Construction of the classical $R$-matrices for the Toda and Calogero models.

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Abstract
We use the definition of the Calogero-Moser models as Hamiltonian reductions of geodesic motions on a group manifold to construct their $R$-matrices. In the Toda case, the analogous construction yields constant $R$-matrices. By contrast, for Calogero-Moser models they are dynamical objects.

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1 Introduction

The unification by Faddeev and his school \cite{1,4} of the classical and quantum inverse scattering method with the Yang-Baxter equation was one of the important achievements in the modern theory of integrable systems. This lead to the concept of classical $R$-matrix \cite{2,3} which encodes the Hamiltonian structure of Lax equations \cite{5}.

In the Lax representation of a dynamical system \cite{5}, the equations of motion can be written

$$\dot{L} = [M, L]$$

where $L$ and $M$ are elements of a Lie algebra $\mathcal{G}$. The primary interest of this representation is to provide us with conserved quantities

$$I_n = \text{Tr} (L^n)$$

Integrability in the sense of Liouville \cite{7,8} requires the existence of a sufficient number of conserved quantities in involution under the Poisson bracket. Hence a dynamical system represented by a Lax pair will be a natural candidate to integrability if the $I_n$’s commute.

The commutation of the $I_n$’s is equivalent to the existence of an $R$ matrix \cite{9}

$$\{L_1, L_2\} = [R_{12}, L_1] - [R_{21}, L_2]$$

(1.1)

where the notation is as follows. If $e_i$ is a basis of the Lie algebra $\mathcal{G}$, then $L = \sum_i L^i e_i$ and

$$L_1 = \sum_i L^i e_i \otimes 1 \quad L_2 = \sum_i L^i 1 \otimes e_i$$

$$\{L_1, L_2\} = \sum_{i,j} \{L^i, L^j\} e_i \otimes e_j$$

$$R_{12} = \sum_{i,j} R^{ij} e_i \otimes e_j \quad R_{21} = \sum_{i,j} R^{ij} e_j \otimes e_i$$

Remarks.

1) The Lax operator $L$ and the $R$-matrix encode all the information about the dynamical system. In particular when the Hamiltonian is chosen to be $I_n$, the $M$ matrix of the corresponding flow reads

$$M_n = -n \text{Tr}_2 R_{12} L_2^{n-1}$$

(1.2)

2) The $R$-matrix in general is a non-constant function on the phase space. The first examples of such $R$-matrices occurred in \cite{10}.

3) Since the above formula expresses only the involution property of the eigenvalues of the Lax matrix $L$, every conjugate matrix $L^g = g^{-1} L g$, where $g$ is any group valued function on the phase space, also admits an $R$-matrix which can be explicitly computed.

4) Although this theorem guarantees the existence of an $R$-matrix if one knows that the $I_n$’s are in involution, it does not provide a practical way to find it. Experienced scholars in this domain know that it is in general not an easy task to find an $R$-matrix, and it is desirable to have a constructive way to obtain them.

We present here such a constructive scheme to obtain the $R$-matrices for the Calogero-Moser models. The standard Calogero-Moser model \cite{11,13,12} describes a set of $n$ particles submitted to the equations of motion

$$\ddot{q}_i = \sum_{j \neq i} \frac{\cosh(q_i - q_j)}{\sinh^3(q_i - q_j)}$$
This model admits a Lax representation with

\[ L = p + \sum_{k \neq l} \frac{i}{\sinh(q_k - q_l)} E_{kl} \]

where \( p \) is a traceless diagonal matrix containing the momenta. The dynamical \( R \)-matrix for this model was first found in [17].

It is well known [14, 16, 15] that both the Toda models and the Calogero-Moser models are obtained by Hamiltonian reduction of the geodesic motion on the cotangent bundle \( T^*G \) of a Lie group \( G \). Denoting by \( g, \xi \) the coordinates on \( T^*G \), we will see that the Poisson structure on \( T^*G \) implies the existence of an \( R \)-matrix for \( \xi \) i.e.

\[ \{ \xi_1, \xi_2 \} = [C_{12}, \xi_1] - [C_{21}, \xi_2] \]

where \( C_{12} \) is the quadratic Casimir element in \( G \otimes G \) and we have used an invariant bilinear form to identify \( G^* \) and \( G \). We shall then use the fact that in the reduction process, the Lax matrix of the reduced system is expressed in terms of \( \xi \) by a formula of the type

\[ L = h \xi h^{-1} \]

where \( h \) is some element in \( G \). The remark then enables us to compute the \( R \)-matrix of the reduced system.

In contrast with the Toda case, where the \( R \)-matrix is a constant, this method gives us dynamical \( R \)-matrices for the Calogero-Moser models. We recover the previously known \( R \)-matrix for the standard Calogero-Moser model which corresponds to the symmetric space \( Sl(N, \mathbb{C})/SU(n) \).

We will also consider a generalization of the standard Calogero-Moser model associated to the symmetric space \( SU(n, n)/S(U(n) \times U(n)) \). The Hamiltonian of this system reads

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i \neq j}^{N} \left\{ \frac{1}{\sinh^2(q_i - q_j)} + \frac{1}{\sinh^2(q_i + q_j)} \right\} + \sum_{i=1}^{N} \frac{(1 + \gamma)^2}{\sinh^2(2q_i)} \]

where \( \gamma \) is an arbitrary real coupling constant. We obtain straightforwardly the \( R \)-matrix for this case which had eluded a direct computation scheme.

The construction presented here applies to a Lax representation without spectral parameter. It is known that there exists another Lax representation for the Calogero models depending on a spectral parameter [18]. The corresponding \( R \)-matrices were recently computed by E.K. Sklyanin [19]. It would be interesting to find the equivalent scheme yielding these \( R \)-matrices.

2 Hamiltonian reduction on cotangent bundles.

2.1 Hamiltonian reduction.
We begin by recalling some well-known facts concerning the Hamiltonian reduction of dynamical systems whose phase space is a cotangent bundle [8]. Let \( M \) be a manifold and
\( N = T^*M \) its cotangent bundle. \( N \) is equipped with the canonical 1-form \( \alpha \) whose value at the point \( p \in T^*M \) is \( \pi^*p \) where \( \pi \) is the projection of \( N \) on \( M \). Hence any group of diffeomorphisms of \( M \), lifted naturally to \( N \), namely \( \phi \in \text{Diff}(M) \) lifted to \((\phi^*)^{-1} \in \text{Diff}(N)\), leaves \( \alpha \) invariant. We shall be especially interested in the case in which a Lie group \( G \) acts on \( M \). Each element \( X \in \mathfrak{g} \) generates a vector field on \( M \) that we shall denote \( X.m \) at the point \( m \in M \). It is the derivative of the application \( g \rightarrow g.m \) at the unit element of \( G \). Since each map \( m \rightarrow g.m \) is a diffeomorphism of \( M \) this lifts to a vector field on \( N \) leaving \( \alpha \) invariant. We shall also denote \( X.p \in T_p(N) \) the value at \( p \in N \) of this vector field, so that the Lie derivative \( \mathcal{L}_{X.p} \alpha \) of the canonical 1-form vanishes.

Notice that \( N \) is a symplectic manifold equipped with the canonical 2-form
\[
\omega = -d\alpha. \tag{2.1}
\]
To any function (or Hamiltonian) \( H \) on \( N \) we associate a vector field \( X_H \) such that:
\[
dH = i_{X_H} \omega \tag{2.2}
\]
and conversely since \( \omega \) is non–degenerate. It is easy to find the Hamiltonian associated to the above vector field \( X.p, X \in \mathfrak{g} \). As a matter of fact we have \( 0 = \mathcal{L}_{X.p} \alpha = d\iota_{X.p} \alpha + \iota_{X.p} d\alpha \) hence:
\[
H_X(p) = \iota_{X.p} \alpha = \alpha([X,p]) \tag{2.3}
\]

For any two functions \( F, G \) on \( N \) one defines the Poisson bracket \( \{F,G\} \) as a function on \( N \) by:
\[
\{F,G\} = \omega(X_F,X_G) \tag{2.4}
\]
The Poisson bracket of the Hamiltonians associated to the group action has a simple expression. This follows from the:

**Lemma 2.1** For any \( X, Y \) in \( \mathfrak{g} \) and \( p \in N \) we have:
\[
\omega(X.p,Y.p) = \alpha([X,Y],p) \text{ and } [X.p,Y.p] \equiv \mathcal{L}_{X.p} Y.p = -[X,Y].p
\]

**Proof.** By definition of the Lie bracket:
\[
\mathcal{L}_{X.p} Y.p = \frac{d}{dt} e^{-X.t} \left( Y.(e^{X.t} p) \right) \bigg|_{t=0} = -[X,Y].p
\]
Then from the equations (2.2,2.3) and the invariance of \( \alpha \) we get:
\[
\omega(X.p,Y.p) = \langle dH_X, Y.p \rangle = \mathcal{L}_{X.p} \alpha(X.p) = \alpha([X,Y].p)
\]
Noticing that \( \omega(X.p,Y.p) = \{H_X,H_Y\} \) we have shown that the group action is Poissonian, i.e.:
\[
\{H_X,H_Y\} = H_{[X,Y]} \tag{2.5}
\]

Obviously the application \( X \in \mathfrak{g} \rightarrow H_X(p), \text{ any } p \in N, \) is a linear map from \( \mathfrak{g} \) to the scalars and so defines an element \( \mathcal{P}(p) \) of \( \mathfrak{g}^\ast \) which is called the momentum at \( p \in N \). For any flow induced by a Hamiltonian \( H \) invariant under \( G \) the momentum is conserved (this is Noether’s theorem):
\[
\frac{d}{dt} \langle \mathcal{P}, X \rangle = -\{H,H_X\} = \mathcal{L}_{X.p} H = 0, \forall X \in \mathfrak{g}
\]
Hence in this case one can restrict oneself to the submanifold $N_\mu$ of $N$ with fixed momentum $\mu$ i.e. such that:

$$N_\mu = \mathcal{P}^{-1}(\mu)$$

(2.6)

assuming this is a well-defined manifold.

