The fermion determinant
in (4,4) 2d lattice super-Yang-Mills

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Abstract

We find that the fermion determinant is not generally positive in a class of lattice actions recently constructed by Cohen et al. [hep-lat/0307012]; these are actions that contain an exact lattice supersymmetry and have as their target (continuum) theory (4,4) 2-dimensional super-Yang-Mills. We discuss the implications of this finding for lattice simulations and give some preliminary results for the phase of the determinant in the phase-quenched ensemble.

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1 Introduction

A major motivation for efforts to latticize supersymmetric models is that some non-perturbative aspects of supersymmetric field theories are not accessible by the usual techniques, most often relying on holomorphy. Indeed, phenomenological applications of softly broken $\mathcal{N} = 1$ 4d supersymmetric gauge theories such as the minimal supersymmetric standard model are generally regarded as effective field theories valid for the TeV regime, derived from a more fundamental supersymmetric theory valid at higher energy scales. To give rise to spontaneous supersymmetry breaking, the more fundamental theory must contain other sectors that are responsible for this effect. Often the additional sectors include a strongly-interacting super-Yang-Mills theory. The most general effective theory consistent with symmetry constraints typically involves nonholomorphic quantities, determined by the Kähler potential. It is of interest to understand the nonperturbative corrections that this potential may receive, particularly if the corresponding dynamics are expected to play an important role in determining qualitative features of the effective theory; in addition to breaking supersymmetry, nonperturbative effects are generally considered to be important to moduli stabilization (vacuum selection).

Thus, one hope for lattice supersymmetry is that it would lead to simulations that would provide further data on nonperturbative aspects of supersymmetric field theories, especially those that include super-Yang-Mills. However, the lattice regulator generically breaks supersymmetry. This is a natural consequence of the fact that the supersymmetry algebra is embedded into the super-Poincaré algebra, which involves both rotations and translations. Only a discrete subgroup of the Poincaré group survives, so it is not surprising that the regulator is not supersymmetric. Thus the target theory is obtained by fine-tuning the bare parameters of the lattice action—supplemented by counterterms; a considerable amount of work has been done in this direction. Early efforts in this direction can be found in [1, 2]; recent reviews with extensive references are given in [3, 4]. In the case of pure 4d $\mathcal{N} = 1$ super-Yang-Mills, a gauge invariant formulation with chiral lattice fermions suffices to guarantee the correct continuum limit; for example, simulations using domain wall fermions have been performed [5].

It is worth noting, however, that the discrete rotations and translations that are preserved in a typical (isotropic, hypercubic) latticization guarantee that in the continuum limit the only relevant and marginal operators that can appear are those that preserve the full Poincaré invariance of the target theory. It is of interest to explore whether or not it is possible to realize something analogous for the super-Poincaré group. Some amount of exact lattice supersymmetry—i.e., a fermionic symmetry that relates lattice bosons and lattice fermions and closes with the discrete subgroup of the Poincaré group that is preserved by the lattice—would presumably allow for the target theory to be obtained in a more controlled fashion. Remarkably, in some recent
examples possessing an exact lattice supersymmetry, it has been found that certain super-Yang-Mills theories may be obtained from a lattice theory without the need for fine-tuning; additionally, other types of super-Yang-Mills theories may be obtained with less fine-tuning than would occur in a naive discretization \cite{6, 7, 8}. It is worth noting that other sorts of models with exact lattice supersymmetry have been discussed in the literature by a few groups: supersymmetric quantum mechanics \cite{9, 10}, the 2d Wess-Zumino model \cite{11, 10}, pure super-Yang-Mills using overlap fermions \cite{12}, and direct constructions in the spirit of the Ginsparg-Wilson relation\footnote{Lüscher first suggested that the perfect action approach might be used to identify an analogue of the Ginsparg-Wilson relation for lattice supersymmetry \cite{13}. This idea was worked out in noninteracting examples in \cite{14, 15}. (An approach very similar to \cite{15} has been applied in \cite{16}, yielding slightly different expressions.)} have all been considered. In some cases, the constructions have been related to topological field theory \cite{17}.

