Knot Topology of QCD Vacuum

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We show that one can express the knot equation of Skyrme theory completely in terms of the vacuum potential of SU(2) QCD, in such a way that the equation is viewed as a generalized Lorentz gauge condition which selects one vacuum for each class of topologically equivalent vacua. From this we show that there are three ways to describe the QCD vacuum (and thus the knot), by a non-linear sigma field, a complex vector field, or by an Abelian gauge potential. This tells that the QCD vacuum can be classified by an Abelian gauge potential with an Abelian Chern-Simon index.

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The non-Abelian gauge theory has been well known to have a non-trivial topology. In particular it has infinitely many topologically distinct vacua which can be connected by vacuum tunneling through the instantons. The existence of topologically distinct vacua and the vacuum tunneling has played a very important role in quantum chromodynamics (QCD). In a totally independent development the Skyrme theory has been shown to admit a topologically stable knot which can be interpreted as a twisted magnetic vortex ring made of helical baby skyrms. And very interestingly, this knot is shown to describe the topologically distinct QCD vacua.

This is puzzling because the knot is a physical object which carries a nonvanishing energy. So it appears strange that the knot can be related to a QCD vacuum. On the other hand this is understandable since the Skyrme theory is closely related to QCD, and both the knot and the QCD vacuum are described by the same topology \( \pi_3(S^3) \). Under this circumstance one need to know in exactly what sense the QCD vacuum can be identified as the knot. Since there exists one knot solution for each topological quantum number (up to the trivial space-time translation and the global SU(2) rotation), one might suspect that the knot equation could be viewed as a gauge condition for the topologically equivalent vacua. In fact it has been suggested that the knot equation can be viewed as a non-local gauge condition which describes the maximal Abelian gauge in SU(2) QCD. The purpose of this Letter is to show that the knot equation is nothing but a generalized Lorentz gauge condition which selects one representative vacuum for each class of topologically equivalent QCD vacua. This allows us to interpret the knot as a complex vector field which couples to an Abelian gauge field, and the knot equation as an Abelian gauge condition for the complex vector field. We first obtain a most general expression of the vacuum, and write the knot equation completely in terms of the vacuum potential. With this we prove that the knot equation is nothing but a generalized Lorentz gauge condition of the QCD vacuum. From this we show that the knot equation can be viewed as an Abelian gauge condition for a complex vector field. Moreover, we show that this complex vector field is uniquely determined by the Abelian gauge potential. This allows a new interpretation of the knot, the knot as a complex vector field or an Abelian gauge potential. As importantly, this tells that one can classify the topologically different QCD vacua by an Abelian Chern-Simon index.

A best way to describe the QCD vacuum is to introduce a local orthonormal frame in the non-Abelian group space and obtain a potential which parallelizes the local orthonormal frame. Consider the SU(2) QCD and let \( \hat{n}_i \) \((i=1,2,3)\) be a right-handed local orthonormal frame. A vacuum potential must be the one which parallelizes the local orthonormal frame. Imposing the condition to the gauge potential \( \hat{A}_\mu \)

\[
D_\mu \hat{n}_i = (\partial_\mu + g \hat{A}_\mu \times) \hat{n}_i = 0, \quad (i = 1, 2, 3)
\]

we obtain a most general vacuum potential

\[
\hat{\Omega}_\mu = -C^k_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} = -C^k_\mu \hat{n}_k,
\]

\[
\frac{1}{g} \hat{n} \times \partial_\mu \hat{n} = C^1_\mu \hat{n}_1 + C^2_\mu \hat{n}_2,
\]

\[
C^k_\mu = \frac{1}{2g} \epsilon^{kj}_{ij} (\hat{n}_i \cdot \partial_\mu \hat{n}_j),
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\]

\[
C^k_\mu = \frac{1}{2g} \epsilon^{kj}_{ij} (\hat{n}_i \cdot \partial_\mu \hat{n}_j),
\]
where $\hat{n}$ is $\hat{n}_3$ and $C_{\mu}$ is $C_{\mu}^3$. One can easily check that $\hat{\Omega}_n$ describes a vacuum

