Conditional Lower Bound for Inclusion-Based Points-to Analysis

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Abstract

Inclusion-based (i.e., Andersen-style) points-to analysis is a fundamental static analysis problem. The seminal work of Andersen gave a worst-case cubic $O(n^3)$ time points-to analysis algorithm for C, where $n$ is proportional to the number of variables in the input program. From an algorithmic perspective, an algorithm is truly subcubic if it runs in $O(n^{3-\delta})$ time for some $\delta > 0$. Despite decades of extensive effort on improving points-to analysis, the cubic bound remains unbeaten. The best combinatorial analysis algorithms have a “slightly subcubic” $O(n^3/\log n)$ complexity which improves Andersen’s original algorithm by only a log factor. It is an interesting open problem whether inclusion-based points-to analysis can be solved in truly subcubic time.

In this paper, we prove that a truly subcubic $O(n^{3-\delta})$ time combinatorial algorithm for inclusion-based points-to analysis is unlikely: a truly subcubic combinatorial points-to analysis algorithm implies a truly subcubic combinatorial algorithm for Boolean Matrix Multiplication (BMM). BMM is a well-studied problem, and no truly subcubic combinatorial BMM algorithm has been known. The fastest combinatorial BMM algorithms run in time $O(n^3/\log^4 n)$.

Our conditional lower bound result includes a simplified proof of the BMM-hardness of Dyck-reachability. The reduction is interesting in its own right. First, it is slightly stronger than the existing BMM-hardness results of Dyck-reachability because our reduction only requires one type of parenthesis in Dyck-reachability ($D_1$-reachability). Second, we formally attribute the “cubic bottleneck” of points-to analysis to the need to solve $D_1$-reachability, which captures the semantics of properly-balanced pointer references/dereferences. This new perspective enables a more general reduction that applies to programs with arbitrary types of pointer statements. Last, our reduction based on $D_1$-reachability shows that demand-driven points-to analysis is as hard as the exhaustive counterpart. The hardness result generalizes to a wide variety of demand-driven interprocedural program analysis problems.

1 Introduction

Points-to analysis is a fundamental static analysis problem. Points-to information is a prerequisite for many practical program analyses. Points-to analysis is computationally hard. It is well-known that computing the precise points-to information is undecidable [29]. Even for the simplest (i.e., context- and flow-insensitive) variant, the precise analysis problem is known to be NP-hard [22]. Any practical points-to analysis must approximate the exact solution. In the literature, two predominant frameworks for computing sound points-to information are equality-based (i.e., Steensgaard-style) [38] and inclusion-based (i.e., Andersen-style) points-to analyses [4].

Inclusion-based points-to analysis [5] is more precise than equality-based analysis. A study by Blackshear et al. [9] shows that “The precision gap between Andersen’s and precise flow-insensitive analysis is non-existent in practice.” An inclusion-based points-to analysis collects a set of inclusion constraints from the input program and constructs a constraint graph. It obtains the points-to information by computing a fixed point solution over the corresponding constraint graph [15, 18, 36]. All existing points-to analysis algorithms are combinatorial in the sense that they are discrete and graph-theoretic. However, computing the closure of the constraint graph is quite expensive. Despite decades of research, the fastest algorithm for inclusion-based points-to analysis exhibits an “slightly subcubic” $O(n^3/\log n)$ time complexity [14, 36]. In practice, many studies have observed a quadratic scaling behavior for inclusion-based pointer analysis [36]. It is open whether inclusion-based points-to
analysis problem admits truly subcubic algorithms, i.e., algorithms with running time $O(n^{3-\delta})$ for some constant $\delta > 0$.

In this paper, we prove a conditional lower bound for the inclusion-based points-to analysis problem, which shows that a truly subcubic combinatorial algorithm is unlikely to exist. Our hardness result is based on the popular Boolean Matrix Multiplication (BMM) conjecture:

**Conjecture 1** (Boolean Matrix Multiplication $[1, 42]$). For all $\delta > 0$, there exists no combinatorial algorithm that computes the product of two $n \times n$ Boolean matrices in time $O(n^{3-\delta})$.

The conjecture states that in the RAM model with $O(\log n)$ bit words, any combinatorial BMM algorithm requires $n^{3-o(1)}$ time $[41]$. The BMM conjecture has been utilized to prove fine-grained lower bounds of many problems in theoretical computer science $[1, 21, 42]$ and program analysis/verification $[11, 12, 23]$. Note that BMM can be solved in truly subcubic time using the heavy machinery of fast matrix multiplication (FMM) originated by Strassen $[39]$. The current fastest algorithms run in $O(n^{2.373})$ time $[17, 40]$. However, those algorithms based on FMM are algebraic which rely on the ring structure of matrices over the field. Algebraic approaches have enjoyed little success in practice $[2, 41]$.

In particular, we are unaware of any practical program analyses that are based on FMM. On the other hand, inclusion-based points-to analysis has been formulated as a combinatorial problem of resolving inclusion constraints (via a sequence of set-union and table-lookup operations). All existing pointer analysis algorithms are combinatorial in nature. Moreover, combinatorial algorithms are practical and avoid FMM or other “Strassen-style” methods. Thus, lower bounds for combinatorial algorithms are of particular interest from both theoretical and practical perspectives.

Our proof of the BMM-hardness of inclusion-based points-to analysis involves two steps. In the first step, we reduce BMM to Dyck-Reachability with one type of parentheses ($D_1$-reachability). Dyck-reachability is a graph reachability problem where the edges in the input graph are labeled with $k$ types of open and close parentheses $[12, 32, 46]$. The goal is to compute all reachable nodes that can be joined by paths with properly-matched parentheses. Dyck-reachability has been utilized to express many static analysis problems, such as interprocedural data flow analysis $[34]$, program slicing $[33]$, shape analysis $[31]$, and type-based flow analysis $[28, 30]$. It has also been widely used in practical analysis tools such as Soot $[10]$. In the second step of our proof, we reduce $D_1$-reachability to inclusion-based points-to analysis. In particular, we introduce a novel Pointer Expression Graph (PEG) representation for C-style programs and formulate the points-to analysis as a $Pt$-reachability problem on PEGs. The key insight in our reduction is to leverage the pointer reference and dereference in $Pt$-reachability formulation to express the properly-matched parentheses in $D_1$-reachability. Our two-step reduction yields two conditional lower bounds on both $D_1$-reachability and inclusion-based points-to analysis:

**Theorem 1** (BMM-hardness of Dyck-reachability). For any fixed $\delta > 0$, if there is a combinatorial algorithm that solves $D_1$-reachability in $O(n^{3-\delta})$ time, then there is a combinatorial algorithm that solves Boolean Matrix Multiplication in truly subcubic time.

**Theorem 2** (BMM-hardness of inclusion-based points-to analysis). For any fixed $\delta > 0$, if there is a combinatorial algorithm that solves inclusion-based points-to analysis in $O(n^{3-\delta})$ time, then there is a combinatorial algorithm that solves Boolean Matrix Multiplication in truly subcubic time.

Our key insight on $D_1$-reachability-based reduction yields several interesting implications. We formally summarize the results as two corollaries. In particular, Corollary 1 considers any non-trivial C-style programs with pointers and Corollary 2 applies to programs with pointer dereferences.

**Corollary 1** (Universality). Inclusion-based points-to analysis is BMM-hard for non-trivial programs under arbitrary combinations of statement types “$a = b$”, “$a = \&b$”, “$a = b$”, and “$a = \ast b$.”

1In many combinatorial structures such as Boolean semiring, there is no inverse under addition. In practice, algebraic methods have large hidden constants and generally considered impractical. See more detailed discussions in the work of Ballard et al. $[3]$, Bansal and Williams $[8]$ and Henzinger et al. $[21]$. 

2Non-trivial C-style programs always contain address-of statements “$a = \&b$” and at least one of the three types of statements “$a = b$”, “$a = b$”, and “$a = \ast b$”. The address-of statements “initialize” the points-to sets of all variables. Without such statements, all points-to sets in the program are empty sets. For programs with only address-of statements, all points-to sets can be obtained via a simple linear-time scan. Both types of programs are trivial for pointer analysis.
Corollary 2 (BMM-hardness of demand-driven analysis). In the presence of pointer dereferences, the demand-driven inclusion-based points-to analysis is BMM-hard.

The two corollaries offer a comprehensive view on the cubic bottleneck of inclusion-based points-to analysis. Prior to our work, it is folklore that the complexity of inclusion-based points-to analysis is related to computing the “dynamic transitive closure” (DTR). Unfortunately, the “dynamic” aspect of DTR, which informally refers to the process of adding inclusion-constraint edge to the graph during constraint resolution, has not been rigorously defined. The work by Sridharan and Fink [36] establishes a relation between points-to analysis and transitive closure. However, the reduction is restrictive because it applies to programs with no pointer dereferences (i.e., programs consist of only two types of statements “\(a = b\)” and “\(a = &b\)”). Our universality corollary (Corollary 1), which generalizes to all types of program statements, is more principled. Specifically, the \(D_1\)-reachability used in our reduction formally captures the properly-matched pointer references/dereferences. Moreover, this insight also enables Corollary 2 which shows that demand-driven pointer analysis is as hard as the exhaustive counterpart. Note the traditional transitive-closure-based reduction can not yield Corollary 2 as the single-source-single-target graph reachability can be trivially solved in \(O(n)\) time via a depth-first search.

To sum up, the significance of our results is threefold.

- Theorem 1 gives a simplified proof to establish the BMM-hardness of Dyck-reachability. It is slightly stronger than the existing hardness results because previous proofs [12, 19] require Dyck languages of \(k \geq 2\) types of parentheses.

- Theorem 2 formally establishes a cubic lower bound for inclusion-based points-to analysis conditioned on the popular BMM conjecture.

- The proof of Theorem 2 is based on the reduction from \(D_1\)-reachability. The key insight of depicting the well-balanced pointer references/dereferences with \(D_1\)-reachability yields two interesting corollaries:
  - Corollary 1 permits all types of constraints in inclusion-based points-to analysis and makes no assumption of the input programs. Previous transitive-closure-based reduction [36] is restricted to programs with no pointer dereferences, which cannot be generalized to programs with pointer dereferences.
  - Corollary 2 generalizes to interprocedural program analysis. It demonstrates that the bottleneck of demand-driven interprocedural analysis is due to matching the well-balanced properties such as procedure calls/returns and pointer references/dereferences, as opposed to computing the transitive closure.

