Non-planar spin bits beyond two loops

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Abstract

We study higher-loop orders of spin bit models underlying the non-planar dynamics of $\mathcal{N} = 4$ SYM gauge theory. In particular, we derive a "tower" of non-planar identities involving products of site permutation operators. Such identities are then applied in the formulation of planarly consistent, testable conjectures for the full non-planar, higher-loop Hamiltonian of the $su(2)$ spin-chain.

1 Introduction

Large $N$ physics [1] gained considerable interest in recent years (see [2] for a recent review and references) due to the AdS/CFT conjecture enlightenment [3] [4] and, more recently, by considering various limits of this correspondence [5, 15]. This led to an intensive study of the anomalous dimensions of local gauge invariant composite operators in 4-dimensional (4-d.) $\mathcal{N} = 4$ Super Yang–Mills (SYM) model [16]. A real breakthrough was the discovery of the integrability of the Hamiltonians governing anomalous dimensions in the planar limit, $N \to \infty$ [17] [18]. These results were extended to two and higher loops [19] [20].

As it is now clear, there is a one-to-one correspondence between single trace operators in SYM theory and spin states in spin-chain models. It was enough
to consider the planar limit of SYM theory. If the non-planar contribution is considered, the single trace sector is not conserved anymore, and one ends up with trace splitting and joining in the operator mixing \[21\]. Even in this case one can still consider a one-to-one map between local gauge invariant operators and a spin system \[22, 23\]. In this case one has to introduce a set of new degrees of freedom, beyond the spin states, which describes the linking structure of the sites in the spin-chain. This new field takes values in the spin bits permutation group, and introduces a new gauge degree of freedom.

In this paper we extend the analysis of \[24\] to formulate (planarly consistent) conjectures about the explicit form of the full non-planar \(su(2)\) spin-chain Hamiltonian beyond the 2-loop order. We will extensively make use of some identities between (products of) permutation operators, holding true at the non-planar level and here rigorously obtained, clarifying their relation with the already known formulae in the literature (see, e.g., \[25\]).

The plan of the paper is as follows. In the next Section we introduce the notations; then in Section 3 we obtain the above mentioned non-planar permutational identities by simple antisymmetrization procedures. Thence, in Section 4 we face the problem to go beyond 2-loops in non-planar spin bit models. We formulate a set of (planarly consistent) higher-loop Ansätze in the \(su(2)\) sector of \(\mathcal{N} = 4\) SYM theory, by using proper "deplanarizing lifts" of site permutation operators. Finally, in the Section 5 we draw some conclusions and perspectives for further future developments.

Three Appendices conclude the paper: in the Appendix I we specialize the previously obtained results on non-planar permutational identities to the \(su(2)\) sector, particularly in relation to ultra-localization by Pauli \(\sigma\)-matrices; thence, in Appendix II the planar limit of the permutational identities and their relation with known results are clarified; finally, in Appendix III we consider in detail some "spin edge-differences" Ansätze for the higher-loop non-planar \(su(2)\) (closed) spin-chain Hamiltonian. However, despite their intrinsic geometrical elegance, such conjectures appear to fail, stressing the procedure based on operator "deplanarizing lifts" as being the only planarly consistent approach to
non-planar $su(2)$ sector of 4-d. $\mathcal{N} = 4$ SYM theory.

In this paper we use conventions and notations of [22, 23, 24].

2 The setup

We consider the $su(2)$ sector of local gauge invariant SYM operators which are generated by two holomorphic (multi)trace operators built out of two complex SYM scalars $\phi = \phi_5 + i\phi_6$ and $Z = \phi_1 + i\phi_2$, of typical form

$$O = \text{Tr}(\phi Z \phi Z \ldots) \text{Tr}(\phi \phi Z \ldots) \text{Tr}(\ldots) \ldots$$

This trace can be written in the following explicit form, using a permutation group element $\gamma \in S_L$:

$$O = \phi^{a_1 \gamma_1}_{a_1} \phi^{a_2 \gamma_2}_{a_2} \ldots \phi^{a_L \gamma_L}_{a_L},$$

(1)

where $L$ is the total number of “letters” $\phi = (\phi, Z)$ in $O$ which are numbered by a label $k = 1, \ldots, L$. The permutation group action $\gamma_k$ on the $k$-th label gives the next multiplier to the $k$-th one:

$$\gamma \equiv (\gamma_1 \gamma_2 \ldots \gamma_k \ldots \gamma_L) : \begin{pmatrix} 1 & 2 & \ldots & k & \ldots & L \\ \gamma_1 & \gamma_2 & \ldots & \gamma_k & \ldots & \gamma_L \end{pmatrix} \in S_L.$$

Obviously, the reshuffling of the labels $k \rightarrow \sigma_k$ accompanied by a conjugation of $\gamma$ with the same group element $\sigma^{-1} \cdot \gamma \cdot \sigma$ leaves the trace form of $O$ unchanged. Therefore, the configurations related by such a transformation should be considered as equivalent

$$(\phi_k, \gamma) \sim (\phi_{\sigma k}, \sigma^{-1} \cdot \gamma \cdot \sigma).$$

(2)

Now, we should map the space of such operators to the system of $L \, su(2)$ 1/2-spins (spin bits). The map is completed by associating to each bit the spin value $|{-1/2}\rangle$ when we find in the respective place the letter $\phi$, and $|{+1/2}\rangle$, when we find $Z$. 

3
In perturbation theory the anomalous dimension matrix is given by
\[ \Delta(g) = \sum_k H_{2k} \lambda^{2k}, \quad (3) \]
with \( \lambda^2 = \frac{g^2 N}{16 \pi^2} \) being the ’t Hooft coupling. The coefficients in this expansion are given in terms of effective vertices, i.e. the operators \( H_{2k} \). In principle, they can be completely determined by an explicit evaluation of the divergencies of two-point function \( \langle \mathcal{O}(0)\mathcal{O}(x) \rangle \) Feynman amplitudes; but such an approach is hardly feasible, specially for higher-loop orders. The procedure based on spin bit models represents a much simpler method to perform calculations, also at the non-planar level.

The 0-, 1- and 2- loop, \( su(2) \) anomalous dimension matrices are given by the following expressions \(^1\) \[25]:

\[ H_0 = \text{Tr}(\phi\phi + Z\tilde{Z}), \quad (4) \]
\[ H_2 = -\frac{2}{N} : \text{Tr}(\{\phi, Z\}[\phi, \tilde{Z}]) :, \quad (5) \]
\[ H_4 = \frac{1}{N^2} \left( 2 : \text{Tr}(\{Z, \phi\}[\tilde{Z}, [Z, \phi]]) : + 2 : \text{Tr}(\{Z, \phi\}[\phi, \{Z, \phi]\}) : + 2N : \text{Tr}(\{\phi, Z\} [\phi, \tilde{Z}]) : \right) , \quad (6) \]

where the checked letters \( \tilde{\phi} \) and \( \tilde{Z} \) correspond to derivatives with respect to the matrix elements
\[ \tilde{Z}_{ij} = \frac{\partial}{\partial Z^{ji}}, \quad \tilde{\phi}_{ij} = \frac{\partial}{\partial \phi^{ij}} \quad (7) \]
and colons denote the ordering, in which all checked letters in the group are assumed to stay on the right of the unchecked ones.

In order to find the “pull back” of the Hamiltonian \( (3) \) to the spin description, one has to apply it on a (multi)trace operator corresponding to the spin bit state \(|s, \gamma\rangle\) and map the result back to the corresponding spin bit state. This can be done term-by-term in the perturbation theory expansion series.

