On the Computational Complexity of Defining Sets

Hamed Hatami† Hossein Maserrat‡

Abstract

Suppose we have a family \( F \) of sets. For every \( S \in F \), a set \( D \subseteq S \) is a defining set for \((F, S)\) if \( S \) is the only element of \( F \) that contains \( D \) as a subset. This concept has been studied in numerous cases, such as vertex colorings, perfect matchings, dominating sets, block designs, geodetics, orientations, and Latin squares.

In this paper, first, we propose the concept of a defining set of a logical formula, and we prove that the computational complexity of such a problem is \( \Sigma_2 \)-complete.

We also show that the computational complexity of the following problem about the defining set of vertex colorings of graphs is \( \Sigma_2 \)-complete:

**INSTANCE:** A graph \( G \) with a vertex coloring \( c \) and an integer \( k \).

**Question:** If \( \mathcal{C}(G) \) be the set of all \( \chi(G) \)-colorings of \( G \), then does \( (\mathcal{C}(G), c) \) have a defining set of size at most \( k \)?

Moreover, we study the computational complexity of some other variants of this problem.

Keywords: defining sets; complexity; graph coloring; satisfiability.

1 Introduction

In this paper we consider a unification of the concepts already known as critical sets, forcing sets, and defining sets, where we formulate different natural problems in this regard. Specially, through considering such problems for 3SAT, by introducing suitable reductions, we prove that the decision problem related to the minimum defining set problem of graph coloring\(^1\) is \( \Sigma_2 \)-complete.

\(^*\)This research is done in Discrete Mathematics Laboratory of Department of Mathematics in Sharif University of Technology.

\(^†\)Department of Computer Science, University of Toronto, email: hamed@cs.toronto.edu

\(^‡\)Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

\(^1\)Defined formally below
Defining sets were studied for Latin squares [3, 7], perfect matchings [1, 2, 11], orientations [5], geodetics [6], vertex colorings [13], designs [9], and dominating sets [4].

Let $F$ be a family of sets. For every $S \in F$, a set $D \subseteq S$ is a defining set of $(F, S)$, if $S$ is the only element in $F$ which contains $D$ as a subset. By abuse of language every defining set of $(F, S)$ is also called a defining set of $F$.

In what follows, we try to introduce a general formulation for the type of problems we are going to consider in the rest of this paper.

Suppose an input $I$ is given. The input $I$ might be a graph, a number, or any other mathematical object. Then let $F(I)$ be a family of sets which is defined according to the set $I$. In this paper we are interested in the computational complexity of the following three general types of questions for specified inputs and definitions of $F$.

1. • Q1
   
   **Instance:** $I$, a set $S \in F(I)$, and a set $D \subseteq S$.
   
   **Question:** Is $D$ a defining set of $(F(I), S)$?

2. • Q2
   
   **Instance:** $I$, a set $S \in F(I)$, and an integer $k$.
   
   **Question:** Does $S$ have a defining set of size at most $k$?

3. • Q3
   
   **Instance:** $I$ and an integer $k$.
   
   **Question:** Does $F(I)$ have a defining set of size at most $k$?

The computational complexity of the problems related to defining sets was first studied by Colbourn in [7]. He studied Q1 when $F(n)$ is the set of Latin squares of order $n$, and proved that this question is CoNP-complete. Recently Adams, Mahdian, and Mahmoodian [1] studied Q2 when $F(G)$ is the set of perfect matchings of a graph $G$, and proved that the question is NP-complete. In [2] it is shown that the question Q3 for this family is NP-complete. It is not hard to see that the question Q1 for this family is in P. Hatami and Tusserkani in [12] studied Q2 and Q3 when $F(G)$ is the set of vertex colorings of a graph $G$, and proved that both of the questions are NP-hard. In this paper we improve their result by showing that these problems are both $\Sigma_2$-complete. In this regard we consider the family of all proper assignments to the variables of a $k$CNF where a $k$CNF is a Boolean expression in conjunctive normal form such that every clause has exactly $k$ variables. Let Q1-$k$SAT, Q2-$k$SAT, and Q3-$k$SAT stand for the three questions Q1, Q2, and Q3 in this case, respectively. We show that Q1-$3$SAT is CoNP-Complete, and Q2-$3$SAT and Q3-$3$SAT are both $\Sigma_2$-complete. We also refer the reader to the recent paper [10] for some other computational complexity results on the defining sets of vertex colorings.
We determine the computational complexity of Q1-3SAT, Q2-3SAT, and Q3-3SAT in Section 2. Section 3 is devoted to the study of the computational complexity of the questions Q1, Q2, and Q3 for the set of vertex colorings of a graph.