Due to equation (2.3) and the invariance of $\alpha$ the action of the group $G$ on $N$ is transformed by $\mathcal{P}$ into the coadjoint action of $G$ on $G^*$

$$\mathcal{P}(g.p)(X) = \alpha(g.g^{-1}Xg.p) = \text{Ad}^*_g \mathcal{P}(p)(X)$$

(2.7)

where the coadjoint action on an element $\xi$ of $G^*$ is defined as:

$$\text{Ad}^*_g \xi(X) = \xi(g^{-1}Xg)$$

The stabilizer $G_\mu$ of $\mu \in G^*$ acts on $N_\mu$. The reduced phase space is precisely obtained by taking the quotient (assumed well-behaved):

$$\mathcal{F}_\mu = N_\mu/G_\mu$$

(2.8)

It is known that this is a symplectic manifold. As a matter of fact, for any two tangent vectors $\zeta, \eta$ at a point $f \in \mathcal{F}_\mu$ one defines:

$$\omega_f(\zeta, \eta) = \omega_p(\zeta', \eta')$$

where $\zeta', \eta'$ are any tangent vectors to $N_\mu$ projecting to $\zeta, \eta$, at some point $p \in N_\mu$ above $f$. This definition is independent of the choices.

In the following we shall need to compute the Poisson bracket of functions on $\mathcal{F}_\mu$. These functions are conveniently described as $G_\mu$ invariant functions on $N_\mu$. To compute their Poisson bracket we first extend them arbitrarily in the vicinity of $N_\mu$. Two extensions differ by a function vanishing on $N_\mu$. The difference of the Hamiltonian vector fields of two such extensions is controlled by the following:

**Lemma 2.2** Let $f$ be a function defined in a vicinity of $N_\mu$ and vanishing on $N_\mu$. Then the Hamiltonian vector field $X_f$ associated to $f$ is tangent to the orbit $G.p$ at any point $p \in N_\mu$.

**Proof.** The subvariety $N_\mu$ is defined by the equations $H_{X_i} = \mu_i$ for some basis $X_i$ of $\mathcal{G}$. Since $f$ vanishes on $N_\mu$ one can write $f = \sum (H_{X_i} - \mu_i)f_i$ for some functions $f_i$ defined in the vicinity of $N_\mu$. For any tangent vector $v$ at a point $p \in N_\mu$ one has:

$$< df(p), v> = \sum_i < dH_{X_i}(p), v> f_i(p) = \omega(\sum_i f_i(p) X_i.p, v)$$

since the Hamiltonian vector field associated to $H_{X_i}$ is $X_i.p$ and $\sum (H_{X_i} - \mu_i) df_i$ vanishes on $N_\mu$. Hence $X_f = \sum f_i(p) X_i.p \in \mathcal{G}.p$.

As a consequence of this lemma we have a method to compute the reduced Poisson bracket. We take two functions defined on $N_\mu$ and invariant under $G_\mu$ and extend them arbitrarily. Then we compute their Hamiltonian vector fields on $N$ and project them on the tangent space to $N_\mu$ by adding a vector tangent to the orbit $G.p$. These projections are independent of the extensions and the reduced Poisson bracket is given by the value of the symplectic form on $N$ acting on them.
Proposition 2.1 At each point \( p \in N_\mu \) one can choose a vector \( V_f.p \in \mathcal{G}.p \) such that \( X_f + V_f.p \in T_p(N_\mu) \) and \( V_f.p \) is determined up to a vector in \( \mathcal{G}_\mu.p \).

Proof. Let us notice that the symplectic orthogonal of \( \mathcal{G}.p \) is exactly \( T_p(N_\mu) \). This is because \( \omega(\xi, X.p) = 0 \) for any \( X \in \mathcal{G} \) means \( dH_X(\xi) = 0 \) hence \( \xi \in T_p(N_\mu) \) since \( N_\mu \) is defined by the equations \( H_X = \mu(X) \). Hence \( T_p(N_\mu) \cap \mathcal{G}.p = \mathcal{G}_\mu.p \) is the kernel of the symplectic form restricted to \( \mathcal{G}.p \). We want to solve \( \omega(X_f + V_f.p, X.p) = 0, \forall X \in \mathcal{G} \) i.e.

\[
\chi(X, V_f) = (X.p).f, \forall X \in \mathcal{G}
\]

where we have introduced:

\[
\chi(X, Y) = \omega(X.p, Y.p)
\]

Let us remark that this form on \( N_\mu \) only depends on the momentum \( \mu \) since assuming that the the group action is Poissonian \((2.5)\) we have:

\[
\chi(X, Y) = \{H_X, H_Y\} = H_{[X,Y]} = \mathcal{P}([X,Y]) = \mu([X,Y])
\]

Since \( \mathcal{G}_\mu = \{X \in \mathcal{G} | \text{ad}^*_X \mu = 0\} \) \( \chi \) defines a non–degenerate skew–symmetric bilinear form on \( \mathcal{G}/\mathcal{G}_\mu \). Finally \( \chi \) induces a canonical isomorphism \( \hat{\mu}: \mathcal{G}/\mathcal{G}_\mu \to (\mathcal{G}/\mathcal{G}_\mu)^* \) by setting \( (\hat{\mu}(Y))(X) = \chi(X, Y) \). Since \( f \) is \( \mathcal{G}_\mu \)-invariant on \( N_\mu \) the right–hand side of equation \((2.5)\) defines an element \( \lambda_f: X \to (X.p).f \) in \((\mathcal{G}/\mathcal{G}_\mu)^*\). Then \( V_f \) may be seen as an element of \( \mathcal{G}/\mathcal{G}_\mu \) given by \( \hat{\mu}^{-1}(\lambda_f) \).

For any such functions \( f, g \) the reduced Poisson bracket is given by:

\[
\{f, g\}_{\text{reduced}} = \omega(X_f + V_f.p, X_g + V_g.p) = \{f, g\} - \omega(V_f.p, V_g.p)
\]

Notice that if \( f|_{N_\mu} = 0 \) we have \( X_f + V_f.p \in \mathcal{G}_\mu.p \) hence \( \{f, g\}_{\text{reduced}} = \omega(X_f + V_f.p, X_g + V_g.p) = 0 \) so that equation \((2.12)\) indeed defines a Poisson bracket on the reduced phase space. From eq. \((2.11)\) we have:

\[
\omega(V_f.p, V_g.p) = <\mu, [V_f, V_g]>
\]

which can be further simplified by substituting \( X = V_g \) in equation \((2.9)\). By antisymmetrization one gets: \( \omega(V_f.p, V_g.p) = (1/2) \left( (V_f.p) \cdot g - (V_g.p) \cdot f \right) \).

Assuming in particular that \( f \) and \( g \) are \( G \)-invariant extensions of our given functions on \( N_\mu \) it is obvious that \( \{f, g\} \) is \( G \)-invariant (invariance of \( \omega \)) hence its restriction to \( N_\mu \) is \( G_\mu \)-invariant and independent of the choices. Moreover the associated Hamiltonian vector fields \( X_f, X_g \) are tangent to \( N_\mu \) since the \( G \)-invariance of \( f \) implies:

\[
0 = df(X.p) = \omega(X_f, X.p) = -dH_X(X_f), \ X \in \mathcal{G}
\]

therefore the functions \( H_X \) are constant along \( X_f \) i.e. \( X_f \) is tangent to \( N_\mu \). As a result the symplectic form defined above on \( \mathcal{F}_\mu \) yields the same Poisson brackets for \( f \) and \( g \) as computed by this method.

We have shown the:

**Proposition 2.2** The reduced Poisson bracket of two functions on \( \mathcal{F}_\mu \) can be computed using any extensions \( f, g \) in the vicinity of \( N_\mu \) according to:

\[
\{f, g\}_{\text{reduced}} = \{f, g\} + \frac{1}{2} ((V_g.p).f - (V_f.p).g)
\]

This is equivalent to the Dirac bracket.
2.2 The case $N = T^*G$

If $M = G$ is a Lie group, one can use the left translations to identify $N = T^*G$ with $G \times G^*$.

$$\omega \in T_g^*(G) \rightarrow (g, \xi) \quad \text{where} \quad \omega = L_{g^{-1}}^* \xi$$

(2.14)

If $(v, \kappa)$ is a vector tangent to $T^*G$ at the point $(g, \xi)$, the canonical 1-form, invariant under both left and right translations, is given by:

$$\alpha(v, \kappa) = \xi(g^{-1} \cdot v)$$

(2.15)

The right action of $G$ on $G$ produces left invariant vector fields $g.X$, $X \in G$ which can be lifted to $T^*G$. The associated Hamiltonians are simply:

$$H_X = \xi(X)$$

(2.16)

The equation (2.5) for right actions becomes

$$\{H_X, H_Y\} = -H_{[X,Y]}$$

as may be seen by considering the left action $g' \cdot g = gg'\cdot 1$ hence:

$$\{\xi(X), \xi(Y)\} = -\xi([X,Y])$$

(2.17)

i.e. the Poisson bracket of the $\xi$'s is just the Kirillov bracket. Moreover, since $H_X$ generates a right translation, we have

$$\{H_X, g\} = \omega(g.X, X_g) = -dg(g.X) = -g.X$$

(2.18)

Finally, we have a complete description of Poisson brackets with:

$$\{g, g\} = 0$$

(2.19)

Geodesics on the group $G$ correspond to left translations of 1-parameter groups (the tangent vector is transported parallel to itself), therefore

$$\frac{d}{dt}(g^{-1} \dot{g}) = 0$$

(2.20)

This is a Hamiltonian system whose Hamiltonian is:

$$H = \frac{1}{2} (\xi, \xi)$$

(2.21)

where we have identified $G^*$ and $G$ through the invariant Killing metric.

