The logarithmic contribution to the QCD
static energy at N^4LO

Nora Brambilla^a, Xavier Garcia i Tormo^b,c, Joan Soto^c,
Antonio Vairo^a

^aDipartimento di Fisica dell’Università di Milano and INFN, via Celoria 16,
20133 Milan, Italy
^bHigh Energy Physics Division, Argonne National Laboratory, 9700 South Cass
Avenue, Argonne, IL 60439, USA
^cDepartament d’Estructura i Constituents de la Matèria, Universitat de Barcelona,
Diagonal 647, E-08028 Barcelona, Catalonia, Spain

Abstract

Using pNRQCD and known results for the field strength correlator, we calculate the
ultrasoft contribution to the QCD static energy of a quark-antiquark pair at short
distances at N^4LO in α_s. At the same order, this provides the logarithmic terms of
the singlet static potential in pNRQCD and the log α_s terms of the static energy.

1 Introduction

The ground state energy, E_0(r), of a static quark and a static antiquark sepa-
rated by a distance r is a key object for the understanding of the QCD
dynamics. It is also a basic ingredient of the Schrödinger-like formulation of
heavy quarkonium systems [1]. Its linear behavior at long distances is a signal
for confinement [2], but also at short distances (r ≪ 1/Λ_{QCD}), where weak cou-
lping calculations are reliable, it shows a non-trivial behavior. Indeed, when
calculated in perturbation theory, infrared divergences are found starting at
three loops [3,4]. These are due to the virtual emission of ultrasoft gluons with
energy of the order E_0(r), which turn a color singlet quark-antiquark pair into

1 Current address
a color octet one and vice-versa. The proper treatment of the ultrasoft emissions requires the resummation of an infinite class of diagrams, which produces a non-analytic dependence on $\alpha_s$ (typically logarithms of it). We shall focus here on this short distance behavior.

The current knowledge of $E_0(r)$ at short distance may be summarized as follows

$$E_0(r) = -\frac{C_F\alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} \left[ a_1 + 2\gamma_E\beta_0 \right] \right. \right.$$

$$\left. + \left( \frac{\alpha_s(1/r)}{4\pi} \right)^2 \left[ a_2 + \left( \frac{\pi^2}{3} + 4\gamma_E^2 \right) \beta_0^2 + \gamma_E \left( 4a_1\beta_0 + 2\beta_1 \right) \right] \right. \right.$$

$$\left. + \left( \frac{\alpha_s(1/r)}{4\pi} \right)^3 \left[ \frac{16\pi^2}{3} C_A^2 \log \frac{C_A\alpha_s(1/r)}{2} + \bar{a}_3 \right] \right. \right.$$

$$\left. + \left( \frac{\alpha_s(1/r)}{4\pi} \right)^4 \left[ a_4^{L2} \log^2 \frac{C_A\alpha_s(1/r)}{2} + a_4^L \log \frac{C_A\alpha_s(1/r)}{2} + \bar{a}_4 \right] \right. \right.$$

$$\left. + \cdots \right\}, \tag{1}$$

where $C_F = T_F(N_c^2 - 1)/N_c$, $C_A = N_c$, $T_F = 1/2$, $N_c$ is the number of colors, $\beta_0 = 11C_A/3 - 4T_Fn_f/3$ and $\beta_1 = 34C_A^2/3 - 20C_AT_Fn_f/3 - 4C_FT_Fn_f$ are the first two coefficients of the beta function, $n_f$ is the number of (massless) flavors, $\gamma_E$ is the Euler constant and $\alpha_s$ is the strong coupling constant in the \textit{MS} scheme. The one-loop coefficient $a_1$ is given by [5-6]

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_f, \tag{2}$$

and the two loop coefficient $a_2$ by [7-8][9][10]

$$a_2 = \left( \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right) C_A^2 - \left( \frac{1798}{81} + \frac{56}{3} \zeta(3) \right) C_A T_F n_f$$

$$- \left( \frac{55}{3} - 16\zeta(3) \right) C_F T_F n_f + \left( \frac{20}{9} T_F n_f \right)^2. \tag{3}$$

The logarithmic piece of the third-order correction was calculated in [11], whereas the non-logarithmic piece $\bar{a}_3$ has not been calculated yet. However, $\bar{a}_3$ is believed to be dominated by contributions which are known from renormalization group arguments [12]. If we write $\bar{a}_3 = a_3 + a_3^{\text{RG}}$, $a_3^{\text{RG}} \gg a_3$. $a_3^{\text{RG}}$ has a known expression in terms of the coefficients of the beta function and of those entering in the potential at lower orders (see [12], where $\bar{a}_3 = -48\pi^3 \times V_3$, $a_3 = 64c_0$). Estimates of $a_3$ have been carried out using Padé approximations.
and renormalon dominance [13,14,15,16], which are consistent with the inequality above, and give similar results. The double logarithmic fourth-order correction may be obtained from [17], where higher-order contributions of the form $\alpha_s^{n+3} \log^n \alpha_s$ were resummed using renormalization group techniques. It reads

$$a_4^L = \frac{16\pi^2}{3} C_A^3 \left( -\frac{11}{3} C_A + \frac{4}{3} T_F n_f \right).$$  \hfill (4)$$

