Speed of Excited Random Walks with Long Backward Steps

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Abstract
We study a model of multi-excited random walk with non-nearest neighbour steps on \( \mathbb{Z} \), in which the walk can jump from a vertex \( x \) to either \( x + 1 \) or \( x - i \) with \( i \in \{1, 2, \ldots, L\} \), \( L \geq 1 \). We first point out the multi-type branching structure of this random walk and then prove a limit theorem for a related multi-type Galton–Watson process with emigration, which is of independent interest. Combining this result and the method introduced by Basdevant and Singh (Probab Theory Relat Fields 141:3–4, 2008), we extend their result (w.r.t. the case \( L = 1 \)) to our model. More specifically, we show that in the regime of transience to the right, the walk has positive speed if and only if the expected total drift \( \delta > 2 \). This confirms a special case of a conjecture proposed by Davis and Peterson.

Keywords Excited random walks · Non-nearest neighbour random walks · Multi-type branching processes with emigration

Mathematics Subject Classification 60K35 · 60J80 · 60J85

1 Introduction
Excited random walk is a model of non-markovian random walk in a cookie environment, in which the walker consumes a cookie (if available) upon reaching a site and makes a jump with transition law dynamically depending on the number of remaining cookies at its current position. The model of nearest-neighbour excited random walks has been extensively studied in recent years. Benjamini and Wilson [3] first studied once-excited random walks with a focus on higher-dimensional integer lattice. Later, Zerner [26] extended this model to multi-excited random walks and established a criterion for recurrence/transience of the model on \( \mathbb{Z} \). There are also notable results for asymptotic behaviour of the multi-excited model including criteria non-ballisticity/ballisticity [2] as well as characterization of the limit distribution in such specific regimes [7, 14, 17]. See also [15, 18, 20, 22]. For a literature review, we refer the reader to [16].
1.1 Description of the Model and the Main Result

We define a non-nearest-neighbour random walk $X := (X_n)_{n \geq 0}$, which describes the position of a particle moving in a cookie environment on the integers $\mathbb{Z}$ as follows. For any integer $n$, set $[n] = \{1, 2, \ldots, n\}$. Let $M$ and $L$ be positive integers, $v$ and $(q_j)_{j \in [M]}$ be probability measures on $\Lambda := \{-L, -L + 1, \ldots, -1, 1\}$. Initially, each vertex in $\mathbb{Z}$ is assigned a stack of $M$ cookies and we set $X_0 = 0$. Suppose that $X_n = x$ and by time $n$ there are exactly remaining $M - j + 1$ cookie(s) at site $x$ with some $j \in [M]$. Before the particle jumps to a different location, it eats one cookie and jumps to site $x + i$, $i \in \Lambda$, with probability $q_j(i)$. On the other hand, if the stack of cookies at $x$ is empty then it jumps to site $x + i$, $i \in \Lambda$ with probability $v(i)$. More formally, denote by $(\mathcal{F}_n)_{n \geq 0}$ the natural filtration of $X$. For each $i \in \Lambda$,

$$\mathbb{P}(X_{n+1} = X_n + i | \mathcal{F}_n) = \omega(\mathcal{L}(X_n, n), i)$$

where $\mathcal{L}(x, n) = \sum_{i=0}^{n} \mathbb{I}_{X_i = x}$ is the number of visits to vertex $x \in \mathbb{Z}$ up to time $n$, and $\omega : \mathbb{N} \times \Lambda \to [0, 1]$ is the cookie environment given by

$$\omega(j, i) = \begin{cases} q_j(i), & \text{if } 1 \leq j \leq M, \\ v(i), & \text{if } j > M. \end{cases}$$

Throughout this paper, we make the following assumption.

**Assumption A**

(i) The distribution $v$ has zero mean.

(ii) For each $j \in [M]$, the distribution $q_j$ has positive mean and $q_j(1) < 1$.

We call the process $X$ described above $(L, 1)$ non-nearest neighbors excited random walk ($(L, 1)$-ERW, for brevity). It is worth mentioning that $(L, 1)$-ERW is a special case of excited random walks with non-nearest neighbour steps considered by Davis and Peterson in [5] in which the particle can also jump to non-nearest neighbours on the right and $\Lambda$ can be an unbounded subset of $\mathbb{Z}$. In particular, Theorem 1.6 in [5] implies that the process studied in this paper is

- Transient to the right if the expected total drift $\delta$, defined as

$$\delta := \sum_{j=1}^{M} \sum_{\ell \in \Lambda} \ell q_j(\ell)$$

is larger than 1, and

- Recurrent if $\delta \in [0, 1]$.

Additionally, Davis and Peterson conjectured that the limiting speed of the random walk exists if $\delta > 1$ and it is positive when $\delta > 2$. (see Conjecture 1.8 in [5]).

Recently, a sufficient condition for once-excited random walks with long forward jumps to have positive speed has been shown in [4]. However, the coupling method introduced in [4] seems to not be applicable to models of multi-excited random walks.

In the present paper, we verify Davis-Peterson conjecture for $(L, 1)$-ERW. More precisely, we show that $\delta > 2$ is a sufficient and necessary condition for $(L, 1)$-ERW to have positive limiting speed, under Assumption A.

**Theorem 1.1** Under Assumption A,

(a) if $\delta > 1$ the speed of $(L, 1)$-ERW $X$ exists, i.e. $X_n/n$ converges a.s. to a non-negative constant $v$, and

(b) if $\delta > 2$ we have that $v > 0$. If $\delta \in (1, 2]$ then $v = 0.$
1.2 Summary of the Proof of Theorem 1.1.

Our proof strategy relies on the connection between non-nearest neighbor excited random walks and multi-type branching processes with migration. The idea can be traced back to the branching structures of nearest-neighbor excited random walks [2] and random walks in a random environment (see e.g. [13] and [10]). In the present paper, we introduce multi-type branching process with emigration and develop techniques from [2] to deal with various higher dimensional issues in our model.

The remaining parts of the paper are organized as follows. We first describe in Sect. 2 the multi-type branching structure of the number of backward jumps. This branching structure is formulated by a multi-type branching process with (random) migration $Z$ defined in Proposition 2.1. In Sect. 3, we next demonstrate a limiting theorem for a class of critical multi-type Galton–Watson processes with emigration. We believe that this result is of independent interest. In Sect. 4, we derive a functional equation related to limiting distribution of $Z$ (Propositions 4.2 and 4.3). Combining these results together with a coupling between $Z$ and a critical multi-type branching process with emigration (which is studied in Sect. 3), we deduce the claim of Theorem 1.1.

It is worth mentioning that the techniques introduced in this paper is unfortunately not applicable to the case of excited random walks having non-nearest-neighbour jumps to the right. We refer the reader to [4] for a recent work studying the speed of once-excited random walks with long forward steps.

2 Multi-branching Structure of Excited Random Walks

For any pair of functions $f$ and $g$ of one real or discrete variable, we write

- $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x \to x_0} f(x)/g(x) = 1$,
- $f(x) = O(g(x))$ as $x \to x_0$ if $\limsup_{x \to x_0} |f(x)/g(x)| < \infty$ and
- $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x \to x_0} f(x)/g(x) = 0$.

Denote $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$. For any $m, n \in \mathbb{Z}$, $m \leq n$, set $[m, n]_\mathbb{Z} := \{m, \ldots, n\}$ and $[n] = \{1, 2, \ldots, n\}$.

For each $n \in \mathbb{Z}_+$, let $T_n = \inf\{k \geq 0 : X_k = n\}$ be the first hitting time of site $n$. For $i \leq n - 1$, define $V^n_i = (V^n_{i,1}, V^n_{i,2}, \ldots, V^n_{i,L})$ where for $\ell \in [L]$,

$$V^n_{i,\ell} = \sum_{k=0}^{T_{n}-1} \mathbb{1}_{\{X_k > i, X_{k+1} = i-\ell+1\}}$$

stands for the number of backward jumps from a site in the set $i + \mathbb{N}$ to site $i - \ell + 1$ before time $T_n$. Notice that $T_n$ is equal to the total number of forward and backward jumps before time $T_n$. In particular, the number backward jumps to site $i$ before time $T_n$ is equal to $V^n_{i,1}$.

On the other hand, between two consecutive forward jumps from $i$ to $i + 1$, there is exactly one backward jump from $i + \mathbb{N}$ to $i - \mathbb{Z}_+$. Furthermore, for $0 \leq i \leq n - 1$, before the first backward jump from $i + \mathbb{N}$ to $i - \mathbb{Z}_+$, the walk must have its first forward jump from $i$ to $i + 1$. Therefore the number of forward jumps from $i$ to $i + 1$ before time $T_n$ is equal to $\mathbb{1}_{\{0 \leq i \leq n-1\}} + \sum_{\ell=1}^{L} V^n_{i,\ell}$. As a result, we obtain

$$T_n = n + 2 \sum_{-\infty < i \leq n-1} V^n_{i,1} + \sum_{-\infty < i \leq n-1} \sum_{\ell=2}^{L} V^n_{i,\ell}. $$
Assume from now that \((X_n)_{n \geq 0}\) is transient to the right. Notice that the walk spends only a finite amount of time on \(-N\) and thus

\[
T_n \sim n + 2 \sum_{i=0}^{n-1} V_{i,1}^n + \sum_{i=0}^{n-1} \sum_{\ell=2}^{L} V_{i,\ell}^n \quad \text{as} \quad n \to \infty, \quad \text{a.s.} \tag{2}
\]

It is worth mentioning that the above hitting time decomposition was mentioned by Hong and Wang [10], in which they studied random walks in random environment with non-nearest-neighbour jumps to the left. The idea can be traced back to the well-known Kesten–Kozlov–Spitzer hitting time decomposition for nearest-neighbour random walks in random environments [13].

Let \((\xi_n)_{n \geq 1}\) be a sequence of independent random unit vectors such that the distribution of \(\xi_n = (\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,L+1})\) is given by

\[
\mathbb{P}(\xi_n = e_\ell) = \begin{cases} 
q_n(-\ell), & \text{if } 1 \leq n \leq M \text{ and } 1 \leq \ell \leq L, \\
q_n(1), & \text{if } 1 \leq n \leq M \text{ and } \ell = L + 1, \\
v(-\ell), & \text{if } n > M, 1 \leq \ell \leq L, \\
v(1), & \text{if } n > M, \ell = L + 1
\end{cases}
\]

where \(e_\ell\) with \(\ell \in [L+1]\) is the standard basis of \(\mathbb{R}^{L+1}\). If \(\xi_n = e_\ell\) with \(\ell \in [L]\), we say that the outcome of the \(n\)-th experiment is an \(\ell\)-th type failure. Otherwise, if \(\xi_n = e_{L+1}\), we say that it is a success.

For \(m \in \mathbb{Z}_+\), we define the random vector \(A(m) = (A_1(m), \ldots, A_L(m))\) such that for \(\ell \in [L]\),

\[
A_\ell(m) := \sum_{i=1}^{\gamma_m} \xi_{i,\ell}, \quad \text{with} \quad \gamma_m = \inf \left\{ n \geq 1 : \sum_{i=1}^{n} \xi_{i,L+1} = m + 1 \right\}. \tag{3}
\]

In other words, the random variable \(A_\ell(m)\) is the total number of \(\ell\)-th type failures before obtaining \(m+1\) successes.

Let \((A^{(n)}(m))_{m \in \mathbb{Z}_+}\), with \(n \in \mathbb{N}\), be i.i.d. copies of the process \((A(m))_{m \in \mathbb{Z}_+}\). We define a \(L\)-dimensional process \(Z = (Z_n)_{n \geq 0} = (Z_{n,1}, Z_{n,2}, \ldots, Z_{n,L})_{n \geq 0}\) such that \(Z_0 \in \mathbb{Z}_+^L\) is independent of \((A^{(n)}(m))_{n \in \mathbb{N}, m \in \mathbb{Z}_+}\) and for \(n \geq 1\),

\[
Z_n = A^{(n)}(\lfloor Z_{n-1} \rfloor) + (Z_{n-1}, Z_{n-1,3}, \ldots, Z_{n-1,L}, 0) \tag{4}
\]

where for \(z = (z_1, z_2, \ldots, z_L) \in \mathbb{R}^L\), we denote \(|z| = z_1 + z_2 + \cdots + z_L\). Therefore \(Z = (Z_n)_{n \geq 0}\) is a Markov chain in \(\mathbb{Z}_+^L\) and its transition law is given by

\[
\mathbb{P}(Z_{n+1} = (k_1, k_2, \ldots, k_L) \mid Z_n = (j_1, j_2, \ldots, j_L)) = \mathbb{P} \left( A_1 \left( \sum_{\ell=1}^{L} j_\ell \right) = k_1 - j_2, \ldots, A_{L-1} \left( \sum_{\ell=1}^{L} j_\ell \right) = k_{L-1} - j_L, A_L \left( \sum_{\ell=1}^{L} j_\ell \right) = k_L \right)
\]

for \((k_1, k_2, \ldots, k_L), (j_1, j_2, \ldots, j_L) \in \mathbb{Z}_+^L\).

