Dynamical system analysis of cosmologies with running cosmological constant from quantum Einstein gravity

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Abstract. We discuss a mechanism that induces a time-dependent vacuum energy on cosmological scales. It is based on the instability-induced renormalization triggered by the low-energy quantum fluctuations in a Universe with a positive cosmological constant. We use the dynamical systems approach to study the qualitative behavior of the Friedmann–Robertson–Walker cosmologies where the cosmological constant is dynamically evolving according with this nonperturbative scaling at low energies. It will be shown that it is possible to realize ‘two regimes’ dark energy phases, where an unstable early phase of power-law evolution of the scale factor is followed by an accelerated expansion era at late times.

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1. Introduction

The idea that quantum gravity (QG) effects can be important at astrophysical and cosmological distances has recently attracted much attention. In particular, the framework of the exact renormalization group (ERG) approach for QG [1] has opened up the possibility of investigating both the ultraviolet (UV) and the infrared (IR) sector of gravity in a systematic manner.

An essential ingredient of this tool is the effective average action \( \Gamma_k[g_{\mu\nu}] \), a Wilsonian coarse grained free energy dependent on an IR momentum scale \( k \) which defines an effective field theory appropriate for the scale \( k \). By construction, when evaluated at tree level, \( \Gamma_k \) correctly describes all gravitational phenomena, including all loop effects, if the typical momentum involved is of the order of \( k \). When applied to the Einstein–Hilbert action the ERG yields renormalization group (RG) flow equations [2], which have made possible detailed investigations of the scaling behavior of Newton’s constant at high energies [3–15]. The scenario emerging from these studies, first demonstrated by Weinberg [16] in \( d = 2 + \epsilon \) dimensions, suggests that the theory could be consistently defined in \( d = 4 \) at a nontrivial UV fixed point where the dimensionless Newton constant, \( g(k) = G(k)k^2 \), does not vanish in the \( k \to \infty \) limit, i.e. \( g(k \to \infty) = g^* \). As a consequence, the dimensionful Newton constant \( G(k) \) is antiscreened at high energies, very much as one would expect based on the intuitive picture that the larger the cloud of virtual particles, the greater the effective mass seen by a distant observer [17].

Recent works have included matter fields [18] and have also considered a growing number of purely gravitational operators in the action. In particular, truncations involving quadratic terms in the curvature have been considered in [11, 19–21], while higher powers of the Ricci scalar have been studied in [22, 23]. In all the investigations the UV critical surface has turned out to be finite dimensional (\( d_{UV} = 3 \)), implying that the theory is nonperturbatively renormalizable. At the UV fixed point the theory has a behavior very similar to QCD, being weakly coupled at high energies, but the running of the dimensionful Newton constant \( G(k) = g(k)/k^2 \) in the deep UV region is a power law, at variance with the logarithmic scaling of QCD.

A weakly coupled gravity at high energies is expected to generate important consequences in several astrophysical and cosmological contexts [24], and in fact the RG flow of the effective average action, obtained by different truncations of theory space, has been the basis of various investigations of ‘RG improved’ black hole spacetimes [25–27] and Early Universe models [28–33].

However, the behavior of the theory is more complicated at low energies, corresponding at cosmological scales, because the \( \beta \)-functions of any local operator of the type \( \sqrt{g} R, \sqrt{g} R^2 \), . . . ,
\( \sqrt{g} R^n \) are singular in the IR due to the presence of a pole at \( \Lambda(k)/k^2 = 1/2 \) with \( \Lambda \) being the cosmological constant. The presence of this pole is signaling that the Einstein–Hilbert truncation is no longer a consistent approximation to the full flow equation, and most probably a new set of IR-relevant operators is emerging in the \( k \to 0 \) limit. This singular behavior in fact appears in the ERG for nearly all cutoff threshold functions in the Einstein–Hilbert truncation, and it is caused by the presence of negative eigenvalues in the spectrum of the fluctuations determinant of the gravitational degrees of freedom. As discussed in [34], the dynamical origin of these strong IR effects is due to the ‘instability-driven renormalization’, a phenomenon well known from many other physical systems [35–37]. We shall see that the low energy domain of the theory is regulated by an IR fixed point which drives the cosmological constant to zero at very late times \( \Lambda(t \to \infty) = 0 \). This new type of ‘decaying \( \Lambda \) cosmologies’ is therefore quite different from previous models where the time-dependent cosmological constant encodes the effect of the matter creation process [38].

Astrophysical consequences of the possible presence of an IR fixed point for QG appeared in [32, 39–42] where it was shown that a solution of the ‘cosmic coincidence problem’ arises naturally without the introduction of a ‘dark energy field’. In particular in the fixed point regime the vacuum energy density \( \rho_\Lambda = \Lambda/8\pi G \) is automatically adjusted so as to equal the matter energy density, i.e. \( \Omega_\Lambda = \Omega_M = 1/2 \), and the deceleration parameter approaches \( q = -1/4 \). Moreover, an analysis of the high-redshift SNe Ia data leads to the conclusion that this IR fixed point cosmology is in good agreement with the observations [41]. Recent works have also considered the possibility that the ‘basin of attraction’ of the IR fixed point can act already at galactic scale, thus providing an explanation for the galaxy rotation curve without dark matter [34, 43–45], although a detailed analysis based on available experimental data is still missing. Cosmologies discussing the complete evolution from the early Universe to the present time have been considered in [46] and [47]. In particular in [47] it has been shown that the ‘RG improved’ Einstein equations admit (power-law or exponential) inflationary solutions and that the running of the cosmological constant can account for the entire entropy of the present universe in the massless sector (see also [48] for a review).

