S1. Numerical values of the parameters used in modeling the experiments of Brangbour et al. [2].

From [2] we obtain

\[ V_0 = 0.42 \text{ nm/sec}, \quad 2R = 1,100 \text{ nm}. \] (i)

- Figure 2(a) of this paper (Figure 3 of [2]): Brangbour et al. give \( N_{GS} = 4000 \) and \( c = 0.2 \pm 0.1 \). Thus we take \( c = 0.3 \) so that

\[ EA = c k_B T \frac{N_{GS}}{4R} = 0.3 \times 4.14 \times \frac{4,000}{2200} = 2.258 \text{ pN}. \] (ii)

They also give the filament length at the end of two stress-free growth periods to be 200 nm and 400 nm. Therefore we take

\[ \ell_R = 200 \text{ nm} \quad \text{and} \quad \ell_R = 400 \text{ nm}. \] (iii)

- Figure 3(a) of this paper (Figure 1c of [2]): Brangbour et al. give \( N_{GS} = 10,000 \). Based on the item above we take \( c = 0.2 \). Then

\[ EA = c k_B T \frac{N_{GS}}{4R} = 0.2 \times 4.14 \times \frac{10,000}{2200} = 3.764 \text{ pN}. \] (iv)

Next we want to calculate the time \( t_0 \) at which the force was applied. Let \( d(t) = \ell(t) + 2R \); it represents the distance between the centers of two adjacent particles in the model in [2]. Then from the formula

\[ \ell(t) = \frac{v_0 t}{1 + \sigma(t)/E} \quad \Rightarrow \quad d(t) = \frac{v_0 t}{1 + \sigma(t)/E} + 2R, \] (v)
and therefore at the instant $t_0^+$ just after the application of the force, one has
\[ d(t_0^+) = \frac{v_0 t_0}{1 + \sigma(t_0^+)/E} + 2R, \tag{vi} \]
and therefore
\[ t_0 = \frac{[d(t_0^+) - 2R][1 + \sigma(t_0^+) A/E A]}{v_0}. \tag{vii} \]

From Figure 1c of [2], with $EA = 3.764 \text{ pN}$, $2R = 1100 \text{ nm}$ and $V_0 = 0.42 \text{ nm/sec}$, we find

| $\sigma(t_0^+) A$ | $d(t_0^+)$ | $t_0$ using (vii) |
|------------------|-----------|------------------|
| 0.5 pN           | 1460 nm   | 971 sec          |
| 3 pN             | 1320 nm   | 941 sec          |
| 17 pN            | 1180 nm   | 1051 sec         |

Therefore we take $t_0 = 1000 \text{ sec}$.

- Figure 2(b) of this paper (Figure 5 of [2]): Brangbour et al. give. $N_{GS} = 4000$, $c = 0.13 \pm 0.02$ and $N_{GS} = 10000$, $c = 0.18 \pm 0.02$. So we take $c = 0.13$ and $c = 0.18$ and then obtain
\[ EA = c k_B T \frac{N_{GS}}{4R} = 0.13 \times 4.14 \times \frac{4000}{2200} = 0.979 \text{ pN}, \tag{viii} \]
\[ EA = c k_B T \frac{N_{GS}}{4R} = 0.18 \times 4.14 \times \frac{10000}{2200} = 3.387 \text{ pN}. \tag{ix} \]

- Figure 3(b) of this paper (Figure 2 of [2]): Brangbour et al. give $N_{GS} = 10,000$. Based on the first item above we take $c = 0.2$ and then
\[ EA = c k_B T \frac{N_{GS}}{4R} = 0.2 \times 4.14 \times \frac{10000}{2200} = 3.764 \text{ pN} \tag{x} \]

They also give $t_1 = 650 \text{ sec}$; $t_2 = 855 \text{ sec}$; the smaller value of force to be $0.8 \text{ pN}$; and the larger value of force to be $39 \text{ pN}$.

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S2. Numerical values of the parameters used in modeling the experiments of Parekh et al. [6]

Table 1:

| Quantity | Description | Value | Source |
|----------|-------------|-------|--------|
| 1        | $k_B T$ Boltzmann constant times absolute temperature | 4.14 pN nm | At 300°K $k_B = 1.381 \times 10^{-23}$ J/K |
| 2        | $E_\infty$ Young’s modulus of polymer network | 0.7 – 6.7 nN/µm² | Marcy et al. [5]. Average 3.7 nN/µm². Parekh et al. [6] refer to Marcy’s data in their supplement |
| 3        | $A$ Specimen cross-sectional area | 381 µm² | Parekh et al. [6] supplementary material |
| 4        | $k_c$ AFM stiffness (force/deflection) | 30 nN/µm | Parekh et al. [6] supplement: two cantilevers. $k_c = 0.03$ nN/µm and $k_c = 0.02$ nN/µm |
| 5        | $\sigma_{\text{stall}}A$ Force in specimen at stall | 294 nN | Parekh et al. [6] Figure 2 |
| 6        | $\ell_0$ Length of unstressed specimen at initial instant $t_0$ | 3000 nm | Parekh et al. [6] Figure 2A. Value of $\ell(t)$ at time $t = 0$ |
| 7        | $E_f$ Filament Young’s modulus | 2.3 GPa | Howard [3] Table 3.2 |
| 8        | $A_f$ Filament cross-sectional area | 19 nm² [$= \pi(2.46)^2$] | Howard [3] Table 7.1 Boal page 24 $\pi(4)^2 = 50$ nm² |
| 9        | $f_{\text{stall}}$ Stall force for one filament | 7 pN 8 pN | Howard [3] page 170 Košmrlj [4] |
| 10       | $a$ Length of a monomer (G-actin) | 2.5 nm | Košmrlj [4] More or less Howard’s $\delta$ |
Table 2:

