Excitonic condensation for the surface states of topological insulator bilayers

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Abstract. We propose a generic topological insulator bilayer (TIB) system to study the excitonic condensation with self-consistent mean-field (SCMF) theory. We show that the TIB system presents the crossover behavior from the Bardeen–Cooper–Schrieffer (BCS) limit to the Bose–Einstein condensation (BEC) limit. Moreover, in comparison with traditional semiconductor systems, we find that for the present system the superfluid property in the BEC phase is more sensitive to electron–hole density imbalance and the BCS phase is more robust. Applying this TIB model to the Bi\(_2\)Se\(_3\)-family material, we find that the BEC phase is most likely to be observed in experiment. We also calculate the critical temperature for the Bi\(_2\)Se\(_3\)-family TIB system, which is \(\sim 100\) K. More interestingly, one can expect this relative high-temperature excitonic condensation, since our calculated SCMF critical temperature is approximately equal to the Kosterlitz–Thouless transition temperature.
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1. Introduction

Recent technological advances in microfabrication have brought about growing interest in the study of exciton condensation in different bilayer physical systems such as semiconductor electron–hole bilayers [$1$–$3$] and graphene bilayers [$4$, $5$]. A number of novel physical phenomena are obtained in these systems, such as the Bardeen–Cooper–Schrieffer (BCS)–Bose–Einstein condensation (BEC) crossover [$6$] as well as the subtle phase transition in the crossover region induced by the density imbalance [$7$], the dark and bright excitonic condensation under spin–orbit coupling [$8$], anomalous exciton condensation in high Landau levels in magnetic field [$5$], room-temperature superfluidity in graphene bilayers [$9$], etc. Conventional electron–hole bilayers are fabricated with semiconductor heterostructures such as GaAs/AlGaAs/GaAs. The character of the semiconductor electron–hole bilayers is that the electron and hole bands are quadratic with different effective masses, which means missing particle–hole symmetry in these kinds of systems and small superfluid density. Hence, in semiconductor electron–hole bilayers, the excitonic condensation needs very low temperature. Another good candidate for electron–hole bilayers is graphene, which has a two-dimensional (2D) massless linear Dirac-band structure in the low-energy limit. However, the coupling between different Dirac-cone structures in the same Brillouin zone brings flaws in graphene to fabricate electron–hole bilayers [$10$].

On the other hand, another topic of growing interest in condensed matter physics is the very recent theoretical prediction [$11$] and experimental verification [$12$] of topological insulators (TIs) [$13$] with strong spin–orbit interaction. Several 3D solids, such as Bi$_{1-x}$Sb$_x$ alloys and Bi$_2$Se$_3$-family crystals, have been identified [$14$–$18$] to be strong TIs possessing anomalous band structures. The energy scale for the surface states of these 3D TIs is dominated by the $k$-linear spin–orbit interaction. In particular, the strong TI surface has single Dirac-cone band structure, which is also different from graphene. As a result, it is expected that the excitonic condensate of these topological surface states probably has new characters.

Inspired by this expectation, in this paper we propose a topological insulator bilayer (TIB) model analogous to [$19$], gated double TI layers separated by an insulating spacer. Using this TIB model, we numerically study the excitonic condensation of TI surface states. We find that the system also presents BCS–BEC crossover along with the change in carrier densities in the zero-temperature limit. However, there are two characters that differ from those of conventional excitonic condensation in semiconductor bilayer systems. The first is that the BCS phase of TIB is more robust than that of the semiconductor bilayer systems; the second is that the superfluidity of the TIB is more sensitive to the electron–hole density imbalance than that of the semiconductor bilayer systems. These two characters are physically rooted in the $k$-linear band...
dispersion of the TIB. Moreover, by putting this TIB model in the Bi$_2$Se$_3$-family material, we investigate the excitonic condensation and only find the BEC phase occurring due to the values of the parameters of the material. The critical temperature of excitonic condensation in Bi$_2$Se$_3$-family TIB is also calculated in the self-consistent mean-field (SCMF) approximation ($\sim$ 100 K), which is found to be higher than that in the traditional semiconductor electron–hole bilayers. More interestingly, we can expect this relative high-temperature excitonic condensation since our calculated SCMF critical temperature is approximately equal to the Kosterlitz–Thouless (KT) transition temperature.

