CLASSIFICATION OF EXCEPTIONAL COMPLEMENTS: ELLIPTIC CURVE CASE

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Abstract. We classify the log del Pezzo surface \((S, B)\) of rank 1 with no 1-, 2-, 3-, 4-, or 6-complements with the additional condition that \(B\) has one irreducible component \(C\) which is an elliptic curve and \(C\) has the coefficient \(b\) in \(B\) with \(\frac{1}{n}\lfloor(n+1)b\rfloor = 1\) for \(n=1, 2, 3, 4,\) and 6.

1. Introduction

This paper is a part of the project to classify “log del Pezzo surfaces with no regular complements”, that is, the pairs \((S, B)\) of surface \(S\) and boundary \(B\) on \(S\) such that:

(EX1) \((- (K + B)\) is nef ((\(S, B\)) is “quasi log del Pezzo”),

(EX2) \((- (K + B)\) has no regular complements i.e. it has no \(n\)-complements for any of \(n \in \{1, 2, 3, 4, 6\}\).

We assume throughout that coefficients of \(B\) are “standard”, i.e.

\[B = \sum b_i C_i\]

where \(m\) natural number, or \(b_i \geq \frac{6}{7}\). An invariant \(\delta\) for such a pair is defined in [Sh2, 5] by

\[\delta(S, B) = \#\{E | E\text{ is an exceptional or non-exceptional divisor with log discrepancy } a(E) \leq \frac{1}{7} \text{ for } K + B\}\]

and it was proved there that \(\delta \leq 2\) ([Sh2, Th.5.1]). We can assume, after crepant blow ups of exceptional \(E\)’s with \(a(E) \leq \frac{1}{7}\), that those \(E\) are all non-exceptional, and thus,

(EX3) \((S, B)\) is \(\frac{1}{7}\)-log terminal.

Now define the divisor \(D\) by \(D = \sum d_i C_i\) where \(d_i = 1\) if \(b_i \geq \frac{6}{7}\) and \(d_i = b_i\) otherwise. And write \(C = \lfloor D \rfloor = \sum_{a(C_i) \leq \frac{1}{7}} C_i\). We know by [Sh2, Lemma 4.2] that if \(\delta \geq 1\) then we can successively contract curves semi-negative with respect to \(K + B\), but not components of \(C\), and thereby assume

(EX4) \(\rho(S) = 1\).

The conditions (EX1),(EX2) and (EX3), as well as the condition on the coefficients, are preserved under this reduction. We form a minimal resolution \(f : (S^{\text{min}}, B^{\text{min}}) \to (S, B)\) where \(B^{\text{min}}\) is a crepant pullback, i.e. \(K_{S^{\text{min}}} + B^{\text{min}} = f^*(K + B) = K + f^{-1}(B) + \Sigma e_j E_j\) satisfies \(K + B^{\text{min}} \cdot E_j = 0\) for all \(j\). From \(S^{\text{min}}\) we contract \((-1)\)-curves successively to get a smooth model \(S'\) which is either \(\mathbb{P}^2\) or \(\mathbb{P}_m\):

\[g : (S^{\text{min}}, B^{\text{min}}) \to (S', B').\]

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If $\delta \geq 1$ we have $p_a(C) \leq 1$, and the same is true for the birational image of $C$ on $S'$ as well ([Sh2, Prop.5.4]). In this paper we consider the case

- $\delta = 1$.

Thus, $C$ is an irreducible curve of arithmetic genus $\leq 1$. We write $C = C_1$, and $B = bC + \sum_{i=2}^{r} b_i C_i = bC + B_1$, with $b_i \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$ and $b \geq \frac{6}{7}$.

The dual graph of a configuration

In the following we use the language of graphs to talk about the configuration of curves.

The dual graph of a configuration of curves is a (weighted-multi) graph where we have a vertex for each curve and an $n$-ple edge for each intersection point with multiplicity $n$ between two curves. Each vertex has a weight $\in \mathbb{Z}$ which is the self-intersection number of the curve.

Graphically, we use • (“b(lack)-vertex”) to represent exceptional curves with self-intersection number $\leq -2$, ◦ (“w(hite)-vertex”) for $(-1)$-curves, and squares for curves with non-negative self-intersection. The weight of a vertex is shown by a number next to each vertex, and multiplicity of an edge by the number of lines joining the two end vertices.

“Blow up of an edge” means the transformation of the graph reflecting the blow up of the corresponding point, that is, introduce a new white vertex, decrease the multiplicity of edge by 1, decrease the weight of the both end vertices of the edge by 1, and join them to the new white vertex by a simple edge. “Blow up of a vertex” reflects the blow up of a point on the curve outside the intersection with neighboring curves: introduce a white vertex, decrease the weight of the vertex by 1, and join it to the new white vertex by a simple edge. Blow up of a complete subgraph of any cardinality $k$ can be defined in the same way.

Types of Singularities on $C$

\textbf{Lemma 1.1.} (i) The singularity of $C$ is at worst a node, and it is outside $\text{Sing}(S) \cup \text{Supp}(B_1)$.

(ii) At most one component $C_i$ of $B_1$ passes through each point $P \in C$. If $P$ in a smooth point of $S$, then the intersection is normal, with one possible exception where $C_i$ has coefficient $\frac{1}{2}$ and has a simple tangency with $C$ at a smooth point $P$ of $S$.

