ON GLOBAL DYNAMICS IN A MULTI-DIMENSIONAL DISCRETE MAP

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Abstract. We derive preliminary results on global dynamics of the multi-dimensional discrete map
\[ F : (x_1, x_2, \ldots, x_{k-1}, x_k) \mapsto (x_1 + af(x_k), x_1, x_2, \ldots, x_{k-1}) \]
where the continuous real-valued function \( f \) is one-sided bounded and satisfying the negative feedback condition, \( x \cdot f(x) < 0, x \neq 0, a \) is a positive parameter. We show the existence of a compact global attractor for map \( F \), and derive a condition for the global attractivity of the zero fixed point.

1. Introduction. This paper deals with several aspects of global dynamics of a family of multi-dimensional discrete maps of the form
\[ F : (x_1, x_2, \ldots, x_{k-1}, x_k) \mapsto (x_1 + af(x_k), x_1, x_2, \ldots, x_{k-1}), \quad k \geq 2, \tag{1} \]
where \( f \) is a continuous real-valued function satisfying the negative feedback condition
\[ x \cdot f(x) < 0 \quad \text{for} \quad x \neq 0, \tag{2} \]
and \( a > 0 \) is a parameter. In addition, the nonlinearity is assumed to be bounded from one side
\[ f(x) \leq M \quad \text{or} \quad f(x) \geq -M \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and some} \quad M > 0. \tag{3} \]
The motivation for the assumptions (2) and (3) comes from related considerations in functional differential equations (see e.g. [4, 5, 8]).

The discrete map \( F \) appears in a study of the differential equation with deviating argument
\[ x'(t) = a(t)f([t - K]), \tag{4} \]
where \( f \) satisfies hypotheses (2) and (3), \( a(t) > 0 \) is continuous periodic, and \([\cdot]\) is the integer value function. Under certain additional assumptions the dynamics of equation (4) are reduced exactly to that of the discrete map \( F \) (see related details in [7]). Besides, the study of dynamical properties of map (1) represents an interesting and challenging problem on its own. The present paper makes first steps into this direction by establishing two basic properties of the map \( F \) in the case \( k \geq 2 \).

The first property is the eventual uniform boundedness of iterations under the map \( F \). There is a constant \( C_0 > 0 \) such that for arbitrary initial point \( x_0 \in \mathbb{R}^k \) there is a positive integer \( N_0 \) such that all subsequent iterations of \( x_0 \) under \( F \) are bounded by \( C_0 \):
\[ |F^n(x_0)| \leq C_0, \quad \forall n \geq N_0. \]
The second one deals with the global attractivity of the zero fixed point \( 0 = [0, \ldots, 0] \). It is shown that there exists a positive value \( a_0 \) such that for every \( 0 < a < a_0 \) the fixed point \( 0 \) is globally attracting: for arbitrary initial point \( x_0 \in \mathbb{R}^k \) one has \( \lim_{n \to \infty} F^n(x_0) = 0 \).

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The case \( k = 1 \), when map \( F \) becomes one-dimensional, was extensively studied in [7]. Its dynamics have been shown to be quite complex in general, as much as an arbitrary interval map can be. It is a natural expectation that the case of multidimensional map \( F \), when \( k \geq 2 \), can potentially exhibit even more complex dynamics. Related detailed studies appear to be highly nontrivial, and they will be incorporated into future research plans. This paper makes a very first step into the study of basic properties of map (1), as outlined in the previous paragraph.

2. Preliminaries. This section includes some basic definitions and fundamentals related to multi-dimensional maps, which are necessary for the exposition in the remainder of the paper. They can be found in standard monographs and textbooks dealing with discrete dynamical systems, including interval maps (such as e.g. [1, 2, 3, 6, 9]).

