Fixed Point Theorem in Fuzzy Automata Normed Linear Space

S.Karpagavalli
Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai, Tamil Nadu, India.
E-mail: valli2061@gmail.com

V.Visalakshi
Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai, Tamil Nadu, India.
E-mail: visalsenthil86@gmail.com

Abstract. The main purpose of this paper is to study the existence of a fixed point in fuzzy automata normed linear space. In particular, fixed point for a self map on compact fuzzy automata normed linear space has been established.

1. Introduction

Zadeh [12] established fuzzy set theory and its logic with vagueness. Any uncertainty can be interpreted by membership function which is consecutively called as a fuzzy set. The problems with ambiguity is mainly concerned with fuzzy set theory.

Fuzzy norm on linear space is introduced first by Katsaras [7] in year 1984. In 1992, Felbin[4] introduced the finite-dimensional fuzzy normed linear space. T.Bag and Samanta [2] aggregated many properties in completeness and compactness on finite dimensional fuzzy normed linear with a restricted choice of t-norm. Cheng and Mordeson [3] introduced fuzzy normed linear space based on the ideas of fuzzy metric space introduced by Karmosil and Michalek [8] have, defined fuzzy normed linear space and aggregated with many properties. The concept of fuzzy metric space introduced by Karmosil and Michalek [8] has been studied with some modifications by George and Veeramani [5]. Ismat Beg, Shaban Sedghi and Nabi Shobe [6] studied the fixed point theorem in fuzzy metric space.

S.P.Tiwari and many authors [9, 10, 11] has studied the algebraic properties of fuzzy automata in lattice ordered monoid and also introduced some topological concepts. B. Amudhambigai and V. Madhuri [1] introduced fuzzy automata normed linear space also introduced the topological concepts on fuzzy automata normed linear structure. The concept of fuzzy automata normed linear space introduced in [1] influenced us to obtain fixed point of self map satisfying a relation defined on non decreasing function in fuzzy automata normed linear space.
2. Preliminaries

2.1. Fuzzy Set

Let $X$ be a non empty set and $I = [0, 1]$. A fuzzy subset in $X$ is a function with domain $X$ and value in $I$, that is an element of $I^X$.

2.2. Fuzzy Normed Linear Space

Let $X$ be a linear space over the field $F$. A fuzzy subset $N$ on $X \times R$ is called a fuzzy norm on $X$ if and only if $x, y \in X$ and $c \in F$.

(i) $\forall \ t \in R\ with t \leq 0, N(x, t) = 0$
(ii) $\forall \ t \in R, t > 0 \ , N(x, t) = 1$ if $x = 0$
(iii) $\forall \ t \in R, t > 0$ then $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$
(iv) $\forall \ s, t \in R, x, y \in X \ , N(x + y, t + s) \geq \min \{N(x, t), N(y, s)\}$
(v) $N(x, \cdot)$ is a non decreasing function of $R$ and $\lim_{t \to \infty} N(x, t) = 1$.

The pair $(X, N)$ will be referred to as a fuzzy normed linear space (FNLS).

2.3. Fuzzy Automata Normed Linear Space

Let $N = (Q, X, \delta)$ be a fuzzy automata. A fuzzy automata normed linear space (FANLS) is a 3-tuple $(Q, N, T)$ where $Q$ is non-empty set of states of $N$ and also it is linear space over the field $F$, $T$ is a $t$-norm and $N$ is a fuzzy set on $Q \times (0, \infty)$ such that $\forall \ p, q \in Q$ and all $\nu, \mu > 0$ the following condition:

(i) $N(p, \mu) > 0$
(ii) $N(p, \mu) = 1 \ \forall \ \mu > 0$ iff $p = 0$
(iii) If $\alpha \neq 0$ then $N(\alpha p, \mu) = N(p, \frac{\mu}{|\alpha|}) \ \forall \ \mu, \alpha \in F$
(iv) $\forall \ p, \mu \in Q, T[N(q, \mu), N(q, \nu)] \leq N(p + q, \mu + \nu) \ \forall \ \mu, \nu \in F$
(v) $N(p, \cdot)$ is a non-decreasing function $F$ and $\lim_{\mu \to \infty} [N(p, \mu)] = 1$
(vi) Assume that for all $p \neq 0$, $N(p, \cdot)$ is a continuous function of $F$ and strictly increasing on the subset $0 < N(p, \mu) < 1$ of $F$.

