FURTHER REFINEMENTS OF THE CAUCHY–SCHWARZ INEQUALITY FOR MATRICES

MOJTABA BAKHERAD

Abstract. Let $A$, $B$ and $X$ be $n \times n$ matrices such that $A$, $B$ are positive semidefinite. We present some refinements of the matrix Cauchy-Schwarz inequality by using some integration techniques and various refinements of the Hermite–Hadamard inequality. In particular, we establish the inequality

$$|||A^{1/2}XB^{1/2}|||^2 \leq |||A^{1-t}XB^s|||^2 \leq \max\{|||AX|||^r, |||AXB|||^r, |||XB|||^r, |||AXB|||^r\},$$

where $s, t \in [0, 1]$ and $r \geq 0$.

1. Introduction and preliminaries

Let $\mathcal{M}_n$ be the $C^*$-algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathcal{M}_n$, we write $A \geq 0$ if $A$ is positive semidefinite, $A > 0$ if $A$ is positive definite, and $A \geq B$ if $A - B \geq 0$. We use $\mathcal{S}_n$ for the set of positive semidefinite matrices and $\mathcal{P}_n$ for the set of positive definite matrices in $\mathcal{M}_n$. A norm $|||\cdot|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. The numerical range of $A \in \mathcal{M}_n$ is $W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ and the numerical radius of $A$ is defined by $\omega(A) = \sup\{\|\langle Ax, x \rangle\| : x \in \mathbb{C}^n, \|x\| = 1\}$. It is well-known [5] that $\omega(\cdot)$ is a weakly unitarily invariant norm on $\mathcal{M}_n$, that is $\omega(U^*AU) = \omega(A)$ for every unitary $U \in \mathcal{M}_n$. The Hadamard product (Schur product) of two matrices $A, B \in \mathcal{M}_n$ is the matrix $A \circ B$ whose $(i, j)$ entry is $a_{ij}b_{ij} (1 \leq i, j \leq n)$. The Schur multiplier operator $S_A$ on $\mathcal{M}_n$ is defined by $S_A = A \circ X (X \in \mathcal{M}_n)$. The induced norm of $S_A$ with respect to the spectral norm is $\|S_A\| = \sup_{X \neq 0} \frac{\|S_A(X)\|}{\|X\|} = \sup_{X \neq 0} \frac{\|A \circ X\|}{\|X\|}$, and the induced norm of $S_A$ with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$

A continuous real valued function $f$ on an interval $J \subseteq \mathbb{R}$ is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all $A, B \in \mathcal{M}_n$ with spectra in $J$. Recall that a real

2010 Mathematics Subject Classification. Primary 15A18, Secondary 15A60, 15A42, 47A60, 47A30. Key words and phrases. convex function, the Cauchy-Schwarz inequality, unitarily invariant norm, numerical radius, Hadamard product.
valued function $F$ defined on $J_1 \times J_2$ is called convex if

$$F(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \leq \lambda F(x_1, y_1) + (1 - \lambda)F(x_2, y_2)$$

for all $x_1, x_2 \in J_1, y_1, y_2 \in J_2$ and $\lambda \in [0, 1]$.

For two sequences $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ of real numbers, the classical Cauchy-Schwarz inequality states that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right) \left(\sum_{j=1}^{n} b_j^2\right)$$

with equality if and only if the sequences $a$ and $b$ are proportional [11]. Horn and Mathias [7] gave a matrix Cauchy-Schwarz inequality as follows

$$||| |A^*B^r||| \leq ||| |(AA^*)^r||| ||| |(BB^*)^r|||$$

$(A, B, X \in \mathcal{M}_n, r \geq 0)$.