The continuous map \( F \) generates a discrete dynamical system in \( \mathbb{R}^k \). Given an initial point \( x \in \mathbb{R}^k \) the trajectory through \( x \) consists of all forward iterations of \( x \) under \( F \): \( \{x, F(x), F^2(x), \ldots, F^n(x), \ldots\} \), where \( F^n(x) = F(F(\ldots F(x) \ldots)) \) stands for the \( n \)-th iteration of \( F \). As usual, a point \( x_0 \in \mathbb{R}^k \) is called a fixed point if \( F(x_0) = x_0 \). It is an attracting fixed point if there exists its open neighborhood \( U(x_0) \) such that \( F(U) \subseteq U \), and for every \( x \in U \) one has \( \lim_{n \to \infty} F^n(x) = x_0 \).

In general, a set \( A \subseteq \mathbb{R}^k \) will be called an attractor if it is invariant under \( F \), \( F(A) = A \), and there exists its open neighborhood \( U \supseteq A \) such that for every \( x \in U \) the sequence \( F^n(x) \) converges to \( A \). This means that \( \lim_{n \to \infty} \text{dist}(F^n(x), A) = 0 \), where the distance function is chosen in the existing norm of \( \mathbb{R}^k \). We shall use the maximum norm here: with \( x = [x_1, \ldots, x_k] \) its norm is given by \( |x| := \max\{|x_1|, \ldots, |x_k|\} \). An attractor \( A \) will be called a global attractor on \( \mathbb{R}^k \) if for arbitrary initial point \( x \in \mathbb{R}^k \) the sequence \( \{F^n(x)\} \) converges to \( A \).

In this paper, an element \( x \in \mathbb{R}^k = [x_1, \ldots, x_k] \) will be defined as positive, and denoted by \( x > 0 \), if \( x_j > 0 \), \( 1 \leq j \leq k \); it will be called negative if \( x_j < 0 \), \( 1 \leq j \leq k \). In the obvious way we will be referring to the non-negative and non-positive points, denoting them by \( x \geq 0 \) or \( x \leq 0 \), respectively. We shall call an iterative sequence \( \{x_n = F^n(x)\} \) to be positive (negative) if \( x_n > 0 \) \((< 0)\) \( \forall n \geq 0 \). It will be called eventually positive/negative if the corresponding inequality holds for all sufficiently large \( n \). Likewise, we talk about the eventually non-negative and non-positive sequences. The sequence \( x_n \) will be called oscillating if it is not eventually sign definite. That is, if it is not eventually non-negative or eventually non-positive.

A point \( x_n = F^n(x) := [y_1, \ldots, y_i, y_{i+1}, \ldots, y_r] \) will be called P-N transitional if there exists an index \( i, 1 \leq i < r \) such that \( y_j < 0 \) for \( 1 \leq j \leq i - 1 \), \( y_i \leq 0 \) and \( y_j > 0 \) for \( i + 1 \leq j \leq r \). Likewise, a point \( x_m = F^m(x) := [y_1, \ldots, y_i, y_{i+1}, \ldots, y_r] \) will be called N-P transitional if there exists an index \( i, 1 \leq i < r \) such that \( y_j > 0 \) for \( 1 \leq j \leq i - 1 \), \( y_i \geq 0 \), and \( y_j < 0 \) for \( i + 1 \leq j \leq r \).

We shall call an iterative sequence \( x_n = F^n(x), n \geq 0 \) slowly oscillating if its every member is either positive, or negative, or transitional. An iterative sequence will be called eventually slowly oscillating if there exists a positive integer \( N_0 \) such that the sequence \( y_n = x_{N_0 + n} = F^{N_0+n}(x), n \geq 0 \) is slowly oscillating. An oscillating iterative sequence which is not eventually slowly oscillating will be called rapidly oscillating. As one can see from Lemma 2.1 below, any slowly oscillating iterative sequence has quite specific structure.

Lemma 2.1. (Structure of slowly oscillating iterations). If the sequence \( x_n = F^n(x) \) is eventually slowly oscillating, with \( F^{N_0}(x) \) \( > 0 \) \((< 0)\), then for every \( n \geq N_0 \) the member \( x_n \) of the iterative sequence is either non-negative, or non-positive, or transitional.

