Products and tensor products of graphs and homomorphisms

Izak Broere  
Department of Mathematics and Applied Mathematics, University of Pretoria  
Johannes Heidema  
Emeritus, Department of Mathematical Sciences, University of South Africa

Comments: 12 pages  
Subjects: Combinatorics (math.CO); Discrete Mathematics (cs.DM)  
MSC-classes: 05C76 (Primary), 05C25 (Secondary)  
Corresponding author: Izak Broere  
Email: izak.broere@up.ac.za  
Telephone: +27-12-420-2611  
Fax: +27-12-420-3893

Abstract

We introduce and study, for a process P delivering edges on the Cartesian product of the vertex sets of a given set of graphs, the P-product of these graphs, thereby generalizing many types of product graph. Analogous to the notion of a multilinear map (from linear algebra), a P-morphism is introduced and utilised to define a P-tensor product of graphs, after which its uniqueness is demonstrated. Congruences of graphs are utilised to show a way to handle projections (being weak homomorphisms) in this context. Finally, the graph of a homomorphism and a P-tensor product of homomorphisms are introduced, studied, and linked to the P-tensor product of graphs.

1 Preliminaries

At least half a dozen types of product graph have been introduced and studied. One of these types, the “direct product”, is known under at least ten different names ([3], p. 36). In the interest of some consolidation towards the identification and study of commonalities in the spectrum of possible products, we introduce the general notion of a “P-product” of graphs, covering a number of cases. One of the names of the direct product is “tensor product”, a well-known concept in linear algebra and category theory – it can be defined by the tensor product of matrices. Rosinger [5] generalizes the notion of tensor product from vector spaces to structures with arbitrary binary operations, and even beyond those to structures with arbitrary “generators” on the underlying sets, with then binary operations as a special case of generators. We show, i.a., that the P-product – and hence all its instances – has all the attributes required of a general “tensor product”. The notion of a P-(tensor) product and its role can be transposed from graphs to (sets of) graph homomorphisms, with analogous results.

For those notions on graphs in general not defined here, we refer the reader to [2]. The Handbook of Product Graphs [3] is a comprehensive treatise on many types of product of mainly finitely many, mainly finite graphs. Except when explicitly stated otherwise, all graphs considered here are simple, undirected and unlabelled, and have non-empty vertex sets. There is, in general, no upper bound on the cardinalities of sets we use; neither on the vertex sets (and edge sets)
of graphs we use nor on index sets (used, amongst others, to describe a set of graphs). More definitions, especially of the concepts “loop-allowing graph”, “loopy graph”, and “congruences” on such graphs, which are new in graph theory, will be given in Section 4 – they were introduced and studied in [1].

A graph $G$ with vertex set $V$ and edge set $E$ will typically be denoted by $G = (V, E)$; when we are dealing with different graphs, we shall use the notation $V_G$ for $V$ when the description of $G$ contains no subscripts, and $V(G)$ otherwise, similarly we shall use $E_G$ or $E(G)$ for $E$. A (graph) homomorphism is an edge preserving mapping from the vertex set of a graph into the vertex set of a graph. When two graphs are isomorphic, one will be called a clone of the other. We shall use the abbreviation “iff” for the logical connective “if and only if”.

If $x$ and $y$ are elements of some set, we shall denote an ordered pair formed by them by $(x, y)$ and the unordered pair $\{x, y\}$ formed by them by $xy$, especially when this pair is an (undirected) edge in some graph. We use the symbol $\bigcup$ for the disjoint union of sets and graphs. The next definition is a standard definition from set theory.

The Cartesian product of sets: The Cartesian product $\prod_{i \in I} V_i$ of a collection of sets \{\(V_i \mid i \in I\)} is the set of all functions \(f\) from \(I\) to $\bigcup_{i \in I} V_i$ which satisfy the requirement that $f(i) \in V_i$ for all $i \in I$.

2 The P-product of graphs

In their approach to the classification of products, the authors of [3] require (on p. 41) that “The edge set of the product should be determined by some definite rule.” We introduce a notion of “rule” corresponding to theirs for our situation in this section. In the sequel, let $\mathcal{G} := \{G_i \mid i \in I\}$ be any given set of graphs, and let $V := \prod_{i \in I} V(G_i)$ be the Cartesian product of their vertex sets.

A (product) process $P$: Throughout this paper $P$ denotes (the description or definition of) a process or rule that delivers a set of unordered pairs of vertices of $V$. This process must be only dependent on the index set $I$ and its elements (including possibly set-theoretical structure on $I$, such as a distinguished subset of $I$ or an order relation on $I$), and the vertex sets $V(G_i)$ and edge sets $E(G_i)$ of the graphs $G_i$ and must have the property that it delivers a unique set of unordered pairs once $\mathcal{G}$ is given; this set of unordered pairs will be denoted by $EP$.

The P-product of graphs: The P-product of the set $\mathcal{G}$ of graphs is the graph $P(\mathcal{G}) := P_{i \in I} G_i := (V, EP)$ with the Cartesian product $V$ as vertex set of which the edge set $EP$ is this set of unordered pairs delivered by $P$.

Examples of P-products: In each example below we assume that a set $\mathcal{G}$ of graphs is given and we describe a process $P$ that leads in each of the first four cases to a well-known product of graphs. In [3] (p. 35) the products in the first three examples are called the “three fundamental products” – the reader is invited to check that our terminology is in line with theirs by applying our definitions to that situation.

