The Characterizing Properties of (Signless) Laplacian Permanental Polynomials of Almost Complete Graphs

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Received 16 August 2021; Accepted 17 September 2021; Published 30 September 2021

Abstract

We use $G$ to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$. The degree matrix of $G$, denoted by $D(G)$, is the diagonal matrix whose $(i,i)$th entry is $d(v_i)$. For a subgraph $H$ of $G$, let $G - E(H)$ denote the subgraph obtained from $G$ by deleting the edges of $H$. Let $c_1(G)$ and $p_1(G)$ denote, respectively, the numbers of $i$-cycles and $i$-vertex paths in $G$. Let $c_1(G_v)$ denote the number of triangles containing the vertex $v$ of $G$. Let $G \cup H$ be the union of two graphs $G$ and $H$ which have no common vertices. For any positive integer $l$, let $I_G$ be the union of $l$ disjoint copies of graph $G$. For convenience, the complete graph, path, cycle, and star on $n$ vertices are denoted by $K_n$, $P_n$, $C_n$, and $K_{1,n-1}$, respectively.

The permanent of the $n \times n$ matrix $X = (x_{ij})$ ($i, j = 1, 2, \ldots, n$) is defined as

$$\text{per}(X) = \sum_{\sigma} \prod_{i=1}^{n} x_{\sigma(i), i},$$

where the sum is taken over all permutations $\sigma$ of $\{1, 2, \ldots, n\}$. Valiant [1] has shown that computing the permanent is \#P-complete even when restricted to $(0, 1)$-matrices.

Let $M$ be an $n \times n$ matrix. The permanent polynomial of $M$, denoted by $\pi(M, x)$, is defined to be the permanent of the characteristic matrix of $M$, i.e.,

$$\pi(M) = \pi(M, x) = \text{per}(xI - M),$$

where $I$ is the identity matrix of size $n$. Let $A(G)$ denote the adjacency matrix of $G$. The Laplacian matrix $L(G)$ and signless Laplacian matrix $Q(G)$ of $G$ are defined by $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. We call $\pi(L(G), x)$ (resp., $\pi(Q(G), x)$) the Laplacian (resp., signless Laplacian) permanental polynomial of $G$.

1. Introduction

In this paper, we show that almost complete graphs are determined by their (signless) Laplacian permanental polynomials.

Editor: Barbara Martinucci

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Let $G$ be a graph with $n$ vertices, and let $L(G)$ and $Q(G)$ denote the Laplacian matrix and signless Laplacian matrix, respectively. The Laplacian (respectively, signless Laplacian) permanental polynomial of $G$ is defined as the permanent of the characteristic matrix of $L(G)$ (respectively, $Q(G)$). In this paper, we show that almost complete graphs are determined by their (signless) Laplacian permanental polynomials.
Laplacian permanental polynomial? Answer to the problem is very hard. Up to now, only a few results are known about the problem. Merris et al. computed the Laplacian permanent polynomials of all connected graphs on 6 vertices, and they found that there exist no nonisomorphic Laplacian copermanental graphs of such graphs. Based on the result, they stated that they do not know of a pair of nonisomorphic Laplacian copermanental graphs. Recently, Liu [19] showed that complete graph $K_n$ and star $S_n$ are determined by their (signless) Laplacian permanental polynomials.

Let $G_n$ denote the set of graphs each of which is obtained from $K_n$ by removing five or fewer edges. Cámara and Haemers [20] showed that all graphs in $G_n$ are determined by their characteristic polynomials of adjacency matrices of these graphs. The authors [21] proved that all graphs in $G_n$ are determined by their $A_n$-spectra. In this paper, our interest is to discuss which graph in $G_n$ is determined by its (signless) Laplacian permanental polynomial. And, we prove the following result.

**Theorem 1.** All graphs in $G_n$ are determined by their (signless) Laplacian permanental polynomial.

The rest of this paper is organized as follows. In Section 2, we present some characterizing properties of the (signless) Laplacian permanental polynomial and give some structural properties of graphs in $G_n$. In Section 3, we give the Proof of Theorem 1.

## 2. Preliminaries

Let $G_n$ denote the set of graphs each of which is obtained from $K_n$ by removing five or fewer edges. For $n \geq 10$, there exist exactly 45 nonisomorphic graphs each of which is obtained from $K_n$ by removing five or fewer edges. These graphs are labeled by $G_{ij}$, $1 \leq i \leq 5$, $0 \leq j \leq 25$, and illustrated in Figure 1. For some properties of graphs in $G_n$, see [21, 22], among others.