2. The topological insulator bilayer model

The TIB system is schematically represented in the left panel of figure 1. Two TI films are separated by an insulating spacer of thickness $d$, and the electron (hole) density can be independently tuned by the external gate voltage $V_1$ ($V_2$). The linear dispersions of the TIs around the Dirac point are depicted in cartoon representation in the right panel of figure 1. The grand-canonical Hamiltonian describing this TIB system can be written as

\[
H = - \sum_{p,k,\sigma} \mu_p \hat{p}_{k\sigma}^\dagger \hat{p}_{k\sigma} + \sum_{p,k} \hbar v_F^p (k_x - ik_y) \hat{p}_{k\uparrow}^\dagger \hat{p}_{k\downarrow} + h.c.
+ \frac{1}{2\Omega} \sum_{p,p',k,k',q,\sigma,\sigma'} V_{qp'}^{pp'} \hat{p}_{k+q\sigma}^\dagger \hat{p}_{k'-q\sigma'}^\dagger \hat{p}_{k'\sigma'} \hat{p}_{k\sigma}.\tag{1}
\]

Here, $k$, $k'$ and $q$ are 2D wave vectors in the layers and $\Omega$ is the quantization volume. $\mu_p$ is the chemical potential for the electron layer ($p = e$) or hole layer ($p = h$). $\hat{p}_{k\sigma}$ indicates the annihilation operator of the electron at the wave vector $k$ and spin $\sigma (= \uparrow, \downarrow)$ for the electron layer ($p = e$) and hole layer ($p = h$). Note that $v_F^e = v_F$ and $v_F^h = -v_F$. The surface states of the strong TI film have the linear dispersion: $\epsilon_{ke,h} = \pm \hbar v_F |k|$. $V_{qp'}^{pp'}$ is the Fourier transform of
the Coulomb interaction: the intralayer Coulomb repulsive interaction $V_{qe}^{ee} (V_{qh}^{hh}) = 2\pi e^2/(q\varepsilon)$ and the interlayer Coulomb attractive interaction \cite{20, 21} $V_{q}^{eh} = -2\pi e^2 \exp(-qd)/(q\varepsilon)$, which indicates that on the one hand, in the limit of $d \to 0$, the interaction between the electron and the hole becomes that in a monolayer; on the other, in the large thickness limit $d \to \infty$, the interactions between the electrons in the upper layer and holes in the lower layer should vanish.

Here, $\varepsilon$ is the background dielectric constant. Furthermore, for the present TIB system, the two TI films are separated by an insulating spacer such as $\text{SiO}_2$, and the spin–orbit interaction in the spacer is obviously negligible. Thus it can be expected that our model is appropriate in neglecting the interlayer hopping coupling.