Proof. Indeed, due to the autonomous nature of (1) and for the sake of definiteness, we can assume that the initial point is positive \( x = [x_1, \ldots, x_k] > 0 \). Since \( F(x) = [x_1 + af(x_k), x_1, \ldots, x_{k-1}] \) then either \( x_1 + af(x_k) > 0 \) or \( x_1 + af(x_k) \leq 0 \). In the first case
we define $F(x) = y > 0$ as a new initial point, and look at its first iteration $F(x)$. If $F(y) > 0$ then we repeat this reassignment of the initial point again, until $F^n(x)$ is not positive for some first $m + 1$. That is, $F^n(x) > 0$, for all $0 \leq n \leq m$ and $F^{m+1}(x)$ is not positive. By setting $F^{n+1}(x) = y = [y_1, \ldots, y_k] > 0$ we then necessarily have that $F^{m+1}(x) = F(y) = [y_1 + af(y_k), y_1, \ldots, y_{k-1}]$ satisfies

$$y_1 + af(y_k) \leq 0 \quad \text{and} \quad y_j > 0 \quad \forall \quad 1 \leq j \leq k - 1,$$

which is the second case to consider. In the second case, assume first that $y_1 + af(y_k) < 0$. Since $x_j > 0$ for all $1 \leq j \leq k$, the iterative element $x_1 = F(x)$ is P-N transitional with $i = 1$. In view of (5) and the negative feedback assumption (2), the iterative element $x_2 = F(x_1) = F^2(x)$ is P-N transitional with $i = 2$. Continuing by induction one sees that $x_n = F^n(x), 1 \leq n \leq k - 1$ is P-N transitional with $i = n$. Also, $x_{k-1} = F^{k-1}(x) = [y_1, \ldots, y_{k-1}, x_1]$ is such that $y_j < 0, 1 \leq j \leq k - 1$ and $x_1 > 0$ (that is, P-N transitional with $i = k - 1$). But then one has $F(x_{k-1}) = F_k(x) = [y_1 + af(x_1), y_1, \ldots, y_{k-1}] < 0$.

If $x_1 + af(x_k) = 0$ then $F(x)$ has the form $F(x) = [0, x_1, \ldots, x_{k-1}]$ with $x_j > 0, 1 \leq j \leq k - 1$. The second iteration $F^2(x)$ is of the form $F^2(x) = [af(x_{k-1}), 0, x_1, \ldots, x_{k-2}]$ with $af(x_{k-1}) < 0$. One easily sees that the $k$-th iteration of $x$ has the form $F_k(x) = [y_1, \ldots, y_{k-1}, 0]$ where $y_j < 0, 1 \leq j \leq k - 1$. But then the next iteration has the form $F_{k+1} = [y_1, y_1, y_2, \ldots, y_{k-1}] < 0$. So one is in a position symmetric to the case of initial point being positive, $x > 0$.

The reasoning in the case $x < 0$ is completely analogous, and the remainder of the proof is left to the reader. The possibilities $x \geq 0$ or $x \leq 0$ are slightly more elaborate than those considered above, $x > 0$ and $x < 0$. However, they are similar to the reasoning in the previous paragraph, dealing with the subcase $x_1 + af(x_k) = 0$. Due to their significant length and the scope of the present paper they are left out as an exercise for the reader.

\[ \square \]

**Remark 1.** As it is easily seen from the details of the proof of Lemma 2.1 the nature of the slow oscillation of an iterative sequence $x_n = F^n(x)$ can be described more precise. Namely:

(i) Every non-negative point $x_m$ is followed by either a non-negative point $x_{m+1}$, or by a P-N transitional point $x_{m+1}$ with $i = 1$;

(ii) Every non-positive point $x_m$ is followed by either a non-positive point $x_{m+1}$, or by a N-P transitional point $x_{m+1}$ with $i = 1$;

(iii) Every P-N transitional point $x_m$ with a particular $i$ is followed by the same type transitional point with the next value $i+1$; any P-N transitional point $x_n$ with $i = k - 1$ is followed by a non-positive point $x_n+1 \leq 0$;

(iv) Every N-P transitional point $x_m$ with a particular $i$ is followed by the same type transitional point with the next value $i + 1$; any N-P transitional point $x_m$ with $i = k - 1$ is followed by a non-negative point $x_{m+1} \geq 0$.