1.1 Deconstruction, orbifolds and lattice supersymmetry

In this letter we will be interested in the super-Yang-Mills constructions that lead to a Euclidean lattice whose target theory contains 8 supercharges \cite{6}. This is a generalization of the 4 supercharge constructions of \cite{7}. The method of building all such models is based on deconstruction of extra dimensions \cite{18, 19}. The corresponding interpretation in terms of the world-volume theory of D-branes has led to the latticizations of 2d, 3d and 4d supersymmetric gauge theories. While a complete latticization of spacetime is studied here, partial latticizations also yield interesting results; for example, the chiral anomalies \cite{20} or instanton solutions \cite{21} of 4d $N = 2$ super-Yang-Mills can be equivalently described by deconstructing one dimension to obtain a 3d $N = 4$ product group (quiver) super-Yang-Mills theory that is an effective latticization of one dimension.

The Euclidean spacetime lattice constructions are all arrived at by orbifold projections of super-Yang-Mills matrix models.\footnote{These matrix models are obtained as 0d reductions of 4d, 6d and 10d $\mathcal{N} = 1$ Euclidean super-Yang-Mills.} I.e., in each case we quotient a matrix model by some discrete symmetry group of the theory. Degrees of freedom that are not invariant with respect to the combined action of the orbifold generators are projected out.\footnote{For a detailed discussion, we refer the reader to \cite{6, 7}.} Following \cite{6, 7}, we will refer to the “nonorbifolded” matrix model as the mother theory and the “orbifolded” matrix model as the daughter theory. The (effective) lattice theory is obtained by studying the daughter theory expanded about a nontrivial minimum of its scalar potential; i.e., a point in its moduli space.
1.2 The fermion determinant

In lattice theories containing fermions, it is well-known that it is of great practical importance that the fermion determinant (or more generally, the Pfaffian), obtained by integrating over the fermion degrees of freedom in the partition function, be positive semi-definite. For let $v$ represent the lattice bosons. Then having integrated out the lattice fermions, one obtains an equivalent effective lattice action ($S_B$ is the bosonic part of the action before integrating out fermions):

$$S_{\text{eff}}(v) = S_B(v) - \ln \det M(v).$$

A positive semi-definite $\det M(v)$ yields a real effective action, thus avoiding the inherent problems of a complex action with respect to estimating correlation functions by Monte Carlo simulation.

However, in [22] we found that the fermion determinant was complex for the constructions of [7], which have (2,2) 2d $U(k)$ super-Yang-Mills as their target theory. Here we study whether or not a similar problem exists in the constructions of [6], which have (4,4) 2d $U(k)$ super-Yang-Mills as their target theory.

1.3 Summary of results

For the reader’s convenience, we now summarize the content of our work:

- In this letter we show that the lattice theory with (4,4) 2d super-Yang-Mills as its target [6], obtained from orbifolded supersymmetric matrix models, possesses a problematic fermion determinant.

- Due to a ever-present fermion zeromode, $\det M(\phi) \equiv 0$, i.e., for all boson configurations. The zero eigenvalue can be factored out in a controlled way in order to exhibit the determinant for the other fermion modes. In the daughter theory with (4,4) 2d $SU(2)$ super-Yang-Mills as its target, we carry out this factorization (numerically).

- Once the zeromode fermion has been factored out, we find that the remaining product of eigenvalues is generically nonzero with arbitrary complex phase.

- We discuss the implications of our results for lattice simulations of the latticized (4,4) 2d super-Yang-Mills theories. We emphasize that the complex action may be turned into a virtue, in that it provides an interesting system in which to study complex action simulation techniques—relevant to lattice QCD at finite temperature and/or baryon density.

4In [22] we also showed that this was not in conflict with the well-known positivity of the fermion determinant in the mother theory [23, 24].
We conclude by presenting some preliminary Monte Carlo results for the distribution of the phase of the fermion determinant when sampled in the phase-quenched distribution; that is, when the determinant in (1) is replaced by its absolute value. We discuss why this may be relevant to the continuum limit. Unfortunately, for the small lattice that we consider (2×2), we find that the phase distribution is essentially uniform in its range (−π, π).

In the remainder of this letter we will discuss various details related to these results.