$$
\hat{\Omega}_{\mu\nu} = \partial_{\mu}\hat{\Omega}_\nu - \partial_{\nu}\hat{\Omega}_\mu + g\hat{\Omega}_\mu \times \hat{\Omega}_\nu
$$

$$
= - (\partial_{\mu} C_{\nu}^k - \partial_{\nu} C_{\mu}^k + g\epsilon_{ijk} C_{\mu}^i C_{\nu}^j) \hat{n}_k = 0. \quad (3)
$$

This tells that both $\hat{\Omega}_n$ and $(C_{\mu}^1, C_{\mu}^2, C_{\mu}^3)$ describe a QCD vacuum. Obviously they are gauge equivalent. Notice that the vacuum is essentially fixed by $\hat{n}$, because $\hat{n}_1$ and $\hat{n}_2$ are uniquely determined by $\hat{n}$ up to a $U(1)$ gauge transformation which leaves $\hat{n}$ invariant.

A nice feature of [2] is that the topological character of the vacuum is naturally inscribed in it. The topology of the $SU(2)$ QCD vacuum has been described by the non-trivial mapping $\pi_3(S^3)$ from the (compactified) three-dimensional space $S^3$ to the group space $S^3$. But $\hat{n}$ can also describe the vacuum topology because it defines the mapping $\pi_3(S^2)$ which can be transformed to $\pi_3(S^3)$ through the Hopf fibering [2]. So one can naturally classify the vacuum topology by $\hat{n}$, which is manifest in [2].

With

$$
\hat{n} = \begin{pmatrix} \sin \alpha \cos \beta \\ \sin \alpha \sin \beta \\ \cos \alpha \end{pmatrix},
$$

one may choose

$$
\hat{n}_1 = \begin{pmatrix} \cos \alpha \cos \beta \\ \cos \alpha \sin \beta \\ -\sin \alpha \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} -\sin \beta \\ \cos \beta \\ 0 \end{pmatrix}. \quad (5)
$$

so that one has

$$
C_{\mu}^1 = -\frac{1}{g} \sin \alpha \partial_{\mu} \beta, \quad C_{\mu}^2 = \frac{1}{g} \partial_{\mu} \alpha,
$$

$$
C_{\mu} = \frac{1}{g} \cos \alpha \partial_{\mu} \beta. \quad (6)
$$

Of course they are uniquely determined up to the $U(1)$ gauge transformation which leaves $\hat{n}$ invariant. Notice that, when $\hat{n}$ becomes the unit radial vector $\hat{r}$, $C_{\mu}$ describes the well-known Dirac’s monopole potential. But when $\hat{n}$ is smooth everywhere, it describes a vacuum.

The vacuum [2] is obtained by three conditions given by [11]. Suppose we impose only one condition

$$
D_{\mu} \hat{n} = 0. \quad (\hat{n} = \hat{n}_3) \quad (7)
$$

This singles out the restricted potential which defines the restricted gauge theory [10][11]

$$
\hat{A}_{\mu} = A_{\mu} \hat{n} - \frac{1}{g} \hat{n} \times \partial_{\mu} \hat{n} = \hat{\Omega} + (A_{\mu} + C_{\mu}) \hat{n}, \quad (8)
$$

where $A_{\mu} = \hat{n} \cdot \hat{A}_{\mu}$ is the chromoelectric potential. This tells that the two extra conditions (for $i = 1, 2$) in [10] uniquely determines $A_{\mu}$ to be

$$
A_{\mu} = -C_{\mu}. \quad (9)
$$

Indeed with this [8] becomes [2], which tells that the restricted QCD has exactly the same multiple vacua. Furthermore, in the absence of [4], one can express the most general $SU(2)$ gauge potential $\hat{A}_{\mu}$ by [10][11]