The main purpose of this paper is to explore the idea that quantum effects could dynamically drive the cosmological constant to zero at late times so that \( \Lambda(t \to \infty) = 0 \) as a result of an explicit dynamical mapping \( k \to k(t) \) of the RG trajectories generated by the unstable IR modes of the gravitational sector.

This idea will then be studied using the so-called dynamical systems approach (DSA), a technique already used in cosmology by Bogoyavlensky [49] and further developed by Collins, and Ellis and Wainwright to analyze nontrivial cosmologies (i.e. Bianchi models) in the context of pure general relativity (GR) [50]. Some work has also been done in the case of (minimally coupled) scalar fields in cosmology [51, 52] and, more recently, of scalar tensor theories of gravity [53], \( f(R) \) theories of gravity [53–57] and the Hořava–Lifschits gravity [58]. Studying cosmologies using the DSA has the advantage of offering a relatively simple method to obtain particular exact solutions and to obtain a (qualitative) description of the global dynamics of these models.

In our specific context, the DSA allows us to prove that the presence of a singular behavior of the RG evolution for the cosmological constant in the IR puts strong constraints on the possible RG trajectories. In particular, we will show that it is possible to realize a scenario where the Universe has a transition from an early unstable phase of power-law evolution of the type \( a \propto t^\beta \) with \( \beta \in ]0, 1[ \) to a de Sitter phase.
The structure of the paper is as follows. In section 2, the basic mechanism of dynamical suppression of the cosmological constant by unstable low-energy modes is discussed. In section 3, the dynamical system analysis of the resulting cosmologies is presented. Section 4 is devoted to the conclusions.

2. Instability-induced renormalization

It is interesting to review in detail the main arguments suggesting that the cosmological constant \( \Lambda \) must have a nontrivial scaling at cosmological distances due to QG effects [34]. As already mentioned, the key mechanism is the so-called ‘instability-induced renormalization’. In order to illustrate this point, let us look at the \( \mathbb{Z}_2 \)-symmetric real scalar field in a simple truncation:

\[
\Gamma_k[\phi] = \int \! d^4x \, \left\{ \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} m^2(k) \phi^2 + \frac{1}{12} \lambda(k) \phi^4 \right\}.
\]

In momentum representation we have

\[
\Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta \phi^2} = p^2 + m^2(k) + \lambda(k) \phi^2.
\]

so that \( \Gamma_k^{(2)} \) is positive if \( m^2 > 0 \); but when \( m^2 < 0 \), it can become negative for \( \phi^2 \) small enough. Of course, the negative eigenvalue for \( \phi = 0 \), for example, indicates that the fluctuations are unstable, and the nonlinear evolution of this instability is a ‘condensation’ which shifts the field from the ‘false vacuum’ to the true one, the phenomenon that produces the instability-induced renormalization. In particular, the \( \beta \)-functions, obtained by \( p \)-integrals over (powers of) the propagator \([p^2 + m^2(k) + k^2]^{-1}\), are regular in the symmetric phase \( m^2 > 0 \), but there is a pole at \( p^2 = -m(k)^2 - k^2 \) provided \( k^2 \) is small enough in the broken phase \( m^2 < 0 \). For \( k^2 \ll |m(k)|^2 \), the \( \beta \)-functions become large and there the instability-induced renormalization occurs. In a reliable truncation, a physically realistic RG trajectory in the spontaneously broken regime will not hit the singularity at \( k^2 = |m(k)|^2 \), but rather make \( m(k) \) run in such a way that \( |m(k)|^2 \) is always smaller than \( k^2 \). This requires that

\[
-m(k)^2 \propto k^2.
\]

In order to avoid the singularity, a mass renormalization is necessary in order to evolve a double-well-shaped symmetry-breaking classical potential into an effective potential that is convex and has a flat bottom, as it emerges from analytical and numerical calculations [37]. However, the truncation implied in equation (1) is not enough to describe the broken phase, because its RG trajectories terminate at a finite scale \( k_{\text{term}} \) with \( k_{\text{term}}^2 = |m(k_{\text{term}})|^2 \) at which the \( \beta \)-functions diverge. Instead, if one allows for an arbitrary running potential \( U_\phi(\phi) \), containing infinitely many couplings, all trajectories can be continued to \( k = 0 \), and for \( k \to 0 \) one finds indeed the quadratic mass renormalization (3) as discussed in [37].

In the case of gravity, we can consider a family of ‘off-shell’, spherically symmetric backgrounds labeled by the radius of the sphere \( \phi \), in order to disentangle the contributions from the two invariants \( \int \! d^4x \, \sqrt{g} \propto \phi^4 \) and \( \int \! d^4x \, \sqrt{g} \, R \propto \phi^2 \) to the Einstein–Hilbert flow. It is then convenient to decompose the fluctuation \( h_{\mu\nu} \) on the sphere into irreducible components [4] and to expand the irreducible pieces in terms of the corresponding spherical harmonics. For \( h_{\mu\nu} \) in the transverse traceless (TT) sector, the operator \( \Gamma_k^{(2)} \) equals, up to a positive constant,

\[
-\nabla^2 + 8 \phi^2 - k^2 - 2 \Lambda(k),
\]

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They lead to strong IR renormalization effects for \( k^2 \) gravity with \( \Lambda_1 > 0 \), denoted by \( \{p^2\} \). In this case, there are only stable, bounded oscillations, leading to a mild fluctuation-induced renormalization. The situation is very different for \( \Lambda > 0 \) where, for \( k^2 \) sufficiently small, equation (4) has negative eigenvalues, i.e. unstable eigenmodes.