2 Defining sets and SAT

Let \( D \) and \( R \) be two sets, and \( f : D \rightarrow R \) be a function. We can refer to \( f \) as the set \( f = \{(x, f(x)) : x \in D\} \). This representation enables us to study the defining set of a family of functions.

Let \( \phi \) be a \( k \)CNF with variables \( V = \{v_1, v_2, \ldots, v_n\} \). For the sake of simplicity we use the notation \( \phi(v) \) instead of \( \phi(v_1, \ldots, v_n) \), where we would think of \( v \) as a vector of \( v_1, v_2, \ldots, v_n \). Since any truth assignment \( t : V \rightarrow \{true, false\} \) is a function, we can study the defining set of a family of assignments. A proper assignment of \( \phi \) is an assignment which makes \( \phi \) true. Let \( S \subseteq V \), be a subset of the variables of \( \phi \). A partial assignment of \( \phi \) over the set \( S \) is a truth assignment \( t : S \rightarrow \{true, false\} \).

The set \( S \) is called the support set of \( t \), and this is denoted by \( S = \text{supp}(t) \). A partial assignment over \( S \) is called proper if every clause of \( \phi \) contains at least one true literal from the variables in \( S \).

For every \( k \)CNF \( \phi \), let \( \mathcal{P}(\phi) \) denote the family of proper assignments of \( \phi \). We study the computational complexity of the general questions Q1, Q2, and Q3 for this special family. Let Q1-kSAT, Q2-kSAT, and Q3-kSAT stand for the three questions Q1, Q2, and Q3 in this case, respectively.

Duplicating a variable in a clause of a CNF does not change the family of its proper assignments. Hence if all clauses of a CNF are of size at most \( k \) (not necessarily equal to), then it can be converted to a \( k \)CNF. Therefore without loss of generality, we may always assume that all such expressions are in \( k \)CNF form.

In this section we show that Q1-3SAT is \textbf{CoNP}-Complete, and Q2-3SAT and Q3-3SAT are both \( \Sigma_2 \)-complete. From [14] we know that the following problem is \( \Sigma_2 \)-complete.

- \( \exists \forall \) 3SAT

\textbf{INSTANCE:} A 3CNF, \( \phi(x, y) \).

\textbf{QUESTION:} Is \( \exists x \forall y \phi(x, y) \)?

Next, we define the \( \exists \forall \exists \) \( k \)SAT problem, and prove that it is \( \Sigma_2 \)-complete. A \( k \)CNF, \( \phi \) consisting of variables \( V = \{x_1, \ldots, x_n, y_1, \ldots, y_m\} \) with a proper partial assignment \( t \) over the set \( \{y_1, y_2, \ldots, y_m\} \) is given. The question is:

"Is there a partial assignment \( t' \) (not necessarily proper) over the set \( \{x_1, \ldots, x_n\} \) such that \( \phi \) has a unique proper assignment \( r \) which satisfies \( r(x_i) = t'(x_i) \) for every \( 1 \leq i \leq n \)扰"
Note that since $t$ is a proper partial assignment, if such $t'$ exists, then $r(y_j) = t(y_j)$ for $1 \leq j \leq m$.