- **The Cartesian product of graphs**: For the Cartesian product $\square_{i \in I} G_i$ of $\mathcal{G}$ (or $\square \mathcal{G}$) we stipulate for the process $P$ that two vertices $f$ and $g$ are adjacent in $\square_{i \in I} G_i$ iff there is an index $j \in I$ (in general depending on $f$ and $g$) such that $f(j)g(j) \in E(G_j)$ while $f(i) = g(i)$ for all $i \in I$ with $i \neq j$.
- **The direct product of graphs**: For the direct product $\times_{i \in I} G_i$ (or $\times \mathcal{G}$) we stipulate for the process $P$ that two vertices $f$ and $g$ are adjacent in $\times_{i \in I} G_i$ iff for every index $i \in I$ we have that $f(i)g(i) \in E(G_i)$. The edge set of the direct product $\times_{i \in I} G_i$ will also be denoted by $E_x$.
- **The strong product of graphs**: For the strong product $\boxdot_{i \in I} G_i$ (or $\boxdot \mathcal{G}$) of $\mathcal{G}$ we stipulate for
the process $P$ that two vertices $f$ and $g$ are adjacent in $\bigotimes_{i \in I} G_i$ iff there is a proper subset $K \subset I$ (in general depending on $f$ and $g$) such that for all $j \in I \setminus K$, $f(j)g(j) \in E(G_j)$, while for all $k \in K$, $f(k) = g(k)$.

- **The lexicographic product of graphs**: For this type of product the process $P$ is co-determined by an order on the factors $G_i$. We assume that $\langle I, \langle \rangle$ is well-ordered, so is order isomorphic to some fixed ordinal $\langle k, \rangle$. For the lexicographic product $\bigotimes_{i \in I} G_i$ (or $\bigotimes G$) of $G$ we then stipulate for the process $P$ that two vertices $f$ and $g$ are adjacent in $\bigotimes_{i \in I} G_i$ iff $J := \{ j \in I \mid f(j)g(j) \in E(G_j) \} \neq \emptyset$ and, when $m$ is then the $\langle -$minimum element of $J$, for every $k \in I$ such that $k < m$ (if such exist) $f(k) = g(k)$.

- **The $D$-product of graphs**: This is new product, as far as we know. Let $D$ be a fixed (distinguished) non-empty subset of the index set $I$. For the $D$-product $\bigotimes_{i \in I} G_i$ (or $\bigotimes G$) we stipulate for the process $P$ that two vertices $f$ and $g$ are adjacent in $\bigotimes_{i \in I} G_i$ iff for every index $j \in D$ we have $f(j)g(j) \in E(G_j)$.

The triples of sets $J(fg), K(fg)$ and $L(fg)$: Given a process $P$ delivering the edge set $E_P$ of $P(G)$ for a set of graphs $G$, every unordered pair $\{ f, g \}$ (with $f, g \in V$) and in particular every edge $fg \in E_P$ determines the following three subsets of the index set $I$:

- $J(fg) := \{ j \in I \mid f(j)g(j) \in E(G_j) \}$
- $K(fg) := \{ k \in I \mid f(k) = g(k) \}$; and
- $L(fg) := \{ l \in I \mid f(l)g(l) \notin E(G_l) \}$ and $f(l) \neq g(l) \} = I \setminus (J(fg) \cup K(fg))$.

Conversely, if for any vertices $f, g \in V$ we know that $fg \in E_P$ and we know the three sets of indices $J(fg), K(fg)$, and $L(fg)$ – even the first two will do – then we know for every $i \in I$ whether $f(i)g(i) \in E(G_i)$, or $f(i) = g(i)$, or neither. Different choices of processes $P$, leading to different decisions on whether for $f, g \in V$ we have $fg \in E_P$ or not, may entail different set-theoretical constraints on the sets $J(fg), K(fg)$, and $L(fg)$ – constraints on their cardinalities, on whether they are empty or not, constraints of inclusion, etc. For a given $P$ we call these constraints the $P$-constraints on the three index sets. For our five examples of $P$-products the $P$-constraints are, respectively, the following for every edge $fg \in E_P$:

- **$\square$**: For the Cartesian product $|J(fg)| = 1$, i.e., $J(fg)$ is a singleton, say $\{ j \} \subseteq I$; $K(fg) = I \setminus J(fg)$, say $I \setminus \{ j \}$; and $L(fg) = \emptyset$.
- **$\times$**: For the direct product $J(fg) \neq \emptyset$;
- $K(fg) = I \setminus J(fg)$; and
- $L(fg) = \emptyset$.
- **$\bigcirc$**: For the lexicographic product the well-ordering $\langle I, \langle \rangle$ is fixed once and for all, and $J(fg) \neq \emptyset$;
- $K(fg) \supseteq \{ k \in I \mid k < m \}$, where $m$ is the $\langle -$minimum of $J(fg)$; and
- $L(fg) = I \setminus (J(fg) \cup K(fg))$.
- **$\bigodot$**: For the $D$-product the non-empty set $D (\subseteq I)$ is fixed once and for all, and $J(fg) \supseteq D$;
- $K(fg) \subseteq I \setminus J(fg)$; and
- $L(fg) = I \setminus (J(fg) \cup K(fg))$.

Furthermore, in all five these examples, $J(fg), K(fg)$, and $L(fg)$ are pairwise disjoint (of course by definition) for every $fg \in E_P$. In fact, in all of these examples $P$ induces “$JKL$” that satisfy the following universal constraints:
For all $fg \in EP$, $J(fg)$, $K(fg)$, and $L(fg)$ are pairwise disjoint, their union is $I$, and $J(fg) \neq \emptyset$.

In [3] (pp. 41–43) an incidence function $\delta$ is introduced for any graph $G$ (not necessarily a product graph) as $\delta : V_G \times V_G \to \{1, \Delta, 0\}$. The incidence functions of two graphs then determine the incidence function – and hence the adjacency relation – on their product of a certain type. It should be clear that the three values of $\delta$, $\delta(g, g') = 1$ if $g \neq g'$ and $gg' \in E_G$; $\delta(g, g') = \Delta$ if $g = g'$; and $\delta(g, g') = 0$ if $g \neq g'$ and $gg' \notin E_G$, function analogously to, respectively, our three subsets $J(fg)$; $K(fg)$; and $L(fg)$ of $I$ in determining the adjacency relation in a $P$-product of any number of graphs.