**Proposition 2.1** Assume that \(Z_{0, \ell} = 0\) for all \(\ell \in [L]\). Then for each \(n \in \mathbb{N}\), we have that \((V_{n-1}^n, V_{n-2}^n, \ldots, V_0^n)\) has the same distribution as \((Z_0, Z_1, \ldots, Z_{n-1})\).

**Proof** A backward jump is called \(\ell\)-th type of level \(i\) if it is a backward jump from a site in \(i + N\) to site \(i - \ell + 1\). Recall that \(V_{i,\ell}^n\) is the number of \(\ell\)-th type backward jumps of level \(i\) before time \(T_n\). Assume that \((V^n_{i,\ell} = (V^n_{i,1,\ell}, \ldots, V^n_{i,L,\ell}) = (j_1, j_2, \ldots, j_L))\). The number of forward jumps from \(i\) to \(i+1\) before time \(T_n\) is thus equal to \(1 + \sum_{\ell=1}^{L} V_{i,\ell}^n = 1 + \sum_{\ell=1}^{L} j_\ell\).
For each \( i \in \mathbb{Z} \), denote by \( T_i^{(k)} \) the time for \( k \)-th forward jump from \( i - 1 \) to \( i \) and also set \( T_i^{(0)} = 0 \). We have that \( T_i^{(1)} = T_i \). Moreover, as the process \( X \) is transient, we have that only finitely many \( (T_i^{(k)})_k \) are finite, and conditioning on \( \{V_i^n = (j_1, j_2, \ldots, j_L)\} \), we have that \( T_i^k < \infty \) for \( k \leq 1 + \sum_{s=1}^L f_s \).

Note that \( V_{i-1,\ell} \), i.e. the number of \( \ell \)-th type backward jumps of level \( i - 1 \) before time \( T_n \), is equal to the sum of \( \ell \)-th type backward jumps of level \( i - 1 \) during \([T_{i+1}^{(k-1)}, T_i^{(k)} - 1] \) for \( k \in [1 + \sum_{k=1}^L f_k] \).

By the definition of \( T_i^{(k)} \), the walk will visit \( i \) at least once during the time interval \([T_{i+1}^{(k-1)}, T_i^{(k)} - 1] \). Whenever the walk visits \( i \), it will make a forward jump from \( i \) to \( i + 1 \) (which corresponds to a success) or a backward jump from \( i \) to \( i - \ell \), i.e. a \( \ell \)-th type jump of level \( i - 1 \), with \( \ell \in [L] \). If the latter happens, then \( i \) will be visited again during \([T_{i+1}^{(k-1)}, T_i^{(k)} - 1] \). Moreover, an \( \ell \)-th type backward jump of level \( i \) is also an \((\ell - 1)\)-th type backward jump of level \( i - 1 \). Thus conditionally on \( \{(V_{i,1}^n, \ldots, V_{i,L}^n) = (j_1, j_2, \ldots, j_L)\} \), the random vector \( (V_{i-1,1}^n, V_{i-1,2}^n, \ldots, V_{i-1,L}^n) \) has the same distribution as

\[
\left( A_1 \left( \sum_{\ell=1}^L j_\ell \right) + j_2, \ldots, A_{L-1} \left( \sum_{\ell=1}^L j_\ell \right) + j_L, A_L \left( \sum_{\ell=1}^L j_\ell \right) \right).
\]

\( \square \)

Recall that \( M \) is the total number of cookies initially placed on each site. By the definition of sequence \( (A(m))_{m \in \mathbb{Z}^+} \) given in (3), we can easily obtain the following.

**Proposition 2.2** For \( m \geq M - 1 \), we have

\[ A(m) = A(M - 1) + \sum_{k=1}^{m-M+1} \eta_k \]

where \( (\eta_k)_{k \geq 1} \) are i.i.d. random vectors independent of \( A(M - 1) \) with multivariate geometrical law

\[
\mathbb{P}(\eta_1 = (i_1, i_2, \ldots, i_L)) = \frac{v(1)}{(i_1 + i_2 + \cdots + i_L)!} \prod_{k \in [L]} i_k! v(-k)^k. \tag{5}
\]

In the above formula, we use the convention that \( 0^0 = 1 \).

**Remark 2.1** The multivariate Markov chain \( Z \) defined in (4) can be interpreted as a multi-type branching process with (random) migration as follows. Let \( M' \) be a fixed integer such that \( M' \geq M - 1 \) and suppose that \( Z_{n-1} = j = (j_1, j_2, \ldots, j_L) \). Then

- If \( |j| = j_1 + j_2 + \cdots + j_L \geq M' \), we have

\[
Z_n = A^{(n)}(M') + \sum_{k=M'-M+2}^{|j|-M+1} \eta_k^{(n)} + \tilde{\eta}
\]

where \( \tilde{\eta} := (j_2, \ldots, j_L, 0) \); \( \eta_k^{(n)} \) with \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \) are i.i.d. random vectors with the multivariate geometrical law defined in (5); and for each \( n \in \mathbb{N} \), \( \eta_k^{(n)} \) is independent of \( A^{(n)}(M') \) and \( (Z_k)_{0 \leq k \leq n-1} \). In this case, there is an emigration of \( M' \) particles (each particle of any type has the same possibility to emigrate) while all the remaining \(|j| - M' \) particles reproduce according to the multivariate geometrical law defined in (5). For each
\( \ell \in [L] \), there is also an immigration of \( A^{(n)}_\ell (M') + j_{\ell + 1} \) new \( \ell \)-th type particles (here we use the convention that \( j_{L + 1} = 0 \)).

- If \(|j| < M'\), we have \( Z_n = A^{(n)}(|j|) + \tilde{j} \). In this case, for each \( \ell \in [L] \), all \( j \) particles of \( \ell \)-th type emigrate while \( A^{(n)}_\ell (|j|) + j_{\ell + 1} \) new particles of \( \ell \)-th type immigrate.

**Proposition 2.3** The Markov chain \( Z \) is ergodic.

**Proof** Recall from (4) that for each \( n \in \mathbb{Z} \),

\[ Z_n = A^{(n)}(|Z_{n - 1}|) + (Z_{n - 1}, 2, Z_{n - 1}, 3, \ldots, Z_{n - 1}, L, 0). \]

Taking \( L \) iterations, we have that for \( n \geq L \),

\[ Z_n = \left( \sum_{k=1}^{L} A^{(n-k+1)}_k(|Z_{n-k}|), \ldots, A^{(n)}_{L-1}(|Z_{n-1}|) \right) \]

Define \( L' = \max(\ell \in [L] : \nu(-\ell) > 0) \) and \( d_\ell = \text{Card}\{k \in [M] : q_k(-\ell) > 0\} \). Set \( S = S_1 \times S_2 \times \cdots \times S_L \) where

\[ S_\ell = \begin{cases} \mathbb{Z}_+ & \text{for } 1 \leq \ell \leq L', \\ \{0, 1, \ldots, \sum_{s=\ell}^{L} d_s\} & \text{for } L' + 1 \leq \ell \leq L. \end{cases} \]

It is evident from from (6) that for \( n \geq L \), the support of \( Z_n \) is equal to \( S \). In particular, \( \mathbb{P}(Z_L = j | Z_0 = i) > 0 \) and \( \mathbb{P}(Z_{L+1} = j | Z_0 = i) > 0 \) for any \( j \in S, i \in \mathbb{Z}_+^L \). Hence \( Z \) is irreducible and aperiodic. Using Proposition 2.1, we have that conditional on \( \{Z_0 = (0, 0, \ldots, 0)\} \), \( Z_{n-1} \) has the same distribution as \( V_0^n \). As the process \( X \) is transient, \( V_0^n \) convergences almost surely to \( V_0^\infty = (V_0^\infty, V_0^\infty, \ldots, V_L^\infty) \) where \( V_0^\infty \) is the total number of jumps from a site in \( \mathbb{N} \) to site \( -\ell + 1 \). Hence \( Z_n \) converges in law to some a.s. finite random vector \( Z_\infty \) as \( n \to \infty \). This implies that \( Z \) is positive recurrent. Hence \( Z \) is ergodic. \( \square \)

### 3 Critical Multi-Type Galton–Watson Branching Process with Emigration

In this section, we prove a limit theorem (see Theorem 3.1 below) for critical multi-type Galton–Watson processes with emigration. This result will be used to solve the critical case \( \delta = 2 \) in Sect. 4.

**Definition 1** Let \( N = (N_1, N_2, \ldots, N_L) \) be a vector of \( L \) deterministic positive integers and \((\psi(k, n))_{k,n \in \mathbb{N}}\) be a family of i.i.d. copies of a random matrices \( \psi \) such that \( \psi \) takes values in \( \mathbb{Z}_+^{L \times L} \) and its rows are independent. Let \((U(n))_{n \geq 0}\) be a Markov chain in \( \mathbb{Z}_+^L \) defined recursively by

\[ U_j(n) = \sum_{i=1}^{L} \sum_{k=1}^{\psi_i(U(n-1))} \psi_{i,j}(k, n) \quad \text{for } j \in [L], n \in \mathbb{N} \]

where \( \psi_i(s) = (s_i - N_i) \mathbb{I}_{\{s_j \geq N_j, \forall j \in [L]\}} \). We call \((U(n))_{n \geq 0}\) a multi-type Galton–Watson branching process with \((N_1, N_2, \ldots, N_L)\)-emigration.
We can interpret the branching process $(U(n))_{n \geq 0}$ defined above as a model of a population with $L$ different types of particles in which $U_i(n)$ stands for the number of particles of type $i$ in generation $n$. The number of offsprings of type $j$ produced by a particles of type $i$ has the same distribution as $\psi_{i,j}$. In generation $n$, if $U_i(n) \geq N_i$ for all $i \in [L]$ then there is an emigration of $N_i$ particles of type $i$ for $i \in [L]$, otherwise all the particles emigrate and $U(n + 1) = (0, 0, \ldots, 0)$.

Let $|\cdot|$, $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ stand for the 1-norm, the Euclidean norm and the Euclidean scalar product on $\mathbb{R}^L$ respectively. Denote $\mathbf{0} = (0, 0, \ldots, 0)$, $1 = (1, 1, \ldots, 1)$.

From now on, we always assume that $(U(n))_{n \geq 0}$ satisfies the following condition.

**Assumption B** The branching process $(U(n))_{n \geq 0}$ is critical, i.e. the expected offspring matrix $\mathbb{E}[\psi]$ is positively regular (in the sense that there exists $n \in \mathbb{N}$ such that $(\mathbb{E}[\psi])^n$ is a positive matrix) and $\lambda = 1$ is its largest eigenvalue in modulus.

By Perron-Frobenius theorem, the maximal eigenvalue 1 is simple and has the positive right and left eigenvectors $u = (u_1, \ldots, u_L)$, $v = (v_1, \ldots, v_L)$ which are uniquely defined such that $\langle u, v \rangle = 1$ and $|u| = 1$. Furthermore, other eigenvalue than 1 is strictly smaller than 1 in modulus.

Set

$$\sigma_{i,j}(k) = \mathbb{E}[\psi_{ki}\psi_{kj} - \delta_{i,j}\psi_{kj}], \quad \beta = \frac{1}{2} \sum_{i,j,k \in [L]} v_k u_i \sigma_{ij}(k) u_j \quad \text{and} \quad \theta = \frac{\langle N, u \rangle}{\beta}.$$

We will prove the following theorem, which is a multivariate extension of the limit theorem for critical (one-type) branching processes with emigration obtained in [23], [25] and [12] (see also [24] for a literature review).

