RELATIVE WEAK INJECTIVITY OF OPERATOR SYSTEM PAIRS

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ABSTRACT. The concept of a relatively weakly injective pair of operator systems is introduced and studied in this paper, motivated by relative weak injectivity in the C*-algebra category. E. Kirchberg [12] proved that the C*-algebra $C^*(F_{\infty})$ of the free group $F_{\infty}$ on countably many generators characterises relative weak injectivity for pairs of C*-algebras by means of the maximal tensor product. One of the main results of this paper shows that $C^*(F_{\infty})$ also characterises relative weak injectivity in the operator system category. A key tool is the theory of operator system tensor products [10, 11].

1. Introduction

A pair $(A, B)$ of unital C*-algebras is a relatively weakly injective pair for every unital C*-algebra $C$, $A \otimes_{\text{max}} C$ is a unital C*-subalgebra of $B \otimes_{\text{max}} C$. (In particular, one has that $A$ is a unital C*-subalgebra of $B$.) It is common to say that $A$ is relatively weakly injective in $B$ if the pair $(A, B)$ is a relatively weakly injective pair. Relative weak injectivity for pairs of C*-algebras was introduced by E. Kirchberg [12] and was motivated by the work of E.C. Lance [14] on the weak expectation property for C*-algebras.

The purpose of this paper is to introduce and study a notion of relative weak injectivity for pairs $(S, T)$ of operator systems $S$ and $T$. To do so, one therefore needs to consider operator system tensor products. Although the theory of tensor products [10, 11] in the category $O_1$, whose objects are operator systems and whose morphisms are unital completely positive (ucp) linear maps, shares many similarities with C*-algebraic tensor products, there some significant differences, particularly when considering the operator system analogue of the maximal C*-algebraic tensor product, $\otimes_{\text{max}}$. With the max tensor product, there are two distinct tensor products (denoted by $\otimes_\text{c}$ and $\otimes_{\text{max}}$) in the category $O_1$ that collapse to the maximal C*-algebraic tensor product on the subcategory of unital C*-algebras and unital *-homomorphisms. In this paper an operator system analogue of relative weak injectivity will be developed using the commuting tensor product, $\otimes_\text{c}$. Specifically, a pair $(S, T)$ of operator systems is said to be a relatively weakly injective pair if, for every operator system $R$, $S \otimes_\text{c} R$ is a unital operator subsystem of $T \otimes_\text{c} R$.

The C*-algebra $C^*(F_{\infty})$ of the free group $F_{\infty}$ on countably infinitely many generators is universal in the sense that every unital separable C*-algebra is a quotient of $C^*(F_{\infty})$. Therefore, it is striking that the C*-algebra $C^*(F_{\infty})$ can be used to characterise both the weak expectation property and relative weak injectivity, as demonstrated by two important theorems of Kirchberg. More precisely, $A$ has WEP
if and only if $\mathcal{A} \otimes_{\min} C^*(F_\infty) = \mathcal{A} \otimes_{\max} C^*(F_\infty)$ \cite{12} Proposition 1.1, and $(\mathcal{A}, \mathcal{B})$ is a relatively weakly injective pair if and only if $\mathcal{A} \otimes_{\max} C^*(F_\infty) \subset \mathcal{B} \otimes_{\max} C^*(F_\infty)$ \cite{12} Proposition 3.1.

An operator system analogue of the weak expectation property for $C^*$-algebras—namely the double commutant expectation property—was introduced and studied in \cite{9} \cite{10}, and it was shown that $C^*(F_\infty)$ characterises this property. One of the main results of this paper shows that $C^*(F_\infty)$ also characterises relative weak injectivity of operator system pairs (Theorem \ref{thm:main}). In addition to establishing some alternate characterisations of relative weak injectivity, the existence of relatively weakly injective pairs $(\mathcal{S}, \mathcal{T})$ in the operator system category will be achieved (in Theorem \ref{thm:existence}) in a manner similar to Kirchberg’s result \cite{12} Corollary 3.5 that every unital separable $C^*$-algebra is a unital $C^*$-subalgebra of a unital separable $C^*$-algebra with the weak expectation property. The paper concludes with a selection of examples.

The theory of operator algebraic tensor products is treated in the books \cite{1, 18}, while operator system tensors products are developed in the papers \cite{10} \cite{11}. Standard references for operator systems and completely positive maps are \cite{16} \cite{17}.

2. The Commuting Operator System Tensor Product

If $\mathcal{S}$ and $\mathcal{T}$ are operator systems, then the notation $\mathcal{S} \subset \mathcal{T}$ means that $\mathcal{S}$ is a unital operator subsystem of $\mathcal{T}$. That is, if $1_\mathcal{S}$ and $1_\mathcal{T}$ denote the distinguished Archimedean order units for $\mathcal{S}$ and $\mathcal{T}$ respectively, then $1_\mathcal{S} = 1_\mathcal{T}$. Unless the context is not clear, the order unit for an operator system will be denoted simply by $1$.

The algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ of operator systems $\mathcal{S}$ and $\mathcal{T}$ is a $*$-vector space. An operator system tensor product structure on $\mathcal{S} \otimes \mathcal{T}$ is a family $\tau = \{ C_n \}_{n \in \mathbb{N}}$ of cones $C_n \subset M_n(\mathcal{S} \otimes \mathcal{T})$ such that:

1. $(\mathcal{S} \otimes \mathcal{T}, \tau, 1_\mathcal{S} \otimes 1_\mathcal{T})$ is an operator system, denoted by $\mathcal{S} \otimes_\tau \mathcal{T}$, in which $1_\mathcal{S} \otimes 1_\mathcal{T}$ is an Archimedean order unit,
2. $M_n(\mathcal{S})_+ \otimes M_m(\mathcal{T})_+ \subset C_{nm}$, for all $n, m \in \mathbb{N}$, and
3. if $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive (ucp) maps, then $\phi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{nm}$ is a ucp map.

Recall that a unital completely positive linear (ucp) map $\phi : \mathcal{S} \to \mathcal{T}$ of operator systems is a complete order isomorphism if it is a linear bijection and if both $\phi$ and $\phi^{-1}$ are completely positive. If the ucp map $\phi$ is merely injective, then $\phi$ is a complete order injection if $\phi$ is a complete order isomorphism of between $\mathcal{S}$ and the operator subsystem $\phi(\mathcal{S})$ of $\mathcal{T}$.

If $\mathcal{S}_1 \subset \mathcal{T}_1$ and $\mathcal{S}_2 \subset \mathcal{T}_2$ are inclusions of operator systems, and if $\iota_j : \mathcal{S}_j \to \mathcal{T}_j$ are the inclusion maps, then for any operator system structures $\tau$ and $\sigma$ on $\mathcal{S}_1 \otimes \mathcal{S}_2$ and $\mathcal{T}_1 \otimes \mathcal{T}_2$, respectively, the notation (as used in \cite{6} also)

$$\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_+ \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$$

expresses the fact that the linear vector-space embedding $\iota_1 \otimes \iota_2 : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{S}_1 \otimes_\tau \mathcal{T}_2$ is a ucp map $\iota_1 \otimes \iota_2 : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$. That is, $\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_+ \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$ if and only if $M_n(\mathcal{S}_1 \otimes_\tau \mathcal{S}_2)_+ \subset M_n(\mathcal{T}_1 \otimes_\sigma \mathcal{T}_2)_+$ for every $n \in \mathbb{N}$. If, in addition, $\iota_1 \otimes \iota_2$ is a complete order isomorphism onto its range, then this is denoted by

$$\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_{\coi} \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2.$$ 

Thus, $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_\sigma \mathcal{T}$ means $\mathcal{S} \otimes_\tau \mathcal{T} \subset_{\coi} \mathcal{S} \otimes_\sigma \mathcal{T}$ and $\mathcal{S} \otimes_\sigma \mathcal{T} \subset_{\coi} \mathcal{S} \otimes_\tau \mathcal{T}$. 
The commuting operator system tensor product \( \otimes_c \) was introduced and studied in \([10]\) and will be defined below. A slight simplification in the definition is afforded by the following lemma, which allows one to restrict to ucp maps rather than use all completely positive maps.

**Lemma 2.1.** \([2, \text{Lemma 2.2}], [3, \text{Lemma 5.1.6}]\) Let \( S \subset B(K) \) be an operator system and \( \phi : S \to B(H) \) be a completely positive map. Then there exists a ucp map \( \tilde{\phi} : S \to B(H) \) such that

\[
\phi(\cdot) = \phi(1)^{\frac{1}{2}} \tilde{\phi}(\cdot) \phi(1)^{\frac{1}{2}}.
\]

The proof of the lemma above describes the map \( \tilde{\phi} \) as a strong limit of \( \tilde{\phi}^{(n)} \) in \( B(H) \), where

\[
\tilde{\phi}^{(n)}(s) = \left( \phi(1) + \frac{1}{n} \right)^{-\frac{1}{2}} \phi(s) \left( \phi(1) + \frac{1}{n} \right)^{-\frac{1}{2}} + \langle s\eta, \eta \rangle (1 - P_{\phi(1)}),
\]

for \( \eta \in K \), and \( P_{\phi(1)} \) is the projection onto the closure of the range of \( \phi(1) \). Thus, for operator systems \( S \subset B(K_S) \) and \( T \subset B(K_T) \), if \( \phi : S \to B(H) \) and \( \psi : T \to B(H) \) are completely positive maps with commuting ranges, then the corresponding ucp maps \( \tilde{\phi} \) and \( \tilde{\psi} \) also have commuting ranges.

Denote by ucp\((S, T)\) the set of all pairs \((\phi, \psi)\) of ucp maps from \( S \) and \( T \), respectively, into \( B(H) \) for some Hilbert space \( H \), such that \( \phi(S) \) commutes with \( \psi(T) \). For each \((\phi, \psi) \in \text{ucp}(S, T)\) let \( \phi \cdot \psi : S \otimes T \to B(H) \) be the unique linear map whose value on elementary tensors is given by

\[
\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y).
\]

Define cones by

\[
\mathcal{C}_n^{\text{comm}} = \{ \eta \in M_n(S \otimes T) : (\phi \cdot \psi)^{(n)}(\eta) \geq 0, \text{ for all } (\phi, \psi) \in \text{ucp}(S, T) \}.
\]

It was shown in \([10]\) that the collection of cones above is a matrix ordering on \( S \otimes T \) with Archimedean matrix order unit \([1, 0] \otimes [1, 0] \).