(iii) Singularity $P$ of $S$ on $C$ is a cyclic quotient singularity, i.e. log terminal singularities with resolution graph $A_n$ (a chain), where $C$ meets one end curve of the chain normally. If another component $C_i$ passes through $P$, then it meets the other end curve normally.

\textbf{Proof.} Note that $K + D$, as defined above, is log canonical by the existence of local complements ([Sh1, Cor.5.9.]). Then all the statements follow from the classification of surface log canonical singularities ([Ka], or [Al2]) and $\frac{1}{2}$-log terminal condition. For example, for (iii), if we had a type $D_n$ singularity, (case (6) in [Ka, Th.9.6]) we would have a log discrepancy $\leq \frac{1}{2}$. Note also that the exception in (ii) is the only case where $K + D$ is not log terminal at $P$ ([Sh2, Prop.5.2]).}
As is well known, the singularities mentioned above are isomorphic, analytically, to the origin 0 in the quotient of $\mathbb{C}^2$ by the action of cyclic group $\mu_m$ of order $m$, where the generator $\varepsilon = e^{2\pi i/m}$ acts by $(z_1, z_2) \mapsto (\varepsilon^{-k} \cdot z_1, \varepsilon \cdot z_2)$, where $1 \leq k \leq m$ and $gcd(m, k) = 1$.

The minimal resolution of such a singularity has a chain of rational curves $E_1, E_2, \cdots, E_r$ as its exceptional locus, and the cocontinued fraction expansion

$$\frac{m}{m-k} = w_1 - \frac{1}{w_2 - \frac{1}{w_3 - \cdots}}$$

give their self intersection numbers (cf. for example, [Ful]). We call such a singularity $P$ type $[m, k]$. We extend this correspondence to incorporate the information on the component $C_i$ that passes through $P$ (cf. [Sh1, Cor. 3.10], [Sh2, Lemma 2.22]).

Namely, if the component $C_i$ has the standard coefficient $d-1$ and the singularity $P$ has type $(m', k')$, we represent it by the pair $(m, k) = (dm', dk')$. The “dual graph” of the minimal resolution of this singularity is:

![Diagram of the minimal resolution of a singularity with components labeled accordingly.]

**Figure 1.**

Generalizing the notation of [KM], we may denote the same singularity by $(w_1, w_2, \cdots, w_r)_d$ with the underline indicating the curve meeting $C$.

This singularity has the minimal log discrepancy

$$\text{mld}(P, K + B) = a(E_1) = \frac{1+(m-k)(1-b)}{m} \leq \frac{1}{1 + \frac{1}{m}(m-k)},$$

where $m = d \cdot (\text{index of } P)$. Also we denote the co-discrepancy, or the coefficient, of $P$ by $c(P, K + B) = 1 - \text{mld}(P, K + B)$.

Now the $\frac{1}{7}$-log terminal condition

$$\frac{1 + \frac{1}{7}(m-k)}{m} > \frac{1}{7}$$

is equivalent to $k < 7$. Therefore the possible singularities on $C$ are put into $21 = 6(6+1)/2$ (infinite) series according to the pair $(m(\text{mod } k), k)$ with $1 \leq k \leq 6$. This will be convenient later on.

### 2. Elliptic Curve Case

Now we start the classification of the case $p_a(C) = 1$. Thus, $C \in S$ is a smooth curve of genus 1 or a rational curve with one node. We call it the “elliptic curve case”.

**Lemma 2.1.** In the “elliptic curve case”, the condition (EX2) is equivalent to the condition that $(S, B)$ has log-singularities on $C$. That is, either $S$ has singularities on $C$, or $B$ has components other than $C$ (which intersect $C$ since $\rho(S) = 1$).
Proof. If $(S, B)$ is smooth on $C$, then $(K + f^*(D)).C = (K + C).C = 0$ on $S^{\text{min}}$, so $K + D = K + C \sim 0$ on $S$ and (EX2) is not satisfied. In fact $K + B = K + bC$ has a 1-complement. On the other hand if $(S, B)$ has a singularity on $C$, then $(K + f^*(D)).C > (K + C).C = 0$ so we have $K + D > 0$ on $S$, which implies (EX2).

The case when $S$ is a cone ($\mathbb{P}^2$ or $\mathbb{Q}_m$) has been classified elsewhere and from it we have only one case with $C=$elliptic: $S = \mathbb{F}_2$, $\mathbb{C} = $double section, $B_1 = \frac{1}{2}C_2$ where $C_2$ is a generator of the cone. Then $C \equiv 2H \equiv -K, C_2 \equiv \frac{1}{2}H$. So $K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0$ and $K + B$ has 7-complement $= 0$. It also has the trivial 8-complement: $K + \frac{7}{8}C + \frac{1}{2}C_2 \equiv 0$ (This is the entry #1 in the table at the end).

From now on we assume $S$ is not a cone.

**Lemma 2.2.** $C^2 \geq 3$ on $S^{\text{min}}$. If $(S, B)$ has two singularities on $C$, then $C^2 \geq 6$. On the other hand, the minimum log discrepancy of the singularity $P$ on $C$ with respect to $K + B$ (hence also with respect to $K + bC$) is at least $1 - (C^2/7)$.

