On the Fundamental Limits of Cache-aided Multiuser Private Information Retrieval

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Abstract

We consider the problem of cache-aided Multiuser Private Information Retrieval (MuPIR) which is an extension of the single-user cache-aided PIR problem to the case of multiple users. In MuPIR, each of the $K_u$ cache-equipped users wishes to privately retrieve a message out of $K$ messages from $N$ databases each having access to the entire message library. The privacy constraint requires that any individual database learns nothing about the demands of all users. The users are connected to each database via an error-free shared-link. In this paper, we aim to characterize the optimal trade-off between users’ memory and communication load for such systems. Based on the proposed novel approach of cache-aided interference alignment (CIA), first, for the MuPIR problem with $K = 2$ messages, $K_u = 2$ users and $N \geq 2$ databases, we propose achievable retrieval schemes for both uncoded and general cache placement. The CIA approach is optimal when the cache placement is uncoded. For general cache placement, the CIA approach is optimal when $N = 2$ and 3 verified by the computer-aided approach. Second, when $K, K_u$ and $N$ are general, we propose a new product design (PD) which incorporates the PIR code into the linear caching code. The product design is shown to be order optimal within a multiplicative factor of 8 and is exactly optimal when the user cache memory size is large.

Index Terms
I. INTRODUCTION

Introduced by Chor et al. in 1995 [3], the problem of private information retrieval (PIR) seeks efficient ways for a user to retrieve a desired message from $N$ databases, each holding a library of $K$ messages, while keeping the desired message’s identity private from each database. Sun and Jafar (SJ) recently characterized the capacity of the PIR problem with non-colluding databases [4], [5]. Coded caching was originally proposed by Maddah-Ali and Niesen (MAN) in [6] for a shared-link caching network consisting of a server, which is connected to $K_u$ users through a noiseless broadcast channel and has access to a library of $K$ equal-length files. Each user can cache $M$ files and demands one file. The MAN scheme proposed a combinatorial cache placement design so that during the delivery phase, each transmitted coded message is simultaneously useful to multiple users such that the communication load can be significantly reduced. Under the constraint of uncoded cache placement and for worst-case load, the MAN scheme was proved to be optimal when $K \geq K_u$ [7] and order optimal within a factor of 2 in general [8].

The combination of privacy and coded caching has gained significant attentions recently. Two different privacy models are commonly considered. First, in [9]–[11], the user-against-database privacy model was studied where individual databases are prevented from learning the single-user’s demand. The author in [9] studied the case where a single cache-aided user is connected to a set of $N$ replicated databases and showed that memory sharing between the memory-load pairs $(0, 1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}})$ and $(K, 0)$ (i.e., split the messages and cache memory proportionally and then implement two PIR schemes on the independent parts of the messages) is actually optimal if the databases are aware of the user’s cached content. However, if the databases are unaware of the user’s cached content, then there is a multiplicative “unawareness gain” in capacity in terms of the user memory as shown in [10], [11]. Second, the authors in [12]–[15] considered the user-against-user privacy model where users are prevented from learning each other’s demands. The authors in [12] first formulated the coded caching with private demands problem where a shared-link cache network with demand privacy, i.e., any user can not learn anything about the demands of other users, was considered. The goal is to design efficient delivery schemes such
that the communication load is minimized while preserving such privacy. Order optimal schemes were proposed based on the concept of virtual user.

This paper formulates the problem of cache-aided Multiuser PIR (MuPIR), where each of the $K_u$ cache-equipped users aims to retrieve a message from $N$ distributed non-colluding databases while preventing any one of them from gaining knowledge about the user demands given that the cached content of the users are known to the databases. The contributions of this paper are as follows. First, based on the novel idea of cache-aided interference alignment (CIA), we construct cache placement and private delivery phases achieving the memory-load pairs $(\frac{N-1}{2N}, \frac{N+1}{N})$ and $(\frac{2(N-1)}{2N-1}, \frac{N+1}{2N-1})$ for the MuPIR problem with $K = 2$ messages, $K_u = 2$ users and $N \geq 2$ databases. Different from the existing cache-aided interference alignment schemes in [18]–[21] which were designed for the cache-aided interference channels, the purpose of our proposed private delivery scheme is to let each server send symmetric messages (in order to keep user demand privacy), each of which contains some uncached and undesired symbols (i.e., interference) of each user. The proposed CIA approach effectively aligns these interference for each user and thus facilitates correct decoding. We prove that the proposed scheme is optimal under the constraint of uncoded cache placement. Computer-aided investigation given in [22] also shows that the proposed schemes are optimal for general cache placement when $N = 2$ and $3$. Second, for general system parameters $K, K_u$ and $N$, we propose a Product Design (PD) which incorporates the SJ scheme [5] into the MAN coded caching scheme [6]. Interestingly, the load of the proposed design is the product of the loads achieved by these two schemes and is optimal within a factor of 8. Moreover, the PD is exactly optimal when the users’ cache memory size is beyond a certain threshold. Finally, we characterize the optimal memory-load trade-off for the case of $K = K_u = N = 2$ where users request distinct messages. It is shown that under the constraint of the distinct demands, the achieved load can be smaller than the case without constraint of distinct demands.

The paper is organized as follows. In Section II we present the formal problem formulation. The main results of this paper are given in Section III In Section IV we describe the proposed CIA schemes in detail and in Section V we present the product design for general system.

1 Note that the virtual user strategy and the strategy based on scalar linear function retrieval for coded caching with private demands [12], [16], [17] were designed based on the fact that the user caches are not transparent to each other. Therefore, they cannot be used in the considered MuPIR problem, where databases are aware of the user caches.
Fig. 1: Cache-aided MuPIR system with $N$ replicated databases, $K$ independent messages and $K_u$ cache-equipped users. The users are connected to each DB via an error-free shared-link broadcast channel.

parameters. We discuss some interesting observations for distinct demands in Section VI. Finally, we conclude this paper and present several future directions in Section VII.

Notation Convention: $|·|$ represents the cardinality of a set. $[n] \triangleq \{1, 2, \cdots, n-1, n\}$, $[m:n] \triangleq \{m, m+1, m+2, \cdots, n\}$ and $(m:n) \triangleq (m, m+1, \cdots, n)$ for some integers $m \leq n$. $\mathbb{Z}^+$ denotes the set of non-negative integers. For two sets $A$ and $B$, define the difference set as $A \setminus B \triangleq \{x \in A : x \notin B\}$. For an index set $\mathcal{I}$, the notation $A_\mathcal{I}$ represents the set $\{A_i : i \in \mathcal{I}\}$. When $\mathcal{I} = [m:n]$, we write $A_{[m:n]}$ as $A_{m:n}$ for brevity. For an index vector $I = (i_1, i_2, \cdots, i_n)$, the notation $A_I$ represents a new vector $A_I \triangleq (A_{i_1}, A_{i_2}, \cdots, A_{i_n})$. $\mathbf{0}_n$ denotes the all-zero vector with length $n$, i.e., $\mathbf{0}_n \triangleq (0, 0, 0, \cdots, 0)$. Let $\mathbf{1}_n \triangleq (1, 1, \cdots, 1)$ with length $n$. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. For a matrix $A$, $A(i,:)$ and $A(:,j)$ denote the $i$-th row and $j$-th column of $A$ respectively. $A^T$ represents the transpose of $A$. $\mathbb{F}_2$ represents the binary field. In this paper, the operations (addition and linear combination) are on the binary field.

II. PROBLEM FORMULATION

We consider a system with $K_u$ users, each aiming to privately retrieve a message from $N \geq 2$ replicated non-colluding databases (DBs). Each DB stores $K$ independent messages, denoted by $W_1, W_2, \cdots, W_K$, each of which is uniformly distributed over $[2^L]$. Each user is equipped with a cache memory of size $ML$ bits, where $0 \leq M \leq K$. Let the random variables $Z_1, Z_2, \cdots, Z_{K_u}$ denote the cached content of the users. The system operates in two phases, the cache placement phase followed by the private delivery phase. In the cache placement phase, the users fill up their cache memory without knowledge of their future demands. We assume that the cached content of each user is a function of $W_{1:K}$ and is known to all DBs. In the private delivery phase, each
user $k \in [K_u]$ wishes to retrieve a message $W_{\theta_k}$ where $\theta_k \in [K]$. Let $\theta \triangleq (\theta_1, \theta_2, \ldots, \theta_{K_u})$ denote the demands of the users. We assume that $\theta$ follows an arbitrary distribution with full support over $[K]^{K_u}$. Depending on $\theta$ and $Z_1, Z_2, \ldots, Z_{K_u}$, the users cooperatively generate $N$ queries $Q^{[\theta]}_1, Q^{[\theta]}_2, \ldots, Q^{[\theta]}_N$, and then send the query $Q^{[\theta]}_n$ to DB $n$. Upon receiving the query, DB $n$ responds with an answer $A^{[\theta]}_n$ broadcasted to all users. The answer $A^{[\theta]}_n$ is a deterministic function of $Q^{[\theta]}_n$, $W_{1:K}$ and $Z_{1:K}$, which written in terms of conditional entropy, is

$$H(A^{[\theta]}_n|Q^{[\theta]}_n, W_{1:K}, Z_{1:K}) = 0, \quad \forall n \in [N].$$

After collecting all the answers from the $N$ DBs, the users can recover their desired messages error-free using their cached information, i.e.,

$$H(W_{\theta_k}|Q^{[\theta]}_{1:N}, A^{[\theta]}_{1:N}, Z_k) = 0, \quad \forall k \in [K_u].$$

To preserve the privacy with respect to the DBs, it is required that $^2$

$$I(\theta; Q^{[\theta]}_n, A^{[\theta]}_n, W_{1:K}, Z_{1:K}) = 0, \quad \forall n \in [N].$$

Let $D$ denote the total number of bits broadcasted from the DBs, then the load (or transmission rate) of the MuPIR problem is defined as

$$R \triangleq \frac{D}{L} = \frac{\sum_{n=1}^{N} H(A^{[\theta]}_n)}{L}.$$  (4)

From the privacy constraint (3), the expression of load can be also written as

$$R = \frac{\sum_{n=1}^{N} H(A^{[\theta]}_n)}{L}, \quad \forall i \in [K^{K_u}],$$

where $\theta^i$ represents the $i$-th realization of all the $K^{K_u}$ possible realizations of the demand vector. This is because the load $R$ should not depend on the user demands $\theta$ otherwise it leaks information about the user demands to the DBs and (3) would not be possible.

A memory-load pair $(M, R)$ is said to be achievable if there exists a MuPIR scheme satisfying the decodability constraint (2) and the privacy constraint (3). The goal of the MuPIR problem is to design the cache placement and the corresponding private delivery phases such that the load is minimized. For any $0 \leq M \leq K$, let $R^*(M)$ denote the minimal achievable load. In addition,

\footnote{The privacy constraint (3) can be equivalently written as $I(\theta; Q^{[\theta]}_n, W_{1:K}, Z_{1:K}) = 0, \forall n \in [N]$ since the answer $A^{[\theta]}_n$ is a deterministic function of $Q^{[\theta]}_n$ and $W_{1:K}$, which implies $I(\theta; Q^{[\theta]}_n, A^{[\theta]}_n, W_{1:K}, Z_{1:K}) = I(\theta; Q^{[\theta]}_n, W_{1:K}, Z_{1:K}) + I(\theta, A^{[\theta]}_n|Q^{[\theta]}_n, W_{1:K}, Z_{1:K}) = I(\theta; Q^{[\theta]}_n, W_{1:K}, Z_{1:K})$.}
if the users directly cache a subset of the library bits, the placement phase is said to be *uncoded*. We denote the minimum achievable load under the constraint of uncoded cache placement by $R_{\text{uncoded}}^*(M)$.