**Definition 2.2.** The operator system \((S \otimes T, \{\mathcal{C}_n^{\text{comm}}\}_{n \in \mathbb{N}}, [1_S \otimes 1_T])\) is called the commuting operator system tensor product of \( S \) and \( T \) and is denoted by \( S \otimes_c T \).

The following notation, introduced in \([11]\), will be used.

**Notation 2.3.** If \( \mathcal{X} \) and \( \mathcal{Y} \) are operator systems, then \( \mathcal{X} \otimes_c \mathcal{Y} \) shall denote the norm-completion of \( \mathcal{X} \otimes \mathcal{Y} \). For any subspaces \( \mathcal{X}_0 \subset \mathcal{X} \) and \( \mathcal{Y}_0 \subset \mathcal{Y} \), \( \mathcal{X}_0 \overline{\otimes} \mathcal{Y}_0 \) denotes the closure of \( \mathcal{X}_0 \otimes \mathcal{Y}_0 \) in \( \mathcal{X} \otimes \mathcal{Y} \).

The symbol \( \otimes_{\text{max}} \) is reserved in this paper (unlike in \([10, 11]\)) for the maximal C*-algebra tensor product. An important fact: if two unital C*-algebras \( A \) and \( B \) are considered as operator systems, then \( A \otimes_c B = A \otimes_{\text{max}} B \) \([10, \text{Theorem 6.6}]\).

In principle an abstract operator system \( S \) generates many different C*-algebras. The largest such C*-algebra is called the universal C*-algebra generated by \( S \). That is, a unital C*-algebra \( A \) is universal for \( S \) if:

- (1) there is a unital complete order injection \( \iota_u : S \to A \),
- (2) \( A \) is generated by \( \iota_u(S) \), and
- (3) if \( \phi : S \to B \) is a ucp map into another C*-algebra \( B \), then there is a homomorphism \( \pi : A \to B \) such that \( \phi = \pi \circ \iota_u \).
It was shown in [13, Proposition 8] that every operator system has a universal C*-algebra, unique up to isomorphism, and an explicit construction was given. Therefore, $C_u^*(S)$ shall unambiguously denote the universal C*-algebra generated by $S$.

**Theorem 2.4.** ([11, Lemma 2.5]) For all operator systems $S$ and $T$, $$S \otimes c T \subset_{coi} S \otimes c C_u^*(T) \subset_{coi} C_u^*(S) \otimes_{\text{max}} C_u^*(T).$$

**Corollary 2.5.** For every unital C*-algebra $A$, operator system $S$, and $n \in \mathbb{N}$, the operator systems $M_n(S \otimes_c A)$ and $S \otimes_c M_n(A)$ are completely order isomorphic.

### 3. Preliminary Results

In this section we will use the fact that the matricial order on an operator system $S$ gives rise to a norm $\| \cdot \|_{M_n(S)}$ on each matrix space $M_n(S)$ [10, Chapter 3].

**Lemma 3.1.** Let $S$ be an operator system and $A$ be a unital C*-algebra. A linear map $\phi : S \otimes_c A \to B(H)$ is a ucp map if and only if there is a Hilbert space $K$, homomorphisms $\pi : C_u^*(S) \to B(K)$ and $\rho : A \to B(K)$ with commuting ranges, and an isometry $V : H \to K$ such that $\phi(s \otimes a) = V^* \pi(s) \rho(a) V$ for all $s \in S$ and $a \in A$.

**Proof.** Because $S \otimes_c A \subset_{coi} C_u^*(S) \otimes_{\text{max}} A$ by Proposition 2.4, $\phi$ admits a ucp extension $\Psi : C_u^*(S) \otimes_{\text{max}} A \to B(H)$. Therefore, by [10, Corollary 6.5], the restriction of $\Psi$ to $S \otimes_c A$ has the structure indicated in the statement of the lemma. \qed

**Lemma 3.2.** Let $S$ be a operator system. Let $\{S_i\}_{i \in I}$ be the set of all separable nontrivial operator subsystems of $S$ (that is, $S_i \subset S$). Then, there is a non-trivial ultrafilter $U$ on $I$ such that the map $\Psi : S \to \prod^U C_u^*(S_i)$ given by

$$x \mapsto \{\psi_i(x)\}_{\mu U},$$

where $\psi_i(x) = x$ if $x \in S_i$ or 0 otherwise, is a unital completely positive linear map, where $\prod^U$ denotes the C*-ultraproduct.

**Proof.** Note that the set $I$ is partially ordered by inclusion of the corresponding operator subsystems $S_i$ and $S = \bigcup S_i$. Consider a cofinal ultrafilter $U$ on the directed set $I$. The map $\Psi$ defined in the statement of the lemma is linear because of the structure of C*-ultraproducts (see [7]). To show that $\Psi$ is ucp it is sufficient to show that $\Psi$ is a complete isometry (following the discussion after 17, Remark 2.8.4)).