- $∃∃_*$ $k$SAT

**INSTANCE:** A $k$CNF, $φ(x, y)$ and a proper partial assignment $t$ over the set of the variables $y_j$.

**QUESTION:** Is $∃x∃_t φ(x, y)$?

**Theorem 1** The $∃∃_*$ 4SAT problem is $Σ_2$-complete.

**Proof.** The problem is in $Σ_2$. To prove the completeness, we give a reduction from $∃∃$ 3SAT. Consider a 3CNF, $φ(x, y)$ and the problem $∃x∃y φ(x, y)$. We construct an instance of $∃∃_*$ 4SAT, a 4CNF $µ$ with a proper partial assignment $t$, as in the following. The expression $µ$ has all of the variables of $φ$ plus one more variable $z$. Let $C_1, C_2, \ldots, C_n$ be the clauses of $φ(x, y)$. Then

$$µ(x, y, z) = (C_1 \lor z) \land (C_2 \lor z) \land \ldots \land (C_n \lor z) \land (\bar{z} \lor y_1) \land (\bar{z} \lor y_2) \land \ldots (\bar{z} \lor y_m) \quad (1)$$

The partial assignment $t(z) = true$ and $t(y_j) = true$ ($1 \leq j \leq m$) is given. This partial assignment is proper because there exists a true literal in every clause among $z$ and $y_j$’s. For every proper assignment $u$ of $µ(x, y, z)$, if $u(z) = true$, then $u(y_j) = true$ ($1 \leq j \leq m$). If $u(z) = false$, then by ignoring the variable $z$ in $u$, $u$ is a proper assignment of $φ(x, y)$, and vice versa. So $∃x∃y φ(x, y)$ if and only if $∃x∃_t(y, z) µ(x, y, z)$.

Next, we modify the proof of Theorem 1 so that we can conclude that $∃∃_*$ 3SAT is $Σ_2$-complete. Consider $µ(x, y, z)$, defined in (1). In $µ(x, y, z)$ every clause of the form $(\bar{z} \lor y_j)$ has two literals. But a clause of the form $(C_i \lor z)$ has four literals. Suppose $C_i = a_1 \lor a_2 \lor a_3$, where $a_1, a_2, a_3$ are literals. We replace each clause $(C_i \lor z)$ in $µ$ by $C_i'$ defined as follows,

$$C_i' = (a_1 \lor a_2 \lor v_i) \land (a_3 \lor z \lor \bar{v}_i) \land (a_1 \lor z \lor \bar{v}_i) \land (a_2 \lor a_3 \lor v_i) \land (a_2 \lor a_3 \lor v_i),$$

where $v_i$’s are new variables, and call the new expression as $µ'(x, y, v, z)$. Thus

$$µ'(x, y, v, z) = C_1' \land \ldots \land C_n' \land (\bar{z} \lor y_1) \land (\bar{z} \lor y_2) \land \ldots (\bar{z} \lor y_m).$$

Define the partial assignment as $u(z) = true$, $u(y_i) = true$, $u(v_i) = true$ for $1 \leq i \leq m$.

The following three observations imply that $∃∃_t(y, z) µ(x, y, z)$ if and only if $∃x∃_t(y, v, z) µ'(x, y, v, z)$. 4
(a) \( u(z) = true, u(y_j) = true \ (1 \leq j \leq m), \) and \( u(v_i) = true \ (1 \leq i \leq n) \) is a proper partial assignment of \( \mu'(x, y, v, z) \).

(b) Every truth assignment to \( a_1, a_2, a_3, \) and \( z \) which assigns a true value to at least one of them is extended uniquely to a proper assignment of \( C'_i \).

(c) Since every assignment which assigns a false value to \( a_1, a_2, a_3, \) and \( z \) simultaneously is not a proper assignment of \( C'_i \), every proper assignment of \( \mu'(x, y, v, z) \) leads to a proper assignment of \( \mu(x, y, z) \) by ignoring the values of \( v_i \)’s.

Note that any proper subset of the clauses of \( C'_i \) does not satisfy these properties. For example consider the assignment \( t(a_1) = false, t(a_2) = true, t(a_3) = true, t(z) = false \). In this case regardless of what value is assigned to \( v_i \) the first five clauses are satisfied, and the last clause is necessary to fix the value of \( v_i \).

We conclude the following theorem from (a), (b), and (c).

Theorem 2 The \( \exists \exists! \ast 3\text{SAT} \) problem is \( \Sigma_2 \)-complete.