Note that any converse bound on the worst-case load for the coded caching problem without privacy constraint formulated in [6] is also a converse bound on $R^*(M)$. Suppose $R'(M)$ is a converse bound on the worst-case load for the coded caching problem without privacy. Then for any achievable caching scheme there must be exist one demand vector realization $\theta' \in [K]^{K_u}$ such that in order to satisfy this demand, the total number of broadcasted bits is no less than $R'(M)L$. With the consideration of the privacy constraint in (3) and the definition of the load in (5), it can be seen that

$$R^*(M) = \frac{1}{L} \sum_{n=1}^{N} H(A_n^{\theta'_n}) \geq R'(M). \quad (6)$$

Similarly, any converse bound on the worst-case load under the constraint of uncoded cache placement for the coded caching problem without privacy constraint formulated in [6] is also a converse bound on $R_{\text{uncoded}}^*(M)$. Suppose $R''(M)$ is a converse bound on the worst-case load under the constraint of uncoded cache placement for the coded caching problem without privacy; thus we have

$$R_{\text{uncoded}}^*(M) \geq R''(M). \quad (7)$$

### III. MAIN RESULTS

First, we consider the MuPIR problem with parameters $K = K_u = 2$ and $N \geq 2$. In this case, we propose a novel *cache-aided interference alignment* (CIA) based scheme (see Section [IV]) and the corresponding load is given by Theorem 1.

*Theorem 1:* For the cache-aided MuPIR problem with $K = 2$ messages, $K_u = 2$ users and $N \geq 2$ DBs, the following load is achievable

$$R_{\text{CIA}}(M) = \begin{cases} 
2(1 - M), & 0 \leq M \leq \frac{N-1}{2N}, \\
\frac{(N+1)(3-2M)}{2N+1}, & \frac{N-1}{2N} \leq M \leq \frac{2(N-1)}{2N-1}, \\
\frac{(N+1)(2-M)}{2N}, & \frac{2(N-1)}{2N-1} \leq M \leq 2. 
\end{cases} \quad (8)$$

*Proof:* The proof of Theorem 1 is provided in Section [IV], where we present the CIA approach achieving the memory-load pairs $\left(\frac{N-1}{2N}, \frac{N+1}{N}\right)$ and $\left(\frac{2(N-1)}{2N-1}, \frac{N+1}{2N-1}\right)$. Together with the
two trivial pairs \((0,2)\) and \((2,0)\), we obtain four corner points. By memory sharing among these corner points, the load in Theorem 1 can be achieved.

**Remark 1:** The computer-aided approach given in [22] shows that the achievability result in Theorem 1 is optimal when \(N = 2, 3\). For \(N \geq 4\), the converse remains open. In addition, the achieved load by the CIA based scheme is strictly better than applying twice the single-user cache-aided PIR of [9] for each user, which yields a load of \(\frac{(N+1)(2-M)}{N} \geq R_{\text{CIA}}(M), \forall M \in [0, 2]\) (i.e., the achieved memory-load tradeoff is the memory sharing between \((0, N+1)\) and \((2, 0)\)).

**Corollary 1:** The load \(R_{\text{CIA}}(M)\) in Theorem 1 is optimal when \(M \geq \frac{2(N-1)}{2N-1}\).

**Proof:** When \(M \geq \frac{2(N-1)}{2N-1}\), the load of \(R_{\text{CIA}}(M) = \frac{(N+1)(2-M)}{2N}\) coincides with the converse bound for the single-user cache-aided PIR in [9]. Since increasing the number of users \(K_u\) cannot decrease the load, the achieved load in Theorem 1 is optimal.

**Corollary 2:** For the cache-aided MuPIR problem with \(K = 2\) messages, \(K_u = 2\) users and \(N \geq 2\) databases, the optimal memory-load trade-off under uncoded cache placement is characterized as

\[
R_{\text{uncoded}}^*(M) = \begin{cases} 
2 - \frac{3}{2}M, & 0 \leq M \leq \frac{2(N-1)}{2N-1} \\
\frac{(N+1)(2-M)}{2N}, & \frac{2(N-1)}{2N-1} \leq M \leq 2
\end{cases}
\]  

(9)

**Proof:** For achievability, the corner points in Theorem 2 are the memory-load pairs \((0,2)\), \(\left(\frac{2(N-1)}{2N-1}, \frac{N+1}{2N-1}\right)\), and \((2,0)\), which can be achieved by the same scheme as Theorem 1. It can be seen in Section IV that the achievable schemes for these corner points are uncoded, and by memory sharing, the load of Theorem 2 can be achieved.

Under the assumption of uncoded cache placement, as shown in (7), the converse bound for the shared-link coded caching problem without privacy constraint in [23], [24] is also a converse for our considered cache-aided MuPIR problem. When \(0 \leq M \leq 1\), it was proved in [23], [24] that

\[
R_{\text{uncoded}}^*(M) \geq 2 - \frac{3}{2}M.
\]  

(10)

In addition, the converse bound for the single-user cache-aided PIR problem in [9] is also a converse for the considered problem since increasing the number of users cannot reduce the load. When \(0 \leq M \leq 2\),

\[
R_{\text{uncoded}}^*(M) \geq \left(1 - \frac{M}{2}\right) \left(1 + \frac{1}{N}\right) = \frac{(N+1)(2-M)}{2N}.
\]  

(11)
By combining (10) and (11), (9) can be obtained, which coincides with the achievability.

For general parameter settings, we propose a Product Design (PD) design (Section V). The corresponding achievable load is given by the following theorem.

**Theorem 2:** The proposed product design achieves the load of \( R_{PD}(M) = \min \{ K(1 - \frac{M}{K}), \hat{R}(M) \} \) in which

\[
\hat{R}(M) = \frac{K_u - t}{t + 1} \left( 1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}} \right),
\]

where \( t = \frac{K_u M}{K} \in \mathbb{Z}^+ \). For non-integer values of \( t \), the lower convex envelope the integer points \( \left( \frac{tK}{K_u}, \hat{R}(M) \right) \) in which \( \frac{tK}{K_u} \in [K_u] \) can be achieved. Moreover, \( \frac{R_{PD}(M)}{R^*(M)} \leq 8 \).

**Proof:**

**Achievability:** See Section V for the achievable scheme to achieve \( \hat{R}(M) \). A naive scheme suffices to achieve the load \( K(1 - \frac{M}{K}) \) which is described as follows. Let each user cache the same \( \frac{M}{K} \) portion of each message. In the private delivery phase, the remaining \( 1 - \frac{M}{K} \) portion of each message is broadcasted to the users. It can be seen that each user can correctly decode all \( K \) messages. Hence, the scheme is private and the achievable load is \( K(1 - \frac{M}{K}) \).

**Converse:** We use the converse bound in [25] on the worst-case load for coded caching without privacy constraint, denoted by \( R_{caching}(M) \). As shown in (6), \( R_{caching}(M) \) is also a converse bound for the considered MuPIR problem. In addition, it was proved in [25] that \( R_{caching}(M) \) is no less than the lower convex envelop of \( \frac{1}{4} \min \{ \frac{K_u - t}{t + 1}, K(1 - \frac{M}{K}) \} \) where \( t \in [K_u] \).

Hence, we have

\[
\frac{R_{PD}(M)}{R^*(M)} \leq \frac{R_{PD}(M)}{R_{caching}(M)} \leq 4 \cdot \left( 1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}} \right) \leq 8,
\]

which comes from that \( R_{PD}(M) = \min \{ K(1 - \frac{M}{K}), \hat{R}(M) \} \), and that \( 1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}} \leq 2 \) when \( N \geq 2 \).

**Corollary 3:** The proposed product design is optimal when \( \frac{(K_u - 1)}{K_u} K \leq M \leq K \).

**Proof:** When \( K \geq K_u \) and \( M = \frac{K(K_u - 1)}{K_u} \), (12) becomes \( \hat{R}(M) = \frac{1}{K_u} \left( 1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}} \right) \). On the other hand, the author in [9] showed that when \( K_u = 1 \), the optimal single-user cache-aided PIR load is equal to \( \left( 1 - \frac{M}{K} \right) \left( 1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}} \right) \), which also equals to (12) when \( M = \frac{K(K_u - 1)}{K_u} \). By memory sharing between \( \left( \frac{K(K_u - 1)}{K_u}, \hat{R}(\frac{K(K_u - 1)}{K_u}) \right) \) and \( (K, 0) \), we conclude that the proposed PD is optimal when \( \frac{(K_u - 1)}{K_u} K \leq M \leq K \).
Fig. 2: An illustration of the achievable load $R$ of the MuPIR system with $K = K_u = 2$. (a) $N = 2$. Both the CIA scheme ($R_{CIA}$) and the optimal scheme under distinct demands ($R^*_d(M)$) have four corner points; Both the optimal scheme under uncoded cache placement ($R^*_{uncoded}$) and the product design ($R_{PD}$) have three corner points; (b) $N = 3$. $R_{CIA}$ has four corner points. $R^*_{uncoded}$ and $R_{PD}$ have three corner points.

**Numerical Evaluations:** In Fig. 2 we consider the MuPIR systems with $K = K_u = 2$, where $N = 2$ in Fig. 2a and $N = 3$ in Fig. 2b respectively. We compare the proposed CIA based scheme in Theorem 1, the optimal scheme under uncoded cache placement in Corollary 9, the product design in Theorem 2, and the computer-aided converse in [22]. In addition, it will be clarified in Theorem 3 that for the case $K = K_u = N = 2$, if the users demand distinct messages, the optimal memory-load trade-off is $R^*_d(M)$ given in $(51)$.

In Fig. 2a there are two non-trivial corner points $(\frac{1}{4}, \frac{3}{2})$ and $(\frac{2}{3}, 1)$ associated with the CIA based scheme in Theorem 1. It can be seen that in the case of general demands, the CIA based scheme outperforms both the optimal scheme under uncoded cache placement in Corollary 9 and the product design in Theorem 2. When $\frac{1}{4} \leq M \leq 2$, $R_{CIA}$ coincides with the computer-aided converse [22] and hence is optimal. It also can be seen that a lower load can be achieved when users only have distinct demands. Fig. 2b shows the case when $N = 3$ in which $R_{CIA}$ is optimal when $\frac{1}{3} \leq M \leq 2$ by the computer-aided converse.

In Fig. 3 we compare the load of the product design with the best-known caching bound provided in [8] when $K = K_u = 6, N = 2$. More specifically, Theorem 2 of [8] gives the caching converse as a lower convex envelope of the set of memory-load pairs

$$\left\{ \left( \frac{7 - s}{s}, \frac{s - 1}{2} + \frac{\ell(s - 1)}{2s} \right) \left| \forall s \in [1:6], \forall \ell \in [1:s] \right. \right\} \cup \{(0, 6)\}. \quad (14)$$
Since $R_{PD}(1)$ lies below the line segment connecting the memory-load pairs $(0,6)$ and $(2,R_{PD}(2))$, we can use memory sharing to achieve a better load for $M = 1$.