**Theorem 3.1** Let $K = (K_1, K_2, \ldots, K_L)$ be a deterministic vector such that $K_i \geq N_i$ for all $i \in [L]$. Assume that $U(0) = K$ a.s. and there exists $\epsilon > 0$ such that

$$\mathbb{E}[\psi_{1,\theta}^{1+\lceil \theta \rceil \vee (1+\epsilon)}] < \infty \quad \text{for all} \quad i, j \in [L]$$

(7)

where we denote by $[\theta]$ the ceiling value of $\theta$. Then the followings hold true:

a. There exists a constant $\varrho > 0$ such that

$$\mathbb{P}(U(n) \neq \mathbf{0}) \sim \frac{\varrho}{n^{1+\theta}} \quad \text{as} \quad n \to \infty.$$

b. We have

$$\lim_{n \to \infty} \frac{\mathbb{E}[[U_n | U(n) \neq \mathbf{0}]]}{n} \geq \beta.$$

For each $n \in \mathbb{Z}_+$ and $s = (s_1, s_2, \ldots, s_L) \in [0, 1]^L$, we set

$$F(s, n) := \mathbb{E} \left[ \prod_{\ell=1}^L s_{\ell}^{U_{\ell}(n)} \right], \quad f(s) := (f_1(s), \ldots, f_L(s)) \quad \text{with} \quad f_i(s) := \mathbb{E} \left[ \prod_{\ell=1}^L s_{\ell}^{\psi_{i,\ell}} \right],$$

which stand respectively for the multivariate probability generating functions of $U(n)$ and the random row vectors $(\psi_{1,\ell})_{\ell=1}^L, (\psi_{2,\ell})_{\ell=1}^L, \ldots, (\psi_{L,\ell})_{\ell=1}^L$.

Let $f^k = (f_1^k, f_2^k, \ldots, f_L^k)$ be the $k$-th iteration of $f$, i.e. $f^0(s) = s$ and $f^{k+1}(s) = f(f^k(s))$ for $k \geq 0$. We also set

$$g(s) := \prod_{\ell=1}^L s_{\ell}^{-N_{\ell}} \quad \text{and} \quad \gamma_n(s) := \prod_{k=1}^n g(f^k(s)), \quad \text{with} \quad \gamma_0(s) = 1.$$
In order to prove Theorem 3.1, we will need the following lemmas.

**Lemma 3.2** Assume that there exists \( \varepsilon > 0 \) such that

\[
\mathbb{E}[\psi_{i,j}^{2+\varepsilon}] < \infty, \quad \text{for all } i, j \in [L].
\]

Then there exists a positive constant \( C \) such that

\[
\gamma_n(0) \sim Cn^\theta \quad \text{as } n \to \infty
\]

and

\[
\sum_{n=0}^{\infty} \gamma_n(0)z^n \sim \frac{\Gamma(\theta + 1)C}{(1-z)^{\theta+1}} \quad \text{as } z \to 1^{-}
\]

where \( \Gamma \) is the Gamma function defined by \( \Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du \) for \( x > 0 \).

**Proof** We denote by \( 1 \) the \( L \)-dimensional vector with all entries equal to 1. Note that

\[
1 - f^n(0) \sim \frac{u}{\beta n}, \quad \text{as } n \to \infty.
\]

(see, e.g., Corollary V.5, p. 191 in [1]). Set \( r_n := \|1 - f^n(0) - \frac{u}{\beta n}\| \). We first show that

\[
\sum_{n=1}^{\infty} r_n < \infty.
\]

(10)

Indeed, let \( Q(s) = (Q_1(s), Q_2(s), \ldots, Q_L(s)) \) be a vector of quadratic forms with

\[
Q_k(s) := \frac{1}{2} \sum_{i,j \in [L]} \sigma_{ij}(k)s_is_j
\]

and set

\[
a(s) := \left< v, Q\left( \frac{1 - s}{\langle v, 1 - s \rangle} \right) \right>, \quad d(s) := \frac{1}{\langle v, 1 - s \rangle} + a(s) - \frac{1}{\langle v, 1 - f(s) \rangle}.
\]

We have

\[
\frac{1}{\langle v, 1 - f^n(s) \rangle} - \frac{1}{\langle v, 1 - s \rangle} = \sum_{k=0}^{n-1} a(f^k(s)) - \sum_{k=0}^{n-1} d(f^k(s)).
\]

(11)

In virtue of Taylor’s expansion, we have \( 1 - f(s) = (\mathbb{E}[\psi] - H(s)) \cdot (1 - s) \) with \( H(s) = O(\|1 - s\|) \). It follows that

\[
1 - f^n(0) = (\mathbb{E}[\psi] - H(f^{n-1}(0))) \ldots (\mathbb{E}[\psi] - H(f(0))) (\mathbb{E}[\psi] - H(0)) \cdot 1.
\]

(12)

Using (9), we note that \( \|H(f^n(0))\| = O(\|1 - f^n(0)\|) = O(1/n) \). Moreover, \( \mathbb{E}[\psi]^n = uv^T + O(|\lambda|^n) \) where \( \lambda \) (with \( |\lambda| < 1 \)) is the eigenvalue of \( \mathbb{E}[\psi] \) with the second largest modulus. In what follows, we denote by \( C_{st} \) a positive constant but its value may vary from line to line. Applying inequality (4.11) in [11] to (12), we deduce that

\[
\left\| \frac{1 - f^n(0)}{\langle v, 1 - f^n(0) \rangle} - \frac{u}{n} \right\| \leq C_{st}.
\]

(13)
Since $Q(s)$ is Lipschitz, we thus have
\[
\|a(f^n(0)) - \langle v, Q(u) \rangle\| \leq \|v\| \cdot \|Q\left(\frac{1 - f^n(0)}{\langle v, 1 - f^n(0) \rangle}\right) - Q(u)\| \leq \frac{\text{Cst}}{n}.
\]

As a result, we have
\[
\sum_{k=0}^{n-1} a(f^k(0)) = \langle v, Q(u) \rangle n + O(\log(n)) = \beta n + O(\log(n)). \tag{14}
\]

W.l.o.g., we assume that $\varepsilon \in (0, 1)$ (which satisfies (8)). By Taylor’s expansion, there exists a vector function $E(t, s)$ such that
\[
1 - f(s) = E\left(\psi, 1 - f^n(0)\right) \rightarrow u \text{ as } n \rightarrow \infty. \tag{17}
\]

By reason of (13), we notice that
\[
1 - f^n(0) = \langle v, 1 - f^n(0) \rangle = \beta n + O(\log(n)). \tag{18}
\]

Combining (14) with (15)-(17) and using Cauchy-Schwarz inequality, we obtain
\[
d(f^n(0)) = Cst\|1 - f^n(0)\| = O(n^{-1}) \quad \text{and} \quad d(f^n(0)) \leq Cst\|1 - f^n(0)\|^2 = O(n^{-\varepsilon}). \tag{18}
\]

Combining (11) with (14) and (18), we get
\[
\langle v, 1 - f^n(0) \rangle = \frac{1}{|v|^{-1} + \sum_{k=0}^{n-1} a(f^k(0)) - \sum_{k=0}^{n-1} d(f^k(0))} = \frac{1}{\beta n} + O(n^{-1-\varepsilon}).
\]

Consequently,
\[
1 - f^n(0) = \langle v, 1 - f^n(0) \rangle \cdot \frac{1 - f^n(0)}{\langle v, 1 - f^n(0) \rangle} = \left(\frac{1}{\beta n} + O(n^{-1-\varepsilon})\right)\left(u + O(n^{-1})\right) = \frac{u}{\beta n} + O(n^{-1-\varepsilon}). \tag{19}
\]

Hence $r_n = O(n^{-1-\varepsilon})$ and (10) is thus proved. On the other hand, by Taylor’s expansion, we have
\[
g(f^k(0)) = 1 + \langle N, 1 - f^k(0) \rangle + O(\|1 - f^k(0)\|^2). \tag{20}
\]
Thus
\[ \gamma_n(0) = \prod_{k=1}^{n} \left( 1 + \langle N, 1 - f^k(0) \rangle + O(1 - f^k(0))^2 \right) \]
\[ \sim \text{Cst} \cdot \exp \left( \sum_{k=1}^{n} \left[ \langle N, 1 - f^k(0) \rangle + O(k^{-2}) \right] \right) \]
\[ \sim \text{Cst} \cdot \exp \left( \frac{\langle N, u \rangle}{\beta} \sum_{k=1}^{n} \left( \frac{1}{k} + O(k^{-(1+\varepsilon)}) \right) \right). \]

Since \( \sum_{k=1}^{n} 1/k = \log(n) + O(1) \) as \( n \to \infty \) and \( \theta = \langle N, u \rangle / \beta \), we obtain that \( \gamma_n(0) \sim Cn^\theta \)
for some positive constant \( C \). Furthermore, by Hardy–Littlewood tauberian theorem for power series (see e.g. Theorem 5, Section XIII.5, p. 447 in [8]), we deduce that
\[ \sum_{n=0}^{\infty} \gamma_n(0) z^n \sim \frac{\Gamma(\theta + 1) C}{(1-z)^{\theta+1}} \quad \text{as} \quad z \to 1^- . \]

\[ \square \]

In what follows, for \( x, y \in \mathbb{Z}^L_+ \) we write \( x \succeq y \) if \( x_i \geq y_i \) for all \( i \in [L] \), otherwise we write \( x \prec y \). Set \( S(N) = \{ r \in \mathbb{Z}^L_+ \setminus \{0\} : r \npreceq N \} \). For each \( r \in \mathbb{Z}^L_+ \) and \( s = (s_1, s_2, \ldots, s_L) \in \mathbb{R}^L \) such that \( s_\ell \neq 0 \) for all \( \ell \in [L] \), define
\[ H_r(s) := \left( \prod_{\ell=1}^{L} s_\ell^{r_\ell - N_\ell} - 1 \right) \mathbb{1}_{\{r \in S(N)\}}. \]

For each \( n \in \mathbb{Z}^L_+ \) and \( z \in [0, 1] \), set
\[ \mu_n := \mathbb{P}(U(n) \neq 0) = 1 - F(0, n) \quad \text{and} \quad Q(z) := \sum_{n=0}^{\infty} \mu_n z^n. \]

\textbf{Lemma 3.3} \textit{The generating function of} \( (\mu_n)_{n \geq 0} \) \textit{is given by}
\[ Q(z) = \frac{B(z)}{D(z)}, \]
\textit{in which we define}
\[ B(z) := \sum_{n=0}^{\infty} \left( 1 - F(f^n(0), 0) + \sum_{k=1}^{\infty} \mathbb{E} \left[ H_{U(k-1)}(f^{n+1}(0)) \right] z^k \right) \gamma_n(0) z^n \quad \text{and} \]
\[ D(z) := (1 - z) \sum_{n=0}^{\infty} \gamma_n(0) z^n. \]

\textbf{Proof} \textit{From the definition of} \( U(n) \), \textit{we have}
\[ F(s, n) = \mathbb{E} \left[ \prod_{i=1}^{L} \prod_{k=1}^{\psi_i(U_i(n-1))} \prod_{\ell=1}^{L} s_\ell^{\psi_i(U_i(k,n))} \right] = \mathbb{E} \left[ \prod_{i=1}^{L} (f_i(s))^{\psi_i(U_i(n-1))} \right] \]
\[ = \sum_{r \in \mathbb{Z}^L_+, r \succeq N} \mathbb{P}(U(n-1) = r) \prod_{i=1}^{L} (f_i(s))^{r_i - N_i} + \mathbb{P}(U(n-1) = 0) \]
Denote by \( P \). Note that

\[
\sum_{n=1}^{\infty} \mathbb{P}(U(n-1) = r) = F(f(s), n-1)g(f(s)) - F(0, n-1)(g(f(s)) - 1) \\
- \sum_{r \in \mathbb{Z}_+^L} \mathbb{P}(U(n-1) = r)H_r(f(s)).
\]

Consequently,

\[
F(s, n) = F(f^n(s), 0)\gamma_n(s) - \sum_{k=1}^{n} F(0, n-k)(\gamma_k(s) - \gamma_{k-1}(s)) \\
- \sum_{r \in \mathbb{Z}_+^L} \sum_{k=1}^{n} \mathbb{P}(U(n-k) = r)H_r(f^k(s))\gamma_{k-1}(s).
\]

Note that \( \mu_n = 1 - F(0, n) \). Substituting \( s = 0 \) into (21), we obtain

\[
\mu_n + \sum_{k=1}^{n} \mu_{n-k}(\gamma_k(0) - \gamma_{k-1}(0)) = (1 - F(f^n(0), 0))\gamma_n(0) \\
+ \sum_{r \in \mathbb{Z}_+^L} \sum_{k=1}^{n} \mathbb{P}(U(n-k) = r)H_r(f^k(0))\gamma_{k-1}(0).
\]

