Inclusion Theorems for the Moyal Multiplier Algebras of Generalized Gelfand–Shilov Spaces

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Abstract. We prove that the Moyal multiplier algebras of the generalized Gelfand–Shilov spaces of type $S$ contain Palamodov spaces of type $E$ and the inclusion maps are continuous. We also give a direct proof that the Palamodov spaces are algebraically and topologically isomorphic to the strong duals of the spaces of convolutors for the corresponding spaces of type $S$. The obtained results provide a general and efficient way to describe the algebraic and continuity properties of pseudodifferential operators with symbols having an exponential or super-exponential growth at infinity.

Mathematics Subject Classification. 53D55, 43A22, 35S05, 46F05.

Keywords. Deformation quantization, Weyl symbols, Moyal product, Multiplier algebras, Gelfand–Shilov spaces, Pseudodifferential operators.

1. Introduction

The Gelfand–Shilov spaces of type $S$ provide a natural framework for studying infinite order pseudodifferential operators whose symbols have faster than polynomial growth at infinity. Various classes of pseudodifferential operators of this kind were investigated in recent papers [1,4,5,30] with special attention to their continuity, composition, and invariance properties. Similar issues arise in the context of noncommutative quantum field theory, where spaces of type $S$ were used to characterize the violations of locality and causality [7,34] and to analyze the behavior of propagators in some noncommutative models [13,14,42]. It is crucial for these applications that, under natural restrictions

†Editor-in-Chief’s note: After this paper was accepted on June 23, 2021, with great sadness, we had to learn that Dr. Soloviev had passed away on May 30, 2021, as a consequence of the terrible COVID19 pandemic. Thanks to the kind help of his former student and colleague, Dr. A.G. Smirnov (smirnov@lpi.ru), and of the referee, some minor final amendments of the original submission could be made.
specified in [33], the Gelfand–Shilov spaces of functions on the linear symplectic space $\mathbb{R}^{2d}$ are algebras under the Weyl–Moyal product

$$(f \star_\hbar g)(x) = (\pi \hbar)^{-2d} \int_{\mathbb{R}^{4d}} f(x - x')g(x - x'')e^{(2i/\hbar)x' \cdot Jx''} dx' dx'', \quad (1.1)$$

where $J = \left( \begin{array}{cc} 0 & I_d \\ -I_d & 0 \end{array} \right)$ is the standard symplectic matrix, $x = (q, p)$, $x' \cdot Jx'' = q'p'' - q''p'$, and $\hbar$ is the Planck constant. In this and previous papers [38–40], we study the algebras of multipliers of generalized Gelfand–Shilov spaces with respect to the noncommutative product (1.1). Their importance is due to the fact that these algebras extend this operation to the largest possible class of functions, including some elements of the duals of spaces of type $S$. From the viewpoint of the Weyl symbolic calculus, the multiplier algebras consist of the symbols of the operators that map the corresponding Fourier-invariant spaces of type $S$ continuously into itself. These algebras generalize the Moyal multiplier algebra $\mathcal{M}_\hbar(S)$ for the Schwartz space $S(\mathbb{R}^{2d})$ of all infinitely differentiable rapidly decreasing functions. The algebra $\mathcal{M}_\hbar(S)$ has been studied in many papers starting from [3,19,27,41] and its applications in quantum field theory on noncommutative spaces have been discussed in [16,18]. We treat the product (1.1) as a deformation of the ordinary pointwise product and use the notation $\star_\hbar$, accepted in deformation quantization theory, instead of the notation $\#$ which is used for the composition of Weyl symbols in [15,20,23] and corresponds to $\hbar = 1$.

Typically, applications use Gelfand–Shilov spaces of a particular type, denoted in [17] by $S^2_\alpha$. The algebras of their pointwise multipliers have been explicitly described by Palamodov [29] in terms of spaces of type $E$. Multiplier spaces of such a kind were recently studied in a more general setting by Debrouwere and Neyt [8] and by Albanese and Mele [2], with emphasis on their topological properties. The noncommutative deformation of multiplication violates the isomorphisms established in [29], but does not change some important inclusion relations, which are the subject of our research. The starting point for us is that the Moyal multiplier algebras of the spaces of type $S$ contain the duals of their convolutor spaces. This inclusion was proved for $S^2_\alpha$ in [35] and holds true in the general case, as shown in [38,40]. The spaces of convolutors for Gelfand–Shilov spaces were studied in [11,12] and, most thoroughly, by Debrouwere and Vindas in [9,10], where the short-time Fourier transform and a projective description of inductive limits were used for this purpose. Here we develop an alternative approach based on the continuous extension of the convolutors of spaces of type $S$ to the corresponding spaces of type $E$. A simple and natural way of such extension was proposed earlier in [36] and was also used in [40]. This approach allows us to completely characterize the convolutor spaces for the generalized spaces of type $S$. Significantly, it is also well suited for the cases that have not yet been considered, for example, where a space of type $S$ is nontrivial, whereas its projective counterpart is trivial. In combination with Theorem 4 of [39],
this approach provides a direct and simple proof of the continuous embedding of generalized Palamodov spaces of type $\mathcal{E}$ into the Moyal multipliers algebras of the corresponding spaces of type $S$, which is the main result of this paper. Along the way, we give new proofs and extensions of previous results on the algebraic and continuity properties of pseudodifferential operators with symbols that have an exponential or super-exponential growth at infinity.

The paper is structured as follows. In Sect. 2, we give the basic definitions concerning spaces of type $S$ and type $\mathcal{E}$ and introduce the notation. We try to follow the original notation in [17, 29] and let $a_n$ and $b_n$ denote the sequences defining the function spaces and specifying the behavior at infinity and the degree of smoothness of their elements. But instead of the notation $S_{a_k}^{b_n}$ introduced in [17], we use $S_{\{a\}}^{\{b\}}$, where the curly brackets mean that this space is the inductive limit of a family of normed spaces. The projective limit of the same family is denoted by $S_{\{a\}}^{(b)}$. Such a rule was used by Komatsu [25] and in many subsequent papers, and we apply it also to the spaces of type $\mathcal{E}$.

In Sect. 3, we prove that the convolutor spaces for $S_{\{a\}}^{\{b\}}$ and $S_{\{a\}}^{(b)}$ contain respectively the duals of the Palamodov spaces $\mathcal{E}_{\{a\}}^{\{b\}}$ and $\mathcal{E}_{\{a\}}^{(b)}$ and that these inclusions are continuous. In Sect. 4, we give a complete characterization of the convolutor spaces $C(S_{\{a\}}^{\{b\}})$ and $C(S_{\{a\}}^{(b)})$ in terms of the spaces of type $\mathcal{E}$.

In particular, we show that $\mathcal{E}_{\{a\}}^{\{b\}}$ is canonically isomorphic to the strong dual of $C(S_{\{a\}}^{\{b\}})$ and $\mathcal{E}_{\{a\}}^{(b)}$ is canonically isomorphic to the strong dual of $C(S_{\{a\}}^{(b)})$. As a consequence, we obtain Corollary 4.6 which describes some invariance properties of $\mathcal{E}_{\{a\}}^{\{b\}}$ and $\mathcal{E}_{\{a\}}^{(b)}$ and extends Theorem 4.1 of [5] to more general spaces and to a larger class of operators. In Sect. 5, we define the left, right, and two-sided multipliers for the noncommutative algebras $(S_{\{a\}}^{\{b\}}, \star_\hbar)$ and $(S_{\{a\}}^{(b)}, \star_\hbar)$ and prove that these algebras have approximate identities. This allows us to prove the equivalence of three different definitions of their corresponding multiplier algebras. In Sect. 6, we show that $\mathcal{E}_{\{a\}}^{\{b\}}$ is continuously embedded in the algebra $\mathcal{M}_\hbar(S_{\{a\}}^{\{b\}})$ of two-sided Moyal multipliers for $S_{\{a\}}^{\{b\}}$ and $\mathcal{E}_{\{a\}}^{(b)}$ is continuously embedded in $\mathcal{M}_\hbar(S_{\{a\}}^{(b)})$. In the same section we extend these theorems to other translation invariant star products and prove that $\mathcal{E}_{\{a\}}^{\{b\}}$ and $\mathcal{E}_{\{a\}}^{(b)}$ are topological algebras under these products. This result generalizes Theorem 4.14 of Cappiello and Toft [5]. Additional inclusion relations are established in the case of the Fourier-invariant spaces $S_{\{a\}}^{\{a\}}$ and $S_{\{a\}}^{(a)}$. Section 7 contains concluding remarks. In Appendix, we examine the mapping properties of pseudodifferential operators with symbols in another Palamodov space $\mathcal{E}_{\{a\}}^{\{a\}}$ and correct some statements made in [1,5] regarding operators of this class.
2. Preliminaries and Notation

Let $a = (a_n)_{n \in \mathbb{Z}_+}$ be a sequence of positive numbers such that

\begin{align}
    a_0 &= 1, \quad a_{n+1} \geq a_n, \quad (2.1) \\
    a_n^2 &\leq a_{n-1}a_{n+1}, \quad (2.2) \\
    a_{k+n} &\leq KH^{k+n}a_ka_n, \quad (2.3)
\end{align}

where $K$ and $H$ are positive constants. The logarithmic convexity condition (2.2) coupled with the normalization condition $a_0 = 1$ implies the inequality

\begin{align}
    a_ka_n &\leq a_{k+n}, \quad (2.4)
\end{align}

which will be also used throughout the paper. Let $b = (b_n)_{n \in \mathbb{Z}_+}$ be another sequence of positive numbers satisfying the same conditions. The Gelfand–Shilov space $S^{(b)}{(a)}(\mathbb{R}^d)$ consists of all infinitely differentiable functions $f$ defined on $\mathbb{R}^d$ and satisfying the inequalities

\[ |x^\alpha \partial^\beta f(x)| \leq CA|\alpha|B|\beta|a_\alpha b_\beta |a_{\alpha}|b_{\beta} | \quad \forall \alpha, \beta \in \mathbb{Z}_d^d, \]

where $C$, $A$, and $B$ are positive constants depending on $f$, $\mathbb{Z}^d_+$ is the set of $d$-tuples of nonnegative integers, and the standard multi-index notation is used. In what follows, we write for brevity $S^{(b)}{(a)}$ instead of $S^{(b)}{(a)}(\mathbb{R}^d)$ when this cannot cause confusion. This space is the union of a family of Banach spaces $\{S^{b,B}_{a,A}\}_{A,B>0}$ whose norms are given by

\begin{align}
    \|f\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta f(x)|}{A|\alpha|B|\beta|a_\alpha b_\beta |a_{\alpha}|b_{\beta} |}, \quad (2.5)
\end{align}

and its topology is defined to be the inductive limit topology with respect to the inclusion maps $S^{b,B}_{a,A} \hookrightarrow S^{(b)}{(a)}$. The most frequently used Gelfand–Shilov spaces $S^{\gamma}_{\alpha}$ were defined in [17] by sequences of the form

\begin{align}
    a_n = n^{\alpha n}, \quad b_n = n^{\beta n}, \quad (2.6)
\end{align}

where in this case $\alpha$ and $\beta$ are nonnegative numbers, which should not be confused with the multi-indices in (2.5). We will also consider the spaces

\begin{align}
    S^{(b)}{(a)} = \bigcap_{A \to 0, B \to 0} S^{b,B}_{a,A} \quad (2.7)
\end{align}

equipped with the projective limit topology. If $\lim_{n \to \infty} a_n^{1/n} = 0$, then the spaces $S^{(b)}{(a)}$ and $S^{(b)}{(a)}$ are trivial, i.e., contain only the identically zero function.

If $0 < \lim_{n \to \infty} a_n^{1/n} < \infty$, then all functions in $S^{(b)}{(a)}$ are compactly supported, and this space coincides with the space defined by $a_n \equiv 1$ and $b_n$, which is usually denoted by $D^{(b)}$. The space $S^{(b)}{(a)}$ is trivial in this case, and considering the spaces (2.7) we always assume that

\begin{align}
    \lim_{n \to \infty} a_n^{1/n} = \infty. \quad (2.8)
\end{align}

There are also other non-triviality conditions for the spaces of type $S$, see [17]. Their precise description is not needed for what follows, but we assume
throughout the paper that the spaces under consideration are nontrivial. A non-quasianalyticity condition is often imposed on $b_n$ to ensure that the space contains sufficiently many functions with compact support (see [17, 25]), but this condition is not used in the proofs given below. The Fourier transformation

$$F: f(x) \to \hat{f}(\zeta) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \zeta} f(x) dx$$

maps $S^{(b)}_{\{a\}}$ isomorphically onto $S^{(a)}_{\{b\}}$ and maps $S^{(b)}_{\{a\}}$ isomorphically onto $S^{(a)}_{\{b\}}$. Using (2.4), it is easy to see that $S^{(b)}_{\{a\}}$ and $S^{(a)}_{\{b\}}$ are algebras under pointwise multiplication and that this operation is continuous in their topologies. As a consequence, these spaces are also topological algebras under convolution. The norm (2.5) can be written as

$$\|f\|_{A,B} = \sup_{x,\beta} \frac{w_a(|x|/A)|\partial^\beta f(x)|}{B|\beta|b_\beta},$$

(2.9)

where $|x| = \max_{1 \leq j \leq d} |x_j|$ and

$$w_a(t) := \sup_{n \in \mathbb{Z}_+} \frac{t^n}{a_n}, \quad t \geq 0.$$  

(2.10)

This function is often called a weight function. We note in this connection that the replacement $\sup_x |\cdot| \rightarrow \|\cdot\|_{L^1}$ in (2.9) gives an equivalent system of norms (see, e.g, Lemma A.2 in [38] for a proof) and then $w_a(t)$ plays the role of a weight in the integral. Under condition (2.8), the function $w_a(t)$ is finite and continuous. If $a_n \equiv 1$, then the corresponding function $w_1(t)$ is equal to 1 for $0 \leq t \leq 1$ and is infinite for $t > 1$, i.e., $1/w_1(t)$ is the characteristic function of the interval $[0, 1]$. It follows from the definition and from (2.1) that $w_a(t) \geq 1$ and that this function is convex and monotonically increases faster than $t^n$ for any $n$. Therefore,

$$w_a \left( \frac{t_1 + t_2}{A_1 + A_2} \right) \leq w_a \left( \frac{t_1}{A_1} \right) w_a \left( \frac{t_2}{A_2} \right)$$

(2.11)

for any positive $A_1$ and $A_2$. In particular, $w_a((t_1 + t_2)/2) \leq w_a(t_1) w_a(t_2)$. Setting $t_1 = |x|$ and $t_2 = |y|$, we obtain the inequality

$$w_a \left( \frac{1}{2} |x + y| \right) \leq w_a(|x|) w_a(|y|)$$

(2.12)

which is most often used below. The condition (2.3) implies that

$$w_a(t)^2 \leq Kw_a(Ht).$$

(2.13)

Along with the spaces of rapidly decreasing functions, we will consider spaces of rapidly increasing functions with the same degree of smoothness.
Namely, let \( \mathcal{E}^{b,B}_{a,A} \) be the Banach space of all functions with the finite norm \(^1\)

\[
\|h\|_B^A = \sup_{x,\beta} \frac{|\partial^\beta h(x)|}{w_a(|x|/A)B^{|\beta|/|\beta|}}.
\]

(2.14)

Using the family \( \{\mathcal{E}^{b,B}_{a,A}\}_{A,B > 0} \), we define the spaces

\[
\mathcal{E}^{(b)}_{(a)} := \bigcap_{A \to \infty} \bigcup_{B \to \infty} \mathcal{E}^{b,B}_{a,A}, \quad \mathcal{E}^{(a)}_{(b)} := \bigcap_{B \to 0} \bigcup_{A \to 0} \mathcal{E}^{b,B}_{a,A},
\]