The main result of this letter is the calculation of the logarithmic fourth-order correction to the singlet potential, $a_4^L$; we anticipate that it reads

$$a_4^L = 16\pi^2 C_A^3 \left[ a_1 + 2\gamma_E \beta_0 + T_F n_f \left( -\frac{40}{27} + \frac{8}{9} \log 2 \right) \right.$$

$$\left. + C_A \left( \frac{149}{27} - \frac{22}{9} \log 2 + \frac{4}{9} \pi^2 \right) \right].$$ \hfill (5)$$

The non-logarithmic piece $\tilde{a}_4$ remains unknown.

A convenient method to calculate the logarithmic contributions to Eq. (1), which steam from the dynamics at the ultrasoft scale $E_0(r)$, consists in integrating out from static QCD degrees of freedom at the soft energy scale $1/r$ and working within the effective field theory framework of pNRQCD [18,19] (see [20] for a review). The quark-antiquark system may be in a color singlet or in a color octet configuration, which are encoded in color singlet, $S$, and color octet, $O$, fields in pNRQCD. At leading order in the multipole expansion, the integration of the soft energy scale gives rise to a singlet, $V_s(r; \mu)$, and an octet, $V_o(r; \mu)$, static potential, which depend on $r$ and a factorization scale $\mu$. At next-to-leading order, two more “potentials” appear, $V_A(r; \mu)$ and $V_B(r; \mu)$, which are the matching coefficients of the singlet-octet and octet-octet vertices respectively. At this order, the pNRQCD Lagrangian reads

$$\mathcal{L}_{\text{pNRQCD}} = \mathcal{L}_{\text{light}}$$

$$+ \int d^3 r \left\{ S^\dagger \left[ i\partial_0 - V_s(r; \mu) \right] S + O^\dagger \left[ iD_0 - V_o(r; \mu) \right] O \right\}$$

$$+ V_A(r; \mu) \text{Tr} \left\{ O^\dagger r \cdot gE S + S^\dagger r \cdot gE O \right\}$$

$$+ \frac{V_B(r; \mu)}{2} \text{Tr} \left\{ O^\dagger r \cdot gE O + O^\dagger r \cdot gE O \right\}$$

$$+ \ldots,$$ \hfill (6)$$

where $\mathcal{L}_{\text{light}}$ is the part of the Lagrangian involving gluons and light quarks, and coincides with the QCD one. The dots stand for higher-order terms in the multipole expansion. The static energy calculated from the above Lagrangian
has the form

\[ E_0(r) = V_s(r; \mu) + \delta_{\text{US}}(r, V_s, V_o, V_A, V_B, \ldots; \mu), \]  

(7)

where \( \delta_{\text{US}}(r, V_s, V_o, V_A, V_B, \ldots; \mu) \) (\( \delta_{\text{US}} \) for short) contains contributions from the ultrasoft gluons. \( V_s(r; \mu) \) and \( V_o(r; \mu) \) do not depend on \( \mu \) up to \( N^2\text{LO} \) [19]. The former coincides with \( E_0(r) \) at this order and the latter may be found in [21]. The fact that the \( \mu \) dependence of \( \delta_{\text{US}} \) must cancel the one in \( V_s(r; \mu) \) is the key observation that leads to a drastic simplification in the calculation of the log \( \alpha_s \) terms in \( E_0(r) \). So, for instance, the logarithmic contribution at \( N^3\text{LO} \), which is part of the three-loop contributions to \( V_s(r; \mu) \), may be extracted from a one-loop calculation of \( \delta_{\text{US}} \) [11,19] and the single logarithmic contribution at \( N^4\text{LO} \), which is part of the four-loop contributions to \( V_s(r; \mu) \), may be extracted from a two-loop calculation of \( \delta_{\text{US}} \).

In Sec. 2, we review the calculation of the third-order logarithmic term since it follows the same lines as that of the fourth-order one, which will be presented in Sec. 3. In Sec. 4 we conclude and discuss some applications of this work.

2 Review of the third-order logarithmic correction

In \( d \) dimensions, the order \( r^2 \) contribution due to ultrasoft gluons reads [11,19]

\[ \delta_{\text{US}} = -i g^2 T_F V_A^2 \frac{r^2}{N_c} \int_0^\infty dt \, e^{-i(t(V_o-V_s))} \langle 0|E^{a}_{ab}(t)\phi(t, 0)^{\text{adj}}E^{b}(0)|0 \rangle. \]  

(8)

\( \phi(y, x)^{\text{adj}}_{ab} \) is the Wilson line in the adjoint representation connecting the points \( y \) and \( x \) by a straight line (\( t \) stands for \( (t, 0) \)). We will evaluate Eq. (8) perturbatively in \( \alpha_s \). The dependence on \( \alpha_s \), apart from the trivial \( g^2 \) factor, enters through (i) the \( V_s \) and \( V_o \) potentials, (ii) \( V_A \) and (iii) the field strength correlator of the chromoelectric fields.