| Quantity | Description | Value | Source |
|----------|-------------|-------|--------|
| a        | $\ell_R(0)$ | initial length in ref space | 3000 nm | $\ell_R(0) = \ell(0)/\Lambda(0) = \ell(0)$ |
| b        | $k$         | AFM stiffness (stress/deflection) | 0.0787 nN/µm² | $k = k_c/A = 30/381$ |
| c        | $\sigma_0$ | $= k\ell_0$ | 0.2362 nN/µm² | $\sigma_0 = k\ell_0 = 0.0787 \times 3$ |
| d        | $\sigma_{stall}$ | Stress in specimen at stall | 0.77 nN/µm² | $\sigma_{stall} = 294/A = 294/381$ |
| e        | $V_0$       | $V$ at $f = 0$ | 32 – 2350 nm/min | Min value from Brangbour [2] Max value from Marcy et al. [5] See Remark 1 |
| f        | $\tau_R$   | Time scale for growth at tips | 1.3 – 94 min | $\tau_R = \ell_0/V_0$ with $V_0$ from row-e |
| g        | $\tau_0$   | Time scale for development of new filaments | Arbitrary | Unknown |

– CONTINUED –
Remark 1: Estimating $V_0$ in row-e of Table 2 from other people’s data.

Brangbour et al. [2], Figure 5, gives $V = 0.39 \text{ nm/s at } f = 0.5 \text{ pN}$. If $V = V_0 \exp(-fa/kT)$, then with $f = 0.5 \text{ pN}, a = 2.5 \text{ nm}, kT = 4.14 \text{ pN nm}$, one gets $V_0 = 32 \text{ nm/min}$ (and so $\tau_R = \ell_0/V_0 = 94 \text{ min}$).

Marcy et al. [5], page 5995, right column, top paragraph gives values of $V_0$ (which they call $V_{F=0}$) in the range $1.75 \pm 0.6 \mu\text{m/min} = 1750 \pm 600 \text{ nm/min}$. Therefore $V_0$ can be as large as 2350 nm/min. This means $\tau_R = \ell_0/V_0$ can be as small as 1.28 min.

Therefore we have the ranges $V_0 = 32 - 2350 \text{ nm/min}$ and $\tau_R = 1.28 - 94 \text{ min}$.

Remark 2: Determining which springs were used in the experiments underlying Figure 3 of [6]: Recall from the supplementary material in Parekh et al. [6] that they used 2 springs in their experiments with stiffnesses $k_c = 20 \text{ nN/\mu m}$ and $k_c = 30 \text{ nN/\mu m}$.

From the data in Figure 3a of [6]:

$$\dot{\sigma} = \frac{145 - 120}{381 \times (100 - 93.5)} = 0.0100 \text{ nN min}^{-1} \mu\text{m}^{-2},$$

$$k = \frac{\dot{\sigma}}{\ell} = \frac{0.0100}{129} = 0.0782 \text{ nN/\mu m}^3.$$

By comparing this with the value in the top row of row-b, we infer that they would have used the spring with stiffness $k_c = 30 \text{ nN/\mu m}$ in the experiment related to their Figure 3a.

From the data in Figure 3b of [6]:

$$\dot{\sigma} = \frac{83 - 68}{381 \times 6} = 0.00656 \text{ nN min}^{-1} \mu\text{m}^{-2},$$

$$k = \frac{0.00656}{129} = 0.0508 \text{ nN/\mu m}^3.$$

By comparing this with the value in the bottom row of row-b, we infer that they would have used the spring with stiffness $k_c = 20 \text{ nN/\mu m}$ in the experiment related to their Figure 3b.

Remark 3: Value of $\rho_\infty$. Though we do not need the value of $\rho_\infty$ since it gets nondimensionalized out, it is still worth calculating it in two different ways, as a consistency check of the model. The first estimate is

$$\rho_\infty = \frac{\sigma_{\text{stall}}}{\ell_{\text{stall}}} = \frac{0.77 \text{ nN/\mu m}^2}{8 \text{ pN}} = 96 \mu\text{m}^{-2},$$

and the second follows from $E_\infty = \rho_\infty A_f E_f$:

$$\rho_\infty = \frac{1}{A_f \sqrt{\frac{E_\infty}{E_f}}} = \frac{1}{19 \text{ nm}^2} \sqrt{\frac{3.7 \text{ nN/\mu m}^2}{2.3 \times 10^6 \text{ nN/\mu m}^2}} = 67 \mu\text{m}^{-2}.$$


The upper bound on the number of filaments is
\[ \rho < \frac{1}{A_f} = 53,000 \mu m^{-2}. \]

**Remark 4: Stiffness of the spring used in the experiments underlying Figure 2 of [6]**: From Figure 2a of [6]
\[ \frac{\dot{F}}{100 - 33} = 2.015 nN/min, \quad \frac{\dot{\ell}}{100 - 33} = 0.067 \mu m/min, \]
and therefore
\[ k_c = \frac{\dot{F}}{\dot{\ell}} = \frac{2.015}{0.067} = 30 nN/\mu m. \]

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S3. Calculations underlying Figures 9 and 10.