We have seen that every product process $P$ induces a triple of functions

$$J, K, L : EP \to \mathcal{P}(I); fg \mapsto J(fg), K(fg), L(fg)$$

such that for every $fg \in EP$, the triple $(J(fg), K(fg), L(fg))$ satisfies the $P$-constraints. It may happen that, conversely, the existence of $J, K, L$ triples of subsets of $I$ satisfying certain constraints completely determines $EP$.

**Constraint-determined product processes**: We shall call the product process $P$ constraint-determined if the following holds:

There exists a set $C$ of constraints of the relevant nature (including the universal ones) on triples $J, K, L : \{\{f, g\} \mid f, g \in V \text{ and } f \neq g\} \to \mathcal{P}(I)$ such that for every unordered pair $\{f, g\}, f \neq g$, of elements of $V$, $fg \in EP$ iff there exists some triple $J\{f, g\}, K\{f, g\}, L\{f, g\}$ of subsets of $I$ satisfying $C$. In this case we could then legitimately consider $C$ to be (equivalent to) the $P$-constraints. Careful examination of their definitions and our exposition of the constraints for our five examples of product processes verifies the next result.

**Lemma 1.** All the product processes $\Box$, $\times$, $\otimes$, $\circ$, and $\otimes$ are constraint-determined.

**Permutable product processes**: Beyond the attribute that a $P$ may have of being constraint-determined, we now want to define a more general attribute of $P$ being “permutable”. The process $P$ is permutable if the following holds:

Whenever $p : I \to I$ is a permutation (bijection) of $I$ that respects whatever occasional fixed set-theoretical structure $P$ has imposed upon $I$ (as for $\Box$ and $\otimes$), then, for all $f, g \in V$, $p(f)p(g) \in EP$ iff $fg \in EP$. (Any permutation $p : I \to I$ induces a bijection again (naughtily) called $p : V \to V$; $f \mapsto p(f); (p(f))(i) = f(p(i)).$) The class of permutable processes includes the class of constraint-determined processes:

**Lemma 2.** If product process $P$ is constraint-determined, then $P$ is permutable.

**Proof.** Consider a constraint-determined $P$ and any permutation $p : I \to I$ respecting occasional $P$-imposed structure on $I$. Then $p$ preserves every $P$-constraint in the following sense:

Every set-theoretical constraint on one, two, or three subsets of $I$ (like emptiness, non-emptiness, cardinality, set-theoretical difference, inclusion, etc.) holds intact between the $p$-images of those sets. Since these constraints determine adjacency in $EP$ by the existence of $JKL$ triples, the $p$-images of those triples determine the same $EP$. So the bijection $p : V \to V$ establishes the isomorphism $P_{i \in I}G_i \cong P_{i \in I}G_{p(i)}$.

An immediate consequence is the following.

**Lemma 3.** Any graph product yielded by a permutable (and in particular by a constraint-determined) process is commutative and associative in every possible sense of those attributes.
Occasional provisos should not be forgotten. In the definition of the lexicographic product a fixed well-ordering \( (I, \prec) \) is assumed and the only permutation of \( I \) preserving this order is \( \text{id}_I \), the identity function on the set \( I \). Similarly, for the \( D \)-product only permutations of \( I \) that map the fixed distinguished set \( D \) of indices onto itself should be allowed.

3 **P-morphisms and the P-tensor product of graphs**

In order to advance our study of the \( P \)-product of graphs, we need the notion of a morphism from the set \( V \) to a graph \( H \) that is linked to the process \( P \).

**P-morphisms:** Whenever we have a set of graphs \( G = \{G_i \mid i \in I\} \), a process \( P \) (delivering \( P(G) = (V, EP) \)), and a graph \( H \) available, we define a **P-morphism** to the graph \( H \) as a function \( \delta : V \to V_H \) such that \( \delta \) is a homomorphism \( \delta : P(G) \to H \); we may write \( \delta : V \overset{P}{\to} H \) to indicate this fact. We note already here that \( P \)-morphisms in graph theory are analogous to multilinear mappings in linear algebra. In Section 7 we shall clarify this remark.

Next we define a tensor product of a set of graphs which is also linked to a given process \( P \). This definition follows the idea found in many algebra textbooks, especially in the context of the linear algebra of vector spaces; the one we follow in particular is the so-called “universal property of tensor products” given in Theorem 14.3 of [4], which is there shown to be (for the case of two graphs) equivalent to the definition, formulated in terms of free vector spaces and bilinear maps, given on p. 298 of [4].

**The P-tensor product of graphs:** Assume as given a process \( P \) as above. For any given set of graphs \( G \) (so that the graph \( P(G) \) is uniquely determined), as well as some graph \( T \) together with a fixed \( P \)-morphism \( \varphi : V \overset{P}{\to} T \) (delivering the homomorphism \( \varphi : P(G) \to T \), the pair \((\varphi, T)\) – or just \( T \) when \( \varphi \) is understood – is called a **P-tensor product of** \( G \) when the following holds:

(i) \( \varphi \) is surjective, i.e., \( \varphi(V) = V_T \); and

(ii) if \( H \) is any graph and \( \delta : V \overset{P}{\to} H \) is any \( P \)-morphism, then there exists a homomorphism \( \delta^* : T \to H \) such that \( \delta = \delta^* \circ \varphi \).

![Diagram](https://via.placeholder.com/150)

One may well refer to the requirements in this definition as the *universal factorization condition of \( \delta \) through \( T \).* These morphisms, and the fact that \( \delta = \delta^* \circ \varphi \), are illustrated in the accompanying commutative diagram.