IV. PROOF OF THEOREM 1: DESCRIPTION OF THE CIA SCHEME

In this section, we prove Theorem 1. For the cache-aided MuPIR problem with $K = 2$ messages, $K_u = 2$ users and $N \geq 2$ DBs, we first show the achievability of the memory-load pairs $\left(\frac{N-1}{2N}, \frac{N+1}{N}\right)$ and $\left(\frac{2(N-1)}{2N-1}, \frac{N+1}{2N-1}\right)$ using the proposed CIA scheme. Note that when $M = 0$, we let any one of the DBs broadcast the two messages to the users, so the memory-load pair $(0,2)$ is achievable. When $M = 2$, we let both users store the two messages in the placement phase and there is no need for the DBs to transmit anything. Therefore, $(2,0)$ is achievable. By memory sharing between the above four corner points, the load of Theorem 1 can be achieved. For each of the above two non-trivial corner points, we first describe the general achievable schemes for arbitrary number of DBs and then present an example to highlight the design intuition. Computer-aided investigation [22] shows that the achievable load in Theorem 1 is optimal when $N = 2$ and 3. For general values of $N$, the converse remains open.

A. Achievability of $\left(\frac{N-1}{2N}, \frac{N+1}{N}\right)$

Let $W_1 = A$ and $W_2 = B$ denote the two messages, each consisting of $L = 2N$ bits, i.e., $A = (A_1, A_2, \cdots, A_{2N}), B = (B_1, B_2, \cdots, B_{2N})$. The proposed scheme is described as follows.

1) Cache placement: Each user stores $N - 1$ linear combinations of the message bits in its...
cache (therefore \( M = \frac{N-1}{2N} \)), i.e.,

\[
Z_1 = \{ \alpha_{1,j} A_{(1:N)}^T + \beta_{1,j} B_{(1:N)}^T : j \in [N-1] \},
\]

and

\[
Z_2 = \{ \alpha_{2,j} A_{(N+1:2N)}^T + \beta_{2,j} B_{(N+1:2N)}^T : j \in [N-1] \},
\]

in which the linear combination coefficients \( \alpha_{i,j}, \beta_{i,j} \in \mathbb{F}_2^{1 \times N} \setminus \{0_N\}, \forall i \in [2], \forall j \in [N-1] \) are chosen such that \( \text{rank}([\alpha_{i,1}; \alpha_{i,2}; \cdots ; \alpha_{i,N-1}]) = N-1 \) and \( \text{rank}([\beta_{i,1}; \beta_{i,2}; \cdots ; \beta_{i,N-1}]) = N-1, \forall i \in [2] \). WLOG, \( \forall i = 1, 2 \), we choose the cache coefficients to be \( [\alpha_{i,1}; \alpha_{i,2}; \cdots ; \alpha_{i,N-1}] = [\beta_{i,1}; \beta_{i,2}; \cdots ; \beta_{i,N-1}] = [I_{N-1}, 0_{N-1}^T] \). Furthermore, let \( Z_{1,1}, Z_{1,2}, \cdots, Z_{1,N-1} \) denote the \( N-1 \) linear combinations in \( Z_i, \forall i \in [2] \), i.e., \( \forall j \in [N-1] \), we have \( Z_{1,j} = \alpha_{1,j} A_{(1:N)}^T + \beta_{1,j} B_{(1:N)}^T \) and \( Z_{2,j} = \alpha_{2,j} A_{(N+1:2N)}^T + \beta_{2,j} B_{(N+1:2N)}^T \).

2) Private delivery: In this phase, the two users download an answer from each DB according to their demands \((\theta_1, \theta_2)\). The answers are random linear combinations of certain message bits.

In particular, the answer of DB \( n \in [N-1] \) consists of two random linear combinations, i.e., \( A_n^{[\theta]} \triangleq (A_n^{[\theta,1]}, A_n^{[\theta,2]}) \) in which \( A_n^{[\theta,1]} = u_n A_{(1:N)}^T + v_n B_{(1:N)}^T \), \( A_n^{[\theta,2]} = u_n A_{(N+1:2N)}^T + v_n B_{(N+1:2N)}^T \). The answer of DB \( N \) consists of four random linear combinations, i.e., \( A_N^{[\theta]} \triangleq (A_N^{[\theta,1]}, A_N^{[\theta,2]}, A_N^{[\theta,3]}, A_N^{[\theta,4]}) \) in which \( A_N^{[\theta,1]} = g_1 A_{(1:N)}^T, A_N^{[\theta,2]} = g_2 B_{(1:N)}^T, A_N^{[\theta,3]} = g_3 A_{(N+1:2N)}^T, A_N^{[\theta,4]} = g_4 B_{(N+1:2N)}^T \). The linear coefficients \( g_1, g_2, g_3, g_4, u_n, v_n, n \in [N-1], \forall j \in [2] \) used by the answers are subject to design according to the user demands. Therefore, in total \( 2N + 2 \) random linear combinations will be downloaded in the delivery phase. These answers can be written as

\[
\begin{bmatrix}
A_{n,1}^{[\theta,1]} & u_{n,1} & 0_N & v_{1,1} & 0_N \\
A_{n,2}^{[\theta,1]} & 0_N & u_{1,2} & 0_N & v_{1,2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_{N-1,1}^{[\theta,1]} & u_{N-1,1} & 0_N & v_{N-1,1} & 0_N \\
A_{N-1,2}^{[\theta,1]} & 0_N & u_{N-1,2} & 0_N & v_{N-1,2} \\
A_{N,1}^{[\theta,2]} & g_1 & 0_N & 0_N & 0_N \\
A_{N,2}^{[\theta,2]} & 0_N & g_2 & 0_N & 0_N \\
A_{N,3}^{[\theta,2]} & 0_N & 0_N & g_3 & 0_N \\
A_{N,4}^{[\theta,2]} & 0_N & 0_N & 0_N & g_4 \\
\end{bmatrix}
= \begin{bmatrix}
A_{(1:2N)}^T \\
B_{(1:2N)}^T
\end{bmatrix}.
\]
We next show how the linear coefficients can be designed using the idea of CIA such that the users can correctly recover their desired messages. Due to the space limit, we will only consider \((\theta_1, \theta_2) = (1, 2)\) and \((\theta_1, \theta_2) = (1, 1)\). The cases of \((\theta_1, \theta_2) = (2, 1)\) and \((2, 2)\) follow similarly.

For \((\theta_1, \theta_2) = (1, 2)\), i.e., user 1 and 2 demand messages \(A\) and \(B\) respectively, the following six coefficient matrices should be full-rank:

\[
\begin{bmatrix}
\alpha_{1,1} & \beta_{2,1} & u_{1,2} & v_{1,1} & \alpha_{2,1} & \beta_{1,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{1,N-1} & \beta_{2,N-1} & u_{N-1,2} & v_{N-1,1} & \alpha_{2,N-1} & \beta_{1,N-1}
\end{bmatrix}
\]

\(\text{For correct decoding of } (\theta_1, \theta_2) = (1, 2)\)

Note that only the first four coefficient matrices being full-rank in (18) is mandatory to guarantee the correct decoding for the case of \((\theta_1, \theta_2) = (1, 2)\). The extra two matrices \([\alpha_{2,1}; \cdots; \alpha_{2,N-1}; g_3]\) and \([\beta_{1,1}; \cdots; \beta_{1,N-1}; g_2]\) being full-rank is mandatory for the correct decoding of other user demands. The reason that we require the two extra matrices to be full-rank for \((\theta_1, \theta_2) = (1, 2)\) is that, if the two matrices are not full-rank here (the DBs can check this since we have assumed that the DBs are aware of the users’ cache placement), then DB 2 can know that the actual demands being requested are \((\theta_1, \theta_2) = (1, 2)\) since correct decoding is impossible for any other case of \((\theta_1, \theta_2) \neq (1, 2)\). In fact, any full-rank coefficient matrix consisting of the linear coefficients of one DB and the cache coefficients which are necessary for the correct decoding of one demand vector \((\theta_1, \theta_2)\) must be full-rank for all possible demands \((\theta_1, \theta_2) \in [2]^2\) for the purpose of privacy. This multi-purpose full-rank requirement holds for all user demands. The required alignment is

\[
\begin{align*}
\text{Alignment condition: } & g_1 = u_{1,1} = u_{2,1} = \cdots = u_{N-1,1}, \\
& g_4 = v_{1,2} = v_{2,2} = \cdots = v_{N-1,2}.
\end{align*}
\]

(19a)

(19b)

We next show that with the above full-rank and alignment conditions, the two users can correctly decode messages \(A\) and \(B\) respectively.

Due to the alignment condition of (19a), we have

\[
A^{[1,2]}_{N,4} = g_4 B^T_{(N+1:2N)} = v_{1,2} B^T_{(N+1:2N)} = \cdots = v_{N-1,2} B^T_{(N+1:2N)},
\]

e.g., the message bits \(B_{(N+1:2N)}\) are aligned among the linear combinations \(A^{[1,2]}_{1,2}, A^{[1,2]}_{2,2}, \ldots, A^{[1,2]}_{N-1,2}\). Subtracting \(A^{[1,2]}_{N,4}\) from \(A^{[1,2]}_{1,2}, A^{[1,2]}_{2,2}, \ldots, A^{[1,2]}_{N-1,2}\) in (17),
we obtain
\[
\begin{bmatrix}
A_{1,2}^{[1,2]} - A_{N,4}^{[1,2]} \\
\vdots \\
A_{N-1,2}^{[1,2]} - A_{N,4}^{[1,2]} \\
A_{N,3}^{[1,2]}
\end{bmatrix}
= 
\begin{bmatrix}
u_{1,2} \\
\vdots \\
u_{N-1,2} \\
g_3
\end{bmatrix}
\begin{bmatrix}
A_{(N+1:2N)}^T
\end{bmatrix}.
\]
(20)

Since the coefficient matrix on the RHS of (20) is full-rank, \(A_{(N+1:2N)}\) can be solved by matrix inversion as
\[
A_{(N+1:2N)}^T = \begin{bmatrix}
u_{1,2} \\
\vdots \\
u_{N-1,2} \\
g_3
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{1,2}^{[1,2]} - A_{N,4}^{[1,2]} \\
\vdots \\
A_{N-1,2}^{[1,2]} - A_{N,4}^{[1,2]} \\
A_{N,3}^{[1,2]}
\end{bmatrix}.
\]
(21)

Therefore, both users can decode \(A_{(N+1:2N)}\). Similarly, due to the alignment condition of (19b), we have \(A_{N,1}^{[1,2]} = g_1 A_{(1:N)}^T = u_{1,1} A_{(1:N)}^T = \cdots = u_{N-1,1} A_{(1:N)}^T\), i.e., the message bits \(A_{(1:N)}\) are aligned among the linear combinations \(A_{1,1}^{[1,2]}, A_{2,1}^{[1,2]}, \ldots, A_{N-1,1}^{[1,2]}\). Subtracting \(A_{N,1}^{[1,2]}\) from \(A_{1,1}^{[1,2]}; A_{2,1}^{[1,2]}, \ldots, A_{N-1,1}^{[1,2]}\), we obtain
\[
\begin{bmatrix}
A_{1,1}^{[1,2]} - A_{N,1}^{[1,2]} \\
\vdots \\
A_{N-1,1}^{[1,2]} - A_{N,1}^{[1,2]} \\
A_{N,2}^{[1,2]}
\end{bmatrix}
= 
\begin{bmatrix}
v_{1,1} \\
\vdots \\
v_{N-1,1} \\
g_2
\end{bmatrix}
\begin{bmatrix}
B_{(1:N)}^T
\end{bmatrix}.
\]
(22)

Since the linear coefficient matrix on the RHS of (22) is full-rank, \(B_{(1:N)}\) can be solved as
\[
B_{(1:N)}^T = \begin{bmatrix}
v_{1,1} \\
\vdots \\
v_{N-1,1} \\
g_2
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{1,1}^{[1,2]} - A_{N,1}^{[1,2]} \\
\vdots \\
A_{N-1,1}^{[1,2]} - A_{N,1}^{[1,2]} \\
A_{N,2}^{[1,2]}
\end{bmatrix}.
\]
(23)

Therefore, both users can correctly decode \(B_{(1:N)}\).

