On the Convergence Time of the Best Response Dynamics in Player-specific Congestion Games

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Abstract

We study the convergence time of the best response dynamics in player-specific singleton congestion games. It is well known that this dynamics can cycle, although from every state a short sequence of best responses to a Nash equilibrium exists. Thus, the random best response dynamics, which selects the next player to play a best response uniformly at random, terminates in a Nash equilibrium with probability one. In this paper, we are interested in the expected number of best responses until the random best response dynamics terminates.

As a first step towards this goal, we consider games in which each player can choose between only two resources. These games have a natural representation as (multi-)graphs by identifying nodes with resources and edges with players. For the class of games that can be represented as trees, we show that the best-response dynamics cannot cycle and that it terminates after $O(n^2)$ steps where $n$ denotes the number of resources. For the class of games represented as cycles, we show that the best response dynamics can cycle. However, we also show that the random best response dynamics terminates after $O(n^2)$ steps in expectation.

Additionally, we conjecture that in general player-specific singleton congestion games there exists no polynomial upper bound on the expected number of steps until the random best response dynamics terminates. We support our conjecture by presenting a family of games for which simulations indicate a super-polynomial convergence time.

*Parts of the results presented here already appeared in the Proceedings of the 4th Symposium on Stochastic Algorithms, Foundations, and Applications (SAGA) in 2007 [1].
1 Introduction

We study the convergence time of the best response dynamics to pure Nash equilibria\footnote{In the following, the term \textit{Nash equilibrium} always refers to a pure one.} in player-specific singleton congestion games. In such games, we are given a set of resources and a set of players. Each player is equipped with a set of non-decreasing, player-specific delay functions which measure the delay the player experiences from allocating a particular resource and sharing it with a certain number of other players. A player’s goal is to allocate a \textit{single} resource with minimum delay given fixed choices of the other players. Milchtaich \cite{Milchtaich12}, who was the first to consider player-specific singleton congestion games, proves that every such game possesses a Nash equilibrium which can be computed efficiently. However, he also observes that these games are no potential games \cite{Rosenthal73}, that is, the best response dynamics, in which players consecutively change to resources with minimum delay, can cycle. This is in contrast to congestion games with common delay functions in which all players sharing a resource observe the same delay. In the following, we refer to congestion games with common delay functions as \textit{standard congestion games}. Rosenthal \cite{Rosenthal73}, who introduces standard congestion games, proves that they always admit a potential function guaranteeing the existence of Nash equilibria and that the best response dynamics cannot cycle. Ieong et al. \cite{Ieong09} consider the convergence time of the best response dynamics to Nash equilibria in standard singleton congestion games. They observe that the delay values can be replaced by their ranks in the sorted list of these values without affecting the best responses dynamics. By applying Rosenthal’s potential functions to these new delay functions they observe that after at most $n^2m$ best responses a Nash equilibrium is reached, where $n$ equals the number of players and $m$ the number of resources. This result is independent of any assumption on the ordering according to which players change their strategies.

Since the best response dynamics in player-specific singleton congestion games can cycle, we propose to study random best response dynamics in such games. This approach is motivated by the following observation due to Milchtaich \cite{Milchtaich12}: From every state of a player-specific singleton congestion game there exists a polynomially long sequence of best responses leading to a Nash equilibrium. Thus, the random best response dynamics selecting the next player to play a best response at random terminates with probability one after a finite number of steps. Milchtaich’s analysis leaves open the question how long it takes until the random best response dynamics terminates in expectation. In this paper, we address this question as we think that it is a natural and interesting one. Currently, we are not able to analyze the convergence time in arbitrary player-specific singleton congestion games. However, our experimental results support the following conjecture.

\textbf{Conjecture 1.} There exist player-specific singleton congestion games in which the expected number of steps until the random best response dynamics terminates is super-polynomial.

In order to gain insights into the random best response dynamics, we begin with very simple yet interesting classes of games, and consider games in which each player chooses between only two alternatives. These games can be represented as multi-graphs: each resource corresponds to a node and each player to an edge. In the following, we call games that can be represented as graphs with topology $t$ player-specific congestion games on topology $t$. First, we consider games on trees and circles.

We prove that player-specific congestion games on trees admit a potential function from which we derive an upper bound of $O(n^2)$ on the maximum number of best responses until a Nash equilibrium is reached. Note, that this result is independent of the initial state and any assumption on the ordering according to which players change their strategies.

The result bases on the observation that one can replace the player-specific delay functions by common delay functions without changing the players preferences. Thus, player-specific congestion games on trees are isomorphic to standard congestion games on trees and we can apply the result
of Ieong et al. [9] to upper bound the convergence time. We proceed with player-specific congestion games on circles, and show that these games are the simplest games in which the best response dynamics can cycle. As we are only given four different delay values per player, we characterize with respect to the ordering of these four values in which cases the best response dynamics can cycle. We observe that only one such case exists. Finally, we analyze the convergence time of the random best response dynamics in such games, and prove a bound of $O(n^2)$ on the expected number of steps until the dynamics terminates. In order to prove this result we introduce the notion of over- and underload tokens. An overload token indicates that a resource is shared by two players, an underload token indicates that it is unused. We observe that the number of tokens cannot increase, and that once in a while tokens get stuck or vanish.

Based on the insights gained by analyzing player-specific congestion games on circles we present a family of games and conjecture that there exists no polynomial upper bound on the expected time until the random best response dynamics terminates. Obviously, this depends on the initial state, and so we implicitly assume that the initial configuration is chosen appropriately. Our conjecture is motivated by a slightly different notion of over- and underload tokens. Now their definition depends on the fact that every resource has a fixed congestion that it takes in every state, and so we implicitly assume that the initial configuration is chosen appropriately. Intuitively one may think of the number of tokens as a measure of derangement of order. In games on circles this measure can only decrease whereas it can also increase in general games. We fail to give a rigorous proof of a super-polynomial lower bound. However, we support our conjecture by empirical results obtained from simulations.