(i) The difference \( V_o - V_s \) is given at leading-order by \( \frac{C_A \alpha_s (1/r)}{2} \). Note that at leading and next-to-leading order \( V_s \) and \( V_o \) only differ by an overall color factor.

(ii) At tree level \( V_A = 1 \).

We denote \( N^n\text{LO} \), contributions to the potential of order \( \alpha_s^{n+1} \) and \( N^n\text{LL} \), contributions of order \( \alpha_s^{n+2} \log^{n-1} \alpha_s \).
(iii) The two-point field strength correlator

\[ D_{\mu\nu\lambda\omega}(z) \equiv \langle 0 \vert T \left\{ G_{\mu\nu}^a(y)\phi(y, x)_{ab}^\text{adj} G_{\lambda\omega}^b(x) \right\} \vert 0 \rangle \]

(9)

can be parameterized in terms of two scalar functions \( D(z^2) \) and \( D_1(z^2) \) according to

\[
D_{\mu\nu\lambda\omega}(z) = (g_{\mu\lambda}g_{\nu\omega} - g_{\mu\omega}g_{\nu\lambda}) \left( D(z^2) + D_1(z^2) \right) \\
+ (g_{\mu\lambda}z_\nu z_\omega - g_{\mu\omega}z_\nu z_\lambda - g_{\nu\lambda}z_\mu z_\omega + g_{\nu\omega}z_\mu z_\lambda) \frac{\partial D_1(z^2)}{\partial z^2},
\]

(10)

where \( z = y - x \) [22]. In (8), \( x \) and \( y \) only differ in the time component, hence \( z = t \). Furthermore, in \( d \) dimensions, the chromoelectric component is given by

\[
\langle 0 \vert E^a(y)\phi(y, x)_{ab}^\text{adj} E^b(x) \vert 0 \rangle = D_{1010}(z) = -(d - 1) \left[ D(z^2) + D_1(z^2) \\
+ z^2 \frac{\partial D_1(z^2)}{\partial z^2} \right].
\]

(11)

The leading-order contribution to the field strength correlator is given by the diagram shown in Fig. 1. In \( d = 4 - 2\epsilon \), the result is

\[
D_1^{(0)}(z^2) = \mu^{2\epsilon}(N_c^2 - 1) \frac{\Gamma(2 - \epsilon)}{\pi^{2-\epsilon}(-z^2)^{2-\epsilon}}, \quad D^{(0)}(z^2) = 0.
\]

(12)

Note that keeping \( \epsilon \neq 0 \) in the chromoelectric correlator provides a regularization for the integral over \( t \) in Eq. (8).

We now insert (i), (ii) and (iii) into Eq. (8). Since at the ultrasoft scale \( t(V_o - V_s) \sim 1 \), the integral in \( t \) is performed without expanding the exponential
and making use of
\[ \int_0^\infty dt t^n e^{-at} = \frac{\Gamma(n+1)}{a^{n+1}}. \] (13)

The final result reads:
\[ \delta_{US} = C_F \frac{C_A^3}{24} \frac{1}{r} \frac{1}{\pi} \alpha_s^3(1/r) \left( \frac{1}{\hat{\epsilon}} - 2 \log \frac{V_o - V_s}{\mu} + \frac{5}{3} - 2 \log 2 \right), \] (14)

where \( \frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} - \gamma_E + \log(4\pi) \). Note that the \( \alpha_s \) coming from the potential is evaluated at the soft scale \( 1/r \), while the \( \alpha_s \) coming from the ultrasoft coupling is evaluated at the scale \( \mu \). This will become relevant in the next section. The ultraviolet divergence in (14) can be reabsorbed by a renormalization of the potential. In the \( \overline{\text{MS}} \) scheme, in coordinate space, we have:
\[ V_s(r; \mu) \to Z V_s(r; \mu), \quad Z = 1 + C_F^3 \frac{1}{24} \frac{\alpha_s^2(1/r)}{\pi} \alpha_s^2(1/r) \frac{1}{\hat{\epsilon}}. \] (15)

Since the static energy is \( \mu \) independent, from the calculation above we infer that the logarithmic contribution to \( V_s(r; \mu) \) at order \( \alpha_s^4 \) must be
\[ \delta V_s(r; \mu) = -C_F \frac{C_A^3}{12} \frac{1}{r} \frac{1}{\pi} \alpha_s^3(1/r) \log(r\mu), \] (16)

which added to the renormalized \( \delta_{US} \) contribution (from (14)) gives the log \( \alpha_s \) term displayed in the third line of Eq. (1). This term was first calculated in [11], where the cancellation between the IR cut-off of \( V_s(r; \mu) \) and the UV cut-off of the pNRQCD expression was checked explicitly by calculating the relevant Feynman diagrams in the Wilson loop.

