Nonribbon 2-links
all of whose components are trivial knots and
some of whose band-sums are nonribbon knots

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Abstract. There is a nonribbon 2-link all of whose components are trivial 2-knots and one of whose band-sums is a nonribbon 2-knot.

1 Main result

We work in the smooth category.

An $m$-component 2-(dimensional) link is a closed oriented 2-submanifold $L = (K_1, ..., K_m) \subset S^4$ such that $K_i$ is diffeomorphic to $S^2$. If $m = 1$, $L$ is called a 2-knot We say that 2-links $L_1$ and $L_2$ are equivalent if there exists an orientation preserving diffeomorphism $f : S^4 \to S^4$ such that $f(L_1) = L_2$ and that $f|_{L_1} : L_1 \to L_2$ is an order and orientation preserving diffeomorphism.

Take a 3-ball $B^3$ in $S^4$. Then $\partial B^3$ is a 2-knot. We say that a 2-knot $K$ is a trivial knot if $K$ is equivalent to the 2-knot $\partial B^3$.

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A 2-link \( L = (K_1, ..., K_m) \) is called a \textit{ribbon} 2-link if \( L \) satisfies the following properties. (See e.g. [1].)

1. There is a self-transverse immersion \( f : D_1^3 \sqcup ... \sqcup D_m^3 \to S^4 \) such that \( f(\partial D_i^3) = K_i \).
2. The singular point set \( C (\subset S^4) \) of \( f \) consists of double points. \( C \) is a disjoint union of 2-discs \( D_i^2 (i = 1, ..., k) \).
3. Put \( f^{-1}(D_j^2) = D_j^2 \sqcup D_j^2 \). The 2-disc \( D_j^2 \) is trivially embedded in the interior \( \text{Int} D_\alpha^3 \) of a 3-disc component \( D_\alpha^3 \). The circle \( \partial D_\beta^2 \) is trivially embedded in the boundary \( \partial D_\beta^3 \) of a 3-disc component \( D_\beta^3 \). The 2-disc \( D_j^2 \) is trivially embedded in the 3-disc component \( D_\beta^3 \). (Note that there are two cases, \( \alpha = \beta \) and \( \alpha \neq \beta \)).

There are nonribbon 2-knots. (See e.g. [1].) It is trivial that, if a component of a 2-link is a nonribbon 2-knot, the 2-link is a nonribbon 2-link. It is natural to ask:

**Question** Is there a nonribbon 2-link all of whose components are ribbon knots? In particular, is there a nonribbon 2-link all of whose components are trivial knots?

We give an affirmative answer to this question.

**Theorem 1.1** There is a nonribbon 2-link \( L = (K_1, K_2) \) such that \( K_i \) is a trivial 2-knot \( (i = 1, 2) \).

**Note.** The announcement of Theorem 1.1 is in [4].

## 2 Band-sums

Let \( L = (K_1, K_2) \) be a 2-link. A 2-knot \( K_0 \) is called a \textit{band-sum} of the components \( K_1 \) and \( K_2 \) of the 2-link \( L \) along a \textit{band} \( h \) if we have:

1. There is a 3-dimensional 1-handle \( h \), which is attached to \( L \), embedded in \( S^4 \).
2. There are a point \( p_1 \in K_1 \) and a point \( p_2 \in K_2 \). We attach \( h \) to \( K_1 \sqcup K_2 \) along \( p_1 \sqcup p_2 \). \( h \cap (K_1 \cup K_2) \) is the attach part of \( h \). Then we obtain a 2-knot from \( K_1 \) and \( K_2 \) by this surgery. The 2-knot is \( K_0 \).
3 A sufficient condition of Theorem 1.1

In §4 and §5 we prove:

**Proposition 3.1** There is a 2-link $L = (K_1, K_2)$ such that
(1) $K_i$ is a trivial 2-knot ($i = 1, 2$), and
(2) a band-sum $K_3$ of the components $K_1, K_2$ of the 2-link $L$ is a nonribbon 2-knot.

