WEAK STABILITY FOR INTEGRO-DIFFERENTIAL
INCLUSIONS OF DIFFUSION-WAVE TYPE INVOLVING
INFINITE DELAYS

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ABSTRACT. We deal with the Cauchy problem associated with integro-differential inclusions of diffusion-wave type involving infinite delays. Based on the behavior of resolvent operator associated with the linear part, an explicit estimate for solutions will be established. As a consequence, the weak stability of zero solution is proved in case the resolvent operator is asymptotically stable.

1. Introduction. We consider the following problem

\[ u'(t) \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + F(t, u_t), \quad t > 0, \]

\[ u_0 = \varphi \in B, \]

where the unknown function \( u \) takes values in a Banach space \( (X, \| \cdot \|) \), \( A \) is a closed, linear and unbounded operator, \( F \) is a multi-valued map which will be specified in Section 3, \( B \) is an admissible phase space that will be defined later. Here \( \alpha \in (1, 2) \) and \( u_t \) stands for the history of the state function up to the time \( t \), i.e. \( u_t(s) = u(t+s), s \leq 0 \).

It is noted that the linear part of (1) can be rewritten as

\[ u(t) = u(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)ds. \]

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The last equation was studied in [15, 16] in the case $A = \Delta$ (the Laplacian) with the mention that it interpolates the heat equation ($\alpha = 1$) and the wave equation ($\alpha = 2$). In addition, the authors in [22, 23, 24] described the following equation

$$\frac{\partial u}{\partial t} = \int_0^t (t-s)^{\alpha-2} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \Delta u(s, x) ds + F(t, x, u(t, x)),$$

$t > 0, x \in \Omega \subset \mathbb{R}^N,$

as a model for anomalous diffusion processes and wave propagation in viscoelastic materials. We also refer the reader to [1, 6, 8, 9] for recent results on the existence of the so-called asymptotically almost automorphic, $S$-asymptotic $\omega$-periodic solutions to diffusion-wave type equations.

In the recent paper [7], we studied (1) in the case of single-valued and obtained the existence of a unique decay solution under the assumption that $F$ is Lipschitzian and the unknown function is subject to impulsive effect and nonlocal condition. Our problem in this paper involves the multi-valued nonlinearity, which derives from control systems with multi-valued feedback ([21]), and various problems such as regularizing differential equations with discontinuous right-hand side ([14]), converting differential variational inequalities ([25]). The main aim of this work is to address, for the first time, weak stability of solutions to (1)-(2). We adopt the following concept of weakly asymptotic stability. Let $\Sigma(\varphi)$ be the solution set of (1)-(2) with respect to the initial datum $\varphi$. Assume that $0 \in \Sigma(0)$, that is (1) admits zero solution. The zero solution of (1) is said to be weakly asymptotically stable if

1. It is stable, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi|_B < \delta$ then $|u_t|_B < \varepsilon$ for all $u \in \Sigma(\varphi)$;

2. It is weakly attractive, i.e. for each $\varphi \in B$, there exists $u \in \Sigma(\varphi)$ such that $|u_t|_B \to 0$ as $t \to \infty$.

Let us give a short description for our approach. By analyzing the behavior of $\alpha$-resolvent $\{S_\alpha(t)\}_{t \geq 0}$ generated by the linear part, we construct appropriate solution spaces, on which the solution operator has a fixed point. To this end, we define a suitable measure of noncompactness (MNC) on the solution spaces, in which the solution operator is condensing. It is worth mentioning that the fixed point approach for studying stability of ordinary/functional differential equations was introduced by Burton and Furumochi in [4, 5] as an alternative for the Lyapunov functional method.

The rest of our paper is organized as follows. In Section 2, we recall the theory of $\alpha$-resolvent for the linear part of (1) and the concept of phase spaces, using for differential systems with infinite delays. We also give the notion of MNC introduced in [21]. Especially, we define a new MNC on solution spaces $C_g([0, \infty); X)$ for $g$ being a non-decreasing function, which will be used to determine a compact criterion on these spaces. Section 3 deal with the case when the $\alpha$-resolvent has an exponential growth. We will show that the problem (1)-(2), in this case, has an exponentially bounded solution. In section 4, we prove the weakly asymptotic stability of the zero solution when the $\alpha$-resolvent is asymptotically stable. The last section is devoted to an application, in which a control problem governed by partial integro-differential equations (PDEs) and multi-valued feedback is considered.
2. Preliminaries.

2.1. Resolvent operators. Let $\mathcal{L}(X)$ be the space of bounded linear operators on $X$. We recall some notions and results on resolvent operators related to our problem.

**Definition 2.1.** Let $A$ be a closed and linear operator with domain $D(A)$ on a Banach space $X$. We say that $A$ is the generator of an $\alpha$-resolvent if there exists $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to \mathcal{L}(X)$ such that 
$$ \{ \lambda^\alpha : \text{Re} \lambda > \omega \} \subset \rho(A) \quad \text{and} \quad \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \text{Re} \lambda > \omega, x \in X. $$

It is known that, in the case $\alpha = 1$, $S_\alpha(\cdot) = S_1(\cdot)$ is a $C_0$-semigroup while if $\alpha = 2$, we have a cosine family $S_2(\cdot)$. By the Subordination Principle (see [3]), if $A$ generates a $\beta$-resolvent with $\beta > \alpha$ then it also generates an $\alpha$-resolvent. In particular, if $A$ is the generator of a cosine family, there exists an $\alpha$-resolvent generated by $A$ with $\alpha \in (1, 2)$.

Another case ensuring the existence of $\alpha$-resolvent was discussed in [10]. Specifically, let $A$ be a closed and densely defined operator. Assume that $A$ is sectorial of type $(\omega, \theta)$, that is, there exist $\omega, \theta \in \mathbb{R}$, with $\omega < \theta$, $M > 0$ such that its resolvent lies in $\mathbb{C} \backslash \Sigma_{\omega, \theta}$ and
$$ \|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \Sigma_{\omega, \theta}, $$
here
$$ \Sigma_{\omega, \theta} = \{ \omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta \}. $$
In the case $0 \leq \theta < \pi(1 - \alpha/2)$, $S_\alpha(\cdot)$ exists and has the following formulation:
$$ S_\alpha(t) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} d\lambda, \quad t \geq 0, $$
where $\gamma$ is a suitable path lying outside $\Sigma_{\omega, \theta}$. Furthermore, we have the following assertion for the behavior of $S_\alpha(\cdot)$:

**Theorem 2.2.** ([10]) Let $A : D(A) \subset X \to X$ be a sectorial operator of type $(\omega, \theta)$ with $0 \leq \theta < \pi(1 - \alpha/2)$. Then there exists $C > 0$ independent of $t$ such that
$$ \|S_\alpha(t)\| \leq \begin{cases} C(1 + \omega t^\alpha)e^{\omega^{1/\alpha} t}, & \omega \geq 0, \\ \frac{C}{1 + |\omega|^{1/\alpha}}, & \omega < 0, \end{cases} \quad (3) $$
for $t \geq 0$.

In what follows, for $J \subset \mathbb{R}$ we denote by $L^1(J; X)$ the space of functions defined on $J$, taking values in $X$, which are integrable in the sense of Bochner. If $X = \mathbb{R}$ then we write $L^1(J)$ for brevity. In addition, by $C(J; X)$ we mean the space of continuous functions $v : J \to X$.