Now the message bits \(A_{(N+1:2N)}, B_{(1:N)}\) are available to both users. User 1 still needs \(A_{(1:N)}\) and user 2 still needs \(B_{(N+1:2N)}\). Removing the interference of \(B_{(1:N)}\) from \(Z_1\), user 1 obtains \(N\)–
1 linear combinations of $A_{(1:N)}$. Together with $A_{N,1}^{[1,2]} = g_1 A_{(1:N)}^T$, user 1 obtains $N$ independent linear combinations of $A_{(1:N)}$, from which $A_{(1:N)}$ can be solved as

$$
A_{(1:N)}^T = \begin{bmatrix}
\alpha_{1,1} \\
\vdots \\
\alpha_{1,N-1} \\
g_1
\end{bmatrix}^{-1}
\begin{bmatrix}
Z_{1,1} - \beta_{1,1} B_{(1:N)}^T \\
\vdots \\
Z_{1,N-1} - \beta_{1,N-1} B_{(1:N)}^T \\
A_{N,1}^{[1,2]}
\end{bmatrix}.
$$

(24)

Therefore, user 1 can correctly decode all the $2N$ bits of the desired message $A$. Similarly, user 2 can also correctly decode all the $2N$ bits of the desired message $B$.

For $(\theta_1, \theta_2) = (1, 1)$, the following six coefficient matrices

$$
\begin{bmatrix}
\alpha_{1,1} \\
\alpha_{1,2} \\
\vdots \\
\alpha_{1,N-1} \\
g_1
\end{bmatrix},
\begin{bmatrix}
\alpha_{2,1} \\
\alpha_{2,2} \\
\vdots \\
\alpha_{2,N-1} \\
g_3
\end{bmatrix},
\begin{bmatrix}
\beta_{1,1} \\
\beta_{1,2} \\
\vdots \\
\beta_{1,N-1} \\
g_2
\end{bmatrix},
\begin{bmatrix}
\beta_{2,1} \\
\beta_{2,2} \\
\vdots \\
\beta_{2,N-1} \\
g_4
\end{bmatrix},
\begin{bmatrix}
u_{1,1} \\
u_{1,2} \\
\vdots \\
u_{N-1,1} \\
g_1
\end{bmatrix},
\begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
\vdots \\
u_{N-1,2} \\
g_3
\end{bmatrix}
$$

(25)

are required to be full-rank. The alignment condition is

$$
g_2 = v_{1,1} = v_{2,1} = \cdots = v_{N-1,1},
$$

(26a)

$$
g_4 = v_{1,2} = v_{2,2} = \cdots = v_{N-1,2}.
$$

(26b)

With the above full-rank and alignment conditions, we show that both users can correctly decode message $A$.

Due to the alignment of (26a), we have $A_{N,2}^{[1,1]} = g_2 B_{(1:N)}^T = v_{1,1} B_{(1:N)}^T = \cdots = v_{N-1,1} B_{(1:N)}^T$. Subtracting $A_{N,2}^{[1,1]}$ from $A_{n,1}^{[1,1]}$, $\forall n \in [N-1]$, and due to the full-rank condition of (25), the users can solve $A_{(1:N)}$ from

$$
A_{(1:N)}^T = \begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
\vdots \\
u_{N-1,1} \\
g_1
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{1,1}^{[1,1]} - A_{N,2}^{[1,1]} \\
A_{2,1}^{[1,1]} - A_{N,2}^{[1,1]} \\
\vdots \\
A_{N-1,1}^{[1,1]} - A_{N,2}^{[1,1]} \\
A_{N,1}^{[1,1]}
\end{bmatrix}.
$$

(27)
Therefore, both users can correctly decode message \( A \). Subtracting \( A_{N,4}^{(1,1)} \) from \( A_{n,2}^{(1,1)} \), \( \forall n \in [N-1] \), and due to the full-rank condition of (25), the users can solve \( A_{(N+1):2} \) from

\[
A_{(N+1):2}^T = \begin{bmatrix} u_{1,1} \\ \\ \\ u_{N-1,2} \\ g_3 \end{bmatrix}^{-1} \begin{bmatrix} A_{1,2}^{(1,1)} - A_{N,4}^{(1,1)} \\ \\ \\ A_{N-1,2}^{(1,1)} - A_{N,4}^{(1,1)} \\ A_{N,3}^{(1,1)} \end{bmatrix}.
\]

(28)

Therefore, both users can correctly decode message \( A \).

Remark 2: Note that for the case for identical demands, i.e., \((\theta_1, \theta_2) = (1, 1), (2, 2)\), the cached contents of the users are actually not used in the decoding process.

With the above full-rank and alignment conditions, we now employ a randomized specification of the linear combination coefficients used by each DB and formally describe the delivery scheme.

We first introduce some necessary notations. Let \( Y_N \in \mathbb{F}_2^{N \times N} \) be a full-rank binary matrix defined as \( Y_N \equiv \begin{bmatrix} I_{N-1} & 1^T_{N-1} \\ 0_{N-1} & 1 \end{bmatrix} \). Let \( \mathcal{Y}_N \) be a set containing the rows of \( Y_N \), i.e., \( \mathcal{Y}_N \) contains \( N \) linearly independent binary row vectors. Also define two binary matrices \( M(u_{.,i}, g_j), M(v_{.,i}, g_j) \in \mathbb{F}_2^{N \times N}, \forall i \in [2], \forall j \in [4] \) as \( M(u_{.,i}, g_j) \equiv [u_{1,i}; u_{2,i}; \cdots; u_{N-1,i}; g_j] \), \( M(v_{.,j}, g_j) \equiv [v_{1,j}; v_{2,j}; \cdots; v_{N-1,j}; g_j] \). The private delivery schemes for \((\theta_1, \theta_2) = (1, 2)\) and \((\theta_1, \theta_2) = (1, 1)\) are given by

1) \((\theta_1, \theta_2) = (1, 2)\): Let \( g_1 \) and \( g_4 \) be chosen randomly and uniformly i.i.d. from the rows in \( \mathcal{Y}_N \). Then let \( M(u_{.,2}, g_3) \) and \( M(v_{.,1}, g_2) \) be two independent random permutations of the rows of \( Y_N \). It can be seen that with such a specification of the answer linear coefficients and the previously defined cache coefficients, the full-rank condition of (18) can be satisfied. Therefore, both users can correctly decode their desired messages.

2) \((\theta_1, \theta_2) = (1, 1)\): Let \( g_2 \) and \( g_4 \) be chosen randomly and uniformly i.i.d. from the rows in \( \mathcal{Y}_N \). Then let \( M(u_{.,1}, g_1) \) and \( M(u_{.,2}, g_3) \) be two independent random permutations of the rows of \( Y_N \). It can be seen that the full-rank condition (25) is satisfied.

3) \((\theta_1, \theta_2) = (2, 1)\): Let \( g_2 \) and \( g_3 \) be chosen randomly and uniformly i.i.d. from the rows in \( \mathcal{Y}_N \). Then let \( M(u_{.,1}, g_1) \) and \( M(u_{.,2}, g_4) \) be two independent random permutations of the rows of \( Y_N \). It can be easily checked that the corresponding full-rank condition is satisfied.
4) \((\theta_1, \theta_2) = (2, 2)\): Let \(g_1\) and \(g_3\) be chosen randomly and uniformly i.i.d. from the rows in \(Y_N\). Then let \(M(v_{i,1}, g_2)\) and \(M(v_{i,2}, g_4)\) be two independent random permutations of the rows of \(Y_N\). It can be easily checked that the corresponding full-rank condition is satisfied.

We next prove the correctness and privacy of the above delivery scheme.

**Correctness:** For any demands \((\theta_1, \theta_2)\), the random specification of the answer linear coefficients satisfies the corresponding full-rank and alignment conditions, implying the decodability.

**Privacy:** WLOG, we prove that the above delivery scheme is private from DB 1’s perspective, i.e., the demand vector \(\theta\) is equally likely to be \((1, 2), (2, 1), (1, 1)\) or \((2, 2)\). More specifically, let \(x \triangleq [u_{1,1}, u_{1,2}, v_{1,1}, v_{1,2}] \in Y^4_N\) be a random realization of the answer linear coefficients of DB 1. Let \(\Gamma(u_{1,j}, \theta)\) denote a random query of the value of \(u_{1,j}, j = 1, 2\) to DB 1 when the user demand vector is \(\theta\). Other notations follow similarly. Let \(X(\theta) \triangleq [\Gamma(u_{1,1}, \theta), \Gamma(u_{1,2}, \theta), \Gamma(v_{1,1}, \theta), \Gamma(v_{1,2}, \theta)]\) represent the random query to DB 1 when the user demand vector is \(\theta\). Then the probability that \(x\) is generated for \(\theta\) (i.e., \(X(\theta) = x\)) is

\[
P(X(\theta) = x) = P(\Gamma(u_{1,1}, \theta) = u_{1,1}) \times P(\Gamma(u_{1,2}, \theta) = u_{1,2})
\]

\[
\times P(\Gamma(v_{1,1}, \theta) = v_{1,1}) \times P(\Gamma(v_{1,2}, \theta) = v_{1,2})
\]

\[
= \begin{cases} 
\frac{1}{N} \times \frac{(N-1)!}{N!} \times \frac{(N-1)!}{N!} \times \frac{1}{N} = \frac{1}{N^4}, & \text{if } \theta = (1, 2) \\
\frac{(N-1)!}{N!} \times \frac{1}{N} \times \frac{1}{N} \times \frac{(N-1)!}{N!} = \frac{1}{N^4}, & \text{if } \theta = (2, 1) \\
\frac{(N-1)!}{N!} \times \frac{(N-1)!}{N!} \times \frac{1}{N} \times \frac{1}{N} = \frac{1}{N^4}, & \text{if } \theta = (1, 1) \\
\frac{1}{N} \times \frac{1}{N} \times \frac{(N-1)!}{N!} \times \frac{(N-1)!}{N!} = \frac{1}{N^4}, & \text{if } \theta = (2, 2)
\end{cases}
\]  

(29)

Since \(P(X(\theta) = x)\) does not depend on \(\theta\), from DB 1’s perspective, the coefficient realization \(x\) is equally likely to be generated for \(\theta = (1, 2), (2, 1), (1, 1)\) or \((2, 2)\). Therefore, the scheme is private from DB 1’s point of view. Due to symmetry, the scheme is also private from any other individual DB’s perspective. As a result, the proposed delivery scheme is private.