A consistent calculation including all the components of the metric fluctuation \( h_{\mu\nu} \) explicitly illustrates this scenario. Following [10], the \( \beta \)-functions for the dimensionless Newton constant \( g(k) \equiv k^{d-2}G(k) \) and the dimensionless cosmological constant \( \lambda(k) \equiv k^{-2}\Lambda(k) \) can be introduced:

\[
\begin{align*}
\partial_k g &= \beta_k(g, \lambda, \gamma) \equiv [d - 2 + \eta_N]g, \\
\partial_k \lambda &= \beta_k(g, \lambda),
\end{align*}
\]

where

\[
\beta_k \equiv -(2 - \eta_N)\lambda + 4(4\pi)^{1-(d/2)} \left[ \frac{d(d+1)}{4} (1 - 2\lambda)^{(d/2)-n-1} - d \right] g \frac{\Gamma(n+1-d/2)}{\Gamma(n+1)},
\]

with \( \eta_N \equiv -G(k)^{-1}\partial_k G(k) \) being the anomalous dimension. Its explicit expression reads

\[
\eta_N = 8(4\pi)^{1-(d/2)} \left[ \frac{d(d-5d)}{24} (1 - 2\lambda)^{(d/2)-n-2} - \frac{d+6}{6} \right] g \frac{\Gamma(n+2-d/2)}{\Gamma(n+1)},
\]

where \( n > 1 \) is an integer related to the regulator and \( d \) is the dimension of the spacetime [10].

Clearly, the allowed part of the \( g-\Lambda \)-plane (\( \lambda < 1/2 \)) in equations (7) and (8) corresponds to the situation \( k^2 > 2\Lambda/k \) where the singularity is avoided thanks to the large regulator mass. When \( k^2 \) approaches \( 2\Lambda \) from above, the \( \beta \)-functions become large and strong renormalizations set in, driven by the modes that would go unstable at \( k^2 = 2\Lambda \). In this respect the situation is completely analogous to the scalar theory discussed above: its symmetric phase \( (m^2 > 0) \) corresponds to gravity with \( \Lambda < 0 \); in this case all fluctuation modes are stable and only small renormalization effects occur. Conversely, in the broken phase \( (m^2 < 0) \) and in gravity with \( \Lambda > 0 \), there are modes which are unstable in the absence of the IR regulator. They lead to strong IR renormalization effects for \( k^2 \sim [m(k)]^2 \) and \( k^2 \sim 2\Lambda(k) \), respectively.

We are thus led to the conclusion that the instability-induced renormalization should occur also in this framework as \( k \to 0 \), so that to avoid the singularity the cosmological constant must run proportional to \( k^2 \),

\[
\Lambda(k) = \lambda^{IR}_2 k^2 + \text{subleading terms}, \quad k \to 0,
\]

with the constant \( \lambda^{IR}_2 < 1/2 \) being an IR fixed point of the \( \lambda \)-evolution. If the behavior (9) is actually realized, the renormalized cosmological constant observed at very large (cosmological) distances, \( \Lambda(k \to 0) \), vanishes regardless of its bare value. It is important to stress that recent investigations based on a conformal reduction of Einstein gravity have actually found a new IR fixed point which could represent the counterpart, in the reduced theory, of the physical IR fixed point present in the full theory [59].

As in the case of the scalar field, the presence of an IR pole is signaling that the Einstein–Hilbert truncation is no longer a consistent approximation to the full flow equation near the IR singularity, and most probably, a new set of IR-relevant operators is emerging at \( k \to 0 \). Although we do not have an explicit solution for the RG flow near the unstable phase in
the case of gravity, near the IR fixed point we can always write
\[ \frac{\Lambda(k)}{k^2} = \lambda^{\text{IR}}_0 + h_2 k^{2\theta}, \quad k \to 0, \]
(10)
with \( \theta \) being a critical exponent and \( h_2 \) a constant related with the eigenvalue of the stability matrix around the IR fixed point. Its precise value cannot be determined within the linearized theory, but we shall see that our analysis will not depend on the actual value of \( h_2 \). For \( \theta > 0 \), the IR fixed point is attractive, while for \( \theta < 0 \) it is repulsive. In the latter case, the IR fixed point is a high-temperature fixed point where \( \beta \)-functions are suppressed as \( k \to 0 \) and the flows stops before reaching \( \lambda^{\text{IR}}_0 \).

In order to close the system, we must map the RG flow onto the Universe dynamics so that \( k \to k(t) \). Clearly, the Hubble scale \( H(t) \) would be a natural choice for the IR cutoff \( k \) because in cosmology the Hubble length \( 1/H(t) \) measures the size of the ‘Einstein elevator’ outside which curvature effects become appreciable [46, 47] and therefore \( k \sim H(t) \). For actual calculations, we will set
\[ k(t) = \xi H(t), \]
(11)
with \( \xi \) being a positive number that we expect to be of the order of unity. We are thus led to the following \( \Lambda(H) \) dependence:
\[ \Lambda = H^2 \xi^2 \lambda_0 + h_2 \xi^\alpha \equiv A H^2 + B H^\alpha, \quad \alpha = 2 + 2 \theta. \]
(12)
Here \( 0 < \alpha < 2 \) corresponds to a repulsive IR fixed point, while \( \alpha > 2 \) represents an attractive IR fixed point. The parameters \( A = \xi^2 \lambda_0 \) and \( B = \xi^\alpha h_2 \) are introduced for notation simplicity instead of \( \xi \) and \( h_2 \) in the following discussion. Remarkably, we shall see that our conclusions will be independent of the actual value of \( B \), while the parameters \( A \) and \( \alpha \) will be the only free parameters of our theory.