Given $T > 0$, consider the operator $\mathcal{W} : L^1(0, T; X) \to C([0, T]; X)$ given by
$$ \mathcal{W}(f)(t) = \int_0^t S_\alpha(t - s) f(s) ds, \quad f \in L^1(J; X). \quad (4) $$
A subset $\Omega \subset L^1(0, T; X)$ is said to be integrably bounded if there exists a function $\nu \in L^1(0, T)$ such that for all $f \in \Omega$,
$$ \|f(t)\| \leq \nu(t), \text{ for a.e. } t \in [0, T].$$
**Definition 2.3.** A sequence \( \{f_n\} \subset L^1(0,T;X) \) is said to be semicompact if it is integrably bounded and \( \{f_n(t)\} \subset K(t) \) for a.e. \( t \in [0,T] \) where \( K(t) \) is a family of compact sets.

The following property holds for the operator \( W \).

**Proposition 1.** Let \( S_\alpha (\cdot) \) be norm continuous, i.e. the map \( t \mapsto S_\alpha (t) \) is continuous on \((0, \infty)\). Then the operator \( W \) defined by (4) maps any integrably bounded set in \( L^1(0,T;X) \) into an equicontinuous one in \( C([0,T];X) \). If \( \{f_n\} \subset L^1(0,T;X) \) is a semicompact sequence then \( \{W(f_n)\} \) is relatively compact in \( C([0,T];X) \), moreover the weak convergence \( f_n \rightharpoonup f \) in \( L^1(0,T;X) \) implies the strong convergence \( W(f_n) \rightarrow W(f) \) in \( C([0,T];X) \).

**Proof.** The operator

\[
\Phi(t,s) = S_\alpha (t-s)
\]

satisfies the assumptions of [28, Lemma 1]. It follows that \( W(\Omega) \) is equicontinuous for each integrably bounded set \( \Omega \subset L^1(0,T;X) \). The rest of proposition can be proved by the same arguments as in [21, Theorem 5.1.1]. The proof is complete. \( \Box \)

2.2. Phase spaces. In this work, we will deploy the axiomatic definition of the phase space \( B \) introduced by Hale and Kato in [18]. The space \( B \) is a linear space of functions mapping \((-\infty, 0)\) into \( X \) endowed with a seminorm \( |\cdot|_B \) and satisfying the following fundamental axioms. If a function \( y : (-\infty, T+\sigma] \rightarrow X \) is such that \( y|_{[\sigma,T+\sigma]} \in C([\sigma,T+\sigma];X) \) and \( y_\sigma \in B \), then

(B1) \( y_t \in B \) for \( t \in [\sigma,T+\sigma] \);

(B2) the function \( t \mapsto y_t \) is continuous on \([\sigma,T+\sigma] \);

(B3) \( |y_t|_B \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |y_s|_B + M(t-\sigma)|y_\sigma|_B \), where \( K,M : [0,\infty) \rightarrow [0,\infty) \), \( K \) is continuous, \( M \) is locally bounded, and they are independent of \( y \).

In this work, we need an additional assumption on \( B \):

(B4) there exists \( \varrho > 0 \) such that \( ||\varphi(0)|| \leq \varrho |\varphi|_B \), for all \( \varphi \in B \).

We give here some examples of phase spaces. For more examples, we refer to the book by Hino, Mukarami and Naito [20].

The first one is given by

\[
C_\gamma = \{ \varphi \in C((-\infty,0];X) : \lim_{\theta \rightarrow -\infty} e^{\gamma \theta} \varphi(\theta) \text{ exists in } X \},
\]

where \( \gamma \) is a positive number. This phase space satisfies (B1)-(B3) with

\[
K(t) = 1, \; M(t) = e^{-\gamma t},
\]

and it is a Banach space with the norm

\[
|\varphi|_B = \sup_{\theta \leq 0} e^{\gamma \theta} ||\varphi(\theta)||.
\]

Considering another typical example, suppose that \( 1 \leq p < \infty, 0 \leq r < \infty \) and \( g : (-\infty, -r) \rightarrow \mathbb{R} \) is a nonnegative, Borel measurable function on \((-\infty, -r)\). Let \( CL^p_g \) is a class of functions \( \varphi : (-\infty,0] \rightarrow X \) such that \( \varphi \) is continuous on \([-r, 0] \) and \( g(\cdot)||\varphi(\cdot)||^p \in L^1((-\infty, -r)) \). A seminorm in \( CL^p_g \) is given by

\[
|\varphi|_{CL^p_g} = \sup_{-r \leq \theta \leq 0} ||\varphi(\theta)|| + \left( \int_{-\infty}^{-r} g(\theta)||\varphi(\theta)||^p \, d\theta \right)^{\frac{1}{p}}.
\]
Assume further that
\begin{equation}
\int_{s}^{t-r} g(\theta) d\theta < +\infty, \text{ for every } s \in (-\infty, -r) \text{ and } (8)
g(s + \theta) \leq G(s)g(\theta) \text{ for } s \leq 0 \text{ and } \theta \in (-\infty, -r), \tag{9}
\end{equation}
where \( G : (-\infty, 0] \rightarrow \mathbb{R}^+ \) is locally bounded. We know from [20] that if (8)-(9) hold true, then \( CL^p_g \) satisfies (B1)-(B3). Moreover, one can take
\begin{equation}
K(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq r, \\
1 + \left[ \int_{-t}^{-r} g(\theta) d\theta \right]^\frac{1}{p} & \text{for } t > r;
\end{cases} \tag{10}
\end{equation}
\begin{equation}
M(t) = \begin{cases} 
\max \left\{ 1 + \left[ \int_{-t}^{-r} g(\theta) d\theta \right]^\frac{1}{p}, G(t)^\frac{1}{p} \right\} & \text{for } 0 \leq t \leq r, \\
\max \left\{ \left[ \int_{-t}^{-r} g(\theta) d\theta \right]^\frac{1}{p}, G(t)^\frac{1}{p} \right\} & \text{for } t > r.
\end{cases} \tag{11}
\end{equation}

2.3. Measures of compactness and condensing multi-valued maps. Let \( E \) be a Banach space. Denote
\begin{align*}
\mathcal{P}(E) &= \{ Y \subset E : Y \neq \emptyset \}, \\
\mathcal{P}_b(E) &= \{ Y \in \mathcal{P}(E) : Y \text{ is bounded} \}, \\
K(E) &= \{ Y \in \mathcal{P}(E) : Y \text{ is compact} \}, \\
Kv(E) &= \{ Y \in K(E) : Y \text{ is convex} \}.
\end{align*}

We will use the following definition of measure of noncompactness (see, e.g. [21]).

**Definition 2.4.** A function \( \beta : \mathcal{P}_b(E) \to \mathbb{R}^+ \) is called a measure of noncompactness (MNC) on \( E \) if
\[ \beta(\overline{\Omega}) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{P}_b(E), \]
where \( \overline{\Omega} \) is the closure of convex hull of \( \Omega \). An MNC \( \beta \) is said to be:

(i) monotone if for each \( \Omega_0, \Omega_1 \in \mathcal{P}_b(E) \) such that \( \Omega_0 \subseteq \Omega_1 \), we have \( \beta(\Omega_0) \leq \beta(\Omega_1) \);

(ii) nonsingular if \( \beta(\{a\} \cup \Omega) = \beta(\Omega) \) for any \( a \in E, \Omega \in \mathcal{P}_b(E) \);

(iii) invariant with respect to the union with a compact set, if \( \beta(K \cup \Omega) = \beta(\Omega) \) for every relatively compact set \( K \in E \) and \( \Omega \in \mathcal{P}_b(E) \);

(iv) algebraically semi-additive if \( \beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1) \) for any \( \Omega_0, \Omega_1 \in \mathcal{P}_b(E) \);

(v) regular if \( \beta(\Omega) = 0 \) is equivalent to the relative compactness of \( \Omega \).