$$
\hat{A}_{\mu} = \hat{X}_{\mu} + \hat{n} \hat{C}_{\mu}, \quad (\hat{n} \cdot \hat{X}_{\mu} = 0) \quad (10)
$$

where $\hat{X}_{\mu}$ is a gauge covariant vector field. This is because under the infinitesimal gauge transformation

$$
\delta \hat{n} = -\hat{\alpha} \times \hat{n}, \quad (11)
$$

one has

$$
\delta A_{\mu} = \frac{1}{g} \hat{n} \cdot \partial_{\mu} \hat{\alpha}, \quad \delta C_{\mu} = -\delta A_{\mu}, \quad (12)
$$

where $\hat{\alpha}$ is the infinitesimal gauge parameter. This means that one can interpret QCD as a restricted gauge theory which has a gauge covariant valence gluon $\hat{X}_{\mu}$ as the colored source [10][11]. The importance of the decompositions is that they are gauge independent. Once $\hat{n}$ is given the decomposition follows automatically, independent of the choice of a gauge.

The above analysis shows that $\hat{A}_{\mu}$ by itself describes an $SU(2)$ connection which enjoys the full non-Abelian gauge degrees of freedom. More importantly, it has a dual structure [10][11]

$$
\hat{F}_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu}) \hat{n}, \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \quad (13)
$$

This tells that $C_{\mu}$ in [2] is nothing but the chromomagnetic potential of the field strength $H_{\mu\nu}$ (Since $H_{\mu\nu}$ forms a closed two-form, it admits a potential). Moreover $A_{\mu}$ and $C_{\mu}$ transform equally but oppositely under the gauge transformation. In particular, the Abelian gauge group which leaves $\hat{n}$ invariant acts on both $A_{\mu}$ and $C_{\mu}$. This shows that the restricted QCD has a manifest electromagnetic duality [10][11].

Now, let’s review the knot in Skyrme theory [3][4]. Let $\hat{\omega}$ and $\hat{n}$ (with $\hat{n}^2 = 1$) be the Skyrme field and the non-linear sigma field, and let

$$
U = \exp\left[\frac{\omega}{2} \hat{\sigma} \cdot \hat{n}\right] = \cos \frac{\hat{\omega}}{2} - i(\hat{\sigma} \cdot \hat{n}) \sin \frac{\hat{\omega}}{2},
$$

$$
L_{\mu} = U \partial_{\mu} U^\dagger. \quad (14)
$$

With this one can write the Skyrme Lagrangian as

$$
\mathcal{L}_{S} = \frac{\mu^2}{4} \text{tr} L_{\mu}^2 + \frac{1}{32} \text{tr} \left[\left[L_{\mu}, L_{\nu}\right]\right]^2
$$

$$
= -\frac{g^2}{4} (1 - \sigma^2) \hat{F}_{\mu\nu}^2 - \frac{\mu^2}{2} \frac{g^2}{2} (1 - \sigma^2) \hat{C}_{\mu}^2
$$

$$
- \frac{\mu^2 (\partial_{\mu} \sigma)^2}{2} + \frac{g^2}{4} \left(\partial_{\mu} \sigma \partial_{\mu} \sigma - \partial_{\mu} \sigma \partial_{\mu} \sigma\right)^2. \quad (15)
$$
where

\[ \sigma = \cos \frac{\omega}{2}, \]

\[ \hat{H}_{\mu\nu} = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu + g \hat{C}_\mu \times \hat{C}_\nu = H_{\mu\nu} \hat{n}, \]

\[ \hat{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}. \]

The Lagrangian has an obvious global SU(2) symmetry, but it also has a (hidden) U(1) gauge symmetry [5]. This is because the invariant subgroup of \( \hat{n} \) can be viewed as a U(1) gauge group. Notice that \( \hat{C}_\mu \) is nothing but the magnetic part of the restricted potential (8). This provides the crucial link between QCD and Skyrmium theory. From this link one can argue that the Skyrmium theory is a theory of monopole which describes the chromomagnetic dynamics of QCD [7].