Bhatia and Davis [2] showed that

$$||| |A^*XB^r||| \leq ||| |AA^*X^r||| ||| |XBB^r|||$$

$(A, B, X \in \mathcal{M}_n, r \geq 0)$, (1.1)

which is equivalent to

$$||| |A^{1/2}XB^{1/2}^r||| \leq ||| |AX^r||| ||| |XB^r|||$$

$(A, B \in \mathcal{S}_n, X \in \mathcal{M}_n, r \geq 0)$. (1.2)

In [6] it is proved that the function $f(t) = ||| |A^tXB^{1-t}^r||| ||| |A^{1-t}XB^t^r|||$ is convex on the interval $[0, 1]$, when $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and attains its minimum at $t = 1/2$. In view of the fact that the function $f$ is decreasing on the interval $[0, 1/2]$ and increasing on the interval $[1/2, 1]$. In particular, we have a refinement of the Cauchy-Schwarz inequality [6] as follows

$$||| |A^{1/2}XB^{1/2}^r||| \leq ||| |A^{1/2}XB^{1/2}^r||| ||| |A^{1-t}XB^t^r||| \leq ||| |AX^r||| ||| |XB^r|||$$

$$(1.3)$$

where $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and $\mu \in [0, 1]$.

Applying the convexity of the function $f(t) = ||| |A^tXB^{1-t}^r||| ||| |A^{1-t}XB^t^r|||$ $(t \in [0, 1])$, we show some refinements of inequality (1.3). we also show the convexity of the function $f(s, t) = ||| |A^sXB^{1-t}^r||| ||| |A^{1-s}XB^t^r|||$ and present some other refinements of inequality (1.3). In the last section we show some related numerical radius inequalities.

2. Norm inequality involving the Cauchy-Schwarz

In this section, we establish some refinements of inequality (1.3). To this end, we need the following Hermite-Hadamard inequality.
Lemma 2.1. [4] Let $g$ be a real-valued convex function on $[a,b]$. Then

$$
g \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b g(s)ds \leq \frac{1}{4} \left[ g(a) + 2g \left( \frac{a+b}{2} \right) + g(b) \right] \leq \frac{g(a) + g(b)}{2}.\tag{2.1}
$$

Applying Lemma 2.1 we have following result.

Proposition 2.2. Suppose that $A, B \in S_n$, $X \in M_n$ and $r \geq 0$. Then

$$
\left\| \left\| A^\frac{s}{2} X B^\frac{s}{2} \right\| \right\|^2 \leq \frac{1}{1-2\mu} \left| \int_{\mu}^{1-\mu} \left( \left\| A^s X B^{1-s} \right\| \right)^r \left( \left\| A^{1-s} X B^s \right\| \right)^r ds \right|
$$

$$
\leq \frac{1}{2} \left[ \left\| A^\frac{s}{2} X B^\frac{s}{2} \right\|^2 + \left\| A^s X B^{1-s} \right\| \left\| A^{1-s} X B^s \right\| \right]
$$

$$
\leq \left\| A^s X B^{1-s} \right\| \left\| A^{1-s} X B^s \right\|
$$

for all $0 \leq \mu \leq 1$ and all unitarily invariant norms $\left\| . \right\|$.\tag{2.2}

Proof. Let $f(t) = \left\| A^t X B^{1-t} \right\| \left\| A^{1-t} X B^t \right\|$. First assume that $0 \leq \mu < \frac{1}{2}$. It follows from Lemma 2.1 that

$$
f \left( \frac{\mu + 1 - \mu}{2} \right) \leq \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} f(s)ds
$$

$$
\leq \frac{1}{4} \left[ f(\mu) + 2f \left( \frac{\mu + 1 - \mu}{2} \right) + f(1-\mu) \right]
$$

$$
\leq \frac{f(1-\mu) + f(\mu)}{2},
$$

whence

$$
f \left( \frac{1}{2} \right) \leq \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} f(s)ds \leq \frac{1}{2} \left[ f(\mu) + f \left( \frac{1}{2} \right) \right] \leq f(\mu).
$$

Hence

$$
\left\| \left\| A^\frac{s}{2} X B^\frac{s}{2} \right\| \right\|^2 \leq \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} \left( \left\| A^{1-s} X B^s \right\| \right) \left( \left\| A^s X B^{1-s} \right\| \right) ds
$$

$$
\leq \frac{1}{2} \left[ \left\| A^\frac{s}{2} X B^\frac{s}{2} \right\|^2 + \left\| A^s X B^{1-s} \right\| \left\| A^{1-s} X B^s \right\| \right]
$$

$$
\leq \left\| A^s X B^{1-s} \right\| \left\| A^{1-s} X B^s \right\|. \tag{2.1}
$$