2 Construction

2.1 Mother theory

The action of the mother theory is that of a 6d → 0d reduction of $U(kN^2) \mathcal{N} = 1$ super-Yang-Mills:

$$S = -\frac{1}{4g^2} \text{Tr} (v_m, v_n)[v_m, v_n] + \frac{1}{2g^2} \epsilon_{ij} \text{Tr} (\Psi^T_i C \Sigma_m [v_m, \Psi_j])$$

(2)

where $\epsilon = i\sigma_2, v_m = v_m^\alpha T^\alpha, \Psi_i = \Psi_i^\alpha T^\alpha (i = 1, 2)$ with $T^\alpha = (1, T^a)$ a Hermitian basis for the generators of $U(kN^2)$. Each of the two $\Psi_i$ is a 4-component fermion. The 4×4 matrices $\Sigma_m$ are components in the construction of the 6d (Euclidean) Clifford algebra, and $C$ is a charge conjugation matrix. For further details we refer the reader to [6].

2.2 Orbifold to daughter theory

In addition to the $U(kN^2)$ gauge invariance, the mother theory possesses an $SO(6)_E$ Euclidean invariance group and an $SU(2)_R$ chiral R-symmetry group. From the invariance group $SO(6)_E \otimes SU(2)_R$ one isolates a $U(1)_1 \otimes U(1)_2$ subgroup, with corresponding generators $r_1$ and $r_2$. The bosons and the fermions of the mother theory are written in a basis with well-defined $r_1, r_2$ charges. In the construction of [6] this basis is:

$$\Psi_1^T = (\lambda, \xi_1, \xi_2, \xi_3), \quad \Psi_2^T = (\chi, \psi_1, \psi_2, \psi_3),$$

(3)

$$C\Sigma_m v_m = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & -y & z \\ -y & z & 0 & x \\ -z & y & -x & 0 \end{pmatrix}$$

(4)

5Our lattice boson notation is related to that of [6] by $(x, y, z) \equiv (z_1, z_2, z_3)$. 
The $r_1, r_2$ charges are given in Table 1 of [6]; we will not need them here. All that is important is that the bosons and fermions (denoted collectively by $\Phi$) of the mother theory are subjected to a projection with respect to a $Z_N \otimes Z_N$ subgroup of $U(1)_1 \otimes U(1)_2$, together with a nontrivial embedding into the gauge group $U(kN^2)$. That is, we keep only fields that satisfy

$$\Phi \equiv e^{2\pi ir_a/N}C_a\Phi C_a^{-1}$$  \hspace{1cm} (5)$$

where $C_a$ are generators of a $Z_N \otimes Z_N$ subgroup of $U(kN^2)$. This breaks the gauge group down to $U(k)^{N^2} = \bigotimes_{m_1, m_2=1}^N U(k)_{m_1, m_2}$, corresponding the $U(k)$ gauge invariance of an $N \times N$ lattice theory. Bosons and fermions associated with a single factor $U(k)_{m_1, m_2}$ of $U(k)^{N^2}$ are interpreted as site variables; in an appropriate basis, the other surviving bosons and fermions are charged with respect to 2 factors and are therefore interpreted as link variables.

### 3 Fermion action

Here we examine the fermion determinant for the daughter theory. Expanded about the chosen point in moduli space, it is the fermion determinant for the lattice theory. We find it convenient to define

$$\text{Tr} (T^\mu T^\nu T^\rho) = \tilde{N} t^{\mu\nu\rho}, \hspace{1cm} t^{\mu\nu\rho}_{m,n} = \delta_{m,n} t^{\mu\nu\rho}$$  \hspace{1cm} (6)$$

where $\tilde{N}$ is an overall normalization that may be chosen as seems convenient, and $m = (m_1, m_2)$ labels sites on a 2d square lattice; $\hat{i} = (1, 0)$ and $\hat{j} = (0, 1)$ are unit vectors in the two directions.