**Lemma 1** (see [22]). Let $H \subseteq K_n$ be a graph with $l$ edges and let $G = K_n - E(H)$. Then,

$$c_{4}(G) = \binom{n}{3} - l(n - 2) + \sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H). \quad (3)$$

In [22], the first author calculated the number of quadrangles of some $G \in G_n$, see Table 2.

**Lemma 3** (see [21]). Let $c_3(G_v)$ denote the number of triangles containing the vertex $v$ of $G$. Using the principle of inclusion-exclusion, we can obtain the following result. Let $H \subseteq K_n$ be a graph with $k$ edges and let $G = K_n - E(H)$. Let $v \in V(G)$ and let $v$ be an endpoint of $l$ edges in $E(H)$. Then,

$$c_3(G_v) = \binom{n - 1}{2} - (k - l) - (n - 1 - l) - \binom{l}{2} + c_3(G(v_l)) + |P|, \quad (5)$$

**Lemma 4** (see [19]). Let $G$ be a graph with $n$ vertices and $m$ edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Suppose that $\pi(L(G), x) = \sum p_i (G)x^{n-i}$. Then,

(i) $p_0(G) = 1$
(ii) $p_1(G) = -2m$
(iii) $p_2(G) = 2m^2 + m - (1/2)\sum d_i^2$
(iv) $p_3(G) = - (1/2) \sum d_i^2 + (m + 1) \sum d_i - (4/3) m^3 - 2m^2 + 2c_1(G)$
(v) $p_4(G) = - (1/4) \sum d_i^2 + ((2/3)m + 1) \sum d_i^3 - (1/2)$
\hspace{1cm} \begin{align*}
&\quad (2m^2 + 5m + 1) \sum d_i^3 + (1/8) (\sum d_i^2)^2 + \sum_{(v,y) \in E(G)} d_y d_j d_{c_3(G_v)} + 2c_4(G) - 4mc_3(G) + (2/3) \\
&\quad m^4 + 2m^3 + (1/2)m^2 + (1/2)m
\end{align*}$

**Lemma 5** (see [19]). Let $G$ be a graph with $n$ vertices and $m$ edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Suppose that $\pi(Q(G), x) = \sum q_i (G)x^{n-i}$. Then,

(i) $q_0(G) = 1$
(ii) $q_1(G) = -2m$
(iii) $q_2(G) = 2m^2 + m - (1/2)\sum d_i^2$
(iv) $q_3(G) = - (1/3) \sum d_i^2 + (m + 1) \sum d_i - (4/3) m^3 - 2m^2 + 2c_1(G)$
(v) $q_4(G) = - (1/4) \sum d_i^2 + ((2/3)m + 1) \sum d_i^3 - (1/2)$
\hspace{1cm} \begin{align*}
&\quad (2m^2 + 5m + 1) \sum d_i^3 + (1/8) (\sum d_i^2)^2 + \sum_{(v,y) \in E(G)} d_y d_j d_{c_3(G_v)} + 2c_4(G) + 4mc_3(G) + (2/3) \\
&\quad m^4 + 2m^3 + (1/2)m^2 + (1/2)m$
Figure 1: The graphs obtained from $K_n$ by deleting five or fewer edges drawn as lines in a disk.

Table 1: The numbers of triangles of some graphs in $\mathcal{G}_n$.

| Graph       | $c_4(G)$         |
|-------------|------------------|
| $G_{30}$    | $\left(\frac{n}{3}\right) - 3n + 9$ |
| $G_{32}, G_{33}$ | $\left(\frac{n}{3}\right) - 3n + 8$ |
| $G_{41}, G_{47}$ | $\left(\frac{n}{3}\right) - 4n + 11$ |
| $G_{44}, G_{46}, G_{410}$ | $\left(\frac{n}{3}\right) - 4n + 12$ |
| $G_{50}, G_{51}, G_{514}$ | $\left(\frac{n}{3}\right) - 5n + 12$ |
| $G_{52}$    | $\left(\frac{n}{3}\right) - 5n + 11$ |
| $G_{54}$    | $\left(\frac{n}{3}\right) - 5n + 20$ |
| $G_{53}, G_{57}, G_{518}, G_{519}, G_{522}$ | $\left(\frac{n}{3}\right) - 5n + 14$ |
| $G_{51}$    | $\left(\frac{n}{3}\right) - 3n + 7$ |
| $G_{40}$    | $\left(\frac{n}{3}\right) - 4n + 14$ |

Table 1: Continued.

| Graph       | $c_4(G)$         |
|-------------|------------------|
| $G_{41}$    | $\left(\frac{n}{3}\right) - 4n + 9$ |
| $G_{42}, G_{45}, G_{48}$ | $\left(\frac{n}{3}\right) - 4n + 10$ |
| $G_{59}, G_{524}$ | $\left(\frac{n}{3}\right) - 5n + 17$ |
| $G_{55}, G_{512}, G_{520}, G_{523}$ | $\left(\frac{n}{3}\right) - 5n + 15$ |
| $G_{58}, G_{511}, G_{513}, G_{517}$ | $\left(\frac{n}{3}\right) - 5n + 13$ |
| $G_{50}, G_{510}, G_{515}, G_{516}, G_{525}$ | $\left(\frac{n}{3}\right) - 5n + 16$ |

By Lemmas 4 and 5, we have the following.