Now, assume that $\frac{1}{2} < \mu \leq 1$. By the symmetry property of (2.1) with respect to $\mu$, if we replace $\mu$ by $1 - \mu$, then

$$
\left\| \left\| A^\frac{s}{2} X B^\frac{s}{2} \right\| \right\|^2 \leq \frac{1}{2\mu} \int_{1-\mu}^{\mu} \left( \left\| A^{1-s} X B^s \right\| \right) \left( \left\| A^s X B^{1-s} \right\| \right) ds
$$

$$
\leq \frac{1}{2} \left[ \left\| A^\frac{s}{2} X B^\frac{s}{2} \right\|^2 + \left\| A^s X B^{1-s} \right\| \left\| A^{1-s} X B^s \right\| \right]
$$

$$
\leq \left\| A^s X B^{1-s} \right\| \left\| A^{1-s} X B^s \right\|. \tag{2.2}
$$
Since \( \lim_{\mu \to \frac{1}{2}} \frac{1}{|2\mu - 1|} \int_{\mu}^{1-\mu} |||A^sXB^{1-s}|^r||| |||A^{1-s}XB^s|^r||| ds = |||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r|||^2 \), inequalities (2.1) and (2.2) yield the desired result.

Now, we show the convexity of the function

\[
F(s, t) = |||A^{1-t}XB^{1+s}|^r||| |||A^{1+t}XB^{1-s}|^r|||
\]

and we use the convexity of \( F \) to prove some Cauchy-Schwarz type inequalities.

**Theorem 2.3.** Suppose that \( A, B \in \mathcal{S}_n \), \( X \in \mathcal{M}_n \) and \( r \geq 0 \). Then the function

\[
F(s, t) = |||A^{1-t}XB^{1+s}|^r||| |||A^{1+t}XB^{1-s}|^r|||
\]

is convex on \([-1, 1] \times [-1, 1]\) and attains its minimum at \((0, 0)\).

**Proof.** The function \( F \) is continuous and \( F(s, t) = F(-s, -t) \) \((s, t \in [0, 1])\). Thus it is enough to show that

\[
F(s_1, t_1) \leq \frac{1}{2} [F(s_1 + s_2, t_1 + t_2) + F(s_1 - s_2, t_1 - t_2)],
\]

where \( s_1 \pm s_2, t_1 \pm t_2 \in [-1, 1] \times [-1, 1] \).

Let \( s_1 \pm s_2, t_1 \pm t_2 \in [-1, 1] \times [-1, 1] \). Applying inequality (1.1) we obtain

\[
|||A^{1-t_1}XB^{1+s_1}|^r||| = |||A^{t_2}(A^{1-t_1-t_2}XB^{1+s_1-s_2})B^{s_2}|^r|||
\]

\[
\leq \left\{ |||A^{1-(t_1-t_2)}XB^{1+(s_1-s_2)}|^r||| |||A^{1-(t_1+t_2)}XB^{1+(s_1+s_2)}|^r||| \right\}^{1/2} \tag{2.3}
\]

and

\[
|||A^{1+t_1}XB^{1-s_1}|^r||| = |||A^{t_2}(A^{1+t_1-t_2}XB^{1-s_1-s_2})B^{s_2}|^r|||
\]

\[
\leq \left\{ |||A^{1+(t_1+t_2)}XB^{1-(s_1+s_2)}|^r||| |||A^{1+(t_1-t_2)}XB^{1-(s_1-s_2)}|^r||| \right\}^{1/2}. \tag{2.4}
\]

Applying (2.3), (2.4) and the arithmetic-geometric mean inequality we get

\[
F(s_1, t_1) = |||A^{1-t_1}XB^{1+s_1}|^r||| |||A^{1+t_1}XB^{1-s_1}|^r|||
\]

\[
\leq [F(s_1 + s_2, t_1 + t_2)F(s_1 - s_2, t_1 - t_2)]^{1/2}
\]

\[
\leq \frac{1}{2} [F(s_1 + s_2, t_1 + t_2) + F(s_1 - s_2, t_1 - t_2)].
\]

\(\square\)