(2.15)

\[
\tilde{\mathcal{E}}^{(b)}_{(a)} := \bigcup_{B \to \infty} \bigcap_{A \to \infty} \mathcal{E}^{b,B}_{a,A}, \quad \tilde{\mathcal{E}}^{(a)}_{(b)} := \bigcup_{A \to 0} \bigcap_{B \to 0} \mathcal{E}^{b,B}_{a,A},
\]

(2.16)

where the intersections are endowed with the projective limit topology and the unions are endowed with the inductive limit topology. The spaces (2.15) and (2.16) play the same role in the theory of ultradistributions defined on \( \mathcal{S}^{(a)}_{(b)} \) and \( \mathcal{S}^{(b)}_{(a)} \) as the spaces \( \mathcal{O}_M \) and \( \mathcal{O}_C \) in the Schwartz theory of tempered distributions [32]. They were introduced by Palamodov [29] (with somewhat different notation) for the case (2.6) and were called spaces of type \( \mathcal{E} \). This terminology is quite natural because the notation \( \mathcal{E}(\mathbb{R}^d) \) was used by Schwartz for the space of all infinitely differentiable functions on \( \mathbb{R}^d \). In the case where \( a_n \equiv 1 \), we write \( \mathcal{E}^{(b)}_{(a)} \) instead of \( \mathcal{E}^{(b)}_{(1)} \) and \( \tilde{\mathcal{E}}^{(b)}_{(a)} \) instead of \( \tilde{\mathcal{E}}^{(b)}_{(1)} \), which is in agreement with the notation in [25]. Obviously, we have the continuous inclusions

\[
\tilde{\mathcal{E}}^{(b)}_{(a)} \hookrightarrow \mathcal{E}^{(b)}_{(a)} , \quad \tilde{\mathcal{E}}^{(a)}_{(b)} \hookrightarrow \mathcal{E}^{(a)}_{(b)}.
\]

(2.17)

It should be noted that the spaces \( \mathcal{O}^{(M,p)}_{a_n,A_n}(\mathbb{R}^d) \) and \( \mathcal{O}^{(M,p)}_{a_n,A_n}(\mathbb{R}^d) \) considered in [10] are respectively \( \mathcal{E}^{(M)}_{(A)} \) and \( \mathcal{E}^{(M)}_{(A)} \) in our notation, and the symbol classes \( \Gamma_{s}^{\infty}(\mathbb{R}^d) \) and \( \Gamma_{0,s}^{\infty}(\mathbb{R}^d) \) studied in [5] coincide respectively with \( \tilde{\mathcal{E}}^{(a)}_{(a)} \) and \( \mathcal{E}^{(a)}_{(a)} \), where \( a_n = n^{sn} \). The spaces \( \mathcal{S}^{(a)}_{(b)} \) and \( \mathcal{S}^{(a)}_{(b)} \) with \( a = n^{sn} \) are denoted there by \( \mathcal{S}_a(\mathbb{R}^d) \) and \( \Sigma_a(\mathbb{R}^d) \). We also note that the notation in [38–40] is related to the notation used here as follows: \( S_a^{b,b} \leftrightarrow S^{(b)}_{a(a)} \), \( \mathcal{O}^{b,b} \leftrightarrow S_{a(a)}^{(b)} \),

\[
E^{b}_{a} \leftrightarrow \mathcal{E}^{(b)}_{a(a)}, \quad E^{b}_{a} \leftrightarrow \mathcal{E}^{(b)}_{a(a)}.
\]

The spaces of type \( S \) and type \( \mathcal{E} \) have nice topological properties which follow from the fact that the inclusion maps \( S^{b,B}_{a,A} \hookrightarrow S^{b,B}_{a,A} \) and \( \mathcal{E}^{b,B}_{a,A} \hookrightarrow S^{b,B}_{a,A} \), where \( A < A \) and \( B < B \), are compact. This is well known for the spaces of type \( S \) and is proved in the same manner as in [17] for \( S_{a}^{b} \). In the case of spaces of type \( \mathcal{E} \), the compactness of the inclusion maps can also be proved analogously, see, e.g., Lemma 2 in [39]. It follows that \( S^{(b)}_{a(a)} \) is an (FS)-space (Fréchet-Schwartz space) and \( \mathcal{S}^{(b)}_{a(a)} \) belongs to the dual class of (DFS)-spaces (see [24,28] for the basic properties of (FS) and (DFS)-spaces). In particular, these spaces are complete, barrelled, reflexive, and Montel. An important consequence is that the inductive limit \( S^{(b)}_{a(a)} \) is regular, i.e., every bounded

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\(^1\)In the case where \( a_n \equiv 1 \), the functions in \( \mathcal{E}^{b,B}_{a,A} \) are defined only for \( |x| \leq A \) and are regarded as zero if they are zero in this domain.
subset of it is contained and bounded in some $S_{a,A}^{b,B}$. We let $S_{(a)}^{(b),\prime}$ and $S_{(a)}^{(b)}$ denote the dual spaces of $S_{\{a\}}^{(b)}$ and $S_{\{a\}}^{(b)}$ and assume that the duals are endowed with the strong topology. Then the first of them is an (FS)-space and the second is a (DFS)-space. The inductive system of spaces $E_{a,a,A}^{b,B} = \bigcap_{A \to \infty} E_{a,a,A}^{b,B}$, $B > 0$, is equivalent to the system $E_{a,a,A}^{b,B+} = \bigcap_{A \to \infty, \epsilon \to 0} E_{a,a,A}^{b,B+\epsilon}$, because there are continuous inclusions $E_{a,a,A}^{b,B} \subset E_{a,a,A}^{b,B+} \subset E_{a,a,A}^{b,B}$ for $B < B$. In turn, the inductive system $E_{a,a,A}^{b,B} = \bigcap_{A \to \infty} E_{a,a,A}^{b,B}$, $A > 0$, is equivalent to the system $E_{a,a,A}^{b,B} = \bigcap_{B \to 0, \epsilon \to 0} E_{a,a,A}^{b,B}\cdot$. As a consequence, the projective system of the duals $(E_{(a)}^{b,B})'$ is equivalent to the system $(E_{(a)}^{b,B+\epsilon})'$ and the projective system $(E_{a,a,A}^{b,B})'$ is equivalent to the system $(E_{a,a,A}^{b,B})'$. By Lemma 2 in [39], $E_{(a)}^{b,B+\epsilon}$ and $E_{a,a,A}^{b,B}$ are (FS)-spaces and their strong duals are hence (DFS)-spaces. The representations

\[
\hat{E}_{(a)}^{b,B} = \lim_{B \to \infty} E_{a,a,A}^{b,B+}, \quad \hat{E}_{(a)}^{b,B} = \lim_{A \to 0} E_{a,a,A}^{b,B-}
\]

play an essential role in our study.

For any topological vector space $E$, we let $L(E)$ denote the space of all continuous linear maps $E \to E$, equipped with the topology of uniform convergence on bounded subsets. If $E$ is an (FS)-space or a (DFS)-space, then $L(E)$ is complete (Sect. 39.6 in [26]). If $E$ is a test-function space on $R^d$, then a functional $u \in E'$ is called a convolutor for $E$ if the function

\[
(u * f)(x) = \langle u, f(x - \cdot) \rangle
\]

belongs to $E$ for any $f \in E$ and the map $f \mapsto u * f$ is continuous on $E$. In the case of the spaces of type $S$, the continuity condition is automatically satisfied by the closed graph theorem (Theorem 8.5 in Ch. IV in [31]). The set of all convolutors for $E$ is denoted by $C(E)$ and is equipped with the topology induced by that of $L(E)$. The Fourier transformation maps $C(S_{\{a\}}^{b})$ and $C(S_{\{a\}}^{(b)})$ isomorphically onto the spaces of pointwise multipliers for $S_{\{a\}}^{b}$ and $S_{\{a\}}^{(b)}$, which we respectively denote by $M(S_{\{a\}}^{b})$ and $M(S_{\{a\}}^{(b)})$.

### 3. The Duals of Spaces of Type $E$ as Spaces of Convolutors

Clearly, we have the canonical continuous inclusions

\[
S_{\{a\}}^{b} \hookrightarrow E_{\{a\}}^{b}, \quad S_{\{a\}}^{(b)} \hookrightarrow E_{\{a\}}^{(b)}.
\]

**Proposition 3.1.** If $S_{\{a\}}^{b}$ and $S_{\{a\}}^{(b)}$ are nontrivial, then the inclusion maps (3.1) have dense ranges.

This was proved in [38] and we reproduce the proof below when proving Theorems 4.1 and 4.3. It follows that the adjoint maps $E_{\{a\}}^{(b),\prime} \to S_{\{a\}}^{b}$ and
\[ \mathcal{E}_{(a)}^{(b)} \rightarrow S_{(a)}^{(b)} \] are injective. Hence \( \tilde{\mathcal{E}}_{(a)}^{(b)} \) and \( \tilde{\mathcal{E}}_{(a)}^{(b)} \) are naturally identified with vector subspaces of \( S_{(a)}^{(b)} \) and \( S_{(a)}^{(b)} \), respectively.

**Proposition 3.2.** The space \( \tilde{\mathcal{E}}_{(a)}^{(b)} \) is contained in the convolutor space \( C(S_{(a)}^{(b)}) \) and \( \tilde{\mathcal{E}}_{(a)}^{(b)} \) is contained in \( C(S_{(a)}^{(b)}) \).

**Proof.** Standard arguments show that for any \( u \in S_{(a)}^{(b)} \) and \( f \in S_{(a)}^{(b)} \), the convolution \( (u \ast f)(x) \) is infinitely differentiable and \( \partial^\beta(u \ast f) = u \ast \partial^\beta f \) (see Lemma A.4 in [38]). If \( u \) belongs to \( \tilde{\mathcal{E}}_{(a)}^{(b)} \), then it is continuous on every space \( \mathcal{E}_{(a)}^{b,B} = \lim_{A \to \infty} \mathcal{E}_{a,A}^{b,B} \), \( B > 0 \), and is bounded in some norm \( \| \cdot \|_B^A \), where \( A \) depends on \( B \). Hence,

\[
|\partial^\beta(u \ast f)(x)| \leq \|u\|_B^A \|\partial^\beta f(x - \cdot)\|_B^A. \tag{3.2}
\]

Let \( f \in \mathcal{E}_{a,A_0}^{b,B_0} \) and \( B \geq HB_0 \). Using (2.3) applied to \( b_0 \) and (2.11), we obtain

\[
\|\partial^\beta f(x - \cdot)\|_B^A = \sup_{y,\gamma} \frac{|\partial^\gamma f(x - y)|}{w_a(|y|/A)B^{\gamma}|b_\gamma|} 
\leq \|f\|_{A,B_0} \sup_{y,\gamma} \frac{B_0^{\gamma+\gamma}b_\gamma}{w_a(|y|/A)w_a(|x - y|/A_0)B^{\gamma}|b_\gamma|} 
\leq K\|f\|_{A,B_0} \frac{(HB_0)^\beta|b_\beta|}{w_a(|x|/(A + A_0))}. \tag{3.3}
\]

It follows from (3.2) and (3.3) that \( u \ast f \in S_{a,A_0}^{h,B} \) and

\[
\|u \ast f\|_{A,A_0,H,B_0} \leq K\|u\|_B^A \|f\|_{A_0,B_0}. \tag{3.4}
\]

Hence, \( \tilde{\mathcal{E}}_{(a)}^{(b)} \subset C(S_{(a)}^{(b)}) \). The proof holds true for \( a_n \equiv 1 \). In this case, the supports of the functions \( 1/w_1(|y|/A) \) and \( 1/w_1(|x - y|/A_0) \) are disjoint for \( |x| > A + A_0 \), and \( (u \ast f)(x) = 0 \) for each \( x \) in this region.

If \( u \in \tilde{\mathcal{E}}_{(a)}^{(b)} \), then \( u \) is continuous on every space \( \mathcal{E}_{a,A}^{B} = \lim_{B \to 0} \mathcal{E}_{a,A}^{b,B} \) and is bounded in some norm \( \| \cdot \|_B^A \), where \( B \) depends on \( A \). For \( f \in S_{(a)}^{(b)} \), the norm \( \|f\|_{A_0,B_0} \) is finite for arbitrarily small \( A_0 \) and \( B_0 \). Choosing \( B_0 \leq B/H \), we again arrive at (3.4), which implies the inclusion \( \tilde{\mathcal{E}}_{(a)}^{(b)} \subset C(S_{(a)}^{(b)}) \). \( \square \)

**Proposition 3.3.** The inclusion maps

\[
\tilde{\mathcal{E}}_{(a)}^{(b)} \hookrightarrow C(S_{(a)}^{(b)}), \quad \tilde{\mathcal{E}}_{(a)}^{(b)} \hookrightarrow C(S_{(a)}^{(b)}), \tag{3.5}
\]

are continuous.

**Proof.** By the duality between projective and inductive limits (Sect. IV.4.5 in [31]), it follows from (2.18) that \( \tilde{\mathcal{E}}_{(a)}^{(b)} \) can be algebraically identified with \( \lim_{B \to \infty} (\mathcal{E}_{(a)}^{b,B}) \). A base of neighborhoods of the origin in \( \mathcal{E}_{(a)}^{(b)} \) is formed by the polars of the bounded subsets of \( \mathcal{E}_{(a)}^{(b)} \), and this topology is finer than the projective limit topology because every bounded subset of \( \mathcal{E}_{(a)}^{b,B} \) is bounded.
in $\tilde{E}^{(b)}_{(a)}$. We show that the projective limit topology is in turn finer than the topology induced by $C(S^{(b)}_{(a)})$. Let $Q$ be a bounded set in $S^{(b)}_{(a)}$, let $V$ be a 0-neighborhood in $S^{(b)}_{(a)}$, and let $W_{Q,V}$ be the set of operators in $L(S^{(b)}_{(a)})$ that map $Q$ into $V$. The family of all sets $W_{Q,V}$ with various $Q$ and $V$ forms a base of 0-neighborhoods in $L(S^{(b)}_{(a)})$. We assert that for every $W_{Q,V}$, there exists a 0-neighborhood $U$ in $\lim_{B \to \infty} (\mathcal{E}^{(b),B+}_{(a)})$ such that all operators of convolution by elements of $U$ are contained in $W_{Q,V}$. Taking the projective limit $\mathcal{E}^{b,B+}_a = \lim_{B \to \infty} (\mathcal{E}^{(b),B+}_a)$ in the reduced form, i.e., replacing every space $\mathcal{E}^{b,B+}_{a,A}$, $B > 0$, with the closure of $\mathcal{E}^{b,B+}_{a,A}$ in this space and letting $E^{b,B+}_{a,A}$ denote this closure, we have $\left(\mathcal{E}^{b,B+}_a\right)' = \lim_{\to A \to \infty, \epsilon \to 0} \left(E^{b,b+}_{a,A}\right)'$. By Theorem 11 in [24], this identity holds algebraically and topologically because the inclusion maps $E^{b,B}_{a,A} \hookrightarrow E^{b,B}_{a,A}$, where $A < A$ and $B < B$, are compact. To simplify the notation, we use an injective sequence of spaces that is equivalent to the system $S^{(b)}_{a,A}$. Since the inductive limit $S^{(b)}_{(a)}$ is regular, the set $Q$ is contained in $S^{(b)}_{a,k}/H$ with sufficiently large $k \in \mathbb{N}$ and is bounded in its norm by a constant $C$. By the definition of the inductive topology, $V$ contains a set of the form $\bigcup_{n \in \mathbb{N}} \sum_{m \leq n} V_m$, where $V_m = \{ f \in S^{b,m}_{a,m} : \Vert f \Vert_{m,m} \leq \varepsilon_m \}$ with some $\varepsilon_m > 0$. Let $U$ be the intersection of $\tilde{E}^{(b)}_{(a)}$ with a 0-neighborhood in $(\mathcal{E}^{(b),k+}_a)' = \lim_{m \to \infty} (E^{b,k+1/m}_{a,m})'$ which has the form $\bigcup_{n \in \mathbb{N}} \sum_{m \leq n} U_m$, where $U_m = \{ u \in (\mathcal{E}^{(b),k+}_a)' : \Vert u \Vert_{k+1/m} \leq \varepsilon_{k+m}/(KC) \}$. It follows from (3.4) that $u \ast f \in S^{b,k}_{a,k+m}$ and $\Vert u \ast f \Vert_{k+m,k} \leq \varepsilon_{k+m}$ for all $f \in Q$ and $u \in U_m$. Because $\Vert u \ast f \Vert_{k,m,k} \leq \Vert u \ast f \Vert_{k+m,k}$, we conclude that the operator of convolution by $u \in U_m$ maps $Q$ into $V_{k+m}$. Therefore, all the operators of convolution by elements of $U$ map $Q$ into $V$, as claimed.