**S3.1. Figure 9**: Figure 3a of Parekh et al. [6] shows that when the specimen grows under spring loading the force increases linearly from the value 120 nN to 145 nN in 6.5 min. Assuming that the force increased linearly from the start, and extrapolating backwards, we conclude that the force was zero at time \( t_0 = 62.3 \) min. Thus we take as initial conditions
\[ \sigma(t_0) = 0, \quad \rho(t_0) = 0.5 \rho_\infty, \quad \ell(t_0) = \ell_R(t_0) = 3000 \text{ nm} \quad \text{at } t_0 = 62.3 \text{ min}, \]
where \( 0.5 \rho_\infty \) is the arbitrarily chosen initial condition for the filament density and 3000 nm is the distance between the AFM cantilever and the support when the cantilever is not deflected. In this experiment the specimen grows under spring loading for \( t_0 < t < t_2 = 100 \) min; at time \( t_2 \) the stress is suddenly decreased to \( 0.315 \text{ nN/\mu m}^2 \); and thereafter it is held at \( \sigma(t) = 0.315 \text{ nN/\mu m}^2 \) for \( t > t_2 \). The differential equation (4.24) with the preceding initial conditions and \( \sigma = k(\ell - \ell_0) \) can now be solved to find \( \sigma(t), \dot{\ell}(t) \) and \( \ell(t) \) for \( t_0 < t < t_2 \). In particular one obtains
\[ \sigma(t_2^-) = 0.373 \text{ nN/\mu m}^2, \quad \dot{\ell}(t_2^-) = 111.6 \text{ nm/min}, \quad \ell(t_2^-) = 7742 \text{ nm}. \]

Next, the conditions at time \( t_2^+ \) can be found by first calculating \( \ell(t_2^+) \) from (4.26) (keeping in mind that we are given \( \sigma(t_2^-) = 0.315 \text{ nN/\mu m}^2 \) and we know \( \rho(t_2) \) from (4.6)) and \( \ell(t_2^+) \) from (4.23):
\[ \sigma(t_2^+) = 0.315 \text{ nN/\mu m}^2, \quad \dot{\ell}(t_2^+) = 212 \text{ nm/min}, \quad \ell(t_2^+) = 8329 \text{ nm}. \]

\(^1\) Presumably prior to that, for \( 0 < t < t_0 \), the actin filaments did not extend all the way from the AFM cantilever to the other support and so the specimen was growing freely under stress-free conditions without engaging the AFM spring.
Finally, the differential equation (4.23) is solved for \( t > t_2 \) using the known information at \( t_2^+ \) as initial conditions. Figure 9 shows plots of \( \sigma(t) \) and \( \dot{\ell}(t) \) versus \( t \) resulting from these calculations. This figure is to be compared with Figure 3a of [6].

**S3.2: Figure 10.** As seen in Figure 3b of [6], in their second experiment Parekh et al. kept the stress fixed at the value \( \sigma(t) = 0.178 \text{nN/\mu m}^2 \) for \( t_0 < t < t_1 = 73 \text{ min} \); the force clamp was then released at time \( t_1 \) and the specimen allowed to grow under spring loading conditions for \( t_1 < t < t_2 = 79 \text{ min} \); at the instant \( t_2 \) the value of the stress was suddenly decreased back to the value \( 0.178 \text{nN/\mu m}^2 \), and held there for \( t > t_2 \). In order to calculate the response predicted by our model, the first task is to estimate the time \( t_0 \) at which the stress was initially applied on the specimen (which we assume was done at the instant when the filaments extended all the way from the AFM cantilever to the other support).

In order to determine \( t_0 \) we proceed as follows: according to Figure 3b of [6] the stress varies continuously at the instant \( t_1 \) when the force clamp is released and therefore \( \sigma(t_1^+) = 0.178 \text{nN/\mu m}^2 \). Thus from \( \sigma(t_1^+) = k(\ell(t_1^+) - \ell_0) \) we find \( \ell(t_1^+) = 6390 \text{ nm} \). When \( \sigma \) varies continuously it follows from (4.26) that \( \ell \) varies continuously (since \( \rho \) varies continuously). Thus \( \ell(t_1^-) = 6390 \text{ nm} \). Now focus on the time interval \( t_0 < t < t_1 \). At the instant \( t_0 \) we have the initial conditions\(^2\) \( \ell_R(t_0^+) = 3000 \text{ nm} \) and \( \rho(t_0) = 0.7\rho_\infty \) where the initial value of the filament density has been chosen to be consistent with the dissipation inequality; see Section S4.4.2. Thus integrating (4.22) with respect to time from \( t_0 \) to \( t_1 \) and using \( \ell_R(t_0^+) = 3000 \text{ nm} \) and \( \ell_R(t_1^-) = \ell(t_1^-)/\Lambda(\sigma(t_1^-), \rho(t_1^-)) = 6390/\Lambda(0.178, \rho(t_1^-)) \) with \( \rho(t_1^-) \) given by (4.6) leads to a nonlinear algebraic equation for \( t_0 \). This yields \( t_0 = 48.23 \text{ min} \). Thus we have the initial conditions

\[
\sigma(t_0) = 0.178 \text{nN/\mu m}^2, \quad \rho(t_0) = 0.7\rho_\infty, \quad \ell_R(t_0) = 3000 \text{ nm} \quad \text{at} \quad t_0 = 48.23 \text{ min}.
\]

Determining \( \sigma(t) \) and \( \dot{\ell}(t) \) is now straightforward. We first determine \( \rho(t) \) for all time from (4.6) with \( \rho(t_0) = 0.7\rho_\infty \). We then integrate (4.22) with respect to \( t \) and use (4.22) and the preceding initial conditions to find the stress and elongation-rate for \( t_0 < t < t_1 \). In particular we obtain

\[
\sigma(t_1^-) = 0.178 \text{nN/\mu m}^2, \quad \dot{\ell}(t_1^-) = 186 \text{ nm/min}, \quad \ell(t_1^-) = 6390 \text{ nm}.
\]

By the aforementioned continuity of the stress and elongation at time \( t_1 \) and (4.24) we have

\[
\sigma(t_1^+) = 0.178 \text{nN/\mu m}^2, \quad \dot{\ell}(t_1^+) = 124 \text{ nm/min}, \quad \ell(t_1^+) = 6390 \text{ nm}.
\]

