ON THE MOTIVE OF KAPUSTKA–RAMPAZZO’S CALABI-YAU THREEFOLDS

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ABSTRACT. Kapustka and Rampazzo have exhibited pairs of Calabi-Yau threefolds $X$ and $Y$ that are $L$–equivalent and derived equivalent, without being birational. We complete the picture by showing that $X$ and $Y$ have isomorphic Chow motives.

1. INTRODUCTION

Let $\text{Var}(\mathbb{C})$ denote the category of algebraic varieties over the field $\mathbb{C}$. The Grothendieck ring $K_0(\text{Var}(\mathbb{C}))$ encodes fundamental properties of the birational geometry of varieties. The intricacy of the ring $K_0(\text{Var}(\mathbb{C}))$ is highlighted by the result of Borisov [7], showing that the class of the affine line $\mathbb{A}$ is a zero–divisor in $K_0(\text{Var}(\mathbb{C}))$. Following on Borisov’s pioneering result, a great many people have been hunting for Calabi–Yau varieties $X, Y$ that are not birational (and so $[X] \neq [Y]$ in the Grothendieck ring), but

$$([X] - [Y])\mathbb{L}^r = 0 \quad \text{in} \quad K_0(\text{Var}(\mathbb{C})),$$

i.e., $X$ and $Y$ are “$L$–equivalent” in the sense of [19]. In many cases, the captured varieties $X$ and $Y$ are also derived equivalent [13], [14], [23], [18], [27], [8], [19], [10], [22], [17], [16].

According to a conjecture made by Orlov [26, Conjecture 1], derived equivalent smooth projective varieties should have isomorphic Chow motives. This conjecture is true for $K3$ surfaces [12], but is still open for Calabi–Yau varieties of dimension $\geq 3$. In [20], I verified Orlov’s conjecture for the Calabi–Yau threefolds of Ito–Miura–Okawa–Ueda [13]. The aim of the present note is to check that Orlov’s conjecture is also true for the threefolds constructed recently by Kapustka–Rampazzo:

**Theorem** (= theorem 4.1). Let $X, Y$ be two derived equivalent Calabi–Yau threefolds as in [17]. Then there is an isomorphism of Chow motives

$$h(X) \cong h(Y) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.$$

To prove theorem 4.1, we exploit the “homological projective duality–style” diagram given in [17] relating $X$ and $Y$. One key ingredient in the proof that might be of independent interest is a result (theorem 3.3) concerning higher Chow groups of certain fibrations; this is a variant of a result of Vial’s [33].

Conventions. In this note, the word variety will refer to a reduced irreducible scheme of finite type over the field of complex numbers $\mathbb{C}$. For any variety $X$, we will denote by $A_j(X)$ the

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Chow group of dimension \( j \) cycles on \( X \) with \( \mathbb{Q} \)-coefficients. For \( X \) smooth of dimension \( n \), the notations \( A_j(X) \) and \( A^{n-j}(X) \) will be used interchangeably.

The notation \( A^j_{\text{hom}}(X) \) will be used to indicate the subgroups of homologically trivial cycles. For a morphism between smooth varieties \( f: X \to Y \), we will write \( \Gamma_f \in A^*(X \times Y) \) for the graph of \( f \), and \( \Gamma_f^\top \in A^*(Y \times X) \) for the transpose correspondence.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in \([30], [25]\)) will be denoted \( M_{\text{rat}} \).

2. The Calabi–Yau threefolds

**Theorem 2.1** (Kapustka–Rampazzo \([17]\)). Let \( X, Y \) be a general pair of Calabi–Yau threefolds in the family \( \tilde{X}_{25} \) that are dual to one another (in the sense of \([17\) Section 2]). Then \( X \) and \( Y \) are not birational, and so

\[
[X] \neq [Y] \quad \text{in} \quad K_0(\text{Var}(\mathbb{C})).
\]

However, one has

\[
([X] - [Y])L^2 = 0 \quad \text{in} \quad K_0(\text{Var}(\mathbb{C})).
\]

Moreover, \( X \) and \( Y \) are derived equivalent, i.e. there is an isomorphism of bounded derived categories

\[
D^b(X) \cong D^b(Y).
\]

In particular, there is an isomorphism of polarized Hodge structures

\[
H^3(X, \mathbb{Z}) \cong H^3(Y, \mathbb{Z}).
\]

**Proof.** Everything but the last phrase is in \([17]\). The isomorphism of Hodge structures is a corollary of the derived equivalence, in view of \([27\) Proposition 2.1 and Remark 2.3]. \(\square\)

**Remark 2.2.** As explained in \([17]\), the threefolds \( X, Y \) in the family \( \tilde{X}_{25} \) are a limit case of the Calabi–Yau threefolds in the family \( X_{25} \) studied in \([8], [27]\). A pair of dual varieties \( X, Y \) in the family \( X_{25} \) are also derived equivalent and L-equivalent (the exponent of \( L \) is, however, higher than in theorem 2.1).

