Quantum X-waves in Kerr media and the progressive undistorted squeezed vacuum

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A quantum theory of 3D X-shaped optical bullets in Kerr media is presented. The existence of progressive undistorted squeezed vacuum is predicted. Applications to quantum non-demolition experiments, entanglement and interferometers for gravitational waves detection are envisaged.

The standard for a non-monochromatic quantum light packet is a superposition of photons with different angular frequencies and wave-vectors. This approach has been adopted for dispersive and nonlinear media, including all the spatial dimensions (see e.g. [2, 4, 5, 6]). However, its application to particle-like three dimensional objects, like self-trapped optical bullets, is not trivial.

Conversely, 1D optical solitons propagating in a fiber can be completely described at a quantum level. This circumstance led to a series of extensively studied macroscopic quantum effects, like quantum non-demolition and squeezing. Fibers solitons are one-dimensional objects, with transversal profile given by the waveguide mode, their propagation invariant nature has favored long range interactions, up to the recent investigation of entangled pulses for quantum communications and computing.

The extension of these results to multi-dimensional self-trapped wave-packets is an interesting enterprise. Classically, the 3D counterpart of solitons are the mentioned optical bullets, which propagate without diffraction and dispersion and are somehow generated in a nonlinear medium. Among the various proposal of 3D light bullets, due to different mechanisms and optically nonlinear processes in bulk media, the only experimentally demonstrated are the so-called “nonlinear X-waves”. The latter have a distinctive bi-conical shape, which appears as an “X” in some section plane containing the propagation direction. Notably X-waves do not need the nonlinearity for being self-sustained, while a self-action may favor their spontaneous formation. In essence, 3D solitonic waves destroy themselves after exiting the nonlinear medium, conversely an X-wave may propagate undistorted even in vacuum (it also named “progressive undistorted wave”, for a review see, e.g., [24, 25]). It is an ideal candidate for (quantum) information channels in air, previously investigated by Lu and He at a classical level.

Having in mind these properties, it is naturally argued which is (if any) the quantum counterpart of nonlinear X-waves. Is it possible to provide a fully quantized version at any photon number, as in the case of optical solitons? This is the subject of this Letter. I will report a quantum theory of nonlinear X-waves and some of its corollaries; in particular the possibility of generating a very exotic (but at the same time intriguing for many applications) state of light: the progressive undistorted squeezed vacuum. Quantum nonlinear X-waves turn out to have the same properties of quantum fiber solitons: they can propagate with a well defined photon number, thus enabling sequential measurements on the same quantum state. All the applications of quantum fiber solitons may be translated in the quantum X-waves world, with the non trivial benefits of a 3D space.

The basic idea underlying this work is using 3D wave packets parametrized by their velocity, instead of angular frequency, for a quantization procedure. Since it is interesting to be as much as possible near to realm of experiments, the starting point is the equation for the motion of an optical pulse, with complex amplitude $A$, traveling with diffraction in a normally dispersive medium. It can be equivalently cast as an evolution problem with respect to the direction of propagation $z$, or to time $t$, which I adopt here to be consistent with standard quantum mechanics:

$$i\frac{\partial A}{\partial t} + i\omega_0 J_0(\sqrt{\frac{k_0}{\omega_0}}r)e^{i(\alpha - \beta)z} = \frac{\delta \mathcal{H}_I}{\delta A^*}. \tag{1}$$

$\omega_0$ and $\omega''_0$ are the first order and the modulus of the second order dispersion terms ($\omega''_0 > 0$); $k = n_0(\omega_0)\omega_0/c$ and $\omega_0$ are the wave-number and the carrier angular frequency, respectively, and $n = n(\omega_0)$ is the refractive index. $\mathcal{H}_I$ is a classical interaction Hamiltonian taking into account nonlinear effects.

In the linear case ($\mathcal{H}_I = 0$) the general radially symmetric ($r^2 = x^2 + y^2$, in the following) solution can be expressed as a superposition, parametrized by the velocity $v$, of a special class of undistorted progressive waves, the radially symmetric X-waves, given by three dimensional complex profile:

$$\psi_q^{(p)}(z, r) = \int_0^\infty f_q(\alpha) J_0(\sqrt{\frac{\omega''_0 k_0}{\omega_0}}r)e^{i(\alpha - \beta)z}d\alpha, \tag{2}$$

with $f_q = \sqrt{k_0/\pi} \sqrt{\omega''_0} (\alpha \Delta) L_q^{(1)}(2\alpha \Delta) \exp(-\alpha \Delta)$ (related details are also given in [26]), $\Delta$ is parameter measuring the transversal spatial extent of the beam, $L_q^{(1)}$ the generalized Laguerre polynomials ($q = 0, 1, \ldots$), and having introduced, for later convenience, the "momentum
Dirac operators be standardly quantized, and the resulting positive fre-
ingation/dispersion process is reduced to a one-dimensional
energy is

\[ E = \int \int \int |A|^2 \, dx \, dy \, dz = \sum_q \left| C_q \right|^2 dp. \]  

