Electric-circuit realization of a hyperbolic drum

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The Laplace operator encodes the behaviour of physical systems on vastly different scales, describing heat flow, fluids, as well as electric, gravitational, and quantum fields. A key input for the Laplace equation is the curvature of space. Here we demonstrate that the spectral ordering of Laplacian eigenstates for hyperbolic (negative curvature) and flat (zero curvature) two-dimensional spaces has a universally different structure. We use a lattice representation of hyperbolic space in an electric-circuit network to measure the eigenstates of a ‘hyperbolic drum’, and to analyze signal propagation along the curved geodesics. Our experiments showcase a versatile platform to emulate hyperbolic lattices in tabletop experiments, which can be utilized to explore propagation dynamics as well as to realize topological hyperbolic matter.

INTRODUCTION

Holographic principles elevate the behaviour of systems on curved spaces to fundamental importance. Most famously, in the AdS/CFT correspondence1,2, the description of a bulk theory in negatively curved Anti-de-Sitter (AdS) space is conjectured to be dual to that of a strongly-coupled conformal field theory (CFT) on its boundary. Remarkably, several strongly correlated electronic systems have been analyzed with holographic tools as representative states that could hypothetically live on a holographic boundary3–11. More recently, major advancements in the mathematical characterization of classical and quantum states in negatively curved spaces12–15 sparked a resurgence of interest in hyperbolic lattices16–18, ushering the research of hyperbolic topological matter19,20. These rapid developments call for new experimental platforms to implement tabletop simulations of hyperbolic toy-models.

However, systems that furnish negatively curved space21,22 are hard to realize experimentally. The mathematical reason for this is encompassed in Hilbert’s theorem: even the lowest dimensional model of a hyperbolic space, the hyperbolic plane, cannot be embedded in three-dimensional Euclidean (flat) laboratory space. We cannot build a hyperbolic drum. This is in sharp contrast to the case of positive curvature: a sphere can be embedded in three-dimensional space, and we can study the standing waves (hereafter called eigenmodes) of a spherical membrane, which directly relate to quantum numbers of atomic orbitals. Despite such obstacles, hyperbolic space can be emulated experimentally. For instance, it has been suggested23 that a non-trivial metric can be implemented in metamaterials via spatial variations of the electromagnetic permittivity of continuous media. However, it is very challenging to induce these variations in a controlled manner, which limits the applicability of such approaches.

Electric circuits24–31 and similar systems, e.g., coplanar waveguide resonator31, overcome these experimental limitations by relying on a discretization of space. In electric circuit networks, the physical distances between the nodes are fundamentally decoupled from the metric that enters the long-wavelength description of its degrees of freedom, namely the voltages and currents that pass through the circuit nodes. The latter depend merely on the circuit elements that connect the nodes. Compared to other experimental platforms, electric circuits significantly excel in their flexibility of design, ease of fabrication, and high accessibility to measurements.

In this work we demonstrate the efficiency of electric circuits in experimentally emulating the physics of negatively curved spaces. For concreteness, we consider the most fundamental differential operator on curved spaces, the Laplace-Beltrami operator, which generalizes the notion of the Laplace operator on flat space. The first key result of our work is the experimental observation of negative curvature in the spectral ordering of the eigenmodes of the Laplace-Beltrami operator in hyperbolic space. To paraphrase the words of Ref. 32, our measurements confirm that a hyperbolic drum has a sound distinct from a Euclidean drum. Second, since electric circuits allow for time-resolved measurements, we can study not only static, but also dynamic properties. Our measurements confirm that signals in the present realization travel along hyperbolic geodesics, a smoking gun signature for the negative curvature of space. Based on our results, we infer that electric circuit networks could be readily utilized to implement and to experimentally verify the predicted features of the recently studied hyperbolic models ofRefs. 13–19.

RESULTS
Spectra of Euclidean and hyperbolic drums

We start by comparing the eigenmodes of Euclidean and hyperbolic drums in the continuum. The hyperbolic plane, characterized by a constant negative Gaussian curvature $\kappa<0$, is naturally embedded in $(2+1)$-dimensional Minkowski space.
where \( \Delta \) is the Laplace-Beltrami operator. It is convenient to set \( \kappa = -1 \) and to employ the stereographic projection (Fig. 1a), which maps the hyperbolic plane onto the \textit{Poincaré disk}, i.e., the unit disk with length element \( ds^2 = (1 - x^2 - y^2)^{-2}(dx^2 + dy^2) \).

The eigenmodes of the hyperbolic drum with \( x^2 + y^2 \leq r_0^2 < 1 \) correspond to the spectrum \( \lambda^\ell_n \) of the Laplace-Beltrami operator

\[
\Delta_H = (1 - x^2 - y^2)^2 \Delta_E, \tag{1}
\]

where \( \Delta_E = (\partial^2/\partial x^2 + \partial^2/\partial y^2) \) is the usual Laplace operator in the Euclidean plane. Adopting Dirichlet boundary conditions, which yield vanishing amplitude on the disk boundary, the spectrum of the drum is given by solutions to

\[
- \Delta_g u^{\ell_n} = \lambda_n^{\ell} u^{\ell_n} \quad \text{with} \quad u^{\ell_n} |_{x^2+y^2=r_0^2} = 0, \tag{2}
\]

where \( g \in \{E, H\} \) indicates the geometry, and \( \lambda_n^{\ell} \) is the frequency of the mode with angular momentum \( \ell \) and with \( n \) radial zeroes. Solutions to Eq. (2) are superpositions of Bessel functions (associated Legendre functions) in the Euclidean (hyperbolic) case.\(^{35}\)

We plot in Fig. 1b the first few solutions to Eq. (2) on the Euclidean vs. Poincaré disk for \( r_0 = 0.94 \), which corresponds to our experimental realization discussed below. We observe a significant reordering of the eigenmodes characterized by \( (n, \ell) \): while in the Euclidean case the first eigenmode with \( n = 1 \) is the fourth (not counting degenerate eigenmodes separately), in the hyperbolic case, it is only the sixth mode. This reordering becomes even more apparent when considering the angular momentum dispersion \( \lambda_n^{\ell} \) vs. \( \ell \) displayed in Fig. 2a. In both the Euclidean and the hyperbolic case, several branches (corresponding to different values of \( n \), indicated by red numbers) are discernible. The spectral reordering manifests as a reduced slope of the branches (relative to their spacing) compared to their behaviour for the Euclidean drum. Consequently, eigenmodes with large \( \ell \) and small \( n \) appear much earlier in the spectrum in hyperbolic compared to Euclidean space. The spectral reordering is stronger for larger radii \( r_0 \). This is intuitively understood from the fact that the circumference of a hyperbolic drum grows superlinearly with its radius, such that oscillations in the angular direction stretch over larger distances. This makes them energetically favorable over oscillations in the radial direction, resulting in the observed reordering.

