Environmental dependence in the ellipsoidal collapse model

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ABSTRACT

N-body simulations have demonstrated a correlation between the properties of haloes and their environment. In this paper, we assess whether the ellipsoidal collapse model, whose dynamics includes the tidal shear, can produce a similar dependence. First, we explore the statistical correlation that originates from Gaussian initial conditions. We derive analytic expressions for a number of joint statistics of the shear tensor and estimate the sensitivity of the local characteristics of the shear to the global geometry of the large-scale environment. Next, we concentrate on the dynamical aspect of the environmental dependence using a simplified model that takes into account the interaction between a collapsing halo and its environment. We find that the tidal force exerted by the surrounding mass distribution alters the axes collapse and causes haloes embedded in overdense regions to virialize earlier. The environment density is the key parameter in determining the virialization redshift, while the environment asphericity primarily contributes to the increase in the scatter of the critical collapse density. An effective density threshold whose shape depends on the large-scale density provides a good description of this environmental effect. Such an interpretation has the advantage that the excursion set formalism can be applied to quantify the environmental dependence of halo properties. We show that, using this approach, a correlation between formation redshift, large-scale bias and environment density naturally arises. The strength of the effect is comparable, albeit smaller, to that seen in simulations. It is largest for low-mass haloes \( M \ll M_\star \), and decreases as one goes to higher mass objects \( M > M_\star \). Furthermore, haloes that formed early are substantially more clustered than those that assembled recently. On the other hand, our analytic model predicts a decrease in median formation redshift with increasing environment density, in disagreement with the trend detected in overdense regions. However, our results appear consistent with the behaviour inferred in relatively underdense regions. We argue that the ellipsoidal collapse model may apply in low-density environments where non-linear effects are negligible.

Key words: gravitation – galaxies: haloes – cosmology: theory – dark matter.

1 INTRODUCTION

In standard scenarios of structure formation, dark matter haloes grow hierarchically from initially small, Gaussian fluctuations. Properties of haloes can be studied in great detail using both N-body simulations and analytic models. The remarkably useful extended Press–Schechter (EPS) theory predicts halo mass functions (Press & Schechter 1974; Bond et al. 1991), merging histories (Lacey & Cole 1993; Sheth & Lemson 1999b; Van den Bosch 2002; Neistein, Van den Bosch & Dekel 2006) and spatial clustering (Mo & White 1996; Mo, Jing & White 1997; Catelan et al. 1998; Sheth & Lemson 1999a) that are in reasonable agreement with the simulations. This analytic approach is based on the spherical collapse model (Gunn & Gott 1972). In this Lagrangian approximation, haloes are identified in the initial conditions and a single parameter, the initial density contrast, is needed to characterize their epoch of formation (Press & Schechter 1974). The collapse of a halo occurs when the linear density reaches a critical threshold. The fundamental properties of dark matter haloes are then obtained from the statistics of trajectories of the linear density field as a function of the smoothing scale (e.g. Bond et al. 1991; Bower 1991; Kauffmann & White 1993; Lacey & Cole 1993; Kitayama & Suto 1996; Sheth & van de Weygaert 2004). Although this spherical approximation works well until the first orbit crossing, it may not be accurate since perturbations in Gaussian density fields are inherently triaxial (Doroshkevich 1970; Bardeen et al. 1986; Jing & Suto 2002). Furthermore, the initial shear field rather than the density has been shown to play a crucial role in the formation of non-linear structures (e.g. Hoffman 1986, 1988; Peebles 1990; Dubinski 1992;
Bertschinger & Jain 1994; Audit & Alimi 1996; Audit, Teyssier & Alimi 1997).

Unlike the spherical model whose dynamics depends on a single parameter only (the density), the ellipsoidal model that follows the evolution of triaxial perturbations can be used to ascertain the influence of the (external) tidal shear on the properties of collapsed regions. The gravitational collapse of homogeneous ellipsoids has been investigated by numerous authors over the past decades (e.g. Lynden-Bell 1964; Lin, Mestel & Shu 1965; Fujimoto 1968; Zeldovich 1970; Icke 1973; White & Silk 1979; Barrow & Silk 1981; Lemson 1993; Eisenstein & Loeb 1995; Hui & Bertschinger 1996). In the formulation of Bond & Myers (1996), initial conditions and external tides are chosen to recover the Zel’dovich approximation in the linear regime. The dynamic of ellipsoidal collapse can be incorporated in various ways in the Press–Schechter formalism to predict the properties of haloes (Monaco 1995, 1997a,b; Lee & Shandarin 1998; Chieu & Lee 2001; Sheth & Tormen 2002). As pointed out by Sheth, Mo & Tormen (2001), the inclusion of non-sphericity in the dynamics introduces a simple dependence of the critical collapse density on the halo mass. The resulting first crossing distribution yields a better fit to the halo mass functions measured in N-body simulation (Sheth & Tormen 2002). However, other modifications to the original excursion set approach, such as the inclusion of non-radial degrees of freedom, might also improve the theoretical mass function (Audit et al. 1997; Del Popolo & Gambera 1998). It would thus be very desirable to identify additional distinctive predictions of the ellipsoidal collapse model beyond the mass function and bias to further test this theory.

While earlier numerical studies have not provided any conclusive evidence for a dependence of halo properties on environment (Lemson & Kauffmann 1999; Percival et al. 2003; Zentner et al. 2005), recent numerical investigations indicate that, at fixed halo mass, haloes in dense regions form at (slightly) higher redshift than in low-density environments (Sheth & Tormen 2004; Avila-Reese et al. 2005; Harker et al. 2006). Using the Millennium Run (Springel et al. 2005), Gao, Springel & White (2005) have convincingly shown that the clustering of haloes of a fixed mass depends on formation time. This dependence is strong for haloes with mass less than the typical collapsing mass $M_\ast$, and fades rapidly for $M > M_\ast$. Subsequent studies have demonstrated that many other halo properties, such as spin parameter or concentration, correlate with the halo assembly history (Maulbetsch et al. 2006; Wechsler et al. 2006; Zhu et al. 2006; Gao & White 2007; Jing, Suto & Mo 2007; Wetzel et al. 2007). Also, haloes that have undergone major mergers may be more strongly clustered relative to other haloes of the same mass (e.g. Furlanetto & Kamionkowski 2006). These results call into question the simplest descriptions of structure formation based on the simulation of random walks (Bond et al. 1991; Lacey & Cole 1993; White 1996). However, relaxing the assumption of sphericity and/or sharp k-space filtering can introduce a dependence on environment (Bond et al. 1991; White 1996; Sandvik et al. 2007). In the ellipsoidal model, the time required for a given overdensity to virialize increases monotonically with the initial shear (Sheth et al. 2001). Therefore, as recognized by Wang, Mo & Jing (2007), the ellipsoidal dynamics should give an environmental effect owing to the tidal field generated by the large-scale environment.

In the present paper, we take an analytic approach and assess whether the ellipsoidal collapse model can produce an environmental dependence (also termed ‘assembly bias’) similar to that seen in N-body simulations. We start with the statistical dependence that arises in correlated (Gaussian) initial conditions. We extend the results of Doroshkevich (1970) to the joint statistics of the shear tensor. We derive conditional distributions and quantify the extent to which the asymmetry of initially triaxial perturbations is sensitive to the geometry of the large-scale environment. Next, we investigate the dynamical aspect of the environmental dependence using a simplified model that takes into account the interaction between a triaxial protohalo and its environment. We find that the tidal force exerted by the surrounding mass distribution affects the axes collapse and causes haloes embedded in large overdensities to virialize earlier. A moving barrier whose shape depends on the environment density provides a good description of this environmental effect. This enables us to apply the EPS formalism in order to estimate the environmental dependence of halo properties. Our approach thus is very different than the multidimensional extension presented in Sandvik et al. (2007).

The paper is organized as follows. Section 2 briefly reviews the basic concepts associated with the ellipsoidal collapse model. Section 3 is devoted to the statistical correlation between the local properties of the shear and the large-scale environment (Appendix B details a delicate step of the calculation). Section 4 investigates the dynamical origin of the environmental dependence, focusing on the distribution of halo formation redshift, large-scale bias and alignment of spin parameter. Non-Markovianess and tidal interactions as potential sources of environmental dependence are discussed in Section 5. A final section summarizes our results.

2 THEORETICAL BACKGROUND

We emphasize the role played by the shear in current theories of structure formation and introduce the basic definitions and relations relevant to the study of environmental effects in the ellipsoidal collapse model.

2.1 Shear tensor

The comoving Eulerian position of a particle can be generally expressed as a mapping $x = q + S(q, t)$, where $q$ is the Lagrangian (initial) position and $S$ is the displacement field. In the Zel’dovich approximation (1970), the displacement field is $S(q, t) = -D(t) \Phi(q)$, where $\Phi(q) = \phi(q, t)/4\pi G \rho_m(t) a^2 D(t)$ is the perturbation potential ($\phi(q, t)$ is the Newtonian gravitational potential), $\rho_m$ is the average matter density and $D(t)$ is the linear growth factor (Peebles 1980). The second derivatives of the perturbation potential define the deformation tensor (or strain field) $D_{ij} = \partial_i \partial_j \Phi$. For convenience, we introduce the real, symmetric tensor $\xi_{ij}(q) = \frac{1}{\sigma} D_{ij}(q) = \frac{1}{\sigma} \frac{\partial^2 \Phi}{\partial q_i \partial q_j}(q)$, where $\sigma = \bar{\sigma}(R)$ is the rms variance of density fluctuations smoothed on scale $R$ (see Section 3.1 below). We will henceforth refer to $\xi_{ij}$ as the shear tensor. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ designate the order eigenvalues of $\xi_{ij}$. An important quantity is the probability distribution of the ordered set $(\lambda_1, \lambda_2, \lambda_3)$, first derived for Gaussian random fields by Doroshkevich (1970),

$$P(\lambda_1, \lambda_2, \lambda_3) = \frac{15^3}{8 \pi \sqrt{5}} \Delta(\lambda) e^{-3\xi_1^2/15(15/2\xi_2)\lambda_3},$$

where $\Delta(x) = \det \left( x_i^{N-j} \right) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ is the Vandermonde determinant in the arguments $x_i, i = 1, \ldots, N$, and $\xi_1(x)$ are the elementary symmetric functions of degree $n$...
If $x_1, \ldots, x_N$ are the eigenvalues of a matrix $X$, the functions $s_k(x)$ can be written in terms of the traces of power of $X$, $\text{tr}^k X$, with $k, l = 0, 1, \ldots$. For instance, $s_2(x) = (1/2) [(\text{tr}X)^2 - \text{tr}(X^2)]$, etc.

### 2.2 Geometry of the initial density field

As shown by Bond, Kofman & Pogosyan (1996), the filamentary pattern seen in $N$-body simulations (e.g. Park 1990; Bertschinger & Gelb 1991; Cen & Ostriker 1993; Springel et al. 2005, for a recent example) is a consequence of the initial spatial coherence of the shear tensor. In this Cosmic Web paradigm, the correspondence between large-scale structures in the evolved density field and local properties of the shear tensor in the initial conditions, and the knowledge of the probability $P(\lambda_1, \lambda_2, \lambda_3)$, allows us estimate the morphology of the large-scale matter distribution.

The geometry of the primeval density field depends on the signature of the ordered sequence of shear eigenvalues. If, in a given region, the largest eigenvalue only is positive ($++-$), there is contraction along one direction and expansion in the other two so that a pancake will form. If two eigenvalues are positive while the third one is negative ($+-+$), collapse occurs along two directions and a filament will form. The probability for these two configurations, ~0.04, is much larger than the probability that all three eigenvalues are positive. $P(+-+) = 0.08$. However, these values depend strongly on the density enhancement of the region under consideration. While filaments or sheet-like configurations are favoured when $\nu \lesssim 1.5$, one encounters predominantly spherical-like mass concentrations above $\nu \simeq 1.5$ (e.g. Bernardeau 1994; Pogosyan et al. 1998). Note that several methods have recently been proposed to quantify precisely the geometry and topology of density fields (e.g. Sahni, Sathyaprakash & Shandarin 1998; Colombi, Pogosyan & Souradeep 2000; Hanami 2001; Gleser et al. 2006; Novikov, Colombi & Doré 2006; Aragón-Calvo et al. 2007a).

The sequence in which these structures form is still an open problem. We briefly note that, in the pancake picture, the collapse proceeds in the order pancakes–filaments–clusters (Lin et al. 1965; Zeldovich 1970; Arnold, Shandarin & Zeldovich 1982). Supporting evidence comes from several $N$-body simulations showing first pancake-like collapse (e.g. Shandarin et al. 1995).

Identifying the precursors of haloes (protohaloes) in the initial conditions is another unresolved issue, despite the major advance made in the analysis of (Gaussian) random fields (Doroshkevich 1970; Adler 1981; Peacock & Heavens 1985; Bardeen et al. 1986; Bond & Myers 1996). Shandarin & Klypin (1984) have shown by means of simulations that massive clusters with $M \gtrsim 10^{15} M_\odot h^{-1}$ are initially located close to local maxima of the smallest eigenvalue, with $\lambda_3 > 0$. On the other hand, recent $N$-body investigation (Porciani, Dekel & Hoffman 2002a,b) indicate that a large fraction of small-mass haloes $M \lesssim M_\ast$ are rather associated with primeval configurations of signature $(++-)$. Since, at present, there is no reliable alternative to the Press–Schechter prescription, we assume that haloes form out of initially spherical patches of size $R$, so that the relevant averages are taken over spherical Lagrangian regions. This assumption is quite unrealistic in the light of numerical results, but it greatly facilitates the calculation. We shall adopt the usual critical density criterion issued from the spherical collapse (see below). This restrictive collapse condition does not guarantee a strictly positive signature $(+++)$. But in the ellipsoidal collapse model, the formation of a bound object can occur even if $\lambda_3 < 0$: once the shortest axis has collapsed, the non-linear density causes the other two axes to collapse very rapidly (Bond & Myers 1996).