In the proof of Theorem 2 the problem \( \exists \forall 3\text{SAT} \) is reduced to \( \exists \exists! \ast 3\text{SAT} \). In that proof by assuming that there are no variables \( x_i \)’s in \( \phi(x, y) \) (i.e. the number of variables after the first quantifier of \( \exists \forall 3\text{SAT} \) is zero), we can obtain a reduction form \( \exists \forall 3\text{SAT} \) to the problem which asks whether a given proper assignment of a 3CNF is its only proper assignment. This problem is a restriction of \( Q1-3\text{SAT} \) in which \( D \), the set which is asked to be the defining set, is the empty set. Since \( \forall 3\text{SAT} \) is \( \text{CoNP} \)-complete, we have:

Theorem 3 \( Q1-3\text{SAT} \) is \( \text{CoNP} \)-complete.

The next theorem determines the computational complexity of \( Q2-3\text{SAT} \).

Theorem 4 \( Q2-3\text{SAT} \) is \( \Sigma_2 \)-complete.

Proof. The problem is in \( \Sigma_2 \). We reduce \( \exists \exists! \ast 3\text{SAT} \) to this problem. Let \( \exists x \exists y \mu(x, y) \) be an instance of \( \exists \exists! \ast 3\text{SAT} \), where \( \mu(x, y) \) is a 3CNF with variables \( x_1, x_2, \ldots, x_k \) and \( y_1, y_2, \ldots, y_m \), and \( t \) is a proper partial assignment over variables \( y_j \). We construct an instance of \( Q2-3\text{SAT} \), a 3CNF \( \phi \) with a proper assignment \( t' \), such that \( (P(\phi), t') \) has a defining set of size at most \( k \), the number of the variables \( x_i \), if and only if \( \exists x \exists y \mu(x, y) \). In the following we describe how \( \phi \) is obtained from \( \mu(x, y) \).

For every \( 1 \leq i \leq k \), consider two new variables \( v_i \) and \( v'_i \), and replace every \( x_i \) in each clause of \( \mu(x, y) \) with \( v_i \) and every \( \bar{x}_i \) with \( v'_i \).
For every $1 \leq j \leq m$, a literal $a_j$ is defined as follows. If $t(y_j) = \text{false}$, then $a_j$ is $y_j$ and otherwise $a_j$ is $\bar{y}_j$. We add the following clauses to the expression in which $w_i$ are new variables.

$$(a_1 \lor a_2 \lor w_1) \land (\bar{w}_1 \lor a_3 \lor w_2) \land \ldots \land (\bar{w}_{m-2} \lor a_m \lor w_{m-1}) \quad (2)$$

Note that by setting $y_j$'s according to the given assignment $t$, $w_i$'s are forced to take the truth value $true$. The following clauses are also added to the expression.

$$(\bar{w}_{m-1} \lor v_1 \lor \bar{v}_1') \land (\bar{w}_{m-1} \lor \bar{v}_1 \lor v_1') \land \ldots \land (\bar{w}_{m-1} \lor v_k \lor \bar{v}_k') \land (\bar{w}_{m-1} \lor \bar{v}_k \lor v_k')$$

Call this new 3CNF, $\phi(v, v', y, w)$. Let $t'$ be the assignment $t'(v_i) = t'(v'_i) = false$ ($1 \leq i \leq k$), $t'(w_i) = true$ ($1 \leq i \leq m - 1$), and $t'(y_j) = t(y_j)$ ($1 \leq j \leq m$). Note that $t'$ is a proper assignment of $\phi$.

We claim that $(P(\phi), t')$ has a defining set of size at most $k$, if and only if $\exists x \exists y \mu(x, y)$.

Suppose that $\exists x \exists y \mu(x, y)$. This means that there is a partial assignment $u$ over $x_1, x_2, \ldots, x_k$ such that the only proper values for $y_j$ are the values that are assigned to them by the partial assignment $t$. If $u(x_i) = true$, we choose $(v'_i, false)$, and if $u(x_i) = false$, we choose $(v_i, false)$. Call this set $S$. We claim that $S$ is a defining set of $(P(\phi), t')$.