Next we use these as initial conditions to solve (4.24) together with \( \sigma = k(\ell - \ell_0) \) to determine \( \sigma(t) \) and \( \dot{\ell}(t) \) for \( t_1 < t < t_2 \). In particular we find

\[
\sigma(t_2^-) = 0.216 \text{nN/\mu m}^2, \quad \dot{\ell}(t_2^-) = 119 \text{ nm/min}, \quad \ell(t_2^-) = 7118 \text{ nm}.
\]

\(^2\text{Keep in mind that the length} \ell(t_0^+) \neq 3000 \text{ nm} \text{ since the specimen length will change suddenly as the stress is applied.}\)
Turning to the instant $t^+_2$, we know the stress $\sigma(t^+_2) = 0.178 \text{nN/\mu m}^2$ and so use (4.26) to find $\ell(t^+_2)$. This together with (4.23) gives

$$\sigma(t^+_2) = 0.178 \text{nN/\mu m}^2, \quad \dot{\ell}(t^+_2) = 187 \text{ nm/min}, \quad \ell(t^+_2) = 7508 \text{ nm}.$$ 

Finally we solve (4.23) to find the stress and elongation-rate for $t > t_2$ using the above information at $t^+_2$ as initial conditions. Figure 10 shows plots of $\sigma(t)$ and $\dot{\ell}(t)$ versus $t$ resulting from these calculations. This figure is to be compared with Figure 3b of [6].

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S4. Dissipation, the driving force and the dissipation inequality

In order to set the stage, first consider the $\rho, \sigma$-plane. Each radial line $\sigma = f\rho$ on this plane corresponds to a constant filament force $f$; see Figure S.1. We are interested in the range $0 \leq f \leq f_{\text{stall}}, 0 \leq \rho \leq \rho_\infty$, corresponding to the wedge-shaped shaded region in the figure. The kinetic relation for surface growth has the form $V = V(f) \text{ and so each radial line also corresponds to a constant growth speed } V$. Since $V(f_{\text{stall}}) = 0$ and $V(f_{\text{stall}}) > 0$ for $0 \leq f < f_{\text{stall}}$, the growth speed vanishes on the bold red line $\sigma = f_{\text{stall}}\rho$ and is positive below it. Now consider a generic initial-value problem for a spring-loaded specimen. In this case one solves the differential equations $\dot{\sigma} = S(\sigma, \rho), \dot{\rho} = R(\rho)$ subject to initial conditions, say $\sigma(t_0) = 0, \rho(t_0) = \rho_0$, where $S$ is given by (4.18), (4.15)2,3 and $R$ by (4.5). The solution $\sigma = \sigma(t), \rho = \rho(t), t \geq t_0$, of this problem describes a trajectory in the $\rho, \sigma$-plane shown schematically by the dashed blue curve in Figure S.1. It starts at $(\rho, \sigma) = (\rho_0, 0)$ and terminates at $(\rho, \sigma) = (\rho_\infty, \sigma_{\text{stall}})$ corresponding to stall.

![Figure S.1: The radial straight lines $\sigma = f\rho$ are lines of constant filament force. The kinetic relation $V = V(f)$ yields $V = 0$ on the bold red line $\sigma = f_{\text{stall}}\rho$, and $V > 0$ on the shaded region below it. The trajectory defined by a solution $(\rho(t), \sigma(t)), t \geq t_0$, of a generic initial-value problem is depicted schematically by the blue dashed curve. It starts at the initial point $(\rho_0, 0)$ and terminates at stall corresponding to $(\rho_\infty, \sigma_{\text{stall}})$.](image)
In this section we examine the implications of the dissipation inequality ("second law of thermodynamics") on the kinetic law for surface growth $V = V(f)$. Recall that the kinetic laws (3.3) and (4.9) imply that the growth speed $V$ is $> 0$ when the filament force is in the range $0 \leq f < f_{\text{stall}}$. As we shall see, the dissipation inequality requires $f_{\text{driv}} \geq 0$ where $f_{\text{driv}}(\rho, \sigma)$ is the driving force for growth (which will be identified below). Our task therefore is to examine the implications of the inequality $f_{\text{driv}} \geq 0$ on the inequality $0 \leq f < f_{\text{stall}}$, or in terms of stress and filament density, the connection between $f_{\text{driv}}(\rho, \sigma) \geq 0$ and $0 \leq \sigma < \rho f_{\text{stall}}, 0 \leq \rho \leq \rho_{\infty}$.

S4.1 The driving force for growth.

As shown schematically in Figure 5, the specimen occupies the interval $[y_0(t), y_1(t)]$ in physical space and its associated length is $\ell = y_1 - y_0$. Its left end is attached to an AFM cantilever of stiffness $k_c$. The elastic energy stored in the cantilever (modeled as a Hookean spring) is $\frac{1}{2} k_c (y_0 - Y_0)^2$ where $Y_0$ is the position of the cantilever when it is undeflected. In reference space the specimen occupies the interval $[x_0(t), x_1]$ and its associated length is $\ell_R = x_1 - x_0$. Let $W$ be the free energy of the specimen per unit reference volume. Then, the rate of increase of the energy stored in the specimen and spring is

$$\frac{d}{dt}[WA(x_1 - x_0)] + \frac{d}{dt} \left[ \frac{1}{2} k_c (y_0 - Y_0)^2 \right] = \dot{W} \ell_R - WA \dot{x}_0 - \sigma A \dot{y}_0,$$

having equated the compressive force $\sigma A$ in the specimen to the force $-k_c (y_0 - Y_0)$ in the spring. The rate at which work is being done on the specimen by the compressive force $\sigma A$ at the right-hand end is

$$= -\sigma A \dot{y}_1.$$  \hspace{1cm} (xii)