**Lattice regularization of the hyperbolic plane**

To experimentally realize a hyperbolic drum in an electric circuit network, we use a lattice regularization constructed by tessellating the hyperbolic plane. The metric of the underlying continuous space is then manifested only in the connectivity of the lattice sites. Different tessellations of the hyperbolic plane...
Figure 3. Experimental data. a, Schematic of hyperbolic tessellation (left half) with the unit circle in blue and the circle with radius $r_0 = 0.94$ in red, and photograph of the electric circuit (right half). b, Measurement of impedance to ground $Z_a$ of the circuit at node $a$ as a function of input frequency $f$ for different nodes (see inset legend and panel a for an identification of the nodes). Each impedance peak indicates an eigenmode at that corresponding frequency, which can be excited at the corresponding input node. The highest six frequencies are indicated by vertical grey lines and the corresponding eigenmodes are shown in panel c. c, Measurement of the voltage profile of the first six eigenmodes (only one mode is shown for each pair of degenerate modes). The saturation encodes the magnitude as a fraction of the voltage (white denotes 0 and full saturation 1) at the input node (black dots), and the color encodes the phase relative to the reference voltage (see legend).

are possible, and they generally differ in their symmetries and in how densely their vertices cover the disk. Regular tilings with $q$ copies of $p$-sided polygons meeting at each vertex are conventionally denoted by the Schl"{a}fli symbol $\{p, q\}$, where a hyperbolic lattice is obtained for $(p - 2)(q - 2) > 4$. In contrast to the Euclidean case, hyperbolic tessellations uniquely fix the distance between neighboring sites.

We analyze and compare several different tessellations in supplementary text S3.36 For a given coverage $r_0$ of the disk, a good approximation of the continuum is naturally achieved by tessellations that feature small area per vertex (i.e., which tile the hyperbolic plane densely).17 However, for our experiments, three different aspects of the modelled lattice are important. First, for a fixed number of vertices, tessellations with larger area per vertex cover a larger fraction of the Poincaré disk, leading to stronger signatures of the negative curvature. Second, a vertex at the origin of the disk allows for easy excitation and identification of $\ell = 0$ modes. And third, a high order of rotation symmetry prevents $\ell \neq 0$ modes to have non-vanishing weight at the origin of the disk, which would impede the identification of $\ell = 0$ modes. These considerations favour the $\{3, 7\}$ tessellation, which exhibits a seven-fold rotation symmetry with respect to a site at the centre, and which covers a disk with radius $r_0 = 0.94$ with only 85 sites, see Fig. 3a.

The lattice regularization of the continuum Laplace-Beltrami operator is given by the graph Laplacian matrix, which is fully determined by the connectivity or topology of the lattice sites.17,35 Eigenvectors of that matrix, therefore, correspond to eigenmodes of the Laplace-Beltrami operator, and the absolute value of the angular momentum $\ell$ can be systematically determined by a Fourier transform of the eigenvector components on the outermost sites. Due to the discreteness of the lattice, this analysis is only reliable for modes with sufficiently small $\ell$ and $n$, i.e., in the long-wavelength limit. We extract the angular momentum dispersion for the chosen tessellation, and in Fig. 2b compare it to the corresponding Euclidean $\{3, 6\}$ tessellation with the same number of sites. As in the continuum, a strong spectral reordering is observed. This reordering is a universal feature of the spatial curvature and does, therefore, not rely on the details of the tessellation, as long as it adequately approximates the continuum.

**Implementation in an electric circuit**

In our experiments, the tessellation is realized as an electric circuit network (right half of Fig. 3a) with a node at each site. Nodes are coupled capacitively among each other and inductively to ground. The boundary conditions are implemented by additional capacitive coupling of the nodes in the outermost shell to ground. Effectively, this corresponds to adding one more shell with all nodes shorted to ground, i.e., it represents the lattice equivalent of the Dirichlet boundary conditions. A generic electric circuit network is described by Kirchoff’s law

$$I_a = \sum_b J_{ab}(\omega)V_b,$$

where $I_a$ and $V_a$ are the input current and voltage amplitude (for angular frequency $\omega$) at node $a$, respectively. The matrix $J(\omega)$ is called the grounded circuit Laplacian, and generally depends on $\omega$. In the continuum limit, the input current $I$ at some position is related to the divergence of the current density $\mathbf{j}$ via $I = \nabla \cdot \mathbf{j}$, with $\mathbf{j} = \sigma \mathbf{E} = \sigma \nabla V$, $\sigma$ the conductivity, $\mathbf{E}$ the electric field due to an applied voltage $V$, and $\nabla$ the del operator (for brevity, we dropped the subscript “g” indicating the geometry). Hence, $I = \nabla \cdot (\sigma \nabla V) = \sigma \Delta V$, establishing the interpretation of $J$ as the restriction of the continuum Laplace operator to the grounded circuit. The impedance to ground of node $a$, $Z_a(\omega) = V_a/I_a$, is fully determined by $J$ and its resonances correspond to eigenmodes of $J$ with eigenvalues $\lambda \propto 1/\omega^2$.23 Note that this relationship could be changed to $\lambda \propto \omega^2$ by exchanging the roles of capacitors and inductors in implementing the connections between the nodes resp. to the ground.

Three types of experiments are performed. First, an impedance analyzer is used to measure $Z_a$ as a function of
frequency \( f = \omega/2\pi \) for each node \( a \). The data for three input nodes are shown in Fig. 3b. Second, these eigenmodes are resonantly excited and their voltage profile is measured using lock-in amplifiers. For the modes at the highest six frequencies, both magnitude (relative to the voltage at the input node) and phase (relative to a reference signal) are shown in Fig. 3c. In the final experiment, the circuit is stimulated by the broadband voltage pulse shown in Fig. 4a fed into the circuit as a current pulse at a node close to the boundary. Subsequently, the voltage is measured as a function of time at each node. We observe the pulse to propagate in the Poincaré disk (the full time dependence is shown in Supplementary Movie S1 and discussed in supplementary text S5\(^{(2)}\)). A snapshot of the instantaneous phase profile is shown in Fig. 4b, which visualizes the propagation of the pulse.

**Discussion of the experimental data**

We proceed with discussing the results of these three measurements. Comparing the impedance of input node 1 (blue curve) to nodes 14 and 18, see Fig. 3b, we clearly observe the spectral reordering discussed in the previous section: there are four additional peaks for input node 14 and 18 located between the two highest-frequency peaks for input node 1. This implies that the second \( \ell = 0 \) mode (i.e., the first mode with \( n > 0 \)) is the sixth eigenmode. The explicit values of \( \ell \) and \( n \) for specific modes can be deduced from the voltage profiles of the eigenmodes obtained in the second experiment, see Fig. 3c.

