Braided Supersymmetry and (Co-)Homology

Bernhard Drabant

Department of Mathematics and Computer Science
University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam, Netherlands
email: drabant@fwi.uva.nl

Abstract. Within the framework of braided or quasisymmetric monoidal categories braided \( \hat{O} \)-supersymmetry is investigated, where \( \hat{O} \) is a certain functorial isomorphism \( \hat{O} : \otimes \cong \otimes \) in a braided symmetric monoidal category. For an ordinary (co-)quasitriangular Hopf algebra \((H, \mathcal{R})\) a braided monoidal category of \(H\)-(co-)modules with braiding induced by the \(\mathcal{R}\)-matrix is considered. It can be shown for a specific class of \( \hat{O} \)-supersymmetries in this category that every braided \( \hat{O} \)-super-Hopf algebra \(B\) admits an ordinary \( \hat{O} \)-super-Hopf algebra structure on the cross product \(B \rtimes H\) such that \(H\) is a sub-Hopf algebra and \(B\) is a subalgebra in \(B \rtimes H\). Applying these results to the quantum Koszul complex \((K(q, g), d)\) of the quantum enveloping algebra \(U_q(g)\) for Lie algebras \(g\) associated with the root systems \(A_n\), \(B_n\), \(C_n\) and \(D_n\) one obtains a classical super-Hopf algebra structure on \((K(q, g), d)\) where the structure maps are morphisms of modules with differentiation.

*Supported in part by the Deutsche Forschungsgemeinschaft (DFG) through a research fellowship
1. Introduction

In [DSO] a deformation of the (co-)homology of Lie algebras was found for quantum enveloping algebras $U_q(g)$ of Lie algebras $g$ associated with the root systems $A_n$, $B_n$, $C_n$ and $D_n$. The deformed Koszul complex $(K(q, g), d)$ is the cross product of $U_q(g)$ and the $q$-exterior algebra of forms $\Lambda(X)$ together with a derivative $d$ obeying the usual properties. With regard to the results of [Ma1, Ma2] the question now arises: Does the algebra $K(q, g)$ admit a Hopf algebra structure (which respects the original complex properties)? An answer to this question is given by the concept of braided $\hat{O}$-supersymmetry which can be considered as an extension of a given braided symmetry and is itself a quasisymmetry – braided symmetries were introduced in [JS]. It then turns out that the Koszul complex $(K(q, g), d)$ is a usual super-Hopf algebra [Man], and that the differentation $d$ is compatible with this super-Hopf algebraic structure. In some sense this “dualizes” the results of M. Schlieker and B. Zumino who found that the algebra of the bicovariant differential calculus on the quantum group $A_q(G)$ is a super-Hopf algebra where the super-Hopf structure is respected by the differentiation $d$ [SZ]. Super-Hopf algebra structures on cross products of Hopf algebras have also been found in [SWZ]. These examples are instructive applications of the more general $\hat{O}$-super-bosonization in certain braided $\hat{O}$-supersymmetric monoidal categories.

The paper is organized as follows. In Section 2 general facts on braided symmetric monoidal categories are reviewed [JS, Ma1] and the concept of braided $\hat{O}$-supersymmetric monoidal categories is introduced. By a braided $\hat{O}$-supersymmetric monoidal category we understand a braided symmetric monoidal category with quasisymmetry $\Psi : \otimes \cong \otimes^\circ$ together with a certain functorial isomorphism $\hat{O} : \otimes \cong \otimes$ yielding $\hat{O}$-supersymmetry in the category. $\hat{O}$-Supersymmetric extensions of results of [Ma1, Ma2] are derived. Especially $\hat{O}$-super bosonization is investigated. In a category of $H$-left modules of a quasitriangular Hopf algebra $(H, \mathcal{R})$ with braided symmetry induced by the $\mathcal{R}$-matrix [Dri, Ma3], $\hat{O}$-super-bosonization for a specific class of functorial isomorphisms will be carried out which converts any braided $\hat{O}$-super-Hopf algebra $\mathcal{B}$ in this category to an “ordinary” $\hat{O}$-super-Hopf algebra $\mathcal{B} \rtimes H$, which is the cross product of $\mathcal{B}$ and $H$. $\mathcal{B} \rtimes H$ contains $\mathcal{B}$ as subalgebra and $H$ as sub-Hopf algebra. For the functorial isomorphism $\hat{O}^{(1)}$ of “classical” supersymmetry these results are then used in Section 3 to show that the deformed Koszul complex $(K(q, g), d)$ for Lie algebras $g$ associated with the root systems $A_n$, $B_n$, $C_n$ and $D_n$ [DSO] is a super-Hopf subalgebra of the cross product of the braided super-Hopf algebra $\Lambda(\mathcal{X})$ (in the category of $\mathbb{Z}_2$-graded $D(U_q(g))$-modules) with the quantum double $D(U_q(g))$ of $U_q(g)$ (see also [SZ]). $\Lambda(\mathcal{X})$ is the algebra of quantum exterior forms on the quantum group. Furthermore it is derived that the original algebraic structure of $(K(q, g), d)$ [DSO] enters into the super-Hopf algebra and that the structure maps are morphisms of modules with differentiation [CE].

Acknowledgements: I would like to thank T.H. Koornwinder and M. Schlieker for stimulating discussions.
2. Braided Supersymmetry . . .

For categorical notations we refer to [Mac], for the notion of braided monoidal categories, bosonization, transmutation, braided Hopf algebras etc. we refer to [JS, Ma1, Ma2]. We begin with the definition of braided \( \hat{O} \)-supersymmetry.

**Definition 2.1.** Let \((C, \otimes, 1, \Psi)\) be a braided monoidal category† with bi-functor \(\otimes\), unit object 1 and braided symmetry \(\Psi\), and let \(\hat{O} : \otimes \cong \otimes\) be a functorial isomorphism such that

\[
\hat{O} \circ \Psi = \Psi \circ \hat{O},
\]

and

\[
(\hat{O}\Psi)_{\underline{\underline{\otimes}}} = (\id_{\underline{\underline{\otimes}}} \otimes \hat{O}\underline{\underline{\otimes}}) \circ \Psi_{\underline{\underline{\otimes}}} \circ (\hat{O}\underline{\underline{\otimes}} \otimes \id_{\underline{\underline{\otimes}}}),
\]

\[
(\hat{O}\Psi)_{\underline{\underline{\otimes}}} = (\hat{O}\underline{\underline{\otimes}} \otimes \id_{\underline{\underline{\otimes}}}) \circ \Psi_{\underline{\underline{\otimes}}} \circ (\id_{\underline{\underline{\otimes}}} \otimes \hat{O}\underline{\underline{\otimes}})
\]

\(\forall \underline{\underline{\otimes}} \in \Ob(C)\). (2.1)

Then \((C, \otimes, 1, \hat{O}\Psi)\) is called *braided \(\hat{O}\)-supersymmetric monoidal category* w.r.t. the braided symmetry \(\Psi\). If \(\Psi^2 = \id\) then we speak of an \(\hat{O}\)-supersymmetric monoidal category w.r.t. \(\Psi\).

**Remark.** The id-supersymmetry w.r.t. \(\Psi\) is just \(\Psi\) itself.

The following proposition shows that the unit object is “bosonic” and that \(\hat{O}\Psi\) is again a braiding such that (2.1) holds. This is equivalent to Definition 2.1.