We now show that the inclusion map $\tilde{E}^{(b)}_{(a)} \to C(S^{(b)}_{(a)})$ is continuous. Each bounded set $Q \subseteq S^{b}_{(a)}$ is bounded in the norm of every space $S^{b,1/(Hm)}_{a,1/m}$ by a constant $C_m$. Any 0-neighborhood $V$ in $S^{(b)}_{(a)}$ contains the intersection of $\mathcal{E}^{(b)}_{(a)}$ with a set of the form $V_{k,\varepsilon} = \{ f \in E^{b,1/k}_{a,1/k} : \Vert f \Vert_{1/k,1/k} \leq \varepsilon \}$, where $k \in \mathbb{N}$ and $\varepsilon > 0$. We take $U$ to be the intersection of $\tilde{E}^{(b)}_{(a)}$ with the 0-neighborhood in $(\mathcal{E}^{(b)}_{a,1/k})'$, which is the absolutely convex hull of the union of the sets $U_m = \{ u \in (\mathcal{E}^{(b),1/k}_{a,1/k})' : \Vert u \Vert_{1/k,1/k} \leq \varepsilon/(KC_m) \}$, $m > k$. It follows from (3.4) that $\Vert u \ast f \Vert_{1/k,1/m} \leq \varepsilon$ for all $f \in Q$ and $u \in U_m$. Because $\Vert u \ast f \Vert_{1/k,1/k} \leq \Vert u \ast f \Vert_{1/k,1/m}$ for $m > k$, we conclude that the operators of convolution by elements of $U_m$ map $Q$ into $V_{k,\varepsilon}$. The set $V_{k,\varepsilon}$ is absolutely convex, and all the operators of convolution by elements of $U$ hence belong to $W_{Q,V}$. This completes the proof. \qed
4. Complete Characterization of the Convolutor Spaces for the Spaces of Type $S$

Theorem 1 in [40] establishes that every functional $u \in C(S^{(b)}_{(a)})$ has a unique continuous extension to $\mathcal{E}^{(b)}_{(a)}$ and every $u \in C(S^{(b)}_{(a)})$ extends uniquely to a continuous linear functional on $\mathcal{E}^{(b)}_{(a)}$. Therefore, there are natural inclusion maps $C(S^{(b)}_{(a)}) \to \mathcal{E}^{(b)}_{(a)}$ and $C(S^{(b)}_{(a)}) \to \mathcal{E}^{(b)}_{(a)}$. Clearly, their compositions with the respective inclusions (3.5) are the identity on $C(S^{(b)}_{(a)})$ and on $C(S^{(b)}_{(a)})$. Since the extensions are unique, the compositions of these maps in reverse order are the identity on $\mathcal{E}^{(b)}_{(a)}$ and on $\mathcal{E}^{(b)}_{(a)}$. Hence, the convolutor space $C(S^{(b)}_{(a)})$ consists of the same elements of $S^{(b)}_{(a)}$ as $\mathcal{E}^{(b)}_{(a)}$, and $C(S^{(b)}_{(a)})$ coincides algebraically with $\mathcal{E}^{(b)}_{(a)}$. We now show that the extension procedure used in [36,40] makes clear the relation between the topologies of these spaces.

**Theorem 4.1.** The space $C(S^{(b)}_{(a)})$ of convolutors for $S^{(b)}_{(a)}$ is canonically isomorphic to $\lim_{A \to 0} (\mathcal{E}^{(b)}_{A})$.

**Proof.** If $u$ belongs to $C(S^{(b)}_{(a)})$, then it can be extended to a continuous functional on $\mathcal{E}^{(b)}_{(a)}$ in the following way. Let $f_0$ be a function in $S^{(b)}_{(a)}$ such that $\int f_0(\xi)d\xi = 1$ and let $h \in \mathcal{E}^{(b)}_{(a)}$. We set

$$\langle \tilde{u}, h \rangle := \int \langle u, h(\cdot)f_0(\xi - \cdot) \rangle d\xi \equiv \int (u * h_\xi)(\xi)d\xi, \quad \text{ (4.1)}$$

where

$$h_\xi(x) := h(\xi - x)f_0(x).$$

The integrand in (4.1) is well-defined and continuous in $\xi$ because translations act continuously on $S^{(b)}_{(a)}$ and $h$ is a pointwise multiplier of this space by Theorem 2 in [39]. We examine the behaviour of $(u * h_\xi)(\xi)$ at infinity. The norm $\|f_0\|_{A_0,B_0}$ is finite for any $A_0,B_0 > 0$ and there is an $A$ such that $\|h\|_B < \infty$ for any $B > 0$. Using (2.4) and (2.12), we obtain

$$|\partial^\beta_x h_\xi(x)| \leq \sum_{\gamma \leq \beta} \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) |\partial^\gamma h(x - x)\partial^{\beta - \gamma} f_0(x)|$$

$$\leq \|h\|_B^A \|f_0\|_{A_0,B_0} \sum_{\gamma \leq \beta} \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) B^{[\gamma]} B_0^{[\beta - \gamma]} b_{[\beta - \gamma]} \frac{w_a(|\xi - x|/A_0)}{w_a(|x|/A_0)}$$

$$\leq K \|h\|_B^A \|f_0\|_{A_0,B_0} (B + B_0)^{|\beta|} b_{|\beta|} \frac{w_a(2|x|/A_0)}{w_a(|x|/A_0)} w_a(2|x|/A_0). \quad \text{ (4.2)}$$

Choosing $A_0 \leq A/(2H)$ and using (2.13), the $x$-dependent factor in (4.2) is estimated as follows

$$\frac{w_a(2|x|/A)}{w_a(|x|/A_0)} \leq \frac{w_a(|x|/HA_0)}{w_a(|x|/A_0)} \leq \frac{K}{w_a(|x|/HA_0)}.$$
Consequently, \( h_\xi \in S_{a,H A_0}^{b,B+B_0} \) and
\[
\| h_\xi \|_{H A_0,B+B_0} \leq K \| h \|_B^A \| f_0 \|_{A_0,B_0} w_a(2|\xi|/A).
\] (4.3)
Let \( U = \{ f \in S_{(a)}^{(b)} : \| f \|_{A/2H,1} < 1 \} \). Since the map \( f \to u * f \) is continuous on \( S_{(a)}^{(b)} \), there is a neighborhood \( U_1 = \{ f \in S_{(a)}^{(b)} : \| f \|_{A_1,B_1} \leq \varepsilon \} \) such that \( u * f \in U \) for all \( f \in U_1 \). Choose \( A_0, B_0 \) and \( B \) such that \( HA_0 \leq A_1 \) and \( B + B_0 \leq B_1 \). Then \( \| f \|_{A_1,B_1} \leq \| f \|_{H A_0,B+B_0} \) and (4.3) implies that
\[
\| u * h_\xi \|_{A/2H,1} \leq \varepsilon^{-1} K \| h \|_B^A \| f_0 \|_{A_0,B_0} w_a(2|\xi|/A).
\]
Using (2.13) again, we obtain
\[
\left| (u * h_\xi)(\xi) \right| \leq \varepsilon^{-1} K \| h \|_B^A \| f_0 \|_{A_0,B_0} \frac{w_a(2|\xi|/A)}{w_a(2|\xi|/A)} \leq \varepsilon^{-1} K^2 \| h \|_B \| f_0 \|_{A_0,B_0} \frac{1}{w_a(2|\xi|/A)}. \tag{4.4}
\]
Therefore, the integral in (4.1) is absolutely convergent and defines a linear functional \( \tilde{u} \) on \( \mathcal{E}_{(a)}^{(b)} \). Since the right-hand side of (4.4) contains \( \| h \|_B^A \), this functional is continuous on every space \( \mathcal{E}_{(a),A}^{(b)}, A > 0 \), and therefore on \( \mathcal{E}_{(a)}^{(b)} \).

We now show that \( \tilde{u}|_{S_{(a)}^{(b)}} = u \). If \( h \in S_{(a)}^{(b)} \), then for any positive \( A_0, B_0, A_1, \) and \( B_1 \), we have the inequality
\[
|\partial_x^\beta(h(x)f_0(\xi - x))| \leq \| h \|_{A_1,B_1} \| f_0 \|_{A_0,B_0} \frac{(B_1 + B_0)^{|\beta|}b_{|\beta|}}{w_a(|x|/A_1)w_a(|\xi - x|/A_0)}. \tag{4.5}
\]
If \( HA_1 \leq A_0 \), then \( 1/w_a(|x|/A_1) \leq K/w_a^2(|x|/A_0) \) and (4.5) coupled with (2.12) gives
\[
\| (h(\cdot)f_0(\xi - \cdot)) \|_{A_0,B_1+B_0} \leq K \| h \|_{A_1,B_1} \| f_0 \|_{A_0,B_0} \frac{1}{w_a(|\xi|/2A_0)}. \tag{4.6}
\]
We see that in this case, the integral in (4.1) remains absolutely convergent when \( u \) is replaced with any functional in \( S_{(a)}^{(b)} \) and hence the sequence of Riemann sums
\[
s_n(x) = \sum_{\alpha \in \mathbb{Z}^d, |\alpha| \leq n^2} h(x)f_0(\alpha/n - x)/n^d \tag{4.6}
\]
for \( \int_{\mathbb{R}^d} h(x)f_0(\xi - x)d\xi \) is weakly Cauchy in \( S_{(a)}^{(b)}(\mathbb{R}^d) \). Since \( S_{(a)}^{(b)} \) is a Montel space, it is weakly sequentially complete and weak sequential convergence implies convergence in its topology. So, the sequence \( s_n(x) \) converges in \( S_{(a)}^{(b)} \). Its limit cannot be anything other than \( h(x) \), because the topology of this space is finer than the topology of simple convergence. Hence, \( \langle \tilde{u}, h \rangle = \lim_{n \to \infty} \langle u, s_n \rangle = \langle u, h \rangle \) if \( h \in S_{(a)}^{(b)} \). Similar arguments show that \( S_{(a)}^{(b)} \) is dense in \( \mathcal{E}_{(a),A}^{(b)} \). Namely, if \( h \in \mathcal{E}_{(a),A}^{(b)} \), then for any positive \( A_0, B_0, \) and \( B \) we have
\[
|\partial_x^\beta(h(x)f_0(\xi - x))| \leq \| h \|_B \| f_0 \|_{A_0,B_0} (B + B_0)^{|\beta|}b_{|\beta|} \frac{w_a(|x|/A)}{w_a(|\xi - x|/A_0)}, \tag{4.7}
\]
where \( w_a(|x|/A) \leq Kw_a(H|x|/A)/w_a(|x|/A) \). Choosing \( A_0 \leq A \) and using (2.12), we obtain
\[
\|h(\cdot)f_0(\xi - \cdot)\|_{B+B_0}^{A/H} \leq K\|h\|^A_B\|f_0\|_{A_0s,B_0} \frac{1}{w_a(|\xi|/2A)}. \tag{4.8}
\]
Hence in this case, the integral in (4.1) is absolutely convergent for any \( u \) in the dual of the Montel space \( \mathcal{E}^{(b)}_{a,A/H} \) and the sequence (4.6) converges to \( h \) in this space and, a fortiori, in \( \mathcal{E}^{(b)}_{\{a\}} \). This proves Proposition 3.1, or more precisely, its part concerning the inclusion \( S^{(b)}_{\{a\}} \hookrightarrow \mathcal{E}^{(b)}_{\{a\}} \) and shows that the continuous extension of \( u \) to \( \mathcal{E}^{(b)}_{\{a\}} \) is unique. In combination with Proposition 3.2, this also completes the proof of the algebraic isomorphism between \( C(S^{(b)}_{\{a\}}) \) and \( \tilde{\mathcal{E}}^{(b)}_{\{a\}} \).

It remains to show that the map \( u \mapsto \tilde{u} \) from \( C(S^{(b)}_{\{a\}}) \) to \( \lim_{A \to 0} (\mathcal{E}^{(b)}_{a,A})' \) is continuous, since the continuity of its inverse was proved in the proof of Proposition 3.3. Let \( \mathcal{B}_A \) be the set of all bounded subsets of \( \mathcal{E}^{(b)}_{a,A} \) and \( \mathcal{B} = \bigcup_{A > 0} \mathcal{B}_A \). The polars of sets in \( \mathcal{B} \) form a 0-neighborhood base for \( \lim_{A \to 0} (\mathcal{E}^{(b)}_{a,A})' \). Therefore it suffices to show that for any given \( G \in \mathcal{B}_A \), there are a bounded set \( Q \) and a 0-neighborhood \( V \) in \( S^{(b)}_{\{a\}} \) such that the inclusion \( u * Q \subset V \) implies \( \sup_{h \in G} |\langle u, h \rangle| \leq 1 \). It follows from (4.3) that the set \( Q = \{ w_a(2|\xi|/A)^{-1}h_\xi : h \in G, \xi \in \mathbb{R}^d \} \) is bounded in \( S^{(b)}_{\{a\}}(\mathbb{R}^d) \). Let \( V = \{ f \in S^{(b)}_{\{a\}} : \|f\|_{A/2H,1} < \varepsilon \} \) and let \( \varepsilon^{-1} = K \int_{\mathbb{R}^d} w_a(2|\xi|/A)^{-1}d\xi \). If \( u * Q \subset V \), then
\[
|\langle u * h_\xi \rangle(\xi)\rangle| \leq \varepsilon\frac{w_a(2|\xi|/A)}{w_a(2H|\xi|/A)} \leq \frac{\varepsilon K}{w_a(2|\xi|/A)}.
\]
Hence, \( |\langle u, h \rangle| \leq \int_{\mathbb{R}^d} |\langle u * h_\xi \rangle(\xi)\rangle|d\xi \leq 1 \), which completes the proof. \( \square \)

We turn to the case of spaces \( S^{(b)}_{\{a\}} \). To prove a theorem analogous to Theorem 4.1, we need the following simple lemma.