**Corollary 2.4.** Suppose that \( A, B \in \mathcal{S}_n \), \( X \in \mathcal{M}_n \) and \( r \geq 0 \). Then

\[
|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r|||^2 \leq |||A^tXB^{1-s}|^r||| |||A^{1-t}XB^s|^r|||
\]

\[
\leq \max\{|||AX|^r|||, |||XB|^r|||, |||AXB|^r|||, |||X|^r|||\},
\]

where \( s, t \in [0, 1] \).
Proof. If we replace \(s, t, A, B\) by \(2s - 1, 2t - 1, A^\frac{1}{2}, B^\frac{1}{2}\), respectively, in Theorem 2.3, we get the function \(G(s, t) = |||A^sXB^{1-s}|r||| |||A^{1-t}XB^t|r|||\) is convex on \([0, 1] \times [0, 1]\) and attains its minimum at \((\frac{1}{2}, \frac{1}{2})\). Hence

\[
|||A^\frac{1}{2}XB^\frac{1}{2}|r|||^2 \leq |||A^sXB^{1-s}|r||| |||A^{1-t}XB^t|r|||.
\]

In addition, since the function \(G\) is continuous and convex on \([0, 1] \times [0, 1]\), it follows that \(G\) attains its maximum at the vertices of the square. Moreover, due to the symmetry there are two possibilities for the maximum. \(\square\)

Dragomir [3, p. 316] proved that

\[
F \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b F(x, \frac{c + d}{2}) \, dx + \frac{1}{d - c} \int_c^d F(\frac{a + b}{2}, y) \, dy \right]
\]

\[
\leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d F(x, y) \, dy \, dx
\]

\[
\leq \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4},
\]

whenever \(F\) is a convex function on \([a, b] \times [c, d] \subseteq \mathbb{R}^2\). Applying inequality (2.5) for the convex function \(G(s, t) = |||A^sXB^{1-s}|r||| |||A^{1-t}XB^t|r|||\) on \([0, 1] \times [0, 1]\) we get the following result.

**Corollary 2.5.** Suppose that \(A, B \in S_n, X \in \mathcal{M}_n\) and \(r \geq 0\). Then

\[
2|||A^\frac{1}{2}XB^\frac{1}{2}|r|||^2 \leq \frac{1}{1 - 2\alpha} \int_0^{1-\alpha} |||A^sXB^{1-s}|r||| |||A^{1-s}XB^s|r||| \, ds
\]

\[
+ \frac{1}{1 - 2\beta} \int_\beta^{1-\beta} |||A^\frac{1}{2}XB^{1-t}|r||| |||A^\frac{1}{2}XB^t|r||| \, dt
\]

\[
\leq \frac{2}{(1 - 2\alpha)(1 - 2\beta)} \int_0^{1-\alpha} \int_\beta^{1-\beta} |||A^sXB^{1-t}|r||| |||A^{1-s}XB^t|r||| \, ds \, dt
\]

\[
\leq |||A^\alpha XB^{1-\beta}|r||| |||A^{1-\alpha}XB^{1-\beta}|r||| + |||A^{1-\alpha}XB^{1-\beta}|r||| |||A^\alpha XB^\beta|r|||
\]

for all \(\alpha, \beta \in [0, \frac{1}{2})\) and

\[
2|||A^\frac{1}{2}XB^\frac{1}{2}|r|||^2 \leq \frac{1}{2\alpha - 1} \int_0^\alpha |||A^sXB^{1-s}|r||| |||A^{1-s}XB^s|r||| \, ds
\]

\[
+ \frac{1}{2\beta - 1} \int_0^\beta |||A^\frac{1}{2}XB^{1-t}|r||| |||A^\frac{1}{2}XB^t|r||| \, dt
\]

\[
\leq \frac{2}{(2\alpha - 1)(2\beta - 1)} \int_0^\alpha \int_0^\beta |||A^sXB^{1-t}|r||| |||A^{1-s}XB^t|r||| \, ds \, dt
\]

\[
\leq |||A^\alpha XB^{1-\beta}|r||| |||A^{1-\alpha}XB^{1-\beta}|r||| + |||A^{1-\alpha}XB^{1-\beta}|r||| |||A^\alpha XB^\beta|r|||
\]