It is easy to see that conditions (i) and (ii) of the definition of the \( P \)-tensor product of graphs are trivially met for each set of graphs \( G \) and each process \( P \) by the pair \((\text{id}_V, P(G))\) (since condition (ii) will then be satisfied by choosing \( \delta^* \) as \( \delta \)); this of course includes all the examples in the list in the previous section. Hence each pair \((\text{id}_V, P(G))\) described in that list is a \( P \)-tensor product for the process \( P \) chosen in the example. We now proceed to show that this is effectively the only type of example of a \( P \)-tensor product of graphs, i.e., we show that \((\text{id}_V, P(G))\) is up to isomorphism the only \( P \)-tensor product of \( G \). This will justify us in calling \((\text{id}_V, P(G))\) the **canonical** \( P \)-tensor product of \( G \).
The next lemma uses notation anticipating its later employment.

**Lemma 4.** Consider two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ and two homomorphisms $\delta_1^*: H_1 \to H_2$ and $\delta_2^*: H_2 \to H_1$ such that $\delta_2^* \circ \delta_1^* = id_{V_1}$ and $\delta_1^* \circ \delta_2^* = id_{V_2}$. Then $\delta_1^*$ and $\delta_2^*$ are both bijective isomorphisms (and inverses of each other), making $H_1$ and $H_2$ clones of each other.

**Proof.** We first prove that $\delta_1^*$ and $\delta_2^*$ are injective, and surjective onto $H_2$ and $H_1$ respectively. The fact that $\delta_2^* \circ \delta_1^* = id_{V_1}$, implies that $\delta_1^*$ is injective. (Suppose not, and that $v_1, v'_1 \in V_1$, $v_1 \neq v'_1$, while $\delta_1^*(v_1) = \delta_1^*(v'_1) := v_2 \in V_2$. Then $\delta_2^*(v_2)$ has to be both $v_1$ and $v'_1$, which is impossible.) By the symmetry of the conditions on $\delta_1^*$ and on $\delta_2^*$, $\delta_2^*$ is also injective.

Next we show that $\delta_1^*$ is surjective onto $V_2$. Suppose not, and that there is a $v_2 \in V_2$ such that for all $v_1 \in V_1$, $\delta_1^*(v_1) \neq v_2$, while $\delta_2^*(v_2) := v'_1 \in V_1$. Then $\delta_1^*(v'_1) = (\delta_1^* \circ \delta_2^*)(v_2) = v_2$, a contradiction. Similarly, $\delta_2^*$ is surjective onto $V_1$. So, both $\delta_1^*$ and $\delta_2^*$ are bijective homomorphisms.

The homomorphism $\delta_1^*$ (and by symmetry also $\delta_2^*$) is an isomorphism, since it preserves not only adjacency, but also non-adjacency. Suppose $v_1, v'_1 \in V_1$ with $v_1 v'_1 \notin E_1$. Then $\delta_1^*(v_1) \delta_1^*(v'_1) \notin E_2$, for were $\delta_1^*(v_1) \delta_1^*(v'_1) \in E_2$, then (since $\delta_2^*$ is a homomorphism) we would have $[(\delta_2^* \circ \delta_1^*)(v_1)][(\delta_2^* \circ \delta_1^*)(v'_1)] = v_1 v'_1 \in E_1$. 

The result of Lemma 4 is useful in the proof of Theorem 1.

We now show, for a given process $P$ and a given set $G$ of graphs, that $(id_V, P(G))$ is “up to isomorphism” the only pair $(\varphi, T)$ which has the properties required by the definition of a $P$-tensor product of these graphs. The exact meaning of the phrase “up to isomorphism” will be strengthened by a discussion after the proof of the theorem.

**Theorem 1.** $(id_V, P(G))$ is (up to isomorphism) the unique $P$-tensor product of $G$.

**Proof.** We have already remarked that conditions (i) and (ii) of the definition of the $P$-tensor product of $G$ are trivially met by the pair $(id_V, P(G))$ by choosing $\delta^*$ as $\delta$.

To show uniqueness, assume that $(\varphi, T)$ is any $P$-tensor product of $G$. We first apply the definition of “$(\varphi, T)$ is a $P$-tensor product of $G$” and use in it the graph $P(G)$ for $H$ and the identity map $id_V$ for $\delta$ (which is a $P$-morphism) to conclude that there exists a homomorphism $id_V^*: T \to P(G)$ such that

$$id_V = id_V^* \circ \varphi. \quad (1)$$

On the other hand, since $\varphi: V \to T$ is a $P$-morphism and $(id_V, P(G))$ is a $P$-tensor product of $G$, we can apply the definition again (with $T$ for $H$) to conclude that there exists a homomorphism $\varphi^*: P(G) \to T$ such that $\varphi = \varphi^* \circ id_V$; so

$$\varphi = \varphi^* \quad (2)$$

since $id_V$ is an identity map. By equations (1) and (2), it now follows that

$$id_V = id_V^* \circ \varphi^*. \quad (3)$$

Now consider any $y \in V_T$. Since $\varphi$ is surjective, there exists at least one $x \in V$ with $\varphi(x) = y$. Then $id_V^*(y) \in V$ and, since $id_V$ is an identity map, $id_V^*(y) = id_V(id_V^*(y))$. Furthermore,

$$\varphi^*(id_V^*(\varphi(x))) = \varphi^*(id_V(x)) \quad (1)$$

$$= \varphi^*(x) \text{ since } id_V \text{ is an identity map}$$

$$= \varphi(x) \quad (2),$$

and by (3),
so that \( \varphi^*(id_{V_T}(y)) = y \), for all \( y \in V_T \). Hence

\[
id_{V_T} = \varphi^* \circ id_{V_T}.
\]

It is now clear that \( \varphi^* \) and \( id_{V_T} \) satisfy the premises of Lemma 4 for \( \delta_1 \) and \( \delta_2 \) respectively and hence we can conclude that \( T \) is a clone of \( P(G) \).