We further plot (orange squares in Fig. 2b) the extracted dispersion of the Laplacian frequencies \( \lambda_{n,\ell}^H \) with the angular momentum \( |\ell| \), obtained by a circular Fourier transform of the measured signal. We observe an almost perfect match with the theoretically predicted values (blue dots in Fig. 2b) for the first few measured modes. However, higher modes are increasingly difficult to excite and detect, due to the finite resolution in frequency and space. We remark that the boundary sites of the present experimental realization of a hyperbolic lattice could be used to probe holographic dualities. For each eigenmode of the system, only its angular distribution on the boundary is important (cf. the angular momentum dispersion in Fig. 2b), yielding a novel and universal one-dimensional physical system on the boundary. We leave a detailed examination of these intriguing edge modes to future studies.

Finally, we discuss the time-resolved measurements. We excite the densest region of the frequency spectrum (Fig. 3b) using a current pulse (Fig. 4b) of mean frequency 500 kHz (Fig. 4c). By exciting a large number of modes, we approximate the continuum response. The propagation of the pulse through the circuit network leads to the profile of instantaneous phases depicted in Fig. 4d, where the phase fronts can be easily identified by the positions of equal instantaneous phase. Since the connectivity of the nodes implements the metric of the Poincaré disk, these phase fronts form concentric hyperbolic circles, highlighted by black circles in Fig. 4d. This agrees with the theoretical expectation that the signal emanates from the excited node along geodesics, which are the generalization of “straight lines” in curved space (red lines in Fig. 4a).

Wave fronts are perpendicular to these geodesics and thus constitute concentric circles (black circles in Fig. 4a) up to corrections due to boundary reflections. In Fig. 4d-f, we have chosen an early time during the excitation such that contributions from such reflections do not have a significant impact on the measured phases. Finally, when plotting the phase vs. hyperbolic \( d_H \) and Euclidean \( d_E \) distance in Fig. 4e and f,
respectively, we observe that the correlation of the phase with $d_{1}$ is stronger than with $d_{9}$. This manifests that the propagation of the signal indeed follows hyperbolic rather than Euclidean geodesics, thereby verifying that the system realizes the hyperbolic rather than Euclidean metric.

**DISCUSSION**

Our work demonstrates that electric circuits provide a versatile and highly scalable platform for emulating hyperbolic models, eclipsing the previously considered methods\(^{16,23}\) in several aspects. First, electric circuits provide easy means for embedding hyperbolic lattices on a flat physical geometry, while allowing for unconnected wire crossings. Such flexibility could be utilized to include coupling beyond nearest neighbors and to implement the plethora of other hyperbolic tessellations\(^{1,22}\). In particular, going beyond the presented emulation of the Laplace operator in a negatively curved space, the platform allows to emulate much more complex tight-binding models. These could, for example, be used to test the recently emerging concepts of hyperbolic band theory\(^{12-14}\), hyperbolic crystallography\(^{35}\), and hyperbolic topological insulators\(^{19}\). Our results also demonstrate that signatures of the hyperbolic geometry are easily accessible in terms of experimental probes. This is in stark contrast to the realization of a hyperbolic lattice in coupled waveguide resonators where those properties are experimentally barely detectable\(^{6}\).

Furthermore, including non-linear and non-reciprocal elements in the network, such as transistors and diodes, is trivial\(^{17}\). We also remark that electric circuits are easily fabricable using existing chip manufacturing technology and commercially available components, such that building lattices with $\sim 10^{4}$ sites should be within reach. Therefore, electric circuits could be manufactured with the goal to experimentally study non-linear dynamics of systems with sizes that are unwieldy for numerical simulations. Staying instead within the linear regime, there is a relationship between particles moving freely on geodesics of negatively curved space and deterministic chaos, as illustrated by the Hadamard system\(^{38}\). In combination with our experimental verification of the signal propagation along the geodesics, this relationship designates electric circuits a promising experimental platform to investigate classical models of chaos.

Finally, it is conceivable\(^{17}\) that superconducting qubits could be combined with electric circuits, suggesting exciting future generalizations of our work towards quantum models. We expect such an integration will inspire a new paradigm for designing and measuring holographic toy-models and topological or conformal boundary field theories in discrete geometries. In this context, it is worth reminding that theoretical models of hyperbolic quantum systems were proposed\(^{39}\), which still await experimental implementation, including MERA tensor networks\(^{8,3}\) and topological quantum memories\(^{40,41}\). These efforts have the potential to fundamentally alter our understanding of physics in curved spaces and imply novel views on problems in condensed matter theory, quantum gravity, cosmology, and holography.

**MATERIALS AND METHODS**

**Eigenmodes of the Laplace-Beltrami operator**

The solutions to Eq. (2) on the disk $D_{r_{0}}$ of radius $r_{0} < 1$ correspond to the eigenmodes of a drum of radius $r_{0}$ in the corresponding geometry. They can be conveniently expressed using special functions. Going to polar coordinates $(r, \theta)$, one finds (cf. supplementary text S1) for the Euclidean metric

$$u_{E}^{\ell}(r, \theta) = \mathcal{J}_{\ell}(k_{\ell}r)e^{i\ell\theta}, \tag{4}$$

where $\mathcal{J}_{\ell}(z)$ are the Bessel functions of the first kind and $k_{\ell}$ is the $(n + 1)$-th zero of $k \mapsto \mathcal{J}_{\ell}(kr_{0})$. From the angular part of the solution it follows that $\ell$ can be interpreted as the angular momentum. Furthermore, $k_{n\ell} = \frac{2\ell n + 1}{r_{0}}$, where $z_{n\ell}$ is the $n$th zero of $\mathcal{J}_{\ell}(z)$. The radial zeroes $r_{m}$ of $u_{E}^{\ell}(r, \theta)$ are then given by

$$r_{m} = \frac{z_{n\ell, m}}{k_{n\ell}} = r_{0} \frac{z_{n\ell, m}}{z_{n\ell, n+1}}, \tag{5}$$

such that $m = 1, 2, \ldots, n$ for the non-trivial zeroes $r_{m} < r_{0}$. Thus, $u_{E}^{\ell}$ has exactly $n$ non-trivial radial zeroes.

For the hyperbolic metric, on the other hand, one finds (cf. supplementary text S1)

$$u_{H}^{\ell}(r, \theta) = P^{\ell}_{\frac{1}{2}}(1 + ik_{\ell}r) \left( 1 + r^{2} \right)^{-\frac{1}{2}} e^{i\ell\theta} \tag{6}$$

with $P^{\ell}_{\frac{1}{2}}(z)$ the associated Legendre functions and $k_{n\ell}$ the $(n + 1)$-th zero of $k \mapsto P^{\ell}_{\frac{1}{2}}(1 + ikr) \left( 1 + r^{2} \right)^{-\frac{1}{2}}$. Again we can interpret $\ell$ as the angular momentum and $n$ as the number of radial zeroes of $u_{H}^{\ell}$.