**Lemma 4.2.** The weight function \( w_a(t) \) satisfies, for any \( n \in \mathbb{N} \), the inequality
\[
t^n w_a(t) \leq C_n w_a(Ht), \quad t \geq 0. \tag{4.9}
\]

*Proof.* It follows from the definition (2.10) and from (2.3) that
\[
t^n w_a(t) = \sup_{k \in \mathbb{Z}_+} \frac{t^{k+n}}{a_k} \leq Ka_n \sup_{k \in \mathbb{Z}_+} \frac{(Ht)^{k+n}}{a_{k+n}} \leq Ka_n w_a(Ht).
\]
Hence, (4.9) is satisfied with \( C_n = Ka_n \). \( \square \)

**Theorem 4.3.** The space \( C(S^{(b)}_{\{a\}}) \) of convolutors for \( S^{(b)}_{\{a\}} \) is canonically isomorphic to \( \lim_{B \to \infty} (\mathcal{E}^{(b)}_{\{a\}})' \).
Proof. Every $u \in C\left(S_{\{a\}}^{(b)}\right)$ can be extended to a functional $\tilde{u} \in \mathcal{E}_{\{a\}}^{\{b\}}$ in the same manner as given by (4.1), using $f_0 \in S_{\{a\}}^{(b)}$ with $\int f_0(\xi)d\xi = 1$. In this case there are $A_0, B_0 > 0$ such that $\|f_0\|_{A_0, B_0} < \infty$ and there is a $B > 0$ such that $\|h\|_B^A < \infty$ for every $A > 0$. If $A \geq 2HA_0$, then the inequality (4.3) holds. The unit ball of $S_{\{a\}}^{\{b\}, B^+A_0}$ is bounded in $S_{\{a\}}^{\{b\}}$ and its image under the continuous map $f \to u * f$ is also bounded in $S_{\{a\}}^{\{b\}}$. Since the inductive limit $S_{\{a\}}^{\{b\}}$ is regular, this image is contained and bounded in some $S_{\{a\}}^{\{b\}}, B_1$, where $A_1$ and $B_1$ are independent of $A$. Hence, there exists a constant $C > 0$ such that

$$\|u * h\xi\|_{A_1, B_1} \leq C\|h\|_B^A w_a(2|\xi|/A).$$

(4.10)

In particular, for $A \geq 2HA_1$, we have

$$|(u * h_\xi)(\xi)| \leq C\|h\|_B^A w_a(2|\xi|/A) \leq CK\|h\|_B^A \frac{1}{w_a(|\xi|/HA_1)}.$$  

(4.11)

Therefore, the integral in (4.1) is absolutely convergent and the functional $\tilde{u}$ is well defined on $\mathcal{E}_{\{a\}}^{\{b\}}$. Since the right-hand side of (4.11) contains $\|h\|_B^A$, this functional is continuous.

We now show that $\tilde{u}$ coincides with $u$ on $S_{\{a\}}^{\{b\}}$. If $h \in S_{\{a\}}^{\{b\}, B_1}$, then the inequality (4.5) holds and, choosing $A' \geq \max(2HA_1, A_0)$ and using (2.12) and (2.13), we obtain

$$\|h(\cdot)f_0(\xi - \cdot)\|_{A', B_1 + B_0} \leq \|h\|_{A_1, B_1}\|f_0\|_{A_0, B_0}\sup_x w_a(|x|/A') \frac{w_a(|\xi - x|/A_0)}{w_a(|\xi - x|/A_0)} \leq K\|h\|_{A_1, B_1}\|f_0\|_{A_0, B_0} w_a(|\xi|/A') \frac{1}{w_a(|\xi|/2A')}.$$  

Hence in this case, the integral in (4.1) is absolutely convergent for any $u \in S_{\{a\}}^{\{b\}}$ and the sequence (4.6) of Riemann sums converges to $h$ in $S_{\{a\}}^{\{b\}}$ by the same argument as for $S_{\{a\}}^{\{b\}}$ because $S_{\{a\}}^{\{b\}}$ is also a Montel space. In a similar manner we find that $S_{\{a\}}^{\{b\}}$ is dense in $\mathcal{E}_{\{a\}}^{\{b\}}$. Indeed, if $h \in \mathcal{E}_{\{a\}}^{\{b\}, B}$, then (4.7) holds for any $A > 0$ and (4.8) holds for $A \geq A_0$. Therefore, in this case, the integral in (4.1) is absolutely convergent for any $u$ in the dual of the Montel space $\mathcal{E}_{\{a\}}^{\{b\}, (B+B_0)^+}$, and the sequence (4.6) converges to $h$ in this space and, a fortiori, in $\mathcal{E}_{\{a\}}^{\{b\}}$.

It remains to show that the map $u \to \tilde{u}$ from $C\left(S_{\{a\}}^{\{b\}}\right)$ to $\lim_{B \to \infty} \left(\mathcal{E}_{\{a\}}^{\{b\}, B}\right)'$ is bounded and is contained in $\mathcal{E}_{\{a\}}^{\{b\}, B_0}$. Since $\mathcal{E}_{\{a\}}^{\{b\}, B}$ is bounded and $B$ is independent of $A$, there is a bounded set $Q$ and a 0-neighborhood $V$ in $S_{\{a\}}^{\{b\}}$ such that $u * Q \subset V$ implies $\sup_{h \in G} |\langle u, h \rangle| \leq 1$. If $A \geq 2HA_0$, then it follows from (4.3) that

$$\sup_{h \in G} \|h\|_{HA_0, B + B_0} \leq K\|f_0\|_{A_0, B_0}\sup_{h \in G} \|h\|_B^A w_a(2|\xi|/A).$$

(4.12)
Let \( w(\xi) \) be defined by
\[
w(\xi) = \inf_{A \geq 2H A_0} \sup_{h \in G} \| h \|_B^A w_a(2|\xi|/A).
\]
This function is measurable, locally integrable and bounded from below by a positive constant. It follows from (4.12) that the set of functions \( h_\xi(x)/w(\xi) \), where \( h \) runs over \( G \) and \( \xi \) runs over \( \mathbb{R}^d \), is bounded in \( S^{(a)}_{\{b\}}(\mathbb{R}^d) \). We take this set as \( Q \) and note that
\[
\left\{ f \in S^{(b)}_{\{a\}}(\mathbb{R}^d): \sup_{x \in \mathbb{R}^d} |f(x)w(x)(1 + |x|^{d+1})| < \epsilon \right\} \quad (\epsilon > 0) \tag{4.13}
\]
is a 0-neighborhood in \( S^{(b)}_{\{a\}}(\mathbb{R}^d) \) because by Lemma 4.2 we have
\[
\sup_x |f(x)w(x)(1 + |x|^{d+1})| \leq \sup_{h \in G} \| h \|_B^A \sup_x |f(x)(1 + |x|^{d+1})w_a(2|x|/A)|
\leq C_{d,A} \sup_{h \in G} \| h \|_B^A \sup_x |f(x)w_a(2H|x|/A)|
\]
for any \( A \geq 2H A_0 \) and therefore the norm \( \| f \|_w = \sup_x |f(x)w(x)(1 + |x|^{d+1})| \) is weaker than the norm of any space \( S^{(b),B_1}_{\{a\},A_1} \), \( A_1, B_1 > 0 \). Taking for \( V \) the neighborhood (4.13) with \( \epsilon^{-1} = \int_{\mathbb{R}^d} (1 + |\xi|^{d+1})^{-1} d\xi \), we conclude that for each \( h \in G \), the inclusion \( u * Q \subset V \) implies
\[
|\langle u, h \rangle| \leq \int_{\mathbb{R}^d} |(u * h_\xi)(\xi)| d\xi = \int_{\mathbb{R}^d} \left| \left( u * \frac{h_\xi}{w(\xi)} \right)(\xi) w(\xi) \right| d\xi \leq \int_{\mathbb{R}^d} \frac{\epsilon d\xi}{1 + |\xi|^{d+1}} = 1.
\]

The proof is somewhat simpler in the case where \( a_n \equiv 1 \). Returning to the second line of (4.2), we see that if \( A \geq 2A_0 \) and \( |\xi| \leq A/2 \), then \( w_1(|\xi - x|/A) = 1 \) on the support of \( 1/w_1(|x|/A_0) \). Hence, the inequality (4.3) is replaced by \( \| h_\xi \|_{A_0,B_0 + B_0} \leq \| h \|_B^A \| f_0 \|_{A_0,B_0} \), where \( |\xi| \leq A/2 \). Instead of (4.10), we obtain \( \| u * h_\xi \|_{A_1,B_1} \leq C \| h \|_B^A \) for all \( \xi \) in this region. The function \( u * h_\xi \) has compact support in this case and (4.11) holds with \( K = 1, H = 1 \), the integration in (4.1) is over a bounded domain and \( \tilde{u} \) is well defined and continuous on \( \tilde{E}^{(b)}_{\{1\}} \). An easy adaptation of the above arguments shows that \( \tilde{u} = u \) on \( S^{(b)}_{\{1\}} = D^{(b)} \) and \( D^{(b)} \) is dense in \( \tilde{E}^{(b)}_{\{1\}} \). Finally, let \( G \) be a bounded set in \( \tilde{E}^{b,B} \). Since \( \| h \|_B^A \) increases monotonically with \( A \), we have for any \( \xi \)
\[
\sup_{h \in G} \| h_\xi \|_{A_0,B_0 + B_0} \leq \| f_0 \|_{A_0,B_0} \sup_{h \in G} \| h \|_B^{2A_0 + 2|\xi|}.
\]
Setting \( w(\xi) = \sup_{h \in G} \| h \|_B^{2A_0 + 2|\xi|} \), we see that \( Q = \{ h_\xi(x)/w(\xi): h \in G, \xi \in \mathbb{R}^d \} \) is bounded in \( D^{(b)}(\mathbb{R}^d) \). The set defined by (4.13) is a 0-neighborhood in this space since \( S^{(b),B_1}_{\{1\},A_1} \) consists of functions supported in \( \{ x \in \mathbb{R}^d: |x| \leq A_1 \} \) and is continuously embedded into the normed space with the norm \( \| \cdot \|_w \). Taking for \( V \) this set, we conclude as before that \( u * Q \subset V \) implies \( \sup_{h \in G} |\langle \tilde{u}, h \rangle| \leq 1 \). The proof is complete. \( \square \)

**Corollary 4.4.** The spaces \( C(S^{(b)}_{\{a\}}) \) and \( C(S^{(b)}_{\{a\}}) \) are complete and semi-reflexive.
This directly follows from the heredity properties of projective limits (Sect. IV.5.8 in [31]). The same conclusion can be made taking into account that $C(S^{(b)}_{\{a\}})$ and $C(S^{(b)}_{\{a\}})$ are closed subspaces of $L(S^{(b)}_{\{a\}})$ and $L(S^{(b)}_{\{a\}})$, respectively, and that $L(E)$ is complete and nuclear (and therefore semi-reflexive) for any nuclear (FS)- or (DFS)-space $E$ (Sect. 39.6 in [26] and Sect. IV.9.7 in [31]).

Corollary 4.5. The space $\tilde{\mathcal{E}}^{(b)}_{\{a\}}$ is canonically isomorphic to the strong dual of $C(S^{(b)}_{\{a\}})$ and $\mathcal{E}^{(b)}_{\{a\}}$ is canonically isomorphic to the strong dual of $C(S^{(b)}_{\{a\}})$.

Proof. The projective limit $\lim_{B \to \infty} (\mathcal{E}^{b,B+}_{\{a\}})'$ is reduced because $S^{(b)}_{\{a\}}$ is contained and dense in each of the spaces $(\mathcal{E}^{b,B+}_{\{a\}})'$, $B > 0$. Indeed, the map $\mathcal{E}^{b,B+}_{\{a\}} \to S^{(b)}_{\{a\}} : h \mapsto (f \mapsto \int h(x)f(x)dx)$ is the adjoint of the natural inclusion $S^{(b)}_{\{a\}} \to (\mathcal{E}^{b,B+}_{\{a\}})'$ with respect to the dualities $\langle S^{(b)}_{\{a\}}, S^{(b)}_{\{a\}} \rangle$ and $\langle (\mathcal{E}^{b,B+}_{\{a\}})', (\mathcal{E}^{b,B+}_{\{a\}}) \rangle$ and is injective because $S^{(b)}_{\{a\}}$ has sufficiently many functions. Therefore $S^{(b)}_{\{a\}}$ is weakly dense in $(\mathcal{E}^{b,B+}_{\{a\}})'$ and is also strongly dense because $(\mathcal{E}^{b,B+}_{\{a\}})'$ is a (DFS)-space. By Theorem 4.4 in Ch. IV of [31], the Mackey dual of $\lim_{B \to \infty} (\mathcal{E}^{b,B+}_{\{a\}})'$ is identified with $\lim_{B \to \infty} (\mathcal{E}^{b,B+}_{\{a\}})^\prime\prime$, i.e., with $\tilde{\mathcal{E}}^{(b)}_{\{a\}}$, because the spaces $\mathcal{E}^{b,B+}_{\{a\}}$ are reflexive. Since $\lim_{B \to \infty} (\mathcal{E}^{b,B+}_{\{a\}})'$ is semi-reflexive, the strong topology on its dual coincides with the Mackey topology (Sect. IV.5.5 of [31]). Hence the strong dual of $\lim_{B \to \infty} (\mathcal{E}^{b,B+}_{\{a\}})'$ is identified with $\tilde{\mathcal{E}}^{(b)}_{\{a\}}$ and the adjoint of the map $u \mapsto \tilde{u}$ in Theorem 4.3 is an algebraic and topological isomorphism of $\mathcal{E}^{(b)}_{\{a\}}$ onto $C^\prime(S^{(b)}_{\{a\}})$. In the same way we see that $\tilde{\mathcal{E}}^{(b)}_{\{a\}}$ is isomorphic to the strong dual of $C(S^{(b)}_{\{a\}})$.

The statement of Corollary 4.5 is an analogue of the well-known fact established by Grothendieck (Sect. 4.4 in Ch. II of [21]) that the strong dual of the space of convolutors for the Schwartz space $S(\mathbb{R}^d)$ is isomorphic to the space $\mathcal{O}_C(\mathbb{R}^d)$ of very slowly increasing smooth functions.

Under an additional condition on the defining sequences $a$ and $b$, it follows from Corollary 4.5 combined with Theorem 1 in [38] that the spaces $\mathcal{E}^{(b)}_{\{a\}}$ and $\tilde{\mathcal{E}}^{(b)}_{\{a\}}$ are invariant under an important class of ultradifferential operators. Following Komatsu [25], we write $b_n \subset a_n$ if there exist constants $C$ and $L$ such that

$$b_n \leq CL^na_n \quad \forall n \in \mathbb{Z}_+. \quad (4.14)$$

Corollary 4.6. Let $Q = (Q^{jk})$ be a $d \times d$ matrix with real entries and let $b_n \subset a_n$. If the space $S^{(b)}_{\{a\}}(\mathbb{R}^d)$ is nontrivial, then the operator $e^{iQ^{jk}\partial_j\partial_k}$ is a homeomorphism of $\tilde{\mathcal{E}}^{(b)}_{\{a\}}(\mathbb{R}^d)$ onto itself, and if $S^{(b)}_{\{a\}}(\mathbb{R}^d)$ is nontrivial, then $e^{iQ^{jk}\partial_j\partial_k}$ is a homeomorphism of $\mathcal{E}^{(b)}_{\{a\}}(\mathbb{R}^d)$ onto itself.
Proof. The operator \( e^{i\mathcal{Q}^k \partial_x \partial_k} \) is defined via the inverse Fourier transform of \( e^{-i\mathcal{Q}^k \zeta_j \zeta_k} \), as in the case with the Schwartz space \( S(\mathbb{R}^d) \) considered in Sect. 7.6 of [22]. It follows from Corollary 4.5 that the statement of Corollary 4.6 amounts to saying that the multiplication by \( e^{-i\mathcal{Q}^k \zeta_j \zeta_k} \) is a self-homeomorphism of the multiplier spaces \( M(S_{\{b\}}^{(a)}) \) and \( M(S_{\{a\}}^{(a)}) \). This in turn follows from Theorem 1 in [38] which states that under condition (4.14) the function \( e^{-i\mathcal{Q}^k \zeta_j \zeta_k} \) is a pointwise multiplier of \( S_{\{b\}}^{(a)} \) and \( S_{\{a\}}^{(a)} \). \( \square \)

For the case of spaces \( S_{\beta}^\alpha \), \( \beta \leq \alpha \), defined by sequences of the form (2.6), a simple proof of the fact that \( e^{-i\mathcal{Q}^k \zeta_j \zeta_k} \) belongs to \( M(S_{\beta}^\alpha) \) was given in the course of proving Theorem 1 in [33]. Similar invariance properties of the symbol spaces \( \Gamma_{\infty}^S \) and \( \Gamma_{0,\infty}^S \) (which coincide respectively with \( \hat{\mathcal{E}}_{\{a\}}^S \) and \( \hat{\mathcal{E}}_{\{a\}}^S \)) for \( a_n = n^{sn} \), as noted in Sect. 2) were also proved by different methods in Theorem 4.1 of [5], and in Proposition 4.4 of [6].