\(^1\) Notice that, in order to obtain the correct planar limit, the expression for \( H_4 \) (corresponding, modulo a \( \frac{1}{N} \) overall factor, to Eq.(5.5) of \[25]) must be properly modified, by changing \( N \) into \( \frac{1}{N} \) in front of the third term : \( \text{Tr}(\{\phi, Z\} [\phi, \tilde{Z}]) : \).
The 1-loop Hamiltonian was found earlier \cite{22, 23} and reads\(^2\)

\[
H_2 = \frac{1}{N} \sum_{k_1, k_2 = 1, (k_1 \neq k_2)}^{L} \left( \Sigma_{\gamma_1, k_2} + \Sigma_{k_1, \gamma_2} - \Sigma_{k_1, k_2} - \Sigma_{\gamma_1, \gamma_2} \right) = \frac{1}{N} \sum_{k_1, k_2 = 1}^{L} \left( (P_{k_1, k_2} - 1) \left( \Sigma_{k_1, k_2} + \Sigma_{\gamma_1, \gamma_2} - \Sigma_{\gamma_1, k_2} - \Sigma_{k_1, \gamma_2} \right) \right), \tag{8}
\]

where the site index permutation and chain “twist” operators are respectively defined in the following way \((k_1, k_2 = 1, ..., L)\):

\[
P_{k_1, k_2} \left| \{ \ldots A_{k_1} \ldots A_{k_2} \ldots \} \right> = \left| \{ \ldots A_{k_2} \ldots A_{k_1} \ldots \} \right>, A_{k_2}, A_{k_1} = \phi, Z \tag{9}
\]

\[
\Sigma_{k_1, k_2} \left| \gamma \right> = \begin{cases} 
\left| \gamma \sigma_{k_1, k_2} \right> & \text{if } k_1 \neq k_2 \\
N \left| \gamma \right> & k_1 = k_2.
\end{cases} \tag{10}
\]

\(\Sigma_{k_1, k_2}\) acts as a chain splitting and joining operator. The factor \(N\) in the case \(k_1 = k_2\) in Eq. \((9)\) appears because the splitting of a trace at the same place leads to a chain of length zero, whose corresponding trace is \(\text{Tr} 1 = N\). It is important to note that the operator \(\Sigma_{k_1, k_2}\) acts only on the linking variable, while the two-site \(su(2)\) 1-loop spin bit Hamiltonian \(H_{k_1, k_2} = (1 - P_{k_1, k_2})\) acts on the spin space. Therefore, the two operators commute.

\(^2\)Because of the periodic boundary conditions (p.b.c.) assumed for the closed spin-chain, the subscript site indices may equivalently range in \(\{1, ..., L\}\) or in \(Z_L\), i.e. in the integer numeric field with period \(L\).

\(H_{k_1, k_2} \equiv 1 - P_{k_1, k_2}\) is the two-site, planar \(su(2)\) 1-loop spin bit Hamiltonian; it is nothing but twice the site index antisymmetrizer \(\frac{1}{2}(1 - P_{k_1, k_2})\), and therefore it makes the constraint \(k_1 \neq k_2\) redundant.
3 Identities of permutation operators at the non-planar level

Given three site indices $k_1, k_2, k_3$ all different from each other (this assumption will be denoted, here and further below, by the notation $k_1 \neq k_2 \neq k_3$), the following identity holds ($(k_1, k_2, k_3) \in \{1, ..., L\}^3, k_1 \neq k_2 \neq k_3)$:

$$P_{k_1k_2}P_{k_1k_3} = P_{k_2k_3}P_{k_1k_2} = P_{k_1k_3}P_{k_2k_3},$$  \hspace{1cm} (11)

expressing the redundancy in the product of two permutation operators with one repeated index ($L$ may be here considered as the total number of spin-chain sites, and also the spin-chain length, if a unit distance between adjacent sites is assumed).

Notice that, differently from the identities reported below, (11) is always true, even out of the $su(2)$ symmetry operatorial sector of the considered 4-d. $\mathcal{N} = 4$ SYM theory.

Many other identities, holding at the non-planar level, may be obtained by simply antisymmetrizing tensors of rank $\geq M$ in operatorial sectors with spin(-site indices) having range(s) with cardinality $\leq M - 1$ on each site. In the following treatment, we will mainly focus on the $su(2)$ symmetry sector, with a spin $s = \frac{1}{2}$ irreducible representation on each site, whence $M - 1 = 2s + 1 \Leftrightarrow M = 3$.

Thus, from the antisymmetrization of a 3-index tensor we get (1 denotes, here and further below, the permutational identity operator and, as before, $(k_1, k_2, k_3) \in \{1, ..., L\}^3, k_1 \neq k_2 \neq k_3)$:

$$1 - P_{k_1k_2} - P_{k_1k_3} - P_{k_2k_3} + P_{k_1k_2}P_{k_2k_3} + P_{k_2k_3}P_{k_1k_2} = 0,$$  \hspace{1cm} (12)

$^3$Round brackets may equivalently be put or not, because of $P_{k_1k_2} = P_{\hat{S}_{k_1}\hat{S}_{k_2}}$, $\hat{S}_{k_1}$ and $\hat{S}_{k_2}$ denoting the set of spin operators on the $k_1$-th and $k_2$-th spin-chain sites, respectively.
holding true in general in each symmetry sector with spin(-site indices) ranging in sets with cardinality \( \leq 2 \).

Analogously, antisymmetrizing a 4-index tensor, we get \(((k_1, k_2, k_3, k_4) \in \{1, \ldots, L\}^4, k_1 \neq k_2 \neq k_3 \neq k_4,)\):

\[
1 - P_{k_1k_2} - P_{k_1k_3} - P_{k_1k_4} - P_{k_2k_3} - P_{k_2k_4} - P_{k_3k_4} +
+ P_{k_3k_4} P_{k_2k_3} + P_{k_3k_4} P_{k_2k_4} + P_{k_3k_4} P_{k_1k_2} + P_{k_3k_4} P_{k_1k_3} + P_{k_3k_4} P_{k_1k_4} +
+ P_{k_2k_3} P_{k_1k_2} + P_{k_2k_3} P_{k_1k_3} + P_{k_2k_3} P_{k_1k_4} +
+ P_{k_2k_4} P_{k_1k_2} + P_{k_2k_4} P_{k_1k_3} + P_{k_2k_4} P_{k_1k_4} +
- P_{k_3k_4} P_{k_2k_3} P_{k_1k_2} + P_{k_3k_4} P_{k_2k_3} P_{k_1k_3} + P_{k_3k_4} P_{k_2k_3} P_{k_1k_4} +
- P_{k_3k_4} P_{k_2k_4} P_{k_1k_2} + P_{k_3k_4} P_{k_2k_4} P_{k_1k_3} + P_{k_3k_4} P_{k_2k_4} P_{k_1k_4} = 0,
\]

(13)

which is true in general, in each symmetry sector with spin(-site indices) ranging in sets with cardinality \( \leq 3 \).

Looking at (12) and (13) and using Eq. (11), it can be shown by explicit calculations that such identities have, respectively, the following structure:

\[
\sum_{\pi \in S_3} (\sigma)^{\sigma_{\pi}} P_{\pi} = 0, \tag{14}
\]

\[
\sum_{\pi \in S_4} (\sigma)^{\sigma_{\pi}} P_{\pi} = 0; \tag{15}
\]

Here, as previously defined, \( S_3 \) and \( S_4 \) are the permutation groups of 3 and 4 different spin-chain sites, and \( \pi \) stands for one element of such groups; its realization in terms of (a product of) spin-chain site pair permutation operators \( P \)'s is denoted with \( P_{\pi} \). \( \sigma_{\pi} \) is defined "length of permutation \( \pi \)" and depends on the realization through \( P \)'s: \( \pi \) will have \( \sigma_{\pi} = 0, 1, 2, 3, \ldots \) if it is proportional to identity operator, linear, quadratic, cubic and so on in \( P \)'s, respectively. Finally, \( (\sigma)^{\sigma_{\pi}} \) is called "parity" of the \( \sigma_{\pi} \)-lengthed permutation \( \pi \). Notice that, because of the cardinality of discrete groups \( S_3 \) and \( S_4 \) is respectively 6 and 24, Eqs. (12) and (13) do contain the right number of terms in their l.h.s.s.

Therefore, it is possible to argue the general structure of the identities on permutation operators \( P \)'s, arising from the antisymmetrization procedure of
\( M(\geq 3) \)-index tensors in symmetry sectors of 4-d. \( \mathcal{N} = 4 \) SYM theory having spin(-site indice)s ranging in sets with cardinality \( \leq M - 1 \):

\[
\sum_{\pi \in S_M} (-)^{\sigma_\pi} P_{\pi} = 0,
\]

with the l.h.s. containing \( M! \) independent permutations (corresponding to all the elements of the group \( S_M \)), realized through (product of) site pair permutation operators \( P \)'s, acting on a set \( \{k_1, k_2, \ldots, k_M\} \) of \( M \) different spin-chain site indices.