Notice that $H$ is bi–invariant, so one can attempt to reduce this dynamical system using Lie subgroups $H_L$ and $H_R$ of $G$ of Lie algebras $\mathcal{H}_L$ and $\mathcal{H}_R$, acting respectively on the left and on the right on $T^*G$ in order to obtain a non–trivial result.

Using the coordinates $(g, \xi)$ on $T^*G$ this action reads:

$$((h_L, h_R), (g, \xi)) \rightarrow (h_L gh_R^{-1}, \text{Ad}_{h_R}^* \xi)$$

We have written this action as a left action on $T^*G$, in order to apply the formalism developed in Section (2.1). The infinitesimal version of this action is given by:

$$(X_L, X_R) \cdot (g, \xi) = (X_L \cdot g - g.X_R, [X_R, \xi])$$

(2.22)

so that the corresponding Hamiltonian can be written:

$$H_{(X_L, X_R)}(p) = \alpha((X_L, X_R), p) = \langle X_L > g^{-1}, X_L > - \langle \xi, X_R >$$

This means that the moments are:

$$\mathcal{P}^L(g, \xi) = P_{\mathcal{H}_L} \cdot \text{Ad}^*_g \xi \quad \mathcal{P}^R(g, \xi) = -P_{\mathcal{H}_L} \cdot \xi \quad \mathcal{P} = (\mathcal{P}^L, \mathcal{P}^R)$$

(2.23)

where we have introduced the projector on $\mathcal{H}$ of forms in $G^*$ induced by the restriction of these forms to $\mathcal{H}$. 
3 The Toda model

Let us now apply this construction to obtain the $R$-matrix of the Toda models. This was first done by Ferreira and Olive [20], but we wish to present here a short discussion of this case since it is a very simple illustration of the general scheme which we shall use in the more complicated case of Calogero models.

3.1 Iwasawa decomposition.

Let $G$ be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let $\{H_i\}$ be the generators of a Cartan subalgebra $\mathcal{H}$, and let $\{E_{\pm \alpha}\}$ be the corresponding root vectors, chosen to form a Weyl basis, i.e. all the structure constants are real. The real normal form of $G$ is the real Lie algebra $G_0$ generated over $\mathbb{R}$ by the $H_i$ and $E_{\pm \alpha}$. Let $\sigma$ be the Cartan involution: $\sigma(H_i) = -H_i$, $\sigma(E_{\pm \alpha}) = -E_{\mp \alpha}$. The fixed points of $\sigma$ form a Lie subalgebra $K$ of $G_0$ generated by $\{E_\alpha - E_{-\alpha}\}$. We have the decomposition:

$$G_0 = K \oplus M$$

where $M$ is the real vector space generated by the $\{E_\alpha + E_{-\alpha}\}$ and the $\{H_i\}$. Notice that due to the choice of the real normal form $A = \mathcal{H} \cap G_0$ is a maximal abelian subalgebra of $G_0$ and it is entirely contained in $M$. Finally we need the real nilpotent subalgebras $N_{\pm}$ generated respectively by the $\{E_{\pm \alpha}\}$.

Let $G_0$ be the connected Lie group corresponding to the Lie algebra $G_0$, and similarly $K$ corresponding to $K$ and $N_{\pm}$ corresponding to $N_{\pm}$. Notice that $G_0/K$ is a symmetric space of the non–compact type. Finally the Cartan algebra $A$ exponentiates to $A$.

The connected Lie group $G_0$ admits the following Iwasawa decomposition:

$$G_0 \simeq N_+ \times A \times K$$

as a manifold

that is any element $g$ in $G_0$ can be written uniquely $g = nQk$. We shall perform the reduction of the geodesic motion on $T^*G_0$ by the action of the group $N_+$ on the left and $K$ on the right.

3.2 The moment map.

The reduction is obtained by a suitable choice of the momentum. We take:

$$P_{N^*_+}(g^{-1}\xi g) = \mu^L = \sum_{\alpha \text{ simple}} E_{-\alpha} \quad (3.2)$$

where we have identified $N^*_+$ with $N_-$ through the Killing form.

The isotropy group $G_\mu$ is $N_+ \times K$. This is obvious for the right component since $\mu^R = 0$. The isotropy group of $\mu^L$ is by definition the set of elements $g \in N_+$ such that:

$$< \mu^L, g^{-1}Xg > = < \mu^L, X > \quad \forall X \in N_+$$

Since $\mu^L$ only contains roots of height -1, the only contribution to $< \mu^L, X >$ comes from $X^{(1)}$, the level one component of X. But $(g^{-1}Xg)^{(1)} = X^{(1)} \forall g \in N_+$. Hence the isotropy group of $\mu^L$ is $N_+$ itself.
3.3 The submanifold $N_\mu$.

Let us first compute the dimension of the reduced phase space. Let $d = \dim G_0$ and $r = \dim A$. Then we have:

$$\dim \mathcal{K} = \dim \mathcal{N}_+ = \frac{d-r}{2} \quad \dim T^* G_0 = 2d$$

The dimension of the submanifold $N_\mu$ defined by the equations (3.1, 3.2) is

$$\dim N_\mu = 2d - \dim \mathcal{K} - \dim \mathcal{N}_+ = 2d - \frac{d-r}{2} = \frac{d+r}{2}$$

and the dimension of the reduced phase space is

$$\dim F_\mu = \dim N_\mu - \dim \mathcal{K} - \dim \mathcal{N}_+ = 2d - \frac{d-r}{2} = \frac{d+r}{2}$$

which is the correct dimension of the phase space of the Toda chain.

We now construct a section of the bundle $N_\mu$ over $F_\mu$. Since the isotropy group is the whole of $N_+ \times K$ any point $(g, \xi)$ of $N_\mu$ can be brought to the form $(Q, L)$ with $Q \in A$ (due to the Iwasawa decomposition) by the action of the isotropy group. In this subsection we shall identify $G_0$ and $G_0^*$ under the Killing form for convenience. Equation (3.1) implies that $L \in M$ which is the orthogonal of $K$. Thus we can write:

$$L = p + \sum_\alpha l_\alpha (E_\alpha + E_{-\alpha}), \quad p \in A$$

Inserting this form into equation (3.2) and setting $Q = \exp(q)$ we get:

$$P_{N_\mu} Q^{-1} L = \sum_\alpha l_\alpha \exp(\alpha(q)) E_{-\alpha} = \sum_{\alpha \text{ simple}} E_{-\alpha}$$

hence $l_\alpha = \exp(-\alpha(q))$ for $\alpha$ simple and $l_\alpha = 0$ otherwise. We have obtained the standard Lax matrix of the Toda chain:

$$L = p + \sum_{\alpha \text{ simple}} e^{-\alpha(q)} (E_\alpha + E_{-\alpha})$$

(3.3)

The set of the $(Q, L)$ is obviously a $(2r)$-dimensional subvariety $S$ of $N_\mu$ forming a section of the above-mentioned bundle. This means that for any point $(g, \xi)$ in $N_\mu$ one can write uniquely $g = nQk$ and $\xi = k^{-1}Lk$.

3.4 The $R$–matrix of the Toda model.

The function $L(X)$ (for any $X \in \mathcal{M}$) defined on the section $S$ has a uniquely defined extension on $T^* G_0$, invariant under the action of the group $N_+ \times K$:

$$F_X (g, \xi) = \langle \xi, k^{-1}Xk \rangle$$

(3.4)

where $k = k(g)$ is uniquely determined by the Iwasawa decomposition $g = nQk$. In this situation equation (2.13) has no term $\omega(V_f.p, V_g.p)$ corresponding to a projection on $TN_\mu$ and we have simply:

$$\{L(X), L(Y)\} = \{F_X, F_Y\}$$

This is evaluated immediately with the help of equations (2.17, 2.18, 2.19) and leads on the section $S$ to:

$$\{L(X), L(Y)\} = -L([X, Y]) + \langle L, [X, \nabla_g k(Y)]_S \rangle + \langle [\nabla_g k(X)]_S, Y \rangle$$
where the derivatives of $k$ are defined as:

$$\nabla g k(X) = \frac{d}{dt}k(g \exp(tX))|_{t=0}$$

Notice that $[X,Y] \in \mathcal{K}$ since $G_0/K$ is a symmetric space hence $L([X,Y]) = 0$. From this equation follows immediately an $R$–matrix structure for the Toda system given by:

$$RX = \nabla g k(X)|_S$$

We can compute the derivatives as follows: due to the Iwasawa decomposition we have uniquely

$$QX = X_+ Q + X_a Q + QX_K, \quad X_+ \in \mathcal{N}_+, \; X_a \in \mathcal{A}, \; X_K \in \mathcal{K}$$

and $\nabla k_g(X)|_S = X_K$. Then multiplying this equation by $Q^{-1}$ on the left and noticing that $Q^{-1}X_+Q \in \mathcal{N}_+$ we get $X_K = \sum x_\alpha (E_\alpha - E_{-\alpha})$ when $X = \sum x_\alpha (E_\alpha + E_{-\alpha}) + \sum i x_i H_i$.