4 Making each \( \pi_i \) a homomorphism using congruences

In this section we use the theory of congruences on graphs developed in \[1\]. This development is given there in full detail for simple graphs, while the last section is devoted to graphs which have a loop at every vertex, called loopy graphs there. A loop-allowing (hence generally non-simple) graph allows loops, but does not (like a loopy graph) prescribe them at every vertex. We remark that the definition of a congruence of a loop-allowing graph (to follow) is simpler than that of a simple graph and also from that of a loopy graph.

While in the rest of this article all graphs are simple, in this section and at the end of the next section only, we shall allow a graph construction (defined on all loop-allowing graphs) which, when applied to a simple graph, may yield a loop-allowing graph as a result. To describe this construction – forming the quotient of a graph modulo a congruence – we first define the notion of “congruence” on a loop-allowing graph.

**Congruences and quotients:** A congruence on a loop-allowing graph \( G = (V, E) \) is a pair \( \theta = (\sim, \hat{E}) \) such that

(i) \( \sim \) is an equivalence relation on \( V \) (hence \( id_V \subseteq \sim \));

(ii) \( \hat{E} \) is a set of unordered pairs of elements from \( V \) with \( E \subseteq \hat{E} \); and

(iii) when \( x, y, x', y' \in V, x \sim x', y \sim y', \) and \( xy \in \hat{E} \), then \( x'y' \in \hat{E} \), i.e., \( \hat{E} \) is substitutive with respect to \( \sim \).

The congruence \( \iota_G := (id_V, E) \) on \( G \) is the smallest congruence on \( G \), i.e., \( \subseteq \)-smallest in both components.

Given any congruence \( \theta = (\sim, \hat{E}) \) on a (loop-allowing) graph \( G = (V, E) \), we define a new graph, denoted by \( G/\theta \) and called (the quotient of) \( G \) modulo \( \theta \), as follows:

\[
G/\theta := (V_{G/\theta}, E_{G/\theta}) := (V/\sim, \{[x][y] \mid xy \in \hat{E}\}),
\]

where \([x] \in V_{G/\theta}\) denotes the \( \sim \)-equivalence class of \( x \in V \).

We note that the surjective mapping \( x \mapsto [x] \) from \( V_G \) onto \( V_{G/\theta} \) establishes the natural or canonical homomorphism \( G \to G/\theta \). When \( \theta = \iota_G \), this natural homomorphism is the mapping \( x \mapsto \{x\}, x \in V_G \) and it is an isomorphism, i.e., \( G \cong G/\iota \). Making the distinction between \( x \) and \( \{x\} \) in such a case is so superficial that we shall not always bother to do so and thus treat them as
if they are equal; using the phrase “identity function” in the last result in this section is the first instance where this remark is applied fully.

Given any graph homomorphism between loop-allowing graphs, say $\varphi : G \to H$, we define a congruence on $G$, denoted by $\theta_\varphi$ and called the congruence induced by $\varphi$ or the kernel of $\varphi$, by

$$
\theta_\varphi := \langle \sim_\varphi, \widehat{E}_\varphi \rangle := \langle \{ (x, y) \in V_G^2 \mid \varphi(x) = \varphi(y) \}, \{ uv \mid u, v \in V_G \text{ and } \varphi(u)\varphi(v) \in E_H \} \rangle = \langle \varphi^{-1}[\text{id}_{V(G)}], \varphi^{-1}[E(H[\varphi(V_G)])] \rangle.
$$

It should be immediately clear that $\theta_\varphi$ is a congruence on $G$.

Now back to $\mathbf{P}$-products: Assume as given the set $\mathcal{G} = \{ G_i \mid i \in I \}$ of loop-allowing graphs with $G_i = (V_i, E_i)$ for each $i \in I$ and a process $\mathbf{P}$. Consider the $\mathbf{P}$-tensor product $\mathbf{P}(\mathcal{G})$ of $\mathcal{G}$, any $i \in I$, and the projection $\pi_i : V \to V_i$ which, for each $f \in V$ maps $f$ to $f(i)$. We remark that, for the direct product $\times \mathcal{G}$, the projection $\pi_i$ is a homomorphism $\pi_i : \times \mathcal{G} \to G_i$ for every $i \in I$, but for other products this does not hold in general. Can we somehow, for every $\mathbf{P}$-product, restore each $\pi_i$ to its status as a homomorphism by transforming $G_i$ in a uniform way? We shall now demonstrate how we may, by taking a suitable quotient graph, to be called $\mathbf{P}(G_i)$, indeed reach the conclusion that $\pi_i : \mathbf{P}(\mathcal{G}) \to \mathbf{P}(G_i)$ is a homomorphism. Let

$$
\theta_i := \langle \text{id}_{V_i}, \widehat{E}_i \rangle, \quad \text{where} \quad \widehat{E}_i := \{ xy \mid x, y \in V_i, \text{ and } (xy \in E_i \text{ or there exist } f, g \in V \text{ such that } fg \in EP, f(i) = x, \text{ and } g(i) = y \}.
$$

It is easy to see that $\theta_i$ is a congruence on $G_i$ for each $i \in I$ (according to the above definition). With respect to this congruence, we now define the quotient graph $\mathbf{P}(G_i) := (V_i, \widehat{E}_i) = G_i/\theta_i$. Hence $\mathbf{P}(G_i)$ is a well-defined graph – but it is loop-allowing (even if $G_i$ is simple) since $\mathbf{P}$ may be such that $fg \in EP$ while $x = f(i) = g(i) = y$, giving $xx \in \widehat{E}_i$ and hence $xx = \{ x \} \{ x \} = [x][x] \in E(G_i/\theta_i)$. This is our way to surmount a problem that Hammack, Imrich and Klavžar handle in [3] by introducing “weak homomorphisms” (functions allowed to map adjacent vertices to the same vertex) for other than direct products. The next result is now trivial.

