A NOTE ON POLYNOMIAL EQUATIONS OVER ALGEBRAS

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Abstract. We provide sufficient conditions for systems of polynomial equations over general (real or complex) algebras to have a solution. This generalizes known results on quaternions, octonions and matrix algebras. We also generalize the fundamental theorem of algebra for quaternions to polynomials with two monomials in the leading form, while showing that it fails for three.

1. Introduction

By the fundamental theorem of algebra, every non-constant polynomial has a zero over $\mathbb{C}$. Over the reals, only polynomials of odd degree are guaranteed to have zeros, by the intermediate value theorem. But how about other real or complex algebras?

A beautiful result for quaternions and octonions was proven in [1], see also [4]: Every non-constant polynomial, whose leading form consists of only one monomial, has a zero. So in this regard, quaternions/octonions behave like complex numbers, although they are non-commutative/even non-associative. Another such result is [2]: Every odd-degree Hermitian polynomial over the complex matrix algebra, whose leading form is non-degenerate, has a zero in Hermitian matrices. This resembles exactly the behavior of real numbers.

In this note, we study and extend these results to general algebras over algebraically closed and real closed fields, and to systems of multivariate polynomial equations.

First, for algebras over an algebraically closed field, Theorem 3.1 gives a sufficient condition for a system of non-constant polynomial equations to have a solution. The two crucial requirements are non-degenerateness of the leading form, and some finite-dimensionality condition on the polynomials. Both conditions are necessary for the result to hold. In the proof we reduce the problem to a suitable system of classical polynomial equations, whose solvability is well-known.

Second, we prove the same result for algebras over real closed fields, given that the polynomials are of odd degree (Theorem 3.3). Here we give two different proofs. One reduces the problem again to classical polynomials over the ground field, the other uses methods from algebraic topology as in [1, 2] and one of the standard proofs of the fundamental theorem of algebra. This proof is a simplification of the one in [2], and proves a much more general result. In particular, it also applies to quaternions and octonions, and yields a significant generalization of [1] in case of odd degree.

Finally, we consider quaternions and octonions in more detail. For even degree polynomials, we extend the results from [1, 4] to polynomials with two monomials in the leading form.
form. Maybe surprisingly, we then show that the result fails for leading forms of three monomials. So there are indeed (non-degenerate) polynomials over quaternions/octonions without a zero.

This paper is structured as follows. In Section 2 we explain the setup of polynomials over algebras. In Section 3 we prove the general results Theorem 3.1 and Theorem 3.3. Section 4 contains the results about quaternions and octonions.

2. Polynomials over algebras

Let $\mathcal{A}$ be an algebra over the infinite field $k$. By definition, this means that $\mathcal{A}$ is a $k$-vector space, equipped with a $k$-bilinear map $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, called the multiplication of $\mathcal{A}$. We do not assume the multiplication to be associative or commutative, nor do we assume the existence of a unit 1. For the product of two elements $a, b \in \mathcal{A}$ we will just write $ab$.

In this setup we now define (multivariate) polynomial maps.

**Definition 2.1.** (i) Monomial maps are defined recursively:

- All constant maps $\mathcal{A} \rightarrow \mathcal{A}$ are monomial.
- For all $n \geq 1$ and $i = 1, \ldots, n$, the projection $\pi_i: \mathcal{A}^n \rightarrow \mathcal{A}$ to the $i$-th component is monomial (note that this includes the identity $id_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A}$).
- The multiplication $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is monomial.
- If $p_1, \ldots, p_m: \mathcal{A}^n \rightarrow \mathcal{A}$ and $q: \mathcal{A}^m \rightarrow \mathcal{A}$ are monomial, then so is

  \[ q(p_1, \ldots, p_m): \mathcal{A}^n \rightarrow \mathcal{A} \]

  \[ a \mapsto q(p_1(a), \ldots, p_m(a)). \]

(ii) A polynomial map is a finite $k$-linear combination of monomial maps (inside the $k$-vector space of maps from $\mathcal{A}^n$ to $\mathcal{A}$).

(iii) A zero of a map $p: \mathcal{A}^n \rightarrow \mathcal{A}$ is an element $a \in \mathcal{A}^n$ with $p(a) = 0_\mathcal{A}$.

