DISPERSION RELATIONS
AND INCONSISTENCY OF $\rho$ DATA

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**Abstract.** The high energy elastic nucleon cross section is treated from the viewpoint of the basic principles of local field theory. The connection between the energy dependence of $\sigma_{\text{tot}}$ and the $\rho$ - ratio of the real to imaginary parts of the forward elastic scattering amplitude is examined in the framework of dispersion relations, derivative dispersion relations and crossing symmetry.

1. **Introduction**

Interest in diffractive processes is now revived due, in most part, to the discovery of rapidity gaps at HERA, i.e. of processes where diffractive interactions contribute a sizeable part of the whole amplitude. The diffraction interaction is defined by the multigluon colorless exchange named pomeron [1]. In the perturbative regime of QCD, it can be considered as a compound system of the two Reggeized gluons [2] in the approximation where one sums the leading ln’s in energy, though its nonperturbative structure is basically unknown. Research on the nature of pomeron requires the knowledge of the parameters of purely diffractive processes, such as the total cross section and the phase of the elastic scattering amplitude. These quantities are closely related with the first principles of field theory such as causality, polynomial boundedness, crossing symmetry etc. They are also important for modern nuclear physics at high energies, as these quantities underlie many modern descriptions of nuclear interactions when we study such effects as nuclear shadowing, transparency or broken parity [3].

The energy dependence of the cross section and of the parameter $\rho(s)$ - the ratio of the real to imaginary part of the scattering amplitude in the high energy domain - is a much discussed question which still remains without a definite answer. Quite many efforts were spent to understand
the high-energy hadron scattering from such general principles of relativistic quantum field theory as Lorenz invariance, analyticity, unitarity and crossing symmetry. Analyticity, which arises from the principle of causality, leads to dispersion relations which give the real part of the scattering amplitude at \( t = 0 \) as an energy integral involving \( \sigma_{\text{tot}} \).

The forward dispersion relations for nucleon-nucleon scattering have not been proven, although they were written down a long time ago [4]. Numerous papers have been devoted to calculations of the real parts of the \( pp \) and \( \bar{p}p \) forward scattering amplitude, using variety of dispersion relations and different representation for the energy dependence of the imaginary part of the scattering amplitude at \( t = 0 \).

The dispersion relations for scattering amplitudes depend on three properties, analyticity in the energy variable, the optical theorem, and polynomial boundedness. The prediction of the dispersion relation is very important for the discovery of new physical phenomena at superhigh energies. For example, in the case of potential scattering by nonlocal potentials the polynomial boundedness can be broken [5]. With some additional assumptions connected with string models [6], the effect of non-local behavior will decrease the phase, and hence increase \( \rho = \text{Re}F/\text{Im}F \). A different behavior can be obtained in the case of extra internal dimension [7]. So, such a remarkable effect (the deviation of \( \rho(s) \) from the prediction of the dispersion relation) could be discovered already at the LHC in the TOTEM experiment.

2. Different connections between the real and the imaginary part of the scattering amplitude

The optical theorem connects the imaginary part of the forward elastic scattering amplitude with the total cross section. \( \sigma_{\text{tot}}(s) = 4\pi\text{Im}F(s, t = 0) \). The crossing property of the scattering amplitude is used very often to relate the imaginary part to the real part of the scattering amplitude with the substitution \( S \rightarrow \tilde{S} = S\exp(-i\pi/2) \). In the case of the maximal behavior, the crossing-even and crossing-odd amplitudes at \( t = 0 \) are [10]

\[
\begin{align*}
F_+(s, t = 0)/is &= F_+(0) \left[ \text{ln}(s e^{-i\pi/2}) \right]^2 \\
F_-(s, t = 0)/is &= iF_-(0) \left[ \text{ln}(s e^{-i\pi/2}) \right]^2
\end{align*}
\]