Multiplying both sides of the above equation by \( z^n \) and summing over all \( n \geq 0 \), we get

\[
(1 - z)\left( \sum_{n=0}^{\infty} \gamma_n(0)z^n \right)\left( \sum_{n=1}^{\infty} \mu_n z^n \right) = \sum_{n=0}^{\infty} (1 - F(f^n(0), 0))\gamma_n(0)z^n \\
+ \sum_{r \in \mathbb{Z}_+^L} \sum_{k=1}^{n} \mathbb{P}(U(k-1) = r)z^k \sum_{n=0}^{\infty} H_r(f^{n+1}(0))\gamma_n(0)z^n.
\]

This ends the proof of the lemma. \( \square \)

Let \( (p_n)_{n \geq 0} \) be a non-negative sequence such that its generating function \( P(z) = \sum_{n=0}^{\infty} p_n z^n \) has radius of converge 1. For each \( k \in \mathbb{Z}_+ \), we define the sequence \( (p_n^{(k)})_{n \geq 0} \) recursively by

\[
p_n^{(0)} = p_n \quad \text{and} \quad p_n^{(k+1)} = \sum_{j=n+1}^{\infty} p_j^{(k)} \quad \text{for} \quad k \in \mathbb{Z}_+.
\]

Denote by \( P^{(k)}(z) = \sum_{n=0}^{\infty} p_n^{(k)} z^n \) the generating function of \( (p_n^{(k)})_{n \geq 0} \). Assume that \( P^{(j)}(1^-) < \infty \) for all \( 0 \leq j \leq k \). By Abel’s theorem, we notice that \( P^{(j)}(1^-) = P^{(j)}(1) = \sum_{n=0}^{\infty} p_n^{(j)} \) and

\[
P^{(j+1)}(z) = \frac{P^{(j)}(1^-) - P^{(j)}(z)}{1-z} \quad \text{for} \quad z \in (-1, 1), 0 \leq j \leq k.
\]

The next lemma is derived directly from Corollary 2 and Lemma 5 in [23].

**Lemma 3.4** Let \( (p_n)_{n \geq 0} \) be a non-negative non-increasing sequence and \( p_n \to 0 \) as \( n \to \infty \).
Lemma 3.5 We have
\[ \sum_{k=1}^{\infty} k p_k \sim n^{2-\alpha} \text{ as } n \to \infty \text{ for some } 0 < \alpha < 2. \]

By induction, we obtain that
\[ \sum_{k=1}^{\infty} k p_k \sim \sigma n^{\alpha-1} \text{ as } n \to \infty. \]

This ends the proof of the lemma.

Lemma 3.6 Assume that the condition (7) is fulfilled. Then:

i. For each \( k \in \{0, 1, \ldots, \tilde{\theta} - 1\} \), there exist power series \( R_D^{(k)}(z) \), \( D_m^{(k)}(z) \) and non-zero constants \( d_m^{(k)} \), \( m \in \{k+1, \ldots, \tilde{\theta}\} \) such that as \( z \to 1^- \),

\[
R_D^{(k)}(z) \sim \begin{cases} 
\text{Cst} \cdot \ln \left( \frac{1}{1-z} \right) & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise}, 
\end{cases}
\]

\[
D_m^{(k)}(z) \sim \Gamma(\theta - m + 1)d_m^{(k)}(1-z)^{-(\theta-m+1)} \text{ and }
\]

\[
(1-z)^k D(z) = \sum_{m=k+1}^{\tilde{\theta}} D_m^{(k)}(z) + R_D^{(k)}(z).
\]
ii. There exist power series \( R_B(z), B_m(z) \) and \( b_m(z) \) with \( m \in \{1, 2, \ldots, \bar{\theta}\} \) such that \( |b_m(1^-)| < \infty \),

\[
R_B(z) \sim \begin{cases} 
\text{Cst} \cdot \ln\left(\frac{1}{1-z}\right), & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise},
\end{cases} 
\]

(27)

\[
B_m(z) \sim \Gamma(\theta - m + 1)b_m(z)(1-z)^{-(\theta-m+1)} \quad \text{and}
\]

(28)

\[
B(z) = \sum_{m=1}^{\bar{\theta}} B_m(z) + R_B(z)
\]

(29)

as \( z \to 1^- \).

**Proof** For \( k \in \mathbb{N}, r \in \mathbb{Z}^L \setminus \{0\} \) and \( j = (j_1, \ldots, j_L) \in \mathbb{Z}_+^L \) with \( |j| \leq \bar{\theta} + 1 \), we set

\[
c_{j,r,k} := \frac{1}{j_1! \cdots j_L!} \left. \frac{\partial^{j_1+\cdots+j_L} \prod_{\ell=1}^{L} (f_{\ell}^{n-\bar{\theta}+k}(0))^r_{\ell} \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\bar{\theta}}(0))^{j_{\ell}}}{\partial s_1^{j_1} \cdots \partial s_d^{j_d}} \right|_{s=1}
\]

(30)

which is well-defined thanks to the condition (7). Using Taylor’s expansion, we have that for \( r = (r_1, \ldots, r_L) \in \mathbb{Z}^L \setminus \{0\}, n \geq \bar{\theta} \) and \( k \geq 1, \)

\[
\prod_{\ell=1}^{L} (f_{\ell}^{n-\bar{\theta}+k}(0))^r_{\ell} = 1 + \sum_{1 \leq |j| \leq \bar{\theta}} (-1)^{|j|} c_{j,r,k} \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\bar{\theta}}(0))^{j_{\ell}} + o_{n,r,k}
\]

with \( o_{n,r,k} \sim (-1)^{\bar{\theta}+1} \sum_{|j| = \bar{\theta}+1} c_{j,r,k} \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\bar{\theta}}(0))^{j_{\ell}} \) as \( n \to \infty \).

By reason of (9), we notice that

\[
o_{n,r,k} \sim (-1/\beta)^{\bar{\theta}+1} n^{-\bar{\theta}+1} \sum_{|j| = \bar{\theta}+1} c_{j,r,k} \prod_{\ell=1}^{L} u_{\ell}^{j_{\ell}}.
\]

(32)

Recall from Lemma 3.2 that \( \gamma_n(0) \sim C n^\theta \). Using Hardy–Littlewood tauberian theorem for power series, we thus have that as \( z \to 1^- \),

\[
\sum_{n=0}^{\infty} o_{n,r,k} \gamma_n(0) z^n \sim \begin{cases} 
\text{Cst} \cdot \ln\left(\frac{1}{1-z}\right), & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise}.
\end{cases}
\]

(33)

**Part i.** Notice that for \( n \in \mathbb{N}, \gamma_n(0) - \gamma_{n-1}(0) = \gamma_n(0) \left(1 - \prod_{\ell=1}^{L} (f_{\ell}^{n}(0))^{N_{\ell}}\right) \) and thus

\[
D(z) = (1-z) \sum_{n=0}^{\infty} \gamma_n(0) z^n = \gamma_0(0) + \sum_{n=1}^{\infty} (\gamma_n(0) - \gamma_{n-1}(0)) z^n
\]

\[
= \sum_{n=0}^{\infty} \left(1 - \prod_{\ell=1}^{L} (f_{\ell}^{n}(0))^{N_{\ell}}\right) \gamma_n(0) z^n.
\]

(34)

By induction, we easily obtain that for \( k \in \{0, 1, \ldots, \bar{\theta} - 1\}, \)

\[
(1-z)^k D(z) = (1-z)^{k+1} \sum_{n=0}^{\infty} \gamma_n(0) z^n = \sum_{n=0}^{k} \sum_{m=0}^{k} \left(1 - \prod_{\ell=1}^{L} (f_{\ell}^{n-m}(0))^{N_{\ell}}\right) \gamma_n(0) z^n
\]

(35)
where we use the convention that \( f^h(0) = 0 \) for \( h \leq 0 \). Using (31) and (35), we have that for \( k \in \{0, 1, \ldots, \tilde{\theta} - 1 \),

\[
(1 - z)^k D(z) = \sum_{n=\tilde{\theta}}^{\infty} \left[ \sum_{k+1 \leq j \leq \tilde{\theta}+1} \chi_j^{[k]} \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\tilde{\theta}}(0))^{j_\ell} + o(\|1 - f^{n-\tilde{\theta}}(0)\|^{1 + \tilde{\theta}}) \right] \gamma_n(0) z^n
\]

\[
+ \sum_{n=0}^{\tilde{\theta}-1} \prod_{m=0}^{k} \left( 1 - \prod_{\ell=1}^{L} (f_{\ell}^{m-n}(0))^{N_\ell} \right) \gamma_n(0) z^n
\]

where for \( k \in \mathbb{Z}_+ \) and \( j \in \mathbb{Z}_+^L \), we set

\[
\chi_j^{[k]} := (-1)^{|j|+k+1} \sum_{\substack{j^{(m)} \in \mathbb{Z}_+^L \setminus \{0\}, \ j^{(0)} + \cdots + j^{(k)} = j \}} \prod_{m=0}^{k} c_{j^{(m)}, m, \tilde{\theta} - m}.
\]

In virtue of Lemma 3.2 and (9), we note that

\[
\gamma_n(0) \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\tilde{\theta}}(0))^{j_\ell} \sim C \beta^{-|j|} \left( \prod_{\ell=1}^{L} u_{\ell}^{j_\ell} \right) n^{\tilde{\theta} - |j|} \quad \text{as } n \to \infty.
\]

For \( 1 \leq m \leq \tilde{\theta} + 1 \) and \( 0 \leq k \leq \tilde{\theta} - 1 \), set

\[
d_m^{[k]} := C \beta^{-m} \sum_{|j|=m} \chi_j^{[k]} \prod_{\ell=1}^{L} u_{\ell}^{j_\ell} \quad \text{and} \quad D_m^{[k]} := \gamma_n(0) \sum_{|j|=m} \chi_j^{[k]} \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\tilde{\theta}}(0))^{j_\ell}.
\]

Note that \( D_m^{[k]} \sim d_m^{[k]} n^{\tilde{\theta} - m} \) as \( n \to \infty \). For \( m \in \{k + 1, \ldots, \tilde{\theta} \} \), define \( D_m^{[k]}(z) := \sum_{n=\tilde{\theta}}^{\infty} D_m^{[k]}(0) z^n \). In view of Hardy–Littlewood tauberian theorem for power series, we thus obtain (24), (25) and (26).

Part ii. Recall that

\[
B(z) = \sum_{n=0}^{\infty} \left[ 1 - F(f_{\ell}^{n}(0), 0) \right] + \sum_{r \in \mathcal{S}(N)} H_r(f_{\ell}^{n+1}(0)) \Pi_r(z) \right] \gamma_n(0) z^n
\]

in which we have

\[
F(f_{\ell}^{n}(0), 0) = \prod_{\ell=1}^{L} (f_{\ell}^{n}(0))^{K_\ell} \quad \text{and} \quad H_r(f_{\ell}^{n+1}(0)) = \prod_{\ell=1}^{L} (f_{\ell}^{n+1}(0))^{r_\ell - N_\ell} - 1 \quad \text{for } r \in \mathcal{S}(N).
\]

Using (31), we obtain that

\[
B(z) = \sum_{n=\tilde{\theta}}^{\infty} \left[ \sum_{0 \leq |j| \leq \tilde{\theta}} a_j(z) \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\tilde{\theta}}(0))^{j_\ell} + o_R(z) \right] \gamma_n(0) z^n
\]

\[
+ \sum_{n=0}^{\tilde{\theta}-1} \left[ 1 - F(f_{\ell}^{n}(0), 0) \right] + \sum_{r \in \mathcal{S}(N)} H_r(f_{\ell}^{n+1}(0)) \Pi_r(z) \right] \gamma_n(0) z^n
\]

where we set

\[
a_j(z) := (-1)^{|j|-1} c_{j, K, \tilde{\theta}} + (-1)^{|j|} \sum_{r \in \mathcal{S}(N)} c_{j, r - N_\ell, \tilde{\theta} + 1} \Pi_r(z),
\]
\[ \phi_n^B(z) := -\phi_{n,K,\tilde{\theta}} + \sum_{r \in S(N)} \phi_{n,r-N,\tilde{\theta}+1} \Pi_r(z). \]

Here we notice that for  \( s \in \mathbb{R}_+^* \) with  \(|s| < 1\) we have
\[
\left| \sum_{r \in S(N)} H_r(f_{\ell}^{\tilde{\theta}+1}(s)) \Pi_r(1^-) \right| \leq \left( \prod_{\ell=1}^{L} (f_{\ell}^{\tilde{\theta}+1}(0))^{-N_\ell} + 1 \right) \sum_{r \in S(N)} \Pi_r(1^-) < \infty.
\]

It follows that
\[
\left| \sum_{r \in S(N)} c_{j,r-N,\tilde{\theta}+1} \Pi_r(1^-) \right| < \infty \quad \text{and thus} \quad |a_j(1^-)| < \infty \quad \text{for all} \quad 1 \leq |j| \leq \tilde{\theta} + 1 (37)
\]

By (32) and (37), we also note that  \(|\phi_n^B(1^-)| < \infty\). Furthermore, in virtue of (33), we notice that as  \( z \to 1^- \),
\[
\sum_{n=0}^{\infty} \phi_n^B(z) \gamma_n(0) z^n = \begin{cases} 
\text{Cst} \cdot \ln \left( \frac{1}{1-z} \right), & \text{if} \quad \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise.} \end{cases} (38)
\]

For  \( m \in \{1, 2, \ldots, \tilde{\theta}\} \), set
\[
b_m(z) := (-1)^m C \beta^{-m} \sum_{|j|=m} a_j(z) \prod_{\ell=1}^{L} u_\ell^{j_\ell} \quad \text{and}
\]
\[
B_m,n(z) := \gamma_n(0) \sum_{|j|=m} a_j(z) \prod_{\ell=1}^{L} (1 - f_{\ell}^{n-\tilde{\theta}}(0))^{j_\ell} \sim b_m(z)n^{\theta-m} \quad \text{as} \quad n \to \infty
\]

and define  \( B_m(z) := \sum_{n=0}^{\infty} B_{m,n}(z) z^n \). By (37), we note that  \(|b_m(1^-)| < \infty\). In view of Hardy–Littlewood tauberian theorem and (38), we obtain (27), (28) and (29).

