Thermal activation by power-limited coloured noise

Peter Jung¹, Alexander Neiman, Muhammad K N Afghan, Suhita Nadkarni and Ghanim Ullah
Department of Physics and Astronomy and Quantitative Biology Institute, Ohio University, Athens, OH 45701, USA
E-mail: jung@helios.phy.ohiou.edu

New Journal of Physics 7 (2005) 17
Received 10 September 2004
Published 31 January 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/017

Abstract. We consider thermal activation in a bistable potential in the presence of correlated (Ornstein–Uhlenbeck) noise. Escape rates are discussed as a function of the correlation time of the noise at a constant variance of the noise. In contrast to a large body of previous work, where the variance of the noise decreases with increasing correlation time of the noise, we find a bell-shaped curve for the escape rate with a vanishing rate at zero and infinite correlation times. We further calculate threshold crossing rates driven by energy-constrained coloured noise.

Contents

1. Introduction 1
2. Coloured noise driven bistable flow 4
   2.1. Escape rates for small correlation times 5
   2.2. Escape rates for large correlation times 5
3. Stochastic simulations 7
4. Threshold crossing driven by energy-limited coloured noise 7
5. Summary and discussion 10
References 10

1. Introduction

Ever since the pioneering work of Smoluchowski [1], scientists have been fascinated by the conceptual richness and predictive power of the evolving theory of stochastic processes. Applications of this theory are ubiquitous and range from chemical reaction rates (for an excellent

¹ Author to whom any correspondence should be addressed.
and comprehensive review see [2]), through quantum optics and atomic optics to models of neuroscience [3] and cell biology [4, 5].

The general subject is rooted in the study of the motion of small particles suspended in a fluid and moving under the influence of random forces that result from collisions with molecules of the fluid propelled by thermal fluctuations or, in short, the phenomenon of Brownian motion [6]. In early studies, the fluctuations occur on a time scale which is very much shorter than that of the Brownian particle. On the time scale of the Brownian particle, the correlation function of the random force \( \xi(t) \) driving the Brownian particle can then be assumed to be a \( \delta \)-function, i.e.

\[
\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t) \xi(t') \rangle = 2\delta(t - t').
\] (1)

This assumption considerably simplifies the problem, because it allows one to treat the stochastic dynamical motions as a Markov process for which many methods and approximation schemes are available [7]–[10]. Fluctuations that can be treated under this assumption are often termed ‘white noise’. The hallmark of white noise is its infinite variance. Although at first glance this seems unphysical, it is not. It reflects the compression of the originally finite collision time \( t_c \) during which the Brownian particle interacts with a molecule into an infinitesimally small time interval conserving the interaction energy.

In the physical world, the assumption that the fluctuations are fast in comparison with the relevant system time scales may not always be true. In a neuronal system, for example, fluctuations associated with the opening and closing of calcium channels are slow in comparison with those associated with the opening and closing of sodium and potassium channels. To approximate such fluctuations by noise-terms that are \( \delta \)-correlated would be inappropriate, and the resulting model would produce inaccurate predictions. That is why the topic of coloured noise in dynamical systems has attracted much attention (for a comprehensive review see [11]). One of the simplest examples of time-correlated noise is Gaussian noise \( \varepsilon(t) \) with zero mean and an exponential correlation function, i.e.

\[
\langle \varepsilon(t) \rangle = 0 \quad \text{and} \quad \langle \varepsilon(t) \varepsilon(t') \rangle = \frac{D}{\tau} \exp\left(-\frac{|t - t'|}{\tau}\right),
\] (2)

with variance \( \sigma^2 \equiv \langle \varepsilon^2 \rangle = D/\tau. \) In the limit \( \tau \to 0 \), the correlation function approaches the \( \delta \)-function and thus \( \varepsilon(t) \) is white noise if the noise strength \( D \) is kept constant. The linear Ornstein–Uhlenbeck process (with zero mean)

\[
\dot{\varepsilon} = -\frac{1}{\tau} \varepsilon + \frac{\sqrt{D}}{\tau} \xi(t),
\] (3)

with \( \xi(t) \) being characterized by (1) is consistent with the correlation function in equation (2) (transients neglected) and can be used as a generator of exponentially correlated noise. Equation (3) can be integrated, i.e.