for all \(\alpha, \beta \in (\frac{1}{2}, 1]\).
Proof. Let \( G(s, t) = ||| A'X B^{1-t} ||| \cdot ||| A^{1-t} X B^t ||| \). If we replace \( a \) by \( 1 - \alpha \), \( c \) by \( \beta \) and \( d \) by \( 1 - \beta \) \((\alpha, \beta \in [0, \frac{1}{2}])\) for the convex function \( G \) in (2.5) we reach the first inequality and if we replace \( a \) by \( 1 - \alpha \), \( b \) by \( \alpha \), \( c \) by \( 1 - \beta \) and \( d \) by \( \beta \) \((\alpha, \beta \in (\frac{1}{2}, 1])\) in (2.5) we obtain the second inequality. \( \square \)

The spacial case \( \alpha = \beta = 1 \) of Theorem 2.5 reads as follows.

**Corollary 2.6.** Suppose that \( A, B \in S_n, X \in M_n \) and \( r \geq 0 \). Then

\[
2 ||| A^{\frac{1}{2}} X B^{\frac{1}{2}} ||| ||| A^{\frac{1}{2}} X B^{\frac{1}{2}} ||| ds \\
\quad + \int_0^1 ||| A^{\frac{1}{2}} X B^{1-t} ||| ||| A^{\frac{1}{2}} X B^t ||| dt \\
\leq 2 \int_0^1 \int_0^1 ||| A^{\frac{1}{2}} X B^{1-t} ||| ||| A^{\frac{1}{2}} X B^t ||| dt \ ds \\
\leq ||| AX ||| ||| XB ||| + ||| X ||| ||| AXB |||.
\]

3. Further refinements of the Cauchy-Schwarz inequality

In this section, we establish some refinements of the Cauchy-Schwarz inequality. The following result, derived in the recent papers [8, 9].

**Lemma 3.1.** [8] Let \( f : [a, b] \to \mathbb{R} \) be a convex function and \( \delta \in [a, b], p \in (0, 1) \) be fixed parameters. Then the function \( \varphi : [a, b] \to \mathbb{R} \), defined by

\[
\varphi(t) = (1 - p)f(\delta) + pf(t) - f((1 - p)\delta + pt)
\]

is decreasing on \([a, \delta]\) and is increasing on \([\delta, b]\).

In the next result, we show a refinement of the right side of inequality (1.2).

**Theorem 3.2.** Let \( A, B \in S_n, X \in M_n \), \( r \geq 0 \), \( \mu \in [0, 1] \), \( p \in (0, 1) \) and let \( ||| \cdot \cdot \cdot \| \) be any unitarily invariant norm. Then

\[
\| ||| AX ||| ||| XB ||| - ||| AXB^{1-\mu} ||| ||| A^{1-\mu} X B^\mu ||| \| \\
\quad \geq \frac{1}{p} \left( f\left( \frac{1-p}{2} \right) - f\left( \frac{1-p}{2} + p\mu \right) \right) \geq 0, \quad (3.1)
\]

where \( f(t) = ||| A' X B^{1-t} ||| ||| A^{1-t} X B^t ||| \) \((t \in [0, 1])\).

**Proof.** Assume that the functions \( f(t) = ||| A' X B^{1-t} ||| ||| A^{1-t} X B^t ||| \) \((t \in [0, 1])\) and \( \varphi(\mu) = (1 - p)f\left( \frac{1}{2} \right) + pf(\mu) - f\left( \frac{1-p}{2} + p\mu \right) \) \((\mu \in [0, 1])\). Using Lemma 3.1, we see
that \( \varphi \) is decreasing on \([0, \frac{1}{2}]\) and increasing on \([\frac{1}{2}, 1]\). Let that \( \mu \in [0, \frac{1}{2}] \). Since \( \varphi \) is decreasing on \([0, \frac{1}{2}]\), we have \( \varphi(0) \geq \varphi(\mu) \), that is,

\[
pf(0) - f\left(\frac{1-p}{2}\right) \geq pf(\mu) - f\left(\frac{1-p}{2} + p\mu\right),
\]

whence

\[
f(0) - f(\mu) \geq \frac{1}{p}\left[f\left(\frac{1-p}{2}\right) - f\left(\frac{1-p}{2} + p\mu\right)\right],
\]

which yields desired inequality. Note, the right hand side of (3.2) is decreasing and 
\[
\frac{1-p}{2} + p\mu \geq \frac{1-p}{2}.
\]