1.1 Definitions and Notations

A player-specific singleton congestion game $\Gamma$ is a tuple $(\mathcal{N}, \mathcal{R}, (\Sigma_r)_{r \in \mathcal{N}}, (d_r^i)_{r \in \mathcal{R}})$ where $\mathcal{N}$ denotes the set of $n$ players, $\mathcal{R}$ the set of $m$ resources, $\Sigma_r \subseteq \mathcal{R}$ the strategy space of player $i$, and $d_r^i : \mathbb{N} \rightarrow \mathbb{N}$ a strictly increasing delay function associated with player $i$ and resource $r$. In the following, we assume that ties are broken arbitrarily. That is, for every pair of resources $r_1, r_2 \in \Sigma_i$ and every pair $n_{r_1}, n_{r_2} \in \mathbb{N}$, $d_{r_1}^i(n_{r_1}) \neq d_{r_2}^i(n_{r_2})$. We denote by $S = (r_1, \ldots, r_n)$ the state of the game in which player $i$ allocates resource $r_i \in \Sigma_i$. For a state $S$, we define the congestion $n_r(S)$ on resource $r$ by $n_r(S) = |\{i \mid r = r_i\}|$, that is, $n_r(S)$ equals the number of players sharing resource $r$ in state $S$. We assume that players act selfishly and wish to allocate resources minimizing their individual delays. The delay of player $i$ from allocating resource $r$ in state $S$ is given by $d_r^i(n_r(S))$. Given a state $S = (r_1, \ldots, r_n)$, we call a resource $r' \in \Sigma_i \setminus \{r_i\}$ a best response of player $i$ to $S$ if, for all $r' \in \Sigma_i \setminus \{r_i\}$, $d_{r'}(n_r(S) + 1) \leq d_{r_i}(n_r(S) + 1)$, and if $d_{r_i}(n_r(S) + 1) \leq d_{r_i}(n_r(S))$. Furthermore, we call $r_i$ a best response of player $i$ to $S$ if, for all $r' \in \Sigma_i \setminus \{r_i\}$, $d_{r_i}(n_r(S)) \leq d_{r_i}(n_r(S) + 1)$. The standard solution concept in player-specific singleton congestion games are Nash equilibria. A state $S$ is a Nash equilibrium if for each player $i$ the resource $r_i$ is a best response.

In this paper, we consider games that have natural representations as graphs. We assume that each player chooses between only two resources. In this case, we can represent the resources as the nodes of a graph and the players as the edges. If different players choose between the same two resources, then the corresponding graph has multi-edges. The direction of an edge naturally corresponds to the strategy the player currently plays.

In the following, we will sometime refer to standard singleton congestion games. Standard singleton congestion games are defined in the same way as player-specific singleton congestion games except that we are not given player-specific delay functions $d_r^i$, $r \in \mathcal{R}, i \in \mathcal{N}$, but common delay functions $d_r$, $r \in \mathcal{R}$. Ieong et al. [9] observe that in standard singleton congestion games one can always replace the delay values $d_r(n_r)$ with $r \in \mathcal{R}$ and $1 \leq n_r \leq n$ by their ranks in the sorted list of these values without affecting the players preferences in any state of the game. Note that this approach is not restricted to standard singleton congestion games but also applies to player-specific
singleton congestion games. That is, given a player-specific congestion game \( \Gamma \), fix a player \( i \) and consider a list of all delays \( d_r^i(n_r) \) with \( r \in R \) and \( 1 \leq n_r \leq n \). Assume that this list is sorted in a non-decreasing order. For each resource \( r \), we define an alternative player-specific delay function \( \tilde{d}_r^i : \mathbb{N} \to \mathbb{N} \) where, for each possible congestion \( n_r \), \( \tilde{d}_r^i(n_r) \) equals the rank of the delay \( d_r^i(n_r) \) in the aforementioned list of all delays. Due to our assumptions on the delay functions, all ranks are unique. In the following, we define the \textit{type of a player} \( i \) by the ordering of the player-specific delays \( d_r^i(1), \ldots, d_r^i(n) \) of the resources \( r \in \Sigma_i \).

We define the \textit{transition graph} \( TG(\Gamma) \) of a player-specific singleton congestion game \( \Gamma \) as the graph that contains a vertex for every state of the game. Moreover, there is a directed edge labeled with \( i \) from state \( S \) to state \( S' \) if we obtain \( S' \) from \( S \) by permitting player \( i \) to play a best response. We call the dynamics in which players iteratively play best responses the \textit{best response dynamics}. Furthermore, we use the term \textit{best response schedule} to denote an algorithm that selects, given a state \( S \), the next player to play a best response. We assume that such a player is always selected among those players who have an incentive to change their strategy. The convergence time \( t(n, m) \) of a best response schedule is the maximum number of steps to reach a Nash equilibrium in any game with \( n \) players and \( m \) resources and for any initial state. If the schedule selects the next player to play a best response uniformly at random then \( t(n, m) \) refers to the maximum expected convergence time. We use the term \textit{random best response dynamics} to denote the resulting dynamics. Additionally, we use the terms best response dynamics and best response schedule interchangeably.

1.2 Related Work

We already mentioned that every player-specific singleton congestion game possesses a Nash equilibrium which can be computed efficiently. Moreover, we mentioned that such games are no potential game, even though from every state there exists a polynomially long sequence of best responses leading to a Nash equilibrium. These results are due to Milchtaich [12]. Milchtaich also observes that player-specific network congestion games, i.e., games in which each player wants to allocate a path in a network, do not possess Nash equilibria in general [13]. He proposes to characterize those games with respect to their networks which always possess Nash equilibria. Such a characterization should be independent of further assumptions on the delay functions. Ackermann, Röglin, and Vöcking [4] extend the results presented in [12] to player-specific matroid congestion games, and prove that the matroid property is the maximal property with respect to the combinatorial structure of the players’ strategy spaces guaranteeing the existence of Nash equilibria. In such games, the players’ strategy spaces are sets of bases of matroids over the resources. Gairing, Monien, and Tiemann [7] consider player-specific singleton congestion games with linear delay functions without offsets and prove among other results that such games are potential games.

A model closely related to player-specific congestion games are standard congestion games. Rosenthal [15] proves that these games are potential games. Ieong et al. [9] address the convergence time of the best responses dynamics in standard singleton congestion games. They show that the best response dynamics converges quickly. Fabrikant, Papadimitriou, and Talwar [6] show that in general standard congestion games players do not convergence quickly. Their result holds especially in the case of network congestion games, in which players seek to allocate paths in a network. Later, Ackermann, Röglin, and Vöcking [3] extended the result of Ieong et al. [9] to matroid congestion games, and prove that the matroid property is the maximal property on the players’ strategy spaces guaranteeing polynomial convergence time. Even-Dar et al. [5] consider the convergence time in standard singleton congestion games with weighted players.

Another model which possesses similar properties as player-specific singleton congestion games are two-sided markets. In these games, we are given a set of resources and a set of players, and for every resource and every player a preference list of the elements of the other set. Given such a
game, one seeks for a stable matching assigning players to resources such that there exists no pair of player and resource that are not matched to each other but prefer each other to their current matches. Gale and Shapely [8] prove that stable matchings always exist. Knuth [10] proposes to study better or best response dynamics in such games and observes that they can cycle. However, Roth and Vande Vate [16] observe that short better response paths to stable matchings always exists. Ackermann et al. [2] follow this line of research and prove an exponential lower bound on the expected time until the random better (best) response dynamics terminates.

