STEVIN NUMBERS AND REALITY

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ABSTRACT. We explore the potential of Simon Stevin’s numbers, obscured by shifting foundational biases and by 19th century developments in the arithmetisation of analysis.

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1. FROM DISCRETE ARITHMETIC TO ARITHMETIC OF THE CONTINUUM

Simon Stevin (1548-1620) initiated a systematic approach to decimal representation of measuring numbers, marking a transition from a discrete arithmetic as practiced by the Greeks, to the arithmetic of the continuum taken for granted today, see A. Malet [63] and Naets [60].

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For over two centuries now, such numbers have been called *real*. Concerns about the *reality* of numbers generally preoccupy cognitive scientists and philosophers more than mathematicians. Thus, cognitive scientists view mathematical infinity as necessarily a metaphor [54], while philosophers such as G. Hellman [39] have attempted nominalistic reconstructions that seek to diminish an investigator’s reliance on ontological assumptions that provoke tensions with philosophical examinations of the foundations. Such reconstructions have, in turn, been criticized by other philosophers [15], see also [21] for a response. It is interesting to note in this context that C. S. Peirce thought of the Weierstrassian doctrine of the limit as a nominalistic reconstruction, see J. Dauben [24] and Section 7 below.

Many mathematicians regard such ontological questions as of limited relevance to the practice of mathematics. They feel that a mathematician reasons the same way, whether or not he thinks mathematical objects actually pass any reality check, if such were possible. At the same time, they readily admit serious negative effects in the past caused by an undue influence of a preoccupation with whether such things as complex numbers really *exist*.

Our goal here is neither to pursue the cognitive thread, nor to endorse any nominalistic reconstruction, but rather to focus on the reception of Stevin’s ideas, and how such reception was influenced by received notions of what a continuum should, or rather should not, be. We also examine the effects of Platonist perceptions of the real numbers on the practice of both mathematics and the history of mathematics, as well as the attitude toward infinitesimal-enriched extensions of the traditional number system. Some related issues are analyzed by Katz and Tall in [50].

2. **Stevin’s construction of the real numbers**

Stevin created the basis for modern decimal notation in his 1585 work *De Thiende* (“the art of tenths”). He argued that quantities such as square roots, irrational numbers, surds, negative numbers, etc., should all be treated as numbers and not distinguished as being different in nature. He wrote that “there are no absurd, irrational, irregular, inexplicable or surd numbers.” He further commented as follows:

> It is a very common thing amongst authors of arithmetics to treat numbers like $\sqrt{8}$ and similar ones, which

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1C. Boyer refers to Cantor, Dedekind, and Weierstrass as “the great triumvirate”, see [13, p. 298]. The triumvirate reconstruction of analysis as a nominalistic project is explored in our text [44].
they call absurd, irrational, irregular, inexplicable or surds etc and which we deny to be the case for number which turns up.

Thus, Stevin explicitly states that numbers that are *not* rational have equal rights of citizenship with those that are. According to van der Waerden, Stevin’s general notion of a real number was accepted, tacitly or explicitly, by all later scientists [84, p. 69].

D. Fearnley-Sander wrote that

the modern concept of real number [...] was essentially achieved by Simon Stevin, around 1600, and was thoroughly assimilated into mathematics in the following two centuries [31, p. 809].

D. Fowler points out that

Stevin [...] was a thorough-going arithmetizer: he published, in 1585, the first popularization of decimal fractions in the West [...]; in 1594, he described an algorithm for finding the decimal expansion of the root of any polynomial, the same algorithm we find later in Cauchy’s proof of the intermediate value theorem [32, p. 733].

The algorithm is discussed in more detail in [77, §10, p. 475-476]. Unlike Cauchy, who *halves* the interval at each step (see Section 3), Stevin subdivides the interval into *ten* equal parts, resulting in a gain of a new decimal digit of the solution at every iteration of the algorithm.² Thus, while Cauchy’s algorithm can be described as a binary search, Stevin’s approach is a more general divide-and-conquer algorithm.

