additional cuts $\gamma_a$ on the surface $\tilde{\Gamma}$ that run from $P_0$ to $P_a$. Let $\gamma^\pm_a$ denote both sides of the cut, then

$$\Omega(P)|_{\gamma^+_a} - \Omega(P)|_{\gamma^-_a} = 2\pi i \text{ res}_{P_0} d\Omega^a_{P_0} = -2\pi i \text{ res}_{P_0} d\Omega^a_{P_0} = 2\pi i .$$  \hspace{1cm} (A.3)

Most of the manipulations involved in the proofs of the consistency relations among derivatives of the prepotential rely heavily on the next result: let $d\Omega$ and $d\Omega'$ be two abelian differentials, then

$$\frac{1}{2\pi i} \oint_{\partial \tilde{\Gamma}} \Omega d\Omega' = \sum_{k=1}^{g} \oint_{A^k} d\Omega \oint_{B_k} d\Omega' - \oint_{B_k} d\Omega \oint_{A^k} d\Omega' .$$  \hspace{1cm} (A.4)

Notice that this is also true for $d\Omega$ and $d\Omega'$ being, just, closed differentials. Applying the residue theorem to the left hand side, we obtain various relations for the periods of abelian integrals:

(i) If $d\Omega$ and $d\Omega'$ are meromorphic of the first kind (i.e. holomorphic), then

$$\sum_{k=1}^{g} \oint_{A^k} d\Omega \oint_{B_k} d\Omega' - \oint_{B_k} d\Omega \oint_{A^k} d\Omega' = 0 .$$  \hspace{1cm} (A.5)

In particular, for a canonical basis of holomorphic differentials $d\omega_i$, $\oint_{A^i} d\omega_j = \delta^i_j$, we get that

$$\tau_{ij} = \tau_{ji} ,$$  \hspace{1cm} (A.6)

where $\tau_{ij} = \oint_{B_i} d\omega_j$ is the period matrix of $\Gamma_g$. 

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(ii) For $d\Omega = d\omega_j$ a holomorphic differential in a canonical basis, and $d\Omega' = d\Omega_n^{P_a} = (\xi^{-n-1} + O(1))d\xi$ a meromorphic differential of the second kind normalized as in (2.6) and (2.8), we find
\[
\oint_{B_j} d\Omega_n^{P_a} = \frac{1}{2\pi i n} \oint_{P_a} \xi^{-n} d\omega_j(\xi) = \frac{1}{2\pi i n} \text{res}_{P_a}(\xi^{-n} d\omega_j(\xi)). \tag{A.7}
\]

(iii) Again let $d\Omega = d\omega_j$ be holomorphic in a canonical basis, and $d\Omega_0^{P_a,P_0}$ a meromorphic differential of the third kind, then
\[
\oint_{B_j} d\Omega_0^{P_a,P_0} = \frac{1}{2\pi i} \oint_{P_a} \omega_j (\xi^{-1} + O(1))d\xi - \frac{1}{2\pi i} \oint_{P_0} \omega_j (\xi_0^{-1} + O(1))d\xi_0
\]
\[
= \frac{1}{2\pi i} \oint_{P_0} d\omega_j. \tag{A.8}
\]

(iv) If both $d\Omega = d\Omega_n^{P_a}$ and $d\Omega' = d\Omega_m^{P_b}$ are of the second or third kind ($m, n = 0, 1, 2, \ldots$), and normalized as in (2.6) and (2.8), or (2.9), then the r.h.s. of (A.4) vanishes and we obtain the symmetry relations
\[
\frac{1}{n} \text{res}_{P_a} \xi^{-n} d\Omega_m^{P_b} = \frac{1}{m} \text{res}_{P_b} \xi^{-m} d\Omega_n^{P_a}, \tag{A.9}
\]
\[
\frac{1}{n} \oint_{P_a} \xi^{-n} d\Omega_0^{P_b,P_0} = \oint_{P_0} d\Omega_n^{P_a}. \tag{A.10}
\]

B Miscellaneous proofs

Let us show here, for completeness, some of the propositions claimed in this article.