The fermion matrix depends on bosons

$$x_m = x^\mu_m T^\mu, \hspace{1cm} y_m = y^\mu_m T^\mu, \hspace{1cm} z_m = z^\mu_m T^\mu$$  \hspace{1cm} (7)$$

as well as conjugates $\overline{x^\mu_m} = (x^\mu_m)\dagger$, etc. The fermion action can be written in the form

$$S_F = -\frac{\tilde{N}\sqrt{2}}{g^2} \left( \psi^\mu_{1,m}, \psi^\mu_{2,m}, \psi^\mu_{3,m}, \chi^\mu_m \right) \cdot M^\mu_{mn} \cdot \left( \begin{array}{c} \xi^\rho_{1,n} \\ \xi^\rho_{2,n} \\ \xi^\rho_{3,n} \\ \lambda^\rho_n \end{array} \right)$$  \hspace{1cm} (8)$$

The elements of the fermion matrix that follow from the expressions of [6] and the
the soft mass bosonic zeromode of the theory. This zeromode eigenvalue of the daughter theory can be factored out following See also (18) below.

\[ (M_{\mu \nu}^{\mu \rho})_{1,1} = (M_{\mu \nu}^{\mu \rho})_{2,2} = (M_{\mu \nu}^{\mu \rho})_{3,3} = (M_{\mu \nu}^{\mu \rho})_{4,4} = 0, \]
\[ (M_{\mu \nu}^{\mu \rho})_{1,2} = -t^{\mu \nu}_{\mu \mu} n^{-i \nu} + t^{\mu \nu}_{\mu \nu} n^{-i \nu}, \]
\[ (M_{\mu \nu}^{\mu \rho})_{1,4} = t^{\mu \nu}_{\mu \mu} n^{-i \nu} - t^{\mu \nu}_{\mu \nu} n^{-i \nu}, \]
\[ (M_{\mu \nu}^{\mu \rho})_{2,3} = -t^{\mu \nu}_{\mu \mu} n^{i j} + t^{\mu \nu}_{\mu \nu} n^{i j}, \]
\[ (M_{\mu \nu}^{\mu \rho})_{3,1} = -t^{\mu \nu}_{\mu \mu} n^{i j} + t^{\mu \nu}_{\mu \nu} n^{i j}, \]
\[ (M_{\mu \nu}^{\mu \rho})_{3,4} = t^{\mu \nu}_{\mu \mu} n^{i j} - t^{\mu \nu}_{\mu \nu} n^{i j}, \]
\[ (M_{\mu \nu}^{\mu \rho})_{4,2} = -t^{\mu \nu}_{\mu \mu} n^{i j} + t^{\mu \nu}_{\mu \nu} n^{i j}, \]
\[ (M_{\mu \nu}^{\mu \rho})_{4,3} = -t^{\mu \nu}_{\mu \mu} n^{i j} + t^{\mu \nu}_{\mu \nu} n^{i j}. \]

3.1 Fermion zeromode

The mother theory fermion modes \( \lambda = \lambda^a T^a \) that appear in (3) are \( U(1)_1 \otimes U(1)_2 \) neutral; that is, they have \( r_1 = r_2 = 0 \) in (3). As a consequence, the surviving parts correspond to \( T^a \) that are nothing but the Cartan subalgebra of the mother theory, including the operator \( T^0 = 1 \). Because of the commutator in (2), this fermion mode \( \lambda^0 \) disappears from the action. Thus it corresponds to an ever-present zeromode in both the mother theory and the daughter theory.

This zeromode eigenvalue of the daughter theory can be factored out following the method used in [22]. We deform the fermion matrix appearing in (8) according to

\[ M \rightarrow M_\epsilon \equiv M + \epsilon 1_{N_f} \]  

where \( N_f = 4k^2 N^2 \) is the dimensionality of the fermion matrix and \( \epsilon \ll 1 \) is a deformation parameter that we will eventually take to zero. We factor out the zero mode through the definition

\[ \tilde{M}(0) = \lim_{\epsilon \rightarrow 0^+} \tilde{M}(\epsilon), \quad \tilde{M}(\epsilon) \equiv \epsilon^{-1/N_f} M_\epsilon \Rightarrow \det \tilde{M}(0) = \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} \det M_\epsilon. \]

If this deformation is added to the action, it explicitly breaks the exact lattice supersymmetry and gauge invariance. This infrared regulator could be removed in the continuum limit, say, by taking \( \epsilon a \ll N^{-1} \). Noting that \( L = Na \) is the physical size of the lattice, the equivalent requirement is that \( \epsilon \ll L^{-1} \) be maintained as \( a \rightarrow 0 \), for fixed \( L \). Thus in the thermodynamic limit \( (L \rightarrow \infty) \), the deformation is removed. The parameter \( \epsilon \) is a soft infrared regulating mass, and is quite analogous to the soft mass \( \mu \) introduced by Cohen et al. [3] in their Eq. (1.2) to control the bosonic zeromode of the theory.\(^6\)