**Performance:** Since \(D = 2N + 2\) linear combinations, each containing one bit, are downloaded in total, the achieved load is \(R = \frac{D}{L} = \frac{N+1}{N}\).

We provide the following example to briefly illustrate how we choose the coefficients in the above proposed scheme.

**Example 1:** (Achievability of \((\frac{1}{4}, \frac{3}{2})\) for \(N = 2\)) Consider the cache-aided MuPIR problem...
with $K = 2$ messages, $K_u = 2$ users and $N = 2$ DBs.

1) **Cache placement:** Assume that each message consists of $L = 4$ bits, i.e., $A = (A_1, A_2, A_3, A_4)$, $B = (B_1, B_2, B_3, B_4)$. Each user stores a linear combination of the message bits which are $Z_1 = \alpha_1, A_1, A_2$ and $Z_2 = \alpha_2, A_3, A_4$. Thus we let $A_i = \alpha_i, A_1, A_2$ and $B_i = \alpha_i, B_1, B_2$ without using the cache. For user 1, the interference of the message bits each user receives $g_1, g_2$ and $3$ are linearly independent. Thus each user can decode $u_1$ and $v_1$ as shown in (19). In order to choose the coefficients in (30), we introduce $3$ with a slight abuse of notation, here we use $Z_1, Z_2$ to denote the caches bits of the users despite they are defined as sets.

2) **Private delivery:** For any demand vector, we construct the answers of the DBs as

$$
\begin{bmatrix}
A_{1,1}^g \\
A_{1,2}^g \\
A_{2,1}^g \\
A_{2,2}^g \\
A_{2,3}^g \\
A_{2,4}^g
\end{bmatrix} =
\begin{bmatrix}
u_{1,1} & 0 & v_{1,1} & 0 \\
0 & u_{1,2} & 0 & v_{1,2} \\
g_1 & 0 & g_2 & 0 \\
0 & 0 & g_3 & 0 \\
0 & 0 & g_4 & 0 \\
0 & 0 & 0 & g_4
\end{bmatrix}
\begin{bmatrix}
A_{1,2}^T \\
A_{2,3}^T \\
B_{1,2}^T \\
B_{2,3}^T \\
B_{3,4}^T
\end{bmatrix}.
$$

(30)

Now suppose the demand vector is $(\theta_1, \theta_2) = (1, 2)$. For this demand vector, we let $u_{1,1} = g_1$ and $v_{1,2} = g_4$ as shown in (19). In order to choose the coefficients in (30), we introduce the matrix $Y_2 = [1, 1; 0, 1]$ which is independent of the demand vector. Since the demand vector is $(\theta_1, \theta_2) = (1, 2)$, we let $g_1$ and $g_4$ be chosen randomly and uniformly i.i.d. from $\mathcal{Y}_2 = \{[1, 1], [0, 1]\}$. In addition, we let $[u_{1,2}; g_3]$ and $[v_{1,1}; g_2]$ be two independent random permutations of the rows of $Y_2$. The purpose of the above coefficient selection is such that the message bits $A_3, A_4$ and $B_1, B_2$ can be directly decoded from the answers by both users without using the cache. For user 1, the interference of $B_1, B_2$ will be eliminated and a linear combination of $A_1$ and $A_2$ is left. Together with the one aligned equation of $A_1$ and $A_2$ from the answers, user 1 can decode $A_1$ and $A_2$. Similarly, user 2 can eliminate the interference of $A_3$ and $A_4$, left with a linear combination of $B_3$ and $B_4$ in the cache. Together with the one received equation of $B_3$ and $B_4$, user can decode $B_3$ and $B_4$.

**Correctness:** From $A_{1,2}^g = A_{2,4}^g$, each user can decode $u_{1,2}[A_3, A_4]^T$. In addition, from $A_{2,3}^g$, each user receives $g_3[A_3, A_4]^T$. Since $u_{1,2}$ and $g_3$ are two different rows of $Y_2$, we can see they are linearly independent. Thus each user can decode $A_3$ and $A_4$. From $A_{1,1}^g - A_{2,1}^g$, each user
can decode $v_{1,1}[B_1, B_2]^T$. In addition, from $A_{2,1}^{[\theta]}$, each user receives $g_2[B_1, B_2]^T$. Since $v_{1,1}$ and $g_2$ are two different rows of $Y_2$, we can see they are linearly independent. Thus each user can decode $B_1$ and $B_2$. Since user 1 caches $A_1 + B_1$ and has decoded $B_1$, it can then recover $A_1$. From $A_{2,1}^{[\theta]}$, user 1 receives $g_1[A_1, A_2]^T$ where $g_1$ is one row in $Y_2$. No matter which row is $g_1$, user 1 can always recover $A_1$ and $A_2$ from $g_1[A_1, A_2]^T$. Hence, user 1 can recover the whole file $A$. Similarly, since user 2 caches $A_3 + B_3$ and has decoded $A_3$, it can then recover $B_3$. From $A_{2,3}^{[\theta]}$, user 2 receives $g_4[B_3, B_4]^T$ where $g_4$ is one row in $Y_2$. No matter which row is $g_4$, user 2 can always recover $B_3$ and $B_4$ from $g_4[B_3, B_4]^T$. Hence, user 2 can recover the whole file $B$.

**Privacy:** Intuitively, from the viewpoint of DB 1 whose sent linear combinations are $g_1[A_1, A_2]^T + v_{1,1}[B_1, B_2]^T$ and $u_{1,2}[A_3, A_4]^T + g_4[B_3, B_4]^T$, the vectors $g_1, g_4, v_{1,1}, u_{1,2}$ are randomly and independently chosen from the rows of $Y_2$. Thus, the sent linear combinations are independent of the demand vector. From the viewpoint of DB 2 whose sent linear combinations are $g_1[A_1, A_2]^T$, $g_2[B_1, B_2]^T$, $g_3[A_3, A_4]^T$, and $g_4[B_3, B_4]^T$, the vectors $g_1, g_2, g_3, g_4$ are randomly and independently chosen from the rows of $Y_2$. Thus, the sent linear combinations are independent of the demand vector. Therefore, each DB cannot get any information about the demand vector from its sent linear combinations and the user cache.

**Performance:** The achieved load is $R = \frac{P}{T} = \frac{6}{4} = \frac{3}{2}$.

B. **Achievability of** $\left(\frac{2(N-1)}{2N-1}, \frac{N+1}{2N-1}\right)$

1) **Cache placement:** Let $W_1 = A, W_2 = B$ be the two messages each of which consists of $L = 2N - 1$ bits, i.e., $A = (A_1, A_2, \cdots, A_{2N-1}), B = (B_1, B_2, \cdots, B_{2N-1})$. Each user stores $2(N-1)$ uncoded bits from each message, i.e.,

$$Z_1 = \{A_{1:N-1}, B_{1:N-1}\},$$  \hspace{1cm} (31)  

$$Z_2 = \{A_{N:2N-2}, B_{N:2N-2}\}.  \hspace{1cm} (32)$$

Note that neither of the two users store the message bits $A_{2N-1}$ and $B_{2N-1}$.

2) **Private delivery:** We first construct the answers from the DBs. The answer of DB $n \in [N-1]$ is a linear combination of certain message bits, $A_n^{[\theta]} = u_n A_{(1:2N-1)}^T + v_n B_{(1:2N-1)}^T$. The answer of DB $N$ consists of two linear combinations, i.e., $A_N^{[\theta]} = (A_N^{[\theta,1]}, A_N^{[\theta,2]})$ where $A_N^{[\theta,1]} = g_1 A_{(1:2N-1)}^T, A_N^{[\theta,2]} = g_2 B_{(1:2N-1)}^T$. The linear combination coefficients $u_n \triangleq [u_{n,1}, u_{n,2}, \cdots, u_{n,2N-1}], v_n \triangleq [v_{n,1}, v_{n,2}, \cdots, v_{n,2N-1}], \forall n \in [N-1], g_{j} \triangleq [g_{j,1}, g_{j,2}, \cdots, g_{j,2N-1}], j = 1, 2,$ belong
to $\mathbb{F}_2^{1 \times (2^N - 1)} \setminus \{0_{2^N - 1}\}$ and are subject to design according to different user demands. These answers can be written in a more compact form as

$$
\begin{bmatrix}
A_1^{[\theta]} \\
A_{N-1}^{[\theta]} \\
A_N^{[\theta]} \\
A_{N,1}^{[\theta]} \\
A_{N,2}^{[\theta]}
\end{bmatrix} =
\begin{bmatrix}
u_1 & v_1 \\
\vdots & \vdots \\
u_{N-1} & v_{N-1} \\
g_1 & 0_{2^N-1} \\
0_{2^N-1} & g_2
\end{bmatrix}
\begin{bmatrix}
A^T_{(1:2^N-1)} \\
B^T_{(1:2^N-1)}
\end{bmatrix} =
\begin{bmatrix}
u_{N-1,(1:2^N-2)} & v_{N-1,(1:2^N-2)} \\
g_{1,(1:2^N-2)} & 0_{2^N-2} \\
0_{2^N-2} & g_{2,(1:2^N-2)}
\end{bmatrix}
\begin{bmatrix}
A^T_{(1:2^N-1)} \\
B^T_{(1:2^N-1)}
\end{bmatrix},
$$

(33)

where we let

$$
g_{1,2^N-1} = g_{2,2^N-1} = u_{n,2^N-1} = v_{n,2^N-1} = 1, \quad \forall n \in [N-1].
$$

(34)

We next consider the four possible demands $(\theta_1, \theta_2)$ and show that the full-rank and alignment conditions are necessary for correct decoding. For simplicity, we will focus on the cases of $(\theta_1, \theta_2) = (1, 2)$ and $(\theta_1, \theta_2) = (1, 1)$. The cases of $(\theta_1, \theta_2) = (2, 1)$ and $(2, 2)$ follow similarly.