3. Dynamical system analysis

3.1. Basic equations

From the previous discussion, it is then clear that in this framework the cosmological constant \( \Lambda \) is promoted to the status of dynamical variable by equation (12), so that \( \Lambda = \Lambda(t) \).

Let us then specialize \( g_{\mu\nu} \) to describe a generic Friedman–Robertson–Walker metric with scale factor \( a(t) \), where we take \( T_{\mu}^{\nu} = \text{diag}[\rho, -p, -p, -p] \) to be the energy momentum tensor of an ideal fluid with the equation of state \( p = w\rho \), \( w \geq 0 \) being a constant.\(^6\) Then the QG ‘improved’ Einstein equations reduce to the following set of cosmological equations:
\[ \frac{\kappa}{a^2} + H^2 = \frac{8\pi G \rho}{3} + \frac{1}{3} \Lambda, \]
(13)
\[ \dot{H} + H^2 = -\frac{4}{3} \pi G \rho (3w + 1) + \frac{1}{3} \Lambda, \]
(14)
\(^5\) Even for the simple scalar theory the non-perturbative investigation of the IR instability requires the use of special numerical techniques to correctly resolve the singularity; see [37] for details.
\(^6\) The dynamical systems analysis that will follow can be easily generalized to the case of non-constant \( w \) by, for example, adding a new variable corresponding to the pressure. However, this information does not add any further understanding of the relation between RG flow and dark energy and we will therefore limit ourselves to the case of a single fluid with a constant barotropic factor \( w \).
\[ \dot{\rho} = -3(1 + w)H\rho - \frac{\rho G}{G} - \frac{\dot{\Lambda}}{8\pi G}, \]  
(15)

where \( H = \dot{a}/a \) and \( \kappa = -1, 0, 1 \) with the usual meaning.

It is difficult to determine the running of \( G \) near the infrared fixed point since the \( \beta \)-functions are singular near \( k \rightarrow 0 \), as discussed before. However, one expects that if an infrared fixed point is also present in the running of \( g \) so that \( g(k \rightarrow 0) = g_* \), the dimensionful coupling constant \( G(k) \) grows without bound as \( k \rightarrow 0 \), so that \( G(t \rightarrow \infty) = \infty \) at late times.

In the following, we will assume that the energy momentum tensor of matter field is conserved so that we are led to the following ‘consistency condition’:

\[ \dot{G} = -\frac{\dot{\Lambda}}{8\pi \rho}, \]  
(16)

and the dynamical evolution of \( G \) is fixed by this request as a function of \( \Lambda(t) \).

We shall see that the late-time behavior of \( G \) obtained by the assumption (16) is actually consistent with the possibility that running of \( g \) in the IR is determined by an IRFP.

By using equation (12) in the Friedmann equation, we find that

\[ \frac{3\kappa}{a^2} + (3 - A)H^2 = BH^\alpha + 8\pi G\rho, \]  
(17)

which resembles the standard GR one. Assuming \( H \neq 0 \) and \( A \neq 3 \), we define dimensionless variables

\[ x = \frac{8\pi G\rho}{(3 - A)H^2}, \quad y = \frac{BH^{\alpha - 2}}{(3 - A)}, \quad K = \frac{3\kappa}{(3 - A)H^2a^2}, \]  
(18)

which will be considered to be functions of the logarithmic time \( N = \ln(a/a_0) \), where \( a_0 \) is the value of the scale factor at some reference time. With some algebra the cosmological equations can be written as the first order autonomous system

\[ x' = \frac{2A}{3} - \left[ \frac{1}{3}(A + 3) - (A - 3)w \right]x - \frac{1}{3}(A - 3)(1 + 3w)x^2 - \frac{1}{3}[\alpha(A - 3) + 2A]y \]
\[ + \frac{1}{3}\alpha(A - 3)y^2 + \frac{1}{6}(A - 3)(4 - \alpha(1 + 3w))xy, \]  
(19)

\[ y' = \left( 1 - \frac{A}{3} \right)(2 - \alpha) \left[ 1 + \frac{(1 + 3w)x}{2} - y \right]y, \]  
(20)

\[ K' = K \left[ \left( 1 - \frac{A}{3} \right)(1 + 3w)x - 2 \left( 1 - \frac{A}{3} \right)y - \frac{2}{3}A \right], \]  
(21)

with the constraint

\[ 1 + K - x - y = 0. \]  
(22)

In the above equations, the prime stands for the derivative with respect to \( N \).

Equation (22) allows us to eliminate one of the equations (19)–(21) and obtain a two-dimensional (2D) phase space. We choose here to eliminate equation (21) to obtain

\[ x' = \frac{2A}{3} - \left[ \frac{1}{3}(A + 3) - (A - 3)w \right]x - \frac{1}{3}(A - 3)(1 + 3w)x^2 - \frac{1}{3}[\alpha(A - 3) + 2A]y \]
\[ + \frac{1}{3}\alpha(A - 3)y^2 + \frac{1}{6}(A - 3)(4 - \alpha(1 + 3w))xy, \]  
(23)
Table 1. Coordinates of the fixed points, eigenvalues and solutions for the system (19)–(21).

| Point | Coordinates $(x, y)$ | Eigenvalues | $\Lambda(t)$ |
|-------|----------------------|-------------|-------------|
| $A$   | $[1, 0]$             | $\left\{ \frac{1}{2} (A - 3) (\alpha - 2) (1 + w), 1 + 3 w - A (1 + w) \right\}$ | $\Lambda = \frac{4A^2}{(A-3)(A+4)}$ |
| $B$   | $\left[ \frac{2A}{(3-A)(1+3w)}, 0 \right]$ | $\{ \alpha(1 + 3 w) - 2 A (w + 1) + \lambda, \alpha(1 + 3 w) - 2 A (w + 1) - \lambda, \}$ | $\Lambda = \frac{4A^2}{(A-3)(A+4)}$ |
| $C$   | $[0,1]$              | $[-2, \frac{1}{2} (3 - A) (\alpha - 2) (1 + w)]$ | $\Lambda = \frac{A}{(t-t_0)^2}$ |