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It is well-known that \( \sigma = 0 \) is a classical solution of (15), independent of \( \hat{n} \). When \( \sigma = 0 \) the Skyrmium Lagrangian is reduced to

\[ \mathcal{L}_S \rightarrow -\frac{1}{4} \hat{H}_{\mu\nu}^2 - \frac{\mu^2}{2} \hat{C}_\mu^2, \]  

(16)

whose equation is given by

\[ \frac{1}{g} \hat{n} \times \partial^2 \hat{n} - \frac{1}{\mu^2} (\partial_\mu H_{\mu\nu}) \partial_\nu \hat{n} = 0. \]  

(17)

It is this equation that allows the monopole, the baby skymion, and the knot in Skyrmium theory [5, 6].

With (17) the knot equation is written as

\[ \partial^2 \alpha - \sin \alpha \cos(\partial_\mu \beta)^2 - \frac{g}{\mu^2} \sin \alpha (\partial_\mu H_{\mu\nu}) \partial_\nu \beta = 0, \]

\[ \sin \alpha \partial^2 \beta + 2 \cos \alpha (\partial_\mu \alpha \partial_\mu \beta) + \frac{g}{\mu^2} (\partial_\mu H_{\mu\nu}) \partial_\nu \alpha = 0, \]  

(18)

where

\[ H_{\mu\nu} = -\frac{1}{g} \sin \alpha (\partial_\mu \alpha \partial_\nu \beta - \partial_\nu \alpha \partial_\mu \beta). \]

But this can neatly be expressed by the vacuum potential \( C_\mu \). To see this, notice that the knot equation (14) can be understood as a conservation equation of an SU(2) current \( \hat{j}_\mu \)

\[ \hat{j}_\mu = \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} - \frac{1}{\mu^2} H_{\mu\nu} \partial_\nu \hat{n} = (C_\mu^1 - \frac{1}{\mu^2} H_{\mu\nu} C_\nu^2) \hat{n}_1 + (C_\mu^2 + \frac{1}{\mu^2} H_{\mu\nu} C_\nu^1) \hat{n}_2, \]

\[ \partial_\mu \hat{j}_\mu = 0. \]  

(19)

The origin of this conserved current, of course, is the global SU(2) symmetry of the Skyrme Lagrangian [5]. From (14) one can express the knot equation by

\[ \partial_\mu C_\mu^1 - g C_\mu C_\mu^2 - \frac{g}{\mu^2} (\partial_\mu H_{\mu\nu}) C_\nu^2 = 0, \]

\[ \partial_\mu C_\mu^2 + g C_\mu C_\mu^1 + \frac{g}{\mu^2} (\partial_\mu H_{\mu\nu}) C_\nu^1 = 0. \]  

(20)

With

\[ \omega_\mu = \frac{C_\mu^1 + i C_\mu^2}{\sqrt{2}}, \]  

(21)

this can be put into a single complex equation

\[ \bar{D}_\mu \omega_\mu = 0, \]

\[ \bar{D}_\mu = \partial_\mu + ig(C_\mu + \frac{1}{\mu^2} \partial_\nu H_{\alpha\mu}). \]  

(22)

This tells that the knot equation (14) can be expressed completely in terms of the QCD vacuum potential, as an Abelian gauge condition for the complex vector field \( \omega_\mu \). In this form the U(1) gauge symmetry of the knot equation (and the Skyrmium theory) becomes manifest [7].

Now let us go back to QCD, and consider the following constraint equation for the vacuum

\[ \bar{D}_\mu \tilde{\Omega}_\mu = 0, \]  

(23)

where now

\[ \bar{D}_\mu \tilde{\omega}_\mu = 0, \quad \partial_\mu C_\mu = 0. \]  

(24)

This tells that the equation (24) not only describes the knot, but also fixes the U(1) gauge degree of the knot equation. This proves that the knot equation can be interpreted as a generalized Lorentz gauge condition of the QCD vacuum which selects one vacuum for each topologically equivalent class of vacua.