In the basis $(\hat{e}_1, \hat{e}_2, \hat{h}_1, \hat{h}_2)^T$, the Hamiltonian (1) can be decoupled to $H_{\text{MF}}$ under the mean-field approximation: $\Delta_{\sigma\sigma'}(\mathbf{k}) = \sum_{q} V_{q}^{eh}(\hat{e}_{k+q,\sigma}^\dagger \hat{h}_{k+q,\sigma'})$, $\Sigma_{\sigma\sigma'}^{(p)}(\mathbf{k}) = -\sum_{q} V_{q}^{pp}(\mathbf{q}) \times \langle \hat{P}_{k+q,\sigma} \hat{P}_{k,\sigma'} \rangle$. Then, $H_{\text{MF}}$ can be diagonalized with a $4 \times 4$ unitary matrix $U(\mathbf{k})$, $U^\dagger(\mathbf{k}) H_{\text{MF}}(\mathbf{k}) U(\mathbf{k}) = \text{diag}(E_1(\mathbf{k}), E_2(\mathbf{k}), E_3(\mathbf{k}), E_4(\mathbf{k}))$. The unitary matrix $U(\mathbf{k})$ is constructed by the normalized eigenfunctions of the Hamiltonian (1), which can be numerically calculated by diagonalizing the Hamiltonian matrix (1) in the basis $(\hat{e}, \hat{e}, \hat{h}, \hat{h})^T$. Explicitly, the elements $U_{ij}(\mathbf{k})$ denote the $i$th component of the eigenfunction corresponding to the eigenvalue $E_j$. The relevant mean-field equations to be solved for the variables $\mu_e$, $\mu_h$ and the gap functions $\Delta_{\sigma\sigma'}(\mathbf{k})$ and self-energies $\Sigma_{\sigma\sigma'}^{(p)}(\mathbf{k})$ are

$$\Delta_{ji}(\mathbf{k}) = -\frac{1}{\Omega} \sum_{i=1}^{4} \sum_{q} V_{q}^{eh}(\mathbf{k} + \mathbf{q}) U_{ji}(\mathbf{k} + \mathbf{q}) f(E_i(\mathbf{k} + \mathbf{q})), \quad (2)$$

$$\Sigma_{ji}^{(e)}(\mathbf{k}) = \frac{1}{\Omega} \sum_{i=1}^{4} \sum_{q} V_{q}^{ee}(\mathbf{k} + \mathbf{q}) U_{ji}(\mathbf{k} + \mathbf{q}) f(E_i(\mathbf{k} + \mathbf{q})), \quad (3)$$

$$\Sigma_{ji}^{(h)}(\mathbf{k}) = \frac{1}{\Omega} \sum_{i=1}^{4} \sum_{q} V_{q}^{hh}(\mathbf{k} + \mathbf{q}) U_{ji}(\mathbf{k} + \mathbf{q}) f(E_i(\mathbf{k} + \mathbf{q})), \quad (4)$$

$$n_e = \frac{1}{\Omega} \sum_{i=1}^{2} \sum_{j=2}^{3} \sum_{\mathbf{k}} |U_{ij}(\mathbf{k})|^2 f(E_j(\mathbf{k})), \quad (5)$$

$$n_h = \frac{1}{\Omega} \sum_{i=3}^{4} \sum_{j=2}^{3} \sum_{\mathbf{k}} \left[ 1 - |U_{ij}(\mathbf{k})|^2 f(E_j(\mathbf{k})) \right], \quad (6)$$

where $f(E_i(\mathbf{k})) = 1/(1 + e^{E_i(\mathbf{k})/k_B T})$ is the Fermi distribution function and $E_i(\mathbf{k}) (i = 1, \ldots, 4)$ are the eigenenergies of $H_{\text{MF}}(\mathbf{k})$. In table 1, we give an explicit correspondence between $\Delta_{\sigma\sigma'}(\mathbf{k})$, $\Sigma_{\sigma\sigma'}^{(p)}(\mathbf{k})$ and $\Delta_{ji}(\mathbf{k})$, $\Sigma_{ji}^{(p)}(\mathbf{k})$.

In addition, for the present 2D case the average interparticle spacing is given by \cite{7}

$$r_s = \frac{1}{\sqrt{2 (n_e + n_h)}}. \quad (7)$$

Many meaningful physical quantities, including the order parameters, can be obtained by self-consistently solving four-band equations (2)–(6) with the confinement of the electron and hole number densities. We numerically calculate the exciton’s energy spectrum and the order

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Table 1. Correspondence between $\Delta_{\sigma\sigma'}(k)$, $\Sigma^{(p)}_{\sigma\sigma'}(k)$ and $\Delta_{jl}(k)$, $\Sigma^{(p)}_{jl}(k)$.