3. Higher Chow groups and fibrations

**Definition 3.1** (Bloch \([4], [5]\)). Let \( \Delta^j \cong A_j(\mathbb{C}) \) denote the standard \( j \)-simplex. For any quasi-projective variety \( M \) and any \( i \in \mathbb{Z} \), let \( z_i^{\text{simp}}(M, *) \) denote the simplicial complex where \( z_i(X, j) \) is the group of \((i + j)\)-dimensional algebraic cycles in \( M \times \Delta^j \) that meet the faces properly. Let \( z_i(M, *) \) denote the single complex associated to \( z_i^{\text{simp}}(M, *) \). The higher Chow groups of \( M \) are defined as

\[
A_i(M, j) := H^j(z_i(M, *) \otimes \mathbb{Q}).
\]

**Remark 3.2.** Clearly one has \( A_i(M, 0) \cong A_i(M) \). Higher Chow groups are related to higher algebraic \( K \)-theory: there are isomorphisms

\[
\text{Gr}^{\gamma-i}_K J_j (M)_{\mathbb{Q}} \cong A_i(M, j) \quad \text{for all} \quad i, j
\]

where \( K_j(M) \) is Quillen’s higher \( K \)-theory group associated to the category of coherent sheaves on \( M \), and \( \text{Gr}^{\gamma}_K \) is a graded for the \( \gamma \)-filtration \([4]\). Higher Chow groups are also related to
Voevodsky’s motivic cohomology (defined as hypercohomology of a certain complex of Zariski sheaves) [9], [24].

For later use, we establish the following result, which is a variant of a result of Vial’s [33]:

**Theorem 3.3.** Let $\pi : M \to B$ be a flat projective morphism between smooth quasi–projective varieties of relative dimension $m$. Assume that for every $b \in B$, the fibre $M_b := \pi^{-1}(b)$ has

$$A_i(M_b) = \mathbb{Q} \quad \forall i .$$

(i) The maps

$$\Phi_* := \sum_{k=0}^{m} h^{m-k} \circ \pi^*: \bigoplus_{k=0}^{m} A_{\ell-k}(B, j) \to A_{\ell}(M, j)$$

and

$$\Psi_* := \sum_{k=0}^{m} \pi_* \circ h^k: A_{\ell}(M, j) \to \bigoplus_{k=0}^{m} A_{\ell-k}(B, j)$$

are both isomorphisms, for any $\ell$ and $j$. (Here $h^k$ denotes the operation of intersecting with the $k$–th power of a hyperplane section $h \in A^1(M)$.)

(ii) Set $V_k := (\Psi_*)^{-1} A_{\ell-k}(B, j) \subset A_{\ell}(M, j)$. Then

$$(\Phi_* \Psi_*)|_{V_m} = \lambda \text{id},$$

for some non–zero $\lambda \in \mathbb{Q}$.

**Proof.** (i) For $j = 0$ (i.e., for usual Chow groups), this is exactly [33, Theorem 3.2]. For arbitrary $j$ (i.e., for higher Chow groups), a straightforward although laborious proof would consist in convincing the reader that everything Vial does in the proof of [33, Theorem 3.2] also applies to higher Chow groups. Indeed, all formal properties of Chow groups exploited in loc. cit. also hold for higher Chow groups.

Under the simplifying assumption that all fibres $M_b$ are isomorphic to $\mathbb{P}^m$ (which will be the case when we apply theorem 3.3 in this note), a quick proof could be as follows. Let $H \subset M$ be a general hyperplane section, and let $U = \mathbb{P}^m \cap B$ be the open over which the fibres of the restricted morphism $\pi|_U: H \to B$ are isomorphic to $\mathbb{P}^{m-1}$. Let $M_U := \pi^{-1}(U)$, and let us consider the restricted morphism

$$\pi|_U : M_U \to U .$$

Using the localization sequence for higher Chow groups and noetherian induction, we are reduced to proving (i) for $\pi|_U$. Let us consider the open $M'_U := M_U \setminus (H \cap M_U)$. The fibres of the morphism $\pi': M'_U \to U$ are isomorphic to $\mathbb{A}^m$. There is a commutative diagram with exact rows