Corollary 1. Let $G$ be a graph with $n$ vertices and $m$ edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Suppose that $\pi(L(G), x) = \sum_i r_i \pi_i(G) x^{n-i}$ and $\pi(Q(G), x) = \sum_i q_i \pi_i(G) x^{m-i}$. Then,
Lemma 8. We give some lemmas to prove Theorem 1 before. First, by Lemma 4 (iv) and Table 1, we have

\[ \sum_{i=1}^{n} d_i^2(G_{10}) = \frac{1}{3} \left( \sum_{i=1}^{n} d_i^3(G_{30}) - \sum_{i=1}^{n} d_i^3(G_{32}) \right) + 2(c_3(G_{30}) - c_3(G_{32})) = 4. \]
Lemma 10. The following statements hold:

(i) Graphs $G_{41}, G_{43}$, and $G_{47}$ are not pairwise (signless) Laplacian copemanental

(ii) Graphs $G_{42}$ and $G_{48}$ are not (signless) Laplacian copemanental

(iii) Graphs $G_{44}$ and $G_{410}$ are not (signless) Laplacian copemanental

Proof

(i) By Lemma 4 (iv) and Table 1, we get that $p_4(G_{41}) - p_4(G_{43}) = 4, p_4(G_{43}) - p_4(G_{47}) = 2$ and $p_4(G_{42}) - p_4(G_{47}) = 2$. Furthermore, by Corollary 1 (i), Table 1 and the equations above, we obtain $q_4(G_{41}) - q_4(G_{43}) = 4 - 4(3 + 4) = 0, q_4(G_{43}) - q_4(G_{47}) = 2 - 4(3 + 4) = 2$, and $q_4(G_{42}) - q_4(G_{47}) = 2 - 4(3 + 4) = -6$. Furthermore, by Lemma 5 (v) and Tables 1–3, we have $q_4(G_{41}) - q_4(G_{43}) = -2$. These mean that graphs $G_{41}, G_{43}$, and $G_{47}$ are not pairwise (signless) Laplacian copemanental.

(ii) From Table 4, we obtain that $\sum_{i=1}^n d_i^2(G_{42}) = \sum_{i=1}^n d_i^2(G_{48}) = n^2 - 2n^2 - 15n + 28$. By Lemma 4 (v) and Tables 1–3, we have $p_4(G_{48}) = p_4(G_{42}) = 7$. By Corollary 1 (ii), $p_4(G_{48}) - p_4(G_{42}) = -7$. Tables 1 and 3, and the equation above, we obtain that $q_4(G_{48}) - q_4(G_{42}) = -7 - 4\sum_i d_i c_i(G_{48}) - \sum_i d_i c_i(G_{42}) + 8m(3 + 4) = 1$. This implies that $G_{42}$ and $G_{48}$ are not (signless) Laplacian copemanental.

(iii) Similarly, by Lemma 4 (iv) and Table 1, we have $p_4(G_{44}) = p_4(G_{410}) = -2$. By Corollary 1 (i), Table 1 and the equation above, we obtain $q_4(G_{44}) - q_4(G_{410}) = -2 - 4(3 + 4) = -2$. These indicate that $G_{44}$ and $G_{410}$ are not (signless) Laplacian copemanental.

Lemma 11. The following statements hold:

(i) Graphs $G_{50}$ and $G_{51}$ are not (signless) Laplacian copemanental

(ii) Graphs $G_{52}, G_{57}, G_{513}, G_{519}, G_{522}$ are not pairwise (signless) Laplacian copemanental

(iii) Graphs $G_{58}, G_{511}, G_{514}$, and $G_{517}$ are not pairwise (signless) Laplacian copemanental

(iv) Graphs $G_{55}, G_{519}, G_{520}$, and $G_{523}$ are not pairwise (signless) Laplacian copemanental