**Proof.** Because $- (K + B)$ is nef,

\[
0 \geq (K + B).C = K^{\text{min}} + bC + \sum b_iC_i + \Sigma d_jE_j \cdot C \\
\geq -(1 - b)C^2 + \sum b_i + \Sigma P(1 - \text{mld}(P)) \\
\geq -(1 - b)C^2 + \text{min}\{b_i, 1 - \text{mld}(P)\} \\
\geq -\frac{1}{7}C^2 + \frac{6}{7}.
\]

Note that, because of Lemma 1.1, $1 - \text{mld}(P) = d_j$ for the exceptional curve $E_j$ meeting $C$. The last inequality holds because we have at least one nonzero $b_i$ or $1 - \text{mld}(P)$ by Lemma 2.1 and the minimum nonzero value for $b_i$ is $\frac{1}{2}$, that for $1 - \text{mld}(P)$ is $\frac{1}{2} \cdot \frac{6}{7} = \frac{3}{7}$, the latter being attained when $P$ in duVal of type $A_4$. Therefore, $C^2 \geq 3$. By the same calculation, if there are two singularities on $C$ we have $0 \geq -\frac{1}{7}C^2 + \frac{6}{7}$. On the other hand, the second inequality in particular implies that $1 - \text{mld}(P) \leq (1 - b)C^2 \leq \frac{4}{7}C^2$, whence the second assertion.

**Reduction to $\mathbb{F}_2$**

We need the following

**Lemma 2.3.** Let $E$ be a $(-1)$-curve on $S^{\text{min}}$. Then on its image $f_* (E)$, $S$ has either at least two singularities, or one singularity that is not log-terminal for $K + E$.

**Proof.** If $E$, on $E$, $S$ had at most one singularity $P$ that is log-terminal for $K + E$, i.e. a cyclic quotient singularity such that $E$ meets one end curve $E_1$ of the chain of the resolution, then we would have

\[
(f_* (E))^2 = E.f^*f_* (E) = E.(E + (1 - a(E_1))E_1) = -1 + (1 - a(E_1)) < 0
\]

which is absurd since $\rho(S) = 1$.

Now we can prove the

**Lemma 2.4.** We can always obtain $\mathbb{F}_2$ as a smooth model of $S$ (and $C$ as a double section).
Proof. $p_a(C) = 1$ means that after reconstruction, $C$ is either a cubic in $\mathbb{P}^2$, curve of bidegree $(2, 2)$ on $\mathbb{P}_0$, or a double section of $\mathbb{F}_2$. Suppose $S'$ is $\mathbb{P}^2$ and $C$ is a cubic, since there are no irreducible curves with arithmetic genus 1 on $\mathbb{F}_m$, with $m \geq 3$. If $g : S' \to \mathbb{P}^2$ contracts two or more exceptional curves to a point $P \in \mathbb{P}^2$, then we can choose different contractions to get $S' = \mathbb{F}_2$. Therefore we may assume that we have only one exceptional curve for $g$ over each center $P \in \mathbb{P}^2$, and we shall derive a contradiction.

Since all the curves contracted by $g$ are $(-1)$-curves on $S'\min$, no exceptional curve $E_i$ for $f$ are contracted and all of them are present on $\mathbb{P}^2$ as divisors. Thus we have an inequality

$$0 \geq \text{deg}(K_{2} + bC + B_1') = -3 + \frac{6}{7} \cdot 3 + \text{deg}(B_1') = -\frac{3}{7} + \text{deg}(B_1')$$

Therefore, since the coefficients are standard, no component $C_i$ other than $C$ are present on $\mathbb{P}^2$. And we have

$$(*) \quad \Sigma d_j \leq \Sigma d_j \cdot \text{deg}(E_j) = \text{deg}(B_1') \leq \frac{3}{7}$$

We have the two possibilities:

1. $B$ has at least one component, say $C_2$, other than $C$. Then by the above, $C_2$ must be contracted on $\mathbb{P}^2$ and is a $(-1)$-curve on $S'\min$. Therefore, by Lemma 2.3, $S$ must have either at least two singularities on $C_2$, or a singularity that is not log-terminal for $K + C_2$. In the former case, then, we would have $\Sigma d_j \geq (\frac{1}{2} + \frac{1}{2})b_2 \geq \frac{7}{2} > \frac{3}{7}$, contradicting $(*)$. In the latter case, we have an exceptional curve $E$ with $a(E, K + C_2) \leq 0$. Then because $a(E, K + b_2C_2)$ is a linear function of $b_2$ and we also have $a(E, K + 0 \cdot C_2) = a(E, K) \leq 1$, we have $a(E, K + b_2C) \leq 1 - b_2$. Thus $d_2 = 1 - a(E, K + B) \geq 1 - a(E, K + b_2C) \geq b_2 \geq \frac{1}{2} > \frac{3}{7}$, again a contradiction to $(*)$.