Suppose that $S$ is not a defining set of $(P(\phi), t')$. Then there is a proper assignment $t'' \neq t'$ which is an extension of $S$. Since $t''(v_i) = false$ and $t''(v'_i) = false$ for $v_i, v'_i \in supp(S)$, it can be easily seen that the assignment $r$ defined as $r(x_i) = u(x_i)$ ($1 \leq i \leq k$) and $r(y_j) = t''(y_j)$ ($1 \leq j \leq m$) is a proper assignment to the variables of $\mu(x, y)$ which is a contradiction. So all $y_j$ take the values that are assigned to them by the assignment $t$. Hence $w_i$'s are true for all $1 \leq i \leq m - 1$. Since the two clauses $(\bar{w}_{m-1} \lor v_1 \lor \bar{v}_1')$ and $(\bar{w}_{m-1} \lor \bar{v}_1 \lor v_1')$ are in $\phi$, and exactly one of $v_i$ or $v'_i$ is in $supp(S)$, the value of the other one is also determined to be false, and this is the assignment $t'$.

Next suppose that $(P(\phi), t')$ has a defining set $S$ of size at most $k$. Then for every $1 \leq i \leq k$, at least one of $v_i$ or $v'_i$ is in $supp(S)$. Otherwise we can change the values of both $v_i$ and $v'_i$ to true, and still have a proper assignment. So a defining set of size at most $k$ includes exactly one of $(v_i, false)$ or $(v'_i, false)$ for every $1 \leq i \leq k$. Let $u$ be a partial assignment of $\mu(x, y)$ such that $u(x_i) = true$ if $v_i \in supp(S)$, and $u(x_i) = false$ if $v'_i \in supp(S)$.

We claim that $\mu(x, y)$ has a unique proper assignment $r$ such that $r(x_i) = u(x_i)$ for every $1 \leq i \leq k$. Suppose that there is a proper assignment $r$ for $\mu(x, y)$ such that $r(x_i) = u(x_i)$ for all $1 \leq i \leq k$, but there exists at least one $1 \leq i_0 \leq m$ such that $r(y_{i_0}) \neq t(y_{i_0})$.

Consider $\phi(v, v', y, w)$, and let $r'(y_j) = r(y_j)$ ($1 \leq j \leq k$). Since $r(y_{i_0}) \neq t(y_{i_0})$, it is possible to assign values $r'(w_i)$ ($1 \leq i \leq m - 1$) such that $r'(w_{m-1}) = false$ and the clauses in (2) are true.
Note that exactly one of $v_i$ or $v'_i$ is in supp($S$). For every $1 \leq i \leq k$, if $v_i \in$ supp($S$), then define $r'(v_i) = f\text{alse}$, $r'(v'_i) = t\text{rue}$; and if $v'_i \in$ supp($S$), then define $r'(v_i) = t\text{rue}$, $r'(v'_i) = f\text{alse}$.

Since $t(w_{m-1}) = f\text{alse}$, the values assigned by $r'$ do not make $(\bar{w}_{m-1} \lor v_i \lor \bar{v}'_i) \land (\bar{w}_{m-1} \lor \bar{v}_i \lor v'_i)$ false.

Now all clauses are satisfied. So there exists another proper assignment containing the defining set, which is a contradiction.

\section*{Theorem 5} $Q3$-\textsc{3SAT} is $\Sigma_2$-complete.

\textbf{Proof.} The problem is in $\Sigma_2$. We give a reduction from Q2-\textsc{3SAT}. Consider an instance of Q2-\textsc{3SAT}, a 3CNF $\phi$ with a proper assignment $t$ and an integer $k$. Let the variables of $\phi$ be $x_1, x_2, \ldots, x_n$. We add $n(k+1)$ new variables $y_{ij}$ ($1 \leq i \leq n$ and $1 \leq j \leq k+1$). For every $x_i$, if $t(x_i) = t\text{rue}$, then we add the following clauses:

\[(\bar{x}_i \lor y_{i1}) \land (\bar{x}_i \lor y_{i2}) \land \ldots \land (\bar{x}_i \lor y_{i(k+1)})\]

and if $t(x_i) = f\text{alse}$, then we add the following clauses:

\[(x_i \lor y_{i1}) \land (x_i \lor y_{i2}) \land \ldots \land (x_i \lor y_{i(k+1)})\]

The new 3CNF consists of $\phi$ and these $n(k+1)$ new clauses. Denote this 3CNF by $\phi'$. We claim that $P(\phi')$ has a defining set of size at most $k$, if and only if $(P(\phi), t)$ has a defining set of size at most $k$. Every defining set of $(P(\phi), t)$ is also a defining set of $P(\phi')$, because the assignment $t$ forces all of the $y_{ij}$ to take a true value.