**Proposition 2.2.** \(\hat{O}\Psi\) is a braided symmetry, i.e. \((C, \otimes, 1, \hat{O}\Psi)\) is a braided tensor category and

\[
\hat{O} \otimes 1 = \id_{\otimes 1}, \quad \hat{O} 1 \otimes = \id_{1 \otimes} \quad \forall \mathfrak{X} \in \Ob(C).
\]

(2.3)

**Proof.** \(\hat{O}\Psi : \otimes \cong \otimes\) is a functorial isomorphism and the further properties of a braided symmetry can be derived simply by applying Definition 2.1. Since \(\hat{O}\Psi\) and \(\Psi\) are braidings we obtain \((\hat{O}\Psi)_{\otimes 1} = \hat{O} 1 \otimes \Psi_{\otimes 1} = \Psi_{1 \otimes} 1\). This completes the proof. \(\square\)

In the braided monoidal category \((C, \otimes, 1, \hat{O}\Psi)\) we can define algebras, coalgebras, bi- and Hopf algebras w.r.t. \(\Psi\). We call them *braided \(\hat{O}\)-super-bi-, braided \(\hat{O}\)-super-Hopf algebras* etc. If it is clear from the context we omit the addendum “w.r.t. \(\Psi\)”.

**Lemma 2.3.** Let \((\mathfrak{B}, \eta_\mathfrak{B}, m_\mathfrak{B})\) and \((\mathfrak{A}, \varepsilon_\mathfrak{A}, \Delta_\mathfrak{A})\) be an algebra and a coalgebra in the braided \(\hat{O}\)-supersymmetric monoidal category \((C, \otimes, 1, \hat{O}\Psi)\) respectively. Then \(\forall \mathfrak{U} \in \Ob(C)\)

\[
\hat{O}_{\mathfrak{B} \otimes} (\eta_\mathfrak{B} \otimes \id_{\mathfrak{B}}) = \eta_\mathfrak{B} \otimes \id_{\mathfrak{B}},
\]

\[
\hat{O}_{\mathfrak{A} \otimes} (\id_{\mathfrak{A}} \otimes \eta_\mathfrak{B}) = \id_{\mathfrak{A}} \otimes \eta_\mathfrak{B}
\]

(2.4)

† For convenience we omit throughout the paper the functorial isomorphisms which govern the associativity of the tensor product and the unital property of the unit object.
and

\[(\varepsilon_R \otimes \text{id}_\mathfrak{M}) \hat{\mathcal{O}}_{\mathfrak{M}} = (\varepsilon_R \otimes \text{id}_\mathfrak{M}),\]

\[(\text{id}_\mathfrak{M} \otimes \varepsilon_R) \hat{\mathcal{O}}_{\mathfrak{M}} = (\text{id}_\mathfrak{M} \otimes \varepsilon_R).\]

I.e. the unit and the counit are “bosonic”.

**Proof.** Since \(\hat{\mathcal{O}}\) is a functorial isomorphism and \(\eta_\mathfrak{M}, \varepsilon_R\) and \(\text{id}_\mathfrak{M}\) are morphisms in the category, the lemma can be proved using Proposition 2.2. \(\square\)

From now on we restrict our considerations to monoidal categories \((\mathcal{M}, \otimes, k)\) where \(k\) is a field, the objects and morphisms in \(\mathcal{M}\) are in particular \(k\)-vector spaces and \(k\)-vector space homomorphisms, and the usual tensor transposition \(\tau\) is a symmetry in \(\mathcal{M}\). Let \((H, \mathcal{R})\) be a quasitriangular Hopf algebra in \((\mathcal{M}, \otimes, k)\) and let \((H, \mathcal{M}, \otimes, k)\) be a monoidal category of \(H\)-left modules and \(H\)-left module morphisms in \(\mathcal{M}\) containing \(H\) as a module. In \(H\mathcal{M}\) the braided symmetry \(\Psi_0\) induced by the \(\mathcal{R}\)-matrix \([\text{Dri}, \text{Ma3}]\) is supposed to exist.

\[
\Psi_{0\mathfrak{M}}(u \otimes v) := \sum \mathcal{R}_2 \triangleright v \otimes \mathcal{R}_1 \triangleright u
\]

where \(\mathfrak{U}, \mathfrak{V} \in \text{Ob}(H\mathcal{M})\), \(u \in \mathfrak{U}, v \in \mathfrak{V}\), \(\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \in H \otimes H\) and \(\triangleright\) is the action of \(H\) on \(\mathfrak{U}\) and \(\mathfrak{V}\) respectively\(^\dagger\). Now we investigate a certain class of functorial isomorphisms \(\hat{\mathcal{O}}: \otimes \cong \otimes\) with the following properties.

**Definition 2.4.**

i. \((\mathcal{M}, \otimes, k, \hat{\mathcal{O}}\tau)\) and \((H\mathcal{M}, \otimes, k, \hat{\mathcal{O}}\Psi_0)\) are braided \(\hat{\mathcal{O}}\)-supersymmetric monoidal categories.  
ii. \(\hat{\mathcal{O}}_{\mathfrak{M}} = \text{id}_{H\mathfrak{M}}\) and \(\hat{\mathcal{O}}_{\mathfrak{M}H} = \text{id}_{\mathfrak{M}H} \forall \mathfrak{M} \in H\mathcal{M}\), i.e. \(H\) is “bosonic”.
iii. \([(\text{id}_\mathfrak{U} \otimes h \triangleright), \hat{\mathcal{O}}_{\mathfrak{M}H}] = 0\) and \([(h \triangleright \otimes \text{id}_\mathfrak{V}), \hat{\mathcal{O}}_{\mathfrak{M}\mathfrak{V}}] = 0 \forall \mathfrak{U}, \mathfrak{V} \in H\mathcal{M}, h \in H\).

Then for these supersymmetries the \(\hat{\mathcal{O}}\)-bosonization theorem can be stated.

**Theorem 2.5.** Let \(\mathfrak{R}\) be a braided \(\hat{\mathcal{O}}\)-super-Hopf algebra in the category \(_H\mathcal{M}\). Then the space \(\mathfrak{R} \otimes H\) can be equipped with an \(\hat{\mathcal{O}}\)-super-Hopf algebra structure in the category \(\mathcal{M}\), which is the cross product of \(\mathfrak{R}\) and \(H\) and is denoted by \(\mathfrak{R} \times_{\hat{\mathcal{O}}} H\). Explicitly \(\mathfrak{R} \times_{\hat{\mathcal{O}}} H\) has the structure

\[
\hat{\eta} = \eta_\mathfrak{R} \otimes \eta_H,  
\hat{m}((u \otimes a) \otimes (v \otimes b)) = \sum u (a(1) \triangleright v) \otimes a(2) b,  
\hat{\varepsilon} = \varepsilon_\mathfrak{R} \otimes \varepsilon_H,  
\hat{\Delta}(u \otimes a) = \sum u(1) \otimes \mathcal{R}(2)a(1) \otimes \mathcal{R}(1) \triangleright u(2) \otimes a(2),  
\hat{\hat{S}}(u \otimes a) = \sum (S_H(\mathcal{R}(a)(1)) \mathcal{R}(1) \triangleright S_\mathfrak{R}(u) \otimes S_H(\mathcal{R}(a)(2)),
\]

where \(u \otimes a, v \otimes b \in \mathfrak{R} \otimes H\), \(\Delta_H(a) = \sum a(1) \otimes a(2) \forall a \in H\) and \(\Delta_\mathfrak{R}(u) = \sum u(1) \otimes u(2) \forall u \in \mathfrak{R}\).