**Remark 2.2.** (i) Monomial maps can also be defined as follows. A non-associative word is a finite string of variables and elements from $\mathcal{A}$, but equipped with a sensible bracketing. Sensible in this context means that it possible to actually compute the expression, whenever the variables are replaced by algebra elements. One example is

\[ (xy)(((ax)(gb))x) \]

where $a, b \in \mathcal{A}$ and $x, y$ are variables. Plugging in elements for the variables then defines a map, and the so-defined maps are precisely the monomial maps in our above sense. Since the notion of a non-associative word is pretty technical to define exactly, and also different words can define the same map, we will stick to the above definition of monomial and polynomial maps.

(ii) If $\mathcal{A}$ is associative, monomial maps become easier to express. One can use a word composed of variables and algebra elements without brackets. If the algebra is even unital, each monomial map has a representation as

\[ (a_1, \ldots, a_n) \mapsto c_0 a_{i_1} c_1 a_{i_2} c_2 \cdots c_{d-1} a_{i_d} c_d \]
with coefficients $c_j \in \mathcal{A}$. If the algebra is not unital, one also has to allow $a_i$'s without separation by coefficients $c_i$.

(iii) If $\mathcal{A}$ is associative and commutative, monomial maps take the classical form

$$(a_1, \ldots, a_n) \mapsto ca_1^{d_1} \cdots a_n^{d_n}$$

with $c \in \mathcal{A}$. Again, if $\mathcal{A}$ is not unital, one also has to allow the same rule without a coefficient $c$ in front.

**Definition 2.3.** Using that $k$ is infinite, it's easy to see that nonzero monomial maps are **homogeneous** of a unique degree $d$, i.e. fulfill

$$p(\lambda a) = \lambda^dp(a)$$

for all $\lambda \in k, a \in \mathcal{A}^n$. In fact they are even **multi-homogeneous**, i.e. homogeneous with respect to every single argument separately. The zero map is defined to be homogeneous of every degree. Every polynomial map is a unique finite sum of nonzero homogeneous polynomial maps of different degrees, called its homogeneous parts or forms. The degree of a polynomial map is the largest degree among its homogeneous parts, and its leading form is the homogeneous part of largest degree.

**Example 2.4.** A polynomial map in one variable over an associative algebra $\mathcal{A}$ is for example given by

$$a \mapsto c_1ac_2ac_3 + c_4ac_5ac_6 + c_7a + ac_8 + c_9$$

where all $c_i \in \mathcal{A}$. The first two terms constitute the leading form, i.e. the homogeneous part of degree 2, the third and fourth term are the homogeneous part of degree 1, and $c_9$ is the homogeneous part of degree 0. In two variables, one example of a polynomial map is

$$(a, b) \mapsto c_1ac_2bc_3ac_4 + bc_5b + ac_6b + c_7.$$ 

**Remark 2.5.** Let $p: \mathcal{A}^n \to \mathcal{A}$ be a polynomial map. It is easily seen along the recursive definition, that for any finite-dimensional subspace $\mathcal{H} \subseteq \mathcal{A}$, there exists another finite-dimensional subspace $\mathcal{H}' \subseteq \mathcal{A}$ with $p(\mathcal{H}') \subseteq \mathcal{H}'$. Also, if $b_1, \ldots, b_d$ is a basis of $\mathcal{H}$ and $b'_1, \ldots, b'_e$ is a basis of $\mathcal{H}'$, then

$$p(a) = \sum_{j=1}^{e} h_j(a)b'_j$$

for all $a \in \mathcal{H}^n$, where the $h_j$ are classical polynomials over $k$ in the variables $\lambda_{k\ell}$, when $a$ is expressed in the basis as follows:

$$\mathcal{H}^n \ni a = (a_1, \ldots, a_n) = \left( \sum_{k=1}^{d} \lambda_{k1}b_k, \ldots, \sum_{k=1}^{d} \lambda_{kn}b_k \right).$$

The homogeneous parts of $p$ correspond to the homogeneous parts of the $h_j$ of the same degree.
3. Solutions to polynomial equations

The following is our main result on solutions of polynomial equations over algebraically closed fields.