Next suppose that there is a defining set of $P(\phi')$ which fixes a proper assignment $t'$. For every $1 \leq x \leq n$, if $t'(x_i) \neq t(x_i)$, then since it is possible to assign every arbitrary values to $y_{i1}, y_{i2}, \ldots, y_{i(k+1)}$, all these $k+1$ variables are in the defining set. Hence in every defining set of size at most $k$, all $x_i$ take the same values in $t'$ and $t$. Now, since $t'(x_i) = t(x_i)$, by fixing the value of $x_i$, the values of $y_{ij}$'s are determined to be true, for $1 \leq j \leq k+1$. So if $(y_{ij}, t'(y_{ij}))$ is in the defining set, then it is possible to replace it by $(x_i, t'(x_i))$. Thus a defining set of size at most $k$ of $P(\phi')$ can be modified so that all its elements are in $\{(x_i, t'(x_i)) : i = 1, \ldots, n\}$, and $t'(x_i) = t(x_i)$. This is also a defining set of $(P(\phi), t)$.

\section{Vertex Coloring}

For every graph $G$ with vertex set $V = \{v_1, \ldots, v_n\}$, every vertex coloring $c$ of $G$ is a function which maps every vertex $v_i$ to a color $c(v_i)$. For every partial coloring $c$ of $G$, define supp($c$) as the set of the vertices that $c$ assigns a color to them. Denote the family of all $\chi(G)$-vertex colorings of $G$ by $\mathcal{C}(G)$. In \cite{unique-colorability} it is shown that the uniqueness of colorability is CoNP-complete. This implies the following theorem.
Theorem 6 The problem $Q_1$ - Vertex coloring is CoNP-complete.

In this section we show that both of the problems $Q_2$ and $Q_3$ for this family are $\Sigma_2$-complete.

- **Q2- Vertex coloring**

**INSTANCE:** A graph $G$ with a $\chi(G)$-vertex coloring $c$, and an integer $k$.

**QUESTION:** Does $(C(G), c)$ have a defining set of size at most $k$?

- **Q3- Vertex coloring**

**INSTANCE:** A graph $G$, and an integer $k$.

**QUESTION:** Does $C(G)$ have a defining set of size at most $k$?

Theorem 7 $Q_2$- Vertex coloring is $\Sigma_2$-complete for graphs with $\chi = 3$.

**Proof.** The problem is in $\Sigma_2$. To prove the completeness, we introduce a reduction from $Q_2$-3SAT. Consider an instance of $Q_2$-3SAT: A proper assignment $t$ of $\phi(x_1, x_2, \ldots, x_n)$ and an integer $k$. We construct a graph $G_\phi$ with chromatic number 3 and a 3-vertex coloring $c_t$ of $G_\phi$ such that $(P(\phi), t)$ has a defining set of size at most $k$ if and only if $(C(G_\phi), c_t)$ has a defining set of size at most $k + 4$.

We begin by considering a cycle of size 3 with vertices $w_0$, $w_1$, and $w_2$ which are connected to four vertices $w'_1$, $w'_2$, $w'_3$, and $w'_4$ as it is shown in Figure 1(a). For every variable $x_i$, add two vertices $u_{x_i}$ and $u_{\bar{x}_i}$ and edges $\{u_{x_i}, u_{\bar{x}_i}\}$, $\{u_{x_i}, w_2\}$, and $\{u_{\bar{x}_i}, w_2\}$ to the graph. This is illustrated in Figure 1(a).

