ON HERMITE–HADAMARD TYPE INEQUALITIES FOR $F$–CONVEX FUNCTIONS

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Abstract. In this paper we give two different Hermite-Hadamard type inequalities for $F$–convex functions. As special cases of it we get known and new Hermite-Hadamard type inequalities for different concepts of convexity.

1. Introduction

In this paper by $I$ we denote a nonempty and open interval of $\mathbb{R}$. It is well known that for a convex function $f : I \to \mathbb{R}$ the following inequality is true

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{2}$$

(1)

for all $x,y \in I \ (x \neq y)$. This is the classical Hermite-Hadamard inequality [7] (see also [9] for interesting historical remarks). This inequality constitutes a crucial element of convex analysis and it has a vast literature concerning its generalizations, refinements, applications and concepts of convexity (cf. e.g. [5, 8, 12] with the references therein). One of the concepts of convex functions was introduced by Polyak [16]. Namely, a function $f : I \to \mathbb{R}$ is called strongly convex with modulus $c > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in I \ (x \neq y)$. Since strong convexity is an essential strengthening of convexity (cf. [14]), we can expect a better estimation of the integral mean for strongly convex functions than (1). In [10] the authors proved that for a strongly convex function with modulus $c > 0$, $f : I \to \mathbb{R}$, the following Hermite-Hadamard type inequality is true

$$f\left(\frac{x+y}{2}\right) + \frac{c}{12} (x-y)^2 \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{2} - \frac{c}{6} (x-y)^2$$

for all $x, y \in I \ (x \neq y)$. Notice that by following the proof of this result, the assumption ”$c > 0$” is not essential – we can assume that $c \in \mathbb{R}$. In [1], as a generalization of strongly convex functions, we can find the concept of $F$–convex functions. We adopt the following definition.

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**Definition 1.** Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. A function $f : I \to \mathbb{R}$ is called $F$-convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)F(x-y)$$

(2)

for all $x, y \in I$ and $t \in [0, 1]$.

Such functions were defined in the context of strongly convex functions. But note that from $F$-convexity we can also obtain another concepts of convexity:

- for $F(x) = -cx^2$ we get the definition of $c$-convex functions introduced by J.P. Vial (see [18]);
- for $F(x) = -c|x|$ with $c > 0$ we get approximate convex functions introduced by H.V. Ngai, D.T. Luc and M. Théra (see [11]);
- for $F(x) = -c|x|^p$ with $c > 0$ and $p > 0$ we get approximately convex functions of order $p$ introduced by K. Nikodem and Zs. Páles (see [13]);
- for $F(x) = -|x|\omega(|x|)$ with nondecreasing, upper-semicontinuous function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$ we obtain the definition of semiconvex functions introduced by G. Alberti, L. Ambrosio and P. Cannarsa (see [4]).

**2. Useful tools**

From the result presented in [2] (Theorem 1 and its proof) we conclude:

**Theorem 1.** Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If an $F$-convex function $f : I \to \mathbb{R}$ is one-sided differentiable at a point $x_0 \in I$ and $f'_-(x_0) \leq f'_+(x_0)$, then the following inequality is true

$$f(x) \geq F(x-x_0) + a(x-x_0) + f(x_0), \quad x \in I,$$

where $a$ is an arbitrary number such that $f'_-(x_0) \leq a \leq f'_+(x_0)$.

In paper [3] we find the following results:

**Theorem 2.** Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is $F$-convex, then it is continuous.

**Theorem 3.** Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If

$$\liminf_{x \to 0} \frac{F(x)}{x^2} > -\infty,$$

then every $F$-convex function $f : I \to \mathbb{R}$ has one-sided derivatives at each point $x \in I$ and $f'_-(x) \leq f'_+(x)$. 
Theorem 4. Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If
\[
-\infty < \liminf_{x \to 0} \frac{F(x)}{x^2} < +\infty,
\]
then every $F$-convex function $f : I \to \mathbb{R}$ is $c$-convex (in the sense of Vial), where $c = \liminf_{x \to 0} \frac{F(x)}{x^2}$.

