CATEGORIES OF COMODULES AND CHAIN COMPLEXES OF MODULES

A. ARDIZZONI, L. EL KAOUTIT, AND C. MENINI

Abstract. Let $\mathcal{L}(A)$ denote the coendomorphism left $R$-bialgebroid associated to a left finitely generated and projective extension of rings $R \rightarrow A$ with identities. We show that the category of left comodules over an epimorphic image of $\mathcal{L}(A)$ is equivalent to the category of chain complexes of left $R$-modules. This equivalence is monoidal whenever $R$ is commutative and $A$ is an $R$-algebra. This is a generalization, using entirely new tools, of results by B. Pareigis and D. Tambara for chain complexes of vector spaces over fields. Our approach relies heavily on the non commutative theory of Tannaka reconstruction, and the generalized faithfully flat descent for small additive categories, or rings with enough orthogonal idempotents.

1. Introduction

1.1. Methodology and motivation overviews. The starting point of this paper is a result due to B. Pareigis [24, Theorem 18] which asserts that the category of unbounded complexes of vector spaces is monoidally equivalent to the category of left comodules over a certain Hopf algebra which is neither commutative nor cocommutative. Later on, in [28, Theorem 4.4], D. Tambara associated to every finite dimensional algebra $A$ over a field $k$, a bialgebra $\mathcal{L}(A)$ (termed coendomorphism bialgebra) such that the category of left comodules $\mathcal{L}(A)\text{Comod}$ is monoidally equivalent to the category $\text{Ch}_+(k)$ of chain complexes of $k$-vector spaces. The Hopf algebra considered by B. Pareigis is recovered by choosing $A = k \oplus kt$ with $t^2 = 0$, i.e. the trivial extension of $k$, and localizing the bialgebra $\mathcal{L}(A)$ using a multiplicative set generated by a single grouplike element. The equivalence of categories established by Tambara relies on the use of a variant of the equivalence between simplicial $k$-vector spaces and chain complexes of $k$-vector spaces, provided by the normalization functor, due to Dold and Kan, see [10, Theorem 1.9, Corollary 1.12] or [20, Theorem 2.4]. The functor that provides such equivalence is given, in some sense, by tensoring chain complexes with the augmented cochain complex $Q_{\bullet}$ constructed using the Amitsur cosimplicial vector space attached to the $k$-algebra $A$. Note that $Q_{\bullet}$ is the universal differential graded $k$-algebra of $A$, given by $Q_0 = k$, $Q_1 = A$ and $Q_n = K \otimes_A \cdots \otimes_A K$, $(n-1)$-times for $n \geq 2$, where $K$ is the kernel of the multiplication of $A$. The construction of this functor will be clarified in Section 3 see also the forthcoming subsection. A different approach to Pareigis’s result, using Tannaka reconstruction for several-objects coalgebras, was also given by Paddy McCrudden in [23, Examples 6.6, 6.9], where the same coendomorphism bialgebra $\mathcal{L}(A)$ was constructed for a commutative base ring $k$ instate of a field.

Let $R$ be an algebra over a commutative ring $k$. The purpose of this paper is to investigate the relationship between the category of left comodules over certain left $R$-bialgebroids, termed coendomorphism bialgebroids coming from the $\times_R$-bialgebra defined in [28, Remark 1.7], and the category...
of chain complexes of left $R$-modules. Tambara’s results, and in particular Pareigis’s one, are then immediate consequences of the general theory developed here. It is noteworthy that our methods can be seen as new and more conceptual even for the case of vector spaces. Indeed, we will see why concretely the trivial extension of rings, already considered by B. Pareigis, induces the above equivalence of categories. Our approach makes use of the ”non commutative” Tannakian categories theory following the spirit of [8, 6] and [15], as well as of the generalized faithfully flat descent for rings with enough orthogonal idempotents stated in [13]. We mean that all (left) bialgebroids arising here come in fact from the non commutative version of Tannaka reconstruction process which in our approach involves rings with enough orthogonal idempotents.

In the setting of non commutative Tannakian categories, one basically starts with a small $k$-linear monoidal category $(\mathcal{A}, \otimes, 1)$ and a faithful monoidal functor $\omega : \mathcal{A} \to R\text{-Mod}_R$ (the fiber functor), valued in the category $\text{add}(R)\text{-mod}$ of finitely generated and projective left $R$-modules (i.e. locally free sheaves of finite rank). There are several objects under consideration:

$$\Sigma(\omega) = \bigoplus_{p \in \mathcal{A}} \omega(p), \quad \check{\Sigma}(\omega) = \bigoplus_{p \in \mathcal{A}} \check{\omega}(p), \quad \mathcal{G}(\mathcal{A}) = \bigoplus_{p, p' \in \mathcal{A}} \text{Hom}_{\mathcal{A}^{op}}(p, p').$$

Here the second is the right $R$-module direct sum of the left duals while the third is Gabriel’s ring with enough orthogonal idempotents, introduced in [17], attached to the opposite category $\mathcal{A}^{op}$ of $\mathcal{A}$. Using the canonical actions, we consider $\mathcal{L}(\omega) : = \Sigma(\omega) \otimes_{\mathcal{G}(\mathcal{A})} \check{\Sigma}(\omega)$ as an $R^{e} \text{-mod}$-bimodule, where $R^{e} : = R \otimes_k R^{\circ}$ denotes the enveloping ring. A well known argument in small additive categories says that the object $\mathcal{L}(\omega)$ solves the following universal problems in $R\text{-bi}$$

$$\text{Nat} \left(\omega, - \otimes_R \omega\right) \cong \text{Hom}_{R-R}(\mathcal{L}(\omega), -),$$

$$\text{Nat} \left(\omega \otimes_R \omega, - \otimes_R (\omega \otimes_R \omega)\right) \cong \text{Hom}_{R-R}(\mathcal{L}(\omega) \otimes_{R^{e}} \mathcal{L}(\omega), -),$$

where the $R$-bimodule structures of $\mathcal{L}(\omega)$ have been chosen properly. It is indeed this solution which allows us to construct a left $R$-bialgebroid (or a Hopf bialgebroid if desired). Of course there is an obvious (monoidal) functor connecting left unital $\mathcal{G}(\mathcal{A})$-modules and left $\mathcal{L}(\omega)$-comodule, namely

$$\Sigma(\omega) \otimes_{\mathcal{G}(\mathcal{A})} - : \mathcal{G}(\mathcal{A})\text{-Mod} \longrightarrow \mathcal{L}(\omega)\text{-Comod.}$$

In the case when each of the left $R$-modules $\omega(p)$ is endowed with a structure of left $\mathcal{C}$-comodule for some $R$-coring $\mathcal{C}$ (or certain left $R$-bialgebroid), there is a map of $R$-corings, known as a canonical map,$$
can_{\mathcal{G}(\mathcal{A})} : \mathcal{L}(\omega) \longrightarrow \mathcal{C}
$$
defined by using the left $\mathcal{C}$-coaction of the $\omega(p)$’s. This homomorphism of corings is not in general bijective, see [13] for more discussions. The associated coinduction functor of the canonical map leads to the following composition of functors

$$\mathcal{G}(\mathcal{A})\text{-Mod} \xrightarrow{\Sigma(\omega) \otimes_{\mathcal{G}(\mathcal{A})} -} \mathcal{L}(\omega)\text{-Comod} \xrightarrow{(-)\text{can}_{\mathcal{G}(\mathcal{A})}} \mathcal{C}\text{-Comod.}$$

Indeed this is a conceptual framework that allows us to compare certain categories of $k$-linear functors with the categories of comodules over some corings (or leftbialgebroids). For instance, take $R = k$ to be a field and $A$ a finite dimensional $k$-algebra. Consider the associated cochain complex $Q_{\bullet}$ mentioned above and the monoidal $k$-linear category $k\langle N \rangle$ generated by the natural number $N$. There is a fiber functor $\chi : k\langle N \rangle \to \text{Mod}_k$ defined by $\chi(n) = Q_n$ on objects and sending the morphism

\footnote{Our setting requires an isomorphism only at the level of unit. That is, $R \cong \omega(1)$, while $\omega(- \otimes -) \to \omega(-) \otimes_R \omega(-)$ is not necessarily a natural isomorphism.}
Then the following statements are equivalent
\(\text{(3)}\) of equation
\[\text{Consider the associated left } R\text{-module } Q,\]
and then show that the restriction of this functor to the category of left \(R\)-comodules whose components are finitely generated and projective left \(A\)-module maps (i.e. the category of monoids in \(\mathbf{Mod}_R\)) admits a left adjoint which we denoted by \(\mathcal{L} : \mathbf{Rings} \to \mathbf{R}\text{-Rings}\). Then \(\mathcal{L}(A)\) the image of \(A\) by the functor \(\mathcal{L}\), admits a structure of left \(R\)-bialgebroid (termed a coendomorphism bialgebroid) such that \(A\) is a left \(\mathcal{L}(A)\)-comodule ring, see Proposition 2.1.1 and Corollary 2.1.2. Explicitly, the underlying \(k\)-module \(\mathcal{L}(A)\) is given by the following quotient of the tensor \(R\)-ring of \(A \otimes_k *A\):

\[
\mathcal{L}(A) := \frac{\mathcal{F}_R\left(A \otimes *A\right)}{\left\langle \sum_i (a \otimes e_i \varphi) \otimes_R (a' \otimes *e_i) - (aa' \otimes \varphi), (1 \otimes \varphi) - 1 \otimes \varphi(1)^o \right\rangle_{\{a, a' \in A, \varphi \in *A\}}}
\]

On the other hand, we consider the augmented cochain complex of the universal differential graded ring:

\[
Q_* : R \to A \xrightarrow{\partial} K \xrightarrow{\partial_2} K \otimes_A K \xrightarrow{\partial_3} K \otimes_A K \otimes_A K \to \cdots
\]

where \(K\) denotes the kernel of \(A \otimes_R A \to A\), the multiplication of \(A\). We check that this is in fact a cochain complex of left \(\mathcal{L}(A)\)-comodules whose components are finitely generated and projective left \(R\)-modules. This leads to a fiber functor \(\chi : \mathbf{k}(N) \to \mathbf{Mod}_R\) defined in the obvious way, as well as to a canonical map \(\text{can}_B : Q \otimes_B \mathcal{L}(A) \to \mathcal{L}(A)\), where \(B = \mathbf{k}(N) \oplus \mathbf{k}(N)\) (direct sum of copies of \(k\)) is the ring with enough orthogonal idempotents attached to the small category \(\mathbf{k}(N)\) (this is the \(k\)-linear category generated by the set of natural numbers \(N\)). Using the fiber functor \(\chi\), we first endow \(Q \otimes_B \mathcal{L}(A)\) with a structure of left \(R\)-bialgebroid, and then show that \(\text{can}_B\) is an isomorphism of left \(R\)-bialgebroids. This means, in the sense of [13], that \(Q\) is actually a Galois object in the category of left comodules. In this way we arrive to our first main result stated below as Theorem 3.1.3.

**Theorem A.** Let \(R \to A\) be a \(k\)-algebra map with \(A\) finitely generated and projective as left \(R\)-module. Consider the associated left \(R\)-bialgebroid \(\mathcal{L}(A)\) (see equation \(\text{(2)}\) above) and the cochain complex \(Q_*\) of equation \(\text{(3)}\) with its canonical right unital \(B\)-action and left \(\mathcal{L}(A)\)-coaction, where \(B = \mathbf{k}(N) \oplus \mathbf{k}(N)\). Then the following statements are equivalent

1. The right \(R\)-module \(Q \otimes_R \mathcal{L}(A)\) is flat and the functor \(Q \otimes_B : \mathbf{BMod} \to \mathcal{L}(A)\mathbf{Comod}\) is an equivalence of monoidal categories;
2. \(Q_B\) is a faithfully flat unital module.
Obviously, the category of left unital $B$-module $_B\text{Mod}$ is isomorphic to the category of chain complexes of $\mathbb{k}$-modules. Thus, Theorem A allows one freely to transfer the monoidal model structure described in [10] §2.3 for chain complexes of $\mathbb{k}$-modules, to the categories of left comodules over coendomorphism bialgebroids. This suggests that certain categories of comodules can be equipped with a (monoidal) model structure. Indeed this is one of the main motivations to further investigate categories of comodules over this class of bialgebroids.

Clearly the unit map $\mathbb{k} \to R$ can be extended to a morphism of rings with the same set of orthogonal idempotents: $B = \mathbb{k}^{(n)} \oplus \mathbb{k}^{(n)} \to R^{(n)} \oplus R^{(n)} = C$. By [11], this enables us to consider the usual adjunction between the scalars-restriction functor and the tensor product functor and, in particular, to define a canonical map $\text{can}_C$ with codomain a suitable quotient of $L(A)$. Thus one can try to extend Theorem A to left unital $C$-modules. In this way we arrive to our second main theorem which is stated below as Theorem 3.2.9

**Theorem B.** Let $R \to A$ be a $\mathbb{k}$-algebra map with $A$ finitely generated and projective as left $R$-module. Consider $L(A)$ the associated left $R$-bialgebroid (see equation (2) above) and $\mathcal{J}$ the coideal of $L(A)$ generated by the set of elements $\{1_{L(A)}(r \otimes 1^p - 1 \otimes r^p)\}_{r \in R}$; denote by $\overline{L(A)} = L(A)/\mathcal{J}$ the corresponding quotient $R$-coring. Consider the cochain complex $Q_\bullet$ of equation (3) with its structures of right unital $C$-module and left $\overline{L(A)}$-comodule. Then the following statements are equivalent

1. The right $R$-module $1 \otimes_{\mathbb{k}} \overline{L(A)}$ is flat and the functor $Q \otimes_C - : \text{CMod} \to \text{Comod}$ is an equivalence of categories;

2. $Q_C$ is a faithfully flat unital module.

The problem of obtaining an equivalence of categories as above, is then closely linked to the faithfully flat condition on the right unital module $Q$. This is in fact not at all easy to check. Our third main result, which is a combination of Theorem 3.3.10 and Proposition 3.4.13 gives certain homological conditions under which $Q$ becomes flat (or faithfully flat).

**Theorem C.** The notations and assumptions are that of Theorem B. Assume further that $A_R$ is finitely generated and projective, and the cochain complex $Q_\bullet$ is exact and splits, in the sense that, for every $m \geq 1$, $Q_m = \partial Q_{m-1} \oplus Q_m = \text{Ker}(\partial) \oplus \overline{Q}_m$ as right $R$-modules, for some right $R$-module $\overline{Q}_m$. Then $Q_C$ is a flat module. In particular, $Q_C$ is faithfully flat in either one of the following cases.