Notice that all identities obtainable by varying \( M \geq 3 \) in (16) are independent, i.e. they cannot be obtained one from the other by applying one or more permutation operators. This is trivially evident by comparing (12) with (13), because the application of \( P \)'s carrying new spin(-site indice)s does not allow to obtain the correct number of independent higher-order products of \( P \)'s.

At this point, it would be very interesting, and we leave it for further future investigations, to see if general recursive (possibly algorithmical) formulae for (16) as function of \( M \geq 3 \) may exist and be given; if so, probably they quite intriguingly link the identities among (products of) permutation operators \( P \)'s with the random graph theory in a discrete set of sites. An interesting task would be to study such a connection, depending on the imposed boundary conditions (periodic in our case), intimately related to the kind of string spin bits yield a dynamical discretization of.

4 "Deplanarizing operator lifts" and planarly consistent higher-loop Ansätze in the \( su(2) \) sector of 4-d. \( \mathcal{N} = 4 \) SYM theory

Let us now consider the non-planar spin-chain Hamiltonian in the \( su(2) \) sector of the 4-d. \( \mathcal{N} = 4 \) SYM theory.

As already reported in Eq. (8), at the 1-loop level in perturbation theory it reads \[22, 23\] (for simplicity’s sake, we omit to "check" the symbols denoting
spin operators, as instead rigorously done in Appendix I)

\[ \begin{align*}
H_2 &= \frac{2}{N} \sum_{(k_1, k_2) \in \mathbb{Z}_L^2} (1 - P_{k_1, k_2}) \Sigma_{k_1 \gamma k_2} = \frac{2}{N} \sum_{(k_1, k_2) \in \mathbb{Z}_L^2} H_{k_1, k_2} \Sigma_{k_1 \gamma k_2} = \\
&= \frac{2}{N} \sum_{(k_1, k_2) \in \mathbb{Z}_L^2, \ k_1 \neq k_2} H_{k_1, k_2} \Sigma_{k_1 \gamma k_2} = \\
&= \frac{2}{N} \sum_{(k_1, k_2) \in \mathbb{Z}_L^2, \ k_1 \neq k_2} \left[ (S_{k_1}^{-} - \overrightarrow{S}_{k_2})^2 - 1 \right] \Sigma_{k_1 \gamma k_2},
\end{align*} \tag{17}\]

where \( H_{k_1, k_2} \) is the previously defined two-site, planar, 1-loop \( su(2) \) spin chain Hamiltonian (coinciding with twice the site antisymmetrizer), and in the last passage we used Eqs. \((12)\) and \((54)\), implying that

\[ P_{k_1, k_2} - 1 = 1 - (S_{k_1}^{-} - \overrightarrow{S}_{k_2})^2 - \delta_{k_1, k_2}, \quad \forall \ (k_1, k_2) \in \mathbb{Z}_L^2. \tag{18}\]

Notice that \( H_2 \) is completely symmetric under the exchange \((k_1, k_2) \leftrightarrow (k_2, k_1)\), as it has to be from the site-index structure obtained from the linking part \( \Sigma_{k_1 \gamma k_2} \) of the Hamiltonian.

Analogously, at the 2-loop level, in \([24]\) the following form for the non-planar \( su(2) \) spin-chain Hamiltonian was obtained:

\[ \begin{align*}
H_4 &= \frac{2}{N^2} \sum_{(k_1, k_2, k_3) \in \mathbb{Z}_L^3} (2P_{k_1, k_2} - P_{k_1, k_3} + 2P_{k_2, k_3} - 3) \Sigma_{k_1 \gamma k_2} \Sigma_{k_2 \gamma k_3} = \\
&= -\frac{2}{N^2} \sum_{(k_1, k_2, k_3) \in \mathbb{Z}_L^3, \ k_1 \neq k_2 \neq k_3} (2H_{k_1, k_2} - H_{k_1, k_3} + 2H_{k_2, k_3}) \Sigma_{k_1 \gamma k_2} \Sigma_{k_2 \gamma k_3} = \\
&= \frac{2}{N^2} \sum_{(k_1, k_2, k_3) \in \mathbb{Z}_L^3, \ k_1 \neq k_2 \neq k_3} \left( \overrightarrow{S}_{k_1} - 2\overrightarrow{S}_{k_2} + \overrightarrow{S}_{k_3} \right)^2 \Sigma_{k_1 \gamma k_2} \Sigma_{k_2 \gamma k_3}. \tag{19}\]

Notice that \( H_4 \) is completely symmetric under the exchange \((k_1, k_2, k_3) \leftrightarrow (k_3, k_2, k_1)\), as it has to be from the site-index structure obtained from the linking part \( \Sigma_{k_1 \gamma k_2} \Sigma_{k_2 \gamma k_3} \) of the Hamiltonian.

By comparing Eqs. \((17)\) and \((19)\), it appears reasonable to formulate an elegant and simply-meaning set of conjectures, named "spin edge-differences" \( Ansätze \), for the explicit form of the \( n \)-loop, non-planar \( su(2) \) spin-chain Hamiltonian \( H_{2n} \); they are treated in some detail in Appendix III. Such \( Ansätze \) unfortunately appear to fail; therefore, if we want to go beyond the 2-loop level in
(non-planar) spin bit models, we have to find other ways to formulate consistent Ansätze for the explicit form of $H_{2n}$, for $n \geq 3$.

To achieve this, let us start introducing the compact notation \[25\]

$$\{n_1, n_2, \ldots\} \equiv \sum_{k=1}^{L} P_{k+n_1,k+n_1+1} P_{k+n_2,k+n_2+1}, \quad n_1, n_2, \ldots \in \mathbb{Z},$$

and let us report the currently known planar $su(2)$ spin-chain Hamiltonians \[26\], \[27\]:

- **tree level**: $H_{0,\text{planar}} = \{\}$
- **1-loop level**: $H_{2,\text{planar}} = 2\{\} - 2\{0\}$
- **2-loop level**: $H_{4,\text{planar}} = -8\{\} + 12\{0\} - 2 (\{1, 0\} + \{0, 1\})$

- **3-loop level**: $H_{6,\text{planar}} = 60\{\} - 104\{0\} + 24 (\{1, 0\} + \{0, 1\}) + 4\{0, 2\} - 4 (\{0, 1, 2\} + \{2, 1, 0\})$

- **4-loop level**: $H_{8,\text{planar}} (\beta) = -560\{\} + (1036 + 4\beta)\{0\} + (-266 - 4\beta)(\{0, 1\} + \{1, 0\}) + (-66 - 2\beta)\{0, 2\} - 4\{0, 3\} + 4(\{0, 1, 3\} + \{0, 2, 3\} + \{0, 3, 2\} + \{1, 0, 3\})$

where in the last 4-loop expression $\beta$ is an undetermined real parameter; it is unphysical, because it corresponds to rotations of the space of the states of the $su(2)$ spin-chain system, and therefore it changes the (unphysical) eigenstates, but not the (physical) eigenvalues \[27\].

10
As already mentioned in [24], from the definition (10) of the "twist" operator Σ_{kl} the following decomposition holds:

\[ Σ_{kl} = Nδ_{kl} + (1 - δ_{kl}) \tilde{Σ}_{kl}, \]  

(26)

where \( \tilde{Σ}_{kl} \) is the "real" chain splitting and joining operator, spoiled of its degeneracy in the case of coinciding sites. Whence the planar limit \( N \to \infty \) affects just the "twist" operator, and it does in the following way:

\[ \lim_{N \to \infty} \frac{1}{N} Σ_{kl} = δ_{kl}. \]  

(27)

Consequently, using the fact that, from Eq. (2) and from the periodic boundary conditions for the closed spin-chain, the following identity trivially holds (\( ∀γ ∈ S_L \)):

\[ \sum_{k \in \mathbb{Z}_L} P_{kγ_k} = \sum_{k \in \mathbb{Z}_L} P_{γ_kγ_k^2}, \]  

(28)

and passing to a canonical form (in which \( γ_k = k + 1, γ_k^2 = k + 2, \) and so on), it is easy to check that the planar limits of Eqs. (17) and (19) perfectly match (22) and (23), respectively.