**Theorem 3.1.** Let $A$ be an algebra over the algebraically closed field $k$, and let $p_1, \ldots, p_n: A^n \to A$ be polynomial maps of positive degree. Assume $H \subseteq A$ is a finite-dimensional subspace, such that the leading forms of the $p_i$ are non-degenerate on $H$ (i.e. they have no common zero in $H^n \setminus \{0\}$). Further assume there exists another subspace $H' \subseteq A$ with $\dim(H') \leq \dim(H)$ and $p_i(H^n) \subseteq H'$ for $i = 1, \ldots, n$.

Then $p_1, \ldots, p_n$ have a common zero in $H^n$.

**Proof.** Choose bases $b_1, \ldots, b_d$ of $H$ and $b'_1, \ldots, b'_e$ of $H'$, and obtain

$$p_i(a) = \sum_{j=1}^e h_{ji}(a) b'_j,$$

where all $h_{ji}$ are classical polynomials over $k$, when expressed in the coefficients $\lambda_{k\ell}$ of $a \in H^n$ with respect to $b_1, \ldots, b_d$.

Now setting all $p_i$ to zero on $H^n$ means setting all $h_{ji}$ to zero, i.e. we solve a system of $ne \leq nd$ many polynomial equations in $nd$ variables. The homogeneous parts of degree $\deg(p_i)$ of each $h_{ji}$ must be nonzero, because otherwise we had strictly less than $nd$ of these nontrivial homogeneous equations, which would admit a nontrivial solution over $k$, see for example [5], Chapter 1, Section 6. This would give rise to a common zero of the leading forms of the $p_i$ in $H^n \setminus \{0\}$, which we have excluded. So all of our $ne$ many polynomial equations have positive degree (we also see here that non-degeneratenes in fact implies $\dim(H) = d = e = \dim(H')$).

After homogenizing the whole system with an additional variable, we have $nd$ many (nonconstant) equations in $nd+1$ variables, and this system has a nontrivial solution over $k$, as above. But the value of the additional variable in such a solution must be nonzero, because otherwise we again had a common zero in $H^n \setminus \{0\}$ of the leading forms. Thus we can assume that the additional variable takes the value 1, and this gives rise to a common zero of the $p_i$ in $H^n$, as desired. \hfill $\square$

An obvious corollary of Theorem 3.1 is the following:

**Corollary 3.2.** Let $A$ be a finite-dimensional algebra over the algebraically closed field $k$, and let $p_1, \ldots, p_n: A^n \to A$ be polynomial maps of positive degree, whose leading forms are non-degenerate on $A^n$. Then $p_1, \ldots, p_n$ have a common zero in $A^n$.

Our next main result is a version for real closed fields.

**Theorem 3.3.** Let $A$ be an algebra over the real closed field $k$, and let $p_1, \ldots, p_n: A^n \to A$
be polynomial maps of odd degree. Assume \( \mathcal{H} \subseteq \mathcal{A} \) is a finite-dimensional subspace, such that the leading forms of the \( p_i \) are non-degenerate on \( \mathcal{H} \) (i.e. they have no common zero in \( \mathcal{H}^n \setminus \{0\} \)). Further assume there exists another subspace \( \mathcal{H}' \subseteq \mathcal{A} \) with \( \dim(\mathcal{H}') \leq \dim(\mathcal{H}) \) and
\[
p_i(\mathcal{H}^n) \subseteq \mathcal{H}' \quad \text{for } i = 1, \ldots, n.
\]
Then \( p_1, \ldots, p_n \) have a common zero in \( \mathcal{H}^n \).

1st proof of Theorem 3.3. One proceeds exactly as in the proof of Theorem 3.1. This time one uses Theorem 4.3 in [5] to obtain a solution of the derived polynomial equations over \( k \). Note that the result is stated only for the field of real numbers there, but it immediately transfers to any real closed field by Tarski’s Transfer Principle (see for example [3]). □

The above used argument on real solutions of polynomial equations of odd degree relies on Bézout’s Theorem. But it can also be proven with methods from algebraic topology. Since this proof applies directly in our more general setup, we include it here as an alternative. It is also the approach used in [1, 2] and a standard proof of the fundamental theorem of algebra. Note that it is again enough to prove the result for \( k = \mathbb{R} \), since it transfers to any real closed field by Tarski’s Transfer Principle.