3. Main results

We start with three lemmas. The first one sets an upper bound of integral mean for an $F$-convex function. Further lemmas give two different lower bounds of integral mean for an $F$-convex function, respectively.

Lemma 1. Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is $F$-convex, then
\[
\frac{1}{y-x} \int_x^y f(u)du \leq \frac{f(x) + f(y)}{2} - \frac{1}{6}F(x-y)
\]
for all $x, y \in I (x \neq y)$.

Proof. From Theorem 2 each $F$-convex function $f : I \to \mathbb{R}$ must be continuous; thus it is also integrable. Now, integrating side-by-side inequality (2) with respect to $t$ over interval $[0, 1]$ we conclude inequality (3).

Lemma 2. Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function integrable on each compact subinterval of $(-\alpha, \alpha)$, where $\alpha = \frac{\sup I - \inf I}{2}$. If an $F$-convex function $f : I \to \mathbb{R}$ is one-sided differentiable and $f_- \leq f_+$, then we have the inequality
\[
f \left( \frac{x+y}{2} \right) + \frac{1}{y-x} \int_x^y F \left( u - \frac{x+y}{2} \right) du \leq \frac{1}{y-x} \int_x^y f(u)du
\]
for all $x, y \in I (x \neq y)$.

Proof. Fix $x, y \in I (x \neq y)$. From Theorem 1 we conclude the inequality
\[
f \left( \frac{x+y}{2} \right) + a \left( tx + (1-t)y - \frac{x+y}{2} \right) + F \left( tx + (1-t)y - \frac{x+y}{2} \right) 
\leq f \left( tx + (1-t)y \right), \quad t \in [0, 1].
\]
Integrating this inequality side-by-side with respect to $t$ over interval $[0, 1]$ we obtain (4).

Remark 1. Using Theorem 3 we can replace the assumption ”$f : I \to \mathbb{R}$ is one-sided differentiable and $f_- \leq f_+$” in Theorem 2 by ”$\liminf_{x \to 0} \frac{F(x)}{x^2} > -\infty$”.
REMARK 2. For an even function $F$ and its primitive function $G$ such that $G(0) = 0$ inequality (4) takes the form

$$f\left(\frac{x+y}{2}\right) + \frac{2}{y-x} G\left(\frac{y-x}{2}\right) \leq \frac{1}{y-x} \int_{x}^{y} f(u)du$$

for all $x, y \in I$ ($x \neq y$).

Without Theorem 1, but applying methods from paper [6], we get a lower bound of the integral mean other than (4).

**Lemma 3.** Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a fixed function integrable on each compact subinterval of $(-\alpha, \alpha)$, where $\alpha = \sup I - \inf I$. If $f : I \rightarrow \mathbb{R}$ is an $F$-convex function, then

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4(y-x)} \int_{x}^{y} F(x+y-2u)du \leq \frac{1}{y-x} \int_{x}^{y} f(u)du$$

for all $x, y \in I$ ($x \neq y$).

**Proof.** From $F$-convexity of a function $f$ and the identity

$$\frac{x+y}{2} = \frac{tx + (1-t)y + (1-t)x + ty}{2}$$

we get

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{tx + (1-t)y + (1-t)x + ty}{2}\right) \leq \frac{f(tx + (1-t)y) + f((1-t)x + ty)}{2} - \frac{1}{4} F((2t-1)(x-y))$$

for all $x, y \in I$ and $t \in [0,1]$. Thus

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4} F((2t-1)(x-y)) \leq \frac{f(tx + (1-t)y) + f((1-t)x + ty)}{2}$$

(6)

for all $x, y \in I$ and $t \in [0,1]$. Fixing different $x, y \in I$ and integrating side-by-side with respect to $t$ over interval $[0,1]$ inequality (6) we obtain

