On a rank-unimodality conjecture of Morier-Genoud and Ovsienko

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Introduction

Fences with long segments

Fences with at most three segments

Other approaches
Let $\alpha = (a, b, \ldots)$ be a composition. A \textit{fence} is a poset $F = F(\alpha)$ with elements $x_1, \ldots, x_n$ and covers

$$x_1 \triangleleft x_2 \triangleleft \ldots \triangleleft x_{a+1} \triangleright x_{a+2} \triangleright \ldots \triangleright x_{a+b+1} \triangleleft x_{a+b+2} \triangleleft \ldots.$$  

Ex.

$$F(2, 3, 1) =$$

The maximal chains of $F$ are called \textit{segments}. Note that if $\alpha = (\alpha_1, \alpha_2, \ldots)$ then

$$n = \#F(\alpha) = 1 + \sum_i \alpha_i.$$
Let $L = L(\alpha)$ be the distributive lattice of order ideals of $F(\alpha)$. These lattices can be used to compute mutations in a cluster algebra on a surface with marked points.

| Who                      | When   | What                                      |
|--------------------------|--------|-------------------------------------------|
| Propp                    | 2005   | perfect matchings on snake graphs         |
| Yurikusa                 | 2019   | perfect matchings of angles               |
| Schiffler                | 2008, 2010 | $T$-paths                           |
| Schiffler and Thomas     | 2009   | $T$-paths                                 |
| Propp                    | 2005   | lattice paths on snake graphs             |
| Claussen                 | 2020   | lattice paths of angles                   |
| Claussen                 | 2020   | $S$-paths                                 |
Lattice $L(\alpha)$ is ranked with rank function $rk\ I = \# I$. We let

$$R_k(\alpha) = \{ I \in L(\alpha) \mid rk\ I = k \} \quad \text{and} \quad r_k(\alpha) = \# R_k(\alpha).$$

We will also use the rank generating function

$$r(q; \alpha) = \sum_k r_k(\alpha) q^k.$$

This generating function was used by Morier-Genoud and Ovsienko to define $q$-analogues of rational numbers. Call a sequence $a_0, a_1, \ldots$ or its generating function unimodal if there is an index $m$ with

$$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots.$$

**Conjecture (Morier-Genoud and Ovsienko, 2020)**

*For any $\alpha$ we have that $r(q; \alpha)$ is unimodal.*

Previous work: Gansner (1982), Munarini and Salvi (2002), Claussen (2020).
Call sequence $a_0, a_1, \ldots, a_n$ symmetric if, for all $k \leq n/2$, 
$$a_k = a_{n-k}.$$ 

Call the sequence top heavy or bottom heavy if, for all $k \leq n/2$, 
$$a_k \leq a_{n-k} \quad \text{or} \quad a_k \geq a_{n-k},$$ 
respectively.

Call the sequence top interlacing (TI) if 
$$a_0 \leq a_n \leq a_1 \leq a_{n-1} \leq a_2 \leq \ldots \leq a_{\lfloor n/2 \rfloor}$$
or bottom interlacing (BI) if 
$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq a_{n-2} \leq \ldots \leq a_{\lfloor n/2 \rfloor}.$$ 

Note that interlacing implies unimodality and heaviness.

Conjecture (MSS)

Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$.

(a) If $s$ is even, then $r(q; \alpha)$ is BI.

(b) Suppose $s \geq 3$ is odd and let $\alpha' = (\alpha_2, \ldots, \alpha_{s-1})$.

(i) If $\alpha_1 > \alpha_s$ or $\alpha_1 < \alpha_s$ then $r(q; \alpha)$ is BI or TI, respectively.

(iii) If $\alpha_1 = \alpha_s$ then $r(q; \alpha)$ is symmetric, BI, or TI depending on whether $r(q; \alpha')$ is symmetric, TI, or BI, respectively.
A *chain decomposition (CD)* of a poset $P$ is a partition of $P$ into disjoint saturated chains. If $P$ is ranked then the *center* of a chain $C$ is

$$\text{cen } C = \frac{\text{rk}(\min C) + \text{rk}(\max C)}{2}.$$ 

If $\text{rk } P = n$ then a CD is *symmetric (SCD)* if for all chains $C$ in the CD

$$\text{cen } C = \frac{n}{2}.$$ 

A CD is *top centered (TCD)* if for all chains $C$ in the CD

$$\text{cen } C = \frac{n}{2} \text{ or } \frac{n + 1}{2}.$$ 

A *bottom centered CD (BCD)* has $\text{cen } C = n/2$ or $(n - 1)/2$ for all chains $C$. If $P$ has an SCD, TCD, or BCD then its rank sequence is symmetric, top, or bottom interlacing, respectively.