If $x \in S$, note that the set $\{i \mid x \in S_i\} \in U$. To see this, simply observe that $\{i \mid x \in S_i\} = \{i \mid i \geq i_x\}$, where $S_{i_x} = \text{span}\{1, x, x^*\}$. Now, for $n = 1$,

$$\|\Psi(x)\| = \|\psi_i(x)\|_U = \lim_{U} \|\psi_i(x)\| = \|x\|$$

by the preceding comment.

For $n > 1$, we use a similar argument as follows. Let $X = (x_{kl}) \in M_n(S)$. Now, an ultrafilter is closed under finite intersections. So,

$$I_X = \{i \mid x_{kl} \in S_i \forall k,l\} = \bigcap_{k,l} \{i \mid x_{kl} \in S_i\}$$
is in $\mathcal{U}$. Finally, using the identification $M_n(\prod^\mathcal{U} C_u^*(S_i)) = \prod^\mathcal{U} M_n(C_u^*(S_i))$ (see Remark on Pg-60 of [17]) we obtain
\[
\|\Psi(x)\| = \|\Psi(x_{kl})\|_{k,l} = \|\psi_i(x_{kl})\|_{k,l} = \|\psi_i(x_{kl})\|_{k,l} = \lim_{\mathcal{U}} \|\psi_i(x_{kl})\|_{k,l} = \|\psi_i(x_{kl})\|_{M_n(S_i), i \in I_X} = \|x\|,
\]
thereby showing that $\Psi$ is a complete isometry. □

The following result is of central importance in what follows.

**Lemma 3.3.** Assume that $\mathcal{A}$ is a $C^*$-algebra and $\mathcal{T}$ is an operator system, and fix $x \in \mathcal{T} \otimes \mathcal{A}$. If $\{\mathcal{T}_i\}_{i \in I(x)}$ is the directed set of all separable unital operator subsystems of $\mathcal{T}$ for which $x \in \mathcal{T}_i \otimes \mathcal{A}$, then
\[
\|x\|_{\mathcal{T}_i \otimes \mathcal{A}} = \lim_{\mathcal{I}(x)} \|x\|_{\mathcal{T}_i \otimes \mathcal{A}}.
\]

**Proof.** Let us denote by $\|x\|_{(\cdot)}$ the norm $\|x\|_{(\cdot)\otimes \mathcal{A}}$. If $x \in \mathcal{T}_1 \subset \mathcal{T}_2$, then
\[
\mathcal{T}_1 \otimes \mathcal{A} \subset \mathcal{T}_2 \otimes \mathcal{A} \quad \text{implies that} \quad \|x\|_{\mathcal{T}_2} \leq \|x\|_{\mathcal{T}_1}.
\]
Thus, $\lim_{\mathcal{I}} \|x\|_{\mathcal{T}_i}$ exists, since it is a decreasing net, and
\[
\|x\|_{\mathcal{T}} \leq \lim_{\mathcal{I}} \|x\|_{\mathcal{T}_i}.
\]
To establish the opposite inequality, following the techniques in the proof of [13 Proposition 3.4], we proceed as follows.

Assume that $\|x\|_{\mathcal{T}_i} \geq 1$ for all $i \in \mathcal{I}$. Thus, $\|x\|_{\mathcal{T}_i} = \|x\|_{C_u^*(\mathcal{T}_i) \otimes_{\max} \mathcal{A}} \geq 1$. Therefore, there exists representations $\pi_i, \rho_i$ of $C_u^*(\mathcal{T}_i)$ and $\mathcal{A}$ respectively, on $\mathcal{B}(\mathcal{H}_i)$ with commuting ranges such that
\[
\|\pi_i \cdot \rho_i(x)\| \geq 1.
\]
Using the map $\Psi$ from Lemma 3.2 above and the injective $*$-homomorphism $\iota : \mathcal{A} \hookrightarrow \prod^\mathcal{U} \mathcal{A}$, where $\mathcal{U}$ is the same ultrafilter over the same index set $\mathcal{I}$ as in Lemma 3.2 or above, we have ucp maps $\phi : \mathcal{T} \to \mathcal{B}(\mathcal{H}_\mathcal{T})$ and $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\mathcal{T})$ with commuting ranges and such that
\[
\|\phi \cdot \rho(x)\| \geq 1,
\]
where $\mathcal{H}_\mathcal{T} = \prod^\mathcal{U} \mathcal{H}_i, \phi = (\prod^\mathcal{U} \pi_i) \circ \Psi$ and $\rho = (\prod^\mathcal{U} \rho_i) \circ \iota$.