3. Principal Results. This section contains two principal results of the present paper. The first one establishes the property of the eventual uniform boundedness of any iterative sequence $F^n(x), x \in \mathbb{R}^k$, as $n \to \infty$ (subsection 3.1). As a consequence, the corresponding dynamical system possesses a compact bounded global attractor. The second one provides a sufficient condition when the unique fixed point $0$ of the map $F$ is a global attractor on $\mathbb{R}^k$ (subsection 3.2).

3.1. Eventual Uniform Boundedness.

**Theorem 3.1.** (Eventual Uniform Boundedness). Suppose that $f$ satisfies assumptions (2) and (3), and $a > 0$ is arbitrary and fixed. There is a positive constant $C_0 = C_0(f, a)$ such that for every initial point $x_0 \in \mathbb{R}^k$ there exists a positive integer $N_0 = N_0(x_0, f, a)$ such that $|F^n(x_0)| \leq C_0$ for all $n \geq N_0$. 
Proof. Based on the expression (1) for the map $F$ its consecutive iterations are easily found in the explicit form. The second iteration has the form

$$F^2(x) = (x_1 + af(x_k) + af(x_{k-1}), x_1 + f(x_k), x_1, \ldots, x_{k-2}).$$  \hfill (5)

The general iteration $F^m(x) = y = (y_1, \ldots, y_k), 1 \leq m \leq k$, is given by

$$y_1 = x_1 + af(x_k) + \cdots + af(x_{k-m+1})$$

$$y_2 = x_1 + af(x_k) + \cdots + af(x_{k-m+2})$$

$$\cdots$$

$$y_m = x_1 + af(x_k)$$

$$y_{m+1} = x_1$$

$$\cdots$$

$$y_k = x_{k-m}.$$  \hfill (6)

In particular, we will be interested in the $k$-th iteration $F^k(x) = y$, which is given by

$$y_1 = x_1 + af(x_k) + \cdots + af(x_1)$$

$$y_2 = x_1 + af(x_k) + \cdots + af(x_2)$$

$$\cdots$$

$$y_k = x_1 + af(x_k).$$  \hfill (7)

The remainder of the proof is split into three cases, depending on possible behavior of an iterative sequence. To be definite, we shall assume that $f(x)$ is bounded from below in condition (3), $f(x) \geq -M$ for all $x$ and some $M > 0$. The case of the boundedness from above is considered along analogous reasoning.

(i) The iterative sequence $x_n = F^n(x)$ is eventually sign definite. To be specific, we can assume that $x_n > 0$ (the case $x_n < 0$ is treated similarly). Also, due to the autonomous nature of the dynamical system (1), we can assume that $x > 0$ and $x_n > 0$ for all $n \geq 1$. Let $x = (x_1, x_2, \ldots, x_{k-1}, x_k)$ with $x_i > 0 \ \forall i$. Then in view of (1) one sees that $|x_1| = |F(x)| \leq |x|$, therefore $|F^n(x)|$ is a decreasing sequence. If $\lim_{n \to \infty} F^n(x) = 0$ then it is uniformly bounded. Suppose that $\lim_{n \to \infty} |F^n(x)| = L > 0$. Then for arbitrary $\varepsilon > 0$ there exists positive integer $N_0$ such that $L \leq F^n(x) \leq L + \varepsilon$ for all $n \geq N_0$. Let $F^{N_0}(x) := y_0 = [y_1, y_2, \ldots, y_k]$ be a new initial value for the dynamical system (1), and consider its next $k$-th iteration $F^k(y_0) = [u_1, u_2, \ldots, u_k]$. One easily finds from expressions (7) that $0 < u_1 < u_2 < \cdots < u_{k-1} < u_k < L + af(L) \leq L - \delta_0$ for some fixed positive constant $\delta_0 > 0$ (e.g., one can choose $\delta_0 = (L - af(L))/2$). Since $\varepsilon > 0$ is arbitrary, and $N_0$ can be chosen large enough, this contradicts to the fact that the limit of the sequence $F^n(x_0)$ as $n \to \infty$ is number $L > 0$. Therefore, any iterative sequence that maintains the sign (either non-positive or non-negative) for sufficiently large $n$ converges to the fixed point $0 = (0, \ldots, 0)$. The possibility $x \geq 0$ is treated similarly.