An important example of MNC satisfying all properties, is the Hausdorff MNC \( \chi(\cdot) \), which is defined as follows
\[ \chi(\Omega) = \inf \{ \varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - \text{net} \}. \]
We now define two useful MNCs on \( C([0, T]; X) \). For given \( L > 0 \) and \( D \subseteq C([0, T]; X) \), put
\begin{equation}
\omega_T(D) = \sup_{t \in [0, T]} e^{-Lt} \chi(D(t)), \text{ where } D(t) := \{ x(t) : x \in D \}, \tag{12}
\end{equation}
\begin{equation}
\text{mod}_T(D) = \lim_{\delta \to 0} \sup_{x \in D} \max_{t,s \in [0, T], |t-s| < \delta} \| x(t) - x(s) \|. \tag{13}
\end{equation}
According to [21, Example 2.1.2, 2.1.4], \( \omega_T \) and \( \text{mod}_T \) are MNCs which satisfy all properties stated in Definition 2.4, except for regularity. In addition, for \( D \subseteq C([0, T]; X) \),
• \( \omega_T(D) = 0 \) iff \( D(t) \) is relatively compact for all \( t \in [0, T] \);
• \( \text{mod}_T(D) = 0 \) iff \( D \) is equicontinuous.

Let
\[
\chi_T(D) = \omega_T(D) + \text{mod}_T(D),
\]
then \( \chi_T \) is a regular MNC on \( C([0, T]; X) \). Indeed, if \( \chi_T(D) = 0 \) then \( \omega_T(D) = \text{mod}_T(D) = 0 \). This implies that \( D(t) \) is relatively compact for all \( t \in [0, T] \) and \( D \) is equicontinuous. Hence \( D \) is relatively compact by the Arzelà-Ascoli theorem.

Consider the following function space
\[
C_g(\mathbb{R}^+; X) = \{ u \in C([0, \infty); X) : \lim_{t \to \infty} \frac{u(t)}{g(t)} = 0 \},
\]
endowed with the norm
\[
\| u \|_g = \sup_{t \geq 0} \frac{\| u(t) \|}{g(t)},
\]
where \( g : \mathbb{R}^+ \to [1, \infty) \) is a continuous and nondecreasing function.

Then it is easily seen that \( C_g(\mathbb{R}^+; X) \) is a Banach space. We now define an MNC on this space. We make use of the restriction operator \( \pi_t : C_g(\mathbb{R}^+; X) \to C([0, T]; X) \) defined by \( \pi_t(u) = u|_{[0,T]} \). Let \( \Omega \) be a bounded set in \( C_g(\mathbb{R}^+; X) \). Put
\[
\chi_\infty(\Omega) = \sup_{T > 0} \omega_T(\pi_T(\Omega)) + \sup_{T > 0} \text{mod}_T(\pi_T(\Omega)),
\]
\[
d_\infty(\Omega) = \lim_{T \to \infty} \sup_{u \in \Omega} \sup_{t \geq T} \frac{\| u(t) \|}{g(t)},
\]
\[
\chi^*(\Omega) = \chi_\infty(\Omega) + d_\infty(\Omega),
\]
where \( \omega_T \) and \( \text{mod}_T \) is given by (12) and (13), respectively. Then one can check that \( \chi_\infty, d_\infty \) and \( \chi^* \) are monotone, nonsingular MNCs on \( C_g(\mathbb{R}^+; X) \). The following lemma tests the compactness in \( C_g(\mathbb{R}^+; X) \).

**Lemma 2.5.** Let \( \Omega \subset C_g(\mathbb{R}^+; X) \) be a bounded set such that \( \chi^*(\Omega) = 0 \). Then \( \Omega \) is relatively compact in \( C_g(\mathbb{R}^+; X) \).

**Proof.** Let \( \epsilon > 0 \). Since \( d_\infty(\Omega) = 0 \), one can choose \( T > 0 \) such that
\[
\left\| \frac{u(t)}{g(t)} \right\| < \frac{\epsilon}{3}, \forall t \geq T, \forall u \in \Omega.
\]
Let \( \{ u_n \} \) be a sequence in \( \Omega \). Then \( \chi_\infty(\{ u_n \}) = 0 \). This implies \( \omega_T(\pi_T(\{ u_n \})) = \text{mod}_T(\pi_T(\{ u_n \})) = 0 \), and hence \( \{ u_n|_{[0,T]} \} \) has a convergent subsequence in \( C([0, T]; X) \) (still indexed by \( n \)). So there exists \( N(\epsilon) \in \mathbb{N} \) such that
\[
\sup_{t \in [0,T]} \| u_n(t) - u_m(t) \| < \frac{\epsilon}{3}, \forall n, m \geq N(\epsilon).
\]
Accordingly,
\[
\sup_{t \in [0,T]} \left\| \frac{u_n(t)}{g(t)} - \frac{u_m(t)}{g(t)} \right\| < \frac{\epsilon}{3}, \forall n, m \geq N(\epsilon).
\]
Combining (18)-(19), one gets
\[
\sup_{t \geq 0} \left| \frac{u_n(t)}{g(t)} - \frac{u_m(t)}{g(t)} \right| \leq \sup_{t \in [0,T]} \left| \frac{u_n(t)}{g(t)} - \frac{u_m(t)}{g(t)} \right| + \sup_{t \geq T} \left| \frac{u_n(t)}{g(t)} - \frac{u_m(t)}{g(t)} \right| \\
\leq \sup_{t \in [0,T]} \left| \frac{u_n(t)}{g(t)} - \frac{u_m(t)}{g(t)} \right| + \sup_{t \geq T} \left| \frac{u_n(t)}{g(t)} \right| + \sup_{t \geq T} \left| \frac{u_m(t)}{g(t)} \right| \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\]
for all \( n, m \geq N(\epsilon) \). Therefore \( \{u_n\} \) is a Cauchy sequence in \( C_g(\mathbb{R}^+; X) \). The proof is complete. \( \square \)

We are now in a position to recall a basic estimate based on the Hausdorff MNC.

**Proposition 2.** ([2]) Let \( D \subset L^1(0,T; X) \) be such that

(i) \( D \) is integrably bounded,
(ii) \( \chi(D(t)) \leq q(t) \) for a.e. \( t \in [0,T] \), where \( q \in L^1(0,T) \). Then
\[
\chi \left( \int_0^t D(s) \, ds \right) \leq 4 \int_0^t q(s) \, ds,
\]
here \( \int_0^t D(s) \, ds = \left\{ \int_0^t \zeta(s) \, ds : \zeta \in D \right\} \).

We now recall some notions of set-valued analysis and fixed point theory for condensing maps. Let \( Y \) be a metric space.

**Definition 2.6.** ([21]) A multi-valued map (multimap) \( F : Y \to \mathcal{P}(E) \) is said to be:

(i) upper semicontinuous (u.s.c) if \( F^{-1}(V) := \{ y \in Y : F(y) \cap V \neq \emptyset \} \) is a closed subset of \( Y \) for every closed set \( V \subset E \);
(ii) closed if its graph \( \Gamma_F := \{(y, z) : z \in F(y)\} \) is closed subset of \( Y \times E \).

**Definition 2.7.** A multimap \( F : Z \subseteq E \to \mathcal{P}(E) \) is said to be condensing with respect to an MNC \( \beta \) (\( \beta \)-condensing) if for any bounded set \( \Omega \subset Z \), the relation
\[
\beta(\Omega) \leq \beta(F(\Omega))
\]
implies the relatively compactness \( \Omega \).

Let \( \beta \) be a monotone nonsingular MNC in \( E \). The application of the topological degree theory for condensing maps (see, e.g. [21]) yields the following fixed point principle.