| $\Delta_{\sigma\sigma'}(k)$ | $\Delta_{jl}(k)$ | $\Sigma^{(e)}_{\sigma\sigma'}(k)$ | $\Sigma^{(e)}_{jl}(k)$ | $\Sigma^{(b)}_{\sigma\sigma'}(k)$ | $\Sigma^{(b)}_{jl}(k)$ |
|---------------------------|-----------------|-------------------------------|----------------|-----------------------------|----------------|
| $\uparrow\uparrow$      | 13              | $\uparrow\uparrow$            | 11            | $\uparrow\uparrow$        | 33            |
| $\uparrow\downarrow$    | 14              | $\uparrow\downarrow$          | 12            | $\uparrow\downarrow$      | 34            |
| $\downarrow\uparrow$    | 23              | $\downarrow\uparrow$          | 21            | $\downarrow\uparrow$      | 43            |
| $\downarrow\downarrow$  | 24              | $\downarrow\downarrow$        | 22            | $\downarrow\downarrow$    | 44            |

Figure 2. (a) The exciton’s energy spectrum with $r_s = 1.5$ and $\alpha = 0$. The solid and dashed lines correspond to equations (1) and (8), respectively. (b) Exciton density of the states with $r_s = 5$ and $\alpha = 0$.

Figure 3. Wave-vector dependence of the gap function $\Delta(k)$ for $\alpha = 0$ and several values of $r_s$. Inset: calculated $\Delta_{\sigma\sigma'}(k)$ from the original Hamiltonian (1) at $r_s = 1.5$ and 5.0. The solid and dashed lines correspond to $\Delta_{\uparrow\uparrow} (= \Delta_{\downarrow\downarrow})$ and $\Delta_{\downarrow\uparrow} (= \Delta_{\uparrow\downarrow})$, respectively. Comparing with $\Delta_{\sigma\sigma'}(k)$ in the inset, we can approximately use $\Delta(k)$ replacing $\Delta_{\sigma\sigma'}(k)$ to study the general properties of the order parameters without other spin-dependent interactions.

parameters under different exciton number densities: $r_s = 1.5$, $\alpha = 0$ and 5.0, $\alpha = 0$. Here the density imbalance parameter $\alpha$ is defined as $\alpha \equiv (n_e - n_h)/(n_e + n_h)$. The calculated results are correspondingly shown by solid lines in figure 2(a) and the inset in figure 3.

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Because the main goal of this paper is to focus on the general properties of the order parameters and neglect the other spin-dependent physical conditions, such as the effect of the Rashba-type spin–orbit coupling by surface inversion asymmetry, we plan to simplify our TIB model, i.e. to define a single order parameter \( \Delta(k) \), which can approximately replace the four \( \Delta_{\sigma\sigma'}(k) \). Similar to that in the semiconductor case \([7]\), the corresponding simplified grand-canonical Hamiltonian describing this TIB system can be approximately written as

\[
H = \sum_{k,p} (\epsilon_{kp} - \mu_p) c_{kp}^\dagger c_{kp} + \frac{1}{2\Omega} \sum_{k,k',q,p,p'} V_{p,p'}^{pp'} c_{k+k+q/2p}^\dagger c_{k-q/2p'}^\dagger c_{k+q/2p'} c_{k'+q/2p} c_{k'q/2p}. \tag{8}
\]

With the SCMF theory, equation (8) can be rewritten in a \( 2 \times 2 \) matrix in the basis \((e, h)^T\); the relevant mean-field equations to be solved for the variables \( \mu_e, \mu_h \) and the gap function \( \Delta_k \) are

\[
\Delta_k = -\frac{1}{\Omega} \sum_{k'} V_{k-k'}^{eh} \frac{\Delta_{k'}}{2E_{k'}} \left[ f(E_{k'}^e) - f(E_{k'}^h) \right], \tag{9}
\]