On the other hand the integral dispersion relations give us the most powerful connections between \( \sigma_{\text{tot}} \) and \( \rho \). Their form depends on the behavior of the crossing-even and crossing-odd parts of the scattering amplitude. In the case where the crossing-even part of the scattering amplitude saturates the Froissart bound, so that the total cross section does not grow faster than a power of \( \text{ln}(s) \), the dispersion relation requires a single subtraction. If the
crossing-odd part of the scattering amplitude does not grow with energy, so that the difference of the total cross section between hadron-hadron and hadron-antihadron scattering falls with energy, the odd dispersion relation needs no subtraction

\[ Re f^{\text{even}}(E) = Re F^{\text{even}}(0) + P \frac{1}{\pi} \int_{m}^{\infty} dE' \frac{2 E^2}{(E'^2 - E^2)} Im f^{\text{even}}(E'); \] (2)

\[ Re f^{\text{odd}}(E) = P \frac{1}{\pi} \int_{m}^{\infty} dE' \frac{2 E}{(E'^2 - E^2)} Im f^{\text{odd}}(E'), \] (3)

where \( P \) denotes the principal part of the integrals.

On the basis of the integral dispersion relation, one obtained derivative dispersion relations (first used in the context of Regge theory \[11\] and developed in \[12\]) which are more suitable for calculation but, of course, have a more narrow region of validity of scattering amplitudes \[13, 14\]. It is to be noted that in this case we lose the subtraction constant. It means that the derivative dispersion relation can be used only at high energies where this constant is not perceived. The derivative dispersion relations can be written

\[ \frac{R(s)}{s^\alpha} = \tan\left(\frac{\pi}{2}(\alpha - 1) + \frac{d}{dn s} I(s) s^{-\alpha}\right), \] (4)

where the parameter \( \alpha \) is some arbitrary constant modifying the integral. Usually we choose \( \alpha = 1 \) from the sake of simplicity \[14\]. In \[15\] it was shown that if we fit the experimental data, the best value will is \( \alpha = 1.25 \). But in this case, the parameter \( \alpha \neq 1 \) will change the relation between the imaginary and the real parts of the scattering amplitude in the whole energy region. So, the predictions of the integral dispersion relations and of the derivative dispersion relation will not coincide. That means that one or the other is false. From the viewpoint of basic principles, which lead to the integral dispersion relation, we have to give preference to them. Hence, the equality of both representations at high energies requires that \( \alpha \) be equal to 1.

### 3. The experimental data and the check of the integral dispersion relations

Now it is a common belief that the experimental data confirm the validity of the dispersion relations. Of course, there is some ambiguity concerning the energy dependence of the total cross sections. Already the fit by Amaldi et al. \[16\] and by Amos et al. \[17\] has given the tendency for \( \sigma_{\text{tot}}(s) \) and \( \rho(s) \) energy to increase. Khuri and Kinoshita \[5\] have predicted that \( \rho(s)_{pp} \)
should approach zero from positive values. At the time of this prediction, \( \rho(s)_{pp} \) was known to be a small negative quantity implying, thus, that \( \rho(s)_{pp} \) should be zero at some energy, reach a maximum, and then tend to zero at still higher energies.

The comparison of the predicted values of the real part of the \( pp \) scattering amplitude with the experimental ones allows one to conclude that, on the whole, the experimental data agree well with the dispersion relation predictions. However if we take into account the whole set of experimental data, without any exception, we find that the total \( \chi^2 \) is large and that some experimental data contradict others. Some points of one experiment are situated above the theoretical line and some points of other experiment are below the theoretical curve by more than three statistical errors. Moreover, in [8], it was noted that experimental data on \( \bar{p}p \)-scattering around \( p_L = 5 \text{ GeV}/c \) disagreed with the analysis based on the dispersion relations [9].