$$\begin{array}{cccccc}
A_i(M'_U, j + 1) & \rightarrow & A_i(H, j) & \rightarrow & A_i(M_U, j) & \rightarrow \\
\uparrow (\pi')^* & & \uparrow \sum_{k=0}^{m} h^{m-1-k} o(\pi|_U)^* & & \uparrow \sum_{k=0}^{m} h^{m-k} o(\pi|_U)^* & \\
A_i(U, j + 1) & \rightarrow & \bigoplus_{k=0}^{m-1} A_k(U, j) & \rightarrow & \bigoplus_{k=0}^{m} A_k(U, j) & \rightarrow \\
\end{array}$$
Doing an induction on the fibre dimension $m$, it will suffice to prove that $(\pi')^*$ is an isomorphism for all $i, j$. But this follows from the corresponding result for $K$–theory [29, Proposition 4.1], in view of the isomorphism (1) and the fact that the pullback $(\pi')^*: K_j(U) \to K_j(M'_U)$ respects the $\gamma$–filtration. This proves that $\Phi_*$ is an isomorphism. The argument for $\Psi_*$ is similar.

(ii) The direct summand $V_m$ can be identified as

$$V_m = \bigcap_{k=0}^{m-1} \ker(\pi_* \circ h^k) \subset A_\ell(M, j).$$

Using this description, it is readily checked that

$$V_m = \pi^* A_{\ell-m}(X, j).$$

This implies (ii).

\[\square\]

Remark 3.4. In case $B$ and $M$ are smooth projective, theorem 3.3 can be upgraded to a relation of Chow motives [33, Theorem 4.2]. In the more general case where $B$ and $M$ are smooth but only quasi–projective, perhaps one can relate $B$ and $M$ in the category $DM_{eff}^{gm}$ of Voevodsky motives? If so, the relation of higher Chow groups obtained in theorem 3.3 would be an immediate consequence, since higher Chow groups (with $\mathbb{Q}$–coefficients) can be expressed as Hom–groups in $DM_{eff}^{gm}$ [9], [24].

4. MAIN RESULT

Theorem 4.1. Let $X, Y$ be a pair of Calabi–Yau threefolds in the family $\bar{X}_{25}$ that are dual to one another, in the sense of [17, Section 2]. Then there is an isomorphism

$$h(X) \cong h(Y) \text{ in } \mathcal{M}_{\text{rat}}.$$

Proof. First, to simplify matters, let us slightly cut down the motives of $X$ and $Y$. It is known [17] that $X$ and $Y$ have Picard number 1. A routine argument gives a decomposition of the Chow motives

$$h(X) = 1 \oplus 1(1) \oplus h^3(X) \oplus 1(2) \oplus 1(3),$$
$$h(Y) = 1 \oplus 1(1) \oplus h^3(Y) \oplus 1(2) \oplus 1(3) \text{ in } \mathcal{M}_{\text{rat}},$$

where 1 is the motive of the point $\text{Spec}(k)$. (The gist of this “routine argument” is as follows: let $H \in A^1(X)$ be a hyperplane section. Then

$$\pi^{2i}_X := c_i H^{3-i} \times H \in A^3(X \times X), \ 0 \leq i \leq 3,$$

defines an orthogonal set of projectors lifting the Küneth components, for appropriate $c_i \in \mathbb{Q}$. One can then define $\pi^i_X := \Delta_X - \sum_i \pi^{2i}_X \in A^3(X \times X)$, and $h^i(X) = (X, \pi^i_X, 0) \in \mathcal{M}_{\text{rat}}$, and ditto for $Y$.)

To prove the theorem, it will thus suffice to prove there is an isomorphism of motives

$$h^3(X) \cong h^3(Y) \text{ in } \mathcal{M}_{\text{rat}}.$$
We observe that the above decomposition (plus the fact that $H^*(h^3(X)) = H^3(X)$ is odd-dimensional) implies equality

$$A^*(h^3(X)) = A^*_{\text{hom}}(X),$$

and similarly for $Y$.

To construct the isomorphism (2), we need look no further than the construction of the threefolds $X, Y$. As explained in [17, Section 2], the Calabi–Yau threefolds $X, Y$ are related via a diagram

$$
\begin{array}{ccc}
D & \xrightarrow{i} & M \\
\lf \downarrow & & \lf \downarrow g \downarrow \ \\
X & \xleftarrow{f} & G(2, 5) & \xrightarrow{\pi_1} & F & \xrightarrow{\pi_2} & G(3, 5) & \xleftarrow{g_Y} & Y
\end{array}
$$

(3)

Here $G(j, 5)$ denotes the Grassmannian of $j$–dimensional subspaces in a 5–dimensional vector space. The variety $F$ is the flag variety parametrizing pairs $(V, W) \in G(2, 5) \times G(3, 5)$ such that $V \subset W$. The variety $M \subset F$ is a hyperplane section. The Calabi–Yau varieties $X, Y$ are closed subvarieties of $G(2, 5)$ resp. $G(3, 5)$, and the closed subvarieties $D, E$ are defined as $f^{-1}(X)$ resp. $g^{-1}(Y)$. The morphisms $f, g$ are $\mathbb{P}^1$–fibrations over the opens $G(2, 5) \setminus X$ resp. $G(3, 5) \setminus Y$, but the restrictions $f_X, g_Y$ are $\mathbb{P}^1$–fibrations.