Next we model the inflow of chemical energy into the specimen. If $a$ denotes the length of a single stress-free monomer (i.e. its length in a stress-free reference configuration), the number of monomers in a single filament is $\ell_R a$ and the total number of monomers in the specimen is $N \ell_R / a$. Thus the monomer concentration $c$, defined as the number of monomers per unit reference volume, is

$$c = \frac{N \ell_R / a}{\ell_R A} = \frac{N}{a A} = \frac{\rho}{a},$$

where $\rho = N / A$ is the filament density. When the body grows, the left-hand boundary of the specimen moves outwards in reference space at a speed $-\dot{x}_0$, and so the rate at which monomers are added to a filament at that end is $-\dot{x}_0 / a$. Therefore the associated rate of intake of chemical energy is $-\mu \dot{x}_0 / a$ per filament. In addition, there is an intake of chemical energy due to the formation of new filaments. Since the number of monomers in a filament is $\ell_R / a$ and its chemical energy $\mu \ell_R / a$, the rate of intake of chemical energy due to the formation of new filaments is $(\mu \ell_R / a) \dot{N}$. Thus the total inflow of chemical energy into the specimen per unit time is

$$-\mu (\dot{x}_0 / a) \dot{N} + (\mu \ell_R / a) \dot{N} = -\mu c A \dot{x}_0 + \mu c A \ell_R.$$  \hspace{1cm} (xiii)
Therefore the dissipation rate is given by (xi) plus (xiii) less (xi):

\[ D = -\sigma A \dot{y}_1 + \left[ -\mu c A \dot{x}_0 + \mu \dot{c} A \ell_R \right] - \left[ W A \ell_R - W A \dot{x}_0 - \sigma A \dot{y}_0 \right] = \]

\[ = (-\sigma \lambda + \mu c - W) AV + \left( -\sigma \dot{\lambda} + \mu \dot{c} - \dot{W} \right) A \ell_R, \]

where \( \lambda = \ell / \ell_R = (y_1 - y_0) / (x_1 - x_0) \) is the stretch and \( V = -\dot{x}_0 \) is the outward propagation speed of the left-hand boundary of the specimen. Suppose that the material is described by the constitutive characterization \( W = W(\lambda, c) \) together with

\[ -\sigma = \frac{\partial W}{\partial \lambda}, \quad \mu = \frac{\partial W}{\partial c}, \]

keeping in mind that \( \sigma \) is positive in compression. In view of (xv), the dissipation rate (xiv) reduces to

\[ D = (-\sigma \lambda + \mu c - W) AV, \]

and we therefore identify the driving force for growth to be

\[ f_{\text{driv}} := -\sigma \lambda + \mu c - W = -\sigma \lambda + \mu \rho / a - W. \]

Note that \( \mu / a \) is the chemical potential per unit reference length. By (xvi) and (xvii), the dissipation inequality \( D \geq 0 \) requires

\[ f_{\text{driv}} V \geq 0. \]

Growth corresponds to \( V = -\dot{x}_0 > 0 \) which therefore requires \( f_{\text{driv}} \geq 0 \).

**S4.2 The driving force specialized to the constitutive relation in this paper.**

Next we calculate an explicit expression for the driving force associated with the stress-strain-filament density relation

\[ -\sigma = \frac{\partial W}{\partial \lambda} = E \left( 1 - \lambda^{-1} \right), \quad E = E(\rho). \]

Note from (xviii) and \( \lambda > 0 \) that

\[ \sigma > -E. \]

Integrating (xviii) with respect to \( \lambda \) gives the free energy

\[ W(\lambda, \rho) = E(\lambda - \log \lambda - 1) + g(\rho), \]

and the corresponding chemical potential is

\[ \mu = \frac{\partial W}{\partial c} = a \frac{\partial W}{\partial \rho} = a E'(\rho)(\lambda - \log \lambda - 1) + ag'(\rho). \]
The driving force is then given by (xvii), (xviii), (xx) and (xxi) to be

$$f_{\text{driv}} = E(\rho) \log \lambda + \rho E'(\rho)(\lambda - 1 - \log \lambda) + f_0(\rho),$$

where we have set $f_0(\rho) = \rho g'(\rho) - g(\rho)$. This can be written in terms of the stress (and filament density) using $\lambda = (1 + \sigma/E)^{-1}$:

$$f_{\text{driv}} = f_{\text{driv}}(\rho, \sigma) = -E \log \left(1 + \frac{\sigma}{E}\right) + \rho E' \left[\frac{1}{1 + \sigma/E} - 1 + \log \left(1 + \frac{\sigma}{E}\right)\right] + f_0.$$

The driving force is a function of both $\sigma$ and $\rho$ since both $E$ and $f_0$ depend on $\rho$. All functions of $\rho$ including $f_0(\rho)$ here and $\sigma_{st}(\rho)$ below are defined for $0 \leq \rho \leq \rho_{\infty}$.

When $\sigma = 0$ (or equivalently $\lambda = 1$), the driving force reduces to $f_{\text{driv}} = f_0(\rho)$. This suggests that $f_0(\rho)$ is determined entirely by chemistry and so we refer to it as the chemical driving force. Recall that according to the kinetic relations (3.3), (1.1) and (4.9), the growth speed $V$ vanishes when the filament force $f = f_{\text{stall}}$. Since $f = \sigma/\rho$, this means that the growth speed vanishes at the stress $\sigma = f_{\text{stall}}/\rho$. We now make the assumption that the driving force vanishes when the growth-rate vanishes:

$$f_{\text{driv}}(\rho, \sigma_{st}(\rho)) = 0 \quad \text{where} \quad \sigma_{st}(\rho) := f_{\text{stall}}/\rho \quad \text{for} \quad 0 \leq \rho \leq \rho_{\infty}. \quad (\text{xxiv})$$