The rest of the paper is organized as follows. Section 2 introduces problem definitions. Section 3 presents an overview of our reductions. Sections 4 and 5 prove the reduction correctness. Section 6 discusses the implications. Section 7 surveys related work and Section 8 concludes.

## 2 Preliminaries

This section formally defines inclusion-based points-to analysis (Section 2.1) and Dyck-reachability (Section 2.2) involved in our conditional hardness result.

### 2.1 Inclusion-Based Points-to Analysis

Our work focuses on flow-insensitive inclusion-based (i.e., Andersen-style) points-to analysis [18, 36]. The control flow between assignment statements in \(P\) is irrelevant. Given a program \(P\), a points-to analysis determines the set of variables that a pointer variable might point to during program execution.

We consider a simple C-style language that contains the assignment statements of the form “\(e_1 = e_2\)”, where the expressions \(e_1\) and \(e_2\) are defined by the following context-free grammar:

\[
\begin{align*}
  e_1 & \rightarrow \text{ID} \mid *e_1 \\
  e_2 & \rightarrow \&\text{ID} \mid e_1.
\end{align*}
\]
The assignment statements permit arbitrary pointer dereferences (e.g., \(x = y\)). Typically, all assignment statements in the input program \(P\) are normalized. The normalization procedure replaces the statements that involve multiple levels of dereferencing by a sequence of statements that contain only one level of dereferencing \(p = d\). After the normalization, each statement has one of the four forms: “\(ID = \&ID\)”, “\(ID = ID\)”, “\(ID = \*ID\)” or “\(\*ID = ID\)”.

Inclusion-based points-to analysis \(PA(P, V)\) is a set-constraint problem \([13, 18]\). It generates four types of inclusion constrains w.r.t. the normalized statements in program \(P\). For each pointer variable \(v \in V\), the goal of points-to analysis is to compute a points-to set \(pt(v)\) which contains all variables that \(v\) may point to during execution. Table 1 gives the four constraints as well as their corresponding statements and meanings.

**Definition 1** (Inclusion-based points-to analysis). Given a normalized program \(P\) and a collection of input constraints based on Table 1, the inclusion-based points-to analysis problem is to solve the inclusion constraints and to determine if \(p \in P\) can point to \(q \in P\), i.e., determine if \(loc(q) \in pt(p)\).

Algorithm 1 gives an algorithm for inclusion-based points-to analysis based on computing dynamic transitive closure \([18]\).

**Example 1.** Consider the following simple C-style program: \(a = \&b\); \(b = \&d\); \(c = \*a\). We illustrate the Andersen-style pointer analysis by computing the \(pt\) set for each variable. According to the semantics introduced in Table 1 these three statements represent \(loc(b) \in pt(a)\), \(loc(d) \in pt(b)\) and \(\forall v \in pt(a) : pt(v) \subseteq pt(c)\). After the fixed-point computation, we get the analysis result \(pt(a) = \{loc(b)\}\), \(pt(b) = \{loc(d)\}\) and \(pt(c) = \{loc(d)\}\).
2.2 Dyck-Reachability

Dyck-Reachability is a subclass of context-free language (CFL) reachability [12, 32, 46]. A CFL-reachability problem instance contains a context-free grammar \(CFG = (\Sigma, N, P, S)\) and an edge labeled digraph \(G\). Each edge \(u \xrightarrow{l} v \in G\) is labeled by a symbol \(l \in L(u, v) \subseteq \Sigma \cup N\). Each path \(p = v_0, v_1, v_2, \ldots, v_m \in G\) realizes a string \(R(p)\) over the alphabet \(\Sigma\) by concatenating the edge labels in the path in order, i.e., \(R(p) = L(v_0, v_1)L(v_1, v_2) \ldots L(v_{m-1}, v_m)\). A path \(p = v_0, \ldots, v_m \in G\) is an \(l\)-path if its realized string \(R(p)\) is either a terminal \(l \in \Sigma\) or it can be derived from a nonterminal \(l \in N\). We represent an \(l\)-path from node \(u\) to \(v\) as a summary edge \(u \xrightarrow{l} v \in G\). Moreover, we say node \(v\) is \(l\)-reachable from node \(u\) if there exists a summary edge \(u \xrightarrow{l} v\).

Definition 2 (L-reachability). Given a grammar \(CFG\) of a context-free language \(L\) and an edge labeled digraph \(G\), the \(L\)-reachability problem is to compute all \(S\)-paths in \(G\), where \(S\) is the start symbol in the \(CFG\).

Dyck language \(D_1\) is a context-free language that generates the strings of one kind of properly matched parentheses. Formally, \(D_1\) is specified by a \(CFG = (\Sigma, N, P, S)\) where \(\Sigma = \{[\cdot], \cdot\}\), \(N = \{D_1, S\}\), and \(S = \{D_1\}\). The production rules \(P\) are \(\{D_1 \rightarrow S, S \rightarrow [S] | SS | \varepsilon\}\).

Definition 3 (\(D_1\)-reachability). Given a grammar of Dyck language \(D_1\) and an edge labeled digraph \(G\), the \(D_1\)-reachability problem is to compute all \(D_1\)-paths in \(G\), where \(D_1\) is the start symbol in the grammar.

3 BMM-Hardness of \(D_1\)-Reachability and Points-to Analysis

We give two reductions to establish the conditional lower bounds of \(D_1\)-reachability and inclusion-based points-to analysis, respectively. In particular, the first reduction reduces Boolean Matrix Multiplication (BMM) to \(D_1\)-reachability and the second reduction reduces \(D_1\)-reachability to points-to analysis. Our hardness results are conditioned on a widely believed BMM conjecture about the complexity of multiplying two Boolean matrices as described in Conjecture \([1]\). Note that the cubic lower bound of \(D_k\)-reachability is indeed known in the literature. For instance, the work by Heintze and McAllester \([19]\) has proven a cubic lower bound for a \(D_2\)-reachability data-flow analysis problem conditioned on the hardness for solving \(2\text{NPDA}\) problems. A recent result of Chatterjee et al. \([12]\) proves the BMM-hardness of Dyck-reachability by giving a reduction from context-free language (CFL) parsing which requires multiple kinds of parentheses. Our hardness result on \(D_1\)-reachability is slightly stronger since a lower bound on \(D_1\)-reachability implies a lower bound on \(D_k\)-reachability, for any \(k \geq 2\).

This section gives a high-level overview of our main results. In particular, Section 3.1 gives a running example. Sections 3.2 and 3.3 give the detailed description of the two reductions, respectively.

3.1 Overview

Figure \([1]\) gives a concrete example to illustrate our reduction. We briefly describe the three problem instances in Figure \([1]\) as a gentle introduction to our reduction.

- **BMM to \(D_1\)-reachability.** Figure \([1a]\) gives a BMM instance with two \(4 \times 4\) matrices \(A\) and \(B\). Their product is given in matrix \(C\). Figure \([1b]\) depicts the transformed digraph \(G\) as a \(D_1\)-reachability instance. Our reduction maps each non-zero \(a_{ij} \in A\) to \(x_i \xrightarrow{\varepsilon} y_j \in G\) and each non-zero \(b_{ij} \in B\) to \(y_i \xrightarrow{\varepsilon} z_j \in G\). Our reduction guarantees that non-zero \(c_{ij} \in C \iff x_i \xrightarrow{D_1} z_j \in G\) (Theorem \([5]\)).

- **\(D_1\)-reachability to points-to analysis.** Our second reduction takes as input the \(D_1\)-reachability problem instance in Figure \([1b]\). Note that the second reduction is general which does not rely on the output of the first reduction. In Figure \([1]\) we reuse the output graph \(G\) in Figure \([1b]\) for brevity. Figure \([1c]\) shows a normalized C-style program \(P\) as a points-to analysis problem instance. The program \(P\) contains three sets of variables \(Var_b\), \(Var_w\), and \(Var_y\). Our reduction maps each node \(v \in G\) to a variable \(v \in Var_y\). We also map each node \(v \in G\) to a new address-taken variable \&\(v'\) \(\in Var_y\). Moreover, we construct program statements in \(P\) to make variable
(a) A BMM problem instance.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(b) A \(D_1\)-reachability problem instance.

\[
\begin{array}{c}
A \\
B \\
C
\end{array}
\begin{array}{c}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{array}
\begin{array}{c}
y_0 \\
y_1 \\
y_2 \\
y_3
\end{array}
\begin{array}{c}
z_0 \\
z_1 \\
z_2 \\
z_3
\end{array}
\]

(c) A C-style program \(P\).

Figure 1: A running example.

\[v \text{ point to variable } v', \text{ i.e., } loc(v') \in pt(v)\]. Set \(Var_w\) contains auxiliary variables \(t_i\) for program statement construction. Our reduction guarantees that \(u \xrightarrow{D_1} v \in G \iff loc(v') \in pt(u)\) in program \(P\) (Theorem 4).

3.2 Reducing BMM to \(D_1\)-Reachability

**Reduction 1: From BMM to \(D_1\)-Reachability**

**Input:** Two \(n \times n\) Boolean matrices \(A\) and \(B\);  
**Output:** An edge-labeled digraph \(G = (V, E)\), where \(|V| = 3n\) and \(|E|\) is equal to the number of non-zero entries in both \(A\) and \(B\).

**Intuition.** A Boolean matrix is a matrix with entries from the set \(\{0, 1\}\). Given two \(n \times n\) Boolean matrices \(A\) and \(B\), the BMM problem is to compute the product \(C = A \times B\), whose entries are defined by \(c_{ij} = \bigvee_{k=1}^{n}(a_{ik} \land b_{kj})\). That is, \(c_{ij} = 1\) if and only if there exists an integer \(k \in [0, n-1]\) such that \(a_{ik} = b_{kj} = 1\). Our basic reduction idea is to treat every non-zero element \(a_{ij} \in A\) as a directed edge labeled by an open parenthesis “[” in the output digraph \(G\). Similarly, we treat every non-zero element \(b_{ij} \in B\) as a “]”-labeled edge in \(G\). Our reduction guarantees that every non-zero element \(c_{ij} \in C\) corresponds to a pair of nodes joined by a balanced-parenthesis path (i.e., \(D_1\)-path) in graph \(G\).