1. $A = R \oplus Rt$ ($t^2 = 0$), the trivial extension of $R$.

2. $\mathbb{k}$ is a field and $R$ is a division $\mathbb{k}$-algebra.

As a consequence of Theorems B and C, we get that for every $\mathbb{k}$-algebra $R$, there is a left $R$-bialgebroid $L$ such that the category of chain complexes of left $R$-modules is equivalent to the category of left comodules over an epimorphic image of $L$. In particular, if $R$ is commutative, then this equivalence is in fact a monoidal equivalence.

### 1.3. Basic notions and notations.

Given any Hom-set category $\mathcal{C}$, the notation $X \in \mathcal{C}$ means that $X$ is an object of $\mathcal{C}$. The identity morphism of $X$ will be denoted by $X$ itself. The set of all morphisms $f : X \to X'$ in $\mathcal{C}$ is denoted by $\text{Hom}_\mathcal{C}(X, X')$. The identity functor of $\mathcal{C}$ is denoted by $id_\mathcal{C}$. We denote the dual (or opposite) category of $\mathcal{C}$ by $\mathcal{C}^o$. The class of all natural transformations between two functors $F$ and $G$ is denoted by $\text{Nat}(F, G)$.

We work over a ground commutative ring with 1 denoted by $\mathbb{k}$. Up to Section 3 all rings under consideration are $\mathbb{k}$-algebras, and morphisms of rings are morphisms of $\mathbb{k}$-algebras. Modules are assumed to be unital modules and bimodules are assumed to be central $\mathbb{k}$-bimodules. For every ring $R$, these categories are denoted by $R\text{Mod}$ (left modules), $\text{Mod}_R$ (right modules) and $R\text{Mod}_R$ (bimodules).
respective. The tensor product over \( R \), is denoted as usual by \(- \otimes R -\). The unadorned symbol \( \otimes \) stands for \( \otimes_\mathbb{k} \) the tensor product over \( \mathbb{k} \).

We denote by \( Ch(R) \) the category of chain complexes of left \( R \)-modules. That is, complexes of left modules of the form:

\[
(M_\bullet, d_\bullet) : \cdots \longrightarrow M_n \xrightarrow{d_n} \cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \longrightarrow \cdots \longrightarrow M_{-n} \xrightarrow{d_{-n}} \cdots.
\]

Let \( Ch_+(R) \) denote the full subcategory of \( Ch(R) \) consisting of positive chain complexes i.e. of complexes of the form:

\[
(M_\bullet, d_\bullet) : \cdots \longrightarrow M_n \xrightarrow{d_n} \cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0.
\]

From now on, chain complex of left \( R \)-modules will stands for an object of the category \( Ch_+(R) \). When \( R \) is commutative (i.e. commutative \( \mathbb{k} \)-algebra), we will considered this category in a standard way as a monoidal category with unit object the chain complex \( R[0]_\bullet \), where \( R[0]_0 = R \), and \( R[0]_n = 0 \), for \( n > 0 \).

Given an \( R \)-bimodule \( X \), its \( \mathbb{k} \)-submodule of \( R \)-invariant elements is denoted by

\[
X^R := \{ x \in X \mid xr = rx, \forall r \in R \}.
\]

This in fact defines a functor \((-)^R : R\text{Mod}_R \rightarrow \text{Mod}_{\mathbb{Z}(R)} \), where \( \mathbb{Z}(R) \) is the centre of \( R \). As usual, we use the symbols \( \text{Hom}_{R-}(-, -) \), \( \text{Hom}_{-R}(-, -) \) and \( \text{Hom}_{R-R}(-, -) \) to denote the Hom-functor of left \( R \)-linear maps, right \( R \)-linear maps and \( R \)-bilinear maps, respectively.

For two bimodules \( R_P \) and \( R_Q \) over rings \( R \) and \( S \), we will consider the \( \mathbb{k} \)-modules of \( R \)-linear maps \( \text{Hom}_{R-}(P, Q) \) as an \( S \)-bimodule with actions:

\[
sf : p \mapsto f(ps), \quad \text{and} \quad fs' : p \mapsto f(p)s', \quad \text{for every } f \in \text{Hom}_{R-}(P, Q), s, s' \in S, \quad \text{and } p \in P.
\]

Similarly, \( \text{Hom}_{-S}(P, Q) \) is considered as an \( R \)-bimodule with actions:

\[
sf : p \mapsto rg(p), \quad \text{and} \quad gr' : p \mapsto g(rp), \quad \text{for every } g \in \text{Hom}_{-S}(P, Q), r, r' \in R, \quad \text{and } p \in P.
\]

Under these considerations, the left dual \( ^*X = \text{Hom}_{R-}(X, R) \) of an \( R \)-bimodule \( X \), is an \( R \)-bimodule, as well as its right dual \( X^* = \text{Hom}_{R}(X, R) \).

Let \( R \) be a ring, for any \( r \in R \), we denote by \( r^o \) the same element regarded as an element in the opposite ring \( R^o \). Let \( R^e := R \otimes R^o \) be the enveloping ring of \( R \). Next, we recall the Sweedler-Takeuchi’s \([26, 27]\) product on the category of \( R^e \)-bimodules, usually denoted by \(- \times_R -\). So, given an \( R^e \)-bimodule \( M \), the underlying \( \mathbb{k} \)-module \( M \) admits several structures of \( R \)-bimodule. Among them, we will select in the forthcoming step the following two ones. The first structure is that of the opposite \( R^e \)-bimodule \( 1_{\otimes R} M_{\otimes R} \) which we denote by \( M^o \). That is, the, \( R \)-biacton on \( M^o \) is given by

\[
rm^o = \left( m (1 \otimes r^o) \right)^o, \quad m^o s = \left( (1 \otimes s^o) m \right)^o, \quad m^o \in M^o, \quad r, s \in R.
\]

The second structure is defined by the left \( R^e \)-module \( R \otimes M ^. \) That is, the \( R \)-bimodule \( M^l = R_{\otimes 1} \text{M}^l_R \) whose \( R \)-biacton is defined by

\[
rm^l = \left( (r \otimes 1^o)m \right)^l, \quad m^l s = \left( (1 \otimes s^o)m \right)^l, \quad m^l \in M^l, \quad r, s \in R.
\]

Now, given \( M \) and \( N \) two \( R^e \)-bimodules, we set

\[
M \times_R N := \left( M \otimes_R N \right)^R.
\]
where \( R \mathbf{M}_R = M^o \) and \( R \mathbf{N}_R = R \otimes \mathbf{1} \mathbf{v} \mathbf{N} \mathbf{R} \otimes \mathbf{1} \mathbf{v} \). The elements of \( M \times_R N \) are denoted by \( \sum_i m_i \times_R n_i \), for \( m_i \in M \) and \( n_i \in N \). Henceforth, using these notations and given an element \( m \times_R n \in M \times_R N \), we have the following equalities

\[
(6) \quad \left( m(1 \otimes r^o) \right) \times_R n = m \times_R n(r \otimes 1^o), \quad \text{and} \quad \left( 1 \otimes r^o \right) m \times_R n = m \times_R (r \otimes 1^o) n,
\]

for every \( r, s \in R \). The \( k \)-module \( M \times_R N \) is actually an \( R^e \)-bimodule with actions:

\[
(7) \quad \left( p \otimes q^o \right) . (m \times_R n). (r \otimes s^o) := \left( (p \otimes 1^o)(r \otimes 1^o) \right) \times_R \left( (1 \otimes q^o)(1 \otimes s^o) \right),
\]

for every \( r, s, p, q \in R \) and \( m \times_R n \in M \times_R N \).

On the other hand, since we have \( M^e_R = M^l_R \) for every \( R^e \)-bimodule \( M \), there is a canonical natural transformation (injective at least as \( k \)-linear map)

\[
(8) \quad \Theta_{M,N} : M \times_R N \longrightarrow M^l \otimes_R N^l.
\]

With this product, the \( R^e \)-bimodule \( S \times_R T \) is an \( R^e \)-ring whenever \( S \) and \( T \) are. The multiplication of \( S \times_R T \) is defined componentwise, and the identity element is given by \( 1_S \times_R 1_T \).

An \( R \)-ring \( S \) is a monoid in the monoidal category of \( R \)-bimodules, equivalently, a \( k \)-algebra map \( R \rightarrow S \). Dually, an \( R \)-coring is a comonoid in \( R \mathbf{Mod}_R \), which is by definition a three-tuple \((\mathcal{C}, \Delta, \varepsilon)\) consisting of \( R \)-bimodule \( \mathcal{C} \) and two \( R \)-bilinear maps \( \Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes_R \mathcal{C} \) (comultiplication), \( \varepsilon : \mathcal{C} \rightarrow R \) (counit) satisfying the usual coassociativity and counitary constraints. In contrast with coalgebras, corings admit several convolution rings. For instance, the right convolution of an \( R \)-coring \( \mathcal{C} \), is the right dual \( R \)-bimodule \( \mathcal{C}^* \) whose multiplication is defined by

\[
\sigma \cdot \sigma' = \sigma \circ (\sigma' \otimes_R \mathcal{C}) \circ \Delta,
\]

for all \( \sigma, \sigma' \in \mathcal{C}^* \), and its unit is the counit \( \varepsilon \) of \( \mathcal{C} \). A morphism of \( R \)-corings is an \( R \)-bilinear map \( \phi : \mathcal{C} \rightarrow \mathcal{C}' \) such that \( \Delta' \circ \phi = (\phi \otimes_R \phi) \circ \Delta \) and \( \varepsilon' \circ \phi = \varepsilon \). A left \( \mathcal{C} \)-comodule is pair \((N, \lambda_N)\) consisting of left \( R \)-module \( N \) and left \( R \)-linear map \( \lambda_N : N \rightarrow \mathcal{C} \otimes_R N \) (coaction) compatible in the canonical way with comultiplication and counit. A morphism of left \( \mathcal{C} \)-comodules is a left \( R \)-linear map which is compatible with coactions. We denote by \( \mathcal{C} \mathbf{Comod} \) the category of left \( \mathcal{C} \)-comodules. Right comodules are similarly defined. Given any morphism of \( R \)-corings \( \phi : \mathcal{C} \rightarrow \mathcal{C}' \) one can define, in the obvious way, a functor \((-)_{\phi} : \mathcal{C} \mathbf{Comod} \rightarrow \mathcal{C}' \mathbf{Comod} \) referred to as the coinduction functor.

For more information on comodules as well as the definitions of bicomodules and cotensor product over corings, the reader is referred to \cite{1}. For the notions of bialgebroids and their basic properties, the reader is referred to \cite{5}.

We will also consider here rings with enough orthogonal idempotents. These are central \( k \)-modules \( B \) with internal multiplication which admit a decomposition of \( k \)-modules \( B = \oplus_{p \in P} B_1 p = \oplus_{p \in P} 1_p B \), where \( \{1_p\}_{p \in P} \subseteq B \) is a set of orthogonal idempotents. Module over a ring with enough orthogonal idempotents stands for \( k \)-central and unitary module. Recall that \( M \) is a left unital \( B \)-module provided that \( M \) has an associative left \( B \)-action which satisfies \( M = \oplus_{p \in P} 1_p M \). We denote by \( B \mathbf{Mod} \) the category of left unital \( B \)-modules.

2. Coendomorphism and comatrices bialgebroids.

2.1. Coendomorphism bialgebroid and \( \times_R \)-comodules. In this subsection we recall the construction of coendomorphism bialgebroids attached to any finitely generated and projective extension of rings. We also recall from \cite{25} the monoidal structure of the category of comodules over the underlying coring of a given left bialgebroid.
A \times_R\text{-coalgebra} is an \( R\)-bimodule \( C \) together with two \( R\)-bilinear maps \( \Delta : C \to C \times_R C \) (comultiplication) and \( \varepsilon : C \to \text{End}_k(R) \) (counit) which satisfy the coassociativity and counitary properties in the sense of [27, §4, Definition 4.5], see also [3] and [25]. A \( \times_R\text{-coalgebra} \) \( C \) is said to be an \( \times_R\text{-bialgebra} \) provided that comultiplication and counit are morphisms of \( R\)-rings.

A left \( \times_R\text{-C-comodule} \), is a pair \( (X, \lambda_X) \) consisting of an \( R\)-bimodule \( X \) and an \( R\)-bilinear map \( \lambda_X : X \to C \times_R X \) satisfying, in some sense, the coassociativity and counitary axioms. Morphism between left \( \times_R\text{-C-comodules} \) are \( R\)-bilinear maps compatible in the obvious way with the left \( \times_R\text{-C-coactions} \). This leads to the definition of the category of left \( \times_R\text{-C-comodules} \). When \( C \) is a \( \times_R\text{-bialgebra} \), this category becomes a monoidal category [25, Proposition 5.6], and the forgetful functor to the category of \( R\)-bimodules is a monoidal functor. There is a strong relation which will be clarified in the sequel, between the category of left \( \times_R\text{-comodules} \) over an \( \times_R\text{-bialgebra} \) and the category of left comodules over the underlying \( R\)-coring whose structure maps are

\[
\begin{array}{ccc}
C & \xrightarrow{\Theta_{\epsilon,C}} & C \otimes_R C \\
\end{array}
\]

where \( \Theta_{\epsilon,-} \) is the natural transformation of equation [8].

Let \( A \) be an \( R\)-ring, that is, a \( k\)-algebra map \( R \to A \), and denote by \( ^*A \) the dual of the \( k\)-module \( A_k \), i.e. \( ^*A = \text{Hom}_k(A, k) \). We consider the tensor product \( A \otimes \ ^*A \) as an \( R^e\)-bimodule in the following way

\[
(r \otimes s') \cdot (a \otimes \varphi) \cdot (p \otimes q^o) = (rap) \otimes (q \varphi s), \quad p, q, r, s \in R, \text{ and } a, \in A, \varphi \in \ ^*A,
\]

where \( A \) and \( ^*A \) are considered as \( R\)-bimodules in the usual way.