Whence, comparing Eqs. (22) and (23) with their full non-planar versions, respectively given by Eqs. (17) and (19), it is easy to see that they are related in the following simple way: each term of the full non-planar spin part, once one identifies (without loss of generality) \( k_2 = γ_k \) and \( k_3 = γ_k^2 \), gives rise, by the use of (28), to the corresponding planar term of the spin part, with the correct coefficients.

The generalization of such a procedure to the 3- and 4-loop levels gives a (planarly consistent) way to formulate higher-loop Ansätze for the spin part of the non-planar, \( su(2) \) spin-chain Hamiltonian. For what concerns the linking variable part, i.e. the \( Σ \)'s, the same conjectures used in the "spin edge-differences" approach explained in Appendix III will be assumed; this amounts to identifying the linking variable part of the \( n \)-loop non-planar Hamiltonian with the \( (γ\)-dependent) "splitting and joining chain operator of order \( n \)”, defined as

\[ Σ_{k_0 k_1 \ldots k_n} (γ) ≡ Σ_{k_0 γ_k_1} Σ_{k_1 γ_k_2} \ldots Σ_{k_{n-2} γ_k_{n-1}} Σ_{k_{n-1} γ_k_n}, \quad γ ∈ S_L. \]  

(29)
Such a definition necessarily implies the invariance of the spin part under the "site-index inversion"

\[(k_0, k_1, k_2, ..., k_{n-1}, k_n) \leftrightarrow (k_n, k_{n-1}, k_{n-2}, ..., k_1, k_0).\]  (30)

It is worth remarking that such an assumption, beside being well-motivated from the knowledge of the first loop orders, does not take into account possible contributions from higher-loop Feynman’s diagrammatics. Indeed, a priori one should consider that the linking variable part of the \(m\)-loop non-planar Hamiltonian should include all \(\Sigma k_0 k_1 ... k_n (\gamma)\), with \(1 \leq n \leq m\); this is due to the fact that usually all structures arising at a given loop order reappear at higher orders, because an insertion of an internal loop into a lower-order diagram produces structurally equivalent contribution to the Hamiltonian. Nevertheless, as it is evident by comparing Eqs. (17) and (19), this does not happen at 1- and 2-loops, where the above-formulated conjecture based on the \((\gamma\text{-dependent})\) "splitting and joining chain operator of order \(n\)” defined in Eq. (29) perfectly matches the already known (and independently obtained) non-planar results; therefore we assume it to hold true also at higher-loop orders.

Thus, we have to start from the planar, 3- and 4-loop expressions of the \(su(2)\) spin-chain Hamiltonian, respectively given by Eqs. (24) and (25), and consider all possible products of \(P\)’s that, in the planar limit, would give the considered planar permutational term. All such non-planar permutational terms will come with free (real) coefficients, constrained by two request:

\(i)\) their sum must give the right numerical known coefficient of the considered planar permutational term;

\(ii)\) they must make the spin part of the non-planar Hamiltonian completely symmetric under the proper exchange of site indices (30), as requested by the site index structure determined by the linking part.

Then, making the (non-reductive) conventional site identifications \(k_2 = \gamma_k, \ k_3 = \gamma^2_k\) and so on, and adding the linking variable using the definition (29), a complete Ansatz for the full non-planar, \(su(2)\) spin-chain Hamiltonian at the
considered loop order is obtained$^4$.

Let us consider an explicit example of the described method, in order to build a consistent Ansatz for the expression of the full non-planar, 3-loop $su(2)$ spin-chain Hamiltonian.

We start from the expression of $H_{6,\text{planar}}$ given by (24). By conventionally identifying (without loss of generality) the spin chain-site indices in the following way:

\[ k_1 \equiv k, \quad k_2 \equiv \gamma_k, \quad k_3 \equiv \gamma_k^2, \quad k_4 \equiv \gamma_k^3, \quad \text{(31)} \]

we therefore have to find all possible non-planar permutational terms giving rise, in the planar limit $N \to \infty$, to each of the permutational terms of $H_{6,\text{planar}}$ given by Eq. (24). We have that:

\begin{enumerate}
\item The planar term $P_{k\gamma_k}$ receives three kind of contributions from the non-planar level, respectively from $P_{k\gamma_k} = P_{k_1k_2}$, $P_{k_2k_3} = P_{k_1k_2}$, and $P_{k_3k_4} = P_{k_1k_2}$, whence the proper "deplanarizing operator lift" of $P_{k\gamma_k}$ reads ($\xi_1, \xi_2 \in \mathbb{R}$)

\[ -26P_{k\gamma_k} \to \xi_1 P_{k_1k_2} + \xi_2 P_{k_2k_3} - (26 + \xi_1 + \xi_2)P_{k_3k_4}; \quad \text{(32)} \]

\item The planar-level product $P_{k\gamma_k}P_{\gamma_k \gamma_k}$ instead receives contribution just from two non-planar terms, i.e. $P_{k\gamma_k}P_{\gamma_k \gamma_k} = P_{k_1k_2}P_{k_2k_3}$ and $P_{\gamma_k \gamma_k}P_{\gamma_k \gamma_k} = P_{k_2k_3}P_{k_3k_4}$, whence the proper "deplanarizing operator lift" of the term $P_{k\gamma_k}P_{\gamma_k \gamma_k}$ reads ($\xi_3 \in \mathbb{R}$)

\[ 6P_{k\gamma_k}P_{\gamma_k \gamma_k} \to \xi_3 P_{k_1k_2}P_{k_3k_3} + (6 - \xi_3)P_{k_2k_3}P_{k_3k_4}; \quad \text{(33)} \]

Analogously, for the other terms of $H_{6,\text{planar}}$ we obtain the following proper "deplanarizing operator lifts" ($\xi_4 \in \mathbb{R}$):

\[ 6P_{\gamma_k \gamma_k}P_{\gamma_k \gamma_k} \to \xi_4 P_{k_2k_3}P_{k_3k_4} + (6 - \xi_4)P_{k_3k_4}P_{k_2k_3}, \]

\[ P_{k\gamma_k}P_{\gamma_k \gamma_k} \to P_{k_1k_2}P_{k_3k_4}, \]

\[ P_{\gamma_k \gamma_k}P_{\gamma_k \gamma_k} \to P_{k_1k_2}P_{k_2k_3}P_{k_3k_4}, \]

\[ P_{\gamma_k \gamma_k}P_{\gamma_k \gamma_k} \to P_{k_3k_4}P_{k_2k_3}P_{k_1k_2}. \quad \text{(34)} \]
\end{enumerate}

$^4$It should be noticed that we assume that (eventually rather structurally complicated) non-planar permutational terms, such that their planar limit is zero, do not exist; indeed, for the time being, their existence may not be guessed by an inferring approach starting from the planar level, such as the one adopted in this paper.
Finally, considering $\frac{1}{N^3} \Sigma_{k_1,k_2,k_3,k_4} (\gamma)$ as the linking variable part, and therefore imposing the symmetry of the spin part under the site index exchange

$$(k_1, k_2, k_3, k_4) \leftrightarrow (k_4, k_2, k_3, k_1), \quad (35)$$

we may write the following expression for the non-planar, 3-loop $su(2)$ spin-chain Hamiltonian ($\alpha_1, \alpha_2 \in R$):

$$H_6 (\alpha_1, \alpha_2) = \frac{4}{N^3} \sum_{(k_1,k_2,k_3,k_4) \in \mathbb{Z}_L^4} \left[ 15 + \alpha_1 (P_{k_1k_2} + P_{k_3k_4}) - 2 (\alpha_1 + 13) P_{k_2k_3} + \alpha_2 (P_{k_1k_2}P_{k_3k_4} + P_{k_2k_3}P_{k_3k_4}) + (6 - \alpha_2) (P_{k_2k_3}P_{k_1k_2} + P_{k_2k_3}P_{k_3k_4}) + P_{k_1k_2}P_{k_3k_4} - P_{k_1k_2}P_{k_2k_3}P_{k_3k_4} - P_{k_3k_4}P_{k_2k_3}P_{k_1k_2} \right] \Sigma_{k_1,k_2,k_3,k_4} (\gamma). \quad (36)$$

Notice that $H_6$ is completely symmetric under the site index exchange

$$(k_1, k_2, k_3, k_4) \leftrightarrow (k_4, k_2, k_3, k_1), \quad (37)$$

as it has to be from the site-index structure obtained from the linking part $\Sigma_{k_1\gamma_{k_2}} \Sigma_{k_2\gamma_{k_3}} \Sigma_{k_3\gamma_{k_4}}$ of the Hamiltonian.