The flag variety $F$ has trivial Chow groups (i.e. $A^*_{\text{hom}}(F) = 0$), and so $F$ has a Chow–Künneth decomposition (this is a general fact for any smooth projective variety with trivial Chow groups; since all cohomology is algebraic, a Künneth decomposition exists; since $F \times F$ again has trivial Chow groups, the Künneth decomposition is a Chow–Künneth decomposition). By a standard trick (cf. for instance [15, Lemma 5.2]), this induces a Chow–Künneth decomposition $\{\pi^{i}_M\}$ for the hyperplane section $M \subset F$, with the property that

$$(M, \pi^{i}_M) \cong \oplus \mathbb{I}(\ast) \quad \text{in } \mathcal{M}_{\text{rat}} \quad \text{for all } j \neq 7 = \dim M.$$

In particular, we have that

$$A^i_{\text{hom}}(M) = A^i(h^7(M)) := (\pi^7_M)_*A^i(M) \quad \text{for all } i.$$ 

We now make the following claim:

**Claim 4.2.** There are isomorphisms

$$
\begin{align*}
\Gamma_1: & \quad h^3(X) \xrightarrow{\cong} h^7(M)(2), \\
\Gamma_2: & \quad h^3(Y) \xrightarrow{\cong} h^7(M)(2) \quad \text{in } \mathcal{M}_{\text{rat}}.
\end{align*}
$$

This claim obviously suffices to prove (2). To prove the claim, let us treat the isomorphism $\Gamma_1$ in detail (the same argument applies to $\Gamma_2$, upon replacing $X$ and $G(2, 5)$ by $Y$ resp. $G(3, 5)$). To prove the claim for $\Gamma_1$, it will suffice to find correspondences $\Gamma_1 \in A^5(X \times M), \Psi_1 \in A^5(M \times X)$ with the property that

$$
\begin{align*}
(\Xi_1)_*(\Gamma_1)_* = \text{id}: & \quad A^i_{\text{hom}}(X) \rightarrow A^i_{\text{hom}}(X), \\
(\Gamma_1)_*(\Xi_1)_* = \text{id}: & \quad A^i_{\text{hom}}(M) \rightarrow A^i_{\text{hom}}(M).
\end{align*}
$$

(4)
(Indeed, let us assume one has correspondences $\Gamma_1, \Xi_1$ satisfying (4). By what we have said above, this means that
\begin{align}
(\pi_X^3 \circ \Xi_1 \circ \pi_M^7 \circ \Gamma_1 \circ \pi_X^3)_* &= (\pi_X^3)_*: \ A^i(X) \to A^i(X), \\
(\pi_M^7 \circ \Gamma_1 \circ \pi_X^3 \circ \Xi_1 \circ \pi_M^7)_* &= (\pi_M^7)_*: \ A^i(M) \to A^i(M). 
\end{align}

There exists a field $k \subset \mathbb{C}$, finitely generated over $\mathbb{Q}$, such that $X, M, \pi_X^3, \pi_M^7 \Gamma_1, \Xi_1$ are defined over $k$. Because $\mathbb{C}$ is a universal domain, for any finitely generated field extension $K \supset k$, there is an inclusion $K \subset \mathbb{C}$. Thus, the natural maps $A^i(X_K) \to A^i(X_\mathbb{C})$ and $A^i(M_K) \to A^i(M_\mathbb{C})$ are injections [3, Appendix to Lecture 1]. This implies that the relations (5) also hold over $K$. Manin’s identity principle then gives that
\[ \Gamma_1: \ h^3(X_k) \to h^7(M_k)(2) \text{ in } M_{\text{rat}} \]
is an isomorphism, and so $\Gamma_1$ induces an isomorphism of motives over $\mathbb{C}$ as claimed.)

Before proving the claim, let us introduce some lemmas.

**Lemma 4.3.** Set-up as above. The composition
\[ A^i_{\text{hom}}(X) \xrightarrow{(f_X)^*} A^i_{\text{hom}}(D) \xrightarrow{i_*} A^{i+2}_{\text{hom}}(M) \]
is surjective, for any $i$.

**Proof.** Let us write $U := M \setminus D$, and $G := G(2, 5)$. By assumption, $U$ is a $\mathbb{P}^1$–fibration over $V := G \setminus X$.