Assume that \( R\_A \) is finitely generated and projective module, and fix a left dual basis \( \{ e_i, \ ^*e_i \}_{1 \leq i \leq n} \). Define the \( R^e\)-ring \( \mathcal{L}(A) \) by the quotient algebra

\[
\mathcal{L}(A) = \mathcal{L}_{R^e}\left( A \otimes \ ^*A \right) / \mathcal{J}
\]

where \( \mathcal{L}_{R^e}\left( A \otimes \ ^*A \right) = \bigoplus_{n \in \mathbb{N}} \left( A \otimes \ ^*A \right)^{\otimes n} \) is the tensor \( R^e\)-ring of the \( R^e\)-bimodule \( A \otimes \ ^*A \) and where \( \mathcal{J} \) is the two-sided ideal generated by the set

\[
\left\{ \sum_i \left( (a \otimes e_i \varphi) \otimes_R (a' \otimes \ ^*e_i) - (aa' \otimes \varphi); 1_R \otimes \varphi(1_A)^o - (1_A \otimes \varphi) \right) \right\}
\] \( a, a' \in A, \varphi \in \ ^*A \)

We denote by

\[
\pi_A : \mathcal{L}_{R^e}(A \otimes \ ^*A) \to \mathcal{L}(A)
\]

the canonical projection. From now on, given a homogeneous elements \( (a \otimes \varphi) \in \mathcal{L}_{R^e}(A \otimes \ ^*A) \) of degree one, we denote by \( \pi_A(a \otimes \varphi) \) its image in \( \mathcal{L}(A) \). Next, we recall the structure of \( \times_R\text{-bialgebra} \) of the object \( \mathcal{L}(A) \), which is denoted by \( a_R(A, A) \) in [25, Remark 1.7]. The underlying structure of an \( R^e\)-ring, is given by the following composition of algebra maps

\[
R^e \xrightarrow{\iota_m} \mathcal{L}_{R^e}(\mathcal{L}(A)) \xrightarrow{\pi_A} \mathcal{L}(A),
\]

where \( \iota_m \) denotes the canonical \( R^e\)-bilinear injection in degree \( m \geq 0 \).
Proposition 2.1.1. Let $A$ be an $R$-ring which is finitely generated and projective as left $R$-module with dual basis $\{(*e_i, e_i)\}_i$. Then $\mathcal{L}(A)$ is a $\times_R$-bialgebra with structure maps

\[
\begin{align*}
\Delta : & \mathcal{L}(A) \to \mathcal{L}(A) \times_R \mathcal{L}(A), \\
\pi_A(a \otimes \varphi) \mapsto & \sum_j \pi_A(a \otimes *e_j) \times_R \pi_A(e_j \otimes \varphi) \\
\varepsilon : & \mathcal{L}(A) \to \text{End}_k(R), \\
\pi_A(a \otimes \varphi) \mapsto & \left[ r \mapsto \varphi(ar) \right].
\end{align*}
\]

The relation between the $R$-ring structure of $A$ and the $\times_R$-bialgebra structure of $\mathcal{L}(A)$, is expressed as follows.

Corollary 2.1.2. Let $A$ be an $R$-ring such that $RA$ is finitely generated and projective and $\mathcal{L}(A)$ the associated $\times_R$-bialgebra defined in Proposition 2.1.1. Then $A$ is a left $\times_R$-$\mathcal{L}(A)$-comodule $R$-ring, that is, $A$ admits a left $\times_R$-$\mathcal{L}(A)$-coaction

\[
\lambda_A : A \to \mathcal{L}(A) \times_R A, \quad \left( a \mapsto \sum_j \pi_A(a \otimes *e_j) \times_R e_j \right)
\]

which is also a morphism of $R$-rings.

The $\times_R$-bialgebra $\mathcal{L}(A)$ defined in Proposition 2.1.1 is referred to as coendomorphism $R$-bialgebroid since by [3, Theorem 3.1], $\mathcal{L}(A)$ is in fact a (left) bialgebroid whose structure of $R^e$-ring is the map

\[
\pi_A \circ \iota_0 : R^e \to \mathcal{L}(A),
\]

and its structure of $R$-coring is given as follows. The underlying $R$-bimodule is $\mathcal{L}(A)^l = R^e \mathcal{L}(A)$, the comultiplication and counit are given by

\[
\begin{align*}
\Delta : & \mathcal{L}(A)^l \to \mathcal{L}(A)^l \otimes_R \mathcal{L}(A)^l, \\
\pi_A(a \otimes \varphi) \mapsto & \sum_i \pi_A(a \otimes *e_i) \otimes_R \pi_A(e_i \otimes \varphi) \\
\varepsilon : & \mathcal{L}(A)^l \to R, \\
\pi_A(a \otimes \varphi) \mapsto & \varphi(a).
\end{align*}
\]

Example 2.1.3. Let $A = R \oplus Rt$ be the trivial generalized $R$-ring, i.e. the $R$-ring which is free as left $R$-module with basis $1 = (1, 0)$ and $t = (0, t)$ such that $t^2 = 0$. Using [10] and Proposition 2.1.1 we can easily check that $\mathcal{L}(A)$ is an $R$-bialgebroid generated by the image of $R^e$ and two $R^e$-invariant elements $\{x, y\}$ subject to the relations $xy + yx = 0$, $x^2 = 0$. The comultiplication and counit of it underlying $R$-coring are given by

\[
\begin{align*}
\Delta(x) & = x \otimes_R 1 + y \otimes_R x, \quad \varepsilon(x) = 0 \\
\Delta(y) & = y \otimes_R y, \quad \varepsilon(y) = 1.
\end{align*}
\]

The ring $A$ is a left $\mathcal{L}(A)$-comodule ring with coaction: $\lambda : A \to \mathcal{L}(A) \otimes_R A$ sending

\[
\lambda(1_A) = 1_{\mathcal{L}(A)} \otimes_R 1_A, \quad \lambda(t) = x \otimes_R 1_A + y \otimes_R t,
\]

extended by $R$-linearity to the whole set of elements of $A$.

In [25] it was shown that the category of left $\times_R$-comodules over an $\times_R$-bialgebra is a monoidal category such that the forgetful functor to the category of $R$-bimodules is a monoidal functor. What we will need in the sequel is a monoidal structure on the category of left $\mathcal{L}(A)$-comodules where $\mathcal{L}(A)$ is viewed as an $R$-coring with structure maps [13] and [14]. The following lemma is a consequence of [25, Proposition 5.6], see also [3, 3.6].
Thus, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}^l \text{Comod} & \xrightarrow{U} & R \text{Mod} \\
\downarrow & & \downarrow \\
R \text{Mod}_R & \rightarrow & \end{array}
\]

where the dashed arrow is a monoidal functor.

Summing up, given two left \( \mathcal{L}^l \)-comodules \((X, \lambda_X)\) and \((Y, \lambda_Y)\), using Lemma 2.1.4 we can consider \((X \otimes_R Y, \lambda_{X \otimes_R Y})\) as a left \( \mathcal{L}^l \)-comodule with coaction

\[
\lambda_{X \otimes_R Y} : X \otimes_R Y \rightarrow \mathcal{L}^l \otimes_R X \otimes_R Y,
\]

where we have considered \(X\) as \(R\)-bimodule with the right \(R\)-action given by the action

\[
x \cdot r = \sum_{(x)} \varepsilon \left( x_{(-1)} (r \otimes 1^0) \right) x_{(0)}, \quad \text{for every } r \in R \text{ and } x \in X.
\]

2.2. The complex of left \( \mathcal{L} \)-comodules \( Q_* \). Keep the assumptions and notations of subsection 2.1 that is, we are considering an \(R\)-ring \(A\) over a fixed \(k\)-algebra \(R\). Let us denote by

\[
K = \text{Ker} \left( A \otimes_R A \xrightarrow{\mu} A \right)
\]

the kernel of the multiplication \(\mu\) of \(A\) with canonical derivation

\[
\begin{array}{ccc}
A & \xrightarrow{\partial} & K \\
\xrightarrow{a} & & \xrightarrow{\partial a = 1 \otimes_R a - a \otimes_R 1} \\
\end{array}
\]

The associated cochain complex is denoted by

\[
Q_* : R \xrightarrow{\partial_0 = 1} A \xrightarrow{\partial_1 = \partial} K \xrightarrow{\partial_2} K \otimes_A K \xrightarrow{\partial_3} K \otimes_A K \otimes_A K \xrightarrow{\partial_4} \cdots
\]

where \(\partial_n : Q_n \rightarrow Q_{n+1}\) sends \(a_0 \partial a_1 \otimes_A \cdots \otimes_A a_{n-1}\) to \(\partial a_0 \otimes_A \partial a_1 \otimes_A \cdots \otimes_A a_{n-1}\), \(n \geq 2\).

The following lemma, which will play a key role in Subsection 3.3 characterizes a split ring extension \(R \rightarrow A\) (in \(\text{Mod}_R\)) in terms of the cochain complex \(Q_*\).

**Lemma 2.2.5.** Let \(A\) be any \(R\)-ring. Then the following conditions are equivalent.

(i) The unit \(u : R \rightarrow A\) is a split monomorphism in \(\text{Mod}_R\).

(ii) The cochain complex \(Q_*\) is exact and splits, in the sense that, for every \(m \geq 1\), there is a right \(R\)-module \(Q_m\) such that \(Q_m = \partial Q_{m-1} \oplus \overline{Q}_m = \text{Ker}(\partial) \oplus \overline{Q}_m\), as right \(R\)-modules.

**Proof.** (ii) \(\Rightarrow\) (i) It is trivial.

(i) \(\Rightarrow\) (ii). Let us denote by \(u^c : A \rightarrow \overline{A}\) the cokernel of \(u : R \rightarrow A\) in \(\text{Mod}_R\). Put \(\Omega_0 := R\), \(\Omega_1 := A\),
and \( \Omega_n := A \otimes_R \overline{A} \otimes_R \cdots \otimes_R \overline{A} \), \((n - 1)\)-fold \( \overline{A} \), for \( n \geq 2 \). Consider now the following split exact sequence of right \( R \)-modules

\[
0 \to \overline{A}^{\otimes R_n} \xrightarrow{\gamma_n} A \otimes_R \overline{A}^{\otimes R_n} \to \overline{A}^{\otimes R_{n+1}} \to 0,
\]

where \( \gamma_n = u \otimes_R \overline{A}^{\otimes R_n} \), for \( n \geq 1 \). In view of this, we have a split exact cochain complex of right \( R \)-modules

\[
\Omega_\bullet : \Omega_0 \xrightarrow{d_0} \Omega_1 \xrightarrow{d_1} \Omega_2 \xrightarrow{d_2} \Omega_3 \xrightarrow{d_4} \cdots,
\]

with differential \( d_0 = u, d_1 = \gamma_1 \circ u^c, d_n = \gamma_n \circ (u^c \otimes_R \overline{A}^{\otimes R_{n-1}}) \), for \( n \geq 2 \). Since \( \Omega_2 \) is the cokernel of the map \( A \otimes_R u \), and the later split by \( \mu \) the multiplication of \( A \), we obtain the following split exact sequence of \( R \)-bimodules

\[
0 \to A \xrightarrow{A^{\otimes R_u}} A \otimes_R A \xrightarrow{A^{\otimes R_{u^c}}} \Omega_2 \to 0.
\]

This gives the split exact sequence

\[
0 \to \Omega_2 \to A \otimes_R A \xrightarrow{\mu} A \to 0.
\]

Thus we have an \( R \)-bilinear isomorphism \( \omega_2 : \Omega_2 \to Q_2 = K \). Henceforth, there is an unique \( A \)-bimodule structure on \( \Omega_2 \) which renders \( \omega_2 \) an \( A \)-bilinear isomorphism, namely

\[
a \cdot (x \otimes_R y) \cdot b = ax \otimes_R yb - axy \otimes_R b, \quad \text{for every } a, x, y, b \in A,
\]

wherein the notation \( u^c(z) = \overline{z} \), for every \( z \in A \), have been used. Define iteratively \( \omega_n : \Omega_n \to Q_n \), for all \( n \geq 3 \), as the composition

\[
\Omega_n = \Omega_{n-1} \otimes_R \overline{A} \cong \Omega_{n-1} \otimes_A (A \otimes_R \overline{A}) = \Omega_{n-1} \otimes_A \Omega_2 \xrightarrow{\omega_{n-1} \otimes_A \omega_2} Q_{n-1} \otimes_A K = K^{\otimes_A n-1} = Q_n.
\]

By construction, \( \omega_\bullet : (\Omega_\bullet, d_\bullet) \to (Q_\bullet, \partial_\bullet) \) is a morphism of complexes of \( R \)-bimodules. We leave to the reader to check that \( \omega_\bullet \) is in fact an isomorphism of cochain complexes. Now, since \( (\Omega_\bullet, d_\bullet) \) is split exact in right \( R \)-modules, then so is \( (Q_\bullet, \partial_\bullet) \). \( \square \)

**Remark 2.2.6.** In the case of finitely generated and projective extension of rings, the left version of condition \((i)\) in Lemma 2.2.5 implies that \( RA \) is in fact faithfully flat module (see, for example [2 Chap. I, Proposition 9, page 51]). In this case, one can easily show that \( Q_\bullet \otimes_R A \) is homotopically trivial which by [18 Théorème 2.4.1] gives condition \((ii)\). In this way, Lemma 2.2.5 can be seen as a generalization of [H Propositions 6.1, 6.2].

The convolution product on the left dual chain complex of \( Q_\bullet \) is given as follows: For every \( \varphi \in S Q_n \) and \( \psi \in S Q_m \) with \( n, m \geq 1 \), we have a left \( R \)-linear map

\[
(17) \quad \varphi \ast \psi : Q_{n+m} \longrightarrow R
\]

\[
x \otimes_A \partial(a) \otimes_A y \longmapsto \varphi \left( x\psi(ay) \right) - \varphi \left( xa\psi(y) \right),
\]

where \( x \in Q_n, y \in Q_m \), and \( a \in A \). The convolution product with zero degree element is just the left and right \( R \)-actions of \( S Q_n \), for every \( n \geq 1 \), namely

\[
(18) \quad r \ast \varphi : Q_n \longrightarrow R \quad \varphi \ast s : Q_n \longrightarrow R
\]

\[
x \longmapsto \varphi(xr), \quad x \longmapsto \varphi(x) s,
\]

for every elements \( r, s \in R \) and \( \varphi \in S Q_n \).
Remark 2.2.7. The convolution product defined in (17) and (18) derives from the structure of comonoid of the cochain complex $Q_\bullet$ viewed as an object in the monoidal category of cochain complexes of $R$-bimodules. Precisely, the identity map $A \otimes R \cdots \otimes R A = A \otimes_R n = A \otimes_R p \otimes_R A \otimes_R q$, for $p + q = n$, reed as a map $Q_n \to Q_p \otimes R Q_q$ sending $x \otimes_A \partial_A y \mapsto x \otimes_R a y - xa \otimes_R y$, for every $x \in Q_p$, $a \in A$ and $y \in Q_q$. Thus $\oplus_{n \geq 0} Q_n$ has a structure of differential $R$-coring in the sense of [5] pages 6, 7). Since each $Q_n$ is finitely generated and projective left $R$-module (see Lemma 2.2.8 below, of course under the same assumption for the left module $R A$), the comultiplication of $Q$ is transferred to the graded left dual $V Q = \oplus_{n \geq 0}^* Q_n$ which gives a multiplication defined explicitly by (17) and (18). A comonoidal structure on $Q_\bullet$ could also be obtained by transferring some comonoidal structure of the Amitsur cosimplicial object of $R$-bimodules induced by $A$ (see (1)), using for this the normalization functor and it structure of comonoidal functor obtained from Eilenberg-Zilber Theorem, see [22] Theorem 8.1, Exercise 4. p. 244] (of course in their dual form). It seems that Tambara’s approach [25] runs in this direction. Anyway this approach uses a slightly variant of the category of cosimplicial groups endowed with some monoidal structure which is not the usual one. Since our methods run in a different way, we will not make use of the normalization process here.