2. $B$ has no other components than $C$, i.e. $B = bC$, and $S$ has a singularity on $C$. Then $(*)$ implies that and we have $K + B > 0$ except in the following case: $S$ has only one duVal singularity $P$ of type $A_1$ on $C$, the exceptional curve $E_1$ of the resolution of $P$ is a line on $\mathbb{P}^2$, $b = \frac{6}{7}$, and $B_1'$ has no other component than $E_1$, so that $K + B = K + \frac{6}{7}C + \frac{3}{7}E_1 \sim 0$. In particular all the singularities on $S$ are duVal so

$$f^*(K + B) = K_{S'min} + B_{min} = K + \frac{6}{7}C + \frac{3}{7}E_1$$

Also, the triviality of $K + B$ means that pull back $g^*$ is crepant so that the above is also equal to $g^*(K + B')$. On the other hand, since $E_1.C = 3$ on $\mathbb{P}^2$ and $E_1.C = 1$ on $S'\min$, two of the intersection points of $E_1$ and $C$ has to be blown up on $S'\min$. The exceptional curve $E$ for the first of such blowups would have the coefficient $\frac{6}{7} + \frac{3}{7} - 1 = \frac{3}{7}$ in $K_{S'min} + B_{min} = g^*(K + B')$. Contradicting the explicit form of $B_{min}$ given above.

If we have a model $S' = \mathbb{F}_0$, then we have had at least one contraction of $(-1)$-curve so we can get $S' = \mathbb{P}^2$ by choosing other contractions, and we are reduced to the previous case.

Therefore we have a $\mathbb{P}^1$-fibration $p : S'\min \to \mathbb{F}_2 \to \mathbb{P}^1$. Now our strategy for the classification is to start from $\mathbb{F}_2 = S'$, make blow ups to construct $S'\min$, choose $B_{min}$ on it so that resulting $(S, B)$ would have singularities on $C$ ($\Leftrightarrow$ (EX2) by Lemma 2.1) and would satisfy (EX1),(EX3), and (EX4).

The conditions (EX1) and (EX4) implies that the number of $(-n)$-curves, $n \geq 2$, on $S'\min$ must equal $\rho(S'\min) - 1$, and they are all exceptional for the resolution $f$. These curves are
either in the fibres of \( p \), or they are not, i.e. they are (multi-) sections of \( p \). As for the number of curves of each type, we have the following:

**Lemma 2.5. ([Zhang, Lemma 1.5 ])** We have

\[
  r = \# \{ \text{Exceptional curves } E_i \text{'s of the resolution } f \text{ that are not in the fibres of } p \} - 1 \\
  = \# \{ (-1)\text{-curves on } S_{\text{min}} \text{ that are in the fibres of } p. \} \\
  - \# \{ \text{Singular fibres of } p \}
\]

**Proof.** Add \((2 + \# \{ E_i \text{'s that are in the fibres of } p \})\) to both sides, and we get two expressions for \( \rho(S_{\text{min}}) \). \( \square \)

**The search for exceptions**

**Case 1.** \( r = 0 \), i.e. minimal section \( \Sigma \) on \( \mathbb{F}_2 \) is the only \( E_i \) with \( p(E_i) = \mathbb{P}^1 \).

Then there is only one \((-1)\text{-curve in each singular fibre of } p \). Therefore on each fibre \( F \) modified we have to have initially two blow ups at the same point \( P \). Suppose \( C^2 = w \) before the modification, then according as the intersection multiplicity \( i = I(P; F \cap C) = 2, 1, \text{ or } 0, \) i.e. according as \( P = \text{tangency of } F \text{ and } C, \text{ normal intersection of } F \text{ and } C, \) or \( P \in F \setminus (F \cap C), \) we get one of the three dual graphs in the Figure 2 below.

![Figure 2](attachment:image.png)

**Figure 2.**

In the figure the b-vertex at the bottom is the minimal section \( \Sigma \in \mathbb{F}_2 \). In the case (III), the curve \( C \) and neighboring \((-2)\text{-curve (} = F \text{)) have either two normal intersections, one simple tangency, or \( C \) has a node on \( F \).

Case(III) gives a non log canonical point (cf. Lemma 1.1(i)) and is excluded. Case(II) gives one example with trivial complement (entry \#2 in the table at the end):

\[
  S = \text{Gorenstein del Pezzo surface with singularities } A_1 + A_2, \\
  C = \text{elliptic curve through } A_1 \text{ and } A_2 \text{ points,} \\
  K + B = K + \frac{6}{7} C \equiv 0 \\
  7(K_{S_{\text{min}}} + B_{\text{min}}) = 7(K + \frac{6}{7} C + \frac{3}{7} E_1 + \frac{4}{7} E_2 + \frac{2}{7} E_2) \sim 0
\]

(Following [MZ], we denote the Gorenstein del Pezzo surfaces of rank 1 by its singularity type, for example, \( S(A_1 + A_2) \) for the surface above, and their resolution by e.g. \( \tilde{S}(A_1 + A_2). \))
Since we already have $K + B \equiv 0$, if we make any more blow ups (which have to be on the unique (-1)-curve) or add other components to $B$, we would have $K + B > 0$ and $(S, B)$ will violate (EX1). So we need not consider this case any longer. Thus we are left with case (I), i.e. two initial blow ups at the ramification point of $C \to \mathbb{P}^1$ (tangency of $C$ and a fibre).