Consider a clause $C_i = (a_1 \lor a_2 \lor a_3)$ of $\phi$, where $a_j$ ($j = 1, 2, 3$) is a literal. Since $t$ is a proper assignment of $\phi$, without loss of generality we can assume that $t(a_2) = \text{true}$. For every such clause, we add a copy of the graph shown in Figure 1(b) to the graph, and connects its vertices to the other vertices as it is shown in Figure 1(b). Notice that $u_{a_j}$ ($j = 1, 2, 3$) is one of the vertices $u_{x_1}, u_{x_2}, \ldots, u_{x_n}$ or $u_{\bar{x}_1}, u_{\bar{x}_2}, \ldots, u_{\bar{x}_n}$. Call this new graph as $G_\phi$.

One can easily check that assigning a 3-coloring $c_t$ to $u_{a_1}$, $u_{a_2}$, and $u_{a_3}$ such that $c_t(w_0) = 0$, $c_t(w_1) = 1$, and $c_t(w_2) = 2$ and also $c_t(u_{a_2}) = 1$ determines the colors of $v_1, v_2, \ldots, v_8$ uniquely. Let $c_t$ be a 3-coloring of $G_\phi$ defined as in the following:

- $c_t(w_0) = 0$, $c_t(w_1) = 1$, and $c_t(w_2) = 2$.
- $c_t(w'_1) = 1$, $c_t(w'_2) = 2$, $c_t(w'_3) = 0$, and $c_t(w'_4) = 2$.
- For every $1 \leq i \leq n$ if $t(x_i) = \text{true}$, then $c_t(u_{x_i}) = 1$ and $c_t(u_{\bar{x}_i}) = 0$, and otherwise $c_t(u_{x_i}) = 0$ and $c_t(u_{\bar{x}_i}) = 1$.
- Colors of $v_{ij}$ are determined uniquely by the colors of the vertices above.
Figure 1: (a) The vertices $u_{x_i}$ and $u_{\bar{x}_i}$ are connected to $w_2$. (b) For every clause we add a copy of this graph to $G_\phi$.

The vertices $w'_1, w'_2, w'_3$, and $w'_4$ are in every defining set (otherwise we can change their colors). The colors of these four vertices determine the colors of $w_0, w_1,$ and $w_2$ uniquely.

We claim that the size of the smallest defining set of $(C(G_\phi), c_t)$ is equal to the size of the smallest defining set of $(P(\phi), t)$ plus 4. Note that any partial coloring which only assigns 0 or 1 to $u_{a_1}, u_{a_2}, u_{a_3}$ and does not assign 0 to all of them can be extended to a proper coloring of the graph in Figure 1(b). Moreover if all the vertices $u_{a_1}, u_{a_2}, u_{a_3}$ are colored by 0, then it can be easily seen that $v_{i_8}$ is also forced to be colored by 0. Since $v_{i_8}$ is connected to $w_0$ and $w_2$, $G_\phi$ admits a 3-coloring, if and only if $\phi$ has a proper assignment.

Suppose $(C(\phi), t)$ has a defining set consists of $k$ variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$. Then assigning colors of $k + 4$ vertices $w'_{i_1}, w'_{i_2}, w'_{i_3}, w'_{i_4}$ and $u_{x_{i_1}}, u_{x_{i_2}}, \ldots, u_{x_{i_k}}$ constitutes a defining set of $(C(G_\phi), c_t)$.

Next suppose that $S$ is the smallest defining set of $(C(G_\phi), c_t)$. Then $w'_{i_1}, w'_{i_2}, w'_{i_3}, w'_{i_4}$ are in $\text{supp}(S)$. By assigning the colors of these vertices the colors of $w_1, w_2, w_3,$ and all $v_{i_8}$'s are determined uniquely. It can be verified easily that for every clause $C_i = (a_1, a_2, a_3)$ of $\phi$, since $c_t(u_{a_2}) = 1$, the colors of $v_{i_1}, v_{i_2}, \ldots, v_{i_7}$ are determined uniquely by fixing the color of $u_{a_2}$, and the color of $u_{x_i}$ determines the color of $u_{\bar{x}_i}$. Hence we can assume that $\text{supp}(S)$ contains $w'_{i_1}, w'_{i_2}, w'_{i_3}, w'_{i_4}$, and some of $u_{x_i}$. Using
the fact that any partial coloring which only assigns 0 or 1 to \(u_1, u_2, u_3\) and does not assign 0 to all of them can be extended to a proper coloring of the graph in Figure 3(b), we conclude that the corresponding variables of these \(u_i\) constitute a defining set of \((C(\phi), t)\).