**Conjecture (MSS)**

*For any $\alpha$, the lattice $L(\alpha)$ admits an SCD, TCD, or BCD consistent with the previous conjecture.*
Theorem (MSS)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ and suppose that for some $t$ we have

$$\alpha_t > \sum_{i \neq t} \alpha_i.$$ 

Then $r(q; \alpha)$ is unimodal.

Theorem (MSS)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ where for some $t$

$$\alpha_t = 1 + \sum_{i \neq t} \alpha_i.$$ 

If $L(\alpha)$ has an SCD, TCD, or BCD then so does $L(\beta)$ where

$$\beta = (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + a, \alpha_{t+1}, \ldots, \alpha_s)$$

for any $a \geq 0$.

Theorem (MMS)

If $\alpha$ has at most three parts then $L(\alpha)$ has an SCD, TCD, or BCD.
The following recursion also has a version for \( s \) even.

**Lemma**

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \). Then for \( s \) odd

\[
 r(q; \alpha) = r(q; \alpha_1, \ldots, \alpha_{s-1}, \alpha_{s-1}) + q^{\alpha_{s+1}} \cdot r(q; \alpha_1, \ldots, \alpha_{s-2}, \alpha_{s-1}-1). 
\]

**Proof.**

If \( l \in L(\alpha) \) then either \( x_n \notin l \) or \( x_n \in l \) where \( n = \#F(\alpha) \). So

\[
 l = x_n \quad \text{or} \quad l = x_n
\]

Using the lemma as well as induction on \( \alpha_2 + \cdots + \alpha_s \):

**Theorem**

We have \( r(q; \alpha) \) unimodal if \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \) satisfies

\[
 \alpha_1 \geq \alpha_2 + \alpha_3 + \cdots + \alpha_s.
\]

A similar result hold when \( \alpha_s \) plays the role of \( \alpha_1 \).
Lemma

Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$, $n = \#F(\alpha)$, and for some $t$

$$\alpha_t \geq 1 + \sum_{i \neq t} \alpha_i.$$  

(1)

Let $S$ be the segment of length $\alpha_t$, $F' = F - S$,

$$m = \#F' \text{ and } \ell = \#L(F').$$

Then the maximum size of a rank of $L = L(\alpha)$ is $\ell$ and this maximum occurs at ranks $m + 1$ through $n - m - 1$.

Proof.

If $I \in L(\alpha)$ then $I = J \cup K$ where $J \in L(F')$ and $K \in L(S)$. Since $L(S)$ is a chain, given $J$ and $rk I$, there is $\leq 1$ choice for $K$. \qed

This lemma permits us to prove that if (1) holds then $L(\alpha)$ is rank unimodal. It also has an SCD, TCD or BCD as long as that is true for the base case of equality in (1).
Claussen proved that if $\alpha$ has at most four parts then $L(\alpha)$ is rank unimodal. We are able to prove the stronger SCD, TCD, and BCD conditions when $\alpha$ has at most three parts using a variant of the Greene-Kleitman core. If $P$ is a poset on $[n] = \{1, \ldots, n\}$ then we can associate with any $I \subseteq P$ a zero-one word $w_I = w_1 \ldots w_n$ which is the indicator function of $I$. Form the Greene-Kleitman core, $\text{GK}(w_I)$, by pairing any $w_i = 0$ with $w_{i+1} = 1$, then pair zeros and ones separated only by pairs, etc.