5. The Moyal Multiplier Algebras for Spaces of Type \( S \)

Theorem 2 in [38] shows that if (4.14) is satisfied, then the spaces \( S_{\{b\}}^{(b)}(\mathbb{R}^d) \) and \( S_{\{a\}}^{(b)}(\mathbb{R}^d) \) are topological algebras under the twisted multiplication (1.1). (For \( S_{\beta}^\alpha \), \( \beta \leq \alpha \), this has been proved in [33].) The Fourier transform converts the Weyl–Moyal product into the twisted convolution

\[
(f \ast_h \tilde{g})(\zeta) = \int_{\mathbb{R}^d} \tilde{f}(\zeta') \hat{g}(\zeta - \zeta') e^{(ih/2)\zeta \cdot J \zeta'} d\zeta'
\]

(5.1)

(multiplied by \((2\pi)^{-d}\)). Under the same condition (4.14), the spaces \( S_{\{b\}}^{(a)} = F[S_{\{a\}}^{(b)}] \) and \( S_{\{a\}}^{(a)} = F[S_{\{a\}}^{(b)}] \) are topological algebras with the twisted convolution as multiplication. We will consider them in parallel with \( (S_{\{b\}}^{(b)}, \ast_h) \) and \( (S_{\{a\}}^{(a)}, \ast_h) \).

The products \( \langle f \ast_h u \rangle \) and \( \langle u \ast_h f \rangle \) of a function \( f \in S_{\{a\}}^{(b)} \) with an element \( u \) of the dual space \( S_{\{a\}}^{(b)\prime} \) are defined by

\[
\langle f \ast_h u, h \rangle := \langle u, h \ast_h f \rangle, \quad \langle u \ast_h f, h \rangle := \langle u, f \ast_h h \rangle \quad \forall h \in S_{\{a\}}^{(b)}
\]

(5.2)

(and analogously for the dual pair \( (S_{\{b\}}^{(a)}, S_{\{a\}}^{(b)\prime}) \)). Since the right-hand sides in (5.2) are linear and continuous in \( h \), these products are well defined as elements of \( S_{\{a\}}^{(b)\prime} \). The formulas (5.2) agree with the definition of the operation \( \ast_h \) in \( S_{\{a\}}^{(b)} \) due to the identity

\[
\int (f \ast_h g)(x) dx = \int f(x) g(x) dx,
\]

(5.3)

which is equivalent to the obvious identity \( (\hat{f} \ast_h \hat{g})(0) = (\hat{f} \ast \hat{g})(0) \). Indeed, if \( f, g, h \in S_{\{a\}}^{(b)} \), then using (5.3) and the associativity of the Weyl–Moyal
product, we obtain
\[ \langle f \ast h g, h \rangle \equiv \int (f \ast h g)(x)h(x)dx = \int (h \ast (f \ast h g))(x)dx \]
\[ = \int ((h \ast h f) \ast h g)(x)dx = \langle g, h \ast h f \rangle. \]
Under condition (4.14), the twisted convolution product of \( g \in S_{\{a\}}^{(a)} \) and \( v \in S_{\{b\}}^{(a)} \) can also be defined by duality, namely:
\[ \langle v \ast h g, h \rangle := \langle v, \hat{g} \ast h h \rangle, \quad \langle g \ast h v, h \rangle := \langle v, h \ast h \hat{g} \rangle \quad \forall h \in S_{\{b\}}^{(a)}; \]
where \( \hat{g}(\xi) = g(-\xi) \). Then we clearly have the relations
\[ \hat{u} \ast h \hat{f} = (2\pi)^{-d} \hat{u} \ast h f, \quad \hat{f} \ast h u = (2\pi)^{-d} \hat{f} \ast h \hat{u}. \]
The spaces of left and right multipliers for the algebra \( S_{\{a\}}^{(b)} \) are defined as follows:
\[ \mathcal{M}_{h,L}(S_{\{a\}}^{(b)}) := \left\{ u \in S_{\{a\}}^{(b)} : u \ast h f \in S_{\{a\}}^{(b)} \quad \forall f \in S_{\{a\}}^{(b)} \right\}, \quad (5.4) \]
\[ \mathcal{M}_{h,R}(S_{\{a\}}^{(b)}) := \left\{ u \in S_{\{a\}}^{(b)} : f \ast h u \in S_{\{a\}}^{(b)} \quad \forall f \in S_{\{a\}}^{(b)} \right\}. \quad (5.5) \]
The definitions of \( \mathcal{M}_{h,L}(S_{\{a\}}^{(b)}) \) and \( \mathcal{M}_{h,R}(S_{\{a\}}^{(b)}) \) are similar. The mappings
\[ f \mapsto u \ast h f \text{ and } f \mapsto f \ast h u \text{ of } S_{\{a\}}^{(b)} \text{ into itself and of } S_{\{a\}}^{(b)} \text{ into itself} \]
are continuous by the closed graph theorem, and the multiplier spaces are naturally endowed with the respective topologies induced by \( \mathcal{L}(S_{\{a\}}^{(b)}) \) and \( \mathcal{L}(S_{\{a\}}^{(b)}) \). Theorem 3 in [38] establishes that under condition (4.14), the spaces
\[ \mathcal{M}_{h,L}(S_{\{a\}}^{(b)}), \quad \mathcal{M}_{h,R}(S_{\{a\}}^{(b)}), \quad \mathcal{M}_{h,R}(S_{\{a\}}^{(b)}), \quad \text{and } \mathcal{M}_{h,L}(S_{\{a\}}^{(b)}), \]
are unital algebras with separately continuous multiplication \( \ast_h \). (For the case of spaces \( S_{\{a\}}^{(b)}, \beta \leq \alpha \), this was proved in [35].) The Fourier transformation maps
\[ S_{\{a\}}^{(b)} \] onto the algebras of left and right twisted convolution multipliers
\[ C_{h,L}(S_{\{a\}}^{(b)}) = \left\{ v \in S_{\{a\}}^{(b)} : v \ast h g \in S_{\{b\}}^{(a)} \quad \forall g \in S_{\{b\}}^{(a)} \right\}, \quad (5.6) \]
\[ C_{h,R}(S_{\{a\}}^{(b)}) = \left\{ v \in S_{\{a\}}^{(b)} : g \ast h v \in S_{\{b\}}^{(a)} \quad \forall g \in S_{\{b\}}^{(a)} \right\}. \quad (5.7) \]
Analogously, \( \mathcal{M}_{h,L}(S_{\{a\}}^{(b)}) = C_{h,L}(S_{\{a\}}^{(b)}) \) and \( \mathcal{M}_{h,R}(S_{\{a\}}^{(b)}) = C_{h,R}(S_{\{a\}}^{(b)}) \).

**Lemma 5.1.** Let \( b_n \subset a_n \). The algebras \( (S_{\{a\}}^{(b)}, \ast_{h}) \), \( (S_{\{a\}}^{(b)}, \ast_{h}) \), \( (S_{\{b\}}^{(a)}, \ast_{h}) \), and \( (S_{\{b\}}^{(a)}, \ast_{h}) \), have sequential approximate identities.

**Proof.** Since the algebra \( (S_{\{a\}}^{(b)}, \ast_{h}) \) is isomorphic, via the Fourier transform, to the algebra \( (S_{\{a\}}^{(b)}, \ast_{h}) \) and \( (S_{\{a\}}^{(b)}, \ast_{h}) \) is isomorphic to \( (S_{\{b\}}^{(a)}, \ast_{h}) \), it suffices to consider the case of twisted convolution. Every nontrivial space \( S_{\{b\}}^{(a)}(\mathbb{R}^{2d}) \) contains a function \( e_1(\xi) \) such that \( \int_{\mathbb{R}^{2d}} e_1(\xi)d\xi = 1 \) and \( e_1(\xi) \geq 0 \), because
it is an algebra under pointwise multiplication. We claim that the sequence 
\( e_n(\zeta) = n^{2d} e_1(n\zeta) \), \( n \in \mathbb{N} \), is an approximate identity for \( (S_{\{b\}}^{(a)}, *_h) \), i.e.,
that for any \( g \in S_{\{b\}}^{(a)} \), the limit relations \( e_n *_h g \to g \) and \( g *_h e_n \to g \) hold in the topology of \( S_{\{a\}}^{(b)} \) as \( n \to \infty \). Clearly, \((e_n *_h g)(\zeta) \to g(\zeta)\) and 
\((g *_h e_n)(\zeta) \to g(\zeta)\) at every \( \zeta \). We show that the sequence \( e_n *_h g \) is bounded in \( S_{\{a\}}^{(b)} \). Using (2.4) and the inequality \( t^k \leq A^k a_k w_a(t/A') \) and (4.14) in the form \( w_a(t) \leq Cw_b(Lt) \), we obtain
\[
|\partial^\alpha (e_n *_h g)(\zeta)| \leq \int_{\mathbb{R}^{2d}} e_n(\zeta') |\partial^\alpha \left(e^{(ih/2)\zeta' \cdot J'} g(\zeta - \zeta')\right)| d\zeta'
\leq \|g\|_{A,B} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} A^{\alpha-\gamma} d_{\gamma} \int_{\mathbb{R}^{2d}} e_n(\zeta') \left(h|\zeta'|/2\right)^{|\gamma|} w_b(|\zeta - \zeta'|/B) d\zeta'
\leq C\|g\|_{A,B} (A + A'h/2)^{\alpha} \int_{\mathbb{R}^{2d}} e_n(\zeta') \frac{w_b(L|\zeta'|/A)}{w_b(|\zeta - \zeta'|/B)} d\zeta'.
\tag{5.8}
\]
The function \( e_1 \) belongs to \( S_{\{b,B_1/H\}}^{a,A_1} \) with sufficiently large \( A_1 \) and \( B_1 \) and by (2.11) we have
\[
\frac{1}{w_b(|\zeta - \zeta'|/B)} \leq \frac{w_b(|\zeta'|/HB_1)}{w_b(|\zeta|/(B + HB_1))}.
\tag{5.9}
\]
Let \( A' = LHB_1 \). Then it follows from (5.8), (5.9) and (2.13) that
\[
|\partial^\alpha (e_n *_h g)(\zeta)| \leq CK\|g\|_{A,B} \left(\frac{A + LHB_1h/2}{w_b(|\zeta|/(B + HB_1))}\right)^{\alpha} \int_{\mathbb{R}^{2d}} e_n(\zeta') \frac{w_b(|\zeta'|/B_1)}{d\zeta'}.
\tag{5.10}
\]
Since \( e_n(\zeta') \leq C_n/w_b(H|\zeta'|/B_1) \), the integral in the right-hand side is finite for any \( n \) in view of (2.13) and tends to \( w_b(0) = 1 \) as \( n \to \infty \). Therefore, the sequence \( e_n *_h g \) is contained and bounded in \( S_{\{b,B_2\}}^{a,A_2} \), where \( A_2 = A + LHB_1h/2 \) and \( B_2 = B + HB_1 \). This sequence has at least one limit point in the Montel space \( S_{\{b\}}^{(a)} \). The topology of \( S_{\{b\}}^{(a)} \) is finer than the topology of simple convergence, and only \( g \) can be the limit point. Hence \( e_n *_h g \) converges to \( g \) in \( S_{\{b\}}^{(a)} \), because otherwise it would have a limit point other than \( g \), which is impossible. Similarly \( g *_h e_n \to g \) in \( S_{\{b\}}^{(a)} \). The proof is simpler in the case where \( b_n \equiv 1 \). Then \( e_1 \in D^{(a)} = S_{\{1\}}^{(a)} \) is supported in a compact set \( \{ x : |x| \leq B_1 \} \) and the integral in the second line of (5.8) is clearly less than \( (B_1h/2)^{|\gamma|} d_{\gamma} \int_{|\zeta - \zeta'| \leq B} e_n(\zeta') d\zeta' \), which immediately implies the boundedness of the sequence \( e_n *_h g \) in \( D^{(a)} \).

In the case of spaces \( S_{\{b\}}^{(a)} \), the norm \( \|g\|_{A,B} \) is finite for any \( A, B > 0 \), and \( e_1 \) belongs to \( S_{\{b,B_1/H\}}^{a,A_1} \) for arbitrarily small \( A_1 \) and \( B_1 \). Hence, the same estimate (5.10) shows that the sequence \( e_n *_h g \) is bounded in \( S_{\{b\}}^{(a)} \). The rest of the proof is the same as for \( S_{\{b\}}^{(a)} \), because \( S_{\{b\}}^{(a)} \) is also a Montel space. \( \square \)
The above proof generalizes and simplifies a proof given in [35] for $(S^β_{α}, \star_ℏ), \ β \leq α$.

**Theorem 5.2.** The algebra $\mathcal{M}_{h,L}(S^{(b)}_{\{a\}})$ is canonically identified with the closure in $\mathcal{L}(S^{(b)}_{\{a\}})$ of the set of all operators of the left $\star_ℏ$-multiplication by elements of $S^{(b)}_{\{a\}}$. This closure consists of all $V \in \mathcal{L}(S^{(b)}_{\{a\}})$ such that

$$V(f \star_ℏ g) = V(f) \star_ℏ g \quad \forall f, g \in S^{(b)}_{\{a\}}. \quad (5.11)$$

An analogous statement holds for $\mathcal{M}_{h,R}(S^{(b)}_{\{a\}})$, with the replacement of the left $\star_ℏ$-multiplication by the right $\star_ℏ$-multiplication and with the condition $V(f \star_ℏ g) = f \star_ℏ V(g)$ instead of (5.11). Similar statements are true for $\mathcal{M}_{h,L}(S^{(b)}_{\{a\}})$ and $\mathcal{M}_{h,R}(S^{(b)}_{\{a\}})$.

**Proof.** Let $u \in \mathcal{M}_{h,L}(S^{(b)}_{\{a\}})$ and let $L_u$ denote the map $f \mapsto u \star f$ from $S^{(b)}_{\{a\}}$ to itself. (For brevity, we temporarily suppress the index $ℏ$ in the notation of the star product). Using an approximate identity $e_n$ for $(S^{(b)}_{\{a\}}, \star)$ and passing to the limit in $\langle u, e_n \star f \rangle = \langle u \star e_n, f \rangle$, we find that the map $u \mapsto L_u$ of $\mathcal{M}_{h,L}(S^{(b)}_{\{a\}})$ to $\mathcal{L}(S^{(b)}_{\{a\}})$ is injective. It follows from the definition (5.2) and from the associativity of the $\star$-multiplication in $S^{(b)}_{\{a\}}$ that

$$u \star (f \star g) = (u \star f) \star g, \quad (f \star g) \star u = f \star (g \star u). \quad (5.12)$$

In terms of $L_u$, the first of these relations takes the form $L_u(f \star g) = L_u(f) \star g$. On the other hand, to every $V \in \mathcal{L}(S^{(b)}_{\{a\}})$ there corresponds a unique $v \in S^{(b)}_{\{a\}}$ such that

$$\langle v, f \rangle = \int V(f) \, dx.$$  

If $V$ satisfies (5.11), then (5.2) and (5.3) give

$$\langle v \star f, g \rangle = \langle v, f \star g \rangle = \int V(f \star g) \, dx = \int V(f) g \, dx.$$  

Hence, $v \star f = V(f), v \in \mathcal{M}_{h,L}(S^{(b)}_{\{a\}})$, and $L_v = V$. The relation $V(e_n \star f) = V(e_n) \star f$ implies that the sequence of operators of the left $\star$-multiplication by $V(e_n) \in S^{(b)}_{\{a\}}$ converges pointwise to $V$. Since $S^{(b)}_{\{a\}}$ is barrelled, the convergence is uniform on every precompact subset of $S^{(b)}_{\{a\}}$ by the Banach-Steinhaus theorem (Theorem 4.6 in Ch. III in [31]). Since, further, $S^{(b)}_{\{a\}}$ is a Montel space, every its bounded subset is precompact, and we conclude that the sequence in question converges to $V$ in the topology of $\mathcal{L}(S^{(b)}_{\{a\}})$. On the other hand, if $V \in \mathcal{L}(S^{(b)}_{\{a\}})$ is the limit of a net of operators of the left $\star$-multiplication by $h_ν \in S^{(b)}_{\{a\}}$, then

$$V(f \star g) = \lim_ν h_ν \star (f \star g) = \lim_ν (h_ν \star f) \star g = V(f) \star g.$$
and hence $V$ satisfies (5.2). The case of right multipliers is treated similarly, using the second relation in (5.12). The same arguments apply to $\mathcal{M}_{h,L}(S^{(b)}_{\{a\}})$ and $\mathcal{M}_{h,R}(S^{(b)}_{\{a\}})$, which completes the proof.