For $(\theta_1, \theta_2) = (1, 2)$, the following two coefficient matrices are required to be full-rank, i.e.,

$$
\text{Full-rank condition: } \begin{bmatrix}
u_{1,(N:2^N-1)} \\
\vdots \\
u_{N-1,(N:2^N-1)} \\
g_{1,(N:2^N-1)}
\end{bmatrix}, \begin{bmatrix}[v_{1,(N-1)}, v_{1,2^N-1}] \\
\vdots \\
[v_{N-1,(N-1)}, v_{N-1,2^N-1}] \\
g_{2,(N-1)}
\end{bmatrix}.
$$

(35)

For alignment, we let $\forall n \in [N-1]:$

$$
\text{Alignment condition: } [u_{n,(1:N-1)}, u_{n,2^N-1}] = [g_{1,(1:N-1)}, g_{1,2^N-1}],
$$

(36a)

$$
v_{n,(N-1)} = g_{2,(N-1)};
$$

(36b)

i.e., the message bits $A_{(1:N-1)}, A_{2^N-1}$ are aligned among the linear combinations $A_1^{[(1,2)]}, A_2^{[(1,2)]}, \ldots, A_{N-1}^{[(1,2)]}$ and $A_{N,1}^{[(1,2)]}$ and the bits $B_{(N:2^N-1)}$ are aligned among the linear combinations $A_1^{[(1,2)]}, A_2^{[(1,2)]}, \ldots, A_{N-1}^{[(1,2)]}, A_{N,2}^{[(1,2)]}$. We next show that the above two conditions guarantee correct decoding.
For user 1, due to the alignment of $B_{(N:2N-1)}$, we have
\[ A_{N,2}^{(1,2)} - g_{2,(1:N-1)}B_{(1:N-1)}^T = g_{2,(N:2N-1)}B_{(N:2N-1)}^T = v_{n,(N:2N-1)}B_{(N:2N-1)}^T, \forall n \in [N-1]. \tag{37} \]
Subtracting $A_{N,2}^{(1,2)} - g_{2,(1:N-1)}B_{(1:N-1)}^T$ (this is known to user 1 since $B_{1:N-1}$ are already cached by user 1) from $A_1^{(1,2)}, A_2^{(1,2)}, \ldots, A_{N-1}^{(1,2)}$ in (33), together with $A_{N,1}^{(1,2)} = g_1A_{1:2N-1}^T$, we obtain $N$ independent linear combinations of $A_{(N:2N-1)}$, which can be solved as
\[ A_{(N:2N-1)}^T = \begin{bmatrix} u_{1,(N:2N-1)} \\ \vdots \\ u_{N-1,(N:2N-1)} \\ g_{1,(N:2N-1)} \end{bmatrix}^{-1} y, \tag{38} \]
in which
\[ y \triangleq \begin{bmatrix} A_1^{(1,2)} - (A_{N,2}^{(1,2)} - g_{2,(1:N-1)}B_{(1:N-1)}^T) \\ \vdots \\ A_{N-1}^{(1,2)} - (A_{N,2}^{(1,2)} - g_{2,(1:N-1)}B_{(1:N-1)}^T) \\ A_{N,1}^{(1,2)} \end{bmatrix} - \begin{bmatrix} u_{1,(1:N-1)} & v_{1,(1:N-1)} \\ \vdots & \vdots \\ u_{N-1,(1:N-1)} & v_{N-1,(1:N-1)} \\ g_{1,(1:N-1)} & 0_{N-1} \end{bmatrix} \begin{bmatrix} A_{(1:N-1)}^T \\ B_{(1:N-1)}^T \end{bmatrix}. \tag{39} \]
Since the message bits $A_{(1:N-1)}, B_{(1:N-1)}$ are already cached by user 1, it can decode the bits $A_{(N:2N-1)}$ and correctly recover message $A$. Similarly, user 2 can decode the bits $B_{(1:N-1)}, B_{2N-1}$ and correctly recover message $B$.

For $(\theta_1, \theta_2) = (1, 1)$, the following two coefficient matrices
\[ \begin{bmatrix} u_{1,(N:2N-1)} \\ u_{2,(N:2N-1)} \\ \vdots \\ u_{N-1,(N:2N-1)} \\ g_{1,(N:2N-1)} \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} u_{1,(1:N-1)} & u_{1,2N-1} \\ u_{2,(1:N-1)} & u_{2,2N-1} \\ \vdots & \vdots \\ u_{N-1,(1:N-1)} & u_{N-1,2N-1} \\ g_{1,(1:N-1)} & g_{1,2N-1} \end{bmatrix} \end{bmatrix}, \tag{40} \]
are required to be full-rank and the alignment is:
\[ g_2 = v_n, \quad \forall n \in [N-1]. \tag{41} \]
The decoding is explained as follows. Due to the alignment of (41), we have
\[ A_{N,2}^{(1,1)} = g_2B_{(1:2N-1)}^T = v_1B_{(1:2N-1)}^T = \cdots = v_{N-1}B_{(1:2N-1)}^T, \forall n \in [N-1]. \tag{42} \]
Subtracting $A_{N,2}^{[(1,1)]}$ from all $A_n^{[(1,1)]}$, $\forall n \in [N - 1]$, we obtain

$$A_{(N:2N-1)}^T = \begin{bmatrix}
  u_{1,(N:2N-1)} \\
  u_{2,(N:2N-1)} \\
  \vdots \\
  u_{N-1,(N:2N-1)} \\
  g_{1,(N:2N-1)}
\end{bmatrix} - \begin{bmatrix}
  A_1^{[(1,1)]} - A_{N,2}^{[(1,1)]} \\
  A_2^{[(1,1)]} - A_{N,2}^{[(1,1)]} \\
  \vdots \\
  A_{N-1}^{[(1,1)]} - A_{N,2}^{[(1,1)]} \\
  A_{N,1}^{[(1,1)]}
\end{bmatrix} \begin{bmatrix}
  u_{1,(1:N-1)} \\
  u_{2,(1:N-1)} \\
  \vdots \\
  u_{N-1,(1:N-1)} \\
  g_{1,(1:N-1)}
\end{bmatrix}. \tag{43}
$$

Since the message bits $A_{1:N-1}$ are cached by user 1, it can decode the desired bits $A_{N:2N-1}$ and correctly recover $A$. Similarly, user 2 can decode the desired bits $A_{1:N-1}, A_{2N-1}$. Therefore, both users can correctly recover the desired message $A$.

With the above full-rank and alignment conditions, we now employ a randomized specification of the linear combination coefficients used by each DB and formally describe the delivery scheme.

We first introduce some necessary notations. Let $Y'_N \triangleq [I_{N-1}; 0_{N-1}]$ be a binary matrix with dimension $N \times (N - 1)$ and let $\mathcal{Y}'_N$ be a set containing the rows of $Y'_N$. For an index vector $(m : n) = (m, m + 1, \ldots, n - 1, n)$ where $m \leq n$, define $\forall j = 1, 2$:

$$M'(u, g_j, (m : n)) \triangleq [u_{1,(m:n)}; u_{2,(m:n)}; \cdots; u_{N-1,(m:n)}; g_{j,(m:n)}] \in \mathbb{F}_2^{N \times (n-m+1)}, \tag{44a}$$

$$M'(v, g_j, (m : n)) \triangleq [v_{1,(m:n)}; v_{2,(m:n)}; \cdots; v_{N-1,(m:n)}; g_{j,(m:n)}] \in \mathbb{F}_2^{N \times (n-m+1)}. \tag{44b}$$

The delivery schemes for different $(\theta_1, \theta_2)$ are then as follows.

1) $(\theta_1, \theta_2) = (1, 2)$: Let $g_{1,(1:N-1)}$ and $g_{2,(N:2N-2)}$ be chosen randomly and uniformly i.i.d. from the rows in $\mathcal{Y}'_N$. Also, let $M'(u, g_1, (N : 2N - 2))$ and $M'(v, g_2, (1 : N - 1))$ be two independent random permutations of the rows of $Y'_N$. It can be easily seen that the full-rank condition of (35) is satisfied, implying the decodability.

2) $(\theta_1, \theta_2) = (1, 1)$: Let $g_{2,(N:2N-2)}$ and $g_{2,(1:N-1)}$ be chosen randomly and uniformly i.i.d. from the rows in $\mathcal{Y}'_N$. Also, let $M'(u, g_1, (N : 2N - 2))$ and $M'(u, g_1, (1 : N - 1))$ be two independent random permutations of the rows of $Y'_N$.

3) $(\theta_1, \theta_2) = (2, 1)$: Let $g_{1,(N:2N-2)}$ and $g_{2,(1:N-1)}$ be chosen randomly and uniformly i.i.d. from the rows in $\mathcal{Y}'_N$. Also, let $M'(u, g_1, (1 : N - 1))$ and $M'(v, g_2, (N : 2N - 2))$ be two independent random permutations of the rows of $Y'_N$.

4) $(\theta_1, \theta_2) = (2, 2)$: Let $g_{1,(N:2N-2)}$ and $g_{1,(1:N-1)}$ be chosen randomly and uniformly i.i.d. from the rows in $\mathcal{Y}'_N$. Also, let $M'(v, g_2, (1 : N - 1))$ and $M'(v, g_2, (N : 2N - 2))$ be two
independent random permutations of the rows of $\mathbf{Y}_N'$. 

**Correctness:** Decodability is straightforward since the randomized specifications of the linear coefficients for different user demands guarantee the corresponding full-rank and alignment conditions.

**Privacy:** By the similar argument in (29), it can be seen that this scheme is private.

**Performance:** Since $D = N + 1$ linear combinations, each containing one bit, are downloaded in total, the achieved load is $R = \frac{D}{L} = \frac{N + 1}{2N - 1}$.

We provide the following example to briefly illustrate how we choose the coefficients in the above proposed scheme.

**Example 2:** (Achievability of $(\frac{2}{3}, 1)$ for $N = 2$) Consider the same setting as Example 1 where $K = K_u = N = 2$. In this example we show the achievability of the memory-load pair $(\frac{2}{3}, 1)$.

1) **Cache placement:** Assume that each message consists of $L = 3$ bits, i.e., $A = (A_1, A_2, A_3)$, $B = (B_1, B_2, B_3)$. Each user stores two message bits, i.e., $Z_1 = \{A_1, B_1\}$, $Z_2 = \{A_2, B_2\}$.

2) **Private delivery:** For any demand vector, the answers of the DBs are constructed as

$$
\begin{bmatrix}
A_1^\theta \\
A_2^\theta \\
A_2^\theta
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{u}_1 & \mathbf{v}_1 \\
\mathbf{g}_1 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{g}_2
\end{bmatrix}
\begin{bmatrix}
A_{(1:3)}^T \\
B_{(1:3)}^T
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{u}_{1,1} & \mathbf{u}_{1,2} & 1 & \mathbf{v}_{1,1} & \mathbf{v}_{1,2} & 1 \\
\mathbf{g}_{1,1} & \mathbf{g}_{1,2} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{g}_{2,1} & \mathbf{g}_{2,2} & 1
\end{bmatrix}
\begin{bmatrix}
A_{(1:3)}^T \\
B_{(1:3)}^T
\end{bmatrix}.
$$

(45)

Now consider $(\theta_1, \theta_2) = (1, 2)$. For this demand vector, we let $u_{1,1} = g_{1,1}$, $v_{1,2} = g_{2,2}$, as shown in (36). Thus the answers of the DBs in (45) become

$$
\begin{bmatrix}
A_1^\theta \\
A_2^\theta \\
A_2^\theta
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{g}_{1,1} & \mathbf{u}_{1,2} & 1 & \mathbf{v}_{1,1} & \mathbf{g}_{2,2} & 1 \\
\mathbf{g}_{1,1} & \mathbf{g}_{1,2} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{g}_{2,1} & \mathbf{g}_{2,2} & 1
\end{bmatrix}
\begin{bmatrix}
A_{(1:3)}^T \\
B_{(1:3)}^T
\end{bmatrix}.
$$

(46)

In order to specify the coefficients in (45), we choose the matrix $\mathbf{Y}_2' \triangleq [1, 0]^T$ and thus $\mathcal{Y}_2' = \{0, 1\}$, both of which are independent of the demand vector. Since the demand vector is $(\theta_1, \theta_2) = (1, 2)$, we let $g_{1,1}$ and $g_{2,2}$ be chosen randomly and uniformly i.i.d. from $\mathcal{Y}_2'$. Also, we let $[u_{1,2}; g_{1,2}]$ and $[v_{1,1}; g_{2,1}]$ be two independent random permutations of the rows of $\mathbf{Y}_2'$.