We notice that
\[
\sum_{n=0}^{\infty} \mu_n = Q(1^-) = \lim_{z \to 1^-} \frac{B(z)(1-z)^\theta}{D(z)(1-z)^\theta} = \frac{b_1(1^-)}{d_1(0)} < \infty.
\]

**Proof of Theorem 3.1** Part a. We adopt an idea by Vatutin (see [23]) as follows. Recall that  \( \tilde{\theta} := [\theta] \) is the ceiling value of  \( \theta \). We will prove that for all  \( k \in \{0, 1, \ldots, \tilde{\theta} - 1\} \),
\[
Q^{(k)}(1^-) = \sum_{n=0}^{\infty} \mu^{(k)}_n < \infty \quad (39)
\]

and as  \( z \to 1^- \),
\[
Q^{(\tilde{\theta})}(z) = \sum_{n=0}^{\infty} \mu^{(\tilde{\theta})}_n z^n \sim \begin{cases} 
\text{Cst} \cdot \ln \left( \frac{1}{1-z} \right), & \text{if} \quad \theta \in \mathbb{N}, \\
\text{Cst} \cdot (1-z)^{-1(\tilde{\theta}-\theta)} & \text{otherwise,} \end{cases} (40)
\]

\[
\frac{d}{dz} Q^{(\tilde{\theta})}(z) \sim \text{Cst} \cdot (1-z)^{-1} \quad \text{if} \quad \theta \in \mathbb{N}. \quad (41)
\]

We first consider the case when  \( \theta \) is not an positive integer. By Hardy–Littlewood tauberian theorem for power series, it follows from (40) that  \( \mu^{(\tilde{\theta})}_n \sim \text{Cst} \cdot n^{\tilde{\theta}-\theta-1} \) as  \( n \to \infty \). If  \( \theta \in \mathbb{N} \) then it follows from (41) and Hardy–Littlewood tauberian theorem for power series that...
\[
\sum_{k=1}^{n} k \mu_k^{(\theta)} \sim \text{Cst} \cdot n \text{ as } n \to \infty. \text{ In the latter case, by Lemma 3.4.ii, we obtain that } \\
\mu_k^{(\theta)} \sim \text{Cst} \cdot n^{-1} \text{ as } n \to \infty. \text{ In both cases, using Lemma 3.4.i, we deduce that } \mu_n \sim \varrho n^{\theta - 1} \text{ for some constant } \varrho > 0 \text{ as } n \to \infty.
\]

Hence, to finish the proof of Theorem 3.1(a), we only have to verify (39), (40) and (41).

Set \( B^{[k]}(z) = B(z) \) and \( B^{[k]}(z) = Q^{(k-1)}(1-z)D(z) - B^{[k-1]}(z) \) for \( 1 \leq k \leq \tilde{\theta} \). By induction on \( k \), we notice that if \( Q^{(k-1)}(1^-) < \infty \) then

\[
Q^{(k)}(z) = \sum_{n=0}^{\infty} \mu_n^{(k)} z^n = \frac{Q^{(k-1)}(1^-) - Q^{(k-1)}(z)}{1-z} = \frac{B^{[k]}(z)}{(1-z)^k D(z)}.
\]

Assume that up to some \( k \in \{1, 2, \ldots, \tilde{\theta} \} \), the power series \( R^{[k-1]}_B(z) \), \( B^{[k-1]}_m(z) \), and \( b^{[k-1]}_m(z) \) are defined for all \( m \in \{k, k+1, \ldots, \tilde{\theta} \} \) such that \( |b^{[k-1]}_m(1^-)| < \infty \) and as \( z \to 1^- \),

\[
R^{[k-1]}_B(z) \sim \begin{cases} 
\text{Cst} \cdot \ln \left( \frac{1}{1-z} \right) & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise,}
\end{cases}
\]

\[
B^{[k-1]}_m(z) \sim \Gamma(\theta - m + 1)b^{[k-1]}_m(z)(1-z)^{-(\theta - m + 1)},
\]

\[
B^{[k-1]}(z) = \sum_{m=k}^{\tilde{\theta}} B^{[k-1]}_m(z) + R^{[k-1]}_B(z)
\]

yielding that \( Q^{(k-1)}(1^-) = b^{[k-1]}_k(1^-)/d^{[k-1]}_k \) is finite. The above statement holds for \( k = 1 \) thanks to Lemma 3.6. We next prove that this statement also holds true when replacing \( k \) by \( k + 1 \). Indeed, by reason of Lemma 3.6, notice that as \( z \to 1^- \),

\[
B^{[k]}(z) = \sum_{m=k}^{\tilde{\theta}} \left( Q^{(k-1)}(1^-)D^{[k-1]}_m(z) - B^{[k-1]}_m(z) \right) + Q^{(k-1)}(1^-)R^{[k-1]}_D(z) - R^{[k-1]}_B(z).
\]

We also have that as \( z \to 1^- \),

\[
R^{[k]}_B(z) := Q^{(k-1)}(1^-)R^{[k-1]}_D(z) - R^{[k-1]}_B(z) \sim \begin{cases} 
\text{Cst} \cdot \ln \left( \frac{1}{1-z} \right) & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise,}
\end{cases}
\]

and \( Q^{(k-1)}(1^-)D^{[k-1]}_k(z) - B^{[k-1]}_k(z) \sim \Gamma(\theta - k + 1) \left( b^{[k-1]}_k(1^-) - b^{[k-1]}_k(z) \right)(1-z)^{\theta} \)

\[
= \Gamma(\theta - k + 1)\hat{b}_k(z)(1-z)^{-(\theta - 1)},
\]

where we set \( \hat{b}_k(z) := (b^{[k-1]}_k(1^-) - b^{[k-1]}_k(z))/(1-z). \) For \( m \in \{k+1, \ldots, \tilde{\theta} \} \), set

\[
B^{[k]}_m(z) := \begin{cases} 
\sum_{n=k+1}^{k+1} \left( Q^{(k-1)}(1^-)D^{[k-1]}_n(z) - B^{[k-1]}_n(z) \right) & \text{if } m = k + 1, \\
Q^{(k-1)}(1^-)D^{[k-1]}_m(z) - B^{[k-1]}_m(z) & \text{if } k + 2 \leq m \leq \tilde{\theta}
\end{cases}
\]

and

\[
b^{[k]}_m(z) := \begin{cases} 
(\theta - k)\hat{b}_k(z) + Q^{(k-1)}(1^-)d^{[k-1]}_{k+1}(z) - b^{[k-1]}_{k+1}(z) & \text{if } m = k + 1, \\
Q^{(k-1)}(1^-)d^{[k-1]}_m(z) - b^{[k-1]}_m(z) & \text{if } k + 2 \leq m \leq \tilde{\theta}.
\end{cases}
\]
By the recurrence relation of $b_m^{[k]}(z)$ and Lemma 3.5, we note that $|b_m^{[k]}(1^-)| < \infty$. Therefore, as $z \to 1^-$,

$$B^{[k]}(z) = \sum_{m=k+1}^{\hat{d}} B_m^{[k]}(z) + R_B^{[k]}(z) \quad \text{with} \quad R_B^{[k]}(z) \sim \begin{cases} \text{Cst} \cdot \ln \left( \frac{1}{1-z} \right) & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise} \end{cases}$$

and $B_m^{[k]}(z) \sim \Gamma(\theta - m + 1)b_m^{[k]}(z). (1-z)^{-(\theta-m+1)}$

and thus $Q^{(k)}(1^-) = b_{k+1}^{[k]}(1^-)/d_{k+1}^{[k]}$ is finite. By the principle of mathematical induction, we deduce that (39) holds true for all $k \in \{0, 1, \ldots, \hat{d} - 1\}$.

We now have

$$Q^{(\hat{d})}(z) = \frac{B^{[\hat{d}]}(z)}{(1-z)^{\hat{d}} D(z)}.$$

By Lemma 3.2, we notice that

$$D(z) = (1-z)\sum_{n=0}^{\infty} \gamma_n(0)z^n \sim \text{Cst} \cdot (1-z)^{-\theta} \quad \text{as } z \to 1^-.$$  \hspace{1cm} (42)

We also note that as $z \to 1^-$,

$$B^{[\hat{d}]}(z) = R_B^{[\hat{d}]}(z) = Q^{(\hat{d}-1)}(1^-)R_D^{[\hat{d}-1]}(z) - R_B^{[\hat{d}-1]}(z) \sim \begin{cases} \text{Cst} \cdot \ln \left( \frac{1}{1-z} \right) & \text{if } \theta \in \mathbb{N}, \\
\text{Cst} & \text{otherwise} \end{cases}.$$  \hspace{1cm} (43)

Hence (40) is verified.

Assume now that $\theta \in \mathbb{N}$. One can easily show by induction that $\frac{\text{d}}{\text{d}z} R_B^{[k]}(z) \sim \text{Cst} \cdot (1-z)^{-1}$ as $z \to 1^-$ for all $k \in \{0, 1, \ldots, \theta\}$. Hence

$$\frac{\text{d}}{\text{d}z} B^{[\theta]}(z) = \frac{\text{d}}{\text{d}z} R_B^{[\theta]}(z) \sim \text{Cst} \cdot (1-z)^{-1} \quad \text{as } z \to 1^-.$$  \hspace{1cm} (44)

On the other hand, using (34), we have

$$\frac{\text{d}}{\text{d}z}((1-z)^{\theta} D(z)) = (1-z)^{\theta} \left( -\sum_{n=0}^{\infty} \gamma_n(0)z^n + \sum_{n=0}^{\infty} n(1-g(f_n(0)))\gamma_n(0)z^{n-1} \right).$$