2 Player-specific Congestion Games on Trees

In this section, we consider player-specific congestion games on trees. Note that in such games the number of resources equals the number of players. First, we observe that one can always replace the player-specific delay functions by common delay functions such that the players’ types are preserved. Hence, we obtain a standard singleton congestion game, whose transition graph equals the transition graph of the player-specific game. We prove the following theorem.

**Theorem 2.** In every player-specific congestion game on a tree with $n$ nodes, every best response schedule terminates after at most $2n^2$ steps.

**Proof.** Let $\Gamma$ be a player-specific congestion game $\Gamma$ on a tree. In the following, we describe how to replace the player-specific delay functions of $\Gamma$ by common delay functions $d_r : \mathbb{N} \rightarrow \mathbb{N}, r \in \mathcal{R}$, with the following property: For every player $i$ its type with respect to the player-specific delay functions equals its type with respect to the standard delay functions. Remember that the types completely describe the preferences of the players, and hence, the transition graph of $\Gamma$ is not affected by replacing the player-specific delay functions by common ones. Since the resulting game is a standard singleton congestion game, $\Gamma$ is a potential game and we can apply the result of Ieong et al. [9] to upper bound the convergence time. Obviously, the same bound holds in $\Gamma$. Thus, by applying the proof of the convergence time in standard singleton congestion game as presented in [3], we conclude that every best response schedule for player-specific congestion games on trees terminates after at most $2n^2$ steps.

We prove the theorem by induction on the number of players and describe how to construct a sequence of player-specific congestion games $\Gamma_1, \ldots, \Gamma_n$ on trees with the following properties. $\Gamma_1$ is obtained from $\Gamma$ by removing the players 2 to $n$ from the game. The set of resources in $\Gamma_0$ is the set of the two resources the first player is interested in. Now $\Gamma_1$ is obtained from $\Gamma_{i-1}$ by adding one player and one resource to $\Gamma_i$. The player and the resource is chosen in such a way that $\Gamma_i$ is a player-specific congestion game on a tree. That is, we choose a player $i$ who is interested in resource $r$ of $\Gamma_{i-1}$, and add the additional resource $r'$ the player is interested in to $\Gamma_i$.

Obviously $\Gamma_1$, the player-specific congestion game with a single player and two resources, is a standard congestion game. As induction hypothesis assume, that we already replaced the player-specific delay functions in $\Gamma_{i-1}$ by common ones without affecting the players’ types. For ease of notation let $\Gamma_{i-1}^*$ be this game. In the following, we assume that for every resource $r$ in $\Gamma_i^*$ its delay functions is defined for all possible congestion values $n_r$ between 1 and $n$ and not only for the maximum number of players that are interested in $r$ in $\Gamma_i^*$. Now given $\Gamma_{i-1}^*$, we describe how to choose the delay functions $d_r$ of the resources in $\Gamma_i^*$ such that the players in $\Gamma_i^*$ and $\Gamma_i$ have the same types. The delay functions of the resources $r$ that belong to $\Gamma_{i-1}^*$ are the same as in $\Gamma_{i-1}^*$. Additionally, we assume that for every such resource $r$ and every congestion $1 < n_r \leq n$, $d_r(n_r) - d_r(n_r - 1) \geq n$. If this is not the case, then due to our assumption that the delay functions are strictly increasing, we can scale all delays by a factor of $n$ in order to achieve the desired goal. Thus, it remains to choose a delay function of the additional resource $r'$ that does not belong to $\Gamma_{i-1}^*$. Since the gap between consecutive values of the delay function $d_r$ is large enough, we can realize every type for the additional player by choosing the delay function $d_{r'}$ appropriately.
Applying the result from [3] to the game $\Gamma_n^*$ directly implies the theorem.

3 Player-specific Congestion Games on Circles

In this section, we consider player-specific congestion games on circles. Without loss of generality, we assume that the resources are enumerated from $0, \ldots, n-1$, and that they are arranged in increasing order clockwise. Furthermore, we assume w.l.o.g. that for every player $i$, $\Sigma_i = \{r_i, r_{i+1} \mod n\}$. In the following, we call $r_i$ the 0- and $r_{i+1} \mod n$ the 1-strategy of player $i$. Furthermore, we drop the mod $n$ terms and assume that all indices are computed modulo $n$. Due to our assumptions on the delay functions, there are six different types of players in such games:

- type 1: $d_{r_i}(1) < d_{r_i}(2) < d_{r_{i+1}}(1) < d_{r_{i+1}}(2)$
- type 2: $d_{r_i}(1) < d_{r_i+1}(1) < d_{r_i}(2) < d_{r_{i+1}}(2)$
- type 3: $d_{r_i}(1) < d_{r_{i+1}}(1) < d_{r_i+1}(2) < d_{r_i}(2)$

We call the three other types, which can be obtained by exchanging the identities of the resources $r_i$ and $r_{i+1}$ in the above inequalities, type 1', type 2', and type 3'. Furthermore, we call two players $i$ and $j$ consecutive, if they share a resource, that is, if $j = i + 1$ or $i = j + 1$. Given a state $S$, we call two consecutive players synchronized, if both play the same strategy, that is, if both either play their 0- or their 1-strategy. Moreover, we call a set of consecutive players synchronized if all players play the same strategy.

3.1 Cycles in the Transition Graphs and a Lower Bound

We present an infinite family of games possessing cycles in their transition graphs. From this construction we derive a lower bound of $\Omega(n^2)$ on the convergence time of the random best response dynamics in player-specific congestion games on circles.

Consider a game on a circle with $n$ players which are all of type 3. It is not difficult to verify that this game possesses only two Nash equilibria: either all players play their 0-strategy or their 1-strategy. Consider now a state $S$ with the following properties: In $S$ we can partition the players into two non-empty sets $S_0$ and $S_1$ of synchronized players. Players in $S_0$ all play their 0-strategy, whereas players in $S_1$ all play their 1-strategy. Again, it is not difficult to verify that in every such state there are exactly two players who have an incentive to change their strategies. From both sets only the first player clockwise has an incentive to change its strategy. Thus, there exist cycles in the transition graphs of these games. We obtain such a cycle by selecting players from the two sets alternately, and letting them play best responses.

