Geometric barriers for the existence of hypersurfaces with prescribed curvatures in \(M^n \times \mathbb{R}\).

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Abstract

We show the existence of a deformation process of hypersurfaces from a product space \(M_1 \times \mathbb{R}\) into another product space \(M_2 \times \mathbb{R}\) such that the relation of the principal curvatures of the deformed hypersurfaces can be controlled in terms of the sectional curvatures or Ricci curvatures of \(M_1\) and \(M_2\). In this way, we obtain barriers which are used for proving existence or non existence of hypersurfaces with prescribed curvatures in a general product space \(M \times \mathbb{R}\).

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1 Introduction.

Our main objective is to describe a simple method for obtaining barriers in a product space \(M^n \times \mathbb{R}\). To that end, we consider a hypersurface \(S\) in a product space \(M_1 \times \mathbb{R}\) and obtain a new hypersurface \(S^*\) in a different product space \(M_2 \times \mathbb{R}\) such that the principal curvatures of \(S\) and \(S^*\) can be related in terms of the sectional curvatures or Ricci curvatures of \(M_1\) and \(M_2\).

The previous method has a special interest when \(S\) is a hypersurface with constant mean curvature \(H\), or in general with constant \(r\)-mean curvature \(H_r\), and \(M_1\) has constant sectional

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curvature $c$. In such a case, we obtain barriers for the existence of hypersurfaces with constant r-mean curvature $H_r$ in a general $M_2 \times \mathbb{R}$ if the sectional (or Ricci) curvature of $M_2$ is bounded from above or below by $c$.

Thus, we will generalize different known results in the homogeneous spaces $\mathbb{H}^n \times \mathbb{R}, \mathbb{R}^{n+1}$ or $\mathbb{S}^n \times \mathbb{R}$ to general product spaces $M^n \times \mathbb{R}$. For the case $n = 2$ the analogous method was described in [GL].

The paper is organized as follows. Section 2 is devoted to the calculus of the principal curvatures of certain graphs in a general product $M^n \times \mathbb{R}$ in terms of Jacobi fields on $M^n$ (Lemma 1). In Section 3 we describe in detail our method for obtaining barriers in a product space $M^n \times \mathbb{R}$, and obtain two comparison results. The first one relates the principal curvatures of two graphs $S$ in $M_1 \times \mathbb{R}$ and $S^*$ in $M_2 \times \mathbb{R}$ with the sectional curvatures of $M_1$ and $M_2$ (Theorem 1). The second comparison result relates the mean curvatures of $S$ and $S^*$ with the Ricci curvatures of $M_1$ and $M_2$ (Theorem 2).

In Section 4 we show different examples of how these barriers can be used. Thus, in Theorem 3 we prove that given a closed geodesic ball $B_r \subseteq M^n$ of radius $r$ then there is an explicit constant $H_0$, which only depends on the radius $r$ and the minimum of its Ricci curvature, such that there exists no vertical graph over $B_r$ in $M^n \times \mathbb{R}$ with minimum of its mean curvature greater than or equal to $H_0$. This generalizes a previous result by Espinar and Rosenberg in [ER] for $n = 2$. In Theorem 4 we obtain an analogous result for Gauss-Kronecker curvature, or in general for $r$-mean curvature, depending on the sectional curvatures of the closed geodesic ball.

Moreover, in Theorem 5, we prove that, under certain restrictions on the ambient space $M^n \times \mathbb{R}$, for every properly embedded hypersurface $\Sigma \subset M^n \times \mathbb{R}$ with mean curvature $H \geq H_0 > 0$, its mean convex component cannot contain a certain geodesic ball of radius $r$, where $r$ only depends on $H_0$ and the infimum of the Ricci curvature of $M$. In particular, this shows the non existence of entire horizontal graphs over a Hadamard manifold for certain values of the mean curvature (Corollary 1).

In Theorem 6 we also prove the existence of vertical graphs in $M^n \times \mathbb{R}$ with boundary on a horizontal slice and constant mean curvature $H_0$ for any $H_0 \in [0, (n - 1)/n]$, when $M^n$ is a Hadamard manifold with sectional curvature pinched between $-c^2$ and $-1$. In fact, we show that any compact hypersurface with constant mean curvature $H_0$ and the same boundary must be the previous graph or its reflection with respect to the slice. This generalizes previous results in $\mathbb{H}^n \times \mathbb{R}$ (see [NSST] and [BE]).

Finally, in Theorem 7 we give a result of existence for vertical graphs with positive constant Gauss-Kronecker curvature in $M^n \times \mathbb{R}$, which solves the Dirichlet problem for the associated Monge-Ampère equation with zero boundary values.
2 The principal curvatures of the graph.

Let $\mathcal{H}$ be an $(n-1)$-dimensional manifold and $(\mathbb{M}, g)$ be an $n$-dimensional Riemannian manifold. Consider two smooth maps $i : \mathcal{H} \rightarrow \mathbb{M}$ (non necessarily an immersion) and $n : \mathcal{H} \rightarrow T\mathbb{M}$ such that $n(x) \in T_{i(x)}\mathbb{M}$ is a unit vector with $g(di_{x}(v), n(x)) = 0$ for all $v \in T_{x}\mathcal{H}$. Here, for instance, $T_{p}\mathbb{M}$ denotes the tangent space to $\mathbb{M}$ at the point $p \in \mathbb{M}$.

Let $I$ be an open real interval such that 0 is in its closure $\bar{I}$, and assume that the map

$$\varphi(x, t) = \exp_{i(x)}(t \, n(x)), \quad (x, t) \in \mathcal{H} \times \bar{I}, \quad (1)$$

is smooth and $\varphi|_{\mathcal{H} \times I}$ a global diffeomorphism onto its image; where $\exp$ denotes the exponential map in $\mathbb{M}$.

Observe that $\varphi|_{\mathcal{H} \times I}$ can be seen as a certain parametrization of an open set of $\mathbb{M}$, and the parameter $t$ can be considered as a distance function to $i(x)$.