By reason of (19) and (20), we note that $1 - g(f_n(0)) \sim \theta n^{-1} + O(n^{-(1+\varepsilon)})$ for some $\varepsilon \in (0, 1)$ as $n \to \infty$. Hence

$$\frac{\text{d}}{\text{d}z}((1-z)^{\theta} D(z)) = O((1-z)^{\theta} \sum_{n=1}^{\infty} n^{-\varepsilon} \gamma_n(0)z^n) = O((1-z)^{-(1-\varepsilon)}) \quad \text{as } z \to 1^-.$$  \hspace{1cm} (45)

where in the last equality we use Hardy–Littlewood tauberian theorem and the fact that $\gamma_n(0) \sim C n^\theta$. Combining (42), (43), (44) and (45), we have

$$\frac{\text{d}}{\text{d}z} Q^{(\theta)}(z) = \frac{(1-z)^{\theta} D(z) \frac{\text{d}}{\text{d}z} B^{[\theta]}(z) - B^{[\theta]}(z) \frac{\text{d}}{\text{d}z}((1-z)^{\theta} D(z))}{((1-z)^{\theta} D(z))^2}$$

$$\sim \text{Cst} \cdot (1-z)^{-1} \quad \text{as } z \to 1^-.$$

Hence (41) is verified.
Part b. Recall from the proof of Lemma 3.3 that
\[
F(s, n) = F(f(s), n - 1)g(f(s)) - F(0, n - 1)(g(f(s)) - 1)
\]
\[
- \sum_{r \in S(N)} \mathbb{P}(U(n - 1) = r)H_r(f(s)).
\]
Differentiating both sides of the above equation at \( s = 1 \), we obtain that
\[
\mathbb{E}[U(n)] = \left( \mathbb{E}[U(n - 1)] - \mu_n N + \sum_{r \in S(N)} \mathbb{P}(U(n - 1) = r)(N - r) \right) \cdot \mathbb{E}[\psi].
\]
Iterating the above equality and taking the scalar product with \( u \), where we recall that \( \varsigma \) where
\[
\sum_{\text{hand, we notice that}} \exists
\]
By reason of (23), we have that
\[
\mathbb{E}[\langle U(n), u \rangle] = \langle K, u \rangle + \sum_{k=1}^{n-1} \left( - \mu_k \langle N, u \rangle + \sum_{r \in S(N)} \mathbb{P}(U(k - 1) = r)\langle N - r, u \rangle \right) \tag{46}
\]
where we recall that \( U(0) = K \) a.s. and \( u \) is the right eigenvector of \( \mathbb{E}[\psi] \) w.r.t. the maximal eigenvalue 1. Taking \( n \to \infty \) in (46) and using (22), we get
\[
\sum_{n=0}^{\infty} \mu_n = Q(1^-) = \frac{\langle K, u \rangle + \sum_{r \in S(N)} \Pi_r(1^-)\langle N - r, u \rangle}{\langle N, u \rangle}.
\]
Hence we obtain
\[
\mathbb{E}[\langle U(n), u \rangle] = \langle N, u \rangle \sum_{k=n}^{\infty} \mu_k - \sum_{k=n}^{\infty} \sum_{r \in S(N)} \mathbb{P}(U(k - 1) = r)\langle N - r, u \rangle \geq \langle N, u \rangle \left[ \sum_{k=n}^{\infty} \mu_k - \sum_{k=n}^{\infty} \sum_{r \in S(N)} \mathbb{P}(U(k - 1) = r) \right].
\]
By reason of (23), \( \sum_{k=n}^{\infty} \sum_{r \in S(N)} \mathbb{P}(U(k - 1) = r) \leq \mu_{n-1} = O(n^{-1-\theta}). \) On the other hand, we notice that \( \sum_{k=n}^{\infty} \mu_k \sim \frac{\theta}{\gamma} n^{-\theta} \) and \( \langle U(n), u \rangle \leq |U(n)| \cdot |u| = |U(n)|. \) Hence
\[
\liminf_{n \to \infty} \frac{\mathbb{E}[|U(n)| | |U(n)| \neq 0]}{n} = \liminf_{n \to \infty} \frac{\mathbb{E}[|U(n)|]}{n \mu_n} \geq \liminf_{n \to \infty} \frac{\langle N, u \rangle \left( \frac{\theta}{\gamma} n^{-\theta} + O(n^{-1-\theta}) \right)}{n \cdot \theta n^{-\theta} - 1} = \beta.
\]
\[\square\]

4 Phase Transition for the Speed of ERW

We first prove the following result which in particular implies Part (a) of Theorem 1.1.

Theorem 4.1 Under Assumption A, we have that
\[
\lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{1 + \langle \mathbb{E}[Z_{\infty}], \varsigma \rangle} \quad \text{almost surely,} \tag{47}
\]
where \( \varsigma = (2, 1, \ldots, 1). \)

\[\square\] Springer
Proof In virtue of Proposition 2.3, the distribution of \( Z_\infty \) does not depend on \( Z_0 \) and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \langle Z_i, \varsigma \rangle = \langle E[Z_\infty], \varsigma \rangle \quad \text{almost surely.}
\]
Assume w.l.o.g. that \( Z_0 = 0 \) almost surely. By reason of Proposition 2.1, \( \sum_{i=0}^{n-1} \langle Z_i, \varsigma \rangle \) is equal to \( \sum_{i=0}^{n-1} \langle V^n_i, \varsigma \rangle \) in distribution. Moreover, recall from (2) that \( T_n \sim n + \sum_{i=0}^{n-1} \langle V^n_i, \varsigma \rangle \) as \( n \to \infty \). Hence
\[
\lim_{n \to \infty} \frac{T_n}{n} = 1 + \langle E[Z_\infty], \varsigma \rangle \quad \text{in probability.}
\]
Using an argument by Zerner (see the proof of Theorem 13 in [26]), we next show that \( X_n/n \) and \( \lim_{n \to \infty} n/T_n \) converge in probability to the same limit. Indeed, for \( n \geq 0 \), set \( S_n = \sup \{ k : T_k \leq n \} \). We have \( T_{S_n} \leq n < T_{S_n+1} \). It immediately follows that
\[
\lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{T_{S_n}}{S_n} = \lim_{n \to \infty} \frac{T_n}{n} = 1 + \langle E[Z_\infty], \varsigma \rangle \quad \text{in probability.}
\]
Since \( n < T_{S_n+1} \) and before time \( T_{S_n+1} \), the walk is always below level \( S_n + 1 \), we must have \( X_n \leq S_n \). On the other hand \( X_n = S_n + X_n - X_{T_{S_n}} \geq S_n - L(n - T_{S_n}) \). It follows that
\[
\frac{S_n}{n} - L \left( 1 - \frac{T_{S_n}}{n} \right) \leq \frac{X_n}{n} \leq \frac{S_n}{n}.
\]
Using (48), we note that
\[
\lim_{n \to \infty} \frac{T_{S_n}}{S_n} = \lim_{n \to \infty} \frac{T_{S_n} S_n}{S_n} = 1 \quad \text{in probability.}
\]
As a result,
\[
\lim_{n \to \infty} \frac{X_n}{n} = \lim_{n \to \infty} \frac{S_n}{n} = \lim_{n \to \infty} \frac{T_n}{n} = \frac{1}{1 + \langle E[Z_\infty], \varsigma \rangle} \quad \text{in probability.}
\]
Furthermore, \( X_n/n \) converges almost surely as \( n \to \infty \) thanks to Proposition 3.1 in [4].
We note that the proof of Proposition 3.1 in [4] generally holds for transient multi-excited random walk with long jumps both to the left and the right assuming that
\[
\sup \left( \sup \{ \text{supp}(\nu) \} \cup \bigcup_{i=2}^{M} \text{supp}(q_i) \right) \leq \sup(\text{supp}(q_1)) < \infty
\]
where \( \text{supp}(\mu) \) stands for the support of measure \( \mu \). This ends the proof of the theorem. \( \square \)

Theorem 4.1 also implies that Part (b) of Theorem 1.1 is equivalent to the following theorem.

**Theorem 4.2** \( \mathbb{E}[Z_\infty, \ell] < \infty \) for all \( \ell \in [L] \) if and only if \( \delta > 2 \).

To prove the above theorem, we will need some preliminary results. The next proposition is immediate from the proof of Proposition 3.6 in [2].

**Proposition 4.1** Suppose that for \( s \in [0, 1] \)
\[
1 - G \left( \frac{1}{2 - s} \right) = a(s)(1 - G(s)) + b(s)
\]
where
\[
G(t) = \int_0^t \frac{1}{1 + \frac{1}{2 - s}} ds
\]
I. \( a(s) \) and \( b(s) \) are analytic functions in some neighborhood of 1 such that \( a(1) = 1, a'(1) = \delta \) for some \( \delta > 1 \) and \( b(1) = 0 \);

II. \( G \) is a function defined on \([0, 1]\) such that \( G \) is left continuous function at 1, \( G'(1^-) \in (0, \infty) \) and there exists \( \epsilon \in (0, 1) \) such that \( G^{(i)}(s) > 0 \) for each \( s \in (1 - \epsilon, 1) \) and \( i \in \mathbb{N} \).

Then, the following statements hold true:

(i) \( b'(1) = 0 \).
(ii) If \( \delta > 2 \) then \( b''(1) > 0 \) and \( 1 - G(1 - s) = \frac{b''(1)}{2(\delta - 2)} s + O(s^{2\wedge(\delta - 1)}) \) as \( s \downarrow 0 \).
(iii) If \( \delta = 2 \) and \( b''(1) = 0 \) then \( G^{(i)}(1^-) < \infty \) for all \( i \in \mathbb{N} \).
(iv) If \( \delta = 2 \) and \( b''(1) \neq 0 \) then \( 1 - G(1 - s) \sim C s |\ln(s)| \) as \( s \downarrow 0 \) for some constant \( C > 0 \).

In this section, we consider the following function

\[
G(s) := \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{Z_{\infty, \ell}} \right], \quad s \in [0, 1].
\]  

(50)

Notice that \( G'(1) = \mathbb{E}\left[ \sum_{\ell=1}^{L} \ell Z_{\infty, \ell} \right] \). For \( s_0 \in \left( \frac{L-1}{L}, 1 \right) \), we have

\[
G(s) = \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 - \ell(1-s_0) + \ell(s-s_0))^{Z_{\infty, \ell}} \right]
\]

\[
= \sum_{k \in \mathbb{Z}^L_+} \mathbb{P}(Z_{\infty} = k) \prod_{\ell=1}^{L} \sum_{j=0}^{k_{\ell}} \binom{k_{\ell}}{j} (1 - \ell(1-s_0))^{j} (\ell(s-s_0))^{k_{\ell}-j}.
\]

It follows that \( G(s) \) can be expanded as a power series of \( s - s_0 \) with all positive coefficients.

As a consequence, \( G^{(i)}(s) > 0 \) for all \( s \in \left( \frac{L-1}{L}, 1 \right) \) and \( i \in \mathbb{N} \). Hence the condition II of Proposition 4.1 is verified. We next show (in Proposition 4.2 and Proposition 4.3 below) that there exist functions \( a \) and \( b \) such that the condition I and the functional equation (49) (with \( G \) given by (50)) are fulfilled.

Recall that \((\eta_n)_{n \geq 1}\) is a sequence of i.i.d. random vectors with multivariate geometrical law defined in (5). For \( \ell \in [L] \), set \( \rho_\ell = v(-\ell)/v(1) \). Notice that the probability generating function of \( \eta_1 \) is given by

\[
\mathbb{E}\left[ \prod_{\ell=1}^{L} s_{\ell}^{\eta_1, \ell} \right] = \frac{1}{1 + \sum_{\ell=1}^{L} \rho_\ell (1 - s_\ell)}
\]  

(51)

for \( s = (s_1, s_2, \ldots, s_L) \) such that \( \sum_{\ell=1}^{L} s_\ell v(-\ell) < 1 \).

**Lemma 4.3** For \( \ell \in [L] \),

\[
\mathbb{E} [A_\ell(M - 1)] = \sum_{i=1}^{M} (q_i(-\ell) + \rho_\ell (1 - q_i(1)))
\]

**Proof** Set \( S = \sum_{i=1}^{M} \xi_{i, L+1} \). We have that \( \mathbb{E}[S] = \sum_{i=1}^{M} q_i(1) \), and

\[
A_\ell(M - 1) = \sum_{i=1}^{M} \xi_{i, \ell} + \sum_{i=M+1}^{M-1} \xi_{i, \ell}.
\]
On the other hand, recall that
\[ \gamma_{M-1} = \inf \left\{ k \geq 1 : \sum_{i=1}^{k} \xi_{i, L+1} = M \right\} = M + \inf \left\{ k \geq 1 : \sum_{i=M+1}^{M+k} \xi_{i, L+1} = M - S \right\}. \]

Hence \( \sum_{i=M+1}^{\gamma_{M-1}} \xi_{i, \ell} \) has the same distribution as \( \sum_{i=1}^{M-S} \eta_{i, \ell} \). Therefore
\[
\mathbb{E}[A_{\ell}(M - 1)] = \sum_{i=1}^{M} \mathbb{E}[\xi_{i, \ell}] + \mathbb{E}[\eta_{1, \ell}] \mathbb{E}[M - S] = \sum_{i=1}^{M} q_i(-\ell) + \rho_{\ell} \sum_{i=1}^{M} (1 - q_i(1)).
\]

We can combine Proposition 2.2 with Proposition 4.3 to compute \( \mathbb{E}[A(m)] \) for \( m \geq M \). We also note from (51) that \( \mathbb{E}[\eta_{1, \ell}] = \rho_{\ell} \) for \( \ell \in [L] \). As a result, for \( \ell \in [L] \) and \( k \in \mathbb{Z}_+^L \) such that \( |k| = k_1 + \cdots + k_L \geq M - 1 \), we have
\[
\mathbb{E}[Z_{1, \ell} | Z_0 = k] = \mathbb{E}[A_{\ell}(|k|)] + k_{\ell+1} = \mathbb{E}[A_{\ell}(M - 1)] + \sum_{i=1}^{M} \mathbb{E}[\xi_i] + k_{\ell+1}
= \sum_{i=1}^{M} q_i(-\ell) + \rho_{\ell} \left( |k| + 1 - \sum_{i=1}^{M} q_i(1) \right) + k_{\ell+1}
\]
where we use the convention \( k_{L+1} = 0 \).