In order to prove a lower bound on the convergence time of the random best response dynamics, observe that with probability $1/2$ the total number of players playing their 0-strategy increases or decreases by one whenever a player is selected uniformly at random. After the strategy change either all players are synchronized, and therefore the random best response dynamics terminates, or again we are in a state $S'$ with two sets of synchronized players. Observe now that this process is isomorphic to a random walk on a line with nodes $v_0, \ldots, v_n$. The node $v_i$ corresponds to the fact that $i$ players play their 0-strategy. As the expected time of a random walk on a line with $n+1$ nodes to reach one of the two ends of the line is $\Theta(n^2)$ if the walk starts in the middle of the line [11], we obtain a lower bound of $\Omega(n^2)$.

Corollary 3. There exists an infinite family of instances of player-specific congestion games on circles with corresponding initial states such that the number of steps until the random best response dynamics terminates is lower bounded by $\Omega(n^2)$.
3.2 An Upper Bound

In this section, we present an upper bound on the convergence time of the random best response dynamics in player-specific congestion games on circles. We prove the following theorem which matches the lower bound presented in Corollary 3.

**Theorem 4.** In every player-specific congestion game on a circle the random best response dynamics terminates after $O(n^2)$ steps in expectation.

The remainder of this section is organized as follows. We characterize with respect to the types of the players in which cases there are cycles in the transition graphs of such games. We show that cycles only exist if all players are of type 3 or type 3'. We analyze the convergence time of deterministic best response dynamics in games with acyclic transition graphs by developing a general framework that allows to derive potential functions from which one can easily derive upper bounds. Finally, we analyze the convergence time of the random best response dynamics in the case of games with players of type 3 or type 3'.

3.2.1 The Impact of Type 1 Players

First, we investigate the impact of type 1 players on the existence of cycles in the transition graphs and on the convergence time of the best response dynamics. We claim that games with at least one player of type 1 do not possess cycles in their transition graphs. Intuitively, this is true since every player of type 1 changes its strategy at most once, whereas in a cycle every player changes its strategy at least twice.

**Lemma 5.** Let $\Gamma$ be a player-specific congestion game on a circle. If there exists at least one player of type 1 or 1', then $TG(\Gamma)$ is acyclic. Moreover, every best response schedule terminates after at most $4n^2$ steps.

In order to prove Lemma 5, we first prove the following one.

**Lemma 6.** Let $\Gamma$ be a player-specific congestion game on a circle whose transition graph contains cycles. Then every player changes its strategy at least twice in every cycle of $TG(\Gamma)$.

**Proof.** The fact that players being involved in the cycle change their strategy an even number of times is obvious. Thus, it remains to show that every player changes its strategy. For contradiction, assume that there exists a player $i$ and a cycle in $TG(\Gamma)$ such that player $i$ does not change its strategy on that cycle. In this case, we could remove the player from the game, and artificially increase the congestion on the resource the player allocates by one. We would then obtain a player-specific congestion game on a tree which cannot have a cycle in the transition graph due to Theorem 2.

Next we prove Lemma 5 for type 1 players. The proof for type 1' players is essentially the same.

**Proof of Lemma 5.** Without loss of generality, let player 0 be of type 1. Then observe that player 0 will never play its 1-strategy again, once it played its 0-strategy. Thus, by Lemma 6 $TG(\Gamma)$ is acyclic.

In order to prove the convergence time, observe that if we fix player 0 to one of its strategies, then we obtain a player-specific congestion game on a tree. Due to Theorem 2, the convergence time of such games is upper bounded by $2n^2$. Furthermore, observe that the transition graphs of these games are isomorphic to disjoint subgraphs of $TG(\Gamma)$. The first subgraph contains all nodes of $TG(\Gamma)$ in which player 0 plays its 0-strategy, the second one contains all nodes in which player 0 plays its 1-strategy. Finally, as all edges between these two subgraphs are directed from the second one to the first one, and as the maximal length of any best response sequence in each of these subgraphs is upper bounded by $2n^2$, the lemma follows.
In the following, we will assume that there exists no player of type 1 or 1’, as otherwise we could apply Lemma 5.

3.3 A Framework to Analyze the Convergence Time

In this section, we present a framework to analyze the convergence time of best response schedules in player-specific congestion games on circles. Let \( \Gamma \) be a game such that there is no player of type 1 or 1’. First, we investigate whether there is a sufficient condition such that player \( i \) does not want to change its strategy in a state \( S \) of \( \Gamma \).

Observation 7. Suppose that player \( i \) is not of type 1 or 1’. Then if it is synchronized with the players \( i - 1 \) and \( i + 1 \) in \( S \), it has no incentive to change its strategy.

In the following, we call a resource \( r \) overloaded in state \( S \) if two players share \( r \). Additionally, we call a resource \( r’ \) underloaded in state \( S \) if no player allocates \( r’ \). Obviously in every state of \( \Gamma \), the total number of overloaded resources equals the total number of underloaded resources. From Observation 7 we conclude that in every state \( S \) only players who allocate a resource that is currently overloaded or who could allocate a resource that is currently underloaded might have an incentive to change their strategy.

Based on this observation, we now present a general framework to analyze the convergence time of best response schedules. First, we introduce the notion of over- and underload tokens. Given an arbitrary state \( S \) of \( \Gamma \), we place an overload token on every overloaded resource. Additionally, we place an underload token on every underloaded resource. Obviously over- and underload tokens alternate on the circle. Furthermore, note that a legal placement of tokens uniquely determines the strategies the players play. A placement of tokens is legal if no two tokens share a resource, and if the tokens alternate on the circle.

In the following, we investigate in which directions tokens move if players play best responses. Consider first a sequence of resources \( r_i, \ldots, r_j \) and assume that players \( i, \ldots, j - 1 \) are of the same type \( t \). Additionally, assume that an overload token is placed on resource \( r_k \), and that an underload token is placed on resource \( r_l \) with \( i < k < l < j \). The scenario we consider is depicted in Figure 1.

Assume first, that the distance (number of edges) between the two tokens is at least two, i.e., \( |l - k| \geq 2 \). In this case, each token can only move in one direction. The directions are uniquely
determined by the type of the players. They can be derived from investigating, with respect to the players’ type $t$, which players have incentives to change their strategy. The directions are stated in Figure 1 too. Assume now that the distance between the two tokens is one. That is, $k = l - 1$. Then, there exists a player who is interested in the over- and underloaded resource, and who currently allocates the overloaded one. It is not difficult to verify that this player always has an incentive to change its strategy. Note that this holds regardless of the player’s type since we assumed that there are no players of type 1 and $1'$. Observe that after the strategy change of this player all players $i, \ldots, j - 1$ are synchronized and therefore there exist no over- and underloaded resources anymore. In the following, we call such an event a collision of tokens.