### 2.3 Ellipticity, prolateness and critical collapse density

The eigenvalues $\lambda_i$ can be equivalently parametrized in terms of the shear ellipticity $e$ and prolateness $p$, where

$$e = \frac{\lambda_1 - \lambda_3}{2\nu}, \quad p = \frac{\lambda_1 - 2\lambda_2 + \lambda_3}{2\nu}$$

and $v = \delta/\sigma = \lambda_1 + \lambda_2 + \lambda_3$ is the density contrast of the region under consideration. The ordering constraint implies that $e \geq 0$ if $v > 0$ and $e \leq 0$ if $v < 0$. In all case, the shear prolateness is $-e \leq p \leq e$. In this parametrization, extreme sheet-like (oblate) structures have $p \lesssim e$ while extreme filaments (prolate) have $p \gtrsim -e$. Doroshkevich’s formula can be used to write down the distribution $g(e, p | v)$ of ellipticity $e$ and prolateness $p$ for a given density contrast $v$ (e.g. Bardeen et al. 1986),

$$g(e, p | v) = \frac{1125}{\sqrt{10\pi}} e^{e^2 - p^2} e^{-3/2(e^2 + p^2)}.$$  

For all $v$, the maximum of this distribution occurs at $e_m(v) = 1/(\sqrt{5}v)$, $p_m(v) = 0$, while the variances of $e$ and $p$ are (for $v > 0$ only)

$$\langle e^2 | v \rangle - \langle e | v \rangle^2 = (19\pi - 54)/(60\pi v^2), \quad \langle p^2 | v \rangle = 19/(20v^2).$$

Note that the most probable value $e_m$ is comparable to the mean ellipticity $\langle e | v \rangle = 3/(\sqrt{10}v) \approx 1.2e_m$. It depends on the smoothing scale $r$ through $v \propto \delta/\sigma(r)$. At fixed $r$, denser regions are more likely to be spherical than less dense regions while, at fixed $v$, larger regions are more likely to be spherical than smaller ones. The scatter in the asymmetry parameters increases strongly with decreasing density and/or halo mass.

In the excursion set formalism (Bond et al. 1991; Lacey & Cole 1993), the critical collapse density encodes the details of the collapse dynamics. In the spherical collapse model, the dynamics in a given cosmological background is governed by a single parameter, namely the density. A top-hat perturbation of overdensity $v = \delta_{sc}/\sigma$ (linearly extrapolated to present epoch) collapses at redshift $z = 0$. The (linear) critical density threshold $\delta_{sc}$ depends on the cosmology. We have $\delta_{sc} = 1.673$ for the cosmological parameters considered here (Eke, Cole & Frenk 1996; Navarro, Frenk & White 1997; Lokas & Hoffman 2001). On the other hand, in the ellipsoidal collapse dynamics, the evolution of a perturbation depends on the values of $e$, $p$ and $v$. The critical density threshold $\delta_{sc}$ is always larger than the spherical value $\delta_{sc}$ and is very sensitive to the initial shear (Sheth et al. 2001).

### 3 Environmental Effect from the Statistics of the Initial Shear

The statistics of the shear tensor for Gaussian random fields has been pioneered by Doroshkevich (1970) to study the formation of large-scale structures. In his seminal paper, Doroshkevich calculated the probability distribution of the shear eigenvalues and ascertained the amount of material being incorporated in a pancake. Later, Doroshkevich & Shandarin (1978) re-examined the formation
of sheet-like structures and derived a distribution function for the largest eigenvalue of the shear tensor. Recently, Lee & Shandarin (1998) computed conditional probability distributions for individual shear eigenvalues to obtain an analytic approximation to the halo mass function.

Here, we derive expressions for a number of joint statistics of the shear tensor upon the assumption of Gaussianity. This enables us to quantify the importance of the statistical correlation between the asphericity of triaxial collapsing regions and the shape of their large-scale environment.

3.1 Analytic considerations

We confine our calculation to the case in which the components $\xi_{ij}(x)$ and $\xi_{ij}(x')$ are smoothed on different scales, but the joint distribution is evaluated at a single comoving position $x = x'$. This is most relevant to the issues considered in this paper. The central results of this section are the joint probability distribution of shear eigenvalues, equation (23), and the conditional probability for the shear ellipticity and prolateness, equation (27).

3.1.1 Spectral parameter

We begin with the two-point correlation functions of the shear tensor. The general form of these correlations is given in Appendix A. Evaluated at a single comoving position $x$, they take the simple form

$$\langle \xi_{ij}(x)\xi_{kl}(x) \rangle = \frac{\gamma}{\sigma_{\xi}^2} \left( \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right),$$

where $\xi_{ij}$ and $\xi_{kl}$ are smoothed on comoving scale $R_0$ and $R_1$, respectively. The spectral parameter

$$\gamma \equiv \frac{1}{\sigma_{\xi}^2} \int_0^\infty \Delta^2_{\xi}(k) \frac{W(R_0,k)}{W(R_1,k)} \, dk,$$

$$0 \leq \gamma \leq 1,$$

is a measure of the correlation between these scales. Here, $\Delta^2_{\xi}(k) \equiv k^4 P_{\xi}(k)/2\pi^2$ is the dimensionless, linear density power spectrum (Peebles 1980) and $\sigma_{\xi}$ and $\sigma_\delta$ are the rms variances of density fluctuations smoothed on scale $R_0$ and $R_1$, respectively. $R$ is the comoving characteristic scale of the spherically symmetric window function $W(R,k)$. Many choices are possible for this filtering function. We will adopt a top-hat filter throughout this paper. The top-hat smoothing radius $R$ defines a mass scale $M = (4\pi/3)\rho_m R^3$ so that, for a given power spectrum, $\sigma_\delta$, $M$, and $R$ are equivalent variables. Note also that, instead of writing explicitly the smoothing radius, we will use subscripts to distinguish quantities at different smoothing lengths. We will reserve the subscript 1 for haloes and the subscript 0 for the environment, assumed uncollapsed at $z = 0$.

Fig. 1 shows the correlation strength $\gamma$ as a function of $R_0$ for three different values of $R_1$: 0.5, 2 and $6 \, h^{-1}$ Mpc (curves from top to bottom). The curves have been computed for a Lambda cold dark matter (ΛCDM) model of spectral index $n_s = 0.96$ and normalization $\sigma_8 = 0.83$ using the fitting formulae of Eisenstein & Hu (1999). This choice is consistent with the constraints inferred from the latest CMB measurements (WMAP3, see Spergel et al. 2007). For the special case of a power-law spectrum $P_\xi(k) \propto k^n$, the parameter $\gamma$ scales with the smoothing lengths $R_0$ and $R_1$ as

$$\gamma \propto \left( \frac{R_1}{R_0} \right)^{(3+n)/2}.$$  

Strictly speaking, this expression is valid for a power-law spectrum only. However, it provides a reasonable approximation in the ΛCDM cosmology considered here if the spectral index $n$ is replaced by an effective index $n_{\text{eff}}(k) \equiv \ln P_\xi(k)/\ln k$ evaluated on scale $k \sim 1/R_0$. The scaling $\gamma \propto (R_1/R_0)^{(3+n_{\text{eff}})/2}$ is plotted as dotted curves in Fig. 1. Recall that the spectral index is close to $n_{\text{eff}} \approx -2$ on comoving scales $1 \sim 10 \, h^{-1}$ Mpc.

3.1.2 Joint statistics of the shear tensor

Owing to the symmetry of $\xi_{ij}$, only six components are independent. We adopt the notation of Bardeen et al. (1986) and label them by $\xi_{1A}$, where the components $A = 1, \ldots, 6$ of the six-dimensional vector refer to the components $ij = 11, 22, 33, 12, 13, 23$ of the tensor. The joint probability distribution $P(\xi_{0A}, \xi_{1A})$ of the shear tensor $\xi_{0A}$ and $\xi_{1A}$, smoothed on scale $R_0$ and $R_1$, respectively, is given by a multivariate Gaussian whose covariance matrix $M$ has 12 dimensions. This $12 \times 12$ matrix may be partitioned into four $6 \times 6$ block matrices, $M_1 = \langle \xi_{1A} \xi_{1A}^\dagger \rangle$ in the top-left hand corner, $M_2 = \langle \xi_{0A} \xi_{1A}^\dagger \rangle$ in the bottom-right hand corner, $B = \langle \xi_{0A} \xi_{1A}^\dagger \rangle$ in the bottom-left hand corner and its transpose $B^\dagger$ in the top-right hand corner. Following Bardeen et al. (1986), we transform the six dimensions $\{\xi_{0A}, \xi_{1A}, A = 1, 2, 3\}$ to a new set of variables $\{u_k, v_k, w_k, k = 0, 1\}$, where

$$u_k = \xi_{1k} + \xi_{2k} + \xi_{3k},$$

$$v_k = \frac{1}{2}(\xi_{1k} - \xi_{2k} - \xi_{3k}),$$

$$w_k = \frac{1}{2}(\xi_{1k} - 2\xi_{2k} + \xi_{3k}).$$

With these definitions,

$$\langle u_k^2 \rangle = 1, \quad \langle v_k^2 \rangle = \frac{1}{15}, \quad \langle w_k^2 \rangle = \frac{1}{5},$$

$$\langle u_0 u_1 \rangle = \gamma, \quad \langle v_0 v_1 \rangle = \frac{\gamma}{15}, \quad \langle w_0 w_1 \rangle = \frac{\gamma}{5}.$$  

(13)

The other correlations are zero. For the six remaining components $\{\xi_{0A}, \xi_{1A}, A = 4, 5, 6\}$, the correlations functions are

$$\langle \xi_{0A} \xi_{0B} \rangle = \langle \xi_{0A} \xi_{1B} \rangle = \frac{1}{15} \delta_{AB},$$

$$\langle \xi_{0A} \xi_{1B} \rangle = \frac{\gamma}{15} \delta_{AB}.$$  

(14)
All the cross-correlations between \( u_i, v_j, w_k \) and \( \xi_{0,\alpha}, \xi_{1,\alpha} \) vanish. The block matrices \( M_1, M_2 \) and \( B \) are diagonal in the basis introduced above,

\[
M_1 = B^1 = \begin{pmatrix} C & 0 \\ 0 & 1/15 \end{pmatrix}, \quad B = \begin{pmatrix} \gamma C & 0 \\ 0 & \gamma \gamma /115 \end{pmatrix},
\]

where \( C = \text{diag}(1, 1/15, 1/5) \) and \( I \) is the \( 3 \times 3 \) identity matrix. The quadratic form which appears in the joint probability distribution

\[
P(\xi_0, \xi_1) d\xi_0 d\xi_1 = \frac{1}{(2\pi)^3} \frac{1}{\det M} \exp \left\{ -\frac{1}{\beta \epsilon \delta} \xi^T \xi \right\} d\xi_0 d\xi_1,
\]

(16)

where \( \det M \) is the determinant of the covariance matrix \( M \), can be computed easily using Schur’s identities. The result may be expressed in terms of the elementary symmetric functions \( s_i \) or, equivalently, in terms of traces,

\[
Q(\xi_0, \xi_1) = \frac{3}{4(1 - \gamma^2)} \left\{ \frac{1}{s_1^2} + \frac{1}{s_2^2} + \frac{1}{s_3^2} - \frac{1}{s_1 s_2 + s_3} \right\},
\]

and the square root of the determinant is given by \( |\det M|^{1/2} = (20/15)^{3/2}(1 - \gamma^2)^{3/2} \). The results are described by the correlation strength \( \gamma \) solely. The invariance under rotation \( P(\xi_0, \xi_1) = P(R \xi_0, R \xi_1) \), where \( R \) is a real orthogonal symmetric \( 3 \times 3 \) matrix, requires that \( P \) be a symmetric function of the eigenvalues, and thus a function of the trace \( \xi^T \xi \), \( I \), \( r = 0, 1, \ldots \), regardless of the statistical properties of \( \xi_j \). The expression (17) follows from our assumption of Gaussianity. Note also that no assumptions have been made so far about the coordinates.

### 3.1.3 Joint distribution of the eigenvalues

Lee & Shandarin (1998, see their appendix B) have computed the joint probability distribution of the eigenvalues of the deformation tensor for a sharp \( k \)-space filter. However, they assume that both principal axis frames are aligned, which is not true in general.

To obtain the joint probability distribution of the ordered eigenvalues of the shear tensor, we choose a coordinate system such that the coordinate axes are aligned with the principal axes of \( \xi_0 \). Let \( \alpha \) and \( \lambda \) be the diagonal matrices consisting of the three ordered eigenvalues \( \alpha_1 \geq \alpha_2 \geq \alpha_3 \) and \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \) of the deformation tensors \( \xi_0 \) and \( \xi_1 \), respectively. The principal axes are labelled according to this ordering. With this choice of coordinate, \( \xi_0 = \alpha \) and \( \xi_1 = R \lambda R^T \), where \( R \) is an orthogonal matrix that is the function of the eigenvalues of \( \xi_1 \) relative to those of \( \xi_0 \). To preserve the orientation of the principal axis frames, we further impose the condition that the determinant of \( R \) must be \( +1 \). Namely \( R \) belongs to the special orthogonal group \( SO(3) \). The properties of the trace imply that \( \text{tr} \xi_1 = \text{tr} \alpha \) and \( \text{tr} (\xi_1^2) = \text{tr} (\lambda \lambda ^T) \). We note, however, that the term \( \text{tr} (\xi_0 \xi_1) = \text{tr} (R \lambda R^T \alpha) \) depends on the rotation matrix.