**Lattice regularization**

The graph Laplacian of a simple (i.e., undirected) graph is given by

$$Q = A - D, \tag{7}$$

where $D$ is the degree matrix (the diagonal matrix containing the number of adjacent sites for each site as entries) and $A$ the adjacency matrix ($A_{ab} = 1$ if sites $a$ and $b$ are adjacent and zero otherwise). Assuming the graph represents a lattice regularization of a continuum, then any function $u(x, y)$ induces a function on the lattice, via $a \mapsto u(x_{a}, y_{a}) =: u_{a}$, and the action of the Laplacian matrix, $\sum_{b} Q_{ab}u_{b}$, can be expressed in terms of the continuum Laplace-Beltrami operator, e.g., following the steps outlined in Ref. 17.

Tessellations of the Euclidean or hyperbolic plane constitute a lattice regularization of the continuum\(^{1}\), and the boundaries of the tiles (i.e., vertices and edges) can be interpreted as forming a graph. If only a finite segment of the plane is tiled, the tessellation has a boundary, which corresponds to vertices of the graph that are attached to fewer edges than the bulk vertices. Naturally, this is reflected both in the adjacency matrix $A$ as well as in the degree matrix $D$. However, if we impose Dirichlet boundary conditions for $u(x, y)$ as we do in the main text, then $u$ vanishes on the boundary sites, which allows us to drop them from the matrix description.
Consequently, we are left only with the bulk part of $Q$. For a Euclidean $\{3, 6\}$ tessellation, we find (cf. supplementary text S2)

$$
\sum_b Q_{ab} u_b = \frac{3}{2} d^2 \Delta_E u_a + \mathcal{O}(d^3),
$$

(8)

where $d$ is the distance between sites. For the hyperbolic tessellation $\{3, 7\}$, on the other hand, we find (cf. supplementary text S2)

$$
\sum_b Q_{ab} u_b = \frac{7}{4} h^2 \Delta_H u_a + \mathcal{O}(h^3),
$$

(9)

where $h = \tanh(d_0) = 0.496970$, and $d_0 = 0.545275$ is the hyperbolic distance between two neighboring sites in the Poincaré disk representation. For both tessellations, the leading contribution is the Laplace-Beltrami operator for the appropriate metric, such that eigenstates of $Q$ correspond to $u^{\text{aff}}_k$ from Eq. (2) and the eigenvalues are proportional to $\lambda_k^{n_f}$ (up to higher-order corrections).

**Extraction of angular momentum dispersion**

The angular momentum dispersion, $\lambda_k^{n_f}$ vs. $|\ell|$, shown in Fig. 2b is extracted from the spectrum and eigenstates of the graph Laplacian using Fourier analysis on shells of the graph, i.e. sites that have approximately the same distance from the disk center and form a circle. A shell can therefore be considered as a one-dimensional system with periodic boundary conditions with the polar angle taking the role of position. For each eigenvector $u$, its components on one of the shells, therefore, define a periodic function $u_{\text{shell}}(\theta)$ defined at discrete $\theta$. By first interpolating $u_{\text{shell}}(\theta)$ and then performing a discrete Fourier transform on regular samples, we determine the dominant Fourier component which is interpreted as the angular momentum $|\ell|$ of $u$. For the eigenstates shown in Fig. 2b it is sufficient to consider the outermost shell, but for higher eigenstates, considering additional shells can improve the results.

**Theoretical description of electric circuit**

In our circuit network, nodes are coupled with capacitance $C$, each node is coupled to ground via an inductance $L$ and the boundary conditions are implemented by adding additional capacitive couplings to ground such that each node is capacitively coupled to seven other components. The grounded circuit Laplacian is then given by the graph Laplacian $Q$ of the underlying (buck) lattice and a contribution from the inductive grounding (neglecting resistances and other parasitic effects):

$$
J(\omega) = i\omega C Q + \frac{1}{i\omega L} \mathbb{1}.
$$

(10)

The spectral decomposition is therefore given by the eigenstates $\psi^\beta$ and eigenvalues $q^\beta$ of the Laplacian matrix, $-Q \psi^\beta = q^\beta \psi^\beta$, with eigenvalues

$$
\lambda^\beta(\omega) = \frac{1 - q^\beta \omega^2 LC}{i\omega L}.
$$

(11)

The eigenmode index can be decomposed into the principal and orbital index, $\beta = (n, \ell)$, to match the analytic solution in the continuum.

The inverse of $J$ is called the circuit Green function and can be obtained by expanding $J$ into eigenmodes (here we assume that $J$ is Hermitian and the circuit grounded, as is the case for our circuit) $J(\omega) = \sum_{\beta} j^\beta(\omega) |\psi^\beta\rangle \langle \psi^\beta|$, then,

$$
G(\omega) = \sum_{\beta} \frac{1}{j^\beta(\omega)} |\psi^\beta\rangle \langle \psi^\beta|.
$$

(12)

Assuming current fed into node $a$, i.e., $I_a = \sum_c I_{c,a}$, the impedance of that node to ground can be written in terms of the eigenmodes of $J$

$$
Z_a(\omega) = G_{aa}(\omega) = \sum_{\beta} \frac{1}{j^\beta(\omega)} |\psi^\beta_a|^2,
$$

(13)

and the stationary voltage response, i.e., after equilibration, at some other node $b$ is given by

$$
V_b = G_{ba}(\omega) I_a = \sum_{\beta} \frac{1}{j^\beta(\omega)} \psi^\beta_b \psi^\beta_a^*.
$$

(14)

We observe that in both cases the result is a superposition of eigenmodes of $J$ with the weight proportional to $1/j^\beta(\omega)$, which has a resonance at

$$
\omega^\beta = \frac{1}{\sqrt{LCq^\beta}}.
$$

(15)

By combining this result with Eq. (9) for the bulk-to-lattice correspondence, it follows that a resonance of $Z_a$ at frequency $f^\beta = \omega^\beta/(2\pi)$ corresponds to an eigenmode of the hyperbolic drum with eigenvalue

$$
\lambda^\beta = \frac{1}{7\pi^2 h^2 LC} \frac{1}{(f^\beta)^2}.
$$

(16)

This results in a spectral reversal where the lowest-frequency (small $\lambda$) eigenmodes of the Laplace-Beltrami operator correspond to the highest-frequency (large $f$) oscillations of the electric circuit. Equation (16) is used to plot the experimental data in Fig. 2. If the circuit is probed at one of the resonance frequencies, $\omega^\beta$, then the dominant contribution to $V_b$ is

$$
V_b \approx \frac{1}{j^\beta(\omega^\beta)} \psi^\beta_b \psi^\beta_a^* \approx \psi^\beta_b V_a,
$$

(17)

where $j^\beta$ does not diverge in practice due to the presence of small resistive terms (see supplementary text S4b for a discussion of the impact of parasitic resistances). This implies that the voltage profile encodes the eigenvectors $\psi^\beta_b$.