\textbf{Remark 5.3.} Similar theorems hold for $\mathcal{C}_{h,L}(S^{(a)}_{\{b\}})$, $\mathcal{C}_{h,L}(S^{(a)}_{\{b\}})$, $\mathcal{C}_{h,R}(S^{(a)}_{\{b\}})$, and $\mathcal{C}_{h,R}(S^{(a)}_{\{b\}})$. These algebras can also be characterized in another way. Proposition 2 in [36] shows that $\mathcal{C}_{h,L}(S^{(a)}_{\{b\}})$ and $\mathcal{C}_{h,L}(S^{(a)}_{\{b\}})$ can be identified with the respective sets of those operators in $\mathcal{L}(S^{(a)}_{\{b\}})$ and in $\mathcal{L}(S^{(a)}_{\{b\}})$ that commute with the twisted translations $\tau_\xi: g(\zeta) \mapsto e^{(ih/2)\xi \cdot J_\xi} g(\zeta - \xi), \xi \in \mathbb{R}^{2d}$. Analogous statements are valid for $\mathcal{C}_{h,R}(S^{(a)}_{\{b\}})$ and $\mathcal{C}_{h,R}(S^{(a)}_{\{b\}})$, but with $\bar{\tau}_\xi: g(\zeta) \mapsto e^{-i(h/2)\xi \cdot J_\xi} g(\zeta - \xi)$ in place of $\tau_\xi$. Theorem 5.2 and the above characterizations hold true at $h = 0$, i.e., in the case of pointwise multiplication and ordinary convolution.

In the sequel, we consider the spaces of two-sided multipliers

\begin{equation}
\mathcal{M}_h(S^{\{b\}}_{\{a\}}) = \mathcal{M}_{h,L}(S^{\{b\}}_{\{a\}}) \cap \mathcal{M}_{h,R}(S^{\{b\}}_{\{a\}}),
\end{equation}

\begin{equation}
\mathcal{M}_h(S^{(b)}_{\{a\}}) = \mathcal{M}_{h,L}(S^{(b)}_{\{a\}}) \cap \mathcal{M}_{h,R}(S^{(b)}_{\{a\}}),
\end{equation}

\begin{equation}
\mathcal{C}_h(S^{\{a\}}_{\{b\}}) = \mathcal{C}_{h,L}(S^{\{a\}}_{\{b\}}) \cap \mathcal{C}_{h,R}(S^{\{a\}}_{\{b\}}),
\end{equation}

\begin{equation}
\mathcal{C}_h(S^{(a)}_{\{b\}}) = \mathcal{C}_{h,L}(S^{(a)}_{\{b\}}) \cap \mathcal{C}_{h,R}(S^{(a)}_{\{b\}}).
\end{equation}

The space $\mathcal{M}_h(S^{\{b\}}_{\{a\}})$ is naturally endowed with the initial topology with respect to the inclusion maps $\mathcal{M}_h(S^{\{b\}}_{\{a\}}) \rightarrow \mathcal{M}_{h,L}(S^{\{b\}}_{\{a\}})$ and $\mathcal{M}_h(S^{\{b\}}_{\{a\}}) \rightarrow \mathcal{M}_{h,R}(S^{\{b\}}_{\{a\}})$. The spaces $\mathcal{M}_h(S^{(b)}_{\{a\}})$, $\mathcal{C}_h(S^{\{a\}}_{\{b\}})$, and $\mathcal{C}_h(S^{(a)}_{\{b\}})$ are topologized in the same manner. All these spaces are unital involutive algebras with separately continuous multiplication (for more detail, see [38] and also [35] for the case of spaces $S^{(b)}_{\{a\}}$). We note that multiplication in $\mathcal{M}_h(S^{\{b\}}_{\{a\}})$ and in $\mathcal{M}_h(S^{(b)}_{\{a\}})$ can be defined by either of the two formulas

$\langle u \ast v, f \rangle := \langle u, v \ast f \rangle$, $\langle u \ast v, f \rangle := \langle v, f \ast u \rangle$.

Indeed, replacing $f$ by $e_n \ast f$ and using (5.12) and then (5.2), we can write the right-hand sides as $\int (f \ast u)(v \ast e_n) dx$. Passing to the limit as $n \rightarrow \infty$ and using the continuity of the maps $f \rightarrow v \ast f$ and $f \rightarrow f \ast u$, we see that these definitions are equivalent.

6. Inclusion Relations Between the Moyal Multiplier Algebras and Spaces of Type $\mathcal{E}^{a}$

Along with the inclusions (3.1), we have the continuous inclusions

$S^{\{b\}}_{\{a\}} \hookrightarrow \mathcal{E}^{\{a\}}_{\{b\}}$, $S^{(a)}_{\{b\}} \hookrightarrow \mathcal{E}^{(a)}_{\{b\}}$.

(6.1)
where $\mathcal{E}_{(b)}^{(a)}$ and $\mathcal{E}_{\{b\}}^{(a)}$ are defined by (2.15). (Recall that the upper index determines the smoothness of the space elements, and the lower index determines their behavior at infinity.) Lemmas 3 and 4 in [39] show that these inclusions are dense. Therefore, $\mathcal{E}_{(b)}^{(a)}$ and $\mathcal{E}_{\{b\}}^{(a)}$ are naturally identified with the respective vector subspaces of $S_{\{b\}}^{(a)}$ and $S_{(b)}^{(a)}$. It follows from (2.17) that $\mathcal{E}_{(b)}^{(a)} \subset \mathcal{E}_{\{b\}}^{(a)}$ and $\mathcal{E}_{\{b\}}^{(a)} \subset \mathcal{E}_{(b)}^{(a)}$. The noncommutative deformation of convolution violates the inclusion relations (3.5), but Theorem 4 in [39] shows that under condition (4.14), the inclusions

$$
\mathcal{E}_{(b)}^{(a)} \subset C_h(S_{\{b\}}^{(a)}), \quad \mathcal{E}_{\{b\}}^{(a)} \subset C_h(S_{(b)}^{(a)})
$$

are valid. They are the starting point for deriving other inclusion relationships in this section.

**Theorem 6.1.** The inclusions (6.2) are continuous.

**Proof.** As in the case of spaces (2.16), it is useful to represent $\mathcal{E}_{(b)}^{(a)}$ and $\mathcal{E}_{\{b\}}^{(a)}$ as limits of families of spaces with nice topological properties, namely:

$$
\mathcal{E}_{(b)}^{(a)} = \lim_{B \to \infty} \mathcal{E}_{b,B+}^{(a)}, \quad \mathcal{E}_{\{b\}}^{(a)} = \lim_{A \to 0} \mathcal{E}_{b,B+}^{a,A},
$$

where

$$
\mathcal{E}_{b,B+}^{(a)} := \lim_{A \to -\infty,\epsilon \to 0} \mathcal{E}_{b,B+}^{a,A}, \quad \mathcal{E}_{b,B+}^{a,A} := \lim_{B \to 0,\epsilon \to 0} \mathcal{E}_{b,B+}^{a,A - \epsilon}.
$$

By Lemma 2 in [39], $\mathcal{E}_{b,B+}^{(a)}$ and $\mathcal{E}_{\{b\}}^{a,A}$ are (DFS)-spaces. It follows that the projective limits (6.3) are semi-reflexive. Furthermore, Lemmas 3 and 4 in [39] show that they are reduced. The duals $(\mathcal{E}_{b,B+}^{(a)})'$ and $(\mathcal{E}_{\{b\}}^{(a)})'$ are (FS)-spaces and therefore Mackey spaces. Using, as in the proof of Corollary 4.5, the duality between projective and inductive limits, we conclude that

$$
\mathcal{E}_{(b)}^{(a)}' = \lim_{B \to \infty} (\mathcal{E}_{b,B+}^{(a)})', \quad \mathcal{E}_{\{b\}}^{(a)}' = \lim_{A \to 0} (\mathcal{E}_{b,B+}^{a,A})',
$$

because the semi-reflexivity of $\mathcal{E}_{(b)}^{(a)}$ and $\mathcal{E}_{\{b\}}^{(a)}$ implies that the strong topology on $\mathcal{E}_{(b)}^{(a)}$ and $\mathcal{E}_{\{b\}}^{(a)}$ coincides with the Mackey topology. It suffices to show now that the maps $(\mathcal{E}_{b,B+}^{(a)})' \to C_h(S_{\{b\}}^{(a)})$ and $(\mathcal{E}_{b,B+}^{a,A})' \to C_h(S_{(b)}^{(a)})$ are continuous for every $B > 0$ and every $A > 0$. We note that for any fixed $g \in S_{\{b\}}^{(a)}$, the graphs of the maps

$$
(\mathcal{E}_{b,B+}^{(a)})' \to S_{\{b\}}^{(a)}: v \mapsto v \ast h g, \quad (\mathcal{E}_{b,B+}^{a,A})' \to S_{\{b\}}^{(a)}: v \mapsto g \ast h v
$$

are closed. Indeed, if $v_{\nu}$ is a net in $(\mathcal{E}_{b,B+}^{(a)})'$ such that $v_{\nu} \to v \in (\mathcal{E}_{b,B+}^{(a)})'$ and $v_{\nu} \ast h g \to f \in S_{\{b\}}^{(a)}$, then for any $h \in S_{\{b\}}^{(a)}$, we have

$$
\int f(\zeta)h(\zeta)d\zeta = \lim_{\nu} \langle v_{\nu} \ast h g, h \rangle = \lim_{\nu} \langle v_{\nu}, \hat{g} \ast h h \rangle = \langle v, \hat{g} \ast h h \rangle = \langle v \ast h g, h \rangle,
$$

hence $f = v \ast h g$. Consequently, the maps (6.4) are continuous by the closed graph theorem. The rest of the proof is similar to that of Theorem 5.2. Let $v_{n}$
be a sequence in \((E^{(a)}_{h,B^+})'\) converging to zero. Then \(v_n *_{h} g \to 0\) and \(g *_{h} v_n \to 0\) for any \(g \in S^{(a)}_{\{b\}}\) and the convergence is uniform on every precompact subset of \(S^{(a)}_{\{b\}}\) by the Banach-Steinhaus theorem. Since \(S^{(a)}_{\{b\}}\) is a Montel space, every its bounded subset is precompact. Hence, \(v_n \to 0\) in the topology of \(C_{\overline{0}}(S^{(a)}_{\{b\}})\). Analogous arguments show that the second inclusion in (6.2) is also continuous, which completes the proof.

\[\square\]

It is clear from the definitions (2.14)-(2.16) that the Palamodov spaces \(E^{(b)}_{(a)}\) and \(E^{(b)}_{(a)}\) are naturally embedded in \(S^{(b)}_{\{a\}}\), whereas \(E^{(b)}_{\{a\}}\) and \(\tilde{E}^{(b)}_{\{a\}}\) are naturally embedded in \(S^{(b)}_{\{a\}}\).

**Theorem 6.2.** If \(b_n \subset a_n\), then the following continuous embeddings hold:

\[\tilde{E}^{(b)}_{\{a\}} \hookrightarrow \mathcal{M}_{\overline{h}}(S^{(b)}_{\{a\}}), \quad \tilde{E}^{(b)}_{\{a\}} \hookrightarrow \mathcal{M}_{\overline{h}}(S^{(b)}_{\{a\}}).\]  

**Proof.** Theorem 2 in [39] shows that the functions of \(E^{(a)}_{(b)}\) are pointwise multipliers for \(S^{(a)}_{\{b\}}\), the functions of \(E^{(a)}_{\{b\}}\) are pointwise multipliers for \(S^{(a)}_{\{b\}}\), and the inclusion maps

\[E^{(b)}_{\{a\}} \to M(S^{(a)}_{\{b\}}), \quad E^{(a)}_{\{b\}} \to M(S^{(a)}_{\{b\}})\]  

are continuous. These maps clearly have dense ranges (and are even surjective, but we will not prove this here), and their adjoints

\[M'(S^{(a)}_{\{b\}}) \to E^{(a)}_{\{b\}}', \quad M(S^{(a)}_{\{b\}}) \to E^{(a)}_{\{b\}}'\]  

are continuous and injective. The compositions of the canonical inclusions \(E^{(a)}_{\{b\}}' \hookrightarrow S^{(a)}_{\{b\}}'\) and \(E^{(a)}_{\{b\}}' \hookrightarrow S^{(a)}_{\{b\}}'\) with their respective maps in (6.7) are the adjoints of the canonical inclusions \(S^{(a)}_{\{b\}} \hookrightarrow M(S^{(a)}_{\{b\}})\) and \(S^{(a)}_{\{b\}} \hookrightarrow M(S^{(a)}_{\{b\}})\). If \(b_n \subset a_n\), then (6.7) in combination with Theorem 6.1 gives

\[M'(S^{(a)}_{\{b\}}) \to C_{\overline{h}}(S^{(a)}_{\{b\}}), \quad M'(S^{(a)}_{\{b\}}) \to C_{\overline{h}}(S^{(a)}_{\{b\}}).\]  

After Fourier transforming we obtain

\[C'(S^{(b)}_{\{a\}}) \hookrightarrow \mathcal{M}_{\overline{h}}(S^{(b)}_{\{a\}}), \quad C'(S^{(b)}_{\{a\}}) \hookrightarrow \mathcal{M}_{\overline{h}}(S^{(b)}_{\{a\}}).\]  

By Corollary 4.5, \(\tilde{E}^{(b)}_{\{a\}}\) is isomorphic to \(C'(S^{(b)}_{\{a\}})\) and this isomorphism is implemented by the adjoint of the map \(i: u \to \tilde{u}\) from Theorem 4.3. Let \(j\) denote the natural embedding of \(S^{(b)}_{\{a\}}\) into \(C(S^{(b)}_{\{a\}})\). The composition \(j' \circ i'\) of the adjoint maps is precisely the natural embedding of \(\tilde{E}^{(b)}_{\{a\}}\) into \(S^{(b)}_{\{a\}}\), because \(\langle (j' \circ i')(h), f \rangle = \langle h, \tilde{i}(j(f)) \rangle\) for any \(h \in \tilde{E}^{(b)}_{\{a\}}\) and \(f \in S^{(b)}_{\{a\}}\) and by the definition (4.1) we have for \(u = j(f)\)

\[\langle \tilde{i}(j(f)), h \rangle = \int \int f(x)h(x)f_0(\xi - x)dx \, d\xi = \int f(x)h(x)dx.\]

Similarly, the second embedding in (6.9), together with Corollary 4.5, implies the second embedding in (6.5).
Remark 6.3. Unlike $\mathcal{E}^{(b)}_{\{a\}}$, the space $\mathcal{E}^{(b)}_{\{a\}}$ is not contained in $\mathcal{M}_h(S^{(b)}_{\{a\}})$ and $\mathcal{E}^{(a)}_{\{b\}}$ is not contained in $\mathcal{M}_h(S^{(b)}_{\{a\}})$. In particular, the function $e^{(2i/h)p-q}$, where $(p, q)$ are symplectic coordinates on $\mathbb{R}^d$, belongs to $\mathcal{E}^{(b)}_{\{a\}}(\mathbb{R}^d)$, but does not belong to $\mathcal{M}_h(S^{(b)}_{\{a\}}(\mathbb{R}^d))$, see Proposition 6 in [37].