Fowler makes the following additional points (see [32]). The belief that all arithmetic operations, as well as extracting roots, etc., should follow the “same” rules as the rationals, originates precisely with Stevin. The rigorous justification of such a belief had to await Dedekind’s contribution at the end of the 19th century. The “existence” of multiplication of the real numbers was first proved by Dedekind. The widespread belief that there exists an *algorithm* for determining the digits of the result of multiplying real numbers in terms of finite pieces of the decimal string, is unfounded (namely, there is no such

²Stevin’s numbers were anticipated by E. Bonfils in 1350, see S. Gandz [34]. Bonfils says that “the unit is divided into ten parts which are called Primes, and each Prime is divided into ten parts which are called Seconds, and so on into infinity” [34, p. 39].
algorithm). This thread concerning the precise nature of Dedekind’s contribution is pursued further in Section 7.

In his *L’Arithmétique*, Stevin grounds a transition from the classical arithmetic of the discrete, to a continuous arithmetic, by means of his well-known “water-and-wetness” metaphor. Numbers are measures, and measures of continuous magnitudes are by their nature continuous: as well as to continuous water corresponds a continuous wetness, so to a continuous magnitude corresponds a continuous number” (Stevin, 1585, see [76, p. 3]; quoted in Malet [63]).

3. A Stevin-Cauchy Proof of the Intermediate Value Theorem

How are we to understand van der Waerden’s contention that Stevin numbers were accepted by all later scientists? To illustrate the issue at stake, consider Cauchy’s proof of the intermediate value theorem [16] (the proof was forshadowed in a text of Stevin’s, see Section 2). Cauchy constructs an increasing sequence $a_n$ and a decreasing sequence $b_n$ of successive approximations, $a_n$ and $b_n$ becoming successively closer than any positive distance. At this stage, the desired point is considered to have been exhibited, by Cauchy. A modern mathematician may object that Cauchy has not, and could not have, proved the existence of the limit point.

But imagine that the perspicacious *polytechnicien* Auguste Comte had asked *M. le Professeur* Cauchy the following question:

Consider a decimal rank, say $k > 0$. What is happening to the $k$-th decimal digit $a^k_n$ of $a_n$, and $b^k_n$ of $b_n$?

*M. le Professeur* would have either sent Comte to the library to read Simon Stevin, or else provided a brief argument to show that for $n$ sufficiently large, the $k$-th digit stabilizes, noting that special care needs to be taken in the case when $a_n$ is developing a tail of 9s and $b_n$ is developing a tail of 0s. Clearly the arguments appearing in Cauchy’s

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4 To illustrate the point, consider a computer multiplying a decimal .333... by 3, where we are deliberately vague about what the ellipsis stands for. A computer programmer can spend an arbitrarily large time thinking that the resulting decimal will start with a long string of 9s. It suffices for a single digit greater than 3 to appear at the trillionth rank to prove it wrong, showing that in this calculation, at no time can the programmer be sure of any given digit.

4 Cauchy’s notation for the two sequences is $x_0, x_1, x_2, \ldots$ and $X, X', X'', \ldots$ [16, p. 462].

5 Comte’s notes of Cauchy’s lectures are extant, see [72, p. 437].
textbook are sufficient to identify the Stevin decimal expression of the limit point.

From the modern viewpoint, the only item missing is the remark that a Stevin decimal is a number, by definition (modulo the technical detail of the identification of the pair of tails).

In the same spirit, D. Laugwitz points out that in France after 1870, the main objective of French mathematics professors, under the leadership of the Ecole Polytechnique ... was to prepare students of engineering and the sciences for useful jobs ... With decimal expansions of real numbers at hand, nobody was bothered by theories of irrational numbers [58, p. 274].

The incoherence of triumvirate scholarship in relation to Cauchy was already analyzed in 1973 by Hourya Benis Sinaceur [74], and is further analyzed in [12, 15] (see also Section 9).