**Electric circuit parameters**

The capacitances of the electric circuit are implemented by ceramic capacitors with $C = 1 \mu\text{F}$ and 1% tolerance, the inductances as power inductors with $L = 10 \mu\text{H}$, 5% tolerance and a minimal quality factor of 40 at 1 MHz. Nodes on the
boundary have additional capacitors $C$ to ground such that each node is connected to seven identical capacitors in total. Finally, each node is made accessible for in- and output via SMB connectors.

**Measurement details**

The impedance measurements were performed in a two-terminal measurement configuration using a Zurich Instruments MFIA 5 MHz impedance analyzer. A short/open compensation routine was used to remove the residual impedance and stray capacitance of the test fixture. The impedance of all 85 circuit nodes has been recorded for frequencies in the range from 250 kHz to 1.75 MHz. To exclude transmission line effects in the measurement, the maximum cable length was restricted to be below 1.8 m.

For the measurement of the voltage profiles of the eigen-modes, a reference voltage signal and phase sensitive detection is needed. This was achieved by using three Zurich Instrument MFIA 5 MHz impedance analyzers as lock-in amplifiers synchronized in frequency and phase. Each mode was excited by a current signal of the corresponding frequency fed into the circuit. The impedance measurements were performed in a two-terminal measurement configuration using a Zurich instrument. The impedance measurements were performed in a two-terminal measurement configuration using a Zurich instrument. The impedance measurements were performed in a two-terminal measurement configuration using a Zurich instrument.

Finally, each node is made accessible for in- and output via SMB connectors.

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SUPPLEMENTARY MATERIALS
Supplementary Text S1 to S5
Figures S1 to S4
Movie S1

ACKNOWLEDGEMENTS
SUPPLEMENTARY TEXT

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S1. EIGENMODES OF THE LAPLACE-BELTRAMI OPERATOR

The general Laplace-Beltrami operator for a given metric tensor $g_{ij}$ is

$$\Delta_g = \frac{1}{\sqrt{\det(g)}} \partial_i \left( \sqrt{\det(g)} g^{ij} \partial_j \right), \quad (S1)$$

where $g^{ij}$ is the matrix inverse of $g_{ij}$. In Euclidean space, we have $(g_E)_{ij} = \delta_{ij}$, such that $\Delta_E = \partial_x^2 + \partial_y^2$, the usual Laplace operator. In contrast, for the Poincaré disk representation of the hyperbolic plane ($D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = r^2 < 1\}$ with length element $ds^2 = (1 - x^2 - y^2)^{-2}(dx^2 + dy^2)$) corresponding to constant negative curvature $\kappa = -1$,

$$(g_H)_{ij} = (1 - r^2)^{-2} \delta_{ij}, \quad (S2)$$

such that the Laplace-Beltrami operator is

$$\Delta_H = (1 - (x^2 + y^2))^2 \left( \partial_x^2 + \partial_y^2 \right). \quad (S3)$$
We now consider the disk \( \mathcal{D}_{r_0} := \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r_0^2 \} \subset \mathcal{D} \) and rewrite the Laplace-Beltrami operator in polar coordinates \( x = r \cos(\theta), y = r \sin(\theta) \):

\[
\Delta_E = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2, \\
\Delta_H = (1 - r^2)^2 \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right).
\]

(S4) \hspace{1cm} (S5)

We are interested in eigenmodes of \(-\Delta_g\), where \( g \in \{ E, H \} \) indicates the geometry, i.e. solutions to the Dirichlet problem

\[
(\Delta_g + \lambda) u(x, y) = 0, \quad u(x, y)|_{(x, y) \in \partial \mathcal{D}_{r_0}} = 0.
\]

(S6)

a. Euclidean space

We first discuss the solutions to Eq. (S6) in the Euclidean case. The differential equation is separable, such that we can make the ansatz \( u(x, y) = R(r)\Theta(\theta) \) and find

\[
- \frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{r^2 R''(r) + r R'(r) + r^2 \lambda R(r)}{R(r)}.
\]

(S7)

Since, we are on the disk, \( \Theta(\theta + 2\pi) = \Theta(\theta) \), such that \( \Theta(\theta) = e^{i\ell \theta} \) for \( \ell \in \mathbb{Z} \) and

\[
r^2 R''(r) + r R'(r) + (k^2 r^2 - \ell^2) R(r) = 0,
\]

(S8)

where we substituted \( \lambda = k^2 \). With the further substitution \( \rho = kr \), we obtain

\[
\rho^2 R''(\rho) + \rho R'(\rho) + (\rho^2 - \ell^2) R(\rho) = 0,
\]

(S9)

which is the Bessel equation, such that the solutions are given by the Bessel functions of the first kind

\[
u^\ell_E(x, y) = J_{\ell}(k_n r) e^{i\ell \theta}
\]

(S10)

where \( k_n = z_n/r_0 \) and \( z_n \) is the \( n \)-th root of \( J_{\ell} \).

b. Hyperbolic space

We proceed analogously in the hyperbolic case, where the same ansatz \( u(x, y) = R(r)\Theta(\theta) \) results in

\[
(1 - r^2)^2 r^2 R''(r) + (1 - r^2)^2 r R'(r) + (\lambda r^2 - \ell^2 (1 - r^2)^2) R(r) = 0.
\]

(S11)

Introducing \( s := (1 + r^2)/(1 - r^2) \), this can be rewritten as

\[
2s R'(s) - 4r^2 (1 - s^2) R''(s) + (\lambda r^2 - \ell^2 (1 - r^2)^2) R(s) = 0
\]

(S12)

Dividing by \(-4r^2\) and setting \( \lambda = -4q(q + 1) = 1 + k^2 \), we find

\[
\left( (1 - s^2) \partial_s^2 - 2s \partial_s + \left( q(q + 1) - \ell^2 \frac{1}{1 - s^2} \right) \right) R(s) = 0,
\]

(S13)

whose solutions are the associated Legendre functions \( P^\ell_q(s) \), such that we obtain

\[
u^\ell_H(x, y) = P^\ell_{\ell/2(-1 + ik_{n\ell})} \left( \frac{1 + r^2}{1 - r^2} \right) e^{i\ell \theta}
\]