The joint probability distribution \( P(\alpha, \lambda) \) is obtained by integrating over the rotations that define the orientations of the orthonormal eigenvectors of \( \xi_0 \) and \( \xi_1 \). The volume measure \( d \xi \) for the space of real \( 3 \times 3 \) symmetric matrices can be expressed in terms of the non-increasing sequence of eigenvalues \( t_i \) as

\[
d\xi = 8 \pi^3 \Delta (t) d^3 t dR.
\]

Here, \( dR \) is the invariant measure on the group \( SO(3) \) normalized to \( \int dR = 1 \). \( \Delta (t) \) is the Vandermonde determinant (equation 3) and \( d^3 t = dt_1 dt_2 dt_3 \). Since the quadratic form \( Q \) depends only on the relative orientation of the two orthonorm triads, we can immediately integrate over one of the \( SO(3) \) manifolds. The relevant volume is \( 8 \pi^2 / 4 = 2 \pi^2 \). The factor 4 comes from not caring whether the rotated axis points in the positive or negative direction (see e.g. Bardeen et al. 1986). The essential problem is the calculation of the integral over the rotations that define relative, distinct triad orientations. Appendix B shows that the integral over the second \( SO(3) \) manifold can be cast into the form

\[
\frac{1}{4} \int_{SO(3)} dR \exp \left\{ \beta \text{tr} (R \lambda R^T \alpha) \right\} = \frac{e^{\beta \epsilon}}{4} W (\beta \epsilon, \epsilon, \lambda),
\]

(19)

where the function \( W \) depends on \( \gamma \) through the parameter \( \beta = (15/2) / (1 - \gamma^2) \). The four independent variables \( \epsilon_+, \epsilon_-, \epsilon_\alpha, \epsilon_\lambda \) are combinations of the six eigenvalues of \( \alpha \) and \( \lambda \) (Wei & Eichinger 1990). We choose \( \epsilon_+ = (1/3) \text{tr} (\lambda \lambda), \epsilon_- = (1/3) \text{tr} (\lambda \lambda), \epsilon_\alpha = (\alpha_1 - \alpha_2) / \text{tr} \lambda, \epsilon_\lambda = (\lambda_1 - \lambda_2) / \text{tr} \lambda, \text{tr} \lambda = \text{tr} - 3 \alpha_3 \), and \( \lambda_1 \). With this parametrization, \( -\infty < \epsilon_+ < \infty, \epsilon_- < 0, 0 < \epsilon_\alpha < 1, 0 < \epsilon_\lambda < 1 \). The function \( W(\beta \epsilon, \epsilon_\alpha, \epsilon_\lambda) \) can be written down as a double integral (equation B9). We have found that a fifth-order expansion about \( \epsilon_- = 0 \) (equation B12) is accurate to within 2% in the range \( 0 \leq \epsilon_- \leq 1.5 \). We use this truncated series in the computation of the group integral (19). The joint probability distribution \( P(\alpha, \lambda) \) can now be formulated as

\[
P(\alpha, \lambda) d\alpha d\lambda = \frac{15^6}{320 \pi^2} (1 - \gamma^2)^{-3} W(\beta \epsilon, \epsilon_\alpha, \epsilon_\lambda) \frac{e^{\beta \epsilon}}{4} \Delta (\alpha) \Delta (\lambda) d\alpha d\lambda,
\]

(20)

where the quadratic form \( Q_{01} \) is a function of \( \alpha \) and \( \lambda \),

\[
Q_{01}(\alpha, \lambda) = \frac{3}{4(1 - \gamma^2)} \left\{ \frac{1}{s_1^2} + \frac{1}{s_2^2} + \frac{1}{s_3^2} - \frac{1}{s_1 s_2 + s_3} \right\},
\]

and

\[
\frac{15^4}{8 \pi^3} (1 - \gamma^2)^{-3} W(\beta \epsilon, \epsilon_\alpha, \epsilon_\lambda) e^{\beta \epsilon} \Delta (\lambda) \times \exp \left\{ -\frac{15^2 \gamma^2 (\alpha) + 15^2 (\lambda) - 3(\gamma (\text{tr} \alpha) - \text{tr} \lambda)^2}{4(1 - \gamma^2)} \right\}.
\]

(22)

It can be verified by direct numerical integration that the probability distribution (22) is normalized to unity (we have used the multidimensional integrator DCUHRE described in Bernsten, Espelid & Genz 1991).

### 3.1.4 Joint distribution of the shear ellipticity and prolateness

The results of Section 3.1.3 can be conveniently expressed in terms of the density contrast \( \nu \), shear ellipticity \( e \) and shear prolateness \( p \). The joint probability distribution for these variables, \( g(\epsilon_1, p_1, \nu_1, e_0, p_0, \nu_0) \), is readily obtained from the following coordinate transformation:

\[
\begin{align*}
\alpha_1 &= \frac{\nu_0}{3} (1 + 3 e_0 + p_0), & \lambda_1 &= \frac{\nu_1}{3} (1 + 3 e_1 + p_1), \\
\alpha_2 &= \frac{\nu_0}{3} (1 - 2 p_0), & \lambda_2 &= \frac{\nu_1}{3} (1 - 2 p_1), \\
\alpha_3 &= \frac{\nu_0}{3} (1 - 3 e_0 + p_0), & \lambda_3 &= \frac{\nu_1}{3} (1 - 3 e_1 + p_1).
\end{align*}
\]

(23)
We have, for example, $\text{tr}(\alpha) = v_0^2$, $d^3\alpha = (2/3)v_0^3 \, dv_0 \, dp_0 \, dp_0$ and $\Delta(\alpha) = 2v_0^3 \, e_0(e_0^2 - p_0^2)$. With this parametrization, the quadratic form $Q_{01}$ becomes

$$Q_{01} = \frac{5}{2} \left( 1 - \gamma^2 \right) \left[ v_0^2 \left( 3e_0^2 + p_0^2 \right) + v_1^2 \left( 3e_1^2 + p_1^2 \right) + \gamma v_0 v_1 \right] + \frac{1}{2} \left[ \left( v_1 - \gamma v_0 \right)^2 + v_0^2 \right].$$

(24)

The other variables are simply $e_+ = (1/3)v_0 v_1$, $e_- = (1/3)v_0 v_1 (3e_0 - p_0)(3e_1 - p_1)$, $e_u = (e_0 + p_0)/(3e_0 - p_0)$ and $e_1 = (e_1 + p_1)/(3e_1 - p_1)$.

Let us introduce the notational shorthands $0 \equiv (e_0, p_0, v_0)$ and $1 \equiv (e_1, p_1, v_1)$. We wish now to calculate the conditional probability $g(e_1, p_1|v_1, 0)$ of having an ellipticity $e_1$ and prolateness $p_1$ given a density $v_1$ on scale $R_1$, and given the values $(e_0, p_0, v_0)$ on scale $R_0$. This probability will be useful to estimate the effect of the large-scale environment on the primordial distribution of shear ellipticity and prolateness. Bayes’ theorem implies that

$$g(e_1, p_1|v_1, 0) = \frac{g(0, 1)}{g(v_1, 0)}.$$

(25)

Since $(v_1e_0, a) = \gamma/2$ if $a = 1, 2, 3$ and the other cross-correlations are zero, the calculation of the denominator is straightforward. We have $g(v_1, 0) = g(v_1|v_0)g(0)$, where

$$g(v_1|v_0) = \frac{1}{\sqrt{2\pi}} \left( 1 - \gamma^2 \right)^{-1/2} \exp \left[ -\frac{1}{2} \left( \frac{v_1 - \gamma v_0}{1 - \gamma^2} \right)^2 \right].$$

(26)

as the density contrast $v$ is independent of the asymmetry parameters $e$ and $p$. With this information, the conditional probability distribution can be expressed as

$$g(e_1, p_2|v_1, 0) = b(0, 1) g(e_1, p_1|v_1),$$

(27)

where

$$g(e_1, p_1|v_1) = \frac{1}{\sqrt{10\pi}} \, v_1^3 e_1 \left( e_1^2 - p_1^2 \right) e^{-5/2v_1^2(3e_1^2 + p_1^2)}$$

(28)

is the distribution without the environmental constraint (see e.g. Sheth et al. 2001), and

$$b(0, 1) = (1 - \gamma^2)^{-5/2} W(\beta e_+, e_-, e_+) \times \exp \left\{ -\frac{5\gamma^2}{2(1 - \gamma^2)} \left[ v_0^2 \left( 3e_0^2 + p_0^2 \right) + v_1^2 \left( 3e_1^2 + p_1^2 \right) \right] \right\}$$

(29)

is a correction factor which results from the constraint ‘0’. As expected, in the limit where the correlation becomes weak, the conditional distribution $g(e_1, p_1|v_1, 0)$ reduces to the unconditional distribution, equation (2), derived by Doroshkevich. It vanishes outside the domain defined by $|p_1| \leq e_1$. We stress that these expressions are valid for any window function. In the particular case of a sharp $k$-space filter however, the spectral parameter is simply $\gamma = \sigma_0/\sigma_1$.

The distribution (27) is sufficient to estimate the magnitude of the environmental effect which arises from the statistics of the initial shear field.

### 3.2 Statistical correlation between protohalo and environment in Gaussian initial conditions

We illustrate how the local characteristics of the shear tensor smoothed on the scale of collapsing haloes depend on the initial geometry of their large-scale environment.

**Figure 2.** The 20 and 68 per cent contours of the conditional probability distribution $g(e_1, p_1|v_1, 0)$ for several values of the correlation strength, $\gamma = 0, 0.4, 0.6$ and 0.8 (from top to bottom). The values of $(e_0, p_0, v_0)$ are typical of a (prolate) filament. The dashed lines indicate the boundary of the domain $|p| \geq e$.

#### 3.2.1 Shear ellipticity and prolateness

First, it is worthwhile studying how the conditional probability distribution changes with the correlation strength $\gamma$. To this purpose, we evaluate the probability (27) for a density $\delta_1 = 2$, a value sufficiently large so that the small-scale overdensity collapses at moderate to high redshift regardless of the asymmetry parameters. For a filtering scale $R_1 = 2h^{-1}\text{Mpc}$, i.e. a halo mass $M_1 \approx 2 \times 10^{12}\,\text{M}_\odot\,h^{-1}$, this corresponds to a density contrast $v_1 = 1.28$. On large scale $R_0 > R_1$, we take the shear ellipticity, prolateness and density to be $(e_0, p_0, v_0) = (1.4, -0.8, 0.6)$. These values are appropriate for a filament-like configuration. In Fig. 2, the 20 and 68 percentiles of the conditional probability distribution $g(e_1, p_1|v_1, 0)$ are plotted for $\gamma = 0, 0.4, 0.6$ and 0.8 (contours from top to bottom). A larger value of $\gamma$ increases the asymmetry of the distribution and sharpens its maximum. Recall that, at fixed value of $R_1$, the correlation strength $\gamma$ decreases with increasing $R_0$ (see Fig. 1). For instance, with $R_1 = 2h^{-1}\text{Mpc}$ and a reasonable environment size $R_0 \sim 10h^{-1}\text{Mpc}$, we have $\gamma \sim 0.6$. The resultant distribution is displayed third from the top. In this case, the most probable values of the shear ellipticity and prolateness are $(e_{m}, p_{m}) = (0.48, -0.15)$, significantly different than those in the limit $\gamma \rightarrow 0$, namely $(0.35, 0)$. Note that the top-hat smoothing artificially reduces the asphericity of the large-scale environment. Therefore, the values of $e_{m}$ and $p_{m}$ certainly underestimate the asymmetry that could be measured from $N$-body simulations (Bardeen et al. 1986). Finally, unless otherwise stated, we shall adopt $R_0 = 10h^{-1}\text{Mpc}$ ($M_0 = 2.7 \times 10^{14}\,\text{M}_\odot\,h^{-1}$) as the environment radius in the remaining of this paper.

Fig. 3 further illustrates the environmental dependence which arises from the statistical properties of the shear tensor. Contours are plotted for three different configuration shapes of the large-scale environment. The halo mass is now $M_1 = 2.2 \times 10^{12}$ (left-hand panel) and $M_1 = 3.4 \times 10^{13}\,\text{M}_\odot\,h^{-1}$ (right-hand panel), and the resultant correlation coefficient is $\gamma = 0.58$ and 0.85, respectively. Also, we choose $v_1$ so that $\delta_1 = 2$ in all cases. The configuration shape of the large-scale region is either a protocluster of signature
Figure 3. The 20 and 68 per cent percentiles of \( g(e_1, p_1|\nu_1, 0) \) for two different halo mass \( M_1 = 2.2 \times 10^{12} \) (left-hand panel) and \( M_1 = 3.4 \times 10^{13} \ M_\odot h^{-1} \) (right-hand panel). These mass scales correspond to a smoothing radius \( R_1 = 2 \) and \( 5 h^{-1} \) Mpc, respectively. On large scale, the shear is smoothed on a fixed radius \( R_0 = 10 h^{-1} \) Mpc (\( M_0 = 2.7 \times 10^{13} \ M_\odot h^{-1} \)). The shape of the large-scale region is either a protocluster, the precursor of a filament or a sheet-like structure, characterized by \((e_{\alpha}, p_{\alpha}, \nu_{\alpha}) = (0.5, -0.1, 1), (1.5, -1.3, 0.5) \) and \((1.3, 1.1, 0.4)\), respectively. The dashed lines indicate the boundary of the domain \( |p_1| \geq e_1 \). The interior of the triangle bounded by \((e_1, p_1) = (0, 0), (1, 1/4, -1/4)\) and \((1/2, 1/2)\) shows the region where \( \lambda_1 > 0 \). For a given mass \( M_1 \), protohaloes which collapse within filament-like or sheet-like structures are, on average, initially more asymmetric than those which will form in spherical-like environment. The average asphericity and the scatter increase with decreasing halo mass.

(+ + +), the precursor of a filament (+ + –), or a sheet-like structure (+ –). Clearly, at fixed halo mass \( M_1 \), the protohaloes that collapse within the pancake or the filament are initially more asymmetric than those that will form in the spherical-like region. Furthermore, the asymmetry increases noticeably with decreasing halo mass. For \( M_1 = 2.2 \times 10^{12} \ M_\odot h^{-1} \), the most probable values of \( e_1 \) and \( p_1 \) are \((e_{\alpha}, p_{\alpha}) = (0.32, 0.0), (0.48, -0.14) \) and \((0.39, 0.05)\), respectively, whereas, for \( M_1 = 3.4 \times 10^{13} \ M_\odot h^{-1} \), we find \((e_{\alpha}, p_{\alpha}) = (0.17, -0.02), (0.37, -0.2) \) and \((0.28, 0.14)\). In the high-mass halo case, the conditional distributions exhibit much less scatter as a result of the larger values of \( e \) and \( p \).

Since the density contrast is independent of the shear ellipticity and prolateness, the probability of finding a virialized halo in a given environment of initial shape \((e_0, p_0, \nu_0)\) is modulated by \( g(\nu_1|\nu_0) \), which is very sensitive to \( \nu \). For the low-mass halo considered here, this probability is 0.38, 0.31 and 0.17 for the cluster, filament and pancake configurations, respectively. For the massive protohalo however, \( g(\nu_1|\nu_0) \) is significantly non-zero (0.10) for the spherical-like environment solely. Thus, low-mass haloes form in the mildly overdense structures of the primeval density field, while the most massive collapse mainly in the densest, weakly aspherical regions. This is a distinctive feature of hierarchical formation models (e.g. Kaiser 1984; Mo & White 1996) that adds to decrease the scatter in the relation between the environment and the local properties of the shear with increasing halo mass. This statistical effect may provide an explanation for the strong environmental dependence of low-mass haloes.