A comment is in order concerning the scheme dependence of the calculation of \( \delta_{US} \). This is not important if we are only interested in the logarithmic contribution, but it is if we wish eventually to combine our result with a (yet to be done) calculation of \( V_s(r; \mu) \) at N^3LO and get the non-logarithmic pieces of the static energy right. We will assume that such a calculation will be done in momentum space and that dimensional regularization and the \( \overline{\text{MS}} \) scheme will be used to renormalize the UV divergences, like in the N^2LO calculation [7,8,9]. The result will still be IR divergent when \( d \to 4 \), and the question is how one should proceed in order to combine that result with ours in a consistent way.\(^3\)

\(^3\) Note that the \( \overline{\text{MS}} \) subtraction of (15) in coordinate space is not equivalent to the \( \overline{\text{MS}} \) subtraction in momentum space.
We propose to convert the (UV renormalized) momentum-space potential to coordinate space (in \(d\) dimensions) in that calculation, and together to use \(d\)-dimensional expressions for all the objects in our calculation, namely also for \(V_s(r; \mu)\) and \(V_o(r; \mu)\). \((V_A(r; \mu)\) remains the same in \(d\) dimensions). This guarantees that the IR behavior of the regulated effective theory is exactly the same as the one of the fundamental theory. Had we expanded \(V_o - V_s\) in Eq. (8) we would have obtained zero, which means that the UV divergences, which remain after renormalization by the \(\overline{\text{MS}}\) QCD counterterms (and by that of the color octet field wave function) in the effective theory, cancel exactly the IR divergences. Therefore, as a consequence of the fact that the IR behavior of the regulated effective theory is the same as the one of the fundamental one, the UV divergences in (14) cancel exactly the IR divergences in \(V_s(r; \mu)\), the \(\mu\) dependence disappears, and the non-logarithmic pieces are correctly calculated. This procedure would be analogous to the one employed in [24]. Alternatively, one could use \(\overline{\text{MS}}\) for the IR divergences of \(V_s(r; \mu)\) in momentum space, work out the momentum space expressions for the \(d\)-dimensional version of (14) and make the \(\overline{\text{MS}}\) UV subtraction accordingly.

In the following section, we will use the same procedure employed here to obtain the next-to-leading IR logarithmic dependence of the static potential. That is the logarithmic \(\alpha_s^5\) contribution to the potential, which is part of the \(N^4\)LO contribution.

3 Fourth-order logarithmic correction

Equation (8) does not rely on an expansion in \(\alpha_s\), therefore it also provides NLO contributions to \(\delta_{US}\). In fact, as we argue next, it provides the full contribution to this order.

In principle, we may have diagrams with more insertions of the operators in (6) and diagrams with operators of higher order in the multipole expansion that contribute to \(\delta_{US}\) at NLO. Concerning the former, for symmetry reasons we need at least two more operator insertions, which implies a suppression of \(\alpha_s^3\) with respect to the leading-order \(\delta_{US}\). Concerning the latter, operators of higher order in the multipole expansion may be found in [25,26]. Their contributions are suppressed by \(\alpha_s^2\) with respect to the leading-order \(\delta_{US}\). To see this just recall that the ultrasoft fields (and derivatives acting on them) must be counted as \(E_0(r) \sim \alpha_s/r\). Then, any insertion of the kind \(\int dt \, r \cdot E\) implies an \(\alpha_s\) suppression (with an extra \(\alpha_s\) suppression for any \(r^j D^i\) acting on the chromoelectric field). For a given diagram, additional suppressions may appear due to the coupling constants in front of the chromoelectric fields.
Fig. 2. Static QCD diagrams for the leading-order matching of $V_A$. The solid lines stand for a static quark and antiquark, the dashed line for a longitudinal gluon.

The NLO contribution to $\delta_{US}$ is then provided by Eq. (8) evaluated at relative order $\alpha_s$. Since the dependence in $\alpha_s$ enters through $V_A$, $V_s$, $V_o$ and the chromoelectric correlator, we need the $O(\alpha_s)$ corrections to all these quantities. These will be given in the following two sections. Finally, in Sec. 3.3, we will obtain the fourth-order logarithmic correction to the potential.