At low energy, the definition of the subtraction constant is very important. The spin-independent amplitude could be continued into the low-energy region of \( pp \) scattering and into unphysical region. However, because of the lack of the data for the low energy \( pp \) scattering, such a continuation (e.g., by means of the effective range approximation) cannot be carried out. This is why in the dispersion relation calculations the imaginary part of the amplitude in the unphysical and low-energy region is sometimes decomposed into a series in which the coefficients are determined by comparison of the calculated real parts with the experimentally measured ones.

An alternative way of handling these regions is to replace the continuum of states with a set of bound states at fixed energies. Mathematically, this is equivalent to replacing a cut of an analytic function by a sum of poles (resonances). The values of their residues (coupling constant) are either a priori fixed, or are found by comparing the calculated values of the real part with the experimental data. One should note that the replacement of the cut with the poles is, in itself, an arbitrary procedure. Besides, both the exact number of poles-resonances and the values of their coupling constants are unknown.

Therefore, it seems to us that the apparent agreement of theoretical (i.e. by the dispersion relation) calculations with experimental data on the real parts of the \( pp \) forward scattering amplitude means only that the parameters in various approximations of unphysical and low-energy region of \( pp \) scattering can be chosen so as to obtain consistency between the theoretical calculations and the experimental data. Really, the dispersion relations for \( pp, \bar{p}p \) forward scattering have been tested at low energy, at best, only quantitatively.
4. The calculation of $\rho$ through different methods

As noted above, we can obtain the value of $\rho$ by different methods. The simplest one is to use the crossing symmetry properties of the scattering amplitude and to obtain the $ReF(s,t)$ by changing $S \rightarrow S exp(-i\pi/2)$ in all energy dependent parameters of a model, see for example [10]. In that case, the real part of the scattering amplitude is obtained straight away.

Another way to obtain the $ReF(s,t)$ is to use the derivative dispersion relations. The total cross section can be taken in form

$$\sigma_{tot} = Z_{pp} + A \ln(s/s_0) + Y_1 s^{-\eta_1} - Y_2 s^{-\eta_2}. \quad (5)$$

Hereafter we suppose, beside the special separate cases, that $s$ has the coefficient $s_1^{-1} = 1 \text{ GeV}$. The derivative dispersion relation gives us [18]

$$\rho(s,t) \sigma_{tot} = \frac{\pi}{2} A \ln\left(\frac{s}{s_0}\right) - Y_1 s^{-\eta_1} [\tan\left(\frac{1-\eta_1}{2}\pi\right)]^{-1} - Y_2 s^{-\eta_2} [\cot\left(\frac{1-\eta_2}{2}\pi\right)]^{-1}. \quad (6)$$

To accurately compare the results of the different calculations, let us examine carefully the calculations with the integral dispersion relation. There were many works on the methods of calculating the dispersion integrals [19]. From recent works, one should note the paper [20] where it was shown that for the most common parametrizations of the total cross section, the principal value integrals can be calculated in terms of rapidly convergent series. The advantages are not restricted to a reduction of the computer time.

Let us consider, as in [21], the standard dispersion relations for $pp \rightarrow pp$ (+) and $\bar{p}p \rightarrow p\bar{p}$ (−)

$$\rho(E) \pm \sigma_{\pm}(E) = \frac{B}{p} + \frac{E}{\pi p} P \int_{m}^{\infty} dE' p' \left[\frac{\sigma_{\pm}(E')}{E'(E' - E)} - \frac{\sigma_{\mp}(E')}{E'(E' + E)}\right], \quad (7)$$

where $E$ and $p$ are the energy and the momentum in the laboratory frame, $m$ is the proton mass and $B$ is a subtraction constant. In terms of the variable $s = 2m(E + m)$, a parametrization of the cross section can be written in the form (5).