So far, we considered sequences of players of the same type and observed that there is a unique direction in which tokens of the same kind move. In sequences with multiple types of players such unique directions do not exist any longer, i.e., overload as well as underload tokens can move in both directions. However, if two players of different types share a resource and if due to best responses of both players an over- or underload token moves onto this resource, then the token could stop there. In the following, we formalize this observation with respect to overload tokens and introduce the notion of termination points.

**Definition 8.** We call a resource $r_i$ a termination point of an overload token if the following conditions are satisfied.

1. The players $i - 1$ and $i$ have different types. Let these types be $t_{i-1}$ and $t_i$.

2. In sets of consecutive players of type $t_{i-1}$ overload tokens move clockwise, whereas they move anticlockwise in sets of consecutive players of type $t_i$.

We illustrate the definition in Figure 2(a). Let player $i - 1$ be of type 3, and let player $i$ be of type 2. In this case, the requirements of the definition are satisfied. Assume, that player $i - 1$ plays its 1-strategy and that it is synchronized with player $i - 2$. Additionally, assume that player $i$ plays its 0-strategy and that it is synchronized with player $i + 1$. Observe now that the token cannot move as neither player $i - 1$ nor player $i$ has an incentive to change its strategy. Suppose now that initially all players along the path play their 0-strategy. Then an overload token that moves from the left to the right along the path stops at $r_i$. The token may only move on if one of the two players is not synchronized with its neighbor any longer. In this case, this player always has an incentive to change its strategy as it can allocate a resource that is currently underloaded. Thus, an underload and an overload token collide. Additionally, if initially all players play their 1-strategy and an overload token moves from the right to the left along the path, we observe the same phenomenon. The token cannot pass the resource $r_i$ unless it collides with an underload token.

Note that the definition of a termination point can easily be adopted to underload tokens. A list of all termination points is given in Figure 2(b). In the left column we present all termination points for overload tokens, in the right one for underload tokens.

![Orientation of the players](example.png)

| type 3 | type 2 |
|--------|--------|
| overloaded |         |

(a) Example of a termination point

| 2' 2' | 2 2' |
| 3 3' | 3 3' |
| 3 2 | - |
| - | 2 3' |
| - | 3 2' |
| 2' 3' | - |

(b) List of all termination points
3.4 Analyzing the Convergence Time

In this section, we analyze the convergence time in player-specific congestion games on circles. We distinguish between the following four cases.

**Case 1:** For both kinds of tokens there exists at least one termination point.

**Case 2:** Only for one kind of tokens there exists at least one termination point.

**Case 3:** There exist no termination points but over- and underload tokens move in opposite directions.

**Case 4:** There exist no termination points and over- and underload tokens move in the same direction.

In the first two cases, we present potential functions and prove that the transition graphs of such games are acyclic and that every best response schedule terminates after $O(n^2)$ steps. In the third case, we can do slightly better and prove an upper bound of $O(n)$ on the convergence time. In all cases one can easily construct matching lower bounds. Only in the fourth case deterministic best response schedules can cycle. In this case, we prove that the random best response schedule terminates after $O(n^2)$ steps in expectation.

Before we take a closer look at the different cases, we discuss which games with respect to their players’ types belong to which case. Games only with players of type 2 and 2’ or only with players of type 3 and 3’ belong to the first case. Additionally, some games with more than two types of players belong to this case. The second case covers all games with at least three different kinds of players which do not belong to the first case. Furthermore, it covers games with type 2 and type 3 players, with type 2’ and type 3’ players, type 2’ and type 3 players, and with type 2 and type 3’ players. Games with type 2 players only, or games with type 2’ players only belong to the third case. Finally, games with type 3 players only and games with type 3’ players only belong to the fourth case. These observations can easily be derived from Figure 2(b).

### 3.4.1 Case 1

**Lemma 9.** Let $\Gamma$ be a player-specific congestion game on a circle with termination points for both kinds of tokens. Then $\Gamma$ is a potential game, and every best response schedule terminates after $O(n^2)$ steps.

*Proof.* Let $S$ be a state of $\Gamma$ and consider the mapping that maps every token in $S$ to the next termination point lying in the direction in which the token moves. In the following, we define $d(t, S)$ as the distance of a token $t$ in state $S$ to its corresponding termination point. Obviously $d(t, S) \leq n$. Consider now the potential function $\phi(S) = \sum_{\text{token } t} d(t, S)$ and suppose that a player plays a best response. Then either one token moves closer to its termination point or two tokens collide. In both cases $\phi(S)$ decreases by at least 1. Thus, $\phi(S)$ strictly decreases if a player plays a best response and therefore, $TG(\Gamma)$ is acyclic. Moreover, as $\phi(S)$ is upper bounded by $O(n^2)$, every best response schedule terminates after $O(n^2)$ steps. \qed

### 3.4.2 Case 2

**Lemma 10.** Let $\Gamma$ be a player-specific congestion game on a circle with termination points only for one kind of token. Then $\Gamma$ is a potential game, and every best response schedule terminates after $O(n^2)$ steps.

*Proof.* Without loss of generality, assume that termination points only exist for overload tokens. In this case, we define $d(t_o, S)$ for every overload token $t_o$ as in the proof of Lemma 9. For every underload token $t_u$ we define $d(t_u, S)$ as follows. Let $t_o$ be the first overload token lying in the same direction as $t_u$ moves.
1. If \( t_o \) moves in the opposite direction than \( t_u \), then we define \( d(t_u, S) \) as the distance between the two tokens. The distance of two tokens moving in opposite directions is defined as the number of moves of these tokens until they collide.

2. If \( t_o \) moves in the same directions as \( t_u \) then we define \( d(t_u, S) \) as the distance between \( t_u \) and \( t_o \) plus the distance between \( t_o \) and the first termination point at which \( t_o \) has to stop. Thus, \( d(t_u, S) \) equals the maximum number of moves of these two tokens until they collide.