**Ex.** Let $F = F(2, 3, 1)$ be labeled starting with the middle segment, then the last, and finally the first, and let $I = \{1, 4, 5\}$.

$$w_I = 1001100 \quad \text{and} \quad \text{GK}(w_I) = \overset{\wedge}{*0011\overset{*}{*}} = \{(2, 5), (3, 4)\}.$$
Let \( \preceq \) be the order relation in \( F(\alpha) \). Call an unpaired \( f \in [n] \) \textit{frozen with respect to} \( I \) if there is \((i, j) \in GK(w_I)\) with

\[ f \triangleright i \quad \text{or} \quad f \triangleleft j. \]

We now define the \textit{core of} \( w_I \) to be

\[ \text{core } w_I = GK(w_I) \cup \{ f \mid f \text{ is frozen with respect to } I \}. \]

The elements \( i \notin \text{core } w_I \) are \textit{free with respect to} \( I \). Given \( \kappa = \text{core } w_I \) we form a saturated chain \( C_\kappa \) by starting with \( w_0 \) which has core \( \kappa \) and all free positions equal to zeros. We then change each free zero to a one from left to right.

\textbf{Ex.} Let \( F = F(2, 3, 1) \) be labeled as before \( I = \{1, 4, 5\} \).

\[ w_I = 1001100 \quad \text{and} \quad GK(w_I) = \{(2, 5), (3, 4)\} \]

\[ 7 \triangleright 2 \in (2, 5) \text{ so } 7 \text{ frozen} \quad \text{and} \quad 1 \triangleleft 4 \in (3, 4) \text{ so } 1 \text{ frozen} \]

\[ \kappa = \text{core } w_I = \{1, (2, 5), (3, 4), 7\} \quad \text{and} \quad 6 \text{ is free} \]

\[ C_\kappa = \{1001100, 1001110\} \]

\textbf{Theorem}

\textit{If} \( \kappa = (a, b, c), a \geq c, \text{ then the } C_\kappa \text{ form an SCD or TCD of } L(\alpha).}
(1) An explicit formula. There is an explicit formula for $r(q; \alpha)$ in terms of powers of $q$ and $[n]_q = 1 + q + \cdots + q^{n-1}$. This expression appeared in a more complicated form in Claussen’s thesis. Let

$$Z(\alpha) = \{z_1, z_2, \ldots, z_u\}$$

be the set of maxima of $F(\alpha)$ written in order from left to right.

**Theorem (Claussen)**

If $\alpha = (\alpha_1, \ldots, \alpha_{2u-1})$ then

$$r(q; \alpha) = \sum_{Z \subseteq Z(\alpha)} q^{|Z|} r(Z, 1)r(Z, 2) \cdots r(Z, u)$$

where, letting $\alpha_0 = 1$,

$$r(Z, i) = \begin{cases} 
q^{\alpha_{2i-2} + \alpha_{2i-1} - 1} & \text{if } z_{i-1}, z_i \in Z, \\
q^{\alpha_{2i-2}}[\alpha_{2i-1}]_q & \text{if } z_{i-1} \in Z \text{ and } z_i \not\in Z, \\
q^{\alpha_{2i-1}}[\alpha_{2i-2}]_q & \text{if } z_{i-1} \not\in Z \text{ and } z_i \in Z, \\
1 + q^{[\alpha_{2i-2}]_q[\alpha_{2i-1}]_q} & \text{if } z_{i-1}, z_i \not\in Z.
\end{cases}$$
2. **Lexicographic CDs.** Suppose $P$ is a poset on $[n]$. We construct the chains $C_1, C_2, C_3, \ldots$ of a CD of $L = L(P)$ as follows. Suppose $C_1, \ldots, C_{i-1}$ have been constructed. Since $P = [n]$ as sets, we can consider any ideal $I$ of $P$ as a subset of $\{1, \ldots, n\}$ and we will not make any distinction between an ideal and its subset. So given two ideals, we can compare them in the lexicographic order on subsets. Now we form $C_i$ by starting with the unique ideal $I_0$ which has minimum rank and is also lexicographically least among all elements of $L' = L - (C_1 \cup \cdots \cup C_{i-1})$. We now consider all ideals of $L'$ which cover $I_0$ and take the lexicographically least of them to be the next element $I_1$ on $C_i$. We continue in this manner until we come to an ideal which has no cover in $L'$ at which point $C_i$ terminates. We have the following conjecture which we have verified for all compositions $\alpha$ with $\sum_i \alpha_i \leq 6$.

**Conjecture**

*For every $\alpha$ there is a labeling of $F(\alpha)$ with $[n]$ such that the corresponding lexicographic CD of $L(\alpha)$ is an SCD, TCD, or BCD.*
THANKS FOR LISTENING!