The polar geodesic parameters at a point $p \in \mathbb{M}$ are examples of the previous situation. For that, one can consider $\mathcal{H}$ as the unit sphere of $T_{p}\mathbb{M}$, $i$ as the constant map $i(x) = p$ and $n(x) = x$.

Now, let us consider the product space $\mathbb{M} \times \mathbb{R}$ with the standard product metric and let us call $h$ to the parameter in $\mathbb{R}$. Let $\psi(x, t)$ be the graph given by the height function $f(t)$ which only depends on the distance function, that is, the graph in $\mathbb{M} \times \mathbb{R}$ parameterized as

$$\psi(x, t) = (\exp_{i(x)}(t \, n(x)), f(t)) = (\varphi(x, t), f(t)). \quad (2)$$

Then, one has

$$\overline{\partial}_{x_{i}} = \partial_{x_{i}}, \quad \overline{\partial}_{t} = \partial_{t} + f'(t)\partial_{h},$$

where $x = (x_{1}, \ldots, x_{n-1})$ are local coordinates in $\mathcal{H}$. Here, for instance, $\partial_{x_{i}}$ denotes the vector field $\frac{\partial}{\partial x_{i}}$ in $\mathbb{M} \times \mathbb{R}$ and $\overline{\partial}_{x_{i}}$ the corresponding vector field in the graph.

If $\langle \cdot, \cdot \rangle = g + dh^{2}$ stands for the product metric in $\mathbb{M} \times \mathbb{R}$, then from the Gauss lemma we obtain

$$\langle \overline{\partial}_{x_{i}}, \overline{\partial}_{t} \rangle = g(\partial_{x_{i}}, \partial_{t}) = 0.$$

Hence, the pointing upwards unit normal of the graph is

$$N = \frac{1}{\sqrt{1 + f'(t)^{2}}(-f'(t)\partial_{t} + \partial_{h})}.$$

So, if we denote by $\nabla$ the Levi-Civita connection in $\mathbb{M} \times \mathbb{R}$, it is easy to see that

$$-\nabla_{\overline{\partial}_{t}}N = \frac{f''(t)}{(1 + f'(t)^{2})^{3/2}} \overline{\partial}_{t}.$$
In particular, $\partial_t$ is a principal direction with associated principal curvature

$$k_n = \frac{f''(t)}{(1 + f'(t)^2)^{3/2}}.$$  

Observe that this principal curvature does not depend on either $\mathcal{H}$, or $\mathbb{M}$, or its metric.

In order to compute the rest of principal curvatures of the graph we will focus on the directions which are orthogonal to $\partial_t$, that is, the ones generated by $\partial_x$.

Let $\gamma(t) = \varphi(x_0, t)$ be a geodesic in $\mathbb{M}$ and $J(t)$ a Jacobi field along $\gamma(t)$ with $g(J(t), \gamma'(t)) = 0$. If we denote by $\nabla$ the Levi-Civita connection in $\mathbb{M}$ then the second fundamental form of the graph satisfies

$$II(J, J) = \langle -\nabla_J N, J \rangle = g(-\nabla_J \left( \frac{-f'(t)}{\sqrt{1 + f'(t)^2}} \partial_t \right), J) = \frac{f'(t)}{\sqrt{1 + f'(t)^2}} g(\nabla_J \partial_t, J).$$

On the other hand, since $J$ is a Jacobi field then

$$\frac{D^2 J}{dt^2} + R(J, \gamma') \gamma' = 0,$$

where, as usual, we use the notation $\frac{DJ}{dt}$ for $\nabla_{\gamma'(t)} J$ and $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.

Moreover, since $\frac{DJ}{dt} = \nabla_J \partial_t$ because $J$ is a Jacobi field, we obtain that

$$g(\nabla_J \partial_t, J)|_{t_0} = g(J(0), \frac{DJ}{dt}(0)) + \int_{t_0}^{t_0} \frac{d}{dt} g(\frac{DJ}{dt}, J) \ dt$$

$$= g(J(0), \frac{DJ}{dt}(0)) + \int_{t_0}^{t_0} \left( \left| \frac{DJ}{dt} \right|^2 - g(R(J, \gamma') \gamma', J) \right) dt.$$

Observe that since $\partial_t(x, 0) = n(x)$ we have that if $i$ is an immersion then

$$g(J(0), \frac{DJ}{dt}(0)) = g(\nabla_J \partial_t, J)(0) = -II_{\mathcal{H}}(J(0), J(0)), \quad (3)$$

where $II_{\mathcal{H}}$ will denote the second fundamental form of $i : \mathcal{H} \rightarrow \mathbb{M}$ in the direction of $n$. On the other hand, the amount $g(J(0), \frac{DJ}{dt}(0))$ vanishes if $i$ is constant (as in the polar geodesic coordinates).

Given a vector field $V(t)$ along $\gamma_{[0, t_0]}$, with $g(V, \gamma'(t)) = 0$, we will define the index form of $V$ as

$$\mathcal{I}_{(t_0, H)}(V, V) = g(V(0), \frac{DV}{dt}(0)) + \int_{t_0}^{t_0} \left( \left| \frac{DV}{dt} \right|^2 - g(R(V, \gamma') \gamma', V) \right) dt. \quad (4)$$

With all of this, we obtain
Lemma 1. In the previous conditions the graph $\psi(x, t)$ given by (2) has a principal curvature $k_n$, with principal direction $\bar{\partial}_t$, given by

$$k_n = \frac{f''(t)}{(1 + f'(t)^2)^{3/2}}.$$  

Moreover, the second fundamental form of the graph at $\psi(x_0, t_0)$ for a tangent vector $v_0$ perpendicular to $\bar{\partial}_t$ can be computed as follows: Take a perpendicular Jacobi field $J(t)$ along the geodesic $\gamma(t) = \varphi(x_0, t)$ with $J(t_0) = v_0$ then the second fundamental form is given by

$$II(v_0, v_0) = II(J(t_0), J(t_0)) = \frac{f'(t_0)}{\sqrt{1 + f'(t_0)^2}} T_{(t_0, H)}(J, J).$$  

(5)

3 Comparison results.

The above lemma will help us to compare the principal curvatures of two graphs $\psi_j(x, t), j = 1, 2$, with the same height function in two different product spaces $\mathbb{M}_j \times \mathbb{R}$. For that, we need to relate the index forms in the manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$.