\[
\varepsilon(t + \delta t) = \varepsilon(t) \exp\left(-\frac{\delta t}{\tau}\right) + \frac{\sqrt{D}}{\tau} \exp\left(-\frac{t + \delta t}{\tau}\right) \int_{t}^{t+\delta t} \exp\left(\frac{s}{\tau}\right) \xi(s) \, ds.
\] (4)

The philosophy behind the parametrization of the Ornstein–Uhlenbeck noise by the noise strength \( D \) is to preserve the ‘action’ of the noise on the position of a Brownian particle \( x(t) \) driven by \( \varepsilon(t) \) in the limit \( \tau \to 0 \). For example, in the case of simple Brownian motion

\[
\dot{x} = \varepsilon(t),
\] (5)
where $x(t)$ is the position of the Brownian particle, we define the action of the noise $\epsilon(t)$ as the change of the position of the Brownian particle within the time interval $\delta t$, i.e.

$$
\delta x = x(t + \delta t) - x(t) = \int_t^{t + \delta t} \epsilon(s) \, ds.
$$

Integrating equation (3) from $t$ to $t + \delta t$ one finds

$$
\epsilon(t + \delta t) - \epsilon(t) = -\frac{1}{\tau} \int_t^{t + \delta t} \epsilon(s) \, ds + \frac{\sqrt{D}}{\tau} \int_t^{t + \delta t} \xi(s) \, ds
$$

and, thus, by inserting equations (7) and (4) into equation (6)

$$
\delta x = \int_t^{t + \delta t} \epsilon(s) \, ds = -\tau \epsilon(t) \left( \exp\left(\frac{-\delta t}{\tau}\right) - 1 \right) + \sqrt{D \left( 2\delta t - 3\tau + 4\tau \exp\left(\frac{-\delta t}{\tau}\right) - \tau \exp\left(\frac{-2\delta t}{\tau}\right) \right)} G(1),
$$

where $G(a)$ is a random number drawn from a Gaussian distribution with variance $a$. In the limit $\tau \to 0$ (at constant $D$), only the first term under the square root of equation (8) is nonzero, yielding the action

$$
\delta x = \sqrt{2D\delta t} G(1),
$$

consistent with the white-noise Langevin equation $\dot{x} = \xi(t)$.

The philosophy for our approach in this paper is different. The starting point is a system of interest, represented by the system variable $x(t)$, which is driven by a fluctuating force $\epsilon(t)$ with correlation time $\tau$. The variance $\sigma^2$ of the noise $\epsilon(t)$ is considered to be fixed and the effects of such noise on $x(t)$ is studied as the correlation time $\tau$ is changed. The white-noise limit $\tau \to 0$ is now different from the white-noise limit in the interpretation above, as the variance of noise—as in a real physical system—does not grow to infinity. The action of such noise on a Brownian particle with position $x(t)$, described by equation (5) vanishes in the limit $\tau \to 0$ since $D = \sigma^2 \tau$ vanishes in equation (8).

This kind of coloured noise is best parametrized by the correlation time $\tau$ and the variance $\sigma^2$. The generating stochastic differential equation reads

$$
\dot{\epsilon} = -\frac{1}{\tau} \epsilon + \sqrt{\frac{\sigma^2}{\tau}} \xi(t),
$$

where $\xi(t)$ is characterized by (1). Since the equation is linear, the autocorrelation function can be easily determined as

$$
K(t - t') \equiv \langle \epsilon(t)\epsilon(t') \rangle = \sigma^2 \exp\left(\frac{-|t - t'|}{\tau}\right).
$$
The spectral density, given through the Wiener–Khintchine theorem by the Fourier transform of the autocorrelation function

\[ S(\omega) = \int_{-\infty}^{\infty} \langle \varepsilon(t)\varepsilon(0) \rangle \exp(-i\omega t) \, dt = 2\sigma^2 \frac{\tau}{1 + \omega^2 \tau^2}, \quad (12) \]

vanishes for \( \tau = 0 \) and for \( \tau \to \infty \) at a given value of the variance of the noise \( \sigma^2 \). As \( \tau \to 0 \), the spectral density becomes flat, i.e. the noise is more white, but the total power, i.e. the area under the spectral density

\[ \int_{-\infty}^{\infty} S(\omega) \, d\omega = \int_{-\infty}^{\infty} S(\omega) \exp(i\omega t) \bigg|_{t=0} \, d\omega = 2\pi K(0) = 2\pi \sigma^2, \quad (13) \]

is finite and conserved. The spectral density is a maximum when \( \omega \tau = 1 \).