In all what follows, we will fix a (left) finitely generated and projective extension $R \to A$ with dual basis $\{e_i, \ast e_j\}_{1 \leq i \leq n}$. We will denote by $L := L(A)$ the corresponding left $R$-bialgebroid coming from Proposition 2.1.1 and by $\pi$ the projection $\pi_A$ defined in (12).

Using this dual basis, one can check that $R Q_2 = RK$ is finitely generated and projective module whose dual basis is given by the set $\{(e_i \partial e_j, \ast e_i \ast e_j)\}_{i, j}$. Moreover, we have

**Lemma 2.2.8.** Each $Q_n$, $n \geq 0$, is finitely generated and projective as left $R$-module. Furthermore, if $\{\{\omega_{n,\alpha}, \ast \omega_{n,\alpha}\}\}_{\alpha}$ is a dual basis for $Q_n$ with $n \geq 1$, then $\{\{\omega_{n,\alpha} \otimes_A \partial \omega_{n,\beta}, \ast \omega_{n,\alpha} \ast \omega_{n,\beta}\}\}_{\alpha, \beta}$ is a dual basis for $Q_{n+m}$, while $\{\{\omega_{n,\alpha} \otimes_A \partial \omega_{n,\beta}, \ast \omega_{n,\alpha} \ast \partial \omega_{n,\beta}\}\}_{\alpha, \beta}$ is a dual basis for $Q_{n+m-1}$ when $m \geq 2$.

**Proof:** Straightforward.

The cochain complex $Q_\bullet$ is actually a complex of left $L$-comodules.

**Proposition 2.2.9.** The cochain complex $Q_\bullet$ is a complex of left $L$-comodules. For $n = 0$, the coaction is given by $(R \to \mathcal{L}, r \mapsto \pi(r \otimes 1^0))$ and, for $n \geq 1$, by $\lambda_n : Q_n \to \mathcal{L} \otimes_R Q_n$ sending

$$
(19) \quad a_0 \partial a_1 \otimes_A \cdots \otimes_A a_{n-1} \sum_{i_0, i_1, \ldots, i_{n-1}} \pi(a_0 \otimes e_{i_0}) \cdots \pi(a_{n-1} \otimes e_{i_{n-1}}) \otimes_R (e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}).
$$

**Proof:** The statement is trivial for $n = 0$. For $n \geq 1$, the coassociativity of $\lambda_n$ is deduced using Lemma 2.2.8 which assert that $\{\{e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}, e_{i_0} \ast \cdots \ast e_{i_{n-1}}\}\}_{i_0, i_1, \ldots, i_{n-1}}$ is a dual basis for $Q_n$. Here each $e_{i_0} \ast \cdots \ast e_{i_{n-1}}$ is the $n$-fold convolution product defined in (17). The rest of the proof uses the fact that each coaction $\lambda_n, n \geq 1$, satisfies the equality

$$
(20) \quad \lambda_n \left(\partial b_1 \otimes_A \cdots \otimes_A \partial b_{n-1}\right) = \sum_{i_1, \ldots, i_{n-1}} \pi(b_{i_1} \otimes e_{i_1}) \cdots \pi(b_{i_{n-1}} \otimes e_{i_{n-1}}) \otimes_R (\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}).
$$

**Lemma 2.2.10.** Given two elements $u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A a_{n-1} \in Q_n$ and $u_m = b_0 \partial b_1 \otimes_A \cdots \otimes_A b_{m-1} \in Q_m$ with $n, m \geq 1$. Then

$$
\lambda_n+m-1(u_n \otimes_A u_m) = \sum_{i_0, \ldots, i_{n-1}, j_0, \ldots, j_{m-1}} \left(\pi(a_0 \otimes e_{i_0}) \cdots \pi(a_{n-1} \otimes e_{i_{n-1}}) \pi(b_0 \otimes e_{j_0}) \cdots \pi(b_{m-1} \otimes e_{j_{m-1}})\right).
$$

The following lemma will be used in the sequel.
Furthermore, for every \( u \in Q_n, n \geq 1 \) and \( v \in Q_m, m \geq 1 \), we have
\[
\lambda_{n+m-1}(u \otimes_A v) = \sum u_{(-1)}v_{(-1)} \otimes_R (u_{(0)} \otimes_A v_{(0)}),
\]
and
\[
\lambda_{n+m}(u \otimes_A v) = \sum u_{(-1)}v_{(-1)} \otimes_R (u_{(0)} \otimes_A \partial v_{(0)}),
\]
where Sweedler’s notation for coactions is used.

\[\text{Proof.}\] The proof of the first claim is based upon the observation that the coaction of any \( Q_k = K \otimes_A \cdots \otimes_A K \) \((k-1)\)-times), with \( k \geq 2 \), is induced from that of \( A \otimes R \cdots \otimes_R A \) \(k\)-times). The later is a left \( \mathcal{L}\)-comodule, by Corollary 2.1.2 and Lemma 2.1.4 using the coactions described in [15]. The last statement is deduced from the first one by left \( R\)-linearity. \(\square\)

2.3. The infinite comatrix bialgebroid induced by \( Q_* \). Let \( Q_* \) be the cochain complex of \( \mathcal{L}\)-comodules considered in Proposition 2.2.4. In this subsection we will construct a left bialgebroid associated to \( Q_* \) and a canonical map from this left bialgebroid to \( \mathcal{L}\). First we recall from [12, 13] the notion of infinite comatrix coring and the canonical map. A different approach to this notion can be found in [29, 7] and [19]. We should mention here that this object coincides with the one already constructed in the context of Tannaka-Krein duality over fields or commutative rings, see [9, 6, 21] and [13], see also [24]. However, the description given in [13] in terms of tensor product over a ring with enough orthogonal idempotents, seems easier to handle from a computational point of view.

Let \( A \) be a small full sub-category of an additive category. Following [17, page 346], we can associate to \( A \) the ring with enough orthogonal idempotents \( S = \oplus_{p \in A} \text{Hom}_A(p, p') \), where \( A' \) is the opposite category of \( A \). The category of left unital \( S\)-module is denoted by \( s\text{Mod} \).

Let us denote by \( \text{add}(R) \) the full sub-category of \( R\text{Mod} \) consisting of all finitely generated and projective left \( R\)-modules. Let \( \chi : A \to \text{add}(R) \) be a faithful functor, refereed to as fiber functor. We denote by \( p^X \) the image of \( p \in A \) under \( \chi \) or by \( p \) itself if no confusion arises. Consider the left \( R\)-module direct sum of the \( p\)'s: \( \Sigma = \oplus_{p \in A} p \) (i.e. \( \Sigma = \oplus_{p \in A} p^X \)) and the right \( R\)-module direct sum of their duals: \( \Sigma^\vee = \oplus_{p \in A} p^\vee \). It is clear that \( \Sigma^\vee \) is a left unital \( S\)-module while \( \Sigma \) is a right unital \( S\)-module. In this way \( \Sigma \) becomes an \((R, S)\)-bimodule and \( \Sigma^\vee \) an \((S, R)\)-bimodule. Then \( \Sigma \otimes_S \Sigma^\vee \) is now an \( R\)-bimodule whose elements are described as a finite sum of diagonal ones, i.e. of the form \( u_p (u_p) \otimes_S \iota_p(\varphi_p) \) where \( (u_p, \varphi_p) \in p^X \times (p^X) \) and \( \iota_p \) are the canonical inclusions in \( \Sigma^{\vee} \) and \( \Sigma \). From now on, we will write \( u_p \otimes_S \varphi_p \) instead of \( u_p (u_p) \otimes_S \iota_p(\varphi_p) \) to denote a generic element of \( \Sigma \otimes_S \Sigma^{\vee} \).

This bimodule admits a structure of an \( R\)-coiring given by the following comultiplication
\[
(21) \quad \Delta : \Sigma \otimes_S \Sigma^\vee \longrightarrow (\Sigma \otimes_S \Sigma^{\vee}) \otimes_R (\Sigma \otimes_S \Sigma^{\vee})
\]
\[
\otimes_{i} u_{p,i} \otimes S^* u_{p,i} \otimes_R u_{p,i} \otimes_S \varphi_{p},
\]
where, for a fixed \( p \in A \), the finite set \( \{(u_{p,i}, \varphi_{p,i})\}_i \subset p \otimes S^* p \) is a left dual basis of the left \( R\)-module \( p \). The counit is just the evaluating map. Note that this comultiplication is independent from the chosen bases. With this structure \( \Sigma \otimes_S \Sigma^{\vee} \) is referred to as the \emph{infinite comatrix coring} associated to the small category \( A \) and the fiber functor \( \chi \). On the other hand, each of the left \( R\)-modules \( p^X \) is actually a left \((\Sigma \otimes_S \Sigma^{\vee})\)-comodule with coaction, using the above notation is given by
\[
(22) \quad \bar{\lambda}_p : p \longrightarrow \Sigma \otimes_S \Sigma^{\vee} \otimes_R p,
\]
\[
u \longrightarrow \sum_{i} u \otimes S^* u_{p,i} \otimes_R u_{p,i} \right).
\]
Another description of the infinite comatrices is given in [13, Proposition 5.2] which establishes an isomorphism of $R$-bimodules

\[
\Sigma \otimes_B \Sigma \cong \bigoplus_{p \in A} p \otimes_{T_p} \ast p
\]

where $T_p := \text{End}_{A^e}(p)$ and $T_{p,q} := \text{Hom}_{A^e}(p, q)$, for every objects $p, q$ in $A$.

Now, let $\mathcal{C}$ be an $R$-coring and let $\mathcal{Q}$ be a small full sub-category of the category of comodules $\mathcal{C}\text{-Comod}$ whose underlying left $R$-modules are finitely generated and projective. Denote by $\lambda_q$ the coaction of $q \in \mathcal{Q}$. Then one can directly apply the above constructions, by putting $\chi(q) = U(q)$, where $U : \mathcal{C}\text{-Comod} \to \mathcal{K}\text{Mod}$ is the left forgetful functor. In this case, the left $\mathcal{C}$-coaction of $\Sigma = \bigoplus_{q \in \mathcal{Q}} q$ is right $S$-linear, while the right $\mathcal{C}$-coaction of $\Sigma$ is left $S$-linear. Moreover, there is a canonical morphism of $R$-corings defined by

\[
\text{can}_S : \Sigma \otimes_S \Sigma \xrightarrow{\sim} \mathcal{C}
\]

\[u_q \otimes_S \varphi_q \mapsto (\mathcal{C} \otimes_R \varphi_q) \circ \lambda_q(u_q).
\]

Here $S$ is the induced ring from the category $\mathcal{Q}$, that is,

\[
S = \bigoplus_{q,p \in \mathcal{Q}} \text{Hom}_\mathcal{C}(q, p).
\]

However, the construction of the infinite comatrix coring, as well as the canonical map $\text{can}$, can be also performed for any sub-ring of $S$ with the same set of orthogonal idempotents (i.e. the $q$’s identities).

Let us consider the $k$-linear category $k(\mathbb{N})$ whose objects are the natural numbers $\mathbb{N}$, and homomorphisms sets are defined by

\[
\text{Hom}_{k(\mathbb{N})}(n, m) = \begin{cases} 
0, & \text{if } m \notin \{n, n+1\} \\
k.1_n, & \text{if } n = m \\
k.j_{n+1}, & \text{if } m = n + 1,
\end{cases}
\]

where the last two terms are free $k$-modules of rank one. The induced ring with enough orthogonal idempotents is the free $k$-module $B = k^{(\mathbb{N})} \oplus k^{(\mathbb{N})}$ generated by the set $\{h_n, v_n\}_{n \in \mathbb{N}}$, where $h_n$ and $v_n$ corresponds to $1_n$ and $j_{n+1}$ respectively, subject to the following relations:

\[
\begin{align*}
\delta_{n,m} h_n &= h_m, \quad \forall m, n \in \mathbb{N} \quad \text{(Kronecker delta)} \\
v_n v_m &= v_m v_n = 0, \quad \forall m, n \in \mathbb{N} \\
v_n h_{n+1} &= v_n = h_n v_n, \quad \forall m, n \in \mathbb{N}.
\end{align*}
\]

In other words $B$ is the ring of $(\mathbb{N} \times \mathbb{N})$-matrices over $k$ of the form

\[
\begin{pmatrix}
k & k & 0 & 0 \\
0 & k & k & 0 \\
0 & 0 & k & k \\
& \ddots & \ddots & \ddots \\
& & & & k & k \\
& & & \ddots & \ddots & \ddots
\end{pmatrix}
\]

consisting of matrices with only possibly two non-zero entries in each row: $(i, i)$ and $(i, i + 1)$. It is clear that the category of unital left $B$-modules is isomorphic to the category $Ch_+(k)$ of chain
complexes of \( k \)-modules. Precisely, this isomorphism functor \( \mathcal{O} \) sends every chain complex \((V, \partial^V)\) to its associated differential graded \( k \)-module \( \mathcal{O}(V) = \oplus_{n \geq 0} V_n \) with the following left \( B \)-action

\[
\mathfrak{h}_n \left( \sum_{n \geq 0} v_i \right) = v_n, \quad \text{and} \quad v_n \left( \sum_{n \geq 0} v_i \right) = \partial^V(v_{n+1})
\]

and acts in the obvious way on morphisms of chain complexes. The inverse functor is clear.

By Proposition 2.2.9, we have a faithful functor \( \chi: k[N] \to \mathcal{C} \text{Comod} \) sending \( n \to Q_n \), whose composition with the left forgetful functor gives rise to a fiber functor \( \chi: k[N] \to \text{add}(R) \). Therefore, we can apply the above process to construct an infinite comatrix \( R \)-coring \( Q \otimes_B \mathcal{V}Q \) where \( Q = \oplus_{n \in \mathbb{N}} Q_n \) and \( \mathcal{V}Q = \oplus_{n \in \mathbb{N}}^* Q_n \) are given by the cochain complex of Subsection 2.2.