Proof of Lemma A: Taking into account (3.7) and (4.4), we obtain
\[
\left. \frac{\partial d\hat{\Omega}_n}{\partial u_k} \right|_w = \left( \partial_{u_k} [W^+]_+ + [W^+]_+ \partial_{u_k} \lambda - \partial_{u_k} \kappa_n \right) \frac{dw}{w}
\]
\[
= \sqrt{F} \left( W' \partial_{u_k} [W^+]_+ - [W^+]_+ \partial_{u_k} W - W' \partial_{u_k} \kappa_n \right) \frac{d\lambda}{y} \tag{B.1}
\]
\[
= \left( \left( P' - \frac{1}{2} \frac{F'}{F} \right) \partial_{u_k} \left( [W^+]_+ - \kappa_n \right) - [W^+]_+ \partial_{u_k} P \right) \frac{d\lambda}{y}, \tag{B.2}
\]
which exhibits poles at $m_j^\pm$ with residue (cf. Eq.(4.6))
\[
\text{res}_{m_j^\pm} \left. \frac{\partial d\hat{\Omega}_n}{\partial u_k} \right|_w = \pm \frac{\partial b^\pm_{m_j}}{\partial u_k}. \tag{B.3}
\]
To see what is the behaviour at $\infty^\pm$, we recast (B.1) in the following form:

$$\frac{\partial d\hat{\Omega}_n}{\partial u_k} \bigg|_w = \sqrt{F} \left( W' [\partial_{u_k} W^\pm]_+ - [W^\pm]' + \partial_{u_k} W - W' \partial_{u_k} \kappa_n \right) \frac{d\lambda}{y}$$

$$= \sqrt{F} \left( -W' [\partial_{u_k} W^\pm]_- + [W^\pm]' \partial_{u_k} W - W' \partial_{u_k} \kappa_n \right) \frac{d\lambda}{y}. \quad \text{(B.4)}$$

The highest order in $\lambda$ of the first two terms is $N_c - 2 = g - 1$, so they yield holomorphic differentials. Only the last one has a pole with residue

$$\text{res}_{\infty^\pm} \frac{\partial d\hat{\Omega}_n}{\partial u_k} \bigg|_w = \mp N \partial \kappa_n = \pm \sum_{f=1}^{N_f} \frac{\partial b_n^f}{\partial u_k} \quad \text{(B.5)}$$

Altogether, these results imply that we can expand in a canonical basis

$$\frac{\partial d\hat{\Omega}_n}{\partial u_k} \bigg|_w = \sum_i \left( \frac{\partial c_i}{\partial u_k} \right) d\omega_i + \sum_f \left( \frac{\partial b_f^i}{\partial u_k} \right) d\Omega_{0,f} \quad \text{(B.6)}$$

The coefficients in front of the $d\omega_i$ are fixed by first contour integrating (cf. Eq.(4.5)) and, afterwards, taking the derivative $\partial u_k$. This is the desired result and a similar analysis can be performed concerning the derivatives of this object with respect to the parameters $m_f$. Indeed, we can compute

$$\frac{\partial d\hat{\Omega}_n}{\partial m_f} \bigg|_w = \sqrt{F} \left( W' \partial_{m_f} [W^\pm]_+ - [W^\pm]' \partial_{m_f} W - W' \partial_{m_f} \kappa_n \right) \frac{d\lambda}{y} \quad \text{(B.7)}$$

$$= \left( P' - \frac{1}{2} \frac{F'}{F} P \right) \left( \partial_{m_f} [W^\pm]_+ (\lambda) - \partial_{m_f} \kappa_n \right) - \frac{[W^\pm]' (\lambda)}{(\lambda - m_f)} \frac{d\lambda}{y} \quad \text{(B.8)}$$