Recall that Stevin’s algorithm involved partitioning the interval into ten parts, and produces an additional rank of the decimal expansion of the solution at each step of the iteration (see Section 2). Much has been said about the proof of the “existence” of the real numbers by the great triumvirate at the end of the 19th century. But who needs such an existence proof when the Stevin-Cauchy method gives an algorithm that produces a concrete infinite decimal string?

4. Peirce’s framework

The customary set-theoretic framework (e.g., the language of equivalence classes of Cauchy sequences; Dedekind cuts; etc.) has become the reflexive litmus test of mathematical rigor in most fields of modern mathematics (with the possible exception of the field of mathematical logic). Such a framework makes it difficult to analyze Cauchy’s contribution to the foundations of analysis, particularly Cauchy’s use of the concept of an infinitesimal, and to evaluate its significance. We will therefore use a conceptual framework proposed by C. S. Peirce in 1897 (going back to his text How to make our ideas clear of 1878, see [67, item (5.402)]), in the context of his analysis of the concept of continuity and continuum, which, as he felt at the time, is composed of infinitesimal parts, see [37, p. 103]. Peirce identified three stages in creating a novel concept:

6Note that Cauchy exploited decimal notation on occasion; see, for instance, [20 p. 34].
there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of the word with familiar experience. . . . The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. . . . The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems [68] (see [37, p. 87]).

The “three grades” can therefore be summarized as

1. familiarity through experience;
2. abstract definition with an eye to future applications;
3. fruitful reasoning “made to turn” upon it, with applications.

A related taxonomy was developed by D. Tall [81], in terms of his three worlds of mathematics, based on embodiment, symbolism and formalism in which mathematical proof develops in each world in terms of recognition, description, definition and deduction.

To apply Peirce’s framework to Cauchy’s concept of an infinitesimal, we note that the perceptual stage (1) is captured in Cauchy’s description of continuity of a function in terms of “varying by imperceptible degrees”. Such a turn of phrase occurs both in his letter to Coriolis of 1837, and in his 1853 text [20, p. 35].

At stage (2), Cauchy describes infinitesimals as generated by null sequences (see [14]), and defines continuity in terms of an infinitesimal $x$-increment resulting in an infinitesimal change in $y$.

Finally, at stage (3), Cauchy fruitfully applies the crystallized concept of an infinitesimal both in Fourier analysis and in evaluation of singular integrals. Thus, Cauchy exploits a “Dirac” delta function defined in terms of what would be called today the Cauchy distribution with an infinitesimal scaling parameter, see Cauchy [17, p. 188], [18], Freudenthal [33, p. 136], and Laugwitz [56, p. 219] and [57].

Peirce’s flexible framework allows us to appreciate Cauchy’s foundational contributions and their fruitful application. How do Cauchy’s infinitesimals fare in a triumvirate framework? This issue is explored in the Section 5.

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Note that both Cauchy’s original French “par degrés insensibles”, and its correct English translation “by imperceptible degrees”, are etymologically related to sensory perception.
5. A CASE STUDY IN TRIUMVIRATE STRAWMANSHIP

Cauchy’s 1821 Cours d’Analyse [16] presented only a theory of infinitesimals of polynomial rate of growth as compared to a given “base” infinitesimal $\alpha$. The shortcoming of such a theory is its limited flexibility. Since Cauchy only considers infinitesimals behaving as polynomials of a fixed infinitesimal, called the “base” infinitesimal in 1823, his framework imposes obvious limitations on what can be done with such infinitesimals. Thus, one typically can’t extract the square root of such a “polynomial” infinitesimal.

What is remarkable is that Cauchy did develop a theory to overcome this shortcoming. Cauchy’s astounding theory of infinitesimals of arbitrary order (not necessarily integer) is analyzed by Laugwitz [55, p. 271].

