The Risk-Sharing Problem Under Limited Liability Constraints in a Single-Period Model

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Abstract
This work provides analysis of a variant of the Risk-Sharing Principal-Agent problem in a single period setting with additional constant lower and upper bounds on the wage paid to the Agent. The effect of the extra constraints on optimal contract existence is studied and leads to conditions on the underlying utility functions under which an optimum may be attained. Solution characterization is then provided along with the derivation of a Borch rule for Limited Liability. Finally, some applications, including the CARA utility case, are discussed.

Keywords Principal-Agent problems · Risk-Sharing · Limited liability · Optimization on Hilbert spaces

Mathematics Subject Classification 91B41 · 91B43 · 49J55

1 Introduction

A central question in economics involves analysis of the mechanics of incentives. This question has many applications, from finding an optimal wage structure that motivates employees to understanding the behavior of subcontractors to whom a company may wish to delegate a task. The Principal-Agent problem plays a key role in the analysis of such a question. Indeed, it allows the study of the effect of Moral Hazard on contracting situations by providing a model for incentive analysis when there exists a difference in interest between two parties. (For example, an employee’s primary motivation may not always be maximizing his employer’s revenue.) Founding works on the topic

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include the papers by Ross [20], Mirrlees [16], Hölmstrom [6] and Sannikov [21]. However, analysis of the Moral Hazard case is often paired with a comparison to the Risk-Sharing benchmark, solution to the less constrained Risk-Sharing problem which models incentive analysis when there is no difference in interest. As well as having less real-life applications than the Moral Hazard problem, this problem is said to be easier to solve and has thus attracted less attention. Some specific literature does nevertheless exist with, for example, the works of Müller on the Risk-Sharing problem in the exponential utility case [17], or the monograph of Cvitanic and Zhang [3].

Many studies of the Principal-Agent problem allow the Principal to pay the Agent an unbounded wage. For example, the optimal wage in the standard Risk-Sharing problem analyzed by Müller in [17] may be negative with positive probability and the Agent may work on behalf of the Principal before also owing him money. A natural extension to the Risk-Sharing involves analyzing the effect of adding exogenous bounds on the wage paid by the Principal. This would prevent situations such negative pay in exchange for positive effort, as well as the implementation of wage bands for example. This is said to introduce limited liability to the problem: limited liability for the Agent when the wage is bounded below and for the Principal when the wage is bounded above.

Limited liability in the Principal-Agent problem has been introduced before and under different forms, mostly in a Moral Hazard setting. In [6] Hölmstrom introduced wage bounds as a by-product of an existence proof for solutions to a Principal-Agent problem. Sappington then analyzed the case of a risk-neutral Principal and a risk-averse Agent, computing the optimal wage for an Agent who chooses his action after the state of nature is realized [22]. The particular and important case of limited liability and debt contracts was studied by Innes in [7] and extended in [15] and [4], with a possibility for negotiation. The risk-neutral Agent case was studied by Park [19] and Kim [11]. More recently in [8], Jewitt et al. provided a proof of existence and uniqueness of an optimal action and wage for a risk-neutral Principal and risk-averse Agent under general limited liability bounds and Moral Hazard. The paper provides a characterization of the optimal wage, which is of option form, but not of the optimal action. In fact an open question brought up by the authors in this paper is the effect of a lower bound on the Agent’s action: this is unclear due to the lack of a closed-form expression for the optimum yet it is a vital economic question. Some answers are provided by Kadan et al. in [10]. Indeed, in the setting of [8], the paper provides sufficient conditions under which an increase in a lower wage bound increases the Agent’s action.

The effect of introducing limited liability into the Moral Hazard problem has therefore been quite thoroughly studied with some key open questions remaining. It seems that an equivalent level of analysis of the Risk-Sharing problem is lacking. Some questions are answered in [11] where Kim considers the case of a risk-neutral Principal and risk-averse Agent with a bounded output process. For any fixed action $a$, the author is able to characterize a class of wages that satisfy a set of relevant constraints. Thus rather than fully solving the limited liability problem, Kim proves that for every fixed action there exists a class of wages that satisfy all of the constraints necessary to be an optimum. Limited liability in a Risk-Sharing setting is also discussed by Ćvitanić and Zhang in [3]. They derive an implicit characterization of the optimal wage for a
Risk-Sharing problem in a continuous setting with a lower bound constraint and for an Agent who has the ability to control volatility. This paper aims to complement these two works. It provides insight into the effect of limited liability in a single period setting without volatility control. Crucially it allows for general utility functions and thus extends beyond the realms of CARA utility and risk-neutral utility functions whose expressions simplify the reasoning required to analyze the optimal wage and action (for example in the CARA case one may find necessary and sufficient optimality conditions through a Lagrangian taking advantage of explicit calculations). As a consequence, in this work the analysis of maximizer existence is split from that of its characterization. The question of existence is tackled first using a calculus of variations approach. Once existence is established, maximizer characterization is approached using Luenberger’s generalized K.K.T. theorem [13]. Note that a full characterization of both the optimal wage and action is obtained, which contrasts with [11] and even [8]. Remark that the setting considered is different from (and complementary to) that of Page’s important work on solution existence [18]. Indeed Risk-Sharing only concerns itself with the Principal’s optimization problem, and this work includes analysis of wages that are unbounded above. Also remark that this work differs from the key existence proofs for Principal-Agent problems provided in [9]. Indeed, they consider the Principal’s problem across the subset of so-called incentive compatible contracts. These contracts verify some optimality properties for the Agent and such optimality helps for topological analysis.

The rest of the document is structured as follows. In Sect. 2, the setting and the limited liability Risk-Sharing Principal-Agent problem are both presented. Section 3 then deals with maximizer existence. Finally in Sect. 4, characterization of the related optima is presented in general settings before some examples are provided.

2 The Model

Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \(L^2(\Omega)\) the set of square integrable random variables. A Principal employs an Agent to perform a task in exchange for some compensation. The two parties are modeled through their utility functions, \(U_P\) and \(U_A\), that verify \(U_P' > 0, U_P'' \leq 0, U_A' > 0\) and \(U_A'' \leq 0\), and are continuous on \(\mathbb{R}\) with \(\lim_{x \to -\infty} U(x) = -\infty\). The Agent’s action is modeled as some real number \(a\) which affects the Principal’s production process, \(X^a := x_0 + a + B\), where \(B\) belongs to \(L^2(\Omega)\). Performing the action costs the Agent some effort modeled through \(\kappa(a) := K \frac{a^2}{2}\) for \(K > 0\) some fixed parameter. As a reward for his effort the Agent is paid a wage denoted \(W\) and belonging to \(L^2(\Omega)\). In such a context, one may analyze the Risk-Sharing Principal-Agent problem or “first-best” problem 1:

---

1 This “one-sided” optimization for the Principal may, with good reason, seem unfair. This problem is in fact a benchmark problem that is often used as a measure of comparison with other Principal-Agent problems such as Moral Hazard where optimization for the Agent also comes into play. One may also note that in our increasingly digitalized economies with an ever growing use of machines, analysis of optimal contracting without Moral Hazard (and thus in a Risk-Sharing setting) is increasingly relevant in itself too.
\[
\sup_{(W,a) \in L^2(\Omega) \times \mathbb{R}} \mathbb{E} \left[ U_P(X^a - W) \right],
\]

such that the Agent’s participation constraint is satisfied:

\[
\mathbb{E} \left[ U_A(W - \kappa(a)) \right] \geq U_A(y), \quad y \in \mathbb{R}^+(\text{fixed}).
\]

(2.2)

The aim of this work is to add limited liability constraints:

\[
m \leq W \quad \mathbb{P} - a.s.,
\]

(2.3)

or:

\[
m \leq W \leq M \quad \mathbb{P} - a.s.,
\]

(2.4)

where \( m \) and \( M \) are two fixed positive parameters with \( m \leq y \leq M \). Such an assumption ensures that the constant contract \((y, 0)\) satisfies both the participation constraint (2.2) and the relevant limited liability constraint. It also has economic meaning as the upper bound for \( y \) assures the Principal that he is not over investing in human capital. In this setting, two variants of the limited liability problem are considered: (2.1)-(2.2)-(2.3) where limited liability for the Agent is enforced, and (2.1)-(2.2)-(2.4) where limited liability for the Principal is also enforced.

Assume from now on the following utility/process compatibility assumption.

**Assumption 2.1** The constant contract \((y, 0)\) provides the Principal with a finite utility:

\[
\mathbb{E} \left[ |U_P(X^0 - y)| \right] < +\infty.
\]

In other words, the random variable \( B \) and the Principal’s utility are compatible.

The goal of this work is to consider general utility functions such as:

- A \( C^2 \) extension of the logarithmic utility:

\[
U(x) = \log(x)1_{x \geq 1} - \frac{1}{2}(x^2 - 4x + 3)1_{x < 1}.
\]

(2.5)

- A partially IARA and \( C^2 \) variant of the CARA utility:

\[
U(x) = -e^{-x}1_{x \geq 0} - \left( \frac{1}{2}x^2 - x + 1 \right)1_{x < 0}, \quad \text{for } y \text{ fixed}.
\]

(2.6)

This utility partially disrupts the constant risk aversion of the CARA utility.\(^2\)

\(^2\) These are variants on classical utility functions whose validity has been extensively studied.

\(^3\) \( x \mapsto \frac{U''(x)}{U'(x)} \) is increasing up to 0 and is worth 1 beyond 0. On \( \mathbb{R}^- \) it thus exposes “IARA” (increasing absolute risk aversion).
A power type utility:

\[ U(x) = 2x - \sqrt{x^2 + 1}. \]  

(2.7)

### 3 Results on Maximizer Existence

The limited liability problems involve optimizing across two subsets of the Hilbert space \( E := L^2(\Omega) \times \mathbb{R} \). Denote:

\[
\langle (W_1, a_1); (W_2, a_2) \rangle_E := E[W_1 W_2] + a_1 a_2, \quad \text{and} \quad ||(W, a)||_E := \sqrt{E[W^2] + |a|^2}.
\]

and \( d_E \) the induced distance on \( E \). Below some key elements for optimizing on Hilbert spaces are recalled (see [12] for more information).

**Definition 3.1** A functional \( f : E \to \mathbb{R} \) is weakly upper semi-continuous (or weakly u.s.c.) at \((W, a)\) in \( E \) if for any sequence \((W_n, a_n)_{n \in \mathbb{N}}\) such that \((W_n, a_n) \rightharpoonup (W, a)\):

\[
f(W, a) \geq \limsup_{n \to +\infty} f(W_n, a_n).
\]

(3.1)

The following theorems provide existence conditions on Hilbert spaces.

**Theorem 3.1** Suppose that \( f : M \subseteq \mathbb{R} \) is weakly u.s.c. over a weakly closed subset \( M \) of \( E \). Then, if \( M \) is bounded or \( f \) is norm-coercive, meaning that \( ||(W, a)||_E \to +\infty \Rightarrow f(W, a) \to +\infty \), then:

\[
\exists (W_0, a_0) \in M, \; \text{such that} \; f(W_0, a_0) = \sup_{(W, a) \in M} f(W, a).
\]

The goal below is to apply this theorem to the Risk-Sharing model.

### 3.1 Preliminary Results on the Model

First some key preliminary results are derived. Consider the subsets of \( E \):

\[
C^M_m := \{(W, a) \in E, \; m \leq W \leq M \; \mathbb{P}\text{-a.s.}, \; E[U_A(W - \kappa(a))] \geq U_A(y)\},
\]

and

\[
C_m := \{(W, a) \in E, \; m \leq W \; \mathbb{P}\text{-a.s.}, \; E[U_A(W - \kappa(a))] \geq U_A(y)\}.
\]

Both sets contain the constant contract \((y, 0)\) and are thus non-empty. The limited liability optimization problems may be rewritten as follows:

\[
\sup_{(W, a) \in C^M_m} E[U_P(X^a - W)],
\]

(3.2)
and
\[
\sup_{(W,a) \in C_m} \mathbb{E} \left[ U_P \left( X^a - W \right) \right]. \tag{3.3}
\]

**Remark 3.1** For any \((W, a)\) in \(C_m^M\) or \(C_m\), using Jensen’s inequality (as \(U_A\) is concave) and as \(U_A\) is increasing:
\[
\mathbb{E}[U_A(W - \kappa(a))] \geq U_A(y) \Rightarrow \mathbb{E}[W] \geq y + \kappa(a).
\]
Therefore (again through Jensen’s inequality for \(U_P\) and as \(U_P\) is increasing):
\[
\mathbb{E}[U_P(X^a - W)] \leq U_P(x_0 + \mathbb{E}[B] + a - \mathbb{E}[W]) \leq U_P(x_0 - y + \mathbb{E}[B] + a^* - \kappa(a^*)),
\]
where \(a^* = \frac{1}{K} = \text{argsup}_{x \in \mathbb{R}} x - \kappa(x)\), and the Principal’s utility is upper bounded across both constraint sets.

**Remark 3.2** For any \((W, a)\) in \(E\), \(\mathbb{E}[U_A(W - \kappa(a))] < +\infty\). Indeed, as \(W\) belongs to \(L^2(\Omega)\), \(\mathbb{E}[W] - \kappa(a)\) is some finite number. Applying Jensen’s inequality, as \(U_A\) is concave:
\[
\mathbb{E}[U_A(W) - \kappa(a)] \leq U_A(\mathbb{E}[W] - \kappa(a)) < +\infty.
\]
As the participation constraint also lower bounds the Agent’s utility, one may deduce that for any \((W, a)\) in \(C_m^M\) or \(C_m\), \(\mathbb{E}[U_A(W - \kappa(a))]\) exists.
A few of the preliminary results below hold for both \(C_m\) and \(C_m^M\). The notation \(C\) is used when one subset can be trivially substituted for the other.

**Remark 3.3** For any \((W, a)\) in \(C\) where \(a\) belongs to \(\mathbb{R}_+\), \((W, |a|)\) also belongs to \(C\) through the symmetry of the quadratic cost \(\kappa\). Furthermore,
\[
U_P(X^a - W) \leq U_P(X^{|a|} - W) \quad \mathbb{P} - a.s.
\]
thus an optimal action (if it exists) will be non-negative.