In calculating the principal value of the dispersion integral, we face two difficulties. One is the singularity of the integral at the point $E' = E$ and the other is the infinity of the upper bound of the integral which is especially important in the case of a total cross section growing with energy. It is possible to solve both problems if the integral is divided in three parts. Let us cut the integral at the singularity point and select the last part which
contains the tail of the integral
\[
\int_m^\infty dE' p'[\frac{\sigma_\pm(E')}{E'(E' - E)} - \frac{\sigma_\mp(E')}{E'(E' + E)}] = I_1 + I_2 + I_3,
\] (8)
where
\[
I_1 = \int_m^{E-\epsilon} dE' p'[\frac{\sigma_\pm(E')}{E'(E' - E)} - \frac{\sigma_\mp(E')}{E'(E' + E)}],
\] (9)
\[
I_2 = \int_{E+\epsilon}^{kE} dE' p'[\frac{\sigma_\pm(E')}{E'(E' - E)} - \frac{\sigma_\mp(E')}{E'(E' + E)}],
\] (10)
\[
I_3 = \int_{kE}^\infty dE' p'[\frac{\sigma_\pm(E')}{E'(E' - E)} - \frac{\sigma_\mp(E')}{E'(E' + E)}].
\] (11)
Here \(k\) is a number, for example, \(k = 10\). The third integral can easily be calculated analytically in some of the most important cases, for example, if we take the nonfalling part of the total cross section in the form,
\[
\sigma_{\text{tot}}(s) = Z + A \ln^n(s/s_0).
\] (12)
Here we take for simplicity \(s_0 = 1\) and \(n = 1\) or \(n = 2\). The other cases are slightly more complicated, but do not lead to any principal difficulty.

Let us take \(k = 10\). For the constant term the integral is
\[
\int_{kE}^\infty \frac{2Z E}{E'(E'^2 - E^2)} dE' = 2Z \left[ \frac{1}{2kE} (ln(1 + a) - ln(1 - a) + ln(-\frac{1}{E}) - ln(\frac{1}{E})) \right] = 2.00671 \frac{Z}{k} = 0.200671 Z.
\] (13)
For the \(n = 1\) the growing term is
\[
\int_{kE}^\infty \frac{2E A \ln E'}{E'(E'^2 - E^2)} dE' = A \left[ 0.662285 + 0.20067 \ln E \right].
\] (14)
In the case of maximal, allowed by analyticity, growth of the total cross section, \(I_3\) is given by
\[
\int_{kE}^\infty \frac{2EA \ln^2 E'}{E'(E'^2 - E^2)} dE' = \frac{A}{12} \left[ 28.6337 + \ln E (15.8948 + 2.40805 \ln E) \right].
\] (15)
As a result, we can compare our computer calculations with our analytical calculation to find a suitable upper bound of the dispersion integral for
computer calculation. Such a comparison is shown in Figs. 1 and 2 where the line is the analytical calculations, and the box and triangles represent computer calculations. It is clear that both calculations coincide with high accuracy in all energy region.

Now return to the first two integrals. The numerical calculations show that, after removing the tail of the whole integral, the sum of the integrals $I_1$ and $I_2$ very fastly tends to its limit and cancels out the divergent parts. Usually, a precision of 1% is sufficient. As a result, we can easily control the accuracy of our calculations. The calculations of $\rho(s)$ by the dispersion relation are shown in Fig. 1 for the case $\sigma_{tot}(s) \approx \ln(s)$ and in Fig. 2 for the case $\sigma_{tot}(s) \approx \ln^2(s)$.

The calculations by the derivative dispersion relation are shown in these figures also. At high energies, both calculations coincide and give the same predictions. If we choose the corresponding value of the subtracting constant $B$, both calculations coincide with high accuracy at low energy too. The calculation using crossing-symmetry also coincides with both calculations in the high energy region and can coincide with the integral dispersion relations if we choose the corresponding size of the subtraction constant. But it will differ from the subtraction constant required for the derivative dispersion relations.