**Correctness:** We first focus on user 1 who caches $A_1, B_1$ and desires message $A$. From $A_2^\theta$ and its cached content $A_1$, user 1 decodes $g_{1,2}A_2 + A_3$. From $A_1^\theta - A_2^\theta$, user 1 decodes $g_{1,1}A_1 + u_{1,2}A_2 + A_3 + (v_{1,1} - g_{2,1})B_1$, and then decodes $u_{1,2}A_2 + A_3$. Since $u_{1,2}$ and $g_{1,2}$ are
different elements in $[0, 1]^T$, it can be seen that user 1 can recover $A_2$ and $A_3$ from $g_{1,2}A_2 + A_3$ and $u_{1,2}A_2 + A_3$. Hence, user 1 can recover message $A$. We then focus on user 2 who caches $A_2, B_2$ and desires message $B$. From $A_2^{[\theta]}$ and its cached content $B_2$, user 2 decodes $g_{2,1}B_1 + B_3$. From $A_1^{[\theta]} - A_{2,1}^{[\theta]}$, user 2 decodes $(u_{1,2} - g_{1,2})A_2 + v_{1,1}B_1 + g_{2,2}B_2 + B_3$, and then decodes $v_{1,1}B_1 + B_3$. Since $v_{1,1}$ and $g_{2,1}$ are different elements in $[0, 1]^T$, it can be seen that user 2 can recover $B_1$ and $B_3$ from $g_{2,1}B_1 + B_3$ and $v_{1,1}B_1 + B_3$. Hence, user 2 can recover message $B$.

Privacy: Intuitively, from the viewpoint of DB 1 whose sent linear combination is $g_{1,1}A_1 + u_{1,2}A_2 + A_3 + v_{1,1}B_1 + g_{2,2}B_2 + B_3$, the coefficients $g_{1,1}, u_{1,2}, v_{1,1}, g_{2,2}$ are chosen randomly and independently from the rows of $Y'_2$. Thus the sent linear combination is independent of the demand vector. From the viewpoint of DB 2 whose sent linear combinations are $g_{1,1}A_1 + g_{1,2}A_2 + A_3$ and $g_{2,1}B_1 + g_{2,2}B_2 + B_3$, the coefficients $g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}$ are chosen randomly and independently from the rows of $Y'_2$. Thus the sent linear combinations are independent of the demand vector. Therefore, each DB cannot get any information about the demand vector from its sent linear combinations and the user cache. Performance: The achieved load is $R = \frac{D}{L} = 1$.

V. PROOF OF THEOREM 2

In this section, we present a general scheme for arbitrary system parameters, called the Product Design, which is inspired by both coded caching [6] and the SJ PIR schemes [5] and enjoys combined coding gain. By comparing with the already established converse bounds for coded caching in [25], we show that the PD is optimal within a factor 8 in general as indicated in Corollary 3.

For general system parameters $K, K_u$ and $N \geq 2$, we assume that $t = \frac{K_uM}{K} \in [1 : K_u]$. Each message is assumed to have $L = \binom{K_u}{t}N^K$ bits. The cache placement and private delivery phases are described as follows.

1) Cache placement: The MAN cache placement is used. More specifically, each message $W_k, k \in [K]$ is split into $\binom{K_u}{t}$ disjoint and equal-size packets, i.e., $W_k \triangleq \{W_{k,T} : T \subseteq [K_u], |T| = t\}$. Therefore, each packet consists of $\frac{L}{\binom{K_u}{t}} = N^K$ bits. Each user $u \in [K_u]$ caches

\footnote{In fact, any capacity-achieving PIR scheme can be used to combine with the linear caching code to produce a corresponding product design.}
all the message packets $W_{k,T}$ such that $u \in T$, i.e.,

$$Z_u = \{W_{k,T} : T \subseteq [K_u], |T| = t, u \in T, \forall k \in [K]\}. \quad (47)$$

Therefore, each user stores $KL^{(K_u-1)\frac{t}{t+1}} = ML$ bits, satisfying the cache memory constraint.

2) Private Delivery: Suppose the user demands are $\theta = (\theta_1, \theta_2, \cdots, \theta_{K_u}) \in [K]^{K_u}$. We first construct $\binom{K_u}{t+1}$ different coded messages

$$\{X_S^{[\theta]} \triangleq (A_{1,S}^{[\theta]}, A_{2,S}^{[\theta]}, \cdots, A_{N,S}^{[\theta]}): S \subseteq [K_u], |S| = t + 1\}, \quad (48)$$

each of which being useful to a subset of $t + 1$ users in $S$. The $n$-th component of $X_S^{[\theta]}$, $A_{n,S}^{[\theta]}$, represents the answer from DB $n$. For each $S$, the components of the coded message $X_S^{[\theta]}$ are constructed according to the user demands as

$$A_{n,S}^{[\theta]} = \sum_{u \in S} A_{n}^{[\theta_u]} (W_{1:K,S\setminus\{u\}}) , \forall n \in [N], \quad (49)$$

where $W_{1:K,S\setminus\{u\}} \triangleq \{W_{1,S\setminus\{u\}}, W_{2,S\setminus\{u\}}, \cdots, W_{K,S\setminus\{u\}}\}$. The term $A_{n}^{[\theta_u]} (W_{1:K,S\setminus\{u\}})$ in the summation of $[49]$ represents the answer from DB $n$ in the SJ PIR scheme for the single-user PIR problem when the messages are (First message, second message, $\cdots$, $K$-th message) $=(W_{1,S\setminus\{u\}}, W_{2,S\setminus\{u\}}, \cdots, W_{K,S\setminus\{u\}})$ and the user $u$ demands $W_{\theta_u,S\setminus\{u\}}$. To make the scheme private, the users employ a randomly and uniformly distributed permutation (not known to the DBs) of the $N^K$ bits for each packet $W_{\theta_u,S\setminus\{u\}}$. For different coded messages $X_S^{[\theta]}$, the set of the random bit permutations of the corresponding packets are also independent from each other.

To ensure demand privacy, the ordering of the message bits in the query to the corresponding answer is preserved as in the SJ PIR scheme in the summation of $[49]$. Next we show that the proposed PD is both correct and private.

Correctness: We show that for any user $u \in [K_u]$, it can correctly recover its desired message $W_{\theta_u}$ from all the answers received. Since the packets $\{W_{k,T} : |T| = t, u \in T, \forall k \in [K]\}$ are cached by user $u$ in the placement phase, it needs to recover the packets $\{W_{\theta_u,T} : |T| = t, u \notin T\}$. For each $n \in [N], S \subseteq [K_u]$ such that $|S| = t + 1, u \in S$, we can write $[49]$ as

$$A_{n,S}^{[\theta]} = A_{n}^{[\theta_u]} (W_{1:K,S\setminus\{u\}}) + \sum_{u' \in S \setminus \{u\}} A_{n}^{[\theta_u']}(W_{1:K,S\setminus\{u'\}}), \quad (50)$$

from which user $u$ can decode the desired term $A_{n}^{[\theta_u]}(W_{1:K,S\setminus\{u\}})$ since all the packets $\{W_{1:K,S\setminus\{u'\}} : u' \neq u\}$ are cached by user $u$ because $u \in S \setminus \{u'\}$. Therefore, user $u$ obtains a set of desired
answers \( A^{[\theta_u]}_n \{ W_{1:K,S\setminus\{u\}} \} : \forall n \in [N] \) from which the desire packet \( W_{\theta_u,S\setminus\{u\}} \) can be decoded due to the decodability of the SJ PIR scheme. Going through all different such user subsets \( S \), user \( u \) can decode all the \((K_u - 1)\) desired packets. As a result, user \( u \) can correctly recover its desired message \( W_{\theta_u} \).

**Privacy:** It can be seen that each \( A^{[\theta_u]}_n \{ W_{1:K,S\setminus\{u\}} \} : \forall n \in [N] \) is independent of the demands of the users in \( S \) and that of the users in \([K_u]\setminus S\) from the perspective of each individual DB. The reason is explained as follows. For any user \( u \in S \), we see that the first term in (50), i.e., \( A^{[\theta_u]}_n \{ W_{1:K,S\setminus\{u\}} \} : \forall n \in [N] \) is independent of \( \theta_u \) by the privacy of the SJ PIR scheme. Also, each item in the summation of the second term in (50) is independent of \( \theta_u \) because for each \( A^{[\theta_u']}_n \{ W_{1:K,S\setminus\{u'\}} \} \), a set of random and independent permutations are employed to the bits of the set of packets \( \{ W_{k,S\setminus\{u'\}} : \forall k \in [K] \} \). Therefore, \( A^{[\theta_u]}_n \) is independent of the demands of the users in \( S \). Moreover, for \( S \) where \( u \notin S \), due to the employment of the random and independent permutations on the packet bits, \( A^{[\theta_u]}_n \) is independent of the demands of the users in \([K_u]\setminus S\). As a result, \( A^{[\theta_u]}_n \) is independent of \( \theta \) from DB \( n \)’s perspective for any \( S \subseteq [K_u], |S| = t + 1 \), which completes the privacy proof for the PD.

**Performance:** By the SJ PIR scheme, each \( X_{S}^{[\theta]} \) has \((1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}})N^K \) bits. Therefore, \( D = \binom{K_u}{t+1} (1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}})N^K \) bits are downloaded from the DBs in total. As a result, the achieved load is \( \hat{R}(M) = \frac{D}{L} = \frac{K_u-t}{t+1} (1 + \frac{1}{N} + \cdots + \frac{1}{N^{K-1}}) \).

**VI. Discussion: MuPIR with Distinct Demands**

In this section, we consider an interesting scenario where the users have distinct demands. Recall that \( R^*_d \) denotes the minimum load. We obtain the following theorem.

**Theorem 3:** For the cache-aided MuPIR problem with \( K = 2 \) messages, \( K_u = 2 \) users and \( N = 2 \) DBs, where the users demand distinct messages in a uniform manner, the optimal memory-load trade-off is characterized as

\[
R^*_d(M) = \begin{cases} 
2(1 - M), & 0 \leq M \leq \frac{1}{3} \\
\frac{5}{3} - M, & \frac{1}{3} \leq M \leq \frac{2}{3} \\
\frac{3(2-M)}{4}, & \frac{2}{3} \leq M \leq 2 
\end{cases}
\]  

(51)

**Proof:** For achievability, we show that the memory-load pairs \( (\frac{1}{3}, \frac{4}{3}) \) and \( (\frac{2}{3}, 1) \) are achievable using the idea of CIA in Section VI-A. Together with the two trivial pairs \( (0, 2) \) and \( (2, 0) \), we
obtain four corner points. By memory sharing among these corner points, the load of Theorem 3 can be achieved. For the converse, when $M \leq \frac{1}{3}$, the load of $R_d(M) = 2(1 - M)$ coincides with the caching bound without demand privacy and hence is optimal. When $M \geq \frac{2}{3}$, the load $R_d(M) = \frac{3(2-M)}{4}$ is optimal since it coincides with the single-user cache-aided PIR bound of [9]. A novel converse bound is derived for the cache memory regime $\frac{1}{3} \leq M \leq \frac{2}{3}$ to show the optimality of $R(M) = \frac{5}{3} - M$ when the users have distinct demands. □

**Remark 3:** It can be seen that the load in (51) is lower than the one in (1) achieved by the CIA based scheme. Thus the optimal load under the constraint of distinct demands can be strictly lower than the optimal load without such constraint (See Fig. 2a).

