Hidden Messenger Revealed in Hawking Radiation: A Resolution to the Paradox of Black Hole Information Loss

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Using standard statistical method, we discover the existence of correlations among Hawking radiations (of tunnelled particles) from a black hole. The information carried by such correlations is quantified by mutual information between sequential emissions. Through a careful counting of the entropy taken out by the emitted particles, we show that the black hole radiation as tunneling is an entropy conservation process. While information is leaked out through the radiation, the total entropy is conserved. Thus, we conclude the black hole evaporation process is unitary.

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Since Hawking radiation was first discovered [1,2], its inconsistency with quantum theory has been widely noticed. Irrespective of what initial state a black hole starts with before collapsing, it evolves eventually into a thermal state after being completely exhausted into emitted radiations. Such a scenario violates the principle of unitarity as required for quantum mechanics and brings a serious challenge to the foundation of modern physics. Many groups [3-11] have attempted addressing this puzzle of the so-called paradox of black hole information loss. None has been successful. Most discussions treat Hawking radiation as thermally distributed without considering energy conservation or self-gravitation effect. Recently, Parikh and Wilczek point out that Hawking radiation is completely non-thermal when energy conservation is enforced [12]. Making use of their result, we discover the existence of non-trivial correlations amongst Hawking radiations. A queue of correlated radiation can transmit encoded information. By carefully counting the entropy embedded in the sequentially emitted (tunnelled out) radiations/particles, we show that the process of Hawking radiation is entropy conserved, contrary to entropy growth by the thermal spectrum [13]. While information is carried away by Hawking radiation, the total entropy of the black hole and the radiation is conserved. Our work thus implies that the black hole evaporation process, whereby Hawking radiation is emitted, is unitary.

In the past few decades, several approaches have been suggested for resolving the paradox of black hole information loss. Hawking initially proposed to accept information loss when quantum theory is unified with gravity [3]. He has since renounced this proposal and admits “elementary quantum gravity interactions do not lose information or quantum coherence” [4]. A second approach focuses on the black hole remnant [3], stemmed from the idea of correlation or entanglement of the radiation and the black hole. It also fails because of the infinite degeneracy, which is hard to reconcile with causality and unitarity [5]. A third idea is related to “quantum hair” on a black hole [6] that is found to be capable of store more information than one expects. To resolve the paradox with this approach, a projection onto local quantum fields of low energies is required, and no one knows how this can be done. A fourth approach is from Bekenstein [8] who suggests that if the radiation spectrum is analyzed in detail, enough non-thermal features might exist to encode all lost information. Recently a new approach is brought forward along the lines of quantum teleportation and the so-called final state projection [9]. The quantum information is estimated to be capable of escaping with a fidelity of about 85% on average [10], although whether the final state projection exists or not and, how it can be justified, remains a mystery. Finally, another recent work attracted serious attention after it ruled out the possibility that information about the infallen matter could hide in the correlations between the Hawking radiation and the internal states of a black hole [11]. The current state of affairs is a direct confrontation: either unitarity or Hawking radiation being thermal must break down.

In the original treatment, Hawking considered a fixed background geometry without enforcing energy conservation [1,2]. In contrast, energy conservation is crucial in an improved treatment by Parikh and Wilczek that considers s-wave outgoing particles, or the Hawking radiation, as due to quantum tunneling, and obtains a non-thermal spectrum for the Schwarzschild black hole [12]. The non-thermal probability distribution is related directly to the change of entropy in a black hole [12]. In this Letter, we show that the non-thermal distribution implies information can be coded into the correlations of sequential emissions. We find that entropy remains conserved in the radiation process; which leads naturally to the conclusion that the process of Hawking radiation is unitary, and no information loss occurs. This implies that even in a semiclassical treatment of the Hawking radiation process, unitarity is not violated. The so-called
black hole information loss paradox arises from the neglect of energy conservation or self-gravitational effect.