3.1 $O(\alpha_s)$ corrections of $V_A$, $V_s$ and $V_o$

The $O(\alpha_s)$ corrections to $V_s$ and $V_o$ are well known. In particular, we have

$$V_o - V_s = \frac{C_A}{2} \frac{1}{r^2} \alpha_s (1/r) \left[ 1 + (a_1 + 2\gamma_E\beta_0) \frac{\alpha_s (1/r)}{4\pi} \right]. \quad (17)$$

The matching coefficient $V_A$ can be obtained by matching static QCD to pN-RQCD at order $r$ in the multipole expansion. At leading order in $\alpha_s$, we have to calculate the diagrams shown in Fig. 2. They give the tree level result $V_A = 1$. One may naively expect the first correction to be $O(\alpha_s)$, but it is not. This becomes clear if we perform the calculation in dimensional regularization and in Coulomb gauge. Indeed, the diagrams that we can draw at $O(\alpha_s)$ correspond either to self-energy corrections or to iterations of the Coulomb potential, which are identical in the effective theory and hence do not contribute to the matching. Then, the first non-vanishing correction to the tree level result may possibly come from diagrams like the one in Fig. 3, which is $O(\alpha_s^2)$ and, therefore, unimportant here.

3.2 $O(\alpha_s)$ correction of the field strength correlator

The $O(\alpha_s)$ correction to the QCD field strength correlator was calculated in [23]. It is given by the diagrams in Fig. 4. Here we need the expression in $d$

4 The vanishing of the anomalous dimension of $V_A$ at one loop has been observed in [17].
5 The potentials are independent on the gauge used in the matching. Therefore, we can use the most convenient one to do the computation.
dimensions because in (8) the integral over $t$ is singular. The $d$-dimensional result for the $\alpha_s$ correction is [27]

$$D(1)(z^2) = N_c(N_c^2 - 1)\frac{\alpha_s(\mu)}{\pi} \frac{\mu^{4\epsilon}}{4\pi^2 - 2\epsilon} \Gamma^2(1 - \epsilon) \left( \frac{1}{z^2} \right)^{2-2\epsilon} g(\epsilon),$$  \hspace{1cm} (18)

$$D_1(1)(z^2) = N_c(N_c^2 - 1)\frac{\alpha_s(\mu)}{\pi} \frac{\mu^{4\epsilon}}{4\pi^2 - 2\epsilon} \Gamma^2(1 - \epsilon) \left( \frac{1}{z^2} \right)^{2-2\epsilon} g_1(\epsilon),$$  \hspace{1cm} (19)

with

$$g(\epsilon) = \frac{-3 + 8\epsilon - 6\epsilon^2 + 2\epsilon^3}{\epsilon(3 - 5\epsilon + 2\epsilon^2)} + 2\epsilon \frac{B(-1 + 2\epsilon, -2 + 2\epsilon)}{3 - 2\epsilon},$$  \hspace{1cm} (20)

$$g_1(\epsilon) = \frac{6 - 18\epsilon + 17\epsilon^2 - 6\epsilon^3}{\epsilon^2(3 - 5\epsilon + 2\epsilon^2)} - 2(1 - \epsilon + \epsilon^2) \frac{B(-1 + 2\epsilon, -2 + 2\epsilon)}{\epsilon(3 - 2\epsilon)},$$  \hspace{1cm} (21)

where

$$B(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u + v).$$

Since the external points $x$ and $y$ are fixed, the divergences that we encounter in $D_{\Box\Box\Box}$ coming from the expressions above should cancel against the vertex and gluon and octet field propagator counterterms. The counterterm for the vertex is zero, since, as seen in the previous section, the first correction to $V_A$ is of order $\alpha_s^2$. The counterterm for the gluon propagator is the usual one in QCD. The counterterm for the octet propagator coincides with the counterterm for the quark propagator in the Heavy Quark Effective Theory [28] but with the quark in the adjoint representation. We can represent the counterterm contributions by the diagrams of Fig. 5. We have checked that: (i) the divergence coming from the first diagram in Fig. 4 is canceled by the counterterm of the gluon propagator, (ii) the diagram (b) of Fig. 4 does not give a divergent contribution (as one would expect from the fact that the gluons are attached to the external fixed points only) and (iii) when we sum the remaining diagrams the divergence that we obtain is exactly canceled by
Fig. 4. Next-to-leading order contributions to the field strength correlator. The gluonic string is represented by a double line. The shaded blob represents the insertion of the one-loop gluon self-energy. Symmetric graphs are understood for (c) and (d).
Fig. 5. $\mathcal{O}(\alpha_s)$ counterterm diagrams for the chromoelectric correlator. The gluonic string (which comes from the octet propagator) is represented by a double line.

the counterterm of the octet propagator. In the $\overline{\text{MS}}$ scheme, at $\mathcal{O}(\alpha_s)$, the contributions of the counterterms are given by

\[
\mathcal{D}_{c:t}(z^2) = 0
\]

\[
\mathcal{D}_{1:t}(z^2) = N_c(N_c^2 - 1) \alpha_s(\mu) \frac{\mu^{2\epsilon}}{4\pi^{2-\epsilon}} \Gamma(2 - \epsilon) \frac{1}{(-z^2)^{2-\epsilon}} \frac{1}{\epsilon} 
\]

\[
\times \left(-2 - \frac{5}{3} + \frac{4}{3} T_F \frac{n_f}{N_c}\right)
\]

where in the brackets we have kept separated the $-2$ coming from the octet propagator counterterm from the $-\frac{5}{3} + \frac{4}{3} T_F \frac{n_f}{N_c}$ coming from the gluon propagator one. The renormalized $d$-dimensional result for the $\alpha_s$ correction to the chromoelectric correlator is

\[
\mathcal{D}_{d:t}(z^2) = -(3 - 2\epsilon) \left[\mathcal{D}_{c:t}(z^2) + (-1 + 2\epsilon)\mathcal{D}_{1:t}(z^2) \right.
\]

\[
+ \mathcal{D}_{c:t}(z^2) + (-1 + \epsilon)\mathcal{D}_{1:t}(z^2) \left.\right]
\]

which, indeed, is finite for $\epsilon \to 0$.