In the important case of the Fourier-invariant spaces of type $S$, we obtain the following additional result.

Corollary 6.4. If $h \neq 0$, then $\mathcal{E}^{(a)}_{\{b\}}$ is contained in $\mathcal{C}_h(S^{(a)}_{\{a\}})$ and $\mathcal{E}^{(a)}_{\{b\}}$ is contained in $\mathcal{C}_h(S^{(a)}_{\{a\}})$ with continuous inclusions.

Proof. Here we use the symplectic Fourier transforms defined by
\[
(F_J f)(y) := (\pi h)^{-d} \int_{\mathbb{R}^d} f(x) e^{-(2i/h)x \cdot J y} dx,
\]
\[
(f \star_h g)(x) = (\pi h)^{-d} \int_{\mathbb{R}^d} (F_J f)(x') g(x-x') e^{(2i/h)x' \cdot J x'} dx'.
\]
Performing the integration over one of the variables $x'$ or $x''$ in (1.1), we obtain
\[
(f \star h g)(x) = (\pi h)^{-d} \int_{\mathbb{R}^d} (F_J f)(x'') g(x-x'') e^{(2i/h)x \cdot J x''} dx''
\]
\[
= (\pi h)^{-d} \int_{\mathbb{R}^d} f(x-x') (\tilde{F}_J g)(x') e^{(2i/h)x' \cdot J x'} dx'
\]
\[
= (\pi h)^{-d} f \star_{1/h} (\tilde{F}_J g) (x). (6.10)
\]

The maps $v \mapsto f \star v$ and $v \mapsto v \star g$ of $S^{(a)}_{\{a\}}$ into itself are the adjoints of the respective continuous maps $h \mapsto h \star f$ and $h \mapsto g \star h$ of $S^{(a)}_{\{a\}}$ into itself and are therefore continuous. The twisted convolution products in (6.10) have similar continuity properties. Furthermore, $S^{(a)}_{\{a\}}$ is dense in $S^{(a)}_{\{a\}}$. Hence, it follows from (6.10) that
\[
f \star_h v = (\pi h)^{-d} (F_J f) \star_{1/h} v, \quad v \star_h g = (\pi h)^{-d} v \star_{1/h} (\tilde{F}_J g)
\]
for all $f, g \in S^{(a)}_{\{a\}}$ and $v \in S^{(a)}_{\{a\}}$. If $v \in \mathcal{E}^{(a)}_{\{a\}}$, then $f \star_h v \in S^{(a)}_{\{a\}}$ and $v \star_h g \in S^{(a)}_{\{a\}}$ for any $h$ by Theorem 6.2. Since $F_J$ and $\tilde{F}_J$ are automorphisms of $S^{(a)}_{\{a\}}$, we conclude that $\mathcal{E}^{(a)}_{\{a\}} \subset \mathcal{C}_h(S^{(a)}_{\{a\}})$ for $h \neq 0$. The reasoning used in proving Theorem 6.1 shows that this inclusion is continuous. The proof for the case of spaces $S^{(a)}_{\{a\}}$ is entirely analogous. \qed

Theorem 6.2 can be extended to the multiplier algebras associated with the quantization map
\[
f \mapsto \text{Op}_S(f) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\zeta) e^{(i\hbar/4)\zeta \cdot S \zeta} T_{\zeta}^h d\zeta, \quad (6.11)
\]
where \( S \) is a real symmetric matrix and \( T^h_\zeta \) is a Weyl system of unitary operators satisfying
\[
T^h_\zeta T^h_{\zeta'} = e^{-(ih/2) \zeta \cdot J \zeta'} T^h_{\zeta + \zeta'}.
\] (6.12)

This generalization covers, in particular, various operator orderings that differ from the Weyl (totally symmetric) ordering corresponding to \( S = 0 \) when we use the notation \( \text{Op}(f) \). The quantization map (6.11) implies the following composition law for symbols
\[
(f \ast_{h,S} g)(x) = (\pi h)^{-2d} \int_{\mathbb{R}^{4d}} f(x - (J + S)x') g(x - x'') e^{(2i/\hbar)x' \cdot x''} dx' dx''.
\] (6.13)

The Fourier transform converts (6.13) into the deformed convolution
\[
(\hat{f} \ast_{h,S} \hat{g})(\zeta) = \int_{\mathbb{R}^{2d}} \hat{f}(\zeta') \hat{g}(\zeta - \zeta') e^{-(ih/2)\zeta' \cdot (J + S)(\zeta - \zeta')} d\zeta'
\] (multiplied by \((2\pi)^{-d}\)). It follows from the definition (6.11) and relation (6.12) that
\[
\text{Op}_S(f \ast_{h,S} g) = \text{Op}_S(f) \text{Op}_S(g),
\]
as can be verified by using the symmetry of the matrix \( S \) and the antisymmetry of \( J \) (see, e.g., Sect. 4 in [38] for more details). Let \( j_S \) be the operator of multiplication by \( e^{(ih/4)\zeta \cdot S \zeta} \). It is easy to see that the deformed convolution (6.14) and the twisted convolution (5.1) are connected by the relation
\[
j_S(\hat{f} \ast_{h,S} \hat{g}) = j_S(\hat{f}) \ast_{h,S} j_S(\hat{g}).
\]

As already noted above, Theorem 1 in [38] shows that under the condition \( b_n \subset a_n \), the function \( e^{(ih/4)\zeta \cdot S \zeta} \) is a pointwise multiplier for \( S^{(a)}_{\{b\}} \) and for \( S^{(a)}_{\{b\}} \). Hence these spaces are algebras under the deformed convolution (6.14) and the map \( g(\zeta) \rightarrow e^{(ih/4)\zeta \cdot S \zeta} g(\zeta) \) is an algebraic and topological isomorphism of \((S^{(a)}_{\{b\}}, \ast_{h,S})\) onto \((S^{(a)}_{\{b\}}, \ast_{h,S})\) and of \((S^{(a)}_{\{b\}}, \ast_{h,S})\) onto \((S^{(a)}_{\{b\}}, \ast_{h,S})\). The spaces of multipliers with respect to the products \( \ast_{h,S} \) and \( \ast_{h,S} \) are defined in complete analogy with (5.4)–(5.7), (5.14), (5.14), and we obtain another corollary of Theorem 6.2.

**Corollary 6.5.** Suppose \( b_n \subset a_n \). Let \( S \) be a real symmetric matrix, and let \( \mathcal{M}_h^S(S^{(a)}_{\{b\}}) \) and \( \mathcal{M}_h^S(S^{(a)}_{\{a\}}) \) be, respectively, the algebras of two-sided multipliers for \((S^{(a)}_{\{b\}}, \ast_{h,S})\) and \((S^{(a)}_{\{b\}}, \ast_{h,S})\), where \( \ast_{h,S} \) is defined by (6.13). Then we have the continuous embeddings
\[
\tilde{\mathcal{E}}^{(b)}_{\{a\}} \hookrightarrow \mathcal{M}_h^S(S^{(b)}_{\{a\}}), \quad \tilde{\mathcal{E}}^{(b)}_{\{a\}} \hookrightarrow \mathcal{M}_h^S(S^{(b)}_{\{a\}}).
\] (6.15)

**Proof.** Since \( \mathcal{C}_h^S(S^{(a)}_{\{b\}}) \) and \( \mathcal{C}_h^S(S^{(a)}_{\{b\}}) \) are obtained from the algebras (5.14) by multiplication by \( e^{-(ih/4)\zeta \cdot S \zeta} \) and the spaces \( M'(S^{(a)}_{\{b\}}) \) and \( M'(S^{(a)}_{\{b\}}) \) are invariant under this operation, it follows from (6.8) that
\[
M'(S^{(a)}_{\{b\}}) \hookrightarrow \mathcal{C}_h^S(S^{(a)}_{\{b\}}), \quad M'(S^{(a)}_{\{b\}}) \hookrightarrow \mathcal{C}_h^S(S^{(a)}_{\{b\}}).
\]

Fourier transforming and applying Corollary 4.5, we obtain (6.15). \( \square \)
Theorem 6.6. The multiplication (6.13) in $\mathcal{M}^{(a)}_{\{a\}}(\mathbb{R}^2)$ and $\mathcal{M}^{(a)}_{\{b\}}(\mathbb{R}^2)$ can be uniquely extended to a separately continuous multiplication in $\tilde{\mathcal{M}}^{(a)}_{\{a\}}(\mathbb{R}^2)$ and $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$, respectively.

**Proof.** We give a proof for $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$; the argument for $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$ is similar. We consider the composition of the following five maps:

- $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2) \times \tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2) \xrightarrow{\circ} \tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$, $(h_1, h_2) \mapsto h_1 \otimes h_2$,
- $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2) = C'(S_{\{a\}}^{(b)}(\mathbb{R}^4)) \xrightarrow{\mathcal{F}} M'(S_{\{b\}}^{(a)}(\mathbb{R}^4))$,
- $M'(S_{\{b\}}^{(a)}(\mathbb{R}^4)) \to M'(S_{\{b\}}^{(a)}(\mathbb{R}^4))$, $u \mapsto u \cdot e^{-(ih/2)(\zeta \cdot (J+S))'}$,
- $M'(S_{\{b\}}^{(a)}(\mathbb{R}^4)) \xrightarrow{\mathcal{F}^{-1}} C'(S_{\{a\}}^{(b)}(\mathbb{R}^4)) = \tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^4)$,
- $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^4) \circ \triangle \to \tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$, $h \mapsto h \circ \triangle$,

where $\triangle$ is the diagonal map on $\mathbb{R}^2$, i.e., $\triangle(x) = (x, x)$. It is easy to see that the first map is separately continuous. Indeed, from (2.4) and (2.13), we have

$$|\partial^\beta h_1(x)\partial^\gamma h_2(x')| \leq \|h_1\|_B^A \|h_2\|_B^A \|B^{[\beta]} + |\beta'|_{\beta} b_{[\beta]} b_{[\beta']} w_a(x)|w_a(x')|$$

$$\leq K\|h_1\|_B^A \|h_2\|_B^A \|B^{[\beta]} + |\beta'|_{\beta} b_{[\beta]} b_{[\beta']} w_a(H \max(|x|, |x'|)/A).$$

Hence, $\|h_1 \otimes h_2\|_{A+B} \leq K\|h_1\|_B^A \|h_2\|_B^A$ and for every fixed $h_2$, the map $h_1 \mapsto h_1 \otimes h_2$ is continuous from $\mathcal{M}^{(b)}_{\{a\}}(\mathbb{R}^2)$ to $\mathcal{M}^{(b)}_{\{a\}}(\mathbb{R}^4)$ and therefore is continuous from $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$ to $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^4)$ by the definition of the inductive topology. The second and fourth maps are well defined and continuous by Corollary 4.5 of Theorem 4.3. The third map is continuous because $e^{-(ih/2)(\zeta \cdot (J+S))'}$ is a pointwise multiplier of $S_{\{b\}}^{(a)}(\mathbb{R}^4)$ by Theorem 1 in [38]. The fifth map is also continuous because

$$|\partial^\beta h(x, x)| \leq \sum_{\gamma \leq \beta} \left(\begin{array}{c} \beta \\ \gamma \end{array}\right) |\partial_1^{\beta_1} \partial_2^{\beta_2-\gamma} h(x, x)| \leq \|h\|_B^A(2B)^{|\beta_1|} b_{[\beta]} w_a(|x|/A)$$

and so $\|h \circ \triangle\|_B^A \leq \|h\|_{2B}^A$. The composition under consideration is an extension of the bilinear map $(f_1, f_2) \mapsto f_1 \ast_{h,S} f_2$ from $S_{\{b\}}^{(a)}(\mathbb{R}^2) \times S_{\{b\}}^{(a)}(\mathbb{R}^2)$ to $S_{\{a\}}^{(b)}(\mathbb{R}^2)$, as is readily seen if we write $f_1 \ast_{h,S} f_2$ in the form

$$(f_1 \ast_{h,S} f_2)(x) = (2\pi)^{-2d} \int_{\mathbb{R}^{2d}} e^{i(\zeta \cdot \zeta')} f_1(\zeta') f_2(\zeta') e^{-(ih/2)(\zeta \cdot (J+S))'} d\zeta d\zeta'.$$

Proposition 3.1 implies that this extension is unique and also shows that the definition of the $\ast_{h,S}$-multiplication in $\tilde{\mathcal{M}}^{(b)}_{\{a\}}(\mathbb{R}^2)$ via this composition agrees with the definition of multiplication in $\mathcal{M}_{\{a\}}^{(b)}(\mathbb{R}^2)$, because the latter is also separately continuous and the topology of the algebra $\mathcal{M}_{\{a\}}^{(b)}(\mathbb{R}^2)$ is weaker than that of $\tilde{\mathcal{M}}^{(b)}_{\{a\}}$. We conclude that $\tilde{\mathcal{M}}^{(b)}_{\{a\}}$ is a subalgebra of $\mathcal{M}_{\{a\}}^{(b)}(\mathbb{R}^2)$, and every $\mathcal{M}_{\{a\}}^{(b')}$ with $b'_a \subset b_a$ is also its subalgebra. \qed
Remark 6.7. Theorem 6.6 extends in several directions Theorem 4.14 in [5], which concerns the spaces $\mathcal{E}_{(a)}^{(a)}$ and $\mathcal{E}_{(a)}^{(a)}$ with $a_n = n^{sn}$ (denoted there respectively by $\Gamma_{s}^{\infty}$ and $\Gamma_{0,s}^{\infty}$) and the product (6.13) with $S$ of a special form. We note that the $\star_{h,S}$-multiplication in $S^{(b)}_{(a)}$ and $S^{(b)}_{(a)}$ is continuous because separate continuity implies joint continuity in the case of Fréchet spaces and barrelled (DF) spaces (see Sect. 40.2 in [26]), but $\tilde{\mathcal{E}}^{(b)}_{(a)}$ and $\tilde{\mathcal{E}}^{(b)}_{(a)}$ do not belong to these classes.

7. Concluding Remarks

The quantization map (6.11) extends uniquely to a continuous bijection of $S^{(b)}_{(a)}(\mathbb{R}^{2d})$ onto the space $\mathcal{E}(S^{(b)}_{(a)}(\mathbb{R}^{d}), S^{(b)}_{(a)}(\mathbb{R}^{d}))$, as well as to a continuous bijection of $S^{(b)}_{(a)}(\mathbb{R}^{2d})$ onto $\mathcal{E}(S^{(a)}_{(a)}(\mathbb{R}^{d}), S^{(a)}_{(a)}(\mathbb{R}^{d}))$ (for a proof, see Theorem 2 in [37]). These extensions are analogous to the extension of the Weyl map to tempered distributions in [15, 23]. In this way $\text{Op}_{S}(u)$ is well defined for any $u \in S^{(a)}_{(a)}(\mathbb{R}^{2d})$ as a continuous linear map of $S^{(a)}_{(a)}(\mathbb{R}^{d})$ into $S^{(a)}_{(a)}(\mathbb{R}^{d})$ and coincides with the operator whose Weyl symbol is $F^{-1}[\hat{u} e^{-(ih/4)\xi_{c}S^{c}_{\xi}}]$. It follows directly from the definitions that the algebra $\mathcal{M}_{h}^{S}(S^{(a)}_{(a)}(\mathbb{R}^{2d}))$ is transformed by the extended map (6.11) into the same set of operators as $\mathcal{M}_{h}(S^{(a)}_{(a)}(\mathbb{R}^{2d}))$ by the Weyl map. In addition, Corollary 4.6 implies that the image of $\tilde{\mathcal{E}}^{(a)}_{(a)}(\mathbb{R}^{2d})$ under the map $u \rightarrow \text{Op}_{S}(u)$ is the same as its image under the Weyl map. Analogous statements are true regarding $\mathcal{M}_{h}^{S}(S^{(a)}_{(a)}(\mathbb{R}^{2d}))$ and $\tilde{\mathcal{E}}^{(a)}_{(a)}(\mathbb{R}^{2d})$.