In particular, in all the remaining cases, $C^2 \leq 6$, because $C^2 = 8$ on $\mathbb{P}_2$.

This implies that a smooth fibre $F$ cannot be a component of $B'_1$, because if it were, we would have $0 \geq (K + bC + B_1).C \geq -(1 - b)C^2 + \frac{1}{2} FC \geq -\frac{1}{b} + \frac{1}{2} \times 2 = \frac{1}{2}$, a contradiction. Therefore only singularities on $C$ are those coming from the intersection of $C$ and the singular fibres.

After (I), we can only blow up a point on the unique (-1)-curve on each fibre: otherwise we would introduce more than one (-1)-curves in a fibre, violating $r = 0$. There are two types of such blow ups. One is the blow ups of the intersection of $C$ and the (-1)-curve, (blow up of the edge between the white vertex and $C$) which decrease $C^2$. The other is the blow ups of a point of (-1)-curve outside $C$.

We start from the first type of blow ups and get the resolutions of Gorenstein log del Pezzos of rank 1 with $K^2 = C^2 \geq 3$ (Lemma 2.2):

\[
\begin{align*}
\hat{S}(A_1 + A_2) & \to \hat{S}(A_4) \to \hat{S}(D_5) \to \hat{S}(E_6) \\
\downarrow & \quad \downarrow \quad \downarrow \\
\hat{S}(2A_1 + A_3) & \to \hat{S}(A_1 + A_5).
\end{align*}
\]

Each “$\to$” represents one blow up, and each “$\downarrow$” two blow ups on a new fibre.

Then, starting from one of these, we make the second type of blow ups, which decrease the minimal log discrepancy of $S$, until either (EX1) or (EX3) is violated (see below). The Gorenstein rank 1 surfaces listed above are the image of $S^{\min}$ under the morphism $\phi_{|C|}$ defined by the linear system $|C|$ on it. We denote it by $S_C$, and its resolution (one of the above) by $\hat{S}_C$.

Note that $C$ meets every (-1)-curve $E$ on $S_C$ since $C \sim -K_{S_C}$ and $-K.E = 1$. Consider blow ups on one fibre starting at one such $E$. By Lemma 2.3, on $E$, $S$ has either at least two singularity or one singularity that is not log-terminal for $K + E$. That is, on $S_C$, either $E$ meets at least two trees $T_1, T_2$ of b-vertices, or one tree $T_3$ that gives non-log-terminal point for $K + E$.

Now consider the transformation of the subgraph consisting or $C$, $E$, and trees of b-vertices $T_i$ meeting $E$ on $S_C$. It should always contain a unique w-vertex.

If we blow up the vertex $E$, i.e. blow up a point on $E$ other than the intersection points with neighboring exceptional curves, then after the transformation $C$ would meet the b-vertex $E$ in the black graph $T_1 - E - T_2$ or $E - T_3$. Either of these would contracts to a non-log-terminal point on $S$ for $K + C$, contradicting Lemma 1.1. (For an example of the first situation, consider blow up of the white vertex in the configuration (I) in the Figure 2 above. For the second, consider the same in the configuration of the table 9.) Therefore the first blow up has to be at the intersection point of $E$ and one of the neighboring b-vertices, i.e. blow up of the edge joining $E$ and one of its neighbors.

The same argument, repeated for the new white vertex $E_1$ at each stage, shows that successive blow ups also must be at the edge joining $E_1$ and a neighboring b-vertex, because the trees now meeting $E_1$ are even bigger than those that met $E$. Thus, by induction, we see that the full inverse image of $E$ is of the form $E - T - E_1 - T'$, where $T$ and $T'$ are
chains of b-vertices \( T \), or \( T' \) may be a part of a larger tree. And \( E \) may meet another tree \( T'' \) in which case \( T \) should be empty — Remember that \( C \) meets \( E \), and \( E_1 \) is a w-vertex. The blow up described above either increases the weight of an end vertex of \( T \) next to \( E_1 \), or adds one \((-2\)-curve \( E_1 \) to it, depending on which side of \( E_1 \) we blow up. Either of such transformations (those which preserve log-terminal property), if repeated infinitely many times, make the log-discrepancy with respect to \( K + bC \) of the resulting singularity on \( C \) monotonically decrease toward \( 1 - b \leq \frac{4}{7} \). Hence by Lemma 2.2, after finite number of steps, (EX3) will be violated (or perhaps, (EX1) may be violated first). Therefore this procedure of successive blowups must terminate.

We can now refine the lemma 2.2 as follows:

If \( C^2 < 6 \) then we have only one singularity by Lemma 2.2. But on the other hand, if \( C^2 \geq 5 \) we can have only one singular fibre, which means that in every case we have only one singularity of \((S, B)\) on \( C \). (EX1) restricts the possible types of singularities \([m, k]\) on \( C \) as follows:

\[
0 \geq K + bC + B' \cdot C = (1 - b)C^2 + \frac{(k-1) + b(m-k)}{m} + \frac{6}{7} \cdot \frac{2 + d}{m},
\]

or

\[
(6 - C^2) m \leq 7 - k.
\]

In this way, we find that there are 20 possible \( S \)'s, with a few different \( B \)'s for some of the \( S \)'s. These are summarized in the table below.