(S14)

with \( k_{n\ell} \) being the \( n \)-th root of

\[
k \mapsto P^\ell_{\ell/2(-1 + ik)} \left( \frac{1 + r_0^2}{1 - r_0^2} \right)
\]

(S15)

and \( \ell \in \mathbb{Z} \) as in the Euclidean case.
S.2. LATTICE REGULARIZATION OF THE GRAPH LAPLACIAN

As discussed in the Methods, the graph Laplacian can be approximated by the continuum Laplace-Beltrami operator in the leading order in the distance between lattice sites, see e.g., Eq. (8) and Eq. (9). Here we present a detailed derivation of this expansion for regular tessellations with equivalent sites where all distances between adjacent sites are equal (in the corresponding metric). Such tessellations are called Archimedean. They are generally denoted by their vertex configuration “\(n_1, n_2, \ldots, n_q\)”, where \(n_1, n_2, \ldots, n_q\) give the number of sides of the \(q\) regular polygons meeting at each vertex. The tessellations considered in the main text are a special case called Platonic tessellations, because they have \(q\) copies of the same regular \(p\)-gon meeting at each vertex.

We now consider Archimedean tessellations of the unit disk for both Euclidean and hyperbolic space (in the Poincaré disk representation) with Dirichlet boundary conditions imposed. For convenience, we parametrize the coordinates \((x, y)\) of the Euclidean plane and the Poincaré disk using complex numbers \(z := x + iy\), where \(z\) lies in the infinite complex plane for the Euclidean and in the complex unit disk for the hyperbolic case. The boundary condition implies that all vertices, including those on the non-vanishing boundary, are equivalent. Recall that the graph Laplacian is a matrix \(Q = A - D\) with entries \(Q_{ab}\) and any test function \(u(z)\) on the complex unit disk induces a function on the lattice, via \(a \mapsto u(z_a) = u_a\). We closely follow Appendix B of Ref. 17 to express the action of \(Q\) on \(u_a\) in terms of the Laplace-Beltrami operator. The action of the graph Laplacian \(Q\) on the test function \(a \mapsto u_a\) at an arbitrary site \(a\) (using Einstein’s summation convention) is then

\[
Q_{ab} u_b = A_{ab} u_b - D_{ab} u_b = \sum_{i=1}^{q} u(z_{a+e_i}) - qu(z_a),
\]

where \(z_{a+e_i}\) denote the position of the sites adjacent to site \(a\).

a. Euclidean space

We first discuss a Euclidean tessellation with coordination \(q\), i.e., where each site has \(q\) adjacent sites. Let \(d\) be the distance between two adjacent sites, then

\[
z_{a+e_i} = z_a + de^{i\phi_a} e^{i\frac{2\pi}{q}(i-1)} = z_a + dw_{ai},
\]

where \(\phi_a\) is a site-dependent phase factor, and we can expand \(u(z_{a+e_i})\) in powers of \(d\):

\[
u(z_{a+e_i}) = u(z_a) + \frac{d}{dd} u(z_{a+e_i})\bigg|_{d=0} d + \frac{1}{2} \frac{d^2}{dd^2} u(z_{a+e_i})\bigg|_{d=0} d^2 + \mathcal{O}(d^3)
\]

\[
u(z_a) + \left( w_{ai} \partial \bar{z} + \bar{w}_{ai} \partial z \right) u(z)\bigg|_{z=z_a} d + \frac{1}{2} \left( w_{ai} \partial \bar{z} + \bar{w}_{ai} \partial z \right)^2 u(z)\bigg|_{z=z_a} d^2 + \mathcal{O}(d^3)
\]

with \(\partial_z = \partial / \partial z\) and \(\partial \bar{z} = \partial / \partial \bar{z}\) and \(\bar{\cdot}\) denoting complex conjugation. Note that for any \(m \in \mathbb{Z}\)

\[
\sum_{i=1}^{q} w_{ai}^m = e^{i\phi_a} \sum_{i=1}^{q} e^{i\frac{2\pi m}{q}(i-1)} = 0,
\]

and \(|w_{ai}| = 1\), such that \(\phi_a\) drops from the subsequent calculations:

\[
\sum_{i=1}^{q} u(z_{a+e_i}) = qu(z_a) + q d^2 \partial_z \bar{\partial}_z u(z)\bigg|_{z=z_a} + \mathcal{O}(d^3).
\]

Since \(\Delta_E = 4 \partial_z \bar{\partial}_z\), we finally find

\[
Q_{ab} u_b = \frac{q}{4} d^2 \Delta_E u(z_a) + \mathcal{O}(d^3).
\]

With \(q = 6\) for a \(\{3, 6\}\) tessellation this reproduces Eq. (8).
b. Hyperbolic space

We proceed analogously for hyperbolic tessellations with coordination $q$ and hyperbolic distance $d_0$ between adjacent sites. Here it is helpful to first transform the Poincaré disk by the automorphism

$$z \mapsto v(z) = \frac{z_n - z}{1 - z \bar{z}_n}. \tag{S22}$$

This transformation corresponds to a $\pi$-rotation that exchanges $z_n$ and the origin. In particular, note that it squares to identity, implying that $z \mapsto v(z)$ and its inverse $z \mapsto v^{-1}(z)$ are equivalent. Recall further that the hyperbolic distance between the origin and an arbitrary point $z$ in the unit disk takes the form $d = \text{arctanh}(|z|)$. In the transformed coordinates, $z_n + e_i$ takes the simple form

$$v_{n+e_i} = v(z_{n+e_i}) = h e^{i\phi_n e^{i\frac{2\pi}{3}(i-1)}} = h w_{ai} \tag{S23}$$

with $h = \tanh(d_0)$, $u(z_{n+e_i}) = u(z(v_{n+e_i}))$, and $\phi_n$ being again a site-dependent phase factor that subsequently drops out from the calculations. Expanding in powers of $h$, we obtain

$$u(z_{n+e_i}) = u(z_n) + \frac{1}{\frac{d^2}{dh^2} u(z(v_{n+e_i}))} \bigg|_{h=0} h + \frac{1}{2} \frac{d^2}{dh^2} u(z(v_{n+e_i})) \bigg|_{h=0} h^2 + O(h^3)$$

$$= u(z_n) - (1 - |z_n|^2) (w_{ai} \partial_z + \bar{w}_{ai} \partial_{\bar{z}}) u(z) \bigg|_{z=z_n} h + \frac{1}{2} (1 - |z_n|^2)^2 (w_{ai} \partial_z + \bar{w}_{ai} \partial_{\bar{z}})^2 u(z) \bigg|_{z=z_n} h^2 + O(h^3) \tag{S24}$$

Since $w_{ai}$ are still the same as in the Euclidean case, Eq. (S16) becomes

$$Q_{ab} u_b = \frac{q}{4} h^2 (1 - |z|^2)^2 \Delta_E u(z) \bigg|_{z=z_n} + O(h^3) \tag{S25}$$

and recalling that $(1 - |z|^2)^2 \Delta_E = \Delta_H$, we finally arrive at

$$Q_{ab} u_b = \frac{q}{4} h^2 \Delta_H u(z_n) + O(h^3). \tag{S26}$$

With $q = 7$ for a $\{3,7\}$ tessellation this reproduces Eq. (9).