3.2.2 Alignment of principal axes

It is fairly straightforward to derive probability distributions for the relative orientation of the principal axis frame from the results of Section 3.1. Appendix B provides the details of the calculation. Briefly, we parametrize the rotation matrix \( R \) in terms of the Euler angles \( 0 \leq \varphi, \psi \leq 2\pi \) and \( 0 \leq \vartheta \leq \pi \). We adopt the XXZ convention so that \( \vartheta \) is the angle between the two minor (third) axes. The trace \( tr (R \lambda R^\top \alpha) \) is then decomposed into a sum of rotation matrices \( D_{\alpha, \beta} ^{\varphi, \vartheta, \psi} \). The integral over the variable \( \psi \) can be performed easily and yields the conditional probability \( P(\vartheta, \varphi|0, 1) \) given the values of \( e_\alpha, e_\beta, e_\gamma \), etc. In Fig. 4, contours of constant \( P(\vartheta, \varphi|0, 1) \) are plotted for the cluster, filament and pancake-like configurations described above. They are shown for the halo mass \( M_1 = 2.2 \times 10^{12} \) (left-hand panels) and \( M_1 = 3.4 \times 10^{13} \ M_\odot h^{-1} \) (right-hand panels) considered in Fig. 3. In all cases, the asymmetry parameters \( e_\alpha \) and \( p_1 \) assume their most probable values. Unsurprisingly, the alignment between the minor axes of the shear smoothed on scale \( R_0 \) and \( R_1 \) is weaker for the low-mass halo as a result of the smaller values of \( e \) and \( p \). At fixed halo mass however, the alignment is substantially stronger in the case of the filamentary structure (middle panels). This mostly follows from the fact that, for a configuration shape with one or two positive eigenvalues, the alignment is strongest along the axis of symmetry, which coincides with the minor axis for a filamentary structure. Conversely, the alignment between the major axes is much stronger in the pancake configuration. These results show that the principal axis frame of the tidal tensor smoothed on the protohalo mass scale cannot be assumed independent of that defined by the environment. This is especially true when the large-scale configuration shape is highly asymmetric. Accounting for this correlation has a large impact on the probability distribution of the spin parameter (see Section 4.4.3).

To summarize, a correlation between the local properties of the shear and the configuration shape of the environment is expected in Gaussian initial conditions because fluctuations on different scales are correlated. Our results nicely demonstrate the large magnitude of this statistical effect. At fixed halo mass, the shear of a perturbation that collapse within filaments or pancakes is on average more asymmetric than in spherical regions. The principal axis frame tends to be aligned along the axis of symmetry of the external mass distribution. Furthermore, the scatter in the relation between the
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4 ENVIRONMENTAL DEPENDENCE IN THE ELLIPSOIDAL COLLAPSE DYNAMICS

We now concentrate on the dynamical origin of the environmental dependence in the ellipsoidal collapse model. We employ a simplified model based on the collapse of homogeneous ellipsoids that takes into account the interaction between a collapsing halo and its surroundings. To anticipate the results of this section, we find that haloes residing in large overdensities virialize earlier. The environment density is the key parameter in determining the virialization redshift. We incorporate this dynamical effect into the excursion set formalism by means of a collapse barrier whose height depends on the environment density. This approach greatly simplifies the calculation of the environmental dependence of halo properties. It also predicts a clear correlation between formation redshift, large-scale bias and environment.

4.1 Homogeneous ellipsoidal dynamics

4.1.1 Equation of motion

We consider the collapse of a triaxial perturbation embedded in a uniform background assumed to be a \( \Lambda \)CDM cosmology. We neglect the influence of non-linear substructures on the gravitational evolution. The initial (Lagrangian) volume occupied by this over-
dense fluctuation is a (uniform) sphere of comoving radius \( R_1 \).

Following Peebles (1980) and Eisenstein & Loeb (1995), the (proper) position of any point interior to the ellipsoid is conveniently described as \( r^\alpha = A^{\alpha \beta} q^\beta \), where \( q^\alpha q \leq 1 \) and repeated indices are summed. The matrix \( A \) is a function of time solely. At all time, the equation defining the outer shell of the ellipsoid is \( r^\beta (A A^{\beta \gamma})^{-1} r^\gamma = 1 \). The principal axis lengths \( \{ A_k, k = 1, 2, 3 \} \) and directions of the ellipsoid can be found by diagonalizing \( A A^{\top} = Q \hat{A} Q^{\top} \), where \( Q \) is orthogonal and \( \hat{A} \) is a positive definite diagonal matrix. In this model, the potential is quadratic in the coordinates, \( \Phi(r) = 1/2 \Phi^{\alpha \beta} r^\alpha r^\beta \), so that the external and internal forces preserve the homogeneity of the protohalo at all time. Introducing the time variable \( \tau = \ln(z) \) instead of \( t \) (e.g. Barrow & Silk 1981; Nusser & Colberg 1998), the equation of motion reads

\[
\dot{A}^{\alpha \beta} - (1 + q(\tau)) A^{\alpha \beta} = -\frac{3}{2} \Omega_m(\tau) \sum_{\nu} \Phi^{\nu \gamma} A^{\gamma \beta},
\]

and is manifestly independent of the Hubble constant. Upper dots denote derivatives with respect to \( \tau \), \( q(\tau) = \Omega_m(\tau) \Omega_\Lambda(\tau) \) is the deceleration parameter \( (\tau = 1/2 \) in a EdS universe) and the gravitational potential \( \Phi \) is in unit of \( 4 \pi G \bar{\rho}_m(\Omega_\Lambda(z) + \Omega_m(z) \) are the matter and vacuum density in unit of the critical density, respectively. The initial conditions are set by the Zeldovich approximation (see below). Virialization occurs when the three axes have collapsed. To prevent axis \( k \) from shrinking to arbitrary small sizes, we halt its collapse when \( A_k / a R_1 = f_k \equiv 0.177 \). This freeze-out radius is chosen so that the virialized object is 178 times denser than the background (Bond & Myers 1996). When axis \( k \) has collapsed, we set the radial component of the velocity and acceleration in that direction to \( \dot{A}_k = A_k = f_k a R_1 \). This way we end the radial collapse but leave the tangential velocity unchanged, so that the angular momentum of the protohalo is conserved (see e.g. Eisenstein & Loeb 1995, for details).

4.1.2 External shear field

The external force exerted by the rest of the universe on the triaxial perturbation may be generically written in terms of the Green function \( G(r, r') = |r - r'|^{-1} \). It is natural to adopt spherical coordinates as perturbations grow from an initially homogeneous and isotropic background. The potential integral can thus be expanded as a multipole series (Binney & Tremaine 1987),

\[
\Phi(r) = \sum_{lm} (2l+1)^{-1} r^l \hat{q}_{lm}^{(e)}(\hat{n}) Y_l^m(\hat{n}),
\]

where \( r = r \hat{n}, Y_l^m(\hat{n}) \) are the spherical harmonic functions, and the coefficients

\[
\hat{q}_{lm}^{(e)} = \int d^3 r' Y_l^m(\hat{n}) \delta(r') (r')^{-l-1} \]

are the multipole moments that characterize the potential at any (interior) point \( r \). The constant and the dipole \( l = 1 \) term, which corresponds to a translation, do not alter the shape of the whole region and can be dropped out. The quadrupole \( l = 2 \) term describes the force distorting the central region. Analytic calculation (Quinn

Figure 4. Contours of constant probability \( P(\theta, \phi|0, 1) \) for the cluster, filament and pancake-like configurations described in the text. They are shown for a halo mass \( M_1 = 2.2 \times 10^{12} \) (left-hand panels) and \( 3.4 \times 10^{13} M_\odot h^{-1} \) (right-hand panels). The asymmetry parameters \( e_1 \) and \( p_1 \) assume their most probable value. Levels of contours decrease by a factor of 2. \( \phi \) is the angle between the minor axes. In all cases, the probability is highest along the vertical line \( \cos \theta = 1 \). The alignment is strongest in the case of the filament, whose minor axis coincides with the axis of symmetry.
& Binney 1992) indicate that it dominates the higher order terms ignored here. This justifies to some extent the assumption of a quadratic potential.

The quadrupole $q^{ij}_{\text{sel}}$ of the external shear will generally not be aligned with that of the protohalo region (see Section 3). Although one could treat the non-linear evolution of the external mass distribution with concentric shells (Chandrasekhar 1969; Binney & Tremaine 1987; Ryden & Gunn 1987; Eisenstein & Loeb 1995), we adopt a simpler approach and assume that the protohalo is embedded in a single, large-scale triaxial region of initial comoving radius $R_0 > R_1$, not necessarily lined up with the protohalo region. In principle, the protohalo will cause the large-scale perturbation to warp, producing in return a non-quadratic potential that breaks the homogeneity of the small-scale ellipsoid. To avoid this problem, we assume that the large-scale triaxial perturbation and the background are unaffected by the collapse of the protohalo and remain homogeneous at all time (Icke 1973; White & Silk 1979; Eisenstein & Loeb 1995). We can therefore evolve the large-scale ellipsoid independently using the model of Bond & Myers (1996).

4.1.3 Gravitational potential of the protohalo

In the principal axis frame of the large-scale ellipsoid, we write down the total gravitational potential of the protohalo as

$$\Phi^{\alpha\beta} = \Phi_{\text{FRW}}^{\alpha\beta} + \Phi_{E,0}^{\alpha\beta} + \Phi_{\text{vel}}^{\alpha\beta} + \Phi_{\text{fit}}^{\alpha\beta},$$

(33)

where $\Phi_{\text{FRW}}^{\alpha\beta} = [1 - 2\Omega_1(\tau)/\Omega_m(\tau)]/3$ is the contribution of the smooth background and

$$\Phi_{E,0}^{\alpha\beta} = \frac{1}{2} b_{\alpha\beta} \delta^{\alpha\beta},$$

$$\Phi_{\text{vel}}^{\alpha\beta} = \frac{1}{2} (\delta_1 - \delta_0) \sum \beta_1 \beta_2 Q^{\gamma\eta} Q^{\delta\gamma},$$

(34)

are the gravitational potential associated with the large-scale triaxial perturbation and with the remaining mass in the protohalo, respectively. The matrix elements $Q^{\alpha\beta}(\tau)$ depend generally on $\tau$ because the protohalo may be rotating. The functions $b_{\alpha\beta}(\tau)$ are defined in Appendix C, which provides details on the potential of an homogeneous, triaxial ellipsoid. It is worth emphasizing that, since $b_{\alpha\beta}(\tau)$ are homogeneous of degree zero, the potentials (34) do not depend on the value of $R_0$ and $R_1$, $\delta_0(\tau)$ and $\delta_1(\tau)$ are the relative overdensity of the large-scale environment and the protohalo, respectively. Since the contribution of the traceless part of both $\Phi_{E,0}^{\alpha\beta}$ and $\Phi_{\text{vel}}^{\alpha\beta}$ is of second order only, we include the linear approximation for the external shear field,

$$\Phi_{\text{vel}}^{\alpha\beta} = \sum_{\gamma} \lambda_{\alpha\beta\gamma} Q^{\alpha\gamma} Q^{\beta\gamma},$$

(35)

where $\lambda_{\alpha\beta\gamma} = D(\tau)/(\delta_1 - \delta_1/3)$ is a linear function of the initial eigenvalues $\lambda_\gamma$ of the shear tensor smoothed on scale $R_1$, $Q^{\alpha\beta}(\tau)$ describes the initial orientation of the protohalo relative to that of the large-scale region. This ensures that the evolution consistently reduces to the Zeldovich approximation at early times (Bond & Myers 1996). The initial conditions are explicitly

$$A^{\alpha\beta}(t_i) = a(t_i) R_1 (1 - D(t_i) \lambda_\gamma) Q^{\alpha\beta},$$

$$A^{\alpha\beta}(t_i) = A^{\alpha\beta}(t_i) - a(t_i) D(t_i) \lambda_\gamma Q^{\alpha\beta},$$

(36)

where $a_i = a(t_i)$ and $D_i = D(t_i)$. Note that the initial tangential velocities are zero.

The simplifications of this model notwithstanding, our calculation should provide a quantitatively useful description of the impact of environment on the collapse of dense, triaxial regions.

4.2 Effect of environment on the redshift and critical density for collapse

Our first task is to study the dynamical effect of the large-scale environment on the collapse of the protohalo. Once the cosmological background is chosen, the evolution of the protohalo is governed by the initial values of $(\epsilon_1, p_1, \delta_1, \epsilon_0, p_0, \delta_0)$ and the Euler angles $(\varphi, \theta, \psi)$, which describe its initial orientation relative to that of the large-scale ellipsoid. For simplification, we choose $p_0 = p_1 = 0$ and set $(\varphi, \theta, \psi) = (0, 0, 0, 0, 0, 0)$ (in radian). Note that this particular orientation has a reasonable probability of occurring regardless of the configuration shape of the environment (see Fig. 4). The starting redshift is taken to be a hundred. We store the collapse redshift $z_c$ of the three axes, as well as the (linear) critical density $\delta_{c*}$ at virialization. Results are presented for a $\Lambda$CDM universe with $\Omega_m = 0.238$ and $\Omega_\Lambda = 0.762$.

Fig. 5 examines how the collapse redshifts $z_c$ of the three protohalo axes change with the model parameters. For reasons that should become apparent below, $z_c$ is plotted against the shear ellipticity $\epsilon_1$. In all panels, the dotted curve is the reference case $(\epsilon_0, \delta_0) = (0, 0)$. The solid and dashed curves indicate the collapse redshifts for $(\epsilon_0, \delta_0) = (0, \pm 1)$ and $(\epsilon_0, \delta_0) = (0.3, \pm 1)$, respectively. To highlight the effect of changing the relative orientation, we have also plotted as dot-dashed curves the collapse redshifts for $(\epsilon_0, \delta_0) = (0.3, \pm 1)$ and a weaker initial shear alignment, $(\varphi, \theta, \psi) = (1.5, 1.5, 1.5)$. Also shown are the values of $\delta_0$ and $\delta_1$, linearly extrapolated to present epoch. For positive values of $\delta_0$, the tidal force exerted by the large-scale perturbation delays the collapse along the first axis but enhances it along the third, reducing thereby the anisotropy that arises from the linear term, equation (35). By contrast, the difference between the collapse redshift of the major and minor axis is enhanced for $\delta_0 < 0$. In all cases however, the collapse redshift of the intermediate axis is barely affected and remains close to the value predicted by the spherical collapse model (see appendix of Shen et al. 2004).

Figure 5. Collapse redshift $z_c$ for the protohalo axes as a function of the initial ellipticity $\epsilon_1$. We have assumed a $\Lambda$CDM universe. The panels show how $z_c$ varies with the initial protohalo overdensity $\delta_1$ and the large-scale density $\delta_0$ and shear ellipticity $\epsilon_0$. In all panels, $p_0 = p_1 = 0$ and the dotted curve is the reference case $(\epsilon_0, \delta_0) = (0, 0)$. The solid and dashed curves show the collapse redshifts for $(\epsilon_0, \delta_0) = (0, \pm 1)$ and $(\epsilon_0, \delta_0) = (0.3, \pm 1)$, respectively. The dot-dashed curve also has $(\epsilon_0, \delta_0) = (0.3, \pm 1)$, but the initial alignment of the shear is weaker (see text).
et al. 2006). Clearly, at fixed overdensity $\delta_1$, the haloes embedded in the large density environment virialize earlier. The strength of this effect increases with the shear ellipticity $e_1$. It is very sensitive to the initial conditions. For $\delta_0 = -1$, an external quadrupole shear $e_0 > 0$ delays the halo virialization as compared to the spherical case. This delay is still present for $\delta_0 = +1$ when the halo collapse is initially close to spherical (the delay is most significant for the low value of $\delta_1$), but the trend reverses when the ellipticity gets larger than $e_1 > 0.1$. In addition, changing the initial alignment can also increase or decrease the difference in collapse redshift. Note also that, at low redshift, the increasing contribution of the cosmological constant to the energy density slows down the collapse of the third axis noticeably.