\(^\dagger\) In the dual language of coquasitriangular Hopf algebras and comodules \([\text{Ma1}, \text{Wei}]\) the definitions and results can be formulated similarly.
Proof. The definitions in (2.7) yield morphisms in \( \mathcal{M} \). It is verified immediately that \( \hat{\Delta} \) is an algebra morphism. The proofs of the algebra and coalgebra properties of \( \hat{\Delta} \) follow rather analogously like the corresponding proofs in [Ma1, Wei]. It remains to show that \( \hat{\Delta} \) is an algebra morphism in the category \( \mathcal{M} \). In both cases we used the fact that \( \hat{\Delta} \) is an algebra morphism.

In Corollary 2.6. An easy calculation shows that \( \hat{\Delta}(1) = 1 \otimes 1 \). We sketch the proof of the multiplicativity of \( \hat{\Delta} \). One observes that

\[
(\hat{\Delta})((\mathcal{M}, \otimes, \mathcal{I}, \hat{\Delta})) \Rightarrow \hat{\Delta}(\mathcal{M}, \otimes, \mathcal{I}, \hat{\Delta})
\]

In both cases we used the fact that \( H \) is a quasitriangular Hopf algebra and \( \hat{\Delta} \) is a braided Hopf algebra in the category \( (\mathcal{M}, \otimes, \mathcal{I}, \hat{\Delta}) \). Comparing the two results yields the statement and thus Theorem 2.5 is proved. \( \square \)

A straightforward consequence of Theorem 2.5 is the following corollary.

Corollary 2.6. In \( \hat{\mathfrak{h}} \times \hat{\mathfrak{h}} \) the Hopf algebra \( H \) is embedded through the Hopf algebra isomorphism \( H \cong \mathfrak{h} \otimes H \subset \hat{\mathfrak{h}} \times \hat{\mathfrak{h}} \), and \( \hat{\mathfrak{h}} \) considered as an algebra is embedded through the algebra isomorphism \( \hat{\mathfrak{h}} \cong \mathfrak{h} \otimes 1_H \subset \hat{\mathfrak{h}} \times \hat{\mathfrak{h}} H \). \( \square \)

Theorem 2.5 can be formulated for weaker conditions than those listed in Definition 2.4. Let \( (\mathcal{M}, \otimes, \mathcal{I}, \mathcal{K}, \Psi^d) \) be a braided monoidal category which contains \( H \) as a (quasitriangular) Hopf algebra, and let \( (\mathcal{M}, \otimes, \mathcal{I}, \mathcal{K}, \Psi^d) \) be a braided monoidal category of \( H \)-left modules in \( \mathcal{M} \) containing \( H \) as canonical module. Assume that there exist two further monoidal categories \( (\mathcal{M}, \otimes, \mathcal{I}, \mathcal{K}), \) and \( (\mathcal{M}, \otimes, \mathcal{I}, \mathcal{K}) \) which consists of \( H \)-left modules in \( \mathcal{M}, \) and two forgetful monoidal functors \( U \) and \( HU \) such that the diagramm

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{U} & \mathcal{M}^0 \\
V \downarrow & & \downarrow V^0 \\
\mathcal{M}^0 & \xrightarrow{HU} & \mathcal{M}^0
\end{array}
\]

(D-1)
is commutative, where the forgetful functors $V$ and $V^\circ$ are canonical. Suppose that for any $\mathcal{U}, \mathcal{V} \in \text{Ob}(\mathcal{M})$ a morphism $\hat{O} \mathcal{U} \mathcal{V} : \mathcal{U} \otimes \mathcal{V} \to \mathcal{U} \otimes \mathcal{V}$ in $\mathcal{M}^\circ$ exists and let the usual tensor transposition $\tau_{\mathcal{U} \mathcal{V}}$ be a morphism in $\mathcal{M}^\circ$ and the $\mathcal{R}$-matrix induced transposition $\Psi_{\mathcal{U} \mathcal{V}}$ according to eq. (2.6) be a morphism in $\mathcal{M}^\circ$ such that

i. $\tau_{\mathcal{U} \mathcal{V}}^d = \hat{O} \mathcal{V} \mathcal{U} \circ \tau_{\mathcal{U} \mathcal{V}}$ and $\Psi_{\mathcal{U} \mathcal{V}}^d = \hat{O} \mathcal{V} \mathcal{U} \circ \Psi_{\mathcal{U} \mathcal{V}}$.

ii. $\hat{O} \mathcal{U} \mathcal{V} = \text{id}_{\mathcal{U} \otimes \mathcal{V}}$ and $\hat{O} \mathcal{V} \mathcal{U} = \text{id}_{\mathcal{V} \otimes \mathcal{U}}$.

iii. $[(\text{id}_{\mathcal{U}} \otimes h \gg), \hat{O} \mathcal{V} \mathcal{U}] = 0$ and $[(h \ll \otimes \text{id}_{\mathcal{V}}), \hat{O} \mathcal{V} \mathcal{U}] = 0$.

Then a theorem analogous to Theorem 2.5 holds and a counterpart of Corollary 2.6 can be deduced similarly. An example for this construction which is not a braided $\hat{O}$-supersymmetry is provided by the category of complexes $\text{[Par, Ma1]}$. The category $\mathcal{M} := H$-$\text{Comp}_2$ with objects being complexes which have a canonical $\mathbb{Z}_2$-grading and with morphisms being $\mathbb{Z}_2$-graded homomorphisms of modules with differentiation, i.e. $d \circ f = f \circ d$, admits a braided symmetry

$$\tau_{\mathcal{U} \mathcal{V}}^d(u \otimes v) = (-1)^{\hat{u} \hat{v}} v \otimes u + \lambda (-1)^{(\hat{u} - 1) \hat{v}} \gamma_{\mathcal{V}}(v) \otimes \gamma_{\mathcal{U}}(u)$$

(2.11)

where $\mathcal{U}, \mathcal{V} \in \text{Ob}(\mathcal{M}), u \in \mathcal{U}, v \in \mathcal{V}$ are homogeneous elements of degree $\hat{u}, \hat{v} \in \mathbb{Z}$ respectively, $\lambda \in \mathcal{K}$ and $\gamma_{\mathcal{U}}, \gamma_{\mathcal{V}}$ are the corresponding derivatives. The tensor derivative is given through $d_{\mathcal{U} \otimes \mathcal{V}} = d_{\mathcal{U}} \otimes \text{id}_{\mathcal{V}} + \gamma_{\mathcal{U}} \otimes d_{\mathcal{V}}$ with grade indicating vector space homomorphism $\gamma_{\mathcal{U}}(u) := (-1)^{\hat{u}} u$ (see Proposition 3.1). In the category $H \mathcal{M} := H \mathcal{K}$-$\text{Comp}_2$ the objects being the $H$-left modules in $\mathcal{M}$ where the derivative $d$ is $H$-module morphism and with the action is $\mathbb{Z}_2$-graded, the morphisms in $H \mathcal{M}$ are $\mathbb{Z}_2$-graded $H$-module morphisms of modules with differentiation. There exists a braiding $\Psi_0^d$ in $H \mathcal{M}$.

$$\Psi_{\mathcal{U} \mathcal{V}}^d(u \otimes v) = \sum (-1)^{\hat{u} \hat{v}} \mathcal{R}_2 \gg v \otimes \mathcal{R}_1 \gg u + \lambda (-1)^{(\hat{u} - 1) \hat{v}} \mathcal{R}_2 \gg \gamma_{\mathcal{V}}(v) \otimes \mathcal{R}_1 \gg \gamma_{\mathcal{U}}(u)$$