Since each of the \( Q_n \)‘s has a structure of \( R \)-bimodule for which the differential \( \partial_n \) is \( R \)-bilinear, we deduce that \( Q \otimes_B \mathcal{V}Q \) is an \( R^e \)-bimodule with actions

\[
(r \otimes s^o) \cdot (u_n \otimes_B \varphi_n) \cdot (p \otimes q^o) = (ru_n p) \otimes_B (s \varphi_n q),
\]

for every \( p, q, r, s \in R \) and \( u_n \in Q_n \) and \( \varphi_n \in \mathcal{V}Q_n \). In view of this \( R^e \)-bialgebra, the infinite comatrix \( R \)-coring has \( R^0(Q \otimes_B \mathcal{V}Q) \) as its underlying \( R \)-bimodule.

Next we will construct an \( R^e \)-ring structure on the \( R^e \)-bimodule \( (Q \otimes_B \mathcal{V}Q) \). Part of this construction needs the following general Lemma which can be found, under a slightly different form, in [9], [6], and [15]. We adopt the following general notations: For any small \( k \)-linear category \( C \), we denote by \( \text{Funct}(C, \text{add}(R)) \) the category of \( k \)-linear faithful functors valued in \( \text{add}(R) \), i.e. of fiber functors on \( C \). For any object \( \chi: C \to \text{add}(R) \), we denote by \( \mathcal{L}(\chi) \) the associated infinite comatrix \( R \)-coring defined by the isomorphism of (23). Lastly, we consider \( \Sigma: \text{Funct}(C, \text{add}(R)) \to \text{Mod}_{S(C)} \) the canonical functor to the category of right unital \( S(C) \)-modules (recall that \( S(C) \) is the induced ring of \( C^o \)). That is,

\[
\Sigma(\chi) := \bigoplus_{\epsilon \in \mathcal{C}} \mathcal{C}^\epsilon, \quad \Sigma(\gamma) := \bigoplus_{\epsilon \in \mathcal{C}} \gamma_\epsilon
\]

for every fiber functor \( \chi \) and natural transformation \( \gamma \) between fibred functors.

**Lemma 2.3.11.** Let \( \mathcal{A} \) be a small \( k \)-linear category and let \( \chi_1, \chi_2: \mathcal{A} \to \text{RMod}_R \) be functors with images in \( \text{add}(R) \). Define \( (\chi_1 \otimes_R \chi_2): \mathcal{A} \times \mathcal{A} \to \text{RMod}_R \) by setting

\[
(\chi_1 \otimes_R \chi_2)(p, q) = \chi_1(p) \otimes_R \chi_2(q), \quad \text{for} \quad p, q \in \mathcal{A}.
\]

Then

(i) There is a left \( R^e \)-linear isomorphism \( \mathcal{L}(\chi_1 \otimes_R \chi_2) \cong \mathcal{L}(\chi_1) \otimes_R \mathcal{L}(\chi_2) \).

(ii) For every \( R \)-bimodule \( M \), there is a natural isomorphism

\[
\begin{align*}
\text{Nat} \left( (\chi_1 \otimes_R \chi_2), M \otimes_R (\chi_1 \otimes_R \chi_2) \right) & \xrightarrow{\sigma} \text{Hom}_{R-R} \left( \mathcal{L}(\chi_1) \otimes_R \mathcal{L}(\chi_2), M \right) \\
\end{align*}
\]

where \( \sigma(p, q)(u \otimes_R v) = \sum_i m_i \otimes_R p_i \otimes_R q_i \in M \otimes_R p \otimes_R q, \quad \text{for every} \quad u \in p, \varphi \in \mathcal{V}p, v \in q, \psi \in \mathcal{V}q \) and \((p, q) \in \mathcal{A} \times \mathcal{A}\).

**Proof.** Straightforward.

Let us come back to our situation. We are considering the functor

\[
\chi: k[N] \to \mathcal{C} \text{Comod}, \quad \text{sending} \quad n \mapsto Q_n.
\]

On the one hand, we already observed that the composition of \( \chi \) with the left forgetful functor gives rise to a fiber functor \( k[N] \to \text{add}(R) \). On the other hand, we can consider also the fiber functor
\( \chi : \mathbb{N} \to R\text{Mod}_R \) obtained by composing the functor \( \chi : \mathbb{N} \to \mathcal{C}\text{Comod} \) with the functor \( \mathcal{C}\text{Comod} \to R\text{Mod}_R \) stated in Lemma 2.14. Therefore, it is clear from Lemma 2.3.11 that there is a bijective correspondence between multiplications on \( \mathcal{L}(\chi) = (Q \hat{\otimes}_B \check{\otimes} Q) \) and natural transformations \((\chi \otimes\tau) \to \mathcal{L}(\chi) \otimes_R (\chi \otimes\tau)\). One of those natural transformations can be constructed using the left \( \mathcal{L}(\chi) \)-coaction on the \( Q \)'s, as defined in (22). Thus we have the following statement.

**Lemma 2.3.12.** Let \( Q_* \) be the cochain complex of Subsection 2.2 and \((Q \hat{\otimes}_B \check{\otimes} Q)\) the associated \( R \)-coring. Then there is a natural transformation \((\chi \otimes\tau) \to \mathcal{L}(\chi) \otimes_R (\chi \otimes\tau)\) given by: \( \tilde{\lambda}_{n,m} : Q_n \otimes_R Q_m \to (Q \hat{\otimes}_B \check{\otimes} Q) \otimes_R (Q_n \otimes_R Q_m) \)

\[
\sum_{\alpha,\beta} \left[ (u_n \otimes_A u_m) \otimes_B (\omega_{n,\alpha} \otimes \omega_{m,\beta}) + (u_n \otimes_A \partial u_m) \otimes_B (\omega_{n,\alpha} \otimes \omega_{m,\beta}) \right] \otimes_R \left( \omega_{n,\alpha} \otimes_R \omega_{m,\beta} \right)
\]

for every \( n, m \geq 1 \), and by \( \tilde{\lambda}_{0,n} = \lambda_{0,n} : Q_n \to (Q \hat{\otimes}_B \check{\otimes} Q) \otimes_R Q, \) sending \( u_n \mapsto \sum_{\alpha}(u_n \otimes_B \omega_{n,\alpha} \otimes \omega_{m,\alpha}) \) is a dual basis for \( RQ_n, n \geq 1 \).

**Proof.** This is a routine computation using definitions and dual bases notions. \( \square \)

The following lemma will be used in the sequel.

**Lemma 2.3.13.** Let \( \{\omega_{n,\alpha}, \omega_{n,\alpha}^{\ast}\}_\alpha \) be a dual basis for \( RQ_n \) with \( n > 0 \). Then, for every element \( u_n \in Q_n, u_m \in Q_m \), and \( \varphi_n \in Q_n, \varphi_m \in Q_m \), we have

\[
\sum_{\alpha,\beta} \left[ (u_n \otimes_A \varphi_n) \otimes_B (\omega_{n,\alpha} \otimes \omega_{m,\beta}) \right] \times_R \left( \omega_{n,\alpha} \otimes_A \varphi_m \right) = 0
\]

and

\[
\sum_{\alpha,\beta} \left[ (u_n \otimes_A \varphi_n) \otimes_B (\omega_{n,\alpha} \otimes \omega_{m,\beta}) \right] \times_R \left( \omega_{n,\alpha} \otimes_A \varphi_m \right) = 0
\]

as elements in the \( R^e \)-bimodule \((Q \hat{\otimes}_B \check{\otimes} Q) \times_R (Q \hat{\otimes}_B \check{\otimes} Q)\).

**Proof.** Straightforward. \( \square \)

We then arrive to the \( R^e \)-ring structure of \((Q \hat{\otimes}_B \check{\otimes} Q)\).

**Proposition 2.3.14.** There is a structure of \( R^e \)-ring on \( \mathcal{D} := (Q \hat{\otimes}_B \check{\otimes} Q) \) given by the extension of rings \( R^e \to \mathcal{D} \) sending \( r \otimes s^* \to (r \otimes_B s) \) (i.e. \( \iota_0(r) \otimes_B \iota_0(s), \iota_0 \) is the canonical injection), where the multiplication of \( \mathcal{D} \) is defined by the following rules: for every pair of generic elements \( (u_n \otimes_B \varphi_n) \) and \( (u_m \otimes_B \varphi_m) \) of \( \mathcal{D} \) with \( n, m > 0 \), we set

\[
(u_n \otimes_B \varphi_n) \cdot (u_m \otimes_B \varphi_m) = \left( u_n \otimes_A \varphi_n \right) \otimes_B \left( u_m \otimes_B \varphi_m \right) \]

and

\[
(u_n \otimes_B \varphi_n) \cdot (r \otimes_B s) = (u_n r \otimes_B s \varphi_n), \quad (r \otimes_B s) \cdot (u_m \otimes_B \varphi_m) = (ru_m \otimes_B \varphi_n) \quad \forall r, s \in R.
\]

**Proof.** Using Lemmas 2.2.8 and 2.3.13 one can show that each of the maps \( \tilde{\lambda}_{n,m} \) given in Lemma 2.3.12 is coassociative with respect to the comultiplication of \((Q \hat{\otimes}_B \check{\otimes} Q)\). Hence, its image by the natural isomorphism of Lemma 2.3.11 leads to the stated associative multiplication. The unitary property is clear.

**Remark 2.3.15.** As we have seen, the construction of an \( R^e \)-ring structure on \( \mathcal{D} \) is not an immediate task. Part of this difficulty is clearly due to the fact that the natural transformations which lead to the multiplications on \( \mathcal{D} \) are not easy to construct. The other part is probably due to the fact that, although the category \( \mathbb{N} \) is a monoidal category, the fiber functor \( \chi : \mathbb{N} \to R\text{Mod}_R \) given by the complex \( Q_* \) is not only monoidal since the local ”comultiplication” maps \( Q_{n+m} \to Q_n \otimes_R Q_m \),
2.4. The isomorphism between comatrices and coendomorphisms bialgebroids. Now, we come back to the canonical map. As mentioned in the preamble of the previous subsection, there is a canonical map given explicitly by \((24)\). Thus, using the \(\mathcal{L}\)-coactions of Proposition 2.2.9 we have a morphism of \(R\)-corings \(\text{can}_B : \mathcal{D}^l \rightarrow \mathcal{L}^l\) sending

\[
(u_n \otimes_B \varphi_n) \mapsto \sum_{i_0, i_1, \ldots, i_{n-1}} \pi(a_0 \otimes e_{i_0}) \cdots \pi(a_{n-1} \otimes e_{i_{n-1}}) \varphi_n(e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}),
\]

where \(u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \in Q_n\), and \(\text{can}_B(r \otimes_B s) = \pi(r \otimes s^*)\), for \(r, s \in R\).

Our next goal is to show that \(\text{can}_B\) is an isomorphism of left \(R\)-bialgebroids. To this end, we will need the following proposition.

**Proposition 2.4.17.** For every \(n \geq 1\), \(u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \in Q_n\) and \(\varphi_n \in \ast Q_n\), we have the following equality

\[
(u_n \otimes_B \varphi_n) = \sum_{i_0, i_1, \ldots, i_{n-1}} \left[ (a_0 \otimes_B e_{i_0}), (a_1 \otimes_B e_{i_1}) \cdots (a_{n-1} \otimes_B e_{i_{n-1}}) \right] \varphi_n(e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}})
\]

viewed as elements in the left \(R^*\)-module \(\mathcal{D}^l\). In particular, \(\mathcal{D}\) is generated, as an \(R^*\)-ring, by the image of \(R^*\) and the set of elements \(\{(e_i \otimes_B e_j)\}_{i, j}\) (recall that \(\{(e_i, e_i)\}\) is a dual basis of \(RA\)).

**Proof.** It follows by induction, using the dual basis of the \(Q\)’s given in Lemma 2.2.8.

**Theorem 2.4.18.** The canonical map \(\text{can}_B : \mathcal{D} \rightarrow \mathcal{L}\) of \((31)\) is an isomorphism of left \(R\)-bialgebroids.

**Proof.** First we will show that \(\text{can}_B\) is a multiplicative map. By Proposition 2.4.17 this is equivalent to show that

\[
\text{can}_B(a \otimes_B \varphi) \text{can}_B(u_n \otimes_B \varphi_n) = \text{can}_B\left( (a \otimes_B \varphi) \otimes_B (u_n \otimes_B \varphi_n) \right),
\]

for every \(a \in A\), \(\varphi \in \ast A\), \(u_n \in Q_n\), \(\varphi_n \in \ast Q_n\) with \(n \geq 1\). Equality \((32)\), is proved by direct computation. Since \(\text{can}_B\) preserves the unit, we deduce that \(\text{can}_B\) is a morphism of \(R^*\)-rings. The inverse of \(\text{can}_B\) is constructed as follows. It is clear that the map \(\zeta : (A \otimes \ast A) \rightarrow \mathcal{D}\) sending \(a \otimes \varphi \mapsto a \otimes_B \varphi\) is an \(R^*\)-bilinear map. Therefore, it is canonically extended to the tensor algebra \(\zeta : \mathcal{T}_{R^*}((A \otimes \ast A)) \rightarrow \mathcal{D}\), as \(\mathcal{D}\) is an \(R^*\)-ring. Now, for every \(a, b \in A\) and \(\varphi \in \ast A\), one shows that

\[
\zeta\left( \sum_i (a \otimes e_i \varphi) \otimes_R (b \otimes e_i) \right) = \zeta(ab \otimes \varphi),
\]

where \(\{(e_i, e_i)\}\) is the dual basis of \(RA\). This means that \(\zeta\) factors throughout the canonical projection \(\pi : \mathcal{T}_{R^*}((A \otimes \ast A)) \rightarrow \mathcal{L}\), and so we have an algebra map \(\zeta : \mathcal{L} \rightarrow \mathcal{D}\). Given \(a \in A\) and \(\varphi \in \ast A\), we have

\[
\text{can}_B \circ \zeta(\pi(a \otimes \varphi)) = \text{can}_B(a \otimes_B \varphi) = \sum_i \pi(a \otimes_B e_i) \varphi(e_i)
\]

\[
= \pi\left( \sum_i a \otimes e_i \varphi(e_i) \right) = \pi(a \otimes \varphi).
\]
This implies that \( \text{can}_B \circ \zeta = \text{id}_\mathcal{L} \). Now, take \( u_n \in Q_n, n \geq 1, \) of the form \( u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \) and \( \varphi_n \in \ast Q_n. \) Then, by Proposition 2.4.17 we have
\[
\zeta \circ \text{can}_B(u_n \otimes_B \varphi_n) = u_n \otimes_B \varphi_n,
\]
and this shows that \( \zeta \circ \text{can}_B = \text{id}_\mathcal{L}. \)

**Corollary 2.4.19.** Let \((\mathcal{L}\ell)^*\) be the right convolution ring of the \(R\)-coring \(\mathcal{L}_1\). Then there is an isomorphism of rings \((\mathcal{L}\ell)^* \cong \text{End}(Q_B)\).