Studies of Brownian particles that surmount fluctuating barriers [12]–[15] have addressed a similar problem. The potential fluctuates according to an Ornstein–Uhlenbeck process while the particle driven by thermal noise tries to surmount the barrier. While the limit of small correlation times is similar in this paper and the studies referenced above, the limit of large correlation times is very different as discussed below.

2. Coloured noise driven bistable flow

Here we consider a one-dimensional bistable flow driven by a coloured noise. This can be envisioned as the overdamped motion of a Brownian particle in a bistable potential \( V(x) \). The position \( x(t) \) of the particle obeys the differential equation

\[ m\gamma \ddot{x} + V'(x) = \varepsilon(t), \quad (14) \]

with the potential

\[ V(x) = \frac{b}{4}x^4 - \frac{a}{2}x^2, \quad (15) \]

where, \( a \) and \( b \) are the positive-valued parameters. For further discussion, it is useful to bring equations (14) and (15) together with equation (10) in a dimensionless form. With the new dimensionless variables

\[ \tilde{x} \equiv \frac{1}{\alpha}x \equiv \sqrt{\frac{b}{a}}x, \quad \tilde{t} \equiv \frac{1}{\beta}t \equiv \frac{a}{m\gamma}t, \quad \tilde{\varepsilon} \equiv \sqrt{\frac{b}{a^3}}\varepsilon, \quad \tilde{\xi}(\tilde{t}) \equiv \sqrt{\beta\xi(\beta\tilde{t})}, \quad (16) \]

and parameters

\[ \tilde{\sigma}^2 \equiv \frac{b}{a^3}\sigma^2, \quad \tilde{\tau} = \frac{a}{m\gamma}\tau, \quad (17) \]

and the subsequent omission of all bars in the new variables and parameters, one finds

\[ \dot{\tilde{x}} = \tilde{x} - \tilde{x}^3 + \tilde{\varepsilon}, \quad \dot{\tilde{\varepsilon}} = -\frac{1}{\tilde{\tau}}\tilde{\varepsilon} + \sqrt{\frac{\tilde{\sigma}^2}{\tilde{\tau}}}\tilde{\xi}(t), \quad (18) \]

with Gaussian white noise \( \tilde{\xi}(t) \) given by equation (1).
The solution for an unnormalized system, specified by any set of parameters $a$, $b$, $\gamma$ and $m$, can be obtained from the single solution of the scaled system by using the above transformation rules. For example, from escape rates $\bar{r}(\bar{\tau}, \bar{\sigma}^2)$ obtained from the scaled system, one can obtain the specific escape rates with parameters $a$, $b$, $\gamma$ and $m$ by

$$r(a, b, m, \gamma, \tau, \sigma^2) = \frac{1}{\beta(\gamma, a, m)} \bar{r}(\bar{\tau}(\tau, a, \gamma, m), \bar{\sigma}^2(\sigma^2, b, a))$$.

(19)

The Fokker–Planck equation for the probability density in the extended $x$–$\varepsilon$ phase space is given by

$$\frac{\partial P(x, \varepsilon, t)}{\partial t} = -\frac{\partial}{\partial x} \left( x - x^3 + \varepsilon \right) P(x, \varepsilon, t) + \frac{1}{\tau} \frac{\partial}{\partial \varepsilon} \varepsilon P(x, \varepsilon, t) + \frac{\sigma^2}{\tau} \frac{\partial^2}{\partial \varepsilon^2} P(x, \varepsilon, t)$$.

(20)

2.1. Escape rates for small correlation times

In the limit of small correlation times, i.e. $\tau \ll 1$ (in the dimensionless variables introduced above), the variable $\varepsilon$ in the second equation of (18) approaches a steady state at a rate $1/\tau$, much faster than the variable $x$. On the time scale of the variable $x$, we can thus set $\dot{\varepsilon} = 0$ thus finding $\varepsilon = \sqrt{\sigma^2 \tau} \xi(t)$ which can be inserted into the first equation of (18), yielding the white-noise Langevin equation for $x$

$$\dot{x} = x - x^3 + \sqrt{\sigma^2 \tau} \xi(t)$$.