We start with a brief review of Hawking radiation as due to quantum tunneling [12]. Unlike the Schwarzschild coordinate, the derivation that makes use of the Painlevé coordinate system is regular at the horizon and thus is particularly convenient for tunneling calculation. Particles are supplied by considering the geometrical limit because of the infinite blueshift of the outgoing wave-packets near the horizon. The barrier is created by the outgoing particle itself, which is ensured by energy conservation. The radial null geodesic motion is considered, and the WKB approximation is adopted to arrive at the tunneling probability

\[
\Gamma \sim \exp \left[ -2 \text{Im}(I) \right] = \exp \left[ -8\pi E \left( M - \frac{E}{2} \right) \right] = \exp (\Delta S), \quad (1)
\]

as the imaginary part of the action, and is related to the change of the black hole’s Bekenstein-Hawking entropy on the second line, as was shown in Ref. [12]. This result Eq. (1) is clearly distinct from the thermal distribution: \( \Gamma(E) = \exp(-8\pi EM) \), thus subsequent Hawking radiation emissions must be correlated and capable of carrying away information encoded within. Further insight can be gained if we compare with the general form of a quantum transition probability [14], expressed as

\[
\Gamma \sim \frac{e^{S_{\text{final}}}}{e^{S_{\text{initial}}}} = \exp (\Delta S),
\]

in terms of the entropy change \( \Delta S = S_f - S_i \) between the final and initial entropies \( S_f \) and \( S_i \). This is in agreement with the tunneling probability, up to a factor containing the square of the amplitude of the process. In other words, the non-thermal Hawking radiation Eq. (1) reveals the possibility of unitarity and no information loss.

We will find out whether or not there exist statistical correlations between quanta of Hawking radiation. This was first discussed in Refs. [14, 15] by considering two emissions with energies \( E_1 \) and \( E_2 \), or one emission with energy \( E_1 + E_2 \). The function

\[
C(E_1 + E_2, E_1, E_2) = \ln \Gamma(E_1 + E_2) - \ln [\Gamma(E_1) \Gamma(E_2)]
\]

was used to measure statistical correlation between the two emissions. With the following function forms

\[
\Gamma(E_1) = \exp \left[ -8\pi E_1 \left( M - \frac{E_1}{2} \right) \right],
\]

\[
\Gamma(E_2) = \exp \left[ -8\pi E_2 \left( M - E_1 - \frac{E_2}{2} \right) \right],
\]

\[
\Gamma(E_1 + E_2) = \exp \left[ -8\pi (E_1 + E_2) \left( M - \frac{E_1 + E_2}{2} \right) \right],
\]

\[
\Gamma(E_1 + E_2, E_1, E_2) = 0 \text{ is found, and Refs. [14, 15] wrongly conclude that no correlation exists, including the case of tunneling through a quantum horizon [15].}
\]

This makes no sense. The notations used in the above (adopted from Refs. [14, 15]) for \( \Gamma(E_1), \Gamma(E_2) \), and \( \Gamma(E_1 + E_2) \) are incorrect. In particular, the form of the function \( \Gamma(E_2) \) Eq. (2) is misleading because it is different from Eq. (1). To properly evaluate statistical correlation [16], it is important to distinguish between statistical dependence or independence. If the probability of two events arising simultaneously is identically the same as the product probabilities of each event occurring independently, these two events are independent or non-correlated. Otherwise, they are dependent or correlated. Because of the non-thermal nature, the probability \( \Gamma(E_2) \) used in Eq. (2) is not independent; instead, it is conditioned on the emission with energy \( E_1 \).

The proper forms for the probabilities \( \Gamma(E_1) \) and \( \Gamma(E_2) \) are derived in the appendix using the standard approach: \( \Gamma(E_1) = \int \Gamma(E_1, E_2) dE_2 \) and \( \Gamma(E_2) = \int \Gamma(E_1, E_2) dE_1 \), where the probability for simultaneously two emissions with energies \( E_1 \) and \( E_2 \) is \( \Gamma(E_1, E_2) = \Gamma(E_1 + E_2) \). We find both independent probabilities take the expected functional form of Eq. (1).

\[
\Gamma(E_2) = \exp \left[ -8\pi E_2 \left( M - \frac{E_2}{2} \right) \right],
\]

which then gives

\[
\ln \Gamma(E_1 + E_2) - \ln [\Gamma(E_1) \Gamma(E_2)] = 8\pi E_1 E_2 \neq 0, \quad (4)
\]

unlike what was concluded previously [14, 15].