(2.12)

where $\mathcal{U}, \mathcal{V} \in \text{Ob}(H \mathcal{M})$ and $u \in \mathcal{U}, v \in \mathcal{V}$ are homogeneous elements of degree $\hat{u}, \hat{v} \in \mathbb{Z}$ respectively. Let $\mathcal{M}^\circ$ be the category of $\mathbb{Z}_2$-graded $\mathcal{K}$-vector spaces and let $H \mathcal{M}^\circ$ be the category of $\mathbb{Z}_2$-graded $H$-left modules in $\mathcal{M}^\circ$. Then for all objects $\mathcal{U}, \mathcal{V} \in \mathcal{M}$ a morphism $\hat{O} \mathcal{U} \mathcal{V} : \mathcal{U} \otimes \mathcal{V} \to \mathcal{U} \otimes \mathcal{V}$ in $\mathcal{M}^\circ$ can be defined.

$$\hat{O} \mathcal{U} \mathcal{V}(u \otimes v) = (-1)^{\hat{u} \hat{v}} u \otimes v + \lambda (-1)^{(\hat{u} - 1) \hat{v}} \gamma_{\mathcal{U}}(u) \otimes \gamma_{\mathcal{V}}(v)$$

(2.13)

where $\mathcal{U}, \mathcal{V} \in \text{Ob}(\mathcal{M})$ and $u \in \mathcal{U}, v \in \mathcal{V}$ are homogeneous elements of degree $\hat{u}, \hat{v} \in \mathbb{Z}$ respectively. In this setting the above mentioned extension of Theorem 2.5 and of Corollary 2.6 applies. The notation is compatible. However there exists no braided $\hat{O}$-supersymmetry since the morphisms $\hat{O} \mathcal{U} \mathcal{V}$ in eq. (2.13) do not induce a functorial morphism $\hat{O} : \otimes \to \otimes$. Explicit examples of braided $\hat{O}$-supersymmetries are given at the end of Section 2.

For simplicity let in the following $(\mathcal{M}, \otimes, \mathcal{K}, \tau)$ be the monoidal category of $\mathcal{N}_0$-graded $\mathcal{K}$-vector spaces and $(H \mathcal{M}, \otimes, \mathcal{K}, \Psi_0$) be the monoidal category of $\mathcal{N}_0$-graded $H$-left modules in $\mathcal{M}$. The quasitriangular Hopf algebra $(H, \mathcal{R})$ is an object in $\mathcal{M}$ through the identification $H = H_0$. The $H$-module structure on $H$ is the canonical one. In these categories we consider $\hat{O}$-supersymmetries according to Definition 2.4.
Lemma 2.7. [Ma2, Wei] For any \( \mathcal{X} \in \text{Ob}(H\mathcal{M}) \) the tensor space \( \mathcal{X} \otimes \mathcal{X} \) is an \( H \)-left module in \( H\mathcal{M} \) and the tensor algebra \( T(\mathcal{X}) = \bigoplus_{n=0}^{\infty} \mathcal{X} \otimes \mathcal{X} \) is an algebra in the category \( H\mathcal{M} \) with the usual tensor multiplication \( m_T \), unit \( 1_T = 1 \in \mathcal{K} \) and module action according to

\[
h \triangleright 1_T := \varepsilon_H(h) 1_T,
\]

\[
h \triangleright (x_1 \otimes \ldots \otimes x_n) := \sum h_{(1)} \triangleright x_1 \otimes \ldots \otimes h_{(n)} \triangleright x_n, \tag{2.14}
\]

where \( h \in H, x_i \in \mathcal{X} \) \( \forall i \in \{1, \ldots, n\} \), \( x_n \in \mathcal{X} \otimes \mathcal{X} \) \( \forall n \in \{0, 1, 2, \ldots\} \).

For the functorial isomorphisms \( \hat{O} \) under consideration we suppose henceforth that for any \( \mathfrak{U} \in \text{Ob}(H\mathcal{M}) \) it holds

\[
\hat{O}_{T(\mathcal{X})} \mathfrak{U} = \bigoplus_{n=0}^{\infty} \hat{O}_{\mathcal{X} \otimes \mathcal{X}} \mathfrak{U}, \quad \hat{O}_{\mathfrak{U} T(\mathcal{X})} = \bigoplus_{n=0}^{\infty} \hat{O}_{\mathfrak{U} \otimes \mathcal{X}} \mathcal{X}^n. \tag{2.15}
\]

Then we obtain the following proposition which can be proved in the same way as the corresponding statement in [Ma2, Wei] exploiting strongly the braided symmetry properties of \( \hat{O}\Psi_0 \).

Proposition 2.8. \( T(\mathcal{X}) \) is a braided \( \hat{O} \)-super-Hopf algebra in the category \( (H\mathcal{M}, \otimes, \mathcal{K}, \hat{O}\Psi_0) \) with the algebra structure like in Lemma 2.7. Comultiplication, counit and antipode are defined for any \( x \in \mathcal{X} \) through

\[
\Delta_T(x) = x \otimes 1_T + 1_T \otimes x, \quad \varepsilon_T(x) = 0, \quad S_T(x) = -x \tag{2.16}
\]

and by multiplicative continuation according to

\[
\Delta_T(\mathfrak{v} \mathfrak{y}) = (\text{id}_T \otimes (\hat{O}\Psi_0)_T)(\Delta_T(\mathfrak{v}) \otimes \Delta_T(\mathfrak{y})),
\]

\[
\varepsilon_T(\mathfrak{v} \mathfrak{y}) = \varepsilon_T(\mathfrak{v}) \varepsilon_T(\mathfrak{y}),
\]

\[
S_T(\mathfrak{v} \mathfrak{y}) = m_T((\hat{O}\Psi_0)_T((S_T(\mathfrak{v}) \otimes S_T(\mathfrak{y}))),
\]

\[
\forall \mathfrak{v}, \mathfrak{y} \in T(\mathcal{X}). \tag{2.17}
\]

for the whole algebra \( T(\mathcal{X}) \).

Assume furthermore that \( (\hat{O}\Psi_0)_{\mathcal{X} \mathcal{X}} \) decomposes into projectors as follows.

\[
(\hat{O}\Psi_0)_{\mathcal{X} \mathcal{X}} = \sum_{i=1}^{n} \alpha_i P_i, \quad P_i P_j = \delta_{ij} P_j, \quad \sum_{i=1}^{n} P_i = \text{id}_{\mathcal{X} \otimes \mathcal{X}} \tag{2.18}
\]

where \( i, j \in \{1, \ldots, n\} \), \( \alpha_i \in \mathcal{K}, \alpha_i \neq \alpha_j \) for \( i \neq j \). Then each projector \( P_i \) is a morphism in \( H\mathcal{M} \). It is easy to see that for any \( i \in \{1, \ldots, n\} \) the ideal \( J_i \) generated by \( m_T P_i(\mathcal{X} \otimes \mathcal{X}) \) is an object in \( H\mathcal{M} \) which does not contain \( 1_T \) and is invariant under \( \hat{O}\Psi_0 \), i.e. \( (\hat{O}\Psi_0)_T(\mathfrak{V} \otimes J_i) \subset \mathfrak{V} \otimes J_i \) and \( (\hat{O}\Psi_0)_{\mathfrak{U} T}(\mathfrak{V} \otimes J_i) \subset J_i \otimes \mathfrak{V} \forall \mathfrak{U} \in \text{Ob}(H\mathcal{M}) \). Therefore

\[
\bar{T}^i := T(\mathcal{X})/J_i \quad \forall i \in \{1, \ldots, n\} \tag{2.19}
\]
is an $H$-module algebra in $\mathcal{H}M$ and $\hat{O}\Psi_0$ can be defined canonically for $\bar{T}^i$. Suppose now that

$$
(\hat{O}\Psi_0)_{T^i} = (\hat{O}\Psi_0)_{\bar{T}^i}, \quad (\hat{O}\Psi_0)_{\bar{T}^i} = (\hat{O}\Psi_0)_{\bar{T}^i}
$$