Theorem 3 in [37] shows that the Weyl map transforms the algebra $\mathcal{M}_{h,L}(S^{(a)}_{(a)}(\mathbb{R}^{2d}))$ of left Moyal multipliers for $S^{(a)}_{(a)}(\mathbb{R}^{2d})$ into the algebra of operators mapping $S^{(a)}_{(a)}(\mathbb{R}^{d})$ continuously into itself. In [40], this result is extended to the general case of spaces $S^{(a)}_{(a)}$ and $S^{(a)}_{(a)}$, and it implies, in particular, that the pseudodifferential operators whose Weyl symbols belong to $\tilde{\mathcal{E}}^{(a)}_{(a)}$ are continuous on $S^{(a)}_{(a)}$ and the operators with Weyl symbols in $\tilde{\mathcal{E}}^{(a)}_{(a)}$ are continuous on $S^{(a)}_{(a)}$. Pseudodifferential operators with symbols in the spaces $\Gamma_{s}^{\infty}$ and $\Gamma_{0,s}^{\infty}$, which coincide with $\tilde{\mathcal{E}}^{(a)}_{(a)}$ and $\tilde{\mathcal{E}}^{(a)}_{(a)}$ for $a_n = n^{sn}$, were studied by Cappiello and Toft in [5]. Their continuity properties are proved there by a different method based on using modulation spaces and the short time Fourier transform. Similar results on the continuity properties of pseudodifferential operators in the Gelfand–Shilov setting were also derived in another way by Prangoski [30], but for slightly smaller symbol classes than $\tilde{\mathcal{E}}^{(a)}_{(a)}$ and $\tilde{\mathcal{E}}^{(a)}_{(a)}$. It follows from the above considerations that the continuity properties of operators obtained from $\tilde{\mathcal{E}}^{(a)}_{(a)}$ and $\tilde{\mathcal{E}}^{(a)}_{(a)}$ by the map (6.11) with $S \neq 0$ are the same as operators obtained by applying the Weyl map and do not require a separate examination.
Besides the spaces (2.15) and (2.16), Palamodov has introduced in [29] two more classes of spaces, which in our notation are defined by $E_{(a)}^{(b)} = \bigcap_{A \to -\infty} B \to 0 E_{a,A}^{b,B}$ and $E_{\{a\}}^{(b)} = \bigcup_{A \to 0 B \to -\infty} E_{a,A}^{b,B}$. The dual of $E_{(a)}^{(b)}$ is the space of convolutors for $S_{(a)}^{(b)} = \bigcap_{B \to 0} \bigcup_{A \to -\infty} S_{a,B}^{b,B}$ and the dual of $E_{\{a\}}^{(b)}$ is the space of convolutors for $S_{(a)}^{(b)} = \bigcap_{A \to 0} \bigcup_{B \to -\infty} S_{a,B}^{b,B}$. We note that the symbol class denoted in [5] by $\Gamma_{a,A}^{B}$ is, in our notation, $E_{\{a\}}^{(b)}$ for $a_B = n^{s_B}$. It is clear that $E_{(a)}^{(b)} \subset E_{\{a\}}^{(b)} \cap \hat{E}_{(a)}^{(b)}$. Therefore, the pseudodifferential operators with symbols in $E_{(a)}^{(b)}(\mathbb{R}^{2d})$ are continuous from $S_{(a)}^{(b)}(\mathbb{R}^d)$ to $S_{\{a\}}^{(b)}(\mathbb{R}^d)$ and from $S_{(a)}^{(b)}(\mathbb{R}^d)$ to $S_{\{a\}}^{(b)}(\mathbb{R}^d)$. In the case of symbols in $E_{(a)}^{\{a\}}(\mathbb{R}^{2d})$, the situation is completely different because $S_{(a)}^{\{a\}}(\mathbb{R}^{2d})$ is not closed under the Weyl–Moyal product.

Since $E_{\{a\}}^{(b)} \subset S_{(a)}^{(b)}$, its corresponding operators are well defined as elements of $L(S_{(a)}^{(b)}(\mathbb{R}^d), S_{\{a\}}^{(b)}(\mathbb{R}^d))$, but as we show in Appendix, $E_{\{a\}}^{(a)}(\mathbb{R}^{2d})$ includes functions $u$ such that the image of $S_{(a)}^{(a)}(\mathbb{R}^d)$ under $\text{Op}(u)$ is not contained in $L^2(\mathbb{R}^d)$.

Declarations

Conflicts of interest The author has no conflicts of interest to declare that are relevant to the content of this article.

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Appendix

The arguments of the proof of Proposition 3.2 are easily adapted to show that the elements of $E_{\{a\}}^{(b)}$ are convolutors of $S_{(a)}^{(b)}$. To analyse the mapping properties of pseudodifferential operators with symbols in $E_{\{a\}}^{(a)}$ we use the isomorphism $F\left[S_{(a)}^{(b)}\right] = S_{(b)}^{(a)}$. As in the case of the spaces $S_{(a)}^{(b)}$ and $S_{(b)}^{(a)}$, this isomorphism is ensured by condition (2.3) and is easily proved by using the inequality

$$\int_{\mathbb{R}^d} |\partial^\beta x^\alpha| |f(x)| dx \leq \sqrt{2} \int_{\mathbb{R}^d} |x^\alpha| |\partial^\beta f(x)| dx,$$

which holds for any function $f$ in the Schwartz space $S(\mathbb{R}^d)$ and for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$ (see Lemma A.1 in [38]).

**Lemma A.1.** If the space $S_{\{a\}}^{(a)}(\mathbb{R}) = \bigcap_{B \to 0} \bigcup_{A \to -\infty} S_{a,A}^{a,B}(\mathbb{R})$ is nontrivial, then there is a nonnegative even function $\varphi \in S_{\{a\}}^{(a)}(\mathbb{R})$ satisfying

$$\varphi(s) \geq 1/w_a(|s|), \quad s \in \mathbb{R}. \quad (A.1)$$
Proof. It is readily seen that $S_{\{a\}}(\omega)$ is an algebra under multiplication and is invariant under translations. Therefore, there is a nonnegative even function $\omega \in S_{\{a\}}(\mathbb{R})$ such that $\int_0^1 \omega(s)ds = 1$. We set

$$\varphi(s) = \int_{-\infty}^{\infty} \frac{\omega(s-t)}{w_a(|t|/2)} dt.$$ 

The norm $||\omega||_{A,B}$ is finite for any $B > 0$ and some $A$ depending on $B$. Applying (2.13) and (2.11) gives

$$|\partial^n \varphi(s)| \leq \int_{-\infty}^{\infty} \frac{||\omega||_{A,B}B^n a_n}{w_a(|t|/2)w_a(|s-t|/A)} dt \leq \int_{-\infty}^{\infty} \frac{K||\omega||_{A,B}B^n a_n}{w_a(|t|/2H)^2 w_a(|s-t|/A)} dt \leq \frac{C||\omega||_{A,B}B^n a_n}{w_a(|s|/(2H + A))},$$

where $C = 2K \int_0^\infty w_a(t/2H)^{-1} dt$. Hence, $\varphi \in S_{\{a\}}$. Using (2.12), we obtain

$$\varphi(s) \geq \int_0^1 \frac{\omega(t)}{w_a(|s-t|/2)} dt \geq \int_0^1 \frac{\omega(t)}{w_a(|s+1|/2)} dt \geq \frac{1}{w_a(|s|)w_a(1)} = \frac{1}{w_a(|s|)},$$

as claimed.

□

Proposition A.2. The space $S_{\{a\}}(\mathbb{R}^{2d})$ is not closed under the Weyl–Moyal product.

Proof. It suffices to consider the case where $d = 1$ and $h = 2$. Let $f(q,p) = f_1(q)f_2(p)$ and $f_1, f_2 \in S_{\{a\}}(\mathbb{R})$. By the definition (1.1) we have

$$(f \ast f)(q,p) = (2\pi)^{-2} \int_{\mathbb{R}^4} f_1(q-q')f_2(p-p')f_1(q-q'')f_2(p-p'')e^{i(q'p''-q''p')} dq'dq'' dp' dp''$$

$$= (2\pi)^{-1} \int_{\mathbb{R}^2} \hat{f}_1(p'')(\hat{f}_2(p-p')\hat{f}_1(-p')f_2(p-p'')e^{i(qp''-qp')}) dp' dp''.$$ 

If $f_1$ and $f_2$ are even, then

$$(f \ast f)(q,0) = \left| \mathcal{F}^{-1} \left( \hat{f}_1 \cdot \hat{f}_2 \right) (q) \right|^2 = (2\pi)^{-1} \left[ \left( f_1 \ast \hat{f}_2 \right) (q) \right]^2.$$ 

The function $f_1$ can be taken nonnegative and such that $\int_0^1 f_1(q)dq = 1$. We set $\hat{f}_2(q) = \varphi(q/2)$, where $\varphi$ is the function constructed in Lemma A.1. Then

$$\left| \left( f_1 \ast \hat{f}_2 \right) (q) \right| \geq \frac{1}{w_a(|q|)} \quad (A.2)$$
by the same argument as in deriving (A.1). Assume that \( f \ast f \in S^{(a)}_0(\mathbb{R}^2) \). Then we have
\[
|(f \ast f)(q)| \leq \frac{C_A}{wa(|q|/A)}
\]
for any \( A > 0 \) and therefore, in view of (2.13),
\[
\left| \left( f_1 \ast f_2 \right)(q) \right| \leq \frac{C_A'}{wa(|q|/HA)}.
\]
But this contradicts (A.2) for \( H^2 A \leq 1 \) since \( w_a(t)/w_a(t/HA) \leq K/w_a(t) \to 0 \) as \( t \to \infty \).

We now turn to the mapping properties of the operators with Weyl symbols in \( \mathcal{E}^{(a)}_{(a)}(\mathbb{R}^{2d}) \). The symbol class \( \Gamma_1^{(a)} \), which coincides with \( \mathcal{E}^{(a)}_{(a)} \) for \( a_n = n^{2n} \), was studied in \([5]\) using the technique of short time Fourier transform and modulation spaces and the authors came to the conclusion that if \( u \in \Gamma_1^{(a)}(\mathbb{R}^{2d}) \), then \( \text{Op}(u) \) is continuous from \( \Sigma_{a}(\mathbb{R}^d) \) to \( S_s(\mathbb{R}^d) \), which are respectively \( S^{(a)}_0(\mathbb{R}^d) \) and \( S^{(a)}_s(\mathbb{R}^d) \) with \( a_n = n^{2n} \) in our notation. A similar conclusion was made in \([1]\) for a more general case of anisotropic Gelfand–Shilov spaces. Here we show that these conclusions are incorrect. We use the standard representation of the Weyl system
\[
\left( T^h_\zeta \psi \right)(x) = e^{i(h/2)\eta \xi} e^{ipx} \psi(x + h \xi), \quad \psi \in L^2(\mathbb{R}^d), \quad \zeta = (\eta, \xi),
\]
and assume that \( S^{(a)}_0(\mathbb{R}^{2d}) \) is nontrivial. If \( u \in S^{(a)'}_{(a)}(\mathbb{R}^{2d}) \), then applying the operator \( \text{Op}(u) \) to \( f \in \mathcal{E}^{(a)'}_{(a)}(\mathbb{R}^d) \) yields an element of \( S^{(a)'}_{(a)}(\mathbb{R}^d) \) such that
\[
\langle \text{Op}(u) f, g \rangle = (2\pi)^{-d} \langle \hat{u}, \hat{g}, T^h_\zeta, f \rangle_{L^2}, \quad g \in S^{(a)'}_{(a)}(\mathbb{R}^d).
\]
Explicitly
\[
\langle \text{Op}(u) f, g \rangle = (2\pi)^{-d} \langle u, \int_{\mathbb{R}^d} g(q - h \xi/2) f(q + h \xi/2) e^{-ip\xi} d\xi \rangle.
\]
Setting \( h = 1 \), this can also be expressed by saying that the kernel of \( \text{Op}(u) \) is given by
\[
\mathcal{K}(x, y) = (2\pi)^{-d/2} \left( F_2^{-1} u \right) \left( (x + y)/2, x - y \right),
\]
which is taken as a starting definition in \([1, 5]\).

Proposition A.3 provides a counterexample to Theorem 4.12 in \([5]\) and to Theorem 3.16 in \([1]\).

**Proposition A.3.** There is a function \( u \in \mathcal{E}^{(a)}_{(a)}(\mathbb{R}^{2d}) \) such that the image of \( S^{(a)}_0(\mathbb{R}^d) \) under \( \text{Op}(u) \) is not contained in \( L^2(\mathbb{R}^d) \).

**Proof.** We set \( d = 1, h = 1, \) and \( u(q, p) = u_1(q) u_2(p) \). Let \( \omega_1 \) be a nonnegative even function in \( S^{(a)}_0(\mathbb{R}) \) with \( \int_0^\infty \omega_1(t) dt = 1 \), and let
\[
u_1(q) = \int_{-\infty}^\infty \omega_1(q - t) w_a(|t|) dt.
\]
The function \( u_1 \) belongs to \( \mathcal{E}_{\{a\}}(\mathbb{R}) \) because \( \|\omega_1\|_{A,B} < \infty \) for any \( A,B > 0 \) and taking \( A \leq 1/(2H) \) and using (2.13) and (2.12) we obtain

\[
|\partial^n u_1(q)| \leq \|\omega_1\|_{A,B} B^n a_n \int_{-\infty}^{\infty} \frac{w_a(|t|)}{w_a(|q-t|/A)} dt \leq K \|\omega_1\|_{A,B} B^n a_n \int_{-\infty}^{\infty} \frac{w_a(|q-t|)}{w_a(2|t|)^2} dt \leq C \|\omega_1\|_{A,B} B^n a_n w_a(2|q|),
\]

where \( C = 2K \int_0^{\infty} w_a(2t)^{-1} dt \). Furthermore,

\[
u_1(q) \geq \omega_a(|q|) \quad \text{for all } q \in \mathbb{R}. \quad \text{(A.3)}
\]

Indeed, this function is even and for \( q \geq 0 \), we have

\[
u_1(q) \geq \int_{q}^{\infty} \omega_1(t-q)w_a(t) dt \geq \omega_a(q) \int_{0}^{\infty} \omega_1(t) dt = \omega_a(q).
\]

Next, let \( \hat{u}_2(s) = \varphi(s/\Lambda) \), where \( \varphi \) is the function constructed in Lemma A.1 and \( \Lambda \) is a positive parameter. Clearly, \( u(q,p) = u_1(q)u_2(p) \) belongs to \( \mathcal{E}_{\{a\}}(\mathbb{R}^2) \) and

\[
(\text{Op}(u)f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_1((x+y)/2)\varphi((x-y)/\Lambda)f(y)dy.
\]

Let \( f \) be an arbitrary nonzero nonnegative function in \( S_{\{a\}}(\mathbb{R}) \). From (A.1), (A.3), and (2.12), we obtain

\[
(\text{Op}(u)f)(x) \geq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{w_a(|x+y|/2)f(y)}{w_a(|x-y|/\Lambda)} dy \geq \frac{1}{\sqrt{2\pi}} \frac{w_a(|x|/4)}{w_a(2|x|/\Lambda)} \int_{-\infty}^{\infty} \frac{f(y)}{w_a(|y|/2)w_a(2|y|/\Lambda)} dy.
\]

If \( \Lambda \geq 8H \), then (2.13) implies \( w_a(t/4)/w_a(2t/\Lambda) \geq K^{-1} w_a(t/4H) \), and we conclude that \( \text{Op}(u)f \notin L^2(\mathbb{R}) \). □

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Received: December 22, 2020.
Revised: July 23, 2021.