Using this formulation a non-monochromatic pulsed beam is written as an integral sum of wave-packets with different velocities, or momentum \( p \), and the diffraction/dispersion process is reduced to a one-dimensional evolution.

Eq. 6 is a superposition of harmonic oscillators, with angular frequencies \( \omega_q(p) \), each weighting a traveling mode (i.e. corresponding to the usual cavity mode, with the difference here that it is rigidly moving). It can be standardly quantized, and the resulting positive frequency field operator is (in the Heisenberg picture for the Dirac operators \( a_p \))

\[ E = \sum_p \int \frac{2 \hbar \omega_q(p)}{\epsilon_0 n^2 m} a_q(p,t) e^{ik_0 z} \psi_q(p)[z - (\omega' + \frac{p}{m})t, r] dp \]  
The energy of each elementary excitation is given by \( \hbar \omega_q(p) = \hbar \omega_0 + p^2 / 2m \), and the Hamiltonian \( H_0 = \sum_q \int \hbar \omega_q(p) a_q^\dagger(p) a_q(p) dp \) (the zero-point energy has been renormalized, as usual in quantum field theory 28).

This suggestive way of quantizing the propagating optical field, is equivalent to the standard plane wave expansion. It is in essence a change of basis, which enables to represent the 3D evolution as a 1D quantum gas of particles with momentum \( p \), and mass \( m \).

In the presence of a nonlinear coupling between the quasi-particles, due to the Kerr effect, the contribution to the classical energy is

\[ H_I = \frac{\chi}{2} \int \int |A|^4 \, dx \, dy \, dz, \]  

with \( \chi < 0 \) in a focusing medium. The corresponding interaction Hamiltonian is, after some manipulations and with obvious notation,

\[ H_I = \frac{1}{2} \sum_{l m n o} \int \sqrt{\omega_0(p_1) \omega_0(p_2) \omega(p_1) \times \chi_{lmno}(p_4 + p_3 - p_2 - p_1) a_{l}^\dagger(p_4) a_{m}^\dagger(p_3) a_{n}(p_2) a_{o}(p_1) d^4 p}. \]  

The interaction kernel, the “vertex,” \( \chi_{lmno}(\nu) \) turns out to be the Fourier transform of the spatial transversal superposition of the component X-waves profiles:

\[ \kappa_{lmno}(z) = \frac{\chi h^2}{m^2} \int \int (\psi_m(0))^{*} \psi_n(0) \psi_{a}(0) \psi_{o}(0) dx \, dy. \]  

As an application, consider the “device” shown in figure 1 an X-wave, generated by the axicon, travels in a Kerr medium. The axicon must be intended in a generalized sense, i.e. either a linear device, as those typically employed in linear experiments, 29, 30 or a nonlinear process that furnishes the required spatio-temporal reshaping of the laser pulsed beam into an X-wave. 22, 23

By this approach, only a \( p \)-superposition of a single basis element, denoted by index \( q \), must be taken into account and \( 7 \) becomes [with \( a_m(v) = \sigma_{q m} a(v) \) and omitting hereafter the index \( q \) (e.g. \( \chi_{lmno} \to \chi \))]

\[ H_I = \frac{1}{2} \int \sqrt{\omega(p_1) \omega(p_2) \omega(p_1) \times \chi(p_4 + p_3 - p_2 - p_1) a_{l}^\dagger(p_4) a_{m}^\dagger(p_3) a(p_2) a(p_1) d^4 p}. \]  

The analysis is further simplified in the low-momentum approximation, typically adopted in the physics of weakly interacting bosons. 51 This corresponds to assume that the velocities are all in proximity of the linear group velocity, such that \( (p_1 \approx 0) \sqrt{\omega(p_1) \omega(p_2) \omega(p_1) \omega(p_1)} \approx \omega_0^2 \). Introducing the particle operator \( \phi(z) = (1/2\pi \hbar) \int a(p) \exp(ipz/\hbar) dp, \) the interaction Hamiltonian is given by

\[ H_I = \frac{1}{2} \int \sigma(\zeta) \phi^\dagger(z) \phi^\dagger(z) \phi(z) \phi(z) d\zeta, \]  

with

\[ \sigma(\zeta) = (2\pi \hbar)^3 \frac{\chi \omega^2 h^2}{m^2} \int \int |\psi(0)|^4 dxdy. \]  