**Case 2.** \( r = 1 \), i.e. we have one exceptional curve, say \( E \), other than \( \Sigma \) that is a section of \( \mathbb{P}^1 \)-fibration \( p \).

Thus, exactly one fibre contains two \((-1)\)-curves in it. If we modify at any other fibre it has to start like (I) of the Figure 2 (two blow ups at the tangency with the fibre) because (II) and (III) have been eliminated. In particular each time we blow up on a new fibre we decrease \( C^2 \) by at least 2.

**Claim:** Any exceptional curve \( E \) that is a (multi-)section of \( p \) is in fact a 1-section that is disjoint from \( \Sigma \).

**Proof.** Let \( E \) be a (multi-)section, and \( d = \text{mult}_E(B'_1) \). Then if \( F \) is a fibre of \( p \), we have

\[
0 \geq (K + B').F \geq (K + bC + dE).F \leq -2 + \frac{6}{7} \cdot 2 + d.
\]

Hence \( d \leq \frac{2}{7} < \frac{3}{7} \). So \( E \) cannot intersect \( C \) on \( S_{\text{min}} \). Therefore all the intersection point of \( C \) and \( E \) have to be blown up on \( S_{\text{min}} \). If \( E \) is not a 1-section disjoint from \( \Sigma \), then we have \( C.E \geq 6 \) on \( \mathbb{F}_2 \). So we would have \( C^2 \leq 8 - 6 = 2 \) on \( S_{\text{min}} \) which is impossible according to the lemma 2.2. This proves the claim.

So let \( E \) be a simple section disjoint from \( \Sigma \). Then \( E.C = 4 \).

Suppose \( E \) intersects \( C \) at one point with multiplicity 4. Then after four blowups at this point we get \( \tilde{S}(A_1 + A_3) \) (the configuration of the table \#17, with a different choice of fibration), which has already been studied in the case 1 above.

If \( E \) intersects \( C \) at two points with multiplicity 3 and 1 respectively, then by the above observation we have at least \( 3 + 2 = 5 \) blow ups on \( C \), which gives \( \tilde{S}(3A_2) \), with \( C \) passing
through three \((-1)\)-curves joining three \(A_2\) points. Since \(C^2 = 3\) by Lemma 2.2 \(C\) can have at worst \(\mathbb{A}_1\) (= “type [2,1]”) point on it, but that cannot be attained: We could at best choose \(B_1 = \frac{1}{2}C_2\) where \(C_2 = \text{image of one of the } (-1)\text{-curves meeting } C\) and thus get type [2,2] point on \(C\), which is worse than \(\mathbb{A}_1 = \text{type [2,1]}\).

If \(E\) intersects \(C\) at more than 3 points, then we have at least six blow ups on \(C\) thus \(C^2 \leq 2\), which is impossible by Lemma 2.2.

Finally, if \(E\) intersects \(C\) at two points with multiplicity 2 each, we would have two \((-1)\)-curves in each fibre, and this violates (EX4).

Thus, we get no new examples from case 2.

**Case 3.** \(r \geq 2\), i.e. we have at least two exceptional curves, say, \(E_1\) and \(E_2\), other than \(\Sigma\), that are sections of \(p\).

\(E_i\) are simple sections. Then because \(C.E_i = 4\) and \(E_1.E_2 = 2\), we must have at least \(4 + 4 - 2 = 6\) blow ups on \(C\) in order to separate \(E_i\)’s from \(C\). Then \(C^2 \leq 2\) and by Lemma 2.2, this is impossible.

It turns out that in every case \(K + B\) has a 7-complement. Moreover, we can choose \(g\) so that in every case \(B_1'\) has only one component which is a fibre of \(\mathbb{F}_2\).

**Table**

Thus we get the following table. Here,

- The first column shows the configuration on \(S_{\text{min}}\) of the exceptional curve \(E_i\)’s, \((-1)\)curves, and the components of \(B\). ‘c’ denote \((-1)\)-curve, ‘•’ are the \(E_i\)’s with self intersection number \((\leq -2)\) attached, with ‘←’ indicating (one possible) \(\Sigma \subset \mathbb{F}_2\) after a suitable sequence of contractions of \((-1)\)-curves. Squares are curves with non negative self intersection.
- The second column gives the fractional part \(B_1\) of the boundary \(B\), or rather, of \(D\).
- The third column gives the number \((\frac{6}{7} \leq \max\{b|K + bC + B_1 \leq 0\}) \ (< 1)\)
- The fourth column gives an example of \(n\)-complements.
- The last column lists numerical relations between some relevant divisors on \(S\), with \(H\) being the generator of \(\text{Pic}(S)\).

Note that we can compute intersection numbers on \(S_{\text{min}}\) using the crepant pullbacks, and a divisor on \(S\) is Cartier iff its crepant pullback is Cartier, i.e. iff it is integral (cf.[Sakai]).

The table is organized according to \(S_C\), the image of \(S_{\text{min}}\) under the morphism defined by the linear system \(|C|\).