Equation (4) is the central result of this work. To better understand its implications we can make connection to a closely related topic in quantum information. Our result Eq. (4) shows that subsequent emissions are statistically dependent, and correlations must exist between them. For sequential emissions of energies \( E_1 \) and \( E_2 \), the tunneling probability for the second emission with energy \( E_2 \) should be understood as conditional probability given the occurrence of tunneling of the particle with energy \( E_1 \). Thus, instead of the misleading Eq. (2), a proper notation is

\[
\Gamma(E_2|E_1) = \exp \left[ -8\pi E_2 \left( M - E_1 - \frac{E_2}{2} \right) \right],
\]

defined according to \( \Gamma(E_1, E_2) = \Gamma(E_1) \cdot \Gamma(E_2|E_1) \). The Bayesian law \( \Gamma(E_2|E_1) = \Gamma(E_1|E_2)\Gamma(E_2) \) then self-consistently connects between different probabilities.

Analogously, the conditional probability \( \Gamma(E_1|E_f) = \exp \left[ -8\pi E_1 \left( M - E_f - \frac{E_1}{2} \right) \right] \) corresponds to the tunneling probability of a particle with energy \( E_f \) conditional on radiations with a total energy of \( E_f \). The entropy taken away by the tunneling particle with energy \( E_i \) after the black hole has emitted particles with a total energy \( E_f \).
entropy of three emissions at energies
is then given by
\[ S(E_i|E_f) = -\ln \Gamma(E_i|E_f). \] (6)

In quantum information theory \([17]\), \( S(E_i|E_f) \) denotes conditional entropy, and it measures the entropy of \( E_i \) given that the values of all the emitted particles with a total energy \( E_f \) are known. Quantitatively, it is equal to the decrease of the entropy of a black hole with mass \( M - E_f \) upon the emission of a particle with energy \( E_i \). Such a result is consistent with the thermodynamic second law of a black hole \([18]\): the emitted particles must carry entropies in order to balance the total entropy of the black hole and the radiation. In what follows we show that the amount of correlation Eq. (4) hidden inside Hawking radiation is precisely equal to mutual information.

The mutual information \([17]\) in a composite quantum system composed of sub-systems \( A \) and \( B \) is defined as
\[ S(A : B) = S(A) + S(B) - S(A, B) = S(A) - S(A|B), \]
where \( S(A|B) \) is the conditional entropy. It is a legitimate measure for the total amount of correlations between any bi-partite system. For sequential emission of two particles with energies \( E_1 \) and \( E_2 \), we find
\[ S(E_2 : E_1) = S(E_2) - S(E_2|E_1) = -\ln \Gamma(E_2) + \ln \Gamma(E_2|E_1). \] (7)

Using Eqs. (3) and (5), we obtain \( S(E_2 : E_1) = 8\pi E_1 E_2 \), \( i.e., \) the correlation of Eq. (4) is exactly equal to the mutual information between the two sequential emissions.

We now count the entropy carried away by Hawking radiations. The entropy of the first emission with an energy \( E_1 \) from a black hole of mass \( M \) is
\[ S(E_1) = -\ln \Gamma(E_1) = 8\pi E_1 \left( M - \frac{E_1}{2} \right). \] (8)

The conditional entropy of a second emission with an energy \( E_2 \) after the \( E_1 \) emission is
\[ S(E_2|E_1) = -\ln \Gamma(E_2|E_1) = 8\pi E_2 \left( M - E_1 - \frac{E_2}{2} \right). \] (9)