Thus, let $(\mathbb{M}_1, g_1), (\mathbb{M}_2, g_2)$ be two Riemannian manifolds with $\dim(\mathbb{M}_1) \leq \dim(\mathbb{M}_2)$ and $\mathcal{H}_j, j = 1, 2$, two smooth manifolds with $\dim(\mathcal{H}_j) = \dim(\mathbb{M}_j) - 1$. Consider smooth maps $i_j : \mathcal{H}_j \to \mathbb{M}_j$ and $n_j : \mathcal{H}_j \to T\mathbb{M}_j$ such that $n_j(x) \in T_{i_j(x)}\mathbb{M}_j$ is a unit vector with $g(d(i_j)_x(v), n_j(x)) = 0$ for all $v \in T_x\mathcal{H}_j$.

Moreover, assume that the maps

$$\varphi_j(x, t) = \exp_{i_j(x)}(t n_j(x)), \quad (x, t) \in \mathcal{H}_j \times \bar{I},$$

are smooth and $\varphi_j|_{\mathcal{H}_j \times \bar{I}}$ a global diffeomorphism onto its image, where $I$ is an open real interval such that 0 is in its closure $\bar{I}$. Then, we consider the two graphs

$$\psi_j(x, t) = (\exp_{i_j(x)}(t n_j(x)), f(t)) = (\varphi_j(x, t), f(t)), \quad (6)$$

for the same height function $f(t)$ with $f'(t) \geq 0$.

In order to compare the second fundamental forms of both graphs, let $\Gamma : \mathcal{H}_1 \to \mathcal{H}_2$ be an immersion, $x_0 \in \mathcal{H}_1$ and $\gamma_j : [0, t_0] \to \mathbb{M}_j$ be the geodesics

$$\gamma_1(t) = \varphi_1(x_0, t), \quad \gamma_2(t) = \varphi_2(\Gamma(x_0), t).$$

Theorem 1. In the previous conditions assume $K^1_{\gamma_1(t)}(\pi_1) \leq K^2_{\gamma_2(t)}(\pi_2), t \in [0, t_0], \text{for each plane } \pi_j, \text{ where } K^j_{\gamma_j(t)}(\pi_j) \text{ is the sectional curvature of a plane } \pi_j \text{ in } \mathbb{M}_j \text{ containing } \gamma_j(t)$. If:

1. $i_j$ are constant, or
2. \( i_j \) are immersions and \( II_{\mathcal{H}_1}(v, v) \leq II_{\mathcal{H}_2}(w, w) \) for all \( v \in di_1(T_{x_0}\mathcal{H}_1), \ w \in di_2(T_{\pi(x_0)}\mathcal{H}_2) \) with \( |v| = |w| \),

then the second fundamental forms of the graphs satisfy

\[ II_1(V, V) \geq II_2(W, W) \]

for all tangent vectors \( V, W \) such that \( |V| = |W| \) and \( \langle V, \gamma'_1(t_0) \rangle = 0 = \langle W, \gamma'_2(t_0) \rangle \).

In particular, every principal curvature of the graph \( \psi_1 \) at \( \varphi_1(x_0, t_0) \) is greater than or equal to every principal curvature of the graph \( \psi_2 \) at \( \varphi_2(\pi(x_0), t_0) \).

**Proof.** Let \( V \in T_{\gamma_1(t_0)}\mathbb{M}_1 \) and \( W \in T_{\gamma_2(t_0)}\mathbb{M}_2 \) with

\[ |V| = 1 = |W| \quad \text{and} \quad g_1(V, \gamma'_1(t_0)) = 0 = g_2(W, \gamma'_2(t_0)). \]

As above, \( V \) and \( W \) are identified as tangent vectors to the graphs at the points \( \psi_j(\gamma'_j(t_0), f(t_0)), j = 1, 2, \) respectively.

There exist unique perpendicular Jacobi fields \( J_j : [0, t_0] \rightarrow T\mathbb{M}_j, j = 1, 2, \) along the geodesics \( \gamma_1 \) and \( \gamma_2 \) respectively, such that \( J_1(t_0) = V, \ J_2(t_0) = W \) with the additional property: \( J_j(0) = 0 \) if \( i_j \) is constant, or \(-dn(J_j(0)) + \frac{DdJ_j}{dt}(0)\) is proportional to \( \gamma'_j(0) \) if \( i_j \) is an immersion (see, for instance, [Wa]).

Let \( \{e_k(t)\}_{k=1}^m \) be an orthonormal basis of parallel vector fields along \( \gamma_1 \), which are perpendicular to \( \gamma'_1(t) \), with \( m = \dim(\mathbb{M}_1) - 1 \), and such that \( e_1(t_0) = J_1(t_0) \). In a similar way, let \( \{b_k(t)\}_{k=1}^m \) be an orthonormal basis of parallel vector fields along \( \gamma_2 \), which are perpendicular to \( \gamma'_2(t) \), with \( n = \dim(\mathbb{M}_2) - 1 \), and such that \( b_1(t_0) = J_2(t_0) \).