**Proposition 4.2** The function \( G \) defined by (50) satisfies the functional equation (49) where we define
\[
a(s) := \frac{1}{\mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s-1))^{A_{\ell}(M-1)} \right] (2-s)^{M-1}},
\]
\[
b(s) := \sum_{|k| \leq M-1} \mathbb{P}(Z_{\infty} = k) \left( a(s) \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s-1))^{A_{\ell}(|k|) + k_{\ell+1}} \right] - \frac{\prod_{\ell=1}^{L} (1 + \ell(s-1))^{k_{\ell+1}}}{(2-s)^{|k|}} \right) - a(s) + 1.
\]

**Proof** We have that
\[
G(s) = \sum_{k \in \mathbb{Z}_+^L} \mathbb{P}(Z_{\infty} = k) \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s-1))^{Z_{1, \ell}} \mid Z_0 = k \right].
\]

Recall that given \( \{Z_0 = (k_1, k_2, \ldots, k_L)\} \), the random vector \( Z_1 = (Z_{1,1}, Z_{1,2}, \ldots, Z_{1,L}) \) has the same distribution as \( (A_1(|k|) + k_2, \ldots, A_{L-1}(|k|) + k_L, A_L(|k|)) \). Using Proposition 2.2, we thus have
\[
G(s) = \sum_{|k| \leq M-1} \mathbb{P}(Z_{\infty} = k) \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s-1))^{A_{\ell}(|k|)} \right] \prod_{\ell=1}^{L-1} (1 + \ell(s-1))^{k_{\ell+1}}
\]
\[
+ \sum_{|k| \geq M-1} \mathbb{P}(Z_\infty = k) \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(M-1) + \eta_1,\ell + \cdots + \eta_{|k|},\ell - M + 1, \ell} \right] \prod_{\ell=1}^{L-1} (1 + \ell(s - 1))^{k_\ell+1}.
\]

On the other hand, using (51) and the fact that \(\sum_{\ell=1}^{L} \ell \rho_\ell = 1\), we obtain
\[
\mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{\eta_1,\ell} \right] = \frac{1}{2 - s}.
\]

Hence
\[
\sum_{k \in \mathbb{Z}_+^L} \mathbb{P}(Z_\infty = k) \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(M-1)} \right] \prod_{\ell=1}^{L-1} (1 + \ell(s - 1))^{k_\ell+1} = \frac{\mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(M-1)} \right] \prod_{\ell=1}^{L-1} (1 + \ell(s - 1))^{k_\ell+1}}{(2 - s)^{|k| - M + 1}}
\]
\[
= \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(M-1)} \right] (2 - s)^{M-1} \sum_{k \in \mathbb{Z}_+^L} \mathbb{P}(Z_\infty = k) \prod_{\ell=1}^{L} \left( 1 + \ell \left( \frac{1}{2 - s} - 1 \right) \right)
\]
\[
= \frac{1}{a(s)} G \left( \frac{1}{2 - s} \right).
\]
Therefore
\[
G(s) = \sum_{k \in \mathbb{Z}_+^L} \mathbb{P}(Z_\infty = k) \frac{\mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(M-1)} \right] \prod_{\ell=1}^{L-1} (1 + \ell(s - 1))^{k_\ell+1}}{(2 - s)^{|k| - M + 1}}
\]
\[
+ \sum_{|k| \leq M-2} \mathbb{P}(Z_\infty = k) \left( \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(|k|)} \right] - \mathbb{E}\left[ \prod_{\ell=1}^{L} (1 + \ell(s - 1))^{A_\ell(M-1)} \right] \right)
\]
\[
\times \prod_{\ell=1}^{L-1} (1 + \ell(s - 1))^{k_\ell+1} = \frac{1}{a(s)} G \left( \frac{1}{2 - s} \right) + 1 + \frac{b(s) - 1}{a(s)}.
\]

Recall from (1) that the expected total drift \(\delta\) of the cookie environment is given by
\[
\delta := \sum_{j=1}^{M} \left( q_j(1) - \sum_{\ell=1}^{L} \ell q_j(-\ell) \right).
\]
We must have that \(\delta > 1\) as \(X\) is assumed to be transient to the right.
Proposition 4.3 The functions $a(s)$ and $b(s)$ defined in Proposition 4.2 satisfies the condition (I) of Proposition 4.1. More specifically,

$$a(1-s) = 1 - (\delta - 1) s + o(s) \quad \text{and} \quad b(1-s) = b'(1)s + o(s) \quad \text{as } s \to 0$$

where

$$b'(1) = (\delta - 1) - \sum_{|k| \leq M-2} \mathbb{P}[Z_\infty = k] \left( \delta - 1 - |k| + \sum_{\ell=1}^L \ell \mathbb{E}[A_\ell(|k|)] \right).$$

Proof Using Taylor’s expansion, we have

$$a(1-s) = \frac{1}{\mathbb{E} \left[ \prod_{\ell=1}^L (1 - \ell s)^{A_\ell(M-1)} \right] (1 + s)^M}$$

$$= 1 - \left( M - 1 - \sum_{\ell=1}^L \ell \mathbb{E}[A_\ell(M-1)] \right) s + o(s)$$

$$= 1 - \left( \sum_{i=1}^M (q_i(1) - \ell q_i(-\ell)) - 1 \right) s + o(s) = 1 - (\delta - 1)s + o(s)$$

as $s \to 0$, in which the third identity follows from Lemma 4.3 and the fact that $\sum_{\ell=1}^L \ell \rho_\ell = 1$. Furthermore,

$$b(1-s) = 1 - a(1-s) +$$

$$+ \sum_{|k| \leq M-2} \mathbb{P}[Z_\infty = k] \left( a(1-s) \mathbb{E} \left[ \prod_{\ell=1}^L (1 - \ell s)^{A_\ell(|k|)} \right] - (1 + s)^{-|k|} \right) \prod_{\ell=1}^{L-1} (1 - \ell s)^{k_{\ell+1}}$$

$$= (\delta - 1)s - \sum_{|k| \leq M-2} \mathbb{P}[Z_\infty = k] \left( \delta - 1 - |k| + \sum_{\ell=1}^L \ell \mathbb{E}[A_\ell(|k|)] \right) s + o(s).$$

Remark 4.4 Note that the functional equation (49) has the same form with the one in [2]. However the coefficient function $b$ defined in Proposition 4.2 is more complicated and strongly depends on the distribution of $Z_\infty$ while the function $G$ defined by (50) is not a probability generating function when $L \geq 2$ and it will not give us the full information to compute $b$. Nevertheless, Proposition 4.1 still play a crucial role in the proof of Theorem 1.1.

Let $\eta = (\eta^1, \eta^2, \ldots, \eta^L)$ be a $L$-dimensional random vectors with multivariate geometric law defined by

$$\mathbb{P} (\eta = (i_1, i_2, \ldots, i_L)) = \frac{v(1)}{(i_1 + i_2 + \cdots + i_L)!} \prod_{k \in [L]} i_k! v(-k)^{i_k},$$

for each $i = (i_1, i_2, \ldots, i_L) \in \mathbb{Z}_+^L$. Recall that the probability generating function of $\eta$ is given by

$$\mathbb{E} \left[ \prod_{\ell=1}^L s_\ell^{\eta_\ell} \right] = \frac{1}{1 + \sum_{\ell=1}^L \rho_\ell (1 - s_\ell)} \quad \text{with } \rho_\ell = v(-\ell)/v(1).$$
In particular, we have \( \mathbb{E}[^{\eta}] = (\rho_1, \rho_2, \ldots, \rho_L) \).

Let \((\vartheta(k, n))_{k, n \in \mathbb{N}} = (\vartheta_{i, j}(k, n), i, j \in [L])_{k, n \in \mathbb{N}}\) be a sequence of i.i.d. \(L \times L\) random matrices such that its rows are i.i.d. copies of \(\eta\). Let \((W(n))_{n \geq 0} = (W_1(n), \ldots, W_L(n))_{n \geq 0}\) be a multi-type branching process with \((N_1, N_2, \ldots, N_L)\)-emigration such that for each \(n \in \mathbb{N}\) and \(j \in [L]\),

\[
W_j(n) = \sum_{i=1}^{L} \sum_{k=1}^{\infty} \chi_{i, j}(k, n)
\]

where

\[
\chi_{i, j}(k, n) = \vartheta_{i, j}(k, n) + \delta_{i-1, j}
\]

(here \(\delta_{i, j}\) stands for the Kronecker delta) and \(\vartheta_i(w) := (w_i - N_i) I_{\{w_i \geq N_i, \forall \ell \in [L]\}}\).

**Lemma 4.5** Assume \(\nu(-L) > 0\). Then \((W(n))_{n \geq 0}\) is a critical multi-type Galton–Watson process with \((N_1, N_2, \ldots, N_L)\)-emigration according to Definition 1 and Assumption B.

**Proof** We have

\[
\overline{\lambda} := \mathbb{E}\left[\left(\chi_{i, j}(1, 1)\right)_{i, j \in [L]}\right] = \begin{pmatrix}
\rho_1 & \rho_2 & \cdots & \rho_{L-1} & \rho_L \\
\rho_1 + 1 & \rho_2 & \cdots & \rho_{L-1} & \rho_L \\
\rho_1 & \rho_2 + 1 & \cdots & \rho_{L-1} & \rho_L \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_1 & \rho_2 & \cdots & \rho_{L-1} + 1 & \rho_L
\end{pmatrix}
\]

and notice that all the entries of

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \rho_L \\
1 & 0 & \cdots & 0 & \rho_L \\
0 & 1 & \cdots & 0 & \rho_L \\
0 & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \rho_L
\end{pmatrix}_L
\]

are positive as \(\rho_L = \nu(-L)/\nu(1) > 0\). Hence \(\overline{\lambda}\) is positively regular.

Applying the determinant formula (see, e.g., Theorem 18.1.1 in [9])

\[
\det\left(\Sigma + x^T y\right) = \left(1 + x^T \Sigma^{-1} y\right) \det(\Sigma)
\]

(in which \(\Sigma\) is an invertible matrix, \(x\) and \(y\) are column vectors) to \(x = (\rho_1, \rho_2, \ldots, \rho_L)^T\),

\[
y = (1, 1, \ldots, 1)^T
\]

we obtain

\[
\Phi(\lambda) := \det(\overline{\lambda} - \lambda I) = (-1)^L \left(\lambda \sum_{j=0}^{L-1} \left(\sum_{\ell=L-j}^{L} \rho_\ell\right) \lambda^j\right)
\]
\[ = (-1)^L \sum_{j=0}^{L-1} \left( \sum_{\ell=L-j}^L \rho_{\ell} \right) (\lambda^L - \lambda^j) \]

where the last equality follows from the fact that
\[ \sum_{j=0}^{L-1} \sum_{\ell=L-j}^L \rho_{\ell} = \sum_{\ell=1}^L \ell \rho_{\ell} = 1. \]

As a result, \( \Phi(1) = 0 \)

and
\[ \Phi'(1) = (-1)^L \sum_{j=0}^{L-1} (L - j) \left( \sum_{\ell=L-j}^L \rho_{\ell} \right) \neq 0. \]
yielding that \( \lambda = 1 \) is a simple eigenvalue of \( \mathcal{X} \). Using the triangle inequality, we have that
\[ |\Phi(\lambda)| \geq |\lambda^L| - \sum_{j=1}^{L-1} \left( \sum_{\ell=L-j}^L \rho_{\ell} \right) |\lambda|^j = \sum_{j=1}^{L-1} \left( \sum_{\ell=L-j}^L \rho_{\ell} \right) (|\lambda|^L - |\lambda|^j) > 0 \quad \text{if} \quad |\lambda| > 1. \]

Therefore all the eigenvalues of \( \mathcal{X} \) must lie in the closed unit disk. Hence \( \lambda = 1 \) is also the largest eigenvalue in modulus. This ends the proof of the lemma. \( \square \)

**Remark 4.6** The right and left eigenvectors of the maximal eigenvalue \( \lambda = 1 \) are given respectively by
\[
\begin{align*}
    u &= \frac{2}{L(L+1)} (1, 2, \ldots, L - 1, L), \\
    v &= \frac{1}{\rho_L + \frac{1}{L+1} \sum_{\ell=1}^{L-1} \ell(\ell+1) \rho_{\ell}} \left( \rho_L + L \sum_{\ell=1}^{L-1} \rho_{L-\ell}, \ldots, \rho_L + L(L-2) + \rho_{L-1}, \rho_L \right).
\end{align*}
\]

It is also clear that (7) holds true since \( \mathbb{E}[\chi_i^k] < \infty \) for all \( k \geq 1 \) and \( i, j \in [L] \).