Now, $\phi \cdot \rho$ is a ucp map of $\mathcal{T} \otimes \mathcal{A}$. By Lemma 3.1, there exist representations $\pi_0$ and $\rho_0$ of $C_u^*(\mathcal{T})$ and $\mathcal{A}$ with commuting ranges and an isometry $V$ such that
\[
\phi \cdot \rho(x) = V^* \pi_0 \cdot \rho_0(x) V.
\]
Since $\|\phi \cdot \rho(x)\| \geq 1$, we have $\|\pi_0 \cdot \rho_0(x)\| \geq 1$ because $V$ is an isometry. But then
\[
\|x\|_{\mathcal{T}} = \|x\|_{C_u^*(\mathcal{T}) \otimes_{\max} \mathcal{A}} \geq 1,
\]
thereby showing that $\|x\|_{\mathcal{T}} = \lim_{\mathcal{I}} \|x\|_{\mathcal{T}_i}$. □

**Remark 3.4.** Lemma 3.3 is also true if $\mathcal{A}$ is only an operator system, as in that case, one may simply carry out the argument above with $C_u^*(\mathcal{A})$ and arrive at the conclusion by virtue of Proposition 2.4.
4. Main Results

Recall that a pair \((S, T)\) of operator systems is a relatively weakly injective pair if, for every operator system \(R\),
\[
S \otimes_c R \subset_{\text{coi}} T \otimes_c R.
\]
It is also convenient to say that \(T\) is relatively weakly injective in \(T\) if \((S, T)\) is relatively weakly injective pair.

The first main result is an operator system version of Kirchberg’s theorem [12, Proposition 3.1].

**Theorem 4.1.** The following statements are equivalent for operator systems \(S\) and \(T\) for which \(S \subset T\):

1. \((S, T)\) is a relatively weakly injective pair of operator systems;
2. \(S \otimes_c C^*(\mathcal{F}_\infty) \subset_{\text{coi}} T \otimes_c C^*(\mathcal{F}_\infty)\);
3. For any ucp map \(\phi : S \to \mathcal{B}(\mathcal{H})\), there exist a ucp map \(\Phi : T \to \phi(S)'\) such that \(\Phi|_S = \phi\);
4. \((C_u^*(S), C_u^*(T))\) is a relatively weakly injective pair of C*-algebras.

**Proof.** The order of implications to be proved is (4) \(\Rightarrow\) (2) \(\Rightarrow\) (1) \(\Rightarrow\) (3) \(\Rightarrow\) (4).

(4) \(\Rightarrow\) (2). Assume that \(X \in M_n(S \otimes C^*(\mathcal{F}_\infty))\) is positive in \(M_n(T \otimes_c C^*(\mathcal{F}_\infty))\). We need to show that \(X \in M_n(S \otimes C^*(\mathcal{F}_\infty))_{+}\). Because \(X \in M_n(T \otimes_c C^*(\mathcal{F}_\infty))_{+} \subset M_n(C_u^*(T) \otimes_{\text{max}} C^*(\mathcal{F}_\infty))_{+}\), hypothesis (4) implies \(X \in M_n(C_u^*(S) \otimes_{\text{max}} C^*(\mathcal{F}_\infty))_{+}\), and so \(X\) is positive in \(M_n(S \otimes_c C^*(\mathcal{F}_\infty))\) because \(S \otimes_c C^*(\mathcal{F}_\infty) \subset_{\text{coi}} C_u^*(S) \otimes_c C^*(\mathcal{F}_\infty)\).

(2) \(\Rightarrow\) (1). Let \(R\) be an arbitrary operator system. By Theorem 2.4, \(W \otimes_c R \subset_{\text{coi}} W \otimes_c C_u^*(R)\) for every operator system \(W\); thus, if we can show that \(S \otimes_c C_u^*(R) \subset_{\text{coi}} T \otimes_c C_u^*(R)\), then we deduce immediately that \(S \otimes_c R \subset_{\text{coi}} T \otimes_c R\).

To begin, assume that \(R\) is separable. Hence, there is an ideal \(K\) of \(C^*(\mathcal{F}_\infty)\) such that \(C_u^*(R) = C^*(\mathcal{F}_\infty)/K\). By [11] Corollary 5.17, and using Notation 2.3,
\[
S \otimes_c C_u^*(R) \subset_{\text{coi}} S \hat{\otimes}_c C_u^*(R) = \frac{S \hat{\otimes}_c C^*(\mathcal{F}_\infty)}{S \hat{\otimes}_c K}.
\]

The hypothesis \(S \otimes_c C^*(\mathcal{F}_\infty) \subset_{\text{coi}} T \otimes_c C^*(\mathcal{F}_\infty)\) implies that \(S \otimes_c C^*(\mathcal{F}_\infty) \subset_{\text{coi}} C_u^*(T) \otimes_c C^*(\mathcal{F}_\infty)\), again by Theorem 2.4. Therefore, [11] Proposition 5.14 yields
\[
\frac{S \hat{\otimes}_c C^*(\mathcal{F}_\infty)}{S \hat{\otimes}_c K} \subset_{\text{coi}} \frac{C_u^*(T) \hat{\otimes}_c C^*(\mathcal{F}_\infty)}{C_u^*(T) \hat{\otimes}_c K} = C_u^*(T) \otimes_{\text{max}} C_u^*(R).
\]

Thus, \(S \otimes_c C_u^*(R) \subset_{\text{coi}} C_u^*(T) \otimes_c C_u^*(R)\), which implies \(S \otimes_c C_u^*(R) \subset_{\text{coi}} T \otimes_c C_u^*(R)\) and, hence, \(S \otimes_c R \subset_{\text{coi}} T \otimes_c R\).