Note that $\sigma_{st}(\rho_{\infty}) = \sigma_{st}$. From (xxiii) and (xxiv) we now obtain the following expression for the chemical driving force:

$$f_0(\rho) = E \log \left(1 + \frac{\sigma_{st}}{E}\right) - \rho E' \left[\frac{1}{1 + \frac{\sigma_{st}}{E}} - 1 + \log \left(1 + \frac{\sigma_{st}}{E}\right)\right], \quad \sigma_{st} = \sigma_{st}(\rho), \quad (\text{xxv})$$

where $E = E(\rho)$ as well. Equation (xxv) can be used to eliminate $f_0(\rho)$ in favor of $\sigma_{st}(\rho)$ from (xxiii) allowing the driving force to be written as

$$f_{\text{driv}}(\rho, \sigma) = E \log \left(1 + \frac{\sigma_{st}}{1 + \frac{\sigma_{st}}{E}}\right) + \rho E' \left[\frac{1}{1 + \frac{\sigma_{st}}{E}} - 1 + \frac{\sigma_{st}}{1 + \frac{\sigma_{st}}{E}} - \log \left(1 + \frac{\sigma_{st}}{1 + \frac{\sigma_{st}}{E}}\right)\right]. \quad (\text{xxvi})$$

The driving force $f_{\text{driv}}(\rho, \sigma)$ is defined on a subdomain of the $\rho, \sigma$-plane. It, and the growth speed $V$, vanish on the bold red line $\sigma = \sigma_{st}(\rho) = f_{\text{stall}}/\rho$ shown in Figure S.1. The shaded region below that line corresponds to $0 \leq f < f_{\text{stall}}$ and therefore to $V > 0$ by the kinetic relation. It remains to examine the consequences of the dissipation inequality $f_{\text{driv}}(\rho, \sigma) \geq 0$ on this region.

S4.3 The driving force further specialized.

\[ ^3\text{but not necessarily the converse. Recall for example from (4.9) that } V(0) = V_0 > 0. \]
It order to examine where on the shaded region of Figure S.1 one has $f_{\text{driv}}(\rho, \sigma) \geq 0$ we limit attention to the particular forms of the elastic modulus used in this paper, specifically, $E(\rho) = E_0 \rho^n$ where $n = 0, 1$ and 2.

**Case $E(\rho) = E_0$:** This case is applicable to a specimen involving a few parallel actin filaments and the expression (xxvi) for the driving force specializes to

$$f_{\text{driv}} = E \log \left( \frac{1 + \sigma_{st}/E}{1 + \sigma/E} \right), \quad \sigma_{st}(\rho) = f_{\text{stall}} \rho, \quad E(\rho) = E_0.$$  \hspace{1cm} (xxvii)

Keeping (xix) in mind, we conclude from (xxvii) that if $0 < \sigma < \sigma_{st}$ then necessarily $f_{\text{driv}} > 0$. Therefore the dissipation inequality requires $V \geq 0$ on the shaded region of Figure S.1 and so all processes obeying the kinetic law $V = V(f)$ are admissible. In addition, (xxvii) shows that $f_{\text{driv}}$ decreases monotonically with increasing $\sigma$ on this interval, or said differently, the driving force decreases as the stress becomes progressively more compressive.

**Case $E(\rho) = \rho E_0$:** This case pertains to our model of the Brangbour et al. [2] experiments and the expression (xxvi) for the driving force reduces to

$$f_{\text{driv}} = E \left[ \frac{1}{1 + \frac{\sigma}{E}} - \frac{1}{1 + \frac{\sigma_{st}}{E}} \right], \quad \sigma_{st}(\rho) = f_{\text{stall}} \rho, \quad E(\rho) = E_0 \rho.$$  \hspace{1cm} (xxviii)

It follows from (xxviii) in view of (xix) that $f_{\text{driv}} > 0$ when $0 < \sigma < \sigma_{st}$. Therefore in this case also the dissipation inequality requires $V \geq 0$ on the shaded region of Figure S.1 and so all processes satisfying the kinetic law $V = V(f)$ are admissible on this region. Moreover, $f_{\text{driv}}$ decreases monotonically with increasing $\sigma$ (at each fixed $\rho$) as can be seen from (xxviii), and so the driving force for polymerization decreases as the stress becomes increasingly compressive.

**Case $E(\rho) = \rho^2 E_0$:** This case is applicable to the experiments of Parekh et al. [6] and accounts for filament bending. Recall from (4.7) that in Section 4 we wrote $E = E_\infty \rho^2 / \rho_{\infty}^2$, so that in terms of those parameters, $E_0 = E_\infty / \rho_{\infty}^2$.

In this case the expression (xxvi) for the driving force specializes to

$$\frac{f_{\text{driv}}}{E} = \frac{2}{1 + \sigma/E} - \frac{2}{1 + \frac{\sigma_{st}}{E}} - \log \left( \frac{1 + \frac{\sigma_{st}}{E}}{1 + \frac{\sigma}{E}} \right), \quad \sigma_{st}(\rho) = f_{\text{stall}} \rho, \quad E(\rho) = E_\infty \rho^2 / \rho_{\infty}^2.$$  \hspace{1cm} (xxix)

The driving force $f_{\text{driv}}(\rho, \sigma)$ is defined on the wedge shaped region, $0 \leq \rho \leq \rho_{\infty}, 0 \leq \sigma \leq \sigma_{st} = f_{\text{stall}} \rho$, of the $\rho, \sigma$-plane; see Figure S.2. We want to know where $f_{\text{driv}}$ is positive on this region.