**Theorem 2.8.** ([21, Corollary 3.3.1]) Let \( M \) be a bounded convex closed subset of \( E \) and let \( F : M \to K^c(M) \) be a closed and \( \beta \)-condensing multimap. Then
\( \text{Fix}(F) := \{ x \in M : x \in F(x) \} \) is nonempty.

3. Exponentially bounded solutions. Concerning problem (1)-(2), we give the following assumptions:

(A) The operator \( A \) is sectorial of type \( (\omega, \theta) \) with \( \omega \geq 0 \) and \( 0 \leq \theta < \pi(1 - \alpha/2) \) so that the \( \alpha \)-resolvent \( S_\alpha(\cdot) \) generated by \( A \) is norm continuous.
(B) The phase space \( B \) obeys (B1)-(B4) such that the functions \( K \) and \( \frac{M}{g} \) are uniformly bounded.
(F) The multimap \( F : \mathbb{R}^+ \times B \to K^c(X) \) satisfies that:
In this section, we consider the space \( C \) as a closed subset of \( R^+ \). Then, clearly for each \( \psi \in B \) we have
\[
\|F(t, \psi)\| := \sup\{\|\xi\| : \xi \in F(t, \psi)\} \leq m(t)|\psi|_B + p(t)
\]
for a.e. \( t \in \mathbb{R}^+ \).

(1) For any \( \psi \in B \), the multimap \( F(\cdot, \psi) : \mathbb{R}^+ \rightarrow K\psi(X) \) admits a locally strongly measurable selector, i.e. for each \( T > 0 \) one can find a strongly measurable function \( f : [0, T] \rightarrow X \) such that \( f(t) \in F(t, \psi) \) for a.e. \( t \in [0, T] \);

(2) For a.e. \( t \in \mathbb{R}^+ \), the multimap \( F(t, \cdot) : B \rightarrow K\psi(X) \) is u.s.c on \( B \);

(3) There exist nonnegative functions \( m, p \) such that \( m, p \in L^1(\mathbb{R}^+) \), and for every \( \psi \in B \) we have
\[
\|F(t, \psi)\| := \sup\{\|\xi\| : \xi \in F(t, \psi)\} \leq m(t)|\psi|_B + p(t)
\]
for a.e. \( t \in \mathbb{R}^+ \);

(4) There exists a function \( k \in L^\infty(\mathbb{R}^+) \) such that, for every bounded set \( D \subset B \), we have \( \chi(F(t, D)) \leq k(t) \sup_{s \leq 0} \chi(D(s)) \) for a.e. \( t \in \mathbb{R}^+ \).

In view of (3), by (A) we get
\[
\lim_{t \rightarrow \infty} \frac{\|S_\alpha(t)\|}{g(t)} = 0.
\]

For \( \varphi \in B \), we define
\[
C_{g, \varphi} = \{u \in C_g(\mathbb{R}^+ ; X) : u(0) = \varphi(0)\}
\]
as a closed subset of \( C_g(\mathbb{R}^+ ; X) \). For any \( v \in C_{g, \varphi} \), we define the function \( v[\varphi] : \mathbb{R} \rightarrow X \) as follows
\[
v[\varphi](t) = \begin{cases} 
\varphi(t), & -\infty < t \leq 0, \\
v(t), & t > 0.
\end{cases}
\]
Then, clearly
\[
v[\varphi]_t(\theta) = \begin{cases} 
\varphi(t + \theta), & -\infty < \theta < -t, \\
v(t + \theta), & \theta \in [-t, 0].
\end{cases}
\]

For \( v \in C_{g, \varphi} \) put
\[
\mathcal{P}_F(v) = \{f \in L^1_{loc}(\mathbb{R}^+ ; X) : f(t) \in F(t, v[\varphi]_t) \text{ for a.e. } t \in \mathbb{R}^+ \}.
\]
Using the assumption (F)(1)-(F)(3) and the arguments as in [17], one gets \( \mathcal{P}_F(v) \neq \emptyset \) for each \( v \in C_{g, \varphi} \), that is the multimap \( \mathcal{P}_F \) is well-defined.

By the arguments in [7], we adopt the following definition of solution to (1)-(2).

**Definition 3.1.** Given \( \varphi \in B \). A function \( u : \mathbb{R} \rightarrow X \) is said to be an integral solution of problem (1)-(2) if there exists \( f \in \mathcal{P}_F(u[\varphi]_t) \) such that
\[
u(t) = \begin{cases} 
\varphi(t), & t \leq 0, \\
\int_0^t S_\alpha(t-s)f(s)ds, & t > 0.
\end{cases}
\]

Now we consider the multi-valued operator
\[
\mathcal{F} : C_{g, \varphi} \rightarrow \mathcal{P}(C_{g, \varphi}),
\]
\[
\mathcal{F}(v)(t) = S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s)f(s)ds : f \in \mathcal{P}_F(v)
\]
It is clear that if \( v \) is a fixed point of \( \mathcal{F} \) then \( v[\varphi] \) is an integral solution to (1)-(2).
Thanks to the formulation of the operator \( \mathcal{W} \) given by (4), \( \mathcal{F} \) can be rewritten as
\[
\mathcal{F}(v) = S_\alpha(\cdot, \varphi(0)) + \mathcal{W} \circ \mathcal{P}_F(v).
\]
To determine the closedness of \( \mathcal{F} \), we need the following property for \( \mathcal{P}_F \).

**Lemma 3.2.** Let (F) hold. If \( \{v_k\} \subset C_{g, \varphi} \) with \( v_k \to v^* \) and \( f_k \to f^* \) in \( L^1_{loc}(\mathbb{R}^+; X) \) with \( f^* \in \mathcal{P}_F(v^*) \).

**Proof.** Let \( \{v_k\} \subset C_{g, \varphi} \) be such that \( v_k \to v^* \), \( f_k \in \mathcal{P}_F(v_k) \). We see that \( \{f_k(t)\} \subset C(t) := F(t, \{v_k|\varphi|\}) \), and \( C(t) \) is a compact set for a.e. \( t \in \mathbb{R}^+ \). Let \( T > 0 \) be given. From (F)(3), we see that \( \{f_k|0, T\} \) is integrably bounded. Thus by [11, Corollary 3.3], \( \{f_k\} \) is weakly compact in \( L^1(0, T; X) \). So one can assume that \( f_k \rightharpoonup f^{1*} \in L^1(0, T; X) \). Then, by Mazur’s lemma (see, e.g. [13]), there exists a sequence \( \tilde{f}_k \in \text{co}\{f_i : i \geq k\} \) such that \( \tilde{f}_k \to f^{1*} \) in \( L^1(0, T; X) \) and then \( \tilde{f}_k(t) \to f^{1*}(t) \) for a.e. \( t \in [0, T] \). Since \( F \) has compact values, the upper semicontinuity of \( F(t, \cdot) \) means that for \( \epsilon > 0 \)
\[
F(t, v_k|\varphi|_t) \subset F(t, v^*|\varphi|_t) + B_\epsilon
\]
for all large \( k \), here \( B_\epsilon \) denote the ball in \( X \) at origin with radius \( \epsilon \). So
\[
f_k(t) \in F(t, v^*|\varphi|_t) + B_\epsilon, \text{ for a.e. } t \in [0, T].
\]
By the convexity of \( F(t, v^*|\varphi|_t) + B_\epsilon \), we have the same inclusion for \( \tilde{f}_k(t) \). Consequently, \( F^{1*}(t) \in F(t, v^*|\varphi|_t) + B_\epsilon \), for a.e. \( t \in [0, T] \). Since \( \epsilon \) is arbitrary, we obtain the inclusion \( F^{1*}(t) \in F(t, v^*|\varphi|_t) \) for a.e. \( t \in [0, T] \).