Observe, that for every underload token \( t_u \), \( d(t_u, S) \leq 2n \). Consider, the potential function \( \phi: \Sigma_1 \times \ldots \times \Sigma_n \rightarrow \mathbb{N} \times \mathbb{N} \) with \( \phi(S) = (\phi_1(S), \phi_2(S)) \), where \( \phi_1(S) \) equals the total number of overload tokens in \( S \) and \( \phi_2(S) \) equals the sum of all \( d(t, S) \) for all under- and overload tokens. Suppose now that a player plays a best response. Obviously if two tokens collide, then \( \phi_1(S) \) decrease by one. Moreover, if there is no collision, then \( \phi_2(S) \) decreases. Note that in the first case \( \phi_2 \) may increase. This may happen if, due to the collision, \( d(t_u, S) \) of a remaining underload token \( t_u \) has to be recomputed as its associated overload token has been removed. The new value is upper bounded by the sum of the old values of \( t_u \) and the collided underload token plus 1. Now consider the lexicographic ordering \( <_\phi \) of the states of \( \Gamma \) with respect to \( \phi \). Let \( S \) and \( S' \) be two states of \( \Gamma \). Then

\[
S <_\phi S' \Leftrightarrow \begin{cases} 
\phi_1(S) < \phi_1(S') \\
\phi_1(S) = \phi_1(S') \quad \text{and} \quad \phi_2(S) < \phi_2(S')
\end{cases}
\]

Observe that \( \phi \) strictly decreases if a player plays a best response. Thus, \( TG(\Gamma) \) is acyclic. Additionally, observe that \( \phi_1 \) is upper bounded by \( n \), and that \( \phi_2 \) is upper bounded by \( n^2 \). However, as \( \phi_2 \) only increases by one when \( \phi_1 \) decreases, we conclude that every best response schedule terminates after \( O(n^2) \) steps.

\[3.4.3 \text{ Case 3}\]

**Lemma 11.** Let \( \Gamma \) be a player-specific congestion game on a circle with no termination points in which over- and underload tokens move in opposite directions. Then \( \Gamma \) is a potential game, and every best response schedule terminates after \( O(n) \) steps.

**Proof.** Let \( S \) be a state of \( \Gamma \) and consider the one-to-one mapping that maps every overload token to the next underload token lying in the direction in which the token moves. We define the distance of such a pair of tokens as the maximum number of moves of these two tokens till they collide.

Suppose now that a player plays a best response. Then either the number of overload tokens or the distance between one pair of tokens decreases by one. Consider now the potential function \( \phi: \Sigma \rightarrow \mathbb{N} \times \mathbb{N} \) with \( \phi(S) = (\phi_1(S), \phi_2(S)) \), where \( \phi_1(S) \) equals the number of overload tokens in \( S \), and \( \phi_2(S) \) equals the sum of all distances of pairs of tokens. Observe now that in the case of a best response, \( \phi_1 \) either decreases by 1 or remains unchanged. In the first case, \( \phi_2 \) may increase by 1. This is true as tokens from different pairs may collide. However, this can happen at most \( n \) times. If this happens, the remaining two tokens form a new pair whose distance equals the sum of the distances of the previous pairs plus 1. In the second case, \( \phi_2 \) decreases by 1. Then by similar arguments as in the proof of Theorem 10 we conclude that \( TG(\Gamma) \) is acyclic. Finally, observe that \( \phi_1 \) is upper bounded by \( n \). Moreover, \( \phi_2 \) is upper bounded by \( n \), too. Finally, as \( \phi_2 \) only increases by one when \( \phi_1 \) decreases, we conclude that every best response schedule terminates after \( O(n) \) steps.

\[3.4.4 \text{ Case 4}\]

In the following, we present a proof of the fourth case for players of type 3. By symmetry of the types 3 and 3', the same result holds for games with players of type 3', too.
Lemma 12. Let $\Gamma$ be a player-specific congestion game on a circle in which all players are of type 3. Then the random best response schedule terminates after $O(n^2)$ steps in expectation.

Proof. In order to prove the lemma, we prove the following lemma.

Lemma 13. In every state $S$ of $\Gamma$ the number of players who want to change from their 0- to their 1-strategy equals the number of players who want to change from their 1- to their 0-strategy.

Proof. In the following, we call a synchronized set of consecutive players maximal if the next players to both ends of the set play different strategies than the synchronized players. Obviously in every state $S$ of $\Gamma$ which is not an equilibrium the number of maximal synchronized sets of players playing their 0-strategy equals the number of maximal synchronized sets of players playing their 1-strategy.

We now prove that in every maximal synchronized set of consecutive players only the first player clockwise has an incentive to change its strategy. Thus, in every maximal set, there is only a single player who wants to change its strategy. Note that this suffices to prove the lemma.

First, consider a maximal, synchronized subset of consecutive players $N' = \{i, \ldots, j\}$ which all play their 0-strategy. Then player $i - 1$ plays its 1-strategy, and therefore the players $i - 1$ and $i$ share resource $r_i$. In this case, player $i$ can decrease its delay by changing to her 1-strategy. Other players $k \in N', k \neq i$, do not have an incentive to change their strategy as this would increase their delay.

Second, consider a maximal synchronized subset of consecutive players $N' = \{i, \ldots, j\}$ which all play their 1-strategy. Then player $i - 1$ plays its 0-strategy and therefore no player currently allocates resource $r_i$. Observe now that player $i$ may decrease its delay by changing to its 0-strategy. Again, all other players $k \in N', k \neq i$, do not have an incentive to change their strategy as this would increase their delay. This is especially true for the last player, who currently allocates an overloaded resource.

Consider now the random best response schedule activating an unsatisfied player uniformly at random. From Lemma 13 we conclude that the total number of players playing their 0-strategy increases or decreases by $1$ with probability $1/2$. Combining this with the observation that in a Nash equilibrium all players play the same strategy, we conclude that the random best response schedule is isomorphic to a random walk on a line with $n + 1$ vertices. Vertex $v_i$ corresponds to the fact that $i$ players play their 0-strategy. As the time of such a random walk to reach one of the two ends of the line is $O(n^2)$, the lemma follows.

4 Player-specific Congestion Games on General Graphs

In this section, we consider player-specific congestion games on general graphs and present evidence supporting Conjecture 1 by constructing a family of instances for which experimental results clearly show a super-polynomial convergence time. Our analysis of player-specific congestion games on circles is based on the notion of over- and underload tokens, and there is no straightforward extension of this notion to player-specific singleton congestion games on general graphs. The instances we construct have, however, the property that every resource has a fixed congestion that is taken in every Nash equilibrium, and we can define tokens with respect to these congestions. To be precise, if the congestion on a resource deviates by $x$ from the equilibrium congestion, we place $x$ overload tokens in the case $x > 0$ and we place $-x$ underload tokens in the case $x < 0$. Note that for circles with type 3 players this definition coincides with the former definition of tokens.