In 1823, and particularly in 1829, Cauchy develops a more flexible theory, where an infinitesimal is represented by an arbitrary function (rather than merely a polynomial) of a base infinitesimal, denoted “$i$”. This is done in Cauchy’s 1829 textbook [19, Chapter 6]. The title of the chapter is significant. Indeed, the title refers to the functions as “representing” the infinitesimals; more precisely, “fonctions qui représentent des quantités infiniment petites”. Here is what Cauchy has to say in 1829:

Designons par $a$ un nombre constant, rationnel ou irrationnel; par $i$ une quantité infiniment petite, et par $r$ un nombre variable. Dans le système de quantités infiniment petites dont $i$ sera la base, une fonction de $i$ représentée par $f(i)$ sera un infiniment petit de l’ordre $a$, si la limite du rapport $f(i)/i^r$ est nulle pour toutes les valeurs de $r$ plus petite que $a$, et infinie pour toutes les valeurs de $r$ plus grandes que $a$ [19, p. 281].

Laugwitz [55, p. 271] explains this to mean that the order $a$ of the infinitesimal $f(i)$ is the uniquely determined real number (possibly $+\infty$, as with the function $e^{-1/i^2}$) such that $f(i)/i^r$ is infinitesimal for $r < a$ and infinitely large for $r > a$.

Laugwitz [55, p. 272] notes that Cauchy provides an example of functions defined on positive reals that represent infinitesimals of orders $\infty$ and 0, namely

$$e^{-1/i} \quad \text{and} \quad \frac{1}{\log i}$$

(see Cauchy [19, p. 326-327]).

Note that according to P. Ehrlich’s detailed 2006 study [27], the development of non-Archimedean systems based on orders of growth
was pursued in earnest at the end of the 19th century by such authors as Stolz and du Bois-Reymond. These systems appear to have an antecedent in Cauchy’s theory of infinitesimals as developed in his texts dating from 1823 and 1829. Indeed, already in 1966, A. Robinson pointed out that

Following Cauchy’s idea that an infinitely small or infinitely large quantity is associated with the behavior of a function $f(x)$, as $x$ tends to a finite value or to infinity, du Bois-Raymond produced an elaborate theory of orders of magnitude for the asymptotic behavior of functions ... Stolz tried to develop also a theory of arithmetical operations for such entities [69, p. 277-278].

Robinson traces the chain of influences further, in the following terms:

It seems likely that Skolem’s idea to represent infinitely large natural numbers by number-theoretic functions which tend to infinity (Skolem [1934]), also is related to the earlier ideas of Cauchy and du Bois-Raymond [69, p. 278].

The reference is to Skolem’s 1934 work [15].

The material presented in the present section, including a detailed discussion of the chain of influences from Cauchy via Stolz and du Bois-Reymond and Skolem to Robinson, is contained in an article entitled “Who gave you the Cauchy-Weierstrass tale? The dual history of rigorous calculus”. At no point did the article claim that Cauchy’s approach is a variant of Robinson’s approach. Indeed, such a claim would be preposterous, as Cauchy was not in the possession of the mathematical tools required to either formulate or justify the ultrapower construction, requiring as it does a set-theoretic framework (dating from the end of the 19th century) together with the existence of ultrafilters (not proved until 1930 by Tarski [83]).

The article was submitted to the periodical “Revue d’histoire des sciences” on 5 October 2010. The article was rejected by editor Michel Blay five months later, in a letter dated 11 March 2011. Blay based his decision on two referee reports. Referee 1 summarized the article as follows in his third sentence:

Our author interprets A. Cauchy’s approach as a formation of the idea of an infinitely small - a variant of the approach which was developed in the XXth century in the framework of the nonstandard analysis (a hyperreal version of E. Hewitt, J. Los, A. Robinson).
Based on such a strawman version of the article’s conception, the referee came to the following conclusion:

From my point of view the author’s arguments to support this conception are quite unconvincing.