3. Fixed Point Theorem on compact FANLS

Let $(Q, N, T)$ be a FANLS and $\phi \neq Y \subseteq Q$. Define $\delta_N(Y, \mu) = \inf \{N(q - r, \mu) : q, r \in Y \}$ $\forall \ \mu > 0$. For an $B_n = \{q_n, q_{n+1}, \ldots\}$ in a FANLS $(Q, N, T)$. Let $c_n(\mu) = \delta_N(B_n, \mu)$. Then $c_n(\mu)$ is finite $\forall \ n \in N$, $\{c_n(\mu)\}$ is a non-increasing $c_n(\mu) \to c(\mu)$ for some $0 \leq c(\mu) \leq 1$ and also $c_n(\mu) \leq N(q_i - q_j, \mu)$ $\forall i, j \geq n$. Let $\mathcal{F}$ be the set all continuous function on $F : \mathcal{F} \to [-1, 1]$ where $\mathcal{F}$ is $(0, 1)^3 \times (0, 1)$ such that $F$ is non-decreasing continuous function such that $\mathcal{F}(t, t, t), s \leq 0$ it follows that $s \geq \beta(t)$ where $\beta$ is non decreasing continuous function on $[0, 1]$ such that $\beta(s) > s$ for $s \in [0, 1]$.

3.1. Continuous

Let $(Q, N, T)$ be a FANLS $N$ is said to be continuous on $Q \times (0, \infty)$ if $\lim_{n \to \infty} N(q_n - r_n, \mu) = N(q - r, \mu)$ whenever $\{q_n - r_n, \mu_n\}$ is a sequence in $Q \times (0, \infty)$ which is converges to a point $(q - r, \mu) \in Q \times (0, \infty)$ that is $\lim_{n \to \infty} N(q_n - q, \mu) = \lim_{n \to \infty} N(r_n - r, \mu) = 1$.

3.2. Convergent

Let $(Q, N, T)$ be a FANLS, let $\{q_n\}$ be a sequence in $Q$. Then $\{q_n\}$ is said to be convergent if there exist $q \in Q$ such that $\lim_{n \to \infty} N(q_n - q, \mu) = 1 \ \forall \ \mu > 0$. Then $q$ is called the limit of the sequence $\{q_n\}$. The limit of a sequence $\{q_n\}$ is denoted by $q = \lim q_n$. 
3.3. Compact
A FANLS in which every sequence has a convergent subsequence is said to be compact FANLS.

3.4. Theorem
Let \((Q, N, \delta)\) be a compact FANLS and \(S\) is a continuous self map of \(Q\) satisfying \(\forall q-r \in Q\) with \(F(N(q-r, \mu), N(Sq - q, \mu), N(Sq-r, \mu)) < 0\) where \(F \in \mathcal{F}\). Then \(S\) has atmost one fixed point \(x\) in \(Q\) such that \(S(x) = x\)

**Proof:**
For \(n = 1, 2, 3, \ldots\), \(S^nQ\) is Compact and also \(S^{n+1}Q \subset S^nQ\). Let \(Q_0 = \bigcap_{n=1}^{\infty} S^nQ\),then \(Q_0\) is a non empty compact subset of \(Q\) and \(SQ_0 = Q_0\). It is enough to show that \(Q_0\) is a Set with one element . Suppose \(Q_0\) is set with more then one element. Then the function \(N : Q^2 \times (0, \infty) \rightarrow [-1, 1]\) has a unique point \(x\) in \(Q\) such that \(N(q-r, \mu) \leq N(q-r, \mu) \forall q-r \in Q_0\). Since \(SQ_0 = Q_0\) \(\exists q_1 - r_1 \in Q_0\) such that \(S(q_1 - r_1, \mu) = 0\),\(S(r_1, \mu)\) \(\mu\) and \(\mu\) satisfies all the hypothesis and hence \(S\) has a fixed point in \(Q\).