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4} \int_{0}^{1} F((2t-1)(x-y))dt \leq \frac{1}{2} \int_{0}^{1} \left(f(tx + (1-t)y) + f((1-t)x + ty)\right)dt.$$

Which with substitutions "$(1-t)x + ty = u$" for the integrals

$$\int_{0}^{1} F((2t-1)(x-y))dt, \quad \int_{0}^{1} f((1-t)x + ty)dt$$

and "$tx + (1-t)y = u$" for the integral

$$\int_{0}^{1} f(tx + (1-t)y)dt$$. 
gives
\[ f\left(\frac{x+y}{2}\right) + \frac{1}{4(y-x)} \int_x^y F(x+y-2u)du \leq \frac{1}{y-x} \int_x^y f(u)du. \]

**Remark 3.** For an even function \( F \) and its primitive function \( G \) such that \( G(0) = 0 \) inequality (5) takes the form
\[ f\left(\frac{x+y}{2}\right) + \frac{G(y-x)}{4(y-x)} \leq \frac{1}{y-x} \int_x^y f(u)du \]
for all \( x, y \in I (x \neq y) \).

The presented lemmas result in two theorems of Hermite-Hadamard type inequalities for \( F \)-convex functions.

**Theorem 5.** Let \( F : \mathbb{R} \to \mathbb{R} \) be a fixed function integrable on each compact subinterval of \((-\alpha, \alpha)\), where \( \alpha = \sup I - \inf I \). If an \( F \)-convex function \( f : I \to \mathbb{R} \) is one-sided differentiable and \( f^- \leq f^+ \), then we have the inequality
\[
\begin{align*}
f\left(\frac{x+y}{2}\right) + \frac{1}{y-x} \int_x^y F\left(\frac{u-x+y}{2}\right)du &\leq \frac{1}{y-x} \int_x^y f(u)du \\
&\leq \frac{f(x) + f(y)}{2} - \frac{1}{6} F(x-y)
\end{align*}
\]
for all \( x, y \in I (x \neq y) \).

**Theorem 6.** Let \( F : \mathbb{R} \to \mathbb{R} \) be a fixed function integrable on each compact subinterval of \((-\alpha, \alpha)\), where \( \alpha = \sup I - \inf I \). If \( f : I \to \mathbb{R} \) is an \( F \)-convex function, then
\[
\begin{align*}
f\left(\frac{x+y}{2}\right) + \frac{1}{4(y-x)} \int_x^y F(2u-x-y)du &\leq \frac{1}{y-x} \int_x^y f(u)du \\
&\leq \frac{f(x) + f(y)}{2} - \frac{1}{6} F(x-y)
\end{align*}
\]
for all \( x, y \in I (x \neq y) \).

Notice that for the zero function \( F \) we get the classical Hermite-Hadamard inequality, for \( F(x) = cx^2 \) with \( c > 0 \) we get a Hermite-Hadamard type inequality for strongly convex functions, and inequalities (7) and (8) are the same. In general, one of them could be better than the other – it depends on the function \( F \). More precisely, it depends on the expressions:
\[
\frac{1}{y-x} \int_x^y F\left(\frac{u-x+y}{2}\right)du
\]
and
\[
\frac{1}{4(y-x)} \int_x^y F(2u-x-y)du.
\]
In particular, for power functions \( F(x) = c|x|^p \) with \( c \in \mathbb{R} \) and \( p > 0 \) they take the forms:

\[
\frac{1}{y-x} \int_x^y F \left( u - \frac{x+y}{2} \right) \, du = \frac{c}{4(p+1)} |y-x|^p
\]

and

\[
\frac{1}{4(y-x)} \int_x^y F \left( 2u - x - y \right) \, du = \frac{c}{2p(p+1)} |y-x|^p.
\]