When all the parameters but $\delta_1$ are held fixed, then there is a unique value of $\delta_1 = \delta_{ec}(e_1, z)$ which leads to the collapse of the minor axis (virialization) at redshift $z$. In Fig. 6, the critical density for collapse at $z = 0$ is plotted in unit of $\delta_{ec}$ for the initial collapse configurations considered in Fig. 5. ‘WA’ labels the configurations for which the initial orientation of the shear principal axis frame is given by $(\varphi, \theta, \psi) = (1.5, 1.5, 1.5)$ (see text). The dotted curve shows the reference case $(e_0, \delta_0) = (0, 0)$. Note that $p_0 = p_1 = 0$ in all cases.

![Figure 6](image_url)

**Figure 6.** Critical density $\delta_{ec}(e_1, z)$ for the collapse of a protohalo perturbation at $z = 0$ in unit of $\delta_{ec} = 1.673$, the critical density for a spherical collapse in the ΛCDM cosmology considered here. $\delta_{ec}$ is plotted as a function of the shear ellipticity $e_1$. The various curves indicate how $\delta_{ec}$ varies for the initial configurations considered in Fig. 5. ‘WA’ labels the configurations for which the initial orientation of the shear principal axis frame is given by $(\varphi, \theta, \psi) = (1.5, 1.5, 1.5)$ (see text). The dotted curve shows the reference case $(e_0, \delta_0) = (0, 0)$. Note that $p_0 = p_1 = 0$ in all cases.

In what follows, we shall omit writing the other variables, but recall that $\delta_{ec}$ is generally a function of the 12 variables that parametrize the shear eigenvalues and the relative orientation of the principal axis frames on scale $R_0$ and $R_1$.

should provide a good description of this environmental effect if the scatter around it is not too large. We investigate this possibility in the remainder of this section.

### 4.3 Environmental dependence as moving barriers

As seen in Section 3, the local properties of the shear depend substantially on the large-scale environment. Here, we consider a large ensemble of initial halo environment configurations and examine the resultant distribution of critical density $\delta_{ec}$. We find that it is a reasonable approximation to use an average critical density $B(e_1, z, \delta_0)$ and neglect the scatter around it. We provide a fitting formula to $B(e_1, z, \delta_0)$ which facilitates the inclusion of the environmental dependence of the kind considered here into the excursion set formalism.

#### 4.3.1 Monte Carlo simulations

Chiueh & Lee (2001) have shown that random realizations of the linear deformation tensor can be simulated by drawing six independent Gaussian variables. Their algorithm can be easily extended to generate joint realizations of the shear tensor which satisfy the correlation property (9). In the basis defined in equation (12), the $12 \times 12$ covariance matrix $\mathbf{M}$ decomposes into a direct sum of $2 \times 2$ block diagonal matrices. As a result, the variables $X = \{u, v, w, \xi, \eta, \xi_0, \xi_1, \xi_2, \xi_3\}$ can be simulated using the following linear transformation

$$X_0 = \frac{\sigma_x}{\sqrt{2}}(\sqrt{1 + 2x} - 1)$$

$$X_1 = \frac{\sigma_y}{\sqrt{2}}(\sqrt{1 + 2y} - 1)$$

where, again, the subscripts 0 and 1 indicate that the shear is smoothed (with a top-hat filter) on scale $R_0$ and $R_1$, respectively. $x_0$ and $y_0$ are two Gaussian random deviate of dispersion unity. $\sigma_x^2 = \langle X^2 \rangle$ is the rms variance in the variable $X$. For instance, $\sigma_x = 1/\sqrt{15}$ when $X = v$. The remaining components of the shear tensor, $\xi_1, \xi_2$ and $\xi_3$ are readily obtained from the linear relations (12).

Since the small-scale perturbation is identified as a halo at some redshift $z > 0$, while the exterior ellipsoid is assumed uncollapsed at that redshift, we constrain $\delta_0$ so that it does not exceed 0.98. We also enforce the constraint $\delta_1 \geq 1.6$. For a radius $R_1 = 2 h^{-1}$ Mpc, this corresponds to a density threshold $\delta_{c1} \geq 1$. In other words, the vast majority of our haloes form out of one standard deviation fluctuations. In practice, we generate random realizations of the initial conditions and reject the cases that do not satisfy these constraints.

#### 4.3.2 Collapse barriers

We generate a large ensemble of initial collapse configurations. We evolve each realization separately using the model described in Section 4.1 and store the value of the density contrast $\delta_1 \equiv \delta_{ec}$ which corresponds to collapse at redshift $z$.

The distribution of critical density $\delta_{ec}/\delta_{ec}$ and shear ellipticity $e_1$ is plotted in Fig. 7 for several values of the primordial environment density, evenly spaced in the range $-2 \leq \delta_0 \leq 1$. Collapse occurs at $z = 0$. The crosses indicate the actual values of collapse densities and shear ellipticities of (a small subset of the) individual realizations. Also shown as the solid curve is our approximation to the critical collapse boundary $B \equiv \delta_{ec}(e_1, z)$ defined by the implicit
The average density for collapse is plotted as the solid curve. The crosses indicate the actual values of the critical collapse density with ellipticity and to its dependence on both the redshift and the environment density. For three different values of $R_1$, the average density for collapse, equation (38), is plotted as a function of the environment density. Figure 7 shows the distribution of critical densities $\delta_{ec}$ and shear ellipticities $e_1$ as a function of the environment density $\delta_0$ for a collapse redshift $z = 0$. The solid curves show our approximation, equation (38), to the critical collapse boundary. The dashed curves indicate the (approximate) boundaries of the domain $(e_1, \delta_0)$ sampled by the random realizations. They are the barrier shape (38) with $\delta_0 = \pm 5$.

The exponential factor ensures that $\beta_1$ and $\beta_2$ are strictly positive. In addition, we have enforced the constraint $d_i > 0$ to account for the (slight) decrease of the environmental dependence with redshift. Note also that, in the limit of large environment densities, the moving barrier $B(e_1, z)$ tends towards the constant spherical barrier $B = \delta_{ec}(z)$. The coefficients $b_i, c_i$ and $d_i$ are found by fitting the barrier shape (38) to the critical collapse densities of $3 \times 10^5$ realizations with environment density in the range $-2 \leq \delta_0 \leq 1$. We find

$$b_1 \approx 0.412, \quad c_1 \approx 0.113, \quad d_1 \approx 0.0576$$
$$b_2 \approx 0.618, \quad c_2 \approx 0.0451, \quad d_2 \approx 0.0485.$$  

(40)

For $\delta_0 = 0$ and $a = 1$, we obtain $\beta_1 = 0.412$ and $\beta_2 = 0.618$, in good agreement with the values inferred by Sheth, Mo & Tormen ($\beta_1 = 0.47, \beta_2 = 0.615$). A visual inspection of Fig. 7 has convinced us that the barrier shape (38) provides a good approximation to the increase of the critical collapse density with ellipticity and to its dependence on the environment density in the range $0 \leq e_1 \leq 0.45$. To guide the eye, we have also plotted as dashed curves the (approximate) upper and lower boundaries of the domain $(e_1, \delta_0)$ sampled by the random realizations. These boundaries are the barrier shape (38) with an environment density $\delta_0 = \pm 5$.

Figure 8. Top panel: distribution of critical density and shear ellipticity for three different protohalo radii $R_1 = 1$ (circle), 2 (square) and 5 $h^{-1}$ Mpc (triangle). The collapse redshift is $z = 0$ and the environment density is $\delta_0 = 0$. The average density for collapse, equation (38), is plotted as a function of the environment density $\delta_0$. Figure 8 shows the distribution of critical densities $\delta_{ec}$ for three different values of $R_1$. The average density for collapse is plotted in terms of the halo mass $M_1$. The average density for collapse depends on the shape parameters in a complex way. This is the reason why we resort to the Sheth et al. functional form.

Following Sheth et al. (2001), we shall interpret (38) as a ‘moving’ barrier. Such an interpretation has the advantage that, once the barrier shape is known, the excursion set formalism can be used to quantify the dependence of halo properties on environment. Moreover, it is computationally more efficient that studying the first crossing distribution of multidimensional random walks (e.g. Chiueh & Lee 2001; Sheth & Tormen 2002; Sandvik et al. 2007).

4.3.3 Mass scale–ellipticity relation

Before we examine the impact of this dynamical interaction on the properties of collapsed haloes, we need to express the critical density (38) in terms of the halo mass $M_1$. Thus far, we have only considered the collapse of regions with initial radius $R_1 = 2 h^{-1}$ Mpc. The top panel of Fig. 8 shows the distribution of critical densities $\delta_{ec}$ for three different values of $R_1$. The average density for collapse is plotted as a function of the environment density $\delta_0$. Figure 8 shows the distribution of critical densities $\delta_{ec}$ for three different values of $R_1$. The average density for collapse is plotted as a function of the environment density $\delta_0$. The crosses indicate the actual values of the critical collapse density with ellipticity and to its dependence on both the redshift and the environment density.
as the solid curve. Results are shown for an environment density \( \delta_0 = 0 \) only. Note, however, that we have repeated this calculation for other values of \( \delta_0 \) and found good agreement between the critical densities of individual realizations and the approximation (38). This confirms that most of the environmental effect seen here arises from variation in the large-scale density \( \delta_0 \). At fixed \( R_0 \), changing \( R_1 \) merely affects the scatter around the collapse boundary \( B(e_1, z) \), unsurprising since the functions \( b_1(\tau) \) that characterize the potential of the protohalo and its environment are independent of \( R_0 \) and \( R_1 \).

The decrease in scatter with increasing \( R_1 \) is a direct consequence of the statistical correlation explored in Section 3. This is clearly seen in the bottom panel of Fig. 8, where the distribution of initial shear ellipticity and prolateness is plotted as a function of \( R_1 \). The filled symbols indicate the actual, most probable values \( (v_m, p_m) \).

They increase (monotonically) with the protohalo radius \( R_1 \). This suggests relating the mass scale \( M_1 \) to the expectation value of the asymmetry parameters.

In line with the interpretation of Sheth et al. (2001; see also Shen et al. 2006; Sandvik et al. 2007), we use the average values \( \langle \rangle \) to translate the product \( e_1 \delta_1 \) into a peak height \( v(M_1, z) = \delta_a(z)/\sigma(M_1) \).\(^2\) Gao & White (2007) have shown that the properties of haloes in the Millennium Simulation obey this scaling relation over a large redshift range. In terms of this scaled variable, the collapse boundary equation (38) becomes

\[
B(v, z) = \delta_a(z) \left[ 1 + \beta_1(v^{1/2}) \right].
\]

Sheth & Tormen (2002) provide an analytic fit for the first crossing distribution associated to this barrier shape, which allows us to easily calculate halo properties such as formation redshift and bias.

### 4.4 Environmental dependence of halo properties

The main focus is to quantify the environmental dependence of halo properties arising from the moving barrier, equation (38). Even though the ellipsoidal collapse model does not provide an excellent description of the simulations, it does nevertheless a much better job than the spherical collapse (Sheth & Tormen 1999c, 2002). Therefore, the validity of a comparison between predicted and observed environmental effects should be preserved. We calculate the distribution halo of halo formation redshift and large-scale bias associated to that collapse boundary. We end this section with a discussion of the halo spin parameter.

#### 4.4.1 Environment density and formation redshift

The halo formation redshift \( z_{\text{form}} \) is commonly defined as the epoch at which the main progenitor has accumulated half of its final mass. According to the EPS theory, the probability that the formation redshift of present-day haloes of mass \( M \) (we will henceforth drop the subscript \( 1 \)) is larger than \( z \) is given by \( \text{(Lacey & Cole 1993)} \)

\[
P(>z_f) = \int_z^{z_f} \frac{M}{M^*(S')} f(S'|S),
\]

where \( S_f = 5(M/2) \). When the excursion set theory is combined to the ellipsoidal collapse dynamics, the conditional first crossing distribution \( f(S'|S) \) shall be replaced by the following analytic formula:

\[
f(S'|S) = \frac{|T(S'|S)|}{\sqrt{2\pi(S-S')/2}} \exp \left[ -\frac{(B(S') - B(S))^2}{2(S-S')} \right],
\]

where

\[
T(S'|S) = \sum_{n=0}^{\infty} \frac{(S-S')^n}{n!} \frac{\partial^n}[B(S') - B(S)] \frac{\delta^m}{\delta S^m}.
\]

This Taylor expansion provides a good fit to the conditional up-crossing probability for moving barriers of the form (41) (Sheth & Tormen 2002). Note that the ellipsoidal collapse model predicts too many haloes with high formation redshift. The distributions are also broader than seen in the simulations (Lin, Jing & Lin 2003; Giocoli et al. 2007).

The differential probability distribution \( P(z_{\text{form}}) \equiv dP(>z_{\text{form}})/dz_{\text{form}} \) is plotted in Fig. 9 for various halo mass \( M \) and (Lagrangian) environment density \( \delta_0 \). Clearly, the dependence of halo formation redshift on environment increases with decreasing halo mass. Equation (41) indeed shows that, for haloes of mass \( M \geq M_* \), the critical density for collapse is \( B(v, z) \approx \delta_a(z) \), weakly dependent on environment density. This is the reason why the formation redshift of massive haloes is nearly insensitive to \( \delta_0 \). By contrast, random walks associated to small-mass haloes \( M \ll M_* \) cross the collapse boundary \( B(v, z) \) at relatively larger values of \( v \) and thus induce a stronger environmental effect. For \( M = 0.01M_* \), we find a median formation redshift of \( z_{\text{form}} = 1.19 \) for an environment density \( \delta_0 = 1 \). This should be compared to \( z_{\text{form}} = 1.42 \) when \( \delta_0 = -2 \).