\(^6\)See also [18] below.
3.4 of [6], our $\epsilon$ does not modify the result of the quantum continuum limit. The essence of the argument is that we have introduced a vertex that will be proportional to the dimensionless quantity $g_2^2 \epsilon a^3 \ll g_2^2 a^3 / L$, where $g_2$ is the 2d coupling constant. Such contributions to the operator coefficients $C, C$ in Eq. (3.29) of [6] vanish in the thermodynamic limit. Because the target theory is super-renormalizable, we are assured that the perturbative power counting arguments are reliable and the correct continuum limit is obtained.

We have studied the convergence of $\det \hat{M}(\epsilon) \to \det \hat{M}(0)$. Indeed, we find that the convergence is rapid and that a reliable estimate for $\det \hat{M}(0)$ can be obtained in this way. As a check, we have computed the eigenvalues of the undeformed matrix $M$, using the math package Maple, for a subset of 10 of the random boson configurations studied below. We find that the product of nonzero eigenvalues agrees with $\det \hat{M}(0)$ in magnitude and phase to within at least 5 significant digits in each of the 10 cases.

### 3.2 $U(2)$ fermion determinant

In our numerical work, we specialize to $U(2)$. In notation introduced above, we choose

$$
T^\mu = (1_2, \sigma^a), \quad \text{Tr} \left( T^{\mu} T^{\nu} T^{\rho} \right) = 2 t^{\mu\nu\rho} \Rightarrow t^{000} = 1, \quad t^{a00} = 0, \quad t^{ab0} = \delta^{ab}, \quad t^{abc} = i \epsilon^{abc}
$$

(12)

where underlining of indices indicates that all permutations are to be taken.

The bosons have components

$$
x_m = x_m^0 1_2 + x_m^a \sigma^a, \quad y_m = y_m^0 1_2 + y_m^a \sigma^a, \quad z_m = z_m^0 1_2 + z_m^a \sigma^a.
$$

(13)

The lattice theory with lattice spacing $a$ is obtained by expansion about a particular point in moduli space:

$$
x_m^0 = \frac{1}{a \sqrt{2}} + \cdots, \quad y_m^0 = \frac{1}{a \sqrt{2}} + \cdots,
$$

(14)

where $\cdots$ represent the quantum fluctuations and all other bosons are expanded about the origin. For this reason, in our study of $\det \hat{M}(0)$ we scan over a Gaussian distribution where $x_m^0, y_m^0$ have a nonzero mean $1 / a \sqrt{2} \equiv 1$. The remainder of the bosons are drawn with mean zero. All bosons are taken from distributions with unit variance.

For a set of 5000 draws on the bosons of a $2 \times 2$ lattice, we have extrapolated to $\epsilon \to 0$ and binned $\hat{\phi} \equiv \arg \det \hat{M}(0)$ over its range, with 20 bins of size $\pi / 10$. In Fig. 1 we show the frequency for each bin, as a fraction of the total number of draws. It can be seen that once the zeromode eigenvalue is factored out, the product of the nonzero eigenvalues has arbitrary phase and that within statistical errors the distribution $F(\hat{\phi})$ is uniform: $F(\hat{\phi}) \approx 1/20$. Consequently, the effective lattice action [cf. [11]]
Figure 1: Average frequency distribution $F(\phi)$ for $\phi = \arg \det \hat{M}(0)$, for 5000 random (Gaussian) draws, binned into intervals of $\pi/10$. Data was arranged into 50 blocks of 100 draws to estimate errors. The distribution of $\phi$ is seen to be, within errors, uniform. These results are for the $U(2)$ lattice theory, with $2 \times 2$ lattice.

is complex, with the phase of the fermion determinant a field-dependent quantity. This is to be contrasted with the observed reality of the fermion determinant in the mother theory [23]. Presumably the orbifold projection does not commute with the conjugation operator that guarantees this reality in the mother theory. Indeed, it is not difficult to check that the orbifold projection operator does not commute with the usual conjugation operator that is involved in establishing the hermiticity of the $\mathcal{N} = 1$ 6d Minkowski spacetime action. Thus it is not surprising that the fermion determinant in the daughter theory is not real.