The crucial property of games on circles with type 3 players leading to polynomial convergence is that the number of tokens cannot increase. The instances we construct in this section are in some sense similar to circles with type 3 players, but we attach additional gadgets to the nodes which can occasionally increase the number of tokens. We start with a circle with $n$ type 3 players and
replace each edge by \( n \) parallel edges. This modification allows each node to store more than one token of the same kind if the preferences of the players are adjusted accordingly. Other properties are not affected by this modification, that is, over- and underload tokens still move in the same direction with approximately the same speed and if an overload and an underload token meet, they both vanish. Each time a node contains at least two tokens of the same kind, the gadget attached to the node is triggered with constant probability. If a gadget is triggered, it can emit a new pair of overload and underload token into the circle. Usually, this new pair is stored in the gadget and only emitted after the triggering tokens have moved on a linear number of steps. The new tokens are not emitted simultaneously but the second is usually only emitted after the first one has moved on a linear number of steps in order to prevent the new tokens from canceling each other out immediately.

Initially we introduce two overload tokens at node 0 and two underload tokens at node \( n/2 \). The two overload tokens move independently through the circle starting at the same node. Typically they meet a couple of times before they meet the underload tokens and vanish. The same is true for the underload tokens as well, meaning that typically a couple of gadgets get triggered before the initial tokens vanish. Hence, the number of tokens has a tendency to increase. Since the triggered gadgets emit the stored tokens in a random order, the random process soon becomes unwieldy and we fail to rigorously prove that it takes super-polynomial time in expectation until all tokens vanish. This conjecture is, however, strongly supported by simulations.

### 4.1 Our Construction

Given \( n \in \mathbb{N} \) we construct a player-specific congestion game \( \Gamma_n \) consisting of \( n \) gadgets \( G_0, \ldots, G_{n-1} \) as follows. In the following, the notion of a gadget differs from the notion used in the previous discussion. Previously, we described how to attach gadgets to a circle in order to illustrate the relation to games on circles. Next gadgets are arranged on a circle. A single gadget \( G_i \) is depicted in Figure 2(c). It consists of 4 resources \( r_{i,0}, \ldots, r_{i,3} \) and \( 5n \) players. Each edge in the figure represents \( n \) of them. The gadgets are arranged on a circle, such that for every \( i \) the resources \( r_{i,3} \) and \( r_{i+1,0} \) coincide. Thus, for every \( i \), \( 6n \) players are interested in \( r_{i,0} \) and \( r_{i,3} \), and \( 2n \) players are interested in \( r_{i,1} \) and \( r_{i,2} \).

For every player who chooses between the two resources \( r_{i,k} \) and \( r_{i,l} \) with \( l < k \) we call \( r_{i,l} \) the 0-strategy and \( r_{i,k} \) the 1-strategy of that player. In the following, we refer by the term type \( j \) player to a player represented by an edge \( e_{i,j} \). The player-specific delay functions are defined as follows. All players of the same type \( j \) have the same functions for the two resources they choose between. We define these functions in terms of a threshold \( t \) for their 0-strategies, meaning that the 0-strategy is a best response if and only if the total number of other players allocating the 0-strategy resource is less or equal to the threshold \( t \). Otherwise the 1-strategy is the best response. The thresholds are defined as depicted in Figure 2(d).

![Figure 2: The lower bound construction](image)
In the next sections, we prove that every resource has the same congestion in every Nash equilibrium. We proceed with a description of how gadgets can generate new tokens. Finally, we present results obtained from simulations.

4.2 Properties of Nash Equilibria

In order to simplify our proceeding discussion, we introduce the term $c^i_{j,b}(S) \in \mathbb{N}$, $b \in \{0,1\}$, to denote the number of type $j$ players in gadget $i$ who play their $b$-strategy in state $S$. Furthermore, we define $n_{i,j}(S) = n_{i,j}(S^*)$. In the following, let $S^*$ be a Nash equilibrium of $\Gamma_n$. For ease of notation, we use $c^i_{j,b} = c^i_{j,b}(S^*)$ and $n_{i,j} = n_{i,j}(S^*)$. The following observation is true because $S^*$ is a Nash equilibrium.

**Observation 14.** Let $j \in \{1,3\}$ and $b \in \{0,1\}$. Then for every $0 \leq i < n$ the number of type $j$-players playing their $b$-strategy in gadget $G_i$ in $S^*$ is uniquely determined by the number of type $j-1$ players playing their $b$-strategy in gadget $G_i$ in $S^*$, i.e., $c^i_{b,j-1} = c^i_{b,j}$.

Next, we prove that every resource has the same congestion in every Nash equilibrium.

**Lemma 15.** For every Nash equilibrium $S^*$ of $\Gamma_n$ and every $0 \leq i < n$,
\[ n_{i,0} = 3 \cdot n \quad \text{and} \quad n_{i,1} = n_{i,2} = n. \]

**Proof.** First observe that for every gadget $G_i$, it holds
\[ c^i_{0,0} \geq c^i_{1,0} \geq c^i_{0,2}. \]

If the first inequality were not true, then there exist type 0 players in $G_i$ playing their 1-strategy and type 4 players playing their 0-strategy. However, since $S^*$ is a Nash equilibrium, all type 4 players in $G_i$ who play their 0-strategy are satisfied and thus $n_{i,0} \leq 3n$. We observe that all type 0 players currently playing their 1-strategy have an incentive to change their strategy. A similar argument proves the second inequality. Essentially, the same arguments prove the following implications:
\[ c^i_{0,0} < n \implies c^i_{1,4} = c^i_{0,4} = 0, \]
\[ c^i_{0,4} < n \implies c^i_{0,2} = 0. \]

Now consider an arbitrary gadget $G_i$ and let $3n - k_{i-1}$ be the number of players from gadget $G_{i-1}$ allocating resource $r_{i,0}$. In the following, we discuss how the parameter $k_{i-1}$ affects the choices of the players in gadget $G_i$ in the Nash equilibrium $S^*$. We prove that the best responses of the players in $G_i$ are uniquely determined by the parameter $k_{i-1}$. In order to do so, we distinguish 6 cases.

1. **Case $k_{i-1} = 0$:** All type 1, type 3, and type 4 players in gadget $G_{i-1}$ play their 1-strategy. Due to Observation 14, we conclude that all type 0 and type 2 players in $G_{i-1}$ play their 1-strategy as well, and therefore the congestion on $r_{i-1,0}$ is at most $3n$. In this case, however, all type 0 players in $G_{i-1}$ have an incentive to play a best response. We conclude that this case does not appear in a Nash equilibrium.

2. **Case $1 \leq k_{i-1} < n$:** $k_{i-1} + 1$ type 0 and $k_{i-1} + 1$ type 1 players in $G_i$ play their 0-strategy.