The author indeed finds such such a strawman conception unconvincing, but the conception was the referee’s, not the author’s. The referee concluded as follows:

The fact that the actual infinitesimals lived somewhere in the consciousness of A. Cauchy (as in many another mathematicians of XIXth - XXth centuries as, for example, N.N. Luzin) does not abolish his (and theirs) constant aspiration to dislodge them in the subconsciousness and to found the calculus on the theory of limit.

The notion of a Cauchy as a pre-Weierstrassian, apparently espoused by the referee, is just as preposterous as the notion of Cauchy as a pre-Robinsonian. Such a notion is a reflection of a commitment to a triumvirate ideology, elevated to the status of a conditioned reflex. Felix Klein knew better: fifty years before Robinson, he clearly realized the potency of the infinitesimal approach to the foundations (see Section 9).

Cauchy did not aspire to dislodge infinitesimals; on the contrary, he used them with increasing frequency in his work, including his 1853 article [20] where he relies on infinitesimals to express the property of uniform convergence.

The pdf version of the submitted article “Who gave you, etc.”, as well as the two referee reports, may be found at the following web page: http://u.cs.biu.ac.il/~katzmik/straw.html

Similarly, in his 2007 anthology [38], S. Hawking reproduces Cauchy’s infinitesimal definition of continuity on page 639–but claims on the same page, in a comic non-sequitur, that Cauchy “was particularly concerned to banish infinitesimals”.

6. Weierstrassian epsilontics

If we are to take at face value van der Waerden’s evaluation of the significance of Stevin numbers, what is, then, the nature of Weierstrass’s contribution? After describing the formalisation of the real continuum usually associated with the names of Cantor, Dedekind, and Weierstrass on pages 127-128 of his retiring presidential address in 1902, E. Hobson remarks triumphantly as follows:
It should be observed that the criterion for the convergence of an aggregate\(^8\) is of such a character that no use is made in it of infinitesimals \([41, \text{ p. 128}]\).

Hobson reiterates:

In all such proofs [of convergence] the only statements made are as to relations of finite numbers, no such entities as infinitesimals being recognized or employed. Such is the essence of the \(\epsilon, \delta\) proofs with which we are familiar \([41, \text{ p. 128}]\).

The tenor of Hobson’s remarks, is that Weierstrass’s primary accomplishment was the elimination of infinitesimals from foundational discourse in analysis. If our students are being dressed to perform multiple-quantifier epsilon-logical stunts on the pretense of being taught infinitesimal calculus, it is because infinitesimals are assumed to be either metaphysically dubious or logically unsound, see D. Sherry \([73]\).

The significance of the developments in the foundations of the real numbers at the end of the 19th century was a rigorous proof of the existence of arithmetic operations, as discussed at the end of Section 2.

7. **DEDEKIND AND PEIRCE**

The first use of the term *real* to describe Stevin numbers seems to date back to Descartes, who distinguished between real and imaginary roots of polynomials. Thus, the term was used as a way of contrasting what were thought of, ever since Stevin, as measuring numbers, on the one hand, and imaginary ones, on the other. Gradually the meaning of the term *real number* has shifted, to a point where today it is used in the sense of “genuine, objective, true number”. How legitimate is such usage?

Let us consider what natural science tells us about the physical line. Descending below the threshold of sensory perception, quantum physicists tell us that lengths smaller than \(10^{-30}\) meters are not accessible, even theoretically, to any, existing or future, physical electron microscope. This is because the minutest entities considered even theoretically, such as strings in Witten’s M-theory, are never smaller than a barrier of \(10^{-30}\) (up to a few orders of magnitude). Thus, the infinite divisibility taken for granted in the case of the real line, only holds within a suitable range, even in principle, in the spatial line\(^9\).

\(^8\) i.e. an equivalence class defining a real number

\(^9\) To put it another way, physical theories such as quantum mechanics testify to a graininess of physical matter, that is at odds with the infinite divisibility postulated
real numbers, similarly to infinitesimal quantities, appear too small to see. M. Moore\textsuperscript{10} notes that the minute size of strings does not disqualify strings from admission to our scientific ontology; and . . . the difference, on this score, between things as small as strings and distances infinitely small, is one of degree and not of kind.