**Lemma 5.** Every projection $\pi_i : V \to V_i$ is a homomorphism $\pi_i : \mathbf{P}(\mathcal{G}) \to \mathbf{P}(G_i)$. For the special case of the direct product the allowance of loops in $\widehat{E}_i$ can be removed as unnecessary, with $\widehat{E}_i = E_i$, $\theta_i$ the identity congruence $\iota_{G_i}$ on $G_i$, and $\mathbf{P}(G_i) = G_i$.

This Lemma immediately entails the next result, which is trivial when $\mathbf{P} = \times$.

**Theorem 2.** For all sets $\mathcal{G}$ of graphs and all processes $\mathbf{P}$ to construct their $\mathbf{P}$-product $\mathbf{P}(\mathcal{G})$, the identity function on $V$ is a bijective homomorphism from $\mathbf{P}(\mathcal{G})$ to the direct product $\times_{i \in I} \mathbf{P}(G_i)$ of the loop-allowing graphs $\mathbf{P}(G_i)$, i.e., $\mathbf{P}(\mathcal{G}) \subseteq \times_{i \in I} \mathbf{P}(G_i)$.

**Proof.** If $fg \in EP$, then, (by the definition of adjacency in $\mathbf{P}(G_i)$ and Lemma 5), $f(i)g(i) \in \widehat{E}_i$ for every $i \in I$, and hence $fg$ is an edge of the direct product $\times_{i \in I} \mathbf{P}(G_i)$ of the loop-allowing graphs $\mathbf{P}(G_i)$.

\[ \square \]

### 5 $\mathbf{P}$-tensor products of homomorphisms

Beyond the furtive entry of loop-allowing graphs in the previous section, we now, until near the end of this section, restrict ourselves to simple graphs. Remember that the Cartesian product
\[ X := \prod_{i \in I} X_i \] of sets is the set consisting of all functions \( f : I \rightarrow \bigcup_{i \in I} X_i \) satisfying \( f(i) \in X_i \) for all \( i \in I \) and that the \( i \)’th projection \( \pi_i \) is the function \( \pi_i : X \rightarrow X_i \) with \( \pi_i(f) = f(i) \) for each \( i \in I \). Hence if, for a function \( f : I \rightarrow \bigcup_{i \in I} X_i \), the value of \( \pi_i(f) \) is given for each \( i \in I \), then the function is uniquely determined and if, furthermore, this value is in \( X_i \) for each \( i \in I \), then this function \( f \) is in the Cartesian product \( \Pi_{i \in I} X_i \).

We now assume as given a product process \( P \), a fixed index set \( I \), two sets of graphs \( G = \{ G_i \mid i \in I \} \) and \( H = \{ H_i \mid i \in I \} \), as well as a set \( \Phi = \{ \varphi_i \mid i \in I \} \) of functions in which, for each \( i \in I \), \( \varphi_i : V(G_i) \rightarrow V(H_i) \). For ease of notation we let \( V_i := V(G_i) \), \( W_i := V(H_i) \), for each \( i \in I \), and, for the Cartesian products of these vertex sets, \( V := \Pi_{i \in I} V_i \) while \( W := \Pi_{i \in I} W_i \).

Note that, for the given set \( \Phi \) of functions, there exists a unique function \( \varphi : V \rightarrow W \) such that, for each \( f \in V \) and each \( i \in I \), \( \pi_i(\varphi(f)) = \varphi_i(\pi_i(f)) \); this equation can also be written (and will be utilised) as \( \varphi(f)(i) = \varphi_i(f(i)) \), thereby mitigating the venial sin of abusing the name \( \pi \) for two different projections. The fact that there is such a function, and its uniqueness, follows from the remark that its domain and function values are completely specified by the above defining conditions. The function \( \varphi \) and its properties are depicted in the accompanying commutative diagram.

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & W \\
\downarrow{\pi_i} & & \downarrow{\pi_i} \\
V_i & \xrightarrow{\varphi_i} & W_i
\end{array}
\]

How can we, when each \( \varphi_i \) is a homomorphism \( \varphi_i : G_i \rightarrow H_i \), ensure that such a function \( \varphi \) is also a homomorphism \( \varphi : P(G) \rightarrow P(H) \)? Let us call a product process \( P \) hom-preserving if for each index set \( I \), every choice of two sets of graphs \( G \) and \( H \), as well as each set \( \Phi = \{ \varphi_i \mid i \in I \} \) of homomorphisms, the function \( \varphi \) is a homomorphism from \( P(G) \) to \( P(H) \) too. It seems reasonable to name the homomorphism \( \varphi \) resulting in such a way from a hom-preserving process \( P \) the \( P \)-tensor product of \( \Phi \) and write \( \varphi = \boxtimes P \Phi \). Section 6 takes this further by linking it to the tensor product of graphs associated with the given homomorphisms.

**Example:** Suppose an index set \( I \), two sets of graphs \( G \) and \( H \) and a set \( \Phi \) of homomorphisms as above are given. Let \( P \) be the process through which the direct product of a set of graphs is formed. Then \( P \) is hom-preserving: If \( fg \in E_{\times(G)} \), the edge set of \( \times_{i \in I} G_i \), then \( f(i)g(i) \in E(G_i) \) for every \( i \in I \). Hence, since each \( \varphi_i \) is (given as) a homomorphism, \( \varphi_i(f(i))g(i) \in E(H_i) \) for every \( i \in I \). The defining conditions of \( \varphi \) (which do not require projections \( \pi_i \) to be homomorphisms) allow us to conclude that \( \varphi(f)(i)g(i) \in E(H_i) \) for every \( i \in I \). But this means that \( \varphi(f)g \in E_{\times(H)} \), the edge set of \( \times_{i \in I} H_i \), by the choice of \( P \). Hence \( P = \times \) is a hom-preserving process.

We now generalize the above example to a class of constraint-determined product processes.

**Theorem 3.** Every constraint-determined product processes \( P \) for which one of the \( P \)-constraints is that \( L(fg) = \emptyset \) for all \( fg \in EP \) (briefly: \( L = \emptyset \), as for \( \Box \), \( \times \), and \( \boxtimes \)) is hom-preserving.