A. **Achievability**

First, we consider the achievability of the memory-load pair $(\frac{1}{3}, \frac{4}{3})$. Assume that the users have distinct demands, i.e., the demand vector $\theta = (\theta_1, \theta_2)$ can only be $(1, 2)$ or $(2, 1)$. Let $A_{1,1}$ and $A_{1,2}$ be two different answers from DB 1, and let $A_{2,1}$ and $A_{2,2}$ be two different answers from DB 2. Assume that each message contains $L = 3$ bits, i.e., $W_1 = (A_1, A_2, A_3), W_2 = (B_1, B_2, B_3)$. The cache placement is $Z_1 = \{A_1 + B_1\}, Z_2 = \{A_2 + B_2\}$ and the answers are $A_{1,1} = (A_3, B_1 + B_2 + B_3), A_{1,2} = (A_1 + A_2 + A_3, B_3), A_{2,1} = (A_2 + A_3, B_2 + B_3), A_{2,2} = (A_1 + A_3, B_1 + B_3)$.

The private delivery scheme is that the users randomly choose $A_{1,1}$ or $A_{1,2}$ to request from DB 1 with equal probabilities. We then consider the following two cases. When $(\theta_1, \theta_2) = (1, 2)$, if $A_{1,1}$ is chosen, then go to DB 2 to download $A_{2,1}$. Otherwise, if $A_{1,2}$ is chosen, go to DB 2 to download $A_{2,2}$. When $(\theta_1, \theta_2) = (2, 1)$, if $A_{1,1}$ is chosen, then go to DB 2 to download $A_{2,2}$. Otherwise if $A_{1,2}$ is chosen, go to DB 2 to download $A_{2,1}$.

For the correctness of this scheme, one can check that $(A_{1,1}, A_{2,1}, Z_1) \rightarrow W_1, (A_{1,1}, A_{2,1}, Z_2) \rightarrow W_2 (A_{1,2}, A_{2,2}, Z_1) \rightarrow W_1, (A_{1,2}, A_{2,2}, Z_2) \rightarrow W_2, (A_{1,1}, A_{2,2}, Z_1) \rightarrow W_2, (A_{1,1}, A_{2,2}, Z_2) \rightarrow W_1, (A_{1,2}, A_{2,1}, Z_1) \rightarrow W_2, and (A_{1,2}, A_{2,1}, Z_2) \rightarrow W_1$. Therefore, all users can decode their desired messages. For privacy of this scheme, note that the answer from DB 1 is equally likely to be $A_{1,1}$ or $A_{1,2}$, and the answer from DB 2 is also equally likely to be $A_{2,1}$ or $A_{2,2}$. Therefore, we have $P(\theta = (1, 2)) = P((A_{1,1}, A_{2,1})) + P((A_{1,2}, A_{2,2})) = \frac{1}{2}$ and $P(\theta = (2, 1)) = P((A_{1,1}, A_{2,2})) +$

\footnote{The notation $(A_{1,1}, A_{2,1}, Z_1) \rightarrow W_1$ means that the message $W_1$ can be recovered from $A_{1,1}, A_{2,1},$ and $Z_1$. Other notations follow similarly.}
\[ P((A_{1,2}, A_{2,1})) = \frac{1}{2} \] such that the privacy constraint (3) is satisfied (for distinct demands). Since \( D = 4 \) bits are downloaded in total, the achieved load is \( R_d = \frac{D}{L} = \frac{4}{3} \).

Second, we consider the achievability of the memory-load pair \((\frac{2}{3}, 1)\). Assume that each message has \( L = 3 \) bits, i.e., \( W_1 = (A_1, A_2, A_3), W_2 = (B_1, B_2, B_3) \). The cache placement is \( Z_1 = \{A_1, B_1\}, Z_2 = \{A_2, B_2\} \) and the answers from the DBs are \( A_{1,1} = (A_3 + B_3 + B_1 + B_2), A_{1,2} = (A_3 + B_3 + A_1 + A_2), A_{2,1} = (A_2 + A_3, B_2 + B_3) \) and \( A_{2,2} = (A_1 + A_3, B_1 + B_3) \).

The private delivery phase works similarly to the above corner point \((\frac{1}{3}, \frac{4}{3})\). The correctness of this scheme can be checked straightforwardly using the above transmitted codewords and cached information. The privacy argument is similar to the previous case, i.e., from each DB’s perspective, the demand vector \( \theta \) is equally likely to be \((1, 2)\) or \((2, 1)\). Since \( D = 3 \) bits are downloaded in total, the achieved load is \( R_d = \frac{D}{L} = 1 \).

**B. Converse**

The converse curve consists of three piece-wise linear segments corresponding to different cache memory regimes \( 0 \leq M \leq \frac{1}{3}, \frac{1}{3} \leq M \leq \frac{2}{3}, \) and \( \frac{2}{3} \leq M \leq 2 \). We prove the converse for each segment respectively.

1) \( 0 \leq M \leq \frac{1}{3} \): In this regime, the cut-set bound without privacy constraint is tight (see Corollary 1).

2) \( \frac{1}{3} \leq M \leq \frac{2}{3} \): Let \( A_{1,1} = A_{1,1}^{(1,2)} = A_{1,1}^{(2,1)} \) be an answer of DB 1 and let \( A_{2,1} = A_{2,1}^{(1,2)} = A_{2,1}^{(2,1)} \) be an answer of DB 2. It is clear that the message \( W_1 \) can be recovered from \( \{A_{1,1}, A_{2,1}^{(1,2)}, Z_1\} \) while \( W_2 \) can be recovered from \( \{A_{1,1}, A_{2,1}^{(1,2)}, Z_2\} \) for which we use a shorthand notation as \( (A_{1,1}^{(1,2)}, A_{2,1}^{(2,1)}, Z_1) \rightarrow W_1 \), \( (A_{1,1}^{(1,2)}, A_{2,1}^{(2,1)}, Z_2) \rightarrow W_2 \). For privacy of DB 1, there must exist another answer \( A_{2,1}^{(2,1)} \) of DB 2 such that the demands \( \theta = (2, 1) \) can be decoded, i.e., \( (A_{1,1}^{(2,1)}, A_{2,2}^{(2,1)}, Z_1) \rightarrow W_2 \) and \( (A_{1,1}^{(2,1)}, A_{2,2}^{(2,1)}, Z_2) \rightarrow W_1 \). Also, for privacy of DB 2, there must exist another answer \( A_{1,2}^{(2,1)} \) of DB 1 such that \( (A_{1,2}^{(2,1)}, A_{2,1}^{(2,1)}, Z_1) \rightarrow W_2 \) and \( (A_{1,2}^{(2,1)}, A_{2,1}^{(2,1)}, Z_2) \rightarrow W_1 \). Note that \( R_d = (H(A_{1,i}) + H(A_{2,j})) / L \) for any \( i, j \in [2] \) (Load
does not depend on \( \theta \). Denote \( X_{i,j,k}^{[\theta]} \triangleq (A_{1,i}^{[\theta]}, A_{2,j}^{[\theta]}, Z_k), \forall i, j, k \in [2] \). We then have

\[
R_d(M)L + 3ML \\
\geq H(X_{1,1,1}^{[(1,2)]}) + H(X_{2,1,2}^{[(2,1)]}) + H(X_{1,2,2}^{[(2,1)]}) \\
\overset{(a)}{=} 3L + H(X_{1,1,1}^{[(1,2)]}|W_1) + H(X_{2,1,2}^{[(2,1)]}|W_1) + H(X_{1,2,2}^{[(2,1)]}|W_1) \\
\overset{(b)}{=} 3L + H(X_{1,1,1}^{[(1,2)]}|W_1) + H(Z_2|W_1) + H(A_{2,1}^{[(2,1)]}|W_1, Z_2) + H(X_{1,2,2}^{[(2,1)]}|W_1) \\
\geq 3L + H(X_{1,1,1}^{[(1,2)]}, Z_2|W_1) + H(A_{2,1}^{[(2,1)]}|W_1, Z_2) + H(X_{1,2,2}^{[(2,1)]}|W_1) \\
\overset{(c)}{=} 4L + H(A_{2,1}^{[(2,1)]}|W_1, Z_2) + H(X_{1,2,2}^{[(2,1)]}|W_1) \\
\geq 4L + H(A_{2,1}^{[(2,1)]}|W_1, Z_2) + H(Z_2|W_1) + H(A_{1,1}^{[(2,1)]}|W_1, Z_2) \\
\geq 4L + H(A_{2,1}^{[(2,1)]}, A_{1,1}^{[(2,1)]}|W_1, Z_2) + H(Z_2|W_1) \\
= 4L + H(A_{2,1}^{[(2,1)]}, A_{1,1}^{[(2,1)]}, Z_2|W_1) \\
= 4L + H(A_{1,1}^{[(1,2)]}, A_{2,1}^{[(1,2)]}, Z_2|W_1) \\
= 5L, 
\]

where (a) is due to \( X_{1,1,1}^{[(1,2)]} \to W_1, X_{2,1,2}^{[(2,1)]} \to W_1 \) and \( X_{1,2,2}^{[(2,1)]} \to W_1 \); in (b) we used the chain rule and non-negativity of mutual information; (c) is due to the fact that both \( W_1 \) and \( W_2 \) can be decoded from \( X_{1,1,1}^{[(1,2)]} \) and \( Z_2 \). (52) implies \( R_d(M) \geq \frac{5}{3} - M \), which completes the converse proof.

3) \( \frac{2}{3} \leq M \leq 2 \): In this regime, the achievable load \( R_d(M) = \frac{3(2-M)}{4} \) coincides with the single-user cache-aided PIR bound given in [9] as stated in Corollary [1]. Since increasing the number of users while keeping the user demands private from the DBs can only possibly increase the load, we conclude that \( R(M) \geq \frac{3(2-M)}{4} \). This completes the converse proof of Theorem 3.

VII. Conclusions and Future Directions

In this paper, we introduced the problem of cache-aided Multiuser Private Information Retrieval (MuPIR) problem, which generalized the single-user PIR to the case of multiple users. We provided achievability for the MuPIR problem with two messages, two users and arbitrary number of databases utilizing the novel idea of cache-aided interference alignment (CIA). The proposed scheme is shown to be optimal when the cache placement is uncoded. For general systems parameters, inspired by both single-user PIR and coded caching, we proposed a product design
which is order optimal within a factor of 8. Moreover, when the demands are constrained to be distinct, the optimal memory-load tradeoff is characterized for a system with two messages, two users and two databases. Due to the strong connection to both PIR and coded caching, our result on the cache-aided MuPIR problem provides useful insights to understanding the role of side information (i.e., cache) in multiuser and multi-message PIR. Besides the proposed achievability and converse results, the cache-aided MuPIR problem still remains open for arbitrary system parameters in terms of the optimal memory-load trade-off. For example, utilizing the idea of CIA, we expect more achievability results to come. Also, based on the well-established converse results of coded caching and PIR, a systematic approach to characterize the converse is needed.

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