We define the functions \( a_k(t) \) as the ones given by the equality

\[ J_1(t) = \sum_{k=1}^m a_k(t)e_k(t). \]

And consider a new vector field along \( \gamma_2(t) \) given as

\[ W(t) = \sum_{k=1}^m a_k(t)b_k(t). \]

Since \( W(t_0) = J_2(t_0) \) the minimizing property of the Jacobi fields (see [Wa]) gives us

\[ \mathcal{I}_{(t_0, \mathcal{H}_2)}(J_2, J_2) \leq \mathcal{I}_{(t_0, \mathcal{H}_2)}(W, W). \]

Hence,

\[ II_2(J_2(t_0), J_2(t_0)) = \frac{f'(t_0)}{\sqrt{1 + f'(t_0)^2}} \mathcal{I}_{(t_0, \mathcal{H}_2)}(J_2, J_2) \leq \frac{f'(t_0)}{\sqrt{1 + f'(t_0)^2}} \mathcal{I}_{(t_0, \mathcal{H}_2)}(W, W). \]
Since \(|J_1(t)| = |W(t)|, |\frac{dJ_1}{dt}(t)| = |\frac{dW}{dt}(t)|\) and \(II_{\mathcal{H}_1}(J_1(0), J_1(0)) \leq II_{\mathcal{H}_2}(W(0), W(0))\) when \(i_j\) is an immersion, then we obtain from (3), (4), (5) and the previous inequality that
\[
II_2(J_2(t_0), J_2(t_0)) \leq II_1(J_1(t_0), J_1(t_0)),
\]
as we wanted to show. \(\square\)

A different proof of this result was given in [GL] when \(\dim(\mathbb{M}_1) = \dim(\mathbb{M}_2) = 2\) as well as many applications.

Theorem 1 gives us a criterium for comparing all the principal curvatures of a graph at a point with all the principal curvatures of another graph at the corresponding point. Now, we look for some weaker conditions in order to compare the mean curvature of both graphs.

**Theorem 2.** In the previous conditions assume \(\dim(\mathbb{M}_1) = \dim(\mathbb{M}_2)\) and the metric of \(\mathbb{M}_1\) can be written as
\[
g_1 = dt^2 + G(t)g_0, \tag{7}
\]
where \(g_0\) is the \((n-1)\)-dimensional metric of a space form. If \(\text{Ric}^1(\gamma'_1(t)) \leq \text{Ric}^2(\gamma'_2(t))\), where \(\text{Ric}^j(\gamma'_j(t))\) denotes the Ricci curvature in the direction of the unit vector \(\gamma'_j(t)\), and

1. \(i_j\) are constant, or
2. \(i_j : \mathcal{H}_j \rightarrow \mathbb{M}_j\) are immersions and their mean curvatures \(H_{\mathcal{H}_j}\) satisfy \(H_{\mathcal{H}_1}(x_0) \leq H_{\mathcal{H}_2}(\bar{t}(x_0))\),

then the mean curvatures \(H_j\) of the graphs in \(\mathbb{M}_j \times \mathbb{R}\) satisfy
\[
H_1(\gamma_1(t_0)) \geq H_2(\gamma_2(t_0)).
\]

**Proof:** In order to compare the mean curvatures of the graphs given by (6), we use the trace of the second fundamental forms at points \(\psi_1(x_0, t_0)\) and \(\psi_2(\bar{t}(x_0), t_0)\). For this, from (5) and the fact that the function \(f(t)\) is increasing, it is sufficient to compare the corresponding sums of the index forms.

Let \(\{e_k(t)\}_{k=1}^{n-1}\) be an orthonormal basis of parallel fields along \(\gamma_1(t)\), orthogonal to \(\gamma'_1(t)\), and let \(\{b_k(t)\}_{k=1}^{n-1}\) be an orthonormal basis of parallel fields along \(\gamma_2(t)\), orthogonal to \(\gamma'_2(t)\) with \(t \in [0, t_0]\). Then, there exist unique perpendicular Jacobi fields \(J^1_k : [0, t_0] \rightarrow TM_j\), with \(j = 1, 2\) and \(k = 1, \ldots, n-1\) along \(\gamma_1(t)\) and \(\gamma_2(t)\) respectively, such that

1. \(J^1_k(t_0) = e_k(t_0), J^2_k(t_0) = b_k(t_0), \quad k = 1, \ldots, n-1.\)
2. \(J^j_k(0) = 0\) if \(i_j\) are constant maps, and so
\[
\sum_{k=1}^{n-1} g_j(J^j_k(0), \frac{D J^j_k}{dt}(0)) = 0, \quad j = 1, 2; \quad \text{or}
\]
\[-dn(J^1_k(0)) + \frac{D J_k^1}{dt}(0)\] is proportional to $\gamma'_j(0)$ if $i_j$ are immersions (see, for instance, [Wa]), and so
\[
\sum_{k=1}^{n-1} g_1(J^1_k(0), \frac{D J_k^1}{dt}(0)) = -(n - 1) H_{\mathcal{H}_1}(x_0),
\]
and
\[
\sum_{k=1}^{n-1} g_2(J^2_k(0), \frac{D J_k^2}{dt}(0)) = -(n - 1) H_{\mathcal{H}_2}(\mathcal{P}(x_0)).
\]

From (7) the previous Jacobi fields $J^1_k$ in $\mathcal{M}_1$ satisfy
\[
|J^1_k(t)| = |J^1_i(t)| \quad \text{and} \quad \left| \frac{D J^1_k}{dt}(t) \right| = \left| \frac{D J^1_i}{dt}(t) \right|, \quad \text{with} \quad i, k = 1, \ldots, n - 1.
\]

Now, we define the functions $a_{ik}$ as the ones given by the equalities
\[
J^1_i(t) = \sum_{k=1}^{n-1} a_{ik}(t) e_k(t), \quad i = 1, \ldots, n - 1.
\]

Consider the new vector fields $W_i(t)$ along $\gamma_2(t)$ given by
\[
W_i(t) = \sum_{k=1}^{n-1} a_{ik}(t) b_k(t), \quad i = 1, \ldots, n - 1.
\]

In these conditions, $a_{ik}(t_0) = \delta_{ik}$ and $W_i(t_0) = J^2_i(t_0)$. Moreover, by construction,
\[
|J^1_i(t)| = |W_i(t)| \quad \text{and} \quad \left| \frac{D J^1_i}{dt}(t) \right| = \left| \frac{D W_i}{dt}(t) \right|.
\]

Observe now that if $i_j$ are constant maps
\[
\sum_{i=1}^{n-1} g(J^2_i(0), \frac{D J^2_i}{dt}(0)) = \sum_{i=1}^{n-1} g(W_i(0), \frac{D W_i}{dt}(0)) = \sum_{i=1}^{n-1} g(J^1_i(0), \frac{D J^1_i}{dt}(0)) = 0.
\]