Historians Eves and Newsom make the following claim in Dedekind’s name:

Dedekind perceived that the essence of continuity of a straight line lies in the property that if all the points of the line are divided into two classes, such that every point in the first class lies to the left of every point in the second class, then there exists one and only one point of the line which produces this severance of the line into the two classes \textsuperscript{30} p. 222 [emphasis added–authors].

However, their claim is at variance with the fact that Dedekind himself specifically downplayed his claims as to such essence, in the following terms:

If space has at all a real existence, it is not necessary for it to be [complete] . . . if we knew for certain that space was [incomplete], there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it [complete] (as cited in \textsuperscript{65} p. 73).

Dedekind maintains that his completeness principle gives the essence of continuity, but denies that we can know that space is continuous in that sense. M. Moore points out that Dedekind shows commendably little sympathy for the idea that we know by intuition that the line is complete \textsuperscript{65} p. 73].

Moore further points out that Dedekind admits that his is “utterly unable to adduce a proof of [his account’s] correctness, nor has anyone the power” (as quoted in \textsuperscript{65} p. 73).

At variance with the great triumvirate of Cantor, Dedekind, and Weierstrass\textsuperscript{11} American philosopher Charles Sanders Peirce felt that a construction of a true continuum necessarily involves infinitesimals. He wrote as follows:

\textsuperscript{10}See \textsuperscript{65} p. 82 as well as \textsuperscript{64}.

\textsuperscript{11}See footnote \textsuperscript{1}. 

\textsuperscript{30}See also Moore’s discussion of Dedekind’s position, below.
But I now define a *pseudo-continuum* as that which modern writers on the theory of functions call a continuum. But this is fully represented by [...] the totality of real values, rational and irrational numbers in the syllabus (CP 1.185) of his lectures on Topics of Logic. Thus, Peirce’s intuition of the continuum corresponded to a type of a B-continuum (see Section 9), whereas an A-continuum to him was a pseudo-continuum.

While Peirce thought of a continuum as being made up of infinitesimal increments, other authors pursuing B-continuum foundational models (see Section 9) thought of a real number $x$ as having a cluster of infinitesimals around it, more precisely a cluster of points infinitely close to $x$, i.e. differing from $x$ by an infinitesimal amount. The alternative to Dedekind’s view that a cut on the rationals corresponds to a single number, is to view such a cut as being defined by a cluster of infinitely close numbers.

Peirce had a theory of infinitesimals that in many ways anticipated 20th century developments, see J. Dauben [23]. Havenel [37] argues that Peirce’s conception was closer to Lawvere’s approach than to Robinson’s. This is corroborated by Peirce’s opposition to both the law of excluded middle, and to a view of the continuum as being reducible to points. Both of these points are borne out by the category-theoretic framework of Lawvere’s theory (see J. Bell [6]), in the context of intuitionistic logic. On the other hand, Peirce does not seem to have anticipated the notion of a nilsquare infinitesimal, which had been anticipated already by Nieuwentijdt, and implemented in Lawvere’s theory, see J. Bell [7] for details.

8. OH NUMBERS, NUMBERS SO REAL

In an era where no implementation of an infinitesimal-enriched continuum as yet existed, the successful implementation of real analysis on the basis of the real number system and $\epsilon, \delta$ arguments went hand-in-hand with an attempt to ban the infinitesimals, thought of as an intellectual embarrassment at least since George Berkeley’s time [8].

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12See CP 6.176, 1903 marginal note. Here (and below) CP x.y stands for Collected Papers of Charles Sanders Peirce, volume x, paragraph y.
Hobson is explicit in measuring the significance of Weierstrass’s contribution by the yardstick of the elimination of infinitesimals.\footnote{Today, the didactic value of infinitesimals is becoming increasingly evident, see \cite{29, 42, 53, 5, 26, 62}. On the foundational side, an implementation of an infinitesimal-enriched number system can be presented in a traditional set-theoretic framework by means, for example, of the ultrapower construction, see e.g., Keisler \cite{51}; Goldblatt \cite{36}; M. Davis \cite{25}. Some philosophical implications are explored by B/\protect\textsuperscript{2}laszczyzk \cite{9}.}

Today we can perhaps appreciate more clearly, not Weierstrass’s, but Stevin’s contribution toward the implementation of quantities that may have been called Stevin numbers, and that generally go under the reassuring name of numbers so real.