First consider two limiting cases. If $\sigma$ is close to $\sigma_{st}$ at fixed $\rho$, one can approximate (xxix) to read

$$\frac{f_{\text{driv}}}{E} = \frac{\left( 1 - \frac{\sigma_{st}}{E} \frac{E_\infty}{\rho_{\infty}} \right) \sigma_{st} - \sigma}{(1 + \frac{\sigma_{st}}{E})^2} + \text{higher order terms},$$
showing that in this limit the driving force is positive for \( \rho/\rho_{\infty} > \sigma_{\text{stall}}/E_{\infty} \) and negative for \( 0 \leq \rho/\rho_{\infty} < \sigma_{\text{stall}}/E_{\infty} \). Note also that the driving force is proportional to \( f_{\text{stall}} - f \) in this case. In the other limit where, at fixed \( \rho \), the \( \sigma_{\text{st}}/E \leq \sigma_{f} \) in Figure S.2. Finally, we find that \( \Sigma = \Sigma_{\text{st}}/E > 1 \) corresponds to the upper and lower boundaries of the shaded region in Figure S.2. Therefore from these two limiting cases we conclude that the driving force is positive on the upper and lower boundaries of the shaded region in Figure S.2.

Returning to the general expression (xxix) for the driving force, keep in mind that we are concerned with the range \( 0 \leq \sigma/E \leq \sigma_{\text{st}}/E \) corresponding to filament force values in the range \( 0 \leq f \leq f_{\text{stall}} \). In the next paragraph we will show that (at each fixed \( \rho \)), the equation \( f_{\text{driv}}(\rho, \sigma) = 0 \) has two roots \( \sigma \), the smaller of which we denote by \( \Sigma_{\text{st}} \). For \( \sigma_{\text{stall}}/E_{\infty} \leq \rho/\rho_{\infty} \leq 1 \) one has \( \Sigma_{\text{st}} = \sigma_{\text{st}} \), whereas \( \Sigma_{\text{st}} < \sigma_{\text{st}} \) for \( 0 \leq \rho/\rho_{\infty} < \sigma_{\text{stall}}/E_{\infty} \). Furthermore \( \Sigma_{\text{st}} > 0 \) provided \( \rho/\rho_{\infty} > \alpha \sigma_{\text{stall}}/E_{\infty} \) where \( \alpha \approx 0.255 \). (One cannot write a closed form expression for \( \Sigma_{\text{st}} \) on this range.) The curve \( \sigma = \Sigma_{\text{st}}(\rho) \) for \( \alpha \sigma_{\text{stall}}/E_{\infty} \leq \rho/\rho_{\infty} \leq 1 \) is shown in Figure S.2. Finally, we find that \( f_{\text{driv}}(\rho, \sigma) > 0 \) on the shaded region of the figure between the \( \rho \)-axis and the curve \( \sigma = \Sigma_{\text{st}}(\rho) \). The figure has been drawn assuming \( \sigma_{\text{stall}}/E_{\infty} < 1 \). If \( \sigma_{\text{stall}}/E_{\infty} > 1 \) the curve \( \sigma = \Sigma_{\text{st}}(\rho) \) lies below the straight line \( \sigma = \sigma_{\text{st}}(\rho) \) throughout the range of interest.

In order to establish the results described in the preceding paragraph, let \( \rho \) be fixed at any value in the interval \( (0, \rho_{\infty}) \), in which event \( E(\rho) \) and \( \sigma_{\text{st}}(\rho) \) are also fixed. Consider the graph of \( f_{\text{driv}}/E \) versus \( \sigma/E \). First observe that \( f_{\text{driv}}/E \to \infty \) when both \( \sigma/E \to -1^{+} \) and \( \sigma/E \to \infty \). Second, the slope of this curve is negative for \( -1 < \sigma/E < 1 \), positive for \( \sigma/E > 1 \) and vanishes at \( \sigma/E = 1 \). Third, it intersects the horizontal axis at \( \sigma/E = \sigma_{\text{st}}/E \). It follows that this curve necessarily intersects the horizontal axis at precisely two points\(^4\) corresponding to two values of \( \sigma/E (> -1) \), one of them less than unity, the other greater. Let \( \sigma/E = \Sigma_{\text{st}}/E \leq 1 \) correspond to the left-most intersection point. It then follows that \( f_{\text{driv}} > 0 \) for \( -1 < \sigma/E < \Sigma_{\text{st}}/E \), and that \( f_{\text{driv}} \) decreases monotonically with increasing \( \sigma \) on this interval. It is not difficult to show that \( \Sigma_{\text{st}} = \sigma_{\text{st}} \) if \( \rho/\rho_{\infty} > \sigma_{\text{stall}}/E \) and that \( \Sigma_{\text{st}}/E < 1 < \sigma_{\text{st}}/E \) for \( 0 \leq \rho/\rho_{\infty} < \sigma_{\text{stall}}/E \).

The kinetic law \( V = V(f) \) gives \( V > 0 \) on the wedge shaped region \( 0 \leq \rho \leq \rho_{\infty}, 0 \leq \sigma < \sigma_{\text{st}} = f_{\text{stall}}/\rho \), in Figure S.2. The dissipation inequality requires \( f_{\text{driv}}V \geq 0 \) which reduces

\(^4\)except if \( \sigma_{\text{st}}/E = 1 \) in which case the two intersection points coalesce.
Figure S.2: Case $E = E_0 \rho^2$: The driving force vanishes on the curve $\sigma = \Sigma_{st}(\rho)$ for $\alpha \sigma_{stall}/E_\infty \leq \rho/\rho_\infty \leq 1$ where $\alpha \approx 0.255$. Note that $\Sigma_{st}(\rho) = \sigma_{st}(\rho)$ for $\sigma_{stall}/E_\infty \leq \rho/\rho_\infty \leq 1$. The driving force is positive on the shaded region between the curve $\sigma = \Sigma_{st}(\rho)$ and the $\rho$-axis. Therefore the dissipation inequality implies that $V \geq 0$ indicating that growth is permitted on this region. Recall that the kinetic law $V = V(f)$ also gives $V > 0$ here. The trajectory defined by a solution $(\rho(t), \sigma(t)), t \geq t_0$, of a generic initial-value problem is depicted schematically by the blue dashed curve starting at the initial condition $(\rho_0, 0)$ and terminating at stall corresponding to $(\rho_\infty, \sigma_{stall})$. Only solutions in the shaded region conform to the dissipation inequality.

to $f_{driv} \geq 0$ when $V > 0$. Therefore a solution $\sigma(t), \rho(t), t \geq t_0$, of an initial-value problem involving growth will conform to the dissipation inequality only if the trajectory defined by it lies in the shaded region of Figure S.2 corresponding to $f_{driv} \geq 0$. Below, we confirm this to be true in the case of the particular solutions determined in Section 4 of this paper.