On the other hand, if the maps $i_j$ are immersions, then
\[
\sum_{k=1}^{n-1} g(J^2_k(0), \frac{D J^2_k}{dt}(0)) = -(n - 1) H_{\mathcal{H}_2}(\mathcal{P}(x_0)) \leq -(n - 1) H_{\mathcal{H}_1}(x_0) = \sum_{k=1}^{n-1} g(J^1_k(0), \frac{D J^1_k}{dt}(0)).
\]

In addition, the fields $W_i(t)$ are also orthogonal on $[0, t_0]$ by construction, and $|W_i(t)| = |W_1(t)|, i = 1, \ldots, n - 1$. Hence
\[
- \sum_{i=1}^{n-1} g_2(R(W_i, \gamma'_2) \gamma'_2, W_i) = -(n - 1)|W_1|^2 \text{Ric}^2(\gamma'_2(t)) \leq
\]
\[ \leq -(n-1)|J_1|^2 Ric^1(\gamma'_1(t)) = - \sum_{i=1}^{n-1} g_1(R(J^1_i, \gamma'_1)\gamma_1, J^1_i), \]

In these conditions, and by the minimizing property of the Jacobi fields, it is obtained

\[ \sum_{i=1}^{n-1} \mathcal{I}_{(t_0, \mathcal{H}_2)}(J^2_i(t_0), J^2_i(t_0)) = \]

\[ = \sum_{i=1}^{n-1} g_2(J^2_i(0), \frac{D J^2_i}{dt}(0)) + \int_0^{t_0} \left( \sum_{i=1}^{n-1} \left| \frac{D J^2_i}{dt} \right|^2 - \sum_{i=1}^{n-1} g_2(R(J^2_i, \gamma'_2)\gamma_2, J^2_i) \right) dt \]

\leq \sum_{i=1}^{n-1} g_2(W_i(0), \frac{D W_i}{dt}(0)) + \int_0^{t_0} \left( \sum_{i=1}^{n-1} \left| \frac{D W_i}{dt} \right|^2 - \sum_{i=1}^{n-1} g_2(R(W_i, \gamma'_2)\gamma_2, W_i) \right) dt \]

\leq \sum_{i=1}^{n-1} g_1(J^1_i(0), \frac{D J^1_i}{dt}(0)) + \int_0^{t_0} \left( \sum_{i=1}^{n-1} \left| \frac{D J^1_i}{dt} \right|^2 - \sum_{i=1}^{n-1} g_1(R(J^1_i, \gamma'_1)\gamma_1, J^1_i) \right) dt \]

\[ = \sum_{i=1}^{n-1} \mathcal{I}_{(t_0, \mathcal{H}_2)}(J^1_i(t_0), J^1_i(t_0)) \]

as we wanted to show. \( \square \)

**Remark 1.** Observe that a manifold \( \mathbb{M}_1 \) whose metric is described by (7) in geodesic polar coordinates is classically known as a model manifold (see [GW]).

### 4 Existence of barriers in \( \mathbb{M}^n \times \mathbb{R} \).

Our comparison results will allow us to extend some results only known for \( \mathbb{M}^n \times \mathbb{R} \) when \( \mathbb{M}^n \) is a space form to general ambient spaces \( \mathbb{M}^n \times \mathbb{R} \). Thus, in this section we will follow our approach in [GL] for obtaining some existence and non existence results for hypersurfaces in \( \mathbb{M}^n \times \mathbb{R} \).

From now on, we denote by \( \mathbb{M}^n(c) \) the complete simply connected n-dimensional space form of constant curvature \( c \), that is a hyperbolic space if \( c < 0 \), the Euclidean space if \( c = 0 \) or a sphere if \( c > 0 \). Let \( s_{c,n} = \frac{n-1}{n} \sqrt{-c} \) be the infimum of the mean curvature of the topological spheres of constant mean curvature in \( \mathbb{M}^n(c) \times \mathbb{R} \) when \( c < 0 \). Also, for each \( H_0 > 0 \) (\( H_0 > s_{c,n} \) if \( c < 0 \)) we denote by \( r_{c,n}(H_0) \) the radius of the topological sphere of constant mean curvature \( H_0 \) in \( \mathbb{M}^n(c) \times \mathbb{R} \), and for each \( K_0 > 0 \) we will denote by \( r_{c,n}^*(K_0) \) the radius of the topological sphere of constant Gauss-Kronecker curvature \( K_0 > 0 \) in the same ambient space.

Let us start with a topological sphere \( S \) of constant mean curvature \( H_0 \) in \( \mathbb{M}^n(c) \times \mathbb{R} \) (see, for instance, [AR, AEG, BE, HH, PR]). Observe that \( S \) is unique up to isometries of the ambient.
space and only exists for $H_0 > s_{c,n}$ if $c < 0$. Moreover, $S$ is rotational with respect to a vertical axis and symmetric with respect to a horizontal slice. In particular, $S$ is a bigraph over a geodesic ball of $\mathbb{M}^n(c)$ of radius $r_{c,n}(H_0) > 0$.

Thus, let $p \in \mathbb{M}^n(c)$ and $(x, t)$ be geodesic polar coordinates around $p$. Since $S$ is a rotational surface, the lower part of $S$ can be considered as a graph over the geodesic ball centered at $p$ and radius $r_{c,n}(H_0)$, with height function $h(t)$ which only depends on the distance function $t$ to the point $p$. Moreover, $h(t)$ is strictly increasing. Hence, this part of the hypersurface $S$ of constant mean curvature can be described as

$$\psi_1(x, t) = (x, t, h(t)) \in \mathbb{M}^n(c) \times \mathbb{R}.$$ 

Note that, for convenience, we have deleted the parametrization $\varphi$ (given by (1)) in the previous expression.