This remark is sound as it makes sense for the Principal to align the effect of the Agent’s action with positive variations of the production process. It also means that one can consider that the Agent’s actions take their values in \(\mathbb{R}_+\).

**Lemma 3.1** The Principal’s value function is coercive in \(a\) across \(C\).

**Proof** Consider a sequence \((W_n, a_n)_{n \in \mathbb{N}}\) in \(C\) such that \(a_n \to +\infty\). Applying Jensen’s inequality to the Participation Constraint (2.2), \(\mathbb{E}[W_n] \geq y + \kappa(a_n)\). Now applying Jensen to the Principal’s value function, coercivity is obtained:
\[
\mathbb{E} \left[ U_P(X^{a_n} - W_n) \right] \leq U_P(x_0 + a_n + \mathbb{E}[B] - y - \kappa(a_n)) \to +\infty - \infty.
\]
\[\square\]
Remark 3.4  By construction, $U'_P$ and $U'_A$ are positive and decreasing mappings. Therefore, $0 \leq U'_A(W - \kappa(a)) \leq U'_A(m - \kappa(a))$, and thus, $U'_A(W - \kappa(a))$ belongs to $L^1$ over $C$. In the following we also assume that $U_P$ is differentiable over $C$, with $U'_P$ integrable.

Lemma 3.2  $C$ is a convex subset of $E$, and weakly closed.

Proof  Weakly closed property. $(W_n, a_n) \rightarrow (W, a) \in E$. First one may prove that $W \geq m \mathbb{P} - a.s.$ (the reasoning trivially extends to the upper bound in $C^M_m$). To do so suppose that $\mathbb{P}(W < m) > 0$. The weak-convergence of $(W_n, a_n)$ implies in particular that:

$$
\mathbb{E}[W_n \Phi] \rightarrow_{n \rightarrow +\infty} \mathbb{E}[W \Phi] \quad \forall \Phi \in L^2(\Omega),
$$

and thus:

$$
\mathbb{E}[(W_n - m) \Phi] \rightarrow_{n \rightarrow +\infty} \mathbb{E}[(W - m) \Phi] \quad \forall \Phi \in L^2(\Omega). \tag{3.4}
$$

Now set $\Phi = 1_{W < m}$. Then, $\Phi$ belongs to $L^2(\Omega)$ and using Eq (3.4):

$$
\mathbb{E}[(W_n - m) 1_{W < m}] \rightarrow_{n \rightarrow +\infty} \mathbb{E}[(W - m) 1_{W < m}].
$$

By construction, $\mathbb{E}[(W_n - m) 1_{W < m}] \geq 0$ and $\mathbb{E}[(W - m) 1_{W < m}] \leq 0$ and so the convergence cannot hold. Thus, $W \geq m \mathbb{P} - a.s.$.

Now it remains to deal with the participation constraint. As $(W_n, a_n) \rightarrow (W, a) \in E$ and $\kappa$ is continuous on $\mathbb{R}$, it holds that $(W_n, \kappa(a_n)) \rightarrow (W, \kappa(a))$ and in particular for all $Z$ in $L^2(\Omega)$:

$$
\mathbb{E}[Z(W_n - W - \kappa(a_n) + \kappa(a))] \rightarrow_{n \rightarrow +\infty} 0.
$$

As $U''_A \leq 0$ (meaning that $U_A$ is concave), the following inequality holds:

$$
\mathbb{E}[U_A(W_n - \kappa(a_n))] \leq \mathbb{E}[U_A(W - \kappa(a))] + \mathbb{E}[U'_A(W - \kappa(a))(W_n - W - \kappa(a_n) + \kappa(a))],
$$

and thus as soon as $U'_A$ belongs to $L^2(\Omega)$, we have:

$$
\lim sup_{n \rightarrow +\infty} \mathbb{E}[U_A(W_n - \kappa(a_n))] \leq \mathbb{E}[U_A(W - \kappa(a))].
$$

In particular, as $U_A(y) \leq \mathbb{E}[U_A(W_n - \kappa(a_n))]$ it follows that $U_A(y) \leq \mathbb{E}[U_A(W - \kappa(a))]$. Therefore, $C$ is a weakly closed subset of $E$. 

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**Convexity** Let \((W_1, a_1)\) and \((W_2, a_2)\) belong to \(C\) and let \(\lambda > 0\). Then,

\[
m \leq \lambda W_1 + (1 - \lambda) W_2 \quad \mathbb{P} \text{- a.s.}
\]

This trivially extends to the upper bound in \(C^M_m\). It remains to deal with the Participation Constraint:

\[
\mathbb{E}[U_A(\lambda W_1 + (1 - \lambda) W_2 - \kappa(\lambda a_1 + (1 - \lambda) a_2))] \\
\geq \mathbb{E}[U_A(\lambda W_1 + (1 - \lambda) W_2 - \lambda \kappa(a_1) + (1 - \lambda) \kappa(a_2))] \\
\geq \lambda \mathbb{E}[U_A(W_1 - \kappa(a_1))] + (1 - \lambda) \mathbb{E}[U_A(W_2 - \kappa(a_2))] \geq U_A(y).
\]

Thus, \((\lambda W_1 + (1 - \lambda) W_2, \lambda a_1 + (1 - \lambda) a_2)\) belongs to \(C\) and convexity holds. \(\square\)

Note that the proof of Lemma 3.2 includes proof that the Agent’s expected utility is weakly u.s.c. on \(C\) through construction of the constraint sets.

**Remark 3.5** Mazur’s theorem may be applied to deduce from Lemma 3.2 that \(C\) is also strongly closed.

**Lemma 3.3** *The Principal’s value function is concave.*

**Proof** Let \((W_1, a_1)\) and \((W_2, a_2)\) belong to \(C\) and let \(\lambda > 0\). Then,

\[
\mathbb{E}[U_P(X^{\lambda a_1 + (1 - \lambda) a_2} - \lambda W_1 - (1 - \lambda) W_2)] \\
= \mathbb{E}[U_P(\lambda X^{a_1} - \lambda W_1 + (1 - \lambda) X^{a_2} + (1 - \lambda) W_2)] \\
\geq \lambda \mathbb{E}[U_P(X^{a_1} - W_1)] + (1 - \lambda) \mathbb{E}[U_P(X^{a_2} - W_2)].
\]

\(\square\)

The following result provides sufficient conditions for upper semi-continuity of the Principal’s value function across \(C\).

**Lemma 3.4** *The Principal’s value function is weakly u.s.c.*

**Proof** Let \((W_n, a_n)\) be a sequence in \(C\) that converges weakly to \((W, a)\). As \(U_P\) is concave,

\[
\mathbb{E}[U_P(X^{a_n} - W_n)] \leq \mathbb{E}[U_P(X^a - W)] + \mathbb{E}[U'_P(X^a - W)(a_n - W_n - a + W)]
\]

But weak-convergence yields:

\[
\mathbb{E}[U'_P(X^a - W)(a_n - W_n - a + W)] \xrightarrow{n \to +\infty} 0
\]

and thus:

\[
\limsup_{n \to +\infty} \mathbb{E}[U_P(X^{a_n} - W_n)] \leq \mathbb{E}[U_P(X^a - W)].
\]

\(\square\)
A final result is related to the coercivity in $W$ of the Principal’s value function. This is needed in order to guarantee existence in the lower bound case.