S3. COMPARISON OF TESSELLATIONS OF HYPERBOLIC SPACE

In Section S2 we have derived that corrections of the graph Laplacian to the continuum Laplace-Beltrami operator are of third order in $h = \tanh(d_0)$ with $d_0$ being the hyperbolic distance between adjacent lattice sites. This agrees with the intuition that the density of the tessellation determines the accuracy of the approximation of the continuum. We therefore introduce the area per vertex for a general Archimedean tessellation $n_1,n_2,\cdots,n_q$:

$$\alpha = -\pi \left(2 - \sum_{i=1}^q \frac{n_i - 2}{n_i}\right). \tag{S27}$$

In this section we study three tessellations of the hyperbolic plane: (i) $\{3,7\}$, (ii) $\{7,3\}$, and (iii) $6.6.7$ (also called the “hyperbolic soccerball”), illustrated in panels a–c of Fig. S1. The area per vertex for those is (i) $\pi/3$, (ii) $\pi/7$, and (iii) $\pi/21$. We compare these three tessellations with respect to three properties: First, in Section S3a we consider the accuracy of approximating the continuum. This is determined by the density of the tessellation, which in turn depends on the vertex configuration (or equivalently on $\alpha$). Subsequently, in Section S3b, we examine the total (i.e., integrated over the area) curvature that can be obtained in a finite lattice with a fixed number of vertices. Finally, in Section S3c, we discuss the effect of rotation symmetry with respect to a central vertex on the spectrum and the profile of the eigenmodes.

a. Approximating the continuum

Here, we analyze how well the three tessellations shown in Fig. S1 approximate the continuum. All three tessellations cover approximately the same disk of radius $r_0 < 1$; however, due to them having different area per vertex $\alpha$, cf. Eq. (S27), the number
Figure S1. Comparison of hyperbolic tessellations for fixed disk radius. a,b,c, Tessellations \( \{3, 7\} \), \( \{7, 3\} \) and \( \{6.6.7\} \) of the hyperbolic plane in Poincaré disk representation (the unit circle is shown in blue), respectively, covering a disk of radius \( r_0 = 0.99 \) (red circle). This results in 589, 1197, and 3857 vertices, respectively. d,e,f, Angular momentum dispersion of the eigenstates of the graph Laplacian for the three tessellations (orange diamonds) compared to the same for the eigenmodes of the continuum Laplace-Beltrami operator on the hyperbolic drum with the same radius \( r_0 \) (blue disks).

of vertices varies between the three cases. More specifically, for each of the three tessellations we compare the spectrum of the graph Laplacian \( Q \) to the spectrum of the Laplace-Beltrami operator with Dirichlet boundary conditions for a disk of the same radius \( r_0 \).

We have already discussed this problem analytically in Section S2 and have found that the graph Laplacian \( Q \) is approximated by the Laplace-Beltrami operator up to corrections of order \( h^3 \), where \( h = 0.496970 \) for \( \{3, 7\} \), \( h = 0.275798 \) for \( \{7, 3\} \), and \( h = 0.165657 \) for \( \{6.6.7\} \). We therefore anticipate these correction to be smallest for the \( \{6.6.7\} \) tessellation, in agreement with the area per vertex being smallest for this tessellation. We now verify this explicitly for the three tessellations on disks of radius \( r_0 = 0.99 \) by numerically computing the spectrum (eigenvalue as a function of angular momentum) and comparing the dispersion to the one obtained from the continuum, as we did in Fig. 2 in the main text. Note that because of finite-size effects, our method fails to correctly identify the angular momentum of certain highly excited states (see the corresponding discussion in the Methods section). The results are shown in Fig. S1 and we observe that the difference between lattice (orange diamonds) and continuum (blue disks) dispersion is smaller for tessellations with small area per site, i.e., when the total number of sites is larger.

b. Signatures of negative curvature

Above we have answered the question which of the three tessellations gives the best approximation of the continuum for a disk with fixed radius \( r_0 < 1 \). Experimentally, however, we are interested in a different question: For a given number of vertices (sites), which tessellation gives the strongest signatures of negative curvature? Naturally, we expect tessellations that cover a larger area of the Poincaré disk to exhibit stronger signatures of the negative curvature. Therefore, a large area per vertex is desirable. According to Eq. (S27) and the values given in the paragraph following that equation, the \( \{3, 7\} \) tessellation is the one with the largest area per vertex out of the three under consideration.

We fix the (approximate) number of sites to 275 and construct the tessellations such that they consist of full shells. The resulting lattices are shown in Fig. S2. Here, we compare the angular momentum dispersion to the dispersion obtained from the eigenmodes of the continuum Laplace-Beltrami operator on the Euclidean drum of the same radius \( r_0 \), each. The signature of
Figure S2. Comparison of hyperbolic tessellations for fixed number of vertices. a,b,c, Tessellations \{3, 7\}, \{7, 3\} and 6.6.7 of the hyperbolic plane in Poincaré disk representation (the unit circle is shown in blue), respectively, with approximately 275 vertices (the exact numbers of vertices are 274, 273 and 280, respectively). The red circle indicates the bounding circle of each tessellation with radii $r_0 = 0.98$, 0.955 and 0.88, respectively. d,e,f, Angular momentum dispersion of the eigenstates of the graph Laplacian for the three tessellations (orange diamonds) compared to the same for the eigenmodes of the continuum Laplace operator on the Euclidean drum with the same radius $r_0$ (blue disks). Horizontal lines of the corresponding color indicate the eigenvalue of the $(n, \ell) = (1, 0)$ mode in each geometry. The difference in the number of modes below the orange and the blue line quantifies the spectral reordering between the hyperbolic and the Euclidean disk. Note that the eigenvalues $\lambda_E$ of the Euclidean drum are rescaled and shifted to allow for a better qualitative comparison to the hyperbolic dispersion, i.e., to emphasize the reordering of eigenstates.