(1) \(S = S(A - 1) = \mathbb{Q}_2\) (= quadratic cone \(\subset \mathbb{P}^3\), \(S_C = \) its Veronese image)

| configuration | \(B_1\) | \(\max(b)\) | complements | divisors \((\text{Pic}(S) = \mathbb{Z}[H])\) |
|---------------|---------|-------------|-------------|----------------------------------|
| 1             | \(\frac{1}{2}C_2\), \(0\) \(\square\) \(\square\) \(8\) \(\bullet\) \(-2\) \(\bullet\) \(-2\) \(-2\)| \(\frac{7}{8}\) | - 7-compl. = 0 \((K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)\) \| trivial 8-compl. \| \(-K \equiv C \equiv 2H\) \| \(C_2 \equiv \frac{1}{2}H\) |

(2) \(S_C = S_7\) (= a del Pezzo with degree 7)
| 2 | 0 | 6 | trivial 7-compl. $(K + \frac{6}{7}C \equiv 0)$ | $-K \equiv H$  
$C \equiv \frac{1}{5}H$ |
|---|---|---|----------------------------------------|-----------------|
(3) $S_C = S(A_1 + A_2)$

| configuration | $B_1$     | max($b$) | complements                              | divisors $(\text{Pic}(S) = \mathbb{Z}[H])$ |
|---------------|-----------|----------|------------------------------------------|------------------------------------------|
| $3$           | $\frac{1}{2} C_2$ | $\frac{9}{10}$ | trivial 10-compl. $(K + \frac{9}{10} C + \frac{1}{2} C_2 \equiv 0)$ | $-K \equiv C \equiv H$ $C_2 \equiv \frac{1}{6} H$ |
|               | $\frac{2}{3} C_2$ | $\frac{8}{9}$ | trivial 9-compl. $(K + \frac{8}{9} C + \frac{2}{3} C_2 \equiv 0)$ | $\cdot$                                  |
|               | $\frac{3}{4} C_2$ | $\frac{7}{8}$ | trivial 8-compl. $(K + \frac{7}{8} C + \frac{3}{4} C_2 \equiv 0)$ | $\cdot$                                  |
|               | $\frac{4}{5} C_2$ | $\frac{13}{15}$ | 7-compl.$=0$ $(K + \frac{6}{7} C + \frac{4}{5} C_2 \equiv 0)$ | $\cdot$                                  |
|               | $\frac{5}{6} C_2$ | $\frac{31}{36}$ | 7-compl.$=0$ $(K + \frac{6}{7} C + \frac{5}{6} C_2 \equiv 0)$ | $\cdot$                                  |
| $4$           | 0         | $\frac{8}{9}$ | $\cdot$ 7-compl.$=C_2$ $(K + \frac{6}{7} C + \frac{3}{5} C_2 \equiv 0)$ | $-K \equiv \frac{8}{12} H$ $C \equiv \frac{9}{12} H$ $C_2 \equiv \frac{1}{12} H$ |
|               | $\cdot$ 7-compl.$=C_2$ $(K + \frac{6}{7} C + \frac{3}{5} C_2 \equiv 0)$ | $\cdot$ trivial 9-compl. | $\cdot$                                  |
| $5$           | 0         | $\frac{9}{10}$ | $\cdot$ 7-compl.$=3C_2$ $(K + \frac{6}{7} C + \frac{3}{5} C_2 \equiv 0)$ | $-K \equiv \frac{2}{15} H$ $C \equiv \frac{10}{15} H$ $C_2 \equiv \frac{1}{15} H$ |
|               | $\cdot$ 7-compl.$=3C_2$ $(K + \frac{6}{7} C + \frac{3}{5} C_2 \equiv 0)$ | $\cdot$ trivial 10-compl. | $\cdot$                                  |
| $6$           | 0         | $\frac{7}{8}$ | $\cdot$ 7-compl.$=2C_2$ $(K + \frac{6}{7} C + \frac{2}{5} C_2 \equiv 0)$ | $-K \equiv \frac{14}{40} H$ $C \equiv \frac{16}{40} H$ $C_2 \equiv \frac{1}{40} H$ |
|               | $\cdot$ 7-compl.$=2C_2$ $(K + \frac{6}{7} C + \frac{2}{5} C_2 \equiv 0)$ | $\cdot$ trivial 8-compl. | $\cdot$                                  |
| configuration | $B_1$ | $\max(b)$ | complements | divisors $(\Pic(S) = \mathbb{Z}[H])$ |
|---------------|-------|-----------|-------------|-------------------------------------|
| 7             | 0     | $\frac{13}{15}$ | $7$-compl.$=C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ | $-K \equiv \frac{13}{15}H$  
|               |       |           |             | $C \equiv \frac{15}{35}H$  
|               |       |           |             | $C_2 \equiv \frac{1}{35}H$  |
| 8             | 0     | $\frac{19}{22}$ | $7$-compl.$=C_2$ $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$ | $-K \equiv \frac{19}{22}H$  
|               |       |           |             | $C \equiv \frac{22}{35}H$  
|               |       |           |             | $C_2 \equiv \frac{1}{77}H$  |