Again, there are just poles at $m_g^\pm$ whose residues are

$$\text{res}_{m_g^\pm} \frac{\partial d\hat{\Omega}_n}{\partial m_f} \bigg|_w = \mp \left( \partial_{m_f} [W^\pm]_+ (m_g) \mp \partial_{m_f} \kappa_n \pm [W^\pm]' (m_g) \delta_{fg} \right)$$

$$= \mp \partial_{m_f} \left( [W^\pm]_+ (\lambda = m_g) - \kappa_n \right)$$

$$= \pm \frac{\partial b_g^f}{\partial m_f} \quad \text{(B.9)}$$

At $\infty^\pm$, the same trick is in order, namely from (B.4) one easily gets

$$\frac{\partial d\hat{\Omega}_n}{\partial m_f} \bigg|_w = \sqrt{F} \left( -W' [\partial_{m_f} W^\pm]_- + [W^\pm]' \partial_{m_f} W - W' \partial_{m_f} \kappa_n \right) \frac{d\lambda}{y}, \quad \text{(B.10)}$$
and, as in (B.4), this expression is holomorphic at \( \lambda = \infty \) except for the pole in the last term which produces a residue

\[
\mathrm{res}_{\infty \pm} \frac{\partial d \hat{\Omega}_n}{\partial m_f} \bigg|_w = \pm N \partial_{m_f} \kappa_n = \mp \sum_{g=1}^{N_f} \frac{\partial b_g^m}{\partial m_f}.
\]  

(B.11)

This proves that the decomposition given in Eq. (4.9) is correct.

**Proof of Lemma B:** We want to compute the partial derivative of the generating meromorphic differential \( dS \) with respect to the flat moduli and Whitham slow times. We have

\[
\frac{\partial dS}{\partial T^n} = \frac{\partial}{\partial T^n} \sum_{m \geq 1} T^m d \hat{\Omega}_m
\]

\[
= d \hat{\Omega}_n + \sum_{m \geq 1} T^m \left( \frac{\partial u_k}{\partial T^n} \frac{\partial d \hat{\Omega}_m}{\partial u_k} + \frac{\partial m_s}{\partial T^n} \frac{\partial d \hat{\Omega}_m}{\partial m_s} \right),
\]

(B.12)

which, after (4.8)–(4.9) and (4.15)–(4.16), reads

\[
\frac{\partial dS}{\partial T^n} = d \hat{\Omega}_n + \sum_{m \geq 1} T^m \left( \frac{\partial u_k}{\partial T^n} \frac{\partial c_m^i}{\partial u_k} d \omega_i + \frac{\partial u_k}{\partial T^n} \frac{\partial b_f^m}{\partial u_k} d \Omega_{0,f} + \frac{\partial m_s}{\partial T^n} \frac{\partial c_m^i}{\partial m_s} d \omega_i + \frac{\partial m_s}{\partial T^n} \frac{\partial b_f^m}{\partial m_s} d \Omega_{0,f} \right)
\]

\[
= d \hat{\Omega}_n - c_n^i d \omega_i - b_f^i d \Omega_{0,f}
\]

\[
= d \Omega_n.
\]

(B.13)

Following similar steps, it is straightforward to prove the remaining propositions, \( \frac{\partial dS}{\partial \alpha^i} = d \omega_i \) and \( \frac{\partial dS}{\partial T^{0,f}} = d \Omega_{0,f} \).

### C Meromorphic differentials and the Szegö kernel

In this appendix we shall give the details leading to expressions (5.22) and (5.26) in the main text. As we say in Section 5 the residues appearing in the calculations of the second derivatives of the prepotential with respect to \( T^n \) and \( T^m \) and with respect to \( T^n \) and \( T^{0,f} \) involve differentials defined with respect to \( \lambda \) (not to \( w \)). This differentials can be computed using the Szegö kernel, as will be shown in the next subsections.
C.1 Second Kind Differentials

Meromorphic differentials of second kind, $\tilde{\Omega}_n(\lambda)$, are generated by some bi-differential $W(\lambda, \mu)$ upon expanding it around $\mu \to \infty_{\pm}$.

$$W(\lambda, \mu) \rightarrow_{\mu \to \infty_{\pm}} \sum_{p \geq 1} d\tilde{\Omega}_p^\pm(\lambda) \frac{d\mu}{\mu^{p+1}}. \quad (C.1)$$

The key ingredient is the so-called Szegö kernel $\Psi_e(\lambda, \mu) = \Theta_e(\vec{\lambda} - \vec{\mu}) \Theta_e(\vec{0}) E(\lambda, \mu) \quad (C.2)$
where $E(\lambda, \mu)$ is the Prime form. In terms of the Szegö kernel,

$$W(\lambda, \mu) = \Psi_e(\lambda, \mu) \Psi_{-e}(\lambda, \mu) - d\omega_i(\lambda) d\omega_j(\mu) \left( \frac{1}{i\pi} \frac{\partial}{\partial r_{ij}} \log \Theta_E(\vec{0}) \right). \quad (C.3)$$