It is now convenient to introduce a non-planar generalization of the shorthand notation defined in Eq. (F.1) of [25] and previously reported in Eq. (20), in the following way:

$$\left\{ (k_1k_2, k_2k_3, k_3k_4, k_1k_4, ...)_{P}; (k_1k_3, k_1k_4, ...)_{\gamma} \right\} \equiv \sum_{k_1,k_2,k_3,k_4,\ldots \in \mathbb{Z}_L} P_{k_1k_2}P_{k_2k_3}P_{k_3k_4}P_{k_1k_4}\ldots \Sigma_{k_1\gamma_{k_3}} \Sigma_{k_3\gamma_{k_4}}\ldots, \quad (38)$$

where we have the notational identification

$$\left\{ ; (k_1k_3,k_1k_4,\ldots)_{\gamma} \right\} \equiv \left\{ 1; (k_1k_3,k_1k_4,\ldots)_{\gamma} \right\} \equiv \sum_{k_1,k_3,k_4,\ldots \in \mathbb{Z}_L} \Sigma_{k_1\gamma_{k_3}} \Sigma_{k_1\gamma_{k_4}}\ldots \cdot \quad (39)$$
With such a notation the Ansatz \(36\) for the full non-planar, 3-loop \(su(2)\) spin-chain Hamiltonian becomes

\[
H_6(\alpha_1, \alpha_2) = \frac{4}{N^3} \left[ 15 + \alpha_1 (k_1 k_2 + k_3 k_4)_\gamma - 2 (\alpha_1 + 13) (k_2 k_3)_\gamma + \alpha_2 [(k_1 k_2, k_2 k_3)_\gamma + (k_3 k_4, k_2 k_3)_\gamma] + (6 - \alpha_2) [(k_1 k_2, k_1 k_2)_\gamma + (k_2 k_3, k_3 k_4)_\gamma] + (k_1 k_2, k_3 k_4)_\gamma - (k_1 k_2, k_2 k_3, k_3 k_4)_\gamma - (k_3 k_4, k_2 k_3, k_1 k_2)_\gamma : \right]
\]

(40)

A completely analogous approach with the same steps can be performed for the 4-loop order, the only difference being that, already at the planar level, expressed by Eq. (25), the Hamiltonian depends on an undetermined real parameter \(\beta\); the final result for the Ansatz on the full non-planar, 4-loop \(su(2)\)
spin-chain Hamiltonian is the following\textsuperscript{5} $(\eta_1, \eta_2, \eta_3, \eta_4, \beta \in R)$:

\[
H_{8} (\eta_1, \eta_2, \eta_3, \eta_4; \beta) = \frac{1}{N^4} \left\{ -560 + (518 + 2\beta - \eta_1) (k_1 k_2 + k_4 k_5) + \eta_1 (k_2 k_3 + k_4 k_4) + 
+ \eta_2 [(k_1 k_2, k_2 k_3) + (k_4 k_5, k_3 k_4)] + \eta_3 [(k_2 k_3, k_3 k_4) + (k_3 k_4, k_2 k_3)] + 
- (266 + 4\beta + \eta_2 + \eta_3) [(k_3 k_4, k_4 k_5) + (k_2 k_3, k_1 k_2)] + 
- (33 + \beta) [(k_1 k_2, k_3 k_4) + (k_2 k_3, k_4 k_5)] - 4 (k_1 k_2, k_4 k_5) + 
+ 4 \left[ (k_1 k_2, k_2 k_3, k_4 k_5) + (k_1 k_2, k_3 k_4, k_4 k_5) + 
+ (k_1 k_2, k_4 k_5) \right] + 
+ \eta_4 [(k_1 k_2, k_2 k_3, k_3 k_4) + (k_4 k_5, k_3 k_4, k_2 k_3)] + 
+ (78 + 2\beta - \eta_4) [(k_2 k_3, k_3 k_4, k_4 k_5) + (k_3 k_4, k_2 k_3, k_1 k_2)] + 
+ \beta - 9 \left[ (k_1 k_2, k_3 k_4, k_4 k_5) + (k_2 k_3, k_3 k_4, k_4 k_5) + 
+ (k_1 k_2, k_3 k_4) + (k_2 k_3, k_4 k_5) \right] + 
+ (1 - \beta) \left[ (k_1 k_2, k_2 k_3, k_4 k_5, k_3 k_4) + (k_1 k_2, k_4 k_5, k_3 k_4, k_2 k_3) + 
+ (k_2 k_3, k_3 k_4, k_4 k_5) + (k_3 k_4, k_2 k_3, k_1 k_2, k_4 k_5) \right] + 
+ (3 - \beta) [(k_2 k_3, k_1 k_2, k_3 k_4, k_2 k_3) + (k_3 k_4, k_2 k_3, k_1 k_2, k_4 k_5)] + 
+ 2\beta [(k_1 k_2, k_3 k_4, k_2 k_3, k_4 k_5) + (k_2 k_3, k_1 k_2, k_4 k_5, k_3 k_4)] + 
- 10 [(k_1 k_2, k_3 k_3, k_4 k_5) + (k_4 k_5, k_3 k_4, k_2 k_3, k_1 k_2)] ; 
(k_1 k_2, k_2 k_3, k_3 k_4, k_4 k_5) \right\}
\]

(41)

Notice that $H_{8}$ is completely symmetric under the site index exchange

\[
(k_1, k_2, k_3, k_4) \leftrightarrow (k_5, k_4, k_3, k_2, k_1)
\]

(42)

as it has to be from the site-index structure obtained from the linking part $\Sigma_{k_1 \gamma_2} \Sigma_{k_2 \gamma_3} \Sigma_{k_3 \gamma_4} \Sigma_{k_4 \gamma_5}$ of the Hamiltonian.

Whence, trivially noticing that the above considered \textit{deplanarization} procedure has no effects on the tree level, we may say that Eqs. \textsuperscript{17}, \textsuperscript{15}, \textsuperscript{10}.

\textsuperscript{5}We use the semicolon in the arguments of $H_{8}$ to indicate the fact that the real parameter $\beta$ survives also at the 4-loop, planar level.
and (41) are a consistent full non-planar generalization of the corresponding planar formulae, respectively given by Eqs. (22), (23), (24) and (25).

However, while Eqs. (17) and (19) (respectively corresponding to the full non-planar 1- and 2-loop level) are exactly determined, notice that Eqs. (40) and (41) (respectively corresponding to the full non-planar 3- and 4-loop level) contain undetermined free real parameters. Therefore are actually not completely fixed. Such free real parameters (surviving, in the 4-loop case, also at the planar level) should be fixed by matching with some known results about the non-planar, 3- and 4-loop level for the su(2)-symmetric operatorial sector of the considered 4-d. $\mathcal{N} = 4$ SYM theory; unfortunately, at the moment such (independently obtained) full non-planar higher-loop results are unavailable.

By the way, in order to completely fix the free parameters introduced by the general deplanarization procedure, we may formulate an additional assumption, that we are going to call hypothesis of "symmetrization of deplanarizing operator splittings". This conjecture has to be applied after the symmetrization of the non-planar terms with respect to the peculiar renaming of spin-chain site indices, which is given by (30) and determined by the linking part of the Hamiltonian expressed by (29); it amounts to say that each of the sets of non-planar terms arising from a considered planar term in the deplanarization procedure will equally contribute in the planar limit $N \to \infty$. For example, if, after the symmetrization with respect to (30), a planar term $\mathcal{I}$ is deplanarized by the 3-fold splitting

$$\mathcal{I} \rightarrow a_1 \mathcal{I}_{M_1} + a_2 \mathcal{I}_{M_2} + a_3 \mathcal{I}_{M_3},$$

(43)

where $\mathcal{I}_{M_1}$, $\mathcal{I}_{M_2}$ and $\mathcal{I}_{M_3}$ are sets consisting of $M_1$, $M_2$ and $M_3$ non-planar terms made by permutation operators, then we will assume that

$$M_1 a_1 = M_2 a_2 = M_3 a_3.$$

(44)

Hence the contribution of $\mathcal{I}_{M_1}$, $\mathcal{I}_{M_2}$ and $\mathcal{I}_{M_3}$ to the planar limit $\mathcal{I}$ is the same, and therefore the operator splitting given by (43) may be considered symmetric.