\[
\Sigma_k^e = \frac{1}{\Omega} \sum_{k'} V_{k-k'}^{ee} \left[ u_{k}^2 f(E_{k'}^e) + v_{k}^2 f(E_{k'}^h) \right], \tag{10}
\]

\[
\Sigma_k^h = \frac{1}{\Omega} \sum_{k'} V_{k-k'}^{hh} \left[ v_{k}^2 f(E_{k'}^h) + u_{k}^2 f(E_{k'}^e) \right], \tag{11}
\]

\[
n_e = \frac{1}{\Omega} \sum_k \left\{ u_{k}^2 f(E_{k}^e) + v_{k}^2 f(E_{k}^h) \right\}, \tag{12}
\]

\[
n_h = \frac{1}{\Omega} \sum_k \left\{ u_{k}^2 \left[ 1 - f(E_{k}^e) \right] + v_{k}^2 \left[ 1 - f(E_{k}^h) \right] \right\} , \tag{13}
\]

where \( u_{k}^2 = 1 - v_{k}^2 = \frac{1}{2}(1+\xi_k/E_k) \) and \( E_k^\pm = \delta \xi_k \pm E_k \) with \( \delta \xi_k = \frac{1}{2}(\xi_{ke}+\xi_{kh}) \) and \( E_k = \sqrt{\xi_k^2 + \Delta_k^2} \) that are given by \( \xi_{kp} = \epsilon_{kp} - \mu_p + \Sigma_k^p \) (\( p = e, h \)).

We also self-consistently calculate the exciton’s energy spectrum from two-band equations (9)–(13). The result, for comparison with the original exact four-band results from equations (2)–(6), is plotted in figure 2(a) with red dashed lines under the same parameters as those used in four-band calculations. The corresponding density of states (DOS) is shown in figure 2(b). From figure 2(a), one can clearly see that the exciton’s energy spectrum within the two-band approximation is wonderfully consistent with that within the exact four-band formalism. Another character found in figures 2(a) and (b) is that there is an evident stable energy gap protecting the excitonic condensation. In addition, we would like to point out that the parity of the linear dispersion relations of the particles and holes is odd, whereas the parity of the particle–particle and hole–hole Coulomb interactions is even. This parity asymmetry results in the energy shift in figure 2(a) and the corresponding DOS asymmetry in figure 2(b) as well as the asymmetry in figure 5 below. In the rest of this paper, all the numerical results except those shown in the inset of figure 3 are calculated from two-band SCMF equations (9)–(13).
3. Numerical results and the application to the Bi\textsubscript{2}Se\textsubscript{3}-family material

First, we calculate the wave vector dependence of $\Delta(k)$ for equal densities ($\alpha = 0$) and several values of $r_s$. The results are shown in figure 3. We can find the generic feature of the BCS–BEC crossover behavior similar to that in the semiconductor bilayers. However, the striking character in the TI bilayers is that the maximum value of $\Delta(k)$ in the BCS limit is much larger than that in traditional semiconductor electron–hole bilayers [7]. This prominent difference means that the BCS phase of TIB is more robust than that of the semiconductor bilayer for the equal-density case. Also shown in figure 3 (inset) are the calculated four-band gap functions $\Delta_{\alpha\sigma}(k)$ at $r_s = 1.5$ and 5.0, which show the approximate coincidence in amplitude with the two-band result of $\Delta(k)$.

Since there is no obvious interface between the BCS and BEC regimes in terms of the density, we plot in figure 4 the calculated momentum magnitude $k$ at which the order parameter takes its maximum value $k_{\text{max}}$ versus $r_s$ at $\alpha = 0$. From this figure, one can see that as $r_s \to 0$, the number density $n_e$ ($n_h$) and $k_{\text{max}}$ tend to infinity, the exciton’s phase is in the BCS regime. On the other hand, as $r_s \to \infty$, the number density $n_e$ ($n_h$) and $k_{\text{max}}$ tend to 0, and the exciton’s phase is now in the BEC regime. As $r_s$ takes a moderate value, the system is in a mixed regime.