**Reduction.** Algorithm 2 gives the reduction procedure. Given two \(n \times n\) Boolean matrices \(A\) and \(B\), we introduce \(3n\) nodes in our graph (line 1). We then insert edges based on the input matrices (lines 2-7). Let \(m\) be the number of non-zero entries in the input matrices. Algorithm 2 outputs a digraph with \(m\) edges.
Proof. We show that each non-zero element $v \in \text{Var}$ introduce a graph $G$ generates a graph either “1” or “1”.

Our reduction takes as input a generic digraph $D$.

Intuition. Our reduction takes as input a program $P$ with null pointer values.

Let $C = A \times B$. Element $c_{ij} \in C$ is non-zero iff $z_j \in V_z$ is $D_1$-reachable from $x_i \in V_x$ in $G$.

Proof. We show that each non-zero element $c_{ij}$ corresponds to a $D_1$-reachable pair in the constructed graph $G$.

The “⇒” direction. Based on the BMM definition, if $c_{ij} = 1$ there exists at least one $k$ such that $a_{ik} = b_{kj} = 1$. Algorithm 2 inserts $x_i \xrightarrow{k} y_k$ (line 4) and $y_k \xrightarrow{j} z_j$ (line 7) to $G$. We have a path string $L(x_i, y_k) L(y_k, z_j) = [1]_1 \in D_1$. Therefore, node $z_j$ is $D_1$-reachable from $x_i$.

The “⇐” direction. Our constructed graph $G$ is a 3-layered graph, i.e., graph $G$ contains three node sets $V_x, V_y, V_z$. All “1”-labeled edges go from nodes in $V_x$ to nodes in $V_y$. All “1”-labeled edges go from nodes in $V_y$ to nodes in $V_z$. Therefore, the length of every $D_1$-path in $G$ is 2. Specifically, every $D_1$-path begins with a node $x_i \in V_x$, passes through a node $y_k \in V_y$ and ends at a node $z_j \in V_z$. Based on Algorithm 2 such a path corresponds to $a_{ik} = b_{kj} = 1$. Therefore, we have a non-zero element $c_{ij} \in C$.

\[\Box\]

3.3 Reducing $D_1$-Reachability to Inclusion-Based Points-to Analysis

\textbf{Algorithm 2: Reduction from Boolean matrix multiplication to $D_1$-reachability.}

\begin{tabular}{ll}
\textbf{Input:} & Two $n \times n$ Boolean matrices $A$ and $B$;
\textbf{Output:} & An edge-labeled graph $G = (V, E)$;\
1 & Introduce nodes $x_i, y_j, z_k$ to $G$ where $i \in [0, n-1]$
2 & \textbf{foreach} element $a_{ij} \in A$ do
3 & \hspace{1em} if $a_{ij} = 1$ then
4 & \hspace{2em} Insert edge $x_i \xrightarrow{i} y_j$ to $G$
5 & \textbf{foreach} element $b_{ij} \in B$ do
6 & \hspace{1em} if $b_{ij} = 1$ then
7 & \hspace{2em} Insert edge $y_j \xrightarrow{j} z_k$ to $G$
\end{tabular}

\textbf{Correctness.} Let $V_x = \{x_0, \ldots, x_{n-1}\}$, $V_y = \{y_0, \ldots, y_{n-1}\}$, and $V_z = \{z_0, \ldots, z_{n-1}\}$. It is clear that Algorithm 2 is a linear-time reduction. Specifically, given two $n \times n$ matrices, the reduction generates a graph $G = (V, E)$ with $3n$ nodes where $V = V_x \cup V_y \cup V_z$. To show $D_1$-reachability is BMM-hard, it suffices to prove the following theorem on reduction correctness.

\textbf{Theorem 3 (Reduction Correctness).} Algorithm 2 is a linear-time reduction which takes as input two Boolean matrices $A$ and $B$ and outputs a digraph $G = (V, E)$ where $V = V_x \cup V_y \cup V_z$. Let $C = A \times B$. Element $c_{ij} \in C$ is non-zero iff $z_j \in V_z$ is $D_1$-reachable from $x_i \in V_x$ in $G$.

\textbf{Reduction 2: From $D_1$-Reachability to Inclusion-Based Points-to Analysis}

\begin{tabular}{ll}
\textbf{Input:} & An edge-labeled digraph $G = (V, E)$;
\textbf{Output:} & A normalized C-style program with $4|V| + 2|E|$ statements.
\end{tabular}

\textbf{Intuition.} Our reduction takes as input a generic digraph $G = (V, E)$ where each edge is labeled by either “1” or “1”. Note that in $D_1$-reachability, every node is $D_1$-reachable from itself by an empty path since the $D_1$ nonterminal is nullable. However, according to Definition 1 the points-to relation is not reflexive and we cannot generally assume that variable $p$ points to its own location $kp$, i.e., $loc(p) \notin pt(p)$ without any proper ADDRESS-OF statement in Table 1. To handle the reflexivity, we introduce a pointer variable set $Var_p$ and a pointer address set $Var_g$ in program $P$. For each node $v \in V$, we construct a pointer variable $v \in Var_p$ and $v' \in Var_g$ in program $P$. In particular, we use $loc(v')$ to “replace” $loc(v)$ in our reduction and insert additional program statements to make $loc(v') \in pt(v)$ in program $P$. We also introduce auxiliary nodes $Var_w$ in $P$ to facilitate edge construction for $G$. Our key insight is that the statements involving a dereferencing $*a$ and a referencing $a$ exhibit a balanced-parentheses property. As a result, our reduction constructs a C-style program $P$ with those
Algorithm 3: Reduction from $D_1$-reachability to inclusion-based points-to-analysis.

```
Input: An edge-labeled digraph $G = (V, E)$;
Output: A C-style program $P$.
1 Introduce $|E| + 3|V|$ temporary variables $t_i \in Var_w$ to program $P$.
2 $i \leftarrow 1$
3 foreach node $v \in V$ do
   4 // construct $Var_b$ and $Var_g$.
   5 Introduce variables $v \in Var_b$ and $v' \in Var_g$ to program $P$
   6 // make $v$ point to $v'$, i.e., $loc(v') \in pt(v)$.
   7 Insert a statement $"v = *t_i;"$ to $P$
   8 Insert a statement $"t_i = t_{i+1};"$ to $P$
   9 Insert a statement $"t_{i+1} = &t_{i+2};"$ to $P$
 10 Insert a statement $"t_{i+2} = &v;"$ to $P$
 11 $i \leftarrow i + 3$
4 foreach edge $(u, v) \in E$ do
   12 if edge label $L(u, v)$ is $[1]$ then
      13 // encode open parentheses.
      14 Insert a statement $"t_i = &u;"$ to $P$
      15 Insert a statement $"*v = t_i;"$ to $P$
      16 $i \leftarrow i + 1$
   17 else
      18 // encode close parentheses.
      19 Insert a statement $"u = &t_i;"$ to $P$
      20 Insert a statement $"*t_i = y;"$ to $P$
      21 $i \leftarrow i + 1$
```

statements to express open- and close-parenthesis edges in $G$. Finally, the reduction guarantees that each $D_1$-reachable pair $u \xrightarrow{D_1} v$ in $G$ corresponds to the fact that $u \in Var_b$ points to $v' \in Var_g$ in program $P$, i.e., $loc(v') \in pt(u)$.

**Reduction.** Algorithm 3 gives the reduction from $D_1$-reachability to inclusion-based points-to-analysis. It inserts statements to program $P$ based on the nodes (lines 3-11) and edges (lines 12-22) in the input graph $G$. Line 3 maps every node $v \in V$ to a unique element in $Var_b$ and $Var_g$, respectively. Therefore, the mapping between any two of sets $V$, $Var_b$ and $Var_g$ is a bijection.

**Correctness.** Algorithm 3 is a linear-time reduction. Specifically, given a graph $G = (V, E)$ with $|V|$ nodes and $|E|$ edges, our reduction outputs a normalized C-style program with $5|V| + |E|$ variables and $4|V| + 2|E|$ statements. Algorithm 3 maps every node $v \in V$ in the input graph to a unique variable $v \in Var_b$ and a unique variable $v' \in Var_g$ in the output program. Based on Theorem 4, to show that points-to analysis is BMM-hard, it suffices to prove the following theorem on reduction correctness.

**Theorem 4 (Reduction Correctness).** Algorithm 3 takes as input a digraph $G = (V, E)$ and outputs a C-style program $P$ with $O(E)$ variables and $O(E)$ statements. All nodes $v \in V$ are represented as variables $v$ and $v'$ in $P$. Node $v$ is $D_1$-reachable from node $u$ in $G$ iff $loc(v') \in pt(u)$ (i.e., variable $u$ points to variable $v'$) in program $P$ based on solving the inclusion constraints given in Table 7.

We discuss reduction correctness in Sections 4 and 5.

## 4 Inclusion-Based Points-to Analysis via Pt-Reachability

This section describes a graphical representation of any normalized C-style program $P$. We denote the representation as a **pointer expression graph** (PEG) $G_P$. Each PEG node corresponds to a pointer expression $e \in \{a, a, \ast a\}$. Each PEG edge $(u, v)$ is labeled by a letter $L(u, v) \in \{r, r, s, \ast s, as, sa, \ast sa, d, \ast d\}$. Specifically, PEGs contain two kinds of edges defined as follows.

- **Program Edge:** For each ADDRESS-OF statement "a = &b" in $P$, we insert an edge $a \xrightarrow{r} &b$ in $G_P$. For each ASSIGNMENT statement "a = b" in $P$, we insert an edge $a \xrightarrow{s} b$ in $G_P$. For each ASSIGN-STAR statement "a = *b" in $P$, we insert an edge $a \xrightarrow{\ast s} *b$ in $G_P$. For each STAR-ASSIGN statement "*a = b" in $P$, we insert an edge $*a \xrightarrow{sa} b$ in $G_P$. 