For any $i$, there is a commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \to & A_i(V, 1) & \to & A_i(X) & \to & A_i(G) & \to & A_i(V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & W^{-2i}H_{2i-1}(V, \mathbb{Q}) \cap F_{-i} & \to & H_{2i}(X, \mathbb{Q}) \cap F_{-i} & \to & H_{2i}(G, \mathbb{Q}) & \to & W^{-2i}H_{2i}(V, \mathbb{Q})
\end{array}
\]
where vertical arrows are (higher) cycle class maps into Borel–Moore homology, and $W^*, F_*$ denote the weight filtration resp. the Hodge filtration on Borel–Moore homology [28]. (The upper row is exact thanks to localization for higher Chow groups [4], [5], [21]. The lower row is exact because the category of polarizable pure Hodge structures is semisimple [28]. For the cycle class map from higher Chow groups into Borel–Moore homology, cf. [31, Section 4.]) The Grassmannian $G$ has trivial Chow groups. Using the fact that the Hodge conjecture is true for the threefold $X$, this implies that the cycle class map induces isomorphisms
\[ A^i(V) \xrightarrow{\cong} W^{-2i}H^{2i}(V, \mathbb{Q}), \]
and the higher cycle class map induces a surjection
\[ A_i(V, 1) \to W^{-1-2i}H_{2i-1}(V, \mathbb{Q}) \cap F_{1-2i}. \]
(These two facts together can be paraphrased by saying that $V$ satisfies a variant\(^1\) of the “strong property” of Totaro’s [31, Section 4.]) Using theorem 3.3 plus the corresponding property of cohomology, this implies that $U$ has the same property (i.e., $U$ satisfies the strong property).

\(^1\)It is a variant, because in [31] only the weight filtration and not the Hodge filtration is taken into account. This works fine for the linear varieties considered in [31], but not for the varieties $U, V$ under consideration here.
For any $i$, there is a commutative diagram with exact rows
\[
\begin{array}{cccccc}
\to & A_i(U, 1) & \to & A_i(D) & \to & A_i(M) & \to & A_i(U) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
W^{-2i}H_{2i-1}(U, \mathbb{Q}) \cap F^{-i} & \to & H_{2i}(D, \mathbb{Q}) \cap F^{-i} & \to & H_{2i}(M, \mathbb{Q}) & \to & W^{-2i}H_{2i}(U, \mathbb{Q})
\end{array}
\]
By what we have just observed (the strong property for $U$), the left vertical arrow is a surjection and the right vertical arrow is an isomorphism. A quick diagram chase then reveals that pushforward induces a surjection
\[
(6) \quad i_* : \ A_i^{\hom}(D) \to A_i^{\hom}(M) \quad \forall i .
\]
Next, since $f_X : D \to X$ is a $\mathbb{P}^2$-fibration, theorem 3.3 ensures that there are isomorphisms
\[
\Phi_* := \sum_{k=0}^2 h^{2-k} \circ (f_X)^*: \bigoplus_{k=0}^2 A_i^{\hom}(X) \xrightarrow{\cong} A_i^{\hom}(D) ,
\]
\[
\Psi_* := \sum_{k=0}^2 (f_X)_* \circ h^k: \ A_i^{\hom}(D) \xrightarrow{\cong} \bigoplus_{k=0}^2 A_i^{\hom}(X) .
\]
We write
\[
A_i^{\hom}(D) = V_0 \oplus V_1 \oplus V_2
\]
\[
(7) \quad := (\Psi^*)^{-1}A_i^{\hom}(X) \oplus (\Psi^*)^{-1}A_{i-1}^{\hom}(X) \oplus (\Psi^*)^{-1}A_{i-2}^{\hom}(X) .
\]
To prove the lemma, it remains to understand the pushforward map (6). Precisely, we will show that one summand of the decomposition (7) already surjects onto $A_i^{\hom}(M)$:
\[
\text{Im}(V_0 \oplus V_1 \xrightarrow{i_*} A_i^{\hom}(M)) \subset \text{Im}(V_2 \xrightarrow{i_*} A_i^{\hom}(M)) \quad \text{for all } i .
\]
To see this, we observe that there is a commutative diagram of complexes
\[
\begin{array}{ccccccc}
z_i(D, \ast) & \to & z_i(M, \ast) & \to & z_i(U, \ast) \\
\downarrow & & \downarrow & & \downarrow & \\
z_i(X, \ast) & \to & z_i(G(2, 5), \ast) & \to & z_i(V, \ast)
\end{array}
\]
(where the vertical arrows are proper pushforward maps). This gives rise to a commutative diagram with long exact rows
\[
\begin{array}{cccccc}
\to & A_i(U, 1) & \delta' & \to & A_i(D) & \to & A_i(M) & \to \\
\downarrow & (f_{U^*})_* & & \downarrow & (f_X)_* & & \downarrow & f_* & \\
\to & A_i(V, 1) & \delta' & \to & A_i(X) & \to & A_i(G(2, 5)) & \to
\end{array}
\]
Let us now assume $b \in A_i^{\hom}(D)$ lies in the summand $V_0$ of the decomposition (7). Then $(f_X)_*(b)$ is in $A_i^{\hom}(X)$. Since $A_i^{\hom}(G(2, 5)) = 0$, this means that $(f_X)_*(b)$ is in the image of the map $\delta'$, say $(f_X)_*(b) = \delta'(c')$. In view of theorem 3.3, the element $c' \in A_i(V, 1)$ comes from an element $c \in A_i(U, 1)$ lying in the direct summand (isomorphic to) $A_i(V, 1)$. Using sublemma 4.4 below, this means that there is equality
\[
\delta(c) = b - b_2 \quad \text{in } A_i(D) ,
\]
for some \( b_2 \in A_i(D) \) lying in the summand (isomorphic to) \( A_{i-2}(X) \). It follows that
\[
i_*(b) = i_*(b_2) \in \text{Im} \left( A_{i-2}(X) \to A_i(D) \to A_i(M) \right).
\]