*Proof.* We know that each \(h_n \ast Q = \ast Q_n\) is a finitely generated and projective right \(R\)-module, where the \(h\)'s are defined in (26). The same property holds true for each right \(R\)-module of the form \(e_{i_1n} \ast Q\), where \(e_{i_1n} = h_{i_1} + \cdots + h_{in}\). This means that the unital bimodule \(B \ast Q_R\) satisfies the second condition of [11 Proposition 5.1] for each idempotent which belongs to the set of local units of \(B\). Therefore we have, as in the proof of [11 Proposition 5.1], that the functor \(- \otimes_B \ast Q\) is left adjoint to \(- \otimes_R Q\). Hence
\[
\text{Hom}_B(\mathcal{L}, R) = \text{Hom}_R(Q \otimes_B \ast Q, R) \cong \text{Hom}_B(Q, Q).
\]
Now, we conclude by Theorem 2.4.18.

### 3. Categories of comodules and chain complexes of modules.

This section contains our main results, namely Theorems 3.1.3 and 3.3.10. As a consequence of these results, we obtain that the category of chain complexes of \(R\)-modules is always equivalent to the category of left comodules over a quotient \(R\)-coring of the left \(R\)-bialgebroids \(\mathcal{L}(A)\) constructed in Example 2.1.3 When \(R\) is commutative, this quotient inherits a left \(R\)-bialgebroid structure from \(\mathcal{L}(A)\), and the stated equivalence is actually a monoidal equivalence. This will clarify the equivalence of categories already constructed by Pareigis and Tambara, [24, 28].

#### 3.1. Monoidal equivalence between chain complexes of \(k\)-modules and left \(\mathcal{L}\)-comodules.

In this subsection we will use the isomorphism of bialgebroids stated in Theorem 2.4.18 to show that the following are equivalent: 1) \(Q_B\) is faithfully flat, 2) the underlying module \(R \otimes \mathcal{L}\) of \(\mathcal{L}\) is flat and the functor \(Q \otimes_B - : B\text{-Mod} \to \mathcal{L}\text{-Comod}\) is a monoidal equivalence of categories. This is our first main result, and stated below as Theorem 3.1.3.

**Remark 3.1.1.** Let \(B = k^{(N)} \oplus k^{(N)}\) be the ring with enough orthogonal idempotents associated to the small \(k\)-linear category \(k^{(N)}\) considered in Subsection 2.3. We have already observed in Subsection 2.3 that the category of unital left \(B\)-modules \(B\text{-Mod}\) is in a canonical way isomorphic to the category \(Ch_+(k)\) of chain complexes of \(k\)-modules. Therefore, \(B\text{-Mod}\) inherits a structure of monoidal category. Recall that \(B\) is generated as a free \(k\)-module by the set of elements \(\{h_n, v_n\}_{n \in \mathbb{N}}\) with \(\{h_n\}_{n \in \mathbb{N}}\) as a set of orthogonal idempotents given by (26). The multiplication of two object \(X, Y \in B\text{-Mod}\), is then given by
\[
X \otimes Y = \bigoplus_{n \in \mathbb{N}} \left( \bigoplus_{i+j=n} h_i X \otimes h_j Y \right)
\]
That is, \(h_n(X \otimes Y) = \bigoplus_{i+j=n} h_i X \otimes h_j Y, \) for every \(n \in \mathbb{N}\), and for every \(k \geq 1, l \geq 1\) with \(k + l = m\), we have
\[
v_{m-1}(h_k x \otimes h_l y) = v_{k-1} x \otimes h_l y + (-1)^k h_k x \otimes v_{l-1} y,
\]
(i.e. the Leibniz rule), and
\[
v_{m-1}(h_0 x \otimes h_m y) = h_0 x \otimes v_{m-1} y, \quad v_n(h_n x \otimes h_0 y) = v_{n-1} x \otimes h_0 y
\]
for every $x \in X$, $y \in Y$, and $m, n \geq 1$. The multiplication of $B$-linear maps is obvious. The unit object is the left unital $B$-module $\mathbb{k}[0]$ whose underlying $k$-module is $k$, and whose $B$-action is given by

$$h_n \mathbb{k}[0] = \begin{cases} 0, & \text{if } n \neq 0 \\ k, & \text{if } n = 0. \end{cases}$$

We know that the cochain complex $Q_\bullet$ of Subsection 2.2 induces an $\mathcal{L}$-comodule $Q = \oplus_{n \in \mathbb{N}} Q_n$ whose coaction is easily seen to be right $B$-linear. Thus, $Q \otimes_B : \mathcal{B} \text{Mod} \to \mathcal{F} \text{Comod}$, acting in the obvious way, is a well-defined functor. This functor is in fact monoidal with structure $\mathcal{B}$-object is the left unital $R$-module.

**Lemma 3.1.2.** Consider the monoidal categories $\mathcal{B} \text{Mod}$ and $\mathcal{F} \text{Comod}$, with structure, respectively, given in Remark 2.1.1 and Lemma 2.1.4. Then $Q \otimes_B : \mathcal{B} \text{Mod} \to \mathcal{F} \text{Comod}$ is a monoidal functor, with structure

$$\Gamma^2_{X,Y} : (Q \otimes_B X) \otimes_R (Q \otimes_B Y) \to Q \otimes_B (X \otimes Y),$$

explicitly given by

$$\Gamma^2_{X,Y} \left( (u_n \otimes_B h_n x) \otimes_R (u_m \otimes_B h_m y) \right) = \begin{cases} (u_n \otimes_A u_m) \otimes_B (h_n x \otimes y_{m-1}) + (u_n \otimes_A \partial u_m) \otimes_B (h_n x \otimes h_m y), & n, m \geq 1 \\ u_n u_m \otimes_B (h_n x \otimes h_m y), & n = 0 \text{ or } m = 0, \end{cases}$$

for every $u_n \in Q_n$, $u_m \in Q_m$, $x \in X$ and $y \in Y$, and $\Gamma^0 : R \to Q \otimes_B \mathbb{k}[0]$ sending $r \mapsto r \otimes_B h_0 1$.

**Proof.** The fact that $\Gamma^2_{X,Y}$ is a well-defined map comes from the observation that the right $R$-action of $Q \otimes_B X$ as left $\mathcal{L}$-comodule is given by the right $R$-action of $Q$ viewed as left $\mathcal{L}$-comodule. That is, the one given by the rule (16). Now, it is easily seen that the right $R$-action of $Q$ given by (16) is exactly the right $R$-action of $Q$ we started with (i.e. that which comes from the inclusion $R K_R \subset A \otimes_R A$). A direct computation, using Lemma 2.2.10 shows that $\Gamma^2_{X,Y}$ is left $\mathcal{L}$-colinear, for each $X, Y$. We leave to the reader the proof of the associativity and unitary properties of the pair $(\Gamma^2_{X,Y}, \Gamma^0)$. □

Our first main result is the following.

**Theorem 3.1.3.** Let $R$ be an algebra over a commutative ground ring $k$, and $A$ an $R$-ring which is finitely generated and projective as left $R$-module. Consider the associated left $R$-bialgebroid given in Proposition 2.1.1 and let $B = k^{(R)} \oplus k^{(R)}$ be the ring with enough orthogonal idempotents of (27). Consider the cochain complex $Q_\bullet$ of Subsection 2.2 with its canonical right unital $B$-action and left $\mathcal{L}$-coaction. Then the following statements are equivalent

1. The right module $\mathcal{L}^R_R$ is flat and the functor $Q \otimes_B : \mathcal{B} \text{Mod} \to \mathcal{F} \text{Comod}$ is an equivalence of monoidal categories;
2. $Q_B$ is a faithfully flat unital module.

**Proof.** The monoidal condition is, by Lemma 3.1.2, always satisfied, so it can be omitted in the proof. Henceforth, we only need to show that $\mathcal{L}^R_R$ is flat and $Q \otimes_B$ is an equivalence, if and only if $Q_B$ is a faithfully flat module. By the left version of the generalized faithfully flat descent Theorem 5.9 of [13, Theorem 5.9], we know that $Q_B$ is faithfully flat if and only if $Q^R_R = 1 \otimes_{R^e}(Q \otimes_B \mathcal{V}(Q)$ is flat and $Q \otimes_B : \mathcal{B} \text{Mod} \to \mathcal{F} \text{Comod}$ is an equivalence of category. We then conclude by Theorem 2.4.18. □

Notice that, when $Q_B$ is faithfully flat, the inverse of the functor of $Q \otimes_B : \mathcal{B} \text{Mod} \to \mathcal{F} \text{Comod}$ is given by the cotensor product $\mathcal{V} Q \square_{\mathcal{F}} : \mathcal{F} \text{Comod} \to \mathcal{B} \text{Mod}$. The structure of bicomodule on $\mathcal{V} Q$ is
can not affirm that the forgetful functor
By the foregoing observations, this is a direct consequence of Theorem 3.1.3. Proof.

a flat module, see [14, Proposition 2.1].

the canonical isomorphism of categories explicitly given in Subsection 2.3. In the case when

Ch

here is to extend the result of Theorem 3.1.3 to the category

Equivalence between chain complexes of

modules and the category of left

R

modules. In other words, we are interested in relating the category of chain complexes of left

Q

a field, it is known that

non commutative field, that is, a division ring is given in Theorem 3.3.10 below). We thus obtain the

following corollary

COROLLARY 3.1.4. [28, Theorem 4.4] Let k be a field and A an k-algebra such that 1 < dim

Consider the associated coendomorphism k-bialgebra L given in Proposition 2.1.1. Then the category

Ch+ (k) of chain complexes of k-modules and denote by Ω : Ch+ (k) → BMod the canonical isomorphism of categories explicitly given in Subsection 2.3.

In the case when R = k is a field, it is known that Qk is always faithfully flat wherever dimk(A) < ∞ (a complete proof for a non commutative field, that is, a division ring is given in Theorem 3.3.10 below). We thus obtain the following corollary

Explicitly, the composition of the functor given in Corollary 3.1.4 with the forgetful functor χComod → kMod is given as follows. For any chain complex V• in Ch+ (k), we have

\[ Q \otimes_B Ω(V•) = \bigoplus_{n \geq 0} (Q_n \otimes V_n) / (\partial u_n \otimes x_{n+1} - u_n \otimes \partial x_{n+1}) \] for n ≥ 0.

3.2. Equivalence between chain complexes of R-modules and \( L \)-comodules. Our main aim here is to extend the result of Theorem 3.1.3 to the category Ch+ (R) of chain complexes over left R-modules. In other words, we are interested in relating the category of chain complexes of left R-modules and the category of left \( L(A) \)-comodules over the left \( R \)-bialgebroid of Proposition 2.1.1.

Precisely, we show an analogue of Theorem 3.1.3 where \( L \) is replaced by its quotient \( R \)-coring \( L := L(A) / \langle L(r \otimes 1^o - 1 \otimes r^o) \rangle_{r \in R} \) and the ring \( B \) by its extension \( C = R(\oplus) \oplus R(\oplus). \) This is our second main result, see Theorem 3.2.5. Of course, in this case, the monoidal equivalence of categories is reduced to an equivalence, unless the base ring \( R \) is commutative and the extension \( A \) is an \( R \)-algebra.

Let \( A \) be an \( R \)-ring and assume that \( RA \) is a finitely generated and projective module. Fix a dual basis \( \{e_i, *e_i\}_{i} \) for \( RA \), and consider \( L := L(A) \) the left \( R \)-bialgebroid of Proposition 2.1.1. We denote by \( \pi : L(R) ((A \otimes ^\ast A)) \rightarrow L \) the canonical projection.

**LEMMA 3.2.5.** Let \( I \) be the left ideal of \( L \) generated by the following set of elements

\[ \left\{ \pi(a \otimes \varphi) - \pi(a \otimes r \varphi) \right\}_{a \in A, \varphi \in ^\ast A, r \in R} \]

Then \( I \) is a coideal of the underlying \( R \)-coring \( L^I \).

**Proof.** An easy computation shows that

\[ \pi(a \otimes \varphi) - \pi(a \otimes r \varphi) = \pi(a \otimes \varphi) \left( r \otimes 1^o - 1 \otimes r^o \right) \]
for every elements \( a \in A, \varphi \in \ast A \) and \( r \in R \). Thus, \( \mathcal{J} \) as left \( R^e \)-bimodule is generated by the set \( \{ \mathcal{g}_r := 1_{\mathcal{X}}(r \otimes 1^o - 1 \otimes r^o) \}_{r \in R} \). For arbitrary elements \( x \in \mathcal{L} \) and \( r \in R \), we get
\[
\varepsilon(x \mathcal{g}_r) = \varepsilon\left( x(1 \otimes \varepsilon(\mathcal{g}_r)^o) \right) = 0,
\]
as \( \varepsilon(\mathcal{g}_r) = 0 \). Hence, \( \varepsilon(\mathcal{J}) = 0 \). On the other hand, for every \( r \in R \), we have
\[
\Delta(\mathcal{g}_r) = (1_{\mathcal{X}} \otimes_R 1_{\mathcal{X}})(r \otimes 1^o) - (1_{\mathcal{X}} \otimes_R 1_{\mathcal{X}})(1 \otimes r^o).
\]
Using these equalities we can show that, for every \( x \in \mathcal{L} \) and \( r \in R \), we have
\[
\Delta(x \mathcal{g}_r) = \sum_{(x)} x_{(1)} \otimes_R x_{(2)}(r \otimes 1^o) - \sum_{(x)} x_{(1)} \otimes_R x_{(2)}(1 \otimes r^o)
= \sum_{(x)} x_{(1)} \otimes_R x_{(2)}(r \otimes 1^o - 1 \otimes r^o),
\]
where \( \Delta(x) = \sum_{(x)} x_{(1)} \otimes_R x_{(2)} \). Therefore, \( (\pi \otimes_R \pi) \circ \Delta(x \mathcal{g}_r) = 0 \), for every \( x \in \mathcal{L} \) and \( r \in R \), where \( \pi : \mathcal{L} \to \mathcal{L}/\mathcal{J} \) is the canonical projection. Thus \( \mathcal{J} \) is a coideal of \( \mathcal{L} \).