Now, given an $n$-dimensional Riemannian manifold $\mathbb{M}^n$ and geodesic polar coordinates $(x, t)$ around a point $q \in \mathbb{M}^n$, which are well defined for $0 < t \leq r_{c,n}(H_0)$, we can consider the new immersion

$$\psi_2(x, t) = (x, t, h(t)) \in \mathbb{M}^n \times \mathbb{R}.$$ 

Applying the same process for the upper part of $S$, we obtain a sphere $S^*$ in $\mathbb{M}^n \times \mathbb{R}$ which is a bigraph over the geodesic ball of radius $r_{c,n}(H_0)$ centered at $q$.

We remark that $S^*$ is symmetric with respect to a horizontal slice as $S$, and any vertical translation of $S^*$ is congruent to $S^*$. However, $S^*$ depends strongly on the point $q \in \mathbb{M}^n$, i.e. if we start with another point $\tilde{q} \in \mathbb{M}^n$ and obtain a new surface $\tilde{S}^*$ following the same process then $S^*$ and $\tilde{S}^*$ are not isometric in general. If the Ricci curvature in the radial directions on $B_r(q) \subset \mathbb{M}$ are greater than or equal to $c$, for all the geodesics $\gamma(t)$ in $\mathbb{M}$ emanating from $q$, using Theorem 2, we have that the mean curvature of $S^*$ satisfies $H(S^*) \leq H_0$. With all of this we obtain

**Theorem 3.** Let $B_r$ be a closed geodesic ball of radius $r > 0$ in an n-dimensional Riemannian manifold $\mathbb{M}^n$, and $c$ the minimum of the Ricci curvature in the radial directions of unit vectors $\gamma'(t)$ on $B_r$, for all the geodesics $\gamma(t)$ in $\mathbb{M}^n$ emanating from the center of $B_r$. Consider $H_0 > 0$ such that $r_{c,n}(H_0) = r$. Then, there is no vertical graph over $B_r$ with minimum of its mean curvature satisfying $\min(H) \geq H_0$.

**Proof.** Assume $\Sigma$ is a graph over $B_r$ with $\min(h) \geq H_0$ for a unit normal $N$. Without loss of generality, we assume that the unit normal $N$ points upwards.

Let $q \in \mathbb{M}^n$ be the center of the geodesic disk $B_r$ and consider the sphere $S^*$ centered at $q$ previously obtained, which has mean curvature smaller than or equal to $H_0$ for its inner normal.

Move the sphere $S^*$ up until $\Sigma$ is below $S^*$, and go down until $S^*$ intersects $\Sigma$ for the first time. Then, the classical maximum principle for mean curvature asserts that both surfaces must agree locally. In particular, $\Sigma$ and $S^*$ have constant mean curvature $H_0$ and $\Sigma$ agrees with the lower hemisphere of $S^*$. However, this is a contradiction because $S^*$ is not a strict graph over the boundary of $B_r$ since its unit normal is horizontal at those points. \qed
**Theorem 4.** Let $B_r$ be a closed geodesic ball of radius $r > 0$ in an $n$-dimensional Riemannian manifold $\mathbb{M}^n$, $c := \min\{K_p(\pi) : \partial_i \in \pi, p \in B_r\}$ be the minimum of the radial sectional curvatures on $B_r$, and $K_0 > 0$ such that $r_{c,n}(K_0) = r$. Then, there is no vertical graph over $B_r$ with minimum of its Gauss-Kronecker curvature satisfying $\min(K) \geq K_0$ and a point with definite second fundamental form.

**Proof.** The proof follows the same process that in Theorem 3, taking now a sphere with constant Gauss-Kronecker curvature in $\mathbb{M}^n(c) \times \mathbb{R}$ (see, for instance, [EGR, ES]), and using Theorem 1. The requirement of the graph of having a point with definite second fundamental form is now needed for using the maximum principle.

**Remark 2.** It should be observed that a similar result to Theorem 4 is possible for any $r$-mean curvature $H_r$, with $2 \leq r \leq n$, and not only for the Gauss-Kronecker curvature $H_n$.

**Theorem 5.** Let $\mathbb{M}^n$ be a complete, simply connected Riemannian manifold with injectivity radius $i > 0$ and $c \in \mathbb{R}$ the infimum of its Ricci curvature on $\mathbb{M}^n$. Consider a properly embedded hypersurface $\Sigma$ in $\mathbb{M}^n \times \mathbb{R}$ with mean curvature $H \geq H_0 > 0$, $(H_0 > s_{c,n}$ if $c < 0)$. If $r_{c,n}(H_0) < i$ then the mean convex component of $\Sigma$ cannot contain a closed geodesic ball in $\mathbb{M}^n \times \mathbb{R}$ of radius greater than or equal to the extrinsic semi-diameter of a sphere with constant mean curvature $H_0$ in $\mathbb{M}^n(c) \times \mathbb{R}$.

**Proof.** The proof is a consequence of the maximum principle and follows the same process that [GL, Theorem 2], using now Theorem 2.

Also observe that a weaker version of Theorem 5 is possible for the $r$-mean curvatures $H_r$, $2 \leq r \leq n$, using Theorem 1.

Let $\mathbb{M}^n$ be a Hadamard manifold, that is, a complete simply connected Riemannian manifold with non-positive sectional curvature. Since its injectivity radius is $i = \infty$, we obtain as a consequence of the previous result:

**Corollary 1.** Let $H_0 > 0$ and $\mathbb{M}^n$ be a Hadamard manifold with infimum of its Ricci curvature $c > -\infty$. Then, there exists no entire horizontal graph in $\mathbb{M}^n \times \mathbb{R}$ with mean curvature $H \geq H_0 > s_{c,n}$.

Let us denote by $S_{n-1}$ the simply connected rotational entire vertical graph with constant mean curvature $H = \frac{n-1}{n}$ in $\mathbb{H}^n \times \mathbb{R}$. This graph has been described in [BE]. Again, as a consequence of our comparison results, if we consider the corresponding entire vertical graph $S^*_{n-1}$ in $\mathbb{M}^n \times \mathbb{R}$, we have:

**Corollary 2.** Let $\mathbb{M}^n$ be an $n$-dimensional Hadamard manifold with Ricci curvature smaller than or equal to $-1$. Assume $\Sigma$ is an immersed hypersurface in $\mathbb{M}^n \times \mathbb{R}$ with mean curvature $H \leq \frac{n-1}{n}$ and cylindrically bounded vertical ends. Then $\Sigma$ must have more than one end.
We obtain now a generalization to $\mathbb{H}^n \times \mathbb{R}$ of a theorem proven in [BE] for the product space $\mathbb{H}^n \times \mathbb{R}$.