By applying such a simple additional conjecture to the 3- and 4-loop orders, we obtain that the Ansätze (40) and (41) for the non-planar, 3- and 4-loop su(2)
spin-chain Hamiltonians become \textit{completely determined}, respectively reading as follows:

\[ H_{6, \text{non-planar}} = \]
\[ = \frac{4}{N^3} \left[ 15 - \frac{15}{2} (k_1 k_2 + k_3 k_4)_p - 13 (k_2 k_3)_p + 
+ 3 [(k_1 k_2, k_2 k_3) + (k_3 k_4, k_2 k_3)_p + 
+ (k_2 k_3, k_1 k_2)_p + (k_2 k_3, k_3 k_4)_p] + 
+ (k_1 k_2, k_3 k_4)_p - (k_1 k_2, k_2 k_3, k_3 k_4)_p - (k_3 k_4, k_2 k_3, k_1 k_2)_p ; 
(k_1 k_2, k_2 k_3, k_3 k_4)_\gamma \right]_p ; (45) \]

\[ H_{8, \text{non-planar}} (\beta) = \]
\[ = \frac{1}{N^4} \left\{ -560 + (259 + \beta) (k_1 k_2 + k_4 k_5 + k_2 k_3 + k_3 k_4)_p + 
- \left( \frac{266}{45} + \frac{1}{3} \beta \right) \left[ (k_1 k_2, k_2 k_3)_p + (k_4 k_5, k_4 k_5)_p + 
+ (k_3 k_4, k_1 k_2)_p + (k_2 k_3, k_1 k_2)_p \right] + 
- (33 - \beta) (k_1 k_2, k_3 k_4)_p - (k_2 k_3, k_4 k_5)_p - 4 (k_1 k_2, k_4 k_5)_p + 
+ 4 \left[ (k_1 k_2, k_2 k_3, k_4 k_5)_p + (k_3 k_4, k_1 k_2)_p \right] + 
+ (39 + \beta) \left[ (k_1 k_2, k_2 k_3, k_3 k_4)_p + (k_4 k_5, k_3 k_4, k_2 k_3)_p + 
+ (k_1 k_2, k_2 k_3, k_4 k_5, k_3 k_4)_p + (k_1 k_2, k_4 k_5, k_1 k_2, k_4 k_5)_p \right] + 
+ (\beta - 9) \left[ (k_1 k_2, k_3 k_4)_p + (k_4 k_5, k_3 k_4, k_4 k_5)_p + 
+ (k_2 k_3, k_1 k_2, k_4 k_5)_p + (k_3 k_4, k_2 k_3, k_4 k_5)_p \right] + 
+ (1 - \beta) \left[ (k_1 k_2, k_2 k_3, k_4 k_5, k_3 k_4, k_4 k_5)_p + (k_1 k_2, k_4 k_5, k_1 k_2, k_4 k_5)_p \right] + 
+ (3 - \beta) (k_1 k_2, k_3 k_4, k_2 k_3, k_4 k_5)_p + (k_3 k_4, k_2 k_3, k_4 k_5, k_3 k_4)_p + 
+ 2 \beta [(k_1 k_2, k_2 k_3, k_4 k_5, k_3 k_4)_p + (k_2 k_3, k_1 k_2, k_4 k_5, k_3 k_4)_p] + 
- 10 [(k_1 k_2, k_2 k_3, k_3 k_4, k_4 k_5)_p + (k_4 k_5, k_3 k_4, k_2 k_3, k_1 k_2)_p] ; \right\} (46) \]
5 Conclusions and perspectives

In summary, we worked out a general procedure of full "deplanarization" of planar results about $su(2)$ closed spin-chain Hamiltonian, describing the dynamics of the $su(2)$ sector of the 4-d. $\mathcal{N} = 4$ SYM theory. Such a method is based on proper "deplanarizing lifts" of (products of) site permutation operators $P$’s of the kind

$$\zeta P_{k,k+r} \rightarrow \sum_{i=1}^{n \text{ (loop order)}} \zeta_i P_{k_i,k_i+r}, \text{ such that } \sum_{i=1}^{n \text{ (loop order)}} \zeta_i = \zeta, \; r \in \mathbb{Z}_L, \zeta \in \mathbb{R},$$

and in general

$$\zeta P_{k,k+r_1}P_{k,k+r_2}... \rightarrow \sum_{i=1}^{n \text{ (loop order)}} \zeta_i P_{k_i,k_i+r_1}P_{k_i,k_i+r_2}...,$$

such that

$$\sum_{i=1}^{n \text{ (loop order)}} \zeta_i = \zeta, \; r_1, r_2, ... \in \mathbb{Z}_L, \zeta_i \in \mathbb{R}. \quad (47)$$

The number of free parameters $\{\zeta_i\}$ may be decreased by observing that the assumed Ansatz on the linking part of the higher-loop, non-planar Hamiltonian determines a symmetry under a certain exchange of site indices, expressed by Eq. (40) and implied by $\Sigma_{k_0k_1...k_n}(\gamma)$ defined in (29).

Furthermore, performing an (extremely reasonable) additional conjecture of "symmetrization of deplanarizing operator splittings", we were able to completely fix all free parameters introduced by the proposed "deplanarization procedure". Thus, we wrote down some completely determined (up to the free planar parameter $\beta$ at 4-loop order) expressions for the full non-planar, 3- and 4-loop $su(2)$ spin-chain Hamiltonians, respectively given by Eqs. (45) and (46).

Such a "deplanarizing" procedure is perfectly compatible with (independently obtained) known results at the 1- and 2-loop, non-planar level $^{22,23,24}$; moreover, by construction, the above-formulated Ansätze are planarly consistent, i.e. they have the correct planar limit, matching the known results reported in the literature (see e.g. $^{25,26}$).

It is worth noticing that, imposing only invariance under $^{30}$, the general
3- and 4-loop Ansätze (40) and (41) do contain free parameters, easily fixable by matching procedures with (independently obtained) non-planar higher-loop results, achieved by using Feynman diagrams approach on the gauge theory side, or performing perturbative string calculations. Unfortunately, at the moment such results to match with are unavailable, but nonetheless the above-proposed Ansätze express the most general form of non-planar Hamiltonians. Also, notice that such free undetermined parameters should not affect the properties and structure of relevant physical quantities, such as the spectrum; as previously mentioned, this is claimed in the known literature (see e.g. [27]) to hold at the planar level and, due to the the planar consistence, it seems natural and reasonable to conjecturally extend such a claim also at the non-planar level.

Nonetheless, due to the failure of the elegant "spin edge-differences" Ansätze (71) for the explicit form of the \( n \)-loop, non-planar \( su(2) \) spin-chain Hamiltonian, the "deplanarizing" method here proposed seems, at the moment, the only one capable of guaranteeing planar consistency and, in principle, full testability for the higher-loop, non-planar Ansätze.

Also notice that, by construction, Ansätze (40) and (41) show an explicit full factorization in the spin and chain-splitting parts; as already pointed out in [24], such a property is expected to hold at every loop order, since the Hilbert space of the spin bit model is always given by the direct product (modulo the action of the permutation group \( S_L \)) of the spin space and the linking variable \( \gamma \)-space.

As previously considered, attention must also be paid to the fact that, while

Notice also that the proposed Ansätze for the 3- and 4-loop, non-planar Hamiltonians can always be cast in terms of the two-site planar one-loop Hamiltonian \( H_{kl} \). This fact allows us to claim that such Ansätze do formally hold at least also for the \( su(3) \) sector of the 4-d. \( N = 4 \) SYM theory. Similarly to the \( su(2) \) one, such a sector is made out by local gauge invariant SYM operators which are generated by three holomorphic (multi)trace operators built out of two complex SYM scalars \( X = \phi_5 + i\phi_6, \ Y = \phi_3 + i\phi_4 \) and \( Z = \phi_1 + i\phi_2 \), of typical form

\[
\mathcal{O} = \text{Tr}(XYZYZ\ldots)\text{Tr}(XYXZ\ldots)\text{Tr}(\ldots)\ldots
\]
(both at non-planar and planar levels) the 1- and 2-loop formulae for the \( su(2) \) spin-chain Hamiltonian are linear in the site permutation operators \( P \)'s, the 3- and 4-loop level expressions, both in the planar case (see (24) and (25)) and in the non-planar case (see Eqs. (40) and (41)), do seem to show a non-linearity (and non-linearizability) in \( P \)'s. For example, the non-linearizability of the planar, 3-loop \( su(2) \) spin-chain Hamiltonian \( [24] \) caused the failure of the elegant and geometrically meaningful "spin edge-differences" approach to higher-loop Ansätze.