$\lambda = \sqrt{(\alpha(1 + 3 w) - 2 A (w + 1))^2 - 4(3w + 1)^2(\alpha - 2)((A - 3)w + A - 1)}$

$v = \left( 1 - \frac{A}{3} \right) (2 - \alpha) \left[ 1 + \frac{(1 + 3 w) x}{2} - y \right] y$, \hspace{1cm} (24)

$0 = 1 + K - x - y$. \hspace{1cm} (25)

Note that if $y = 0$, the above system implies $y' = 0$ and the $x$-axis is an invariant submanifold for the phase space. This means that if the initial condition for the cosmological model is $y \neq 0$ a general orbit can approach $y = 0$ only asymptotically. As a consequence, there is no orbit that crosses the $x$-axis and no global attractor can exist in the phase space.

3.2. Finite analysis

Setting $x' = 0$, $y' = 0$, we obtain the three fixed points in table 1.

Two of these points ($A, C$) do not depend on the values of the parameters, but one ($B$) has the $x$-coordinate which is a function of $A$ and the barotropic factor $w$. This fact influences also the value of $K$, i.e. the sign of the spatial curvature index $\kappa$. Merging will occur between $A$ and $B$ for

$$A = \frac{1 + 3 w}{1 + w}$$

and between $B$ and $C$ for $A = 0$. The first value for $A$ represents a bifurcation for the dynamical system and we will not treat it here; the second corresponds to the case in which $\Lambda$ does not contain any quadratic term as a function of $H$.

The solutions associated with the above fixed points can be obtained integrating the equation

$$\dot{H} = \left( 1 - \frac{A}{3} \right) \left( -1 - \frac{1}{2} (1 + 3 w) x + y \right) H^2$$

and are listed in table 1. We can see that $A$ is a power-law solution whose index resembles the standard Friedmann solution, but it is modified by the parameter $A$. It is easy to see that this
Table 2. Stability of the fixed points for the system (19)–(21). When the stability depends on the barotropic factor both the characters of the fixed points are shown. See the text for more details.

| Point | $\alpha < 2$ | $\alpha > 2$ |
|-------|---------------|---------------|
| $A$   | Repeller      | Saddle        |
| $B$   | Saddle        | Repeller      |
| $C$   | Attractor     | Saddle        |

Point represents an expansion if $A < 3$ and a contraction if $A > 3$. The point $B$ corresponds to a Milne solution and $C$ corresponds to a de Sitter solution.

The behavior of the energy density and of the gravitational ‘constant’ can only be achieved when an assumption on $G$ has been made. Using equation (16) we find that the behavior of $G$ is in general given by a combination of powers of $t$ which depend on $w$ and $\alpha$, as emerges from the results shown in table 1. However, note in particular that for $0 < A < 3$ for the case $A$ and for the case $B$ the evolution of $G$ implies a strongly coupled gravity at late times, as implied by the IRFP point model [39]. Substitution into the field equations reveals that the point $A$ represents a physical solution (i.e. satisfies equations (13)–(15)) only if $\kappa = 0$ and $B = 0$, while $A$ can take any value.

The solution for $B$ is physical only for $B = 0$; the space curvature is given by

$$\kappa = a_0^2 \left[ \frac{A(1 + w)}{(1 + 3w)} - 1 \right].$$

so that this solution is not flat in general (i.e. $\kappa \neq 0$). In the special case $\alpha = 2$, $B$ is not constrained and $A \rightarrow A + B$ in equation (27). The solution associated with the point $C$ is instead physical only if

$$\rho_0 = 0, \quad B = (3 - A)H_0^{2-\alpha}.$$  

It is interesting to note here that, since at the fixed points $A$ and $B$ the parameter $B$ is zero, these points represent states for the cosmology that are indistinguishable from standard GR. This, as we will see, will be an interesting feature of the physical interpretation of the orbits. The stability of the fixed points can be inferred with the standard techniques of dynamical system analysis and is summarized in table 2.

A global picture of the phase space that summarizes the results above can be found in figures 1 and 2.

There are also other global properties of the phase space that we can deduce from equation (26). Specifically, the deceleration parameter $q$ can be written in terms of the dynamical variables as

$$q = -1 - \frac{\dot{H}}{H^2} = -\frac{1}{6} [2A + 2(A - 3)y + (A - 3)x (1 + 3w)].$$

This means that the line

$$y = \frac{A}{A - 3} + \frac{1 + 3w}{2} x$$

\cite{NJP12}
Figure 1. Phase plots of the system (19)–(21) for $\alpha < 2$ and dust ($w = 0$).
(a) The case $A < 1$, (b) the case $1 < A < 3$ and (c) the case $A > 3$.

divides the phase plane into two regions

\begin{align*}
A > 3 & \quad \left\{ \begin{array}{ll}
y > \frac{A}{A-3} + \frac{1+3w}{2} \quad (q < 0), \\
y < \frac{A}{A-3} + \frac{1+3w}{2} \quad (q > 0), \\
\end{array} \right. \\
A < 3 & \quad \left\{ \begin{array}{ll}
y > \frac{A}{A-3} + \frac{1+3w}{2} \quad (q > 0), \\
y < \frac{A}{A-3} + \frac{1+3w}{2} \quad (q < 0), \\
\end{array} \right. 
\end{align*}

in which the decelerating factor is only positive or only negative. This allows us to understand if a specific orbit includes the transition between a decelerating and an accelerating phase typical of the dark energy. In addition, substituting the coordinates of the fixed points, we can check if they represent a decelerating or accelerating solution. For point $C$ the deceleration factor is always negative as expected by the nature of the associated solution. The point $B$ instead lies
Figure 2. Phase plots of the system (19)–(21) for $\alpha > 2$ and dust ($w = 0$). (a) The case $A < 1$, (b) the case $1 < A < 3$ and (c) the case $A > 3$.

always on the line (30) and this is again expected by the associated solution. Finally, the point $A$ represents an accelerated expansion if

$$\frac{1 + 3w}{1 + w} < A < 3,$$

a decelerated expansion if

$$A < \frac{1 + 3w}{1 + w},$$

and an accelerated contraction if $A > 3$. 