Now let \( \mu \in \left[\frac{1}{2}, 1\right] \). So \( 0 \leq 1 - \mu \leq \frac{1}{2} \). By the symmetry property of (3.2) with respect to \( \mu \), if we replace \( \mu \) by \( 1 - \mu \), then

\[
f(0) - f(1 - \mu) \geq \frac{1}{p}\left[f\left(\frac{1-p}{2}\right) - f\left(\frac{1-p}{2} - p\mu\right)\right],
\]

which is reduce to (3.1) since \( f(1 - \mu) = f(\mu) \), \( (\mu \in [0, 1]) \).

By the same strategy as in the proof of Theorem 3.3, we get a refinement of the left side inequality (1.2).

**Theorem 3.3.** Let \( A, B \in \mathcal{S}_n \), \( X \in \mathcal{M}_n \), \( r \geq 0 \), \( \mu \in [0, 1] \), \( p \in (0, 1) \) and let \( \|\| \cdot \|\| \) be any unitarily invariant norm. Then

\[
\|\| A^{\mu}XB^{1-\mu} \|\| + \|\| A^{1-\mu}XB^\mu \|\| - \|\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \|\| \|\|^2 \geq \frac{1}{p}\left(f\left(\frac{1-p}{2} + p\mu\right) - \|\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \|\|\|^2\right) \geq 0,
\]

where \( f(t) = \|\| A^tXB^{1-t} \|\| \) \( \|\| A^{1-t}XB^t \|\| \) \( (t \in [0, 1]) \).

4. **Some inequalities involving numerical radius**

In this section we show inequalities involving Heinz type numerical radius. A continuous real valued function \( f \) defined on an interval \((a, b)\) with \( a \geq 0 \) is called Kwong function if the matrix

\[
\left(\frac{f(a_i) + f(a_j)}{a_i + a_j}\right)_{i,j=1}^n
\]

is positive semidefinite for any distinct real numbers \( a_1, \ldots, a_n \) in \((a, b)\).

**Lemma 4.1.** \([1, \text{Corollary 4}]\) Let \( A = [a_{ij}] \in \mathcal{M}_n \) be positive semidefinite. Then

\[
\|S_A\|_\omega = \max_i a_{ii}.
\]


Lemma 4.2. [14, Theorem 3.4] (Spectral Decomposition) Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then $A$ is normal if and only if there exists a unitary matrix $U$ such that

$$U^*AU = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

In particular, $A$ is positive definite if and only if the $\lambda_j$ ($1 \leq j \leq n$) are positive.

Theorem 4.3. Suppose that $A \in \mathcal{P}_n$, $X \in \mathcal{M}_n$, $\alpha \in [0,1]$ and $\frac{f}{g}$ be a Kwong function such that $f(t)g(t) \leq t$ ($t \geq 0$). Then

$$\omega(f(A)Xg(A) + g(A)Xf(A)) \leq \omega(AX + XA).$$

Proof. Applying Lemma 4.2, we can assume that $A = \text{diag}(a_1, a_2, \cdots, a_n)$ is diagonalize, where $a_j$ ($j = 1,2,\cdots, n$) are positive numbers. Let $Z = [z_{ij}] \in \mathcal{M}_n$ with the entries $z_{ij} = \frac{f(a_i)g(a_j) + f(a_j)g(a_i)}{a_i + a_j}$ ($1 \leq i, j \leq n$). Since $\frac{f}{g}$ is a Kwong function,

$$Z = S \left( \frac{f(a_i)g^{-1}(a_i) + f(a_j)g^{-1}(a_j)}{a_i + a_j} \right)^n_{i,j=1} S$$

is positive semidefinite where $S = \text{diag}(g(a_1), \cdots, g(a_n))$. It follows from Lemma 4.1 that

$$\|SZ\|_{\omega} = \max_i z_{ii} = \frac{f(a_i)g(a_i)}{a_i} \leq 1,$$

or equivalently, $\frac{\omega(Z \circ X)}{\omega(X)} \leq 1$ ($0 \neq X \in \mathcal{M}_n$). Let $E = [\frac{1}{a_i + a_j}]$ and $D = [f(a_i)g(a_i) + f(a_j)g(a_j)] \in \mathcal{M}_n$. Hence

$$\omega(D \circ E \circ X) = \omega(Z \circ X) \leq \omega(X) \quad (X \in \mathcal{M}_n).$$