4 Re-weighting

It is worthwhile to explore whether or not the complex phase can be overcome for the purposes of simulation. A typical approach would be to compute averages of an operator $\mathcal{O}$ from the re-weighting identity:

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\phi} \rangle_{p.q.}}{\langle e^{i\phi} \rangle_{p.q.}}$$

(15)

Here, $\phi = \arg \det \hat{M}(0)$, as above, and “p.q.” indicates phase-quenching: expectation values are computed with the replacement $\det \hat{M}(0) \rightarrow |\det \hat{M}(0)|$. Thus the effective bosonic action

$$S_{p.q.} = S_B - \ln |\det \hat{M}(0)|$$

(16)
is used to generate the phase-quenched ensemble by standard Monte Carlo techniques. However, it is well-known that this tends to suffer efficiency problems: the number of configurations required to get an accurate estimate for, say, $\langle \exp(i\phi) \rangle_{p.q.}$, grows like $\exp(\Delta F \cdot N_f^2)$. Here, $\Delta F$ is the difference in free energy densities between the full ensemble and the phase-quenched ensemble. Recall that $N_f$ is the dimensionality of the fermion matrix. It has been recently suggested how difficulties with this re-weighting approach might be surmounted by distribution factorization techniques \[24\]. Indeed, this method has been fruitfully applied in some other systems with complex action, some of which are quite similar to that studied here \[26, 27, 28\]. It would be interesting to see what progress may be made in the present context by applying this method. We are currently examining this idea and hope to report on it in a later publication.

However, it is also possible that in the continuum limit the phase-quenched distribution is sharply peaked at a value $\phi_0$ of $\phi$. It follows that

$$\langle O \rangle \approx \frac{\langle O \rangle_{p.q.}}{\langle 1 \rangle_{p.q.}} = \langle O \rangle_{p.q.} \quad (17)$$

Thus it may be that the phase-quenched ensemble gives a good estimation in the continuum limit. For this reason it is of interest to study the distribution of $\phi$ as determined by the phase-quenched ensemble as a function of the lattice spacing. We next report results of a preliminary study of this distribution using Monte Carlo techniques.

## 5 Phase-quenched distribution

We generate a set of configurations, updating using the Metropolis algorithm applied to \[16\]. In this way, we sample the phase-quenched ensemble and estimate the corresponding distribution of $F_{p.q.}(\phi)$. For this purpose we need the bosonic action of the daughter theory. We specialize again to the $U(2)$ case and introduce the notation $\text{Tr}(T^\mu T^\nu T^\rho T^\lambda) = 2\delta_{\mu\rho}\delta_{\nu\lambda}$. Then referring to \[9\], the bosonic action is given by

$$S_B = \frac{2}{g^2} \mu_\mu \rho \lambda \sum_n \left[ \frac{1}{2} (\tau^\mu_{n-i} x^\nu_{n-i} - x^\mu_n x^\nu_n + y^\mu_n y^\nu_n - y^\mu_n y^\nu_n) \right. \\
\times (\tau^\mu_{n-i} x^\nu_{n-i} - x^\mu_n x^\nu_n + y^\mu_n y^\nu_n + y^\mu_n y^\nu_n - y^\mu_n y^\nu_n) \\
+ 2(x^\mu_n y^\nu_n + y^\mu_n x^\nu_n) (\tau^\nu_{n+i} \tau^\mu_n - \tau^\nu_{n+i} \tau^\mu_n) + 2(y^\mu_n z^\nu_n + z^\mu_n y^\nu_n) (\tau^\nu_{n+i} \tau^\mu_n - \tau^\nu_{n+i} \tau^\mu_n) \\
+ 2 (z^\mu_n z^\nu_n - x^\mu_n x^\nu_n) (\tau^\nu_{n+i} \tau^\mu_n - \tau^\nu_{n+i} \tau^\mu_n) + \frac{a^2 \mu_2}{2} (x^\mu_n x^\nu_n x^\nu_n + y^\mu_n y^\nu_n y^\nu_n) \right] \\
+ \frac{\mu_2}{g^2} \sum_n \left[ 2 (z^\mu_n z^\nu_n - x^\mu_n x^\nu_n) + \text{const.} \right. \quad (18)$$
with an implied sum over repeated superscripts. The parameter \( \mu \) is a soft supersymmetry breaking mass that is inserted to stabilize the vacuum expectation values \( \langle \phi \rangle \) and is tuned to zero as the infinite volume limit is taken. In particular, in our simulations we are careful to respect the relation
\[
\frac{g a^2}{\mu a N} \ll 1.
\] (19)

Further details on this deformation of the bosonic action may be found in [6].