### 3.3 Calculation of the fourth-order logarithmic correction

The results of the two preceding sections provide all the necessary ingredients to compute $\delta_{US}$ at NLO. Let us split $\delta_{US}$ as follows

\[
\delta_{US} = \mathcal{G}_{(EE)}^{(r^2)}|_{\mathcal{O}(\alpha_s)} + \mathcal{G}_{(V - V_s)}^{(r^2)}|_{\mathcal{O}(\alpha_s)} + \mathcal{G}_{\mu \to 1/r}|_{\mathcal{O}(\alpha_s)}
\]

where the first and second terms stand for the $\alpha_s$ corrections to the field strength correlator and to the potentials respectively, and the last term accounts for the contribution induced by a change of scale in the $\text{N}^3\text{LO}$ calculation.
First, we shall consider the contribution (24) to the field strength correlator. After integration over \( t \), which can be done using Eq. (13), we obtain

\[
G_{(EE)_{O(\alpha_s)}}(r^2) = \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \alpha_s^3(1/r) C_F C_A^3 \frac{1}{8} \frac{1}{r} \times \left[ A + \frac{B}{\dot{\epsilon}} + C_1 \log^2 \frac{V_o - V_s}{\mu} + C_2 \log \frac{V_o - V_s}{\mu} + D \right],
\]

with

\[
A = \frac{1}{24} \left( \frac{4 T_F n_f}{3} - \frac{11}{3} C_A \right), \quad (27)
\]
\[
B = \frac{1}{108} \left[ -10 T_F n_f + C_A \left( 6 \pi^2 + 47 \right) \right], \quad (28)
\]
\[
C_1 = \frac{1}{6} \left( - \frac{4 T_F n_f}{3} + \frac{11}{3} C_A \right), \quad (29)
\]
\[
C_2 = \frac{1}{54} \left[ 4 T_F n_f (10 - 6 \log 2) + C_A \left( -149 + 66 \log 2 - 12 \pi^2 \right) \right], \quad (30)
\]
\[
D = \frac{1}{9} \left[ T_F n_f \right. \left( - \frac{67}{9} + \frac{5}{6} \gamma_E + 6 \log \frac{V_o - V_s}{\mu} - 2 \log^2 \frac{V_o - V_s}{\mu} - \frac{5}{6} \log \pi - \frac{\pi^2}{3} \right)
\]
\[
+ \frac{9}{2} \pi^2 - \frac{\gamma_E}{2} \left. \pi^2 - \pi^2 \log 2 + \frac{\pi^2}{2} \log \pi \right], \quad (31)
\]

Next, we display the contribution that we obtain if in (8) we use the leading-order expression for the chromoelectric correlator but the \( O(\alpha_s) \) correction for \( V_o - V_s \):

\[
G_{(EE)_{O(\alpha_s)}}^{(r^2)}(r^2) = \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \alpha_s^4(1/r) C_F C_A^3 \frac{1}{8} \frac{1}{r} \left( a_1 + 2 \gamma_E \beta_0 \right) \times \left[ \frac{1}{2} \dot{\epsilon} - \log \left( \frac{V_o - V_s}{\mu} \right) + \frac{5}{6} - \log 2 \right].
\]

The ultraviolet divergences in the expressions (26) and (32) come from the integration over time in (8). They can be absorbed by a renormalization of the potential, analogous to (15).

Finally, we obtain another contribution if in the renormalized version of (14) we change \( \alpha_s(\mu) \) to \( \alpha_s(1/r) \) (we want all \( \alpha_s \) evaluated at the scale \( 1/r \)):
\[ G^{(r^2)}_{\mu \to 1/r|\mathcal{O}(\alpha_s)} \frac{\alpha_s^5(1/r)}{\pi^2} C_F \frac{C_A^3}{24} \frac{1}{r} \beta_0 \log(r\mu) \]
\[ \times \left[ \log \left( \frac{V_o - V_s}{\mu} \right) + \log 2 - \frac{5}{6} \right]. \] (33)