Table 2: Constraints for points-to analysis via graph reachability.

| Type        | Statement | Constraint          | Meaning          |
|-------------|-----------|---------------------|------------------|
| ADDRESS-OF  | a = &b    | Pt \rightarrow S r \mid r | Pt \rightarrow S |
| ASSIGNMENT  | a = b     | S \rightarrow S S   | S \rightarrow s  |
| ASSIGN-STAR | a = *b    | S \rightarrow asdPt d| S \rightarrow dPt |
| STAR-ASSIGN | *a = b    | S \rightarrow dPt dsa| S \rightarrow dPt |

- **Dereference Edge**: For each variable $a \in P$, we insert two edges $\&a \rightarrow a$ and $a \rightarrow \&a$ in $G_P$.

PEGs are bidirected. Let $t$ be an edge label. For each edge $u \rightarrow v$, there always exists a reverse edge $v \rightarrow u$ in $G_P$. Similarly, for each edge $u \rightarrow v$, there always exists a reverse edge $v \rightarrow u$. Based on Table 1, there are four types of program statement in a normalized program $P$. In PEG $G_P$, each program edge corresponds to a statement in $P$. The dereference edges respect the pointer semantics described in Section 2.1. Thus, it is straightforward to see that the mapping between $G_P$ and $P$ is bijective.

**Lemma 1.** The pointer expression graph $G_P$ is equivalent to program $P$.

Next, we describe the PA constraint resolution rules given in Table 1 using CFL-reachability. We define a context-free grammar $Pt = (\Sigma_g, N_g, P_g, S_g)$ for points-to analysis. Specifically, the alphabet $\Sigma_g = \{d, d, r, f, s, as, as, sa, sa, sv\}$ contains the edge labels in PGE $G_P$. The grammar contains three nonterminals with a start symbol $Pt$, i.e., $N_g = \{S, S, Pt, P\}$ and $S_g = \{Pt\}$.

Table 2 describes four types of constraints. In particular, the “Meaning” column in Table 2 shows that the four constraints can be expressed in terms of the points-to and subset constraints. Let $Var$ and $Addr$ be the sets of pointer variables and variable addresses in program $P$, respectively. The points-to constraint defines a binary relation $Pt \subseteq Var \times Addr$ and the subset constraint defines a binary relation $S \subseteq Var \times Var$. For instance, $(p, q) \in Pt$ corresponds to the points-to constraint $loc(q) \in pt(p)$ in Table 1, i.e., $p$ points to $q$. Similarly, $(p, q) \in S$ corresponds to the subset constraint $pt(q) \subseteq pt(p)$ in Table 1, i.e., $q$’s points-to set is a subset of $p$’s.

In our $Pt$-reachability formulation, we use two nonterminals $Pt$ and $S$ to express the two relations $Pt \subseteq$ and $S \subseteq$, respectively. That is, $(p, q) \in Pt$ if $p \rightarrow Pt$ and $q \in PEGs$ and $(p, q) \in S$ if $p \rightarrow S$ and $q \in PEGs$. Nonterminal $S$ and $Pt$ denote the inverses of $S$ and $Pt$, respectively. Next, we discuss the semantics of the four types of constrains in Table 2 and express them using a CFG $Pt$.

- **Assignment**: The assignment statement $a = b$ in program $P$ is represented as an edge $a \rightarrow b$ in the PEG. The Assignment constraint means the $b$’s points-to set is a subset of $a$’s. The resolution in Table 1 adds an inclusion constraint between $pt(b)$ and $pt(a)$, i.e., $(a, b) \in S \subseteq$. Therefore, we have a summary edge $a \rightarrow b$ in PEG. Recall that the new summary is generated based on edge $a \rightarrow b$, we encode it using a rule $S \rightarrow s$. Since inclusion constraints are always transitive and reflexive, we describe this using a rule $S \rightarrow S S \mid \epsilon$.

- **Address-of**: The address-of statement $a = &b$ in program $P$ is represented as an edge $a \rightarrow \&b$ in the PEG. The Address-of constraint means variable $b$ belongs to $a$’s points-to set. The resolution in Table 1 assigns $b$’s location $loc(b)$ to $a$’s points-to set $pt(a)$, i.e., $(a, \&b) \in Pt$. Therefore, we have a summary edge $a \rightarrow Pt \rightarrow \&b$ based on the PEG edge $a \rightarrow \&b$. Note that
&b should also belong to all supersets of pt(a), i.e., (c, &kb) ∈ PtE for all c such that (c, a) ∈ \text{SUBSET}_E. In PEG, we insert a summary edge c \xrightarrow{Pt} &b based on c \xrightarrow{S} a and a \xrightarrow{\delta} &kb. Since nonterminal S is nullable, we can combine these cases by describing them using a rule Pt \xrightarrow{S} r.

- **Assign-star**: The assign-star statement a = *b in program P is represented as an edge a \xrightarrow{as} *b in the PEG. The Assign-star constraint means that, for all variables v that point to (i.e., for all v such that (b, &v) ∈ PtE), v’s points-to set should be a subset of a’s (i.e., (a, v) ∈ \text{SUBSET}_E).

  The relation (b, &v) ∈ PtE is expressed as a summary edge b \xrightarrow{Pt} &v. The newly generated relation (a, v) ∈ \text{SUBSET}_E can be described as a summary edge a \xrightarrow{S} v in the PEG. To sum up, we insert a summary edge a \xrightarrow{S} v based on a \xrightarrow{as} *b and b \xrightarrow{Pt} &v. In the PEG, there are two deference edges *b \xrightarrow{\delta} b and &v \xrightarrow{d} v that bridge the gaps. Note that the deference edges always exist among pointer expressions w.r.t. the pointer semantics. Therefore, in the PEG, we generate a new summary edge a \xrightarrow{S} v based on four summary edges a \xrightarrow{as} *b \xrightarrow{\delta} b \xrightarrow{Pt} &v \xrightarrow{d} v.

  Finally, we describe it using a rule S \xrightarrow{\delta} Pt d.

- **Star-assign**: The star-assign statement *a = b in program P is represented as an edge *a \xrightarrow{sa} b in the PEG. The Star-assign constraint means that, for all variables v that a points to (i.e., for all v such that (a, &v) ∈ PtE), b’s points-to set should be a subset of v’s (i.e., (v, b) ∈ \text{SUBSET}_E).

  The relation (a, &v) ∈ PtE is expressed as a summary edge a \xrightarrow{Pt} &v. The newly generated relation (v, b) ∈ \text{SUBSET}_E can be described as a summary edge v \xrightarrow{S} b in the PEG. To sum up, we insert a summary edge v \xrightarrow{S} b based on a reverse edge &v \xrightarrow{Pt} a and *a \xrightarrow{sa} b. Like the Assign-star case, there are two deference edges v \xrightarrow{\delta} &v and a \xrightarrow{d} *a in the PEG. Therefore, we generate a new summary edge v \xrightarrow{S} b based on four summary edges v \xrightarrow{\delta} &v \xrightarrow{Pt} a \xrightarrow{d} *a \xrightarrow{sa} b. We describe it using a rule S \xrightarrow{\delta} Pt d sa.

The context-free grammar described in Table 2 fully captures the constraint resolution in Table 1. Specifically, each terminal summary edge depicts either a program statement in P or a pointer dereference (i.e., initial constraint) and each nonterminal summary edge describes a new constraint generated during constraint resolution. And all constraints are encoded using summary edges.

To distinguish itself from the other grammars used in our discussion, we rename the Pt nonterminal in Table 2 to P?1. We give the full productions in Figure 55a by expanding all inverse nonterminals. Combined with Lemma 1, we have the following equivalence result.

**Lemma 2.** Given a program P and its representative PEG, computing all-pairs Pt-reachability in PEG is equivalent to computing inclusion-based pointer analysis of P. Specifically, node &b is Pt-reachable from node a in PEG iff \text{loc}(b) ∈ pt(a).

4.1 PEG \(G_P\) Generated in Reduction

To facilitate the reduction from \(D_1\)-reachability, the C-style program P generated in Section 3.3 has some structural properties. Those properties play a pivotal role to show the correctness described in Theorem 3.

Unless otherwise noted, we refer to \(G_P\) as the PEG that corresponds to the output program P of Algorithm 3 in our discussion. Algorithm 3 partitions the variables in the output program P into three disjoint sets Var_w, Var_g, and Var_u. Specifically, each node \(v \in G\) in the \(D_1\)-reachability instance becomes two variables \(v \in \text{Var}_b\) and \(v' \in \text{Var}_g\) (line 5). Each variable \(t \in \text{Var}_w\) is an auxiliary variable which has been used in either node-processing (lines 11-11) or in edge-processing (line 2). We further partition \(\text{Var}_w\) into two disjoint sets \(\text{Var}_w1\) and \(\text{Var}_w2\) based on node-processing and edge-processing, respectively. In the constructed PEG \(G_P\), we associate each node with a color and a shape:

- A **white node** (i.e., ◯ or □) represents a variable \(t \in \text{Var}_w1 \cup \text{Var}_w2\). White nodes correspond to auxiliary variables used in P.
Lemma 3. Based on Figure 2 Algorithm maps each node \( v \in G \) to a black square and a gray circle node in PEG \( G_P \). Both mappings are bijective.

Language \( Pt \). To see the connection between \( D_1 \) and \( Pt \), we expand the \( Pt \) rules in Table 2. In particular, we keep the start nonterminal \( Pt \) and replace any other occurrence of \( Pt \) with \( Sr \). Figure 5a gives the rewritten grammar. Since nonterminal \( S \) is nullable, the rewritten grammar in Figure 5a is equivalent to the original grammar in Table 2.

Example 2. Figure 3 gives the generated PEG for program \( P \) in Figure 1c. In the graph, we can see that there is a \( Pt \)-path from \( w_3 \) to \&\( w_3' \). The realized string of path \( w_3 \rightarrow *t_{13} \rightarrow t_{13} \rightarrow \)
Isomorphism
(Section 5.2)

Non-transitivity Lem.
(Section 5.3)

Isolation Lemma
(Section 5.2)

Graph $G$

PEG $G_P$

PEG $G_P$

PEG $G_P$

Figure 4: Overview of reductions.

t_{14} \rightarrow s t_{15} \rightarrow t_{15} \rightarrow & w_3'$ is “as $\overline{d s r d r}$”. According to the production rules (1-a) and (2-a), the realized string belongs to the $Pt$ language. And node $& w_3'$ is $Pt$-reachable from $w_3$ in $G_P$. We further observe that in the original program (Figure 1c), the last four lines of code are related to $w_3$ and $w_3'$. According to Table 4, we have constraints $\forall v \in pt(t_{13}) : pt(v) \subseteq pt(w_3)$, $pt(t_{14}) \subseteq pt(t_{13})$, $loc(t_{15}) \in pt(t_{14})$ and $loc(w_3') \in pt(t_{15})$. Therefore, we have $loc(w_3') \in pt(w_3)$. Finally, we have $w_3 \rightarrow & w_3' \in G_P \Rightarrow loc(w_3') \in pt(w_3) \in P$ (Lemma 2).