By expressing \( a \) in terms of \( \phi \) in 6, the previous results can be reformulated as follows. The whole 3D evolution of the electric field is given, in the Heisenberg picture for the particle operators \( \phi \) and \( \phi^\dagger \), 32 by

\[ E = \sqrt{\frac{2\hbar \omega_0}{\epsilon_0 n^2 m}} e^{ik_0 z - i\omega_0 t} \int \xi(s, z, t, r) \phi(s, t) ds \]  

with

\[ \xi(s, z, t, r) = \int \sqrt{\frac{\omega(p)}{\omega_0}} \psi(p)[z - (\omega' + \frac{p}{m})t, r] e^{-i\nu s} dp. \]
The Heisenberg evolution equation for $\phi$ is the generalized nonlinear quantum Schrödinger equation:

$$i\hbar \frac{\partial \phi}{\partial t}(t, z) = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial z^2}(t, z) + \sigma(z) \phi^\dagger(t, z) \phi(t, z) \phi(t, z).$$

(14)

Eq. (12) shows that the whole evolution is the composition of the deterministic propagation of the $X$-wave $\psi$, embedded in the $\xi$ kernel, and of the quantum one obeying (13). Taking for $\psi$ the fundamental $X$-wave ($q = 0$), $\sigma(z)$ is a bell shaped function. It is possible to show that, if the classical dispersion length is much smaller than the diffraction length, it can be treated as a constant $\sigma$ and the model reduces to the integrable quantum nonlinear Schrödinger equation. Hence the whole 3D quantum dynamics is reduced to an exactly solvable model. Conversely when the diffraction length is smaller than the dispersion length, the dispersion is negligible, and the model is still integrable, representing self-phase modulation.

Summarizing, the 3D nonlinear quantum propagation of $X$-waves can be treated in terms of a well known approach. All the experiments concerning quantum solitons, involving quantum non-demolition, squeezing and entanglement can be re-stated in terms of undistorted progressive 3D wave-packets. Quantum non-demolition experiments by collision of $X$-waves with different velocities, generated by different axicons, can be envisaged and analyzed with the same techniques previously developed, which will not be reported here (for a review see [18] and references therein).

In figure 2 an idealized interferometer for the generation of squeezed nonlinear $X$-waves is sketched. More elaborated setups may be readily drawn from previously developed fiber solitons schemes. From a single bell shaped 3D wave-packet in air, two identical $X$-waves are generated, propagate in a Kerr medium, and then interfere at a symmetric beam splitter. At one output port a squeezed $X$-wave is obtained; the degree of squeezing can be monitored by balanced photo-detection. With the modifications used for fiber solitons, entangled 3D (progressive undistorted) pulsed beams can be generated.

The electromagnetic field attained at the other output port is peculiar. It is the so-called squeezed vacuum, which can be interpreted as the nonlinearly induced quantum noise, “dissected” from the pump beam, with no average electric field but with spatio-temporal correlation. Furthermore it has the additional, remarkable, property of having the propagation characteristics of a progressive invariant 3D wave. Hence its spectral properties are X-shaped, and can be distinguished inside standard vacuum fluctuations. This property makes such a state appealing for applications in interferometry, where the squeezed vacuum can be used for enhancing the performances, as in the framework of the LIGO (or even MIGO) projects for the detection of gravitational waves, or for geophysical studies. To achieve sensitivity below the so-called quantum limits, it has been proposed to use squeezed vacuum as input in one of the arms of the interferometer used to detect gravitational waves. The effective absence of diffraction for progressive undistorted waves may provide the elements for a reduction of the very high power levels needed to reduce shot noise. Furthermore the $X$-shaped squeezed vacuum is expected to be very robust with respect to the contamination of the standard vacuum fluctuation, a significant problem in the future generations of LIGO. In some sense, the spatio-temporal modulation, characterizing the progressive invariant waves, may be used to encode useful signals and discern them from noise.

In conclusion, the quantum propagation of nonlinear $X$-waves has been investigated. All the quantum effects, which have been previously considered for fiber quantum solitons, do have a counterpart in the 3D realm of progressive undistorted packets. Thus, a variety of new experiments may be conceived, with applications ranging from quantum information to gravitational waves detection.

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