(ii) The iterative sequence $x_n = F^n(x_0)$ is eventually slowly oscillating. In other words, there exists positive integer $N_0$ such that the sequence $x_n$ is slowly oscillating for $n \geq N_0$. Without loss of generality, and due to the autonomous nature of the dynamical system (1), we can assume that $x > 0$ and the corresponding iteration sequence $x_n, n \geq 1$ is slowly oscillating. We can also assume that $x_1 = F(x)$ is P-N transitional with $i = 1$. Otherwise the preceding element to the first P-N transitional point of the iterative sequence $F^n(x), n \geq 1$ should be chosen as the initial point $x$. With $x = (x_1, \ldots, x_k)$, from the expression (1) for $F(x)$ one sees that

$$x_1 + af(x_k) < 0 \quad \text{and} \quad x_j > 0, \ 1 \leq j \leq k - 1,$$
that is, \( x_1 \) is P-N transitional with \( i = 1 \). The second iteration \( F^2(x) \) given by (5) then satisfies
\[
x_1 + af(x_k) + f(x_{k-1}) < 0, \quad x_1 + af(x_k) < 0 \quad \text{and} \quad x_j > 0, \quad 1 \leq j \leq k - 2,
\]
since \( x_1 + af(x_k) < 0, af(x_{k-1}) < 0 \), due to the negative feedback assumption (2). Therefore, \( F^2(x) \) is P-N transitional with \( i = 2 \). Continuing by induction, one arrives at the conclusion that the \( k \)-th iteration \( F^k(x) \) given by formula (7) is negative, \( F^k(x) := (v_1, \ldots, v_k) < 0 \). Since \( x_1 > 0 \), \( f \) satisfies (2), (3) and is bounded from below, \( f(x) \geq -M \), one easily derives the lower bound for \( v_j \) as \( v_j \geq -kaM \), \( 1 \leq j \leq k \).

Consider next subsequent iterations \( F^n(v_0) \) where \( v_0 = (v_1, \ldots, v_k) = F^k(x) \). Since
\[
F(v_0) = (v_1 + af(v_k), v_1, \ldots, v_{k-1})
\]
the negative feedback condition (2) implies that \( v_1 + af(v_k) \geq v_1 \). Therefore, in the case \( v_1 + af(v_k) \leq 0 \) the norm \( |F(v_0)| \) satisfies the inequality \( |F(v_0)| \leq |v_0| \). Let \( F^m(v_0) \) be the last negative member of the iterative segment \( F^j(v_0), 0 \leq j \leq m \), so that \( F^{m+1}(v_0) \) is non-positive. Then
\[
|v_0| \geq |F(v_0)| \geq |F^2(v_0)| \geq \cdots \geq |F^m(v_0)|.
\]
Thus, each coordinate \( x_j, 1 \leq j \leq k \) of every point \( F^n(v_0), 0 \leq n \leq m \), satisfies the inequality \( x_j \geq -akM \).