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Table 3. Asymptotic fixed points, $\theta$ coordinates and solutions for the system (19)–(21).

| Point | $\theta$ | Behavior |
|-------|----------|----------|
| $A_\infty$ | 0 | $a = a_0 \exp[\frac{1}{2}C_1^2 (t - t_0)]$ |
| $B_\infty$ | $\pi$ | $a = a_0 \exp[\frac{1}{2}C_1^2 (t - t_0)]$ |
| $C_\infty$ | $3\pi/4$ | $a = a_0 \exp \left[ -\frac{(\alpha - 1)(t - t_0)}{2 - \alpha} \right]$ |
| $D_\infty$ | $7\pi/4$ | $a = a_0 \exp \left[ -\frac{(\alpha - 1)(t - t_0)}{2 - \alpha} \right]$ |
| $E_\infty$ | $\arctan \left( \frac{1 + 2w}{2} \right)$ | $a_0 \exp \left[ (t - t_0)^r \left( \frac{-3(2 - w)(3w + 2) - 3w + 2}{3w - 2(w + 1)(w - 1)} \right) \right]$ |
| $F_\infty$ | $\arctan \left( \frac{1 + 2w}{2} \right) + \pi$ | $a_0 \exp \left[ (t - t_0)^r \left( \frac{-3(2 - w)(3w + 2) - 3w + 2}{3w - 2(w + 1)(w - 1)} \right) \right]$ |

$r = 1 + \frac{4}{\sqrt{4(a - 2)w^2 - 4(a - 2)w - 3a + 2}}$

3.3. Asymptotic analysis

Looking at equation (25) it is clear that the phase plane is not compact and it is possible that the dynamical system (23) and (24) has a nontrivial asymptotic structure. Thus the above discussion would be incomplete without checking the existence of fixed points at infinity (i.e. when the variables $x$ or $y$ diverge) and calculating their stability. Such points represent regimes in which one or more of the terms in the Friedmann equation (17) become dominant and should not be confused with the time asymptotics, i.e. $t \to \infty$.

The asymptotic analysis can be easily performed by compactifying the phase space using the so-called Poincaré method [60]. The compactification can be achieved by transforming to polar coordinates $(r(N), \theta(N))$:

$$x \to r \cos \theta, \quad y \to r \sin \theta,$$

and substituting $r \to \frac{R}{1 - R}$ so that the regime $r \to \infty$ corresponds to $R \to 1$. Using the coordinates (33) and taking the limit $R \to 1$, equations (23) can be written as

$$R' = \frac{1}{12} (A - 3)(\alpha \sin 2\phi + \alpha \cos 2\phi - \alpha + 4)(3w \cos \phi - 2 \sin \phi + \cos \phi),$$

$$\theta' = -\frac{\alpha(A - 3)}{4(\rho - 1)} [(1 + 3w) \cos \theta - 2 \sin \theta] (-1 + \cos 2\theta - \sin 2\theta).$$

It can be proved that the existence and stability of the fixed points can be derived analyzing the fixed point of the equation for $\theta'$ [60]. Setting $\theta' = 0$, we obtain six fixed points that are given in table 3. The solution associated with the asymptotic fixed points can be derived in the same way as the ones for the finite case and are also shown in table 3. For details of this derivation, see the detailed discussion in [54]. The solutions associated with the asymptotic fixed points are of two basic types. The first one has exponential growth, i.e. a de Sitter phase (points $A$–$B$), and the second one’s growth and decay depend on the values of $\alpha$ and $w$. In particular, they can represent bounces or cosmologies in which the deceleration parameter changes sign.

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Table 4. Stability of the asymptotic fixed points for the system (19)–(21).

| Point | $A < 3$ | $A > 3$ |
|-------|---------|---------|
|       | $\alpha < 0$ | $0 < \alpha < 2$ | $\alpha > 2$ |
| $A_\infty$ | Attractor | Saddle | Saddle |
| $B_\infty$ | Repeller | Saddle | Saddle |
| $C_\infty$ | Saddle | Repeller | Saddle |
| $D_\infty$ | Saddle | Attractor | Saddle |

Finally, the stability of the asymptotic fixed points can be obtained with the standard methods of the dynamical system. The results for the first four fixed points ($A_\infty$–$D_\infty$) are given in table 4.

The same is not true for the last two fixed points ($E_\infty$, $F_\infty$) whose stability does depend on $w$. The character of these points is complicated by the fact that the value of $R'$, whose sign is connected to the stability in the radial direction, is zero. Fortunately, this problem can be solved noting that the next to leading term in the full equation for $R'$ is finite and different from zero and can give information on the behavior of the function $R'$ near $R = 1$. The stability thus obtained is shown in tables 5 and 6.

Now that the asymptotic fixed points and their stability have been determined, let us examine in more detail their physical interpretation.