In the dualized formalism defined by $RX = \text{Tr}_2(R_{12} 1 \otimes X)$ this $R$–matrix reads:

$$R_{12} = \frac{1}{2} \sum_{\alpha > 0} (E_\alpha - E_{-\alpha}) \otimes (E_\alpha + E_{-\alpha})$$

(3.5)

The classical form of the $R$-matrix for the Toda model reads [21, 22, 23]

$$R_{12}^{\text{standard}} = \sum_{\alpha > 0} E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha$$

Since $L$ in (3.3) is by construction invariant under the Cartan automorphism, one can write the Poisson brackets of $L \otimes 1$ with $1 \otimes L$ using the $R$-matrix $R_{12} = \sigma \otimes 1 \; R_{12}^{\text{standard}}$. Adding the two $R$-operators gives back (3.3). This construction is an application of a general formalism used for instance in the construction of rational multipoles and trigonometric $R$-matrices [23, 24, 25].

4 The Calogero models

Olshanetski and Perelomov [14] have shown that the Calogero–Moser models can be obtained by applying a Hamiltonian reduction to the geodesic motion on some suitable symmetric space.

4.1 Symmetric spaces.

Let us consider an involutive automorphism $\sigma$ of a simple Lie group $G$ and the subgroup $H$ of its fixed points. Then $H$ acts on the right on $G$ defining a principal fiber bundle of total space $G$ and base $G/H$, which is a global symmetric space. Moreover $G$ acts on the left on $G/H$ and in particular so does $H$ itself. We shall consider the situation described in Section (2.2) when $H_L = H_R = H$. The Hamiltonian of the geodesic flow on $G/H$ is invariant under the $H$ action allowing to construct the Hamiltonian reduction which under suitable choices of the momentum leads to the Calogero–Moser models. As a matter of fact since the phase space of the Calogero model is non compact one has to start from a non compact Lie group $G$ and quotient it by a maximal compact subgroup $H$ so that the symmetric space $G/H$ is of the non compact type.
The derivative of $\sigma$ at the unit element of $G$ is an involutive automorphism of $\mathcal{G}$ also denoted $\sigma$. Let us consider its eigenspaces $\mathcal{H}$ and $\mathcal{K}$ associated with the eigenvalues $+1$ and $-1$ respectively. Thus we have a decomposition:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{K} \quad (4.1)$$

in which $\mathcal{H}$ is the Lie algebra of $H$ which acts by inner automorphisms on the vector space $\mathcal{K}$ ($h \mathcal{K} h^{-1} = \mathcal{K}$).

Let $\mathcal{A}$ be a maximal commuting set of elements of $\mathcal{K}$. It is called a Cartan algebra of the symmetric space $G/H$. It is known that every element in $\mathcal{K}$ is conjugated to an element in $\mathcal{A}$ by an element of $H$. Moreover $\mathcal{A}$ can be extended to a maximal commutative subalgebra of $\mathcal{G}$ by adding to it a suitably chosen abelian subalgebra $\mathcal{B}$ of $\mathcal{H}$. We shall use the radicial decomposition of $\mathcal{G}$ under the abelian algebra $\mathcal{A}$:

$$\mathcal{G} = \mathcal{A} \bigoplus \mathcal{B} \bigoplus \bigoplus_{e_{\alpha}, \alpha \in \Phi} \mathbb{R} e_{\alpha} \quad (4.2)$$

As a matter of fact the set of $\alpha \in \Phi$, is a non reduced root system in $\mathcal{A}^*$ known as the root system of the symmetric space $G/H$. Hence the root spaces are not generically of dimension one. In the following $\sum_{\alpha}$ will denote the $\sum_{e_{\alpha}, \alpha \in \Phi}$. Equivalently we have the:

**Proposition 4.1** If $e_{\alpha} \neq 0$ is a root vector associated to the root $\alpha$, $\sigma(e_{\alpha})$ is a root vector associated to some other root denoted $\sigma(\alpha)$. Then $\Phi$ is the disjoint union of a subset $\Phi'$ and $\sigma(\Phi')$.

**Proof.** The root vector $e_{\alpha}$ is defined by the equation $[q, e_{\alpha}] = \alpha(q)e_{\alpha}$, $q \in \mathcal{A}$. Applying to it the automorphism $\sigma$ one gets:

$$[q, \sigma(e_{\alpha})] = \sigma(\alpha)\sigma(e_{\alpha}), \sigma(\alpha)(q) = \alpha(\sigma(q)) \quad (4.3)$$

This shows that $\sigma(e_{\alpha})$ is a non zero root vector associated to the linear form $\sigma(\alpha)$ which therefore belongs to $\Phi$. Obviously $\sigma$ acts as an involutive bijection of $\Phi$ allowing to separate it into $\Phi'$ and $\sigma(\Phi')$.

These decompositions of $\mathcal{G}$ exponentiate to similar decompositions of $G$. First $G = KH$ where $K = \exp(\mathcal{K})$. It is known that for a simply connected Lie group $G$ and a non–compact symmetric space $G/H$, $K$ is diffeomorphic to $G/H$ and $K \times H \to G$ is a diffeomorphism (uniqueness of the so–called Cartan decomposition).

Then $A = \exp(\mathcal{A})$ is a maximal totally geodesic flat submanifold of $G/H$ and any element of $K$ can be written as $k = hQh^{-1}$ with $Q \in A$ and $h \in H$. It follows that any element of $G$ can be written as $g = h_1Qh_2$ with $h_1, h_2 \in H$.

Of course this decomposition is non unique. This non–uniqueness is described in the following:

**Proposition 4.2** If $g = h_1Qh_2 = h'_1Q'h'_2$ we have: $h'_1 = h_1d^{-1}h_0^{-1}$, $h'_2 = h_0dh_2$ and $Q' = h_0Qh_0^{-1}$ where $d \in \exp(\mathcal{B}) = B$ and $h_0 \in H$ is a representative of an element of the Weyl group of the symmetric space. So if we fix $Q = \exp(q)$ such that $q$ be in a fundamental Weyl chamber, the only ambiguity resides in the element $d \in B$.

**Proof.** Setting $h = h'_1h_1$ and $h' = h'_2h^{-1}$ the equation reads: $h'^{-1}hQ = h'^{-1}Q'h' \in K$. By uniqueness of the Cartan decomposition $h'^{-1}h = 1$ hence $Q' = hQh^{-1}$. The adjoint action
of $h$ sending the centralizer of $Q$ to that of $Q'$ (when $Q$ hence $Q'$ are assumed regular), both being equal to the maximal “torus” $AB$, we see that $h$ is in the normalizer of $AB$. But it is known that the quotient of this normalizer in $H$ by the centralizer $AB$ is the so–called Weyl group of the symmetric space $G/H$. Hence we can write $h = h_0d$ where $h_0 \in H$ is a representative of this quotient, and $d \in B$. The conclusion follows.

4.2 The moment map

The reduction is obtained by an adequate choice of the momentum $\mu = (\mu^L, \mu^R)$ such that $\mathcal{P} = \mu$. We take $\mu^R = 0$ so that the isotropy group of the right component is $H_R$ itself.

The choice of the moment $\mu^L$ is of course of crucial importance. We will consider $\mu^L$'s such that

- their isotropy group $H_\mu$ is a maximal proper Lie subgroup of $H$, so that the phase space of the reduced system be of minimal dimension but non trivial.

- In order to ensure the unicity of the decomposition introduced in the Proposition \[4.2\] on $N_\mu$ we shall need:

$$\mathcal{H}_\mu \cap \mathcal{B} = \{0\} \quad \text{(4.4)}$$

Obviously $\mathcal{B}$ is an isotropic subspace of the skew–symmetric form $\chi$ introduced in (2.11). We shall require that it is a maximal isotropic subspace. We choose a complementary maximal isotropic subspace $\mathcal{C}$ so that

$$\mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{B} \oplus \mathcal{C} \quad \text{(4.5)}$$

and $\chi$ is a non–degenerate skew–symmetric bilinear form on $\mathcal{B} \oplus \mathcal{C}$, hence $\dim \mathcal{B} = \dim \mathcal{C}$. Notice that $\mathcal{C}$ is defined up to a symplectic transformation preserving $\mathcal{B}$.

- The reduced phase space $\mathcal{F}_\mu$ has dimension $2 \dim \mathcal{A}$

This is a constraint on the choice of $\mu$ that will be verified in the specific examples below.

4.3 The submanifold $N_\mu$

We now give an explicit description of $N_\mu$ i.e. we construct a section $S$ of the bundle $N_\mu$ over $\mathcal{F}_\mu$ so that one can write:

$$N_\mu = H_\mu S H \quad \text{(4.6)}$$

To construct this section we take a point $Q$ in $A$ and an $L \in \mathcal{G}^*$ such that the point $(Q, L)$ is in $N_\mu$. In this subsection we shall for convenience identify $\mathcal{G}$ and $\mathcal{G}^*$ under the Killing form assuming that $G$ is semi–simple. Moreover since the automorphism $\sigma$ preserves the Killing form, $\mathcal{H}$ and $\mathcal{K}$ are orthogonal, and $P_{\mathcal{H}^*}$ reduces to the orthogonal projection on $\mathcal{H}$. Since $\mu^R = 0$ we have $L \in \mathcal{K}$ and one can write:

$$L = p + \sum_{e_\alpha, \alpha \in \Phi'} l_\alpha (e_\alpha - \sigma(e_\alpha)) \quad \text{(4.7)}$$

where $p \in \mathcal{A}$. From equation (2.23) one gets:

$$\mu^L = P_{\mathcal{H}} \left( p + \sum_{\alpha} l_\alpha (Qe_\alpha Q^{-1} - Q\sigma(e_\alpha)Q^{-1}) \right)$$
Since $Q = \exp(q)$, $q \in \mathcal{A}$ we have $Qe_{\alpha}Q^{-1} = \exp(\alpha(q))e_{\alpha}$ and similarly $Q\sigma(e_{\alpha})Q^{-1} = \exp(-\alpha(q))\sigma(e_{\alpha})$ by exponentiating equation (4.3).