The knot equation (25) contains both \( \omega_\mu \) and \( C_\mu \). But they are not independent. To see this, notice that the vacuum condition (3) tells that

\[ D_\mu \omega_\mu - D_\nu \omega_\mu = 0, \]

\[ H_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu = ig(\omega_\mu^* \omega_\nu - \omega_\nu^* \omega_\mu), \]  

(25)

where

\[ D_\mu = \partial_\mu + ig C_\mu. \]

So \( \omega_\mu \) and \( C_\mu \) are determined by each other. This tells that \( \omega_\mu \) alone can describe the knot. Equivalently, this means that the knot can also be described by an Abelian gauge potential \( C_\mu \). So we have three different ways to describe the knot and thus the QCD vacuum, by \( \hat{n} \), \( \omega_\mu \), and \( C_\mu \).

With (26) we have

\[ \bar{D}_\mu \omega_\mu = D_\mu \omega_\mu + i \frac{g}{\mu^2} [\partial_\mu (H_{\mu\nu} \omega_\nu) + ig H_{\mu\nu} C_\mu \omega_\nu], \]  

(26)

so that we can simplify (22) to

\[ \bar{D}_\mu \omega_\mu = 0, \]

\[ \omega_\mu = \omega_\mu + i \frac{g}{\mu^2} H_{\mu\nu} \omega_\nu. \]  

(27)
This tells that the knot equation can be expressed as a covariant Lorentz gauge condition of the complex vector field \( \tilde{\omega}_\mu \).

The knot quantum number is given by the Abelian Chern-Simon index of the magnetic potential \( C_\mu \) \[1\] \[2\],

\[ Q = \frac{g^2}{32\pi^2} \int \epsilon_{ijk} C_i H_{jk} d^3x, \] (28)

which describes the non-trivial topology \( \pi_3(S^2) \) defined by \( \tilde{n} \). The preimage of the mapping from the compactified space \( S^3 \) to the target space \( S^2 \) defined by \( \tilde{n} \) forms a closed circle, and any two preimages of the mapping are linked together when \( \pi_3(S^2) \) is non-trivial. This linking number is given by the Chern-Simon index. Obviously our analysis tells that exactly the same description applies to the QCD vacuum. In particular, this means that the QCD vacuum can also be classified by an Abelian Chern-Simon potential with the Abelian Chern-Simon index \[3\] \[4\].

Conversely, with \[2\] we can transform the knot quantum number \[25\] to a non-Abelian form

\[ Q = \frac{g^2}{32\pi^2} \int \epsilon_{ijk} C_i H_{jk} d^3x = \frac{-g^3}{96\pi^2} \int \epsilon_{abc} \epsilon_{ijk} C^a_i C^b_j C^c_k d^3x, \] (29)

which proves that the knot quantum number can also be expressed by a non-Abelian Chern-Simon index. More significantly, this tells that the Abelian Chern-Simon index is actually identical to the non-Abelian Chern-Simon index. They have been thought to be two different things, but our analysis tells that they are one and the same thing which can be transformed to each other through the vacuum condition \[24\].

The fact that the knot can be described by an Abelian gauge potential \( C_\mu \) raises a totally unexpected and very interesting possibility that, under a proper circumstance, one could create the knot in a condensed matter. Indeed it has been conjectured that a superconducting knot could exist in the ordinary superconductor \[12\]. It has long been assumed that this is impossible, because the Abelian gauge theory is thought to be too trivial to allow the knot topology \[13\]. Our analysis tells that this is not true. There exists a well-defined knot topology in the Abelian gauge theory.

Our analysis could have important applications in QCD. For example, the decomposition \[10\] plays an important role in the discussion of the Abelian dominance of color in QCD \[14\] \[15\]. Moreover it plays a crucial role for us to study the geometry of the principal fiber bundle, in particular the Deligne cohomology of the non-Abelian gauge theory \[16\] \[17\]. Further interesting applications of our analysis to QCD will be published elsewhere \[18\].

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