(21)

equivalent to the Smoluchowski equation for the probability density $P(x, t)$

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x} \left( x - x^3 \right) P(x, t) + \sigma^2 \tau \frac{\partial^2}{\partial x^2} P(x, t)$$.

(22)

Following Kramer’s theory (for a review see [2]), the activation rate out of one of the potential minima at $x = \pm 1$ (in dimensionless units) is given by

$$r(\tau, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{4\tau \sigma^2} \right)$$.

(23)

In the limit $\tau \to 0$, the escape rate thus vanishes exponentially (see also figure 3). This is clearly a consequence of the fact that the spectral density of $\varepsilon(t)$ vanishes at $\tau = 0$.

2.2. Escape rates for large correlation times

For large correlation times, i.e. $\tau \gg 1$, the variable $\varepsilon$ is much slower than the system variable $x$. Thus, the system variable $x$ approaches a quasi-steady state $x_{ss}$

$$x_{ss} - x_{ss}^3 + \varepsilon = 0$$.

(24)

within a time interval in which the auxiliary variable $\varepsilon$ does not change. In other words, the system variable $x$ is relaxing to the minima of the potential

$$V(x, \varepsilon) = \frac{1}{4} x^4 - \frac{1}{2} x^2 - x\varepsilon$$.

(25)
given by the roots of equation (24). A transition of the system variable, say from the left to the right, happens when the left potential minimum disappears (see figure 1), i.e. when \( \varepsilon \) exceeds the critical value

\[
\varepsilon_{\text{crit}} = \frac{2}{\sqrt{27}}.
\]  

Thus, the mean first passage time \( T_{\text{MFPT}} \) is given by the mean first passage time of the auxiliary variable \( \varepsilon \), governed by equation (10) to reach the critical value \( \varepsilon_{\text{crit}} \) starting with an initial value \( \varepsilon_0 = -\varepsilon_{\text{crit}} \). The solution of this problem is standard [2, 16] and reads

\[
T_{\text{MFPT}} = \frac{\tau}{\sigma^2} \int_{-\varepsilon_{\text{crit}}}^{\varepsilon_{\text{crit}}} \exp \left( \frac{y^2}{2\sigma^2} \right) dy \int_{-\infty}^{y} \exp \left( -\frac{x^2}{2\sigma^2} \right) dx.
\]  

Most importantly, we can see from equation (27) that the escape rate \( r = 1/(T_{\text{MFPT}}) \) decreases proportional to \( \tau^{-1} \) as \( \tau \to \infty \). We evaluate the integrals numerically and find that the result compares favourably with simulations (see next section) shown in figure 3. The approach presented here is similar to the one put forward in [17] and [18] except that we do fully evaluate the mean first passage time without a weak-noise approximation.

Additional white thermal noise—as reported by Reimann et al [15]—results in a rate that also exhibits the bell-shaped curve as a function of the correlation time \( \tau \) but does not seem to decrease to zero proportional to \( \tau^{-1} \) (see figure 3 in [15]). As a matter of fact it does not seem to decrease to 0 for the values of the correlation times \( \tau \) presented.

Thus, the escape rate, at small \( \tau \), first increases with increasing correlation time, reaches a maximum and then decreases. For \( \tau \to 0 \), the spectral density of the fluctuations becomes flat and decreases in magnitude all over the frequency range relevant for barrier crossing, thus leading to a decreasing escape rate. For \( \tau \to \infty \), the system slows down and thus also escape events.
3. Stochastic simulations

Several numerical approaches were developed to simulate bistable systems driven by coloured noise. A precise approach based on the numerical calculation of the smallest eigenvalue of the corresponding Fokker–Planck equation (20) (see e.g. [10]) under appropriate boundary conditions was utilized by Jung and Hänggi [19]. An alternative approach based on the direct numerical simulation of the Langevin equations was introduced by Fox et al [20] and Fox [21]. Here we follow the latter procedure to simulate equations (18) and determine the mean time $\langle T \rangle$ in between two subsequent escape events over the barrier. The escape rate can be obtained as $r = 1/\langle T \rangle$. The scheme in equation (8) is exact to all orders in the time-step $\delta t$ but only valid for free Brownian motion. In the presence of a force field systematic schemes up to higher orders can be derived [21], but are cumbersome and no more efficient than first order schemes.