Thus, the non-linearity (and non-linearizability) in site permutation operators seems to be a crucial and fundamental feature, starting to hold at the 3-loop order, of the spin part of the \( su(2) \) spin-chain Hamiltonian for the spin bit model, underlying the dynamics of the \( su(2) \) sector of the 4-d. \( \mathcal{N} = 4 \) SYM theory. Such a "breakdown" of "permutational linearizability" at the 3-loop level would give rise, by means of the AdS/CFT correspondence [3, 4], to some "new" features in the dynamics of the (closed) superstrings in the bulk of \( AdS_5 \times S^5 \) (for the first evidences from 3-loop calculations, see e.g. [25, 26]; for further subsequent developments see e.g. [2, 8, 28, 29]).

Finally, we notice that it would be interesting, following recent research directions, to extend such a "deplanarization" method to other operatorial sectors of the 4-d. \( \mathcal{N} = 4 \) SYM theory [30, 31]; unfortunately, even restricting the consideration to the planar level, only \( su(2) \) anomalous dimension operators are presently known beyond 1-loop.

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Appendix I

$su(2)$ sector: applying permutational identities to ultra-localization by Pauli matrices

In the $su(2)$-symmetry operatorial sector of the 4-d. $\mathcal{N} = 4$ SYM theory the spin operators form a spin $s = \frac{1}{2}$ (representation of the) $su(2)$ algebra on each site of the spin-chain\(^7\)

$$\left[ \hat{S}^i, \hat{S}^j \right] = i\epsilon^{ijk} \hat{S}^k,$$  \hspace{1cm} (49)

where here and in the following supescript indices range in \{1, 2, 3\} (3-d. spatial spin degrees of freedom), and eigenvalues of spin operators $\hat{S}^i$ (denoted with $S^i$) take values in $\mathbb{Z}_2$, i.e. in the integer numeric field with period 2.

Imposing that such an $su(2)$ symmetry be an ultra-local one, i.e. that spin operators belonging to different (also nearest-neighbouring) sites of the spin-chain always commute, it is then possible to say that in the case in which (subscript denotes the site position)

$$S^i_k \in \mathbb{Z}_2 \forall (k, i) \in \mathbb{Z}_L \times \{1, 2, 3\},$$  \hspace{1cm} (50)

we can irreducibly represent the ultra-localized $s = \frac{1}{2}$ $su(2)$ algebra

$$\left[ \hat{S}^i_k, \hat{S}^j_{k'} \right] = i\delta_{kl}\epsilon^{ijm} \hat{S}^m_k$$  \hspace{1cm} (51)

($\delta_{kl}$ denoting the usual Krönecker delta) by the usual Pauli $\sigma$-matrices, defined as

$$\hat{S}^i_k =: \frac{1}{2}\sigma^i_k.$$  \hspace{1cm} (52)

Therefore, Eq. 51 has the irreducible matrix representation

$$\left[ \sigma^i_k, \sigma^j_l \right] = 2i\delta_{kl}\epsilon^{ijm}\sigma^m_k,$$  \hspace{1cm} (53)

expressing the ultra-localization of the Pauli $\sigma$-matrix algebra (i.e. the irreducible representation of the ultra-localized $s = \frac{1}{2}$ $su(2)$ algebra) on the spin-chain sites.

\(^7\)As usual, we will respectively denote with $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ the commutator and anticommutator of (matrix representations of) operators.
Using Eqs. \[51\] - \[53\], the action of the permutation operator \( P_{kl} = P_{\hat{S}_k \hat{S}_l} \) may be represented in the following way

\[
P_{kl} = \frac{1}{2} (1 + \vec{\sigma}_k \cdot \vec{\sigma}_l), \quad (k, l) \in \{1, ..., L\}^2, \ k \neq l,
\]

with \( \vec{\sigma}_k \) being the matrix 3-vector \((\sigma^1_k, \sigma^2_k, \sigma^3_k)\) of Pauli \(\sigma\)-matrices on the \(k\)-th spin-chain site, and \(\cdot\) denoting the usual Euclidean scalar product between such matrix 3-vectors (based on the standard "row-columns" matrix product, denoted in the following with \(\times\)).

At this point, using Eq. \[54\], we may reformulate the permutational identities obtained in Section 2, specializing them to the case of the ultra-localized \(su(2)\)-symmetric operatorial sector of the 4-d. \(\mathcal{N} = 4\) SYM theory, with such a symmetry irreducibly represented by Pauli \(\sigma\)-matrices.

Starting from the identity \[11\], we thence get \((k_1 \neq k_2 \neq k_3 \text{ throughout})\)

\[
\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2} + \vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3} + (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}) (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3}) = \\
\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3} + \vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2} + (\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3}) (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}) = \\
\vec{\sigma}_{k_3} \cdot \vec{\sigma}_{k_1} + \vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3} + (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3}) (\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3}),
\]

implying

\[
\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2} - \vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3} = (\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3}) (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}) - (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}) (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3}). \tag{56}
\]

Since Eq. \[56\] implies (\(\wedge\) denotes the vector product between matrix 3-vectors)

\[
[(\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}), (\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3})] = 2i (\vec{\sigma}_{k_1} \wedge \vec{\sigma}_{k_2}) \cdot \vec{\sigma}_{k_3}, \tag{57}
\]

Eq. \[56\] may be rewritten as

\[
\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2} (1 - \vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}) - (1 - \vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3}) \vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3} = \\
2i [(\vec{\sigma}_{k_1} \wedge \vec{\sigma}_{k_3}) \cdot \vec{\sigma}_{k_2} + (\vec{\sigma}_{k_2} \wedge \vec{\sigma}_{k_3}) \cdot \vec{\sigma}_{k_1}]. \tag{58}
\]

Furthermore, considering the permutational identity \[12\], we obtain \((k_1 \neq k_2 \neq k_3 \text{ throughout})\)

\[
\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3} + \frac{1}{2} \{(\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2}), (\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3})\} = 0, \tag{59}
\]

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which, by means of Eq. (57), may be rewritten in the following way

$$\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3} + (\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2})(\vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3}) - i (\vec{\sigma}_{k_1} \wedge \vec{\sigma}_{k_3}) \cdot \vec{\sigma}_{k_2} = 0. \quad (60)$$

Finally, using Eqs. (54) and (57), the permutational identity (13) may be put in the form ($k_1 \neq k_2 \neq k_3 \neq k_4$ throughout)

$$(1 - \vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2})(\vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_3} + \vec{\sigma}_{k_2} \cdot \vec{\sigma}_{k_3})(1 + \vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2} + \vec{\sigma}_{k_3} \cdot \vec{\sigma}_{k_3} + \vec{\sigma}_{k_3} \cdot \vec{\sigma}_{k_3}) = 0,$$

or, equivalently, by the use of Eq. (60):

$$[2\vec{\sigma}_{k_2} \cdot (\vec{\sigma}_{k_3} + \vec{\sigma}_{k_4}) + i (\vec{\sigma}_{k_3} \wedge \vec{\sigma}_{k_3}) \cdot \vec{\sigma}_{k_4} + i (\vec{\sigma}_{k_3} \wedge \vec{\sigma}_{k_3}) \cdot \vec{\sigma}_{k_3}] \times (1 + \vec{\sigma}_{k_1} \cdot \vec{\sigma}_{k_2} + \vec{\sigma}_{k_3} \cdot \vec{\sigma}_{k_3} + \vec{\sigma}_{k_3} \cdot \vec{\sigma}_{k_3}) = 0. \quad (61)$$

In general, we may obtain many other equations involving Pauli $\sigma$-matrices ultra-localized on the spin-chain sites, by considering the general form (16) of the permutational identities previously obtained, and using Eq. (54):

$$\left( \sum_{\pi \in S_M} (-)^{\sigma_{\pi}} P_{\pi} \right)_{\text{with } P_{kl} = \frac{1}{2}(1 + \vec{\sigma}_{kl} \cdot \vec{\sigma}_{kl})} = 0. \quad (63)$$

It is therefore possible to argue that such a "tower" of (reciprocally independent) matrix equations, becoming more and more involved with the increasing of $k_3$, represents a sort of "generalized Fierz identities" for Pauli $\sigma$-matrices, related to the spin $s = \frac{1}{2}$ irreducible representation of the ultra-localized $su(2)$ algebra.