As \( i_*(b_2) = i_*(b) \) is homologically trivial, the surjection (6) above shows that we may suppose \( b_2 \) is homologically trivial, and so we have found \( b_2 \) lying in the summand denoted \( V_2 \) (isomorphic to \( A_{i-2}^{\text{hom}}(X) \)). This shows that
\[
i_*(b) \in \text{Im} \left( A_{i-2}^{\text{hom}}(X) \to A_i(D) \to A_i(M) \right) =: \text{Im}(V_2 \to A_i(M)).
\]

Let us next assume that \( b \in A_i^{\text{hom}}(D) \) lies in the summand \( V_1 \) of the decomposition (7). The commutative diagram of complexes up to quasi–isomorphism
\[
z_i(D, *) \to z_i(M, *) \to z_i(U, *) \downarrow h \downarrow h \downarrow h
\]
\[
z_{i-1}(D, *) \to z_{i-1}(M, *) \to z_{i-1}(U, *) \downarrow \downarrow \downarrow \downarrow
\]
\[
z_{i-1}(X, *) \to z_{i-1}(G(2, 5), *) \to z_{i-1}(V, *)
\]
gives rise to a commutative diagram with exact rows
\[
\begin{array}{cccc}
A_i(U, 1) & \delta & A_i(D) & \to A_i(M) & \to \\
\downarrow (f|_U)_* \text{oh} & & \downarrow (f_X)_* \text{oh} & & \downarrow f_* \text{oh} \\
A_{i-1}(V, 1) & \to A_{i-1}(X) & \to A_{i-1}(G(2, 5)) & \\
\end{array}
\]

Reasoning just as above, we can find \( c \in A_i(U, 1) \) lying in the summand (isomorphic to) \( A_{i-1}(V, 1) \) such that
\[
\delta(c) = b - b_2 \quad \text{in } A_i(D),
\]
where \( b_2 \in A_i(D) \) is in the summand (isomorphic to) \( A_{i-2}(X) \). It follows once more that
\[
i_*(b) = i_*(b_2) \in \text{Im} \left( A_{i-2}(X) \to A_i(D) \to A_i(M) \right),
\]
and (using the surjectivity (6)) that
\[
i_*(b) \in \text{Im} \left( A_{i-2}^{\text{hom}}(X) \to A_i(D) \to A_i(M) \right) =: \text{Im}(V_2 \to A_i(M)).
\]

We have now proven the inclusion (8).

Combining (6), (8) and theorem 3.3(ii), we see that there is a surjection
\[
A_{i-2}^{\text{hom}}(X) \to A_i^{\text{hom}}(M),
\]
which is given by \( i_*(f_X)^* \). This proves the lemma.

In the proof of lemma 4.3 we have used the following sublemma:
Sublemma 4.4. Given \( i \in \mathbb{Z} \), let
\[
\Psi_*: \quad A_i(D) = \bigoplus_{k=0}^2 A_{i-k}(X), \quad \Psi_*: \quad A_i(U, 1) = \bigoplus_{k=0}^1 A_{i-k}(V, 1)
\]
be the decompositions of theorem 3.3. Let \( \delta: A_i(U, 1) \to A_i(D) \) be the boundary map of the localization exact sequence for the inclusion \( D \subset M \). Then
\[
\delta(A_i(V, 1)) \subset A_i(X) \oplus A_{i-2}(X),
\]
\[
\delta(A_{i-1}(V, 1)) \subset A_{i-1}(X) \oplus A_{i-2}(X).
\]

Proof. For the first inclusion, we consider the commutative diagram (10). In view of theorem 3.3, the direct summand of \( A_i(U, 1) \) isomorphic to \( A_i(V, 1) \) is exactly the kernel of the map \( (f|_U)_* \circ h \). As such, the image under \( \delta \) is contained in
\[
\ker \left( A_i(D) \xrightarrow{(f|_U)_* \circ h} A_{i-1}(X) \right).
\]
Again applying theorem 3.3, this kernel coincides with the two summands isomorphic to \( A_i(X) \) resp. to \( A_{i-2}(X) \), as claimed.