Which means that for functions \( F(x) = c|x|^p \) with \( c < 0 \) and \( p < 2 \) inequality (8) is stronger than inequality (7); for \( c > 0 \) and \( p < 2 \) inequality (8) seems to be weaker than inequality (7) – but in this case, there are no \( F \)-convex function (see [3]); if \( c < 0 \) and \( p > 2 \) inequality (8) also seems to be weaker than inequality (7) – but in this case each \( F \)-convex functions must be convex (see [3]) and we have the classical Hermite-Hadamard inequality which in such case is stronger than inequality (7); for \( c \in \mathbb{R} \) and \( p = 2 \) (also \( c = 0 \) and \( p > 0 \)) inequalities are equivalent. So, for power function \( F(x) = cx^p \) with \( c \in \mathbb{R} \) and \( p > 0 \) inequality (8) is better.

Observe that for \( H(z) = \frac{1}{2z} \int_{-z}^z F(t) \, dt \) the integrals obtained on the left sides of (7) and (8) take the forms

\[
\frac{1}{y-x} \int_x^y F \left( u - \frac{x+y}{2} \right) \, du = H \left( \frac{y-x}{2} \right)
\]

and

\[
\frac{1}{4(y-x)} \int_x^y F \left( 2u - x - y \right) \, du = \frac{1}{4} H(y-x),
\]

which may be better for further analysis of Hermite-Hadamard type inequalities.

In [3] the author proved that an \( F \)-convex function \( f : I \to \mathbb{R} \) is \( c \)-convex (in the sense of Vial) as long as

\[
-\infty < \liminf_{x \to 0} \frac{F(x)}{x^2} < \infty,
\]

moreover, the postulated real number \( c \) in the definition of \( c \)-convex functions is equal \( -\liminf_{x \to 0} \frac{F(x)}{x^2} \). Thus, we conclude that if (9) holds, then an \( F \)-convex function is also with \( c(\cdot)^2 \)-convex with \( c = \liminf_{x \to 0} \frac{F(x)}{x^2} \). Therefore, we get (from Theorem 5 or Theorem 6) the following corollary.

**Corollary 1.** Let \( F : \mathbb{R} \to \mathbb{R} \) be a fixed function. If a function \( f : I \to \mathbb{R} \) is \( F \)-convex and

\[
-\infty < \liminf_{x \to 0} \frac{F(x)}{x^2} < \infty,
\]

then

\[
f \left( \frac{x+y}{2} \right) + \frac{c}{12} (x-y)^2 \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x)+f(y)}{2} - \frac{c}{6} (x-y)^2
\]

for all \( x, y \in I \) (\( x \neq y \)) and \( c = \liminf_{x \to 0} \frac{F(x)}{x^2} \).

Notice that inequality (10) was obtained for strongly convex functions with modulus \( c > 0 \) by N. Merentes and K. Nikodem in [10]. Their result is also derived from the above corollary – it is enough to take \( F(x) = cx^2 \).
Having regard to considerations for inequalities (7) and (8) and power functions $F$ we conclude that for approximately convex functions of order $p$ (in the sense of Nikodem and Páles) inequality (8) is better than (7). Therefore, from Theorem 6 we get the following Hermite-Hadamard type inequality for approximately convex functions of order $p$.

**COROLLARY 2.** Let $c > 0$ and $p > 0$. If $f : I \rightarrow \mathbb{R}$ is an approximately convex functions of order $p$ i.e.

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + ct(1 - t)|x - y|^p$$

for all $x, y \in I$ and $t \in [0, 1]$, then

$$f \left( \frac{x+y}{2} \right) - \frac{c}{4(p+1)}|y-x|^p \leq \frac{1}{y-x} \int_x^y f(u)du \leq \frac{f(x) + f(y)}{2} + \frac{c}{6}|x - y|^p$$

(11)

for all $x, y \in I$ ($x \neq y$).

Comparing Corollary 1 and Corollary 2, we conclude that for approximately convex functions of orders $p > 2$ we have a stronger inequality than inequality (11), namely the classical Hermite-Hadamard inequality – which in this case is stronger than inequality (11).

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