The total entropy for the two emissions \( E_1 \) and \( E_2 \) then becomes
\[ S(E_1, E_2) = S(E_1) + S(E_2|E_1), \]
and the mass of the black hole reduces to \( M - E_1 - E_2 \) while it proceeds with the emission of energy \( E_3 \) with an entropy \( S(E_3|E_1, E_2) = -\ln \Gamma(E_3|E_1, E_2) \). The total entropy of three emissions at energies \( E_1 \), \( E_2 \), and \( E_3 \) is
\[ S(E_1, E_2, E_3) = S(E_1) + S(E_2|E_1) + S(E_3|E_1, E_2). \]

Repeating the process until the black hole is completely exhausted, we find
\[ S(E_1, E_2, \ldots, E_n) = \sum_{i=1}^{n} S(E_i|E_1, E_2, \ldots, E_{i-1}), \] (10)
where \( M = \sum_{i=1}^{n} E_i \) equals to the initial black hole mass due to energy conservation and \( S(E_1, E_2, \ldots, E_n) \) denotes the joint entropy of all emissions while \( S(E_i|E_1, E_2, \ldots, E_{i-1}) \) is the conditional entropy. Equation (10) then corresponds to nothing but the chain rule of conditional entropies in quantum information theory \([17]\). In the appendix, we find the total entropy \( S(E_1, E_2, \ldots, E_n) = 4\pi M^2 \) exactly equals the black hole’s Bekenstein-Hawking entropy. This result is independently verified by counting of microstates of Hawking radiations as shown in the appendix.

The reason information can be carried away by black hole radiation is the probabilistic nature of the emission itself. Given the emission rate \( \Gamma(E) \sim \exp[-\pi E(M - \frac{E}{2})] \), one knows definitively that a radiation of energy \( E \) may occur with a probability \( \Gamma(E) \). In other words, the uncertainty of the event (an emission with an energy \( E \)) or the information we can gain, on average, from the event is \( S(E) = -\ln \Gamma(E) \). When an emission with an energy \( E_1 \) is received, the potential gain in information is \( S(E_1) = -\ln \Gamma(E_1) \). When the next emission with an energy \( E_2 \) is received, an additional information \( S(E_2|E_1) = -\ln \Gamma(E_2|E_1) \) can be gained, which is conditional on already receiving the emission of an energy \( E_1 \). Continuing on, we compute the information gained from all emissions until the black hole is exhausted. The total entropy carried out by radiations is then found to be \( S(E_1, E_2, \ldots, E_n) = 4\pi M^2 \), which means all the entropy of the black hole is taken out by its Hawking radiations. Putting together our earlier result that the entropy carried away by an emission is the same as the entropy reduction of the accompanying black hole during each emission, we conclusively show that entropies of Hawking radiations and their accompanying black holes are conserved during black hole radiation. According to quantum mechanics, a unitary process does not change the entropy of a closed system. This implies that the process of Hawking radiation is unitary in principle, and no information loss is expected.

In conclusion, through a careful reexamination of Hawking radiation, we discover and quantify correlations amongst radiated particles in terms of Eq. (4). Our result for the first time provides a clear picture of how and how much information can be carried away by Hawking radiation from a black hole. Although the prospect for information hidden inside Hawking radiation has been discussed time and again, earlier works do not enforce energy conservation strictly and assumed a thermal distribution for the radiated particles (please see \([6]\) and references therein). In contrast, our study is built on
the principle of energy conservation, where the effect of self-gravitation plays a crucial role, and the spectrum of radiated particles is non-thermal. Making connection with information theory, we find that entropy is strictly conserved during Hawking radiation, i.e., the entropy of a black hole is the same as the entropy of all emitted radiations upon its exhaustion.

Our conclusions show the information is not lost, and unitarity is held in the process of Hawking radiation although based on results within a semiclassical treatment for s-wave emissions where energy conservation is enforced \[12\]. For more elaborate treatments, e.g., those involving coding information in the correlations, a complete quantum gravity theory may still be needed. However, our analysis confirms that the energy conservation or self-gravitational effect remains crucial for approaches based on self-consistent quantum gravity theories.