**Proof.** Consider a constraint-determined product processes \( P \) and the function \( \varphi : V \rightarrow W \) as determined by \( \Phi = \{ \varphi_i \mid i \in I \}, \varphi_i : G_i \rightarrow H_i \). We need to prove that \( \varphi : P(G) \rightarrow P(H) \). Consider all \( fg \in E_{P(G)} \). This is equivalent to considering all the triples \( J(fg), K(fg), L(fg) \) of subsets of \( I \) satisfying the \( P \)-constraints – which determine \( E_{P(G)} \). We now assume that always \( L(fg) = \emptyset \),
and hence \( I = J(fg) \cup K(fg) \) for every \( fg \in EP \). Pick any \( fg \in E_{P(G)} \), \( f, g \in V \). If \( j \in J(fg) \), then \( f(j)g(j) \in E(G_j) \) and (since \( \varphi_j : G_j \to H_j \) \( \varphi_j(f(j))\varphi_j(g(j)) = \varphi(f(j))\varphi(g(j)) \in E(H_j) \), ensuring that \( j \in J(\varphi(f)\varphi(g)) \). And if \( k \in K(fg) \), \( f(k) = g(k) \) in \( V(G_k) \) and hence (since \( \varphi_k : V(G_k) \to V(H_k) \) is a function) \( \varphi_k(f(k)) = \varphi_k(g(k)) \), i.e. \( \varphi(f)(k) = \varphi(g)(k) \) in \( V(H_k) \), ensuring that \( k \in K(\varphi(f)\varphi(g)) \). This means that the two index sets \( J(fg) \) and \( K(fg) \) are respectively equal to the two index sets \( J(\varphi(f)\varphi(g)) \) and \( K(\varphi(f)\varphi(g)) \) determining \( \varphi(f)\varphi(g) \in E_{P(H)} \).

This confirms that \( \varphi \) is a homomorphism and thus that \( P \) is hom-preserving: \( \varphi = \otimes_{P} \Phi : P(G) \to P(H) \). 

When \( P \) is constraint-determined with the index set \( L(fg) \) empty for all \( fg \in EP \), and hence hom-preserving with \( \varphi = \otimes_{P} \Phi \) indeed a homomorphism, we may link the commutative square of this section to the procedure in the previous section of transforming each \( G_i \) to \( P(G_i) = G_i/\theta_i \) (and each \( H_i \) to \( P(H_i) = H_i/\theta_i \)) in order to make each projection \( \pi_i \) a homomorphism. (So far in this section \( \pi_i \) need not be a homomorphism at all.) Consider the commutative square of functions in which \( \varphi \) and the two projections \( \pi_i \) are now homomorphisms. It gives satisfaction that this is indeed now a commutative square of homomorphisms, as we now show.

\[
\begin{array}{ccc}
P(G) & \xrightarrow{\varphi} & P(H) \\
\pi_i \downarrow & & \pi_i \downarrow \\
P(G_i) & \xrightarrow{\varphi_i} & P(H_i)
\end{array}
\]

**Lemma 6.** \( \varphi_i : P(G_i) \to P(H_i) \).

**Proof.** We know that \( \varphi_i : G_i \to H_i \). Suppose that \( xy \in \hat{E}_i(G_i) \), then there are two possibilities:

(i) \( xy \in E_i(G_i) \) and, since \( \varphi_i : G_i \to H_i \), we have \( \varphi_i(x)\varphi_i(y) \in E_i(H_i) \subseteq \hat{E}_i(H_i) \); or

(ii) there exist \( f, g \in V_{P(G)} \) such that \( fg \in E_{P(G)} \), \( f(i) = x, g(i) = y \) and \( i \in J(fg) \). As in the proof (for \( P \)) of Theorem 3 it is clear that \( J(fg) = J(\varphi(f)\varphi(g)) \) and \( K(fg) = K(\varphi(f)\varphi(g)) \) (and \( L \) always empty). Hence \( i \in J(\varphi(f)\varphi(g)) \) and \( \varphi(f)(i)\varphi(g)(i) = \varphi_i(f(i))\varphi_i(g(i)) = \varphi_i(x)\varphi_i(y) \in \hat{E}_i(H_i) \). 

\section{Graphs, products, and homomorphisms intertwined}

We start this section by constructing a graph from any homomorphism. Consider a homomorphism \( \varphi : G \to H \). We define a graph \( \Gamma(\varphi) \), called the graph of \( \varphi \), by stipulating that

\[
V_{\Gamma(\varphi)} := \{ (x, \varphi(x)) \mid x \in V_G \} \subseteq V_G \times V_H; \quad \text{while} \quad E_{\Gamma(\varphi)} := \{ (x, \varphi(x))(y, \varphi(y)) \mid x, y \in V_G \text{ and } \varphi(x)\varphi(y) \in E_H \}.
\]

Note that \( \Gamma(\varphi) \) is a simple graph. Also, in terms of the two projection mappings \( \pi_1 : (x, \varphi(x)) \mapsto x \) and \( \pi_2 : (x, \varphi(x)) \mapsto \varphi(x) \), the above definition ensures that \( \pi_1 \) is surjective and \( \pi_2 \) is a homomorphism \( \Gamma(\varphi) \to H \). Furthermore, it is clear that all the information in the configuration \( \varphi : G \to H \), the set-theoretical structure \( \Gamma(\varphi) \) encodes \( V_G, \varphi, \) and \( H[\varphi(V_G)] \), but that \( E_G \) and the rest of \( H \) is irretrievable from \( \Gamma(\varphi) \).