The second inclusion is proven similarly, reasoning in the diagram (9).

Lemma 4.5. Set–up as above. There is equality
\[
D = \lambda h^2 + h \cdot f^*(d_1) + f^*(d_2) \quad \text{in} \quad A^2(M),
\]
for some non–zero \( \lambda \in \mathbb{Q} \) and some \( d_i \in A^i(G(2, 5)), \ i = 1, 2 \).

Proof. Let us consider the restriction \( h^2|_U \) of \( h^2 \in A^2(M) \) to the open \( U := M \setminus D \). Let \( f_U: U \to V \) be the restriction of the morphism \( f \), where \( V := G(2, 5) \setminus X \). As we have seen, \( f_U \) is a \( \mathbb{P}^1 \)-fibration. It thus follows from theorem 3.3 that
\[
h^2|_U = h \cdot (f_U)^*(c_1) + (f_U)^*(c_2) \quad \text{in} \quad A^2(U),
\]
for some \( c_i \in A^i(V), \ i = 1, 2 \). Let \( \bar{c}_i \in A^i(G(2, 5)) \) be elements such that \( \bar{c}_i|_U = c_i \) for \( i = 1, 2 \). The localization exact sequence (plus the fact that \( D \) is irreducible of codimension 2 in \( M \)) then implies that
\[
h^2 = h \cdot f^*(\bar{c}_1) + f^*(\bar{c}_2) + \mu D \quad \text{in} \quad A^2(M),
\]
for some \( \mu \in \mathbb{Q} \).

Let us assume, for a moment, that \( \mu = 0 \). Then relation (11) would imply in particular that
\[
h^2|_D = \left( h \cdot f^*(\bar{c}_1) + f^*(\bar{c}_2) \right)|_D \quad \text{in} \quad A^2(D).
\]
But this is absurd, for the right hand side maps to 0 under pushforward \( (f_X)_* \), whereas the left hand side maps to a non–zero multiple of \( [X] \in A_3(X) \) under pushforward \( (f_X)_* \). It follows that \( \mu \neq 0 \).

Relation (11) proves the lemma; it suffices to define \( \lambda := 1/\mu \) and \( d_i := \lambda \bar{c}_i \in A^i(G(2, 5)) \), \( i = 1, 2 \).
Armed with these lemmas, we are now ready to prove the claim\(^\text{[4.2]}\) (and hence close the proof of the theorem). Let \(d \in \mathbb{Z}\) be the non–zero integer such that \((f_X)_*(h^d) = d[X]\) in \(A_3(X)\). We define correspondences \(\Gamma_1, \Xi_1\) as follows:

\[
\Gamma_1 := \Gamma_i \circ i^! \Gamma_f \quad \in \quad A^5(X \times M),
\]

\[
\Xi_1 := \frac{1}{d\lambda} i^! \Gamma_1 = \frac{1}{d\lambda} \Gamma_f \circ i^! \Gamma_i \quad \in \quad A^5(M \times X)
\]

(where \(\lambda\) is the non–zero constant of lemma\(^\text{[4.5]}\).

Let us show these correspondences \(\Gamma_1, \Xi_1\) verify the relations \((4)\). By construction, the composition \(\Xi_1 \circ \Gamma_1\) acts on Chow groups in the following way:

\[
(\Xi_1 \circ \Gamma_1)_* : \quad A_i(X) \xrightarrow{(f_X)_*} A_{i+2}(D) \xrightarrow{i^*} A_{i+2}(M) \xrightarrow{i^!} A_i(D) \xrightarrow{(f_X)_*} A_i(X)
\]

Thanks to lemma\(^\text{[4.3]}\) the map

\[
\frac{1}{d\lambda} i^* i_* : \quad A_{i+2}(D) \to A_i(D)
\]

is the same as intersecting with

\[
\frac{1}{d} \left( h^2 + \frac{1}{\lambda} (f_X)^*(d_1|X) : h + \frac{1}{\lambda} (f_X)^*(d_2|X) \right) \in A^2(D).
\]

In particular, if \(b \in A_i(X)\) then

\[
\frac{1}{d\lambda} i^* i_* (f_X)^*(b) = \frac{1}{d} \left( h^2 \circ (f_X)^*(b) + \frac{1}{\lambda} h \circ (f_X)^*(b \cdot d_1|X) + \frac{1}{\lambda} (f_X)^*(b \cdot d_2|X) \right) \quad \text{in} \quad A_i(D).
\]