Repeat the above arguments for \( t \in [(n-1)T, nT], n = 1, 2, \ldots \) we get that \( f_k \rightharpoonup f^{n*} \in L^1((n-1)T, nT; X) \) with \( f^{n*}(t) \in F(t, v^*|\varphi|_t) \) for a.e. \( t \in [(n-1)T, nT] \). Defining the function \( f^* \in L^1_{loc}(\mathbb{R}^+; X) \) as follows
\[
f^*(t) = f^{n*}(t) \text{ if } t \in [(n-1)T, nT],
\]
we obtain the conclusion of the lemma.

By Lemma 3.2, we have the following result concerning the closedness of the solution operator \( \mathcal{F} \).

**Lemma 3.3.** Under the assumptions (A) and (F), the solution operator \( \mathcal{F} \) is closed.

**Proof.** Let \( \{v_n\} \subset C_{g, \varphi}, v_n \to v^*, z_n \in \mathcal{F}(v_n) \) and \( z_n \to z^* \). We need to prove that \( z^* \in \mathcal{F}(v^*) \). Indeed, take \( f_n \in \mathcal{P}_F(v_n) \) such that
\[
z_n(t) = S_\alpha(t, \varphi(0)) + \mathcal{W}(f_n)(t), \quad t > 0, \quad \tag{22}
\]
where \( \mathcal{W} \) is defined in (4). Thanks to Lemma 3.2, we can assume that \( f_n \to f^* \in L^1_{loc}(\mathbb{R}^+; X) \) and \( f^* \in \mathcal{P}_F(v^*) \). In addition, let \( t \in [0, T] \), then \( \{f_n|0, T\} \) is a semicompact sequence. By Proposition 1, we have \( \mathcal{W}(f_n) \to \mathcal{W}(f^*) \) in \( C([0, T]; X) \). In particular, \( \mathcal{W}(f_n)(t) \to \mathcal{W}(f^*)(t) \). Hence it follows from (22) that
\[
z^*(t) = S_\alpha(t, \varphi(0)) + \mathcal{W}(f^*)(t),
\]
where \( f^* \in \mathcal{P}_F(v^*) \). Thus \( z^* \in \mathcal{F}(v^*) \) and the proof is complete.

Considering the MNC \( \omega_T \) defined in (12), we choose \( L > 0 \) such that
\[
4 \sup_{t \geq 0} \int_0^t e^{-L(t-s)} \|S_\alpha(t-s)\|k(s)ds < 1.
\]

**Lemma 3.4.** Let the assumptions (A), (B) and (F) hold. Then the solution operator \( \mathcal{F} \) is \( \chi^*-\)condensing.
Proof. Let Ω ⊂ C_{r,φ} be a bounded set. We will show that if χ^*(F(Ω)) ≥ χ^*(Ω) then Ω is relatively compact. Put Ω_T = π_T(Ω) and Θ_T = π_T(F(Ω)). Then by the formulation of $F$ we have

$$\chi(Θ_T(t)) = \chi \left( S_α(t)φ(0) + \int_0^t S_α(t-s)P_F(Ω)(s)ds \right)$$

$$\leq \chi \left( \int_0^t S_α(t-s)P_F(Ω)(s)ds \right)$$

$$\leq 4 \int_0^t \chi \left( S_α(t-s)P_F(Ω)(s) \right) ds,$$

thanks to Proposition 2. By using (F)(4) one gets

$$\chi \left( S_α(t-s)P_F(Ω)(s) \right) \leq \|S_α(t-s)\| \chi \left( F(s, Ω[φ]) \right)$$

$$\leq \|S_α(t-s)\|k(s) \sup_{θ ≤ 0} \chi \left( Ω[φ](s+θ) \right)$$

$$\leq \|S_α(t-s)\|k(s) \sup_{0 ≤ r ≤ s} \chi \left( Ω(r) \right),$$

thanks to the fact that $Ω[φ](s+θ) = \{φ(s+θ)\}$ for $θ < -s$, which is a singleton. Using the last inequality in (23) we get

$$e^{-Lt} \chi(Θ_T(t)) \leq 4 \int_0^t e^{-L(t-s)}\|S_α(t-s)\|k(s)e^{-Ls} \sup_{0 ≤ r ≤ s} \chi \left( Ω(r) \right) ds$$

$$\leq 4 \int_0^t e^{-L(t-s)}\|S_α(t-s)\|k(s) \sup_{0 ≤ r ≤ s} e^{-Lr} \chi \left( Ω(r) \right) ds$$

$$\leq \left( 4 \int_0^t e^{-L(t-s)}\|S_α(t-s)\|k(s)ds \right) \omega_T(Ω_T).$$

Hence

$$\omega_T(Θ_T) \leq \left( 4 \sup_{t ≥ 0} \int_0^t e^{-L(t-s)}\|S_α(t-s)\|k(s)ds \right) \omega_T(Ω_T).$$

(24)

On the other hand, one can write

$$Θ_T = [S_α(·)φ(0) + \mathcal{W}(P_F(Ω))]|_{[0,T]}.$$

By (F)(3) we have $P_F(Ω)|_{[0,T]}$ is integrably bounded. Then by Proposition 1 we see that $\mathcal{W}(P_F(Ω))|_{[0,T]}$ is equicontinuous. This implies $Θ_T$ is equicontinuous as well. Equivalently,

$$\text{mod}_T(Θ_T) = 0.$$  (25)

Combining (24)-(25) yields

$$\chi_∞(F(Ω)) = \sup_{T > 0} \omega(Θ_T) + \sup_{T > 0} \text{mod}_T(Θ_T)$$

$$\leq \left( 4 \sup_{t ≥ 0} \int_0^t e^{-L(t-s)}\|S_α(t-s)\|k(s)ds \right) \sup_{T > 0} \omega_T(Ω_T)$$

$$\leq \left( 4 \sup_{t ≥ 0} \int_0^t e^{-L(t-s)}\|S_α(t-s)\|k(s)ds \right) \chi_∞(Ω).$$

(26)

We are in a position to estimate $d_∞(F(Ω))$. Let $z ∈ F(Ω)$, then we can take $v ∈ Ω$ and $f ∈ P_F(v)$ such that

$$z(t) = S_α(t)φ(0) + \int_0^t S_α(t-s)f(s)ds.$$
Let $||v||_g \leq R$. So

\[
\frac{\|z(t)\|}{g(t)} \leq \frac{\|S_\alpha(t)\|}{g(t)} ||\varphi(0)|| + \frac{1}{g(t)} \int_0^t \frac{\|S_\alpha(t-s)\|}{g(t)} ||f(s)||\,ds
\]

\[
\leq \frac{\|S_\alpha(t)\|}{g(t)} ||\varphi(0)|| + \int_0^t \frac{\|S_\alpha(t-s)\|}{g(t)} \left( \frac{m(s)}{g(s)} ||v[\varphi]_s||_B + \frac{p(s)}{g(s)} \right)\,ds. \tag{27}
\]

For $s \geq 0$ we have

\[
\frac{1}{g(s)} ||v[\varphi]_s||_B \leq \frac{K(s)}{g(s)} \sup_{r \in [0,s]} \|v(r)|| + \frac{M(s)}{g(s)} ||\varphi||_B
\]

\[
\leq K(s) \sup_{r \in [0,s]} \|v(r)|| + \frac{M(s)}{g(s)} ||\varphi||_B
\]

\[
\leq K_\infty R + M_g ||\varphi||_B, \tag{28}
\]

where $K_\infty = \sup_{s \geq 0} K(s)$, $M_g = \sup_{s \geq 0} \frac{M(s)}{g(s)}$.