**Proposition 4.4** Assume that \( \delta = 2 \). Then there exists a positive integer \( \kappa \) such that
\[ \mathbb{E}[|Z_\infty|^\kappa] = \infty. \]

**Proof** Define \( \tau = \inf\{n \geq 1 : Z_n = 0\} \). Notice that \( \mathbb{E}[\tau | Z_0 = 0] < \infty \) as \( Z \) is positive recurrent. Furthermore, for any function \( \pi : \mathbb{Z}^L \to \mathbb{R}_+ \), we have (see e.g. Theorem 1.7.5 in [21])
\[ \mathbb{E}[\pi(Z_\infty)] = \frac{\mathbb{E} \left[ \sum_{n=0}^{\tau-1} \pi(Z_n) | Z_0 = 0 \right]}{\mathbb{E}[\tau | Z_0 = 0]}. \]

Let \( L' = \max\{\ell \in [L] : \nu(-\ell) > 0\} \) and \( z^* = (z_1^*, \ldots, z_L^*) \) with \( z_{\ell}^* = M \) for \( 1 \leq \ell \leq L' \) and \( z^* = 0 \) for \( L' + 1 \leq \ell \leq L \). Let \( \kappa \) be a fixed positive integer that we will choose later.

By setting \( \pi(z) = \left( \sum_{\ell=1}^L z_{\ell} \right)^\kappa \), we obtain
\[ \mathbb{E}[|Z_\infty|^\kappa] = \frac{1}{\mathbb{E}[\tau | Z_0 = 0]} \mathbb{E} \left[ \sum_{n=0}^{\tau-1} |Z_n|^\kappa | Z_0 = 0 \right]. \]

\( \square \)
Suppose that \( Z_n(\eta(\cdot)) \). Assuming \( \tau > L, Z_0 = 0 \) = \( Z_n(\eta(\cdot)) \) for all \( 1 \leq k \leq L - 1, Z_0 = 0 \) > 0 and thus

\[
P(Z_L = z^*, \tau > L|Z_0 = 0) > 0.
\]

We next use a coupling argument to estimate the order of \( \mathbb{E}[|Z_n \wedge \tau|^k |Z_0 = z^*] \) as \( n \to \infty \). Recall from Remark 2.1 that

\[
Z_n = \begin{cases} 
A^{(n)}(ML - 1) + \sum_{k=(L-1)M+1}^{[Z_n-1]-M+1} \eta_k^{(n)} + \tilde{Z}_{n-1} & \text{if } |Z_n| \geq ML - 1, \\
A^{(n)}(|Z_{n-1}|) + \tilde{Z}_{n-1} & \text{if } |Z_n| < ML - 1
\end{cases}
\]

where we denote \( \tilde{z} = (z_2, z_3, \ldots, z_{L+1}, 0) \) for each \( z = (z_1, z_2, \ldots, z_L) \in \mathbb{R}^L \); and \( \eta_k^{(n)} = (\eta_{k,1}^{(n)}, \ldots, \eta_{k,L}^{(n)}) \) with \( k, n \in \mathbb{N} \) are i.i.d. copies of the random vector \( \eta \) which are independent of \( Z_0 \). Let \( (W(n))_{n \geq 0} \) be the multi-type branching process with \( N \)-emigration defined by (52), in which we set \( N = (N_1, N_2, \ldots, N_L) = (M - 1, M, \ldots, M), W(0) = (M, M, \ldots, M) \) and \( \vartheta_i, j(k, n) = \eta_{(L-1)M+i,k,j}^{(n)} \) for \( i, j \in [L] \) and \( k, n \in \mathbb{N} \). For the sake of simplicity, we assume from now on that \( L = L' \) (otherwise, by reducing the dimension of \( W(n) \), one can easily handle the case \( L < L' \) by the same argument used in the case \( L = L' \)). In this case, we note that \( z^* = (M, M, \ldots, M) \).

Recall that for \( L \)-dimensional vectors \( x \) and \( y \), we write \( x \succeq y \) if \( x_\ell \succeq y_\ell \) for all \( \ell \in [L] \). Assuming \( Z_0 = z^* = (M, M, \ldots, M) \), we will show that

\[
Z_n \succeq W(n) \quad \text{for all } n \geq 1.
\]

Indeed, for \( n = 1 \), we have

\[
Z_1 = A^{(1)}(LM - 1) + \eta_{(L-1)M+1}^{(1)} + (M, M, \ldots, M, 0) \geq \eta_{(L-1)M+1}^{(1)} = W(1).
\]

Suppose that \( Z_{n-1} \succeq W(n - 1) \) for some \( n \in \mathbb{N} \). If \( Z_{n-1, \ell} < N_\ell \) for some \( \ell \in [L] \) then \( W_\ell(n - 1) \leq Z_{n-1, \ell} < N_\ell \) and thus \( W(n) = 0 \). On the other hand, if \( Z_{n-1, \ell} \geq N_\ell \) for all \( \ell \in [L] \) then \( |Z_{n-1}| \geq ML - 1 \) and thus for \( \ell \in [L] \),

\[
Z_{n, \ell} \geq |Z_{n-1}-M+1| \eta_{k}^{(n)} + Z_{n-1, \ell+1} - N_{\ell+1} = \sum_{i=1}^{L} \sum_{k=1}^{Z_{n-1,i}-N_i} \vartheta_{i, \ell}(k, n) + \vartheta_{i-1, \ell}(k, n) \geq W_\ell(n),
\]

in which we use the convention that \( Z_{n-1,L+1} = N_{L+1} = 0 \). Hence \( Z_n \succeq W(n) \). By the principle of mathematical induction, we deduce (55). It implies additionally that on the event \( \{Z_0 = z^*\}, W(n) = 0 \) for all \( n \geq \tau \). Hence, for all \( n \in \mathbb{N} \),

\[
\mathbb{E}[|Z_n \wedge \tau|^k |Z_0 = z^*] \geq \mathbb{E}[|W(n)|^k |Z_0 = z^*] = \mathbb{E}[|W(n)|^k]
\]

where the last equality follows from the fact that \( (W(n))_{n \geq 0} \) is independent of \( Z_0 \).

On the other hand, \( (W(n))_{n \geq 0} \) is a critical multi-type Galton–Watson process with \( (M - 1, M, \ldots, M) \)-emigration. By Theorem 3.1, there exist positive constants \( c_1, c_2 \) and \( \theta \) such
that
\[ \mathbb{P}(W(n) \neq 0) \geq \frac{c_1}{n^{\theta+1}}, \quad \mathbb{E}\left[ |W(n)| \mid W(n) \neq 0 \right] \geq c_2n. \]

Choose \( \kappa = \lfloor \theta \rfloor + 1 \). Using Jensen inequality, we thus have
\[ \mathbb{E}\left[ |W(n)|^{\kappa} \mid W(n) \neq 0 \right] \mathbb{P}(W(n) \neq 0) \geq \mathbb{E}\left[ |W(n)| \mid W(n) \neq 0 \right]^{\kappa} \mathbb{P}(W(n) \neq 0) \geq \frac{c_2^\kappa c_1}{n^{\theta-\lfloor \theta \rfloor}}. \quad (57) \]

Combining (53), (54), (56) and (57), we obtain that there exists \( \ell \) such that
\[ \lim_{n \to \infty} \mathbb{P}(\ell) = 0. \]

We now turn to the proof of our main result.

**Proof of Part (b), Theorem 1.1** Remind that this part is equivalent to Theorem 4.2. As the function \( G \) defined by (50) satisfies the functional equation (49) and the conditions I, II of Proposition 4.1, it follows from Proposition 4.1(ii) that if \( \delta > 2 \) then
\[ G'(1^-) = \mathbb{E}\left[ \sum_{\ell=1}^{L} \ell Z_{\infty, \ell} \right] = \frac{b''(1)}{2(\delta - 2)} < \infty. \]

The above fact and (47) imply that the random walk \( X \) has positive speed in the supercritical case \( \delta > 2 \).

Let us now consider the critical case \( \delta = 2 \). If \( b''(1) \neq 0 \) then by the virtue of Proposition 4.1(iv), we must have \( G'(1^-) = \infty \). Hence, to prove that \( G'(1^-) = \infty \), it is sufficient to exclude the case \( b''(1) = 0 \). Assume now that \( b''(1) = 0 \). By Proposition 4.1(iii), \( G(i)(1^-) < \infty \) for all \( i \in \mathbb{N} \). On the other hand, by Proposition 4.4, there exists a positive integer \( \kappa \) such that \( \mathbb{E}[|Z_{\infty}|^\kappa] = \infty \) and thus \( G^{(\kappa)}(1^-) = \infty \), which is a contradiction. Hence \( G'(1^-) = \infty \) and thus there exists \( \ell \in [L] \) such that \( \mathbb{E}[Z_{\infty, \ell}] = \infty \). It follows that a.s. \( \lim_{n \to \infty} X_n/n = 0 \).

The subcritical case can be solved by showing the monotonicity of the speed as follows. Assume that \( 1 < \delta < 2 \). There exist probability measures \( \tilde{\mathbb{Q}}_1, \tilde{\mathbb{Q}}_2, \ldots, \tilde{\mathbb{Q}}_M \) on \( \{-L, -L + 1, \ldots, -1, 1\} \) such that \( \tilde{\mathbb{Q}}_j(-\ell) \leq q_j(-\ell) \) for each \( \ell \in [L], j \in [M] \) and
\[ \tilde{\delta} := \sum_{j=1}^{M} \left( \tilde{\mathbb{Q}}_j(1) - \sum_{\ell=1}^{L} \ell \tilde{\mathbb{Q}}_j(-\ell) \right) = 2. \]

Let \( \hat{X} = (\hat{X}_n)_n \) be the \( (L, 1) \)-excited random walk w.r.t the cookie environment \( \hat{\omega} \) defined by
\[ \hat{\omega}(j, i) = \begin{cases} \tilde{\mathbb{Q}}_j(i), & \text{if } 1 \leq j \leq M, \\ v(i), & \text{if } j > M. \end{cases} \]

We thus have that a.s. \( \lim_{n \to \infty} \hat{X}_n/n = 0 \). Let \( Z \) and \( \tilde{Z} \) be respectively the Markov chains associated with \( X \) and \( \hat{X} \) as defined by (4). Let \( Z_\infty \) and \( \tilde{Z}_\infty \) be their limiting distributions. For \( h, \hat{h}, k \in \mathbb{Z}_+^L \) with \( h \geq \hat{h} \), we notice that
\[ \mathbb{P}(\tilde{Z}_n \geq k \mid \tilde{Z}_{n-1} = \hat{h}) \leq \mathbb{P}(\tilde{Z}_n \geq k \mid \tilde{Z}_{n-1} = h) \leq \mathbb{P}(Z_n \geq k \mid Z_{n-1} = h). \]

Applying Strassen’s theorem on stochastic dominance for Markov chains (see e.g. Theorem 5.8, Chapter IV, p. 134 in [19] or Theorem 7.15 in [6]), we have that \( \tilde{Z} \) is stochastically dominated by \( Z \). In particular, \( \mathbb{E}[\tilde{Z}_\infty] \leq \mathbb{E}[Z_\infty] \). Combining the above fact and the speed formula (47), we conclude that a.s.
\[ \lim_{n \to \infty} \frac{X_n}{n} \leq \lim_{n \to \infty} \frac{\hat{X}_n}{n} = 0. \]
This ends the proof of our main theorem.

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