Consider another pair of nodes $w_0$ and $w_3$. The path between them realizes the string “$\overline{d r d d s a}$”. Based on the production rules (1-a), (2-a) and (3-a), we can see that it can be derived by nonterminal $S$ in Figure 4a. On the other hand, we can extract the set constraints from the original program, the statements $t_1 = & w_0$; $v_1 = t_1$; $v_1 = & t_3$; $t_3 = w_3$ yield the relation $pt(w_3) \subseteq pt(w_0)$. The $S$-reachability in $PEG_G_P$ and set constraint resolution in $P$ agree on the subset relation.

Finally, we consider the pair of nodes $w_3$ and $w_3$. The path between these two nodes realizes the string “$\overline{d r d d s a}$”. This word cannot be recognized by the $S$ language or the $Pt$ language. From the program statements, the set constraints can not establish a subset relation or a points-to relation between the two corresponding variables.

5 $D_1$-Reachability and $Pt$-Reachability

In this section, we prove that $D_1$-reachability in $G$ is equivalent to $Pt$-reachability among black and gray nodes in $G_P$.

Our basic idea is to simplify the language $Pt$ and convert it to a $D_1$-like language called $D_1'$. Figure 5 gives an overview of our reduction. In particular, in $G_P$, we prove that $Pt$-reachability among black and gray nodes is equivalent to a simplified $Pt'$-reachability with only black nodes. We further simplify $Pt'$-reachability and convert it to $D_1$-reachability in $G_P$. Finally, we show that the $D_1'$-reachability problem in $G_P$ is equivalent to the $D_1$-reachability problem in $G$.

Note that the simplifications mentioned above do not hold for general PEG. However, our reduction in Algorithm 3 emits a specialized PEG. Figure 5 shows all grammars involved in our proof. Our key insight is to leverage the properties in the constructed PEG for simplifying the $Pt$-reachability problem. In particular, our constructed PEG introduces two aspects of restrictions:

- The nodes in the PEG $G_P$ are of different shapes (i.e., square and circle nodes). Figure 4 gives an illustration of our constructed PEG. Our construction in Figure 2 guarantees that the reachability among square nodes does not involve circle nodes. This helps us eliminate a few rules in grammar $Pt$ and obtain a simpler grammar $Pt'$ (Section 5.4).

- The nodes in the PEG $G_P$ are of different colors (i.e., black, white, and gray) as well. The color information and the edge construction forbid certain combinations of nonterminals. We encode
the node color information in grammar $Pt'$ and further simplify the language to $D'_1$ (Section 5.2).

- Finally, the $D'_1$-reachability problem in $G_P$ is isomorphic to the $D_1$-reachability problem in $G$ (Section 5.3).

### 5.1 $Pt$-Reachability and $Pt'$-Reachability

The most notable difference between $D_1$-reachability and $Pt$-reachability is that the $Pt$-reachability problem is bidirectional. For instance, for any summary edge $u \xrightarrow{S} v$ there exists a reversed summary $v \xrightarrow{S} u$ in $G_P$ based on the $Pt$ grammar in Figure 5a. The reversed summaries introduce additional reachability information since a path now can go back and forth at a node.

To cope with the bidirectionality, we introduce reversibility to paths in $G_P$. Formally, we say a path $p = u, \ldots, x, y, x, \ldots, v$ is a reversing path iff there exists at least one node $y \in p$ such that $x \rightarrow y \rightarrow x$ is a subpath of $p$. The node $y$ is called a reversing node of path $p$. For instance, the path $\blacksquare \rightarrow \ldots \rightarrow \blacksquare \rightarrow x \rightarrow y \rightarrow x \rightarrow \blacksquare \rightarrow \ldots \rightarrow \blacksquare$ in Figure 6a is a reversing path with $y$ being the reversing node.

The subpath $x \rightarrow y \rightarrow x$ of a reversing path introduces either a string "t $\overline{t}$" or a string "$\overline{t}$ t". However, most of those strings are invalid in grammar $Pt$. Given a $CFG = (\Sigma, N, P, S)$, we define $\text{Follow}(t)$, for terminal $t \in \Sigma$, to be the set of terminals $w$ that can appear immediately to the right of terminal $t$ in some sentential form, that is, the set of terminals $w$ such that there exists a derivation
of the form $A \rightarrow \alpha t w \beta$ for some $\alpha$ and $\beta$. Note that our definition of Follow set on terminals is similar to the concept of the Follow set on nonterminals in standard compiler text. Table 3 gives the Follow sets of all terminals of grammar $Pt$ in Figure 5a.

**Lemma 4.** No circle node can be a reversing node.

*Proof.* From the $G_P$ construction, we can see that circle nodes represent auxiliary variables in program $P$. They are used in $G_P$ to connect black and gray nodes. Figure 6b shows a path with circle nodes. Based on the Follow sets shown in Table 3 we have only $d \in \text{Follow}(d)$. Therefore, in Figure 6b only nodes 2 and 6 could be the reversing nodes. From Figure 5a we can see that the substring “$d \overline{d}$” can only be generated by rules (6-a) and (7-a):

$$S \overline{a} \rightarrow \overline{a}, \quad S S \overline{a} \rightarrow d S r d S \overline{a} \rightarrow \overline{a} S r d \overline{a} r S d s a;$$

$$\overline{S} \overline{a} \rightarrow \overline{a}, \quad \overline{S} S \overline{a} \overline{a} \rightarrow \overline{a} S r d \overline{a} r S \overline{a} \overline{a} \rightarrow \overline{a} S r d \overline{a} r S d \overline{a} r \overline{a}.$$

We notice that there is always a “$\overline{r}$” symbol that follows a “$\overline{a}$” symbol. Therefore, node 2 in Figure 6b could not be a reversing node. Without node 2 being a reversing node, the realized string of path $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1$ is “$\overline{a} S r d \overline{a} r S d \overline{a} r \overline{a}$”. This realized string cannot be generated by $S$ or $\overline{S}$ discussed above since $as$ cannot be paired with $\overline{as}$. Therefore, node 6 cannot be a reversing node, either.

Consider a $G_P$ path with circle nodes shown in Figure 6b. It corresponds to the edges generated by the last row in Table 4. It contains the variables introduced in Algorithm 3 on lines 7-10, i.e., node 1, 2, 3, 4, 5, 6, 7 represent variables $v$, $t_i$, $t_i$, $t_{i+1}$, $t_{i+2}$, $t_{i+3}$ and $y\overline{y}$ in Algorithm 3, respectively. We say that the black node representing variable $v$ is a root node of the gray circle node representing variable $y\overline{y}$ as well as the white circle nodes representing the auxiliary variables $t_i$ introduced in Algorithm 3 on lines 7-10.

**Lemma 5 (Isolation).** In $G_P$, the Pt- or S-path which joins two square nodes cannot pass through any circle node.

*Proof.* We prove by contradiction. Assume such a path exists. Without loss of generality, we assume the path joining two square nodes $u$ and $v$ is $u \rightarrow \ldots \rightarrow o_x \rightarrow \ldots \rightarrow v$. Note that node $o_x$ can be either a white circle node or a gray circle node. Let the root node of $o_x$ be $\overline{o}_y$. Thus, the path is of the form $u \rightarrow \ldots \rightarrow \overline{o}_y \rightarrow \ldots \rightarrow o_x \rightarrow \ldots \rightarrow \overline{o}_y \rightarrow \ldots \rightarrow v$. The subpath $\overline{o}_y \rightarrow \ldots \rightarrow o_x \rightarrow \ldots \rightarrow \overline{o}_y$ forms a cycle. As shown in Figure 6b with the gray node being one end of the path, there always exits a node $o_y$ such that $\overline{o}_y \rightarrow \ldots \rightarrow o_y \rightarrow \ldots \rightarrow \overline{o}_y$. Thus, $o_y$ is a reversing node. This contradicts the fact that no circle node can be a reversing node (Lemma 4).

**Lemma 6.** Let $Pt$ and $S$ be two nonterminals in grammar $5a$. In PEG $G_P$, Pt-reachability among black and gray nodes is equivalent to S-reachability among black nodes, i.e.,

$$\overline{o}_y \text{ Pt} \overline{*y} x \overline{y} \iff \overline{o}_y \text{ S}

\Rightarrow \overline{o}_x.$$
Proof. Each gray node $*_{k,x'}$ has one unique root node $\mathbf{0}_x$. Due to the $G_P$ construction, there is always a path $\mathbf{0}_x \xrightarrow{\infty} \circ_u \xrightarrow{\tau} \circ_v \xrightarrow{\tau} \circ_w \xrightarrow{\tau} \circ_y \xrightarrow{\tau} \circ_z \xrightarrow{\tau} *_{k,x'}$ between the black node $\mathbf{0}_x$ and the gray node $*_{k,x'}$ shown in Figure 6b. According to the last terminals in rules (2-a) and (3-a), no $S$-path in PEG $G_P$ ends at nodes $\circ_u$, $\circ_v$, and $\circ_y$, respectively. Moreover, there is only one $S$-path $\circ_v \xrightarrow{S} \circ_w$, which ends at $\circ_w$, i.e., there is no node $y$ in $G_P$ such that $y \rightarrow *_x \rightarrow \circ_u \xrightarrow{\tau} \circ_w$ and $y \xrightarrow{S} \circ_u$. Based on rule (2-a), we have $*_x \xrightarrow{\tau} *_{k,x'}$.