Let $\epsilon > 0$. By virtue of (F)(3), there exists $T_1 > 0$ such that

\[
\int_{T_1}^t \frac{p(s)}{g(s)}\,ds < \epsilon,
\]

\[
\int_{T_1}^t m(s)\,ds < \epsilon, \quad \forall t \geq T_1. \tag{29}
\]

Since $\frac{\|S_\alpha(t)\|}{g(t)} \to 0$ as $t \to \infty$, one can find $D_\alpha, T_2 > 0$ such that

\[
\frac{\|S_\alpha(t)\|}{g(t)} \leq D_\alpha, \quad \forall t \geq 0,
\]

\[
\frac{\|S_\alpha(t)\|}{g(t)} < \epsilon, \quad \forall t \geq T_2. \tag{30}
\]

Hence it follows from (27)-(30) that, for all $t \geq T_1 + T_2$

\[
\frac{||z(t)||}{g(t)} \leq \epsilon ||\varphi(0)|| + \int_0^{T_1} \frac{\|S_\alpha(t-s)\|}{g(t-s)} \left[ (K_\infty R + M_g ||\varphi||_B) m(s) + \frac{p(s)}{g(s)} \right]\,ds
\]

\[
+ \int_{T_1}^t \frac{\|S_\alpha(t-s)\|}{g(t-s)} \left[ (K_\infty R + M_g ||\varphi||_B) m(s) + \frac{p(s)}{g(s)} \right]\,ds
\]

\[
\leq \epsilon ||\varphi(0)|| + \epsilon \left[ (K_\infty R + M_g ||\varphi||_B)||m||_{L^1(\mathbb{R}^+)} + \frac{p}{g} \right]_{L^1(\mathbb{R}^+)}
\]

\[
+ \epsilon D_\alpha (K_\infty R + M_g ||\varphi||_B + 1).
\]

We have shown that for any $\epsilon > 0$, there exist $C, T > 0$ such that

\[
\frac{||z(t)||}{g(t)} \leq C\epsilon, \quad \forall t \geq T,
\]

for all $z \in \mathcal{F}(\Omega)$. This implies

\[
d_\infty(\mathcal{F}(\Omega)) = \lim_{T \to \infty} \sup_{z \in \mathcal{F}(\Omega)} \sup_{t \geq T} \frac{||z(t)||}{g(t)} = 0. \tag{31}
\]
Combining (26) and (31) gives
\[ \chi^*(\mathcal{F}(\Omega)) \leq \left( 4 \sup_{t \geq 0} \int_0^t e^{-L(t-s)} \| S_\alpha(t-s) \| k(s) \| ds \right) \chi_\infty(\Omega) \leq \ell \cdot \chi^*(\Omega), \]
with
\[ \ell = 4 \sup_{t \geq 0} \int_0^t e^{-L(t-s)} \| S_\alpha(t-s) \| k(s) \| ds < 1. \]
By the assumption that \( \chi^*(\Omega) \leq \chi^*(\mathcal{F}(\Omega)) \), we get \( \chi^*(\Omega) = 0 \). So \( \Omega \) is relatively compact, due to Lemma 2.5. The proof is complete.

We formulate the main result of this section as follows.

**Theorem 3.5.** Let the assumptions (A), (B) and (F) hold. Then the problem (1)-(2) has at least one integral solution satisfying \( \frac{\| u(t) \|}{g(t)} = o(1) \) as \( t \to \infty \), provided that
\[ \sup_{t \geq 0} \int_0^t \frac{\| S_\alpha(t-s) \|}{g(t-s)} K(s)m(s) \| ds < 1. \] (32)

**Proof.** We will show that the solution operator \( \mathcal{F} \) has a fixed point in \( C_{g,\varphi} \). Obviously \( \mathcal{F}(v) \) is convex for each \( v \in C_{g,\varphi} \), due to the convexity of \( P_F(v) \). On the other hand, by the arguments in the proof of Lemma 3.4 we get
\[ \chi^*(\mathcal{F}(v)) \leq \ell \cdot \chi^*(\{v\}) = 0. \]
Hence \( \mathcal{F}(v) \) is a relatively compact set. Since \( \mathcal{F} \) is closed, \( \mathcal{F}(v) \) is compact. That is, \( \mathcal{F} \) has a fixed point. Taking into account Theorem 2.8, Lemma 3.3 and 3.4, it suffices to prove that \( \mathcal{F}(B_R) \subset B_R \) for some \( R > 0 \), here \( B_R \) is the closed ball with center at origin and radius \( R \). Assume to the contrary that for each \( n \in \mathbb{N} \) there exists \( v_n \in C_{g,\varphi} \) with \( g(v_n) \leq n \) such that \( g(z_n) > n \) for some \( z_n \in \mathcal{F}(v_n) \).

Take \( f_n \in P_F(v_n) \) such that
\[ z_n(t) = S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s)f_n(s) \| ds. \]
Then using the same estimates as in the proof of Lemma 3.4, we have
\[ \| z_n(t) \| \leq \frac{\| S_\alpha(t) \|}{g(t)} \| \varphi(0) \| + \int_0^t \frac{\| S_\alpha(t-s) \|}{g(t-s)} \left( \frac{m(s)}{g(s)} |v_n|_{C} + \frac{p(s)}{g(s)} \right) \| ds, \]
\[ \frac{1}{g(s)}|v_n|_{C} \leq nK(s) + M_g|\varphi|_{B}, \ \forall t, s \geq 0. \]
So
\[ \frac{\| z_n(t) \|}{g(t)} \leq n \int_0^t \frac{\| S_\alpha(t-s) \|}{g(t-s)} K(s)m(s) \| ds + Q(t), \] (33)
where
\[ Q(t) = \frac{\| S_\alpha(t) \|}{g(t)} \| \varphi(0) \| + M_g|\varphi|_{B} \int_0^t \frac{\| S_\alpha(t-s) \|}{g(t-s)} m(s) \| ds + \int_0^t \frac{\| S_\alpha(t-s) \|}{g(t-s)} \frac{p(s)}{g(s)} \| ds, \]
which is uniformly bounded, thanks to (20) and the fact that \( m, \frac{p}{g} \in L^1(\mathbb{R}^+) \). It follows from (33) that
\[ 1 < \frac{1}{n} \sup_{t \geq 0} \frac{\| z_n(t) \|}{g(t)} \leq \sup_{t \geq 0} \int_0^t \frac{\| S_\alpha(t-s) \|}{g(t-s)} K(s)m(s) \| ds + \frac{1}{n} \sup_{t \geq 0} Q(t). \]
Passing to the limit in the last inequality as \( n \to \infty \), we get a contradiction with (32). The proof is complete. \( \square \)

4. Weak stability result. In this section, we consider the case when the operator \( A \) is sectorial of type \((\omega, \theta)\) with \( \omega < 0 \) and \( 0 \leq \theta < \pi(1 - \alpha/2) \), i.e. the \( \alpha \)-resolvent \( S_\alpha(\cdot) \) is asymptotically stable:

\[
\|S_\alpha(t)\| \leq \frac{C}{1 + |\omega|^t}, \quad \forall t \geq 0.
\] (34)

One observes that, by choosing \( g \equiv 1 \) and proceeding as in the previous section, we can prove the existence of attracting solutions to (1)-(2) and then infer the weakly asymptotic stability of zero solution. Precisely, we consider the solution operator \( F \) on the space \( BC_{0,\varphi} = \{ v \in C([0, \infty); X) : v(0) = \varphi(0), \lim_{t \to \infty} \|v(t)\| = 0 \} \), with the norm \( \|v\|_\infty = \sup_{t \geq 0} \|v(t)\| \).