So, we can conclude that the descriptions at low and high energy of
Figure 2. The same as Fig.1 with $n = 2$

$\rho(s)$ are practically independent of each other in the case of the integral dispersion relation. The subtracting constant practically unites these two regions. So, the good description of the data of $\rho$ cannot give valuable information about the validity of our basic principle and weakly impacts on our prediction about the behavior of the total cross section and the parameter $\rho(s)$ at high energies.

5. The experimental data on $\rho$

Let us now consider very shortly the procedure of extracting parameters of the hadronic scattering amplitude from the experimental data $dN/dt$. In fact, in experiment we measure $dN/dt$, as a result of which “experimental” data such as $\sigma_{\text{tot}}$, $B$-slope and $\rho$ are extracted from $dN/dt$ with some model assumptions. Some assumptions are also needed to extrapolate the measured quantities to $t = 0$. Most important in this case is the determination of the normalization of the experimental data. Contributions from the electromagnetic and hadronic interactions exist in the hadron interaction at small angles. We can calculate the Coulomb amplitude with the required accuracy. The contribution of the Coulomb-hadron interference term in $dN/dt$ depends on the parameters of the hadronic amplitudes and the Coulomb-hadron phase also depends on these parameters. After the normalization of the $dN/dt$ data we obtain the differential elastic cross section.

The analysis of the elastic scattering data makes several assumptions. Very often one assumes that the real and imaginary parts of the nuclear amplitude have the same $t$ exponential dependence; the cross-odd part of the
scattering amplitude either has the same behavior of momentum transfer as the cross even part, or it is neglected; the contributions of the spin-flip amplitudes are neglected. Note that the value of $\rho$ is heavily correlated with the normalization of $d\sigma/dt$. Its magnitude weakly impacts on the determination of $\sigma_{tot}$ only in the case where the normalization is known exactly.

There is no experiment which measures the magnitudes of these parameters separately. Sometimes, to reduce experimental errors, the magnitude of some quantities is taken from another experiment. As a result, we can obtain a contradiction between the basic parameters for one energy (for example, if we calculate the imaginary part of the scattering amplitude, we can obtain a nonexponential behavior). It can lead to some errors in the analysis based on the dispersion relations.

For example, let us take one experiment (there are others which are very similar) which was carried out at $\sqrt{s} = 16. \div 27.4$ GeV by the JINR-FNAL Collaboration [22]. To analyze $dN/dt$ in this experiment, the energy dependence of $\sigma_{tot}$ and $B$ were taken in an analytical form supported by the data of other experiments. But the new fit of the $dN/dt$ data with all free parameters gives values of $\rho$ which are systematically above the original values [23]. Such a conclusion was obtained early in [24].

It means that the theoretical line obtained by the dispersion relation will be higher than in the existing descriptions and close to a simple description with the use of only the crossing symmetric properties of the scattering amplitude (see, Figs.1,2). But it does not change essentially our prediction for high energies, only the value of the subtraction constant will be changed.

6. Conclusions

The magnitudes of $\sigma_{tot}$, $\rho$ and slope $B$ have to be determined in one experiment and determinations of their magnitudes depend on each other. The procedure of extrapolation of the imaginary part of the scattering amplitude is significant for determining $\sigma_{tot}$. There can also exist some additional hypotheses which lead to a deviation of differential cross section at very small momentum transfer. For example, the analysis of experimental data shows a possible manifestation of the hadron spin-flip amplitudes at high energies. The research into spin effects will be a crucial stone for different models and will help us understand the interaction and structure of particles, especially at large distances. All this raises the question about the measurement of spin effects in elastic hadron scattering at small angles at future accelerators. Especially, we would like to note the programs at RHIC where the polarizations of both collider beams will be constructed. Additional information on polarization data will help us only if we know sufficiently well the beam polarizations. So, the normalisations of $dN/dt$
and $A_N$ are most important for the determination of the magnitudes of these values. New methods of extracting the magnitudes of these quantities are required.

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