- The $\Rightarrow$ direction. Due to the construction, every $P_t$-path $\mathbf{0}_y \xrightarrow{P_t} *_{k,x'}$ passes through node $\mathbf{0}_x$, i.e., there must be a path $\mathbf{0}_y \xrightarrow{\tau} \mathbf{0}_x \xrightarrow{\tau} \circ_z \xrightarrow{\tau} *_{k,x'}$. Based on rule (1-a), we have $\mathbf{0}_y \xrightarrow{\tau} \circ_z \xrightarrow{\tau} *_{k,x'}$. The nodes $\circ_u$, $\circ_v$, $\circ_w$, $\circ_y$ between $\mathbf{0}_x$ and $\circ_z$ are not $S$-reachable from $\mathbf{0}_y$. Finally, we have $\mathbf{0}_y \xrightarrow{\tau} \mathbf{0}_x$ based on rule (6-a).

- The $\Leftarrow$ direction. For each $S$-path $\mathbf{0}_y \xrightarrow{S} \mathbf{0}_x$, there is a path $\mathbf{0}_y \xrightarrow{\tau} \mathbf{0}_x \xrightarrow{\tau} \circ_z \xrightarrow{\tau} *_{k,x'}$. Therefore, we have a path $\mathbf{0}_y \xrightarrow{\tau} *_{k,x'}$ based on rules (1-a) and (6-a).

\[\square\]

Language $P_t'$ Based on Lemma 6, we are able to compute $S$-reachability with only black nodes. Due to Lemma 6, we can discard all circle nodes when computing $S$-reachability. Therefore, we can discard all rules in Figure 5a that contain symbols associated with circle nodes. As a result, we can safely remove rules (2-a), (4-a), $S \rightarrow s$ and $S \rightarrow s$. Figure 5b gives the simplified grammar with a new start symbol $P_t'$. Based on the discussion, it is immediate that $P_t'$-reachability is equivalent to $S$-reachability.

Lemma 7. In PEG $G_P$, $P_t$-reachability among black and gray nodes is equivalent to $P_t'$-reachability among black nodes, i.e.,

$$\mathbf{0}_y \xrightarrow{P_t} *_{k,x'} \iff \mathbf{0}_y \xrightarrow{P_t'} \mathbf{0}_x.$$

5.2 $P_t'$-Reachability and $D_1'$-Reachability

Our basic idea is to “extract” a $D_1$ grammar from the $P_t'$ grammar. Every $D_1$ string can be generated by either rule $D_1 \rightarrow [1 \ D_1 \ ]_1$ or rule $D_1 \rightarrow D_1 \ D_1$. Lemma 7 considers $P_t'$-paths with only black nodes. However, in $G_P$, the $P_t'$-paths can also join white nodes. In our proof, we need to make sure that our extracted $D_1$ grammar only involves black nodes.

Lemma 8. Based on grammar $P_t'$, the $S$-paths join only same-color square nodes in $G_P$.

Proof. We prove by contradiction. From Figure 5b it is clear that all $S$-paths are of length $4k$ for some $k \geq 0$. In our $G_P$ construction, there are three white nodes between a pair of black nodes. We label the three white nodes as $\square_1 \rightarrow \square_2 \rightarrow \square_3 \rightarrow \square$. Assume an $S$-path joins a black node and a white node. The path could be depicted as one of the followings: $\square \xrightarrow{S} \square_1$, $\square \xrightarrow{S} \square_2$, or $\square \xrightarrow{S} \square_3$. The path lengths are $4k + 1$, $4k + 2$, and $4k + 3$, respectively. Similarly, the path lengths of $\square_1 \xrightarrow{S} \square$, $\square_2 \xrightarrow{S} \square$, and $\square_3 \xrightarrow{S} \square$ are $4k + 1$, $4k + 2$, and $4k + 3$ as well. It contradicts the fact that $S$-paths are of length $4k$.

\[\square\]

Corollary 3. Based on grammar $P_t'$, the $\overline{S}$-paths join only same-color square nodes in $G_P$.

We augment the $P_t'$ grammar in Figure 5b with node color information in PEG $G_P$. Consider a symbol $\tau$ in Table 4. From Table 3, we can see that symbol $\tau \in \text{Follow}(\tau)$. Therefore, a $\tau$-edge can be followed by an $\overline{\tau}$-edge. Now, consider an edge $u \xrightarrow{\tau} \tau \rightarrow y$ in $G_P$. In Table 3 there are three types of $\tau$-edges, i.e., $\square \xrightarrow{\tau} \square$, $\square \xrightarrow{\tau} \square$, and $\square \xrightarrow{\tau} \square$. It is interesting to note that no $\tau$-edge can follow $\square \xrightarrow{\tau} \square$ since all $\tau$-edges start with a white node $\square$. To bridge the gap between the color constraints in $G_P$ and the $P_t'$ grammar, we introduce a colored form of grammar $P_t'$. Specifically, we augment the $P_t'$ grammar in Figure 5b with node color information in graph $G_P$. 

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Table 4: All edges

| Edge type | GP edge | P statement |
|-----------|---------|-------------|
| O1        | □ → □ | $t_i = \& x; \ast y = t_i;$ |
| R1        | $\square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square$ | $x = \& t_i; \ast t_i = y;$ |
| O2        | □ → □ □ → □ → □ → □ → □ → □ | |
| R2        | □ → □ □ → □ → □ → □ → □ → □ | |

Colored Grammar $\square \rightarrow \square$  Given a CFG $=(\Sigma, N, P, S)$ and a PEG $G_P$, we define a colored grammar $CFG_c = (\Sigma_c, N_c, P_c, S_c)$ where every symbol $i_t \in \Sigma_c \cup N_c$ is annotated with two colors $l, r \in \{\square, \blacksquare\}$ iff $l \rightarrow r \in G_P$ for all $t \in \Sigma \cup N$. For each production rule $C \rightarrow AB$ in $P$, we construct a colored rule $i_A \rightarrow i_B i_C$ in $P_c$ iff $i = k, j = n, and l = m$. We denote it as $i_A \rightarrow i_B i_C$ for brevity. The start symbols in $S_c$ could be constructed accordingly. We then give detailed steps to construct the production rules.

Step 1: From grammar $5b$, we can see that $S$ always begins with a $\square$. In Table 4 there is a unique $\square \rightarrow \square$ in the type O1 edge. Therefore, we have “$S \rightarrow \square \square \ldots$”.

Step 2: In the type O1 edge, the unique $\square \rightarrow \square$ is followed by $\square \rightarrow \square$. Therefore, we have “$S \rightarrow \square \square \ldots$”.

Step 3: Based on grammar $5b$ and our current construction, there should be an $S$ symbol that begins with a $\square$. Based on Lemma 8 it should be a $\square S \square$. Our production rule becomes “$S \rightarrow \square d \square S \square \ldots$”. Next, we look into the last symbols of rule $(3b)$ to complete the construction.

Step 4: The production rule of $S \rightarrow \square$ should end with a $\square \square \square$. Therefore, we have “$S \rightarrow \square \square \square \ldots$”.

Step 5: The edge $\square \rightarrow \square$ is unique in Table 4. It immediately follows a $\square \rightarrow \square$ in type O2 edge. As a result, we have “$S \rightarrow \square \square \square \ldots$”.

Step 6: Combining steps 3 and 5, we have a complete production rule “$S \rightarrow \square d \square S \square \square \square$”.

Similarly, we could construct rule “$\square S \square \rightarrow \square d \square S \square S \square d \square$”. With the beginning $\square$ and the terminal $\square$, we could uniquely locate the $\square S \square$ portion in the type O1 edge of Table 4. With the beginning $\square$ and the terminal $\square$, we could also uniquely locate the $\square S \square$ portion in the type O2 edge in Table 4. Based on the first and last symbol in $S \square$, it is immediate that $S \rightarrow S \square$.

Lemma 9 (Non-transitivity). $\square S \square$-paths are not transitive.

Proof. We prove by contradiction. Assume that there is an $\square S \square$-path in $G_P$ which is generated by two consecutive $\square S \square$-paths. With the color constraints in Table 4 we can see that an $\square S \square$ path can only begin with $\square S \square$ and end with $\square S \square$. Putting the two parts together yields a “$\square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square$”. It it clear that the constructed path does not belong to any of the four edge types in Table 4. It contradicts the fact that the $\square S \square$-path is a valid path in $G_P$.

We give the production rules of language $P_t'$ in the colored form in Figure 5c. Note that, based on Lemma 9 we can eliminate rule $7b$ in Figure 5b.

Lemma 10. In PEG $G_P$, $P_t'$-reachability among black nodes is equivalent to $P_t'$-reachability, i.e.,

$\square y \rightarrow \square x \iff \square y \rightarrow \square x.$
Language $D'_1$. The colored from of language $Pt'$ in Figure 5c defines the $Pt'$-reachability among black nodes in PEG $G_P$. We can simplify the set of rules in Figure 5c by eliminating the nonterminal $\square$. We obtain the language $D'_1$ in Figure 5d. It is immediate that $D'_1$ is equivalent to the colored form $Pt'_1$ in Figure 5c. As a result, $D'_1$-reachability is equivalent to $Pt'_1$-reachability.

5.3 $D'_1$-Reachability and $D_1$-Reachability

Let $Pt$-reachability refer to the $Pt$-reachability among black and gray nodes in $G_P$ and $Pt'_1$-reachability refer to the $Pt'_1$-reachability among black nodes in $G_P$. Based on Lemma 7 and Lemma 10, we have:

$$Pt\text{-reachability} \iff Pt'_1\text{-reachability} \iff Pt'_1\text{-reachability} \iff D'_1\text{-reachability}.$$  

Recall that an $L$-reachability problem instance defined in Definition 2 contains a dagraph $G$ and a context-free language $CFG$. The $D'_1$-reachability in PEG $G_P$ is isomorphic to the $D_1$-reachability in $G$, i.e., there is a bijective mapping between $G_P$ and $G$. In particular, each node $v \in G$ has been mapped to a variable in $v \in Var_k$ in program $P$ based on Algorithm 3. The variable has been constructed as a black square node in $G_P$. Figure 2 establishes the bijective mapping between edges. From Figure 5c, it is clear that $D'_1$ and $D_1$ are isomorphic. Therefore, we have the following theorem:

**Theorem 5.** Algorithm 3 takes as input a digraph $G = (V, E)$ and outputs a C-style program $P$ with $O(E)$ variables and $O(E)$ statements. All nodes $v \in V$ are represented as variables $v$ and $v'$ in $P$. Node $v$ is $D_1$-reachable from node $u$ in $G$ iff the gray node $v'$ is $Pt$-reachable from the black node $u$ in $G_P$.