3.5. Corollary
Let \((Q, N, \delta)\) be a Compact FANLS, \(m \in N\) and \(S^m\) is a continuous self map of \(Q\) satisfying \(\forall q-r \in Q\) with \(F(N(q-r, \mu), N(S^m q - q, \mu), N(S^m q-r, \mu), N(S^m q-S^m r, \mu)) < 0\) where \(F \in \mathcal{F}\). Then, \(S\) has atmost one fixed point \(x\) in \(Q\) such that \(S(x) = x\)

**Proof:**
\(S^m\) has a unique point \(x\) in \(Q\). Since \(Sx = S^m x = S^m Sx\)
\(Sx\) is also a fixed point of \(S^m\). By the previous theorem, it follows \(Sx = x\).

3.6. Example
Let \(Q \subseteq R\), \(Q = \left\{ \frac{1}{n} \right\} \mid n \in N \cup \{0\}\). Define \(N : Q \times (0, \infty) \rightarrow [0, 1]\) by \(N(p, \mu) = \frac{\mu}{\mu + |p|} \forall p \in Q\) and \(\mu \in [0, \infty]\). Then \((Q, N, T)\) is a compact FANLS. Define a map \(S : Q \rightarrow Q\) by \(S(q) = q - r \\forall q \in Q\) and \(F : \mathcal{F} \rightarrow [-1, 1]\) where \(\mathcal{F}\) is \((0, 1)^3 \times (0, 1)\) defined by \(F((u_1, u_2, u_3), u_4) = \beta(\min\{u_1, u_2, u_3\}) - u_4\) and non decreasing function \(\beta\) on \([0, 1]\) is defined by \(\beta(s) = 1 - \sqrt{s}\), \(s \in [0, 1]\) and \(\beta(s) > s\), \(s \in (0, 1)\). \(S\) satisfies all the hypothesis and hence \(S\) has atmost one fixed point.

4. Reference
[1] Amudhambigai.B and Madhuri.V, A new view on fuzzy automata normed linear structure space, *Iranian journal of fuzzy system*. (accepted)
[2] Bag .T and Samanta .S, 2003, Finite dimensional fuzzy normed linear space, *The Journal of fuzzy Mathematics*, Vol.3, 687-705.
[3] Cheng.S.C and Mordeson.J.N., 1994, Fuzzy linear operators and fuzzy normed linear spaces, *Bulletin of the calcutta mathematical society*, Vol.86, pp.429-436.
[4] Felbin .C, 1992, Finite dimensional fuzzy normed linear space, *Fuzzy sets and system*, Vol.48, pp.239-248.
[5] George.A and Veeramani.P,1994, On some result in fuzzy metric space, *Fuzzy set and systems*, Vol.64, No.3.
[6] Ismat Beg, Shaban Sedghi and Nabi Shobe,2013, Fixed point theorem in fuzzy metric space, *International journal of analysis*, Article ID 934145.
[7] Katsaras.A.K,1984, Fuzzy topological vector space, *Fuzzy sets and system*, Vol.12, No.2, pp.143-154.
[8] Kramosil.A.K and Michakel.J, 1975, Fuzzy metric and statistical metric space, Kybernetica, Vol.11, pp.326-334.
[9] Tiwari.S.P and Sharan.S, 2012, Fuzzy automata based on lattice ordered monoids with algebraic and topological aspects, Fuzzy information and engineering, Vol.2, pp.155-164.
[10] Tiwari.S.P, Singh.A.K, Sharan . S, 2012, Fuzzy automata based on lattice ordered monoid and associated topology, Journal of uncertain system, Vol.2, pp.155-164.
[11] Tiwari.S.P, Singh .A.K, Sharan.S, Yadev.V.K, 2015, Bifuzzy core of fuzzy automata, Iranian journal of fuzzy system, Vol.12, pp.63-73.
[12] Zadeh.L.A, 1965, Fuzzy sets, Information and Control, Vol.8, No.3, pp.338-358.