Then the above equation becomes:

$$\mu^L = \sum_{\alpha} l_{\alpha} \sinh \alpha(q) \left( e_{\alpha} + \sigma(e_{\alpha}) \right)$$  \hspace{1cm} (4.8)

One can choose the momentum of the form: $\mu^L = \sum_{\alpha} g_{\alpha}(e_{\alpha} + \sigma(e_{\alpha}))$ namely $\mu^L$ has no component in $\mathcal{B}$, where the $g_{\alpha}$ are such that $H_{\mu}$ is of maximal dimension (we shall see that it essentially fixes them, and obviously if $g_{\alpha} \neq 0$ for any $\alpha$ equation (4.4) is automatically satisfied) and we have shown the:

**Proposition 4.3** The couples $(Q, L)$ with $Q = \exp(q)$ and

$$L = p + \sum_{\alpha} \frac{g_{\alpha}}{\sinh \alpha(q)} (e_{\alpha} - \sigma(e_{\alpha}))$$

with $p, q \in \mathcal{A}$ form a submanifold in $N_{\mu}$ of dimension $2\dim \mathcal{A}$.

Notice that $L$ is just the Lax operator of the Calogero model and that the section $\mathcal{S}$ depends of $2\dim \mathcal{A}$ parameters in an immersive way. Hence one can identify $N_{\mu}$ with the set of orbits of $\mathcal{S}$ under $H_{\mu} \times H$ i.e. the set of points $(g = h_1 Qh_2, \xi = h_2^{-1}Lh_2)$ with $h_1 \in H_{\mu}$ and $h_2 \in H$ uniquely defined due to condition (1.4). The variables $p$ and $q$ appearing in $Q$ and $L$ are the dynamical variables of the Calogero model.

Let us remark that the reduced symplectic structure on $\mathcal{F}_{\mu}$ may be seen as the restriction on the section $\mathcal{S}$ of the symplectic form $\omega$ on $N$. According to equation (2.13) the restriction to $\mathcal{S}$ of the canonical 1–form is $< L, dq > = \text{Tr} (pdq)$ since the root vectors $e_{\alpha}$ are orthogonal to $\mathcal{A}$ under the Killing form. Hence the coordinates $(p, q)$ form a pair of canonically conjugate variables.

### 4.4 The $R$–matrix of the Calogero model

We want to compute the Poisson bracket of the functions on $\mathcal{F}_{\mu}$ whose expressions on the section $\mathcal{S}$ are $L(X)$ and $L(Y)$ for $X, Y \in \mathcal{K}$. These functions have uniquely defined $H_{\mu} \times H$ invariant extensions to $N_{\mu}$ given respectively by:

$$F_X(g, \xi) = < \xi, h_2^{-1}Xh_2 >, \quad F_Y(g, \xi) = < \xi, h_2^{-1}Yh_2 > \quad \text{where } g = h_1 Qh_2$$

Notice that $h_2$ is a well–defined function of $g$ in $N_{\mu}$ due to condition (1.4). According to the prescription given in the section (2.1) we choose extensions of these functions in the vicinity of $N_{\mu}$. We define these extensions at the point $p = (g, \xi) \in T^*G$ by the same formulae in which $h_2$ is chosen to be a function depending only on $g$ and reducing to the above–defined $h_2$ when $p \in N_{\mu}$. Because of the non–uniqueness of the decomposition $g = h_1 Qh_2$ outside of $N_{\mu}$ one cannot assert that the functions $F_X$, $F_Y$ are invariant under the action of $H \times H$ and we must appeal to the general procedure to compute the reduced Poisson brackets.

According to the theory developed in section (2.1) it is necessary to compute the projection on $T(N_{\mu})$ of the Hamiltonian vector field associated to a function $F_Z(g, \xi) = \xi(h_2^{-1}Zh_2)$. In order to compute the corresponding vector $V_F$ at the point $p = (g, \xi)$ it is convenient to consider only left action of the group $H \times H$ in the forms given in equation (2.22). Then equation (2.9) becomes:

$$< \xi, g^{-1}[X_L, V_L]g - [X_R, V_R] > = < \nabla_g F_Z, g^{-1}X_Lg - X_R > + < [X_R, \xi], \nabla_\xi F_Z >$$  \hspace{1cm} (4.9)
where the $F$-derivatives are defined as:

$$< \nabla_g F, X > = \frac{d}{dt} F(g \exp(tX), \xi)|_{t=0} \quad < \eta, \nabla\xi F > = \frac{d}{dt} F(g, \xi + t\eta)|_{t=0}$$

For the above–defined function $F_Z$ these derivatives are immediately calculated and the equation (4.9) becomes:

$$< \xi, g^{-1}[X_L, V_L]g - [X_R, V_R] > =$$

$$< \xi, h_2^{-1}[Z, \nabla_g h_2 (g^{-1} X_L g - X_R)h_2^{-1} + h_2 X_R h_2^{-1}] h_2 >$$  \hspace{2cm} (4.10)

The equation (4.10) decomposes into two independent equations for the left and right translations, which written on $S$ read:

$$< L, Q^{-1}[X_L, V_L]Q > = < L, [Z, \nabla_g h_2 (Q^{-1} X_L Q)] >$$  \hspace{2cm} (4.11)

$$< L, [X_R, V_R] > = < L, [Z, \nabla_g h_2 (X_R) - X_R] >$$  \hspace{2cm} (4.12)

In order to further study these equations we first compute $\nabla_g h_2 (X)$ and $\nabla_g h_2 (Q^{-1} XQ)$, $X \in \mathcal{H}$.

**Lemma 4.1** We have on $S$:

- a) \( \nabla_g h_2 (X) = X, \forall X \in \mathcal{H} \).
- b) \( \nabla_g h_2 (Q^{-1} XQ) = D_Q(X), \forall X \in \mathcal{H} \).

where $D_Q(X)$ takes its values in $B$ and vanishes on $\mathcal{H}_\mu$. Moreover $D_Q(X) = X$ for $X \in B$.

**Proof.** Right translations of an element of $N_\mu$ by elements of $H$ always give elements on $N_\mu$ on which the decomposition is unique. Hence $h_2(Q \exp(tX)) = h_2(Q) \exp(tX)$ directly leading to $\nabla_g h_2 (X) = X$ on $S$. Moreover due to Proposition (4.12) we have $h_2(hg) = d_g(h)h_2(g)$ with $d_g(h) \in B$ and $d_g(h) = 1$ if $(g, \xi) \in N_\mu$ and $h \in \mathcal{H}_\mu$. Taking $h$ infinitesimal yields the result. Finally if $X \in B$ we have $D_Q(X) = \nabla_g h_2 (X) = X$ since $Q$ and $X$ commute.

Let us notice that equation (4.12) is identically satisfied for any $V_R$ as it should be since the isotropy group is $H_\mu \times H$. As a matter of fact $[X_R, V_R] \in \mathcal{H}$ while $L \in K^*$ and $\nabla_g h_2 (X_R) - X_R = 0$ by Lemma (4.1).

On the other hand equation (4.11) reads:

$$< \mu, [X, V_Z] > = < L, [Z, D_Q(X)] >$$  \hspace{2cm} (4.13)

This is exactly equation (2.3) for the appropriate function $F_Z$ and its solution is given by:

$$V_Z = \mu^{-1}(\lambda_Z) \text{ where } \lambda_Z : X \rightarrow < L, [Z, D_Q(X)] >$$  \hspace{2cm} (4.14)

Here $V_Z$ is an element of $(\mathcal{H}/\mathcal{H}_\mu)$ depending linearly on $Z$.

We are now in a position to prove the existence of an $R$-matrix for the Calogero model.

**Theorem 4.1** There exists a linear mapping $R : K \rightarrow \mathcal{H}$ such that:

$$\{ L(X), L(Y) \}_{\text{reduced}} = L ([X, RY] + [RX, Y])$$  \hspace{2cm} (4.15)

and $R$ is given by:

$$R(X) = \nabla_g h_2 (X) + \frac{1}{2} D_Q (V_X)$$  \hspace{2cm} (4.16)

Hence the Calogero model is integrable.
Proof. One uses the equation (2.13) and first compute the unreduced Poisson bracket \( \{ F_X, F_Y \} \). Using equations (2.17, 2.18, 2.19) one gets:

\[
\{ \xi(h^{-1}_2Xh_2), \xi(h^{-1}_2Yh_2) \} = -\xi(h^{-1}_2[X, Y]h_2) + \xi(h^{-1}_2 \left( [X, \nabla g h_2(h^{-1}_2Yh_2)h^{-1}_2] + [\nabla g h_2(h^{-1}_2Xh_2)h^{-1}_2, Y] \right)h_2)
\]

Taking the value of this expression on \( S \) yields:

\[
\{ F_X, F_Y \}|_S = < L, [X, \nabla g h_2(Y)] + [\nabla g h_2(X), Y] > (4.17)
\]

Here we have taken into account the fact that \( X, Y \in \mathcal{K} \) hence \( [X, Y] \in \mathcal{H} \) and therefore \( L([X, Y]) = 0 \) since \( L \in \mathcal{K}^* \). We now evaluate the second term. Replacing \((V_Y, p).F_X = \lambda F_X(V_Y)\) in equation (2.13) by its expression (4.14) gives:

\[
\frac{1}{2} (\lambda F_X(V_Y) - \lambda F_Y(V_X)) = \frac{1}{2} < L, [X, D_Q(V_Y)] + [D_Q(V_X), Y] >
\]

Adding the two terms yields the result.