**Theorem 6.** Let $\mathbb{M}^n$ be an $n$-dimensional Hadamard manifold with sectional curvature pinched between $-c^2$ and $-1$, for a constant $c \geq 1$. Let $\Omega$ be a bounded domain in $\mathbb{M}^n \times \{0\}$, with boundary given by a compact embedded hypersurface $\Gamma$. Assume all the principal curvatures of $\Gamma$ are greater than $c$, then for any $H_0 \in [0, \frac{c^2}{n}]$ there exists a graph $h$ over $\Omega$ with constant mean curvature $H_0$ and zero boundary data.

Moreover, if $\Sigma$ is a compact hypersurface immersed in $\mathbb{M}^n \times \mathbb{R}$ with boundary $\Gamma$ and constant mean curvature $H_0$ then, up to a symmetry with respect to $\mathbb{M}^n \times \{0\}$, $\Sigma$ agrees with the previous graph.

**Proof.** Observe that $\Omega$ must be a convex bounded domain in $\mathbb{M}^n$ and homeomorphic to a ball, and $\Gamma$ must be homeomorphic to a sphere (see [Al]).

Let $m_0$ be the minimum of the principal curvatures of $\Gamma$. Since $m_0 > c$ we can take a radius $R_0$ big enough such that for every $R > R_0$ the geodesic spheres of radius $R$ in the hyperbolic space of sectional curvature $-c^2$ have principal curvatures smaller than $m_0$. As the sectional curvature of $\mathbb{M}^n$ is bigger than or equal to $-c^2$, the geodesic spheres in $\mathbb{M}^n$ of radius $R \geq R_0$ have principal curvatures smaller than $m_0$.

Let $p \in \Gamma$ and $\gamma_p(t)$ be the geodesic in $\mathbb{M}^n$ starting at $p$ with initial speed given by the unit normal to $\Gamma$ pointing to $\Omega$. It is clear that the geodesic sphere $S_p(R) \subseteq \mathbb{M}^n$ centered at $\gamma_p(R)$ and radius $R$ is tangent to $\Gamma$ at $p$. Moreover, if $R \geq R_0$ the open geodesic ball bounded by $S_p(R)$ contains to punctured neighborhood of $p \in \Gamma$ because the principal curvatures of $S_p(R)$ are bigger than the principal curvatures of $\Gamma$ at $p$ for the same interior unit normal.

Let $S_0 \subseteq \mathbb{M}^n$ be a geodesic sphere such that $\Gamma$ is contained in the geodesic ball bounded by $S_0$, and the distance from $S_0$ to $\Gamma$ is greater than or equal to $R_0$. Then, consider the map $G : \Gamma \rightarrow S_0$ defined in the following way: given $p \in \Gamma$ the point $G(p)$ is given by the intersection of the geodesic $\gamma_p(t)$ for $t \geq 0$ with $S_0$. It is well known that the previous intersection is given by a unique point due to the convexity of the geodesic spheres (see, for instance, [Al]).

Thus, if we denote by $S_p$ the geodesic sphere centered at $G(p)$ passing across $p \in \Gamma$, then we have shown that $S_p$ is tangent to $\Gamma$ at $p$ and a punctured neighborhood of $p$ in $\Gamma$ is contained in the open geodesic ball bounded by $S_p$. In fact, we assert

**Claim:** For every $p \in \Gamma$ the closed geodesic ball bounded by $S_p$ contains to $\Gamma$.

Observe that if $G : \Gamma \rightarrow S_0$ is injective then the Claim would be proven. Indeed, if there existed $p_1 \in \Gamma$ such that $\Gamma \not\subseteq S_{p_1}$ then there would be a point $p_2 \neq p_1$ such that $d(p_2, G(p_1)) > d(p, G(p_1))$ for all $p \in \Gamma$. Thus, the geodesic sphere centered at $G(p_1)$ passing across $p_2$ is tangent to $\Gamma$ and contains $\Gamma$ in its interior, and so $G(p_2) = G(p_1)$.

Hence, assume there exist two points $p_1, p_2 \in \Gamma$ such that $G(p_1) = G(p_2)$. In such a case, we have shown that $p_1$ and $p_2$ are two strict local maxima for the distance function $g(p)$ from $p \in \Gamma$ to the fixed point $G(p_1) = G(p_2)$. Now, we distinguish two cases depending on the dimension of $\Gamma$:
1. If \( \text{dim}(\Gamma) \geq 2 \) then we can use the mountain pass lemma for the function \( \varphi \) and there must exist a third point \( p_3 \) which is a saddle point for \( \varphi \). Thus, the geodesic sphere \( S_{p_3} \) centered at \( G(p_1) \) passing across \( p_3 \) is tangent to \( \Gamma \). Therefore, depending on the orientation of the unit normal to \( \Gamma \), we have that \( S_{p_3} = S_{p_3}^* \) or \( S_{p_3} \cap S_{p_3}^* = \{ p_3 \} \). But, this contradicts that \( p_3 \) is a saddle point, because a punctured neighborhood of \( p_3 \) is contained in the interior of the geodesic ball bounded by \( S_{p_3} \).

2. If \( \text{dim}(\Gamma) = 1 \) then we can consider the two closed arcs \( \Gamma_1 \) and \( \Gamma_2 \) of \( \Gamma \) joining \( p_1 \) and \( p_2 \). Since \( p_1 \) and \( p_2 \) are strict local maxima for the function \( \varphi \), there must exist \( p_3 \in \Gamma_1 \) and \( p_4 \in \Gamma_2 \) different from \( p_1 \) and \( p_2 \) which are local minima for \( \varphi \). Assume \( \varphi(p_3) \leq \varphi(p_4) \), then from the convexity of \( \overline{\Omega} \) the geodesic arc \( \Lambda \) joining \( p_3 \) and \( p_4 \) is contained in \( \overline{\Omega} \). But, \( \Lambda \setminus \{ p_3, p_4 \} \) is contained in the open geodesic ball centered at \( G(p_1) \) and radius \( \varphi(p_4) \) from the convexity of the geodesic ball. This contradicts that \( p_4 \) is a minimum for \( \varphi \).