Note that the probability distribution \( P(z_{\text{form}}) \) is more sensitive to the exponent \( \beta_1 \) than the multiplicative factor \( \beta_1 \). The former contributes about two thirds of the environmental effect (see Section 5.1 for more details).
To allow for a direct comparison of our results with the analyses of N-body simulations, we need to express the environmental dependence of halo formation redshift in terms of the evolved Eulerian density. Mo & White (1996) and Sheth (1998) have shown how this may be accomplished within the spherical collapse model. The spherical collapse dynamics provides a relation between the Lagrangian radius $R_0$ and density $\delta_0$ and their Eulerian counterparts $R$ and $\delta$. In this model, the mass interior to each perturbation is constant, giving $R_0 = R (1 + \delta)^{1/3}$ if one assumes that the primeval density fluctuations are small. Furthermore, the linear density $\delta_0$ is a monotonically increasing function of the present overdensity fluctuation $\delta$ only. Mo & White (1996) have obtained an accurate approximation to this relation for an EdS universe,

$$\delta_0(\Delta) = \frac{\delta_0}{1.686} \left[ 1.686 - \frac{1.35}{\Delta^{3/2}} - \frac{1.124}{\Delta^{1/2}} + \frac{0.788}{\Delta} \right], \quad (45)$$

where $\Delta \equiv 1 + \delta$. This interpolation formula is also valid in the $\Lambda$CDM cosmology considered here provided that $\delta$ is not too large ($\delta \lesssim 10$). Ideally, we should calculate the relative numbers of patches with $(R_0, \delta_0)$ that have now evolved into regions $(R, \delta)$ (see Sheth 1998). We should also take into account the triaxiality of the large-scale environment in the conversion of $\delta$ into $\delta_0$. Henceforth however, we will neglect these complications and use the spherical approximation to relate the Lagrangian density to the Eulerian density at $z = 0$. This is sensible since, as we have seen, the linear overdensity $\delta_0$ is the key parameter governing the correlation between collapse densities and environment. For illustration, the Lagrangian density is in the range $-3 \lesssim \delta_0 \lesssim 1$ when the Eulerian density varies between $0.3 \leq (1 + \delta) \leq 5$.

Upon these assumptions, we find that the median formation redshift changes by

$$\Delta \bar{z}_{\text{form}} \approx 0.07 \quad \text{when} \quad M = M_\ast,$$

$$\Delta \bar{z}_{\text{form}} \approx 0.33 \quad \text{when} \quad M = 0.01 M_\ast, \quad (46)$$

when the evolved density varies in the range $-0.7 \leq \delta \leq 4$. Again, the statistical correlation induces an effect greater for low-mass haloes $M \leq M_\ast$ while, for $M \gg M_\ast$, the offset in median formation redshift is barely discernible. In addition, the difference is of the magnitude seen in N-body simulations, where $\Delta \bar{z}_{\text{form}} \sim 0.3 - 0.5$ at $M \sim 0.01 M_\ast$ for a large-scale overdensity varying in that same range (Harker et al. 2006). On the other hand, our model predicts that haloes in denser regions have a lower formation redshift than those in less dense regions. This is opposite to the behaviour reported by Harker et al. (2006), who find that, in overdense regions, haloes in the densest environment assemble earliest. A better treatment of the relation between Lagrangian and Eulerian regions should not affect these conclusions.

Interestingly, however, these authors find that, in the most underdense regions $\delta \lesssim -0.4$, the average formation redshift increases with decreasing environment density. Despite the small number of haloes and the large scatter in formation redshift, the trend is robust, yet smaller than the effects present in the high-density regions (Geraint Harker, private communication). This may be a manifestation of the environmental dependence discussed here.

### 4.4.2 The age dependence of halo bias

Having established a correlation between formation redshift and environment density, we now turn to the age dependence of the halo bias.

Mo & White (1996) and Sheth & Tormen (1999c) have shown that there is a direct relation between the halo bias and the shape of the collapse barrier. For a barrier of the form (41), the large-scale bias in Eulerian space can be approximated by (Sheth et al. 2001)

$$b(\nu) = 1 + \frac{1}{\delta_0} \left[ \nu^2 + \beta_1 \nu^{2-2\nu} \right] - \frac{\nu^{2\nu}}{\nu^{2\nu} + \beta_1 (1 - \beta_2)(1 - \beta_2/2)}, \quad (47)$$

This bias relation is plotted in Fig. 10 as a function of the peak height $\nu$. The dashed curve shows the halo bias at mean Eulerian density $\delta = 0$, namely in the case of the ellipsoidal collapse of Sheth et al. (2001), while the dotted line is the prediction of the spherical collapse. The shaded area indicates the amplitude of the large-scale bias when the Eulerian environment density varies in the range $-0.7 \leq \delta \leq 4$. The dashed curve is $b(\nu)$ when $\delta = 0$, whereas the dotted line indicates the prediction of the spherical collapse. The upper and lower solid curves show the bias factor of haloes which have relatively high and low formation redshifts (see text for details). Haloes that assemble relatively early are more clustered and populate the less dense regions.

![Figure 10](https://academic.oup.com/mnras/article-abstract/388/2/638/977306)
our ‘old’ (high $z_{\text{form}}$) and ‘young’ (low $z_{\text{form}}$) haloes. Of course, this is a crude approximation to the 10 (20) per cent tails considered by Gao et al. (2005), but we have not found any better alternative.

In Fig. 10, we show the resultant bias relations as the solid curves labelled by ‘low $z_{\text{form}}$’ and ‘high $z_{\text{form}}$’. The relative bias of our old versus young haloes increases smoothly with decreasing halo mass. The effect becomes large for $n \lesssim 1$ because of the considerably stronger dependence of the barrier shape on the peak height. In this regime, the large-scale bias of the old haloes is $\sim 50$ per cent larger than for the young ones, an increase roughly twice as small as seen in the simulations. Overall, the behaviour is similar to that reported in Gao et al. (2005), where the correlation between halo clustering and formation history is strong for haloes less massive than $M_\star$ only. A better modelling of the properties of haloes lying in the tails of the formation redshift distribution is required to make a more quantitative comparison between our predictions and the measurements of Gao et al. (2005). The point of the present analysis is to show that the moving barrier (41) could lead to a correlation similar to that seen in $N$-body simulations if the scatter in collapse density is taken into account.

In contrast to the behaviour seen in $N$-body simulations, the predicted bias $b(n)$ decreases with increasing environment density, reflecting the flattening of the barrier shape for large values of $\delta$. In the limit $\delta \gg 1$, $b(n)$ tends towards the value predicted by the spherical collapse. Since, on average, dense regions at the present time formed from relatively dense regions in the primeval fluctuation field, an anticorrelation between halo bias and large-scale Eulerian density is expected in our model. This seems at first surprising but, as recognized by Abbas & Sheth (2007), the large-scale bias is not necessarily a monotonically increasing function of environment density. Their results strongly suggest that haloes in extremely underdense environment are more clustered than the mean.

4.4.3 Alignment of halo spin parameter

The dimensionless spin parameter $\lambda$ measures the amount of rigid rotation acquired by the triaxial perturbation before virialization. It is defined as (Peebles 1969)

$$\lambda = \frac{L^2}{GM^2/2},$$

where $L = L \hat{L}$ is the halo angular momentum, $\hat{L}$ is a unit vector, and $E = E_\text{pot} + E_\text{kin}, M$ are the total energy and mass of the ellipsoid, respectively. The quantities $L$ and $E$ can be expressed in terms of the matrix elements $A^{ij}$ (Peebles 1980; Binney & Tremaine 1987; Eisenstein & Loeb 1995). In general, the energy $E$ of the collapsing region is not conserved, notably because the kinetic energy of the ellipsoid is altered when an axis collapses. For simplicity, we use the last value of the energy before the first axis collapses in the calculation of $\lambda = |\lambda|$ (Eisenstein & Loeb 1995). This is a good approximation as the change in total energy is usually small during the collapse. Note also that the resulting error should be relatively small because $\lambda$ depends on $\sqrt{E}$ solely.

The top panel of Fig. 11 illustrates the effect of including the correlation between the initial orientation of the protohalo and its large-scale environment on the probability distribution of the spin parameter $\lambda$. The left-hand histogram shows $P(\lambda)$ when the correlations in the initial alignment are included, the right-hand histogram when they are not. Bottom panels: probability distribution of the cosine of the angle between the angular momentum of the collapsed halo and both filament direction (bottom left-hand side) and sheet normal vector (bottom right-hand side). A random distribution would be a flat line at $P(\cos \theta) = 1$. Results are shown for haloes that collapse at $z = 0$. The histograms were drawn from $\sim 10^3$ random realizations of the initial conditions.

Our median spin value is $\lambda_{\text{med}} \approx 0.005$, an order of magnitude lower than those found in numerical simulations, where $\lambda_{\text{med}} \sim 0.03-0.05$ for haloes of mass $M \gtrsim 10^7 M_\odot h^{-1}$ (Barnes & Efstathiou 1987; Bullock et al. 2001; Bett et al. 2007). There are several reasons for this discrepancy. First, the number density of haloes of a given mass depends noticeably on the environment density $\delta$: haloes are preferentially found in mildly overdense regions. Here, however, the evolution of density $\delta$ associated to the random realizations of Fig. 11 is not representative of a fair halo sample: it peaks around $\delta = 0$, where the average spin at collapse is much lower than in high-density regions $\delta \gtrsim 1$. Secondly, the angular momentum of a collapsing region satisfies the equation

$$\dot{\lambda}_i = -H^{-1} e_{ij} T^{ij} I^j,$$

where $T^{ij} = (3/2) H^2 \Omega_0 (\Phi_{\text{kin}} + \Phi_{\text{pot}})$ is the torque and $I^j = (1/5) M (\text{AA}^{-1})^j$ is the inertia tensor. We note that the angular momentum vanishes to first order if the inertia tensor is zero at initial time, or if $L$ and $T$ are perfectly aligned (White 1984; Catelan & Theuns 1996). In this limit, the growth of angular momentum is of second order only, $L \propto A^{5/2}$, and is dominated by non-linear effects after turnaround (Peebles 1969). This is indeed the case here as we are considering perturbations that are initially spherical. A more realistic treatment should include first-order tidal torquing. Thirdly,
our axis collapse condition strongly reduces the amount of angular momentum gained by the halo after the second axis collapses. A different prescription, such as the one adopted by Eisenstein & Loeb (1995), would increase the magnitude of the spin parameter at third axis collapse. Finally, analytic calculations indicate that a significant fraction of the angular momentum is acquired through the collapse of the outermost shells (Ryden 1988; Quinn & Binney 1992). This can only be taken into account by a detailed modelling of the matter density and velocity profiles around the collapsing haloes.

In spite of these limitations, it is worthwhile looking at the alignment between halo spin and environment as this quantity does not depend on the magnitude of the angular momentum gained during the collapse. In the bottom panels of Fig. 11, the histogram shows the alignment distribution at collapse time. The angle is measured between the angular momentum of the collapsing region and the symmetry axis/plane of the mass distribution. Clearly, haloes show a strong tendency to have their spin aligned perpendicularly to the filament or parallel to the mass sheet. This is in good agreement with the findings of Hahn et al. (2007) and Aragón-Calvo et al. (2007a), although the former authors found a clear correlation for haloes residing in sheet-like structures only and the latter a mass-dependent spin orientation in filaments (see also Trujillo, Carretero & Patiri 2006; Sousbie et al. 2008). More precisely, the alignment is strongest along the second principal axis of the shear tensor. This owes to the fact that, once the first axis has collapsed, \( L_1 \sim \epsilon_{ijk} \Phi^i_{E,\theta} F^j \). The growth rate is largest for the intermediate axis, \( L_2 \propto (a_2 - a_3) F^3 \), since the difference \( a_2 \approx a_3 \) dominates the other two (see also Lee & Pen 2000). It is unclear whether the initial alignment between inertia and deformation tensors reported by Lee & Pen (2001) and Porciani et al. (2002b) can produce a similar correlation. We have not investigated this issue any further.

To summarize, we have demonstrated that our simplified model, which takes into account both the dynamical and statistical aspects of the ellipsoidal collapse, produces a clear correlation between formation redshift, large-scale bias and environment density. The strength of the effect is of the same magnitude as seen in simulations. It is largest for low-mass haloes, \( M \ll M_{\star} \), and fades as we go to high masses, \( M > M_{\star} \). Haloes that formed at high redshift are substantially more clustered than those that assembled recently. This is precisely the behaviour reported by Gao et al. (2005). On the other hand, this model predicts a negative correlation between formation redshift and environment density, in contradiction with the trend measured by Harker et al. (2006). However, simulations indicate that halo properties depend on environment in a complex way: in relatively underdense regions, the average formation redshift increases with decreasing environment density (Harker et al. 2006) while haloes are more strongly clustered than the mean (Abbas & Sheth 2007). Our model produces an effect in the right sense. This suggests that the ellipsoidal collapse may apply in underdense regions where non-linear effects are weak or absent.

5 DISCUSSION

In this section, we discuss the sensitivity of the environmental effects considered above to the shape of the collapse barrier, as well as non-Markovianity and tidal interactions as potential sources of environmental dependence.

\( ^3 \) Here the \( a_i \) values are the (ordered) eigenvalues of the large-scale potential \( \Phi^i_{E,\theta} \).

5.1 Sensitivity to the barrier shape

The strength of the environmental effect explored in Section 4.4 may somewhat depend on the parametrization adopted for the moving barrier. While this is probably true for haloes of mass \( M > M_{\star} \), the detailed shape of the barrier should have a little impact when \( M \lesssim M_{\star} \). In this mass range, the average collapse density \( B(v, z) \), a Taylor series without loss of generality, should be dominated by the term of largest degree. For the simple case considered here, \( B(v, z) \propto \beta_1 v^{-3\delta_1 \phi} \) when \( v \gg 1 \).

The exponent \( \beta_2 \) has a great influence on the correlation between formation redshift and environment density. This is clearly seen in Fig. 12, where contours of constant \( z_{\text{form}} \) are plotted for a halo mass \( M = 0.1 M_{\star} \). Shown for illustration are the median formation redshifts of two extreme cases: \( \delta_0 = 1 \) and \( -3 \) (filled symbol) when \( \beta_1 \) and \( \beta_2 \) assume the functional form (39). The effect of varying \( \beta_1 \) vanishes along a critical line \( \beta_1 = \beta_2 \), where \( \beta_2 \approx 0.58 \) (dotted curve).

Regarding the environmental dependence of halo bias, note that, in the limits \( v \gg 1 \) and \( v \ll 1 \), the bias offset is \( \Delta b(v) \approx -2 \beta_1 \Delta \beta_2 \ln v v^{2-2\phi}/\delta_0 \), to first order in \( \Delta \beta_1 \) and \( \Delta \beta_2 \). This regime, changing the value of \( \beta_2 \), has a large effect on the bias of haloes lying at the extreme ends of the mass range. However, unlike the environmental dependence of formation redshift, it is the parameter \( \beta_1 \) that influences most the bias when the halo mass is in the range \( 0.1 \lesssim M/M_{\star} \lesssim 0.1 \). We also note that the fitting formula (47) of Sheth et al. (2001) holds for \( \beta_2 \) strictly less than one only. It would be prudent to check again their approximation, namely derive the first crossing distribution from a large ensemble of random...
trajectories when the values of $\beta_1$, $\beta_2$ differ significantly from those at mean density ($\delta_0 = 0$).