Of course, due to the general theory \( L(\[X, RY\] + \[RX, Y\]) \) does not depend on the choice of the extension of \( h_2 \) out of \( N_\mu \) but the \( R \)-matrix depends on it through the choice of the function \( D_Q \). In order to get a simple form it is convenient to fix the choice of this function. As a consequence of Lemma (4.1) all the indetermination reduces to the value of \( D_Q \) on the subspace \( \mathcal{C} \) introduced in eq. (4.5), and this can be chosen arbitrarily since \( X.p \) is not tangent to \( N_\mu \) when \( p \in S \) and \( X \in \mathcal{C} \). The most natural choice is:

\[
D_Q(X) = 0, \quad X \in \mathcal{C}
\]

This choice has the important consequence:

**Proposition 4.4** For any \( Z \in \mathcal{K} \) we have \( V_Z \in \mathcal{C} \) hence \( D_Q(V_Z) = 0 \). The \( R \)-matrix is then simply given by:

\[
RZ = \nabla g h_2(Z)
\]

**Proof.** Since \( V_Z \) is defined by the equation (4.13) we see that \( \mu ([X, V_Z]) = 0 \quad \forall X \in \mathcal{C} \). But \( \mathcal{C} \) is a maximal isotropic subspace hence \( V_Z \in \mathcal{C} \).

To proceed we need to compute the variation of the function \( h_2(g) \) induced by a variation of \( g \). It is given in the:

**Proposition 4.5** On the section \( S \) with \( Q = \exp(q) \in A \) we have:

For \( X \in \mathcal{K} \) i.e. \( X = X_0 + \sum X_\alpha (e_\alpha - \sigma e_\alpha) \), \( X_0 \in \mathcal{A} \)

\[
\nabla g h_2(X) = -h_0(X) + \sum_\alpha X_\alpha \coth(\alpha(q))(e_\alpha + \sigma e_\alpha)
\]

Here \( h_0(X) \) is a linear function from \( \mathcal{G} \) to \( \mathcal{B} \) which is fixed by the condition:

\[
X_L \equiv h_0(X) - \sum_\alpha \frac{X_\alpha}{\sinh(\alpha(q))} (e_\alpha + \sigma e_\alpha) \in \mathcal{H}_\mu \oplus \mathcal{C}
\]

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Proof. Since according to Proposition (1.2) any group element can be written in the form 
\[ g = h_1(g)Q(g)h_2(g) \]
any tangent vector \( v \) to \( G \) at \( Q \) can be written uniquely as:

\[ v = X_L.Q + Q.T + Q.X_R \quad \text{with} \quad X_L \in \mathcal{H}_\mu \oplus \mathcal{C}, \quad T \in \mathcal{A}, \quad X_R \in \mathcal{H} \quad (4.22) \]

Let us remark that \( X_L = \nabla_g h_1(\mathcal{A}^{-1}v), \) \( X_R = \nabla_g h_2(\mathcal{A}^{-1}v). \) If \( v = X.Q \) with \( X \in \mathcal{B} \) we have \( X_L = 0, T = 0, X_R = X \) as in Lemma (1.1). On the other hand if \( v = X.Q \) with \( X \in \mathcal{C} \) we take \( X_L = X, T = 0, X_R = 0. \) Comparing with Lemma (1.1) we see that this is equivalent to \( D_Q(X) = 0, \forall X \in \mathcal{C} \) and we are in the situation described in equation (1.1). If \( v = X.Q \) with \( X \in \mathcal{H}_\mu \) we have \( X_L = X, T = 0, X_R = 0. \) If \( v = Q.X \) with \( X \in \mathcal{H} \) one has \( X_L = 0, T = 0, X_R = X \) and finally if \( v = Q.T \) with \( T \in \mathcal{A} \) we have \( X_L = 0, X_R = 0. \) We have completely described \( \nabla_g h_1 \) and \( \nabla_g h_2 \) at the point \( Q \) and we have found that that the choice (1.18) is equivalent to \( \nabla_g h_1 \in \mathcal{H}_\mu \oplus \mathcal{C}. \)

Let us now assume that \( v = Q.X \) with \( X \in \mathcal{K}. \) Decomposing \( v \) as in equation (4.22) we get \( X = Q^{-1}.X_L.Q + T + X_R \) and writing \( X_L = h_0(X) + \sum_\alpha h_\alpha(X)(e_\alpha + \sigma e_\alpha) \) we have:

\[ Q^{-1}.X_L.Q = h_0 + \sum_\alpha h_\alpha \cosh \alpha(q)(e_\alpha + \sigma e_\alpha) - \sum_\alpha h_\alpha \sinh \alpha(q)(e_\alpha - \sigma e_\alpha) \]

so that projecting on \( \mathcal{H} \) and \( \mathcal{K} \) yields:

\[
X = -\sum_\alpha h_\alpha \sinh \alpha(q)(e_\alpha - \sigma e_\alpha) + T \\
0 = h_0 + \sum_\alpha h_\alpha \cosh \alpha(q)(e_\alpha + \sigma e_\alpha) + X_R
\]

This system is uniquely solved by:

\[
T = X_0 \quad h_\alpha(X) = \frac{X_\alpha}{\sinh \alpha(q)} \quad X_R = -h_0(X) + \sum_\alpha X_\alpha \coth(\alpha(q))(e_\alpha + \sigma e_\alpha)
\]

Notice that \( h_0 \) is uniquely determined by the condition \( X_L \in \mathcal{H}_\mu \oplus \mathcal{C} \) knowing the \( h_\alpha, \) since \( X_L \) is uniquely determined in equation (1.22). This fixes the \( R \)-matrix corresponding to the choice (1.18). We shall in the next section compute \( R \) in a concrete case by applying Proposition (1.3).

5 The \( R \)-matrix of the standard Calogero model

The standard Calogero model can be obtained as above starting from the non compact group \( G = SU(n, \mathbb{C}) \) and its maximal compact subgroup \( H = SU(n) \) as first shown by [14]. We choose the momentum \( \mu_L \) as described in Section (3.1) so that the isotropy group \( H_\mu \) be a maximal proper Lie subgroup of \( H. \) Obviously one can take \( \mu_L \) of the form:

\[ \mu_L = i(vv^+ - 1) \quad (5.1) \]

where \( v \) is a vector in \( \mathbb{C}^n \) such that \( v^+ v = 1, \) hence \( \mu_L \) is a traceless antihermitian matrix. Then \( g\mu_L g^{-1} = \mu_L \) if and only if \( gv = cv \) where \( c \) is a complex number of modulus 1. Hence \( H_\mu = S(U(n - 1) \times U(1)) \) which has the above-stated property.

In this case the automorphism \( \sigma \) is given by \( \sigma(g) = (g^+)^{-1} \) (notice that we consider only the real Lie group structure), \( B \) is the group of diagonal matrices of determinant 1 with
pure phases on the diagonal and $A$ is the group of real diagonal matrices with determinant $1$. The property (4.4) is then satisfied as soon as the vector $v$ has no zero component. As a matter of fact, $v$ is further constrained by $\mu_L$ being a value of the moment map. Considering equation (4.8) we see that $\mu_L$ has no diagonal element, which implies that all the components of $v$ are pure phases $v_j = \exp(i\theta_j)$. These extra phases which will appear in the Lax matrix can however be conjugated out by the adjoint action of a constant matrix $\text{diag}(\exp(j\theta_j))$ hence we shall from now on set $v_j = 1$ for all $j$. This is the solution first considered by Olshanetskii and Perelomov.

We now show that the reduced phase space has the correct dimension $2 \dim A = 2 \dim B$. Counting real parameters any $g \in SL(n, \mathbb{C})$ involves $(2n^2 - 2)$ parameters (notice that $\det g = 1$ gives 2 conditions) so that $T^*G$ involves $2(2n^2 - 2)$ parameters. The surface $N_\mu$ in $T^*G$ is defined by $2 \dim H$ equations namely $P_{\mathcal{H}}\xi = 0$ and $P_{\mathcal{H}} \text{Ad}_g^* \xi = \mu_L$. Since $\dim \mathcal{H} = n^2 - 1$ we see that $N_\mu$ is of dimension $(2n^2 - 2)$. Finally $H_\mu$ is of dimension $(n - 1)^2 + 1 + 1$ and $H_\mu \times H$ of dimension $(2n^2 - 2n)$, hence the reduced phase space is of dimension $(2n - 2)$ which exactly corresponds to the Calogero model.

**Proposition 5.1** We have:

$$\mathcal{H}_\mu = \{ M | M^+ = -M, \ Tr M = 0, \ Mv = 0 \} \oplus i\mathbb{R} \mu_L$$

One can take $\mathcal{C} = (\mathcal{H}_\mu \oplus \mathcal{B})^\perp$ (here we take the orthogonal under the Killing form) so that:

$$\mathcal{C} = \{ M | M^+ = -M, \ M_{ij} = u_i - u_j, \ u_i \in \mathbb{R}, \ \sum u_i = 0 \}$$

Finally $\mathcal{B}$ and $\mathcal{C}$ are a pair of maximal isotropic subspaces of $\mathcal{H}/\mathcal{H}_\mu$.