**Lemma 3.5** Suppose that $\text{supp}(B) \subset (-\infty, b)$, for $b \in \mathbb{R}$. Suppose that $U''''_p \geq 0$. Then, for any $0 \leq a \leq a_{\text{max}}$ where $a_{\text{max}}$ some positive constant, the Principal’s value function is coercive in $W$ across $C_m$.

**Proof** Let $(W_n, a_n)_n$ be a sequence in $C_m$ such that $0 \leq a_n \leq a_{\text{max}}$ and $\mathbb{E}[W_n^2] \rightarrow +\infty$. One may consider two cases.

- If $\mathbb{E}[W_n] \rightarrow +\infty$, then through Jensen’s inequality applied to the Principal’s value function:

  $$
  \mathbb{E}\left[U_p(X_{a_n} - W_n)\right] \leq U_p(x_0 + a_n + \mathbb{E}[B] - \mathbb{E}[W_n]) \xrightarrow{n \to +\infty} -\infty.
  $$

- If not, suppose $\mathbb{E}[W_n] \leq K$ with $K$ some real constant. Through a Taylor expansion:

  $$
  \mathbb{E}\left[U_p(X_{a_n} - W_n)\right] = \mathbb{E}[U_p(X_0 - y)] + \mathbb{E}[U'_p(X_0 - y)(a_n - W_n + y)] + \mathbb{E}[U''_p(X_{a_n} - W_\epsilon)(a_n - W_n + y)^2],
  $$

  where $(W_\epsilon, a_\epsilon)$ is some convex combination of $(W_n, a_n)$ and $(y, 0)$ (and belongs to $C_m$ through Lemma 3.2). Now as $U'_p > 0$, it holds that:

  $$
  \mathbb{E}[U'_p(X_0 - y)(a_n - W_n + y)] \leq \mathbb{E}[U'_p(X_0 - y)(a_{\text{max}} - m + y)].
  $$

  It remains to deal with the second-order term. First note that:

  $$
  U''_p(X_{a_n} - W_\epsilon) \leq U''_p(X_{a_{\text{max}}} - m) \leq \sup_{B \leq b} U''_p(X_{a_{\text{max}}} - m) \mathbb{P} - a.s.,
  $$

  where $S := \sup_{B \leq b} U''_p(X_{a_{\text{max}}} - m)$ is a strictly negative real number and exists through the boundedness of $B$ and the DARA characteristic of $U_p$. Therefore:

  $$
  \mathbb{E}[U''_p(X_{a_n} - W_\epsilon)(a_n - W_n + y)^2] \leq S (\mathbb{E}[a_n + y]^2 + \mathbb{E}[W_n]^2 - 2\mathbb{E}[W_n](a_n + y)) \xrightarrow{n \to +\infty} -\infty,
  $$

  and the conclusion is obtained: $\mathbb{E}\left[U_p(X_{a_n} - W_n)\right] \xrightarrow{n \to +\infty} -\infty$.

\[\square\]

**Remark 3.6** The hypothesis $U''''_p \geq 0$ is satisfied for example by DARA utility functions. Empirical analysis mostly validates the DARA hypothesis (see [5]).
3.2 A General Existence Theorem

The following theorem guarantees existence of maximizers in general settings.

Theorem 3.2 Suppose that the Principal’s utility function is weakly upper semi-continuous (e.g., through Lemma 3.4). Then,

1. Problem (3.2) has a maximizer.
2. Problem (3.3) has a maximizer in the two following cases:
   - \( \text{supp}(B) \subseteq [-\infty, b), b \in \mathbb{R} \) and \( U''_p \geq 0 \).
   - there exists a positive mapping \( f \) such that:
     1. \( \mathbb{E}[W^2] \to +\infty \iff \mathbb{E}[f(W)] = +\infty \), and:
     2. \( \mathbb{E}[U_P(X^a - W)] - \epsilon \mathbb{E}[f(W)] \) has a unique maximizer \((W^\epsilon, a^\epsilon)\) on \( C_m \).

Remark 3.7 It is apparent from Theorem 3.2 that existence of a maximizer in the solely lower bounded case requires heavier assumptions than the double bounded case. However, let \( B_K = \{ W \in L^2(\Omega), \mathbb{E}[W^2] \leq K^2 \} \). Then, \( \sup_{(W, a) \in C_{m,K}} \mathbb{E}[U_P(X^a - W)] \) admits a solution as soon as the Principal’s utility function is strongly upper semi-continuous.

Proof 1. Through Lemma 3.1, the Principal’s value function is coercive in \( a \). Through positivity of any optimizer, optimization in \( a \) may be restricted to some interval \([0, a_{max}]\). Then, \( C^M_m \) is both bounded in \( W \) (by construction) and in \( a \). Through Lemma 3.2, \( C^M_m \) is weakly closed. Finally, through Lemmas 3.4 and 3.3, the Principal’s value function is weakly u.s.c. The result follows through Theorem 3.1.

2. (first case). Through Lemma 3.1, the Principal’s value function is coercive in \( a \). Through Lemma 3.5, whose assumptions are satisfied, the value function is coercive in \( W \). It is thus jointly coercive in \( W \) and \( a \). Indeed, let \((W_n, a_n)\) in \( C_m \) be such that \(||(W_n, a_n)||_{E} \to +\infty \), then it holds that that:
   - either \( a_n \to +\infty \) and the Principal’s value function is coercive through coercivity in \( a \),
   - or \( 0 \leq a_n \leq a_{max} \) where \( a_{max} \) is some positive constant, and \( \mathbb{E}[W^2_n] \to +\infty \). The Principal’s value function is then coercive through coercivity in \( W \). Now through Lemmas 3.2, \( C^M_m \) is weakly closed. Finally, through Lemmas 3.4 and 3.3, the Principal’s value function is weakly u.s.c. The result follows through Theorem 3.1.