The effect of $\alpha$ on $k_{\text{max}}$ is shown in figure 5, where $k_{\text{max}} = \max\{k\}$. It is evident that the density imbalance actually suppresses $k_{\text{max}}$ and has different effects on the two sides of the crossover. In the BEC regime, the main effect of the density imbalance is to reduce the number of electron–hole pairs, which results in superfluid properties less sensitive to density imbalance. In the BCS regime, the density imbalance leads to the mismatch of the Fermi surfaces of electrons and holes and finite momentum pairing, which is more easily broken. However, compared with that in the traditional semiconductor bilayers, we find that the superfluid property in the BEC phase in our case is more sensitive to electron–hole density imbalance. As an example, for $r_s = 20$ the maximum of the gap function $k_{\text{max}}$ for TIB disappears as $\alpha$ takes a value smaller than 0.5, while it always takes finite values as $\alpha$ varies in the whole zone ($-1, 1$) for the traditional semiconductor electron–hole bilayers [7].

Now we apply this TIB model to study the condensation of electron–hole pairs for the topological surface states of the Bi\textsubscript{2}Se\textsubscript{3}-family material. The two TI films in the left panel...
Figure 5. Maximum value $\Delta_{\text{max}} = \max\{\Delta_k\}$ as a function of $\alpha$ for $d = 1$ and several values of $r_s$.

Of figure 1 now are two ultrathin TI Bi$_2$Se$_3$-family films [22] (about 80 Å thick). With the adopted experimental [23] lattice constants $a = 4.143$ Å and $c = 28.636$ Å, we calculate the first-principles surface band structure of the Bi$_2$Se$_3$-family [24] by a simple supercell approach with spin–orbit coupling included and obtain the approximate Hamiltonian form describing the gapless surface states of the Bi$_2$Se$_3$-family as follows:

$$H(k) = \gamma k^2 + \hbar v_F (k_x \sigma_y - k_y \sigma_x).$$

(14)

Although this Hamiltonian has the same form as that of the conventional two-dimensional electron gas (2DEG) system with Rashba spin–orbit coupling, the intrinsic difference between these two kinds of systems is that the $k$-linear spin–orbit interaction is primary to the TI surface states, while the parabolic term is dominant in the conventional 2DEG. By fitting the first-principles results, the parameters in equation (14) are given as $\gamma = 0.21$ eV nm$^2$ and $\hbar v_F = 0.2$ eV nm (e.g. $v_F = 3.04 \times 10^5$ m s$^{-1}$). That means the energy dispersion around the Dirac point can be accurately described by $\epsilon_k = \pm \hbar v_F |k|$ when the wave vector $|k|$ is much smaller than 1.0 nm$^{-1}$. For numerical calculation, we choose nm as the length unit and 0.2 eV as the energy unit in the following discussion. The dielectric constant $\varepsilon = 1$ and the spacer width $d = 10$ Å. In fact, the condition that the wave vector $|k|$ is much smaller than 1.0 nm$^{-1}$ requires that only for $r_s \geq 5$ is the TIB model valid for the Bi$_2$Se$_3$-family material. This means that the BEC phase is most likely to emerge in the Bi$_2$Se$_3$-family bilayer system.

Now, let us discuss the critical temperature of this TIB system. The relation between the $\Delta_{\text{max}}$ and temperature $T$ is shown in figure 6(a) for $d = 1$, $\alpha = 0$ and several values of $r_s$, and in figure 6(b) for $d = 1$, $r_s = 5.0$ and several values of $\alpha$. From figure 6(a), we can find that the critical temperature $T_c$ decreases as $r_s$ increases (i.e. as the particle density decreases). For the Bi$_2$Se$_3$-family bilayer at $r_s = 5.0$, the critical temperature $T_c$ is calculated as 0.05 in units of 0.2 eV. That means the critical temperature $T_c$ is about 8–10 meV (100 K), which is much higher than that in traditional semiconductor electron–hole bilayers. Although the Bi$_2$Se$_3$-family TIB system is in the BEC phase ($r_s = 5.0, 20.0$), the numerical calculated results shown in figure 6(a) are consistent with the general relation of the BCS superconductor,