**Claim 3.2** Proposition 3.1 implies Theorem 1.1.

**Proof of Claim 3.2.** By the definition of ribbon links, we have the following fact: If $L = (K_1, K_2)$ is a ribbon 2-link, then any band sum of $L = (K_1, K_2)$ is a ribbon 2-knot. The contrapositive proposition of this fact implies Claim 3.2.

4 $Q(K)$

Let $K$ be a 2-knot $\subset S^4$. We define a 2-knot $Q(K)$ for $K$. The 2-knot $Q(K)$ plays an important role in our proof as we state in the last paragraph of this section.

Let $K \times D^2$ be a tubular neighborhood of $K$ in $S^4$, where $D^2$ is a disc. In $\text{Int } D^2$, take a compact oriented 1-dimensional submanifold $[-1, 1]$. Take $K \times [-1, 1] \subset K \times \text{Int } D^2$. We give an orientation to $K \times [-1, 1]$. Let $D(K)$ be the 2-component 2-link $(K \times \{-1\}, K \times \{1\})$. Then $K \times [-1, 1]$ is a Seifert hypersurface of the 2-link $D(K)$, where we give an orientation to $D(K)$ so that the orientation of $D(K)$ is compatible with that of $K \times [-1, 1]$.

In order to prove our main theorem, we construct some 2-knots, 2-links, and some subsets in $S^4$ from $D(K)$. For this purpose, we prepare the following $B^4$ and $F_\theta$.

Let $B^4$ be a 4-ball $\subset S^4$. Put $B^4 = \{(x, y, z, w)| 0 \leq x \leq 1, 0 \leq y \leq 1, z = r \cdot \cos \theta, w = r \cdot \sin \theta, 0 \leq r \leq 1, 0 \leq \theta < 2\pi \}$.

Let $F_0 = \{(x, y, z, 0)| 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \} \subset B^4$.

Let $A = \{(x, y, 0, 0)| 0 \leq x \leq 1, 0 \leq y \leq 1 \} \subset F_0 \subset B^4$.

We regard $B^4$ as the result of rotating $F_0$ around the axis $A$. For each $\theta$, we put $F_\theta = \{(x, y, r \cdot \cos \theta, r \cdot \sin \theta)| 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq r \leq 1, \theta: \text{ fixed} \}$.

We suppose that $B^4 \cap D(K)$ satisfies the condition that, for each $\theta$, $F_\theta \cap D(K)$ is drawn as in Figure 4.1.
Note. In Figure 4.1, we suppose the following hold: The intersection $B^4 \cap D(K)$ is a disjoint union of two 2-discs. Call them $D_1^2$ and $D_2^2$. The intersection $F_0 \cap D(K)$ is two arcs. Call them $E_1$ and $E_2$. The boundary $\partial E_i$ is a set of two points $a_i \sqcup b_i$, where $a_i$ is in $A$ and $b_i$ is in $F_0 - A$. The 2-disc $D_1^2$ is the result of rotating $E_1$ around the axis $A$. The result of rotating $b_i$ is $\partial D_1^2$. Since $a_i$ is in the axis $A$, the result of rotating $a_i$ is the point $a_i$ itself. The point $b_i$ is in the boundary of $D_1^2$. The point $a_i$ is in the interior of $D_1^2$. 
Let \( Q(K) \) be a band-sum of the components \( K \times \{-1\} \) and \( K \times \{1\} \) of the 2-link \( D(K) \) with the following properties.