2nd proof of Theorem 3.3. Assume for contradiction that the polynomial maps \( p_i \) do not have a common zero in \( \mathcal{H}^n \). Then the continuous map
\[
p: \mathcal{H}^n \to \mathcal{H}'^n
a \mapsto (p_1(a), \ldots, p_n(a))
\]
does not attain the value zero. Thus for \( t \in [0, \infty) \) we can consider the well-defined map
\[
p^{(t)}: S \to S'
a \mapsto \frac{p(ta)}{\|p(ta)\|}
\]
where \( \| \cdot \| \) is any norm on the real vector space \( \mathcal{H}'^n \), and \( S, S' \) are the unit spheres (choose an arbitrary norm in \( \mathcal{H}'^n \) as well). Denote the leading form of \( p_i \) by \( p_i^{\max} \), consider the map
\[
p^{\max} = (p_1^{\max}, \ldots, p_n^{\max})
\]
and
\[
p^{\infty}: S \to S'
a \mapsto \frac{p^{\max}(a)}{\|p^{\max}(a)\|}
\]
which is also well-defined, since the leading forms are non-degenerate. The family \( p^{(t)} \) clearly defines a homotopy between \( p^{(0)} \) and \( p^{\infty} \). Since \( p^{(0)} \) is constant, it has topological degree 0. On the other hand, \( p^{\infty} \) is an odd map, which by Borsuk’s Antipodal Theorem has odd degree, see for example [6], Theorem 10.6.3. This is a contradiction. □
Remark 3.4. (i) The non-degenerateness of the leading forms is necessary. Consider the algebra $\mathcal{A} = \text{Mat}_2(k)$ of $2 \times 2$-matrices, and the equation
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} X + \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = 0.
\]
The leading form is degenerate, and the polynomial indeed has no zero in $\mathcal{A}$.

(ii) For infinite-dimensional subspaces $\mathcal{H}$, the above results fail. For example, in the commutative algebra $\mathcal{A} = C([-1,1], \mathbb{K})$ of continuous functions (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), the equation
\[
f x - 1 = 0
\]
has no solution, if $f$ has a zero. But if $f$ has only isolated zeros, then the leading form is non-degenerate.

For $\mathcal{A} = \text{Mat}_m(\mathbb{C})$ and its real subspace $\mathcal{H} = \text{Her}_m(\mathbb{C})$ of Hermitian matrices, we obtain the main result of [2]. Note that the condition self-adjoint means $p(a)^* = p(a^*)$ for all $a \in \mathcal{A}$, which implies that $p$ maps $\mathcal{H}$ to itself.

Corollary 3.5 ([2]). Let $p: \text{Mat}_m(\mathbb{C}) \rightarrow \text{Mat}_m(\mathbb{C})$ be a self-adjoint polynomial map of odd degree, whose leading form is non-degenerate on $\text{Her}_m(\mathbb{C})$. Then $p$ has a zero in $\text{Her}_m(\mathbb{C})$.

We also obtain a multivariate generalization of the odd-degree case of the fundamental theorem of algebra for quaternions/octonions [1,4]. We will elaborate on this in more detail in the following section.

Corollary 3.6. Let $\mathcal{A} \in \{\mathbb{H}, \mathbb{O}\}$ be the real quaternion or octonion algebra. Let $p_1, \ldots, p_n: \mathcal{A}^n \rightarrow \mathcal{A}$ be polynomial maps of odd degree, whose leading forms are non-degenerate on $\mathcal{A}$ (i.e. have no common zero in $\mathcal{A}^n \setminus \{0\}$). Then $p_1, \ldots, p_n$ have a common zero in $\mathcal{A}^n$.

In particular, if the leading form of a single odd-degree polynomial map $p: \mathcal{A} \rightarrow \mathcal{A}$ consists of one monomial, then $p$ has a zero in $\mathcal{A}$.