**Proof.** First of all $Mv = 0$ is equivalent to $v^TM = 0$ since $v$ is real and the Killing form is simply $(X,Y) = \text{Tr}(XY)$. Hence the orthogonal of the space of matrices such that $Mv = 0$ is the space $\{(uv^T - vu^T) | \forall u \in \mathbb{C}^n \}$. We then ask that such a matrix be orthogonal to any element of $\mathcal{B}$. This immediately gives $u_i - u_j^* = \lambda \forall i$. Finally we ask that this matrix be orthogonal to $\mu_L$. This implies $\lambda = 0$. Hence this matrix takes the form $M_{ij} = u_i - u_j$ $u_i \in \mathbb{R}$ and one can set $\sum u_i = 0$.

The skew-symmetric form $\chi$ on $\mathcal{B} \oplus \mathcal{C}$ can be written:

$$\chi(X,Y) = <\mu_L, [X,Y]> = iv^+ [X,Y]v$$

leading to:

$$\chi(X,Y) = 2i \sum (\rho_i \gamma_i - \kappa_i \beta_i)$$

where $X = \rho + \beta$, $Y = \kappa + \gamma$, $\rho_{ij} = \rho_i \delta_{ij}$, $\beta_{ij} = \beta_i - \beta_j$, $\kappa_{ij} = \kappa_i \delta_{ij}$, $\gamma_{ij} = \gamma_i - \gamma_j$ and $\sum \rho_i = \sum \beta_i = \sum \kappa_i = \sum \gamma_i = 0$. Hence $\mathcal{B}$ and $\mathcal{C}$ are a pair of complementary Lagrangian subspaces.

The root vectors appearing in the radicial decomposition (4.2) are the $E_{kl}$ and $iE_{kl}$ with $k \neq l$ where $(E_{kl})_{ab} = \delta_{ka} \delta_{lb}$ and $\sigma(E_{kl}) = -E_{lk}$, $\sigma(iE_{kl}) = iE_{lk}$. In this basis $\mu_L$ given by (5.1) reads $\mu_L = \sum_{k<l} (iE_{kl} + iE_{lk})$.

The Lax matrix $L$ is then given by Proposition (1.3) and therefore

$$L = p + \sum_{k<l} \frac{1}{\sinh(q_k - q_l)} (iE_{kl} - iE_{lk})$$

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More explicitly:

\[
L = \begin{pmatrix}
p_1 & \frac{i}{\sinh(q_i - q_j)} \\
\frac{i}{\sinh(q_j - q_i)} & \ddots \\
& & \ddots \\
& & & p_n
\end{pmatrix}
\]

The \(R\)-matrix can now be deduced straightforwardly from Proposition (4.3). We compute \(RX\) for \(X \in \mathcal{K}\) of the form:

\[
X = \sum_{k<l} x_{kl}(E_{kl} + E_{lk}) + \sum_{k<l} y_{kl}(iE_{kl} - iE_{lk}) + \sum_k z_k E_{kk}, \quad x_{kl}, y_{kl}, z_k \in \mathbb{R}
\]

The element \(X_L\) appearing in equation (4.21) reads:

\[
X_L = h_0(X) + \sum_{k<l} \frac{1}{\sinh(q_k - q_l)} \left[ - (x_{kl} + iy_{kl})E_{kl} + (x_{kl} - iy_{kl})E_{lk} \right]
\]

where \(h_0\) is a pure imaginary diagonal traceless matrix. The matrix \(X_L\) belongs to \(\mathcal{H}_\mu \oplus \mathcal{C}\) and this condition uniquely determines \(h_0\). Since the action of an element \(M_{ij} = u_i - u_j\) of \(\mathcal{C}\) on the vector \(v\) defining \(\mu\) gives \(i\theta v, \theta \in \mathbb{R}\) we get the condition \(\sum_k (X_L)_{kk} = i\theta + nu_k\). Separating the real and imaginary parts in this equation the real part immediately determines the \(u_i\) which are of no concern to us, and the imaginary part gives:

\[
h_0(X)_{kk} = i\theta + \sum_{l>k} \frac{y_{kl}}{\sinh(q_k - q_l)} - \sum_{l<k} \frac{y_{lk}}{\sinh(q_k - q_l)}
\]

Of course \(\theta\) is determined by \(\sum_k (h_0)_{kk} = 0\). Finally equation (4.20) produces the \(R\)-matrix:

\[
RX = -h_0(X) + \sum_{k<l} \coth(q_k - q_l) \left[ (x_{kl} + iy_{kl})E_{kl} - (x_{kl} - iy_{kl})E_{lk} \right]
\]

(5.2)

In order to recognize the form of the \(R\)-matrix first found in [17] we write \(RX = \text{Tr}_2 R_{12} \mathbf{1} \otimes X\) and we find:

\[
R_{12} = \sum_{k \neq l} \coth(q_k - q_l)E_{kl} \otimes E_{lk} + \frac{1}{2} \sum_{k \neq l} \frac{1}{\sinh(q_k - q_l)}(E_{kk} - \frac{1}{n} \mathbf{1}) \otimes (E_{kl} - E_{lk})
\]

This is exactly the correct \(R\)-matrix of the Calogero model for the potential \(1/\sinh(x)\), and the other potentials \(1/\sin(x)\) and \(1/x\) have similar \(R\)-matrices obtained by analytic continuation.

6 The \(R\)-matrix of the \(SU(n, n)\) Calogero model.

The \(SU(n, n)\) Calogero model is obtained by starting from the non compact group \(G = SU(n, n)\). This is the subgroup of \(SL(2n, \mathbb{C})\) which leaves invariant the sesquilinear quadratic form defined by

\[
Q((u_1, v_1), (u_2, v_2)) = (u^+_1 v^+_1) J \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = u^+_1 v_2 + v^+_1 u_2
\]

(6.1)
where \( u_i, v_i \) are vectors in \( \mathbb{C}^n \) and \( J \) is the matrix
\[
J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The Lie algebra of \( SU(n, n) \) therefore consists of block matrices
\[
\mathcal{G} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a = -d^+, \ Tr (a + d) = 0, b^+ = -b, c^+ = -c \right\}
\tag{6.2}
\]
where \( a, b, c, d \) are \( n \times n \) complex matrices.

We consider again the automorphism \( \sigma : \sigma(g) = (g^+)^{-1} \), which can be consistently restricted to \( SU(n, n) \). Its fixed points at the Lie algebra level consist of block matrices
\[
\mathcal{H} = \left\{ \begin{pmatrix} a & c \\ c & a \end{pmatrix} \middle| a^+ = -a, \ Tr (a) = 0, c^+ = -c \right\}
\tag{6.3}
\]
This Lie algebra is isomorphic to the Lie algebra of \( S(U(n) \times U(n)) \), the two \( u(n) \)'s being realized respectively by \( a + c \) and \( a - c \).

The subalgebra \( \mathcal{B} \) consists of matrices of the form (6.3) with \( c = 0 \), and \( a \) is a diagonal matrix of zero trace and purely imaginary coefficients. The Abelian subalgebra \( \mathcal{A} \) consists of matrices of the form (6.2) with \( b = c = 0 \) and \( a = -d \) is a real diagonal matrix.

To perform the reduction, we choose as above \( \mu^R = 0 \) and
\[
\mu^L = i(vv^+ - 1) + i\gamma J
\tag{6.4}
\]
The vector \( v \) has \( 2n \) components all equal to 1, and \( J \) is the matrix defining the quadratic form (6.1). Remark that \( J \) is invariant under the adjoint action of \( \mathcal{H} \). Then \( g \mu^L g^{-1} = \mu^L \forall g \in H \) is equivalent to \( gv = e^{i\theta} v \). Writing an element of \( \mathcal{H} \) as
\[
\begin{pmatrix} u + w \\ u - w + i\lambda^1 \end{pmatrix} \; ; \; u^+ = -u, w^+ = -w; \; Tr (u) = Tr (w) = 0.
\]
the subalgebra \( \mathcal{H}_\mu \) consists of matrices
\[
\begin{pmatrix} \tilde{u} + w \\ \tilde{u} - w + i\lambda^1 \end{pmatrix} \; ; \; \tilde{u}^+ = -\tilde{u}, w^+ = -w; \; Tr (\tilde{u}) = Tr (w) = 0.
\tag{6.5}
\]
and \( \tilde{u}\tilde{v} = i\theta\tilde{v} \) where \( \tilde{v} \) is an \( n \) component vector with all entries equal to one. This brings us back to the \( SL(n, \mathbb{C}) \) case.

Hence \( H_\mu = SU(n - 1) \times U(1) \times SU(n) \times U(1) \). The factor \( SU(n) \times U(1) \) is generated by \( w + i\lambda^1 \), while the factor generated by \( SU(n - 1) \times U(1) \) is generated by \( \tilde{u} \). The isotropy group \( H_\mu \) is indeed a maximal subgroup of \( S(U(n) \times U(n)) \).

Notice that in equation (6.4) the parameter \( \gamma \) is an arbitrary real number. This will lead to existence of a second coupling constant in the corresponding Calogero model.