We have shown that each of the shape parameters $\beta_1$ and $\beta_2$ has a distinct impact on the halo formation redshift and large-scale bias. Therefore, an environmental dependence of formation redshift and bias will be present regardless the exact values of these variables.

5.2 Environmental effect from non-Markovianity

In the excursion set formalism, a spherical symmetric window function $\hat{W}(R, k)$ is used to define the trajectories of the linear density field $\delta(R)$ as a function of smoothing scale. When the sharp $k$-space filter is adopted, $\delta(R)$ executes a random walk. The property of Markovianity has been exploited extensively to obtain analytic expressions for the first crossing distribution, etc. This is the reason why we have considered that particular window function in the computation of halo formation redshift and bias. However, the Markov nature of Brownian walks prevents any correlation between large-scale environment and assembly history (see White 1996, for a discussion). Halo merger trees are indeed non-Markovian across short time steps (Neistein & Dekel 2008). A natural way to introduce correlations would be to use another window function. This possibility has been considered by several authors (e.g. Peacock & Heavens 1990; Bond et al. 1991; White 1996; Nagashima 2001; Schücker et al. 2001; Amosov & Schücker 2004; Zentner 2007).

Here we argue that non-Markovianess should lead to an effect that is larger for high-mass haloes.

For concreteness, let us consider the collapse of a perturbation of comoving size $R_1$. To investigate the impact of environment on its formation history, we restrict ourselves to trajectories that obey the following two constraints:

$$C_0 : \delta(R_0) = \nu_0 \sigma_0,$$

$$C_1 : \delta(R_1) = \nu_1 \sigma_1,$$  \hspace{1cm} (50)

where, again, $\sigma_1 = \sigma(R)$ and $R_0 = 10 h^{-1}$ Mpc is the scale of the large-scale environment. We neglect triaxiality and choose $\nu_1 = \delta_{\text{sc}} / \sigma_1$, to ensure that the halo has just collapsed by redshift $z = 0$. $\nu_0$ can be positive or negative, depending on whether the halo forms in a high- or low-density region.

We can calculate the most probable trajectory $\bar{\delta}(R)$ given the constraints $\{C_i\}$. Since these are not Markovian, the probability of realizing $\delta(R)$ can be expressed as a shifted Gaussian around an ensemble mean field (see Adler 1981; Hoffman & Ribak 1991; Van de Weygaert & Bertschinger 1996, for a rigorous treatment)

$$\bar{\delta}(R) = \zeta_i(R) \zeta_j^{-1} c_j,$$  \hspace{1cm} (51)

where $\zeta_i(R) = \langle \delta(R) C_i \rangle$ is the cross-correlation between the field and the $i$th constraint, and $\zeta_j = \langle C_i C_j \rangle$ is the constraints’ correlation matrix. The residual field $\tilde{\delta} = \bar{\delta} - \bar{\delta}$ is a Gaussian random field which is not homogeneous nor isotropic, but whose statistical properties are independent of the $\{C_i\}$ (Hoffman & Ribak 1991). We define a normalized cross-correlation between the residual field $\zeta_i$ and the field $\bar{\delta}$, as

$$\zeta_i(R) = \frac{1}{\sigma_1} \int_0^\infty d \ln k \Delta_\zeta(k) \tilde{W}(R, k) \tilde{W}(R, k),$$  \hspace{1cm} (52)

where $\sigma = \sigma(R)$ and $\tilde{W}(R, k)$ is the top-hat filter. The calculation of the matrix elements $\zeta_{ij}$ is immediate. The mean field $\bar{\delta}$ can be expressed as

$$\bar{\delta}(R) = \frac{\nu_0 \sigma}{(1 - \gamma^2)} \left\{ \zeta_1 \left( \frac{v_1}{w_0} - \gamma \right) + \zeta_0 \left( 1 - \gamma \frac{v_1}{w_0} \right) \right\}.$$  \hspace{1cm} (53)

This difference is negative (positive) when the smoothing radius is $R < R_c(R > R_c)$. In other words, the density profile around $\delta(R)$ is steeper in low-density regions. Note that, for a sharp $k$-space filter, $\delta_+ - \delta_-$ vanishes when $R < R_c$ since $\zeta_1 = \tilde{c}_0 / \gamma$, but is generally non-zero in the range $R > R_c$.

Fig. 13 shows the mean field $\bar{\delta}(R)$ for the top-hat filter. $\bar{\delta}(R)$ is plotted as a function of the smoothing radius for two different halo mass $M_1 = 2.7 \times 10^{11}$ and $3.4 \times 10^{11} M_\odot h^{-1}$. On larger scale $R_0 = 10 h^{-1}$ Mpc, the density contrast is set to $\nu_1 = 1$ (dashed curves) and $\nu_- = -1$ (solid curves). To ensure that the trajectories $\delta(R)$ describe the formation history of $M = M_1$ haloes, we should also have constrained the density so that $\delta(R)$ does not cross the barrier $\delta = \delta_{\text{sc}}$ on scale larger than $R_1$. Here we ignore this constraint since its implementation is not straightforward when the window function differs from a sharp $k$-space filter. This is the reason why a substantial fraction of the trajectories penetrate the barrier $\delta = \delta_{\text{sc}}$ on scale $R > R_c$. This caveat notwithstanding, the present calculation is adequate to understand, at least qualitatively, the effect of non-Markovianity. Note that the difference $\delta_+ - \delta_-$ is approximately constant throughout most of the halo formation history ($R/R_c < 1$). Most importantly, the effect is about twice as large for the high-mass halo while, at fixed $R/R_c$, the scatter of the residual field is lower by $\sim 60$ per cent.

This suggests that the environmental dependence which arises from non-Markovianess is stronger for high-mass haloes. This has also been pointed out by Zentner (2007). Therefore, it is unlikely to explain the trend seen in overdense regions where, undoubtedly, correlations are stronger for low-mass haloes (e.g. Gao & White 2007). However, it may apply for isolated haloes in voids or underdense regions. It should also leave a signature distinct from the ellipsoidal...
collapse which, as we have seen, induces a stronger dependence for low-mass haloes.

5.3 On tidal interactions

Irregularities in the mass distribution induce non-radial motions that slow down the collapse (Peebles & Groth 1976; Davis & Peebles 1977; Peebles 1990). This ‘pre-virialization’ conjecture is supported by the numerical investigations of e.g. Barrow & Silk (1981), Szalay & Silk (1983), Villumsen & Davis (1986), Lokas et al. (1996); and by the analytic calculations of Del Popolo & Gambera (1998) and Del Popolo, Ercan & Xia (2001), which indicate that tidal heating can counterbalance the effect of the shear and delay the collapse. Recently, Avila-Reese et al. (2005), Maulbetsch et al. (2006), Wang et al. (2006) and Diemand, Kuhlen & Madau (2007) have proposed that the assembly bias seen in $N$-body simulations originates from tidal interactions with a larger neighbour. They have shown that, at late time, the tidal field of massive neighbouring clumps halts the growth of haloes and, in many cases, even reduces their mass. Furthermore, tidal effects appear to have a larger influence on small-mass haloes. Therefore, this could also explain why the age dependence of clustering is stronger for low-mass haloes.

The large effect measured by Diemand et al. (2007) indicates that, in high-density regions, tidal interactions are likely to overwhelm the environmental dependence arising from anisotropic collapse, and increase the average formation redshift in overdense regions.

Although the impact of tidal stripping can only be rigorously quantified with numerical simulations, the suppression of mass accretion through tidal heating could also be addressed analytically. The spherical collapse model, in which the collapsing object is divided into a series of concentric shells, seems better suited than the ellipsoidal collapse. A thorough discussion of tidal heating is beyond the scope of the present paper. Note, however, that the torque imparted by the (external) mass distribution on a thin spherical shell is $τ(x) \propto \int \Omega dΩ \delta(x) \wedge \nabla \phi(x)$ (e.g. Ryden & Gunn 1987), where $δ(x) = δ(x) - δ(x)$ is the deviation from the spherically symmetric distribution $δ(x)$. This is $δ$ which pulls the infalling matter out of its purely radial motion. In the linear regime, $δ$ is a Gaussian density field statistically independent of the spherical average $δc$. Hence, tidal heating does not induce any environmental dependence at first order. Consequently, modelling the growth of non-linearities in the surrounding mass distribution will be crucial to ascertain analytically the importance of tidal heating in the environmental dependence of halo collapse.

6 CONCLUSION

The ellipsoidal collapse model is an extension of the spherical dynamics that takes into account the anisotropic collapse of triaxial perturbations. This non-spherical dynamics provides a substantially better description of halo statistics such as mass function and large-scale bias. It is, however, unclear whether the ellipsoidal collapse can induce environmental effects similar to that seen in $N$-body simulations.

In this paper, we have attempted to address this issue, paying special attention to both the statistical and dynamical origin of the environmental dependence. In a first part, we have explored the statistical correlation that arises in (Gaussian) initial conditions between the local properties of the shear and the configuration of the large-scale environment. To this purpose, a number of joint statistics for the shear tensor have been derived, thereby extending the previous analysis of Doroshkevich (1970). In a second part, we have examined the dynamical aspect of the environmental dependence using a simplified model that takes into account the interaction between a collapsing, ellipsoidal perturbation and its large-scale environment. Relaxing the assumption of sphericity (at the heart of the spherical collapse) introduces a dependence of collapse redshift on environment. The tidal force exerted by the surrounding mass distribution alters the collapse of the major and minor axes, and causes haloes embedded in large overdensities to virialize earliest. We have found that the environment density is a key parameter in determining the virialization redshift, the large-scale asphericity contributing mostly to increase the scatter in collapse density. An effective barrier whose shape depends on the large-scale density provides a good description of this environmental effect. Such an interpretation has the advantage that the EPS formalism can be applied to estimate the environmental dependence of halo properties like formation redshift and large-scale bias.

We have shown that, using this moving barrier approach, a correlation between formation redshift, large-scale bias and environment density naturally arises. The magnitude of the effect is similar, albeit smaller, to that seen in $N$-body simulations. It is large for low-mass haloes $M \ll M_\star$, and fades as we go to high masses $M > M_\star$ as a result of a genuine statistical effect, namely the decrease in average asymmetry and stochasticity with increasing halo mass. Haloes that formed at high redshift are found to be more clustered than those that assembled recently. This is precisely the behaviour reported by Sheth & Tormen (2004) and Gao et al. (2005). However, haloes in denser regions are predicted to assemble later. This result is inconsistent with the trend measured in overdense environments $δ > 0$. It calls into question the role of the ellipsoidal collapse in shaping the halo mass function. Nevertheless, several lines of recent evidence indicate that, in relatively underdense regions, the average formation redshift increases with decreasing environment density (e.g. Harker et al. 2006), while haloes may be more strongly clustered than the mean (Abbas & Sheth 2007). Our model predicts and effect in the right sense. This suggests that the ellipsoidal collapse model may be applicable in underdense regions, where tidal interactions are weak or absent. Conditional halo mass functions $n(M|δ)$ could provide another testable prediction since, in the moving barrier interpretation adopted here, it is expected that $n(M|δ)$ in underdense regions should be (slightly) biased towards high-mass haloes as compared to the prediction of Sheth & Tormen (2002).

Recently, Sandvik et al. (2007) have discussed a multidimensional extension of the EPS formalism that takes into account ellipsoidal collapse (Chiueh & Lee 2001). They find a very weak correlation between halo assembly history and environment, presumably because their implementation includes only the statistical aspect of environmental effects. In our scenario, the dynamical interaction between the external mass distribution and the collapsing halo plays a crucial role in the environmental dependence of halo properties. The statistical correlations contribute mostly to increase the strength of the effect with decreasing halo mass. Keselman & Nusser (2007) have shown that the age dependence of halo bias persists in a simplified description of gravitational dynamics, suggesting thereby that (at least some of) the environmental dependence arises in the early stages of the collapse. They find that halo formation redshift strongly correlates with the dimensionless parameter $η \propto (1 + 6c_τ^2 + 2p_τ^2)^{-1/2}$, young haloes forming from fluctuations with higher $η$ than old ones. They argue that this follows from the dependence of first axis collapse on configuration shape, namely a planar perturbation collapses faster than a spherical perturbation with the same initial density (Bertschinger & Jain 1994). Their results are, however, difficult to reconcile with the above
interpretation since it implies that, for a fixed halo mass, haloes forming in regions of larger \( \eta \) are older (because they grow less rapidly). This discrepancy can be alleviated if one assumes that the formation of virialized haloes corresponds to the collapse of the third axis. In this case, the collapse is always faster for spherical perturbations (Audit et al. 1997) so that young haloes tend to form in regions of higher \( \eta \), i.e. more spherical or denser environments, in agreement with our findings.

A serious shortcoming of our model is the neglect of anisotropies beyond the quadrupole term in the external density field. Furthermore, apart from a global rotation, it ignores the non-radial degrees of freedom in the collapsing protohalo. It would be of great interest to ascertain whether non-radial motions created by a clumpy, growing large-scale mass distribution can induce an effect similar to that seen in simulations. A priori, tidal heating should be more efficient in high-density environments. It may plausibly reverse the trend found in this paper, namely increase the critical collapse density in large-scale overdensities. If this happens to be true, a moving barrier approach would naturally predict haloes in dense regions to assemble relatively early and to be more strongly clustered than the mean. Alternatively, Wang, Mo & Jing (2007) have suggested that tidal heating causes haloes to appear less massive than expected from their initial density field. This may also provide an explanation for the relatively large/low bias of old/young haloes. Clearly, large \( N \)-body simulations are needed to understand the complex relations reported by e.g. Harker et al. (2006), Wechsler et al. (2006) and Gao & White (2007). This caveat notwithstanding, analytic models can provide an elegant route to capturing the essential features of the environmental dependence.

In standard galaxy formation models, the correlation between the haloes and their large-scale environment introduces a correlation between the properties of galaxies and the regions they occupy (e.g. Navarro, Abadi & Steinmetz 2004; Abbas & Sheth 2005, 2006; Berlind et al. 2005). The observational results of Skibba et al. (2006), Blanton et al. (2006) and Tinker et al. (2007) support this prediction and leave little room for a galaxy assembly bias (see however Croton, Gao & White 2007). Nevertheless, the influence of the environmental dependence of halo properties on the galaxy population remains unclear. It would be valuable to assess whether environmental effects produced by the anisotropic collapse of haloes can leave a detectable signature in the properties of field galaxies.