Adding up the renormalized versions of (26) and (32) and equation (33), we obtain the contribution of \( \delta_{US} \) to \( E_0(r) \) at order \( \alpha_s^5 \). The complete calculation of \( E_0(r) \) at this order requires the knowledge of \( V_s(r; \mu) \) at the same order. However, to obtain the terms proportional to \( \log \alpha_s \) it is enough to enforce \( E_0(r) \) to be independent of the factorization scale \( \mu \). This constrains the terms \( \alpha_s^5 \log^2 r\mu \) and \( \alpha_s^5 \log r\mu \) of the singlet static potential to be

\[ \delta V_s(r; \mu) = -\frac{C_F \alpha_s(1/r)}{r} \left( \frac{\alpha_s(1/r)}{4\pi} \right)^4 \]
\[ \times \left\{ 16\pi^2 \frac{C_A^3}{3} \left( \frac{11}{3} C_A + \frac{4}{3} T_F n_f \right) \log^2 r\mu \right. 
\[ + 16\pi^2 C_A^3 \left[ a_1 + 2\gamma_E \beta_0 - \frac{20}{27} T_F n_f + C_A \left( \frac{94}{27} + \frac{4}{3} \pi^2 \right) \right] \log r\mu \right\}. \] (34)

Summing (34) with (25) provides the coefficients \( a_{L2}^4 \) and \( a_4^L \) of the static energy \( E_0(r) \) given in Eqs. (4) and (5) respectively.

Note that: (i) in order to cancel the \( \mu \) dependence of the two double logarithms in \( \delta_{US} \), \( \log(r\mu) \log((V_o - V_s)/\mu) \) and \( \log^2((V_o - V_s)/\mu) \), against the single double logarithm in \( \delta V_s \), \( \log^2 r\mu \), the coefficient of \( \log(r\mu) \log((V_o - V_s)/\mu) \) must be twice the one of \( \log^2((V_o - V_s)/\mu) \). (ii) The coefficient of the double logarithm \( \log^2 r\mu \) in \( \delta V_s \) should coincide with the one obtained expanding the renormalization group improved static potential of [17]. (iii) The coefficients \( a_{L2}^4 \) and \( a_4^L \) must be renormalization scheme independent.\(^6\) We have explicitly checked that our result satisfies these requirements.

4 Conclusions

We have calculated the ultrasoft contribution to the QCD static energy of a quark-antiquark pair at order \( \alpha_s^5 \). This is sufficient to obtain the logarithmic contribution to the pNRQCD singlet static potential at N^4LO, which, in turn, provides the \( \alpha_s^5 \log^2 \alpha_s/r \) and \( \alpha_s^5 \log \alpha_s/r \) terms of the static energy of a quark-antiquark pair at distance \( r \). The calculation heavily relies on effective field

\(^6\) For a calculation of \( \delta V_s(r; \mu) \) in the subtraction scheme \( 1/\epsilon - \gamma_E + \log \pi \) we refer to [29].
theory techniques and uses the result of Ref. [23] as a key ingredient.

Possible applications of the result include precision comparisons with lattice data, heavy quarkonium spectra and $t\bar{t}$ production near threshold.

At short distances, the perturbative expression of the QCD static energy has been compared with lattice data at N$^2$LO in [30,31] and at N$^2$LL in [14]. Our analysis provides a key ingredient for a N$^3$LL analysis.

Starting from the N$^3$LO in $\alpha_s$, the quarkonium mass becomes sensitive to the ultrasoft scale, if the ultrasoft scale is assumed to be much larger than $\Lambda_{\text{QCD}}$ [32,33,24,34]. Our result also provides an important ingredient for the calculation of the quarkonium mass at N$^3$LL accuracy.

Top-quark pair production near threshold, which will become an important production process at the ILC, is presently known at N$^2$LO [35]. The cross section at N$^2$LL (see e.g. [36,37]) and at N$^3$LO (see e.g. [38,39]) is computed presently by several different groups. Our result will contribute to the cross section at N$^3$LL. The third-order renormalization group improved expression will be needed to resum logarithms potentially as large as the N$^3$LO and reduce the scale dependence of the cross section.

Acknowledgments

We are grateful to Matthias Jamin for making us available the details of the calculations reported in Ref. [23]. N.B. and A.V. thank Yu-Qi Chen, Carlo Ewerz and Yu Jia for discussions. Part of this work has been carried out at ECT*, Trento, in August 2006, during the program “Heavy quarkonium and related heavy quark systems”. We acknowledge financial support from “Azioni Integrate Italia-Spagna 2004 (IT1824)/Acciones Integradas Españ&-Italia (HI2003-0362)”, and from the cooperation agreement INFN05-04 (MEC-INFN). X.G.T. and J.S. are also supported by MEC (Spain) grant CYT FPA 2004-04582-C02-01, the CIRIT (Catalonia) grant 2005SGR00564 and the network Euridice (EU) HPRN-CT2002-00311. X.G.T. acknowledges financial support from the DURSI of the Generalitat de Catalunya and the Fons Social Europeu. The work of X.G.T. was also supported in part by the U.S. Department of Energy, Division of High Energy Physics, under contract W-31-109-ENG-38. A.V. acknowledges the financial support obtained inside the Italian MIUR program “incentivazione alla mobilità di studiosi stranieri e italiani residenti all’estero.”
References

[1] N. Brambilla et al., *Heavy quarkonium physics*, CERN-2005-005, (CERN, Geneva, 2005) [arXiv:hep-ph/0412158].