9. Rival Continua

The historical roots of infinitesimals go back to Cauchy, Leibniz, and ultimately to Archimedes. Cauchy’s approach to infinitesimals is not a variant of the hyperreals. Rather, Cauchy’s work on the rates of growth of functions anticipates the work of late 19th century investigators such as Stolz, du Bois-Reymond, Veronese, Levi-Civita, Dehn, and others, who developed non-Archimedean number systems against virulent opposition from Cantor, Russell, and others, see Ehrlich \cite{27} and Katz and Katz \cite{44} for details. The work on non-Archimedean systems motivated the work of T. Skolem on non-standard models of arithmetic \cite{75}, which stimulated later work culminating in the hyperreals of Hewitt, Los, and Robinson.

Having outlined the developments in real analysis associated with Weierstrass and his followers, Felix Klein pointed out in 1908 that

The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries \cite{53}, p. 214] [emphasis added—authors].
Klein further points out that such a parallel conception of calculus harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of [...] infinitely small parts [53, p. 214] [emphasis added—authors].

The rival theories of the continuum evoked by Klein can be summarized as follows. A Leibnizian definition of the derivative as the infinitesimal quotient

$$\frac{\Delta y}{\Delta x},$$

whose logical weakness was criticized by Berkeley, was modified by A. Robinson by exploiting a map called the standard part, denoted “st”, from the finite part of a “thick” B-continuum (i.e., a Bernoullian continuum)\(^\text{14}\) to a “thin” A-continuum (i.e., an Archimedean continuum), as illustrated in Figure 1.

This section summarizes a 20th century implementation of an alternative to an Archimedean continuum, namely an infinitesimal-enriched continuum. Such a continuum is not to be confused with incipient notions of such a continuum found in earlier centuries. Johann Bernoulli was one of the first to exploit infinitesimals in a systematic fashion as a foundational tool in the calculus. We will therefore refer to such a continuum as a Bernoullian continuum, or B-continuum for short.

We illustrate the construction by means of an infinite-resolution microscope in Figures 2 and 3. We will denote such a B-continuum by the new symbol \(\mathbb{IR}\) (“thick-R”). Such a continuum is constructed in

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\(^{14}\)Schubring [72, p. 170, 173, 187] attributes the first systematic use of infinitesimals as a foundational concept, to Johann Bernoulli.
Figure 3. Zooming in on Wallis’s infinitesimal $\frac{1}{\infty}$, which is adequal to 0 in Fermat’s terminology.

We will also denote its finite part, by

$$\mathbb{R}_{<\infty} = \{ x \in \mathbb{R} : |x| < \infty \},$$

so that we have a disjoint union

$$\mathbb{R} = \mathbb{R}_{<\infty} \cup \mathbb{R}_{\infty},$$

where $\mathbb{R}_{\infty}$ consists of unlimited hyperreals (i.e., inverses of nonzero infinitesimals).

The map “st” sends each finite point $x \in \mathbb{R}$, to the real point $st(x) \in \mathbb{R}$ infinitely close to $x$, see Figure 2. Namely, we have

$$\begin{array}{ccc}
\mathbb{R}_{<\infty} & \xrightarrow{st} & \mathbb{R} \\
\downarrow & & \\
\mathbb{R} & & \\
\end{array}$$

Robinson’s answer to Berkeley’s logical criticism (see D. Sherry [73]) is to define the derivative as

$$st \left( \frac{\Delta y}{\Delta x} \right),$$

instead of $\Delta y/\Delta x$.