Let \( P := \max\{f(x), x \in [-kaM, 0]\} \). With \( F^m(v_0) := (w_1, \ldots, w_k) \) its first iteration \( F(w) = (w_1 + af(w_1), w_1, \ldots, w_{k-1}) \) has the property that \( w_1 + af(w_1) \geq 0 \) and \( w_j < 0, 1 \leq j \leq k - 1 \). This means that \( F(w) \) is N-P transitional with \( i = 1 \), and \( 0 \leq w_1 + af(w_1) \leq af(w_1) \leq aP \). By considering its \( k \)-th iteration \( F^k(w) := (z_1, \ldots, z_k) := z \), and applying the reasoning similar to that above when the initial point satisfies \( x > 0 \), one concludes that \( z_j \geq 0 \) and \( z_j \leq akP \) for all \( 1 \leq j \leq k \). Therefore, each member of the iterative segment \( w_n = F^n(w), 1 \leq n \leq k \) is bounded from above, \( |w_n| \leq arP \).

Considering now the iterative sequence \( z_n = F^n(z), n \geq 0 \), one can conclude, in a complete analogy with the case of the negative initial point \( v \), that as long as it remains non-negative, \( z_n \geq 0 \), its norm in non-increasing in \( n \). Thus, one has the estimate \( |z_n| \leq arP \) for all those \( z_n \) such that \( z_n \geq 0, 0 \leq n \leq m \). If \( m + 1 \) is the first positive integer such that \( F^{m+1}(z) \) is P-N transitional, then one sets \( x := F^m(z) \) as a new positive initial value, and repeats the entire sequence of reasoning above.

Thus, by using the induction, one sees that any iterative sequence \( F^n(x) := [y_1, \ldots, y_k] \), where \( x \geq 0 \), has the property that its every coordinate satisfies the inequality
\[
-akM \leq y_j \leq akP, \quad 1 \leq j \leq k,
\]
for all sufficiently large \( n \geq N_0 \). To be precise, \( N_0 \) is the first value such that \( x_n \) becomes positive again, after the sequence \( x_n \) having become negative once. This proves the statement in the case (ii), as then one has the estimate
\[
|F^n(x)| \leq Q := \max\{kaM, kaP\}
\]
for all \( n \geq N_0 \).

(iii) The iterative sequence \( x_n = F^n(x_0) \) is rapidly oscillating. This means that given an initial point \( x \) none of its subsequent iterations \( x_n = F^n(x), n \geq 0 \), satisfies either the inequality \( x \geq 0 \) or the inequality \( x \leq 0 \). Without loss of generality we can assume that the initial point \( x = (x_1, \ldots, x_k) \) is such that \( x_1 > 0 \) and there is at least one index \( j, 2 \leq j \leq k \), such that \( x_j < 0 \). In addition, we can also assume that \( F(x) = (y_1, y_2, \ldots, y_k) \) is such that \( y_1 \leq 0 \). Consider new initial point \( \tilde{x} \) defined by \( \tilde{x} := ([x_1], [x_2], \ldots, [x_k]) \). By direct comparison using the explicit expression (1) one can see that their first iterations \( F(x) = (p_1, \ldots, p_k) \) and \( F(\tilde{x}) = (q_1, \ldots, q_k) \) are such that inequalities \( p_j \geq q_j \) hold for all \( 1 \leq j \leq k \). Continuing by induction, one can conclude that these inequalities hold for any \( n \)-th iterations \( F^n(x) \) and \( F^n(\tilde{x}) \) for all \( 1 \leq n \leq k \). This implies that any coordinate \( y_j \)
of any such iteration $F^n(x) = (y_1, \ldots, y_k), 1 \leq n \leq k$ satisfies the inequality $y_j \geq -akM$, since this is the case for the slowly oscillating sequence $F^n(\tilde{x}), 1 \leq n \leq k$. In a similar way an estimate from above $y_j \leq akP$ can be established for every iteration $F^n(x)$ for all $k + 1 \leq n \leq 2k$. Thus, the inequality (8) of part (ii) can also be established in this case of rapid oscillation. The remainder of the proof is very similar to that in part (ii); its details are left to the reader.