But then, it follows that

\[
(f_X)_* \frac{1}{d\lambda} i^* i_* (f_X)^*(b) = \frac{1}{d} (f_X)_* \left( h^2 \circ (f_X)^*(b) + \frac{1}{\lambda} h \circ (f_X)^*(b \cdot d_1|X) + \frac{1}{\lambda} (f_X)^*(b \cdot d_2|X) \right)
\]

\[
= \frac{1}{d} (f_X)_* (h^2 \circ (f_X)^*(b))
\]

\[
= \frac{1}{d} (f_X)_* (h^2) \cdot b = b \quad \text{in} \quad A_i(X).
\]

That is, \(\Xi_1 \circ \Gamma_1\) acts as the identity on \(A_i(X)\), which proves the first half of the claimed result\(^\text{[4]}\).

It remains to prove the second half of \((4)\). The composition \(\Gamma_1 \circ \Xi_1\) acts on Chow groups in the following way:

\[
(\Gamma_1 \circ \Xi_1)_* : \quad A_{i,b}^\text{hom}(M) \xrightarrow{i^!} A_{i-2}^\text{hom}(D) \xrightarrow{(f_X)_*} A_{i-2}^\text{hom}(X) \xrightarrow{i^*} A_i^\text{hom}(D) \xrightarrow{i_*} A_i^\text{hom}(M).
\]

Let \(a \in A_i^\text{hom}(M)\). In view of lemma\(^\text{[4.3]}\) we may suppose \(a = i_*(f_X)^*(b)\), for some \(b \in A_{i-2}^\text{hom}(X)\). But we have just checked that \((\Xi_1 \circ \Gamma_1)_*(b) = b\) for any \(b \in A_{i-2}(X)\), which means that

\[
(f_X)_* \frac{1}{d\lambda} i^* (a) = (f_X)_* \frac{1}{d\lambda} i^* i_* (f_X)^*(b) = b \quad \text{in} \quad A_{i-2}^\text{hom}(X).
\]
Applying $i_*(f_X)^*$ on both sides, we conclude that
\[
(\Gamma_1 \circ \Xi_1)_*(a) = i_*(f_X)^*(f_X)_* \frac{1}{d\lambda} i^* i_*(f_X)^*(b) = i_*(f_X)^* b = a \quad \text{in } A^i_{\hom}(M),
\]
i.e., $\Gamma_1 \circ \Xi_1$ acts as the identity on $A^i_{\hom}(M)$ as claimed.

We have now established the equalities (4), and so we have proven the first half of claim 4.2. The second half of claim 4.2 (i.e., the existence of the isomorphism $\Gamma_2$) is proven by the same argument, the only difference being that $X$ and $G(2,5)$ should be replaced by $Y$ resp. $G(3,5)$. \hfill \square

**Remark 4.6.** It would be interesting to refine theorem 4.1 to an isomorphism with $\mathbb{Z}$–coefficients. Is it true that there are isomorphisms
\[
A^i(X)_{\mathbb{Z}} \xrightarrow{\cong} A^i(Y)_{\mathbb{Z}} \quad \forall i
\]
of Chow groups with $\mathbb{Z}$–coefficients?

The problem, in proving this, is that the fibration result (theorem 3.3) is a priori only valid for (higher) Chow groups with rational coefficients.

**Remark 4.7.** It would also be interesting to prove theorem 4.1 for a dual pair $(X,Y)$ of Calabi–Yau threefolds in the family $\mathcal{X}_{25}$ of [8], [27]. In the absence of a nice diagram like (3) linking $X$ and $Y$, this seems considerably more difficult than theorem 4.1.

5. A COROLLARY

**Corollary 5.1.** Let $X, Y$ be the Calabi–Yau threefolds constructed as in [17]. Let $M$ be any smooth projective variety. Then there are isomorphisms
\[
N^j H^i(X \times M, \mathbb{Q}) \cong N^j H^i(Y \times M, \mathbb{Q}) \quad \text{for all } i, j.
\]
(Here, $N^*$ denotes the coniveau filtration [5].)

**Proof.** Theorem 4.1 implies there is an isomorphism of Chow motives $h(X \times M) \cong h(Y \times M)$. As the cohomology and the coniveau filtration only depend on the motive [2], [32], this proves the corollary. \hfill \square

**Remark 5.2.** It is worth noting that for any derived equivalent threefolds $X, Y$, there are isomorphisms
\[
N^j H^i(X, \mathbb{Q}) \cong N^j H^i(Y, \mathbb{Q}) \quad \text{for all } i, j;
\]
this is proven in [1].

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