In this circumstance, \((A), (B)\) and \((F)\) are replaced by the following assumptions.

\((A')\) The operator \( A \) is sectorial of type \((\omega, \theta)\) with \( \omega < 0 \) and \( 0 \leq \theta < \pi(1 - \alpha/2) \) so that the \( \alpha \)-resolvent \( S_\alpha(\cdot) \) generated by \( A \) is norm continuous.

\((B')\) The phase space \( B \) obeys (B1)-(B4) such that the function \( K \) is uniformly bounded and \( M \) satisfies \( M(t) = o(1) \) as \( t \to \infty \).

\((F')\) The multimap \( F : [0, \infty) \times B \to Kv(X) \) satisfies that:
1. for any \( \psi \in B \) the multimap \( F(\cdot, \psi) : [0, \infty) \to Kv(X) \) admits a locally strongly measurable selector, i.e. for each \( T > 0 \) one can find a strongly measurable function \( f : [0, T] \to X \) such that \( f(t) \in F(t, \psi) \) for a.e. \( t \in [0, T] \);
2. for a.e. \( t \in [0, \infty) \) the multimap \( F(t, \cdot) : B \to Kv(X) \) is u.s.c on \( B \);
3. there exists a function \( m \in L^1([0, \infty)) \) such that for every \( \psi \in B \) we have
   \[
   \|F(t, \psi)\| := \sup\{\|\xi\| : \xi \in F(t, \psi)\} \leq m(t)\|\psi\|_B,
   \]
   for a.e. \( t \in [0, \infty) \);
4. there exists a function \( k \in L^\infty([0, \infty)) \) such that, for every bounded set \( D \subset B \) we have \( \chi(F(t, D)) \leq k(t) \sup_{s \leq 0} \chi(D(s)) \) for a.e. \( t \in [0, \infty) \).

As a consequence, we have the following result.

**Theorem 4.1.** Let the hypotheses \((A'), (B')\) and \((F')\) hold. If the following condition

\[
\Lambda_\infty = \sup_{t \geq 0} \int_0^t \|S_\alpha(t-s)\|K(s)m(s)ds < 1,
\]
(35)
is satisfied, then the zero solution of the problem (1)-(2) is weakly asymptotically stable.

**Proof.** Let \( \Sigma(\varphi) \) be the set of solutions to (1) with respect to the initial datum \( \varphi \). By Theorem 3.5, one obtains \( \Sigma(\varphi) \neq \emptyset \) and we can find \( z \in \Sigma(\varphi) \) such that \( \|z(t)\| = o(1) \) as \( t \to \infty \). By \((F')(3)\) we see that \( u = 0 \) is a solution of (1).
We first verify that this solution is stable, i.e. for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for \( |\varphi|_B < \delta \), we have \( |z_t|_B < \varepsilon \), whenever \( z \in \Sigma(\varphi) \). Indeed, for \( z \in \Sigma(\varphi) \) we can take \( f \in P_F(z|_{\mathbb{R}^+}) \) such that

\[
z(t) = S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s) f(s) ds, \quad t \geq 0.
\]

Then by \((\text{F'})\)(3) and (B3)-(B4) we get

\[
\|z(t)\| \leq C\|\varphi(0)\| + \int_0^t \|S_\alpha(t-s)\| m(s) |z_s|_B ds
\]

\[
\leq C \varphi_B + \int_0^t \|S_\alpha(t-s)\| m(s) K(s) \sup_{r \in [0,s]} \|z(r)\| ds
\]

\[
+ \int_0^t \|S_\alpha(t-s)\| m(s) M(s) |\varphi|_B ds
\]

\[
\leq C \varphi_B + A_\infty \sup_{s \geq 0} \|z(s)\| + \Upsilon_\infty |\varphi|_B,
\]

where

\[
\Upsilon_\infty = \sup_{t \geq 0} \int_0^t \|S_\alpha(t-s)\| M(s) m(s) ds < \infty.
\]

Estimate (36) implies

\[
\sup_{t \geq 0} \|z(t)\| \leq \frac{C \varphi_B + \Upsilon_\infty}{1 - A_\infty} |\varphi|_B.
\]

So one has, for all \( t \geq 0 \)

\[
|z_t|_B \leq K(t) \sup_{r \in [0,t]} \|z(r)\| + M(t) |\varphi|_B
\]

\[
\leq K_\infty \sup_{r \geq 0} \|z(r)\| + M_\infty |\varphi|_B
\]

\[
\leq \left[ \frac{K_\infty (C \varphi_B + \Upsilon_\infty)}{1 - A_\infty} + M_\infty \right] |\varphi|_B,
\]

where \( K_\infty = \sup_{t \geq 0} K(t) \) and \( M_\infty = \sup_{t \geq 0} M(t) \). This ensures the stability of the zero solution.

It remains to show that the zero solution is weakly attractive. Taking \( z \in \Sigma(\varphi) \) such that \( ||z(t)|| = o(1) \) as \( t \to \infty \), we show that \( |z_t|_B \to 0 \) as \( t \to \infty \). Deploying (B3) again, but now with \( \sigma = \frac{t}{2} \), we have

\[
|z_t|_B \leq K\left( \frac{t}{2} \right) \sup_{r \in [\frac{t}{2},t]} \|z(r)\| + M\left( \frac{t}{2} \right) \|z_{\frac{t}{2}}\|_B
\]

\[
\leq K_\infty \sup_{r \geq \frac{t}{2}} \|z(r)\| + M\left( \frac{t}{2} \right) \left[ \frac{K_\infty (C \varphi_B + \Upsilon_\infty)}{1 - A_\infty} + M_\infty \right] |\varphi|_B,
\]

thanks to (38). Let \( \varepsilon > 0 \), then there is \( T > 0 \) such that \( \|z(r)\| < \varepsilon \) and \( M(r) < \varepsilon \) for all \( r \geq T \), thanks to \((\text{B'})\). Therefore, inequality (39) implies that, for all \( t \geq 2T \)

\[
|z_t|_B < C^* \varepsilon,
\]

where

\[
C^* = K_\infty + \left[ \frac{K_\infty (C \varphi_B + \Upsilon_\infty)}{1 - A_\infty} + M_\infty \right] |\varphi|_B.
\]

The proof is complete.
5. Application. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. We consider the following PDE

$$\frac{\partial u}{\partial t}(t,x) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Delta x u(s,x) ds + f(t,x), \quad t > 0, \quad x \in \Omega,$$

where $\alpha \in (1,2)$, the unknown function $u$ satisfies the boundary condition

$$u(t,x) = 0, \quad x \in \partial \Omega, \quad t \geq 0,$$

and the initial condition

$$u(t,x) = \varphi(t,x), \quad x \in \Omega, \quad t \leq 0;$$

the function $f$ is subject to

$$f(t,x) \in b(t,x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta,y) \left[ f_1(u(t + \theta,y)), f_2(u(t + \theta,y)) \right] d\theta d\nu,$$

here $\{f_1, f_2\} = \{\tau f_1 + (1-\tau)f_2 : \tau \in [0,1]\}$. The system (40)-(43) can be seen as a control problem with multi-valued feedback. Let $A = \Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, then the resolvent set of $A$, $\rho(A) \subset \mathbb{C}\setminus(-\infty,-\lambda_1]$, where $\lambda_1 = \sup\{\|\nabla u\|^2_{L^2(\Omega)} : \|u\|_{L^2(\Omega)} = 1\}$. So $A$ is sectorial of type $(-\lambda_1,\theta)$ with any $\theta \in (0, \frac{\pi}{2})$.