The remaining players in $G_i$ play their 1-strategy. Thus $k_i = k_{i-1} + 1$.

3. **Case $k_{i-1} = n$:** All type 0 and all type 1 players in $G_i$ play their 0-strategy; all other players in $G_i$ play their 1-strategy. Thus $k_i = k_{i-1}$.

4. **Case $n < k_{i-1} \leq 2n$:** All type 0 and all type 1 players in $G_i$ play their 0-strategy. Additionally, $k_{i-1} - n$ type 4 players in $G_i$ play their 0-strategy. The remaining players in $G_i$ play their 1-strategy. Thus $k_i = k_{i-1}$.
5. Case $2n < k_{i-1} < 3n$: All type 0, all type 1 and all type 4 players in $G_i$ play their 1-strategy. Additionally, $k_{i-1} - 2n - 1$ type 3 and $k_{i-1} - 2n - 1$ type 4 players in $G_i$ play their 0-strategy. The remaining players in $G_i$ play their 1-strategy. Thus $k_i = k_{i-1} - 1$.

6. Case $k_{i-1} = 3n$: Similar arguments as in the first case show that this case does not appear in a Nash equilibrium.

As an intermediate observation we conclude that the lemma is true if at least one gadget $G_i$ exists for which $n \leq k_i \leq 2n$ holds. In this case $k_{i-1} = k_i$ for every $1 \leq i < n$ and the players play the strategies as described above.

Next we take a closer look at the second and fifth case. We begin with the second one in which $1 \leq k_{i-1} < n$ implies $k_i = k_{i-1} + 1$ which implies $k_{i+1} = k_{i-1} + 2$ and so on until $k_j = n$. In this case we enter the third case which implies $k_{j+1} = n$ and so on. Obviously this leads to a contradiction since $k_{i-1} < n$. Thus, whenever there exists a gadget for which $k_{i-1} < n$ holds, $S^*$ is not a Nash equilibrium. Similar arguments show that the fifth case does not appear in a Nash equilibrium either.

4.3 Generating New Tokens

Motivated by Lemma 15 we are now ready to introduce a new notion of tokens.

**Definition 16.** Let $S$ be an arbitrary state of $\Gamma_*$ and let $n_r^*$ be the congestion on a resource $r$ in every Nash equilibrium. Then, we place over- and underload on the resources according to the following rules.

1. If $n_r(S) = n_r^* + k$, $k \in \mathbb{N}$, then we place $k$ overload tokens on $r$.
2. If $n_r(S) = n_r^* - k$, $k \in \mathbb{N}$, then we place $k$ underload tokens on $r$.

Next we describe how the number of overload and underload tokens can increase. This can happen if there are either at least two overload or at least two underload tokens on $r_{i,0}$. In the following, we discuss the first case in detail. The second case in only depicted in Figure 5.

Consider a single gadget $G_i$ as depicted in Figure 4(a). Numbers attached to resources correspond to the number of tokens lying on them. Positive numbers indicate that overload tokens are present, negative numbers indicate that underload tokens are present. Numbers $a$ attached to edges indicate that $a$ players represented by that edge play their 0-strategy, whereas $n - a$ players play their 1-strategy.

**Configuration 4(a)**: Initially, there are two overload tokens on $r_{i,0}$. In this case, all type 0 and all type 4 players have an incentive to change to their 1-strategies. All other players are satisfied. With probability $2/3$, given that a player from $G_i$ is selected, a type 0 player is selected and the configuration 4(b) is reached, in which there is one overload token on $r_{i,0}$ and one on $r_{i,1}$.

**Configuration 4(b)**: All type 1 and all type 4 players have an incentive to change to their 1-strategy. With probability $2/3$ configuration 4(c) is reached in which there is one overload token on $r_{i,0}$ and one on $r_{i,3}$.

**Configuration 4(c)**: Still all type 4 players have an incentive to change to their 1-strategy. However, we assume that the overload token which currently lies on $r_{i,0}$ moves on due to a best response of a player in gadget $G_{i+1}$. In this case, configuration 4(d) is reached in which there is still one overload token on $r_{i,0}$. Additionally, one overload token is in gadget $G_{i+1}$.

**Configuration 4(d)**: Again, all type 4 players have an incentive to change to their 1-strategy. Now one of these players is selected and configuration 4(e) is reached in which there is one overload token on $r_{i,4}$.
Configuration 4(e): In this configuration, the overload token on \( r_{i,4} \) can move to the next gadget. Observe that this event is much more likely than the next one, in which the only type 0 player playing its 1-strategy switches back to its 0-strategy. All other players are satisfied. If both events take place configuration 4(f) is reached. Note, that in this case additional tokens are generated. There is a new underload token on \( r_{i,1} \) and a new overload token on \( r_{i,0} \).

Configuration 4(f): Finally, all \( n - 1 \) type 4 players playing their 0-strategy have an incentive to change to their 1-strategy. Additionally, the only type 1 player playing its 1-strategy wants to change back to its its 0-strategy.

4.4 Simulations

We simulated the random best response dynamics in games \( \Gamma_n \) and obtained the results shown in Figure 3. On the \( x \)-axis we plotted the parameter \( n \), on the \( y \)-axis the average number of best responses until the random best response dynamics terminated. Observe that the \( y \)-axis is plotted in log-scale. For every \( n \in \{5, 10, \ldots, 180, 185\} \) we started the random best response dynamics from the following initial configuration: all type 0 and all type 1 players play their 0-strategies; all type 2 and all type 3 players play their 1-strategies. Additionally, \( n/2 \) type 4 players in the gadgets \( G_0, \ldots, G_{n/2-1} \) and \( n/2 + 2 \) type 4 players in the gadgets \( G_{n/2}, \ldots, G_{n-1} \) play their 1-strategy. All other type 4 players play their 0-strategy. This initial configuration corresponds to placing two overload tokens on \( r_{0,0} \) and two underload tokens on \( r_{n/2,0} \). For \( n \leq 160 \) we took the average over 400 runs, and for larger \( n \) we took the average over 100 runs.

![Figure 3: Average number of best responses](image)

Unfortunately, it does not seem feasible to simulate the best response dynamics for much larger values of \( n \). We believe, however, that the results in Figure 3 are a clear indication for a super-polynomial, maybe even exponential, convergence time.

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Figure 4: The number of tokens increases along the upper path.
(a) Initial configuration. (b) One underload token detours to the lower path.

(c) It continues on the lower path (d) ... and moves to the next gadget.

(e) The second overload token moves. (f) New tokens are generated.

Figure 5: The number of tokens increases along the lower path.