$\Psi_e(\lambda, \mu)$ is a 1/2-bidifferential that has a simple hyperelliptic representation whenever $e$ denotes an even non-singular characteristic $e = -e = E$. Such characteristics are in one-to-one correspondence with the partitions of the set of $2g + 2$ ramification points into two equal subsets, $\{ r_\alpha^\pm, \alpha = 1, 2, ..., g + 1 \}$, such that $y^2(\lambda) = \prod_{\alpha=1}^{g+1} (\lambda - r_\alpha^+) (\lambda - r_\alpha^-) = Q_+(\lambda) Q_-(-\lambda)$. In this particular case $\Psi_{E}(\lambda, \mu) = U_{E}(\lambda) + U_{E}(\mu) \sqrt{d\lambda d\mu} \quad (C.4)$

where

$$U_{E}(\lambda) = \sqrt{\prod_{\alpha=1}^{g+1} \frac{\lambda - r_\alpha^+}{\lambda - r_\alpha^-}} = \frac{1}{y(\lambda)} \prod_{\alpha=1}^{g+1} (\lambda - r_\alpha^+) = \frac{Q_+(\lambda)}{y(\lambda)}. \quad (C.5)$$

An explicit calculation yields

$$\Psi^2_{E}(\lambda, \mu) = \frac{Q_+(\lambda)Q_-(\mu) + Q_+(\mu)Q_-(\lambda) + 2y(\lambda) y(\mu)}{4y(\lambda) y(\mu)} \frac{d\lambda d\mu}{(\lambda - \mu)^2}. \quad (C.6)$$

If we now assume the symmetric scenario, i.e. values of $m_f$ come in pairs, then there is as privilegiate choice for $Q_{\pm} = P \pm 2\sqrt{F}$. The characteristic $E$ appearing in $\Psi_{E}$ is associated with the splitting of the roots of the discriminant $[27]$ $[28][12]$ and for this particular case we have that $E = \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \quad \rightarrow \quad \vec{\alpha} = (0, \ldots, 0) \quad \text{and} \quad \vec{\beta} = (1/2, \ldots, 1/2), \quad (C.7)$
and the theta function involved in the above equations is then
\[
\Theta_{E=[\vec{a}, \vec{b}]}(\vec{z}|\tau) = \sum_{\vec{n} \in \mathbb{Z}} e^{i\pi r_{ij} n_i n_j + 2\pi i \sum_k (z_k + \frac{1}{2}) n_k},
\]
(C.8)
so we have in (C.3) that \( \frac{1}{(2\pi i)^2} \partial_{ij}^2 \log \Theta_E = \frac{1}{\pi i} \partial_{r_{ij}} \log \Theta_E \).

For this particular choice of \( Q_\pm \)
\[
\Psi_E^2(\lambda, \mu) = \frac{P(\lambda)P(\mu) - 4\sqrt{F(\lambda)F(\mu)} + y(\lambda)y(\mu)}{2y(\lambda)y(\mu)} \frac{d\lambda d\mu}{(\lambda - \mu)^2}
\]
\[
\rightarrow_{\mu \rightarrow \infty} \sum_{p=1}^{2N} \left( \frac{\pm P(\lambda) + y(\lambda)}{2y(\lambda)} \right) \lambda^{p-1} d\lambda
\]
\[
\pm \theta(p - N - 1) \sum_{k=0}^{p-N-1} c_k \frac{2(N + k - p)}{p} \sqrt{\frac{F(\lambda)}{y(\lambda)}} \lambda^{p-k-N-1} d\lambda \frac{pd\mu}{\mu^{p+1}}
\]
\[
+ {\mathcal{O}}(\mu^{-2N-2}) \quad (C.9)
\]
where we have expanded \( W^{-1}(\mu) = \sum_{k=0}^{\infty} c_k / \mu^{k+N} \). Next, expand \( d\omega_j(\mu) \) around \( \infty^\pm \)
\[
d\omega_j(\mu) \rightarrow_{\mu \rightarrow \infty} \pm \sum_{p=1}^{2N-1} \left( \frac{1}{p} \frac{\partial h_{p+1}}{\partial a_j} \right) \frac{pd\mu}{\mu^{p+1}} + {\mathcal{O}}(\mu^{-2N-1}) \quad (C.10)
\]
With this setup in mind, comparing with (C.9) and (C.10) with (C.1) we obtain for \( p < 2N \)
\[
d\tilde{\Omega}_p(\lambda) = d\tilde{\Omega}_p^+(\lambda) - d\tilde{\Omega}_p^-(\lambda)
\]
\[
= -\lambda^{p-1} \frac{P(\lambda)}{y(\lambda)} d\lambda + d\omega_i(\lambda) \frac{2}{p} \frac{\partial h_{p+1}}{\partial a_j} \frac{1}{\pi i} \partial_{r_{ij}} \log \Theta_E
\]
\[
- \theta(p - N - 1) \sum_{k=0}^{p-N-1} c_k \frac{4(N + k - p)}{\lambda} \sqrt{\frac{F(\lambda)}{y(\lambda)}} \lambda^{p-k-N-1} d\lambda . \quad (C.11)
\]