We have studied the distribution \( F_{p,q}(\phi) \) on small lattices for a few choices of the bare parameters. It is convenient to rewrite these parameters in terms of physically meaningful quantities with dimensions of length:
\[
a, \quad g^{-1}_a = (g a)^{-1}, \quad L = N a, \quad \mu^{-1}.
\] (20)

We expect that \( g^{-1}_a \) is a rough measure of the correlation length for the system. Thus, we anticipate that systematics due to latticization and working at finite volume are kept to a minimum provided
\[
a \ll g^{-1}_a \ll L.
\] (21)

The effects of supersymmetry breaking in the infrared are expected to be negligible provided \( g^{-1}_a \ll \mu^{-1} \). In the our simulations, we have examined the following parameter sets:
\[
(N; a, g^{-1}_a, L, \mu^{-1}) = (2; 1/8, 8, 1/4, 1/2), \quad (2; 1/20, 40, 1/10, 4/3), \\
(3; 1/20, 40, 3/20, 2), \quad (3; 2/3, 1, 2, 2), \\
(4; 1/2, 1, 2, 2).
\] (22)

The first three sets suffer from extreme finite volume effects, but the effects of discretization are expected to be rather small since \( a g_2 \leq 1/64 \). Supersymmetry breaking is expected to be small in the second and third sets since \( \mu g^{-1}_2 \leq 3/160 \). The last two cases respect a weaker version of the “physical” constraint \( \mu^{-1} \), namely \( a < g^{-1}_2 < L \), and simultaneously a weaker version of \( \mu^{-1} \), namely \( g a^2 < \mu a N = 1 \). (The conditions were of necessity weakened, due to the small lattices we are presently working with.) In all cases, we find that the distribution is uniform; that is, up to statistical fluctuations, the results are indistinguishable from those of Fig. [1].

6 Conclusions

Naturally we would like to study the phase-quenched distribution \( F_{p,q}(\phi) \) on larger lattices where we can better approach the continuum limit. However, these simulations are rather demanding since they involve a fermion determinant; we leave such
an effort to future work, as it will take some time to accumulate the necessary lattice
data and develop efficient algorithms. Moreover, it is not at all clear that the phase of
the determinant over all lattice fermion modes, over all nonzero measure boson con-
figurations in the phase-quenched ensemble, is entirely relevant to the phase of the
continuum limit fermion determinant. Only those modes that have a significant over-
lap with the low energy interpolating fields are expected to have physical meaning.
The other modes integrate out without significantly affecting correlation functions of
physical operators. This is closely related to the distribution factorization method
mentioned above, in the case where the operator $O$ appearing in (15) corresponds to
a good approximation to an operator in the target theory with external momentum
scales well below $a^{-1}$. Research in this direction is underway.

Alternatively, it may be hoped that a different orbifold may preserve the reflection
positivity of the mother theory while still producing an effective lattice theory with
exact supersymmetry. We suspect that the fermion determinant would be positive in
such a construction. In the present construction it seems that reflection positivity is
only recovered in the continuum limit. We are also pursuing this notion in ongoing
research.

The complex action problem encountered here provides an amusing opportunity.
Some exact results are available for the (4,4) 2d super-Yang-Mills theories. In partic-
ular, the 1-loop effective action has been determined [29], modulo higher derivative
terms. It would be interesting to compare the results of complex action simulation
techniques to analytic results such as these.\footnote{Exacting tests of the simulation techniques would of course require reliable nonperturbative analytic results in the (4,4) 2d super-Yang-Mills theories. This should be possible by a dimensional reduction of the Seiberg-Witten results [30].} It may be that the supersymmetric
lattice actions proposed here provide an independent check on Monte Carlo methods
that have been suggested in the simulation of QCD at finite temperature and baryon
density.

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