Negative curvature which we have identified in the main text, i.e., the reordering of the eigenstates compared to the Euclidean case, is with a difference of four states strongest for the \{3, 7\} tessellation (panels a, d) and reduced to only a single state for 6.6.7 (panels c, f). Therefore, to reveal the spectral reordering in an experimental realization with a limited number of sites, it may be desirable to opt for the \{3, 7\} tessellation.

c. Role of rotation symmetry

Finally, we discuss the role of rotation symmetry. The tessellations shown in Fig. S1b,c can be shifted such that they have a vertex at the centre of the disk. As we argued in the main text, this is advantageous in order to to excite and detect $\ell = 0$ modes which have a maximum amplitude at the centre of the disk. However, in the case of the \{7, 3\} tessellation, the seven-fold rotation symmetry is broken down to a three-fold rotation symmetry, while in the case of the 6.6.7 tessellation no rotation symmetry is remaining at all. These shifted tessellations are displayed in Fig. S3b,c.

In Fig. S3d–f we show for each of the first 20 eigenmodes of the graph Laplacian of each considered tessellation their angular momentum $\ell$ and their weight at the central vertex. From the continuum we expect that only $\ell = 0$ modes have non-vanishing weight at that vertex. This, indeed, holds on the \{3, 7\} lattice for all $|\ell| \leq 6$. After that, we observe that eigenvalues of the two $|\ell| = 7$ modes (and similarly for integer multiples of 7) are split (cf. Fig. S1d), in stark contrast with the continuum case where such modes are degenerate. We also observe that one of these two modes acquires a non-zero weight at the central vertex. An analogous feature is observed for the \{7, 3\} tessellation, where the modes with $|\ell|$ being integer multiples of 3 are similarly contaminated. Finally, the situation for the 6.6.7 tessellation is even less ideal as here most of the modes acquire a non-vanishing weight at the central vertex.

Therefore, we conclude that a small order of rotation symmetry leads to a larger number of $\ell \neq 0$ eigenmodes with non-vanishing weight at the central vertex. This in turn prevents us from easily detecting (and exciting) $\ell = 0$ modes via the central
Figure S3. Comparison of hyperbolic tessellations with different order of rotation symmetry. a,b,c, Tessellations \( \{3,7\} \), \( \{7,3\} \) and \( \{6,7\} \) of the hyperbolic plane in Poincaré disk representation (the unit circle is shown in blue), respectively, with approximately 275 vertices (the exact numbers of vertices are 274, 271 and 271, respectively) and a vertex at the centre. The red circle indicates the bounding circle of each tessellation with radii \( r_0 = 0.98, 0.96 \) and 0.88, respectively. d,e,f, For the first 20 eigenmodes (counting degenerate modes), the absolute value of their weight at the central vertex (blue disks, left vertical axis) and absolute value of their angular momentum \( |\ell| \) (orange diamonds, right vertical axis) are shown.

S4. PARASITIC RESISTANCES

Resolving individual peaks in the resonance spectrum requires a sufficiently high Q factor for all circuit elements. With increasing parasitic resistances, the resonance peaks of individual modes widen and flatten, making them harder to identify in an impedance sweep. In practice, inductors are the main source of parasitic resistances in our circuit. Fig. S4 compares simulated impedance sweeps of node 18 for Q values of the inductors. For Q-values of 50 and 20, all relevant impedance peaks can be easily identified, while at a Q-value of 10, the Peak at 0.906 MHz is no longer recognizable.

Figure S4. Comparison of the simulated impedance spectrum at node 18 for different Q factors of the inductors.
In this section we briefly explain the time-dependent behavior of the hyperbolic circles of constant phases in Supplementary Movie S1. While we have already discussed the results at fixed times in the main text, the time-dependence requires some additional explanation. In particular, we observe that the hyperbolic circles of constant phases are falling into the input node, which is a consequence of our specific electric circuit network being a negative-index metamaterial (also called left-handed), i.e., having a negative refractive index.

The propagation of waves on a drum is generally given by the following differential equation involving the Laplace-Beltrami operator:

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(t, x, y) - \Delta_g u(t, x, y) = 0
\]

with the wave speed \(c\) determined by the medium. In the Euclidean case, for example, the wave-equation leads to the dispersion \(\omega(k) = c |k|\) with the two-dimensional momentum vector \(k\); thus, phase and group velocity are equal: \(v_p = v_g = c\). The situation in the experiment corresponds to an additional inhomogenous source term \(S(x, y, t) \sin(\omega t)\), where \(S\) is localized both in space (where the excitation happens) and in time (pulse-like), i.e.,

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(t, x, y) - \Delta_g u(t, x, y) = S(x, y, t) \sin(\omega t).
\]

The source term leads to an excitation of eigenmodes of the drum, i.e., of \(-\Delta_g\), according to its frequency spectrum and the pulse propagates across the drum with speed \(c\).

For an electric circuit there are some important differences to the dynamics, even though the situation is conceptually the same. According to Kirchhoff’s law, the differential equation governing our electric circuit network is

\[
\frac{\partial}{\partial t} I_a = C Q_{ab} \frac{\partial^2}{\partial t^2} V_b - \frac{1}{L} V_a,
\]

where \(I_a(t)\) and \(V_a(t)\) are the input current and voltage at node \(a\), \(C\) the capacitance coupling two adjacent nodes, \(L\) the inductance to ground for each node and \(Q_{ab}\) the graph Laplacian describing the (capacitive) connections between the nodes. The continuum limit therefore is

\[
C \frac{\partial^2}{\partial t^2} \Delta_g V(t, x, y) - \frac{1}{L} V(t, x, y) = \frac{\partial}{\partial t} I(t, x, y)
\]

where the voltage field \(V(t, x, y)\) takes the role of \(u(t, x, y)\) and the time-derivative of the input current \(\frac{\partial}{\partial t} I(t, x, y)\) the role of the source.

The modified wave-equation

\[
LC \frac{\partial^2}{\partial t^2} \Delta_g u(t, x, y) - u(t, x, y) = 0
\]

results in the group velocity \(v_g\) having the opposite sign compared to the phase velocity \(v_p\). In the Euclidean case, for example, the dispersion is

\[
\omega(k) = \frac{1}{\sqrt{LC}} \frac{1}{k},
\]

which implies

\[
v_g = \frac{d\omega}{dk} = -\frac{1}{\sqrt{LC}} \frac{1}{k^2} = -v_p.
\]

This explains the observation in Supplementary Movie S1 that the hyperbolic circles of constant phase seem to fall into the input node (with \(v_p\)), while the excited voltage pulse is propagating away from it (with \(v_g\)).

**SUPPLEMENTARY MOVIE S1**

**Measured signal propagation in the electric circuit.** The application of a short and spatially localized pulse applied to node 31 (blue curve in the left panel) leads to a wave propagating through the circuit. The voltage response at node 31 is shown as an orange curve in the left panel and the instantaneous phase at each node in the right panel. The nodes are indicated by black dots, and concentric hyperbolic circles with center at node 31 are shown in black to illustrate the hyperbolic metric.