2. (second case) The aim is to consider the perturbed solutions \((W_\epsilon, a_\epsilon)\) and to analyze their behavior as \( \epsilon \) goes to 0. As such consider some sequence \((\epsilon_n)_{n \in \mathbb{N}}\) with \( \forall n \in \mathbb{N}, 0 < \epsilon_n < 1 \) and \( \lim_{n \to +\infty} \epsilon_n = 0 \). The reasoning from Lemma 3.1 and Remark 3.3 still holds, and thus, there exists some \( a_{max} \) such that \( a_{\epsilon_n} \in [0, a_{max}] \).
for any \( n \in \mathbb{N} \). Also by assumption, \( \mathbb{E} \left[ W_{\epsilon_n}^2 \right] \leq K \), for any \( n \in \mathbb{N} \). Using the Banach–Alaoglu theorem, one can find a subsequence of \( \epsilon \) such that:

\[
\epsilon_n \xrightarrow{n \to +\infty} 0, \quad W_{\epsilon_n} \xrightarrow{n \to +\infty} W^*, \quad a_{\epsilon_n} \xrightarrow{n \to +\infty} a^*,
\]

where \((W^*, a^*)\) belongs to \( C_m \) through Lemma 3.2. Finally, recall that \((W, a) \mapsto \mathbb{E}[U_p(X^a - W)]\) is weakly u.s.c. Thus, for any \((W, a) \in C_m\), it holds that:

\[
\mathbb{E}[U_p(X^{a^*} - W^*)] \geq \limsup_{n \to +\infty} \mathbb{E}[U_p(X^{a_{\epsilon_n}} - W_{\epsilon_n})]
\]

\[
\geq \limsup_{n \to +\infty} \mathbb{E}[U_p(X^{a_{\epsilon_n}} - W_{\epsilon_n})] - \epsilon_n \mathbb{E}[f(W_{\epsilon_n})] \quad \text{as } f \text{ positive}
\]

\[
\geq \limsup_{n \to +\infty} \mathbb{E}[U_p(X^a - W)] - \epsilon_n \mathbb{E}[f(W)] \quad \text{as } (W_{\epsilon_n}, a_{\epsilon_n}) \text{ maximizer}
\]

\[
= \mathbb{E}[U_p(X^a - W)],
\]

\((W^*, a^*)\) is therefore an optimal contract for Problem (3.3).

\[\square\]

Theorem 3.2 uses assumptions on the Principal’s utility function in order to ensure existence of solutions to the limited liability problem. However, it does not use any assumptions on the Agent’s utility function beyond it being increasing and concave. This allows for substantial freedom in modeling the Agent’s utility.

**Example 3.1** In the following examples, supposing that the Agent has any concave and increasing utility function, different limited liability existence results are provided.

1. **Logarithmic utility** Let a Principal have extended logarithmic utility, as defined in (2.5). Suppose that the underlying uncertainty \( B \) has a uniform distribution on \([-5; 5]\). Set \( m = 0 \) and \( M = 2 \). Then, the contracting problem \( \sup_{(W, a) \in C_0^2} \mathbb{E} \left[ U_p(X^a - W) \right] \) admits a maximizer.

2. **CARA utility** Let a Principal have CARA utility. Suppose that the underlying uncertainty \( B \) has a bounded distribution. Set \( m = 0 \). Then, the contracting problem \( \sup_{(W, a) \in C_0} \mathbb{E} \left[ U_p(X^a - W) \right] \) admits a maximizer.

### 3.3 Some Comments on the Results

#### 3.3.1 An Application to CARA Utilities

The following result is a corollary to Theorem 3.2.

**Corollary 3.1** (Existence theorem for lower limited liability and CARA utilities) Suppose that \( U_p(x) := -e^{-\gamma_p x} \) and \( U_A(x) := -e^{-\gamma_A x} \), for \( \gamma_p > 0 \) and \( \gamma_A > 0 \), then there exists a solution to Problem (3.3).

**Remark 3.8** Crucially Corollary 3.1 lifts the need for a bounded \( B \) for existence of a solution to the lower bound limited liability problem in a CARA setting.
The rest of this section is dedicated to proving this result.

**Lemma 3.6** Set $U_P(x) := -e^{-\gamma_P x}$ and $U_A(x) := -e^{-\gamma_A x}$. Then, there exists a solution to the following problem:

$$\sup_{(W, a) \in C_m} F_\epsilon(W, a),$$

where $F_\epsilon(W, a) = \mathbb{E}[U_P(X^n - W)] + \epsilon \mathbb{E}[U_P(-W)]$.

**Proof** Through Lemma 3.2, $C_m$ is weakly closed. Through Lemmas 3.4 and 3.3, the Principal’s value function is weakly upper semi-continuous. Lemma 3.1 shows $F_\epsilon$ is coercive in $a$. Finally, coercivity in $W$ is obtained through the construction of $F_\epsilon$. Indeed, for any $x$ on $\mathbb{R}^+$ $e^x - x^2 \geq 0$. Therefore, consider a sequence $(W_n, a_n)_n$ in $C_m$ with $a_n \in [0, a_{max}]$ for any $n$ in $\mathbb{N}$. As $W \geq m \mathbb{P} - a.s.$ (with $m \geq 0$):

$$\lim_{n \to +\infty} \mathbb{E} \left[ W_n^2 \right] = +\infty \Rightarrow \lim_{n \to +\infty} \mathbb{E} \left[ (\gamma_P W_n)^2 \right] = +\infty.$$

As a consequence to the Participation Constraint, $\mathbb{E}[W_n] \geq y + \kappa(a_n)$. Through Jensen’s inequality (and denoting $\tilde{a} := \frac{1}{\kappa}$):

$$F_\epsilon(W_n, a_n) = -\mathbb{E} \left[ e^{-\gamma_P (X^n - W_n)} \right] - \epsilon \mathbb{E} \left[ e^{\gamma_P W_n} \right]$$

$$\leq -e^{-\gamma_P (x_0 + \mathbb{E}[B] - y + a_n - \kappa(a_n))} - \epsilon \mathbb{E} \left[ e^{\gamma_P W_n} \right]$$

$$\leq -e^{-\gamma_P (x_0 + \mathbb{E}[B] - y + \tilde{a} - \kappa(\tilde{a}))} - \epsilon \mathbb{E} \left[ e^{\gamma_P W_n} \right].$$

Therefore, using (3.6), $\lim_{n \to +\infty} \mathbb{E} \left[ W_n^2 \right] \Rightarrow \lim_{n \to +\infty} F_\epsilon(W_n, a_n) = -\infty$. Using coercivity in $a$, this yields $\lim_{n \to +\infty} F_\epsilon(W_n, a_n) = +\infty$.

Existence of maximizers is then obtained through Theorem 3.1.

Denote as $(W_\epsilon, a_\epsilon)$ the optimal contract for perturbed Problem (3.5). As the perturbed value function is strongly convex, the contract is unique. The following lemma is key and upper bounds the $L^2$ norm of the perturbed wage.

**Lemma 3.7** Let $(W_\epsilon, a_\epsilon)$ be an $\epsilon$-solution to Problem (3.5) for a fixed $0 < \epsilon < 1$. Then, $\mathbb{E} \left[ W_\epsilon^2 \right] \leq C_{max}$, for $C_{max}$ some constant.