1. The band \( h \) is in \( K \times \text{Int}D^2 \).
2. \( \{h-(\text{the attach part of } h)\} \cap (K \times [-1,1]) = \emptyset \).
3. \( Q(K) - B^4 = D(K) - B^4 \).
4. \( B^4 \cap (D(K) \cup h) \) (\( = B^4 \cap (K \times \{-1\} \cup h \cup K \times \{1\}) \)) satisfies the following conditions. (We summarize the conditions in Table 1.)
   - For \( \pi \leq \theta < 2\pi \) and \( \theta = 0 \), \( F_\theta \cap (D(K) \cup h) \) is drawn as in Figure 4.2.
   - For \( 0 < \theta < \pi \), \( F_\theta \cap (D(K) \cup h) \) is drawn as in Figure 4.1.
5. \( B^4 \cap h \) satisfies the following conditions.
   - For \( \pi \leq \theta < 2\pi \) and \( \theta = 0 \), \( F_\theta \cap h \) is drawn as in Figure 4.3.
   - For \( 0 < \theta < \pi \), \( F_\theta \cap h \) is empty.
6. \( B^4 \cap (\text{the attach part of } h) \) is as follows.
   - For \( \pi \leq \theta < 2\pi \) and \( \theta = 0 \), \( F_\theta \cap (\text{the attach part of } h) \) is drawn as in Figure 4.4.
   - For \( 0 < \theta < \pi \), \( F_\theta \cap (\text{the attach part of } h) \) is empty.
7. \( B^4 \cap Q(K) \) satisfies the following conditions. (We summarize the conditions in Table 1.)
   - For \( \pi < \theta < 2\pi \), \( F_\theta \cap Q(K) \) is drawn as in Figure 4.5.
   - For \( \theta = 0, \pi \), \( F_\theta \cap Q(K) \) is drawn as in Figure 4.2.
   - For \( 0 < \theta < \pi \), \( F_\theta \cap Q(K) \) is drawn as in Figure 4.1.
Figure 4.2
Figure 4.3
Note. The following hold:
(I) $h \cup (K \times [-1,1])$ is a Seifert hypersurface of the 2-knot $Q(K)$.
(II) Let $A$ be a Seifert matrix of $Q(K)$ associated with the Seifert hypersurface $h \cup (K \times [-1,1])$. Since $h \cup (K \times [-1,1])$ is diffeomorphic to $(S^1 \times S^2) - B^3$, $A$ is a $(1 \times 1)$-matrix. By the construction of $Q(K)$, $A = (2)$ or $A = (-1)$ holds. Recall that whether $A$ is a $1 \times 1$-matrix $(2)$ or $(-1)$ depends on which orientation we give $Q(K)$. Recall that the orientation of $Q(K)$ is determined by that of $D(K)$.
(III) $2t - 1$ or $2 - t$ represents for the Alexander polynomial of the 2-knot $Q(K)$. (See §F,G,H of §7 of [8] for Seifert matrices of 2-knots and the Alexander polynomial of 2-knots.) Hence $Q(K)$ is a nontrivial 2-knot.

In §5 we prove:

**Lemma 4.1** Let $K$ be a 2-knot. There is a 2-link $L = (K_1, K_2)$ such that
(1) $K_i$ is a trivial 2-knot ($i = 1, 2$), and
(2) $Q(K)$ is a band-sum $K_3$ of the components $K_1, K_2$ of the 2-link $L$.

The above $Q(K)$ is ‘a 2-knot $D(J, \gamma)$ whose $\gamma$ is sufficiently complicated’ in §4 of [I]. Corollary 4.3 in §4 of [I] or Example after Corollary 4.3 in §4 of [I] says that, for a 2-knot $K$, the above $Q(K)$ is a nonribbon 2-knot. Hence Lemma 4.1 implies Proposition 3.1.

5 Proof of Lemma 4.1

Let $L = (K_1, K_2)$ be a 2-link with the following conditions.
(1) $(S^4 - B^4) \cap L = (S^4 - B^4) \cap D(K)$.
(2) $B^4 \cap L$ satisfies the condition that, for each $\theta$, $F_\theta \cap L$ is drawn as in Figure 4.5. (We summarize the conditions in Table 1.)
Note. In Figure 4.5, the following hold: The two arcs are called $l_1$ and $l_2$. $l_i$ is a trivial arc. $l_i \cap A \neq \emptyset$. $l_2 \cap A = \emptyset$. $K_i$ is made from $l_i$ by the rotation.