(2.20)

for any $i \in \{1, \ldots, n\}$ and any $\mathfrak{V} \in \text{Ob}(\mathcal{H}M)$. Then one finds like in [Ma2, Wei]

**Proposition 2.9.** If there exists an $i_0 \in \{1, \ldots, n\}$ such that $\alpha_{i_0} = -1$ in the projector decomposition (2.18), then $\bar{T}^{i_0}$ is a braided $\hat{O}$-super-Hopf algebra in the category $(\mathcal{H}M, \otimes, IK, \hat{O}\Psi_0)$. The structure maps of $\bar{T}^{i_0}$ are canonically induced by the corresponding maps of $T(X)$.

□

**Remark.** Through the assignment

$$
\mathfrak{U} \times \mathfrak{V} \mapsto \hat{O}^{(\alpha)}_{\mathfrak{U} \otimes \mathfrak{V}}, \quad \hat{O}^{(\alpha)}_{\mathfrak{U} \otimes \mathfrak{V}} : \left\{ \begin{array}{l}
\mathfrak{U} \otimes \mathfrak{V} \to \mathfrak{U} \otimes \mathfrak{V} \\
u_i \otimes v_j \mapsto (-\alpha)^{ij} u_i \otimes v_j
\end{array} \right.
$$

(2.21)

where $\alpha \in IK \setminus \{0\}$, $\mathfrak{U}, \mathfrak{V} \in \text{Ob}(\mathcal{H}M)$, $u_i \in \mathfrak{U}_i$, $v_j \in \mathfrak{V}_j$, $i, j \in \mathbb{N}_0$, a functorial isomorphism $\hat{O}^{(\alpha)} : \otimes \cong \otimes$ is defined on $(\mathcal{H}M, \otimes, IK)$ which is nice enough to fulfill all the properties supposed in this section†. Thus if one identifies an arbitrary $H$-left module $X$ with $X \cong X_1$ then $X \in \text{Ob}(\mathcal{H}M)$ and

$$
(\hat{O}^{(\alpha)}\Psi_0)_{XX} = (-\alpha) \Psi_0_{XX}
$$

(2.22)

which simply rescales the braiding. In the context of [Ma5, SWW, Wei] rescaling yields a new rescaling generator to obtain an ordinary Hopf algebra while in the context of $\hat{O}$-supersymmetry rescaling yields $\hat{O}$-super-Hopf structures. This fact will be used in Section 3 in the case of “classical” supersymmetry, i.e. where $\alpha = 1$, to obtain the desired results (see also [SZ]). When we speak henceforth of “braided super-...” without any further indication, we are working with braided $\hat{O}^{(1)}$-supersymmetry.

3. . . . and (Co-)Homology

The results of Section 2 are now applied to the complex $(K(q, g), d)$ which is a deformation of the Koszul complex of Lie algebras $g$ associated with the root systems $A_n$, $B_n$, $C_n$ or $D_n$. Here we suppose $q \neq \text{"root of unity"}$. We begin by recalling the most important results of [DSO]. It is known that the adjoint representation $\text{ad} : U_q(g) \otimes \mathfrak{X} \to \mathfrak{X}$ of the quantum enveloping algebra $U_q(g)$ in the $\mathfrak{G}$-vector space $\mathfrak{X}$ of $U_q(g)$-generating vector fields is defined through

$$
\text{ad}_u(x) = \sum u_{(1)} x S_u(u_{(2)}) \quad \forall x \in \mathfrak{X}, u \in U_q(g)
$$

(3.1)

† This functorial isomorphism also appears in [Ko1, Ma4].
where $S_U$ is the antipode of $U_q(\mathfrak{g})$. The vector space $\mathfrak{X}$ corresponds to the right invariant vector fields associated with a certain bico\-variant differential calculus on quantized simple Lie groups, such that $\mathfrak{X}$ generates $U_q(\mathfrak{g})$ [CSWW, DJSWZ, Jur, Wor]. A $\mathfrak{C}$-basis in $\mathfrak{X}$ is given through [DSO]

\[ X^a_b = \delta^a_b \mathbf{1}_U - \sum_{k=1}^{N} L^+_k \mathcal{U}_U(L^-_k) \]  

where $N = n + 1$ for $A_n$, $N = 2n + 1$ for $B_n$ and $N = 2n$ for $C_n$ and $D_n$, $a, b \in \{1, \ldots, N\}$, and $(L^\pm s)_{r, s=1, \ldots, N}$ are the regular functionals of the corresponding quantum group [FRT]. The space $\mathfrak{X}$ is dual to the space $\Gamma_{inv}$ of right invariant one-forms [Wor] with basis $\{\eta^a_b | a, b \in \{1, \ldots, N\}\}$ and with left adjoint coaction $\Phi_* : \Gamma_{inv} \to A_q \otimes \Gamma_{inv}$, where $A_q$ is the quantum group dual to $U_q(\mathfrak{g})$. It holds [Jur, Wor]

\[ \langle X^a_b, \eta^d_c \rangle = \delta^a_d \delta^{cb}_e, \quad \Phi_*(\eta), u \otimes x = \langle \eta, \text{ad}_u(x) \rangle, \quad \Phi_*(\eta), s_t = \sum_{t, u} S_A(t^u_r) t^s_t \otimes \eta^t_u \]  

where $\eta \in \Gamma_{inv}$ and $u \otimes x \in U_q(\mathfrak{g}) \otimes \mathfrak{X}$. The action of $\mathfrak{X}$ on $\mathfrak{X}$ can also be written as a deformed Lie bracket [Wor].

\[ \text{ad}_x(y) = xy - m_U \sigma(x \otimes y) \]  

where $x, y \in \mathfrak{X} \subset U_q(\mathfrak{g})$, $m_U$ is the multiplication in $U_q(\mathfrak{g})$ and $\sigma : \mathfrak{X} \otimes \mathfrak{X} \to \mathfrak{X} \otimes \mathfrak{X}$ is a linear transformation with very specific properties [DSO, Jur, Wor]. For the vector fields under consideration, $\sigma$ can be written as a projector decomposition

\[ \sigma = \sum_{\rho \in R} \rho P_\rho, \quad P_\rho P_\rho' = \delta_{\rho \rho'} P_\rho \quad \text{and} \quad \sum_{\rho \in R} P_\rho = \text{id}_{\mathfrak{X} \otimes \mathfrak{X}}. \]  

where $\rho, \rho' \in R$. The set $R$ consists of complex numbers and it contains 1. The space $\Lambda(\mathfrak{X})$ is given by

\[ \Lambda(\mathfrak{X}) = T(\mathfrak{X})/(m_T P_1(\mathfrak{X} \otimes \mathfrak{X})) = \bigoplus_{j=0}^{\infty} \Lambda_j(\mathfrak{X}). \]  