Proof. The first statement is clear from Theorem 3.3. For the second, note that for $c_i \neq 0$ we have
\[
c_1 a c_2 a \cdots a c_d = 0
\]
only if $a = 0$, since $\mathcal{A}$ is a division algebra. So a single nonzero monomial map is automatically non-degenerate. $\square$

4. QUATERNIONS AND OCTONIONS

For the real division algebras $\mathbb{H}, \mathbb{O}$ of quaternions and octonions, Corollary 3.6 provides solutions to systems of polynomial equations of odd degree. But the fundamental theorem of algebra from [1] (see also [4]) provides solutions also in the even-degree case, at least for univariate polynomials with a single monomial leading form. We generalize this now to two monomials. We make use of the canonical norms on $\mathbb{H}, \mathbb{O}$ throughout.
Theorem 4.1. Let $A \in \{\mathbb{H}, \mathbb{O}\}$ and let $p: A \to A$ be a polynomial map of positive even degree, whose leading form is a sum of at most two monomials, and which is nondegenerate (i.e. has no zero in $A \setminus \{0\}$). Then $p$ has a zero in $A$.

Proof. We proceed as in the topological proof of Theorem 3.3, and only need to show that $p^\infty: S \to S$ has non-zero degree. So let

$$ p^{\text{max}}: a \mapsto m_1(a) + m_2(a) $$

be the leading form of $p$, where $m_1, m_2$ are monomial maps. Let $\|m_i\|$ denote the product of the norms of all coefficients in $m_i$, so that

$$ \|m_i(a)\| = \|m_i\| \cdot \|a\|^{\deg(p)} $$

holds for all $a \in A$. We first assume $\|m_1\| \geq \|m_2\| > 0$ and show that for $t \in [0, 1]$ the map

$$ h_t: a \mapsto m_1(a) + (1 - t)m_2(a) $$

has no zero in $A \setminus \{0\}$. For $t = 0$ this follows from our assumption. For $t > 0$ and $h_t(a) = 0$ we get

$$ \|m_1\| \|a\|^{\deg(p)} = \|m_1(a)\| = \|(t - 1)m_2(a)\| = (1 - t)\|m_2\| \|a\|^{\deg(p)}. $$

For $a \neq 0$ this contradicts $\|m_1\| \geq \|m_2\| > 0$ and thus proves the claim.

So $h_t$ can be restricted to $S$ and normalized, and thus provides a homotopy between $p^\infty$ and the map

$$ a \mapsto \frac{m_1(a)}{\|m_1(a)\|}. $$

We have thus reduced our problem to the case of a single monomial leading form. It was shown in [1] that such maps are homotopic to

$$ a \mapsto a^{\deg(p)} $$

which has degree $\deg(p) \neq 0$ (the proof applies to $A = \mathbb{O}$ as well, see also [4]). \hfill \square

Maybe surprisingly, the last result does not extend to more than two monomials in the leading form.

Example 4.2. On $A = \mathbb{H}$ consider the polynomial map

$$ p: a \mapsto c_0 a^2 + ac_1 a + c_2 ac_3 a + c_4 $$

where $c_0 = -1 - i + k, c_1 = -1 - i + j - k, c_2 = -i - j + k, c_3 = -1 + i + j + k, c_4 = 6i$. To see that the leading form of $p$ is non-degenerate, we can cancel one copy of $a$ from the right, and consider the linear map

$$ a \mapsto c_0 a + ac_1 + c_2 ac_3. $$

Its coefficient matrix in real coordinates $a = a_1 + a_2 i + a_3 j + a_4 k$ is

$$ M = \begin{pmatrix}
-1 & -1 & 0 & 1 \\
-3 & -1 & 1 & 0 \\
4 & 3 & -1 & -1 \\
-1 & 0 & 1 & -5
\end{pmatrix}. $$
which is nonsingular.

In real coordinates, the full map $p$ has the form

$$p: \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \mapsto \begin{pmatrix} -a_2^2 + a_2^2 + a_2^2 + 5a_2^2 + 2a_1a_2 - 4a_1a_3 - 4a_2a_3 + 2a_1a_4 \\ -3a_1^2 - a_2^2 - a_3^2 - a_4^2 - 4a_1a_2 + 2a_1a_3 + 4a_1a_4 + 4a_2a_4 + 4a_3a_4 + 6 \\ 4a_1^2 + 2a_1a_2 - 2a_1a_3 + 2a_1a_4 - 4a_2a_4 \\ a_1^2 + a_3^2 - 4a_1a_2 - 3a_2^2 - 2a_1a_3 - 6a_1a_4 \end{pmatrix}.$$