Denote by \( \overline{\mathcal{L}} := \mathcal{L}/\mathcal{J} \) the quotient \( R \)-coring and by \( \pi : \mathcal{L} \to \overline{\mathcal{L}} \) the canonical projection. Notice that \( \pi \) is also left \( \mathcal{L} \)-colinear. Consider the cochain complex \( Q_* \) of Subsection 2.2. We know, by Proposition 2.2.9 that each \( Q_n \) is a left \( \mathcal{L} \)-comodule. Hence each of them is a left \( \overline{\mathcal{L}} \)-comodule with coaction
\[
\overline{\lambda}_n : Q_n \to \mathcal{L} \otimes_R Q_n \to \overline{\mathcal{L}} \otimes_R Q_n, \quad n \geq 0.
\]

**Lemma 3.2.6.** The \( \mathcal{L} \)-coaction \( \overline{\lambda}_n \) is right \( R \)-linear. That is, \( Q_n \) is an \( (\overline{\mathcal{L}}, R) \)-bicodule (here \( R \) is considered as a the trivial \( R \)-coring).

**Proof.** For \( n = 0 \) the statement is trivial since \( \overline{\lambda}_0(r) = (r \otimes 1^o)\pi(1_{\mathcal{X}}) = \pi(1_{\mathcal{X}})(1 \otimes r^o) \), for every \( r \in R \). Take \( n \geq 1 \) and an element \( u_n \in Q_n \) of the form \( u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \). Then, for every \( r \in R \), we have
\[
\overline{\lambda}_n(u_n r) = \sum_{\alpha, i_n} \pi \left( \pi(a_0 \otimes \ast e_{i_0}) \cdots \pi(a_{n-1} r \otimes \ast e_{i_{n-1}}) \right) \otimes_R \omega_{n, \alpha},
\]
where \( \alpha = (i_0, \cdots, i_{n-1}) \), and \( \omega_{n, \alpha} = e_{i_0} \partial e_1 \otimes_A \cdots \otimes_A \partial e_{i_{n-1}} \)
\[
= \sum_{\alpha} \pi \left( \pi(a_0 \otimes \ast e_{i_0}) \cdots \pi(a_{n-1} \otimes r \ast e_{i_{n-1}}) \right) \otimes_R \omega_{n, \alpha},
\]
\[
= \sum_{\alpha} (\pi \otimes_R Q_n) \left[ \pi(a_0 \otimes \ast e_{i_0}) \cdots \pi(a_{n-1} \otimes r \ast e_{i_{n-1}}) (1 \otimes r^o) \otimes_R \omega_{n, \alpha} \right]
= \sum_{\alpha} \pi \left( \pi(a_0 \otimes \ast e_{i_0}) \cdots \pi(a_{n-1} \otimes r \ast e_{i_{n-1}}) \right) \otimes_R \omega_{n, \alpha} r
= \overline{\lambda}_n(u_n) r,
\]
where in the fourth equality we have used that each \( Q_n \) is in fact a left \( \times_R^e \mathcal{L} \)-comodule, that is, \( \lambda_n(Q_n) \subseteq \mathcal{L} \times_R Q_n \). We then conclude by linearity. \( \square \)
Remark 3.2.7. The quotient $R$-coring $\mathcal{L}$ does not admit, in general, a structure of left $R$-bialgebroid. However, if we assume that $R$ is commutative (i.e., a commutative $k$-algebra) and that $A$ is an $R$-algebra, then the left ideal $\mathcal{J}$ is in fact a two-sided ideal, since in this case we have the following equalities

$$g_r \pi (a \otimes \varphi) = \pi (a \otimes \varphi) g_r,$$

for every $r \in R$, $a \in A$, and $\varphi \in A^*$. In view of this, a direct verification shows that $\mathcal{L}$ is an $R$-bialgebroid such that the canonical surjection $\pi : \mathcal{L} \to \mathcal{L}$ is a morphism of $R$-bialgebroids. Notice, that here the prefix "left" was removed before bialgebroid. This is due to the fact that $\mathcal{L}$ is actually an $(R \otimes R)$-algebra, that is, there is only one structure of $(R \otimes R)$-module.

Let us consider the $k$-linear category $R(N)$ whose objects are the natural numbers $N$ and homomorphisms sets are defined by

$$\text{Hom}_{R(N)}(n, m) = \begin{cases} 0, & \text{if } m \notin \{n, n + 1\} \\ R_1 = 1_n R, & \text{if } n = m \\ R \cdot j_n^{n+1} = j_n^{n+1} R, & \text{if } m = n + 1. \end{cases}$$

The last two terms are copies of $R_R R$ viewed as an $R$-bimodule which is free as left and right $R$-module of rank one, generated by an invariant element. The composition is defined using the regular $R$-bimulations of $R_R R$. The induced ring with enough orthogonal idempotents is the free left $R$-module of rank one, generated by an invariant element. The composition is defined using the regular $R$-bimulations $(35)$ generated by elements $\{h_n, u_n\}_{n \in \mathbb{N}}$ subject to the following relations:

$$\begin{align*} h_n h_m &= \delta_{n,m} h_n, \forall n, m \in \mathbb{N} \quad \text{(Kronecker delta)} \\ u_n u_m &= u_m u_n = 0, \forall n, m \in \mathbb{N} \\ u_n h_{n+1} &= u_n = h_n u_n, \forall n, m \in \mathbb{N}. \end{align*}$$

In other words $C$ is the ring of $(N \times N)$-matrices over $R$ of the form

$$C = \begin{pmatrix} R & R & 0 & 0 \\ 0 & R & R & 0 \\ 0 & 0 & R & R \\ \vdots & \ddots & \ddots & \ddots \\ 0 & R & R \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

i.e. with possibly non-zero entries in each row: $(i, i)$ and $(i, i + 1)$. $C$ is also free as right $R$-module, since the generators are invariant. One can easily check that the category of chain complexes of left $R$-modules $Ch_+(R)$ is equivalent to the category of unital left $C$-modules. Let $B$ be the ring with enough orthogonal idempotents of $(27)$. There is a morphism of rings $B \to C$ with the same set of orthogonal idempotents. In this way, we have by [11, page 733] the usual adjunction between left unital $B$-modules and $C$-modules using restriction of scalars and the tensor product functor $C \otimes_B -$.

By Lemma 3.2.6 we have a morphism of rings $R \to \text{End}_C(Q_n)$, for every $n \geq 0$. This leads to a faithful functor from the category $R(N)$ to the category of $(\mathcal{L}, R)$-bicomodules (here $R$ is considered as a trivial $R$-coring) $\chi' : R(N) \to \mathcal{L} \text{Comod}_R$. The composition of $\chi'$ with the forgetful functor gives rise then to a fiber functor $\omega : R(N) \to \mathcal{L} \text{Mod}_R$ whose image is in $\text{add}(R)$. Therefore, we can apply the constructions performed in Subsection 2.3. Thus, we have an infinite comatrix $R$-coring $Q \otimes_C \chi' Q$ together with a canonical map $\text{can}_C : Q \otimes_C \chi' Q \to \mathcal{L}$ sending

$$u_n \otimes_C \varphi_n \xrightarrow{\text{can}_C} \sum_{i_0, \ldots, i_{n-1}} \pi \left( \pi (a_0 \otimes \ast e_{i_0}) \cdots \pi (a_{n-1} \otimes \ast e_{i_{n-1}}) \right) \varphi (e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}).$$
Clearly we have a surjective map $\vartheta : Q \otimes_B \C^\vee Q \to Q \otimes_C \C^\vee Q$. Moreover, we have a commutative diagram with exact rows relating the two $R$-corings morphisms $\text{can}_B$ and $\text{can}_C$ (see equations (31) and (30))

\[\begin{array}{ccccccccc}
0 & \to & \ker(\vartheta) & \to & Q \otimes_B \C^\vee Q & \xrightarrow{\vartheta} & Q \otimes_C \C^\vee Q & \to & 0 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{J} & \to & \mathcal{L} & \xrightarrow{\pi} & \mathcal{I} & \to & 0
\end{array}\]  

\text{(37)}

**Proposition 3.2.8.** In diagram (37), we have the following equality $\text{can}_B(\ker(\vartheta)) = \mathcal{J}$. In particular, the map $\text{can}_C$ of equation (36) is an isomorphism of $R$-corings.

**Proof.** The inclusion $\text{can}_B(\ker(\vartheta)) \subseteq \mathcal{J}$ is clear from the commutative diagram (37). Conversely, consider arbitrary elements $y \in \mathcal{L}$ and $r \in R$. We need to show that $y g_r \in \text{can}_B(\ker(\vartheta))$, where $g_r$ are as in the proof of Lemma 3.2.5. There is no loss of generality if we assume that $y = x \pi(a \otimes \varphi)$, for some $x \in \mathcal{L}$ and $a \in A$, $\varphi \in A$. Since $\text{can}_B$ is, by Theorem 2.4.18, bijective, there exists $u \in Q \otimes_B \C^\vee Q$ such that $x = \text{can}_B(u)$. In view of this, $y g_r = \text{can}_B(u(\ar \otimes_B \varphi - a \otimes_B r \varphi))$, as $\text{can}_B$ is multiplicative. We need to check that $\vartheta(u(\ar \otimes_B \varphi - a \otimes_B r \varphi)) = 0$. However, this is directly obtained from the following equality

$$\vartheta \left( (u_n \otimes_B \varphi_n) (\ar \otimes_B - a \otimes_B r \varphi) \right) = 0,$$

for every $u_n \in Q_n$, $\varphi_n \in A_n$, whose proof follows by induction on $n$. The last statement to prove is a consequence of the first one, since the diagram (37) has exact rows. \hfill $\Box$

Our second main result is the following

**Theorem 3.2.9.** Let $R$ be an algebra over a commutative ground ring $k$, and $A$ an $R$-ring which is finitely generated and projective as left $R$-module. Consider the associated left $R$-bialgebroid $\mathcal{L}$ stated in Proposition 2.1.1 and $\mathcal{J}$ the coideal of $\mathcal{L}$ generated by the set of elements $\{1 \nabla (r \otimes 1 - 1 \otimes r)\}_{r \in R}$. Denote by $\mathcal{Z} = \mathcal{L} / \mathcal{J}$ the corresponding quotient $R$-coring. Let $C = R^{(3)} \otimes R^{(3)}$ be the ring with enough orthogonal idempotents induced from the small $k$-linear category $R(N)$ defined by relations (33). Consider the cochain complex $Q$, given in Subsection 2.2 with its canonical right unital $C$-action and left $\mathcal{Z}$-coaction as in Lemma 3.2.6. Then the following statements are equivalent

1. The right module $\mathcal{Z}_R$ is flat and the functor $Q \otimes_C - : \text{CMod} \to \mathcal{Z}\text{Comod}$ is an equivalence of categories;
2. $Q_C$ is a faithfully flat unital module.

**Proof.** By the left version of the generalized faithfully flat descent Theorem 1.3, Theorem 5.9, we know that $(Q \otimes_C \C^\vee)_{R}$ is flat and $Q \otimes_C - : \text{CMod} \to \text{qComod}$ is an equivalence of categories, if and only if $Q_C$ is faithfully flat. We then deduced the stated equivalence by using the isomorphism of $R$-corings $\text{can}_C : Q \otimes_C \C^\vee Q \cong \mathcal{Z}$ established in Proposition 3.2.8. \hfill $\Box$

Notice that, if $Q_C$ is faithfully flat, then the inverse functor of $Q \otimes_C - : \text{CMod} \to \mathcal{Z}\text{Comod}$ is given by the cotensor product $\C^\vee Q \square_{\mathcal{Z}} : \mathcal{Z}\text{Comod} \to \text{CMod}$. Here the structure of $(C, \mathcal{Z})$-bicomodule of $\C^\vee Q$ is deduced, as was observed in Subsection 5.1, from that of $Q$ using the fact that each of the $Q_n$'s is finitely generated and projective left $R$-module.
3.3. Conditions under which $Q_C$ is faithfully flat. As was seen in Theorems 3.1.3 and 3.2.9 a sufficient and necessary condition for establishing an equivalence of categories of left comodules and chain complexes, is the faithfully flatness of the unital right module $Q$. The proof of this fact is actually the most difficult task in this theory. In this subsection we will analyze assumptions under which $Q_C$ is faithfully flat.

The following is our third main result.

**Theorem 3.3.10.** The notations and assumptions are that of Theorem 3.2.9. Assume further that $A$ is finitely generated and projective, and the cochain complex $Q_*$ is exact and splits, in the sense that, for every $m \geq 1$, $Q_m = \partial Q_{m-1} \oplus \overline{Q}_m = \text{Ker}(\partial) \oplus \overline{Q}_m$ as right $R$-modules, for some right $R$-module $\overline{Q}_m$. Then $Q_C$ is a flat module. Furthermore, if $k$ is a field and $R$ is a division $k$-algebra, then $Q_C$ is faithfully flat.

**Proof.** We first consider the following family of right $R$-modules

$$Q^{(m)} = \begin{cases} \partial Q_m \oplus \overline{Q}_m, & \text{for } m \geq 1 \\ \partial Q_0 \oplus Q_0, & \text{for } m = 0 \end{cases}$$

which we claim to be a family of right unital flat $C$-modules. Using this claim we can easily deduce that $Q_C$ is a flat module since we know that $Q_C = \bigoplus_{m \geq 0} Q_C^{(m)}$. The structure of unital right $C$-module of each $Q^{(m)}$ is given as follows: Denote by $i_m : \partial Q_m \to Q^{(m)}$, $\overline{i}_m : \overline{Q}_m \to Q^{(m)}$ the canonical injections and by $j_m$, $\overline{j}_m$ their canonical projections. For every element $u^{(m)} \in Q^{(m)}$, we set

$$u^{(m)}_m = \begin{cases} 0, & \text{if } n \notin \{m, m+1\} \\ \overline{i}_m j_m (u^{(m)}), & \text{if } n = m \\ i_m j_m (u^{(m)}), & \text{if } n = m + 1 \end{cases}$$

where $\gamma_m : \overline{Q}_m \to Q_m \to \partial Q_m$. That is, the obtained cochain complexes have the following form

$$Q^{(m)} : 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \overline{Q}_m \rightarrow \cdots \rightarrow \partial Q_m \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

Put $e_{n,n+1} = h_n + h_{n+1}$, for every $n \geq 0$. These are idempotents elements in $C$, and the induced rings, i.e. $e_{n,n+1}C e_{n,n+1}$ are all isomorphic to the upper-triangular matrices over $R$. That is, of the form

$$C_{n,n+1} = e_{n,n+1}C e_{n,n+1} = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}, \quad \text{for every } n \in \mathbb{N}.$$ 

It is clear that, for every $m \geq 0$, we have $Q^{(m)} e_{m,m+1} = Q^{(m)}$. Therefore, there is an isomorphism of right unital $C$-modules

$$Q^{(m)} e_{m,m+1} \otimes_{C_{m,m+1}} e_{m,m+1} C \cong Q^{(m)}.$$

Next we will show that each of the right $C_{m,m+1}$-modules $Q^{(m)} e_{m,m+1} = Q^{(m)}$ is finitely generated and projective. This fact, combined with the isomorphisms (39), establish the above claim.