As stated before, these points are characterized by one or more variables to become infinite. In terms of the definitions (18), this corresponds to the fact that either the quantities in the numerators are infinite or the ones in the denominators approach zero. In the first case we are probably seeing some kind of singularity of the model. In the second we are seeing a change in sign of the expansion, which in turn corresponds to a maximum or a minimum of the scale factor. It is important to bear in mind that, because of our definition of the time variable, when an orbit ‘reaches’ an infinite point the time coordinate changes sign and the Universe follows the orbit with a reversed orientation. In some sense, one can picture this transition as the fact that the asymptotic points are the doorway to a mirror phase space in which the orbit orientation and stability of the fixed points are reversed.
Table 6. Stability of the asymptotic fixed point $F_\infty$.

| Attractor | Repeller | Saddle |
|-----------|----------|--------|
| $w = 0$   | $\alpha < -10 \land A < 3$ | $\alpha < -10 \land A > \frac{3a - 6}{\alpha + 10}$ | Otherwise |
|           | $-10 < \alpha < 0 \land \frac{3a - 6}{\alpha + 10} < A < 3$ | $\alpha > 0 \land A < \frac{3a - 6}{\alpha + 10}$ | |
|           | $\alpha > 0 \land A > 3$ | $\alpha > 0 \land A < 0$ | Otherwise |
| $w = \frac{1}{3}$ | $\alpha < 0 \lor 0 < A < 3$ | $\alpha < 0 \land A < 0$ | Otherwise |
|           | $\alpha > 0 \land A > 3$ | $\alpha > 0 \land A < 0$ | |
| $w = 1$   | $\alpha < 0 \land \frac{3a - 6}{a - 5} < A < 3$ | $0 < \alpha < 5 \land A < \frac{3a - 6}{a - 5}$ | Otherwise |
|           | $0 < \alpha \leq 5 \land A > 3$ | $\alpha > 5 \land 3 < A < \frac{3a - 6}{a - 5}$ | |

In the pure GR framework, where the cosmological equation predicts (almost) always a monotonic behavior of the scale factor, extrema of the scale factors can occur only at the origin of time. We usually talk then of ‘bounces’ or ‘(re)collapsing universes’ depending on the sign of the second derivative of the scale factor (or $\ddot{a}$). In more complex cosmological models, such as e.g. $f(R)$ gravity [61], the behavior is more complicated and the scale factor can have in principle a series of extrema located at a generic instant. This is the case also for the RG cosmologies. However, we will retain the traditional names to indicate such features of the scale factor.

Let us focus, for example, on the conditions to have a bounce, i.e. a situation in which $\dot{a} = 0$ and $\ddot{a} > 0$. In terms of the dynamical variables these can be translated in the requirement that one has an asymptotic attractor characterized by $H = 0 = \dot{a}$ and that this attractor lies in the part of the phase space for which $-(1 + q) < 0$, which is determined via equation (30). Using this condition it turns out, for example, that for $\alpha > 0$, only $D_\infty$ can represent a bounce. The results for the other points can be found in table 7.

4. Conclusions

In this paper, we have investigated the effect on cosmological scale of a QG-related decaying of the cosmological constant. This investigation was performed using the DSA that allows a semi-quantitative treatment of the model via the construction of a phase space directly connected to the behavior of the equations. The phase space is two dimensional and contains a total of nine fixed points, of which three are finite and six are asymptotical. The stability of these points is found to depend on the parameters $A$ and $\alpha$, but not on $B$, which has a marginal position in the entire analysis. The determination of the fixed points and their stability allows us to infer the shape of the phase space orbits and to deduce the qualitative features of the cosmic histories possible in this model.

Among these, one is particularly relevant in terms of the problem of dark energy domination, i.e. the presence of a cosmic history characterized by a first phase in which the scale factor grows as a power law with exponent included in $[0, 1]$, followed by a second phase with a faster growth. In terms of the features of the phase space such a scenario can be realized.
Table 7. Physical interpretation of the asymptotic attractors.

| Point | Bounce | (Re)collapsing | Singularity |
|-------|--------|----------------|-------------|
| $A_\infty$ | No | No | Yes |
| $B_\infty$ | No | No | Yes |
| $C_\infty$ | No | $0 < \alpha < 2$, $A > 3$ Otherwise |
| $D_\infty$ | $0 < \alpha < 2$, $A > 3$ | No | Otherwise |
| $E_\infty (w = 0)$ | $\alpha \leq -10 \land 3 < A < \frac{3w - 6}{w + 10}$ | $0 < \alpha < \frac{610A}{3 - A} \land -\frac{3}{5} < A < 0$ | Otherwise |
| $F_\infty (w = 0)$ | $\alpha \leq -10 \land 3 < A < \frac{3w - 6}{w + 10}$ | $0 < \alpha < \frac{610A}{3 - A} \land -\frac{3}{5} < A < 0$ | Otherwise |
| $E_\infty (w = 1/3)$ | $\alpha < 0 \land A > 3$ | $0 < \alpha < 2 \land 0 < A < 3$ | Otherwise |
| $F_\infty (w = 1/3)$ | $\alpha < 0 \land A > 3$ | $0 < \alpha < 2 \land 0 < A < 3$ | Otherwise |
| $E_\infty (w = 1)$ | $\alpha \leq -10 \land 3 < A < \frac{3w - 6}{w + 10}$ | $0 < \alpha < \frac{610A}{3 - A} \land -\frac{3}{5} < A < 0$ | Otherwise |
| $F_\infty (w = 1)$ | $\alpha \leq -10 \land 3 < A < \frac{3w - 6}{w + 10}$ | $0 < \alpha < \frac{610A}{3 - A} \land -\frac{3}{5} < A < 0$ | Otherwise |

If one has an unstable fixed point representing the first phase and an attractor representing the second phase\(^7\). Looking at tables 2–6 one can easily see that there is only one set of the parameters $(A, \alpha)$ for which this can be realized: $(A < \frac{1 + 3w}{1 + w}$, $\alpha < 2)$, for which $A$ is a repeller. It is reassuring to notice that in this case $A$ is of the order of unity and, as $A = \xi^2 \lambda_*$, we deduce that also $\xi$ is of the order of unity being $\lambda_* < 1/2$.