Now assume that \(R\) is an arbitrary nonseparable operator system. We have proved above that \(S \otimes_c R_0 \subset_{\text{coi}} T \otimes_c R_0\) for every separable operator system \(R_0\). Fix \(x \in S \otimes R\) and choose a separable operator subsystem \(R_1 \subset R\) such that \(x \in S \otimes R_1\). Thus, \(S \otimes_c R_1 \subset T \otimes_c R_1\). By the beginning of the proof of Lemma 3.3, we have the inequality
\[
\|x\|_{S \otimes_c R} \leq \|x\|_{S \otimes_c R_1} = \|x\|_{T \otimes_c R_1}.
\]
This inequality above holds for any separable operator subsystem \(R_1 \subset R\) for which \(x \in S \otimes R_1\). Lemma 3.3 (or Remark 3.4) thus implies \(\|x\|_{S \otimes_c R} \leq \|x\|_{T \otimes_c R}\), which
in turn implies
\[ \|x\|_{S \otimes_c R} = \|x\|_{T \otimes_c R}. \]

Next, for \( n > 1 \), fix \( X \in M_n(S \otimes R) \subset M_n(S \otimes C_u^*(R)) \cong S \otimes M_n(C_u^*(R)) \). One also has \( M_n(S \otimes_c C_u^*(R)) \cong S \otimes_c M_n(C_u^*(R)) \). Now, just as in the \( n = 1 \) case, there exists a separable operator system \( R_n^0 \subset M_n(C_u^*(R)) \) such that \( X \in S \otimes R_n^0 \) and therefore, for any separable operator system \( R_n \subset M_n(C_u^*(R)) \) for which \( X \in S \otimes R_n \), we have the inequality
\[ \|X\|_{M_n(S \otimes_c C_u^*(R))} = \|X\|_{S \otimes_c M_n(C_u^*(R))} \leq \|X\|_{T \otimes_c R_n}. \]

This implies (as in case of \( n = 1 \)) that
\[ \|X\|_{M_n(S \otimes_c C_u^*(R))} \leq \|X\|_{T \otimes_c M_n(C_u^*(R))}, \]
which in turn implies that \( \|X\|_{M_n(S \otimes_c C_u^*(R))} = \|X\|_{M_n(T \otimes_c C_u^*(R))} \). That is, the inclusion map \( S \otimes R \to T \otimes R \) is a unital complete isometry \( S \otimes_c R \to T \otimes_c R \) and, hence, is a complete order injection.

(1) \( \Rightarrow \) (3). Let \( \phi : S \to B(H) \) be a ucp map. Since \((S, T)\) is a relatively weakly injective pair, and because the commutant \( \phi(S)' \subset B(H) \) of \( \phi(S) \) is a C*-algebra,
\[ S \otimes_c \phi(S)' \subset \text{coi } T \otimes_c \phi(S)' \subset \text{coi } C_u^*(T) \otimes_{\max} \phi(S)'. \]

By the definition of commuting tensor product, \( \phi \cdot \text{id}_{\phi(S)'} \) is a ucp map on \( S \otimes_c \phi(S)' \) with values in \( B(H) \). Take an Arveson extension \( \Psi \) of \( \phi \cdot \text{id}_{\phi(S)'} \) to \( C_u^*(T) \otimes_{\max} \phi(S)' \) and define a ucp map \( \Phi \) on \( T \) by
\[ \Phi(t) = \Psi(t \otimes 1), \]
for all \( t \in T \). Obviously, \( \Phi|_S = \phi \). Finally, to see that \( \Phi \) takes values in \( \phi(S)'' \), one invokes the usual multiplicative domain argument for completely positive maps.

This concludes our claim (1) \( \Rightarrow \) (3).

(3) \( \Rightarrow \) (4). Since \( S \subset T \), \( C_u^*(S) \) is a unital C*-subalgebra of \( C_u^*(T) \) [13 Proposition 9]. Let \( \pi_U : C_u^*(S) \to B(H_U) \) be the universal representation of \( C_u^*(S) \). Then \( \pi_U|_S : S \to B(H_U) \) is a ucp map. By hypothesis, \( \pi_U|_S \) extends to \( \phi : T \to (\pi_U|_S(S))'' \subset (\pi_U(C_u^*(S))'' \). Now, since \( C_u^*(T) \) is generated as an algebra by \( T \), the unique homomorphism from \( C_u^*(T) \) extending \( \phi \) takes values in \( (\pi_U(C_u^*(S)))'' \). Further, since this homomorphism extends \( \pi_U|_S \), it fixes \( \pi_U \), which completes the proof.

The second main result shows the abundant existence of pairs of relatively weakly injective operator systems and is a generalisation of [12] Lemma 3.4.

**Theorem 4.2.** If \( S \) is a separable operator subsystem of an operator system \( T \), then there exists a separable operator system \( R \) such that \( S \subset \text{coi } R \subset \text{coi } T \) and \( R \) is relatively weakly injective in \( T \).