C.2 Third kind differential

The third kind meromorphic differential \( d\Omega_0^{P,Q}(\lambda) \) with vanishing \( A^i \)-cycles can be written in terms of the Prime form \([27, 28]\) as follows
\[
d\Omega_0^{P,Q}(\lambda) = d\log \frac{E(\lambda, P)}{E(\lambda, Q)} . \quad (C.12)
\]
Also, an explicit representation in terms of the Szegö kernel \([24]\) can be found (see Proposition 2.10 in Ref.\([27]\)),
\[
d\Omega_0^{P,Q}(\lambda) = \frac{\Psi_e(\lambda, P)\Psi_{-e}(\lambda, Q)}{\Psi_e(P, Q)} - d\omega_i(\lambda) \frac{1}{2\pi i} \frac{\partial}{\partial z^i} \left[ \log \Theta_e(\vec{z}_{P,Q}|\tau) - \log \Theta_e(0|\tau) \right] , \quad (C.13)
\]
where
\[ \tilde{z}_{P,Q} = \frac{1}{2\pi i} \int_{P}^{Q} d\tilde{z}, \]
is the image of the divisor \( Q - P \) under the Abel map. When \( e = -e = E \) is an even half-integer characteristic, \( \partial_i \log \Theta(\tilde{z} \mid \tau) \) is odd under \( \tilde{z} \to -\tilde{z} \) so \( \partial_i \log \Theta_E(0 \mid \tau) = 0. \) Letting \( P = m_f^\pm \) and \( Q = \infty^\pm, \) and making use of (C.4) and (C.5) as well as \( U(\infty^\pm) = U(m_f^\pm) = \pm 1 \) the third kind differential reads
\begin{equation}
\frac{d\Omega_{m_f^\pm,\infty^\pm}}{d\lambda} = \pm \frac{(U_E(\lambda) \pm 1)^2}{4U_E(\lambda)} \frac{d\lambda}{\lambda - m_f} \pm d\omega_i(\lambda) \frac{1}{2\pi i} \frac{\partial}{\partial z^i} \log \Theta_E(\tilde{z}_f \mid \tau),
\end{equation}
where \( \tilde{z}_f \) is given in Eq.(5.23). This third kind differential is easily seen to have simple poles at \( m_f^\pm \) with residue +1 and \( \infty^\pm \) with residue −1, while being regular everywhere else. Hence, we find that \( d\Omega_{0,f} = d\Omega_{0,m_f^\pm,\infty^\pm} - d\Omega_{0,\infty^\pm,m_f^\pm} \) can be written as
\begin{equation}
d\Omega_{0,f}(\lambda) = \frac{P}{y} \frac{d\lambda}{\lambda - m_f} + d\omega_j(\lambda) \frac{1}{2\pi i} \frac{\partial}{\partial z^j} \log \Theta_E(\tilde{z}_f \mid \tau),
\end{equation}
which is the result we need to evaluate the second derivative of the prepotential \( F \) with respect to \( T^n \) and \( T^{0,f}. \)

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