Appendix II

The planar limit of permutational identities

Always focussing on the $su(2)$ sector of 4-d. $N = 4$ SYM theory, the result may be considered as a full non-planar generalization of previously known planar permutational identities, obtained in [23].
To explicitly show this, let us report Eq. (F.2) of \[25\]: in the notation specified by \[20\] it is
\[
\{..., n, n \pm 1, n, \ldots\} + \{..., ..., \} + \{..., n, \ldots\} + \{..., n \pm 1, \ldots\} + \{..., n, n \pm 1, \ldots\} - \{..., n, n \pm 1, \ldots\} - \{..., n \pm 1, n, \ldots\} = 0, \quad n \in \mathbb{Z}.
\]
(64)

Therefore, it may be explicitly checked that
\[
\{0, \pm 1, 0\} - \{\} + \{0\} + \{\pm 1\} - \{0, \pm 1\} - \{\pm 1, 0\} = 0 \Leftrightarrow \sum_{k \in Z_L} [1 - P_{k,k+1} - P_{k,k+2} - P_{k+1,k+2} + P_{k,k+1}P_{k+1,k+2} + P_{k+1,k+2}P_{k,k+1}] = 0,
\]
(65)

which is nothing but the canonical planar limit of Eq. (12), summed on all \(k \in Z_L\). This result is perfectly consistent, because, as it is claimed in \[25\], the permutational identity (64) is "due to the impossibility of antisymmetrizing three sites in \(SU(2)\)". Whence Eq. (65) simply implies that Eq. (12) is nothing but the non-canonical, non-planar generalization of Eq. (64).

Furthermore, using periodic boundary conditions on spin-chain, and eventually applying (iterated) "pull-back" or "push-forward" of site indices by a permutation \(\gamma \in S_L\), it may be shown that Eq. (64) actually does not depend on \(n \in \mathbb{Z}\) and on the choice of "\(\pm\" at all, and it is always possible (without loss of generality) to consider \(n = 0\), and choose "\(+\)."

Thus, Eq. (65) may be rewritten as (\(\forall n \in \mathbb{Z}\))
\[
\{n, n \pm 1, n\} - \{\} + \{n\} + \{n \pm 1\} - \{n, n \pm 1\} - \{n \pm 1, n\} = 0 \Leftrightarrow \sum_{k \in Z_L} [1 - P_{k,k+1} - P_{k,k+2} - P_{k+1,k+2} + P_{k,k+1}P_{k+1,k+2} + P_{k+1,k+2}P_{k,k+1}] = 0.
\]
(66)

Summarizing, we can therefore say that Eqs. (16) express a wide and rich variety of permutational identities, holding true at the non-canonical, non-planar level, and arising from the antisymmetrization of \(M (\geq 3)\)-index tensors, e.g. in the \(su(2)\) sector of the 4-d. \(N = 4\) SYM theory. The canonical, planar limit of the case \(M = 3\), summed on all \(k \in Z_L\), coincides with the previously known relation (64) \[25\].

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Appendix III

Geometric ”spin edge-differences” Ansätze

Defining the ”spin edge-difference of order \( n \)” in the following way:

\[
\Delta S_{k_0k_1...k_n} = \Delta S_{k_0k_1...k_{n-1}} - \Delta S_{k_1...k_n} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \Delta S_{k_i},
\]

and introducing the (\( \gamma \)-dependent) ”splitting and joining chain operator of order \( n \)” as in (29), i.e. defining

\[
\Sigma_{k_0k_1...k_n} (\gamma) \equiv \Sigma_{k_0\gamma_k_1} \Sigma_{k_1\gamma_k_2} \cdots \Sigma_{k_{n-2}\gamma_k_{n-1}} \Sigma_{k_{n-1}\gamma_k_n}, \quad \gamma \in S_L,
\]

it is possible to rewrite Eqs. (17) and (19) respectively as

\[
H_2 = \frac{2}{N} \sum_{(k_1,k_2) \in Z_L^2, k_1 \neq k_2} \left[ \left( \Delta S_{k_1k_2} \right)^2 - 1 \right] \Sigma_{k_1k_2} (\gamma),
\]

\[
H_4 = -\frac{2}{N^2} \sum_{(k_1,k_2,k_3) \in Z_L^3, k_1 \neq k_2 \neq k_3} \left( \Delta S_{k_1k_2k_3} \right)^2 \Sigma_{k_1k_2k_3} (\gamma).
\]

Therefore, it is completely reasonable to formulate the following ”spin edge-differences” Ansätze for the \( n \)-loop, non-planar \( su(2) \) spin-chain Hamiltonian (\( n \in N \)):

\[
H_{2n} = (-1)^{n+1} \frac{2}{N^n} \sum_{(k_0,k_1,...,k_n) \in Z_L^{n+1}, k_0 \neq k_1 \neq ... \neq k_n} \left[ \left( \Delta S_{k_0k_1...k_n} \right)^2 + \alpha_n \right] \Sigma_{k_0k_1...k_n} (\gamma),
\]

where \( \alpha_n \in Z \) is such that the spin part of \( H_{2n} \) ia a linear homogeneous function of the quantities \( (1 - P) \), and \( P \) is a permutation operator coming from the \( n \)-th order ”spin edge-difference” \( \Delta S_{k_0k_1...k_n} \). For example, from Eqs. (69) and (70) we respectively get \( \alpha_1 = -1 \) and \( \alpha_2 = 0 \). Notice that the linking part \( \Sigma_{k_0k_1...k_n} (\gamma) \) of the Hamiltonian \( H_{2n} \) will determine a symmetry under the inversion of the order of the \( (n+1) \)-tet \( (k_0, k_1, ..., k_n) \), i.e. under the exchange of site indices expressed by Eq. (30).
As already stressed in [24], the "spin edge-differences" Ansätze (71) for the explicit form of the $n$-loop, non-planar $su(2)$ spin-chain Hamiltonian have a simple meaning: i.e. $\Sigma_{k_0k_1...k_n}(\gamma)$ cyclically exchanges the incoming and outgoing ends of the chains adjacent to the sites $k_0, k_1, ..., k_n$. At the same time, the spin part (modulo the constant $\alpha_n$) acts as the (square of the) discrete $n$-th derivative of the spin operatorial 3-vector along the new chain. In the continuum limit, such Ansätze are compatible with the BMN conjecture ([5]-[15]), yielding a term $\sim \lambda^{2n}(\partial^n\phi)^2$ as the $n$-loop contribution.

As it is evident from Eq. (71), independently on the choice to have an homogeneous dependence of the spin part of $H_{2n}$ on terms $(1 - P)$, and thus, independently on the choice of the constant $\alpha_n \in Z$, the "spin edge-differences" Ansätze imply a linear dependence of $H_{2n}$ on terms $(1 - P)$, and thus, on the permutation operators $P$'s at any loop order (i.e. for any $n \in N$). But this does not seem to be the case.

Indeed, as already briefly mentioned in [24], the attempts to check linearity in permutation operators, even by using the rich set of permutational identities previously treated, fail already for $n = 3$, i.e. for the 3-loop level.

Finally, it is reasonable to claim that the elegant and simply-meaning "spin edge-differences" Ansätze (71) for the explicit form of the $n$-loop, non-planar $su(2)$ spin-chain Hamiltonian $H_{2n}$ unfortunately fail, because their main consequence on the structure (of the spin part) of $H_{2n}$, namely the linearity in the permutation operators $P$'s, fails to be checked also in the first non-trivial case, i.e. at the 3-loop level.

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