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15 An alternative implementation of a B-continuum has been pursued by Lawvere, John L. Bell [6, 7], and others.

16 This is the Fermat-Robinson standard part whose seeds are found in Fermat’s adequality.
Note that both the term “hyper-real field”, and an ultrapower construction thereof, are due to E. Hewitt in 1948, see [40, p. 74]. In 1966, Robinson referred to the theory of hyperreal fields (Hewitt [1948]) which ... can serve as non-standard models of analysis [69, p. 278].

The transfer principle is a mathematical implementation of Leibniz’s heuristic law of continuity: “what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa”, see [69, p. 266]. The transfer principle, allowing an extension of every first-order real statement to the hyperreals, is a consequence of the theorem of J. Loś in 1955, see [61], and can therefore be referred to as a Leibniz-Loś transfer principle. A Hewitt-Loś framework allows one to work in a B-continuum satisfying the transfer principle. To elaborate on the ultrapower construction of the hyperreals, let \( \mathbb{Q}^N \) denote the ring of sequences of rational numbers. Let

\[
(\mathbb{Q}^N)_C
\]
denote the subspace consisting of Cauchy sequences. The reals are by definition the quotient field

\[
\mathbb{R} := \frac{(\mathbb{Q}^N)_C}{\mathbb{F}_{null}}, \tag{9.2}
\]

where \( \mathbb{F}_{null} \) contains all null sequences. Meanwhile, an infinitesimal-enriched field extension of \( \mathbb{Q} \) may be obtained by forming the quotient

\[
\mathbb{Q}^N/\mathbb{F}_u.
\]

Here a sequence \( \langle u_n : n \in \mathbb{N} \rangle \) is in \( \mathbb{F}_u \) if and only if the set of indices

\[
\{n \in \mathbb{N} : u_n = 0\}
\]
is a member of a fixed ultrafilter\footnote{In this construction, every null sequence defines an infinitesimal, but the converse is not necessarily true. Modulo suitable foundational material, one can ensure that every infinitesimal is represented by a null sequence; an appropriate ultrafilter (called a \textit{P-point}) will exist if one assumes the continuum hypothesis, or even the weaker Martin’s axiom. See Cutland \textit{et al} [22] for details.} See Figure 4.

To give an example, the sequence

\[
\langle (-1)^n \rangle \tag{9.3}
\]
represents a nonzero infinitesimal, whose sign depends on whether or not the set \( 2\mathbb{N} \) is a member of the ultrafilter. To obtain a full hyperreal field, we replace \( \mathbb{Q} \) by \( \mathbb{R} \) in the construction, and form a similar quotient

\[
\mathbb{I} \mathbb{R} := \frac{\mathbb{R}^N}{\mathbb{F}_u}. \tag{9.4}
\]
FIGURE 4. An intermediate field $\mathbb{Q}^N/F_u$ is built directly out of $\mathbb{Q}$

We wish to emphasize the analogy with formula (9.2) defining the A-continuum. Note that, while the leftmost vertical arrow in Figure 4 is surjective, we have

$$(\mathbb{Q}^N/F_u) \cap \mathbb{R} = \mathbb{Q}.$$ 

A more detailed discussion of this construction can be found in the book by M. Davis [25]. See also P. Błaszczyk [9] for some philosophical implications. More advanced properties of the hyperreals such as saturation were proved later, see Keisler [52] for a historical outline. A helpful “semicolon” notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [60]. See also P. Roquette [70] for infinitesimal reminiscences. A discussion of infinitesimal optics is in K. Stroyan [78], J. Keisler [51], D. Tall [79], and L. Magnani and R. Dossena [62, 26].

Applications of the B-continuum range from aid in teaching calculus [29, 42, 43, 80, 82] to the Boltzmann equation (see L. Arkeryd [3, 4]); modeling of timed systems in computer science (see H. Rust [71]); mathematical economics (see Anderson [2]); mathematical physics (see Albeverio et al. [1]); etc.

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