$$\frac{2\Delta(0)}{T_c} = 2\pi e^{-\gamma} \approx 3.53,$$

(15)
where $\Delta(0)$ is the energy gap at zero temperature. The introduced electron–hole density imbalance ($\alpha \neq 0$) can reduce the critical temperature. This character is clearly shown in figure 6(b): with increasing density imbalance $\alpha$, the critical temperature $T_c$ decreases. As is known, in 2D superfluids, the critical temperature is often substantially overestimated by mean-field theory. It is ultimately limited by entropically driven vortex and antivortex proliferation at the Kosterlitz–Thouless (KT) transition temperature $T_{KT} = \frac{\pi^2}{2} \rho_s(T_{KT})$ with $\rho_s(T)$ being the superfluid density (the phase stiffness). Min et al [9] give an approximate formula to calculate the counterflow current, which is read as

$$\rho_s(T) \approx \frac{\nu^2 \hbar^2}{16 \pi T} \int k dk \left[ \text{sech}^2 \left( \frac{\Delta^z}{2T} \right) - \text{sech}^2 \left( \frac{\Delta}{2T} \right) \right],$$

where $\Delta^z = -\frac{\mu_e + \Sigma_k + \Sigma_h}{2}$ and $\Delta = \sqrt{(\Delta^z)^2 + \Delta_k^2}$. We adopt this formula to calculate the superfluid density. The temperature dependence of superfluid density is shown in figure 7 at $r_s = 5$ and $\alpha = 0$. From figure 7, it is easy to estimate that the KT transition temperature $T_{KT}$ is about 0.05 in units of 0.2 eV. Comparing with the critical temperature $T_c$ in figure 6 at $r_s = 5$ and $\alpha = 0$, a striking conclusion is reached: $T_c \approx T_{KT}$, which means that high-temperature ($\sim 100$ K) excitonic condensation may occur in the Bi$_2$Se$_3$-family TIB system. On the other hand, we can estimate the KT temperature with the zero-temperature phase stiffness $\rho_s(T = 0) \approx E_F/4\pi$, which is similar to the graphene bilayers [9]. Considering the case shown in figure 2, the Fermi energy $E_F$ can be numerically calculated and is given as $\sim 0.4$ (in units of 0.2 eV). Hence, the KT temperature is estimated as $T_{KT} \approx E_F/8 \approx 0.05$ in units of 0.2 eV. This means that the two estimation methods are consistent and high-temperature excitonic condensation can emerge in the Bi$_2$Se$_3$-family TIB system.
Figure 7. The calculated $T_{KT}$ at $r_s = 5$ and $\alpha = 0$.

4. Summary and conclusions

In summary, we have presented a generic TIB model to study the excitonic condensation with the SCMF theory for the topological surface states. Similar to the traditional semiconductor electron–hole bilayers, the TIB system presents the crossover behavior from the BCS limit to the BEC limit by changing the exciton’s density. However, two prominent novel characters different from traditional semiconductor electron–hole bilayers are found. One is that the superfluid property in the BEC phase is more sensitive to electron–hole density imbalance. The other is that the BCS phase is more robust than that of the semiconductor bilayer. Applying this TIB model to the Bi$_2$Se$_3$-family material, we find that the BEC phase is most likely observed in experiment. Moreover, we theoretically estimate the critical temperature for the Bi$_2$Se$_3$-family TIB system and find that it is much higher than that in traditional semiconductor electron–hole bilayers. For example, at $r_s = 5$ and $\alpha = 0$, the critical temperature $T_c$ is obtained as about 100 K. We have also studied the phase stiffness and find that the KT transition does not suppress the critical temperature for the Bi$_2$Se$_3$ family in SCMF approximation.

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