The assumptions for our last result are now stipulated. We have a fixed index set \( I \) and two \( I \)-indexed sets of graphs \( G = \{ G_i \mid i \in I \} \) and \( H = \{ H_i \mid i \in I \} \). \( \Phi = \{ \varphi_i \mid i \in I \} \) is a set of homomorphisms \( \varphi_i : G_i \to H_i \), while \( P \) is a constraint-determined product process with \( L = \emptyset \) as one of the \( P \)-constraints. Theorem 3 in the previous section then assures us that
With respect to the product $i \in K$ the latter with respect to the product $i \in J$ by using the defining equation above backwards.

If $f = \varphi(f)$ then $f \neq g$ and hence $f(i) \neq g(i)$ for some $i \in I$. But then $f^* \neq g^*$.

Given any function $h \in \Pi_i \varphi_i$, one can describe a function $f$ for which $(f, \varphi(f)) \in \varphi$ and $f^* = h$ by using the defining equation above backwards.

Now consider any two vertices $(f, \varphi(f)), (g, \varphi(g)) \in \varphi = V_{\Gamma(\varphi)}$. By the definition of adjacency of vertices in $\Gamma(\varphi)$,

$$(f, \varphi(f))(g, \varphi(g)) \in E_{\Gamma(\varphi)} \iff \varphi(f)\varphi(g) \in E(\mathbf{P}(\{H_i \mid i \in I\})).$$

With respect to the product $\mathbf{P}(\{H_i \mid i \in I\})$, for every $j \in I$,

$$j \in J_{\{f, \varphi(f), \varphi(g)\}} \iff \varphi(f)(j)\varphi(g)(j) \in E(H_j)$$
$$\varphi_j(f(j))\varphi_j(g(j)) \in E(H_j)$$
$$\varphi(f)(j)\varphi(g)(j) \in E(\Gamma(\varphi_j))$$
$$f^*(j)g^*(j) \in E(\Gamma(\varphi_j))$$
$$j \in J_{\{f^*, g^*\}},$$

the latter with respect to the product $\mathbf{P}(\{\Gamma(\varphi_i) \mid i \in I\})$. So, as subsets of $I$, $J_{\{f, \varphi(f), \varphi(g)\}} = J_{\{f^*, g^*\}}$. According to the assumed properties of $\mathbf{P}$, in particular since $L = \emptyset$, then also $K_{\{f, \varphi(f), \varphi(g)\}} = K_{\{f^*, g^*\}}$, and hence $\varphi(f)\varphi(g) \in E(\mathbf{P}(\{H_i \mid i \in I\}))$ iff $f^*g^* \in E(\mathbf{P}(\{\Gamma(\varphi_i) \mid i \in I\}))$. From what was said previously it follows that $(f, \varphi(f))(g, \varphi(g)) \in E_{\Gamma(\varphi)}$ iff $f^*g^* \in E(\mathbf{P}(\{\Gamma(\varphi_i) \mid i \in I\}))$.

Hence $\alpha$ is an isomorphism and $\Gamma(\varphi) \cong \mathbf{P}(\{\Gamma(\varphi_i) \mid i \in I\})$ now follows.

### 7 The analogy to linear algebra

We now return to redeem an early promise. In what sense could we claim that “$\mathbf{P}$-morphisms in graph theory are analogous to multilinear mappings in linear algebra”? To clarify the analogy we display the correspondences in two parallel columns:

**Graph Theory**

We have a set $\mathcal{G} = \{G_i\}_{i \in I}$ of graphs $G_i$ and form the Cartesian product $V = \Pi_i V_i$ of the underlying sets $V_i$ of (vertices of) the graphs $G_i$. One may consider arbitrary graphs $H$ and arbitrary $\mathbf{P}$-morphisms $\delta : V \xrightarrow{\mathbf{P}} H$.

**Linear Algebra**

We have a set $\mathcal{G} = \{G_i\}_{i \in I}$ of vector spaces $G_i$ and form the Cartesian product $V = \Pi_i V_i$ of the underlying sets $V_i$ of (vectors of) the vector spaces $G_i$. One may consider arbitrary vector spaces $H$ and arbitrary multilinear mappings $\delta : V \to H$. 
The canonical $\mathbf{P}$-tensor product of $\mathcal{G}$ is the pair $(id_V, \mathbf{P}(\mathcal{G}))$ in which $id_V$ is the fixed identity $id_V : V \xrightarrow{\mathbf{P}} \mathbf{P}(\mathcal{G})$, satisfying the defining tensor conditions:

(i) $id_V$ is surjective; and

(ii) if $H$ is any graph and $\delta : V \xrightarrow{\mathbf{P}} H$ any $\mathbf{P}$-morphism, then the function $\delta$ itself is a graph homomorphism which we may write as $\delta^* : \mathbf{P}(\mathcal{G}) \to H$, – and, of course, $\delta = \delta^* \circ id_V$.

A tensor product of $\mathcal{G}$ is a pair $(\varphi, T)$ in which $\varphi$ is a fixed multilinear mapping from $V$ to the vector space $T$, satisfying the defining tensor conditions:

(i) $\varphi(V)$ spans $T$; and

(ii) if $H$ is any vector space and $\delta : V \to H$ any multilinear mapping, then there exists a linear mapping (i.e., vector space homomorphism) $\delta^* : T \to H$ such that $\delta = \delta^* \circ \varphi$.

8 Acknowledgement

Izak Broere is supported in part by the National Research Foundation of South Africa (Grant Number 90841).

References

[1] Broere, I., Heidema, J. and Pretorius, L.M.: Graph congruences and what they connote. Submitted (2015)

[2] Diestel, R.: Graph Theory, Fourth Edition. Graduate Texts in Mathematics 173, Springer, Heidelberg (2010)

[3] Hammack, R., Imrich, W., Klavžar, S.: Handbook of Product Graphs, Second Edition. CRC Press, Boca Raton (2011)

[4] Roman, S.: Advanced Linear Algebra. Graduate Texts in Mathematics 135, Springer, Heidelberg (1992)

[5] Rosinger, E.E.: Two Generalizations of Tensor Products, Beyond Vector Spaces. arXiv:0807.1436v4 (2008)