Finally we remark that if the kinetic relation for growth had the form $V = V(f_{driv})$ with $V(f_{driv})f_{driv} \geq 0$ (rather than $V = V(f)$ with $V(f)f \geq 0$), all solutions of an initial-value problem would automatically conform with the dissipation inequality.

S4.4 The solutions in Sections 4(b) and 4(c) obey the dissipation inequality.

S4.4.1 Spring-loaded solution.

The solution $\rho(t), \sigma(t), t \geq t_0$, found in Section 4(b)(ii) for the spring loaded specimen is shown on the $\rho, \sigma$-plane in Figure S.3. The trajectory starts from $(\rho(t_0), \sigma(t_0)) = (0.5 \rho_\infty, 0)$ and terminates at stall corresponding to $(\rho_\infty, \sigma_{stall})$. Since $\sigma_{stall} = 0.77 \text{nN}/\mu\text{m}^2$ and $E_\infty = 3.7 \text{nN}/\mu\text{m}^2$, the figure has been drawn with $\sigma_{stall}/E_\infty = 0.115$. We are only concerned with the region on and below the straight line $\sigma = f_{stall}\rho$ since the filament force then lies in the range $0 \leq f \leq f_{stall}$. The driving force is positive in the lightly shaded region below this line, and therefore on this region both the dissipation inequality $f_{driv} V \geq 0$ and the kinetic relation $V = V(f)$ give $V > 0$. The solution associated with the trajectory shown therefore satisfies the dissipation inequality.

\[5 \text{The solutions in Section 3 pertain to the case } E = E_0 \rho, \text{ and per the earlier discussion, obey the dissipation inequality.}\]
Figure S.3: The specimen is spring-loaded per Section 4.2. The dissipation inequality and kinetic relation both give $V > 0$ in the lightly shaded region. The reader is referred to Section S4.3 for the definition of the stress $\Sigma_{st}(\rho)$.

S4.4.2. Loading programs involving discontinuous stress.

Next consider the loading programs studied in Section 4(c) involving a discontinuous change in stress at a certain instant $t_2$. Since $\sigma_{stall} = 0.77 \text{nN/}\mu\text{m}^2$ and $E_\infty = 0.7 \text{nN/}\mu\text{m}^2$, in this case we have $\sigma_{stall}/E_\infty = 1.1$.

In the first calculation, the specimen initially grows under spring loading conditions and follows the trajectory shown in Figure S.4 for $t_0 < t < t_2$. It starts from $(\rho(t_0), \sigma(t_0)) = (0.5 \rho_\infty, 0)$ and evolves towards stall. However, when the stress reaches the value $0.49 \sigma_{stall}$ at time $t_2$, it is suddenly decreased to the value $0.41 \sigma_{stall}$ and held constant at that value for $t > t_2$. The trajectory therefore drops vertically at this instant and follows a rightward pointing horizontal line thereafter; this part of the trajectory falls within a very narrow sliver and is barely visible in Figure S.4. Again, the trajectory lies in the lightly shaded part of the figure where $f_{driv} > 0$ and $V > 0$ and therefore the solution satisfies the dissipation inequality.

In the second calculation, initially, for a period $t_0 \leq t \leq t_1$, the stress is held constant at the value $0.23 \sigma_{stall}$ and so the trajectory of $(\rho(t), \sigma(t))$ follows a rightward pointing horizontal line on the $\rho, \sigma$-plane. In order for this trajectory to be admissible by the dissipation inequality it must lie in the lightly shaded region in Figure S.5 and therefore $\rho(t_0)$ must exceed the value $\approx 0.6765$. We take $\rho(t_0) = 0.7 \rho_\infty$. At the instant $t_1$ the force clamp is released and
Figure S.4: The specimen is spring-loaded for $t_0 < t < t_2$. The stress is decreased suddenly at time $t_2$ and then held constant for $t > t_2$. Here $t_0 = 0$ and $t_2 = 100$ min. The stress $\Sigma_{st}(\rho)$ was defined in Section S4.3.

The specimen now grows under spring loading conditions. The trajectory for $t_1 < t < t_2$ is determined as in Section 4(b) with the initial condition $\sigma(t_1)/\sigma_{stall} = 0.23$, $\rho(t_1)/\rho_{\infty} = 0.919$ where the value of $\rho(t_1)$ was determined using equation (4.6). The solution now follows the curved trajectory shown in Figure S.5. At time $t_2$, when the stress has reached the value $0.28\sigma_{stall}$, it is suddenly decreased back to $0.23\sigma_{stall}$ and held constant at that value from then on. The trajectory lies in the lightly shaded region of the figure and therefore satisfies the dissipation inequality.

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Figure S.5: The stress is kept constant at the value $0.23\sigma_{\text{stall}}$ for time $t_0 < t < t_1$. The force clamp is released at time $t_1$ and the specimen is spring-loaded for $t_1 < t < t_2$. At time $t_2$ the stress is decreased suddenly back to its original value, and kept constant for $t > t_2$. Here $t_0 = 0$, $t_1 = 73$ min and $t_2 = 79$ min. The reader is referred to the appendix for the definition of the stress $\Sigma_{\text{st}}(\rho)$.

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