**Proof.** Let \( \{s_k\}_{k \in \mathbb{N}} \) be a dense sequence in \( S \otimes_c C^*(F_\infty) \). Using Lemma [9] we choose separable operator subsystems \( S_n \) of \( T \) such that, \( S \subset S_1 \subset S_2 \subset \ldots \) and \( \|s_k\|_{S_n} \leq \|s_k\|_T + \frac{1}{n} \) for \( 1 \leq k \leq n \). Let \( S^{(1)} = \bigcup S_n \). Then \( S^{(1)} \) is a separable operator system containing \( S \), such that, for all \( x \in S \otimes_c C^*(F_\infty) \), one has \( \|x\|_{S^{(1)}} = \|x\|_T \). By iterating the argument above with \( S^{(1)} \) instead of \( S \) we obtain a sequence of separable operator systems \( S \subset S^{(1)} \subset S^{(2)} \subset \ldots \) such that \( \| \cdot \|_{S^{(n)}} = \| \cdot \|_T \) on \( (S^{(n-1)} \otimes_c C^*(F_\infty)) \). Define \( X_1 = \bigcup S^{(n)} \). Thus, \( X_1 \) is a separable operator system containing \( S \) such that \( \| \cdot \|_{X_1} = \| \cdot \|_T \).
Replacing $X_1$ for $S$ and $M_2(C^*(F_\infty))$ for $C^*(F_\infty)$, repeat the procedure described above to obtain a separable operator system $X_2$ such that, for all $x \in X_2 \otimes_m M_2(C^*(F_\infty))$, we have

$$\|x\|_{X_2 \otimes_m M_2(C^*(F_\infty))} = \|x\|_{T \otimes_m M_2(C^*(F_\infty))}.$$  

In other words, using the identification $W \otimes_e M_2(C^*(F_\infty)) = M_2(W \otimes_e C^*(F_\infty))$ for operator systems $W$, we have that the inclusion map $X_2 \otimes_e C^*(F_\infty) \to T \otimes_e C^*(F_\infty)$ is a 2-isometry.

Further iterations of the procedure above gives us $S \subset X_1 \subset X_2 \subset X_3 \subset \ldots \subset T$ such that the inclusion map $X_k \otimes_e C^*(F_\infty) \to T \otimes_e C^*(F_\infty)$ is a $k$-isometry.

Finally, set $R = \bigcup X_k$. To show that $R$ is relatively weakly injective in $T$, it is enough, by Theorem 4.1 to show that the inclusion map $R \otimes_e C^*(F_\infty) \to T \otimes_e C^*(F_\infty)$ is a complete isometry.

For $Y \in R \otimes M_n(C^*(F_\infty))$ there exists an integer $k_Y > n$ such that $Y \in X_k \otimes M_n(C^*(F_\infty))$ for all $k > k_Y$. Now recall the fact that the inclusion maps $X_k \otimes_e C^*(F_\infty) \to T \otimes_e C^*(F_\infty)$ are $k$-isometries. As a consequence, for $n < k_Y < k$ the inclusions $X_k \otimes C^*(F_\infty) \to T \otimes C^*(F_\infty)$ are also $n$-isometries. Therefore, by Lemma 5.3 we have

$$\|Y\|_{R \otimes_e M_n(C^*(F_\infty))} = \lim_{k} \|Y\|_{X_k \otimes M_n(C^*(F_\infty))} = \lim_{k > k_Y} \|Y\|_{X_k \otimes M_n(C^*(F_\infty))} = \|Y\|_{T \otimes M_n(C^*(F_\infty))}.$$  

This shows that $R$ is relatively weakly injective in $T$, contains $S$, and is separable, thereby concluding the proof. 

5. Remarks on relative weak injectivity with respect to the operator system maximum tensor product

The maximal $C^*$-tensor product has two distinct generalizations in the $O_1$ category, namely the commuting tensor product and the operator system maximal tensor product. See [10, 11] for details. This article focuses on relative weak injectivity with respect to the former. A natural question would be to seek characterizations of relatively weakly injective operator system pairs with respect to the operator system maximal tensor product. Let us denote the operator system maximal tensor product by $\otimes_m$.

**Proposition 5.1.** Let $S \subset_{coi} T$. The following statements are equivalent:

1. For any operator system $R$, $S \otimes_m R \subset_{coi} T \otimes_m R$.
2. There exists a ucp map $\Phi : T \to S^{**}$, such that $\Phi(s) = s$ for all $s \in S$.

**Proof.** (1) $\Rightarrow$ (2). Consider the bidual inclusion $S^{**} \subset_{coi} T^{**} \subset_{coi} B(H)$, where the second inclusion is weak*-WOT homeomorphic exactly as in the proof of [8, Theorem 4.1]. Repeating the proof of [8, Theorem 4.1 (iii) $\Rightarrow$ (iv)] verbatim gives the required result.

(2) $\Rightarrow$ (1). For $X \in M_n(T \otimes_m R)^+ \cap M_n(S \otimes R)$, one has $X = (\Phi \otimes id)^{(n)}(X) \in M_n(S \otimes_m R) \subset_{coi} M_n(S^{**} \otimes_m R)$, where the last inclusion is due to [11, Lemma 6.5].

**Remark 5.2.** Comparing Proposition 5.1 and Theorem 4.1 it is unlikely that a universal characterisation of the likes of Theorem 4.1(2) exists in the $\otimes_m$ case. As
a consequence, it cannot be ascertained that an existence result similar to Theorem 4.2 holds for the maximal operator system tensor product.