**Proof** Consider any $0 < \epsilon < 1$. The unique $\epsilon$-solution $(W_\epsilon, a_\epsilon)$ can be characterized by applying a generalized K.K.T. as detailed in Sect. 4. As such there exists some $\lambda_\epsilon \geq 0$ and $Z_\epsilon$ in $\mathcal{V}$ where $\mathcal{V} := \{ X \in L^2(\Omega), \ X \geq 0 \ \mathbb{P}\text{-a.s.} \}$, such that:

$$\gamma_P e^{-\gamma_P (X_\epsilon^a - W_\epsilon)} + \epsilon \gamma_P e^{\gamma_P W_\epsilon} - Z_\epsilon - \lambda_\epsilon \gamma_A e^{-\gamma_A (W_\epsilon - \kappa(a_\epsilon))} = 0 \quad (1)$$

$$-\gamma_P \mathbb{E}[e^{-\gamma_P (X_\epsilon^a - W_\epsilon)}] + a_\epsilon K \lambda_\epsilon \mathbb{E}[e^{-\gamma_A (W_\epsilon - \kappa(a_\epsilon))}] = 0 \quad (2)$$
\[ \mathbb{E}[Z_\epsilon(W_\epsilon - m)] = 0 \quad \text{and} \quad \lambda_\epsilon \left( \mathbb{E}[e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}] - e^{-\gamma_A Y} \right) = 0. \quad (3) \]

First suppose that \( a_\epsilon = 0 \) and/or \( \lambda_\epsilon = 0 \). Then, in (2), \( \mathbb{E}\left[e^{-\gamma_P(X^{ae} - W_\epsilon)}\right] = 0 \), which is absurd. In particular, it must hold that \( \mathbb{E}[e^{-\gamma_P(X^{ae} - W_\epsilon)}] \geq R \) where \( -R < 0 \) is the corresponding Risk-Sharing optimum (see for example [14]). Therefore,

\[ \gamma_P \mathbb{E}[e^{-\gamma_P(X^{ae} - W_\epsilon)}] = a_\epsilon K \lambda_\epsilon \mathbb{E}[e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}], \]

implying that for any fixed \( \epsilon > 0 \), \( a_\epsilon \lambda_\epsilon = \gamma_P \mathbb{E}[e^{-\gamma_P(X^{ae} - W_\epsilon)}] \geq \frac{R}{K e^{-\gamma_A Y}} > 0 \).

Recall that by construction \( \lambda_\epsilon \geq 0 \). It follows that for any \( \epsilon > 0 \), \( a_\epsilon > 0 \) and \( \lambda_\epsilon > 0 \) and as a consequence the Participation Constraint binds. Through (1) and as (y, 0) belongs to \( C_m \) it holds that: \( 0 \leq \lambda_\epsilon \leq \frac{\gamma_P \mathbb{E}[e^{-\gamma_P(X^0 - y)}] + e^{\gamma_P y}}{\gamma_A e^{-\gamma_A Y}} \) := \( C \), and it follows that \( a_\epsilon \geq \frac{R}{K e^{-\gamma_A Y}} > 0 \). Now, as both \( Z_\epsilon \) and \( W_\epsilon - m \) are positive:

\[ \mathbb{E}[Z_\epsilon(W_\epsilon - m)] = 0 \Rightarrow Z_\epsilon(W_\epsilon - m) = 0 \ \mathbb{P} - a.s.. \]

Therefore, as soon as \( Z_\epsilon > 0 \), it must hold that \( W_\epsilon = m \) and as a consequence, \( 0 \leq Z_\epsilon \leq Z_\epsilon 1_{W_\epsilon = m} \ \mathbb{P} - a.s.. \) Using (1):

\[ 0 \leq Z_\epsilon \leq Z_\epsilon 1_{W_\epsilon = m} \leq \gamma_P e^{-\gamma_P(X^{ae} - m)} + \epsilon \gamma_P e^{\gamma_P m} \leq \gamma_P e^{-\gamma_P(X^0 - m)} + \gamma_P e^{\gamma_P m}, \ \mathbb{P} - a.s., \]

and finally:

\[ 0 \leq m \leq W_\epsilon = \frac{1}{\gamma_P} \ln \left( \frac{Z_\epsilon + \lambda_\epsilon e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}}{\frac{\gamma_P}{e + e^{-\gamma_P X^{ae}}}} \right) \leq \frac{1}{\gamma_P} \ln \left( \frac{Z_\epsilon + \lambda_\epsilon e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}}{\gamma_P e^{-\gamma_P X^{ae}}} \right) \leq \frac{1}{\gamma_P} \ln \left( \frac{\gamma_P e^{-\gamma_P(X^0 - m)} + \gamma_P e^{\gamma_P m} + C e^{-\gamma_A(m - \kappa(a_{max}))}}{\gamma_P e^{-\gamma_P X^{a_{max}}}} \right), \]

and for any \( 0 < \epsilon < 1 \) it holds that:

\[ \mathbb{E}\left[W_\epsilon^2\right] \leq \frac{1}{\gamma_P^2} \mathbb{E}\left[\ln \left( \frac{\gamma_P e^{-\gamma_P(X^0 - m)} + \gamma_P e^{\gamma_P m} + C e^{-\gamma_A(m - \kappa(a_{max}))}}{\gamma_P e^{-\gamma_P X^{a_{max}}}} \right)^2 \right] = C_{max}. \]

□

With these tools in mind the proof of Corollary 3.1 is quite direct. Indeed, the assumptions of the second part of Theorem 3.2 (point 2.) are satisfied and the result follows.
The perturbation method used in this proof resembles that of the Tikhonov method for ill-posed problems ([23]), albeit for a different perturbation term. In fact \( (W_\epsilon, a_\epsilon) \) makes \( \mathbb{E}[\exp(-\gamma p(X^{a_\epsilon} - W_\epsilon))] \) as small as possible without \( \mathbb{E}[\exp(\gamma p W_\epsilon)] \) becoming too big, and thus without \( \mathbb{E}[W^2_\epsilon] \) becoming too big. Lemma 3.7 formalizes this by showing that the \( L^2 \)-norm of \( W_\epsilon \) is upper-bounded for \( \epsilon \) smaller than 1.

### 3.3.2 More General Perturbations of \( B \)

**Modifying the effect of \( a \)** This work has focused on perturbations of the form \( X^a = x_0 + a + B \). Note that given a differentiable mapping \( f \) that is concave and such that:

\[
\lim_{|x| \to +\infty} f(x) - \kappa(x) = -\infty, \tag{3.7}
\]

and setting \( X^a = x_0 + f(a) + B \), the existence results of Theorem 3.2 still hold. Indeed, such a modification only affects the Principal’s expected utility. First note that the Principal’s expected utility remains upper bounded: Remark 3.1 still holds with \( a^* \) now the maximizer of \( f - \kappa \). The crucial ingredients for existence and related to the Principal’s utility are Lemma 3.1 on coercivity in \( a \) and Lemma 3.3 on concaveness (note that the two other results, Lemmas 3.4 and 3.5, follow as a consequence to Lemma 3.3). Coercivity in \( a \) extends through 3.7 and as \( f \) is concave, the Principal’s expected utility remains concave in \( (W, a) \). Thus, the existence result may be extended to such a setting, and the optimum will be characterizable through the results of Sect. 4.

**Modifying the effect of \( X^a \)** Echoing this natural extension, one may wish to model a setting where the wealth of the Principal is given by \( g(X^a) - W \) (for example, the Principal may only earn part of \( X^a \)), where \( g \) is a differentiable mapping. As such a function affects \( B \), we require that:

\[
\mathbb{E}[U_P(g(X^0) - y)] = \mathbb{E}[U_P(g(x_0 + B) - W)] < +\infty.
\]

Once again, such an extension holds as soon as \( g \) is concave and: \( \lim_{|x| \to +\infty} g(x) - \kappa(x) = -\infty \), and the optimum will be characterizable through the results of Sect. 4.