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\[ \frac{1}{2} \int_{-1}^{1} dx_1 \mu \frac{e^{i k r}}{1 - e^{i k r}} = \frac{1}{3} j_0(kr) - \frac{2}{3} j_2(kr) \]

\[ \langle \xi_{ij}(x) \rangle = P_1(k) \hat{k}_i \hat{k}_j \hat{k}_n, \]

\[ \langle \xi_{ij}(k) \xi_{lm}(k) \rangle = P_2(k) \hat{k}_i \hat{k}_j \hat{k}_l \hat{k}_m, \]
where $j_i(x)$ are spherical Bessel functions of the first kind. With this information, the functions $\Psi_i$ may be conveniently expressed as

$$
\Psi_1(r) = \int_0^\infty d\ln k \, \Delta^2(k) \, j_0(kr)
$$

$$
\Psi_2(r) = \int_0^\infty d\ln k \, \Delta^2(k) \, \left[ -\frac{1}{2j_2(kr)} \right]
$$

$$
\Psi_3(r) = \int_0^\infty d\ln k \, \Delta^2(k) \, \left[ \frac{1}{12} j_1(kr) + \frac{2}{21} j_3(kr) + \frac{1}{35} j_5(kr) \right].
$$

In the limit $r \to 0$, both $\Psi_1$ and $\Psi_3$ vanish, but $\Psi_2 = \xi_0/15$ (Fig. A1). Note that the functions $\Psi_i$ can also be cast in the form given in e.g. Lee & Pen (2001), Crittenden et al. (2001) and Catelan & Porciani (2001), which involves $J_0 \equiv n^{\alpha} \int_0^\infty d\xi \, \xi^{\alpha-1}$ (this is best seen by integrating $J_0$ by part).

**APPENDIX B: AN INTEGRAL OVER THE SO(3) MANIFOLD**

In this appendix, we present the calculation of the integral equation (19) over the special orthogonal group SO(3), where $\alpha$ and $\lambda$ are real symmetric matrices.

Let $R \equiv R_\phi$ be the rotation by an angle $\phi$ about the axis $\hat{h}$. Since the Haar measure $dR$ is invariant under both left- and right-hand transformations, we can redefine $R$ so as to diagonalize $\lambda$ and $\alpha$. We will therefore assume that the matrices $\lambda$ and $\alpha$ are diagonal, $\lambda = \text{diag}(\lambda_i)$ and $\alpha = \text{diag}(\alpha_i)$. It is worth noting that $\alpha$ and $\lambda$ are not orthogonal matrices, but belong to the (real) linear group GL(3, R). Consider now the rotation of $\pi$ around the first axis, $\sigma_1 = \text{diag}(1, -1, -1)$, and its distinct permutations $\sigma_2 = \text{diag}(-1, 1, -1)$ and $\sigma_3 = \text{diag}(-1, -1, 1)$. Since the compositions $\sigma_1 R$ characterize equivalent principal axis frames, the integral (19) should in principle be performed over the quotient space SO(3)/(\{\sigma_1, \sigma_2, \sigma_3\}). However, the $\sigma_i$ values are diagonal and we find

$$
\text{tr} \left[ (\sigma_i R) \lambda (\sigma_i R)^\top \alpha \right] = \text{tr} \left( R \lambda R^\top \alpha \right).
$$

Therefore, one can also perform the integration over the whole orthogonal group SO(3) and multiply the final result by 1/4.

Integrals of the form

$$
F(\alpha, \lambda) = \int_R d\Omega \, e^{i\alpha(\Omega R^\top \lambda \Omega)},
$$

(B2)

where $G (\Omega \in G)$ is a compact Lie group and $\alpha, \lambda$ belong to the linear group GL, appear in quantum field theory, chemistry as well as in harmonic analysis (e.g. Hua 1963; Itzykson & Zuber 1980; Wei & Eichinger 1989). When $G$ is the unitary group U(n), group theoretical techniques such as character expansion can be used to obtain a closed form determinantal evaluation of (B2) (e.g. Harish Chandra 1958; Balantekin 2000). Unfortunately, these methods cannot be easily applied when $G$ is the orthogonal group SO(N), essentially because the representations of SO(N) and GL(N, R) are very different. Instead, one generally relies on a specific parametrization of the rotation matrices and express the result in terms of a series of orthogonal functions.

Here we follow the approach outlined in Wei & Eichinger (1990) and parametrize the special rotation matrices $R$ in terms of the Euler angles

$$
\psi = 2\pi n, \theta \leq \alpha \leq \pi.
$$

(R

(B3)

where $c_\psi = \cos \psi$, $s_\psi = \sin \psi$, etc. Note that this representation becomes singular when $\theta = 0$ or $\pi$ [such singularities are expected owing to the topology of SO(3)]. The original integral (19) over the SO(3) manifold can be written as a triple integral with the (normalized) invariant measure $dR = 1/(8\pi^2) \sin \theta d\psi d\theta d\psi$,

$$
F(\alpha, \lambda) = \int_{0}^{2\pi} d\theta \int_{0}^{\pi} \sin \theta d\psi \int_{0}^{2\pi} d\psi e^{i\alpha(\Omega R \Omega^\top \lambda \Omega)}. \quad (B4)
$$

Since the integral is invariant under the transformations $(\alpha, \lambda) \rightarrow (k \alpha, \frac{1}{k} \lambda)$ and $(\alpha, \lambda) \rightarrow (\alpha + k I, \lambda - \frac{k}{\sin \theta} \sin \theta I)$, where $k$ is a real number and $I$ is the 3 x 3 identity matrix, $F(\alpha, \lambda)$ can be expressed as a function of four variables instead of the original six matrix eigenvalues. This can also seen by rewriting the trace as (Wei & Eichinger 1990)

$$
\text{tr} \left( R \lambda R^\top \alpha \right) = \frac{1}{3} \left( \text{tr} \lambda - \text{tr} \tilde{\lambda} - \text{tr} \tilde{\lambda} \right) + \text{tr} \left( R \lambda R^\top \alpha \right).
$$

where, for instance, $\tilde{\alpha} = \text{diag}(\alpha_{11}, \alpha_{22}, 0)$, $\tilde{\lambda} = \text{diag}(\lambda_{11}, \lambda_{22}, 0)$, $\alpha_{ij} = \alpha_{ij} - \alpha_{ji}$ and $\lambda_{ij} = \lambda_{ij} - \lambda_{ji}$. We enforce the ordering $\alpha_i \geq \alpha_k > 0$ and $\lambda_i \geq \lambda_k > 0$, so that $\alpha_{11}, \alpha_{22}, \lambda_{11}, \lambda_{22} \geq 0$. The integral depends now on four distinct combinations of the $\alpha$ and $\lambda$ values: $\alpha_+ = (1/3) \text{tr} \lambda$, $\alpha_- = (1/3) \text{tr} \tilde{\lambda}$, $\alpha_\lambda = \alpha_{12}/\text{tr} \tilde{\lambda}$, $\alpha_\tilde{\lambda} = \lambda_{12}/\text{tr} \tilde{\lambda}$. With this parametrization, $-\infty < \alpha_+ \leq \alpha_- \leq \infty$, $\alpha_+ \geq 0$ and $0 \leq \alpha_\lambda, \alpha_\tilde{\lambda} \leq 1$.

We can expand the trace in terms of the Wigner D-functions $D^\sigma_{m_\sigma m'_{\sigma'}} \Gamma^I$ being the index of the representation (see e.g. Sakurai 1985). These three-dimensional harmonics generate irreducible representations of the three-dimensional rotation group and, therefore, form a complete orthogonal set of functions defined on SO(3) itself. Unsurprisingly, only the quadrupole ($l = 2$) rotation matrices appear in the trace decomposition.

$$
\text{tr} \left( R \lambda \tilde{R}^\top \alpha \right) = \frac{\epsilon_\lambda}{2} \left[ \frac{2}{3} + D^2_{0,0} \right]
$$

$$
- \frac{3}{2} \epsilon_\alpha \left( D^2_{0,2} + D^2_{0,-2} \right) - \frac{3}{2} \epsilon_\lambda \left( D^2_{2,0} + D^2_{-2,0} \right)
$$

$$
+ \frac{3}{2} \epsilon_\alpha \epsilon_\lambda \left( D^2_{2,-2} + D^2_{-2,2} + D^2_{-2,-2} + D^2_{2,2} \right). \quad (B6)
$$

Environmental dependence

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The explicit form of these \( l = 2 \) harmonics is given in Table B1. Note that \( \text{tr}(\mathbf{R}^T \mathbf{R} \Delta) \) depends on the three ‘shape’ parameters \( \epsilon_-, \epsilon_u \) and \( \epsilon_i \) solely, because points on \( \text{SO}(3) \) truly have only three degrees of freedom. The integral over the variable \( \psi \) can be performed using the following identity (see section 3.937 of Gradsteyn & Ryzhik 2000),

\[
\frac{1}{2\pi} \int_0^{2\pi} d\psi \, \exp(i \psi) \sin^2 \frac{3}{4} g(r, \varphi, \epsilon_u) = I_0(\sqrt{r^2 + y^2}),
\]

where \( I_0 \) is a modified Bessel function of the first kind. The result can be written

\[
F(\alpha, \lambda) = e^{\delta_+} W(\beta \epsilon_-, \epsilon_u, \epsilon_i),
\]

where the function \( W(\beta \epsilon_-, \epsilon_u, \epsilon_i) \) is a double integral which can be arranged such that

\[
W(\beta \epsilon_-, \epsilon_u, \epsilon_i) = e^{-\beta \epsilon_-} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} d\varphi \, \exp \left[ \frac{3\beta \epsilon_-}{4} g(r, \varphi, \epsilon_u) \right] \times I_0 \left[ \frac{3\beta \epsilon_-}{4} \sqrt{h(r, \varphi, \epsilon_u)} \right] \right\},
\]

where \( r = -\cos \vartheta \). The functions \( g(r, \varphi, \epsilon_u) \) and \( h(r, \varphi, \epsilon_u) \) are defined as

\[
g(r, \varphi, \epsilon_u) = 1 + r^2 + \epsilon_u \left( 1 - r^2 \right) \cos(2\varphi),
\]

\[
h(r, \varphi, \epsilon_u) = g^2 - 4 \left( 1 - \epsilon_u^2 \right) r^2.
\]

They are periodic of period \( \pi \) in the argument \( \varphi \). Furthermore, on the domain defined by \( 0 \leq r \leq 1 \) and \( 0 \leq \varphi \leq 2\pi \), the function \( g \) is bounded by \( 1 - \epsilon_u \leq g(r, \varphi, \epsilon_u) \leq 1 + \epsilon_u \). Consequently, the double integral that appears in equation (B9) is always larger than or equal to 1. The equality holds only when \( \alpha \) and/or \( \lambda \) is proportional to the identity matrix.

Expanding the integrand of (B9) about \( \beta \epsilon_- = 0 \) gives, upon integration, the following series:

\[
W(\beta \epsilon_-, \epsilon_u, \epsilon_i) \approx 1 + \frac{\beta^2 \epsilon_-^2}{40} p_1(\epsilon_u) p_1(\epsilon_i) + \frac{\beta^3 \epsilon_-^3}{340} p_2(\epsilon_u) p_2(\epsilon_i) + \frac{\beta^4 \epsilon_-^4}{4480} p_1^2(\epsilon_u) p_1^2(\epsilon_i) + \frac{\beta^5 \epsilon_-^5}{73920} p_1(\epsilon_u) p_1(\epsilon_i) p_2(\epsilon_u) p_2(\epsilon_i),
\]

where \( p_1(x) = (3x^2 + 1) \) and \( p_2(x) = (9x^2 - 1) \). We have found that this truncated, fifth-order expansion is accurate to within 2 per cent in the range \( 0 < \beta \epsilon_- \lesssim 1.5 \) and can be used efficiently in the computation of the group integral (19). Conditional probability distributions for the relative orientation can be derived in a straightforward way from the preceding results.

Noteworthy is the special case \( G = \text{SO}(2) \), in which the integral (B2) can be written in closed form,

\[
F(\alpha, \lambda) = e^{\delta_+} I_0 \left( \frac{1}{2} \alpha \epsilon_1 \lambda \right),
\]

where the parameter \( \epsilon_+ \) is now \( \epsilon_+ = (1/2) \text{tr} \lambda \).

### Appendix C: Potential of a Homogeneous Ellipsoid

For a homogeneous ellipsoid defined such that \( \rho(r) = \delta \rho \) if \( r^T (\mathbf{AA}^{-1}) r \leq 1 \), and zero otherwise, the potential at any exterior point \( r = (r_1, r_2, r_3) \) is (Kellog 1929; Chandrasekhar 1969)

\[
\Phi_i(r) = \pi G \delta \rho \left( C(\rho) - \sum_i b_i(\rho) r_i^2 \right),
\]

while the potential inside and on the boundary of the ellipsoid is

\[
\Phi_i(r) = \pi G \delta \rho \left( C(0) - \sum_i b_i(0) r_i^2 \right).
\]

\( \mathbf{AA}^{-1} \) is a positive definite matrix whose eigenvalues are the square of the principal axis lengths \( A_1 \geq A_2 \geq A_3 > 0 \). The variable \( \rho \) is defined as the algebraically largest root of the following cubic equation

\[
\frac{x^2}{A_1^2 + \rho} + \frac{y^2}{A_2^2 + \rho} + \frac{z^2}{A_3^2 + \rho} = 1.
\]

The functions \( C(\rho) \) and \( b_i(\rho) \) are given by

\[
C(\rho) = A_1 A_2 A_3 \int_0^\infty \frac{du}{\Delta(u)},
\]

\[
b_i(\rho) = A_1 A_2 A_3 \int_0^\infty \frac{du}{(A_1^2 + u) \Delta(u)},
\]

where \( \Delta(u) = \prod_{i=1}^3 \left( A_i^2 + u \right)^{1/2} \). These functions can be expressed in terms of the Legendre’s incomplete elliptic integrals of first and second kind (Binney & Tremaine 1987). The gravitational potential energy of this self-gravitating ellipsoid is

\[
E_{\text{pot}} = -\frac{3}{10} G M^2 \int_0^\infty \frac{du}{\Delta(u)}
\]

and is proportional to the potential at \( x = 0, C(0) \). Note also that Poisson’s equation implies \( \sum_i b_i(0) = 2 \). Equation (C5) corrects a typographical error in equation (24) of Eisenstein & Loeb (1995).

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