For $m = 0$, it is clear that the right $C_{0,1}$-module

$$Q^{(0)} = R \oplus R = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} C_{0,1}$$
is finitely generated and projective. Now take \( m \geq 1 \), under the hypothesis \( A_R \) is finitely generated and projective, we can show, as in Lemma 2.2.8, that each right \( R \)-module \( Q_m \) is also finitely generated and projective. Thus, we can consider a dual basis \( \{ (q_{m,k}, p_{m,k}) \} \) for each right \( R \)-module \( Q_m \). In this way, we have a right \( C_{m,m+1} \)-linear map

\[
\theta^*_{m,k} : Q^{(m)} \to C_{m,m+1}, \quad \left[ u^{(m)} \mapsto \begin{pmatrix} q_{m,k}(u_m) & p_{m,k}(u_m) \\ 0 & 0 \end{pmatrix} \right],
\]

where \( q_{m,k} \in Q_m \) is the projection of \( x_m \in Q_m = \partial Q_{m-1} \oplus Q_m \), defined by \( j_m(u^{(m)}) = \partial x_m = \partial y_m \). We should mention that, under our assumptions, the maps \( \theta^*_{m,k} \) are well defined. Effectively, if there is some other element \( y_m \in Q_m \) such that \( j_m(u^{(m)}) = \partial x_m = \partial y_m \), then \( x_m - y_m \in \text{Ker}(\partial_m) = \partial Q_{m-1} \), which means that they have equal image \( q_{m,k} = \overline{q}_m \) in \( Q_m \cap Q_m \). It is convenient to check that \( \theta^*_{m,k} \) are right \( C_{m,m+1} \)-linear. But first we will identify the right module \( Q_m \) with the quotient of \( Q_m \), \( \overline{Q}_m = Q_m / \partial Q_{m-1} \). The right \( C_{m,m+1} \)-action of \( Q^{(m)} \) is given as follows: Take an element \( u^{(m)} \in Q^{(m)} \) and write it in the form \( u^{(m)} = (q_m, \partial p_m) \) for some elements \( q_m, p_m \in Q_m \). Here \( j_m(u^{(m)}) = \partial p_m \) and \( j_m(u^{(m)}) = \overline{q}_m \).

\[
(\overline{q}_m, \partial p_m) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \left( q_m r_{11}, \partial q_m r_{12} + \partial p_m r_{22} \right),
\]

for every element \( \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \in C_{m,m+1} \). Therefore,

\[
\theta^*_{m,k} \left( \overline{q}_m, \partial p_m \right) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \theta^*_{m,k} \left( q_m r_{11}, \partial q_m r_{12} + \partial p_m r_{22} \right)
\]

\[
= \begin{pmatrix} q_m^* \left( q_m^* r_{11} \right) & q_m^* \left( \overline{p}_m r_{22} \right) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q_m^* \left( q_m^* r_{11} \right) & q_m^* \left( q_m^* r_{12} + \overline{p}_m r_{22} \right) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q_m^* \left( q_m^* r_{11} \right) & q_m^* \left( \overline{q}_m r_{22} \right) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \theta^*_{m,k} \left( \overline{q}_m, \partial p_m \right) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}.
\]

Take an arbitrary element \( (\overline{q}_m, \partial p_m) \in Q^{(m)} \), we have

\[
(\overline{q}_m, \partial p_m) = (\overline{q}_m, 0) + (0, \partial p_m) = (\overline{q}_m, 0) + (\overline{p}_m, 0) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right).
\]
For instance, it is clear from Lemma 2.2.5 and Remark 2.2.6 that it is satisfied by assuming that the ring extension $R$ is a division ring and each component of $Q^{(m)}$ is by assumption finite dimensional with $d = \dim_R(Q_m) \leq \dim_R(\partial Q_m) = d'$, we can split $Q^{(m)}$ as

$$Q^{(m)} \cong (eT)^d \oplus ((1 - e)T)^{d-d},$$

and this shows that $Q^{(m)}$ is a progenerator. Let $f : X \to Y$ be a morphism of right unital $C$-modules such that $Q \otimes_C f = 0$. Hence $Q^{(m)} \otimes_C f = 0$, for every $m \geq 0$, as $Q_C = \oplus_{m \geq 0} Q^{(m)}$. Therefore, we have

$$0 = Q^{(m)} \otimes_C f \cong Q^{(m)}_{C, m+1} \otimes_{C, m+1} C \otimes_C f; \quad \forall m \geq 0 \implies e_{m, m+1}C \otimes_C f = 0, \forall m \geq 0.$$

This means that $h_m C \otimes_C f = 0$, for every $m \geq 0$, and so $f = 0$. This shows that $Q \otimes_C -$ is a faithful functor, which completes the proof.

**Remark 3.3.11.** As one can see, the hypothesis on the complex $Q_\bullet$ in Theorem 3.3.10 is not easy to check. However, under further conditions on the ring extension $R \to A$, this hypothesis is satisfied. For instance, it is clear from Lemma 2.2.5 and Remark 2.2.6 that it is satisfied by assuming that the ring extension $R \to A$ splits either in the category of right or left $R$-modules. Obviously this includes the case when $A$ is free as right (or left) $R$-module with $1_A$ as an element of the canonical basis. In particular, this is the case when $R$ is a division ring.

**Corollary 3.3.12.** Let $D$ be a division $k$-algebra over a field $k$, and $A$ a $D$-ring which is finite dimensional as left and right $D$-vector space with dimension $\geq 2$. Consider the associated left $D$-bialgebroid $\mathcal{L}$ given by Proposition 2.1.1 and its coideal $\mathcal{J}$ of Lemma 3.2.5. Then the category $Ch_+(D)$ of chain complexes of left $D$-vector spaces is equivalent to the category of left $(\mathcal{L}/\mathcal{J})$-comodules.
Proof. It follows from Theorems 3.2.9 and 3.3.10. □

3.4. The main example. Here we will explain why Pareigis’s example [24], even in the noncommutative case, always works. Thus, we will check using the first statement of Theorem 3.3.10 that the cochain complex $Q_\bullet$ associated to the example of the $R$-ring $A$ considered in 2.1.3, always satisfies condition (2) of Theorem 3.2.9. In this way the category $Ch_+(R)$ of chain complexes of left $R$-modules is always equivalent to the category of left $\mathcal{L}(A)$-comodules, where $\mathcal{L}(A)$ is the left $R$-bialgebroid described in Example 2.1.3.

Recall from Example 2.1.3 the $R$-ring $A = R \oplus Rt$ which is the trivial generalized ring extension of $R$. Set $1_A = (1, 0)$ and $t = (0, t)$, so we have $t^2 = 0$. It is easily seen that the kernel of the multiplication of $A$, i.e. $K = \text{Ker}(A \otimes_R A \to A)$ is free as a left and right $R$-module with basis $\{\partial t, t\partial t\}$. In fact $K$ is a free $A$-module with rank one and basis $\partial t$. We summarize the properties of the cochain complex $Q_\bullet$, as follows.

**Proposition 3.4.13.** The cochain complex $Q_\bullet$ associated to the trivial generalized ring $A = R \oplus Rt$, fulfills the following properties:

(i) For every $m \geq 2$, $Q_m$ is free as a left and right $R$-module with rank two, and its basis (on both sides) is given by the set $\{t\partial A \cdots \partial A \partial t, \partial A \cdots \partial A \partial t\}$.

(ii) $Q$ is a homotopically trivial complex.

(iii) $Q_C$ is faithfully flat module.

**Proof.** (i) This is proved by induction on $m$.

(ii) The homotopy is given by switching the dual basis. Let $q_m = \partial A \cdots \partial A \partial t, ((m - 1)$-times) and $q_1 = 1_A$ be the generating element of $Q_m$. Then we define a homotopy $h_m : Q_{m+1} \to Q_m$ by sending $q_{m+1} \mapsto t q_m$ and $t q_{m+1} \mapsto q_m$, $h_0$ is the first projection.

(iii) The fact that $Q_C$ is flat follows from Theorem 3.3.10, since we know that $Q_\bullet$ is exact and splits either by Lemma 2.2.5 or by item (ii) and [18, Théorème 2.4.1]. Following the notations of the proof of Theorem 3.3.10 we can easily see that each right $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$-module $Q^{(m)} = \partial Q_{m-1} \oplus t q_m R$ is isomorphic to $e T$, where $e$ is the canonical idempotent of $T$. Henceforth, the same argument of the last part of the proof of Theorem 3.3.10 serves to deduce that $Q_C$ is actually a faithfully flat module. □

**Corollary 3.4.14.** Let $R$ be any $k$-algebra and $A = R \oplus Rt$ its trivial generalized extension. Consider the left $R$-bialgebroid $\mathcal{L}(A)$ described in Example 2.1.3 and its quotient $R$-coring $\mathcal{L}(A)$ by the left ideal $\langle 1_{\mathcal{L}(A)}(r \otimes 1^o - 1 \otimes r^o) \rangle$. Then the functor $Q \otimes_C$ establishes an equivalence between the categories of chain complexes of left $R$-modules and the category of left $\mathcal{L}(A)$-comodules. In particular, if $R$ is a commutative ring, then $Q \otimes_C$ establishes in fact a monoidal equivalence.

**Proof.** The main claim is an immediate consequence of Proposition 3.4.13 and Theorem 3.2.9. In the last statement, the functor in question can be shown to be monoidal using a similar proof of Lemma 3.1.2. □

**Acknowledgements** A. Ardizzoni would like to thank all the members of the Department of Algebra of the University of Granada for the warm hospitality during his visit. L. El Kaoutit would like to thank all the members of the Department of Mathematics of University of Ferrara for a warm hospitality during his visit.
REFERENCES

[1] M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra, **11** (1969), 532–563.

[2] N. Bourbaki, *Éléments de Mathématique. Algèbre commutative. Chapithres 1 à 4*, Masson, Paris 1985.

[3] T. Brzeziński and G. Militaru, *Bialgebroids, ×_R-Bialgebras and duality*, J. Algebra, **251** (2002), 279–294.

[4] T. Brzeziński and R. Wisbauer, *Corings and comodules*, LMS, vol. 309, Cambridge University Press, 2003.

[5] G. Böhm, *Hopf algebroids*. Handbook of algebra. Vol. 6, 173–235. Elsevier/North-Holland, Amsterdam, 2009.

[6] A. Bruguières, *Théorie tannakiennne non commutative*, Commun. in Algebra **22** (1994), 5817–5860.

[7] S. Caenepeel, E. De Groot and J. Vercruysse, *Constructing infinite comatrix corings from colimits*. Appl. Cat. Structures **14** no.5-6 (2006), 539–565.

[8] Henri Cartan, *Homologie et cohomologie d’une algèbre graduée*. Séminaire Henri Cartan, tome 11 n°. 2 (1958-1959), exp. n°. 15, p. 1-20.

[9] P. Deligne, *Catégories tannakiennes*. In *The Grothendieck Festschrift* (P. Cartier et al., eds), Progr. math., 87, vol. II, Birkhäuser, Boston, MA. 1990, pp. 111–195.

[10] A. Dold, *Homology of symmetric products and other functors of complexes*. Ann. Math. **68**, No. 1 (1958), 54–80.

[11] L. El Kaoutit, *Corings over rings with local units*, Math. Nachr. **282** (2009), no. 5, 726–747.

[12] L. El Kaoutit and J. Gómez-Torrecillas, *Comatrix corings: Galois coring, Descent theory, and a structure theorem for cosemisimple comodules*, Math. Z. **244** (2003), 887–906.

[13] L. El Kaoutit and J. Gómez-Torrecillas, *Infinite comatrix corings*. IMRN, **39** (2004), 2017–2037.

[14] L. El Kaoutit, J. Gómez-Torrecillas, and F. J. Lobillo, *Semisimple corings*, Algebra Colloq. **11** (2004), no. 4, 427–442.

[15] Phùng Hồ Hai, *Tannaka-Krein duality for Hopf algebroids*, Israel J. Math. **167** (2008), 193–225.

[16] M. Hovey, *Model categories*. Mathematical surveys and Monographs. Vol. **63**, AMS. 1999.

[17] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[18] R. Godement, *Topologie algébrique et théorie des faisceaux*. Troisième édition. Hermann, Paris 1973.

[19] J. Gómez-Torrecillas and J. Vercruysse, *Comatrix corings and Galois comodules over firm rings*, Alg. Rep. Theory, **10** (2007), 271–306.

[20] Paul G. Goerss, John F. Jardine, *Simplicial Homotopy Theory*. Progress in Mathematics, Vol. 174. Birkhäuser, Basel; Boston; Berlin. (1999).

[21] A. Joyal and R. Street, *An Introduction to Tannaka duality and quantum groups*. In "Category Theory Proceedings, Como 1990 ", Lecture Note in Math. Vol. 1488, pp. 411–492, Springer-Verlag, Berlin 1991.

[22] S. Mac Lane, *Homology*. Grundlehren der math. Wissensch. **114**, Academic Press. Inc., Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[23] Paddy McCrudden, *Categories of Representations of coalgebroids*, Adv. Math. **154**, 299–332 (2000).

[24] B. Pareigis, *A non-commutative non-cocommutative Hopf algebra in "nature"*, J. Algebra **70** (1981), 356–374.

[25] P. Schauenburg, *Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules*. App. Catg. Struct. **6** (1998), 193–222.

[26] M. Sweedler, *Groups of simple algebras*. I. H. E. S. Publ., **44** (1975), 79–189.

[27] M. Takeuchi, *Groups of algebras over A ⊗ A*, J. Math. Soc. Japan **29** (1977), 459–492.

[28] D. Tambara, *The coendomorphism bialgebra of an algebra*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), no. 2, 425–456.

[29] R. Wisbauer, *On Galois comodules*, Commun. Algebra **34** (2006), 2683–2711.

**University of Ferrara**, **Department of Mathematics**, **Via Machiavelli 35**, **Ferrara, I-44121, Italy**

*E-mail address*: rdzlsn@unife.it

*URL*: http://www.unife.it/utenti/alessandro.ardizzoni

**Universidad de Granada**, **Departamento de Álgebra. Facultad de Educación y Humanidades de Ceuta. El Greco N. 10. E-51002 Ceuta, Spain**

*E-mail address*: kaoutit@ugr.es

**University of Ferrara**, **Department of Mathematics**, **Via Machiavelli 35**, **Ferrara, I-44121, Italy**

*E-mail address*: men@unife.it

*URL*: http://www.unife.it/utenti/claudia.menini