Combining the Lemma 2 on the equivalence between $Pt$-reachability and inclusion-based points-to analysis, we prove Theorem 4.

6 Implications of $D_1$-Reachability-Based Reduction

Our BMM-hardness result of inclusion-based points-to analysis is based on a reduction from $D_1$-reachability (Section 5). As mentioned in Section 1, the work by Sridharan and Fink [36] gives a reduction from transitive closure to inclusion-based points-to analysis. It is well-known that Boolean matrix multiplication (BMM) is computationally equivalent to transitive closure [16]. A natural question that arises is: does the $D_1$-reachability-based reduction yield any new insights? This section discusses two important implications of our reduction.

• **Generality.** Based on Table 1 points-to analysis on C-style programs contains four types of constraints: ADDRESS-OF, ASSIGNMENT, ASSIGN-STAR and STAR-ASSIGN. The Sridharan-Fink reduction only permits ADDRESS-OF and ASSIGNMENT constraints. Based on pointer semantics, real-world C-style programs can have pointers without using any assignment statements of the form “$a = b$”, i.e., all assignments are of the forms “$a = &b$”, “$\text{ star } a = b$” and “$a = *b$”. Therefore, the Sridharan-Fink reduction does not apply to the points-to analysis problem without ASSIGNMENT constraints. Our reduction based on $D_1$-reachability can be generalized to non-trivial C-style programs with any types of constraints. It applies to more practical programs (Section 6.1).

• **Expressiveness.** Based on $D_1$-reachability, we can establish a more interesting result that the demand-driven points-to analysis is no easier than the exhaustive counterpart. As noted in Section 1 the transitive-closure-based reduction does not imply such results because the demand-driven version of graph reachability can be trivially solved by a linear-time depth-first search. Dyck-reachability is a fundamental framework to formulate many interprocedural program-analysis problems. Our results demonstrate that the bottleneck of demand-driven interprocedural analysis is due to matching the well-balanced properties such as procedure calls/returns and pointer references/dereferences in programs, as opposed to computing the transitive closure (Section 6.2).
6.1 Generality of Reduction

Consider the four types of constraints Address-of (R), Assignment (S), Assign-star (As) and Star-assign (Sa) in points-to analysis. Among the four constraints, the Address-of constraint is essential. As discussed in Section 1, without Address-of, all points-to sets are empty sets and the points-to-analysis problem becomes trivial. Moreover, if a program contains only Address-of statements, the points-to sets can be trivially decided in linear time as there are not any subset constraints. The other three constraints can be arbitrarily combined in any practical C-style programs. Therefore, to prove Corollary 1, we need to discuss \((\binom{3}{1}) + (\binom{3}{2}) + (\binom{3}{3}) = 7\) combinations. The work by Sridharan and Fink [36, §3] has already established the reduction based on S. This section focuses on the remaining six cases.

- **Case 1 with constraints Sa, As, and S.** Section 5 has established the construction. It gives a reduction based on all three constraints, which is the main contribution of this paper. The other five cases are indeed extended from this case. We briefly summarize the reduction and the correctness to facilitate the discussions on other cases.

  - **Reduction.** Figure 2 gives the construction. For each edge in \(G\), we use Edge-with-as to construct edges in \(G_P\) with square (□ or ■) nodes. For each node in \(G\), we construct paths based on Node-with-as-s using circle (○ or ◦) nodes.

  - **Correctness.** Based on Section 5, the key steps to prove the correctness (Theorem 5) include establishing the isolation lemma (Lemma 5) and the non-transitivity lemma (Lemma 9). From Figure 4, we can see that the isolation lemma guarantees that \(Pt\)-reachability is equivalent to \(Pt'\)-reachability among only black square ■ nodes. The non-transitivity lemma ensures that the mapping between \(Pt'\)-paths and \(D_1'\)-paths is bijective. Finally, due to the grammar construction, \(D_1'\)-reachability in \(G_P\) is always isomorphic to \(D_1\)-reachability in \(G\).

- **Case 2 with constraints Sa and S.** Case 1 includes all three constraints. In Case 2, we need to construct a program without any As constraint. Based on Figure 2, we can see that the As constraint only appears at the paths for node construction (Node-with-as-s). Therefore, in Case 2, we only need to remove the as-related nodes/edges when constructing nodes in \(G_P\).

  - **Reduction.** For each edge in \(G\), we adopt Edge-with-sa shown in Figure 2. For each node in \(G\), we construct a path with Node-with-s shown in Figure 7.
\[ \begin{align*}
Pt' & \rightarrow S & (1-b') \\
S & \rightarrow as \overline{d} r d & (2-b') \\
\overline{S} & \rightarrow d r \overline{S} d \overline{a} & (3-b') \\
S & \rightarrow S S & (4-b') \\
S & \rightarrow \epsilon & (5-b') \\
\overline{S} & \rightarrow \epsilon & (6-b') \\
\end{align*} \]

(a) Rules for language \( Pt' \).

\[ \begin{align*}
Pt'_c & \rightarrow S & (1-c') \\
S & \rightarrow as \overline{d} r d & (2-c') \\
\overline{S} & \rightarrow d r \overline{S} d \overline{a} & (3-c') \\
S & \rightarrow S S & (4-c') \\
S & \rightarrow \epsilon & (5-c') \\
\overline{S} & \rightarrow \epsilon & (6-c') \\
\end{align*} \]

(b) Rules for language \( Pt'_c \).

\[ \begin{align*}
D'_1 & \rightarrow S & (1-c') \\
\overline{S} & \rightarrow as \overline{d} r d & (2-c') \\
S & \rightarrow S S & (3-c') \\
S & \rightarrow \epsilon & (4-c') \\
\overline{S} & \rightarrow \epsilon & (5-c') \\
\end{align*} \]

(c) Rules for language \( D'_1 \).

Figure 8: Grammars used for extended cases. The \( Pt' \) grammar in Figure 8a is obtained from the \( Pt \) grammar in Figure 5a by removing two rules related to \( sa \) and \( \overline{sa} \).

**Node-with-as-a** in Case 1, **Node-with-s** removes the edges representing \( As \) and the corresponding reference/dereference.

**Correctness.** According to Case 1, it suffices to show that Lemmas 5 and 9 hold for Case 2. The nodes used in **Node-with-s** construction are circle (\( o \) or \( e \)) nodes. Lemma 4 holds for all cycle nodes. The **Node-with-s** construction only removes edges in **Node-with-as-a**. Therefore, it holds for the **Node-with-s** construction as well. This concludes that Lemma 5 holds. Note that the **Edge-with-sa** construction is identical to the edge construction in Case 1 (Lemma 9).

- **Case 3 with constraints \( Sa \) and \( As \).** Similar to Case 2, we only need to avoid using the \( S \) constraint when constructing \( G_P \).

  **Reduction.** For each edge in \( G \), we use **Edge-with-sa** in Figure 2 as the previous two cases. For each node in \( G \), we construct a path with **Node-with-as** in Figure 7. Comparing with **Node-with-as-s** in Case 1, we remove the edges representing \( S \) in Case 3.

  **Correctness.** Both Cases 2 and 3 avoid using one constraint compared with the node construction in Case 1. Following a similar argument as in Case 2, Lemmas 5 and 9 hold.

- **Case 4 with constraints \( As \) and \( S \).** In Case 1, we use \( sa \)-edges in \( G_P \) to model \( [1] \) - and \( [1] \) -edges in \( G \). However, Case 4 does not contain any \( Sa \) constraints. We need to use \( as \)-edges to encode \( [1] \) - and \( [1] \) -edges in \( G \). The principal idea is based on Case 1. Indeed, due to the \( Pt \) grammar, the reduction based on \( as \)-edges is simpler than Case 1 which is based on \( sa \)-edges. Let us revisit the \( Pt \) grammar in Figure 5a. The reduction in Section 5.1 uses rules 3-a and 5-a to encode the \( D_1 \) rule \( S \rightarrow [1] S [1] \) shown in Figure 5d. Without the \( sa \) terminal, we can still use rule 2-a. It is interesting to note that, unlike rule 3-a, the \( S \) nonterminal in rule 2-a only depends on \( S \) itself. Rule 2-a is immediately isomorphic to \( D_1 \) rule \( S \rightarrow [1] S [1] \). Consequently, the reduction based on \( as \)-edges is significantly simpler because rule 2-a does not contain the \( \overline{S} \) nonterminal. Figure 8 gives all grammars used in the new reduction for Case 4. Figure 8c gives the grammar that is isomorphic to the \( D_1 \) grammar.

  **Reduction.** For each labeled edge in \( G \), we use **Edge-with-as** in Figure 7 to construct edges in \( G_P \). For each node in \( G \), we construct a path based on **Node-with-s** in Figure 7.

  **Correctness.** It is clear from Figure 8a that rules 1-b' and 7-b' are unreachable from the start symbol, which can be safely eliminated. Due to the **Edge-with-as** and grammar \( D'_1 \) in Figure 8c, there is no \( \square \overline{S} \) symbol, i.e., no white \( \square \) nodes \( S \)-reachable in the new reduction. Therefore, the non-transitive Lemma 9 holds immediately. The **Node-with-s** construction has been used in Case 2 and Lemma 5 holds.
contains a triangle iff node

We introduce two unique nodes Intuition.

The four layers with three edges in

Therefore, to establish the BMM-hardness of demand-driven points-to analysis, it suffices to establish

Section 5 gives a reduction from

Putting everything together, we prove Corollary 1 in Section 1.

6.2 Hardness of Demand-Driven Analysis

Section 4 gives a reduction from $D_1$-reachability to points-to analysis. Corollary 2 further states that $D_1$-reachability can be reduced to points-to analysis under arbitrary combinations of statement types. Therefore, to establish the BMM-hardness of demand-driven points-to analysis, it suffices to establish the BMM-hardness of single-source-single-target $D_1$-reachability ($s$-$t$ $D_1$-reachability). Due to the subcubic fine-grained equivalence of BMM and Triangle Detection [12, Thm 1.3], the BMM conjecture is equivalent to:

Conjecture 2. Any combinatorial algorithm for Triangle Detection in graphs with $n$ nodes requires $n^{3-o(1)}$ time in the Word-RAM model of computation with $O(\log n)$ bit words.