We now compute the dimension of the reduced phase space. The real dimension of \( SU(n, n) \) is \( 4n^2 - 1 \), and so \( \dim T^*G = 8n^2 - 2 \). The dimension of \( H = S(U(n) \times U(n)) \) is \( 2n^2 - 1 \). Hence, the dimension of \( N_\mu \) is \( 4n^2 \). Now \( \dim G_\mu = 4n^2 - 2n \) and therefore the dimension of the phase space is \( 2n = 2 \dim \mathcal{A} \) as it should.
Proposition 6.1 We have the decomposition
\[ \mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{B} \oplus \mathcal{C} \]
where one can take \( \mathcal{C} = (\mathcal{H}_\mu \oplus \mathcal{B})^\perp \) so that
\[ \mathcal{C} = \{ M | M = \begin{pmatrix} c & c \\ c & c \end{pmatrix}; \ c_{ij} = u_i - u_j, \ u_i \in \mathbb{R}, \ \sum u_i = 0 \} \]
Moreover \( \mathcal{B} \) and \( \mathcal{C} \) are a pair of maximally isotropic subspaces of the bilinear form \( \chi \) defined in (2.11).

Proof. In the parametrization (6.5) of \( \mathcal{H} \) we decompose the matrix \( u \) according to proposition (5.1)
\[ u = \tilde{u} + iD + c \]
where \( \tilde{u}v = i\theta v \), \( D \) is a real traceless diagonal matrix and \( c_{ij} = u_i - u_j \) with \( u_i \) real. Hence, any element of \( \mathcal{H} \) can be written as
\[ \begin{pmatrix} \tilde{u} + w' & \tilde{u} - w' + i\lambda \mathbf{1} \\ \tilde{u} - w' + i\lambda \mathbf{1} & \tilde{u} + w' \end{pmatrix} + \begin{pmatrix} 2iD & 0 \\ 0 & 2iD \end{pmatrix} + \begin{pmatrix} c & c \\ c & c \end{pmatrix} \]
where \( w' = w - iD \). The first matrix parametrizes \( \mathcal{H}_\mu \) as in equation (6.5). The second matrix parametrizes \( \mathcal{B} \) and the third matrix parametrizes a supplementary subspace \( \mathcal{C} \) of dimension \( n - 1 \). The rest of the proof is identical to the \( SL(n, \mathbb{C}) \) case.

The root vectors in (4.2) are \( (i, j) = 1 \cdots n \):
\[
\begin{align*}
E_{ij} - E_{j+n,i+n} & \quad iE_{ij} + iE_{j+n,i+n} \quad i \neq j \\
E_{i,j+n} - E_{j,i+n} & \quad iE_{i,j+n} + iE_{j,i+n} \\
E_{i+n,j} - E_{j+i,n} & \quad iE_{i+n,j} + iE_{j+i,n}
\end{align*}
\]
The automorphism \( \sigma \) is the same as before: \( \sigma(E_{kl}) = -E_{lk} \) and \( \sigma(iE_{kl}) = iE_{lk} \). In this basis the momentum \( \mu_L \) reads
\[
\mu_L = \sum_{i<j}(1 + \sigma)(iE_{ij} + iE_{j+n,i+n}) + \sum_{i<j}(1 + \sigma)(iE_{i,j+n} + iE_{j,i+n}) + (\gamma + 1) \sum_i(1 + \sigma)(iE_{i,i+n})
\]
Then, from proposition (4.3), the Lax matrix becomes
\[
L = p + \sum_{i<j} \frac{1}{\sinh(q_i - q_j)}(1 - \sigma)(iE_{ij} + iE_{j+n,i+n}) + \sum_{i<j} \frac{1}{\sinh(q_i + q_j)}(1 - \sigma)(iE_{i,j+n} + iE_{j,i+n}) + (\gamma + 1) \sum_i \frac{1}{\sinh(2q_i)}(1 - \sigma)(iE_{i,i+n})
\]
where \( p \) is a generic element of \( A \) of the form \( \text{diag} \ p_i, -\text{diag} \ p_i \).

The \( R \)-matrix is then computed straightforwardly:

\[
\begin{align*}
R_{12} &= \frac{1}{2} \sum_{k \neq l} \coth(q_k - q_l) (E_{kl} + E_{k+n,l+n}) \otimes (E_{lk} - E_{l+n,k+n}) \\
    &\quad + \frac{1}{2} \sum_{k,l} \coth(q_k + q_l) (E_{k,l+n} + E_{k+n,l}) \otimes (E_{l+n,k} - E_{l,k+n}) \\
    &\quad + \frac{1}{2} \sum_{k \neq l} \frac{1}{\sinh(q_k - q_l)} (E_{kk} + E_{k+n,k+n} - \frac{1}{n} 1) \otimes (E_{kl} - E_{k+n,l+n}) \\
    &\quad + \frac{1}{2} \sum_{k,l} \frac{1}{\sinh(q_k + q_l)} (E_{kk} + E_{k+n,k+n} - \frac{1}{n} 1) \otimes (E_{lk} - E_{l+k+n} - E_{k+n,l})
\end{align*}
\]

Using this result, one can compute the \( M \) operator in the Lax equation from formula (1.2). We get

\[
M = \sum_{k \neq l} \frac{\cosh(q_k - q_l)}{\sinh^2(q_k - q_l)} (E_{kl} + E_{k+n,l+n}) + \sum_{k,l} \frac{\cosh(q_k + q_l)}{\sinh^2(q_k + q_l)} (E_{k,l+n} + E_{k+n,l}) \\
    + \gamma \sum_k \frac{\cosh(2q_k)}{\sinh^2(2q_k)} (E_{k,k+n} + E_{k+n,k}) \\
    + \sum_{k \neq l} \left( \frac{1}{\sinh^2(q_k - q_l)} + \frac{1}{\sinh^2(q_k + q_l)} \right) (E_{kk} + E_{k+n,k+n} - \frac{1}{n} 1) \\
    + \gamma \sum_k \frac{1}{\sinh^2(2q_k)} (E_{kk} + E_{k+n,k+n} - \frac{1}{n} 1)
\]

This is precisely the \( M \) matrix found by Olshanetsky and Perelomov [15].

**References**

[1] E.K. Sklyanin, L.A. Takhtadjan, L.D. Faddeev, “Quantum Inverse Problem Method I.” Theor.Math.Phys. 40 (1979) pp.194-220.

[2] E.K. Sklyanin. “On the complete integrability of the Landau-Lifchitz equation”. Preprint LOMI E-3-79. Leningrad 1979.

[3] M. Semenov Tian Shansky. “What is a classical \( r \)-matrix”. Funct. Anal. and its Appl. 17, 4 (1983), p. 17.

[4] L.D. Faddeev, “Integrable Models in 1+1 Dimensional Quantum Field Theory”. Les Houches Lectures 1982. Elsevier Science Publishers (1984).

[5] L.D. Faddeev, L.A. Takhtajan. “Hamiltonian Methods in the Theory of Solitons”. Springer 1986.

[6] P.D. Lax. “Integrals of non linear equations of evolution and solitary waves”. Comm. Pure Appl. Math. 21 (1968), p. 467.

[7] V. Arnold. “Méthodes mathématiques de la mécanique classique”. MIR 1976. Moscou.
[8] R. Abraham, J.E. Marsden. “Foundations of Mechanics”. Benjamin/Cummings 1978.

[9] O. Babelon, C.M. Viallet “Hamiltonian Structures and Lax Equations” Phys.Lett. 237B (1989) 411.

[10] J.M. Maillet. “Kac-Moody algebra and extended Yang-Baxter relations in the O(n) non-linear σ-model” Phys. Lett. B162 , (1985), p. 137;

[11] F. Calogero, “Exactly Solvable One-Dimensional Many-Body Problems.” Lett.Nuovo Cim. 13 (1975) pp.411-416.

[12] F. Calogero, “On a functional equation connected with integrable many-body problems.” Lett.Nuovo Cim. 16 (1976) pp.77-80.

[13] J. Moser, “Three integrable Hamiltonian systems connected with isospectral deformations.” Adv.Math. 16 (1975) pp.1-23.

[14] M.A. Olshanetsky, A.M. Perelomov. “Completely integrable systems connected with semi-simple Lie algebras”. Inv. Math. 37 (1976), p. 93.

[15] M.A. Olshanetsky, A.M. Perelomov, “Classical Integrable Finite Dimensional Systems Related to Lie Algebras.” Physics Reports 71 (1981) pp.313-400.

[16] D. Kazhdan, B. Kostant, S. Sternberg, “Hamiltonian Group Actions and Dynamical Systems of Calogero Type.” Commun. on Pure and Appl. Math. 31 (1978) pp.481-507.

[17] J. Avan, M. Talon, “Classical R-matrix Structure for the Calogero Model”. Phys.Lett. B303 (1993) pp.33-37.

[18] I.M. Krichever, “Elliptic solutions of the Kadomtsev-Petviashvili equation and many body problems.” Func.Anal.Appl. 14 (1980) p.45.

[19] E.K. Sklyanin, private communication.

[20] L.A. Ferreira, D.I. Olive, “Non-Compact Symmetric Spaces and the Toda Molecule Equations”. Commun.Math.Phys. 99 (1985) pp.365-384.

[21] A.A. Belavin, V.G. Drinfeld. “Solutions of the classical Yang-Baxter equations for simple Lie algebras”. Funct. Anal. and its Appl. 16, 3 (1982), p. 159, and 17, 3 (1983), p.220.

[22] D. Olive, N. Turok. “Algebraic structure of Toda systems”. Nucl. Phys. B220 FS8 (1983), p.491.

[23] A.A. Belavin, V.G. Drinfeld. “Triangle equations and simple Lie algebras”. Soviet Scientific Reviews C4 (1984), p.93.

[24] N.Yu. Reshetikin, L.D. Faddeev. “Hamiltonian structures for integrable models of field theory”. Theor. Math. Phys. 56 (1983), p. 847.

[25] J. Avan, M. Talon, “Graded R-matrices for integrable systems.” Nucl. Phys. B352 (1991) pp.215-249.