[2] K. G. Wilson, Phys. Rev. D 10 (1974) 2445.

[3] T. Appelquist, M. Dine and I. J. Muzinich, Phys. Lett. B 69 (1977) 231.

[4] T. Appelquist, M. Dine and I. J. Muzinich, Phys. Rev. D 17 (1978) 2074.

[5] W. Fischler, Nucl. Phys. B 129 (1977) 157.

[6] A. Billoeire, Phys. Lett. B 92 (1980) 343.

[7] M. Peter, Phys. Rev. Lett. 78 (1997) 602 [arXiv:hep-ph/9610209].

[8] M. Peter, Nucl. Phys. B 501 (1997) 471 [arXiv:hep-ph/9702245].

[9] Y. Schröder, Phys. Lett. B 447 (1999) 321 [arXiv:hep-ph/9812205].

[10] B. A. Kniehl, A. A. Penin, M. Steinhauser and V. A. Smirnov, Phys. Rev. D 65 (2002) 091503 [arXiv:hep-ph/0106135].

[11] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D 60 (1999) 091502 [arXiv:hep-ph/9903355].

[12] F. A. Chishtie and V. Elias, Phys. Lett. B 521 (2001) 434 [arXiv:hep-ph/0107052].

[13] A. Pineda, JHEP 0106 (2001) 022 [arXiv:hep-ph/0105008].

[14] A. Pineda, J. Phys. G 29 (2003) 371 [arXiv:hep-ph/0208031].

[15] G. Cvetic, J. Phys. G 30 (2004) 863 [arXiv:hep-ph/0309262].

[16] T. Lee, JHEP 0310 (2003) 044 [arXiv:hep-ph/0304185].

[17] A. Pineda and J. Soto, Phys. Lett. B 495 (2000) 323 [arXiv:hep-ph/0007197].

[18] A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. 64 (1998) 428 [arXiv:hep-ph/9707481].

[19] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B 566 (2000) 275 [arXiv:hep-ph/9907240].

[20] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. 77 (2005) 1423 [arXiv:hep-ph/0410047].

[21] B. A. Kniehl, A. A. Penin, Y. Schröder, V. A. Smirnov and M. Steinhauser, Phys. Lett. B 607 (2005) 96 [arXiv:hep-ph/0412083].

[22] H. G. Dosch and Yu. A. Simonov, Phys. Lett. B 205 (1988) 339.
[23] M. Eidemüller and M. Jamin, Phys. Lett. B 416 (1998) 415 [arXiv:hep-ph/9709419].

[24] B. A. Kniehl, A. A. Penin, V. A. Smirnov and M. Steinhauser, Nucl. Phys. B 635 (2002) 357 [arXiv:hep-ph/0203166].

[25] N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D 67 (2003) 034018 [arXiv:hep-ph/0208019].

[26] N. Brambilla, D. Gromes and A. Vairo, Phys. Lett. B 576 (2003) 314 [arXiv:hep-ph/0306107].

[27] M. Jamin, private communication.

[28] M. Neubert, Phys. Rept. 245 (1994) 259 [arXiv:hep-ph/9306320].

[29] X. Garcia i Tormo, Ph.D. Thesis (Barcelona, 2006) [arXiv:hep-ph/0610145].

[30] Y. Sumino, Phys. Rev. D 65 (2002) 054003 [arXiv:hep-ph/0104259].

[31] S. Necco and R. Sommer, Nucl. Phys. B 622 (2002) 328 [arXiv:hep-lat/0108008].

[32] B. A. Kniehl and A. A. Penin, Nucl. Phys. B 563 (1999) 200 [arXiv:hep-ph/9907489].

[33] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Lett. B 470 (1999) 215 [arXiv:hep-ph/9910238].

[34] A. A. Penin and M. Steinhauser, Phys. Lett. B 538 (2002) 335 [arXiv:hep-ph/0204290].

[35] A. H. Hoang et al., Eur. Phys. J. directC 2 (2000) 1 [arXiv:hep-ph/0001286].

[36] A. H. Hoang, Phys. Rev. D 69 (2004) 034009 [arXiv:hep-ph/0307376].

[37] A. Pineda and A. Signer, Nucl. Phys. B 762 (2007) 67 [arXiv:hep-ph/0607239].

[38] M. Beneke, Y. Kiyo and K. Schuller, Nucl. Phys. B 714 (2005) 67 [arXiv:hep-ph/0501289].

[39] A. A. Penin, V. A. Smirnov and M. Steinhauser, Nucl. Phys. B 716 (2005) 303 [arXiv:hep-ph/0501042].