(v) Graphs $G_{59}$ and $G_{516}$ are not (signless) Laplacian copemanental

(vi) Graphs $G_{510}$ and $G_{524}$ are not (signless) Laplacian copemanental

(vii) Graphs $G_{56}, G_{512}, G_{515}$, and $G_{525}$ are not pairwise (signless) Laplacian copemanental
Proof. By Table 4, we have that

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{50}) = \sum_{i=1}^{n} d_{i}^{2}(G_{51}) = n^3 - 2n^2 - 19n + 34,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{53}) = \sum_{i=1}^{n} d_{i}^{2}(G_{55}) = \sum_{i=1}^{n} d_{i}^{2}(G_{513}) = \sum_{i=1}^{n} d_{i}^{2}(G_{519}) = \sum_{i=1}^{n} d_{i}^{2}(G_{522}) = n^3 - 2n^2 - 19n + 38,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{58}) = \sum_{i=1}^{n} d_{i}^{2}(G_{518}) = \sum_{i=1}^{n} d_{i}^{2}(G_{515}) = \sum_{i=1}^{n} d_{i}^{2}(G_{523}) = n^3 - 2n^2 - 19n + 40,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{56}) = \sum_{i=1}^{n} d_{i}^{2}(G_{512}) = \sum_{i=1}^{n} d_{i}^{2}(G_{515}) = \sum_{i=1}^{n} d_{i}^{2}(G_{525}) = n^3 - 2n^2 - 19n + 42,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{5a}) = \sum_{i=1}^{n} d_{i}^{2}(G_{511}) = \sum_{i=1}^{n} d_{i}^{2}(G_{514}) = \sum_{i=1}^{n} d_{i}^{2}(G_{517}) = n^3 - 2n^2 - 19n + 36,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{5b}) = \sum_{i=1}^{n} d_{i}^{2}(G_{516}) = n^3 - 2n^2 - 19n + 46,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{5c}) = \sum_{i=1}^{n} d_{i}^{2}(G_{512}) = \sum_{i=1}^{n} d_{i}^{2}(G_{515}) = \sum_{i=1}^{n} d_{i}^{2}(G_{525}) = n^3 - 2n^2 - 19n + 42,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{5d}) = \sum_{i=1}^{n} d_{i}^{2}(G_{511}) = \sum_{i=1}^{n} d_{i}^{2}(G_{514}) = \sum_{i=1}^{n} d_{i}^{2}(G_{517}) = n^3 - 2n^2 - 19n + 36,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{5e}) = \sum_{i=1}^{n} d_{i}^{2}(G_{516}) = n^3 - 2n^2 - 19n + 46,
\]

\[
\sum_{i=1}^{n} d_{i}^{2}(G_{5f}) = \sum_{i=1}^{n} d_{i}^{2}(G_{524}) = n^3 - 2n^2 - 19n + 44.
\]
and the equation above, we have $q_3(G_{510}) - q_3(G_{512}) = 2$. These mean that graphs $G_{510}$ and $G_{512}$ are not (signless) Laplacian coprmental.

(vi) By Lemma 4 (iv) and Table 1, we have that $p_3(G_{510}) - p_3(G_{512}) = -6$. By Corollary 1 (i), Table 1, and the equation above, we obtain that $q_3(G_{510}) - q_3(G_{512}) = -2$. Obviously, graphs $G_{510}$ and $G_{512}$ are not (signless) Laplacian coprmental.

(vii) By Lemma 4 (iv) and Table 1, we have that $p_3(G_{510}) - p_3(G_{512}) = 8, p_3(G_{515}) - p_3(G_{510}) = 6, p_3(G_{512}) - p_3(G_{515}) = 4, p_3(G_{512}) - p_3(G_{513}) = -2, p_3(G_{512}) - p_3(G_{515}) = -4$, and $p_3(G_{510}) - p_3(G_{515}) = -2$. Furthermore, by Corollary 1, Tables 1–3, and the equations above, we have $q_3(G_{510}) - q_3(G_{512}) = 4, q_3(G_{512}) - q_3(G_{515}) = 6, q_3(G_{512}) - q_3(G_{513}) = 4, q_3(G_{512}) - q_3(G_{515}) = 2, q_3(G_{512}) - q_3(G_{512}) = 6, q_3(G_{512}) - q_3(G_{512}) = -2$, and $q_3(G_{512}) - q_4(G_{512}) = 24$. Obviously, graphs $G_{510}, G_{512}, G_{515}$, and $G_{515}$ are not pairwise (signless) Laplacian coprmental. □

Proof of Theorem 1. From Lemmas 6–11, we directly obtain Theorem 1. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11761056), by Natural Science Foundation of Qinghai Province (no. 2020-ZJ-920), and the Scientific Research Innovation Team in Qinghai Nationalities University.

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