6. Examples

6.1. Operator systems generated by free unitaries. Denote the generators of the free group $F_\infty$ by $\{u_j\}_{j \in \mathbb{N}}$. In $C^*(F_\infty)$, each $u_j$ is a unitary and so, for each $n \in \mathbb{N}$, define

$$S_n = \text{span}\{u_{-n}, \ldots, u_{-1}, 1, u_1, \ldots, u_n\},$$

which is an operator subsystem of $C^*(F_n)$.

Example 6.1. For $n \in \mathbb{N}$, the pair $(S_n, C^*(F_n))$ is a relatively weakly injective pair of operator systems.

The proof of this assertion is adapted from the proof of [5, Lemma 4.1] and makes use of our main result, Theorem 4.1. Let $\tilde{T} : S_n \to B(\mathcal{H})$ be a ucp map. By Theorem 4.1, it is enough to show that $\tilde{T}$ extends to $C^*(F_n)$, taking values in $\tilde{y}(S_n)^\prime\prime$. For each contraction $T(u_i)$, $1 \leq i \leq n$, consider its Halmos unitary dilation $\tilde{T}_i$ on $\mathcal{H} \oplus \mathcal{H}$ given by

$$W_i = \begin{bmatrix} \phi(u_i) & (1 - \phi(u_{-1})\phi(u_i))^{\frac{1}{2}} & (1 - \phi(u_{-1})\phi(u_i))^{\frac{1}{2}} \\ 0 & \phi(u_{-1}) & -\phi(u_{-1}) \end{bmatrix}.$$

Let $T \in \tilde{y}(S_n)'$ and consider the operator $\tilde{T} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$. Now, by functional calculus, $\tilde{T}$ commutes with $W_i$ for all $1 \leq i \leq n$. Since $u_1, \ldots, u_n$ are universal unitaries in $C^*(F_n)$, there is a unique homomorphism $\pi : C^*(F_n) \to B(\mathcal{H} \oplus \mathcal{H})$, such that $\pi(u_i) = W_i$ for $1 \leq i \leq n$. Let $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Define ucp map $\tilde{P} : C^*(F_n) \to B(\mathcal{H})$ by $\tilde{P}(\cdot) = P\pi(\cdot)|_\mathcal{H}$. Note that, $\tilde{P}$ extends $\phi$ and $\tilde{T}$ commutes with $P$. Since, $\tilde{T}$ commutes with every $W_i$, it commutes with $\pi(C^*(F_n))$. Thus, for $x \in C^*(F_n)$ we have

$$\tilde{P}(x)T = P\pi(x)P\tilde{T}P = P\pi(x)\tilde{T}P = P\tilde{T}\pi(x)P = P\tilde{T}\pi(x)P = T\tilde{P}(x).$$

So, $\tilde{P}(x) \in \phi(S_n)^\prime\prime$ as $T$ was chosen arbitrarily in $\phi(S_n)'$. This concludes our claim.

6.2. Operator systems generated from universal relations. Let

$$\mathcal{G} = \{h_1, \ldots, h_n\} \text{ and } \mathcal{R} = \{h_j^* = h_j, \|h_j\| \leq 1, 1 \leq j \leq n\},$$

be a set of relations in the set $\mathcal{G}$, and let $C^*(\mathcal{G} | \mathcal{R})$ denote the universal unital $C^*$-algebra generated by $\mathcal{G}$ subject to $\mathcal{R}$. The operator system

$$NC(n) = \text{span}\{1, h_1, \ldots, h_n\} \subset C^*(\mathcal{G} | \mathcal{R}).$$

is called the operator system of the non-commuting $n$-cube.

It was shown in [6] that the $C^*$-envelope of $NC(n)$ is $C^*(*_n \mathbb{Z}_2)$, where $*_n \mathbb{Z}_2$ is the free product of $n$-copies of $\mathbb{Z}_2$. The following example is from [6, Lemma 6.2] and can be proved exactly along the lines of the previous example.

Example 6.2. For $n \in \mathbb{N}$, the pair $(NC(n), C^*(*_n \mathbb{Z}_2))$ is a relatively weakly injective pair of operator systems.
6.3. **Inclusion in the double dual.** The dual $S^*$ of an operator system is a matricially normed space, but the double dual $S^{**}$ is an operator system containing $S$ as an operator subsystem \[11\]. The following example is established in \[11\] Corollary 6.6.

**Example 6.3.** $(S, S^{**})$ is a relatively weakly injective pair of operator systems, for every operator system $S$.

6.4. **Operator systems with DCEP.** An operator system $S$ is said to have the **double commutant expectation property** (DCEP) if, for every complete order embedding $S \to B(\mathcal{H})$, there exists a completely positive linear map $\Phi : B(\mathcal{H}) \to S'' \subset B(\mathcal{H})$, fixing $S$.

**Example 6.4.** If $S$ has the double commutant expectation property, then $(S, T)$ is a relatively weakly injective pair of operator systems, for every operator system $T$ that contains $S$ as an operator subsystem.

This assertion above is a consequence of \[11\] Theorem 7.3, Theorem 7.1, which states that if $S \subset T$ and $S$ has the double commutant expectation property, then $S \otimes_c R \subset_{ca} T \otimes_c R$ for every operator system $R$.

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