Once the Claim is proven, consider a compact hypersurface \( \Sigma \) immersed in \( \mathbb{M}^n \times \mathbb{R} \) with boundary \( \Gamma \) and constant mean curvature \( H_0 \).

Let \( \tilde{S} \) be the rotational entire graph with constant mean curvature \( H_0 \) in \( \mathbb{H}^n \times \mathbb{R} \) for its unit normal pointing upwards. Consider a point \( p \in \Gamma \subseteq \mathbb{M}^n \) and the associated entire graph \( S^* = M^n \times \mathbb{R} \) when we use geodesic polar coordinates at \( G(p) \in \mathbb{M}^n \). Up to a vertical translation we can assume that \( S^* \cap M^n \times \{ 0 \} \) is the geodesic sphere centered at \( G(p) \) and containing to \( p \) in \( M^n \times \{ 0 \} \). Thus, from the previous Claim, \( \Gamma \times \{ 0 \} \) is contained in the closed mean convex component of \( S^* \), and \( (p, 0) \in S^* \).

Let us also denote by \( \overline{S^*} \) the reflection of \( S^* \) with respect to the horizontal slice \( M^n \times \{ 0 \} \). The entire graphs \( S^* \) and \( \overline{S^*} \) are congruent, and from Theorem 1 we obtain that they have mean curvature \( H \geq H_0 \) for its unit normal pointing to the mean convex component.

As \( \Sigma \) is compact we can move vertically \( S^* \) in such a way that \( \Sigma \) is completely contained in the mean convex component of \( S^* \). Now, from the maximum principle, if we move back \( S^* \) then the surfaces \( \Sigma \) and \( S^* \) do not intersect until \( S^* \) is in its initial position. The same is true for \( \overline{S^*} \).

Therefore, for every \( p \in \Gamma \) the hypersurface \( \Sigma \) is contained in the compact domain determined by the intersection of the mean convex components of \( S^* \) and \( \overline{S^*} \). In particular, the interior of \( \Sigma \) is contained in the solid cylinder \( \Omega \times \mathbb{R} \), and if \( \Sigma \) was given by the graph of a function \( h \) then its height is bounded a priori and so is its gradient at the boundary \( \Gamma \).

Now we can prove that there exists a graph \( h \) over \( \Omega \) with constant mean curvature \( H_0 \in [0, \frac{n-1}{n}] \) and zero boundary data. That is, we want to solve the following Dirichlet problem

\[
\begin{align*}
\text{div} \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) &= n \, H_0, \quad \text{in } \Omega \\
h &= 0 \quad \text{on } \Gamma
\end{align*}
\]

where the divergence and gradient \( \nabla h \) are taken with respect to the metric on \( \mathbb{M}^n \) (see [Sp]).
We have proven the existence of height estimates and gradient estimates at the boundary. Hence, from [Sp], we also have global gradient estimates, and the existence of $h$ follows from the classical elliptic theory (see [GT] and [Sp]).

Finally, we want to show that if $\Sigma$ is a compact hypersurface in $\mathbb{M}^n \times \mathbb{R}$ with constant mean curvature $H_0$ and boundary $\Gamma$, then it is the graph of the previous function $h$ or $-h$.

First, let us observe that $\Sigma$ is a vertical graph. In fact, we have shown that $\Sigma$ is contained in the cylinder $\Omega \times \mathbb{R}$ and $\Sigma$ has no interior point in $\Gamma \times \mathbb{R}$. Thus, we can use the maximum principle with respect to horizontal slices from the highest point of $\Sigma$ to the lowest point of $\Sigma$, which proves that $\Sigma$ is a graph.

Moreover, let $\Sigma_0$ be the graph of $h$ or $-h$ which points in the same direction (upwards or downwards) as $\Sigma$. Moving $\Sigma$ vertically upwards until $\Sigma$ and $\Sigma_0$ are disjoint, and coming down again we observe that, from the maximum principle, $\Sigma$ cannot touch $\Sigma_0$ till the boundaries agree. Hence, $\Sigma$ is above $\Sigma_0$. Repeating the same process, but moving now $\Sigma$ vertically downwards, one has $\Sigma$ is below $\Sigma_0$. Therefore, $\Sigma$ and $\Sigma_0$ agree, as we wanted to show.

Remark 3. It is an interesting open question if, under the previous conditions on $\mathbb{M}^n$, there exists an entire vertical graph over $\mathbb{M}^n$ for every constant mean curvature $H_0 \in (0, (n-1)/n]$.

Theorem 7. Let $B_r$ be a closed geodesic ball of radius $r > 0$ in $\mathbb{M}^n$, $c := \max\{K_p(\pi) : \partial_t \in \pi, p \in B_r\}$ be the maximum of the radial sectional curvatures on $B_r$, and $K_0 > 0$ such that $r_{c,n}(K_0) = r$. Then, there exists a strictly convex graph $h_K$ over $B_r$ of constant Gauss-Kronecker curvature $K > 0$ in $\mathbb{M}^n \times \mathbb{R}$ by $h_K|_{\partial B_r} = 0$, for any $K < K_0$.

Proof. Let us consider a sphere $S$ in $\mathbb{M}^n(c) \times \mathbb{R}$ with positive constant Gauss-Kronecker curvature $K < K_0$. From Theorem 1, the corresponding sphere $S^*$ in $\mathbb{M}^n \times \mathbb{R}$, using polar coordinates at the center of $B_r$, has Gauss-Kronecker curvature greater than or equal to $K$, and so it is a subsolution for the existence of the graph we are looking for. Thus, from [Gu] (see also [Sp]) there exists a strictly convex graph of constant Gauss-Kronecker curvature $K$ and zero boundary data.

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