A first-order solver of equation (18) is given by

$$x(t + \delta t) = x(t) + \delta t[x(t) - x^3(t) + \varepsilon(t)], \quad \varepsilon(t + \delta t) = \varepsilon(t) \left(1 - \frac{\delta t}{\tau}\right) + \sqrt{\frac{2\sigma^2}{\tau}} G(1). \quad (28)$$

For a given set of parameters $(\tau, \sigma)$, equations (28) were iterated with a time step of $\delta t = 10^{-3}$ until $10^4$ escape events occurred. An escape event was identified by a transition between $x = -1$ and $x = 1$ (see figure 2). Time intervals between successive events are recorded and the average time interval, an approximation for $\langle T \rangle$ that becomes exact for infinitely long recording times is shown in figure 3. As predicted by the theory above, the escape rate for small $\tau$ increases, reaches a maximum, and then falls off $\propto 1/\tau$. The approximate results for small and large $\tau$ presented above compare favourably with the simulations.

4. Threshold crossing driven by energy-limited coloured noise

In this section we consider the effects of noise on a simple device that responds with a pulse $\delta(t - t')$ when its input $\varepsilon(t)$ crosses a threshold-value, $\varepsilon_0$, from below. The device is driven by Gaussian coloured noise with constant total power. In order for the threshold crossing rate to be non-infinite due to jitter around the threshold, at least the second moment $M_2$ of the spectral density $S(\omega)$, i.e.

$$M_n = \int_{-\infty}^{\infty} \omega^n S(\omega) \, d\omega, \quad (29)$$

or equivalently, the second derivative of the two-point autocorrelation function has to exist [22]. A possible realization of such a stochastic process $\varepsilon(t)$ is twice low-pass filtered Gaussian white noise, i.e. [23]

$$\dot{\varepsilon} = -\frac{1}{\tau_2} \varepsilon + \frac{1}{\tau_2} h, \quad \dot{h} = -\frac{1}{\tau_1} h + \frac{\sqrt{\sigma^2 (\tau_1 + \tau_2)}}{\tau_1} \xi(t), \quad (30)$$

with white Gaussian noise $\xi(t)$ given by equations (1):

$$\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = 2\delta(t - t')$$
with $\langle \varepsilon \rangle = 0$ and $\langle \varepsilon^2 \rangle = \sigma^2$ and cut-off frequencies $1/\tau_1$ and $1/\tau_2$, which we assume to be equal here, i.e. $\tau_2 = \tau_1 = \tau$. The two-point autocorrelation function and the spectral density of $\varepsilon(t)$ can be readily obtained as

$$\langle \varepsilon(t)\varepsilon(t') \rangle = \sigma^2 \left( 1 + \frac{|t-t'|}{\tau} \right) \exp \left(-\frac{|t-t'|}{\tau}\right)$$

(31)

and

$$S(\omega) = \frac{2\tau\sigma^2}{(1 + \tau^2\omega^2)^2}.$$  

(32)
Figure 3. Numerical calculation of the escape rate versus $\tau$ for indicated values of $\sigma^2$. Asymptotic predictions for small $\tau$ (equation (23)) and large $\tau$ (equation (27)) are shown by dotted and dashed lines, respectively.

The threshold crossing rate can be obtained readily [22],

$$r_{th} = \frac{1}{2\pi\tau} \exp \left( -\frac{\varepsilon_0^2}{2\sigma^2} \right).$$

(33)

Rewriting (1) as

$$\ddot{\varepsilon} + \frac{2}{\tau} \dot{\varepsilon} + \frac{1}{\tau^2} \varepsilon = \frac{\sqrt{2\sigma^2\tau}}{\tau^2} \xi(t),$$

(34)

the average energy is given by

$$E = \frac{1}{2} \langle \dot{\varepsilon}^2 \rangle + \frac{1}{2\tau^2} \langle \varepsilon^2 \rangle = \frac{\sigma^2}{\tau^2}.$$

(35)

The threshold crossing rate in terms of the correlation time $\tau$ and the average energy $E$ of the noise is given by

$$r_{th} = \frac{1}{2\pi\tau} \exp \left( -\frac{\varepsilon_0^2}{2\tau^2 E} \right).$$

(36)

Thus, at constant average energy $E$, the escape rate increases with increasing correlation time for small $\tau$, reaches a maximum at $\tau_m = \varepsilon_0/E^{1/2}$, and decreases to zero for $\tau \gg \tau_m$.