\[\text{Theorem 8 Q3- Vertex coloring is } \Sigma_2 \text{-complete for graphs with } \chi = 3.\]

\[\text{Proof.}\] The problem is in \(\Sigma_2\). We give a reduction from Q2- Vertex coloring when \(\chi = 3\). Consider an instance \((C(G), c)\) of Q2- Vertex coloring, where \(G\) is a graph and \(c\) is a 3-vertex coloring of \(G\). Assume that the range of \(c\) is the set \(\{0, 1, 2\}\). An integer \(k\) is given, and it is asked that "Is there a defining set of size at most \(k\) for \((C(G), c)\)?" We construct a new graph \(H\) as follows:

1. First let \(H\) be the disjoint union of \(G\) and a cycle \(w_0w_1w_2\) of size 3. Then

2. for every vertex \(u_i\) of \(G\), let \(c_1\) and \(c_2\) be the two colors other than \(c(u_i)\). Add \(2k + 2\) vertices \(v_{u_i,c_j,t} (1 \leq j \leq 2)\) to \(H\). For every \(1 \leq t \leq k + 1\), connect \(v_{u_i,c_j,t}\) to both \(u_i\) and \(w_{c_j}\). (Notice that \(w_{c_j}\) is one of \(w_0, w_1,\) or \(w_2\).)

3. Add four new vertices \(w'_1, w'_2, w'_3,\) and \(w'_4\) to \(H\), and connect \(w'_1\) and \(w'_2\) to \(w_0\), and also \(w'_3\) and \(w'_4\) to \(w_1\).

Now we claim that \(C(H)\) has a defining set of size at most \(k + 4\) if and only if \((C(G), c)\) has a defining set of size at most \(k\).

First consider a defining set of size at most \(k\) for \((C(G), c)\), say \(D\). If we fix the colors of the vertices in \(D\) and assign the colors 1 to \(w'_1\), 2 to \(w'_2\), 0 to \(w'_3\), and 2 to \(w'_4\), then these \(k + 4\) vertices constitute a defining set of \(C(H)\).

Next suppose that \(D\) is the smallest defining set of \(C(H)\) which has at most \(k + 4\) vertices. Without loss of generality assume that in the extension of \(D\) to a 3-vertex coloring \(c'\) of \(H\), \(w_i\) (\(0 \leq i \leq 2\)) is colored by \(i\). Since the degrees of \(w'_1, w'_2, w'_3,\) and \(w'_4\) are equal to one, they are in \(\text{supp}(D)\). Suppose that in the extension of \(D\) to a 3-coloring of \(H\), a vertex \(u_i\) of \(G\) is colored by \(c'(u_i)\) which is not equal to \(c(u_i)\). The vertices \(v_{u_i,c'(u_i),t} (1 \leq t \leq k + 1)\) are only connected to \(u_i\) and \(w_{c'(u_i)}\). Since these two vertices are colored by the same colors, all these \(k + 1\) vertices are in the defining set, and with the four vertices \(w'_i\), the size of the defining set is at least \(k + 5\). Since \(D\) is of size at most \(k + 4\), for every vertex \(u_i\), \(c'(u_i) = c(u_i)\).

We can suppose that \(w'_1\) and \(w'_2\) (and so \(w'_3\) and \(w'_4\)) are colored by different colors. Otherwise by changing the color of \(w'_2\) (and so \(w'_4\)), \(D\) still remains a defining set of \(C(H)\). Since \(w'_1\) and \(w'_2\) and also \(w'_3\) and \(w'_4\) are colored by different colors, they determine the colors of \(w_1, w_2,\) and \(w_3\) uniquely. Therefore since \(D\) is the smallest defining set, none of \(w_1, w_2,\) and \(w_3\) is in \(\text{supp}(D)\). Also if a vertex \(v_{u_i,c,t} (c_j \neq c(u_i))\) is in \(D\), then we can replace it with \(u_i\). Now, if we remove the four vertices \(w'_i\) from the defining set, we obtain a defining set of \((C(G), c)\).
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