We pick the problem of Triangle Detection because the reduction is more intuitive. Similar to BMM, Triangle Detection has widely been used in fine-grained complexity proofs [1, 26, 42]. This section gives a reduction from Triangle Detection to $s$-$t$ $D_1$-reachability.

Reduction: From Triangle Detection to $s$-$t$ $D_1$-Reachability

Input: An (un)directed graph $G$ with $n$ nodes and $m$ edges;

Output: An edge-labeled digraph $G' = (V', E')$, where $|V'| = 4n + 6m + 2$ and $|E'| = 2n + 12m$.

Intuition. We introduce two unique nodes $s$ and $t$ in the output graph $G'$. The input graph $G$ contains a triangle if node $t$ is $D_1$-reachable from $s$. We arrange all nodes based on a particular ordering and split each node $u \in G$ into four copies $u_0$, $u_1$, $u_2$ and $u_3$ that span four layers in $G'$. The four layers with three edges in $G'$ accompany a triangle in $G$. The graph structure is informally known as a tripartite graph [1, 26, 42]. In particular, the graph contains two parts:

- **Triangle finding part:** For each $u \rightarrow v$ in $G$, we connect $u$ and $v$ in two adjacent layers in $G'$, i.e., $u_j \overset{D_1}{\rightarrow} v_{j+1}$ and $v_j \overset{D_1}{\rightarrow} u_{j+1}$ for all $j \in [0, 3)$. Therefore, we have a bijective map between $u \rightarrow v \rightarrow w \rightarrow u$ in $G$ and $u_0 \overset{D_1}{\rightarrow} v_1 \overset{D_1}{\rightarrow} w_2 \overset{D_1}{\rightarrow} u_3$ in $G'$.

- **Existential testing part:** To test the existence of a triangle in $G$, we connect all nodes in layer 0 via “$i$-edges” and all nodes in layer 3 via “$j$-edges”. Since all nodes in $G$ are arranged based on a random ordering, let $x$ be the first node that appears in the particular ordering. We construct $s \overset{v}{\rightarrow} x_0$ and $x_3 \overset{v}{\rightarrow} t$. Therefore, if graph $G$ has a triangle that contains node $u$, there exists a corresponding $D_1$-path $s \overset{v}{\rightarrow} \ldots \overset{v}{\rightarrow} u_0 \overset{D_1}{\rightarrow} u_3 \overset{v}{\rightarrow} \ldots \overset{v}{\rightarrow} t$ in $G'$, and vice versa.
Lemma 11. Algorithm 4 is a linear-time reduction which takes as input an undirected graph \( G \) and outputs a digraph \( G' \) with two unique nodes \( s \) and \( t \). Graph \( G \) has a triangle iff node \( t \) is \( D_1 \)-reachable from \( s \) in \( G' \).

Proof. A triangle in graph \( G \) corresponds to a \( D_1 \)-path from \( s \) to \( t \) in \( G' \) and vice versa.

- The “⇒” direction. Without loss of generality, we assume a triangle \( x \rightarrow y \rightarrow z \) in \( G \). Algorithm 4 (line 13) introduces a corresponding path \( x_0 \xrightarrow{Di} y_1 \xrightarrow{Di} z_2 \xrightarrow{Di} x_3 \) in \( G' \). Therefore, \( x_3 \) is \( D_1 \)-reachable from \( x_0 \) in \( G' \). From lines 5-9 we can see that Algorithm 4 introduces a path \( p_1 \) from \( s \) to any node \( u_0 \) using “\( [1] \)” labels and a path \( p_2 \) from the corresponding node \( u_3 \) to \( t \) using “\( ]1] \)” labels. The number of open brackets in \( p_1 \) equals to the number of close brackets in \( p_2 \). Since \( x_3 \) is \( D_1 \)-reachable from \( x_0 \) and the brackets in \( p_1 \) and \( p_2 \) are properly matched, node \( t \) is \( D_1 \)-reachable from \( s \) in \( G' \).
The “⇐” direction. Similar to the construction in Section 3.2, our constructed graph $G'$ is a 4-layered graph, i.e., it contains four node sets $V_0$, $V_1$, $V_2$, and $V_3$. Algorithm 4 (lines 13 and 14) always introduces paths with properly-matched brackets from nodes in $V_0$ to nodes in $V_3$. Suppose there exists a $D_1$-path from $s$ to $t$ in $G'$. The path contains three sub-paths: (1) sub-path $p_1$ from $s$ to some node $u_0$; (2) sub-path $p_2$ from $u_0$ to $v_3$; and (3) sub-path $p_3$ from $v_3$ to $t$. Moreover, $p_1$ contains unmatched open brackets and $p_3$ contains unmatched close brackets. Due to Algorithm 2 (lines 52-59), the brackets in $p_1$ and $p_3$ can match if $u_0 = v_0$ or $v_3 = u_3$. As a result, the $D_1$-path is of the form $s \xrightarrow{1} \ldots \xrightarrow{3} u_0 \ldots u_3 \xrightarrow{1} \ldots \xrightarrow{3} t$. Because graph $G'$ does not contain any cycle, the path joining $u_0$ and $u_3$ must be of the form $u_0 \xrightarrow{D_1} x_1 \xrightarrow{D_1} y_2 \xrightarrow{D_1} u_3$. It corresponds to a triangle $u \xrightarrow{1} x \xrightarrow{1} y \xrightarrow{1} u$ in $G$.

Section 5 gives a reduction from all-pairs $D_1$-reachability to all-pairs $Pt$-reachability (Theorem 5). The all-pairs $Pt$-reachability is equivalent to exhaustive points-to analysis (Lemma 2). This section establishes the BMM-hardness of $s$-$t$ $D_1$-reachability (Lemma 11). Putting everything together, we prove that demand-driven points-to analysis is BMM-hard (Corollary 2). $D_1$-reachability essentially captures the balanced-parenthesis property of pointer references and dereferences. It is worth noting that Corollary 2 only holds for non-trivial programs with pointer dereferences. For programs without dereferences, the demand-driven points-to analysis can be solved via a linear-time depth-first search based on the Sridharan-Fink reduction [36].

**Demand-Driven Interprocedural Program Analysis.** Lemma 11 establishes the BMM-hardness of $s$-$t$ $D_1$-reachability. It is immediate that Dyck-reachability with $k$ kinds of parenthesis ($s$-$t$ $D_k$-reachability) is BMM-hard. $D_k$-reachability is a fundamental framework to describe interprocedural program analysis problems [32, 33, 34]. In particular, procedure calls and returns can be depicted as “[$k$]- and “$i_k$”-labeled edges in a graph. Interprocedural static analyses need to ensure that the procedure calls and returns are properly matched. Based on Lemma 11, we have:

**Theorem 6.** Demand-driven interprocedural program analysis is BMM-hard.

### 7 Related Work

Despite extensive work [13, 18, 20], the worst-case complexity of inclusion-based pointer analysis remains cubic [13, 36]. In the literature, many pointer analyses have been formulated as a CFL-reachability problem [32, 33, 46, 47, 48]. Traditional CFL-reachability algorithm also exhibits a cubic time complexity [32]. The subcubic CFL-reachability algorithm was proposed by Chaudhuri [13], improving the cubic complexity by a factor of log $n$. Asymptotically better algorithms exist for special cases. For instance, Sridharan and Fink proposed a quadratic algorithm when the input graph is restricted to be $k$-sparse [36]. Chaudhuri [13] gave $O(n^3/\log^2 n)$-time and $O(n^\omega)$-time algorithms for bounded-stack recursive state machines and hierarchical state machines, respectively. For CFL-reachability-based approach, Zhang et al. [40] proposed an $O(n+m \log m)$ algorithm for alias analysis if the underlying CFL is restricted to be a Dyck language. Zhang and Su [43] also gave an $O(mn)$ time algorithm for computing sound solutions for a class of interleaved Dyck-reachability. The fastest algorithm for solving Dyck-Reachability is due to Chatterjee et al. [12] which runs in time $O(m+n \cdot \alpha(n))$ where $\alpha(n)$ is the inverse Ackermann function. When restricted to graphs with bounded treewidth, Chatterjee et al. [13] gave faster algorithms for solving demand-driven queries in the presence of graph changes. When restricted to directed acyclic graphs, Yannakakis [41] noted that CFL-reachability could be solved in $O(n^\omega)$ time. The work of McAllester [24] established a framework for determining the time complexity of static analysis.

The work of Chatterjee et al. [12] established a conditional cubic lower bound of Dyck-Reachability. Their work gave a reduction from CFL parsing [23] which required a Dyck language of $k$ kinds of parentheses. The class of 2NPDA represents the languages (or problems) definable by a two way nondeterministic pushdown automaton. Alho et al. [4] showed that any problem in the class 2NPDA can be solved in cubic time. Rytter [33] improved the cubic bound by a logarithmic factor leveraging the well-known Four Russians’ Trick [6] to speed up set operations under the random access machine...
(RAM) model. In the work of Heintze and McAllester [19], it was shown that the $D_2$-reachability problem is $2\text{NPDA}$-complete — it is both in $2\text{NPDA}$ and $2\text{NPDA}$-hard. Based on the Four Russians’ Trick, many subcubic algorithms have been proposed for solving program analysis problems such as CFL-reachability [14], control flow analysis [25] and pointer analysis [47]. In the literature, the best known algorithm for solving the $2\text{NPDA}$-complete problems is due to Rytter [35] which exhibits an $O(n^3/\log n)$ time complexity. Recently, Pavlogiannis [27] gave an independent result on the BMM-hardness of inclusion-based points-to analysis via different proof techniques. Pavlogiannis [27]’s reduction enables improved algorithms on restricted cases. Our result based on $D_1$-reachability sheds light on the hardness of analyzing unrestricted non-trivial C-style programs as well as general demand-driven interprocedural program-analysis problems. The two results offer complementary insights.

8 Conclusion

This paper has presented a formal proof to establish the hardness of inclusion-based points-to analysis. Our result shows that it is unlikely to have a truly subcubic time algorithm for inclusion-based points-to analysis in practice. We have also discussed two interesting implications based on our reduction.

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