It becomes an $\mathbb{N}_0$-graded and thus $\mathbb{Z}_2$-graded $U_q(\mathfrak{g})$-module algebra through the action $\text{ad}^\wedge$ which is induced by $\text{ad}$ and therefore

\[ K(q, \mathfrak{g}) := \Lambda(\mathfrak{X}) \text{ ad}^\wedge \otimes U_q(\mathfrak{g}) \]  

is an $\mathbb{N}_0$-graded algebra which contains $\Lambda(\mathfrak{X})$ and $U_q(\mathfrak{g})$ as subalgebras [DSO]. Both in $\Lambda(\mathfrak{X})$ and in $U_q(\mathfrak{g})$ the vector fields $\mathfrak{X}$ are contained. For distinction it is written $\mathfrak{X} \subset \Lambda(\mathfrak{X})$ and $\tilde{\mathfrak{X}} \subset U_q(\mathfrak{g})$. On $K(q, \mathfrak{g}) = \bigoplus_{j=0}^{\infty} \Lambda_j(\mathfrak{X}) U_q(\mathfrak{g}) = \bigoplus_{j=0}^{\infty} K_j(q, \mathfrak{g})$ a grade indicating algebra isomorphism $\gamma : K(q, \mathfrak{g}) \to K(q, \mathfrak{g})$ exists such that $\gamma(a_j) = (-1)^j a_j \forall a_j \in K_j(q, \mathfrak{g})$. The central theorem of [DSO] states that on $K(q, \mathfrak{g})$ a derivative $d$ with the following properties can be uniquely defined.

\[ d \text{ is } U_q(\mathfrak{g})-\text{module morphism}, \]
\[ d(x) = \tilde{x} \in U_q(\mathfrak{g}) \quad \forall \ x \in \Lambda(\mathfrak{X}), \]
\[ d(k l) = d(k) l + \gamma(k) d(l) \quad \forall \ k, l \in K(q, \mathfrak{g}). \]
From this it follows that $d^2 = 0$ and $d\gamma + \gamma d = 0$. The complex $(K(q, g), d)$ is a deformation of the Koszul complex of the Lie algebras $g$ associated with the root systems $A_n, B_n, C_n$ and $D_n$. We also need a chain complex structure on the tensor product $K^\otimes(q, g) := K(q, g) \otimes K(q, g) = \bigoplus_{j=0}^\infty K^\otimes_j(q, g)$ where $K^\otimes_j(q, g) = \bigoplus_{n+m=j} K^\otimes_n(q, g) \otimes K^\otimes_m(q, g)$. This is guaranteed by the next proposition.

**Proposition 3.1.** The tensor space $K^\otimes(q, g)$ is a super complex algebra with multiplication map $m^\otimes = (m_K \otimes m_K) \circ (id_K \otimes (\hat{\mathcal{O}}^{(1)} \tau)_{KK} \otimes id_K)$, with grade indicating algebra isomorphism $\Gamma = \gamma \otimes \gamma$, and with derivative $D = (d \otimes id_K) + (\gamma \otimes d)$ such that

$$D(K^\otimes_j(q, g)) \subset K^\otimes_{j-1}(q, g),$$

$$D(u v) = D(u) v + \Gamma(u) D(v),$$

$$D^2 = 0,$$

$$D\Gamma + \Gamma D = 0$$

where $j \in \mathbb{N}_0$, $u, v \in K^\otimes(q, g)$ and $u_j \in K^\otimes_j(q, g)$.

**Proof.** All the statements can be checked directly on homogeneous elements using the properties of $d$ and $\gamma$. After linear continuation the statements follow. □

The quantum enveloping algebra $U_q(g)$ is a quasitriangular Hopf algebra with $\mathcal{R}$-matrix $\mathcal{R}$ [Dri, Ma3, Res]. Similar as in [SZ] a representation of the quantum double $D(U_q(g))$ of $U_q(g)$ [CEJSZ, Dri] in the space $\Lambda(\mathfrak{X})$ will be constructed. Here the $\mathcal{R}$-matrix

$$\mathcal{R}^\otimes = \sum \mathcal{R}^{-1}_{23} \otimes \mathcal{R}_{12} \otimes \Delta_U(\mathcal{R}^{-1}_{12} \mathcal{R}_{23})$$

will be used which makes $D(U_q(g))$ a quasitriangular Hopf algebra. The structure maps are defined through

$$m^\otimes = (m_U \otimes m_U) \circ (id_U \otimes \tau_{UU} \otimes id_U),$$

$$\eta^\otimes = \eta_U \otimes \eta_U,$$

$$\Delta^\otimes = \mathcal{R}^{-1}_{23} (id_U \otimes \tau_{UU} \otimes id_U) \circ (\Delta_U \otimes \Delta_U)(.),$$

$$\varepsilon^\otimes = \varepsilon_U \otimes \varepsilon_U,$$

$$S^\otimes = \mathcal{R}^{-1}_{21} (S_{UU} \otimes S_{UU})(.) \mathcal{R}^{-1}_{21}$$

with obvious notation [FRT]. We consider the mapping

$$\Phi^\otimes: \{ \Gamma_{inv} \rightarrow D(A_q) \otimes \Gamma_{inv} \}
\eta^s \rightarrow \sum_{t, u} S_A(t^* u^*) \otimes t^* \otimes \eta^t,$$

(3.12)

\[\text{The matrix } \mathcal{R} \text{ is an element of a certain completion of the tensor product of } U_q(g) \text{ [Dri, Ma3, Res]. In this more general setting of quasitriangular Hopf algebras the notations and results especially of Section 2 remain unchanged.}\]
where $D(A_q)$ is the dual quantum double of $D(U_q(\mathfrak{g}))$ [CEJSZ, Dri]. This defines a left coaction of $D(A_q)$ on $\Gamma_{inv}$ [FRT, Ko2] since

$$\Delta^\circ(S_A(t^u_r) \otimes t^s_t) = \sum_{k,l} S_A(t^k_r) \otimes t^l_t \otimes S_A(t^u_k) \otimes t^s_t,$$

$$\varepsilon^\circ(S_A(t^u_r) \otimes t^s_t) = \delta^u_r \delta^s_t$$

where $\Delta^\circ$ and $\varepsilon^\circ$ are the (dual) comultiplication and counit respectively of $D(A_q)$ according to Theorem 2 in [CEJSZ]. Since the spaces $\mathfrak{X}$ and $\Gamma_{inv}$ are dual (and finite dimensional) the map $\Phi^\circ_{\Gamma}$ in eq. (3.12) induces a left action of the quasitriangular Hopf algebra $(D(U_q(\mathfrak{g})), R^\circ)$ on $\mathfrak{X}$.

$$\text{ad}^\circ := (\cdot) \circ \Phi^\circ_{\Gamma} : D(U_q(\mathfrak{g})) \otimes \mathfrak{X} \to \mathfrak{X}.$$ (3.14)

From the definition it is obvious that

$$\text{ad} = \text{ad}^\circ \circ (\Delta \otimes \text{id}_\mathfrak{X}).$$ (3.15)