When these conditions are satisfied there can exist, depending on initial conditions, one orbit that starts at $A$ and either goes directly to $C$ or bounces at $E_\infty$. The first case can be seen as a ‘classical’ Friedmann–de Sitter transition. In the second case the cosmological evolution could be richer because the solution $\tilde{E}_\infty$ can indicate, depending on the sign of $\alpha$, a growth whose rate saturates or an expansion phase comparable with a de Sitter one. One could interpret this as a ‘two regimes’ dark energy phase. Note that, as we have seen, at the fixed points $A$ and $B$ the parameter $B$ is zero; these points represent states for the cosmology that are indistinguishable from standard GR. This means that our model in the neighborhood of these points is indistinguishable from the standard cosmology. An example of the evolution of some key cosmological quantities compared with the standard de Sitter cosmology is given in figure 3, and in figure 4 we compare the additional RG term in equation (17) with the standard

\(^7\) The second point could be unstable as well, but then the set of the initial conditions able to realize such a scenario would be smaller and more difficult to calculate.
Figure 3. Evolution of some key cosmological quantities in RG cosmologies with $w = 0$, $\Lambda = 1/2$, $\alpha = 1$ (dashed) compared with the ones of GR $\Lambda$ (solid). We have chosen as the initial condition the phase space point $(x = 1.1, y = 0.1)$ present in both the phase spaces. The index 0 is associated with the value of all the quantities in this point. Upper panel: semi-logarithmic plot of $H / H_0$ as a function of $\ln (a / a_0)$. Lower panel: logarithmic plot of $\frac{8}{3} \pi G \rho$ as a function of $\ln (a / a_0)$. The energy density is not plotted because its behavior is exactly matched in the two models.

cosmological constant. It is important to emphasize that the standard experimental value of Newton’s constant, $G_{\text{exp}}$, does not coincide with the value $G(k = \xi H_0)$ which is relevant for cosmology today. $G_{\text{exp}}$ is measured (today) at $k_{\text{exp}} \propto \ell^{-1}$ where the length $\ell \equiv \ell_{\text{sol}}$ is a typical solar system length scale, say $10^{12}$ m. Thus, in terms of the running Newton constant, $G_{\text{exp}} = G(k = \xi' / \ell_{\text{sol}})$, since $\ell_{\text{sol}} \ll H_0$. It is only the cosmological quantity $G(k = \xi H)$ that dynamically evolves in the fixed point regime, not $G_{\text{exp}}$. This remark entails that the dynamical evolution of the cosmological Newton constant in the recent past does not ruin the predictions about primordial nucleosynthesis, which requires that $G(k = \xi H_{\text{nucl}})$ coincides with $G_{\text{exp}}$ rather precisely. In fact, at the time $t = t_{\text{nucl}}$ of nucleosynthesis for which $H = H(t = t_{\text{nucl}})$, the cosmological Newton constant was indeed $G(k = \xi H_{\text{nucl}}) \approx G_{\text{exp}}$ since $ct_{\text{nucl}} \approx H_{\text{nucl}}^{-1}$ and $\ell_{\text{sol}}$ are of the same order of magnitude. The phase space structure also sheds light on aspects of the type of RG cosmologies not directly related to the problem of dark energy. For example, it is clear that it is possible to interpret the transition to a dark era as the beginning of a
Figure 4. Evolution of the term $\frac{B H^\alpha}{A}$ in RG cosmologies with $w = 0$, $A = 1/2$, $\alpha = 1$, $B = 2$ (dashed) compared with the $\Lambda$ term in GR (solid). We have chosen as the initial condition the phase space point $(x = 1.1, y = 0.1)$ and the index 0 is associated with the value of all the quantities in this point.

primordial inflationary phase. Our results then imply that there is only one set of values that allows a ‘graceful exit’, i.e. the transition from inflation to a Friedmann cosmology. Specifically in the case ($A < \frac{1+3w}{1+4w}$, $\alpha > 2$), we can have this kind of scenario. However, since there is only one point which represents accelerating expansion, one can see that the RG model can be used to represent either the inflationary era or the dark energy era, but not both. Finally, the analysis of the asymptotic fixed points gives information on the possibility of changes in the sign of expansion rates in this type of cosmology. In GR, such phases are normally associated either with the so-called ‘bounces’ and are associated with cyclic Universes (when $H > 0$) or recollapsing Universes (when $H < 0$). In modified theories of gravity, changes in the sign of $H$ can occur also when the size of the Universe is large, because the scale factor does not have to be monotonic [61]. In terms of the phase space, this kind of behavior is associated with the presence of asymptotic attractors and the sign of the quantity $-(1+q)$ as expressed in equation (29) in the fixed point. In particular, we have that if $-(1+q) > 0$ we have a deceleration followed by an acceleration and if $-(1+q) < 0$ the opposite situation. A quick analysis shows, for example, that only some of the asymptotic attractors for very specific values of the parameters can give rise to a bounce.

The previous results point toward a cosmology with interesting features that, in our opinion, deserves further study. In particular, one could try to test the transition to the Dark Age we have found against the SnIa data to obtain some more constraints on the free parameters. This, and other issues, will be discussed in a forthcoming work.

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