Let $X = L^2(\Omega)$. In order to get the $\alpha$-resolvent $S_\alpha(\cdot)$, consider the linear system

$$\frac{\partial u}{\partial t}(t,x) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Delta x u(s,x) ds, \quad x \in \Omega, \quad t > 0,$$

$$u(t,x) = 0, \quad x \in \partial \Omega, \quad t \geq 0,$$

$$u(0,x) = u_0(x), \quad x \in \Omega,$$

where $u_0 \in X$. Integrating (44), we get

$$u(t,x) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Delta x u(s,x) ds + u_0(x).$$

This means

$$\partial_t^\alpha u(t,x) = \Delta x u(t,x),$$

$$u(t,x) = 0, \quad x \in \partial \Omega, \quad t \geq 0,$$

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = 0,$$

here $\partial_t^\alpha$ stands for the Caputo fractional derivative of order $\alpha \in (1,2)$ with respect to $t$. Obviously, the solution of (47)-(49) is given by

$$u(t,x) = S_\alpha(t)u_0(x).$$

Referring to [27, Theorem 2.3], one sees that $S_\alpha(\cdot)$ is differentiable on $(0, \infty)$ and

$$\left\| \frac{d}{dt} S_\alpha(t)u_0 \right\| \leq C_\alpha t^{-1} \|u_0\|, \quad t > 0,$$

where $C_\alpha$ is a positive constant. In particular, $S_\alpha(\cdot)$ is norm continuous. So $(A')$ is just fulfilled.

Consider the phase space $B = CL^2_\theta$ with $g(\theta) = e^{h\theta}, h \in (0,1)$, where the semi-norm in $B$ is given by

$$|w|_B = \sup_{-\pi \leq \theta \leq 0} \|w(\theta)\|_X + \left( \int_{-\pi}^{-r} e^{h\theta}\|w(\theta)\|_X^2 d\theta \right)^{\frac{1}{2}},$$
for $r = -\frac{1}{r} \ln(1 - h)$. Thanks to the formulas (10)-(11), we have

$$K(t) = \begin{cases} 
1, & 0 \leq t \leq r, \\
1 + \frac{1}{\sqrt{r}} \sqrt{e^{-ht} - e^{-ht}}, & t > r,
\end{cases}$$

$$M(t) = \begin{cases} 
\max \left\{ e^{-\frac{1}{2}ht}, 1 + \sqrt{\frac{e^{-ht}}{r}(1 - e^{-ht})} \right\}, & 0 \leq t \leq r, \\
e^{-\frac{1}{2}ht}, & t > r.
\end{cases}$$

So one observes that $K(t) = O(1), M(t) = o(1)$ as $t \to \infty$, and then $(B')$ is satisfied.

We will show that the system (40)-(43) is weakly asymptotically stable under a suitable setting. Specifically, we assume that

(N1) $b \in L^1(\mathbb{R}^+; L^2(\Omega))$;
(N2) $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

$$|f_i(z)| \leq C_f |z|, \forall z \in \mathbb{R},$$

for some $C_f > 0$;
(N3) $\nu : (-\infty, 0] \times \Omega \to \mathbb{R}$ is a continuous function such that

$$|\nu(\theta, y)| \leq C_\nu e^{h\theta}, \forall y \in \Omega, \theta \leq 0,$$

where $C_\nu > 0$.

Let $F : \mathbb{R}^+ \times \mathcal{B} \to \mathcal{P}(X)$ be a multimap defined by

$$F(t, w)(x) = b(t, x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta, y) \left[ f_1(w(\theta, y)), f_2(w(\theta, y)) \right] dyd\theta.$$

Then we see that, for each $\tau \in [0, 1]$ the function

$$f(t, x) = b(t, x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta, y) \left[ \tau f_1(w(\theta, y)) + (1 - \tau)f_2(w(\theta, y)) \right] dyd\theta$$

is an integrable selector of $F(t, w)$. The assumption $(F')(1)$ is just satisfied.

Now for each bounded set $W \subset \mathcal{B}$, $F(t, W) \subset \text{span}\{b(t, \cdot)\}$, which is an one-dimension subspace of $L^2(\Omega)$. In addition, by (N2) $F(t, W)$ is also bounded and then relatively compact. Hence

$$\chi(F(t, W)) = 0,$$

and the assumption $(F')(4)$ is verified with $k = 0$. In particular, $F(t, w)$ is relatively compact for each $w \in \mathcal{B}$. In view of the continuity of $f_1, f_2$ and the Lebesgue dominated convergence theorem, we can testify the closedness of $F(t, \cdot)$. Thus $F(t, \cdot)$ has compact values and is u.s.c due to [21, Theorem 1.1.5]. The assumption $(F')(2)$ is satisfied.

Regarding $(F')(3)$, we have the following estimates

$$\|F(t, w)\|^2 \leq C_f^2 C_f^2 \|b(t, \cdot)\|^2 \left( \int_{-\infty}^0 e^{h\theta} \int_{\Omega} |w(\theta, y)| dy d\theta \right)^2$$

$$\leq \frac{1}{h} C_f^2 C_f^2 \|b(t, \cdot)\|^2 \int_{-\infty}^0 e^{h\theta} \left( \int_{\Omega} |w(\theta, y)| dy \right)^2 d\theta$$

$$\leq \frac{1}{h} C_f^2 C_f^2 \mu(\Omega) \|b(t, \cdot)\|^2 \int_{-\infty}^0 e^{h\theta} \|w(\theta, \cdot)\|^2 d\theta$$
\[
\frac{1}{h} C^2_\nu C^2_f \mu(\Omega) \left\{ \left( \int_{-r}^{0} + \int_{-\infty}^{-r} \right) e^{h\theta} \| w(\theta, \cdot) \|^2 d\theta \right\} \\
\leq \frac{1}{h} C^2_\nu C^2_f \mu(\Omega) \left\{ \left\| w \right\|^2_{C([-r,0];X)} + \int_{-\infty}^{-r} e^{h\theta} \| w(\theta, \cdot) \|^2 d\theta \right\} \\
\leq \frac{1}{h} C^2_\nu C^2_f \mu(\Omega) \left\| b(t, \cdot) \right\|^2 |w|^2_B.
\]

thanks to the Hölder inequality and the fact that \( r = -\frac{1}{h} \ln(1-h) \), here \( \mu(\Omega) \) is the Lebesgue measure of \( \Omega \). Then

\[
\| F(t, w) \| \leq \frac{1}{\sqrt{h}} C_\nu C_f \sqrt{\mu(\Omega)} \| b(t, \cdot) \| |w|_B.
\]

Thus \((F')(3)\) is testified with \( m(t) = \frac{1}{\sqrt{h}} C_\nu C_f \sqrt{\mu(\Omega)} \| b(t, \cdot) \| \).

Finally, if we assume, in addition, that \( b(t, x) = b_1(t) b_2(x) \) with \( b_1 \in L^1(\mathbb{R}^+) \), \( b_2 \in L^2(\Omega) \) then

\[
\| b(t, \cdot) \| = \| b_1(t) \|_2 \| b_2 \|,
\]

and we can verify condition (35) as follows

\[
A_\infty \leq \sup_{t \geq 0} \int_0^t C_0 K_\infty \| b_2 \| \| b_1(s) \|_1 ds \leq C_0 K_\infty \| b_2 \| \| b_1 \|_{L^1(\mathbb{R}^+)},
\]

where \( C_0 = \frac{1}{\sqrt{h}} C C_\nu C_f \sqrt{\mu(\Omega)} \), thanks to (3). Therefore (35)-(36) hold with \( \| b_2 \| \), \( \| b_1 \|_{L^1(\mathbb{R}^+)} \) small.

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