Finally, we hope to point out that our analysis can be extended to charged black holes, Kerr black holes, and Kerr-Neumann black holes. Even for the situations involving quantum gravity effects or the noncommutative black holes, our method remains effective in providing consistent resolutions \[19\]. We show that due to self-gravitational effect, information can come out in the form of correlated emissions from a black hole, and our work thus resolves the black hole information loss paradox.

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APPENDIX

This appendix contains some details for a few key steps supporting our results as given in the main text.

The joint probability distribution of two simultaneous emissions of energies \(E_1\) and \(E_2\) is

\[
\Gamma(E_1, E_2) = \Gamma(E_1 + E_2) = \exp\left[-8\pi(E_1 + E_2)\left(M - \frac{E_1 + E_2}{2}\right)\right],
\]

subjected to a normalization factor \(\Lambda\), determined by \(\Lambda \int_0^M \exp\left[-8\pi e(M - \frac{E}{2})\right]dE = 1\).

The independent probability distributions for a single emission \(\Gamma(E_1)\) or \(\Gamma(E_2)\) are \(\Gamma(E_1) = \Lambda \int_0^{M-E_1} \Gamma(E_1, E_2)dE_2 = \exp\left[-8\pi E_1(M - \frac{E_1}{2})\right]\) and \(\Gamma(E_2) = \Lambda \int_0^{M-E_2} \Gamma(E_1, E_2)dE_1 = \exp\left[-8\pi E_2(M - \frac{E_2}{2})\right]\) and are identical in their function forms. In the main text, our result Eq. \(\textbf{(6)}\) reveals that Hawking radiations are correlated and carry away that much entropy from the black hole. We now show that the initial entropy of a black hole is the same as the entropy of all emitted radiations upon its exhaustion.

Assuming the tunneling/emission probability is given by Eq. \(\textbf{(1)}\), when the black hole is exhausted due to emissions, we can find the entropy of our system by counting the number of its microstates. For example, one of the microstates is \((E_1, E_2, \cdot \cdot \cdot, E_n)\) and \(\sum E_i = M\). Within such a description, the order of \(E_i\) cannot be changed, the distribution of each \(E_i\) is consistent with the discussion in the main text. The probability for the specific microstate \((E_1, E_2, \cdot \cdot \cdot, E_n)\) to occur is given by

\[
P = \Gamma(M; E_1) \times \Gamma(M - E_1; E_2) \times \cdots \times \Gamma(M - \sum_{j=1}^{n-1} E_j; E_n),
\]

with

\[
\Gamma(M; E_1) = \exp\left[-8\pi E_1(M - E_1/2)\right],
\]

\[
\Gamma(M - E_1; E_2) = \exp\left[-8\pi E_2(M - E_1 - E_2/2)\right],
\]

\[
\cdots,
\]

\[
\Gamma(M - \sum_{j=1}^{n-1} E_j; E_n) = \exp\left[-8\pi E_n(M - \sum_{j=1}^{n-1} E_j - E_n/2)\right] = \exp(-4\pi E_n^2),
\]

where \(\Gamma(M; E_i)\) denotes the probability Eq. \(\textbf{(1)}\) for a emission with energy \(E_1\) by a black hole with mass \(M\). Proceeding with a detailed calculation, we find that \(P = \exp(-4\pi M^2) = \exp(-S_{BH})\), where \(S_{BH}\) is the entropy of the black hole. According to the fundamental postulate of statistical mechanics that all microstates of an isolated system are equally likely, we find the number of microstates \(\Omega = \frac{1}{P} = \exp(S_{BH})\). On the other hand, according to the Boltzmann’s definition, the entropy of a system is given by \(S = \ln \Omega = S_{BH}\) (where the Boltzmann constant \(k = 1\) is taken.) Thus we prove that after a black hole is exhausted due to Hawking radiation, the entropy carried away by all emissions is precisely equal to the entropy in the original black hole.

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