(4) $S_C = S(A_4)$

| configuration | $B_1$ | $\max(b)$ | complements | divisors $(\Pic(S) = \mathbb{Z}[H])$ |
|---------------|-------|-----------|-------------|-------------------------------------|
| 9             | $\frac{1}{2}C_2$ | $\frac{9}{10}$ | $7$-compl.$=C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ | $-K \equiv C \equiv H$  
|               |       |           |              | $C_2 \equiv \frac{1}{5}H$  |
|               | $\frac{2}{3}C_2$ | $\frac{13}{15}$ | $7$-compl.$=0$ $(K + \frac{6}{7}C + \frac{5}{7}C_2 \equiv 0)$ |                                |
| 10            | 0     | $\frac{10}{11}$ | $7$-compl.$=4C_2$ $(K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0)$ | $-K \equiv \frac{10}{11}H$  
|               |       |           |              | $C \equiv \frac{11}{22}H$  
|               |       |           |              | $C_2 \equiv \frac{1}{22}H$  |
|               | $\frac{1}{2}C_2$ | $\frac{19}{22}$ | $7$-compl.$=0$ $(K + \frac{6}{7}C + \frac{3}{7}C_2 \equiv 0)$ |                                |
| 11            | 0     | $\frac{15}{17}$ | $7$-compl.$=3C_2$ $(K + \frac{6}{7}C + \frac{3}{7}C_2 \equiv 0)$ | $-K \equiv \frac{15}{17}H$  
|               |       |           |              | $C \equiv \frac{17}{34}H$  
|               |       |           |              | $C_2 \equiv \frac{1}{17}H$  |
| configuration | $B_1$ | $\max(b)$ | complements | divisors $(\text{Pic}(S) = \mathbb{Z}[H])$ |
|---------------|-------|----------|-------------|------------------------------------------|
| 12            | 0     | $\frac{7}{8}$ | $-\cdot 7$-compl. = $2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ | $-K \equiv \frac{12}{123}H$ |
|               |       |          | $\cdot \text{trivial 8-compl.}$ | $C_1 \equiv \frac{18}{43}H$ |
|               |       |          |               | $C_2 \equiv \frac{4}{12}$H |
| 13            | 0     | $\frac{20}{23}$ | $-\cdot 7$-compl. = $2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ | $-K \equiv \frac{20}{123}H$ |
|               |       |          | $\cdot \text{trivial 8-compl.}$ | $C_1 \equiv \frac{1}{42}H$ |
|               |       |          |               | $C_2 \equiv \frac{4}{123}H$ |
| 14            | 0     | $\frac{25}{29}$ | $-\cdot 7$-compl. = $C_2$ $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$ | $-K \equiv \frac{25}{5-29}H$ |
|               |       |          | $\cdot \text{trivial 9-compl.}$ | $C_1 \equiv \frac{5}{18}H$ |
|               |       |          |               | $C_2 \equiv \frac{15}{5-29}H$ |

(5) $S_C = S(D_5)$

| 15            | $\frac{1}{2}C_2$ | $\frac{7}{8}$ | $-\cdot 7$-compl. = $0$ $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$ | $-K \equiv C \equiv H$ |
|               |                   |               | $\cdot \text{trivial 8-compl.}$ | $C_2 \equiv \frac{1}{4}H$ |
| 16            | 0     | $\frac{8}{9}$ | $-\cdot 7$-compl. = $2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ | $-K \equiv \frac{8}{15}H$ |
|               |                   |               | $\cdot \text{trivial 9-compl.}$ | $C_1 \equiv \frac{1}{9}H$ |
|               |                   |               |               | $C_2 \equiv \frac{15}{18}H$ |

(6) $S_C = S(A_3 + 2A_1)$

| 17            | $\frac{1}{2}C_2$ | $\frac{7}{8}$ | $-\cdot 7$-compl. = $0$ $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$ | $-K \equiv C \equiv H$ |
|               |                   |               | $\cdot \text{trivial 8-compl.}$ | $C_2 \equiv \frac{1}{4}H$ |
| 18            | 0     | $\frac{6}{7}$ | $\text{trivial 7-compl.}$ $(K + \frac{6}{7}C \equiv 0)$ | $-K \equiv \frac{12}{12H}$ |
|               |                   |               |               | $C_1 \equiv \frac{12}{12H}$ |
|               |                   |               |               | $C_2 \equiv \frac{12}{12H}$ |
|               |                   |               |               | $C_3 \equiv \frac{12}{12H}$ |
(7) $S_C = S(E_6)$

| 19 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | C  |
|----|----|----|----|----|----|----|----|----|
| 0  |   |   |   |   |   |   |   | 0  |
| 6  |   |   |   |   |   |   |   | $rac{7}{7}$ |
| trivial 7-compl. | $(K + rac{6}{7}C \equiv 0)$ |
| $-K \equiv \frac{6}{14}H$ |
| $C \equiv \frac{1}{14}H$ |
| $C_2 \equiv \frac{1}{14}H$ |

(8) $S_C = S(A_5 + A_1)$

| 20 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | C  |
|----|----|----|----|----|----|----|----|----|
| 0  |   |   |   |   |   |   |   | 0  |
| 6  |   |   |   |   |   |   |   | $rac{7}{7}$ |
| trivial 7-compl. | $(K + rac{6}{7}C \equiv 0)$ |
| $-K \equiv \frac{6}{14}H$ |
| $C \equiv \frac{1}{14}H$ |
| $C_2 \equiv \frac{1}{14}H$ |
| $C_3 \equiv \frac{1}{14}H$ |

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