5. Summary and discussion

We have shown that the escape rate of a bistable system driven by coloured noise with constant power exhibits a bell-shaped dependence on the correlation time of the noise. This behaviour is
different from previous studies, where the power (or equivalently variance) of the noise decreases with increasing correlation time. There, as to be expected, the escape rate decreases monotonically with increasing correlation time of the noise \[19\]. In the limit \(\tau \to 0\), the escape rate vanishes since the spectral density vanishes uniformly (white noise with finite variance), while in the limit \(\tau \to \infty\), the escape slows down decreasing the escape rate. Our study is related to the problem of surmounting fluctuating barriers, whereby escape rates were studied versus correlation time of barrier fluctuations given its constant variance and additional thermal white noise \[12, 13, 15\]. A bell-shaped dependence of the escape rate versus correlation time was observed in \[15\]. However, the existence of additional white noise leads to a different asymptotic behaviour of the rate for larger \(\tau\), which depends on the ratio of white and coloured noise intensities. Threshold crossing rates in coloured-noise-driven threshold detectors also exhibit a bell-shaped dependence on the correlation time, if the average energy of the noise is kept constant.

References

[1] Smoluchowski M V 1913 Krakauer Berichte 418
    Smoluchowski M V 1915 Ann. Physik 48 1103
    Smoluchowski M V 1916 Z. Phys. 17 557
[2] Hänggi P, Talkner P and Borkovec M 1990 Rev. Mod. Phys. 62 251
[3] Kemptner R, Gerstner W, van Hemmen J L and Wagner H 1998 Neural Computation 10 1987
[4] Falcke M, Tsimring L and Levine H 2000 Phys. Rev. E 62 2636
[5] Shuai J W and Jung P 2002 Biophys. J. 83 87
[6] Einstein A 1905 Ann. Physik 17 549
    Einstein A 1926 Investigations on the Theory of Brownian Movement (London: Methuen; republication by New York: Dover, 1956)
[7] van Kampen N G 1992 Stochastic Processes in Physics and Chemistry (Amsterdam: North-Holland)
[8] Stratonovich R L 1963 Topics in the Theory of Random Noise (New York: Gordon and Breach)
[9] Wax N (ed) 1954 Selected Papers on Noise and Stochastic Processes (New York: Dover)
[10] Risken H 1984 The Fokker–Planck Equation: Methods of Solution and Applications (Berlin: Springer)
[11] Hänggi P and Jung P 1995 Adv. Chem. Phys. 89 239
[12] Doering C R and Gadoua J C 1992 Phys. Rev. Lett. 69 2318
[13] Pechukas P and Hänggi P 1994 Phys. Rev. Lett. 73 2772
[14] Madureira A, Hänggi P, Buonomano V and Rodrigues W A 1995 Phys. Rev. E 51 3849
    Madureira A, Hänggi P, Buonomano V and Rodrigues W A 1995 Phys. Rev. E 52 3301
[15] Reimann P, Bartussek R and Hänggi P 1998 Chem. Phys. 235 113
[16] Gardiner C W 1985 Handbook of Stochastic Methods (Berlin: Springer)
[17] de La Rubi F J, Peacook-Lopez E, Tsironis G P, Lindenberg K, Ramirez-Piscina L and Sancho J M 1988 Phys. Rev. A 38 3827
[18] Hänggi P, Jung P and Marchesoni F 1989 J. Stat. Phys. 54 1367
[19] Jung P and Hänggi P 1988 Phys. Rev. Lett. 61 11
[20] Fox R F, Gatland I R, Roy R and Vemuri G 1988 Phys. Rev. A 38 5938
[21] Fox R F 1991 Phys. Rev. A 43 2649
[22] Jung P and Mayer-Kress G 1995 Nuovo Cimento 17 827
[23] Jung P 1994 Phys. Rev. E 50 2513

New Journal of Physics 7 (2005) 17 (http://www.njp.org/)