**Corollary 1.** Under the assumptions of Theorem 3.1 the dynamical system (1) possesses a compact bounded global attractor.

**Proof.** Indeed, if one defines the interval $I := [-arM, arP]$, and the box $B_I := I^k$ in $\mathbb{R}^k$, then the inequality (8) and the preceding reasoning show that for every initial point $x$ there exists $N_0$ such that $F^n(x) \in B_I$ for all $n \geq N_0$. The set $A := \cap_{n \geq 0} F^n(B_I)$ is then a compact bounded global attractor for the dynamical system (1).

3.2. Global Attractivity.

**Theorem 3.2.** (Global Attractivity). Suppose that $f$ satisfies conditions (2) and (3), is Lipschitz continuous, and $f'(0)$ exists. There is a positive constant $a_0 = a_0(f)$ such that for every $a \in (0, a_0)$ the fixed point $0$ is globally attracting. That is, for every initial point $x_0 \in \mathbb{R}^k$ one has $\lim_{n \to \infty} |F^n(x_0)| = 0$.

**Proof.** We shall provide complete details of the proof in the case $n = 2$.

Let an initial point $x = [u, v]$ be given. The subsequent iterations $x_n = F^n(x)$ are easily found from the general expressions (6). In particular

$$F([u, v]) = [u + af(v), u], \quad F^2([u, v]) = [u + af(v) + af(u), u + af(v)].$$

Since $f$ is Lipschitz continuous, $|f(u) - f(v)| \leq L|u - v|$ for all $u, v \in \mathbb{R}$, one also has the inequality $|f(u)| \leq L|u|, \forall u \in \mathbb{R}$ holding.

Suppose first that $x = [u, v]$ is arbitrary, and the iterative sequence $x_n = F^n(x)$ is eventually positive or negative. Then, exactly as in part (i) of the proof of Theorem 3.1, one concludes that $\lim_{n \to \infty} |F^n(x)| = 0$. This implies that $F^n(x)$ is attracted by the fixed point $0 = [0, 0]$ as $n \to \infty$.

Suppose next that the sequence $F^n(x)$ is oscillatory. We can assume first that $u > 0$ and $v > 0$. If the first iteration $F(x) = [u + af(v), u]$ is such that $u + af(v) \geq 0$ then $|F(x)| \leq |x|$. Likewise, if $u + af(v) + af(u) \geq 0$ then $|F^2(x)| \leq |F(x)| \leq |x|$. Thus, the norm $|F^n(x)|$ is nonincreasing in $n$ as long as $F^n(x)$ remains nonnegative:

$$|x| \geq |F(x)| \geq |F^2(x)| \geq \cdots \geq |F^m(x)|,$$

where $F^m(x)$ is the last element of the segment $F^n(x) > 0, 0 \leq n \leq m$, with all positive members.

If $F(x) = [u + af(v), u]$ is such that $u + af(v) < 0$ then $F^2(x) = [u + af(v) + af(u), u + af(v)] < 0$ is also negative since $af(u) < 0$. Its norm satisfies the estimate

$$|F^2(x)| \leq \max\{|u + af(v) + af(u)|, |u + af(v)|\} \leq \max\{|af(v) + af(u)|, |af(v)|\} \leq 2aL \max\{u, v\} \leq 2aL|x|.$$  \hspace{1cm} (9)