### 4 Characterization of the Optima

With the existence of solutions in mind, one may turn to their characterization. The theory used here for characterizing optimizers on convex subsets of Hilbert spaces is described in Luenberger’s monograph “Optimization by Vector Space methods” ([13]). The aim is to characterize the limited liability optima in situations where its existence is ensured and existence is supposed as a preliminary as well as the Gâteaux-differentiability of the expected utility functions.

Let \( \bar{U}_P = -\bar{U}_P, \bar{U}_A = -U_A \) and \( \mathcal{P}_W := \{ W \in L^2(\Omega), W \geq 0 \ \mathbb{P}\text{-a.s.} \} \). Characterization of the existence optimum will be obtained by exploiting the necessary optimality conditions (from [13]).
4.1 One-Sided Limited Liability

Suppose that Problem (3.3) admits a solution. Let \((W^*, a^*, \lambda^*, Z^*)\) in \(E \times \mathbb{R}^+ \times P_W\) be a solution to Problem (3.3) with the two related Lagrange multipliers. The following four equations are satisfied:

\[
\begin{align*}
\mathbb{E} \left[ \tilde{U}'_p \left( X^{a^*} - W^* \right) \right] - \lambda^* \kappa'(a) \mathbb{E} \left[ \tilde{U}'_A \left( W^* - \kappa(a^*) \right) \right] &= 0, \\
-\tilde{U}'_p \left( X^{a^*} - W^* \right) + \lambda^* \tilde{U}'_A \left( W^* - \kappa(a^*) \right) - Z^* &= 0, \\
\lambda^* \left( \mathbb{E} \left[ U_A(W^* - \kappa(a^*)) \right] - U_A(y) \right) &= 0, \\
\mathbb{E} \left[ Z^* \left( W^* - m \right) \right] &= 0. \tag{4.1}
\end{align*}
\]

**Lemma 4.1** The optimal \(\lambda^*\) satisfies \(\lambda^* > 0\), meaning that the Participation Constraint binds at the optimum.

**Proof** Using the second equality of (4.1) and as \(Z^* \in P_W\), it holds that:

\[-\tilde{U}'_p \left( X^{a^*} - W^* \right) + \lambda^* \tilde{U}'_A \left( W^* - \kappa(a^*) \right) \geq 0,
\]

implying that:

\[\lambda^* \geq \frac{\tilde{U}'_p \left( X^{a^*} - W^* \right)}{\tilde{U}'_A \left( W^* - \kappa(a^*) \right)} > 0. \square
\]

**Lemma 4.2** The optimal \(Z^*\) and \(W^*\) satisfy:

\[Z^* \left( W^* - m \right) = 0 \quad \mathbb{P} - a.s. \quad \text{and} \quad \mathbb{P} \left( W^* = m \right) < 1.
\]

**Proof** By definition it holds that \(Z^* \geq 0 \quad \mathbb{P} - a.s.\) and \(W^* - m \geq 0 \quad \mathbb{P} - a.s..\) Using the final equality of (4.1):

\[\mathbb{E} \left[ Z^* \left( W^* - m \right) \right] = 0 \implies Z^* \left( W^* - m \right) = 0 \quad \mathbb{P} - a.s.
\]

For the second assertion, the first equality of (4.1) states that:

\[\mathbb{E} \left[ \tilde{U}_A \left( W^* - \kappa(a^*) \right) \right] = \tilde{U}_A(y).
\]

Now suppose for contradiction that \(W^* = m \quad \mathbb{P} - a.s..\) Then, \(a^*\) must verify \(y = m - \kappa(a^*)\) for any \(y \geq 0\). This is impossible as for any \(x\) in \(\mathbb{R}^+_+\), \(\kappa(x) \geq 0\), and a contradiction is reached. \(\square\)

**Proposition 4.1** (General characterization for one-sided limited liability) Let \((W^*, a^*, \lambda^*, Z^*)\) be a solution to Problem (3.3). Then,

- \((W^*, a^*)\) saturates the participation constraint:

\[\mathbb{E} \left[ U_A(W^* - \kappa(a^*)) \right] = U_A(y).
\]
(Borch rule for one-sided limited liability). On the event \( \{ Z^* = 0 \} \), the ratio of marginal utilities of the Principal and the Agent is constant:

\[
\lambda^* = \frac{\bar{U}'_p (X^{a^*} - W^*)}{\bar{U}'_A (W^* - \kappa(a^*))} \mathbb{1}_{Z^* = 0}.
\]

**Proof** – The first statement is a direct consequence to Lemma 4.1.

– Lemma 4.2 implies that \( \mathbb{P}(Z^* = 0) > 0 \), so setting oneself on the event \( \{ Z^* = 0 \} \) and using the second equation of Lemma 4.1 it holds that

\[
-\bar{U}'_p (X^{a^*} - W^*) + \lambda^* \bar{U}'_A (W^* - \kappa(a^*)) = 0,
\]

and the result follows.

\( \square \)

### 4.2 Double-Sided Limited Liability

Suppose that Problem (3.2) admits a solution. Let \( (W^*, a^*, \lambda^*, Z^*, Y^*) \) in \( E \times \mathbb{R}^+ \times \mathcal{P}_W \times \mathcal{P}_W \) be a solution to Problem (3.2) with the three related Lagrange multipliers. The following four equations are satisfied:

\[
\begin{align*}
\mathbb{E} \left[ \bar{U}'_p \left( X^{a^*} - W^* \right) \right] - \lambda^* \kappa'(a^*) \mathbb{E} \left[ \bar{U}'_A (W^* - \kappa(a^*)) \right] &= 0 \\
-\bar{U}'_p (X^{a^*} - W^*) + \lambda^* \bar{U}'_A (W^* - \kappa(a^*)) - Z^* + Y^* &= 0 \\
\lambda^* \left( \mathbb{E} [UA(W^* - \kappa(a^*))] - U_A(y) \right) &= 0 \\
\mathbb{E}[Z^*(W^* - m)] &= 0 \quad \text{and} \quad \mathbb{E}[Y^*(M - W^*)] = 0.
\end{align*}
\]

(4.2)

**Lemma 4.3** Let \( \omega \) be such that \( Z^*(\omega) = Y^*(\omega) \), then \( Z^*(\omega) = 0 \) and \( Y^*(\omega) = 0 \).

**Proof** Let \( \omega \) be such that \( Z^*(\omega) = Y^*(\omega) = c \) for some non-negative \( c \). For reasons mirroring those of Lemma 4.2, it must hold that:

\[
Z^*(\omega)W^*(\omega) = Z^*(\omega)m \quad \text{and} \quad Y^*(\omega)W^*(\omega) = Y^*(\omega)M,
\]

which rewrites as \( cW^*(\omega) = cm \) and \( cW^*(\omega) = cM \). As \( m \neq M \), this is only possible for \( c = 0 \).