By the construction of $L = (K_1, K_2)$, $K_1$ satisfies the conditions:

1. $K_1 \subset B^4$.
2. For each $\theta$, $F_\theta \cap K_1$ is drawn as in Figure 5.1.

We prove $K_1$ is a trivial knot. Because: Since $l_1$ is a trivial arc, $K_1$ is a spun knot of a trivial 1-knot. See [10] for spun knots.

By the construction of $L = (K_1, K_2)$, $K_2$ satisfies the following conditions.

1. $(S^4 - B^4) \cap K_2 = (S^4 - B^4) \cap D(K)$.
2. For each $\theta$, $F_\theta \cap K_2$ is drawn as in Figure 5.2.

We prove: $K_2$ is a trivial knot. Because: Let $P$ be a subset $(K \times [-1, 1]) - B^4$. Then $P$ is diffeomorphic to a 3-ball. Hence $\partial P$ is a trivial 2-knot. Since $l_2$ is a trivial arc, $K_2$ is equivalent to $\partial P$. Hence $K_2$ is a trivial 2-knot.

Let $K_3$ be a band-sum of the components $K_1$ and $K_2$ of the 2-link $L$ with the following conditions.

1. The band $h'$ is in $B^4$.
2. $B^4 \cap (L \cup h') = B^4 \cap (K_1 \cup h' \cup K_2)$ satisfies the following conditions. (We summarize the conditions in Table 1.)

For $\pi < \theta < 2\pi$, $F_\theta \cap (L \cup h')$ is drawn as in Figure 4.5.
For $0 \leq \theta \leq \pi$, $F_\theta \cap (L \cup h')$ is drawn as in Figure 4.2.
3. $B^4 \cap h'$ satisfies the following conditions.
For $\pi < \theta < 2\pi$, $F_\theta \cap h'$ is empty.
For $0 \leq \theta \leq \pi$, $F_\theta \cap h'$ is drawn as in Figure 4.3.
4. Note that $h$ and $h'$ are dual handles each other.
5. $B^4 \cap K_3$ satisfies the following conditions. (We summarize the conditions in Table 1.)
For $\pi < \theta < 2\pi$, $F_\theta \cap K_3$ is drawn as in Figure 4.5.
For $\theta = 0, \pi$, $F_\theta \cap K_3$ is drawn as in Figure 4.2.
For $0 < \theta < \pi$, $F_\theta \cap K_3$ is drawn as in Figure 4.1.

By the construction of this knot $K_3$ and the construction of $Q(K)$ in §4, $K_3$ is identical to $Q(K)$. This completes the proof of Lemma 4.1, Proposition 3.1, Theorem 1.1.
Figure 5.1
6 Related problems

Problem 6.1. Let $L = (K_1, K_2)$ be a 2-link. Then do we have $\mu(L) = \mu(K_1) + \mu(K_2)$?

See [9] [6] [7] for the $\mu$ invariant of 2-links and related topics.

Problem 6.2. Is there a 2-link which is not an SHB link?

See [2] [3] for SHB links.

The author proved in [5]: if $L = (K_1, K_2)$ is an SHB link, then the answer to Problem 6.1 is affirmative.

Problem 6.3. Let $K_1, K_2, K_3$ be arbitrary 2-knots. Is there a 2-component 2-link $L = (L_1, L_2)$ such that $L_1$ (resp. $L_2$) is equivalent to $K_2$ (resp. $K_2$) and that a band-sum of $L$ is $K_3$?

If the answer to Problem 6.1 is affirmative, then the answer to Problem 6.3 is negative.

In [5] the author gives the negative answer to the $n$-dimensional knot version of Problem 6.3 and proves the $n$-dimensional version of Theorem 1.1. The announcement of them is in [4].

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