Set $F^m(x) := [y_1, y_2] = y < 0$ ($y_1 < 0$ and $y_2 < 0$). Similar to the case $u > 0, v > 0$ above, one can show that the norm $|F^n(y)|$ is non-increasing as long as $F^n(y)$ remains negative. That is

$$|y| \geq |F(y)| \geq |F^2(y)| \geq \cdots \geq |F^m(y)|,$$

when $F^n(y) < 0$ for all $0 \leq n \leq m$ and some $m$. Since $F^n(x)$ is oscillatory, such finite positive integer $m$ always exists.
Suppose \( F^n(y) = [z_1, z_2] = z < 0 \) is the last negative point in the iterative sequence starting with \( y \). Then its first and second iterations,
\[
F([z_1, z_2]) = [z_1 + af(z_2), z_1], \quad F^2([z_1, z_2]) = [z_1 + af(z_2) + af(z_1), z_1 + af(z_2)]
\]
are such that \( z_1 + af(z_2) > 0 \) and \( z_1 + af(z_2) + af(z_1) > 0 \). Moreover, following a sequence of estimates as in (9) one arrives at
\[
|F^2(z)| \leq 2aL|z|.
\] (10)
Set \( a_0 := 1/(2L) \). If one chooses any \( a \in (0, a_0) \), so that \( q := 2aL < 1 \), then by using (9) and (10) one concludes that
\[
|F^n(z)| \leq (2aL)^n|x| := q^n|x|.
\]
Since \( F^n(x) \) oscillates as \( n \to \infty \), by using the induction one shows that there exists a sequence of positive integers \( n_f \to \infty, f \in \mathbb{N} \), such that
\[
|F^n(x)| \leq (2aL)^n|x| := q^n|x|, \quad \forall \ n \geq n_f.
\]
The last inequality implies that \( F^n(x) \) is attracted by the fixed point \( 0 = [0, 0] \) as \( n \to \infty \).

The case of an initial point being negative, \( x = [u, v] < 0 \), is treated in a complete analogy with the above case \( x > 0 \). If \( x = [0, v] \) then \( F(x) = [af(v), 0] \), and \( F^2(x) = [af(v), af(v)] < 0 \). So it is reduced to the case of an initial point \( x < 0 \). Likewise, one gets a similar reduction when \( x = [u, 0] \) with \( u > 0 \). If \( x = [u, v] > 0 \) and \( F(x) = [u + f(v), u] \) is such that \( u + f(v) = 0 \) then this case can be viewed as the subcase above of \( x = [0, v] \). In general, the iterative sequence \( F^n(x), \forall n \geq 0 \), may have infinitely many members of the form \( [u, 0], u \neq 0 \). This clearly does not affect the global asymptotic attractivity of \( 0 = [0, 0] \) for any \( 0 < a < a_0 \).

The case \( x = [u, v] \) with \( u > 0 \) and \( v < 0 \) results in \( F(x) = [u + af(v), u] > 0 \). If \( u > |v| \) then \( |F(x)| > |x| \). This is the only instance when the norm \( |x| \) is increased under the action of \( F \). However, in this case one can obtain an upper estimate as follows
\[
|F(x)| = |y| \leq (1 + a)|x| \leq (1 + a_0)|x|,
\] (11)
uniformly in \( x \in \mathbb{R}^2 \) and for all \( 0 < a \leq a_0 \). Since \( F(x) = [u + af(v), u] := [y_1, y_2] := y \) is now positive, the sequence \( F^n(y) \) is slowly oscillating. As it was shown above, in this case one has that lim_{n \to \infty} |F^n(y)| = 0. In view of (11) this immediately implies that lim_{n \to \infty} |F^n(x)| = 0.

This completes the proof for the case \( n = 2 \). The case of general \( k \geq 2 \) is similar but more involved with details of possible subcases. We leave it to the reader.

**Remark 2.** Note that the assumption in Theorem 3.2 about the existence of \( f'(0) < \infty \) can be relaxed, by assuming that \( \limsup_{x \to 0} |f(x)/x| < \infty \). The former is more convenient in applications, in particular when considering the linearization of the map \( F \). The latter, however, cannot be further relaxed. As it follows from [7] for the case \( k = 1 \), when \( \lim_{x \to 0} |f(x)/x| = -\infty \) the fixed point \( x = 0 \) of the one-dimensional map \( F(x) = x + af(x) \) is always repelling for any non-zero value \( a > 0 \).

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