The ideal generated by these four coordinate functions contains the univariate polynomial

$$216 + 324a_4^2 - 927a_4^4 - 148a_4^6 + 2578a_4^8,$$

which is in fact an element of the reduced Gröbner basis with respect to the lexicographic ordering. If this polynomial had a real zero, then the polynomial

$$q := 216 + 324x - 927x^2 - 148x^3 + 2578x^4$$

would have a positive real zero. But the solution formula for quartic polynomials shows that $q$ has no real zero at all. Another way to show this is to realize that $q$ is in fact a sum of squares:

$$q = q_1^2 + q_2^2 + q_3^2$$

where

$$q_1 = -\frac{49212319}{478950\sqrt{6}}x^2 + 9\sqrt{\frac{3}{2}}x + 6\sqrt{6}$$

$$q_2 = \frac{124008757}{10\sqrt{188107618886}}x^2 + \frac{1}{5\sqrt{2}}\frac{29456251}{6386}x$$

$$q_3 = \frac{\sqrt{17668412385769586257}}{88368753}x^2.$$

Now $q$ vanishes at a real point if and only if all three $q_i$ vanish, which is clearly not true for any real point.

So in total we have shown that the polynomial $p$ has no zero in $\mathbb{H}$, although its leading form is non-degenerate.

Remark 4.3. The proofs of [1] and Theorem 4.1 provide explicit homotopies of the leading form of $p$ to the form $x^{\deg(p)}$, whose degree was shown to be $\deg(p)$ in [1]. It is crucial to have only non-degenerate forms along the homotopy. The multivariate resultant of the four coordinate functions decides non-degenerateness (at least over $\mathbb{C}$). One might expect that its zero locus divides the space of coefficients of forms into several connected components, and thus homotopies to $x^{\deg(p)}$ cannot be expected in general. This is in fact true, as we conclude from the above example. But since homotopies are possible in certain simple cases, the resultant will sometimes show a different behavior.

We illustrate this in a simple example. Consider linear forms of type

$$\ell = c_0x + xc_1$$
with \( c_0, c_1 \in \mathbb{H} \). When expressing the coefficients as \( c_i = c_{i1} + c_{i2}i + c_{i3}j + c_{i4}k \) and the variable as \( x = x_1 + x_2i + x_3j + x_4k \), the four coordinate functions of \( \ell \) are linear in the \( x_i \), and the resultant is thus the determinant of the coefficient matrix

\[
M = \begin{pmatrix}
c_{01} + c_{11} & -(c_{02} + c_{12}) & -(c_{03} + c_{13}) & -(c_{04} + c_{14}) \\
(c_{02} + c_{12}) & c_{01} + c_{11} & c_{14} - c_{04} & c_{03} - c_{13} \\
c_{03} + c_{13} & c_{04} - c_{14} & c_{01} + c_{11} & c_{12} - c_{02} \\
c_{04} + c_{14} & c_{13} - c_{03} & c_{02} - c_{12} & c_{01} + c_{11}
\end{pmatrix}.
\]

A direct computation shows that

\[
\det(M) = (c_{01} + c_{11})^2 \left( (c_{01} + c_{11})^2 + 2(c_{02}^2 + c_{03}^2 + c_{04}^2 + c_{12}^2 + c_{13}^2 + c_{14}^2) \right)
+ (c_{02}^2 + c_{03}^2 + c_{04}^2 - c_{12}^2 - c_{13}^2 - c_{14}^2)^2
\]

is a sum of squares. Thus the form \( \ell \) is degenerate if and only if

\[
c_{01} = -c_{11} \quad \text{and} \quad c_{02}^2 + c_{03}^2 + c_{04}^2 = c_{12}^2 + c_{13}^2 + c_{14}^2.
\]

This defines an algebraic subset of the coefficient space \( \mathbb{R}^8 \) of codimension 2, and does not divide it into several connected components. A similar behavior must occur for forms of higher degree, as long as there are at most two monomials.

For non-degenerate forms of odd degree, the proof of Theorem 3.3 shows that the degree of the induced mapping is non-zero, independent of the number of monomials. It does however not show that the forms are homotopic to \( x^{\deg(p)} \), and thus does not imply that the resultant must show such a behavior.

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