To calculate the braiding $\Psi_{0,\mathfrak{X}}$ (w.r.t. $D(U_q(\mathfrak{g}))$) one observes that

$$< \Psi_{0,\mathfrak{X}}(x \otimes y), \eta \otimes \zeta > = < x \otimes y, (R^\circ \otimes \text{id}_\Gamma \otimes \text{id}_\Gamma)(\Phi^\circ_{\Gamma} \otimes \Phi^\circ_{\Gamma})(\eta \otimes \zeta) > = < x \otimes y, \Psi^t_{0,\mathfrak{X}}(\eta \otimes \zeta) >.$$ (3.16)

where $x \otimes y \in \mathfrak{X} \otimes \mathfrak{X}$ and $\eta \otimes \zeta \in \Gamma_{inv} \otimes \Gamma_{inv}$. For the basis elements $\eta_{b}^a \otimes \eta_{c}^d \in \Gamma_{inv} \otimes \Gamma_{inv}$ one obtains

$$\Psi^t_{0,\mathfrak{X}}(\eta_{b}^a \otimes \eta_{c}^d) = \sum_{t,u,v,w} L^+ U^c S_A(L^{-d}_t) (S_A(t^w_a)t^b_t) \cdot \eta_t^u \otimes \eta_w^v$$ (3.17)

and comparing with [DSO] yields

$$\Psi_{0,\mathfrak{X}} = \sigma.$$ (3.18)

The results of Section 2 can now be applied. Through the identification $\mathfrak{X} \cong \mathfrak{X}_1$ and for $\alpha = 1$ in (2.21) one finds that

$$(\hat{O}^{(1)}(\Psi_0 ))_{\mathfrak{X}, \mathfrak{X}} = -\Psi_{0,\mathfrak{X}} = -\sigma$$ (3.19)

in the category of $\mathbb{Z}_2$-graded $D(U_q(\mathfrak{g}))$-left modules. Thus according to Proposition 2.9 and eqs. (3.6) and (3.19) the space $\Lambda(\mathfrak{X})$ is a braided super-Hopf algebra in the category $(D(U_q(\mathfrak{g})), \mathcal{M}, \otimes, K, \hat{O}^{(1)}(\Psi)_0)$ of $\mathbb{Z}_2$-graded $D(U_q(\mathfrak{g}))$-left modules. Theorem 2.5 then yields the following corollary.

**Corollary 3.2.** $\Lambda(\mathfrak{X}) \rtimes \hat{O}^{(1)}(D(U_q(\mathfrak{g})))$ is a super-Hopf algebra with structure maps defined through the eqs. (2.7).

For further argumentation we need
Lemma 3.3. The comultiplication $\Delta_U: U_q(\mathfrak{g}) \to D(U_q(\mathfrak{g}))$ is an injective Hopf algebra homomorphism.

Proof. Since $U_q(\mathfrak{g})$ is a Hopf algebra, $\Delta_U$ is an injective $C$-vector space homomorphism. With the help of (3.11) it is straightforward to verify that

\[
\begin{align*}
\Delta_U \circ m_U &= m^\circ \circ (\Delta_U \otimes \Delta_U), \\
\Delta_U \circ \eta_U &= \eta^\circ, \\
(\Delta_U \otimes \Delta_U) \circ \Delta_U &= \Delta^\circ \circ \Delta_U, \\
\varepsilon_U &= \varepsilon^\circ \circ \Delta_U, \\
\Delta_U \circ S_U &= S^\circ \circ \Delta_U.
\end{align*}
\]

(3.20)

This concludes the proof. $\square$

The injective linear mapping $\phi := (\text{id}_\Lambda \otimes \Delta_U): \Lambda(\mathfrak{X}) \otimes U_q(\mathfrak{g}) \to \Lambda(\mathfrak{X}) \otimes D(U_q(\mathfrak{g}))$ enables us to formulate

Theorem 3.4. $\phi(\Lambda(\mathfrak{X}) \otimes U_q(\mathfrak{g}))$ is a super-Hopf subalgebra of $\Lambda(\mathfrak{X}) \rtimes \hat{\phi}(1) D(U_q(\mathfrak{g}))$ and thus $\Lambda(\mathfrak{X}) \otimes U_q(\mathfrak{g})$ becomes a super-Hopf algebra through the mappings

\[
\begin{align*}
m_K := \phi^{-1} \circ \hat{m} \circ (\phi \otimes \phi), \\
\eta_K := \phi^{-1} \circ \hat{\eta}, \\
\Delta_K := (\phi^{-1} \otimes \phi^{-1}) \circ \hat{\Delta} \circ \phi, \\
\varepsilon_K := \hat{\varepsilon} \circ \phi, \\
S_K := \phi^{-1} \circ \hat{S} \circ \phi.
\end{align*}
\]

(3.21)

The algebra $(\Lambda(\mathfrak{X}) \otimes U_q(\mathfrak{g}), m_K, \eta_K)$ coincides with the algebra $K(q, \mathfrak{g})$. Therefore the space $(K(q, \mathfrak{g}), \Delta_K, \varepsilon_K, S_K)$ is a super-Hopf algebra.

Proof. $L := \phi(\Lambda(\mathfrak{X}) \otimes U_q(\mathfrak{g}))$ is $\mathbb{N}_0$-graded w.r.t. the grading in $\Lambda(\mathfrak{X}) = \bigoplus_{j=0}^\infty \Lambda_j(\mathfrak{X})$. If one can show that $\hat{m}(L \otimes L) \subset L$, $\hat{\Delta}(L) \subset L \otimes L$, $\hat{S}(L) \subset L$ the proof is done. Some calculation yields

\[
\begin{align*}
\hat{m}((\lambda \otimes u) \otimes (\mu \otimes v)) &= \sum \lambda \cdot (u(1) \triangleright \mu) \otimes \Delta_U(u(2)v), \\
\hat{\Delta}(\lambda \otimes u) &= \sum \lambda(1) \otimes \Delta_U(R_1^{-1}R_2 u(1)) \otimes (R_2^{-1} \otimes R_1) \triangleright \lambda(2) \otimes \Delta_U(u(2)), \\
\hat{S}(\lambda \otimes u) &= \sum (\Delta_U S_U(R_1^{-1}R_2 u(1)) (R_2^{-1} \otimes R_1) \triangleright S\lambda(\lambda) \otimes \Delta_U S_U(R_1^{-1}R_2 u(2))
\end{align*}
\]

(3.22)

where we used the quasitriangularity of $(D(U_q(\mathfrak{g})), R^c)$, and eqs. (2.7) and (3.10). In the above relations $\triangleright$ is the action $\text{ad}^\wedge: U_q(\mathfrak{g}) \otimes \Lambda(\mathfrak{X}) \to \Lambda(\mathfrak{X})$ and $\triangleright$ is the action $\text{ad}^\wedge: D(U_q(\mathfrak{g})) \otimes \Lambda(\mathfrak{X}) \to \Lambda(\mathfrak{X})$ which are induced from $\text{ad}$ and $\text{ad}^\wedge$ respectively. $\square$
In the last part of this section we will show that $m_K, \eta_K, \Delta_K, \varepsilon_K$, and $S_K$ from the definition in (3.21) respect the chain complex structure induced by the derivation $d$.