\( \square \)

**Proposition 4.2** (General characterization for double-sided limited liability) Let \( (W^*, a^*, \lambda^*, Z^*, Y^*) \) be a solution to Problem (3.2). Then, \( (W^*, a^*, \lambda^*, Z^*, Y^*) \) satisfies a Borch rule for double-sided limited liability. Indeed, on the event \( \{ Z^* = Y^* \} \), the ratio of marginal utilities of the Principal and the Agent is constant:

\[
\lambda^* = \frac{\bar{U}'_p (X^{a^*} - W^*)}{\bar{U}'_A (W^* - \kappa(a^*))} \mathbb{1}_{Z^* = 0} \quad \text{and} \quad Y^* = 0.
\]
Proof The proof of Proposition 4.1 may be adapted to this setting to show here that:

\[
\lambda^* = \frac{\tilde{U}'_p(X^{a*} - W^*)}{\tilde{U}'_A(W^* - \kappa(a^*))} \mathbf{1}_{Z^* = Y^*},
\]

and the final result is obtained through Lemma 4.3. \qed

4.3 Some Applications

Example 4.1 Logarithmic utilities Let a Principal and an Agent have extended logarithmic utility, as defined in (2.5), meaning that:

\[
U_p(x) = U_A(x) = U(x)
\]

where:

\[
U(x) = \log(x)I_{x \geq 1} - \frac{1}{2}(x^2 - 4x + 3)I_{x < 1},
\]

Suppose that the underlying uncertainty \( B \) has a distribution with compact support \([-b_{\min}; b_{\max}]\). Fix some lower bound \( m \geq 0 \), some upper bound \( M \) and some reservation parameter \( y \in [m; M] \). Then, through Theorem 3.2 the contracting problem:

\[
\sup_{(W, a) \in C^M_m} \mathbb{E}[U_p(X^a - W)],
\]

admits a maximizer. Before providing characterization one must study the Gâteaux differentiability of the Principal and the Agent’s expected utilities. Note that \( U \) is a \( C^2 \) function across \( \mathbb{R} \) with derivatives:

\[
U'(x) = \frac{1}{x}I_{x \geq 1} - \frac{1}{2}(2x - 4)I_{x < 1} \quad \text{and} \quad U''(x) = -\frac{1}{x^2}I_{x \geq 1} - I_{x < 1}.
\]

For any \((W, a)\) in \( C^M_m \), \( W \) belongs to \([m, M]\) and \( B - W \) belongs to \([b_{\min} - M; b_{\max} - m]\). As a consequence, \( U(X^a - W), U'(X^a - W), U''(X^a - W) \) as well as \( U(W - \kappa(a)), U'(W - \kappa(a)), U''(W - \kappa(a)) \) are bounded random variables. Let any \( H \in L^2(\Omega) \) such that \((W + \tau H, a)\) belongs to the convex set \( C^M_m \) for \( \tau \leq \epsilon \) for some \( \epsilon > 0 \). Now fix \( \tau > 0 \):

\[
\mathbb{E}[U_p(X^a - (W + \tau H))] - \mathbb{E}[U_p(X^a - W)] = \mathbb{E}[U'_p(X^a - W)\tau H] + \frac{1}{2}\mathbb{E}[U''_p(X^a - W^{\tau H})\tau^2 H^2],
\]

where \( W^{\tau H} \) is a convex combination of \( W \) and \( W + \tau H \). Boundedness of the terms ensures that the expectations exist and thus:

\[
\lim_{\tau \to 0} \frac{\mathbb{E}[U_p(X^a - (W + \tau H))] - \mathbb{E}[U_p(X^a - W)]}{\tau} = \mathbb{E}[U'_p(X^a - W)H]
\]

and one may identify the Gâteaux derivative \( U'_p(X^a - W) \). Similar reasoning can be performed for \( U_A \).
Now, let \((W^*, a^*)\) in \(E\) be a maximizer, then one may characterize it. Indeed, there exists \((\lambda^*, Z^*, Y^*)\) in \(\mathbb{R}^+ \times P_W \times P_W\) such that:

\[
a^* = \kappa^{-1} \left( \frac{\mathbb{E}[Z^* - Y^*]}{\lambda^* \mathbb{E}[U'(W^* - \kappa(a^*))]} - 1 \right),
\]

and the wage:

\[
W^* = m I_{\tilde{W} < m} + \tilde{W} I_{m \leq \tilde{W} \leq M} + M I_{\tilde{W} > M},
\]

satisfies the Borch rule:

\[
\lambda^* = \tilde{U}'(Xa^* - W^*) \frac{I_{Z^* = Y^*}}{\tilde{U}'(W^* - \kappa(a^*))}, \quad \text{with} \quad U'(x) = \frac{1}{x} I_{x \geq 1} - \frac{1}{2} (2x - 4) I_{x < 1}.
\]

This example illustrates the wide scope of the previously proven results. Indeed, this work provides general existence results for limited liability problems in which the underlying utilities lead to relatively unusable expressions. In fact with such utilities the method which involves proving simultaneously existence and characterizing the optima by solving the K.K.T. conditions would be difficult to use, and have to be done on more of a case by case basis. One can nevertheless also provide some general information on the characterization.

**Example 4.2 The CARA setting** Characterization may be furthered in the CARA setting. For example, let \(U_P(x) := -e^{-\gamma_P x}\), \(U_A(x) := -e^{-\gamma_A x}\), and \(B \sim \mathcal{N}(0, 1)\). The problem \(\sup_{(W, a) \in C_0} \mathbb{E} [U_P(Xa - W)]\) admits a solution. Let \((\lambda^*, Z^*)\) be Lagrange multipliers. The optimal wage \(W^*\) is of the form:

\[
W^* = \left( \frac{\gamma_P}{\gamma_P + \gamma_A} Xa^* + \frac{\gamma_A}{\gamma_P + \gamma_A} \kappa(a^*) + \frac{1}{\gamma_P + \gamma_A} \ln \left( \frac{\gamma_A \lambda^*}{\gamma_P} \right) \right) + \ln \left( \frac{\gamma_A \lambda^*}{\gamma_P} \right)
\]

and the optimal action \(a^*\) is of the form: \(a^* = \kappa^{-1} \left( 1 + e^{\gamma_A \mathbb{E}[Z^* \lambda^*] / \gamma_P} \right)\). In particular, \(a^* \geq \frac{1}{\kappa}\). In particular, the participation constraint is saturated. The wage is very close to the one obtained when no wage bounds are imposed: it is a truncated version with a different intercept to account for the participation constraint. The optimal action is higher. Numerically it is observed that the action and the non-truncated part of the wage are very close, except when the Principal is very risk-averse. This makes economic sense: a risk-averse Principal will wish for the Agent to carry the consequences of the wage bound.

**5 Conclusion**

In this paper, the problem of enforcing constant liability constraints in the benchmark Principal-Agent problem in the context of a single-period setting is tackled. General existence results are provided through a calculus of variations approach. For completeness, existence in the case of CARA utility and an unbounded production process
with a lower bound constraint is proven through a perturbation approach. Characterizations of the optima and a Borch rule for the optimal wage are then provided with some examples. In a CARA setting the optima is very closely linked to the standard Risk-Sharing optimum.

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