Let the matrix $C$ be the entrywise inverse of $E$, i.e., $C \circ E = J$. Thus $\omega(D \circ X) \leq \omega(C \circ X)$ ($X \in \mathcal{M}_n$). Hence

$$\omega(f(A)Xg(A) + g(A)Xf(A)) \leq \omega(AX + XA).$$

□

Using $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in Theorem 4.3 we get the following Heinz type inequality in the following result.

Corollary 4.4. Suppose that $A \in \mathcal{P}_n$, $X \in \mathcal{M}_n$ and $\alpha \in [0,1]$. Then

$$\omega(A^\alpha XA^{1-\alpha} + A^{1-\alpha}XA^\alpha) \leq \omega(AX + XA).$$

Kwong [10] showed that the set Kwong functions on $(0, \infty)$ includes all non-negative operator monotone functions $f$ on $(0, \infty)$. 
Example 4.5. The function \( f(t) = \log(t + 1) \) is operator monotone on the interval \((0, \infty)\) [13]. If \( g(t) = \frac{t}{f(t)} \), then, by Theorem 4.3, for every unitarily invariant norm \( \mathcal{||} \cdot \mathcal{||} \), \( A \in \mathcal{P}_n \) and \( X \in \mathcal{M}_n \) we have

\[
\omega \left( \log(A + 1)XA\log(A + 1)^{-1} + A\log(A + 1)^{-1}X\log(A + 1) \right) \leq \omega(AX +XA).
\]

References

[1] T. Ando and K. Okubo, Induced norms of the Schur multiplication operator, Linear and Multilinear Algebra Appl., 147, (1991), 181–199.
[2] R. Bhatia and C. Davis, A Cauchy-Schwarz inequality for operators with applications, Linear Algebra Appl. 223/224 (1995) 119–129.
[3] S.S. Dragomir and Pearce, C. E. M. Selected Topics on Hermite-Hadamard Inequalities and Applications RGMIA Monographs, Victoria University, 2000.
[4] Y. Feng, Refinements of the Heinz inequalities, J. Inequal. Appl. (2012), Art. no. 18 (6 pp.).
[5] K. E. Gustafson and D. K.M. Rao, Numerical Range, The Field of Values of Linear Operators and Matrices Springer, New York, 1997.
[6] F. Hiai and X. Zhan, Inequalities involving unitarily invariant norms and operator monotone functions, Linear Algebra Appl. 341 (2002), 151–169.
[7] R.A. Horn and R. Mathias, Cauchy-Schwarz inequalities associated with positive semidefinite matrices, Linear Algebra Appl. 142 (1990) 63–82.
[8] M. Krnić, N. Lovričević and J.O. Pečarić, Jensens operator and applications to mean inequalities for operators in Hilbert space, Bull. Malays. Math. Sci. Soc., 2012, 35(1), 114
[9] M. Krnić and J.O. Pečarić , Improved Heinz inequalities via the Jensen functional, Cent. Eur. J. Math., 11(9), 2013, 1698-1710
[10] M.K. Kwong, Some results on matrix monotone functions, Linear Algebra Appl. 118 (1989), 129–153.
[11] D.S. Mitrinovic, J. Pecaric and A.M Fink, Classical and New Inequalities in Analysis Kluwer Academic, 1993.
[12] M.S. Moslehian, Matrix Hermite-Hadamard type inequalities, Houston J. Math. 39 (2013), no. 1, 177–189.
[13] J.O. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, Mond Pečarić method in operator inequalities Zagreb, 2005.
[14] F. Zhang, Matrix Theory Springer-Verlag New York, 2011.

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.

E-mail address: mojtaba.bakherad@yahoo.com; bakherad@member.ams.org