**Theorem 3.5.** If the tensor product $K^\otimes(q, g) = K(q, g) \otimes K(q, g)$ is supplied with the chain complex structure according to Proposition 3.1 then the mappings

$$
\begin{align*}
m_K &: (K^\otimes(q, g), D) \to (K(q, g), d), \\
\eta_K &: (\mathcal{C}, 0) \to (K(q, g), d), \\
\Delta_K &: (K(q, g), d) \to (K^\otimes(q, g), D), \\
\varepsilon_K &: (K(q, g), d) \to (\mathcal{C}, 0), \\
S_K &: (K(q, g), d) \to (K(q, g), d)
\end{align*}
$$

(3.23)

are $\mathbb{Z}_2$-graded morphisms of complexes with differentiation [CE], i.e. the diagrams

$$
\begin{array}{c}
\xymatrix{ K(q, g) \ar[d]^{d} \ar[r]^{\varepsilon_K} & \mathcal{C} \ar[d]^{0} \ar[r]^{\eta_K} & K(q, g) \ar[d]^{d} & K(q, g) \ar[d]^{d} \ar[r]^{S_K} & K(q, g) \\
K(q, g) \ar[r]^{\varepsilon_K} & \mathcal{C} \ar[r]^{\eta_K} & K(q, g) & K(q, g) \ar[r]^{S_K} & K(q, g) \}
\end{array}
$$

(D-2)

$$
\begin{array}{c}
\xymatrix{ K^\otimes(q, g) \ar[d]^{D} \ar[r]^{m_K} & K(q, g) \ar[d]^{d} & K(q, g) \ar[d]^{d} \ar[r]^{\Delta_K} & K^\otimes(q, g) \ar[d]^{D} \\
K^\otimes(q, g) \ar[r]^{m_K} & K(q, g) & K(q, g) \ar[r]^{\Delta_K} & K^\otimes(q, g) 
\end{array}
$$

and the corresponding diagrams where $d, D$ and 0 are replaced by $\gamma$, $\Gamma$ and $\text{id}\mathcal{C}$ respectively, are commutative.

**Proof.** The several proofs are either very similar or very simple. We only sketch the proof of the identity $S_K \circ d = d \circ S_K$. Let $X^l_k \in \mathfrak{K}$ be a basis element of $\mathfrak{K} \subset \Lambda(\mathfrak{K})$. Then using the notation of [DSO] one obtains $S_K d(X^l_k) = S_U(X^l_k) \in U_q(g)$. On the other side one gets $d S_K(X^l_k) = -\sum_{a, b} S_U(\Theta^{l_a b_k}) \bar{X}^b_a$ where $\Theta^{l_a b_k} = L^{l_a} b_k S_U(L^{-a}_{-k})$. Inserting the definition of $\bar{X}^b_a$ according to eq. (3.2) yields $d S_K(X^l_k) = S_U(\bar{X}^l_k)$. For $u \in U_q(g) \subset K(q, g)$ the identity $d S_K(u) = S_K d(u)$ is trivially valid. Now let $a, b \in K(q, g)$ be homogeneous elements of degree $\hat{a}$ and $\hat{b}$ respectively which obey the relations $d S_K(a) = S_K d(a)$ and $d S_K(b) = S_K d(b)$. Then $S_K d(a b) = (-1)^{(\hat{a} - 1) \hat{b}} S_K(b) d S_K(a) + (-1)^{\hat{a} \hat{b}} d S_K(b) S_K(a) = d S_K(a b)$ where the fact has been used that $K(q, g)$ is a super-Hopf algebra and that $d$ decreases the degree by 1. Since $\mathfrak{K}$ and $U_q(g)$ generate $K(q, g)$ it follows that $S_K \circ d = d \circ S_K$. \(\square\)
References

[CE] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton Mathematical Series, Vol. 19, Princeton (1956).

[CEJSZ] Chryssomalakos, C., Engelking, R., Jurčo, B., Schlieker, M., Zumino, B.: Complex Quantum Enveloping Algebras as Twisted Tensor Products. Preprint LMU-TPW 93-2 (1993).

[CSWW] Carow-Watamura, U., Schlieker, M., Watamura, S., Weich, W.: Bicovariant differential calculus on quantum groups $SU_q(N)$ and $SO_q(N)$. Commun. Math. Phys. 142, 605 (1991).

[DJSWZ] Drabant, B., Jurčo, B., Schlieker, M., Weich, W., Zumino, B.: The Hopf Algebra of Vector Fields on Complex Quantum Groups. Lett. Math. Phys. 26, 91 (1992).

[Dri] Drinfel’d, V. G.: Quantum groups. Proceedings of the International Congress of Mathematicians, Berkeley 1986, 798 (1986).

[DSO] Drabant, B., Schlieker, M., Ogievetsky, O.: Cohomology of Quantum Enveloping Algebras. Preprint MPI-Ph/93-57, LMU-TPW 1993-19 (1993). Submitted to Commun. Math. Phys.

[FRT] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie Groups and Lie Algebras. Leningrad Math. J. 1, 193 (1990).

[JS] Joyal, A., Street, R.: Braided Monoidal Categories. Mathematics Reports 86008, Macquarie Univ. (1986).

[Jur] Jurčo, B.: Differential Calculus on Quantized Simple Lie Groups. Lett. Math. Phys. 22, 177 (1991). Lett. Math. Phys. 22, 177 (1991).

[Ko1] Koornwinder, T.H.: Orthogonal Polynomials in Connection with Quantum Groups. NATO ASI Series C 294, P. Nevali (ed.), Kluver Acad. Publishers (1990).

[Ko2] Koornwinder, T.H.: General Compact Quantum Groups, a Tutorial. Preprint FWI-Report 94-06 (1994). To appear in “Representations of Lie Groups and quantum Groups”, M. Picardello (ed.), Longman.

Dijkhuiizen, M.S., Koornwinder, T.H.: CQG Algebras: A direct Algebraic Approach to Compact Quantum Groups. Preprint CWI-Report AM-R 9401 (1994). To appear in Lett. Math. Phys.

[Mac] Mac Lane, S.: Categories for the Working Mathematician. Graduate Texts in Mathematics 5, Springer (1972).

[Man] Manin, Yu.I.: Topics in Noncommutative Geometry. Princeton Univ. Press, Princeton (1991).

[Ma1] Majid, S.: Braided Groups. J. Pure Appl. Algebra 86, 187 (1993).

[Ma2] Majid, S.: Cross Products by Braided Groups and Bosonization. J. Algebra 163, 165 (1994).

[Ma3] Majid, S.: Quasitriangular Hopf Algebras and Yang-Baxter Equations. Int. J. Mod. Phys. A5 Vol. 1, 1 (1990).

[Ma4] Majid, S.: Quantum and Braided Linear Algebra. J. Math. Phys. 34, 1176 (1993).
[Ma5] Majid, S.: Braided Momentum in the $q$-Poincaré Group. J. Math. Phys. 34, 2045 (1993).

[Par] Pareigis, B.: A Non-Commutative Non-Cocommutative Hopf Algebra in “Nature”. J. Algebra 70, 356 (1981).

[Res] Reshetikhin, N.Yu.: Quantized Universal Enveloping Algebras, the Yang-Baxter Equation and Invariants of Links I. Preprint LOMI E–4–87 (1987).

[SWW] Schlieker, M., Weich, W., Weixler, R.O.: Inhomogeneous Quantum Groups. Z. Phys. C 53, 79 (1992).

[SWZ] Schupp, P., Watts, P., Zumino, B.: Bicovariant Quantum Algebras and Quantum Lie Algebras. Commun. Math. Phys. 157, 305 (1993).

[SZ] Schlieker, M., Zumino, B.: Braided Hopf Algebras and Differential Calculus. Preprint LBL-35299, UCB-PTH-94/03 (1994).

[Wei] Weixler, R.O.: Inhomogene Quantengruppen. Thesis. Shaker, Aachen (1994).

[Wor] Woronowicz, S.L.: Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups). Commun. Math. Phys. 122, 125 (1989).